Random Graph Gauge Theories as Toy Models for Non-perturbative String Theories

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Abstract

We present simple models which exhibit some of the remarkable features expected to hold for the yet unknown non-perturbative formulation of string theories. Among these are (1) the absence of a background or embedding space for the full theory, (2) perturbative ground states (local minima of the action) having the characteristics of spaces of different dimension, (3) duality transformations between large and small coupling expansions, and (4) perturbative excitations of these ground states which can be interpreted as string world sheets or p-brane world volumes. In this context we formulate gauge theories on arbitrary graphs and speculate about actions for graphs which in a continuum and/or thermodynamic limit might be related to the Einstein-Hilbert action.

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1 Introduction

One of the most striking aspects of M-theory is that it seems to have perturbative regimes with background spaces of different dimensions and geometrically different objects (strings, p-branes etc.) as the corresponding elementary excitations (see e.g. [1]). One way to relate spaces of different dimensions is t-duality: let one of the dimensions of the background space be compactified to a cylinder of radius $R$ then in the presence of strings there exist duality relations to a model with a background space having a compactified dimension being a cylinder of radius $1/R$. Therefore, in the limit $R \to \infty$ (and may be some tuning of other couplings) one finds a duality relation between background spaces of different dimension. However, not all relations between different regimes and dimensions can be understood this way. So it may be helpful to look for objects which include regular “spaces” of given dimensions as special cases, but which are more general. Graphs are one example for such objects.

Graphs have a relatively simple structure (they are sets with a symmetric relation, details will be explained in sec. 2) but they allow for more complex derivatives. Any regular lattice for instance, can be considered as a graph. Lattices serve as background “spaces” in lattice models like lattice gauge theories. Therefore, although graphs may be defined without reference to a background space or a dimension, they can dynamically provide background spaces with a fixed dimension. So one might wonder, if there exist actions or weights on the set of all graphs such that lattices (amongst others) are the local minima of these actions. Or, to be more speculative and ambitious, does there exist an “Einstein” action on the set of graphs such that the local minima correspond to discretized solutions of Einsteins equations in different dimensions. In this case the local minima would provide the background spaces, and perturbations around these minima (defects like dislocations or disclinations etc.) might be considered as fluctuations of this background space with the physical interpretation depending on the geometric structures of these fluctuations. The appealing feature of this picture is that background space and excitations are the same objects - parts of a graph. This reminds us that in general relativity the splitting of the metric for perturbative expansions, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, introduces two objects which seem to be of completely different nature in the perturbative picture (a static background space and a particle propagating in this space), but are in principle the same objects on a non-perturbative level.

Another striking feature of some of the perturbative regimes of M-theory is s-duality: the perturbation expansion in some coupling $g$ is related to some other perturbation expansion in $1/g$ (or, more general, expansions for small $g$ are related to expansions for large $g$). This kind of duality has a long history in statistical mechanics, the most famous example being the self-duality of the Ising model on a 2-dimensional square lattice [2], see e.g. [3, 4]. In 1971 Wegner [5] found more general duality relations among a whole class of spin models including spin-gauge theories on lattices of higher dimension. The first example of duality in quantum field theory was the Coleman-Mandelstam duality between the Sine-Gordon theory and the massive Thirring model in 2 space-time dimensions [6, 7]. In this case the dual fields could be constructed explicitly. In later generalizations of duality to non-abelian gauge theories [8] this was not possible, although there was a general believe that the “defects” of one model are

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[1] Lattices are usually defined as invariant representation spaces of some group of discrete translations (up to boundary conditions) and, therefore, by definition regular. However, in the context of discretized space-time structures one sometimes reads about “random” lattices, meaning combinatorial triangulations or cell complexes. For this reason, we emphasise at this point the regularity of a lattice. In the following, lattices will always be assumed to be regular, otherwise we will speak of graphs.
related to the fundamental excitations of the dual model. More recently it is the AdS-CFT
duality (see, e.g., [9] and references therein), which seems to be of direct relevance to string
theory.

In this paper we consider the spin models and spin-gauge theories of Wegner as the most
simple examples of models with duality. But, why might gauge theories be appealing candidates
in the search for M-theory? Usually the argumentation goes the other way round: string theory
leads naturally to gauge theories as well as to general relativity, because (super)-string theory
includes massless spin-1 and spin-2 particles in a covariant way. But the argument may be
turned around, at least partly. Gauge theories (most explicitly on the lattice, see e.g. [10]
and references therein) allow for a strong coupling expansion in terms of closed surfaces -
“string world-sheets”. Apart from free gauge theories (like electrodynamics) a whole spectrum
of particles - glueballs - can be obtained from loop-loop correlation functions, including spin-1
and spin-2 particles. These particles, however, are massive and therefore no candidates for a
gauge theory or general relativity. Another difference is that the expansion parameter in string
theory is the weight for string sheet topologies while in gauge theories the weight for closed
surfaces depends on their area and, possibly (depending the gauge group), on local intersections
of surfaces. It is not clear how to formulate a gauge theory such that in the large coupling
expansion the weight for the surface topologies can be tuned. Nevertheless, a non-trivial gauge
theory with massless spin-1 and spin-2 “glueballs” might be an interesting alternative to (or
candidate for) non-perturbative string theories. These remarks merely serve as a motivation
for studying gauge theories in the context of non-perturbative string theory.

