Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderón-Zygmund operators

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Abstract Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$. In this article, we introduce the weak Hardy-type space $WH_X(\mathbb{R}^n)$, associated with $X$, via the radial maximal function. Assuming that the powered Hardy-Littlewood maximal operator satisfies some Fefferman-Stein vector-valued maximal inequality on $X$ as well as it is bounded on both the weak ball quasi-Banach function space $WX$ and the associated space, we then establish several real-variable characterizations of $WH_X(\mathbb{R}^n)$, respectively, in terms of various maximal functions, atoms and molecules. As an application, we obtain the boundedness of Calderón-Zygmund operators from the Hardy space $H_X(\mathbb{R}^n)$ to $WH_X(\mathbb{R}^n)$, which includes the critical case. All these results are of wide applications. Particularly, when $X := M_{pq}^{\infty}(\mathbb{R}^n)$ (the Morrey space), $X := L_{p}\bar{p}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space) and $X := (E_{pq}^{\infty}(\mathbb{R}^n)$ (the Orlicz-slice space), which are all ball quasi-Banach function spaces rather than quasi-Banach function spaces, all these results are even new. Due to the generality, more applications of these results are predictable.

Keywords ball quasi-Banach function space, weak Hardy space, Orlicz-slice space, maximal function, atom, molecule, Calderón-Zygmund operator

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1 Introduction

It is well known that the real-variable theory of the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which was introduced by Stein and Weiss [75] and further developed by Fefferman and Stein [26], plays a key role in harmonic analysis and partial differential equations. These works [26, 75] inspire many new ideas for the real-variable theory of function spaces. It is worth pointing out that the real-variable characterizations of classical Hardy spaces reveal the intrinsic connections among some important notions in harmonic
and proved that the aforementioned operator Hardy space in the critical case. Recently, He [35] and Grafakos and He [34] further studied the vector-valued weak atomic characterization of classical Hardy spaces Musielak-Orlicz type Hardy space unified real-variable theory for Hardy spaces associated with ball quasi-Banach function spaces on very recently, by Sawano et al. [70]. Moreover, Sawano et al. [70] and Wang et al. [79] established a unified real-variable theory for Hardy spaces associated with ball quasi-Banach function spaces on \( \mathbb{R}^n \) and gave some applications of these Hardy-type spaces to the boundedness of Calderón-Zygmund operators and pseudo-differential operators. More function spaces based on ball quasi-Banach function spaces can be found in [69].

Recall that ball quasi-Banach function spaces generalize quasi-Banach function spaces. Compared with quasi-Banach function spaces, ball quasi-Banach function spaces contain more function spaces. For example, the Morrey spaces are ball quasi-Banach function spaces, which are not quasi-Banach function spaces and hence the class of quasi-Banach function spaces is a proper subclass of ball quasi-Banach function spaces (see [70] for more details). Let \( X \) be a ball quasi-Banach function space (see [70] or Definition 2.3 below). Sawano et al. [70] introduced the Hardy space \( H_X(\mathbb{R}^n) \) via the grand maximal function (see [70] or Definition 6.1 below). Assuming that the Hardy-Littlewood maximal function is bounded on the \( p \)-convexification of \( X \), Sawano et al. [70] established several different maximal function characterizations of \( H_X(\mathbb{R}^n) \). On the other hand, Coifman [18] and Latter [50] found the most useful atomic characterization of classical Hardy spaces \( H^p(\mathbb{R}^n) \), which plays an important role in developing the real-variable theory of Hardy spaces. Sawano et al. [70] found that these atomic characterizations strongly depend on the Fefferman-Stein vector-valued maximal inequality and the boundedness on the associated space of the powered Hardy-Littlewood maximal operator.

Recall that, to find the biggest function space \( \mathcal{A} \) such that Calderón-Zygmund operators are bounded from \( \mathcal{A} \) to \( W^{1,1}(\mathbb{R}^n) \), Fefferman and Soria [27] originally introduced the weak Hardy space \( WH^1(\mathbb{R}^n) \) and they did obtain the boundedness of the convolutional Calderón-Zygmund operator with kernel satisfying the Dini condition from \( WH^1(\mathbb{R}^n) \) to \( W^{1,1}(\mathbb{R}^n) \) by using the \( \infty \)-atomic characterization of \( WH^1(\mathbb{R}^n) \). It is well known that the classic Hardy spaces \( H^p(\mathbb{R}^n) \), with \( p \in (0, 1] \), are good substitutes of Lebesgue spaces \( L^p(\mathbb{R}^n) \) when studying the boundedness of some Calderón-Zygmund operators. For example, if \( \delta \in (0, 1] \) and \( T \) is a convolutional \( \delta \)-type Calderón-Zygmund operator, then \( T \) is bounded on \( H^p(\mathbb{R}^n) \) for any given \( p \in (n/(n + \delta), 1] \) (see [4]). However, this is not true when \( p = n/(n + \delta) \) which is called the critical case or the endpoint case. Liu [54] introduced the weak Hardy space \( WH^p(\mathbb{R}^n) \) with \( p \in (0, 1] \) and proved that the aforementioned operator \( T \) is bounded from \( H^{n/(n+\delta)}(\mathbb{R}^n) \) to \( WH^{n/(n+\delta)}(\mathbb{R}^n) \) by first establishing the \( \infty \)-atomic characterization of the weak Hardy space \( WH^p(\mathbb{R}^n) \). Thus, the classical weak Hardy spaces \( WH^p(\mathbb{R}^n) \) play an irreplaceable role in the study of the boundedness of operators in the critical case. Recently, He [35] and Grafakos and He [34] further studied the vector-valued weak Hardy space \( WH^{p, \infty}(\mathbb{R}^n, \ell^2) \) with \( p \in (0, \infty) \). In 2016, Liang et al. [52] (see also [82]) considered the weak Musielak-Orlicz type Hardy space \( WH^p(\mathbb{R}^n) \), which covers both the weak Hardy space \( WH^{p, \infty}(\mathbb{R}^n) \) and the weighted weak Hardy space \( WH^p_\delta(\mathbb{R}^n) \) from [65], and obtained various equivalent characterizations of \( WH^p(\mathbb{R}^n) \), respectively, in terms of maximal functions, atoms, molecules and Littlewood-Paley functions, as well as the boundedness of Calderón-Zygmund operators in the critical case. Meanwhile, Yan et al. [81] developed a real-variable theory of variable weak Hardy spaces \( WH^{p, \infty}(\mathbb{R}^n) \) with \( p(\cdot) \in C^0(\mathbb{R}^n) \).

Let \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \) introduced by Sawano et al. [70]. In this article, we introduce the weak Hardy-type space \( WH_X(\mathbb{R}^n) \), via the radial maximal function, associated with \( X \). Assuming that the powered Hardy-Littlewood maximal operator satisfies some Fefferman-Stein vector-valued maximal inequality on \( X \) as well as it is bounded on both the weak ball quasi-Banach
function space $WX$ and the associated space, we then establish some real-variable characterizations of $W^Y_X(\mathbb{R}^n)$, respectively, in terms of various maximal functions, atoms and molecules. Using the atomic characterization of $H_X(\mathbb{R}^n)$, we further obtain the boundedness of Calderón-Zygmund operators from the Hardy space $H^+_X(\mathbb{R}^n)$ to $W^Y_X(\mathbb{R}^n)$, which includes the critical case. All these results are of wide applications and, particularly, when $X := M^\Phi(\mathbb{R}^n)$ (the Morrey space) introduced by Morrey [61] (see Definition 7.1 below), $X := L^\Phi(\mathbb{R}^n)$ (the mixed-norm Lebesgue space) (see, for example, [8, 41] or Definition 7.20 below) and $X := (E^\Phi_q)_s(\mathbb{R}^n)$ (the Orlicz-slice space) introduced in [86] (or see Definition 7.42 below), all these results are even new.

Recall that the atomic and the molecular characterizations of the Hardy-type space $H^+_X(\mathbb{R}^n)$ obtained in [70] strongly depend on the boundedness on the associate space $(X^{1/r})'$ of the powered Hardy-Littlewood maximal operator (see (4.16)) which requires that $X^{1/r}$ be a ball Banach function space. However, this approach is no longer feasible for $W^+H_X(\mathbb{R}^n)$ because, generally, $(WX)^{1/r}$ for any given $r \in (0, \infty)$ is not a ball Banach function space and hence we cannot assume that the powered Hardy-Littlewood maximal operator is bounded on $[(WX)^{1/r}]'$. Another challenge is that we cannot use the method in [70] to prove that the distribution in $W^+H_X(\mathbb{R}^n)$ vanishes weakly at infinity because $WX$ does not have an absolutely continuous quasi-norm (see [9, Definition 3.1] or [36, Definition 2.4] for the definition of absolutely continuous quasi-norms) which is crucial. Recall that $f \in S'((\mathbb{R}^n)\omega)$ vanishing weakly at infinity is a necessary condition to establish a Calderón reproducing formula (see Lemma 4.4 below). Moreover, since $X$ may not have an absolutely continuous quasi-norm, when we prove that the convolutional $\delta$-type and the non-convolutional $\gamma$-order Calderón-Zygmund operators are bounded from $H^+_X(\mathbb{R}^n)$ to $W^+H_X(\mathbb{R}^n)$ including the critical case, we cannot apply the standard density argument used in [81, Theorem 7.4] to define the Calderón-Zygmund operators on $W^+H_X(\mathbb{R}^n)$, which is also crucial. To overcome all these challenges, we need to employ several different methods and all of them need to use the weighted Lebesgue space $L^\omega_s(\mathbb{R}^n)$ with $\omega := [M(1_B(\mathbb{R}^n))]'$ and $s, \epsilon \in (0, 1)$. Roughly speaking, one can embed $X$ into $L^\omega_s(\mathbb{R}^n)$ (see Lemma 2.17 below) which has a reverse Hölder property and an absolutely continuous quasi-norm, and hence can help us to escape all the above difficulties.

Also, to limit the length of this article, applying these characterizations of $W^+H_X(\mathbb{R}^n)$ in this article, Wang et al. [80] established various Littlewood-Paley function characterizations of $W^+H_X(\mathbb{R}^n)$ and proved that the real interpolation intermediate space $(H^+_X(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta, \infty)}$, between $H^+_X(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, is $W^+H^+_{X, 1/(1-\theta)}(\mathbb{R}^n)$, where $\theta \in (0, 1)$. These results in [80] are also of wide applications; particularly, when $X := M^\Phi(\mathbb{R}^n)$ (the Morrey space), $X := L^\Phi(\mathbb{R}^n)$ (the mixed-norm Lebesgue space) and $X := (E^\Phi_q)_s(\mathbb{R}^n)$ (the Orlicz-slice space), all these results are even new; when $X := L^\Phi_\omega(\mathbb{R}^n)$ (the weighted Orlicz space), the result on the real interpolation is new and, when $X := L^\Phi(\mathbb{R}^n)$ (the variable Lebesgue space) and $X := L^\Phi_\omega(\mathbb{R}^n)$, the Littlewood-Paley function characterizations of $W^+H^+_X(\mathbb{R}^n)$ obtained in [80] improve the existing results by weakening the assumptions on the Littlewood-Paley functions (see [80] for more details). It is easy to see that, due to the generality, more applications of these results obtained both in the present article and [80] are predictable.

To be precise, the rest of this article is organized as follows.

In Section 2, we recall some notions concerning the ball (quasi)-Banach function space $WX$ and the weak ball (quasi)-Banach function space $WX$. Then we state the assumptions of the Fefferman-Stein vector-valued maximal inequality on $X$ (see Assumption 2.18 below) and the boundedness on the $p$-convexification of $WX$ for the Hardy-Littlewood maximal operator (see Assumption 2.20). Finally, in Definition 2.21 below, we introduce the weak Hardy space $W^HX(\mathbb{R}^n)$ via the radial grand maximal function.

Under the assumption about the boundedness on the $p$-convexification of $WX$ for the Hardy-Littlewood maximal operator (see (4.16)), we establish various real-variable characterizations of $W^HX(\mathbb{R}^n)$ in Theorem 3.2 below, respectively, in terms of the radial maximal function, the grand maximal function, the non-tangential maximal function, the maximal function of Peetre type and the grand maximal function of Peetre type (see Definition 3.1 below). If $WX$ satisfies an additional assumption (3.7) (namely, the $WX$-norm of the characteristic function of any unit ball of $\mathbb{R}^n$ has a low bound), we then characterize $W^HX(\mathbb{R}^n)$ by means of the non-tangential maximal function with respect to Poisson kernels in
Theorem 3.3 below. Moreover, the relations between $WX$ and $WH_X(\mathbb{R}^n)$ are also clarified in this section.

Section 4 is devoted to establishing the atomic characterization of $WH_X(\mathbb{R}^n)$. Under the assumption that $X$ satisfies the Fefferman-Stein vector-valued inequality and is $\vartheta$-concave for some $\vartheta \in (1, \infty)$, we show that any $f \in WH_X(\mathbb{R}^n)$ has an atomic decomposition in terms of $(X, \infty, d)$-atoms in Theorem 4.2 below. Recall that the atomic decomposition of $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ was obtained via a dense argument which does not work for the atomic decomposition of $WH^p(\mathbb{R}^n)$ due to the lack of a suitable dense subset of $WH^p(\mathbb{R}^n)$. We have the same problem for $WH_X(\mathbb{R}^n)$. To overcome this difficulty, we obtain the atomic decomposition of $WH_X(\mathbb{R}^n)$ via using some ideas from [12, 52, 81], namely, in the proof of Theorem 4.2, we need to use the fact that $X$ continuously embeds into $L_p^\infty(\mathbb{R}^n)$ (see Lemma 2.17 below), the global Calderón reproducing formula in $S'(\mathbb{R}^n)$ (see Lemma 4.4 below), the generalized Campanato space, and the Banach-Alaoglu theorem. To obtain the reconstruction theorem in terms of $(X, q, d)$-atoms (see Theorem 4.7), we need to further assume that $X$ is strictly $r$-convex for any $r \in (0, p_-)$, where $p_-$ is as in Assumption 2.18, and the boundedness on the associate space of the powered Hardy-Littlewood maximal operator (4.16), besides the Fefferman-Stein vector-valued inequality.

In Section 5, we establish the molecular characterization of $WH_X(\mathbb{R}^n)$ in Theorems 5.2 and 5.3 below with all the same assumptions as in the atomic decomposition theorem (see Theorem 4.2) and the reconstruction theorem (see Theorem 4.7). Since each atom of $WH_X(\mathbb{R}^n)$ is also a molecule of $WH_X(\mathbb{R}^n)$, to prove Theorem 5.3, it suffices to show that the weak molecular Hardy space $WH_{mol}^{X,q,d,\epsilon}(\mathbb{R}^n)$ is continuously embedded into $WH_X(\mathbb{R}^n)$ due to Theorems 4.2 and 4.7. To this end, a key step is to prove that an $(X, q, d, \epsilon)$-molecule can be divided into an infinite linear combination of $(X, q, d)$-atoms. We show this via borrowing some ideas from the proof of [81, Theorem 5.3].

Section 6 is devoted to proving that both the convolutional $\delta$-type Calderón-Zygmund operator and the non-convolutional $\gamma$-order Calderón-Zygmund operator are bounded from $H_X(\mathbb{R}^n)$ to $WH_X(\mathbb{R}^n)$ in the critical case when $p_- = \frac{n}{n+\gamma}$ or when $p_- = \frac{n}{n+\gamma}$ (see Theorems 6.3 and 6.4 below). In this case, any convolutional $\delta$-type or any non-convolutional $\gamma$-order Calderón-Zygmund operator may not be bounded on $H_X(\mathbb{R}^n)$ even when $X = L^p(\mathbb{R}^n)$ with $p \in (0, 1]$. In this sense, the space $WH_X(\mathbb{R}^n)$ is a proper substitution of $H_X(\mathbb{R}^n)$ in the critical case for the study on the boundedness of some operators.

In Section 7, we apply the above results to Morrey spaces, mixed-norm Lebesgue spaces and Orlicz-slice spaces, respectively, in Subsections 7.1–7.3. They are ball Banach function spaces rather than Banach function spaces.

Recall that, due to the applications in elliptic partial differential equations, the Morrey space $M^p_q(\mathbb{R}^n)$ with $0 < q \leq p < \infty$ was introduced by Morrey [61] in 1938. In recent decades, there exists an increasing interest in applications of Morrey spaces to various areas of analysis, such as partial differential equations, potential theory and harmonic analysis (see, for example, [2, 3, 16, 47, 51, 59, 85]). Particularly, Jia and Wang [47] introduced the Hardy-Morrey spaces and established their atomic characterizations. Later, based on the Morrey space, various variants of Hardy-Morrey spaces have been introduced and developed, such as weak Hardy-Morrey spaces (see [40]), variable Hardy-Morrey spaces (see [38]) and Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces (see [68]). Observe that, as was pointed out in [70, p. 86], $M^p_q(\mathbb{R}^n)$ with $1 \leq q < p < \infty$, which violates (2.1) below (see [72, Example 3.3]), is not a Banach function space as in Definition 2.1, but it does be a ball Banach function space as in Definition 2.3. In Subsection 7.1, We first recall some of the useful properties of Morrey spaces. Borrowing some ideas from [77], we establish a weak-type vector-valued inequality of the Hardy-Littlewood maximal operator $\mathcal{M}$ from the Morrey space $M^p_q(\mathbb{R}^n)$ to the weak Morrey space $W M^p_q(\mathbb{R}^n)$ with $p \in (1, \infty)$ (see Proposition 7.16 below), which may be of independent interest and applicable to many other analysis problems. From this and the results in [16,38,40], we can easily show that all the assumptions of main theorems in Sections 3–6 are satisfied. Thus, applying these theorems, we obtain the atomic and the molecular characteristics of weak Hardy-Morrey spaces and the boundedness of Calderón-Zygmund operators from the Hardy-Morrey spaces to the weak Hardy-Morrey spaces including the critical case.

The study of the mixed-norm Lebesgue space $L^\vec{p}(\mathbb{R}^n)$ with $\vec{p} \in (0, \infty]^n$ originated from Benedek and Panzone [8] in the early 1960s, which can be traced back to Hörmander [41]. Later on, in 1970, Lizorkin [55] further developed both the theory of multipliers of Fourier integrals and estimates of con-
olutions in the mixed-norm Lebesgue spaces. Particularly, in order to meet the requirements arising in the study of the boundedness of operators, partial differential equations and some other fields, the real-variable theory of mixed-norm function spaces, including mixed-norm Morrey spaces, mixed-norm Hardy spaces, mixed-norm Besov spaces and mixed-norm Triebel-Lizorkin spaces, has rapidly been developed in recent years (see, for example, [17, 31, 43–45, 64]). Observe that \( L^\vec{p} (\mathbb{R}^n) \) when \( \vec{p} \in (0, \infty)^n \) is a ball quasi-Banach function space, but, it is not a quasi-Banach function space (see Remark 7.21 below).

In Subsection 7.2, to establish a vector-valued inequality of the Hardy-Littlewood maximal operator \( M \) on the weak mixed-norm Lebesgue space \( WL^\vec{p} (\mathbb{R}^n) \) with \( \vec{p} \in (1, \infty)^n \) (see Theorem 7.25 below), we first establish an interpolation theorem of sublinear operators on the space \( WL^\vec{p} (\mathbb{R}^n) \). Then, via an extrapolation theorem (see Lemma 7.34 below) which is a slight variant of a special case of [21, Theorem 4.6], we establish a vector-valued inequality of the Hardy-Littlewood maximal operator \( M \) from \( L^\vec{p} (\mathbb{R}^n) \) to \( WL^\vec{p} (\mathbb{R}^n) \) with \( \vec{p} \in [1, \infty)^n \) (see Proposition 7.33 below).

Since all the assumptions of main theorems in Sections 3–6 are satisfied, applying these theorems, we obtain the atomic and the molecular characterizations of weak Hardy-Morrey spaces and the boundedness of Calderón-Zygmund operators from the mixed-norm Hardy spaces to the weak mixed-norm Hardy spaces including the critical case.

In Subsection 7.3, let \( q, t \in (0, \infty) \) and \( \Phi \) be an Orlicz function. Recall that the Orlicz-slice space \( (E^\tau_q)_t(\mathbb{R}^n) \) introduced in [86] generalizes both the slice space \( E^\tau_p(\mathbb{R}^n) \) (in this case, \( \Phi(\tau) := \tau^2 \) for any \( \tau \in [0, \infty) \)), which was originally introduced by Auscher and Mourougoglou [6] and has been applied to study the classification of weak solutions in the natural classes for the boundary value problems of a time independent elliptic system in the upper plane, and \( (E^\tau_p)_t(\mathbb{R}^n) \) (in this case, \( \Phi(\tau) := \tau^r \) for any \( \tau \in [0, \infty) \)), which was originally introduced by Auscher and Prisuelos-Arribas [7] and has been applied to study the boundedness of operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator and the Riesz potential. The Orlicz-slice space \( (E^\tau_q)_t(\mathbb{R}^n) \) is a ball quasi-Banach function space; however, they may not be a quasi-Banach function space (see Remark 7.43(i) for more details). Moreover, Zhang et al. [86] introduced the Orlicz-slice Hardy space \( (HE^\tau_q)_t(\mathbb{R}^n) \) and obtained real-variable characterizations of \( (HE^\tau_q)_t(\mathbb{R}^n) \), respectively, in terms of various maximal functions, atoms, molecules and Littlewood-Paley functions, and the boundedness on \( (HE^\tau_q)_t(\mathbb{R}^n) \) for convolutional \( \delta \)-order and non-convolutional \( \gamma \)-order Calderón-Zygmund operators. Naturally, this new scale of Orlicz-slice Hardy spaces contains the variant of the Hardy-amalgam space (in this case, \( t = 1 \) and \( \Phi(\tau) := \tau^p \) for any \( \tau \in [0, \infty) \) with \( p \in (0, \infty) \)) of de Paul Ablé and Feuto [23] as a special case. Note that amalgam spaces become more and more important in partial differential equations (see, for example, [19, 20, 60]). Moreover, the results in [86] indicate that, similarly to the classical Hardy space \( H^p(\mathbb{R}^n) \) with \( p \in (0, 1] \), \( (HE^\tau_q)_t(\mathbb{R}^n) \) is a good substitute of \( (E^\tau_q)_t(\mathbb{R}^n) \) in the study of the boundedness of operators. On another hand, observe that \( (E^\tau_q)_t(\mathbb{R}^n) \) when \( p = t = 1 \) goes back to the amalgam space \( (L^\Phi, \ell^1)(\mathbb{R}^n) \) introduced by Bonami and Feuto [10], where \( \Phi(t) := \frac{1}{\|t\|_{\ell^1}} \) for any \( t \in [0, \infty) \), and the Hardy space \( H^\Phi(\mathbb{R}^n) \) associated with the amalgam space \( (L^\Phi, \ell^1)(\mathbb{R}^n) \) was applied by Bonami and Feuto [10] to study the linear decomposition of the product of the Hardy space \( H^1(\mathbb{R}^n) \) and its dual space \( \text{BMO}(\mathbb{R}^n) \).

Another main motivation to introduce \( (HE^\tau_q)_t(\mathbb{R}^n) \) in [86] exists in that it is a natural generalization of \( H^\Phi(\mathbb{R}^n) \) in [10]. In the last part of this section, we focus on the weak Orlicz-slice Hardy space \( (WHE^\tau_q)_t(\mathbb{R}^n) \) built on the Orlicz-slice space \( (E^\tau_q)_t(\mathbb{R}^n) \), which is actually the starting point of this article. We first recall some of the useful properties of Orlicz-slice spaces. To obtain the atomic characterization of \( (WHE^\tau_q)_t(\mathbb{R}^n) \), we only need to show that the powered Hardy-Littlewood maximal operator is bounded on the weak Orlicz-slice space \( (WHE^\tau_q)_t(\mathbb{R}^n) \) (see Definition 7.44 below), because \( (E^\tau_q)_t(\mathbb{R}^n) \), as a ball quasi-Banach space, has been proved, in [86], to satisfy all the other assumptions required in Theorems 4.2 and 4.7. To this end, we first establish an interpolation theorem of Marcinkiewicz type for sublinear operators on \( (WHE^\tau_q)_t(\mathbb{R}^n) \) (see Theorem 7.46 below). As a corollary, we immediately obtain the vector-valued inequality of the Hardy-Littlewood maximal operator \( M \) on \( (WHE^\tau_q)_t(\mathbb{R}^n) \). To prove Theorem 7.46, differently from the proofs of [52, Theorem 2.5] and [81, Theorem 3.1], we cannot directly apply the Fubini theorem. We overcome this difficulty by establishing a Minkowski type inequality mixed with the norms of both the Lebesgue space \( L^1(\mathbb{R}^n) \) and the Orlicz space \( L^\Phi(\mathbb{R}^n) \) with the lower type \( p_{\Phi} \in (1, \infty) \) (see Lemma 7.45 below). As an application, we obtain the boundedness of Calderón-
Zygmund operators from the Orlicz-slice Hardy space \( (HE^q_{q'})_i(\mathbb{R}^n) \) to \( (WHE^q_{q'})_i(\mathbb{R}^n) \) in the critical case. To this end, applying Theorems 6.3 and 6.4, we only need to establish the Fefferman-Stein vector-valued inequality for the Hardy-Littlewood maximal operator from \( (E^q_{q'})_i(\mathbb{R}^n) \) to \( (WE^q_{q'})_i(\mathbb{R}^n) \). We do this by borrowing some ideas from [86].

Finally, we make some conventions on notation. Let \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \) and \( \mathbb{Z}_+^n := (\mathbb{Z}_+)^n \). We always denote by \( C \) a positive constant which is independent of the main parameters, but it may vary from line to line. We also use \( C_{(\alpha, \beta, \ldots)} \) to denote a positive constant depending on the indicated parameters \( \alpha, \beta, \ldots \). The symbol \( f \asymp g \) means that \( f \leq Cg \) \( \text{and} \ f \geq Cg \). If \( f \asymp g \) and \( g = h \) \( \text{or} \ g \leq h \), then we write \( f \approx g \). We also use the following convention: If \( f \leq Cg \) \( \text{and} \ g = h \) \( \text{or} \ g \leq h \), rather than \( f \approx g = h \) or \( f \approx g \leq h \). The symbol \[ \lceil s \rceil \] (resp. \( \lfloor s \rfloor \)) for any \( s \in \mathbb{R} \) denotes the maximal (resp. minimal) integer not greater (resp. less) than \( s \). We use \( \delta_n \) to denote the origin of \( \mathbb{R}^n \) and let \( \mathbb{R}^{n+1} := \mathbb{R}^n \times (0, \infty) \). If \( E \) is a subset of \( \mathbb{R}^n \), we denote by \( 1_E \) its characteristic function and by \( E^c \) the set \( \mathbb{R}^n \setminus E \). For any cube \( Q := Q(x_Q, l_Q) \subset \mathbb{R}^n \), with center \( x_Q \in \mathbb{R}^n \) \( \text{and} \ l_Q \in (0, \infty) \), and \( \alpha \in (0, \infty) \), let \( \alpha Q := Q(x_Q, \alpha l_Q) \). Denote by \( \mathcal{Q} \) the set of all cubes having their edges parallel to the coordinate axes. For any \( \theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n \), let \( |\theta| := \theta_1 + \cdots + \theta_n \). Furthermore, for any cube \( Q \) in \( \mathbb{R}^n \) \( \text{and} \ j \in \mathbb{Z}_+^n \), let \( S_j(Q) := (2^{j+1}Q) \setminus (2^jQ) \) with \( j \in \mathbb{N} \) \( \text{and} \ S_0(Q) := 2Q \). Finally, for any \( q \in [1, \infty] \), we denote by \( q' \) its conjugate exponent, namely, \( 1/q + 1/q' = 1 \).

## 2 Preliminaries

In this section, we present some notions and preliminary facts on ball quasi-Banach function spaces.

### 2.1 Ball quasi-Banach function spaces

Denote by the symbol \( \mathcal{M}(\mathbb{R}^n) \) the set of all measurable functions on \( \mathbb{R}^n \). Let us first recall the notion of Banach function spaces (see, for example, [9, Chapter 1, Definitions 1.1 and 1.3]).

**Definition 2.1.** A Banach space \( Y \subset \mathcal{M}(\mathbb{R}^n) \) is called a Banach function space if the norm \( \| \cdot \|_Y \) is a Banach function norm, i.e., for all measurable functions \( f, g \) \( \text{and} \ \{f_m\}_{m \in \mathbb{N}} \), the following properties hold true:

1. \( \|f\|_Y = 0 \) if and only if \( f = 0 \) almost everywhere;
2. \( \|g\| \leq |f| \) almost everywhere implies that \( \|g\|_Y \leq \|f\|_Y \);
3. \( 0 \leq f_m \uparrow f \) almost everywhere implies that \( \|f_m\|_Y \uparrow \|f\|_Y \);
4. \( 1_E \in Y \) for any measurable set \( E \subset \mathbb{R}^n \) with finite measure;
5. \( \text{For any measurable set} \ E \subset \mathbb{R}^n \text{with finite measure, there exists a positive constant} \ C_{(E)} \text{, depending on} \ E, \text{such that for any} \ f \in Y, \int_E |f(x)| \, dx \leq C_{(E)} \|f\|_Y. \) (2.1)

**Remark 2.2.** It was pointed out in [70, p.9] that \( \| \cdot \|_Y \) is sometimes denote the quality of functions via some function spaces beyond Banach function spaces, e.g., Morrey spaces \( \mathcal{M}^{p,q}(\mathbb{R}^n) \) with \( 1 \leq q < p < \infty \), which violates (2.1) (see [72, Example 3.3]).

For any \( x \in \mathbb{R}^n \) \( \text{and} \ r \in (0, \infty) \), let \( B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\} \) and

\[ \mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\} \] (2.2)

(namely, the set of all balls in \( \mathbb{R}^n \)).

**Definition 2.3.** A quasi-Banach space \( X \subset \mathcal{M}(\mathbb{R}^n) \) is called a ball quasi-Banach function space if it satisfies

1. \( \|f\|_X = 0 \) implies that \( f = 0 \) almost everywhere;
2. \( \|g\| \leq |f| \) almost everywhere implies that \( \|g\|_X \leq \|f\|_X \);
Moreover, a ball quasi-Banach function space $X$ is called a ball Banach function space if the norm of $X$ satisfies the triangle inequality: for any $f, g \in X$,

$$
\|f + g\|_X \leq \|f\|_X + \|g\|_X
$$

and for any $B \in \mathcal{B}$, there exists a positive constant $C(B)$, depending on $B$ such that for any $f \in X$,

$$
\int_B |f(x)| \, dx \leq C(B) \|f\|_X.
$$

Observe that, in Definition 2.3, if we replace any ball $B$ by any bounded measurable set $E$, we obtain its another equivalent formulation.

Recall that a quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a quasi-Banach function space if it is a ball quasi-Banach function space and it satisfies Definition 2.3(iv) with the ball $B$ replaced by any measurable set $E$ of finite measure.

It is easy to see that every Banach function space is a ball Banach function space. As was mentioned in [70, p.9], the family of ball Banach function spaces includes Morrey type spaces, which are not necessarily Banach function spaces.

For any ball Banach function space $X$, the associate space (Köthe dual) $X'$ is defined by setting

$$
X' := \{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup \{ \|fg\|_{L^1(\mathbb{R}^n)} : g \in X, \|g\|_X = 1 \} < \infty \},
$$

where $\| \cdot \|_{X'}$ is called the associate norm of $\| \cdot \|_X$ (see, for example, [9, Chapter 1, Definitions 2.1 and 2.3]).

**Remark 2.4.** (i) By [70, Proposition 2.3], we know that, if $X$ is a ball Banach function space, then its associate space $X'$ is also a ball Banach function space.

(ii) A ball quasi-Banach function space $Y \subset \mathcal{M}(\mathbb{R}^n)$ is called a quasi-Banach function space (see, for example, [70, Definition 2.4.7]), if for any measurable set $E \subset \mathbb{R}^n$ with finite measure, $1_E \in Y$.

