Homological eigenvalues of graph $p$-Laplacians

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Abstract

Inspired by persistent homology in topological data analysis, we introduce the homological eigenvalues of the graph $p$-Laplacian $\Delta_p$, which allows us to analyse and classify non-variational eigenvalues. We show the stability of homological eigenvalues, and we prove that for any homological eigenvalue $\lambda(\Delta_p)$, the function $p \mapsto p(2\lambda(\Delta_p))^\frac{1}{p}$ is locally increasing, while the function $p \mapsto 2^{-p}\lambda(\Delta_p)$ is locally decreasing. As a special class of homological eigenvalues, the min-max eigenvalues $\lambda_1(\Delta_p), \lambda_2(\Delta_p), \cdots$, are locally Lipschitz continuous with respect to $p \in [1, +\infty)$. We also establish the monotonicity of $p(2\lambda_k(\Delta_p))^\frac{1}{p}$ and $2^{-p}\lambda_k(\Delta_p)$ with respect to $p \in [1, +\infty)$.

These results systematically establish a refined analysis of $\Delta_p$-eigenvalues for varying $p$, which lead to several applications, including: (1) settle an open problem by Amghibech on the monotonicity of some function involving eigenvalues of $p$-Laplacian with respect to $p$; (2) resolve a question asking whether the third eigenvalue of graph $p$-Laplacian is of min-max form; (3) refine the higher order Cheeger inequalities for graph $p$-Laplacians by Tudisco and Hein, and extend the multi-way Cheeger inequality by Lee, Oveis Gharan and Trevisan to the $p$-Laplacian case.

Furthermore, for the 1-Laplacian case, we characterize the homological eigenvalues and min-max eigenvalues from the perspective of topological combinatorics, where our idea is similar to the authors’ work on discrete Morse theory.

Keywords: $p$-Laplacian, min-max principle, variational eigenvalue, homological critical value, simplicial complex, zonotope

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1 Introduction

As a discrete version of $p$-Laplacian, the graph $p$-Laplacian has been successfully used in various applications, including data and image processing problems and spectral clustering. Furthermore, eigenvalue problems involving the graph $p$-Laplacian are also important in machine learning, especially in the fields of semi-supervised learning and unsupervised learning. Plenty of recent works indicate that the graph $p$-Laplacian may enhance the performance of classical algorithms based on the standard graph Laplacian [3, 25]. This has contributed to several progresses on both the theoretical and the numerical aspects of $p$-Laplacians on graphs and networks [1, 10, 38, 39, 41, 42]. A remarkable development is that the second eigenvalue has a mountain-pass characterization and thus it is a min-max eigenvalue, and more importantly, the second eigenvalue satisfies the Cheeger inequality [3, 22, 38].

Another important result says that the second and the largest eigenvalues satisfy certain monotonicity. Precisely, $p(2\lambda_p)^{\frac{1}{p}}$ and $p(2\gamma_p)^{\frac{1}{p}}$ are increasing with respect to $p$, where $\lambda_p$ and $\gamma_p$ represent the second and the largest eigenvalues of $p$-Laplacians, respectively. But, it is unknown whether the other eigenvalues satisfy this property. So, Amghibech asked in his original paper [1]:

**Question 1.** Is there a version of the monotonicity for other eigenvalues for general graphs?

To the best of the author’s knowledge, this question has not been answered. In fact, this problem is quite difficult because the eigenvalues of $p$-Laplacian are full of mysteries — we even don’t know the number of the eigenvalues of $p$-Laplacian.

In the spectra of $p$-Laplacians, the variational eigenvalues have attracted particular attention because they possess Rayleigh-type quotient reformulation, good nodal domain properties, and multi-way Cheeger inequalities [3, 10, 15, 38, 39]. Here are some important questions on the min-max (variational) eigenvalues of $p$-Laplacian:

**Question 2.** Can we find an eigenvalue of $p$-Laplacian that is not in the list of min-max (variational) eigenvalues?

**Question 3.** Is there an eigenvalue of $p$-Laplacian larger than the second eigenvalue but smaller than the third min-max eigenvalue?

**Question 4.** Is there a graph such that for both $p = 1$ and some $p > 1$, its $p$-Laplacian has a non-variational eigenvalue?

In the setting of $p$-Laplacian on Euclidean domains, Question 2 is a major long-standing open problem on the higher order eigenvalues of $p$-Laplacian [14, 16, 22]. The only known case is that the domain is of one-dimension, i.e., an interval (see e.g., [17]).

While, in the setting of discrete $p$-Laplacian on connected graphs with $p \not\in \{1, 2\}$, the only known cases are tree graphs and complete graphs (see [1, 15]). Precisely, Amghibech observed that there are more eigenvalues than the number of vertices for $p$-Laplacian on a complete graph [1]. That is a positive answer to Question 2 on complete graphs with $p \not\in \{1, 2\}$. However, it is surprising that the variational spectrum of 1-Laplacian on complete graphs is the entire spectrum (see Section 4.3). Moreover, interestingly, based on the advanced nodal domain theory [15], a recent breakthrough by Deidda, Putti and Tudisco states that all the eigenvalues of $p$-Laplacian on a tree are variational eigenvalues [15], that is, Question 2 has a negative answer on forests.

Nevertheless, we do not yet know the answer to Question 2 on connected graphs other than trees and complete graphs. We also don’t know if Question 2 has a positive answer for both $p = 1$ and some $p > 1$ on certain graphs. In addition, whether the third eigenvalue is in the sequence of min-max eigenvalues is still waiting to be explored. From these perspectives, Questions 3 and 4 are very natural, and they indeed first appeared in the field of $p$-Laplacians on Euclidean domains [16, 18, 35].

---

1Deidda, Putti and Tudisco [15] also construct a graph of order 4 which possesses non-variational $p$-Laplacian eigenvalues for $p \not\in \{1, 2\}$. Their new example shows the smallest simple graph whose unnormalized $p$-Laplacian has a non-variational eigenvalue. If we work on generalized graphs that have repeated edges with different real incidence coefficients, then we can give a graph with 2 vertices and 2 edges, whose normalized $p$-Laplacian has a non-variational eigenvalue (see Remark 4.3).
The difficulty to study the above questions as well as other relevant aspects on p-Laplacian eigenvalue problem is twofold: on one hand, we cannot compute all the eigenvalues of p-Laplacian for general graphs whenever \( p \not\in \{1, 2\} \); on the other hand, the feature and structure of the spectra of p-Laplacians are less known [1, 3, 18, 25, 38]. In other words, the structure of the eigenspaces of p-Laplacian is unclear.

In this paper, we answer Questions 1, 2, 3 and 4 by introducing the homological eigenvalues of graph p-Laplacian. More specifically, we overcome the difficulty by offering the following new contributions: (1) establishing the upper semi-continuity of the spectra of \( \hat{\Delta} \) and \( \Delta \); (2) introducing homological eigenvalues for p-Laplacians and showing certain local monotonicity on those eigenvalues with respect to \( p \); (3) bringing the ideas from persistent homology theory and discrete Morse theory to graph 1-Laplacian, and characterizing its homological eigenvalues and variational eigenvalues from the perspective of topological combinatorics.

To state our result precisely, we first present the eigenvalue problem of graph p-Laplacian. In this paper, we are working on a finite simple graph \( G = (V, E) \) with the vertex set \( V = \{1, \cdots, n\} \) and the edge set \( E \). For \( p > 1 \), the p-Laplacian \( \Delta_p : \mathbb{R}^n \to \mathbb{R}^n \) is defined by

\[
(\Delta_p x)_i = \sum_{j \in V: (j,i) \in E} |x_i - x_j|^{p-2}(x_i - x_j), \quad \forall i \in V, \quad \forall x = (x_1, \cdots, x_n) \in \mathbb{R}^n.
\]

Following the definition in [38], the (normalized) eigenvalue problem for \( \Delta_p \) is to find \( \lambda \in \mathbb{R} \) and \( x \neq 0 \) such that\(^2\)

\[
(\Delta_p x)_i = \lambda \deg(i)|x_i|^{p-2}x_i, \quad \forall i \in V,
\]

where \( \deg(i) \) indicates the number of edges that are incident to \( i \). For the case of \( p = 1 \), we refer to Definition 2.1.

The Lusternik-Schnirelman theory allows to define a sequence of eigenvalues of the \( p \)-Laplacian:

\[
\lambda_k(\Delta_p) := \inf_{\gamma(S) \geq k} \sup_{x \in S} F_p(x), \quad k = 1, \cdots, n,
\]

where

\[
F_p(x) := \frac{\sum_{(j,i) \in E} |x_i - x_j|^p}{\sum_{i \in V} \deg(i)|x_i|^p} \quad \text{for} \quad x \neq 0,
\]

and \( \gamma(S) \) represents the Yang index of a centrally symmetric compact subset \( S \) in \( \mathbb{R}^n \setminus \{0\} \) (see Definition 2.3). If we use the Kransnoselskii genus \( \gamma^- \) instead of the Yang index \( \gamma \) in (1), then we actually define the so-called variational eigenvalues \( \lambda_1(\Delta_p), \cdots, \lambda_n(\Delta_p) \). Similarly, if we replace the Yang index \( \gamma \) in (1) by the index-like quantity \( \gamma^+ \) probably first studied by Conner and Floyd [12], then we define a sequence of min-max eigenvalues \( \lambda_1^-(\Delta_p), \cdots, \lambda_n^-(\Delta_p) \), which was first introduced for Dirichlet \( p \)-Laplace eigenproblem by Drábek and Robinson [19].

Since \( \lambda_k(\Delta_p), \lambda_k^-(\Delta_p) \) and \( \lambda_k^+(\Delta_p) \) are all in min-max form, we call them min-max eigenvalues. These three sequences of min-max eigenvalues were originally defined for the continuous \( p \)-Laplacian (see Section 0.7 in [33]). In this paper, we would mainly focus on \( \{\lambda_k(\Delta_p)\}_{k=1}^n \) and \( \{\lambda_k^-(\Delta_p)\}_{k=1}^n \), which are often referred to as the ‘the min-max eigenvalues’ and ‘the variational eigenvalues’, respectively. These eigenvalues satisfy the relations [1, 11, 25, 28, 29, 38]:

\[
\{\text{min-max } \Delta_p\text{-eigenvalues}\} \subset \{\text{critical values of } F_p\} = \{\Delta_p\text{-eigenvalues}\} \subset [0, 2^{p-1}]
\]

for \( p > 1 \), and

\[
\{0, h, 1\} \subset \{\text{min-max } \Delta_1\text{-eigenvalues}\} \subset \{\text{critical values of } F_1\} \subset \{\Delta_1\text{-eigenvalues}\} \subset [0, 1],
\]

where \( h \) indicates the usual Cheeger constant of the graph. It is worth to note that the critical points of \( F_1 \) are to be understood in the sense of Clarke subdifferential calculus.

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\(^2\)We can also use the normalized \( p \)-Laplacian \( \hat{\Delta}_p = \text{diag}(1/\deg(1), \cdots, 1/\deg(n))\Delta_p \) instead of \( \Delta_p \), and write the eigenvalue problem as \( \hat{\Delta}_p x = \lambda(|x_1|^{p-2}x_1, \cdots, |x_n|^{p-2}x_n) \).
We refine the above inclusion relations by employing the homological critical values [4, 6, 24]:

\[ 0 \in \{ \lambda_1(\Delta_p), \cdots, \lambda_n(\Delta_p) \} \subset \text{homological critical values of } F_p \]
\[ \subset \text{critical values of } F_p \subset \{ \text{eigenvalues of } \Delta_p \} \subset [0, 2^{p-1}] \]

where the penultimate inclusion ‘\{critical values of } F_p \subset \{eigenvalues of } \Delta_p \}' is indeed an equality if \( p > 1 \), while for \( p = 1 \), all the sets above are mutually different in general. For convenience, we call such homological critical values of \( F_p \) the **homological eigenvalues** of \( \Delta_p \), and we refer to Section 2.2 for the definitions.

Our most significant new contribution is the following theorem on monotonicity:

**Theorem 1.1.** For any \( p \geq 1 \) and any isolated homological eigenvalue (resp., any eigenvalue produced by homotopical linking) \( \lambda(\Delta_p) \) of \( \Delta_p \), and any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( q \in (p - \delta, p + \delta) \), there exists \( \Delta_q \)-eigenvalue \( \lambda(\Delta_q) \in (\lambda(\Delta_p) - \epsilon, \lambda(\Delta_p) + \epsilon) \), such that \( \lambda(\Delta_q) \) is a homological eigenvalue (resp., an eigenvalue produced by homotopical linking) with the property:

\[
\begin{align*}
 p(2\lambda(\Delta_p))^\frac{1}{p} &\leq q(2\lambda(\Delta_q))^\frac{1}{q} \quad \text{and} \quad 2^{-p}\lambda(\Delta_p) \geq 2^{-q}\lambda(\Delta_q), \quad \text{if } q \in (p, p + \delta), \\
 p(2\lambda(\Delta_p))^\frac{1}{p} &\geq q(2\lambda(\Delta_q))^\frac{1}{q} \quad \text{and} \quad 2^{-p}\lambda(\Delta_p) \leq 2^{-q}\lambda(\Delta_q), \quad \text{if } q \in (p - \delta, p).
\end{align*}
\]

For any \( k \), the min-max eigenvalue \( \lambda_k(\Delta_p) \) is locally Lipschitz continuous with respect to \( p \), and moreover, we have the following monotonicity of certain functions involving min-max eigenvalues:

- the function \( p \mapsto p(2\lambda_k(\Delta_p))^\frac{1}{p} \) is increasing on \([1, +\infty)\);
- the function \( p \mapsto 2^{-p}\lambda_k(\Delta_p) \) is decreasing on \([1, +\infty)\).

All the claims above hold if we use \( \lambda^{-}_k(\Delta_p) \) or \( \lambda^{+}_k(\Delta_p) \) instead of \( \lambda_k(\Delta_p) \).

**Remark 1.1.** The monotonicity of the function \( p \mapsto p(2\lambda_k(\Delta_p))^\frac{1}{p} \) also holds under the domain setting, but the monotonicity of the function \( p \mapsto 2^{-p}\lambda_k(\Delta_p) \) only holds for the graph setting. It is worth noting that if \( \lambda_k(\Delta_p) \neq 0 \) for some \( p \geq 1 \), then \( p \mapsto p(2\lambda_k(\Delta_p))^\frac{1}{p} \) is strictly increasing on \([1, +\infty)\), and \( p \mapsto 2^{-p}\lambda_k(\Delta_p) \) is strictly decreasing on \([1, +\infty)\). It is noteworthy that the local monotonicity and stability for homomorphism eigenvalues in Theorem [17] do not hold for non-homological eigenvalues (see Section 2.2 and Example 4.2). The eigenvalue produced by homotopical linking is introduced in Section 2.2.

Theorem 1.1 establishes asymptotic behaviors for homological eigenvalues of \( \Delta_p \) with respect to \( p \). It not only answers a question by Amghibech [1], but also provides an elegant solution to Question 2 about higher order eigenvalues: we can find an eigenvalue that is not in the list of min-max eigenvalues, and in fact, the third one satisfies the requirement. Roughly speaking, we construct a graph and prove that there is a homological nonvariational eigenvalue of \( \Delta_p \) for some \( p > 1 \).

**Theorem 1.2.** Let \( G = (V, E) \) be the simple graph on \( V = \{1, 2, 3, 4, 5, 6\} \) shown as:

![Graph](image)

Then, for any \( 0 < \epsilon < \frac{1}{10} \), there exists \( 0 < \delta < 1 \) such that for any \( p \in [1, 1 + \delta) \), there is a \( p \)-Laplacian eigenvalue \( \lambda(\Delta_p) \in (\frac{5}{6} - \epsilon, \frac{5}{6} + \epsilon) \) which is not in the lists of min-max eigenvalues \( \{\lambda_k(\Delta_p)\}_{k=1}^{n}, \{\lambda^{-}_k(\Delta_p)\}_{k=1}^{n}, \{\lambda^{+}_k(\Delta_p)\}_{k=1}^{n} \).
In order to prove Theorem 1.2 we just need to carefully select homological eigenvalues from nonvariational eigenvalues of $\Delta_1$, and then apply Theorem 1.1 to complete the verification. It remains to construct a homological eigenvalue of $\Delta_1$ which is not a variational eigenvalue. But it is very difficult to check whether a $\Delta_1$-eigenvalue is a homological eigenvalue. Fortunately, we establish the following characterization of homological eigenvalues of 1-Laplacian.

For any subset $A \subset V$, we use $1_A$ to denote the characteristic vector of $A$. Let $K_n$ be the simplicial complex on the vertex set $\{ -1, 0, 1 \}^n$ with $n! \cdot 2^n$ maximal simplexes, where each maximal simplex is a $(n-1)$-dimensional simplex of the form $\text{conv} \{1_A, -1_B : i = 1, \cdots, n\}$, whenever $A_1 \cdots \subset A_n$ and $B_1 \cdots \subset B_n$ and $A_i \cap B_i = \emptyset$ and $A_i \cup B_i = A_{i+1} \cup B_{i+1}$, $i = 1, \cdots, n - 1$. Then, $K_n$ is a pure simplicial complex and its geometric realization $|K_n|$ is actually the boundary of the hypercube $\{ x \in \mathbb{R}^n : \| x \|_\infty \leq 1 \}$ which is a PL manifold. Thus, we regard $K_n$ as a triangulation of the unit $l^\infty$-sphere, and we also identify $|K_n|$ with $\{ x \in \mathbb{R}^n : \| x \|_\infty = 1 \}$. In addition, we can also see $K_n$ as the order complex on $\mathcal{P}_2(V) = \{(A, B) : A \cap B = \emptyset, A \cup B \neq \emptyset, A, B \subset V \}$ (see Section 2.3 for details).

**Theorem 1.3.** $\lambda \in \mathbb{R}$ is a homological eigenvalue of $\Delta_1$ if and only if the subcomplex of $K_n$ induced by the vertices located in the open sublevel set $\{ x \in |K_n| : F_1(x) < \lambda \}$ and the subcomplex of $K_n$ induced by the vertices located in the closed sublevel set $\{ x \in |K_n| : F_1(x) \leq \lambda \}$ have different homology groups. Particularly, if there exists $A \neq \emptyset$ such that:

1. $F_1(1_A) = \lambda$, $F_1(v) \neq \lambda$ for any vertex $v$ located in $\text{link}(1_A)$, and

2. the subcomplex of $K_n$ induced by the vertices located in the open sublevel set $\{ x \in \text{link}(1_A) : F_1(x) < \lambda \}$ is homologically non-trivial.

then $\lambda$ is a homological eigenvalue of $\Delta_1$.

This result is the second ingredient of the proof of Theorem 1.2 by which we only need to check the vertices of $K_n$.

More significantly, based on Theorem 1.1, we obtain Cheeger-type inequalities which not only refine and strengthen the higher-order Cheeger inequality for graph $p$-Laplacian by Tudisco and Hein [33], but also establish the first nonlinear multi-way Cheeger inequality for $p$-Laplacian which generalizes the related works by Miclo [33], Lee, Oveis Gharan and Trevisan [32]. Recall the multi-way Cheeger constant [32, 33]:

$$h_k := \min_{\text{non-empty disjoint } A_1, \cdots, A_k} \max_{1 \leq i \leq k} \frac{|\partial A_i|}{\text{vol}(A_i)}, \quad k = 1, 2, \cdots, n, \quad \text{(2)}$$

where $|\partial A|$ is the number of edges connecting $A$ and $V \setminus A$, and $\text{vol}(A) := \sum_{i \in A} \deg(i)$.

We introduce the modified combinatorial $k$-way Cheeger constant (see Section 3.2 for details)

$$\hat{h}_k = \min_{\mathcal{A} \in \mathcal{S}_k} \max_{(A, B) \in \mathcal{A}} \frac{|\partial A| + |\partial B|}{\text{vol}(A \cup B)},$$

where $\mathcal{S}_k := \{ A \subset \mathcal{P}_2(V) : \text{the Yang index of the subcomplex of } K_n \text{ induced by } A \text{ is at least } k \}$. We similarly define $\hat{h}_k^-$ and $\hat{h}_k^+$ by using $\mathcal{S}_k^- := \{ A \subset \mathcal{P}_2(V) : \text{the Krasnoselskii genus of the subcomplex of } K_n \text{ induced by } A \text{ is at least } k \}$ and $\mathcal{S}_k^+ := \{ A \subset \mathcal{P}_2(V) : \text{the } \gamma^- \text{-index of the subcomplex of } K_n \text{ induced by } A \text{ is at least } k \}$ instead of $\mathcal{S}_k$ in the definition of $\hat{h}_k$. We would devote some words to clarify how to go from a family $A \subset \mathcal{P}_2(V)$ to its corresponding subcomplex, which will be explained at length in Section 2.3. For convenience, we would use a partial order $\prec$ on $\mathcal{P}_2(V)$ defined as $(A, B) \prec (A', B')$ if $A \subset A'$ and $B \subset B'$. This induces the partial order $\prec$ on a subfamily $A \subset \mathcal{P}_2(V)$ in the same way, from which we obtain a partial order set $(A, \prec)$. The subcomplex of $K_n$ induced by $A$ is defined as the order complex on $A$ with respect to the partial order for set-pairs, i.e., the faces of the corresponding subcomplex are the chains (totally ordered subsets) of the partial order set $(A, \prec)$.

