On the geometry of dark energy

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Abstract
Experimental evidence suggests that we live in a spatially flat, accelerating universe composed of roughly one-third of matter (baryonic + dark) and two-thirds of a negative-pressure dark component, generically called dark energy. The presence of such energy not only explains the observed accelerating expansion of the universe but also provides the remaining piece of information connecting the inflationary flatness prediction with astronomical observations. However, despite its good observational indications, the nature of dark energy still remains an open question. In this paper we explore a geometrical explanation for such a component within the context of braneworld theory without mirror symmetry, leading to a geometrical interpretation for dark energy as a warp in the universe given by the extrinsic curvature. In particular, we study the phenomenological implications of the extrinsic curvature of a Friedmann–Robertson–Walker universe in a five-dimensional constant curvature bulk, with signatures (4,1) or (3,2), as compared with the x-matter (XCDM) model. From the analysis of the geometrically modified Friedmann’s equations, the deceleration parameter and the weak energy condition, we find a consistent agreement with the presently known observational data on inflation for the de Sitter bulk, but not for the anti-de Sitter case.

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1. Introduction

The possibility of an accelerating universe, as indicated by measurements of SNe Ia, has led to one of the most important debates of modern cosmology, which involves the conception of a late time dominant ‘dark energy’ component with negative pressure [1]. The nature of such
dark energy constitutes an open and tantalizing question connecting cosmology and particle physics. Currently, we do not have a complete scheme capable of explaining such phenomenon or why it is happening now. The simplest and most appealing proposal considers a relic cosmological constant $\lambda$. However, a reasonable explanation for the large difference between astrophysical estimates of this constant $\lambda / 8\pi G \approx 10^{-47}$ GeV$^4$ and theoretical estimates for the average vacuum energy density $\rho_v \approx 10^{71}$ GeV$^4$ is unknown, except through an extreme fine tuning of 118 orders of magnitude between these values [2, 3].

Other more elaborate explanations for dark energy have been proposed. One example is the so-called ‘quintessence’ model featuring a slowly decaying scalar field associated with a phenomenological potential with energy scales of the order of the present day Hubble constant $\sim 10^{-42}$ GeV [4]. Yet, it seems difficult to reconcile such a small repulsive force with any attempt to solve the hierarchy problem for fundamental interactions [5].

A phenomenological explanation based on current observational data is given by the ‘x-matter’ or XCDM model which is associated with an exotic fluid characterized by an equation of state like $p_x = \omega_x \rho_x$, where the parameter $\omega_x$ can be a constant or, more generally a function of the time [6]. The presence of such a fluid is consistent with the observed acceleration rate, without conflicting with the abundance of light elements resulting from the big-bang nucleosynthesis [5]. Although interesting from the phenomenological point of view, the XCDM model lacks an explanation from first principles.

On the other hand, we have witnessed a growing interest in the cosmological implications of braneworld theory. Generally speaking, this is a gravitational theory defined in a higher-dimensional bulk space whose geometry is defined by the Einstein–Hilbert principle. In such a scenario, standard gauge interactions remain confined to the four-dimensional spacetime (the braneworld generated by a 3-brane) embedded in the higher-dimensional bulk, but gravitons are free to probe the extra dimensions at the TeV scale of energies [7, 8] (see [9, 10] for recent reviews on braneworld gravity). If such expectations are confirmed, the impact of strong and quantized gravity at the same energy level of the standard gauge interactions will be considerable. Not only does it eliminate the hierarchical obstacle for a consistent unification programme but it also suggests a possible laboratory and cosmic ray generation of branes and short-lived black holes by collision phenomenology [11–14]. A review of high energy braneworld phenomenology can be found in [15].

Braneworld theory originated from M-theory, particularly in connection with the derivation of the Horava–Witten heterotic $E_8 \times E_8$ string theory in the space AdS$_5 \times S^5$, through the compactification of one extra dimension on the orbifold $S^1 / Z_2$ as in the Z$_2$ (or mirror) symmetry on the circle $S^1$. The presence of the five-dimensional anti-de Sitter AdS$_5$ space is mainly motivated by the prospects of the AdS/CFT correspondence between the superconformal Yang–Mills theory in four dimensions and the anti-de Sitter gravity in five dimensions.

The same $Z_2$ symmetry has been used as an argument to implement braneworld cosmology in the AdS$_5$ bulk, particularly in the popular Randall–Sundrum model II, where that symmetry is applied across a background braneworld taken as a boundary embedded in that bulk. In this case, the extrinsic curvature of the background boundary is completely determined by the confined matter energy–momentum tensor, through an algebraic relation known as the Israel–Darmois–Lanczos condition (IDL for short).

As it happens, when applied to a homogeneous and isotropic cosmology defined in the AdS$_5$ bulk, the IDL condition leads to a modification of Friedmann’s equations, which includes a term proportional to the square of the energy density of the confined perfect fluid of the universe. The presence of such a quadratic density was initially welcomed as a possible solution to the accelerated expansion of the universe. However, soon it was seen to be incompatible with
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the big-bang nucleosynthesis, requiring additional fixes [16–18]. More recently it has been shown that high-energy inflationary regimes are also constrained by the presence of the same quadratic term, as compared with the recent data from the SDSS/2dF/WMAP experiments [19–22]. It has also been argued that gravitational waves generated by the bulk geometry may produce vector perturbations on the braneworld, whose modes disagree with the data from the same experiments [23].

