General quantum error-correcting code with entanglement based on codeword stabilized quantum code

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In this paper, we introduce a unified framework to construct entanglement-assisted quantum error-correcting codes, including additive and nonadditive codes, based on the codeword stabilized framework on subsystems. The codeword stabilized (CWS) framework is a scheme to construct quantum error-correcting codes (QECCs) including both additive and nonadditive codes, and gives a method to construct a QECC from a classical error-correcting code in standard form. Entangled pairs of qubits (ebits) can be used to improve capacity of quantum error correction. In addition, it gives a method to overcome the dual-containing constraint. Operator quantum error correction (OQEC) gives a general framework to construct quantum error-correcting codes. We construct OQEC codes with ebits based on the CWS framework. This new scheme, entanglement-assisted operator codeword stabilized (EAOCWS) quantum codes, is the most general framework we know of to construct both additive and nonadditive codes from classical error-correcting codes. We describe the formalism of our scheme, demonstrate the construction with examples, and give several EAOCWS codes.

I. INTRODUCTION

Quantum error correction (QEC) plays an important role in quantum information processing and communication. Without QEC it is impossible to maintain a quantum state against the corrupting effects of decoherence for long enough to carry out nontrivial quantum computations or communication protocols. Since Shor introduced a method to encode information qubits into a highly entangled state [1], the field of QEC has developed rapidly into a large and diverse field of study.

In Refs. [2, 3] it was shown that quantum error-correcting codes (QECCs) can be constructed from classical binary linear codes that satisfy a dual-containing constraint. Stabilizer codes [4] are a general framework to construct quantum codes analogous to classical additive codes.

Recently, a more general method to construct QECC was introduced. It was shown that both additive and nonadditive quantum codes can be constructed from classical codes using the codeword stabilized (CWS) framework [5]. CWS codes consist of a unique base state, specified by a word stabilizer group, and a set of word operators that produce the other basis states of the code from the base state. For a CWS code in standard form, the base state is a graph state [6], with stabilizer generators corresponding to each vertex of the graph. Using these stabilizer generators, all single-qubit Pauli errors acting on a codeword state can be transformed into errors consisting only of $Z$ and identity operators. CWS codes therefore correspond to classical codes that can correct a particular set of binary errors induced by the word stabilizer.

The word operators produce a set of basis states that spans the code space of the CWS code. From the associated classical code, word operators of a CWS code that can correct the given set of Pauli errors can be identified. The same procedure can be done with a set of multi-qubit Pauli errors to construct codes with higher distances. Therefore, a CWS code in standard form is specified by a graph, whose vertices correspond to the qubits of the codeword, and a classical binary code [7].

In QEC, maximally entangled pairs of qubits (ebits) shared between the sender (Alice) and receiver (Bob) can be used to improve the parameters of quantum codes, such as the minimum distance and/or code rate [8]. The use of ebits also allows stabilizer codes to be constructed from classical codes without the dual-containing restriction [8]. In addition, Ref. [9] showed that QECCs based on the CWS framework, and having minimum distances greater than 3, can be constructed by using ebits. Without shared entanglement, the highest value of minimum distance for a CWS code constructed so far, based on the ring topology, is less than 4.

Operator quantum error correction (OQEC) [10, 11] is a more general scheme to construct QECCs. In this scheme, quantum information can be encoded into either a subspace (as in a standard QECC) or into a subsystem. OQEC unifies both passive error-avoiding schemes and active error correction. CWS codes, including the stabilizer codes and entanglement-assisted QECCs (EAQECCs) described above, encode quantum information into a subspace. Recently, however, there has been work on constructing EAQECCs and CWS codes on subsystems. A theory of entanglement-assisted operator QECCs was developed in [12] in the stabilizer...
formalism, and it was shown how to construct CWS codes on subsystems in [13].