The models I shall discuss in this paper exhibit some of the remarkable features expected
for the non-perturbative formulation of string theories, among which are:

1. The formulation of these models make no explicit reference to a background or embed-
ding space, a dimension, or a geometric structure (apart from the structures inherent to
graphs).

2. Depending on the values of the coupling constants, the actions for these models have
enumerous local minima, some of which correspond to lattices with different dimensions
for the different ground states.

3. Not all local minima of the action correspond to flat spaces but also other forms of
“background spaces” - e.g., compactified dimensions - exist.

4. In some cases, perturbations around these ground states can be interpreted as summations
of closed surfaces, the euclidean equivalent of world sheets of strings, or closed volumes
of higher dimension, the analoga of world volumes of p-branes.

5. In some cases duality transformations exist which transform the perturbation expansion
for a small coupling constant into a different perturbative regime where this coupling is
large.

In this paper I shall mainly concentrate on two special models: $\mathbb{Z}_2$-spin (Ising) model and
$\mathbb{Z}_2$-gauge theory coupled to random, non-directed and simple graphs. The generalization
of these models to arbitrary scalar fields or gauge groups is straightforward and will be sketched.
However, the implementation of duality transformations for these generalized cases (especially
for non-abelian gauge groups) is more difficult.
In section 2, we review some simple structures of graph theory, formulate actions for graphs (weights on the set of graphs), and discuss their properties.

Section 3 will summarize the formulation of $\mathbb{Z}_2$-spin, $\mathbb{Z}_2$ gauge, and $\mathbb{Z}_2 p$-gauge (analog of gauge theories of $p$-forms) theories on hypercubic (regular) lattices. In this context, we also review the duality transformations among these models, first formulated by Wegner, as relations between the small and high temperature perturbation expansions, which can be interpreted as summations over objects of different geometrical structure.

In section 4, scalar and gauge field theories are formulated on arbitrary graphs, and the features of the $\mathbb{Z}_2$ cases are discussed in more detail. Finally, in sec. 5, I will speculate about further generalizations.

## 2 Random Graphs

In this section, I shall first review some of the algebraic properties of graph theory, in particular the notion of the adjacency matrix and related structures (Sect. 2.1). Next, I will formulate different contributions to an action for random graphs and discuss their properties (Sect. 2.2). I will also speculate about an action for graphs which might be related to the Einstein-Hilbert action in general relativity (Sect. 2.3).

### 2.1 The adjacency matrix

The following notions can be found in any good book on graph theory, e.g. in [11] where the algebraic properties of graphs are emphasized.

We will only consider undirected graphs without self-loops and multiple connections between vertices. In this case, a graph can be defined as a set $V$ (the set of vertices) together with a symmetric, non-reflexive relation stating which of the vertices are neighbours. The adjacency matrix $A$ is the matrix of this relation, i.e.

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$A$ is symmetric and its diagonal elements are 0. Two graphs are considered to be equivalent, if they differ only by a relabeling of their vertices. Therefore, two adjacency matrices $A_1$ and $A_2$ correspond to the same graph, if

$$A_2 = P^+ A_1 P$$

where $P$ is some permutation matrix, acting on the functions over the set of vertices. If, for an adjacency matrix $A$, we have

$$A = P^+ AP \quad \text{or} \quad [A, P] = 0 \quad (2)$$

for some permutation $P$, we call $P$ a symmetry of $A$. The set of all $P$’s which are symmetries of $A$ generate the symmetry group of that graph. (The size of this group determines the symmetry factors for Feynman diagrams in a QFT perturbation expansion.)

