Unified covariant treatment of hyperfine splitting for heavy and light mesons.

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This paper aims at proving the fundamental role of a relativistic formulation for quarkonia models. We present a completely covariant description of a two-quark system interacting by the Cornell potential with a Breit term describing the hyperfine splitting. Using an appropriate procedure to calculate the Breit correction, we find heavy meson masses in excellent agreement with experimental data. Moreover, also when applied to light quarks and even taking average values of the running coupling constant, we prove that covariance properties and hyperfine splitting are sufficient to explain the light mesons spectrum and to give a very good agreement with the data.

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INTRODUCTION

Potential models of interacting quark systems have a long history and are still a very lively subject of investigation: this is witnessed by the large number of research papers and reviews that keep being published [1], which we refer to for bibliography and exhaustive details on the subject. Since the first papers that gave a rather complete overall picture of the subject [2], the starting point is often a Schrödinger equation with a potential having a Coulomb behavior at the origin and confining at infinity; the relativistic corrections together with the spin-orbit and the spin-spin contributions are taken into account by adding terms which are treated perturbatively. Attempts have also been made to overcome the limitations of a potential model due to asymptotic freedom at short distances and to light quark creation: a description of these effects has been tried by means of screened potentials softening the Coulomb interaction at the origin, which in many cases yields difficulties in explaining the hyperfine splittings of the spectra. Although this approximation may be good for heavy mesons, a smearing of the δ-function has been proposed to get a better description of the small distance behavior: recent results [3], however, show that this point has not been settled.

A major point of discussion has always been the relevance of relativistic properties of the systems, not only in the obvious case of light mesons, but also for heavy mesons. A truly covariant formulation going beyond the “relativized” treatment has often been invoked and approaches in such direction have been actually worked out [4, 5]. Many of them are connected with field theory along the lines of the Bethe-Salpeter equation and the spectra of the resulting equations are not of straightforward computation. Few models deal with a consistent relativistic description. In [3] a full spinor treatment is presented. The confinement is essentially obtained by a cutoff of the wave function at a fixed interparticle separation, the Breit interaction is differently treated for light and heavy mesons and an ad hoc contact interaction is introduced: the approach is interesting but not fully covariant. A covariant formulation is given in [4]; however, since the main subject of investigation are the Regge trajectories, the assumed potential is just linear in the radial variable. The papers in [7] study a well formulated relativistic model with a two-body Dirac equation derived from constraint dynamics. The interaction is first introduced by a relativistic extension of the Adler-Piran potential and then improved by the addition of a time-like confining vector potential, yielding very good results.

We present here a canonical description of quarkonium, focusing on the complete covariance of the formulation and on the fermionic nature of the elementary constituents. The formulation originates from a wave equation for two relativistic fermions with arbitrary masses obtained from two Dirac operators coupled by the interaction [6]. We refer to those papers for the proofs of the full covariance, of the Schrödinger and the one-particle Dirac limits, as well as of the cyclicity of the relative time that avoids the difficulties of relative energy excitations. We observe that our construction has different assumptions from [7], so that the final equations and the results also are somewhat different. In [3] the hyperfine splitting of Positronium was calculated, finding an agreement better than up to the fourth power of the fine structure constant with the results obtained by QED semi-classical expansions. In the present context we will use the simplest Cornell potential with a Breit term for the spin-spin interaction. Our purpose is to show that the full relativistic description and a proper perturbation treatment of the Breit term, avoiding the evaluation of a delta function at the origin, are already sufficient to give results in excellent agreement with the experimental data both for heavy and light mesons, contrary to some diffused ideas. Further improvements of the poten-
tial are an important issue which should be developed at a more phenomenological level of the investigation. For instance in our calculations we have used average values of the running coupling constant (rcc) for the different families of mesons, verifying ex post that the ratios of the assumed values are in agreement with those obtained from the well known $\alpha_S$ curve [8]; a fine tuning of the rcc, modeled according to the $\alpha_S$ curve, should produce much better results.

