Notes on the Neighborhood Polynomials

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Abstract

The neighborhood polynomial of graph $G$, denoted by $N(G, x)$, is the generating function for the number of vertex subsets of $G$ which are subsets of open neighborhoods of vertices in $G$. For any graph polynomial, it can be useful to generate a new family of polynomials by introducing some restrictions and characterizations. In this paper, we investigate two new graph polynomials that are obtained from $N(G, x)$ by adding independence or connectivity restrictions to the vertex subsets or to the subgraphs induced by the vertex subsets which are generated by $N(G, x)$. These new polynomials are not only related to $N(G, x)$, but also having strong connections to other known graph polynomials of $G$ or its subgraphs, such as independence polynomials or subgraph component polynomials.

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1 Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph. For a vertex $v \in V$, the open neighborhood of $v$, denoted by $N(v)$, is defined by $N(v) = \{u \mid \{u, v\} \in E\}$. 
The neighborhood complex of $G$, denoted by $\mathcal{N}(G)$ and introduced in [1], is the family of all subsets of open neighborhoods of vertices of $G$,

$$\mathcal{N}(G) = \{A \mid A \subseteq V, \exists v \in V : A \subseteq N(v)\}.$$ 

The neighborhood polynomial of graph $G$, denoted by $N(G, x)$ and introduced in [2], is the ordinary generating function for the neighborhood complex of $G$.

$$N(G, x) = \sum_{U \in \mathcal{N}(G)} x^{|U|}.$$ 

The neighborhood polynomial of graph $G$ can be represented as $N(G, x) = \sum_{k=0}^{n} n_k(G)x^k$ where $n$ is the order of $G$ and $n_k(G)$ is the number of vertex subsets of cardinality $k$ in $G$ which are subsets of open neighborhoods of vertices in $G$. In other words $n_k(G) = |\{A \mid A \in \mathcal{N}(G), |A| = k\}|$.

A dominating set of a graph $G$ is a vertex subset $W$ of $G$ such that the closed neighborhood of $W$ is equal to $V$, where the closed neighborhood of $W \subseteq V$ is defined by $N[W] = \bigcup_{w \in W} N(w) \cup W$. If the family of all dominating sets of $G$ is denoted by $\mathcal{D}(G)$, the domination polynomial of a graph, introduced in [3], can be defined as the ordinary generating function for $\mathcal{D}(G)$. In other words,

$$D(G, x) = \sum_{W \in \mathcal{D}(G)} x^{|W|}.$$ 

As it is proved in [4], the following relation between the domination polynomial of $G$ and the neighborhood polynomial of the complement of $G$, $N(\bar{G}, x)$, is satisfied which motivates the study of neighborhood polynomials.

$$D(G, x) + N(\bar{G}, x) = (1 + x)^{|V|}.$$ 

In Section 2 of this paper, we investigate how the addition of the independence restriction to the vertex subsets in $\mathcal{N}(G)$ changes the neighborhood polynomial of $G$. We give the title of independent neighborhood polynomial to this new graph polynomial and then explore some of its properties, the effect of several graph operations on it and study its relation with other known graph polynomials.

In Section 3 we consider the addition of connectivity restriction to the subgraphs induced by the vertex subsets in $\mathcal{N}(G)$ and call the new polynomial connected neighborhood polynomial. Then, we investigate the properties of this polynomial and its characteristics as well.
2 Independent Neighborhood Polynomial

The independence polynomial of graph $G$, denoted by $I(G, x)$ and introduced in [5], is the polynomial

$$I(G, x) = \sum_{k=0}^{n} i_k(G) x^k,$$

where $n$ is the order of $G$ and $i_k(G)$ is the number of independent vertex subsets of cardinality $k$ in $G$.

Here, we introduce the independent neighborhood polynomial of $G$ which is the ordinary generating function for the number of subsets of open neighborhoods of vertices of $G$ which form independent sets in $G$.

