Hadamard-Type Fractional Integro-Differential Problem: A Note on Some Asymptotic Behavior of Solutions

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Abstract: As a follow-up to the inherent nature of Hadamard-Type Fractional Integro-differential problem, little is known about some asymptotic behaviors of solutions. In this paper, an integro-differential problem involving Hadamard fractional derivatives is investigated. The leading derivative is of an order between one and two whereas the nonlinearities may contain fractional derivatives of an order between zero and one as well as some non-local terms. Under some reasonable conditions, we prove that solutions are asymptotic to logarithmic functions. Our approach is based on a generalized version of Bihari-LaSalle inequality, which we prove. In addition, several manipulations and crucial estimates have been used. An example supporting our findings is provided.

Keywords: asymptotic behavior; fractional differential equation; Hadamard fractional derivative

1. Introduction

Of concern is the following general class of initial value problems modelled by:

\[
\begin{align*}
\left( \, H^{\alpha} I_0^\beta u \right)(t) &= f\left( t, \left( H^{\alpha} D_0^\beta u \right)(t), \int_0^t h\left( t, s, \left( H^{\alpha} D_0^\beta u \right)(s) \right) \, ds \right), \quad t > t_0 \geq 0, \\
\left( \, H^{\alpha - \beta} I_0^{\alpha - \beta} u \right)(t_0^+) &= u_1, \quad \left( H^{\alpha} D_0^\beta u \right)(t_0^+) = u_2, \quad u_1, u_2 \in \mathbb{R},
\end{align*}
\]

(1)

where \( H^{\alpha} I_0^\beta \) and \( H^{\alpha} D_0^\beta \) are the Hadamard fractional derivatives of orders \( \alpha, \alpha_1 \) and \( \alpha_2 \), respectively, \( 0 \leq \alpha_1 < \alpha < 1 \) and \( 0 \leq \alpha_2 < \alpha < 1 \). The operator \( H^{\alpha - \beta} I_0^{\alpha - \beta} \) is the Hadamard fractional integral of order \( \rho \geq 0 \). The definitions of these operators are given in Section 2.

We shall investigate the asymptotic behavior of solutions for Problem (1). Sufficient conditions on the nonlinear source term guaranteeing the convergence of solutions to logarithmic functions, for large values of time, are established. The importance of using analytical techniques to study the asymptotic behavior of solutions for Problem (1) arises from the lack of explicit solutions.

It is known that solutions for many kinds of (integer-order) ordinary differential equations may approach a certain function as time goes to infinity; in particular they may decay to zero, oscillate, or blow up in finite time. Many results in this regard exist in the literature. For example, we refer the reader to the papers [1–6], in which various classes of linear and nonlinear ordinary differential equations have been studied. Generalizing the existing results from integer orders to non-integer fractional orders is of great importance due to their numerous applications; see for instance, [7–10]. Unfortunately, imitating the techniques verbatim is not straightforward. Many difficulties arise when trying to do so. Some of these difficulties are due to the nature of the fractional derivatives themselves as they involve by definition all the past memory of solutions as well as nonregular kernels. In addition, many fundamental properties of integer-order derivatives are not valid for
fractional-order derivatives. The chain rule is an example of such invalid properties. We will go around these difficulties by utilizing some adequate estimations, like desingularization methods, to deal with singular terms and by modifying and/or generalizing some versions of Bihari–LaSalle inequality.

The study of the asymptotic behavior of solutions for fractional differential equations, with Riemann–Liouville or Caputo fractional derivatives, has been investigated by many researchers, see e.g., [11–20]. The authors of [15] considered the fractional differential equation:

\[
(D_{0+}^{\alpha}u)'(t) + f(t, u) = 0, \quad t > 0,
\]

under the condition

\[
|f(t, u)| \leq \phi \left( t, \frac{|u|}{(1 + t)^{\alpha}} \right), \quad t \geq 0, \quad u \in \mathbb{R},
\]

where \( D_{0+}^{\alpha} \) is the Riemann–Liouville fractional derivative of order \( \alpha, 0 < \alpha < 1 \). The function \( f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is continuous and the function \( \phi : [0, \infty) \times [0, \infty) \to [0, \infty) \) is continuous in each argument and nondecreasing in the second one. They proved that the solutions can be represented asymptotically as \( a_1 + a_2 t^\alpha + O(t^{\alpha-1}), a_1, a_2 \in \mathbb{R} \).

In [20], the case when the source function \( f \) depends on the solution and its sub-first-order fractional derivative has been considered, namely,

\[
\begin{cases}
(CD_{0+}^{\alpha}u)(t) = f(t, u(t), CD_{0+}^{\alpha}u(t)), & t > t_0 > 0, \\
u(t_0) = u_0,
\end{cases}
\]

where \( CD_{0+}^{\alpha} \) is the Caputo fractional derivative of order \( \alpha, 0 < \alpha < 1 \). The authors showed that any global solution of Problem (2) is asymptotic to \( ct^\alpha \) for some real number \( c \).

The present authors investigated the boundedness, power-type decay and asymptotic behavior of solutions for the initial value problems:

\[
\begin{cases}
(CD_{0+}^{\alpha}u)(t) = f(t, u(t), \int_0^t h(t, s, u(s))ds), & t > 0, \quad 0 < \alpha \leq 1 \\
u(0) = u_0, \quad u_0 \in \mathbb{R},
\end{cases}
\]

and

\[
\begin{cases}
(D_{0+}^{\alpha+1}u)(t) = f(t, (D_{0+}^{\alpha}u)(t), \int_0^t h(t, s, (D_{0+}^{\alpha}u)(s))ds), & t > 0, \quad 0 \leq \alpha_1, \alpha_2 \leq \alpha < 1 \\
(l_{0+}^{1-\alpha}u)(0^+) = u_0, \quad (D_{0+}^{\alpha}u)(0^+) = u_1, \quad u_0, u_1 \in \mathbb{R}.
\end{cases}
\]

Several different classes of source functions \( f \) such as

\[
|f(t, u, v)| \leq k(t)P(|u|) + l(t)Q(|v|), \quad |f(t, u, v)| \leq k(t)P(t^{\alpha}u)|v|, \\
or |f(t, u, v)| \leq k(t)P(t^{1-\alpha}u)v + l(t)Q(|v|),
\]

and on the kernel \( h \) such as:

\[
|h(t, s, u)| \leq w(s)K(|u|), \quad |h(t, s, u)| \leq w(s)K(s^{\alpha}u), \quad |h(t, s, u)| \leq w(t,s)K(|u|),
\]

or \( |h(t, s, u)| \leq w(s)K(1^{1-\alpha}u) \)

for some functions \( k, l, P, Q, w \) and \( K \) have been treated, see [11,12,21–23].

For fractional differential equations with Hadamard-type fractional derivatives, we found relatively few results in the literature tackling the long-time behavior of solutions of
fractional initial value problems, see, [24–28]. The authors of [28] studied the stability and decay rate of the zero solution of the fractional differential problem:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( H D_{t_0}^\alpha u \right)(t) &= f(u(t)), \quad t > t_0 > 0, 0 < \alpha < 1 \\
\left( H I_{t_0}^{1-\alpha} u \right)(t_0^+) &= u_0, \quad u_0 \in \mathbb{R}.
\end{array} \right.
\end{align*}
\]  

(3)

They considered first the linear case, \( f(u) = cu, \ c \in \mathbb{R} \), and established a criteria for the decay rate of solutions and the Lyapunov stability of the zero equilibrium. For the nonlinear case, they obtained the stability and decay rate of the hyperbolic zero equilibrium. They used a modified Laplace transform to express solutions by Mittag–Leffler functions and then used the asymptotic expansions of these functions to discuss the stability and logarithmic decay of the solutions. In [25], the authors discussed the stability of logarithmic type for the initial value problem:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( H D_{t_0}^\alpha u \right)(t) &= f(t, u(t), \left( H D_{t_0}^{\alpha_1} u \right)(t)), \quad t > t_0 > 0, 0 < \alpha_1 < \alpha < 1, \\
\left( H D_{t_0}^{\alpha-1} u \right)(t_0^+) &= u_0, \quad u_0 \in \mathbb{R}.
\end{array} \right.
\end{align*}
\]  

(4)

Under some sufficient growth conditions of \( f \), it has been shown that the solutions decay to zero as the logarithmic function \( \left( \ln \frac{t}{t_0} \right)^{\alpha-1} \). Recently, the same authors considered Problem (4) in [26] with \( \frac{d}{dt} \left( H D_{t_0}^\alpha u \right)(t) \) on the left hand side and the additional initial condition \( \left( H D_{t_0}^\alpha u \right)(t_0^+) = u_1 \in \mathbb{R} \). They showed that solutions approach a logarithmic function as time goes to infinity.

