PRIMORDIAL MAGNETIC FIELD AND NON-GAUSSIANITY OF THE ONE-YEAR WILKINSON MICROWAVE ANISOTROPY PROBE DATA

PAVEL D. NASELSKY,1 LUNG-YIH CHIANG,1 POUŁ OLESEN,1 AND OLEG V. VERKHODANOV2

Received 2004 May 10; accepted 2004 July 5

ABSTRACT

Alfvén turbulence caused by statistically isotropic and homogeneous primordial magnetic field induces correlations in the cosmic microwave background (CMB) anisotropies. The correlations are specifically between the spherical harmonic modes $a_{l-1,m}$ and $a_{l+1,m}$. In this paper we approach this issue from phase analysis of the CMB maps derived from the WMAP data sets. Using circular statistics and return phase mapping, we examine phase correlation of $\Delta l = 2$ for the primordial non-Gaussianity caused by the Alfvén turbulence at the epoch of recombination. Our analyses show that such specific features from the power-law Alfvén turbulence do not contribute significantly in the phases of the maps and could not be a source of primordial non-Gaussianity of the CMB.

Subject headings: cosmic microwave background — cosmology: observations — methods: data analysis

1. INTRODUCTION

After the release of the one-year WMAP data for analysis (Bennett et al. 2003a, 2003b, 2003c; Hinshaw et al. 2003a, 2003b), one of the most exciting areas of investigation is the statistical characterization of the cosmic microwave background (CMB) signal. After the WMAP science team’s report on the observational constraint on the quadratic non-Gaussianity of the anisotropy of the CMB (Komatsu et al. 2003), detections of non-Gaussianity by various methods for the CMB maps derived from the WMAP data have been reported (Chiang et al. 2003; Coles et al. 2004; Naselsky et al. 2003, 2004; Vielva et al. 2004). Unfortunately, the origin of such non-Gaussian features, detected by different methods, is still unknown. Chiang et al. (2003), Naselsky et al. (2003, 2004), and Eriksen et al. (2004a) point out that the non-Gaussian features might come from the foregrounds, whereas Hansen et al. (2004) argue in favor of systematic effects. Moreover, different methods detect various properties of non-Gaussian features from localized peculiar spots (Vielva et al. 2004) to global north-south asymmetry in the WMAP signal and the COBE map as well (Eriksen et al. 2004a).

There is another point of view on the issue of the WMAP non-Gaussianity. It comes from the theory of turbulence in magnetohydrodynamics (MHD). Using the so-called extended self-similarity (ESS) method, Bershadskii & Sreenivasan (2003, 2004) point out that the statistical properties of angular increments $\delta T_l^p = T(r + r) - T(R)$, where $r$ is the vector connecting two pixels $R + r$ and $R$ of the map, have the following relation in MHD: $\langle |\delta T_l^p|^2 \rangle \propto r^2$ for high-order moments $p = 4, 6, \ldots, 12$. For those moments, the ESS is clearly detected in both the COBE and the WMAP data: $\langle |\delta T_l^p|^2 \rangle \propto \langle S^2 \rangle_{\zeta_p}$, where $\zeta_p = p^{8/3} - 1 - (1/2)(p^2/4)$. Moreover, Brandenburg et al. (1996), Durrer et al. (1998), Mack et al. (2002), Subramanian et al. (2003), and Chen et al. (2004) showed that the primordial magnetic field, one of the most significant relics from inflation, can generate vorticity of Alfvén waves before and during the epoch of hydrogen recombination. It interacts with the CMB photons, producing the nonadiabatic tail of the CMB anisotropy and polarization. Assuming statistical homogeneity and isotropy on primordial magnetic field, it is shown that such Alfvén turbulence will have non-Gaussian properties, because of the quadratic dependence of the vorticity amplitude on the magnetic strength $B$. Namely, such a magnetic field induces correlation between the $a_{l-1,m}$ and $a_{l+1,m}$ multipole coefficients of the CMB temperature anisotropy expansion by the spherical harmonics.

An intriguing issue then follows: has WMAP observed the cosmological Alfvén turbulence? Recently this issue is discussed in Chen et al. (2004), which exploits the correlations between $a_{l-1,m}$ and $a_{l+1,m}$ of the CMB caused by the vorticity of Alfvén waves in order to put constraints on the strength of the magnetic field: $|B| < 15$ nG for the spectral power index $n = -5$ and $|B| < 1.7$ nG for the spectral power index $n = -7$.

In this paper we discuss this issue from another aspect: could the detected non-Gaussianity from the WMAP data be related to the Alfvén turbulence? The basic idea is that even if the Alfvén turbulence is small in amplitude in the CMB signal, it will manifest itself in the WMAP maps between the phases $\phi_{l-1,m}$ and $\phi_{l+1,m}$. Phase correlation between Fourier modes has been investigated in relation to large-scale structure formation of the universe (Scherrer et al. 1991; Chiang & Coles 2000; Chiang et al. 2002) and also applied as a test of non-Gaussianity (Chiang et al. 2004, 2003; Coles et al. 2004) based on the random phase hypothesis as a practical definition of Gaussian random fields (Bardeen et al. 1986; Bond & Efstathiou 1987). Naselsky et al. (2002) use neighboring phase correlations to extract extragalactic point sources.

