On the extremal graphs in generalized Turán problems

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Abstract

Given two graphs $H$ and $F$, the generalized Turán number $\text{ex}(n, H, F)$ is the largest number of copies of $H$ in an $n$-vertex $F$-free graph. For every $F$ and sufficiently large $n$, we present an extremal graph for a generalized Turán problem, i.e., an $F$-free $n$-vertex graph $G$ that for some $H$ contains exactly $\text{ex}(n, H, F)$ copies of $H$.

1 Introduction

In ordinary Turán problems, we are interested in $\text{ex}(n, F)$, which is the largest number of edges in $n$-vertex graphs that do not contain $F$ as a subgraph. Turán’s theorem [18] states that $\text{ex}(n, K_{r+1}) = |E(T(n, r))|$, where $T(n, r)$ is the $r$-partite Turán graph, which is the complete $r$-partite graph with each part having order either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. The Erdős-Stone-Simonovits theorem [5, 6] states that for any graph $F$ with $\chi(F) = r + 1$ we have $\text{ex}(n, F) = |E(T(n, r))| + o(n^2)$. This gives the asymptotics of $\text{ex}(n, F)$ if $r > 1$, but only gives the bound $o(n^2)$ for bipartite $F$. For many bipartite graphs $F$, we do not even know the order of magnitude of $\text{ex}(n, F)$, see [7] for a survey.

In generalized Turán problems, we are interested in $\text{ex}(n, H, F)$, which is the largest number of copies of $H$ in $n$-vertex graphs that do not contain $F$ as a subgraph. More formally, let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. We let $\text{ex}(n, H, F) = \max\{\mathcal{N}(H, G) : G$ is an $n$-vertex $F$-free graph}. After several sporadic results, the systematic study of generalized Turán problems was initiated by Alon and Shikhelman [1].

Given $H$ and $F$, the extremal graphs are $n$-vertex $F$-free graphs with $\text{ex}(n, H, F)$ copies of $H$. In this paper, for every graph $F$ with more than one edge, we show a graph $H$ and an extremal $n$-vertex graph $G$ for $\text{ex}(n, H, F)$ if $n$ is sufficiently large. More precisely, we show infinitely many graphs $H$ that do not contain $F$ and for each such $H$, for sufficiently large $n$ we give an $n$-vertex $F$-free graph $G$ with $\text{ex}(n, H, F) = \mathcal{N}(H, G)$.

Note that in the more precise formulation we avoided the trivial example: in the case $H$ contains $F$, all the $F$-free graphs are extremal.

Let $\chi(F) = r + 1$. We denote by $\sigma(F)$ the smallest color class that can appear in an $(r+1)$-coloring of $F$. Given graphs $G$ and $G'$, we denote by $G + G'$ the graph obtained by taking
vertex-disjoint copies of $G$ and $G'$ and adding the edges $uv$ for each $u \in V(G), v \in V(G')$. The following theorem constructs an extremal graph for non-bipartite $F$.

**Theorem 1.1.** If $F$ has chromatic number $r + 1 > 2$ and $\sigma(F) = s$, $H$ is the complete $r$-partite graph $K_{a_1, \ldots, a_r}$ with a sufficiently large $t$, then for sufficiently large $n$ we have $\text{ex}(n, H, F) = \mathcal{N}(H, K_{s-1} + T(n - s + 1, r))$.

Note that the case $s = 1$ was proved by the author [8].

Let us assume now that $F$ is bipartite. Let $K_{s,t}$ denote the graph we obtain from $K_{s,t}$ by adding all the edges inside the part of order $s$, i.e., $K_{s,t} = K_s + T(t, 1)$. If $F$ is not a star, then similar to Theorem 1.1, $K_{s-1} + T(n - s + 1, 1)$ is an extremal graph. Clearly this graph is $F$-free. On the other hand, $F$ is contained in $K_{s,t}$ for $t = |V(F)| - s$. It was shown by Gerbner and Patkós [13] that $\text{ex}(n, K_{1,q}, K_{1,t}) = \mathcal{N}(K_{1,q}, K_{s-1} + T(n - s + 1, 1))$ if $q \geq 2t - 1$. This implies that $\text{ex}(n, K_{1,q}, F) = \mathcal{N}(K_{1,q}, K_{s-1} + T(n - s + 1, 1))$.

