Contemplating some invariants of the Jaco Graph, \( J_n(1), n \in \mathbb{N} \)

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Abstract

Kok et.al. [7] introduced Jaco Graphs (order 1). In this essay we present a recursive formula to determine the independence number \( \alpha(J_n(1)) = |I| \) with, \( I = \{ v_{i,j} | v_1 = v_1, v_i = v_i,j = v(d^+ (v_{m,(j-1)}+m+1)) \} \). We also prove that for the Jaco Graph, \( J_n(1), n \in \mathbb{N} \) with the prime Jaconian vertex \( v_i \) the chromatic number, \( \chi(J_n(1)) \) is given by:

\[
\begin{align*}
\chi(J_n(1)) &= (n - i) + 1, \quad \text{if and only if the edge } v_i v_n \text{ exists}, \\
&= n - i, \quad \text{otherwise}.
\end{align*}
\]

We further our exploration in respect of domination numbers, bondage numbers and declare the concept of the murtage number\(^1\) of a simple connected graph \( G \), denoted \( m(G) \). We conclude by proving that for any Jaco Graph \( J_n(1), n \in \mathbb{N} \) we have that \( 0 \leq m(J_n(1)) \leq 3 \).

Keywords: Jaco graph, Hope graph, Independence number, Covering number, Chromatic number, Domination number, Bondage number, Murtage number, \( d_{am} \)-sequence, Compact \( \gamma \)-set, Murtage partition.

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1 Introduction

Let \( \mu(G) \) be an arbitrary invariant of the simple connected graph \( G \). The \( \mu \)-stability number of \( G \) is conventionally, the minimum number of vertices whose removal changes \( \mu(G) \). If

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\(^2\)In honour of U.S.R. Murty, co-author of [2].
the removal of the minimum vertices results in a decrease of the invariant the result is conventionally denoted, $\mu^-(G)$ and if the change is to the contrary the change is denoted $\mu^+(G)$. We note that the domination number, $\gamma(G')$, of a subgraph $G'$ of $G$ can be larger or smaller than $\gamma(G)$. Note that a subgraph may result from the removal of vertices and/or edges from $G$. Furthermore, we note that the removal of edges only from the graph $G$ to obtain $G'$ can only result in $\gamma(G') \geq \gamma(G)$.

The minimum number of edges whose removal from $G$ results in a graph $G'$ with $\gamma(G') > \gamma(G)$, is called the bondage number $b(G)$, of $G$.

2 Some invariants of a Jaco Graph, $J_n(1), n \in \mathbb{N}$

The infinite directed Jaco graph (order 1) was introduced in [7], and defined by $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i-d^-(v_i) \geq j$. The graph has four fundamental properties which are; $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and, if $v_j$ is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if $v_k$, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to $v_j$ and finally, the degree of vertex $k$ is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

2.1 Independence number of a Jaco Graph, $J_n(1), n \in \mathbb{N}$

Consider the underlying graph of the finite directed Jaco Graph, $J_n(1), n \in \mathbb{N}$. Obviously the graph has vertices $v_1, v_2, v_3, ..., v_n$. Because the independence number is defined to be the number of vertices in a maximum independent set [1], it is optimal to choose non-adjacent vertices recursively, each of minimum indice. This observation leads to the next theorem. Observe that $v_{i,j} = v_i$ as calculated on the $j$-th step of a recursive formula applied to the vertices of a simple connected graph.

**Theorem 2.1.** The cardinality of the set $\mathcal{I} = \{v_{i,j} | v_1 = v_{1,1} \in \mathcal{I} \text{ and } v_i = v_{i,j} = v_{d^+(v_{m,(j-1)+m+1})}\}$, derived from the underlying graph of the Jaco Graph $J_n(1), n \in \mathbb{N}$ is equal to the independence number, $\alpha(J_n(1))$.

