An upper bound of the density for packing of congruent hyperballs in hyperbolic $3$–space

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Abstract. In Szirmai (Ars Math Contemp 16:349–358, 2019) we proved that to each saturated congruent hyperball packing there exists a decomposition of the $3$-dimensional hyperbolic space $H^3$ into truncated tetrahedra. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices. In this paper we prove, using the above results and results of the papers Miyamoto (Topology 33(4): 613–629, 1994) and Szirmai (Mat Vesn 70(3): 211–221, 2018), that the density upper bound of the saturated congruent hyperball (hypersphere) packings related to the corresponding truncated tetrahedron cells is realized in regular truncated tetrahedra with density $\approx 0.86338$. Furthermore, we prove that the density of locally optimal congruent hyperball arrangement in a regular truncated tetrahedron is not a monotonically increasing function of the height (radius) of the corresponding optimal hyperball, unlike the ball (sphere) and horoball (horosphere) packings.

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1. Preliminary results

Let $X$ denote a space of constant curvature, either the $n$-dimensional sphere $S^n$, the Euclidean space $E^n$, or the hyperbolic space $H^n$ with $n \geq 2$. An important question of discrete geometry is to find the highest possible packing density in $X$ by congruent non-overlapping balls of a given radius [1,5].

The Euclidean cases are the best explored. One major recent development has been the settling of the long-standing Kepler conjecture, part of Hilbert’s 18th problem, by Thomas Hales at the turn of the 21st century. Hales’ computer-assisted proof was largely based on a program set forth by L. Fejes Tóth in the 1950s [9].

In $n$-dimensional hyperbolic geometry there are several new questions concerning the packing and covering problems, e.g. in $H^n$ there are 3 kinds of
“generalized balls (spheres)”: the usual balls (spheres), horoballs (horospheres) and hyperballs (hyperspheres). Moreover, the definition of packing density is crucial in hyperbolic spaces as shown by Böröczky [3], for standard examples also see [5,22]. The most widely accepted notion of packing density considers the local densities of balls with respect to their Dirichlet–Voronoi cells (cf. [3,12]). In order to consider ball packings in $\mathbb{H}^n$, we use an extended notion of such local density.

In space $X^n$ let $d_n(r)$ be the density of $n + 1$ mutually touching spheres or horospheres of radius $r$ (for a horosphere $r = \infty$) with respect to the simplex spanned by their centres. L. Fejes Tóth and H. S. M. Coxeter conjectured that the packing density of balls of radius $r$ in $X^n$ cannot exceed $d_n(r)$. This conjecture has been proved by C. A. Rogers for the Euclidean space $\mathbb{E}^n$. The 2-dimensional spherical case was settled by L. Fejes Tóth [8].

Ball (sphere) and horoball (horosphere) packings:

In [3,4] K. Böröczky proved the following theorem for ball and horoball packings for any $n$ ($2 \leq n \in \mathbb{N}$):

**Theorem 1.1. (K. Böröczky)** In an $n$-dimensional space of constant curvature consider a packing of spheres of radius $r$. In the spherical space suppose that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet-Voronoi cell cannot exceed the density of $n + 1$ spheres of radius $r$ mutually touching one another with respect to the simplex spanned by their centers.

The above greatest density in $\mathbb{H}^3$ is $\approx 0.85328$ which is not realized by packing with any equal balls. However, it is attained by the horoball packing (in this case $r = \infty$) of $\mathbb{H}^3$ where the ideal centers of horoballs lie on the absolute figure of $\mathbb{H}^3$. This ideal regular tetrahedron tiling is given with the Coxeter-Schläfli symbol $\{3,3,6\}$. Ball packings of hyperbolic $n$-space and of other Thurston geometries are extensively discussed in the literature see e.g. [1,3,6,7,20,36], where the reader finds further references as well.

