ON SCHRODINGER OPERATORS WITH DYNAMICALLY DEFINED POTENTIALS

MICHAEL GOLDSTEIN AND WILHELM SCHLAG

1. Introduction

The purpose of this article is to review some of the recent work on the operator

\[(H_x \psi)_n = -\psi_{n-1} - \psi_{n+1} + \lambda V(T^n x) \psi_n \]

on \(l^2(\mathbb{Z})\), where \(T : X \to X\) is an ergodic transformation on \((X, \nu)\) and \(V\) is a real-valued function. \(\lambda\) is a real parameter called coupling constant. Typically, \(X = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d\) with Lebesgue measure, and \(V\) will be a trigonometric polynomial or analytic. We shall focus on the papers \(\text{[GolSch1]}\) and \(\text{[GolSch2]}\) by the authors, as well as other work which was obtained jointly with Jean Bourgain. Our goal is to explain some of the methods and results from these references. Some of the material in this paper has not appeared elsewhere in print.

Even more specifically, we will be mostly concerned with the distribution of the eigenvalues of (1.1), both on finite intervals \([-N, N]\) as well as in the limit \(N \to \infty\) (and not so much with Anderson localization). In more technical terms, we are referring here to the integrated density of states or IDS. It is a nondecreasing, deterministic function \(N(E)\), and it is related to the Lyapunov exponent \(L(E)\) by means of the Thouless formula

\[L(E) = \int \log |E - E'| N(dE')\]

As usual, we set

\[L(E) = \lim_{n \to \infty} \frac{1}{n} \int_X \log ||M_n(x, E)|| \, \nu(dx)\]

where \(M_n\) are the transfer matrices

\[M_n(x, E) = \begin{bmatrix} V(T^k x) - E & -1 \\ 1 & 0 \end{bmatrix}\]

of (1.1), i.e., the column vectors of \(M_n\) are a fundamental system of the equation \(H_x \psi = E \psi\).

One of the most basic problems related to (1.1) concerns the positivity of the Lyapunov exponent \(L(E)\). More specifically, there are dynamical systems for which \(L(E)\) exhibits a “phase transition” from the region \(L(E) > 0\) to that where \(L(E) = 0\) when \(\lambda\) varies, and there are systems for which \(L(E) > 0\) for all values of \(\lambda \neq 0\). For instance, one expects that for the skew-shift \(T : \mathbb{T}^2 \to \mathbb{T}^2, T(x, y) = (x + \omega, y + x)\), the Lyapunov exponent is positive for all \(\lambda \neq 0\). A rigorous description of phase transitions for \(L(E)\) or the proof of the absence of such transitions is a primary objective in the study of (1.1).

In the study of (1.1) much attention has traditionally been given to the fine properties of the distribution of the eigenvalues of (1.1), i.e., the IDS. As observed by Avron and Simon \(\text{[AvrSim]}\), and by Craig and Simon \(\text{[CraSim]}\) the Thouless formula implies that the IDS is log-Hölder continuous. In \(\text{[GolSch1]}\) it was shown that for positive Lyapunov exponents the IDS is Hölder continuous, and their argument was refined in \(\text{[Bou2]}\) to show that the Hölder exponent remains bounded below by a positive constant when the Lyapunov exponent approaches zero. For the almost Mathieu potential \(V(\theta) = \cos(\theta)\), Sinai \(\text{[Sin]}\) and Bourgain \(\text{[Bou1]}\) observed perturbatively (i.e., when \(\lambda\) is large), that the IDS is Hölder \(\frac{1}{2} - \epsilon\) continuous for any \(\epsilon > 0\). Moreover,
2. LARGE DEVIATION THEOREMS

It was shown by Fürstenberg and Kesten [FurKes] that
\[ \lim_{n \to \infty} \frac{1}{n} \log \| M_n(x, E) \| = L(E) \]
for a.e. \( x \in X \). To quantify this convergence, set
\[ L_n(E) = \frac{1}{n} \int_X \log \| M_n(x, E) \| \, \nu(dx). \]

For certain types of dynamics and \( V \) the following large deviation theorems (or LDTs) are known to hold for some choice of \( 0 < \sigma, \tau < 1 \):
\[ \nu(\{x \in X : \| M_n(x, E) \| - nL_n(E) > n^{1-\sigma}\}) \leq C \exp(-n^{\tau}). \]

For definiteness, let \( X = \mathbb{T}, \, dv = dx \) the Lebesgue measure, and \( Tx = x + \omega \mod \mathbb{Z} \) with an irrational \( \omega \). Moreover, we shall assume that \( V : \mathbb{T} \to \mathbb{R} \) is analytic and nonconstant. The LDTs are known to hold in this case.

To motivate (2.1), consider first a commutative model case, namely
\[ u(x) = \sum_{k=1}^q \log|e(x) - e(k\omega)| \]
with \( \omega = \frac{\Lambda}{q} \) and \( e(x) = e^{2\pi ix} \). Then \( u(x) = \log|e(xq) - 1| \) and \( \int_{\mathbb{T}} u(x) \, dx = 0 \) so that for \( \lambda < 0 \)
\[ \text{mes} \left( \{x \in \mathbb{T} : u(x) < \lambda \} \right) = \text{mes} \left( \{x \in \mathbb{T} : |e(x) - 1| < e^\lambda \} \right) \]
which is of size \( e^\lambda \) (here mes stands for Lebesgue measure). In this model case, \( u(x + 1/q) = u(x) \).

Returning to \( u(x) = \log \| M_n(x, E) \| \), this exact invariance needs to be replaced by the almost invariance
\[ \sup_{x \in \mathbb{T}} |u(x) - u(x + k\omega)| \leq Ck \quad \text{for any} \quad k \geq 1. \]
The logarithm in our model case is a reasonable choice because of Riesz’s representation theorem for subharmonic functions (see Levin [Lev]) applied to the function \( u(z) = \log \| M_n(z, E) \| \):

Let \( u(z) \) be a subharmonic function on some domain \( \Omega \subset \mathbb{C} \). Then there exist a positive measure \( \mu \) (called the Riesz measure), finite on all compact sub-domains \( \Omega' \subset \Omega \) so that
\[ u(z) = \int_{\Omega'} \log|z - \zeta| \, \mu(d\zeta) + h(z) \quad \forall z \in \Omega' \]
where \( h \) is harmonic on \( \Omega' \).

If \( u \in C^2(\Omega) \) then subharmonicity is the same as \( \Delta u \geq 0 \) and \( \mu = \Delta u \). Moreover, in this case (2.4) is an instance of Green’s formula. The general case follows by taking limits. In what follows, we will also make use of the following bounds on \( \mu \) and \( h \): If (2.4) holds and \( |u| \leq K \) on \( \Omega' \), then
\[ \mu(\Omega^\prime) + \|h\|_{L^\infty(\Omega^\prime)} \lesssim K \]

\[ \text{This is defined to mean upper semicontinuous and satisfying the sub-mean value property.} \]
where $\Omega'' \subset \Omega'$ (as a compact sub-domain).

2.1. Cartan estimates. One way of deriving (2.11) from (2.3) and (2.4) is via Cartan’s estimate for subharmonic functions:

Given a finite positive measure $\mu$ on $\mathbb{C}$ and $H > 0$ there is a (possibly infinite) collection of disks $\{D(z_j, r_j)\}$ so that

$$\sum_j r_j < CH \text{ and } u(z) > -C\|\mu\| \log H \quad \forall z \in \mathbb{C} \setminus \bigcup_j D(z_j, r_j)$$

The most obvious example here is of course $\mu = n\delta_0$. More generally, for the measures $\mu = \sum_k \delta_{z_k}$ with some finite collection $\{z_k\}_k \subset \mathbb{C}$ of (not necessarily distinct) points, CARTAN’S theorem becomes a statement about polynomials which already captures all the main features. In order to prove (2.11) we will impose the Diophantine condition

$$\|n\omega\| \geq \frac{c}{n^a} \quad \forall n \geq 1$$

where $c = c(\omega) > 0$ and $a > 1$. Here $\| \cdot \|$ measures the distance to the nearest integer.

Proof of (2.1). Write

$$u(z) = \log \|M_u(z, E)\| = \int \log |z - \zeta| \mu(d\zeta) + h(z)$$

on some open rectangle $R$ which contains $[0,1]$. Then $0 \leq u(z) \leq n$ and thus $\mu(R') \lesssim n$ where $R' \subset R$ is a slightly smaller rectangle, as well as $\|h\|_{L^\infty(R')} \lesssim n$, see (2.5). Fix a small $\delta > 0$ and take $n$ large. Then there is a disk $D_0 = D(x_0, n^{-2\delta})$ with the property that $\mu(D_0) \lesssim n^{1-2\delta}$. Write

$$u(z) = u_1(z) + u_2(z) = \int_{D_0} \log |z - \zeta| \mu(d\zeta) + \int_{\mathbb{C} \setminus D_0} \log |z - \zeta| \mu(d\zeta)$$

Set $D_1 = D(x_0, n^{-3\delta})$. Then

$$|u_2(z) - u_2(z')| \lesssim n^{1-\delta} \quad \forall z, z' \in D_1$$

Cartan’s theorem applied to $u_1(z)$ yields disks $\{D(z_j, r_j)\}_j$ with $\sum_j r_j \lesssim \exp(-2n^\delta)$ and so that

$$u_1(z) \gtrsim -n^{1-\delta} \quad \forall z \in \mathbb{C} \setminus \bigcup_j D(z_j, r_j)$$

Since also $u_1 \leq 0$ on $D_1$ as well as $|h(z) - h(z')| \lesssim n|z - z'|$, it follows that

$$|u(z) - u(z')| \lesssim n^{1-\delta} \quad \forall z, z' \in D_1 \setminus \bigcup_j D(z_j, r_j)$$

From the Diophantine property (with $a < 2$), for any $x, x' \in \mathbb{T}$ there are positive integers $k, k' \lesssim n^{4\delta}$ such that

$$x + k\omega, x' + k'\omega \in D_1 \mod \mathbb{Z}$$

In order to avoid the Cartan disks $\bigcup_j D(z_j, r_j)$ we need to remove a set $B \subset \mathbb{T}$ of measure $\lesssim \exp(-n^\delta)$. Then from the almost invariance (2.3), for any $x, x' \in \mathbb{T} \setminus B$,

$$|u(x) - u(x')| \lesssim n^{4\delta} + n^{1-\delta} \lesssim n^{1-\delta}$$

This implies (2.1) with $\sigma = \tau = \delta$ and we are done. \hfill $\square$

This proof generalizes to other types of dynamics as well. For example, let $Tx = x + \omega \mod \mathbb{Z}^d$, $d \geq 2$, be a higher-dimensional shift.

---

5 Exact numerical values of the constants $C$ are known, see Levin. Also, one can replace $\sum r_j$ with $\sum r_j^\varepsilon$ for any $\varepsilon > 0$, which implies that $\{u = -\infty\}$ has Hausdorff dimension zero.

6 The upper bound of 2 here is more of a cosmetic nature.

7 Recall that $u(x) = \log \|M_u(x, E)\|$ is a one-periodic function.
Definition 2.1. Let $0 < H < 1$. For any subset $\mathcal{B} \subset \mathbb{C}$ we say that $\mathcal{B} \in \text{Car}_1(H)$ if $\mathcal{B} \subset \bigcup_j D(z_j, r_j)$ with
\begin{equation}
\sum_j r_j \leq C_0 H.
\end{equation}
If $d$ is a positive integer greater than one and $\mathcal{B} \subset \mathbb{C}^d$ we define inductively that $\mathcal{B} \in \text{Car}_d(H)$ if there exists some $\mathcal{B}_0 \in \text{Car}_{d-1}(H)$ so that
\[ \mathcal{B} = \{ (z_1, z_2, \ldots, z_d) : (z_2, \ldots, z_d) \in \mathcal{B}_0 \text{ or } z_1 \in \mathcal{B} (z_2, \ldots, z_d) \text{ for some } \mathcal{B} (z_2, \ldots, z_d) \in \text{Car}_1(H) \}. \]
We refer to the sets in $\text{Car}_d(H)$ for any $d$ and $H$ collectively as Cartan sets.

Using the following theorem from [GoSch1], the previous proof of (2.1) easily generalizes. We state the case $d = 2$, with $d > 2$ being similar (see also [Sch1]).

Theorem 2.2. Let $u$ be a continuous function on $D(0,2) \times D(0,2) \subset \mathbb{C}^2$ so that $|u| \leq 1$. Suppose further that
\[ \begin{cases}
z_1 \mapsto u(z_1, z_2) & \text{is subharmonic for each } z_2 \in D(0,2) \\
z_2 \mapsto u(z_1, z_2) & \text{is subharmonic for each } z_1 \in D(0,2).
\end{cases} \]
Fix some $\gamma \in (0, \frac{1}{2})$. Given $r \in (0,1)$ there exists a polydisk $\Pi = D(x_{01}, r^{1-\gamma}) \times D(x_{02}, r) \subset D(0,1) \times D(0,1)$ with $x_{01}, x_{02} \in [-1, 1]$ and a set $\mathcal{B} \in \text{Car}_2(H)$ so that
\begin{align}
|u(z_1, z_2) - u(z_1', z_2')| &< C_\gamma r^{1-2\gamma} \log \frac{1}{r} \quad \text{for all } (z_1, z_2), (z_1', z_2') \in \Pi \setminus \mathcal{B} \\
H &\equiv \exp(-r^{-\gamma}).
\end{align}

The point of this theorem is that it takes the place of (2.12) in the previous proof.

2.2. Fourier series. An alternative approach to (2.1) is based on Fourier series. Indeed, one writes
\[ u(x) - \langle u \rangle = \frac{1}{k} \sum_{j=1}^k u(x + j\omega) - \langle u \rangle + O(k) = \sum_{\nu \neq 0} \hat{u}(\nu) e(x\nu) \frac{1}{k} \sum_{j=1}^k e(j\nu\omega) + O(k) \]
Then one has that
\[ \left| \frac{1}{k} \sum_{j=1}^k e(j\nu\omega) \right| \lesssim \min(1, k^{-1} \| \nu\omega \|^{-1}) \]
for all $\nu \geq 1$. Also, it follows from (2.14) that $|\hat{u}(\nu)| \lesssim n|\nu|^{-1}$ which in turn implies that
\[ |u(x) - \langle u \rangle| \lesssim \frac{1}{k} \sum_{j=1}^k \sum_{|\nu| > K} \hat{u}(\nu) e(\nu(x + k\omega)) + \sum_{0 < |\nu| \leq K} n|\nu|^{-1} \min(1, k^{-1} \| \nu\omega \|^{-1}) \]
Clearly,
\[ \left\| \frac{1}{k} \sum_{j=1}^k \sum_{|\nu| > K} \hat{u}(\nu) e(\nu(x + k\omega)) \right\|_{L^2_p} \lesssim n^{K^{-1/2}} \]
Taking $K = e^n$ it follows from the Diophantine condition that
\[ \sum_{0 < |\nu| \leq K} n|\nu|^{-1} \min(1, k^{-1} \| \nu\omega \|^{-1}) \lesssim nk^{-\frac{1}{2}} \log K \lesssim n^{1+\sigma} k^{-\frac{1}{2}} \]
Choosing $\tau > 0$ small and $k = n^\tau$, say, yields (2.1).

For applications related to the study of fine properties of the IDS it turns out to be important to obtain sharp versions of (2.1). The commutative model example suggests that the optimal relation is $0 \leq \sigma = \tau < 1$. This is indeed the case, see Section 3 below.