To the best of our knowledge, the long-time behavior of solutions for the class of fractional integrodifferential equations with Hadamard fractional derivatives (1) has not been investigated so far. In this paper, we prove that solutions of (1) are asymptotic to the logarithmic function \( \left( \ln \frac{t}{t_0} \right)^{\alpha} \) where \( \alpha \) is the order of the involved fractional derivative. Under sufficient growth conditions, we show that there exist a real number \( r \) such that any solution for (1) in the space \( u \in C^{\alpha+1}_{1-\alpha, \ln}[t_0, \infty) \), see (16), has the following property

\[ \lim_{t \to \infty} \frac{u(t)}{\left( \ln \frac{t}{t_0} \right)^{\alpha}} = r. \]

The rest of this paper is organized as follows. In the next section, Section 2, we give some notations from fractional calculus and present some preliminary results. In Section 3, we introduce and prove our main results. Section 4 is devoted to an example that supports our results. A brief conclusion is presented at the end of the study in Section 5.

2. Preliminaries

This section is devoted to briefly introduce some basic definitions, notions, and properties from fractional calculus and fractional differential equations theory which will be used in further considerations.

**Definition 1 ([7]).** We denote by \( C_{\gamma, \ln}[t_0, T], \ 0 \leq \gamma < 1 \), the following weighted space of continuous functions:

\[ C_{\gamma, \ln}[t_0, T] = \left\{ \tilde{\xi} : (t_0, T] \to \mathbb{R} : \left( \ln \frac{t}{t_0} \right)^{\gamma} \tilde{\xi}(t) \in C[t_0, T] \right\}, \]

(5)

with the norm

\[ \| \tilde{\xi} \|_{C_{\gamma, \ln}} = \left\| \left( \ln \frac{t}{t_0} \right)^{\gamma} \tilde{\xi}(t) \right\|_{C[t_0, T]}, \]
where $C[t_0, T] = C_{0,\ln}[t_0, T]$ is the space of continuous functions on $[t_0, T]$.

**Definition 2 ([7])**. Let $\delta = \frac{dt}{dt}$ be the $\delta$-derivative. For $n \in \mathbb{N}$ and $0 \leq \gamma < 1$, the weighted space of continuously $\delta$-differentiable functions up to order $n - 1$ with $n$th $\delta$-derivative in $C_{T,\ln}[t_0, T]$, is denoted by $C^\gamma_{\delta \gamma}[t_0, T]$ and defined by:

$$C^\gamma_{\delta \gamma}[t_0, T] = \left\{ \xi : (t_0, T) \rightarrow \mathbb{R} \mid \delta^i \xi \in C[t_0, T], \ i = 0, 1, 2, \ldots, n - 1, \ \delta^n \xi \in C_{T,\ln}[t_0, T] \right\},$$

with the norm

$$\|\xi\|_{C^\gamma_{\delta \gamma}} = \sum_{i=0}^{n-1} \|\delta^i \xi\|_C + \|\delta^n \xi\|_{C_{T,\ln}}.$$  \hspace{1cm} (6)

In particular, $C^\gamma_{\delta \gamma}[t_0, T] = C_{T,\ln}[t_0, T]$.

A characterization of the space $C^\gamma_{\delta \gamma}[t_0, T]$ is given as follows [7]: The functions $\xi$ in the space $C^\gamma_{\delta \gamma}[t_0, T]$, $n \in \mathbb{N}$ and $0 \leq \gamma < 1$, can be represented as:

$$\xi(t) = \frac{1}{(n - 1)!} \int_{t_0}^t \left( \ln \frac{t}{s} \right)^{n-1} \frac{h(s)}{s} ds + \sum_{i=0}^{n-1} b_i \left( \ln \frac{t}{t_0} \right)^i,$$

where $h \in C_{T,\ln}[t_0, T]$ and $b_i$, $i = 0, 1, 2, \ldots, n - 1$, are arbitrary constants. In fact, $h(t) = (\delta^n \xi)(t)$ and $b_i = \frac{(\delta^n \xi)(t_0)}{n!}$.

**Definition 3 ([7])**. The Hadamard left-sided fractional integral of order $\alpha > 0$ is defined by:

$$(H I^\alpha_{t_0} w)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left( \ln \frac{t}{s} \right)^{\alpha-1} \frac{w(s)}{s} ds, \ t > t_0, \hspace{1cm} (7)$$

provided the right-hand side exists. We define $H I^0_{t_0} w = w$. The function $\Gamma$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$, where $t^{\alpha-1} = e^{(\alpha-1) \ln t}$.

**Definition 4 ([7])**. The Hadamard left-sided fractional derivative of order $\alpha > 0$, is defined by:

$$(H D^\alpha_{t_0} w)(t) = \frac{1}{\Gamma(n - \alpha)} \delta^n \int_{t_0}^t \left( \ln \frac{t}{s} \right)^{n-\alpha-1} \frac{w(s)}{s} ds, \ t > t_0,$$

that is,

$$(H D^\alpha_{t_0} w)(t) = \delta^n \left( H I^{\alpha-n}_{t_0} w \right)(t), \ t > t_0, \hspace{1cm} (8)$$

where $\delta^n = \left( \frac{d}{dt} \right)^n$, $n = -[\alpha]$. In particular, when $\alpha = m \in \mathbb{N}_0$, we have $H D^m_{t_0} w = \delta^m w$.

The next lemma shows that the Hadamard fractional derivative (or integral) of a logarithmic function results in a multiple of the same logarithmic function with the order of the fractional derivative (or integral) subtracted from (or added to) its power.

**Lemma 1 ([7])**. If $\alpha > 0$, $\beta > 0$, then:

$$\left( H I^{\alpha}_{t_0} \left( \ln \frac{s}{t_0} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} \left( \ln \frac{t}{t_0} \right)^{\beta + \alpha - 1}, \ t > t_0,$$

$$\left( H D^{\alpha}_{t_0} \left( \ln \frac{s}{t_0} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} \left( \ln \frac{t}{t_0} \right)^{\beta - \alpha - 1}, \ t > t_0.$$

In particular, when $0 < \alpha < 1$, $\beta = 1$, then:

$$(H^\alpha I^\beta_{t_0} 1)(t) = \frac{1}{\Gamma(1 + \alpha)} \left( \ln \frac{t}{t_0} \right)^\alpha, \quad t > t_0,$$

$$\left(H^\alpha D^\beta_{t_0} 1\right)(t) = \frac{1}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{t_0} \right)^{-\alpha}, \quad t > t_0.$$  

The last property shows that the Hadamard-type derivative of a constant is not zero.

The composite of the Hadamard operators of fractional differentiation and integration with different orders is given next.

**Lemma 2** ([7]). Let $0 < \beta < \alpha$ and $0 \leq \gamma < 1$, then:

$$H^\alpha D^\beta_{t_0} H^\gamma_{t_0} w = H^\beta_{t_0} H^\gamma_{t_0} w$$

at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$ and at every point in $[t_0, T]$ if $w \in C_{\gamma}[t_0, T]$. In particular, when $\alpha > \beta = m \in \mathbb{N}$, we obtain:

$$H^\alpha D^m_{t_0} H^\alpha_{t_0} w = H^\alpha_{t_0} w.$$  

The Hadamard differentiation operator is the left inverse to the associated Hadamard integration operator [7]. That is, $H^\alpha D^\alpha_{t_0} H^\alpha_{t_0} w = w$ at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$. This property is not valid when the Hadamard fractional derivative and the Hadamard fractional integral are inverted as shown in the lemma below.

**Lemma 3** ([7]). Let $\alpha \geq 0$ and $n = -[\alpha]$. If $w \in C_{\gamma,\ln}[t_0, T]$, $0 \leq \gamma < 1$ and $H^\alpha I^{n-\alpha}_{t_0} w \in C^\gamma_{\delta,\gamma}[t_0, T]$, then:

$$\left(H^\alpha I^{n}_{t_0} H^\alpha_{t_0} w\right)(t) = w(t) - \sum_{i=1}^{n} \frac{(\delta^{n-i}(H^\alpha I^{n-\alpha}_{t_0} w)(t_0^+))}{\Gamma(\alpha - i + 1)} \left( \ln \frac{t}{t_0} \right)^{\alpha - i},$$  

(9)  

for all $t \in (t_0, T]$. In particular, for $0 \leq \alpha < 1$, we have:

$$\left(H^\alpha I^{n}_{t_0} H^\alpha_{t_0} w\right)(t) = w(t) - \frac{(H^\alpha I^{1-\alpha}_{t_0} w)(t_0^+)}{\Gamma(\alpha)} \left( \ln \frac{t}{t_0} \right)^{\alpha - 1}$$  

(10)  

at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$ and at every point in $[t_0, T]$ if $w \in C_{\gamma}[t_0, T]$ and $H^\alpha I^{1-\alpha}_{t_0} w \in C^\gamma_{\delta,\gamma}[t_0, T]$.

