How Hartree type nonlinearity takes effect on the properties for the solution of a quasilinear Schrödinger equation

Xianfa Song\textsuperscript{a}\textsuperscript{*} and Zhi-Qiang Wang\textsuperscript{b,c}\textsuperscript{†}

\textsuperscript{a} Department of Mathematics, School of Mathematics, Tianjin University, Tianjin, 300072, P. R. China
\textsuperscript{b} Center of Applied Mathematics, School of Mathematics, Tianjin University, Tianjin, 300072, P. R. China
\textsuperscript{c} Department of Mathematics and Statistics, Utah State University, Logan, UT 84322, USA

November 22, 2018

Abstract

In this paper, we deal with the Cauchy problem of the quasilinear Schrödinger equation

\[
\begin{align*}
  iu_t &= \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + (W(x) * |u|^2)u, \quad x \in \mathbb{R}^N, \quad t > 0 \\
  u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{align*}
\]

Here \( h(s) \) and \( W(x) \) are some real functions. We focus on how the potential \( W(x) \) takes effect on the blowup in finite time and global existence of the solution. In some cases, we even can obtain the watershed condition on \( W(x) \) in the following sense: If \( W(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \), then exist \( q_c \) and \( q_s \) such that the solution is global existence for any initial data when \( q > q_c \) while the solution maybe blow up in finite time for some initial data when \( q_s < q \leq q_c \). Especially, if \( W \in L^q(\mathbb{R}^N) \), \( a[h(s) + s^{\frac{2}{s}}] \geq \max(s^{\frac{2}{s}}, s^\alpha) \) for \( s \geq 1 \), where \( 0 < \alpha < \frac{N-1}{N} \), then \( q_c = \frac{2}{2 - \frac{2}{\alpha}} \) and

\[
a^{^2}2 + C_s^2 \|W\|_{L^q(\mathbb{R}^N)} \|u_0\|_{L^2(\mathbb{R}^N)}^{\min(4 - 4\alpha, 2)} = 1
\]

can be regarded as the watershed for the initial data \( u_0 \) which determines whether the solution is global existence or not, while the exponent \( q_s = \max(\frac{\max(a, 1)}{\max(2\alpha, 1)}2^\frac{2}{2 - \frac{2}{\alpha}}, 1) \) likes Sobolev critical exponent \( 2^* \). We also establish the pseudo-conformal conservation laws, give some asymptotic behavior results on the global solution and lower bound for the blowup rate of the blowup solution.

\textsuperscript{*}E-mail: songxianfa@tju.edu.cn (X.F. Song)

\textsuperscript{†}Corresponding author, Email: zhi-qiang.wang@usu.edu (Z. Q. Wang)
1 Introduction

In this paper, we consider the following Cauchy problem:

\[
\begin{aligned}
  iu_t &= \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + (W(x) \ast |u|^2)u, \quad x \in \mathbb{R}^N, \quad t > 0 \\
  u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]  

(1.1)

Here \( h(s) \) and \( W(x) \) are some real functions. \( h(s) \geq 0 \) for \( s \geq 0 \) and \( W(x) \) is even. \( N \geq 3 \). (1.1) can be used to model a lot of physical phenomena, such as the self-channelling of a high-power ultra short laser in matter. It often appears in plasma physics and dissipative quantum mechanics, and in condensed matter theory, see [1, 3, 4, 13, 17, 18, 20].

The motivations of this paper are as follows. First, there are many literatures considered the Cauchy problem of Schrödinger equation with Hartree type nonlinearity and dealt with the global existence and other behaviors for the solution. We can refer to [5, 7, 8, 15, 22] and the references therein. Naturally, we hope to study problem (1.1) which also contains Hartree type nonlinearity in the equation. Second, letting \( H(x,|u|^2) = W(x) \ast |u|^2 \),

\[
H(x,|u|^2) = [W(x) \ast |u|^2],
\]

(1.2)

(1.1) can be written in the following general form

\[
\begin{aligned}
  iu_t &= \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + F(x,|u|^2)u \quad \text{for } x \in \mathbb{R}^N, \quad t > 0 \\
  u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]

(1.3)

Recently, the local well-posedness result on (1.3) had been established by many authors, see [9, 16, 19]. In [21], the authors of this paper gave qualitative analysis for (1.3) with \( F(x, s) = F(s) \) and obtained the key conditions on the global existence and blowup in finite time for the solution, which are the explicit relationship between \( sF(s) \) and \( G(s)(\text{for example, } c_NG(s) \leq sF(s)) \), the explicit relationship between \( G(s)(\text{or } sF(s)) \) and \( h(s)(\text{for example, } [G(s)]^{\theta} \leq c[h(s)]^{2\theta}) \), where \( G(s) = \int_0^s F(\eta)d\eta \), \( 2^* = \frac{2N}{N-2} \). By the classic results in Section 3.2 of [6], if the even function \( W(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) with \( q \geq 1 \), \( q > \frac{N}{4} \), then \( (W \ast |u|^2)u \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N)) \) and \( (W \ast |u|^2)u \in C(L^r(\mathbb{R}^N), L^{r'}(\mathbb{R}^N)) \) for \( r = \frac{4q}{2q-4} \). Consequently, the local well-posedness result on (1.1) can be contained in these framework of [9, 16, 19]. However, letting \( \tilde{H}(x,s) = \int_0^s H(x,\eta)d\eta \), we cannot find the explicit relationship between \( sH(x,s) \) and \( \tilde{H}(x,s) \) now. Yet we don’t care whether we can find the explicit relationship between \( sH(x,s) \) and \( \tilde{H}(x,s) \) or not, we hope to establish the key condition on the global existence and blowup in finite time for the solution from another viewpoint. In fact, we will focus...
on how the potential \( W(x) \) takes effect on the properties for the solution, especially for the blowup in finite time and global existence of the solution. In some cases, we even got the watershed condition. It is the first time that we let the potential \( W(x) \) be the criterion of the conditions on the blowup in finite time and global existence of the solution to a quasilinear Schrödinger equation.

Now we give the definition of the global existence and blowup in finite time for the solution of (1.1) as follows.

**Definition 1.** Let \( u(x,t) \) be the solution of (1.1). We say that \( u(x,t) \) exists globally if the maximum existence interval for \( t \) is \([0, +\infty)\), while \( u(x,t) \) will blow up in finite time if there exists a time \( 0 < T < +\infty \) such that

\[
\lim_{t \to T^-} \int_{\mathbb{R}^N} [||\nabla u(x,t)||^2 + |\nabla h(|u(x,t)|^2)|^2] dx = +\infty. \tag{1.4}
\]

About the topic on the global existence and blowup phenomena of nonlinear Schrödinger equation, the following Cauchy problem

\[
\begin{aligned}
  &i u_t = \Delta u + F(|u|^2)u \quad \text{for } x \in \mathbb{R}^N, \ t > 0 \\
  &u(x,0) = u_0(x), \quad x \in \mathbb{R}^N
\end{aligned} \tag{1.5}
\]

was considered by Glassey in his famous paper [12]. \( sF(s) \geq c_N G(s) \) for some constant \( c_N > 1 + \frac{2}{N} \) and all \( s \geq 0 \) is the key condition on the blowup of the solution to (1.5). In [2], Berestycki and Cazenave obtained a sharp threshold on the blowup of the solution. Other related results can be found in [6, 23] and the references therein. In [14], the Cauchy problem of quasilinear Schrödinger equation

\[
\begin{aligned}
  &-i u_t + \Delta u + 2(\Delta |u|^2)u + |u|^{q-2}u = 0 \quad \text{for } x \in \mathbb{R}^N, \ t > 0 \\
  &u(x,0) = u_0(x), \quad x \in \mathbb{R}^N
\end{aligned} \tag{1.6}
\]

was studied by Guo, Chen and Su. They showed that the solution of (1.6) will blow up in finite time if \( 4 + \frac{4}{N} < q < 2 \cdot 2^* \) for some initial data.

The mass and energy of (1.1) are defined by

(i) Mass:

\[
m(u) = \left( \int_{\mathbb{R}^N} |u(\cdot, t)|^2 dx \right)^{\frac{1}{2}} := [M(u)]^{\frac{1}{2}};
\]

(ii) Energy :

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} [||\nabla u||^2 + |\nabla h(|u|^2)|^2] dx - \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2) |u|^2 dx.
\]

In the sequels, we will use \( C, C' \), and so on, to denote some constants, the values of it may vary line to line.

Our first result will establish the sufficient conditions on the global existence of the solution to (1.1).
Theorem 1. Let \( u(x,t) \) be the solution of (1.1) with \( u_0 \in X \),
\[
X = \{ w \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx < +\infty \}. \tag{1.7}
\]

(1). If \( W(x) \leq 0 \) for all \( x \in \mathbb{R}^N \), then \( u \) is global existence for any \( u_0 \in X \).

(2). Assume that the even function \( W(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) or changes sign, and there exist \( a > 0 \) and \( \alpha > 0 \) such that \( \max(s^{\frac{2}{N}}, s^{\alpha}) \leq a[h(s) + s^{\frac{2}{N}}] \) for \( s \geq 1 \),
\[
(C1) \quad W(x) = W_1(x) + W_2(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad q > 1, \quad q > \frac{\max(\alpha, \frac{1}{2}) 2^*}{\max(2\alpha, 1) 2^* - 2}
\]

We have
Case (i) \( 0 < \alpha < \frac{N-1}{N} \), if \( q > \frac{2^*}{\max(2\alpha, 1) 2^* - 2} \), then the solution is global existence for any initial \( u_0 \in X \);

or

Case (ii) \( 0 < \alpha < \frac{N-1}{N} \), if \( q = \frac{2^*}{\max(2\alpha, 1) 2^* - 2} \), then the solution is global existence for the initial \( u_0 \in X \) satisfying
\[
a^2 2^{\frac{(q-1)N+4q}{N}} C_s^2 \|W\|_{L^q(\mathbb{R}^N)} \|u_0\|_{L^2(\mathbb{R}^N)}^{\min(4-4\alpha, 2)} = 1;
\]

or

Case (iii) \( \alpha \geq \frac{N-1}{N} \), if \( q > 1 \), then the solution is global existence for any initial \( u_0 \in X \).