These observational constraints have suggested that braneworld cosmology using the $Z_2$ symmetry and/or the IDL condition should be somehow modified. For example, adding a Gauss–Bonnet term to the five-dimensional action, while still keeping the $Z_2$ symmetry [21]. Another explored possibility is to remove that symmetry and the IDL condition altogether [24–26]. In a different approach to the problem, the $Z_2$ symmetry is broken but some form of junction condition (including the IDL) is maintained [27–30]. This has evolved to a more general idea, where the extrinsic curvature should be governed by a dynamical equation, rather than just being specified at a background braneworld [31–33].

Therefore, the application of the IDL condition on the braneworld cosmology either with the $Z_2$ symmetry or not, has led to an extensive and still ongoing debate, involving some unresolved issues. One of these is related to the fact that the IDL condition expresses the extrinsic curvature in terms of the confined matter, producing the mentioned inflationary constraint. So, we may well ask if this is a problem of the IDL condition itself, or if it is a problem inherent to the extrinsic curvature and its embedding properties. If the IDL condition is dropped, do we improve the agreement with the inflationary data? If so, can we infer from these data an alternative, perhaps dynamical, condition on the extrinsic curvature? Is the IDL condition an independent postulate? Finally, is the $Z_2$ symmetry compatible with the embedding requirement of the braneworld structure?

The purpose of this paper is to investigate the phenomenology of the dark energy hypothesis in the braneworld context, without using the $Z_2$ symmetry, or without postulating any junction condition separately, at least for the time being. In this case, the braneworld dynamics follows essentially from three basic postulates: the Einstein–Hilbert principle applied to the bulk geometry, the confinement hypothesis and the probing of the extra dimensions by the gravitational field.

Under these conditions, Friedmann’s equation is modified by a geometrical term which is defined by the extrinsic curvature [34, 35]. In order to evaluate the compatibility of the resulting cosmology with the observations, we make an analogy with the phenomenological XCDM dark energy model. On the basis of the analysis of the deceleration parameter, we find that the expansion of the universe described by geometrically modified Friedmann’s equation can match today’s observable data. We also find that for the case where the inflation driving energy is positive, then the universe expands in a bulk with signature $(4, 1)$, compatible with the de Sitter cosmology.

As shown in appendix A, the covariant equations of motion are derived from the Einstein–Hilbert action, in accordance with the embedding equations, in the most general case. Appendix B is specific to five dimensions, where we review the derivation of the IDL condition, showing how the $Z_2$ symmetry specifies the value of the extrinsic curvature out of Lanczos jump condition. We also discuss the limitations imposed by $Z_2$ symmetry on the differentiable embedding of the braneworld in a constant curvature bulk.

2. The FRW brane

Based on very general theorems on differentiable manifold embeddings [36, 37], a five-dimensional bulk with constant curvature would have limited degrees of freedom. However, in
the particular case of the Friedmann–Robertson–Walker (FRW) universe seen as a braneworld, a five-dimensional bulk with constant curvature is sufficient as it does not require any additional conditions. The constant curvature bulk is characterized by the Riemann tensor

$$R_{ABCD} = K_s (G_{AC} G_{BD} - G_{AD} G_{BC}),$$

where $G_{AB}$ denotes the bulk metric components in arbitrary coordinates and where $K_s$ is either zero (for a flat bulk), or proportional to a positive or negative bulk cosmological constant, respectively corresponding to two possible signatures: $(4, 1)$ for the de Sitter $dS_5$ bulk and $(3, 2)$ for the anti-de Sitter bulk $AdS_5$. Accordingly, we take in the embedding equations (A.1) in appendix A, $g^{55} = \varepsilon = \pm 1$. In any of these cases, the integrability conditions for the embedding, equations (A.10)–(A.12), become simply

$$R_{\alpha\beta\gamma\delta} = \frac{1}{\varepsilon} (k_{\alpha\beta} k_{\gamma\delta} - k_{\alpha\delta} k_{\beta\gamma}) + K_s (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

(1)

$$k_{\alpha\beta\gamma\delta} = 0,$$

(2)

where the sign of the last term in (1) depends on the sign of $K_s$. The equations of motion derived in appendix A can be adjusted to the present case, but it is just as easy to derive directly from (1) and (2). The result is essentially Einstein’s equations as modified by the presence of the extrinsic curvature (for the covariant equations in five dimensions in a more general setting, see [38]):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} + Q_{\mu\nu},$$

(3)

where we have denoted by $\lambda$ the effective cosmological constant in four dimensions, the confined vacuum energy. The last term in (3) derived from (A.18) in appendix A is completely geometrical:

$$Q_{\mu\nu} = \frac{1}{\varepsilon} \left( k^\rho k_{\rho\nu} - h k_{\mu\nu} - \frac{1}{2} (K^2 - h^2) g_{\mu\nu} \right).$$

(4)

Here we have denoted $h = g^{\mu\nu} k_{\mu\nu}$ and $K^2 = k^{\mu\nu} k_{\mu\nu}$.