In general, $\hat{h}_k \leq h_k$, and for $k = 2$, $\hat{h}_2$ always agrees with the usual Cheeger constant $h_2$. Then, we have:
Theorem 1.4. For any $p \geq 1$, and $k = 1, \cdots, n$,

\[
\begin{array}{llllll}
\frac{h_k^2}{C_k^p} & \leq & \frac{2p-1}{p} h_{k}^- & \leq & \frac{2p-1}{p} h_{k}^+ & \leq & 2p-1 h_{k}^-
\end{array}
\]

where $C > 0$ and $C_p > 0$ are universal constants, and the quantity at the end of each arrow is greater than or equal to the quantity at the corresponding arrowhead, i.e., $a \leftarrow b$ means $a \leq b$. In the above diagram,

\[
s_k := \max_{x \in \bigcup_{q \in [1,p]} S_k(\Delta_q)} \mathcal{G}(x), \quad s_k^- := \max_{x \in \bigcup_{q \in [1,p]} S_k^-(\Delta_q)} \mathcal{G}(x) \quad \text{and} \quad s_k^+ := \max_{x \in \bigcup_{q \in [1,p]} S_k^+(\Delta_q)} \mathcal{G}(x),
\]

where $\mathcal{G}(x)$ is the number of strong nodal domains of $x$,

\[
\begin{array}{l}
S_k(\Delta_q) := \{ x \neq 0 \mid x \text{ is an eigenvector corresponding to some eigenvalue } \lambda \text{ with } \lambda \leq \lambda_k(\Delta_q) \}, \\
S_k^-(\Delta_q) := \{ x \neq 0 \mid x \text{ is an eigenvector corresponding to some eigenvalue } \lambda \leq \lambda_k^-(\Delta_q) \}.
\end{array}
\]

Remark 1.2. By the above theorem, for any $k$, the quantities on the left-hand-side and the right-hand-side of the higher order Cheeger-type inequality

\[
\sqrt{\frac{1}{2p-1} \lambda_k^+(\Delta_p)} \leq \hat{h}_k^+ \leq \frac{p}{2} (2\lambda_k^-(\Delta_p))^{\frac{1}{p}} \quad (3)
\]

are decreasing and increasing with respect to $p \in [1, +\infty)$, respectively.

Taking $p = 2$ and $k = 2$ in Theorem 1.4, we get the usual Cheeger inequality on graphs, while taking $p = 1$ and $k = 2$, we recover the identity $\hat{h}_2^+ = \hat{h}_2^-$ proved by Hein-Bühler [23] and Chang [9] independently. More importantly, taking $p = 1$ and $k \geq 1$, we indeed establish the equality $\hat{h}_k^+ = \lambda_k^+(\Delta_1)$ which means that the $k$-th min-max eigenvalue of graph 1-Laplacian is actually the $k$-th modified Cheeger constant (a combinatorial quantity). In particular, note that our findings in combination with the recent work [13] show that $\hat{h}_k = \hat{h}_k^+ = \hat{h}_k^-$ holds on a tree. In addition, Theorem 1.4 and (3) refine Tudisco-Hein’s higher order Cheeger inequality for $p$-Laplacian [32], and establish the first $p$-Laplacian version of Lee-Oveis Gharan-Trevisan’s multi-way Cheeger inequality [32]. Theorem 1.4 implies $\hat{h}_k \geq \frac{h_k^2}{C_k^p}$. We further conjecture that there exists a universal constant $C > 0$ such that for any $k$, $\hat{h}_k \geq h_k^2/C_k^p$. If such a conjecture has a positive answer, then by Theorem 1.1, we can obtain a strict refinement of the famous multi-way Cheeger inequality proved by Lee, Oveis Gharan and Trevisan [32].

Corollary 1.1. For any $k$, $\lambda_k^-(\Delta_2) = \lambda_k^+(\Delta_2) = \lambda_k^+(\Delta_2)$. For any $p \geq 1$, $\lambda_1^-\Delta(p) = \lambda_k^-(\Delta_1) = \lambda_k^+(\Delta_1) = \lambda_k^+(\Delta_1) = 0$, $\lambda_n^-(\Delta_1) = \lambda_n^+(\Delta_1) = 1$ and $\lambda_2^-(\Delta_1) = \lambda_2^+(\Delta_1) = \lambda_2^+(\Delta_1)$. Moreover, on forests, we further have $\lambda_k^-(\Delta_1) = \lambda_k^+(\Delta_1) = \hat{h}_k^+ = \hat{h}_k^+ = \hat{h}_k^+ = \hat{h}_k$, $\forall k = 1, \cdots, n$.

Furthermore, Theorem 1.1 allows to bound some min-max eigenvalues in a novel way:

Theorem 1.5. For any connected graph, there is a min-max eigenvalue of $\Delta_p$ in

\[
\begin{cases}
(2p-3, 2p-1) & \text{if } p < 2, \\
\left[\frac{1}{2}, \frac{3}{2}\right], & \text{if } p = 2, \\
\left[\frac{2p-1}{p}, 3 \times 2p-3\right], & \text{if } p > 2.
\end{cases}
\]
For any graph $G$, there are at least $\alpha_*(G)$ min-max eigenvalues of $\Delta_p$ larger than $\frac{2^{p-1}}{p}$ when $p > 1$, and there are at least $\alpha_*(G)$ min-max eigenvalues of $\Delta_1$ equal to 1, where $\alpha_*(G)$ is the pseudo-independence number introduced in [43] (see also Corollary 3.1 in Section 3.1 for details).

**Remark 1.3.** The first statement in Theorem 1.5 is nontrivial when $p > \sqrt{3}$, and the case of $p = 2$ is the main result established in the author’s recent work [30].

**Remark 1.4.** The inequality $\lambda_{n-\alpha_*(G)+1}(\Delta_p) > \frac{2^{p-1}}{p}$ for $p > 1$ in Theorem 1.5 can be compared with the inertia bound for $p$-Laplacian [17, 29], i.e., $\lambda_{n-\alpha(G)+1}(\Delta_p) \geq 1$, where $\alpha(G)$ is the standard independence number of $G$. In some cases, it is better than the inertia bound. For example, considering a triangle graph whose Laplacian eigenvalues are $0, \frac{3}{2}, \frac{3}{2}$, by the inertia bound for $p = 2$, there is an eigenvalue larger than or equal to 1 which is not optimal, while by Theorem 1.5 there are two eigenvalues larger than $\frac{1}{2}$ which is sharp.

Other new results that we would like to highlight in this paper are:

- We show that the graph 1-Laplacian is zonotope-valued, which reveals that the graph 1-Laplacian is closely related to combinatorial geometry. See Section 2.1 for details.

- We derive that for any graph of order larger than or equal to 4, there exists a number $c \in (\lambda_2, \lambda_n)$ which is not an eigenvalue of $\Delta_p$ for any $p$ sufficiently close to 1. And for particular graphs like trees and complete graphs (see Section 4.3), for any $p \geq 1$, there always exists a number in $(\lambda_2, \lambda_n)$ that is not in the list of the $p$-Laplacian eigenvalues. This partially answers the question: is there a number between the second and the largest $\Delta_p$-eigenvalues that is not an eigenvalue? For the $p$-Laplacian on a domain, the question asks whether there exists one number $c > \lambda_2$ that is not an eigenvalue [16]. Our method is expected to solve this open problem in the domain setting.

- We obtain a characterization for the homological eigenvalues of $\Delta_1$ (see Theorem 1.3), which is inspired by the discrete Morse theory on simplicial complexes [23] and the PL Morse theory on triangulated manifolds [5, 20]. We also note that there are infinitely many graphs whose 1-Laplacian eigenvalues are more than their orders. For details, see Section 4.4.

- We offer new perspectives that $\Delta_1$ is a combinatorial operator because it encodes so many combinatorial properties involving graphs, while $\Delta_p$ induces a nonlinear evolution from the linear operator $\Delta_2$ to the combinatorial operator $\Delta_1$, and this nonlinear evolution essentially implies Cheeger-type inequalities (see Remarks 3.2 and 3.3 for details).

The general theory on the eigenvalues of graph $p$-Laplacians that we are exploring can be compactly represented in the following diagram:
The organization of this paper is as follows. We present in Section 2 auxiliary lemmas and more relevant results on continuity and monotonicity of $p$-Laplacian, and we establish sharp estimates for variational eigenvalues in Section 3. We refer to Section 2.2 and Proposition 2.3 for the proof of Theorem 1.1. We prove Theorem 1.2 in Section 4.1 and Theorem 1.3 in Section 2.3. The proofs of Theorem 1.4 and Theorem 1.5 are established in Section 3.2 and Section 3.3, respectively. In order to make the paper accessible to experts in graph $p$-Laplacian theory as well as those in homology theory we include in the paper somewhat more than the usual amount of background material.

## 2 The spectrum of $\Delta_p$

First, we recall the definition of graph 1-Laplacian, which has been systematically studied in \[9, 10, 25, 28\].

**Definition 2.1** (1-Laplacian for graphs). Given a simple, unweighted, undirected, finite graph $G = (V, E)$ with $V = \{1, \cdots, n\}$, the 1-Laplacian $\Delta_1$ is a set-valued map on $\mathbb{R}^n$ defined by

$$\Delta_1 x_i = \left\{ \sum_{j \in V: (j, i) \in E} z_{ij} \in \text{Sgn}(x_i - x_j), z_{ij} = -z_{ji} \right\}, \quad i \in V,$$

in which

$$\text{Sgn}(t) := \begin{cases} 
1 & \text{if } t > 0, \\
[-1, 1] & \text{if } t = 0, \\
\{-1\} & \text{if } t < 0.
\end{cases}$$

The 1-Laplacian eigenvalue problem is to find $\lambda \in \mathbb{R}$ and $x \neq 0$ such that

$$(\Delta_1 x)_i \cap \lambda \deg(i) \text{Sgn}(x_i) \neq \emptyset, \quad \forall i \in V,$$

i.e., there exist $z_{ij} \in \text{Sgn}(x_i - x_j)$ with $z_{ij} = -z_{ji}$, $\forall \{i, j\} \in E$, such that

$$\sum_{j \in V: (j, i) \in E} z_{ij} \in \lambda \deg(i) \text{Sgn}(x_i), \quad \forall i \in V. \quad (4)$$

It is known that the critical points and critical values\(^4\) of the Rayleigh quotient

$$F_1(x) := \frac{\sum_{(i, j) \in E} |x_i - x_j|}{\sum_{i \in V} \deg(i)|x_i|}$$

are eigenvectors and eigenvalues of the graph 1-Laplacian, respectively.

Before recalling the variational eigenvalues for 1-Laplacians, we give the following preliminary definition.

**Definition 2.2.** For a centrally symmetric compact set $S$ in $\mathbb{R}^n \setminus \{0\}$, its Krasnoselskii genus is

$$\gamma^-(S) := \begin{cases} 
\min\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous } \varphi : S \to S^{k-1}\} & \text{if } S \neq \emptyset, \\
0 & \text{if } S = \emptyset.
\end{cases}$$

We let $\gamma^-(S) = 0$ if $S$ is not centrally symmetric with respect to the origin 0.

The constants

$$\lambda_k^- (\Delta_1) := \inf_{\gamma^-(S) \geq k, x \in S} \sup F_1(x), \quad k = 1, \cdots, n,$$

define a sequence of critical values of the Rayleigh quotient $F_1$, which are called the variational eigenvalues of $\Delta_1$.

The recent work \[15\] shows that on a tree graph, the variational eigenvalues coincide with the multi-way Cheeger constants exactly, i.e., $\lambda_k^- (\Delta_1) = h_k, \ k = 1, \cdots, n$.

\(^4\)The critical points and critical values of the locally Lipschitz function $F_1$ are defined by means of the Clarke subdifferential.
Definition 2.3. Given a centrally symmetric compact set $S$ in $\mathbb{R}^n \setminus \{0\}$, define

$$
\gamma^+(S) := \begin{cases} 
\max\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous } \varphi : \mathbb{S}^{k-1} \to S\} & \text{if } S \neq \emptyset, \\
0 & \text{if } S = \emptyset.
\end{cases}
$$

Let $\gamma^+(S) = 0$ when $S$ is not centrally symmetric with respect to the origin $0$.

We shall give an brief introduction for the Yang index \cite{40}. First, let $-\cdot$ be the antipodal map from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any centrally symmetric set $S \subset \mathbb{R}^n \setminus \{0\}$ with respect to the origin $0$, $-\cdot$ is a continuous involution without fixed point, and $-S = S$. Let $C_*(S)$ be the singular chain complex with $\mathbb{Z}_2$-coefficients, and denote by $-\cdot#$ the chain map of $C_*(S)$ induced by the antipodal map $-\cdot$. We say that a $q$-chain $c$ is symmetric if $-\cdot # (c) = c$. The symmetric $q$-chains form a subgroup $C_q(S, -)$ of $C_q(S)$, and the boundary operator $\partial_q$ maps $C_q(S, -)$ to $C_{q-1}(S, -)$. Then, these subgroups form a subcomplex $C_*(S, -)$, and we can define the corresponding cycles $Z_q(S, -)$, boundaries $B_q(S, -)$, and homology groups $H_q(S, -)$, respectively. Let $\nu : Z_q(S, -) \to \mathbb{Z}_2$ be homomorphisms inductively defined by

$$
\nu(z) = \begin{cases} 
\text{In}(c), & \text{if } q = 0, \\
\nu(\partial_zc), & \text{if } q \geq 1
\end{cases}
$$

if $z = -\cdot # (c) + c$, where the index of a 0-chain $c = \sum n_i \sigma_i$ is defined by $\text{In}(c) := \sum n_i$. It is known that $\nu$ is well-defined and $\nu B_*(S, -) = 0$, and thus it induces the index homomorphism $\nu_* : H_q(X, -) \to \mathbb{Z}_2$ by $\nu_*(z) = \nu(z)$ (see \cite{40}).

Definition 2.4. The Yang index of a centrally symmetric compact set $S$ in $\mathbb{R}^n \setminus \{0\}$ is defined as

$$
\gamma(S) := \begin{cases} 
\min\{k \in \mathbb{Z}^+ : \nu_* H_k(S, -) = 0\} & \text{if } S \neq \emptyset, \\
0 & \text{if } S = \emptyset.
\end{cases}
$$

And we take $\gamma(S) = 0$ when $S$ is not centrally symmetric with respect to the origin $0$.

By using $\gamma^+$ and $\gamma$ instead of $\gamma^-$ in the definition of the variational eigenvalue $\lambda_k^-(\Delta_1)$, we can define $\lambda_k^+(\Delta_1)$ and $\lambda_k(\Delta_1)$, respectively. It is actually known that for any symmetric set $S$, $\gamma^+(S) \leq \gamma(S) \leq \gamma^-(S)$, and thus for any $p \geq 1$,

$$
\lambda_k^- (\Delta_p) \leq \lambda_k (\Delta_p) \leq \lambda_k^+ (\Delta_p), \quad k = 1, \ldots, n. \tag{5}
$$

For the sake of completeness, we give a detailed proof of $\gamma^+(S) \leq \gamma(S) \leq \gamma^-(S)$ below.

Proof. For a given symmetric set $S$, if there exist odd continuous maps $\varphi_+ : \mathbb{S}^{k+1} \to S$ and $\varphi_- : S \to \mathbb{S}^{k+1}$, then Proposition 2.4 in \cite{33} implies $k^+ = \gamma^+ (\mathbb{S}^{k+1}) \leq \gamma(S) \leq \gamma^+ (\mathbb{S}^{k+1}) = k^-$. According to Example 3.4 in \cite{33}, one has $\gamma^+ (\mathbb{S}^{k+1}) = k^+$ and $\gamma^- (\mathbb{S}^{k+1}) = k^-$. Thus, we obtain $k^+ \leq \gamma(S) \leq k^-$. By the definition of $\gamma^+$ and $\gamma^-$, there holds $\gamma^+ (S) \leq \gamma(S) \leq \gamma^- (S)$.

2.1 Upper semi-continuity of the spectra of $p$-Laplacians when $p$ varies

In this subsection, we mainly prove the upper semi-continuity for the set of eigenvalues of $\Delta_p$, namely:

Lemma 2.1. Given a graph, denote by $\text{spec}(\Delta_p)$ the set of all the eigenvalues of $\Delta_p$. Then the set-valued map $p \mapsto \text{spec}(\Delta_p)$ is upper semi-continuous on $[1, +\infty)$, i.e., for any $p \geq 1$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $p' \in (p - \delta, p + \delta)$ with $p' \geq 1$,

$$
\text{spec}(\Delta_{p'}) \subset \bigcup_{\lambda \in \text{spec}(\Delta_p)} (\lambda - \epsilon, \lambda + \epsilon).
$$

For two points (or vectors) $a$ and $b$ in $\mathbb{R}^n$, we use $[a, b]$ to denote the segment with the endpoints $a$ and $b$. A zonotope is the Minkowski summation of finitely many segments. For convenience, we also regard a point (resp., a segment) as a zonotope of dimension 0 (resp., dimension 1).
Proposition 2.1. The graph 1-Laplacian maps each vector to a zonotope in the following way:

\[ \Delta_1 x = \lim_{p \to 1^+} \Delta_p x + \sum_{\{i,j\} \in E; x_i = x_j} [e_i - e_j, e_j - e_i] \]

where the addition ‘+’ is in the sense of Minkowski summation, \((e_i)_{i=1}^n\) is the standard orthogonal base of \(\mathbb{R}^n\).

Proof. Note that

\[ \Delta_1 x = \left\{ \sum_{i=1}^n \sum_{j \in V: \{j,i\} \in E} z_{ij} e_i \mid z_{ij} \in \text{Sgn}(x_i - x_j), z_{ij} = -z_{ji} \right\} \]

\[ = \left\{ \sum_{\{i,j\} \in E} z_{ij} (e_i - e_j) \mid z_{ij} \in \text{Sgn}(x_i - x_j) \right\} \]

\[ = \left\{ \sum_{\{i,j\} \in E; x_i > x_j} (e_i - e_j) + \sum_{\{i,j\} \in E; x_i = x_j} z(e_i - e_j) \mid z \in [-1, 1] \right\} \]

\[ = \sum_{\{i,j\} \in E; x_i > x_j} (e_i - e_j) + \sum_{\{i,j\} \in E; x_i = x_j} [e_i - e_j, e_j - e_i], \]

where the addition ‘+’ in the last equality is in the sense of Minkowski summation.

We note that \( \lim_{p \to 1^+} (\Delta_p x)_i = \sum_{j \in V: \{j,i\} \in E} \text{sign}(x_i - x_j) \), and thus

\[ \lim_{p \to 1^+} \Delta_p x = \sum_{i=1}^n \sum_{j \in V: \{j,i\} \in E} \text{sign}(x_i - x_j) e_i = \sum_{\{i,j\} \in E; x_i > x_j} (e_i - e_j), \]

where

\[ \text{sign}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0, \end{cases} \]

indicates the standard sign function. The proof is completed. \(\square\)

It is known that the 1-Laplacian can be regarded as the limit of the \(p\)-Laplacians in some sense. Interestingly, we have the following exact description of the limit of \(\Delta_p\) as \(p\) tends to 1.

Proposition 2.2. For any \(x \in \mathbb{R}^n\),

\[ \Delta_1 x = \lim_{\substack{x \to x, \delta \to 0^+}} \Delta_p \delta = \lim_{\substack{p \to 1^+, \delta \to 0^+}} \Delta_p(\mathcal{B}_\delta(x)), \]

and it is interesting that

\[ \lim_{\substack{x \to x, \delta \to 0^+}} \Delta_p \delta = \lim_{\substack{p \to 1^+, \delta \to 0^+}} \Delta_p(\mathcal{B}_\delta(x)) = \lim_{\substack{p \to 1^+}} \Delta_p x = \text{the center point of } \Delta_1 x, \]

\[ \lim_{\substack{x \to x, p \to 1^+, \delta \to 0^+}} \Delta_p \delta = \lim_{\substack{p \to 1^+, \delta \to 0^+}} \Delta_p(\mathcal{B}_\delta(x)) = \bigcup_{\delta \text{ near } x} \lim_{\substack{p \to 1^+}} \Delta_p \delta = \{ \text{the centers of the faces of } \Delta_1 x \}, \]

where \(\mathcal{B}_\delta(x)\) is the \(\delta\)-neighborhood of \(x\), i.e., the \(\delta\)-ball centered at \(x\). The last expression ‘the centers of the faces of \(\Delta_1 x\)’ depends on the fact that a convex polytope is a zonotope if and only if every face is centrally symmetric (see \([7, 37]\)).

In addition, the limit points of the eigenvalues of \(\Delta_p\) are eigenvalues of \(\Delta_1\), as \(p\) tends to 1.
Proof. We study the limit \( \lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) \) when \( \delta \) is sufficiently small. In fact, for any \( x \), taking \( 0 < \delta < \frac{1}{2} \min\{\{|x_i - x_j| : \{i,j\} \in E\} \setminus \{0\}\} \), then for any \( y \in \mathbb{B}_\delta(x) \), for \( \{i,j\} \in E \), \( x_i > x_j \) implies \( y_i > y_j \), and thus \( \Delta_1y \) is a face of \( \Delta_1x \). In consequence, by Proposition 2.1, \( \lim_{p \to 1^+} \Delta_py = \) the center of \( \Delta_1y \), and therefore, \( \lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) = \) the centers of the faces of \( \Delta_1x \).

By the definition of the \( p \)-Laplacian, for \( p > 1 \), \( \Delta_p : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous map, and thus \( \lim_{p \to 1^+} \Delta_py = \lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) = \Delta_py \). Combining with Proposition 2.1, \( \lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) = \) the center point of \( \Delta_1x \).