**Proof.** Clearly for $J_1(1)$ the cardinality of $\mathcal{I} = \{v_1\}$ equals 1 and it is indeed the maximum independent set. It is equally easy to see that the set $\mathcal{I} = \{v_1\}$ is indeed a maximum in-
dependent set of $J_2(1)$ as well. Considering $J_3(1)$ the derived maximum independent set is, $\mathbb{I} = \{v_1, v_3\}$. It easily follows that $v_3 = v_{3,2} = v_{d+(v_1)+1+1} = v_{d+(v_1, (2-1))+1+1}$. It follows that this maximum independent set (not unique) remains valid for $J_3(1), J_4(1), J_5(1)$. Hence, $\alpha(J_i(1)) = 2, \text{ for } 3 \leq i \leq 5$.

Assume on the $\ell$-th step we have the maximum independent set $\{v_1, v_3, v_6, \ldots, v_{d+(v_m, (\ell-1))}+m+1\}$ in respect of the Jaco Graphs $J_i(1)$ for $k = (d+(v_m, (\ell-1)) + m + 1) \leq i \leq k + d+(v_k)$.

Considering the Jaco Graph $J_{(k+d+(v_k)+1)}(1)$ will yield a maximum independent set, $\{v_1, v_3, v_6, \ldots, v_{d+(v_m, (\ell-1))}+m+1, v_{(k+d+(v_k)+1)}\}$. So the result holds for the $(\ell + 1)$-th step. Through mathematical induction the result holds in general.

**Corollary 2.2.** It follows that the covering number, $\beta(J_n(1)) = n - \alpha(J_n(1))$.

### 2.2 Chromatic number of a Jaco Graph, $J_n(1), n \in \mathbb{J}$

From the definitions provided in [7] the Hope Graph of the Jaco Graph, $J_n(1)$ is the complete graph on the vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ if and only if $v_i$ is the prime Jaconian vertex of $J_n(1)$. Hence, $\mathbb{H}_n(1) \simeq K_{n-i}$. The reader is reminded that a $t$-colouring of a graph $G$ is a map $\lambda : V(G) \to [c] := \{1, 2, 3, \ldots, c, c \geq 0\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u, v \in V(G)$ are adjacent in $G$. The chromatic number of $G$ denoted $\chi(G)$ is the minimum $c$ such that $G$ is $c$-colourable. Now the following theorem can be settled.

**Theorem 2.3.** For the Jaco Graph, $J_n(1), n \in \mathbb{N}$ with the prime Jaconian vertex $v_i$ we have that the chromatic number, $\chi(J_n(1))$ is given by:

$$
\chi(J_n(1)) = \begin{cases} 
(n - i) + 1, & \text{if and only if the edge } v_iv_n \text{ exists,} \\
(n - i), & \text{otherwise.} 
\end{cases}
$$

**Proof.** (a(i)) If the edge $v_iv_n$ exists the largest complete subgraph of $J_n(1)$ is given by $\mathbb{H}_n(1) + v_i \simeq K_{n-i+1}$. Since it is known that $\chi(K_{n-i+1}) = (n - i) + 1$, it follows that $\chi(J_n(1)) \geq (n - i) + 1$. For $J_1(1)$ we have that the prime Jaconian vertex is $v_1$ and inherently connected to itself. One may imagine the imaginary edge "$v_1v_1$" to find $\chi(J_1(1)) = (1 - 1) + 1 = 1$ to be true. For $J_2(1)$ the prime Jaconian vertex is $v_1$ and the Hope Graph, $\mathbb{H}_2(1) \simeq K_1$. Also, the edge $v_1v_2$, exists. Thus, $\chi(J_2(1)) = (2 - 1) + 1 = 2$, which is true.
Now assume the result holds for any \( J_n(1), n > 2 \) for which the edge \( v_i v_n \) exists and \( v_i \) is the prime Jaconian vertex. Label the \((n - i) + 1\) colours used to colour the vertices \( v_i, v_{i+1}, v_{i+2}, \ldots, v_n \), consecutively, \( c_i, c_{i+1}, c_{i+2}, \ldots, c_n \). From definitions 1.3 and 1.4 and Lemma 1.1 [7] it follows that if the prime Jaconian vertex \( v_i \) is unique, the Jaco Graph \( J_{n+1}(1) \) will be the smallest Jaco Graph larger than \( J_n(1) \) with prime Jaconian vertex \( v_{i+1} \) for which the edge \( v_{i+1} v_{n+1} \), exists. It also implies that \( H_{n+1}(1) \cong H_n(1) \). Since the edge \( v_i v_{n+1} \) does not exists, the colouring of \( v_{n+1} \) with \( c_1 \) suffices, whilst the colouring of the rest of the graph \( J_{n+1}(1) \) remains the same as that of \( J_n(1) \). So clearly the result \( \chi(J_{n+1}(1)) = ((n + 1) - (i + 1)) + 1 = (n - i) + 1 = \chi(J_n(1)) \) holds.

