Higher dimensional Scherk’s hypersurfaces

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November 2, 2018

Abstract

In 3-dimensional Euclidean space, Scherk second surfaces are singly periodic embedded minimal surfaces with four planar ends. In this paper, we obtain a natural generalization of these minimal surfaces in any higher dimensional Euclidean space \( \mathbb{R}^{n+1} \), for \( n \geq 3 \). More precisely, we show that there exist \((n-1)\)-periodic embedded minimal hypersurfaces with four hyperplanar ends. The moduli space of these hypersurfaces forms a 1-dimensional fibration over the moduli space of flat tori in \( \mathbb{R}^{n-1} \). A partial description of the boundary of this moduli space is also given.

1 Introduction

In 3-dimensional Euclidean space Scherk second surfaces come in a 1-parameter family \((S_\varepsilon)_{\varepsilon \in (0, \frac{\pi}{2})}\) which can be described in many different ways. For example it can be described via its Weierstrass representation data

\[
X_\varepsilon(\omega) := \Re \int_{\omega_0}^\omega \left( \frac{1}{2} \left( \frac{1}{g} - g \right), \frac{i}{2} \left( \frac{1}{g} + g \right), 1 \right) \, dh_\varepsilon,
\]

where

\[
g(\omega) := \omega \quad \text{and} \quad dh_\varepsilon := 4 \sin \varepsilon (\omega^4 + 1 - 2 \cos \varepsilon \omega^2)^{-1} \, d\omega.
\]

Or even more simply as the zero set of the function

\[
F_\varepsilon(x_1, x_2, z) := (\cos \varepsilon)^2 \cosh \left( \frac{x_1}{\cos \varepsilon} \right) - (\sin \varepsilon)^2 \cosh \left( \frac{z}{\sin \varepsilon} \right) - \cos x_2.
\]  

Indeed, it is well known that, the zero set of a function \( F \) is a minimal surface if and only if 0 is a regular value of \( F \) and

\[
\text{div} \left( \frac{\nabla F}{|\nabla F|} \right) = 0,
\]

on the zero set of \( F \). Using this, it is straightforward to check that the zero set of \( F_\varepsilon \) is a minimal surface.

In any of these descriptions, the parameter \( \varepsilon \) belongs to \((0, \pi/2)\). Observe that we do not consider any dilation, translation or rotation of a minimal surface, in other words we are only interested in the space of surfaces modulo isometries and dilations. Now, we would like to point out a few properties of Scherk’s second surfaces which will enlighten our construction of their higher dimensional analogues.

(i) - Periodicity : Observe that Scherk’s second surfaces are singly periodic and, in the above description, their common period has been normalized to be equal to \((0, 2\pi, 0)\). Hence, if we define \( T^1 := \mathbb{R}/2\pi\mathbb{Z} \), we can consider \( S_\varepsilon \) to be a minimal surface embedded in \( \mathbb{R} \times T^1 \times \mathbb{R} \).

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(ii) - Asymptotic behavior as \( \varepsilon \) tends to 0: Another feature which will be very important for us is the study of the behavior of Scherk’s second surfaces as the parameter \( \varepsilon \) tends to 0 (a similar analysis can be performed when the parameter \( \varepsilon \) tends to \( \pi/2 \)). To this aim, we write for all \((x_1, x_2)\) in some fixed compact subset \(\mathbb{R}^2 - \{0\} \times 2\pi\mathbb{Z}\) and for all \(\varepsilon\) small enough

\[
    z = \pm \sin \varepsilon \cosh \left( (\tan \varepsilon)^{-2} \cos \left( \frac{x_1}{\cos \varepsilon} \right) - (\sin \varepsilon)^{-2} \cos x_2 \right).
\]

Using this, we readily see that, away from the set \(\{0\} \times 2\pi\mathbb{Z}\), the one parameter family of surfaces \(S_\varepsilon\) converges to the union of two horizontal planes, as \(\varepsilon\) tends to 0. In other words, the sequence of surfaces \(S_\varepsilon\) converges, away from the origin, to two copies of \(\mathbb{R} \times T^1 \times \{0\}\) in \(\mathbb{R} \times T^1 \times \mathbb{R}\), as the parameter \(\varepsilon\) tends to 0.

As already mentioned, a similar analysis can be carried out as the parameter \(\varepsilon\) tends to \(\pi/2\) and, this time, we find that the sequence of surfaces \(S_\varepsilon\) converges, away from the origin, to two copies of \(\{0\} \times T^1 \times \mathbb{R}\) in \(\mathbb{R} \times T^1 \times \mathbb{R}\).

(iii) - Blow down analysis: For each fixed \(\varepsilon \in (0, \pi/2)\), the surface \(S_\varepsilon\) has four planar ends which are asymptotic to

\[
    V^\pm_\varepsilon := \{(x_1, x_2, z) \in \mathbb{R} \times T^1 \times \mathbb{R} : z = \pm (\tan \varepsilon |x_1| - 2 \sin \varepsilon \log \tan \varepsilon)\}.
\]

More precisely, away from a compact set in \(\mathbb{R} \times T^1 \times \mathbb{R}\), the surface \(S_\varepsilon\) is a normal graph over \(V^\pm_\varepsilon\) for some function which is exponentially decaying as \(x_1\) tends to \(\pm\infty\). Another way to understand this would be to say that, the sequence of surfaces \(\lambda S_\varepsilon\) converges, as \(\lambda\) tends to 0, to \(W^+_\varepsilon \cup W^-_\varepsilon\), where

\[
    W^\pm_\varepsilon := \{(x_1, x_2, z) \in \mathbb{R} \times T^1 \times \mathbb{R} : z = \pm \tan \varepsilon |x_1|\}.
\]

(iv) - Blow up analysis: Instead of blowing down the surfaces \(S_\varepsilon\) as we have done in (iii), we can blow up the surfaces \(S_\varepsilon\) by considering the sequence of scaled surfaces \(\varepsilon^{-1} S_\varepsilon\). As \(\varepsilon\) tends to 0 this sequence converges on compact to a vertical catenoid. To see this, just define the new set of coordinates

\[
    (\hat{x}_1, \hat{x}_2, \hat{z}) := \frac{1}{2\sin \varepsilon} (x_1, x_2, z),
\]

and, in (i), we expend both \(\cos x_2\) and \(\cosh(x_1/\cos \varepsilon)\), in terms of powers of \(\varepsilon\). We find with little work

\[
    (\cos \varepsilon)^2 \left(1 + 2(\tan \varepsilon)^2 \hat{x}_2^2\right) - (\sin \varepsilon)^2 \cosh(2 \hat{z}) = 1 - 2(\sin \varepsilon)^2 \hat{x}_2^2 + \mathcal{O}(\varepsilon^4).
\]

Hence

\[
    \hat{x}_1^2 + \hat{x}_2^2 = \varepsilon^2 + \mathcal{O}(\varepsilon^2).
\]

Clearly, as \(\varepsilon\) tends to 0, this converges, uniformly on compact sets, to an implicit parameterization of a vertical catenoid.

To complete this brief description, let us mention that Scherk’s second surfaces have recently been used as one of the building blocks of some desingularization procedure, to produce new embedded minimal surfaces in 3-dimensional Euclidean space. We refer to the work of M. Traizet [11] and also to the recent work of N. Kapouleas [4] for further details.

In order to state our result properly, we need to introduce two ingredients which will be fundamental in our analysis. First observe that, in higher dimensions, there is a natural generalization of the catenoid in Euclidean 3-space. This hypersurface, which we will call the unit \(n\)-catenoid, is a hypersurface of revolution with two hyperplanar ends. It can be parameterized by

\[
    \mathbb{R} \times S^{n-1} \ni (s, \theta) \mapsto (\varphi(s) \theta, \psi(s)) \in \mathbb{R}^{n+1},
\]
where the function $\varphi$ is defined by the identity $\varphi^{n-1}(s) = \cosh((n-1)s)$ and where the function $\psi$ is given by

$$\psi(s) := \int_0^s \varphi^{2-n}(t) \, dt.$$ 

Using this $n$-catenoid, S. Fakhi and the author have produced examples of complete immersed minimal hypersurfaces of $\mathbb{R}^{n+1}$ which have $k \geq 2$ hyperplanar ends \[3\]. These hypersurfaces have the topology of a sphere with $k$ punctures and they all have finite total curvature, they generalize the well known $k$-noids in 3-dimensional Euclidean space \[3\].

Another ingredient in our analysis is the moduli space of flat tori in $\mathbb{R}^m$, for $m \geq 1$. We recall a few well known facts about this moduli space and refer to \[13\] for further details. Any flat torus in $\mathbb{R}^m$ can be identified with $\mathbb{R}^m/\mathbb{Z}^m$ where $A \in GL(m, \mathbb{R})$. The volume of the $m$-dimensional torus $T^m := \mathbb{R}^m/\mathbb{Z}^m$ is then given by

$$\text{vol}(T^m) = |\det A|.$$ 

It is a simple exercise to check that two tori $\mathbb{R}^m/\mathbb{Z}^m$ and $\mathbb{R}^m/B\mathbb{Z}^m$ are isometric if and only if there exist $M \in O(m, \mathbb{R})$ and $N \in GL(m, \mathbb{Z})$ such that $A = M B N$. The moduli space of flat tori $T^m$ is defined to be the space of flat tori $\mathbb{R}^m/\mathbb{Z}^m$, normalized by asking that

$$\text{vol}(T^m) = \text{vol}(S^m),$$

modulo isometries. For later use, it will be convenient to identify any torus $T^m \in T^m$ with a subset of $\mathbb{R}^m$. To this aim, if

$$T^m = \mathbb{R}^m/\mathbb{Z}^m,$$

for some $A \in GL(n, \mathbb{R})$, we identify $T^m$ with the image of $[-\frac{1}{2}, \frac{1}{2}]^m$ by $A$. In particular, we will talk about the origin $0 \in T^m$, simply referring to the origin in $A[-\frac{1}{2}, \frac{1}{2}]^m \subset \mathbb{R}^m$. We will also consider $B^m \subset \mathbb{R}^{n-m} \times T^m$ as the $n$-dimensional ball of radius $\rho$ in $\mathbb{R}^{n-m} \times A[-\frac{1}{2}, \frac{1}{2}]^m$. And so on. Also observe that, granted this identification, $T^m$ is invariant under the action of the following subgroup of $O(m, \mathbb{R})$

$$\mathcal{D}_m := \{ D := \text{diag}(\eta_1, \ldots, \eta_m) \mid \eta_i = \pm 1 \}.$$ 

