Anti-plane stressed state uniformly piece-homogeneous space with a periodic system of parallel semi-infinite interfacial cracks

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Abstract. In the present work, the exact solutions for two problems of the antiplane stress state of a uniformly piecewise homogeneous space formed by alternately joining two heterogeneous layers of equal thickness from different materials, which is weakened on the junction planes of layers by a periodic system of parallel semi-infinite tunnel cracks are obtained. In the first of these problems, it is assumed that stresses are given on the crack banks, and in the second problem, that stresses are given on one of the crack banks and displacements on the other.

Introduction
The study of the stress-strain state of elastic homogeneous and composite massive bodies with stress concentrators of various types is one of the evolving directions of the theory of contact and mixed problems of the mathematical theory of elasticity. The main results obtained in this direction for homogeneous massive bodies are presented in monographs [1–3]. The study of similar problems for piecewise homogeneous, layered massive bodies, which, in our opinion, are relevant from the point of view of layered composites, began relatively recently. In this regard, we cite the papers [4–7]. We also point out the papers [8–10], where the exact and effective solutions of some antiplane periodic and doubly periodic problems were constructed for a uniformly piecewise homogeneous space with interphase and internal finite cracks, which are closely related to the problems studied here.

1. Statement of problems and the derivation of governing equations
Let a piecewise-homogeneous space made by alternately joining two heterogeneous layers with thickness $2h$ with shear moduli $G_1$ and $G_2$, respectively, under conditions of antiplane deformation, on the junction planes of heterogeneous layers $y = 2nh$ ($n \in \mathbb{Z}$) along infinite strips $(-\infty, -a) \cup (a, \infty)$ ($-\infty < z < \infty$), weakened by a periodic system of parallel tunnel cracks. In the first problem, it is assumed that oppositely directed shear stresses of intensity factors $\tau_1(x)$ on one of the crack banks belonging to one of the heterogeneous layers, and oppositely directed shear stresses of intensity factor $\tau_2(x)$ on the other side of the cracks belonging to the second of the heterogeneous layers. In the second problem, it is considered that oppositely directed shear stresses with intensity factor $\tau_1(x)$ on one of the crack banks belonging to one of the...
heterogeneous layers, and zero displacements are given on the other banks of the crack faces belonging to the second of the heterogeneous layers.

The goal is to build the exact solutions to the problem and obtain simple formulas for the contact stresses acting at the junction of the layers and the intensity factor of the destructive stresses at the end points of the cracks.

First, consider the first task. Obviously, with such a statement of the problem, the planes \( y = (2n + 1)h \) \((n \in \mathbb{Z})\) are antisymmetry planes and the stress state in the composite layers located between the symmetry planes \( y = (2k - 1)h \) and \( y = (2k + 1)h \) will be the same and, therefore, it is sufficed to consider only a two-component layer (base layer) between antisymmetry planes \( y = \pm h \).

For the base layer, the problem can be mathematically formulated as the following boundary value problem:

\[
\begin{align*}
W_j(x, 0) = W_0, \quad & -\infty < x < \infty, \\
\tau_{yz}^{(1)}(x, 0) = \tau_{yz}^{(2)}(x, 0), \quad & x \in (-a, a), \\
\tau_{yz}(x, 0) = \tau_0, \quad & x \notin (-a, a), \\
\tau_{yz}(x, 0) = \tau(x), \quad & x \notin (-a, a).
\end{align*}
\]

Here \( W_j(x, y) \) \((j = 1, 2)\) be the components of the displacements of the points of the corresponding layers, each of which in the domain of its definition satisfies the Laplace equation and are related to the stress components by the formulas

\[
\tau_{yz}(x, y) = G_j \frac{\partial W_j(x, y)}{\partial y}.
\]

To obtain the governing equation of the problem, in the mind’s eye separate the layers from each other and introduce into consideration the function of unknown contact stresses acting at the junction of the layers

\[
\tau_{yz}^{(1)}(x, 0) = \tau_{yz}^{(2)}(x, 0) = \tau(x), \quad x \in (-a, a)
\]

and consider the following auxiliary boundary value problems for each layer:

\[
\begin{align*}
W_1(x, h) = 0, \quad & -\infty < x < \infty, \\
\tau_{yz}^{(1)}(x, 0) = \tau(x), \quad & x \in (-a, a), \quad j = 1, 2, \\
\tau_{yz}^{(2)}(x, 0) = \tau_j(x), \quad & x \notin (-a, a).
\end{align*}
\]