From definitions 1.3 and 1.4 and Lemma 1.1 [7] it follows that if the prime Jaconian vertex \( v_i \) of \( J_n(1) \) is not unique, the Jaco Graph \( J_{n+2}(1) \) will be the smallest Jaco Graph larger than \( J_n(1) \) with prime Jaconian vertex \( v_{i+1} \) for which both the edge \( v_{i+1} v_{n+1} \) and \( v_{i+1} v_{n+2} \), exist (also see the Fisher Table for illustration). Since the edge \( v_i v_{n+1} \) does not exist, colour vertices \( v_{n+1}, v_{n+2} \) respectively \( c_1 \) and \( c_{n+1} \). Since \( H_{n+2}(1) \) has \((n - i) + 1\) vertices we must consider the colouring of \( K_{(n-i)+2} \). We however, have that \( \chi(K_{(n-i)+2}) = (n - i) + 2 = ((n - i) + 1) + 1 = ((n + 1) - i) + 1 = ((n + 2) - (i + 1)) + 1 = \chi(J_{n+2}(1)). \)

Assume that for some Jaco Graph \( J_n(1) \) with the edge \( v_i v_n \) existing we have that \( \chi(J_n(1)) > (n - i) + 1 \). Clearly this contradicts the definition on minimality of the colouring set so we safely conclude that \( \chi(J_n(1)) \leq (n - i) + 1 \).

Since all cases have been considered the necessary condition follows through mathematical induction.

(a(ii)) Consider the converse statement namely, if \( \chi(J_n(1)) = (n - i) + 1 \) then the edge \( v_i v_n \) exists and assume it is not true for some Jaco Graph \( J_n(1) \) by assuming that the edge \( v_i v_n \) does not exists. The Hope Graph \( H_n(1) \cong K_{n-i} \) requires \( n - i \) colours. Since, the edge \( v_i v_n \) does not exists, colouring \( v_i \) the same as \( v_n \) will suffice. It implies that using \( (n - i) + 1 \) colours contradicts the definition on minimality of the colouring set. Hence, the sufficient condition follows thus, the result.

(b)\( ^3 \) The result follows directly from the proof of result (a) and the definition on minimality of the colouring set.

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\(^3\)Reader can formalise the proof as an exercise.
2.3 Introduction to the murtage number \( m(G) \) of a simple connected graph \( G \)

Note that if vertices \( u \) and \( v \) are not adjacent in \( G \), then \( \gamma(G + uv) \leq \gamma(G) \). The significance of this concept becomes apparent in the application of domination theory. In a situation where a \( \gamma \)-set of a graph is to represent costly facilities in a network \( N \), it may be preferable to establish additional links (edges) between vertices of \( N \) rather than constructing facilities at all vertices of a \( \gamma \)-set.