Finally, if $F$ is a star $K_{1,t}$, let $G$ be the vertex-disjoint union of $\lfloor n/t \rfloor$ copies of $K_t$ and a $K_t$ with $t' = n - t\lfloor n/t \rfloor$. A theorem of Chase [4] shows that $\text{ex}(n, K_k, K_{1,t}) = \mathcal{N}(K_k, G)$ for any $k \geq 2$, giving us an extremal graph if $t > 1$. We also have a simple observation of Cambie, de Verclos and Kang [3], which states that for any tree $T$, $\text{ex}(n, T, K_{1,t}) = \mathcal{N}(T, G)$ for every $n$-vertex graph $G$ that have girth more than $|V(T)|$ and at least $n - 1$ vertices of degree $t - 1$, where the last vertex has degree $t - 1$ or $t - 2$. This gives another extremal graph if $t > 2$.

**2 Proofs**

We will use a theorem of the author [11]. Given graphs $H$ and $F$ with $\chi(H) < \chi(F)$, we say that $H$ is weakly $F$-Turán-stable if the following holds. Any $F$-free $n$-vertex graph $G$ with $\mathcal{N}(H, G) = \text{ex}(n, H, F) - o(n^{V(H)})$ can be turned into a complete $r$-partite graph by adding and removing $o(n^2)$ edges. Results from [3] imply that $K_{a_1, \ldots, a_r}$ is $F$-Turán-stable. Given a graph $F$ with chromatic number $r + 1$, $\mathcal{D}(F)$ denotes the set of bipartite graphs that appear as two chromatic classes in an $(r + 1)$-coloring of $F$.

**Theorem 2.1.** Let $r + 1 = \chi(F) > \chi(H)$ and assume that $H$ is weakly $F$-Turán-stable. Then for every $n$-vertex $F$-free graph $G$ with $\mathcal{N}(H, G) = \text{ex}(n, H, F)$ there is an $r$-partition of $V(G)$ to $A_1, \ldots, A_r$, a constant $K = K(F)$ and a set $B$ of at most $rK(\sigma(F) - 1)$ vertices such that each member of $\mathcal{D}(F)$ inside the parts shares at least two vertices with $B$, every vertex of $B$ is adjacent to $\Omega(n)$ vertices in each part, every vertex of $A_i \setminus B$ is adjacent to $o(n)$ vertices in $A_i$ and all but $o(n)$ vertices in $A_j$ with $j \neq i$.

We will use some results of Gerbner and Patkós [13].

**Theorem 2.2** (Gerbner and Patkós [13]). Let $p + 1 < s \leq t < q$ and $n$ is large enough. If $p + q \geq s + t$ or $p = 1$ and $q$ is large enough, then $\text{ex}(n, K_{p,q}, K_{s,t}) = \mathcal{N}(K_{p,q}, K_{s-1,n-s+1})$ and every extremal $n$-vertex $K_{s,t}$-free graph contains $K_{s-1,n-s+1}$. Moreover, if an $n$-vertex $K_{s,t}$-free graph does not contain $K_{s-1,n-s+1}$, then it has at most $\text{ex}(n, K_{p,q}, K_{s,t}) - \Omega(n^{p-1})$ copies of $K_{p,q}$.
We will also need the following structural information about $K_{s,t}$-free graphs.

**Lemma 2.3.** Let $p < s \leq t$, $s < q$, $n$ large enough and $G$ be a $K_{s,t}$-free $n$-vertex graph and let $U$ be the set of vertices in $G$ with degree at most $\varepsilon n$. Then the number of copies of $K_{p,q}$ such that the smaller part contains a vertex of $U$ is at most $\varepsilon n^{p+q-1}$.

**Proof.** Consider first the vertices of degree at most $\varepsilon' n^{\frac{q-1}{s}}$ for some sufficiently small $\varepsilon' > 0$. There are at most $\varepsilon n^{q-1}/2$ ways to choose $q$ neighbors of them, thus there are at most $\varepsilon n^{p+q-1}/2$ copies of $K_{p,q}$ that contains such a vertex in the smaller part.