For the purpose of the embedding of the FRW universe in a five-dimensional bulk with maximal symmetry it is convenient to parametrize the FRW metric as [39]

$$dS^2 = g_{ab} dx^a dx^b = -dr^2 + a^2 [d\theta^2 + f(r)(d\phi^2 + \sin^2 \theta d\rho^2)]$$

where $f(r) = \sin r$, $r$, $\sinh r$ corresponding to the spatial curvature $k = 1, 0, -1$, respectively, and where the confined source is the perfect fluid given in co-moving coordinates by

$$T_{ab} = (p + \rho) U_a U_b + \rho g_{ab}, \quad U_a = \delta^4_a.$$  

(5)

Using York’s relation (equation (A.9) in appendix A) it follows that in the FRW spacetime $k_{ab}$ is diagonal. After separating the spatial components we find that Codazzi’s equations (2) reduce to (here $\mu, \nu, \rho, \sigma = 1 \ldots 3$)

$$k_{\mu\nu,\rho} - k_{\nu\sigma} \Gamma^\sigma_{\mu\rho} = k_{\mu\rho,\nu} - k_{\rho\sigma} \Gamma^\sigma_{\nu\mu},$$

$$k_{\mu\nu,\lambda} - k_{\mu\lambda,\nu} \frac{\dot{a}}{a} = -a \ddot{a} (\delta^1_{\mu} \delta^1_{\nu} + f^2 \delta^2_{\mu} \delta^2_{\nu} + f^2 \sin^2 \theta \delta^3_{\mu} \delta^3_{\nu}) k_{44}$$

5 A curly $\mathcal{R}$ denotes bulk curvatures while a straight $\mathcal{R}$ denotes braneworld curvatures. Capital Latin indices refer to the bulk dimensions. Small case Latin indices refer to the extra dimensions and all Greek indices refer to the brane. The semicolon denotes the covariant derivative with respect to $g_{ab}$. For generality we denote $G = |\det(G_{AB})|$. 
where \( a(t) \) is the scale factor and the dot means derivative with respect to \( t \). The first equation gives \( k_{11,v} = 0 \) for \( v \neq 1 \), so that \( k_{11} \) does not depend on the spatial coordinates. Denoting \( k_{11} = b(t) \), the second equations give [40]

\[
k_{44} = -\frac{1}{\varepsilon} \frac{d}{dt} \left( \frac{b}{a} \right).
\]

Repeating the same procedure for \( \mu, \nu = 2, 3 \) we obtain \( k_{22} = b(t) f^2 \) and \( k_{33} = b(t) f^2 \sin^2 \theta \).

Thus, (6) and (7) represent the general solution of (2) for the FRW metric in a five-dimensional constant curvature bulk. Note that as a consequence of the homogeneity of (2), the function \( b(t) = k_{11} \) representing the extrinsic curvature component along the radial direction \( r \), remains arbitrary. Denoting \( B = \dot{b}/b e^H = \dot{a}/a \), we find from (4) that

\[
K^2 = \frac{b^2}{a^2} \left( \frac{B^2}{H^2} - 2 \frac{B}{H} + 4 \right), \quad h = \frac{b}{a^2} \left( \frac{B}{H} + 2 \right)
\]

\[
Q_{\mu\nu} = \frac{1}{\varepsilon} \frac{b^2}{a^2} \left( \frac{2B}{H} - 1 \right) g_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 3
\]

\[
Q_{44} = -\frac{1}{\varepsilon} \frac{3b^2}{a^2}
\]

\[
Q = g^{\alpha\beta} Q_{\alpha\beta} = \frac{1}{\varepsilon} \frac{6b^2}{a^2} \frac{B}{H}
\]

Replacing these in (3) and separating the space and time components it follows that

\[
\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{k}{a^2} = 4\pi G (\rho - p) + \frac{\lambda}{2} + \frac{1}{\varepsilon} \frac{b^2}{a^2} \frac{1}{ab} \frac{d}{dt} (ab)
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\lambda}{6} + \frac{1}{\varepsilon} \frac{b^2}{a^2} \frac{1}{ab} \frac{d}{dt} \left( \frac{b}{a} \right)
\]

and finally, after eliminating \( \ddot{a} \), we obtain the modified Friedmann’s equation for the FRW braneworld

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\lambda}{3} + \frac{1}{\varepsilon} \frac{b^2}{a^2}
\]

where we see that the correction term with respect to the standard Friedmann’s equation is given by the component of the extrinsic curvature\(^6\). Note also the presence of \( \varepsilon \) which marks the effects of the bulk signature on the expansion of the universe.

\(^6\) Just for comparison purposes, it is illustrative to see how the \( \rho^2 \) term may emerge from (12) when the IDL condition (equation (B.6) in appendix B) is postulated. For the perfect fluid with energy density \( \rho \) that condition gives \( k_{11} = b(t) = -a^2 \rho a^2 \). Replacing this in (12) we obtain

