1. Bounded extension of operators. 1. A separable case.

The isomorphic analogue of the class $\mathcal{P}_1(X^f)$ is the class $\mathcal{P}^\sim(X^f)$, which consists of all such spaces $E <_I X$, whose isomorphic image in every space from $X^f$ admits a bounded projection.

This class essentially differs from $\mathcal{P}_1(X^f)$ but has much common with $\mathfrak{R}(X^f)$. Below it will be shown (in difference to the classical case $\mathcal{P}_2(\ell_0^f)$) there are such classes $X^f$ that $\mathcal{P}^\sim(X^f) = \emptyset$ and such classes $Y^f$ that $\mathcal{P}^\sim(Y^f) \neq \emptyset$ but $\mathcal{P}_1(X^f) \nsubseteq \mathcal{P}^\sim(X^f)$.

It is more natural to study the (formally more wide) class $\mathcal{P}^\sim(X^f)$, define below.

**Definition 1.** Let $X$ be a Banach space, $X^F$ be the corresponding class of crudely finite equivalence. A space $X_0$ is said to be boundedly universally complemented in $X^F$ provided every space $Z \in X^F$, which contains a subspace isometric to $X_0$, admits a bounded projection $P : Z \to X_0$.

Surely, this definition is equivalent to the formally stronger property of existence of a bounded projection $P' : Z \to iX_0$ where $i : X_0 \to Z$ is an isomorphic embedding.

Certainly, every finite-dimensional Banach space $A$ generates the class $A^F$, which consists of all finite-dimensional Banach spaces $B$ of dimension $\dim B = \dim A = n$.

Since for every finite-dimensional Banach space $A$ and every space $Y$ that contains a subspace $A$ there exists a projection $P : Y \to A$ of norm $\|P\| \leq \sqrt{\dim A}$ (cf. [25]), all finite-dimensional Banach spaces are universally complemented in each class $X^F$, which is generated by an infinite-dimensional space $X$. So, such a case will be eliminated as trivial.

A class of all infinite-dimensional Banach spaces, which are boundedly universally complemented in the class $X^F$ will be denoted by $\mathcal{P}^\sim(X^F)$.

Similarly to the classes $\mathcal{A}(X^f)$, $\mathcal{A}_0(X^f)$ and $\mathcal{E}(X^f)$ there may be defined their isomorphic analogues – classes $\mathcal{A}^\sim(X^F)$, $\mathcal{A}_0^\sim(X^F)$ and $\mathcal{E}^\sim(X^F)$.

**Definition 2.** The class $\mathcal{A}^\sim(X^F)$ consists of those (infinitely-dimensional) Banach spaces $X_0$ that have the property:

- If $Z \in X^F$ and $X_0 \hookrightarrow Z$ then the annihilator $(X_0)^\perp \hookrightarrow Z^*$ is a complemented subspace of $Z^*$.

The class $\mathcal{A}_0^\sim(X^F)$ is given by $\mathcal{A}_0^\sim(X^F) = \mathcal{A}^\sim(X^F) \cap X^F$.

**Definition 3.** Let $Z \in \mathcal{B}; W \hookrightarrow Z; \lambda < \infty$. $W$ is said to be a $\lambda$-reflecting subspace of $Z$ if for every finitely dimensional subspace $A \hookrightarrow Z$ there exists an isomorphic embedding $u : A \to W$ with $\|u\|\|u^{-1}\| \leq \lambda$, such that $u|_{A \cap Z} = Id_{A \cap Z}$.

Certainly if $W$ is a $\lambda$-reflecting subspace of $Z$ for all $\lambda > 1$ then $W \prec_\omega Z$.

**Definition 4.** The class $\mathcal{E}^\sim(X^F)$ consists of those (infinitely-dimensional) Banach spaces $W$ that have the property:

- If $Z \in X^F$ and $W$ is isomorphic to a subspace $iW$ of $Z$ then there exists a such $\lambda < \infty$ that the image $iW$ is a $\lambda$-reflecting subspace of $Z$.

The following theorem is proved similarly to the isometric analogue. Here $\mathfrak{B}^\sim$ denotes the class of all Banach spaces $X$ that have a complement in the second dual $X^{**}$ (under the canonical embedding $k_X : X \to X^{**}$).
Theorem 1. \( \Psi^\sim(X^F) = A^\sim(X^F) \cap \emptyset^\sim. \)

If \( Y \in A^\sim(X^F) \) then \( Y^{**} \in \Psi^\sim(X^F). \)

If \( X \) is superreflexive then \( \Psi^\sim(X^F) = A^\sim(X^F). \)

This theorem is of restricted interest.

Indeed, in difference with the isometric case, there is no analogue to the non-emptyness of the class \( E^\sim(X^F) \subseteq A^\sim(X^F) \). Moreover, in some cases even the widest class \( A_0^\sim(X^F) \) also may be empty.

