HILBERT MODULARITY OF SOME DOUBLE OCTIC CALABI–YAU THREEFOLDS

SLAWOMIR CYNK, MATTHIAS SCHÜTT, AND DUCO VAN STRATEN

Abstract. We exhibit three double octic Calabi–Yau threefolds over the quadratic fields \( \mathbb{Q}(\sqrt{2}) \), \( \mathbb{Q}(\sqrt{5}) \), \( \mathbb{Q}(\sqrt{-3}) \) and prove their modularity. The non-rigid threefold \( \mathbb{Q}(\sqrt{2}) \) has two conjugate Hilbert modular forms of weight \([4,2]\) and \([2,4]\) attached while the two rigid threefolds correspond to a Hilbert modular form of weight \([4,4]\) and to the twist of the restriction of a classical modular form of weight 4.

1. Introduction

After the modularity of elliptic curves over \( \mathbb{Q} \) was proven, much emphasis has been put on Calabi–Yau threefolds. Indeed, for rigid Calabi–Yau threefolds over \( \mathbb{Q} \), the modularity has been established in the meantime independently in [6], [8]. Over number fields other than \( \mathbb{Q} \), however, there seems to be only one (Hilbert) modular example to date (outside the CM case), the Consani–Scholten quintic from [1], [7]. This paper will provide three more examples of modular Calabi-Yau threefolds, each of which is defined over some quadratic field \( K \). In detail, we will exhibit

1. a non-rigid Calabi–Yau threefold \( X \) over \( \mathbb{Q} \) (with \( b_3(X) = 4 \)) admitting an endomorphism over \( \mathbb{Q}(\sqrt{2}) \) which can be used to split \( H^3(X) \) into two 2-dimensional eigenspaces; the corresponding Galois representations are proved to correspond to a Hilbert modular form over \( \mathbb{Q}(\sqrt{2}) \) (Theorem 4.1, Remark 4.2);
2. a rigid Calabi–Yau threefold \( Y \) over \( \mathbb{Q}(\sqrt{5}) \) such that the Galois representation on \( H^3(Y) \) corresponds to a Hilbert modular form over \( \mathbb{Q}(\sqrt{5}) \) of weight \([4,4]\) (Theorem 5.1);
3. a rigid Calabi–Yau threefold \( Z \) over \( \mathbb{Q}(\sqrt{-3}) \) such that the Galois representation on \( H^3(Z) \) corresponds to a twist of the restriction of a classical modular form of weight 4 and level 72 (Theorem 6.2).

The Calabi–Yau threefolds will be constructed as crepant resolutions of certain double octics. More precisely, we will choose the branch locus to consist of 8 distinct planes. The construction, following [13], will be reviewed in Section 3. Then the proof of the mentioned results amounts to comparing the two-dimensional Galois representations attached to the motive of the third cohomology of the Calabi–Yau threefolds (or a given submotive in the first case) on the one hand and to the Hilbert or classical modular forms in question on the other. In practice, this can be achieved by working with the 2-adic representations and applying a method going back to Faltings and Serre and worked out in detail by Livné in [11]. This technique will be explained, with a view towards the given base fields, in Section 2. In essence, it reduces the proof of modularity to comparing a few traces and determinants of the Galois representations at certain Frobenius elements; these, in turn, can be obtained from extensive point counting using the Lefschetz fixed point formula, and from Hilbert modular forms calculations as incorporated in MAGMA.

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In this section we study special cases of [11 Thm. 4.3]; they will be instrumental in establishing the modularity of the three two-dimensional Galois representations attached to the Calabi–Yau threefolds $X, Y, Z$ to be introduced in the next sections.

Throughout this section, the set-up comprises continuous two-dimensional 2-adic Galois representations of the absolute Galois group of a specified number field $K$ which are unramified outside a given finite set $S$ of prime ideals in the integer ring $\mathcal{O}_K$ of $K$. For simplicity, we list prime ideals just by a single generator. The norms will be included in the tables to follow.

**Proposition 2.1.** Let $K = \mathbb{Q}[\sqrt{2}]$ and $E = \mathbb{Q}_2[\sqrt{2}]$ and let $\mathcal{P} := \sqrt{2}\mathbb{Z}_2$ be the maximal ideal of the ring of integers of $E$. Let $S := \{\sqrt{2}, 3\}$ and

$$T = \{5, 11, \sqrt{2} + 3, \sqrt{2} - 3, 3\sqrt{2} - 1, \sqrt{2} + 5, \sqrt{2} - 5, 4\sqrt{2} - 1, 4\sqrt{2} + 1, 5\sqrt{2} - 3, \sqrt{2} - 7, \sqrt{2} + 7, 4\sqrt{2} - 11, 1 - 7\sqrt{2}\}$$

$$U = \{5, 11, 13, \sqrt{2} - 3, 3\sqrt{2} - 1, \sqrt{2} - 5, 4\sqrt{2} - 1, 5\sqrt{2} - 3\}$$

be two sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \text{Gal}(\bar{K}/K) \to \text{GL}_2(E)$ are continuous Galois representations unramified outside $S$ and satisfying

1. $\text{Tr}(\rho_1(Frob_p)) \equiv \text{Tr}(\rho_2(Frob_p)) \equiv 0 \pmod{\mathcal{P}}$ for $p \in U$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,
3. $\text{Tr}(\rho_1(Frob_p)) = \text{Tr}(\rho_2(Frob_p))$ and $\det(\rho_1(Frob_p)) = \det(\rho_2(Frob_p))$ for $p \in T$.

Then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications.

**Proof.** Following the arguments of [11] we first verify that assumption 1. implies that $\text{Tr}(\rho_1) \equiv \text{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$. Indeed, suppose that $\text{Tr}(\rho_1) \not\equiv 0 \pmod{\mathcal{P}}$ and denote by $L/K$ the Galois extension cut out by the kernel $\text{Ker} \tilde{\rho}_i$ of the reduction $\tilde{\rho}_i$ of $\rho_i$ modulo $\mathcal{P}$. By inspection, we have $\text{im}(\tilde{\rho}_i) \subset \text{GL}_2(\mathbb{F}_2)$ where the elements of odd trace are exactly those of order 3 (which by assumption will correspond to some Frobenius elements). Hence, the Galois group of the extension $L/K$ is isomorphic to $S_3$ or $C_3$, so it is the Galois closure of a degree 3 extension $M/K$.

