THE QUARK-ANTIQUARK WILSON LOOP FORMALISM IN THE NRQCD POWER COUNTING SCHEME

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ABSTRACT

The quark-antiquark interaction from the NRQCD Lagrangian is studied in the Wilson loop formalism.

1. The NRQCD Lagrangian

Large radii quarkonia (i.e. excited heavy mesons like \( \bar{b}b \) and \( \bar{c}c \)) cannot be described in terms of perturbative QCD with the addition of leading nonperturbative effects encoded into local condensates. This happens since in this situation the gluonic correlation length cannot be considered infinitely large with respect to the other scales of the system. A successful description needs therefore a systematic inclusion of non-local condensates. A solution is provided by the so-called Wilson loop formalism. The non-local quantities are field strength insertions in the Wilson loop made up by the quark trajectories.

While a full relativistic QCD formulation in this formalism is still lacking such a formulation is possible for heavy quark bound states which are essentially described by non relativistic dynamics. These systems are characterized by a dynamical adimensional parameter, the quark velocity \( v \), which is small and allows a classification of the energy scales of the problem in hard (\( \sim m \)), soft (\( \sim mv \)) and ultrasoft (\( \sim mv^2 \)). Moreover, this provides a power counting scheme for the operators in the Lagrangian. The relation between \( v \) and the QCD parameters is unknown (for infinitely heavy quarks \( v \) coincides with \( \alpha_s \), for realistic quarks it is the result of perturbative and nonperturbative effects) but irrelevant once the power counting scheme works. Due to the heavy quark mass \( m \), at a scale \( \mu \) between \( m \) and \( mv \) the physics is still dominated by perturbative effects. Therefore, in order to describe heavy quark bound states, it is possible to substitute the QCD Lagrangian with an effective non relativistic Lagrangian via perturbative matching at that scale. The new Lagrangian is simpler, since the hard degrees of freedom have been integrated out explicitly, but equivalent to the QCD one at a given order in \( \alpha_s \) and \( v \). This effective Lagrangian is known as the NRQCD Lagrangian. At order \( O(v^4) \) the NRQCD Lagrangian describing a bound state between a quark of mass \( m_1 \) and an antiquark of mass \( m_2 \) is:

\[
L = Q_1 \left( iD_0 + \frac{c_2(1)}{2m_1} D_1^2 + \frac{c_4(1)}{8m_1^3} D_1^4 \right) + \frac{c_F(1)}{2m_1} g \frac{\sigma \cdot B}{2m_1} + \frac{c_D(1)}{8m_1^2} g \frac{D \cdot E - E \cdot D}{8m_1^2}
\]
where \( Q_j \) are the heavy quark fields. The coefficients \( c_2^{(j)}, c_4^{(j)}, \ldots \) are evaluated at the matching scale \( \mu \) for a particle of mass \( m_j \). They encode the ultraviolet regime of QCD order by order in \( \alpha_s \). The explicit expressions and a numerical discussion can be found in [1]. The power counting rules for the operators of Eq. (1) are \( Q \sim (mv)^{3/2}, \ D \sim mv, \ gA_0 \sim mv^2, \ gA \sim mv^3, \ gE \sim m^2v^3 \) and \( gB \sim m^2v^4 \). Four quark operators which are apparently of order \( v^3 \) are actually suppressed by additional powers in \( \alpha_s \) in the matching coefficients and the octet contributions by an additional power in \( v^2 \) on singlet states. Therefore in the following we will neglect these contributions with the exception of a term which mixes under RG transformation with the chromomagnetic operator contribution to the spin-spin potential. We will call the corresponding matching coefficient \( d \).