Let $U' = \{u_1, \ldots, u_{|U'|}\}$ denote the vertices in $U$ of degree more than $\varepsilon' n^{\frac{q-1}{s}}$. Consider now the copies of $K_{p,q}$ where the smaller part contains a vertex in $U'$ and call such a vertex the center of the $K_{p,q}$. Let $S_j$ denote the neighborhood of $u_j$. We consider the sets that are the intersections of $s-1$ sets $S_\ell$ with $\ell < j$, then there are $\binom{s-1}{\ell}$ such sets. Each such set shares at most $t-1$ vertices with $S_j$ because of the $K_{s,t}$-free property. Therefore, $S_j$ creates at most $(t-1)\binom{s-1}{\ell}$ vertices that are covered at least $s$ times. This implies that after the $j$th set, at most $\sum_{\ell=s}^{j-1}(t-1)\binom{s-1}{\ell} \leq (t-1)s^j$ vertices are covered at least $s$ times. At $j = 2sn^{\frac{1}{s}}/\varepsilon'$, at most $cn^\varepsilon$ vertices are covered at least $s$ times for some $c$ depending on $s$ and $\varepsilon'$. This shows that $\sum_{\ell=1}^{j}|S_\ell| \leq (s-1)n + jcn^{\frac{1}{s}}$. On the other hand, $\sum_{\ell=1}^{j}|S_\ell| \geq j\varepsilon' n^{\frac{q-1}{s}} \geq 2sn$, a contradiction if $n$ is large enough.

We obtained that $|U'| < 2sn^{\frac{1}{s}}/\varepsilon'$. Moreover, we have that $\sum_{\ell=1}^{j}|S_\ell| \leq (s-1)n + jcn^{\frac{1}{s}} \leq sn$. We will use this above property and that $0 \leq |S_\ell| \leq \varepsilon n$. By the convexity of the binomial function, $\sum_{\ell=1}^{j}\binom{|S_\ell|}{q}$ is maximal when we have some sets $S_j$ of order $\varepsilon n$ and the others of order 0, i.e., $\sum_{\ell=1}^{j}\binom{|S_\ell|}{q} \leq s(\varepsilon n)/\varepsilon \leq \varepsilon n^q/2$. Now we count the copies of $K_{p,q}$ that contain a vertex of $U'$. First we pick the center and $q$ of its neighbors, at most $\varepsilon n^{q}/2$ ways, then we pick the remaining vertices at most $n^{p-1}$ ways. This completes the proof.

We will use the following simple observation multiple times.

**Proposition 2.4.** Let $V(G) = A_1 \cup \ldots \cup A_r$ with $|A_i| \geq |V(F)|^2$. Assume that each vertex of $A_i$ is adjacent to all but at most $|A_j|/|V(F)|^2$ vertices in $A_j$ for each $j \neq i$. Let $F$ be an $(r+1)$-colorable graph such that there is a coloring with a color class of order $s$ and another class of order $t$. Assume that there is a $K_{s,t}$ inside $A_i$. Then there is an $F$ in $G$.

**Proof.** We will embed two color classes into $A_i$, and each other color class greedily into the other sets $A_j$. Each time, when we want to embed some vertices into $A_j$, the already embedded at most $|V(F)|-1$ vertices are adjacent to each but at most $(|V(F)|-1)|A_j|/|V(F)|$ vertices of $A_j$, thus there are enough vertices in their common neighborhood to pick the next color class.

Now we are ready to prove Theorem 2.1. It will be convenient for us to use the $O$ and $o$ notation in most of the proof, but use $\varepsilon$ in some more delicate situation.

