GEOMETRY OF FOUR-FOLDS WITH THREE NON-COMMUTING INVOLUTIONS

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Abstract. In this paper we adapt some techniques developed for K3 surfaces, to study the geometry of a family of projective varieties in \( \mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K \) defined as the intersection of a form of degree \( (2, 2, 2) \) and a form of degree \( (1, 1, 1) \). Members of the family will be equipped with dominant rational self-maps and we will study the actions of those maps on divisors and compute the first dynamical degrees of the composition of any pair.

1. Introduction

As a generalization of the work of Silverman and others [10], [6] on families of K3 surfaces with infinite groups of automorphisms, we study dynamics on a family of varieties \( X^{A,B} \) in \( \mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K \) defined as the intersection of a form of degree \( (2, 2, 2) \) and a form of degree \( (1, 1, 1) \). Individual members of the family \( X^{A,B} \) come equipped with \((2 : 1)\)-projections \( p_1, p_2, p_3 : X^{A,B} \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \) that generate involutions \( \sigma_1, \sigma_2, \sigma_3 \) on \( X^{A,B} \). In this situation however the maps \( \sigma_i \) for \( i = 1, 2, 3 \) are not morphisms of the whole \( X^{A,B} \), but only rational dominant maps. Still it is possible to induce maps \( \sigma_1^* : \text{Pic}(X) \rightarrow \text{Pic}(X) \) and \( \tilde{\sigma}_i^* : \text{NS}(X)_\mathbb{Q} \rightarrow \text{NS}(X)_\mathbb{Q} \), on divisors modulo linear and numerical equivalence. The computations with divisors in the case of three involutions is going to be similar to the K3 surfaces of type \( (2, 2, 2) \) in \( \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \) studied by several authors like Wang [12] and Baragar [2], [3] and [4].

The following degree associated to the dynamics was initially studied by Arnold in [1], and particularly for dominant rational maps by Silverman in [11].

Definition 1.1. Let \( X \) be an algebraic variety and \( \varphi : X \rightarrow X \) a dominant rational map. The first dynamical degree of \( \varphi \) is

\[
\delta_\varphi = \limsup_{n \rightarrow \infty} \rho(\varphi^n)^{1/n},
\]

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where \( \rho(\varphi^n) \) represents the spectral radius or maximal eigenvalue of the map \( \varphi^n : NS(X)_{\mathbb{Q}} \to NS(X)_{\mathbb{Q}} \).

It is also possible to extend the notion of polarization, with respect to one rational map or, more general, in the sense of Kawaguchi [9], associated to several rational maps:

**Definition 1.2.** Let \( X \) be a projective variety and \( \varphi_i : X \to X \) for \( i = 1, \ldots, k \) dominant rational maps. We say that the system \( (X, \{\varphi_1, \ldots, \varphi_k\}, L, d) \) is a polarized dynamical system of \( k \) maps if there exist an ample line bundle \( L \in \text{Pic}(X) \otimes \mathbb{R} \) such that \( \bigotimes_{i=1}^k \varphi_i^* L \cong L^d \) for some \( d > k \).

The action of the maps \( \sigma_1^*, \sigma_2^* \) and \( \sigma_3^* \) on \( \text{Pic}(X) \) will provide a polarization for the system of three maps \( \{\sigma_1, \sigma_2, \sigma_3\} \). Also, under the condition that the Picard number is the least possible value \( p(X) = 3 \), the first dynamical degree of any of the maps \( \sigma_{ij} = \sigma_i \circ \sigma_j \) will be computed. The computations will produce the same dynamical degree as the dynamical degree of the maps on K3 surfaces (Section 12 of [11]).

2. **Four dimensional Varieties with three involutions**

Let \( L^A \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \) be a family of varieties defined over a field \( K \) by a single equation linear on each variable,

\[
L^A = \{ P \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x, y, z) = \sum_{i,j,k=0}^2 a_{i,j,k} x^i y^j z^k = 0 \},
\]

where \( A = (a_{ijk})_{0 \leq i,j,k \leq 2} \). A member of the family \( L \) comes equipped with projections

\[
p_3 = p_{xy} : L \to \mathbb{P}^2 \times \mathbb{P}^2,
\]

\[
p_2 = p_{xz} : L \to \mathbb{P}^2 \times \mathbb{P}^2,
\]

\[
p_1 = p_{yz} : L \to \mathbb{P}^2 \times \mathbb{P}^2,
\]

and the \( \text{Pic}(L) \cong \mathbb{Z}^2 \) from the embedding \( L \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \). Using the adjunction formula we can get its canonical line bundle

\[
\omega_L \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(-3, -3, -3) \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(L) = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(-2, -2, -2).
\]