The following Hölder inequality is a direct corollary of both Definition 2.3(i) and (2.5) (see also [9, Theorem 2.4]); we omit the details.

**Lemma 2.5** (Hölder’s inequality). Let $X$ be a ball Banach function space with the associate space $X'$. If $f \in X$ and $g \in X'$, then $fg$ is integrable and

$$
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
$$

Similarly to [9, Theorem 2.7], we have the following conclusion, whose proof is a slight modification of that of [9, Theorem 2.7].

**Lemma 2.6** (Lorentz-Luxembourg lemma). Every ball Banach function space $X$ coincides with its second associate space $X''$. In other words, a function $f$ belongs to $X$ if and only if it belongs to $X''$ and, in that case,

$$
\|f\|_X = \|f\|_{X''}.
$$

**Proof.** Let $X$ be a ball Banach function space. From this and [70, Proposition 2.3], we deduce that $X'$ and $X''$ are both ball Banach function spaces. Using this and Lemma 2.5 and repeating the proof of [9, Theorem 2.7] via replacing Definition 2.1(iv) by Definition 2.3(iv), we then complete the proof of Lemma 2.6.

We still need to recall the notions of the convexity and the concavity of ball quasi-Banach function spaces, which come from, for example, [53, Definition 1.d.3].
Definition 2.7.  Let $X$ be a ball quasi-Banach function space and $p \in (0, \infty)$.

(i) The $p$-convexification $X^p$ of $X$ is defined by setting

$$X^p := \{ f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X \}$$

equipped with the quasi-norm $\|f\|_{X^p} := \|f^p\|_X^{\frac{1}{p}}$.

(ii) The space $X$ is said to be $p$-concave if there exists a positive constant $C$ such that for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of $X^{1/p}$,

$$\sum_{j \in \mathbb{N}} \|f_j\|_{X^{1/p}} \leq C \left( \sum_{j \in \mathbb{N}} \|f_j\|_{X^{1/p}} \right)^{\frac{1}{p}}.$$

Particularly, $X$ is said to be strictly $p$-concave when $C = 1$.

Now we introduce the notion of weak ball quasi-Banach function spaces in a traditional way as follows.

Definition 2.8.  Let $X$ be a ball quasi-Banach function space. The weak ball quasi-Banach function space $WX$ is defined to be the set of all measurable functions $f$ satisfying

$$\|f\|_{WX} := \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \|_X] < \infty. \quad (2.7)$$

Remark 2.9.  (i) Let $X$ be a ball quasi-Banach function space. For any $f \in X$ and $\alpha \in (0, \infty)$, we have $1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}}(x) \leq |f(x)|/\alpha$ for any $x \in \mathbb{R}^n$, which, together with Definition 2.3(ii), further implies that

$$\sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \|_X] \leq \|f\|_X.$$

This shows that $X \subset WX$.

(ii) Let $f, g \in WX$ with $|f| \leq |g|$. By Definition 2.3(ii), we conclude that $\|f\|_{WX} \leq \|g\|_{WX}$.

Lemma 2.10.  Let $X$ be a ball quasi-Banach function space. Then $\| \cdot \|_{WX}$ is a quasi-norm on $WX$, namely,

(i) $\|f\|_{WX} = 0$ if and only if $f = 0$ almost everywhere.

(ii) For any $\lambda \in \mathbb{C}$ and $f \in WX$,

$$\|\lambda f\|_{WX} = |\lambda| \|f\|_{WX}.$$

(iii) For any $f, g \in WX$, there exists a positive constant $C$ such that

$$\|f + g\|_{WX} \leq C \|f\|_{WX} + \|g\|_{WX}.$$ 

Moreover, if $p \in (0, \infty)$ and $X^{1/p}$ is a ball Banach function space, then

$$\|f + g\|_{WX}^{1/p} \leq 2^\max\{1/p, 1\} \|f\|_{WX}^{1/p} + \|g\|_{WX}^{1/p}.$$ 

Proof.  It is easy to show (i) and (ii); the details are omitted. We now show (iii). We first assume that $X^{1/p}$ is a ball Banach function space for some given $p \in (0, \infty)$. Then for any $f, g \in WX$ and $\alpha \in (0, \infty)$, by Definition 2.7(i), (2.3) with $X$ replaced by $X^{1/p}$ and the well-known inequality that $(a + b)^{1/p} \leq 2^{\max\{1/p - 1, 0\}}(a^{1/p} + b^{1/p})$ for any $a, b \in (0, \infty)$, we have

$$\|f + g\|_{WX} \leq \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| + |g(x)| > \alpha\}} \|_X] = \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| + |g(x)| > \alpha\}} \|_{X^{1/p}}]\]$$

$$\leq \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha/2\}} \|_{X^{1/p}} + \|1_{\{x \in \mathbb{R}^n : |g(x)| > \alpha/2\}} \|_{X^{1/p}}]^{1/p}$$

$$\leq 2^{\max\{1/p - 1, 0\}} \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha/2\}} \|_{X^{1/p}} + \|1_{\{x \in \mathbb{R}^n : |g(x)| > \alpha/2\}} \|_{X^{1/p}}]^{1/p}$$

$$\leq 2^{\max\{1/p, 1\}} \left\{ \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \|_{X^{1/p}}]^{1/p} + \sup_{\alpha \in (0, \infty)} [\alpha \|1_{\{x \in \mathbb{R}^n : |g(x)| > \alpha\}} \|_{X^{1/p}}]^{1/p} \right\}$$

$$= 2^{\max\{1/p, 1\}} \left\{ \|f\|_{WX} + \|g\|_{WX} \right\}. $$
For the ball quasi-Banach function space $X$, the same procedure as above leads us to the desired estimate with the positive constant $C$ depending on the positive constant appearing in the quasi-triangular inequality of the quasi-norm $\| \cdot \|_X$. This finishes the proof of Lemma 2.10.

**Remark 2.11.** Let $X$ be a ball quasi-Banach function space. Then, by the Aoki-Rolewicz theorem (see, for example, [32, Exercise 1.4.6]), one finds a positive constant $\nu \in (0, 1)$ such that for any $N \in \mathbb{N}$ and $\{f_j\}_{j=1}^N \subset \mathcal{M}(\mathbb{R}^n)$,

$$\left\| \sum_{j=1}^N |f_j| \right\|_{W^X}^\nu \leq 4 \left\{ \sum_{j=1}^N \|f_j\|_{W^X} \right\}^\nu.$$

**Lemma 2.12.** Let $X$ be a ball quasi-Banach function space and $\{f_m\}_{m \in \mathbb{N}} \subset W^X$. If $f_m \rightarrow f$ as $m \rightarrow \infty$ almost everywhere in $\mathbb{R}^n$ and if \(\liminf_{m \rightarrow \infty} \|f_m\|_{W^X} < \infty\), then $f \in W^X$ and

$$\|f\|_{W^X} \leq \liminf_{m \rightarrow \infty} \|f_m\|_{W^X}.$$

**Proof.** For any $k \in \mathbb{N}$, letting $h_k := \inf_{m \geq k} |f_m|$, then $0 \leq h_k \uparrow |f|$, $k \rightarrow \infty$, almost everywhere in $\mathbb{R}^n$ and hence for any $\alpha \in (0, \infty)$,

$$1_{\{x \in \mathbb{R}^n : \left|h_k(x)\right| > \alpha\}} \uparrow 1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \text{.}$$

Moreover, by Definition 2.3(iii) and the definition of $h_k$, for any $\alpha \in (0, \infty)$, we have

$$\|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}}\|_X = \lim_{k \rightarrow \infty} \|1_{\{x \in \mathbb{R}^n : |h_k(x)| > \alpha\}}\|_X \leq \liminf_{m \rightarrow \infty} \|1_{\{x \in \mathbb{R}^n : |f_m(x)| > \alpha\}}\|_X.$$

This further implies that for any $\alpha \in (0, \infty)$,

$$\alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}}\|_X \leq \alpha \liminf_{m \rightarrow \infty} \|1_{\{x \in \mathbb{R}^n : |f_m(x)| > \alpha\}}\|_X$$

$$\leq \liminf_{m \rightarrow \infty} \sup_{\alpha \in (0, \infty)} \|1_{\{x \in \mathbb{R}^n : |f_m(x)| > \alpha\}}\|_X = \liminf_{m \rightarrow \infty} \|f_m\|_{W^X},$$

which completes the proof of Lemma 2.12.

From the definition of $W^X$, Remark 2.11, Lemmas 2.10 and 2.12, it is easy to deduce the following lemma and we omit the details.

**Lemma 2.13.** Let $X$ be a ball quasi-Banach function space. Then the space $W^X$ is also a ball quasi-Banach function space.

**Remark 2.14.** Let $X$ be a ball quasi-Banach function space. By Lemma 2.13, we know that $W^X$ is also a ball quasi-Banach function space. For any given $s \in (0, \infty)$, it is easy to show that $X^s$ is also a ball quasi-Banach function space. Thus, $(W^X)^s$ and $W(X^s)$ make sense and coincide with equal quasi-norms. Indeed, for any $f \in (W^X)^s$, by Definitions 2.7(i) and 2.8, we have

$$\|f\|_{(W^X)^s} = \|f^s\|_{W^X} = \|f\|_{W(X^s)}.$$

Now, we recall the notions of Muckenhoupt weights $A_p(\mathbb{R}^n)$ (see, for example, [32]).

**Definition 2.15.** An $A_p(\mathbb{R}^n)$-weight $\omega$, with $p \in [1, \infty)$, is a locally integrable and nonnegative function on $\mathbb{R}^n$ satisfying that, when $p \in (1, \infty)$,

$$\sup_{B \in \mathcal{B}} \left[ \frac{1}{|B|} \int_B \omega(x) dx \right] \left[ \frac{1}{|B|} \int_B \{\omega(x) \}^{\frac{1}{p-1}} dx \right]^{p-1} < \infty$$

and, when $p = 1$,

$$\sup_{B \in \mathcal{B}} \frac{1}{|B|} \int_B \omega(x) dx \|\omega^{-1}\|_{L^{\infty}(B)} < \infty,$$

where $\mathcal{B}$ is as in (2.2). Define $A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n)$. 
Definition 2.16. Let $p \in (0, \infty)$ and $\omega \in A_{\infty}(\mathbb{R}^n)$. The weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that
\[
\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right]^{\frac{1}{p}} < \infty.
\]

The following technical lemma is just [14, Lemma 4.7], which plays a vital role in the proof of Theorems 4.2, 6.3 and 6.4 below.

Lemma 2.17. Let $X$ be a ball quasi-Banach function space. Assume that there exists an $s \in (0, \infty)$ such that $X^{1/s}$ is a ball Banach function space and $M$ is bounded on $(X^{1/s})'$. Then there exists an $\epsilon \in (0,1)$ such that $X$ continuously embeds into $L^\infty_\omega(\mathbb{R}^n)$ with $\omega := |\mathcal{M}(1_{B(0,1)})|^r$, namely, there exists a positive constant $C$ such that for any $f \in X$,
\[
\|f\|_{L^\infty_\omega(\mathbb{R}^n)} \leq C\|f\|_X.
\]

2.2 Assumptions on the Hardy-Littlewood maximal operator

Denote by the symbol $L^1_{\text{loc}}(\mathbb{R}^n)$ the set of all locally integrable functions on $\mathbb{R}^n$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,
\[
\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,
\]
where the supremum is taken over all balls $B \subset \mathbb{B}$ containing $x$.

For any $\theta \in (0, 1)$, the powered Hardy-Littlewood maximal operator $\mathcal{M}^{(\theta)}$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,
\[
\mathcal{M}^{(\theta)}(f)(x) := (\mathcal{M}(|f|^\theta)(x))^{1/\theta}.
\]

To establish atomic characterizations of weak Hardy spaces associated with ball quasi-Banach function spaces $X$, the approach used in this article heavily depends on the following assumptions on the boundedness of the Hardy-Littlewood maximal function on $X^{1/p}$, which is stronger than [70, (2.8)].

Assumption 2.18. Let $X$ be a ball quasi-Banach function space and there exists a $p_- \in (0, \infty)$ such that for any given $p \in (0, p_-)$ and $s \in (1, \infty)$, there exists a positive constant $C$ such that for any $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathbb{R}^n)$,
\[
\left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{1/s} \leq C \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{1/s} \quad X^{1/p}.
\]

Remark 2.19. (i) Let $X$ and $p_-$ be the same as in Assumption 2.18. Let
\[
p := \min\{p_-, 1\}.
\]

Then for any given $r \in (0, p)$ and for any sequence $\{B_j\}_{j \in \mathbb{N}} \subset \mathbb{B}$ and $\beta \in [1, \infty)$, by Definition 2.3(ii), the fact that $1_{\beta B_j} \leq [\beta^n \mathcal{M}(1_{B_j})]^{1/r}$ almost everywhere on $\mathbb{R}^n$ for any $j \in \mathbb{N}$, Definition 2.7(i) and Assumption 2.18, we have
\[
\left\| \sum_{j \in \mathbb{N}} 1_{\beta B_j} \right\|_X \leq \left\| \sum_{j \in \mathbb{N}} [\beta^n \mathcal{M}(1_{B_j})]^{1/r} \right\|_X = \beta^r \left\| \sum_{j \in \mathbb{N}} [\mathcal{M}(1_{B_j})]^{1/r} \right\|_{X^{1/r}} \leq C \beta^r \left\| \sum_{j \in \mathbb{N}} 1_{B_j} \right\|_{X^{1/r}} = C \beta^r \left\| \sum_{j \in \mathbb{N}} 1_{B_j} \right\|_X,
\]
where the positive constant $C$ is independent of $\{B_j\}_{j \in \mathbb{N}}$ and $\beta$.

(ii) In Assumption 2.18, let $X := L^\infty(\mathbb{R}^n)$ with any given $\tilde{p} \in (0, \infty)$. In this case, $p_- = \tilde{p}$ and the inequality (2.10) becomes the well-known Fefferman-Stein vector-valued maximal inequality, which was originally established by Fefferman and Stein [25, Theorem 1(a)].

Assumption 2.20. Let $X$ be a ball quasi-Banach function space. Assume that there exists an $r \in (0, \infty)$ such that $\mathcal{M}$ in (2.8) is bounded on $(WX)^{1/r}$. 

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2.3 Weak Hardy type spaces

In what follows, we denote by $S(\mathbb{R}^n)$ the space of all Schwartz functions, equipped with the well-known topology determined by a countable family of norms, and by $S'(\mathbb{R}^n)$ its topological dual space, equipped with the weak-$*$ topology. For any $N \in \mathbb{N}$, let

$$F_N(\mathbb{R}^n) := \left\{ \varphi \in S(\mathbb{R}^n) : \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \leq N} \sup_{x \in \mathbb{R}^n} [(1 + |x|)^{N+n} |\partial_x^\beta \varphi(x)|] \leq 1 \right\};$$

(2.13)

here and thereafter, for any $\beta :=(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$, $|\beta| := \beta_1 + \cdots + \beta_n$ and

$$\partial_x^\beta := \left( \frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\beta_n}.$$ 

For any given $f \in S'(\mathbb{R}^n)$, the radial grand maximal function $M^g_N(f)$ of $f$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M^g_N(f)(x) := \sup\{ |f \ast \varphi_t(x)| : t \in (0, \infty) \text{ and } \varphi \in F_N(\mathbb{R}^n) \},$$

(2.14)

where for any $t \in (0, \infty)$ and $\xi \in \mathbb{R}^n$, $\varphi_t(\xi) := t^{-n} \varphi(\xi/t)$.

**Definition 2.21.** Let $X$ be a ball quasi-Banach function space. Then the weak Hardy-type space $WH_X(\mathbb{R}^n)$ associated with $X$ is defined by setting

$$WH_X(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{WH_X(\mathbb{R}^n)} := \|M^g_N(f)\|_{WX} < \infty \right\},$$

where $M^g_N(f)$ is as in (2.14) with $N \in \mathbb{N}$ sufficiently large.

**Remark 2.22.**

(i) When $X := L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, the Hardy-type space $WH_X(\mathbb{R}^n)$ coincides with the classical weak Hardy space $WH^p(\mathbb{R}^n)$ (see, for example, [54, p. 114]).

(ii) By Theorem 3.2(ii) below, we know that, if the Hardy-Littlewood maximal operator $M$ in (2.8) is bounded on $(WX)^{1/r}$ and $N \in [\frac{n}{p} + 1, \infty) \cap \mathbb{N}$, then $WH_X(\mathbb{R}^n)$ in Definition 2.21 is independent of the choice of $N$.

3 Maximal function characterizations of $WH_X(\mathbb{R}^n)$ and relations between $WX$ and $WH_X(\mathbb{R}^n)$

The aim of this section is to characterize $WH_X(\mathbb{R}^n)$ via radial or non-tangential maximal functions. We begin with the following notions of the radial functions and the non-tangential maximal functions.

**Definition 3.1.** Let $\psi \in S(\mathbb{R}^n)$, $a, b \in (0, \infty)$, $N \in \mathbb{N}$ and $f \in S'(\mathbb{R}^n)$.

(i) The radial maximal function $M(f, \psi)$ of $f$ associated with $\psi$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M(f, \psi)(x) := \sup_{t \in (0, \infty)} |f \ast \psi_t(x)|.$$ 

(ii) The non-tangential maximal function $M^*_f(f, \psi)$ of $f$ associated with $\psi$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M^*_f(f, \psi)(x) := \sup_{t \in (0, \infty), |y-x|<at} |f \ast \psi_t(y)|.$$ 

(iii) The maximal function $M^*_f(f, \psi)$ of Peetre type is defined by setting, for any $x \in \mathbb{R}^n$,

$$M^*_f(f, \psi)(x) := \sup_{(y,t) \in \mathbb{R}^{n+1}} \frac{|(\psi_t \ast f)(x-y)|}{(1+t^{-1}|y|)^b}.$$ 

(iv) The non-tangential grand maximal function $M_{N,a}(f)$ of $f$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{N,a}(f)(x) := \sup_{0 \in F_N(\mathbb{R}^n), t \in (0, \infty), |y-x|<at} \sup_{\varphi \in F_N(\mathbb{R}^n)} |f \ast \varphi_t(y)|.$$
(v) The grand maximal function $M_{N,b}^*(f)$ of Peetre type is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{N,b}^*(f)(x) := \sup_{\varphi \in F_N(\mathbb{R}^n)} \left\{ \sup_{(y,t) \in \mathbb{R}^{n+1}} \frac{|\varphi_t \ast f(x-y)|}{(1 + t^{-1}|y|^b)} \right\},$$

where $F_N(\mathbb{R}^n)$ is as in (2.13). When $a = 1$, we simply denote $M_{N,a}(f)$ by $M_N(f)$.

The following theorem is the main result of this section, which presents the maximal function characterizations of the space $WH_X(\mathbb{R}^n)$.

**Theorem 3.2.** Let $a$, $b \in (0, \infty)$ and $X$ be a ball quasi-Banach function space. Let $\psi \in S(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi(x) \, dx \neq 0$.

(i) Let $N \geq [b + 1]$ be an integer. Then for any $f \in S'(\mathbb{R}^n)$,

$$\|M(f, \psi)\|_{WX} \lesssim \|M_a^*(f, \psi)\|_{WX}, \quad (3.1)$$

$$\|M(f, \psi)\|_{WX} \lesssim \|M_N(f)\|_{WX} \lesssim \|M_{b+1}(f)\|_{WX} \lesssim \|M_b^*(f, \psi)\|_{WX}, \quad (3.2)$$

$$\|M_b^*(f, \psi)\|_{WX} \sim \|M_{b,N}^*(f)\|_{WX} \quad (3.3)$$

and

$$\|M_b^0(f)\|_{WX} \sim \|M_N(f)\|_{WX}, \quad (3.4)$$

where the implicit positive constants are independent of $f$.

(ii) Let $r \in (0, \infty)$. Assume that $b \in (n/r, \infty)$ and the Hardy-Littlewood maximal operator $M$ in (2.8) is bounded on $(WX)^{1/r}$. Then for any $f \in S'(\mathbb{R}^n)$,

$$\|M_b^*(f, \psi)\|_{WX} \lesssim \|M(f, \psi)\|_{WX}, \quad (3.5)$$

where the implicit positive constant is independent of $f$. In particular, when $N \geq [b + 1]$, if one of the quantities

$$\|M_b^0(f)\|_{WX}, \quad \|M(f, \psi)\|_{WX}, \quad \|M_a^*(f, \psi)\|_{WX}, \quad \|M_N(f)\|_{WX},$$

$$\|M_b^*(f, \psi)\|_{WX} \quad \text{and} \quad \|M_{b,N}^*(f)\|_{WX}$$

is finite, then the others are also finite and mutually equivalent with the positive equivalence constants independent of $f$.

**Proof.** The proof of this theorem is similar to that of [70, Theorem 3.1]. For the convenience of the reader, we present some details.

Let $f \in S'(\mathbb{R}^n)$. We first prove (i). From (i)–(iii) of Definition 3.1, it follows that for any $x \in \mathbb{R}^n$,

$$M(f, \psi)(x) \leq M_a^*(f, \psi)(x) \leq M_b^*(f, \psi)(x),$$

which, together with Remark 2.9(ii), implies (3.1).

Moreover, by (i) and (iv) of Definition 3.1 again, we have, for any $x \in \mathbb{R}^n$,

$$M(f, \psi)(x) \leq M_N(f)(x) \leq M_{b+1}(f)(x). \quad (3.6)$$

In addition, from the proof of [33, Theorem 2.1.4(d)], we deduce that for any $x \in \mathbb{R}^n$,

$$M_{b+1}(f)(x) \lesssim M_b^*(f, \psi)(x),$$

which, together with (3.6) and Remark 2.9(ii), implies (3.2).

It is easy to see that for any $x \in \mathbb{R}^n$,

$$M_b^*(f, \psi)(x) \lesssim M_{b,N}^*(f)(x),$$

which, combined with [70, Lemma 2.13], implies (3.3). By [81, Remark 3.6(i)], we know that there exists a positive constant $C$ such that for any $x \in \mathbb{R}^n$,

$$C^{-1}M_N(f)(x) \leq M_b^0(f)(x) \leq CM_N(f)(x),$$

which, together with (3.6) and (3.2), implies (3.4).
which, together with Remark 2.9(ii), implies that (3.4) holds true. This finishes the proof of (i).

Now we prove (ii). It was proved in [70, p. 35] that, if \( r \in (0, \infty) \) and \( br > n \), then for any \( x \in \mathbb{R}^n \),

\[
M_{t^*}(f, \psi)(x) \lesssim \mathcal{M}^{(r)} \left( \sup_{t \in (0, \infty)} |\psi_t * f| \right)(x) \sim \mathcal{M}^{(r)}(M(f, \psi))(x),
\]

which, combined with Remark 2.9(ii) and the assumption that \( \mathcal{M} \) is bounded on \( WX^{1/r} \), further implies that

\[
\|M_{t^*}(f, \psi)\|_{WX} \lesssim \|\mathcal{M}^{(r)}(M(f, \psi))\|_{WX} \lesssim \|M(f, \psi)\|_{WX}.
\]

Thus, (3.5) holds true. This finishes the proof of Theorem 3.2. \( \square \)

For any given \( t \in (0, \infty) \), the Poisson kernel \( P_t \) is defined by setting, for any \( x \in \mathbb{R}^n \),

\[
P_t(x) := \frac{\Gamma([n + 1]/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |x|^2)^{(n+1)/2}},
\]

where \( \Gamma \) denotes the Gamma function.

Recall that \( f \in S'(\mathbb{R}^n) \) is called a bounded distribution, if for any \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), \( f * \varphi \in L^\infty(\mathbb{R}^n) \). For any given bounded distribution \( f \), its non-tangential maximal function \( \mathcal{N}(f) \), with respect to Poisson kernels \( \{P_t\}_{t \in (0, \infty)} \), is defined by setting, for any \( x \in \mathbb{R}^n \),

\[
\mathcal{N}(f)(x) := \sup_{t \in (0, \infty), |y - x| < t} |f * P_t(y)|.
\]

**Theorem 3.3.** Let \( X \) be a ball quasi-Banach function space satisfying Assumption 2.20. Assume that there exists a positive constant \( C_0 \) such that

\[
\inf_{x \in \mathbb{R}^n} \|1_{B(x, 1)}\|_{WX} \geq C_0. \tag{3.7}
\]

Then \( f \in WH_X(\mathbb{R}^n) \) if and only if \( f \) is a bounded distribution and \( \mathcal{N}(f) \in WX \). Moreover, for any \( f \in WH_X(\mathbb{R}^n) \),

\[
\|f\|_{WH_X(\mathbb{R}^n)} \sim \|\mathcal{N}(f)\|_{WX}
\]

with the positive equivalence constants independent of \( f \).

**Proof.** Assume that \( f \in WH_X(\mathbb{R}^n) \). By Assumption 2.20 and Theorem 3.2(ii), we know that for any given \( N \in ([\frac{n}{2}] + 1, \infty) \cap \mathbb{N} \),

\[
\|M_N(f)\|_{WX} \sim \|f\|_{WH_X(\mathbb{R}^n)}.
\]

It is easy to see that for any fixed \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), there exists a positive constant \( C_{(\varphi)} \) such that \( C_{(\varphi)} \varphi \in \mathcal{F}_N(\mathbb{R}^n) \) with \( \mathcal{F}_N(\mathbb{R}^n) \) as in (2.13). Therefore, for any \( x \in \mathbb{R}^n \),

\[
M_{t^*}(f, C_{(\varphi)} \varphi)(x) \lesssim M_{N}(f)(x),
\]

which, together with Definition 2.3(ii), Remark 2.9(iii), (3.7) and Theorem 3.2(ii), further implies that for any \( x \in \mathbb{R}^n \),

\[
C_{(\varphi)} |(\varphi * f)(x)| \leq \inf_{|y - x| < 1} M_{t^*}(f, C_{(\varphi)} \varphi)(y) = \frac{\|1_{B(x, 1)}\inf_{|y - z| < 1} M_{t^*}(f, C_{(\varphi)} \varphi)(y)\|_{WX}}{\|1_{B(x, 1)}\|_{WX}} \lesssim \|M_{N}(f)\|_{WX} \leq \frac{\|M_{N}(f)\|_{WX}}{C_0} < \infty. \tag{3.8}
\]

This means that \( f \) is a bounded distribution. Next, we show that \( \mathcal{N}(f) \in WX \). From the proof of [33, p. 72], we deduce that for any \( N \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), \( \mathcal{N}(f)(x) \leq C_{(\varphi; N)} M_{N}(f)(x) \), which implies that \( \mathcal{N}(f) \in WX \) and

\[
\|\mathcal{N}(f)\|_{WX} \lesssim \|M_{N}(f)\|_{WX} \sim \|f\|_{WH_X(\mathbb{R}^n)}.
\]
Now, assume that $f$ is a bounded distribution and $\mathcal{N}(f) \in WX$. Then, by [74, p. 99] or [70, p. 35], we know that there exists a $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi_0(x)dx = 1$ such that for any $x \in \mathbb{R}^n$, $M(f, \psi_0)(x) \lesssim \mathcal{N}(f)(x)$, which, combined with $\mathcal{N}(f) \in WX$, Remark 2.9(ii), Assumption 2.20 and Theorem 3.2(ii), implies $f \in WH_X(\mathbb{R}^n)$ and

$$\|f\|_{WH_X(\mathbb{R}^n)} \sim \|M(f, \psi_0)\|_{WX} \lesssim \|\mathcal{N}(f)\|_{WX}.$$ 

This finishes the proof of Theorem 3.3.

Now, we discuss the relation between the spaces $WX$ and $WH_X(\mathbb{R}^n)$.

**Theorem 3.4.** Let $X$ be a ball quasi-Banach function space and $M$ in (2.8) bounded on $(WX)^{1/r}$ for some $r \in (1, \infty)$. Then

(i) $WX \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

(ii) If $f \in WX$, then $f \in WH_X(\mathbb{R}^n)$ and there exists a positive constant $C$, independent of $f$, such that $\|f\|_{WH_X(\mathbb{R}^n)} \leq C\|f\|_{WX}$.

(iii) If $f \in WH_X(\mathbb{R}^n)$, then there exists a locally integrable function $g \in WX$ such that $g$ represents $f$, which means that $f = g$ in $\mathcal{S}'(\mathbb{R}^n)$, $\|f\|_{WH_X(\mathbb{R}^n)} = \|g\|_{WH_X(\mathbb{R}^n)}$ and there exists a positive constant $C$, independent of $f$, such that $\|g\|_{WX} \leq C\|f\|_{WH_X(\mathbb{R}^n)}$.

**Proof.** Observe that

$$\ell_{WX} := \sup\{r \in (0, \infty) : M \text{ is bounded on } (WX)^{1/r}\} > 1.$$ 

Moreover, by Lemma 2.13, we know that the space $WX$ is a ball quasi-Banach function space. Thus, all assumptions of [70, Theorem 3.4] with $X$ and $H_X(\mathbb{R}^n)$ replaced, respectively, by $WX$ and $WH_X(\mathbb{R}^n)$ are satisfied, from which we deduce all the desired conclusions of Theorem 3.4. This finishes the proof of Theorem 3.4.

4 Atomic characterizations

In this section, we establish the atomic characterization of $WH_X(\mathbb{R}^n)$. Now we introduce the notion of atoms associated with $X$, which originates from [70, Definition 3.5].

**Definition 4.1.** Let $X$ be a ball quasi-Banach function space, $q \in (1, \infty]$ and $d \in \mathbb{Z}_+$. Then a measurable function $a$ on $\mathbb{R}^n$ is called an $(X, q, d)$-atom if there exists a ball $B \in \mathcal{B}$ such that

(i) $\text{supp } a := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset B$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq \frac{|B|^{1/q}}{1_q}$;

(iii) $\int_{\mathbb{R}^n} a(x)\alpha(x)dx = 0$ for any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ with $|\alpha| \leq d$; here and thereafter, for any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Now we first formulate a decomposition theorem as follows.

**Theorem 4.2.** Let $X$ be a ball quasi-Banach function space satisfying that for some given $r \in (0, 1)$ and for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$,

$$\left\|\left\{\sum_{j \in \mathbb{N}} |M(f_j)|^{1/r}\right\}^{1/r}\right\|_{X^{1/r}} \leq C\left\|\left\{\sum_{j \in \mathbb{N}} |f_j|^{1/r}\right\}^{1/r}\right\|_{X^{1/r}},$$

(4.1)

where the positive constant $C$ is independent of $\{f_j\}_{j \in \mathbb{N}}$. Assume that $X$ satisfies Assumption 2.20 and there exist $s \in (0, \infty)$, $\theta_0 \in (1, \infty)$ and $p \in (0, \infty)$ such that $X^{1/s}$ is a ball Banach function space, $X$ is $\theta_0$-concave and $M$ is bounded on $(X^{1/s})$ and $X^{1/(\theta_0 p)}$. Let $d \geq \lceil n(1/p - 1) \rceil$ be a fixed nonnegative integer and $f \in WH_X(\mathbb{R}^n)$.

Then there exist $\{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ of $(X, \infty, d)$-atoms supported, respectively, in balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ satisfying that for any $i \in \mathbb{Z}$, $\sum_{j \in \mathbb{N}} \chi_{B_{i,j}} \leq A$ with $c \in (0, 1]$ and $A$ being a positive constant independent of $f$.
and $i$ such that $f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j}$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\lambda_{i,j} := \tilde{A}^n_2 \|1_{B_{i,j}}\|_X$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, with $\tilde{A}$ being a positive constant independent of $i$, $j$ and $f$, and
\[
\sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X \lesssim \|f\|_{WH_X(\mathbb{R}^n)},
\]
where the implicit positive constant is independent of $f$.

Before showing Theorem 4.2, we recall some notions and establish some necessary lemmas. Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity, if for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see, for example, [32, Proposition 7.2.8]).

**Lemma 4.3.** Let $X$ be a ball quasi-Banach function space. Assume that there exists an $s \in (0, \infty)$ such that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$. If $f \in WH_X(\mathbb{R}^n)$, then $f$ vanishes weakly at infinity.

