THREE-DIMENSIONAL SOLVSOLITONS AND THE
MINIMALITY OF THE CORRESPONDING SUBMANIFOLDS

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ABSTRACT. In this paper, we define the corresponding submanifolds to left-invariant Riemannian metrics on Lie groups, and study the following question: does a distinguished left-invariant Riemannian metric on a Lie group correspond to a distinguished submanifold? As a result, we prove that the solvsolitons on three-dimensional simply-connected solvable Lie groups are completely characterized by the minimality of the corresponding submanifolds.

1. INTRODUCTION

1.1. Solvsolitons. Lie groups with left-invariant Riemannian metrics provide a lot of concrete examples of distinguished Riemannian metrics, such as Einstein metrics and Ricci solitons. Recently, such distinguished left-invariant Riemannian metrics have been studied very actively (see, for instance, [1, 7, 9, 10, 13, 15, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27]).

In this paper, we treat solvsolitons as distinguished left-invariant Riemannian metrics. Recall that a left-invariant Riemannian metric \((\cdot, \cdot)\) on a simply-connected solvable Lie group \(G\) is called a solvsoliton if the Ricci operator satisfies

\[
\text{Ric}_{(\cdot)} = cI + D \quad \text{(for some } c \in \mathbb{R} \text{ and } D \in \text{Der}(g)).
\]

A solvsoliton on \(G\) is called a nilsoliton if \(G\) is nilpotent. Solvsolitons have been introduced by Lauret ([17]), and play a key role in the study of homogeneous Ricci solitons. In particular, every solvsoliton on a simply-connected solvable Lie group is a Ricci soliton ([17]), and every left-invariant Ricci soliton on a solvable Lie group is isometric to a solvsoliton ([10]).

In the study of solvsolitons, including left-invariant Einstein metrics on solvable Lie groups, the tools from geometric invariant theory have played very important roles. Among others, Lauret ([17]) obtained structural and uniqueness results for solvsolitons. It enables to classify solvsolitons in low-dimensional cases ([17, 27]). For further information, we refer to ([15] and references therein.

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1.2. **An approach from the submanifold theory.** In this paper, we propose a new framework for studying distinguished left-invariant Riemannian metrics, such as solvsolitons, in terms of the group actions on and the submanifold theory in noncompact symmetric spaces. This paper only concerns simply-connected solvable Lie groups of dimension three, but here we formulate our framework in a general way.

Let $G$ be a Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. Consider the set of all left-invariant Riemannian metrics on $G$, which can be identified with

$$\mathcal{M} := \{\langle \cdot, \cdot \rangle | \text{an inner product on } \mathfrak{g}\} \cong \text{GL}_n(\mathbb{R})/\text{O}(n),$$

where $n = \dim G$. Throughout this paper, this space is assumed to be endowed with the natural $\text{GL}_n(\mathbb{R})$-invariant Riemannian metric (see Subsection 2.1), and hence is a noncompact symmetric space. Let us consider the actions of

$$\mathbb{R}^\times \text{Aut}(\mathfrak{g}) := \{c\varphi \in \text{GL}_n(\mathbb{R}) | c \in \mathbb{R}^\times, \varphi \in \text{Aut}(\mathfrak{g})\}$$

on $\mathcal{M} = \text{GL}_n(\mathbb{R})/\text{O}(n)$. Note that $\mathbb{R}^\times$ denotes the set of nonzero scalar maps on $\mathfrak{g}$, and $\text{Aut}(\mathfrak{g})$ the automorphism group. The group $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ comes from the equivalence relation “isometry up to scaling” in the Lie algebra level (see Definition 2.1). Denote its equivalence class by $[\cdot]$. Then, for each inner product $\langle \cdot, \cdot \rangle$, it follows from [11] that

$$[\langle \cdot, \cdot \rangle] = \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle,$$

which we call the corresponding submanifold to $\langle \cdot, \cdot \rangle$. An important point is that the Riemannian geometric properties of $\langle \cdot, \cdot \rangle$ are preserved by isometry and scaling. Thus we can regard properties of left-invariant Riemannian metrics as properties of the corresponding submanifolds. Therefore, it would be natural to ask the following:

**Question.** Does a distinguished left-invariant Riemannian metric correspond to a distinguished submanifold?

If an answer for this question is positive, then the approach from the corresponding submanifolds would possibly be useful for the study of left-invariant metrics. For example, the existence and nonexistence problem of distinguished left-invariant Riemannian metrics on $G$ can be translated to the problem of the $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$-action, that is, the existence and nonexistence of distinguished orbits.

1.3. **Results of this paper.** Let $G$ be a three-dimensional simply-connected solvable Lie group with Lie algebra $\mathfrak{g}$. In this paper, we present that there is a good relationship between the existence of solvsolitons on $G$ and geometric aspects of the corresponding action of $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ on $\text{GL}_3(\mathbb{R})/\text{O}(3)$. We will see this by using the classification of three-dimensional solvable Lie algebras ([2]), which is summarized in Table E3. Note that Table E3 contains a decomposable one, $\mathfrak{r}_{3,0}$, and our results are true for both decomposable and indecomposable cases.
| Name | Non-zero commutation relation | Solvability |
|------|-------------------------------|-------------|
| $\mathfrak{h}_3$ | $[e_1, e_2] = e_3$ | Nilpotent |
| $\mathfrak{r}_3$ | $[e_1, e_2] = e_2 + e_3$, $[e_1, e_3] = e_3$ | Solvable |
| $\mathfrak{r}_{3,a}$ | $[e_1, e_2] = e_2$, $[e_1, e_3] = ae_3$ ($-1 \leq a \leq 1$) | Solvable |
| $\mathfrak{r}_{3,a}'$ | $[e_1, e_2] = ae_2 - e_3$, $[e_1, e_3] = e_2 + ae_3$ ($a \geq 0$) | Solvable |

Table 1. Three-dimensional solvable Lie algebras

We recall that, for an isometric action on a Riemannian manifold, orbits of maximal dimension are said to be regular, and other orbits singular. An action is said to be of cohomogeneity one if the regular orbits have codimension one. Then, the good relationship we obtain can be summarized as follows.

- Let $\mathfrak{g} = \mathfrak{h}_3$ or $\mathfrak{r}_{3,1}$. Then, $\mathbb{R} \times \text{Aut}(\mathfrak{g})$ acts transitively on $\tilde{M}$, and hence there is the only one orbit. The left-invariant Riemannian metric on $G$ is unique up to isometry and scaling, and the metric is a solvsoliton (nilsoliton for $\mathfrak{h}_3$, and Einstein for $\mathfrak{r}_{3,1}$).

- Let $\mathfrak{g} = \mathfrak{r}_3$. Then, the action of $\mathbb{R} \times \text{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are regular. Furthermore, all orbits are isometrically congruent to each other (namely there are no distinguished orbits). On the other hand, $G$ does not admit a solvsoliton.

- Let $\mathfrak{g} = \mathfrak{r}_{3,a}$ ($-1 \leq a < 1$). Then, the action of $\mathbb{R} \times \text{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are regular. This action has the unique minimal orbit. On the other hand, $G$ admits a solvsoliton, whose corresponding submanifold coincides with this minimal orbit.

- Let $\mathfrak{g} = \mathfrak{r}_{3,a}'$ ($a \geq 0$). Then, the action of $\mathbb{R} \times \text{Aut}(\mathfrak{g})$ is of cohomogeneity one, and has the unique singular orbit. On the other hand, $G$ admits a left-invariant Einstein metric, whose corresponding submanifold coincides with this singular orbit.

By studying the geometry of $\mathbb{R} \times \text{Aut}(\mathfrak{g})$-orbits in more detail, we obtain a positive answer to the above mentioned Question for three-dimensional solvsolitons. Namely, three-dimensional solvsolitons can completely be characterized by the minimality of the corresponding submanifold.

**Main Theorem.** Let $G$ be a three-dimensional simply-connected solvable Lie group, and $\langle \cdot, \cdot \rangle$ be a left-invariant Riemannian metric on $G$. Then, $\langle \cdot, \cdot \rangle$ is a solvsoliton if and only if the corresponding submanifold $[\langle \cdot, \cdot \rangle]$ is a minimal submanifold in $\tilde{M}$ with respect to the natural $\text{GL}_3(\mathbb{R})$-invariant Riemannian metric.

This paper is organized as follows. In Section 2, we recall the necessary background on the corresponding submanifolds $[\langle \cdot, \cdot \rangle]$ to left-invariant Riemannian metrics $\langle \cdot, \cdot \rangle$ on Lie groups. In Section 3, for each three-dimensional solvable Lie algebra $\mathfrak{g}$, we study the orbit space of the action of $\mathbb{R} \times \text{Aut}(\mathfrak{g})$. Expressions of the
orbit spaces will be used in both Sections 4 and 5. In Section 4, we study three-dimensional solvsolitons. In particular, we obtain the “Milnor-type theorems” for each \( g \), and apply them to the reclassification of three-dimensional solvsolitons. In Section 5, we study the actions of \( \mathbb{R}^\times \text{Aut}(g) \). The results of Sections 4 and 5 provide the proof of our Main Theorem.

2. THE CORRESPONDING SUBMANIFOLDS

In this section, we define the notion of the corresponding submanifolds to left-invariant Riemannian metrics on Lie groups. This gives a correspondence between left-invariant Riemannian metrics and \( \mathbb{R}^\times \text{Aut}(g) \)-homogeneous submanifolds.