In this paper, we pursue the quest of higher dimensional generalizations of classical minimal surfaces which we have initiated in \[3\]. More precisely, we obtain a natural generalization of Scherk’s second surfaces in higher dimensional Euclidean spaces. Recall that one can view the moduli space of Scherk’s surfaces as a 1-dimensional fibration over the moduli space of flat tori in $\mathbb{R}$. We will show that, in $\mathbb{R}^{n+1}$, for $n \geq 3$, there exists a finite dimensional family of embedded minimal hypersurfaces satisfying properties which are similar to (i)-(iv). This family, which turn out to be a 1-dimensional fibration over the moduli space of flat tori in $\mathbb{R}^{n-1}$, yields a partial description of the moduli space of what might be called ”higher dimensional Scherk’s hypersurfaces”. More precisely, we obtain a description of the boundary of this moduli space, this boundary turns out to be modeled over the moduli space of tori in $\mathbb{R}^m$ for any $1 \leq m \leq n-1$.

Our main result can be stated as follows:

**Theorem 1** Assume that $n \geq 3$ and $1 \leq m \leq n-1$ are fixed. Let $T^m \in T^m$ be any flat torus of $\mathbb{R}^m$. Then, there exist $\varepsilon_0 > 0$ and $(S_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ a one parameter family of minimal hypersurfaces of $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$ such that:

(i) For all $\varepsilon \in (0, \varepsilon_0)$, the hypersurface $S_\varepsilon$ is embedded in $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$ and is invariant under the action of $O(n-m, \mathbb{R}) \otimes \mathcal{D}_m \otimes \{ \pm I_l \} \subset O(n+1, \mathbb{R})$.

(ii) As $\varepsilon$ tends to 0, the sequence of hypersurfaces $(S_\varepsilon)_\varepsilon$ converges to the union of two copies of $\mathbb{R}^{n-m} \times T^m \times \{ 0 \}$, away from the origin.
(iii) For all \( \varepsilon \in (0, \varepsilon_0) \), there exists \( c_\varepsilon > 0 \) and \( d_\varepsilon > 0 \) such that the hypersurface \( S_\varepsilon \) has four ends which are asymptotic to
\[
V^\pm_\varepsilon := \{(x_1, x_2, z) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R} : z = \pm (c_\varepsilon \zeta_m(x_1) + d_\varepsilon)\},
\]
where \( \zeta_{n-1}(y) := |y|, \zeta_{n-2}(y) := \log |y| \) and \( \zeta_m(y) := 0 \), when \( m \leq n - 3 \). In particular, this means that, up to a translation along the \( z \)-axis, the hypersurface \( S_\varepsilon \) is a normal graph over \( V^\pm_\varepsilon \) for some function which is exponentially decaying in \( |x_1| \) function when \( m = n - 1 \) and for some function which is polynomially decaying in \( |x_1| \) when \( m \leq n - 2 \). Furthermore, when \( m = n - 1 \), we have
\[
\lim_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon^{n-1}} = \frac{1}{2}.
\]

(iv) As \( \varepsilon \) tends to 0, the sequence of rescaled hypersurfaces \( (\varepsilon^{-1} S_\varepsilon)_\varepsilon \) converges, uniformly on compact sets, to a vertical unit \( n \)-catenoid.

This result, when \( m = n - 1 \), yields minimal hypersurfaces which constitute the natural generalization of Scherk’s second surfaces in higher dimensional Euclidean spaces. More precisely, when \( m = n - 1 \), the above result provides a description of part of \( S_n \), the moduli space of \( n \)-dimensional Scherk’s hypersurfaces in \( \mathbb{R}^{n+1} \). As this result shows, this moduli space is locally a 1-dimensional fibration over the moduli space of flat tori in \( \mathbb{R}^{n-1} \). Though we have not been able to prove it, we expect this fibration to extend, as it does when \( n = 1 \), to all \( c_\varepsilon \in (0, \pi/2) \).

The above result, when \( m \leq n - 2 \), yields hypersurfaces which have to be understood as belonging to the boundary of the moduli space \( S_n \), in the same way that any product \( \mathbb{R}^{n-m-1} \times T^m \), for \( m \leq n - 2 \) corresponds to a point in the compactification of the moduli space of flat tori in \( \mathbb{R}^{n-1} \). We expect that the moduli space \( S_n \) can be compactified and that the family of hypersurfaces described in the above result constitute a collar neighborhood of the boundary of \( \overline{S_n} \). In other words, Theorem 1 should provide a local description of \( S_n \), near its boundary.

To conclude, let us briefly describe the strategy of the proof of the result. It should be clear from (ii) and (iv) that, for small \( \varepsilon \), Scherk’s second surfaces can be understood as a desingularization of two copies of \( \mathbb{R} \times T^1 \times \{0\} \) in \( \mathbb{R} \times T^1 \times \mathbb{R} \). Keeping this observation in mind, our strategy will be to show that a similar desingularization is possible for two copies of \( \mathbb{R}^{n-m} \times T^m \times \{0\} \) in \( \mathbb{R}^{n-m} \times T^m \times \mathbb{R} \). The proof of this result is very much in the spirit of [12], however, some aspects are simpler in the present paper thanks to the special geometry of our problem.

Our work has been strongly influenced by the recent work of M. Traizet [12] and the work of N. Kapouleas [4, 5] in their construction of minimal embedded surfaces in \( \mathbb{R}^3 \). Indeed, on the one hand, N. Kapouleas has used Scherk’s second surfaces to desingularize finitely many catenoids or planes having a common axis of revolution and he has produced embedded minimal surfaces with finitely many ends and very high genus. On the other hand, M. Traizet has used finitely many catenoids to desingularize parallel planes and produced minimal surfaces with finitely many ends and any genus (larger than 2). There is a formal link between these two constructions since, in some vague sense, the surfaces constructed by N. Kapouleas on the one hand and the surfaces constructed by M. Traizet, for a genus large enough, on the other hand, should belong to the same moduli space. It was therefore tempting to try to produce Scherk’s second surfaces using some desingularization procedure.

Acknowledgment: This paper was written while author was visiting the Mathematical Sciences Research Institute in Berkeley. He would like to take this opportunity to thank the MSRI for their support and hospitality. The author would also like to thank D. Hoffman, R. Mazzeo and M. Weber, for stimulating discussions.
2 Definitions and notations

In this brief section we record some notations and definitions which will be used throughout the paper.

Eigenfunctions of $\Delta_{T^m}$: Given $m \geq 1$ and $T^m \in T^m$, we will denote by $E_i$, $i \in \mathbb{N}$ the eigenfunctions of the Laplacian on $T^m$ with corresponding eigenvalues $\mu_j$, that is $\Delta_{T^m} E_i = -\mu_i E_i$, with $\mu_i \leq \mu_{i+1}$. We will assume that these eigenfunctions are counted with multiplicity and are normalized so that

$$\int_{T^m} E_i^2 \, dx = 1.$$  

Though the spectral data of $\Delta_{T^m}$ do depend on $T^m$, we will not write this dependence in the notation.

Functions on $T^m$ which are invariant under the action of some group: We will only be interested in functions on $T^m$ and eigenfunctions of $\Delta_{T^m}$ which have some special symmetry. Namely the set of functions and eigenfunctions which are invariant under the action of the following subgroup of $O(m, \mathbb{R})$

$$\mathcal{D}(m) := \{ D := \text{diag}(\eta_1, \ldots, \eta_m) \mid \eta e = \pm 1 \}.$$  

We define $\mathcal{I}(m) \subset \mathbb{N}$ to be the set of indices $i$ corresponding to eigenfunctions $E_i$ which are invariant under the action of $\mathcal{D}(m)$, that is

$$\mathcal{I}(m) := \{ i \geq 0 \mid E_i = E_i \circ D, \text{ for all } D \in \mathcal{D}(m) \}.$$  

Eigenfunctions of $\Delta_{S^{n-1}}$: For all $n \geq 2$, we will denote by $e_j$, $j \in \mathbb{N}$, the eigenfunctions of the Laplacian on $S^{n-1}$ with corresponding eigenvalues $\lambda_j$, that is $\Delta_{S^{n-1}} e_j = -\lambda_j e_j$, with $\lambda_j \leq \lambda_{j+1}$. We will assume that these eigenfunctions are counted with multiplicity and are normalized so that

$$\int_{S^{n-1}} e_j^2 \, d\theta = 1.$$  

Functions on $\mathbb{R}^n$ or on $S^n$ which are invariant under the action of some group: Given $1 \leq m \leq n-1$, we can decompose $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$. We will only be interested in function on $\mathbb{R}^n$ and eigenfunctions of $\Delta_{S^{n-1}}$ which have some special symmetry. Namely functions which are invariant under the action of the following subgroup of $O(n, \mathbb{R})$

$$\mathcal{H}(n, m) := O(n-m, \mathbb{R}) \otimes \mathcal{D}(m).$$  

It will be convenient to define $\mathcal{J}(n, m)$ to be the set of indices $j \in \mathbb{N}$ corresponding to eigenfunctions $e_j$ which are invariant under the action of $\mathcal{H}(n, m)$, that is

$$\mathcal{J}(n, m) := \{ j \geq 0 \mid e_j = e_j \circ R, \text{ for all } R \in \mathcal{H}(n, m) \}.$$  

It will be important to observe that 1, 2, \ldots, $n$ do not belong to $\mathcal{J}(n, m)$ since the eigenfunctions corresponding to the eigenvalues $\lambda_1 = \ldots = \lambda_n$ are not invariant under the action of $-I_n \in \mathcal{H}(n, m)$.

