On critical scaling at the QCD $N_f = 2$ chiral phase transition

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We investigate the critical scaling of the quark propagator of $N_f = 2$ QCD close to the chiral phase transition at finite temperature. We argue that it is mandatory to take into account the back-reaction effects of pions and the sigma onto the quark to observe critical behavior beyond mean field. On condition of self-consistency of the quark Dyson-Schwinger equation we extract the scaling behavior for the quark propagator analytically. Crucial in this respect is the correct pion dispersion relation when the critical temperature is approached from below. Our results are consistent with the known relations for the quark condensate and the pion decay constant from universality. We verify the analytical findings also numerically assuming the critical dispersion relation for the Goldstone bosons.

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I. INTRODUCTION

The phase diagram of strongly interacting quark matter has received a lot of attention, both from theoretical and experimental perspectives. One of the debated issues in this respect is the nature of the phase transition in $N_f = 2$ QCD in the chiral limit. It has long been suggested that the $SU(2)_V \times SU(2)_A \simeq O(4)$ symmetry in this case is broken spontaneously to $SU(2)_V \simeq O(3)$ by a second order chiral phase transition of the $O(4)$-universality class in 3-dimensions [3]. The underlying symmetry breaking pattern results from the assumption that the anomalously broken $U_A(1)$ symmetry of QCD leads to a relatively heavy pseudoscalar flavor singlet meson also for temperatures around $T_c$.

One of the tools to investigate this issue is lattice gauge theory, see e.g. [2, 3]. The problem is delicate since the analysis of a potential second order phase transition in a finite volume is elaborate and light quarks are expensive in lattice computations. As a result there are indications for $O(4)$ scaling from simulations with Wilson fermions [4] and very recently from results with staggered fermion actions [5]. However, since there are also arguments in favor of a first order transition [8, 9], the problem seems not yet completely settled.

The symmetry breaking pattern with an anomalously broken $U_A(1)$ symmetry is a widely used assumption for the construction of effective models as for example the Polyakov-loop quark meson model (PQM), see e.g. [10–13]. Recent constructions of effective models include and investigate the effects of the anomaly [14, 15]. The issue of critical scaling has been addressed in [16–17].

A third approach to the phase diagram are functional methods like the functional renormalization group [18–20] or Dyson-Schwinger equations (DSEs) [21, 22] applied directly to QCD. In principle, these methods are capable to link information on the microscopic degrees of freedom of the theory to the characteristics of the phase transitions. Recent works include the calculation of the non-perturbative Polyakov-loop potential from the propagators of Yang-Mills theory [19], the investigation of the chiral and deconfinement transition [20, 22, 23], quark spectral functions [24] and the exploration of color superconducting phases [25].

Although DSEs are well established they are usually not used to study critical phenomena and it may be even unclear to what extend they are suitable. In fact, up to now critical scaling in DSEs has only been observed on the mean field level [26, 27]. The main goal of this paper is to go beyond these studies. We employ the Dyson-Schwinger equations to explore the $O(4)$ scaling in the quark propagator at the chiral phase transition for $N_f = 2$ QCD. By including the effects of pion and sigma back-reaction onto the quarks along the lines of the zero temperature studies in Refs. [28, 29] we identify a mechanism that generates critical scaling beyond the mean field level. Our results therefore provide a basis for further studies of the chiral phase transition in this framework. They also serve to study the connection between the scaling properties in the effective theory and the behavior of the fundamental microscopic degrees of freedom such as the quark propagator.

The paper is structured as follows: In section II we discuss our approximation scheme for the DSE of the quark propagator at finite temperature. We then analyze the scaling behavior of the DSEs analytically in section III and demonstrate that the mechanism for the onset of critical scaling beyond the mean field level in the quark propagator is located in the pion and sigma dispersion relations. For our numerical investigations we discuss a tractable truncation scheme in section IV A and verify our analytical findings in section IV B under the assumption of a scaling law for the pion velocity. We conclude and summarize in section V.
II. QUARK DYSON-SCHWINGER EQUATION AND EFFECTS FROM GOLDSTONE BOSONS