2.1. The space of left-invariant metrics. First of all, we recall the space of left-invariant Riemannian metrics, which will be the ambient space of the corresponding submanifolds. We refer to [11].

Let \( G \) be an \( n \)-dimensional simply-connected Lie group, and \( g \) be the Lie algebra of \( G \). We consider the set of all left-invariant Riemannian metrics on \( G \), which can naturally be identified with

\[
\tilde{\mathcal{M}} := \{ \langle , \rangle \mid \text{an inner product on } g \}.
\]

We identify \( g \) with \( \mathbb{R}^n \) as vector spaces from now on. Then, since \( \text{GL}_n(\mathbb{R}) \) acts transitively on \( \tilde{\mathcal{M}} \) by

\[
g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle \quad (\text{for } g \in \text{GL}_n(\mathbb{R}), \, \langle \cdot, \cdot \rangle \in \tilde{\mathcal{M}}),
\]

we have an identification

\[
\tilde{\mathcal{M}} = \text{GL}_n(\mathbb{R})/O(n).
\]

Note that \( \tilde{\mathcal{M}} \) equipped with the natural \( \text{GL}_n(\mathbb{R}) \)-invariant Riemannian metric is a noncompact Riemannian symmetric space. In order to describe this natural metric, we recall a general theory of reductive homogeneous spaces. Let \( U/K \) be a reductive homogeneous space, that is, there exists an \( \text{Ad}_K \)-invariant subspace \( m \) of \( u \) satisfying

\[
u = \mathfrak{k} \oplus m.
\]

Note that \( u \) and \( \mathfrak{k} \) are the Lie algebras of \( U \) and \( K \), respectively, and \( \oplus \) is the direct sum as vector spaces. The decomposition (2.4) is called a reductive decomposition. Denote by \( \pi : U \to U/K \) the natural projection, and by \( o := \pi(e) \) the origin of \( U/K \). We identify \( m \) with the tangent space \( T_o(U/K) \) at \( o \) by

\[
d\pi_e|_m : m \to T_o(U/K).
\]

This identification induces a one-to-one correspondence between the set of \( U \)-invariant Riemannian metrics on \( U/K \) and the set of \( \text{Ad}_K \)-invariant inner products on \( m \).
Now one can see that $\widetilde{M} = \text{GL}_n(\mathbb{R})/O(n)$ is a reductive homogeneous space, whose reductive decomposition is given by the subspace
\begin{equation}
\text{sym}(n) := \{ X \in \text{gl}_n(\mathbb{R}) \mid X = ^tX \}.
\end{equation}
We define the $\text{Ad}_{O(n)}$-inner product on $\text{sym}(n)$ by
\begin{equation}
\langle X, Y \rangle := \text{tr}(XY) \quad (\text{for } X, Y \in \text{sym}(n)).
\end{equation}
We call the $\text{GL}_n(\mathbb{R})$-invariant Riemannian metric corresponding to the above $\text{Ad}_{O(n)}$-inner product the \textit{natural Riemannian metric}.

2.2. The corresponding submanifolds. We now define the submanifolds in the space of left-invariant Riemannian metrics, and see that they are homogeneous. These submanifolds come from the equivalence relation “isometric up to scaling”.

Definition 2.1. Two inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on $\mathfrak{g}$ are said to be isometric up to scaling if there exist $k > 0$ and an automorphism $f : \mathfrak{g} \to \mathfrak{g}$ such that $\langle \cdot, \cdot \rangle_1 = k \langle f(\cdot), f(\cdot) \rangle_2$.

Assume that inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on $\mathfrak{g}$ are isometric up to scaling. Then, the corresponding left-invariant Riemannian metrics on $G$, the simply-connected Lie group with Lie algebra $\mathfrak{g}$, are isometric up to scaling as Riemannian metrics (we refer to [11, Remark 2.3]). Therefore, this equivalence relation preserves all Riemannian geometric properties of left-invariant metrics. In particular, it preserves solvsolitons.

Definition 2.2. For each inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, we call its equivalence class $[\langle \cdot, \cdot \rangle]$ the corresponding submanifold to $\langle \cdot, \cdot \rangle$.

Note that $[\langle \cdot, \cdot \rangle]$ is a submanifold in $\widetilde{M} = \text{GL}_n(\mathbb{R})/O(n)$. We here recall that $[\langle \cdot, \cdot \rangle]$ is a homogeneous submanifold. Let us denote by
\begin{equation}
\mathbb{R}^\times := \{ c \cdot \text{id} : \mathfrak{g} \to \mathfrak{g} \mid c \in \mathbb{R} \setminus \{0\} \},
\end{equation}
\begin{equation}
\text{Aut}(\mathfrak{g}) := \{ \phi : \mathfrak{g} \to \mathfrak{g} \mid \text{an automorphism} \}.
\end{equation}
Then, the subgroup $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ of $\text{GL}_n(\mathbb{R})$ acts naturally on $\widetilde{M}$. Let us denote by $\mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle$ the $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$-orbit through $\langle \cdot, \cdot \rangle$.

Proposition 2.3 ([11 Theorem 2.5]). Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$. Then, the corresponding submanifold $[\langle \cdot, \cdot \rangle]$ is a homogeneous submanifold with respect to $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$, that is,
\begin{equation}
[\langle \cdot, \cdot \rangle] = \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle.
\end{equation}
3. Explicit expressions of the moduli spaces

In this section, for each three-dimensional solvable Lie algebra $g$, we give an explicit expression of the “moduli space” of left-invariant Riemannian metrics. The results of this section will be used in Sections 4 and 5.

3.1. Preliminaries on the moduli spaces. In this subsection, we recall some necessary facts on the moduli spaces of left-invariant Riemannian metrics. We refer to [11].

**Definition 3.1.** For a Lie algebra $g$, the quotient space of $\tilde{M}$ by “isometric up to scaling” is called the moduli space of left-invariant Riemannian metrics, and denoted by $PM := \{[\langle \cdot , \cdot \rangle] | \langle \cdot , \cdot \rangle \in \tilde{M} \}$.

(3.1)

In order to determine $PM$ explicitly, we will use the following notion of a set of representatives. Recall that we identify $g \cong \mathbb{R}^n$. Denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$, and by $\langle \cdot , \cdot \rangle_0$ the inner product so that the canonical basis is orthonormal.

**Definition 3.2.** A subset $U \subset GL_n(\mathbb{R})$ is called a set of representatives of $PM$ if it satisfies

(3.2)

In the later arguments, it is convenient to use the double cosets. Note that our double coset $[[g]]$ of $g \in GL_n(\mathbb{R})$ is defined by

(3.3)

**Lemma 3.3 ([6]).** Let $U \subset GL_n(\mathbb{R})$. Then, $U$ is a set of representatives of $PM$ if and only if, for every $g \in GL_n(\mathbb{R})$, there exists $h \in U$ such that $h \in [[g]]$.

In order to obtain a set of representatives of $PM$, one needs $\mathbb{R}^\times Aut(g)$. The Lie algebra of $\mathbb{R}^\times Aut(g)$ coincides with $\mathbb{R} \oplus Der(g)$, where

(3.4)

(3.5)

The Lie algebra $\mathbb{R} \oplus Der(g)$ determines $(\mathbb{R}^\times Aut(g))^0$, the connected component of $\mathbb{R}^\times Aut(g)$ containing the identity.

For each three-dimensional solvable Lie algebra, the moduli space $PM$ has been studied in [11]. We here mention the trivial case, which means that $PM$ consists of one point.

**Proposition 3.4 ([11, 14]).** Let $g = h_3$ or $r_{3,1}$. Then, $\mathbb{R}^\times Aut(g)$ acts transitively on $M$, and hence $PM = \{pt\}$.  

Remark 3.5. One can see that Theorem 1.3 holds for $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{r}_{3,1}$. In fact, it is well-known that any left-invariant Riemannian metrics $\langle \cdot, \cdot \rangle$ on these Lie algebras are solvsolitons (nilsoliton for $\mathfrak{h}_3$, and Einstein for $\mathfrak{r}_{3,1}$). Furthermore, for every $\langle \cdot, \cdot \rangle$, the corresponding submanifold $\mathcal{N}_{\langle \cdot, \cdot \rangle}$ coincides with the ambient space $\mathcal{M}$, which is minimal.

In the following, we will study the remaining three-dimensional solvable Lie algebras.

3.2. A lemma for nontrivial cases. This subsection gives a preliminary to obtain a set of representatives $U$ of $\mathcal{PM}$ for $\mathfrak{g} = \mathfrak{r}_3$, $\mathfrak{r}_{3,a}$ ($-1 \leq a < 1$), and $\mathfrak{r}_{3,a}'$ ($a \geq 0$).

First of all, let us recall a matrix expression of $\text{Der}(\mathfrak{g})$ for these Lie algebras. The following results can be calculated directly, and be found in [11, Section 4].

