GAUSS SUMMATION AND Ramanujan Type Series For $1/\pi$

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Abstract. Using some properties of the gamma function and the well-known Gauss summation formula for the classical hypergeometric series, we prove a four-parameter series expansion formula, which can produce infinitely many Ramanujan type series for $1/\pi$.

1. Introduction

The gamma function $\Gamma (z)$ can be defined by the formula [17, p. 76]

$$\frac{1}{\Gamma (z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where $\gamma$ is the Euler constant defined as

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right).$$

$\Gamma (z)$ is meromorphic in the entire complex plane and has simple poles at $z = 0, -1, -2, \ldots$. It is easy to verify that $\Gamma (1) = 1$ and $\Gamma (z)$ satisfies the recurrence relation $\Gamma (z + 1) = z \Gamma (z)$. It follows that for every positive integer $n$, we have $\Gamma (n) = (n - 1)!$. Using the recurrence relation for the gamma function and $\Gamma (1/2) = \sqrt{\pi}$, we can find the following proposition.

Proposition 1.1. If $n$ is a nonnegative integer, then we have

$$\Gamma (n + 1/2) = \left(\frac{2n!}{4^n n!}\right) \sqrt{\pi}, \quad \Gamma (1/2 - n) = \frac{(-1)^n 4^n n!}{(2n)!} \sqrt{\pi}.$$

One of the most important properties of $\Gamma (z)$ is the Euler reflection formula [1, p. 9], [17, p. 78].

Proposition 1.2. (Euler’s reflection formula).

$$\Gamma (z) \Gamma (1 - z) = \frac{\pi}{\sin \pi z}.$$
Definition 1.1. For any complex \( \alpha \), we define the general rising shifted factorial by
\[
(z)_\alpha = \frac{\Gamma(z + \alpha)}{\Gamma(z)}.
\]

In particular, for every non-negative integer \( n \), we have
\[
(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)} = z(z + 1) \cdots (z + n - 1),
\]
and for every positive integer \( n \),
\[
(z)_{-n} = \frac{1}{(z - 1) \cdot (z - 2) \cdots (z - n)}.
\]

The well-known Gauss summation formula is stated in the following theorem [1, p. 66], [15, p. 102].

Theorem 1.1. (The Gauss summation formula). If \( c \) is not zero or a negative integer and \( \text{Re}(c - a - b) > 0 \), then we have
\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.
\]

In his famous paper [16], Ramanujan recorded a total of 17 series for \( 1/\pi \) without proofs. Ramanujan’s series for \( 1/\pi \) were not extensively studied until around 1987. The Borwein brothers [5] provided rigorous proofs of Ramanujan’s series for the first time and also obtained many new series for \( 1/\pi \). Some remarkable extensions of them were given by the Chudnovsky brothers [10].

It should be pointed out that before Ramanujan, some mathematicians had derived some series expansions for \( 1/\pi \). For example, G. Bauer in 1859 [4] obtained some series expansions for \( 1/\pi \) using Legendre polynomials, and J. W. L. Glaisher in 1905 [11] gave a systematic study on the series expansions for \( 1/\pi \) using the theory of elliptic functions.

Many new Ramanujan type series for \( 1/\pi \) have been published recently, see for example, [2], [6], [9], [12], [13]. One may consult the survey paper [3] for the interesting history of Ramanujan type series for \( 1/\pi \).

Very recently, Chu [8] derived numerous Ramanujan type series for \( 1/\pi \) and \( \pi \) using Dougall’s bilateral \( _2H_2 \) series and the summation by parts formula.

This paper is motivated by [8] and [13]. In this paper we will use the general rising shifted factorial and the Gauss summation formula to prove the following four-parameter series expansion formula for \( 1/\pi \).