Then, using the generalized integral Fourier transform, we solve these auxiliary boundary value problems and determine the derivatives of the displacements of the points of both layers on the plane \( y = 0 \) through a function \( \tau(x) \). We get

\[
W_1'(x, 0) = -\frac{1}{2h G_1} \int_{-a}^a \frac{\tau(s) ds}{\sinh \mu(s - x)} - f_1(x), \quad -\infty < x < \infty,
\]

\[
W_2'(x, 0) = \frac{1}{2h G_2} \int_{-a}^a \frac{\tau(s) ds}{\sinh \mu(s - x)} + f_2(x), \quad -\infty < x < \infty.
\]

Here

\[
f_j(x) = \frac{1}{2h G_1} \left[ \int_{-a}^{-a} \frac{\tau_j(s) ds}{\sinh \mu(s - x)} + \int_{a}^{\infty} \frac{\tau_j(s) ds}{\sinh \mu(s - x)} \right], \quad G = \frac{G_2}{G_1}, \quad \mu = \frac{\pi}{2h}, \quad j = 1, 2.
\]
Further, using relations (4), we satisfy the third of conditions (1), initially differentiating it with respect to \( x \). As a result, to determine the unknown contact stresses \( \tau(x) \), we obtain the following singular integral equations of the first kind:

\[
\frac{1}{2h} \int_{-a}^{a} \frac{\tau(s) ds}{\sinh \mu(s - x)} = F_1(x), \quad x \in (-a, a), \quad F_1(x) = -\frac{G_2}{1 + G_2} (f_1(x) + f_2(x)).
\] (5)

The equation (5) should be considered together with the condition

\[
\int_{-a}^{a} \tau(s) ds = 0,
\] (6)

which is the equilibrium condition of the upper or lower half-spaces.

Thus, the solution of the problem was reduced to the solution of equation (5) under conditions (6).

Now we turn to the second problem. Here, also, the planes \( y = (2n + 1)h \) \((n \in Z)\) are antisymmetry planes and the stress state in the composite layers located between the planes of symmetry \( y = (2k-1)h \) and \( y = (2k+1)h \) will be the same and, therefore, only a two-component layer (base layer) between antisymmetry planes \( y = \pm h \) can be considered.

The stated problem can be mathematically formulated as the following boundary value problem:

\[
\begin{align*}
W_j(x, (-1)^{j+1}h) &= 0, \quad -\infty < x < \infty, \quad j = 1, 2, \\
\tau_{y'z}^{(1)}(x, 0) &= \tau_{y'z}^{(2)}(x, 0), \quad x \in (-a, a), \\
W_1(x, 0) &= W_2(x, 0), \quad x \in (-a, a), \\
\tau_{y'z}^{(1)}(x, 0) &= \tau_1(x), \quad x \notin (-a, a), \\
W_2(x, 0) &= c_0, \quad x \notin (-a, a).
\end{align*}
\] (7)

In order to derive the determining system of equations of the problem, we again mentally separate the layers from each other and introduce the function of the unknown contact stresses acting at the interface of the layers and the function of the displacements of the points of interface of the layers according to the formulas:

\[
\begin{align*}
\tau_{y'z}^{(1)}(x, 0) &= \tau_{y'z}^{(2)}(x, 0) = \tau(x), \quad x \in (-a, a), \\
W_1(x, 0) &= W_1(x, 0) = W(x), \quad x \in (-a, a).
\end{align*}
\] (8)

The following auxiliary boundary value problems for each layer is considered:

\[
\begin{align*}
W_1(x, h) &= 0, \quad -\infty < x < \infty, \\
\tau_{y'z}^{(1)}(x, 0) &= \tau(x), \quad x \in (-a, a), \\
\tau_{y'z}^{(1)}(x, 0) &= \tau_1(x), \quad x \notin (-a, a), \\
W_2(x, 0) &= W(x), \quad x \in (-a, a), \\
W_2(x, 0) &= 0, \quad x \notin (-a, a).
\end{align*}
\] (9)

Further, using the generalized integral Fourier transform, we solve these two auxiliary boundary value problems and determine in the first problem the derivative of the displacements of the junction points of the first layer through the unknown contact stresses \( \tau(x) \), and in the second problem the contact stresses acting at the junction of the second layer through the derivative of the displacements of the contact zone points \( W(x) \). Find:

\[
\begin{align*}
W_1(x, 0) &= -\frac{1}{2hG_1} \int_{-a}^{a} \frac{\tau(s) ds}{\sinh \mu(s - x)} - f_1(x), \quad -\infty < x < \infty, \\
\tau_{y'z}^{(1)}(x, 0) &= -\frac{G_2}{2h} \int_{-a}^{a} \coth \mu(s - x) W'(s) ds, \quad -\infty < x < \infty,
\end{align*}
\] (10)
where the function \( f_1(x) \) is the same as above.