Then $M$ is a degree 6 extension of $\mathbb{Q}$ unramified outside $\{2, 3\}$. The database [10] lists 398 such fields presented as a splitting field of a monic degree 6 polynomial with rational coefficients. The assumption that $M$ contains the subfield $K = \mathbb{Q}[\sqrt{2}]$ implies that the minimal polynomial of any primitive element of the extension $M/\mathbb{Q}$ factors over $\mathbb{Q}[\sqrt{2}]$. We check that exactly 25 of the 398 polynomials from [10] satisfy this condition. For each of these 25 degree 6 polynomials, we determine a prime integer $p$ such that the reduction of the degree 3 polynomial over $K$ modulo a prime $p$ in $\mathcal{O}_K$ over $p$ stays irreducible over $\mathcal{O}_K/p \cong \mathbb{F}_p$. We list these data below.

$$x^6 - 2x^3 - 1 = (x^3 + \sqrt{2} - 1) \times (x^3 - \sqrt{2} - 1), \ p = 5$$

$$x^6 - 12x^4 + 36x^2 - 8 = (x^3 - 6x - 2\sqrt{2}) \times (x^3 - 6x + 2\sqrt{2}), \ p = 5$$

$$x^6 - 2 = (x^3 + \sqrt{2}) \times (x^3 - \sqrt{2}), \ p = 7$$

$$x^6 - 4x^3 + 2 = (x^3 + \sqrt{2} - 2) \times (x^3 - \sqrt{2} - 2), \ p = 5$$

$$x^6 + 6x^4 + 9x^2 - 8 = (x^3 + 3x + 2\sqrt{2}) \times (x^3 + 3x - 2\sqrt{2}), \ p = 11$$

$$x^6 + 6x^4 + 9x^2 - 2 = (x^3 + 3x - \sqrt{2}) \times (x^3 + 3x + \sqrt{2}), \ p = 5$$

$$x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 = (x^3 + 3\sqrt{2}x^2 + 6x + 2\sqrt{2} - 3)$$

$$\times (x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} - 3), \ p = 7$$

$$x^6 - 18 = (x^3 - 3\sqrt{2}) \times (x^3 + 3\sqrt{2}), \ p = 7$$

$$x^6 - 6x^4 - 12x^3 + 12x^2 - 72x + 28 = (x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} - 6)$$

$$\times (x^3 + 3\sqrt{2}x^2 + 6x + 2\sqrt{2} - 6), \ p = 13$$

$$x^6 - 6x^3 - 9 = (x^3 - 3\sqrt{2} - 3) \times (x^3 + 3\sqrt{2} - 3), \ p = 5$$

$$x^6 - 6x^4 - 4x^3 + 9x^2 + 12x - 14 = (x^3 - 3x - 3\sqrt{2} - 2) \times (x^3 - 3x + 3\sqrt{2} - 2), \ p = 5$$
\[x^6 - 18x^4 - 12x^3 + 81x^2 + 108x + 18 = (x^3 - 9x + 3\sqrt{2} - 6) \times (x^3 - 9x - 3\sqrt{2} - 6), \ p = 5\]
\[x^6 + 6x^4 - 4x^3 - 9x^2 + 12x - 4 = \left(x^3 + 3\sqrt{2}x + 3x - 2\sqrt{2} - 2\right) \times (x^3 - 3x + 2\sqrt{2} - 2), \ p = 23\]
\[x^6 + 6x^4 - 4x^3 + 9x^2 - 12x - 4 = \left(x^3 + 3x + 2\sqrt{2} - 2\right) \times \left(x^3 + 3x - 2\sqrt{2} - 2\right), \ p = 31\]
\[x^6 - 6x^4 - 4x^3 + 9x^2 + 12x - 4 = \left(x^3 - 3x - 2\sqrt{2} - 2\right) \times \left(x^3 - 3x + 2\sqrt{2} - 2\right), \ p = 5\]
\[x^6 - 12x^3 + 18 = \left(x^3 + 3\sqrt{2} - 6\right) \times \left(x^3 - 3\sqrt{2} - 6\right), \ p = 5\]
\[x^6 - 12x^3 - 36 = \left(x^3 + 6\sqrt{2} - 6\right) \times \left(x^3 - 6\sqrt{2} - 6\right), \ p = 5\]
\[x^6 - 6x^4 - 4x^3 - 9x^2 - 12x - 4 = \left(x^3 - 3\sqrt{2}x - 3x - 2\sqrt{2} - 2\right) \times \left(x^3 + 3\sqrt{2}x - 3x + 2\sqrt{2} - 2\right), \ p = 41\]
\[x^6 - 12x^3 - 48x - 16 = \left(x^3 - 3\sqrt{2}x - 4\sqrt{2} - 4\right) \times \left(x^3 + 3\sqrt{2}x + 4\sqrt{2} - 4\right), \ p = 7\]
\[x^6 + 6x^4 - 12x^3 + 9x^2 - 36x + 28 = \left(x^3 + 3x - 2\sqrt{2} - 6\right) \times \left(x^3 + 3x + 2\sqrt{2} - 6\right), \ p = 17\]
\[x^6 - 8x^3 - 18x^2 + 24x + 8 = \left(x^3 - 3\sqrt{2}x + 2\sqrt{2} - 4\right) \times \left(x^3 + 3\sqrt{2}x - 2\sqrt{2} - 4\right), \ p = 13\]
\[x^6 - 16x^4 + 18x^2 + 48x + 32 = \left(x^3 + 3\sqrt{2} - 4\sqrt{2} - 8\right) \times \left(x^3 - 3\sqrt{2} + 4\sqrt{2} - 8\right), \ p = 11\]
\[x^6 - 18x^4 - 36x^3 - 81x^2 - 108x + 36 = \left(x^3 - 9\sqrt{2}x - 9x - 12\sqrt{2} - 18\right) \times \left(x^3 + 9\sqrt{2}x - 9x + 12\sqrt{2} - 18\right), \ p = 11\]
\[x^6 - 18x^4 - 12x^3 + 81x^2 + 108x - 36 = \left(x^3 - 9x - 6\sqrt{2} - 6\right) \times \left(x^3 - 9x + 6\sqrt{2} - 6\right), \ p = 11\]
\[x^6 - 18x^4 - 36x^3 + 81x^2 + 324x + 252 = \left(x^3 - 9x - 6\sqrt{2} - 18\right) \times \left(x^3 - 9x + 6\sqrt{2} - 18\right), \ p = 11\]

Given \(M\) and \(p\) as above, it follows that any element in the conjugacy class of \(\text{Frob}_p\) in \(\text{Gal}(L/K)\) has order 3; consequently \(\text{Tr}(\rho_i(\text{Frob}_p)) \equiv 1 \pmod{\mathcal{P}}\), contradicting our assumptions. Thus we see that the set \(U\) was indeed chosen in such a way that condition 1. implies that \(\text{Tr}(\rho_1) \equiv \text{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}\).

The traces being even is the key ingredient to apply [11 Thm. 4.3]. To this end, let \(K_S\) be the compositum of all quadratic extensions of \(K\) unramified outside \(S\). Since the ring \(\mathcal{O}_K\) is a unique factorization domain, the compositum \(K_S\) is obtained by extracting square roots of generators of \(\mathcal{O}_K^*\) and of prime elements \(\alpha \in \mathcal{O}_K\) with norm \(N_K(\alpha)\) divisible only by elements of \(S\). Presently, generators of \(K_S/K\) can be taken as \(\sqrt{-1}, \sqrt{2}, \sqrt{2}, \sqrt{2} - 1, \sqrt{3}\).