Theorem 1.2. For any complex \( \alpha \) and \( \text{Re}(c - a - b) > 0 \), we have
\[
\sum_{n=0}^{\infty} \frac{(\alpha)_{a+n}(1 - \alpha)_{b+n}}{n! \Gamma(c + n + 1)} = \frac{(\alpha)_a (1 - \alpha)_b \Gamma(c - a - b)}{(\alpha)_{c-b} (1 - \alpha)_{c-a}} \times \frac{\sin \pi \alpha}{\pi}.
\]

When \( a, b \) and \( c \) are positive integers, it is obvious that every term of the series on the left hand side of the above equation is a rational function of \( n \). Hence Theorem 1.2 allows us to derive infinitely many series expansion formulas for \( 1/\pi \).
The remainder of the paper is organized as follows. In Section 2, we prove Theorem 1.2 using Definition 1.1 and the Gauss summation formula. Some special cases of Theorems 1.2 are given in Section 3.

2. proof of Theorem 1.2

Using (1.1) we find that the identity in Theorem 1.1 can be rewritten as

$$\sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\Gamma(c+n)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c - a - b)}{\Gamma(c-a)\Gamma(c-b)}.$$  

Replacing \((a, b, c)\) by \((a + \alpha, b + 1 - \alpha, c + 1)\) in the above equation, we find

$$\sum_{n=0}^{\infty} \frac{\Gamma(a+n+\alpha)\Gamma(b+n+1-\alpha)}{n!\Gamma(c+n+1)} = \frac{\Gamma(a+\alpha)\Gamma(b+1-\alpha)\Gamma(c-a-b)}{\Gamma(c-a+1-\alpha)\Gamma(c-b+\alpha)}.$$  

Using the general rising shifted factorial in Definition 1.1 we easily find that

$$\Gamma(a+\alpha) = (\alpha)_a\Gamma(\alpha), \Gamma(b+1-\alpha) = (1-\alpha)_b\Gamma(1-\alpha),$$

$$\Gamma(a+n+\alpha) = (\alpha)_{a+n}\Gamma(\alpha), \Gamma(b+n+1-\alpha) = (1-\alpha)_{b+n}\Gamma(1-\alpha),$$

$$\Gamma(c-a+1-\alpha) = (1-\alpha)_{c-a}\Gamma(1-\alpha), \Gamma(c-b+\alpha) = (\alpha)_{c-b}\Gamma(\alpha).$$

Substituting these equations into (2.1) and dividing both sides by \(\Gamma(\alpha)\Gamma(1-\alpha)\), we find that

$$\sum_{n=0}^{\infty} \frac{(\alpha)_{a+n}(1-\alpha)_{b+n}}{n!\Gamma(c+n+1)} = \frac{(\alpha)_a(1-\alpha)_b\Gamma(c-a-b)}{\Gamma(\alpha)\Gamma(1-\alpha)(\alpha)_{c-a}(1-\alpha)_{c-a}}.$$  

Replacing \(\Gamma(\alpha)\Gamma(1-\alpha)\) by \(\pi/\sin \alpha \pi\) in the right hand side of the above equation, we complete the proof of Theorem 1.2.

3. Some special cases

In this section we will give some interesting special cases of Theorem 1.2.

Corollary 3.1. \((\alpha = \frac{1}{2} \text{ in Theorem 1.2})\). For \(\text{Re}(c-a-b) > 0\),

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{a+n}(\frac{1}{2})_{b+n}}{n!\Gamma(c+n+1)} = \frac{(\frac{1}{2})_a(\frac{1}{2})_b\Gamma(c-a-b)}{\pi(\frac{1}{2})_{c-a}(\frac{1}{2})_{c-b}}.$$  

Special case 3.1. \((a = b = 0 \text{ and } c = k \text{ in Corollary 3.1})\). If \(k\) is a positive integer, then

$$\frac{(k-1)!}{\pi(\frac{1}{2})^k} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!(k+n)!}.$$  

Example 3.1. \((k = 1 \text{ in Special case 3.1})\) Glaisher [11, p. 174]).