It is easy to check that the set of limit points of the function \( (t,p) \mapsto \|t\|p^{-2}t \) as \( p \to 1^+ \) and \( t \to 0 \) is the closed interval \([-1,1]\), which we shall write as \( \lim_{t\to0,p\to1^+} \|t\|p^{-2}t = [-1,1] \). Taking \( t_{ij} = \hat{x}_i - \hat{x}_j = -t_{ij} \) and \( E_x = \{\{i,j\} \in E : x_i = x_j\} \), we have \( \Delta_p(\mathbb{B}_\delta(x)) = \sum_{t \in V} \sum_{j \in V : \{i,j\} \in E} \|t_{ij}\|p^{-2}t_{ij}(e_i - e_j) = \sum_{\{i,j\} \in E} \|t_{ij}\|p^{-2}t_{ij}(e_i - e_j) \) and

\[
\lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) = \lim_{p \to 1^+} \Delta_p(\mathbb{B}_\delta(x)) = \sum_{\{i,j\} \in E} \|t_{ij}\|p^{-2}t_{ij}(e_i - e_j) \]

where the last equality has been shown in the proof of Proposition 2.1.

Let \( (\lambda(\Delta_p),x^p) \) be an eigenpair of \( \Delta_p \) such that \( \lambda(\Delta_p) \to \lambda, x^p \to x \neq 0, p \to 1^+ \). Then the limit points \( \lim_{p \to 1^+} \Delta_py \) are included in \( \Delta_1x \). Similarly, the limit points \( \lim_{p \to 1^+} \lambda(\Delta_p)||x^p||p^{-2}(x^p) \) are included in \( \lambda\text{Sgn}(x_i) \).

Thus, it follows from the eigenvalue equation \( (\lambda(\Delta_p)x^p)_t = \lambda(\Delta_p)\text{deg}(t)||x^p||p^{-2}(x^p)_t \) and the compactness of \( \Delta_1x \) and \( \text{Sgn}(x_i) \) that \( (\Delta_1x)_t \cap \lambda\text{deg}(t)\text{Sgn}(x_i) \neq \emptyset, \forall t \in V \), which indicates that \( (\lambda,x) \) is an eigenpair.

Suppose that \( \lim_{p \to 1^+} \lambda(\Delta_p) = \lambda \). We can always assume that each eigenvector \( x^p \) is normalized, i.e., \( ||x^p||_2 = 1 \). Then \( (x^p)_{p \to 1^+} \) has a convergent subsequence, with a limit point denoted by \( x \). Then, by the above discussions, there is no difficulty to show that \( (\lambda,x) \) is an eigenpair of \( \Delta_1 \).

Denote by \( S_1(\Delta_p) = \{x \neq 0 : (\lambda, x) \) is an eigenpair of \( \Delta_p\} \) the eigenspace corresponding to \( \lambda \). For convenience, we set \( S_1(\Delta_p) = \emptyset \) if \( \lambda \) is not an eigenvalue. And, we usually work on the set of the normalized eigenvectors, i.e., \( \hat{S}_1(\Delta_p) = \{x \in \mathbb{R}^n : (\lambda,x) \) is an eigenpair of \( \Delta_p, ||x||_\infty = 1\} \).

Proposition 2.3. The set-valued map \( (p,\lambda) \mapsto \hat{S}_1(\Delta_p) \) defines an upper semi-continuous map on \([1, +\infty) \times [0, +\infty) \).

Proof. Suppose \( p_n \to p \) and \( \lambda_n \to \lambda, n \to +\infty \). By a slight generalization of Proposition 2.2, if \( \lambda \) is not an eigenvalue of \( \Delta_p \), then \( \lambda_n \) cannot be an eigenvalue of \( \Delta_{p_n} \) when \( n \) is sufficiently large. In this case, \( \hat{S}_1(\Delta_{p_n}) = \emptyset = \hat{S}_1(\Delta_{p_n}) \). So \( \hat{S}_1(\Delta_{p_n}) \) is continuous at \( (p, \lambda) \).

If \( \lambda \) is an eigenvalue of \( \Delta_p \), we shall prove that \( \hat{S}_1(\Delta_p) \) is upper semi-continuous at \( (p, \lambda) \). Suppose the contrary, that \( (p, \lambda) \mapsto \hat{S}_1(\Delta_p) \) is not upper semi-continuous at some \( (p, \lambda) \in [1, +\infty) \times [0, +\infty) \). Then, there exist \( \epsilon > 0 \) and a subsequence \( (p_n, \lambda_n) \to (p, \lambda) \) such that \( \hat{S}_1(\Delta_{p_n}) \not\subset \hat{B}_\epsilon(\hat{S}_1(\Delta_p)) \), where \( \hat{B}_\epsilon(\hat{S}_1(\Delta_p)) \) denotes the \( \epsilon \)-neighborhood of the nonempty compact set \( \hat{S}_1(\Delta_p) \). Then, by a standard technique, we derive that there is a limit point \( x \) of \( \hat{S}_1(\Delta_{p_n}) \) which is not in \( \hat{S}_1(\Delta_p) \). However, similar to the proof of Proposition 2.2, \( x \) must be an eigenvector of \( \lambda(\Delta_p) \), which is a contradiction.

Remark 2.1. Similarly, \( p \mapsto \bigcup_{\lambda \in \text{spec}(\Delta_p)} \hat{S}_1(\Delta_p) \) is also an upper semi-continuous set-valued map.
Proof of Lemma 2.7. We only need to show that \( p \mapsto \text{spec}(\Delta_p) \) is an upper semi-continuous set-valued map. Suppose the contrary, that there exists \( \epsilon > 0 \) and a subsequence \( p_n \rightarrow p \) and a subsequence \( \lambda_{p_n} \in \text{spec}(\Delta_{p_n}) \setminus B_{\epsilon}(\text{spec}(\Delta_p)) \). Without loss of generality, we may assume that \( \lambda_{p_n} \) converges to some number \( \lambda \). Then, by \( \hat{S}_{\lambda_n}(\Delta_{p_n}) \neq \emptyset \) and Proposition 2.3 there holds \( \hat{S}_{\lambda}(\Delta_p) \neq \emptyset \), implying that \( \lambda \in \text{spec}(\Delta_p) \). This is a contradiction. \( \square \)

In addition, we have the upper semi-continuity of the eigenvalue multiplicity. Precisely, denote by \( m(p, \lambda) := \gamma(\hat{S}_{\lambda}(\Delta_p)) \) the \( \gamma \)-multiplicity of \( \lambda \) of \( \Delta_p \) (if \( \lambda \) is not an eigenvalue of \( \Delta_p \), we simply write \( m(p, \lambda) = 0 \)). As an analogous, we can define the \( \gamma^- \)-multiplicity \( m^-(p, \lambda) \) and the \( \gamma^+ \)-multiplicity \( m^+(p, \lambda) \) for an eigenvalue \( \lambda \) of \( \Delta_p \). We remark here that the \( \gamma^- \)-multiplicity was introduced in [29], and independently in [13].

**Proposition 2.4.** The multiplicity function \( m^- : [1, +\infty) \times [0, +\infty) \rightarrow \mathbb{N} \) is upper semi-continuous, i.e., \( \limsup_{(p', \lambda') \rightarrow (p, \lambda)} m^-(p', \lambda') \leq m^-(p, \lambda) \).

**Proof.** By the continuity of the Krasnosel’kii genus \( \gamma^- \), there exists \( \epsilon > 0 \) such that \( \gamma^-(B_\epsilon(\hat{S}_{\lambda}(\Delta_p))) = \gamma^-(\hat{S}_{\lambda}(\Delta_p)) \). By Proposition 2.3, for any \( (p', \lambda') \) sufficiently close to \( (p, \lambda) \), \( \hat{S}_{\lambda'}(\Delta_{p'}) \subset B_\epsilon(\hat{S}_{\lambda}(\Delta_p)) \). Then, by the monotonicity of \( \gamma^- \), we have \( \gamma^-(\hat{S}_{\lambda'}(\Delta_{p'})) \leq \gamma^-(B_\epsilon(\hat{S}_{\lambda}(\Delta_p))) \). Therefore, \( m^-(p', \lambda') = \gamma^-(\hat{S}_{\lambda'}(\Delta_{p'})) \leq \gamma^-(\hat{S}_{\lambda}(\Delta_p)) = m^-(p, \lambda) \).

Finally, we show the locally Lipschitz continuity of variational eigenvalues.

**Proposition 2.5.** For any \( k \in \{1, \cdots, n\} \), the functions \( p \mapsto \lambda_k(\Delta_p), p \mapsto \lambda_k^-(\Delta_p) \) and \( p \mapsto \lambda_k^+(\Delta_p) \), are locally Lipschitz continuous with respect to \( p \in [1, +\infty) \).

**Proof of Proposition 2.5.** It is known that the \( k \)-th min-max eigenvalue \( \lambda_k^-(\Delta_p) \) varies continuously in \( p \). This statement is a direct consequence of the main result in [13], and thus we omit it. Actually, the continuity of \( \lambda_k(\Delta_p), \lambda_k^-(\Delta_p) \) and \( \lambda_k^+(\Delta_p) \) w.r.t. \( p \) can be derived from Theorem 1.1 straightforwardly.

Moreover, since the function \( (p, q) \mapsto \max_{x \neq 0} |F_q(x) - F_p(x)| \) is locally Lipschitz, it follows from Lemma 2.3 that \( p \mapsto \lambda_k(\Delta_p), p \mapsto \lambda_k^-(\Delta_p) \) and \( p \mapsto \lambda_k^+(\Delta_p) \) are also locally Lipschitz. \( \square \)

**Remark 2.2.** The continuity of the \( k \)-th min-max eigenvalue \( \lambda_k(\Delta_p) \) is essentially known. We refer to [13, 30] for the case of the \( p \)-Laplacian on Euclidean domains under Dirichlet boundary conditions.

### 2.2 Homological eigenvalues of \( \Delta_p \) and eigenvalues produced by homotopical linking

The homological critical value is an important concept used in the theory of persistent homology. We shall adopt the definition proposed by Bubenik and Scott [3], which is inspired by the definition suggested by Cohen-Steiner, Edelsbrunner, and Harer [2]:

**Definition 2.5.** A real number \( c \) is a homological regular value of the function \( F \) if there exists \( \epsilon > 0 \) such that for each pair of real numbers \( t_1 < t_2 \) on the interval \( (c - \epsilon, c + \epsilon) \), the inclusion \( \{F \leq t_1\} \hookrightarrow \{F \leq t_2\} \) induces isomorphisms on all homology groups \([\mathbb{Z}]\). A real number that is not a homological regular value of \( F \) is called a homological critical value of \( F \).

It is clear that the following critical value lemma holds:

**Lemma 2.2.** Suppose the function \( F \) has no homological critical values on the closed interval \([x, y]\), then the inclusion \( F^{-1}(-\infty, x) \hookrightarrow F^{-1}(-\infty, y) \) induces isomorphisms on all homology groups.

**Remark 2.3.** A symmetric homological critical value \([6]\) of \( F \) is a real number \( c \) for which there exists an integer \( k \) such that for all sufficiently small \( \epsilon > 0 \), the map \( H_k(\{F \leq c - \epsilon\}) \hookrightarrow H_k(\{F \leq c + \epsilon\}) \) induced by inclusion is not an isomorphism \([6]\). Here \( H_k \) denotes the \( k \)-th singular homology (possibly with coefficients in a field). As mentioned in [4, 24], this widely cited definition given by Cohen-Steiner, Edelsbrunner, and Harer [6] doesn’t imply the critical value lemma in generic scenarios.
In fact, if we replace homological critical values by symmetric homological critical values in Lemma 2.2, the conclusion doesn’t hold even though $F$ is continuous. Govea [24] presented a modified version of the critical value lemma: Suppose $F$ is continuous and has no symmetric homological critical values on the interval $[x, y)$, then the inclusion $F^{-1}(-\infty, x) \hookrightarrow F^{-1}(-\infty, y)$ induces isomorphisms on all homology groups.

We shall simply recall the deformation lemma which will be essentially used many times in the present paper. In critical point theory, the deformation lemma roughly says that if an interval $[a, b] \subset \mathbb{R}$ contains no critical value of a typical continuous function $F$, then there is a continuous deformation from the sublevel sets $\{F \leq b\}$ to $\{F \leq a\}$, which induces a homotopy equivalence between these sublevel sets [8, 33].

**Proposition 2.6.** Let $F$ be a continuous even function on $\{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$. Then, the min-max critical values

$$
\lambda_k := \inf_{\gamma(S) \geq k} \sup_{x \in S} F(x), \quad k = 1, 2, \ldots
$$

of $F$ are homological critical values of $F$.

**Proof.** Suppose $\lambda_k$ is the $k$-th min-max critical value of $F$, and assume $\lambda_k < \lambda_{k+1}$. We shall prove that $\lambda_k$ is a homological critical value. On one hand, there exists $A \subset \{F \leq \lambda_k\}$ with $\gamma(A) \geq k$, which yields $\gamma\{F \leq \lambda_k\} \geq k$. On the other hand, for any $A'$ with $\gamma(A') \geq k + 1$, $A' \nsubseteq \{F \leq \lambda_k\}$, which implies that $\gamma\{F \leq \lambda_k\} < k + 1$. Therefore, $\gamma\{F \leq \lambda_k\} = k$.

Similarly, it is easy to check that $\gamma\{F \leq \lambda_k - \epsilon\} < k$ for any sufficiently small $\epsilon > 0$. Since different Yang indices imply different homology groups (see [40] or Section 0.7 in [35]), the homology of the topology of the sublevel set changes from $\lambda_k - \epsilon$ to $\lambda_k$. Therefore, $\lambda_k$ is a homological critical value of $F$. \qed

Returning to graph $p$-Laplacians, we introduce the concept of homological eigenvalues.

**Definition 2.6.** We say $\lambda$ is a homological eigenvalue of $\Delta_p$, if $\lambda$ is a homological critical value of the $p$-Rayleigh quotient

$$
F_p(x) := \frac{\sum_{(j,i) \in E} |x_i - x_j|^p}{\sum_{i \in V} \deg(i)|x_i|^p}
$$

where $1 \leq p < +\infty$. We say $\lambda$ is an isolated homological eigenvalue of $\Delta_p$, if $\lambda$ is a homological eigenvalue, and $\lambda$ is not a limit point of the eigenvalues of $\Delta_p$.

For simplicity, we use $\{F_p \leq \lambda\}$ to indicate the lower level set $\{x \in \mathbb{R}^n : F_p(x) \leq \lambda, \|x\|_\infty = 1\}$.

**Theorem 2.1.** For any homological eigenvalue $\lambda$ of $\Delta_1$, and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $p \in (1, 1 + \delta)$, $\Delta_p$ has an eigenvalue in $(\lambda - \epsilon, \lambda + \epsilon)$.

Theorem 2.1 overcomes the difficulty that in general, a non-variational $\Delta_1$-eigenvalue may not be a limit point of the $\Delta_p$-eigenvalues. We shall prove it at the end of this section.

The concept of homotopical linking is useful for obtaining critical points.

**Definition 2.7** (Definition 0.17 in [33]). For subsets $Q$, $Q'$ and $S$ of a given topological space $X$ with $Q \subset Q'$, we say that $Q$ homotopically links $S$ with respect to $Q'$, if $Q \cap S = \emptyset$ and for any continuous map $\gamma : Q' \to X$ with $\gamma|_Q = \text{id}$ (i.e., $\gamma(x) = x$ for any $x \in Q$), $\gamma(Q') \cap S \neq \emptyset$.

If $Q$ homotopically links $S$ with respect to $Q'$, and $f : X \to \mathbb{R}$ is continuous with $\min_{x \in S} f(x) > \max_{x \in Q} f(x)$, then by linking theorem (Theorem 0.21 and Proposition 3.21 in [33]),

$$
\inf_{\text{continuous } \gamma : Q' \to X \text{ with } \gamma|_Q = \text{id}} \max_{x \in \gamma(Q')} f(x)
$$

is a critical value of $f$, which is said to be a critical value of $f$ produced by homotopical linking.

The next lemma shows the stability and local monotonicity for both isolated homological critical values and critical values produced by homotopical linking.
Lemma 2.3. Given a function $f$ on a topological space $X$, for any isolated homological critical value $\lambda$ of $f$, and for any sufficiently small $\epsilon > 0$, for any $\epsilon$-perturbation $f_\epsilon$ of $f$, i.e., $\| f_\epsilon - f \|_\infty < \epsilon$, there is a homological critical value of $f_\epsilon$ in $[\lambda - 3\epsilon, \lambda + 3\epsilon]$. This is the stability of isolated homological critical values.

If $f_\epsilon$ is further assumed to an increasing $\epsilon$-perturbation, i.e., $f(x) \leq f_\epsilon(x) \leq f(x) + \epsilon$, $\forall x$, then there is a homological critical value of $f_\epsilon$ in $[\lambda, \lambda + \epsilon]$. This shows the local monotonicity of isolated homological critical values.

All the above statements still hold if we use critical values produced by homotopical linking instead of isolated homological critical values, and if we further assume that both $f$ and $f_\epsilon$ are continuous.

Proof. We concentrate on homological critical values. Without loss of generality, we assume that there is no homological critical value of $f$ in $[\lambda - 3\epsilon, \lambda) \cup (\lambda, \lambda + 3\epsilon)$. Then, we can find $\epsilon_1 > 0$ such that any value in $(\lambda - 3\epsilon - \epsilon_1, \lambda) \cup (\lambda, \lambda + 3\epsilon + \epsilon_1)$ is homological regular.

For the first statement about the stability of isolated homological critical values, we suppose the contrary, that there is no homological critical value of $f_\epsilon$ in $[\lambda - 3\epsilon, \lambda + 3\epsilon]$. Then, there exists $\epsilon_1 \in (0, \epsilon_1]$ such that any value in $(\lambda - 3\epsilon - \epsilon_1, \lambda + 3\epsilon + \epsilon_1)$ is homological regular for $f_\epsilon$. We consider the inclusion relation

$$\{ f \leq \epsilon \} \overset{i_1}{\hookrightarrow} \{ f_\epsilon \leq c + \epsilon \} \overset{i_2}{\hookrightarrow} \{ f \leq c + 2\epsilon \} \overset{i_3}{\hookrightarrow} \{ f \leq c + 3\epsilon \}.$$

Taking $c = \lambda - 3\epsilon - \epsilon_1$ in the above inclusion relation, where $0 < \epsilon_1 < \epsilon_1$, by the critical value lemma, both the inclusion $\{ f \leq c + \epsilon \} \overset{i_2}{\hookrightarrow} \{ f \leq c + 3\epsilon \}$ and the inclusion $\{ f \leq c \} \overset{i_2}{\hookrightarrow} \{ f \leq c + 2\epsilon \}$ induce isomorphisms on all homology groups, and then by Lemma 3.1 in [24], the inclusions $i_1$, $i_2$ and $i_3$ also induce isomorphisms on all homology groups.

In consequence, we get the homological equivalence $\{ f \leq \lambda - 3\epsilon - \epsilon_1 \} \sim \{ f_\epsilon \leq \lambda - \epsilon_1 \}$, i.e., the inclusion $\{ f \leq \lambda - 3\epsilon - \epsilon_1 \} \rightarrow \{ f \leq \lambda - \epsilon_1 \}$ induces isomorphisms on all homology groups.

Similarly, taking $c = \lambda + \epsilon_1$ in the following inclusion chain

$$\{ f \leq c \} \subset \{ f \leq c + \epsilon \} \subset \{ f \leq c + 2\epsilon \} \subset \{ f \leq c + 3\epsilon \},$$

we have the homological equivalence $\{ f_\epsilon \leq \lambda + \epsilon_1 \} \sim \{ f \leq \lambda + 3\epsilon + \epsilon_1 \}$.

Note that by the inclusion chain

$$\{ f \leq \lambda - 3\epsilon - \epsilon_1 \} \overset{i_1}{\hookrightarrow} \{ f_\epsilon \leq \lambda - \epsilon_1 \} \overset{i_2}{\hookrightarrow} \{ f_\epsilon \leq \lambda + \epsilon_1 \} \overset{i_3}{\hookrightarrow} \{ f \leq \lambda + 3\epsilon + \epsilon_1 \}$$

we have the diagram

$$\begin{array}{ccc}
H_*(\{ f \leq \lambda - 3\epsilon - \epsilon_1 \}) & \overset{i_2 \circ i_2 \circ i_3}{\longrightarrow} & H_*(\{ f \leq \lambda + 3\epsilon + \epsilon_1 \}) \\
\bigg\downarrow i_2 & & \bigg\downarrow i_2 \\
H_*(\{ f_\epsilon \leq \lambda - \epsilon_1 \}) & \overset{i_2}{\longrightarrow} & H_*(\{ f_\epsilon \leq \lambda + \epsilon_1 \})
\end{array}$$

Since $f_\epsilon$ has no homological critical value in $(\lambda - 3\epsilon - \epsilon_1, \lambda + 3\epsilon + \epsilon_1)$, the inclusion $i_2$ induces the isomorphisms $i_2^*$ on all homology groups. Then, we have obtained that all the inclusions $i_1^*, i_2^*, i_3^*$ induce isomorphisms $i_1^*, i_2^*, i_3^*$, which leads to an isomorphism

$$(i_3^* \circ i_2^* \circ i_1^*)_* : H_*(\{ f \leq \lambda - 3\epsilon - \epsilon_1 \}) \cong H_*(\{ f \leq \lambda + 3\epsilon + \epsilon_1 \}).$$

That is, we obtain the homological equivalence $\{ f \leq \lambda - 3\epsilon - \epsilon_1 \} \sim \{ f \leq \lambda + 3\epsilon + \epsilon_1 \}$, which is a contradiction with the assumption that $\lambda$ is a homological critical value of $f$.