A. Backcoupling mesons onto quarks

The DSE for the quark propagator is given diagrammatically in Fig. 1. The unknown quantities in this equation are the dressed gluon propagator and the one-particle irreducible (1PI) quark-gluon vertex. We are particularly interested in the behavior of the quarks close to the chiral limit and in the vicinity of the chiral phase transition. There the relevant degrees of freedom are associated with long-range correlations and can be identified as the Goldstone modes and the associated radial excitation. These contribute to the DSE of the quark-

\[ \Sigma_{\pi}(\vec{p},\omega_p) = 2 Z_1 F g^2 C_F T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q,\omega_q) \Gamma_{\mu}^{\pi}(q,\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \]

\[ \Sigma_{\sigma}(\vec{p},\omega_p) = Z_1 F g^2 C_F T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q,\omega_q) \Gamma_{\mu}^{\sigma}(q,\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \]

FIG. 1: The Schwinger-Dyson equation for the fully dressed quark propagator

The calculation of the quark self-energy with meson exchange is in general quite demanding. It contains a two-loop diagram containing the full Bethe-Salpeter amplitudes, \( \Gamma_{\pi,\sigma} \), which can in principle be determined from the corresponding Bethe-Salpeter equation. In order to obtain a tractable truncation that preserves the long-range effects close the chiral phase transition we will use some low-energy properties of pions in the next section.

B. The DSEs at finite temperature

Before we go into the details, we first give the formal expression for the DSE of the quark propagator as shown in Fig. 2. At finite temperature it is given by

\[ S^{-1}(\vec{p},\omega_p) = Z_2 S^{-1}_0(\vec{p},\omega_p) + \Sigma_{\pi}(\vec{p},\omega_p) + \Sigma_{\sigma}(\vec{p},\omega_p) \]

where the self-energies are

\[ \Sigma_{YM}(\vec{p},\omega_p) = Z_1 F g^2 C_F T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q,\omega_q) \Gamma_{\mu}^{YM}(q,\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \]

\[ \Sigma_{\pi}(\vec{p},\omega_p) = 2 Z_1 F g^2 C_F T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q,\omega_q) \Gamma_{\mu}^{\pi}(q,\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \]

\[ \Sigma_{\sigma}(\vec{p},\omega_p) = Z_1 F g^2 C_F T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q,\omega_q) \Gamma_{\mu}^{\sigma}(q,\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \]
with $k = p - q = (\vec{k}, \omega_k)$, the temperature $T$, the Casimir $C_T = (N_T^2 - 1)/(2N_c)$ and the renormalization factors $Z_{1F}$ of the quark gluon vertex and $Z_2$ of the quark propagator. We used the abbreviations $\Gamma^{\nu}_{\nu}(\bar{p}, \omega_p)$ for the triangle meson exchange that is part of the quark-gluon propagator. This should not be confused with the Bethe-Salpeter amplitudes $\Gamma_{\pi, \sigma}$. All color and flavor traces have been carried out already. In this work we only consider $N_f = 2$ and $N_c = 3$. The gluon propagator is denoted by $D_{\mu\nu}$ and the purely gluonic part of the dressed quark-gluon vertex by $\Gamma_{\nu}^{YM}$. Using a decomposition in Dirac components the dressed quark propagator can be parametrized by

$$S(\bar{p}, \omega_p) = \left[ -i\gamma_\mu \bar{q} C(\bar{p}, \omega_p) - i\gamma_\nu \bar{q} A(\bar{p}, \omega_p) + B(\bar{p}, \omega_p) \right]^{-1} = i\gamma_\mu \bar{q} \sigma_C(\bar{p}, \omega_p) + i\gamma_\nu \bar{q} \sigma_A(\bar{p}, \omega_p) + \sigma B(\bar{p}, \omega_p)$$

(5)

with the scalar dressing function $B(\bar{p}, \omega_p)$ and the two vector dressing functions $A(\bar{p}, \omega_p)$ and $C(\bar{p}, \omega_p)$. Its bare counterpart reads $S_0^{-1}(\bar{p}, \omega_p) = -i\gamma_\mu + m$ and we work with very small quark masses $m \to 0$.