**Proof.** Let $f \in WH_X(\mathbb{R}^n)$. From Lemma 2.17, we deduce that there exists an $\epsilon \in (0, 1)$ such that $X$ continuously embeds into $L^s_\omega(\mathbb{R}^n)$ with $\omega := |\mathcal{M}(1_{B(\tilde{a}, 1)}^N)|^s$, which implies that $f \in WH^s_\omega(\mathbb{R}^n)$, where $WH^s_\omega(\mathbb{R}^n)$ denotes the weighted Hardy space as in Definition 2.21 with $X^{1/s}$. By this, we know that for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $t \in (0, \infty)$, $x \in \mathbb{R}^n$ and $y \in B(x, t)$,
\[
|f * \phi_t(x)| \lesssim M^0_N(f)(y),
\]
where $N \in \mathbb{N}$ is large enough. Thus, there exists a positive constant $C(N)$, independent of $x$, $t$ and $f$, such that
\[
B(x, t) \subset \{y \in \mathbb{R}^n : M^0_N(f)(y) \geq C(N) |f * \phi_t(x)| \}.
\]
From this, it follows that for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,
\[
|f * \phi_t(x)| \lesssim \inf_{y \in B(x, t)} M^0_N(f)(y) \lesssim \frac{\|1_{B(x,t)}M^0_N(f)\|_{WL^s_\omega(\mathbb{R}^n)}}{\|1_{B(x,t)}\|_{WL^s_\omega(\mathbb{R}^n)}} \lesssim \frac{\|M^0_N(f)\|_{WL^s_\omega(\mathbb{R}^n)}}{\|1_{B(x,t)}\|_{L^s_\omega(\mathbb{R}^n)}},
\]
where $WL^s_\omega(\mathbb{R}^n)$ denotes the weak weighted Lebesgue space as that in Definition 2.8 with $X$ replaced by $L^s_\omega(\mathbb{R}^n)$. By [32, Theorem 7.2.7], we know that $\omega \in A_1(\mathbb{R}^n)$, which, combined with [32, Proposition 7.2.8], implies that there exists a $\delta \in (0, 1)$ such that for any $t \in (1, \infty)$ and $x \in \mathbb{R}^n$,
\[
\|1_{B(x,t)}\|_{L^s_\omega(\mathbb{R}^n)} \gtrsim \left[\frac{|B(x,t)|}{|B(x,1)|}\right]^{\delta/s} \sim t^{n\delta/s}.
\]
From this, (4.3) and the doubling property of Muckenhoupt weights (see, for example, [32, Proposition 7.1.5(9)]), it follows that for any $\psi \in \mathcal{S}(\mathbb{R}^n)$,
\[
\left| \int_{\mathbb{R}^n} \phi_t * f(x)\psi(x)dx \right| \lesssim \int_{\mathbb{R}^n} \frac{1}{\|1_{B(x,t)}\|_{L^s_\omega(\mathbb{R}^n)}} |\psi(x)|dx
\]
\[
\lesssim \int_{\mathbb{R}^n} \frac{1}{\|1_{B(\tilde{a}, 1)}\|_{L^s_\omega(\mathbb{R}^n)}} \frac{\|1_{B(\tilde{a}, 1)}\|_{L^s_\omega(\mathbb{R}^n)}}{\|1_{B(\tilde{a}, 1)}\|_{L^s_\omega(\mathbb{R}^n)}} \|1_{B(\tilde{a}, 1)}\|_{L^s_\omega(\mathbb{R}^n)} |\psi(x)|dx
\]
\[
\lesssim \int_{\mathbb{R}^n} \frac{\|1_{B(\tilde{a}, 1)}\|_{L^s_\omega(\mathbb{R}^n)}}{\|1_{B(x,t)}\|_{L^s_\omega(\mathbb{R}^n)}} |\psi(x)|dx \sim t^{-n\delta/s} \to 0 \quad \text{as} \quad t \to \infty,
\]
which implies that $f$ vanishes weakly at infinity. This finishes the proof of Lemma 4.3. \qed

In what follows, the symbol $\tilde{a}$ denotes the origin of $\mathbb{R}^n$ and for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\varphi}$ denotes its Fourier transform which is defined by setting, for any $\xi \in \mathbb{R}^n$,
\[
\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \varphi(x)dx.
\]
We also use the symbol $C^\infty(\mathbb{R}^n)$ to denote the set of all infinitely differentiable functions with compact supports, and the symbol $\epsilon \to 0^+$ to denote $\epsilon \in (0, \infty)$ and $\epsilon \to 0$. 

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Combining Calderón [13, Lemma 4.1] and Folland and Stein [28, Theorem 1.64] (see also [12, p. 219] and [81, Lemma 4.6]), we immediately obtain the following Calderón reproducing formula and we omit the details.

**Lemma 4.4.** Let φ be a Schwartz function, and for any $x \in \mathbb{R}^n \setminus \{0\}$, there exists a $t \in (0, \infty)$ such that $\hat{\phi}(tx) \neq 0$. Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi} \in C_\infty^0(\mathbb{R}^n)$ with its support away from $\{0\}$, $\hat{\phi} \geq 0$, and for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^\infty \hat{\phi}(tx) \hat{\psi}(tx) \, dt = 1.$$ 

Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, if $f$ vanishes weakly at infinity, then

$$f = \int_0^\infty f * \phi_t * \psi_t \, dt \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

namely,

$$f = \lim_{A \to 0^+} \int_0^A f * \phi_t * \psi_t \, dt \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

Let $X$ be a ball quasi-Banach function space. For any $q \in [1, \infty)$ and $d \in \mathbb{Z}_+$, a locally integrable function $f$ on $\mathbb{R}^n$ is said to be in the Campanato-type space $\mathcal{L}_{q,X,d}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}_{q,X,d}(\mathbb{R}^n)} := \sup_Q \left\{ \|Q\|_X \left[ \frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q \, dx \right]^{\frac{1}{q}} \right\} < \infty,$$

where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ and $P_Q^d$ denotes the unique polynomial $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that for any polynomial $R \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_Q |f(x) - P(x)| R(x) \, dx = 0$$

(see [62, Definition 6.1]); here and thereafter, the symbol $\mathcal{P}_d(\mathbb{R}^n)$ denotes the set of all polynomials with order at most $d$.

The following lemma is just [56, p. 54, Lemma 4.1] (see also [76, p. 83]).

**Lemma 4.5.** Let $d \in \mathbb{Z}_+$. Then there exists a positive constant $C$ such that for any $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and cube $Q \subset \mathbb{R}^n$,

$$\sup_{x \in Q} |P_Q^d g(x)| \leq \frac{C}{|Q|} \int_Q |g(x)| \, dx.$$

**Lemma 4.6.** Let $q \in [1, \infty)$, $d \in \mathbb{Z}_+$ and $X$ be a ball quasi-Banach function space. Assume that there exists a $p \in (0, \infty)$ such that $\mathcal{M}$ in (2.8) is bounded on $X^{1/p}$. If $p \in \left(\frac{n}{n+d+1}, \infty\right)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in \mathcal{L}_{q,X,d}(\mathbb{R}^n)$.

**Proof.** We first claim that there exists a positive constant $C$ such that for any two cubes $Q_1$ and $Q_2$ with $Q_1 \subset Q_2$,

$$\frac{\|1_{Q_2}\|_X}{\|1_{Q_1}\|_X} \leq C \left[ \frac{|Q_2|}{|Q_1|} \right]^{1/p}.$$  \hfill (4.4)

Indeed, we have $1_{Q_2} \leq \frac{|Q_2|}{|Q_1|} 1_{Q_1}$ and $\mathcal{M}(1_{Q_1}) \leq C 1_{Q_1}^{1/p}$. By this, Definition 2.3(ii) and the assumption that $\mathcal{M}$ is bounded on $X^{1/p}$, we know that there exists a positive constant $C$, independent of $Q_1$ and $Q_2$, such that

$$\|1_{Q_2}\|_X \leq \left[ \frac{|Q_2|}{|Q_1|} \right]^{1/p} \|\mathcal{M}(1_{Q_1})^{1/p}\|_X \leq \left[ \frac{|Q_2|}{|Q_1|} \right]^{1/p} \|1_{Q_1}\|_X^{1/p} \sim \left[ \frac{|Q_2|}{|Q_1|} \right]^{1/p} \|1_{Q_1}\|_X,$$

i.e., the above claim holds true.
For any \( f \in S(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \) and cube \( Q := Q(x_0, r) \subset \mathbb{R}^n \) with \( (x_0, r) \in \mathbb{R}^{n+1} \), to prove this lemma, let
\[
p_Q(x) := \sum_{|\beta| \leq d} \frac{\partial^\beta f(x_0)}{\beta!} (x - x_0)^\beta \in \mathcal{P}_d(\mathbb{R}^n).
\]
Then, from Lemma 4.5 and Hölder’s inequality, it follows that
\[
\left[ \int_Q |f(x) - P_Q^d f(x)|^q dx \right]^\frac{1}{q} \leq \left[ \int_Q |f(x) - p_Q(x)|^q dx \right]^\frac{1}{q} + \left[ \int_Q |P_Q^d (p_Q - f)(x)|^q dx \right]^\frac{1}{q}
\leq \left[ \int_Q |f(x) - p_Q(x)|^q dx \right]^\frac{1}{q} + \left\{ |Q| \left[ \frac{1}{|Q|} \int_Q |p_Q(x) - f(x)| dx \right]^q \right\}^\frac{1}{q}
\leq \left[ \int_Q |f(x) - p_Q(x)|^q dx \right]^\frac{1}{q}.
\]
(4.5)

Now, if \(|x_0| + r \leq 1\), namely, \( Q \subset Q(\bar{0}_n, \sqrt{n}) \), then, by (4.5), the Taylor remainder theorem and (4.4), we conclude that
\[
\frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q dx \right]^\frac{1}{q} \leq \frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q \sum_{|\beta| = d+1} \left| \frac{\partial^\beta f(x)}{\beta!} (x - x_0)^\beta \right|^q dx \right]^\frac{1}{q}
\leq \frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q |x - x_0|^{q(d+1)} dx \right]^\frac{1}{q}
\leq |Q|^{1+(d+1)/n-1/p} \frac{|Q(\bar{0}_n, \sqrt{n})|^{1/p}}{1_{Q(\bar{0}_n, \sqrt{n})||x}} \lesssim 1,
\]
(4.6)
where \( \xi(x) := x_0 + \theta(x - x_0) \) for some \( \theta \in [0, 1] \). If \(|x_0| + r > 1\) and \(|x_0| \leq 2r\), then \( r > 1/3 \) and \(|Q| \sim |Q(\bar{0}_n, \sqrt{n}(|x_0| + r))|\). From Lemma 4.5, Hölder’s inequality, the fact that \( |f(x)| \lesssim (1 + |x|)^{-n}\) for any \( x \in \mathbb{R}^n \), and (4.4), we deduce that
\[
\frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q dx \right]^\frac{1}{q} \leq \frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q |f(x)|^q dx \right]^\frac{1}{q}
\leq \frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_{B(\bar{0}_n, \sqrt{n}(|x_0| + r))} \frac{1}{(1 + |x|)^q} dx \right]^\frac{1}{q}
\leq \left[ \frac{|Q(\bar{0}_n, \sqrt{n}(|x_0| + r))|}{|Q|} \right]^{1/p} \frac{1}{1_{Q(\bar{0}_n, \sqrt{n}(|x_0| + r))||x}} \lesssim 1.
\]
(4.7)
If \(|x_0| + r > 1\) and \(|x_0| > 2r\), then for any \( x \in Q \), we have \(|x| \sim |x_0| \gtrsim 2/3\) and \( 1 + |x| \sim |x_0| + r \). By this, (4.5) and the fact that \( |\partial^\beta f(x)| \lesssim (1 + |x|)^{-n-\varepsilon} \) for any \( x \in \mathbb{R}^n \) with \( |\gamma| = d + 1 \) and some given \( \varepsilon \in (1 + d, \infty) \), and (4.4), we find that
\[
\frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q dx \right]^\frac{1}{q} \leq \frac{|Q|}{1_{Q||x}} \left[ \frac{1}{|Q|} \int_Q \sum_{|\beta| = d+1} \left| \frac{\partial^\beta f(x)}{\beta!} (x - x_0)^\beta \right|^q dx \right]^\frac{1}{q}
\lesssim \frac{|Q|^{1+(d+1)/n} (1 + |x_0|)^{-n-\varepsilon}}{|Q|^{1+(d+1)/n-1/p} \frac{1}{1_{Q(\bar{0}_n, \sqrt{n}(|x_0| + r))||x}} \lesssim 1,
\]
(4.8)
where $\xi(x) := x_0 + \theta(x - x_0)$ for some $\theta \in [0, 1]$. Combining (4.6)–(4.8), we know that $f \in \mathcal{L}_{q,X,d}(\mathbb{R}^n)$, which completes the proof of Lemma 4.5.

Now let us show Theorem 4.2. 

**Proof of Theorem 4.2.** Assume that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\text{supp} \psi \subset B(\overline{0}_n, 1)$ and $\int_{\mathbb{R}^n} \psi(x)x^\gamma dx = 0$ for any $\gamma \in \mathbb{Z}^{n}_{+}$ with $|\gamma| \leq d$. Then, by Lemma 4.4, we know that there exists a $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that the support of $\phi$ is compact and away from the origin, and for any $x \in \mathbb{R}^n \setminus \{\overline{0}_n\}$,

$$
\int_0^\infty \tilde{\psi}(tx)\phi(tx) \frac{dt}{t} = 1.
$$

Let $\eta$ be a function on $\mathbb{R}^n$ such that $\eta(\overline{0}_n) := 1$, and for any $x \in \mathbb{R}^n \setminus \{\overline{0}_n\}$,

$$
\eta(x) := \int_0^\infty \tilde{\psi}(tx)\phi(tx) \frac{dt}{t}.
$$

Then, by [12, p. 219], we know that such an $\eta$ exists and $\eta$ is infinitely differentiable, has compact support and equals 1 near the origin.

Let $x_0 := (2, \ldots, 2) \in \mathbb{R}^n$ and $f \in WH_X(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, let $\tilde{\phi}(x) := \phi(x - x_0)$, $\tilde{\psi}(x) := \psi(x + x_0)$, $F(x, t) := f * \tilde{\phi}(x)$ and $G(x, t) := f * \eta(x)$. Then, due to Assumption 2.20 and Theorem 3.2(ii), for any $f \in WH_X(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$
M_\gamma(f)(x) := \sup_{t \in (0, \infty), |y - x| \leq 3(|x_0| + 1)t} \|F(y, t)\| + \|G(y, t)\| \in W_X,
$$

and $\|M_\gamma(f)\|_{W_X} \sim \|f\|_{WH_X(\mathbb{R}^n)}$. Then, by Lemmas 4.3 and 4.4, we know that

$$
f(\cdot) = \int_0^\infty \int_{\mathbb{R}^n} F(y, t)\tilde{\psi}_t(\cdot - y) \frac{dy \, dt}{t} \text{ in } \mathcal{S}'(\mathbb{R}^n).
$$

For any $i \in \mathbb{Z}$, let $\Omega_i := \{x \in \mathbb{R}^n : M_\gamma(f)(x) > 2^i\}$. Then $\Omega_i$ is open and, by (2.7), we further find that

$$
\sup_{i \in \mathbb{Z}} \{2^i \|1_{\Omega_i} \|_X\} \leq \|M_\gamma(f)\|_{W_X} \lesssim \|f\|_{WH_X(\mathbb{R}^n)}.
$$

Since for any $i \in \mathbb{Z}$, $\Omega_i$ is a proper open subset of $\mathbb{R}^n$, by the Whitney decomposition (see, for example, [32, p. 463]), we know that there exists a sequence of cubes, $\{Q_{i,j}\}_{j \in \mathbb{N}}$, such that for any given $i \in \mathbb{Z}$,

(i) $\bigcup_{j \in \mathbb{N}} Q_{i,j} = \Omega_i$ and $\{Q_{i,j}\}_{j \in \mathbb{N}}$ have disjoint interiors;

(ii) for any $j \in \mathbb{N}$, $\sqrt{n}Q_{i,j} \subseteq \text{dist}(Q_{i,j}, \Omega^0_i) \leq 4\sqrt{n}Q_{i,j}$ here and thereafter, $l_{Q_{i,j}}$ denotes the side length of the cube $Q_{i,j}$ and $\text{dist}(Q_{i,j}, \Omega^0_i) := \inf\{|x - y| : x \in Q_{i,j}, y \in \Omega^0_i\}$;

(iii) for any $j$, $k \in \mathbb{N}$, if the boundaries of two cubes $Q_{i,j}$ and $Q_{i,k}$ touch, then $\frac{1}{4} \leq \frac{l_{Q_{i,j}}}{l_{Q_{i,k}}} \leq 4$;

(iv) for any given $j \in \mathbb{N}$, there exist at most $2^n$ different cubes $\{Q_{i,j}\}_k$ that touch $Q_{i,j}$.

For any $\epsilon \in (0, \infty)$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$
\text{dist}(x, \Omega^0_i) := \inf\{|x - y| : y \in \Omega_i\},
$$

$$
\tilde{\Omega}_i := \{\{x, t\} \in \mathbb{R}^{n+1} : x \in \Omega_i, (x, t) \in \mathbb{R}^n \times (0, \infty), 0 < 2t(|x_0| + 1) < \text{dist}(x, \Omega^0_i)\},
$$

$$
\tilde{Q}_{i,j} := \{\{x, t\} \in \mathbb{R}^{n+1} : x \in Q_{i,j}, (x, t) \in \tilde{\Omega}_i \setminus \tilde{\Omega}_{i+1}\}
$$

and

$$
b_{i,j}^\epsilon(x) := \int_{\mathbb{R}^n} 1_{\tilde{Q}_{i,j}}(y, t) \frac{F(y, t)\tilde{\psi}_t(x - y)}{t} \frac{dy \, dt}{t}.
$$

Then, by the proof of [12, pp. 221–222], we know that there exist positive constants $C_1$ and $C_2$ such that for any $\epsilon \in (0, \infty)$, $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\sup \frac{b_{i,j}^\epsilon \subseteq C_1Q_{i,j}$, $\|b_{i,j}^\epsilon\|_{L_\infty(\mathbb{R}^n)} \leq C_2 2^j$, and $\int_{\mathbb{R}^n} b_{i,j}^\epsilon(y) x^\gamma dx = 0$ for any $\gamma \in \mathbb{Z}^n_{+}$ satisfying $|\gamma| \leq d$. Moreover, for any $\zeta \in \mathcal{S}(\mathbb{R}^n)$, by the Lebesgue dominated convergence theorem and $\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \zeta \tilde{Q}_{i,j} = 1$, we have

$$
\left\langle \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i,j}^\epsilon, \zeta \right\rangle = \int_{\mathbb{R}^n} \zeta(x) \left( \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^n} 1_{\tilde{Q}_{i,j}}(y, t) \frac{F(y, t)\tilde{\psi}_t(x - y)}{t} \frac{dy \, dt}{t} dx \right) dx.
$$
Moreover, since for any \( \epsilon \in (0,1) \), \( i \in \mathbb{Z} \) and \( j \in \mathbb{N} \), \( \|b_{i,j}\|_{L^\infty(\mathbb{R}^n)} \leq C_2^{2^i} \), it follows that \( \{b_{i,j}\}_{i \in (0,1)} \) is bounded in \( L^\infty(\mathbb{R}^n) \). Then by the Banach-Alaoglu theorem (see, for example, [67, Theorem 3.17]), we find that there exist \( \{b_{i,j}\}_{i \in \mathbb{Z},j \in \mathbb{N}} \subset L^\infty(\mathbb{R}^n) \) and a sequence \( \{\epsilon_k\}_{k \in \mathbb{N}} \subset (0,\infty) \) such that \( \epsilon_k \to 0 \) as \( k \to \infty \), and for any \( i \in \mathbb{Z}, j \in \mathbb{N} \) and \( g \in L^1(\mathbb{R}^n) \),

\[
\lim_{k \to \infty} \langle b_{i,j}, g \rangle = \langle b_{i,j}, g \rangle, \tag{4.12}
\]

supp \( b_{i,j} \subset C_1Q_{i,j} \), \( \|b_{i,j}\|_{L^\infty(\mathbb{R}^n)} \leq C_2^{2^i} \), and for any \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| \leq d \),

\[
\int_{\mathbb{R}^n} b_{i,j}(x) x_\gamma dx = \int_{\mathbb{R}^n} b_{i,j}(x) x_\gamma 1_{C_1Q_{i,j}} = \lim_{k \to \infty} \int_{\mathbb{R}^n} b_{i,j}^k(x)x_\gamma dx = 0.
\]

Next, we show that

\[
\lim_{k \to \infty} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i,j}^k = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i,j} \in S'(\mathbb{R}^n). \tag{4.13}
\]

Indeed, by the facts that for any \( i \in \mathbb{Z} \) and \( j, k \in \mathbb{N} \), \( \|b_{i,j}\|_{L^\infty(\mathbb{R}^n)} \leq 2^i \) and \( \|b_{i,j}^k\|_{L^\infty(\mathbb{R}^n)} \leq 2^i \), and for any \( k \in \mathbb{N} \) and \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| \leq d \),

\[
\int_{\mathbb{R}^n} b_{i,j}(x) x_\gamma dx = \int_{\mathbb{R}^n} b_{i,j}^k(x) x_\gamma dx = 0,
\]

we conclude that for any \( N \in \mathbb{N} \) and \( \zeta \in S(\mathbb{R}^n) \),

\[
\sum_{|\gamma| \geq N} \sum_{j \in \mathbb{N}} |\langle b_{i,j}^k, \zeta \rangle| + |\langle b_{i,j}, \zeta \rangle| 
\]

\[
= -\sum_{i = -\infty}^{N-1} \sum_{j \in \mathbb{N}} |\langle b_{i,j}^k, \zeta \rangle| + |\langle b_{i,j}, \zeta \rangle| + \sum_{i = N+1}^{\infty} \sum_{j \in \mathbb{N}} \left( \left| \int_{C_1Q_{i,j}} b_{i,j}^k(x) \zeta(x) dx \right| - P_{C_1Q_{i,j}}^{d} \zeta(x) dx \right)
\]

\[
\leq -\sum_{i = -\infty}^{N-1} 2^i \int_{\mathbb{R}^n} |\zeta(x)| dx + \sum_{i = N+1}^{\infty} \sum_{j \in \mathbb{N}} 2^i \int_{C_1Q_{i,j}} |\zeta(x)| dx - P_{C_1Q_{i,j}}^{d} \zeta(x) dx.
\]

Since \( d \geq \lfloor n(1/p - 1) \rfloor \), it follows that \( p \in (\frac{n}{n(1/p - 1)} + 1) \), which, together with \( X^{1/(\varsigma_0)} = X^{1/\varsigma_0} \), the assumption that \( \mathcal{M} \) in (2.8) is bounded on \( X^{1/(\varphi_0)} \), and Lemma 4.6, further implies that for any \( \zeta \in S(\mathbb{R}^n) \), \( \|\zeta\|_{L^{1,\varsigma_0,\varphi_0}(\mathbb{R}^n)} < \infty \). By this, and the assumption that \( X^{1/\varphi_0} \) is concave with \( \varphi_0 > 1 \), we further conclude that for any \( k, N \in \mathbb{N} \) and \( \zeta \in S(\mathbb{R}^n) \),

\[
\sum_{|\gamma| \geq N} \sum_{j \in \mathbb{N}} |\langle b_{i,j}^k, \zeta \rangle| + |\langle b_{i,j}, \zeta \rangle| 
\]

\[
\lesssim 2^{-N} \|\zeta\|_{L^1(\mathbb{R}^n)} + \sum_{i = N+1}^{\infty} 2^i \|1_{Q_{i,j}}\|_{X^{1/\varphi_0}} \|\zeta\|_{L^{1,\varsigma_0,\varphi_0}(\mathbb{R}^n)} 
\]

\[
\lesssim 2^{-N} \|\zeta\|_{L^1(\mathbb{R}^n)} + \|\zeta\|_{L^{1,\varsigma_0,\varphi_0}(\mathbb{R}^n)} \sum_{i = N+1}^{\infty} 2^i \|1_{Q_{i,j}}\|_{X^{1/\varphi_0}} 
\]

\[
\lesssim 2^{-N} \|\zeta\|_{L^1(\mathbb{R}^n)} + \|\zeta\|_{L^{1,\varsigma_0,\varphi_0}(\mathbb{R}^n)} \left( \sup_{i \in \mathbb{Z}} 2^i \|1_{Q_{i}}\|_{X}^{\varphi_0} \right)^{\frac{\varphi_0}{\varphi_0 - 1}} \sum_{i = N+1}^{\infty} 2^{-i(\varphi_0 - 1)}.
\]
Let \( \lambda \) be the positive constant \( C_{(N, \zeta)} \) such that for any \( k \in \mathbb{N} \),

\[
\sum_{|i| \leq N} \sum_{j \in \mathbb{N}} \left[ |(b_{i,j}^1, \zeta)| + |(b_{i,j}, \zeta)| \right] \leq C_{(N, \zeta)} < \infty.
\]

(4.15)

Therefore, using (4.14) and (4.15), and repeating the argument similar to that used in [52, p. 651], we find that (4.13) holds true.

For any \( i \in \mathbb{N} \) and \( j \in \mathbb{N} \), let \( B_{i,j} \) be the ball with the same center as \( Q_{i,j} \) and the radius \( 5 r \), where the implicit positive constant is independent of \( i,j \).

Then, using the properties of \( b_{i,j} \), we know that \( a_{i,j} \) is an \((X, \infty, d)\)-atom supported in the ball \( B_{i,j} \) satisfying that \( \{cB_{i,j}\}_{j \in \mathbb{N}} \) for any given \( i \in \mathbb{N} \) is finite overlapping for some \( c \in (0, 1) \) and, due to (4.13) and (4.11), \( f = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} \in \mathcal{S}'(\mathbb{R}^n) \). Similarly to (2.12), by (4.1), we conclude that

\[
\left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \lesssim \left\| \sum_{j \in \mathbb{N}} 1_{Q_{i,j}} \right\|_X.
\]

From this and (4.10), we deduce that

\[
\sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X \sim \sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \lesssim \sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} 1_{Q_{i,j}} \right\|_X \lesssim \sup_{i \in \mathbb{N}} \left\| \lambda_{i,j} a_{i,j} \right\|_X \lesssim \|f\|_{WH_X(\mathbb{R}^n)},
\]

which completes the proof of Theorem 4.2.

Next, we present a reconstruction theorem.

**Theorem 4.7.** Let \( X \) be a ball quasi-Banach function space satisfying Assumption 2.18 for some \( p_+ \in (0, \infty) \). Assume that for any given \( r \in (0, p) \) with \( p \) as in (2.11), \( X^{1/r} \) is a ball Banach function space. Assume that there exist an \( r_0 \in (0, p) \) and a \( p_0 \in (r_0, \infty) \) such that for any \( f \in (X^{1/r_0})' \),

\[
\|\mathcal{M}^{(p_0/r_0)}(f)\|_{(X^{1/r_0})'} \lesssim C\|f\|_{(X^{1/r_0})'},
\]

(4.16)

where the positive constant \( C \) is independent of \( f \). Let \( d \in \mathbb{Z}_+ \) with \( d \geq (n/2 - 1) \), \( c \in (0, 1) \), \( q \in (\max\{1, p_0\}, \infty) \) and \( A \), \( \tilde{A} \in (0, \infty) \), and let \( \{a_{i,j}\}_{i \in \mathbb{N}, j \in \mathbb{N}} \) be a sequence of \((X, q, d)\)-atoms supported, respectively, in balls \( \{B_{i,j}\}_{i \in \mathbb{N}, j \in \mathbb{N}} \) satisfying that \( \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \leq A \) for any \( i \in \mathbb{N} \), \( \lambda_{i,j} := \tilde{A}^2 \|1_{B_{i,j}}\|_X \) for any \( i \in \mathbb{N} \) and \( j \in \mathbb{N} \), the series \( f := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} \) converges in \( \mathcal{S}'(\mathbb{R}^n) \) and

\[
\sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X < \infty.
\]

Then \( f \in WH_X(\mathbb{R}^n) \) and

\[
\|f\|_{WH_X(\mathbb{R}^n)} \lesssim \sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X,
\]

where the implicit positive constant is independent of \( f \).

To prove Theorem 4.7, we need the following useful technical lemma.

**Lemma 4.8.** Let \( r \in (0, \infty) \), \( q \in (r, \infty) \) and \( X \) be a ball quasi-Banach function space. Assume that \( X^{1/r} \) is a ball Banach function space and there exists a positive constant \( C \) such that for any \( f \in (X^{1/r})' \),

\[
\|\mathcal{M}^{(q/r)}(f)\|_{(X^{1/r})'} \leq C\|f\|_{(X^{1/r})'},
\]

where the implicit positive constant is independent of \( f \).
Then there exists a positive constant $C$ such that for any sequence $\{B_j\}_{j \in \mathbb{N}}$ of balls, numbers $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and measurable functions $\{a_j\}_{j \in \mathbb{N}}$ satisfying that for any $j \in \mathbb{N}$, $\text{supp}(a_j) \subset B_j$ and $\|a_j\|_{L^p(\mathbb{R}^n)} \leq |B_j|^{1/p}$,
\[
\left\| \left( \sum_{j \in \mathbb{N}} |\lambda_j a_j|^r \right)^{\frac{1}{r}} \right\|_{X} \leq C \left\| \left( \sum_{j \in \mathbb{N}} |\lambda_j 1_{B_j}|^r \right)^{\frac{1}{r}} \right\|_{X}.
\]

**Proof.** By the definition of the associate space, the assumption that $X^{1/r}$ is a ball Banach function space and Lemma 2.6, we have
\[
\left\| \left( \sum_{j \in \mathbb{N}} |\lambda_j a_j|^r \right)^{\frac{1}{r}} \right\|_{X} = \left\| \sum_{j \in \mathbb{N}} \|\lambda_j a_j\|_{X^{1/r}} \right\|_{X^{1/r}} = \sup \left\{ \int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}} |\lambda_j a_j|^r |g(x)| \, dx : g \in (X^{1/r})' \text{ such that } \|g\|_{(X^{1/r})'} = 1 \right\}.
\]

Then, from Hölder’s inequality, we deduce that for any $g \in (X^{1/r})'$ with $\|g\|_{(X^{1/r})'} = 1$,
\[
\int_{\mathbb{R}^n} \sum_{j \in \mathbb{N}} |\lambda_j a_j(x)|^r |g(x)| \, dx = \sum_{j \in \mathbb{N}} |\lambda_j|^r \int_{\mathbb{R}^n} |a_j(x)|^r |g(x)| \, dx \leq \sum_{j \in \mathbb{N}} |\lambda_j|^r \|a_j\|_{L^q(\mathbb{R}^n)} \|g 1_{B_j}\|_{L^{q/r}(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{N}} |\lambda_j|^r \|g\|_{(X^{1/r})'} \leq \sum_{j \in \mathbb{N}} \|\lambda_j 1_{B_j}\|_{X^{1/r}} = K.
\]

Applying Lemma 2.5 and the assumption that $M^{(q/r')}$ is bounded on $(X^{1/r})'$, we conclude that
\[
K \leq \left\| \sum_{j \in \mathbb{N}} |\lambda_j|^r 1_{B_j} \right\|_{X^{1/r}} \left\| \left( \mathcal{M}(g^{(q/r')})(x) \right)^{(1/q/r')} \right\|_{X^{1/r}} \leq \left\| \sum_{j \in \mathbb{N}} |\lambda_j|^r 1_{B_j} \right\|_{X^{1/r}} \|g\|_{(X^{1/r})'} \sim \left\| \sum_{j \in \mathbb{N}} |\lambda_j|^r 1_{B_j} \right\|_{X^{1/r}},
\]

which, together with Definition 2.7(i), further implies the desired conclusion. This finishes the proof of Lemma 4.8. \(\square\)

Now we prove Theorem 4.7.