Then, using relations (9), we satisfy the conditions (8). As a result, to determine the unknown functions, we obtain the following governing system of singular integral equations:

\[
\begin{align*}
\tau(x) + \frac{G_2}{2h} \int_{-a}^{a} \coth \mu(s-x)W'(s) \, ds &= 0, \\
W'(x) + \frac{1}{2hG_1} \int_{-a}^{a} \frac{\tau(s) \, ds}{\sinh \mu(s-x)} &= -f_1(x),
\end{align*}
\]

which must be considered together with the condition of continuity of displacements at the end points of the cracks and the equilibrium of the upper or lower half-spaces

\[
\int_{-a}^{a} W'(s) \, ds = 0, \quad \int_{-a}^{a} \tau(s) \, ds = 0.
\]

Thus, the solution of the second problem is reduced to the solution of the governing system (10) under conditions (11).

2. Exact solutions of the governing equations

The focus of this section is to construct of the exact solutions of the resulting governing equations. Note that the governing system of the first problem with its structure is identical with the system of one of the governing equations obtained in [10], where its exact solution was constructed. Using the results of mentioned paper, the exact solution of equations (5) under conditions (6) can be represented as:

\[
\tau(x) = -\frac{1}{2\omega_e(x)} \left[ \frac{1}{h} \int_{-a}^{a} \frac{\omega_e(\xi)e^{\mu(x-s)F_1(s)}}{\sinh \mu(s-x)} \, ds \right] - \sqrt{2C}, \quad x \in (-a, a),
\]

where

\[
C = \frac{I_1}{I_2}, \quad I_1 = \frac{1}{\pi} \int_{\pi a}^{\pi a} d\eta \sqrt{\eta^2 + \pi^2 a^2} \int_{\pi a}^{\pi a} \frac{\omega(\xi)F_1(\xi)}{\xi - \eta} d\xi, \quad I_2 = \frac{2}{\sqrt{b_s}} F\left(\frac{\pi}{2}, k_1\right),
\]

\[
\omega_e(x) = \sqrt{\cosh(2\mu a) - \cosh(2\mu x)}, \quad \omega(\eta) = \sqrt{(-\eta - a_s)(b_s - \eta)}, \quad k_1 = \sqrt{1 - \frac{a_s}{b_s}},
\]

and the function \( F(a, k_1) \) is an elliptic integral of the first kind.

Using formula (12) for the stress intensity factors of destructive stresses, we obtain the following expression:

\[
K_{III}(\pm a) = \sqrt{2\pi} \lim_{x \to \pm a} \sqrt{|x| + a} |\tau(x)| = \frac{\sqrt{\mu}}{2\sinh(2\mu a)} \left[ \frac{1}{h} \int_{-a}^{a} \frac{\omega_e(s)e^{\mu(x-a)F_1(s)}}{\sinh \mu(s-a)} \, ds \right] - \sqrt{2C}.
\]

Now we construct the exact solution for the governing system of integral equations of the second problem. Here we also note that a closed solution of a similar system of equations was obtained in [8]. Following the mentioned work in (10) and (11), we pass to the new variables by formulas \( \xi = e^{\mu \eta}, \eta = e^{\mu x} (\mu = \pi/(2h)) \) and introduce the notation

\[
\tau_e(\eta) = \frac{\tau(\ln(\eta/\mu))}{G_1 \eta}, \quad W_e(\eta) = \frac{W'(\ln(\eta/\mu))}{\eta}, \quad f_e(\eta) = \frac{f_1(\ln(\eta/\mu))}{G_1 \eta}, \quad a_s = \exp(-\mu a), \quad b_s = \exp(\mu a),
\]
and taking into account the first of conditions (11), we rewrite system (10) in the following form:

\[
\begin{align*}
&\tau_\ast(\eta) + \frac{G}{\pi} \int_{a_\ast}^{b_\ast} \left( \frac{1}{\xi - \eta} - \frac{1}{\xi + \eta} \right) W_\ast(\xi) d\xi = 0, \\
&W_\ast(\eta) + \frac{1}{\pi} \int_{a_\ast}^{b_\ast} \left( \frac{1}{\xi - \eta} + \frac{1}{\xi + \eta} \right) \tau_\ast(\xi) d\xi = -f_\ast(\eta),
\end{align*}
\]