One computes the table of quadratic characters \(\text{Gal}(K_S/K) \rightarrow (\mathbb{Z}/2)^4\) at the primes from \(T\) as follows:

| \(p\) | \(N(p)\) | \(\sqrt{2}\) | 3 | \(\sqrt{2} - 1\) | \(\sqrt{2} - 1\) | \(\sqrt{2} - 1\) |
|------|------|-------|---|----------------|----------------|----------------|
| 5    | 25   | 1     | 0 | 0             | 0              | 0              |
| 11   | 121  | 0     | 0 | 0             | 1              | 0              |
| \(\sqrt{2} + 3\) | 7 | 0 | 1 | 1 | 5\(\sqrt{2} - 3\) | 4\(\sqrt{2} - 1\) | 31 | 1 | 1 | 1 |
| \(\sqrt{2} - 3\) | 7 | 1 | 1 | 1 | 0 | \(\sqrt{2} - 7\) | 47 | 0 | 0 | 1 |
| 3\(\sqrt{2} - 1\) | 17 | 1 | 1 | 0 | 1 | \(\sqrt{2} + 7\) | 47 | 1 | 0 | 1 |
| \(\sqrt{2} + 5\) | 23 | 0 | 0 | 1 | 1 | 4\(\sqrt{2} - 11\) | 89 | 0 | 1 | 0 |
| \(\sqrt{2} - 5\) | 23 | 1 | 0 | 1 | 0 | 1 - 7\(\sqrt{2}\) | 97 | 1 | 0 | 1 |

From the table we infer that the image of the Frobenius elements \(\text{Frob}_t, t \in T\), contains 14 different non-zero elements, hence it is non–cubic in the terminology of [11 Def. 4.1] (see e.g. [14 p. 53]). Thus the assertion that the Galois representations \(\rho_1, \rho_2\) have isomorphic semisimplifications follows from [11 Thm. 4.3].

Note in particular that Proposition 2.1 implies that \(\rho_1, \rho_2\) have the same \(L\)-series, a feature which will be centrally used in the proof of the modularity results stated in the introduction. The next two propositions concern the same kind of problem for different base fields and adjusted ramification sets. Since the arguments are very similar, we give only the essential ingredients.
Proposition 2.2. Let $K = \mathbb{Q}[\sqrt{5}]$, $E = \mathbb{Q}_2$ and let $\mathcal{P} := 2\mathbb{Z}_2$ be the maximal ideal of the ring of integers of $E$. Let $S := \{2\}$ and

$$T = \{3, 13, \sqrt{5} + 4, 2\sqrt{5} + 7, \sqrt{5} + 6, \sqrt{5} - 6, 2\sqrt{5} + 9\}$$

be two sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \text{Gal}(\bar{K}/K) \to \text{GL}_2(E)$ are continuous Galois representations unramified outside $S$ and satisfying

1. $\text{Tr}(\rho_1(\text{Frob}_p)) \equiv \text{Tr}(\rho_2(\text{Frob}_p)) \equiv 0 \pmod{\mathcal{P}}$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,
3. $\text{Tr}(\rho_1(\text{Frob}_p)) = \text{Tr}(\rho_2(\text{Frob}_p))$ and $\det(\rho_1(\text{Frob}_p)) = \det(\rho_2(\text{Frob}_p))$ for $p \in T$.

Then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications.

Proof. By [10] there are 106 degree 6 extensions of $\mathbb{Q}$ unramified outside $\{2, 5\}$. Only one of them contains $\mathbb{Q}[\sqrt{5}]$; it is the splitting field of the following polynomial:

$$x^6 - 2x^5 - 2x - 1 = \left(x^3 - x^2 + \frac{1}{2}(-1 + \sqrt{5})x + \frac{1}{2}(-1 + \sqrt{5}) \right) \left(x^3 - x^2 - \frac{1}{2}(1 + \sqrt{5})x - \frac{1}{2}(1 + \sqrt{5})\right)$$

As each cubic polynomial over $\mathbb{Z}[\sqrt{5}]$ is irreducible modulo 3, assumption 1. implies that $\text{Tr}(\rho_1) \equiv \text{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$ as required.

The compositum $K$ of quadratic extensions of $K$ unramified outside $S$ is obtained from the three quadratic extensions $K[\sqrt{2}], K[\sqrt{-1}], K[\sqrt{5} - 1]$. The table of characters is computed as follows:

| $p$ | $N(p)$ | 2 | $-1$ | $\frac{1}{2}(\sqrt{5} - 1)$ |
|-----|--------|---|------|-------------------|
| 3   | 9      | 1 | 1    | 0                 |
| 13  | 169    | 1 | 1    | 1                 |
| $\sqrt{5} + 4$ | 11 | 0 | 0    | 1                 |
| $2\sqrt{5} + 7$ | 29 | 0 | 1    | 1                 |

From the table, we infer that the image of the Frobenius elements $\text{Frob}_t, t \in T$ equals $(\mathbb{Z}/2\mathbb{Z}) \setminus \{0\}$; in particular, it is non-cubic, and the Proposition follows from [11, Thm. 4.3] as before. □

Proposition 2.3. Let $K = \mathbb{Q}[\sqrt[3]{-3}], E = \mathbb{Q}_2$ and let $\mathcal{P} := 2\mathbb{Z}_2$ be the maximal ideal of the ring of integers of $E$. Let $S := \{2, \sqrt{-3}\}$ and

$$T = \{\sqrt{-3} - 2, \sqrt{-3} + 2, 1 + 2\sqrt{-3}, \sqrt{-3} - 4, \sqrt{-3} + 4, 5 + 2\sqrt{-3}, 5 + 4\sqrt{-3}\}$$

be three sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \text{Gal}(\bar{K}/K) \to \text{GL}_2(E)$ are continuous Galois representations unramified outside $S$ and satisfying

1. $\text{Tr}(\rho_1(\text{Frob}_p)) \equiv \text{Tr}(\rho_2(\text{Frob}_p)) \equiv 0 \pmod{\mathcal{P}}$ for $p \in U$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,
3. $\text{Tr}(\rho_1(\text{Frob}_p)) = \text{Tr}(\rho_2(\text{Frob}_p))$ and $\det(\rho_1(\text{Frob}_p)) = \det(\rho_2(\text{Frob}_p))$ for $p \in T$.

Then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications.

