AMENABILITY AND ERGODIC PROPERTIES OF TOPOLOGICAL GROUPS: FROM BOGOLYUBOV ONWARDS

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ABSTRACT. The purpose of this expository article is to revisit the notions of amenability and ergodicity, and to point out that they appear for topological groups that are not necessarily locally compact in articles by Bogolyubov (1939), Fomin (1950), Dixmier (1950), and Rickert (1967).

1. Introduction

Let $G$ be a topological group and $X$ a non-empty compact space on which $G$ acts continuously by homeomorphisms. Denote by $\mathcal{P}(X)$ the space of probability measures on $X$, by $\mathcal{P}^G(X)$ its subspace of $G$-invariant probability measures, and by $\mathcal{E}^G(X)$ the subspace of indecomposable $G$-invariant probability measures on $X$ (the definitions of indecomposable measures, and of related ergodic notions, are recalled in Section 6). In the classical case of $G$ the additive group $\mathbb{R}$ of real numbers and $X$ a metrizable compact space, Bogolyubov and Krylov established in [KrBo–37] three fundamental results, that we quote essentially as Fomin does in [Fomi–50 § 2]:

(1) $\mathcal{P}^\mathbb{R}(X)$ is a non-empty convex compact subspace in the appropriate topological vector space.

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(2) $\mathcal{P}^\mathbb{R}(X)$ is the convex closure of the space $\mathcal{E}^\mathbb{R}(X)$ of ergodic invariant probability measures; in particular $\mathcal{E}^\mathbb{R}(X)$ is non-empty. (“Ergodic” is used today, but “transitive” is used in [KrBo–37, Bogo–39, Fomi–50].)

(3) For every $\mu \in \mathcal{E}^\mathbb{R}(X)$, there exists an $\mathbb{R}$-invariant Borel subset $E_\mu \subset X$ with the following properties: $\mu(E_\mu) = 1$, and $\mu'(E_\mu) = 0$ for all $\mu' \in \mathcal{E}^\mathbb{R}(X)$, $\mu' \neq \mu$.

See also [Oxito–52] for $G = \mathbb{Z}$ (instead of $\mathbb{R}$).

The generalization from $\mathbb{R}$ to other topological groups raises several problems and justifies new notions, in particular that of amenable groups, for which (1) holds. When (1) holds, (2) holds without further restriction on $G$, as it is indicated in [Bogo–39]; this follows alternatively from the Krein-Milman theorem [KrMi–40]. Result (3) holds more generally when $G$ is a second countable locally compact groups, as exposed in [Vara–63] and [GrSc–00, Theorem 1.1(3)], see Section 6 below; but (3) does not hold for all groups, see the examples of Kolmogorov discussed in Section 7.

Historically, the starting point of the notion of amenability is often proposed to be the article [vNeu–29] of von Neumann, sometimes complemented by a note [Tars–29] of Tarski, developed in [Tars–38]. The prehistoric background includes the Hausdorff paradox [Haus–14, Page 469] as revisited by Banach and Tarski [BaTa–24]; there is a good and comprehensive exposition of this subject in [Wago–85]. In these references, groups do not have topology, in other words they are just “discrete groups” (even if topological groups appear in Wagon’s book, they appear only marginally).

The first point of the present article is to suggest that the notion has at least one other independent origin. Indeed, it was our surprise to discover that topological groups $G$ such that there exists a left-invariant mean on $\mathcal{C}^b(G)$ are already the main subject of the 1939 article of Bogolyubov [Bogo–39], where appears the class of topological groups that we call “strongly amenable” in Definition 2.8 below. Here, $\mathcal{C}^b(G)$ stands for the Banach space of bounded continuous real-valued functions on $G$.

(A possible other origin could be an article [Ahlf–35] on Nevanlinna theory, where Ahlfors defines “regularly exhaustible” open Riemann surfaces, i.e. surfaces $S$ with a nested sequence $\Omega_1 \subset \Omega_2 \subset \cdots$ of domains with compact closures and smooth boundaries, such that $S = \bigcup_{n=1}^{\infty} \Omega_n$, and such that $\lim_{n \to \infty} \frac{\text{length}_{g}(\partial \Omega_n)}{\text{area}_{g}(\Omega_n)} = 0$, where the subscript $g$ refers to some metric in the conformal class defined by the complex structure on $S$; Ahlfors has used such sequences to define averaging
processes, as Følner did later with “Følner sequences” in groups. The notion of regular exhaustion has natural formulations in Riemannian geometry; from several possible references on this subject, we will only quote [Roe–88]. The connection between Ahlfors’ regular exhaustions and amenability seems rather obvious now, but we are not aware of any discussion of it in the literature before the 80’s. Apparently, the connection could only be observed after amenability was recognized as a metric property of both groups and spaces.)

In later publications, attention is often restricted to two particular classes of topological groups: discrete groups (and semi-groups), as in [HeRo–63, § 18], and locally compact groups, as in the influential exposition of Greenleaf [Gree–69]. But the notion has been considered for topological groups in general, as can be seen in contributions by Fomin [Fomi–50], Dixmier [Dixm–50], and Rickert [Rick–67].