This proof also generalized to higher-dimensional tori, see [BonGo] as well as [Bon2]. S. Klein [Kle] has removed the analyticity assumption and obtained estimates as in (2.1) for the Gevrey classes by means of Fourier methods (using higher-order Féjer kernels) and suitable truncations of the Fourier series.
2.3. Other dynamics. The arguments which we have just presented do not depend on positive Lyapunov exponents. The situation is very different for the skew-shift defined by $T(x, y) = (x + n y + n(n - 1) \omega / 2, y + n \omega)$ modulo $\mathbb{Z}^2$. Due to the presence of $n y$ in the first coordinate we are faced with the problem that the Riesz mass of the subharmonic extensions of $\log \|M_n(x, y, E)\|$ is now of size $n^2$, at least if we consider the extensions to a fixed neighborhood\(^6\) of $\mathbb{T}^2 \subset \mathbb{C}^2$. Indeed, in this case $|M_n(x, y + i \varepsilon, E)|$ behaves like a product $\prod_{j=0}^{\infty} e^{j \varepsilon}$ which is of size $e^{C \varepsilon n^2}$. Thus, in this context neither of the two methods discussed so far lead to a bound of the form \((2.1)\) for the skew-shift (the problem is that these methods only gain a factor of $n^{-\delta}$ over the Riesz mass as far as the deviations are concerned – here we would therefore get $n^{2-\delta}$ for the deviations which is useless).

In [BonGolSch] a LDT is proved for the skew-shift but for large disorders. This refers to the fact that the potential has to be of the form $\lambda V$ for large $\lambda$. The method in [BonGolSch] proceeds by induction over the scale $n$, and the first stage requires large $\lambda$. The inductive step is realized by means of the avalanche principle (see the following section) which is a purely deterministic statement about products of $2 \times 2$ matrices. Moreover, the analytic difficulty of having $n^2$ Riesz masses is circumvented by the following splitting lemma from [BonGolSch] (see also [Bon2]):

**Lemma 2.3.** Suppose $u$ is subharmonic on $A_\rho$ (a $\rho$-neighborhood of $\mathbb{T}$), with $\sup_{A_\rho} |u| \leq N$. Furthermore, assume that $u = u_0 + u_1$, where

\begin{equation}
\|u_0 - \langle u_0 \rangle\|_{L^n(\mathbb{T})} \leq \varepsilon_0 \quad \text{and} \quad \|u_1\|_{L^1(\mathbb{T})} \leq \varepsilon_1.
\end{equation}

Then for some constant $C_\rho$ depending only on $\rho$,

\begin{equation}
\|u\|_{BMO(\mathbb{T})} \leq C_\rho \left( \varepsilon_0 + \sqrt{N \varepsilon_1} \right).
\end{equation}

To apply this lemma one uses the avalanche principle to generate the splitting into $u_0$ and $u_1$ with an exponentially small $\varepsilon_1 \sim e^{-n}$. This allows for Riesz masses $N$ which are polynomially large, say $N = n^C$ as is the case for the skew-shift.

Finally, and in a very different vein, we would like to mention that LDTs have also been established for the doubling dynamics $x \mapsto 2x \mod 1$ and for very small disorder $\lambda > 0$ in [BonSch]. The latter is needed in order to apply the Figotin-Pastur formula, see [FigPas].

Generally speaking, it remains an open problem to prove LDTs for Schrödinger cocycles with potentials of the type $\lambda V(T^n x)$ for general classes of dynamics $T$, disorder $\lambda$, as well as wider classes of potentials $V$.

3. Positive Lyapunov exponents

As we have already mentioned of the central problem concerning \((1.1)\) is to decided whether or not $L(E) > 0$. In the case of random i.i.d. potentials this was established by Fürstenberg [Fur]. In case of quasi-periodic potentials, the well-known Herman’s method [Her] establishes this positivity for large disorders provided the potential function $V$ is a trigonometric polynomial. Sorets and Spencer extended this to analytic $V$. Here we present a different approach, which is based on the following avalanche principle (AP) from [GolSch1] (for this version which does not assume that the matrices belong to $SL(2, \mathbb{R})$ see [GolSch2]).

**Proposition 3.1.** Let $A_1, \ldots, A_n$ be a sequence of $2 \times 2$–matrices whose determinants satisfy

\begin{equation}
\max_{1 \leq j \leq n} |\det A_j| \leq 1.
\end{equation}

Suppose that

\begin{equation}
\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n \quad \text{and}
\end{equation}

\begin{equation}
\max_{1 \leq j \leq n} \left[ \log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1} A_j\| \right] < \frac{1}{2} \log \mu.
\end{equation}

\(^6\)Shrinking to a neighborhood to size $O(n^{-1})$ reduces the Riesz-mass to $\lesssim n$, but then there is a price to pay for the small diameter of the neighborhood.
Then

\[(3.4) \quad \left| \log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C \frac{N}{\mu}
\]

with some absolute constant \(C\).

The meaning of (3.3) is that adjacent matrices do not cancel pairwise, whereas (3.2) insures that each matrix is sufficiently large. The conclusion is that the entire product has to be large with the very precise difference bound from (3.4).

As an application of this principle, let us study the rate of convergence of \(L_N(E)\) to \(L(E)\) for the operator (1.1). We will assume that \(L(E) > \gamma > 0\) for some \(E \in \mathbb{R}\). Furthermore, we shall assume that there is a LDT of the form (2.1). We shall make no other assumptions on the dynamics \(T\). Given a large integer \(n\), define \(k = C_0(\log N)^{\frac{1}{2}}\) (here \(\tau\) is as in (2.1)). Then

\[
M_N(x, E) = M_{k'}(T^{k'+\ell}x, E)M_k(T^{\ell}x, E) \prod_{j=1}^{1} M_k(T^{(j-1)k}x, E)
\]

where \(k/2 \leq k', k'' < k\). In view of (2.1) there exists a set \(B \subset X\) of measure \(\leq N^{-10}\), say, so that for all \(x \in X \setminus B\) we can apply the AP to this product. This requires making \(C_0 = C_0(\gamma)\) large. In order to check (3.3) one uses the fact that \(L_N(E) \to L(E)\) as \(n \to \infty\). We can now average (3.4) over \(x \in X\) which yields

\[
(3.5) \quad |L_N(E) - 2L_{2k}(E) + L_k(E)| \lesssim \frac{\log N}{N}
\]

Applying the same reasoning with \(2N\) and the same choice of \(k\) yields

\[
|L_{2N}(E) - 2L_{2k}(E) + L_k(E)| \lesssim \frac{\log N}{N}
\]

and thus also

\[
|L_{2N}(E) - L_N(E)| \lesssim \frac{\log N}{N}
\]

Passing to the limit therefore implies that

\[
0 \leq L_N(E) - L(E) \lesssim \frac{\log N}{N}.
\]

This can be further improved to

\[
0 \leq L_N(E) - L(E) \lesssim N^{-1}
\]

Moreover, this convergence holds uniformly in the energy for all intervals \(I \subset \mathbb{R}\) for which \(\inf_{E \in I} L(E) > \gamma > 0\), see [GolSch1].

The AP can also be used to establish positive Lyapunov exponents. Indeed, let \(V : \mathbb{T}^d \to \mathbb{R}\) be an analytic potential and \(T : \mathbb{T}^d \to \mathbb{T}^d\) be ergodic. The matrix

\[
M_n(x, \lambda, E) = \prod_{j=n}^{1} \begin{bmatrix} \lambda V(T^jx) - E & -1 \\ 1 & 0 \end{bmatrix}
\]

denotes the transfer matrix of the equation (1.1) where the potential is now written as \(\lambda V(T^n x)\). As before,

\[
L_n(\lambda, E) = \frac{1}{n} \int_{\mathbb{T}^d} \log \|M_n(x, \lambda, E)\| \ dx
\]

and \(L(\lambda, E) = \lim_{n \to \infty} L_n(\lambda, E)\) exists. Finally, let \(S(\lambda, E)\) be a number satisfying

\[
(3.6) \quad S(\lambda, E) = \sup_{n \geq 1} \sup_{x \in \mathbb{T}^d} \frac{1}{n} \log \|M_n(x, \lambda, E)\|.
\]

Then the following is shown in [GolSch1]: If the weak large deviation theorem (with some \(\sigma > 0\))

\[
(3.7) \quad \int_{\mathbb{T}^d} \left| \frac{1}{n} \log \|M_n(x, \lambda, E)\| - L_n(\lambda, E) \right| \ dx \leq C S(\lambda, E)n^{-\sigma}
\]
holds for all \( n = 1, 2, \ldots, \) then
\[
\inf_E L(\lambda, E) > 0 \quad \text{for all} \quad \lambda > \lambda_0(V, d, \sigma).
\]

This is proved inductively, the main step being described by the following lemma:

**Lemma 3.2.** Suppose that (3.1) holds for all \( n \) with some choice of \( \sigma > 0 \). Then there exists a positive integer \( \ell_0 = \ell_0(\sigma) \) such that if
\[
(3.8) \quad L_\ell > S(E, \lambda)\ell^{-\sigma/4} \quad \text{and} \quad L_\ell(E, \lambda) - L_{2\ell}(E, \lambda) < \frac{L_\ell(E, \lambda)}{8}
\]
for some \( \ell \geq \ell_0 \), then \( L(E, \lambda) > L_\ell(E, \lambda)/2 \).

This lemma does not require any further information about \( T \) or \( V \) other than (3.7). On the other hand, to insure that the conditions of this lemma are met, one chooses \( \lambda \) large using analyticity of \( V \). It would be interesting to apply this method to non-analytic \( V \) which satisfy some natural non-degeneracy assumption (in particular, one would need to establish (3.7)).

K. Bjerklöv recently used this result to prove positive exponents for some interval of energies for his class of potentials which exhibit mixed behavior (i.e., both zero and positive exponents).

### 4. Regularity of the IDS

Let us now assume that we have the following sharp LDTs for \( u_n(x) = \log \| M_n(x, E) \| \):
\[
(4.1) \quad \nu(\{ x \in X : |u_n(x) - nL_n(E)| > n\delta \}) \leq \exp \left( -c(\delta)n + C(\log n)^A \right).
\]

where \( c(\delta) > 0 \) and \( C, A \) are some constants. For the case of \( X = \mathbb{T} \) and the shift by a Diophantine \( \omega \), such estimates were obtained in [GolSch1] with \( c(\delta) = \delta^2 \). Assuming positive Lyapunov exponents \( L(E) > \gamma \) one can further show that \( c(\delta) = c(\gamma)\delta \) which shows that one can take \( 0 \leq \sigma < 1 \) in (4.1) (the latter is done via the AP).

Then by the arguments of the previous section we obtain the following stronger version of (4.1): \( \nu(\{ x \in X : |u_n(x) - nL_n(E)| > n\delta \}) \leq \exp \left( -c(\delta)n + C(\log n)^A \right) \). This is due to the fact that we can break up \( M_N \) into products of matrices of size \( k = [C\log N] \) when applying the AP. Therefore,
\[
|L_N(E) - 2L_{2k}(E) + L_k(E)| \lesssim \frac{\log N}{N}
\]

Consequently, choosing \( N = |E - E'|^{-\alpha} \) for some \( 0 < \alpha < 1 \), we deduce that \( L(E) \) and therefore also that the IDS (by (1.2)) are Hölder continuous.

This approach to the regularity of \( L(E) \) was subsequently modified by other authors. For example, Bourgain and Jitomirskaya use this very approach from [GolSch1] to show that \( L(E, \omega) \) is jointly continuous away from rational \( \omega \) for the case of the shift on \( T \). Their argument is based on sharp large deviation theorems, the avalanche principle, as well as the difference relation (3.7).

---

5We can consider these estimates as a “black box” without specifying \( T \) or \( V \) further

6This means that \( \|n\omega\| \geq \frac{c(\omega)}{n(\log n)^2} \) with \( a > 1 \)
5. Eigenvalues, localization, and the zeros of \( f_N(z, \omega, E) \) in \( z \)

In \cite{GolSch2} a different approach to the regularity of the IDS was developed which allows us to obtain a lower bound on the Hölder exponent by non-perturbative methods. In fact, for the almost Mathieu model with \( \lambda > 2 \) it is shown that the Hölder regularity is \( \frac{1}{2} - \varepsilon \) for any \( \varepsilon > 0 \). Moreover, off a set of Hausdorff dimension zero the IDS is \( 1 - \varepsilon \) Hölder regular for any \( \varepsilon > 0 \) (the latter requires the removal of a set of \( \omega \) of measure zero). Similar results are obtained for other potentials assuming positive Lyapunov exponents as well as the strong Diophantine condition \( (2.11) \).

It is well-known that in the case of the almost Mathieu model the exponent \( \frac{1}{2} \) cannot be improved. For large disorders this was observed by Sinai \cite{Sin1}, whereas Puig \cite{Pu1, Pu2} has obtained this non-perturbatively. On the other hand, Bourgain \cite{Bou1} has shown for the almost Mathieu operator that the Hölder exponent is no worse than \( \frac{1}{2} - \varepsilon \) for very large disorders.

In the almost Mathieu case the optimality of \( \frac{1}{2} \) is intimately connected with the Cantor structure of the spectrum. In fact, it is at the gap edges that one encounters loss of the Lipschitz behavior. For more general potentials as in \cite{GolSch2} this connection is not clear and it would certainly be of great interest to elucidate the connection between gaps and the regularity of the IDS further.

We now set out to describe some of the basic ingredients of \cite{GolSch2}.

5.1. Large deviation theorems for the entries. Recall that

\[
M_n(x, E) = \begin{bmatrix}
  f_n(x, E) & -f_{n-1}(Tx, E) \\
  f_{n-1}(x, E) & -f_{n-2}(Tx, E)
\end{bmatrix}
\]

where

\[
f_n(x, E) = \det \begin{bmatrix}
  v(1, x) - E & -1 & 0 & 0 & \ldots & 0 \\
  -1 & v(2, x) - E & -1 & 0 & 0 & \ldots & 0 \\
  0 & -1 & v(3, x) - E & -1 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & -1 & v(n, x) - E
\end{bmatrix}
\]

(5.11)

It is customary to denote the matrix on the right-hand side as \( H_{[1,n]}(x) - E \) so that one as \( f_n(x, E) = \det(H_{[1,n]}(x) - E) \). The following result is proved in \cite{GolSch2}. Henceforth, we shall assume that the Lyapunov exponents are positive as well as that \( \omega \) is strongly Diophantine.

**Lemma 5.1.** There exist constants \( A \) and \( C \) depending on \( \omega \) and the potential \( V \), so that for every \( n \geq 1 \)

\[
\left| \int_0^1 \log |\det(H_{[1,n]}(x, \omega) - E)| dx - n L_n(\omega, E) \right| \leq C
\]

(5.12)

\[
\| \log |\det(H_{[1,n]}(x, \omega) - E)| \|_{BMO} \leq C (\log n)^A.
\]

(5.13)

In particular, for every \( n \geq 1 \),

\[
\text{mes} \left[ x \in \mathbb{T} \mid |\log |\det(H_{[1,n]}(x, \omega) - E)| - n L_n(\omega, E)| > H \right] \leq C \exp \left( -\frac{cH}{(\log n)^A} \right)
\]

(5.14)

for any \( H > (\log n)^A \). Moreover, the set on the left-hand side is contained in at most \( \lesssim n \) intervals each of which does not exceed the bound stated in \( (5.14) \) in length.