For the second statement on the local monotonicity of isolated homological critical values, note that

$$\{ f_\epsilon \leq c \} \subset \{ f \leq c \} \subset \{ f_\epsilon \leq c + \epsilon \} \subset \{ f \leq c + \epsilon \}. \quad (6)$$

A similar argument yields that there is a homological critical value of $f_\epsilon$ in $[\lambda - \epsilon, \lambda + \epsilon]$. Suppose the contrary, that there is no homological critical value of $f_\epsilon$ in $[\lambda, \lambda + \epsilon]$. Then, without loss of generality,
we may assume that there is no homological critical value of $f_{\epsilon}$ in $[\lambda - \epsilon_1, \lambda + \epsilon + \epsilon_1]$. Then, for any $0 < \epsilon' < \epsilon_1$, it is not difficult to verify that the inclusion relation
\[
\{ f \leq \lambda - \epsilon - \epsilon' \} \subset \{ f \leq \lambda - \epsilon \} \subset \{ f \leq \lambda - \epsilon' \} \subset \{ f \leq \lambda + \epsilon - \epsilon' \}
\]
implies the homological equivalence $\{ f \leq \lambda - \epsilon \} \sim \{ f \leq \lambda - \epsilon' \}$. However, since there is no homological critical value of $f_{\epsilon}$ in $(\lambda - \epsilon_1, \lambda + \epsilon)$, we have the homological equivalence $\{ f \leq \lambda - \epsilon \} \sim \{ f \leq \lambda + \epsilon \}$. Taking $c = \lambda + \epsilon'$ in (6), we derive the homological equivalence $\{ f \leq \lambda + \epsilon \} \sim \{ f \leq \lambda + \epsilon' \}$ in a similar manner as shown in the proof of the first statement. Therefore, we deduce the homological equivalence $\{ f \leq \lambda - \epsilon \} \sim \{ f \leq \lambda + \epsilon \}$, which contradicts to the assumption that $\lambda$ is a homological critical value of $f$.

Finally, we focus on the critical values produced by homotopical linking. Suppose that $Q$ homotopically links $S$ with respect to $Q'$, where $Q \subset Q' \subset X$, $S \subset X$, and $\min_{x \in S} f(x) > \max_{x \in Q} f(x) + 4\epsilon$. Then, for any continuous function $f : X \to \mathbb{R}$ with $\| f - f_{\epsilon} \|_{\infty} < \epsilon$, $\min_{x \in S} f_{\epsilon}(x) > \max_{x \in Q} f_{\epsilon}(x)$ which implies that
\[
\lambda_{Q,Q',S}(f_{\epsilon}) := \inf_{\gamma : Q' \to X \text{ with } \gamma|_Q = \text{id}} \max_{x \in \gamma(Q')} f_{\epsilon}(x)
\]
is a critical value of $f_{\epsilon}$. It is clear that $|\lambda_{Q,Q',S}(f_{\epsilon}) - \lambda_{Q,Q',S}(f)| < \epsilon$. Similarly, if $f(x) \leq f_{\epsilon}(x) \leq f(x) + \epsilon$, $\forall x \in X$, then $\lambda_{Q,Q',S}(f) \leq \lambda_{Q,Q',S}(f_{\epsilon}) \leq \lambda_{Q,Q',S}(f) + \epsilon$. The proof is completed.

We should note that Theorem 2.1 is a consequence of Lemma 2.3 due to the fact that the homological critical values of $\Delta_1$ are isolated. For the sake of completeness, we write down a brief proof below.

**Proof of Theorem 2.1.** By Theorem 1 in [10], $\Delta_1$ has finitely many eigenvalues. We may assume without loss of generality that $\epsilon < \frac{1}{2} \min\{|\lambda' - \lambda''| : \text{different eigenvalues } \lambda', \lambda'' \text{ of } \Delta_1\}$. Then, there is no eigenvalue of $\Delta_1$ in the set $(\lambda - \epsilon, \lambda) \cup (\lambda, \lambda + \epsilon)$. That is, $\lambda$ is an isolated homological eigenvalue of $\Delta_1$, and thus it is an isolated homological critical value of $F_1$.

Take sufficiently small $\delta > 0$ such that $|F_p(x) - F_1(x)| < \epsilon/4$ for any $1 < p < 1 + \delta$, and for any $x$ with $x \neq 0$. By Lemma 2.3, there exists a homological critical value of $F_p$ in $(\lambda - \epsilon, \lambda + \epsilon)$, which is actually an eigenvalue of $\Delta_p$. The proof is completed.

The next example shows that the local monotonicity doesn’t hold for a non-homological critical value.

**Remark 2.4.** Let $f_1(x) = (x + t)^3 - \frac{t^3}{6}$. Then $f_t > f_0$ when $t > 0$ and $f_t < f_0$ when $t < 0$, while the unique critical value of $f_1$ is $-\frac{t^3}{6}$. This implies that a larger function may have a smaller critical value.

In addition, the stability also fails for a non-homological critical value. For example, let $g_t(x) = x^3 + t^2x$. Then, $g_0$ has the unique critical value 0, but for $t \neq 0$, $g_t$ has no critical value.

From these examples, we actually show that both the stability and local monotonicity do not hold in the case of non-homological critical values.

**Definition 2.8.** We define the eigenvalues of $\Delta_p$ produced by homotopical linking as the critical values of $F_p$ produced by homotopical linking.

**Remark 2.5.** Theorem 2.1 still hold if we use eigenvalues produced by homotopical linking instead of homological eigenvalue. The proof is also based on Lemma 2.3.

### 2.3 Homological eigenvalues of 1-Laplacian

Based on the discussions in Section 3.1, we are able to estimate the variational eigenvalues of $\Delta_1$. To further characterize the homological eigenvalues of $\Delta_1$, we borrow some ideas from PL Morse theory and discrete Morse theory, which are shown in the following proof of Theorem 1.3.

We first recall the simplicial complex $K_n$. Let $P_2(V) = \{(A, B) : A \cap B = \emptyset, A \cup B \neq \emptyset, A, B \subset V\}$ be the collection of all the pairs of disjoint subsets of $V$. Then, there is a natural partial order $\prec$
on \(\mathcal{P}_2(V)\) defined as \((A, B) \prec (A', B')\) if \(A \subset A'\) and \(B \subset B'\). This partial order \(\prec\) is actually the inclusion order for set-pairs. The simplicial complex \(K_n\) is defined as the order complex on \(\mathcal{P}_2(V)\) with respect to the inclusion order for set-pairs, i.e., the faces of \(K_n\) refer to the inclusion chains in \(\mathcal{P}_2(V)\). A natural geometric realization of \(K_n\) is the following triangulation of the unit \(l^\infty\)-sphere \(\{x \in \mathbb{R}^n : \|x\|_\infty = 1\}\):

For any \(A \subset V\) and any permutation \(\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\), we define the simplex

\[
\Delta_{A, \sigma} = \text{conv}\left(\{1_A - 1_{V \setminus A}\} \cup \{1_{A \setminus \{\sigma(1), \ldots, \sigma(i)\}} - 1_{(V \setminus A) \setminus \{\sigma(1), \ldots, \sigma(i)\}} : i = 1, \ldots, n - 1\}\right)
\]

which is of dimension \((n - 1)\). Then, \(\{\Delta_{A, \sigma} : A \subset V, \text{permutation } \sigma \text{ on } V\}\) is the set of maximal simplices of \(K_n\), and we rewrite the vertices of \(K_n\) as \(\{1_A - 1_B : (A, B) \in \mathcal{P}_2(V)\}\). Similarly, for a given subset \(A \subset \mathcal{P}_2(V)\), we regard \(A\) as a poset with the partial order \(\prec\), i.e., for any \((A, B), (A', B') \in A\), \((A, B) \prec (A', B')\) if and only if \(A \subset A'\) and \(B \subset B'\). The subcomplex corresponding to \(A\) is defined to be the order complex of the poset \((A, \prec)\), and we work on its natural geometric realization whose faces are determined by the geometric simplices \(\text{conv}\{1_{A_1} - 1_{B_1}, \ldots, 1_{A_k} - 1_{B_k}\}\) for any chain \((A_1, B_1) \prec \cdots \prec (A_k, B_k)\) in \((A, \prec)\).

Given a vertex \(1_A - 1_B\) of \(A\), we shall use \(\text{star}(1_A - 1_B)\) to denote the closed star of \(1_A - 1_B\), that is, the closure of the union of the simplices in \(|K_n|\) that have \(1_A - 1_B\) as a vertex.

We use \(\text{link}(1_A - 1_B)\) to denote the link of \(1_A - 1_B\), that is, \(\text{star}(1_A - 1_B) \setminus \text{star}(1_A - 1_B)\), where \(\text{star}(1_A - 1_B)\) is just the union of the relatively open simplices in \(|K_n|\) that have \(1_A - 1_B\) as a vertex.

**Proof of Theorem 2.1** Assume that the subcomplexes induced by the two sublevel sets \(\{x : F_1(x) < \lambda\}\) and \(\{x : F_1(x) \leq \lambda\}\) have different homology groups. Then, there exists a pair \((A, B)\) of disjoint subsets of \(V = \{1, \ldots, n\}\) such that \(F_1(1_A - 1_B) = \lambda\), that is, \(\lambda\) is the value of \(F_1\) acting on some vertices of \(K_n\). We shall prove that \(\lambda\) is a homological eigenvalue of \(\Delta_1\). Suppose the contrary, that \(\lambda\) is not a homological eigenvalue of \(\Delta_1\), i.e., \(\lambda\) is not a homological critical value of \(F_1\). Then, by the definition of homological regular values, there exists \(\epsilon > 0\) such that for each pair of real numbers \(t_1 < t_2\) on the interval \((\lambda - \epsilon, \lambda + \epsilon)\), the inclusion \(\{F_1 \leq t_1\} \hookrightarrow \{F_1 \leq t_2\}\) induces isomorphisms on all homology groups. Particularly, for any sufficiently small \(\epsilon' > 0\), \(\{F_1 \leq \lambda - \epsilon'\} \hookrightarrow \{F_1 \leq \lambda\}\) induces isomorphisms on all homology groups. It is clear that the subcomplex of \(K_n\) induced by the vertices in \(\{x \in \mathbb{R}^n : F_1(x) < \lambda\}\) coincides with the subcomplex of \(K_n\) induced by the vertices in \(\{x \in \mathbb{R}^n : F_1(x) \leq \lambda - \epsilon'\}\) exactly, when \(\epsilon' > 0\) is sufficiently small.

Let \(X_1 = \{x \in \mathbb{R}^n : \sum_{i=1}^n \deg(i)|x_i| = 1\}\) and let \(|\tilde{K_n}|\) be the triangulation of \(X_1\) whose \((n - 1)\)-dimensional simplices of \(|K_n|\) possess the form

\[
\text{conv}\left\{\frac{1_{A_1} - 1_{B_1}}{\text{vol}(A_1 \cup B_1)}, \ldots, \frac{1_{A_n} - 1_{B_n}}{\text{vol}(A_n \cup B_n)}\right\}
\]

where \((A_1, B_1) \prec \cdots \prec (A_n, B_n), \#(A_i \cup B_i) = i, (A_i, B_i) \in P_2(V)\). Note that both \(|K_n|\) and \(|\tilde{K_n}|\) are piecewise linear manifolds sharing the same simplicial complex structure, in which \(|\tilde{K_n}|\) is a triangulation of \(X_1\) while \(|K_n|\) is a triangulation of \(X_\infty := \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}\).

Consider a map \(r : X_1 \rightarrow X_\infty\) defined as \(r(x) = \frac{x}{\|x\|_\infty}\). Then \(r\) is a homeomorphism, and map each simplex of \(|\tilde{K_n}|\) to a simplex of \(|K_n|\) by sending each vertex \(\frac{1_{A_i} - 1_{B_i}}{\text{vol}(A_i \cup B_i)}\) of \(|\tilde{K_n}|\) to its corresponding vertex \(1_{A_i} - 1_{B_i}\) of \(|K_n|\). Clearly, \(r^{-1} : |K_n| \rightarrow |\tilde{K_n}|\) is defined as \(r^{-1}(x) = \sum_{i=1} \frac{x}{\|x\|_\infty} \deg(i)|x_i|\). Moreover, \(F_1\) is piecewise linear on \(X_1\), and it is actually linear on every simplex of \(|K_n|\). It follows from the zero-homogeneity of \(F_1\) that \(r(\{x \in |\tilde{K_n}| : F_1(x) \leq t\}) = \{x \in |K_n| : F_1(x) \leq t\}\) for any \(t \in \mathbb{R}\).

We should use the following argument:

**Claim 2.1** (Proposition 2.25 in [2], see also Kühnel [31] and Morozov [34]). Given a PL function \(f^{PL}\) on a simplicial complex \(|K|\), the induced subcomplex of \(K\) on \(\{v \in K_0 : f^{PL}(v) \leq t\}\) is homotopy equivalent to the sublevel set \(\{f^{PL} \leq t\}\), where \(K_0\) is the vertex set of \(K\).

---

5We usually don’t distinguish the simplicial complex \(K_n\) and its geometric realization \(|K_n|\).
Remark 2.6. A similar idea has been used to establish a relationship between Forman’s discrete sense of Brehm and Kühnel [5] (or in the sense of Edelsbrunner [20, 21]).

Thus, we can apply Kühnel’s theorem (i.e., Claim 2.1) to derive that the subcomplex of $K_n$ induced by the vertices in $\{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\}$ is homotopy equivalent to the sublevel set $\{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\}$. Furthermore, we have the commutative diagram:

\[
\begin{array}{ccc}
\{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\} & \xrightarrow{h} & \mathcal{K}_{\{x \in K_n : F_1(x) \leq \lambda - \epsilon'\}} \\
\uparrow_{r^{-1}} & & \downarrow r \\
\{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\} & \xrightarrow{h'} & \mathcal{K}_{\{x \in K_n : F_1(x) \leq \lambda - \epsilon'\}}
\end{array}
\]

where $h$ is a continuous map such that $h \circ i \simeq id$ and $i \circ h \simeq id$, and

\[i : \mathcal{K}_{\{x \in K_n : F_1(x) \leq \lambda - \epsilon'\}} \hookrightarrow \{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\}\]

is the inclusion map. Since $r$ is a homeomorphism, we have $h' \circ i' = (r \circ h \circ r^{-1}) \circ (r \circ i \circ r^{-1}) = r \circ h \circ i \circ r^{-1} \simeq id$, and similarly, there holds $i' \circ h' \simeq id$, where

\[i' : \mathcal{K}_{\{x \in K_n : F_1(x) \leq \lambda - \epsilon'\}} \hookrightarrow \{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\}\]

is the inclusion map. Therefore, the subcomplex $\mathcal{K}_{\{x \in K_n : F_1(x) \leq \lambda - \epsilon'\}}$ is homotopy equivalent to the sublevel set $\{x \in |K_n| : F_1(x) \leq \lambda - \epsilon'\}$.

In consequence,

\[H_*(\mathcal{K}_{F_1 < \lambda}) = H_*(\mathcal{K}_{F_1 \leq \lambda - \epsilon'}) = H_*(\{F_1 \leq \lambda - \epsilon'\}) = H_*(\{F_1 \leq \lambda\}) = H_*(\mathcal{K}_{F_1 \leq \lambda})\]

which is a contradiction. In the above equalities, we use $\mathcal{K}_{F_1 < \lambda}$ to denote the subcomplex of $K_n$ induced by the vertices in the lower level set $\{F_1 < \lambda\}$.

For the converse part, assume that $\lambda$ is a homological eigenvalue. Then, it follows from Kühnel’s Theorem and the definition of homological eigenvalue that the inclusion $\mathcal{K}_{F_1 < \lambda} \hookrightarrow \mathcal{K}_{F_1 \leq \lambda}$ does not induce isomorphisms on homology groups.

Now, we move on to the local case. Assume that there exists $A \neq \emptyset$ with $F_1(1_A) = \lambda$ and $F_1(v) \neq \lambda$ for any vertex $v$ in $\text{link}(1_A)$. Then locally, the subcomplex of $K_n$ induced by the vertices in $\{x \in \text{star}(1_A) : F_1(x) \leq \lambda\}$ is a cone, and therefore it is contractible. It is easy to see that in this situation, the subcomplex of $K_n$ induced by the vertices in $\{x \in \text{star}(1_A) : F_1(x) < \lambda\}$ coincides with the subcomplex of $K_n$ induced by the vertices in $\{x \in \text{link}(1_A) : F_1(x) < \lambda\}$.

Thus, if the subcomplex of $K_n$ induced by the vertices in $\{x \in \text{link}(1_A) : F_1(x) < \lambda\}$ has non-vanishing reduced homology, then

\[H_*(\mathcal{K}_{\{F_1 < \lambda\} \cap \text{star}(1_A)}) \neq H_*(\mathcal{K}_{\{F_1 \leq \lambda\} \cap \text{star}(1_A)})\]

and consequently, it is easy to see that $H_*(\mathcal{K}_{F_1 < \lambda}) \neq H_*(\mathcal{K}_{F_1 \leq \lambda})$, which implies that $\lambda$ is a homological eigenvalue of $\Delta_\lambda$.

Incidentally, $1_A$ is a PL critical point and $\lambda$ is the corresponding PL critical value of $F_1$ in the sense of Brehm and Kühnel [5] (or in the sense of Edelsbrunner [20, 21]). \hfill \square

Remark 2.6. A similar idea has been used to establish a relationship between Forman’s discrete Morse theory and Edelsbrunner’s PL Morse theory in a previous work of the author [22]. Moreover, we have studied the combinatorial structure of the simplicial complex $K_n$ in another work [23].

2.4 Monotonicity of some functions involving eigenvalues of $\Delta_p$ with respect to $p$

Proof of Theorem 2.7. Given $p, q \geq 1$, we define an odd continuous map $\Phi_{q/p} : \mathbb{R}^n \to \mathbb{R}^n$ by

\[\Phi_{q/p}(x) = (|x_1|^p \text{sign}(x_1), \ldots, |x_n|^p \text{sign}(x_n)).\]
Claim 2.2. For any $1 \leq p \leq q$, for any $x \in \mathbb{R}^n$,

$$F_p(\Phi_{q/p}(x)) \geq 2^{p-q} F_q(x). \tag{7}$$

In addition, if $q < p$, then for any $x$ with $F_q(x) > 0$, the inequality in (7) is strict.

Proof. Denote by $\|x\|_p = \left(\sum_{i=1}^n \deg(i) |x_i|^p \right)^{\frac{1}{p}}$, $\forall p \geq 1$. It is clear that $\|\Phi_{q/p}(x)\|_p^p = \|x\|_q^q$. Denote by $TV_p(x) = \sum_{(i,j)\in E} |x_i - x_j|^p$. Then

$$TV_p(\Phi_{q/p}(x)) = \sum_{(i,j)\in E} \left| |x_i|^\frac{q}{p} \text{sign}(x_i) - |x_j|^\frac{q}{p} \text{sign}(x_j) \right|^p$$

$$\geq \sum_{(i,j)\in E} |x_i - x_j|^p \left( \frac{|x_i|^\frac{q}{p} + |x_j|^\frac{q}{p}}{2} \right)^{1-rac{q}{p}}$$

$$\geq \sum_{(i,j)\in E} |x_i - x_j|^p \left( \frac{|x_i| + |x_j|}{2} \right)^{\frac{q}{p}(1-\frac{q}{p})}$$

$$\geq \sum_{(i,j)\in E} |x_i - x_j|^p \left( \frac{|x_i - x_j|}{2} \right)^q$$

$$= 2^{p-q} \sum_{(i,j)\in E} |x_i - x_j|^q = 2^{p-q} TV_q(x)$$

where we used the inequality (see Lemma [A.1])

$$\left| |b|^\frac{q}{p} \text{sign}(b) - |a|^\frac{q}{p} \text{sign}(a) \right| \geq |b - a| \left( \frac{|b|^\frac{q}{p} + |a|^\frac{q}{p}}{2} \right)^{1-\frac{q}{p}}.$$ 

The above inequality is strict whenever $a \neq b$ and $q > p$. This implies that $TV_p(\Phi_{q/p}(x)) > 2^{p-q} TV_q(x)$ if $x_i \neq x_j$ for some $(i, j) \in E$.

Therefore,

$$F_p(\Phi_{q/p}(x)) = \frac{TV_p(\Phi_{q/p}(x))}{\|\Phi_{q/p}(x)\|_p^p} \geq 2^{p-q} TV_q(x) \|x\|_q^q = 2^{p-q} F_q(x),$$

and the inequality is strict whenever $q > p$ and $F_q(x) > 0$.

The proof of the claim is completed.

Similarly, we have another inequality which has been essentially shown in [1]:

Claim 2.3. For any $1 \leq p \leq q$, for any $x \in \mathbb{R}^n$,

$$F_p(\Phi_{q/p}(x)) \leq 2^{\frac{q}{p} - 1} \left( \frac{q}{p} \right)^p F_q(x)^{\frac{p}{q}}$$

or equivalently,

$$p(2F_p \circ \Phi_{q/p})^{\frac{1}{p}} \leq q(2F_q)^{\frac{1}{q}}. \tag{8}$$

Similarly, the inequality is strict whenever $q > p$ and $F_q(x) > 0$.

To prove Theorem [11], we also need the following results:

Claim 2.4. The homological critical values of $F_p$ coincide with that of $F_p \circ \Phi_{q/p}$.

Proof: Since $\Phi_{q/p}$ is a homeomorphism, it is clear that for any $c \in \mathbb{R}$, the topology of $\{F_p \leq c\}$ and the topology of $\{F_p \circ \Phi_{q/p} \leq c\}$ are the same.