In a previous paper [13] we proved that the above known optimal horoball packing arrangement in $\mathbb{H}^3$ is not unique using the notions of horoballs of the same and different types. Two horoballs in a horoball packing are of the “same type” iff the local densities of the horoballs to the corresponding cell (e.g. D-V cell or ideal simplex) are equal, (see [31]). We gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings that yield the above Böröczky–Florian packing density upper bound (see [4]).

We have also found that the Böröczky-Florian type density upper bound for horoball packings of different types is no longer valid for fully asymptotic simplices in higher dimensions $n > 3$ (see [30]). For example in $\mathbb{H}^4$, the density of such optimal, locally densest horoball packing is $\approx 0.77038$ larger than the analogous Böröczky-Florian type density upper bound of $\approx 0.73046$. However, these horoball packing configurations are only locally optimal and cannot be extended to the whole hyperbolic space $\mathbb{H}^4$. 
In the papers [14,15] we continued our previous investigation in $\mathbb{H}^n$ ($n \in \{4,5\}$) allowing horoballs of different types. We gave several new examples of horoball packing configurations that yield high densities ($\approx 0.71645$ in $\mathbb{H}^4$ and $\approx 0.59421$ in $\mathbb{H}^5$) where horoballs are centered at ideal vertices of certain Coxeter simplices, and are invariant under the actions of their respective Coxeter groups.

**Hyperball (hypersphere) packings:**

A hypersphere is the set of all points in $\mathbb{H}^n$, lying at a certain distance, called its *height*, from a hyperplane, on both sides of the hyperplane (cf. [41] for the planar case).

In the hyperbolic plane $\mathbb{H}^2$ the universal upper bound of the hypercycle packing density is $\frac{3}{\pi}$, and the universal lower bound of the hypercycle covering density is $\frac{\sqrt{12}}{\pi}$, proved by I. Vermesi [40–42]. We note here that independently from him in [17] T. H. Marshall and G. J. Martin obtained similar results to hypercycle packings.

In [32,33] we analysed regular prism tilings (simply truncated Coxeter orthoscheme tilings) and the corresponding optimal hyperball packings in $\mathbb{H}^n$ ($n = 3,4$) and we extended the method developed in the paper [33] to the 5-dimensional hyperbolic space (see [34]). In the paper [35] we studied $n$-dimensional hyperbolic regular prism honeycombs and the corresponding coverings by congruent hyperballs and we determined their least dense covering densities. Furthermore, we formulated conjectures for candidates of the least dense hyperball covering by congruent hyperballs in 3- and 5-dimensional hyperbolic spaces.

In [27] we discussed congruent and non-congruent hyperball packings of truncated regular tetrahedron tilings. These are derived from the Coxeter simplex tilings $\{p,3,3\}$ ($7 \leq p \in \mathbb{N}$) and $\{5,3,3,3,3\}$ in 3- and 5-dimensional hyperbolic spaces. We determined the densest hyperball packing arrangement and its density with congruent hyperballs in $\mathbb{H}^5$ and determined the smallest density upper bounds of non-congruent hyperball packings generated by the above tilings in $\mathbb{H}^n$, ($n = 3,5$).

In [26] we deal with packings derived by horo- and hyperballs (briefly hyp-hor packings) in $n$-dimensional hyperbolic spaces $\mathbb{H}^n$ ($n = 2,3$) which form a new class of the classical packing problems. We constructed in the 2– and 3–dimensional hyperbolic spaces hyp-hor packings that are generated by complete Coxeter tilings of degree 1 and we determined their densest packing configurations and their densities. We proved using also numerical approximation methods that in the hyperbolic plane ($n = 2$) the density of the above hyp-hor packings arbitrarily approximate the universal upper bound of the hypercycle or horocycle packing density $\frac{3}{\pi}$ and in $\mathbb{H}^3$ the optimal configuration belongs to the $\{7,3,6\}$ Coxeter tiling with density $\approx 0.83267$. Furthermore, we analyzed the hyp-hor packings in truncated orthoschemes $\{p,3,6\}$ ($6 < p < 7, p \in \mathbb{R}$).
whose density function attains its maximum for a parameter which lies in the interval \([6.05, 6.06]\) and the densities for parameters lying in this interval are larger than \(\approx 0.85397\).