We define a $L$-path on a graph from a vertex $i$ to a vertex $j$ as a sequence of vertices $i = k_0, k_1, k_2, ..., k_{L-1}, j = k_L$, such that $k_n$ and $k_{n+1}$ are neighbored vertices, i.e. $A_{k_n k_{n+1}} = 1$, for all $n = 0, ..., L - 1$. The number of steps, $L$, is called the length of the path. If all vertices of a graph can be connected by paths, we call the graph connected.
The adjacency matrix $A$ may also be considered as the generator of the number of random paths on the graph:

$$(A^L)_{ij} = \text{number of paths from } j \text{ to } i \text{ of length } L = \text{number of } L\text{-paths from } j \text{ to } i.$$  

This follows immediately from the definition of $A$ and the definition of paths, as

$$(A^L)_{ij} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{L-1}} A_{ik_1} A_{k_1 k_2} \cdots A_{k_{L-1} j}.$$  

The only non-zero contributions to this expression come from sequences for which the elements of the adjacency matrix are all 1, i.e. paths from $j$ to $i$. In particular,

$$(A^2)_{ii} = \delta_i = \text{number of incident lines to } i,$$  

i.e. the degree $\delta_i$ of $i$, and

$$\text{tr}(A^2) = 2E,$$

where $E$ is equal to the total number of edges (lines) of the graph. (We mostly consider graphs with a finite number of vertices and lines. However, in statistical mechanics one is usually interested in the thermodynamic limit where $V$ and $E$ become infinite. The problems related to this limit are well known shall not be discussed here.) For later purposes, we also define the valence matrix $V$ of the graph as the diagonal matrix of the degrees of the vertices:

$$V_{ij} = \text{diag}\{\delta_i\}.$$  

If all vertices of a graph have the same degree $\delta$, the graph is called regular of degree $\delta$. Let $K_{ij} = 1$, then

$$[K, A] = 0 \text{ iff } A \text{ is the adjacency matrix of a regular graph.}$$  

The proof is straight forward (see, e.g., [1]).

The geographic distance $r(i,j)$ between two vertices $i$ and $j$ on a graph is defined by the length of the shortest path connecting these two vertices. The number of vertices within a distance $r$ from a given vertex $i$ is called the volume of a ball of radius $r$ around $i$:

$$\text{Vol}_i(r) = |\{j|r(i,j) \leq r\}|.$$  

For infinite graphs, this allows the definition of a dimension $d$,

$$d = \inf \left\{ x | \lim_{r \to \infty} \text{Vol}(r)/r^x < \infty \right\},$$

which is often formulated in terms of the following scaling relation

$$\text{Vol}(r) \xrightarrow{r \to \infty} \text{const } r^d (1 + O(1/r)),$$

although the existence of such a scaling relation is more restrictive. For lattices this definition agrees with the usual notion of the dimension of a lattice. This intrinsic definition of a dimension has long been used in the theory of clusters or percolation models (see e.g. [12]). A recent and more mathematical treatment can be found in [13].
Finally, I will discuss some properties of the graph Laplacian. In order to motivate and justify this name, we consider a scalar Gaussian field defined on the vertices of the graph. The natural action is

\[ S[A; \varphi] = \frac{1}{4} \sum_{ij} A_{ij} (\varphi_i - \varphi_j)^2 \]

(5)

\[ = \frac{1}{4} \sum_{ij} (2A_{ij} \varphi_i^2 - 2A_{ij} \varphi_i \varphi_j) \]

\[ = -\frac{1}{2} \sum_{ij} \varphi_i \Delta_{ij} \varphi_j , \]

(6)

where

\[ \Delta = A - V \]

is the graph Laplacian. \( \Delta \) is a generalization of the usual concept of a Laplacian to arbitrary graphs and is easily shown to be equivalent to the second difference quotients summed over all directions on a lattice. (In the theory of electrical networks \( -\Delta \) is sometimes referred to as the matrix of admittance.) As the action in eq. (5) is always non-negative, \( -\Delta \) is also a non-negative matrix. The number of zero-modes is equal to the number of components of the graph. Hence, for connected graphs the graph Laplacian has exactly one zero mode. As usual

\[ \int \prod_i d\varphi_i \exp - \left( \frac{1}{4} \sum_{ij} A_{ij} (\varphi_i - \varphi_j)^2 + \frac{1}{2} \mu^2 \sum_i \varphi_i^2 \right) \propto (\det(V - A + \mu^2))^{-1/2} . \]

2.2 An action for random graphs

Next, I will write down an action on the set of graphs. For simplicity, I will restrict myself to the set of all graphs with a fixed number of vertices. It should be kept in mind, however, that the proposed action is only one possible choice out of many. The aim is to find an action for which some of the local minima may be interpreted as background spaces (“flat” lattices, or lattices with nontrivial compactifications for some directions). For this purpose, other actions might be more suitable; the one formulated in this section is just a candidate.