It seems that Bogolyubov, and perhaps later Fomin, were not aware of von Neumann’s and Tarski 1929 articles when they wrote [Bogo–39] and [Fomi–50]. Moreover, Bogolyubov’s 1939 article had very little impact at the time (see [Anos–94], in particular bottom of Page 10). Indeed, neither [Bogo–39] nor [Fomi–50] appears in the lists of references of any of [Dixm–50], [HeRo–63], [Rick–67], [Gree–69], or [Wago–85].

Nevertheless, in his 1939 article, Bogolyubov shows that Result (1) above holds for every topological group which is strongly amenable, and the same proof shows that this extends to amenable topological groups. In § 2 of [Fomi–50], Fomin shows that the following classes of groups are strongly amenable: compact groups (his Theorem 3), groups containing a cocompact closed strongly amenable subgroup (Theorem 4), Abelian groups (Theorem 5), and solvable groups (Corollary 2). For Fomin’s work in general during the 50’s, see [Ale+–76].

Among non locally compact topological groups, several important examples of amenable groups have the much stronger property of extreme amenability, a notion that we discuss in Section 5.

Though we will not discuss it here, we note that the notion of amenability has been extended to many other objects than groups, including semi-groups, associative algebras, Banach algebras, operator algebras (nuclearity, exactness, injectivity), metric spaces, equivalence relations, group actions, foliations, and groupoids.

Plan of the article. In Section 2, we review two equivalent classical notions, amenability and the fixed point property, as well as strong amenability (for locally compact groups, the three notions are equivalent). Section 3 is about their hereditary properties. Section 4 contains three examples of amenable topological groups that are not locally
compact: the unitary group of a separable infinite dimensional Hilbert space, the symmetric group \( \text{Sym}(\mathbb{N}) \) of the positive integers, and the general linear group \( \text{GL}(V) \) of a vector space of countable infinite dimension over a finite field (with their Polish topologies). Section 5 provides more examples which have indeed a stronger property: they are extremely amenable. In Section 6, we discuss several definitions of ergodicity for actions on compact spaces; they agree for second countable locally compact groups, but not in general, as shown in Section 7 by an Example of Kolomogorov involving \( \text{Sym}(\mathbb{N}) \). The final Section 8 describes a characterization, due to Fomin, of ergodicity in terms of unitary representations.

It is convenient to agree that all topological spaces and groups appearing in this article are assumed to be Hausdorff.

2. Amenability, fixed point property, and strong amenability

Let \( X \) be a topological space. Let \( C^b(X) \) denote the Banach space of real-valued bounded continuous functions on \( X \), with the sup norm defined by \( \| f \| = \sup_{x \in X} |f(x)| \) for \( f \in C^b(X) \). When \( X \) is compact, every continuous function on \( X \) is bounded, and we rather write \( C(X) \) for \( C^b(X) \).

Let \( G \) be a topological group. For \( a \in G \) and \( f \in C^b(G) \), define \( af \) and \( fa \) by \( af(g) = f(a^{-1}g) \) and \( fa(g) = f(ga^{-1}) \) for all \( g \in G \). Observe that, for a given \( f \in C^b(G) \), the map \( G \to C^b(G) \), \( a \mapsto af \) need not be continuous (example with \( G = \mathbb{R} \) and \( f(t) = \sin(\pi t^2) \) for all \( t \in \mathbb{R} \)). Let \( C^b_{ru}(G) \) denote the subspace of \( C^b(G) \) of those functions \( f \) for which \( af \) depends continuously of \( a \), i.e. the space of bounded right-uniformly continuous functions on \( G \). This is a Banach space, indeed a closed subspace of the Banach space \( C^b(G) \).

[We use \( a^{-1} \) in the definition of \( af \), so that \( (a,f) \mapsto af \) defines a left-action of \( G \) on \( C^b(G) \) and \( C^b_{ru}(G) \); some authors use \( a \) rather than \( a^{-1} \).]

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1Here is a justification of the word “right”. The right-uniform structure on \( G \) is rightfully the uniform structure with entourages of the form \( U_V = \{ (g,h) \in G \times G \mid hg^{-1} \in V \} \), for some neighbourhood \( V \) of 1 in \( G \); this structure is invariant by right multiplications. Hence a function \( f : G \to \mathbb{R} \) is right-uniformly continuous if, for all \( \varepsilon > 0 \), there exists a neighbourhood \( V \) of 1 in \( G \) such that

\[
(g,h) \in U_V \implies |f(h) - f(g)| < \varepsilon,
\]

i.e.

\[
a \in V \implies |f(ah) - f(a)| < \varepsilon \ \forall \ a \in G,
\]

i.e.

\[
a \in V \implies \|af - f\| < \varepsilon.
\]

This is why we use “right” here, as for example in [Rick–67], even though, in the last inequality, \( a \) appears on the left of \( f \); other authors use “left” at the same place [Zimm–84, Page 136].
a^{-1}; this does not change the definition of $C^b_{ru}(G)$. Similarly, the space $C^b_{lu}(G)$ of bounded left-uniformly continuous functions on $G$, i.e. the space of those $f \in C^b(G)$ for which $f_a$ depends continuously on $a$, is a closed subspace of $C^b(G)$. We denote by $C^b_u(G)$ the intersection $C^b_{ru}(G) \cap C^b_{lu}(G)$.