For all $k \in \mathbb{N}$ and all $\alpha \in (0, 1)$, we define $\mathcal{C}^{k, \alpha}(S^{n-1} \setminus \mathcal{J}(n, m))$ to be the subset of functions of $\mathcal{C}^{k, \alpha}(S^{n-1})$ whose eigenfunction decomposition only involves indices belonging to $\mathcal{J}(n, m)$. In other words, $g \in \mathcal{C}^{k, \alpha}(S^{n-1} \setminus \mathcal{J}(n, m))$ if and only if $g \in \mathcal{C}^{k, \alpha}(S^{n-1})$ and

$$g = g_0 + \sum_{j \in \mathcal{J}} g_j e_j.$$
Observe that, by definition, any function of $C^{k,\alpha}(S^{n-1},\mathcal{H}(n,m))$ is orthogonal to $e_1,\ldots,e_{n-1}$ in the $L^2$ sense, on $S^{n-1}$.

**Notations:** Given $1 \leq m \leq n-1$, we will adopt the following notation:

- $x$ or $(x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^m \sim \mathbb{R}^{n-m}$,
- $(x, z) \in \mathbb{R}^n \times \mathbb{R} \sim \mathbb{R}^{n+1}$,
- $\theta$ will denote a point in $S^{n-1}$.

### 3 Minimal hypersurfaces close to a truncated $n$-catenoid

This section is mainly adapted from [2], we first recall some of the technical results of [2] which are needed in this paper and adapt them to our situation.

#### 3.1 The $n$-catenoid

Assume that $n \geq 3$ is fixed. We recall some well known fact concerning the unit $n$-catenoid $C_1$ which is a minimal hypersurface of revolution in $\mathbb{R}^{n+1}$, further details are available in [2]. By definition, $C_1$ is the minimal hypersurface of revolution parameterized by

$$X_0 : (s, \theta) \in \mathbb{R} \times S^{n-1} \rightarrow (\varphi(s) \theta, \psi(s)) \in \mathbb{R}^{n+1}, \quad (4)$$

where $\varphi$ to be the unique, smooth, non constant solution of

$$(\partial_s \varphi)^2 + \varphi^{4-2n} = \varphi^2 \quad \text{with} \quad \varphi(0) = 1,$$

and where the function $\psi$ is the unique solution of

$$\partial_s \psi = \varphi^{2-n} \quad \text{with} \quad \psi(0) = 0.$$

As already mentioned in the introduction, it might be interesting to observe that $\varphi$ is explicitly given by the identity

$$\varphi^{-n}(s) = \cosh((n-1)s)).$$

Using this, it is easy to check that the function $\psi$ converges at $\pm \infty$. We set

$$c_\infty := \lim_{s \to \pm \infty} \psi.$$

The fact that $\psi$ converges at both $\pm \infty$ implies that the hypersurface $C_1$ has two hyperplanar ends and is in fact contained between the two asymptotic hyperplanes defined by $z = \pm c_\infty$. In addition, the upper end (resp. lower end) of the unit $n$-catenoid can be parameterized as a graph over the $z = 0$ hyperplane for some function $u$ (resp. $-u$). It is easy to check that the function $u$ has the following expansion as $r := |x|$ tends to $\infty$

$$u = c_\infty - \frac{1}{n-2} r^{2-n} + O(r^{4-3n}). \quad (5)$$
3.2 The mean curvature operator

Let us assume that the orientation of $C_1$ is chosen so that the unit normal vector field is given by

$$N_0 := \frac{1}{\varphi} (\partial_s \psi, -\partial_s \varphi).$$

(6)

All surfaces close enough to $C_1$ can be parameterized (at least locally) as normal graphs over $C_1$, namely as the image of

$$X_w := X_0 + w \varphi^{\frac{2-n}{2n}} N_0,$$

for some small function $w$. The following technical result is borrowed from [2]. It just states that the mean curvature of the hypersurface parameterized by $X_w$ has some nice expansion in terms of $w$. Observe that, in order to define $X_w$, we have used $w \varphi^{\frac{2-n}{2n}} N_0$ instead of the usual $w N_0$, there is no loss of generality in doing so and this choice will simplify the notations in the forthcoming result which describes the structure of the nonlinear partial differential equation $w$ has to satisfy in order for the hypersurface parameterized by $X_w$ to be minimal.

**Proposition 1** [3]: The hypersurface parameterized by $X_w$ is minimal if and only if the function $w$ is a solution of the nonlinear elliptic partial differential equation

$$\mathcal{L} w = \varphi^{\frac{2-n}{2n}} Q_2 \left( \varphi^{-\frac{n}{2}} w \right) + \varphi^{\frac{2}{n}} Q_3 \left( \varphi^{-\frac{n}{2}} w \right),$$

(7)

where

$$\mathcal{L} := \partial_s^2 + \Delta_{S^{n-1}} - \left( \frac{n-2}{2} \right)^2 + \frac{n(3n-2)}{4} \varphi^{2-2n},$$

where $\xi \to Q_2(\xi)$ is a nonlinear second order differential operator which is homogeneous of degree 2 and where $\xi \to Q_3(\xi)$ is a nonlinear second order differential operator which satisfies

$$Q_3(0) = 0, \quad D_\xi Q_3(0) = 0 \quad \text{and} \quad D_\xi^2 Q_3(0) = 0.$$

Furthermore, the coefficients of $Q_2$ on the one hand and the coefficients in the Taylor expansion of $Q_3$ with respect to the $\xi$, computed at any $\xi$ in some fixed neighborhood of $0$ in $C^{3,\alpha}(\mathbb{R} \times S^{n-1})$ on the other hand are bounded functions of $s$ and so are the derivatives of any order of these functions.

The operator $\mathcal{L}$ is clearly equivariant with respect to any action of the form

$$\mathbb{R} \times S^{n-1} \ni (s, \theta) \longrightarrow (-s, R \theta) \in \mathbb{R} \times S^{n-1},$$

when $R \in \mathfrak{so}(n,m)$. Since in addition the mean curvature is invariant by isometries, we conclude that the nonlinear operator which appears on the right hand side of (6) also enjoys this equivariance property.

It might be useful to rephrase the properties of the nonlinear operators $Q_2$ and $Q_3$ into a slightly weaker form. It follows from the properties of $Q_2$ and $Q_3$ that there exist constants $c, c_0 > 0$ such that, for all $s \in \mathbb{R}$ and all $\xi_1, \xi_2 \in C^{2,\alpha}([s, s+1] \times S^{n-1})$, we have

$$\|Q_2(\xi_1) - Q_2(\xi_2)\|_{C^{0,\alpha}} \leq c \left( \sup_{i=1,2} \|\xi_i\|_{C^{2,\alpha}} \right) \|\xi_2 - \xi_1\|_{C^{2,\alpha}}$$

(8)

and, provided $\|\xi_i\|_{C^{2,\alpha}} \leq c_0$, we also have

$$\|Q_3(\xi_1) - Q_3(\xi_2)\|_{C^{0,\alpha}} \leq c \left( \sup_{i=1,2} \|\xi_i\|_{C^{2,\alpha}} \right)^2 \|\xi_2 - \xi_1\|_{C^{2,\alpha}},$$

(9)
where all norms are understood on \([s, s + 1] \times S^{n-1}\). Since \(Q_2\) is homogeneous of degree 2 no assumptions on \(\xi_i\) are required in order to get the estimate involving \(Q_2\), however they are required for the estimates involving \(Q_3\).

Let us warn the reader that the operator \(L\) which appears in this result is not the Jacobi operator which is defined to be the linearized mean curvature operator when nearby hypersurfaces are normal graphs over the \(n\)-catenoid, that is when they are parameterized by

\[
\tilde{X}_w := X_0 + w N_0,
\]

but \(L\) is conjugate to the Jacobi operator.

### 3.3 Linear analysis

Projecting the operator \(L\) over the eigenspaces spanned by \(e_j\), we are left with the study of the sequence of operators

\[
L_j := \partial_s^2 - \lambda_j - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4} \varphi^{2-2n}, \quad j \in \mathbb{N}.
\]

The indicial roots of \(L\) at both \(+\infty\) or \(-\infty\) are given by \(\pm \gamma_j\) where

\[
\gamma_j := \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}.
\]

Let us recall that these indicial roots appear in the study of the asymptotic behavior at \(\pm \infty\) of the solutions of the homogeneous problem \(L_j w = 0\). More precisely, for each \(j \in \mathbb{N}\), one can find \(w_j^\pm\), two independent solutions of \(L_j w = 0\) such that \(w_j^+(s) \sim e^{\gamma_j s}\) and \(w_j^-(s) \sim e^{-\gamma_j s}\) at \(+\infty\). Observe that the functions \(s \to w_j^\pm(-s)\) are solutions of \(L_j w = 0\) such that \(w_j^+(s) \sim e^{-\gamma_j s}\) and \(w_j^-(s) \sim e^{\gamma_j s}\) at \(-\infty\). These indicial roots will play a crucial rôle in the study of the mapping properties of \(L\).

To keep the notations short, we define the second order elliptic operator

\[
\Delta_0 := \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2,
\]

which acts on functions defined on \(\mathbb{R} \times S^{n-1}\). In particular

\[
L = \Delta_0 + \frac{n(3n-2)}{4} \varphi^{2-2n}.
\]

The indicial roots of \(\Delta_0\) at both \(+\infty\) or \(-\infty\) are also given by \(\pm \gamma_j\).