\(\eta \in (a_\ast, b_\ast).\) (13)

Now, we continue the functions \(\tau_\ast(\eta)\) and the first equation (12) to the interval \((-b_\ast, -a_\ast)\) in an odd way, and the function \(W_\ast(\eta)\) and the second equation (13) in an even way. Then, using notation \(L = (-b_\ast, -a_\ast) \cup (a_\ast, b_\ast),\) we come to the system:

\[
\begin{align*}
&\tau_\ast(\eta) + \frac{G}{\pi} \int_L \psi_j(\xi) d\xi = 0, \\
&W_\ast(\eta) + \frac{1}{\pi} \int_L \psi_j(\xi) \frac{d\xi}{\xi - \eta} = f_\ast(|\eta|),
\end{align*}
\]

\(x \in L.\) (14)

The conditions (11) take the form:

\[
\int_{a_\ast}^{b_\ast} W_\ast(\eta) d\eta = 0, \quad \int_{a_\ast}^{b_\ast} \tau_\ast(\eta) d\eta = \frac{\mu T_\ast}{G_1}.
\]

To construct a closed solution of system (14), we reduce its solution to the solution of two independent equations. To do this, we multiply the second equation (13) by \(\pm \sqrt{G}\) and add it to the first equation. As a result, using notation

\[
\psi_j(\eta) = \tau_\ast(\eta) + (-1)^{j+1} \sqrt{G} W_\ast(\eta),
\]

the following two independent singular integral equations are obtained:

\[
\psi_j(\eta) + \frac{(-1)^{j+1} \sqrt{G}}{\pi} \int_L \psi_j(\xi) \frac{d\xi}{\xi - \eta} = (-1)^{j+1} \sqrt{G} f_\ast(|\eta|), \quad x \in L, \ j = 1, 2.
\]

(15)

To construct the exact solutions of equations (15) we introduce analytic in the entire complex plane, except, perhaps, lines \(L,\) functions [11]

\[
\Omega_j(z) = \frac{1}{2\pi i} \int_L \psi_j(\xi) \frac{d\xi}{\xi - z}, \quad j = 1, 2
\]

(16)

with the help of which equations (15) are formulated in the form of equivalent Riemann problems:

\[
\Omega_j^+(\eta) = q_j \Omega_j^-(\eta) + F_j(\eta), \quad j = 1, 2.
\]

(17)

Here

\[
F_j(\eta) = \frac{(-1)^{j+1} \sqrt{G} f_\ast(\eta)}{1 + (-1)^{j+1} i \sqrt{G}}, \quad q_1 = \bar{q}_2 = \frac{1 - i \sqrt{G}}{1 + i \sqrt{G}}.
\]

The solutions of equations (15) have the form [11]:

\[
\Omega_j(z) = \frac{(-1)^{j+1} \sqrt{G} X_j(z)}{2\pi i [1 + (-1)^{j+1} \sqrt{G}]} \int_L \frac{f_\ast(\xi) d\xi}{X_j^+(\xi)(\xi - z)} + P_j(z) X_j(z).
\]

(18)
Here
\[ X_j(z) = (z + b_*)^{-\gamma_j}(z + a_*)^{\gamma_j-1}(z - a_*)^{-\gamma_j}(z - b_*)^{\gamma_j-1}, \quad P_j(z) = c_1^{(j)} z + c_0^{(j)}, \]
\[ \gamma_1 = \gamma = \frac{1}{2\pi} \arg q_1 = \begin{cases} 
1 - \arctan \frac{2\sqrt{G}}{1-G}, & G < 1, \\
1 + \arctan \frac{2\sqrt{G}}{G-1}, & G > 1
\end{cases} \]
\[ \gamma_2 = 1 - \gamma_1 = 1 - \gamma, \]

\[ X_j^+(\xi) \] is the value of the piecewise holomorphic function \( X_j(z) \) on the upper bank of the line \( L \), and \( P_j(z) \) is the polynomial of the first degree with unknown coefficients to be determined.