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\lambda}{3} + \frac{1}{\varepsilon} \frac{a^2}{a^2} \rho^2
\]

producing the \( \rho^2 \)-dependent cosmology [26].
3. Dark energy as geometry

The additional degrees of freedom offered by braneworld gravity admit a wide range of possibilities for dark energy, beyond the $\Lambda$CDM model [41]. Here we explore the fact that $Q_{\alpha\beta}$ is independently conserved, suggesting an analogy with the energy–momentum tensor of an uncoupled non-conventional energy source. In this analogy we take the XCDM model as a practical example, denoting the ‘geometric pressure’ associated with the extrinsic curvature by $p_{\text{extr}}$ (the suffix ‘extr’ stands for ‘extrinsic’) and the ‘geometric energy density’ by $\rho_{\text{extr}}$. The corresponding geometric energy–momentum is identified to $Q_{\mu\nu}$ as

$$Q_{\mu\nu} \equiv -\frac{1}{8\pi G} \left[ (p_{\text{extr}} + \rho_{\text{extr}}) U_\mu U_\nu + p_{\text{extr}} g_{\mu\nu} \right] \quad \text{(13)}$$

where $U_\alpha = \delta_\alpha^4$. Comparing with the previous components (9)–(11) we obtain

$$p_{\text{extr}} = -\frac{1}{8\pi G} \frac{b^2}{a^4} \left( 2 \frac{B}{H} - 1 \right), \quad \rho_{\text{extr}} = \frac{3}{8\pi G} \frac{b^2}{a^4}. \quad \text{(14)}$$

Note the dependence of these terms on the bulk signature $\epsilon$. The sign ($-$) in (13) was chosen in accordance with the weak-energy condition corresponding to the positive energy $\rho_{\text{extr}} > 0$ and negative pressure $p_{\text{extr}} < 0$ with $\epsilon = 1$.

Like the XCDM model, the geometric ‘dark energy fluid’ can be implemented by a state-like equation

$$p_{\text{extr}} = \omega_{\text{extr}} \rho_{\text{extr}} \quad \text{(15)}$$

where $\omega_{\text{extr}}$ may be a function of time. After replacing the expressions of $B$ and $H$, we obtain the following equation for $b(t)$:

$$\frac{b}{b_0} = \frac{1}{2} (1 - 3\omega_{\text{extr}}) \frac{\dot{a}}{a}. \quad \text{(16)}$$

We cannot readily solve this equation because $\omega_{\text{extr}}$ is not known. However, a simple and useful example is given when $\omega_{\text{extr}} = \omega^0_{\text{extr}} = \text{constant}$. In this case, the general solution of (16) is very simple:

$$b(t) = b_0 \left( \frac{a}{a_0} \right)^{\frac{1}{2}(1-3\omega^0_{\text{extr}})} \quad \text{(17)}$$

where $a_0$ is the present value of the expansion parameter and $b_0$ is an integration constant representing the current warp of the universe. Clearly it must not vanish, otherwise all extrinsic curvature components would also vanish, and the braneworld would behave just as a trivial plane.

Replacing (17) in (12), now expressed in terms of the redshift $z = a/a_0 - 1$ and of the observable parameters $\Omega$, we obtain

$$\left( \frac{\dot{a}}{a} \right)^2 = H_o^2 \left[ \Omega_m (1+z)^3 + \Omega_k + \frac{1}{\epsilon} \Omega_{\text{extr}} (1+z)^{3(1+\omega^0_{\text{extr}})} + \Omega_k (1+z)^2 \right], \quad \text{(18)}$$

where $H_o$ is the present value of the Hubble parameter, $\Omega_m, \Omega_k$ and $\Omega_{\text{extr}}$ are, respectively, the confined matter, cosmological constant and the spatial curvature relative density parameters and where we have denoted the relative density parameter associated with the geometrical dark energy by

$$\Omega_{\text{extr}} = \frac{b_0^2}{H_o^2 a_0^4}. \quad \text{(19)}$$
Figure 1. Deceleration parameter as a function of redshift for a fixed value of $\Omega_x = 0.7$. In both panels the horizontal line labelled decelerating/accelerating ($q_0 = 0$) divides models with a decelerating or accelerating expansion at a given redshift. Note that while the signature $\epsilon = 1$ gives a speed-up scenario at $z_a \simeq 1$, the signature $\epsilon = -1$ implies a slow-down scenario.

Note that equation (16) is identical to the corresponding equation in XCDM [6], except that here $b(t)$ has the geometrical meaning of the radial component of the extrinsic curvature $k_{11}$.

If we had taken the bulk signature to be $(3,2)$ (or, $\epsilon = -1$), equation (16) would represent a fluid with negative energy and positive pressure, producing an unexpected behaviour of the expansion of the universe. In order to better visualize this difference, figures 1(a) and (b) show the behaviour of the deceleration parameter $q(z) = -\dddot{a}/\dot{a}^2$ as a function of redshift for selected values of $\Omega_m$ and $\Omega_{\text{extr}}$. Figure 1(a) shows the plane $q(z) - z$ for the signature $\epsilon = +1$. As it is well known, the acceleration redshift for such models happens around $z_a \simeq 0.7$, which seems to be in good agreement with observational data [42]. However, the case $\epsilon = -1$ in figure 1(b) presents an opposite behaviour, with the deceleration parameter becoming more positive at redshifts of the order of $z \simeq 1$. Therefore, in the light of this simple qualitative analysis and having in mind the recent supernovae (SNe) results [1], it is possible to exclude the bulk signature $(3,2)$ for an acceleration driven by a positive $\rho_{\text{extr}}$.