By choosing a section \( Q = Q^4 \) of \( \mathcal{O}_L(2, 2, 2) \) and consider the variety \( X = \text{Var}(Q) \) we get a variety with trivial canonical divisor \( K_X \sim 0 \). Besides, by the weak lefschetz theorem, we have an injective map \( \mathbb{Z}^3 \cong \text{Pic}(L) \hookrightarrow \text{Pic}(X) \) and we will get three distinct classes even in \( \text{NS}(X) \) and therefore a Picard number \( p(X) \geq 3 \).
By varying the coefficients $A, B$ one obtains a family $X^{A,B}$ defined in $\mathbb{P}_K^2 \times \mathbb{P}_K^2 \times \mathbb{P}_K^2$ by equations

$$L(x, y, z) = \sum_{i,j,k=0}^2 a_{i,j,k}x_iy_jz_k = 0,$$
$$Q(x, y, z) = \sum_{i,j,k,l,m,n=0}^2 b_{i,j,k,l,m,n}x_ix_jy_jy_my_mz_nz_n = 0,$$

where $A = (a_{ijk})$, $B = (b_{i,j,k,l,m,n})$ and all indices are moving in the set $\{0, 1, 2\}$. The projections $p_1, p_2, p_3$ restricted to $X$ represent generically $(2 : 1)$ coverings of $\mathbb{P}^2 \times \mathbb{P}^2$. Indeed when we fix two of the variables we get the intersection on $\mathbb{P}^2$ of a quadric and a line, which is general, will give two points $P_i, P'_i \in X$ for $i = 1, 2, 3$ and will determine involutions $\sigma_1, \sigma_2, \sigma_3 : X \rightarrow X$. The involutions $\sigma_i$ for $i = 1, 2, 3$, will not be in general morphisms but just rational maps defined on certain open sets $U_i \subset X$. We are interesting in studying the dynamics of the maps $\sigma_i$, but first we should devote some time to get familiar with the geometry of $X = X^{A,B}$. We collect the coefficients of our variables using the following notation for $i, j, k$ in the set $\{0, 1, 2\}$

$$L^{x,y}_{k}(x, y) = \sum_{i,j=0}^2 a_{i,j,k}x_iy_j, \quad Q^{x,y}_{k}(x, y) = \sum_{i,j,l,m=0}^2 b_{i,j,k,l,m}x_iy_jy_my_m,$$
$$L^{x,z}_{j}(x, z) = \sum_{i,k=0}^2 a_{i,j,k}x_iz_k, \quad Q^{x,z}_{j}(y, z) = \sum_{i,j,k,l,m=0}^2 b_{i,j,k,l,m}y_jy_my_mz_nz_n,$$
$$L^{y,z}_{i}(y, z) = \sum_{j,k=0}^2 a_{i,j,k}y_jz_k, \quad Q^{y,z}_{i}(x, z) = \sum_{i,j,k,l,m,n=0}^2 b_{i,j,k,l,m,n}x_ix_iz_kz_n.$$

Suppose, with the above notation in mind, that we want to study the action of $\sigma_3$ computing the solutions $(z_0, z_1, 1)$ of the system

$$0 = L^{x,y}_{0}z_0 + L^{x,y}_{1}z_1 + L^{x,y}_{2},$$
$$0 = Q^{x,y}_{0}z_0^2 + Q^{x,y}_{1}z_1^2 + Q^{x,y}_{2}z_2 + Q^{x,y}_{0,1}z_0z_1 + Q^{x,y}_{0,2}z_0 + Q^{x,y}_{1,2}z_1,$$

assuming that $L^{x,y}_{1} \neq 0$ and replacing $z_1 = \frac{-L^{x,y}_{2}z_0 - L^{x,y}_{0}z_0^2}{L^{x,y}_{1}}$ in the second equation gives $G^{x,y}_{0} + H^{x,y}_{0,2}z_0 + G^{x,y}_{2}z_0^2 = 0$ where,