**Definition 1.** The independent neighborhood polynomial of graph $G$ is the polynomial

$$N^{(i)}(G, x) = \sum_{k=0}^{n} n_k^{(i)}(G) x^k,$$

where $n$ is the order of $G$ and $n_k^{(i)}(G)$ is the number of independent vertex subsets of cardinality $k$ which are subsets of open neighborhoods in $G$.

If a graph $G$ is a tree, then every vertex set with a common neighbor in $G$ also forms an independent set in $G$, otherwise $G$ contains a cycle as a subgraph which contradicts the fact that $G$ is a tree. This argument leads to the following result.

**Lemma 1.** If $G$ is a tree, then

$$N^{(i)}(G, x) = N(G, x).$$

Also, if $G$ is a cycle of order $n$ with $n > 3$, clearly it does not contain any cycle of order $k$, with $k < n$, as a subgraph which means any vertex subset with a common neighbor in $G$ forms an independent set in $G$. This argument leads to the following result.

**Lemma 2.** If $G$ is a cycle of order $n$ with $n > 3$, then

$$N^{(i)}(G, x) = N(G, x).$$
The disjoint union of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) with disjoint vertex sets \( V_1 \) and \( V_2 \), denoted by \( G_1 \cup G_2 \), is a graph with the vertex set \( V_1 \cup V_2 \), and the edge set \( E_1 \cup E_2 \).

**Theorem 1.** Suppose \( G_1 \) and \( G_2 \) are vertex disjoint graphs. The independent neighborhood polynomial of the disjoint union \( G_1 \cup G_2 \) of these two graphs satisfies

\[
N^{(i)}(G_1 \cup G_2, x) = N^{(i)}(G_1, x) + N^{(i)}(G_2, x) - 1.
\]

**Proof.** It is clear that every independent vertex subset with a common neighbor in \( G_1 \) forms also an independent vertex subset with a common neighbor in \( G_1 \cup G_2 \). The same argument holds for \( G_2 \). The only set that is counted more than once is the empty set and this over-counting is corrected by subtracting 1.

The join of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) with disjoint vertex sets \( V_1 \) and \( V_2 \), denoted by \( G_1 + G_2 \), is the disjoint union of \( G_1 \) and \( G_2 \) together with all those edges that join vertices in \( V_1 \) to vertices in \( V_2 \).

**Theorem 2.** Suppose \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are vertex disjoint graphs. The independent neighborhood polynomial of the join \( G_1 + G_2 \) of these two graphs satisfies

\[
N^{(i)}(G_1 + G_2, x) = I(G_1, x) + I(G_2, x) - 1.
\]

**Proof.** Let \( X \) be an independent vertex subset of \( G_1 + G_2 \) with a common neighbor in \( G_1 \). We can distinguish the following cases:

- Suppose \( X \subseteq V_1 \) or \( X \subseteq V_2 \). Since in \( G_1 + G_2 \) there is an edge between each vertex in \( V_1 \) and every vertex in \( V_2 \), the vertices in every independent subset of \( V_1 \) have a common neighbor in \( G_2 \). Similarly, the vertices in every independent subset of \( V_2 \) have a common neighbor in \( G_1 \). All such subsets are generated by \( I(G_1, x) + I(G_2, x) - 1 \) in which \( I(G, x) \) is the independence polynomial of graph \( G \). By subtracting 1, we correct the over-counting of the empty set.
Since there is an edge between each vertex in $V_1$ and every vertex in $V_2$, the set $X$ cannot contain vertices from both $V_1$ and $V_2$ and at the same time remains an independent set. This argument completes the proof.

\[ \square \]

**Corollary 1.** For the given graphs $G_1$ and $G_2$ the following holds.

\[ N^{(i)}(G_1 + G_2, x) = I(G_1 + G_2, x). \]

The *Cartesian product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets $V_1$ and $V_2$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2 = \{(u, v) \mid u \in V_1, v \in V_2\}$ where the vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are adjacent in $G_1 \square G_2$ if and only if $x_1 = y_1$ and $\{x_2, y_2\} \in E_2$ or $x_2 = y_2$ and $\{x_1, y_1\} \in E_1$.