Proof. We claim that condition 1. implies $\text{Tr}(\rho_1) \equiv \text{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$. To prove this we have to determine all cubic extensions of $\mathbb{Q}[\sqrt[3]{-3}]$, which are unramified outside $S$; they give degree 6 extensions of $\mathbb{Q}$ unramified outside $\{2, 3\}$ as in the proof of Proposition 2.1. Out of the 398 extensions 16 contain $\mathbb{Q}[\sqrt[3]{-3}]$; they are splitting fields of the degree six polynomials

$$x^6 - x^3 + 1, \quad x^6 - 3x^5 + 5x^3 - 3x + 1, \quad x^6 + 3, \quad x^6 - 3x^5 + 3x^3 + 6x^2 - 9x + 3$$
$$x^6 - 3x^4 - 2x^3 + 9x^2 + 12x + 4, \quad x^6 - 2x^3 + 4, \quad x^6 - 3x^3 + 3, \quad x^6 + 12, \quad x^6 + 48$$
$$x^6 - 3x^4 + 9x^2 - 18x + 12, \quad x^6 - 6x^3 + 12, \quad x^6 - 6x^3 + 36, \quad x^6 + 18x^4 + 81x^2 + 12$$
$$x^6 + 3x^4 - 2x^3 + 9x^2 - 12x + 4, \quad x^6 + 27x^2 - 36x + 12, \quad x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 52$$

each of which factors into a product of degree 3 polynomials over $\mathbb{Q}[\sqrt{-3}]$. One readily verifies that each degree 3 polynomial has irreducible reduction modulo at least one prime from $U$. The evenness of the traces follows.
The compositum $K_S$ of all quadratic extensions of $\mathbb{Q}[\sqrt{-3}]$ unramified outside $\{2, \sqrt{-3}\}$ equals the compositum of $\mathbb{Q}[\sqrt{-3}, \sqrt{2}], \mathbb{Q}[\sqrt{-3}], \sqrt{\frac{1}{2}(\sqrt{-3}+1)}$. From the table of quadratic characters below, we read off that the elements from the Galois group $\text{Gal}(K_S/\mathbb{Q}[\sqrt{-3}])$ corresponding to Frobenius elements at the primes from $T$ form a non-cubic set. Now the proposition follows from \textbf{[11 Thm. 4.3].}

| $p$            | $N(p)$ | $\sqrt{-3}$ | $2$ | $\frac{1}{2}(\sqrt{-3}+1)$ | $p$ | $N(p)$ | $\sqrt{-3}$ | $2$ | $\frac{1}{2}(\sqrt{-3}+1)$ |
|---------------|--------|-------------|-----|---------------------------|-----|--------|-------------|-----|---------------------------|
| $\sqrt{-3}-2$ | 7      | 0           | 0   | 1                         | 19  | 1      | 1           | 1   | 1                         |
| $\sqrt{-3}+2$ | 7      | 1           | 0   | 1                         | 5+2$\sqrt{-3}$ | 37 | 0 | 1 | 0 |
| $1+2\sqrt{-3}$ | 13   | 1           | 1   | 0                         | 5+4$\sqrt{-3}$ | 73 | 1 | 0 | 0 |
| $\sqrt{-3}-4$ | 19   | 0           | 1   | 1                         |     |        |             |     |               |

\[\square\]

3. DOUBLE OCTICS

In this paper we shall study the modularity of three Calabi-Yau threefolds constructed as crepant resolution of a double cover of the projective space $\mathbb{P}^3$ branched along an arrangement of eight planes $S = P_1 \cup \cdots \cup P_8$. If the planes satisfy the following two conditions:

\begin{equation}
\text{(3.1)} \quad \text{no six planes intersect, no four planes contain a common line,}
\end{equation}

then the double cover admits a projective crepant resolution of singularities. One calls the resulting Calabi-Yau threefold a double octic. It is sometimes useful to note that the crepant resolution can be arranged in such a way that it exhibits the double cover as a double cover of a blow-up of the projective space.

One of the key features of double octics is that one can control their invariants, in particular their Hodge numbers. In particular, this is instrumental for constructing rigid double octics or one-dimensional families (accounting for all the infinitesimal deformations of the smooth members). For brevity, we omit the details here and refer the reader to the section 4.2 of C. Meyer’s monograph \textbf{[13].}

4. DOUBLE OCTIC WITH REAL MULTIPLICATION BY $\mathbb{Q}[\sqrt{2}]$

Let $X$ be the double octic Calabi-Yau threefolds constructed as a resolution of the double covering of $\mathbb{P}^3$ branched along the following 8 hyperplanes:

\[u^2 = x(x-z)(x-v)(x-z-v)y(y-z)(y-v)(y+v+2z).\]

By separating the variable $x, y$ on the right-hand side, one realizes that the double octic $X$ admits a fibration by Kummer surfaces (the fibration is induced by the map $(x, y, z, v, u) \mapsto (z, v)$). Following \textbf{[13]}, this Kummer structure arises from the fiber product of rational elliptic surfaces with singular fibers $I_4, I_4, I_2, I_2$ and $I_2, I_2, I_2, I_2, I_2, I_2$ where the singular fibers are located as follows:

\[\infty \quad 0 \quad 1 \quad -1 \quad -\frac{1}{2} \quad -\frac{1}{4}\]

\[I_4 \quad I_4 \quad I_2 \quad I_2\]

\[I_2 \quad I_2 \quad I_2 \quad I_2 \quad I_2\]

The Calabi-Yau threefold $X$ is isomorphic to the element corresponding to $t = -1/2$ of the one parameter family defined by the Arrangement No. 250 \textbf{[13].} In particular

\begin{equation}
\text{(4.1)} \quad h^{11}(X) = 37, \quad h^{12}(X) = 1,
\end{equation}

and the only primes of bad reduction of $X$ are 2 and 3. The Riemann-symbol of the Picard-Fuchs operator of the family of Calabi-Yau threefolds defined by the Arrangement No. 250 is

\[
\begin{pmatrix}
-2 & -1 & -1/2 & 0 & 1 & \infty \\
0 & 0 & 0 & 0 & 0 & 1/2 \\
1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\
1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\
2 & 1 & 4 & 1 & 2 & 3/2
\end{pmatrix}
\]

(for details see \textbf{[5]}). The Picard-Fuchs operator is symmetric with respect to the involution $t \mapsto -1-t$ and its fixed point $t = -\frac{1}{2}$ is an apparent singularity. The family, however, does not seem to be symmetric in
acts on the étale cohomology we have

\[ p \text{ Lefschetz fixed point formula, the trace of } \text{Frob} \]

and \( \bar{p} \). The number of points on \( N \) we count the points

In order to compute the trace of the Frobenius morphism \( \text{Frob} \)

related to the Legendre symbol \(( \frac{a}{p} )\). Hence, if \( p \) is an inert prime in \( \mathbb{Q}[\sqrt{2}] \), the trace of \( \text{Frob}_p \) on \( H^3(X) \) vanishes:

\[
a_p := \text{Tr}(\text{Frob}_p | H^3(X)) = 0;
\]

The pullback by \( \Psi \) of a canonical form \( \omega_X \) is \( \Psi^* \omega_X = \sqrt{2} \omega_X \). In particular the map \( \Psi^* \) acts as multiplication by \( \sqrt{2} \) on \( H^{3,0}(X) \oplus H^{0,3}(X) \). On the other hand, the map \( \Psi^* \) acts as the multiplication by \((-1)\) on the infinitesimal deformation space \( H^1(T_X) \) and hence as multiplication by \((-\sqrt{2})\) on \( H^{1,2}(X) \oplus H^{2,1}(X) \) (by Serre duality there is an isomorphism between vector spaces \( H^{1,2}(X) \) and \( (H^1(T_X) \otimes H^{3,0})^* \) compatible with the action induced by \( \Psi \)). Consequently the map \( \Psi \) decomposes the motive \( H^3(X) \) into a direct sum of two two-dimensional submotives

\[
H^3(X) = H^3_{\text{sym}} \oplus H^3_{\text{skew}}
\]

defined as \((\pm \sqrt{2})\)-eigenspaces of \( \Psi^* \). The restriction of the Galois action to the subgroup \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}]) \) preserves the two submotives and hence decomposes \( H^3(X) \) as the direct sum of two Galois-conjugate Galois representations

\[
\rho, \bar{\rho} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}]) \rightarrow \text{GL}_2(\mathbb{Q}_2[\sqrt{2}]).
\]

We shall study these Galois representations using the Lefschetz fixed point formula (in order to eventually prove their modularity). To this end, we have to inspect the crepant resolution of the double octic more closely.

