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Elliptic Problems with Additional Unknowns in Boundary Conditions and Generalized Sobolev Spaces

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Abstract: In generalized inner product Sobolev spaces we investigate elliptic differential problems with additional unknown functions or distributions in boundary conditions. These spaces are parametrized with a function OR-varying at infinity. This characterizes the regularity of distributions more finely than the number parameter used for the Sobolev spaces. We prove that these problems induce Fredholm bounded operators on appropriate pairs of the above spaces. Investigating generalized solutions to the problems, we prove theorems on their regularity and a priori estimates in these spaces. As an application, we find new sufficient conditions under which components of these solutions have continuous classical derivatives of given orders. We assume that the orders of boundary differential operators may be equal to or greater than the order of the relevant elliptic equation.

Keywords: elliptic problem; generalized Sobolev space; Fredholm operator; regularity of solutions; a priori estimate for solutions

MSC: Primary 35J40; 46E35

1. Introduction

This work is a contribution to the theory of elliptic boundary-value problems in generalized Sobolev spaces founded recently by Mikhailets and Murach [1–8] and developed in [9–16]. These spaces are parametrized with a general enough function of frequency variables (which are dual to spatial variables with respect to the Fourier transform). It characterizes the regularity of distributions more finely than the number parameter used for the Sobolev spaces or other classical distribution spaces. Thus, the order of regularity of generalized Sobolev spaces is a function, not a number. We apply these spaces to elliptic differential problems with additional unknown functions or distributions in boundary conditions. Such problems were introduced by Lawruk [17–19] and appear naturally as formally adjoint problems to nonregular elliptic problems with respect to a relevant Green formula. Important examples of such problems occur, e.g., in hydrodynamics and the theory of elasticity [20–22]. Since these problems belong to the Boutet de Monvel algebra, the theorems on their solvability in Sobolev spaces of large enough orders are contained in the results by Boutet de Monvel [23], Rempel and Schulze (Chapter 4, [24]), Grubb [25,26]. Such a theorem is also proved in (Section 23, Subsection 4, [27]) within Eskin and Vishik’s theory of elliptic pseudodifferential boundary problems. The case of Sobolev spaces of arbitrary orders was investigated by Kozlov, Maz’ya, and Rossmann (Chapters 3 and 4, [28]), I. Roitberg [29,30], Y. Roitberg (Chapter 2, [31]), and A. Kozhevnikov [32] in the framework of special spaces introduced by Y. Roitberg [33,34].

In contrast to the works just mentioned, we study these problems in Hilbert distribution spaces that form the extended Sobolev scale investigated in [35,36] and (Section 2.4, [8]). The regularity orders of such spaces are arbitrary OR-varying (O-regularly varying) functions at infinity. It is remarkable that this scale consists of all Hilbert spaces that are
interpolation ones between inner product Sobolev spaces, which allows the use of the interpolation (with function parameter) between Hilbert spaces in proofs. Unlike the nearest articles [37–41] to the present research, we do not impose any restrictions on the orders of the boundary differential operators involved in the problems and do not require that the regularity orders of the generalized Sobolev spaces being used satisfy any additional (unessential) conditions. The results obtained in this paper are partly announced in [42] (without proofs).

This paper consists of eight sections. Section 1 is Introduction. Section 2 gives the statement of the elliptic problem under investigation. Section 3 presents and discusses generalized Sobolev spaces being used. The main results are formulated in Section 4. They consist of the Fredholm property of bounded operators induced by the problem on appropriate pairs of generalized Sobolev spaces, relevant isomorphisms between some subspaces of finite codimension, conditions for local (up to the boundary) regularity of generalized solutions to the problem, and their a priori estimate in these spaces. The case of the homogeneous elliptic equation is separately considered at the end of this section. Section 5 is devoted to the method of interpolation (with function parameter) between Hilbert spaces and discusses some of its properties used in our proofs. The proofs are given in Section 6. Section 7 is devoted to applications of the extended Sobolev scale to the investigations of classical smoothness of the generalized solutions. We find new sufficient conditions under which components of these solutions have continuous classical derivatives of given orders. Among them are conditions for generalized solutions to be classical. The final Section 8 contains concluding remarks.

2. Statement of the Problem

Let $\Omega$ be a bounded domain in the Euclidean space $\mathbb{R}^n$, with $n \geq 2$, and let $\Gamma$ denote the boundary of $\Omega$. Suppose that $\Gamma$ is an infinitely smooth closed manifold of dimension $n - 1$, with the $C^\infty$-structure on $\Gamma$ being induced by $\mathbb{R}^n$. Let $\nu$ denote the field of the unit inward normal vectors to $\Gamma$.

Choose integers $q \geq 1$, $\kappa \geq 1$, $m_1, \ldots, m_q$, and $r_1, \ldots, r_\kappa$ arbitrarily. We consider the following boundary-value problem in $\Omega$:

$$Au = f \quad \text{in } \Omega,$$

$$B_j u + \sum_{k=1}^{\kappa} C_{jk} v_k = g_j \quad \text{on } \Gamma, \quad j = 1, \ldots, q + \kappa,$$

Here, the unknowns are the distribution $u$ in $\Omega$ and $\kappa$ distributions $v_1, \ldots, v_\kappa$ on $\Gamma$. We suppose that $A := A(x, D)$ is a linear partial differential operator (PDO) on $\Omega := \Omega \cup \Gamma$; each $B_j := B_j(x, D)$ is a linear boundary PDO on $\Gamma$, and every $C_{jk} := C_{jk}(x, D\tau)$ is a linear tangent PDO on $\Gamma$. Their orders satisfy the conditions $\text{ord } A = 2q$, $\text{ord } B_j \leq m_j$, and $\text{ord } C_{jk} \leq m_j + r_k$, and their coefficients are infinitely smooth complex-valued functions of $x \in \Omega$ or $x \in \Gamma$ respectively. (Of course, a PDO of negative order is assumed to be equal zero identically.) We consider complex-valued functions and distributions and use corresponding complex function or distribution spaces.

We put

$$m := \max\{m_1, \ldots, m_q + \kappa\} \quad \text{and} \quad \mu := \max\{2q, m + 1\}$$

and assume that

$$m \geq -r_k \quad \text{for each } k \in \{1, \ldots, \kappa\}. \quad (3)$$

This assumption is natural; indeed, if $m + r_k < 0$ for some $k$, all the operators $C_{1,k}, \ldots, C_{q + \kappa, k}$ will equal zero identically, i.e., the unknown distribution $v_k$ will be absent in the boundary conditions (2). Note that the $m \geq 2q$ case is possible.

We suppose that the boundary-value problem (1), (2) is elliptic in $\Omega$. Let us recall the relevant definition (see, e.g., (Subsection 3.1.3, [28])).
Let $A^\circ(x, \xi), B_j^\circ(x, \xi),$ and $C_{j,k}^\circ(x, \tau)$ denote the principal symbols of the PDOs $A(x, D)$, $B_j(x, D),$ and $C_{j,k}(x, D_\tau)$ respectively, the last two PDOs being considered as that of the form $m_j$ and $m_j + r_k$ respectively. Thus, $A^\circ(x, \xi)$ and $B_j^\circ(x, \xi)$ are homogeneous polynomials in $\xi \in \mathbb{C}^n$ of order $2q$ and $m_j$ respectively, and $C_{j,k}^\circ(x, \tau)$ is a homogeneous polynomial of order $m_j + r_k$ in $\tau$, where $\tau$ is a tangent vector to the boundary $\Gamma$ at the point $x$. Defining the principal symbols, we consider the principal parts of the PDOs as polynomials with respect to $D_\ell := i\partial/\partial x_\ell$, where $\ell = 1, \ldots, n$, and then replace each differential operator $D_\ell$ with the $\ell$-th component $\xi_\ell$ of the vector $\xi$.

The boundary-value problem (1), (2) is called elliptic in $\Omega$ if the following three conditions are satisfied:

(i) The PDO $A(x, D)$ is elliptic at every point $x \in \overline{\Omega}$, i.e., $A^\circ(x, \xi) \neq 0$ whenever $\xi \neq 0 \in \mathbb{R}^n$.

(ii) The PDO $A(x, D)$ is properly elliptic at every point $x \in \Gamma$; i.e., for an arbitrary tangent vector $\tau \neq 0$ to $\Gamma$ at $x$, the polynomial $A^\circ(x, \tau + v(x)\xi)$ in $\xi \in \mathbb{C}$ has $q$ roots with positive imaginary part and $r$ roots with negative imaginary part (of course, these roots are counted with regard for their multiplicity).

(iii) The boundary conditions (2) cover $A(x, D)$ at every point $x \in \Gamma$. This means that, for each vector $\tau \neq 0$ from condition (ii), the boundary-value problem

$$
A^\circ(x, \tau + v(x)D_\ell)\theta(t) = 0 \quad \text{for } t > 0,
$$

$$
B_j^\circ(x, \tau + v(x)D_\ell)\theta(t)|_{t=0} + \sum_{k=1}^\infty C_{j,k}^\circ(x, \tau)\lambda_k = 0, \quad j = 1, \ldots, q + \infty,
$$

$$
\theta(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty
$$

has only the trivial (zero) solution. Here, the function $\theta \in C^\infty([0, \infty))$ and numbers $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$ are unknown. In addition, $A^\circ(x, \tau + v(x)D_\ell)$ and $B_j^\circ(x, \tau + v(x)D_\ell)$ are differential operators with respect to $D_\ell := i\partial/\partial t$. We obtain them putting $\xi := D_\ell$ in the polynomials $A^\circ(x, \tau + v(x)\xi)$ and $B_j^\circ(x, \tau + v(x)\xi)$ in $\xi$, respectively.

As is known (Chapter 2, Sections 1.1 and 1.2, [43]), condition (ii) follows from condition (i) in the $n \geq 3$ case and also in the case where $n = 2$ and where all the leading coefficients of $A(x, D)$ are real-valued. If $\kappa = 0$, condition (iii) is equivalent to the Lopatinski condition for classical elliptic problems.

Examples of elliptic problems of the form (1), (2) are given in (Subsection 3.1.5, [28]).

We supplement them with the following boundary-value problem:

$$
\Delta u = f \quad \text{in } \Omega,
$$

$$
\partial^p u + iav_1 = g_1, \quad \partial^{p+2\ell} u + b\Delta^\ell v_1 = g_2 \quad \text{on } \Gamma.
$$

Here, we arbitrarily choose integers $p \geq 0$ and $\ell \geq 1$ and real-valued functions $a, b \in C^\infty(\Gamma)$ such that $|a(x)| + |b(x)| \neq 0$ for every $x \in \Gamma$. As usual, $\partial_v := \partial/\partial v$, $\Delta$ is the Laplace operator in $\mathbb{R}^n$, and $\Delta_v$ is the Beltrami–Laplace operator on $\Gamma$. This problem takes the form (1), (2), where $q = 2, \kappa = 1$, $m_1 = p$, $m_2 = p + 2\ell$, and $r_1 = -p$. Direct calculation shows that this problem is elliptic in $\Omega$. Note that, if $a(x_0) = 0$ for some point $x_0 \in \Gamma$, it is impossible to exclude the unknown function $v_1$ from the boundary conditions and preserve the smoothness of the coefficients and right-hand side of the boundary condition obtained.

With the problem (1), (2) under investigation, we associate the linear mapping

$$
\Lambda : (u, v_1, \ldots, v_\kappa) \mapsto \left( A u, B_1 u + \sum_{k=1}^\kappa C_{1,k} v_k, \ldots, B_q u + \sum_{k=1}^\kappa C_{q+r,k} v_k \right)
$$

where $u \in C^\infty(\overline{\Omega})$ and $v_1, \ldots, v_\kappa \in C^\infty(\Gamma)$.