The point here is of course that the entries of \( M_n \) satisfy the same bound as \( M_n \) itself. One basic step in the proof of this lemma is to show that

\[
\int_\mathbb{T} \log |f_n(x, E)| dx \geq n L_n(E) - C n^\sigma
\]

(5.15)

with some \( \sigma < 1 \). Then combine this with a uniform upper bound of the form (see \cite{GolSch1})

\[
\sup_{x \in \mathbb{T}} \log |f_n(x, E)| \leq \sup_{x \in \mathbb{T}} \|M_n(x, E)\| \leq n L_n(E) + C n^\sigma
\]
to conclude that
\[ \log |f_n(x, E)| = \int \log |f_n(x, E)| \, dx + u_1(x) \quad \text{with} \quad \|u_1\|_1 \lesssim n^\sigma \]

Thus, by Lemma 2.3 we conclude that
\[ \| \log |f_n(x, E)| \|_{BMO} \lesssim n^{(1 + \sigma)/2} \]

which beats the trivial bound of \( n \). We used here that the Riesz mass of the subharmonic extension of \( \log |f_n(x, E)| \) to a neighborhood of the circle is \( \lesssim n \).

The John-Nirenberg inequality therefore implies the large deviation theorem
\[ \text{mes} \left( \{ x \in \mathbb{T} : |\log |f_n(x, E)| - nL_n(E)| > n^{1-\delta} \} \right) \lesssim \exp(-n^{\delta}) \]

provided \( \delta > 0 \) is sufficiently small.

This is of course considerably weaker than Lemma 5.1. To improve on it, we apply the avalanche principle to the product
\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_n(x, \omega, E) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f_n(x, E) & 0 \\ 0 & 0 \end{bmatrix} \cdot \]

Note that the AP does not require \( SL(2, \mathbb{R}) \) matrices, but rather (3.1) which holds here. Write \( n = \ell_1 + (m-2)\ell + \ell_m \) where \( \ell \asymp (\log n)^c, \ell_1 \asymp \ell_n \asymp \ell \), and set \( s_1 = 0, s_j = \ell_1 + (j-2)\ell \) for \( 2 \leq j \leq m \). Hence,
\[ \begin{bmatrix} f_n(x, E) & 0 \\ 0 & 0 \end{bmatrix} = \prod_{j=m} A_j(x) \]

where \( A_j(x) = M_\ell x + s_j \omega, \) for \( 2 \leq j \leq m-1, \) and \( A_1(x) = M_{\ell_1}(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_m(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{\ell_m}(x + s_m \omega) \).

One checks easily from (5.14) that the conditions (3.2) and (3.3) hold up to a set of \( n^{-100} \), say. Hence, by Proposition 6.1
\[ \log |f_n(x, E)| = -\sum_{j=2}^{m-1} \log \| A_j(x) \| + \sum_{j=1}^{m-1} \log \| (A_{j+1} A_j)(x) \| + O \left( \frac{1}{n} \right) . \]

We now invoke the following LDT for sums of shifts of subharmonic functions, see Theorem 3.8 in [GolSch1]. For any subharmonic function \( u \) on a neighborhood of \( \mathbb{T} \) with bounded Riesz mass and harmonic part
\[ \text{mes} \left[ x \in \mathbb{T} : \{ \sum_{k=1}^n u(x - k\omega) - n\langle u \rangle \} > \delta n \right] < \exp(-c\delta n + r_n) \]

where \( r_n \lesssim (\log n)^A. \) The sums in (5.14) involve shifts by \( \ell \omega \) rather than \( \omega. \) In order to overcome this, note that we can take \( \ell_n > 2\ell, \) say. Repeating the argument that lead to (5.17) \( \ell - 1 \) times with the length of \( A_1 \) increasing by one and that of \( A_m \) decreasing by one, respectively, at each step leads to
\[ \log |f_n(x, \omega, E)| = -\frac{2}{\ell} \sum_{k=0}^{\ell-1} \sum_{j=2}^{m-1} \log \| A_j(x + k\omega) \| + \frac{1}{\ell} \sum_{k=0}^{\ell-1} \sum_{j=2}^{m-2} \log \| (A_{j+1} A_j)(x + k\omega) \| + \frac{1}{\ell} \sum_{k=0}^{\ell-1} u_k(x) + O \left( \frac{1}{n} \right) \]

(5.19)
\[ = -\frac{1}{\ell} \sum_{j=\ell}^{m-1} \log \| M_\ell(x + j\omega) \| + \frac{1}{\ell} \sum_{j=\ell}^{m-1} \log \| M_2\ell(x + j\omega) \| + \frac{1}{\ell} \sum_{k=0}^{\ell-1} u_k(x) + O \left( \frac{1}{n} \right) . \]

The functions \( u_k \) compensate for omitting the terms \( j = 1 \) and \( j = m - 1 \) when summing \( \log \| A_{j+1} A_j \|. \) They are subharmonic, with Riesz mass and harmonic part bounded by \( (\log n)^c. \) Estimating the sums involving \( M_\ell \) and \( M_2\ell \) by means of (6.15), and the sums involving \( u_k \) directly by means of Cartan’s bound shows that there exists \( \mathcal{B} \subset \mathbb{T} \) of measure \( \leq \exp(-\log N)^c, \) so that for all \( x \in \mathbb{T} \setminus \mathcal{B}, \)
\[ \left| \log |f_n(x, \omega, E)| - \langle \log |f_n(x, \omega, E)| \rangle \right| \leq (\log n)^{2c_0}. \]

Thus,
\[ \log |f_n(x, \omega, E)| = u_0(x) + u_1(x) , \]
where
\[ \|u_0 - \langle \log |f_n(\cdot, E)| \rangle \|_{L^\infty(T)} \leq (\log n)^{2C_0}, \]
and
\[ \|u_1 - \langle \log |f_n(\cdot, E)| \rangle \|_{L^1(T)} \lesssim \| \log |f_n(\cdot, E)| \|_{L^2(T)} \sqrt{\text{mes}(B)} \]
\[ \lesssim n \cdot \sqrt{\text{mes}(B)} \lesssim \exp\left( -\frac{1}{4} (\log n)^{C_0} \right). \]

Applying Lemma 2.3 one now obtains that
\[ \| \log |f_n(x, \omega, E)| \|_{\text{BMO}(T)} \leq C \left( (\log n)^{2C_0+1} + \sqrt{n \cdot \exp\left( -\frac{1}{4} (\log n)^{C_0} \right)} \right) \]
\[ \leq C (\log n)^{2C_0+1}, \]
as claimed.

It therefore remains to obtain (5.15). For this, as well as other details of Lemma 5.1 we refer the reader to Section 2 of [GolSch2].

We remark that (5.19) illustrates how the AP allows us to write the determinants \( f_n(x, E) \) as rational functions which are composed of products of shifts of “short” functions (more precisely, of functions with small Riesz mass). This cannot be done for all \( x \) (because of the bad sets in the LDTs) and also leads to certain small errors. This approximate factorization is one of the basic tools of [GolSch2].

5.2. Uniform upper bounds and zeros of determinants. The following result based on Lemma 5.1 improves on these uniform upper bounds. Uniform upper bounds on the norm of the monodromy matrices in terms of \( L(E) \) were found in [BonGol], [GolSch1]. The \( \log N)^A \) error obtained in [GolSch2] (rather than \( N^\alpha \), say, as in [BonGol] and [GolSch1]) is crucial for the study of the fine properties of the integrated density of states.

Lemma 5.2. Let \( \omega \) be as in (2.11). Assume \( L(\omega, E) > 0 \). Then for all large integers \( N \),
\[ \sup_{x \in T} \log \| M_N(x, \omega, E) \| \leq NL_N(\omega, E) + C(\log N)^A, \]
for some constants \( C \) and \( A \).

Proof. We only consider \( x \) and suppress \( \omega \) and \( E \) from most of the notation. Take \( \ell \asymp (\log N)^A \). Write \( N = (n-1)\ell + r, \ell \leq r < 2\ell \) and correspondingly
\[ M_N(x) = M_r(x + (n-1)\ell \omega) \prod_{j=n-2}^0 M_\ell(x + j\ell \omega). \]
The avalanche principle and the LDT (4.1) imply that for every small \( y \) there exists \( B_y \subset \mathbb{T} \) so that \( \text{mes}(B_y) < N^{-100} \) and such that for \( x \in [0,1) \setminus B_y \),
\[ \log \| M_N(x + iy) \| = \sum_{j=0}^{n-3} \log \| M_{2\ell}(x + j\ell \omega + iy) \| - \sum_{j=1}^{n-2} \log \| M_{\ell}(x + j\ell \omega + iy) \| \]
\[ + \log \| M_{r}(x + (n-1)\ell \omega) M_{\ell}(x + (n-2)\ell \omega) \| + O(1) \]
\[ = \sum_{j=0}^{n-3} \log \| M_{2\ell}(x + j\ell \omega + iy) \| - \sum_{j=1}^{n-2} \log \| M_{\ell}(x + j\ell \omega + iy) \| + O(\ell). \]
Combining the elementary almost invariance property
\[ \log \| M_N(x + iy) \| = \ell^{-1} \sum_{0 \leq j \leq \ell-1} \log \| M_N(x + j\omega + iy) \| + O(\ell) \]
with (5.21) yields
\[
\log \| M_N(x + iy) \| = \ell^{-1} \sum_{0 \leq j < N} \log \| M_{2\ell}(x + j\omega + iy) \|
\]
(5.22)
\[-\ell^{-1} \sum_{0 \leq j < N} \log \| M_{\ell}(x + j\omega + iy) \| + O(\ell) ,
\]
for any \( x \in [0, 1] \setminus B'_y \), where \( \text{mes } B'_y < N^{-9} \). Integrating (5.22) over \( x \) shows that
(5.23)
\[ L_N(y, E) = 2L_{2\ell}(y, E) - L_{\ell}(y, E) + O(\ell/N) . \]
This identity is formula (5.3) in [GolSch1] (with \( y = 0 \)). Since the Lyapunov exponents are Lipschitz in \( y \), the sub-mean value property of subharmonic functions on the disk \( D(x, 0; \delta) \) with \( \delta = N^{-1} \) in conjunction with (5.22) and (5.23) implies that, for every \( x \in \Gamma \),
\[
\log \| M_N(x) \| - \int_0^1 \log \| M_N(\xi) \| d\xi
\]
(5.24)
\[ \leq \iint_{D(x, 0; \delta)} \left( \sum_{0 \leq j < N} u(\xi + j\omega + i\eta) - N\langle u(\cdot + i\eta) \rangle \right) d\xi d\eta
\]
\[-\iint_{D(x, 0; \delta)} \left( \sum_{0 \leq j < N} v(\xi + j\omega + i\eta) - N\langle v(\cdot + i\eta) \rangle \right) d\xi d\eta + O(\ell) ,
\]
where \( \iint_{D(x, 0; \delta)} \) denotes the average over the disk,
\[ u(\xi + i\eta) := \ell^{-1} \log \| M_{2\ell}(\xi + i\eta) \| \quad \text{and} \quad v(\xi + i\eta) := \ell^{-1} \log \| M_{\ell}(\xi + i\eta) \| , \]
and \( \langle \cdot \rangle \) denotes averages over the real line. The lemma now follows easily from (5.24). \( \Box \)

The first application of this estimate is as follows:

**Lemma 5.3.** Let \( \omega \) satisfy (2.11). Then for any \( x_0 \in \Gamma \), \( E_0 \in \mathbb{R} \) one has
(5.25)
\[ \# \{ E \in \mathbb{R} : f_N(e(x_0), \omega, E) = 0, \ |E - E_0| < \exp(-\langle \log N \rangle^A) \} \leq \langle \log N \rangle^{A_1} \]
(5.26)
\[ \# \{ z \in \mathbb{C} : f_N(z, \omega, E_0) = 0, \ |z - e(x_0)| < N^{-1} \} \leq \langle \log N \rangle^{A_1} \]
for all sufficiently large \( N \).

**Proof.** It follows from Lemma 5.1 that
\[ \sup \left\{ \log |f_N(e(x), \omega, E)| : x \in \Gamma, \ E \in \mathbb{C}, \ |E - E_1| < \exp(-\langle \log N \rangle^A) \right\} \]
\[ \leq NL_N(\omega, E_1) + (\log N)^B \]
for any \( E_1 \). Due to the large deviation theorem, there exist \( x_1, E_1 \) such that \( |x_0 - x_1| < \exp(-\langle \log N \rangle^{2A}) \), \( |E_0 - E_1| < \exp(-\langle \log N \rangle^{2A}) \) so that
\[ \log |f_N(e(x_1), \omega, E_1)| > NL_N(\omega, E_1) - (\log N)^{4A} . \]
Due to Jensen’s well-known formula, see (5.71) below,
\[ \# \{ E : f_N(e(x_1), \omega, E) = 0, \ |E - E_1| < \exp(-\langle \log N \rangle^A) \} \leq \langle \log N \rangle^{C_1} . \]
Since \( \| H_N^{(D)}(x_0, \omega) - H_N^{(D)}(x_1, \omega) \| \leq \exp(-\langle \log N \rangle^{2A}) \) and since \( H_N^{(D)}(x_0, \omega) \) is self adjoint one has
\[ \# \{ E : f_N(e(x_0), \omega, E) = 0, \ |E - E_0| < \exp(-\langle \log N \rangle^{2A}) \} \leq \# \{ E : f_N(e(x_1), \omega, E) = 0, \ |E - E_1| < \exp(-\langle \log N \rangle^A) \} \leq \langle \log N \rangle^{C_1} . \]
That proves (5.25). The proof of (5.20) similar. \( \Box \)
These estimates of the local number of zeros of the determinants $f_N$ allows one to factorize $f_N$ in each neighborhood of size $\exp(-(\log N)^A)$ using the Weierstrass preparation theorem, with a polynomial factor of degree at most $(\log N)^C$. For example, the following is proved in [GolSch2], see Section 6.

**Proposition 5.4.** Given $z_0 \in A_{\rho_0/2}$, $E_0 \in \mathbb{C}$, and $\omega_0$ as in (2.11), there exist a polynomial

$$P_N(z, \omega, E) = z^k + a_{k-1}(\omega, E)z^{k-1} + \cdots + a_0(E, \omega)$$

with $r_0 \approx N^{-1}$ such that:

- (a) $f_N(z, \omega, E) = P_N(z, \omega, E)g_N(z, \omega, E)$
- (b) $g_N(z, \omega, E) \neq 0$ for any $(z, \omega, E) \in \mathcal{P}$
- (c) For any $(\omega, E) \in \mathcal{D}(\omega_0, r_1) \times \mathcal{D}(E_0, r_1)$, the polynomial $P_N(\cdot, \omega, E)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, r_0)$
- (d) $k = \deg P_N(\cdot, \omega, E) \leq (\log N)^A$.

Another application of the uniform upper estimates is the following analogue of Wegner’s estimate from the random case (see [Weg]). It will be important that there is only a loss of $(\log N)^A$ in (5.27).

**Lemma 5.5.** Suppose $\omega$ satisfies (2.11). Then for any $N \gg 1$, $E \in \mathbb{R}$, $H \geq (\log N)^A$ one has

$$(5.27) \quad \text{mes} \{x \in \mathbb{T} : \text{dist}(\text{sp } H_N(x, \omega), E) < \exp(-H)\} \leq \exp(-H/(\log N)^A).$$

Moreover, the set on the left-hand side is contained in the union of $\lesssim N$ intervals each of which does not exceed the bound stated in (5.27) in length.