Example 1. If \( 1 \leq p < 2 \) then \( A_0^\sim((l_p)^F) = A^\sim((l_p)^F) = \emptyset. \)

If \( 2 < p < \infty \) then \( A_0^\sim((l_p)^F) = \emptyset; A^\sim((l_p)^F) = (l_2)^F. \)

Proof. Notice that every space \( L_p \) \( (1 \leq p \neq 2 < \infty) \) contains an uncomplemented space, isomorphic to \( l_p \). Moreover, if \( 1 < p < 2 \) then \( L_p \) contains an uncomplemented subspace isomorphic to \( l_2 \) (certainly, this is obvious for \( p = 1 \)). For \( p > 2 \) this result follows from [26] as it was noted in [27], p. 145. For \( 1 \leq p < 4/3 \) this is a consequence of one W. Rudin’s result [28]; on the whole interval \((1, 2)\) it was extended in [29]. At last, the existence in \( L_1 \) of an uncomplemented subspace isomorphic to \( l_1 \) was shown by J. Bourgain [30].

If \( W \in A^\sim((l_p)^F) \) then it is either \( L_p \)-space or \( L_2 \)-space (in the sense of [31]) and, hence, contains a complemented subspace that is isomorphic to \( l_p \) (in the case \( L_p; 1 \leq p < \infty \)). It is easy to see that the only possibility is: \( p > 2 \) and \( W \in (l_2)^F. \)

Indeed, in this case any space, which is isomorphic to the Hilbert space, has a complement in every space of the class \((l_p)^F\) that it contains, as it follows from [32].

In any other case there exists an isomorphic embedding \( u \) of \( W \) in the space \( L_p(\mu) \) (for the suitable measure \( \mu \)) such that the image \( uW \) has no complement in \( L_p(\mu). \)

The result of M.I. Kadec and A. Pelczynski [32], mentioned above, was extended by B. Maurey [33] to the wide class of Banach spaces of type 2,.

In [33] it was shown:

- If a Banach space \( X \) is of type 2 and \( W \) is its subspace, isomorphic to the Hilbert space \( H \), then there exists a projection \( P: X \to W \) of norm \( \|P\| \leq f(T_p(X))d(W, H), \)

where the constant \( f(T_p(X)) \) depends only on \( T_p(X). \).

An obvious consequence of this result is the following.

Theorem 2. For any Banach space \( X \) of type 2 the class \( \Psi^\sim(X^F) \) is non-empty.

Proof. As it follows from the aforementioned B. Maurey’s theorem [33], the following inclusion is fulfilled: \( (l_2)^F \subseteq \Psi^\sim(X^F). \)

Notice that in the general (isometric) case classes \( A_0(X^F) \) and \( E(X^F) \) are different. E.g., if \( X = l_p \) and \( p \neq 2 \) then the unique separable space in the class \( E((l_p)^F) \) is the space \( L_p[0, 1]. \) However, spaces \( l_p, l_p \oplus_p L_p[0, 1] \) and a sequence of spaces \( \{l_p^n \oplus_p L_p[0, 1]: n < \infty\} \) are pairwise non isometric and each of them belongs to \( A_0((l_p)^F). \) Notice, also, that in this case \( A_0((l_p)^F) = A((l_p)^F) \) for all
The union of \( P \) is \( Y \). Clearly, \( W \) is a \( \lambda \)-refining subspace of \( Z \) then \( W \) is a \( \lambda \)-reflecting subspace of \( Z \) for some \( \lambda \leq 24 \).

The converse recall the following result, due to J. Stern [9]:

Let \( Z \in X^F \); \( Y \in A^\sim(X^F) \). Without loss of generality it may be assumed that \( Y \in \mathfrak{P}^\sim \) and \( Y \rightarrow Z \). Hence there exists such \( \lambda < \infty \) that there exists a projection \( P : Z \rightarrow Y, \|P\| \leq \lambda \).

Define on \( Z \) a new norm \( \|\cdot\|^\prime \) by taking as its unit ball the convex hull of the union of \( \lambda^{-1}B(Y) \) and \( B(Z) \) (recall that \( B(X) \) denotes the unit ball of a Banach space \( X \)):

\[
B\left(\langle Z, \|\cdot\|^\prime \rangle\right) = \text{conv}\{\lambda^{-1}B(Y) \cup B(Z)\}.
\]

Clearly, \( \|y\|^\prime = \|y\| \) for \( y \in Y \); \( \|z\|^\prime \leq \lambda \|z\| \leq \lambda \|z\|^\prime \) for \( z \in Z \) and there exists a projection of norm one from \( Z' = \langle Z, \|\cdot\|^\prime \rangle \) onto \( Y \).

Hence \( Z' \) may be represented as a direct orthogonal sum \( Z' = Y \oplus W \), where \( W \) is \( C \)-finitely representable in \( Y \) for some \( C < \infty \).

Let \( W \) be represented as a direct limit of the direct systems \( (W_{\alpha})_{\alpha \in A} \) of all its finite-dimensional subspaces: \( W = \lim W_{\alpha} \). Proceeding by induction on the directed set \( A \), chose a set \( (W'_{\alpha})_{\alpha \in A} \) so that it also forms an isometric direct system, spaces \( Y \oplus W'_{\alpha} \) are finitely representable in \( Y \) and \( d(W_{\alpha}, W'_{\alpha}) \leq C' \) for some \( C' < \infty \). Then the space \( Z'' = Z = \lim Y \oplus W'_{\alpha} \) is finitely equivalent to \( Y \), isomorphic to \( Z \) and \( Y \) is an algebraic subspace of \( Z'' \). By the aforementioned Stern's theorem, \( Y \) is a \( \lambda \)-reflecting subspace of \( Z'' \). Using the inverse isomorphisms we see that \( Y \) is a \( \lambda \)-reflecting subspace of \( Z \). Since \( Y \in A^\sim(X^F) \) and \( Z \in X^F \) are arbitrary, this yields that \( A^\sim(X^F) \subseteq \mathcal{E}^\sim(X^F) \). \( \square \)

Now it will be shown that the class \( \mathfrak{P}^\sim(X^F) \) may be non-empty for a wide class of Banach spaces that are not of type 2. It will be needed the following result.