We will investigate properties of an extension (by continuity) of this mapping on appropriate pairs of Hilbert distribution spaces that form extended Sobolev scales over $\Omega$ and $\Gamma$.
To describe the range of this extension, we need the following special Green formula (formula (4.1.10), [28]):

\[
(Au, \omega)_\Omega + \sum_{j=1}^{\mu-2q} (D_v^{j-1} Au, w_j)_\Gamma + \sum_{k=1}^{q+\infty} \left( B_j \nu + \sum_{k=1}^{\infty} C_{j,k} \nu_k, h_j \right)_\Gamma = (u, A^+ \omega)_\Omega + \sum_{k=1}^{\mu} \left( D_v^{k-1} u, K_k \omega + \sum_{j=1}^{\mu-2q} R_{j,k}^{+} w_j + \sum_{j=1}^{q+\infty} Q_{j,k}^{+} h_j \right)_\Gamma + \sum_{k=1}^{\infty} \left( \nu_k, \sum_{j=1}^{q+\infty} C_{j,k} h_j \right)_\Gamma
\]

for arbitrary functions \( u, \omega \in C^\infty(\Omega) \) and

\[
v_1, \ldots, v_{\infty}, w_1, \ldots, w_{\mu-2q}, h_1, \ldots, h_{q+\infty} \in C^\infty(\Gamma).
\]

Of course, if \( \mu = 2q \) (which is equivalent to \( m \leq 2q - 1 \)), the functions \( w_1, \ldots, w_{\mu-2q} \) and the relevant sums will be absent. Here, \((\cdot, \cdot)_\Omega \) and \((\cdot, \cdot)_\Gamma \) stand respectively for the inner products in the Hilbert spaces \( L_2(\Omega) \) and \( L_2(\Gamma) \) of functions square integrable over \( \Omega \) and \( \Gamma \) relative to the Lebesgue measures. We also let \( A^+ \) denote the PDO which is formally adjoint to \( A \) relative to \((\cdot, \cdot)_\Omega \). Moreover, \( C_{j,k}^+ R_{j,k}^+ \) and \( Q_{j,k}^+ \) respectively denote the tangent PDOs which are formally adjoint to \( C_{j,k} \), \( R_{j,k} \), and \( Q_{j,k} \) relative to \((\cdot, \cdot)_\Gamma \), the tangent PDOs \( R_{j,k} \) and \( Q_{j,k} \) appearing in the representation of the boundary PDOs \( D_v^{j-1} \) and \( B_j \) in the form

\[
D_v^{j-1} A(x, D) = \sum_{k=1}^{\mu} R_{j,k} (x, D_x) D_v^{j-1}, \quad j = 1, \ldots, \mu - 2q,
\]

\[
B_j (x, D) = \sum_{k=1}^{\mu} Q_{j,k} (x, D_x) D_v^{j-1}, \quad j = 1, \ldots, q + \infty. \tag{5}
\]

We put \( D_v := i \partial / \partial v \) and understand \( D_v^{j-1} \) as a boundary PDO on \( \Gamma \); specifically, \( D_v^0 \) means the trace operator on \( \Gamma \). Note that \( \text{ord} R_{j,k} \leq 2q + j - k \) and \( \text{ord} Q_{j,k} \leq m_j - k + 1 \). Finally, \( K_k := K_k (x, D) \) is a certain boundary PDO on \( \Gamma \) whose order \( \text{ord} K_k \leq 2q - k \) and whose coefficients belong to \( C^\infty(\Gamma) \).

This Green formula leads us to the following boundary-value problem in \( \Omega \):

\[
A^+ \omega = \theta \quad \text{in} \quad \Omega, \tag{6}
\]

\[
K_k \omega + \sum_{j=1}^{\mu-2q} R_{j,k}^{+} w_j + \sum_{j=1}^{q+\infty} Q_{j,k}^{+} h_j = \psi_k \quad \text{on} \quad \Gamma, \quad k = 1, \ldots, \mu, \tag{7}
\]

\[
\sum_{j=1}^{q+\infty} C_{j,k}^{+} h_j = \psi_{\mu+k} \quad \text{on} \quad \Gamma, \quad k = 1, \ldots, \infty. \tag{8}
\]

Here, the distribution \( \omega \) in \( \Omega \) and the distributions \( w_1, \ldots, w_{\mu-2q}, h_1, \ldots, h_{q+\infty} \) on \( \Gamma \) are unknown. This problem is formally adjoint to the problem (1), (2) with respect to the above Green formula. As is known (Theorem 3.1.2, [28]), the problem (1), (2) is elliptic in \( \Omega \) if and only if the formally adjoint problems (6)–(8) are also elliptic in \( \Omega \).

3. Extended Sobolev Scale

This scale was introduced and investigated in (Section 2.4.2, [8]) and [36], first over \( \mathbb{R}^n \) and then over Euclidean domains and closed infinitely smooth manifolds. The scale consists of Hilbert generalized Sobolev spaces \([44,45]\) whose order of regularity is a function from a certain class \( \text{OR} \).

By definition, the class \( \text{OR} \) consists of all Borel measurable functions \( \alpha : [1, \infty) \to (0, \infty) \) for each of which there exist numbers \( b > 1 \) and \( c > 1 \) such that \( \alpha^{-1} \leq \alpha(t)/\alpha(t) \leq c \) whenever \( t \geq 1 \) and \( 1 \leq \lambda \leq b \). (If we fix \( b \), e.g., choose \( b = 2 \), the class \( \text{OR} \) will not change.)
The number \( c \) depends on \( a \). Such functions were introduced by Avakumović [46], are called OR-varying (or O-regularly varying) at infinity, are well investigated, and have various applications [47–49].

This class admits the following simple description:

\[
\alpha \in \text{OR} \iff \alpha(t) = \exp \left( \beta(t) + \int_1^t \frac{\gamma(\tau)}{\tau} \, d\tau \right) \text{ whenever } t \geq 1; \tag{9}
\]

here, \( \beta \) and \( \gamma \) are bounded Borel measurable real-valued functions on \([1, \infty)\) (see, e.g., (Theorem A.1, [49])).

The next characteristic property of the class OR plays an important role in our paper: A Borel measurable function \( \alpha : [1, \infty) \to (0, \infty) \) pertains to OR if and only if there exist numbers \( s_0, s_1 \in \mathbb{R} \), with \( s_0 \leq s_1 \), and a number \( c_1 \geq 1 \) such that

\[
c_1^{-1} \lambda^{s_0} \leq \frac{\alpha(\lambda t)}{\alpha(t)} \leq c_1 \lambda^{s_1} \text{ for all } t \geq 1 \text{ and } \lambda \geq 1 \tag{10}
\]

(see, e.g., (Theorem A.2(a), [49])). Given \( \alpha \in \text{OR} \), we put

\[
\sigma_0(\alpha) := \sup \{ s_0 \in \mathbb{R} : \text{the left inequality in (10) is true} \},
\]
\[
\sigma_1(\alpha) := \inf \{ s_1 \in \mathbb{R} : \text{the right inequality in (10) is true} \}.
\]

The numbers \( \sigma_0(\alpha) \) and \( \sigma_1(\alpha) \) are called respectively the lower and the upper Matuszewska indices of \( \alpha \) (see [50] and also (Theorem 2.2.2, [47])).

The well-known example of a function \( \alpha \in \text{OR} \) is given by every continuous function \( \alpha : [1, \infty) \to (0, \infty) \) such that

\[
\alpha(t) = t^r (\log t)^{s_1} (\log \log t)^{s_2} \cdots (\log \cdots \log t)^{s_k} \text{ whenever } t \gg 1,
\]

with \( 1 \leq k \in \mathbb{Z} \) and \( s, s_1, \ldots, s_k \in \mathbb{R} \). In this case, \( \sigma_0(\alpha) = \sigma_1(\alpha) = s \).

We obtain a simple example of a function \( \alpha \in \text{OR} \) with the different Matuszewska indices if we put

\[
\alpha(t) := \begin{cases} 
  t^{\theta + \delta \sin((\log \log t)')^j} & \text{if } t > e, \\
  t^\theta & \text{if } 1 \leq t \leq e
\end{cases}
\]

provided that \( \theta \in \mathbb{R}, \delta > 0, \) and \( 0 < r < 1 \). In this case, \( \sigma_0(\alpha) = \theta - \delta \) and \( \sigma_1(\alpha) = \theta + \delta \).

If \( r = 1 \), this function will still belong to the class OR but with \( \sigma_0(\alpha) = \theta - \sqrt{2} \delta \) and \( \sigma_1(\alpha) = \theta + \sqrt{2} \delta \); if \( r > 1 \), then \( \alpha \notin \text{OR} \) (see, e.g., (Section 3, [9])).

Another example of a function \( \alpha \in \text{OR} \) with the different Matuszewska indices \( \sigma_0(\alpha) = r < s = \sigma_1(\alpha) \) is given by Formula (9) in which

\[
\gamma(\tau) := \begin{cases} 
  r & \text{if } \tau \in [\pi_{2j-1}, \pi_{2j}] \text{ for some integer } j \geq 1, \\
  s & \text{otherwise}
\end{cases}
\]

provided that \( \pi_k := \theta_1 \cdots \theta_k \) for an arbitrarily chosen increasing sequence \( (\theta_k)_{k=1}^\infty \subset [1, \infty) \) satisfying \( \theta_1 = 1 \) and \( \theta_k \to \infty \) as \( k \to \infty \). Note that the latter condition is essential; thus, if \( \theta_k = \theta \) for some number \( \theta > 1 \) whenever \( k \gg 1 \), the function \( \alpha \) will be slowly equivalent to the power function \( t^{(r+s)/2} \) on \([1, \infty)\) and hence \( \sigma_0(\alpha) = \sigma_1(\alpha) = (r+s)/2 \) will hold.

Now let us turn to generalized Sobolev spaces that form the extended Sobolev scale. We begin with the spaces given over \( \mathbb{R}^n \), with \( n \geq 1 \). Let \( \alpha \in \text{OR} \). The (complex) linear space \( H^\alpha(\mathbb{R}^n) \) consists of all distributions \( w \in S'(\mathbb{R}^n) \) such that their Fourier transform \( \hat{w} \) is a classical function which is locally Lebesgue integrable over \( \mathbb{R}^n \) and satisfies the condition

\[
\int_{\mathbb{R}^n} |\alpha^2(|\xi|)) |\hat{w}(\xi)|^2 \, d\xi < \infty.
\]
As usual, \( S'(\mathbb{R}^n) \) denotes the linear topological space of tempered distributions on \( \mathbb{R}^n \), and \( \langle \xi \rangle := (1 + |\xi|^2)^{1/2} \) stands for the smoothed absolute value of a vector \( \xi \in \mathbb{R}^n \). We interpret distributions as antilinear continuous functionals on a relevant space of test functions.

The space \( H^\alpha(\mathbb{R}^n) \) is endowed with the inner product
\[
(w_1,w_2)_{\alpha,\mathbb{R}^n} := \int_{\mathbb{R}^n} \alpha^2(\langle \xi \rangle) \overline{w_1}(\xi) \overline{w_2}(\xi) \, d\xi
\]
and the corresponding norm \( \|w\|_{\alpha,\mathbb{R}^n} := (w,w)^{1/2} \). This space is Hilbert and separable, and the set \( C_0^\infty(\mathbb{R}^n) \) of compactly supported test functions is dense in it. We say that \( \alpha \) is the order of regularity of the space \( H^\alpha(\mathbb{R}^n) \) and its versions for \( \Omega \) and \( \Gamma \) considered below.

