Slipping flows and their breaking

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The process of breaking of inviscid incompressible flows along a rigid body with slipping boundary conditions is studied. Such slipping flows are compressible, which is the main reason for the formation of a singularity for the gradient of the velocity component parallel to rigid border. Slipping flows are studied analytically in the framework of two- and three-dimensional inviscid Prandtl equations. Criteria for a gradient catastrophe are found in both cases. For 2D Prandtl equations breaking takes place both for the parallel velocity along the boundary and for the vorticity gradient. For three-dimensional Prandtl flows, breaking, i.e. the formation of a fold in a finite time, occurs for the symmetric part of the velocity gradient tensor, as well as for the antisymmetric part - vorticity. The problem of the formation of velocity gradients for flows between two parallel plates is studied numerically in the framework of two-dimensional Euler equations. It is shown that the maximum velocity gradient grows exponentially with time on a rigid boundary with a simultaneous increase in the vorticity gradient according to a double exponential law. Careful analysis shows that this process is nothing more than the folding, with a power-law relationship between the maximum velocity gradient and its width: \( \max |u_x| \propto \ell^{-2/3} \).

I. INTRODUCTION

Collapse as a process of the singularity formation for smooth initial conditions represents one of the key issues for understanding nature for hydrodynamic turbulence. The Kolmogorov-Obukhov theory [1, 2] of developed hydrodynamic turbulence at large Reynolds numbers, \( \text{Re} \gg 1 \), in inertial interval predicts the divergence of vorticity fluctuations \( \langle \delta \omega \rangle \) with scale \( \ell \) at small \( \ell \) like \( \ell^{-2/3} \), which indicates the connection of Kolmogorov turbulence with collapse.

Different numerical experiments executed in the late 90s seem to indicate the observation of collapse, with a more accurate examination which showed its absence (a discussion of these issues can be found in [3, 4]).

Up to now, the question about existence of blow-up for ideal fluids within the three-dimensional Euler equations remains controversial. It is well-known that in the two-dimensional Euler hydrodynamics for smooth initial conditions, collapse as a process of the singularity formation in a finite time, is forbidden [5–7]. But this, however, does not exclude the singularity appearance in infinite time with exponential growth, as it was evidenced in numerical experiments [8], in which the formation of the vorticity quasi-shocks is accompanied by exponential in time narrowing their widths. In the three-dimensional Euler hydrodynamics numerical experiments also show an exponential increase of the vorticity \( \omega \) in the pancake-type vortex structures for which the exponential narrowing of the pancake thickness \( \ell \) was also observed, but without any tendency to blow-up [9, 11]. The formation of such structures is possible, as shown in [12, 14], for the frozen-in vorticity \( \omega \) within the three-dimensional Euler equations and the vorticity rotor \( \mathbf{B} \) (divorticity) for two-dimensional flows [8]. Due to this property, frozen-in vector fields turn out to be compressible. Moreover, it was found that the formation of such structures can be considered as a folding process when the maximal values of \( \omega_{\text{max}} \) and \( B_{\text{max}} \) are evaluated proportionally to their widths as \( \ell^{-2/3} \) [9, 11, 13] (see also review paper [17] and references therein). The key point for understanding the compressibility property for frozen-in fluid fields is based on the vortex line representation introduced for the first time by Kuznetsov and Ruban [12]. Explanation of compressibility for frozen fields \( \mathbf{B} \) in incompressible fluids follows from a simple observation. The
equation of motion for $B$:

$$\frac{\partial B}{\partial t} = \text{curl} [v \times B], \quad \text{div} \ v = 0,$$

contains the vector product. Therefore the only velocity component $v_\perp$, perpendicular to the field $B$, can change its value. The parallel velocity component obviously has no influence on the motion of $B$. In the general case, however $\text{div} v_\perp \neq 0$. Moreover, due to the frozenness this velocity component represents a velocity of the field $B$. This is a reason of the compressibility of such fields.

It is necessary to mention that all numerical simulations presented in [9–11, 15, 16] were mainly performed for spatially periodical boxes.

However, recent findings for flows of ideal fluids in the presence of boundaries, both analytical and numerical, demonstrate blow-up behavior. For two-dimensional planar flows in the region with non-smooth boundaries Kiselev and Zlatos [18] proved blow-up existence.

In 2015 Luo and Hou [19] in their intriguing numerical experiments for axi-symmetrical Eulerian flows with swirl inside the cylinder of constant radius observed appearance of collapse just on the boundary. It was a challenge why boundaries play so important role in formation of singularities.

In 2019 Elgindi and Jeong [20] proved the existence of finite-energy strong solutions to the axi-symmetric 3D Euler equations outside the cylinder $(1 + |z|^2)^{\frac{1}{2}} \leq x^2 + y^2$ which become singular in finite time. Singularity develops on the cylinder surface at $z = 0, x^2 + y^2 = 1$ corresponding to a corner of this cylinder. The flow geometry, thus, dictates appearance of the singularity point. The latter result correlates with studies of Kiselev and Zlatos [18] for two-dimensional Euler flow inside the region with not-smooth boundaries.

The similar result was obtained recently in [21] by Elgindi and Jeong for finite-time singularity formation for the 2D Boussinesq system. The flow region again was not smooth, contains a corner and singularity develops again at the surface corner. The authors of [21] state that the flow region can be prolonged up to the half-space. However, from our point of view, just the flow geometry in both cases [20] and [21] dictates appearance of the singularity point but in less extent than the initial conditions. For smooth boundary conditions in the case of two-dimensional Euler equations for flows inside a disk Kiselev and Šverák [22] constructed an initial data for which the gradient of vorticity exhibits double exponential growth in time with maximum value on the boundary. Simultaneously the velocity gradient grows on the boundary exponentially in time.