$$G^{x,y}_{0} = (L^{x,y}_{1})^2Q^{x,y}_{2,2} - L^{x,y}_{1}L^{x,y}_{2}L^{x,y}_{1,2}Q^{x,y}_{1,2} + (L^{x,y}_{2})^2Q^{x,y}_{1,1},$$
$$G^{x,y}_{2} = (L^{x,y}_{1})^2Q^{x,y}_{0,0} - L^{x,y}_{1}L^{x,y}_{0}L^{x,y}_{0,1}Q^{x,y}_{0,1} + (L^{x,y}_{1})^2Q^{x,y}_{1,1},$$
$$H^{x,y}_{0,2} = 2L^{x,y}_{0}L^{x,y}_{2}Q^{x,y}_{1,1} - L^{x,y}_{0}L^{x,y}_{1}L^{x,y}_{1,2}Q^{x,y}_{1,2} - L^{x,y}_{1}L^{x,y}_{1}Q^{x,y}_{0,1} + (L^{x,y}_{1})^2Q^{x,y}_{0,2},$$

and

$$Q^{x,y}_{0,1} = L^{x,y}_{1}L^{x,y}_{0}Q^{x,y}_{1,1}.$$
and the map $\sigma_3$ that sends $(z_0, z_1, 1) \mapsto (z'_0, z'_1, 1)$ will be defined unless all the three coefficients $G_{x}^{xy}, H_{x}^{xy}, G_{2}^{xy}$ vanish. So, we are forced, by a codimension checking, to work with rational maps $\sigma_i : X \rightarrow X$ and our first task will be, to locate where are these maps well defined morphisms.

Motivated by the above discussion we define for any permutation $(i, j, k)$ of $(0, 1, 2)$ the $(4, 4)$-bi-homogeneous forms

$$
G_{k}^{x,y} = (L_{i}^{x,y})^{2}Q_{j,j}^{x,y} - L_{i}^{x,y}L_{j}^{x,y}Q_{x,j}^{x,y} + (L_{j}^{x,y})^{2}Q_{i,i}^{x,y},
$$

$$
G_{k}^{y,z} = (L_{i}^{y,z})^{2}Q_{j,j}^{y,z} - L_{i}^{y,z}L_{j}^{y,z}Q_{x,j}^{y,z} + (L_{j}^{y,z})^{2}Q_{i,i}^{y,z},
$$

$$
G_{k}^{x,z} = (L_{i}^{x,z})^{2}Q_{j,j}^{x,z} - L_{i}^{x,z}L_{j}^{x,z}Q_{x,j}^{x,z} + (L_{j}^{x,z})^{2}Q_{i,i}^{x,z},
$$

$$
H_{i,j}^{x,y} = 2L_{i}^{x,y}L_{j}^{x,y}Q_{x,kk}^{x,y} - L_{i}^{x,y}L_{k}^{x,y}Q_{x,jk}^{x,y} - L_{j}^{x,y}L_{k}^{x,y}Q_{x,ik}^{x,y} + (L_{k}^{x,y})^{2}Q_{ij}^{x,y},
$$

$$
H_{i,j}^{y,z} = 2L_{i}^{y,z}L_{j}^{y,z}Q_{x,kk}^{y,z} - L_{i}^{y,z}L_{k}^{y,z}Q_{x,jk}^{y,z} - L_{j}^{y,z}L_{k}^{y,z}Q_{x,ik}^{y,z} + (L_{k}^{y,z})^{2}Q_{ij}^{y,z},
$$

$$
H_{i,j}^{x,z} = 2L_{i}^{x,z}L_{j}^{x,z}Q_{x,kk}^{x,z} - L_{i}^{x,z}L_{k}^{x,z}Q_{x,jk}^{x,z} - L_{j}^{x,z}L_{k}^{x,z}Q_{x,ik}^{x,z} + (L_{k}^{x,z})^{2}Q_{ij}^{x,z},
$$