THE TWO-FERMION WAVE EQUATION WITH CORNELL POTENTIAL AND BREIT TERM

The Dirac operators entering the wave equation prescribe the correct form for the interactions according to their tensorial nature: the Coulomb-like term of the Cornell potential is vectorial and thus minimally coupled to the energy; the linear term is scalar and therefore coupled to the mass. Indeed only a scalar growing potential is actually confining, while an unbounded vector interaction is not [8]. We refer to [8] for the derivation of the radial system of the model. We call $r_a, q_a$ the Wigner vectors of spin one given by the spatial parts of relative coordinates and momenta boosted to the frame with vanishing total spatial momentum and we put $r = (r_ar_a)^{1/2}$ (sum over repeated indexes). We denote by $\gamma_{(i)}$ the gamma matrices acting in the spinor space of the $i$-th fermion of mass $m_{(i)}$, $M = m_{(1)} + m_{(2)}$ and $\rho = |m_{(1)} - m_{(2)}|/M$. The vector and scalar couplings produce the terms $E + b/r$, $\frac{1}{2}(M + \sigma r)$ and the final wave equation reads

$$\left[ \left( \gamma_0 \gamma_{(1)} + \gamma_0 \gamma_{(2)} \right) q_a + \frac{1}{2} \left( \gamma_0 \gamma_{(1)} + \gamma_0 \gamma_{(2)} \right) \left( M + \sigma r \right) + \frac{1}{2} \left( \gamma_0 \gamma_{(1)} - \gamma_0 \gamma_{(2)} \right) M \rho - \left( E + \frac{b}{r} \right) + V_B(r) \right] \Psi(\vec{r}) = 0. \tag{1}$$

where

$$V_B(r) = \frac{b}{2r} \gamma_0 \gamma_{(1)} \gamma_{(2)} + \frac{\delta_{ab} + \frac{r_ar_b}{r^2}}{r} \tag{2}$$

is the Breit term generating the hyperfine splitting. As in [8] the first perturbation order of this term is evaluated by substituting $V_B(r)$ with $\varepsilon V_B(r)$ in (1) and taking the first derivative of the eigenvalues with respect to $\varepsilon$ in $\varepsilon = 0$ from the numerical solutions of the differential equations. This could also be seen as an application of the spectral correspondence to the Feynman-Hellman theorem.

The radial system is obtained by diagonalizing angular momentum and parity. As in [8] it is formed by four algebraic plus four first order differential equations for each parity. Using the algebraic relations and defining the dimensionless variables $\Omega, w, s$ by

$$\sigma = \frac{M^2}{4} \Omega^2, \quad E = \frac{M}{2} (2 + \Omega w), \quad r = \frac{2}{M} \Omega^{-\frac{1}{2}} s, \tag{3}$$

the radial system for (1), replacing $V_B(r)$ by $\varepsilon V_B(r)$, is

$$\begin{bmatrix} u_1'(s) \\ u_2'(s) \\ u_3'(s) \\ u_4'(s) \end{bmatrix} = \begin{bmatrix} 0 & A_0(s) + B_0(s) & 0 & 0 \\ \frac{1}{A_0(s)} & 1/s & 0 & B_0(s) \\ \frac{1}{A_0(s)} & 0 & C_{\varepsilon}(s) & 0 \\ 0 & \frac{1}{A_0(s)} & 0 & D_{\varepsilon}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{bmatrix} = 0.$$

Here $A_0 = A_{\varepsilon|\varepsilon=0}$, $B_0 = B_{\varepsilon|\varepsilon=0}$ and $u'(s) = du(s)/ds$.

Letting $J^2 = j(j+1)$, the even parity coefficients are:

$$A_{\varepsilon}(s) = \frac{2\sqrt{J^2} \rho}{\sqrt{\Omega}(sh(s) - 2b)}, \quad B_{\varepsilon}(s) = \frac{(h(s) - 2 - 2\rho^2/\Omega) s^2 - 2\varepsilon b^2}{s^2 h(s) - 2\varepsilon bs}, \quad C_{\varepsilon}(s) = \frac{h(s) + 2\varepsilon b}{2s} + \frac{2J^2}{2s - 2s h(s)} + \frac{2s k^2(s)}{4\varepsilon b - s h(s)}, \quad D_{\varepsilon}(s) = \frac{4b^2 \varepsilon^2 - s^2 h^2(s) + 4s^2 k^2(s)}{4s^2 b^2 - 2s^2 h(s)} \tag{4}$$

with $h(s) = (2 + \Omega w)/\sqrt{\Omega + b/s}$, $k(s) = (2 + \Omega s)/2\sqrt{\Omega}$. The coefficients for the odd parity system are:

$$A_{\varepsilon}(s) = \frac{2\sqrt{J^2} \kappa(s)}{2\varepsilon b - s h(s)}, \quad B_{\varepsilon}(s) = \frac{4\varepsilon^2 b^2 - s^2 h^2(s) + 4s^2 k^2(s)}{4\varepsilon b - s^2 h(s)}, \quad C_{\varepsilon}(s) = \frac{h(s) + 2\varepsilon b}{2s} + \frac{2J^2}{2s - 2s h(s)} + \frac{2s \rho^2}{\Omega(4\varepsilon b - s h(s))}, \quad D_{\varepsilon}(s) = -\frac{h(s) + 2\varepsilon b}{2s} \frac{2J^2}{4b^2 - s h(s)} - \frac{\varepsilon b}{s} + \frac{2\rho^2}{\Omega(s h(s) - 2b s)} \tag{5}$$