In this paper we show that flat boundary itself introduces some element of compressibility into flow which, from our point of view, can be considered as a reason of the singularity formation on the boundary. We will consider the 2D and 3D inviscid Prandtl equations which describes the dynamics of the boundary layer, and demonstrate that singularity is formed for the velocity gradient on the wall with slipping boundary conditions. This process is nothing more than breaking phenomenon which is well known in gas dynamics since the classical works of famous Riemann. Notice that in 1985 E and Engquist [23] reported some rigorous results about blow-up existence for both inviscid and viscous Prandtl equations for some initial data when the velocity component parallel to a wall vanishes at the whole vertical line. For such initial conditions these authors found sufficient condition for blow-up in the viscous case and exact blow-up solution for the inviscid Prandtl equation depending on this vertical coordinate. It is worth noting that before, in 1980, the blowup appearance in the Prandtl equations was observed in the numerical simulations by Van Dommelen and Shen [24] (see also the review [25] and references therein).

Following to [22], in this paper on the example of flows between two parallel walls we will show numerically and give some analytical arguments that double exponential growth for vorticity gradient and exponential growth for velocity gradient at the boundary, are connected each others. From our point of view, explanation of the results by Lou and Hou [19] as well as by Kiselev and Šverák [22] are connected with slipping flows for smooth boundaries. For such type of flows the normal velocity component vanishes on the boundary and the rest slipping flow along the boundary will be considered as a compressible one. The divergence of the normal velocity, in this
situation, provides incompressibility condition of the flow. In the case of the two-dimensional inviscid Prandtl equation, first time this fact was established by Hong and Hunter \cite{26} for the pressureless conditions. In this paper we show that for the constant pressure gradient it is also possible to find breaking criteria and establish blow-up for the vorticity gradient on the boundary. The latter, as we will show, occurs in the correspondence with double exponential growth of the vorticity gradient for the 2D Euler flows between two parallel plates. We show that the maximum of the velocity gradient $\max |u_x|$ at the wall grows in time exponentially like for a disk \cite{22}. This process can be considered as folding with typical dependence between growing $\max |u_x|$ and its narrowing in time width $\ell$:

$$\max |u_x| \propto \ell^{-2/3}.$$ 

For the 3D Prandtl equations the slipping flow in the pressureless case is defined from the 2D Hopf equation for two velocity components parallel to the wall. The gradient catastrophe here is of the blow-up type for both stress velocity tensor and vorticity.

The plan of the paper is as follows. In the next section we consider 2D ideal flows in the boundary layer and analyze them by means of the so-called Crocco transformation \cite{27} which we slightly modify. It is worth noting that the Crocco transformation can be applied in more general cases, for instance, when pressure depends on coordinate. Note also that the Crocco transformation was widely used by Oleinik \cite{28}.

Based on the Crocco transformation, we show that the 2D inviscid Prandtl equations for incompressible flow with constant pressure can be transformed into the linear relation between velocity component $u$ parallel to the wall and streamfunction $\psi$. This linear relation can be resolved by introducing the generating function $\theta$. For the Prandtl equations $\theta$ is easily found and the general solution can be obtained in an implicit form for the slipping boundary condition. In this case, the singularity of the gradient type develops on the wall due to the compressible character of the slipping flow. The similar character for the singularity formation remains for arbitrary dependence of pressure on the coordinate $x$ along the boundary. The velocity gradient near singular point behaves like $(t_0 - t)^{-1}$ where $t_0$ is the collapse time.

In the third section we present results of numerical simulations for flows within 2D Euler equation between two parallel plates. Careful analysis of the numerical results allows us to say that the growth of the velocity gradient at the boundary can be considered as a folding process.

Section 4 deals with flows in the framework of the 3D inviscid Prandtl equation. In this case, the slipping flow in the pressureless limit is defined from the 2D Hopf equation for two velocity components parallel to the wall. The gradient catastrophe here is of the blow-up type for both stress velocity tensor and vorticity. The last Section is the conclusion.

We also note that the preliminary results of this paper about slipping flows within the inviscid Prandtl equations were presented by Kuznetsov at the International conference "Integrability", September 22 - 24, 2021 \cite{29}.

II. TWO-DIMENSIONAL PRANDTL EQUATIONS

In the inviscid limit the Prandtl equation for 2D flows is written for the velocity component parallel to the blowing plane $y = 0$:

$$u_t + uu_x + vu_y = -P_x, \quad u_x + v_y = 0$$

with the following initial-boundary conditions:

$$u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y),$$

$$\text{and } v|_{y=0} = 0, \quad \lim_{y \to \infty} u(x, y, t) = U(x),$$

where $P_x$ is the pressure gradient.
where \( u \) and \( v \) are \( x \)- and \( y \)-velocity components, respectively. Subscripts here and everywhere below mean the corresponding derivatives.

The Prandtl approximation assumes that the boundary layer has a thickness much smaller than the characterized size along the layer. As a result pressure \( P \) can be considered as a function depending on the longitudinal coordinate \( x \) only.

Introducing stream function \( \psi \),

\[
\psi_y = u, \quad \psi_x = v,
\]

the initial-boundary conditions \( 2 \) are rewritten as

\[
\psi(x, y, 0) = \psi_0(x, y), \psi(x, 0, t) = \text{const}(=0), \psi(x, y, t) \to U(x) y \text{ at } y \to \infty.
\]

Here we assume that the pressure \( P \) being independent on \( y \) and \( t \) satisfies the Bernoulli law at \( y \to \infty \)

\[
\frac{U^2(x)}{2} + P(x) = \text{const}.
\]

The inviscid Prandtl equation \( 1 \) can be used for approximate description of real flows in the boundary layer if one assumes its thickness to be large enough in comparison with a width of the viscous sublayer. In the latter case the influence of viscous sublayer might be modeled by the slip boundary condition, \( v(x, 0) = 0 \). In this paper for the equation \( 1 \) we will use only slip boundary conditions on the wall.

