Cosmological fluctuations: Comparing Quantum and Classical Statistical and Stringy Effects

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Abstract

The theory of cosmological fluctuations assumes that the pre-inflationary state of the universe was the quantum vacuum of a scalar field(s) coupled to gravity. The observed cosmic microwave background fluctuations are then interpreted as quantum fluctuations. Here we consider alternate interpretations of the classic calculations of scalar and tensor power spectra by replacing the quantum vacuum with a classical statistical distribution, and suggest a way of distinguishing the quantum from the classical alternatives. The possibility that the latter is governed by a fundamental length scale as in string theory is also explored.

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1 Introduction

The theory of cosmological fluctuations is considered to be one of the crowning achievements of theoretical cosmology. Given a model for inflationary cosmology this theory enables one to calculate the measured scalar and tensor fluctuation spectra. In particular the standard theory appears to tell us the absolute value of the scale of inflation. In fact if the value of the ratio of tensor to scalar power reported by BICEP2 had held up to scrutiny, the theory would imply that we are effectively looking at energy scales as large as $10^{16}$ GeV, a scale that is practically the same as the Grand Unification (GUT) scale of particle physics. The latter is only two orders of magnitude below the Planck scale - a scale at which quantum gravity effects are necessarily of order one.

On the other this particular conclusion of the theory depends crucially on the absolute normalization of the Fourier modes of the gravitational tensor and scalar field modes. This is fixed by assuming that the initial state of the universe just prior to inflation is the quantum vacuum of free quantum fields - essentially an infinite product of harmonic oscillator ground state wave functions, i.e. the so-called Bunch-Davies vacuum. This is in effect the same as assuming that the short wave length modes are picked from a particular Gaussian distribution which involves $\hbar$ explicitly (see section 4). Indeed this is the only occurrence of the quantum of action in the entire theory of Gaussian cosmological fluctuations.

One might then ask what the consequences are of replacing the assumption of the quantum harmonic oscillator vacuum, by postulating that the initial fluctuations are given by a Gaussian statistical distribution. In this case obviously the factor of $\hbar$ will be replaced by some arbitrary factor with the same units. Consequently the relation between the power spectrum and the scale of inflation will involve an undetermined factor. This much of course is well-known and rather obvious. It should also be remarked that what is truly peculiar to quantum behavior is a) the quantization of energy $E = \hbar \omega$ and momentum $p = \hbar k$ for a wave of angular frequency $\omega$ and wave number $k$ b) the quantization of angular momentum - especially spin c) long range correlations of EPR type signaling the entanglement of quantum states. These are properties that would be very hard if not impossible to reproduce by appealing to classical statistical distributions. None of these crucial properties of quantum behavior is however tested in the cosmological observations.

Moreover the possibility that the initial configuration is described by a statistical distribution determined for instance by string theory has not been considered hitherto as far as we know. What is meant here is not the procedure that has been followed hitherto (in works such as [7, 8, 9, 10, 11]), where the initial state is still regarded as a pure quantum mechanical state and the string theory effects are just viewed as corrections to the background. Here what we will argue instead is that the standard calculations of the inflaton fluctuations may actually be replaced by an initial configuration which is Gaussian distributed whose width is determined by $\hbar$. In other words the initial distribution that should be used is the result of some pre-inflationary process of decoherence of some initial (possibly stringy) pure state, which may have been the quantum mechanical state of the multiverse. The question we are asking is, what if this initial distribution is determined by some other parameter with the same dimensions as $\hbar$. String theory for instance (unlike quantum

\footnote{See for instance [1]-[3].}
\footnote{For a discussion of the relation of quantum to classical distributions that is in the same spirit as this paper see [6].}
field theory) has a natural fundamental length scale $l_s$ which may be defined in terms of the Regge
slope $\alpha'$ as $l_s^2 = 2\pi\alpha'$. Together with the gravitational coupling constant $\kappa^2$ we can then define
(having set $c = 1$) a fundamental unit of action $l_s^2/\kappa^2$. Note that this is a ratio of two classical
constants. Thus if string theory is the fundamental theory of the universe, one might consider as
an alternative to the usual assumption for the initial state, the possibility that it is a Gaussian
distribution involving this unit of action $\hbar^2$ rather than $\hbar$.

In fact it seems unlikely that the standard argument for the initial state of the inflationary
cosmology being the Bunch Davies vacuum is valid in the context of string theory. Within the
context of a scalar field theory coupled to gravity, the analysis of quantized perturbations in the
inflationary background is performed under the assumption that the pre-inflationary primordial
state of the universe continues to be described by this theory - i.e. all the way back to the initial
singularity at the scale factor $a = 0$. An argument for choice of this vacuum has been given in
using contour rotation in an imaginary time direction. This essentially corresponds to the choice of
the Hartle-Hawking “no boundary” proposal for the wave function of the universe which vanishes
at the origin $a = 0$.

There are two issues that need to be considered in connection with this reasoning. Firstly,
it appears that this wave function (in contrast to the so-called tunneling wave function), has
very low probability for leading to inflationary dynamics (see for instance [13]). Secondly, and
more importantly, if the fundamental theory that describes the universe is string theory, then this
simple picture is unlikely to be the whole story particularly in the pre-inflationary stage. It is far
more likely to have been some primordial stringy state, and the stage before inflation may have
been one with a primordial distribution of the decay products of string and Kaluza-Klein states.
Unfortunately in the absence of a solid theoretical construction of such an initial state, all we can
do is to parametrize our ignorance with some simple ansatz. Here we choose to pick the initial
state from a Gaussian statistical distribution. We note that choosing the width of the distribution
to correspond to the free field theory one gives us the usual story, while choosing it to correspond
to one that might plausibly arise from string theory gives us different results which may then
be compared to the usual ones.

If the physics of the universe is governed by string theory, then there is a further reason to
think that the sort of conjecture described in the previous paragraph may be reasonable. This is
because at low energies even at the classical level (i.e. zero string loop level) one expects higher
derivative ($R^2$ etc.) terms in the effective action. These will lead to terms in the effective stress
tensor that will be larger than those that come from quantum effects in quantum field theory.
In fact as is well known (and discussed in detail below) semi-classical string theory has a double
expansion - the $\alpha'$ expansion as well as the semi-classical string loop expansion. It is the latter
which corresponds to the the standard calculations of QFT in curved backgrounds as discussed
for instance in the classic text book by Birrell and Davies [14]. The former is a purely stringy
effect and is usually not considered in cosmological discussions. One of the aims of this paper is
to discuss the consequences for the theory of cosmological fluctuations, of the leading terms in the
classical string theory $\alpha'$ expansion assuming that this is related to the stringy modification of the
initial state that we described earlier. These as we will see below, are actually larger than the

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3If the compactification scale is the string scale and the string coupling $g_s = 1$, then these two units of action
may be identified. However if this is the case we are no longer in the weak coupling large volume regime in which
one is able to analyze the cosmology and phenomenology of string theory.
terms which may be identified with the standard QFT calculation.