The details of the non-perturbative gluon propagator and Yang-Mills quark-gluon vertex are not important for the scaling behavior in the vicinity of the chiral phase transition. This may be expected from the universality argument and will also show up in our analysis presented in the next section. In this section we only state the general tensor decomposition of the gluon propagator and Yang-Mills quark-gluon vertex that is used in this paper. For further details we refer to section IV A where we consider the numerical solution. Throughout this paper we will work in Landau gauge. In this case the most general representation of the gluon propagator at finite temperature is composed of two dressing functions. It can be written as

$$D_{\mu\nu}(\vec{k}, \omega_k) = \frac{Z_T(\vec{k}, \omega_k)}{k^2} \left[ P_{\mu\nu}^T \frac{1}{k^2} + \frac{Z_L(\vec{k}, \omega_k)}{k^2} P_{\mu\nu}^L \right]$$

(6)

with transverse and longitudinal projectors ($i, j = 1 \ldots 3$)

$$P_{\mu\nu}^T(\vec{k}, \omega_k) = \left( \delta_{i,j} - \frac{k_i k_j}{k^2} \right) \delta_{\mu,i} \delta_{\nu,j}$$

$$P_{\mu\nu}^L(\vec{k}, \omega_k) = P_{\mu\nu}(\vec{k}, \omega_k) - P_{\mu\nu}^T(\vec{k}, \omega_k).$$

(7)

and dressing functions $Z_T(\vec{k}, \omega_k)$ and $Z_L(\vec{k}, \omega_k)$. For the Yang-Mills part of the quark-gluon vertex we adopt the rainbow ladder approximation

$$\Gamma_{\nu}^{YM}(\bar{q}, \omega_q, \bar{p}, \omega_p) \to \gamma_\mu \Gamma_{\nu}^{YM}(\vec{k}, \omega_k)$$

(8)

where $\Gamma_{\nu}^{YM}(\vec{k}, \omega_k)$ is a vertex model depending only on the gluon momentum.

Now we need to specify the meson contributions $\Gamma_{\nu}^{\pi, \sigma}$ in Eqs. 6 and 14. These are given by

$$\Gamma_{\nu}^{\pi, \sigma} = T \sum_{n_r} \int \frac{d^3q}{(2\pi)^3} \Gamma_{\pi, \sigma}(\bar{q}, \omega_q, \bar{r}, \omega_r) S(\bar{q} + \bar{r}, \omega_q + \omega_r) \times \gamma_\nu S(\bar{p} + \bar{r}, \omega_p + \omega_r) \Gamma_{\pi, \sigma}(\bar{p}, \omega_p, \bar{r}, \omega_r) D_{\pi, \sigma}(\bar{r}, \omega_r)$$

(9)

with the Bethe-Salpeter amplitudes $\Gamma_{\pi, \sigma}$ and the meson propagators $D_{\pi, \sigma}$. In the following we consider the pion Bethe-Salpeter amplitude and propagator.

In Ref. [30] Son and Stephanov showed that the real part of the pion dispersion relation in the symmetry broken phase is given by

$$\omega^2 = u^2 (\vec{p}^2 + m_\pi^2)$$

(10)

where $u$ denotes the pion velocity and $m_\pi$ the pion screening mass. At $\vec{p} = 0$ the energy of a pion $E_\pi = u m_\pi$ is called the pion pole mass. Furthermore as already noted in Ref. [31] there are two distinct pion decay constants in the medium. At finite temperature these are given by $f_s$ transverse to the heat bath and $f_t$ longitudinal to the heat bath. The ratio of both determines the pion velocity

$$u^2 = \frac{f_s^2}{f_t^2}$$

(11)

At temperature $T = 0$ we have $f_\pi = f_s = f_t$ and consequently $u = 1$, whereas at nonzero temperature this is no longer the case. We stress that the pion decay constants constitute static quantities and are in principle calculable in a thermodynamic equilibrium approach as pointed out in [30]. The relation between the pion decay constant, the full quark propagator and the pion Bethe-Salpeter amplitude in the vacuum is outlined in Ref. [32]. To obtain the decay constants at finite temperatures we may proceed along the same lines considering the in-medium pion propagation

$$D_\pi = \frac{1}{\omega_p^2 + u^2 (\vec{p}^2 + m_\pi^2)}.$$

(12)