TABLE 1. The $\Xi_b$ levels in MeV. First column: term symbol, $I^G(JPC)$ numbers, particle name. $\sigma = 1.111$ GeV/fm, $\alpha = 0.3272$, $m_b = 4725.5$ MeV. Experimental data from [8].
A word about the numerical method we have used is in order. The origin and infinity are the only singular points of the boundary value problem and no further singularities arise from the matrix of the coefficients. The solution was obtained by a double shooting method, the spectral condition being the vanishing of the $4 \times 4$ determinant of the matching conditions at a crossing point. Padé techniques have been used to improve the accuracy of the approximate solutions at zero and infinity. The integration precision has always been kept very high and tested against the stability of the spectral values.

**DISCUSSION OF THE NUMERICAL RESULTS**

As stated in the Introduction, in order to have a test as good as possible of the relevance of the relativistic dynamics in quarkonium models, we have aimed at choosing the least number of fit parameters. Flavor independence could be expected for heavy quarks. In fact, doing separate fits for $b\bar{b}$, $b\bar{s}$ and $c\bar{c}$ we find that the strong tensions turn out to be the same within the computation precision. The same values of $\sigma$ and of the masses are taken for the unique measured $Bc$ state. We introduce $\alpha = (3/4)b$, where $b$ is the parameter of the Cornell potential appearing in [11]. We assume a constant $\alpha$ determined by a separate fit for each family of mesons.

| State | Exp | Num |
|-------|-----|-----|
| $(1s_0)$ $0^+(0^-)$ $\eta_c$ | 2978.40±1.2 | 2978.26 |
| $(1s_1)$ $0^-(1^-)$ $J/\psi$ | 3096.916±0.011 | 3097.91 |
| $(1p_0)$ $0^+(0^+)$ $\chi_{c0}$ | 3414.75±0.31 | 3423.88 |
| $(1p_1)$ $0^+(1^+)$ $\chi_{c1}$ | 3510.66±0.07 | 3502.83 |
| $(1p_2)$ $0^-(1^-)$ $h_c$ | 3525.41±0.16 | 3523.67 |
| $(1s_0)$ $0^+(2^+)$ $\chi_{c2}$ | 3556.20±0.09 | 3555.84 |
| $(2s_0)$ $0^+(0^-)$ $\eta_c$ | 3637±4 | 3619.64 |
| $(2s_1)$ $0^-(1^-)$ $\psi$ | 3686.09±0.04 | 3692.91 |
| $(2d_0)$ $0^-(1^-)$ $\psi$ | 3772.92±0.35 | 3808.48 |
| $(2p_0)$ $0^+(1^+)$ $\chi_{c1}$ | 3871.57±0.25 | - |
| $(2p_1)$ $0^+(2^+)$ $\chi_{c2}$ | 3917.4±2.7 | 3961.21 |
| $(2p_2)$ $0^+(2^+)$ $\chi_{c2}$ | 3927±2.6 | 4003.93 |
| $(3s_0)$ $0^-(0^-)$ $\eta_c$ | 3942±13 | - |
| $(3s_1)$ $0^-(-1^-)$ $\psi$ | 4039±1 | 4122.95 |
| $(3d_0)$ $0^-(-1^-)$ $\psi$ | 4153±3 | 4200.51 |
| $(4s_1)$ $0^-(-1^-)$ $\psi$ | 4421±4 | 4479.22 |

**TABLE II.** The $c\bar{c}$ levels in MeV. $\sigma=1.111$ GeV/fm, $\alpha=0.435$, $m_c=1394.5$ MeV. Experimental data from [9].

| State | Exp | Num |
|-------|-----|-----|
| $(1s_0)$ $0^+(0^-)$ $B_c^+$ | 6277±0.06 | 6277 |
| $(1s_0)$ $0^+(0^-)$ $B_c^-$ | 5366.77±0.24 | 5387.41 |
| $(1s_1)$ $0^-(-1^-)$ $B_c^0$ | 5415.4±2.1 | 5434.34 |
| $(1p_0)$ $0^+(1^-)$ $B_s(5830)^0$ | 5829.4±0.7 | 5817.80 |
| $(1p_1)$ $0^+(2^-)$ $B_s(5840)^0$ | 5839.7±6.6 | 5829.33 |
| $(1s_0)$ $0^+(0^-)$ $D_c^+$ | 1968.49±0.32 | 1961.24 |
| $(1s_1)$ $0^-(-1^-)$ $D_c^0$ | 2112.3±0.50 | 2101.78 |
| $(1p_0)$ $0^+(1^-)$ $D_c(2317)^{\pm}$ | 2317.8±6.6 | 2339.94 |
| $(1p_1)$ $0^+(1^-)$ $D_c(2460)^{\pm}$ | 2459.6±6.6 | 2466.15 |
| $(1p_1)$ $0^+(1^-)$ $D_c(2536)^{\pm}$ | 2535.12±0.13 | 2535.82 |
| $(1p_2)$ $0^+(2^-)$ $D_c(2573)^{\pm}$ | 2571.9±8.2 | 2574.92 |