The use of the bulk signature $(4,1)$ associated with $\rho_{\text{extr}} > 0$ allows us to use the wealth of available data from the recent measurements to determine limits on the values of $\omega_{\text{extr}}^0$ in our geometric model. For example, from the current SNe Ia data (the so-called gold set of 157 events) one finds $\omega_{\text{extr}}^0 < -0.5$ at 95% confidence level (c.l.) for $\Omega_k = \Omega_\Lambda = 0$, regardless of the value of the matter density parameter [43]. When combined with cosmic microwave background (CMB) and large-scale structure (LSS) observations, the same SNe Ia data provide $\omega = -1.02^{+0.07}_{-0.09}$ (and $\omega < -0.76$ at 95% c.l.) [44]. These limits agree with the constraints obtained from a wide variety of different phenomena, using the ‘cosmic concordance method’ [45]. In this case, the combined maximum likelihood analysis suggests $\omega_{\text{extr}}^0 \leq -0.6$, (which incidentally also rules out an unknown component like topological defects of dimension $n$, such as domain walls and cosmic strings, for which we would have $\omega_{\text{extr}}^0 = -\frac{1}{n}$). Other
methods have also contributed to the collection of data. For example, gravitational lens statistics based on the final cosmic lens all sky survey data suggests that $\omega_{\text{extr}} < -0.55^{+0.18}_{-0.11}$ at 68% c.l. [46, 47]. Similarly, distance estimates of galaxy clusters from interferometric measurements of the Sunyaev–Zeldovich effect and x-ray observations along with SNe Ia and CMB data requires $\omega = -1.2^{+0.11}_{-0.15}$ [48–50]. We may also use the measurements of the angular size of high-redshift sources, suggesting that we could take $-1 \leq \omega_{\text{extr}} \leq -0.5$ [51], whereas the use of SNe data and measurements of the position of the acoustic peaks in the CMB spectrum, suggest $-1 \leq \omega_{\text{extr}} \leq -0.93$ at 2$\sigma$ [52]. We therefore conclude that in contrast to the five-dimensional braneworld models with the $\rho^2$ term in Friedmann’s equation which face experimental constraints, the present geometrical model, at least in the simple case where $\omega_{\text{extr}}$ constant, is consistent with the latest experimental observations, within the limits imposed by the weak energy condition, in the de Sitter bulk.

4. Summary

We have provided ample evidence to support the hypothesis that dark energy can be a consequence of the extrinsic curvature in braneworld cosmology. For that purpose, we have taken the FRW universe, seen as a braneworld embedded in a five-dimensional bulk of constant curvature, with undefined signature and without the $Z_2$ symmetry or any form of junction condition. The undefined signature has been motivated mostly by the fact that, except from the theoretical arguments in the application of the AdS/CFT in string theory, there is no experimental argument in favour of the anti-de Sitter signature ($3, 2$) over the de Sitter signature ($4, 1$).

Under these conditions, Friedmann’s equation depends only on the signature of the bulk and on the extrinsic curvature as a possible driver for inflation and in order to evaluate the compatibility of such geometrical model with the present experimental data, we have established a correspondence with the phenomenological XCDM dark energy model.

In the simple example where the factor $\omega_{\text{extr}}$ in the state equation (15) is constant, we found that when the energy density of the geometric fluid is positive and the pressure is negative, the AdS bulk with signature $(3, 2)$ is not compatible with the expected value of the deceleration parameter for the redshift $z \approx 1$, favouring the de Sitter case with signature $(4, 1)$. However, if we had taken the expansion energy to be negative, with positive pressure, then the universe would expand in the bulk with signature $(3, 2)$.

This example suggests not only that the extrinsic curvature can be responsible for the accelerated expansion, but also that in the more general case where $\omega_{\text{extr}}$ is not constant it must be dynamic, much in the sense proposed in the literature. The fact that the extrinsic curvature is a symmetric rank-two tensor suggests that the required dynamical equation should be nonlinear, in fact an Einstein-like equation [53]. Work on such an ultimate dynamical ‘junction’ condition is still in progress.

Appendix A. Equations of motion

With a few exceptions, mostly in the five-dimensional cases, the use of differentiable properties of the braneworld embedding has been largely neglected. Nonetheless, the probing of the extra dimensions by gravitons means that the classical geometry of the braneworld should be subjected to continuous deformations (or perturbations) in response to the Einstein–Hilbert dynamics of the bulk. Such deformable embeddings have been studied by Nash and Greene, concluding that the dimension and signature of the bulk is not a matter of choice, but they
depend on the regularity of the embedding functions [36, 37]. Therefore, restriction of the bulk to the AdS5 geometry requires the use of some additional conditions, such as imposing boundary rigidity, and the application of the $Z_2$ symmetry. However, it is not clear that the full dynamics of the braneworld will be compatible with such limited embedding, except perhaps in some specific cases, such as the FRW example in the main text. The embedding is, of course, a fundamental issue in differential geometry which has been frequently applied to general relativity. A comprehensive reference list on spacetime embedding properties can be found in [54]. In the following we give a short summary on the generation of the differentiable embedding by perturbations of a given background geometry.