Theorem 4. Let a superreflexive class \( X^f \) contains a separable space \( E_X \) of almost universal disposition (with respect to \( \mathfrak{M}(X^f) \)). Let \( Z_1 \) and \( Z_2 \) be isomorphic subspaces of \( E_X \) of equal codimension (in \( E_X \)); \( u : Z_1 \rightarrow Z_2 \) be the corresponding isomorphism. Then there exists an isomorphic automorphism \( \tilde{u} : E_X \rightarrow E_X \), which restriction to \( Z_1 \) is equal to \( u \).

Proof. Let \( Z_1 \) and \( Z_2 \) be subspaces of \( E_X \) of equal codimension in \( E_X \), i.e.

\[
\text{codim}(Z_1) = \text{dim}(E_X/Z_1) = \text{dim}(E_X/Z_2) = \text{codim}(Z_2).
\]

If \( \text{codim}(Z_1) < \infty \) then assertions of the theorem are satisfied independently of a property of \( E_X \) to be a space of almost universal disposition. Indeed, for an arbitrary Banach space \( X \) any its subspaces of equal codimension, say \( Y \) and \( Z \), are isomorphic and each of them has a complement in \( X \). So, \( X \) may be represented both as a direct sum \( X = Y \oplus A \) and as \( X = Z \oplus B \), where \( \text{dim}(A) = \text{dim}(B) < \infty \).

Let \( v : A \rightarrow B \) and \( u : Y \rightarrow Z \) be isomorphisms. Clearly, \( v \oplus u : X \rightarrow X \) is an automorphism of \( X \), which extends \( u \).
Assume that $Z_1$ and $Z_2$ are isomorphic subspaces of $E_X$ of infinite codimension. Let $u : Z_1 \to Z_2$ be the corresponding isomorphism.

Let $Z_1$ be represented as a direct limit (= as the closure of union) of a chain

$$Z'_1 \to Z'_2 \to \ldots \to Z'_n \to \ldots \to Z_1;$$

namely, $Z_1 = \bigcup \{Z'_n : n < \omega \}$ and $\bigcup Z'_n$ be dense in $Z_1$. Clearly,

$$uZ'_1 \to uZ'_2 \to \ldots \to uZ'_n \to \ldots \to uZ_1 = Z_2$$

and the union $\bigcup uZ'_n$ is dense in $Z_2$.

Let $(e_n)_{n<\omega}$ be a sequence of linearly independent elements of $E_X$ of norm one, which linear span is dense in $E_X$.

Denote the restriction $u \mid Z'_1$ by $u_n$ and define by induction two sequences $(f_n)$ and $(g_n)$ of elements of $E_X$ and a sequence $(v_n)$ of isomorphisms.

Let $f_1$ be an element of $(e_n)_{n<\omega}$ with the minimal number, which does not belong to $Z'_1$.

Put $W_1 = \text{span} \left( Z'_1 \cup \{f_1\} \right)$ (recall that $\text{span}(A)$ denotes the closure of the linear span of $A$).

Let $\varepsilon > 0$ and $v_1 : W_1 \to E_X$ be an extension of $u_1$ such that

$$\|v_1\| \|v_1^{-1}\| \leq (1 + \varepsilon) \|u_1\| \|u_1^{-1}\|.$$ 

Put $g_1 = v_1 \circ f_1; U_1 = v_1W_1$.

Let $g_2$ be an element of $(e_n)_{n<\omega}$ with the minimal number, which does not belong to span $(uZ'_1 \cup U_1)$.

Let $U_2 = \text{span} (uZ'_1 \cup U_1 \cup \{g_2\})$.

Since $E_X$ is a space of almost universal disposition, there exists an isomorphism $(v_2)^{-1}$ which extends both $(u_2)^{-1}$ and $(v_1)^{-1}$ and satisfies the inequality

$$\|v_2\| \|v_2^{-1}\| \leq (1 + \varepsilon^2) \max \{\|u_2\| \|u_2^{-1}\| ; \|v_1\| \|v_1^{-1}\|\}.$$ 

Let $f_2 = (v_2)^{-1} \circ g_2$. This close the first step of the induction.