For any $a, b, c \in \mathbb{P}^2$, the fibres of the projections $p_1, p_2$ and $p_3$ will be defined as $X_{a,b}^z = p_1^{-1}(a, b) = L_{a,b}^z \cap Q_{a,b}^z$, $X_{b,c}^x = p_2^{-1}(b, c) = L_{b,c}^x \cap Q_{b,c}^x$, and $X_{a,c}^y = p_3^{-1}(a, c) = L_{a,c}^y \cap Q_{a,c}^y$; where

$$
L_{a,b}^z = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, b, z) = 0\},
$$

$$
Q_{a,b}^z = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, b, z) = 0\},
$$

$$
L_{a,c}^x = \{(x, b, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x, b, c) = 0\},
$$

$$
Q_{a,c}^x = \{(x, c, b) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(x, b, c) = 0\},
$$

$$
L_{b,c}^y = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, y, c) = 0\},
$$

$$
Q_{a,b}^y = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, y, c) = 0\}.
$$

**Definition 2.1.** We say that a fibre $X_{a,b}^z, X_{b,c}^x$ or $X_{a,c}^y$ is degenerate if it has positive dimension.

If the fibres $X_{a,b}^z, X_{b,c}^x$ or $X_{a,c}^y$ are non-degenerate at $(a, b, c)$, they will consist of two points and the maps $\sigma_1, \sigma_2$ and $\sigma_3$ will be well defined morphisms at $(a, b, c) \in X$. Following the outline of [6] we have the following result characterizing the degenerate fibres.

**Proposition 2.2.** Let $[a, b, c] \in \mathcal{X}$.

1. $X_{a,b}^z$ is degenerate if and only if $G_{0}^{x,y}(a, b) = G_{1}^{x,y}(a, b) = G_{2}^{x,y}(a, b) = H_{0,1}^{x,y}(a, b) = H_{1,2}^{x,y}(a, b) = 0$.

2. $X_{a,c}^y$ is degenerate if and only if $G_{0}^{x,z}(a, c) = G_{1}^{x,z}(a, c) = G_{2}^{x,z}(a, c) = H_{0,1}^{x,z}(a, c) = H_{1,2}^{x,z}(a, c) = 0$. 


The proof is identical to the proof of proposition 1.4 in [6].

Now, the proof is divided into two parts, depending on whether or not for the point \([a, b, c]\) \(\in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\) we have \(L(a, b, z) \equiv 0\).

If \(L(a, b, z) \equiv 0\), then \(X^z_{a,b} = Q^z_{a,b}\) and the fibre is degenerate. In this case \(L_0 = L_0^{a,b} = L_0^{a,b} = 0\) will force \(H^{x,y}_{1,3}(a, b) = G^{x,y}_{k}(a, b) = 0\) and the proof is finished.

If \(L(a, b, z) \neq 0\), one of the \(L_i^{x,y}(a, b) \neq 0\) and the fact that \(G^{x,y}_{0}(a, b) = G^{x,y}_{1}(a, b) = G^{x,y}_{2}(a, b) = H^{x,y}_{0,1}(a, b) = H^{x,y}_{0,2}(a, b) = H^{x,y}_{1,2}(a, b) = 0\) forces \(Q(a, b, z) \equiv 0\) \((\text{mod} L(a, b, z))\) and hence \(X^z_{a,b}\) is degenerate containing the entire line \(L^z_{a,b}\).

If \(L(a, b, z) \neq 0\) and the fibre \(X^z_{a,b}\) is degenerate we must have \(L_{a,b}^z \subset Q^z_{a,b}\). We are going to proof that \(G^{x,y}_{0}(a, b) = G^{x,y}_{1}(a, b) = G^{x,y}_{2}(a, b) = H^{x,y}_{0,1}(a, b) = H^{x,y}_{0,2}(a, b) = H^{x,y}_{1,2}(a, b) = 0\). First let’s do \(G^{x,y}_{0}(a, b) = 0\).

If \(L_1^{x,y}(a, b) = L_1^{x,y}(a, b) = 0\), this follows from the definition, otherwise \((0, L_2^{x,y}(a, b), -L_2^{x,y}(a, b)) \in L_{a,b}^z\) and therefore must belong to \(Q^z_{a,b}\), when we evaluate we get

\[
0 = Q^{x,y}_{1,1}(a, b)(L_2^{x,y}(a, b))^2 - Q^{x,y}_{1,2}L_1^{x,y}(a, b)L_2^{x,y}(a, b) + Q^{x,y}_{2,2}(L_1^{x,y}(a, b))^2
\]

So \(G^{x,y}_{0}(a, b) = 0\). In a similar way we do \(G^{x,y}_1(a, b) = G^{x,y}_2(a, b) = 0\). The substitution of the results \(G^{x,y}_{k}(a, b) = 0\) in the equations and evaluations at \(x = a, y = b\) will give

\[
H^{x,y}_{1,2}(a, b)z_1z_2 = H^{x,y}_{0,2}(a, b)z_0z_2 = H^{x,y}_{1,0}(a, b)z_1z_0 = 0
\]

for all points \((z_0, z_1, z_2) \in L^z(a, b)\). If \(L_i^z(a, b)\) is the line \(z_i = 0\), then \(L_0^{x,y}(a, b) = L_2^{x,y}(a, b) = 0\) and \(H^{x,y}_{1,2}(a, b) = 0\) using the definition.