Here \( C_s \) denotes the best constant in the Sobolev’s inequality
\[
\int_{\mathbb{R}^N} w^{2^*} \, dx \leq C_s \left( \int_{\mathbb{R}^N} |\nabla w|^2 \, dx \right)^{\frac{2^*}{2}} \quad \text{for any} \quad w \in H^1(\mathbb{R}^N), \tag{1.8}
\]

Remark 1.1. 1. If \( h(|u|^2) \equiv 0 \), (1.1) becomes the classic semilinear Schrödinger with Hartree type nonlinear, we can take \( a = 1 \) and \( \alpha = \frac{1}{2} \), the solution of (1.1) is global existence if \( q > \frac{N}{2} \).

2. If \( W(x) \equiv c \in \mathbb{R} \), then \( W(x) \in L^\infty(\mathbb{R}^N) \), by the results of Theorem 1, the solution is always global existence for any initial data \( u_0 \).

3. The conclusion of (2)(iii) shows the interaction between the term \( 2uh'(|u|^2)\Delta h(|u|^2) \) and Hartree type nonlinear one. Roughly, if \( h(s) \) increases fast enough when \( s > 1 \), then the solution is global existence for any \( W(x) \in L^q(\mathbb{R}^N) \) \( (q > 1) \) and initial data \( u_0 \).

Our second result is about the sufficient conditions on the blowup in finite time for the solution of (1.1).

Theorem 2. Let \( u(x,t) \) be the solution of (1.1) with \( u_0 \in X \), \( xu_0 \in L^2(\mathbb{R}^N) \),
\[
E(u_0) \leq 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u_0) \, dx > 0.
\]

Assume that there exist constants \( k \) such that
\[
sh''(s) \leq kh'(s) \quad \text{if} \quad h'(s) \geq 0 \quad \text{or} \quad sh''(s) \geq kh'(s) \quad \text{if} \quad h'(s) \leq 0.
\]

(C2) \[
\max((2k+1)N,0) + 2W + (x \cdot \nabla W) \leq 0,
\]
then there exists a finite time $T$ such that
\[
\lim_{t \to T^-} \int_{\mathbb{R}^N} [\nabla u(x, t)]^2 + [\nabla h(|u|^2)(x, t)]^2 dx = +\infty.
\]

**Remark 1.2.** We would like to say something about the conditions
\[
\max((2k + 1)N, 0) + 2W + (x \cdot \nabla W) \leq 0
\]
and $E(u_0) \leq 0$.

1. Obviously, $W(x) \equiv c > 0$ means that $\max((2k + 1)N, 0) + 2W + (x \cdot \nabla W) > 0$, which is the opposite of the condition (C2). $W(x) \equiv c < 0$ implies that $E(u_0) > 0$ for $u_0(x) \neq 0$.

2. It is well known that $L^{q_1}(\mathbb{R}^N) \cap L^{q_2}(\mathbb{R}^N) \neq \emptyset$ for $0 < q_1 < q_2$. If $W(x)$ is a nontrivial radially symmetric function, by the condition
\[
\max((2k + 1)N, 0) + 2W(r) + rW'(r) \leq 0,
\]
then for any $q > \frac{N}{\max((2k+1)N,0)+2}$,
\[
W(r) \geq \frac{W(1)}{r^{\max((2k+1)N,0)+2}} \notin L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N).
\]

**Remark 1.3.** By the results of Theorem 1 and Theorem 2, in some cases, if $W(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, we can get the watershed of the exponent $q_c$ such that the solution exists globally for any initial $u_0$ if $q > q_c$ while the solution may blow up in finite time for some initial $u_0$ if $q < q_c$. Especially, if $W \in L^q(\mathbb{R}^N)$ and $a[h(s) + s^{\frac{2}{q}}] \geq \max(s^\alpha, s^{\frac{2}{q}})$ in the critical case of $q = q_c$, we find that
\[
a^{2(2-N)}C_s^{\frac{2}{q}} \|W\|_{L^q(\mathbb{R}^N)} \|u_0\|_{L^2(\mathbb{R}^N)}^{\min(4-4\alpha,2)} = 1
\]
can be look as the watershed for the initial data $u_0 \in X$ which can determines that the solution is global existence or not, while the exponent $q_s = \frac{\max(\alpha,\frac{4}{q})2^\ast}{\max(2\alpha,1)2^\ast-2}$ likes Sobolev critical exponent $2^\ast$. The details will be given in the Section 4.

In some cases, we even can get the sharp threshold for the global existence and blowup in finite time for the solution of (1.1) as follows.

**Theorem 3.** (Sharp Threshold) Let $u(x,t)$ be the solution of (1.1) with $u_0 \in X$. Assume that:

(i) There exist constant $k \in \mathbb{R}$, $a > 0$, $0 < \alpha < \frac{N-1}{N}$ and $0 < \beta \leq 2$ such that $sh''(s) \leq kh'(s)$ if $h'(s) \geq 0$ or $sh''(s) \geq kh'(s)$ if $h'(s) \leq 0$, $\max(s^\alpha, s^{\frac{2}{q}}) \leq a[h(s) + s^{\frac{2}{q}}]$ for $s \geq 0$, and
\[
(2 - \beta) + 4(N + 2 - \beta)[h'(s)]^2 s + 8Nh''(s)h'(s)s^2 \geq 0;
\]
(ii) \( W(x) \geq 0 \) for \( x \in \mathbb{R}^N \), \( W(x) \in L^q(\mathbb{R}^N), \max(1, \frac{N}{2}) < q \leq \frac{N}{2} \) if \( h(s) \equiv 0 \), \( \max\left(1, \frac{\max(2\alpha, 1)^2}{\max(2\alpha_1, 1)^2 - 2}\right) < q \leq \frac{2}{\max(2\alpha_1, 1)^2 - 2} \) if \( h(s) \neq 0 \) and \( 0 < \alpha < \frac{N - 1}{N} \), and there exist constant \( L > 1 + \frac{\max[(2k + 1)N, 0]}{2} \) and \( C \) such that
\[
LW(x) \leq -\frac{x \cdot \nabla W}{2} \leq CW(x).
\]

Moreover, suppose that there exists \( \omega > 0 \) such that
\[
d_I := \inf_{\{w \in H^1(\mathbb{R}^N) \setminus \{0\}; Q(w) = 0\}} \left( \frac{\omega}{2} \|w\|^2 + E(w) \right) > 0, \tag{1.9}\]
where
\[
Q(w) = 2 \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + (N + 2) \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx + 8N \int_{\mathbb{R}^N} h''(|w|^2)h'(|w|^2)|w|^4 |\nabla w|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) * |w|^2]|w|^2 \, dx,
\]
\[
E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + |\nabla h(|w|^2)|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |w|^2)|w|^2 \, dx, \tag{1.10}\]
and \( u_0 \) satisfies
\[
\frac{\omega}{2} \|u_0\|^2 + E(u_0) < d_I.
\]

Then we have:
1. If \( Q(u_0) > 0 \), the solution of \((1.1)\) exists globally;
2. If \( Q(u_0) < 0 \) and \( \exists \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla \bar{u}_0) \, dx \geq 0, \bar{u}_0 \in L^2(\mathbb{R}^N) \), the solution of \((1.1)\) blows up in finite time.

Inspired by \cite{10, 11}, we also consider the pseudo-conformal conservation laws as follows.

Theorem 4. (Pseudo-conformal Conservation Laws) 1. Assume that \( u \) is the global solution of \((1.1)\), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \). Then
\[
P(t) = \int_{\mathbb{R}^N} |(x - 2it\nabla)u|^2 \, dx + 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx - 2t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx
\]
\[
= \int_{\mathbb{R}^N} |xu_0|^2 \, dx + 4 \int_0^t \tau \theta(\tau) \, d\tau. \tag{1.12}\]

2. Assume that \( u \) is the blowup solution of \((1.1)\) with blowup time \( T \), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \). Then
\[
B(t) := \int_{\mathbb{R}^N} |(x + 2iT - t\nabla)u|^2 \, dx + 4(T - t)^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx
\]
\[\quad - 2(T - t)^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx
\]
\[\quad = \int_{\mathbb{R}^N} |(x + 2iT\nabla)u_0|^2 \, dx + 4T^2 \int_{\mathbb{R}^N} |\nabla h(|u_0|^2)|^2 \, dx - 2T^2 \int_{\mathbb{R}^N} (W * |u_0|^2)|u_0|^2 \, dx
\]
\[\quad + 32E(u_0) \int_0^t (T - \tau) \, d\tau - 4 \int_0^t (T - \tau) \theta(\tau) \, d\tau. \tag{1.13}\]
Here
\[
\theta(t) = \int_{\mathbb{R}^N} -4N[2h''(|u|^2)h'(|u|^2)|u|^2 + (h'(|u|^2))^2]|u|^2|\nabla u|^2 dx \\
- \int_{\mathbb{R}^N} \left(W + \frac{(x \cdot \nabla W)}{2}\right) |u|^2 dx.
\] (1.14)

As the applications of Theorem 4, we give some asymptotic behavior results on the
global solution of (1.1) and the lower bound for the blowup rate the blowup solution
of (1.1) (see Theorem 5 in Section 6). Roughly, under some assumptions, for the global
solution,
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W|*|u|^2)|u|^2 dx \leq \frac{C}{t^l}, \quad \lim_{t \to \infty} \int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx = 2E(u_0)
\]
for some constant 0 < l ≤ 2, while for the blowup solution,
\[
\int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla h(|u|^2)|^2] dx \geq \frac{C}{(T-t)^2}, \quad \int_{\mathbb{R}^N} (W*|u|^2)|u|^2 dx \geq \frac{C}{(T-t)^2}.
\]

The organization of this paper is as follows. In Section 2, we will prove the mass
and energy conservation laws and some equalities. In Section 3, we will prove Theorem
1. In Section 4, we will prove Theorem 2. In Section 5, we will prove Theorem 3. In
Section 6, we will prove Theorem 4 and Theorem 5.