In order to calculate the murtage number of a graph we introduce the concept of a \( d_{om} \)-sequence of a \( \gamma \)-set, \( X_i \) of a graph. Label the vertices of \( X_i \) such that \( V(G) \) can be partitioned into sets \( D_{1,i}, D_{2,i}, ..., D_{\gamma(G),i} \) such that \( D_{j,i} \) contains the vertex \( v_j \in X_i \) and vertices in \( V(G) - X_i \) which are adjacent to \( v_j \) and such that, \( |D_{1,i}| \leq |D_{2,i}| \leq ... \leq |D_{\gamma(G),i}| \) and \( |D_{1,i}| \) is a minimum. We define a \( d_{om} \)-sequence of the \( \gamma \)-set \( X_i \) as \( (|D_{1,i}|, |D_{2,i}|, ..., |D_{\gamma(G),i}|) \). Clearly a \( \gamma \)-set can have more than one \( d_{om} \)-sequence. Assume \( G \) has \( k \) \( \gamma \)-sets namely \( X_1, X_2, ..., X_k \). Let \( \theta = absolute(min|D_{1,j}|) \) for some \( X_j \). All \( \gamma \)-sets, \( X_i \) for which firstly, \( |D_{1,i}| = \theta \) (primary condition) and secondly, \( d(v_1, v_i) \) is minimum for all \( v_i \in X_\ell \) (secondary condition) is said to be compact \( \gamma \)-sets. The partitioning described above in respect of a compact \( \gamma \)-set is called a murtage partition of \( V(G) \).

As example let us consider the path \( P_4 \) with vertices labelled from left to right \( v_1, v_2, v_3 \) and \( v_4 \). Clearly the \( \gamma \)-set \( \{v_2, v_3\} \) is a \( \gamma \)-set with the \( d_{om} \)-sequence = (2,2) and \( d(v_2, v_3) = 1 \). The aforesaid set is however not a compact \( \gamma \)-set because the set \( \{v_1, v_3\} \) has \( d_{om} \)-sequence = (1,3) meaning \( absolute(min|D_{1,i}|) = 1 < 2 \) which is primary in the definition. The fact that \( d(v_1, v_3) = 2 = 1 = d(v_2, v_3) \) is secondary in the definition. The corresponding murtage partition of \( V(P_4) \) is \( \{\{v_1\}, \{v_2, v_3, v_4\}\} \).

Another example will be considering the path \( P_5 \) with the vertices labelled left to right \( v_1, v_2, v_3, v_4 \) and \( v_5 \). Clearly the sets \( \{v_1, v_4\}, \{v_2, v_4\} \) are \( \gamma \)-sets. Both have \( d_{om} \)-sequence (2,3) with set \( \{v_2, v_4\} \) providing \( d(v_2, v_4) = 2 \) hence compact, whilst the set \( \{v_1, v_4\} \) provides \( d(v_1, v_4) = 3 \) hence, non-compact. The murtage partition associated with the compact \( \gamma \)-set \( \{v_2, v_4\} \) is \( \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\} \).

**Definition 2.1.** We define the murtage number, \( m(G) \), of a simple connected graph \( G \) to be the minimum number of edges that has to be added to \( G \) such that the resulting graph \( G' \) has \( \gamma(G') < \gamma(G) \).

It follows from the definition that \( m(G) = 0 \) if and only if \( \gamma(G) = 1 \).
Theorem 2.4. Let $|D_{1,i}| = \theta$ for some compact $\gamma$-set $X_i$ of $G$, then:

$$m(G) = \begin{cases} 
\theta, & \text{if and only if } v_1 \text{ is not adjacent to any } v_j \in X_i, \\
\theta - 1, & \text{if and only if } v_1 \text{ is adjacent to some } v_j \in X_i.
\end{cases}$$

Proof. (a) Assume $v_1$ is not adjacent to any $v_j \in X_i$. Since we are considering a $d_{om}$-sequence of a compact $\gamma$-set of $G$, it is clear that the vertices in $D_{1,i}$ are uniquely dominated by $v_1$ hence, we must join all vertices in $D_{1,i}$ to vertices in $X_i - \{v_1\}$ in order to eliminate $v_1$ from $X_i$. Since, $|D_{1,i}| = \theta$ is an absolute minimum over all minimum number of edges to be added to have a resulting graph $G'$ such that $\gamma(G') = \gamma(G) - 1 < \gamma(G)$, it follows from the definition that $m(G) = \theta$.