Lemma 3.6 ([11]). The matrix expressions of $\text{Der}(\mathfrak{g})$ with respect to the bases $\{e_1, e_2, e_3\}$ in Table 1.3 are given as follows:

1. Let $\mathfrak{g} = \mathfrak{r}_3$. Then, we have

$$\text{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}.$$

2. Let $\mathfrak{g} = \mathfrak{r}_{3,a}$ ($-1 \leq a < 1$). Then, we have

$$\text{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.$$

3. Let $\mathfrak{g} = \mathfrak{r}_{3,a}'$ ($a \geq 0$). Then, we have

$$\text{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}.$$

Let us consider $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ for these Lie algebras $\mathfrak{g}$. One can see from Lemma 3.6 that $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ contain

$$F := \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{22} \end{pmatrix} \mid x_{11}, x_{22} > 0 \right\}.$$

For the later use, we prepare the following lemma, which can be applied for all Lie algebras we have to consider.
Lemma 3.7. Let \( g \) be a three-dimensional Lie algebra, and fix a basis of \( g \). If \( F \subset \mathbb{R} \times \text{Aut}(g) \) holds, then the following \( L' \) is a set of representatives of \( \mathcal{P}^m \):

\[
L' := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix} \mid a_{33} > 0 \right\}.
\]

Proof. Take any \( g \in \text{GL}_3(\mathbb{R}) \). By Lemma 3.3, we have only to show that there exists \( g' \in L' \) such that \( g' \in [[g]] \). First of all, one knows that there exists \( k \in \text{O}(3) \) such that

\[
gk = \begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad g_{11}, g_{22}, g_{33} > 0.
\]

By assumption, we can take

\[
\varphi := \frac{1}{g_{11}g_{22}} \begin{pmatrix} g_{22} & 0 & 0 \\ -g_{21} & g_{11} & 0 \\ -g_{31} & 0 & g_{11} \end{pmatrix} \in F \subset \mathbb{R} \times \text{Aut}(g).
\]

By a direct calculation, one has

\[
[[g]] \ni \varphi gk = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g_{32}/g_{22} & g_{33}/g_{22} \end{pmatrix} =: g'.
\]

Since \( g' \in L' \), we complete the proof. \( \square \)

3.3. Case of \( g = \mathfrak{r}_3 \). In this subsection, we give an explicit expression of \( \mathcal{P}^m \) for \( g = \mathfrak{r}_3 \). We fix a basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{r}_3 \) whose bracket relations are given by

\[
[e_1, e_2] = e_2 + e_3, \quad [e_1, e_3] = e_3.
\]

From Lemma 3.6 we have

\[
\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}.
\]

This yields that

\[
(\mathbb{R} \times \text{Aut}(g))^0 = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{11}, x_{22} > 0 \right\}.
\]

Therefore, we can apply Lemma 3.7 for this case.
Proposition 3.8. Let $g = r_3$. Then the following $U$ is a set of representatives of $\mathcal{P}\mathcal{M}$:

(3.14) \[ U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda > 0 \right\}. \]

Proof. Take any $g \in \text{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $\lambda > 0$ such that

(3.15) \[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \in [[g]]. \]

We use $L'$ defined in Lemma 3.7. One has from (3.13) and Lemma 3.7 that there exists $g' \in L'$ such that $g' \in [[g]]$. Since $g' \in L'$, one can write

(3.16) \[ g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad a_{33} > 0. \]

It follows from (3.13) that

(3.17) \[ \varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a_{32} & 1 \end{pmatrix} \in (\mathbb{R}^* \text{Aut}(g))^0. \]

This shows that

(3.18) \[ [[g]] \ni \varphi g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}. \]

Therefore, by putting $\lambda := 1/a_{33}$, we complete the proof. \qed

3.4. Case of $g = r_{3,a}$ ($-1 \leq a < 1$). In this subsection, we give an explicit expression of $\mathcal{P}\mathcal{M}$ for $g = r_{3,a}$. Throughout this subsection, we fix $a$ satisfying $-1 \leq a < 1$, and a basis $\{e_1, e_2, e_3\}$ of $r_{3,a}$ whose bracket relations are given by

(3.19) \[ [e_1, e_2] = e_2, \quad [e_1, e_3] = ae_3. \]

From Lemma 3.6 we have

(3.20) \[ \mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}. \]

This yields that

(3.21) \[ (\mathbb{R}^* \text{Aut}(g))^0 = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{22}, x_{33} > 0 \right\}. \]
Proposition 3.9. Let \( g = \mathfrak{r}_{3,a} \). Then the following \( U \) is a set of representatives of \( \mathfrak{P} \mathfrak{M} \):
\[
U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}.
\] (3.22)

Proof. Take any \( g \in \text{GL}_3(\mathbb{R}) \). By Lemma 3.3, we have only to show that there exists \( \lambda \in \mathbb{R} \) such that
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \in [[g]].
\] (3.23)

By (3.21) and Lemma 3.7, there exists \( g' \in L' \) such that \( g' \in [[g]] \). Since \( g' \in L' \), one can write
\[
g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad a_{33} > 0.
\] (3.24)

It follows from (3.21) that
\[
\varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a_{33} \end{pmatrix} \in (\mathbb{R}^\times \text{Aut}(g))^0.
\] (3.25)

This yields that
\[
[[g]] \ni \varphi g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32}/a_{33} & 1 \end{pmatrix}.
\] (3.26)

Therefore, by putting \( \lambda := a_{32}/a_{33} \), we complete the proof. \( \square \)

3.5. Case of \( g = \mathfrak{r}_{3,a}' \, (a \geq 0) \). In this subsection, we give an explicit expression of \( \mathfrak{P} \mathfrak{M} \) for \( g = \mathfrak{r}_{3,a}' \). Throughout this subsection, we fix \( a \) satisfying \( a \geq 0 \), and a basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{r}_{3,a}' \) whose bracket relations are given by
\[
[e_1, e_2] = a e_2 - e_3, \quad [e_1, e_3] = e_2 + a e_3.
\] (3.27)

From Lemma 3.6, we have
\[
\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}.
\] (3.28)

This yields that we can also apply Lemma 3.7 for this case.
Proposition 3.10. Let $g = r_{3,a}^j$. Then the following $U$ is a set of representatives of $\mathcal{M}$:

\[ U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda \geq 1 \right\}. \]  

(3.29)

Proof. Take any $g \in \text{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $\lambda \geq 1$ such that

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \in [[g]]. \]  

(3.30)

By (3.28), one can see that $(\mathbb{R} \times \text{Aut}(g))^0$ contains $F$ defined by (3.6). Hence, by Lemma 3.7, there exists $g' \in L'$ such that $g' \in [[g]]$. Since $g' \in L'$, one can write

\[ g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad a_{33} > 0. \]  

(3.31)

Then, from (3.28), one has

\[ R(\theta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \in (\mathbb{R} \times \text{Aut}(g))^0. \]  

(3.32)

It follows from linear algebra (or the theory of Cartan decomposition) that

\[ \text{GL}_2(\mathbb{R}) = \text{SO}(2) \cdot \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x \geq y > 0 \right\} \cdot \text{O}(2). \]  

(3.33)

This yields that there exist $\theta \in \mathbb{R}$ and $k \in \text{O}(3)$ such that

\[ [[g]] \ni R(\theta)g'k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} =: g'', \quad x \geq y > 0 \]  

(3.34)

By using (3.28) again, one has

\[ \varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{pmatrix} \in (\mathbb{R} \times \text{Aut}(g))^0. \]  

(3.35)

This yields that

\[ [[g]] \ni \varphi g'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y/x \end{pmatrix}. \]  

(3.36)

Therefore, by putting $\lambda := x/y \geq 1$, we complete the proof. □
4. THREE-DIMENSIONAL SOLVSOLITONS

In this section, we give a Milnor-type theorem for each three-dimensional solvable Lie algebra \( g \), and apply it to determine which points in the moduli space \( \mathcal{P}M \) are solvsolitons. Note that a classification of three-dimensional solvsolitons has already been obtained by Lauret ([17]), but we here reprove it, since Milnor-type theorems itself and their application would be interesting.

4.1. Preliminaries on curvatures. In this subsection, we recall the notion of solvsolitons introduced by Lauret ([17]), and study the Ricci operators of three-dimensional solvable Lie algebras. Note that we discuss everything on a metric Lie algebra \( (g, \langle \cdot, \cdot \rangle) \), instead of the simply-connected Lie group with Lie algebra \( g \) equipped with the corresponding left-invariant Riemannian metric.

**Definition 4.1.** An inner product \( \langle \cdot, \cdot \rangle \) on a solvable Lie algebra \( g \) is called a solvsoliton if it satisfies

\[
\text{Ric}_{\langle \cdot, \cdot \rangle} \in \mathbb{R} \oplus \text{Der}(g),
\]

where \( \text{Ric}_{\langle \cdot, \cdot \rangle} \) is the Ricci operator of \( \langle \cdot, \cdot \rangle \). If \( g \) is nilpotent, then a solvsoliton on \( g \) is called a nilsoliton.

Here we recall the definition of the Ricci operator of \( (g, \langle \cdot, \cdot \rangle) \). First of all, the Levi-Civita connection \( \nabla : g \times g \to g \) is given by

\[
2\langle \nabla_X Y, Z \rangle = \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle + \langle X, Y \rangle \cdot [Z, X],
\]

The Riemannian curvature \( R \) is defined by

\[
R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.
\]

Let \( \{e_i\} \) be an orthonormal basis of \( g \) with respect to \( \langle \cdot, \cdot \rangle \). The Ricci operator \( \text{Ric}_{\langle \cdot, \cdot \rangle} : g \to g \) is defined by

\[
\text{Ric}_{\langle \cdot, \cdot \rangle}(X) := \sum R(X, e_i)e_i.
\]

Let us consider the equivalence relation, isometry and scaling in the sense of Definition 2.1. Recall that \([\cdot, \cdot]\) denotes the equivalence class of \( \langle \cdot, \cdot \rangle \). Then it is easy to see the following.