It is worth noting that within the Prandtl approximation for inviscid flows it is possible to introduce the vorticity as

\[
\omega = -\frac{\partial u}{\partial y}
\]

which satisfies the equation of the same form as for the 2D Euler fluids:

\[
\omega_t + u\omega_x + v\omega_y = 0.
\]

Thus, \( \omega \) is the Lagrangian invariant. By this reason, its values will be bounded at all \( t > 0 \). However, for another components of the velocity gradient such restrictions are absent. As we will see below, \( u_x \) as well as \( v_y \) can take arbitrary values, in particular, infinite ones.

Firstly we consider the special case \( P = \text{const} \) when the quantity \( u \) in \( 1 \) becomes a Lagrangian invariant as well as, for instance, the density \( n \) in the Boussinesq system.

Consider the equation for \( n \) advected by the fluid,

\[
n_t + un_x + vn_y = 0
\]

with condition

\[
u_x + v_y = 0.
\]

Suppose that we know solution of equation \( 3 \)

\[
n = n(x, y, t).
\]

Assuming this function is smooth and monotonic at each \( x \) with increase (or decrease) of \( y \). In this case one can find

\[
y = y(x, n, t)
\]
inverse to (9). Thus, at fixed $t$ this function defines a mapping from old $(x, y)$ variables to a new independent Lagrangian variable $n$ and old Eulerian coordinate $x$ so that we arrive at the mixed Lagrangian-Eulerian description which can be considered as a *non-complete* Legendre transformation. In gas dynamics, as well known the classical example of a *complete* Legendre transformation represents the Hodograph transformation when instead of density $n(x, t)$ and velocity $v(x, t)$ the inverse functions $t(n, v)$ and $x(n, v)$ are considered.

Fixing $n$ in (6, 7) yields the $n$-level line. Thus, this transformation represents transition to the *curvilinear* system of coordinates movable with level lines of $n$. In terms of the new Lagrangian variable $n$ and the old ones $x$ and $t$, as we see now, the equation (5) transforms into a linear equation.

To rewrite the equation in new variables one can find how derivatives with respect to variables $(x, y, t)$ and derivatives relative to $(x, n, t)$ are connected with each other:

\[
\frac{\partial f}{\partial t} = \frac{1}{y_n} [f_t y_n - f_n y_t], \quad (8)
\]
\[
\frac{\partial f}{\partial x} = \frac{1}{y_n} [f_x y_n - f_n y_x], \quad (9)
\]
\[
\frac{\partial f}{\partial y} = \frac{f_n}{y_n}. \quad (10)
\]

In these relations derivatives standing in the left hand sides are taken against $(x, y, t)$, and, respectively, in the right hand sides relative to $(x, n, t)$. The expression $y_n$ standing in denominator in these relations is nothing more than the Jacobian of mapping (7). This Jacobian is not fixed, for instance, it is not equal to unity, because we use the mixed Lagrangian-Eulerian description. Our change of variables given by (7) will be correct up to the moment of time when the Jacobian vanishes: $J \rightarrow 0$. Generally, it will happen first time in one separate point. If initial data have some symmetry then the situation may be more degenerated. The Jacobian in such situation can vanish simultaneously in a few points or even at some curve, finite or infinite.

Applying the formulas (8), (10) to the equation (5) gives us the kinematic condition, well-known for the free-surface hydrodynamics:

\[
y_t + u y_x = v. \quad (11)
\]

Expressing the velocity through the streamfunction $\psi$ (3) in terms of $(x, n)$ by means of relations (8-10) gives

\[
u = \frac{1}{y_n} \psi_n, \quad v = -\psi_x + \frac{y_x}{y_n} \psi_n. \quad (12)
\]

Substitution of these formulas into the equation (11) results in the linear relation between $y$ and $\psi$:

\[
y_t = -\psi_x. \quad (13)
\]

Note, that in this equation all derivatives are taken for fixed $n$. This equation can be easily resolved by introducing the generating function $\theta(x, n, t)$:

\[
y = \theta_x, \quad \psi = -\theta_t. \quad (13)
\]

To find the function $\theta(x, n, t)$ one needs to know dynamical behavior of the velocity field.
III. SOLUTION TO THE 2D INVISCID PRANDTL EQUATIONS

First consider the case of zeroth pressure gradient, $P_x = 0$. In this case, to find $\theta$ for the inviscid Prandtl equation \[1\] we need to solve the first equation of the system \[12\] (by changing $n$ to $u$):

$$u = \frac{1}{y_u} \psi_u,$$  \hfill (14)

which after substitution of \[13\] transforms into the linear equation

$$\frac{\partial \theta_u}{\partial t} + u \frac{\partial \theta_u}{\partial x} = 0.$$

This equation evidently has the following solution:

$$\theta_u = F(x - ut, u)$$

where $F$ is an arbitrary smooth function which should be determined from the boundary-initial conditions. Integration with respect to $u$ yields

$$\theta = \int_{f(x,t)}^{u} F(x - zt, z)dz + g(x, t)$$

where $f(x, t)$ and $g(x, t)$ are arbitrary functions of two arguments. Using the definition of the generating function \[13\] we have

$$y = \int_{f(x,t)}^{u} \frac{\partial}{\partial x} F[x - zt, z]dz + g_x(x, t) - F [x - f(x,t)t, f(x,t)] f_x(x, t),$$

$$\psi = - \int_{f(x,t)}^{u} \frac{\partial}{\partial t} F[x - zt, z]dz - g_t(x, t) + F[x - f(x,t)t, f(x,t)] f_t(x, t).$$