Claim 2.5. The critical values of $F_p$ produced by homotopical linking coincide with that of $F_p \circ \Phi_{q/p}$.
Proof: Let $Q \subset Q' \subset \mathbb{R}^n \setminus \{0\}$ and $S \subset \mathbb{R}^n \setminus \{0\}$ be such that $Q$ homotopically links $S$ with respect to $Q'$, where $Q, Q', S$ are compact subsets. Suppose $\min_{x \in S} F_p(x) > \max_{x \in Q} F_p(x)$. Then, for any $q \geq 1$, $\Phi_{q/p}^{-1}(Q) \subset \Phi_{q/p}^{-1}(Q') \subset \mathbb{R}^n \setminus \{0\}$, $\Phi_{q/p}^{-1}(S) \cap \mathbb{R}^n \setminus \{0\}$, and $\Phi_{q/p}^{-1}(Q)$ homotopically links $\Phi_{q/p}^{-1}(S)$ with respect to $\Phi_{q/p}^{-1}(Q')$. Clearly, $\min_{x \in \Phi_{q/p}^{-1}(S)} F_p \circ \Phi_{q/p}(x) = \min_{x \in S} F_p(x) > \max_{x \in \Phi_{q/p}^{-1}(Q')} F_p(x) = \max_{x \in \Phi_{q/p}^{-1}(Q)} F_p(x)$.

In consequence,
\[
\inf_{\text{continuous } \gamma: \Phi_{q/p}^{-1}(Q') \to \mathbb{R}^n \setminus \{0\}} \max_{x \in \gamma(\Phi_{q/p}^{-1}(Q'))} F_p \circ \Phi_{q/p}(x) = \inf_{\text{continuous } \gamma: \Phi_{q/p}^{-1}(Q') \to \mathbb{R}^n \setminus \{0\}} \max_{x \in \gamma(\Phi_{q/p}^{-1}(Q'))} F_p(x)
\]
indicates a critical value produced by homotopical linking for both $F_p \circ \Phi_{q/p}$ and $F_p$.

**Claim 2.6.** The min-max (variational) critical values of $F_p$ coincide with that of $F_p \circ \Phi_{q/p}$.

**Proof:** Since $\Phi_{q/p}^{-1} : \text{Ind}_k(\mathbb{R}^n) \to \text{Ind}_k(\mathbb{R}^n)$ is a bijection, for any $k = 1, \ldots, n$,
\[
\inf_{S \in \text{Ind}_k(\mathbb{R}^n)} \sup_{x \in S} F_p \circ \Phi_{q/p}(x) = \inf_{S \in \text{Ind}_k(\mathbb{R}^n)} \sup_{x \in \Phi_{q/p}^{-1}(S)} F_p \circ \Phi_{q/p}(x) = \inf_{S \in \text{Ind}_k(\mathbb{R}^n)} \sup_{x \in S} F_p(x),
\]
where $\text{Ind}_k(\mathbb{R}^n) := \{ S \subset \mathbb{R}^n \setminus \{0\} : \gamma(S) \geq k \}$. This implies that the $k$-th min-max critical value of $F_p \circ \Phi_{q/p}$ agrees with the $k$-th min-max critical value of $F_p$. The same property holds when $\gamma$ is replaced by $\gamma^-$ or $\gamma^+$.

**Claim 2.7.** The critical points of $F_p$, $p(2F_p)^{1/\gamma}$ and $2^{-p}F_p$ are exactly the same, and their (min-max, or homological, or homotopical linking) critical values coincide up to certain scaling and power factors. Precisely, $\lambda$ is a critical value of $F_p$ if and only if $p(2\lambda)^{1/\gamma}$ is a critical value of $p(2F_p)^{1/\gamma}$ if and only if $2^{-p}\lambda$ is a critical value of $2^{-p}F_p$.

This claim is easy to check, and thus we omit the proof.

Together with all the above claims, the local monotonicity of the isolated homological eigenvalues is then derived by Lemma 2.3. In fact, for any isolated homological eigenvalue (or any eigenvalue produced by homotopical linking) $\lambda(\Delta_p)$ of the $p$-Laplacian, by Claims 2.4 and 2.5, $\lambda(\Delta_p)$ is an isolated homological critical value (or a critical value produced by homotopical linking) of $F_p \circ \Phi_{q/p}$.

Since $\lim_{q \to p} \Phi_{q/p} = \text{Id}$ for any $p \geq 1$, we have for any sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that for any $q \in (p - \delta, p + \delta)$, $q(2F_q)^{1/\gamma} - \epsilon \leq p(2F_p \circ \Phi_{q/p})^{1/\gamma}$ and $2^{-p}F_p \circ \Phi_{q/p} \leq 2^{-q}F_q + \epsilon$. By Claims 2.2 and 2.3, for any $q \in (p, p + \delta)$, $p(F_q \circ \Phi_{p/q})^{1/\gamma} \leq q(2F_q)^{1/\gamma}$ and $2^{-q}F_q \leq 2^{-p}F_p \circ \Phi_{q/p}$. In consequence, we get $q(2F_q)^{1/\gamma} - \epsilon \leq p(2F_p \circ \Phi_{q/p})^{1/\gamma} \leq q(2F_q)^{1/\gamma}$ and $2^{-q}F_q \leq 2^{-p}F_p \circ \Phi_{q/p} \leq 2^{-q}F_q + \epsilon$.

Then, it follows from Lemma 2.3 and Claim 2.7 that there is a homological critical value (or a critical value produced by homotopical linking) $\lambda(\Delta_q)$ of $F_q$ satisfying $2^{-q}\lambda(\Delta_q) \leq 2^{-p}\lambda(\Delta_p) \leq 2^{-q}\lambda(\Delta_q) + \epsilon$. The case of $p(2\lambda(\Delta_q))^{1/\gamma} - \epsilon \leq p(2\lambda(\Delta_p))^{1/\gamma}$ is similar.

For the case of min-max eigenvalues, for any $1 \leq p < q$, according to (7), (8), Claims 2.6 and 2.7 we can similarly verify that $p(2\lambda_k(\Delta_q))^{1/\gamma} \leq q(2\lambda_k(\Delta_q))^{1/\gamma}$ and $2^{-p}\lambda_k(\Delta_p) \geq 2^{-q}\lambda_k(\Delta_q)$, and the same inequalities hold when we consider $\lambda^2/\gamma$ instead of $\lambda^2$. Moreover, for positive eigenvalues, these inequalities are strict, whenever $p \neq q$. We complete the whole proof.

---

3 Eigenvalue estimates and refined Cheeger inequalities

3.1 Variational eigenvalues of 1-Laplacian

Given a simple graph $G = (V, E)$, let $P := \{V_1, \ldots, V_k\}$ be a subpartition\(^6\) of $V$ such that each $V_i$ induces a complete subgraph. We denote by $\mathcal{SC}(G)$ the collection of all these subpartitions. For

---

\(^6\)A subpartition of $V$ is a family of pairwise disjoint subsets of $V$. 
any $P \in SC(G)$, let
\[
h_\lambda(P) := h_\lambda(V_1, \cdots, V_k) = \min_{A \subseteq \cup_{i=1}^k V_i : |A \cap V_i| \leq 1} \frac{|\partial A|}{\text{vol}(A)}
\]
and let
\[
c(P) := \sum_{i=1}^k c(V_i), \quad \text{where } c(V_i) = \begin{cases} 1, & \text{if } |V_i| \leq 2, \\ 2, & \text{if } |V_i| \geq 3. \end{cases}
\]

**Lemma 3.1.** For any subpartition $P \in SC(G)$,
\[
\lambda_{n-c(P)+1}(\Delta_1) \geq h_\lambda(P).
\]

**Proof.** We only need to work on a subpartition $P = \{V_1, \cdots, V_k\}$ such that each $V_i$ is either a singleton or a triangle. Here and after, we simply say that a subset $V_i$ is a triangle if it induces a three-order complete subgraph in $G$. In fact, for any subpartition $P = \{V_1, \cdots, V_k\} \in SC(G)$, we take $P' = \{V'_1, \cdots, V'_k\} \in SC(G)$ satisfying $V'_i \subseteq V_i$ and
\[
|V'_i| = \begin{cases} 3, & \text{if } |V_i| \geq 3 \\ 1, & \text{if } |V_i| \in \{1, 2\}. \end{cases}
\]
Then, we get a new subpartition $P' = \{V'_1, \cdots, V'_k\}$ consisting of only singletons and triangles. By the definition of $c(P)$, it is clear that $c(P') = c(P)$. Also, note that
\[
\{A \subseteq \cup_{i=1}^k V_i : |A \cap V_i| \leq 1, i = 1, \cdots, k\} \supseteq \{A \subseteq \cup_{i=1}^k V'_i : |A \cap V'_i| \leq 1, i = 1, \cdots, k\}
\]
which implies that $h_\lambda(P) \leq h_\lambda(P')$. Hence, if we have $\lambda_{n-c(P')}(\Delta_1) \geq h_\lambda(P')$, then $\lambda_{n-c(P)+1}(\Delta_1) \geq h_\lambda(P)$.

Figure 1: The hexagon $\varhexagon(a, b, c)$ is the union of the six red segments.

According to this fact, we may assume without loss of generality that each $V_i$ of the subpartition $P$ is a singleton or a triangle. Before proving Lemma 3.1, we do some preparations.

The hexagon structure corresponding to three linear independent vectors $a, b, c$ is a spatial hexagon defined as
\[
\varhexagon(a, b, c) := [a, -b] \cup [-b, c] \cup [c, -a] \cup [-a, b] \cup [b, -c] \cup [-c, a].
\]
where $[a, -b]$ indicates the segment with the endpoints $a$ and $-b$. The geometric intuition for such a hexagon can be seen in Figure 1.

Let
\[
S(V_i) = \begin{cases} \{1_{\{v\}}, 1_{\{v\}}\}, & \text{if } V_i = \{v\}, \\ \varhexagon(1_{\{u\}}, 1_{\{v\}}, 1_{\{w\}}), & \text{if } V_i = \{u, v, w\}, \end{cases}
\]
and let
\[
S(P) = S(V_1) * S(V_2) * \cdots * S(V_k)
\]
is easy to check that

\[
\min_{x \in S(P)} F_1(x) = h_s(P).
\]

\textbf{Claim 3.1.} \( \min_{x \in S(P)} F_1(x) = h_s(P) \).

Proof: For any \( x \in \mathbb{R}^n \setminus \{0\} \), there exist \( x^+, x^- \in \mathbb{R}^n_+ \) such that \( x = x^+ - x^- \), where \( n = \#V \). It is easy to check that

\[
F_1(x) = \frac{\sum_{i,j} |x^+_i - x^-_j| + \sum_{i,j} |x^-_i - x^-_j|}{\sum_{i \in V} \deg(j) |x^+_j| + \sum_{i \in V} \deg(j) |x^-_j|} \geq \min\{F_1(x^+), F_1(x^-)\}. \tag{10}
\]

Hence, \( \min_{x \in S(P)} F_1(x) = \min_{x \in S(P) \cap \mathbb{R}^n_+} F_1(x) \). Note that \( S(P) \cap \mathbb{R}^n_+ = (S(V_1) \cap \mathbb{R}^n_+) \times \cdots \times (S(V_k) \cap \mathbb{R}^n_+) \) which is due to the definition of \( S(V_i) \). Therefore, by the definition of \( S(V_i) \) in (9), for any \( x \in S(P) \cap \mathbb{R}^n_+ \), \( \#(\text{supp}(x) \cap V_i) \leq 1, \forall i \), where \( \text{supp}(x) := \{i \in V : x_i \neq 0\} \) is the support of \( x \). According to an elementary technique (see [26, 28]), it can be verified that \( F_1(x) \geq \frac{\partial A_{t_0}}{\text{vol}(A_{t_0})} \) for some nonempty subset \( A \subset \text{supp}(x) \). For readers’ convenience, we present a brief proof here. In fact, by the inequality (10) and the relation \( \text{supp}(x^+) \cup \text{supp}(x^-) = \text{supp}(x) \), we can assume that \( x \in \mathbb{R}^n_+ \). Then it can be verified that

\[
F_1(x) = \frac{\int_0^t \|x\| \|\partial A_t\| dt}{\int_0^t \|x\| \text{vol}(A_t) dt} \geq \frac{\|\partial A_{t_0}\|}{\text{vol}(A_{t_0})}
\]

for some \( 0 < t_0 < \|x\|_\infty \), where \( \partial A_{t_0} := \{i \in V : x_i > t\} \). Hence, we can take such subset \( A = A_{t_0} \) which is a subset of \( \text{supp}(x) \). Thus, we have proved \( \min_{x \in S(P)} F_1(x) \geq h_s(P) \).

The opposite inequality is easy to prove. Indeed, given \( P := \{V_1, \ldots, V_k\} \) and \( A \subset \bigcup_{i=1}^k V_i \) such that \( |A \cap V_i| \leq 1, i = 1, \ldots, k \), we can take \( x = 1_A \in S(P) \) and it is easy to check that \( F_1(1_A) = |\partial A|/\text{vol}(A) \). The proof is completed.

\textbf{Claim 3.2.} There is an odd homeomorphism maps \( S(P) \) to \( S^{c(P)-1} \), where \( S^{c(P)-1} \) denotes the unit sphere of dimension \( c(P) - 1 \).

Proof: Suppose that \( V_1, \ldots, V_i \) are triangles, and \( V_{i+1}, \ldots, V_k \) are single point sets. Then, by the definition of \( S(V_i) \) in (9), there exists an odd homeomorphism \( \psi_i : S(V_i) \rightarrow S^1 \hookrightarrow \mathbb{R}^V_i \) if \( i \in \{1, \ldots, l\} \) and an odd homeomorphism \( \psi_j : S(V_j) \rightarrow S^0 \hookrightarrow \mathbb{R}^V_j \) if \( j \in \{l + 1, \ldots, k\} \), where \( S^1 \hookrightarrow \mathbb{R}^V_i \) means that we put the image \( \psi(S(V_i)) = S^1 \) as a one-dimensional unit sphere centered at the origin in \( \mathbb{R}^V_i \). Note that we also have \( S(V_i) \subset \{x \in \mathbb{R}^n : \text{supp}(x) \subset V_i\} \), and

\[
S^1 \times \cdots \times S^1 \times S^0 \times \cdots \times S^0 = S^{2l+k-l-1} = S^{c(P)-1}.
\]

Then, we can define a map \( \psi : S(P) \rightarrow S^{c(P)-1} \) by \( \psi(\sum_{i=1}^k t_i x^i) := \sum_{i=1}^k t_i \psi_i(x^i) \), \( \forall x^i \in S(V_i), \) \( 0 \leq t_i \leq 1, \sum_{i=1}^k t_i = 1, i = 1, \ldots, k \). It is easy to check that \( \psi \) is an odd homeomorphism via the diagram:

\[
\begin{array}{c}
S(P) \downarrow \psi \downarrow S^{c(P)-1} \\
S(V_1) \times \cdots \times S(V_i) \times S(V_{i+1}) \times S(V_k) \downarrow \psi_1 \times \cdots \times \psi_k \downarrow S^1 \times \cdots \times S^1 \times S^0 \times \cdots \times S^0
\end{array}
\]

\( l \) times \( k-l \) times

The proof is completed.

\textbf{Claim 3.3.} For any centrally symmetric compact subset \( S \) with \( \gamma^-(S) \geq n - c(P) + 1 \), we have

\( S \cap \{tx : x \in S(P), t \geq 0\} \neq \emptyset \).

\(^5\)The simplicial join of subsets \( A \) and \( B \) in \( \mathbb{R}^n \) is defined as \( A \ast B := \{ta + (1-t)b : a \in A, b \in B, 0 \leq t \leq 1\} \).
Proof: Continuing the preceding proof, it is not difficult to verify that any \(x \in \mathbb{R}^n\) has a unique decomposition \(x = y + \sum_{i=1}^l t_i 1_i + \sum_{v \in V \setminus \cup_{i=1}^l 1_i} t_v 1_v\) with \(t_i, t_v \in \mathbb{R}\), and \(y/\|y\|_1 \in S(P)\) if \(y \neq 0\). With the help of such a decomposition, we then define a map \(\tilde{\psi} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) by

\[
\tilde{\psi}(x) = \|y\|_1 \psi\left(\frac{y}{\|y\|_1}\right) + \sum_{i=1}^l t_i \xi_i + \sum_{v \in V \setminus \cup_{i=1}^l 1_i} t_v 1_v
\]

where \(\psi\) is the odd homeomorphism introduced in the preceding proof, and \(\xi_i\) is a nonzero vector orthogonal to \(\psi(S(1_i))\) in \(\mathbb{R}^l\), \(i = 1, \ldots, l\).

One can check that \(\tilde{\psi} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is an odd homeomorphism. Also, \(\tilde{\psi}\) is positively one-homogeneous, i.e., \(\tilde{\psi}(tx) = t \tilde{\psi}(x), \forall x \in \mathbb{R}^n, \forall t \geq 0\). In particular, \(\tilde{\psi}(S(P)) = \psi(S(P)) = S^{c(P)-1}\), and thus \(\{tx : x \in \psi(S(P)), t \geq 0\} = \text{span}(\psi(S(P))) = \mathbb{R}^{c(P)}\). For any centrally symmetric compact subset \(S\) with genus(\(S\)) \(\geq n - c(P) + 1\), \(\psi(S)\) is also a centrally symmetric compact subset with genus(\(\psi(S)\)) = genus(\(S\)) \(\geq n - c(P) + 1\). Finally, by the intersection property of Krasnoselskii genus, the subset \(\psi(S)\) intersects with the linear subspace \(\{tx : x \in \psi(S(P)), t \geq 0\}\). Accordingly,

\[
S \cap \{tx : x \in S(P), t \geq 0\} = \tilde{\psi}^{-1}(\tilde{\psi}(S)) \cap \tilde{\psi}^{-1}\{tx : x \in \tilde{\psi}(S(P)), t \geq 0\} = \tilde{\psi}^{-1}\left(\tilde{\psi}(S) \cap \{tx : x \in \psi(S(P)), t \geq 0\}\right) \neq \emptyset
\]

which completes the proof.

We are ready to prove Lemma 3.1. By the zero-homogeneity of \(F_1\), Claims 3.1 and 3.3, we have

\[
\lambda_{n-c(P)+1}^{-}(\Delta_1) = \inf_{\text{genus}(S) \geq n-c(P)+1} \sup_{x \in S} F_1(x) \geq \inf_{\text{genus}(S) \geq n-c(P)+1} \sup_{x \in \text{span}((tx : x \in S(P)), t \geq 0)} F_1(x) \geq \min_{x \in S(P)} F_1(x) = h_*(P).
\]

We complete the whole proof. \(\square\)

**Definition 3.1.** We say that some subsets \(V_1, \ldots, V_k\) of \(V\) are pairwise non-adjacent if every edge \(\{i, j\} \in E\) intersects at most one \(V_l, l = 1, \ldots, k\).

**Corollary 3.1.** Let \(\mathcal{SC}_{na}(G) = \{P \in \mathcal{SC}(G) : \text{the elements in } P \text{ are pairwise non-adjacent}\}\), and denote by

\[
\alpha_*(G) = \max\{c(P) : P \in \mathcal{SC}_{na}(G)\}
\]

the so-called pseudo-independence number introduced in [43]. Then

\[
\lambda_{n-\alpha_*(G)+1}^{-}(\Delta_1) = \cdots = \lambda_{n}^{-}(\Delta_1) = 1.
\]

**Proof.** For any \(P = (V_1, \cdots, V_k) \in \mathcal{SC}_{na}(G)\), \(V_1, \ldots, V_k\) are pairwise non-adjacent. Thus, for any \(A \subset \cup_{i=1}^k V_i\) with \(#(A \cap V_i) \leq 1\), \(A\) is the disjoint union of some pairwise non-adjacent vertices, which implies that \(|\partial A| = \text{vol}(A)|\. Therefore, we have \(h_*(P) = 1\). In particular, taking \(P' \in \mathcal{SC}_{na}(G)\) such that \(c(P') = \alpha_*(G)\), we have \(h_*(P') = 1\). By Lemma 3.1, \(\lambda_{n-c(P') + 1}^{-}(\Delta_1) \geq \alpha_*(P') = 1\). On the other hand, it is known that \(\lambda_{n-c(P)+1}^{-}(\Delta_1) \leq \cdots \leq \lambda_{n}^{-}(\Delta_1) = 1\). Hence, we have \(\lambda_{n-\alpha_*(G)+1}^{-}(\Delta_1) = \lambda_{n-c(P') + 1}^{-}(\Delta_1) = \cdots = \lambda_{n}^{-}(\Delta_1) = 1\). \(\square\)

A set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The independence number of a graph is the cardinality of its largest independent set.

**Corollary 3.1** generalizes and enhances a result in [43]. Moreover, we have a stronger version of the main theorem in [43];
Proposition 3.1. Let $\alpha(G)$ be the independence number of $G$, and let $\gamma(G)$ be the multiplicity of the eigenvalue 1 of $\Delta_1$. Denote by $t(G)$ the largest integer satisfying $\lambda_{n-t(G)+1}^-(\Delta_1) = 1$. Then,

$$\alpha(G) \leq \alpha_s(G) \leq t(G) \leq \gamma(G) \leq \min\{c_s(G), 2\alpha(G)\}$$

where $c_s(G) := \min\{c(P)|P \in SC_p(G)\}$ and $SC_p(G) = \{P \in SC(G) : P$ is a partition of $V\}$.

Proof. The inequalities $\alpha(G) \leq \alpha_s(G)$, $t(G) \leq \gamma(G)$ and $\gamma(G) \leq 2\alpha(G)$ have been established in Theorem 1 in [43]. The inequality $\gamma(G) \leq c_s(G)$ is equivalent to Corollary 1 in [43]. It remains to show $\alpha_s(G) \leq t(G)$.