It is straightforward to check that \(\Delta_0\) satisfies the maximum principle and also that the operator \(L\) does not satisfy the maximum principle because of the presence of the extra potential. Indeed, one can check that the functions

\[
\Psi_0^0 := \partial_s (\varphi \frac{n-2}{2}), \quad \Psi_0^1 := \varphi \partial_s \varphi - \psi \partial_s \psi,
\]

and, for \(j = 1, \ldots, n\), the functions

\[
\Psi_j^0 := \varphi \partial_s (\varphi \partial_s \varphi + \psi \partial_s \psi) e_j, \quad \Psi_j^1 := \varphi \partial_s \varphi e_j,
\]

are Jacobi fields, i.e. are solutions of the homogeneous problem \(L w = 0\), and that the \(\Psi_j^1\) are bounded. Nevertheless, the following result, borrowed from [2], asserts that, the operator \(L\) still satisfies the maximum principle if it is restricted to the higher eigenspaces of the cross-sectional Laplacian \(\Delta_{S^{n-1}}\).
Proposition 2 Assume that $\delta < \frac{n+2}{2}$ is fixed and that $w$ is a solution of
\[ Lw = 0, \]
which is bounded by $\varphi^\delta$ on $(s_1, s_2) \times S^{n-1}$ and which satisfies $w = 0$ on $\{s_1\} \times S^{n-1}$, if any of the $s_i$ is finite. Further assume that, for each $s \in (s_1, s_2)$, the function $w(s, \cdot)$ is orthogonal to $e_0, \ldots, e_n$ in the $L^2$ sense on $S^{n-1}$. Then $w \equiv 0$.

In view of the previous result, it is natural to consider the operator $L$ acting on functions bounded by a constant times a power of the function $\varphi$. As in [1] and [2], we define a family of weighted Hölder spaces by:

Definition 1 For all $\delta \in \mathbb{R}$, the space $C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1})$ is defined to be the space of functions $w \in C^{k,\alpha}_{\text{loc}}(\mathbb{R} \times S^{n-1})$ for which the following norm is finite
\[ \|w\|_{C^{k,\alpha}_\delta} := \sup_{s \in \mathbb{R}} \varphi^{-\delta} |w|_{C^{k,\alpha}([s, s+1] \times S^{n-1})}. \]
Here $|w|_{C^{k,\alpha}([s, s+1] \times S^{n-1})}$ denotes the Hölder norm in $[s, s+1] \times S^{n-1}$.

Moreover, for any $S > 0$, the space $C^{2,\alpha}_\delta([-S, S] \times S^{n-1})$ is defined to be the space of restriction of functions of $C^{2,\alpha}_\delta(\mathbb{R} \times S^{n-1})$ to $[-S, S] \times S^{n-1}$. This space is naturally endowed with the induced norm.

Though this will not be necessary for the remaining of the analysis, we quote here some well known properties of the operator
\[ L : C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}) \to C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}). \]
To keep track of the weighted space over which $L$ is defined, we will denote the above operator by $L_\delta$. The most important fact is that the mapping properties of $L_\delta$ crucially depend on the choice of the weight parameter $\delta$. Indeed, it follows from general arguments that $L_\delta$ has closed range and is even Fredholm if and only if the weight $\delta$ is not equal to any of the indicial roots $\pm \gamma_j$, $j \in \mathbb{N}$, (a fact which, given the special structure of our operator, can be easily proven by separation of variables). The fact that the functions given in (12) are Jacobi fields shows that $L_\delta$ is not injective when $\delta > \frac{n}{2}$ and it can be proven, with the help of Proposition 2, that $L_\delta$ is injective if $\delta < -\frac{n}{2}$. This later fact in turn implies that $L_\delta$ is surjective if $\delta > \frac{n}{2}$ is not equal to any $\gamma_j$, $j \geq 0$ (this uses the fact that the operator $L_\delta$ and $L_{-\delta}$ are, in some sense, dual).

As already mentioned in §2, we will only be interested in functions which are invariant under the action of some group. This is the reason why we introduce the :

Definition 2 For all $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\delta \in \mathbb{R}$, the space $C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}, \mathcal{S}(n, m))$ is defined to be the space of functions $w \in C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1})$ which satisfy
\[ \forall (s, \theta) \in \mathbb{R} \times S^{n-1}, \quad w(s, \theta) = w(-s, \theta), \]
and also
\[ \forall (s, \theta) \in \mathbb{R} \times S^{n-1}, \quad w(s, R\theta) = w(s, \theta), \]
for all $R \in \mathcal{S}(n, m)$. This space is endowed with the induced norm.

Observe that, any function $w \in C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}, \mathcal{S}(n, m))$ can be decomposed as
\[ w(s, \theta) = \sum_{j \in \mathcal{J}} w_j(s) e_j(\theta), \]
where, for all \(j\), all functions \(s \rightarrow w_j(s)\) are even.

Observe that the Jacobi fields defined in (12) are not invariant with respect to the action of \(\mathfrak{h}(n, m)\), hence one can show that

\[
\mathcal{L} : C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}, \mathfrak{h}(n, m)) \rightarrow C^{k,\alpha}_\delta(\mathbb{R} \times S^{n-1}, \mathfrak{h}(n, m))
\]

is injective for all \(\delta < \frac{2-n}{2}\) and surjective for all \(\delta > \frac{2-n}{2}\), which is not equal to any \(\gamma_j\), for \(j \geq 0\). We will not need such general statements but, since we will be working with functions defined on \([-S, S] \times S^{n-1}\).

Among the Jacobi fields defined in (11) and (12),

\[
\Psi^{+,0} = \varphi^{\frac{2-n}{2}}(\varphi \partial_s \psi - \psi \partial_s \varphi).
\]

is the only one which is invariant with respect to the action of \(\mathfrak{h}(n, m)\) and which is an even function of \(s\). It is easy to see that this Jacobi field vanishes for finitely many values of \(s\). Let us define \(s_0 > 0\) to be the largest zero of the function \(\Psi^{+,0}\).

The result we will need reads:

**Proposition 3** Assume that \(\delta \in \left(\frac{2-n}{2}, \frac{n-2}{2}\right)\) and \(\alpha \in (0, 1)\) are fixed. There exists some constant \(c > 0\) and, for all \(S > s_0\), there exists an operator

\[
\mathcal{G}_S : C^{0,\alpha}_\delta([-S, S] \times S^{n-1}, \mathfrak{h}(n, m)) \rightarrow C^{2,\alpha}_\delta([-S, S] \times S^{n-1}, \mathfrak{h}(n, m)),
\]

such that, for all \(f \in C^{0,\alpha}_\delta([-S, S] \times S^{n-1}, \mathfrak{h}(n, m))\), the function \(w = \mathcal{G}_S(f)\) is a solution of

\[
Lw = f
\]

in \((-S, S) \times S^{n-1}\) with \(w = 0\) on \(\{\pm S\} \times S^{n-1}\). Furthermore, \(\|w\|_{C^{2,\alpha}} \leq c \|f\|_{C^{0,\alpha}}\).

**Proof:** Our problem being linear, we can assume without loss of generality that

\[
\sup_{[-S, S] \times S^{n-1}} \varphi^{-\delta} |f| = 1.
\]

Observe that, it follows from Proposition 2 that, when restricted to the space of functions \(w\) such that \(w(s, \cdot)\) is orthogonal to \(e_0, \ldots, e_n\) in the \(L^2\) sense on \(S^{n-1}\), the operator \(\mathcal{L}\) is injective over \((-S, S) \times S^{n-1}\). Also, if \(s > s_0\) then \(\mathcal{L}\) is injective over \((-S, S) \times S^{n-1}\) when restricted to functions which are even and only depend on \(s\). As a consequence, for all \(S > s_0\), we are able to solve \(Lw = f\) in \((-S, S) \times S^{n-1}\), with \(v = 0\) on \(\{\pm S\} \times S^{n-1}\).

We claim that there exists some constant \(c > 0\), independent of \(S > s_0\) and of \(f\), such that

\[
\sup_{[-S, S] \times S^{n-1}} |\varphi^{-\delta} w| \leq c.
\]

Observe that the result is true when \(S > 1\) stays bounded. We argue by contradiction and assume that the result is not true. In this case, there would exist a sequence \(S_k > 1\) tending to \(+\infty\), a sequence of functions \(f_k\) satisfying

\[
\sup_{[-S_k, S_k] \times S^{n-1}} |\varphi^{-\delta} f_k| = 1,
\]

and a sequence \(v_k\) of solutions of \(Lv_k = f_k\) in \((-S_k, S_k) \times S^{n-1}\), with \(v_k = 0\) on \(\{\pm S_k\} \times S^{n-1}\) such that

\[
A_k := \sup_{[-S_k, S_k] \times S^{n-1}} |\varphi^{-\delta} v_k| \rightarrow +\infty.
\]
Let us denote by \((s_k, \theta_k) \in [0, S_k) \times S^{n-1}\), a point where the above supremum is achieved, observe that all the functions we consider are even in the \(s\) variable. We claim that the sequence \(S_k - s_k\) remains bounded away from 0. Indeed, since \(v_k\) and \((\partial^2_s + \Delta_{S^{n-1}}) v_k\) are both bounded by a constant (independent of \(k\)) times \(\varphi^\delta(S_k) A_k\) in \([S_k - 1, S_k] \times S^{n-1}\) and since \(v_k = 0\) on \(\{S_k\} \times S^{n-1}\), we may apply standard elliptic estimates and conclude that the gradient of \(v_k\) is also uniformly bounded by a constant times \(\varphi^\delta(S_k) A_k\) in \([S_k - \frac{1}{2}, S_k] \times S^{n-1}\). As a consequence the above supremum cannot be achieved at a point which is too close to \(S_k\). Therefore, up to some subsequence, we may also assume that the sequence \(S_k - s_k\) converges to \(S^* \in (0, +\infty]\).

We now distinguish a few cases according to be the behavior of the sequence \(s_k\), which, up to a subsequence, can be assumed to converge in \([0, +\infty]\).