Assume that \{f_1, f_2, \ldots, f_n\}; \{g_1, g_2, \ldots, g_n\}; \{v_1, v_2, \ldots, v_n\}; \{U_1, U_2, \ldots, U_n\}$ and \{W_1, W_2, \ldots, W_n\} are already chosen. If $n$ is odd, we choose sequentially: $f_{n+1}$ be an element of $(e_n)_{n<\omega}$ with the minimal number, which does not belongs to $Z'_{n+1}$:

$$W_{n+1} = \text{span} \left( Z'_{n+1} \cup \{f_{n+1}\} \right);$$

$$v_{n+1} : W_{n+1} \to E_X$$

be an extension of $u_{n+1}$ such that

$$\|v_{n+1}\| \|v_{n+1}^{-1}\| \leq (1 + \varepsilon^n) \|u_{n+1}\| \|u_{n+1}^{-1}\| ;$$

$$g_{n+1} = v_{n+1} \circ f_{n+1}; U_{n+1} = v_{n+1}W_{n+1}.$$ 

If $n$ is even, we choose sequentially:

$g_{n+1}$ to be an element of $(e_n)_{n<\omega}$ with the minimal number, which does not belong to span $(uZ'_n \cup U_n)$;

$$U_{n+1} = \text{span} (uZ'_n \cup U_n \cup \{g_{n+1}\});$$

$$(v_{n+1})^{-1}$$, which extends both $(u_{n+1})^{-1}$ and $(v_n)^{-1}$ and satisfies

$$\|v_{n+1}\| \|v_{n+1}^{-1}\| \leq (1 + \varepsilon^2) \max \{\|u_{n+1}\| \|u_{n+1}^{-1}\| ; \|v_n\| \|v_n^{-1}\|\};$$

$$f_{n+1} = (v_{n+1})^{-1} \circ g_{n+1}.$$ 

Since $E_X$ is superreflexive, a sequence of isomorphisms $(v_n)$ converges to an automorphism $V : E_X \to E_X$, which satisfies

$$\|V\| \|V^{-1}\| \leq \prod_{n=1}^{\infty} (1 + \varepsilon^n) \|u\| \|u^{-1}\|.$$
Theorem 5. Let $X^f$ be a superreflexive class that contains a separable space $E_X$ of almost universal disposition. Then $E_X \in \mathcal{E}^\sim(X^f) = \mathfrak{P}^\sim(X^f)$.

Proof. Let $Z \in X^f$, $j : E_X \to Z$ be an isomorphic embedding. Without loss of generality it may be assumed that $Z$ is separable (if $Z$ is non separable, it has a separable complemented subspace $Z_0$, which contains $jE_X$). Also, it may be assumed that there exists an isometric embedding $k : Z \to E_X$ (since $E_X \in \mathcal{E}(X^f)$). Hence, the composition $k \circ j$ is an isomorphic embedding of $E_X$ into $E_X$.

It may be assumed without the loss of generality that there exists an isometric embedding $w$ of $E_X$ into $E_X$.

There exists a bounded projection $P : E_X \to wE_X$. According to the previous theorem, there exists an automorphism $v : E_X \to E_X$, which sends $jE_X$ to $wE_X$. Clearly, this implies that there exists a bounded projection from $Z$ onto $jE_X$. Since $Z$ is arbitrary, $E_X \in \mathfrak{P}^\sim(X^f)$. \hfill $\square$

Analogously to $\mathfrak{P}^\sim(X^f)$ it may be defined the class $\mathfrak{J}^\sim(X^f)$.

Definition 5. Let $X$ be a Banach space and $E \ll_X X$. $E$ belongs to the class $\mathfrak{J}^\sim(X^f)$ provided for any pair $Y_0 \hookrightarrow Y$ where $Y \ll_X X$ every operator $u : Y_0 \to Y$ has a bounded extension $\tilde{u} : Y \to E$.

Theorem 6. Let $X^f$ be superreflexive and contains a separable space $E_X$ of almost universal disposition. Let $W$ be isomorphic to $E_X$. Then $W \in \mathfrak{J}^\sim(X^f)$.

Proof. Let $W$ be isomorphic to $E_X$; $u : E_X \to W$ be the corresponding isomorphism. Let $v : W \to E_X$ be the isomorphic embedding.

Let $Y_0 \hookrightarrow Y$, $Y \ll_X E_X$ and $Y$ be separable. Let $i : Y_0 \to Y$ be an operator. Then $v \circ i : Y_0 \to vW \hookrightarrow E_X$ and this mapping may be extended to the operator $V : Y \to E_X$ of the same norm.

Put $A = VY \cap vW$.

Since $E_X$ is a space of almost universal disposition, there exists a subspace $W_0 \hookrightarrow E_X$ of infinite codimension such that $A \hookrightarrow W_0$ and $W_0 \in \mathcal{E}((E_X)^f)$. Hence, there exists an automorphism $U : E_X \to E_X$ such that $UW_0 = W$. Certainly, this isomorphism may be chosen to fix $(v \circ i)Y_0 \hookrightarrow E_X$.

Clearly, $U \circ V$ extends $v \circ i$ and, hence, $v^{-1} \circ U \circ V$ extends $i$. \hfill $\square$

Corollary 1. Let $X^f$ be a superreflexive class that contains a separable space $E_X$ of almost universal disposition. Let $W$ be isomorphic to $E_X$. Then $W \in \mathfrak{P}^\sim(X^f)$.

