From emission to inertial coordinates: a numerical approach

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Abstract. We numerically implement the coordinate transformation between emission and inertial coordinates recently derived— in Minkowski space-time— by Coll et al. [1]. In order to carry out this transformation, we must know both proper times from four satellites and the inertial coordinates of these satellites when they emitted. It allows us to determine the position and the time coordinate of the receiver. Since we need to know the satellite positions at any time, we have simulated satellite constellations similar to those of GPS and Galileo global positioning system. All the satellite positions are calculated, at any time, with respect to an inertial frame with the origin at the Earth center. The orientation criterion described by the same authors [2] has been also implemented. The code has been appropriately tested and, then it has been applied to perform a preliminary study about the structure of the emission coordinate region and the grid region. The errors due to uncertainties in satellite positions have been also estimated.

1. Introduction and generalities
Nowadays Global Navigation Satellite Systems are: GPS in USA, Glonass in Russia, and Galileo (under construction) in the EU. They provide the spatial coordinates and the universal time of any event on the Earth. Position coordinates are calculated thanks to information received from satellites into orbit around the Earth. The GPS constellation of 24 satellites is arranged in six different orbital planes (four satellites per plane), each of them inclined 55 degrees with respect to the equator. To obtain exactly two orbits per day, the satellites are placed at an altitude of 20200 km. The nominal error in the resulting receiver positions is approximately 15 m, but their RMS can reach 100 m. The Galileo constellation is composed by 27 satellites (plus three spares), located in three equally spaced orbital planes (9 equally spaced satellites in each plane). These planes are inclined 56 degrees and the altitude of the circular orbits is 23222 km; thus, the orbital period is about 14 h. All these positioning systems are based on Newtonian physics, although relativistic post-Newtonian corrections may be performed if necessary.

Relativistic positioning systems are based on relativistic principles from the beginning. The proper times of four satellites (emission coordinates) are sent by means of electromagnetic signals to the receiver, whose inertial coordinates can be found—from the emission ones—by using fully relativistic equations. The general transformation from emission to inertial coordinates—in Minkowski space-time—was given in [1]; so any receiver able to detect the signals from the satellites may use this transformation to get its own space and time coordinates in special relativity. Of course, the formalism must be extended to general relativity to...
include gravitational fields, accelerated frames, and so on. Realistic 4D implementations of the transformation presented by Coll et al. in [1] require numerical calculations. We have designed and tested a numerical code to do these calculations. It is applied in next sections. The mentioned transformation uses the position of the four satellites when they emitted their respective proper times. We have simulated the motion of GPS and Galileo satellites in an inertial frame with origin at the Earth center; thus, the satellite positions may be calculated at any given time.

In paper [1], a system having a quadratic and three linear equations (hereafter called the main system) was solved to obtain the coordinate transformation between emission and inertial coordinates. This system can have up to two solutions. There are two kinds of solutions, one of them corresponds to emission signals travelling from the satellite to the receiver (hereafter emission solutions), whereas the second kind describes reception signals propagating in the opposite direction (reception solutions). We are only interested in emission solutions.

Let us now introduce some concepts and notations (extracted from [1] and [2]), which are necessary to understand the following sections. The parameter $\hat{\epsilon}$ appears in the final expression of the solution for the main system and it can take on the values +1 or -1. Vector $\chi$ is orthogonal to the hyperplane defined by the relative positions of the satellite pairs. It is called configuration vector and $\chi^2$ stands for its square in Minkowski space-time. For $\chi^2 > 0$, the system has two possible solutions, which correspond to two emission signals, i.e., two true physical observers at different positions. There are then criteria to distinguish between both solutions. The criterion proposed in [2] is as follows: consider a conical surface with the receiver at the vertex which contains three of the four satellites and then, take $\hat{\epsilon} = 1$ or $\hat{\epsilon} = -1$ depending upon whether the fourth satellite is inside or outside the cone, respectively. Finally, in the case $\chi^2 \leq 0$, the system of equations has an unique emission solution.

2. Grid Structure
Given an arbitrary grid point $(\tau^1, \tau^2, \tau^3, \tau^4)$, the question is: would these proper times be received in some point of Minkowski space? In other words, are there emission solutions of the main system? If affirmative, the points in Minkowski space (receivers) would belong to the so-called emission region and the chosen grid point to the co-region. Our aim in this section is to do a preliminary study about the grid and co-region structure. In order to do this study, we select a point in the grid where the main system has a known solution and, then, we choose quadruplets of proper times (grid points) covering a straight line, $L$, which contains the selected point. Finally, we study the existence of solutions, for the $L$-points, by using the coordinate transformation given in [1].