The action consists of three terms related to the properties of the adjacency matrix discussed in the previous subsection and is defined to be

\[ S[A] = \alpha S_1[A] + \beta S_2[A; \delta] + S_3[A; \mu] , \]

where

\[ S_1[A] = \sum_P \text{tr}(P^+ AP - A)^2 \]

(7)

\[ S_2[A; \delta] = \frac{1}{2} \sum_i (\delta_i - \delta)^2 \]

(8)

\[ S_3[A; \mu] = \ln \mu - \frac{1}{2} \ln \det(V - A + \mu^2) . \]

(9)

I shall now discuss the meaning of the different terms. The parameters \( \alpha, \beta, \delta \) and \( \mu \) might be tuned in order to enhance or suppress the contribution of graphs with certain properties.
1. The summation runs over all permutations of vertices. Each term \( \text{tr}(P^+ A P - A)^2 \) is a measure as to how far \( A \) deviates from being symmetric under \( P \). If \( P \) is a symmetry of \( A \), the term vanishes. \( S_1[A] \) is zero for two graphs: the empty graph (all points are isolated, there are no lines) and the complete graph. These two are the absolute minima of \( S_1 \). Local minima of \( S_1 \) should correspond to graphs with a large symmetry. “Local” refers to the set of graphs and may be made more explicit with a distance functional

\[
D(A_1, A_2) = \min_P \text{tr}(A_1 - P^+ A_2 P)^2.
\]

Formally, \( S_1[A] \) resembles a nearest neighbor interaction of matrix models. In this case,

\[
S_M[A] = \sum_{i=1}^d \text{tr}(T_i^+ A T_i - A)^2,
\]

where \( T_i \) is the translation matrix for one step in direction \( i \) and the sum runs over all directions. Equation (10) is minimized for adjacency matrices \( A \) having the symmetry of the translation group, i.e., \( A \) corresponds to a lattice graph. Translations form a subgroup of the permutation group. However, using the full permutation group in eq. (7) has the advantage that no dimension is specified.

2. \( \delta_i \) is the degree of vertex \( i \). Therefore, this term suppresses graphs for which the degree at the vertices deviates much from a preferred value \( \delta \). Noticing that

\[
\sum_i \delta_i^2 = \text{tr} A^2 K \quad \text{and} \quad \sum_i \delta_i = \text{tr} A K
\]

we can rewrite the second term also as

\[
S_2[A; \delta] = \frac{1}{2} \text{tr}(A - \delta)^2 K.
\]

We might also have added a term

\[
S_2'[A] = \text{tr}[K, A]^2
\]

which suppresses all graphs which deviate from being regular without reference to a certain degree.

It might seem that specifying a degree \( \delta \) implicitly specifies a dimension for the graph. This is not true. For any given degree \( \delta > 2 \), there exist graphs of any dimension. The two extreme cases are represented, e.g., by the Bethe lattice (Cayley tree) giving \( d = \infty \), and by a linear chain of complete graphs glued together appropriately giving \( d = 1 \).

3. The first two actions in combination will suppress graphs with very irregular degrees (like complete subgraphs embedded in an otherwise empty environment), or with degrees which deviate much from \( \delta \), and they will enhance graphs with high symmetry. There exist very disconnected graphs which satisfy both properties. Consider, e.g., an ensemble of disconnected complete graphs of degree \( \delta \). They minimize action 2 and they also are local minima of action 1. The last action, \( S_3[A] \), may be used to suppress graphs with more than one component. In the limit \( \mu \to 0 \) this term becomes infinite unless the graph is connected.

The determinant of the graph Laplacian may also be used to enhance regular graphs with many small loops \([14]\), again supporting lattices.
2.3 An Einstein action for graphs?

The actions mentioned in the previous section enhance graphs which have a large symmetry, small fluctuations in the valencies, and which are connected. It is not expected that a combination of these terms leads to an action which resembles Einstein’s action or which has the same local minima as Einstein’s action (not even in a certain limit). Therefore, the question arises, if it is possible to formulate an Einstein action for arbitrary graphs which does not refer to a definite dimension. Minima of this action would correspond to solutions of Einstein’s equations in different space-time dimensions.

Discrete Einstein actions have been used in Regge calculus [15] and for combinatorial triangulations [16]. However, in both cases the dimension is explicitly given, in the first case even a metric structure. In the second case curvature is “counted” by the numbers of cells meeting at subcells of codimension 2 [16]. Although the actions (8) and (11) look similar, they are not supposed to be real analogues of an Einstein action. Only if we require additional structures for the graphs (for instance “planarity”, as in the case of 2-dimensional triangulations and matrix models [16]) a relation to Einstein’s action might be proven.

Several approaches come to one’s mind to find an analoga of Einstein’s action for arbitrary graphs. One is related to the following relation known from differential geometry (see e.g. [17]):

\[
\text{Vol}_x(r) = c_1 r^d \left(1 - c_2 r^2 R(x) + O(r^3)\right).
\]  (12)

On the left hand side, \(\text{Vol}_x(r)\) denotes the volume of a ball of radius \(r\) around point \(x\). The leading term on the right hand side (\(c_1\) and \(c_2\) are positive constants) is just the volume of a ball in \(d\)-dimensional flat space, the first correction term contains the scalar curvature \(R(x)\) at point \(x\). However, to apply this formula to graphs one has to deal with the following problem first. Equation (12) holds in the limit \(r \to 0\), while scaling relations of the type (4) hold for the limit \(L \to \infty\). This is related to the fact that in differential geometry manifolds are supposed to approach flat spaces when viewed at sufficiently small regions, while random graphs are expected to approach homogeneous (not necessarily flat) structures when viewed at large scales. Therefore, the best one can hope for is that for graphs a relation like (12) only holds in a “scaling window”, i.e., for \(r\) not too small (where the discrete structure of the graph dominates) and not too large (where higher order corrections to eq. (12) are important).