**Proof.** Let $\delta = 1/|V(F)|$. We let $a$ be large enough compared to $\delta$. We pick $\varepsilon > 0$ small enough compared to $\delta, a$ and $F$. We will apply Theorem 2.1 such a way that every vertex of $A_i \setminus B$ is adjacent to at most $\varepsilon n$ vertices in $A_j$.
Let $G$ be an $n$-vertex $F$-free graph with $\mathcal{N}(H,G) = \text{ex}(n,H,F)$. We apply Theorem 2.1 to obtain vertex sets $A_1, \ldots, A_r$ such that $A_1$ is one of the smallest of these sets. It is easy to see that each part has order $\frac{n}{r} + o(n)$. Indeed, otherwise we lose $\Omega(n^{V(H)})$ copies of $H$ compared to the Turán graph, and the $o(n^2)$ extra edges inside the parts create $o(n^{V(H)})$ copies of $H$. We use this in the form that for each $i$, we have $\frac{n}{r} - \varepsilon n \leq |A_i| \leq \frac{n}{r} + \varepsilon n$. Let $t$ be the order of the second smallest part of $F$ in an $(r+1)$-coloring of $F$, then $A_i \setminus B$ is $K_{s,t}$-free.

Consider now the different types of copies of $H$. There are at most $\mathcal{N}(H,T(n,r))$ copies without any edge inside any $A_i$, $O(n^{ra-2})$ copies containing at least two vertices from $B$ and $O(n^{ra-1})$ copies containing one vertex from $B$.

The remaining copies of $H$ each contain an edge inside some $A_i \setminus B$. Observe that if a copy of $H$ does not contain an edge inside $A_j \setminus B$, then it contains at most $a$ vertices in $A_j \setminus B$. Therefore, there is an $i$ such that $H$ contains an edge and at least $a$ vertices in $A_i \setminus B$. Observe that $H$ intersects $A_i \setminus B$ in a complete multipartite graph $H'$ on, say, $m \geq a$ vertices. In particular, $H'$ contains a complete bipartite graph $K_{b,m-b}$ with $b \leq m - b$ and $b \leq a$. On the other hand, $A_i \setminus B$ is $K_{s,t}$-free and as $a \geq t$, we must have $b < s$. Then we have $\text{ex}(n,K_{b,m-b},K_{s,a}) = O(n^b) = O(n^{m-1})$ by a result from [12]. This implies that for each non-empty subgraph of $H$, there are $O(n^{ra-1})$ copies of $H$ that intersects $A_i \setminus B$ in that subgraph, thus there are $O(n^{ra-1})$ copies of $H$ that contain an edge inside $A_i \setminus B$.

We obtained that $\text{ex}(n,H,F) = \mathcal{N}(H,T(n,r)) + \Theta(n^{ra-1})$. To improve this bound, we will further divide some of the above types of copies of $H$. Let $x$ denote the number of copies of $H$ that intersect $A_i$ in a complete multipartite graph $H'$ on $m \geq a$ vertices with one part being a singleton vertex $v \in A_i \setminus B$. Again, we can just forget about the additional edges and consider the induced subgraph of $H'$ that is $K_{1,m-1}$. By Lemma 2.3, we have that $x = o(n^{V(H')-1})$.

Let $y$ denote the number of copies of $H$ that intersect $A_i$ in a complete multipartite graph $H'$ on $m \geq a$ vertices with one part being a singleton vertex $v \in A_i \cap B$, such that $H' \neq K_{1,a}$ and $H' \neq K_{1,a-1}$. We claim that $y = o(n^{ar-1})$. This statement is obvious if $H'$ contains another vertex of $B$ or an edge inside an $A_j$, since then we can pick $H'$ by picking $v$ and the described vertex or edge, and then any other vertex $n$ ways, to show that there are $o(n^{V(H')-1})$ such copies of $H'$. Otherwise, each part of $H'$ without $v$ is inside a part $A_j$, thus the intersection of $H'$ with $A_j \setminus B$ is an independent set on $a$ or $a-1$ vertices. Moreover, the intersection can have $a-1$ vertices for at most one $j$ (since only the center of $K_{1,a}$ can extend such a set to a part of $H$). Therefore, we must have $m \leq a + 1$, hence $H' = K_{1,a}$ or $K_{1,a-1}$, a contradiction.

Let $v_1, \ldots, v_{|B|}$ be the vertices of $B$ in decreasing order of their degrees.

Claim 2.5. We have that $|B| \geq s - 1$. Moreover, for $j \leq s - 1$, $v_j$ has at least $(1 - \delta)|A_i|$ neighbors in each $A_i$.