Let $E$ be a linear subspace of $C^b(G)$ containing the constant functions. A mean on $E$ is a linear form $M$ on $E$ that is positive, i.e. $M(f) \geq 0$ whenever $f(g) \geq 0$ for all $g \in G$, and normalized, i.e. $M(1) = 1$, where 1 denotes both the number $1 \in \mathbb{R}$ and the corresponding constant function. Equivalently, a mean is a linear form on $E$ such that

$$\inf_{g \in G} f(g) \leq M(f) \leq \sup_{g \in G} f(g) \quad \forall f \in E.$$ 

Observe that a mean $M$ on $C^b(G)$ is bounded of norm 1. Assume moreover that $E$ is such that $a f$ is in $E$ for all $f \in E$ and $a \in G$. A mean $M$ on $E$ is left-invariant if $M(a f) = M(f)$ for all $f \in E$ and $a \in G$. Right-invariant means are defined similarly. Assume furthermore that $E$ is such that $f a \in E$ for all $f \in E$ and $a \in G$; a mean on $E$ is bi-invariant if it is both left-invariant and right-invariant.

**Proposition 2.1.** For a topological group $G$, the following properties are equivalent:

(i) there exists a left-invariant mean on $C^b_{ru}(G)$,
(ii) there exists a right-invariant mean on $C^b_{lu}(G)$,
and they imply

(iii) there exists a bi-invariant mean on $C^b_u(G)$.

**Proof.** For a function $f$ defined on $G$, denote by $f^\lor$ the function $g \mapsto g^{-1}$. Observe that $(f^\lor)^\lor = f$, and that, for $f \in C^b(G)$, we have $f \in C^b_{ru}(G)$ if and only if $f^\lor \in C^b_{lu}(G)$.

If there exists a left-invariant mean $M$ on $C^b_{ru}(G)$, then $f \mapsto M(f^\lor)$ is a right-invariant mean on $C^b_{lu}(G)$. Hence (i) implies (ii). Similarly, (ii) implies (i).

Assume that there exist a left-invariant mean $M_1$ on $C^b_{ru}(G)$ and a right-invariant mean $M_1$ on $C^b_{lu}(G)$. For $f \in C^b_{lu}(G)$, define first $F_f : G \to \mathbb{R}$ by $F_f(a) = M_1(f_a)$; then $F_f \in C^b_u(G)$; define now a linear form $M$ on $C^b_u(G)$ by $M(f) = M_1(F_f)$. Then $M$ is a bi-invariant mean on $C^b_u(G)$. This shows that (i) or/and (ii) implies (iii). 

We do not know if (iii) implies (i) and (ii). It does for locally compact groups: see “References for the proof” after Proposition 2.10.
Definition 2.2. A topological group $G$ is **amenable** if it has Properties (i) and (ii) of the previous proposition.

We write LCTVS as a shorthand for “locally convex topological vector space”, here on the field of real numbers. Let $X$ be a convex subspace of an LCTVS; a transformation $g$ of $X$ is **affine** if $g(\lambda x + (1 - \lambda)y) = \lambda g(x) + (1 - \lambda)g(y)$ for all $x,y \in X$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$. An action of a topological group $G$ on $X$ is **continuous** if the corresponding map $G \times X \to X$ is continuous. Group actions below are **actions from the left**, unless explicitly written otherwise.

Definition 2.3. A topological group $G$ has the **fixed point property (FP)** if every continuous affine action of $G$ on a non-empty compact convex set in an LCTVS has a fixed point.

Definition 2.3 plays a fundamental role in a work of Furstenberg [Furs–63]. It is straightforward to check that compact groups have Property (FP). For abelian groups, the following fixed-point theorem appeared first in [Mark–36] and [Kaku–38]; a convenient reference is [Edwa–65, Theorem 3.2.1].

**Theorem 2.4** (Markov-Kakutani theorem). Every abelian group has Property (FP).

The result carries over from abelian groups to solvable groups by Proposition 3.1(3) and from groups to topological groups by Proposition 3.1(4). Therefore, we have also:

**Corollary 2.5.** Every solvable topological group has Property (FP).

**Proposition 2.6.** For a topological group $G$, the following three properties are equivalent:

(i) $G$ has Property (FP);

(ii) $G$ is amenable;

(iii) for every non-empty compact space $X$ and every continuous action of $G$ on $X$, there exists a $G$-invariant probability measure on $X$.

Before the proof, we recall two standard facts from functional analysis.

**Unit balls in duals of Banach spaces.** For a real Banach space $E$, we denote by $E^*$ its Banach space dual, and $E^*_1$ the unit ball in $E^*$. On $E^*$ and $E^*_1$, we consider also the weak-$*$-topology, for which $E^*$ is a LCTVS, and $E^*_1$ a compact subspace.
Barycentres. Let $C$ be a non-empty compact convex subspace of an LCTVS $E$, and let $\mu$ be a probability measure on $C$. Then there exists a unique point $b_\mu \in E$ such that $f(b_\mu) = \int_C f(x) d\mu(x)$ for every continuous linear form $f$ on $E$; moreover $b_\mu \in C$. The point $b_\mu$ is called the barycentre of $\mu$, or the resultant of $\mu$, and $\mu$ represents $b_\mu$. Recall the following formulation of the Krein-Milman theorem: every point of $C$ is the barycentre of a probability measure on $C$ which is supported by the closure of the set of extreme points of $C$; see [Phel–66, Section 1].