In [25] we proved that if the truncated tetrahedron is regular, then the density of the densest packing is \(\approx 0.86338\). This is larger than the Böröczky-Florian density upper bound but our locally optimal hyperball packing configuration cannot be extended to the entirety of \(\mathbb{H}^3\). However, we described a hyperball packing construction, by the regular truncated tetrahedron tiling under the extended Coxeter group \(\{3, 3, 7\}\) with maximal density \(\approx 0.82251\).

Recently, (to the best of the author’s knowledge) the candidates for the densest hyperball (hypersphere) packings in the 3, 4 and 5-dimensional hyperbolic spaces \(\mathbb{H}^n\) are derived by regular prism tilings which were studied in the papers [32–34].

In [28] we considered hyperball packings in the 3-dimensional hyperbolic space and developed a decomposition algorithm that for each saturated hyperball packing provides a decomposition of \(\mathbb{H}^3\) into truncated tetrahedra. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices.

In [37] we studied hyperball packings related to the truncated regular octahedron and cube tilings that are derived from the Coxeter simplex tilings \(\{p, 3, 4\}\) \((7 \leq p \in \mathbb{N})\) and \(\{p, 4, 3\}\) \((5 \leq p \in \mathbb{N})\) in 3-dimensional hyperbolic space \(\mathbb{H}^3\). We determined the densest hyperball packing arrangement and its density with congruent and non-congruent hyperballs related to the above tilings. Moreover, we prove that the locally densest congruent or non-congruent hyperball configuration belongs to the regular truncated cube with density \(\approx 0.86145\). This is larger than the Böröczky-Florian density upper bound for balls and horoballs. We described a non-congruent hyperball packing construction, by the regular cube tiling under the extended Coxeter group \(\{4, 3, 7\}\) with maximal density \(\approx 0.84931\).

In [39] we examined congruent and non-congruent hyperball packings generated by doubly truncated Coxeter orthoscheme tilings in the 3-dimensional hyperbolic space. We proved that the densest congruent hyperball packing belongs to the Coxeter orthoscheme tiling of parameter \(\{7, 3, 7\}\) with density \(\approx 0.81335\). This density is equal – in our conjecture – with the upper bound density of the corresponding non-congruent hyperball arrangements.

**Remark 1.2.** We can try to define the density of system of sets in hyperbolic space as we did in the Euclidean space, i.e. by the limiting value of the density with respect to a sphere \(C(r)\) of radius \(r\) with a fixed centre \(O\). But since for a fixed value of \(h\) the volume of the spherical shell \(C(r + h) - C(r)\) is of the same order of magnitude as the volume of \(C(r)\), the argument used in the Euclidean space to prove that the limiting value is independent of the
choice of \(O\) does not work in the hyperbolic space. Therefore the definition of packing density is crucial in hyperbolic spaces \(\mathbb{H}^n\) as shown by K. Böröczky [3]. For nice examples also see [5,22]. The most widely accepted notion of packing density considers the local densities of balls with respect to their Dirichlet–Voronoi cells (cf. [3,12]), but in our cases these cells are infinite hyperbolic polyhedra. The other possibility: the packing density \(\delta\) can be defined (see [32,34,41,42]) as the reciprocal of the ratio of the volume of a fundamental domain for the symmetry group of a tiling to the volume of the ball pieces contained in the fundamental domain (\(\delta < 1\)). The covering density \(\Delta > 1\) is defined similarly. In the present paper our aim is to determine a density upper bound for saturated, congruent hyperball packings in \(\mathbb{H}^3\) therefore we use an extended notion of such local density.