**Proposition 4.2.** Let \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle' \) be inner products on a solvable Lie algebra \( g \), and assume that \([\langle \cdot, \cdot \rangle] = [\langle \cdot, \cdot \rangle']\). If \( \langle \cdot, \cdot \rangle \) is a solvsoliton, then so is \( \langle \cdot, \cdot \rangle' \).

This proposition is an easy observation, but has an important conclusion. That is, it is enough to consider \( \mathcal{P}M \) to examine whether \( g \) admits a solvsoliton or not.

**Remark 4.3.** It is worthwhile to mention that the uniqueness of solvsolitons holds. That is, if \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle' \) are solvsolitons on a solvable Lie algebra \( g \), then \([\langle \cdot, \cdot \rangle] = [\langle \cdot, \cdot \rangle']\) holds. This follows from the proof of [17] Theorem 5.1. But, we will not use this in the latter arguments. In particular, for solvsolitons on three-dimensional solvable Lie algebras, the uniqueness can be directly seen from our classification.
At the end of this subsection, we calculate the Ricci curvatures of three-dimensional solvable Lie algebras in a unified way.

**Lemma 4.4.** Let $\mathfrak{g}$ be a three-dimensional solvable Lie algebra, and $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$. Suppose that there exist $a, b, c, d \in \mathbb{R}$ and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $\langle \cdot, \cdot \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = ax_2 + bx_3, \quad [x_1, x_3] = cx_2 + dx_3.$$

Then, the Ricci operator satisfies

$$\text{Ric}_{\langle \cdot, \cdot \rangle}(x_i) = \begin{cases} -(a^2 + b^2 + \frac{1}{2}(b + c)^2) x_1 & (i = 1), \\
-(a(a + d) + (1/2)(b^2 - c^2)) x_2 - (ac + bd) x_3 & (i = 2), \\
-(ac + bd) x_2 - (d(a + d) - (1/2)(b^2 - c^2)) x_3 & (i = 3). \end{cases}$$

**Proof.** First of all, we calculate the Levi-Civita connection $\nabla$. A direct calculation shows that

$$\nabla_{x_i} x_1 = 0, \quad \nabla_{x_i} x_2 = ax_1, \quad \nabla_{x_i} x_3 = dx_1.$$

In order to calculate the other components, we use $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$2\langle U(X, Y), Z \rangle = \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle$$

for every $X, Y, Z \in \mathfrak{g}$. One can easily calculate that

$$U(x_1, x_2) = -(a/2) x_2 - (c/2) x_3.$$

Note that $U$ is symmetric. Hence, one obtains that

$$\begin{align*}
\nabla_{x_1} x_2 &= (1/2)[x_1, x_2] + U(x_1, x_2) = ((b - c)/2) x_3, \\
\nabla_{x_2} x_1 &= (1/2)[x_2, x_1] + U(x_2, x_1) = -ax_2 - ((b + c)/2) x_3.
\end{align*}$$

By changing the roles of $x_2$ and $x_3$, we also have

$$\begin{align*}
\nabla_{x_1} x_3 &= ((c - b)/2) x_2, \\
\nabla_{x_3} x_1 &= -dx_3 - ((b + c)/2) x_2.
\end{align*}$$

A similar calculation shows that $U(x_2, x_3) = ((b + c)/2) x_1$, which concludes

$$\begin{align*}
\nabla_{x_2} x_3 &= ((b + c)/2) x_1, \\
\nabla_{x_3} x_2 &= ((b + c)/2) x_1.
\end{align*}$$

One can thus calculate the Riemannian curvatures $R$. The above calculations of $\nabla$ yield that

$$\begin{align*}
R(x_1, x_2)x_2 &= -(a^2 + (3/4)b^2 - (1/4)c^2 + (1/2)bc)x_1, \\
R(x_1, x_3)x_3 &= -(-(1/4)b^2 + (3/4)c^2 + d^2 + (1/2)bc)x_1.
\end{align*}$$
By summing up them, we obtain the Ricci curvature $\text{Ric}_{\langle \cdot \rangle}(x_1)$. Similarly, one can obtain $\text{Ric}_{\langle \cdot \rangle}(x_2)$ and $\text{Ric}_{\langle \cdot \rangle}(x_3)$ by

\[
R(x_2, x_1)x_1 = -(a^2 + (3/4)b^2 - (1/4)c^2 + (1/2)bc)x_2 - (ac + bd)x_3,
\]

\[
R(x_2, x_3)x_3 = ((1/4)b^2 + (1/4)c^2 - ad + (1/2)bc)x_2,
\]

\[
R(x_3, x_1)x_1 = -(ac + bd)x_2 - (- (1/4)b^2 + (3/4)c^2 + d^2 + (1/2)bc)x_3,
\]

\[
R(x_3, x_2)x_2 = ((1/4)b^2 + (1/4)c^2 - ad + (1/2)bc)x_3.
\]

This completes the proof of the lemma. □

Lemma 4.4 is a slight generalization of some known results. In fact, when $a + d \neq 0$ and $ac + bd = 0$, the Ricci operators were calculated by Milnor ([19, Lemma 6.5]). Note that the Ricci operators are diagonal in this case. Han and Lee ([5]) also calculated the Ricci operators in some cases, which essentially correspond to the case of $a = 0$.

4.2. Preliminaries on Milnor-type theorems. In this subsection, we recall a method for studying all inner products on a given Lie algebra $\mathfrak{g}$. This method is called a Milnor-type theorem in [6], since it generalizes the famous theorem by Milnor ([19]).

**Theorem 4.5.** Let $U$ be a set of representatives of $\mathfrak{P}\mathfrak{M}$. Then, for every inner product $\langle \cdot \rangle$ on $\mathfrak{g}$, we have the following:

1. There exist $h \in U$, $\varphi \in \text{Aut}(\mathfrak{g})$, and $k > 0$ such that $\{\varphi he_1, \ldots, \varphi he_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $k(\cdot)$.

2. The matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{\varphi he_1, \ldots, \varphi he_n\}$ coincides with

\[
\{h^{-1} Dh \in \text{GL}_n(\mathbb{R}) \mid D \in \text{Der}(\mathfrak{g})\}.
\]

**Proof.** The first assertion has been proved in [6]. We show the second assertion. One has that $\{\varphi he_1, \ldots, \varphi he_n\}$ and $\{he_1, \ldots, he_n\}$ have the same bracket relations, since $\varphi \in \text{Aut}(\mathfrak{g})$. This yields that the matrix expressions of $\text{Der}(\mathfrak{g})$ with respect to these two bases are the same. Furthermore, the latter basis and $\{e_1, \ldots, e_n\}$ are related by

\[
(he_1, \ldots, he_n) = (e_1, \ldots, e_n)h.
\]

Therefore, an elementary linear algebra shows that the matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{he_1, \ldots, he_n\}$ coincides with the one in the second assertion. This completes the proof. □

By applying this theorem for a given Lie algebra $\mathfrak{g}$, we can obtain a Milnor-type theorem. More precisely, the basis $\{\varphi he_1, \ldots, \varphi he_n\}$ plays a similar role to the Milnor frames. Note that the bracket relations among elements of this basis depend only on $h \in U$, since $\varphi$ preserves the bracket product.
In the following subsections, we will study the existence of solvsolitons on threedimensional solvable Lie algebras. Note that we can omit the cases of $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{r}_{3,1}$, because of Remark 3.5.

### 4.3. Case of $\mathfrak{g} = \mathfrak{r}_3$

In this subsection, we prove that $\mathfrak{g} = \mathfrak{r}_3$ does not admit solvsolitons. The main tool is the following Milnor-type theorem.

**Proposition 4.6.** For every inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{r}_3$, there exist $\lambda > 0$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle \cdot, \cdot \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = x_2 + \lambda x_3, \quad [x_1, x_3] = x_3. \quad (4.11)$$

Furthermore, the matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$ coincides with

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}. \quad (4.15)$$

**Proof.** Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathfrak{r}_3$. Recall that the bracket relations are given by

$$[e_1, e_2] = e_2 + e_3, \quad [e_1, e_3] = e_3. \quad (4.12)$$

We have proved in Proposition 3.8 that the following $U$ is a set of representatives of $\mathfrak{M}$:

$$U := \left\{ g_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda > 0 \right\}. \quad (4.13)$$

Take any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. By Theorem 4.5, there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\{\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3\}$ is orthonormal with respect to $k\langle \cdot, \cdot \rangle$.

Put $x_i := \varphi g_\lambda e_i$ for $i = 1, 2, 3$. We calculate the bracket relations among them. One has

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = (1/\lambda)e_3. \quad (4.14)$$

We thus obtain

$$[g_\lambda e_1, g_\lambda e_2] = [e_1, e_2] = e_2 + e_3 = g_\lambda e_2 + \lambda g_\lambda e_3,$n
$$[g_\lambda e_1, g_\lambda e_3] = [e_1, (1/\lambda)e_3] = (1/\lambda)e_3 = g_\lambda e_3,$n
$$[g_\lambda e_2, g_\lambda e_3] = [e_2, (1/\lambda)e_3] = 0. \quad (4.15)$$

Therefore, by applying $\varphi \in \text{Aut}(\mathfrak{g})$ to the both sides of these equations, we obtain

$$[x_1, x_2] = [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \varphi [g_\lambda e_1, g_\lambda e_2] = x_2 + \lambda x_3,$n
$$[x_1, x_3] = [\varphi g_\lambda e_1, \varphi g_\lambda e_3] = \varphi [g_\lambda e_1, g_\lambda e_3] = x_3,$n
$$[x_2, x_3] = [\varphi g_\lambda e_2, \varphi g_\lambda e_3] = \varphi [g_\lambda e_2, g_\lambda e_3] = 0. \quad (4.16)$$
This completes the proof of the first assertion. We show the second assertion.