The Calabi-Yau threefold \( X \) is a double covering of a blow-up \( X' \) of the projective space \( \mathbb{P}^3 \), consequently there is an involution \( i : X \rightarrow X \) acting on \( X \). This involution induces a decomposition

\[
\text{Pic}(X) = H^2(X, \mathbb{Z}) = H^2_{\text{sym}}(X, \mathbb{Z}) \oplus H^2_{\text{skew}}(X, \mathbb{Z})
\]

of the Picard group of \( X \) into symmetric and skew-symmetric part. The symmetric part \( H^2_{\text{sym}}(X, \mathbb{Z}) \) is isomorphic to the cohomology group \( H^2(X', \mathbb{Z}) \). The octic arrangement defining the Calabi-Yau threefold \( X \) has 28 double lines and 8 fourfold points, consequently in the process of resolution of singularities of \( X \) we blow-up the doubly covered projective space 36 times and the rank of the cohomology group \( H^2(X', \mathbb{Z}) \) equals 37. It follows from \( (4.1) \) that the cohomology group \( H^2(X, \mathbb{Z}) \) is generated by classes of symmetric divisors defined over \( \mathbb{Q} \). By the comparison theorem for any prime \( p \geq 5 \) the Frobenius morphism \( \text{Frob}_p \), acts on the étale cohomology \( H^3_{\text{et}}(X_p, \mathbb{Q}_l) \) by multiplication with \( p \), where \( X_p \) is the reduction of \( X \) modulo \( p \) and \( \bar{X}_p = X \otimes \mathbb{F}_p \), and likewise for all powers \( \text{Frob}_p \).

In order to compute the trace of the Frobenius morphism \( \text{Frob}_q \) using the Lefschetz fixed point formula, first we count the points \( N_q \) on the singular double octic over \( \mathbb{F}_q \) using a computer program, then we add the correction terms for the crepant resolution of singularities. Over \( \mathbb{Q} \), the exceptional locus of the blow-up of a fourfold point not contained in any triple line (type \( p^3_3 \) in the notation of \( (4.3) \)) is isomorphic to a surface

\[
E = \{ u^2 = \alpha xyz(x + y + z) \subset \mathbb{P}^3(1, 1, 1, 2) \}, \quad \text{where } \alpha \in \mathbb{Q}.
\]

The number of points on \( E \) over \( \mathbb{F}_q \) depends on whether \( \alpha \) is a square (for details see \( (4.2) \) p. 56); in particular, we have \( E(\mathbb{F}_q) = q^2 + 2q + 1 \) for any even \( p \)-power \( q \). Any other blow-up adds \( q^2 + q \) points to \( X \). By the Lefschetz fixed point formula, the trace of \( \text{Frob}_{p^2} \) on \( H^3_{\text{et}}(X_p, \mathbb{Q}_l) \) thus equals

\[
a_{p^2} := \text{Tr}(\text{Frob}_{p^2} | H^3(X_p)) = -N_{p^2} + p^6 + p^4 + 9p^2 + 1
\]

By \( (4.2) \), the Galois representation on \( H^3(X) \) equals its tensor product with the Dirichlet character associated to the Legendre symbol \(( \frac{2}{p} )\). Hence, if \( p \) is an inert prime in \( \mathbb{Q}[\sqrt{2}] \), the trace of \( \text{Frob}_p \) on \( H^3(X) \) vanishes:

\[
a_p := \text{Tr}(\text{Frob}_p | H^3(X)) = 0;
\]
consequently the Frobenius polynomial equals
\[ X^4 - \frac{a_p^2}{2} X^2 + p^6. \]
If \( F_p \in Gal(\mathbb{Q}/\mathbb{Q}[\sqrt{2}]) \) is a Frobenius element, then the trace of value of \( \rho \) and \( \tilde{\rho} \) at \( F_p \) equals
\[ \text{Tr}(\rho(F_p)) = \text{Tr}(\tilde{\rho}(F_p)) = \frac{1}{2}a_p^2. \]
If \( p \) is a split prime, then the trace of \( \text{Frob}_p \) can be computed as before by a point count in \( \mathbb{F}_{p^2} \). In order to compute the trace of \( \text{Frob}_p \) with a point-count we have to take into account the contribution from the eight fourfold points of the arrangement. Using (4.3) we get in this situation
\[ a_p := \text{Tr}(\text{Frob}_p | H^3(X)) = -N_p + p^3 + p(2 + 3(\frac{1}{p}) + 4(\frac{2}{p})) + 1. \]
The Frobenius polynomial equals
\[ \chi(\text{Frob}_p) = X^4 - a_p X^3 - \frac{1}{2}(a_p^2 + a_{p^2}) X^2 - a_p p^3 + p^6. \]
In the following table we collect Frobenius polynomials for the values of \( p \) that we will need to prove modularity.

| \( p \) | \( a_p \) | \( a_{p^2} \) | \( F_p \) |
|-------|--------|------------|--------|
| 5     | 0      | 20         | \( X^4 - 10 X^2 + 15625 \) |
| 7     | 32     | -796       | \( X^4 - 32 X^3 + 510 X^2 - 10976 X + 117649 \) |
|       |        |            | \( (X^2 + 4 \sqrt{2} X - 16 X + 343) \times (X^2 - 4 \sqrt{2} X - 16 X + 343) \) |
| 11    | 0      | -1452      | \( X^4 + 726 X^2 + 1771561 \) |
|       |        |            | \( (X^2 - 44 X + 1331) \times (X^2 + 44 X + 1331) \) |
| 17    | -124   | -10940     | \( X^4 + 124 X^3 + 13158 X^2 + 609212 X + 24137569 \) |
|       |        |            | \( (X^2 + 16 \sqrt{2} X + 62 X + 4913) \times (X^2 - 16 \sqrt{2} X + 62 X + 4913) \) |
| 23    | 80     | -45212     | \( X^4 - 80 X^3 + 25806 X^2 - 973360 X + 148035889 \) |
|       |        |            | \( (X^2 + 8 \sqrt{2} X - 40 X + 12167) \times (X^2 - 8 \sqrt{2} X - 40 X + 12167) \) |
| 31    | 272    | -59068     | \( X^4 - 272 X^3 + 66526 X^2 - 8103152 X + 887503681 \) |
|       |        |            | \( (X^2 - 76 \sqrt{2} X - 136 X + 29791) \times (X^2 + 76 \sqrt{2} X - 136 X + 29791) \) |
| 41    | 84     | -148252    | \( X^4 - 84 X^3 + 77654 X^2 - 5789364 X + 4750104241 \) |
|       |        |            | \( (X^2 - 176 \sqrt{2} X + 42 X + 68921) \times (X^2 + 176 \sqrt{2} X - 42 X + 68921) \) |
| 47    | -64    | -134460    | \( X^4 + 64 X^3 + 69278 X^2 + 6644672 X + 10779215329 \) |
|       |        |            | \( (X^2 + 264 \sqrt{2} X + 32 X + 103823) \times (X^2 - 264 \sqrt{2} X + 32 X + 103823) \) |
| 89    | -2476  | 507556     | \( X^4 + 2476 X^3 + 2811510 X^2 + 1745503244 X + 496981290961 \) |
|       |        |            | \( (X^2 + 256 \sqrt{2} X + 1238 X + 704969) \times (X^2 - 256 \sqrt{2} X + 1238 X + 704969) \) |
| 97    | 1284   | -282268    | \( X^4 - 1284 X^3 + 2235462 X^2 - 1171872132 X + 832972004929 \) |
|       |        |            | \( (X^2 + 32 \sqrt{2} X - 642 X + 912673) \times (X^2 - 32 \sqrt{2} X - 642 X + 912673) \) |