For more about Hadamard fractional integral and derivative, we refer to the books [7,29,30].

The limit of the ratio of the Hadamard fractional integral $\left(H^{\alpha+1}_{t_0} s f(s, u(s), v(s))\right)(t)$ and the power function $\frac{1}{\Gamma(1+\alpha)} \left( \ln \frac{t}{t_0} \right)^{\alpha}$ as $t \to \infty$ is treated in the lemma below.

**Lemma 4.** Let $f \in L^1(t_0, \infty), t_0 > 0$. Suppose that $u$ and $v$ are real-valued functions defined on $[t_0, \infty)$, then:

$$\lim_{t \to \infty} \frac{\left(H^{\alpha+1}_{t_0} s f(s, u(s), v(s))\right)(t)}{\frac{1}{\Gamma(1+\alpha)} \left( \ln \frac{t}{t_0} \right)^{\alpha}} = \lim_{t \to \infty} \frac{1}{\Gamma(1+\alpha)} \left( \ln \frac{t}{t_0} \right)^{\alpha} \int_{t_0}^{t} \left( \ln \frac{s}{t_0} \right)^{\alpha} f(s, u(s), v(s))ds$$  

$$= \int_{t_0}^{\infty} f(s, u(s), v(s))ds, \quad \alpha > 0.$$
Proof. It is easy to see that:

$$\left| \frac{1}{\Gamma(a+1)} \int_{t_0}^{s} f(s, u(s), v(s)) ds \right|$$

where

$$\left( H^{\alpha+1}_{t_0} s f(s, u(s), v(s))(t) \right) = \int_{t_0}^{\infty} f(s, u(s), v(s)) ds$$

$$= \left| \frac{1}{\ln \frac{t_0}{t}} \int_{t_0}^{t} \left( \ln \frac{s}{t} \right)^{a} f(s, u(s), v(s)) ds \right|$$

$$= \left| \frac{\ln \frac{t_0}{t}}{\ln \frac{s}{t}} \int_{t_0}^{t} f(s, u(s), v(s)) ds \right|$$

$$= \left| \frac{\ln \frac{t_0}{t}}{\ln \frac{s}{t}} \int_{t_0}^{\infty} f(s, u(s), v(s)) ds \right|$$

$$= \int_{t_0}^{\infty} \left[ 1 - \frac{\ln \frac{t_0}{t}}{\ln \frac{s}{t}} \right] \chi_{[t_0,t]}(s) \leq 1 \left[ f(s, u(s), v(s)) \right] ds,$$

As $f \in L^1(t_0, \infty)$ and

$$\lim_{t \to \infty} \left| 1 - \frac{\ln \frac{t}{t_0}}{\ln \frac{s}{t}} \right| \chi_{[t_0,t]}(s) = 1, \text{ for } s < t,$$

we get from the Dominated Convergence Theorem [31],

$$\lim_{t \to \infty} \left| \frac{1}{\Gamma(a+1)} \int_{t_0}^{t} \left( \ln \frac{s}{t} \right)^{a} f(s, u(s), v(s)) ds \right|$$

$$\leq \lim_{t \to \infty} \int_{t_0}^{\infty} \left[ 1 - \frac{\ln \frac{t_0}{t}}{\ln \frac{s}{t}} \right] \chi_{[t_0,t]}(s) \leq 1 \left| f(s, u(s), v(s)) \right| ds$$

$$= \int_{t_0}^{\infty} \lim_{t \to \infty} \left[ 1 - \frac{\ln \frac{t_0}{t}}{\ln \frac{s}{t}} \right] \chi_{[t_0,t]}(s) \leq 1 \left| f(s, u(s), v(s)) \right| ds = 0.$$

This completes the proof. \(\square\)

The next lemma can be considered as a Hadamard fractional version of L'Hôpital’s rule when applied to the solution of problem (1).

**Lemma 5.** Let $u$ be a solution of problem (1) with $f \in L^1(0, \infty)$. Then,

$$\lim_{t \to \infty} \frac{u(t)}{\ln \frac{t}{t_0}} = \lim_{t \to \infty} \frac{H\mathcal{D}^\alpha_{t_0}u(t)}{\Gamma(a+1)}$$

$$= \frac{1}{\Gamma(a+1)} \left( u_2 + \int_{t_0}^{\infty} f(s, (H\mathcal{D}^\alpha_{t_0}u)(s), \int_{t_0}^{s} h(s, \tau, (H\mathcal{D}^\alpha_{t_0}u)(\tau)) d\tau) ds \right).$$

Proof. Integrating both sides of the equation in (1) over the interval $[t_0, t]$, we obtain:

$$(H\mathcal{D}^\alpha_{t_0}u)(t) = u_2 + \int_{t_0}^{t} f(s, (H\mathcal{D}^\alpha_{t_0}u)(s), \int_{t_0}^{s} h(s, \tau, (H\mathcal{D}^\alpha_{t_0}u)(\tau)) d\tau) ds. \quad (11)$$
Apply $\mu T_{0}^{a}$ to (11) and use Lemmas 1 and 3, to find that:

$$
u(t) - \frac{u_{1}\left(\ln \frac{t}{t_{0}}\right)^{a-1}}{\Gamma(a)} = \frac{u_{2}\left(\ln \frac{t}{t_{0}}\right)^{a}}{\Gamma(a+1)} + \frac{1}{\Gamma(a)} \int_{t_{0}}^{t} \left(\ln \frac{t}{s}\right)^{a-1}$$

$$\times \int_{t_{0}}^{s} f\left(\tau, \left(H D_{0}^{\alpha}u\right)(\tau), \int_{t_{0}}^{\tau} h(\tau, \sigma, \left(H D_{0}^{\alpha}u\right)(\sigma)) d\sigma\right) d\tau \frac{ds}{s},$$

(12)

for all $t > t_{0} > 0$. Reorder the double integral on the right-hand side and integrate by substitution to have:

$$
u(t) = \frac{u_{1}\left(\ln \frac{t}{t_{0}}\right)^{a-1}}{\Gamma(a)} + \frac{u_{2}\left(\ln \frac{t}{t_{0}}\right)^{a}}{\Gamma(a+1)} + \frac{1}{\Gamma(a)} \int_{t_{0}}^{t} \left(\ln \frac{t}{\tau}\right)^{a-1}$$

$$\times f\left(\tau, \left(H D_{0}^{\alpha}u\right)(\tau), \int_{t_{0}}^{\tau} h(\tau, \sigma, \left(H D_{0}^{\alpha}u\right)(\sigma)) d\sigma\right) d\tau.$$

(13)

Dividing both sides of (13) by $\left(\ln \frac{t}{t_{0}}\right)^{a}$ gives:

$$\frac{\nu(t)}{\left(\ln \frac{t}{t_{0}}\right)^{a}} = \frac{u_{1}}{\Gamma(a) \ln \frac{t}{t_{0}}} + \frac{u_{2}}{\Gamma(a+1) \left(\ln \frac{t}{t_{0}}\right)^{a}} \int_{t_{0}}^{t} \left(\ln \frac{t}{\tau}\right)^{a-1}$$

$$\times f\left(\tau, \left(H D_{0}^{\alpha}u\right)(\tau), \int_{t_{0}}^{\tau} h(\tau, \sigma, \left(H D_{0}^{\alpha}u\right)(\sigma)) d\sigma\right) d\tau,$$

(14)

for all $t > t_{0} > 0$. Taking the limit of the resulting ratio at infinity and applying Lemma 4 leads to the desired result.

In the next lemma, we recall Bihari–LaSalle inequality which is a nonlinear generalization of the well-known Grönwall–Bellman inequality.

**Lemma 6** ([32,33]). Suppose that $w$ and $g$ are nonnegative continuous functions on $[t_{0}, \infty)$ and $\varphi$ is a positive function on $(0, \infty)$, continuous and nondecreasing on $[0, \infty)$. If

$$w(t) \leq c + \int_{t_{0}}^{t} g(\tau) \varphi(w(\tau)) d\tau, \quad t \in [t_{0}, \infty),$$

where $c > 0$, then

$$w(t) \leq \Phi^{-1}\left(\Phi(c) + \int_{t_{0}}^{t} g(\tau) d\tau\right), \quad t \in [t_{0}, T_{1}],$$

where $\Phi^{-1}$ is the inverse function of $\Phi$,

$$\Phi(x) = \int_{x_{0}}^{x} \frac{ds}{\varphi(s)}, \quad x > 0, \quad x_{0} > 0,$$

and $T_{1}$ is chosen so that $\Phi(c) + \int_{t_{0}}^{t} g(\tau) d\tau$ is in the domain of $\Phi^{-1}$ for all $t \in [t_{0}, T_{1}]$.