Proof. As it was shown before, condition on $X^f$ implies that $\mathfrak{P}(X^f) = \mathfrak{J}(X^f)$. Hence, by the previous theorem, $W \in \mathfrak{J}^\sim(X^f)$. Let $Z \in X_F$ and $W \hookrightarrow Z$. Then this identical embedding may be extended to the operator $P : Z \to Z$ such that $P|_W = Id_W$. Clearly, $P$ is a projection. \hfill $\square$

2. Bounded extension of operators. 2. A non-separable case.

Because of every class of crudely finite equivalence $X^f$, which contains a quotient-closed divisible class $Y^f$, contains also its Gurarii compression $\Gamma(X^f)$, and, as it was shown before, every superreflexive class of kind $\Gamma(X^f)$ contains a
separable space of almost universal disposition $E_X$ with respect to $M(\Gamma(X^f))$, it follows that for every quotient-closed superreflexive divisible class $X^f$ both classes $\Phi^s(X^f)$ and $\Psi^s(X^f)$ are equal and non-empty. However, it was proved the existence of only one member of these classes, namely $E_X$. The existence of other spaces without assuming set theoretical hypothesis is doubtful.

Here it will be considered a non-separable case and it will be shown that under certain set-theoretical assumptions (e.g., under assumption of existence an inaccessible cardinal) classes $P \sim_p (X_F)$ and $J \sim_p (X_F)$ are confinal with $X_F$ and are non-empty in the superreflexive case.

Recall some definitions on cardinal numbers.

Ordinals will be denoted by small Greece letters $\alpha, \beta, \gamma$. Cardinals are identified with the least ordinals of given cardinality and are denoted by $\iota, \tau, \kappa$ (may be, with indices). As usual, $\omega$ and $\omega_1$ denote respectively the first infinite and the first uncountable cardinals (= ordinals).

A set of elements (of arbitrary nature) $\{a_\alpha : \alpha < \kappa\}$ will be called the $\kappa$-sequence.

For a cardinal $\tau$ its predecessor (i.e., the least cardinal $\varkappa$, which is strongly greater then $\tau$) is denoted by $\tau^+$. The confinality of $\tau$, $\text{cf}(\tau)$ is the least cardinality of a set $A \subset \tau$ such that $\tau = \sup A$.

Let $A, B$ be sets. The symbol $B^A$ denotes the set of all functions from $B$ to $A$.

In a general case, the cardinality of the set $B^A$ is denoted either by $\text{card}(A)^{\text{card}(B)}$ or by $\varkappa^\tau$, if $\text{card}(A) = \varkappa$; $\text{card}(B) = \tau$.

The symbol $\exp(\tau)$ (or, equivalently, $2^\tau$) denotes the cardinality of the set $\text{Pow}(\tau)$ of all subsets of $\tau$.

A cardinal $\tau$ is said to be regular (resp., singular) if its confinality $\text{cf}(\tau) = \tau$ (resp., if $\text{cf}(\tau) < \tau$ ).

A cardinal $\iota$ is said to be inaccessible provided it is regular and $\exp(\varkappa) \leq \iota$ for every $\varkappa < \iota$.

The continuum hypothesis (CH) is the assumption

$$\omega^+ \overset{\text{def}}{=} \omega_1 = \exp(\omega).$$

The general continuum hypothesis (GCH) is the assumption

$$\tau^+ = \exp(\tau) \text{ for every infinite cardinal } \tau.$$ Let $\tau$ be a cardinal. Put

$$\varkappa^\tau = \sum \{\exp(\tau) : \tau < \varkappa\}$$

It is known that there are arbitrary large cardinals $\varkappa$ having the property

$$\varkappa = \varkappa^\tau.$$ In other words, a class of all cardinals with the property $\varkappa = \varkappa^\tau$ is confinal with the class of all cardinals. However, there known such models of the set theory, in which every cardinal $\varkappa$ with the property $\varkappa = \varkappa^\tau$ is singular.

From the other hand, in assuming the general continuum hypothesis (GCH) every regular cardinal $\varkappa$ has the property $\varkappa = \varkappa^\tau$. Certainly, any inaccessible cardinal $\iota$ is regular and enjoys the property $\iota = \iota^\tau$. 
Recall that the existence of an inaccessible cardinal (as well as the CH or the GCH) cannot be proved in ZFC.

In what follows it will be assumed that there exists a regular cardinal \( \varkappa \) with the property \( \varkappa = \varkappa^* \). Every such cardinal will be called the star-cardinal.

Recall that a Banach space \( X \) is said to be an envelope (of the class \( X^f \)) if every Banach space \( Y \), which is finitely representable in \( X \) and is of dimension \( \dim Y \leq \dim X \) is isometric to a subspace of \( X \). In [34] it was shown that for every cardinal \( \varkappa \) with the property \( \varkappa = \varkappa^* \) (it does not matter regular or singular) every class \( X^f \) of finite equivalence (generated by an infinite-dimensional space \( X \)) contains an envelope of this class of dimension \( \varkappa \).

It will be convenient to introduce one more definition.

**Definition 6.** A Banach space \( X \) is said to be \( f \)-saturated if \( X \) is a space of almost universal disposition with respect to a class \( (X^f)_{\dim X} \) of all Banach spaces that are finitely representable in \( X \) and whose dimension is strictly less then \( \dim X \);

\[
(X^f)_{\dim X} = \{ Y <_f X : \dim Y < \dim X \}.
\]

**Theorem 7.** Let \( X^f \) enjoys the isomorphic amalgamation property. Then for every star-cardinal \( \varkappa \) the class \( X^f \) contains an \( f \)-saturated space \( E_\varkappa \in X^f \) of dimension \( \varkappa \).