To avoid working with four-dimensional Galois representations (as in [1]), we have determine the precise traces of \( \text{Frob}_p \) on \( H^3_k \) and \( H^3_L \) for a prime of \( \mathcal{O}_K \) above \( p \). To this end, we shall exploit the endomorphism \( \Psi \); more precisely, we study the action of \( \text{Frob}_p \circ \Psi \) on \( H^3(X) \). This map preserves the Kummer fibration and transforms the fiber at \( (z, v) \) into the fiber at \( (z + v, z - v) \). This allows us to determine the number of fixed points of \( \Psi \); indeed, we can restrict ourselves to the fibers at \( (1 \pm \sqrt{2}, 1) \). At those points, the fiber is isomorphic to the Kummer surface of the product of the elliptic curves
\[ u^2 = x^3 - 30x + 56 \quad \text{and} \quad u^2 = y^3 - y. \]
and the map $\Psi$ is induced by the complex multiplications given
\[ x \mapsto -\frac{x^2 - 4x + 18}{2(x - 4)} \quad \text{and} \quad y \mapsto -\frac{y + 1}{y - 1}. \]

As the map $\Psi$ acts on $H^{0,3} \oplus H^{2,0}$ as multiplication by $\sqrt{2}$ and on $H^{2,1} \oplus H^{1,2}$ as multiplication by $-\sqrt{2}$, we infer that the trace of the induced map on the third cohomology is zero: $\text{tr}(\Psi^*|H^3) = 0$. Using Magma we computed that the map $\Psi$ has Lefschetz number equal 12, so we get
\[
\text{tr}(\Psi^*|H^0) = 1, \quad \text{tr}(\Psi^*|H^2) + \text{tr}(\Psi^*|H^4) = 9, \quad \text{tr}(\Psi^*|H^6) = 2.
\]

In a similar way we computed the trace of the composition $\text{Frob}_p \circ \Psi$ for split primes $p = 7, 17, 23, 31, 47, 89$. As the Picard group of $X$ is defined by divisors defined over $\mathbb{Q}$, the Frobenius morphism $\text{Frob}_p$ acts on $H^2$ as multiplication by $p^k$, $k = 0, 1, 2, 3$. By direct computations with Magma, we found the Lefschetz numbers listed in the table below.

| $p$   | $3 + \sqrt{2}$ | $3 - \sqrt{2}$ | $3\sqrt{2} - 1$ | $5 + \sqrt{2}$ | $5 - \sqrt{2}$ | $4\sqrt{2} + 1$ |
|-------|----------------|----------------|-----------------|----------------|----------------|----------------|
| $\mathcal{L}(\text{Frob}_p \circ \Psi)$ | 944 | 976 | 11404 | 27104 | 27040 | 64208 |
| $\text{tr}(\text{Frob}_p^*|H^3_+)$ | 16 + $4\sqrt{2}$ | 16 - $4\sqrt{2}$ | $-62 - 16\sqrt{2}$ | 40 - $8\sqrt{2}$ | 40 + $8\sqrt{2}$ | 136 + $76\sqrt{2}$ |
| $\text{tr}(\Psi^*|H^2)$ | $4\sqrt{2} - 1$ | $5\sqrt{2} - 3$ | $\sqrt{2} + 7$ | $\sqrt{2} - 7$ | $4\sqrt{2} - 11$ | $1 - 7\sqrt{2}$ |
| $\mathcal{L}(\text{Frob}_p \circ \Psi)$ | 31 | 41 | 47 | 47 | 89 | 97 |
| $\text{tr}(\text{Frob}_p^*|H^3_+)$ | 136 - $76\sqrt{2}$ | 42 - $176\sqrt{2}$ | $-32 - 264\sqrt{2}$ | $-32 + 264\sqrt{2}$ | $-1238 - 256\sqrt{2}$ | $642 + 32\sqrt{2}$ |

The table also lists the traces of $\text{Frob}_p \circ \Psi$ on $H^3(X)$. These can be determined as follows. Since we do not know which factor of the Frobenius polynomial $F_p$ corresponds to the characteristic polynomial of $\text{Frob}_p$ on $H^3_+$ and which to $H^3_-$, we can determine the trace of $\text{Frob}_p \circ \Psi$ a priori only up to a sign. From the table on page 7 we get the following values of traces of $\text{Frob}_p^*$ on $H^3_+ / H^3_-$.

\[
\begin{array}{cccccccc}
7 & 17 & 23 & 31 & 41 & 47 & 89 & 97 \\
16 + 4\sqrt{2} & -62 + 16\sqrt{2} & 136 \pm 76\sqrt{2} & 42 \pm 176\sqrt{2} & -32 \pm 264\sqrt{2} & -1238 \pm 256\sqrt{2} & 642 \pm 32\sqrt{2} \\
\end{array}
\]

We have for any split prime $p \in \mathbb{Z}$ and any prime $p \in \mathbb{Z}[\sqrt{2}]$ over $p$
\[
\mathcal{L}(\text{Frob}_p^* \circ \Psi) = 1 + p \text{tr}(\Psi^*|H^2) - \sqrt{2} (\text{tr}(\text{Frob}_p^*|H^3_+) - \text{tr}(\text{Frob}_p^*|H^3_-)) + p^2 \text{tr}(\Psi^*|H^4) + 2p^3.
\]
and
\[
\text{tr}(\Psi^*|H^2) + \text{tr}(\Psi^*|H^4) = 9.
\]