The next nonlinear integral inequality can be considered a generalization of Bihari–LaSalle inequality that has been recalled in Lemma 6.
Lemma 7 ([34]). Assume that \( w \) and \( \eta_j, j = 1, \ldots, n \) are nonnegative continuous functions on \([t_0, T]\) and \( \varphi_j, j = 1, \ldots, n \) are nonnegative, continuous and nondecreasing on \([0, \infty)\) such that \( \varphi_1 \preceq \varphi_2 \preceq \cdots \preceq \varphi_n \) (that is, \( \varphi_n/\varphi_{n-1}, \ldots, \varphi_2/\varphi_1 \) are nondecreasing functions). Assume further that \( c \) is a positive constant and

\[
w(t) \leq c + \sum_{j=1}^{n} \int_{t_0}^{t} \eta_j(s) \varphi_j(w(s)) ds, \quad t \in [t_0, T],
\]

then, for all \( t \in [t_0, T] \),

\[
w(t) \leq \Phi_n^{-1} \left( \Phi_n(c_{n-1}) + \int_{t_0}^{t} \eta_n(s) ds \right),
\]

where

1. \( \Phi_j^{-1} \) is the inverse function of \( \Phi_j \) and \( \Phi_j(x) = \int_{x_j}^{x} \frac{ds}{\varphi_j(s)}, x > 0, x_j > 0, \ j = 1, \ldots, n \).
2. The constants \( c_j \) are given by \( c_0 = c \) and \( c_j = \Phi_j^{-1} \left( \Phi_j(c_{j-1}) + \int_{t_0}^{T_1} \eta_j(\tau) d\tau \right) \), \( j = 1, \ldots, n - 1 \).
3. The number \( T_1 \in [t_0, T] \) is the largest number such that

\[
\int_{t_0}^{T_1} \eta_j(\tau) d\tau \leq \int_{c_{j-1}}^{\infty} \frac{d\tau}{\varphi_j(\tau)}, \quad j = 1, \ldots, n.
\]

Lemma 8 ([34]). Suppose that \( w \) and \( \eta_j, j = 1, 2, 3 \) are nonnegative continuous functions on \([t_0, T]\) and \( \varphi_j, j = 1, 2, 3 \) are nonnegative, continuous and nondecreasing on \([0, \infty)\) such that \( \varphi_1 \preceq \varphi_2 \preceq \varphi_3 \), (that is, \( \varphi_3/\varphi_2 \) and \( \varphi_2/\varphi_1 \) are nondecreasing functions). Assume further that \( c \) is a positive constant. If

\[
w(t) \leq c + \int_{t_0}^{t} \eta_1(\tau) \varphi_1(w(\tau)) d\tau + \int_{t_0}^{t} \eta_2(\tau) \varphi_2 \left( \int_{t_0}^{\tau} \eta_3(s) \varphi_3(w(s)) ds \right) d\tau,
\]

then, for all \( t \in [t_0, T_1] \),

\[
w(t) \leq \Phi_3^{-1} \left( \Phi_3(c_2) + \int_{t_0}^{t} \eta_3(\tau) d\tau \right),
\]

where the functions \( \Phi_j^{-1}, \Phi_j, j = 1, 2, 3 \) and the constants \( c_0, c_1, c_2 \) are the same as those given in Lemma 7.

Remark 1. By considering the following functions,

\[
\omega_1(t) := \max_{r \in [0, t]} \varphi_1(r), \quad \omega_j(t) := \max_{r \in [0, t]} \left( \frac{\varphi_j(r)}{\omega_{j-1}(r)} \right)^{\alpha_j-1}(t), \quad j = 2, 3,
\]

we can drop the ordering and monotonicity requirements in Lemma 8. It is clear that \( \varphi_j(t) \leq \omega_j(t) \), \( \omega_j \) are nonnegative and nondecreasing functions on \([0, \infty)\) for all \( t \in [0, \infty) \), \( j = 1, 2, 3 \) and \( \omega_1 \preceq \omega_2 \preceq \omega_3 \).

3. Main Results

In this section, we study the asymptotic behavior of continuable solutions for the problem (1) in the space \( C_{1-a,\ln}^{a+1}[t_0, T] \), \( 0 < t_0 < T \leq \infty \), defined by:

\[
C_{1-a,\ln}^{a+1}[t_0, T] = \left\{ u : (0, T] \to \mathbb{R} \mid u \in C_{1-a,\ln}[t_0, T], \frac{d \left( H^a \Delta_h^n u(t) \right)}{dt} \in C_{1-a,\ln}[t_0, T] \right\},
\]
where the space $C_{1-a,\ln}[t_0, T]$ is defined in (5).

The following two types of functions are used repeatedly in the rest of this paper.

**Definition 5.** A function $g$ is said to be $\Omega_{\kappa}$-type function if it is continuous, nonnegative on $(t_0, \infty)$ and

$$\int_{t_0}^{\infty} \left(\ln \frac{t}{t_0}\right)^{\kappa} g(t)dt < \infty, \quad \kappa = 0 \text{ or } 1.$$  

**Definition 6.** A function $\psi$ is said to be $\Psi$-type function if it is continuous, positive, nondecreasing on $(0, \infty)$,

$$\psi(a) \leq b\psi\left(\frac{a}{b}\right) \text{ for all } a \geq 1, b \geq 1,$$

and

$$\int_{\rho}^{t} \frac{ds}{\psi(s)} \to \infty \text{ as } t \to \infty \text{ for any } \rho > 0.$$  

The two classes of functions introduced above are not empty, see the example provided in Section 4.

Consider the following nonlinear inequality:

$$w(t) \leq a_1 + \left(\ln \frac{t}{t_0}\right) \left(a_2 + a_3 \int_{t_0}^{t} g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{t} g_3(\tau)\varphi_3(w(\tau))d\tau\right)\right)ds,$$  

(17)

for all $t > t_0$, where $a_1, a_2$ and $a_3$ are positive constants, $g_1, g_3$ are $\Omega_1$-type functions, $g_2$ is $\Omega_0$-type function and $\varphi_j, j = 1, 2, 3$ are $\Psi$-type functions with $\varphi_1 \propto \varphi_2 \propto \varphi_3$.

Let

$$K := \Phi_3(b_3) + \int_{t_0}^{t_0e} g_3(s)ds,$$  

(18)

and

$$K_1 := \Phi_3(c_2) + \int_{t_0e}^{\infty} \left(\ln \frac{t}{t_0}\right) g_3(t)dt,$$  

(19)

where

$$b_3 = \Phi_2^{-1}\left(\Phi_2(b_2) + a_3 \int_{t_0}^{t_0e} g_2(s)ds\right), \quad b_2 = \Phi_1^{-1}\left(\Phi_1(a_1 + a_2) + a_3 \int_{t_0}^{t_0e} g_1(s)ds\right),$$  

$$c_2 = \Phi_2^{-1}\left(\Phi_2(c_1) + a_3 \int_{t_0e}^{\infty} g_2(t)dt\right), \quad c_1 = \Phi_1^{-1}\left(\Phi_1(K_2) + a_3 \int_{t_0e}^{\infty} g_1(t)dt\right),$$  

$$K_2 = a_1 + a_2 + a_3 \varphi_1(\Phi_3^{-1}(K)) \int_{t_0}^{t_0e} g_1(s)ds + a_3 \varphi_2\left(\Phi_3^{-1}(K)\right) \int_{t_0e}^{t_0e} g_3(\tau)d\tau \int_{t_0}^{t_0e} g_2(s)ds,$$  

(20)

and $\Phi_j^{-1}$ is the inverse functions of $\Phi_j(x) = \int_{\rho}^{x} \frac{ds}{\psi(s)}, j = 1, 2, 3, x > \rho > 0$.

A generalized version of Lemma 8 is introduced below. We shall provide an estimate for an integral term that arises later in our present problem. Although this estimate is not the best possible, it ensures that such an integral is bounded, which is exactly what we will need to prove our results below.