This space is an isotropic Hilbert case of the spaces \( B_{p,k} \) introduced and investigated by Hörmander in (Section 2.2, [44]) and applied by him to partial differential equations (see also (Section 10.1, [51])). Namely, if \( p = 2 \) and \( k(\xi) = \alpha(\langle \xi \rangle) \) for all \( \xi \in \mathbb{R}^n \), then \( H^\alpha(\mathbb{R}^n) = B_{p,k} \). Note that the Hörmander spaces in the Hilbert case form a subclass of the spaces introduced by Malgrange [52] and coincide with the spaces investigated by Volevich and Paneah (§ 2, [45]).

If \( \alpha(t) \equiv t^s \) for some \( s \in \mathbb{R} \), the space \( H^s(\mathbb{R}^n) \) is the inner product Sobolev space \( H^{(s)}(\mathbb{R}^n) \) of order \( s \). Generally,
\[
(s_0 < \sigma_0(\alpha) \text{ and } \sigma_1(\alpha) < s_1) \implies H^{(s_0)}(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \hookrightarrow H^{(s_1)}(\mathbb{R}^n),
\]
both embeddings being continuous and dense. This property is a direct consequence of the inequality (10) written for \( t = 1 \).

According to (Section 2.4.2, p. 105, [8]), the class
\[
\{ H^\alpha(\mathbb{R}^n) : \alpha \in \text{OR} \}
\]
is called the extended Sobolev scale over \( \mathbb{R}^n \). This class has remarkable interpolation properties; namely, it is obtained by means of the interpolation with function parameter between inner product Sobolev spaces, is closed with respect to the (quadratic) interpolation between Hilbert spaces, and consists of all Hilbert spaces that are interpolation ones between inner product Sobolev spaces [36]. Thus, the class (12) is the maximal extension of the Hilbert scale of Sobolev spaces with the help of the interpolation between Hilbert spaces. These properties of the extended Sobolev scale make it suitable and useful in the study of linear operators induced by elliptic PDEs and elliptic problems (see (Section 2.4.3, [8]) and [9,10,12,13,53]).

The extended Sobolev scales over the domain \( \Omega \) and its boundary \( \Gamma \) are built in a standard way on the base of (12) (see (Section 2, p. 139, [36]) and (Section 2.4.2, p. 106, [8]) respectively). Let us give the necessary definitions. Now that \( s \geq 2 \).

As above, \( \alpha \in \text{OR} \). By definition, the linear space \( H^\alpha(\Omega) \) consists of the restrictions to \( \Omega \) of all distributions \( w \in H^\alpha(\mathbb{R}^n) \). The space \( H^\alpha(\Omega) \) is endowed with the norm
\[
\|u\|_{\alpha,\Omega} := \inf \{ \|w\|_{\alpha,\mathbb{R}^n} : w \in H^\alpha(\mathbb{R}^n), \, w = u \text{ in } \Omega \},
\]
with \( u \in H^\alpha(\Omega) \). The space \( H^\alpha(\Omega) \) is Hilbert and separable with respect to this norm because it is a factor space of the Hilbert space \( H^\alpha(\mathbb{R}^n) \) by its subspace
\[
\{ w \in H^\alpha(\mathbb{R}^n) : \text{supp} \, w \subseteq \mathbb{R}^n \setminus \Omega \}.
\]

The set \( C_\infty(\overline{\Omega}) \) is dense in \( H^\alpha(\Omega) \).

Briefly saying, the space \( H^\alpha(\Gamma) \) consists of all distributions on \( \Gamma \) that yield elements of \( H^\alpha(\mathbb{R}^{n-1}) \) in local coordinates on \( \Gamma \). Let us give a detailed definition. We arbitrarily choose a finite atlas \( \pi_j : \mathbb{R}^{n-1} \leftrightarrow \Gamma_j \), with \( j = 1, \ldots, \lambda \), from \( C_\infty \)-structure on the manifold \( \Gamma \). Here,
the open sets $\Gamma_j$ form a covering of $\Gamma$. Let functions $\chi_j \in C^\infty(\Gamma)$, with $j = 1, \ldots, \lambda$, satisfy the conditions $\chi_1 + \cdots + \chi_\lambda \equiv 1$ and supp $\chi_j \subset \Gamma_j$. By definition, the linear space $H^a(\Gamma)$ consists of all distributions $h$ on $\Gamma$ such that $(\chi_j h) \circ \pi_j \in H^a(\mathbb{R}^{n-1})$ for each $j \in \{1, \ldots, \lambda\}$. Here, $(\chi_j h) \circ \pi_j$ is a representation of the distribution $\chi_j h$ in the local map $\pi_j$. The space $H^a(\Gamma)$ is endowed with the norm
\[
\|h\|_{a, \Gamma} := \left( \sum_{j=1}^\lambda \|(\chi_j h) \circ \pi_j\|_{a, \mathbb{R}^{n-1}}^2 \right)^{1/2}.
\]

This space is Hilbert and separable and does not depend up to equivalence of norms on our choice of $\pi_j$ and $\chi_j$ (Theorem 2.21, [8]). The set $C^\infty(\Gamma)$ is dense in $H^a(\Gamma)$. Thus, we have the extended Sobolev scales
\[
\{H^a(\Omega) : a \in \text{OR}\} \quad \text{and} \quad \{H^a(\Gamma) : a \in \text{OR}\}
\]
over $\Omega$ and $\Gamma$ respectively. They contain Hilbert Sobolev scales; namely, if $a(t) \equiv t^s$ for some $s \in \mathbb{R}$, then $H^a(\Omega) = H^{(s)}(\Omega)$ and $H^a(\Gamma) = H^{(s)}(\Gamma)$ are the inner product Sobolev spaces of order $s$.

The classes (12) and (13) are partially ordered with respect to embedding of spaces. Let $\alpha, \eta \in \text{OR}$ and $G \in \{\mathbb{R}^\alpha, \Omega, \Gamma\}$. The function $\alpha/\eta$ is bounded in a neighbourhood of infinity if and only if $H^a(G) \hookrightarrow H^b(G)$. This embedding is dense and continuous. It is compact in the $G \in \{\Omega, \Gamma\}$ case if and only if $\alpha(t)/\eta(t) \to 0$ as $t \to \infty$. This follows directly from (Theorems 2.2.2 and 2.2.3, [44]). Specifically, property (11) remains true and the relevant embeddings become compact if we replace $\mathbb{R}^\alpha$ with $\Omega$ or $\Gamma$.

Both the classes (13) have the same above-mentioned interpolation properties as (12). We will discuss some of them in Section 5.

4. The Main Results

With the problem (1), (2), we associate the following Hilbert spaces:
\[
D^\eta(\Omega, \Gamma) := H^\eta(\Omega) \oplus \bigoplus_{k=1}^\infty H^\eta q^{1+k-1/2}(\Gamma)
\]
and
\[
E^{q, -2q}(\Omega, \Gamma) := H^{q, -2q}(\Omega) \oplus \bigoplus_{j=1}^{q+k} H^{q, -n_j-1/2}(\Gamma),
\]
where $\eta \in \text{OR}$. In these and similar designations, we use the function parameter $\eta(t) \equiv t$ not to write the argument $t$ in indices. Thus, e.g., the parameter $\eta q^{-2q}$ means the function $\eta(t)t^{-2q}$ of $t \geq 1$. Let $\|\cdot\|_q$ denote the norm in $D^\eta(\Omega, \Gamma)$, and let $\|\cdot\|_{q, -2q}$ stand for the norm in $E^{q, -2q}(\Omega, \Gamma)$. In the Sobolev case where $\eta(t) \equiv t^s$ for certain $s \in \mathbb{R}$, we denote these spaces by $D^{(s)}(\Omega, \Gamma)$ and $E^{(s, -2q)}(\Omega, \Gamma)$ respectively.

Let $\mathcal{N}$ denote the linear space of all solutions
\[
(u, v_1, \ldots, v_\infty) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^\infty
\]
to the problem (1), (2) in the case where $f = 0$ in $\Omega$ and each $g_j = 0$ on $\Gamma$. Similarly, let $\mathcal{N}^+\Gamma$ stand for the linear space of all solutions
\[
(\omega, w_1, \ldots, w_{q+2q}, h_1, \ldots, h_{q+k}) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^\mu q^+\Gamma
\]
to the formally adjoint problems (6)–(8) in the case where $\theta = 0$ in $\Omega$ and all $\psi_k = 0$ and $\psi_{q+k} = 0$ on $\Gamma$. Since both problems are elliptic in $\Omega$, the spaces $\mathcal{N}$ and $\mathcal{N}^+\Gamma$ are finite-dimensional (Consequence 4.1.1, [28]).
Theorem 1. Let $\eta \in \mathbb{R}$ and $c_0(\eta) > m + 1/2$. Then the mapping (4) extends uniquely (by continuity) to a bounded linear operator

$$\Lambda : \mathcal{D}(\Omega, \Gamma) \rightarrow \mathcal{E}^{\eta e^{-2q}}(\Omega, \Gamma).$$  \hspace{1cm} (14)

This operator is Fredholm. Its kernel coincides with $\mathcal{N}$. Its range consists of all vectors

$$(f, g) := (f, g_1, \ldots, g_{q+x}) \in \mathcal{E}^{\eta e^{-2q}}(\Omega, \Gamma)$$  \hspace{1cm} (15)

such that

$$(f, \omega) + \sum_{j=1}^{\mu - 2q} (D_j^{-1} f, w_j) + \sum_{j=1}^{q+x} (g_j, h_j) = 0$$  \hspace{1cm} (16)

for every $(\omega, w_1, \ldots, w_{\mu - 2q}, h_1, \ldots, h_{q+x}) \in \mathcal{N}^+$. The index of the operator (14) is equal to $\dim \mathcal{N} - \dim \mathcal{N}^+$ and hence does not depend on $\eta$.

As to this theorem, we recall that a linear bounded operator $T : X \rightarrow Y$ between Banach spaces $X$ and $Y$ is called Fredholm if its kernel $T := \{ x \in X : Tx = 0 \}$ and cokernel $Y/T(X)$ are finite-dimensional. The Fredholm operator has the closed range $T(X)$ (see, e.g., (Lema 19.1.1, [54])) and the finite index

$$\text{ind} T := \dim \ker T - \dim (Y/T(X)) = \dim \ker T - \dim \ker T^*,$$

where $T^*$ is the adjoint of $T$.

Formula (16) needs commenting. Certainly, if $\mu = 2q$, the first sum with respect to $j$ will be absent in this formula. The first components of the forms $(\cdot, \cdot)_\Gamma$ in (16) belong to $L_2(\Gamma)$. Indeed, since $c_0(\eta) > m + 1/2$, we conclude in the $\mu = m + 1$ case that

$$D_j^{-1} f \in H^{\eta e^{-2q} - 1/2}(\Gamma) \subset L_2(\Gamma)$$

in view of (Proposition 4, [10]) and because

$$c_0(\eta e^{-2q} - 1/2) = c_0(\eta) - 2q - j + 1/2$$

$$> m + 1/2 - 2q - (\mu - 2q) + 1/2 = m - \mu = 0.$$

In addition,

$$g_j \in H^{\eta e^{m_j - 1/2}}(\Gamma) \subset L_2(\Gamma)$$

because $c_0(\eta e^{-m_j - 1/2}) > 0$. Thus, both the sums with respect to $j$ are well defined in (16). If $c_0(\eta) > 2q$, then $f \in L_2(\Omega)$ and the inner product $(f, \omega)_\Omega$ is also well defined. If $m + 1/2 < c_0(\eta) \leq 2q$, we put $(f, \omega)_\Omega = \lim_{k \rightarrow \infty} (f_k, \omega)_\Omega$ where $(f_k)_{k=1}^\infty$ is an arbitrary sequence of functions $f_k \in C^\infty(\overline{\Omega})$ that converges to $f$ in $H^{\eta e^{-2q}}(\Omega)$. The limit $(f, \omega)_\Omega$ exists for every $\omega$ indicated in (16) and does not depend on the choice of $f_k$, which will be shown in the proof of Theorem 1.