In this paper, we provide a new construction method for QECCs on subsystems based on the CWS framework, using shared ebits. This formalism of entanglement-assisted operator CWS (EAOCWS) codes gives a unified scheme for QECCs, including both additive and nonadditive codes. Since EAOCWS codes are based on the CWS framework, these codes are also specified by a graph state and a classical binary error-correcting code, and can correct a set of errors induced by the word gauge group of the code. The word gauge group includes both stabilizer operators (that leave the base state unchanged) and gauge code. The word gauge group includes both stabilizer operators and gauge operators that act only on the noisy subsystem. All Pauli operators (that leave the base state unchanged) and gauge operators corresponding to the vertices of the graph $G$. For a CWS code in standard form, the codeword stabilizer generators $\{S_i\}$ have the following structure:

$$S_i = X_iZ^{r_i},$$

where $r_i$ is the $i$th row vector of the adjacency matrix $A$. That is, each generator $S_i$ has a Pauli $X$ operator on the qubit corresponding to vertex $i$ of the graph, Pauli $Z$ operators on the qubits corresponding to each of the neighbors of $i$, and identity operators $I$ on all the other qubits. The word stabilizer $S$ is generated by the set $\{S_i\}$.

A unique base state $|S\rangle$ is the common $+1$ eigenstate of the word stabilizer $S$ specified by the graph $G$. This state is fixed by any element $S \in S$ of the word stabilizer:

$$|S\rangle = S|S\rangle.$$  

The word operators $\{w_l\}$ are also elements of $P_n$. The code space is spanned by basis states obtained by applying word operators to the base state $|S\rangle$, and each basis state is of the form

$$|w_l\rangle = w_l|S\rangle.$$

Therefore, the number of the word operators determines the dimension of the code space, and the word operators map the base state onto an orthogonal state.

Lemma 2 in [3] specified that any error in a correctible error set $E$, acting on codewords of a CWS code in standard form, can be represented by another form consisting only of $I$ and $Z$ operators. This equivalent error is called the induced error. By multiplying the error by those word stabilizers that have $X$ operators at the same locations as the original error, all factors of $X$ in the correctible errors can be eliminated, leaving only $Z$ operators. With the induced errors, we can map between the error set $E$ and a set of classical binary errors.

II. NOTATION AND BACKGROUND

A. Codeword stabilized quantum codes

Codeword stabilized (CWS) codes [3] are a broad class of quantum error-correcting codes that include both additive and nonadditive quantum codes. Stabilizer codes can be considered a subset of CWS codes (though generally not in standard form).

Nonadditive codes and additive codes have a difference in the dimension of the code space. An additive (stabilizer) code encodes a definite number $k$ of logical qubits into a codeword of $n$ physical qubits. Such a code with minimum distance $d$ is denoted an $[[n, k, d]]$ code. The dimension of the codespace is $K = 2^k$. For a nonadditive code, the dimension $K$ of the code space need not be a power of 2. Thus we introduce a different notation for nonadditive codes; we denote a nonadditive quantum code that encodes a $K$-dimensional code space into $n$ physical qubits with minimum distance $d$ as an $((n, K, d))$ code.
The mapping $\text{Cl}_G(E_a)$ between the quantum error $E_a = Z^nX^n$ and a classical binary error is defined by

$$\text{Cl}_G(E_a = Z^nX^n) = v \oplus \bigoplus_{i=1}^{n} u_i r_i,$$  \hspace{1cm} (3)

where $v$ and $u$ are binary vectors, $r_i$ is the $i$th row vector of the adjacency matrix $A$ for $G$, and $u_i$ is the $i$th bit of $u$.

Theorem 3 of [3] demonstrates the equivalence between the error correction of a CWS code in standard form and a classical code with this definition. A CWS code in standard form, defined by a graph $G$ and a classical binary code $C_b$, detects errors from the set $E$ if and only if $C_b$ detects errors from the set $\text{Cl}_G(E)$, and for each $E_a \in E$,

- either $\text{Cl}_G(E_a) \neq 0$,
- or, for each $i$, $Z^{c_i} E_a = E_a Z^{c_i},$

where the $c_i$ are the binary codewords from $C_b$. So we can see that the word operators $w_i$ of the CWS code in standard form are derived from the codewords $c_l$ of the binary code $C_b$ by

$$W = \{w_i\} = \{Z^{c_i}\}_{i \in c_b}. \hspace{1cm} (4)$$

B. Entanglement-assisted quantum error-correcting codes

The rate of a quantum error-correcting code can be improved by using pairs of maximally entangled qubits (ebits) [14]. Entanglement also allows us to overcome the dual-containing constraint [8].

It is convenient to explain the properties of a code over its initial code space (before the encoding operation), because initial code space is unitarily equivalent to the code space (after the encoding operation) by the unitary encoding operation $U$.