**Proof of Theorem 4.7.** Let $c \in (0, 1]$, $q \in (p_0, p_0/r]$, and $\{a_{i,j}\}_{i, j \in \mathbb{N}}$ be a sequence of $(X, q, d)$-atoms supported, respectively, in balls $\{B_{j,i}\}_{i, j \in \mathbb{N}}$ satisfying that for any $i \in \mathbb{Z}$, $\sum_{j \in \mathbb{N}} 1_{B_{j,i}} \leq A$ with $A$ being a positive constant independent of $i$, $\lambda_{i,j} := A^2 \|1_{B_{j,i}}\|_{X}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ with $\tilde{A}$ being a positive constant independent of $i$ and $j$.
\[
f := \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),
\]

and
\[
\sup_{i \in \mathbb{Z}} 2^{i} \left\| \sum_{j \in \mathbb{N}} 1_{B_{j,i}} \right\|_{X} < \infty.
\]

To prove $f \in WH_X(\mathbb{R}^n)$, by the definition of $WH_X(\mathbb{R}^n)$, it suffices to show that
\[
\sup_{\alpha \in (0, \infty)} \|1_{\{x \in \mathbb{R}^n : M_X^\alpha(f)(x) > \alpha\}}\|_X \leq \sup_{i \in \mathbb{Z}} 2^{i} \left\| \sum_{j \in \mathbb{N}} 1_{B_{j,i}} \right\|_X.
\]

For any fixed $\alpha \in (0, \infty)$, let $i_0 \in \mathbb{Z}$ be such that $2^{i_0} \leq \alpha < 2^{i_0+1}$ and we then write
\[
f = \sum_{i = -\infty}^{i_0 - 1} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} + \sum_{i = i_0}^{\infty} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} =: f_1 + f_2.
\]
From Definition 2.3(ii), it follows that
\[
\|1_{\{x \in \mathbb{R}^n : M_N^0(f)(x) > \frac{q}{r}\}} \|_X \lesssim \|1_{\{x \in \mathbb{R}^n : M_N^0(f)(x) > \frac{q}{r}\}} \|_X + \|1_{\{x \in A_{i_0} : M_N^0(f)(x) > \frac{q}{r}\}} \|_X + \|1_{\{x \in (A_{i_0})^c : M_N^0(f)(x) > \frac{q}{r}\}} \|_X
\]
\[
=: I_1 + I_2 + I_3,
\]
(4.17)
where \(A_{i_0} := \bigcup_{i=i_0}^{\infty} \bigcup_{j \in \mathbb{N}} (2B_{i,j}).\)

For \(I_1\), by Definition 2.3(ii), we further decompose it into
\[
I_1 \lesssim \|1_{\{x \in \mathbb{R}^n : \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) > \frac{q}{r}\}} \|_X
\]
\[
+ \|1_{\{x \in \mathbb{R}^n : \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) > \frac{q}{r}\}} \|_X
\]
\[
=: I_{1.1} + I_{1.2}.
\]
(4.18)
We first estimate \(I_{1.1}\). Let \(\tilde{q} := q/p_0 \in (1, 1/r_0)\) and \(a \in (0, 1 - 1/\tilde{q})\). Then, from Hölder's inequality, we deduce that
\[
\sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \lesssim \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \right)^{1/\tilde{q}} \lesssim \|M_N^0(f)\|_{X^{1/r_0}} \lesssim \|M(f)\|_{X^{1/r_0}}
\]
where \(\tilde{q}^* := \tilde{q}/(\tilde{q} - 1)\) by this, Definitions 2.3(ii) and 2.7(i), \(\tilde{q}_0 \in (0, 1)\) and the fact that \(M_N^0(f) \lesssim M(f)\) and the assumption that \(X^{1/r_0}\) is a ball Banach function space, we conclude that
\[
I_{1.1} \lesssim \|1_{\{x \in \mathbb{R}^n : \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) > \frac{q}{r}\}} \|_X
\]
\[
\lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \right)^{1/\tilde{q}} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x)^{1/\tilde{q}} \right)^{\tilde{q}} \lesssim \|M_N^0(f)\|_{X^{1/r_0}}
\]
\[
\lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \right)^{1/\tilde{q}} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x)^{1/\tilde{q}} \right)^{\tilde{q}} \lesssim \|M_N^0(f)\|_{X^{1/r_0}}
\]
\[
\lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \right)^{1/\tilde{q}} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x)^{1/\tilde{q}} \right)^{\tilde{q}} \lesssim \|M_N^0(f)\|_{X^{1/r_0}}
\]
\[
\lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x) \right)^{1/\tilde{q}} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) 1_{2B_{i,j}}(x)^{1/\tilde{q}} \right)^{\tilde{q}} \lesssim \|M_N^0(f)\|_{X^{1/r_0}}
\]
From \(q = p_0 \tilde{q}\) and the boundedness of \(M\) on \(L^q(\mathbb{R}^n)\) and Definition 4.1(ii), it follows that for any \(i \in \mathbb{Z}\) and \(j \in \mathbb{N},\)
\[
\|1_{B_{i,j}} \mathcal{M}(a_{i,j}) \|_{L^{q}(\mathbb{R}^n)} \lesssim \|1_{B_{i,j}} \|_{X^q} \mathcal{M}(a_{i,j}) \|_{L^{q}(\mathbb{R}^n)} \lesssim \|1_{B_{i,j}} \|_{X^q} \|a_{i,j} \|_{L^{q}(\mathbb{R}^n)} \lesssim |B_{i,j}|^{\frac{1}{q_0}}
\]
which, combined with Lemma 4.8, (2.12) and \((1 - a)\tilde{q} > 1\), further implies that
\[
I_{1.1} \lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} 2^{(1-a)io \tilde{q} r_0} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x) \right)^{\frac{1}{\tilde{q}}} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x)^{\frac{1}{\tilde{q}}} \right)^{\tilde{q}} \right) \lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} 2^{(1-a)io \tilde{q} r_0} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x) \right)^{\frac{1}{\tilde{q}}} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x)^{\frac{1}{\tilde{q}}} \right)^{\tilde{q}} \right) \lesssim 2^{-io \tilde{q}(1-a)} \left( \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} 2^{(1-a)io \tilde{q} r_0} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x) \right)^{\frac{1}{\tilde{q}}} \left( \sum_{j \in \mathbb{N}} 1_{2B_{i,j}}(x)^{\frac{1}{\tilde{q}}} \right)^{\tilde{q}} \right)
\]
\[
\lesssim |B_{i,j}|^{\frac{1}{q_0}} \sum_{j \in \mathbb{N}} \|1_{B_{i,j}} \|_{X^q} \lesssim |B_{i,j}|^{\frac{1}{q_0}} \|1_{B_{i,j}} \|_{X^q}
\]
\[
\lesssim \alpha^{-1} \sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right) \|1_{B_{i,j}} \|_{X^q}
\]
This shows that

$$\alpha I_{1,1} \lesssim \sup_{t \in \mathbb{Z}} \left[ 2^t \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right].$$  \hfill (4.19)

To deal with $I_{1,2}$, we first estimate $M_1^0(f)$ on $(2B_{i,j})^c$. Let $\phi \in \mathcal{F}_N(\mathbb{R}^n)$, and for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $x_{i,j}$ denote the center of $B_{i,j}$ and $r_{i,j}$ its radius. Then, using the vanishing moments of $a_{i,j}$ and the Taylor remainder theorem, we have, for any $i \in \mathbb{Z} \cap [0, \infty)$, $j \in \mathbb{N}$, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$|a_{i,j} \ast \phi_t(x)| = \left| \int_{B_{i,j}} a_{i,j}(y) \left[ \phi \left( \frac{x - y}{t} \right) - \sum_{|\beta| \leq d} \frac{\partial^\beta \phi(x - y)}{\beta!} \frac{(x_{i,j} - y)^\beta}{t^\beta} \right] dy \right| \lesssim \int_{B_{i,j}} |a_{i,j}(y)| \sum_{|\beta| = d+1} \left| \partial^\beta \phi \left( \frac{x_{i,j} - y}{t} \right) \right| \left| x_{i,j} - y \right|^{d+1} \frac{dy}{t^n},$$  \hfill (4.20)

where $\xi := (x - x_{i,j}) + \theta(x_{i,j} - y)$ for some $\theta \in [0, 1]$.

For any $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $x \in (2B_{i,j})^c$ and $y \in B_{i,j}$, it is easy to see that $|x - y| \sim |x - x_{i,j}|$ and $|\xi| \geq |x - x_{i,j}| - |x_{i,j} - y| \gtrsim |x - x_{i,j}|$. By this, (4.20), the fact that $\phi \in S(\mathbb{R}^n)$ and Hölder’s inequality, we conclude that for any $i \in \mathbb{Z} \cap [0, \infty)$, $j \in \mathbb{N}$, $t \in (0, \infty)$ and $x \in (2B_{i,j})^c$,

$$|a_{i,j} \ast \phi_t(x)| \lesssim \int_{B_{i,j}} |a_{i,j}(y)| \left| \frac{y - x_{i,j}}{|x - x_{i,j}|^{n+d+1}} \right|^{d+1} dy \lesssim \left( \frac{r_{i,j}}{|x - x_{i,j}|} \right)^{d+1} \left[ \int_{B_{i,j}} |a_{i,j}(y)| q dy \right]^{1/q} |B_{i,j}|^{1/q'},$$  \hfill (4.21)

which implies that for any $x \in (2B_{i,j})^c$,

$$M_1^0(a_{i,j})(x) \lesssim \left\| |B_{i,j}|^{-1} \mathcal{M}(1_{B_{i,j}})(x) \right\|_X^{n+d+1}.$$  \hfill (4.22)

Observe that $d \geq \lceil n(\frac{1}{p} - 1) \rceil$ implies $p \in (\frac{n}{n+d+1}, 1]$. Let $r_1 \in (0, \frac{n}{n+d+1}) \subset (0, p)$, $q_1 \in \left( \frac{n}{(n+d+1)q_1} - \frac{1}{q_1} \right)$, $a \in (0, 1 - \frac{1}{q_1})$. From Hölder’s inequality, it follows that

$$\sum_{i = -\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c} \leq \frac{2^{2\alpha/a}}{(2^q - 1)^{1/6}} \left\{ \sum_{i = -\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c} \right\}^{1/q_1},$$

where $q_1' := q_1 / (q_1 - 1)$. By this, Definition 2.3(ii), (4.22), the definition of $\lambda_{i,j}$ and the assumption that $X^{1/r_1}$ is a ball Banach function space, we conclude that

$$I_{1,2} \lesssim \left\| \frac{\sum_{i \in \mathbb{N}} 2^{-i \alpha} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c}}{(n+d+1)q_1} \right\|_X \lesssim 2^{-i_0 q_1 (1 - a)} \left\| \sum_{i = -\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c} \right\|_X^{1/q_1} \left\| \mathcal{M}(1_{B_{i,j}}) \right\|^{(n+d+1)q_1 r_1}_X \lesssim 2^{-i_0 q_1 (1 - a)} \left\| \sum_{i = -\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c} \right\|_X^{1/q_1} \left\| \mathcal{M}(1_{B_{i,j}}) \right\|^{(n+d+1)q_1 r_1}_X \lesssim 2^{-i_0 q_1 (1 - a)} \left\| \sum_{i = -\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_1^0(a_{i,j}) 1_{(2B_{i,j})^c} \right\|_X^{1/q_1} \left\| \mathcal{M}(1_{B_{i,j}}) \right\|^{(n+d+1)q_1 r_1}_X.$$
\[
\lesssim 2^{-i_0 q_1 \alpha (1 - \alpha)} \left\{ \sum_{i = -\infty}^{i_0 - 1} 2^{(1 - \alpha)q_1 r_1} \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right\} \frac{1}{r_1}
\]

\[
\lesssim 2^{-i_0 q_1 (1 - \alpha)} \left\{ \sum_{i = 0}^{i_0 - 1} 2^{(1 - \alpha)q_1 r_1 + 1} \left( \left\| \sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \right\|_X \right) \right\} \frac{1}{r_1} \lesssim \alpha^{-1} \sup_{i \in \mathbb{Z}} 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X,
\]

which implies that

\[
\alpha_1 \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right].
\] (4.23)

By this, (4.18) and (4.19), we find that

\[
\alpha_1 \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right].
\] (4.24)

Next, we deal with $I_2$. Let $r_2 \in (0, p)$. Then, by (2.12), Definition 2.7(i), the assumption that $X^{1/r_2}$ is a ball Banach function space and $\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \lesssim A$, we conclude that

\[
I_2 \lesssim \left\| 1_{A_{i_0}} \right\|_X \lesssim \left\| \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} 1_{2B_{i,j}} \right\|_X \lesssim \left\| \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \sim \left\{ \left( \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \right)^{r_2} \right\}^{\frac{1}{r_2}} \left( \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} \left\| 1_{B_{i,j}} \right\|_X \right)^{\frac{1}{r_2}} \lesssim \alpha^{-1} \sup_{i \in \mathbb{Z}} 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X,
\]

which implies that

\[
\alpha_2 \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right].
\] (4.25)

It remains to estimate $I_3$. Recall that $\frac{p}{2} \in \left( \frac{n}{n + d + 1}, 1 \right]$ and hence there exists an $r_3 \in \left( \frac{n}{n + d + 1}, 1 \right)$. By Definitions 2.3(ii) and 2.7(i), the assumption that $X^{1/r_3}$ is a ball Banach function space and (4.22), we conclude that

\[
I_3 \lesssim \left\| 1_{\{ x \in (A_{i_0})^{c} \} \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(a_{i,j}) (x) \geq \frac{n}{n + d + 1} \right\|_X \lesssim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} \right\} \left\| 1_{(A_{i_0})^{c}} \right\|_X
\]

\[
\lesssim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \right\} \sim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \right\}
\]

\[
\lesssim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \right\} \sim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \right\}
\]

Since \( \frac{n}{n + d + 1} \in (0, p) \subset (0, 1) \), from Definition 2.7(i) and Assumption 2.18, it follows that

\[
I_3 \lesssim \alpha^{-r_3} \left\{ \sum_{i = i_0}^\infty \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \right\} \sim \alpha^{-r_3} \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \left\| \lambda_{i,j} M_N^0(a_{i,j}) \right\|_{X^{1/r_3}} 1_{(A_{i_0})^{c}} \right\|_X \lesssim \alpha^{-1} \sup_{i \in \mathbb{Z}} 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X
\]
namely,
\[\alpha I_3 \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right].\] (4.26)

By (4.17) and (4.24)–(4.26), we conclude that
\[\|f\|_{\text{W}^H(X)} = \sup_{\alpha \in (0, \infty)} \{\alpha \|1_{\{x \in \mathbb{R}^n : M_{\alpha}^X(f)(x) > \alpha\}}\|_X\} \lesssim \sup_{\alpha \in (0, \infty)} \{\alpha (I_1 + I_2 + I_3)\} \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_X \right],\]
which completes the proof of Theorem 4.7.

\[\square\]

5 Molecular characterizations

In this section, we establish the molecular characterization of \(\text{W}^H(X)\). We begin with recalling the notion of molecules (see [70, Definition 3.8]).

**Definition 5.1.** Let \(X\) be a ball quasi-Banach function space, \(\epsilon \in (0, \infty)\), \(q \in [1, \infty)\) and \(d \in \mathbb{Z}_+\). A measurable function \(m\) is called an \((X, q, d, \epsilon)\)-molecule associated with some ball \(B \subset \mathbb{R}^n\) if

(i) for any \(j \in \mathbb{N}\),
\[\|m\|_{L^\infty(S_j(B))} \lesssim 2^{-j\epsilon} |S_j(B)|^{\frac{1}{d}} \|1_B\|^{-1}_X,\]
where \(S_0 := B\), and for any \(j \in \mathbb{N}\), \(S_j(B) := (2^j B) \setminus (2^{j-1} B)\);

(ii) \(\int_{\mathbb{R}^n} m(x) x^\beta dx = 0\) for any \(\beta \in \mathbb{Z}_+^d\) with \(|\beta| \leq d\).

**Theorem 5.2.** Let \(X\) and \(p\) be the same as in Theorem 4.2. Let \(d \geq |n(1/p - 1)|\) be a fixed nonnegative integer, \(\epsilon \in (n + d + 1, \infty)\) and \(f \in \text{W}^H(X)\). Then \(f\) can be decomposed into
\[f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j}\]
in \(S'(\mathbb{R}^n)\), where \(\{m_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) is a sequence of \((X, \infty, d, \epsilon)\)-molecules associated, respectively, with balls \(\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) and \(\{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}} := \{A^2 \|1_{B_{i,j}}\|_X\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) with \(A\) being a positive constant independent of \(f\), \(i\) and \(j\), and there exist positive constants \(A\) and \(c\) such that for any \(i \in \mathbb{Z}\), \(\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \leq A\). Moreover,
\[\sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X \lesssim \|f\|_{\text{W}^H(X)},\]
where the implicit positive constant is independent of \(f\).

**Proof.** Observe that every \((X, \infty, d)\)-atom is also an \((X, \infty, d, \epsilon)\)-molecule. Thus, Theorem 5.2 is a direct corollary of Theorem 4.2, which completes the proof of Theorem 5.2.

**Theorem 5.3.** Let \(X\) be a ball quasi-Banach function space satisfying Assumption 2.18 for some \(p_- \in (0, \infty)\). Assume that for any given \(r \in (0, p)\) with \(p\) as in (2.11), \(X^{1/r}\) is a ball Banach function space and assume that there exists a \(p_+ \in [p_-, \infty)\) such that for any given \(r \in (0, p_-)\) and \(p \in (p_+, \infty)\), and any \(f \in (X^{1/r})'\),
\[\|M_r(f)\|_{(X^{1/r})'} \leq C\|f\|_{(X^{1/r})'},\]
where the positive constant \(C\) is independent of \(f\). Let \(d \in \mathbb{Z}_+\) be such that \(d \geq \lfloor n(1/p - 1) \rfloor\). Let \(q \in (\max\{p_+, 1\}, \infty), \epsilon \in (n + d + 1, \infty), A, \tilde{A} \in (0, \infty)\) and \(c \in (0, 1]\), and let \(\{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) be a sequence of \((X, q, d, \epsilon)\)-molecules associated, respectively, with balls \(\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) satisfying \(\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \leq A\) for any \(i \in \mathbb{Z}\), \(\{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}} := \{A^2 \|1_{B_{i,j}}\|_X\}_{i \in \mathbb{Z}, j \in \mathbb{N}},\)
\[\sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X < \infty\]
and the series \( f := \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j} \) converges in \( S'(\mathbb{R}^n) \). Then \( f \in WH_X(\mathbb{R}^n) \) and

\[
\|f\|_{WH_X(\mathbb{R}^n)} \leq \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{j,i}} \right\|_X,
\]

where the implicit positive constant is independent of \( f \).

**Proof.** Let \( m \) be any given \((X, q, d, \epsilon)\)-molecule associated with some ball \( B := B(x_B, r_B) \), where \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \). Without loss of generality, we may assume that the center of the ball is the origin. Then we claim that \( m \) is an infinite linear combination of \((X, q, d)\)-atoms both pointwisely and in \( S'(\mathbb{R}^n) \).

To show this, for any \( k \in \mathbb{Z}_+ \), let \( m_k := m 1_{S_k(B)} \) with \( S_k(B) \) as in Definition 5.1(i), and \( P_k \) be the linear vector space generated by the set \( \{ x^\gamma 1_{S_k(B)} \}_{|\gamma| \leq d} \) of “polynomial”. For any given \( k \in \mathbb{Z}_+ \), we know that there exists a unique polynomial \( P_k \in \mathcal{P}_k \) such that for any multi-index \( \beta \) with \(|\beta| \leq d\),

\[
\int_{\mathbb{R}^n} x^\beta [m_k(x) - P_k(x)] dx = 0,
\]

(5.1)

where \( P_k \) is defined by setting

\[
P_k := \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \leq d} \left[ \frac{1}{|S_k(B)|} \int_{\mathbb{R}^n} y^\beta m_k(y) dy \right] Q_{\beta,k},
\]

(5.2)

and for any \( \beta \in \mathbb{Z}_+^n \) and \(|\beta| \leq d\), \( Q_{\beta,k} \) is the unique polynomial in \( \mathcal{P}_k \) satisfying that for any multi-index \( \gamma \) with \(|\gamma| \leq d\),

\[
\int_{\mathbb{R}^n} x^\gamma Q_{\beta,k}(x) dx = |S_k(B)| \delta_{\gamma,\beta},
\]

(5.3)

where \( \delta_{\gamma,\beta} \) denotes the Kronecker delta, namely, when \( \gamma = \beta, \delta_{\gamma,\beta} := 1 \) and, when \( \gamma \neq \beta, \delta_{\gamma,\beta} := 0 \) (see, for example, [76, p.77]).

Using the polynomials \( \{ P_k \}_{k=0}^\infty \), we decompose

\[
m = \sum_{k=0}^\infty m_k = \sum_{k=0}^\infty (m_k - P_k) + \sum_{k=0}^\infty P_k
\]

pointwisely. First, we show that \( \sum_{k=0}^\infty (m_k - P_k) \) can be divided into an infinite linear combination of \((X, q, d)\)-atoms. For any \( k \in \mathbb{Z}_+ \), obviously, \( \text{supp}(m_k - P_k) \subset S_k(B) \) and it was proved in [76, p.83] that

\[
\sup_{x \in S_k(B)} |P_k(x)| \lesssim \frac{1}{|S_k(B)|} \left\| m_k \right\|_{L^1(\mathbb{R}^n)},
\]

which, together with Hölder’s inequality and Definition 5.1(i), implies that

\[
\left\| m_k - P_k \right\|_{L^1(\mathbb{R}^n)} \lesssim \left\| m_k \right\|_{L^q(S_k(B))} + \left\| P_k \right\|_{L^q(S_k(B))} \lesssim \tilde{C} \left\| m \right\|_{L^q(S_k(B))} \lesssim \tilde{C} 2^{-k|\beta|} \|1_B\|_X, \quad (5.4)
\]

where \( \tilde{C} \) is a positive constant independent of \( m, B \) and \( k \).

For any \( k \in \mathbb{Z}_+ \), let

\[
a_k := \frac{2^{k\epsilon} \left\| 1_B \right\|_X (m_k - P_k)}{\tilde{C} \left\| 1_{2^k B} \right\|_X} \quad \text{and} \quad \mu_k := \tilde{C} 2^{-k|\beta|} \left\| 1_{2^k B} \right\|_X.
\]

By (5.4) and (5.1), it is easy to show that for any \( k \in \mathbb{Z}_+ \), \( a_k \) is an \((X, q, d)\)-atom. Therefore,

\[
\sum_{k=0}^\infty (m_k - P_k) = \sum_{k=0}^\infty \mu_k a_k \quad (5.5)
\]

pointwisely is an infinite linear combination of \((X, q, d)\)-atoms.
Now we prove that \( \sum_{k=0}^{\infty} P_k \) can also be pointwisely divided into an infinite linear combination of \((X, q, d)\)-atoms. For any \( j \in \mathbb{Z}_+ \) and \( \ell \in \mathbb{Z}_+^n \), let

\[
N_{\ell}^j := \sum_{k=j}^{\infty} \int_{S_k(B)} m_k(x)x^d \, dx.
\]

Then for any \( \ell \in \mathbb{Z}_+^n \) with \( |\ell| \leq d \), by Definition 5.1(ii), we have

\[
N_{\ell}^0 = \sum_{k=j}^{\infty} \int_{S_k(B)} m_k(x)x^d \, dx = \int_{\mathbb{R}^n} m(x)x^d \, dx = 0. \tag{5.6}
\]

Therefore, from Hölder’s inequality and the assumption that \( \epsilon \in (n + d + 1, \infty) \), combined with Definition 5.1(i), we deduce that for any \( j \in \mathbb{Z}_+ \) and \( \ell \in \mathbb{Z}_+^n \) with \( |\ell| \leq d \),

\[
|N_{\ell}^j| \leq \sum_{k=j}^{\infty} \int_{S_k(B)} |m_k(x)x^d| \, dx \leq \sum_{k=j}^{\infty} (2^{k}\rho_B)^{|\ell|/q} |2^k B|^{1/q'} \|m\|_{L^q(S_k(B))}
\]

\[
\lesssim \sum_{k=j}^{\infty} 2^{-k(\epsilon-n-|\ell|)}|B|^{1+|\ell|/n}1_B \|1_B\|_{X}^{-1} \lesssim 2^{-j(\epsilon-n-|\ell|)}|B|^{1+|\ell|/n}1_B \|1_B\|_{X}^{-1}. \tag{5.7}
\]

Furthermore, by the argument used in [76, p. 77], we know that for any \( j \in \mathbb{Z}_+ \) and \( \beta \in \mathbb{Z}_+^n \) with \( |\beta| \leq d \), \( |Q_{\beta,j}| \leq (2^r\rho_B)^{-|\beta|} \), which, together with (5.7), implies that for any \( j \in \mathbb{Z}_+ \), \( \ell \in \mathbb{Z}_+^n \) with \( |\ell| \leq d \) and \( x \in \mathbb{R}^n \),

\[
|\{j\} |^{-1}|N_{\ell}^j Q_{\ell,j} 1_{S_j(B)}(x)| \lesssim 2^{-jr} \|1_B\|_{X}^{-1}. \tag{5.8}
\]

Moreover, by (5.2), the definition of \( N_{\ell}^j \) and (5.6), we conclude that

\[
\sum_{k=0}^{\infty} P_k = \sum_{\ell \in \mathbb{Z}_+^n, |\ell| \leq d} \sum_{k=0}^{\infty} |S_k(B)|^{-1} Q_{\ell,k} \int_{\mathbb{R}^n} m_k(x)x^d \, dx
\]

\[
= \sum_{\ell \in \mathbb{Z}_+^n, |\ell| \leq d} \sum_{k=0}^{\infty} N_{\ell}^{k+1} ||S_k+1(B)||^{-1} Q_{\ell,k+1} 1_{S_{k+1}(B)} - |S_k(B)|^{-1} Q_{\ell,k} 1_{S_k(B)}
\]

\[
=: \sum_{\ell \in \mathbb{Z}_+^n, |\ell| \leq d} \sum_{k=0}^{\infty} b_{\ell}^k \tag{5.9}
\]

pointwisely. From this, (5.8) and (5.3), it follows that there exists a positive constant \( C_0 \) such that for any \( k \in \mathbb{Z}_+ \) and \( \ell \in \mathbb{Z}_+^n \) with \( |\ell| \leq d \),

\[
\|b_{\ell}^k\|_{L^\infty(\mathbb{R}^n)} \leq C_0 2^{-k\epsilon} \|1_B\|_{X}^{-1} \quad \text{and} \quad \text{supp} \ b_{\ell}^k \subset 2^{k+1}B; \tag{5.10}
\]

moreover, for any \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| \leq d \),

\[
\int_{\mathbb{R}^n} b_{\ell}^k(x)x^\gamma \, dx = 0.
\]

For any \( k \in \mathbb{Z}_+ \) and \( \ell \in \mathbb{Z}_+^n \) with \( |\ell| \leq d \), let

\[
\rho_{\ell}^k := 2^{-k\epsilon} \|1_{2^{k+1}B}\|_X \|1_{B}\|_X \quad \text{and} \quad a_{\ell}^k := 2^{k\epsilon} b_{\ell}^k \|1_{B}\|_X \|1_{2^{k+1}B}\|_X.
\]

By (5.3) and the definitions of \( b_{\ell}^k \) and \( a_{\ell}^k \), we find that for any \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| \leq d \),

\[
\int_{\mathbb{R}^n} a_{\ell}^k(x)x^\gamma \, dx = 0.
\]
For I and Definition 2.3(ii), we have

$$\sum_{k=0}^{\infty} P_k = \sum_{\ell \in \mathbb{Z}_+^{n}, |\ell| \leq d} \sum_{k=0}^{\infty} \mu_k^\ell a_k^\ell$$

(5.11)

pointwisely forms an infinite combination of \((X, q, d)\)-atoms.

Combining (5.5) and (5.11), we obtain

$$m = \sum_{k=0}^{\infty} m_k = \sum_{k=0}^{\infty} (m_k - P_k) + \sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \mu_k a_k + \sum_{\ell \in \mathbb{Z}_+^{n}, |\ell| \leq d} \sum_{k=0}^{\infty} \mu_k^\ell a_k^\ell$$

(5.12)

pointwisely, which shows that any \((X, q, d, \epsilon)\)-molecule is an infinite linear combination of \((X, q, d)\)-atoms both pointwisely and in \(S'(\mathbb{R}^n)\). Therefore, we have proved the above claim.

To show \(f \in WH_X(\mathbb{R}^n)\), it suffices to prove that for any \(\alpha \in (0, \infty)\),

$$\alpha \|1_{\{x \in \mathbb{R}^n: M_N^n(f)(x) > \alpha\}} \|_X \lesssim \sup_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X,$$

(5.13)

where the implicit positive constant is independent of \(f\) and \(\alpha\).