If $\mathcal{N} = \{0\}$ and $\mathcal{N}^+ = \{0\}$, then the operator (14) becomes an isomorphism between the spaces $\mathcal{D}(\Omega, \Gamma)$ and $\mathcal{E}^{\eta e^{-2q}}(\Omega, \Gamma)$. Generally, this operator induces an isomorphism between some of their (closed) subspaces, which have a finite codimension. It is convenient to give this isomorphism with the help of certain decompositions of the source and target spaces of (14) in direct sum of their subspaces. Let $c_0(\eta) > m + 1/2$; then

$$\mathcal{D}(\Omega, \Gamma) = \mathcal{N} + \left\{ (u, v_1, \ldots, v_x) \in \mathcal{D}(\Omega, \Gamma) : (u, u^\circ)_\Omega + \sum_{k=1}^x (v_k, v_k^\circ)_\Gamma \text{ for each } (u^\circ, v_1^\circ, \ldots, v_x^\circ) \in \mathcal{N} \right\}.$$  \hspace{1cm} (17)
This decomposition is well defined because it is a restriction of the relevant orthogonal decomposition of the Hilbert space $L^2(\Omega) \oplus (L^2(\Gamma))^\times$. Note that $\mathcal{D}^\eta(\Omega, \Gamma)$ lies in the above space due to (3). A decomposition of $\mathcal{E}^{\eta=2q}(\Omega, \Gamma)$ is based on the following result:

**Lemma 1.** There exists a finite-dimensional space

\[ \mathcal{G} \subset C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^\times \]

such that \( \dim \mathcal{G} = \dim \mathcal{N}^+ \) and

\[ \mathcal{E}^{\eta=2q}(\Omega, \Gamma) = \mathcal{G} + \{(f, g) \in \mathcal{E}^{\eta=2q}(\Omega, \Gamma) : (16) \text{ is valid}\}. \tag{18} \]

whenever \( \eta \in \text{OR} \) and \( \sigma_0(\eta) > m + 1/2 \). If \( \mu = 2q \), we may take \( \mathcal{G} = \mathcal{N}^+ \).

Let \( P \) and \( P^+ \) denote respectively the projectors of the spaces \( \mathcal{D}^\eta(\Omega, \Gamma) \) and \( \mathcal{E}^{\eta=2q}(\Omega, \Gamma) \) onto the second term in the sums (17) and (18) parallel to the first. The rules that define these projectors do not depend on \( \eta \).

**Theorem 2.** Let \( \eta \in \text{OR} \) and \( \sigma_0(\eta) > m + 1/2 \). The restriction of the operator (14) to the subspace \( P(\mathcal{D}^\eta(\Omega, \Gamma)) \) is an isomorphism

\[ \Lambda : P(\mathcal{D}^\eta(\Omega, \Gamma)) \leftrightarrow P^+(\mathcal{E}^{\eta=2q}(\Omega, \Gamma)). \tag{19} \]

Let us study properties of generalized solutions to the elliptic problem (1), (2) in the spaces used above. Beforehand, we will give a definition of such solutions. Put

\[ \mathcal{D}^{m+1/2+}(\Omega, \Gamma) := \bigcup_{\eta \in \text{OR}: \sigma_0(\eta) > m + 1/2} \mathcal{D}^{\eta}(\Omega, \Gamma) = \bigcup_{s > m + 1/2} \mathcal{D}^{(s)}(\Omega, \Gamma), \]

the last equality being valid due to (11). Let the right-hand sides of the problem (1), (2) satisfy the condition

\[ (f, g) := (f, g_1, \ldots, g_{q+\times}) \in \mathcal{D}'(\Omega) \times (\mathcal{D}'(\Gamma))^\times. \]

As usual, \( \mathcal{D}'(\Omega) \) and \( \mathcal{D}'(\Gamma) \) denote the linear topological spaces of all distributions on \( \Omega \) and \( \Gamma \) respectively. A vector

\[ (u, v) := (u, v_1, \ldots, v_{q+\times}) \in \mathcal{D}^{m+1/2+}(\Omega, \Gamma) \]

is called a generalized solution to this problem if \( \Lambda(u, v) = (f, g) \). Here, \( \Lambda \) means the operator (14) for a certain parameter \( \eta \in \text{OR} \) subject to \( \sigma_0(\eta) > m + 1/2 \). This definition is reasonable because it is independent of \( \eta \).

We investigate local (up to the boundary \( \Gamma \)) regularity of generalized solutions to the problem (1), (2). Let \( V \) be an open subset of \( \mathbb{R}^n \) such that \( \Omega_0 := \Omega \cap V \neq \emptyset \). We put \( \Gamma_0 := \Gamma \cap V \), the \( \Gamma_0 = \emptyset \) case being possible. Given \( \alpha \in \text{OR} \), we introduce local versions of the spaces \( H^\alpha(\Omega) \) and \( H^\alpha(\Gamma) \) as follows:

\[ H^\alpha_{\text{loc}}(\Omega_0, \Gamma_0) := \{u \in \mathcal{D}'(\Omega) : \chi u \in H^\alpha(\Omega) \quad \text{for all } \chi \in C^\infty(\overline{\Omega}) \text{ such that } \text{supp } \chi \subset \Omega_0 \cup \Gamma_0 \} \]

and

\[ H^\alpha_{\text{loc}}(\Gamma_0) := \{h \in \mathcal{D}'(\Gamma) : \chi h \in H^\alpha(\Gamma) \quad \text{for all } \chi \in C^\infty(\Gamma) \text{ such that } \text{supp } \chi \subset \Gamma_0 \}. \]
Given \( \eta \in \text{OR} \), we put
\[
D^\eta_{\text{loc}}(\Omega_0, \Gamma_0) := H^\eta_{\text{loc}}(\Omega_0, \Gamma_0) \times \prod_{k=1}^\kappa H^{q_k-1/2}_{\text{loc}}(\Gamma_0)
\]
and
\[
\mathcal{E}_{\text{loc}}^{q_0-2}\eta(\Omega_0, \Gamma_0) := H_{\text{loc}}^{q_0-2}\eta(\Omega_0, \Gamma_0) \times \prod_{j=1}^{q+\kappa} H^{q_j-m_j-1/2}_{\text{loc}}(\Gamma_0).
\]

**Theorem 3.** Let a vector \((u, v)\) be a generalized solution to the elliptic problem (1), (2) whose right-hand sides satisfy the condition \((f, g) \in \mathcal{E}_{\text{loc}}^{q_0-2}\eta(\Omega_0, \Gamma_0)\) for a certain parameter \(\eta \in \text{OR} \) subject to \(\sigma_0(\eta) > m + 1/2\). Then \((u, v) \in D^\eta_{\text{loc}}(\Omega_0, \Gamma_0)\).

If \(\Omega_0 = \Omega\) and \(\Gamma_0 = \Gamma\), we have the equalities \(D^\eta_{\text{loc}}(\Omega_0, \Gamma_0) = D^\eta(\Omega, \Gamma)\) and \(\mathcal{E}_{\text{loc}}^{q_0-2}\eta(\Omega_0, \Gamma_0) = \mathcal{E}_{\text{loc}}^{q_0-2}\eta(\Omega, \Gamma)\). In this case, Theorem 3 deals with the global regularity of \((u, v)\), i.e., concerns the regularity of \(u\) in \(\bar{\Omega}\) and \(v\) on \(\Gamma\).

We supplement this theorem with the following a priori estimate of \((u, v)\):

**Theorem 4.** Let \(\eta \in \text{OR} \) and \(\sigma_0(\eta) > m + 1/2\), and suppose that a vector \((u, v) \in D^{m+1/2}(\Omega, \Gamma)\) satisfies the hypothesis of Theorem 3. Let functions \(\chi, \xi \in C^\infty(\bar{\Omega})\) be such that \(\text{supp} \chi \subset \text{supp} \xi \subset \Omega_0 \cup \Gamma_0\) and that \(\xi = 1\) in a neighbourhood of \(\text{supp} \chi\). Then
\[
\|\chi(u, v)\|^2_\eta \leq c \left( \|\chi A(u, v)\|_{q_0-2}\eta + \|\xi(u, v)\|_{q_0-1}\right), \quad (20)
\]
where \(c\) is a certain positive number that does not depend on \((u, v)\).

Here, of course,
\[
\chi(u, v) := \langle \chi u, (\chi \mid \Gamma)v_1, \ldots, (\chi \mid \Gamma)v_{\kappa}\rangle
\]
and the expression \(\chi A(u, v)\) is similarly understood.

These theorems were proved in (Sections 4 and 6, [40]) in the special case where the function \(\eta(t)\) varies regularly at infinity in the sense of J. Karamata and on the assumption that \(m \geq 2q\). If \(m < 2q - 1\) and if the function \(\eta \in \text{OR} \) satisfies the stronger condition \(\sigma_0(\eta) > 2q - 1/2\), Theorems 1–3 were proved in (Sections 4 and 6, [39]). (The indicated articles are published in Ukrainian.)

Generally, the conclusions of these theorems are not valid for arbitrary \(\eta \in \text{OR}\). Specifically, if \(s \leq m_j + 1/2\) for certain \(j \in \{1, \ldots, q + \kappa\}\) and if \(Q_{m_j+1} \neq 0\) in the representation of \(B_j(x, D)\) in the form (5), then the mapping \(u \mapsto B_j u\), where \(u \in C^\infty(\bar{\Omega})\), cannot be extended to a continuous linear operator from the whole Sobolev space \(H^{(s)}(\Omega)\) to \(D'(\Gamma)\); this follows from (Chapter 1, Theorem 9.5, [43]). Hence, the bounded linear operator (14) is not well defined in the \(\eta(t) \equiv t^s\) case under these conditions. However, if the elliptic Equation (1) is homogeneous (i.e., \(f = 0\) in \(\Omega\)), certain versions of the above theorems will hold for any \(\eta \in \text{OR}\). We restrict ourselves to a relevant version of the key Theorem 1.

Given \(\eta \in \text{OR}\), we put
\[
H^\eta_A(\Omega) := \{ u \in H^\eta(\Omega) : Au = 0 \text{ in } \Omega \}.
\]

Here, \(Au\) is understood in the sense of the distribution theory. We endow the linear space \(H^\eta_A(\Omega)\) with the inner product and norm in \(H^\eta(\Omega)\). The space \(H^\eta_A(\Omega)\) is complete because the differential operator \(A\) is continuous on \(D'(\Omega)\). The set
\[
C^\infty_A(\bar{\Omega}) := \{ u \in C^\infty(\bar{\Omega}) : Au = 0 \text{ on } \bar{\Omega} \}.
\]
is dense in $H^0_A(\Omega)$ by (Theorem 7.1, [9]).

Consider the linear mapping

$$
\Lambda_A : (u,v_1,\ldots,v_{\kappa}) \mapsto \left( B_1 u + \sum_{k=1}^{\kappa} C_{1,k} v_k, \ldots, B_{q+\kappa} u + \sum_{k=1}^{\kappa} C_{q+\kappa,k} v_k \right)
$$

where $u \in C^\infty(\Omega)$ and $v_1,\ldots,v_{\kappa} \in C^\infty(\Gamma)$.

With this mapping, we associate the Hilbert spaces

$$
\mathcal{D}^\eta_A(\Omega,\Gamma) := H^\eta_A(\Omega) \oplus \bigoplus_{k=1}^{\kappa} H^{\eta\kappa\kappa-1/2}(\Gamma)
$$

and

$$
\mathcal{H}_\eta(\Gamma) := \bigoplus_{j=1}^{q+\kappa} H^{\eta\kappa\kappa-j-1/2}(\Gamma).
$$

Let $\mathcal{N}_0^+$ denote the linear space of all vectors $(h_1,\ldots,h_{q+\kappa}) \in (C^\infty(\Gamma))^{q+\kappa}$ for each of which there exist functions $\omega \in C^\infty(\Omega)$ and $w_1,\ldots,w_{q-2\kappa} \in C^\infty(\Gamma)$ such that

$$(\omega,w_1,\ldots,w_{q-2\kappa},h_1,\ldots,h_{q+\kappa}) \in \mathcal{N}_0^+.$$ 

Certainly, $\dim \mathcal{N}_0^+ \leq \dim \mathcal{N}_0^+$, with the strict inequality being possible (Theorem 13.6.15, [51]).