An $[[n, k, d; c]]$ EAQECC encodes $k$ logical qubits into $n + c$ physical qubits (including $c$ entangled pairs shared between Alice and Bob). The initial state $|\psi\rangle$ of the $[[n, k, d; c]]$ EA-QECC consists of $k$ information qubits $|\phi\rangle$, $m = n - c$ qubits in the state $|0\rangle$ and $c$ entangled states:

$$|\psi\rangle_{EA} = |\Phi_+\rangle^c|0\rangle^{(n-k-c)}|\phi\rangle \hspace{1cm} (5)$$

where $|\Phi_+\rangle$ is the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and is shared by Alice and Bob. The other qubits are all initially on Alice's side.

For the initial state $|\psi\rangle_{EA}$, the stabilizer group is generated by stabilizer generators as follows:

$$Z_i Z_j, \text{ for } i = 1, \cdots, c,$$
$$X_j X_j, \text{ for } j = 1, \cdots, c,$$
$$Z_i I, \text{ for } i = c + 1, \cdots, n - k,$$  \hspace{1cm} (6)

where the operators on the left and right of the $'i'$ act on the qubits on Alice's and Bob's sides, respectively.

The logical operators of the initial state $|\psi\rangle_{EA}$ are

$$Z_{n-k+1} |I, \cdots, Z_n |I,$$
$$X_{n-k+1} |I, \cdots, X_n |I,$$

where all of the operators act on Alice's part.

After an encoding operation $U = U_E \otimes I$, the encoded state is

$$|\psi\rangle_{EA} = U(|\Phi_+\rangle^c|0\rangle^{(n-k-c)}|\phi\rangle).$$

The stabilizer generators of the encoded state $|\psi\rangle_{EA}$ are

$$g_i = U_E Z_i^A (U_E^\dagger) |I\rangle, \text{ for } i = c + 1, \cdots, n - k,$$
$$g_j = U_E Z_j^A (U_E^\dagger) |Z_j\rangle, \text{ for } j = 1, \cdots, c,$$
$$h_j = U_E X_j^A (U_E^\dagger) |X_j\rangle, \text{ for } j = 1, \cdots, c,$$

and the logical operators on $|\psi\rangle_{EA}$ are represented as follows:

$$U_E Z_n |I\rangle \text{ for } i = 1, \cdots, k,$$
$$U_E X_n |I\rangle \text{ for } i = 1, \cdots, k.$$  

The stabilizer group $S$ is generated by two subgroups: the isotropic subgroup $S_I$ and the symplectic subgroup $S_S$:

$$S = \langle S_I, S_S \rangle.$$  

The symplectic subgroup is generated by $S_S = \langle \{g_1, \cdots, g_n, h_1, \cdots, h_c\} \rangle$, and the isotropic subgroup is generated by $S_I = \langle \{g_{k+1}, \cdots, g_{n-k}\} \rangle$. The minimum distance $d$ is equal to the minimum weight of the operators in $N(S) - S_I$.

C. Operator code

Operator quantum error correction (OQEC) [10, 11] generalizes the theory of quantum error correction (QEC) and gives a unified framework to construct both active error correction and passive error avoiding schemes such as decoherence-free subspaces and noiseless subsystems. In OQEC, quantum information is encoded into a subsystem rather than a subspace. Consider a fixed partition of a system’s Hilbert space:

$$\mathcal{H} = (\mathcal{A} \otimes \mathcal{B}) \oplus \mathcal{K}.$$  

Here, the Hilbert space is partitioned into two subspaces, $\mathcal{K}$ and $\mathcal{A} \otimes \mathcal{B}$. The subspace $\mathcal{A} \otimes \mathcal{B}$ is orthogonal to $\mathcal{K}$, and factors into two subsystems by the tensor product.