**Proof.** Let \( F \) be an envelope of \( X^f \) of dimension \( \varkappa \). Let \( (f_i)_{i<\varkappa} \) be dense in the unit sphere of \( F \).

For every \( A \subset \varkappa \) of cardinality \( A < \varkappa \) put \( F_A = \text{span}\{f_i : i \in A\} \). Surely, there are just \( \varkappa \) different spaces of kind \( F_A \).

For \( A, B \subset \varkappa \), \( \max\{\text{card} A, \text{card} B\} < \varkappa \) let \( u_{A,B} : F_A \to F_B \) be an isomorphic embedding (if it exists). The set of all such embeddings is of cardinality \( \tau \leq \varkappa \).

Let \( \varepsilon < 1 \); \( n \in \mathbb{N} \); \( i_{A,n} : F_A \to F \) be an isomorphic embedding of norm \( \|i_{A,n}\| \|i_{A,n}^{-1}\| \leq 1 + \varepsilon^n \). The set of all such embeddings is of cardinality that does not exceed \( \varkappa \) as well.

So, the set \( T \) of all \( V \)-formations of kind \( t = (F_A, F_B, i_{A,n}, u_{A,B}) \) may be indexed by elements of \( \varkappa \). Let \( T = \{t_\alpha : \alpha < \varkappa\} \).

Construct by induction an \( \varkappa \)-sequence of spaces \( (E_\alpha^f)_{\alpha<\varkappa} \) as a chain

\[
E_0^f \to E_1^f \to ... E_\alpha^f \to ...
\]

Put \( E_0^f = F \). If \( \alpha = \beta + 1 \) then as \( E_\alpha^f \) it will be chosen an amalgam of the \( V \)-formation \( \langle F_A, E_\beta^f, F_B, v_{A,n,u_{A,B}} \rangle \), where \( \langle F_A, F_B, i_{A,n,u_{A,B}} \rangle \) is just \( t_\beta \) in our indexation and \( v_{A,n} = j \circ i_{A,n} \), where \( j : F \to E_\beta^f \) is a natural embedding.

If \( \alpha \) is a limit ordinal (i.e. \( \alpha \) has no representation \( \alpha = \beta + 1 \) for any \( \beta \)) put \( E_\alpha^f = \text{span}\{E_\beta^f : \beta < \alpha\} \).

Put \( E_{(1)} = \text{span}\{E_\alpha^f : \alpha < \varkappa\} \).

Now we repeat this inductive procedure \( \varkappa \) times starting at the \( \gamma \)-th step with \( E_{(\gamma)} \). Sequentially there will be obtained \( E_{(1)} \to E_{(2)} \to ... \)

Put \( E_{(\gamma)} = \text{span}\{E_{(\gamma)} : \gamma < \varkappa\} \).

Certainly, \( \dim E_\varkappa = \varkappa \). To show that \( E_\varkappa \) is \( f \)-saturated chose a pair \( G, H \) of subspaces of \( E_\varkappa \), \( \dim G \leq \dim H < \varkappa \) and an isomorphic embedding \( u : G \to H \).

Since \( F \) is an envelope, \( G \) and \( H \) almost isometric to corresponding subspaces of \( F \) (say, \( FG \) and \( FH \)). Let \( i : G \to E_\varkappa \) be an isomorphic embedding, \( \varepsilon > 0 \) be fixed.
Since $\kappa$ is regular, there exists such $\gamma < \kappa$ that is $(1 + \varepsilon)$-embedding of $G$ into $E(\gamma)$ and, hence, it may be extended to the embedding $u$ of $B$ into $E(\gamma)$ with $\|u\|\|u^{-1}\| \leq (1 + \varepsilon)\|u\|\|u^{-1}\|$

Since $G$, $H$ and $u : G \to H$ are arbitrary, this shown that $E_{\lambda}$ is $f$-saturated. $\blacksquare$

According to [35] a Banach space $X$ is said to be *subspace-homogeneous* if for every pair $A$, $B$ of its isomorphic subspaces of equal (finite or infinite codimension) the isomorphism $v : A \to B$ may be extended to an isomorphism $v$ of $X$ (i.e., $v|_A = v$). Using this terminology it may be said that the theorem 33 proves that every superreflexive separable space $E$ of almost universal disposition (with respect to $\mathfrak{M}(E')$) is subspace-homogeneous. This theorem, that was proved firstly in the author’s preprint [36], solves the problem of [35] on existence subspace-homogeneous Banach spaces that are different from $l_2$ and $c_0$.

The preceding theorem makes possible to prove the existence of non-separable subspace-homogeneous spaces (in assumption on existence of star-cardinals). The question on existence of non-separable subspace-homogeneous spaces also was posed in [35].

**Corollary 2.** Let $E_{\lambda}$ be a dual $f$-saturated space of the class $X^f$ (this means that $E_{\lambda} = W^*$ for some $W \in \mathcal{B}$). Then $E_{\lambda}$ is subspace-homogeneous.