**Theorem 5.** For every $\eta \in \text{OR}$, the mapping (21) extends uniquely (by continuity) to a bounded linear operator

$$
\Lambda_A : \mathcal{D}^\eta_A(\Omega,\Gamma) \to \mathcal{H}_\eta(\Gamma).
$$

This operator is Fredholm. Its kernel coincides with $\mathcal{N}$. Its range consists of all vectors

$$
\mathcal{G} := (g_1,\ldots,g_{q+\kappa}) \in \mathcal{H}_\eta(\Gamma)
$$

such that

$$
\sum_{j=1}^{q+\kappa} (g_j, h_j)_\Gamma = 0 \quad \text{for every} \quad (h_1,\ldots,h_{q+\kappa}) \in \mathcal{N}_0^+.
$$

The index of the operator (22) equals $\dim \mathcal{N} - \dim \mathcal{N}_0^+$ and hence does not depend on $\eta$.

Since the function parameter $\eta \in \text{OR}$ is arbitrary in this theorem, components of the vector (23) may be irregular distributions on $\Gamma$. We therefore interpret the expression $(g_j, h_j)_\Gamma$ in (24) as the value of the distribution $g_j \in \mathcal{D}'(\Gamma)$ on the test function $h_j \in C^\infty(\Gamma)$ and consider the space $\mathcal{D}'(\Gamma)$ as the dual of $C^\infty(\Gamma)$ with respect to the inner product in $L_2(\Gamma)$.

This theorem was given (without a complete proof) in [37,38] in the special case where the function $\eta(t)$ varies regularly at infinity, paper [38] treating the $m \leq 2q - 1$ case.

**5. The Interpolation between Hilbert Spaces**

As has been mentioned in Section 3, the extended Sobolev scale possesses an important interpolation property, which will play a decisive role in the proof of Theorems 1 and 5. Namely, each space $H^\alpha(G)$, where $G \in \{\mathbb{R}^n, \Omega, \Gamma\}$ and $\alpha \in \text{OR}$, can be obtained by the interpolation (with an appropriate function parameter) between inner product Sobolev spaces $H^{(q)}(G)$ and $H^{(p)}(G)$ such that $s_0 < s_0(\alpha)$ and $s_1 > s_1(\alpha)$. Therefore, we will recall the definition of the interpolation between Hilbert spaces and formulate its properties being used in our proofs.

The interpolation method we need was introduced by C. Foiaş and J.-L. Lions in (p. 278, [55]). Expounding it, we mainly follow monograph (Section 1.1, [8]), which gives
its various applications to elliptic operators and elliptic boundary-value problems. It is sufficient for our purposes to restrict ourselves to separable Hilbert spaces.

Let $X := [X_0, X_1]$ be an ordered pair of separable complex Hilbert spaces $X_0$ and $X_1$ such that $X_1$ is a manifold in $X_0$ and that $\|w\|_{X_0} \leq c\|w\|_{X_1}$ whenever $w \in X_1$, with the number $c > 0$ not depending on $w$. This pair is called regular. As is known, for $X$ there exists a positive-definite self-adjoint operator $J$ given in the Hilbert space $X_0$ and such that $X_1$ is the domain of $J$ and that $\|Jw\|_{X_0} = \|w\|_{X_1}$ for all $w \in X_1$. This operator is uniquely determined by the pair $X$ and is called the generating operator for this pair. The operator $J$ sets an isometric isomorphism between $X_1$ and $X_0$.

Let $B$ denote the set of all Borel measurable functions $\psi : (0, \infty) \to (0, \infty)$ such that $\psi$ is bounded on each compact interval $[a, b]$, with $0 < a < b < \infty$, and that $1/\psi$ is bounded on every set $(r, \infty)$, with $r > 0$.

Given $\psi \in B$ and applying the spectral theorem to the self-adjoint operator $J$, we obtain the (generally, unbounded) operator $\psi(J)$ on $X_0$. Let $[X_0, X_1]_\psi$ or, briefly, $X_\psi$ denote the domain of $\psi(J)$ endowed with the inner product $(u_1, u_2)_{X_\psi} := (\psi(J)u_1, \psi(J)u_2)_{X_0}$ and the corresponding norm $\|u\|_{X_\psi} = \|\psi(J)u\|_{X_0}$. The space $X_\psi$ is Hilbert and separable and is continuously embedded in $X_0$.

We call a function $\psi \in B$ an interpolation parameter if the following condition is satisfied for all regular pairs $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ of Hilbert spaces and for an arbitrary linear mapping $T$ given on whole $X_0$: If the restriction of $T$ to $X_1$ is a bounded operator from $X_1$ to $Y_j$ for every $j \in \{0, 1\}$, then the restriction of $T$ to $X_\psi$ is also a bounded operator from $X_\psi$ to $Y_j$. We say in this case that $X_\psi$ is obtained by the interpolation, with the function parameter $\psi$, of the pair $X$ (or, in other words, between $X_0$ and $X_1$) and that the bounded operator $T : X_\psi \to Y_j$ is the result of the interpolation applied to the operators $T : X_j \to Y_j$ with $j \in \{0, 1\}$.

A function $\psi \in B$ is an interpolation parameter if and only if $\psi$ is pseudoconcave in a neighbourhood of infinity, i.e., $\psi \asymp \psi_1$ there for a certain positive concave function $\psi_1(t)$ of $t \gg 1$. (As usual, $\psi \asymp \psi_1$ means that the functions $\psi/\psi_1$ and $\psi_1/\psi$ are bounded on the indicated set). This fundamental fact follows from J. Peetre’s [36] description of all interpolation functions of positive order. Specifically, the power function $\psi(t) \equiv t^s$ is an interpolation parameter if and only if $0 \leq s \leq 1$.

It is useful for us to formulate the above-mentioned interpolation property of the extended scale as follows:

**Proposition 1.** Let $\eta \in \text{OR}$, and suppose that real numbers $s_0$ and $s_1$ satisfy $s_0 < c_0(\eta)$ and $s_1 > c_1(\eta)$. Define a function $\psi \in B$ by the formula

$$\psi(t) := \begin{cases} t^{-s_0/(s_1-s_0)} \eta(t^{1/(s_1-s_0)}) & \text{whenever } t \geq 1, \\ \eta(1) & \text{otherwise.} \end{cases}$$  \hfill (25)

Then $\psi$ is an interpolation parameter, and

$$[H^{(s_0+\theta)}(G), H^{(s_1+\theta)}(G)]_{\psi} = H^{\eta(\theta)}(G) \quad \text{for every } \theta \in \mathbb{R}$$  \hfill (26)

up to equivalence of norms provided that $G \in \{\Omega, \Gamma\}$. If $G = \mathbb{R}^n$, then (26) holds true with equality of norms.

This property is proved in (Theorems 2.19 and 2.22, [8]) for $G \in \{\mathbb{R}^n, \Gamma\}$ and in (Theorem 5.1, [36]) for $G = \Omega$.

Proving Theorem 5, we will use the following interpolation property of the space $H_A^p(\Omega)$ (Theorem 7.8(i), [9]):
Proposition 2. Let \( \eta \in \mathcal{OR}, s_0, s_1 \in \mathbb{R} \), and \( \psi \in \mathcal{B} \) satisfy the hypothesis of Proposition 1. Then
\[
[H^{(s_0)}_A(\Omega), H^{(s_1)}_A(\Omega)] = H^\eta_A(\Omega)
\]
up to equivalence of norms.

We also need two general interpolation properties given below.

Proposition 3. Let \( X = [X_0, X_1] \) and \( Y = [Y_0, Y_1] \) be regular pairs of Hilbert spaces. Suppose that a linear mapping \( T \) on \( X_0 \) satisfies the following condition: The restrictions of \( T \) to the spaces \( X_j \), with \( j = 0, 1 \), are bounded and Fredholm operators \( T : X_j \to Y_j \) that have a common kernel and the same index. Then, for an arbitrary interpolation parameter \( \psi \in \mathcal{B} \), the bounded operator \( T : X_\psi \to Y_\psi \) is Fredholm with the same kernel and index, and the range of the last operator equals \( Y_\psi \cap T(X_0) \).

Proposition 4. Let \( [X_0^{(k)}, X_1^{(k)}] \), where \( k = 1, \ldots, p \), be a finite number of regular pairs of Hilbert spaces. Then
\[
\left[ \bigoplus_{k=1}^p X_0^{(k)}, \bigoplus_{k=1}^p X_1^{(k)} \right]_{\psi} = \bigoplus_{k=1}^p [X_0^{(k)}, X_1^{(k)}]_{\psi}
\]
with equality of norms norms whatever \( \psi \in \mathcal{B} \).

The proofs of these propositions are given, e.g., in (Subsections 1.1.7 and 1.1.5, resp., [8]).

6. Proofs of the Main Results

Proof of Theorem 1. We first consider the Sobolev case where \( \eta(t) \equiv t^s \) and \( s > m + 1/2 \). In this case, Theorem 1 is essentially contained in Grubb’s result (Corollary 5.5, [25]). As compared with the last two sentences of Theorem 1, Grubb proved that there exists

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According to Theorem 1, the restriction of the operator \( (14) \) to \( m \) is continuously embedded in \( H^{q \rho - 2q} (\Omega) \) by passing to the limit on functions of class \( C^0 (\overline{\Omega}) \) because \( H^{q \rho - 2q} (\Omega) \) is continuously embedded in \( H^{(s - 2q)} (\Omega) \) for certain \( s > m + 1/2 \) due to (11). Choose real numbers \( s_0 \) and \( s_1 \) such that \( m + 1/2 < s_0 < c_0 (\eta) \) and \( s_1 > c_0 (\eta) \), and define the interpolation parameter \( \psi \) by (25). Applying the interpolation with the parameter \( \psi \) to the Fredholm bounded operators \( \Lambda_{s_0} \) and \( \Lambda_{s_1} \), we obtain the Fredholm bounded operator

\[
\Lambda : [D^{(s_0)}(\Omega, \Gamma), D^{(s_1)}(\Omega, \Gamma)]_\psi \to [E^{(s_0 - 2q)}(\Omega, \Gamma), E^{(s_1 - 2q)}(\Omega, \Gamma)]_\psi
\]  

(27)
due to Proposition 3. Owing to Propositions 1 and 4, this operator acts between the spaces

\[
[D^{(s_0)}(\Omega, \Gamma), D^{(s_1)}(\Omega, \Gamma)]_\psi
= [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi \oplus \bigoplus_{k=1}^\infty [H^{(s_0 + r_k - 1/2)}(\Gamma), H^{(s_1 + r_k - 1/2)}(\Gamma)]_\psi
= D^\eta(\Omega, \Gamma)
\]

and

\[
[E^{(s_0 - 2q)}(\Omega, \Gamma), E^{(s_1 - 2q)}(\Omega, \Gamma)]_\psi
= [H^{(s_0 - 2q)}(\Omega), H^{(s_1 - 2q)}(\Omega)]_\psi \oplus \bigoplus_{j=1}^{q+\infty} [H^{(s_0 - m_j - 1/2)}(\Gamma), H^{(s_1 - m_j - 1/2)}(\Gamma)]_\psi
= E^{q \rho - 2q}(\Omega, \Gamma).
\]

The operator is an extension by continuity of (4) because the set \( C^0 (\overline{\Omega}) \times (C^0 (\Gamma))^\kappa \) is dense in \( D^\eta(\Omega, \Gamma) \). Owing to Proposition 3, the kernel and index of this operator coincide with the common kernel \( N \) and the index \( \dim N - \dim N^+ \) of the operators \( \Lambda_{s_0} \) and \( \Lambda_{s_1} \). In addition,

\[
\Lambda(D^\eta(\Omega, \Gamma)) = E^{q \rho - 2q}(\Omega, \Gamma) \cap \Lambda(D^{(s_0)}(\Omega, \Gamma))
= \{(f, g) \in E^{q \rho - 2q}(\Omega, \Gamma) : (16) \text{ is satisfied}\}.
\]

Thus, we have proved all the properties of the operator \( (14) \) indicated in Theorem 1. \( \square \)

**Proof of Lemma 1.** We separately consider the cases where \( m \geq 2q \) and where \( m \leq 2q - 1 \). If \( m \geq 2q \), this lemma is contained in (Lemma 1, [40]) provided that \( \eta(t) \equiv t^q \) and \( s > m + 1/2 \). Hence, the required formula (18) follows directly from the decomposition

\[
E^{(s - 2q)}(\Omega, \Gamma) = G + \{(f, g) \in E^{(s - 2q)}(\Omega, \Gamma) : (16) \text{ is valid}\}
\]
in which \( m + 1/2 < s < c_0 (\eta) \).