Quantum information can be encoded into subsystem $\mathcal{A}$ by preparing the information state $\rho^A$ in subsystem $\mathcal{A}$:

$$\rho = \rho^A \otimes \rho^B \otimes 0^K,$$

where $\rho^B$ is an any arbitrary state on the subsystem $\mathcal{B}$. This subsystem is called the noisy or gauge subsystem; operations that affect only the gauge subsystem leave the encoded information unchanged.
It is possible to extend the stabilizer formalism to include OQEC codes [11]. In this case, we encode a state of \( k \) logical qubits into \( n \) physical qubits. Let \( \mathcal{P}_n \) be the \( n \)-fold Pauli group. The initial state before encoding can be represented by

\[
|C\rangle = |0\rangle^{\otimes s}|\psi\rangle|\phi\rangle,
\]

where \(|\phi\rangle\) is the \( k \)-qubit state we wish to encode into a subsystem, \(|\psi\rangle\) is an arbitrary \( r \)-qubit state (which will correspond to the gauge subsystem), and the remaining \( s = n - k - r \) qubits are ancillas in the state \(|0\rangle\). Even if \(|C\rangle\) and \(|C'\rangle = |0\rangle^{\otimes s}|\psi'\rangle|\phi\rangle\) are different (because \(|\psi\rangle \neq |\psi'\rangle\)), both states are considered to encode the same information. Therefore, \(|C\rangle\) and \(|C'\rangle\) are equivalent by a gauge transformation:

\[
|C\rangle = g|C'\rangle
\]

where \( g \) is an operator in the algebra generated by the gauge group \( G \).

The gauge group \( G \) of this OQEC code is a nonabelian subgroup of \( \mathcal{P}_n \) generated by

\[
Z_1, \ldots, Z_s, X_{s+1}, \ldots, X_{s+r}.
\]

Defined in this way, the gauge group includes the stabilizer group of this code, \( S \), that is generated by \( Z_1, \ldots, Z_s \).

The algebraic structure of this trivial code carries over to the OQEC after encoding. The initial state is encoded by a unitary operator \( U \) in the Clifford group. After encoding, the generators of the gauge group are \( \{S_1, \ldots, S_s, g_{s+1}, \ldots, g_{s+r}\} \), where \( S_i \) and \( g_j \) are isomorphic to \( Z_i \) and \( X_j \) on the unencoded state:

\[
S_i = UZ_iU^\dagger, \quad g_j = UX_jU^\dagger.
\]

With this definition of the gauge group, the error set \( \mathcal{E} \) is correctable if and only if

\[
E_aE_b \notin N(S) - G
\]

for all \( E_a, E_b \in \mathcal{E} \). We characterize an operator code by the parameters \( n, k, \) and \( d \) (just as for a standard stabilizer code), but also the number of gauge qubits \( r \); we write this as \([n, k, r, d]\).

### III. ENTANGLEMENT-ASSISTED OPERATOR CWS CODES

In this section, we introduce a framework for entanglement-assisted CWS codes that encode quantum information into a subsystem. We call these entanglement-assisted operator codeword stabilized (EAOCWS) codes. In an EAOCWS code, it is supposed that Alice and Bob share \( c \) pairs of maximally entangled states, typically the Bell state \(|\Phi_+\rangle\). Furthermore, it is assumed that the halves of ebits held by Bob do not suffer from errors, since they do not pass through the noisy channel. An EAOCWS code is defined by a word gauge group \( G \) (including the word stabilizer \( S \)), a base state \(|S\rangle\), and word operators \( W \).

#### A. Initial base state of EAOCWS codes

In a CWS code, the unique base state is defined by the word stabilizer. Similarly, the base state of an EAOCWS code is specified by the word gauge group. For convenience, we first consider the initial base state before applying the encoding unitary. The initial base state \(|S_b\rangle\) of the \((n, K, r, d; c)\) EAOCWS code consists of \( s = n - r - c \) qubits in the state \(|0\rangle\), \( c \) ebits and \( r \) gauge qubits in an arbitrary state:

\[
|S_b\rangle = |0\rangle^{\otimes s}|\Phi_+\rangle^{\otimes c}|\psi\rangle
\]

where \(|\psi\rangle\) is an arbitrary \( r \)-qubit state. Each maximally entangled pair \(|\Phi_+\rangle\) is shared by Alice and Bob, and it is assumed that the halves of ebits on Bob’s side do not suffer from errors. All other qubits are on Alice’s side.

Because we are defining a code on a subsystem, the “base state” is not really a unique state; it is only defined up to the arbitrary state of the gauge subsystem. We therefore identify an equivalence class of base states that can be turned into each other by gauge transformations \( G \):

\[
|S_b\rangle \sim |S_b'\rangle \Leftrightarrow |S_b\rangle = G|S_b'\rangle,
\]

where \( G \in G \). This equivalence class of initial base states is stabilized by a stabilizer group \( S \), i.e.,

\[
|S_b\rangle = S|S_b\rangle,
\]

where \( S \in S \).