Assuming that the CMB signal is composed of a pure Gaussian signal and a subdominant vorticity tail, we use circular statistics on phases for such a signal in order to place a constraint on the power spectrum of Alfvén turbulence at each multipole number $l$. We show that circular statistics allow us to decrease contamination of the dominant (Gaussian) part of the signal in order to investigate possible contamination from the non-Gaussian subdominant part.
2. CIRCULAR STATISTICS

OF THE ALFVEN TURBULENCE

For statistical characterization of temperature fluctuations on a sphere we express each signal (either CMB or foreground components) as a sum over spherical harmonics:

$$\Delta T(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |a_{lm}| e^{i\phi_{lm}} Y_{lm}(\theta, \varphi),$$

where $|a_{lm}|$ and $\phi_{lm}$ are the moduli and phases of the coefficients of the expansion, respectively.

Homogeneous and isotropic CMB Gaussian random fields (GRFs), as a result of the simplest inflation paradigm, possess Fourier modes whose real and imaginary parts are Gaussian and mutually independent. The statistical properties are then completely specified by the power spectrum $C_l^{\text{CMB}}$:

$$\left\langle a_{lm}^* (a_{lm}')^* \right\rangle = C_l^{\text{CMB}} \delta_{ll'} \delta_{mm'}.$$  

In other words, from the central limit theorem their phases

$$\Psi_{lm}^{\text{CMB}} = \tan^{-1} \frac{\text{Im}(a_{lm})}{\text{Re}(a_{lm})}$$

are randomly and uniformly distributed at the range $[0,2\pi]$. We denote with $G$ the pure Gaussian tail of the CMB signal. For the combined signal,

$$a_{lm} = G_{lm} + V_{lm},$$

where $G_{lm}$ is the Gaussian tail and $V_{lm}$ is the vortex tail. We can write down

$$a_{lm} = |a_{lm}| \exp(i\Psi_{lm}),$$

where

$$|a_{lm}|^2 = |G_{lm}|^2 + |V_{lm}|^2 + 2|G_{lm}||V_{lm}| \cos(\Psi_{lm}^G - \Psi_{lm}^V),$$

$$\tan \Psi_{lm} = \frac{|G_{lm}| \sin \Psi_{lm}^G + |V_{lm}| \sin \Psi_{lm}^V}{|G_{lm}| \cos \Psi_{lm}^G + |V_{lm}| \cos \Psi_{lm}^V}$$

and $\Psi_{lm}^G, \Psi_{lm}^V$ are the phases of Gaussian and vortex at the $(l, m)$ harmonics, respectively.

Durrer et al. (1998, hereafter DKY98) show that for the CMB anisotropy produced by the Alfvén turbulence, the properties of the power spectrum and phases are different from the adiabatic modes, which corresponds to the Gaussian tail of the signal. Namely, for the power spectrum $C_l(m) = \langle a_{l-1,m} a_{l+1,m}^* \rangle$ from DKY98 we get

$$C_l(m) = A \frac{\Gamma(-n-1) \Gamma(l+n/2+3/2)}{\Gamma(-n/2) \Gamma(l-n/2+1/2)} \frac{2l^4 + 4l^3 - l^2 - 3l + (6 - 2l^2 - 2l^2) m^2}{(2l-1)(2l+3)} \times$$

$$\left. \frac{2A}{[n+1]} \frac{\Gamma(-n-1) \Gamma(l+n/2+3/2)}{\Gamma(-n/2) \Gamma(l-n/2+1/2)} \frac{(l+m+1)(l-m+1)(l+m)(l-m)}{(2l-1)(2l+1)^2(2l+3)^2} \right)^{1/2},$$

where $A$ is the normalization constant and $-7 \leq n \leq -1$ is the power spectral index of the Alfvén turbulence. Below we discuss the models with $n = -3, -5, -7$, for which the mean power spectra over the $m$-direction in equation (7) are $\bar{C}_l, D_l \propto l^{n+3}$. Note that the model of $n = -5$ corresponds to the power spectrum $\bar{C}_l \propto l^{-2}$, which, at the multipole range of $5 < l < 20$, has the same feature as that of the CMB for adiabatic perturbation. For the model of $n = -7$, on the other hand, the power of the vortex tail is $\bar{C}_l \propto l^{-4}$ and increases rapidly if $l \to 0$, which is typical for the diffuse extragalactic foregrounds. For $n = -3$ the vortex power $\bar{C}_l \sim constant$ mimics the power of the extragalactic point source component. Therefore, simple statistics based on the estimation of the $D_l$-th moment from the WMAP data is potentially misleading, because of possible contribution from foregrounds such as subdominant components to the signal.

The method we propose in this paper is based on the statistical characterization: non-Gaussianity of the vortex part of the CMB signal and related with correlation of the phases of $(l-1, m), (l+1, m)$ harmonics. The correlation vanishes for a pure Gaussian CMB signal, whereas it is significant for the vortex component. We therefore introduce some specific functions of phases that minimize the contribution from non-correlated Gaussian signal and maximize the non-Gaussian tail of the phases. For this purpose we use trigonometric moment statistics to counter the circular nature of phases (Fisher 1993; see also Naselsky et al. 2003). Let us define the following trigonometric moments:

$$C(l) = \frac{1}{l} \sum_{m=1}^{l-1} \cos(\Psi_{l-1,m} - \Psi_{l+1,m}),$$

$$S(l) = \frac{1}{l} \sum_{m=1}^{l-1} \sin(\Psi_{l-1,m} - \Psi_{l+1,m}),$$

$$r^2(l) = C^2(l) + S^2(l),$$

$$R(l) = \frac{1}{l_{\text{max}} - l_{\text{min}}} \sum_{l_{\text{min}}}^{l_{\text{max}}} r(l).$$