Proof. $[i] \Rightarrow [ii]$ The set $\text{Mean}(G)$ of all means on $C^{b}(\mu)^{r}$ is a subspace of the unit ball $C^{b}(\mu)^{r}$, and is closed for the weak-∗-topology, so that $\text{Mean}(G)$ is a compact convex set. Moreover the natural action of $G$ on $\text{Mean}(G)$ is affine and continuous (in the sense recalled just before Definition 2.3).

If $G$ has Property [i] there exists a $G$-invariant probability measure $\mu$ on $\text{Mean}(G)$. Such a measure has a barycentre $M \in \text{Mean}(G)$. Since $M$ is $G$-invariant (by unicity of the barycentre), $G$ has Property [ii].

[It is even more straightforward to check that [i] implies [iii] because if $G$ acts on $X$, then $G$ acts on the space $\mathcal{P}(X)$ of probability measures on $X$, that is naturally a compact convex set.]

$[ii] \Rightarrow [iii]$ We reformulate the argument of [Bogo–39], written there for $\mathcal{C}^{b}(G)$; the same argument applies equally well to $\mathcal{C}^{b}(\mu)^{r}$, as can be read for example in [Rick–67].

Let $\nu$ be a probability measure on $X$. For $f \in \mathcal{C}(X)$, define a function

$$F_f : G \rightarrow \mathbb{R}, \quad g \mapsto \int_X f(gx) d\nu(x).$$

Then $F_f$ is bounded and right-uniformly continuous on $G$. Let moreover $a \in G$. Then

$$F_{(a^{-1})}(g) = \int_X f(a^{-1}gx) d\nu(x) = F_{f(a^{-1}g)} = (a(F_f))(g)$$

for all $g \in G$.

Let $M$ be left-invariant mean on $\mathcal{C}^{b}(\mu)^{r}$. Define a linear form

$$\mu : \mathcal{C}(X) \rightarrow \mathbb{R}, \quad f \mapsto M(F_f).$$

Then $\mu$ is a normalized positive linear form on $\mathcal{C}(X)$, i.e. $\mu$ can be seen as a probability measure on $X$. Since $M$ is left-invariant, we have

$$\mu(af) = M(F_{(a^{-1})}) = M((a(F_f)) = M(F_f) = \mu(f)$$

for all $f \in \mathcal{C}(X)$ and $a \in G$, i.e. the measure $\mu$ is $G$-invariant.
Let $C$ be a compact convex set on which $G$ acts continuously, by affine transformations. If $G$ has Property (iii), there exists a $G$-invariant probability measure $\tilde{\mu}$ on $C$. The barycentre $\mu \in C$ of $\tilde{\mu}$ is fixed by $G$.

Remark 2.7. (1) The equivalence of (i) and (ii) appears in many places, for example in [Eyma–75], who quotes [Day–61] (as much as we know the first place where the arguments are found, but applied to abstract groups only) and [Rick–67] (as much as we know the first place where the arguments are applied to topological groups); see more precisely [Rick–67, Theorem 4.2].

(2) For a direct proof of (ii) $\Rightarrow$ (i) see also [BeHV–08, Theorem G.1.7].

(3) There is a natural embedding $G \hookrightarrow C^b_{ru}(G)^*_1$, $g \mapsto (F \mapsto F(g))$, which is continuous for the weak-$^*$-topology on the range (recall that the subscript 1 stands for “unit ball”). By definition, the universal equivariant compactification $\gamma_u G$ of $G$ is the closure of the image of this embedding, and the natural action of $G$ on $\gamma_u G$ is continuous. The properties of Proposition 2.6 are moreover equivalent to

(iv) the natural action of $G$ on its universal equivariant compactification $\gamma_u G$ has an invariant probability measure,

as it is shown in [BaBo–11].

(4) Finally, Properties (i) to (iv) are equivalent to

(v) every continuous action of $G$ on the Hilbert cube has an invariant probability measure,

as has been established in [AMP–11], and previously in [BoFe–07] for countable groups. For a countable group $\Gamma$, amenability is also equivalent to the following property [GiHa–97]:

(vi) every action by homeomorphisms of $\Gamma$ on the Cantor middle-third space has an invariant probability measure.

(5) In [Fomi–50, Section 2.3], Fomin proves Property (iii) of Proposition 2.6 for compact groups, abelian groups, solvable groups, and topological groups with an abelian cocompact closed normal subgroup; his proof for abelian groups does not quote the Markov-Kakutani fixed point theorem.

The end of this section is devoted to a notion stronger than amenability. It was introduced by Bogolyubov; we call it strong amenability (Definition 2.8). Historically, strong amenability came first and
amenability later; amenability, equivalent to Property (FP) of Definition 2.3, has proved to be more convenient than strong amenability.

The shortest expression, i.e. “amenable” as in Definition 2.2, is rightfully used for the notion equivalent with Property (FP). However, this does not agree with the definition of Rickert: the “left amenable” groups of 

\[ \text{Rick–67} \]

are the groups that have the property of our Definition 2.8.

**Definition 2.8.** A topological group \( G \) is **strongly amenable** if there exists a left-invariant mean on \( \mathcal{C}^b(G) \).

**Proposition 2.9.** A strongly amenable topological group is amenable.

**Proof.** If \( G \) is a topological group, the space \( \mathcal{C}^b_{lu}(G) \) is a \( G \)-invariant subspace of \( \mathcal{C}^b(G) \), and the restriction to this subspace of a \( G \)-invariant mean on \( \mathcal{C}^b(G) \) is a \( G \)-invariant mean on \( \mathcal{C}^b_{lu}(G) \).