1.1 Units

We work with units where the velocity of light \( c = 1 \) (so that a unit of time is equivalent to a unit of length \( T = L \)) but do not set \( \hbar = 1 \) so that we keep independent units of length \( L \) and mass \( M \). Also as usual we define \( \kappa^2 = 8\pi G \). Note that \([\hbar] = ML\) and \([\kappa^2] = M^{-1}L\) and the Planck length \( l_P \) defined by \( l_P^2 = \kappa^2\hbar \) has units of length while the Planck mass \( M_P \) defined by \( M_P^2 = \hbar/\kappa^2 \) has dimensions of mass. In quantum field theory (for instance the standard model coupled to gravity treated semi-classically), the Planck length is a derived quantity and is usually regarded as a length scale that goes to zero in the classical limit \( \hbar \to 0 \). In string theory on the other hand there is an independent fundamental length \( l_s \) defined as the fundamental scale of the 2D sigma model.\(^4\) In particular the loop expansion of the gravitational constant will take the form

\[
\frac{1}{2\kappa^2} = \frac{1}{2\kappa_0^2} + \frac{\hbar}{l_s^2} f\left(\frac{\kappa_0^2\hbar}{l_s^2}\right)
\]

(1)

where \( \kappa \) is the physical (i.e. renormalized) gravitational constant \( \kappa_0 \) is the bare constant and \( l_s \) may be naturally identified with the string scale in string theory but is an arbitrary short distance cutoff in QFT. In the rest of the paper the gravitational constant is taken to be the physical constant \( \kappa \).

Note also that we use the mostly positive metric convention and the Ricci tensor is defined as \( R_{jk} = R^i_{jik} \).

2 Inflationary fluctuations

2.1 Review

Inflationary cosmology\(^5\) is usually formulated in terms of a theory of a scalar field coupled to gravity with the action,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{g} R + S_m,
\]

(2)

\[
S_m = -\int d^4x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right).
\]

(3)

Note that \([S] = ML\), \([g_{\mu\nu}] = L^0 M^0\), \([\phi] = M^{1/2} L^{-1/2}\).

The classical Einstein equation for this system is

\[
G_{\mu\nu} = \kappa^2 T_{\mu\nu},
\]

(4)

\[
T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi + V(\phi)).
\]

(5)

\(^4\)Unlike a standard action the string sigma model action has dimensions of \( L^2 \) rather than \( ML \) since the field is the coordinate in the ambient space. So the functional integral is defined by introducing a fundamental length scale \( l_s \).

\(^5\)I’ve followed closely the discussion in the review by [1] in this section.
Since the system is generally covariant the Einstein equation implies the conservation of the
stress tensor and when there is only one scalar field it also implies the equation of motion
\[ \nabla^2 \phi = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = -\frac{\partial V}{\partial \phi}. \] (6)

For an FRW (homogeneous isotropic) background where \( g_{\mu\nu} = \text{diagonal}(-1, a^2(t)\delta_{ij}), i, j = 1, 2, 3; \phi = \phi(t), \) we have two independent equations,
\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \] (7)
\[ \dot{H} = -\frac{1}{2} \kappa^2 \dot{\phi}^2. \] (8)

Inflation requires a period of accelerated expansion which leads to the so-called slow roll conditions
\[ \epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1, \eta \equiv -\frac{\ddot{H}}{2HH} \ll 1. \] (9)

For the single inflaton case above this translates into conditions on the potential:
\[ \epsilon = \frac{1}{2\kappa^2} \left( \frac{\partial V(\phi)}{V(\phi)} \right)^2 \ll 1, \eta = \frac{1}{\kappa^2} \left[ \frac{\partial \phi V(\phi)}{V(\phi)} - \frac{1}{2} \left( \frac{\partial V(\phi)}{V(\phi)} \right)^2 \right]. \] (10)

To analyze fluctuations around this background one typically goes to conformal coordinates in
which the background metric takes the form \( ds^2 = a^2(\tau)(-d\tau^2 + dx^2) \) taking spatial curvature to
be zero. The standard procedure is to impose canonical commutation relations on the scalar field
after writing \( \phi(\tau, x) = \phi(\tau) + \delta \phi(\tau, x) \) where the first term is the classical background field. So
one expands
\[ \delta \phi(\tau, x) = \frac{1}{(2\pi)^{3/2}} \int d^3k (\phi_k(\tau)b_k e^{ik \cdot x} + h.c.) \] (11)

Writing \( \phi_k(\tau) = a^{-1}(\tau)u_k \) we have from (11) (with \( X' \equiv \frac{dX}{d\tau} \)) in the slow roll approximation,
\[ u_k'' + (k^2 - \frac{a''}{a})u_k = 0. \] (12)

Now the standard argument is to identify the fluctuation spectrum of \( \phi, \) with the quantum fluctuations of an essentially free field in the vacuum state. Observing that for \( k^2 \gg \frac{a''}{a} \) one has in effect plane waves, and the physics is that of Minkowski space, so one can follow the standard procedure
for flat space quantization.

One takes the scalar field to be canonically quantized i.e. \( [\phi(x, \tau), \pi(y, \tau)] = i\hbar \delta^3(x - y), \)
with the other two commutators vanishing. If the solutions to (12) are normalized with the usual
Klein-Gordon norm (i.e. \( u_k u^*_{k'} - u^*_k u_{k'} = i \)) then the operators \( b_k \) satisfy the relations
\[ [b_k, b^\dagger_{k'}] = \hbar \delta^3(\mathbf{k} - \mathbf{k'}), \]

\[ ^6 \text{We've assumed for simplicity that the three curvature is zero.} \]
with the other commutators vanishing. It is convenient to define now the scalar power spectrum although this is not what is physically relevant (i.e. related to the temperature fluctuations). This is defined as

\[ P(k, \tau) = \frac{\hbar k^3}{2\pi^2} |\phi_k(\tau)|^2 \]  

in terms of which the autocorrelation function of the scalar field fluctuation (in a spatially translational invariant background) is

\[ \langle \phi(\tau, x)\phi(\tau, y) \rangle = \int \frac{dk}{k} P(k) \frac{\sin k|x-y|}{k|x-y|} \]  

The power spectrum of the scalar curvature fluctuation which is related to the measured temperature fluctuation is then given as (with \( N \) being the number of e-foldings regarded as a function of \( \phi \) with \( dN = Hdt \))

\[ P_R(k) = \left( \frac{\delta N}{\delta \phi} \right)^2 P(k) = \frac{\kappa^2 P(k)}{2\epsilon}. \]  

Note that this is independent of the normalization of \( \phi \) as it should be, since it is directly related to a measurable physical effect.