The reason for such statistics is clear. Simple algebra leads to the following properties of the phases:

$$\cos(\Psi_{l-1,m} - \Psi_{l+1,m}) = |g_{l-1,m}| \frac{|g_{l+1,m}| \cos(\Psi_{l-1,m} - \Psi_{l+1,m})}{|g_{l-1,m}| \cos \Psi_{l-1,m}} + |v_{l-1,m}| \frac{|v_{l+1,m}| \cos(\Psi_{l+1,m} - \Psi_{l-1,m})}{|v_{l+1,m}| \cos \Psi_{l+1,m}} + |g_{l+1,m}| \frac{|g_{l-1,m}| \cos(\Psi_{l+1,m} - \Psi_{l-1,m})}{|g_{l-1,m}| \cos \Psi_{l+1,m}} + |v_{l+1,m}| \frac{|v_{l-1,m}| \cos(\Psi_{l-1,m} - \Psi_{l+1,m})}{|v_{l-1,m}| \cos \Psi_{l-1,m}},$$

where $|g_{lm}| = |G_{lm}|/|a_{lm}|, |v_{lm}| = |V_{lm}|/|a_{lm}|$. As one can see from equation (9), the first term is proportional to the $D_l(m) = \langle a_{l-1,m} a_{l+1,m}^* \rangle$ moment for the pure Gaussian tail of the signal, which should vanish, the second term is $D_l(m)$ for the vortex, and the last two terms correspond to correlations between $G$ and $V$, which should be statistically negligible after summation over $m$. If the vortex component is subdominant, then in equations (6) and (9) $|V_{lm}| \ll |G_{lm}|$ for all $l, m$, and $|a_{lm}|$ is given by equation (6). This means that $|g_{lm}| \approx 1$ and $|v_{lm}| \ll 1$. However, because of the finite number of $m$-modes and especially for low $l$-range, correlations can manifest themselves
between the \((l - 1, m), (l + 1, m)\) harmonics. In order of magnitude this effect can be estimated in the following way. For a pure Gaussian signal with noncorrelated phases \(\phi_{lm}\) one obtains

\[
\frac{1}{l} \sum_{m=1}^{l} \cos \phi_{lm} \simeq \frac{1}{\sqrt{l}}, \quad \frac{1}{l} \sum_{m=1}^{l} \sin \phi_{lm} \simeq \frac{1}{\sqrt{l}}. \tag{10}
\]

Then, for the moduli \(|a_{lm}|\) from equation (6) one gets

\[
|a_{lm}|^2 \simeq \begin{cases} |G_{lm}|^2 + |V_{lm}|^2, & |V_{lm}| \gg |G_{lm}|/\sqrt{l}; \\ |G_{lm}|^2 + 2|G_{lm}||V_{lm}|\cos(\Psi_{lm}^G - \Psi_{lm}^V), & |V_{lm}| \ll |G_{lm}|/\sqrt{l}. \end{cases} \tag{11}
\]

For the asymptotic \(|V_{lm}| \gg |G_{lm}|/\sqrt{l},

\[
C(l) \sim \frac{1}{\sqrt{l}} - \frac{1}{3} \sum_{j=1}^{l} \frac{|V_{l-1,m}|^2 |V_{l+1,m}|}{|G_{l-1,m}| |G_{l+1,m}|} + \frac{1}{l} \sum_{m} \frac{|V_{l-1,m}| |V_{l+1,m}|}{|G_{l-1,m}| |G_{l+1,m}|} \cos(\Psi_{l-1,m}^V - \Psi_{l+1,m}^V). \tag{12}
\]

For the correlated phases of the vortex component, the last term in equation (12) is larger than the second, and if \(\Psi_{l-1,m}^V \approx \Psi_{l+1,m}^V\), then

\[
C(l) \sim \frac{1}{\sqrt{l}} + \frac{\langle D_l \rangle}{G_l^2}. \tag{13}
\]

where \(G_l^2\) is the power spectrum of the Gaussian tail.

For highly correlated vortex perturbations, therefore, the contribution to the \(C\)-function is significant if \((\langle D_l \rangle/G_l^2) \sqrt{l} \geq 1\). At \(l = 100\), \((\langle D_l \rangle/G_l^2) \geq 0.1\), which is exactly the DKY98 criterion for estimation of the magnetic field amplitude for different values of the spectral index \(n\). However, if the observable data sets covering the range of multipoles up to the Silk damping scale \(l_f\) for the vortex perturbations, \(l_f \sim 500\) (Durrer et al. 1998), then non-Gaussian features could be detectable for \(\langle D_l \rangle/G_l^2 \geq 0.045\). Moreover, one interesting feature comes from the power spectral index dependence of the vortex perturbations mentioned above. For \(n = -7\), \(\langle D_l \rangle \propto l^{-4}\), while \(G_l^2\) has the same asymptotic \(l^{-2}\) as the \(\Lambda\)CDM WMAP best-fit model at \(5 < l < 30\). For that model of the power index, the most stringent constraint should come from statistics of the low multipoles from the WMAP data, then from high multipole range \(l \sim l_f\) and simple estimator could be 6–7 times bigger than in DKY98: \(\langle D_l \rangle/G_l^2 \sim 0.6–0.7\).