**Theorem 1.** Suppose that $w(t)$ is a continuous nonnegative function on $(t_0, \infty)$ satisfying Inequality (17) for all $t > t_0$, $a_1, a_2$ and $a_3$ are positive constants, $g_1, g_3$ are $\Omega_1$-type functions, $g_2$ is $\Omega_0$-type function and $\varphi_j, j = 1, 2, 3$, are $\Psi$-type functions with $\varphi_1 \propto \varphi_2 \propto \varphi_3$. Then,

$$w(t) \leq \Phi_3^{-1}(K) \text{ for all } t_0 < t < t_0e,$$  

(21)

and

$$w(t) \leq \Phi_3^{-1}(K_1) \left(\ln \frac{t}{t_0}\right) \text{ for all } t \geq t_0e,$$  

(22)
where \( K \) and \( K_1 \) are given in (18) and (19), respectively.

**Proof.** For \( t_0 < t < t_0e \), the inequality (17) becomes:

\[
\frac{w(t)}{\ln \frac{t}{t_0}} \leq a_1 + a_2 + a_3 \int_{t_0}^t \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^s g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds, t_0 < t < t_0e,
\]

and Lemma 8 is applicable. The relation (21) follows immediately.

For the case \( t \geq t_0e \), we have \( \ln \frac{t}{t_0} \geq 1 \) and (17) may be rewritten as:

\[
\frac{w(t)}{\ln \frac{t}{t_0}} \leq a_1 + a_2 + a_3 \int_{t_0}^t \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^s g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds
\leq a_1 + a_2 + a_3 \int_{t_0}^{t_0e} \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^s g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds
+ a_3 \int_{t_0e}^t \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^s g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds.
\]

In virtue of the estimate (21) together with the continuity and monotonicity of the functions \( \varphi_i, i = 1, 2, 3 \), we get:

\[
\frac{w(t)}{\ln \frac{t}{t_0}} \leq K_2 + a_3 \int_{t_0e}^t \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^s g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds,
\]

where \( K_2 \) is the constant given in (20).

Let

\[
z := z_1 + z_2 + z_3,
\]

where

\[
z_1(t) := K_2 + a_3 \int_{t_0e}^t g_1(s) \varphi_1(w(s)) ds, \quad t \geq t_0e,
\]

\[
z_2(t) := a_3 \int_{t_0e}^t g_2(s) \varphi_2(z_3(s)) ds, \quad t \geq t_0e,
\]

\[
z_3(t) := a_3 \int_{t_0}^t g_3(s) \varphi_3(w(s)) ds, \quad t > t_0.
\]

It is clear that

\[
\frac{w(t)}{\ln \frac{t}{t_0}} < z(t) \quad \text{for all} \; t \geq t_0e.
\]

In light of the types of the functions \( g_i \) and \( \varphi_i, i = 1, 2, 3 \), see Definitions 5 and 6, differentiating \( z \) yields:

\[
z'(t) = a_3 g_1(t) \varphi_1(w(t)) + a_3 g_2(t) \varphi_2(z_3(t)) + g_3(t) \varphi_3(w(t))
\leq a_3 g_1(t) \left( \ln \frac{t}{t_0} \right) \varphi_1 \left( \frac{w(t)}{\ln \frac{t}{t_0}} \right) + a_3 g_2(t) \varphi_2(z(t)) + g_3(t) \left( \ln \frac{t}{t_0} \right) \varphi_3 \left( \frac{w(t)}{\ln \frac{t}{t_0}} \right)
\leq a_3 \left( \ln \frac{t}{t_0} \right) g_1(t) \varphi_1(z(t)) + a_3 g_2(t) \varphi_2(z(t)) + \left( \ln \frac{t}{t_0} \right) g_3(t) \varphi_3(z(t)), \quad t \geq t_0e.
\]

Now, we integrate both sides of (29) over the interval \([t_0e, t]\) to find:

\[
z(t) \leq z(t_0e) + a_3 \int_{t_0e}^t \left( \ln \frac{s}{t_0} \right) g_1(s) \varphi_1(z(s)) ds + a_3 \int_{t_0e}^t g_2(s) \varphi_2(z(s)) ds
+ \int_{t_0e}^t \left( \ln \frac{s}{t_0} \right) g_3(s) \varphi_3(z(s)) ds.
\]
Applying Lemma 7 with \( \eta_1(t) = a_3 \left( \ln \frac{t}{t_0} \right) g_1(t), \eta_2(t) = a_3 g_2(t), \eta_3(t) = \left( \ln \frac{t}{t_0} \right) g_3(t) \) and \( T_1 = \infty \) (by assumption \( \int_{t_0}^{\infty} \frac{d\rho}{\Phi(\rho(t))} = \infty, i = 1, 2, 3 \) for any \( \rho > 0 \)), the inequality (30) leads to:

\[
z(t) \leq \Phi_{3}^{-1}(K_1), \text{ for all } t \geq t_0e,
\]
as desired. \( \square \)

From now on, we assume that the functions \( f \) and \( h \) on the right hand-side of (1), satisfy the conditions below:

(A1) \( f(t, u, v) : (t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R} \) is a \( C_{1-a, \ln} [t_0, \infty) \) function in \( E_1 = \{(t, u, v) : t > t_0 > 0, u, v \in C_{1-a, \ln} [t_0, \infty) \} \).

(A2) \( h(t, s, u) \) is a continuous function in \( E_2 = \{(t, s, u) : t_0 \leq t < s < \infty, u \in C_{1-a, \ln} [t_0, \infty) \} \).

(A3) There are \( \Omega_1 \)-type functions \( g_1, g_3 \), an \( \Omega_0 \)-type function \( g_2 \) and \( \Psi \)-type functions \( \phi_i, i = 1, 2, 3 \) with \( \phi_1 < \phi_2 < \phi_3 \) such that:

\[
|f(t, u, v)| \leq g_1(t) \phi_1 \left( \left( \ln \frac{t}{t_0} \right)^{1-a+a_1} |u| \right) + g_2(t) \phi_2 (|v|), (t, u, v) \in E_1,
\]
\[
|h(t, s, u)| \leq g_3(s) \phi_3 \left( \left( \ln \frac{s}{t_0} \right)^{1-a+a_2} |u| \right), (t, s, u) \in E_2,
\]

\( 0 \leq a_1 < a < 1 \) and \( 0 \leq a_2 < a < 1 \).

Functions satisfying the above hypotheses are given in Section 4.

In the next result, we shall show that the solution for Problem (1) satisfies the useful nonlinear inequality below.

Lemma 9. Suppose that \( u(t) \) is a \( C_{1-a, \ln} [t_0, \infty) \)-solution for Problem (1) and the functions \( f \) and \( h \) satisfy (A1), (A2) and (A3). Then,

\[
\left( \ln \frac{t}{t_0} \right)^{1-a} |u(t)| \leq v(t), t > t_0,
\]  \( \text{(31)} \)

and

\[
\left( \ln \frac{t}{t_0} \right)^{1-a+a_j} \left| \left( H D^j_{t_0} u \right) (t) \right| \leq v_j(t), t > t_0,
\]  \( \text{(32)} \)

where

\[
v(t) = \frac{|u_1|}{\Gamma(a)} + \frac{\ln \frac{t}{t_0}}{\Gamma(a + 1)} \left\{ |u_2| + \int_{t_0}^{t} \left[ g_1(s) \phi_1 \left( \left( \ln \frac{s}{t_0} \right)^{1-a+a_1} \left| \left( H D^1_{t_0} u \right) (s) \right| \right) \right] ds + g_2(s) \phi_2 \left( \int_{t_0}^{s} g_3(\tau) \phi_3 \left( \left( \ln \frac{\tau}{t_0} \right)^{1-a+a_2} \left| \left( H D^2_{t_0} u \right) (\tau) \right| \right) d\tau \right] ds \right\},
\]  \( \text{(33)} \)

and

\[
v_j(t) = \frac{|u_1|}{\Gamma(a - j)} + \frac{\ln \frac{t}{t_0}}{\Gamma(a - j + 1)} \left\{ |u_2| + \int_{t_0}^{t} \left[ g_1(s) \phi_1 \left( \left( \ln \frac{s}{t_0} \right)^{1-a+a_1} \left| \left( H D^1_{t_0} u \right) (s) \right| \right) \right] ds + g_2(s) \phi_2 \left( \int_{t_0}^{s} g_3(\tau) \phi_3 \left( \left( \ln \frac{\tau}{t_0} \right)^{1-a+a_2} \left| \left( H D^2_{t_0} u \right) (\tau) \right| \right) d\tau \right] ds \right\},
\]  \( \text{(34)} \)

for all \( t > t_0 \) and \( j = 1, 2 \).
Proof. Firstly, we recall the relations (12) and (13) from the proof of Lemma 5,

\[
\begin{align*}
    u(t) &= \frac{u_1\left(\ln \frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{u_2\left(\ln \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{s}{t}\right)^{\alpha-1} \\
    &\quad \times \int_{t_0}^{s} f \left(\tau, (H^{D_{t_0}^{\alpha_1}}u)(\tau), \int_{t_0}^{\tau} h(\tau, \sigma, (H^{D_{t_0}^{\alpha_2}}u)(\sigma))d\sigma\right) d\tau \frac{ds}{s}, \quad t > t_0,
\end{align*}
\]