2 Preliminaries

In this section, we will prove a lemma as follows.

**Lemma 2.1.** Assume that u is the solution to (1.1). Then in the time interval
[0, t] when it exists, u satisfies

(i) Mass conversation:
\[
M(u) = \int_{\mathbb{R}^N} |u(x,t)|^2 dx = \int_{\mathbb{R}^N} |u_0(x)|^2 dx = M(u_0);
\]

(ii) Energy conversation:
\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla h(|u|^2)|^2] dx - \frac{1}{4} \int_{\mathbb{R}^N} (W*|u|^2)|u|^2 dx = E(u_0);
\]

(iii)
\[
\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2|u|^2 dx = -4\Im \int_{\mathbb{R}^N} \tilde{u}(x \cdot \nabla u) dx;
\]

(iv)
\[
\frac{d}{dt} \Im \int_{\mathbb{R}^N} \tilde{u}(x \cdot \nabla u) dx = -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- 8N \int_{\mathbb{R}^N} h''(|u|^2)h'(|u|^2)|u|^4|\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) * |u|^2]|u|^2 dx.
\]
Proof: (i) Multiplying (1.1) by $2\bar{u}$, taking the imaginary part of the result, we get
\[
\frac{\partial}{\partial t}|u|^2 = \Im(2\bar{u}\Delta u) = \nabla \cdot (2\bar{u}\nabla u). \tag{2.1}
\]
Integrating it over $\mathbb{R}^N \times [0, t]$, we have
\[
\int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u_0|^2 dx.
\]

(ii) Multiplying (1.1) by $2\bar{u}_t$, taking the real part of the result, then integrating it over $\mathbb{R}^N \times [0, t]$, we obtain
\[
\int_{\mathbb{R}^N} [\nabla u]^2 + \nabla h(|u|^2)^2]dx - \frac{1}{2} \int_{\mathbb{R}^N} (W*|u|^2)|u|^2 dx
\]
\[
= \int_{\mathbb{R}^N} [\nabla u_0]^2 + \nabla h(|u_0|^2)^2]dx - \frac{1}{2} \int_{\mathbb{R}^N} (W*|u_0|^2)|u_0|^2 dx.
\]

(iii) Multiplying (2.1) by $|x|^2$ and integrating it over $\mathbb{R}^N$, we get
\[
\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |u|^2 dx = \int_{\mathbb{R}^N} |x|^2 \nabla \cdot (2\bar{u}\nabla u) dx = -4\Im \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx.
\]

(iv) Let $a(x, t) = \Re u(x, t)$ and $b(x, t) = \Im u(x, t)$. Then $u(x, t) = a(x, t) + ib(x, t)$,
\[
\frac{d}{dt} \Im \bar{u}(x \cdot \nabla u) = \sum_{k=1}^{N} \left[ x_k (b_t)_x a - x_k (a_t)_x b \right] + \sum_{k=1}^{N} (x_k b_{x_k} a_t - x_k a_{x_k} b_t).
Lemma 2.1 is proved. And

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx = \int_{\mathbb{R}^N} \sum_{k=1}^{N} [x_k (b_t)_{x_k} a - x_k (a_t)_{x_k} b] dx + \int_{\mathbb{R}^N} \sum_{k=1}^{N} (x_k a_{x_k} \Delta a + x_k b_{x_k} \Delta b) dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \sum_{k=1}^{N} x_k (|u|^2)_{x_k} [2h'(|u|^2) \Delta h(|u|^2) + (W \ast |u|^2)] dx
\]

\[
= N \int_{\mathbb{R}^N} (a_t b - ab_t) dx + \int_{\mathbb{R}^N} \sum_{k=1}^{N} (x_k b_{x_k} a_t - x_k a_{x_k} b_t) dx \\
+ \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- \frac{1}{2} \int_{\mathbb{R}^N} [(NW + \frac{(x \cdot \nabla W)}{2})^2 \ast |u|^2] |u|^2 dx
\]

\[
= N \int_{\mathbb{R}^N} (|a \Delta a + b \Delta b| + 2|u|^2 h'(|u|^2) \Delta h(|u|^2) + |u|^2 (W \ast |u|^2)) dx \\
+ (N-2) \int_{\mathbb{R}^N} |\nabla u|^2 dx + (N-2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- \int_{\mathbb{R}^N} [(NW + \frac{(x \cdot \nabla W)}{2})^2 \ast |u|^2] |u|^2 dx
\]

\[
= -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N+2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- 8N \int_{\mathbb{R}^N} h'(|u|^2) h''(|u|^2) |u|^4 |\nabla u|^2 dx - \int_{\mathbb{R}^N} \frac{(x \cdot \nabla W)}{2} |u|^2 |u|^2 dx
\]

(2.2)

Lemma 2.1 is proved. \qed

Remark 2.1. If \( u \) is the solution of (1.1), similar to [21], we have

\[
\int_{\mathbb{R}^N} |u|^{p_2} dx \leq \left( \int_{\mathbb{R}^N} |u|^{p_1} dx \right)^{\frac{p_1 - p_2}{p_1 - 2}} \left( \int_{\mathbb{R}^N} |u|^{p_1} dx \right)^{\frac{p_2 - 2}{p_1 - 2}}
\]

\[
= \left( \int_{\mathbb{R}^N} |u_0|^{p_1} dx \right)^{\frac{p_1 - p_2}{p_1 - 2}} \left( \int_{\mathbb{R}^N} |u_0|^{p_1} dx \right)^{\frac{p_2 - 2}{p_1 - 2}}
\]

for \( p_1 > p_2 > 2 \) by mass conservation law.

3 The proofs of Theorem 1

In this section, we will prove Theorem 1 and establish the sufficient conditions on the global existence of the solution to (1.1).

Proof of Theorem 1: Case (1). \( W(x) \leq 0 \) for \( x \in \mathbb{R}^N \). The global existence of the solution is a direct result of the energy conversation law of Lemma 2.1(ii) because

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (|W| \ast |u|^2) |u|^2 dx = 2E(u_0),
\]
which implies that \( \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \) is uniformly bounded for all \( t > 0 \).

Case (2). There exist constants \( a > 0 \) and \( \alpha > 0 \) such that \( a[h(s) + s^{\frac{4}{\alpha}}] \geq \max(s^{\frac{1}{\alpha}}, s^\alpha) \) for \( s \geq 1 \), \( W(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) or changes sign, \( W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \), \( q > 1 \), \( q > \frac{\max(\alpha, \frac{1}{\alpha})^2}{\max(2\alpha, 1)^2 - 2} \).

Obviously, \( q > \frac{\max(\alpha, \frac{1}{\alpha})^2}{\max(2\alpha, 1)^2 - 2} \) is equivalent to \( \frac{4q}{2q - 1} < \max(2\alpha, 1)^2 \).

Denoting

\[
\tau_1 = \frac{(2q - 1)[\max(2\alpha, 1)2^* - 2]}{(2q - 1)[\max(2\alpha, 1)2^* - 2] - 2}, \quad \tau_2 = \frac{(2q - 1)[\max(2\alpha, 1)2^* - 2]}{2},
\]

(3.1)

then

\[
\frac{2q - 1}{q\tau_1} = \frac{(2q - 1)[\max(2\alpha, 1)2^* - 2] - 2}{q[\max(2\alpha, 1)2^* - 2]}, \quad \frac{2q - 1}{q\tau_2} = \frac{2}{q[\max(2\alpha, 1)2^* - 2]}. \]

(3.2)

Suppose that \( W(x) = W_1(x) + W_2(x) \), where \( W_1(x) \in L^q(\mathbb{R}^N) \) and \( W_2(x) \in L^\infty(\mathbb{R}^N) \). Denote

\[
C_W = \frac{1}{2} \left( \int_{\mathbb{R}^N} |W_1|^q dx \right)^{\frac{1}{q}}.
\]

Using the mass and energy conversation laws of Lemma 2.1(ii), using Hölder’s inequality,
Young’s inequality, then Sobolev’s inequality, we have