For any given \(\alpha \in (0, \infty)\), we know that there exists an \(i_0 \in \mathbb{Z}\) such that \(2^{i_0} \leq \alpha < 2^{i_0+1}\). Then we decompose \(f\) into

$$f = \sum_{i=\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j} + \sum_{i=\infty}^{i_0} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j} =: f_1 + f_2.$$

By the fact that

$$1_{\{x \in \mathbb{R}^n: M_N^n(f)(x) > \alpha\}} \lesssim 1_{\{x \in \mathbb{R}^n: M_N^n(f_1)(x) > \alpha/2\}} + 1_{\{x \in \mathbb{R}^n: M_N^n(f_2)(x) > \alpha/2\}}$$

and Definition 2.3(ii), we have

$$\|1_{\{x \in \mathbb{R}^n: M_N^n(f)(x) > \alpha\}} \|_X \lesssim \|1_{\{x \in \mathbb{R}^n: M_N^n(f_1)(x) > \alpha/2\}} \|_X + \|1_{\{x \in \mathbb{R}^n: M_N^n(f_2)(x) > \alpha/2\}} \|_X =: I_1 + I_2.$$

(5.14)

To deal with \(I_1\), we first need an estimate of \(M_N^0(m_{i,j})\). By (5.4), (5.10) and (5.12), for any \(i \in \mathbb{Z}\) and \(j \in \mathbb{N}\), we have a sequence of multiples of \((X, q, d)\)-atoms, \(\{a^{(i)}_{i,j}\}_{i \in \mathbb{Z}^+}\), supported, respectively, in balls \(\{2^{i+1}B_{i,j}\}_{i \in \mathbb{Z}^+}\) such that

$$\|a^{(i)}_{i,j}\|_{L^q(\mathbb{R}^n)} \lesssim \|2^{-i}|2^{i+1}B_{i,j}|^{1/q}\|_X$$

and \(m_{i,j} = \sum_{i \in \mathbb{Z}^+} a^{(i)}_{i,j}\) pointwisely in \(\mathbb{R}^n\). Then for any \(i \in \mathbb{Z} \cap (-\infty, i_0 - 1]\) and \(j \in \mathbb{N}\),

$$M_N^0(m_{i,j}) \lesssim \sum_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} M_N^0(a^{(i)}_{i,j}) \leq \sum_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} M_N^0(a^{(i)}_{i,j}) 1_{S_k(2^{i+1}B_{i,j})} =: \sum_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} J^{(i,j)}_{i,k},$$

(5.15)

where \(S_k(2^{i+1}B_{i,j}) := (2^{i+1}B_{i,j}) \setminus (2^{k+i}B_{i,j})\). From this, we deduce that

$$I_1 \lesssim \|1_{\{x \in \mathbb{R}^n: \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \lambda_{i,j} M_N^0(m_{i,j})(x) > \frac{\alpha}{2}\}} \|_X$$

$$\lesssim \|1_{\{x \in \mathbb{R}^n: \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} \lambda_{i,j} J^{(i,j)}_{i,k}(x) > \frac{\alpha}{2}\}} \|_X$$

$$+ \|1_{\{x \in \mathbb{R}^n: \sum_{i=-\infty}^{i_0-1} \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} \lambda_{i,j} J^{(i,j)}_{i,k}(x) > \frac{\alpha}{2}\}} \|_X$$

$$=: I_{1,1} + I_{1,2}.$$

(5.16)

For \(I_{1,1}\), by an argument similar to that used in the estimation of (4.19), we obtain

$$\alpha I_{1,1} \lesssim \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_X.$$

(5.17)
For $I_{1,2}$, we first estimate every term $J^{(i,j)}_{l,k}$. By an argument similar to that used in the estimation of (4.21), we conclude that for any $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $k \in [3, \infty) \cap \mathbb{Z}_+$ and $x \in S_k(2^lB_{i,j})$,
\[
J^{(i,j)}_{l,k}(x) \lesssim \int_{2^{l+1}B_{i,j}} \frac{|y - x_{i,j}|^{d+1}}{|x - x_{i,j}|^{n+d+1}} |a_{i,j}^{(l)}(y)| dy \|1_{S_k(2^lB_{i,j})}(x)\|
\]
\[
\lesssim \frac{(2^{l+1}r_{i,j})^{d+1}}{(2^{l}r_{i,j})^{n+d+1}} \|a_{i,j}^{(l)}\|_{L^q(\mathbb{R}^n)} 2^{l+1}B_{i,j}^{1/q'} \|1_{S_k(2^lB_{i,j})}(x)\|
\]
\[
\lesssim 2^{-l(n+c)-k(n+d+1)} r_{i,j}^{d+1} \|1_{B_{i,j}}\| X 2^{-l-c(k(n+d+1))} \|1_{S_k(2^lB_{i,j})}(x)\|.
\]
(5.18)

Let $r \in \left(\frac{n}{n+d+1}, p\right)$ By Definition 2.7(i), we have
\[
\|1_{\{x \in \mathbb{R}^n : M_{\alpha}^{(l)}(f)(x) > \alpha\}}\| X \equiv \|1_{\{x \in \mathbb{R}^n : M_{\alpha}^{(l)}(f)(x) > \alpha\}}\|_{X^{1/r}}.
\]
This, together with Definition 2.3(ii), (5.15), (5.18), the assumption that $X^{1/r}$ is a ball Banach function space, (2.12) and the fact that $c \in (n + d + 1, \infty)$, implies that
\[
\alpha I_{1,2} \lesssim \alpha^{-1/r} \left[ \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{l}2^{-l}2^{-k(n+d+1)} \|1_{S_k(2^lB_{i,j})}\|_{X^{1/r}} \right]
\]
\[
\lesssim \alpha^{-1/r} \left[ \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{-l}2^{-k(n+d+1)} \sum_{i=0}^{n-1} \|1_{S_k(2^lB_{i,j})}\|_{X^{1/r}} \right]^{1/r}
\]
\[
\lesssim \alpha^{-1/r} \left[ \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{-l}2^{-k(n+d+1)} \frac{2^{n(k+i)}}{2^{n+1}} \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \|1_{B_{i,j}}\|_{X^{1/r}} \right]^{1/r}
\]
\[
\lesssim \alpha^{-1/r} \sup_{i \in \mathbb{Z}} 2^{l} \left[ \sum_{j \in \mathbb{N}} \|1_{B_{i,j}}\|_{X} \left( \sum_{i=0}^{n-1} 2^{l(1-r)} \right)^{1/r} \right] \lesssim \sup_{i \in \mathbb{Z}} 2^{l} \left[ \sum_{j \in \mathbb{N}} \|1_{B_{i,j}}\|_{X^{1/r}} \right].
\]
By this, (5.16) and (5.17), we conclude that
\[
\alpha I_{1} \lesssim \sup_{i \in \mathbb{Z}} 2^{l} \left[ \sum_{j \in \mathbb{N}} \|1_{B_{i,j}}\|_{X} \right].
\]
(5.19)

Next, we turn to estimate $I_2$. To this end, by (5.15) and Definition 2.3(ii), we know that
\[
I_2 \lesssim \|1_{\{x \in \mathbb{R}^n : \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j})(x) > \frac{4}{\alpha}\}}\|_{X}
\]
\[
+ \|1_{\{x \in \mathbb{R}^n : \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j})(x) > \frac{4}{\alpha}\}}\|_{X}
\]
\[
=: I_{2,1} + I_{2,2}.
\]
(5.20)

We first deal with $I_{2,1}$. For any $\tilde{q} \not\in (0, 1)$, we have
\[
\sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) \|1_{S_k(2^lB_{i,j})}\|^\tilde{q} \lesssim \left\{ \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \|\lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) 1_{S_k(2^lB_{i,j})}\|^\tilde{q} \right\}^{1/\tilde{q}}.
\]
Let $r \in \left(\frac{n}{n+d+1}, p\right)$ and choose $\tilde{q} \in (0, 1)$ such that \(r \tilde{q} > \frac{n}{n+d+1}\). By Definition 2.3(ii), \(\lambda_{i,j} := A \|1_{B_{i,j}}\|_{X}\), and the assumption that $X^{1/r}$ is a ball Banach function space, we conclude that
\[
I_{2,1} \lesssim 2^{-\lambda_{i,j}} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \|\lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) 1_{S_k(2^lB_{i,j})}\|^\tilde{q} \|1_{B_{i,j}}\|_{X} \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \|\lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) 1_{S_k(2^lB_{i,j})}\|^\tilde{q} \|1_{B_{i,j}}\|_{X}
\]
\[
\sim 2^{-\lambda_{i,j}} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \|\lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) 1_{S_k(2^lB_{i,j})}\|^\tilde{q} \|1_{B_{i,j}}\|_{X} \sum_{i=0}^{n-1} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=0}^{\infty} \|\lambda_{i,j} M_{\alpha}^{(l)}(a^{(l)}_{i,j}) 1_{S_k(2^lB_{i,j})}\|^\tilde{q} \|1_{B_{i,j}}\|_{X}
\]
(5.21)
\[
\sum_{i=0}^{\infty} \sum_{l \in \mathbb{Z}_+} 2^{i \tilde{q}} \left\| \sum_{j \in \mathbb{N}} 2^{i \tilde{q}} \right\| X \cdot M^0_N(a_{i,j}) \cdot S_k(2^l B_{i,j}) \right\|^\tilde{q} \right\|_{X^{1/r}} \right)^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} 2^{-i \lambda r \tilde{q}} \sum_{j \in \mathbb{N}} 2^{i \lambda r \tilde{q}} \left\| X M^0_N(a_{i,j}) S_k(2^l B_{i,j}) \right\|^\tilde{q} \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sum_{l \in \mathbb{Z}_+} 2^{-i \lambda r \tilde{q}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sum_{l \in \mathbb{Z}_+} 2^{-i \lambda r \tilde{q}} 2^{(l+1)n} \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \lesssim \alpha^{-1} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \\
\right.
\]

Let \( p_0 := \tilde{q}/\tilde{q} \). Then \( p_0 \in (p_+, \infty) \) and, from this and the boundedness of \( M \) on \( L^p(\mathbb{R}^n) \), we deduce that for any \( i \in \mathbb{N}, j \in \mathbb{N}, l \in \mathbb{Z} \) and \( k \in \{0, 1, 2\} \),

\[
\left\| X M^0_N(a_{i,j}) S_k(2^l B_{i,j}) \right\|_{L^p(\mathbb{R}^n)} \leq 2^{i \lambda r \tilde{q}} \left\| \sum_{j \in \mathbb{N}} X M^0_N(a_{i,j}) \right\|_{L^p(\mathbb{R}^n)} \lesssim |2^{i+1} B_{i,j}|^{1/p_0}.
\]

By Lemma 4.8, (2.12), the fact that \( \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \leq A \) and \( \tilde{q} \in (0, 1) \), we further conclude that

\[
I_{2,1} \lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sum_{l \in \mathbb{Z}_+} 2^{-i \lambda r \tilde{q}} \left\| X \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sum_{l \in \mathbb{Z}_+} 2^{-i \lambda r \tilde{q}} 2^{(l+1)n} \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \lesssim \alpha^{-1} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \\
\right.
\]

which implies that

\[
\alpha I_{2,1} \lesssim \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \] (5.21)

To estimate \( I_{2,2} \), for any \( a \in (0, 1) \), we also have

\[
\sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j}J^{i,j}(l,k) \leq \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j}J^{i,j}(l,k) \] (5.22)

Then, by Definition 2.3(ii), (5.22), (5.18) and \( \lambda_{i,j} := A \tilde{q} \| 1_{B_{i,j}} \| X \), we know that

\[
I_{2,2} \leq \left\| \left\{ x \in \mathbb{R}^n \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j}J^{i,j}(l,k) \alpha^{n+2} \right\} \right\|_X \lesssim 2^{-i \tilde{q}} \left\| \sum_{i=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j}J^{i,j}(l,k) \right\|_X \]

Let \( r \in \left( \frac{n}{n+d+1} - 1, p \right) \). We choose \( a \in (0, 1) \) such that \( ar > \frac{n}{n+d+1} - 1 \). From Definition 2.7(i), the assumption that \( X^{1/r} \) is a ball Banach function space, (2.12), \( ar(n + d + 1) \) - \( n > 0 \) and \( \varepsilon > n + d + 1 \), we further deduce that

\[
I_{2,2} \lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{-i \lambda r \tilde{q}} 2^{k(n+1)} 2^{i \lambda r \tilde{q}} \left\| X \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \left\| \sum_{l \in \mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{k(n+1)} 2^{i \lambda r \tilde{q}} \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \left\| \mathbb{I}_{2^{i+1} B_{i,j}} \right\|_X \right\}^{1/r} \\
\lesssim 2^{-i \tilde{q}} \left\{ \sum_{i=0}^{\infty} 2^{i \tilde{q}} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \leq \alpha^{-1} \sup_{i} \left[ 2^i \left\| \sum_{j \in \mathbb{N}} \mathbb{I}_{2^{i+1} B_{i,j}} \left\| X \right\|_X \right\] \\
\right.
\]
which implies that
\[
\alpha I_{2,2} \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^{j} \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_{X} \right].
\]
This, combined with (5.21) and (5.20), implies that
\[
\alpha I_{2} \lesssim \sup_{i \in \mathbb{Z}} \left[ 2^{j} \left\| \sum_{j \in \mathbb{N}} 1_{B_{i,j}} \right\|_{X} \right].
\]
By this, (5.14) and (5.19), we know that (5.13) holds true and hence complete the proof of Theorem 5.3.

6 Boundedness of Calderón-Zygmund operators

In this section, as an application of the weak Hardy type space \( WH_{X}(\mathbb{R}^{n}) \), we establish the boundedness of Calderón-Zygmund operators from the Hardy type space \( H_{X}(\mathbb{R}^{n}) \) to \( WH_{X}(\mathbb{R}^{n}) \). We begin with recalling the notion of the Hardy type space \( H_{X}(\mathbb{R}^{n}) \) (see [70, Definition 2.22]).

**Definition 6.1.** Let \( X \) be a ball quasi-Banach function space. The Hardy space \( H_{X}(\mathbb{R}^{n}) \) associated with \( X \) is defined to be the set of all \( f \in S'(\mathbb{R}^{n}) \) such that
\[
\|f\|_{H_{X}(\mathbb{R}^{n})} := \|M_{\ast \ast}^{b}(f, \psi)\|_{X} < \infty,
\]
where \( M_{\ast \ast}^{b}(f, \psi) \) is as in Definition 3.1(iii) with \( b \) sufficiently large and \( \psi \in \mathcal{S}(\mathbb{R}^{n}) \) satisfying \( \int_{\mathbb{R}^{n}} \psi(x)dx \neq 0 \).

In what follows, we assume that the ball quasi-Banach function space \( X \) satisfies the following assumption: For some \( \theta, s \in (0, 1] \), there exists a positive constant \( C \) such that for any \( \{f_{j}\}_{j=1}^{\infty} \subset \mathcal{M}(\mathbb{R}^{n}) \),
\[
\left\| \left\{ \sum_{j=1}^{\infty} [M_{(\theta)}^{b}(f_{j})]_{s} \right\}^{1/s} \right\|_{X} \leq C \left\| \left\{ \sum_{j=1}^{\infty} |f_{j}|^{s} \right\}^{1/s} \right\|_{X}. \tag{6.1}
\]

Let \( X \) be a ball quasi-Banach function space satisfying (6.1) for some \( \theta, s \in [0, 1] \). Let \( d \geq \lfloor n(1/\theta - 1) \rfloor \) be a fixed integer and \( q \in (1, \infty] \). Assume that for any \( f \in \mathcal{M}(\mathbb{R}^{n}) \),
\[
\|M_{d}^{(q/s)}(f)\|_{(X^{1/s})'} \lesssim \|f\|_{(X^{1/s})'}, \tag{6.2}
\]
where the implicit positive constant is independent of \( f \). The atomic Hardy space \( H_{atom}^{X,q,d}(\mathbb{R}^{n}) \) is defined to be the set of all \( f \in S'(\mathbb{R}^{n}) \) such that \( f = \sum_{j \in \mathbb{N}} \lambda_{j} a_{j} \) in \( S'(\mathbb{R}^{n}) \), where \( \{\lambda_{j}\}_{j \in \mathbb{N}} \) is a sequence of nonnegative numbers and \( \{a_{j}\}_{j \in \mathbb{N}} \) is a sequence of \((X, q, d)\)-atoms as in Definition 4.1, and
\[
\|f\|_{H_{atom}^{X,q,d}(\mathbb{R}^{n})} := \inf \left\{ \left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_{j} 1_{B_{j}}}{\|1_{B_{j}}\|_{X}} \right]^{s} \right\}^{1/s} \right\|_{X} \right\} < \infty,
\]
where the infimum is taken over all the decompositions of \( f \) as above.

The following atomic characterization of \( H_{X}(\mathbb{R}^{n}) \) is just [70, Theorems 3.6 and 3.7].

**Lemma 6.2.** Let \( \theta, s \in (0, 1] \), \( q \in (1, \infty] \) and \( d \geq \lfloor n(1/\theta - 1) \rfloor \) be a fixed integer. Assume that \( X \) is a ball quasi-Banach function space satisfying (6.1), (6.2) and that \( X^{1/s} \) is a ball Banach function space. Then \( H_{X}(\mathbb{R}^{n}) = H_{atom}^{X,q,d}(\mathbb{R}^{n}) \) with equivalent quasi-norms.

Recall that for any given \( \delta \in (0, 1] \), a linear operator \( T \) is called a convolutional \( \delta \)-type Calderón-Zygmund operator \( T \) (see, for example, [4]) if \( T \) is a linear bounded operator on \( L^{2}(\mathbb{R}^{n}) \) with kernel \( k \in S'(\mathbb{R}^{n}) \) coinciding with a locally integrable function on \( \mathbb{R}^{n} \setminus \{0_{n}\} \) and satisfying that for any \( x, y \in \mathbb{R}^{n} \) with \( |x| > 2|y| \),
\[
|k(x - y) - k(x)| \leq C \frac{|y|^{\delta}}{|x|^{n+\delta}},
\]
and for any $f \in L^2(\mathbb{R}^n)$, $Tf = p.v. k \ast f$.

The boundedness from $H_X(\mathbb{R}^n)$ to $WH_X(\mathbb{R}^n)$ of convolutional $\delta$-type Calderón-Zygmund operators is stated as follows.

**Theorem 6.3.** Let $\theta, s, \delta \in (0, 1]$ and $q \in (1, \infty)$. Assume that $X$ is a ball quasi-Banach function space satisfying (6.1), (6.2) and Assumption 2.20. Assume that $X^{1/s}$ is a ball Banach function space. Let $T$ be a convolutional $\delta$-type Calderón-Zygmund operator. If there exists a positive constant $C_0$ such that for any $\alpha \in (0, \infty)$ and any sequence $\{f_j\}_{j \in \mathbb{N}} \subset M(\mathbb{R}^n)$,

$$
\alpha \|1_{\{x \in \mathbb{R}^n : |\mathcal{M}(f_j(x))|^{\alpha \frac{n+1}{n+\alpha}} > \alpha\}} \|_X^{\frac{n+1}{n+\alpha}} \leq C_0 \left(\sum_{j \in \mathbb{N}} |f_j|^{\frac{n+1}{n+\alpha}}\right)^{\frac{n+\alpha}{n+1}},
$$

(6.3)

then $T$ has a unique extension on $H_X(\mathbb{R}^n)$ and, moreover, there exists a positive constant $C$ such that for any $f \in H_X(\mathbb{R}^n)$,

$$
\|Tf\|_{WH_X(\mathbb{R}^n)} \leq C\|f\|_{H_X(\mathbb{R}^n)}.
$$

**Proof.** Let $\theta, s$ and $d$ be as in Lemma 6.2 and $f \in H_X(\mathbb{R}^n)$. Then, by [70, Theorem 3.7], we find that there exist a sequence $\{a_j\}_{j=1}^\infty$ of $(X, \infty, d)$-atoms supported, respectively, in a sequence $\{Q_j\}_{j=1}^\infty$ of cubes, and a sequence $\{\lambda_j\}_{j=1}^\infty$ of nonnegative numbers, independent of $f$ but depending on $s$, such that

$$
f = \sum_{j=1}^\infty \lambda_j a_j \quad \text{in } S'(\mathbb{R}^n)
$$

and

$$
\left\|\left\{\sum_{j=1}^\infty \left(\frac{\lambda_j}{1_{Q_j}}\right)^s 1_{Q_j}\right\}^{1/s}\right\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
$$

From Lemma 2.17, we deduce that there exists an $\epsilon \in (0, 1)$ such that $X$ is continuously embedded into $L^s_\epsilon(\mathbb{R}^n)$ with $\omega := [\mathcal{M}(1_{B(0,1)})]^{\epsilon\frac{n}{n+1}}$, which implies that

$$
\left\|\left\{\sum_{j=1}^\infty \left(\frac{\lambda_j}{1_{Q_j}}\right)^s 1_{Q_j}\right\}^{1/s}\right\|_L^s(\mathbb{R}^n)
$$

Moreover, since for any $j \in \mathbb{N}$, $a_j$ is an $(X, \infty, d)$-atom, it follows that for any $j \in \mathbb{N}$, $\frac{1_{Q_j}}{1_{Q_j}^{s\frac{n+1}{n+\alpha}}}$ is an $(L^s_\epsilon(\mathbb{R}^n), \infty, d)$-atom, where an $(L^s_\epsilon(\mathbb{R}^n), \infty, d)$-atom is as in Definition 4.1 with $X$ replaced by $L^s_\epsilon(\mathbb{R}^n)$. By [32, Theorem 7.2.7], we know that $\omega \in A_1(r_n)$, which, combined with [79, Remarks 2.4(b) and 2.6(b)], implies that $L^s_\epsilon(\mathbb{R}^n)$ satisfies all the assumptions of [70, Theorem 3.6 and Corollary 3.11]. Using [70, Theorem 3.6 and Corollary 3.11] and (6.4), we conclude that

$$
\sum_{j=1}^\infty \lambda_j \left[\int_{Q_j} \left|\mathcal{M}(f_j)|_{L^s_\epsilon(\mathbb{R}^n)}\right|^s \frac{1_{Q_j}}{1_{Q_j}^{s\frac{n+1}{n+\alpha}}}
\right] \left[\int_{Q_j} \frac{1_{Q_j}}{1_{Q_j}^{s\frac{n+1}{n+\alpha}}}
\right] \|
\leq \sum_{j=1}^\infty \lambda_j a_j = f \quad \text{in } S'(\mathbb{R}^n) \quad \text{and } H^s_\epsilon(\mathbb{R}^n),
$$

(6.5)

where $H^s_\epsilon(\mathbb{R}^n)$ denotes the weighted Hardy space as in Definition 6.1 with $X$ replaced by $L^s_\epsilon(\mathbb{R}^n)$. Furthermore, from [52, Theorem 5.2], we deduce that $T$ is bounded from $H^s_\epsilon(\mathbb{R}^n)$ to $WH^s_\epsilon(\mathbb{R}^n)$, where $WH^s_\epsilon(\mathbb{R}^n)$ denotes the weak weighted Hardy space as in Definition 2.21 with $X$ replaced by $L^s_\epsilon(\mathbb{R}^n)$, and hence

$$
Tf = \sum_{j=1}^\infty \lambda_j T(a_j) \quad \text{in } WH^s_\epsilon(\mathbb{R}^n) \quad \text{and } S'(\mathbb{R}^n).
$$

(6.6)
Let \( \psi \in S(\mathbb{R}^n) \) satisfy \( \int_{\mathbb{R}^n} \psi(x) dx \neq 0 \). Then, to prove Theorem 6.3, by Assumption 2.20 and Theorem 3.2(ii), we only need to show that for any \( f \in H_X(\mathbb{R}^n) \),

\[
\|M(Tf, \psi)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H_X(\mathbb{R}^n)},
\]

where \( M(Tf, \psi) \) is as in Definition 3.1(i) with \( f \) replaced by \( Tf \). For any \( \alpha \in (0, \infty) \), by (6.6), Lemma 2.10(iii) and Remark 2.9(i), we have

\[
\alpha \|1_{\{x \in \mathbb{R}^n, M(Tf, \psi)(x) > \alpha\}}\|_X \leq \alpha \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)(x) > \alpha\}}\|_X
\]

\[
\lesssim \alpha \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)(x) > 2\}}\|_X + \alpha \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)(x) > 2\}}\|_X
\]

\[
\lesssim \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)1_{4B_j} + \alpha \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)(x) > 2\}}\|_X
\]

\[
= I + II.
\]

We first estimate I. Observing that \( M(Ta_j, \psi) \leq M(Ta_j) \) and \( a_j \in L^q(\mathbb{R}^n) \), by the fact that \( T \) is bounded on \( L^q(\mathbb{R}^n) \) (see, for example, [24, Theorem 5.1]) and the size condition of \( a_j \), we conclude that

\[
\|M(Ta_j, \psi)\|_{L^q(\mathbb{R}^n)} \lesssim \|M(Ta_j)\|_{L^q(\mathbb{R}^n)} \lesssim \|Ta_j\|_{L^q(\mathbb{R}^n)} \lesssim \|a_j\|_{L^q(\mathbb{R}^n)} \lesssim \|B_j\|_{B_j}^{1/q} \|1_{B_j}\|_X,
\]

which, combined with Lemma 4.8, (6.1) and [70, Theorem 3.6], implies that

\[
I \lesssim \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j 1_{B_j}}{\|1_{B_j}\|_X} \right]^\frac{1}{q} \right\} \frac{1}{q} \leq \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j 1_{B_j}}{\|1_{B_j}\|_X} \right]^\frac{1}{q} \right\} \frac{1}{q} \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
\]

To deal with the term II, for any \( t \in (0, \infty) \), let \( k^{(t)} := k * \psi_t \) with \( \psi_t(\cdot) := t^{-n} \psi(\cdot / t) \). By [81, p. 2881], we know that \( k^{(t)} \) satisfies the same conditions as \( k \). From this, together with the vanishing moments of \( a_j \), Hölder’s inequality and the size condition of \( a_j \), we deduce that for any \( x \in (4B_j)^c \),

\[
M(Ta_j, \psi)(x) = \sup_{t \in (0, \infty)} |\psi_t * (k * a_j)(x)| = \sup_{t \in (0, \infty)} |k^{(t)} * a_j(x)|
\]

\[
\leq \sup_{t \in (0, \infty)} \int_{\mathbb{R}^n} |k^{(t)}(x-y) - k^{(t)}(x-x_j)| |a_j(y)| dy
\]

\[
\lesssim \int_{B_j} \frac{|y-x_j|^\delta}{|x-x_j|^{n+\delta}} |a_j(y)| dy \lesssim \frac{r_j^\delta}{|x-x_j|^{n+\delta}} \|a_j\|_{L^q(\mathbb{R}^n)} \|B_j\|_X^{1/q'} \lesssim \frac{1}{|x-x_j|^{n+\delta}} \frac{\|M(1_{B_j})(x)\|_X^{\frac{n+\delta}{n}}} \lesssim \frac{1}{\|1_{B_j}\|_X}.
\]

This shows that for any \( x \in (4B_j)^c \),

\[
M(Ta_j, \psi)(x)1_{(4B_j)^c}(x) \lesssim \|M(1_{B_j})(x)\|_X^{\frac{n+\delta}{n}} \frac{1}{\|1_{B_j}\|_X}.
\]

Therefore, by this and (6.3), we find that

\[
II \lesssim \frac{\alpha}{2} \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(1_{B_j})(x) > \alpha\}}\|_X
\]

\[
\lesssim \frac{\alpha}{2} \|1_{\{x \in \mathbb{R}^n, \sum_{j \in \mathbb{N}} \lambda_j M(1_{B_j})(x) > \alpha\}}\|_X \leq \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j 1_{B_j}}{\|1_{B_j}\|_X} \right]^\frac{n+\delta}{n} \right\} \frac{1}{\|1_{B_j}\|_X} \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
\]
Finally, combining (6.8) and (6.9), we conclude that for any $\alpha \in (0, \infty)$,
\[ \alpha \|1_{\{x \in \mathbb{R}^n : M(Tf, \psi)(x) > \alpha\}}\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}, \]

namely, (6.7) holds true. This finishes the proof of Theorem 6.3.

Recall that for any given $\gamma \in (0, \infty)$, a linear operator $T$ is called a non-convolutional $\gamma$-order Calderón-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^n)$ and its kernel

\[ k : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{C} \]
satisfies that there exists a positive constant $C$ such that for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq \lfloor \gamma \rfloor - 1$ and $x, y, z \in \mathbb{R}^n$ with $|x - y| \geq 2|y - z|$,

\[ |\partial^\alpha_x k(x, y) - \partial^\alpha_x k(x, z)| \leq C \frac{|y - z|^{\lfloor \gamma \rfloor + 1}}{|x - y|^{n+\gamma}}, \]

and for any $f \in L^2(\mathbb{R}^n)$ having compact support and $x \notin \text{supp} f$,
\[ T(f)(x) = \int_{\text{supp} f} k(x, y)f(y)dy. \]

Here and thereafter, for any $\beta \in (0, \infty)$, the symbol $[\beta]$ denotes the minimal integer not less than $\beta$.

For any given $m \in \mathbb{N}$, an operator $T$ is said to have the vanishing moments up to order $m$, if for any $a \in L^2(\mathbb{R}^n)$ having compact support and satisfying that for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq m$, $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, it holds true that $\int_{\mathbb{R}^n} x^\beta T(a)(x)dx = 0$.

We now have the following boundedness of non-convolutional $\gamma$-order Calderón-Zygmund operators from $H_X(\mathbb{R}^n)$ to $WH_X(\mathbb{R}^n)$.

**Theorem 6.4.** Let $\theta, s \in (0, 1]$. Assume that $X$ is a ball quasi-Banach function space satisfying (6.1) and (6.2) with $q = 2$ and Assumption 2.20. Assume that $X^{1/s}$ is a ball Banach function space. Let $\gamma \in (0, \infty)$ and $T$ be a non-convolutional $\gamma$-order Calderón-Zygmund operator having the vanishing moments up to order $\lfloor \gamma \rfloor - 1$ with $\gamma$ satisfying $\lfloor \gamma \rfloor - 1 \leq n(1/\theta - 1)$. If there exists a positive constant $C_0$ such that for any $\alpha \in (0, \infty)$ and any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$,

\[ \alpha \|1_{\{x \in \mathbb{R}^n : \sum_{j \in \mathbb{N}} |\mathcal{M}(f_j)(x)|^{\frac{n+\gamma}{n}} > \alpha\}}\|_X^{\frac{n}{n+\gamma}} \leq C_0 \left( \sum_{j \in \mathbb{N}} |f_j|_1^{\frac{n+\gamma}{n+1}} \right)^{\frac{n}{n+\gamma}}, \]

then $T$ has a unique extension on $H_X(\mathbb{R}^n)$ and, moreover, there exists a positive constant $C$ such that for any $f \in H_X(\mathbb{R}^n)$,
\[ \|Tf\|_{WH_X(\mathbb{R}^n)} \leq C \|f\|_{H_X(\mathbb{R}^n)}. \]

**Remark 6.5.** (i) Recall that for any given $\delta \in (0, 1]$, a linear operator $T$ is called a non-convolutional $\delta$-type Calderón-Zygmund operator if $T$ is a linear bounded operator on $L^2(\mathbb{R}^n)$ and there exist a kernel $k$ on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) : x \in \mathbb{R}^n\}$ and a positive constant $C$ such that for any $x, y, z \in \mathbb{R}^n$ with $|x - y| > 2|y - z|$,

\[ |k(x, y) - k(x, z)| \leq C \frac{|y - z|^\delta}{|x - y|^{n+\delta}}, \]

and for any $f \in L^2(\mathbb{R}^n)$ having compact support and $x \notin \text{supp} f$,
\[ T(f)(x) = \int_{\text{supp} f} k(x, y)f(y)dy. \]

Observe that, when $\gamma := \delta \in (0, 1]$, the operator $T$ in Theorem 6.4 coincides with a non-convolutional $\delta$-type Calderón-Zygmund operator. Thus, the operators in Theorem 6.4 include non-convolutional $\delta$-type Calderón-Zygmund operators as special cases. By this, we know that the critical index of non-convolutional $\delta$-type Calderón-Zygmund operators is $\frac{n}{n+\delta}$ (see Remark 7.19 for more details).
(ii) Theorems 6.3 and 6.4 obtain the boundedness of convolutional \( \delta \)-type and non-convolutional \( \gamma \)-order Calderón-Zygmund operators from \( H^q_X(\mathbb{R}^n) \) to \( WH_X(\mathbb{R}^n) \). Since for any \( q \in (2, \infty) \), the boundedness of non-convolutional \( \gamma \)-order Calderón-Zygmund operators on \( L^q(\mathbb{R}^n) \) cannot be guaranteed by our assumptions on \( T \), it follows that (6.2) for some \( q \in (1, \infty) \) in Theorem 6.3 is weaker than (6.2) with \( q = 2 \) in Theorem 6.4.

**Proof of Theorem 6.4.** By an argument similar to that used in the proof of Theorem 6.3, to show Theorem 6.4, it suffices to prove that for any \( \alpha \in (0, \infty) \) and \( f \in H^q_X(\mathbb{R}^n) \),

\[
\alpha \|1_{\{x \in \mathbb{R}^n : \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \xi)(x) 1_{\{a_j \in B_j\}}(x) > \frac{\gamma}{\beta} \}} \|_X \lesssim \|f\|_{H^q_X(\mathbb{R}^n)},
\]

where for any \( j \in \mathbb{N} \), \( \lambda_j \), \( a_j \) and \( B_j \) are the same as in the proof of Theorem 6.3.