Conversely we assume that $m(G) = \theta$ and that $v_1$ is adjacent to some $v_j \in X_i$. Since we are considering a $d_{om}$-sequence of a compact $\gamma$-set of $G$, it is clear that the vertices in $D_{1,i}$ are uniquely dominated by $v_1$ hence, we must join all vertices in $D_{1,i} - \{v_1\}$ to vertices in $X_i - \{v_1\}$ in order to eliminate $v_1$ from $X_i$. However, it required only $\theta - 1$ edges to be added hence, $m(G) = \theta - 1$. The latter is a contradiction, implying $v_1$ is not adjacent to any vertex $v_j \in X_i$.

(b) The proof follows in a similar way as part (a).

Proposition 2.5. For any graph $G$ for which $m(G) \geq 1$ we have that $m(G) = \gamma^-(G)$.

Proof. Since $m(G) \geq 1$ it follows that $\gamma(G) \geq 2$. Consider any compact $\gamma$-set $X_i$ of $G$. From the definition it follows that $m(G) = |D_{1,i}| = \theta$. If $\gamma^-(G) = k < \theta$, let $Y \subseteq V(G)$ be a $\gamma^-$-set of $G$ with $|Y| = k$. Since $\gamma(G - Y) < \gamma(G)$ there exists at least one vertex $v_j \in X_i$ such that every vertex of $Y \cup X_i \cup \{v_j\}$ is joined to a vertex in $X_i - \{v_1\}$. Join every vertex in $Y$ to a vertex $v_i \in X_i, v_i \neq v_j$ to obtain $G'$. Clearly $\gamma(G') < \gamma(G)$ and it follows that $m(G) \leq k < \theta$, which is a contradiction.

If $\theta < |Y| = \gamma^-(G)$ we consider the graph $G - D_{1,i}$ which has $\gamma$-set, $X_i - \{v_1\}$. Since $\gamma(G - D_{1,i}) < \gamma(G)$ we have that $\gamma^-(G) \leq \theta < |Y|$ which renders a contradiction.

Hence $m(G) = \gamma^-(G)$.

Although the two invariants differ conceptually, the result is very useful. We only have to
investigate one of the invariants and all the results will hold for the other.

**Theorem 2.6.** Any simple connected graph $G$ has a spanning subtree $T$ such that: 
$\Delta(T) = \Delta(G)$, $\gamma(T) = \gamma(G)$ and $m(T) = m(G)$.

*Proof. Consider a compact $\gamma$-set, $X_i = \{v_1, v_2, v_3, ..., v_{\gamma(G)}\}$ of $G$ and an associated murtagh partitioning of $V(G)$. Consider the forest $\cup \langle D_{j,i} \rangle$ with $\langle D_{j,i} \rangle$ the star with edges $\{v_jv_k | v_k \in D_{j,i}\}$. If in $\langle D_{\gamma(G),i} \rangle$ we have $d(v_{\gamma(G)}) = \Delta(G)$, then join all $\langle D_{j,i} \rangle$, $j = 1, 2, ..., (\gamma(G) - 1)$ to $\langle D_{\gamma(G),i} \rangle$ with one edge $uv$ if and only if $u \in D_{\gamma(G),i}$, $v \in D_{j,i}$ and $uv \in E(G)$. Label the tree $T^\star$. If any of the stars $\langle D_{j,i} \rangle$ has not been joined to $\langle D_{\gamma(G),i} \rangle$ we join them to $T^\star$ with one edge $uv$ if and only if $u \in V(T^\star)$, $v \in D_{j,i}$ and $uv \in E(G)$. Label this successor tree $T^\star$. Since $G$ is connected it is evident that recursively all stars will eventually be connected. Clearly $\Delta(T) = \Delta(G)$.