**Proof.** By Cramer’s rule

$$\left((H_N(x, \omega) - E)^{-1}(k, m) = \frac{|f_{[1,k]}(e(x), \omega, E)| |f_{[m+1,N]}(e(x), \omega, E)|}{f_N(e(x), \omega, E)}\right.$$  

By Lemma 5.2

$$\log |f_{[1,k]}(e(x), \omega, E)| + \log |f_{[m+1,N]}(e(x), \omega, E)| \leq NL(\omega, E) + (\log N)^A$$

for any $x \in \mathbb{T}$. Therefore,

$$\|\left((H_N(x, \omega) - E)^{-1}\right) \leq N^2 \frac{\exp(NL(\omega, E) + (\log N)^A)}{|f_N(e(x), \omega, E)|}$$

for any $x \in \mathbb{T}$. Since

$$\text{dist}(\text{sp } H_N(x, \omega), E) = \|\left((H_N(x, \omega) - E)^{-1}\|^{-1},$$

the lemma follows. \hfill \Box

### 5.3. Elimination of resonances and the separation of zeros.

Given arbitrary $E \in \mathbb{R}$, the typical distance from $E$ to the eigenvalues of equation (1.1) on a finite interval $[-N, N]$ should be at least $\text{const} \cdot N^{-1}$. If for some $E \in \mathbb{R}$ and $x \in \mathbb{T}$ this distance $\rho$ is considerably smaller$^9$ than $N^{-1}$, then we say that $(E, x)$ are in resonance and we refer to $\rho^{-1}$ as the magnitude of the resonance. Clearly, the $x$-averaged distribution of the eigenvalues of (1.1) on the interval $[-N, N]$ controls the probability of resonances. For more accurate estimates the fine properties of this distribution are very important.

Assume that for some $E \in \mathbb{R}$, $x \in \mathbb{T}$, $n \in \mathbb{Z}$, both $(E, x)$ and $(E, T^nx)$ are in resonance. In this case we say that this pair forms a double resonance. Double resonances play a crucial role in any proof of Anderson localization, i.e., that the eigenfunctions of (1.1) decay exponentially as $|n| \rightarrow \infty$. This was found in Sinai’s classical work [Sin1], where Anderson localization was established for (1.1) with $V(x) = \cos(2\pi x)$ and large $|\lambda|$. A novel, non-perturbative approach to the study of double resonances was found by Bourgain and the first author in [BouGol]. It is based on the following notion:

**Definition 5.6.** A set $S \subset \mathbb{R}^m$ is called semi-algebraic if it is a finite union of sets defined by a finite number of polynomial inequalities.

$^9$Technically speaking, this means $\exp(-N^b)$ with $b < 1$
For instance, let $V(x), x \in \mathbb{T}$ be a trigonometric polynomial and consider the dynamics of the shift on $\mathbb{T}$. Then for fixed $x$ the double resonances can be included into a semialgebraic set (in the $\omega, E$ plane). An important parameter of a semialgebraic set is its degree which equals

$$(\text{number of polynomials involved}) \times (\text{maximal degree of these polynomials})$$

If $V$ is a trigonometric polynomial, then the degree of the set of resonances on the interval $[-N, N]$ is at most $N^C$ for some absolute constant $C$. On the other hand, the only “dangerous resonances” for the localization are those of magnitude $\exp(cN)$. That allows one to eliminate double resonances for the case of the shift $x \rightarrow x + \omega$ and skew-shifts $(x, y) \rightarrow (x + \omega, y + x)$ using simple geometrical ideas related to semialgebraic sets, see [Bou2], Section 9. In [GolSch2] we develop a more quantitative method for analyzing the resonances, which is based on the theory of resultants and discriminants of polynomials. The polynomials in question are those which arise in the factorization of the determinants $f_N$ via the Weierstrass preparation theorem in the phase variable $x$. This method turns the set of resonances into a set on which some analytic function (in the case of a double resonance it is the resultant) attains very small values. Cartan’s estimate from above applied to this analytic function then leads to bounds on the measure and complexity of this set. The logic of this is captured by the following lemma from Section 7 in [GolSch2]. We also use the following notation:

Assume that $k > 0$, $s = 1, 2$, and which satisfies the following properties: $S$ has multiplicity $\leq 1$, cardinality $\#(S) \leq t \exp((\log \ell_1)^A)$ and for each $m$, there exists a subset $\Omega_{1, t, z, t, H, m} \subset D(\omega_m, rt^{-1}/2)$ with

$$(5.31) \quad \text{dist} \left( \left\{ \text{zeros of } P_1(\cdot, w) \right\}, \left\{ \text{zeros of } P_2 \left( \cdot + t(w_1 - w_0), w \right) \right\} \right) \geq e^{-CHk}.$$

It is instructive for the reader to first consider the meaning of the previous lemma for the case where neither $P_1$ nor $P_2$ depend on $w$. Lemma 5.7 is the principal tool for eliminating resonant phases and energies in the paper [GolSch2]. As a typical application of it we mention the following lemma on the separation of the zeros of determinants from Section 8 of [GolSch2]. Let $T(x) = x + \omega$ be a shift and $f_N(z, \omega, E)$ be the Dirichlet determinants defined as in (5.11) with $v(n, x) = V(x + n\omega)$. Also, $Z(f, z_0, r_0)$ denotes the set of zeros of $f$ in the disk $D(z_0, r_0)$.

Lemma 5.8. Let $C_1 > 1$ be an arbitrary constant. Given $\ell_1 \geq \ell_2 \gg 1$, $t > \exp((\log \ell_1)^4)$, $H \gg 1$, there exists a cover of $\mathbb{T} \times \mathbb{T}_{c,a} \times [-C_1, C_1]$ by a system $S$ of polydisks

$$D(x_m, r) \times D(\omega_m, rt^{-1}) \times D(E_m, r), \quad x_m \in \mathbb{T}, \quad E_m \in [-C_1, C_1],$$

with $\omega_m \in \mathbb{T}_{c,a}$, and $r = \exp((-(\log \ell_1)^4))$, and which satisfies the following properties: $S$ has multiplicity $\leq 1$, cardinality $\#(S) \leq t \exp((\log \ell_1)^A)$ and for each $m$, there exists a subset $\Omega_{1, t, z, t, H, m} \subset D(\omega_m, rt^{-1}/2)$ with

$$(5.31) \quad \text{dist} \left( \left\{ \text{zeros of } P_1(\cdot, w) \right\}, \left\{ \text{zeros of } P_2 \left( \cdot + t(w_1 - w_0), w \right) \right\} \right) \geq e^{-CHk}.$$

such that for any $w \in D(\omega_m, rt^{-1}/2) \setminus \Omega_{1, t, z, t, H, m}$ there exists a subset

$$S_{\omega_m, \#(S) \leq t \exp((\log \ell_1)^A)}$$

such that for any $E \in D(E_m, r) \setminus \Omega_{1, t, z, t, H, m}$ one has

$$\text{dist} \left( Z(f, \omega, E), D(x_m, r) \right) > e^{-H((\log \ell_1)^C)}.$$
5.4. Localization. Let \( f_N(z, \omega, E) \) be the Dirichlet determinants for equation \([\text{Eq}]\) on the interval \([1, N]\) with the dynamics \( T x = x + \omega \). The separation of the zeros of \( f_N(z, \omega, E) \) and \( f_N(x + n\omega, \omega, E) \) described in Lemma\([\text{Lemma}]\) is achieved due to the dynamics \( z \to z \varepsilon(\omega) \). It is shown in \([\text{GolSch2}]\) that this separation property implies that for most energies the associated eigenfunctions on a finite interval \([-N, N]\) are exponentially localized. Furthermore, the authors also show that this localization on a finite interval can be used to obtain a lower bound on the minimal distance between a typical pair of eigenvalues, i.e., between the zeros of \( E \leftrightarrow f_N(x, \omega, E) \). This application of Anderson localization will be described in the following subsection. Here we outline how one can find the localized eigenfunctions.

Any solution of the equation
\[
-\psi(n + 1) - \psi(n - 1) + v(n)\psi(n) = E\psi(n), \quad n \in \mathbb{Z},
\]
obey the relation
\[
\psi(m) = \mathcal{G}_{[a, b]}(E)(m, a - 1)\psi(a - 1) + \mathcal{G}_{[a, b]}(E)(m, b + 1)\psi(b + 1), \quad m \in [a, b]
\]
where \( \mathcal{G}_{[a, b]}(E) = (H_{[a, b]} - E)^{-1} \) is the Green function, \( H_{[a, b]} \) being the linear operator defined by \([\text{Eq}']\) for \( n \in [a, b] \) with zero boundary conditions.

In view of \([\text{GolSch2}]\), one can prove exponential localization on \([1, N]\) by showing that outside of some subinterval \([n_0 - L, n_0 + L]\) (where \( \log L \ll \log N \)) all Green functions of a much smaller scale (say, scale \( (\log N)^C \)) have exponential off-diagonal decay. Indeed, \([\text{5.42}]\) would then imply that any eigenfunction is exponentially small outside of the window \([n_0 - L, n_0 + L]\). This strategy was introduced by Fröhlich and Spencer in their fundamental work (see \([\text{FroSpe1}], \text{FroSpe2}, \text{FroSpeWit}\) on Anderson’s model. The following simple lemma shows that the question of exponential off-diagonal decay for the Green function is intimately related to the LDTs for the determinants.

**Lemma 5.9.** Let \( \omega \in T_{c, a} \). Suppose \( L(\omega, E_0) = \gamma > 0 \),
\[
\log |f_\ell(z_0, \omega, E_0)| > \ell L(\omega, E_0) - K/2
\]
for some \( z_0 = e(x_0) \), \( x_0 \in T, E_0 \in \mathbb{R}, \ell \gg 1, K > (\log \ell)^4 \). Then
\[
|\mathcal{G}_{[1, \ell]}(z_0, \omega, E)(j, k)| \leq \exp\left(-\frac{\gamma}{2}(k - j) + K\right)
\]
\[
|\mathcal{G}_{[1, \ell]}(z_0, \omega, E)| \leq \exp(K)
\]
where \( \mathcal{G}_{[1, \ell]}(z_0, \omega, E) = (H_{[1, \ell]}(z_0, \omega) - E_0)^{-1} \) is the Green’s function, \( 1 \leq j \leq k \leq \ell \).

**Proof.** By Cramer’s rule applied to \((H_{[1, \ell]} - E)^{-1}\), the uniform upper bound of Lemma\([\text{5.2}]\) as well as the rate of convergence estimate \( |L - L_\ell| \lesssim \ell^{-1} \) from \([\text{GolSch1}]\),
\[
|\mathcal{G}_{[1, \ell]}(z_0, \omega, E)(j, k)| = |f_{j-1}(z_0, \omega, E)| \cdot |f_{\ell-k}(z_0 e(K\omega), \omega, E_0)| \cdot |f_{\ell}(z_0, \omega, E_0)|^{-1}
\]
\[
\leq |f_{\ell}(z_0, \omega, E_0)|^{-1} \exp(\ell L(\omega, E_0) - (k - j)L(\omega, E_0) + (\log \ell)^C),
\]
and the lemma follows.

The idea behind implementing the aforementioned Fröhlich-Spencer scheme is now as follows: First, if \( \psi \) is an \( \ell^2 \)-normalized eigenfunction of \( H_{[1, N]}(x_0, \omega) \) with eigenvalue \( E \), then \([\text{1.43}]\) must fail with \( \ell = (\log N)^C \) for some \( z_0 = e(x_0 + k_0 \omega) \). In other words, it must fail for some determinant \( f_{[k_0, k_0 + \ell]}(x_0, \omega, E) \). Second, if it were to fail for another determinant \( f_{[k_1, k_1 + \ell]}(x_0, \omega, E) \) where \( |k_0 - k_1| = t \) for a sufficiently large \( t \), then this would lead to a contradiction of the separation of zeros property described above. Note that the latter dictates the size of \( t \) (here it turns out to be \( t = \exp((\log \log N)^C) \)) and therefore also the size of the localization window. Moreover, note that we are forced to eliminate a set of energies and \( \omega \) to achieve this separation of the zeros. Hence, we can only hope to obtain the localization property of the eigenfunctions if the energy falls outside a set of exceptional energies. We conclude that outside of some window all determinants of a smaller scale satisfy \([\text{1.43}]\), and thus the eigenfunction \( \psi \) has to be exponentially small there (due to an application of the avalanche principle and \([\text{5.42}]\)). For further details we refer the reader to Section 10 of \([\text{GolSch2}]\).
5.5. Distances between eigenvalues on a finite interval. Dirichlet eigenvalues on a finite interval are simple. However, the eigenvalues can be as close as \( \exp(-cN) \), where \( N \) is the size of the interval. It follows for instance from the analysis of \( f \) in Sinai’s work \[Sin1\] that this is the case if \( x \) belongs to some subset of \( \mathbb{T} \) of measure \( \exp(-\gamma N) \). In \[GolSch2\] the localized eigenfunctions were used to improve upon the \( e^{-cN} \) bound. Indeed, the “typical” distance between the eigenvalues turns out to be \( e^{-N^\delta} \).

It will be convenient for us to work with the operators \( H_{[-N,N]}(x,\omega) \) instead of \( H_{[1,N]}(x,\omega) \) as we did in the previous section. We use the symbols \( E_j^{(N)}, \psi_j^{(N)} \) to denote the eigenvalues and normalized eigenfunctions of \( H_{[-N,N]}(x,\omega) \). We denote the sets of \( \omega \) and energy, which we needed to remove in the previous section, by \( \Omega_N, \mathcal{E}_{N,\omega} \). They are of measure \( \lesssim \exp(-(\log N)^{A_1}) \) and complexity \( \lesssim (\log N)^{A_2} \) where \( A_2 \gg A_1 \).

**Lemma 5.10.** For any \( \omega \in \mathbb{T}_{c,a} \setminus \Omega_N \) and all \( x \) one has for all \( j,k \) and any small \( \delta > 0 \)

\[
|E_j^{(N)}(x,\omega) - E_k^{(N)}(x,\omega)| > e^{-N^\delta}
\]

provided \( E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega} \) and \( N \geq N_0(\delta) \).