This leads to the following expression for the decay constants

$$\bar{P}_\mu f_t = 3 \text{tr} g T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \Gamma_\pi(\bar{q}, q, P) S(q + P) \gamma_\mu \gamma_\nu S(q)$$

(13)

valid on the pion mass shell. We used the abbreviation $\bar{P}_\mu = (u\bar{p}, \omega_p)$. In the chiral limit $m_\pi^2 = 0$ and the expression evaluated in the limit $\bar{P}_\mu \to 0$ determines the decay constants: The longitudinal decay component $f_t$ is obtained from the time component of Eq. (13), whereas the transverse component $f_s = u f_t$ can be extracted from the spatial components. In order to be able to compute them

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1 Note that the flavor trace for the self-energy part due to the pion backreaction results in a factor of two, since the flavor of the external quark is fixed. This has been overlooked in Refs. 28, 29, where a factor of three has been used leading to a slight overestimation of the pion backcoupling effects.
and to specify the pion contribution $\Gamma_\pi$ we need the pion Bethe-Salpeter amplitude $\Gamma_\pi$. It satisfies a homogenous Bethe-Salpeter equation which in principle needs to be solved at finite temperature. This is a tremendous task beyond the scope of the present work. However from the axial vector Ward-Takahashi identity in the chiral limit it follows that the Bethe-Salpeter amplitude for $P^2 = 0$ may be written as

$$\Gamma_\pi(q, 0) = \gamma_5 \frac{B(q)}{f_L}.$$  

This constitutes a Goldberger-Treiman like relation for the quark-pion coupling and is known to represent the leading term also for $P^2 \neq 0$ at zero temperature. In the following we approximate the Bethe-Salpeter amplitude for general momenta by

$$\Gamma_\pi(q, P) \approx \gamma_5 \frac{B(q)}{f_L}.$$  

For our purposes this approximation seems particularly reasonable since it incorporates the long-range interaction in Eq. (9) which is important for the critical behavior. Using (12) in (13) yields the Pagel-Stokar approximation at finite temperature for the pion decay constants.

Following the above considerations and knowledge of the gluon and Yang-Mills quark-gluon vertex dressing functions provided we obtain a closed system of equations for the quark propagator. This truncation also includes effects of pion backcoupling. Up to now we did not consider the contribution of the scalar flavor singlet meson $\Gamma_\sigma$. We know that its correlator is degenerate with the pion correlator for momenta large compared to its mass. Near the transition temperature and close to the chiral limit $m \to 0$ the mass $m_\sigma$ tends to zero and therefore sufficiently close to $T_c$ we expect the above relations to hold also for the sigma meson, with $\gamma_5$ in (15) replaced by one. This will be exploited in the following.

III. ANALYTICAL SCALING ANALYSIS

Having outlined general properties of pions in the symmetry broken phase we will now perform a scaling analysis of the quark DSE given in Eq. (1) for $T < T_c$ close to the critical temperature. We concentrate on the scalar dressing function $B(\vec{p}, \omega_p)$ and work with very small but nonzero bare quark mass $m$. The reduced temperature is given by $t = (T_c - T)/T_c$.

The scaling behavior of the pion velocity and decay constants close to the critical temperature can be obtained from a matching of an effective theory with QCD at the scale $m_\sigma$. As shown by Son and Stephanov in Ref. 30 this matching reveals

$$u \sim f_\pi \sim t^{\nu/2}$$

using the known scaling relations for the inverse correlation length and the order parameter

$$m_\sigma \sim t^{\nu}, \quad \langle \bar{\psi}\psi \rangle \sim t^{\nu/2(1-\eta)}.$$  

In $d = 3$ dimensions the values for the exponents of the $O(4)$-universality class are $\nu \approx 0.73$ and $\eta \approx 0.03$, see e.g. 33, 34.