Proof of Claim. We first show that $\sum_{i=1}^{s} d(v_i) \leq sn - |A_i| + \varepsilon n/|V(F)|$. Indeed, otherwise these $s$ vertices have at least $\varepsilon n/|V(F)|$ common neighbors in each $A_i$. Then we can embed $F$ into $G$ greedily as in Proposition 2.4. We embed first a color class of $F$ of order $s$ into $v_1, \ldots, v_s$ and then we go through the sets $A_i$ one by one to embed the other color classes of
not decrease when we change $H$. The other less than $|V(F)|$ vertices already picked each avoid at most $\varepsilon n/|V(F)|$ neighbors, thus we have at least $\varepsilon n/|V(F)|$ common neighbors. This way we find a copy of $F$ in $G$, a contradiction. This implies that vertices $v_j$ with $j \geq s$ have degree at most $n - |A_1|/s + \varepsilon n/s|V(F)|$, thus they have at most $|A_t|-|A_i|/sr + 2\varepsilon n$ neighbors in their part $A_i$.

Let $z(v)$ denote the number of copies of $H$ that intersect $A_i$ in a $K_{1,a}$ or $K_{1,a-1}$ where $v \in B$ is the center.

By the above part of the proof, there are $o(n^{ar-1})$ other copies of $H$ using at least one edge inside parts. Let $v$ be of degree $n - 1$ in $G$, then

$$z(v) = (1+o(1)) \left((|A_1|-1)/a\right) N(H_0, T(n - |A_i|, r - 1)) + \left((|A_1|-1)/a-1\right) N(H_1, T(n - |A_i|, r - 1)),$$

where $H_0$ is the complete $(r-1)$-partite graph $K_{a-1,a,...,a}$ and $H_1$ is the complete $r$-partite graph $K_{1,a-1,a,...,a}$. Let $v' \in B$ a vertex with at most $(1 - \delta)|A_i|$ neighbors in $A_i$, then

$$z(v') \leq (1+o(1)) \left((|1-\delta|A_i|)/a\right) N(H_0, T((r-1)n/r, r-1)) + \left((1-\delta)|A_i|)/a-1\right) N(H_1, T((r-1)n/r, r-1)).$$

If $a$ is large enough compared to $\delta$, then $z(v)/z(v')$ can be arbitrarily large. In particular, if $v' = v_j$ with $j \geq s$, then $z(v)/z(v') \geq rK(s-1) \geq |B|$.

This implies that if $|B| < s - 1$, then we have at most $N(H, T(n, r)) + (s - 2)z(v) + |B|z(v') + o(n^{ar-1}) < N(H, K_{s-1} + T(n - s + 1, r))$ copies of $H$, a contradiction. Thus we have $|B| \geq s - 1$. For $s \leq s - 1$, we have that $v_j$ has more than $(1 - \delta)|A_i|$ neighbors in $A_i$, otherwise $\sum_{t=1}^{|B|} z(v_t) \leq (s - 2)z(v) + |B|z(v') < (1+o(1))(s-1)z(v)$. 

The above claim implies that there is no $K_{s,t}$ in $B \cup A_i$ using Proposition 2.4. Moreover, if $|B \cap A_i| = s - s_i$, then there is no $K_{s_i,t}$ in $A_i$ by the same reasoning.

Let $B' = \{v_1, ..., v_{s-1}\}$. Let us delete now the edges inside each $A_i \setminus B'$ and add all the edges between parts and all the edges incident to $B'$ to obtain the graph $G'$, which is isomorphic to $K_{s-1} + T$ for some complete $r$-partite graph $T$. Let $H_0$ denote the set of copies of $H$ in $G$ that contain only edges between parts. Let $H_1$ be the set of copies of $H$ in $G$ that intersect an $A_i$ in a star $K_{1,a}$ and contain only edges between parts otherwise. Let $H_2$ be the set of copies of $H$ in $G$ that intersect an $A_i$ in a $K_{2,a}$ or $K_{1,1,a}$ or $K_{1,1,a-1}$ or intersect both $A_i$ and $A_j$ in $K_{1,a}$ or $K_{1,a-1}$ for some $i \neq j$ such that the 1-element parts and the 2-element part is in $B$, and the copy of $H$ contains only edges between parts otherwise. Let $H_3$ be the set of other copies of $H$ in $G$. Let $H'_i$ denote the set of copies of $H$ in $G'$ defined analogously to $H_i$.