For any given \( j \in \mathbb{N} \), we first estimate \( M(Ta_j, \xi) \), which is as in Definition 3.1(i) with \( f \) replaced by \( Ta_j \). By the vanishing moments of \( T \) and the fact that \( \gamma - 1 \leq n(1/\theta - 1) \) implies that \( \gamma - 1 \leq d \), we know that for any \( j \in \mathbb{N} \), \( t \in (0, \infty) \) and \( x \in (4B_j)^c \),

\[
|\psi_t \ast T(a_j)(x)| \leq \frac{1}{t^n} \int_{\mathbb{R}^n} \left| \psi \left( \frac{x-y}{t} \right) - \sum_{|\beta| = \gamma} \frac{\partial^\beta \psi(\frac{x-y}{t})}{\beta!} \left( \frac{x_j - y_j}{t} \right)^\beta \right| |T(a_j)(y)| dy
\]

\[= \frac{1}{t^n} \left( \int_{|y-x_j| < 2r_j} + \int_{2r_j \leq |y-x_j| < 3r_j} + \int_{|y-x_j| \geq 3r_j} \right) \times \left| \psi \left( \frac{x-y}{t} \right) - \sum_{|\beta| = \gamma} \frac{\partial^\beta \psi(\frac{x-y}{t})}{\beta!} \left( \frac{x_j - y_j}{t} \right)^\beta \right| |T(a_j)(y)| dy
\]

\[=: I_1 + I_2 + I_3.\]

For \( I_1 \), the Taylor remainder theorem guarantees that for any \( j \in \mathbb{N} \) and \( y \in \mathbb{R}^n \) with \( |y - x_j| < 2r_j \), there exists an \( \xi_1(y) \in 2B_j \) such that

\[
I_1 \lesssim \frac{1}{t^n} \int_{|y-x_j| < 2r_j} \left| \sum_{|\beta| = \gamma} \partial^\beta \psi \left( \frac{x - \xi_1(\gamma)}{t} \right) \left( \frac{y - x_j}{t} \right)^\gamma \right| |T(a_j)(y)| dy,
\]

which, together with Hölder’s inequality and the fact that \( T \) is bounded on \( L^2(\mathbb{R}^n) \), further implies that for any \( t \in (0, \infty) \) and \( x \in (4B_j)^c \),

\[
I_1 \lesssim \frac{1}{r_j^n} \int_{|y-x_j| < 2r_j} \frac{t^{n+\gamma}}{|x-x_j|^{n+\gamma}} \frac{|y-x_j|^\gamma}{t^\gamma} \left| T(a_j)(y) \right| dy
\]

\[\lesssim \frac{r_j^{n+\gamma}}{|x-x_j|^{n+\gamma}} \|T(a_j)||L^2(\mathbb{R}^n)|B_j|^{1/2} \lesssim \frac{1}{t^n} \frac{r_j^{n+\gamma}}{|x-x_j|^{n+\gamma}} \|1_{B_j} \|_X.\]

For \( I_2 \), from the Taylor remainder theorem, the vanishing moments of \( a_j \), \( \gamma - 1 \leq \lfloor n(1/\theta - 1) \rfloor \leq d \), (6.10) and Hölder’s inequality, it follows that for any \( z \in B_j \), there exists a \( \xi_2(z) \in B_j \) such that for any \( t \in (0, \infty) \) and \( x \in (4B_j)^c \),

\[
I_2 \lesssim \int_{2r_j \leq |y-x_j| < 3r_j} \frac{|y-x_j|^\gamma}{|x-x_j|^{n+\gamma}} \left| \sum_{|\beta| = \gamma} \frac{\partial^\beta k(y, x_j)}{\beta!} (z-x_j)^\beta \left| dz \right| dy
\]

\[\lesssim \frac{1}{|x-x_j|^{n+\gamma}} \int_{2r_j \leq |y-x_j| < 3r_j} \frac{|y-x_j|^\gamma}{|x-x_j|^{n+\gamma}} \left| \sum_{|\beta| = \gamma} \frac{\partial^\beta k(y, x_j)}{\beta!} (z-x_j)^\beta \left| dz \right| dy
\]

\[\times \int_{B_j} |a_j(z)| \sum_{|\beta| = \gamma} \frac{\partial^\beta k(y, x_j) - \partial^\beta k(y, \xi_2(z))}{\beta!} (z-x_j)^\beta \left| dz \right| dy.
\]
For $I_3$, by the vanishing moments of $a_j$, $[\gamma] - 1 \leq [n(1/\theta - 1)] \leq d$, (6.10) and Hölder’s inequality, we know that for any $z \in B_j$, there exists a $\xi(z) \in B_j$ such that for any $t \in (0, \infty)$ and $x \in (4B_j)^\circ$,

\[
I_3 \lesssim \int_{|y-x_j| \geq |x-x_j|/4} \left[ \frac{1}{\rho^n} \left| \psi\left( \frac{y-x}{\rho} \right) - \sum_{|\beta| \leq [\gamma]-1} \frac{\partial^n \psi(y, x)}{\beta!} (x-y)^\beta \right| \right] dy \\
\lesssim \int_{|y-x_j| \geq |x-x_j|/4} \left[ \frac{1}{\rho^n} \left| \psi\left( \frac{y-x}{\rho} \right) - \sum_{|\beta| \leq [\gamma]-1} \frac{\partial^n \psi(y, x)}{\beta!} (x-y)^\beta \right| \right] dy \\
\times \left\{ \int_{B_j} |a_j(z)| \left| k(y, z) - \sum_{|\beta| \leq [\gamma]-1} \frac{\partial^n k(y, x_j)}{\beta!} (z-x_j)^\beta \right| dz \right\} dy \\
\lesssim \int_{|y-x_j| \geq |x-x_j|/4} \left[ \frac{1}{\rho^n} \left| \psi\left( \frac{y-x}{\rho} \right) - \sum_{|\beta| \leq [\gamma]-1} \frac{\partial^n \psi(y, x)}{\beta!} (x-y)^\beta \right| \right] dy \\
\times \int_{B_j} |a_j(z)| \left| \frac{\partial^n k(y, x_j)}{\beta!} (z-x_j)^\beta \right| dz dy \\
\lesssim \|a_j\|_{L^2(\mathbb{R}^n)|B_j}^{1/2} \left[ \frac{r_j^n}{|x-x_j|^{n+\gamma}} \right]^{1/2} \left[ \frac{1}{|x-x_j|^{n+\gamma}} \right] \int_{|y-x_j| \geq |x-x_j|/4} |\psi(x-y)| dy \\
+ \sum_{|\beta| \leq [\gamma]-1} \frac{r_j^n}{|x-x_j|^{n+|\beta|}} \left[ \frac{1}{|y-x_j|^{n+\gamma} - |y-x_j|^{n+|\beta|}} \right] dy \\
\lesssim \frac{1}{|x-x_j|^{n+\gamma}} \|1_{B_j}\|_{X}.
\]

(6.16)

Combining (6.13)–(6.16), we obtain that for any $x \in (4B_j)^\circ$,

\[
M(Ta_j, \psi)(x) = \sup_{t \in (0, \infty)} |\psi * Ta_j(x)| \lesssim \frac{r_j^{n+\gamma}}{|x-x_j|^{n+\gamma}} \frac{1}{\|1_{B_j}\|_X} \lesssim \frac{1}{\|1_{B_j}\|_X},
\]

which implies that

\[
M(Ta_j, \psi)(x)1_{(4B_j)^\circ}(x) \lesssim \|M(1_{B_j})(x)\|^{n+\gamma} \frac{1}{\|1_{B_j}\|_X}.
\]

Therefore, by (6.11) and an argument similar to that used in the estimation of (6.9), we conclude that

\[
\alpha \|1_{\{x \in \mathbb{R}^n : \sum_{j \in \mathbb{N}} \lambda_j M(Ta_j, \psi)(x)1_{(4B_j)^\circ}(x) > \varphi \}}\|_X \\
\lesssim \frac{\alpha}{2} \left[ \|1_{\{x \in \mathbb{R}^n : \sum_{j \in \mathbb{N}} \lambda_j M(1_{B_j})(x) \times \|M(1_{B_j})(x)\|^{n+\gamma} \times (\varphi) \times (\varphi) \times \|M(1_{B_j})(x)\|^{n+\gamma} \} \|_X^{n+\gamma} \right]^{\frac{n+\gamma}{n+\gamma}} \lesssim \|f\|_{H^\gamma(\mathbb{R}^n)}.
\]

This shows that (6.12) holds true and hence finishes the proof of Theorem 6.3. \qed
7 Applications

In this section, we apply all above results to three concrete examples of ball quasi-Banach function spaces, namely, Morrey spaces, mixed-norm Lebesgue spaces and Orlicz-slice spaces, respectively, in Subsections 7.1–7.3.

7.1 Morrey spaces

We begin with recalling the notion of Morrey spaces.

**Definition 7.1.** Let $0 < q \leq p < \infty$. The Morrey space $M^p_q(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{M^p_q(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} |B|^{1/p-1/q} \|f\|_{L^q(B)} < \infty,$$

where $\mathcal{B}$ is as in (2.2) (the set of all balls of $\mathbb{R}^n$).

The space $M^p_q(\mathbb{R}^n)$ was introduced by Morrey [61]. Furthermore, the following Fefferman-Stein vector-valued maximal inequalities for $M^p_q(\mathbb{R}^n)$ hold true (see, for example, [77, Lemma 2.5]), which shows that the Morrey space $M^p_q(\mathbb{R}^n)$ satisfies Assumption 2.18.

**Lemma 7.2.** Let $0 < q \leq p < \infty$. Assume that $r \in (1, \infty)$ and $s \in (0, q)$. Then there exists a positive constant $C$ such that for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{R}^n)$,

$$\left\{ \sum_{j=1}^\infty |\mathcal{M}(f_j)|^r \right\}^{1/r} \leq C \left\{ \sum_{j=1}^\infty |f_j|^r \right\}^{1/r},$$

where $[M^p_q(\mathbb{R}^n)]^{1/s}$ denotes the $\frac{1}{s}$-convexification of $M^p_q(\mathbb{R}^n)$ as in Definition 2.7(i) with $X$ and $p$ replaced, respectively, by $M^p_q(\mathbb{R}^n)$ and $1/s$.

Now we recall the notion of the weak Morrey space $WM^p_q(\mathbb{R}^n)$.

**Definition 7.3.** Let $0 < q \leq p < \infty$. The weak Morrey space $WM^p_q(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{WM^p_q(\mathbb{R}^n)} := \sup_{\alpha \in (0, \infty)} \|\alpha \mathbb{1}_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \|_{M^p_q(\mathbb{R}^n)} < \infty.$$

**Remark 7.4.** Let $0 < q \leq p < \infty$. The weak Morrey space $WM^p_q(\mathbb{R}^n)$ is just the weak Morrey space $M^{q, \infty}_u(\mathbb{R}^n)$ in [40] with $u(B) := |B|^{1/q-1/p}$ for any $B \in \mathcal{B}$, where $\mathcal{B}$ is as in (2.2).

The following Fefferman-Stein vector-valued maximal inequalities for $WM^p_q(\mathbb{R}^n)$ hold true (see, for example, [40, Theorem 3.2]), which shows that the Morrey space $M^p_q(\mathbb{R}^n)$ satisfies Assumption 2.20.

**Lemma 7.5.** Let $0 < q \leq p < \infty$. Assume that $r \in (1, \infty)$ and $s \in (0, q)$. Then there exists a positive constant $C$ such that for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{R}^n)$,

$$\left\{ \sum_{j=1}^\infty |\mathcal{M}(f_j)|^r \right\}^{1/r} \leq C \left\{ \sum_{j=1}^\infty |f_j|^r \right\}^{1/r},$$

where $[WM^p_q(\mathbb{R}^n)]^{1/s}$ denotes the $\frac{1}{s}$-convexification of $WM^p_q(\mathbb{R}^n)$ as in Definition 2.7(i) with $X$ and $p$ replaced, respectively, by $WM^p_q(\mathbb{R}^n)$ and $\frac{1}{s}$.

Similarly to [38, Lemma 5.7] and [71, Theorem 4.1], we can easily show the following conclusion and we omit the details here.

**Lemma 7.6.** Let $0 < q \leq p < \infty$, $r \in (0, q)$ and $s \in (q, \infty]$. Then there exists a positive constant $C$ such that for any $f \in \mathcal{M}(\mathbb{R}^n)$,

$$\|\mathcal{M}^{(q/r')}((f)(M^p_q(\mathbb{R}^n))^{1/r'}) \|_{(M^p_q(\mathbb{R}^n))^{1/r'}} \leq C \|f\|_{(M^p_q(\mathbb{R}^n))^{1/r'}}.$$
Now we introduce the notion of the weak Morrey Hardy space $WHM^p_q(\mathbb{R}^n)$.

**Definition 7.7.** Let $0 < q \leq p < \infty$. The weak Morrey Hardy space $WHM^p_q(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{WHM^p_q(\mathbb{R}^n)} := \|M^0_N(f)\|_{WM^p_q(\mathbb{R}^n)} < \infty,
$$

where $M^0_N(f)$ is as in (2.14) with $N$ sufficiently large.

**Remark 7.8.** Let $1 < q \leq p < \infty$. By Lemma 7.5, we conclude that for any $r \in (1, q)$, $\mathcal{M}$ in (2.8) is bounded on $(WM^p_q(\mathbb{R}^n))^{1/r}$, which, combined with Theorem 3.4, implies that $WHM^p_q(\mathbb{R}^n) = WM^p_q(\mathbb{R}^n)$ with equivalent norms.

By Lemma 7.5 and Theorem 3.2(ii), we obtain the following maximal function characterizations of the weak Morrey Hardy space $WHM^p_q(\mathbb{R}^n)$.

**Theorem 7.9.** Let $0 < q \leq p < \infty$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Assume that $b \in (n/q, \infty)$ and $N \geq \lceil b + 1 \rceil$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, if one of the following quantities

$$
\|M^0_N(f)\|_{WM^p_q(\mathbb{R}^n)}, \quad \|M(f, \psi)\|_{WM^p_q(\mathbb{R}^n)}, \quad \|M^*_N(f, \psi)\|_{WM^p_q(\mathbb{R}^n)}, \quad \|M_N(f)\|_{WM^p_q(\mathbb{R}^n)},
$$

$$
\|M^*_N(f)\|_{WM^p_q(\mathbb{R}^n)}, \quad \|M^*_{N,b}(f)\|_{WM^p_q(\mathbb{R}^n)} \quad \text{and} \quad \|N(f)\|_{WM^p_q(\mathbb{R}^n)}
$$

is finite, then the others are also finite and mutually equivalent with the implicit positive equivalence constants independent of $f$.

**Remark 7.10.** Let $0 < q \leq p < \infty$ and $p = q$. Then we know that $M^p_q(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $WM^p_q(\mathbb{R}^n) = WH^p_q(\mathbb{R}^n)$, where $WH^p_q(\mathbb{R}^n)$ denotes the classical weak Hardy space, and the characterizations of $WH^p_q(\mathbb{R}^n)$ in terms of all the maximal functions except for $M^*_N(f, \psi)$ and $M^*_{N,b}(f)$ in Theorem 7.9 were obtained in [52, Theorems 2.10 and 2.11] or [81, Theorem 3.7 and Corollary 3.8] as a special case. Moreover, in this case, Theorem 7.50 extends the range of $N \subset (\frac{n}{q} + n + 1, \infty) \cap \mathbb{N}$ in [81, Theorem 3.7 and Corollary 3.8] into $N \subset (\frac{n}{q} + 1, \infty) \cap \mathbb{N}$.

Using Lemmas 7.2, 7.5 and 7.6 and Theorems 4.2 and 4.7, we immediately obtain the atomic characterization of $WHM^p_q(\mathbb{R}^n)$ (see Theorem 7.11 below) and the molecular characterization of $WHM^p_q(\mathbb{R}^n)$ (see Theorem 7.13 below) as follows.

**Theorem 7.11.** Let $0 < q \leq p < \infty$. Assume that $r \in (\max\{1, q\}, \infty)$ and $d \in \mathbb{Z}_+$ satisfies $d \geq \lceil n/\min\{1, q\} - 1 \rceil$. Then $f \in WHM^p_q(\mathbb{R}^n)$ if and only if

$$
f = \sum \sum \lambda_{i,j} \eta_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i,j \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} \mathbf{1}_{B_{i,j}} \right\|_{M^p_q(\mathbb{R}^n)} < \infty,
$$

where $\{\eta_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ are $(M^p_q(\mathbb{R}^n), r, d)$-atoms supported, respectively, in balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ such that for any $i \in \mathbb{Z}$, $\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \leq A$ with $c \in (0, 1]$ and $A$ being a positive constant independent of $f$ and $i$, and for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := A^2 \|\mathbf{1}_{B_{i,j}}\|_{M^p_q(\mathbb{R}^n)}$ with $A$ being a positive constant independent of $f$ and $i$.

Moreover, for any $f \in WHM^p_q(\mathbb{R}^n)$,

$$
\|f\|_{WHM^p_q(\mathbb{R}^n)} \sim \inf \left\{ \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} \mathbf{1}_{B_{i,j}} \right\|_{M^p_q(\mathbb{R}^n)} \right\},
$$

where the infimum is taken over all the decompositions of $f$ as above and the positive equivalence constants are independent of $f$.

**Remark 7.12.** We should point out that, when $q \in (0, 1]$ and $r = \infty$, Theorem 7.11 was obtained by Ho [40, Theorems 4.2 and 4.3].

**Theorem 7.13.** Let $p, q, r$ and $d$ be the same as in Theorem 7.11, and $\epsilon \in (n + d + 1, \infty)$. Then $f \in WHM^p_q(\mathbb{R}^n)$ if and only if

$$
f = \sum \sum \lambda_{i,j} n_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} \mathbf{1}_{B_{i,j}} \right\|_{M^p_q(\mathbb{R}^n)} < \infty,
$$

where $\{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ are $(M^p_q(\mathbb{R}^n), r, d)$-atoms supported, respectively, in balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ such that for any $i \in \mathbb{Z}$, $\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \leq A$ with $c \in (0, 1]$ and $A$ being a positive constant independent of $f$ and $i$, and for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := A^2 \|\mathbf{1}_{B_{i,j}}\|_{M^p_q(\mathbb{R}^n)}$ with $A$ being a positive constant independent of $f$ and $i$.
where \( \{m_{i,j}\}_{i,j \in \mathbb{Z}, j \in \mathbb{N}} \) are \( (M^p_q(\mathbb{R}^n), r, d, \epsilon) \)-molecules associated, respectively, with balls \( \{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}} \) such that for any \( i \in \mathbb{Z}, \sum_{j \in \mathbb{N}} 1_{eB_{i,j}} \leq A \) with \( c \in (0, 1) \) and \( A \) being a positive constant independent of \( f \) and \( i \), and for any \( i \in \mathbb{Z} \) and \( j \in \mathbb{N}, \lambda_{i,j} := A2^j \|1_{B_{i,j}}\|_{M^p_q(\mathbb{R}^n)} \) with \( \lambda \) being a positive constant independent of \( f, i \) and \( j \).

Moreover, for any \( f \in WHM^p_q(\mathbb{R}^n) \),

\[
\|f\|_{WHM^p_q(\mathbb{R}^n)} \sim \inf \left[ \sup_{j \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{M^p_q(\mathbb{R}^n)} \right],
\]

where the infimum is taken over all the decompositions of \( f \) as above and the positive equivalence constants are independent of \( f \).

**Remark 7.14.** Let \( 0 < q \leq p < \infty \) and \( p = q \). In this case, for any \( \tau \in (0, \infty), r \in [1, \infty] \) and \( d \in \mathbb{Z}_+ \), any \( (M^p_q(\mathbb{R}^n), r, d) \)-atom and any \( (M^p_q(\mathbb{R}^n), r, d, \epsilon) \)-molecule just become, respectively, a well-known classical atom (see, for example, [56, Definition 1.1] or [74, p. 112]) and a well-known classical molecule (see, for example, [42, Definition 1.2] with \( X := L^q(\mathbb{R}^n) \)). In this case, Theorem 7.11 was obtained by [52, Theorem 3.5] and [81, Theorem 4.4] as a special case; Theorem 7.13 was obtained by [52, Theorem 3.9] and [81, Theorem 5.3] as a special case.

Now, we recall the notion of the Morrey Hardy space \( HM^q_p(\mathbb{R}^n) \) as follows.

**Definition 7.15.** Let \( 0 < q \leq p < \infty \). The Morrey Hardy space \( HM^q_p(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{HM^q_p(\mathbb{R}^n)} := \|M^p_N(f)\|_{M^q_p(\mathbb{R}^n)} < \infty,
\]

where \( M^p_N(f) \) is as in (2.14) with \( N \) sufficiently large.

To obtain the boundedness of Calderón-Zygmund operators from \( HM^q_p(\mathbb{R}^n) \) to \( WHM^q_p(\mathbb{R}^n) \), we need the following vector-valued inequality of the Hardy-Littlewood maximal operator \( M \) in (2.8) from \( M^p_q(\mathbb{R}^n) \) to \( WHM^p_q(\mathbb{R}^n) \).

**Proposition 7.16.** Let \( p \in [1, \infty) \) and \( r \in (1, \infty) \). Then there exists a positive constant \( C \) such that for any \( \{f_j\}_{j \in \mathbb{N}} \subset M^p_q(\mathbb{R}^n) \),

\[
\left\| \left\{ \sum_{j=1}^{\infty} |M(f_j)|^r \right\}^{\frac{1}{r}} \right\|_{WHM^p_q(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{M^p_q(\mathbb{R}^n)}.
\]

**Proof.** Let \( B := B(x_0, R) \in \mathcal{B} \) with \( x_0 \in \mathbb{R}^n \) and \( R \in (0, \infty) \), where \( \mathcal{B} \) is as in (2.2) (the set of all balls of \( \mathbb{R}^n \)). For any given \( j \in \mathbb{N} \), we decompose \( f_j \) into

\[
f_j = f_{j}^{(0)} + \sum_{k=1}^{\infty} f_{j}^{(k)},
\]

where \( f_{j}^{(0)} := f_j 1_{2B}, \) and for any \( k \in \mathbb{N}, f_{j}^{(k)} := f_j 1_{2^{k+1}B \setminus 2^kB} \). From this and the Minkowski inequality, we deduce that

\[
\left\{ \sum_{j=1}^{\infty} |M(f_j)|^r \right\}^{\frac{1}{r}} \leq \left\{ \sum_{j=1}^{\infty} |M(f_{j}^{(0)})|^r \right\}^{\frac{1}{r}} + \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{\infty} |M(f_{j}^{(k)})|^r \right\}^{\frac{1}{r}}.
\]

For any given \( \lambda \in (0, \infty) \), we find that

\[
\left\| 1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} |M(f_{j}(x))|^{\frac{1}{r}} \geq \lambda \} \right\|_{L^1(B)} \leq \left\| 1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} |M(f_{j}^{(0)})(x)|^{\frac{1}{r}} \geq \lambda/2 \} \right\|_{L^1(B)} + \left\| 1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |M(f_{j}^{(k)})(x)|^{\frac{1}{r}} \geq \lambda/2 \} \right\|_{L^1(B)}
\]

\[
\leq \left\| 1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} |M(f_{j}^{(0)})(x)|^{\frac{1}{r}} \geq \lambda/2 \} \right\|_{L^1(B)} + \lambda^{-1} \left\| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{\infty} |M(f_{j}^{(k)})|^r \right\}^{\frac{1}{r}} \right\|_{L^1(B)}
\]
From this and the Minkowski inequality, we deduce that for any $r > \lambda/2$,

$$\|1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} (M(f_j^{(k)})(x))^r > \lambda\}}\|_{L^1(B)} + \lambda^{-1} \sum_{k=1}^{\infty} \left\| \left\{ \left( \sum_{j=1}^{\infty} [M(f_j^{(k)})]^r \right) \right\}^{\frac{1}{r}} \right\|_{L^1(B)} =: I + II.$$

From the Fefferman-Stein vector-valued inequality (see [25, Theorem 1(2)]), it follows that

$$I \lesssim \lambda^{-1} \left\| \left[ \sum_{j=1}^{\infty} |f_j^{(0)}|^r \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)} \sim \lambda^{-1} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}.$$

For any given $j, k \in \mathbb{N}$ and $x \in B$, it is easy to find that

$$M(f_j^{(k)})(x) = \sup_{t > 0} \frac{1}{|B(x, t)|} \int_{B(x, t)} |f_j^{(k)}(y)| dy \sim \sup_{t > 2R} \frac{1}{|B(x, t)|} \int_{B(x, t)} |f_j^{(k)}(y)| dy \lesssim (2^k R)^{-n} \int_{\mathbb{R}^n} |f_j^{(k)}(y)| dy.$$

From this and the Minkowski inequality, we deduce that for any $k \in \mathbb{N}$ and $x \in B$,

$$\left\{ \sum_{j=1}^{\infty} [M(f_j^{(k)})(x)]^r \right\}^{\frac{1}{r}} \lesssim \left\{ \sum_{j=1}^{\infty} (2^k R)^{-n} \int_{\mathbb{R}^n} |f_j^{(k)}(x)| dx \right\}^{\frac{1}{r}} \lesssim (2^k R)^{-n} \int_{\mathbb{R}^n} \left[ \sum_{j=1}^{\infty} |f_j^{(k)}(x)| ^r \right]^{\frac{1}{r}} dx \lesssim (2^k R)^{-n} \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \left\| \left[ \sum_{j=1}^{\infty} [M(f_j^{(k)})(x)]^r \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}.$$

which implies that

$$II \lesssim \lambda^{-1} \sum_{k=1}^{\infty} |B|^2 (2^k R)^{-n} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)} \sim \lambda^{-1} \sum_{k=1}^{\infty} 2^{-kn} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{L^1(2^k+1B)}.$$

By the estimates of I and II, we conclude that

$$\left| B \right|^{\frac{1}{p}-1} \|1_{\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} (M(f_j)(x))^r > \lambda\}}\|_{L^1(B)} \lesssim \lambda^{-1} \sum_{k=0}^{\infty} 2^{-\frac{k}{p}} \|2^k B\|_{L^1(2^k+1B)}^{\frac{1}{p}-1} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{L^1(2^k+1B)} \lesssim \lambda^{-1} \sum_{k=0}^{\infty} 2^{-\frac{k}{p}} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{M^p_q(\mathbb{R}^n)} \lesssim \lambda^{-1} \left\| \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{\frac{1}{r}} \right\|_{M^p_q(\mathbb{R}^n)}.$$
Remark 7.19. Let $0 < q < p < \infty$ and $p = q$. In this case, we know that $M_t^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $W^tM_t^q(\mathbb{R}^n) = W^qL^q(\mathbb{R}^n)$. Thus, by Theorem 7.17, we recover that the convolutional $\delta$-type Calderón-Zygmund operator $T$ is bounded from $H^{p,\alpha}(\mathbb{R}^n)$ to $WH^{p,\alpha}(\mathbb{R}^n)$, which is just [54, Theorem 1] (see also [52, Theorem 5.2] and [81, Theorem 7.4]). Here, $\frac{n}{p+\gamma}$ is called the critical index. Also, by Theorem 7.18, we recover that any $\gamma$-order Calderón-Zygmund operator is bounded from $H^{p,\alpha}(\mathbb{R}^n)$ to $WH^{p,\alpha}(\mathbb{R}^n)$, which is a special case of [81, Theorem 7.6]. Yan et al. [81] pointed out that the critical index of $\gamma$-order Calderón-Zygmund operators is $\frac{n}{\alpha+\gamma}$.

7.2 Mixed-norm Lebesgue spaces

We begin with recalling the notion of mixed-norm Lebesgue spaces.

Definition 7.20. Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$. The mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right)^{\frac{p_1}{p}} \cdots dx_n \right\}^{\frac{1}{p}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \ldots, n\}$.

The space $L^{\vec{p}}(\mathbb{R}^n)$ was studied by Benedek and Panzone [8] in 1961, which can be traced back to Hörmander [41]. From the definition of $\| \cdot \|_{L^{\vec{p}}(\mathbb{R}^n)}$, it is easy to deduce that the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is a ball quasi-Banach function space. Let $\vec{p} := (p_1, \ldots, p_n) \in [1, \infty]^n$. Then for any $f \in L^{\vec{p}}(\mathbb{R}^n)$ and $g \in L^{\vec{p}}(\mathbb{R}^n)$, it is easy to know that

$$\int_{\mathbb{R}^n} |f(x)|^{p_i} dx \leq \|f\|_{L^{\vec{p}}(\mathbb{R}^n)} \|g\|_{L^{\vec{p}}(\mathbb{R}^n)}$$

where $\vec{p}'$ denotes the conjugate vector of $\vec{p}$, namely, for any $i \in \{1, \ldots, n\}$, $1/p_i + 1/p_i' = 1$. This implies that $L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in [1, \infty)^n$ is a ball Banach function space, which is not a Banach function space (see the following remark).

Remark 7.21. It is worth pointing out that $L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in [1, \infty]^n$ may not be a Banach function space. For example, let $\vec{p} := (2, 1)$ and $n := 2$. In this case, $L^{\vec{p}}(\mathbb{R}^n) = L^{(2,1)}(\mathbb{R}^2)$. Let

$$E := \bigcup_{m \in \mathbb{N}} [m, m + 1/m] \times [m, m + 1/\sqrt{m}]$$

Then it is easy to show that $|E| < \infty$, but

$$\|1_E\|_{L^{(2,1)}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} 1_E(x_1, x_2) dx_1 \right)^{\frac{1}{2}} dx_2 = \sum_{m \in \mathbb{N}} \int_m^{m+1/\sqrt{m}} \left( \int_m^{m+1/m} dx_1 \right)^{\frac{1}{2}} dx_2 = \infty$$

Thus, $L^{(2,1)}(\mathbb{R}^2)$ does not satisfy Definition 2.1(iv), which means that $L^{(2,1)}(\mathbb{R}^2)$ is not a Banach function space.

Furthermore, the following Fefferman-Stein vector-valued maximal inequalities for $L^{\vec{p}}(\mathbb{R}^n)$ hold true (see, for example, [43, Lemma 3.7] or [39, Theorem 4.3]), which shows that the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ satisfies Assumption 2.18. For any $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$, we always let $p_- := \min\{p_1, \ldots, p_n\}$ and $p_+ := \max\{p_1, \ldots, p_n\}$.

Lemma 7.22. Let $\vec{p} \in (0, \infty)^n$. Assume that $r \in (1, \infty)$ and $s \in (0, p_-)$. Then there exists a positive constant $C$ such that for any $\{f_j\}_{j=1}^{\infty} \subset M(\mathbb{R}^n)$,

$$\left\{ \sum_{j=1}^{\infty} \left[ M(f_j)^{p_i} \right]^r \right\}^{1/r} \left\|L^{\vec{p}(\mathbb{R}^n)} \right\|^{1/r} \leq C \left\{ \sum_{j=1}^{\infty} \left| f_j \right|^{p_i} \right\}^{1/r} \left\|L^{\vec{p}(\mathbb{R}^n)} \right\|^{1/r}$$

where $[L^{\vec{p}(\mathbb{R}^n)}]^{1/s}$ denotes the $\frac{1}{s}$-convexification of $L^{\vec{p}(\mathbb{R}^n)}$ as in Definition 2.7(i) with $X$ and $p$ replaced, respectively, by $L^{\vec{p}(\mathbb{R}^n)}$ and $1/s$. 
Now we introduce the weak mixed-norm Lebesgue space $WL^p(R^n)$.

**Definition 7.23.** Let $p \in (0, \infty)^n$. The weak mixed-norm Lebesgue space $WL^p(R^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{WL^p(R^n)} := \sup_{\alpha \in (0, \infty)} \left[ \alpha \| 1_{\{ x \in R^n : |f(x)| > \alpha \} } \|_{L^p(R^n)} \right] < \infty.$$  

Let $T$ be an operator defined on $\mathcal{M}(R^n)$. Then $T$ is called a sublinear operator, if for any $f, g \in \mathcal{M}(R^n)$ and any $\lambda \in C$,  

$$|T(f + g)| \leq |T(f)| + |T(g)| \quad \text{and} \quad |T(\lambda f)| = |\lambda| |T(f)|.$$  

The interpolation theorem of operators on the mixed-norm Lebesgue space $L^p(R^n)$ is stated as follows.

**Theorem 7.24.** Let $p \in (1, \infty)^n$. Let $r_1 \in \left( \frac{1}{p_1}, 1 \right)$ and $r_2 \in (1, \infty)$. Assume that $T$ is a sublinear operator defined on $L^{r_1, p}(R^n) + L^{r_2, p}(R^n)$ satisfying that there exist positive constants $C_1$ and $C_2$ such that for any $i \in \{ 1, 2 \}$ and $f \in L^{r_i, p}(R^n)$,

$$\|T(f)\|_{L^{r_i, p}(R^n)} \leq C_i \|f\|_{L^{r_i, p}(R^n)},$$  

(7.1)

where $r_i p := (r_i p_1, \ldots, r_i p_n)$ for any $i \in \{ 1, 2 \}$. Then $T$ is bounded on $WL^p(R^n)$ and there exists a positive constant $C$ such that for any $f \in WL^p(R^n)$,

$$\|T(f)\|_{WL^p(R^n)} \leq C \|f\|_{WL^p(R^n)}.$$  

**Proof.** Let $f \in WL^p(R^n)$ and

$$\lambda := \|f\|_{WL^p(R^n)} = \sup_{\alpha \in (0, \infty)} \left[ \alpha \| 1_{\{ x \in R^n : |f(x)| > \alpha \} } \|_{L^p(R^n)} \right].$$

We need to show that for any $\alpha \in (0, \infty)$,

$$\alpha \| 1_{\{ x \in R^n : |f(x)| > \alpha \} } \|_{L^p(R^n)} \lesssim \lambda$$  

with the implicit positive constant independent of $\alpha$ and $f$.