**Theorem 3.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint graphs with no isolated vertex. Then

\[
N^{(i)}(G_1 \square G_2, x) = \sum_{(u,v) \in V_1 \times V_2} (I(G_1[N_{G_1}(u)], x) - 1)(I(G_2[N_{G_2}(v)], x) - 1) - |V_1||V_2|x - 2|E_1||E_2|x^2.
\]

**Proof.** Clearly, $G_1 \square G_2$ contains $|V_1|$ copies of $G_2$ and at the same time $|V_2|$ copies of $G_1$. Every independent vertex subset of each copy of $G_1$ with a common neighbor in that copy of $G_1$ can be transformed to an independent vertex subset of $G_1 \square G_2$ with a common neighbor in $G_1 \square G_2$ and the same argument holds for the copies of $G_2$. All such vertex subsets are generated by $1 + |V_1|(N^{(i)}(G_2, x) - 1) + |V_2|(N^{(i)}(G_1, x) - 1)$ and since any set containing a vertex $(u, v)$ with $u \in V_1$ and $v \in V_2$ is counted twice, once in $N^{(i)}(G_1, x)$ as the vertex $u$ in a copy of $G_1$ and once more in $N^{(i)}(G_2, x)$ as the vertex $v$ in a copy of $G_2$, by subtracting $|V_1||V_2|x$ we correct this over-counting.
By $\sum_{(u,v)\in V_1 \times V_2} (I(G_1[N_{G_1}(u)], x) - 1)(I(G_2[N_{G_2}(v)], x) - 1)$, we generate all non-empty independent subsets of the open neighborhood of the vertex $(u,v)$ in $G_1 \square G_2$ which contain at least one vertex of the form $(u',v)$ with $u' \in N_{G_1}(u)$, and at least one vertex of the form $(u,v')$ with $v' \in N_{G_2}(v)$ for all $(u,v) \in V_1 \times V_2$.

Every copy of an edge $\{a,b\} \in E_1$ together with every copy of an edge $\{x,y\} \in E_2$ forms a four cycle in $G_1 \square G_2$ in which, each of the vertex subsets $\{(a,x),(b,y)\}$ and $\{(b,x),(a,y)\}$ are counted twice and the double-counting of such subsets is corrected by subtracting $2|E_1||E_2|x^2$.

Let $G = (V,E)$ be a graph. For any positive integer $r$ the $r$-expansion of $G$, denoted by $exp(G,r)$, is the graph obtained from $G$ by replacing every vertex $v \in V$ with an independent set $I_v$ of size $r$ and replacing every edge $\{u,v\} \in E$ with a complete bipartite graph $K_{r,r}$ with the bipartite sets $I_u$ and $I_v$.

**Theorem 4.** Let $G = (V,E)$ be a graph, $r$ be a positive integer and $exp(G,r)$ be the $r$-expansion of $G$. Then,

$$N^{(i)}(exp(G,r), x) = N^{(i)}(G, ((1+x)^r - 1)).$$

**Proof.** Suppose $X$ is an independent vertex subset with a common neighbor $v$ in $G$. Then any vertex subset in $exp(G,r)$ which contains at least one vertex from at least one $I_x$ forms an independent set with all vertices in $I_v$ as common neighbors. On the other hand, any independent vertex subset with a common neighbor in $exp(G,r)$ is of this form and all such sets are generated by $N^{(i)}(G, (1+x)^r - 1)$.