If in $\langle D_{\gamma(G),i} \rangle$ we have $d(v_{\gamma(G)}) < \Delta(G)$, join all $\langle D_{j,i} \rangle$, $j = 1, 2, ..., (\gamma(G) - 1)$ to $\langle D_{\gamma(G),i} \rangle$ with one edge $uv_{\gamma(G)}$ if and only if $u \in D_{j,i}$ and $uv_{\gamma(G)} \in E(G)$. Label the tree $T^\star$. Note that $\Delta(T^\star) = \Delta(G)$. All other stars $\langle D_{j,i} \rangle$ which have not been joined at this first iteration can recursively be joined as described above. Hence, in all cases a spanning subtree $T$ can be constructed with $\Delta(T) = \Delta(G)$.

To complete the proof we note that $\gamma(G) \leq \gamma(T)$ and the set $X_i$ is a $\gamma$-set of $T$, hence $\gamma(T) = \gamma(G)$.
It is also clear that $X_i$ is a compact $\gamma$-set of $T$ hence, $m(T) = m(G)$. $\square$

Furthermore, let $\mathbb{G} = \{G_1, G_2, G_3, ..., G_\ell\}$ with each $G_i$, a simple connected graph. It follows easily that $\gamma(\cup_{\forall i} G_i) = \sum_{\forall i} \gamma(G_i)$ and similarly, $m(\cup_{\forall i} G_i) = \sum_{\forall i} m(G_i)$. Also if $\gamma(G_i) \leq \gamma(H_i)$, $i = 1, 2, 3, ..., n$ then $\gamma(\cup_{\forall i} G_i) = \sum_{\forall i} \gamma(G_i) \leq \sum_{\forall i} \gamma(H_i) = \gamma(\cup_{\forall i} H_i)$.

### 2.4 Murtagh number of a Jaco Graph, $J_n(1)$, $n \in \mathbb{N}$

In this subsection, reference to a Jaco Graph will mean we consider the undirected underlying graph of the Jaco Graph. Hence we peel off the orientation of the Jaco Graph. From the definition of a Jaco Graph it follows that all Jaco Graphs on $n \geq 2$ has at least one leaf (vertex with degree = 1). Hence, the bondage number is $b(J_n(1))_{n \geq 2} = 1$. 

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The fact that $m(J_n(1))_{n \in \mathbb{N}} \geq 0$ follows from the definition.

From the definition of a Jaco Graph it follows easily that vertex $v_1$ dominates $J_1(1)$ and $J_2(1)$ and vertex $v_2$ dominates $J_3(1)$ hence, $m(J_1(1)) = m(J_2(1)) = m(J_3(1)) = 0$.

For $J_4(1)$ and $J_5(1)$ it follows that the set $\{v_1, v_3\}$ is a compact $\gamma$-set with the $d_{om}$-sequences, $(1, 2)$ and $(1, 3)$ hence, $m(J_4(1)) = m(J_5(1)) = 1$.

For the Jaco Graphs $J_6(1)$ and $J_7(1)$ we have sets $\{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_2, v_7\}$ being $\gamma$-sets with only $\{v_2, v_4\}$ and $\{v_2, v_5\}$ the compact $\gamma$-sets. The corresponding $d_{om}$-sequences are $(2, 4)$ and $(2, 5)$ hence, $m(J_6(1)) = m(J_7(1)) = 2$. For $J_8(1)$ we have that the sets $\{v_2, v_5\}, \{v_2, v_6\}, \{v_2, v_7\}$ are $\gamma$-sets with $\{v_2, v_5\}$ the unique compact $\gamma$-set. The unique corresponding $d_{om}$-sequence is $(2, 6)$ so, $m(J_8(1)) = 2$.

