On the signed 2-independence number of graphs

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Abstract

In this paper, we study the signed 2-independence number in graphs and give new sharp upper and lower bounds on the signed 2-independence number of a graph by a simple uniform approach. In this way, we can improve and generalize some known results in this area.

Keywords: domination number, limited packing, tuple domination, signed 2-independence number

Mathematics Subject Classification : 05C69

DOI:10.5614/ejgta.2017.5.1.4

1. Introduction

Throughout this paper, let $G$ be a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use [13] as a reference for terminology and notation which are not defined here. The open neighborhood of a vertex $v$ is denoted by $N(v)$, and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degree of $G$ are respectively denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$.

Let $S \subseteq V$. For a real-valued function $f : V \to R$ we define $f(S) = \sum_{v \in S} f(v)$. Also, $f(V)$ is the weight of $f$. A signed 2-independence function, abbreviated S2IF, of $G$ is defined in [14] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \leq 1$, for every $v \in V$. The signed 2-independence number, abbreviated S2IN, of $G$ is $\alpha_2^s(G) = \max\{f(V) | f \text{ is a S2IF of } G\}$. This concept was

Received: 9 January 2015, Revised 15 January 2017, Accepted: 26 January 2017.
defined in [14] as a certain dual of the signed domination number of a graph [3] and has been studied by several authors including [8, 10, 11, 12].

A set $S \subseteq V$ is a dominating set if each vertex in $V \setminus S$ has at least one neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set [7]. A subset $B \subseteq V$ is a 2-packing in $G$ if for every pair of vertices $u, v \in B$, $d(u, v) \geq 3$. The 2-packing number (or packing number) $\rho(G)$ is the maximum cardinality of a 2-packing in $G$.

Gallant et al. [5] introduced the concept of limited packing in graphs. They exhibited some real-world applications of it to network security, NIMBY, market saturation and codes. In this paper we exhibit an application of it to signed 2-independence number in graphs. In fact as it is defined in [5], a set of vertices $B \subseteq V$ is called a $k$-limited packing in $G$ provided that for all $v \in V$, we have $|N[v] \cap B| \leq k$. The limited packing number, denoted $L_k(G)$, is the largest number of vertices in a $k$-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. In [6], Harary and Haynes introduced the concept of tuple domination in graphs. A set $D \subseteq V$ is a $k$-tuple dominating set in $G$ if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The $k$-tuple domination number, denoted $\gamma_{\times k}(G)$, is the smallest number of vertices in a $k$-tuple dominating set. When $k = 2$, $D$ is called a double dominating set and the 2-tuple domination number is called the double domination number and is denoted by $dd(G)$. In fact the authors showed that every graph $G$ with $\delta \geq k - 1$ has a $k$-tuple dominating set and hence a $k$-tuple domination number.

By a simple uniform approach, we derive many new sharp bounds on $\alpha^2_s(G)$ in terms of several different graph parameters. Some of our results improve some known bounds on the S2IN of graphs in [8, 11, 12].

The authors noted that most of the existing bounds on $\alpha^2_s(G)$ are lower bounds. In section 2, we prove that $\alpha^2_s(G) \geq 2\left[\frac{\delta + 2\rho(G)}{2}\right] - n$, for a graph $G$ of order $n$. Also in section 3, by a simple connection between the concepts of limited packing and tuple domination, we obtain the exact value of the signed 2-independence numbers of regular graphs. In particular, we bound the signed 2-independence numbers of cubic graphs from below and above just in terms of order as, $-\frac{n}{3} \leq \alpha^2_s(G) \leq 0$.

2. Main results

At this point we are going to present some sharp upper bounds on $\alpha^2_s(G)$. First, let us introduce some notation. Let $f : V \rightarrow \{-1, 1\}$ be a maximum S2IF of $G$. We define $V_+ = \{v \in V | f(v) = 1\}$, $V_- = \{v \in V | f(v) = -1\}$, $G_+ = G[V_+]$ and $G_- = G[V_-]$ where $G_+$ and $G_-$ are the subgraphs of $G$ induced by $V_+$ and $V_-$, respectively. For convenience, let $[V_+, V_-]$ be the set of edges having one end point in $V_+$ and the other in $V_-$. Finally, $deg_{G_+}(v) = |N(v) \cap V_+|$ and $deg_{G_-}(v) = |N(v) \cap V_-|$. Obviously, $|V_+| = \frac{n + \alpha^2_s(G)}{2}$ and $|V_-| = \frac{n - \alpha^2_s(G)}{2}$.