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx = 2E(u_0) + \frac{1}{2} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx
\]

\[
= 2E(u_0) + \frac{1}{2} \int_{\mathbb{R}^N} (W_1 * |u|^2)|u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (W_2 * |u|^2)|u|^2 \, dx
\]

\[
\leq C + \frac{1}{2} \left( \int_{\mathbb{R}^N} |W_1|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |u|^{\frac{4q}{2q-1}} \, dx \right)^{\frac{2q-1}{2}} + \frac{1}{2} \|W_2\|_{L^\infty} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{2}
\]

\[
\leq C + \frac{1}{2} \|W_2\|_{L^\infty} (M(u_0))^2 + 2^{\frac{q-1}{q}} C_W \left( \int_{\{|u| \leq 1\}} |u|^{\frac{4q}{2q-1}} \, dx \right) + 2^{\frac{q-1}{q}} C_W \left( \int_{\{|u| > 1\}} |u|^2 \, dx \right) \left( \int_{\{|u| > 1\}} |u|^{\max(2\alpha,1)2^*} \, dx \right)^{\frac{2q-1}{2q}}
\]

\[
\leq C + 2^{\frac{q-1}{q}} C_W \left( \int_{\{|u| \leq 1\}} |u|^2 \, dx \right)^{\frac{2q-1}{q} \frac{1}{q_1}} + 2^{\frac{q-1}{q}} C_W \left( \int_{\{|u| > 1\}} |u|^2 \, dx \right)^{\frac{2q-1}{q_1}} \left( \int_{\{|u| > 1\}} |u|^{\max(2\alpha,1)2^*} \, dx \right)^{\frac{2q-1}{2q}}
\]

\[
\leq C' + 2^{\frac{q-1}{q}} C_W \left( \int_{\mathbb{R}^N} |u_0|^2 \, dx \right)^{\frac{2q-1}{q} \frac{1}{q_1}} \left( \int_{\{|u| > 1\}} \max(|u|^{2\alpha2^*}, |u|^{2^*}) \, dx \right)^{\frac{2q-1}{q}}
\]

\[
\leq C' + a^{\frac{q-1}{q}} C_W \left( \int_{\mathbb{R}^N} |u_0|^2 \, dx \right)^{\frac{2q-1}{q} \frac{1}{q_1}} \left( \int_{\{|u| > 1\}} a^{2^*} [h(|u|^2) + |u|^{2^*}] \, dx \right)
\]

\[
\leq C' + a^{\frac{q-1}{q}} C_W \left( \frac{q-1}{2} \frac{(2q-1)(2q-1)}{2} \left( \int_{\mathbb{R}^N} (|h(|u|^2)|^{2^*} + |u|^{2^*}) \, dx \right) \frac{2q-1}{q_1}
\]

\[
\leq C' + a^{\frac{q-1}{q}} \frac{(2q-1)^2}{q_1} C_W \left( \frac{q-1}{2} \frac{(2q-1)}{2} \left( \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |\nabla u|^2 \, dx \right) \right)^{\frac{q-1}{q_1}}
\]

We give the estimate for (3.3) in three subcases.

Subcase (i) 0 < \alpha < \frac{N-1}{N} = \frac{2^*+2}{22} and q > \frac{2^*}{\max(2\alpha,1)2^*}. In this subcase,

\[
\frac{2^*(2q-1)}{2q} < 1.
\]

For any initial data u_0, using Young’s inequality, we have

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \leq C + C'(C_W, u_0, q) + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |\nabla u|^2 \, dx,
\]

which implies that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \leq C.
\]
Subcase (ii) \(0 < \alpha < \frac{N - 1}{N} = \frac{2^* - 2}{2^* + 2}\) and \(q = \frac{2^*}{\max(2\alpha, 1)2^* - 2}\). In this subcase,

\[
\frac{2(2q - 1)}{q \tau_1} = 2 - \max(2\alpha, 1) = \min(2 - 2\alpha, 1), \quad \frac{2^*(2q - 1)}{2q \tau_2} = 1.
\]

For the initial \(u_0\) satisfying

\[
a^2 \frac{2^*(2q - 1)}{q \tau_2} (2 - \frac{3 - 2\alpha}{2} C_s^\frac{2}{q}) \frac{(2q - 1)}{q} \| W_1 \|_{L^q} \| u_0 \|_{L^q} \frac{2(2q - 1)}{2q \tau_2} < 1,
\]

i.e.,

\[
a^2 \frac{2^*(q - 1)N + 4q}{q \tau_2} C_s^\frac{2}{q} \| W_1 \|_{L^q} \| u_0 \|_{L^q} < 1,
\]

we have

\[
(1 - a^2 \frac{2^*(q - 1)N + 4q}{q \tau_2} C_s^\frac{2}{q} \| W_1 \|_{L^q} \| u_0 \|_{L^q}) \int_{\mathbb{R}^N} [\| \nabla h(|u|^2) \|^2 + |\nabla u|^2] dx \leq C.
\]

Note that in the two subcases above, if \(0 < \alpha \leq \frac{2}{2^*}\), then

\[
\frac{2^*}{\max(2\alpha, 1)2^* - 2} > \frac{\max(\alpha, \frac{1}{2})2^*}{\max(2\alpha, 1)2^* - 2} \geq 1,
\]

while if \(\frac{2}{2^*} < \alpha < \frac{2^* + 2}{2^* + 2}\), then

\[
\frac{2^*}{\max(2\alpha, 1)2^* - 2} > \frac{\max(\alpha, \frac{1}{2})2^*}{\max(2\alpha, 1)2^* - 2}.
\]

Subcase (iii) \(\alpha \geq \frac{N - 1}{N} = \frac{2^* + 2}{2^* + 2}\). In this subcase,

\[
\frac{2^*}{\max(2\alpha, 1)2^* - 2} \leq 1, \quad \frac{\max(\alpha, \frac{1}{2})2^*}{\max(2\alpha, 1)2^* - 2} < 1.
\]

Then for any \(q > 1\), we have

\[
\frac{4q}{2q - 1} < \max(2\alpha, 1)2^*, \quad \frac{2^*(2q - 1)}{2q \tau_2} < 1.
\]

Similar the proof of Subcase (i), applying Young's inequality to (3.3), we get

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \leq C
\]

for all \(t > 0\).

\[\square\]

**Remark 3.1.** The typical even function which satisfies condition (C1) is \(W(x) = \frac{1}{|x|^p}\). If \(h(|u|^2) = b|u|^{2\alpha} (b \geq 0), 0 < \alpha < \frac{N - 1}{N}, p < \frac{N \max(2\alpha, 1)2^* - 2}{2^* - 2}\), letting \(W_1(x) = \frac{1}{|x|^p}\) for \(|x| < 1\) and \(W_1(x) = 0\) for \(|x| \geq 1\), while \(W_2(x) = 0\) for \(|x| < 1\) and \(W_2(x) = \frac{1}{|x|^p}\) for \(|x| \geq 1\), then \(W_1(x) \in L^q(\mathbb{R}^N)\) and \(W_2(x) \in L^\infty(\mathbb{R}^N)\), we have \(W(x) \in \)
\( L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) for \( \frac{2^*}{\max(2\alpha,1)^2} < q < \frac{N}{p} \). The solution of (1.1) is global existence if \( p < \frac{N[\max(2\alpha,1)\cdot 2^*-2]}{2^*} \), especially, \( p < 2 \) when \( h(|u|^2) \equiv 0 \).

Similar to Proposition 3.1 in [21], noticing that \( \|u(\cdot,t)\|_{L^2} = \|u_0\|_{L^2} \), using the results of Theorem 1, we give a related result below without proof.

**Proposition 3.1.** Assume that \( u \) is the global solution of (1.1). Besides the other conditions of Theorem 1, suppose that the function \( f(s) \) satisfying the following conditions: There exist \( 0 < \alpha_1 < 1 \) and \( \beta_1 > 1 \) such that

\[(EC) \quad \|f(s)\|^{\alpha_1} \leq c_1 s, \quad \|f(s)\|^{\beta_1} \leq C_1 [h(s)]^{2^*}\]

for \( s \geq 0 \), where \( c_1 \) and \( C_2 \) are positive constants. Then

\[\int_{\mathbb{R}^N} |f(|u|^2)|dx \leq C. \quad (3.4)\]

Similarly to Proposition 3.2 in [21], recalling that \( \|\nabla u(\cdot,t)\|_{L^2} \leq C \) uniformly for all \( t > 0 \), we give the following propositions without proof.

**Proposition 3.2.** Assume that \( u \) is the global solution of (1.1). Besides the other conditions of Theorem 1, suppose that \( f(s) \) satisfies

\[(\|f'(s)\|^2)^{2^*} \leq C[h'(s)]^{2^*} s\]

for \( s \geq 0 \), where the constant \( \tilde{\tau} > 1 \). Then

\[\|\nabla f(|u|^2)\|_{L^2}^2 = \int_{\mathbb{R}^N} |\nabla f(|u|^2)|^2 dx \leq C. \quad (3.5)\]

4 The Proof of Theorem 2

In this section, we will give the proof of Theorem 2 and deal with the sufficient conditions on blowup in finite time for the solution by using the results of Lemma 2.1.

**Proof of Theorem 2:** Wherever \( u \) exists, let

\[y(t) = \Im \int_{\mathbb{R}^N} \tilde{u}(x \cdot \nabla u) dx.\]

We discuss it in two cases:
Case 1. $h(s) \equiv 0$ or $h(s) \neq 0$ and $k \leq -\frac{1}{2}$. We have

$$
\dot{y}(t) = -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 8N \int_{\mathbb{R}^N} h''(|u|^2)h'(|u|^2)|u|^4 |\nabla u|^2 dx \\
- \int_{\mathbb{R}^N} \left( \frac{x \cdot \nabla W}{2} \right) * |u|^2 |u|^2 dx \\
\geq -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2 + 2kN) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - \int_{\mathbb{R}^N} \left( \frac{x \cdot \nabla W}{2} \right) * |u|^2 |u|^2 dx \\
= -4E(u_0) - (2k + 1) N \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( [2W + (x \cdot \nabla W)] * |u|^2 \right) |u|^2 dx \\
\geq 0,
$$

which means that $y(t) \geq y(0) > 0$ for $t > 0$.

Case 2. $h(s) \neq 0$ and $k > -\frac{1}{2}$. We have

$$
\dot{y}(t) \geq -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2 + 2kN) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - \int_{\mathbb{R}^N} \left( \frac{x \cdot \nabla W}{2} \right) * |u|^2 |u|^2 dx \\
= (2k + 1) N \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2[(2k + 1)N + 2]E(u_0) \\
- \frac{1}{2} \int_{\mathbb{R}^N} \left( [(2k + 1)N + 2)W + (x \cdot \nabla W)] * |u|^2 \right) |u|^2 dx \\
\geq 0,
$$

which also means that $y(t) \geq y(0) > 0$ for $t > 0$.

Setting

$$
J(t) = \int_{\mathbb{R}^N} |x|^2 |u|^2 dx,
$$

we have $J'(t) = -4y(t) < -4y(0) < 0$. Then

$$
0 \leq J(t) = J(0) + \int_{0}^{t} J'(\tau) d\tau < J(0) - 4y(0)t,
$$

which implies that the maximum existence interval of time for $u$ is finite, and $u$ will blow up before $\frac{J(0)}{4y(0)}$.