Lemma 3.6 yields that, for every $D \in \text{Der}(\mathfrak{g})$, the matrix expression of $D$ with respect to $\{e_1, e_2, e_3\}$ is given by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix}. \tag{4.17}$$

A direct calculation shows that

$$g_{\lambda}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} g_{\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ \lambda x_{31} & \lambda x_{32} & x_{22} \end{pmatrix}. \tag{4.18}$$

Note that $\lambda x_{31}$ and $\lambda x_{32}$ can take any real numbers, and are independent of the other components. Therefore, by Theorem 4.5 (2), one can obtain the matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$. This completes the proof of the second assertion. □

By applying the Milnor-type theorem, Proposition 4.6, we prove that $\mathfrak{r}_3$ does not admit solvsolitons.

**Proposition 4.7.** The Lie algebra $\mathfrak{g} = \mathfrak{r}_3$ does not admit solvsolitons.

**Proof.** Take any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. We show that this is not a solvsoliton. By Proposition 4.6 there exist $\lambda > 0$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle \cdot, \cdot \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = x_2 + \lambda x_3, \quad [x_1, x_3] = x_3. \tag{4.19}$$

We can assume $k = 1$ without loss of generality, since solvsolitons are preserved by scaling. Then, from Lemma 4.4 the matrix expression of $\text{Ric}_{\langle \cdot, \cdot \rangle}$ with respect to the orthonormal basis $\{x_1, x_2, x_3\}$ is given by

$$\text{Ric}_{\langle \cdot, \cdot \rangle} = -\begin{pmatrix} 2 + (\lambda^2/2) & 0 & 0 \\ 0 & 2 + (\lambda^2/2) & \lambda \\ 0 & \lambda & 2 - (\lambda^2/2) \end{pmatrix}. \tag{4.20}$$

On the other hand, by Proposition 4.6 one knows the matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$. By looking at the $(2, 3)$-component, we have

$$\text{Ric}_{\langle \cdot, \cdot \rangle} \not\in \mathbb{R} \oplus \text{Der}(\mathfrak{g}). \tag{4.21}$$

This proves that $\langle \cdot, \cdot \rangle$ is not a solvsoliton. □

### 4.4. Case of $\mathfrak{g} = \mathfrak{r}_{3,a}$ ($-1 \leq a < 1$).

In this subsection, we classify solvsolitons on $\mathfrak{g} = \mathfrak{r}_{3,a}$. Throughout this subsection, we fix $a$ satisfying $-1 \leq a < 1$. Recall that, for the canonical basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}_{3,a}$, the bracket relations are given by

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = ae_3. \tag{4.22}$$
Proposition 4.8. For every inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{r}_{3,a}$, there exist $\lambda \in \mathbb{R}$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle \cdot, \cdot \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = x_2 + \lambda(a - 1)x_3, \quad [x_1, x_3] = ax_3.$$ 

Furthermore, the matrix expression of $\text{Det}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$ coincides with

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.$$ 

Proof. The proof is similar to that of Proposition 4.6. Take any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{r}_{3,a}$. By Proposition 3.9 the following $U$ is a set of representatives of $\mathfrak{F}_M$:

$$U := \left\{ g_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}.$$ 

By Theorem 4.5 there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \text{Aut}(\mathfrak{g})$ such that

$$x_1, x_2, x_3 := (\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3)$$
forms an orthonormal basis with respect to $k\langle \cdot, \cdot \rangle$. We have only to check the bracket relations. By definition, we have

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2 + \lambda e_3, \quad g_\lambda e_3 = e_3.$$ 

One can thus calculate that

$$[g_\lambda e_1, g_\lambda e_2] = [e_1, e_2 + \lambda e_3] = e_2 + a\lambda e_3 = (g_\lambda e_2 - \lambda g_\lambda e_3) + a\lambda g_\lambda e_3$$

$$= g_\lambda e_2 + \lambda(a - 1)e_3,$$

$$[g_\lambda e_1, g_\lambda e_3] = [e_1, e_3] = ae_3 = a g_\lambda e_3,$$

$$[g_\lambda e_2, g_\lambda e_3] = [e_2 + \lambda e_3, e_3] = 0.$$ 

By applying $\varphi \in \text{Aut}(\mathfrak{g})$, one completes the proof of the first assertion. The second assertion follows from Lemma 3.6 and Theorem 4.5. In fact, one has

$$g_\lambda^{-1} \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} g_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ -\lambda x_{21} + x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix}.$$ 

This completes the proof, since $-\lambda x_{21} + x_{31}$ can take any real number and is independent of the other components. \hfill \Box

By applying the Milnor-type theorem, Proposition 4.8 one can classify solvsolitons on $\mathfrak{g} = \mathfrak{r}_{3,a}$. Recall that $\langle \cdot, \cdot \rangle_0$ is the inner product on $\mathfrak{g}$ so that the canonical basis $\{e_1, e_2, e_3\}$ is orthonormal.

Proposition 4.9. An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{r}_{3,a}$ is a solvsoliton if and only if $[\langle \cdot, \cdot \rangle] = [\langle \cdot, \cdot \rangle_0]$. 

Proof. First of all, we show the “if”-part. We have only to show that $\langle , \rangle_0$ is a solvsoliton. By Lemma 4.4, one knows

\[
\text{Ric}_{\langle , \rangle_0} = -\begin{pmatrix} 1 + a^2 & 0 & 0 \\ 0 & 1 + a & 0 \\ 0 & 0 & a(1 + a) \end{pmatrix},
\]

One also knows by Lemma 3.6 that

\[
\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.
\]

Then we have $\text{Ric}_{\langle , \rangle_0} \in \mathbb{R} \oplus \text{Der}(g)$, that is, $\langle , \rangle_0$ is a solvsoliton.

We show the “only if”-part. Take any inner product $\langle , \rangle$ on $g = \mathfrak{r}_{3,a}$, and assume that it is a solvsoliton. Proposition 4.8 yields that there exist $\lambda \in \mathbb{R}$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by

\[
[x_1, x_2] = x_2 + \lambda(a - 1)x_3, \quad [x_1, x_3] = ax_3.
\]

We can assume $k = 1$ without loss of generality. Hence $\{x_1, x_2, x_3\}$ is orthonormal. For simplicity of the notation, we put

\[
T := (1/2)\lambda^2(a - 1)^2.
\]

Then, from Lemma 4.4 one obtains the matrix expressions of $\text{Ric}_{\langle , \rangle}$ with respect to the basis $\{x_1, x_2, x_3\}$ as follows:

\[
\text{Ric}_{\langle , \rangle} = -\begin{pmatrix} 1 + a^2 + T & 0 & 0 \\ 0 & 1 + a + T & \lambda(a - 1) \\ 0 & \lambda(a - 1) & a + a^2 - T \end{pmatrix}.
\]

On the other hand, Proposition 4.8 gives the matrix expression with respect to $\{x_1, x_2, x_3\}$ as follows:

\[
\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix} \right\}.
\]

We here claim that $\lambda = 0$. Recall that $\langle , \rangle$ is a solvsoliton. Hence, by looking at the $(2, 3)$-component, we have

\[
\lambda a(a - 1) = 0.
\]

Assume that $\lambda \neq 0$. Since $-1 \leq a < 1$, one has $a = 0$. Then, by looking at the $(3, 2)$-component, we have

\[
0 = \lambda(-T - (1 + T)) = \lambda(-1 - \lambda^2) \neq 0.
\]

This is a contradiction, which shows the claim.
For every inner product $(\langle \cdot, \cdot \rangle_0)$, the proof is similar to that of Proposition (4.37).