**TABLE IV.** The $B_c, B_s$ and $D_s$ levels in MeV. $\sigma=1.111, 1.111, 1.227$ GeV/fm and $\alpha=0.3591, 0.3975, 0.5348$ respectively.
TABLE V. The \( J^3 \) has the two possible assignments \( X \) and \( \Upsilon \), with the two possible assignments \( X \) and \( \Upsilon \). On the contrary, the calculated values for \( \Delta q \) are very different from the experimental values. We find that the model could suggest a \( (3915) \) resonance \( [9] \), staying just below the experimental values. We would like to thank our colleagues Stefano Catani and Francesco Bigazzi for useful discussions and interest in our work.

From Table II, for instance, as the resonance \( X(3782) \) has the two possible assignments \( J^{PC} = 1^{+}\bar{+} \) and \( 2^{+}\bar{+} \), the model could indicate a \( \chi_{c1} \) classification. Nothing can be suggested for \( X(3915) \) and \( X(3940) \), having no accepted quantum numbers. The situation is simpler in Table I, where there are no unclassified physical states.

We point out the good estimate of the recently discovered \( \chi_b(3P) \) resonance \( [8] \), staying just below the \( B \) production threshold. On the contrary, the calculated values for \( \Gamma(4^0 S_1) \) and \( \Gamma(5^0 S_1) \) exceed the experimental values.

We next consider the \( s\bar{s} \) system, for which there are few accepted experimental states. The much lighter mass of the \( s \) quark highly enhances the relativistic character of the \( s\bar{s} \) composite system and the fundamental role of the Breit corrections, giving rise to large hyperfine splittings. Due to these reasons the string tension \( \sigma \) has not been given the same value of the previous systems but has been considered a free parameter, finding a value larger than in \( b\bar{b} \). We report our results in Table III, where we have also included the unassigned \( f_1(1420) \), \( X(1750) \), \( \phi_3(1850) \) and \( \phi(2170) \). Although we cannot have a complete phenomenological confidence in the numerical results, still a fair number of experimental data can be accommodated with a pretty good accuracy. For instance the model could suggest a \( (1^3 d_1) \) assignment for \( X(1750) \). We then use the mass of the \( s \)-quark together with the \( b \) and \( c \) masses to determine the levels of the \( B_s \) and \( D_s \) mesons, reported in Table IV. Even for different quark masses the agreement with the data is very good.

We finally look at the lightest \( u\bar{d} \) mesons, for which the Breit correction, as usually calculated, is commonly accepted to be insufficient to reproduce the data. We have again fitted the data with a constant rcc. The fit includes also the very light \( \rho(770) \), but obviously excludes the \( \pi^\pm \) for which the use of a higher \( \alpha \) cannot be avoided, due to the steepness of the \( \alpha_S \) curve for very low masses.

The results are not very sensitive to the mass ratio \( \rho \) that we fix at the physical value 0.53; the string tension appears to be the same found for \( s\bar{s} \). Finally, the \( u \) and \( d \) masses are found close to current algebra masses as opposed to constituent masses (see also \( [3] \)), normally much higher in potential models. The exact mass 139.5 MeV of \( \pi^\pm \) is got with \( \alpha = 0.99 \).

To conclude we give some values of the Breit corrections \( \Delta q \) for different states. For \( s\bar{s} \) the values in MeV of \( (\Delta b, \Delta c, \Delta s, \Delta u) \) respectively are \( (92.31, 155.22, 296.81, 600.12) \). For \( 1^3 s_1 \) \( (18.09, 38.80, 94.37, 106.21) \). For \( 1^3 p_2 \), \( (7.51, 21.10, 55.93, 63.72) \). As expected, the corrections are of increasing values of \( j \) and become more and more important for decreasing quark masses. We thus believe that the results we have presented show that the covariant formulation based on Dirac equations, in addition to being conceptually very simple, is also extremely effective in quarkonium models.

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