The minimal set of operators that can generate the gauge group of an EAOCWS code comprises three types of operators. First, the word stabilizer \( S_b^c \) of the initial base state corresponds to a fixed \( s \)-qubit state in subsystem \( A \), and acts trivially on subsystem \( B \). For the initial base state Eq. (10), the fixed state is \(|0\rangle^{\otimes s}\), and the word stabilizer \( S_b^c \) is generated by the operators

\[
Z_1I\cdots I|I^{\otimes c}
\]

\[
\vdots
\]

\[
I\cdots IZ_mI\cdots I|I^{\otimes c}
\]

The operators on the left and right of the ‘\(|\rangle\rangle' act on the qubits on Alice’s and Bob’s sides, respectively.

The initial base state includes \( c \) maximally entangled pairs of qubits between subsystem \( A \) and \( c \) qubits on Bob’s side. The Bell state \(|\Phi_+\rangle\) is stabilized by two operators \( XX \) and \( ZZ \). Therefore, the word stabilizer \( S_b^c \) acting on the \( c \) ebits is generated by

\[
I\cdots IZ_{s+1}I\cdots IZ_1I\cdots I,
I\cdots IX_{s+1}I\cdots IX_1I\cdots I,
\]

\[
\vdots
\]

\[
I\cdots IZ_{s+c}I\cdots I|I\cdots IZ_c,
I\cdots IX_{s+c}I\cdots I|I\cdots IX_c.
\]

The groups \( S_b^c \) and \( S_b^c \) stabilize the fixed state \(|0\rangle^{\otimes s}|\Phi_+\rangle^{\otimes c} \) on subsystem \( A \).
Since the state $|\psi\rangle$ in the initial base state $|S\rangle_b$, Eq. (10), is an arbitrary state on subsystem $B$, any operators acting only on subsystem $B$ are elements of the gauge subgroup $S_B^g$ acting on $|\psi\rangle$. Therefore, the gauge subgroup $S_B^g$ is generated by

\[
\begin{align*}
I \cdots IZ_{n-r+1}I \cdots I|I^{\otimes c}, \\
I \cdots IX_{n-r+1}I \cdots I|I^{\otimes c}, \\
\vdots \\
I \cdots IZ_{n}I^{\otimes c}, \\
I \cdots IX_{n}I^{\otimes c}.
\end{align*}
\]

(15)

From the above three subgroups of the word gauge group, we generate the word gauge group $G_b$ of the initial base state $|S_b\rangle$ for an EAOCWS code:

\[
G_b = \langle \{S_b^s, S_b^c, S_b^g\} \rangle
\]

(16)

The word gauge operators act trivially on subsystem $A$ and leave the equivalence class of initial base states invariant.

The code space is spanned by basis states given by applying the set of word operators $W$ to the base state. The encoding operation must be performed only on Alice’s side. Therefore, the word operators act only on Alice’s qubits as well. For the initial base state $|S_b\rangle$, the initial word operators $W_b = \{w_l\}$ of an EAOCWS code can be represented as

\[
w_l = X^x \otimes Z^u X^w \otimes I^{\otimes r} I^{\otimes c},
\]

(17)

where $x$, $u$, and $w$ are binary vectors of length $m$, $c$ and $c$, respectively. We could similarly define $w''$ having non-trivial operators on subsystem $B$ as a word operator for the initial base state. The operator $w''$ is equivalent to $w_l$ by a gauge transformation $G$. In [13], it is shown that without loss of generality we can define the word operators to have the form in Eq. (17).

The number of word operators is equal to the dimension of the code subsystem. To encode $K$ logical states, we need $K$ word operators. The basis state is

\[
w_l' |S'_l\rangle \equiv w_l' |\psi\rangle = |x\rangle \otimes Z^u X^w |\Phi_+\rangle^{\otimes c} \otimes |\psi\rangle.
\]

(18)