2. The Wilson Loop Formalism

The use of the Wilson loop formalism on the Lagrangian (1) was first suggested in [1]. Let us sketch the derivation of the heavy quark potential. The 4-point gauge invariant Green function \( G \) associated with the Lagrangian (1) is defined as

\[
G(x_1, y_1, x_2, y_2) = \langle 0|Q_1^\dagger(x_2)\phi(x_2, x_1)Q_1(x_1)Q_1^\dagger(y_1)\phi(y_1, y_2)Q_2(y_2)|0\rangle,
\]

where \( \phi(x_2, x_1) \equiv \exp \left\{ -ig \int_0^1 ds (x_2 - x_1)^\mu A_\mu(x_1 + s(x_2 - x_1)) \right\} \) is a Schwinger line added to ensure gauge invariance. After integrating out the heavy quark fields, \( G \) can be expressed as a quantum-mechanical path integral over the quark trajectories:

\[
G(x_1, y_1, x_2, y_2) = \int_{y_1}^{x_1} Dz_1 Dp_1 \int_{y_2}^{x_2} Dz_2 Dp_2 \exp \left\{ i \int_{-T/2}^{T/2} dt \sum_{j=1}^2 p_j \cdot z_j - m_j - c_2^{(j)} \frac{p_j^2}{2m_j} \right. \\
+ c_4^{(j)} \frac{p_j^4}{8m_j^2} \left) \frac{1}{N_c} \right\} \text{Tr} \text{P} \left\{ 1 \right\} \exp \left\{ -ig \int_{-T/2}^{T/2} dz \mu A_\mu(z) + i \int_{-T/2}^{T/2} dz_{0j} c_4^{(j)} \frac{\sigma \cdot B}{2m_j} \\
+ i c_2^{(j)} g \frac{D \cdot E - E \cdot D}{8m_j^2} + i c_2^{(j)} g \frac{\sigma(D \times E - E \times D)}{8m_j^2} \right) \right\}
\]

\[
\times \exp \left\{ i \int_{-T/2}^{T/2} dt g^2 T^{a(1)}(T^{a(1)}T^{a(2)} + \delta^3(z_1 - z_2)) \right\}
\]

\[
= \int_{y_1}^{x_1} Dz_1 Dp_1 \int_{y_2}^{x_2} Dz_2 Dp_2 \exp \left\{ i \int_{-T/2}^{T/2} dt \sum_{j=1}^2 p_j \cdot z_j - m_j - \frac{p_j^2}{2m_j} + \frac{p_j^4}{8m_j^2} - i \int_{-T/2}^{T/2} dt U \right\},
\]

where the bracket means the Yang–Mills average over the gauge fields, \( \Gamma \) is the Wilson loop made up by the quark trajectories \( z_1 \) and \( z_2 \) and the endpoints Schwinger strings.
and $y_0^0 = y_1^0 \equiv -T/2$, $x_0^0 = x_1^0 \equiv T/2$. Reparameterization invariance fixes $c_2 = c_4 = 1$. Assuming that the limit exists, we define the heavy quark-antiquark potential $V$ as

$$\lim_{T \rightarrow \infty} \int_0^{T/2} dt U. $$

For a discussion on the existence of a quark-antiquark potential we refer to [1, 2]. Expanding in $v$ (following the power counting given in the previous section) or in the inverse of the mass we get

$$V(r) = \lim_{T \rightarrow \infty} \frac{i \log \langle W(\Gamma) \rangle}{T} + \left( \frac{S^{(1)} \cdot L^{(1)}}{m_1^2} + \frac{S^{(2)} \cdot L^{(2)}}{m_2^2} \right) \frac{2c_F^2V_1'(r) + c_F^2V_0'(r)}{2r} + \frac{S^{(1)} \cdot L^{(2)} + S^{(2)} \cdot L^{(1)}}{m_1m_2} \frac{c_F^2V_2'(r)}{r}$$

$$+ \left( \frac{S^{(1)} \cdot L^{(1)}}{m_1^2} - \frac{S^{(2)} \cdot L^{(2)}}{m_2^2} \right) \frac{2c_F^2V_1'(r) + c_F^2V_0'(r)}{2r} + \frac{S^{(1)} \cdot L^{(2)} - S^{(2)} \cdot L^{(1)}}{m_1m_2} \frac{c_F^2V_2'(r)}{r}$$

$$+ \frac{1}{8} \left( \frac{c_D^{(1)}}{m_1^2} + \frac{c_D^{(2)}}{m_2^2} \right) \Delta V_0(r) + \frac{1}{8} \left( \frac{c_F^{(1)}}{m_1^2} + \frac{c_F^{(2)}}{m_2^2} \right) \Delta V_a^{E}(r)$$

$$+ \frac{c_F^{(1)}}{m_1m_2} \left( \frac{S^{(1)} \cdot r S^{(2)} \cdot r}{r^2} - \frac{S^{(1)} S^{(2)}}{3} \right) V_3(r) + \frac{S^{(1)} S^{(2)}}{3m_1m_2} \left( c_F^{(1)} c_F^{(2)} V_4(r) - 48 \pi \alpha_s C_F d \delta^3(r) \right).$$