2. Saturated hyperball packings in \(\mathbb{H}^3\) and their density upper bound

We use for \(\mathbb{H}^3\) (and analogously for \(\mathbb{H}^n\), \(n \geq 3\)) the projective model in the Lorentz space \(\mathbb{E}^{1,3}\) that denotes the real vector space \(\mathbb{V}^4\) equipped with the bilinear form of signature \((1,3)\), \(\langle\mathbf{x}, \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3\), where the non-zero vectors \(\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbb{V}^4\) and \(\mathbf{y} = (y^0, y^1, y^2, y^3) \in \mathbb{V}^4\), are determined up to real factors, for representing points of \(\mathbb{P}^3(\mathbb{R})\). Then \(\mathbb{H}^3\) can be interpreted as the interior of the quadric \(Q = \{(x) \in \mathcal{P}^3 | \langle\mathbf{x}, \mathbf{x}\rangle = 0\} =: \partial \mathbb{H}^3\) in the real projective space \(\mathcal{P}^3(\mathbb{V}^4, \mathbb{V}^4)\) (here \(\mathbb{V}^4\) is the dual space of \(\mathbb{V}^4\)). Namely, for an interior point \(\mathbf{y}\) we have \(\langle\mathbf{y}, \mathbf{y}\rangle < 0\).

Points of the boundary \(\partial \mathbb{H}^3\) in \(\mathcal{P}^3\) are called points at infinity, or at the absolute of \(\mathbb{H}^3\). Points lying outside \(\partial \mathbb{H}^3\) are said to be outer points of \(\mathbb{H}^3\) relative to \(Q\). Let \((\mathbf{x}) \in \mathcal{P}^3\), a point \((\mathbf{y}) \in \mathcal{P}^3\) is said to be conjugate to \((\mathbf{x})\) relative to \(Q\) if \(\langle\mathbf{x}, \mathbf{y}\rangle = 0\) holds. The set of all points which are conjugate to \((\mathbf{x})\) form a projective (polar) hyperplane \(pol(\mathbf{x}) := \{(\mathbf{y}) \in \mathcal{P}^3 | \langle\mathbf{x}, \mathbf{y}\rangle = 0\}\). Thus the quadric \(Q\) induces a bijection (linear polarity \(\mathbb{V}^4 \rightarrow \mathbb{V}^4\)) from the points of \(\mathcal{P}^3\) onto their polar hyperplanes.

A point \(X(\mathbf{x})\) and a hyperplane \(\alpha(a)\) are incident if \(\mathbf{x}a = 0\) \((\mathbf{x} \in \mathbb{V}^4 \setminus \{0\}, a \in \mathbb{V}^4 \setminus \{0\})\).

The hypersphere (or equidistance surface) is a quadratic surface at a constant distance from a plane (base plane) in both halfspaces. The infinite body of the hypersphere, containing the base plane, is called hyperball.

The half hyperball with distance \(h\) to a base plane \(\beta\) is denoted by \(\mathcal{H}^h_+\). The volume of a bounded hyperball piece \(\mathcal{H}^h_+(A)\), delimited by a 2-polygon \(A \subset \beta\), and its prism orthogonal to \(\beta\), can be determined by the classical formula (2.1) of J. Bolyai [2].

\[
\text{Vol}(\mathcal{H}^h_+(A)) = \frac{1}{4} \text{Area}(A) \left[ k \sinh \frac{2h}{k} + 2h \right],
\]
The constant $k = \sqrt{-1/K}$ is the natural length unit in $\mathbb{H}^3$, where $K$ denotes the constant negative sectional curvature. In the following we may assume that $k = 1$.

Let $B^h$ be a hyperball packing in $\mathbb{H}^3$ with congruent hyperballs of height $h$.

The notion of saturated packing follows from that fact that the density of any packing can be improved by adding further packing elements as long as there is sufficient room to do so. However, we usually apply this notion for packings with congruent elements.

In [28] we modified the classical definition of saturated packing for non-compact ball packings with generalized balls (horoballs, hyperballs) in the $n$-dimensional hyperbolic space $\mathbb{H}^n$ ($n \geq 2$ integer parameter):

**Definition 2.1.** A ball packing with non-compact generalized balls (horoballs or/and hyperballs) in $\mathbb{H}^n$ is saturated if no new non-compact generalized ball can be added to it.