The base state of an EAOCWS code doesn’t include information qubits. So we have to consider how to encode an information state $|\phi\rangle$ into a state $|\phi'\rangle$ in the code space spanned by the states $|w_l\rangle$. If we assume that $|\phi\rangle$ is a $K$-dimensional system state

\[
|\phi\rangle = \sum_{l=0}^{K-1} \alpha_l |l\rangle,
\]

we prepare the base state $|S'\rangle$ and define a unitary transformation $U_w$ [9] that swaps the state $|\phi\rangle$ into the codeword:

\[
U_w(|\phi\rangle \otimes |S'\rangle) = |0\rangle \otimes \sum_{l=0}^{K-1} \alpha_l |w_l\rangle \equiv |0\rangle \otimes |\phi'\rangle.
\]

(19)
If an error $E = IXIX|I$ occurs on a codeword, the effective error induced from $E$ can be represented as a binary vector by Eq. (3):

$$C_{IG}(E = IXIX|I) = 10000|1.$$  (23)

This binary vector is determined by applying operators $s_2$, $s_3$ and $g_1$ to the error $E$ to eliminate the $X$ operators and leave only $Z$s.

As stated above, it is assumed that the physical errors do not affect Bob’s qubits. However, as shown in Eq. (23), when we convert to an effective error, there can be $Z$ operators acting on Bob’s side resulting from the $h_i$ operators. Since the word operators correspond to codewords from a binary code designed to correct this set of effective errors, this means that the word operators will also include operators that act on both Alice’s and Bob’s qubits, in general. In [9], it was shown that $Z$ operators on Bob’s side in the word operators can be removed by applying elements of the word stabilizer. Therefore, it is possible to construct word operators that act only on Alice’s side.

This is not quite the end of the story. As we will see in our examples below, even a set of word operators that act only on Alice’s side may not allow the encoding procedure to be carried out. Since the operator $U_w$ is a generalized swap operation, it can at most encode a state whose dimension is no greater than that of the set of ancillas in the initial base state. If two word operators act on the base state in a way that is indistinguishable on Alice’s side alone, they cannot both be used in the encoding circuit. Therefore, we may be able to use only a subset of the word operators. We will see below how this can arise.

### B. Examples

We now show some examples of EAOCWS codes based on the construction introduced in the previous section. All these codes use a base state based on the ring graph, and the particular binary codes and word operators were found by numerical search.

A ring graph $G$ consists of $n$ vertices arranged in a closed loop, so each vertex has exactly two neighbors. From this ring graph, the base state of the EAOCWS can be defined. For example, suppose an EAOCWS code is defined on 9 physical qubit state with $c = 3$ and $r = 1$. The initial base state of this code (before encoding) is given by

$$|S_b⟩ = |0⟩|Ph⟩^⊗3|ψ⟩,$$  (24)

where $|ψ⟩$ is an arbitrary 1-qubit state. The word gauge group of this code is generated by the operators which is

$$s_1 = XZIIIZ|III, s_2 = NZIII|III, s_3 = IZIXII|ZII, h_1 = IIYIII|XII, s_4 = IIIZXI|IZI, h_2 = IIIIZII|XI, s_5 = IIIIZXI|IZI, h_3 = IIIIZII|IX, s_6 = ZIIIXZ|III, g_1 = IIIIZII|III.$$  (23)

Using this group, all single errors that occur in the channel are mapped onto induced errors consisting only of $Z$ and $I$ operators. First, by applying $s_i$ to all single-qubit errors, all possible $X$, $Y$ and $Z$ errors can be represented as the following induced errors:

$$ZIII|II IIIZIII|II ZZIII|II,$$  (23)

$$IZII|II ZIII|II ZZIII|II,$$  (23)

$$IZII|II IZII|II ZZII|II,$$  (23)

$$IZII|II IIIZ|II ZZII|II,$$  (23)

$$IZII|II ZZII|II ZZII|II.$$  (23)

After that, we apply $g_1$ to the induced errors to remove any $Z$ operators located on the 6th qubit:

$$ZIIII|II IIIIIZ|II ZZIIIZ|II$$  (23)

$$IZIIII|II ZZIIII|II ZZIIII|II. $$  (23)

A classical binary code that can correct binary errors corresponding to these effective errors has codewords

$$00000|000 110100|010 110100|101 1110100|101 111100|011 000010|101 010100|111 101000|100.$$  (23)

From the above binary vectors, the word operators of this code are $w_l = Z^c$:

$$IIIIII|I I ZIIII|I ZZIIII|II ZZIII|II ZZZIII|II ZZIII|II ZZIII|II ZZIII|II.$$  (23)