**Remark 4.1.** If $W(x) = \frac{1}{|x|^p}$, the condition $[\max((2k+1)N, 0) + 2]W + (x \cdot \nabla W) \leq 0$ implies that $p \geq \max((2k+1)N, 0) + 2$. If $h(s) \equiv 0$, we take $k = -\frac{1}{2}$ and get $p \geq 2$, if $h(s) = s^a$, we can take $k = a - 1$ and get $p \geq \frac{N\max(2a, 1)2^*}{2^*-2}$.

**Remark 4.2.** We would like to say something about the conditions on the initial $u_0$ in the critical case $q = q_c$ if $W \in L^q(\mathbb{R}^N)$ and $a[h(s) + s^{\frac{q}{2}}] \geq \max(s^a, s^{\frac{q}{2}})$. By the results of Theorem 1, if $0 < \alpha < \frac{N-1}{N} = \frac{2^*+2}{2^*}$, then $q_c = \frac{2^*}{\max(2a, 1)2^* - 2}$ and

$$
a^{\frac{4}{2^*}} C_s^{\frac{2}{2^*}} \|W\|_{L^q} u_0^{\min(4\alpha, 2)} < 1,
$$

which implies that the maximum existence interval of time for $u$ is finite, and $u$ will blow up before $\frac{J(0)}{4y(0)}$. \qed
then the solution is global existence. By the results of Theorem 2, if \( q = q_c \), \( E(u_0) \leq 0 \) and \( \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u_0) dx > 0 \), then the solution will blow up in finite time. But \( E(u_0) \leq 0 \) implies that

\[
\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u_0|^2)|^2 dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx \\
\leq a^2 2^{\frac{4}{q}} C_s^{\frac{2}{q}} \left( \int_{\mathbb{R}^N} |W|^q dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |u_0|^2 dx \right)^{\frac{2q-1}{q+1}} \\
\times \left( \int_{\mathbb{R}^N} ([|\nabla h(|u_0|^2)|^2 + |\nabla u_0|^2] dx \right)^{\frac{2q(2q-1)}{q+1}},
\]

which means that

\[
a^2 2^{\frac{4}{q}} C_s^{\frac{2}{q}} \|W\|_{L^q} \|u_0\|_2^{\min(4-4\alpha, 2)} \geq 1.
\]

Obviously, \( S_{gl}(u_0) \cap S_{bl}(u_0) = \emptyset \), where

\[
S_{gl}(u_0) = \{ u_0 \in X, a^2 2^{\frac{4}{q}} C_s^{\frac{2}{q}} \|W\|_{L^q} \|u_0\|_2^{\min(4-4\alpha, 2)} < 1 \}
\]

and

\[
S_{bl}(u_0) = \{ u_0 \in X, E(u_0) \leq 0 \} \\
\subseteq \{ u_0 \in X, a^2 2^{\frac{4}{q}} C_s^{\frac{2}{q}} \|W\|_{L^q} \|u_0\|_2^{\min(4-4\alpha, 2)} \geq 1 \}.
\]

That is, if \( W \in L^q(\mathbb{R}^N), a[h(s) + s^{\frac{1}{2}}] \geq \max(s^\alpha, s^{\frac{1}{2}}) \) in the critical case of \( q = q_c \),

\[
a^2 2^{\frac{4}{q}} C_s^{\frac{2}{q}} \|W\|_{L^q} \|u_0\|_2^{\min(4-4\alpha, 2)} = 1 \tag{4.3}
\]

can be regarded as the watershed for the initial data \( u_0 \) which determines whether the solution is global existence or not.

The similar conclusion for the initial data \( u_0 \) is also true for the following system, which is a special case of that in [21].

\[
\begin{cases}
  iu_t = \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + F(|u|^2)u & \text{for } x \in \mathbb{R}^N, \ t > 0 \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\tag{4.4}
\]

Assume that there exist positive constants \( c_1, c_2, 0 < \theta < 1, q > 1 \) satisfying the critical condition \( (2 - 2^\theta)\theta + 2q = 2^\theta \) such that \( |G(s)|^\theta \leq c_1 s \) and \( |G(s)|^q \leq c_2 [s^{\frac{1}{2}} + h(s)]^{2^\theta} \) for \( s \geq 0 \). Here \( c_1 \) and \( c_2 \) are taken the values in the sense of the supremum. Then the solution is global existence if

\[
2^{\frac{3q^* - q}{2q^*}} c_1^\theta (c_2 C_s)^{\frac{1}{q}} \|u_0\|_2^{\frac{2}{q^*}} < 1,
\]

15
while the solution will blow up in finite time if $E(u_0) \leq 0$, $\exists \int_{\mathbb{R}^N} u_0(x \cdot \nabla u_0)dx > 0$, $|x|u_0 \in L^2(\mathbb{R}^N)$ with other assumptions on $h(s)$ and $G(s)$. Here

$$\frac{1}{\tau_1} = \frac{q - 1}{q - \theta}, \quad \frac{1}{\tau_2} = \frac{1}{q - \theta}.$$  

But $E(u_0) \leq 0$ implies that

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u_0|^2)|^2 dx$$

$$\leq \int_{\mathbb{R}^N} G(|u_0|^2)dx$$

$$\leq \left( \int_{\mathbb{R}^N} c_1 |u_0|^2 dx \right)^{\frac{1}{\tau_1}} \left( \int_{\mathbb{R}^N} c_2 |u_0| + h(|u_0|^2)^{2^*} dx \right)^{\frac{1}{\tau_2}}$$

$$\leq 2^{\frac{2^* - 2}{2^*} - \frac{1}{\tau_2}} c_1 \left( c_2 C_s \right)^{\frac{1}{\tau_2}} \int_{\mathbb{R}^N} |\nabla u_0|^2 + |\nabla h(|u_0|^2)|^2 dx$$

which means that

$$2^{\frac{2^* - 2}{2^*} - \frac{1}{\tau_2}} c_1 \left( c_2 C_s \right)^{\frac{1}{\tau_2}} \|u_0\|_2^{\frac{2}{\tau_2}} \geq 1.$$  

Obviously, $S_{gl}(u_0) \cap S_{vl}(u_0) = \emptyset$, where

$$S_{gl}(u_0) = \{ u_0 \in X, \quad 2^{\frac{2^* - 2}{2^*} - \frac{1}{\tau_2}} c_1 \left( c_2 C_s \right)^{\frac{1}{\tau_2}} \|u_0\|_2^{\frac{2}{\tau_2}} < 1 \}$$

and

$$S_{vl}(u_0) = \{ u_0 \in X, E(u_0) \leq 0 \} \subseteq \{ u_0 \in X, \quad 2^{\frac{2^* - 2}{2^*} - \frac{1}{\tau_2}} c_1 \left( c_2 C_s \right)^{\frac{1}{\tau_2}} \|u_0\|_2^{\frac{2}{\tau_2}} \geq 1 \}.$$  

That is, in the critical case of $(2 - 2^*)\theta + 2q = 2^*$,

$$2^{\frac{2^* - 2}{2^*} - \frac{1}{\tau_2}} c_1 \left( c_2 C_s \right)^{\frac{1}{\tau_2}} \|u_0\|_2^{\frac{2}{\tau_2}} = 1 \quad (4.5)$$

can be regarded as the watershed for the initial data $u_0$ which determines whether the solution is global existence or not.

**Remark 4.3.** If $h(|u|^2) = b|u|^{2\alpha}$, we would like to compare the results of [21] with those of this paper when $W(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, and discuss the similar roles of the exponents. Consider

$$\left\{ \begin{array}{l}
iu_t = \Delta u + 2b^2 \alpha |u|^{2\alpha - 2} u \Delta(|u|^{2\alpha}) + (|u|^{2\alpha - 2}) u \quad \text{for } x \in \mathbb{R}^N, \quad t > 0 \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \end{array} \right. \quad (4.6)$$

and

$$\left\{ \begin{array}{l}
iu_t = \Delta u + 2b^2 \alpha |u|^{2\alpha - 2} u \Delta(|u|^{2\alpha}) + (\frac{1}{|x|^p} * |u|^2) u \quad \text{for } x \in \mathbb{R}^N, \quad t > 0 \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \end{array} \right. \quad (4.7)$$

16
1. (i) If $b = 0$, $h(s) \equiv 0$, the watershed exponent of $p$ for (4.6) in [21] is $\tilde{p}_c = 1 + \frac{2}{N}$ in the means that the solution is global existence for any $u_0 \in X$ if $\tilde{p} < \tilde{p}_c$ and the solution will blow up in finite time under certain conditions if $\tilde{p} \geq \tilde{p}_c$, while the watershed exponent of $q$ for (4.7) in this paper is $q_c = \frac{N}{2}$ (or $p_c = 2$) in the means that the solution is global existence for any $u_0 \in X$ if $q > q_c$ (or $p < 2$) and the solution will blow up in finite time under certain conditions if $q \geq q_c$ (or $p \geq 2$).

(ii) If $b > 0$, $0 < \alpha < \frac{N-1}{N}$ and $h(s) \neq 0$, the watershed exponent of $p$ in (4.6) is $\tilde{p}_c = \max\{2, 2\alpha - 1 + \frac{2}{N}\}$ in the means that the solution is global existence for any $u_0 \in X$ if $\tilde{p} < \tilde{p}_c$ and the solution will blow up in finite time under certain conditions if $\tilde{p} \geq \tilde{p}_c$, while the watershed exponent of $q$ for (4.7) is $q_c = \frac{2^*}{\max(2\alpha,1)2^*-2}$(or $p_c = (N-2)\max(2\alpha,1)2^*-2$) in the means that the solution is global existence for any $u_0 \in X$ if $q > q_c$ (or $p < p_c$) and the solution will blow up in finite time under certain conditions if $q \leq q_c$ (or $p \geq p_c$).

