ON THE POINCARÉ FUNCTIONAL EQUATION

SERGEI KAZENAS

Abstract. In this paper, a formula for the solution of the Poincaré functional equation in algebra of formal power series and its application to continuous iteration are presented.

The Poincaré functional equation and associated Schröder’s equation arise naturally in analytic iteration theory and have been studied by a great many authors (see, for instance, [1]). In the following, Poincaré’s equation is treated algebraically; and the formal solution representation is derived using finite symbolic manipulations with homogeneous quantum difference operators.

Here the formal power series treatment is near to that used by Niven [2], but the notation is slightly different.

Let $F$ denote the algebra of formal power series over field $\mathbb{C}$ of complex numbers. The algebra $\mathcal{P}$ of formal polynomials is a subalgebra of $F$. Any $f \in F$ is determined by the vector of its coefficients. Denote by $(f_j)_{j \geq 0}$ or by $(f_0, f_1, \ldots)$ that vector.

Also denote by $x$ the monomial determined by $x_j = \delta_{1,j}$.

Recall that for any $\{f, g\} \subset F$, two basic operations, addition and multiplication, are defined by $(f + g)_j = f_j + g_j$ and $(fg)_j = \sum_{0 \leq l \leq j} f_l g_{j-l}$, accordingly. If $f$ is not polynomial, then the composition is generally defined by $(f \circ g)_j = \sum_{0 \leq l \leq j} f_l (g^l)_j$ only when $g_0 = 0$. If $f$ is polynomial, there are no restrictions on $g$.

In addition, it is convenient to implement one outer operation; for any element $a$ of some algebra $A$, which is isomorphic to $F$ or $\mathcal{P}$, define $f[a] := \sum_{j \geq 0} f_j a^j$. For example, $f[xg] \equiv f \circ (xg)$, when $\{f, g\} \subset F$. If $a$ is number, then $f[a]$ may only be a number. The composition operator $\hat{q}x$ defined by $\hat{q}xf = f \circ (qx)$, where $q \in \mathbb{C}$, will also be needed later on. Clearly, the expression $p[\hat{q}x]$ represents a linear operator, when $p$ is polynomial.

Recall that $f$ is invertible if and only if $f_0 = 0$ and $f_1 \neq 0$, or $f_0 \neq 0$ and $f = f_0 + f_1 x$. Let $f \in xF$ be invertible and $q \neq 0$ not be a root of unity. Let $f$ also satisfy Poincaré’s equation

$$f \circ (qx) = p \circ f. \quad (1)$$

Then it immediately follows that $f^{\circ -1}$ satisfies Schröder’s equation $q f^{\circ -1} = f^{\circ -1} \circ p$.

It also follows that an invertible solution exists if and only if $p_0 = 0$ and $p_1 = q$. Therefore, equation (1) can be rewritten in the equivalent form: $f \circ (q^{-1}x) = p^{\circ -1} \circ f$. Without loss of generality, the nonzero coefficient $f_1$ can be fixed by $f_1 = 1$ and the other coefficients $f_2, f_3, \ldots$ can be calculated recursively from

$$f_j = \frac{1}{q^{j-1} - 1} \sum_{l=1}^{j-1} p_l (f^l)_j. \quad (2)$$
However, there are approaches to represent the coefficients of $f$ nonrecursively. One such approach is described below.

For an arbitrary nonzero polynomial $g$ define a homogeneous quantum difference operator in $\mathcal{F}$ by

$$D_{g;q} := (x^{-n})g[qx],$$

where operator $(x^{-n})$ is naturally defined in $x^n\mathcal{F}$, and $n$ is the smallest nonnegative integer such that $g[q^n] \neq 0$; let call that number the order of operator $D_{g;q}$ or $q$-difference order of $g$. For example, $D_{1+q}^{-1}$ is the first order operator, which can be viewed as Euler-Jackson difference operator in $\mathcal{F}$.

From $D_{g;q}x^m = (x^{-n})g[qx]x^m = (x^{-n})g[q^m]x^m = 0$ ($0 \leq m < n$) Maclaren-like expansion comes:

$$f_{j} = \frac{(D_{g;q}f)[0]}{g_j[q^j]},$$

where $D_{g;q}$ is of order $j$.

Now the preparation is complete to derive a nonrecursive companion to (2).

**Theorem 1.** Let $f$ satisfy Poincaré’s equation (1). $f_0 = 0$, $f_1 = 1$; $g_j$ stands for an arbitrary nonzero polynomial of $j$-th $q$-difference order, then the coefficients of $f$ can be calculated by

$$f_j = \frac{1}{g_j[q^j]} \sum_{i \geq 0} g_j; i(p^{oi})_{j}.$$  

(4)

**Proof.** Calculate $q$-differences:

$$D_{g_j;q} f = (x^{-j})g_j[qx]f = x^{-j} \sum_{i \geq 0} g_j; i f_{i[q^i]} x^i = x^{-j} \sum_{i \geq 0} g_j; i p^{oi} [f] x^i.$$ 

Note that $\frac{f - x}{x} [0] = 0$ and continue:

$$\langle D_{g_j;q} f \rangle [0] = \left( x^{-j} \sum_{i \geq 0} g_j; i p^{oi} \left[ x \left(1 + \frac{f - x}{x}\right)\right] \right) [0]$$

$$= \left( x^{-j} \sum_{i \geq 0} g_j; i p^{oi} [x] \right) [0] = \sum_{i \geq 0} g_j; i (p^{oi})_{j}.$$ 

Taking in account (3) completes the proof. 