Clearly we have $|H_0| \leq |H'_0|$. By Theorem 2.2, the number of copies of $K_{1,a}$ in $G$ is at most $N(K_{1,a}, K_{s-1, |A_i| \cup B| - s_i + 1})$. This clearly implies that the number of such copies of $K_{1,a}$ does not decrease when we change $G$ to $G'$. Observe that for each such $K_{1,a}$, the remaining part of $H$ contains only edges between parts and we added all the missing such edges when created $G'$, thus the number of ways to extend such stars also does not decrease, showing that $|H_1| < |H'_1|$. We also have $|H_2| < |H'_2|$ since we can pick the vertices in $B$ (the same in $G$ and $G'$) and then the remaining vertices can be picked from their neighborhoods in $A_i$ or
$A_j$. We did not delete any edges incident to vertices of $B$, thus the number of ways to pick such vertices does not decrease.

**Claim 2.6.** $|\mathcal{H}_3| \leq \varepsilon cn^{ar-2}$ for some constant $c = c(a, r)$.

**Proof of Claim.** The copies of $H$ in $\mathcal{H}_3$ that contain edges from only one part $A_i$ have the intersection containing $K_{p,a}$ for some $p \leq s$. As we deal with $\mathcal{H}_3$ now, we have that $p \geq 3$. Clearly there are $O(n^{ar-3})$ such copies of $H$ where each vertex in the part of order $p$ is in $B$. There are at most $\varepsilon n^{p+a-2}$ copies of $K_{p,a}$ inside $A_i$ with a center outside $B$ by Lemma 2.3. This is multiplied by at most $2^{a+2p}$ for picking the intersection (a graph containing $K_{p,a}$), by $s$ for picking $p$, by $r$ for picking $i$, and by $n^{ar-a-p}$ for picking the other vertices, thus there are at most $\varepsilon rsa^{2a+2p}$ such copies of $H$.

Consider now the copies of $H$ in $\mathcal{H}_3$ that contain edges inside exactly two parts $A_i$ and $A_j$. Let $H_i$ denote the intersection of $H$ with $A_i$ and $H_j$ denote the intersection of $H$ with $A_j$. Recall that there are at most $a$ vertices of $H$ in each other part, thus there are at least $2a$ vertices of $H$ in $A_i \cup A_j$. As $A_i$ and $A_j$ avoid $K_{s,t}$, each of them contains at most $(r-1)(s-1) + a$ vertices of any $H$. Therefore, each of them contains at least $a - (r-1)(s-1) \geq 2t + 1$ vertices of $H$. In particular, both $H_i$ and $H_j$ contain a complete bipartite graph with larger part of order more than $t$ (thus smaller part less than $s$). By Theorem 2.2, there are $O(n^{V(H_i)|-1})$ copies of $H_i$ inside $A_i$ and $O(n^{V(H_j)|-1})$ copies of $H_j$ inside $A_j$.

If $H_i$ or $H_j$ contains a smaller part of order more than 1, or contains more than two parts, then there are $O(n^{ar-2})$ copies of $H_i$ inside $A_i$ (or the same holds for $H_j$), thus there are $O(n^{ar-3})$ such copies of $H$.

If $H_i$ and $H_j$ are both stars and the center of $H_i$ is in $A_i \setminus B$, then Lemma 2.3 shows that there are at most $\varepsilon(n^{V(H_i)|-1})$ copies of $H_i$ in $A_i$. The analogous statement holds for $H_j$, this gives at most $2r(r-1)a^2$ copies of $H$, where we get additional factors for choosing $i$, $j$, which one or both contain a center outside $B$, and how many leaves they contain.

If $H_i$ and $H_j$ are both stars with center in $B$, at least one of them has at most $a - 2$ leaves because this copy of $H$ is not in $\mathcal{H}_2$. Recall that this copy of $H$ contains $r - 2$ independent sets in addition to $H_1$ and $H_2$. This is clearly impossible.

Finally, there are at most $O(n^{ar-3})$ copies of $H$ that contain edges in at least three parts.