To obtain a hyperball (hypersphere) packing upper bound it obviously suffices to study saturated hyperball packings (using the above definition) and in what follows we assume that all packings are saturated unless otherwise stated.

We take the set of hyperballs $\{\mathcal{H}_i^h\}$ of a saturated hyperball packing $B^h$ (see Definition 2.1). Their base planes are denoted by $\beta_i$. Thus in a saturated hyperball packing the distance between two ultraparallel base planes $d(\beta_i, \beta_j)$ is at least $2h$ (where for the natural indices we have $i < j$ and $d$ is the hyperbolic distance function).

In [28] we described a procedure to get a decomposition of the 3-dimensional hyperbolic space $\mathbb{H}^3$ into truncated tetrahedra corresponding to a given saturated hyperball packing whose main steps were the following:

1. Using the radical planes of the hyperballs $\mathcal{H}_i^h$, similarly to the Euclidean space, we can construct the unique Dirichlet-Voronoi (in short D-V) decomposition of $\mathbb{H}^3$ to the given hyperball packing $B^h$.
2. We consider an arbitrary proper vertex $P \in \mathbb{H}^3$ of the above $D - V$ decomposition and the hyperballs $\mathcal{H}_i^h(P)$ whose D-V cells meet at $P$. The base planes of the hyperballs $\mathcal{H}_i^h(P)$ are denoted by $\beta_i(P)$, and these planes determine a non-compact polyhedron $\mathcal{D}_i(P)$ with the intersection of their halfspaces containing the vertex $P$. Moreover, denote with $A_1, A_2, A_3, \ldots$ the outer vertices of $\mathcal{D}_i(P)$ and cut off $\mathcal{D}_i(P)$ with the polar planes $\alpha_j(P)$ of its outer vertices $A_j$. Thus, we obtain a convex compact polyhedron $\mathcal{D}(P)$. This is bounded by the base planes $\beta_i(P)$ and “polar planes” $\alpha_j(P)$. Applying this procedure to all vertices of the above Dirichlet-Voronoi decomposition, we obtain an other decomposition of $\mathbb{H}^3$ into convex polyhedra.
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3. We consider $D(P)$ as a tile of the above decomposition. The planes from the finite set of base planes $\{\beta_i(P)\}$ are called adjacent if there is a vertex $A_k$ of $D^i(P)$ that lies on each of the above planes. We consider non-adjacent planes $\beta_{k_1}(P), \beta_{k_2}(P), \beta_{k_3}(P), \ldots \beta_{k_m}(P) \in \{\beta_i(P)\}$ ($k_l \in \mathbb{N}^+$, $l = 1, 2, 3, \ldots$) that have an outer point of intersection denoted by $A_{k_1 \ldots k_m}$. Let $N_{D(P)} \in \mathbb{N}$ denote the finite number of the outer points $A_{k_1 \ldots k_m}$ related to $D(P)$. It is clear, that its minimum is 0 if $D^i(P)$ is tetrahedron. The polar plane $\alpha_{k_1 \ldots k_m}$ of $A_{k_1 \ldots k_m}$ is orthogonal to the planes $\beta_{k_1}(P), \beta_{k_2}(P), \ldots \beta_{k_m}(P)$ (thus it contains their poles $B_{k_1}, B_{k_2}, \ldots B_{k_m}$) and divides $D(P)$ into two convex polyhedra $D_1(P)$ and $D_2(P)$.

4. If $N_{D_1(P)} \neq 0$ and $N_{D_2(P)} \neq 0$ then $N_{D_1(P)} < N_{D(P)}$ and $N_{D_2(P)} < N_{D(P)}$ then we apply point 3 to the polyhedra $D_i(P)$, $i \in \{1, 2\}$.