Equation (36) leads to the following bound on \(|u(t)||

\[
|u(t)| \leq \frac{|u_1|\left(\ln \frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|u_2|\left(\ln \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\ln \frac{t}{t_0}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{s}{t}\right)^{\alpha-1} \\
\quad \times \int_{t_0}^{s} f \left(\tau, (H^{D_{t_0}^{\alpha_1}}u)(\tau), \int_{t_0}^{\tau} h(\tau, \sigma, (H^{D_{t_0}^{\alpha_2}}u)(\sigma))d\sigma\right) d\tau, \quad t > t_0,
\]

or

\[
\left(\ln \frac{t}{t_0}\right)^{-\alpha} |u(t)| \leq \frac{|u_1|\left(\ln \frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|u_2|\left(\ln \frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\ln \frac{t}{t_0}}{\Gamma(\alpha+1)} \\
\quad \times \int_{t_0}^{s} f \left(\tau, (H^{D_{t_0}^{\alpha_1}}u)(\tau), \int_{t_0}^{\tau} h(\tau, \sigma, (H^{D_{t_0}^{\alpha_2}}u)(\sigma))d\sigma\right) d\tau, \quad t > t_0,
\]

for all \(t > t_0\), which, in the light of the hypothesis (A3), leads to (31) as desired.

Now, apply the Hadamard differentiation operator \(H^{D_{t_0}^{\alpha_j}}\), \(j = 1, 2\) to (35), then employ Lemmas 1 and 2 (with \(\beta = \alpha_j\) and \(\gamma = 1 - \alpha\), see (16)) to get:

\[
\begin{align*}
    (H^{D_{t_0}^{\alpha_j}}u)(t) &= \frac{u_1\left(\ln \frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha - \alpha_j)} + \frac{u_2\left(\ln \frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha - \alpha_j + 1)} + \frac{1}{\Gamma(\alpha - \alpha_j)} \int_{t_0}^{t} \left(\ln \frac{s}{t}\right)^{\alpha-\alpha_j-1} \\
    &\quad \times \int_{t_0}^{s} f \left(\tau, (H^{D_{t_0}^{\alpha_1}}u)(\tau), \int_{t_0}^{\tau} h(\tau, \sigma, (H^{D_{t_0}^{\alpha_2}}u)(\sigma))d\sigma\right) d\tau \frac{ds}{s}, \quad t > t_0.
\end{align*}
\]

In a similar way to that used to obtain (36), the relation (39) reduces to:

\[
\begin{align*}
    (H^{D_{t_0}^{\alpha_j}}u)(t) &= \frac{u_1\left(\ln \frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha - \alpha_j)} + \frac{u_2\left(\ln \frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha - \alpha_j + 1)} + \frac{1}{\Gamma(\alpha - \alpha_j + 1)} \int_{t_0}^{t} \left(\ln \frac{s}{t}\right)^{\alpha-\alpha_j} \\
    &\quad \times f \left(\tau, (H^{D_{t_0}^{\alpha_1}}u)(\tau), \int_{t_0}^{\tau} h(\tau, \sigma, (H^{D_{t_0}^{\alpha_2}}u)(\sigma))d\sigma\right) d\tau, \quad t > t_0,
\end{align*}
\]

and consequently,
\[
\left| \left( \mathcal{H}^\alpha_{b_0} u \right)(t) \right| \leq |u_1| \left( \frac{\ln \left( \frac{t}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j)} \right)^{a_{a_j}} + \left| u_2 \right| \left( \frac{\ln \left( \frac{t}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j + 1)} \right)^{a_{a_j}} \int_{b_0}^t \left( \frac{\ln \left( \frac{\tau}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j + 1)} \right)^{a_{a_j}} d\tau
\]
\[
\times \left| f \left( \tau, \left( \mathcal{H}^\alpha_{b_0} u \right)(\tau), \int_{b_0}^\tau h(\tau, \sigma, \left( \mathcal{H}^\alpha_{b_0} u \right)(\sigma)) \sigma \right) d\tau \right| \leq \left| u_1 \right| \left( \frac{\ln \left( \frac{t}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j)} \right)^{a_{a_j}} + \left( \frac{\ln \left( \frac{t}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j + 1)} \right)^{a_{a_j}} \left| u_2 \right| + \left| 1 \right| \left( \frac{\ln \left( \frac{t}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j + 1)} \right)^{a_{a_j}} \int_{b_0}^t \left( \frac{\ln \left( \frac{\tau}{b_0} \right)^{a_{a_j}}}{\Gamma(a - a_j + 1)} \right)^{a_{a_j}} d\tau , t > b_0 ,
\]
which yields, in light of the hypothesis (A3), the desired result in (32) and completes the proof. 

The main result of this section is given below.

**Theorem 2.** Suppose that the hypotheses (A1), (A2) and (A3) are satisfied, then there exists a real number \( r \) such that any solution \( u \in C_{a+1}^a [b_0, \infty) \) of Problem (1), has the following asymptotic property \( \lim_{t \to \infty} \frac{u(t)}{\left( \ln \frac{t}{b_0} \right)^r} = r \).

**Proof.** Let us start by recalling the relations (31) and (32) in Lemma 9,

\[
\left( \frac{\ln \left( \frac{t}{b_0} \right)^{1-a}}{\Gamma(a) \Gamma(a)} \right) \left| u(t) \right| \leq v(t), \ t > b_0 ,
\]

and

\[
\left( \frac{\ln \left( \frac{t}{b_0} \right)^{1-a+j}}{\Gamma(a+1) \Gamma(a-a_j)} \right) \left| \left( \mathcal{H}^\alpha_{b_0} u \right)(t) \right| \leq v_j(t), \ t > b_0 , \ j = 1, 2,
\]

where \( u \) is a \( C_{a+1}^a [b_0, \infty) \)-solution for Problem (1), \( v(t) \) and \( v_j(t), \ j = 1, 2 \) are given in (33) and (34), respectively.

Let

\[
a_1 = \left| u_1 \right| \max_{j=1,2} \left\{ \frac{1}{\Gamma(a)}, \frac{1}{\Gamma(a-a_j)} \right\}, \ a_2 = \left| u_2 \right| \max_{j=1,2} \left\{ \frac{1}{\Gamma(a+1)}, \frac{1}{\Gamma(a-a_j+1)} \right\}, \ a_3 = \left| u_3 \right| \max_{j=1,2} \left\{ \frac{1}{\Gamma(a+1)}, \frac{1}{\Gamma(a-a_j+1)} \right\},
\]

and

\[
w(t) = a_1 \left( \ln \left( \frac{t}{b_0} \right)^{1-a} \right) \left\{ a_2 + a_3 \int_{b_0}^t g_1(s) \varphi_1 \left( \left( \frac{\ln \left( \frac{s}{b_0} \right)^{1-a+j}}{\Gamma(a+1)} \right)^{a_{a_j}} \right) \right\} + s_2(s) \varphi_2 \left( \int_{b_0}^s g_3(s) \varphi_3 \left( \left( \frac{\ln \left( \frac{\tau}{b_0} \right)^{1-a+j}}{\Gamma(a+1)} \right)^{a_{a_j}} \right) d\tau \right) ds, t > b_0 ,
\]

then,

\[
v(t) \leq w(t) \text{ and } v_j(t) \leq w(t) \text{ for all } t > b_0 , \ j = 1, 2.
\]

From the nondecreasingness property of the functions \( \varphi_i, i = 1, 2, 3 \), we have:
\[
\begin{align*}
w(t) &= a_1 + \left( \ln \frac{t}{t_0} \right) \left\{ a_2 + a_3 \int_{t_0}^{t} \left[ g_1(s) \varphi_1(v_1(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^{t} g_3(\tau) \varphi_3(v_2(\tau)) d\tau \right) \right] ds \right\}, \\
&\leq a_1 + \left( \ln \frac{t}{t_0} \right) \left\{ a_2 + a_3 \int_{t_0}^{t} \left[ g_1(s) \varphi_1(w(s)) + g_2(s) \varphi_2 \left( \int_{t_0}^{t} g_3(\tau) \varphi_3(w(\tau)) d\tau \right) \right] ds \right\},
\end{align*}
\]

for all \( t > t_0 \), which is Inequality (17). Therefore, we get from Theorem 1 that:

\[
w(t) \leq \left( \ln \frac{t}{t_0} \right) \Phi_3^{-1}(K_1) \text{ for all } t \geq t_0e,
\]

and as a consequence of the estimates (42) and (46), we find:

\[
\left( \ln \frac{t}{t_0} \right)^{1-a} |u(t)| \leq \left( \ln \frac{t}{t_0} \right) \Phi_3^{-1}(K_1) \text{ for all } t > t_0e,
\]

or

\[
\frac{|u(t)|}{\left( \ln \frac{t}{t_0} \right)^a} \leq K_3 := \Phi_3^{-1}(K_1) \text{ for all } t > t_0e,
\]

where \( K_3 \) is as in (19).