Let us return to the proof of the theorem. Assume that $G[A_i \cup B]$ is not isomorphic to $\overline{K}_{s-1,|A_i \cup B|^{-s+1}}$. Recall that we have shown $|\mathcal{H}_1| \leq |\mathcal{H}_1|$. Now we use the moreover part of Theorem 2.2. It gives that the number of copies of $K_{1,a}$ is at most $\mathcal{N}(K_{1,a}, \overline{K}_{s-1,|A_i \cup B|^{-s+1}}) - c_0|A_i \cup B|^{a-1}$ for some $c_0$ that depends on $a$, $s$ and $t$. This implies that $|\mathcal{H}_1| \leq |\mathcal{H}_1'| - c_1 n^{ar-2}$ for some $c_1$ that depends on $a$, $s$, $t$ and $r$. As $|\mathcal{H}_0| \leq |\mathcal{H}_0'|$, $|\mathcal{H}_2| \leq |\mathcal{H}_2'|$ and $|\mathcal{H}_3| \leq c' \varepsilon n^{ar-2}$, we obtain that there are less than $|\mathcal{H}_0'| + |\mathcal{H}_1'| + |\mathcal{H}_2'|$ copies of $H$ in $G$, a contradiction.

We obtained that each vertex of $B'$ has degree $n - 1$. Now assume that a vertex $z \notin B'$ has at least $t$ neighbors in its part $A_i$ in $G$. Then with the vertices of $B'$ and its neighbors in $A_i$ they form a $K_{s,t}$ inside $A_i$, thus we can use Proposition 2.4 to find an $F$ in $G$, a contradiction.
Assume now that a copy of $H$ in $G$ contains an edge not in $G'$, i.e., an edge inside a part that does not contain a vertex from $B'$. Clearly, $H$ contains at most $a$ vertices from each $A_i \setminus B$ if $H$ does not contain an edge inside $A_i$. Therefore, $H$ contains at least $a$ vertices of $A_j \cup B$ for some $j$ such that $H$ contains an edge inside $A_j \setminus B$. It implies that $H$ contains at least $a - s > r(t + s)$ vertices of $A_j$. Recall that $H$ intersects $A_j \setminus B$ in a complete $r'$-partite graph $H'$ with $2 \leq r' \leq r$. Then there is a part of $H'$ of order more than $t + s$ in it, thus vertices in the other parts of $H'$ each have more than $t$ neighbors inside their part $A_j$, a contradiction.

We obtained that $G$ is contained in $G'$ which is of the form $K_{s-1} + T$ for some complete $r$-partite graph on $n - s + 1$ vertices, and the other edges do not create further copies of $H$. It is left to show that $T$ is the Turán graph. Assume not, i.e., without loss of generality $|A_2 \setminus B'| > |A_1 \setminus B'|$. Then we move $[(|A_2| - |A_1|)/2]$ vertices from $A_2$ to $A_1$ in $G'$, to obtain $G''$. It is well-known and easy to see that the number of edges increases this way. We will use a result of Ma and Qiu [16] that determines when the $m$-vertex complete bipartite graph with the most copies of $K_{p,q}$ is $T(m, 2)$ (an equivalent statement was proved by Brown and Sidorekno [2] in a different setting earlier). The result implies that for $a$ large enough, this holds for $K_{a,a}$, $K_{a-1,a}$, $K_{a-1,a-1}$ and $K_{a-2,a}$. For $K_{a,a}$ and $K_{a-1,a}$, this result also appears in a paper [15] by Győri, Pach and Simonovits.

We claim that the number of copies of $K_{a,a}$ inside $(A_1 \cup A_2) \setminus B'$ increases by $\Omega(n^{2a-2})$. Indeed, for each edge, we take $K_{a-1,a-1}$ from the remaining part of $(A_1 \cup A_2) \setminus B'$. Let $x$ (resp. $x'$) denote the number of copies of $K_{a-1,a-1}$ inside the remaining part of $(A_1 \cup A_2) \setminus B'$ in $G''$ (resp $G''$). Clearly we have $x' \geq x = \Omega(n^{2a-2})$. As each $K_{a,a}$ is counted $a^2$ ways as the number of edges increases by at least 1, we prove our claim.