Propositions 2.10 and 4.2 establish that the converse holds for locally compact groups, but not in general.

**Proposition 2.10.** Let \( G \) be a locally compact group. Then \( G \) is strongly amenable if and only if \( G \) is amenable.

**Reference for the proof.** See [Gree–69, Theorem 2.2.1 Page 26, and Page 29]. Greenleaf shows there that the following properties of a locally compact group \( G \) are equivalent: existence of a left-invariant mean on each of

(i) the space \( L^\infty(G) \) of essentially bounded Borel measurable functions (modulo equality in complements of locally null sets),

(ii) the space \( \mathcal{C}^b(G) \),

(iii) the space \( \mathcal{C}^b_{lu}(G) \),

(iv) the space \( \mathcal{C}^b_{u}(G) \).

Moreover, Properties (i) to (iv) are equivalent to

(*) there exists a bi-invariant mean on \( E \),

for \( E \) one of the spaces in (i) to (iv) above.

Indeed, a substantial part of the early theory of amenability on locally compact groups is to show that infinitely many definitions are equivalent with each other.

**Example 2.11.** Compact groups are strongly amenable.

Indeed, the normalized Haar measure on a compact group \( G \) provides a mean on \( \mathcal{C}^b(G) \) that is both left- and right-invariant.

**Proposition 2.12.** For a topological group \( G \), the following two properties are equivalent:
(i) $G$ is strongly amenable,
(ii) for all $n \geq 1$, $f^{(1)}, \ldots, f^{(n)} \in C^b(G)$, $a_1, \ldots, a_n \in G$, and $t \in \mathbb{R}$, such that $f^{(1)} - a_1 f^{(1)} + \cdots + f^{(n)} - a_n f^{(n)} \geq t$, we have $t \leq 0$.

On the proof. The non-trivial implication is (ii) $\Rightarrow$ (i). It is a consequence of the Hahn-Banach theorem; see [Dixm–50, Théorème 1].

This has the following consequence, for which we refer again to Dixmier [Dixm–50, Théorème 2(α)]:

**Proposition 2.13.** Abelian topological groups are strongly amenable.

By Proposition 3.3, we have also:

**Corollary 2.14.** Solvable topological groups are strongly amenable.

Compare with Corollary 2.5.

3. Hereditary properties

In most of this section, we address topological groups in general. However, groups are assumed to be locally compact in Proposition 3.4, and metrizable in Corollary 3.6.

A topological group $G$ is the directed union of a family $(H_\alpha)_{\alpha \in A}$ of closed subgroups of $G$ if the following conditions hold: (1) $G = \bigcup_{\alpha \in A} H_\alpha$; (2) for every $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $H_\alpha \cup H_\beta \subset H_\gamma$; (3) $G$ has the topology of the inductive limit of the $H_\alpha$ ’s. [Note that the index set $A$ is a directed set for the preorder defined by $\alpha \leq \beta$ if $H_\alpha \subset H_\beta$.]

**Proposition 3.1** (on amenability). Let $G$ be a topological group.

1. If $G$ is amenable, then every open subgroup of $G$ is amenable.
2. If $G$ is a directed union of a family $(H_\alpha)_{\alpha \in A}$ of closed subgroups, and if each $H_\alpha$ is amenable, then $G$ is amenable.
3. If $G$ has an amenable closed normal subgroup $N$ such that the quotient $G/N$ is amenable, then $G$ is amenable.
4. Let $H$ be a topological group such that there exists a continuous homomorphism $H \longrightarrow G$ with dense image; if $H$ is amenable, then so is $G$.
5. If $H$ is a dense subgroup of $G$, then $G$ is amenable if and only if $H$, endowed with the induced topology, is amenable.

On the proof. The proof can use any of the three properties of Proposition 2.6 and is in any case rather straightforward; see [Rick–67, Section 4].

As a sample, let us check Claim 5. Since right-uniformly continuous functions can be extended from $H$ to $G$, the restriction of functions
provides an isomorphism of Banach spaces $C^b_{ru}(G) \rightarrow C^b_{ru}(H)$, by which these spaces can be identified; let us denote it (them) by $C^b_{ru}$.

On the one hand, a left-$G$-invariant mean on $C^b_{ru}$ is obviously a left-$H$-invariant mean on $C^b_{ru}$. On the other hand, since the action of $G$ on $C^b_{ru}$ is continuous, a left-$H$-invariant mean on this space is also left-$G$-invariant. Claim (5) follows. □

**Remark 3.2.** (1) In the first claim of Proposition 3.1, “open” cannot be replaced by “closed” (Corollary 4.4). See however Propositions 3.4 and 3.5.

(2) Let $G$ be a group with two Hausdorff topologies $\mathcal{T}_s, \mathcal{T}_w$ such that $G_s := (G, \mathcal{T}_s)$ and $G_w := (G, \mathcal{T}_w)$ are topological groups, and $\mathcal{T}_s$ stronger than $\mathcal{T}_w$. If $G_s$ is amenable, then $G_w$ is amenable, as it follows from Claim (4) applied to the continuous identity homomorphism $\text{id} : G_s \rightarrow G_w$.