By Corollary 3.1, $\lambda_{n-\alpha_s(G)+1}^-(\Delta_1) = 1$. Since $t(G)$ is the largest number such that $\lambda_{n-t(G)+1}^-(\Delta_1) = 1$, we have $t(G) \geq \alpha_s(G)$. The proof is completed. \hfill \Box

3.2 Refined multi-way Cheeger inequalities

In this section, we shall prove Theorem 14. Before proving this theorem, we collect here some useful claims.

Claim 3.4 (Theorem 8 in [10]). Let $x$ be an eigenvector corresponding to an eigenvalue $\lambda$ of $\Delta_1$ with $\lambda \leq \lambda_k^-(\Delta_1)$, and assume that $x$ has $m$ (strong) nodal domains. Then,

$$h_m \leq \lambda_k^-(\Delta_1) \leq h_k.$$

The above inequality still holds when we use $\lambda_k^+(1)$ or $\lambda_k^+(\Delta_1)$ instead of $\lambda_k^-(\Delta_1)$ with the same proof as that of Theorem 8 in [10].

Claim 3.5 (Theorem 5.1 in [38]). For $p > 1$, let $x$ be an eigenvector corresponding to the eigenvalue $\lambda_k^-(\Delta_p)$, and assume that $x$ has $m$ strong nodal domains. Then,

$$\frac{2^{p-1}}{p^p}h_m^p \leq \lambda_k^-(\Delta_p) \leq 2^{p-1}h_k.$$

Moreover, if there is an eigenpair $(\lambda, x)$ of $\Delta_p$ with $\lambda \leq \lambda_k^-(\Delta_p)$ and $\Theta(x) = m$, then we still have $\frac{2^{p-1}}{p^p}h_m^p \leq \lambda_k^-(\Delta_p)$, and this inequality holds when we use $\lambda_k^+(\Delta_1)$ instead of $\lambda_k^-(\Delta_1)$. The proofs of these versions of $\lambda_k^-(\Delta_1)$ and $\lambda_k^+(\Delta_1)$ are the same as that of Theorem 5.1 in [38].

Claim 3.6 (Theorem 1.1 in [32]). For every graph, and each natural number $k$,

$$\frac{h_k^2}{Ck^3} \leq \lambda_k(\Delta_2) \leq 2h_k$$

where $C$ is a universal constant.

Convention: We say that the index of a subcomplex of $K_n$ is $k$, if the geometric realization of such subcomplex in $|K_n|$ is centrally symmetric and its index is $k$, where the word ‘index’ can be Yang index, Krasnoselski genus or the $\gamma^+$-index.

Claim 3.7. The $k$-th min-max eigenvalue of graph 1-Laplacian has the following combinatorial characterization:

$$\lambda_k(\Delta_1) = h_k = \min_{A \in S_k} \max_{(A,B) \in A} \frac{|\partial A| + |\partial B|}{\text{vol}(A \cup B)} \quad (11)$$

where $S_k := \{A \subset P_2(V) : \text{the Yang index of the subcomplex of } K_n \text{ induced by } A \text{ is at least } k\}$.

Proof. We shall prove a general statement: for any odd piecewise linear function $F : |K_n| \to \mathbb{R}$ (resp., $F : |K_n| \to |K_n|$) that is linear on each simplex of $|K_n|$ (resp., $|K_n|$), we have

$$\inf_{\gamma(S) \geq k} \sup_{x \in S} F(x) = \min_{A \in S_k} \max_{(A,B) \in A} F(A - 1_B), \quad (12)$$
where \( \overline{K}_n \) is another geometric realization of \( K_n \) and is also a triangulation of \( X_1 := \{ x \in \mathbb{R}^n : \sum_{i=1}^n \deg(i)|x_i| = 1 \} \). We refer the reader to the proof of Theorem 13 for the construction of \( |K_n| \).

To show the proof of the above statement, we can assume without loss of generality that \( F \) is injective on the vertices of any simplex of \( |K_n| \), that is, \( F(v) \neq F(u) \) whenever \( v \) and \( u \) are two different vertices of a simplex in \( |K_n| \). In fact, if \( \text{(12)} \) holds for all such 'injective' \( F \), it holds also neglecting the injectivity. The reason is as follows. In fact, let \( F := \{ \text{odd continuous function } F : |K_n| \to \mathbb{R} \text{ that is linear on each simplex of } |K_n| \} \) and \( F_{in} := \{ F \in F \mid F \text{ is injective on the vertices of } |K_n| \} \).

Clearly, \( F_{in} \) is an open dense subset of \( F \) where we use the topology induced by the maximum norm \( \| \cdot \|_\infty \). Note that the min-max quantities on both side of \( \text{(12)} \) are preserved under uniform convergence, that is, if \( f_n \in F \) and \( f_n \to f \) as \( n \to +\infty \), then

\[
\lim_{n \to \infty} \sup_{x \in S} f_n(x) = \sup_{x \in S} f(x) \quad \text{and} \quad \lim_{n \to \infty} \max_{A \in S_k(A,B) \in A} f_n(1_A - 1_B) = \max_{A \in S_k(A,B) \in A} f(1_A - 1_B).
\]

Thus, to prove the equality \( \text{(12)} \) for any \( F \in F \), it suffices to prove \( \text{(12)} \) for any \( F \in F_{in} \).

Now, given a function \( F \in F_{in} \), for any centrally symmetric compact subset \( S \subset |K_n| \) with \( \gamma(S) \geq k \), denote by \( S_F \) the set of the maximizers of \( F|S \) (i.e., the restriction of \( F \) on \( S \)). According to the linearity of \( F \) on each simplex of \( |K_n| \), and the injectivity of \( F \) on the vertex set of any simplex of \( |K_n| \), it is not difficult to show that the critical points of \( F \) must be vertices of \( |K_n| \).

So, if \( S_F \) doesn’t contain any vertex of \( |K_n| \), there is no critical point of \( F \) in \( S_F \). Thus, by the deformation theorem in nonlinear analysis, we can take a small perturbation \( S' \) of \( S \) such that they are odd-homotopy equivalent and

\[
\sup_{y \in S'} F(y) < F(s) = \sup_{x \in S} F(x).
\]

At this time, we also have \( \gamma(S') \geq k \).

Therefore, we only need to consider such \( S \) with the additional property that \( S_F \) contains some vertices of \( |K_n| \). In other words, we assume that \( \max_{x \in S} F(x) \) is achieved at some vertex \( v \) of \( |K_n| \), i.e., \( F(v) \geq F(x), \forall x \in S \). Let \( \lambda = F(v) \). Consider the sublevel set \( \{ F \leq \lambda \} \). It is clear that

\[
\gamma(\{ F \leq \lambda \}) \geq \gamma(S) \geq k \quad \text{and} \quad \max_{x \in \{ F \leq \lambda \}} F(x) = \lambda = F(v) = \max_{x \in S} F(x).
\]

By Claim 2.1, for any \( c \in \mathbb{R} \), there is a homotopy equivalence between the sublevel set \( \{ F \leq c \} \) and the induced subcomplex \( K_n|\{ F \leq c \} \) of \( |K_n| \), where \( K_n|\{ F \leq c \} \) denotes the induced closed subcomplex of \( |K_n| \) on the vertices lying in the sublevel set \( \{ F \leq c \} \). Thus, \( \gamma(\{ F \leq c \}) = \gamma(K_n|\{ F \leq c \}) \).

In consequence, we have the following identities

\[
\inf_{S \in \text{Ind}_k} \max_{x \in S} F(x) = \inf_{c \in \mathbb{R}} \max_{S \text{ s.t. } \{ F \leq c \} \in \text{Ind}_k} \sup_{x \in \{ F \leq c \}} F(x) = \min_{c \in \mathbb{R} \text{ s.t. } K_n|\{ F \leq c \} \in \text{Ind}_k} \max_{\text{vertex } v \text{ of } K_n|\{ F \leq c \}} F(v) = \min_{A \in S_k(A,B) \in A} \max_{\text{vertex } v \text{ of } K_n|\{ F \leq c \}} F(1_A - 1_B),
\]

where \( \text{Ind}_k := \{ S \subset |K_n| : \gamma(S) \geq k \} \), and in the last equality, we identify the collection of the induced subcomplexes in \( \text{Ind}_k \) with \( S_k \), because any vertex \( v \) of \( K_n \) is in the form of \( 1_A - 1_B \) which can be identified with the set-pair \((A,B) \in \mathcal{P}_2(V)\). Also, we have used the fact that on each induced subcomplex of \( |K_n| \), \( F \) reaches the maximum at some vertices. The proof of \( \text{(12)} \) is then completed.

Now, taking \( F = F_1 \) in the above equality \( \text{(12)} \), it is easy to check that

\[
F_1(1_A - 1_B) = \frac{|\partial A| + |\partial B|}{\text{vol}(A \cup B)},
\]

which implies

\[
\lambda_k(\Delta_1) = \inf_{S \in \text{Ind}_k} \sup_{x \in S} F_1(x) = \min_{A \in S_k(A,B) \in A} \frac{|\partial A| + |\partial B|}{\text{vol}(A \cup B)} = \hat{h}_k.
\]

The proof of \( \text{(11)} \) is completed. \( \square \)
Remark 3.1. The above proof of Claim 3.7 indeed uses a similar approach developed in the author’s previous work [22]. Also, we should note that any \( S \in \text{Ind}_k \) realizing \( c_k := \inf_{S \in \text{Ind}_k} \sup_{x \in S} F(x) \) contains a critical point of \( F \) corresponding to the critical value \( c_k = \max_{x \in S} F(x) \). Moreover, we can similarly prove that

\[
\lambda_k^\pm(\Delta_1) = h_k^\pm = \min_{A \in \mathcal{S}^k_h} \max_{(A,B) \in A} \frac{|\partial A| + |\partial B|}{\text{vol}(A \cup B)}
\]

where \( \mathcal{S}^k_h := \{ A \subset P_2(V) : \gamma^\pm(\text{the subcomplex of } K_n \text{ induced by } A) \geq k \} \), as both \( \gamma^- \) and \( \gamma^+ \) are homotopy invariants.

Proof of Theorem 7.4. The inequality \( \lambda_k^\pm(\Delta_1) \leq \lambda_k(\Delta_p) \leq \lambda_k^\pm(\Delta_p) \) has been shown in (5). Hence, \( \mathcal{S}^-_k(\Delta_q) \subset \mathcal{S}_k(\Delta_q) \subset \mathcal{S}^+_k(\Delta_q) \) and therefore \( s_k^- \leq s_k \leq s_k^+ \). Analogously, the relation \( \gamma^+ \leq \gamma \leq \gamma^- \) simply implies \( \mathcal{S}^+_k \subset \mathcal{S}_k \subset \mathcal{S}^-_k \) and thus \( h_k^- \leq h_k \leq h_k^+ \).

By Theorem 1.1, the function \( p \mapsto 2^{-p} \lambda_k(\Delta_1) \) is decreasing on \([1, +\infty)\), which implies \( 2^{-p} \lambda_k(\Delta_1) \geq 2^{-p} \lambda_k(\Delta_p), \forall p \geq 1 \). So, we have \( \lambda_k(\Delta_p) \leq 2^{-p} \lambda_k(\Delta_1) \). Together with \( \hat{h}_k = \lambda_k(\Delta_1) \) by Claim 3.7 and \( \lambda_k(\Delta_1) \leq h_k \) by Claim 3.4, we obtain the upper bound estimate:

\[
\lambda_k(\Delta_p) \leq 2^{p-1} \hat{h}_k \leq 2^{p-1} h_k.
\]

Let us move on to the lower bound estimate. By Theorem 1.1, the function \( p \mapsto p(2\lambda_k(\Delta_p))^\frac{1}{p} \) is increasing on \([1, +\infty)\), which yields \( 2\lambda_k(\Delta_1) \leq p(2\lambda_k(\Delta_p))^\frac{1}{p} \) for any \( p \geq 1 \). Hence, \( \frac{2^{p-1}}{p^p} \lambda_k(\Delta_1)^p \leq \lambda_k(\Delta_p) \). Since \( \hat{h}_k = \lambda_k(\Delta_1) \), we get the inequality \( \lambda_k(\Delta_p) \geq \frac{2^{p-1}}{p^p} \hat{h}_k^p \).

Let \( x \) be an eigenvector corresponding to some eigenvalue \( \lambda \) of \( \Delta_1 \) with \( \lambda \leq \lambda_k(\Delta_1) \), and assume that \( x \) has \( m \) strong nodal domains. Then, by Claims 3.4 and 3.7, \( \hat{h}_k = \lambda_k(\Delta_1) \geq h_m \). Therefore, we have

\[
\lambda_k(\Delta_p) \geq \frac{2^{p-1}}{p^p} \hat{h}_k^p \geq \frac{2^{p-1}}{p^p} h_m^p.
\]

For \( p \geq q > 1 \), Theorem 1.1 also implies \( p(2\lambda_k(\Delta_q))^\frac{1}{p} \geq q(2\lambda_k(\Delta_q))^\frac{1}{q} \), which can be reformulated as \( \lambda_k(\Delta_p) \geq 2^{\frac{p-1}{p}} \frac{q}{p} \lambda_k(\Delta_q) \frac{q}{p} \) for any \( q \leq p \). Let \( x \) be an eigenvector corresponding to an eigenvalue \( \lambda \) of \( \Delta_q \) with \( \lambda \leq \lambda_k(\Delta_q) \), and let \( m \) be the number of the strong nodal domains of \( x \). Then, by Claim 3.5, \( \lambda_k(\Delta_q) \geq \frac{q^{p-1}}{p^p} h_m^q \). Thus, we have

\[
\lambda_k(\Delta_p) \geq 2^\frac{p-1}{p} \left( \frac{q}{p} \right)^p \lambda_k(\Delta_q) \frac{q}{p} \geq 2^\frac{p-1}{p} \left( \frac{q}{p} \right)^p \left( \frac{2^{p-1}}{p^p} h_m^q \right)^\frac{q}{p} = \frac{2^{p-1}}{p^p} h_m^p.
\]

Since \( h_1 \leq h_2 \leq \cdots \leq h_n \), we may take \( s_k \) to be the largest \( m \) such that \( m \) is the number of the strong nodal domains of some eigenvector corresponding to some eigenvalue less than or equal to \( \lambda_k(\Delta_q) \) for some \( q \in [1, p] \). Then, we simply have \( \lambda_k(\Delta_p) \geq \frac{2^{p-1}}{p^p} h_s^p \) by the inequalities (11) and (15). The analogous inequalities on \( \lambda_k^-(\Delta_p) \) and \( \lambda_k^+(\Delta_p) \) can be derived in the same way.

Note that for the 2-Laplacian, we always have \( \lambda_k^-(\Delta_2) = \lambda_k(\Delta_2) = \lambda_k^+(\Delta_2) \), \( \forall k \). To complete the whole proof of Theorem 1.1, we should use the multi-way Cheeger inequality proposed in [32] (see Claim 3.6): for every graph, and each natural number \( k \),

\[
\lambda_k^-(\Delta_2) = \lambda_k(\Delta_2) \geq \frac{h_k^2}{C k^4},
\]

where \( C \) is a universal constant.

Again, using Theorem 1.1, we have the following simple estimates:

Case 1. \( 1 \leq p \leq 2 \)

In this case, we use the fact that the function \( p \mapsto 2^{-p} \lambda_k^-(\Delta_p) \) is decreasing. This implies \( 2^{-p} \lambda_k^-(\Delta_p) \geq 2^{-2} \lambda_k^-(\Delta_2) \). Thus, \( \lambda_k^-(\Delta_p) \geq 2^{p-2} \lambda_k^-(\Delta_2) \geq 2^{p-2} \frac{h_k^2}{C k^4} \geq \lambda_k^-(\Delta_2) \frac{h_k^2}{C k^4} \). Taking \( C = 2\Omega \), we have \( \lambda_k^-(\Delta_p) \geq h_k^2/2C k^4 \).
Case 2. \( p \geq 2 \)

In this case, we use the fact that the function \( p \mapsto p(2\lambda_k^-(\Delta_p))^\frac{1}{p} \) is increasing. This implies \( p(2\lambda_k^-(\Delta_p))^\frac{1}{p} \geq 2(2\lambda_k^-(\Delta_2))^\frac{1}{p} \). Hence,

\[
\lambda_k^-(\Delta_p) \geq 2^{\frac{p}{2}} - 1 \left( \frac{2}{p} \right)^p \lambda_k^-(\Delta_2)^\frac{1}{p} \geq 2^{\frac{p}{2}} - 1 \left( \frac{2}{p} \right)^p \left( \frac{h_k}{Ck^4} \right)^\frac{1}{p} = \frac{h_k^p}{2^{1-\frac{2}{3}p}p^\frac{3}{2}k^2}. 
\]

Taking \( C_p = 2^{1-\frac{2}{3}p}p^\frac{3}{2}k^2 \), we have \( \lambda_k^-(\Delta_p) \geq \frac{h_k^p}{c_pk^{2p}} \).

In summary, we obtain

\[
\lambda_k^-(\Delta_p) \begin{cases} \frac{h_k^p}{c_pk^{2p}}, & \text{if } 1 \leq p \leq 2, \\ \frac{h_k^p}{c_pk^{2p}}, & \text{if } p \geq 2. \end{cases}
\]

We have completed the whole proof. \( \square \)

**Proof of Corollary 3.1.** The \( \lambda_k^-(\Delta_1) = h_k \) has been shown in [13], due to the nodal domain estimates on forests. The proof of the equality \( \lambda_k^-(\Delta_p) = \lambda_k^-(\Delta_p) \) is standard, due to the mountain pass characterization, and thus we omit it.

Together with Theorem 1.4 and Corollary 3.1, we complete the proof of these equalities. \( \square \)

**Remark 3.2.** We note that Cheeger-type inequalities essentially reflect the connections between the spectra of \( \Delta_p \) and \( \Delta_1 \). For example, the usual Cheeger inequality on graphs is nothing but a relation between the principal eigenvalues of \( \Delta_2 \) and \( \Delta_1 \). Lee-Oveis Gharan-Trevisan’s multi-way Cheeger inequality reveals a certain relationship between the higher-order eigenvalues of \( \Delta_2 \) and \( \Delta_1 \). Tudisco-Hein’s higher-order Cheeger inequality for the graph \( p \)-Laplacian establishes some estimates between the higher-order variational eigenvalues of \( \Delta_p \) and \( \Delta_1 \). In addition, the monotonicity problem proposed by Amghibech is intended to find a comparison theorem for the eigenvalues of \( \Delta_p \) and \( \Delta_q \), for any given \( p, q > 1 \).

**Remark 3.3.** The graph \( 2 \)-Laplacian is a linear operator, while the graph \( 1 \)-Laplacian (resp., graph \( \infty \)-Laplacian) can be viewed as a combinatorial operator. It is interesting that we can regard the graph \( p \)-Laplacian as a non-linear evolution from the linear case (i.e., \( 2 \)-Laplacian) to the combinatorial case (i.e., \( 1 \)-Laplacian and \( \infty \)-Laplacian).

### 3.3 Distribution of eigenvalues for \( p \)-Laplacians

In this section, we shall prove Theorem 1.5.

**Proof of Theorem 1.5.** By Corollary 3.1, \( \lambda_{n-\alpha_s(G)+1}(\Delta_1) = \cdots = \lambda_n(\Delta_1) = \lambda_{n-\alpha_s(G)+1}(\Delta_1) = \cdots = \lambda_n(\Delta_1) = 1 \), which implies that there are at least \( \alpha_s(G) \) min-max eigenvalues of \( \Delta_1 \) equal to 1. Here, \( \alpha_s(G) \) is the pseudo-independence number introduced in [3] (see Corollary 3.1 in Section 3.1 for the definition).

Then, by Theorem 1.1 (or Theorem 1.4), for \( p > 1 \),

\[
\lambda_n(\Delta_p) \geq \cdots \geq \lambda_{n-\alpha_s(G)+1}(\Delta_p) > \frac{2^{p-1}}{p^p}
\]

meaning that there are at least \( \alpha_s(G) \) eigenvalues (counting multiplicity) of \( \Delta_p \) larger than \( \frac{2^{p-1}}{p^p} \) when \( p > 1 \).

To show the spectral gap for min-max eigenvalues of \( \Delta_p \), on connected graphs, we recall the following surprising result on the largest Laplacian spectral gap from 1:

\[*The graph \( \infty \)-Laplacian \( \Delta_\infty \) is defined as \( \Delta_\infty x = \partial \max_{(i,j) \in E} |x_i - x_j| \), where \( \partial \) indicates the Clarke subgradient. We do not study \( \infty \)-Laplacian in this paper, but we would like to present here that \( \Delta_\infty \) can also be seen as a limit of \( \Delta_p \), as \( p \to +\infty \). Precisely, \( \Delta_\infty x = \lim_{p \to +\infty} \left( \sum_{(i,j) \in E} |x_i - x_j|^p \right)^{\frac{1}{p}-1} \Delta_p x. \]*

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Claim 3.8. For any connected graph on $n \geq 3$ nodes,

$$\min_{1 \leq k \leq n} |\lambda_k(\Delta_2) - 1| \leq \frac{1}{2}.$$ 

By Claim 3.8, for any connected graph with $n \geq 3$ vertices, there exists $l \in \{1, \ldots, n\}$ such that $\frac{1}{2} \leq \lambda_l(\Delta_2) \leq \frac{3}{2}$. By employing Theorem 1.3, we simply have:

$$\lambda_l(\Delta_p) > \begin{cases} 2^{p-2} \lambda_l(\Delta_2) \geq 2^{p-3}, & \text{if } 1 \leq p < 2, \\ 2^{\frac{p}{2}} - \left(\frac{2}{p}\right)^p \lambda_l(\Delta_2) \geq 2^{\frac{p}{2}} - \left(\frac{2}{p}\right)^p \left(\frac{1}{2}\right)^{\frac{p}{2}} = \frac{2^{p-1}}{p^p}, & \text{if } p > 2, \end{cases}$$

and

$$\lambda_l(\Delta_p) < \begin{cases} 2^{\frac{p}{2}} - \left(\frac{2}{p}\right)^p \lambda_l(\Delta_2) \leq 2^{\frac{p}{2}} - \left(\frac{2}{p}\right)^p \left(\frac{3}{2}\right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \cdot \frac{2^{p-1}}{p^p}, & \text{if } 1 \leq p < 2, \\ 2^{p-2} \lambda_l(\Delta_2) \leq 3 \times 2^{p-3}, & \text{if } p > 2. \end{cases}$$

That is, $2^{p-3} < \lambda_l(\Delta_p) < 3^{\frac{p}{2}} \cdot \frac{2^{p-1}}{p^p}$ if $1 \leq p < 2$, and $2^{p-3} < \lambda_l(\Delta_p) < 3 \cdot 2^{p-3}$ if $p > 2$. This completes the proof of Theorem 1.3.

\[\Box\]

4 Non-variational eigenvalues

4.1 Homological non-variational eigenvalues

For readers’ convenience, we redraw the picture shown in Theorem 1.2 again:

![Diagram of a graph with vertices labeled 1 to 6 and edges connecting them.]