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n+1)n!^2}.$$
Example 3.2. \( k = 2 \) in Special case 3.1.

\[
\frac{16}{9\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^2_n}{(n+1)(n+2)n!^2}.
\]

Special case 3.2. \((a = b = -1, c = k, \text{in Corollary 3.1})\). If \( k \) is a nonnegative integer, then

\[
\frac{4(k+1)!}{\pi(\frac{1}{2})^2_{k+1}} = 4k + 5 + (k+1)! \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2_n}{(n+1)(k+n+1)!}.
\]

Proof. If \((a, b, c) = (-1, -1, k)\), then Corollary 3.1 becomes

\[
(\frac{1}{2})^2_1 \Gamma(k+2) \pi(\frac{1}{2})^2_{k+1} = \frac{(\frac{1}{2})^2_1}{k!} + \frac{1}{(k+1)!} + \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^2_{n-1}}{n!\Gamma(k+n+1)}.
\]

Using (1.2), we easily find \((1/2)_2 = -2\). Thus the above equation becomes

\[
\frac{4(k+1)!}{\pi(\frac{1}{2})^2_{k+1}} = 4 \frac{1}{k!} + \frac{1}{(k+1)!} + \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^2_{n-1}}{n!\Gamma(k+n+1)}.
\]

Multiplying both sides by \((k+1)!\), and making the variable change \(n \to n+1\), we arrive at Special case 3.2.

Example 3.3. \((k = 0 \text{ in Special case 3.2: Glaisher} [11, p. 174])\).

\[
\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2_n}{(n+1)!^2}.
\]

Example 3.4. \((k = 1 \text{ in Special case 3.2})\).

\[
\frac{256}{9\pi} = 9 + 2 \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2_n}{(n+2)(n+1)!^2}.
\]

Special case 3.3. \((a = b = -2, c = k, \text{in Corollary 3.1})\). If \( k \) is a nonnegative integer, then

\[
\frac{32(k+2)!(k+3)!}{\pi(\frac{1}{2})^2_{k+2}} = 32k^2 + 168k + 217 + 18(k+2)! \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2_n}{(n+2)(k+n+2)!}.
\]

Proof. If \(a = b = -2, c = k\), then Corollary 3.1 becomes

\[
(\frac{1}{2})^2_{-2} \Gamma(k+4) \pi(\frac{1}{2})^2_{k+2} = (\frac{1}{2})^2_{-2} \frac{1}{k!} + (\frac{1}{2})^2_{-1} \frac{1}{(k+1)!} + \frac{1}{2(k+2)!} + \sum_{n=3}^{\infty} \frac{(\frac{1}{2})^2_{n-2}}{n!\Gamma(k+n+1)}.
\]

Using (1.2), we find that \((1/2)_{-2} = 4/3\). Thus we conclude that

\[
\frac{16(k+3)!}{9\pi(\frac{1}{2})^2_{k+2}} = \frac{16}{9k!} + \frac{4}{(k+1)!} + \frac{1}{2(k+2)!} + \sum_{n=3}^{\infty} \frac{(\frac{1}{2})^2_{n-2}}{n!\Gamma(k+n+1)}.
\]

Multiplying both sides of the above equation by \(18(k+2)!\), we arrive at Special case 3.3.
Example 3.5. \((k = 0\) in Special case \([3,3]\)).

\[
\frac{2048}{3\pi} = 217 + 36 \sum_{n=1}^{\infty} \frac{\left(\frac{1}{9}\right)^n}{(n + 2)!}.
\]

Corollary 3.2. \((\alpha = \frac{1}{3}\) in Theorem \([1,2]\)). For \(\text{Re}(c - a - b) > 0\),

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{9}\right)_{a+n}(\frac{2}{9})_{b+n}}{n!(c+n+1)} = \frac{\sqrt{3}(\frac{1}{3})_{a}(\frac{2}{3})_{b}\Gamma(c - a - b)}{2\pi(\frac{2}{3})_{c-a}(\frac{1}{3})_{c-b}}.
\]