Here both functions $f(x, t)$ and $g(x, t)$ should be determined from the boundary conditions. It turns out that the function $f(x, t)$ coincides with $u$ at $y = 0$ and $g(x, t) = 0$ so that

$$y = \int_{f(x,t)}^{u} \frac{\partial}{\partial x} F[x - zt, z]dz$$

$$\psi = - \int_{f(x,t)}^{u} \frac{\partial}{\partial t} F[x - zt, z]dz.$$ 

The formulas \[13\] and \[14\] coincide with the Crocco transformation \[27\] which was applied first time to the 2D Prandtl equation but written here in the different form.

A. The Hopf equation

Because at $y = 0$ (at the wall) the normal velocity component $v$ vanishes, the Prandtl equation \[1\] at $P_x = 0$ becomes nothing more than the Hopf equation (i.e., this is an autonomous equation!),

$$u_t + uu_x = 0.$$  \hfill (15)
This equation is well-known and has been discussed many times in the literature (see, for instance [30]). This is the simplest equation describing breaking, in particular, in a one-dimensional gas flow with zero pressure. We briefly show how the corresponding solution of this type can be obtained.

It is well known that solution to (15) is written in the following implicit form

\[ u = u_0(a), \]

\[ x = a + u_0(a)t \]

or

\[ u = u_0(x - ut). \]  \hspace{1cm} (16)

This means that on the boundary the breaking, i.e. the formation of singularity in a finite time, becomes possible. It happens when the derivative

\[ \frac{\partial u}{\partial x} = \frac{u'_0(a)}{1 + u'_0(a)t} = \frac{1}{t + (u'_0(a))^{-1}} \]

at some point \( x = x_*, t = t_* \), becomes infinite. It is evident that in this case

\[ t_* = \min_a \left[ -1/u'_0(a) \right] > 0. \]  \hspace{1cm} (17)

If, for instance, \( u_0(a) = A \cos a \) with \( a \in [0, \pi] \) the singular point \( a_* = \pi/2 \) and \( t_* = 1/A \). Near the breaking point the derivative \( \frac{\partial u}{\partial x} \) behaves like

\[ \frac{\partial u}{\partial x} \approx -\frac{1}{\tau + \beta(\Delta a)^2} \]  \hspace{1cm} (18)

where

\[ \tau = t_* - t, \ \Delta a = a - a_*, \]

\[ \beta = 1/2 \frac{d^2}{da^2} [1/u'_0(a)] |_{a=a_*}. \]

Thus, asymptotically this dependence demonstrates a self-similar compression, \( \Delta a \propto \tau^{1/2} \). Notice, the denominator in (18), up to the constant multiplier \( C \), coincides with the Jacobian,

\[ J = \frac{\partial x}{\partial a} = C \left( \tau + \beta a^2 \right). \]  \hspace{1cm} (19)

where for simplicity we put \( a_* = 0 \). Integration of this equation gives the cubic dependence relative to \( a \),

\[ x = C \left( \tau a + \beta a^3/3 \right). \]

Respectively, in the physical space we get more rapid compression than in the space of the Lagrangian markers: \( x \propto \tau^{3/2} \).

At distances \( \beta a^2 \gg \tau \), for the Jacobian we have the time-independent asymptotics,

\[ J \sim a^{2/3}. \]
Thus, as $\tau \to 0$ we arrive at the singularity for the derivative $u_x$,
\[ u_x \propto x^{-2/3}. \tag{20} \]

Any time changes of $u_x$ happen at the narrowing region in the $a$-space, $a \propto \tau^{1/2}$, or equivalently at $x \propto \tau^{3/2}$. In the physical space, thus, we have the following self-similar asymptotics,
\[ u_x \simeq \frac{1}{\tau} F(\xi), \quad \xi = \frac{x}{\tau^{3/2}} \tag{21} \]
where function $F(\xi)$ at large $\xi$ behaves proportionally to $\xi^{-2/3}$. Hence the maximum value of $u_x$ and its width $\ell$ are connected each other by the power-type law:
\[ \max u_x \propto \ell^{-2/3}. \]

It should be emphasized that this is a general asymptotic behavior for folding, independently whether the singularity formation happens in finite or infinite time.

It is interesting to note that for the constant pressure gradient, $P_x = \text{const}$, the equation for slipping flow (at $y = 0$)
\[ u_t + uu_x = -P_x \tag{22} \]
can be solved by the same means as (15) (compare with [31]):
\[ u = u_0(a) - P_x t, \]
\[ x = a + u_0(a)t - P_x t^2 / 2. \]

Breaking moment of time is given formally by the same formula (17),
\[ \frac{\partial u}{\partial x} = \frac{u_0'(a)}{1 + u_0'(a)t}, \]
\[ t_* = \min_a \left[ -1 / u_0'(a) \right]. \]

For arbitrary dependence $P(x)$ a solution of (22) is found by means of the method of characteristics. The equations for characteristics
\[ \frac{d}{dt} u = -P_x, \quad \frac{d}{dt} x = u. \]
reduce to the Newton equation for $x(t)$:
\[ \frac{d^2 x}{dt^2} = -P_x. \]

The first integral is the energy,
\[ E(a) = \frac{x^2}{2} + P(x) = \frac{u^2_0(a)}{2} + P(a), \]
that allows one to integrate the equation in quadratures
\[ \int_a^x \frac{dz}{\sqrt{2(E - P(z))}} = t, \]
that defines the mapping $x = x(a,t)$. In this case, the breaking time $t_*$ is found as a minimal root $T(> 0)$ for equation $J(a,T) = 0$,

$$t_* = \min_a T(a),$$

where $J = \partial x/\partial a$ is a Jacobian of the mapping.