**Theorem 2.1.** Let $G$ be a graph of order $n$. Then

$$\alpha^2_s(G) \leq \left(\left\lfloor\frac{\Delta}{2}\right\rfloor - \left\lceil\frac{\delta}{2}\right\rceil + 1\right)n$$

and this bound is sharp.
**Theorem 2.5.** Let \( f \) be a maximum S2IF of \( G \). Let \( v \in V_+ \). Since \( f(N[v]) \leq 1 \), the vertex \( v \) has at least \( \lceil \frac{\deg(v)}{2} \rceil \geq \lceil \frac{\delta}{2} \rceil \) neighbors in \( V_- \). Therefore \( |[V_+, V_-]| \geq \lceil \frac{\delta}{2} \rceil |V_+| \). Now let \( v \in V_- \). Since \( f \) is a S2IF, the vertex \( v \) has at most \( \lceil \frac{\deg(v)}{2} \rceil + 1 \leq \lceil \frac{\delta}{2} \rceil + 1 \) neighbors in \( V_+ \). Therefore \( |[V_+, V_-]| \leq (\lceil \frac{\Delta}{2} \rceil + 1)|V_-| \).

In fact

\[
\left\lfloor \frac{\delta}{2} \right\rfloor |V_+| \leq |[V_+, V_-]| \leq (\lceil \frac{\Delta}{2} \rceil + 1)|V_-|.
\]

Using \( |V_+| = \frac{n + \alpha^2_s(G)}{2} \) and \( |V_-| = \frac{n - \alpha^2_s(G)}{2} \), we obtain the desired upper bound. For sharpness it is sufficient to consider the complete graph \( K_n \).

In [8] the author established a relationship between the signed 2-independence number and the domination number of a graph as follows.

**Theorem 2.2.** ([8]) If \( G \) is a connected graph of order \( n \geq 2 \), then \( \alpha^2_s(G) + 2\gamma(G) \leq n \), and this bound is sharp.

Now we are going to improve Theorem 2.2. We shall need the following result, which can be found implicit in [4] and explicit in [2] as Corollary 81.

**Theorem 2.3.** ([2],[4]) If \( G \) is a graph with \( \delta \geq k - 1 \), then \( \gamma \times k(G) \geq \gamma(G) + k - 1 \).

**Theorem 2.4.** If \( G \) is a connected graph of order \( n \), then \( \alpha^2_s(G) + 2\gamma(G) \leq n - 2\lceil \frac{\delta}{2} \rceil + 2 \), and this bound is sharp.

**Proof.** Let \( f \) be a maximum S2IF of \( G \). We have shown that \( |N[v] \cap V_-| \geq \lceil \frac{\delta}{2} \rceil \) for all \( v \in V_+ \). On the other hand, if \( v \in V_- \), then \( \deg_{G_-}(v) \geq \lceil \frac{\deg(v)}{2} \rceil - 1 \geq \lceil \frac{\delta}{2} \rceil - 1 \). Therefore \( |N[v] \cap V_-| \geq \lceil \frac{\delta}{2} \rceil \). This shows that \( V_- \) is a \( \lceil \frac{\delta}{2} \rceil \)-tuple dominating set in \( G \). This implies, \( |V_-| \geq \gamma \times \lceil \frac{\delta}{2} \rceil(G) \) and hence \( \alpha^2_s(G) \leq n - 2\gamma \times \lceil \frac{\delta}{2} \rceil(G) \). Now by Theorem 2.3, we have \( \alpha^2_s(G) \leq n - 2(\gamma(G) + \lceil \frac{\delta}{2} \rceil) - 1 \). Therefore \( \alpha^2_s(G) + 2\gamma(G) \leq n - 2\lceil \frac{\delta}{2} \rceil + 2 \). For sharpness it is sufficient to consider the complete graph \( K_n \).