Special case 3.4. \((a = b = -1\) and \(c = k\) in Corollary \([3,2]\)). If \(k \geq 0\) is an integer, then

\[
\frac{\sqrt{3}k!(k+1)!}{2\pi(\frac{2}{3})_{k+1}(\frac{1}{3})_{k+1}} = 1 + \frac{2}{9}k! \sum_{n=0}^{\infty} \frac{\left(\frac{1}{9}\right)n(\frac{2}{9})n}{(n+1)!(n+k+1)!}.
\]

Proof. If \(a = b = -1\) and \(c = k\), then Corollary \([3,2]\) becomes

\[
\frac{\sqrt{3}(\frac{1}{3})_{-1}(\frac{2}{3})_{-1}\Gamma(k+2)}{2\pi(\frac{2}{3})_{k+1}(\frac{1}{3})_{k+1}} = \frac{1}{k!} + \sum_{n=1}^{\infty} \frac{(\frac{1}{9})n(\frac{2}{9})n}{n!(n+k)!}.
\]

Using \([1,2]\), we find that \((1/3)_{-1} = -3/2\) and \((2/3)_{-1} = -3\). It follows that

\[
\frac{9\sqrt{3}(k+1)!}{4\pi(\frac{2}{3})_{k+1}(\frac{1}{3})_{k+1}} = 9 + \frac{2}{k!} \sum_{n=0}^{\infty} \frac{(\frac{1}{9})n(\frac{2}{9})n}{(n+1)!(n+k+1)!}.
\]

Multiplying both sides of the above equation by \(2k!/9\), we arrive at Special case \([3,4]\). \(\square\)

Example 3.6. \((k = 0\) in Special case \([3,4]\)).

\[
\frac{9\sqrt{3}}{4\pi} = 1 + \frac{2}{9} \sum_{n=0}^{\infty} \frac{(\frac{1}{9})n(\frac{2}{9})n}{(n+1)!2}.
\]

Corollary 3.3. \((\alpha = \frac{1}{4}\) in Theorem \([1,2]\)). For \(\text{Re}(c - a - b) > 0\),

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_{a+n}(\frac{3}{4})_{b+n}}{n!(c+n+1)} = \frac{(\frac{1}{4})_{a}(\frac{3}{4})_{b}\Gamma(c - a - b)}{\sqrt{2}\pi(\frac{3}{4})_{c-a}(\frac{1}{4})_{c-b}}.
\]

Special case 3.5. \((a = b = -1\) and \(c = k\) in Corollary \([3,3]\)). If \(k \geq 0\) is an integer, then

\[
\frac{k!(k+1)!}{\sqrt{2}\pi(\frac{2}{4})_{k+1}(\frac{1}{4})_{k+1}} = 1 + \frac{3}{16}k! \sum_{n=0}^{\infty} \frac{(\frac{1}{4})n(\frac{3}{4})n}{(n+1)!(n+k+1)!}.
\]

Example 3.7. \((k = 0\) in Special case \([3,5]\)).

\[
\frac{8\sqrt{2}}{3\pi} = 1 + \frac{3}{16} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})n(\frac{3}{4})n}{(n+1)!2}.
\]
Corollary 3.4. \((\alpha = \frac{1}{6})\) in Theorem 1.2. For \(\text{Re}(c - a - b) > 0\),

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_{a+n} \left(\frac{5}{6}\right)_{b+n}}{n! \Gamma(c + n + 1)} = \frac{\left(\frac{1}{6}\right)_{a} \left(\frac{5}{6}\right)_{b} \Gamma(c - a - b)}{2 \pi \left(\frac{5}{6}\right)_{c-a} \left(\frac{1}{6}\right)_{c-b}}.
\]