Suppose for example that $\mathcal{T}_s$ is the discrete topology and that $G_s$ is amenable; this is the case if $G$ is abelian, or more generally solvable, by Theorem 2.4. Then $G_w$ is amenable for every topology $\mathcal{T}_w$ making $G$ a topological group.

(3) The proof of Claim (5) cannot be adapted to Proposition 3.3. Indeed the analogue of Claim (5) does not hold for strong amenability (Remark 4.3(1)).

(4) The following result of Calvin C. Moore (1979) is reminiscent of Claim (5), but the proof uses completely different notions. Let $G$ be the group of real points of an $\mathbb{R}$-algebraic group, $H$ an amenable group, and $\varphi : H \rightarrow G$ a continuous homomorphism; if $H$ is amenable and $\varphi(H)$ Zariski-dense in $G$, then $G$ is amenable. We refer to [Zimm–84, Theorem 4.1.15].

**Proposition 3.3** (on strong amenability). Let $G$ be a topological group.

(1) If $G$ is strongly amenable, then every open subgroup of $G$ is strongly amenable.

(2) If $G$ is a directed union of a family $(H_\alpha)_{\alpha \in A}$ of closed subgroups, and if each $H_\alpha$ is strongly amenable, then $G$ is strongly amenable.

(3) If $G$ has a strongly amenable closed normal subgroup $N$ such that the quotient $G/N$ is strongly amenable, then $G$ is strongly amenable.

(4) Let $H$ be a topological group such that there exist a continuous epimorphism $H \twoheadrightarrow G$; if $H$ is strongly amenable, then so is $G$.
Reference for the proof. As for Proposition 3.1, proofs are straightforward. In [Rick–67], see respectively Theorems 3.2, 2.4, 2.6, and 2.2. There are related results in the older article by Dixmier [Dixm–50]. □

Amenability of closed subgroups of amenable groups is more subtle. Let us first recall the following result in the classical setting of locally compact groups.

**Proposition 3.4.** Let $G$ be a locally compact group and $H$ a closed subgroup. If $G$ is amenable, then so is $H$.

*On the proof.* We mention here two proofs of this statement.

The proof of [Zimm–84, Proposition 4.2.20] uses induction from $H$ to $G$ for actions on compact convex sets. Since Haar measure is an essential ingredient of induction, this cannot be used for groups that are not locally compact.

The proof of [Gree–69, Theorem 2.3.2] has three steps. We denote by $H \setminus G$ the space of $H$-cosets of the form $Hg$; we could use $G/H$ instead, but this would impose on us right-invariant means on $C^b(\cdot)$, rather than left-invariant means.

For the first step, $G$ is assumed to be second countable. Then there exists a Borel transversal $T$ for $H \setminus G$, so that the multiplication map $H \times T \to G$, $(h, t) \mapsto ht$, is a Borel isomorphism. This can be used first to extend functions on $H$ to functions on $G$, and then to show that amenability is inherited from $G$ to $H$ (we refer to Greenleaf’s book for details). For the second step, $G$ is assumed to be $\sigma$-compact. Then $G$ has a compact normal subgroup $N$ such that $G/N$ is second countable (Kakutani-Kodaira theorem [KaKo–44]). The first step and Proposition 3.1(3) imply again that amenability is inherited from $G$ to $H$. The final step makes use of the fact that every locally compact group contains an open $\sigma$-compact subgroup.

In Rickert’s article, the proposition is proved with additional hypothesis only, essentially that the quotient of $G$ by its connected component is a compact group [Rick–67, Section 7]. □

Corollary 4.4 below shows that Proposition 3.4 does not carry over to topological groups. In particular this proposition is unlikely to have a completely straightforward proof.

However, with appropriate extra hypothesis, strong amenability is inherited by closed subgroups. This is shown by the following proposition, which is Theorem 3.4 in [Rick–67].

**Proposition 3.5.** Let $G$ be a topological group, and $H$ a closed subgroup. Assume that $H \setminus G$ is paracompact and the fibration $\pi : G \to H \setminus G$ is locally trivial.
If $G$ is strongly amenable, then so is $H$.

Proof. Since $G \rightarrow H \setminus G$ is locally trivial, there exist an open cover $(U_i)_{i \in I}$ of $H \setminus G$ and a family of continuous sections $(\sigma_i : U_i \rightarrow G)_{i \in I}$ such that $\pi \sigma_i(x) = x$ for all $i \in I$ and $x \in U_i$. Since $H \setminus G$ is paracompact, there is a partition of unit $(\varphi_i)_{i \in I}$ subordinate do $(U_i)_{i \in I}$. For $f \in C^b(H)$, define $F_f : G \rightarrow \mathbb{R}$ by

$$F_f(g) = \sum_{i \in I} \varphi_i(\pi(g)) \ f (g(\sigma_i(g))^{-1}).$$

The function $F_f$ is well-defined, continuous because the family of the supports of the $\varphi_i$'s is locally finite, and obviously bounded. The assignment $f \mapsto F_f$ is a linear map from $C^b(H)$ to $C^b(G)$, that respects positivity and constant functions. Moreover, for $f \in C^b(H)$ and $a \in H$, we have

$$a(F_f) = F_{af}.$$ 

Indeed

$$(a(F_f))(g) = \sum_{i \in I} \varphi_i(\pi(a^{-1}g)) \ f (a^{-1}g(\sigma_i(a^{-1}g))^{-1})$$

$$= \sum_{i \in I} \varphi_i(\pi(g)) \ f (a^{-1}g(\sigma_i(g))^{-1}) = (F_{af})(g)$$

for all $g \in G$.