If \(L_1^z(a, b)\) is the line \(z_1 = 0\), then \(L_1^{x,y}(a, b) = L_2^{x,y}(a, b) = 0\) and \(H^{x,y}_{1,2}(a, b) = 0\) will be again equal to zero. Otherwise if \(L_{a,b}^z\) is none of the lines \(z_1 = 0\) or \(z_2 = 0\), then \(H^{x,y}_{1,2}(a, b) = 0\) from the previous line.

The other cases for \(H^{x,y}_{1,2}(a, b) = 0\) are solved similarly.
The prove of all parts will be analogous and straightforward. We have the following equivalences of divisors in $\text{Pic}(\sigma)$, representing the two fundamental classes in $\text{Pic}(\mathbb{P}^3)$:

$$H_{0,1}^x(a, c) = H_{0,2}^x(a, c) = H_{1,2}^x(a, c) = 0,$$

$$H_{0,1}^y(a, b) = H_{0,2}^y(a, b) = H_{1,2}^y(a, b) = 0.$$

The maps $\sigma_1, \sigma_2, \sigma_3$ induce maps on divisors: Let's consider $\sigma_i^*: \text{Pic}(X) \to \text{Pic}(X)$ and $\hat{\sigma}_i^*: \text{NS}(X)_Q \to \text{NS}(X)_Q$.

To study the action of the $\sigma_i^*$ on Pic($X$) we denote by $H, H'$ hyperplane sections representing the two fundamental classes in $\text{Pic}((\mathbb{P}^2 \times \mathbb{P}^2)$, $H = \{(a_0 : a_1 : a_2), (b_0 : b_1 : b_2) \in \mathbb{P}^2 \times \mathbb{P}^2 : a_0 = 0\}$, $H' = \{(a_0 : a_1 : a_2), (b_0 : b_1 : b_2) \in \mathbb{P}^2 \times \mathbb{P}^2 : b_0 = 0\}$.

The pullbacks of $H, H'$ by the different projections give back the $D_x, D_y, D_z$:

$$p_{xy}^*H = p_{xz}^*H = D_x, \quad p_{xy}^*H' = p_{xz}^*H' = D_y,$$

$$p_{yz}^*H = p_{yx}^*H = D_y, \quad p_{yz}^*H' = p_{yx}^*H' = D_z.$$

Lemma 2.3. We have the following equivalences of divisors in div($X$):

(a) $p_{1*}p_2^*H \sim 4H + 4H'$;

(b) $p_{2*}p_1^*H \sim 4H + 4H'$;

(c) $p_{3*}p_1^*H' \sim 4H + 4H'$.

Proof. The prove of all parts will be analogous and straightforward from the definition of $H, H'$ and the $p_i$'s. Let's see for example the proof of (a). The pull-back $p_2^*H = \{P \in X : x_0 = 0\}$ is given by the two equations

$$L_1^y x_1 + L_2^y x_2 = 0, \quad Q_{1,1}^y x_1^2 + Q_{1,2}^y x_1 x_2 + Q_{2,2}^y x_2^2 = 0.$$
When we project onto \((y, z)\) we eliminate \(x_1, x_2\) and get the equation

\[
G_0^{y,z} = (L_0^{y,z})^2Q_2^{y,z} - L_1^{y,z} L_2^{y,z} Q_1^{y,z} + (L_2^{y,z})^3Q_1^{y,z} = 0.
\]

where \(G_0^{y,z}\) is a \((4,4)\)-bihomogeneous form in \(y\) and \(z\), and therefore \(p_1, p_2 H \sim 4H + 4H'\).