**Proof.** Fix \( x \in \mathbb{T} \), \( E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega} \). Let \( Q \approx \exp((\log\log N)^C) \). By the Anderson localization property (see the previous subsection as well as Section 9 in \[GolSch2\] ) there exists

\[
\Lambda_Q := \left[ \psi_j^{(N)}(x,\omega) - Q, \psi_j^{(N)}(x,\omega) + Q \right] \cap [-N,N]
\]

so that

\[
\sum_{n \in [-N,N] \setminus \Lambda_Q} \left| f_{[-N,N]}(e(x),\omega; E_j^{(N)}(x,\omega)) \right|^2 \leq e^{-2Q\gamma} \sum_{n=-N}^N \left| f_{[-N,N]}(e(x),\omega; E_j^{(N)}(x,\omega)) \right|^2.
\]

Here we used that with some \( \mu = \text{const} \)

\[
\psi_j^{(N)}(x,\omega; n) = \mu \cdot f_{[-N,N-1]}(e(x),\omega; E_j^{(N)}(x,\omega))
\]

for \( -N \leq n \leq N \) and the convention that

\[
f_{[-N,N-1]} = 0, \quad f_{[-N,N]} = 1.
\]

One can assume \( \psi_j^{(N)}(x,\omega) \geq 0 \) by symmetry. It follows from the avalanche principle (recall that all determinants of scale \((\log N)^C\) which fall outside of the window of localization are non-resonant) and a simple stability bound in the energy that

\[
\sum_{n=-N}^{n-N} \left| f_{[-N,N]}(e(x),\omega; E) - f_{[-N,N]}(e(x),\omega; E_j^{(N)}(x,\omega)) \right|^2 \leq e^{-2\gamma Q} \| E - E_j^{(N)}(x,\omega) \|^{(\log N)^C} \sum_{n \in \Lambda_Q} \left| f_{[-N,N]}(e(x),\omega; E_j^{(N)}(x,\omega)) \right|^2
\]

Let \( n_1 = \psi_j^{(N)}(x,\omega) - Q - 1 \). Furthermore, we bound the difference on the window of localization simply by

\[
\left\| \begin{pmatrix} f_{[-N,N+1]}(e(x),\omega, E) \\ f_{[-N,N]}(e(x),\omega, E) \end{pmatrix} - \begin{pmatrix} f_{[-N,N+1]}(e(x),\omega, E_j^{(N)}(x,\omega)) \\ f_{[-N,N]}(e(x),\omega, E_j^{(N)}(x,\omega)) \end{pmatrix} \right\|
\]

\[
= \left\| M_{n_1+1}(e(x),\omega, E) \begin{pmatrix} f_{[-N,N+1]}(e(x),\omega, E) \\ f_{[-N,N]}(e(x),\omega, E) \end{pmatrix} - M_{n_1+1}(e(x),\omega, E_j^{(N)}(x,\omega)) \begin{pmatrix} f_{[-N,N+1]}(e(x),\omega, E_j^{(N)}(x,\omega)) \\ f_{[-N,N]}(e(x),\omega, E_j^{(N)}(x,\omega)) \end{pmatrix} \right\|
\]

\[
\leq e^{C(n-n_1)} e^{-\gamma Q} \| E - E_j^{(N)}(x,\omega) \|^{(\log N)^C} \left( \sum_{n \in \Lambda_Q} \left| f_{[-N,N]}(e(x),\omega; E_j^{(N)}(x,\omega)) \right|^2 \right)^{1/2}.
\]
Now suppose there is \( E_k^{(N)}(x, \omega) \) with \( |E_k^{(N)}(e(x), \omega) - E_j^{(N)}(x, \omega)| < e^{-N^\delta} \) for some small \( \delta > 0 \). Then, by the preceding,

\[
\sum_{n=-N}^{N} \left| f_{[-N,n]}(e(x), \omega; E_j^{(N)}(x, \omega)) - f_{[-N,n]}(e(x), \omega; E_j^{(N)}(x, \omega)) \right|^2 < e^{\frac{1}{2}N^\delta} \sum_{n \in \Lambda_Q} \left| f_{[-N,n]}(e(x), \omega; E_j^{(N)}(x, \omega)) \right|^2 \]

provided \( N^\delta > \exp((\log \log N)^A) \). That contradicts the orthogonality of the eigenfunctions. \( \square \)

### 5.6. Simplicity of the zeros of \( f_N(\cdot, \omega, E) \)

An estimate for the minimal distance between the zeros of \( f_N(\cdot, \omega, E) \) is crucial for the analysis of the IDS in \( \text{[GoSch2]} \). In contrast with the eigenvalues of \( H_N(x, \omega) \), \( x \in \mathbb{T} \), the real zeros of the discriminant \( f_N(\cdot, \omega, E) \), \( E \in \mathbb{R} \), can be degenerate. However, that happens only for special values of the spectral parameter \( E \). This follows from the simplicity of the zeros of \( f_N(x, \omega, \cdot) \) by means of Sard-type arguments. To turn this statement into a quantitative estimate one has to make use of the estimate for the minimal distance between the Dirichlet eigenvalues from the previous subsection. The following general assertion, which is a combination of Sard’s theorem and Cartan’s estimate for analytic functions, allows one to do that.

**Lemma 5.11.** Let \( f(z, w) \) be an analytic function defined in \( D(0,1) \times D(0,1) \). Assume that one has the following properties:

1. \( f(z, w) = (w - b_0(z))\chi(z, w) \), for any \( z \in D(0, r_0) \), \( w \in D(0, r_1) \), where \( b_0(z) \) is analytic in \( D(0, r_0) \), \( r_0 < r_1 < \frac{1}{2} \), \( \sup \left| b_0(z) \right| \leq 1 \), \( \chi(z, w) \) is analytic and non-vanishing on \( D(0, r_0) \times D(0, r_1) \), and all the zeros of \( P(z, w) \) belong to \( D(0,1/2) \).
2. \( f(z, w) = P(z, w)\theta(z, w) \), for any \( z \in D(0, r_0) \), \( w \in D(0, r_1) \) where

\[
P(z, w) = z^k + c_{k-1}(w)z^{k-1} + \cdots + c_0(w),
\]

\( c_j(w) \) are analytic in \( D(0, r_0) \), and \( \theta(z, w) \) is analytic and non-vanishing on \( D(0, r_0) \times D(0, r_1) \), and all the zeros of \( P(z, w) \).

Then, given \( H \gg k^2 \log[(r_0r_1)^{-1}] \) one can find a set \( S_H \subset D(w_0, r_1) \) with the property that

\[
\text{mes}(S_H) \lesssim r_1^2 \exp \left( -cH/k^2 \log[(r_0r_1)^{-1}] \right), \quad \text{and} \quad \text{compl}(S_H) \lesssim k^2 \log[(r_0r_1)^{-1}]
\]

such that for any \( w \in D(0, r_1/2) \setminus S_H \) and \( z \in D(0, r_0) \) for which \( w = b_0(z) \) one has

\[
\left| b_0'(z) \right| > e^{-kH}2^{-k}r_1.
\]

Moreover, for those \( w \) the distance between any two zeros of \( P(\cdot, w) \) exceeds \( e^{-H} \).

**Proof.** Assume that \( k \geq 2 \) and set \( \psi(w) = \text{disc} P(-, w) \). If \( k = 1 \), then skip to \( \text{Lemma 5.6} \). Then \( \Psi(w) \) is analytic in \( D(0, r_1) \). Assume that \( |\psi(w)| < \tau \) for some \( \tau > 0 \), \( w \in D(0, r_1) \). Recall that for any \( w \)

\[
(5.61) \quad \psi(w) = \prod_{i \neq j} (\zeta_i(w) - \zeta_j(w)),
\]

where \( \zeta_i(w), i = 1, 2, \ldots, k \) are the zeros of \( P(\cdot, w) \). Then \( |\zeta_i(w) - \zeta_j(w)| < \tau^{2/k(k-1)} \) for some \( i \neq j \). Set \( \zeta_i = \zeta_i(w), \zeta_j = \zeta_j(w) \). Assume first \( \zeta_i \neq \zeta_j \). Then

\[
f(\zeta_i, w) = 0 \quad f(\zeta_j, w) = 0, \quad 0 < |\zeta_i - \zeta_j| < \tau^{2/k(k-1)}.
\]

Due to (i) one has \( w = b_0(\zeta_i) = b_0(\zeta_j) \). Hence,

\[
(5.62) \quad \left| b_0'(\zeta_i) \right| \leq \frac{1}{2}|\zeta_i - \zeta_j| \max\left| b_0''(z) \right| \lesssim |\zeta_i - \zeta_j| r_0^{-2} < r_0^{-2} \tau^{2/k(k-1)}.
\]

If \( \zeta_i = \zeta_j \) then \( P(\zeta_i, w) = 0, \partial_w P(\zeta_i, w) = 0 \). Then \( f(\zeta_i, w) = 0, \partial_w f(\zeta_i, w) = 0 \) due to the representation (ii). Then \( w - b_0(\zeta_i) = 0, b'(\zeta_i) = 0 \) due to the representation (i). Thus \( \text{Lemma 5.6} \) holds at any event. If \( \varphi(z) \) is analytic function in \( D(0, r) \), then it follows from the general change of variables formula that

\[
\text{mes} \{ w : w = \varphi(z), z \in D(0, r), |\varphi'(z)| < \eta \} \leq \pi \eta^2.
\]

In view of the preceding one obtains

\[
(5.63) \quad \text{mes} \{ w \in D(0, r_1) : |\psi(w)| < \tau \} \lesssim r_0^{-2} \tau^{2/k(k-1)}.
\]
On the other hand, due to (5.61) one obtains
\[
\sup \{|\psi(w)| : w \in \mathcal{D}(0, r_1)\} \leq 1.
\]
Take \(\tau < (r_0 r_1)^{k(k-1)/2}\). Then one obtains from (5.63) that
\[
|\psi(w)| \geq \tau
\]
for some \(|w| < \frac{r_0}{\tau}\). By Cartan’s estimate there exists a set \(\mathcal{T}_H \subset \mathcal{D}(0, \frac{r_0}{\tau})\) with
\[
\text{mes } \mathcal{T}_H \lesssim r_1^2 \exp \left(-cH/k^2 \log((r_0 r_1)^{-1})\right)
\]
and of complexity \(\lesssim k^2 \log((r_0 r_1)^{-1})\) such that
\[
(5.64) \quad \log|\psi(w)| > -H
\]
for any \(w \in \mathcal{D}(0, \frac{r_0}{\tau}) \setminus \mathcal{T}_H\).

In particular, (5.64) implies that
\[
|\zeta_i(w) - \zeta_j(w)| > e^{-H}
\]
for any \(w \in \mathcal{D}(0, \frac{r_0}{\tau}) \setminus \mathcal{T}_H, i \neq j\). Take arbitrary \(w_0\) such that \(\text{dist}(w_0, \mathcal{T}_H) > 2e^{-H}, w_0 = b_0(z_0)\) for some \(z_0 \in \mathcal{D}(0, r_0)\). Then
\[
|P(z, w_0)| \geq (2e^H)^{-k} \quad \text{for all } |z - z_0| = e^{-H}/2
\]
by the separation of the zeros (5.65). By our assumption on the zeros of \(P(z, w)\),
\[
\sup_{z \in \mathcal{D}(0, r_0)} \sup_{w \in \mathcal{D}(0, r_1)} |\partial_w P(z, w)| \lesssim r_1^{-1}.
\]
Thus,
\[
|P(z, w)| > \frac{1}{2} 2^{-k} e^{-kH} \quad \text{if } |z - z_0| = e^{-H}/2, \quad |w - w_0| \ll 2^{-k} e^{-kH} r_1.
\]
Then due to the Weierstrass preparation theorem,
\[
(5.66) \quad P(z, w) = (z - \zeta(w)) \lambda(z, w)
\]
for any \(z \in \mathcal{D}(z_0, r_0')\), \(w \in \mathcal{D}(w_0, r_1')\), where \(r_0' = e^{-H}/2, r_1' \ll e^{-kH} 2^{-k} r_1\), and \(\zeta(w)\) is an analytic function in \(\mathcal{D}(w_0, r_1')\), \(\lambda(z, w)\) is analytic and non-vanishing on \(\mathcal{D}(z_0, r_0') \times \mathcal{D}(w_0, r_1')\). Comparing the representation (i) and (5.66) one obtains
\[
(5.67) \quad \begin{cases} w - b_0(z) = 0 \quad \text{iff} \\ z - \zeta(w) = 0 \end{cases}
\]
for any \(z \in \mathcal{D}(z_0, r_0'), w \in \mathcal{D}(w_0, r_1')\). It follows from (5.67) that
\[
|b_0'(\zeta(w))| \geq |\zeta'(w)|^{-1} r_1' \gtrsim e^{-kH} 2^{-k} r_1,
\]
as claimed. \(\square\)

5.7. Harnack’s inequality and Jensen’s formula for the logarithm of the norms of monodromy matrices. The logarithm of the norm of an analytic matrix-function is a subharmonic function. Harnack’s estimate in this context is not as sharp as for the logarithm of the modulus of an analytic function. The same comment applies to Jensen’s averages. The latter here refers to the following: the Jensen formula states that for any function \(f\) analytic on a neighborhood of \(\mathcal{D}(z_0, R)\), see [Lev],
\[
(5.71) \quad \int_0^1 \log|f(z_0 + R e(\theta))| d\theta - \log|f(z_0)| = \sum_{\zeta : f(\zeta) = 0} \log \frac{R}{|\zeta - z_0|}
\]
provided \(f(z_0) \neq 0\). We showed above how to combine this fact with the large deviation theorem and the uniform upper bounds to bound the number of zeros of \(f_N\) which fall into small disks, in both the \(z\) and \(E\)
variables. In what follows, we will refine this approach further. For this purpose, it will be convenient to average over $z_0$ in (5.74). Henceforth, we shall use the notation

$$\nu_f(z_0, r) = \{z \in \mathcal{D}(z_0, r) : f(z) = 0\}$$

(5.72)

$$J(u, z_0, r_1, r_2) = \int_{\mathcal{D}(z_0, r_1)} \int_{\mathcal{D}(z, r_2)} dx \, dy \int_{\mathcal{D}(z, r)} d\xi \, d\eta \, [u(\zeta) - u(z)].$$

(5.73)

The following simple lemma is proved in [GolSch2]. It is our main tool for counting zeros.

**Lemma 5.12.** Let $f(z)$ be analytic in $\mathcal{D}(z_0, R_0)$. Then for any $0 < r_2 < r_1 < R_0 - r_2$

$$\nu_f(z_0, r_1 - r_2) \leq 4 \frac{r_2^2}{r_1^2} J(\log |f|, z_0, r_1, r_2) \leq \nu_f(z_0, r_1 + r_2)$$

We now describe how the aforementioned technical issues were addressed in Section 12 of [GolSch2] for the transfer matrices of the Schrödinger co-cycles. More precisely, we state the two main results of that section. The reader should not be distracted by technicalities, but rather notice how the norms of the matrices mimic the behavior of the entries. For the latter the crucial piece of information is the number of zeros in various disks. In that respect, we emphasize the quadratic estimate in (5.74). The linear estimate (i.e., the one where the scalar logarithm is not subtracted) would be too weak for the study of the IDS in [GolSch2].

**Proposition 5.13.** (i) Suppose that one of the Dirichlet determinants

$$f^{[1, N]}(\cdot, \omega, E), f^{[1, N-1]}(\cdot, \omega, E), f^{[2, N]}(\cdot, \omega, E), f^{[2, N-1]}(\cdot, \omega, E)$$

has no zeros in $\mathcal{D}(z_0, r_1)$, $\exp(-\sqrt{N}) \leq r_1 \leq \exp(-(\log N)^C)$. Then

$$\|M_N(z_0, \omega, E)\| - |1 + a_0(z - z_0)| \leq |z - z_0|^2 r_1^{-2}$$

for any $z \in \mathcal{D}(z_0, r_2)$, $r_2 = r_1 \exp(-(\log N)^{2C})$, and with $|a_0| \leq r_2^{-1}$.

(ii) Assume that the following conditions are valid

(a) each of the determinants $f^{[a, N-b]}(\cdot, \omega, E)$, $a = 1, 2$, $b = 0, 1$ has at least one zero in $\mathcal{D}(z_0, \rho_0)$, where $e^{-\sqrt{N}} \leq \rho_0 \leq \exp(-(\log N)^{B_0})$

(b) no determinant $f^{[a, N-b]}(\cdot, \omega, E)$ has a zero in $\mathcal{D}(z_0, \rho_1) \setminus \mathcal{D}(z_0, \rho_0)$, $\rho_1 \geq \exp((\log N)^{B_1}) \rho_0$, $B_0 \gg B_1 + A$.