5. If $N_{D_1(P)} \neq 0$ or $N_{D_2(P)} = 0$ ($i \neq j$, $i, j \in \{1, 2\}$) then we consider the polyhedron $D_i(P)$ where $N_{D_1(P)} = N_{D(P)} - 1$ because the vertex $A_{k_1 \ldots k_m}$ is left out and apply point 3.

6. If $N_{D_1(P)} = 0$ and $N_{D_2(P)} = 0$ then the procedure is over for $D(P)$. We continue the procedure with the next cell.

7. We have seen in steps 3, 4, 5 and 6 that the number of the outer vertices $A_{k_1 \ldots k_m}$ of any polyhedron obtained after the cutting process is less than the original one, and we have proven in step 7 that the original hyperballs form packings in the new polyhedra $D_1(P)$ and $D_2(P)$, as well. We continue the cutting procedure described in step 3 for both polyhedra $D_1(P)$ and $D_2(P)$. If a derived polyhedron is a truncated tetrahedron then the cutting procedure does not give new polyhedra, thus the procedure will not be continued. Finally, after a finite number of cuttings we get a decomposition of $D(P)$ into truncated tetrahedra, and in any truncated tetrahedron the corresponding congruent hyperballs from $\{H^n_i\}$ form a packing. Moreover, we apply the above method to the other cells.

From the above algorithm we obtained the following

Theorem 2.2. (J. Sz. [28]) The algorithm described in [28] provides for each congruent saturated hyperball packing a decomposition of $\mathbb{H}^3$ into truncated tetrahedra. □

Remark 2.3. Przeworski, A. proved a similar theorem in [21] but it was true only for cases where the base planes of hyperspheres form “symmetric cocompacts arrangements” in $\mathbb{H}^n$.

In [18] Y. Miyamoto proved the analogue theorem of K. Böröczky’s theorem (Theorem 1.1):

Theorem 2.4. (Y. Miyamoto, [18]) If a region in $\mathbb{H}^n$ bounded by hyperplanes has a hyperball (hypersphere) packing of height (radius) $r$ about its boundary,
then in some sense, the ratio of its volume to the volume of its boundary is at least that of a regular truncated simplex of (inner) edgelength $2r$.

**Remark 2.5.** Independently from the above paper A. Przeworski proved a similar theorem with other methods in [21].

Therefore, in order to get density upper bound related to the saturated hyperball packings it is sufficient to determine the density upper bound of hyperball packings in truncated regular simplices (see Fig. 1).

Thus, in the following we assume that the ultraparallel base planes $\beta_i$ of $H_i^{h(p)}$ ($i = 1, 2, 3, 4$, and $6 < p \in \mathbb{R}$) generate a “regular truncated tetrahedron” $S(p)$ with outer vertices $B_i$ (see Fig. 1) whose non-orthogonal dihedral angles are equal to $\frac{2\pi}{p}$, and the distances between two base planes $d(\beta_i, \beta_j) =: e_{ij}$ ($i < j \in \{1, 2, 3, 4\}$) are equal to $2h(p)$ depending on the angle $\frac{\pi}{p}$.

The truncated regular tetrahedron $S(p)$ can be decomposed into 24 congruent simply truncated orthoschemes; one of them $O = Q_0Q_1Q_2P_0P_1P_2$ is illustrated in Fig. 1 where $P_0$ is the centre of the “regular tetrahedron” $S(p)$, $P_1$ is the centre of a hexagonal face of $S(p)$, $P_2$ is the midpoint of a “common perpendicular” edge of this face, $Q_0$ is the centre of an adjacent regular triangle face of $S(p)$, $Q_1$ is the midpoint of an appropriate edge of this face and one of its endpoints is $Q_2$.