A second approach is based on the following asymptotic expansion for the heat kernel of the Laplace operator \(\Delta[g]\) on a manifold of dimension \(d\) with metric \(g\) (for simplicity we assume that the manifold is compact and has no boundary) [18, 19]:

\[
\text{tr} \ e^{\Delta[g] t} \xrightarrow{t \to 0} c'_1 \frac{\text{Vol}}{t^{d'/2}} - c'_2 \frac{\int \sqrt{g} R}{t^{(d'-2)/2}} + \cdots.
\]  (13)

The leading term is simply the total volume of the manifold and corresponds to a cosmological term in the Einstein-Hilbert action. (Note that the spectral dimension \(d'\) in eq. (13) may differ from the Hausdorff dimension \(d\) in eq. (12). Requiring them to be equal again suppresses “irregular” graphs.) The first correction is the integrated scalar curvature, i.e., the Einstein term. Therefore, it is conceivable to use the asymptotic behavior of the spectral density of the graph Laplacian for a construction of an action related to the Einstein-Hilbert action. Much is known about the spectrum of the adjacency matrix and the graph Laplacian (see e.g. [20]), so in this case even analytical investigations might be possible.
A third approach is closely related to the previous one. Let \((A^L)_{ii}\) be the number of paths of length \(L\) starting and ending at vertex \(i\), and let \(N_i(L) = \sum_j (A^L)_{ij}\) be the total number of paths of length \(L\) starting at \(i\). Then
\[
R_i(L) = \frac{(A^L)_{ii}}{N_i(L)} \xrightarrow{L \to \infty} \frac{c}{L^{d/2}} + \cdots
\]
defines a recurrence probability of paths of length \(L\). The leading term is again given by the volume and dimension of a ball around \(i\) on the graph and the corrections are again expected to be related to the scalar curvature. The analogy with the previous approach will be obvious if one interprets the diagonal elements of the heat kernel, \((e^{\Delta g}t)_{ii}\), as the recurrence probabilities of a diffusion process on the graph.

These are only a few examples for expressions which for regular manifolds are known to have an expansion where the leading terms are related to the volume(density) and the dimension of the manifold and the correction terms are related to the scalar curvature, and which all have analogues also for arbitrary graphs. Therefore, expressions of this type have a good chance to define an action for graphs which in a sensible continuum and thermodynamic limit give rise to a cosmological term and an Einstein action. However, much analytical and, presumably, even more numerical work will be necessary until an adequate action for graphs will be found.

\section{\(Z_2\)-models on hypercubic lattices}

The \(Z_2\)-spin model and the \(Z_2\)-gauge theory are the simplest examples among a class of models discussed intensively by Wegner \cite{5}. Of special relevance are the different duality relations among these models on \(d\)-dimensional lattices. In this section we will restrict ourself to the self-dual hypercubic lattices.

Let us consider \(Z_2\) variables \((z \in \{+1, -1\})\) associated to the \(p\)-dimensional elements (cells) of a hypercubic lattice. These are the vertices \((p = 0)\), the lines \((p = 1)\), the plaquettes \((p = 2)\), the cubes \((p = 3)\), etc. The \(p + 1\)-dimensional cells are bounded by \(2(p + 1)\) \(p\)-cells. Define the action to be
\[
S_p = -\sum_{(p+1)\text{-cells}} \left( \prod_{\partial(p+1)\text{-cell}} z_p \right),
\]
where the summation extends over all \((p + 1)\)-cells of the lattice and for each cell we take the product of spin variables on its boundary. The partition function is given by
\[
Z_p = \sum_{\{z\}} e^{-\beta S_p}.
\]
For \(p = 0\) we obtain the Ising model with the spins \(z\) attached to the vertices of the lattice and the action given by
\[
S_0[z] = -\sum_{\langle i,j \rangle} z_i z_j.
\]
The summation extends over all links (cells of dimension 1) of neighboring vertices \(i\) and \(j\).
The case \( p = 1 \) yields the standard \( \mathbb{Z}_2 \)-gauge theory with the group variables \( z = \pm 1 \) associated to the links and the action given by

\[
S_1[z] = - \sum_{p_{ijkl}} z_{ij} z_{jk} z_{kl} z_{li} ,
\]

where now the sum extends over all plaquettes (cells of dimension 2) of the lattice and for each plaquette we take the product of the four spin variables on the links which form the boundary of this plaquette.