Special case 3.6. \((a = b = -1 \text{ and } c = k)\) in Corollary 3.4. If \(k \geq 0\) is an integer, then

\[
\frac{k!(k+1)!}{2\pi \left(\frac{1}{6}\right)_{k+1} \left(\frac{5}{6}\right)_{k+1}} = 1 + \frac{5}{36} k! \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_{n} \left(\frac{5}{6}\right)_{n}}{(n+1)! (n+k+1)!}.
\]

Example 3.8. \((k = 0)\) in Special case 3.6.

\[
\frac{18}{5\pi} = 1 + \frac{5}{36} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_{n} \left(\frac{5}{6}\right)_{n}}{(n+1)!^2}.
\]

Corollary 3.5. \((\alpha = \frac{1}{10})\) in Theorem 1.2. For \(\text{Re}(c - a - b) > 0\),

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{10}\right)_{a+n} \left(\frac{9}{10}\right)_{b+n}}{n! \Gamma(c + n + 1)} = \frac{\left(\sqrt{5} - 1\right) \left(\frac{1}{10}\right)_{a} \left(\frac{9}{10}\right)_{b} \Gamma(c - a - b)}{4 \pi \left(\frac{9}{10}\right)_{c-a} \left(\frac{1}{10}\right)_{c-b}}.
\]

Special case 3.7. \((a = b = -1 \text{ and } c = k)\) in Corollary 3.5. If \(k \geq 0\) is an integer, then

\[
\frac{\left(\sqrt{5} - 1\right) k!(k+1)!}{4 \pi \left(\frac{1}{10}\right)_{k+1} \left(\frac{9}{10}\right)_{k+1}} = 1 + \frac{9}{100} k! \sum_{n=0}^{\infty} \frac{\left(\frac{1}{10}\right)_{n} \left(\frac{9}{10}\right)_{n}}{(n+1)! (n+k+1)!}.
\]

Example 3.9. \((k = 0)\) in Special case 3.7.

\[
\frac{25 \left(\sqrt{5} - 1\right)}{9\pi} = 1 + \frac{9}{100} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{10}\right)_{n} \left(\frac{9}{10}\right)_{n}}{(n+1)!^2}.
\]

Corollary 3.6. \((\alpha = \frac{1}{5})\) in Theorem 1.2. For \(\text{Re}(c - a - b) > 0\),

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{5}\right)_{a+n} \left(\frac{4}{5}\right)_{b+n}}{n! \Gamma(c + n + 1)} = \frac{\left(\sqrt{10} - 2\sqrt{5}\right) \left(\frac{1}{5}\right)_{a} \left(\frac{4}{5}\right)_{b} \Gamma(c - a - b)}{4 \pi \left(\frac{4}{5}\right)_{c-a} \left(\frac{1}{5}\right)_{c-b}}.
\]

Special case 3.8. \((a = b = -1 \text{ and } c = k)\) in Corollary 3.6. If \(k \geq 0\) is an integer, then

\[
\frac{\left(\sqrt{10} - 2\sqrt{5}\right) k!(k+1)!}{4 \pi \left(\frac{1}{5}\right)_{k+1} \left(\frac{4}{5}\right)_{k+1}} = 1 + \frac{4}{25} k! \sum_{n=0}^{\infty} \frac{\left(\frac{4}{5}\right)_{n} \left(\frac{5}{5}\right)_{n}}{(n+1)! (n+k+1)!}.
\]

Example 3.10. \((k = 0)\) in Special case 3.8.

\[
\frac{25 \left(\sqrt{10} - 2\sqrt{5}\right)}{16\pi} = 1 + \frac{4}{25} \sum_{n=0}^{\infty} \frac{\left(\frac{4}{5}\right)_{n} \left(\frac{5}{5}\right)_{n}}{(n+1)!^2}.
\]

Using the same method as used in this paper, in [14] we use Dougall’s \(\genfrac{[}{]}{0pt}{}{5}{4}\) summation for the classical hypergeometric functions to derive a general series expansion formula which can produce infinitely many Ramanujan type series for \(1/\pi^2\).
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