In our case the essential dihedral angles of orthoschemes $O$ are the following: $\alpha_{01} = \frac{\pi}{p}$, $\alpha_{12} = \frac{\pi}{3}$, $\alpha_{23} = \frac{\pi}{3}$. Therefore, the volume $\text{Vol}(O)$ of the orthoscheme $O$ and the volume $\text{Vol}(S(p)) = 24 \cdot \text{Vol}(O)$ can be computed for any given parameter $p$ ($6 < p \in \mathbb{R}$) by Theorem 2.6 of R. Kellerhals [11] (extending the brilliant formula of N. I. Lobachevsky [16] to classical orthoschemes):
Theorem 2.6. (R. Kellerhals, [11], Theorem II.) The volume of a three-dimensional hyperbolic complete orthoscheme (except for Lambert cube cases, i.e. complete orthoschemes of degree $m = 2$ with outer edge) $O \subset \mathbb{H}^3$ is expressed with the essential angles $\alpha_{01}, \alpha_{12}, \alpha_{23}$, $(0 \leq \alpha_{ij} \leq \frac{\pi}{2})$ in the following form:

$$\text{Vol}(O) = \frac{1}{4}\{\mathcal{L}(\alpha_{01} + \theta) - \mathcal{L}(\alpha_{01} - \theta) + \mathcal{L}(\frac{\pi}{2} + \alpha_{12} - \theta) + \mathcal{L}(\frac{\pi}{2} - \alpha_{12} - \theta) + \mathcal{L}(\alpha_{23} + \theta) - \mathcal{L}(\alpha_{23} - \theta) + 2\mathcal{L}(\frac{\pi}{2} - \theta)\},$$

where $\theta \in [0, \frac{\pi}{2})$ is defined by:

$$\tan(\theta) = \frac{\sqrt{\cos^2 \alpha_{12} - \sin^2 \alpha_{01} \sin^2 \alpha_{23}}}{\cos \alpha_{01} \cos \alpha_{23}},$$

and where $\mathcal{L}(x) := -\int_0^x \log |2 \sin t| \, dt$ denotes the Lobachevsky function.

In this case for a given parameter $p$ the length of the common perpendiculars $h(p) = \frac{1}{2}e_{ij}$ ($i < j$, $i, j \in \{1, 2, 3, 4\}$) can be determined by the machinery of projective metric geometry. (In the following $\mathbf{x} \sim c \cdot \mathbf{x}$ with $c \in \mathbb{R}\{0\}$ represents the same point $X = (\mathbf{x} \sim c \cdot \mathbf{x})$ of $\mathbb{P}^3$.)

The points $P_2(p_2)$ and $Q_2(q_2)$ are proper points of the hyperbolic 3-space and $Q_2$ lies on the polar hyperplane $\text{pol}(B_1)(\mathbf{b}^1)$ of the outer point $B_1$.

Thus the hyperbolic distance $h(p)$ can be calculated by the following formula (see [25]):

$$\cosh h(p) = \cosh P_2Q_2 = -\frac{\langle q_2, p_2 \rangle}{\sqrt{\langle q_2, q_2 \rangle \langle p_2, p_2 \rangle}} = \frac{h_{23}^2 - h_{22}h_{33}}{\sqrt{h_{22} \langle q_2, q_2 \rangle}} = \sqrt{\frac{h_{22} h_{33} - h_{23}^2}{h_{22} h_{33}}},$$

where $h_{ij}$ is the inverse of the Coxeter-Schläfi matrix

$$(e^{ij}) := \begin{pmatrix}
1 & -\cos \frac{\pi}{p} & 0 & 0 \\
-\cos \frac{\pi}{p} & 1 & -\cos \frac{\pi}{3} & 0 \\
0 & -\cos \frac{\pi}{3} & 1 & -\cos \frac{\pi}{3} \\
0 & 0 & -\cos \frac{\pi}{3} & 1
\end{pmatrix} \quad (2.2)$$

of the orthoscheme $O$. We get that the volume $\text{Vol}(S(p))$, the maximal height $h(p)$ of the congruent hyperballs lying in $S(p)$ and $\text{Vol}(H^h(p) \cap S(p))$ all depend only on the parameter $p$ of the truncated regular tetrahedron $S(p)$.