3 Connected Neighborhood Polynomial

In this section, we introduce the connected neighborhood polynomial of a graph which is the ordinary generating function for the number of subsets of open neighborhoods of vertices of a given graph which induce a connected subgraph.
Definition 2. The connected neighborhood polynomial of graph $G$ is the polynomial

$$N^{(c)}(G, x) = \sum_{k=0}^{n} n^{(c)}_k(G) x^k,$$

where $n$ is the order of $G$ and $n^{(c)}_k$ is the number of vertex subsets of cardinality $k$ which are subsets of open neighborhoods of vertices in $G$ and induce a connected subgraph in $G$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple graphs. An isomorphism from $G$ to $H$ is a bijection $\phi : V(G) \rightarrow V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$. If such an isomorphism from $G$ to $H$ exists we write $G \cong H$.

The following lemma can be easily proved.

Lemma 3. Let $G$ be a graph of order $n$. Then the following statements are satisfied.

- Suppose $K_n$ is the complete graph of order $n$ and $G \cong K_n$, then $N^{(c)}(G, x) = N(G, x) = (1 + x)^n - x^n$.
- Suppose $T_n$ is a tree of order $n$ and $G \cong T_n$, then $N^{(c)}(G, x) = 1 + nx$.
- Suppose $C_n$ is a cycle of order $n$ with $n > 3$ and $G \cong C_n$, then $N^{(c)}(G, x) = 1 + nx$.

Theorem 5. Suppose $G_1$ and $G_2$ are vertex disjoint graphs. The connected neighborhood polynomial of the disjoint union $G_1 \cup G_2$ of these two graphs satisfies

$$N^{(c)}(G_1 \cup G_2, x) = N^{(c)}(G_1, x) + N^{(c)}(G_2, x) - 1.$$

Proof. It is clear that every vertex subset with a common neighbor in $G_1$ which induces a connected subgraph in $G_1$ forms a subset with a common neighbor in $G_1 \cup G_2$ which also induces a connected subgraph in $G_1 \cup G_2$. The same argument holds for $G_2$. The only set that is counted twice is the empty set which is corrected by subtracting 1. □
Let $G = (V, E)$ be a simple graph of order $n$. The subgraph component polynomial of graph $G$, denoted by $Q(G; x, y)$ and introduced in [6], is defined by
\[ Q(G; x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} q_{ij}(G)x^iy^j, \]
where
\[ q_{ij}(G) = |\{X \subseteq V \mid k(G[X]) = j \land |X| = i \}| \]
in which $k(G[X])$ gives the number of connected components of the graph induced by the vertex subset $X \subseteq V(G)$.

The coefficient of $y^k$ in $Q(G; x, y)$, written as $[y^k]Q(G; x, y)$ and denoted by $Q_k(G; x)$, is the ordinary generating function for the number of vertex subsets in $G$ that induce a subgraph in $G$ with exactly $k$ components: $Q_k(G; x) = [y^k]Q(G; x, y)$. It is clear that $Q_1(G; x)$ is the generating function for the number of vertex subsets that induce a connected subgraph. This particular case is renamed to subgraph polynomial and rephrased as follows:
\[ S(G, x) = Q_1(G; x) = \sum_{k=0}^{n} s_k(G)x^k, \]
in which $s_k(G)$ gives the number of vertex subsets of size $k$ which induce a connected subgraph in $G$. For further information on subgraph polynomial, see [6].

**Theorem 6.** Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are vertex disjoint graphs. The connected neighborhood polynomial of the join $G_1 + G_2$ of these two graphs satisfies
\[ N^{(c)}(G_1 + G_2, x) = S(G_1, x) + S(G_2, x) + (N(G_1, x) - 1)((1 + x)^{|V_2|} - 1) \]
\[ + (N(G_2, x) - 1)((1 + x)^{|V_1|} - 1) \]
\[ - (N(G_1, x) - 1)(N(G_2, x) - 1). \]

**Proof.** Let $X$ be a vertex subset in $G_1 + G_2$ inducing a connected subgraph in $G_1 + G_2$ with a common neighbor in $G_1 + G_2$. We can distinguish the following cases:
- Suppose $X \subseteq V_1$ or $X \subseteq V_2$. Since any subset of $V_1$ that induces a connected subgraph in $G_1$ forms a vertex subset of $G_1 + G_2$ which induces a connected subgraph in $G_1 + G_2$ and obviously has all vertices in $V_2$ as common neighbors in $G_1 + G_2$. So, we just need to count all vertex subsets in $G_1$ that induce connected subgraphs in $G_1$. The same argument holds for $G_2$. All such sets are generated by $S(G_1, x) + S(G_2, x)$.