Let us go into the details of the computation of $\lambda_k(\Delta_1)$. First, it is known that $\lambda_1(\Delta_1) = 0$ and $\lambda_2(\Delta_1) = h_2(G) = \frac{5}{9}$ because we generally have

$$\lambda_2^-(\Delta_1) = \lambda_2(\Delta_1) = \lambda_2^+(\Delta_1) = \hat{h}_2^- = \hat{h}_2 = \hat{h}_2^+ = h_2$$

due to the Cheeger equality $\lambda_2^-(\Delta_1) = h_2$ in [4, 23] and Theorem 1.4. Here and after, $h_k(G)$ indicates the $k$-th multi-way Cheeger constant on the graph $G$ (see [2] for the definition).

By Corollary 3.7, we have $\lambda_4(\Delta_1) = \lambda_5(\Delta_1) = \lambda_6(\Delta_1) = 1$. It remains to compute $\lambda_3(\Delta_1)$.

Let $P = \{V_1, V_2\}$ be a partition of $V$ with $V_1 = \{2, 3, 4\}$ and $V_2 = \{1, 5, 6\}$. Then $c(P) = 4$, and $h_s(P) = \frac{5}{9}$. Thus, by Lemma 3.1, $\lambda_3(\Delta_1) \geq \lambda_3^-(\Delta_1) = \lambda_3^+(\Delta_1) = \lambda_6-c(P)+1(\Delta_1) \geq h_s(P) = \frac{5}{9}$. On the other hand, by the multi-way Cheeger inequality (see Claim 3.7), $\lambda_3(\Delta_1) \leq \lambda_3^-(\Delta_1) \leq h_3(G) = \frac{5}{9}$, which implies $\lambda_3(\Delta_1) = \frac{5}{9}$.

Therefore, we have determined all the min-max eigenvalues of $\Delta_1$. Note that on the graph $G$ shown above, we have actually proved that $\lambda_k(\Delta_1) = \lambda_k(\Delta_1) = \lambda_k^+(\Delta_1)$, $k = 1, 2, 3, 4, 5, 6$.

Now, we are able to prove our main result in this section.

Proof of Theorem 1.2. Since we have written down the min-max eigenvalues of $\Delta_1$, $\frac{5}{9}$ is not a min-max eigenvalue. The rest of the verification is to prove that $\frac{5}{9}$ is a homological eigenvalue of $\Delta_1$.

Note that $F_1(1_{(2,5,6)}) = \frac{5}{9}$. In order to use Theorem 1.3, we list all the values of $F_1$ acting on the vertices of link$(1_{(2,5,6)})$:

- $F_1(1_{(2,5,6,3,4)} - 1_{\{1\}}) = \frac{1}{2}$, $F_1(1_{(2,5,6,3,4)} - 1_{\{4\}}) = F_1(1_{(2,5,6,3,4)} - 1_{\{3\}}) = \frac{5}{10}$,
\[ F_1(1_{\{2,5,6,1\}} - 1_{\{3,4\}}) = \frac{\delta}{5}, \quad F_1(1_{\{2,5,6,3\}} - 1_{\{1,4\}}) = F_1(1_{\{2,5,6,4\}} - 1_{\{1,3\}}) = \frac{\delta}{5}, \]
\[ F_1(1_{\{2,5,6,1\}} - 1_{\{3\}}) = F_1(1_{\{2,5,6,1\}} - 1_{\{4\}}) = \frac{\delta}{5}, \quad F_1(1_{\{2,5,6,3\}} - 1_{\{4\}}) = F_1(1_{\{2,5,6,4\}} - 1_{\{3\}}) = \frac{\delta}{5}, \]
\[ F_1(1_{\{2,5,6,3\}} - 1_{\{1\}}) = F_1(1_{\{2,5,6,4\}} - 1_{\{1\}}) = \frac{\delta}{17}, \]
\[ F_1(1_{\{2,5,6,3\}}) = F_1(1_{\{2,5,6,4\}}) = \frac{\delta}{2}, \quad F_1(1_{\{2,5,6,1\}}) = \frac{\delta}{2}, \quad F_1(1_{\{2,5,6,3,4\}}) = \frac{\delta}{3}. \]
\[ F_1(1_{\{2,5,6,4,1\}}) = F_1(1_{\{2,5,6,4,3\}}) = 0, \]
\[ F_1(1_{\{2,5,6\}} - 1_{\{3,1\}}) = F_1(1_{\{2,5,6\}} - 1_{\{4,1\}}) = \frac{\delta}{17}, \quad F_1(1_{\{2,5,6\}} - 1_{\{3,4\}}) = \frac{\delta}{2}, \]
\[ F_1(1_{\{2,5,6\}} - 1_{\{3\}}) = F_1(1_{\{2,5,6\}} - 1_{\{4\}}) = \frac{\delta}{2}, \quad F_1(1_{\{2,5,6\}} - 1_{\{1\}}) = \frac{\delta}{2}, \quad F_1(1_{\{2,5,6\}} - 1_{\{1,3,4\}}) = \frac{\delta}{2}, \]
\[ F_1(1_{\{2\}}) = F_1(1_{\{5\}}) = F_1(1_{\{6\}}) = F_1(1_{\{2\}}) = \frac{\delta}{7}, \quad F_1(1_{\{5\}}) = \frac{\delta}{7}, \quad F_1(1_{\{6\}}) = \frac{\delta}{7}. \]

Thus, all the vertices in \( \{x \in \text{link}(1_{\{2,5,6\}}) : F_1(x) < F_1(1_{\{2,5,6\}}) \} \) are:

\[ (S1) \quad 1_{\{2,5,6\}} - 1_{\{3,4\}} \]
\[ (S2) \quad 1_{\{2,5,6,3\}}, \quad 1_{\{2,5,6,4\}}, \quad 1_{\{2,5,6,1\}}, \quad 1_{\{2,5,6,3,4\}}, \quad 1_{\{2,5,6,3,1\}}, \quad 1_{\{2,5,6,4,1\}}, \quad 1_{\{2,5,6,4,3\}} - 1_{\{1\}}, \quad 1_{\{2,5,6,3,1\}} - 1_{\{4\}}, \quad 1_{\{2,5,6,4,1\}} - 1_{\{3\}}, \quad 1_{\{2,5,6,1\}} - 1_{\{3,4\}}, \quad 1_{\{2,5,6,1\}} - 1_{\{3\}}, \quad 1_{\{2,5,6,1\}} - 1_{\{4\}} \]

It is easy to see that these subcomplexes are disconnected simplicial complexes in \( K_n \). In fact, it is clear that the subcomplex has two connected components, one is the singleton in \( (S1) \), and the other is the subcomplex induced by the vertices listed in \( (S2) \).

Therefore, \( \frac{\delta}{2} \) is a homological eigenvalue of \( \Delta_1 \). Note that \( \lambda_2(\Delta_1) = \frac{\delta}{2} \), and for any \( 0 < \epsilon < \frac{1}{17} \) and \( 0 < \epsilon' < \frac{1}{20} \), \( (\frac{\delta}{2} - \epsilon, \frac{\delta}{2} + \epsilon) \) doesn’t intersect with \( (\frac{\delta}{2} - \epsilon', \frac{\delta}{2} + \epsilon') \). Then, by Proposition \( \ref{prop:delta} \) and Theorem \( \ref{thm:epsilon} \) there exists \( 0 < \delta < 1 \) such that for any \( p \in [1,1 + \delta] \), \( \lambda_2(\Delta_p) \in (\frac{\delta}{2} - \epsilon', \frac{\delta}{2} + \epsilon') \). Then, by Proposition \( \ref{prop:delta} \) and Theorem \( \ref{thm:epsilon} \) there exists a \( \lambda_3(\Delta_p) \in (\frac{\delta}{2} - \epsilon', \frac{\delta}{2} + \epsilon') \). Then, Theorem \( \ref{thm:epsilon} \) is applicable to prove the existence of an eigenvalue \( \Delta_1 \) which implies that such an eigenvalue \( \lambda_3(\Delta_p) \) is not a (variationsal) min-max eigenvalue of \( \Delta_p \) on the graph \( G \), \( \forall p \in (1,1 + \delta) \). The proof is completed.

\[ \square \]

**Remark 4.1.** The graph presented in Theorem \( \ref{thm:two} \) (or Section \( \ref{sec:two} \)) was first studied in a previous work of the author \( \cite{14} \). But in that paper, we only get partial results on the 1-Laplacian eigenvalues (for example, we didn’t even know the value of \( \lambda_3(\Delta_1) \) for the graph in that paper).

**Remark 4.2.** Note that \( \lambda_3(\Delta_1) \) equals the third Cheeger constant \( h_3 \). However, from the multi-way Cheeger inequality, we can only get \( \lambda_3(\Delta_1) \leq h_3 \). In fact, it can be verified that every eigenvector of \( \lambda_3(\Delta_1) \) has at most two nodal domains, and thus both the nodal domain theorem and the multi-way Cheeger inequality in \( \cite{14} \) are not sharp on this example. They are not powerful enough to obtain the results that we desired. To this end, we establish Lemma \( \ref{lem:epsilon} \) which is special but strong.

The picture in Fig. 2 shows a sketch of the proof for Theorem \( \ref{thm:two} \). In fact, we can determine all the eigenvalues and all the min-max eigenvalues of \( \Delta_1 \) by some auxiliary results established in Section \( \ref{sec:two} \) and then we apply Theorem \( \ref{thm:two} \) to select the eigenvalue \( \frac{\delta}{2} \) which is homological but not in the min-max form. Note that the min-max eigenvalues of \( \Delta_p \) should be far away from \( \frac{\delta}{2} \) when \( p \) is sufficiently close to 1, due to Proposition \( \ref{prop:delta} \). Then, Theorem \( \ref{thm:epsilon} \) is applicable to prove the existence of a non-minimally homological eigenvalues of \( \Delta_p \) near \( \frac{\delta}{2} \), when \( p \) is sufficiently close to 1.

To some extent, the picture suggests us to consider the ‘persistent p-Laplacian’ for varying \( p \), which is able to record the birth (appearance) and death (disappearance) of the spectra of \( \Delta_p \) when \( p \) goes from 1 to 10, and thus extra combinatorial information is embedded in it.

**Remark 4.3.** If we allow that the graph possesses repeated edges with different real incidence coefficients, then we can find non-variational eigenvalues for p-Laplacians on certain graphs of order 2. Precisely, let \( V = \{1,2\} \) and \( E = \{e_1, e_2\} \), consider a generalized graph \( (V,E,\varphi) \) with repeated edges and vertex-edge incidence coefficients \( \varphi : V \times E \rightarrow \mathbb{R} \) defined by \( \varphi(1,e_1) = \varphi(2,e_1) = \varphi(1,e_2) = 1 \), \( \varphi(2,e_2) = 2 \), \( \varphi(1,e_2) = 3 \), \( \varphi(2,e_1) = 4 \), and \( \varphi(1,\emptyset) = \varphi(2,\emptyset) = 0 \). Then, the Laplacian operator \( \Delta_p \) on such graphs is defined as

\[ \varphi(1,\emptyset) = \varphi(2,\emptyset) = 0 \]
\[ \varphi(1,e_1) = \varphi(2,e_1) = \varphi(1,e_2) = 1 \]
\[ \varphi(2,e_2) = 2 \]
\[ \varphi(1,\emptyset) = \varphi(2,\emptyset) = 0 \]
\[ \varphi(1,e_2) = \varphi(2,e_2) = 3 \]
\[ \varphi(2,e_1) = \varphi(1,e_1) = 4 \]

The Laplacian operator is given by

\[ \Delta_p = A \]
Figure 2: This picture illustrates the variance of the eigenvalues of $\Delta_p$ ($2 \geq p \geq 1$) for the graph presented in Theorem 1.2. The left six numbers are exactly the eigenvalues of $\Delta_2$, while the right nine numbers are exactly the eigenvalues of $\Delta_1$. All the black points are the variational (min-max) eigenvalues, while the red one is the homological eigenvalue which is non-variational. The blue points are also non-variational eigenvalues of $\Delta_1$, but we haven’t checked whether they are homological eigenvalues.
and $\varphi(2, e_2) = -2$. Then, similar to the usual $p$-Laplacian eigenvalue problem on simple graphs, the generalized $p$-Laplacian eigenvalue problem is to find $(\lambda, x)$ such that

$$\begin{cases}
|x_1 + x_2|^{p-2}(x_1 + x_2) + |x_1 - 2x_2|^{p-2}(x_1 - 2x_2) = 2\lambda|x_1|^{p-2}x_1 \\
|x_1 + x_2|^{p-2}(x_1 + x_2) - 2|x_1 - 2x_2|^{p-2}(x_1 - 2x_2) = 3\lambda|x_2|^{p-2}x_2
\end{cases}$$

which includes the critical values and critical points of the corresponding Rayleigh quotient

$$F_p(x) = \frac{|x_1 + x_2|^p + |x_1 - 2x_2|^p}{2|x_1|^p + 3|x_2|^p}.$$

Since

$$F_p(1, 0) = F_p(0, 1) = F_p(-1, 0) = F_p(0, -1) = 1 > F_p(1, -1) = \frac{3^p}{3} > F_p(2, 1) = \frac{3^p}{2^{p+1} + 3}$$

whenever $1 \leq p < \log_3 5$, $F_p$ has a local minimizer in the first quadrant $\{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, and a local minimizer in $\{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\}$. Since $(1, -1)$ and $(2, 1)$ are local minimizers of $F_1$, and $F_1(1, -1) = \frac{3}{5} > F_1(2, 1) = \frac{3}{7}$, $F_1$ has exactly three different critical values, and the minimum of $F_p$ restricted on $\{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ is smaller than the minimum of $F_p$ restricted on $\{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\}$ if $p$ is sufficiently close to 1. Therefore, it is easy to see that $F_p$ has at least three different critical values, and thus the $p$-Laplacian has at least three distinct eigenvalues when $1 \leq p < \log_3 5$.

4.2 Non-homological eigenvalues

**Example 4.1.** Using the same technique in Section 4.1, we can show that $1_{\{1, 2\}}$ is an eigenvector of the 1-Laplacian on the following graph:

![Graph Image]

It can be verified that $\frac{1}{2}$ is an eigenvalue of the 1-Laplacian on the graph above, while by Theorem 1.1, it is not a homological eigenvalue.

A non-homological eigenvalue does not possess the stability and local monotonicity, which means that Theorem 1.1 cannot be improved. We give an example on path graphs.

**Example 4.2.** By Theorem 3.7 in [13], for $p > 1$, the $p$-Laplacian on a tree admits only variational eigenvalues, and by the discussion in [13] and Corollary 1.1, the min-max eigenvalues of the 1-Laplacian on a tree coincide exactly with the multi-way Cheeger constants. In particular, on the path graph $P_6$ with six vertices, we have $h_1 = \lambda_1(\Delta_1) = 0$, $h_2 = \lambda_2(\Delta_1) = 1/5$, $h_3 = \lambda_3(\Delta_1) = 1/2$, and $h_4 = h_5 = h_6 = \lambda_4(\Delta_1) = \lambda_5(\Delta_1) = \lambda_6(\Delta_1) = 1$, and by Theorem 1.1, for any $0 < \epsilon < \frac{1}{100}$, there exists $\delta > 0$ such that for any $1 < p < 1 + \delta$, $\text{spec}(\Delta_p) \subset [0, \epsilon) \cup (\frac{1}{5} - \epsilon, \frac{1}{5} + \epsilon) \cup (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \cup (1 - \epsilon, 1]$.

By Theorem 2 in [14], the set of 1-Laplacian eigenvalues of the path graph $P_6$ is $\{0, 1/5, 1/3, 1/2, 1\}$.

Therefore, the 1-Laplacian eigenvalue $1/3$ is non-variational. Furthermore, we can show that the eigenvalue $1/3$ must also be non-homological. If fact, if $1/3$ is a homological eigenvalue of $\Delta_1$ on $P_6$, then by the stability of homological eigenvalues (Theorem 1.1), for any $p > 1$ which is sufficiently close to 1, $\Delta_p$ on $P_6$ has an eigenvalue that is sufficiently close to $1/3$, which is a contradiction to the discussions above. In consequence, $\frac{1}{3}$ is a non-homological eigenvalue of $\Delta_1$ on $P_6$. Moreover, from this example, we see that in contrast to homological eigenvalues of $\Delta_p$, the stability and local monotonicity with respect to $p$, do not hold on non-homological eigenvalues of $\Delta_p$. 

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4.3 A note on complete graphs

Based on the results in [1] and [10], we have:

**Proposition 4.1.** For \( p \notin \{1, 2\} \), and \( n \geq 4 \), the complete graph of order \( n \) has exactly \( \lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor) + 1 \) non-variational eigenvalues of \( p \)-Laplacian.

While for any \( n \geq 1 \), all the eigenvalues of \( 1 \)-Laplacian on the complete graph of order \( n \) are variational eigenvalues.

**Proof.** By Theorem 6 in [1], for \( n \geq 3 \) and \( p > 1 \), the nonzero eigenvalues of \( \Delta_p \) on the complete graph of order \( n \) are:

\[
\frac{1}{n-1} \left( n - i - j + (i^{p-1} + j^{p-1})^{p^{-1}} \right), \quad i, j \in \mathbb{Z}_+, i + j \leq n.
\]

It can be checked that the number of the eigenvalues is \( \lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor) + 1 \), and every eigenvalue has \( \gamma^- \)-multiplicity 1 (due to the proof of Theorem 6 in [1]), whenever \( p \neq 1, 2 \).

By Proposition 8 and Theorem 5 in [10], for \( n \geq 3 \), the nonzero eigenvalues of \( \Delta_1 \) on the complete graph of order \( n \) are

\[
\frac{n - i}{n - 1}, \quad i \in \mathbb{Z}_+, i \leq \lfloor \frac{n}{2} \rfloor.
\]

Moreover, the \( \gamma^- \)-multiplicity of the eigenvalue \( \frac{n-i}{n-1} \) is 2, whenever \( 1 \leq i < \lfloor \frac{n}{2} \rfloor \). The smallest positive eigenvalue (i.e., the Cheeger constant) \( \frac{n-\lfloor \frac{n}{2} \rfloor}{n-1} \) has \( \gamma^- \)-multiplicity

\[
\begin{cases} 
1, & \text{if } n \text{ is even}, \\
2, & \text{if } n \text{ is odd}.
\end{cases}
\]

Thus, there is no difficulty to verify that all the eigenvalues are variational eigenvalues, that is, they can be listed as \( \lambda_1^\top(\Delta_1), \cdots, \lambda_n^\top(\Delta_1) \).

\[\square\]

4.4 Further remarks on nonvariational eigenvalues of the 1-Laplacian

We have shown in Section 4.1 that for some graphs,

\[\{\text{the min-max } \Delta_p\text{-eigenvalues } \lambda_1(\Delta_p), \cdots, \lambda_n(\Delta_p)\} \subseteq \{\text{homological eigenvalues of } \Delta_p\} .\]

In this section, we show that for \( p = 1 \), the size of the difference set

\[\{\Delta_1\text{-eigenvalues}\} \setminus \{\text{min-max } \Delta_1\text{-eigenvalues}\}\]

can be very large for some graphs. For convenience, given a simple graph \( G \), we shall use \( \Delta_1(G) \) to denote the 1-Laplacian on the graph \( G \).

- For sufficiently large integer \( n \), there exists a connected graph \( G_n \) on \( n \) vertices with \( O(n \ln n) \) eigenvalues (counting multiplicity) of \( \Delta_1(G_n) \).

**Example 4.3.** Let \( G_n \) be the \( n \)-order cycle graph, i.e., \( V(G_n) = \{1, \cdots, n\} \) and \( E(G_n) = \{\{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}, \{n, 1\}\} \). According to Theorem 4 in [10], we know that the eigenvalues of \( \Delta_1(G_n) \) are \( 1, \frac{1}{2}, \cdots, \frac{1}{\lfloor \frac{n}{2} \rfloor} \), 0, and it is not difficult to verify that the corresponding multiplicities are \( \lfloor \frac{n}{2} \rfloor \), \( \lfloor \frac{n}{2} \rfloor \), \cdots, \( \lfloor \frac{n}{2} \rfloor \), 1, respectively. Here and after, we use \( \lfloor \cdot \rfloor \) to denote the floor function.

Therefore, the number of eigenvalues (counting multiplicity) of \( \Delta_1(G_n) \) is

\[
1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{i}{2} \rfloor = \frac{n}{2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{\lfloor \frac{n}{2} \rfloor} \right) + O(n) = \frac{n}{2} \ln(\frac{n}{2}) + O(n),
\]

where we used the logarithmic growth property of harmonic series.