For our scaling analysis we assume the scalar function to fulfill a scaling law, $B \sim t^\xi$, with some unknown exponent $x$. On the other hand there is no indication for the vector dressing functions $A$ and $C$ to scale with reduced temperature. The main idea is to utilize self-consistency of the Dyson-Schwinger equation. This requires left- and right-hand side of (11) to scale with $t$ in the same way. For simplicity we ignore the anomalous dimension $\eta$ which would be only a small quantitative correction anyway. Focusing on $B$ it remains to analyze the scalar projection of the self-energies given in Eqs. (2), (3) and (4):

$$B(t) = \Sigma_{\pi,\sigma}^B(t) + \Sigma_{\pi,\sigma}^B(t) + \Sigma_{\pi,\sigma}^B(t) ,$$  

with $\Sigma_{\pi,\sigma}^B = \text{Tr} \Sigma_{\pi,\sigma}^B/4$ respectively $\Sigma_{\pi,\sigma}^B = \text{Tr} \Sigma_{\pi,\sigma}^B/4$. The important quantities of the analysis are the scalar dressing function $B$ and the transverse pion decay constant $f_\pi$. In $\Sigma_{\pi,\sigma}^B$ and $\Sigma_{\pi,\sigma}^B$ the scalar dressing function $B$ occurs in the non-perturbative quark propagator and in the Bethe-Salpeter amplitude. The pion decay constant $f_\pi$ only occurs in $\Sigma_{\pi,\sigma}^B$ in the pion velocity $u = f_\pi/f_L$. We stress that the scalar dressing function in the denominator of the quark propagator can safely be ignored for our analysis, since $B$ is very small close to $T_c$ and therefore dominated by the large, non-vanishing fermionic Matsubara frequencies in the denominator.

For the first contribution we find $\Sigma_{\pi,\sigma}^B(t) \sim B(t)$, since only $B(\vec{q}, \omega_q)$ in the integral kernel of $\Sigma_{\pi,\sigma}^B$ scales with reduced temperature$^2$. Therefore self-consistency is trivially fulfilled and no constraint for $x$ can be derived from this. On the other hand, the projection of the self-energies $\Sigma_{\pi,\sigma}^B$ yields

$$B(t) \sim \Sigma_{\pi,\sigma}^B(t) \sim \frac{B^3(t)}{f_L^2(t)} + c \frac{B_5(t)}{f_L^2(t)} ,$$

where $c$ is a dimensionful constant with respect to $t$. The numerator is obtained from the fact that the trace over Dirac matrices is only non-vanishing for an even number of matrices. The denominator stems from the zero frequency meson propagator with $\omega_p^2 = 0$. For small $B$, i.e.

2 Here we already use that the Yang-Mills vertex dressing and the gluon dressing function are independent of $t$. The latter assumption can be justified by a scaling analysis of the quark-loop contribution to the gluon propagator along the lines followed here. One consistently obtains scaling with $t$ proportional to $B^2/u^2 \sim \text{const.}$, cf. Eq. (20). For the Yang-Mills part of the vertex one can show that close to $T_c$ diagrammatic contributions containing only dressing functions $A, C$ dominate and consequently this part of the vertex also does not scale.
close to $T_c$, the cubic term is leading. With the scaling of
the transverse pion decay constant given in Eq. (16) we find
\[ t^x \sim t^{3x} \Rightarrow x = \nu/2. \] (20)
Thus our analysis suggests that in case of a second order
phase transition of $N_f = 2$ QCD the quark scalar dress-
ing function should be found to behave like $t^{\nu/2}$ with
$\nu \approx 0.73$. Our result is supported by the fact that this
solution is not only consistent with the quark propaga-
tor Dyson-Schwinger equation but also with the known
relations for the quark condensate (17) and for the trans-
verse pion decay constant (16). To see this we recall that
the quark condensate can be obtained from the full quark
propagator by
\[ \langle \bar{\psi} \psi \rangle = Z_2 N_c T \sum_{n_p} \int \frac{d^3 p}{(2\pi)^3} \sigma_B(\vec{p}, \omega_p) \] (21)
with $\sigma_B(\vec{p}, \omega_p) = B/\omega_p^2 C^2(\vec{p}, \omega_p) + \vec{p}^2 A^2(\vec{p}, \omega_p) + B^2(\vec{p}, \omega_p)$. Therefore
\[ \langle \bar{\psi} \psi \rangle(t) \sim B(t) \sim t^{\nu/2} \] (22)
consistent with (17) in the approximation of vanishing
anomalous dimension. For the pion decay constant we con-
clude by inserting $B \sim t^{\nu/2}$ in Eq. (22) and evaluation
for $P = (\vec{p}, 0) \to 0$:
\[ B \sim t^{\nu/2} \Rightarrow f_s^2 \sim t^\nu. \] (23)
Hence, we obtain a closed system in terms of scaling with
reduced temperature $t$. The above analysis relies on self-consistency of the
quark-DSE. Given a non-trivial scaling behavior of $f_s$, we find scaling of the scalar part of the quark propagator,
which is induced via the pion dispersion relation. We there-
fore identified the mechanism for non-trivial scaling of the
quark. However, the emergence of non-mean-field critical exponents in $f_s$ is not explained. This issue needs to be further clarified in future studies. In the following
we verify the analytical findings of this section also in a
numerical treatment of the quark-DSE.