Applying lemma 2.3 we obtain the pushforwards:

\[
p_1^*(D_x) \sim 4H + 4H', \quad p_2^*(D_y) \sim 4H + 4H', \quad p_3^*(D_z) \sim 4H + 4H',
\]

and the action of the \(\sigma^*_i\)'s on the divisors \(D_x, D_y, D_z\):

\[
\begin{align*}
\sigma_1^*(D_x) &= p_1^* p_1^* D_x - D_x \sim 4D_y + 4D_z - D_x, \\
\sigma_1^*(D_y) &= \sigma_1^* p_1^* H = (p_1 \circ \sigma_1)^* H = D_y, \\
\sigma_1^*(D_z) &= \sigma_1^* p_1^* H' = (p_1 \circ \sigma_1)^* H' = D_z, \\
\sigma_2^*(D_x) &= \sigma_2^* p_2^* H = (p_2 \circ \sigma_2)^* H' = D_x, \\
\sigma_2^*(D_y) &= p_2^* p_2^* D_y - D_y \sim 4D_x + 4D_z - D_y, \\
\sigma_2^*(D_z) &= \sigma_2^* p_2^* H' = (p_2 \circ \sigma_2)^* H' = D_z, \\
\sigma_3^*(D_x) &= \sigma_3^* p_3^* H = (p_3 \circ \sigma_3)^* H = D_x, \\
\sigma_3^*(D_y) &= \sigma_3^* p_3^* H' = (p_3 \circ \sigma_3)^* H' = D_y, \\
\sigma_3^*(D_z) &= p_3^* p_3^* D_z - D_z \sim 4D_x + 4D_y - D_z.
\end{align*}
\]

Using the actions of the \(\sigma^*_i\) we can get a polarizations by a very ample line bundle for the system of involutions \(\sigma_1, \sigma_2, \sigma_3\).

**Proposition 2.4.** Suppose that \(r_x, r_y, r_z\) are positive real numbers and we have the polarization by three maps

\[
\sum_i \sigma_i^*(r_x D_x + r_y D_y + r_z D_z) \sim d(r_x D_x + r_y D_y + r_z D_z),
\]

in \(\text{Pic}(X) \otimes \mathbb{R}\). Then \(d = 9\) and \(r_x = r_y = r_z = 1\).

**Proof.** When we add up the actions of \(\sigma_i^*\) on \(r_x D_x + r_y D_y + r_z D_z\), and equal that to \(d(r_x D_x + r_y D_y + r_z D_z)\) for some \(d > 3\), we get the system of linear equations:

\[
\begin{align*}
r_x + 4r_y + 4r_z &= dr_x, \\
4r_x + r_y + 4r_z &= dr_y, \\
4r_x + 4r_y + r_z &= dr_z.
\end{align*}
\]

The determinant is \((9 - d)(3 + d)^3\) and the value of \(d = 9\) gives \(r_x = r_y = r_z = 1\). 

**Proposition 2.5.** The maps \(\sigma_i\) and \(\sigma_{ij} = \sigma_i \circ \sigma_j\), for \(i, j \in \{0, 1, 2\}\), satisfy the properties:

\[
(1) \quad (\sigma_i \circ \sigma_j)^* = \sigma_j^* \circ \sigma_i^*,
\]
Given two rational maps \( \tau : X \rightarrow X \) and \( \tau' : X \rightarrow X \) defining involutions \( \tau : U_\tau \rightarrow U_\tau \) and \( \tau' : U_{\tau'} \rightarrow U_{\tau'} \) on open sets \( U_\tau \) and \( U_{\tau'} \) respectively, we will have \((\tau \circ \tau')^* = \tau'^* \circ \tau^* \). Let \( Y \) be an irreducible subvariety. If \( P \in \tau(Y \cap U_\tau) \cap U_{\tau'} \), there exist a sequence \( P_n \rightarrow P \), with \( P_n \in \tau(Y \cap U_\tau) \cap U_{\tau'} \). Therefore, \( \tau'(P_n) \rightarrow \tau'(P) \) and \( \tau'(P) \in \tau'(\tau(Y \cap U_\tau) \cap U_{\tau'}) \). In other words \( \tau'(\tau(Y \cap U_\tau) \cap U_{\tau'}) \subset \tau'(\tau(Y \cap U_\tau) \cap U_{\tau'}) \), so this two sets must be equal and \((\tau \circ \tau')^* = \tau'^* \circ \tau^* \). For the first part of the theorem we take \( \sigma = \tau \) and \( \sigma_j = \tau' \).