For short wave lengths \( k\tau \gg 1 \) as in the above discussion the normalized solution (consistent with the Lorentz invariant plane wave solution for \( \phi \)) to (12) is \( u_k = e^{ik\tau}/\sqrt{2k} \) and

\[ P(k, \tau) = \frac{\hbar}{4\pi^2} \frac{k^2}{a^2(\tau)} = \frac{1}{4\pi^2} \frac{\hbar}{\lambda_{\text{physical}}(\tau)}. \]  

For constant \( \epsilon \) it is possible to find the exact solutions to (12) and the solution that asymptotes to the Minkowski solution for short wave lengths is

\[ u_k(\tau) = \frac{1}{2} \sqrt{\frac{\pi}{k}} \sqrt{-k\tau} H^{(1)}_\nu(-k\tau), \]  

where \( H^{(1)}_\nu \) is the Hankel function of the first kind and

\[ \nu = \frac{3 - \epsilon}{2(1 - \epsilon)}. \]  

In the long wave length limit we have

\[ [P(k, \tau)]^{1/2} = h^{3/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1 - \epsilon) \frac{H}{2\pi} \left( \frac{k}{a(\tau)H(1 - \epsilon)} \right)^{\frac{3}{2} - \nu}. \]  

For the exactly deSitter case (\( \epsilon = 0 \))

\[ P(k, \tau) = h \left( \frac{H}{2\pi} \right)^2. \]  

There is a similar formula for the tensor power spectrum. Defining the tensor fluctuations around the FRW background as

\[ \delta g_{ij} \equiv a^2 h_{ij} = a^2 (h_+ e^+_{ij} + h_- e^-_{ij}), \quad h_\lambda(k) = \frac{1}{(2\pi)^3} \int d^3x e^{ikx} h_\lambda(x). \]
Here $e^\lambda_{ij}$, $\lambda = +, \times$ is the polarization 3-tensor satisfying
\begin{align}
e^+_{ij} & = e_{ji}^\times, k^i e^\lambda_{ij} = 0, e_{ii} = 0, e^\lambda_{ij}(-k) = e^\lambda_{ij}(k)^* \\
\sum \lambda e^\lambda_{ij} e^\lambda_{ij} & = 4.
\end{align}

From the Einstein action we then have in the Gaussian approximation the action for the tensor fluctuations
\begin{equation}
\Delta S = \frac{1}{2\kappa^2} \int \sqrt{g_{(0)}} g^{\mu\nu}_{(0)} \frac{1}{2} \partial_\mu h_{ij} \partial_\nu h_{ij}
\end{equation}

This is essentially a sum of four free scalar fields so that defining $v_k \equiv \frac{1}{\sqrt{2\kappa}} a(\tau) h_k$ in analogy with (11) (and the line below it) we have for the power spectrum in tensors,
\begin{equation}
P_T(k) = \frac{k^3}{2\pi^2} \sum \lambda |h_{\lambda k}|^2 = 8\kappa^2 P(k) = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}.
\end{equation}

From (18) and the above we have
\begin{equation}
r \equiv \left. \frac{P_T}{P_R} \right|_{k=aH} = 16\epsilon = -8n_T
\end{equation}

where we’ve parametrized $P_T \propto k^{n_T}$.

### 2.2 Quantum effects in the stress tensor

In this subsection it will convenient to use natural units $c = \hbar = 1$.

The energy density gets a contribution from the inflaton quantum fluctuations:
\begin{equation}
< 0|T_{00}^\phi|0 > = a^{-2}(\tau) < 0|T_{\tau \tau}^\phi|0 > = \frac{1}{2} a^{-2} \int \frac{d^3 k}{(2\pi)^3} (|\phi'_k|^2 + k^2 |\phi_k|^2)
\end{equation}

There is also a contribution from graviton fluctuations:
\begin{equation}
< 0|T_{00}^h|0 > = 8 < 0|T_{00}^\phi|0 >.
\end{equation}

The quantum energy density may be written as a sum of three terms in the different wave length regimes.
\begin{equation}
< 0|T_{00}|0 > = \int_{k \ll a^{-1}}^{k\ll a \tau^{-1}} \frac{dk}{k} \frac{k^4}{4\pi^2 a^4(\tau)} + \int_{k \ll aH}^{k_{IR}} \frac{dk}{k} \frac{k^2}{a^2(\tau)} P(k) + \int_{k_{IR}=aH}^{k_{IR}} \frac{dk}{k} \frac{k^2}{a^2(\tau)} P(k).
\end{equation}

The last term above just gives a contribution (assuming exact deSitter $\epsilon = 0$ for simplicity) corresponding to the Gibbons-Hawking temperature:
\begin{equation}
< T_{00} >_{IR} = \frac{4\pi^2}{2} \left(\frac{H}{2\pi}\right)^4 = 2\pi^2 T_{dS}^4.
\end{equation}
and is negligible compared to the classical energy during inflation \( < T_{00} >_{IR} \ll 3H^2M_P^2 \approx V(\phi) \) since we must necessarily have \( H \ll M_P \) for the validity of the EFT.

The contribution in the deep UV however is as usual divergent. To evaluate it we impose a cutoff at some comoving scale \( k_{UV} \) and evaluate the short distance contribution from \( k^2 \gg a''/a \) by using the Minkowski (BD vacuum) modes \( \phi_k = a^{-1}u_k = a^{-1}e^{ik\tau}/\sqrt{2k} \).

\[
<0|T^0_0|0>_{uv} = \frac{1}{2a^2} \int_{k_{UV}}^{k_{UV}} \frac{d^3k}{(2\pi)^3} \frac{2k^2}{2k} \sim \frac{k_{UV}^4}{16\pi^2a^4} = \frac{k_{UV}^{(ph0)^4}}{16\pi^2} \quad (29)
\]

This must necessarily be smaller than the classical potential energy density at the onset of inflation for the validity of inflationary cosmology\( ^7 \) so that

\[
k_{UV}^4 < 192\pi^2H^2M_P^2 \quad (30)
\]

In a supersymmetric theory on the other hand this quartic divergence will be absent and (for SUSY broken at a gravitino mass scale \( m_{3/2} \) we have instead of (30) the relation

\[
k_{UV}^2 < 192\pi^2\frac{H^2}{m_{3/2}^2}M_P^2. \quad (31)
\]

On the other hand at late times (i.e. after many e-folds of inflation), the UV contribution (assuming a fixed physical cutoff at the onset of inflation), will have been inflated away and only the last contribution in (27) will survive. It is this that will be compared with the string theory contribution below.

### 3 String theory expansion

In this section we revert back to units in which \( c = 1 \) but \( \hbar \) is not set equal to unity.