Let $k_0 = \min_{a, b} \mathcal{Z}(f^{[a, N-b]}(\cdot, \omega, E), z_0, \rho_0)$. Then for any

$$z, \zeta \in \mathcal{D}(z_0, \rho_1) \setminus \mathcal{D}(z_0, \rho_2), \rho_1' = \exp(-(\log N)^{B_1}) \rho_1, \rho_2 = \exp((\log N)^{B_2}) \rho_0, \quad B_1 \gg B_2 \gg 1$$

one has

$$\left| \log \frac{\|M(z)\|}{\|M(\zeta)\|} - k_0 \log \frac{|\zeta - \zeta_0|}{|z - \zeta_0|} \right| \leq \exp(-(\log N)^C)$$

**Proposition 5.14.** (i) Assume that one of the Dirichlet determinants $f^{[a, N-b]}(\cdot, \omega, E)$, $a = 1, 2$, $b = 0, 1$ has no zeros in $\mathcal{D}(z_0, r_1)$, $\exp(-\sqrt{N}) \leq r_1 \leq \exp(-(\log N)^{C_1})$. Then

$$4 \frac{r_2^2}{r_1^2} J(\log \|M_N(\cdot, \omega, E)\|, z_0, \rho_1, \rho_2) \leq \rho_1^2 r_1^{-2} \exp((\log N)^B)$$

for any $r_1 \exp(-\sqrt{N}) \leq \rho_1 \leq r_1 \exp(-(\log N)^4)$, $\rho_2 = \rho_1$.

(ii) Assume that for some $\zeta_0$ the following conditions are valid

(a) each of the determinants $f^{[a, N-b]}(\cdot, \omega, E)$, $a = 1, 2$, $b = 0, 1$ has at least one zero in $\mathcal{D}(z_0, \rho_0)$, $\exp(-\sqrt{N}) \leq \rho_0 \leq \exp(-(\log N)^{B_0})$.

(b) no determinant $f^{[a, N-b]}(\cdot, \omega, E)$ has a zero in $\mathcal{D}(z_0, \rho_1) \setminus \mathcal{D}(z_0, \rho_0)$, $\rho_1 \geq \exp((\log N)^{B_1}) \rho_0$, $B_0 > B_1$. 
Let \( k_0 = \min_{a,b} \# \mathcal{E}(f_{[a,N-b]}(\cdot, \omega, E), \zeta_0, \rho_0) \). Then for any
\[
z_1 \in \mathcal{D}(\zeta_0, \rho_1') \setminus \mathcal{D}(\zeta_0, \rho_2), \quad \rho_1' = \exp(-(\log N)^{B_2}) \rho_1, \quad \rho_2 = \exp((\log N)^{B_2}) \rho_0,
\]
\( B_1 > B_2 \), one has
\[
\left| \frac{r_1^2}{r_2^2} J(\log \| M_N(\cdot, \omega, E) \|, z_1, r_1, r_2) - k_0 \right| \leq \exp(-(\log N)^C)
\]
where \(|z_1 - \zeta_0|(1+2c) < r_1 < \rho_1', \ r_2 = \rho_1^*, \text{ and } 0 < c \ll 1 \) is some constant.

6. The IDS: Lipschitz, Hölder, and absolute continuity

We now sketch the main steps that allow us in to pass from information about the zeros of \( f_N(z, \omega, E) \) in \( z \) and \( E \) to information on the IDS.

6.1. Concatenation terms and the number of eigenvalues falling into an interval. Consider the following expressions which we call \textit{concatenation terms} in view of their role in the avalanche principle expansion:

\[
W_{N,k}(e(x), E + i\eta) = \frac{\| M_{[1,k]}(e(x), \omega, E + i\eta) \| \| M_{[k+1,N]}(e(x), \omega, E + i\eta) \|}{\| M_{[1,N]}(e(x), \omega, E + i\eta) \|}
\]

\( 1 \leq k \leq N \), where \( \omega \) is fixed.

**Lemma 6.1.** Let \( x \in \mathbb{T} \), \( E \in \mathbb{R} \), \( \eta > 0 \), and let

\[
|f_{[a,N-b+1]}(e(x), \omega, E + i\eta)| = \max_{1 \leq a', b' \leq 2} |f_{[a', N-b'+1]}(e(x), \omega, E + i\eta)|
\]

for some \( 1 \leq a, b \leq 2 \). Then

\[
\# \left( \text{sp} \ H_{[a,N-b+1]}(e(x), \omega) \cap (E - \eta, E + \eta) \right) \leq 4\eta \sum_{1 \leq k \leq N} W_{N,k}(e(x), E + i\eta)
\]

**Proof.** By Cramer’s rule

\[
(H_{[a,N]})(e(x), \omega) - E - i\eta)^{-1}(k, k) = \frac{f_{[a,k]}(e(x), \omega, E + i\eta)f_{[k+2,N]}(e(x), \omega, E + i\eta)}{f_{[a,N]}(e(x), \omega, E + i\eta)}
\]

\[
M_{[a,N]}(e(x), \omega, E + i\eta) = \begin{bmatrix} f_{[a,N]}(e(x), \omega, E + i\eta) & -f_{[a+1,N]}(e(x), \omega, E + i\eta) \\ f_{[a,N-1]}(e(x), \omega, E + i\eta) & -f_{[a+1,N-1]}(e(x), \omega, E + i\eta) \end{bmatrix}
\]

Due to \( \textbf{[12]} \)

\[
\| M_N(e(x), \omega, E + i\eta) \| \leq 2|f_{[a,N-b+1]}(e(x), \omega, E + i\eta)|
\]

Combining \( \textbf{[14]}, \textbf{[15]}, \textbf{[16]} \) one obtains

\[
\left| \text{tr} \left( (H_{[a,N-b+1]}(x, \omega) - E - i\eta)^{-1} \right) \right|
\]

\[
\leq \sum_{a \leq k \leq N-b+1} \left| \frac{f_{[a,k]}(e(x), \omega, E + i\eta)}{f_{[a,N-b+1]}(e(x), \omega, E + i\eta)} \right| |f_{[k+2,N-b+1]}(e(x), \omega, E + i\eta)|
\]

\[
\leq \sum_{a \leq k \leq N-b+1} 2W_{N,k}(e(x), E + i\eta)
\]

On the other hand,

\[
\left| \text{tr} \left( H_{[a,N-b+1]}(x, \omega) - E - i\eta \right)^{-1} \right|
\]

\[
\geq (2\eta)^{-1} \# \left( \text{sp} \ H_{[a,N-b+1]}(e(x), \omega) \cap (E - \eta, E + \eta) \right)
\]

and we are done. \( \Box \)
Recall that
\[ (6.18) \quad \left| (H(e(x), \omega) - E - i\eta)^{-1} (k, k) \right| \leq \left\| (H(e(x), \omega) - E - i\eta)^{-1} \right\| \leq \eta^{-1} \]

**Corollary 6.2.** Using the notations of Lemma 6.1 one has
\[
\# \left( \text{sp} \left( H_{[1,N]}(e(x), \omega) \right) \cap (E - \eta, E + \eta) \right) \\
\leq 4 \eta \sum_{k \in K} W_{N,k}(e(x), E + i\eta) + \#(K) + 2
\]

**Proof.** Due to Weyl’s Comparison Lemma, see [Bha],
\[
\# \left( \text{sp} \left( H_{[1,N]}(e(x), \omega) \right) \cap (E - \eta, E + \eta) \right) \leq \\
\# \left( \text{sp} \left( H_{[a,N-b+1]}(e(x), \omega) \right) \cap (E - \eta, E + \eta) \right) + 2
\]

Therefore, the assertion follows from the previous lemma. \( \square \)

**Lemma 6.3.** Let \( A \) be an \( n \times n \) Hermitian matrix. Let \( \Psi^{(1)}, \Psi^{(2)}, \ldots, \Psi^{(n)} \in \mathbb{C}^n \) be an orthonormal basis of eigenvectors of \( A \) and \( E^{(1)}, E^{(2)}, \ldots, E^{(n)} \) be the corresponding eigenvalues. Then for any \( E + i\eta, E \in \mathbb{R}, \eta > 0 \) one has
\[
\sum_{1 \leq k \leq n} \left| \left( (A - E - i\eta)^{-1} e_k, e_k \right) \right|^2 \geq \sum_{1 \leq j \leq n} \left( \sum_{1 \leq k \leq n} \left| (e_k, \Psi^{(j)}) \right|^4 \right) \left( \text{Im} \left( (E^{(j)} - E - i\eta)^{-1} \right) \right)^2
\]
where \( e_1, e_2, \ldots, e_n \) is arbitrary orthonormal basis in \( \mathbb{C}^n \).

**Proof.** One has
\[
\left| \left( (A - E - i\eta)^{-1} e_k, e_k \right) \right|^2 \geq \sum_{1 \leq j \leq n} \left( (e_k, \Psi^{(j)}) \right)^4 \left( \text{Im} \left( (E^{(j)} - E - i\eta)^{-1} \right) \right)^2
\]
Since \( \text{Im} \left( (E^{(j)} - E - i\eta)^{-1} \right)^2 > 0, j = 1, 2, \ldots, n \), the assertion follows (use \( \sum_j a_j^2 \geq \sum_j a_j^2 \) if \( a_j \geq 0 \)). \( \square \)

In the next corollary we show how to use effectively the localized eigenfunction to sharpen the estimate on the number of eigenvalues falling into an interval in an abstract setting.

**Corollary 6.4.** Using the notations of the previous lemma assume that the following condition is valid for some \( E, \eta; \)

\[ (L) \quad \text{for each eigenvector } \Psi^{(j)} \text{ with } |E^{(j)} - E| < \eta \text{ there exists a set } \mathcal{S}(j) \subset \{1, 2, \ldots, n\}, \# \mathcal{S}(j) \leq \ell \text{ such that } \sum_{k \in \mathcal{S}(j)} \left| (e_k, \Psi^{(j)}) \right|^2 \leq 1/2. \]

Then
\[
\# \left\{ j : |E^{(j)} - E| < \eta \right\} \leq 8 \ell \eta^2 \sum_{1 \leq k \leq n} \left| \left( (A - E - i\eta)^{-1} e_k, e_k \right) \right|^2
\]

**Proof.** Recall that for any positive \( \alpha_1, \ldots, \alpha_\ell \) with \( \sum_j \alpha_j = 1 \)
\[
\sum_j \alpha_j^2 \geq \sum_j \frac{1}{\ell^2} = \frac{1}{\ell}
\]
by Cauchy-Schwarz. Due to the assumptions of the corollary
\[
1 = \left( \Psi^{(j)}, \Psi^{(j)} \right) = \sum_{1 \leq k \leq n} \left| (e_k, \Psi^{(j)}) \right|^2 \leq \sum_{k \in \mathcal{S}(j)} \left| (e_k, \Psi^{(j)}) \right|^2 + 1/2
\]
for any $|E(j) - E| < \eta$. Hence, for such $E(j)$ we have
\begin{equation}
\sum_{k \in S(j)} |(e_k, \Psi(j))|^4 \geq 1/4 \ell
\end{equation}
and the assertion follows from the previous lemma.

Consider the following expressions which we also call concatenation terms:

$$w_m(z) = -\log W_m(z) = \log \frac{\|M_{2m}(z, \omega, E)\|}{\|M_m(ze(\omega E), \omega, E)\|} \|M_m(z, \omega, E)\|$$

These terms are simple linear combinations of subharmonic functions. Hence, the Riesz representation theorem allows one to write them in the usual way, albeit with a signed measure rather than a positive one. Nevertheless, it turns out that this measure is an "almost" positive measure. This feature of the concatenation terms combined with the avalanche principle expansion allows one to establish a very sharp relation between these terms for different scales. Assume that the following condition holds:

(I) no determinant $f_{[a,m-b]}(e(nm\omega), \omega, E)$, $f_{[a,2m-b]}(\cdot, \omega, E)$, $a = 1, 2$; $b = 0, 1$; $n = 0, 1$, has a zero in some annulus $D(\zeta_0, \rho_1) \setminus D(\zeta_0, \rho_0)$, where $\rho_0 \lesssim \exp(-m^\delta)$, $\rho_0 < \rho_1 < \exp(-\log m)^A$, $0 < \delta \ll 1$.

Set
\begin{align*}
\bar{k}_n &= \min_{a,b} \nu f_{[a,m-b]}(e(nm\omega), \omega, E)(\zeta_0, \rho_0), \\
n &= 0, 1,
\end{align*}
\begin{align*}
\bar{k} &= \min_{a,b} \nu f_{[a,2m-b]}(\cdot, \omega, E)(\zeta_0, \rho_0).
\end{align*}

Lemma 6.5. Assume that $\rho_1 \geq \rho_0^{(\log m)^{-B_0}}$. Then
\begin{align*}
\bar{k}_0 + \bar{k}_1 \leq \bar{k} \leq \bar{k}_0 + \bar{k}_1 + k_1(\lambda, V)
\end{align*}
provided $B_0 \gg 1$. Here $k_1(\lambda, V)$ is some integer constant.

Proof. Recall that due to the large deviation theorem there exists $z = e(x + iy) \in D(\zeta_0, \rho') \setminus D(\zeta_0, \rho_1/2)$ such that
\begin{align*}
|\log \|M_{[1,m]}(ze(nm\omega), \omega, E)\| - mL(y, E)| \leq m^\delta (\log m)^{-B_2}, \\
|\log \|M_{[1,2m]}(z, \omega, E)\| - 2mL(y, E)| \leq m^\delta (\log m)^{-B_2}
\end{align*}
with $1 \ll B_2 < B_0$. Combining these relations with Proposition B.13 one obtains
\begin{align*}
\frac{\log \|M_{[1,m]}(\zeta e(\omega m), \omega, E)\| \|M_m(\zeta, \omega, E)\|}{\|M_{[1,2m]}(\zeta, \omega, E)\|} - (\bar{k}_0 + \bar{k}_1 - \bar{k}) \log \frac{|\zeta - \zeta_0|}{|z - \zeta_0|} \leq C m^\delta (\log m)^{B_1}
\end{align*}
for any $\zeta \in D(\zeta_0, \rho') \setminus D(\zeta_0, \rho_2)$. Since $|z - \zeta_0| \approx \rho'_1$, $\rho'_1 \approx \exp(-m^\delta (\log m)^{-B_0} - (\log m)^{B_1})$, one can pick $\zeta \in D(\zeta_0, \rho_1') \setminus D(\zeta_0, \rho_2)$ such that $|\zeta - \zeta_0| = |z - \zeta_0|(\log m)^{B_0}/2$. Then
\begin{align*}
\log \frac{\|M_{[1,m]}(\zeta e(\omega m), \omega, E)\| \|M_m(\zeta, \omega, E)\|}{\|M_{[1,2m]}(\zeta, \omega, E)\|}
\end{align*}
\begin{align*}
\leq C m^\delta (\log m)^{B_1}.
\end{align*}
Recall that
\begin{align*}
\|M_{2m}(\zeta, \omega, E)\| \leq \|M_m(\zeta e(\omega m), \omega, E)\| \|M_m(\zeta, \omega, E)\|
\end{align*}
Relations $\eqref{3.10}, \eqref{3.11}$ imply $k_0 + k_1 - \tilde{k} \leq 0$. Removing the absolute values in $\eqref{3.10}$ and $\eqref{3.11}$ and taking Jensen’s averages one obtains the following:

$$\left| 4 \frac{r_2^2}{r_1^2} \left( \log \left\| M_{2m}(\cdot,\omega,E) \right\| \right) \left\| M_{1,m}(e(m\omega),\omega,E) \right\|, z_0, r_1, r_2 \right) - (\tilde{k} - k_1 - \tilde{k}_2) \right| < 2$$

where $r_1 = \rho'_1/2$, $r_2 = cr_1$. Hence, $k - \tilde{k}_1 - \tilde{k}_2 \leq k_1(\lambda, V)$. \hfill \Box

The integers defined in the previous lemma almost perfectly substitute the measure representing the concatenation term. More precisely, the following assertion holds:

**Proposition 6.6.** Given $E \in \mathbb{C}$ and integer $m \gg 1$, there exists a cover of $A_{p_0/2}$ by disks $\mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m})$, $\zeta_{j,m} \in A_{p_0/2}$, $j = 1, 2, \ldots, j_m$ such that the following conditions are valid:

1. $\exp(-m^\delta) \leq \bar{\rho}_{j,m} \leq \exp(-m^{\delta/2})$, $j = 1, 2, \ldots, j_m$, $0 < \delta < 1$,
2. $\text{dist} \left( \mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m}), \mathcal{D}(\zeta_{j',m}, \bar{\rho}_{j',m}) \right) \geq \bar{\rho}_{j',m} + \bar{\rho}_{j,m}$ where $\bar{\rho}_{j,m} = \rho_{j,m}^{(log m)^{a_1}}$, $B_1 \gg 1$, $j = 1, 2, \ldots, j_m$, provided $j_1 \neq j_2$,
3. no determinant $f_{[a,m-b]}(e(m\omega),\omega,E)$ or $f_{[a,2m-b]}(\cdot,\omega,E)$, $a = 1, 2; b = 0, 1; n = 0$, has a zero in $\mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m}) \setminus \mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m}/2)$, where $\bar{\rho}_{j,m} = \rho_{j,m}^{(log m)^{a_1}}, \bar{\rho}_{j,m} = \rho_{j,m}^{(log m)^{-a_1}}$, $j = 1, 2, \ldots, j_m$,
4. for each $\zeta_{j,m}$ there is an integer $k(j,m)$,

$$0 \leq k(j,m) \leq \nu_{2m}(\cdot,\omega,E) \left( \zeta_{j,m}, \bar{\rho}_{j,m} \right)$$

such that for any $z, \zeta \in \mathcal{D}(\zeta_{j,m}, 2\bar{\rho}_{j,m}) \setminus \mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m}/2)$ holds

$$\left| (w_m(z) - w_m(\zeta)) - k(j,m) \log \left| \frac{\zeta - \bar{\zeta}_{j,m}}{z - \zeta_{j,m}} \right| \right| \leq |z - \zeta|^2 \cdot (\bar{\rho}_{j,m})^{-2}.$$ 

Assume that the following condition is valid:

**II.m** no determinant $f_{[a,m-b]}(e(m\omega),\omega,E), f_{[a,2m-b]}(\cdot,\omega,E)$ has more than one zero in any disk

$$\mathcal{D}(z_0, r_m), \quad r_m = \exp \left( -(\log m)^A \right), \quad z_0 \in A_{p_0/2}, \quad n = 0, 1, a = 1, 2, b = 0, 1$$

Consider two concatenation terms $w_m(z)$ and $w_m(\zeta)$ with $m \gg \exp((m^\delta)$, $0 < \delta_1 < 1$. Let $\mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m}), \quad \zeta = 1, 2, \ldots, j_m$ and $\mathcal{D}(\bar{\zeta}_{j,m}, \bar{\rho}_{j,m}), \quad j = 1, 2, \ldots, j_m$ be the disks defined in Proposition 6.6 for $w_m(z)$ and $w_m(\zeta)$, respectively. Note that due to the avalanche principle expansion we can conclude the following:

**Lemma 6.7.** There exists $F_{m,\omega,E} \subset A_{p_0}$ with $\text{mes} F_{m,\omega,E} \leq \exp(-m^{1/2})$ such that

$$\left| w_m(z) - w_m(\zeta (e((m - m)\omega)) \right| < \exp \left( -m^{1/2} \right)$$

for any $z \in A_{p_0/2} \setminus F_{m,\omega,E}$.

Combining this assertion with the preceding one obtains the following Proposition (using the notations of Proposition 6.6).

**Proposition 6.8.** Assume that conditions **II.m**, **II.m** are valid. Then, using the notations of the previous lemma one has

0. if $k(j_1, m) = 0$, then there exists $\zeta_{j_1,m} \in \mathcal{D}(\zeta_{j_1,m}, \bar{\rho}_{j_1,m})$ with $k(j,m) = 0$ such that

$$\left| w_m(z) - w_m(\zeta_1 e((m - m)\omega)) \right| \leq \exp \left( -m^{1/2} \right)$$

for any $z \in \mathcal{D}(\zeta_{j_1,m}, \bar{\rho}_{j_1,m}) \setminus \mathcal{D}(\zeta_{j,m}, \bar{\rho}_{j,m})$ and any

$$\zeta_1 \in \mathcal{D}(\zeta_{j_1,m}, \bar{\rho}_{j_1,m}) \setminus F_{m,\omega,E}, \quad \text{mes} F_{m,\omega,E} < \exp \left( -m^{1/2} \right)$$
Let one can obtain Theorem 6.9. Let can be done all the way down to a scale of unit size. In this fashion one arrives at the following:

\[ E \]

Finally, using the Hölder bound of Theorem 6.11 as well as the measure and complexity bounds on the \( \tau = \min \{ \omega \} \) and fix \( \eta > \tau \). Let \( (1) \) is absolutely continuous on \( \omega \). Assume \( L(H) \) and \( \gamma, V, \lambda > 0 \) so that:

For any \( \varepsilon > 0 \), there exists \( \Omega(\varepsilon) \subset T, \) mes \( \Omega(\varepsilon) \) is such that for any \( \omega \in (\omega_0 - \tau_0, \omega_0 + \tau_0) \cap (\mathbb{T}_{c,a} \setminus \Omega(\varepsilon)) \), there exists \( E_\omega(\varepsilon) \subset \mathbb{R}, \) mes \( E_\omega(\varepsilon) \) is \( \varepsilon \) such that for any \( N > N_0 \) and any \( E \in (E', E'') \setminus E_\omega(\varepsilon) \) and any \( \eta > 1/(N \log N)^{1+b} \), one has

(6.112) \[ \int \#(\text{sp}(H_N(x, \omega)) \cap (E - \eta, E + \eta)) \, dx \leq \exp \left((\log \varepsilon^{-1})^A\right) \eta N . \]

In particular, the IDS satisfies

\[ N(E + \eta) - N(E - \eta) \leq \exp \left((\log \varepsilon^{-1})^A\right) \eta \]

for any \( E \in (E', E'') \setminus E_\omega(\varepsilon), \) \( \eta > 0 \).

The proof of Theorem 6.9 establishes the estimate (6.112) for any \( E \in \mathbb{R} \setminus E_\omega(\varepsilon) \), with very detailed description of \( E_\omega(\varepsilon) \) as a union of intervals of different scales. This allows one to combine the Lipschitz estimate here with the Hölder bound of Theorem 6.11 below to prove the following

Theorem 6.10. For almost all \( \omega \in (\omega_0 - \tau_0, \omega_0 + \tau_0) \) the IDS \( N(E) \) is absolutely continuous on \( (E', E'') \). In particular, if \( L(H) \geq \gamma_0 > 0 \) for all \( E \), then \( N(\cdot) \) is absolutely continuous everywhere.

Proof of Theorem 6.10. Let \( (E_n', E_n'') \), \( 1 \leq n \leq \bar{n} \) be disjoint intervals with \( \varepsilon = \sum_{n} (E_n'' - E_n') \ll 1 \). Set \( \tau = \min_n (E_n'' - E_n') \). Let \( \omega_r = p_r q_r^{-1} \) be a convergent of \( \omega \) with \( q_r > \tau^{-4} \). Let \( m(s), s = 1, 2, \ldots, t + 1 \) be integers such that: (1) \( \log(m(s+1)) = m(s)^4, s = 1, 2, \ldots, t \), (2) \( \varepsilon > \exp(-m(1)) > \sqrt{\varepsilon} \), \( m(1+1) = q_r =: N \).

Using the Lipschitz estimates of the previous theorem applied to each pair of consecutive scales \( m(s), m(s+1) \) one can obtain

\[ \frac{1}{N} \int \#(\text{sp}(H_N(x, \omega)) \cap \left( \frac{\ell}{N^{1/2}} + \frac{1}{N^{1/2}} \right)) \, dx \lesssim m(1) N^{-1/2} \]

for any interval \( \left( \frac{s+1}{N^{1/2}}, \frac{s+1}{N^{1/2}} \right) \subset \mathbb{R} \setminus \bigcup_{s=1}^{t+1} E_\omega^{(s)}, \) where \( \ell \in \mathbb{Z}, \) provided \( \omega \in \mathbb{T}_{c,a} \setminus \bigcup \Omega(s) \). Let \( \{T_\ell : \ell \in \mathcal{L}\} \) be the collection of such intervals. Then

\[ \frac{1}{N} \int \#(\text{sp}(H_N(x, \omega) \cap \bigcup_{\ell \in \mathcal{L}, T_\ell \subset \bigcup(E_n', E_n''))) \, dx \leq m(1) \varepsilon . \]

Let \( \mathcal{L}' = \{ \ell : \ell \in \mathcal{L}, T_\ell \cap \{E_n', E_n'' : n = 1, 2, \ldots, \bar{n}\} \neq \emptyset \} \). Then \( \# \mathcal{L}' \lesssim 2 \bar{n} \). Since \( \bar{n} \lesssim \tau^{-1} \) one obtains:

\[ \frac{1}{N} \int \#(\text{sp}(H_N(x, \omega) \cap \bigcup_{\ell \in \mathcal{L}'}) \, dx \lesssim m(1) N^{-1/2} \bar{n} \lesssim m(1) \tau . \]

Finally, using the Hölder bound of Theorem 6.11 as well as the measure and complexity bounds on the exceptional sets \( E_\omega^{(s)} \) yields

\[ \frac{1}{N} \int \#(\text{sp}(H_N(x, \omega) \cap \bigcup_{s=1}^{t+1} E_\omega^{(s)}) \, dx < \exp(-m(1) \varepsilon) \]

and we are done.
Note that the previous proof exploits detailed information on the size and complexity of those sets on which the IDS is not Lipschitz (the exceptional set). This is necessary, as can be seen from the example of a Cantor staircase function. Indeed, in that case there is a uniform Hölder bound with an exponent that equals the Hausdorff dimension of the Cantor set. However, in our case the exceptional set has Hausdorff dimension zero, whereas the Hölder exponent is fixed and positive.

6.2. The Hölder bound. The following result is proved in [GolSch2].

**Theorem 6.11.** Let \( V_0(e(x)) = \sum_{-k_0}^{k_0} v(k) e(kx) \) be a trigonometric polynomial, \( v(-k) = \overline{v(k)} \), \( -k_0 \leq k \leq k_0 \). Let \( L(E,\omega_0) \) be the Lyapunov exponent for \( V = V_0 \) and some \( \omega_0 \in \mathbb{T}_{c,a} \). Assume that it exceeds \( \gamma_0 \) for all \( E \in (E',E'') \).

1. Given \( \rho_0 > 0 \) there exists \( \tau_0 = \tau_0(\lambda,V_0,\omega_0,\gamma_0,\rho_0) \) with the following property: for any 1-periodic, analytic function \( V(e(x+iy)) \), \( -\rho_0 < y < \rho_0 \) assuming real values when \( y = 0 \) and deviating from \( V_0(e(x)) \) by at most \( \tau_0 \), any \( \omega \in \mathbb{T}_{c,a} \cap (\omega_0 - \tau_0, \omega_0 + \tau_0) \), and any \( E \in (E',E'') \), with \( \eta = N^{-1+\delta} \), \( \delta \ll 1 \), \( N \gg 1 \), one has

\[
(6.21) \quad \int_{\mathbb{T}} \#(\text{sp}(H_N(x,\omega)) \cap (E - \eta, E + \eta)) \, dx \leq \eta^\frac{1}{2\delta} \cdot N
\]

with some constant \( 1 \ll B \) and arbitrary \( \varepsilon > 0 \).

2. The IDS \( \mathcal{N}(\cdot) \) satisfies, for any small \( \varepsilon > 0 \),

\[
\mathcal{N}(E + \eta) - \mathcal{N}(E - \eta) \leq \eta^\frac{1}{2\delta} \cdot \varepsilon,
\]

for all \( E \in (E',E'') \) and all small \( \eta > 0 \).

For the case of the almost Mathieu equation (1.1) (which corresponds to \( k_0 = 1 \)) and large \( \lambda \), Bourgain [Bon1] had previously obtained a Hölder-(\( \frac{1}{2} - \varepsilon \)) result for the IDS, which is known to be optimal in those regimes, see [Sin1]. See also [Bon2].

The proof of Theorem 6.11 is similar to that of Theorem 6.10 above. Recall that the latter result exploited the fact that as long as we remove all energies \( E \) belonging to some bad set \( \mathcal{E}_\omega \), the zeros of \( f_N(z,\omega,E) \) in \( z \) do not cluster. In fact, any small disk (say of size \( e^{-N^\delta} \)) does not contain more than one zero.

The logic is that here we can no longer guarantee this separation property of the zeros since we are not allowed to remove energies. Nevertheless, we will be able to show that zeros cannot cluster too much in any small disk. The argument proceeds by contradiction: If there were too many (in fact, \( 2 \deg(V) + 1 \) many) zeros in a small disk, then we can show that this would have to be the case in a large number of disks of the same size (by exploiting the dynamics). Ultimately, this leads to a contradiction due to the fact that the determinant \( f_N \) cannot have more than \( 2N \deg(V) \) zeros in total. This fact appears to be of independent interest, and is formulated as a theorem in [GolSch2]:

**Theorem 6.12.** Using the notations of Theorem 6.11 there exists \( k_0(\lambda,V) \leq 2 \deg V_0 \) with the following property: for all \( E \in \mathbb{R}, s \in \mathbb{Z} \) and \( \omega \in \mathbb{T}_{c,a} \) and any \( x_0 \in \mathbb{T} \) there exists \( s^-, s^+ \) with \( |s - s^\pm| < \exp((\log s)^{\delta}) \) such that the Dirichlet determinant \( f_{[-s^-,s^+]}(\cdot,\omega,E) \) has no more that \( k_0(\lambda,V) \) zeros in \( \mathcal{D}(e(x_0),r_0), r_0 \asymp \exp(-\exp((\log s)^{\delta})) \).

While we need to refer the reader to [GolSch2] for more details, we do present some basic statements here, which elucidate the role of Jensen averages and the avalanche principle in this context.

**Definition 6.13.** Let \( \ell \gg 1 \) be some integer, and \( s \in \mathbb{Z} \). We say that \( s \) is adjusted to a disk \( \mathcal{D}(z_0,r_0) \) at scale \( \ell \) if for all \( k \gg \ell \)

\[
Z(f_k(e((s + m)\omega),\omega,E),z_0,r_0) = 0 \quad \forall |m| \leq C\ell.
\]

Consider the avalanche principle expansion of \( \log |f_N(z,\omega,E)| \):

\[
(6.22) \quad \log |f_N(z,\omega,E + i\eta)| = \sum_{m=1}^{n-1} \log \|A_{m+1}(z)A_m(z)\| - \sum_{m=2}^{n-1} \log \|A_m(z)\| + O\left(\exp(-\ell^{1/2})\right),
\]
for any $z \in A_{j_0/2} \setminus B_{E,\eta,\omega}$, since $B_{E,\eta,\omega} \leq \exp(-\ell^{1/2})$, where $A_m(z) = M_{\ell}(ze(s_m\omega),\omega,E + i\eta)$, $m = 2, \ldots, n - 1$, $A_1(z) = M_{\ell}(z,\omega,E) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right]$, $A_n(z) = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] M_{\ell}(ze(s_n\omega),\omega,E)$, $\ell_m = \ell$, $m = 1, 2, \ldots, n - 1$, $\ell_n = \ell$, $(n-1)\ell + \ell = N$, $\ell, \bar{\ell} \asymp (\log N)^4$, $s_m = \sum_{j<m} \ell_j$.