The watershed role of $q_c$ (or $p_c$) for (4.7) in this paper is similar to that of $\tilde{p}_c$ for (4.6) in [21].

2. (i) If $b = 0$, $h(s) \equiv 0$, the Sobolev critical exponent of $\tilde{p}$ for (4.6) is $\tilde{p}_s = \frac{2^*}{N}$, while $q_s = \frac{N}{2}$ (or $p_s = 4$) plays the similar role of Sobolev critical exponent for (4.7) when $N \geq 4$. We can establish the blowup results on (4.6) if $\tilde{p}_s < \tilde{p} < \tilde{p}_c$ and those on (4.7) if $\frac{N}{2} = q_s < q < q_c = \frac{N}{2}$ (or $2 = p_c \leq p < p_s = 4$) for some initial $u_0$.

(ii) If $b > 0$, $0 < \alpha < \frac{N-1}{N}$ and $h(s) \neq 0$, the Sobolev critical exponent of $\tilde{p}$ for (4.6) is $\tilde{p}_s = \max(\alpha, \frac{1}{2})2^*$, while $q_s = \max(\frac{\max(2\alpha,1)2^*}{\max(2\alpha,1)2^*-2}, 1)$ (or $p_s = N \min(\frac{\max(2\alpha,1)2^*-2}{\max(\alpha,\frac{1}{2})2^*}, 1)$) likes Sobolev critical exponent for (4.7). We can establish the blowup results on (4.6) if $\tilde{p}_c \leq \tilde{p} < \tilde{p}_s$ and those on (4.7) if $\max(\frac{\max(\alpha,\frac{1}{2})2^*}{\max(2\alpha,1)2^*-2}, 1) = q_s < q < q_c = \max(\frac{\max(2\alpha,1)2^*}{\max(2\alpha,1)2^*-2}, 1)$ (or $2^* = \frac{2\alpha}{\max(2\alpha,1)2^*-2}, p_c \leq p < p_s = N \min(\frac{\max(2\alpha,1)2^*-2}{\max(\alpha,\frac{1}{2})2^*}, 1)$) for some initial $u_0$.

(iii) We also point out that $\tilde{p}_c < \tilde{p}_s$, $p_c < p_s$, while $q_s < q_c$. The Sobolev critical exponent role of $q_s$ (or $p_s$) for (4.7) in this paper is similar to that of $\tilde{p}_s$ for (4.6) in [21].

5 The Proof of Theorem 3

In this section, we will prove Theorem 3 and establish a sharp threshold for the blowup and global existence of the solution to (1.1) under certain conditions.

**The proof of Theorem 3.** We proceed in four Steps.

Step 1. We will prove $d_I > 0$. 

17
Since \( Q(w) = 0 \), \( w \not\equiv 0 \), we have

\[
l(\int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx)
= 2 \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + (N + 2) \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx + 8N \int_{\mathbb{R}^N} h''(|w|^2)h'(|w|^2)|w|^4 |\nabla w|^2 \, dx
- \int_{\mathbb{R}^N} \left( (2 - L) + 4(N + 2 - L)(h'(|w|^2))^2 |w|^2 + 8Nh''(|w|^2)h'(|w|^2)|w|^4 \right) |\nabla w|^2 \, dx
\leq -\frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) \ast |w|^2]u^2 \, dx \leq C \int_{\mathbb{R}^N} (W \ast |w|^2)|w|^2 \, dx
\leq C \left( \left( \int_{\mathbb{R}^N} |W|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |w| \frac{4q}{2q-1} \right)^{\frac{2q-1}{q}} \right)^{\frac{1}{2}} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |w|^2 \, dx \right)^{\frac{2q-1}{q+1}} \left( \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx \right)^{\frac{2^*(2q-1)}{2q^2}} \]  

(5.1)

which implies that

\[
\left( \int_{\mathbb{R}^N} |w|^2 \, dx \right)^{\frac{2q-1}{q+1}} \left( \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx \right)^{\frac{2^*(2q-1)}{2q^2}} \geq C > 0.
\]

Since \( q \leq \frac{2^*}{\max(2\alpha,1)2^*-2} \) equals to \( \frac{2^*(2q-1)}{2q^2} - 1 \geq 0 \), we have

\[
\int_{\mathbb{R}^N} |w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx \geq C > 0.
\]

(5.2)

Here

\[
\frac{1}{\tau_1} = \frac{(2q - 1)[\max(2\alpha,1)2^*-2] - 2}{(2q - 1)[\max(2\alpha,1)2^*-2]}, \quad \frac{1}{\tau_2} = \frac{2}{(2q - 1)[\max(2\alpha,1)2^*-2]}.
\]

On the other hand, using \( Q(w) = 0 \) again, we get

\[
(\max[(2k + 1)N,0] + 2) \int_{\mathbb{R}^N} |\nabla w|^2 + |\nabla h(|w|^2)|^2 \, dx
\geq 2 \int_{\mathbb{R}^N} |\nabla w|^2 + ((2k + 1)N + 2) \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 \, dx
\geq 2 \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + (N + 2) \int_{\mathbb{R}^N} (\nabla h(|w|^2))^2 \, dx + 8N \int_{\mathbb{R}^N} h''(|w|^2)h'(|w|^2)|w|^4 |\nabla w|^2 \, dx
= -\frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) \ast |u|^2]|u|^2 \, dx \geq L \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 \, dx.
\]

(5.3)
Therefore
\[ E(w) = \frac{1}{2} \int_{\mathbb{R}^N} \|\nabla w\|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |w|^2)|w|^2 dx \]
\[ \geq \frac{1}{2} \left( 1 - \frac{(\max[(2k + 1)N, 0] + 2)}{2L} \right) \int_{\mathbb{R}^N} \|\nabla w\|^2 + |\nabla h(|w|^2)|^2 dx. \] (5.4)

(5.7) and (5.8) mean that
\[ \frac{\omega}{2} \int_{\mathbb{R}^N} |w|^2 dx + E(w) \]
\[ \geq \frac{1}{2} \left( \omega, 1 - \frac{(\max[(2k + 1)N, 0] + 2)}{2L} \right) \int_{\mathbb{R}^N} \|w\|^2 + |\nabla w|^2 + |\nabla h(|w|^2)|^2 dx \]
\[ \geq C > 0. \] (5.5)

Therefore \( d_I > 0 \).

Step 2. Denote
\[ K_+ = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ Q(u) > 0, \ \frac{\omega}{2} \|u\|^2 + E(u) < d_I \} \]
and
\[ K_- = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ Q(u) < 0, \ \frac{\omega}{2} \|u\|^2 + E(u) < d_I \}. \]

We will prove that \( K_+ \) and \( K_- \) are invariant sets of (1.1).

Assume that \( u_0 \in K_+ \), i.e., \( Q(u_0) > 0 \) and \( \frac{\omega}{2} \|u_0\|^2 + E(u_0) < d_I \). It is easy to verify that
\[ u(\cdot, t) \in H^1(\mathbb{R}^N) \setminus \{0\}, \ \frac{\omega}{2} \|u(\cdot, t)\|^2 + E(u(\cdot, t)) < d_I. \] (5.6)

because \( \|u\|^2 \) and \( E(u) \) are conservation quantities for (1.1).

We need to show that \( Q(u(\cdot, t)) > 0 \) for \( t \in (0, T) \). Contradictorily, if there exists \( t_1 \in (0, T) \) such that \( Q(u(\cdot, t_1)) < 0 \), then there exists a \( t_2 \in [0, t_1] \) such that \( Q(u(\cdot, t_2)) = 0 \) by the continuity. And
\[ \frac{\omega}{2} \|u(\cdot, t_2)\|^2 + E(u(\cdot, t_2)) < d_I \]
by (5.6), which is a contradiction to the definition of \( d_I \). Hence \( Q(u(\cdot, t)) > 0 \). This inequality and (5.6) imply that \( u(\cdot, t) \in K_+ \), which means that \( K_+ \) is a invariant set of (1.1).

Similarly, we can prove that \( K_- \) is also a invariant set of (1.1). We omit the details here.

Step 3. Assume that \( Q(u_0) > 0 \) and \( \frac{\omega}{2} \|u_0\|^2 + E(u_0) < d_I \). Since \( K \) is invariant set of (1.1), we have \( Q(u(\cdot, t)) > 0 \) and \( \frac{\omega}{2} \|u(\cdot, t)\|^2 + E(u(\cdot, t)) < d_I \). Using \( Q(u(\cdot, t)) > 0 \),
we get
\[
(\max[(2k+1)\mathcal{N},0]+2) \int_{\mathbb{R}^N} [\|
abla u(\cdot,t)\|^2 + |\nabla h(|u(\cdot,t)|^2)|^2]\,dx
\]
\[
= 2 \int_{\mathbb{R}^N} \|
abla u(\cdot,t)\|^2 + (2k+1)\mathcal{N} + 2 \int_{\mathbb{R}^N} |\nabla h(|u(\cdot,t)|^2)|^2\,dx
\]
\[
\geq 2 \int_{\mathbb{R}^N} \|
abla u(\cdot,t)\|^2 + (N+2) \int_{\mathbb{R}^N} |\nabla h(|u(\cdot,t)|^2)|^2\,dx
\]
\[
+ 8N \int_{\mathbb{R}^N} h''(|u(\cdot,t)|^2) h'(|u(\cdot,t)|^2) u(\cdot,t)^4 |\nabla u(\cdot,t)|^2\,dx
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) \ast |u(\cdot,t)|^2] |u(\cdot,t)|^2\,dx \geq L \int_{\mathbb{R}^N} (W \ast |u(\cdot,t)|^2) |u(\cdot,t)|^2\,dx. \tag{5.7}
\]