In quantum field theory the perturbative expansion is an expansion in the number of loops with \( \hbar \) serving as a loop counting parameter. The quantum effective (1PI) action has the expansion,

\[
\Gamma(g_{\mu\nu}, \phi, \hbar) = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \ldots = \sum_{l=0}^{\infty} \hbar^l\Gamma_l, \quad \Gamma_0 = S,
\]

\( S \) being the classical action.

In perturbative string theory\( ^{17,18} \) each loop order is defined through the functional integral \( Z = \int [dX]\exp\{-I\} \) over the embedding coordinates \( X(\sigma) \) defining the world sheet in the ambient space, weighted by the sigma model (dimensionless) action

\[
I = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \gamma^{ab}\partial_aX^\mu\partial_bX^\nu g_{\mu\nu}(X) + \ldots .
\]

The \( l \)th loop order is defined by the action on a Riemann surface of genus \( l \) and \( \alpha' \) is the squared string length so that \( [\alpha'] = L^2 \). The loop counting parameter here is the string coupling \( g_s \).

\( ^7 \) The constraints coming from this for different inflationary scenarios and how they can be mitigated in supersymmetric scenarios will be discussed in a separate publication\( ^{16} \).
Although there is no non-perturbative background independent formulation of string theory valid at arbitrarily high scales, one can still construct a low energy effective action.

However in order to get low energy four-dimensional physics we need to compactify string theory on an internal manifold. For the purpose of discussing inflation in four dimensions we assume that the volume of this manifold is fixed at a value $V$ in string units (i.e. the physical volume is $V(2\pi\sqrt{\alpha'})^6$). Then we have the following standard relations between the four dimensional gravitational coupling $\kappa$, Planck’s constant $\hbar$ and the string theory parameters - namely:

$$\hbar \kappa^2 \equiv l_p^2 = \frac{2\pi\alpha'}{V}g_s^2 = \frac{l_s^2 g_s^2}{V}$$

(33)

It is important to note that both $\kappa^2$ and $l_s^2 \equiv 2\pi\alpha'$ are classical parameters. So the semi-classical expansion in $\hbar$ is equivalent (in string theory) to the expansion in terms of the squared string coupling $g_s^2$. Of course the validity of this expansion requires that the dilaton has been stabilized such that $g_s^2 < 1$.

Now the long distance quantum effective action coming from string theory has a double expansion defined as follows. We have the (quantum) semi-classical expansion as before, i.e. we again have (32), but now the expansion is in terms of $g_s^2$ so we have

$$\Gamma(g_{\mu\nu}, \phi, g_s) = \Gamma_0 + g_s^2 \Gamma_1 + g_s^4 \Gamma_2 + \ldots = \sum_{l=0}^{\infty} g_s^{2l} \Gamma_l.$$ (34)

Each term in this expansion is given at long distances (compared to the string scale $l_s$) as an expansion in powers of $l_s^2$. Thus we may write schematically (keeping only pure metric dependent terms at higher orders),

$$\Gamma_l = \frac{1}{2\kappa^2} \int d^4x \sqrt{g} \left[ \delta_{10}(R + 2\kappa^2 L_m) + l_s^2 (R)^2_l + l_s^4 (R)^3_l + \ldots \right]$$ (35)

In the above $L_m$ is the classical matter lagrangian and the notation $(R)^n_l$ represents all possible contractions of curvatures and covariant derivatives to yield terms with $2n$ derivatives of the metric, with dimensionless loop order dependent coefficients (some of which may be zero), that are determined once the string theory data are given.

The gravitational equation of motion coming from (34) $\partial \Gamma / \partial g_{\mu\nu} = 0$, then takes the form (after moving all string/quantum corrections to the RHS of the equation),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}.$$ (36)

Here the RHS is the effective stress-energy tensor coming from the full quantum theory and is given by the double infinite series,

$$\kappa^2 T_{\mu\nu} = \kappa^2 \left< \dot{T}_{\mu\nu} \right> = \kappa^2 T_{\mu\nu}^{(m)} + l_s^2 (R^2)_{\mu\nu}(c_0^0 + c_2^0 \frac{1}{\sqrt{2^3}} + c_4^0 \frac{1}{V} + \ldots) + O(l_s^4) \text{ classical}$$

$$+ g_s^2 l_s^2 (R^2)_{\mu\nu}(c_0^1 + c_2^1 \frac{1}{\sqrt{2^3}} + c_4^1 \frac{1}{V} + \ldots) + O(g_s^2 l_s^2) \text{ 1-loop}$$

$$+ 2 - \text{loop} + \ldots.$$ (37)
The first line represents the classical contributions to the stress tensor including all classical string corrections. Note however that \( c_0^0 \neq 0 \) only for type I and Heterotic strings. For type II strings \( c_0^0 = 0 \) and the leading classical term is the \( c_0^2 \) term. The second line is the leading quantum correction \( O(h) \) and so on. The subscripts on the \( c \)'s is half the number of derivatives in the six compact space directions in the original 10D action, from which the corresponding 4D term is extracted. The powers of \( V \) come from the contractions in the internal directions which scale like \( g_{ij} = V^{-1/3} \hat{g}_{ij} \) where \( \hat{g} \) is a fiducial metric normalized such that the volume of the internal space is \( (2\pi)^6 \alpha' \). Correspondingly we expect the internal curvature in the hatted metric to be \( O(l_s^{-2}) \). The superscripts on the \( c \)'s give the loop order. In principle given a string theory and its compactification data these coefficients are determined.

4 Comparison with cosmological calculation

The standard expressions for the scalar and tensor curvature fluctuation are

\[
P_R = \frac{\hbar \kappa^2}{2\epsilon} \left( \frac{H}{2\pi} \right)^2 = \frac{g_s^2 l_s^2}{2\epsilon V} \left( \frac{H}{2\pi} \right)^2, \quad P_T = 8\hbar \kappa^2 \left( \frac{H}{2\pi} \right)^2 = 8 \frac{g_s^2 l_s^2}{V} \left( \frac{H}{2\pi} \right)^2, \tag{38}
\]

where in the second equalities in each of the above we have reexpressed \( \hbar \kappa^2 \) in terms of the string coupling and length scale using (33). The corresponding contribution to the stress tensor is given by (see (28))

\[
\kappa^2 < T_{00} > \sim 2\pi^2 \hbar \kappa^2 \left( \frac{H}{2\pi} \right)^4 = 2\pi^2 g_s^2 l_s^2 \left( \frac{H}{2\pi} \right)^4, \tag{39}
\]

and can only come from the first term in the second line of (37) and is therefore equivalent to a one-loop string effect. However as discussed in the previous subsection, string theory may also have somewhat larger contributions corresponding to some of the terms in the first (classical) line of (37) as well the leading term in the second line. In fact the first term in the first line would be larger than (39) by a factor \( V/g_s^2 \). However it should be emphasized that this term is present (i.e. \( c_0^0 \neq 0 \) in (37)) only for type I or Heterotic strings where the volume cannot be larger than about a factor of 20 so this is only a factor \( \gtrsim 10 \). In type II strings on the other hand the volume could be much larger \( V > 10^3 \). In this case the leading classical term in (37) is the \( c_0^2 \) term and gives a contribution which is a factor \( V^{1/3}/g_s^2 \gg 10 \) larger than (39).