By the concept of limited packing we can present a sharp lower bound on \( \alpha^2_s(G) \) that involves the packing number.

**Theorem 2.5.** Let \( G \) be a connected graph of order \( n \). Then

\[
\alpha^2_s(G) \geq 2\left\lfloor \frac{\delta + 2\rho(G)}{2} \right\rfloor - n
\]

and this bound is sharp.

**Proof.** Let \( B \) be a \( \lceil \frac{\delta}{2} \rceil \)-limited packing set in \( G \). Obviously, \( L_{\lfloor \frac{\delta}{2} \rfloor}(G) \leq L_{\lfloor \frac{\delta}{2} + 1 \rfloor}(G) \). We claim that \( B \neq V \). If \( B = V \) and \( v \in V \) such that \( \deg(v) = \Delta \), then \( \Delta + 1 = |N[v] \cap B| \leq \lceil \frac{\delta}{2} \rceil \leq \Delta \), a contradiction. Now let \( u \in V - B \). It is easy to check that \( |N[v] \cap (B \cup \{u\})| \leq \lceil \frac{\delta}{2} \rceil + 1 \), for all \( v \in V(G) \). Therefore \( B \cup \{u\} \) is a \( \lceil \frac{\delta}{2} \rceil + 1 \)-limited packing set in \( G \). Hence

\[
L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) \geq |B \cup \{u\}| = |B| + 1 = L_{\lfloor \frac{\delta}{2} \rfloor}(G) + 1.
\]
Repeating these inequalities, we have

\[ L_{\lfloor \delta/2 \rfloor + 1}(G) \geq L_{\lfloor \delta/2 \rfloor}(G) + 1 \geq \ldots \geq L_1(G) + \left\lfloor \frac{\delta}{2} \right\rfloor = \rho(G) + \left\lfloor \frac{\delta}{2} \right\rfloor. \]  

(1)

Now let \( B \) be a maximum \( \left\lfloor \frac{\delta}{2} \right\rfloor + 1 \)-limited packing set in \( G \). We define \( f : V \to \{-1, 1\} \) by

\[ f(v) = \begin{cases} 
1 & \text{if } v \in B \\
-1 & \text{if } v \in V - B.
\end{cases} \]

We deduce that

\[ f(N[v]) = |N[v] \cap B| - |N[v] \cap (V - B)| = 2|N[v] \cap B| - |N[v]| \leq 2\left\lfloor \frac{\delta}{2} \right\rfloor - \delta + 1 \leq 1, \]

for all \( v \in V \). Therefore, \( f \) is a S2IF of \( G \). This implies

\[ \alpha_s^2(G) \geq f(V) = |B| - |V - B| = 2|B| - n = 2L_{\lfloor \delta/2 \rfloor + 1}(G) - n. \]

Now (1) implies

\[ \alpha_s^2(G) \geq 2L_{\lfloor \delta/2 \rfloor + 1}(G) - n \geq 2(\rho(G) + \left\lfloor \frac{\delta}{2} \right\rfloor) - n, \]

as desired. Considering the graph \( K_n \) we can see that this bound is sharp.

Volkmann in [11] proved that if \( G \) is a graph of order \( n \), then \( 2 - n \leq \alpha_s^2(G) \). Moreover if \( n \geq 3 \), then \( 4 - n \leq \alpha_s^2(G) \). Obviously, the lower bound in Theorem 2.5 is an improvement of the first inequality and when \( \delta \geq 2 \) this improves the second, as well.

At the end of this section we exhibit a short comment about signed 2-independence number of bipartite graphs. The following upper bound on \( \alpha_s^2(G) \) of a bipartite graph was obtained by Wang [12].

**Theorem 2.6.** ([12]) If \( G \) is a bipartite graph of order \( n \geq 2 \), then

\[ \alpha_s^2(G) \leq n + 6 - 2\sqrt{2n} + 9. \]

Furthermore, the bound is sharp.