Proof. Proposition 4.10. If $F$ is a linear map $F : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying
\begin{equation}
F(e_i) = x_i \quad (i = 1, 2, 3)
\end{equation}
gives an isometry from $(\mathfrak{g}, \langle \cdot, \cdot \rangle_0)$ onto $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. This proves $[\langle \cdot, \cdot \rangle] = [\langle \cdot, \cdot \rangle_0]$. \hfill \Box

4.5. Case of $\mathfrak{g} = \mathfrak{r}'_{3,a}$ $(a \geq 0)$. In this subsection, we classify solvsolitons on $\mathfrak{g} = \mathfrak{r}'_{3,a}$. Throughout this subsection, we fix $a$ satisfying $a \geq 0$. Recall that, for the canonical basis $\{e_1, e_2, e_3\}$, the bracket relations are given by
\begin{equation}
[e_1, e_2] = a e_2 - e_3, \quad [e_1, e_3] = e_2 + a e_3.
\end{equation}

Proposition 4.10. For every inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{r}'_{3,a}$, there exist $\lambda \geq 1$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k \langle \cdot, \cdot \rangle$ such that the bracket relations are given by
\begin{equation}
[x_1, x_2] = ax_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda)x_2 + ax_3.
\end{equation}
Furthermore, the matrix expression of $\text{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$ coincides with
\begin{equation}
\left\{ \begin{pmatrix}
0 & 0 & 0 \\
x_{21} & x_{22} & x_{23} \\
x_{31} & -\lambda^2 x_{23} & x_{22}
\end{pmatrix} \mid x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}.
\end{equation}

Proof. The proof is similar to that of Proposition 4.6. Take any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{r}'_{3,a}$. By Proposition 3.10, the following $U$ is a set of representatives of $\mathfrak{PM}$:
\begin{equation}
U := \left\{ g_\lambda := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/\lambda
\end{pmatrix} \mid \lambda \geq 1 \right\}.
\end{equation}

By Theorem 4.5, there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \text{Aut}(\mathfrak{g})$ such that
\begin{equation}
(x_1, x_2, x_3) := (\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3)
\end{equation}
forms an orthonormal basis with respect to $k \langle \cdot, \cdot \rangle$. We have only to check the bracket relations. By definition, we have
\begin{equation}
g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = (1/\lambda)e_3.
\end{equation}
One can thus calculate that
\begin{align}
[g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2] = ae_2 - e_3 = ag_\lambda e_2 - \lambda g_\lambda e_3, \\
[g_\lambda e_1, g_\lambda e_3] &= [e_1, (1/\lambda)e_3] = (1/\lambda)(e_2 + ae_3) = (1/\lambda)g_\lambda e_2 + ag_\lambda e_3, \\
[g_\lambda e_2, g_\lambda e_3] &= [e_2, (1/\lambda)e_3] = 0.
\end{align}
By applying $\varphi \in \text{Aut}(g)$, one completes the proof of the first assertion. The second assertion follows from Lemma 3.6 and Theorem 4.5. In fact, one has

$$g^{-1}_\lambda \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} g_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -(1/\lambda)x_{23} \\ \lambda x_{31} & \lambda x_{23} & x_{22} \end{pmatrix}.$$  

This completes the proof by changing $\lambda x_{31}$ to $x_{31}$, and $-(1/\lambda)x_{23}$ to $x_{23}$. □

By applying the Milnor-type theorem, Proposition 4.10, one can classify solvsolitons on $g = r_{3,a}'$. In fact, this admits a left-invariant Einstein metric. Recall that $\langle \cdot, \cdot \rangle_0$ is the inner product so that the canonical basis $\{e_1, e_2, e_3\}$ is orthonormal.

**Proposition 4.11.** An inner product $\langle \cdot, \cdot \rangle$ on $g = r_{3,a}'$ is a solvsoliton if and only if $[\langle \cdot, \cdot \rangle] = [\langle \cdot, \cdot \rangle_0]$. In fact, $\langle \cdot, \cdot \rangle_0$ is Einstein.

**Proof.** The proof is similar to that of Proposition 4.9. First of all, we show the “if”-part. By Lemma 4.4, one knows

$$\text{Ric}(\langle \cdot, \cdot \rangle_0) = -\begin{pmatrix} 2a^2 & 0 & 0 \\ 0 & 2a^2 & 0 \\ 0 & 0 & 2a^2 \end{pmatrix}.$$  

This shows that $\langle \cdot, \cdot \rangle_0$ is Einstein, and hence a solvsoliton.

We show the “only if”-part. Take any inner product $\langle \cdot, \cdot \rangle$ on $g = r_{3,a}'$, and assume that it is a solvsoliton. Proposition 4.10 yields that there exist $\lambda \geq 1$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle \cdot, \cdot \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = ax_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda)x_2 + ax_3.$$  

We can assume $k = 1$ without loss of generality. Hence $\{x_1, x_2, x_3\}$ is orthonormal. For simplicity of the notation, we put

$$S := \lambda - (1/\lambda).$$  

Then, from Lemma 4.4, one obtains the matrix expressions of $\text{Ric}(\langle \cdot, \cdot \rangle)$ with respect to the basis $\{x_1, x_2, x_3\}$ as follows:

$$\text{Ric}(\langle \cdot, \cdot \rangle) = -\frac{1}{2} \begin{pmatrix} 4a^2 + S^2 & 0 & 0 \\ 0 & 4a^2 + (\lambda^2 - (1/\lambda)^2) & -2aS \\ 0 & -2aS & 4a^2 - (\lambda^2 - (1/\lambda)^2) \end{pmatrix}.$$  

On the other hand, Proposition 4.10 gives the matrix expression with respect to $\{x_1, x_2, x_3\}$ as follows:

$$\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & -\lambda^2 x_{23} & x_{22} \end{pmatrix} \right\}.$$  

We here show that $\lambda = 1$. Recall that $\langle, \rangle$ is a solvsoliton. Hence, By looking at the $(2,2)$ and $(3,3)$-components, we have

$$4a^2 + (\lambda^2 - (1/\lambda)^2) = 4a^2 - (\lambda^2 - (1/\lambda)^2).$$

Since $\lambda \geq 1$, this yields that

$$\lambda = 1.$$  \hspace{1cm} (4.50)

Since $\lambda = 1$, one can see that \{e_1, e_2, e_3\} and \{x_1, x_2, x_3\} have the same bracket relations. Thus, a linear map $F : g \rightarrow g$ satisfying

$$F(e_i) = x_i \ (i = 1, 2, 3)$$

(4.51)

gives an isometry from $(g, \langle, \rangle_0)$ onto $(g, \langle, \rangle)$. This proves $[\langle, \rangle] = [\langle, \rangle_0]$. \hspace{1cm} □

5. The minimality of the corresponding submanifolds

In this section, we study the actions of $\mathbb{R}^\times \text{Aut}(g)$ and examine the minimality of its orbits, the corresponding submanifolds to left-invariant metrics. After some necessary preliminaries in Subsection 5.1, we study the cases of $g = r_3, r_{3,a}$ ($-1 \leq a < 1$), and $r_{3,a}$ ($a \geq 0$) in Subsections 5.2, 5.3, and 5.4, respectively. We have only to study these cases, since the actions of $\mathbb{R}^\times \text{Aut}(g)$ is transitive for the remaining cases $g = h_3$ and $r_{3,1}$.

5.1. Preliminary. In this subsection, we review some of the standard facts on reductive homogeneous spaces and homogeneous submanifolds. We refer to \cite{13}.

Let $U/K$ be a reductive homogeneous space with a reductive decomposition

$$u = \mathfrak{k} \oplus \mathfrak{m}.$$ \hspace{1cm} (5.1)

As in Subsection 2.1, denote by $\pi : U \rightarrow U/K$ the natural projection, and by $o := \pi(e)$ the origin of $U/K$. We identify $\mathfrak{m}$ with the tangent space $T_o(U/K)$ at $o$ by

$$d\pi_e|_{\mathfrak{m}} : \mathfrak{m} \rightarrow T_o(U/K).$$ \hspace{1cm} (5.2)

In the following, we equip a $U$-invariant Riemannian metric $g$ on $U/K$.

We here recall a formula for the Levi-Civita connection $\nabla$ of $g$. For any $X \in u$, we define the fundamental vector field $X^* \in U/K$ by

$$X^*_p = \frac{d}{dt}(\exp tX).p|_{t=0} \quad (\text{for } p \in U/K).$$ \hspace{1cm} (5.3)

Let $X, Y, Z \in u$. Then one knows

$$X^*_o = d\pi_e(X),$$ \hspace{1cm} (5.4)

$$[X^*, Y^*] = -[X, Y]^*,$$ \hspace{1cm} (5.5)

$$2g(\nabla_X Y^*, Z^*) = g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*) + g(X^*, [Y^*, Z^*]).$$ \hspace{1cm} (5.6)

We now consider homogeneous submanifolds in $(U/K, g)$. Let $U'$ be a Lie subgroup of $U$, and consider the orbit $U'.o$ through the origin $o$. Let $u'$ be the
Lie algebra of $U'$, and denote by $\langle , \rangle$ the inner product on $m$ corresponding to $g$. We define

\begin{equation}
(5.7) \quad m' := d\pi_e(u') \cong T_o(U'.o).
\end{equation}

Denote by $m \ominus m'$ the orthogonal complement of $m'$ in $m$ with respect to $\langle , \rangle$. Then, the second fundamental form $h : m' \times m' \to m \ominus m'$ of $U'.o$ at $o$ is defined by

\begin{equation}
(5.8) \quad h(X_\circ^*, Y_\circ^*) := \langle \nabla X \circ^* - \nabla' X \circ^*, Y_\circ^* \rangle \quad \text{for } X, Y \in u',
\end{equation}

where $\nabla'$ is the Levi-Civita connection of $U'.o$ with respect to the induced metric. Take $Z \in u$ satisfying $Z_\circ^* \in m \ominus m'$. From (5.5) and (5.6), one obtains

\begin{equation}
(5.9) \quad 2\langle h(X_\circ^*, Y_\circ^*), Z_\circ^* \rangle = \langle [Z, X_\circ^*], Y_\circ^* \rangle + \langle X_\circ^*, [Z, Y_\circ^*] \rangle.
\end{equation}

The mean curvature vector of $U'.o$ at $o$ is defined by

\begin{equation}
(5.10) \quad H := -(1/k)\text{tr}(h) = -(1/k)\sum h(E'_i, E'_i),
\end{equation}

where $\{E'_i\}$ is an orthonormal basis of $m'$, and $k$ is the dimension of $U'.o$. We call $U'.o$ minimal if its mean curvature vector is equal to zero.