To understand where such a contribution might come from in the context of the usual arguments, we should revisit the assumptions for inflationary initial conditions.

Suppose that the initial value (at \( \tau \to \tau_0 \gg k^{-1} \) for all relevant comoving scales \( k \)) of the field \( \phi \) is Gaussian distributed with a probability density

\[
p(\phi)d\phi = \lim_{\tau \to \tau_0} \exp \left[ -\frac{1}{2} \int d^3x \int d^3y \phi(x, \tau) K(x, y; \tau) \phi(y, \tau) \right] d\phi \equiv e^{-\frac{1}{2} \phi.K.\phi}. \tag{40}
\]

All initial correlation functions are then given in terms of the two point function and are computed from the generating formula

\[
< e^{J.\phi} > = e^{W[J]} = e^{W(0)} e^{\frac{1}{2} J.K^{-1}.J}. \tag{41}
\]

---

8 This comes from a \("R^4\) term in the 10D long distance effective action.
Here $K^{-1} = \int d^3 y K^{-1}(x, y) K(y, z) = \delta^3((x - z)$ and initial two point function is given by

$$< \phi(x, -\infty) \phi(y, -\infty) > = K^{-1}(x, y; -\infty). \quad (42)$$

Now the crucial assumption of the theory of cosmological fluctuations is that the initial probability distribution is the same as that corresponding to a initial quantum mechanical state given by the free field (harmonic oscillator) vacuum. This corresponds to choosing (after setting $a(\tau = \tau_0) = 1$ for convenience)

$$K = \frac{2\mathcal{E}}{\hbar} = \frac{1}{\hbar} \int \frac{d^3 k}{(2\pi)^3} e^{ik(x-y)} 2k \quad (43)$$

Then we have from (12) for the initial value of the two point function

$$< \phi(x, \tau_0) \phi(y, \tau_0) > = \hbar \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik(x-y)}}{2k} = \hbar \int \frac{dk}{k} \frac{k^3}{2\pi^2} \frac{1}{2k} \frac{\sin k|x-y|}{k|x-y|} \quad (44)$$

This in fact is the initial value of the standard calculation (see (13) and (14)) which gives

$$< \phi(x, \tau) \phi(y, \tau) > = \hbar \int \frac{dk}{k} \frac{k^3}{2\pi^2} |\phi_k(\tau)|^2 \frac{\sin k|x-y|}{k|x-y|}$$

when the limit $\tau \to \tau_0$ is taken since $\phi_k(\tau) \to e^{-ik\tau}/\sqrt{2k}$ (recall that we set $a(\tau = \tau_0) = 1$). The point is that the dependence on $\hbar$ and hence the supposed quantum nature of the cosmological perturbations, just comes from the normalization derived from the assumption that the initial distribution of short wave length fluctuations is given by the product of quantum harmonic oscillator ground state wave functions.

It would be nice to have some criterion for actually testing this hypothesis. But in any case we should entertain also the possibility that the initial state of inflation is simply a classical statistical distribution given by (40). In fact using (11) the probability distribution (40) becomes (after using (43) and averaging over the initial time $\tau_0$ (with $|k\tau_0| \gg 1$) so as to get rid of the oscillatory pieces,

$$p(\phi) d\phi = \exp \left( -\frac{1}{\hbar} \int d^3 k b_k b_k^* \right) \prod_q dq b_q \prod_p db_p^* \quad (45)$$

The usual free quantum field theory calculation is thus completely equivalent to the above classical distribution which gives

$$< b_k b_q^* > = \hbar \delta^3(k - q), \quad < b_k b_q > = < b_k^* b_q^* > = 0 \quad (46)$$

Here we will consider the consequences of assuming that the initial distribution is defined by replacing the quantum unit of action $\hbar$ in (43) by some other unit of action $\mathcal{A}$. Now in classical physics there is no fundamental unit of action but in string theory one can define such a unit,

$$\mathcal{A} = \frac{l_s^2}{\kappa^2}, \quad (47)$$

Note that $[b_k] = M^{1/2} L^2$. 

9
where \( l_s \) is the string scale defined after (33). In this case the initial probability distribution is given again by (40), but now with the kernel being given by

\[
K = \frac{2\mathcal{E}}{\mathcal{A}} = \frac{2\kappa^2}{l_s^2} \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)}k
\]

(48)
corresponding to having the correlator

\[
\langle b_k b_q^* \rangle = \frac{l_s^2}{\kappa^2} \delta^3(k - q). \quad (14) \quad \text{(with (13)) is replaced by}
\]

\[
\langle \phi(x, \tau) \phi(y, \tau) \rangle = \frac{l_s^2}{\kappa^2} \int \frac{dk}{k^2} \frac{k^3}{\sqrt{2\pi^2}} |\phi_k(\tau)|^2 \text{sin} k|x-y| \]

(49)
and the scalar field power spectrum (for simplicity we take \( \epsilon = 0 \)) becomes,

\[
P(k) = \frac{l_s^2}{\kappa^2} \left( \frac{H}{2\pi} \right)^2.
\]

The physical power spectra for scalar curvature and tensor fluctuations is now,

\[
P_R = \frac{l_s^2}{2\epsilon} \left( \frac{H}{2\pi} \right)^2, \quad P_T = 8l_s^2 \left( \frac{H}{2\pi} \right)^2.
\]

These are a factor \( \mathcal{V}/g_s^2 \) larger than the standard values quoted in (38). Correspondingly a given observed power spectrum will imply a lower scale of inflation (by a factor \( g_s^2/\mathcal{V} \)) compared to the standard result. Also this power spectrum corresponds to a contribution to the stress tensor at late times that is of the same order as the leading string correction in line one of (37).