For the second part we proceed by induction and use the result to proof the induction step. If we suppose that \((\sigma_{ij}^n)^* = (\sigma_{ij}^n)^* \) is true, then \((\sigma_{ij}^{n+1})^* = \sigma_{ij}^n((\sigma_{ij}^n)^*) = \sigma_{ij}^n((\sigma_{ij}^n)^*)\). By our result above with \( \tau = \sigma_{ij} \) and \( \tau' = \sigma_{ij}^n \), the last equals to \((\sigma_{ij}^{n+1})^* \).

### 2.1. Computation of dynamical degree

In this subsection we study the action induced by the maps \( \sigma_{ij} = \sigma_i \circ \sigma_j \) on the subspace \( V = \text{Span}(D_x, D_y, D_z) \) of \( \text{Pic}(X) \otimes \mathbb{R} \). As an application we will be able to get the dynamical degree of those maps for members of the family with Picard number \( p(X) = 3 \).

#### Theorem 2.6. Let \( \sigma_{ij} \) be the rational dominant map \( \sigma_i \circ \sigma_j : X \rightarrow X \).

Let \( V \) be the subspace of \( \text{Pic}(X) \otimes \mathbb{R} \) spanned by \( D_x, D_y, D_z \) and consider the action of \( \sigma_{ij}^n : V \rightarrow V \). The eigenvalues of \( \sigma_{ij}^n | V \) belong to the set \( \{1, \beta^n, \beta'^n\} \), where \( \beta = 7 + 4\sqrt{3} \) and \( \beta' = \frac{1}{\beta} \).

#### Proof. The action of the maps \( \sigma_{12}^*, \sigma_{31}^*, \sigma_{31}^*, \sigma_{13}^*, \sigma_{13}^*, \sigma_{23}^* \) with respect to that base \( \{D_x, D_y, D_z\} \) is given respectively by the matrices

\[
\sigma_{12}^* = \begin{pmatrix} -1 & -4 & 0 \\ 4 & 15 & 0 \\ 4 & 20 & 1 \end{pmatrix}, \quad \sigma_{13}^* = \begin{pmatrix} 15 & 0 & 4 \\ 20 & 1 & 4 \\ -4 & 0 & -1 \end{pmatrix}, \\
\sigma_{12}^* = \begin{pmatrix} 15 & 4 & 0 \\ -4 & -1 & 0 \\ 20 & 4 & 1 \end{pmatrix}, \quad \sigma_{23}^* = \begin{pmatrix} 1 & 20 & 4 \\ 0 & 15 & 4 \\ 0 & -4 & -1 \end{pmatrix}, \\
\sigma_{31}^* = \begin{pmatrix} -1 & 0 & -4 \\ 4 & 1 & 20 \\ 4 & 0 & 15 \end{pmatrix}, \quad \sigma_{32}^* = \begin{pmatrix} 1 & 4 & 20 \\ 0 & -1 & -4 \\ 0 & 4 & 15 \end{pmatrix},
\]

With the help of SAGE we find that the six matrices are sharing the same characteristic polynomial \( p = -(\lambda - 1)(\lambda^2 - 14\lambda + 1) \). The roots of \( p(\lambda) \) are \( \{1, \beta, \beta'\} \) with \( \beta = 7 + 4\sqrt{3} \) and \( \beta' = 1/\beta \), therefore all the six matrices are diagonalizable and the eigenvalues of the the powers are from the set \( \{1, \beta^n, \beta'^n\} \).
Corollary 2.7. Suppose that the Picard number $p(X) = 3$, then the first dynamical degree $\delta_{\sigma_{ij}}$ of $\sigma_{ij}$ is $\delta_{\sigma_{ij}} = \beta$.

Proof. The divisors $D_x, D_y, D_z$ represent three distinct classes in $NS(X)_Q$. If the Picard number $p(X) = 3$, then we have $NS(X)_Q \cong V_3$. The first dynamical degree of any of the maps $\sigma_{ij}$ is:

$$\delta_{\sigma_{ij}} = \limsup_{n \to \infty} \rho((\sigma_{ij}^n)^*)^{1/n} = \limsup_{n \to \infty} \rho((\sigma_{ij}^*)^n)^{1/n} = \limsup_{n \to \infty} (\beta^n)^{1/n} = \beta.$$ 

\[ \square \]

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