$W(\Gamma) \equiv \exp \left\{ -ig \int_T^t dz^\mu A_\mu(z) \right\}$ is the non-static Wilson loop. The expansion of it around the static Wilson loop $W(\Gamma_0)$ ($\Gamma_0$ is a $r \times T$ rectangle) gives the static potential $V_0 = \lim_{T \rightarrow \infty} i \log \langle W(\Gamma_0) \rangle / T$ plus velocity (non-spin) dependent terms. If $S^{(j)}$ and $L^{(j)}$ are the spin and orbital angular momentum operators of the particle $j$. The matching coefficients are defined as $2c_{F,S}^\pm \equiv c_{F,S}^{(1)} \pm c_{F,S}^{(2)}$. The “potentials” $V_1, V_2, \ldots$ are scale dependent gauge field averages of electric and magnetic field strength insertions in the static Wilson loop:

$$\Delta V_a^{E}(r) = 2 \lim_{T \rightarrow \infty} \int_0^T dt \langle [E(0, 0), E(0, t)] \rangle - \langle [E(0, 0)] \rangle \langle [E(0, t)] \rangle,$$

$$\Delta V_a^{B}(r) = 2 \lim_{T \rightarrow \infty} \int_0^T dt \langle [B(0, 0), B(0, t)] \rangle,$$

$$\frac{r^k}{r} V_1'(r) = \epsilon_{ijk} \lim_{T \rightarrow \infty} \int_0^T dt t \langle B_i(0, 0) E_j(0, t) \rangle,$$

$$\frac{r^k}{r} V_2'(r) = \frac{1}{2} \epsilon_{ijk} \lim_{T \rightarrow \infty} \int_0^T dt t \langle B_i(0, 0) E_j(r, t) \rangle,$$

$$\left( \frac{r_i r_j}{r^2} - \frac{\delta_{ij}}{3} \right) V_3(r) = 2 \lim_{T \rightarrow \infty} \int_0^T dt \left[ \langle [B_i(0, 0), B_j(r, t)] \rangle - \frac{\delta_{ij}}{3} \langle [B(0, 0), B(r, t)] \rangle \right],$$

$$V_4(r) = 2 \lim_{T \rightarrow \infty} \int_0^T dt \langle [B(0, 0), B(r, t)] \rangle,$$

where $\langle \rangle \equiv \langle W(\Gamma_0) \rangle / \langle W(\Gamma_0) \rangle$. 


3. Conclusions

Two comments in order to conclude. The explicit expression for the potential \( V_0 \) in terms of field strength insertions in a static Wilson loop is suitable for direct lattice and analytic evaluations. This makes the obtained result of particular importance. Different vacuum models can be easily compared on the heavy quark potential predictions, once the Wilson loop average has been evaluated.

The \( O(v^2) \) leading order NRQCD Lagrangian \( L = Q_1^\dagger (iD_0 + \partial^2/2m_1)Q_1 + \) anti-quark part does not contribute to \( V_0 \) only with a static potential (with the exception of the perturbative contribution if evaluated in Coulomb gauge). Since the corresponding Wilson loop \( P \exp \{ -ig \oint_{\Gamma} dz_0 A_0(z) \} \) is a function of the non-static loop \( \Gamma \), its expansion produces in general velocity dependent terms as well. This is not surprising since the power counting scheme of NRQCD has to be considered as a leading order power counting scheme. An exact value in \( v \) cannot be assigned to each term of the effective Lagrangian at least before the soft and ultrasoft degrees of freedom have been disentangled.

An extensive analysis of the topic discussed here can be found in \cite{1} with particular emphasis on the relevance of the matching procedure in order to have a consistent potential in the perturbative regime.

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