IV. NUMERICAL SOLUTIONS OF THE DSE

In the previous section we presented a self-consistent
approximation for the quark Dyson-Schwinger equation
including meson back-coupling effects. This constitutes a
two loop expression whose numerical computation is still
non-standard and beyond the scope of this work. Therefore
and in order to avoid technical problems arising in such a computation we simplify the expression further. This is done similar to Refs. 20. We first give the de-
tails of this approximation and then check the analysis of
the previous section by setting a power law for the decay constant $f_s$.

FIG. 3: The approximated Schwinger-Dyson equation for the
quark propagator.

A. Truncation scheme

In order to further simplify the diagram in the lower
line of Fig. 2 note that one of the loops involves two bare
quark-gluon vertices, a dressed gluon propagator and a
full Bethe-Salpeter vertex. If the gluon propagator and
the two quark-gluon vertices would be replaced by a full
Bethe-Salpeter kernel this loop would exactly match a
Bethe-Salpeter equation. From this we deduce that the
leading part of this loop is given by a term proportional
to $Z_2^{\gamma_5} \tau^i$ in the pseudoscalar meson sector and $Z_2 \tau^i$
for a scalar meson. In the vacuum it has been shown in
Ref. 28 that good results for meson phenomenology can
already be obtained if the proportionality factor is set to
one. Here such a simple truncation is not possible, since
it would destroy the scaling properties close to $T_c$ ana-
yzed in the previous section. Instead, we need to replace
the loops by $Z_2^{\gamma_5} \tau^i$ and $Z_2 \tau^i$, each modified with their
respective quark-meson couplings $\Gamma_{\gamma_5, \tau}$. The resulting
approximated quark Dyson-Schwinger equation, displayed
in Fig. 3 has the same scaling properties as its two-loop
counterpart in Fig. 2. Furthermore it is sufficient for the
scaling analysis to consider only the zeroth Matsubara
frequency of the mesons and choose $m_\pi = m_\sigma = 0$. The
corresponding equation is then given by
\( S^{-1}(\vec{p},\omega_p) = Z_2 S_0^{-1}(\vec{p},\omega_p) + g^2 C_F Z_{1F} T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3} \gamma_{\mu} S(\vec{q},\omega_q) \Gamma^\nu_{YM}(\vec{q},\omega_q,\vec{p},\omega_p) D_{\mu\nu}(\vec{k},\omega_k) \)
\[ -T \int \frac{d^3q}{(2\pi)^3} \left( 2 \Gamma^T_{\sigma}(\vec{p},\omega_p) S(\vec{q},\omega_p) \Gamma^\nu_{\sigma}(\vec{q},\omega_p) D_{\sigma}(\vec{k},0) + \Gamma^T_{\sigma}(\vec{p},\omega_p) S(\vec{q},\omega_p) \Gamma^\nu_{\sigma}(\vec{q},\omega_p) D_{\sigma}(\vec{k},0) \right) \] . (24)

As an aside, note that this truncation also fulfills the Mermin-Wagner theorem: Using Eq. (14) for the Bethe-Salpeter amplitude we find that in case of spontaneous symmetry breaking the Goldstone boson propagator \( D_{\sigma}(\vec{k},0) \) would lead to an infrared singular integral in Eq. (24) for \( d \leq 2 \) space dimensions. The singularity would shift \( B(\vec{p},\omega_p) \to -\infty \). Therefore in \( d \leq 2 \) space dimensions only the symmetric phase with \( B(\vec{p},\omega_p) = 0 \) can be realized in agreement with Mermin-Wagner.