Suppose that there exists a left-$G$-invariant mean $M$ on $C^b(G)$. The assignment $m : f \mapsto M(F_f)$ is obviously a mean on $C^b(H)$. Since $M$ is invariant, we have

$$m(af) = M(F_{af}) = M(F_f) = m(f)$$

for all $f \in C^b(H)$ and $a \in H$, i.e. $m$ is a left-$H$-invariant mean on $C^b(H)$.

\[\square\]

Corollary 3.6. Let $G$ be a metrizable topological group. If $G$ contains a non-amenable discrete subgroup $\Gamma$, then $G$ is not strongly amenable.

Proof. Since $G$ is metrizable, so is $\Gamma \setminus G$, hence $\Gamma \setminus G$ is paracompact.

Let $p : G \rightarrow \Gamma \setminus G$ denote the canonical projection. Let $V$ be a neighbourhood of 1 in $G$ such that $V^{-1} = V$ and $\Gamma \cap V^2 = \{1\}$. For every $g \in G$, the open subsets $\gamma V g$, for $\gamma \in \Gamma$, are pairwise disjoint; it follows that $p^{-1}(p(\gamma V g))$ is homeomorphic to the product $\Gamma \times V g$, and therefore that the fibration $p$ is locally trivial.

Hence $G$ is not strongly amenable by Proposition 3.5. \[\square\]

Concerning the hypothesis of Corollary 3.6, let us recall a theorem due to Birkhoff and Kakutani; for a topological group $G$, the three following conditions are equivalent:
(i) $G$ is metrizable as a topological space;
(ii) the topology of $G$ can be defined by a left-invariant metric;
(iii) $G$ has a countable basis of neighbourhoods of 1.

See for example [BTG5-10, § 3, no 1, Pages IX.23-24], or [CoHa].

**Example 3.7.** In Proposition 3.5 the hypothesis of local triviality cannot be deleted, as the following example, from [Karu–58], shows.

Consider the circle group $T = \mathbb{R}/\mathbb{Z}$, its subgroup $C$ of order 2, the compact group $G = T^\mathbb{N}$, and the closed subgroup $H = C^\mathbb{N}$ of $G$. Then $G$ is locally connected, as a product of (locally) connected groups, while $H$ and $H \times (H \setminus G)$ are not locally connected (indeed $H$ is totally disconnected). It follows that $G$ and $H \times (H \setminus G)$ are not locally homeomorphic, and in particular that the projection $G \to H \setminus G$ is not locally trivial.

4. **Examples**

Propositions 4.2, 4.5, and 4.6 provide examples of topological groups that are amenable and are not strongly amenable. By necessity, these groups are not locally compact (Proposition 2.10).

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. We denote by $U(\mathcal{H})_{str}$ its unitary group, endowed with the strong topology (equivalently: with the weak topology); as is well-known, this is a topological group, and it is not locally compact. For the strong and weak topologies on sets of operators, and their properties, see e.g. [Dixm–57]. If $\mathcal{H}$ is separable, $U(\mathcal{H})_{str}$ is a Polish group, i.e. it is separable and its topology can be defined by a complete metric. As a curiosity, we note that, for a separable Hilbert space $\mathcal{H}$, the strong topology is the unique topology for which $U(\mathcal{H})$ is a Polish group [AtKa–12].

**Lemma 4.1.** Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space, and $U(\mathcal{H})_{str}$ its unitary group.

Every countable group $\Gamma$ is isomorphic to a discrete subgroup of $U(\mathcal{H})_{str}$.

**Proof.** Suppose first that $\Gamma$ is an infinite group. We can identify $\mathcal{H}$ with the Hilbert space $\ell^2(\Gamma)$ of complex-valued functions $\xi$ on $\Gamma$ such that $\sum_{\gamma \in \Gamma} |\xi(\gamma)|^2 < \infty$. Let

$$\lambda : \Gamma \to U(\ell^2(\Gamma))_{str}$$

be the left-regular representation of $\Gamma$, defined by $(\lambda(\gamma)\xi)(\gamma') = \xi(\gamma^{-1}\gamma')$ for all $\gamma, \gamma' \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. 
For $\gamma \in \Gamma$, let $\delta_{\gamma} \in \ell^2(\Gamma)$ denote the unit vector defined by $\delta_{\gamma}(\gamma) = 1$ and $\delta_{\gamma}(\gamma') = 0$ if $\gamma' \neq \gamma$; observe that $\lambda(\gamma)\delta_1 = \delta_{\gamma}$. Set

$$U_\gamma = \{ g \in U(\ell^2(\Gamma))_{\text{str}} \mid \| (g - \lambda(\gamma))\delta_1 \|^2 < 1 \}.$$ 

On the one hand, $U_\gamma$ is open in $U(\ell^2(\Gamma))_{\text{str}}$ and $\lambda(\gamma) \in U_\gamma$. On the other hand, since $\| \delta_{\gamma} - \delta_{\gamma'} \|^2 = 2$ for $\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'$, the $U_\gamma$'s are pairwise disjoint. Hence $\lambda(\Gamma)$ is a discrete subgroup of $U(\ell^2(\Gamma))_{\text{str}}$.