Now, in the light of Hypothesis (A3), and the inequalities (43) and (46), we deduce that:

\[
\begin{align*}
\mathcal{J} : &= \left| \int_{t_0}^{t} f \left( s, \left( H D_{t_0}^{a_1} u \right)(s), \int_{t_0}^{s} h(s, \tau, \left( H D_{t_0}^{a_2} u \right)(\tau)) d\tau \right) ds \right| \\
&\leq \int_{t_0}^{t} \left| f \left( s, \left( H D_{t_0}^{a_1} u \right)(s), \int_{t_0}^{s} h(s, \tau, \left( H D_{t_0}^{a_2} u \right)(\tau)) d\tau \right) \right| ds \\
&\leq \int_{t_0}^{t} g_1(s) \varphi_1 \left( \left( \ln \frac{s}{t_0} \right)^{1-a+a_1} \left| \left( H D_{t_0}^{a_1} u \right)(s) \right| \right) ds + \int_{t_0}^{t} g_2(s) \\
&\times \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3 \left( \left( \ln \frac{\tau}{t_0} \right)^{1-a+a_2} \left| \left( H D_{t_0}^{a_2} u \right)(\tau) \right| \right) d\tau \right) ds \\
&\leq \int_{t_0}^{t_{0e}} g_1(s) \varphi_1(w(s)) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3(w(\tau)) d\tau \right) ds \\
&+ \int_{t_{0e}}^{t} g_1(s) \varphi_1(w(s)) ds + \int_{t_{0e}}^{t} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3(w(\tau)) d\tau \right) ds \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{align*}
\]

The first integral,

\[
\begin{align*}
\mathcal{J}_1 : &= \int_{t_0}^{t_{0e}} g_1(s) \varphi_1(w(s)) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3(w(\tau)) d\tau \right) ds \\
&\leq \int_{t_0}^{t_{0e}} g_1(s) \varphi_1 \left( \Phi_3^{-1}(K) \right) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3 \left( \Phi_3^{-1}(K) \right) d\tau \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds \\
&\leq \varphi_1(K_4) \int_{t_0}^{t_{0e}} g_1(s) ds + \int_{t_0}^{t_{0e}} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \right) ds
\end{align*}
\]

is finite by (21) from Theorem 1, \( K_4 := \Phi_3^{-1}(K) \) and \( K \) is given in (18).

The second integral,

\[
\begin{align*}
\mathcal{J}_2 : &= \int_{t_{0e}}^{t} g_1(s) \varphi_1(w(s)) ds + \int_{t_{0e}}^{t} g_2(s) \varphi_2 \left( \int_{t_0}^{s} g_3(\tau) \varphi_3(w(\tau)) d\tau \right) ds, t \geq t_0e,
\end{align*}
\]
can be estimated in view of (47), Theorem 1 and the type of the functions \( \varphi_i, i = 1, 2, 3 \) as follows:

\[
\mathcal{J}_2 \leq \int_{t_0e}^{t} \left( \ln \frac{s}{t_0} \right) g_1(s) \varphi_1(K_3) ds + \int_{t_0e}^{t} g_2(s) \varphi_2 \left( \int_{t_0}^{t_0e} g_3(\tau) \varphi_3(K_4) d\tau \right) ds \\
+ \int_{t_0e}^{t} \left( \ln \frac{s}{t_0} \right) g_3(\tau) \varphi_3(K_3) d\tau \leq \varphi_1(K_3) \int_{t_0e}^{t} \left( \ln \frac{s}{t_0} \right) g_1(s) ds + \int_{t_0e}^{t} g_2(s) \varphi_2 \left( \varphi_3(K_4) \int_{t_0}^{t_0e} g_3(\tau) d\tau \right) ds \\
+ \varphi_3(K_3) \int_{t_0e}^{t} \left( \ln \frac{s}{t_0} \right) g_3(\tau) d\tau ds, \text{ for all } t \geq t_0e.
\]

(53)

Since \( g_1 \) and \( g_3 \) are \( \Omega_1 \)-type functions and \( g_2 \) is an \( \Omega_0 \)-type function, we see that the integral \( \mathcal{J}_2 \) is uniformly bounded and so is the integral \( \mathcal{J} \).

Therefore, the integral \( \int_{t_0}^{t} f \left( s, \left( H D_{t_0}^{q_1} u \right)(s), \int_{t_0}^{s} h(s, \tau, \left( H D_{t_0}^{q_2} u \right)(\tau)) d\tau \right) ds \) absolutely convergent and, consequently,

\[
\lim_{t \to \infty} \int_{t_0}^{t} f \left( s, \left( H D_{t_0}^{q_1} u \right)(s), \int_{t_0}^{s} h(s, \tau, \left( H D_{t_0}^{q_2} u \right)(\tau)) d\tau \right) ds < \infty,
\]

(54)

Using Lemma 5, there exits a finite real number \( r \) such that:

\[
\lim_{t \to \infty} \frac{u(t)}{\left( \ln \frac{t}{t_0} \right)^{\alpha}} = \frac{1}{\Gamma(\alpha + 1)} \left( u_2 + \lim_{t \to \infty} \int_{t_0}^{t} f \left( s, \left( H D_{t_0}^{q_1} u \right)(s), \int_{t_0}^{s} h(s, \tau, \left( H D_{t_0}^{q_2} u \right)(\tau)) d\tau \right) ds \right) = r,
\]

(55)

as wanted. \( \square \)

**Remark 2.** If the condition \((A_3)\) is replaced by the condition:

\[ (A_4) \text{ There are } \Omega_1 \text{-type functions } \xi_1, \xi_2 \text{ and } \Psi \text{-type functions } \phi_i, i = 1, 2, 3 \text{ with } \phi_1 \phi_2 \propto \phi_3 \text{ such that } \]

\[
|f(t, u, v)| \leq \xi_1(t) \phi_1 \left( \left( \ln \frac{t}{t_0} \right)^{1-\alpha+\alpha_1} |u| \right) \phi_2(|v|), \ (t, u, v) \in E_1, \ 0 \leq \alpha_1 < \alpha < 1,
\]

and

\[
|h(t, s, u)| \leq \xi_2(s) \phi_3 \left( \left( \ln \frac{s}{t_0} \right)^{1-\alpha+\alpha_2} |u| \right), \ (t, s, u) \in E_2, \ 0 \leq \alpha_2 < \alpha < 1,
\]

then the conclusion of Theorem 2 is still valid. We state this fact below.