To this end, for any $\alpha \in (0, \infty)$, let

$$f(\alpha) := f 1_{\{ x \in R^n : |f(x)| > \alpha \}} \quad \text{and} \quad f_\alpha := f 1_{\{ x \in R^n : |f(x)| \leq \alpha \}}.$$  

We claim that

$$\|f(\alpha)\|_{L^{r_1, p}(R^n)} \lesssim \alpha (\lambda/\alpha)^{1/r_1},$$  

(7.2)

and

$$\|f_\alpha\|_{L^{r_2, p}(R^n)} \lesssim \alpha (\lambda/\alpha)^{1/r_2}.$$  

(7.3)

Assuming that this claim holds true for the moment, then, by the condition that $T$ is sublinear and (7.1), we conclude that for any $\alpha \in (0, \infty)$,

$$\alpha \| 1_{\{ x \in R^n : |T(f(x))| > \alpha \} } \|_{L^p(R^n)}$$

$$\lesssim \alpha \| 1_{\{ x \in R^n : |T(f(\alpha))(x)| > \alpha/2 \} } \|_{L^p(R^n)} + \alpha \| 1_{\{ x \in R^n : |T(f_\alpha)(x)| > \alpha/2 \} } \|_{L^p(R^n)}$$

$$\sim \alpha \| 1_{\{ x \in R^n : |T(f(\alpha))(x)| > \alpha/2 \} } \|_{L^{r_1, p}(R^n)} + \alpha \| 1_{\{ x \in R^n : |T(f_\alpha)(x)| > \alpha/2 \} } \|_{L^{r_2, p}(R^n)}$$

$$\lesssim \alpha^{1-r_1} \| f(\alpha) \|_{L^{r_1, p}(R^n)} + \alpha^{1-r_2} \| f_\alpha \|_{L^{r_2, p}(R^n)} \lesssim \lambda,$$

This implies that $\|T(f)\|_{WL^p(R^n)} \lesssim \|f\|_{WL^p(R^n)}$, which is the desired conclusion.

Therefore, it remains to prove the above claim. To prove (7.2), by the Minkowski inequality, we have

$$\|f(\alpha)/\alpha\|_{L^{r_1, p}(R^n)} = \int_0^\infty \int_{\{ y \in R^n : |f(\alpha)(y)/\alpha| > \alpha/2 \}} |f(\alpha)(y)/\alpha|^{r_1-1} \, dy \, d\alpha \lesssim \int_0^\infty \int_{\{ y \in R^n : |f(\alpha)(y)| > \alpha/2 \}} |f(\alpha)(y)|^{r_1-1} \, dy \, d\alpha.$$
By the definition of \( f^{(a)} \) and Definition 7.23, it is easy to see that
\[
I_1 \lesssim \left( \int_0^\infty \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha \}} \|L^p(\mathbb{R}^n) \, d\tau \right)^{\frac{1}{r_1}} \lesssim \left[ \frac{\alpha}{\lambda} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha \}} \|L^p(\mathbb{R}^n) \right]^{\frac{1}{r_1}} \lesssim 1.
\]
As for \( I_2 \), from the definition of \( f^{(a)} \), Definition 7.23 and \( \frac{1}{r_1} > 1 \), it follows that
\[
I_2 \lesssim \left( \int_0^\infty \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha \} \} \|L^p(\mathbb{R}^n) \, d\tau \right)^{\frac{1}{r_2}} \lesssim \left\{ \int_0^\alpha \left[ \lambda^{-1} \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{r_2}} \right] \lambda \, d\tau \right\}^{\frac{1}{r_2}} \lesssim 1.
\]
Combining the estimates for \( I_1 \) and \( I_2 \), we then obtain (7.2).

To prove (7.3), by a proof similar to that used in the estimation of (7.2), we have
\[
\left\| \frac{|f^{(a)}|/\alpha}{(\lambda/\alpha)^{1/r_2}} \right\|_{L^{2r}(\mathbb{R}^n)} \lesssim \left( \int_0^\infty \|1_{\{y \in \mathbb{R}^n : |f^{(a)}(y)| > \alpha (\frac{\lambda}{\alpha})^{1/r_2} \}} \|L^p(\mathbb{R}^n) \, d\tau \right)^{\frac{1}{r_2}} + \left[ \int_0^\infty \cdots \, d\tau \right]^{\frac{1}{r_2}} =: II_1 + II_2.
\]
From the definition of \( f^{(a)} \), Definition 7.23 and \( 0 < \frac{1}{r_2} < 1 \), we deduce that
\[
II_1 \lesssim \left( \int_0^\infty \|1_{\{y \in \mathbb{R}^n : |f^{(a)}(y)| > \alpha (\frac{\lambda}{\alpha})^{1/r_2} \}} \|L^p(\mathbb{R}^n) \, d\tau \right)^{\frac{1}{r_\alpha}} \lesssim \left\{ \int_0^\alpha \left[ \alpha^{-1} \left( \frac{\lambda}{\alpha} \right)^{-\frac{1}{r_2}} \right] \lambda \, d\tau \right\}^{\frac{1}{r_\alpha}} \lesssim 1.
\]
Observe that, when \( \tau \in \left( \frac{\alpha}{\lambda}, \infty \right) \),
\[
|\frac{|f^{(a)}|/\alpha|^{p_2} \leq 1 < \frac{\tau^\lambda}{\alpha}
\]
and hence \( II_2 = 0 \), which, together with the estimate for \( II_1 \), implies (7.3). Thus, we complete the proof of our claim and hence of Theorem 7.24. \( \square \)

We also need the following Fefferman-Stein vector-valued maximal inequality on \( WL^p(\mathbb{R}^n) \).

**Theorem 7.25.** Let \( \vec{p} \in (1, \infty)^n \) and \( s \in (1, \infty) \). Then there exists a positive constant \( C \) such that for any sequence \( \{f_j\}_{j \in \mathbb{N}} \subset M(\mathbb{R}^n) \),
\[
\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{WL^p(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{WL^p(\mathbb{R}^n)}.
\]

**Proof.** Let \( \{f_j\}_{j \in \mathbb{N}} \) be a given arbitrary sequence of measurable functions, and for any measurable function \( g \) and \( x \in \mathbb{R}^n \), define
\[
A(g)(x) := \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(g \eta_j)(x)]^s \right\}^{\frac{1}{s}},
\]
where \( s \in (1, \infty) \), and for any \( i \in \mathbb{N} \) and \( y \in \mathbb{R}^n \),
\[
\eta_j(y) := \frac{f_j(y)}{\left( \sum_{j \in \mathbb{N}} |f_j(y)|^s \right)^{1/s}} \quad \text{when} \quad \left[ \sum_{j \in \mathbb{N}} |f_j(y)|^s \right]^{1/s} \neq 0,
\]

and $\eta_j(y) := 0$ otherwise. By the Minkowski inequality, we conclude that for any $\lambda \in \mathbb{C}$ and $g_1, g_2 \in \mathcal{M}(\mathbb{R}^n)$,

$$A(g_1 + g_2) \leq A(g_1) + A(g_2) \quad \text{and} \quad A(\lambda g) = |\lambda|A(g).$$

Thus, $A$ is sublinear. For any $\vec{p} \in (1, \infty)^n$ and $s \in (1, \infty)$, from Lemma 7.22, we deduce that

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{L^p(\mathbb{R}^n)}.$$

Using this, we know that for any given $r_1 \in (\frac{1}{p^-}, 1)$ and $r_2 \in (1, \infty)$ and any $h \in \mathcal{M}(\mathbb{R}^n)$,

$$\|A(b)\|_{W^{L^r, \vec{p}}(\mathbb{R}^n)} = \left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(h\eta_j)]^s \right\}^{\frac{1}{s}} \right\|_{W^{L^r, \vec{p}}(\mathbb{R}^n)} \leq \left\| \left\{ \sum_{j \in \mathbb{N}} [h\eta_j]^s \right\}^{\frac{1}{s}} \right\|_{W^{L^r, \vec{p}}(\mathbb{R}^n)} \sim \|h\|_{L^{r, \vec{p}}(\mathbb{R}^n)},$$

which implies that the operator $A$ is bounded from $L^{r, \vec{p}}(\mathbb{R}^n)$ to $W^{L^r, \vec{p}}(\mathbb{R}^n)$, where $i \in \{1, 2\}$. Now, taking $g := [\sum_{j \in \mathbb{N}} |f_j|^s]^{1/s}$, then, by Theorem 7.24, we conclude that

$$\left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \lesssim \|A(g)\|_{W^{L^r, \vec{p}}(\mathbb{R}^n)} \lesssim \|g\|_{W^{L^r, \vec{p}}(\mathbb{R}^n)} \sim \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \|_{W^{L^r, \vec{p}}(\mathbb{R}^n)},$$

which completes the proof of Theorem 7.25.

By [43, Lemma 3.5] and [8, Theorem 1.a], we can easily obtain the following conclusion and we omit the details here.

**Lemma 7.26.** Let $\vec{p} \in (0, \infty)^n$, $r \in (0, p_-)$ and $s \in (p_+, \infty]$. Then there exists a positive constant $C$ such that for any $f \in \mathcal{M}(\mathbb{R}^n)$,

$$\|\mathcal{M}^{((s/r)^r)}(f)\|_{(L^p(\mathbb{R}^n))^{1/r'}} \leq C\|f\|_{(L^p(\mathbb{R}^n))^{1/r'}},$$

where $(L^p(\mathbb{R}^n))^{1/r'}$ is as in (2.5) with $X := [L^p(\mathbb{R}^n)]^{1/r'}$.

Now we give the notion of the weak mixed-norm Hardy space $W^{H^\vec{p}}(\mathbb{R}^n)$.

**Definition 7.27.** Let $\vec{p} \in (0, \infty)^n$. The weak mixed-norm Hardy space $W^{H^\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{W^{H^\vec{p}}(\mathbb{R}^n)} := \|M^\vec{p}(f)\|_{W^{L^p(\mathbb{R}^n)}} < \infty,$$

where $M^\vec{p}(f)$ is as in (2.14) with $N$ sufficiently large.

**Remark 7.28.** Let $\vec{p} \in (1, \infty)^n$. By Theorem 7.25, we conclude that for any $r \in (1, p_-)$, $\mathcal{M}$ in (2.8) is bounded on $(W^{L^p(\mathbb{R}^n)})^{1/r}$, which, combined with Theorem 3.4, implies that $W^{H^\vec{p}}(\mathbb{R}^n) = W^{L^\vec{p}}(\mathbb{R}^n)$ with equivalent norms.

By Lemma 7.25 and Theorem 3.2(ii), we obtain the following maximal function characterizations of the weak mixed-norm Hardy space $W^{H^\vec{p}}(\mathbb{R}^n)$.

**Theorem 7.29.** Let $\vec{p} \in (0, \infty)^n$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Assume that $b \in (n/p_-, \infty)$ and $N \geq [b + 1]$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, if one of the following quantities

$$\|M^\vec{p}(f)\|_{W^{L^p(\mathbb{R}^n)}}, \quad \|M(f, \psi)\|_{W^{L^p(\mathbb{R}^n)}}, \quad \|M^*_\delta(f, \psi)\|_{W^{L^p(\mathbb{R}^n)}}, \quad \|M_N(f)\|_{W^{L^p(\mathbb{R}^n)}},$$

$$\|M^*_\delta(f, \psi)\|_{W^{L^p(\mathbb{R}^n)}}, \quad \|M^*_\delta(f, \psi)\|_{W^{L^p(\mathbb{R}^n)}}$$

is finite, then the others are also finite and mutually equivalent with the positive equivalence constants independent of $f$. 


Using Lemmas 7.22, 7.25 and 7.26, and Theorems 4.2 and 4.7, we immediately obtain the atomic characterization of $WH\tilde{F}(\mathbb{R}^n)$ and the molecular characterization of $WH\tilde{F}(\mathbb{R}^n)$, respectively, as follows.

Theorem 7.30. Let $\vec{\rho} \in (0,\infty)^n$, $r \in (\max\{1,p_+\},\infty)$ and $d \in \mathbb{Z}_+$ with

$$d \geq \left\lfloor n \left( \frac{1}{\min\{1,p_-/\max\{1,p_+\} - 1} \right) \right\rfloor.$$

Then $f \in WH\tilde{F}(\mathbb{R}^n)$ if and only if

$$f = \sum_{i,j} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} \in S'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{L^r(\mathbb{R}^n)} < \infty,$$

where $\{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ are $(\tilde{F}(\mathbb{R}^n),r,d)$-atoms supported, respectively, in balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ such that for any $i \in \mathbb{Z}$ $\sum_{j \in \mathbb{N}} 1_{B_{i,j}} \leq A$ with $c \in (0,1]$ and $A$ being a positive constant independent of $f$ and $i$, and for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := A 2^j 1_{B_{i,j}} 1_{L^r(\mathbb{R}^n)}$ with $A$ being a positive constant independent of $f$ and $i$.

Moreover, for any $f \in WH\tilde{F}(\mathbb{R}^n)$,

$$\|f\|_{WH\tilde{F}(\mathbb{R}^n)} \sim \inf \left\{ \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{L^r(\mathbb{R}^n)} \right\},$$

where the infimum is taken over all the decompositions of $f$ as above and the positive equivalence constant is independent of $f$.

Theorem 7.31. Let $\vec{\rho}$, $r$ and $d$ be the same as in Theorem 7.30, and $\epsilon \in (n + d + 1,\infty)$. Then $f \in WH\tilde{F}(\mathbb{R}^n)$ if and only if

$$f = \sum_{i,j} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j} \in S'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{L^r(\mathbb{R}^n)} < \infty,$$

where $\{m_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ are $(\tilde{F}(\mathbb{R}^n),r,d,\epsilon)$-molecules associated, respectively, with balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ such that for any $i \in \mathbb{Z}$ $\sum_{j \in \mathbb{N}} 1_{B_{i,j}} \leq A$ with $c \in (0,1]$ and $A$ being a positive constant independent of $f$ and $i$, and for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := A 2^j 1_{B_{i,j}} 1_{L^r(\mathbb{R}^n)}$ with $A$ being a positive constant independent of $f$, $i$ and $j$.

Moreover, for any $f \in WH\tilde{F}(\mathbb{R}^n)$,

$$\|f\|_{WH\tilde{F}(\mathbb{R}^n)} \sim \inf \left\{ \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{L^r(\mathbb{R}^n)} \right\},$$

where the infimum is taken over all the decompositions of $f$ as above and the positive equivalence constants are independent of $f$.

Now, we recall the following notion of the mixed-norm Hardy space.

Definition 7.32. Let $\vec{\rho} \in (0,\infty)^n$. The mixed-norm Hardy space $H\tilde{F}(\mathbb{R}^n)$ is defined to be the set of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{H\tilde{F}(\mathbb{R}^n)} := \|M^\rho_{\tilde{F}}(f)\|_{L^r(\mathbb{R}^n)} < \infty,$$

where $M^\rho_{\tilde{F}}(f)$ is as in (2.14) with $N$ sufficiently large.

To discuss the boundedness of Claderón-Zygmund operators from $H\tilde{F}(\mathbb{R}^n)$ to $WH\tilde{F}(\mathbb{R}^n)$, we need the following vector-valued inequality of the Hardy-Littlewood maximal operator $M$ in (2.8) from $\tilde{F}(\mathbb{R}^n)$ to $WL\tilde{F}(\mathbb{R}^n)$.

Proposition 7.33. Let $r \in (1,\infty)$ and $\vec{\rho} \in [1,\infty)^n$ satisfy $p_1 \leq \cdots \leq p_n$. Then there exists a positive constant $C$ such that for any $\{f_j\}_{j \in \mathbb{N}} \subset H(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} |M(f_j)|^r \right\}^{1/r} \right\|_{WL^r(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^r \right\}^{1/r} \right\|_{L^r(\mathbb{R}^n)}.$$
To prove Proposition 7.33, we need the following extrapolation theorem, which is a slight variant of a special case of [21, Theorem 4.6] via replacing Banach function spaces by ball Banach function spaces. Recall that an $A_1(\mathbb{R}^n)$-weight $\omega$ (see, for example, [32, Definition 7.1.1]) is a locally integrable and nonnegative function satisfying that

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \frac{1}{|B|} \int_B \omega(x)dx \left\| \omega^{-1} \right\|_{L_\infty(B)} < \infty,$$

where $\mathcal{B}$ is as in (2.2).

**Lemma 7.34.** Let $X$ be a ball Banach function space and $p_0 \in (0, \infty)$. Let $F$ be the set of all pairs of nonnegative measurable functions $(F,G)$ such that for any given $\omega \in A_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [F(x)]^{p_0} \omega(x)dx \leq C_{(p_0,[\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [G(x)]^{p_0} \omega(x)dx,$$

where $C_{(p_0,[\omega]_{A_1(\mathbb{R}^n)})}$ is a positive constant independent of $(F,G)$, but depends on $p_0$ and $[\omega]_{A_1(\mathbb{R}^n)}$. Assume that there exists a $q_0 \in [p_0, \infty)$ such that $X^{1/q_0}$ is a Banach function space and $M$ is bounded on $(X^{1/q_0})'$. Then there exists a positive constant $C$ such that for any $(F,G) \in F$,

$$\|F\|_X \leq C\|G\|_X.$$

**Proof.** We observe that a key fact used in the proof of [21, Theorem 4.6] is that, if $X$ is a Banach function space as in Definition 2.1, then $X = X''$ with the same norms. However, if $X$ is just a ball Banach function space as in this lemma, by Lemma 2.6, we know that this fact also holds true. Thus, using this fact and repeating the proof of [21, Theorem 4.6], we then complete the proof of Lemma 7.34.

We still need the following weak-type weighted Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator $M$ in (2.8) from [5, Theorem 3.1(a)].

**Lemma 7.35.** Let $\omega \in A_1(\mathbb{R}^n)$ and $r \in (1, \infty)$. Then there exists a positive constant $C$, depending on $p_0$ and $[\omega]_{A_1(\mathbb{R}^n)}$, such that for any $\alpha \in (0, \infty)$ and $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$,

$$\alpha \int_{\mathbb{R}^n} \frac{1}{(\sum_{j \in \mathbb{N}} [M(f_j)(y)]^{q_0})^{1/q_0}} (x) \omega(x)dx \leq C \int_{\mathbb{R}^n} \left\| \sum_{j \in \mathbb{N}} |f_j|^\alpha \right\|^{1/\alpha} \omega(x)dx.$$

**Proof of Proposition 7.33.** For any given $r \in (1, \infty)$, let

$$\mathcal{F} := \left\{ \left( \alpha 1_{\{y \in \mathbb{R}^n : (\sum_{j \in \mathbb{N}} [M(f_j)(y)]^{q_0})^{1/q_0} \leq \alpha \} }, \left\| \sum_{j \in \mathbb{N}} |f_j|^{\alpha} \right\|^{1/\alpha} \right) : \alpha \in (0, \infty), \{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n) \right\}.$$

Then, by Lemma 7.35, we conclude that for any given $\omega \in A_1(\mathbb{R}^n)$ and any $(F,G) \in \mathcal{F}$,

$$\int_{\mathbb{R}^n} F(x)\omega(x)dx \lesssim \int_{\mathbb{R}^n} G(x)\omega(x)dx. \quad (7.4)$$

Let $\vec{p} \in [1, \infty)^n$ satisfy $p_1 \leq \cdots \leq p_n$. From [8, Theorem 1.a] and [43, Lemma 3.5], it follows that $\mathcal{M}$ as in (2.8) is bounded on $(L^{p_0}(\mathbb{R}^n))'$. By this and (7.4), applying Lemma 7.34 with $p_0 := 1$ and the fact that $L^{p_0}(\mathbb{R}^n)$ is a Banach function space, we conclude that for any $(F,G) \in \mathcal{F}$, $\|F\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \|G\|_{L^{p_0}(\mathbb{R}^n)}$. Thus, for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [M(f_j)]^{r} \right\}^{1/\alpha} \right\|_{W(L^{p_0}(\mathbb{R}^n))} \leq \left\{ \sum_{j \in \mathbb{N}} |f_j|^{r} \right\}^{1/\alpha} \right\|_{L^{p_0}(\mathbb{R}^n)},$$

which completes the proof of Proposition 7.33.

Applying Proposition 7.33, Lemmas 7.26 and 7.25, and Theorems 6.3 and 6.4, we immediately obtain the boundedness from $H^{\vec{p}}(\mathbb{R}^n)$ to $WH^{\vec{p}}(\mathbb{R}^n)$ of both convolutional $\delta$-type and $\gamma$-type Calderón-Zygmund operators, respectively, as follows.
Theorem 7.36. Let $\vec{p} \in (0, \infty)^n$ with $p_1 \leq \cdots \leq p_n$ and $\delta \in (0, 1]$. Let $T$ be a convolutional $\delta$-type Calderón–Zygmund operator. If $p_- \in \left[ \frac{n}{n+\delta}, 1 \right]$, then $T$ has a unique extension on $H^p(\mathbb{R}^n)$ and, moreover, there exists a positive constant $C$ such that for any $f \in H^p(\mathbb{R}^n)$,

$$
\|Tf\|_{W H^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}.
$$

Theorem 7.37. Let $\vec{p} \in (0, 2)^n$ with $p_1 \leq \cdots \leq p_n$ and $\gamma \in (0, \infty)$. Let $T$ be a $\gamma$-type Calderón–Zygmund operator and have the vanishing moments up to order $\lfloor \gamma \rfloor - 1$. If $\lfloor \gamma \rfloor - 1 \leq n(1 - 1 - p_-) - 1 \leq \gamma$, then $T$ has a unique extension on $H^\gamma(\mathbb{R}^n)$ and, moreover, there exists a positive constant $C$ such that for any $f \in H^\gamma(\mathbb{R}^n)$,

$$
\|Tf\|_{W H^\gamma(\mathbb{R}^n)} \leq C\|f\|_{H^\gamma(\mathbb{R}^n)}.
$$

7.3 Orlicz-slice spaces

We begin with the notions of both Orlicz functions and Orlicz spaces (see, for example, [66]).

Definition 7.38. A function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is non-decreasing and satisfies $\Phi(0) = 0$, $\Phi(t) > 0$ whenever $t \in (0, \infty)$ and $\lim_{t \to \infty} \Phi(t) = \infty$.

An Orlicz function $\Phi$ is said to be of lower (resp. upper) type $p$ with $p \in (-\infty, \infty)$ if there exists a positive constant $C(p)$, depending on $p$, such that for any $t \in (0, \infty)$ and $s \in (0, 1)$ (resp. $s \in [1, \infty]$),

$$
\Phi(st) \leq C(p)s^p\Phi(t).
$$

An Orlicz function $\Phi : [0, \infty) \to (0, \infty)$ is said to be of positive lower (resp. upper) type $p$ if it is of lower (resp. upper) type $p$ for some $p \in (0, \infty)$.

Definition 7.39. Let $\Phi$ be an Orlicz function with positive lower type $p_-^\Phi$ and positive upper type $p_+^\Phi$. The Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$
\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \leq 1 \right\} < \infty.
$$

Remark 7.40. (i) Let $\Phi$ be an Orlicz function with positive lower type $p_-^\Phi$ and positive upper type $p_+^\Phi$. In what follows, for any given $s \in (0, \infty)$, let $\Phi_s(\tau) := \Phi(\tau^s)$ for any $\tau \in (0, \infty)$. Then $\Phi_s$ is also an Orlicz function with lower type $s^p_-^\Phi$ and upper type $s^p_+^\Phi$. Moreover, for any measurable function $f$ such that $|f|^s \in L^\Phi(\mathbb{R}^n)$, we have

$$
\|f|^s\|_{L^\Phi(\mathbb{R}^n)} = \|f\|_{L^{\Phi^s}(\mathbb{R}^n)}.
$$

(ii) Let $\Phi$ be as in (i) of this remark. By [86, Lemma 2.5], we may always assume that $\Phi$ is continuous and strictly increasing. Let $\Phi^{-1}$ be the inverse function of $\Phi$. Observe that for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$
\|1_{B(x,t)}\|_{L^\Phi(\mathbb{R}^n)} = [\Phi^{-1}(\|B(x,t)\|^{-1})]^{-1} = [\Phi^{-1}(\epsilon_n^{-1}t^{-n})]^{-1} = \bar{C}(\Phi, t)
$$

is independent of $x \in \mathbb{R}^n$, where $\epsilon_n$ denotes the volume of the unit ball of $\mathbb{R}^n$.

We also recall some notions on the Young function. A convex function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if $\Phi$ is non-decreasing, $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$. For any Young function $\Phi$, its complementary function $\Psi : [0, \infty) \to [0, \infty)$ is defined by setting, for any $y \in [0, \infty)$,

$$
\Psi(y) := \sup\{xy - \Phi(x) : x \in [0, \infty]\}.
$$

Remark 7.41. Let $\Phi$ be an Orlicz function with lower type $p_-^\Phi \in [1, \infty)$ and positive upper type $p_+^\Phi$. By [66, p.67, Theorem 10] and [86, Lemma 2.16], we know that $L^\Phi(\mathbb{R}^n)$ coincides with a Banach space in the sense of equivalent norms.

The following notion of Orlicz-slice spaces was introduced by Zhang et al. [86], which is a generalization of the slice spaces proposed by Auscher and Mourougoglou [6] and Auscher and Prisuelos-Arribas [7].
\textbf{Definition 7.42.} Let $t, q \in (0, \infty)$ and $\Phi$ be an Orlicz function with positive lower type $p_{\Phi}^-$ and positive upper type $p_{\Phi}^+$. The Orlicz-slice space $(E^q_{\Phi})_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{(E^q_{\Phi})_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[ \|1_{B(x,t)}\|_{L^q(\mathbb{R}^n)} \right]^q \, dx \right\}^{\frac{1}{q}} < \infty.$$  

\textbf{Remark 7.43.} Let $t, q \in (0, \infty)$ and $\Phi$ be an Orlicz function with positive lower type $p_{\Phi}^-$ and positive upper type $p_{\Phi}^+$.

(i) By [86, Lemma 2.28], we know that the Orlicz-slice space $(E^q_{\Phi})_t(\mathbb{R}^n)$ is a ball quasi-Banach space. It is worth pointing out that $(E^q_{\Phi})_t(\mathbb{R}^n)$ with $q \in (1, \infty)$ and $p_{\Phi}^+ \in (1, \infty)$ may not be a Banach function space. For example, let $t := 1$, $q := 1$, $n := 1$ and $\Phi(\tau) := \tau^2$ for any $\tau \in [0, \infty)$. In this case, by [86, Proposition 2.12], we know that $(E^q_{\Phi})_t(\mathbb{R})$ and $L^1(\mathbb{R})$ coincide with equivalent norms, where $L^1(\mathbb{R})$ denotes the classical amalgam space (see, for example, [23]). Let

$$E := \bigcup_{m \in \mathbb{N}} [m, m + 1/m^2].$$

Then it is easy to show that $|E| < \infty$, but

$$\|1_E\|_{(E^q_{\Phi})_t(\mathbb{R})} \sim \|1_E\|_{L^1(\mathbb{R})} \sim \sum_{k \in \mathbb{Z}} \|1_E\|_{L^2(Q_k)} \sim \sum_{k \in \mathbb{N}} 1/k = \infty,$$

where $Q_k := k + [0, 1)$ for any $k \in \mathbb{Z}$. Thus, $(E^q_{\Phi})_t(\mathbb{R})$ does not satisfy Definition 2.1(iv), which means that $(E^q_{\Phi})_t(\mathbb{R})$ is not a Banach function space.

(ii) Let $\Phi(\tau) := \tau^r$ for any $\tau \in [0, \infty)$ with any given $r \in (0, \infty)$. Then $(E^q_{\Phi})_t(\mathbb{R}^n)$ and $(E^q_{\Phi})_t(\mathbb{R}^n)$, which were introduced in [6,7], coincide with equivalent quasi-norms. Moreover, in this case, if $q \in (0, r]$, then for any $f \in (E^q_{\Phi})_t(\mathbb{R}^n)$, $f \in L^0(\mathbb{R}^n)$ and $\|f\|_{L^0(\mathbb{R}^n)} \leq \|f\|_{(E^q_{\Phi})_t(\mathbb{R}^n)}$; if $r \in (0, q]$, then for any $f \in L^0(\mathbb{R}^n) \cup L^q(\mathbb{R}^n)$, $f \in (E^q_{\Phi})_t(\mathbb{R}^n)$ and

$$\|f\|_{(E^q_{\Phi})_t(\mathbb{R}^n)} \leq \min\{\|f\|_{L^0(\mathbb{R}^n)}, \|f\|_{L^q(\mathbb{R}^n)}\}.$$  

Thus, $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_{(E^q_{\Phi})_t(\mathbb{R}^n)}$ for any $p \in (0, \infty)$ (see [86, Proposition 2.11]).

Now we introduce the notion of weak Orlicz-slice spaces in a traditional way as follows.

\textbf{Definition 7.44.} Let $t, q \in (0, \infty)$ and $\Phi$ be an Orlicz function with positive lower type $p_{\Phi}^-$ and positive upper type $p_{\Phi}^+$. The weak Orlicz-slice space $(WE^q_{\Phi})_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{(WE^q_{\Phi})_t(\mathbb{R}^n)} := \sup_{\alpha \in (0, \infty)} \alpha \|1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}}\|_{(E^q_{\Phi})_t(\mathbb{R}^n)} < \infty.$$  

To establish a Fefferman-Stein vector-valued inequality on $(WE^q_{\Phi})_t(\mathbb{R}^n)$, we first need to establish an interpolation theorem, in the spirit of the Marcinkiewicz interpolation theorem. To this end, we now establish the following Minkowski type inequality.

\textbf{Lemma 7.45.} Let $t \in (0, \infty)$ and $\Phi$ be an Orlicz function with lower type $p_{\Phi}^- \in (1, \infty)$ and positive upper type $p_{\Phi}^+$. Suppose that a measurable function $F$ is defined on $\mathbb{R}^n \times \mathbb{R}^m$. If, for almost every $x \in \mathbb{R}^n$, $F(x, \cdot) \in L^1(\mathbb{R}^m)$ and, for almost every $y \in \mathbb{R}^m$, $F(\cdot, y) \in L^{p_{\Phi}^+}(\mathbb{R}^n)$, then

$$\left\| \int_{\mathbb{R}^m} |F(\cdot, y)| \, dy \right\|_{L^{p_{\Phi}^+}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^m} \|F(\cdot, y)\|_{L^{p_{\Phi}^+}(\mathbb{R}^n)} \, dy.$$  

\textbf{Proof.} Let $\Phi$ be as in this lemma and $\Psi$ the complementary function of $\Phi$. By [66, p. 61, Proposition 4 and p. 81, Proposition 10], we have

$$\left\| \int_{\mathbb{R}^m} |F(\cdot, y)| \, dy \right\|_{L^{p_{\Phi}^+}(\mathbb{R}^n)} \sim \sup \left\{ \left\| \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |F(x, y)| \, dy \, dx \right\| : g \in L^{p_{\Phi}^+}(\mathbb{R}^n) \text{ such that } \|g\|_{L^p(\mathbb{R}^n)} = 1 \right\}.$$
From the Fubini theorem and \[66, \text{p.}58, \text{Proposition} \ 1\], it follows that
\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} F(x,y)dyg(x)dx \right| \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x,y)||g(x)|dydx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x,y)||g(x)|dxdy \\
\leq \int_{\mathbb{R}^m} \|F(\cdot,y)||_{L^p(\mathbb{R}^m)}\|g||_{L^q(\mathbb{R}^n)}dy \sim \int_{\mathbb{R}^n} \|F(\cdot,y)||_{L^p(\mathbb{R}^n)}dy,
\]
which implies the desired conclusion. This finishes the proof of Lemma 7.45.

The interpolation theorem of operators on Orlicz-slice spaces is stated as follows.