Thus the initial conditions with \( h \) replaced by the unit of action (47) seems to correspond to the situation that one might obtain in Heterotic and type I string theories where the coefficient \( c_0^0 \neq 0 \). On the other hand if the string theory is type II then this coefficient is zero and the leading term is \( O(l_s^2/\mathcal{V}^2) \). This would correspond to a initial distribution with a kernel whose normalization factor is given by \( \mathcal{A} = \frac{l_s^2}{\sqrt{2\pi^2} \kappa^2} \) rather than (47). In this case the power spectra become,

\[
P_R = \frac{l_s^2}{2\epsilon \mathcal{V}^{2/3}} \left( \frac{H}{2\pi} \right)^2, \quad P_T = 8\frac{l_s^2}{\mathcal{V}^{2/3}} \left( \frac{H}{2\pi} \right)^2.
\]

(51)

Going back to natural units \( \hbar = 1 \) for simplicity, the bound on the cutoff (31) becomes

\[
k_{\text{UV}}^2 < 192 \pi^2 \frac{l_s^2}{l_s^2} \frac{H^2}{m_{3/2}^2} M_p^2
\]

(52)
for the Heterotic case (i.e. with \( \mathcal{A} = l_s^2/l_s^2 \)), and

\[
k_{\text{UV}}^2 < 192 \pi^2 \mathcal{V}^{2/3} \frac{l_s^2}{l_s^2} \frac{H^2}{m_{3/2}^2} M_p^2,
\]

(53)

\footnote{Recall that the standard normalization corresponds to setting \( \mathcal{A} = \hbar = \frac{l_s^2}{\kappa^2} \).}
for the IIB case with $A = l_s^2/\sqrt{2}l_p^2$. In a string theory set up one expects the $k_{UV} = M_{KK} = M_P/\sqrt{2}$ giving the mild constraints

$$V > \frac{1}{(192\pi^2 g_s^2)^3}, \text{ Heterotic,}$$

$$V > \frac{1}{(192\pi^2 g_s^2)^{1/3}}, \text{ IIB.}$$

The modified scalar curvature and tensor power spectra imply corresponding contributions to the stress tensor from higher derivative terms. In particular they would imply the second term on the first line of (37). As discussed before these classical string contributions to the stress tensor would be larger than the quantum effects of the standard contribution, and from a string theory standpoint perhaps justify the alternative initial state suggested in (48).

We also point out that the quantum corrections to the stress tensor at late times implied by the usual assumptions for the initial state, are consistent with the string theory arguments discussed above, only if the classical stringy corrections (the $c_0$ terms) in the expansion for the effective stress tensor (37) are all absent. This is generically not the case in string theory though it is not inconceivable that there may be compactifications that have this property. Now one may ask why one should choose the precise formula (47) as replacement for $\hbar$ in (46) and hence in (49). Obviously any numerical multiple (say $A \rightarrow \lambda A$) will have the same dimensions and the relations (50) will get multiplied by $\lambda$. This just reflects the fact that at the classical level there is no reason to prefer one value of the unit of action over another. However from a string theory point of view this ambiguity is fixed for a given compactified string theory. As we saw at the end of the last section even the order of magnitude of the normalization factor will change depending on the type of string theory that is being considered. In the classical $\alpha'$ expansion terms in the first line of (37), once a particular string theory and its compactification data are given, the coefficients of the curvature squared terms are determined. This in turn fixes the ambiguity in the value of $\lambda$.

### 5 Higher point functions and quantum vs classical evolution

As discussed above (and shown in more detail in Appendix I) the only thing “quantum” about the usual calculation is the use of the unit of action $\hbar$ in the normalization of the two point function. Now we will address the question of whether the replacement of Planck’s unit of action by some other unit $A$ will give rise to observable consequences - for instance in the Bispectrum, Trispectrum or higher point functions (for a review see [19]). As discussed in detail in the appendix the only possible occurrence of the unit of action is in the expression for the Wick contraction (the two point function).

The expectation value of the observable $A(t)$, a product of field operators at the time $t$, is given by evaluating

$$< \hat{A}(t) >= \int \Pi dbkdb_k^* e^{-\frac{i}{\hbar} \int b_k^* \hat{b}_k \hat{b}_k^* d^3q A[b, b^*; t]},$$

where as discussed in the Appendix the factor $A[b, b^*; t]$ which is usually evaluated using quantum operator equations of motion, can be equally well evaluated (in terms of its initial value) by using
the classical evolution equation (77) or (78). This evolution does not give rise to any factors of \( \hbar \).

All such factors come from the correlators

\[
< b_k b_q^* > = \hbar \delta^3(k - q).
\]  

(55)

Let us now use the notation explained in the Appendix where we write \( z^i \) for a field (for \( i = 1, \ldots, n \)) or its conjugate variable (for \( i = n + 1, \ldots, 2n \)). The interaction Hamiltonian (in the interaction picture) is then at least cubic in the interaction picture fields \( z^i \) which obey free field equations of motion and can be expressed in terms of the classical solution to the free field equations and the \( b_k \)'s. Suppose that we wish to compute \( [54] \) for an \( n \)-point function at equal times, \( A = A_n \).

Consider the \( N \)th term of the expression for \( A_n[b, b^*] \) in equation (78) of Appendix I, namely

\[
\int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{N-1}} dt_N \quad [H^y_I(z_I(t_N), t_N), [H^y_I(z_I(t_{N-1}), t_{N-1}) \cdots [H^y_I(z_I(t_1), t_1), A_{n, I}(t)], \ldots]
\]

\[
\sim A_{n+N, I} + \ldots,
\]

since each commutator removes one factor of \( z_I \) from \( A_n \) and one factor from \( H_I \) and adds two factors of \( z \). \( H_I \) in general has higher than third order terms in \( z \) and the ellipses represent terms which have more factors of \( z \). However in the slow roll approximation these are suppressed so let us ignore them, although of course in evaluating the higher point functions they will be needed since they could become competitive with loop terms involving lower order interactions. Then for \( n = 2r + 1 \), the only non-vanishing terms have \( N \) odd. i.e. writing \( N = 2M + 1 \), the factor of \( \hbar \) coming from evaluating the integral on the RHS of \( [54] \) is \( \hbar^{(n+1)/2} = \hbar^{r+M+1} \). Similarly for \( n = 2s \) the non-vanishing terms are proportional to \( \hbar^{s+P} \) where \( N = 2P \). Thus we have

\[
< \dot{A}_{2r+1}(t) > \sim \hbar^{r+1}(a_{r+1} + a_{r+2} \hbar + \ldots + a_{rM} \hbar^M + \ldots)
\]

(56)

\[
< \dot{A}_{2s}(t) > \sim \hbar^s(b_{s0} + b_{s1} \hbar + \ldots + b_{SP} \hbar^P + \ldots)
\]

(57)

Note that for connected correlation functions some of the leading terms above are absent. For instance the connected four point function has the expansion

\[
< \dot{A}_4(t) > \sim \hbar^2(b_{21} \hbar + \ldots + b_{SP} \hbar^P + \ldots)
\]

provided of course the \( z^4 \) term in \( H_I \) is suppressed. On the other hand the quartic term in \( H_I \) may need to be retained if though suppressed relative to the cubic term, it is competitive with the one-loop \( (b_{21}) \) term. Also the two point function and hence the power spectrum has the expansion,

\[
P \sim \hbar(b_{10} + b_{11} \hbar + \ldots).
\]