We would like to point out that the above-mentioned properties of the \(C\)-statistics provide natural explanations in terms of the cosmic variance limit of error bars for any CMB experiments. Assuming no systematic effect present in the data, for low multipoles \(l \leq 100\) the cosmic variance limit corresponds to \(\delta C_l/C_l \simeq (f_{sky})^{-1/2}\), where \(f_{sky}\) is the sky coverage of the observation. The whole-sky WMAP coverage gives \(f_{sky} = 1\). If some part of the CMB power spectrum is related to vortex perturbations, then (Chen et al. 2004)

\[
C_l = \mathbb{G}_l^2 + \mathbb{C}_l \sim \mathbb{G}_l^2 + |n + 1| \left[ \Gamma\left(\frac{n+1}{2}\right)/\Gamma\left(\frac{n}{2}\right) \right]^2 \langle D_l \rangle. \tag{14}
\]

Using the definition \(\delta C_l/C_l = (C_l - \mathbb{C}_l^2)/\mathbb{C}_l^2\), which corresponds to the contribution of the vortex perturbations to the power spectrum, we have the following threshold of detectability

\[
|n + 1| \left[ \Gamma\left(\frac{n+1}{2}\right)/\Gamma\left(\frac{n}{2}\right) \right]^2 \frac{\langle D_l \rangle}{\mathbb{C}_l} > \frac{1}{\sqrt{l}}, \tag{15}
\]

which is in perfect agreement with our estimation, equation (13).

Let us briefly discuss the properties of the \(S\)-statistics. Similarly to equation (9), one can prove that \(\sin(\Psi_{l-1,m}^V - \Psi_{l+1,m}^V)\) is given by equation (9) with transition from cosine to sine function. However, because of the dependence, if phases of the non-Gaussian tail are highly correlated, we have degradation of the \(|V_{l-1,m}| \gg |V_{l+1,m}| \sin(\Psi_{l-1,m}^V - \Psi_{l+1,m}^V)\) term, even if \(|V_{lm}| \gg |G_{lm}|/\sqrt{l}\). Therefore, if the phases of the vortex perturbations \(\Psi_{l-1,m}^V - \Psi_{l+1,m}^V\) are highly correlated, for the \(S\) statistics we get \(S \sim 1/\sqrt{l}\). Thus, the presence of vortex perturbation (as all non-Gaussian signals) in the CMB data manifests itself as asymmetry of the \(C\)-\(S\) statistics, while for the pure Gaussian signals it should be a symmetrical one. In order of magnitude this asymmetry for each mode \(l\) is

\[
A(l) = \frac{S(l)}{C(l)} = \frac{1}{1 + \sqrt{l} \left(\frac{\langle D_l \rangle}{G_l^2}\right)^{1/2}} < 0.5, \tag{16}
\]

for highly correlated phases of vortex perturbations. In addition to equation (16), we can define the global asymmetry of the signal,

\[
A_g = \sqrt{\frac{\sum_{l} S(l)}{\sum_{l} C(l)}}, \tag{17}
\]

In terms of circular statistical variables (Fisher 1993), this global asymmetry corresponds to the mean angle \(\Theta = \tan^{-1} A_g\) for orientation of all harmonics \((l, m)\) in the CMB map. Needless to say, for vortex perturbations with \(n = -5\) global asymmetry is expected to be, in order of magnitude,

\[
A_g = 1 \left[ 1 + \frac{\langle D_l \rangle}{G_l^2} l_{\max} \right], \tag{18}
\]

and can be detectable, if \((\langle D_l \rangle/G_l^2) l_{\max} \gg 1\). At the end of this section we would like to mention one peculiar asymptotic of the \(A(l)\) and \(A_g\) parameters related with correlations of phases. From equation (12) one can find that the correlation \(\Psi_{l-1,m}^V - \Psi_{l+1,m}^V = \pi/2\). \(3\pi/2\) leads to \(A(l)\), \(A_g \gg 1\). Thus, any peaks of the function \(A(l)\), \(A_g \gg 1\) are the marks of \(\pi/2\), \(3\pi/2\) correlations for the non-Gaussian tail of the CMB signal. Below, in addition to the definition of equation (8), we introduce the phase difference

\[
D_{lm}(\Delta l) = \Psi_{l-\Delta l,m} - \Psi_{l+\Delta l,m} \tag{19}
\]

for even \(\Delta l\) and

\[
D_{lm}(\Delta l) = \Psi_{l+\Delta l,m} - \Psi_{l-\Delta l-1,m} \tag{20}
\]
for odd $\Delta l$ and corresponding circular moments $S_{\Delta l}(l)$ and $C_{\Delta l}(l)$, similar to equation (8). These variables allow us to describe properties of the CMB phases for any $\Delta l$ separations of the modes.