We define the sequence of rescaled functions

\[
\tilde{v}_k(s, \theta) := \frac{\varphi^{-\delta}(s_k)}{A_k} v_k(s + s_k, \theta).
\]

**Case 1:** Assume that the sequence \(s_k\) converges to \(s_* \in \mathbb{R}\). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \(\tilde{v}_k(\cdot, -s_*, \cdot)\) converges on compact to \(v\) some nontrivial solution of

\[
\mathcal{L} v = 0,
\]

in \(\mathbb{R} \times S^{n-1}\). Furthermore

\[
\sup_{\mathbb{R} \times S^{n-1}} \varphi^{-\delta} |v| = \varphi^{-\delta}(s_*).
\]

Moreover, for each \(s \in \mathbb{R}\), the function \(v(s, \cdot)\) is orthogonal in the \(L^2\) sense to \(e_1, \ldots, e_n\) on \(S^{n-1}\). But, the result of Proposition 3 together with the fact that \(\Psi_{\pm, 0}\) are the only solutions of \(\mathcal{L} w = 0\) which only depend on \(s\), implies that \(v \equiv 0\), contradicting (13).

**Case 2:** Assume that the sequence \(s_k\) converges to \(+\infty\). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \(\tilde{v}_k\) converges to \(v\) some nontrivial solution of

\[
\Delta_0 v = 0,
\]

in \((-\infty, S^*) \times S^{n-1}\), with boundary condition \(v = 0\), if \(S^*\) is finite. Furthermore

\[
\sup_{(-\infty, S^*) \times S^{n-1}} e^{-\delta s} |v| = 1.
\]

independently of the fact that \(S^*\) finite or is not, this case is easy to rule out using the eigenfunction decomposition of \(v\)

\[
v = \sum_{j \in \mathbb{Z} - \{0\}} v_j e_j.
\]

Indeed, \(v_j\) has to be a linear combination of the functions \(e^{\pm \gamma_j s}\) (where \(\gamma_j\) has been defined in (10)) and is bounded by \(e^{\delta s}\). Since we have assumed that \(\delta \in (-\frac{n+2}{2}, \frac{n+2}{2})\) and since \(j \geq n + 1\), it is easy to see that all \(v_j \equiv 0\), contradicting (13).

We have reached a contradiction in all cases, hence, the proof of the claim is finished. To complete the proof of the Proposition, it suffices sum the two results we have just obtained and apply Schauder’s estimates in order to get the relevant estimates for all the derivatives. \(\square\)

We will also need some properties of the Poisson operator for \(\Delta_0\) on \([0, \infty) \times S^{n-1}\). The result we will need is standard and a proof can be found, for example, in [2]:

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Lemma 1 There exists $c > 0$ such that, for all $g \in C^{2,\alpha}(S^{n-1}, S(n,m))$, there exists a unique $w \in C^{2,\alpha}_{\text{rad}}((0, +\infty) \times S^{n-1})$ solution of
\[
\begin{cases}
\Delta_0 w = 0 & \text{in } (0, +\infty) \times S^{n-1} \\
w = g & \text{on } \{0\} \times S^{n-1}.
\end{cases}
\]
Furthermore, we have $||w||_{C^{2,\alpha}} \leq c ||g||_{C^{2,\alpha}}$ and, for all $s > 0$ the function $w(s, \cdot)$ is invariant with respect to the action of $\mathcal{S}(n,m)$.

The idea behind the proof of this result is that one can use the eigenfunction decomposition of $g$ to obtain an explicit solution of (15) together with the estimate. In the remaining of the paper, we will denote by $\mathcal{P}(g)$ the solution of (15).

3.4 The nonlinear problem
We fix $\rho \in [0, 1]$ and, for all $\varepsilon \in (0, \rho)$, we define $s_\varepsilon > 0$ by the identity
\[\rho = \varepsilon \varphi(s_\varepsilon) > 0.\]

Let us notice that, as $\varepsilon$ tends to 0, we have
\[s_\varepsilon \sim -\log \varepsilon.\]

In order to parameterize the unit $n$-catenoid we use (4) and define the outer normal $N_0$ as in (4). Let us define a smooth function $\xi_\varepsilon : \mathbb{R} \to [-1, 1]$ which satisfies $\xi_\varepsilon = -1$ for $s \geq s_\varepsilon - 1$, $\xi_\varepsilon = 1$ for $s \leq 1 - s_\varepsilon$ and $\xi_\varepsilon = -\frac{\partial \varphi}{\varphi}$ for $|s| \leq s_\varepsilon - 2$ and which interpolates smoothly between those two functions when $|s| \in [s_\varepsilon - 2, s_\varepsilon - 1]$. We consider the vector field
\[N_\varepsilon(s, \theta) := (\sqrt{1 - \xi_\varepsilon^2(s)} \theta, \xi_\varepsilon(s)).\]

It turns out that this vector field is a perturbation of the unit normal $N_0$, and in fact, we have for all $k \geq 0$
\[|\nabla^k (N_\varepsilon \cdot N_0 - 1)| \leq c_k \varepsilon^{2n-2},\]
for all $|s| \geq s_\varepsilon - 2$.

We now look for all minimal hypersurfaces close to the unit $n$-catenoid which has been rescaled by a factor $\varepsilon$. The hypersurfaces we are looking for will be parameterized by
\[X_w := \varepsilon X_0 + w \varphi^\frac{2}{2-n} N_\varepsilon,\]
for $(s, \theta) \in [-s_\varepsilon, s_\varepsilon] \times S^{n-1}$ and for some small function $w$. It follows from (7) that such an hypersurface is minimal if and only if $w$ satisfies a nonlinear equation of the form
\[\mathcal{L}w = \bar{Q}_\varepsilon(w),\]
where
\[\bar{Q}_\varepsilon(w) := \varepsilon^{2n-2} L_\varepsilon w + \varepsilon \varphi^{\frac{2}{2-n}} \bar{Q}_{2,\varepsilon} \left(\varphi^{-\frac{n}{2}} \varepsilon^{-1} w\right) + \varepsilon \varphi^{\frac{n}{2}} \bar{Q}_{3,\varepsilon} \left(\varphi^{-\frac{n}{2}} \varepsilon^{-1} w\right).\]

Here $\bar{Q}_{2,\varepsilon}$ and $\bar{Q}_{3,\varepsilon}$ enjoy properties which are similar to those enjoyed by $Q_2$ and $Q_3$, namely (8) and (9) still hold uniformly in $\varepsilon \in (0, \rho)$. The linear operator $\varepsilon^{2n-2} L_\varepsilon$ represents the difference between the linearized mean curvature operator for hypersurfaces parameterized using the vector field $N_0$ and those parameterized using the vector field $N_\varepsilon$. The operator $L_\varepsilon$ has coefficients which are supported in $([-s_\varepsilon, 2 - s_\varepsilon] \cup [s_\varepsilon - 2, s_\varepsilon]) \times S^{n-1}$ and which are uniformly bounded in
have obtained so far and translate them in the geometric framework. Let us fix not be useful.

where the operator $\mathcal{L}_\varepsilon$ has been defined in Proposition 4. Using (18) together with Proposition 1 and the properties of $\mathcal{Q}_\varepsilon$, we can estimate

$$\|\varepsilon^{2n-2} L_\varepsilon \tilde{w} - \mathcal{L}_\varepsilon \tilde{w}\|_{C^{0,\alpha}} \leq c \left( 1 + \varepsilon^{\frac{3n-2}{2} + \delta} \right) \|h\|_{C^{2,\alpha}},$$

$$\|\varepsilon^{\frac{2}{n}} \mathcal{Q}_{2,\varepsilon} \left( \varepsilon^{-1} \tilde{w} \right) \|_{C^{0,\alpha}} \leq c \left( \varepsilon^{-1} + \varepsilon^{\frac{3}{2} + \delta} \right) \|h\|_{C^{2,\alpha}},$$

and finally, there exists $\varepsilon_0 > 0$ (which depends on $\kappa$) such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\|\varepsilon^{\frac{2}{n}} \mathcal{Q}_{3,\varepsilon} \left( \varepsilon^{-1} \tilde{w} \right) \|_{C^{0,\alpha}} \leq c \left( \varepsilon^{-2} + \varepsilon^{\frac{2n}{2} + \delta} \right) \|h\|_{C^{2,\alpha}},$$

In the above estimates, the constant $c > 0$ does not depend on $\varepsilon$, nor on $\kappa$. Observe that in order to obtain the last estimate, we have implicitly used that fact that $\|h\|_{C^{2,\alpha}}$ is small enough so that we can apply (4), or rather its counterpart for $\mathcal{Q}_{3,\varepsilon}$. This explains the restriction of $\varepsilon \in (0, \varepsilon_0)$ which is needed.

It is then a simple exercise to show that for any fixed $\kappa > 0$, there exist $c > 0$ and $\varepsilon_0 > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0)$, the nonlinear mapping $\mathcal{N}_\varepsilon$ is a contraction in the ball of radius

$$R(\varepsilon, h) := c \|h\|_{C^{2,\alpha}},$$

in $C^{2,\alpha}([-s_\varepsilon, s_\varepsilon] \times S^{n-1}, \mathcal{F}(n, m))$ into itself, and hence $\mathcal{N}_\varepsilon$ has a unique fixed point $v_\varepsilon$ in this ball. Therefore, the function $\tilde{w} + v_\varepsilon$ is a solution of (14) whose boundary data is given by $h$, up to a constant function. We can even choose the constant $c$ to be independent of $\kappa$, but this will not be useful.