Therefore
\[
E(u(\cdot,t)) = \frac{1}{2} \int_{\mathbb{R}^N} \|
abla u(\cdot,t)\|^2\,dx + |\nabla h(|u(\cdot,t)|^2)|^2\,dx - \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u(\cdot,t)|^2) |u(\cdot,t)|^2\,dx
\]
\[
\geq \frac{2L - \max[(2k+1)\mathcal{N},0]+2}{4L} \int_{\mathbb{R}^N} \|
abla u(\cdot,t)\|^2 + |\nabla h(|u(\cdot,t)|^2)|^2\,dx. \tag{5.8}
\]

By the mass and energy conservation laws, (5.7) and (5.8) mean that
\[
d_t \geq \frac{\omega}{2} \int_{\mathbb{R}^N} |u_0|^2\,dx + E(u_0) = \frac{\omega}{2} \int_{\mathbb{R}^N} |u(\cdot,t)|^2\,dx + E(u(\cdot,t))
\]
\[
\geq \frac{1}{2} \min \left( \omega, 1 - \frac{\max[(2k+1)\mathcal{N},0]+2}{2L} \right) \int_{\mathbb{R}^N} |u(\cdot,t)|^2 + |\nabla u(\cdot,t)|^2 + |\nabla h(|u(\cdot,t)|^2)|^2\,dx
\]
\[
\geq C > 0,
\]

and
\[
\int_{\mathbb{R}^N} |\nabla u(\cdot,t)|^2\,dx + \int_{\mathbb{R}^N} |\nabla h(|u(\cdot,t)|^2)|^2\,dx \leq C < \infty,
\]
i.e., the solution \( u(x,t) \) of (1.1) exists globally.

Step 4. Suppose that \(|x|u_0 \in L^2(\mathbb{R}^N)\), \(Q(u_0) < 0\) and \(\frac{\omega}{2} |u_0|^2 + E(u_0) < d_t\). Since \(K_-\) is an invariant set of (1.1), we have \(Q(u(\cdot,t)) < 0\) and \(\frac{\omega}{2} |u(\cdot,t)|^2 + E(u(\cdot,t)) < d_t\).

Let \(J(t) = \int_{\mathbb{R}^N} |x|^2 |u|^2\,dx\). Then
\[
J''(t) = 4Q(u(x,t)), \quad J'(t) = -4\Re \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u)\,dx.
\]

Since \(J'(0) = -4\Re \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u)\,dx < 0\), we have
\[
J'(t) = J'(0) + \int_0^t J''(\tau)\,d\tau = J'(0) + 4 \int_0^t Q(u(\cdot,\tau))\,d\tau < J'(0) < 0
\]
and
\[
0 \leq J(t) = J(0) + \int_0^t J'(\tau)\,d\tau < J(0) + J'(0)t,
\]
which implies that the maximum existence interval for \(t\) is finite, and the solution blows up in finite time. \(\Box\)
6 The Pseudo-conformal Conservation Laws and Asymptotic Behavior for the Solution

In this section, we will prove two pseudo-conformal conservation laws and consider asymptotic behavior for the solution of (1.1).

Proof of Theorem 4: Assume that $u$ is the global solution of (1.1), $u_0 \in X$ and $xu_0 \in L^2(\mathbb{R}^N)$. Using the conservation of energy, we have

$$P(t) = \int_{\mathbb{R}^N} |xu|^2 dx + 4t \mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 4t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

$$+ 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 2t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx$$

$$= \int_{\mathbb{R}^N} |xu|^2 dx + 4t \mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 8t^2 E(u_0). \quad (6.1)$$

Noticing that

$$\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |u|^2 dx = -4\mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx,$$

we obtain

$$P'(t) = \frac{d}{dt} \int_{\mathbb{R}^N} |xu|^2 dx + 4\mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 4t \frac{d}{dt} \mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 16t E(u_0)$$

$$= 4t \frac{d}{dt} \mathfrak{F} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 16t E(u_0)$$

$$= 4t \left\{ -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx 
- 8N \int_{\mathbb{R}^N} h''(|u|^2) h'(|u|^2) |u|^4 |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} (x \cdot \nabla W) * |u|^2 |u|^2 dx \right\}$$

$$+ 8t \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla h(|u|^2)|^2) dx - 4t \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx$$

$$= 4t \int_{\mathbb{R}^N} -4N [2h''(|u|^2) h'(|u|^2) |u|^2 + (h'(|u|^2))^2] |u|^2 |\nabla u|^2 dx$$

$$- 4t \int_{\mathbb{R}^N} (W + \frac{(x \cdot \nabla W)}{2}) * |u|^2 |u|^2 dx$$

$$= 4t \theta(t). \quad (6.2)$$

Integrating (6.2) from 0 to $t$, we have

$$P(t) = \int_{\mathbb{R}^N} |(x - 2t \nabla u)|^2 dx + 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 2t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx$$

$$= \int_{\mathbb{R}^N} |xu_0|^2 dx + 4 \int_0^t \tau \theta(\tau) d\tau, \quad (6.3)$$

where $\theta(\tau)$ is defined by (1.14).
2. Assume that \( u \) is the blowup solution of (1.1), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \). By the conservation of energy, we have

\[
B(t) := \int_{\mathbb{R}^N} |x + 2i(T - t)\nabla u|^2 dx + 4(T - t)^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx
- 2(T - t)^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx
= \int_{\mathbb{R}^N} |xu|^2 dx - 4(T - t)\int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 4(T - t)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx
+ 4(T - t)^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 2(T - t)^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx
= \int_{\mathbb{R}^N} |xu|^2 dx - 4(T - t)\int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 8(T - t)^2 E(u_0) \quad (6.4)
\]

and

\[
B'(t) = \frac{d}{dt} \int_{\mathbb{R}^N} |xu|^2 dx + 4\int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx
- 4(T - t)\frac{d}{dt} \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 16(T - t)E(u_0)
= -4(T - t)\int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx + 16(T - t)E(u_0)
= 4(T - t) \left\{ \int_{\mathbb{R}^N} 4N[2h''(|u|^2)h'(|u|^2)|u|^2 + (h'(|u|^2))^2]|u|^2|\nabla u|^2 dx
+ \int_{\mathbb{R}^N} (|W + \frac{(x \cdot \nabla W)}{2})*|u|^2)|u|^2 dx \right\} + 32(T - t)E(u_0). \quad (6.5)
\]

Integrating (6.5) from 0 to \( t \), we have

\[
B(t) = B(0) + 32E(u_0) \int_0^t (T - \tau) d\tau - 4 \int_0^t (T - \tau)\theta(\tau) d\tau
= \int_{\mathbb{R}^N} |(x + 2iT\nabla)u_0|^2 dx + 4T^2 \int_{\mathbb{R}^N} |\nabla h(|u_0|^2)|^2 dx
- 2T^2 \int_{\mathbb{R}^N} (W * |u_0|^2)|u_0|^2 dx + 32E(u_0) \int_0^t (T - \tau) d\tau - 4 \int_0^t (T - \tau)\theta(\tau) d\tau,
\]

where \( \theta(\tau) \) is defined by (1.14). \( \square \)

As the application of Theorem 4, we have a theorem as follows.

**Theorem 5.** 1. Assume that \( u \) is the global solution of (1.1), \( u_0 \in X \), \( xu_0 \in L^2(\mathbb{R}^N) \), and \( W(x) \leq 0 \) for \( x \in \mathbb{R}^N \). Then the following properties hold:

1) If \( 2h''(s)h'(s)s + (h'(s))^2 \geq 0 \) for \( s \geq 0 \), and \( 2W + (x \cdot \nabla W) \geq 0 \) for \( x \in \mathbb{R}^N \), then there exists \( C \) such that

\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \leq Ct^{-2} \quad \text{for} \ t \geq 1.
\]
(2) If \(2h''(s)h'(s)s + (h'(s))^2 \geq 0\) for \(s \geq 0\), and \(-cW \leq 2W + (x \cdot \nabla W) \leq 0\) for \(x \in \mathbb{R}^N\) for some \(0 < c < 2\), then there exists \(C\) such that
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)u|^2 dx \leq \frac{C}{t^{2-c}} \text{ for } t \geq 1.
\] (6.7)

(3) If \(-k_1(h'(s))^2 \leq 2h''(s)h'(s)s + (h'(s))^2 \leq 0\) for some \(0 < k_1 < \frac{2}{N}\), and \(2W + (x \cdot \nabla W) \geq 0\) for \(x \in \mathbb{R}^N\), then there exists \(C\) such that
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)u|^2 dx \leq \frac{C}{t^{2-Nk_1}} \text{ for } t \geq 1.
\] (6.8)

(4) If \(-k_1(h'(s))^2 \leq 2h''(s)h'(s)s + (h'(s))^2 \leq 0\) for some \(0 < k_1 < \frac{2}{N}\), and \(-cW \leq 2W + (x \cdot \nabla W) \leq 0\) for \(x \in \mathbb{R}^N\) for some \(0 < c < 2\), then there exists \(C\) such that
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)u|^2 dx \leq \frac{C}{t^{2-\max(Nk_1,c)}} \text{ for } t \geq 1.
\] (6.9)

In all cases above, by the conservation of energy, we have
\[
\lim_{t \to +\infty} \int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx = 2E(u_0), \quad \lim_{t \to +\infty} \|u(\cdot,t)\|_{H^1}^2 = M(u_0) + 2E(u_0).
\] (6.10)

2. Assume that \(u\) is the blowup solution of \((1.1)\), \([(h'(s))^2 + 2h''(s)h'(s)s] \leq 0\) for \(s \geq 0\) and \(2W + (x \cdot \nabla W) \leq 0\) for \(x \in \mathbb{R}^N\), \(u_0 \in X\) and \(xu_0 \in L^2(\mathbb{R}^N)\). If \(E(u_0) \leq 0\) and \(-4T^2E(u_0) - \int_{\mathbb{R}^N} |xu_0|^2 dx - 4T^3 \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u_0) dx > 0\), then
\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla h(|u|^2)|^2) dx \geq \frac{C}{(T-t)^2}, \quad \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx \geq \frac{C}{(T-t)^2}.
\] (6.11)

**Proof of Theorem 5:** 1. Assume that \(u\) is the global solution of \((1.1)\), \(u_0 \in X\) and \(xu_0 \in L^2(\mathbb{R}^N), W(x) \leq 0\).