Further, using the formulas of Plemelj-Sokhotsky and representation (18), we write the general solutions of equations (15) in the following form:
\[ \psi_j(\eta) = (-1)^{j+1} \frac{\sqrt{G} f_*(|\eta|)}{1+G} \int_{L} \frac{f_*(|\eta|) d\xi}{X((-1)^{j+1}\eta)(\xi - \eta)} \]
\[ + (-1)^{j} 2i \sqrt{\frac{G}{1+G}} X((-1)^{j+1}\eta) P_j(\eta) \]
(19)

Here
\[ X(\eta) = -\text{sign}\eta |b_* + \eta|^{-\gamma} |a_* + \eta|^{\gamma-1} |\eta - a_*|\gamma |b_* - \eta|^{-\gamma-1}, \quad \eta \in L. \]

This shows that for a homogeneous space, when \( G = 1, \arg q = 3\pi/2, \) and \( \gamma = 3/4. \) Using the values of the functions \( \psi_j(\eta) \) from (19) and taking into account that the function \( \tau_*(\eta) \) is even and the function \( W_*(\eta) \) is odd, we obtain:
\[ W_*(\eta) = \frac{f_*(|\eta|)}{1+G} - \frac{\sqrt{G}}{2(1+G)} \left( \frac{1}{\pi} \int_{L} \frac{X(\eta)}{X(\xi)} - \frac{X(-\eta)}{X(-\xi)} \right) \frac{f_*(|\eta|) d\xi}{\xi - \eta} \]
\[ + 2i \sqrt{\frac{1+G}{G}} \left\{ c_1 \eta [X(\eta) - X(-\eta)] + c_0 [X(\eta) + X(-\eta)] \right\}, \]
(20)
\[ \tau_*(\eta) = -\frac{\sqrt{G}}{2(1+G)} \left( \frac{\sqrt{G}}{\pi} \int_{L} \frac{X(\eta)}{X(\xi)} + \frac{X(-\eta)}{X(-\xi)} \right) \frac{f_*(|\eta|) d\xi}{\xi - \eta} \]
\[ + 2i \sqrt{1+G} \left\{ c_1 \eta [X(\eta) + X(-\eta)] + c_0 [X(\eta) - X(-\eta)] \right\}. \]
(21)

Here \( c_1 = -c_1^- = c_1^+, \quad c_0 = c_0^- = c_0^+. \)

Comparing the expansion coefficients of \( 1/\xi \) for the functions \( \Omega_j(z) \) \((j = 1, 2)\) from (16) and (18) at infinity, using relations (14), taking into account that the function \( W_*(\eta) \) is even and the function \( \tau_*(\eta) \) is odd, for constants, we obtain the following values:
\[ c_1 = 0, \quad c_0 = \frac{i}{L_-} \left\{ \frac{\mu T_*}{G_1} \sqrt{1+G} + \frac{\sqrt{G}}{2\pi \sqrt{1+G}} \int_{a_*}^{b_*} d\eta \int_{L} \frac{X(\eta)}{X(\xi)} + \frac{X(-\eta)}{X(-\xi)} \right\} \frac{f_*(|\xi|) d\xi}{\xi - \eta} \]
where
\[ L_- = \int_{a_*}^{b_*} [X(\eta) - X(-\eta)] d\eta. \]

Given that \( \gamma > 1/2, \) it is not difficult to verify that the function \( X(\eta) \) is dominant at the point \( a_*, \) and the function \( X(-\eta) \) at the point \( b_*, \) for the dimensionless destructive stress coefficients
at the end points of the cracks we obtain

\[
K_{\text{III}}(-a) = K_{\text{III}}(a_*) = \sqrt{2\pi} \lim_{x \to a_*+0} |\eta - a_*|^\gamma \tau(\eta)
\]

\[
= \frac{AG}{\sqrt{2\pi(1 + G)}} \left\{ \int_L f_*(|\xi|) d\xi - 2\pi ic_0 \sqrt{1 + G} \right\},
\]

\[
K_{\text{III}}(a) = K_{\text{III}}(b_*) = \sqrt{2\pi} \lim_{x \to b_*-0} |\eta - b_*|^\gamma \tau(\eta)
\]

\[
= -\frac{BG}{\sqrt{2\pi(1 + G)}} \left\{ \int_L f_*(|\xi|) d\xi - 2\pi ic_0 \sqrt{1 + G} \right\}.
\]

Here

\[
A = (a_* + b_*)^{-\gamma} (2a_*)^{-1}(b_* - a_*)^{-1}, \quad B = (a_* + b_*)^{-\gamma} (2b_*)^{-1}(b_* - a_*)^{-1}.
\]

**Conclusion**

Thus, the exact solutions of two problems are constructed for a uniformly piecewise-homogeneous space in a state of antiplane deformation with two semi-infinite tunnel parallel interfacial cracks. Simple formulas are obtained for determining the tangential contact stresses of the heterogeneous layers acting at the junction and the intensity coefficients of the breaking stresses at the end points of the cracks.

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