Therefore, the locally optimal density of the congruent hyperball packing related to the regular truncated tetrahedron of parameter $p$ is

$$\delta(S(p)) := \frac{4 \cdot \text{Vol}(H^h(p) \cap S(p))}{\text{Vol}(S(p))},$$
Figure 2. The density function \( \delta(S(p)), \ p \in (6, 10) \)

and \( \delta(S(p)) \) depends only on \( p \ (6 < p \in \mathbb{R}) \). Moreover, the total volume of the parts of the four hyperballs lying in \( S(p) \) can be computed by formula (2.1), and the volume of \( S(p) \) can be determined by Theorem 2.6.

Finally, we obtain the plot after careful analysis of the smooth density function (cf. Fig. 2) and we obtain the following

**Theorem 2.7.** (J. Sz. [25]) The density function \( \delta(S(p)), \ p \in (6, \infty) \) attains its maximum at \( p_{opt} \approx 6.13499 \), and \( \delta(S(p)) \) is strictly increasing in the interval \((6, p_{opt})\), and strictly decreasing in \((p_{opt}, \infty)\). Moreover, the optimal density \( \delta_{opt}(S(p_{opt})) \approx 0.86338 \) (see Fig. 2).

**Remark 2.8.**
1. In our case \( \lim_{p \to 6} (\delta(S(p))) \) is equal to the Böröczky-Florian upper bound of the ball and horoball packings in \( \mathbb{H}^3 \) [4] (observe that the dihedral angles of \( S(p) \) for the case of the horoball equal \( 2\pi/6 \)).
2. \( \delta_{opt}(S(p_{opt})) \approx 0.86338 \) is larger than the Böröczky-Florian upper bound \( \delta_{BF} \approx 0.85328 \); but these hyperball packing configurations are only locally optimal and cannot be extended to the entire hyperbolic space \( \mathbb{H}^3 \).

We obtain the next theorem as a direct consequence of the previous statements:

**Theorem 2.9.** The density upper bound of the saturated congruent hyperball packings related to the corresponding truncated tetrahedron cells is realized in a regular truncated tetrahedra belonging to parameter \( p_{opt} \approx 6.13499 \) with density \( \approx 0.86338 \).

We get from the above theorem directly the denial of A. Przeworski’s conjecture [21]:
Corollary 2.10. The density function $\delta(S(p))$, is not an increasing function of $h(p)$ (the height of hyperballs).

Remark 2.11. The hyperball packings in the regular truncated tetrahedra under the extended reflection groups with Coxeter-Schläfli symbol $\{3,3,p\}$, investigated in paper [25], can be extended to the entire hyperbolic space if $p$ is an integer parameter bigger than 6. They coincide with the hyperball packings given by the regular $p$-gonal prism tilings in $\mathbb{H}^3$ with extended Coxeter-Schläfli symbols $\{p,3,3\}$, see in [32]. As we know, $\{3,3,p\}$ and $\{p,3,3\}$ are dually isomorphic extended reflection groups, just with the above frustum of orthoscheme as fundamental domain (Fig. 1, matrix $(c^{ij})$ in formula (2.2)).

In [25] we studied these tilings and the corresponding hyperball packings. Moreover, we computed their metric data for some integer parameters $p$ ($6 < p \in \mathbb{N}$), where $\mathcal{A}$ is a trigonal face of the regular truncated tetrahedron, cf. Fig. 1. In Table 1 we recalled from [25] important metric data of some “realizable hyperball packings”.

In hyperbolic spaces $\mathbb{H}^n$ ($n \geq 3$) the problems of the densest ball, horoball and hyperball packings have not been settled yet, in general (see e.g. [10,14,23,24,29–31]). Moreover, the optimal sphere packing problem can be extended to other homogeneous Thurston geometries, e.g. $\text{Nil}$, $\text{Sol}$, $\text{SL}_2 \mathbb{R}$. For these non-Euclidean geometries only very few results are known (e.g. [19,36,38] and the references given there).
By the above we can say that the revisited Kepler problem still holds several interesting open questions.

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