Denoting by $\bar{g}_{ab}$ the metric of a given four-dimensional manifold $\bar{V}_4$ (the background) and by $\bar{g}_{AB}$ the metric of the Riemannian bulk $V_D$, the isometric embedding of $\bar{V}_4$ is given by a map $\chi : \bar{V}_4 \rightarrow V_D$, with $D = 4 + N$ components $\chi^A$ such that

$$\bar{\chi}\,^{A}\,_{\mu} \bar{\chi}\,^{B}_{\nu} \bar{g}_{AB} = \bar{g}_{\mu\nu}, \quad \bar{\chi}\,^{A}_{\mu} \bar{\chi}\,^{B}_{\nu} \bar{g}_{AB} = 0, \quad \bar{\eta}^A_{a} \bar{\eta}^B_{b} \bar{g}_{AB} = \bar{g}_{ab} \quad (A.1)$$

where $\bar{\eta}^A_{a}$ are the components of the $N$-independent normal vectors to $\bar{V}_4$. According to Nash, we may continuously deform $\bar{V}_4$ along a normal direction in the bulk, to obtain another submanifold of the same bulk, provided the embedding functions remain regular (in a generalized sense). Denoting a generic orthogonal direction by $\eta = \eta^a \bar{\eta}_a$, the deformation (in fact a perturbation) of the embedding can be expressed as

$$Z^A = \bar{\chi}\,^A + (\xi a \bar{\chi})^A, \quad \eta^A = \bar{\eta}^A + (\xi a \bar{\eta}_a)^A = \bar{\eta}^A.$$  

The components $Z^A$ and the normal $\eta_a$ must satisfy embedding equations similar to (A.1) (now dependent on $\eta^a$):

$$Z^A_{\mu} Z^B_{\nu} \bar{g}_{AB} = g_{\mu\nu}, \quad Z^A_{\mu} \bar{\eta}^B_{a} \bar{g}_{AB} = g_{\mu a}, \quad \bar{\eta}^A_{a} \bar{\eta}^B_{b} \bar{g}_{AB} = g_{ab}. \quad (A.2)$$

From these equations it follows that

$$g^{\mu\nu} Z^B_{\mu} Z^A_{\nu} = G^{AB} - g_{ab} \bar{\eta}^A_{a} \bar{\eta}^B_{b} \quad (A.3)$$

and also the components of the perturbed geometry

$$g_{\mu\nu}(x, y) = Z^A_{\mu} Z^B_{\nu} \bar{g}_{AB} = \bar{g}_{\mu\nu} - 2 y^a \bar{k}_{\mu a} + y^a y^b g^a_{\mu b} \bar{k}_{\nu b} + y^a y^b y^c \bar{A}_{\mu a} \bar{A}_{\nu b}, \quad (A.4)$$

$$g_{\mu a}(x, y) = Z^A_{\mu} \bar{\eta}^B_{a} \bar{g}_{AB} = y^b A_{\mu a b}, \quad (A.5)$$

$$g_{ab}(x, y) = \bar{\eta}^A_{a} \bar{\eta}^B_{b} \bar{g}_{AB} = \bar{g}_{ab} \quad (A.6)$$

$$k_{\mu a}(x, y) = -\bar{\eta}^A_{a} \bar{\eta}^B_{b} \bar{g}_{AB} = \bar{k}_{\mu a} - y^b \bar{g}_{\mu b} \bar{k}_{\nu a} - y^b y^c \bar{A}_{\mu a b} \bar{A}_{\nu b}, \quad (A.7)$$

$$A_{\mu a b}(x, y) = \bar{\eta}^A_{a} \bar{\eta}^B_{b} \bar{g}_{AB} = \bar{A}_{\mu a b}(x). \quad (A.8)$$

From (A.4) and (A.7) we obtain York’s relation for the extrinsic curvature (extended to the extra variables $y^a$):

$$k_{\mu a} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y^a} \quad (A.9)$$

so that when the braneworld gravitational field represented by the metric $g_{\mu\nu}$ propagates in the bulk, so does the extrinsic curvature.

The components of the Riemann tensor of the bulk written in the embedding vielbein $\{Z^A_a, \bar{\eta}^A_a\}$, give the Gauss, Codazzi and Ricci equations, respectively [55].

$$R_{\alpha\beta\gamma\delta} = g^{m\alpha} \bar{k}_{[\gamma m] a} k_{[\delta] b] a} + R_{\alpha\beta\gamma\delta} Z^A a Z^B b Z^C b Z^D b \quad (A.10)$$

$$k_{[\gamma m] a} = g^{m\alpha} A_{[\gamma m] a} + R_{\alpha\beta\gamma\delta} Z^A a Z^B b Z^C b Z^D b \quad (A.11)$$
\[ 2A_{[\nu a b]} = -2g^{mn}A_{[\nu ma]b} - g^{mn}k_{[\nu ma]b} - R_{ABCD}\eta^A_d \eta^B_c \xi^C_v \xi^D_n. \]  
(A.12)

To proceed, we now impose that the bulk geometry is a solution of Einstein’s equations. Denoting \( K^2 = g^{\mu \nu}k_{\mu \nu}, h_\mu = g^{\mu \nu}k_{\mu \nu} \) and \( h^2 = g^{\mu \nu}h_\mu h_\nu \) and using (A.3), we obtain from the contraction of Gauss’ equations with \( \alpha \)
\[ R_{\mu \nu} = g^{cd}(g^{ab}k_{\mu a}c_{\nu d} - h_\mu k_{\mu \nu}) + R_{AB}\eta^A_d \eta^B_c \xi^C_v \xi^D_n. \]  
(A.13)