3. CIRCULAR STATISTICS OF THE ILC, FC, AND WFC MAPS DERIVED FROM THE WMAP DATA

In this section we apply the $S$- and $C$-statistics for the three high-resolution whole-sky maps: the internal linear combination (ILC) map by the WMAP science team, the Tegmark et al. (2003, hereafter TDH03) foreground-cleaned (FC) map, and the TDH03 Wiener-filtered (WF) map. The ILC and the FC maps are obtained from a weighted combination of the five WMAP maps at frequencies of 23, 33, 41, 61, and 94 GHz in order to separate the microwave foreground. The WF map is produced after foreground residual cleaning by Wiener filtering. All the maps formally claim resolution up to $l = 1024$, but none of them is correct for investigation of the CMB properties. The ILC map, as is mentioned at the WMAP Web site, is applicable for the foreground property investigation at the range of multipoles $l < 300$. The FC and WF maps are smoothed for the multipole range $l > 300$ by a Gaussian window function, and the CMB signal is completely erased by the smoothing. However, looking at the $V$ and $Q$ bands of the WMAP data and combining all the foregrounds (synchrotron, free-free, and dust emission) and the point sources from the WMAP catalog, one can find that this map reproduces the CMB signal outside the Kp2 Galactic cut mask extremely well at the range of multipoles $l \leq 50$.

The FC and WF maps have well-defined scales of smoothing in the image domain. However, smoothing does not change the phases of the signal at all, even down to the limit $l = 1024$. Thus, using FC and WF maps, we can examine the phases at the whole range of multipoles. One additional reason for using the FC and WF maps is that they are tested by the ESS method, from which the manifestation of the Alfvén turbulence properties could be clearly seen (Bershadskii & Sreenivasan 2003, 2004).

In Figure 1 we plot for the three maps (ILC, FC, and WF) the $S$-$C$ phase diagrams for $\Delta l = (l + 1) - (l - 1) = 2$. For comparison we include the statistics of Gaussian random signal, which reproduce our expectation of the phase properties. From the second row in Figure 1 one can see that the ILC phases perfectly reproduce the properties of Gaussian signal phases in terms of the $S(l)$, $C(l)$, and $A(l)$ variables. One can see that on the third and the fourth rows, which correspond to the FC and WF maps, starting from $l > 100$ the $S$ symmetry is broken, and while $S$ follows Gaussian statistics, its $C$ and $A(l)$ appear non-Gaussian.

According to Figure 1, the phases of the FC and WF maps are in agreement with the expectation, described in § 2. We have significant increase of the $C$ statistics, while $S$ statistics are nearly the same as for the Gaussian signal. However, next three figures clearly display non-Gaussianity. Figure 2 shows the circular statistics $S(l)$, $C(l)$, and $A(l)$ for $\Delta l = 1$, and Figures 3 and 4 are for $\Delta l = 3$ and 4, respectively. According to DKY98, for all $\Delta l \neq 2$ the phases of the signal should not display any correlated features. One can also find remarkable symmetry in the FC and WF maps between even

![Figure 1](image-url)
Fig. 2.—Same as in Fig. 1, but for $\Delta l = 1$.

Fig. 3.—Same as in Fig. 1, but for $\Delta l = 3$. 


and odd \(\Delta l\) statistics (e.g., \(C\) in Figs. 1 and 4 and \(S\) in Figs. 2 and 3).

\[
\text{sign} C_{\Delta l} = -\text{sign} C_2 \cos \left(\frac{\pi}{2} \Delta l \right), \quad \text{for even } \Delta l
\]

\[
\text{sign} S_{\Delta l} = \text{sign} S_1 \sin \left(\frac{\pi}{2} \Delta l \right), \quad \text{for odd } \Delta l.
\]

This symmetry reflects the symmetry of the phases \(D_{\Delta l}(l)\), namely,

\[
\Psi_{l+\Delta l/2,m} - \Psi_{l-\Delta l/2,m} = \frac{\pi}{2} \Delta l + \pi
\]

for even \(\Delta l\) and

\[
\Psi_{l+\Delta l,m} - \Psi_{l-\Delta l-1,m} = \frac{\pi}{2} \Delta l
\]

for odd \(\Delta l\) and for all values of \(m\).

4. NONLINEAR STATISTICS

In this section in order to investigate the properties of non-Gaussianity, we apply nonlinear statistics \(r^2(l)\) for the ILC, FC, and WF map phases. From equation (8) one obtains

\[
r^2(l) = \frac{1}{l-1} + \sum_{m} \sum_{\substack{n \neq m}} \frac{\cos[D_{\Delta l}(\Delta l) - D_{\Delta l}(\Delta l)]}{(l-1)^2}.
\]

That means that \(r\)-statistics are sensitive to \(m\) dependence of the phase difference for given values \(l\) and \(\Delta l\). This statistic has well-defined properties. If the signal is highly correlated and \(D_{\Delta l}(\Delta l) \to 0\) then \(r^2(l) \to 1\). If the phases are non-correlated, however, \(r^2(l) \to l^{-1}\). In Figures 5 and 6 we plot \(r^2(l)\) versus \(l\) for the same model and the same order as in Figure 1. For all even \(\Delta l\), the corresponding \(r^2(l)\) are similar to Figure 5 and for all odd \(\Delta l\) the \(r^2\) statistics follow Figure 6.

Like \(S\) and \(C\) statistics, \(r^2(l)\) statistics reflect non-Gaussian features for \(\Delta l = 1, 2, 3, \ldots\), which allow us to conclude that the non-Gaussianity of the FC and WF maps is not related to the vortex turbulence at the epoch of recombination. Moreover, in § 5 we show how the symmetry of the \(S\) and \(C\) statistics allow us to detect the source of non-Gaussianity of the FC and WF maps.