We now improve the bound in the previous theorem.

**Theorem 2.7.** Let \( G \) be a bipartite graph of order \( n \). Then

\[ \alpha_s^2(G) \leq n + 2(2 + \left\lceil \frac{\delta}{2} \right\rceil) - 2\sqrt{(2 + \left\lceil \frac{\delta}{2} \right\rceil)^2 + 2\left\lceil \frac{\delta}{2} \right\rceil n} \]

and this bound is sharp.
Proof. Let $f$ be a maximum $S2IF$ of $G$. Let $X$ and $Y$ be the partite sets of $G$. For convenience we define $X_+ = X \cap V_+$, $X_- = X \cap V_-$ and let $Y_+$ and $Y_-$ be defined, analogously. Obviously, $V_+ = X_+ \cup Y_+$ and $V_- = X_- \cup Y_-$. Since every vertex in $X_+$ has at least $\lceil \frac{\delta}{2} \rceil$ neighbors in $Y_-$, by the pigeonhole principle, there exists a vertex $v$ in $Y_-$ that is joined to at least $\lceil \frac{\delta}{2} \rceil |X_+|$ vertices in $X_+$. This implies

$$\frac{\lceil \frac{\delta}{2} \rceil |X_+|}{|Y_-|} - |X_-| - 1 \leq |N[v] \cap X_+| - |N[v] \cap X_-| - 1 = f(N[v]) \leq 1,$$

and hence

$$\frac{\delta}{2} |X_+| \leq |Y_-|(|X_-| + 2). \quad (2)$$

A similar argument shows that

$$\frac{\delta}{2} |Y_+| \leq |X_-|(|Y_-| + 2). \quad (3)$$

Using inequalities (2) and (3) we have

$$\frac{\delta}{2} |V_+| \leq 2|X_-||Y_-| + 2|V_-| \leq \frac{1}{2}(|X_-| + |Y_-|)^2 + 2|V_-| = \frac{1}{2}|V_-|^2 + 2|V_-|.$$

Using $|V_+| = n - |V_-|$, we obtain

$$|V_-|^2 + (4 + 2\lceil \frac{\delta}{2} \rceil)|V_-| - 2|V_-|n \geq 0.$$ This yields to $|V_-| \geq \frac{-4 - 2\lceil \frac{\delta}{2} \rceil + \sqrt{(4 + 2\lceil \frac{\delta}{2} \rceil)^2 + 8\lceil \frac{\delta}{2} \rceil n}}{2}$. Now, by using the value of $|V_-|$ we derive the desired bound. \hfill \Box

Using calculus we can see that $g(x) = n + 2(x + 2) - 2\sqrt{(x + 2)^2 + 2nx}$ is a decreasing function for $x \geq 0$. So, for $\delta \geq 1$, $\lceil \frac{\delta}{2} \rceil \geq 1$ implies that

$$n + 2(2 + \lceil \frac{\delta}{2} \rceil) - 2\sqrt{(2 + \lceil \frac{\delta}{2} \rceil)^2 + 2\lceil \frac{\delta}{2} \rceil n} \leq n + 6 - 2\sqrt{2n + 9}$$

and therefore Theorem 2.7 is an improvement of Theorem 2.6.

3. Remarks on signed 2-independence in regular graphs

Zelinka [14] obtained the following sharp upper bound on $\alpha_s^2(G)$ for regular graphs $G$.

**Theorem 3.1.** ([14]) If $G$ is an $r$-regular graph of order $n$, then $\alpha_s^2(G) \leq \frac{n}{r+1}$ when $r$ is even and $\alpha_s^2(G) \leq 0$ when $r$ is odd.

We note that the bound in Theorem 2.1 implies the previous result. The authors in [9] proved the following result.
Lemma 3.1. ([9]) Let $G$ be a graph. Then the following statements hold.

(i) Let $\delta \geq k - 1$. If $B \subseteq V$ is a $k$-limited packing set, then $V - B$ is a $(\delta - k + 1)$-tuple dominating set in $G$.