This expansion allows us to control the number of zeros of the large scale object (in this case $f_N$) by means of the number of zeros (or rather, the Jensen averages) of the small-scale objects (here $w_j$) and vice versa. Surprisingly, it turns out that the most effective way to implement this idea is to obtain an estimate for the local number of zeros at a smaller scale in terms of the total number of zeros at a larger scale. The all-important quadratic (more precisely, super-linear) error estimate here is due to \cite{GoSc1} above.

**Lemma 6.14.** Assume that $\{s_m\}_{j=1}^{j_0}$ is adjusted to $D(z_0, r_0)$ at scale $\ell$. Set $m_0 = 0$, $m_{j_0+1} = n$, and

$$w_j(z) = \log \left| \prod_{m = m_{j+1}}^{m_j+1} A_m(z) \right| \quad \text{for any} \quad 0 \leq j \leq j_0$$

Then

$$4 \frac{r_2^2}{r_1^2} J \left( \log \left| f_N(\cdot, \omega, E) \right|, z_0, r_1, 2r_2 \right) - \sum_{j=0}^{j_0} J(\omega,\cdot, z_0, r_1, 2r_2) \leq N \exp((\log \ell)^4) r_1^2 r_0^{-2}$$

for any $e^{-\sqrt{\ell}} < r_1 \leq \exp(-\log \ell)^2 r_0$, and $r_2 = cr_1$. In particular,

$$4 \frac{r_2^2}{r_1^2} J \left( \log \left| f_N(\cdot, \omega, E) \right|, z_0, r_1, 2r_2 \right) \geq \sum_{j \in \mathcal{J}} J(\omega,\cdot, z_0, r_1, 2r_2) - N \exp((\log \ell)^4) r_1^2 r_0^{-2}$$

for any $\mathcal{J} \subset [0, j_0]$.

For the remaining details (in particular, the crucial notion of ”contributing” terms) we refer the reader to \cite{GoSc1}.

### 7. Generic $C^3$ potentials

In this section we review some recent work of Jackson Chan, see \cite{Ch}. Given any function $V : \mathbb{T} \to \mathbb{R}$, we have a family of quasi-periodic discrete Schrödinger equations

$$-\varphi(n+1) - \varphi(n-1) + \Lambda V(x + n\omega) \varphi(n) = E \varphi(n), \quad n \in \mathbb{Z}$$

where $(x, \omega) \in \mathbb{T} \times \mathbb{T}$ are parameters. Equation \eqref{eq:71} can be rewritten as a first order difference equation:

$$\begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix} = \begin{pmatrix} \Lambda V(x + n\omega) - E & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(n) \\ \varphi(n-1) \end{pmatrix}. $$

Given a $C^3$ potential $V$, any $C^3$ function $\tilde{V}$ satisfying the conditions

$$\max_{x \in \mathbb{T}} |V(x) - \tilde{V}(x)| < \delta$$

$$\max_{x \in \mathbb{T}} |V'(x) - \tilde{V}'(x)| < \delta$$

$$\max_{x \in \mathbb{T}} |V''(x) - \tilde{V}''(x)| < \delta$$

can be written, near $x = 0$, in the form

$$\tilde{V}(x) = V(x) + \eta + \xi x + \frac{1}{2} \theta x^2 + x^3 R(x)$$

where $|\eta|, |\xi|, |\theta| < \delta$, $R \in C^3$, $|\partial_x R| \lesssim 1$ for any index $|\alpha| \leq 2$. More generally, since $\mathbb{T}$ is compact, we can find some large integer $T$ so that

$$\tilde{V}(x) = V(x) + \sum_{m=1}^{T} \left[ \eta_m + \xi_m \left( x - \frac{m}{T} \right) + \frac{1}{2} \theta_m \left( x - \frac{m}{T} \right)^2 + \left( x - \frac{m}{T} \right)^3 R_m \left( x - \frac{m}{T} \right) \right]$$

(7.2)
for all \( x \in \mathbb{T} \), where \( \eta = (\eta_1, \ldots, \eta_T) \), \( \xi = (\xi_1, \ldots, \xi_T), \) \( \theta = (\theta_1, \ldots, \theta_T) \) \( \in \prod_1^T [-\delta, \delta] \), and \( R_m \in C^3 \), \( |\partial_\alpha R_m| \lesssim 1 \) for any index \( |\alpha| \leq 2 \). This motivates the following definition.

**Definition 7.1.** Let \( T \) be a large integer, \( 0 < \delta \ll \frac{1}{T^3} \). Suppose \( R_m(\eta_m, \xi_m, \theta_m; x) \) are \( C^3 \) functions, \( m = 1, 2, \ldots, T \), \( (\eta, \xi, \theta) \) \( \in \prod_1^T [-\delta, \delta], \) \( x \in \mathbb{T} \), satisfying the following conditions:

\[
|\partial_\alpha R_m(\eta_m, \xi_m, \theta_m; x)| \lesssim \frac{1}{T} \quad \text{for any } |\alpha| \leq 3
\]

\[
R_m(0, 0, 0; x) \equiv 0
\]

\[
R_m(\eta_m, \xi_m, \theta_m; x) = -x^{-3}(\eta_m + \xi_m x + \frac{1}{2} \theta_m x^2) \quad \text{for } |x| \geq \frac{1}{2T}
\]

Define a \((T, \delta)\)-variation of the potential by

\[
W(\eta, \xi, \theta, \{R_m\}; x) = \sum_{m=1}^T v_m(\eta_m, \xi_m, \theta_m; x - \frac{m}{T})
\]

where

\[
v_m(\eta_m, \xi_m, \theta_m; x) = \eta_m + \xi_m x + \frac{1}{2} \theta_m x^2 + x^3 R_m(\eta_m, \xi_m, \theta_m; x)
\]

By the preceding,

\[
v_m(0, 0, 0; x) \equiv 0
\]

and

\[
v_m(\eta_m, \xi_m, \theta_m; x) = 0 \quad \text{for } |x| \geq \frac{1}{2T}.
\]

Denote the collection of \((T, \delta)\)-variations of the potential by \( \mathcal{S}(T, \delta) \). The set of parameters \((\eta, \xi, \theta)\) has measure \((2\delta)^3T\). We want to define a notion of “typical” potential by using the normalized measure on this set of parameters. Hence, a set \( S \subset \mathcal{S}(T, \delta) \) is called \((1 - \varepsilon)\)-typical if

\[
|S| := \min_{\{R_m\}} \frac{1}{(2\delta)^3T} \text{mes}\{(\eta, \xi, \theta) \in [-\delta, \delta]^{3T}: W(\eta, \xi, \theta, \{R_m\}; x) \in S\} \geq 1 - \varepsilon
\]

**Theorem 7.2.** Given \( V \in C^3(\mathbb{T}) \), there is \( \lambda_0 = \lambda_0(V) \) such that for \( |\lambda| > \lambda_0 \), one has a collection of perturbed potentials \( \{S_\ell = S_\ell(V, \lambda)\}_{\ell=1}^\infty \), \( S_\ell \subset \mathcal{S}(T^{(\ell)} , \delta_\ell) \), \( \log T^{(\ell+1)} \approx (T^{(\ell)})^\alpha \), \( 0 < \alpha \ll 1 \), \( \sum_{\ell=1}^\infty (1 - |S_\ell|) \leq \lambda^{-\beta} \), so that for any potential

\[
\tilde{V}(x) = V(x) + \sum_{\ell=1}^\infty W^{(\ell)}(\eta^{(\ell)}, \xi^{(\ell)}, \theta^{(\ell)}, \{R_m^{(\ell)}\}; x)
\]

where \( W^{(\ell)} \in S_\ell \), there exists \( \Omega = \Omega(\lambda, \tilde{V}) \), \( \text{mes} \Omega \leq \lambda^{-\beta} \), so that the Lyapunov exponent \( L(\omega, E) \geq \frac{1}{4} \log \lambda \) for any \( \omega \in \mathbb{T} \setminus \Omega \), \( E \in \mathbb{R} \). Furthermore, the corresponding eigenfunctions are exponentially localized.

There are two central technical problems which one has to deal with in order to establish this theorem. The first one consists of the splitting of eigenvalues of the problem \((7.1)\) on a finite interval \([-N, N] \). The technology for this splitting developed in [GoSch2] for the case of an analytic potential can be modified for a "generic" smooth potential.

**Proposition 7.3.** Using the notation of Theorem 7.2, there exist integers \( T'_s \), \( \log T'_s \approx \log T^{(s)} \), such that for any nested sequence of intervals \( \mathcal{F}_{s, k_s} = \left[ \frac{s}{T'_s}, \frac{k_s + 1}{T'_s} \right] \), \( x \in \mathbb{T}, \omega \in \mathbb{T} \setminus \Omega \), there is a sequence of integers \( \{N_s = N_s(x, \omega)\} \), \( \log N_s \approx \log T'_s \), so that

\[
|E_1 - E_2| > \exp(-N'_s)
\]

for distinct eigenvalues \( E_1, E_2 \in \text{sp } H_{[-N, N]}(x, \omega) \cap \mathcal{F}_{s, k_s} \).
The second problem is as follows. The eigenvalues of the problem (1.1) on a finite interval [1,N] have a parametrization $E_1(x) < E_2(x) < \ldots < E_N(x)$, $x \in \mathbb{T}$ which are as smooth as the potential $V(x)$. This general result is due to the self-adjointness of the the problem (1.1) and nondegeneracy of the the eigenvalues $E_j$ restricted on a finite interval. The problem is how to evaluate the quantity
\begin{equation}
|\partial_x E_j| + |\partial_{xx} E_j|
\end{equation}
from below.

This problem was also studied in [GS2]: for analytic potentials, the problem was solved using discriminants of polynomials and Sard-type arguments. This method has no modification for smooth potentials. This is the very problem for which the variations of the potential were introduced. The most basic idea of the method in [Chu] is as follows.

“Typical” $C^3$ functions $F(x)$ are Morse functions, i.e., the quantity
\begin{equation}
|\partial_x F| + |\partial_{xx} F|
\end{equation}
has a good lower bound, gauged according to the size of $F$. On the other hand, there is a basic relation between $\partial_x E_j$ and the potential $V(x)$:
\begin{equation}
\partial_x E_j = \sum_{k=1}^{N} V'(x + k\omega)|\varphi_j(x)(k)|^2
\end{equation}
where $\varphi_j(x)(.)$ is a normalized eigenfunction of (7.1) on the interval $[1,N]$ corresponding to $E_j(x)$. The relation (7.6) enables one to express the “genericity” of the potential $V$ in terms of a lower bound, provided $\varphi_j(x)(.)$ is exponentially localized. Ultimately, the bad cases can be eliminated by varying the frequencies $\omega$. The Sard-type arguments allow one to show that the total measure of those $\omega$ for which there is no response in (7.6) under the variations of $V$ is extremely small.

REFERENCES

[AvrSim] Avron, J., Simon, B. Almost periodic Schrödinger operators. II. The integrated density of states. Duke Math. J. 50 (1983), no. 1, 369–391.
[Bje] Bjerklöv, K. Explicit examples of arbitrarily large analytic ergodic potentials with zero Lyapunov exponent, preprint 2005, to appear in GAFA.
[Bha] Bhatia, R. Perturbation bounds for matrix eigenvalues. Pitman research notes in mathematics series 162, Longman, 1987.
[Bou1] Bourgain, J. Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime. Lett. Math. Phys. 51 (2000), no. 2, 83–118.
[Bou2] Bourgain, J. Green’s function estimates for lattice Schrödinger operators and applications. Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, 2005.
[BouGol] Bourgain, J., Goldstein, M. On nonperturbative localization with quasi-periodic potential. Ann. of Math. (2) 152 (2000), no. 3, 835–879.
[BouGolSch] Bourgain, J., Goldstein, M., Schlag, W. Anderson localization for Schrödinger operators on $Z$ with potentials given by the skew-shift. Comm. Math. Phys. 220 (2001), no. 3, 583–621.
[BouJit] Bourgain, J., Jitomirskaya, S. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. J. Statist. Phys. 108 (2002), no. 5-6, 1203–1218.
[BouSch] Bourgain, J., Schlag, W. Anderson localization for Schrödinger operators on $Z$ with strongly mixing potentials. Comm. Math. Phys. 215 (2000), no. 1, 143–175.
[Cha] Chan, J. Method of variations of potential of quasi-periodic Schrödinger equation, preprint 2005.
[CraSim] Craig, W., Simon, B. Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices. Comm. Math. Phys. 90 (1983), no. 2, 207–218.
[DeiSim] Deift, P., Simon, B. Almost periodic Schrödinger operators. III. The absolutely continuous spectrum in one dimension. Comm. Math. Phys. 90 (1983), no. 3, 389–411.
[FigPas] Figotin, A., Pastur, L. Spectra of random and almost-periodic operators. Grundlehren der mathematischen Wissenschaften 297, Springer 1992.
[FroSpe1] Fröhlich, J., Spencer, T. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Comm. Math. Phys. 88 (1983), 151–189.
[FroSpe2] Fröhlich, J., Spencer, T. A rigorous approach to Anderson localization. Phys. Rep. 103 (1984), no. 1–4, 9–25.
[FroSpeWit] Fröhlich, J., Spencer, T., Wittwer, P. Localization for a class of one dimensional quasi-periodic Schrödinger operators. Commun. Math. Phys. 132 (1990), 5–25.
[Fur] Fürstenberg, H. Noncommuting random products. Trans. AMS 108 (1963), 377–428.
[FurKes] Fürstenberg, H., Kesten, H. Products of random matrices. Ann. Math. Statist 31 (1960), 457–469.

[GolSch1] Goldstein, M., Schlag, W. Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions. Ann. of Math. (2) 154 (2001), no. 1, 155–203.

[GolSch2] Goldstein, M., Schlag, W. Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues, preprint 2005.

[GorJitLasSim] Gordon, A., Jitomirskaya, S., Last, Y., Simon, B. Duality and singular continuous spectrum in the almost Mathieu equation. Acta Math. 178 (1997), 169–183.

[Her] Herman, M. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2. Comment. Math. Helv. 58 (1983), no. 3, 453–502.

[Kle] Klein, S. Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. J. Funct. Anal. 218 (2005), no. 2, 255–292.

[Las] Last, Y. Zero measure spectrum for the almost Mathieu operator. Comm. Math. Phys. 164 (1994), 421–432.

[Lev] Levin, B. Ya. Lectures on entire functions. Transl. of Math. Monographs, vol. 150. AMS, Providence, RI, 1996.

[Pui1] Puig, J. Cantor spectrum for the almost Mathieu operator. Comm. Math. Phys. 244 (2004), no. 2, 297–309.

[Pui2] Puig, J., thesis, Barcelona 2004.

[Sch] Schlag, W. On the Integrated Density of States for Schrödinger Operators on $Z^2$ with Quasi Periodic Potential, Comm. Math. Phys. 223, 47–65 (2001).

[Sin1] Sinai, Y. G. Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential. J. Stat. Phys. 46 (1987), 861–909.

[SorSpe] Sorets, E., Spencer, T. Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials. Comm. Math. Phys. 142 (1991), no. 3, 543–566.

[Tho] Thouless, D. Scaling for the discrete Mathieu equation. Comm. Math. Phys. 127 (1990), 187–193.

[Weg] Wegner, F. Bounds on the density of states in disordered systems. Z. Phys. B44 (1981), 9–15.

Dept. of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
E-mail address: gold@math.toronto.edu

253-37 Caltech, Pasadena, CA 91125, U.S.A.
E-mail address: schlag@math.princeton.edu