If \( m \leq 2q - 1 \), then (18) holds true for \( G := N^+ \) because the intersection of the subspaces on its right-hand side is zero space and since the finite dimension of the first subspace equals the co-dimension of the second by Theorem 1. \( \square \)

**Proof of Theorem 2.** According to Theorem 1, the restriction of the operator \( (14) \) to \( P(D^\eta(\Omega, \Gamma)) \) is a continuous and one-to-one mapping between \( P(D^\eta(\Omega, \Gamma)) \) and \( P^+ (E^{q \rho - 2q}(\Omega, \Gamma)) \). Hence, by the Banach theorem on inverse operator, this mapping is the required isomorphism (19). \( \square \)
**Proof of Theorem 3.** Our reasoning is motivated by (Proof of Theorem 7, [13]). Put

\[ Y := \{ \chi \in C^\infty(\Omega) : \text{supp} \chi \subset \Omega_0 \cup \Gamma_0 \}. \]

We first prove that, under the hypothesis of Theorem 3, the implication

\[
(\chi(u,v) \in D^\eta(\Omega, \Gamma) + D^{(s)}(\Omega, \Gamma) \text{ for all } \chi \in Y) \\
\Rightarrow (\chi(u,v) \in D^\eta(\Omega, \Gamma) + D^{(s+1)}(\Omega) \text{ for all } \chi \in Y)
\]

holds true for every \( s > m - 1/2 \). Here, we use algebraic sums of spaces.

We choose a number \( s > m - 1/2 \) arbitrarily and suppose that the premise of the implication (28) is true. Given \( \chi \in Y \), we select a function \( \zeta \in Y \) such that \( \zeta(\cdot) = 1 \) in a neighbourhood of \( \text{supp} \chi \). By the hypothesis, \( \chi(f,g) \in E^{q_0 - 2\lambda}(\Omega, \Gamma) \), with \( (f,g) = \Lambda(u,v) \). Interchanging the operator of the multiplication by \( \chi \) with the component-wise PDO operator \( \Lambda \), we write

\[ \chi(f,g) = \chi \Lambda(\zeta(u,v)) = \Lambda(\chi \zeta(u,v)) = \Lambda'(\zeta(u,v)). \]

Here, \( \Lambda' \) is a certain component-wise PDO of the form of \( \Lambda \), the orders of all components of \( \Lambda' \) being at least in 1 less than the orders of the corresponding components of \( \Lambda \). Thus,

\[ \Lambda(\chi(u,v)) = \chi(f,g) + \Lambda'(\zeta(u,v)). \tag{29} \]

By the premise of the implication (28), we have the sum \( \zeta(u,v) = (u^s, v^s) + (u^+, v^+) \) for certain vectors \( (u^s, v^s) \in D^\eta(\Omega, \Gamma) \) and \( (u^+, v^+) \in D^{(s)}(\Omega, \Gamma) \). In view of (29) we obtain

\[ \Lambda(\chi(u,v)) = (f^s, g^s) + (f^+, g^+) = P^+(f^s, g^s) + P^+(f^+, g^+). \tag{30} \]

where

\[
(f^s, g^s) := \chi(f,g) + \Lambda'(u^s, v^s) \in E^{q_0 - 2\lambda}(\Omega, \Gamma), \tag{31}
\]

\[
(f^+, g^+) := \Lambda'(u^+, v^+) \in E^{(s+1-2\lambda)}(\Omega, \Gamma). \tag{32}
\]

Recall that \( \chi(f,g) \in E^{q_0 - 2\lambda}(\Omega, \Gamma) \) by the hypothesis of the theorem. The boundedness of the operator

\[ \Lambda' : D^\eta(\Omega, \Gamma) \to E^{(s+1-2\lambda)}(\Omega, \Gamma) \]

follows by Proposition 1 from the well-known boundedness of the operators

\[ \Lambda' : D^{(\lambda)}(\Omega, \Gamma) \to E^{(\lambda+1-2\lambda)}(\Omega, \Gamma) \quad \text{with} \quad \lambda \in \{s_0, s_1\}, \]

where \( m - 1/2 < s_0 < \sigma_0(\eta) \) and \( s_1 > \sigma_1(\eta) \).

It follows from (30)–(32) that

\[ \Lambda(\chi(u,v)) = \Lambda((u^s, v^s) + (u^+, v^+)); \]

here, the vectors

\[
(u^s, v^s) \in P(D^\eta(\Omega, \Gamma)) \quad \text{and} \quad (u^+, v^+) \in P(D^{(s+1)}(\Omega, \Gamma)) \tag{33}
\]

are the unique solutions of the operator equations

\[ \Lambda(u^s, v^s) = P^+(f^s, g^s) \in P^+(E^{q_0 - 2\lambda}(\Omega, \Gamma)) \]

and

\[ \Lambda(u^+, v^+) = P^+(f^+, g^+) \in P^+(E^{(s+1-2\lambda)}(\Omega, \Gamma)). \]
due to Theorem 2. By Theorem 1 we hence get
\[ \chi(u, v) = (u', v') + (u'', v'') + (u', v') \]
for a certain vector \((u'', v'') \in N\), which in view of (33) proves the required implication (28).

Since \((u, v) \in D^{m+1/2\ast}(\Omega, \Gamma)\) by the hypothesis of the theorem, the premise of this implication holds true for \(s = m\). Choose an integer \(p \geq 1\) such that \(m + p > c_1(\eta)\). Applying the implication (28) \(p\) times successively for \(s = m, s = m + 1, \ldots\), and \(s = m + p - 1\), we conclude that
\[ \chi(u, v) \in D^{\gamma}(\Omega, \Gamma) + D^{(m+p)}(\Omega, \Gamma) = D^{\gamma}(\Omega, \Gamma) \]
for every \(\chi \in Y\); i.e., \((u, v) \in D^{\gamma}_{\text{loc}}(\Omega_0, \Gamma_0)\).

**Proof of Theorem 4.** Let \(c_1\) and \(c_2\) denote some positive numbers that do not depend on \((u, v)\). According to Theorem 3, we have the inclusion \(\chi(u, v) \in D^{\gamma}(\Omega, \Gamma)\). As is known in the theory of operators (Lemma 3, [57]), it follows from the Fredholm property of the bounded operator (14) and from the compact embedding \(D^{\gamma}(\Omega, \Gamma) \hookrightarrow D^{q_{-1}}(\Omega, \Gamma)\) that
\[ \|\chi(u, v)\|_{q} \leq c_1(\|\Lambda(\chi(u, v))\|_{q_{-2}} + \|\chi(u, v)\|_{q_{-1}}). \]

Owing to (29), we get
\[ \|\Lambda(\chi(u, v))\|_{q_{-2}} \leq \|\chi u\|_{q_{-2}} + \|\Lambda'\chi(u, v)\|_{q_{-2}} \leq \|\chi u\|_{q_{-2}} + c_2\|\chi(u, v)\|_{q_{-1}}. \]

This yields the required estimate (20).

**Proof of Theorem 5.** We first prove this theorem in the Sobolev case where \(\eta(t) \equiv t^s\) and \(s \in \mathbb{Z}\) and then deduce it in the general case with the help of Proposition 2. To prove the theorem in the Sobolev case, we exploit some Hilbert spaces introduced by Roitberg [33,34] and make use of known results (Sections 3.4 and 4.1, [28]) on the solvability of the elliptic problem (1), (2) in these spaces.

Let \(s \in \mathbb{Z}\) and \(r \in \mathbb{N}\), and recall the definition of the Roitberg space \(\widetilde{H}^{s,r}(\Omega)\). We previously introduced the Hilbert space \(\widetilde{H}^{s,0}(\Omega)\) used in the definition of \(\widetilde{H}^{s,r}(\Omega)\). If \(s \geq 0\), then \(\widetilde{H}^{s,0}(\Omega) := H^{s}(\Omega)\); if \(s < 0\), then \(\widetilde{H}^{s,0}(\Omega)\) denotes the dual of \(H^{-s}(\Omega)\) with respect to the inner product in \(L^2(\Omega)\). Let \(\|\cdot\|_{H^{s,0}}\) stand for the norm in \(\widetilde{H}^{s,0}(\Omega)\). By definition, the space \(\widetilde{H}^{s,r}(\Omega)\) is the completion of \(C^\infty(\Omega)\) with respect to the Hilbert norm
\[ \|u\|_{s,r,\Omega} := \left( \sum_{k=1}^r \|B_{k-1}^{-1}u\|_{s-k+1/2,\Omega}^2 \right)^{1/2}; \]
here \(\|\cdot\|_{s,\Gamma}\) denotes the norm in \(H^{(s)}(\Gamma)\), with \(\lambda \in \mathbb{R}\). It follows from the trace theorem for Sobolev spaces that
\[ \widetilde{H}^{s,r}(\Omega) = H^{(s)}(\Omega) \quad \text{with equivalence of norms if } s \geq r. \quad (34) \]

The mapping \(u \mapsto (u, (D_{k-1}^{-1}u)^{2}_{k-1})\), where \(u \in C^\infty(\Omega)\), extends by continuity to an isometric linear operator
\[ I_r: \widetilde{H}^{s,r}(\Omega) \rightarrow \widetilde{H}^{s,0}(\Omega) \bigoplus H^{(s-k+1/2)}(\Gamma). \]

According to (Theorems 3.4.1 and 4.1.4, [28]), the mapping (4) extends by continuity to a Fredholm bounded operator
\[ \Lambda : \widetilde{H}^{s,\mu}(\Omega) \oplus \bigoplus_{k=1}^{\infty} H^{(s+r_k-1/2)}(\Gamma) \to \widetilde{H}^{s-2\mu-2\tilde{q}}(\Omega) \oplus \bigoplus_{j=1}^{\tilde{q}+\infty} H^{(s-m_j-1/2)}(\Gamma) \]  

(35)

for arbitrary \( s \in \mathbb{Z} \). Its kernel coincides with \( \mathcal{N} \), and its range consists of all vectors \( (f_1, g_1, \ldots, g_{q+\infty}) \) that belong to the target space in (35) and satisfy the condition

\[ (f_0, \omega)_\Omega + \sum_{j=1}^{\mu-2\tilde{q}} (f_{\mu}, w_j)_\Gamma + \sum_{j=1}^{q+\infty} (g_j, h_j)_\Gamma = 0 \]

(36)

for every \((\omega, w_1, \ldots, w_{\mu-2\tilde{q}}, h_1, \ldots, h_{q+\infty}) \in \mathcal{N}^+ \).