The $Z$ operators on Bob’s side of the word operators are eliminated by applying word stabilizer elements:

$$IIIIII|II ZZIIIZ|II IIIZXXX|II ZZIIIX|II ZZIIIX|II ZZIIIX|II ZZIIIX|II.$$  (23)
TABLE I. EAOCWS codes with $d = 3$

| r \ c | 1     | 2                      | 3                      | 4     | 5     |
|-------|-------|------------------------|------------------------|-------|-------|
| 1     | -     | [5,2,1,3;2]            | [5,1,1,3;3]            | -     | -     |
| 2     | -     | -                      | -                      | -     | -     |
| 3     | -     | -                      | -                      | -     | -     |
| 1     | [6,1,1,3;1] | [6,2,1,3;2] | (6,4,1,3;3) | [6,1,1,3;4] | -     |
| 2     | -     | [6,2,2,3;2]            | [6,1,2,3;3]            | -     | -     |
| 3     | -     | -                      | -                      | -     | -     |
| 4     | -     | -                      | -                      | -     | -     |
| 5     | -     | -                      | -                      | -     | -     |

TABLE II. EAOCWS codes with $d = 5$

| r \ c | 1     | 2     | 3     | 4     | 5     | 6     |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | -     | -     | -     | -     | [7,2,1,5;4] | [7,1,1,5;5] | -     |
| 2     | -     | -     | -     | -     | -     | -     | -     |
| 3     | -     | -     | -     | -     | -     | -     | -     |
| 4     | -     | -     | -     | -     | -     | -     | -     |
| 5     | -     | -     | -     | -     | -     | -     | -     |

From these operators we can find a set of initial word operators that act on Alice’s side before applying the unitary $U_E$:

$\quad IIIIII \quad XXIYII \quad XXZXZI \quad XXZYII$

$\quad XXXYZI \quad IIIZIY \quad IXZYZI \quad XIYIII.$

But now we run into the encoding limitation that was mentioned above. Consider the two operators $IIIIII$ and $IIIZYI$ acting on the initial base state $|S_b\rangle$ in Eq. (24). These operators do produce two orthogonal states. But these states cannot be reliably distinguished by any measurement on Alice’s side alone. This, in turn, means that it is impossible to define the “swap” operator $U_w$ in (19). Our code cannot include both of these codewords. (Note that these operators could be used to encode classical information, in a manner analogous to superdense coding.)

The reason that these operators produce locally indistinguishable states is because both of them have a same operator (in this case, $II’$) acting on the two ancilla qubits $|00\rangle$. Similarly, the states produced by the word operators

$\quad XXIYII \quad XXZXZI \quad XXZYII \quad XXXYZI$

cannot be distinguished on Alice’s side alone, because they all act with the same operator $XX$ on the two ancilla qubits. Therefore, only one of these four word operators can be used, and only one of the corresponding codewords can be included.

Therefore, the initial word operators of this code—that produce an orthonormal basis for the code space and can be encoded locally by Alice—are

$\quad IIIIIII \quad IXZYZI \quad XIYIII \quad XXIYII,$

and the code space is 4-dimensional. So, this code is a nonadditive $((6,4,1,3;3))$ EAOCWS code.

By a procedure like that above, we were able to construct a number of both additive and nonadditive codes in the EAOCWS framework. Table I shows the parameters of some codes with minimum distance $d = 3$, and Table II shows the parameters of two EAOCWS codes with $d = 5$.

**IV. CONCLUSIONS**

We have presented a unified method to construct both additive and nonadditive EAQECCs on subsystems. Our construction is based on the CWS framework, which can be specified by a graph topology and a classical binary code. Using shared ebits between the sender and the receiver, the code rate and distance is improved.
Because of the word stabilizer elements corresponding to the shared ebits, the induced errors can have $Z$ operators on Bob’s side. We can find equivalent word operators—including more than just $Z$ operators—that act only on Alice’s side. This is necessary for the encoding operation to be possible. However, this requirement means that not all possible word operators may be included in the code; word operators that transform the base state to states that are locally indistinguishable on Alice’s side cannot all be included. In addition, the word gauge operators can remove some of the $Z$ operators in the effective errors, so the weight of the induced errors is reduced.

Finally, we showed an example of how to construct an EAOCWS code, and gave the parameters of several codes in the EAOCWS framework. These codes were found by numerical search, and include CWS codes with distance $d$ greater than 3.

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