5. LOCAL DEFECTS OF THE ILC, FC, AND WF MAPS

In order to detect the nature of the non-Gaussianity of the maps with different powers of signals, we use the power filter \(P(l) = 1/|a_{lm}|^2\), proposed by Górski (1997) and Novikov et al. (2001). This linear filter transforms any maps with \(a_{lm}\) coefficients of spherical harmonic expansions to maps with the same amplitude (=1) at each mode \(l, m\), but preserving all the phases \(\Psi_{lm}\) and their correlations.

\[
N_{lm} = P(l) |a_{lm}| = \exp(i\Psi_{lm}).
\]

In Figure 7 we plot the power-filtered ILC, FC, and WF maps after power filtration for the range of multipoles up to \(l = 1024\) in order to show how the power filter allows us to
detect some of the local defects of the maps close to the Galactic plane. In particular, one can easily see that there are defects in the ILC map that were originally obscured by the amplitudes $|a_{lm}|$. The clear-cut features are obviously not of astrophysical origin. One crude guess for such a peculiar shape in the middle of the ILC map is that the WMAP science team might have used a direct mask at the center of the map, where there are pronounced contaminations from foregrounds, and then used Wiener filtering, combining with the best-fit power spectrum, to recover the signal in that area. However, the morphology is related to phases (Chiang 2001), and such a mask will alter the morphology, which manifests itself in the phases. On the other hand, one can see foreground residuals in both the FC and WF maps. Qualitatively speaking, these are clear signatures of non-Gaussianity.

Using equation (25), one can easily visualize the $N_{l-\Delta l/2,m} N_{l+\Delta l/2,m}$ cross-correlations using the simple definition

$$G_{lm}(\Delta l) = N_{l-\Delta l/2,m} N_{l+\Delta l/2,m} \exp[iD_{lm}(\Delta l)].$$

(26)

For these maps, the common features are typically the size of small clusters (one can find corresponding statistics in Naselsky et al. 2004). In order to show how local defects of the maps can produce corresponding features, let us introduce the following model of subdominant non-Gaussian signal. Suppose that we have a set of points (pixels) with coordinates $\theta_j$, $\phi_j$ in which the signal looks like a combination of $\delta$-functions,

$$\Delta T_{\text{loc}}(\theta, \phi) = \sum_j A_j \delta(\phi - \phi_j) \delta(\cos \theta - \cos \theta_j),$$

(27)

where $A_j$ are the amplitudes of defects. In order to obtain the corresponding $c_{lm}$ coefficients, we convolve $\Delta T$ from equation (27) with conjugated spherical harmonics

$$c_{lm} = \int_0^1 d \cos \theta \int_{-\pi}^\pi d \phi \Delta T_{\text{loc}}(\theta, \phi) Y^*_l(\theta, \phi) = \sum_j A_j Y^*_l(\phi_j, \theta_j).$$

(28)

5.1. Peculiarities of $\Delta l$-even Statistics

To obtain the correlators for $S$- and $C$-statistics, we need to know $\cos[D_{lm}(\Delta l)]$ and $\sin[D_{lm}(\Delta l)]$, which can be found from equation (28) to be

$$C_{lm}(\Delta l) = |c_{l-\Delta l/2,m}c_{l+\Delta l/2,m} \cos[D_{lm}(\Delta l)]$$

$$= \left(c_{l-\Delta l/2,m}c_{l-\Delta l/2,m} + c_{l-\Delta l/2,m}c_{l-\Delta l/2,m}\right)/2,$$

$$S_{lm}(\Delta l) = |c_{l-\Delta l/2,m}c_{l+\Delta l/2,m} \sin[D_{lm}(\Delta l)]$$

$$= \left(c_{l-\Delta l/2,m}c_{l-\Delta l/2,m} - c_{l-\Delta l/2,m}c_{l-\Delta l/2,m}\right)/2i.$$  

(29)

Taking into account equation (28), from equation (29) we have

$$C_{lm}(\Delta l) = \sum_{i,k} B_{lm}(i, m) \cos \left[m(\theta_j - \phi_k)\right]$$

$$\times P_{l-\Delta l/2}(\cos \theta_j) P_{l+\Delta l/2}(\cos \theta_k),$$

(30)

$$S_{lm}(\Delta l) = \sum_{i,k} B_{lm}(i, m) \sin \left[m(\theta_j - \phi_k)\right]$$

$$\times P_{l-\Delta l/2}(\cos \theta_j) P_{l+\Delta l/2}(\cos \theta_k).$$
Fig. 7.—ILC (top), FC (middle), and WF (bottom) power-filtered maps. These whitened images display how foreground cleaning and the residuals manifest themselves in phases, which are otherwise obscured by the amplitudes $|a_m|$. One can see the clear-cut feature in the middle of the ILC map and Galactic and point source contaminations for the FC and WF maps. Qualitatively, these are clear signatures of non-Gaussianity.
where \( P_l^m(\cos \theta) \) are the associated Legendre polynomials and
\[
B_m(l, m) = A_j A_k \left[ \frac{\Gamma(l - \Delta l / 2 - m + 1) \Gamma(l + \Delta l / 2 - m + 1)}{\Gamma(l - \Delta l / 2 + m + 1) \Gamma(l + \Delta l / 2 + m + 1)} \right]^{1/2}.
\]

(31)

Following Naselsky et al. (2003), we describe an ideal situation in which \( \theta_1 = \theta_2 = \pi/2 \), but \( \phi_j \neq \phi_k \). For this model, in equation (30) we get
\[
P_{l-\Delta l/2}(0) = \sqrt{\pi} 2^{m} \left[ \frac{\Gamma(l + \Delta l / 2)}{\Gamma(l - \Delta l / 2)} \right]^{1/2}.
\]