In the case \( p = 2 \) the spin-variables are associated to the plaquettes of the lattice, the action consists of a sum over all cubes and for each cube one takes the product of the spin-variables on the six plaquettes which form the boundary of that cube. All models for \( p \geq 1 \) have a gauge invariance. As the fundamental degrees of freedom are attached to the \( p \)-cells of the lattice, they may be considered as lattice analogues of gauge theories of \( p \)-forms.

For all models there exists a high temperature expansion and a low temperature expansion similar to the one known for the Ising model (see e.g. \([3, 4]\)), the difference only being the geometrical objects on the lattice one has to sum over. In all cases the high temperature expansion (small \( \beta \)) has the following form

\[
Z_p = 2^{E_p(cosh \beta)} E_{p+1} \sum_{L=0}^{\infty} t_p(L)(tanh \beta)^L ,
\]

where \( E_p \) and \( E_{p+1} \) are the total number of \( p \) and \( p + 1 \)-cells, respectively, and \( t_p(L) \) a combinatorical factor whose geometric interpretation depends on \( p \). For \( p = 0 \) (the Ising model), \( t_0(L) \) denotes the number of closed polygons of length \( L \) on the lattice. In this case, the high temperature expansion may be interpreted as a summation over all closed polygons on the lattice where each polygon is weighted by a factor depending only on its length (the number of links).

For the \( \mathbb{Z}_2 \)-gauge theory, \( t_1(L) \) equals the total number of closed surfaces (obtained by pasting together plaquettes) of area \( L \) (number of plaquettes) on the lattice. Similarly for the other models: The high temperature expansion of the theory with variables defined on the \( p \)-dimensional cells of the lattice consists of a summation over all closed \((p + 1)\)-volumes obtained by pasting together \((p + 1)\)-dimensional cells of the lattice, and the weight for each such volume depends only on the number of the \((p + 1)\)-cells it contains. In the same way as the high temperature expansion (large coupling expansion) of a gauge theory is related to strings (summation over string world sheets) the high temperature expansions of Wegner’s \( p \)-models are related to \( p \)-branes in the sense that they generate a summation over world volumes of \( p \)-branes.

All models also have a low temperature expansion (large \( \beta \)) which reads

\[
Z_p = 2 e^{\beta E_{p+1}} \sum_{L=0}^{\infty} c_p(L) e^{-2\beta L} ,
\]

Again these expansions only differ in the geometrical interpretation of the combinatorical factors \( c_p(L) \) (and the exponent of the overall factor). \( c_p(L) \) is equal to the total number of closed \((d - p - 1)\)-volumes on the lattice. For the Ising model and \( d = 2 \) these are again polygons showing the self-duality of the Ising model for \( d = 2 \). For the Ising model on a \( d = 3 \) lattice
these are closed surfaces. This expresses the fact that the Ising model and the $\mathbb{Z}_2$-gauge theory are dual to each other for $d = 3$.

This structure extends to the models where the $\mathbb{Z}_2$-variables are defined on $p$-cells, i.e. $p$-form gauge theories. The low temperature expansion contains a summation over closed objects of dimension $d - (p + 1)$, the high temperature expansion a summation over closed objects of dimension $p + 1$. Two models are dual to each other whenever the dimensions of these objects match, i.e., when the high and low temperature expansion of one model coincides with the low and high temperature expansion of the other model, respectively. In this way we obtain a whole series of duality relations depending on the value for $p$ and the dimension of the lattice. For the lowest dimensions the duality relations are listed in table (1).

| $d$ | duality relation | geometric duality | excitation duality |
|-----|------------------|-------------------|-------------------|
| 2   | Ising ↔ Ising    | $\sum$ polygons $\simeq \sum$ polygons | particle ↔ particle |
| 3   | Ising ↔ 1-form gauge | $\sum$ polygons $\simeq \sum$ surfaces | particle ↔ string |
| 4   | Ising ↔ 2-form gauge | $\sum$ polygons $\simeq \sum$ 3-volumes | particle ↔ 2-brane |
| 4   | 1-form gauge ↔ 1-form gauge | $\sum$ surfaces $\simeq \sum$ surfaces | string ↔ string |
| 5   | Ising ↔ 3-form gauge | $\sum$ polygons $\simeq \sum$ 4-volumes | particle ↔ 3-brane |
| 5   | 1-form gauge ↔ 2-form gauge | $\sum$ surface $\simeq \sum$ 3-volumes | string ↔ 2-brane |

Table 1: Duality relations for the $\mathbb{Z}_2$ models on $p$-cells.