A further contraction gives the Ricci scalar
\[ R = (K^2 - h^2) + 2g^{\alpha \beta}R_{AB}\eta^A_d \eta^B_c + g^{\alpha \beta}g^{\gamma \delta}R_{ABCD}\eta^A_d \eta^B_c \eta^C_v \eta^D_n. \]  
(A.14)

Therefore, the Einstein–Hilbert action for the bulk geometry in \( D \) dimensions can be written in terms of the embedded geometry as
\[ \int \sqrt{g} d^D v = \int (1 - (K^2 - h^2)) \sqrt{g} d^D v + \int \left[ 2g^{ab}R_{AB}\eta^A_d \eta^B_c - g^{ac}g^{bd}R_{ABCD}\eta^A_d \eta^B_c \eta^C_v \eta^D_n \right] \sqrt{g} d^D v = \alpha_s \int L^* \sqrt{g} d^D v \]  
(A.15)

where on the right-hand side we have included the bulk source Lagrangian \( L^* \) and we have denoted by \( \alpha_s \) the \( D \)-dimensional fundamental energy scale.

The covariant equations of motion for a braneworld in a \( D \)-dimensional bulk can be derived by taking the variation of (A.15) with respect to \( g_{\mu \nu}, \eta_{\mu \alpha} \) and \( g_{ab} \), noting that the Lagrangian depends on these variables through \( Z^A_{\alpha} \). Alternatively, we may just calculate the components of Einstein’s equations for the bulk geometry:
\[ R_{AB} - \frac{1}{2} RG_{AB} = \alpha_s T_{AB}^* \]  
(A.16)

in the embedding vielbein \( \{ Z^A_{\alpha}, \eta^B_\alpha \} \). Denoting the vielbein components of the energy–momentum tensor by \( T_{\mu \nu} = T_{AB}^* Z^A_{\alpha} Z^B_{\nu} \), \( T_{\mu \alpha} = T_{AB}^* Z^A_{\alpha} \eta^B_\alpha \) and \( T_{ab} = T_{AB}^* \eta^A_d \eta^B_c \), we obtain from (A.16)
\[ g^{ab}R_{AB}\eta^A_d \eta^B_c = \alpha_s \left( g^{ab}T_{ab}^* - \frac{N}{N+2} T^* \right). \]

From (A.13) and (A.14), the (tangent) components of (A.16) on \( Z^A_{\alpha}, Z^B_{\mu} \) give the equation for \( g_{\mu \nu} \) (sometimes referred to as the gravi–vector equation)
\[ R_{\mu \nu} - \frac{1}{2} RG_{\mu \nu} - Q_{\mu \nu} + \left( W_{\mu \nu} - \frac{1}{2} W g_{\mu \nu} \right) = \alpha_s \left( T_{\mu \nu} - \frac{N}{N+2} T^* g_{\mu \nu} - g^{ab} T_{ab}^* \right), \]  
(A.17)

where we have denoted
\[ Q_{\mu \nu} = g^{ab}k_{\mu a}k_{\nu b} - g^{ab}h_{\mu a}h_{\nu b} - \frac{1}{2}(K^2 - h^2)g_{\mu \nu}, \]
\[ W_{\mu \nu} = g^{ad}R_{ABCD}\eta^B_d \xi^C_v \xi^D_n \]  
(A.18)

On the other hand, again from (A.16), the trace of Codazzi’s equation (A.11) gives the gravi–vector equation
\[ k_{\mu a, \nu}^\rho - h_{a, \mu} - (A_{\rho ca}k_{\mu c} - A_{\rho ca}h^c) + 2W_{\mu a} = \alpha_s \left( T_{\mu a}^* - \frac{1}{N+2} T^* g_{\mu a} \right), \]  
(A.19)

where
\[ W_{\mu a} = g^{ab}R_{ABCD}\eta^B_d \eta^C_v \xi^D_n. \]
Finally, the {\textit{gravi–scalar}} equation is obtained from (A.14) and (A.16)

\[ R - K^2 + h^2 - W = 2\alpha_s \left( g^{ab} T^*_{ab} \right) \frac{N + 1}{N + 2} T^* . \]  

(A.20)

Equations (A.17)–(A.20) represent the most general equations of motion of a braneworld, compatible with its differentiable embedding in a $D$-dimensional bulk defined by Einstein’s equations.

The confinement hypothesis as applied to all perturbed geometries (and not just to the background) can be implemented simply as

\[ \alpha_s T^*_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad T^*_{\mu a} = 0, \quad T^*_{ab} = 0 \]  

(A.21)

where $T_{\mu\nu}$ denotes the energy–momentum tensor of ordinary matter and gauge fields, which remain confined to the braneworld. As we should expect, (A.17) reproduces the ordinary Einstein’s equations when the extrinsic geometry components are neglected.