Here, \((f_0, f_1, \ldots, f_{\mu-2\tilde{q}}) := I_{\mu-2\tilde{q}}f \) if \( m \geq 2\tilde{q} \), and \( f_0 := f \) if \( m \leq 2\tilde{q} - 1 \) (then the first sum with respect to \( j \) is absent). The forms \((\cdot, \cdot)_\Omega \) and \((\cdot, \cdot)_\Gamma \) in (36) are well defined as extensions by continuity of the inner products in \( L_2(\Omega) \) and \( L_2(\Gamma) \), respectively.

Choose a number \( s \in \mathbb{Z} \) arbitrarily. As we see from (35), the mapping \( u \mapsto Au \), where \( u \in C^\infty(\Omega) \), extends by continuity to a bounded linear operator \( A : \widetilde{H}^{s,\mu}(\Omega) \to \widetilde{H}^{s-2\mu-2\tilde{q}}(\Omega) \). Using this operator, we put

\[ \widetilde{H}^{s,\mu}_A(\Omega) := \{ u \in \widetilde{H}^{s,\mu}(\Omega) : Au = 0 \}. \]

We may and do consider \( \widetilde{H}^{s,\mu}_A(\Omega) \) as a Hilbert space with respect to the norm in \( \widetilde{H}^{s,\mu}(\Omega) \). The restriction of the operator (35) to \( \widetilde{H}^{s,\mu}_A(\Omega) \) is a bounded operator

\[ \Lambda_A : \widetilde{H}^{s,\mu}_A(\Omega) \oplus \bigoplus_{k=1}^{\infty} H^{(s+r_k-1/2)}(\Gamma) \to \bigoplus_{j=1}^{q+\infty} H^{(s-m_j-1/2)}(\Gamma). \]

(37)

Let \( \widetilde{D}^{s,\mu}_A(\Omega, \Gamma) \) and \( \mathcal{H}_{(s)}(\Gamma) \) respectively denote the source and target spaces of the operator (37). It follows from the above-mentioned properties of (35) that \( \mathcal{N} \) is the kernel of the operator (37) and that its range consists of all vectors \( g := (g_1, \ldots, g_{q+\infty}) \in \mathcal{H}_{(s)}(\Gamma) \) which satisfy (24). Hence, the range is closed in \( \mathcal{H}_{(s)}(\Gamma) \) and its codimension equals \( \dim \mathcal{N}_0^+ \). Thus, the operator (37) is Fredholm, and its index is equal to \( \dim \mathcal{N}_0^+ = \dim \mathcal{N}_{\eta}^+ \).

Let us show that the spaces \( H^{(s)}_A(\Omega) \) and \( \widetilde{H}^{s,\mu}_A(\Omega) \) are completions of \( C^\infty(\Omega) \) with respect to equivalent norms, which proves Theorem 5 in the Sobolev case where \( \eta(t) \equiv t^\mu \). Of course, \( H^{(s)}_A(\Omega) \) denotes the space \( H^s(\Omega) \) in this case. We first show that \( C^\infty(\Omega) \) is dense in \( H^{(s)}_A(\Omega) \). Given \( u \in H^{(s)}_A(\Omega) \), we form the vector \((u, 0) \in \widetilde{H}^{s,\mu}_A(\Omega, \Gamma) \) and choose a sequence of vectors \( g^{(k)} \in (C^\infty(\Gamma))^{q+\infty} \) such that \( g^{(k)} \to \Lambda_A(u, 0) \) in \( \mathcal{H}_{(s)}(\Gamma) \), all limits in the proof holding as \( k \to \infty \). Consider the decomposition

\[ \mathcal{H}_{(s)}(\Gamma) = \mathcal{N}_\eta^+ + \Lambda_A(\widetilde{D}^{s,\mu}_A(\Omega, \Gamma)), \]

and let \( P_\eta^+ \) denote the projector of \( \mathcal{H}_{(s)}(\Gamma) \) onto the second summand parallel to the first. Then

\[ (C^\infty(\Gamma))^{q+\infty} \ni P_\eta^+ g^{(k)} \to P_\eta^+ \Lambda_A(u, 0) = \Lambda_A(u, 0) \quad \text{in} \quad \mathcal{H}_{(s)}(\Gamma). \]

(38)

Let \( \widetilde{Q}^{s,\mu}_A(\Omega, \Gamma) \) denote the orthogonal complement of \( \mathcal{N} \) in the Hilbert space \( \widetilde{D}^{s,\mu}_A(\Omega, \Gamma) \). The restriction of the operator (37) to \( \widetilde{Q}^{s,\mu}_A(\Omega, \Gamma) \) sets an isomorphism between the Hilbert spaces \( \widetilde{Q}^{s,\mu}_A(\Omega, \Gamma) \) and \( \Lambda_A(\widetilde{D}^{s,\mu}_A(\Omega, \Gamma)) \), the latter space being endowed with the norm in \( \mathcal{H}_{(s)}(\Gamma) \). Let \( (u^{(k)}, v^{(k)}) \) be the preimage of \( P_\eta^+ g^{(k)} \) by this isomorphism. Owing to (38), we conclude that

\[ C^\infty(\Omega) \times (C^\infty(\Gamma))^{q+\infty} \ni (u^{(k)}, v^{(k)}) \to (u^{(k)}, v^{(k)}) \to (u, 0) + (u^\mu, v^\nu) \quad \text{in} \quad \widetilde{D}^{s,\mu}_A(\Omega, \Gamma) \]

(39)
for certain \((u^0, v^0) \in \mathcal{N}\). The inclusion in (39) holds true because

\[
\Lambda_A(u^0, v^0) = P_0^+ \mathcal{S}(k) \in \Lambda_A(\overline{D}_A^{q+\mu}(\Omega, \Gamma)) \cap (C^\infty(\Gamma))^{q+\mu} = \Lambda_A(\overline{D}_A^{q+\mu}(\Omega, \Gamma))
\]

whenever \(l \in \mathbb{N}\). Indeed, the last property implies that

\[
(u^0, v^0) = (u^{k,1}, v^{k,1}) + (u^{k,0}, v^{k,0})
\]

for some \((u^{k,1}, v^{k,1}) \in \overline{D}_A^{q+\mu}(\Omega, \Gamma)\) and \((u^{k,0}, v^{k,0}) \in \mathcal{N}\), which yields

\[
(u^0, v^0) \in \bigcap_{l \geq 1} \overline{D}_A^{q+\mu}(\Omega, \Gamma) \subset C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^{q+\mu}
\]

in view of (34). According to (39), we get

\[
C^\infty(\overline{\Omega}) \ni u^0 - u \to \mathcal{F}_A^q(\Omega).
\]

Thus, the set \(C^\infty(\overline{\Omega})\) is dense in \(\mathcal{F}_A^q(\Omega)\).

Choose \(p \in \mathbb{N}\) such that \(2qp \geq \mu\). Then

\[
\|u\|_{s,\Omega} \leq \|u\|_{s,0,\Omega} \leq \|u\|_{s,\Omega,\Gamma} \leq \|u\|_{2q \mu,\Omega} \asymp \|u\|_{s,\Omega} \quad \text{whenever} \quad u \in C^\infty(\overline{\Omega}). \tag{40}
\]

Here, \(\| \cdot \|_{s,\Omega}\) is the norm in the Sobolev space \(H^s(\Omega)\), and \(\asymp\) denotes equivalence of norms. The first inequality and the equivalence of norms in (40) need explanation. If \(s \geq 0\), this inequality becomes the equality due to the definition of \(\mathcal{F}_A^q(\Omega)\). If \(s < 0\), then it follows from the definition that the norm of any function \(u \in C^\infty(\overline{\Omega})\) in \(H^s(\mathbb{R}^n)\) equals the norm of the extension of \(u\) by zero in \(H^s(\mathbb{R}^n)\) (see, e.g., (Chapter 1, Remark 12.4, [43])). This implies the first inequality in (40) in the case \(s < 0\). If \(s \geq 2qp\), the equivalence in (40) follows from (34). If \(s < 2qp\), this equivalence is due to the isomorphism (4.196) from (Section 4.4.2, [8]), we considering this isomorphism for the properly elliptic PDO \(L := A^p\) (of order \(2qp\)) and the space \(X^s(\Omega) := \{0\}\) where \(\sigma := s - 2qp\). Thus, the spaces \(H^s(\mathbb{R}^n)\) and \(\mathcal{F}_A^q(\Omega)\) coincide as completions of \(C^\infty(\overline{\Omega})\) with respect to equivalent norms.

Let us now deduce Theorem 5 for arbitrary \(\eta \in \mathcal{F}_A^q(\Omega)\) from the Sobolev case just treated. Choose integers \(s_0\) and \(s_1\) such that \(s_0 < \sigma_0(\eta)\) and \(s_1 > \sigma_1(\eta)\), and define the interpolation parameter \(\psi\) by (25). Applying the interpolation with the parameter \(\psi\) to the Fredholm bounded operators

\[
\Lambda_A : D_A^{(s)}(\Omega, \Gamma) \to \mathcal{H}_{(s)}(\Gamma) \quad \text{where} \quad s \in \{s_0, s_1\}, \tag{41}
\]

we obtain the Fredholm bounded operator

\[
\Lambda_A : [D_A^{(s_0)}(\Omega, \Gamma), D_A^{(s_1)}(\Omega, \Gamma)]_\psi \to [\mathcal{H}_{(s_0)}(\Gamma), \mathcal{H}_{(s_1)}(\Gamma)]_\psi \tag{42}
\]

by Proposition 3. According to Propositions 1, 2 and 4, this operator acts between the spaces \(D_A^q(\Omega, \Gamma)\) and \(\mathcal{H}_q(\Gamma)\). It extends the mapping (21) by continuity because the set \(C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^{q+\mu}\) is dense in \(D_A^q(\Omega, \Gamma)\). Owing to Proposition 3, the kernel and index of this operator coincide with the common kernel \(\mathcal{N}\) and the index \(\dim \mathcal{N} - \dim \mathcal{N}_0^q\) of the operators (41). In addition, the range of (42) equals

\[
\mathcal{H}_q(\Gamma) \cap \Lambda_A(\mathcal{D}_A^{(s_0)}(\Omega, \Gamma)) = \{g \in \mathcal{H}_q(\Gamma) : (24) \text{ is satisfied}\}.
\]

We have proved all the properties of the operator (22) stated in Theorem 5. \(\square\)
7. Applications

We will apply Theorem 3 to obtain new sufficient conditions under which components of generalized solutions to the elliptic problem (1), (2) have continuous classical derivatives of a prescribed order. To this end, we also use the following result:

**Proposition 5.** Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}^d \), and \( 0 \leq l \in \mathbb{Z} \). Suppose that \( U \) is an open nonempty subset of \( \mathbb{R}^d \). Then

\[
\int_1^{2^{l+d-1} \alpha^{-2}(t)} dt < \infty \iff \{ w \in H^a(\mathbb{R}^d) : \text{supp} \, w \subseteq U \} \subset C^l(\mathbb{R}^d).
\]

This proposition follows from Hörmander’s embedding theorem (Theorem 2.2.27, [44]) as is shown in (Lemma 2, [53]). The case of \( U = \mathbb{R}^d \) is possible here. As usual, \( C^l(\cdot) \) denotes the space of all \( l \) times continuously differentiable functions on a given set.

Suppose that the sets \( \Omega_0 \) and \( \Gamma_0 \) are the same as those in Theorem 3.