In the following subsections, we will calculate the mean curvature vectors of the corresponding submanifolds in $\text{GL}_3(\mathbb{R})/O(3)$ with respect to the natural Riemannian metric (see Section 2). We will frequently use

\begin{equation}
(5.11) \quad d\pi_e : \mathfrak{gl}_3(\mathbb{R}) \to \text{sym}(3) : X \mapsto (1/2)(X + X^t).
\end{equation}

5.2. Case of $g = \mathfrak{r}_3$. In this subsection, we study the case of $g = \mathfrak{r}_3$. First of all, by direct calculations, one has

\begin{equation}
(5.12) \quad \text{Aut}(g) = \left\{ \begin{pmatrix}
1 & 0 & 0 \\
x_{21} & x_{22} & 0 \\
x_{31} & x_{32} & x_{22}
\end{pmatrix} \middle| x_{22} \neq 0 \right\}.
\end{equation}

This easily yields that

\begin{equation}
(5.13) \quad \mathbb{R}^3 \text{Aut}(g) = \left\{ \begin{pmatrix}
x_{11} & 0 & 0 \\
x_{21} & x_{22} & 0 \\
x_{31} & x_{32} & x_{22}
\end{pmatrix} \middle| x_{11}, x_{22} \neq 0 \right\}.
\end{equation}

From Proposition 3.8 the expression of $\mathfrak{P}M$ is given as follows:

\begin{equation}
(5.14) \quad \mathfrak{P}M = \left\{ [g_\lambda, \cdot] \mid g_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}, \lambda > 0 \right\}.
\end{equation}

For any $\lambda > 0$, one can see that

\begin{equation}
(5.15) \quad g_\lambda^{-1}(\mathbb{R}^3 \text{Aut}(g))g_\lambda = \mathbb{R}^3 \text{Aut}(g).
\end{equation}

This is an easy observation, but very important to get the following lemma.
Lemma 5.1. Let $g = v_3$. Then the action of $\mathbb{R}^\times \text{Aut}(g)$ is of cohomogeneity one, and all orbits are isometrically congruent to each other.

Proof. In order to prove that the action of $\mathbb{R}^\times \text{Aut}(g)$ is of cohomogeneity one, it is enough to show that the orbit through $\langle , \rangle_0$ is of codimension one. From (5.13), it is easy to see that

$$\dim \mathbb{R}^\times \text{Aut}(g) = 5, \quad \dim(\mathbb{R}^\times \text{Aut}(g) \cap O(3)) = 0.$$  \tag{5.16}$$

Therefore $\mathbb{R}^\times \text{Aut}(g).\langle , \rangle_0$ has dimension 5. This completes the proof, since the ambient space $\text{GL}_3(\mathbb{R})/O(3)$ has dimension 6.

Next we prove that all orbits are isometrically congruent to each other. Take any $\langle , \rangle$ and $\langle , \rangle'$. By Proposition 3.8, there exist $\lambda, \lambda' > 0$ such that

$$\mathbb{R}^\times \text{Aut}(g).\langle , \rangle = \mathbb{R}^\times \text{Aut}(g).\langle g_\lambda.\langle , \rangle_0 \rangle,$$  \tag{5.17}$$

$$\mathbb{R}^\times \text{Aut}(g).\langle , \rangle' = \mathbb{R}^\times \text{Aut}(g).\langle g_{\lambda'}.\langle , \rangle_0 \rangle.$$  \tag{5.18}$$

We put $\mu := \lambda'/\lambda > 0$, and take $g_\mu \in \text{GL}_3(\mathbb{R})$. Then (5.15) yields that

$$g_\mu \mathbb{R}^\times \text{Aut}(g).\langle , \rangle = g_\mu \mathbb{R}^\times \text{Aut}(g).\langle g_\lambda.\langle , \rangle_0 \rangle$$
$$= g_\mu (g_\mu^{-1}\mathbb{R}^\times \text{Aut}(g)g_\mu).\langle g_\lambda.\langle , \rangle_0 \rangle$$
$$= \mathbb{R}^\times \text{Aut}(g).\langle g_{\lambda'}.\langle , \rangle_0 \rangle$$
$$= \mathbb{R}^\times \text{Aut}(g).\langle , \rangle'.$$  \tag{5.19}$$

Thus $g_\mu$ maps the first orbit onto the second one, which completes the proof. \qed

We refer to [12] for actions all of whose orbits are isometrically congruent to each other. Our idea of the proof of Lemma 5.1 comes from the arguments in [12].

Proposition 5.2. Let $g = v_3$. Then the action of $\mathbb{R}^\times \text{Aut}(g)$ on $\text{GL}_3(\mathbb{R})/O(3)$ has no minimal orbits.

Proof. Consider the action of $\mathbb{R}^\times \text{Aut}(g)$ on $\text{GL}_3(\mathbb{R})/O(3)$. From Lemma 5.1 all orbits are isometrically congruent to each other. Thus it is sufficient to prove that the orbit through the origin $\langle , \rangle_0$ is not minimal. We calculate the mean curvature vector of $\mathbb{R}^\times \text{Aut}(g).\langle , \rangle_0$. One can see from (3.12) that

$$u' := \mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \right\},$$  \tag{5.20}$$

$$m' := d\pi_e(u') = \left\{ \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{21} & x_{22} & x_{32} \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \right\}.$$  \tag{5.21}$$
Let us denote by $E_{ij}$ the matrix whose $(i, j)$-entry is 1 and others are 0. We define a basis $\{X_1, \ldots, X_5\}$ of $u'$ by
\begin{equation}
X_1 := E_{11}, \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}),
\end{equation}
\begin{equation}
X_3 := \sqrt{2}E_{21}, \quad X_4 := \sqrt{2}E_{31}, \quad X_5 := \sqrt{2}E_{32}.
\end{equation}
Furthermore we put
\begin{equation}
X_i' := (X_i)_o^* = (1/2)(X_i + tX_i), \quad A := (1/\sqrt{2})(E_{22} - E_{33}).
\end{equation}
Then $\{X_1', \ldots, X_5'\}$ is an orthonormal basis of $m'$, and $\{A\}$ is an orthonormal basis of $m \oplus m'$. Recall that the mean curvature vector $H$ is given by
\begin{equation}
H = -(1/5) \sum h(X_i', X_i'), \quad \langle h(X_i', X_i'), A \rangle = \langle [A, X_i]_o, (X_i)_o^* \rangle.
\end{equation}
The bracket products $[A, X_i]$ satisfy
\begin{equation}
[A, X_1] = [A, X_2] = 0, \quad [A, X_3] = E_{21}, \quad [A, X_4] = -E_{31}, \quad [A, X_5] = -2E_{32}.
\end{equation}
Therefore, one has
\begin{equation}
\langle [A, X_3]_o, (X_3)_o^* \rangle = \langle (1/2)(E_{21} + E_{12}), (\sqrt{2}/2)(E_{21} + E_{12}) \rangle = \sqrt{2}/2,
\end{equation}
\begin{equation}
\langle [A, X_4]_o, (X_4)_o^* \rangle = \langle (1/2)(-E_{31} - E_{13}), (\sqrt{2}/2)(E_{31} + E_{13}) \rangle = -\sqrt{2}/2,
\end{equation}
\begin{equation}
\langle [A, X_5]_o, (X_5)_o^* \rangle = \langle (-E_{32} - E_{23}), (\sqrt{2}/2)(E_{32} + E_{23}) \rangle = -\sqrt{2}.
\end{equation}
This yields that
\begin{equation}
H = (\sqrt{2}/5)A \neq 0.
\end{equation}
Therefore, $\mathbb{R}^\times$Aut($g$).{$\cdot$}$_0$ is not minimal, which completes the proof. \qed

5.3. Case of $g = v_{3,a}$ ($-1 \leq a < 1$). In this subsection, we study the case of $g = v_{3,a}$. Throughout this subsection, we fix $a$ satisfying $-1 \leq a < 1$. Recall that, from Lemma 3.6, one has
\begin{equation}
\mathbb{R} \oplus \text{Der}(g) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.
\end{equation}
The expression of $\mathfrak{M}$ is given in Proposition 3.9 as follows:
\begin{equation}
\mathfrak{M} = \left\{ [g_{\lambda}, \cdot]_0 \mid g_{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}.
\end{equation}

Proposition 5.3. Let $g = v_{3,a}$. Then, we have
\begin{enumerate}
\item The action of $\mathbb{R}^\times$Aut($g$) is of cohomogeneity one, and all orbits are hypersurfaces.
\item $\mathbb{R}^\times$Aut($g$).{$\cdot$}$_0$ is the unique minimal orbit.
\end{enumerate}
Proof. Take any \( \langle \cdot, \cdot \rangle \). In order to prove (1), we show that \( \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle \) is a hypersurface, that is, has dimension 5. From the expression of \( \mathfrak{P} \mathfrak{M} \), there exists \( \lambda \in \mathbb{R} \) such that

\begin{equation}
\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle = \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).(\langle g_\lambda \cdot, \cdot \rangle_0).
\end{equation}

Let us define

\begin{equation}
U' := g_\lambda^{-1}(\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}))g_\lambda.
\end{equation}

Then, since \( g_\lambda^{-1} \) gives an isometry, one has an isometric congruence

\begin{equation}
\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).(\langle g_\lambda \cdot, \cdot \rangle_0) \cong U'.\langle \cdot, \cdot \rangle_0.
\end{equation}

Let \( \mathfrak{u}' \) be the Lie algebra of \( U' \). From the expression of \( \mathbb{R} \oplus \operatorname{Der}(\mathfrak{g}) \), one can directly calculate

\begin{equation}
\mathfrak{u}' = g_\lambda^{-1}(\mathbb{R} \oplus \operatorname{Der}(\mathfrak{g}))g_\lambda = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\
x_{21} & x_{22} & 0 \\
x_{31} & -\lambda(x_{22} - x_{33}) & x_{33} \end{pmatrix} \right\}.
\end{equation}

Thus it is easy to check that

\begin{equation}
\dim \mathfrak{u}' = 5, \quad \dim(\mathfrak{u}' \cap \mathfrak{o}(3)) = 0.
\end{equation}

Therefore \( U'.\langle \cdot, \cdot \rangle_0 \) has dimension 5, which completes the proof of (1).