4 $\mathbb{Z}_2$-Gauge Theory on Random Graphs

Now we want to combine the models for random graphs and gauge theories. Random graphs may yield lattices as local minima although no preferred dimension is put into the theory beforehand. Therefore, they dynamically provide background spaces for the perturbative regimes. The spin models satisfy duality relations and have perturbation expansions in terms of closed objects - lines, surfaces, etc.

Most of the models mentioned in the previous section have no natural generalization for arbitrary graphs because there are no natural analogues for $p$-cells apart from $p = 0$ (the vertices) and $p = 1$ (the lines). So, at first sight, only the Ising model seems to have a natural formulation on graphs, its partition function defined by

$$Z = \sum_{\{z\}} \exp \left( \frac{1}{2} \beta \sum_{ij} A_{ij} z_i z_j \right).$$
The high and low temperature expansions are unchanged:

\[ Z_0 = 2^{E_0} (\cosh \beta)^{E_1} \sum_{L=0}^{\infty} t_0(L) (\tanh \beta)^L \]

\[ Z_0 = 2 e^{\beta E_1} \sum_{L=0}^{\infty} c_0(L) e^{-2\beta L}, \]

where now \( t_0(L) \) and \( c_0(L) \) are the number of tiesets and cutsets of length \( L \) of the graph, respectively. Tiesets are again closed polygons, or, to be more precise, subgraphs (subsets of lines together with their incident vertices) where all vertices have even degree. To obtain a cutset one first chooses an arbitrary partition of the sets of vertices into two disjoint subsets (the “spin-up”-set and the “spin-down”-set). The corresponding cutset then consists of all lines of the graph which connect vertices of one set with vertices of the other. On a hypercubic lattice the cutsets are in one-to-one correspondence to the \( d-1 \)-dimensional closed volumes on the dual lattice.

The equivalence of these two expansions provides an easy way to relate the number of cutsets to the number of tiesets and vice versa, showing, e.g., that one set is fixed once the other is given. But in general there will be no “dual” graph, for which the tiesets are equivalent to the cutsets of the original graph.

Although there are no natural analogues for plaquettes - the elementary 2-cells - we can nevertheless define a gauge theory on an arbitrary graph using the observation that for a gauge theory on a regular lattice it is known that even if the action is formulated in terms of elementary loops around plaquettes, a renormalization group transformed action will involve also other loops with the couplings for larger loops suppressed.

The following formulation of a gauge theory on a graph does not make reference to a specific gauge group. For each pair of neighbered vertices \((i, j)\) define an element of the gauge group \( g_{ij} \), such that \( g_{ij} = g_{ji}^{-1} \). As usual, \( g_{ij} \) may be interpreted as the parallel transport from the representation space \( W_j \) (supposed to be associated with vertex \( j \)) to the isomorphic representation space \( W_i \). Consider the graph with adjacency matrix \( A \) and define the matrix \( A^G \) by

\[ (A^G)_{ij} = A_{ij} g_{ij} = \begin{cases} \{g_{ij} & \text{if } i \text{ and } j \text{ are neighbered} \\ 0 & \text{otherwise} \end{cases} \]

\( A^G \) is a linear mapping acting on the representation space \( \hat{W} = W_1 \oplus W_2 \oplus \cdots W_N \), where \( N \) equals the number of vertices of the graph. Note that \( A^G \) will be hermitean in general and for the group \( \mathbb{Z}_2 \) even symmetric. The graph Laplacian in the presence of gauge fields now reads

\[ \Delta^G = A^G - V. \]

This can be used to couple scalar fields to gauge fields on graphs.

As before (sec. 2.1), powers of \( A^G \) generate paths on the graph but this time each path is “weighted” by its parallel transport:

\[ (A^G)^L)_{ij} = \sum_{L-\text{paths } j \to i} g_{ik_1} g_{k_1 k_2} \cdots g_{k_{L-1} j}. \]

The right hand side extends over all allowed paths of length \( L \) from vertex \( j \) to vertex \( i \) on the graph and for each path we obtain the product of group elements on the links of this path, i.e.
the parallel transporter along this path. Taking the trace with respect to the vertices of the graph (denoted by “tr”) yields

$$\text{tr}(A^G)^L = \sum_i ((A^G)^L)_{ii} = \sum \text{closed } L\text{-paths } g_{ik_1} g_{k_2 k_3} \cdots g_{k_{L-1} i},$$

i.e. the sum over all parallel transporters along closed paths on the graph which have length $L$ and which start and end at vertex $i$. Taking also the trace with respect to the representation of the group (denoted by “Tr”) we obtain

$$\text{Tr} \text{tr}(A^G)^L = L \sum \text{closed } L\text{-paths } \text{Tr}(g_{ik_1} g_{k_2 k_3} \cdots g_{k_{L-1} i}),$$

i.e., the sum over all Wilson loops of length $L$ on the graph. (The factor $L$ arises because each closed path occurs $L$ times on the right hand side, as each vertex of the path can act as a starting point.)