**Family of minimal hypersurfaces close to $n$-catenoid**: We summarize the results we have obtained so far and translate them in the geometric framework. Let us fix $\delta \in \left( \frac{2-n}{2}, \frac{n-2}{2} \right)$,
\( \alpha \in (0, 1) \) and \( \kappa > 0 \). There exists \( c > 0 \) and \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and for all \( h \in C^{2, \alpha}(S^{n-1}, \mathcal{S}(n, m)) \) satisfying

\[
||h||_{2, \alpha} \leq \kappa \varepsilon^{n-1},
\]

there exists a minimal hypersurface, which will be denoted by \( C_\varepsilon(h) \subset \mathbb{R}^{n+1} \), and which is parameterized by

\[
X_w = \varepsilon X_0 + w \varphi^{\frac{2-n}{2}} N_\varepsilon \quad \text{in} \quad [-s_\varepsilon, s_\varepsilon] \times S^{n-1},
\]

for some function \( w \) satisfying

\[
\|w\|_{C^{2, \alpha}} \leq c \|h\|_{C^{2, \alpha}}.
\]

This hypersurface is symmetric with respect to the hyperplane \( z = 0 \) and further inherits all the symmetries induced by the symmetries used to define the function spaces in Definition 2, hence it is invariant with respect to the action of \( O(n-m, \mathbb{R}) \otimes \mathcal{D}_m \otimes \{\pm I_1\} \subset O(n+1, \mathbb{R}) \). Furthermore, if we perform the change of variable

\[ r := \varepsilon \varphi(s), \]

we see that near its upper boundary, this hypersurface is the graph of the function

\[ x \in \overline{B_\rho} \setminus B_\rho^{n/2} \rightarrow \varepsilon c_\infty - W_h(x) - V_{\varepsilon, h}(x), \]

over the \( z = 0 \) hyperplane. Here \( W_h \) denotes the (unique) harmonic extension of the boundary data \( h \) in \( B_\rho^n \) and the function \( V_{\varepsilon, h} \) satisfies

\[
\|V_{\varepsilon, h}\|_{C^{2, \alpha}} \leq c_0 \varepsilon^{n-1},
\]

for some constant \( c_0 \) which does not depend on \( \kappa \) nor on \( \varepsilon \). Here the norms are taken over \( \overline{B_\rho^n} - B_\rho^{n/2} \). This last claim, which is a key point of our analysis, follows from (3). Indeed, when \( h = 0 \), \( C_\varepsilon(0) \) is just a rescaled \( n \)-catenoid and, using (3) we see that its upper end is the graph of the function

\[ x \rightarrow \varepsilon c_\infty + \mathcal{O}(\varepsilon^{n-1} r^{2-n}). \]

We have also used the fact that the solution of (19) we have constructed is equal to \( \tilde{w} + v_h \) where \( \tilde{w} \), defined in (17), is linear in \( h \) and where \( v_h \) can be estimated by a constant (independent of \( \varepsilon \) and \( \kappa \)) times \( \|h\|_{C^{2, \alpha}} \varphi^{\delta} \). Essentially the constant \( c_0 \) arises from the term \( \mathcal{O}(\varepsilon^{n-1} r^{2-n}) \) in the above expansion, the contributions of \( v_h \) and the perturbation caused by the change of variable being negligible when \( \varepsilon \) is chosen small enough.

Observe that, reducing \( \varepsilon_0 \) if this is necessary, we can assume that the mapping \( h \rightarrow V_{\varepsilon, h} \) is continuous and in fact smooth. With little work we also find that

\[
\|V_{\varepsilon, h_2} - V_{\varepsilon, h_1}\|_{C^{2, \alpha}} \leq c \varepsilon^{\frac{n-2}{2} - \delta} \|h_2 - h_1\|_{C^{2, \alpha}}, \tag{20}
\]

for some constant \( c > 0 \) which does not depend on \( \varepsilon \). The norm on the left hand side of this inequality is understood to be the norm on \( \overline{B_\rho^n} - B_\rho^{n/2} \). Again the constant \( c \) can be chosen to be independent of \( \kappa \) but this will be irrelevant for the remaining of the analysis.

4 Minimal hypersurfaces which are graphs over an hyperplane

We are now concerned with both the mean curvature and the linearized mean curvature operator for hypersurfaces which are graphs over the \( z = 0 \) hyperplane, in \( \mathbb{R}^{n-m} \times T^m \times \mathbb{R} \).
4.1 The mean curvature operator for graphs

We assume that $n \geq 3$ and $1 \leq m \leq n - 1$ are fixed. Further assume that $T^m \in \mathcal{T}$ is fixed. Then, for any function $u$, defined in $\mathbb{R}^{n-m} \times T^m$, which is at least of class $C^2$, we can define an hypersurface which is the graph of $u$

$$\mathbb{R}^{n-m} \times T^m \ni (x_1, x_2) \mapsto (x_1, x_2, u(x_1, x_2)) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R}.$$  

Recall that the mean curvature of this hypersurface, with downward pointing unit normal, is then given by

$$H_u := -\frac{1}{n} \text{div} \left( \frac{\nabla u}{1 + |\nabla u|^2/2} \right). \quad (21)$$

4.2 Linear analysis

We define the function spaces which are adapted to the analysis of the Laplacian in $T^m \times \mathbb{R}^{n-m}$. Our main concern will be the asymptotic behavior of the functions as $|x_1|$ tends to $+\infty$.

**Definition 3** For all $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\nu \in \mathbb{R}$, the space $C^k_\nu,\alpha(\mathbb{R}^{n-m} \times T^m)$ is defined to be the space of functions $w \in C^k_{\text{loc}}(\mathbb{R}^{n-m} \times T^m)$ for which the following norm is finite

$$\|w\|_{C^k_\nu,\alpha} := |w|_{C^k_\nu,\alpha}(B^{n-m}_1 \times T^m) + \sup_{r > 1/2} r^{-\nu} |w(r \cdot)|_{C^k_{\nu}(B^{n-m}_1 - B^{n-m}_m \times r^{-1} T^m)}.$$  

Here $| \cdot |_{C^k_{\nu}(\Omega)}$ denotes the Hölder norm in $\Omega$.

To get a better understanding of these weighted spaces, if $T_m = \mathbb{R}^m/AZ^m$, we can identify any function defined on $\mathbb{R}^{n-m} \times T^m$ with a function defined on $\mathbb{R}^{n-m} \times \mathbb{R}^m$ which has $\{0\} \otimes AZ^m$ as its group of periods. In which case functions which belong to $C^k_\nu,\alpha(\mathbb{R}^{n-m} \times T^m)$ are identified with functions defined on $\mathbb{R}^{n-m} \times T^m$, which are bounded by a constant times $(1 + |x_1|)^\nu$, whose first derivative is bounded by a constant times $(1 + |x_1|)^{\nu-1}$ (if $k \geq 1$), and so on.

As in the previous section, we will only work with functions having some special symmetry. Therefore, we introduce the :

**Definition 4** For all $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\nu \in \mathbb{R}$, the space $C^k_{\nu,\alpha}(\mathbb{R}^{n-m} \times T^m, \mathcal{J}(n, m))$ is defined to be the space of functions $w \in C^k_{\text{loc}}(\mathbb{R}^{n-m} \times T^m)$ which are invariant under the action of $\mathcal{J}(n, m)$.

Observe that, because of the invariance of our function space with respect to the action of $\mathcal{J}(n, m)$, any function $w \in C^k_{\nu,\alpha}(\mathbb{R}^{n-m} \times T^m, \mathcal{J}(n, m))$ can be decomposed as

$$w(x_1, x_2) = \sum_{i \geq 1} w_i(r_1) E_i(x_2),$$

where $\mathcal{J}(m) \subset \mathbb{N}$ has been defined in (3) and where

$$r_1 := |x_1|.$$  

To begin with let us treat the easy case where $1 \leq m \leq n - 3$. We have the :

**Proposition 4** Assume that $1 \leq m \leq n - 3$. Given $\nu \in (2 + m - n, 0)$ and $\alpha \in (0, 1)$. There exist some constant $c > 0$ and an operator

$$G : C^{0,\alpha}_{\nu - 2}(\mathbb{R}^{n-m} \times T^m, \mathcal{J}(n, m)) \rightarrow C^{2,\alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m, \mathcal{J}(n, m)),$$

such that, for all $f \in C^{0,\alpha}_{\nu - 2}(\mathbb{R}^{n-m} \times T^m, \mathcal{J}(n, m))$, the function $w = G(f)$ is a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n-m} \times T^m$. Furthermore, $\|w\|_{C^{2,\alpha}_{\nu}} \leq c \|f\|_{C^{0,\alpha}_{\nu - 2}}$.  

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The proof of the result is simplified by the fact that
\[ \Delta |x_1|^\nu = -\nu(n-m-2-\nu)|x_1|^{\nu-2} \]
Hence, the function \( w(x_1, x_2) := |x_1|^\nu \), which is defined in \((\mathbb{R}^{n-m} - \{0\}) \times T^m\) can be used as a barrier function to prove, for any \( f \in C^{\beta, \alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathcal{H}(n, m)) \), the existence of a solution of 
\[ \Delta w = f, \]
in \( \mathbb{R}^{n-m} \times T^m \). Furthermore, it also yields the estimate
\[ |w(x_1, x_2)| \leq c \|f\|_{C^{2, \alpha}_{\nu-2}} |x_1|^\nu \]
for some constant which does not depend on \( f \). The maximum principle then implies that
\[ |w(x_1, x_2)| \leq c \|f\|_{C^{2, \alpha}_{\nu-2}} (1 + |x_1|)^\nu. \]
Starting from this, Schauder’s estimates yield
\[ \|w\|_{C^{2, \alpha}} \leq c \|f\|_{C^{2, \alpha}_{\nu-2}}. \]
Details are left to the reader.