We claim that for the following complete $r$-partite graphs: $K = K_{a-1,a,a,\ldots,a}$, $K' = K_{a-2,a,a,\ldots,a}$ and $K'' = K_{a-1,a-1,a,\ldots,a}$, the number of copies of them avoiding $B'$ does not decrease. Observe that each copy of $K$, $K'$ and $K''$ intersects $A_1 \cup A_2 \setminus B'$ in one of $K_{a,a}$, $K_{a-1,a}$, $K_{a-1,a-1}$ and $K_{a-2,a}$. The number of copies of those bipartite graphs does not decrease, and the number of ways to extend them does not change, proving our claim.

The number of copies of $H$ containing exactly one vertex from $B'$ is $(s-1)\mathcal{N}(K, T)$, and the number of copies of $H$ containing exactly two vertices from $B'$ is $(s-2)\mathcal{N}(K', T) + \mathcal{N}(K'', T)$, thus does not decrease. The number of copies of $H$ containing at least three vertices from $B'$ is $O(n^{r-3})$, thus $\mathcal{N}(H, G') > \mathcal{N}(H, G) = \mathcal{N}(H, G)$, a contradiction completing the proof. 

3 Concluding remarks

We have shown for any $F$ an extremal graph $G$ in a generalized Turán problem $\text{ex}(n, H, F)$. For any other graphs $H'$, $G$ can be the first candidate to be an extremal graph. However, we show that for each $F \neq K_{1,2}$ there are more than one extremal graphs.

If $F$ has chromatic number $r + 1 > 2$, one possibility is that for some less balanced complete $r$-partite graph $T$ on $n - s + 1$ vertices, $K_{s-1} + T$ contains more copies of $H'$ than $K_{s-1} + T(n - s + 1, r)$. It is the case if $H'$ itself is not balanced, e.g., $H = K_{1,a}$ with a large
enough. However, $K_{s-1} + T$ is not necessarily extremal either. The author [10] showed a graph $H'$ for every $F$ where complete $r$-partite graphs are not even asymptotically optimal. As $\mathcal{N}(H', T) = (1 + o(1))\mathcal{N}(H, K_{s-1} + T)$, we obtain that $K_{s-1} + T$ is not extremal for $H'$.

If $\mathcal{D}(F)$ does not contain a forest, then we can add superlinear many edges to a part of the Turán graph without creating $F$, thus $K_{s-1} + T$ is not even asymptotically optimal. Finally, $K_{s-1} + T$ does not contain $K_{s+r}$, thus not an extremal graph for $\text{ex}(n, K_{s+r}, F)$ (observe that $K_{s+r}$ does not contain $F$, since $F$ has at least $sr$ vertices).

If $F$ is bipartite but not a star, similarly we have that $K_{s-1} + T(n - s + 1, 1)$ does not contain $K_{s+1}$, which does not contain $F$. Finally, for stars we gave two constructions that are different unless $F = K_{1,2}$, in which case every $F$-free graph is a matching, thus the only extremal graph is a maximal matching.

Gerbner and Palmer [13] gave the following definition. A graph $H$ is $F$-Turán-good if $\text{ex}(n, H, F) = \mathcal{N}(H, T(n, \chi(F) - 1))$ for sufficiently large $n$. It was shown by the author [8] that for a given $H$, there is an $F$ such that $H$ is $F$-Turán-good if and only if $\sigma(H) = 1$. In light of our Theorem 1.1, it would have been better to introduce a name for graphs with $\text{ex}(n, H, F) = \mathcal{N}(H, K_{\sigma(F)} + T(n - \sigma(F) + 1, \chi(F) - 1))$ for sufficiently large $n$.

It is a natural question whether we can construct an extremal graph for any $H$ and some $F$. One can immediately see that it is less useful, as there is no reason to expect it to be an extremal graph for any $F' \neq F$. Nevertheless, we can answer this question using a recent result of Morrison, Nir, Norin, Rzążewski and Wesolek [17]. They (resolving a conjecture from [13]) showed that for any graph $H$, if $r$ is large enough, then $H$ is $K_{r+1}$-Turán-good. This gives infinitely many extremal graphs for any $H$.

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