In particular, $E_{\lambda} \in \mathfrak{P}^\sim(X^f) \ (= \mathfrak{J}^\sim(X^f))$

**Proof.** The proof literally repeats the proof of theorems 33 and 34 with obvious changes: in the present proof the transfinite induction is used. At the last step of the proof is used the $w^*$-compactness of the unit ball of $E_{\lambda}$.

**Theorem 8.** Let $E$ be $f$-saturated and dual (i.e., $E = W^*$ for some $W \in \mathcal{B}$; in particular, $E$ may be separable); $F$ be a complemented subspace of $E$. Then $F \in \mathfrak{P}^\sim(X^f) \ (= \mathfrak{J}^\sim(X^f))$.

**Proof.** Let $A$, $B$ be isomorphic subspaces of $E$; $A \hookrightarrow F \hookrightarrow E$; $B \hookrightarrow F \hookrightarrow E$, which are of equal (finite or infinite) codimension in $F$. Let $u : A \to B$ be the corresponding isomorphism. Let $U : E \to E$ be an automorphism of $E$ that extends $u$; $Ua = ua$ for all $a \in A$.

Let $P : E \to F$ be a projection. Certainly, $UF \hookrightarrow E$ is also a complemented subspace of $E$. Let $Q : E \to UF$ be the corresponding projection.

So, $F \cap UF$ is also a complemented subspace of $E$, which contains $B$. Clearly, $F \cap UF$ is a complemented subspace of both $F$ and $UF$. Let $F'$ be its complement in $F$ (resp., let $F''$ be its complement in $UF$). Clearly, $F'$ and $F''$ are isomorphic.

Let $v : F' \to F''$ be the corresponding isomorphism; $id$ be identical on $F \cap UF$. Then the operator $v \circ id \circ U|_F$ is an automorphism of $F$, which restriction to $A$ is equal to $u$.

**Remark 1.** In [36] it was shown that if $E$ is subspace-homogeneous and its $l^p$-spectrum $S(E)$ contains a point $p \neq 2$ then $E$ and $E \oplus E$ are non-isomorphic. So, from the preceding theorem it follows that there exists a pair of non-isomorphic separable spaces of $\mathfrak{P}^\sim(X^f) \ (= \mathfrak{J}^\sim(X^f))$ when $E$ satisfies the mentioned conditions.

**Corollary 3.** In assumption the GCH every superreflexive quotient-closed divisible class $X^f$ contains spaces from $\mathfrak{P}^\sim(X^f)$ (and, thus, from $\mathfrak{J}^\sim(X^f)$) of arbitrary large cardinality.
The same is true if we assume the existence of inaccesible cardinals.

Now it will be shown that the condition of regularity of $\kappa$ cannot be omitted, at least, in the case when the class $X^f$ is not super-stable in the sense of [37-38]. Recall the definition.

**Definition 7.** ([37]). A Banach space $X$ is said to be stable provided for any two sequences $(x_n)$ and $(y_m)$ of its elements and every pair of ultrafilter $D, E$ over $\mathbb{N}$

$$\lim_{D(n)} \lim_{E(m)} \|x_n + y_m\| = \lim_{E(m)} \lim_{D(n)} \|x_n + y_m\|.$$  

The notations $D(n)$ and $E(m)$ are used here (instead of $D$ and $E$) to underline the variable ($n$ or $m$ respectively) in expressions like $\lim_{D(n)} f(n, m)$.

**Definition 8.** ([38]). A Banach space $X$ is said to be superstable if every its ultrapower $(X)^D$ is stable.

Let $X$ be a Banach space.

A sequence $\{x_n : n < \infty\}$ of elements of $X$ is said to be

- **Spreading**, if for any $n < \infty$, any $\varepsilon > 0$, any scalars $\{a_k : k < n\}$ and any choosing of $i_0 < i_1 < \ldots < i_{n-1} < \ldots; j_0 < j_1 < \ldots < j_{n-1} < \ldots$ of natural numbers

  $$\left\| \sum_{k<n} a_k x_{i_k} \right\| = \left\| \sum_{k<n} a_k x_{j_k} \right\|.$$

- **Symmetric**, if for any $n < \omega$, any finite subset $I \subseteq \mathbb{N}$ of cardinality $n$, any rearrangement $\varsigma$ of elements of $I$ and any scalars $\{a_i : i \in I\}$,

  $$\left\| \sum_{i \in I} a_i z_i \right\| = \left\| \sum_{i \in I} a_{\varsigma(i)} z_i \right\|.$$

**Definition 9.** Let $X$ be a Banach space. Its $I S$-spectrum $IS(X)$ is a set of all (separable) spaces $\langle X, (y_i) \rangle$ with a spreading basis $(y_i)$ which are finitely representable in $X$.

**Theorem 9.** A class $X^f$ is superstable if and only if every member $\langle X, (y_i) \rangle$ of its $I S$-spectrum has a symmetric basis.