**Theorem 3.** Suppose that the functions \( f \) and \( h \) satisfy the conditions \((A_1), (A_2)\) and \((A_4)\). Then, there exists a real number \( r \) such that any solution \( u \in C_{1-\alpha, \ln |t_0, \infty|}^{\alpha+1} \) for Problem (1) satisfies

\[
\lim_{t \to \infty} \frac{u(t)}{\left( \ln \frac{t}{t_0} \right)^{\alpha}} = r.
\]

The proof is skipped as it can be shown in a similar manner to that used in the proof of Theorem 2.
Remark 3. The problem,

\[
\begin{cases}
(\hat{H}D_{t_0}^{\alpha}u)'(t) = f(t, u(t), \int_{t_0}^{t} h(t, s, u(s))ds), \ t > t_0 > 0, \ 0 < \alpha < 1, \\
(\hat{H}I_{t_0}^{1-\alpha}u)(t_0^+) = u_1, \ (\hat{H}D_{t_0}^{\alpha}u)(t_0^+) = u_2, \ u_1, u_2 \in \mathbb{R}.
\end{cases}
\]  

(56)

is a special case of Problem (1) when \(\alpha_1 = \alpha_2 = 0\). Therefore, we can conclude that there exists a constant \(r \in \mathbb{R}\) such that any solution \(u \in C_{1-a,\ln|t_0,\infty|}^{\alpha_1}\) for Problem (56) satisfies

\[
\lim_{t \to \infty} \frac{u(t)}{\ln \frac{t}{t_0}} = r.
\]

4. Example

Consider the initial value problem:

\[
\begin{cases}
(\hat{H}D_{t_0}^{\alpha}u)'(t) = t^{-\gamma_1}(\ln \frac{t}{t_0})^{\beta_1} (\hat{H}D_{t_0}^{\alpha_1}u(t))^{\gamma_1} + t^{-\gamma_2}(\ln \frac{t}{t_0})^{\beta_2} \int_{t_0}^{t} (t+s)^{-\gamma_3} \\
\quad \times (\ln \frac{s}{t_0})^{\beta_3} (\hat{H}D_{t_0}^{\alpha_2}u(s))^{\gamma_3} ds, \ t > t_0 > 0, \\
(\hat{H}I_{t_0}^{1-\alpha}u)(t_0^+) = u_1, \ (\hat{H}D_{t_0}^{\alpha}u)(t_0^+) = u_2, \ u_1, u_2 \in \mathbb{R},
\end{cases}
\]

(57)

where \(0 \leq \alpha_1 < \alpha < 1, 0 \leq \alpha_2 < \alpha < 1, \gamma_j > 1, j = 1, 2, 3, \beta_1 > (1 - \alpha + \alpha_1)\gamma_1 - 1, \beta_2 > -1, \beta_3 > (1 - \alpha + \alpha_2)\gamma_3 - 1\) and \(0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 < 1\).

The source function,

\[
f(t, u, v) = t^{-\gamma_1}(\ln \frac{t}{t_0})^{\beta_1} u^{\gamma_1} + t^{-\gamma_2}(\ln \frac{t}{t_0})^{\beta_2} v^{\gamma_2}, \ t > t_0 > 0,
\]

(58)

satisfies

\[
\left| f(t, u, v) \right| = \left| t^{-\gamma_1}(\ln \frac{t}{t_0})^{\beta_1} \right| \left( \hat{H}D_{t_0}^{\alpha_1}u(t) \right)^{\gamma_1} + t^{-\gamma_2}(\ln \frac{t}{t_0})^{\beta_2} \int_{t_0}^{t} (t+s)^{-\gamma_3} \\
\times (\ln \frac{s}{t_0})^{\beta_3} (\hat{H}D_{t_0}^{\alpha_2}u(s))^{\gamma_3} ds \leq g_1(t)\varphi_1 \left( \left( \ln \frac{t}{t_0} \right)^{1-\alpha_1} \right) + g_2(t)\varphi_2 \left( \int_{t_0}^{t} g_3(s) \right),
\]

with

\[
h(t, s, u) = (t+s)^{-\gamma_3} \left( \ln \frac{s}{t_0} \right)^{\beta_3} u^{\gamma_3}, \ g_1(t) = t^{-\gamma_1}(\ln \frac{t}{t_0})^{\beta_1+(\alpha-\alpha_1-1)\gamma_1},
\]

\[
g_2(t) = t^{-\gamma_2}(\ln \frac{t}{t_0})^{\beta_2}, \ g_3(t) = t^{-\gamma_5}(\ln \frac{t}{t_0})^{\beta_3+(\alpha-\alpha_2-1)\gamma_3}, \ \varphi_j(t) = t^{\gamma_j}, \ t > t_0 > 0,
\]

where \(\beta_1 > (1 - \alpha + \alpha_1)\gamma_1 - 1, \beta_2 > -1, \beta_3 > (1 - \alpha + \alpha_2)\gamma_3 - 1, \gamma_j > 1, j = 1, 2, 3\) and \(0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 < 1\).
The functions \( g_j, j = 1, 2, 3 \) are continuous, nonnegative on \( (t_0, \infty) \),
\[
\int_{t_0}^{t_\infty} g_1(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_1 + (a - a_1 - 1)\gamma_1} \frac{dt}{t^\alpha} \leq \frac{1}{t_0^{\gamma_1-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_1 + (a - a_1 - 1)\gamma_1} \frac{dt}{t} < \infty,
\]
\[
\int_{t_0}^{t_\infty} g_3(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_3 + (a - a_2 - 1)\gamma_1} \frac{dt}{t^\alpha} \leq \frac{1}{t_0^{\gamma_3-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_3 + (a - a_3 - 1)\gamma_1 + 1} \frac{dt}{t} < \infty,
\]
and
\[
\int_{t_0}^{t_\infty} g_2(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_2} \frac{dt}{t^{\gamma_2}} \leq \frac{1}{t_0^{\gamma_3-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_2} \frac{dt}{t} = \frac{1}{(2 \beta_1 + 1) t_0^{\gamma_3-1}} < \infty.
\]
The function \( g_2 \) is an \( \Omega_0 \)-type because:
\[
\int_{t_0}^{t_\infty} g_2(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_2} \frac{dt}{t^{\gamma_2}} \leq \frac{1}{t_0^{\gamma_2-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_2} e^{-(c_2-1)\ln \frac{t}{t_0}} \frac{dt}{t} = \frac{1}{t_0^{\gamma_2-1}} \int_{0}^{\infty} \sigma_2 e^{-(c_2-1)} \frac{ds}{s} \leq \frac{1}{t_0^{\gamma_2-1}} \int_{0}^{\infty} \sigma_2 e^{-\gamma \tau} d\tau = \frac{1}{(c_2-1)^2 \sigma_2 - 1} \frac{1}{t_0^{\gamma_2-1}} < \infty,
\]
whereas the functions \( g_1 \) and \( g_3 \) are \( \Omega_1 \)-type functions since
\[
\int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right) g_1(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_1 + (a - a_1 - 1)\gamma_1 + 1} \frac{dt}{t^\alpha} \leq \frac{1}{t_0^{\gamma_1-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_1 + (a - a_1 - 1)\gamma_1 + 1} e^{-(c_1-1)\ln \frac{t}{t_0}} \frac{dt}{t} < \infty,
\]
and
\[
\int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right) g_3(t) dt = \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_3 + (a - a_2 - 1)\gamma_1 + 1} \frac{dt}{t^\alpha} \leq \frac{1}{t_0^{\gamma_3-1}} \int_{t_0}^{t_\infty} \left( \ln \frac{t}{t_0} \right)^{\beta_3 + (a - a_2 - 1)\gamma_1 + 1} e^{-(c_1-1)\ln \frac{t}{t_0}} ds \leq \frac{1}{t_0^{\gamma_3-1}} \int_{0}^{t_\infty} \sigma_3 e^{-\gamma \tau} d\tau < \infty.
\]

The functions \( \varphi_j(t) = t^{\gamma_j}, j = 1, 2, 3, 0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq 1 \), are \( \Psi \)-type functions with \( \varphi_1 \propto \varphi_2 \propto \varphi_3 \). These three functions are continuous, positive, nondecreasing on \( (0, \infty) \),
\[
\varphi_j(a) = a^{\gamma_j} \leq b \left( \frac{a}{b} \right)^{\gamma_j} = b \varphi_j \left( \frac{a}{b} \right) \quad \text{for all } a \geq 1, b \geq 1, 0 < \gamma_j \leq 1,
\]
and
\[
\int_{0}^{x} \frac{dt}{\varphi_j(t)} = \int_{0}^{x} \frac{dt}{t^{\gamma_j}} \to \infty \text{ as } x \to \infty \text{ for any } \rho > 0.
\]
As \( 0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq 1 \), then \( \varphi_3 / \varphi_2 \) and \( \varphi_2 / \varphi_1 \) are nondecreasing functions and so \( \varphi_1 \propto \varphi_2 \propto \varphi_3 \).
Obviously, Conditions (A1), (A2) and (A3) are fulfilled with these functions and the hypotheses of Theorem 2 as well. Therefore, there exists a real constant \( r \in \mathbb{R} \) such that solutions of Problem (57) satisfy \( \lim_{t \to \infty} \left( \frac{u(t)}{\ln \frac{t}{\alpha}} \right) = r. \)

5. Conclusions

In this article, we considered a fractional integro-differential problem with Hadamard-Type fractional derivatives. The nonlinear source function depends on a lower-order Hadamard fractional derivative of the state and an integral involving another lower-order Hadamard fractional derivative of the state. We assumed the boundedness of the nonlinearities in question by some special kinds of functions in some appropriate spaces. Under these nonlinear growth conditions, we demonstrated that solutions of the initial value fractional problem under consideration are not only bounded by logarithmic functions but actually they converge asymptotically to logarithmic functions.

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