The singularity for the velocity gradient on the boundary, which we obtain, appears as a result of collision of two counter-propagating slipping flows. As it was demonstrated numerically first time by Dommelen and Shen [24] and later confirmed by Hong and Hunter [26] this interaction leads to the formation of jets in perpendicular to the boundary direction (see also [25]). As it will be shown in the next section such behavior remains for flows in the 2D Euler equations with slipping boundaries, where, unlike the 2D Prandtl flows, the maximal velocity gradient grows in time exponentially on the boundary and this growth is accompanied also by the generation of jets in perpendicular direction to the boundary.

B. Behavior for the vorticity gradients on the boundary

As we saw previously, the Hopf equation (15) has a blow-up behavior if initially $u'_0(a) < 0$. In this case, at the singular point $a = a^*$ maximum of the parallel velocity gradient behaves like $u_x \sim (t_* - t)^{-1}$.

Now we will calculate how the vorticity $\omega$ behaves at this point. In the Prandtl approximation, as we mentioned already, the vorticity $\omega = -u_y$ satisfies the equation (4).

We are going to show that the vorticity gradient at the maximal point of the velocity gradient at $y = 0$ has the same behavior as the velocity gradient $u_x$. It is important to remind that the vorticity being the Lagrangian invariant cannot change its value along the trajectories, but the gradient of $\omega$ can blow up on the wall.

For simplicity let us restrict by considering the pressureless case. Firstly, we differentiate (4) with respect to $x$ and then in the result put $y = 0$ where $v = 0$ and respectively $v_x = 0$. This yields the following equation

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial}{\partial x} \omega_x = -u_x \omega_x. \quad (23)$$

Secondly, apply to this equation the characteristics method that gives:

$$\frac{dx}{dt} = u(x,t), \quad (24)$$
$$\frac{d\omega_x}{dt} = -u_x \omega_x. \quad (25)$$

It is easy to see that integration of the second equation of this system at the breaking point where in accordance with (18) $u_x$ can be approximated as

$$u_x \sim \frac{1}{t - t_*}$$

gives at this point the same singular behavior for $\omega_x$: 

$$\omega_x \sim \frac{A}{t - t_*},$$

where $A$ is a constant.

At the end of this section we would like to pay attention to the following important fact. Folding development for slipping flows always results in the formation of a jet in the normal direction to
the boundary. As it was mentioned in [26], breaking (as a folding happening in a finite time) for the slipping flows in the 2D Prandtl equation becomes possible because the pressure gradient normal to the boundary can not prevent the formation of jets. As we will see in the next section for the 2D Euler equations the pressure gradient prevents the singularity formation in a finite time and the velocity gradient growth on the boundary occurs more slowly.

IV. FOLDING OF THE SLIPPING FLOWS FOR THE 2D EULER EQUATION

In this section we consider slipping flows within the 2D Euler equation between two parallel plates. We present results of numerical simulations for folding of such flows which develops on the plate boundary. This part of work was motivated by the paper by Kiselev and Sverák [22] where for the 2D Euler equations the authors constructed an initial data for which the gradient of vorticity exhibits double exponential growth in time with maximum value on the boundary. Our numerical results are in the correspondence with this paper. In particular, we have observed that the maximum of the velocity gradient \( \max |u_x| \) at the wall grows in time approximately exponentially like for a disk. This results in the double exponential growth of the vorticity gradient for the 2D Euler flows. We have established also that this process can be considered as folding with typical dependence between growing \( \max |u_x| \) and its narrowing in time width \( \ell \):

\[
\max |u_x| \propto \ell^{-2/3}.
\]

Numerically we solve the 2D Euler equation for the vorticity \( \omega \)

\[
\omega_t + u \omega_x + v \omega_y = 0
\]  

(26)

for flows between two rigid plates \( y = 0 \) and \( y = h \), with slip boundary conditions on the both plates,

\[
v(x, 0) = v(x, h) = 0,
\]

and 2\(\pi\)-periodical conditions relative to \( x \): \([-\pi \leq x \leq \pi]\). The velocity components \( u \) and \( v \) are found through the streamfunction \( \psi \) by means of standard formulas [3] where the vorticity is expressed as

\[
\omega = -\Delta \psi = -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right).
\]  

(27)

For the streamfunction \( \psi \) we use the zero boundary conditions at \( y = 0 \) and \( y = h \). Such choice corresponds to the absence of the flow with a constant velocity along \( x \)-direction.

For integration of equations (26) and (27) we used two approaches. These are the so-called time marching method (pseudo-transient method) [32] to inverse the Laplace operator and the Peaceman-Rachford scheme [33] (see also [34]) for (26). The accuracy for the first method was \( ||\Delta \psi + \omega|| \leq 10^{-7} \). The kinetic energy \( E = \int dxdy(u^2/2 + v^2/2) \) in our simulations was conserved not worth than 10\(^{-6}\).

Below we present results of our simulations for the following initial conditions (IC):

\[
\psi(x, y, 0) = -By(y - h)^2 \sin x;
\]  

(28)

where \( h = 2 \) and \( B = 0.1 \). These IC were chosen so that the folding point appears at \( x = 0 \) on the boundary \( y = 0 \).