In particular this implies that

\[
\frac{< \dot{A}_3(t) >}{P^2(t)} \sim a_{11} + a_{12} \hbar + \ldots,
\]

(58)

\[
\frac{< \dot{A}_4 >}{P^2(t)} \sim b_{21} \hbar + \ldots.
\]

(59)

\(^{11}\) For a general discussion of the action for fluctuations around the inflationary background in single field inflation see [20].
Let us now replace the “quantum distribution” by some classical distribution i.e. we replace \( h \rightarrow \mathcal{A} \) (= \( l_s^2/\kappa^2 \) for instance) as discussed earlier. What we see from the above is that while the leading (tree level) terms are unaffected, the higher order (loop) effects are changed by factors of \( \mathcal{A}/\hbar \) which in the case of \( \mathcal{A} = l_s^2/\kappa^2 \) can be quite large for large volume string compactifications. Thus in principle at least a classical distribution with a significantly different unit of action (such as that coming from classical string theory) can be distinguished from the “quantum” one with the unit of action \( \hbar \). However as we’ve argued, this by itself does not test uniquely quantum features of quantum mechanics (as opposed to classical distributions governed by the same unit of action).

6 Conclusions

In this paper we have argued that the usual quantum field theory calculation may be replaced by a Gaussian statistical distribution governed with a kernel that is given by (43). We have also explored the consequences of replacing \( \hbar \) in this formula by some other unit of action \( \mathcal{A} \). In particular we discussed the consequences of identifying \( \mathcal{A} \) with a natural unit of action coming from string theory and involving only classical (but string theoretic) parameters. We noted how this could be consistent with the usual double expansion of low energy string theory - namely the \( \alpha' \) expansion and the string loop expansion. One consequence of this replacement is to change the relationship between the power spectra and the height of the inflaton potential.

We also noted that the difference between the standard prescription for calculating the higher point functions and any other distribution that is significantly different (i.e. with \( \mathcal{A}/\hbar \) either \( \ll 1 \) or \( \gg 1 \) such as the classical string theory case), will emerge at higher orders in the loop corrections. In fact the important point that we’ve tried to emphasize, is that what is being tested in observations of the power spectrum and higher point spectra, is a statistical distribution of decohered trajectories. In other words there is no need at all to think of the initial state for inflation as a pure quantum mechanical state. The entire discussion of “quantum fluctuations” can be rephrased in terms of a decohered initial state with a certain statistical weight. Whether or not that weight corresponds to that arising from a pre-inflationary quantum state of a simple QFT (which decohered before the onset of inflation) rather than some distribution that does not necessarily have its origin in QM, cannot be definitively established with current (or foreseeable future) measurements. However distributions with values of \( \mathcal{A} \) that are significantly larger than \( \hbar \) (such as one that may arise from a large volume compactification of string theory), may possibly be ruled out by future observations.

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Appendix 1 Quantum and Classical Hamiltonian evolution

Let us write the canonical dynamical variables $q_i, p_i, i = 1, \ldots, n$ as

$$
\hat{z}_i(t) = \hat{q}_i, \quad i = 1, \ldots, n
= \hat{p}_{i-n}, \quad i = n + 1, \ldots, 2n.
$$

Here we’ve used hats to denote quantum operators satisfying the canonical equal time commutation relations which in this notation read

$$
[\hat{z}_i(t), \hat{z}_j(t')] = i\hbar J_{ij},
$$

where

$$
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}
$$

is the symplectic metric. The Heisenberg equations of motion are

$$
\frac{d\hat{z}_i(t)}{dt} = \frac{i}{\hbar} [H(\hat{z}(t), t), \hat{z}_i(t)].
$$

It is easily checked that the formal solution to this equation is

$$
\hat{z}_i(t) = \hat{U}^{-1}(t, t_0)\hat{z}_i(0)\hat{U}(t, t_0),
$$

where

$$
\hat{U}(t, t_0) = T \exp \left( -\frac{i}{\hbar} \int_{t_0}^{t} H(\hat{z}_0, t')dt' \right)
$$

with $T$ denoting time ordering. It is important to note that the $\hat{z}$ in the Hamiltonian in this expression is evaluated at the initial time $t_0$ ($\hat{z}_0 \equiv \hat{z}(0)$). This is of course only relevant because the Hamiltonians that we deal with have explicit time-dependence. For any dynamical variable $\hat{A}(t)$ that is defined as a product of the canonical variables (with some specified ordering if it involves both $q$’s and $p$’s) there are two alternate forms (see Weinberg [21]) for the solution (62),

$$
\hat{A}(t) = \sum_{N=0}^{\infty} \left( \frac{i}{\hbar} \right)^N \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{N-1}} dt_N
[H(\hat{z}_0, t_N), [H(\hat{z}_0, t_{N-1})[ \cdots [H(\hat{z}_0, t_1), \hat{A}(t_0)], \ldots]].
$$

Now let us separate the Hamiltonian into a quadratic (“free”) part $H_0(t)$ and an interaction part $H_1(t)$. The evolution operator corresponding to the free Hamiltonian is

$$
\hat{U}_0(t, t_0) = T \exp \left( -\frac{i}{\hbar} \int_{t_0}^{t} H_0(\hat{z}_0, t')dt' \right).
$$
The interaction picture operators are defined by
\[ \hat{z}_I(t) = \hat{U}^{-1}_0(t, t_0) \hat{z}_0 \hat{U}_0(t, t_0) \] (66)
and the corresponding evolution operator defined by \( \hat{U}_I = \hat{U}^{-1}_0(t, t_0) \hat{U}(t, t_0) \) satisfies the equation of motion,
\[ i\hbar \frac{d\hat{U}_I}{dt} = H_I(\hat{z}_I(t), t) \hat{U}_I = \hat{H}_I(t) \hat{U}_I. \]
This has the formal solution
\[ \hat{U}_I(t, t_0) = T \exp \left( -\frac{i}{\hbar} \int_{t_0}^{t} H_I(\hat{z}_I(t'), t') dt' \right) \] (67)
and (64)(65) are replaced by (see for example [21]),
\[ \hat{A}(t) = T e^{\frac{i}{\hbar} \int_{t_0}^{t} H_I(\hat{z}_I(t'), t') dt'} \hat{A}_I(t) T e^{-\frac{i}{\hbar} \int_{t_0}^{t} H_I(\hat{z}_I(t'), t') dt'} = \sum_{N=0}^{\infty} \left( \frac{i}{\hbar} \right)^N \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{N-1}} dt_N \]
\[ [H_I(\hat{z}_I(t_N), t_N), [H_I(\hat{z}_I(t_{N-1}), t_{N-1}), \ldots [H_I(\hat{z}_I(t_1), t_1), \hat{A}_I(t)], \ldots]. \] (69)
In the first equation above \( T \) is time ordering while \( \hat{T} \) is anti-time ordering. All this (except perhaps the second form of the expressions for \( \hat{A} \)) are quite familiar. What may not be so familiar is that there is an exact classical analog of all these equations.