To implement Eq. (24) it remains to specify the input for the Yang-Mills part of the quark-gluon interaction and the gluon propagator. Since we are looking at critical scaling related to universality we expect that the details of both are not crucial for our studies. We have checked this by using two different approaches: On the one hand we used a simple rainbow-ladder like phenomenological model, which is detailed in appendix A. On the other hand we followed Ref. [22] and used quenched lattice data for the temperature dependent gluon propagator together with an explicit back-coupling of the quarks onto the gluon propagator as input, see appendix A for more details. Indeed we have found that the results for the critical scaling behavior close to \( T_c \) are similar in both approaches, as expected. In the next section we therefore concentrate on one of the two approximation schemes and present results for the more advanced truncation with temperature dependent gluon only.

**B. Results: scaling of quark dressing and condensate**

In order to study scaling close to the critical temperature we first have to determine \( T_c \). This can be done either from the behavior of the scalar quark dressing function \( B \) or, equivalently, from the chiral condensate. Our results for the former are shown in Fig. 4. We observe a reduction for \( B(\vec{p} = 0,\omega_p = \pi T, T) \) with temperature which results in a phase transition at \( T_c = 162.5 \pm 2.5 \) MeV, where the error reflect the coarseness of our temperature grid. For small temperatures the details of this reduction do depend on the details of the interaction, since the Yang-Mills part of the quark-gluon interaction is leading there. For temperatures close to the critical one, effects from meson back-coupling become more and more important due to the long range nature of the meson dispersion relations. In Fig. 4 we have been using all Matsubara frequencies in the pion propagators. Neglecting all but the zeroth Matsubara frequency (which is sufficient for scaling around \( T_c \)) and considering also the sigma propagator which becomes important close to \( T_c \) we obtain a similar picture with only slightly changed temperature \( T_c = 157 \pm 2.5 \) MeV. We use this value for the following scaling analysis.

As shown analytically in section III we obtain the correct \( O(4) \)-scaling in the quark-DSE provided we feed in the correct scaling properties of the spatial part of the pion decay constant \( f_\pi \). In order to numerically test this scenario we choose the following scaling ansatz for \( f_\pi \)

\[ \tilde{f}_\pi(T) = \frac{f_\pi(T_0)}{(T_c - T_0)^{\nu/2}} \] (25)

where \( T_0 = 150 \) MeV and \( f_\pi(T_0) \) is the result for the decay constant obtained from Eq. (13). The critical temperature is chosen to be \( T_c = 162 \) MeV. With this construction our numerical results smoothly connect with the ones for \( f_\pi(T) \) at \( T \leq 150 \) MeV. The exponent is set to \( \nu = 0.73 \) corresponding to the value of the \( O(4) \) universality class.

In figure 5 the logarithm of the zeroth mode of the scalar dressing function \( B(\vec{p} = 0,\omega_p = \pi T) \) with temperature not too close to the chiral critical temperature \( T_c \).
results for the chiral condensate (blue squares) calculated from Eq. (21). They likewise follow such a scaling behavior as seen by the straight line ∼ ρ^ν/2. The size of the scaling region can be inferred from the deviations from scaling at large \( t \) which set in at roughly \( T_c - T \approx 10 \text{MeV} \).

Finally we investigated, whether the scaling of \( f_s \) can be obtained from its defining equation (13). To this end we abandon the scaling ansatz (25) but calculate \( f_s \) self-consistently from Eq. (13). It turns out, that in this case we do observe scaling around \( T_c \), but the corresponding anomalous dimensions are the mean field ones. We attribute this result to the approximations done in the pion sector, which reduce Eq. (13) to the Pagels-Stokar form which may not be sufficiently accurate close to \( T_c \).

V. CONCLUSIONS AND OUTLOOK

In this paper we investigated the quark propagator close to the second order chiral phase transition for \( N_f = 2 \) massless quarks in the symmetry broken phase. We used the continuum formulation of the theory utilizing the Dyson-Schwinger functional framework. In contrast to other methods we have the ability to work in the chiral and thermodynamic limit within this approach. In order to take into account the relevant degrees of freedom we outlined a truncation scheme for the quark propagator including meson backcoupling effects. In particular we studied the effect of the Goldstone bosons, the pions, and the radial excitation, the sigma meson.