In order to prove (2), we have only to show that \( U'.\langle \cdot, \cdot \rangle \) is minimal if and only if \( \lambda = 0 \). From \( \mathfrak{m}' := \mathfrak{d}\pi_e(\mathfrak{u}') \), one can see that

\begin{equation}
\mathfrak{m}' := d\pi_e(\mathfrak{u}') = \left\{ \begin{pmatrix} x_{11} & x_{21} & x_{31} \\
x_{21} & x_{22} & (-\lambda/2)(x_{22} - x_{33}) \\
x_{31} & (-\lambda/2)(x_{22} - x_{33}) & x_{33} \end{pmatrix} \right\}.
\end{equation}

We define a basis \( \{X_1, \ldots, X_5\} \) of \( \mathfrak{u}' \) by

\begin{align}
X_1 := E_{11}, & \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}), \\
X_3 := (1/\sqrt{2(1 + \lambda^2)})(E_{22} - E_{33} - 2\lambda E_{32}), & \quad X_4 := \sqrt{2}E_{21}, \quad X_5 := \sqrt{2}E_{31}.
\end{align}

Let us put

\begin{equation}
X'_i := \left( X_i \right)_o^* = (1/2)(X_i + \mathfrak{i}X_i).
\end{equation}

Then \( \{X'_1, \ldots, X'_5\} \) is an orthonormal basis of \( \mathfrak{m}' \). Furthermore, define

\begin{equation}
A := (1/\sqrt{2(1 + \lambda^2)})(-\lambda E_{22} + \lambda E_{33} - 2E_{32}), \quad A' := (A)_o^*.
\end{equation}

Then \( \{A'\} \) is an orthonormal basis of \( \mathfrak{m} \oplus \mathfrak{m}' \). Recall that the mean curvature vector \( H \) is given by

\begin{equation}
H = -(1/5)\sum h(X'_i, X'_i), \quad \langle h(X'_i, X'_i), A' \rangle = \langle [A, X_i]_o^*, (X_i)_o^* \rangle.
\end{equation}
The bracket products $[A, X_i]$ satisfy
\begin{align}
[A, X_1] &= [A, X_2] = 0, \quad [A, X_3] = -2E_{32}, \\
[A, X_4] &= -(1/\sqrt{1 + \lambda^2})(\lambda E_{21} + 2E_{31}), \quad [A, X_5] = (\lambda/\sqrt{1 + \lambda^2})E_{31}.
\end{align}

Hence, one has
\begin{align}
\langle [A, X_3]^*, (X_3)^* \rangle &= 2\lambda/\sqrt{2(1 + \lambda^2)}, \\
\langle [A, X_4]^*, (X_4)^* \rangle &= -\lambda/\sqrt{2(1 + \lambda^2)}, \\
\langle [A, X_5]^*, (X_5)^* \rangle &= \lambda/\sqrt{2(1 + \lambda^2)}.
\end{align}

This yields that
\begin{align}
H = -\frac{2\lambda}{\sqrt{2(1 + \lambda^2)}}A'.
\end{align}

Therefore, $H = 0$ if and only if $\lambda = 0$. This completes the proof of (2). \hfill \Box

5.4. Case of $g = r_{3,a}'$ ($a \geq 0$). In this subsection, we study the case of $g = r_{3,a}'$. Throughout this subsection, we fix $a$ satisfying $a \geq 0$. Recall that, from Lemma 3.6, $\mathbb{R} \oplus \text{Der}(g)$ is given by
\begin{align}
\mathbb{R} \oplus \text{Der}(g) &= \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \right\}.
\end{align}

The expression of $\mathcal{P}M$ is given in Proposition 3.10 as follows:
\begin{align}
\mathcal{P}M = \left\{ [g_\lambda, \langle, \rangle_0] \mid g_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}, \lambda \geq 1 \right\}.
\end{align}

**Proposition 5.4.** Let $g = r_{3,a}'$. Then, we have

(1) The action of $\mathbb{R}^\times \text{Aut}(g)$ is of cohomogeneity one, and $\mathbb{R}^\times \text{Aut}(g).\langle , \rangle_0$ is the unique singular orbit.

(2) $\mathbb{R}^\times \text{Aut}(g).\langle , \rangle_0$ is the unique minimal orbit.

**Proof.** Take any $\langle, \rangle$. In order to prove (1), we calculate the dimensions of the orbits. From the expression of $\mathcal{P}M$, there exists $\lambda \geq 1$ such that
\begin{align}
\mathbb{R}^\times \text{Aut}(g).\langle, \rangle = \mathbb{R}^\times \text{Aut}(g).\langle g_\lambda, \langle, \rangle_0 \rangle.
\end{align}

Let us denote by
\begin{align}
U' := g_\lambda^{-1}(\mathbb{R}^\times \text{Aut}(g))g_\lambda.
\end{align}

Then, since $g_\lambda^{-1}$ gives an isometry, one has an isometric congruence
\begin{align}
\mathbb{R}^\times \text{Aut}(g).\langle g_\lambda, \langle, \rangle_0 \rangle \cong U'.\langle, \rangle_0.
\end{align}
Let $u'$ be the Lie algebra of $U'$. From the expression of $\mathbb{R} \oplus \text{Der}(g)$, a direct calculation yields that

$$u' = g_1^{-1}(\mathbb{R} \oplus \text{Der}(g))g_\lambda = \left\{ \begin{pmatrix} x_{11} & x_{21} & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 \\ x_{31} & -\lambda^2 x_{23} & x_{33} \end{pmatrix} \right\}.$$ (5.48)

Then we have

$$\dim u' = 5, \quad \dim(u' \cap \mathfrak{o}(3)) = \begin{cases} 0 & (\text{for } \lambda > 1), \\ 1 & (\text{for } \lambda = 1). \end{cases}$$ (5.49)

This yields that the orbit corresponding to $\lambda = 1$ is the unique singular orbit, (which has codimension two). This completes the proof of (1).

We show (2). It is known that every singular orbit of a cohomogeneity one action is minimal (see [22]). Then we have only to show that $U', \langle \cdot, \cdot \rangle$ is not minimal if $\lambda > 1$. From now on assume that $\lambda > 1$. From (5.48), one can see that

$$m' := d\pi_*(u') = \left\{ \begin{pmatrix} x_{11} & x_{21} & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 \\ x_{31} & (1 - \lambda^2)/2 x_{23} & x_{33} \end{pmatrix} \right\}.$$ (5.50)

We define a basis $\{X_1, \ldots, X_5\}$ of $u'$ by

$$X_1 := E_{11}, \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}), \quad X_3 := \sqrt{2}E_{21},$$

$$X_4 := \sqrt{2}E_{31}, \quad X_5 := (\sqrt{2}/(1 - \lambda^2))(E_{23} - \lambda^2 E_{32}).$$ (5.51)

Furthermore we put

$$X'_i := (X_i)^*_o = (1/2)(X_i + ^t X_i), \quad A := (1/\sqrt{2})(E_{22} - E_{33}).$$ (5.52)

Then $\{X'_1, \ldots, X'_5\}$ is an orthonormal basis of $m'$, and $\{A\}$ is an orthonormal basis of $m \oplus m'$. Recall that the mean curvature vector $H$ is given by

$$H = -(1/5) \sum h(X'_i, X'_i), \quad \langle h(X'_i, X'_i), A \rangle = \langle [A, X_i]^*_o, (X_i)^*_o \rangle.$$ (5.53)

The bracket products $[A, X_i]$ satisfy

$$[A, X_1] = [A, X_2] = 0, \quad [A, X_3] = E_{21}, \quad [A, X_4] = -E_{31},$$

$$[A, X_5] = (2/(1 - \lambda^2))(E_{23} + \lambda^2 E_{32}).$$ (5.54)

Hence, one has

$$\langle [A, X_3]^*_o, (X_3)^*_o \rangle = 1/\sqrt{2},$$

$$\langle [A, X_4]^*_o, (X_4)^*_o \rangle = -1/\sqrt{2},$$

$$\langle [A, X_5]^*_o, (X_5)^*_o \rangle = \sqrt{2}(1 + \lambda^2)/(1 - \lambda^2).$$ (5.55)

This yields that

$$H = -\sqrt{2}(1 + \lambda^2)/(5(1 - \lambda^2))A \neq 0.$$ (5.56)

which completes the proof. \qed
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