(1) \(2h''(s)h'(s)s + (h'(s))^2 \geq 0\) and \(2W + (x \cdot \nabla W) \geq 0\). (1.12) implies that
\[
4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \leq \int_{\mathbb{R}^N} |xu_0|^2 dx,
\]
i.e.,
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \leq Ct^{-2}.
\]

(2) \(2h''(s)h'(s)s + (h'(s))^2 \geq 0\) and \(-cW \leq 2W + (x \cdot \nabla W) \leq 0\) for some \(0 < c < 2\). (1.12) implies that
\[
4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \leq \int_{\mathbb{R}^N} |xu_0|^2 dx + 2c \int_0^t \left( \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \right) d\tau.
\] (6.12)
Let
\[ A_1(t) := 2 \int_0^t \tau \left( \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \right) \, d\tau. \]

(6.12) implies
\[ A_1'(t) \leq \frac{C_0}{t} + \frac{c}{t} A_1(t). \]

Using Gronwall’s inequality, we have
\[ A_1(t) \leq t^c [A_1(1) + C - \frac{C}{t^c}] \leq C't^c. \]

Consequently,
\[ \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx + \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \leq \frac{C}{t^{2-c}}. \]

(3) \(-k_1(h'(s))^2 < 2h''(s)h'(s)s + (h'(s))^2 < 0\) for some \(0 < k_1 < \frac{2}{N}\) and \(2W + (x \cdot \nabla W) \geq 0\). (1.12) implies that
\[ 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \]
\[ \leq \int_{\mathbb{R}^N} |xu_0|^2 \, dx + 4Nk_1 \int_0^t \tau \left( \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx \right) \, d\tau. \quad (6.13) \]

Let
\[ A_2(t) := 4 \int_0^t \tau \left( \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx \right) \, d\tau. \]

(6.13) implies
\[ A_2'(t) \leq \frac{C_0}{t} + \frac{Nk_1}{t} A_2(t). \]

Using Gronwall’s inequality, we have
\[ A_2(t) \leq t^{Nk_1} c [A_2(1) + C - \frac{C}{t^{Nk_1}}] \leq C't^{Nk_1}. \]

Consequently, we have
\[ \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx + \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \leq \frac{C}{t^{2-Nk_1}}. \]

(4) \(-k_1(h'(s))^2 < 2h''(s)h'(s)s + (h'(s))^2 < 0\) for some \(0 < k_1 < \frac{2}{N}\) and \(-cW \leq 2W + (x \cdot \nabla W) \leq 0\) for some \(0 < c < 2\). (1.12) implies that
\[ 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \]
\[ \leq \int_{\mathbb{R}^N} |xu_0|^2 \, dx + 4Nk_1 \int_0^t \tau \left( \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx \right) \, d\tau + 2c \int_0^t \tau \left( \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \right) \, d\tau \]
\[ \leq C + 4 \max(Nk_1, c) \int_0^t \tau \left[ \int_{\mathbb{R}^N} |\nabla h(|u^2|)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (|W| * |u^2|)|u|^2 \, dx \right] \, d\tau. \quad (6.14) \]
Let

\[
A_3(t) := 4 \int_0^t \tau \left[ \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \right] d\tau.
\]

(6.14) implies

\[
A_3(t) \leq \frac{C_0}{t} + \max(Nk_1,c) A_3(t).
\]

Using Gronwell’s inequality, we obtain

\[
A_3(t) \leq t^{max(Nk_1,c)} [A_3(1) + C - \frac{C}{t^{max(Nk_1,c)}}] \leq C't^{max(Nk_1,c)}.
\]

Consequently, we have

\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \leq \frac{C}{t^{2 - max(Nk_1,c)}}.
\]

In all cases above, we have

\[
\lim_{t \to +\infty} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx = 0.
\]

By the conservation of energy, we get

\[
\lim_{t \to +\infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \right) = E(u_0),
\]

which means that

\[
\lim_{t \to +\infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2E(u_0).
\]

By the conservation of mass, we obtain

\[
\lim_{t \to +\infty} ||u(\cdot, t)||_{H^1}^2 = \lim_{t \to +\infty} \left( \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) = M(u_0) + 2E(u_0).
\]

(6.10) is proved.

2. Assume that \( u \) is the blowup solution of (1.1), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \), \( W(x) \geq 0 \) and \( 2W + (x \cdot \nabla W) \leq 0 \), \( 2h''(s)h'(s)s + (h'(s))^2 \leq 0 \). Using (1.13), we have

\[
2(T - t)^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx
\]

\[
= \int_{\mathbb{R}^N} |(x + 2i(T - t)\nabla)u|^2 dx + 4(T - t)^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx
\]

\[
+ 4 \int_{0}^{t} (T - \tau)\theta(\tau)d\tau - 32E(u_0) \int_{0}^{t} (T - \tau)d\tau
\]

\[
- 8T^2E(u_0) - \int_{\mathbb{R}^N} |xu_0|^2 dx - 4T\gamma \int_{\mathbb{R}^N} \tilde{u}_0(x \cdot \nabla u_0)dx.
\]

(6.15)

If

\[
-8T^2E(u_0) - \int_{\mathbb{R}^N} |xu_0|^2 dx - 4T\gamma \int_{\mathbb{R}^N} \tilde{u}_0(x \cdot \nabla u_0)dx > 0,
\]

25
then (6.15) implies that

$$\int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx \geq \frac{C}{(T-t)^2}.$$  

Using energy conservation law $E(u) = E(u_0)$, we get

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx = \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx + E(u_0) \geq \frac{C}{(T-t)^2} + E(u_0).$$

As $t$ close to $T$ enough, we have

$$\frac{C}{(T-t)^2} + E(u_0) \geq \frac{C'}{(T-t)^2}.$$

for some constant $0 < C' < C$. Hence

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \geq \frac{2C'}{(T-t)^2},$$

(6.11) holds.

References

[1] F. G. Bass and N. N. Nasanov, Nonlinear electromagnetic spin waves, Phys. Rep., 189(1990), 165–223.

[2] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaire, C. R. Acad. Sci. Paris, 293(1981), 489–492.

[3] A. V. Borovskii and A. L. Galkin, Dynamical modulation of an ultrashort high-intensity laser pulse in matter, JETP, 77(1993), 562–573.

[4] A. de Bouard, N. Hayashi and J. C. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, Commun. Math. Phys., 189(1997), 73–105.

[5] P. Cao, Global existence and uniqueness for the magnetic Hartree equation, J. Evol. Equ., 11(2011), 811–825.

[6] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics 10, New York University, Courant Institute of Mathematical Sciences, AMS, Providence, RI, 2003.

[7] Y. Cho, H. Hajaiej, G. Hwang and T. Ozawa, On the Cauchy Problem of Fractional Schrödinger Equation with Hartree Type Nonlinearity, Funkcialaj Ekvacioj, 56(2013), 193–224.
[8] Y. Cho, Short-range scattering of Hartree type fractional NLS, *J. Differential Equations*, 262(2017), 116–144.

[9] M. Colin, On the local well-posedness of quasilinear Schrödinger equations in arbitrary space dimension, *Commun. Partial Diff. Eqns*, 27(2002), 325–354.

[10] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations, *J. Funct. Anal.*, 32(1979), 1–71.

[11] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations with nonlocal interaction, *Math. Z.*, 170(1980), 109–136.

[12] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.*, 18(1977), 1794–1797.

[13] M. V. Goldman and M. Porkolab, Upper hybrid solitons ans oscillating two-stream instabilities, *Phys. Fluids*, 19(1976), 872–881.

[14] B. L. Guo, J. Q. Chen and F. Q. Su, The “blow up” problem for a quasilinear Schrödinger equation, *J. Math. Phys.*, 46(2005), 073510, 10 pp.

[15] A. Ivanov and G. Venkov, Existence and Uniqueness Result for the Schrödinger-CPoisson System and Hartree Equation in Sobolev Spaces, *J. Evol. Equ.*, 8(2008) 217–229.

[16] C. E. Kenig, G. Ponce and L. Vega, The Cauchy problem for quasi-linear Schrödinger equations, *Invent. Math.*, 158(2004), 343–388.

[17] A. G. Litvak and A. M. Sergeev, One dimensional collapse of plasma waves, *JETP Lett.*, 27(1978), 517–520.

[18] V. G. Makhankov and V. K. Fedynanin, Non-linear effects in quasi-one-dimensional models of condensed matter theory, *Phys. Rep.*, 104(1984), 1–86.

[19] M. Poppenberg, On the local well posedness of quasi-linear Schrödinger equations in arbitrary space dimension, *J. Diff. Eqns*, 172(2001), 83–115.

[20] B. Ritchie, Relativistic self-focusing and channel formation in laser-plasma interactions, *Phys. Rev. E*, 50(1994), 687–689.

[21] X. F. Song, Z. Q. Wang, Global existence and blowup phenomena as well as asymptotic behavior for the solution of quasilinear Schrödinger equation, preprint.

[22] S. Zagatti, The Cauchy problem for Hartree-Fock time-dependent equations, *Annales de l’I. H. P.*, section A, 56(1992), 357–374.

[23] J. Zhang, Sharp conditions of global existence for nonlinear Schrodinger and Klein-Gordon equations, *Nonlinear Analysis, TMA*, 48(2002), 191–207.