- For sufficiently large integer \( n \), there exists a connected graph \( G_n \) on \( n \) vertices with at least \( \lfloor \frac{n}{2} \rfloor - 2 \) pairwise distinct eigenvalues of \( \Delta_1(G_n) \).
Example 4.4. For any even number $n \geq 8$, let $G_n$ be the graph on the vertex set $\{1, \cdots, n\}$, with the edge set

$$E(G_n) = \{\{i, j\} : i + j \leq n + 1, i, j \in \{1, \cdots, n\}\} \cup \left\{\{n, n - i\} : 1 \leq i \leq \frac{n}{2} - 2\right\}.$$ 

Then, we have $\deg(i) = n - i$ for $1 \leq i \leq \frac{n}{2}$, $\deg(n) = \frac{n}{2} - 1$ and $\deg(j) = n + 2 - j$, for $\frac{n}{2} + 2 \leq j \leq n - 1$. See Fig. 3 for the cases of $n = 8$ and $n = 10$, respectively.

![Figure 3: The graphs in Example 4.4](image)

It is clear that $\deg(i) \geq 3, \forall i \in V$. Thus, applying Proposition 4.2 to $G_n$, the number of distinct eigenvalues of $\Delta_1(G_n)$ is at least

$$2 + \#\{\deg(i) + \deg(j) : \{i, j\} \in E(\Gamma_n)\} = 2 + \#\left(\{n, \cdots, 2n - 3\} \cup \left\{\frac{n}{2} + 2, \cdots, n - 1\right\}\right) = \frac{3}{2}n - 2.$$

Finally, we present our technical results used in the examples above.

Proposition 4.2. For a simple graph $G = (V, E)$, the number of distinct eigenvalues of $\Delta_1(G)$ is larger than or equal to

$$2 + \#\{\deg(i) + \deg(j) : \{i, j\} \text{ is an edge s.t. the induced subgraph on } V \setminus \{i, j\} \text{ has no isolated vertex}\}. $$

In particular, if the minimum degree of $G$ is larger than or equal to 3, then the number of distinct eigenvalues of $\Delta_1(G)$ is larger than or equal to

$$2 + \#\{\deg(i) + \deg(j) : \{i, j\} \in E(G)\}.$$

Proof. Note that 0 and 1 are the minimal and the maximal eigenvalues of $\Delta_1$, respectively. By Proposition 4.3 (I), for an edge $\{i, j\}$, the indicator $1_{\{i, j\}}$ is an eigenvector of $\Delta_1$ if and only if the induced subgraph $G|_{V \setminus \{i, j\}}$ has no isolated vertex; and in this case, the eigenvector $1_{\{i, j\}}$ corresponds to the eigenvalue

$$\lambda = 1 - \frac{2}{\deg(i) + \deg(j)}.$$ 

This proves the first statement.

For the second one, it follows from $\deg(i) \geq 3, \forall i \in V$, that $G|_{V \setminus \{i, j\}}$ has no isolated vertex, for any edge $\{i, j\}$ in $G$. Hence, by the first statement, we complete the proof.

Definition 4.1. For a subset $S \subset V$, the $1$-neighborhood of $S$ is $\bigcup_{\{i, j\} \in E : \{i, j\} \cap S \neq \emptyset} \{i, j\}.$
Definition 4.2. A subset $S \subset V$ of order 2 is a simple nodal set if $S$ is an edge, and the subgraph induced by $V \setminus S$ has no isolated vertex.

Proposition 4.3. (I) Assume that $G$ is connected and $S$ is a connected subset of order 2. Then, $1_S$ is an eigenvector of $\Delta_1$ if and only if $S$ is a simple nodal set.

(II) If $S_1, \ldots, S_k$ are simple nodal sets with pairwise non-adjacent 1-neighborhoods, and $\frac{|\partial S_i|}{\text{vol}(S_i)} = \ldots = \frac{|\partial S_k|}{\text{vol}(S_k)}$, then every nonzero vector $x \in \text{span}(1_{S_1}, \ldots, 1_{S_k})$ is an eigenvector of $\Delta_1$.

(III) If $S$ and $S'$ are simple nodal sets with disjoint 1-neighborhoods, and $\frac{|\partial S|}{\text{vol}(S)} = \frac{|\partial S'|}{\text{vol}(S')}$, then $ttS + t'1_{S'}$ is an eigenvector of $\Delta_1$, whenever $tt' \leq 0$ and $(t, t') \neq (0, 0)$.

Proof. It suffices to check the coordinate equation (1) for the 1-Laplacian eigenvalue problem.

(I) Without loss of generality, we may assume that $S = \{1, 2\}$. Note that $|\partial S| = \text{vol}(S) - 2$ and $\lambda := \frac{|\partial S|}{\text{vol}(S)} = 1 - \frac{2}{\deg(1) + \deg(2)}$. Then $z_{i1} = 1$ for $i \not\in S$ with $\{1, i\} \in E$, and $z_{2j} = 1$ for $j \not\in S$ with $\{2, j\} \in E$.

Assume that $1_S$ is an eigenvector. If $i \in V \setminus S$ is isolated in the subgraph induced by $V \setminus S$, then $\deg(i) \in \{1, 2\}$ and every vertex adjacent to $i$ lies in $\{1, 2\}$. There are only two cases: $i$ connected only to 1, and $i$ connected to both 1 and 2. If $i$ is adjacent to 1 only, then by the definition, the eigenequation for the eigenpair $(\lambda, 1_{\{1,2\}})$ satisfied in $i$ is $z_{i1} = -1 \in \lambda\text{Sgn}(0)$ which implies that $|\lambda| \geq 1$, but since it is known that $0 \leq \lambda \leq 1$, there must hold $\lambda = 1$. Similarly, if $i$ is adjacent to 1 and 2 only, then by the definition, the eigenequation satisfied in $i$ is $z_{i1} + z_{i2} = -1 - 1 \in 2\text{Sgn}(0)$, also implying $\lambda = 1$. Hence, in any case, we obtain $\lambda = 1$, which contradicts to $\lambda = 1 - \frac{2}{\deg(1) + \deg(2)} < 1$. In consequence, the subgraph induced by $V \setminus S$ has no isolated vertex.

For $i \in V \setminus S$, denote by $c_{S,i}$:

$$c_{S,i} = \begin{cases} 2, & \text{if } \{1, i\}, \{2, i\} \in E, \\ 0, & \text{if } \{1, i\}, \{2, i\} \not\in E, \\ 1, & \text{otherwise.} \end{cases}$$

We first prove that, if for any $i \in V \setminus S$,

$$\frac{c_{S,i}}{\deg(i)} \leq \frac{|\partial S|}{\text{vol}(S)} := \lambda, \quad (16)$$

then $1_S$ is an eigenvector of $\Delta_1$.

By letting

$$z_{12} = \lambda \deg(1) - \deg(1) + 1 = \frac{\deg(2) - \deg(1)}{\deg(2) + \deg(1)} \in \text{Sgn}(0) = [-1, 1],$$

we have

$$\sum_{i=1} z_{i1} = \deg(1) - 1 + z_{12} = \lambda \deg(1).$$

Similarly, $\sum_{j=2} z_{2j} = \lambda \deg(2)$.

Taking $z_{ij} = 0$ for any $\{i, j\} \in E$ with $i, j \not\in S$, we have

$$\left| \sum_{j \in V \setminus S, j \sim i} z_{ij} \right| \leq c_{S,i} \leq \lambda \deg(i)$$

$\forall i \in V \setminus S$, where $\lambda = 1 - \frac{2}{\text{vol}(S)}$. Thus, $1_S$ is an eigenvector.
Now, we only need to check the remaining case that there exists \( i \in V \setminus S \) satisfying
\[
c_{S,i} > \text{deg}(i) \left( 1 - \frac{2}{\text{vol}(S)} \right)
\]
that is, there is a vertex \( i \) that doesn’t satisfy the inequality (10).

**Case 1.** \( c_{S,i} = 1 \).

Without loss of generality, we assume that \( \{1,i\} \in E \) and \( \{2,i\} \notin E \).
In this case, \( \text{deg}(i) \geq 2 \), \( \text{deg}(1) + \text{deg}(2) \geq 1 + 2 = 3 \), and
\[
1 > \text{deg}(i) \left( 1 - \frac{2}{\text{deg}(1) + \text{deg}(2)} \right).
\]
These imply \( \text{deg}(i) = 2 = \text{deg}(1) \) and \( \text{deg}(2) = 1 \). Then, by taking \( \lambda = \frac{1}{3} \), \( z_{1i} = 1 \), \( z_{12} = -\frac{1}{2} \), \( z_{ij} = \frac{1}{2} \) for \( j \notin \{1,2\} \) with \( j \sim i \), and \( z_{ij'} = 0 \) otherwise, we get a solution of the 1-Laplacian eigenvalue problem (4), which implies that \( 1_S \) is an eigenvector corresponding to the eigenvalue \( \frac{1}{3} \).

**Case 2.** \( c_{S,i} = 2 \).

Without loss of generality, we assume that \( \text{deg}(1) \geq \text{deg}(2) \) for simplicity.
In this case, we have
\[
2 > \text{deg}(i) \left( 1 - \frac{2}{\text{deg}(1) + \text{deg}(2)} \right),
\]
\( \text{deg}(i) \geq 3 \) and \( \text{deg}(1) + \text{deg}(2) \geq 2 + 2 = 4 \). Hence, we get
\( \text{deg}(i) = 3, \text{deg}(2) = 2 \) and \( \text{deg}(1) \in \{2,3\} \). We denote by \( j \) the unique vertex in \( V \setminus S \) that is adjacent to \( i \).

**Case 2.1.** \( \text{deg}(i) = 3, \text{deg}(2) = 2 = \text{deg}(1) \).

In this case, we take \( \lambda = \frac{1}{3} \), \( z_{21} = 0 \), \( z_{1i} = z_{2i} = 1 \), \( z_{ij} = \frac{1}{2} \), and \( z_{ij'} = 0 \) otherwise. Then, (4) is easy to check. This means that \( 1_S \) is an eigenvector corresponding to the eigenvalue \( \frac{1}{3} \).

**Case 2.2.** \( \text{deg}(i) = 3 = \text{deg}(1), \text{deg}(2) = 2 \).

In this case, let \( \lambda = \frac{3}{5} \), \( z_{21} = \frac{1}{2} = z_{ij}, z_{1i} = z_{2i} = z_{i'i} = 1 \), where \( i' \) is the unique vertex in \( V \setminus \{1,2,i\} \) that is adjacent to \( 1 \), and \( z_{ij} = 0 \) otherwise. Substituting these parameters in (4) implies that \( 1_S \) is an eigenvector corresponding to the eigenvalue \( \frac{3}{5} \).

(II)-(III) These two statements can be verified by solving (4) with the help of (I).

The proof is completed.

At the end of this section, we establish a result which shows that the size of the difference set
\[
\{\text{eigenvectors of } \Delta_1\} \setminus \{\text{critical points of } F_1\}
\]
can be very large on some graphs.

- **The Hausdorff dimension of the eigenspace corresponding to an eigenvalue \( \lambda \) of \( \Delta_1 \) may be larger than the Hausdorff dimension of the set of critical points corresponding to the critical value \( \lambda \) of \( F_1 \)**

![Figure 4: The graph for Example 4.5.](image-url)
Example 4.5. In the graph $G$ shown in Figure 4, applying Proposition 4.4 (III) directly to the subsets $\{3, 4\}$ and $\{5, 6\}$, $t\{3,4\} + s\{5,6\}$ is an eigenvector w.r.t. the eigenvalue $\frac{1}{2}$ of $\Delta_1$, for any $t, s \in \mathbb{R}$ with $ts \leq 0$. In fact, the eigenspace of $\frac{1}{2}$ is $S_\frac{1}{2}(\Delta_1) = \{t\{3,4\} + s\{5,6\} : ts \leq 0\}$. We will show that $t\{3,4\} + s\{5,6\}$ is a critical point of $F_1$ if and only if, for any $t > 0$ or $s = 0$. In fact, if $t > 0 > s$, by taking $x = t\{3,4\} + s\{5,6\}$ and $y = 1\{1\} - 1\{2\}$ in Proposition 4.4, we can prove that $\Phi(x, \xi, y) - \frac{1}{2}\Psi(x, \xi, y) \leq -2$ for any $\xi$, where $\Phi$ and $\Psi$ are introduced in Proposition 4.4. To see this, note that

$$
\Phi(x, \xi, y) = -4 + 2z_{12}^t(\xi) \text{ and } \Psi(x, \xi, y) = 3(z_i^t(\xi) - z_j^t(\xi)),
$$

where $z_{12}^t(\xi) = \begin{cases} 1, & \text{if } \xi(1) \geq \xi(2), \\ 0, & \text{if } \xi(1) < \xi(2), \end{cases}$ $z_i^t(\xi) = \begin{cases} 1, & \text{if } \xi(1) > 0, \\ -1, & \text{if } \xi(1) < 0, \end{cases}$ and $z_j^t(\xi) = \begin{cases} 1, & \text{if } \xi(1) > 0, \\ -1, & \text{if } \xi(1) < 0. \end{cases}$

Now, it is easy to see $\Phi(x, \xi, y) - \frac{1}{2}\Psi(x, \xi, y) \leq -2$, $\forall \xi \in \mathbb{R}^n$. By Proposition 4.4, we obtain the desired conclusion. Similarly, for $t > 0 > s$ we take $y = -1\{1\} + 1\{2\}$; for $t > 0$, take $y = 1\{1\} + 1\{2\}$; and for $t, s < 0$, take $y = -1\{1\} - 1\{2\}$. In summary, for $ts \neq 0$, $t\{3,4\} + s\{5,6\}$ is not a critical point of $F_1$. In contrast to the linear combination of $1\{3,4\}$ and $1\{5,6\}$, it is surprising that both $1\{3,4\}$ and $1\{5,6\}$ are critical points of $F_1$ (also by Proposition 4.4).

**Proposition 4.4.** A nonzero vector $x$ is a critical point of $F_1$ corresponding to the critical value $\lambda$ if and only if for any vector $y \in \mathbb{R}^n$, there exists $\xi \in \mathbb{R}^n$ such that

$$
\Phi(x, \xi, y) \geq \lambda \Psi(x, \xi, y),
$$

where

$$
\Phi(x, \xi, y) = \sum_{(i,j) \in E} z_{ij}(y_i - y_j) + \sum_{\{i,j\} \in E} z_{ij}(\xi)(y_i - y_j) + \sum_{\{i,j\} \in E} |y_i - y_j|,
$$

$$
\Psi(x, \xi, y) = \sum_{i \in V} \deg(i)z_iy_i + \sum_{i \in V} \deg(i)z_i(\xi)y_i + \sum_{i \in V} \deg(i)|y_i|,
$$

$z_{ij} := z_{ij}(x) = \text{sign}(x_i - x_j)$, $z_i := z_i(\xi) = \text{sign}(x_i)$, $z_i(\xi) = \text{sign}(\xi_i - \xi_j)$ and $z_i(\xi) = \text{sign}(\xi_i)$.

**Proof.** By the definition of Clarke derivative, $x$ is a critical point of $F_1$ if and only if, for any $y \in \mathbb{R}^n$,

$$
\limsup_{\xi \to 0, t \to 0^+} \frac{1}{t} \left( \frac{TV(x + \xi + ty)}{\|x + \xi + ty\|_1} - \frac{TV(x + \xi)}{\|x + \xi\|_1} \right) \geq 0,
$$

which is equivalent to

$$
\limsup_{\xi \to 0, t \to 0^+} \frac{TV(x + \xi + ty)}{t} \|x + \xi + ty\|_1 - TV(x + \xi)\|x + \xi\|_1 \geq 0, \quad (17)
$$

where $TV(x) := \sum_{\{i,j\} \in E} |x_i - x_j|$ and $\|x\|_1 := \sum_{i \in V} \deg(i)|x_i|$. Since both $TV(\cdot)$ and $\|\cdot\|_1$ are piecewise-linear and one-homogeneous, for sufficiently small $t > 0$, we have

$$
TV(x + \xi + ty) = TV(x + \xi) + t \sum_{\{i,j\} \in E \atop x_i + \xi \neq x_j + \xi_j} z_{ij}(x + \xi)(y_i - y_j) + t \sum_{\{i,j\} \in E \atop x_i + \xi = x_j + \xi_j} |y_i - y_j|,
$$

and

$$
\|x + \xi + ty\|_1 = \|x + \xi\|_1 + t \sum_{i \in V} \deg(i)z_i(x + \xi)y_i + t \sum_{i \in V} \deg(i)|y_i|.
$$

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Therefore, (17) can be rewritten as

$$\limsup_{\xi \to 0} \Phi(\xi) - \lambda(\xi) \Psi(\xi) \geq 0$$

where $\lambda(\xi) := \frac{TV(x+\xi)}{\|x+\xi\|_1}$,

$$\Phi(\xi) := \sum_{\{i,j\} \in E} z_{ij}(x+\xi)(y_i - y_j) + \sum_{\{i,j\} \in E} |y_i - y_j|,$$

$$\Psi(\xi) := \sum_{i \in V} \deg(i)(x+\xi)y_i + \sum_{i \in V} \deg(i)|y_i|.$$

Note that, for $\xi$ sufficiently close to 0, $\Phi(\xi) = \Phi(x,\xi,y)$ and $\Psi(\xi) = \Psi(x,\xi,y)$. Since $\Phi(\cdot)$ and $\Psi(\cdot)$ are zero-homogeneous, (17) is further equivalent to

$$\max_{\xi \in \mathbb{R}^n} \Phi(x,\xi,y) - \lambda \Psi(x,\xi,y) \geq 0,$$

with $\lambda := \frac{TV(x)}{\|x\|_1}$. This shows that $x$ is a critical point of $F_1$ if and only if

$$\inf_{y \in \mathbb{R}^n} \max_{\xi \in \mathbb{R}^n} \Phi(x,\xi,y) - \lambda \Psi(x,\xi,y) \geq 0.$$

The proof is completed. \qed

5 Open problems

We leave some open questions as possible future works. Theorem 1.4 poses the following question:

Open Question 1. How the nodal domains of the p-Laplacian variational eigenfunctions vary with respect to $p$?

Following Proposition 2.4, we are interested in the similar property for $\gamma$ and $\gamma^+$.

Open Question 2. Are the $\gamma$-multiplicity and the $\gamma^+$-multiplicity upper semi-continuous?

Proposition 2.6 leads us to the following question for $\lambda^{\pm}_k(\Delta_p)$.

Open Question 3. Are the variational eigenvalues $\lambda^{\pm}_k(\Delta_p)$ using the Krasnoselskii genus $\gamma^-$, and the min-max eigenvalue $\lambda^{\pm}_k(\Delta_p)$ involving $\gamma^+$, homological eigenvalues of $\Delta_p$?

Corollary 1.1 encourages us to post the following question:

Open Question 4. Is the equality $\lambda^{\pm}_k(\Delta_p) = \lambda_k(\Delta_p) = \lambda^{\pm}_k(\Delta_p)$ true?

As a possible way to strengthen Theorem 1.4, we ask:

Open Question 5. Is there a universal constant $C > 0$ such that for any $p \geq 1$, and for any $k$,

$$\lambda^{\pm}_k(\Delta_p) \geq \frac{h^p_k}{Ck^{2p}}?$$

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Lemma A.1. For any $t \geq 1$, $b, a \in \mathbb{R}$,

$$||b|^t \text{sign}(b) - |a|^t \text{sign}(a)|| \geq |b - a|(\frac{|b|^t + |a|^t}{2})^{1 - \frac{1}{t}}$$

Proof. To prove this, we only need to show that for any $b > a > 0$,

$$b^t - a^t \geq (b - a)(\frac{b^t + a^t}{2})^{1 - \frac{1}{t}}$$

$$b^t + a^t \geq (b + a)(\frac{b^t + a^t}{2})^{1 - \frac{1}{t}}$$

The second one follows from the mean power inequality $\left(\frac{b^t + a^t}{2}\right)^{\frac{1}{t}} \geq \frac{b + a}{2}$. The first one clearly holds when $t = 1$. Now, suppose $t > 1$. By the convexity of the function $x \mapsto x^{\frac{t}{t-1}}$ for $x \in (0, +\infty)$, we have

$$\frac{b - a}{b} \left(\frac{b^t - a^t}{b - a}\right)^{\frac{t}{t-1}} + \frac{a}{b} \left(\frac{a^t - b^t}{b - a}\right)^{\frac{t}{t-1}} \geq (b^{t-1})^{\frac{t}{t-1}}$$

which implies

$$\left(\frac{b^t - a^t}{b - a}\right)^{\frac{t}{t-1}} \geq \frac{b^{t+1} - a^{t+1}}{b - a}$$

Since $b^{t+1} - a^{t+1} - (b - a)(b^t + a^t) = ab(b^{t-1} - a^{t-1}) > 0$, we have

$$\left(\frac{b^t - a^t}{b - a}\right)^{\frac{t}{t-1}} \geq \frac{b^{t+1} - a^{t+1}}{b - a} \geq b^t + a^t$$

and thus

$$b^t - a^t \geq (b - a)(b^t + a^t)^{1 - \frac{1}{t}} \geq (b - a)\left(\frac{b^t + a^t}{2}\right)^{1 - \frac{1}{t}},$$

which proves the first inequality.

Remark A.1. For any $t \geq 1$, $b, a \in \mathbb{R}$,

$$||b|^t \text{sign}(b) - |a|^t \text{sign}(a)|| \leq t|b - a|(\frac{|b|^t + |a|^t}{2})^{1 - \frac{1}{t}}.$$

This inequality was first established in [2].