In the case $p = 7$, $p = 3 + \sqrt{2}$ we get two possibilities
\[
976 = 1 + 7 \text{tr}(\Psi^*|H^2) - \sqrt{2} (16 - 4\sqrt{2} - (16 + 4\sqrt{2})) + 49 \text{tr}(\Psi^*|H^4) + 686
\]
or
\[
976 = 1 + 7 \text{tr}(\Psi^*|H^2) - \sqrt{2} (16 + 4\sqrt{2} - (16 - 4\sqrt{2})) + 49 \text{tr}(\Psi^*|H^4) + 686,
\]
or equivalently,
\[
273 = 7(\text{tr}(\Psi^*|H^2) + 7 \text{tr}(\Psi^*|H^4)) \quad \text{and} \quad 305 = 7(\text{tr}(\Psi^*|H^2) + 7 \text{tr}(\Psi^*|H^4))
\]
As $7 \nmid 305$, the second option is impossible and consequently
\[
\text{tr}(\Psi^*|H^2) + 7 \text{tr}(\Psi^*|H^4) = 39.
\]
Together with
\[
\text{tr}(\Psi^*|H^2) + \text{tr}(\Psi^*|H^4) = 9,
\]
this yields
\[
\text{tr}(\Psi^*|H^2) = 4, \quad \text{tr}(\Psi^*|H^4) = 5.
\]
Now, we get
\[ \sqrt{2}(\text{tr}(\text{Frob}_p^* | H^3_+) - \text{tr}(\text{Frob}_p^* | H^3_-)) = -L(\text{Frob}_p^* \circ \Psi) + 1 + 4p + 5p^2 + 2p^3, \]
and we can compute the entries of the above table.
Using Magma one finds 3 Hilbert modular forms for \( K = \mathbb{Q}[^{\sqrt{2}}] \) of weight \([4, 2]\) and level \( 6^{\sqrt{2}}\mathcal{O}_K \). For one of them, let us denote it by \( h_1 \), the Hecke eigenvalues agree exactly with the traces of the action of Frobenius on \( H^3_+ \) computed in the above table. (For the reader’s convenience, we provide a table of eigenvalues at \([3]\).)

**Theorem 4.1.** The Galois representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[^{\sqrt{2}}]) \) on the motive \( H^3_+ \) is isomorphic to the Galois representation of the Hilbert modular form \( h_1 \) for \( K = \mathbb{Q}[^{\sqrt{2}}] \) of weight \([4, 2]\) and level \( 6^{\sqrt{2}}\mathcal{O}_K \).

**Proof.** There exist continuous Galois representations \( \rho_1, \rho_2 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[^{\sqrt{2}}]) \to \text{GL}_2(\mathbb{Q}[^{\sqrt{2}}]) \)
defined by the motive \( H^3_+ \) and the Hilbert modular form \( h_1 \). We shall verify that the representations \( \rho_1 \) and \( \rho_2 \) satisfy the assumptions of Proposition \([2.1]\). We have computed the traces of \( \text{Frob}_p | H^3_+ \) for \( p \in T \) and verified with MAGMA that they agree with Hecke eigenvalues of \( h_1 \); for \( p \in U \) we check that both traces are even. Moreover for any \( p \in T \) we check that \( \text{det}(\rho_1(\text{Frob}_p)) = p^3 \). Since \( \text{det}(\rho_1(\text{Frob}_p))p^6 \) for any \( p \geq 5 \) and any prime \( p \) in \( \mathbb{Q}[\sqrt{2}] \) over \( p \), it follows that \( \text{det}(\rho_2(\text{Frob}_p)) \) is odd.
Finally, \( h_1 \) is a Hilbert modular form with trivial character, so \( \text{det}(\rho_2(\text{Frob}_p)) = N(p)^3 \) (which is odd). Thus the assumptions of Proposition \([2.1]\) are satisfied, and applying the proposition concludes the proof. \( \square \)

**Remark 4.2.** It follows that the Galois representation on the motive \( H^3_- \) is isomorphic to the Galois representation of the Hilbert modular form \( h_1 \) for \( \mathbb{Q}[\sqrt{2}] \) of weight \([2, 4]\) and level \( 6^{\sqrt{2}}\mathcal{O}_K \). Observe the divisibility condition \( a_p | p \) for all Hecke eigenvalues of \( h_1 \) in the given range of primes (in agreement with \([9, \S 3]\)). We believe that this could be proven geometrically using the Hodge type \((2, 1) + (1, 2)\) of \( H^3_- \) along the lines of \([12]\). This could then also simplify the determination of the factor of the characteristic polynomial of Frobenius corresponding to \( H^3_- \) at ordinary primes.

5. **Hilbert modular rigid Calabi-Yau threefold over \( \mathbb{Q}[\sqrt{5}] \)**
Let \( Y \) be the double octic defined as a crepant resolution of singularities of the hypersurface
\[ u^2 = x y z v (x + y + z) (\varphi y - z + v) (x + y + \varphi v) ((1 - \varphi)x + y - \varphi z + \varphi v) \subset \mathbb{P}(1, 1, 1, 1, 4), \]
where \( \varphi = \frac{1}{2}(-1 + \sqrt{5}) \). Then \( Y \) is a rigid Calabi-Yau threefold with \( h^{11} = 38 \) by \([4, \text{Prop. 5.4}]\), and one verifies as before that the Picard group is generated by divisors defined over \( K = \mathbb{Q}[\sqrt{5}] \) while the only prime of bad reduction of \( Y \) is 2.
For prime numbers \( p \equiv 1, 4 \pmod{5} \) we computed the numbers \( n_p \) and \( n_{p^2} \) of points of the singular double covering over \( \mathbb{F}_p \) and over \( \mathbb{F}_{p^2} \). The resolution of singularities is blowing-up 28 double lines and 9 fourfold points, and by the Lefschetz fixed point formula we get the following traces on \( H^3(Y_p) \) (where we suppress the cohomology group for ease of notation):
\[ \text{Tr}(\text{Frob}_{p^2}) = -n_p + p^6 + p^4 - 8p^2 + 1, \quad \text{Tr}(\text{Frob}_p) = -n_p + p^3 + p^2 + cp + 1, \quad c \in \{-8, \ldots, 10\}, \]
in a similar way as for the Calabi-Yau threefold \( X \). Moreover, comparing the actions of \( \text{Frob}_p \) and \( \text{Frob}_{p^2} \)
\[ \text{Tr}(\text{Frob}_{p^2}) = \text{Tr}(\text{Frob}_p)^2 - 2p^3. \]
We observe that these equalities often suffice to determine $\text{Tr}(\text{Frob}_p)$. The next table collects the results of the computations for the split primes which we shall need in the proof of modularity.

| $p$ | $p$ | $\varphi$ | $n_p$ | $n_p^\varphi$ | $\text{Tr}(\text{Frob}_p)$ | $\text{Tr}(\text{Frob}_p^2)$ |
|-----|-----|--------|------|-------------|------------------|------------------|
| 11  | $\sqrt{5} + 4$ | 3      | 1459 | 1784297     | 60               | 938              |
|     | $\sqrt{5} - 4$ | 7      | 1461 | 1786601     | 36               | −1366            |
| 29  | $2\sqrt{5} + 7$ | 5      | 25217| 595525129   | −218             | −1254            |
|     | $2\sqrt{5} - 7$ | 23     | 25089| 595564553   | −90              | −40678           |
| 31  | $\sqrt{5} + 6$  | 12     | 30685| 888442233   | 192              | −22718           |
|     | $\sqrt{5} - 6$  | 18     | 31003| 888475001   | −64              | −55486           |
| 61  | $2\sqrt{5} - 9$ | 17     | 230471| 51534519081| 354              | −328646          |
|     | $2\sqrt{5} + 9$ | 43     | 230215| 51534272297| 610              | −81862           |