(32)

As one can see from equations (32) and (30), the major part of the summation over \( m \) is related with \( m = l - \Delta l / 2 \), for which
\[
P_{l-\Delta l/2}(0)P_{l+\Delta l/2}(0)|_{m=l-\Delta l/2} = 2^{2l-\Delta l} \pi \Gamma(l + \Delta l / 2) \Gamma(l - l + \Delta l / 2) \cos \left( \frac{\pi \Delta l}{2} \right).
\]

(33)

Thus, the sign of \( C_{lm}(\Delta l) \) for even \( \Delta l \) is determined by the cosine terms. Moreover, if \( \Delta l = 2n \), \( n = 1, 2, 3, \ldots \), the contribution of the local defects with \( \theta_j = \pi/2 \) to \( C \)-statistics manifest itself as the changes of the sign, mentioned in equation (21). One may ask, why are the \( S \)- and \( C \)-statistics so different, if they depend on \( \theta_j \) in the same way? Taking into account the \( \phi_j \)-dependence of \( C_{lm}(\Delta l) \) and \( S_{lm}(\Delta l) \), one can see that for \( j = k \) the cosine terms in equation (30) \((\cos[m(\phi_j - \phi_k)]) \) goes to unity, while the sine terms for \( S_{l,m}(\Delta l) \) are equal to zero. All \( j \neq k \) modes in equation (30) are represented by highly oscillated functions and look like noise in the phase diagrams (see Figs. 1–4).

5.2. Peculiarities of Odd \( \Delta l \) Statistics

Let us discuss the model \( \Delta l = 2n + 1 \), \( n = 1, 2, 3, \ldots \). There are two possibilities to understand the properties of the phase correlations for odd \( \Delta l \). First we discuss the Galactic plane sources contamination. In a case similar to equation (33) one can find
\[
P_{l-\Delta l}(0)P_{l+\Delta l}(0) \sim \cos \left( \frac{\pi \Delta l}{2} \right),
\]

(34)

and odd \( \Delta l \) does not contribute to the \( S \)- and \( C \)-statistics, if \( \theta_j = \pi/2 \) for all \( j \). However, if for some of the local defects \( \theta_j \neq \pi/2 \), but \( \cos \theta_j \ll 1 \), there should produce significant peculiarities in the \( S \)-statistics in the following way. Close to the Galactic plane we can expand the Legendre polynomials using a Taylor series
\[
P_{l-\Delta l}(\cos \theta_j) = P_{l-\Delta l}(0) + \frac{d P_{l-\Delta l}(\cos \theta_j)}{d \cos \theta_j} \bigg|_{\cos \theta_j = 0} \cos \theta_j + \frac{1}{2} \frac{d^2 P_{l-\Delta l}(\cos \theta_j)}{d (\cos \theta_j)^2} \bigg|_{\cos \theta_j = 0} \cos^2 \theta_j.
\]

(35)

Simple algebra allows us to conclude that the dependence \( S_{lm}(\Delta l) \), \( C_{lm}(\Delta l) \propto \cos^2 \theta_j \sin(\pi \Delta l / 2) \) comes from interference between the first and the last terms in equation (35), after their substitution into equation (30). Thus, this dependence over \( \Delta l \) takes place for some of the multipoles, for which \( \theta_j \ll 1 \), while for \( \theta_j \gg 1 \) \( S_{lm}(\Delta l) \), \( C_{lm}(\Delta l) \) statistics goes to zero for odd \( \Delta l \). This dependence of the \( S_{lm}(\Delta l) \), \( C_{lm}(\Delta l) \) over \( \Delta l \) can be clearly seen in Figures 2 and 3, where symmetry \( S \)- and \( C \)-statistics are restored for high \( l \). However, \( S_{lm}(\Delta l) \), like \( C_{lm}(\Delta l) \), depends on \( \Delta l \) in the same way, while in Figures 2 and 3 one can see significant asymmetry in the \( S \)- and \( C \)-statistics. The reason for that could be localization of the defects in the \( \phi \)-direction close to the Galactic center with the width \( -\pi/2 \ll \delta \phi_j \ll \pi/2 \) in the Galactic coordinates.

6. CONCLUSION

We examine the primordial non-Gaussian features from Alfvén turbulence caused by primordial magnetic field. The distortion of the blackbody CMB spectrum by such a homogeneous magnetic field is not discussed in this paper. Because of the vector nature of the magnetic field, off-diagonal correlations between spherical harmonic modes \( a_{l-1,m} \) and \( a_{l+1,m} \) are induced. To see the \( \Delta l = 2 \) correlations, we apply circular statistics of phases to analyze such non-Gaussian component from the CMB signal. We have applied the statistics on the ILC, FC, and WF maps, using all the available phases. The phase information can help us to test some of the properties of the signal even when the amplitudes of the power spectra were smoothed by Gaussian window functions starting from \( l \approx 50 – 200 \).