Identifying the quantities of our truncation obeying known scaling laws we analytically derived the scaling law of the scalar dressing function \( B(p) \) with reduced temperature. Our analysis reveals that the behavior of the pion velocity \( u \) respectively the pion decay constant \( f_s \) is crucial for the scaling of the scalar dressing function close to \( T_c \). Furthermore within the employed approximation (neglecting \( \eta \)) the result is consistent with the known scaling relations for the chiral condensate and the decay constant.

In addition we also considered the quark Dyson-Schwinger equation numerically. Here we could verify our analytical findings also numerically: using an appropriate scaling ansatz for the pion decay constant we showed that the scalar quark dressing function and the chiral condensate scale with \( O(4) \)-critical exponents in a temperature region of about 10 MeV below the chiral critical temperature. Using two different truncation schemes for the Yang-Mills part of the interaction we explicitly verified, that chiral critical scaling is independent of the details of the gluon propagator and the Yang-Mills part of the quark-gluon vertex, as expected from universality.

Finally we performed a first attempt to self-consistently close the system by determining the pion decay constant from the Pagel-Stokar formula. This, however, turned out not to be sufficient, since the resulting critical exponents returned to their mean field values. We interpret this as a failure of the Pagel-Stokar approximation close to the chiral phase transition. Going beyond this approximation will be a task for future work.

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Appendix A: Two truncations of the Yang-Mills part of the quark-DSE

In the main part of this work we numerically investigate a combination of purely gluonic effects and meson back-coupling effects onto the quark propagator. Since the Yang-Mills part of the quark-gluon interaction does not take part in the critical \( O(4) \)-scaling close to \( T_c \), it is sufficient for our purpose to approximate it in the simple fashion of a rainbow-ladder like model (35). For the Yang-Mills part of the quark-gluon vertex \( \Gamma_{YM} = \gamma_\mu \Gamma_{YM}(z) \) with the squared gluon momentum

\[
z = k^2 = \omega^2 + \bar{\omega}^2 = (\omega_q - \omega_p)^2 + (q - \bar{p})^2
\]

we use the
parameters is universal and thus independent of the details 

\[ \beta_0 = \frac{11N_c - 2N_f}{3} \]

development of the longitudinal and transverse dressing functions is neglected.

As an alternative we also employ the much more complex interaction developed in \[22\]. There, quenched lattice data for the temperature dependent gluon propagator have been combined with the quark loop polarization diagram to obtain a numerical expression for the unquenched gluon propagator with \( N_f = 2 \) quarks back-coupled. Unfortunately, there is no reliable information on the details of the quark-gluon vertex at finite temperature, so also in this approach one has to resort to a model for the vertex. Here we choose

\[
Z(z) = Z_T(z) = Z_L(z)
\]

\[
Z(z) = \left( \frac{z}{z + z} \right)^{2\kappa} \left( \frac{1}{\alpha\mu} \left[ \frac{\alpha_0}{1 + z/A^2_{QCD}} \right] - \frac{1}{z/A^2_{QCD} - 1} \right) \gamma
\]

where \( q = (q, \omega_q) \) denotes the gluon momentum and \( p = (\bar{p}, \omega_p) \) the quark and antiquark momenta, respectively. Furthermore \( 2\delta = -18N_c/(44N_c - 8N_f) \) is the anomalous dimension of the vertex and the parameters are \( d_1 = 15 GeV^2, d_2 = 0.5 GeV^2, \Delta = 1.4 GeV \) and we renormalize at \( \alpha(\mu) = 0.3 \).

The numerical results in section \[4\] of this work have been obtained using the second, more complicated truncation of the Yang-Mills part of the quark-gluon interaction. However, we wish to emphasize once more that the results using the simpler rainbow-ladder form, \([A1]\) and \([A2]\), are similar. Critical scaling of the chiral order parameters is universal and thus independent of the details

\[
\Gamma_\nu(q, k, p) = -Z^\alpha_\nu \left( \frac{d_1}{d_2 + q^2} + \frac{q^2}{\Lambda^2 + q^2} \left( \beta_0 \alpha(\mu) \ln[q^2/\Lambda^2 + 1] \right) \right), \quad \text{(A3)}
\]

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