When \( m = n - 2 \) or \( m = n - 1 \), the previous result has to be modified since \( 2 + m - n \geq 0 \) in these two cases. To this aim, we choose \( \chi \) a cutoff function defined on \( \mathbb{R} \) such that \( \chi \equiv 1 \) for \( t \geq 2 \) and \( \chi \equiv 0 \) when \( t \leq 1 \). When \( m = n - 2 \), we define the space
\[ D_2 := \text{Span}\{\chi(r_1) \log r_1\} \subset C^\infty(\mathbb{R}^2), \]
and when \( m = n - 1 \), we set
\[ D_1 := \text{Span}\{\chi(r_1) r_1\} \subset C^\infty(\mathbb{R}). \]
This time we have the :

**Proposition 5** Assume that \( m = n - 2 \) or \( m = n - 1 \). Given \( \nu \in (-\infty, 0) \) and \( \alpha \in (0, 1) \). There exist some constant \( c > 0 \) and an operator
\[ G : C^{0, \alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathcal{H}(n, m)) \rightarrow C^{2, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m, \mathcal{H}(n, m)) \oplus D_{n-m}, \]
such that, for all \( f \in C^{0, \alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathcal{H}(n, m)) \), the function \( w = G(f) \) is a solution of
\[ \Delta w = f, \]
in \( \mathbb{R}^{n-m} \times T^m \). Furthermore, \( \|w\|_{C^{2, \alpha} \oplus D_{n-m}} \leq c \|f\|_{C^{0, \alpha}_{\nu-2}}. \)

**Proof** : We decompose
\[ f = f_0 + \sum_{i \in \mathcal{I} - \{0\}} f_i E_i, \]
and adopt the notation \( f = f_0 + f' \). We look for a solution \( w \) which will also be decomposed as
\[ w = w_0 + \sum_{i \in \mathcal{I} - \{0\}} w_i E_i, \]
and again we set \( w = w_0 + w' \). For notational convenience, \( f', v', w', \ldots \) will denote functions whose eigenfunction decomposition only involves indices \( i \in \mathcal{I}(m) - \{0\} \).
Observe that, because of the invariance of our problem with respect to the action of \( \mathcal{H}_n \), the Laplacian in \( \mathbb{R}^{n-m} \times T^m \) reduces to the study of the operator

\[
L := \partial_{r_1}^2 + \frac{n-m-1}{r_1} \partial_{r_1} + \Delta_{T^m},
\]

where we have set \( r_1 := |x_1| \).

**Step 1 :** We would like to prove the existence of \( w' \) and also obtain the relevant estimate. Our problem being linear, we may always assume that

\[
\sup_{\mathbb{R}^{n-m} \times T^m} (1 + r_1)^{2-\nu} |f'| = 1.
\]

Obviously \( \Delta \), or \( L \), is injective over any \( B_{R}^{n-m} \times T^m \). As a consequence, for any \( R > 1 \) we are able to solve \( \Delta v' = f' \), in \( B_{R}^{n-m} \times T^m \), with \( v' = 0 \) on \( \partial B_{R}^{n-m} \times T^m \).

We claim that, there exists a constant \( c > 0 \), independent of \( R > 1 \) and of \( f' \), such that

\[
\sup_{B_{R}^{n-m} \times T^m} (1 + r_1)^{-\nu} |v'| \leq c.
\]

Observe that the result is certainly true if we assume that \( R \) remains bounded. We argue by contradiction and assume that the claim is not true. In this case, there would exist a sequence \( R_k > 1 \) tending to \( +\infty \), a sequence of functions \( f_k' \) satisfying

\[
\sup_{B_{R_k}^{n-m} \times T^m} (1 + r_1)^{2-\nu} |f'_k| = 1,
\]

and a sequence \( v'_k \) of solutions of \( L v'_k = f'_k \), in \( B_{R_k}^{n-m} \times T^m \), with \( v'_k = 0 \) on \( \partial B_{R_k}^{n-m} \times T^m \), such that

\[
A_k := \sup_{B_{R_k}^{n-m} \times T^m} (1 + r_1)^{-\nu} |v'_k| \longrightarrow +\infty.
\]

Let us denote by \( (x_{1,k}, x_{2,k}) \in B_{R_k}^{n-m} \times T^m \), a point where the above supremum is achieved. We now distinguish a few cases according to the behavior of the sequence \( r_{1,k} := |x_{1,k}| \) which, up to a subsequence can always be assumed to converge in \([0, +\infty)\). Observe that, as in the proof of Proposition \( \ref{prop1} \), the sequence \( R_k - r_{1,k} \) remains bounded away from 0.

We define the sequence of rescaled functions

\[
v'_k(x_{1}, x_{2}) := \frac{(1 + r_{1,k})^{-\nu}}{A_k} v'_k(r_{1,k} x_{1}, r_{1,k} x_{2}).
\]

**Case 1 :** Assume that the sequence \( r_{2,i} \) converges to \( r_{2,*} \in [0, +\infty) \). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \( v'_k(·/r_{1,k}, ·/r_{1,k}) \) converges to some nontrivial solution of

\[
Lv' = 0, \quad (22)
\]

in \( \mathbb{R}^{n-m} \times T^m \). Furthermore

\[
\sup_{\mathbb{R}^{n-m} \times T^m} (1 + r_1)^{-\nu} |v'| = (1 + r_{1,*})^{-\nu}. \quad (23)
\]

But the maximum principle implies that \( v \) is identically equal to 0. This clearly contradicts (23).

**Case 2 :** Assume that the sequence \( r_{2,k} \) converges to \( +\infty \). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \( v'_k \) converges to some nontrivial solution of

\[
\partial_{r_1}^2 v' + \frac{n-m-1}{r_1} \partial_{r_1} v' + \Delta_{\mathbb{R}^{n-m}} v' = 0,
\]

in \( \mathbb{R}^{n-m} \times T^m \). Furthermore
in \((\mathbb{R}^{n-m} - \{0\}) \times \mathbb{R}^m\), which does not depend on \(x_2\). This last claim follows from the fact that the functions \(x_2 \to \tilde{v}_k'(x_1, x_2)\) have a group of period given by \(r_{1, k}^{-1} A\mathbb{Z}^m\), if \(T^m = \mathbb{R}^m / A\mathbb{Z}^m\). In addition, \(\nabla_x \tilde{v}_k'\) is bounded by a constant depending on \(x_1\). Passing to the limit, we see that \(v'\) does not depend on \(x_2\).

Furthermore
\[
\sup_{(\mathbb{R}^{n-m} - \{0\}) \times \mathbb{R}^m} r_1^{-\nu} |v'| = 1. \tag{24}
\]
It should be clear that \(v' \equiv 0\), contradicting (24).

Since we have obtained a contradiction in both cases, this finishes the proof of the claim.

**Step 2**: We now turn our attention to the existence of \(w_0\) as well as the relevant estimate. Again, our problem reduces to one ordinary differential equation since we now have to solve
\[
\partial^2_{r_1} w_0 + \frac{n-m-1}{r_1} \partial_{r_1} w_0 = f_0.
\]
It is easy so check that \(w_0\) is given by the formula
\[
w_0 = \int_0^{r_2} \zeta^{1+m-n} \int_0^\xi t^{n-m-1} f_0 \, dt \, d\zeta - \int_0^\infty \zeta^{1+m-n} \int_0^\xi t^{n-m-1} f_0 \, dt \, d\zeta.
\]

In order to simplify the exposition, we will restrict our attention to the case where \(m \neq n-2\) since, obvious modifications have to be done to treat the case \(m = n-2\). Granted the above formula, one can directly check that we can decompose, for all \(r_2 > 1\), \(w_0 := a_0 r_1^{2+m-n} + \tilde{w}_0\), where
\[
a_0 := \frac{1}{n-m-2} \int_0^\infty t^{n-m-1} f_0 \, dt \, d\zeta,
\]
\[
\tilde{w}_0 := \int_{r_2}^\infty \zeta^{1+m-n} \int_0^\xi t^{n-m-1} f_0 \, dt \, d\zeta.
\]
Moreover, we have
\[
|a_0| + \sup_{(0, \infty)} (1 + r_1)^{-\nu} |\tilde{w}_0| \leq c \sup_{(0, \infty)} (1 + r_1)^{2-\nu} |f_0|.
\]

To complete the proof of the Proposition, it suffices to sum the two results we have just obtained and apply Schauder’s estimates in order to get the relevant estimates for all the derivatives.

If \(\rho > 0\) is fixed, we define the space \(C^{k, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathcal{H}(n, m))\) as the space of restriction of functions of \(C^{2, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m, \mathcal{H}(n, m))\) to \(\mathbb{R}^{n-m} \times T^m - B^n_\rho\). This space is naturally endowed with the induced norm.

In order to simplify notations, we set
\[
\mathcal{E}^{2, \alpha}_{\nu} := C^{2, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathcal{H}(n, m))
\]
when \(1 \leq m \leq n-3\) and
\[
\mathcal{E}^{2, \alpha}_{\nu} := C^{2, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathcal{H}(n, m)) \oplus \mathcal{D}_{n-m}
\]
when \(m = n-2\) or \(m = n-1\). We also define
\[
\mathcal{E}^{2, \alpha}_{\nu-2} := C^{2, \alpha}_{\nu}(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathcal{H}(n, m))
\]
when \(1 \leq m \leq n-1\). Using the previous result together with a standard perturbation result, we obtain the :
Proposition 6 Assume that $\nu \in (2 + m - n, 0)$ when $1 \leq m \leq n - 3$, $\nu \in (-\infty, 0)$ when $m = n - 2$ or $m = n - 1$, and $\alpha \in (0, 1)$ are fixed. There exist $\rho_0 > 0$, $c > 0$ and, for all $\rho \in (0, \rho_0)$, there exists an operator

$$G_{\rho} : F^{0,\alpha}_{\nu - 2} \to \mathcal{E}^{2,\alpha}_{\nu}$$

such that, for all $f \in F^{0,\alpha}_{\nu - 2}$, the function $w = G_{\rho}(f)$ is a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n - m} \times T^m - B^m_{\rho}$, with $w \in \mathbb{R}$ on $\partial B^m_{\rho}$. Furthermore, $\|w\|_{\mathcal{E}^{2,\alpha}_{\nu}} \leq c \|f\|_{F^{0,\alpha}_{\nu - 2}}$.