**Proof.** The first part of the theorem was proved in [37].

Conversely, let every $(Z, (z_i))$ has a symmetric basis. Suppose that $X$ is not superstable. Then there exists a space from $X^f$ which is not stable (it may be assumed that $X$ is not stable itself). By [37] there are such sequences $(x_n)$ and $(y_m)$ of elements of $X$ that

$$\sup_{m < n} \|x_n + y_m\| > \inf_{m > n} \|x_n + y_m\|.$$  

Let $D$ be a countably incomplete ultrafilter over $\mathbb{N}$. Put

$$X_0 \overset{\text{def}}{=} X; \; X_n \overset{\text{def}}{=} (X_{n-1})_D; \; n = 1, 2, \ldots; \; X_\infty = \cup_{n \geq 1} X_n.$$  

Here is assumed that $X_n$ is a subspace of $X_{n+1} = (X_n)_D$ under the canonical embedding $d_{x_n} : X_n \rightarrow (X_n)_D$.

Let $D, E$ be ultrafilters over $\mathbb{N}$. Their product $D \times E$ is a set of all subsets $A$ of $\mathbb{N} \times \mathbb{N}$ that are given by

$$\{ j \in \mathbb{N} : \{ i \in \mathbb{N} : (i, j) \in A \} \in D \} \in E.$$
Certainly, $D \times E$ is an ultrafilter and for every Banach space $Z$ the ultrapower $(Z)_{D \times E}$ may be in a natural way identified with $((Z)_{D})_{E}$.

So, the sequence $(x_n) \subset X$ defines elements

$$
\begin{align*}
\mathfrak{f}_1 &= (x_n)_D \in (X)_D; \\
\mathfrak{f}_2 &= (x_n)_{D \times D} \in ((X)_D)_D; \\
&\vdots \\
\mathfrak{f}_k &= (x_n)_{D \times D \times \cdots \times D} \in (((X)_D)_{D \times D} \times \cdots \times D; \\
&\vdots
\end{align*}
$$

Notice that $\mathfrak{f}_k \in X_k \setminus X_{k-1}$. It is easy to verify that $(\mathfrak{f}_k)_{k<\infty} \subset X_\infty$ is a spreading sequence. Since $X_\infty \in X^f$, it is symmetric. Moreover, for any $z \in X$, where $X$ is regarded as a subspace of $X_\infty$ under the direct limit of compositions

$$
d_{X_n} \circ d_{X_{n-1}} \circ \cdots \circ d_{X_0} : X \to X_n,$$

the following equality is satisfied: for any pair $m, n \in \mathbb{N}$

$$
\|x_n + z\| = \|x_m + z\|. 
$$

Since $(x_n)$ and $z$ are arbitrary elements of $X$, this contradicts with the inequality $\sup_{m<n} \|x_n + y_m\| > \inf_{m>n} \|x_n + y_m\|.$

\begin{proof}
Since $E^f$ is not super-stable, there exists a space $\langle W, (w_n) \rangle$ with a spreading basis, which is not symmetric one. Consider a vector space $c_{00}(\kappa)$ of all $\kappa$-sequences of reals all but finitely many members of which are vanished.

Let $e_\alpha = (\delta_{\alpha \beta})_{\beta<\kappa} \in c_{00}(\kappa)$ $(\alpha < \kappa)$, where $\delta_{\alpha \beta}$ be the generalized Kronecker symbol: $\delta_{\alpha \beta} = 0$ when $\alpha \neq \beta$; $\delta_{\alpha \beta} = 1$ when $\alpha = \beta$.

Every element $x \in c_{00}(\kappa)$ is of kind $x_A = \sum \{x_\alpha e_\alpha : \alpha \in A\}$, where $A \subset \kappa$ is finite.

Let $A = \{\alpha_0, \alpha_1, ..., \alpha_m\}$ and $\alpha_0 < \alpha_1 < ... < \alpha_m$. Define on $c_{00}(\kappa)$ a norm, which is given by

$$
\|x_A\| = \left\| \sum_{i=0}^{m} x_{\alpha_i} w_i \right\|_{W}.
$$

Let $W(\kappa)$ be a completion of $c_{00}(\kappa)$ under this norm.

Surely, $W(\kappa)$ has a spreading, non symmetric basis, is finitely representable in $E_\kappa$ and is isometric to a subspace of $E_\kappa$ (since $\dim W(\kappa) = \kappa = \dim E_\kappa$ and $E_\kappa$ is an envelope).

Since $\kappa$ is singular, $\text{cf}(\kappa) = \tau < \kappa$. Let $(\kappa_\gamma)_{\gamma<\tau}$ be an increasing $\tau$-sequence of cardinals (that are less then $\kappa$), such that $\sup\{\kappa_\gamma : \gamma < \tau\}$

Consider two subspaces of $W(\kappa)$, say $W_1(\tau)$ and $W_2(\tau)$.

$W_1(\tau)$ is spanned by the first $\tau$ elements of $e_\alpha$;

$$W_1(\tau) = \text{span}\{e_\alpha : \alpha < \tau\}.$$

$W_2(\tau)$ is spanned by those $e_\alpha$’s, whose indices belong to the set $(\kappa_\gamma)_{\gamma<\tau}$;

$$W_1(\tau) = \text{span}\{e_{\kappa_\gamma} : \beta < \tau\}.$$ 

Surely, $W_1(\tau)$ and $W_2(\tau)$ are isometric, but no automorphism $u : E \to E$ sends $W_1(\tau)$ to $W_2(\tau)$.

\end{proof}
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