Results are shown in Fig. 1, where the chosen line is represented. The values of the $\lambda$ parameter are given in the horizontal axis ($\lambda = 0$ for the selected point). There are emission (reception) solutions in the segment marked with stars (triangles), whereas there is no solution in the zone with circles. The segment with stars belongs to the co-region. This result suggests a connected co-region surrounded by grid points where either there are reception solutions or there are no solutions at all (see [3] for a grid theoretical analysis). Many lines containing the selected point must be studied to cover the co-region, but a general study is out of the scope of this paper.

3. Emission region structure
The emission region is the zone of Minkowski space-time where four proper times (emission coordinates) may be received from the satellites. In order to perform a preliminary study of the emission region we proceed as follows: (1) a point on the Earth surface –where the main system has a known solution– is selected, (2) a detector with uniform rectilinear motion leaving from the selected point and meeting one of the four satellites after one hour is considered, and (3)
forty points on the detector line world, separated by the same time interval, are considered to find forty associated points in the co-region. These points are numerically calculated taking into account that the distance between associated points is null. This condition leads to a system of algebraic equations which may be solved by using the Newton-Raphson method; thus, we numerically go from inertial to emission coordinates (inverse of the transformation given in [1]). For each of the forty co-region points, we can calculate the $\chi^2$ value; thus, we know the number of emission solutions at these points. If only one of these solutions exists, it may be verified (as a test) that the resulting point in Minkowski space is just that point initially associated to the co-region point. If there are two emission solutions, we have a second point in the emission region receiving the same proper times from the four satellites. The criterion proposed in [2] to identify the two solutions (see above) has been succesfully used in all cases.

Results are shown in Fig. 2. The lowest empty circle represents the chosen point on Earth. The remaining empty circles are positions of the receiver for which there are no a second point in the emission region receiving the same proper times; however, when the receiver approaches the crossing satellite, there appear two points of the emission region receiving the same proper times. One of these points (on the receiver line world) is represented by a filled circle and the associated one by a filled square. We have verified that $\chi^2$ changes its sign, from negative to positive, along the receiver world line and, accordingly, the number of solutions changes from one to two. Other motions of the receiver should be studied to perform a general systematic study of the emission region. It well be done elsewhere.

**Figure 1.** Co-region structure. Emission solutions at different points of a certain straight line in the grid

**Figure 2.** Emission region structure. Single and double emission solutions.

### 4. Uncertainties in satellite positions and positioning errors

Here we deal with the estimation of positioning errors due to uncertainties in satellite positions. We can choose many groups of four satellites among those of the GPS and Galileo constellations. Moreover, for each group, we can proceed as in previous section; namely, we may find co-region points corresponding to various receiver positions approaching one of the satellites. We first assume that there are no uncertainties in satellite positions and, from each of the above grid points $(\tau_1, \tau_2, \tau_3, \tau_4)$, we get the exact receiver position by using the transformation of [1]. Then, uncertainties in satellite positions may be simulated as follows: the position of each satellite is randomly placed inside a sphere with radius $R$ (e.g., 1 m) centered on the exact satellite position. $N$ different positions of each satellite (close to the exact one) are thus generated. For each of them and the same grid point (proper times are not changed), we may find a perturbed receiver position (inertial coordinates). Hereafter, the distance between the exact and perturbed receiver locations is denoted by $r$. As a result, a distribution of $N$ values of $r$ are obtained in each case. The mean and RMS values of this distribution may be easily calculated. These quantities
characterize the positioning errors. In Figs. 3 and 4 we present two distinct cases. Each of them corresponds to four satellites and an exact receiver position. In order to describe these cases, we can consider a cone, with its vertex at the receiver, whose surface contains satellites 1, 2 and 3. Satellite 4 will not be, in general, on the surface. Then, the angle between the axis of the imaginary cone and the direction of the satellite 1 (4) is denoted $\theta_1$ ($\theta_4$). In the Figures of this section one easily see that: (i) the angles $\theta_1$ and $\theta_4$ are rather different in Fig. 3 and rather similar in Fig. 4, and (ii) the positioning errors of Fig. 4 are one order of magnitude greater than those of Fig. 3 (see means and RMS values of $r$). These facts may be easily understood taking into account that the Jacobian of the transformation from emission to inertial coordinates vanishes for $\theta_4 = \theta_1$; namely, for four satellites observed from the receiver along directions contained in the above cone [4, 5]. Hence, for $\theta_4 \approx \theta_1$, the Jacobian takes on small problematic values and the uncertainties in satellite positions produce large positioning errors. These situations should be prevented by choosing, for every receiver position, four available satellites with $\theta_4 \neq \theta_1$. More study on this subject is being developed.

![Figure 3](image1.png)  ![Figure 4](image2.png)

**Figure 3.** Histogram of the distribution of $r$ deviations for $\theta_1 \neq \theta_4$

**Figure 4.** The same histogram as in Fig. 3 for $\theta_1 \approx \theta_4$

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