4.3 The nonlinear problem

Using (21), one can check that the hypersurface parameterized by

$$\mathbb{R}^{n - m} \times T^m - B^m_{\rho} \ni (x_1, x_2) \mapsto (x_1, x_2, u(x_1, x_2)) \in \mathbb{R}^{n - m} \times T^m \times \mathbb{R},$$

has mean curvature 0 if and only if the function $u$ is a solution of

$$\Delta u - Q(u) = 0,$$ (25)

where we have set

$$Q(u) := -\frac{1}{1 + |\nabla u|^2} \nabla^2 u (\nabla u, \nabla u).$$

Solutions of (25) which are parameterized by their boundary data: Let us assume that

$$\nu \in (2 + m - n, 0) \quad \text{when} \quad 1 \leq m \leq n - 3,$$

or that

$$\nu \in (-2, 0) \quad \text{when} \quad m = n - 2$$

or that

$$\nu \in (-\infty, 0) \quad \text{when} \quad m = n - 1$$

is fixed. The new restriction on $\nu$ when $m = n - 2$ is needed to ensure that the nonlinear operator

$$u \mapsto \Delta u - Q(u),$$

maps $\mathcal{E}^{2,\alpha}_{\nu}$ into $F^{0,\alpha}_{\nu - 2}$. Thanks to the result of Proposition 4, it is possible to apply the implicit function theorem to solve (25) with $w$ on $\partial B^m_{\rho}$ equal to some given function $h \in C^{2,\alpha}(S^{n - 1}, \mathcal{F}(n, m))$ satisfying $\|h\|_{2,\alpha} \leq c_0$ for some fixed constant $c_0 > 0$. The solution of (25) provided by the implicit function theorem will be denoted by $w_h$. By construction, the graph of $w_h$ is a minimal hypersurface whose boundary is parameterized by the boundary data $h$.

Family of minimal hypersurfaces which are close to $\mathbb{R}^{n - m} \times T^m$: Let us summarize what we have proved. We fix $\nu$ according to the above choice, $\alpha \in (0, 1)$ and $\kappa > 0$. There exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$, for all $h \in C^{2,\alpha}(S^{n - 1}, \mathcal{F}(n, m))$ satisfying

$$\|h\|_{2,\alpha} \leq \kappa \varepsilon^{n - 1},$$

we have been able to find a minimal hypersurface, which is a graph over $\Omega_\varepsilon$. This hypersurface, once translated by $\varepsilon c_\infty$ along the $z$-axis, will be denoted by $M_\varepsilon(h)$.

Moreover, there exists a constant $c_{h,t}$ such that $M_\varepsilon(h)$ asymptotic to

$$\{(x_1, x_2, z) \in \mathbb{R}^{n - m} \times T^m \times \mathbb{R} \mid z = c_h \zeta_m(x_1)\},$$

where $\zeta_{n - 1}(y) := |y|$, $\zeta_{n - 2}(y) := \log |y|$ and $\zeta_m(y) := 0$, when $1 \leq m \leq n - 3$. 19
Also observe that the hypersurface $M_\varepsilon(h)$ inherits all the symmetries induced by the symmetries used to define the function spaces in Definition 4, hence it is invariant with respect to the action of $O(n-m, \mathbb{R}) \otimes S_m \otimes \{I_1\} \subset O(n+1, \mathbb{R})$.

Furthermore, near its boundary this hypersurface can be parameterized as the graph of

$$\mathbb{B}_{2\rho}^n - B^m_\rho \ni x \mapsto \varepsilon c_\infty - \hat{W}_h(x) - \hat{V}_h(x),$$

where $\hat{W}_h$ is the unique (bounded) harmonic extension of the boundary data $h$ in $\mathbb{R}^{n-m} \times T^m - B^m_\rho$ which belongs to $C^{2,\alpha}$. Here the function $\hat{V}_h$ satisfies

$$\|\hat{V}_h\|_{C^{2,\alpha}} \leq c_0 \varepsilon^{2n-2},$$

for some constant $c_0$ which depends on $\kappa$ but does not depend on $\varepsilon$. Here the norm is taken over $\mathbb{B}_{2\rho}^n - B^m_\rho$.

Observe that, reducing $\varepsilon_0$ if this is necessary, we can assume that the mapping $h \to \hat{V}_h$ is continuous and in fact smooth. It follows from standard properties of the solutions obtained through the application of the implicit function theorem that

$$\|\hat{V}_h \|_{C^{2,\alpha}} \leq c \varepsilon^{n-1} \|h_2 - h_1\|_{C^{2,\alpha}},$$

for some constant $c > 0$ which does not depend on $\varepsilon$, but depends on $\kappa$. Here the norms are understood on $\mathbb{B}_{2\rho}^n - B^m_\rho$.

5 The gluing procedure

We fix $\kappa > 0$ large enough and apply the results of the previous sections. There exists $\varepsilon_0 > 0$ and for all $g, h, \in C^{2,\alpha}(S^{n-1}, S(n, m))$ satisfying $\|g\|_{2,\alpha} \leq \kappa \varepsilon^{n-1}$ and $\|h\|_{2,\alpha} \leq \kappa \varepsilon^{n-1}$, we define the hypersurface $M_\varepsilon(g)$ and the hypersurface $C_\varepsilon(h)$. Our aim will be now to find $g$ and $h$ in such a way that

$$(M_\varepsilon(g) \cup C_\varepsilon(h)) \cap \mathbb{R}^{n-m} \times T^m \times (0, +\infty),$$

is a $C^1$ hypersurface. Then applying a reflection with respect to the hyperplane $z = 0$, we will obtain a complete $C^1$ hypersurface of $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$. Finally, it will remain to apply standard regularity theory to show that this hypersurface is in fact $C^{\infty}$.

By construction, the two hypersurfaces $M_\varepsilon(g, h_0)$ and $C_\varepsilon(h')$ are graphs over the $z = 0$ hyperplane near their common boundary of the function

$$x \in B^m_{2\rho} \setminus B^m_\rho \mapsto \varepsilon c_\infty - \hat{W}_g(x) - \hat{V}_g(x),$$

for $M_\varepsilon(g)$ and of the function

$$x \in B^m_\rho \setminus B^m_{\rho/2} \mapsto \varepsilon c_\infty - W_h(x) - V_{\varepsilon, h}(x),$$

for $C_\varepsilon(h)$.

Hence, to produce a $C^1$ hypersurface, it remains to solve the equations

$$\begin{align*}
\hat{W}_g + \hat{V}_g &= W_h - V_{\varepsilon, h} \\
\partial_r \hat{W}_g + \partial_r \hat{V}_{\varepsilon, g} &= \partial_r W_h + \partial_r V_{\varepsilon, h},
\end{align*}$$

(27)

where all functions are evaluated on $\partial B^m_\rho$. The first identity is obtained by asking that the Dirichlet data of the two graphs on $\partial B^m_\rho$ coincide and already ensures that the hypersurface is
\(C^0\), while the second is obtained by asking that the Neumann data of the two graphs on \(\partial B^\rho\) coincide and ensures that the hypersurface will be of class \(C^1\).

To this aim, let us recall that the mapping \(U\)

\[
U : h \in C^{2,\alpha}(S^{n-1}, \mathcal{F}(n,m)) \rightarrow \rho \partial_r(W_h - \hat{W}_h)(\rho \cdot) \in C^{1,\alpha}(S^{n-1}, \mathcal{F}(n,m)),
\]

is an isomorphism. Indeed, this mapping is a linear first order elliptic pseudo-differential operator and, in order to check that it is an isomorphism, it is enough to prove that it is injective. Now if we assume that \(U(h) = 0\) then the function \(w\) defined by

\[
w := \hat{W}_h \in \mathbb{R}^n - m \times T_m - B^n_\rho \text{ and } w := W_h \in B^n_\rho
\]

is a global solution of \(\Delta w = 0\) in \(\mathbb{R}^n - m \times T_m\), and furthermore, \(w\) belongs to \(E^{2,\alpha}\). It is easy to check that necessarily \(w \equiv 0\) and, as a consequence, \(h \equiv 0\).

Using the above claim, it is easy to see that (27) reduces to a fixed point problem

\[
(g, h) = C_{\varepsilon}(g, h),
\]

in \(E := (C^{2,\alpha}(S^{n-1}), \mathcal{F}(n,m))^2\). However, (2) and (20) imply that \(C_{\varepsilon} : E \rightarrow E\) is a contraction mapping defined in the ball of radius \(\kappa \varepsilon^{n-1}\) of \(E\) into itself, provided \(\varepsilon\) is chosen small enough. Hence, we have obtained a fixed point of the mapping \(C_{\varepsilon}\). This completes the proof of the existence of the hypersurfaces \(S_\varepsilon\) which are described in the Theorem 1. Most of the properties states in Theorem 1 follow readilly from the construction itself except the derivation of (2).

**Proof of (2)**: This follows from the application of the well known balancing formula for minimal hypersurfaces. In the case where \(m = n - 1\) we know from the construction itself that the hypersurface \(S_\varepsilon\) is, away from the origine, the graph of the function

\[
(x_1, x_2) \rightarrow c_{\varepsilon} |x_1| + c_\infty \varepsilon + \mathcal{O}(\varepsilon^{n-1} |x_1|^\nu)
\]

Moreover, near 0 the hypersurface is a graph over the rescaled \(n\)-catenoid. It remains to identify the constant \(c_{\varepsilon}\). In order to do so, we apply the balancing formula of \(\mathcal{E}\) (Theorem 7.2) between the hyperplane \(z = 0\) and \(z = z_0\) for \(z_0\) tending to \(+\infty\). This yields

\[
\text{Vol}(T^{n-1})c_{\varepsilon} \sim \varepsilon^{n-1} \text{Vol}(S^{n-1}).
\]

And (2) follows at once from our normalization of the volume on an \(n - 1\)-dimensional torus.

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