The FC and WF maps have non-Gaussian signatures registered in the phases, as we have found phase correlations of not only \( \Delta l = 2 \), but also \( \Delta l = 1, 3, \) and \( 4 \). Such strong correlations are related to the residuals from component separation in the Galactic plane and some point sources. None of the maps shows specific features from the vortex perturbations at the last scattering surface. Moreover, the detected non-Gaussianity is completely different from the vortex perturbations properties. Roughly speaking, all the modes with \( \Delta l = 1, 2, 3, \) and \( 4 \) show pronounced correlations, which is not the specific features of the vortex. The correlations such as \( \Delta l = 4, 8, \ldots \), might have been the higher order correlation from \( \Delta l = 2 \), but \( \Delta l = 1 \) and others indicate that such strong correlations are related to the residuals from foreground component separation. We conclude that the Alfvén turbulence does not contribute significantly to the phase information. The method we propose for phase analysis is useful for the upcoming Planck mission, especially for testing the properties of foreground cleaning methods. Our final remark is that the MHD structure claimed by Bershadskii (2003) and Bershadskii & Sreenivasan (2004) in the WFC map is related to Galactic plane residuals but not to the Alfvén turbulence contamination.

We thank Tegmark et al. for providing their processed maps.

We acknowledge the use of HEALPIX3 package (Górski et al. 1999) to produce \( a_{lm} \) from the WMAP data and the use of the GLESP package (Doroshkevich et al. 2003) for data analysis and the whole-sky figures.

3 See http://www.eso.org/science/healpix.
REFERENCES

Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15
Bennett, C. L., et al. 2003a, ApJ, 583, 1
———. 2003b, ApJS, 148, 1
———. 2003c, ApJS, 148, 97
Bershadskaï, A. 2003, Int. J. Mod. Phys. D, 12, 509
Bershadskaï, A., & Sreenivasan, K. R. 2003, Phys. Lett. A, 319, 21
———. 2004, Int. J. Mod. Phys. D, 13, 281
Bond, J. R., & Efstathiou, G. 1987, MNRAS, 226, 655
Branderburg, A., Enqvist, K., & Olesen, P. 1996, Phys. Rev. D, 54, 1291
Chen, G., Mukherjee, P., Kahniashvili, T., Ratra, B., & Wang, Y. 2004, ApJ, 611, 655
Chiang, L.-Y. 2001, MNRAS, 325, 405
Chiang, L.-Y., & Coles, P. 2000, MNRAS, 311, 809
Chiang, L.-Y., Coles, P., & Naselsky, P. D. 2002, MNRAS, 337, 488
Chiang, L.-Y., Naselsky, P. D., & Coles, P. 2004, ApJ, 602, L1
Chiang, L.-Y., Naselsky, P. D., Verkhodanov, O. V., & Way, M. J. 2003, ApJ, 590, L65
Coles, P., Dineen, P., Earl, J., & Wright, D. 2004, MNRAS, 350, 989
Doroshkevich, A. G., Naselsky, P. D., Verkhodanov, O. V., Novikov, D. I., Turchaninov, V. I., Novikov, I. D., & Christensen, P. R. 2003, A&A, submitted (astro-ph/0305537)
Durrer, R., Kahniashvili, T. A., & Yates, A. 1998, Phys. Rev. D, 58, 123004
Eriksen, H. K., Banday, A. J., Górski, K. M., & Lilje, P. B. 2004a, ApJ, 612, 633
Eriksen, H. K., Hansen, F. K., Banday, A. J., Górski, K. M., & Lilje, P. B. 2004b, ApJ, 605, 14
Fisher, N. I. 1993, Statistical Analysis of Circular Data (Cambridge: Cambridge Univ. Press)
Górski, K. M. 1997, in Microwave Background Anisotropies, ed. F. R. Bouchet, R. Gispert, B. Guideroni, & J. Tran Thanh Van (Gif-sur-Yvette: Edition Frontières), 77
Górski, K. M., Hivon, E., & Wandelt, B. D. 1999, in Evolution of Large-Scale Structure: From Recombination to Garching, ed. A. J. Banday, R. S. Sheth, & L. Da Costa (Garching: ESO), 37
Hansen, F. K., Cabella, P., Marinucci, D., & Vittorio, N. 2004, ApJ, 607, L67
Hinshaw, G., et al. 2003a, ApJS, 148, 63
———. 2003b, ApJS, 148, 135
Komatsu, E., et al. 2003, ApJS, 148, 119
Larson, D. L., & Wandelt, B. D. 2004, ApJ, 613, 85
Mack, A., Kahniashvili, T., & Kosowsky, A. 2002, Phys. Rev. D, 65, 13004
Naselsky, P. D., Doroshkevich, A. G., & Verkhodanov, O. V. 2003, ApJ, 599, L53
———. 2004, MNRAS, 349, 695
Naselsky, P. D., Novikov, D. I., & Silk, J. 2002, ApJ, 565, 655
Novikov, D., Naselsky, P., Jørgensen, H. E., Christensen, P. R., Novikov, I., & Nørgaard-Nielsen, H. U. 2001, Int. J. Mod. Phys. D, 10, 245
Park, C.-G. 2004, MNRAS, 349, 313
Scherrer, R. J., Melott, A. L., & Shandarin, S. F. 1991, ApJ, 377, 29
Subramanian, K., Seshadri, T. R., & Barrow, J. D. 2003, MNRAS, 344, L31
Tegmark, M., de Oliveira-Costa, A., & Hamilton, A. 2003, Phys. Rev. D, 68, 123523 (TDH03)
Vielva, P., Martínez-González, E., Barreiro, R. B., Sanz, J. L., & Cayón, L. 2004, ApJ, 609, 22