The classical variables \( z^i \) satisfy the Poisson bracket relations
\[ \{z^i(t), z^j(t)\} = J^{ik} \frac{\partial z^j}{\partial z^k} = J^{ij}. \] (70)
Hamilton’s equation of motion may then be written as,
\[ \dot{z}^i(t) = \{z^i(t), H(z(t), t)\} = J^{ij} \partial_j H(z(t), t) = \partial^i H(z(t), t), \] (71)
where we’ve defined \( \partial^i \equiv J^{ij} \partial_j, \partial_i \equiv \partial/\partial z^i \). This equation can be rewritten in the form of a commutator by introducing the Hamiltonian vector field \( H^V(z(t), t) \equiv \partial^i H(z(t), t) \partial_i \):
\[ \dot{z}^i(t) = [H^V(z(t), t), z^i(t)] \] (72)
The solution to this equation is exactly the same as (64)(65), except that there are no factors of \( i/\hbar \) and the operator Hamiltonian is replaced by the vector field. In other words
\[ \frac{i}{\hbar} \hat{H} \rightarrow H^V, \] (73)
and
\[ A(t) = T e^{\int_{t_0}^{t} H^V(z_0, t') dt'} A(t_0) T e^{-\int_{t_0}^{t} H^V(z_0, t') dt'} = \sum_{N=0}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{N-1}} dt_N \]
\[ [H^V(z_0, t_N), [H^V(z_0, t_{N-1}), \ldots [H^V(z_0, t_1), A(t_0)], \ldots]. \] (74)
Of course all we’ve done here is to reverse the procedure of Dirac who replaced the Poisson brackets of classical mechanics by $-\frac{i}{\hbar}$ times the commutator of the quantum operators. The point of the exercise is simply to show that the evolution of a quantum operator represented as an infinite series in terms of commutators, has an exact analog in the classical theory. In fact up to operator ordering ambiguities the relation between $\hat{A}(t)$ and $A(0)$ is exactly the same as that between their classical versions $A(t)$ and $A(0)$. This is easily seen by comparing (65) and (69). Any commutator term in the first of these is of the form

$$\frac{i}{\hbar}[H(z_0, t'), \hat{z}_j^i] = \frac{\partial H(z_0, t')}{\partial z_0^i} \frac{i}{\hbar}[\hat{z}_0^j, \hat{z}_0^i] = -\frac{\partial H(z_0, t')}{\partial z_0^i} \{z_0^i, \hat{z}_0^j(t)\}.$$  

(75)

The first equality follows from the fact that the canonical equal time commutator of two fields is a $c$-number while the second follows from the Dirac identification between equal time commutators and (equal time) Poisson brackets.

On the other hand the corresponding term in (69) is

$$[j_k^i, \frac{\partial H(z_0, t')}{\partial z_0^j}, z_0^k] = -\frac{\partial H(z_0, t')}{\partial z_0^k} j_{ik} \frac{\partial z_0^j}{\partial z_0^k} = -\frac{\partial H(z_0, t')}{\partial z_0^i} \{z_0^i, z_0^j(t)\}.$$  

(76)

So the two expressions (75) (76) are the same up to the replacement $\hat{z} \to z$ and so verifies the statement above of the equality of the quantum and classical evolutions up to operator ambiguities.

Clearly all the manipulations which led to the interaction picture will survive with the replacement (73), so for the classically evolved field we get exactly the same equations as (68) (69), i.e.

$$A(t) = T e^{\int_{t_0}^{t} H^V(z_I(t'), t') dt'} A_I(t) T e^{-\int_{t_0}^{t} H_I(z_I(t'), t') dt'},$$  

(77)

$$= \sum_{N-0}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{N-1}} dt_N$$

$$[H^V_I(z_I(t_N), t_N), [H^V_I(z_I(t_{N-1}), t_{N-1})], \ldots [H^V_I(z_I(t_1), t_1), A_I(t)], \ldots],$$  

(78)

with

$$z_I(t) = (U^V_I)^{-1}(t, t_0) z_0 U^V_I(t, t_0),$$  

(79)

$$U^V_I(t, t_0) = T \exp \left(-\int_{t_0}^{t} H^V_I(z_0, t') dt'\right).$$  

(80)

These arguments are trivially extended to field theory. As usual in the theory of cosmological fluctuations, one expands the original generally covariant Lagrangian around the (time-dependent) inflationary background and gets a time-dependent Hamiltonian functional of the fluctuations. Denote the latter by

$$z^i(x, t) = \phi^i, i = 1, \ldots, n$$

$$= \pi_{i-n}, i = n + 1, \ldots, 2n,$$  

(81)

(82)

where $\phi, \pi$ are canonically conjugate field, field momentum. The Hamiltonian vector field is

$$H^V(z(t), t) \equiv \int d^3x J_i^j \delta H[z(t), t] \frac{\delta}{\delta z^i(x, t)} \delta z^j(x, t).$$
With this definition one can take over all the classical mechanics formulae above to field theory just as the corresponding QM formulae can be taken over to QFT. To proceed further we replace the expectation values of QFT with statistical expectation values with some initial distribution $p(\phi_0)$, i.e.

$$<\Omega|\hat{A}|\Omega> \rightarrow \int [d\phi_0] p(\phi_0) A(\phi_0, t) \quad (83)$$

where the second factor in the integrand on the RHS is to be calculated using (77) or (78). The point is that in both the left hand side and the right hand side of this relation one evaluates in the interaction picture, using (68) or (69) for the LHS and (77) or (74) for the RHS. Thus one just has to calculate expectation values of free fields and by Wick's theorem it is a sum of products of two point functions determined by the correlator $< b^\dagger_k b_q >$. Hence given what we have just established on the time evolution of classical and quantum operators, the LHS and the RHS of the above relation are actually equal in value when $p(\phi)$ is defined as in (45), as discussed in section (4).

As discussed in section (4) with the distribution $p$ given by (45) we get exactly the usual “quantum” calculation. What is evident from this discussion is that all that is quantum here is the use of Planck’s constant $\hbar$ in the expression for the kernel (43) of the distribution. One could equally well have started with a different initial distribution as pointed out in that section. Also it is clear that the possible operator ordering issues that might account for a difference in the quantum and classical evolution will be irrelevant in a calculation of the above expectation value at least if we start from an initial Gaussian distribution since the interaction picture formulation in either the quantum or the classical case, shows that the final result is given by sums of products of Wick contractions. The difference in the usual computations and one with some arbitrary classical distribution is simply obtained by the replacement of the factor $\hbar$ in each Wick contraction (i.e. two point function) by some other unit of action $\mathcal{A}$.

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