For a gauge action on the graph we now take

$$S[A^G] = \sum_L h_L \text{Tr} \text{tr}(A^G)^L,$$  \hspace{1cm} \text{(14)}$$

where $h_L$ are some coupling constants which for simplicity shall only depend on $L$. Note that this is not the most general gauge action on a graph, which might also include products of Wilson loops or different couplings for different “shapes” of loops (e.g., the number of links which are visited repeatedly by the path, or the maximal distance - in the sense of sec. 2 - between two points of the loop). Equation (14) is just the simplest choice which includes the standard Wilson action for regular lattices. In any case, it is expected that in a critical limit the different choices correspond to the same universality class. In order to suppress contributions from very large loops (or from paths which keep winding around some smaller loop) the coupling $h_L$ should approach zero for $L \to \infty$ faster than a power law, e.g. $h_L \sim 1/L!$, or $h_L = 0$ for $L$ larger than some critical length.

Let us now return the the $\mathbb{Z}_2$-models on random graphs. For the Ising model the high temperature expansion is always an expansion in closed polygons on the graph, representing euclidean “world lines”. The low temperature expansion is with respect to cutsets, and for regular lattices this corresponds to an expansion in terms of $d-1$-dimensional closed objects (the “world volumes” of $d-2$-dimensional objects).

For the $\mathbb{Z}_2$-gauge theory the situation is slightly more complicated. The low temperature expansion on an arbitrary graph is an expansion with respect to sets of closed loops such that an even number of loops meet at each link. If we visualize the closed loops as the boundaries of some surfaces (whose shape is not important and, strictly speaking, meaningless) this would correspond to an expansion in terms of closed surfaces. Note, however, that our gauge action does not only contain elementary loops even for the case of regular lattices. In principle (unless there is a cut-off for the couplings $h_L$) all loops contribute, the larger ones with less weight. So the closed surfaces are not only pasted together by elementary plaquettes but also by more complicated accumulations of plaquettes which can be interpreted as more complicated area elements. Although this makes the expansion more complicated, the fundamental picture of an expansion in terms of closed surfaces remains.
The low temperature expansion of the \( \mathbb{Z}_2 \)-gauge theory on an arbitrary graph may also be formulated. For each configuration of spin variables we consider the set of loops where the curvature (product of parallel transports around these loops) is \(-1\). These sets satisfy certain constraints which, on regular lattices, make them equivalent to objects of codimension 2 on the dual lattice. Again we see that the interpretation of the high temperature expansion in terms of “dual flux-tubes” etc. only works for regular lattices. But on the other hand it is only for these perturbative regimes that we expect duality to hold.

5 Generalizations

The following is a list of generalizations which is far from being complete:

1. The action for the random graphs may be different. The aim would be a kind of Einstein action for which the local minima are the graph equivalents of solutions to the Einstein equations.

2. The gauge group can be generalized even to non-abelian cases. In sect. 4 it has been shown how to couple gauge degrees of freedom to random graphs. Matter fields, associated to the vertices of the graph, may also be coupled to the gauge fields.

3. Besides duality another equivalence of models is known quite well from statistical mechanics: the equivalence between quantum statistical models on \( d \)-dimensional lattices and classical statistical models on \( d + 1 \)-dimensional lattices. So one might wonder, if instead of classical spin models one should consider quantum spin models on random graphs. The construction is straightforward: For graphs with \( V \) vertices consider the \( 2^V \)-dimensional Hilbert space \( \mathcal{H} = \otimes_i^V \mathbb{C}_2 \) and define

\[
\vec{\sigma}_i = 1 \otimes 1 \otimes \cdots \sigma \otimes \cdots \otimes 1
\]



to be the spin variables (Pauli matrices) attached to vertex \( i \). The Hamiltonian for the quantum Ising model on a graph with adjacency matrix \( A \) is then given by

\[
H = \mu \sum_i \sigma_i^x + \frac{1}{2} \lambda \sum_{ij} A_{ij} \sigma_i^z \sigma_j^z.
\]

In a similar way one can formulate the quantum gauge theory Hamiltonian. The duality relations of Wegner also hold for the quantum models (with the dimensions of the lattices reduced by one). It is tentative to assume a relation with spin networks [21] which arise in the context of a canonical quantization prescription for gravity.

4. Instead of graphs one might consider general cell-complexes, or, as an intermediate step, graphs with cliques. A clique is a complete subgraph of a graph and represents a higher dimensional simplex. For such cliques also the spin gauge models for \( p \)-forms of Wegner may be defined, because there exists a natural notion of plaquettes, volumes, etc.

It has been shown that simple models with the properties listed in the introduction exist, but it should be stressed that these are only “toy” models. Other actions and gauge groups might reveal more structures with an even closer relationship to non-perturbative string theories.

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