**Appendix B. The $Z_2$ symmetry and the Israel–Darmois–Lanczos condition**

Here we use essentially the same procedure as in Israel’s paper [56], adapted to the case of a braneworld in a constant curvature bulk. The starting point is Einstein’s equation for the bulk geometry (A.16), now written as

\[ R_{AB} = \alpha_s \left( T^*_{AB} - \frac{1}{2 + N} T^* g_{AB} \right) . \]  

(B.1)

For $D = 4 + N = 5$, the bulk metric written in the embedding vielbein is (just for clarity here we set $g_{55} = \epsilon = 1$)

\[ G_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} . \]

After explicitly writing the vielbein components of the Ricci tensor $R_{AB}$ in the case of the constant curvature bulk, we find from (A.17) that

\[ R_{\mu\nu} - \frac{\partial k_{\mu\nu}}{\partial y} - 2k_\mu^k k_{\nu}^k + h k_{\mu\nu} = \alpha_s \left( T^*_{\mu\nu} - \frac{1}{3} T^* g_{\mu\nu} \right) . \]  

(B.2)

Now, consider that the background $y = 0$ separates two sides of the bulk, labelled by + and − respectively, and find the value of (B.2) as we approach $y = 0$ from each side.

As in [56], we consider two distinct situations. Case (i) is characterized by a continuity of the extrinsic curvature across the boundary $y = 0$: $k^+_\mu^k = k^-_{\mu^k}$, with the supposition that the confined matter is given by a well-defined differentiable energy–momentum tensor. Nothing else is added. Then, in the constant curvature bulk the equations equivalent to the O’Brien–Synge junction conditions (A.19) and (A.20) are just identities. As for the tensor equation (A.17), we note that the value of $R_{\mu\nu}$ is the same on both sides of $\bar{\nu}$. On the other hand, admitting that the braneworld is orientable, the term $\left( -\frac{\partial k_{\mu\nu}}{\partial y} \right)$ and all terms involving the square of $k_{\mu\nu}$ do not change sign across the boundary. Since in this case the confined matter is intrinsic and well defined, it follows that the differences of (B.2) calculated on both sides of the boundary cancel each other as $y \to 0$.

This situation changes in case (ii), characterized by a jump in the extrinsic curvature $k^+_{\mu\nu} \neq k^-_{\mu\nu}$ across a background caused by a confined distributional source. In this case, the derivatives $\left( \frac{\partial k_{\mu\nu}}{\partial y} \right)$ in (B.2) continuously change as it approaches $y = 0$, so that the difference
In particular, for $X$

To obtain the IDL condition, we need to specify how remain an embedded differentiable manifold \([36]\).

defined. Under this condition the perturbations of the background cannot be guaranteed to 
remain an embedded differentiable manifold \([36]\).

opposite directions. This means that the derivatives of the embedding functions are not well 

vectors, one on each side, whose projections on the background give vectors pointing in 
the other side of it. Therefore, to each point of the background corresponds two different tangent 

planes, some other examples are discussed in \([57, 58]\).

This is precisely what 

senses the extra dimension. In particular, the derivatives of the normal 

is 

sense that equation (B

7) implies that the background behaves as a mirror for the derivatives of \(\eta\).

we conclude that

\[ k^+_{\mu\nu} = -k^-_{\mu\nu} \]  \hspace{1cm} (B.5)  

so that equation (B.4) gives at \(y = 0\) the Israel–Darmois–Lanczos condition

\[ \tilde{k}_{\mu\nu} = -\alpha_x (\tilde{T}_{\mu\nu} - \frac{1}{3} \tilde{T}_{\hat{g}_{\mu\nu}}) \]  \hspace{1cm} (B.6)  

specifying the value of the extrinsic curvature of the background in terms of the energy–
momentum tensor of its confined sources.

Reciprocally, using the definition of \(k_{\mu\nu}\), the values calculated on the two sides of the 

background are \(k^+_{\mu\nu} = -\tilde{Z}^A_{\mu} \hat{\eta}^{\mu} B_{\hat{A}\hat{B}}\), and therefore (B.5) implies that \(\eta^{\mu}_{\mu} = -\eta^{\mu}_{\mu}\), or, in 

other words, that the background behaves as a mirror for the derivatives of \(\eta\). We conclude that 

while (B.4) follows from Einstein’s equation of the bulk plus the distributional source of the 
braneworld, the \(Z_2\) symmetry (or any symmetry producing the same mirror effect) completely 

specifies \(k_{\mu\nu}\) in terms of the confined source \(\tilde{T}_{\mu\nu}\).

One aspect that has not been considered is that the IDL condition, which was originally 

applied to two- or three-dimensional hypersurfaces, may imply in a limited class of admissible 

background braneworlds in a higher-dimensional bulk, depending on how general that confined 

source is. For example, if we consider a confined source such as \(T_{\mu\nu} = 1/3 \tilde{T}_{\hat{g}_{\mu\nu}}\), then from 

(B.6) it follows that \(\tilde{k}_{\mu\nu} = 0\), which means that the background is just a plane. But from (A.7) 

it follows that all perturbations also have zero extrinsic curvatures and consequently they are 
also planes. Some other examples are discussed in \([57, 58]\).

Another mathematical aspect which deserves further attention is the fact that with the 

\(Z_2\) symmetry, for each perturbation of the background there will be a mirror image on the 
other side of it. Therefore, to each point of the background corresponds two different tangent 

vectors, one on each side, whose projections on the background give vectors pointing in 

opposite directions. This means that the derivatives of the embedding functions are not well 

defined. Under this condition the perturbations of the background cannot be guaranteed to 

remain an embedded differentiable manifold \([36]\).
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