(ii) Let $\delta \geq k$. If $D \subseteq V$ is a $k$-tuple dominating set, then $V - D$ is a $(\Delta - k + 1)$-limited packing set in $G$.

Now, by the above lemma we are able to obtain the exact value of the signed $2$-independence number of regular graphs, first in terms of order and limited packing number, second in terms of order and tuple domination number. At the end we bound $\alpha_s^2(G)$ of a cubic graph $G$ from above and below, just in terms of the order. First we need the following lemma.

Lemma 3.2. Let $G$ be a graph of order $n$, then

(i) $2L_{\lceil s \rceil + 1}(G) - n \leq \alpha_s^2(G) \leq 2L_{\lfloor \frac{s}{2} \rfloor + 1}(G) - n$,

(ii) $n - 2\gamma_{\lfloor \frac{s}{2} \rfloor}(G) \leq \alpha_s^2(G) \leq n - 2\gamma_{\lfloor \frac{s}{2} \rfloor}(G)$.

Proof. (i) In the proof of Theorem 2.5 we have seen that $2L_{\lfloor \frac{s}{2} \rfloor + 1}(G) - n \leq \alpha_s^2(G)$.

Now let $f$ be a maximum S2IF of $G$. In the proof of Theorem 2.1 we have shown that $|N[v] \cap V_+| \leq \lfloor \frac{n}{2} \rfloor + 1$, for all $v \in V_-$. On the other hand, if $v \in V_+$, then $\deg_{G_+}(v) \leq \lfloor \frac{\deg(v)}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$. Therefore $V_+$ is a $(\lfloor \frac{n}{2} \rfloor + 1)$-limited packing set in $G$. This implies $|V_+| \leq L_{\lceil \frac{s}{2} \rceil + 1}(G)$ and hence $\alpha_s^2(G) \leq 2L_{\lfloor \frac{s}{2} \rfloor + 1}(G) - n$.

(ii) According to the proof of Theorem 2.4, we have $\alpha_s^2(G) \leq n - 2\gamma_{\lfloor \frac{s}{2} \rfloor}(G)$.

Now let $D$ be a minimum $\lfloor \frac{s}{2} \rfloor$-tuple dominating set in $G$. We define $f : V \rightarrow \{-1, 1\}$ by

$$f(v) = \begin{cases} 
-1 & \text{if } v \in D \\
1 & \text{if } v \in V - D.
\end{cases}$$

By the previous lemma, we conclude that $f(N[v]) = |N[v] \cap (V - D)| - |N[v] \cap D| \leq \Delta - \lfloor \frac{n}{2} \rfloor + 1 - \lfloor \frac{n}{2} \rfloor \leq 1$. Therefore $f$ is a S2IF of $G$. This implies $\alpha_s^2(G) \geq f(V) = |V - D| - |D| = n - 2|D| = n - 2\gamma_{\lfloor \frac{s}{2} \rfloor}(G)$.

Considering regular graphs, by the previous lemma, we have the following corollary.

Corollary 3.1. Let $G$ be an $r$-regular graph of order $n$. Then

(i) $\alpha_s^2(G) = 2L_{\lceil \frac{s}{2} \rceil + 1}(G) - n$.

(ii) $\alpha_s^2(G) = n - 2\gamma_{\lfloor \frac{s}{2} \rfloor}(G)$.

As an immediate result of the previous corollary we obtain the following.

Corollary 3.2. If $G$ is a cubic graph of order $n$, then

(i) $\alpha_s^2(G) = 2L_2(G) - n$.

(ii) $\alpha_s^2(G) = n - 2dd(G)$.

In [1], the authors showed that if $G$ is a cubic graph of order $n$, then $\frac{n}{3} \leq L_2(G)$. Moreover, the upper bound $L_2(G) \leq \frac{n}{2}$ was presented in [5] for a cubic graph $G$. Therefore Corollary 3.2 leads to
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\[-\frac{n}{3} \leq \alpha_2^s(G) \leq 0\]

for cubic graphs.

Acknowledgement

The authors are grateful to the referee for his/her valuable suggestions.

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