In respect of $J_9(1)$ and $J_{10}(1)$ we make the interesting observation that exactly two $\gamma$-sets, both being compact $\gamma$-sets namely, $\{v_2, v_6\}$ and $\{v_2, v_7\}$, exist. The corresponding $d_{om}$-sequences are $(3, 6)$ and $(3, 7)$ respectively, meaning, $m(J_9(1)) = m(J_{10}(1)) = 3$.

In the case of $J_{11}(1)$ an unique compact $\gamma$-set $= \{v_2, v_7\}$ exists with the $d_{om}$-sequence $(3, 8)$. So also here we have $m(J_{11}(1)) = 3$.

For $J_{12}(1)$ and $J_{13}(1)$ we note that the sets $\{v_1, v_3, v_8\}, \{v_1, v_3, v_9\}$ and $\{v_1, v_3, v_{10}\}$ are the $\gamma$-sets with $\{v_1, v_3, v_8\}$ the unique compact $\gamma$-set. The corresponding $d_{om}$-sequences are $(1, 3, 8)$ and $(1, 3, 9)$. Hence, $m(J_{12}(1)) = m(J_{13}(1)) = 1$. Further exploratory analysis leads to the next theorem.

**Theorem 2.7.** For any Jaco Graph $J_n(1), n \in \mathbb{N}$ we have $0 \leq m(J_n(1)) \leq 3$. The bounds are obviously sharp as well.

**Proof.** Following from the definition of a finite Jaco Graph $J_n(1), n \in \mathbb{N}$, it follows easily that the *murtage number* can always be found by linking the minimum number of minimum (smallest) indiced vertices labelled $v_i, i \in \{1, 2, 3, \ldots, k\}_{k \leq n}$ to some $v_j \in \text{compact } \gamma$-set of $J_n(1)$.

Assume $m(J_n(1)) \geq 4$. It implies that at least the vertices $v_1, v_2, v_3, v_4$ have to be linked to some vertex $v_j \in \gamma$-set, in order to reduce the value of $m(J_n(1))$ with at least 1. It also implies that $v_1, v_2, v_3, v_4 \notin \text{compact } \gamma$-set else $m(J_n(1)) \leq 3$. Furthermore, the lowest indiced vertex $v\in \gamma$-set is $4 < \ell = 8$. However, the lowest indiced vertex dominated by $v_8$ is $v_5$ implying that vertices $v_1, v_2, v_3, v_4$ were not dominated, hence not adjacent to any
vertex in the compact $\gamma$-set under consideration. The latter is a contradiction in terms of the definition of a $\gamma$-set (therefore, compact $\gamma$-set). So the result follows.

**Corollary 2.8.** For any finite Jaco Graph $J_n(1), n \in \mathbb{N}$ we have that:

$$\gamma(J_n(1)) = \gamma(J_{(n-d-(v_n)-d-(v_{(n-d-(v_n))-1})}) + 1.$$  

**Proof.** Consider the Jaco Graph $J_n(1)$ and let vertex $v_\ell$ be the minimum indexed vertex with the edge $v_\ell v_n \in E(J_n(1))$. Clearly all vertices $v_k \neq \ell \in \{v_\ell - d - (v_\ell), ..., v_n\}$ are adjacent to $v_\ell$. Reducing by one more vertex we consider the Jaco Graph $J_{(n-d-(v_n)-d-(v_{(n-d-(v_n))-1})}$.

Hence if $X_i$ is a compact $\gamma$-set of $J_n(1)$, a compact $\gamma$-set of $J_{(n-d-(v_n)-d-(v_{(n-d-(v_n))-1})}$ is given by $X_i \cup \{v_\ell\}$.

It concludes the result that $\gamma(J_n(1)) = \gamma(J_{(n-d-(v_n)-d-(v_{(n-d-(v_n))-1})}) + 1$.

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