At first, the spatiotemporal dependencies of velocity were found numerically and then the temporal evolution of its gradient was determined. Analysis of the distribution of the velocity gradient showed that for the IC [28] almost from the very beginning its maximum is concentrated on the boundary \( y = 0 \) in the vicinity of the point \( x = 0 \) which corresponds to the folding point.
FIG. 1: Dependencies of the slipping ($y = 0$) velocity as a function of $x$ at different moments of time. Black line corresponds to $t = 0$, red line – $t = 1$, blue line – $t = 2$, green line – $t = 5$.

Around this point the parallel velocity $u$ behaves almost like for overturning describing by the Hopf equation [15]. In Fig.1 one can see that at the initial moment the maximal velocity $u$ increases a little but then becomes more or less constant. The latter behavior is familiar to the breaking process in the 2D Prandtl equations for slipping flow within the Hopf equation.

FIG. 2: Dependencies of the slipping ($y = 0$) velocity gradient $u_x$ as a function of $x$ at different moments of time: $t = 0$ (black line), $t = 1$ (red), $t = 2$ (blue), $t = 5$ (green). With time increasing the profile $u_x$ is seen to transform into a cusp.

shows dependencies for the slipping velocity gradient $u_x$ as a function of $x$ at different moments
of time. These distributions transform with time into a cusp with the self-similar dependence \( u_x \) familiar to that for the breaking for the slipping flow in the Prandtl approximation \[21\]. Note that \( u_x \) becomes more and more negative.

![Graph 3](image3.png)

**FIG. 3**: Time dependence of the maximal value of \( |u_x| \) for the slipping \((y = 0)\) flow (logarithmic scale). The black squares correspond to the numerical results, the red line indicates the slope \( \propto e^{\gamma_1 t} \) with \( \gamma_1 = 0.44 \).

![Graph 4](image4.png)

**FIG. 4**: Spatial thickness of \( |u_x| \) for the slipping \((y = 0)\) flow as a function of time. The black squares correspond to the numerical results, the red line indicates the slope \( \propto e^{-\gamma_2 t} \) with \( \gamma_2 = 0.7 \).

Fig.3 and Fig.4 demonstrate that the maximum \( |u_x| \) grows approximately exponentially in time while its thickness \( \ell \) decreases but also exponentially.
FIG. 5: Maximum velocity gradient $|u_x|$ versus thickness $\ell$. Black dotted line corresponds to numerical results, and the red line is the power dependence $\max |u_x| \propto \ell^{-2/3}$.

Such behavior means that as is seen from Fig.5 between $\max |u_x|$ and $\ell$ the power law dependence arises,

$$\max |u_x| \propto \ell^{-\alpha},$$

with exponent $\alpha \approx 2/3$.

This process for the slipping flow can be considered as a folding, it results in the formation of jet in transverse direction to the boundary $y = 0$ (see Fig.5). The physical reason of the jet appearance is connected with collision of two counter-propagating slipping flows. Unlike the Prandtl case this process for the 2D Euler equation occurs more slowly.

Following arguments to Subsection 3.2 exponential growth of $|u_x|$ should promote the vorticity gradient increase during the folding process. Indeed, it is so, the 2D Euler equation for $\omega_x$ coincides in its form with (23):

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial \omega_x}{\partial x} = -u_x \omega_x.$$  \hspace{1cm} (29)

It is worth noting that this equation is $y$-component of the equation for divorticity [8] taken at $y = 0$. As before, equation (29) can be solved by the method of characteristics:

$$\frac{dx}{dt} = u(x,t),$$

$$\frac{d\omega_x}{dt} = -u_x \omega_x.$$

From the second equation of this system we get that

$$\log \omega_x = - \int^{t} u_x dt'.$$
FIG. 6: Streamlines at $t = 5$ in the neighborhood of $x = y = 0$ show the jet forming as a result of folding of the slipping flow.

FIG. 7: Dependence of $\omega_x$ at $x = 0$ (logarithmic scale) versus time.

As already noted, at the folding region $u_x < 0$. If $\max |u_x|$ increases exponentially in time for the slipping flow then $\omega_x$ will have a double exponential growth in time. Our numerical simulations support this conclusion. In the logarithmic scales as it is seen from Fig.6 initially the growth of $\ln \omega_x$ at the folding point $x = 0$ looks like exponential (direct line) but at the later stage one can see positive deviation from this line and respectively the beginning of the double exponential increase of $\omega_x$. Thus, our numerical results correspond to those by Kiselev and Šverák [22] for the Eulerian flow inside a disk.
V. BREAKING IN THE 3D PRANDTL EQUATIONS

In this section we consider the 3D inviscid Prandtl equations for which as it will be seen below the slipping flow introduce more possibilities for finite-time singularity formation in comparison with the 2D equations. The 3D inviscid Prandtl equations represent a simple generalization of the 2D Prandtl equations:

\[ \mathbf{u}_t + (\mathbf{u} \nabla) \mathbf{u} + v \mathbf{u}_z = -\nabla P(r), \quad (\nabla \cdot \mathbf{u}) + v_z = 0. \quad (30) \]

Here \( r = (x, y) \) and \( \mathbf{u} \) are coordinates and velocity components parallel to the wall, respectively, \( \nabla = (\partial_x, \partial_y) \), \( v \) is the normal (\( \parallel \hat{z} \)) velocity component. Boundary conditions to these equations at the wall are slipping ones:

\[ v|_{z=0} = 0, \]

another boundary condition at \( z \to \infty \) connects the asymptotic velocity \( \mathbf{u} \) with the pressure gradient \( \nabla P(r) \) by the stationary 2D Euler equations.