To apply Proposition 2.2, we also require information at two inert primes. For $p = 3$ we were able to compute the number of points of the singular double cover of $\mathbb{P}^3$ over $\mathbb{F}_{3^2}$ and $\mathbb{F}_{3^3}$, obtaining $n_9 = 815$, $n_{81} = 538617$. Similar computations as in the case of split primes give $\text{Tr}(\text{Frob}_3) = −1262$ and $\text{Tr}(\text{Frob}_{81}) = 14$. For $p = 13$ we computed that the number of points over $\mathbb{F}_{13^2}$ equals 4857691. To obtain the contribution for the exceptional divisors (4.3) over the fourfold points, notice that the values of $\alpha$ in question, $6 - 2 \sqrt{5}$, $\frac{1}{4}(-3 + 3 \sqrt{5})$, $\frac{1}{4}(-3 + 3 \sqrt{5})$, $4 \sqrt{5} - 8$, $-6 \sqrt{5} + 14$, $\frac{1}{4}(3 - 1 \sqrt{5})$, $6 \sqrt{5} - 14$, $\sqrt{5} - 3$, $-1 + \sqrt{5}$, are all squares in $\mathbb{F}_{13^2}$. From the Lefschetz fixed point formula we thus derive the trace of $\text{Frob}_{13}$ on $H^3(Y_{13})$ as

$$\text{Tr}(\text{Frob}_{13}) = -(4857961 + 9 \cdot 13^2) + 1 + 13^2 + 13^4 + 13^6 = -3942.$$ Using Magma we found 24 Hilbert modular forms for $\mathbb{Q}[\sqrt{5}]$ of parallel weight $[4, 4]$ and level $16\mathcal{O}$; one of them, denoted by $h_2$, has exactly the same Hecke eigenvalues as the Frobenius traces above. Proposition 2.2 thus guarantees the modularity of $Y$:

**Theorem 5.1.** The Galois representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{5}])$ on $H^3_{et}(Y, \mathbb{Q}_l)$ is Hilbert modular with corresponding Hilbert modular form $h_2$.

### 6. Modular Calabi-Yau threefold over $\mathbb{Q}[\sqrt{-3}]$

Let $Z$ be the double octic defined as a resolution of singularities of the hypersurface

$$(6.1) \quad u^2 = xyzv (x + y) (x + y + z - v)(\zeta x - y + \zeta z)(y - \zeta z - v) \subset \mathbb{P}(1, 1, 1, 1, 4),$$

where $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ and, of course, $i^2 = -1$. Then $Z$ is a rigid Calabi-Yau threefold with $h^{11} = 46$ as can be checked by considering it as a member of the one-dimensional family of double octics given by arrangement No. 262 in [13]. As before, one verifies that the Picard group is generated by divisors defined over $K = \mathbb{Q}[\sqrt{-3}]$, and that the only prime of bad reduction of $Z$ is 2.

**Proposition 6.1.** $Z$ is birational to a Calabi-Yau threefold defined over $\mathbb{Q}[i]$.

**Proof.** The standard crepant resolution of a double octic proceeds as follows: blow-up successively fivefold points, triple lines, fourfold points and double lines. The resolution depends on the order of blow-ups of double lines; to overcome this subtlety we modify the last step and blow-up the union of all double lines in the singular double cover (cf. 2.4). Then the map

$$(x, y, z, v, u) \mapsto (\zeta x, -\zeta x - \zeta y, - (\zeta + 1) x - y - (\zeta + 1) z, - (\zeta + 1) (x + y + z - t), iu)$$

defines an isomorphism of $Z$ and its Galois conjugate over $K$, hence $Z$ isomorphic to a variety defined over $\mathbb{Q}[i]$ by the Weil Galois Descent Theorem ([16, Thm. 3]).
We can count points over $\mathbb{F}_p$ only if $p \equiv 1 \pmod{6}$, i.e. $p$ is a split prime in $K$. Above a given split prime $p$ there are two prime ideals $p$ in the ring of integers of $\mathbb{Q}[\sqrt{-3}]$; this corresponds to two choices for $\zeta \in \mathbb{F}_p$ and two possibilities for the trace of Frobenius on $H^3(\bar{\mathbb{Z}}_p)$ which we list in the next table.

| $p$  | $\zeta$ | $\text{Tr}(\text{Frob}_p)$ | $\zeta$ | $\text{Tr}(\text{Frob}_p)$ |
|------|---------|-----------------------------|---------|-----------------------------|
| 7    | 4       | -12                         | 2       | 12                          |
| 13   | 3       | -58                         | 9       | -58                         |
| 19   | 11      | -136                        | 7       | 136                         |
| 31   | 25      | 20                          | 5       | -20                         |
| 37   | 26      | -18                         | 10      | -18                         |
| 43   | 6       | -200                        | 36      | 200                         |
| 61   | 47      | -458                        | 13      | -458                        |
| 67   | 29      | -496                        | 37      | 496                         |
| 73   | 64      | 602                         | 8       | 602                         |
| 79   | 55      | 1108                        | 23      | -1108                       |
| 97   | 61      | -206                        | 35      | -206                        |

The computed traces agree up to sign with the Fourier coefficients of a modular form $f$ of weight 4 for $\Gamma_0(72)$ :\((72/1 \text{ in Meyer’s notation in } [13]):

$$
\begin{array}{cccccccccc}
 p & 7 & 13 & 19 & 31 & 37 & 43 & 61 & 67 & 73 & 79 & 97 \\
 a_p & -12 & 58 & -136 & 20 & -18 & -200 & -458 & -496 & -602 & 1108 & 206 \\
\end{array}
$$

We compare the signs with characters from the table in the proof of Proposition 2.3 to notice that the sign changes are governed by the character corresponding to the extension $\mathbb{Q}[\sqrt{-3}]/\mathbb{Q}[\sqrt{-3}]$.

**Theorem 6.2.** Consider the Galois representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{-3}])$ on $H^3_{et}(\bar{\mathbb{Z}}, \mathbb{Q}_l)$ and the one associated to the modular form $f$ restricted to $\mathbb{Q}[\sqrt{-3}]$ and then twisted by the quadratic character associated to the extension $\mathbb{Q}[\sqrt{-3}]/\mathbb{Q}[\sqrt{-3}]$. Then the Galois representations have isomorphic semi-simplifications.

**Proof.** In view of Proposition 2.3 compared to the present data, it suffices to check the following two properties:

- $\sqrt{-3} = 2\zeta + 1$ is a square in $\mathbb{F}_p$ if and only if the corresponding choice of $\text{Tr}(\text{Frob}_p)$ matches the Fourier coefficient $a_p$ of $f$;
- the Galois representations have even trace at $p = 11$.

The latter condition follows easily since $a_{11} = 64$ and any double octic Calabi–Yau threefold given by an arrangement of 8 planes satisfying condition (3.1) is checked to have an even number of points over any finite field of odd parity by an elementary combinatorial argument. □

**References**

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