**Remark.** Suppose $p$ is polynomial and the function $f$ defined by $f(x) = \sum_{j \geq 0} f_j x^j$ is entire. Then $f$ is determined by its values at points $aq^0, aq^1, ...$, where $a$ is such that $f(a)$ is not periodic (or eventually periodic) point of the map $x \mapsto p[x]$. This is clear from the fact that sums in (4) are finite and $p^{oi}$ are determined by its values at points $f [aq^0], f [aq^1], ...$

As a corollary, one can derive a formula for continuous iteration.

Fix $j$. Using Kac and Cheung’s notation, $(x-a)_q^j := (x-a)(x-qa)...(x-q^{j-1}a)$, and taking $x^{n-j}(x-1)_q^n$ or $x^{n-j}(x-1)_q^n (n > j)$ in place of $g_j$ in (4) gives equality:
\[
q^{(n-j)j} \sum_{i=n-j}^{n} (x^{n-j}(x-1)^{j})_{q,i}(p^{o})_{j} = q^{(n-1-j)j} \sum_{i=n-1-j}^{n-1} (x^{n-1-j}(x-1)^{j})_{q,i}(p^{o})_{j}
\]

Using Gauss' binomial, this yields recursive formula:

\[
(p^{o})_{j} = \sum_{i=1}^{j+1} \binom{j+1}{i} (-1)^{i+1} q^{(i-1)/2}(p^{o(n-i)})_{j} = (S^{0} - (S^{0} - S)_{q}^{j})(p^{o})_{j}
\]

where \(S\) is shift operator with respect to \(n\), i.e. \(S(p^{o})_{j} = (p^{o(n-1)})_{j}\).

It follows from this representation that \((p^{o})_{j}\) is a polynomial of degree \(j\) in \(q^{n}\) (recall that \(q = p_{1}\)) and Waring's interpolation process gives:

\[
(p^{o})_{j} = \sum_{i=0}^{j} \frac{(q^{n} - 1)_{q}^{j} (q^{n} - q^{i+1})_{q}^{-i}}{(q^{i} - 1)_{q}^{j} (q^{i} - q^{i+1})_{q}^{-i}}(p^{o})_{i}
\]

(5)

Expanding this polynomial gives:

\[
(p^{o})_{j} = \sum_{i=1}^{j} \sum_{k=0}^{i} (-1)^{i-k} q^{(j-k)(j-k+1)/2} \binom{j+1}{k} \rho_{l,j,i},
\]

where

\[
\rho_{l,j,i} := \sum_{i=0}^{j} \frac{q^{kl-\bar{d}-\bar{l}}(p^{o})_{j}}{(q^{l} - 1)_{q}^{j} (q^{l} - q^{l+1})_{q}^{-i}}.
\]

The behavior of \(\rho_{l,j,i}\) may be the subject of further research.

Jacobi [6] gave an explicit general formula for \((p^{o})_{j}\) (for positive integer \(n\)). Therefore, formula (5) allows calculating fractional functional powers directly. And there are no restrictions on \(q\). For example,

\[
(qx + qx^{2})^{1/2} = q^{1/2}x + \frac{q^{1/2}}{1 + q^{1/2}}x^{2} - \frac{2q}{(1 + q^{1/2})^{2}(1 + q)}x^{3} + ...
\]

Of course, analytic function \(x \mapsto qx + qx^{2}\) \((q \neq 1)\) has two functional square roots. The second root can be obtained by replacing \("q^{n}\) with \("-q^{1/2}\) in the expression (5). (In case \(q = 1\), it has only one functional square root. This case was developed in [3], for instance.)

Note that changing \(g_{j}\) in the expansion (4) does not lead to new properties of iterated formal power series \(p^{o}\). Therefore, by setting \(g_{j} = (x-1)_{q}^{j}\) and using Gauss' binomial, it may be simplified:

\[
f_{j} = \frac{1}{(q-1)^{j}q^{(j-1)j/2}j!_{q}} \sum_{i=0}^{j} q^{(j-i)(j-i-1)/2} \binom{j}{i} (-1)^{j-i}(p^{o})_{j},
\]

where

\[
j!_{q} := \frac{(1 - q)^{j}_{q}}{(1 - q)^{j}}.
\]
References

[1] Derfel G., Grabner P., Vogl F., *Asymptotics of the Poincaré functions*, Conf.: Prob. and Math. Ph., 2007

[2] Niven I., *Formal power series*, Amer. Math. Month., 1989, 76(8), 871-889

[3] Labelle G., *Sur l’inversion et l’itération continue des séries formelles*, Europ. J. Combinatorics, 1980, 1, 113-138

[4] Kac V., Cheung V., *Quantum calculus*, Springer, 2002

[5] Waring E., *Problems concerning interpolations*, Phil. Trans., 1779

[6] Biermann K., *Eine unveröffentlichte Jugendarbeit C. G. J. Jacobis über wiederholte Funktionen*, Crelle, 1961

Email address: kazenas@protonmail.com, sergei.kazenas@gmail.com