For simplicity, we will consider only the pressureless case when equations (30) are written in the form

\[ \mathbf{u}_t + (\mathbf{u} \nabla) \mathbf{u} + v \mathbf{u}_z = 0, \quad (\nabla \cdot \mathbf{u}) + v_z = 0. \]

These equations admit the method of characteristics. However, we will not consider its application and only notice that the solution given in [26] for the 2D inviscid Prandtl equations can be generalized to the 3D case. From our opinion, the most interesting question is connected with slipping flow. Its dynamics (at \( v|_{z=0} = 0 \)) will be defined by the 2D Hopf equation

\[ \mathbf{u}_t + (\mathbf{u} \nabla) \mathbf{u} = 0 \quad (31) \]

which evidently can also describe breaking of slipping flows. However, the conditions for breaking in this case are different than in the 2D case (see, e.g. [37]).

As it was demonstrated in [35] solution of equation (31) can be constructed more or less easy if one considers the velocity gradient \( U_{ij} = \partial u_i / \partial x_j \). Differentiation of equation (31) with respect to \( r \) leads to the following matrix equation

\[ \frac{dU}{dt} = -U^2 \quad (32) \]

where

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \nabla) \]

is the (material) derivative along the trajectory:

\[ \frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t), \quad \mathbf{r}|_{t=0} = \mathbf{a}. \]

On this trajectory

\[ \frac{d\mathbf{u}}{dt} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{a}) \]

that gives the mapping

\[ \mathbf{r} = \mathbf{a} + \mathbf{u}_0(\mathbf{a}) t. \quad (33) \]
Then the solution of equation (32) has the form
\[ U = U_0(a)(1 + U_0(a)t)^{-1} \]
where \( U_0(a) \) is the initial values of \( U \). Here
\[ \hat{J} = 1 + U_0(a)t \]
is the Jacobi matrix for the mapping (33). Expanding the matrix \( U_0(a) \) through its projectors \( P^{(n)} \) yields
\[ U = \sum_k \lambda_k P^{(k)} \]
Here the projections \( P^{(k)} \), being a matrix function of \( a \), are expressed through the eigenvectors for the direct \( (U_0(a)\psi = \lambda\psi) \) and conjugated \( (\varphi U_0(a) = \varphi\lambda) \) spectral problems for the matrix \( U_0(a) \),
\[ P^{(k)}_{ij} = \psi^{(k)}_i \varphi^{(k)}_j, \]
where the vectors \( \psi^{(k)} \) and \( \varphi^{(m)} \) with different \( k \) and \( m \) are mutually orthogonal:
\[ \psi^{(k)}_i \varphi^{(m)}_j = \delta_{km}. \]
In terms of the vectors \( \psi^{(k)} \) and \( \varphi^{(m)} \) the eigen value \( 1 + \lambda_k t \) of the Jacobi matrix can be written as differential equations,
\[ \varphi^{(k)}_i \frac{\partial x_j}{\partial a_i} \psi^{(k)}_j = 1 + \lambda_k t \]
which define connections between the physical \( x \)-space and the space of Lagrangian markers \( a \).

It is easily to see that the Jacobian of the mapping (33) is expressed in terms of the eigenvalues as follows
\[ J = \det \hat{J} = \prod_k (1 + \lambda_k t). \]
From (34) it is seen that the breaking time is given by the following expression (compare with [30, 36]):
\[ t_0 = \min_{k,a} (-\lambda_k^{-1}) > 0. \]
As it was mentioned in [37], positive value of \( t_0 \) imposes a few restrictions on eigen values \( \lambda_k \). First of all, in the 2D case \( \lambda_{1,2} \) as roots of the corresponding characteristic equation (quadratic relative to \( \lambda \)) should be real. The latter means that the equation discriminant takes real values. Secondly, among \( \lambda_{1,2} \) in the \( a \)-space there should be found at least one eigen value which provides positiveness of \( t_0 \).

At \( t \to t_0 \), as it follows from (34), \( U \) transforms into the degenerate matrix when the only one term survives corresponding to minimum (37) at some \( k \) (which we denote further 1) and \( a = a_0 \),
\[ U \approx \frac{P^{(n)}}{\tau + \beta_{ij} \Delta a_i \Delta a_j}, \]
where \( \tau = t_0 - t \), \( \Delta a = a - a_0 \),
\[
2\beta_{ij} = -\frac{\partial^2 \lambda_1^{-1}}{\partial a_i \partial a_j} \bigg|_{a=a_0},
\]
and \( P^{(n)} \) is a constant projector taken at \( a = a_0 \). Respectively, at \( t = t_0 \), \( k = 1 \) and \( a = a_0 \) the Jacobian \( (39) \) vanishes.

This gives simultaneous singularities for both symmetric and antisymmetric parts of \( U \). The symmetric part of \( U \),
\[
S = \frac{1}{2}(U + U^T)
\]
has a meaning of the stress tensor where \( T \) denotes transpose. The antisymmetric part of \( U \),
\[
\Omega = U - U^T,
\]
is the vorticity tensor. The presence of the vorticity in the slipping boundary means a fluid rotation existence. Evidently, that while approaching a singularity the rotation velocity will increase. On the other hand, as we saw already for both 2D Prandtl and Euler equations, the appearance of the breaking/folding for slipping flows leads to the onset of a jet in the perpendicular direction to the boundary. For 3D Prandtl flows such jet also exists. The interference of rotation and jet gives an updraft with rotation which is an analogue of a tornado in this problem.