- Suppose $X$ contains at least one vertex from each of $V_1$ and $V_2$. Let $X$ be the disjoint union of non-empty vertex subsets $A_1$ and $A_2$ where $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$. If there is a common neighbor $v_1$ for the vertices of $A_1$ in $G_1$, then after joining $G_1$ and $G_2$ all vertices in $X$ have $v_1$ as their common neighbor in $G_1 + G_2$ and $X$ induces a connected subgraph in $G_1 + G_2$. A similar argument holds for the case that there is a vertex $v_2 \in V_2$ as a common neighbor for vertices of $A_2$ in $G_2$. Such sets $X$ are generated by $(N(G_1, x) - 1)(1 + x)^{|V_2| - 1} + (N(G_2, x) - 1)(1 + x)^{|V_1| - 1}$.

By subtracting $(N(G_1, x) - 1)(N(G_2, x) - 1)$ we correct the double-counting of those sets $X = A_1 \cup A_2$ where $A_1 \in \mathcal{N}(G_1)$ and $A_2 \in \mathcal{N}(G_2)$.

\[ \square \]

## 4 Conclusions and Open Problems

In this paper, we have defined the independent neighborhood polynomial of a given graph $G$, denoted by $N^{(i)}(G, x)$, as well as the connected neighborhood polynomial of $G$, denoted by $N^{(c)}(G, x)$ which are obtained from the neighborhood polynomial of $G$ after taking particular restrictions into account and then we investigated some of the characteristics of these polynomials.

For a given statement $S$ if $S$ is true $[S] = 1$, otherwise $[S] = 0$. Let $iso(G)$ be the number of isolated vertices in $G$. We can reformulate the neighborhood polynomial of $G$ as follows:
\[ N(G, x) = \sum_{X \in \mathcal{N}(G)} x^{\left| X \right|} \]
\[ = \sum_{X \in \mathcal{N}(G)} [k(G[X]) = \left| X \right|] x^{\left| X \right|} + \sum_{X \in \mathcal{N}(G)} [k(G[X]) = 1] x^{\left| X \right|} \]
\[ + \sum_{X \in \mathcal{N}(G)} [1 < k(G[X]) < \left| X \right|] x^{\left| X \right|} \]
\[ - 1 - (n - iso(G))x. \]

Clearly, \( \sum_{X \in \mathcal{N}(G)} [k(G[X]) = \left| X \right|] x^{\left| X \right|} \) is exactly \( N^{(i)}(G, x) \), and similarly \( \sum_{X \in \mathcal{N}(G)} [k(G[X]) = 1] x^{\left| X \right|} \) is nothing but \( N^{(c)}(G, x) \). By \(-1 - (n - iso(G))x\), we correct the over counting of the empty set and all non-isolated vertices. On the other hand, \( \sum_{X \in \mathcal{N}(G)} [1 < k(G[X]) < \left| X \right|] x^{\left| X \right|} \), which we denote by \( N^{(d)}(G, x) \), generates all vertex subsets of open neighborhoods of vertices in \( G \) which induce a disconnected graph with at least one component containing more than one vertex. To conclude, we have

\[ N(G, x) = N^{(i)}(G, x) + N^{(c)}(G, x) + N^{(d)}(G, x) - 1 - (n - iso(G))x. \]

Can we find an efficient way or a combinatorial approach to calculate \( N^{(d)}(G, x) \) for a graph \( G \), given \( N^{(i)}(G, x) \) or/and \( N^{(c)}(G, x) \)?

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