**Theorem 6.** Let \( 0 \leq l \in \mathbb{Z} \). Assume that a vector \((u, v)\) ∈ \( D^{m+1/2} (\Omega, \Gamma) \) satisfies the hypothesis of Theorem 3 for a certain parameter \( \eta \in \mathbb{R}^d \) such that \( \sigma_0(\eta) > m + 1/2 \) and

\[
\int_1^{2^{l+n-1} \eta^{-2}(t)} dt < \infty.
\]

Then \( u \in C^l(\Omega_0 \cup \Gamma_0) \).

**Proof.** Choose a point \( x \in \Omega_0 \cup \Gamma_0 \) arbitrarily, and take a function \( \chi \in C^\infty(\overline{\Omega}) \) such that \( \text{supp} \, \chi \subseteq \Omega_0 \cup \Gamma_0 \) and that \( \chi = 1 \) in a neighbourhood \( V(x) \) of \( x \) (in the topology on \( \Omega \)). The inclusion \( \chi u \in H^\eta(\Omega) \) holds true by Theorem 3. Let \( w \in H^\eta(\mathbb{R}^n) \) be a certain extension of \( \chi u \). It follows from (43) by Proposition 5, that \( w \in C^l(\mathbb{R}^n) \). Hence, \( u \in C^l(V(x)) \), which implies that \( u \in C^l(\Omega_0 \cup \Gamma_0) \) due to the arbitrariness of \( x \in \Omega_0 \cup \Gamma_0 \). \( \Box \)

**Theorem 7.** Let \( 0 \leq l \in \mathbb{Z}, k \in \{1, \ldots, \alpha \}, \) and \( \Gamma_0 \neq \emptyset \). Assume that a vector \((u, v)\) ∈ \( D^{m+1/2} (\Omega, \Gamma) \) satisfies the hypothesis of Theorem 3 for a certain parameter \( \eta \in \mathbb{R}^d \) such that \( \sigma_0(\eta) > m + 1/2 \) and

\[
\int_1^{2^{l+n-1} \eta^{-2}(t)} dt < \infty.
\]

Then \( v_k \in C^l(\Gamma_0) \).

**Proof.** Choose a point \( x \in \Gamma_0 \) arbitrarily, and take a function \( \chi \in C^\infty(\Gamma) \) such that \( \text{supp} \, \chi \subseteq \Gamma_0 \) and that \( \chi = 1 \) in a neighbourhood \( V(x) \) of \( x \) (in the topology on \( \Gamma \)). By Theorem 3, we have the inclusion \( \chi v_k \in H^{\eta \delta_{k-1/2}}(\Gamma) \). Let \( \pi_j \) and \( \chi_j \), with \( j = 1, \ldots, \lambda \), be the same as those in the definition of \( H^{\alpha}(\Gamma) \) given in Section 3. It follows from (44) by Proposition 5 that

\[
(\chi_j \chi v_k) \circ \pi_j \in H^{\eta \delta_{k-1/2}}(\mathbb{R}^n) \subset C^l(\mathbb{R}^n)
\]

for every \( j \in \{1, \ldots, \lambda \} \). Hence, the distribution \( \chi v_k = (\chi_j + \cdots + \chi_\lambda) \chi v_k \) belongs to \( C^l(\Gamma) \). Thus, \( v_k \in C^l(V(x)) \), which implies the inclusion \( v_k \in C^l(\Gamma_0) \) in view of the arbitrariness of \( x \in \Gamma_0 \). \( \Box \)

**Remark 1.** Conditions (43) and (44) are exact in Theorems 6 and 7. Namely, let \( 0 \leq l \in \mathbb{Z}, \eta \in \mathbb{R}^d, \) and \( \sigma_0(\eta) > m + 1/2 \). Then it follows from the implication

\[
((u, v) \in D^{m+1/2} (\Omega, \Gamma) \text{ and } \Lambda(u, v) \in C^{\eta \delta_{k-1/2}}(\Omega_0, \Gamma_0))^\top \quad u \in C^l(\Omega_0 \cup \Gamma_0)
\]
that \( \eta \) satisfies (43). Let \( k \in \{1, \ldots, \varepsilon\} \), and suppose that \( \Gamma_0 \neq \emptyset \). Then it follows from the implication

\[
((u, v) \in D^{m+1/2}_+ (\Omega, \Gamma) \text{ and } \Lambda(u, v) \in C^{q}_{\text{loc}}(\Omega_0, \Gamma_0)) \implies v_k \in C^1(\Gamma_0)
\]

that \( \eta \) satisfies (44).

**Proof.** Let us show that (45) \( \Rightarrow \) (43). Assume that (45) is true. Let \( \Omega_1 \) be an open ball in \( \mathbb{R}^n \) satisfying \( \Omega_1 \subset \Omega_0 \). We arbitrarily choose a distribution \( w \in H^q(\mathbb{R}^n) \) such that \( \text{supp}(w) \subset \Omega_1 \). We form the vector \( (u, v) \in D^q(\Omega, \Gamma) \) letting \( u \) denote the restriction of \( w \) to \( \Omega_0 \) and putting \( v := 0 \). This vector satisfies the premise of the implication (45). Hence, \( u \in C^1(\Omega_0) \), which implies that \( w \in C^1(\mathbb{R}^n) \). Thus, \( \eta \) satisfies (43) due to Proposition 5.

Let us prove that (46) \( \Rightarrow \) (43). Assume that (46) is true. Let \( \pi_j : \mathbb{R}^{n-1} \to \Gamma_j \) with \( j = 1, \ldots, \lambda \), be an \( C^\infty \)-atlas on \( \Gamma \) such that \( \Gamma_1 \subset \Gamma_0 \). Let \( U \) be an open ball of radius 1 in \( \mathbb{R}^{n-1} \). We arbitrarily choose a distribution \( w \in H^q(\mathbb{R}^{n-1}) \) such that \( \text{supp}(w) \subset U \). We form the vector \( (u, v) \in D^q(\Omega, \Gamma) \) putting \( u := 0, v_k := O(w \circ \pi_1) \) and \( v_j = 0 \) whenever \( j \neq k \); here \( O(w \circ \pi_1) \) denotes the extension of the distribution \( w \circ \pi_1 \) by zero from \( \Gamma_1 \) to \( \Gamma \). This vector satisfies the premise of the implication (46). Hence, \( v_k \in C^1(\Gamma_0) \), which implies that \( w \in C^1(\mathbb{R}^{n-1}) \). Thus, \( \eta \) satisfies (44) due to Proposition 5.

Using Theorems 6 and 7, we will deduce the following sufficient condition under which a generalized solution \( (u, v) \) to the elliptic problem (1), (2) is classical, i.e., \( u \in C^{2l}(\Omega) \cap C^m(\mathbb{R}^n) \) for certain \( \varepsilon > 0 \), and \( v_k \in C^{m+q_2}(\Gamma) \) for each \( k \in \{1, \ldots, \varepsilon\} \). Here,

\[
U_k := \{ x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon \}.
\]

(Note that, providing \( m \leq 2q - 1 \), the condition \( u \in C^{2l}(\Omega) \cap C^m(\mathbb{R}^n) \) is equivalent to \( u \in C^{2l}(\Omega) \cap C^m(\Omega_0) \).) If the solution \( (u, v) \) is classical, the left-hand sides of the problem (1), (2) are calculated with the help of classical derivatives and are continuous functions on \( \Omega \) and \( \Gamma \) respectively.

**Theorem 8.** Suppose that a vector \( (u, v) \in D^{m+1/2}_+ (\Omega, \Gamma) \) is a generalized solution to the elliptic problem (1), (2) whose right-hand sides satisfy the conditions

\[
f \in H^{q_1}_{\text{loc}}(\Omega, \emptyset) \cap H^{q_2}_{\text{loc}}(U_k, \Gamma)
\]

and

\[
g_j \in H^{q_1q_2 - 1/2}_{\text{loc}}(\Gamma), \quad \text{with} \quad j = 1, \ldots, q + \varepsilon,
\]

for certain \( \varepsilon > 0 \) and some parameters \( \eta_1, \eta_2 \in \text{OR} \) such that \( c_0(\eta_1) > m + 1/2, c_0(\eta_2) > m + 1/2 \),

\[
\int_1^{t^{q_1q_2 - 1}} \eta_1^{-2}(t) dt < \infty, \quad \text{and} \quad \int_1^{t^{q_1q_2 - 2}} \eta_2^{-2}(t) dt < \infty.
\]

Then the solution \( (u, v) \) is classical.

**Proof of Theorem 8.** Putting \( l := 2q, \Omega_0 := \Omega, \Gamma_0 := \emptyset, \) and \( \eta := \eta_1 \) in Theorem 6, we conclude that \( u \in C^{2l}(\Omega) \) by (47). In addition, letting \( l := m, \Omega_0 := U_k, \Gamma_0 := \Gamma, \) and \( \eta := \eta_2 \) in this theorem, we see that \( u \in C^m(U_k \cup \Gamma) \) by (47) and (48). Finally, putting \( l := m + r_k, \Omega_0 := U_k, \Gamma_0 := \Gamma, \) and \( \eta := \eta_2 \) in Theorem 7, we conclude by (47) and (48) that \( v_k \in C^{m+q_2}(\Gamma) \) for each \( k \in \{1, \ldots, \varepsilon\} \). Thus, the solution \( (u, v) \) is classical.

Some versions of Theorems 6–8 were proved in (Sections 5 and 6, [40]) in the case where the function \( \eta(t) \) varies regularly at infinity and when \( m \geq 2q \). If \( m \leq 2q - 1 \) and if the function \( \eta \in \text{OR} \) satisfies the stronger condition \( c_0(\eta) > 2q - 1/2 \), these theorems were proved in (Section 7, [39]).
8. Concluding Remarks

The results obtained in this paper form a core of a solvability theory for elliptic problems that have additional unknowns in boundary conditions and are considered in generalized Sobolev spaces. The use of OR-varying function parameters as orders of regularity of distribution spaces allows obtaining more precise results than those received in the framework of classical Sobolev spaces, whose orders of regularity are given by power functions only. This is demonstrated by applications given in Section 7. For example, analyzing Theorem 6, we see that, if the regularity order \( \eta \) takes the form \( \eta(t) \equiv t^s \) for some \( s \in \mathbb{R} \), then (43) is equivalent to \( s > \frac{l+n}{2} \). The latter condition cannot be weakened in the framework of Sobolev spaces. However, using generalized Sobolev spaces, we find, e.g., that the function \( \eta(t) \equiv t^{l+n/2} \log(1+t) \) satisfies (43).

The choice of the function class OR as a set of regularity orders for generalized Sobolev spaces allows using the interpolation technique in our proofs, which facilitates them essentially as compared with proofs based on the Fourier transform approach and theory of pseudodifferential operators. This class seems the broadest one in order that generalized Sobolev spaces be well defined on smooth manifolds. It contains some functions that have not a definite order at infinity (i.e., their lower and upper Matuszewska indices are different). This circumstance specifically complicates the proof of Theorem 3 as compared with the case of power function or regularly varying functions (see, e.g., (Section 5, [5])). Our proof of Theorem 5 involves special Roitberg’s spaces, which allows treating rough boundary data (of arbitrarily low regularity).

It is possible to show that the Fredholm property of the operator (14) will remain valid if the boundary \( \Gamma \) and coefficients of PDOs involved in the elliptic problem be of some finite smoothness and if a certain condition is imposed on the upper Matuszewska index of the regularity order \( \eta \) (compare with (Section 4.4.5, [28]) and (Theorem 4.1.5, [34]) in the case of Sobolev spaces). This may be a subject of another article.

Our approach is applicable to elliptic problems for systems of differential equations, pseudodifferential elliptic problems, and parameter-elliptic problems. It can be extended to generalized \( L_p \)-Sobolev, Besov, and Triebel–Lizorkin spaces by using various methods of interpolation with function parameter between normed spaces, as indicated in (Section 1.3.3, [58]), (Section 4.2, [59]), and (Sections 3 and 6, [60]).

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