**Theorem 7.46.** Let \( t \in (0, \infty), \ q \in (1, \infty) \) and \( \Phi \) be an Orlicz function with positive lower type \( p_\Phi \in (1, \infty) \) and positive upper type \( p_\Phi^* \). Let \( p_1 \in \left( \frac{1}{\min\{p_\Phi, q\}}, 1 \right) \) and \( p_2 \in (1, \infty) \). Assume that \( T \) is a sublinear operator defined on \((E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n) + (E_{\Phi p_2}^{p_1})_r(\mathbb{R}^n)\) satisfying that there exist positive constants \( C_1 \) and \( C_2 \), independent of \( t \), such that for any \( i \in \{1, 2\} \) and \( f \in (E_{\Phi p_i}^{p_j})_i(\mathbb{R}^n) \),
\[
\|T(f)\|_{(W E_{\Phi p_i}^{p_j})_i(\mathbb{R}^n)} \leq C_i\|f\|_{(E_{\Phi p_i}^{p_j})_i(\mathbb{R}^n)}, \tag{7.6}
\]
where \( \Phi_{p_1}(\tau) := \Phi(\tau^{p_1}) \) for any \( \tau \in [0, \infty) \) and \( i \in \{1, 2\} \). Then \( T \) is bounded on \( (W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n) \) and there exists a positive constant \( C \), independent of \( t \), such that for any \( f \in (W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n) \),
\[
\|T(f)\|_{(W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n)} \leq C\|f\|_{(W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n)}. \tag{7.7}
\]

**Proof.** Let \( f \in (W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n) \) and
\[
\lambda := \|f\|_{(W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n)} = \sup_{\alpha \in (0, \infty)} \|\alpha\|_{(E_{\Phi p_1}^{p_2})_i(\mathbb{R}^n)} \tag{7.8}
\]
with the implicit positive constant independent of \( \alpha, f \) and \( t \).

To this end, for any \( \alpha \in (0, \infty) \), let
\[
f^{(\alpha)} := f1_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \quad \text{and} \quad f_{(\alpha)} := f1_{\{x \in \mathbb{R}^n : |f(x)| \leq \alpha\}}. \tag{7.9}
\]
We claim that
\[
\|f^{(\alpha)}\|_{(E_{\Phi p_1}^{p_2})_i(\mathbb{R}^n)} \lesssim \alpha(\lambda/\alpha)^{1/p_1} \tag{7.10}
\]
and
\[
\|f^{(\alpha)}\|_{(E_{\Phi p_2}^{p_1})_i(\mathbb{R}^n)} \lesssim \alpha(\lambda/\alpha)^{1/p_2}. \tag{7.11}
\]
Assuming that this claim holds true for the moment, then, by the condition that \( T \) is sublinear and (7.6), we conclude that for any \( \alpha \in (0, \infty) \),
\[
\|T(f^{(\alpha)})\|_{(E_{\Phi p_1}^{p_2})_i(\mathbb{R}^n)} + \|T(f_{(\alpha)})\|_{(E_{\Phi p_2}^{p_1})_i(\mathbb{R}^n)} \lesssim \lambda.
\]
This implies that \( \|T(f)\|_{(W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n)} \lesssim \|f\|_{(W E_{\Phi p_1}^{p_2})_r(\mathbb{R}^n)}, \) which is the desired conclusion.

Therefore, it remains to prove the above claim. To prove (7.7), by Lemma 7.45, we have
\[
\left\{ \int_{\mathbb{R}^n} \frac{|f^{(\alpha)}|/\alpha}{(\lambda/\alpha)^{1/p_1}} 1_{B(x,t)} \|f\|_{(E_{\Phi p_1}^{p_2})_i(\mathbb{R}^n)} dx \right\}^{1/p_1}.
\]
From the definition of \( f^{(\alpha)} \), Definition 7.44 and (7.5), we deduce that

\[
I_1 \lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^q(\mathbb{R}^n)} \right]^{\frac{\gamma}{p}} dt \right\}^{\frac{p}{\gamma}}
\]

\[
\lesssim \left\{ \frac{\alpha}{\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^q(\mathbb{R}^n)} \right]^{\frac{\gamma}{p}} dt \right\}^{\frac{p}{\gamma}} \lesssim \bar{C}_{(\Phi,t)},
\]

here and thereafter, \( \bar{C}_{(\Phi,t)} \) is the same as in (7.5). As for \( I_2 \), by the definition of \( f^{(\alpha)} \), Definition 7.44, (7.5) and \( \frac{1}{p_1} > 1 \), we conclude that

\[
I_2 \lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^p(\mathbb{R}^n)} \right]^{\frac{p}{p_1}} dt \right\}^{\frac{p_1}{p}}
\]

\[
\lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^p(\mathbb{R}^n)} \right]^{\frac{p}{p_1}} dt \right\}^{\frac{p_1}{p}} \lesssim \bar{C}_{(\Phi,t)}.
\]

From (7.5) and the estimates for \( I_1 \) and \( I_2 \), we then deduce (7.7).

To prove (7.8), by a proof similar to that used in the estimation of (7.7), we have

\[
\left\{ \int_{\mathbb{R}^n} \|f^{(\alpha)}\|_{(\lambda/\alpha)^{1/p_2}B(x,t)} \|_{L^p_{\lambda^{1/p_2}}(\mathbb{R}^n)} \right\}^{\frac{1}{p_2}} \lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^q(\mathbb{R}^n)} \right]^{\frac{\gamma}{p}} dt \right\}^{\frac{p}{\gamma}} + \left\{ \int_0^{\alpha/\lambda} \cdots dt \right\}^{\frac{1}{p_2}} =: I_1 + I_2.
\]

From the definition of \( f^{(\alpha)} \), Definition 7.44, (7.5) and \( 0 < \frac{1}{p_2} < 1 \), we deduce that

\[
I_1 \lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^q(\mathbb{R}^n)} \right]^{\frac{\gamma}{p}} dt \right\}^{\frac{p}{\gamma}}
\]

\[
\lesssim \left\{ \int_0^{\alpha/\lambda} \left[ \int_{\mathbb{R}^n} \|1_{\{y \in \mathbb{R}^n : |f(y)| > \alpha\}} 1_{B(x,t)} \|_{L^q(\mathbb{R}^n)} \right]^{\frac{\gamma}{p}} dt \right\}^{\frac{p}{\gamma}} \lesssim \bar{C}_{(\Phi,t)}.
\]

Observe that, when \( \tau \in \left(\frac{\alpha}{\lambda}, \infty\right) \), \( \|f^{(\alpha)}\|_{\lambda^{1/p_2}} \leq 1 < \frac{\lambda}{\alpha} \) and hence \( \Pi_2 = 0 \), which, together with the estimate for \( I_1 \) and (7.5), implies (7.8). Thus, we complete the proof of our above claim and hence of Theorem 7.46.

Moreover, we can establish the following vector-valued inequality of the Hardy-Littlewood operator \( \mathcal{M} \) in (2.8) on \( (W E^{(\alpha)^q}_{p,q}(\mathbb{R}^n)) \), which shows that \( (E^{(\alpha)^q}_{p,q}(\mathbb{R}^n)) \) satisfies Assumption 2.20.

**Proposition 7.47.** Let \( t \in (0, \infty) \), \( q, s \in (1, \infty) \) and \( \Phi \) be an Orlicz function with positive lower type \( p_\Phi \in (1, \infty) \) and positive upper type \( p_\Phi' \). Then there exists a positive constant \( C \), independent of \( t \), such
that for any sequence \( \{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n) \),
\[
\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{(WE^p_{q,s})_r(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{(WE^p_{q,s})_r(\mathbb{R}^n)}.
\]

**Proof.** Let \( \{f_j\}_{j \in \mathbb{N}} \) be a given arbitrary sequence of measurable functions, and for any measurable function \( g \) and \( x \in \mathbb{R}^n \), define
\[
A(g)(x) := \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(g \eta_j)(x)]^s \right\}^{\frac{1}{s}},
\]
where \( s \in (1, \infty) \), and for any \( i \in \mathbb{N} \) and \( y \in \mathbb{R}^n \),
\[
\eta_j(y) := \frac{f_j(y)}{\left[ \sum_{j \in \mathbb{N}} |f_j(y)|^s \right]^{1/s}} \quad \text{when} \quad \left[ \sum_{j \in \mathbb{N}} |f_j(y)|^s \right]^{1/s} \neq 0,
\]
and \( \eta_j(y) := 0 \) otherwise. It is easy to see that by the Minkowski inequality, for any \( \lambda \in \mathbb{C} \) and \( g_1, g_2 \in \mathcal{M}(\mathbb{R}^n) \),
\[
A(g_1 + g_2) \leq A(g_1) + A(g_2) \quad \text{and} \quad A(\lambda g) = |\lambda| A(g).
\]
Thus, \( A \) is sublinear. For any \( p^-_\Phi, q, s \in (1, \infty) \), from [86, Theorem 2.20], we deduce that
\[
\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{(E^p_{\Phi,q,s})_r(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{(E^p_{\Phi,q,s})_r(\mathbb{R}^n)}.
\]
Using this, we know that for any given \( p_1 \in \left( \frac{1}{\min\{1, q, s\}}, 1 \right) \) and \( p_2 \in (1, \infty) \) and any \( h \in \mathcal{M}(\mathbb{R}^n) \),
\[
\|A(h)\|_{(WE^{p_1}_{q,s})_r(\mathbb{R}^n)} = \left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(h \eta_j)]^s \right\}^{\frac{1}{s}} \right\|_{(WE^{p_1}_{q,s})_r(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(h \eta_j)]^s \right\}^{\frac{1}{s}} \right\|_{(E^{p_2}_{\Phi,q,s})_r(\mathbb{R}^n)}.
\]
which implies that the operator \( A \) is bounded from \( (E^{p_1}_{\Phi,q,s})_r(\mathbb{R}^n) \) to \( (WE^{p_2}_{q,s})_r(\mathbb{R}^n) \), where \( i \in \{1, 2\} \). Now, taking \( g := |\sum_{j \in \mathbb{N}} |f_j|^s|^{1/s} \), then, by Theorem 7.46, we conclude that
\[
\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{(WE^p_{\Phi,q,s})_r(\mathbb{R}^n)} = \|A(g)\|_{(WE^{p_1}_{q,s})_r(\mathbb{R}^n)} \lesssim \|g\|_{(WE^p_{\Phi,q,s})_r(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{(WE^p_{\Phi,q,s})_r(\mathbb{R}^n)},
\]
which completes the proof of Proposition 7.47.

Now we introduce the notion of weak Orlicz-slice Hardy spaces.

**Definition 7.48.** Let \( t, q \in (0, \infty) \), \( N \in \mathbb{N} \) and \( \Phi \) be an Orlicz function with positive lower type \( p^-_\Phi \) and positive upper type \( p^+_\Phi \). The weak Orlicz-slice Hardy space \( (WE^{p_1}_{q,s})_t(\mathbb{R}^n) \) is defined to be the set of all \( f \in S'(\mathbb{R}^n) \) such that \( M^{p_1}_{q,s}(f) \in (WE^p_{\Phi,q,s})_t(\mathbb{R}^n) \), and for any \( f \in (WE^{p}_\Phi,q,s)_t(\mathbb{R}^n) \), let
\[
\|f\|_{(WE^{p}_\Phi,q,s)_t(\mathbb{R}^n)} := \|M^{p_1}_{q,s}(f)\|_{(WE^{p}_\Phi,q,s)_t(\mathbb{R}^n)},
\]
where \( M^{p_1}_{q,s}(f) \) is as in (2.14) with \( N \) sufficiently large.

**Remark 7.49.** Let \( t \in (0, \infty) \), \( q \in (1, \infty) \) and \( \Phi \) be an Orlicz function with positive lower type \( p^-_\Phi \in (1, \infty) \) and positive upper type \( p^+_\Phi \). By Proposition 7.47, we conclude that for any \( r \in (1, \min\{q, p^-_\Phi\}) \), \( \mathcal{M} \) in (2.8) is bounded on \( (WE^{p}_\Phi,q,s)_t(\mathbb{R}^n))^{1/r} \), which, combined with Theorem 3.4, further implies that \( (WE^{p}_\Phi,q,s)_t(\mathbb{R}^n) = (WE^{p}_\Phi,q,s)_t(\mathbb{R}^n) \) with equivalent norms.
Applying Proposition 7.47 and Theorem 3.2(ii), we directly obtain the following maximal function characterizations of the weak Orlicz-slice Hardy space \((WH E^2_\Phi)_t(\mathbb{R}^n)\).

**Theorem 7.50.** Let \(t, a, b, q \in (0, \infty)\). Let \(\Phi\) be an Orlicz function with positive lower type \(p_\Phi^0\) and positive upper type \(p_\Phi^+\). Let \(\varphi \in \mathcal{S}(\mathbb{R}^n)\) satisfy \(\int_{\mathbb{R}^n} \varphi(x)dx \neq 0\). Assume that \(b \in \left(0, \frac{n}{\min\{p_\Phi^0, q\}}\right]\), and \(N \geq b + 1\). For any \(f \in \mathcal{S}'(\mathbb{R}^n)\), if one of the following quantities is finite, then the others are also finite and mutually equivalent with the positive equivalence constants independent of \(f\) and \(t\).

To establish the atomic characterization of weak Orlicz-slice Hardy spaces, although [86, Lemma 4.3] and Proposition 7.47 ensure that \((E^2_\Phi)_t(\mathbb{R}^n)\) satisfies Assumptions 2.18 and 2.20, we still need the following three lemmas, which are just, respectively, [86, Lemmas 4.3, 4.4 and 5.4].

**Lemma 7.51.** Let \(t, q \in (0, \infty)\) and \(\Phi\) be an Orlicz function with positive lower type \(p_\Phi^0\) and positive upper type \(p_\Phi^+\). Let \(\vartheta \in (0, \min\{p_\Phi^+, q\}]\). Then \((E^2_\Phi)_t(\mathbb{R}^n)\) is a strictly \(\vartheta\)-convex ball quasi-Banach function space as in Definition 2.7(ii).

**Lemma 7.52.** Let \(t, q \in (0, \infty)\) and \(\Phi\) be an Orlicz function with positive lower type \(p_\Phi^0\) and positive upper type \(p_\Phi^+\). Let \(r \in (\max\{q, p_\Phi^0\}, \infty]\) and \(s \in (0, \min\{p_\Phi^+, q\}]\). Then there exists a positive constant \(C_{r,s}\), depending on \(s\) and \(r\), but independent of \(t\), such that for any \(f \in \mathcal{M}(\mathbb{R}^n)\),

\[
\| M^{(r,s)}(f) \|_{(E^2_\Phi)_t(\mathbb{R}^n)} \leq C_{r,s} \| f \|_{(E^2_\Phi)_t(\mathbb{R}^n)};
\]

where \(M^{(r,s)}(f)\) is the \(r\)-\(s\)-convexification of \((E^2_\Phi)_t(\mathbb{R}^n)\) as in Definition 2.7(i) with \(X := (E^2_\Phi)_t(\mathbb{R}^n)\) and \(p := 1/s\), and \((E^2_\Phi)_t(\mathbb{R}^n)^{1/s}\) denotes its dual space.

**Lemma 7.53.** Let \(t \in (0, \infty)\), \(q \in (0, 1]\) and \(\Phi\) be an Orlicz function with positive lower type \(p_\Phi^0\) and positive upper type \(p_\Phi^+ \in (0, 1]\). Then there exists a nonnegative constant \(C\) such that for any sequence \(\{f_j\}_{j \in \mathbb{N}} \subset (E^2_\Phi)_t(\mathbb{R}^n)\) of nonnegative functions such that \(\sum_{j \in \mathbb{N}} f_j\) converges in \((E^2_\Phi)_t(\mathbb{R}^n)\),

\[
\left\| \sum_{j \in \mathbb{N}} f_j \right\|_{(E^2_\Phi)_t(\mathbb{R}^n)} \geq C \left\| \sum_{j \in \mathbb{N}} f_j \right\|_{(E^2_\Phi)_t(\mathbb{R}^n)}.
\]

Using Proposition 7.47, Lemmas 7.51–7.53 and Theorems 4.2, 4.7, 5.2 and 5.3, we immediately obtain the following atomic characterization of \((WH E^2_\Phi)_t(\mathbb{R}^n)\) (see Theorem 7.54 below) and the following molecular characterization of \((WH E^2_\Phi)_t(\mathbb{R}^n)\) (see Theorem 7.55 below).

**Theorem 7.54.** Let \(t, q \in (0, \infty)\) and \(\Phi\) be an Orlicz function with positive lower type \(p_\Phi^0\) and positive upper type \(p_\Phi^+\). Let \(p_+ := \max\{1, p_\Phi^+, q\}\) and assume that \(r \in (p_+, \infty]\) and \(d \in \mathbb{Z}_+\) with \(d \geq n\left(\frac{1}{\min\{p_\Phi^0/p_+, q/p_+, 1\}} - 1\right)\). Then \(f \in (WH E^2_\Phi)_t(\mathbb{R}^n)\) if and only if

\[
f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} a_{i,j} \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{(E^2_\Phi)_t(\mathbb{R}^n)} < \infty,
\]

where \(\{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) is a sequence of \((E^2_\Phi)_t)\)-atoms supported, respectively, in balls \(\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) such that for any \(i \in \mathbb{Z}\), \(\sum_{j \in \mathbb{N}} 1_{B_{i,j}} \leq A\) with \(c \in (0,1]\) and \(A\) being a positive constant independent of \(f\) and \(i\), and for any \(i \in \mathbb{Z}\) and \(j \in \mathbb{N}\), \(\lambda_{i,j} := \hat{A}^2 \| 1_{B_{i,j}} \|_{(E^2_\Phi)_t(\mathbb{R}^n)}\) with \(\hat{A}\) being a positive constant independent of \(f\) and \(i\).

Moreover, for any \(f \in (WH E^2_\Phi)_t(\mathbb{R}^n)\),

\[
\| f \|_{(WH E^2_\Phi)_t(\mathbb{R}^n)} \sim \inf \left\{ \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{(E^2_\Phi)_t(\mathbb{R}^n)} \right\},
\]

where the infimum is taken over all the decompositions of \(f\) as above and the positive equivalence constants is independent of \(f\) and \(t\).
We also have the following molecular characterization of \((WHE^q_\Phi)_t(\mathbb{R}^n)\).

**Theorem 7.55.** Let \(t, q, \Phi, r, d, \epsilon, \lambda_0\) be the same as in Theorem 7.54, and \(c \in (n + d + 1, \infty)\). Then \(f \in (WHE^q_\Phi)_t(\mathbb{R}^n)\) if and only if

\[
f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} m_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{(E^q_\Phi)_t(\mathbb{R}^n)} < \infty,
\]

where \(\{m_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) are \((E^q_\Phi)_t\)-molecules associated, respectively, with balls \(\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\) such that for any \(i \in \mathbb{Z}\), \(\sum_{j \in \mathbb{N}} 1_{cB_{i,j}} \leq A \) with \(c \in (0, 1]\) and \(A\) being a positive constant independent of \(f\) and \(i\), and for any \(i \in \mathbb{Z}\) and \(j \in \mathbb{N}\), \(\lambda_{i,j} := \tilde{A}\|1_{B_{i,j}}\|_{(E^q_\Phi)_t(\mathbb{R}^n)}\) with \(\tilde{A}\) being a positive constant independent of \(f\), \(i\) and \(j\).

Moreover, for any \(f \in (WHE^q_\Phi)_t(\mathbb{R}^n)\),

\[
\|f\|_{(WHE^q_\Phi)_t(\mathbb{R}^n)} \approx \inf \left[ \sup_{i \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{N}} \lambda_{i,j} 1_{B_{i,j}} \right\|_{(E^q_\Phi)_t(\mathbb{R}^n)} \right],
\]

where the infimum is taken over all the decompositions of \(f\) as above and the positive equivalence constants are independent of \(f\) and \(t\).

We now recall the notion of Orlicz-slice Hardy spaces introduced in [86].

**Definition 7.56.** Let \(t, q \in (0, \infty)\) and \(\Phi\) be an Orlicz function with positive lower type \(p^-\) and positive upper type \(p^+\). Then the **Orlicz-slice Hardy space** \((HE^q_\Phi)_t(\mathbb{R}^n)\) is defined by setting

\[
(HE^q_\Phi)_t(\mathbb{R}^n) \triangleq \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{(HE^q_\Phi)_t(\mathbb{R}^n)} := \|M(f, \varphi)\|_{(E^q_\Phi)_t(\mathbb{R}^n)} < \infty \},
\]

where \(\varphi \in \mathcal{S}(\mathbb{R}^n)\) satisfies \(\int_{\mathbb{R}^n} \varphi(x)dx \neq 0\). In particular, when \(\Phi(s) \triangleq s^q\) for any \(s \in [0, \infty)\) with any given \(r \in (0, \infty)\), the Hardy-type space \((HE^q_\Phi)_t(\mathbb{R}^n) := (HE^q_\Phi)_t(\mathbb{R}^n)\) is called the slice Hardy space.

Recall that the **centered Hardy-Littlewood maximal operator** \(\mathcal{M}_c\) is defined by setting, for any locally integrable function \(f\) and \(x \in \mathbb{R}^n\),

\[
\mathcal{M}_c(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|dy.
\]

In what follows, for any \(r \in (0, \infty)\), \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) and \(x \in \mathbb{R}^n\), let

\[
\int_{B(x, r)} f(y)dy := \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)dy.
\]

To obtain the boundedness of Calderón-Zygmund operators from \((HE^q_\Phi)_t(\mathbb{R}^n)\) to \((WHE^q_\Phi)_t(\mathbb{R}^n)\), we need to establish the following Fefferman-Stein vector-valued inequality from \((E^q_\Phi)_t(\mathbb{R}^n)\) to \((WHE^q_\Phi)_t(\mathbb{R}^n)\).

**Proposition 7.57.** Let \(t \in (0, \infty)\), \(q \in [1, \infty)\), \(r \in (1, \infty)\) and \(\Phi\) be an Orlicz function with positive lower type \(p^-\) \([1, \infty)\) and positive upper type \(p^+\). Then there exists a positive constant \(C\), independent of \(t\), such that for any \(\{f_j\}_{j \in \mathbb{Z}} \subset \mathcal{M}(\mathbb{R}^n)\),

\[
\left\| \left\{ \sum_{j \in \mathbb{Z}} |M(f_j)|^r \right\}^{\frac{1}{r}} \right\|_{(WHE^q_\Phi)_t(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{(E^q_\Phi)_t(\mathbb{R}^n)}.
\]

**Proof.** Let \(\alpha \in (0, \infty)\) and \(r \in (1, \infty)\). For any sequence \(\{f_j\}_{j \in \mathbb{Z}} \subset \mathcal{M}(\mathbb{R}^n)\) and \(x \in \mathbb{R}^n\), we claim that

\[
\left\| \left\{ \sum_{j \in \mathbb{Z}} |M(f_j)(y)|^{\frac{1}{\alpha}} \right\}_y \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{(E^q_\Phi)_t(\mathbb{R}^n)}.
\]

(7.11)
where $\mathcal{M}_c$ is as in (7.10) and the implicit positive constant is independent of $\{f_j\}_{j \in \mathbb{Z}}$, $x$, $\alpha$ and $t \in (0, \infty)$.

To show this, we write
\[
\|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} \lesssim \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} + \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} =: I + II.
\]

For $I$, since $B(y,s) \subset B(x,2t)$ whenever $s \in (0,t]$ and $y \in B(x,t)$, from the Orlicz Fefferman-Stein vector-valued inequality in [48, Theorem 1.3.1] or in [82, Theorem 2.1.4], it follows that
\[
I \sim \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} \lesssim \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} \lesssim \alpha^{-1} \left\{ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right\}_{L^x(\mathbb{R}^n)}^{q/2}.
\]

As for $II$, observe that for any $\xi, \xi \in B(z,t)$ and $y \in B(x,t)$, from the translation invariance of the Lebesgue measure, $\|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma}$
\[
\lesssim \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} \lesssim \|1_{\{y \in B(x,t) : \sum_{j \in \mathbb{Z}} \mathcal{M}_c(f_j(y)) \}}^{1/\alpha} \|_{L^y(\mathbb{R}^n)}^{1/\gamma} \lesssim \|1_{B(x,t)} \|_{L^x(\mathbb{R}^n)}^{q/2}.
\]

This proves the above claim.

Using (7.11), for any $t \in (0, \infty)$ and any given $q \in [1, \infty)$, we further obtain
\[
\int_{\mathbb{R}^n} \frac{1}{\|1_{B(x,t)} \|_{L^y(\mathbb{R}^n)}^{q/2}} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right]^{q/2} \ dx \lesssim \alpha^{-q} \int_{\mathbb{R}^n} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right]^{q/2} \ dx + \int_{\mathbb{R}^n} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right]^{q/2} \ dx =: III + IV.
\]

Since the closures of both $B(\tilde{0}_n,2t)$ and $B(\tilde{0}_n,t)$ are compact subsets of $\mathbb{R}^n$ with nonempty interiors, it follows that there exist an $N \in \mathbb{N}$ and $\{x_1, \ldots, x_N \} \subset \mathbb{R}^n$, independent of $t$, such that $N \lesssim 1$ and $B(\tilde{0}_n,2t) \subseteq \bigcup_{m=1}^{N} B(x_m,t)$. Thus, by this, (7.5) and the translation invariance of the Lebesgue measure, we conclude that
\[
III \sim \alpha^{-q} \int_{\mathbb{R}^n} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right]^{q/2} \ dx \lesssim \alpha^{-q} \sum_{m=1}^{N} \int_{\mathbb{R}^n} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,2t)}| \right]^{q/2} \ dx \lesssim \alpha^{-q} \int_{\mathbb{R}^n} \left[ \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} |1_{B(x,t)}| \right]^{q/2} \ dx.
\]
\[ \leq \alpha^{-q} \int_{\mathbb{R}^n} \left[ \frac{1}{\|1_{B(x,t)}\|_{L^{q}(\mathbb{R}^n)}} \right]^{q} \left\{ \sum_{j \in \mathbb{Z}} |f_j|^\gamma \right\}^{\frac{q}{\gamma}} 1_{B(x,t)} \left\|1_{B(x,t)}\right\|_{L^{q}(\mathbb{R}^n)}^{q} \, dx, \]

where \( \tilde{C}(\Phi, t) \) is the same as in (7.5), which further implies that

\[ \text{III}^\frac{q}{\gamma} \leq \alpha^{-1} \left\{ \sum_{j \in \mathbb{Z}} |f_j|^\gamma \right\}^{\frac{q}{\gamma}} \left\| \left( E_{\Phi}^{q}\right)_r(\mathbb{R}^n) \right\|. \tag{7.12} \]

We now estimate IV. By the Fefferman-Stein vector-valued inequality from \( B^q \) type, (7.11) and (7.12), then completes the proof of Proposition 7.57.

Let \( r := \frac{r'}{r-1} \). Then there exists a \( \{b_j\}_{j \in \mathbb{Z}} \in \ell^{r'} \), with \( \|\{b_j\}_{j \in \mathbb{Z}}\|_{\ell^{r'}} = 1 \), such that

\[ \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} \left[ \int_{B(x,t)} |f_j(z)| \, dz \right]^r \right)^{\frac{1}{r'}} \, dx = \int_{\mathbb{R}^n} \left[ \sum_{j \in \mathbb{Z}} b_j \int_{B(x,t)} |f_j(z)| \, dz \right]^q \, dx. \]

From [66, p.13, Proposition 1], we deduce that for any ball \( B(x,t) \), \( \Phi^{-1}(|B(x,t)|) \Psi^{-1}(|B(x,t)|) \sim |B(x,t)| \), where the positive equivalence constants are independent of \( x \) and \( t \). This, together with Hölder’s inequality and (7.5), further implies that

\[ \int_{\mathbb{R}^n} \left[ \sum_{j \in \mathbb{Z}} b_j \int_{B(x,t)} |f_j(z)| \, dz \right]^q \, dx \leq \int_{\mathbb{R}^n} \left\{ \left( \sum_{j \in \mathbb{Z}} |f_j(z)|^r \right)^{\frac{1}{r'}} \left( \sum_{j \in \mathbb{Z}} b_j' \right)^{\frac{1}{r}} \, dx \right\} \left( \sum_{j \in \mathbb{Z}} \left[ \int_{B(x,t)} |f_j(z)| \, dz \right]^r \right)^{\frac{1}{r'}} \left( \sum_{j \in \mathbb{Z}} b_j' \right)^{\frac{1}{r}} \left( \sum_{j \in \mathbb{Z}} |f_j(z)|^r \right)^{\frac{1}{r'}} \, dx \]

\[ \leq \int_{\mathbb{R}^n} \left\{ \left[ \sum_{j \in \mathbb{Z}} |f_j(z)|^r \right]^{\frac{1}{r'}} 1_{B(x,t)} \left\| \left( E_{\Phi}^{q}\right)_r(\mathbb{R}^n) \right\| \, dx \right\}^{q} \left( \sum_{j \in \mathbb{Z}} |f_j(z)|^r \right)^{\frac{1}{r'}} \left( \sum_{j \in \mathbb{Z}} b_j' \right)^{\frac{1}{r}} \left( \sum_{j \in \mathbb{Z}} |f_j(z)|^r \right)^{\frac{1}{r'}} \, dx. \]

Thus,

\[ \text{IV}^\frac{q}{\gamma} \leq \alpha^{-1} \left\{ \sum_{j \in \mathbb{Z}} |f_j|^\gamma \right\}^{\frac{q}{\gamma}} \left\| \left( E_{\Phi}^{q}\right)_r(\mathbb{R}^n) \right\|, \]

which, combined with (7.11) and (7.12), then completes the proof of Proposition 7.57. \( \square \)

Applying Proposition 7.57 and Theorems 6.3 and 6.4, we directly obtain the following boundedness from \((HE_\Phi^\alpha)_r(\mathbb{R}^n)\) to \((WHE_\Phi^\alpha)_r(\mathbb{R}^n)\) of both convolutional \( \delta \)-type and \( \gamma \)-type Calderón-Zygmund operators, respectively, as follows.

**Theorem 7.58.** Let \( t \in (0, \infty) \), \( q \in (0, \infty) \), \( \delta \in (0, 1] \) and \( \Phi \) be an Orlicz function with positive lower type \( p_\Phi \) and positive upper type \( p_\Phi^+ \). Let \( T \) be a convolutional \( \delta \)-type Calderón-Zygmund operator. If \( \min\{p_\Phi, q\} \in \left[ \frac{n}{n+\gamma-t} \right] \), then \( T \) has a unique extension on \((HE_\Phi^\delta)_r(\mathbb{R}^n)\) and, moreover, there exists a positive constant \( C \), independent of \( t \), such that for any \( f \in (HE_\Phi^\delta)_r(\mathbb{R}^n) \),

\[ \|Tf\|_{(WHE_\Phi^\delta)_r(\mathbb{R}^n)} \leq C \|f\|_{(HE_\Phi^\delta)_r(\mathbb{R}^n)}. \]

**Theorem 7.59.** Let \( t \in (0, \infty) \), \( q \in (0, 2) \), \( \gamma \in (0, \infty) \) and \( \Phi \) be an Orlicz function with positive lower type \( p_\Phi \) and positive upper type \( p_\Phi^+ \in (0, 2) \). Let \( T \) be a \( \gamma \)-type Calderón-Zygmund operator and have the vanishing moments up to order \( \gamma - 1 \). If \( \gamma - 1 \leq \alpha \left( \frac{1}{\min\{p_\Phi, q\}} - 1 \right) \leq \gamma \), then \( T \) has a unique extension on \((HE_\Phi^\gamma)_r(\mathbb{R}^n)\) and, moreover, there exists a positive constant \( C \), independent of \( t \), such that for any \( f \in (HE_\Phi^\gamma)_r(\mathbb{R}^n) \),

\[ \|Tf\|_{(WHE_\Phi^\gamma)_r(\mathbb{R}^n)} \leq C \|f\|_{(HE_\Phi^\gamma)_r(\mathbb{R}^n)}. \]
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