As in the 2D case (compare with \( (20) \)) for the 3D Prandtl slipping flow it is possible to find a spatial structure of a singularity as \( \tau \to 0 \). Note first that dependence \( (38) \) shows asymptotically the self similar behavior in the space of Lagrangian markers, \( \Delta a \propto \tau^{1/2} \) in all directions. In the physical \( x \)-space, however, the situation will be very different. A singularity turns out to be very anisotropic. Let \( \lambda_1 \) be an eigenvalue giving the expansion \( (38) \) then another \( \lambda_2 \) will be finite at the singular point. In this case in the equation \( (35) \) near singularity we can consider \( \varphi^{(1)} \) and \( \psi^{(1)} \) to be constant vectors (taken at \( a = a_0 \)). Denote as \( \Delta x_\parallel \) projection \( \Delta x \cdot \psi^{(1)} \) and \( \Delta a_\parallel \) projection \( \Delta a \) corresponding to direction \( \varphi^{(1)} \). Differential equation for \( x_\parallel \), according to \( (35) \), at a small vicinity of the singular point can be written in the form
\[
\frac{\partial \Delta x_\parallel}{\partial \Delta a_\parallel} = \lambda_1^{-1} (\tau + \beta_{ij} \Delta a_i \Delta a_j),
\]
where multiplier \( \lambda_1^{-1} \) is considered as a constant. In this equation the main contribution to the sum \( \beta_{ij} \Delta a_i \Delta a_j \) near singularity comes from \( \beta_\parallel (\Delta a_\parallel)^2 \) where \( \beta_\parallel \) is easily calculated through components of the matrix \( \beta \) so that asymptotically we have equation coinciding up to some constants with \( (19) \) for the 1D Hopf equation:
\[
\frac{\partial \Delta x_\parallel}{\partial \Delta a_\parallel} = \lambda_1^{-1} (\tau + \beta_\parallel (\Delta a_\parallel)^2).
\]
Thus, at \( \tau = 0 \) and \( z = 0 \) the singularity for \( U \) has the same structure as for the 1D Prandtl equation:
\[
U \propto (\Delta x_\parallel)^{-2/3}.
\]
This dependence is valid almost for all angles except "transverse" one:
\[
U \propto (\Delta x_\perp)^{-2}.
\]
which follows from equation \( (35) \) for non-singular \( \lambda_2 \). Such a dependence on \( \Delta x_\perp \) means that \( \Delta x_\perp \propto \Delta a_\perp \); it appears for almost transverse direction where two asymptotics \( (39) \) and \( (40) \) are comparable, i.e. at \( \Delta x_\perp \propto (\Delta x_\parallel)^{1/3} \).
VI. CONCLUSION

In this paper, we have developed a new concept of the formation of big velocity gradients with the blow-up behavior or with the exponential in time increase for the slipping flows in incompressible inviscid fluids. These processes develop as a folding due to compressible character of the slipping flows. Namely, the boundary itself introduces some element of compressibility into flow which, from our point of view, can be considered as a reason of the singularity formation on the boundary. We have demonstrated for the 2D and 3D inviscid Prandtl equations that singularities are formed for the velocity gradient on the wall with slipping boundary conditions. This process is nothing more than breaking phenomenon which is well known in gas dynamics since the classical works of famous Riemann. In particular, we have shown that, besides the velocity gradient blow-up on the boundary for the 2D inviscid Prandtl flow, the vorticity gradient also becomes singular in a finite time also on the boundary. It turns out that the similar behavior takes place for slipping flows within the 2D Euler equation between two parallel plates. In this case, we have numerically found that maximum of the velocity gradient is developed also on the plate with exponential increase in time. Simultaneously, the vorticity gradient has been shown to demonstrate the double exponential growth in time on the boundary. These results are in correspondence with the proof given by Kiselev and Šverák [22] for the Eulerian flow inside a disk. We showed numerically, that this process can be considered as a folding with power dependence between the maximum velocity gradient and its thickness \( \ell \): \( \max |u_x| \propto \ell^{-2/3} \). It is interesting to note that such type of regimes with exponential growth were observed in the 3D Euler equations for the vortical pancake structures and within the 2D Euler equations for quasi-one-dimensional (filamentous) structures in the form of quasi-shocks of vorticity [17]. In the 3D Prandtl equations we have shown that the slipping flows demonstrate appearance of the blow-up behavior for both the velocity stress tensor and the vorticity. From our point of view, the latter problem becomes principally important for understanding of which role boundaries play in the collapse formation within the 3D Eulerian flows.

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[1] A.N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR 30, 299-303 (1941).
[2] A.M.Obukhov, Spectral energy distribution in a turbulent flow, Dokl. Akad. Nauk SSSR, 32, 22-24, (1941).
[3] D. Chae, in Handbook of Differential Equations: Evolutionary Equation, ed. by C.M. Dafermos and M. Pokorný, Elsevier, Amsterdam (2008), Vol. 4, p. 1.
[4] J.D. Gibbon, The three-dimensional Euler equations: Where do we stand, Physica D 237, 1894 (2008).
[5] W. Wolibner, Un theoreme sur l’existence du mouvement plan d’un vide parfait, homogene, incompressible, pendant un temps inniment long, Math. Z. 37, 698-726 (1933).
[6] T. Kato, On classical solutions of two-dimensional non-stationary Euler equation, Arch. Rational. Mech. Anal. 25, 189 (1967).
[7] V.I. Yudovich. Nonstationary flows of ideal incompressible fluid, USSR Computational Mathematics and Mathematical Physics 3, 1407-1456 (1963).
[35] E.A. Kuznetsov, *Towards a sufficient criterion for collapse in 3D Euler equations*, Physica D 184, 266-275 (2003).

[36] S. F. Shandarin, and Ya. B. Zeldovich, *The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium*, Reviews of Modern Physics, 61, 185-220 (1989).

[37] B.G. Konopelchenko, and G. Ortenzi, *Homogeneous Euler equation: blow-ups, gradient catastrophes and singularity of mappings*, Journal of Physics A: Mathematical and Theoretical 55, 035203 (2021).