If $\Gamma$ is a finite group, we can apply the previous argument to the direct product $\Gamma \times \mathbb{Z}$. □

**Note.** More generally, for every locally compact group $G$, the left-regular representation $G \to U(L^2(G))_{\text{str}}$ is both a continuous homomorphism and a homeomorphism of $G$ onto a closed subgroup of $U(L^2(G))_{\text{str}}$ [BeHV–08, Exercise G.6.4].

In general, $L^2(G)$ need not be separable. It is when $G$ is separable.

**Proposition 4.2 ([Harp–73]).** Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. The topological group $U(\mathcal{H})_{\text{str}}$ is amenable and is not strongly amenable.

**Remark.** Proposition 5.3 establishes that $U(\mathcal{H})_{\text{str}}$ has a property much stronger than amenability.

**Proof.** Let $(e_n)_{n \geq 1}$ be an orthonormal basis of $\mathcal{H}$. For each $n \geq 1$, we identify the compact Lie group $U(n)_{\text{str}}$ to the subgroup of $U(\mathcal{H})_{\text{str}}$ of those unitary operators leaving invariant the linear span $V_n$ of $\{e_1, \ldots, e_n\}$ and coinciding with the identity on the orthogonal complement $V_n^\perp$. Let $U(\infty)_{\text{str}}$ denote the union of the compact groups in the nested sequence

$$U(1) \subset \cdots \subset U(n) \subset U(n+1) \subset \cdots,$$

with the inductive limit topology. The group $U(\infty)_{\text{str}}$ is dense in $U(\mathcal{H})_{\text{str}}$.

The later claim follows from Kaplansky’s density theorem, essentially the version of [Pede–79 Theorem 2.3.3]. Pedersen formulates his Theorem 2.3.3 in terms of a $C^*$-subalgebra $A$ of $L(\mathcal{H})$. More generally, his proof applies without change to an involutive subalgebra $A$ of $L(\mathcal{H})$ such that $\exp(x) \in A$ for all $x \in A$. In our case, $A$ is the algebra spanned by the identity operator and $\bigcup_{n \geq 1} M_n$, where $M_n$ stands for the the finite-dimensional algebra $M_n = \{ x \in L(\mathcal{H}) \mid x(V_n) \subset V_n$ and $x(V_n^\perp) = \{0\}\}$.

For the inductive limit topology, the group $U(\infty)_{\text{str}}$ is a topological group. It is not locally compact; indeed, it is not a Baire group, because $U(n)$ has empty interior in $U(\infty)_{\text{str}}$ for all $n$. (For other topological
properties of $U(\infty)$, see e.g. [Hans–71, Theorem 4.8]). Since the compact groups $U(n)$ are amenable, so is $U(\infty)$ by Proposition 3.1(3). It follows from Proposition 3.1(5) that $U(\mathcal{H}_{str})$ is amenable.

By Lemma 4.1, the group $U(\mathcal{H}_{str})$ contains non-amenable discrete subgroups. It follows from Corollary 3.6 that $U(\mathcal{H}_{str})$ is not strongly amenable.

The proof of the proposition is complete. Let us however reproduce the argument of [Harp–73], that is a different proof that $U(\mathcal{H}_{str})$ is not strongly amenable.

Suppose ab absurdo that there exists a left-invariant mean $M$ on the space $C^b(U(\mathcal{H}_{str}))$. Let $\xi$ be a unit vector in $H$. For every bounded operator $S$ on $H$, the function $f_{S,\xi} : \{ g \mapsto \text{Re}(\langle g^{-1} S g \xi \mid \xi \rangle) \}$ is bounded and continuous. Let $L(H)$ denote the algebra of all bounded operators on $H$. Observe that, for $h \in U(\mathcal{H}_{str})$, we have $h f_{S,\xi}(g) = \text{Re}(\langle g^{-1} h S h^{-1} g \xi \mid \xi \rangle)$ for all $g \in U(\mathcal{H}_{str})$, i.e. $h f_{S,\xi} = f_{h S h^{-1},\xi}$.

Consider the linear form

$$\tau_\xi : \begin{cases} L(\mathcal{H}) \longrightarrow \mathbb{R} \\ S \longrightarrow M(f_{S,\xi}) \end{cases}.$$ 

Since $M$ is left-invariant, we have for all $S \in L(\mathcal{H})$ and $h \in U(\mathcal{H}_{str})$,

$$\tau_\xi(h S h^{-1}) = M(f_{h S h^{-1},\xi}) = M(h f_{S,\xi}) = M(f_{S,\xi}) = \tau_\xi(S)$$

$$\tau_\xi(h S h^{-1}) = \tau_\xi(S),$$

and therefore also $\tau_\xi(S h) = \tau_\xi(h S)$.

Every operator in $L(\mathcal{H}_{str})$ is a linear combination of unitaries. Hence $\tau_\xi(ST) = \tau_\xi(TS)$ for all $S, T \in L(\mathcal{H})$. Since the identity operator is a sum of two commutators, i.e. since $\text{id}_H$ is of the form $S_1 T_1 - T_1 S_1 + S_2 T_2 - T_2 S_2$ (see e.g. Problem 186 in [Halm–67]), we have $\tau_\xi(\text{id}_H) = 0$. But this is preposterous, because $\tau_\xi(\text{id}_H) = M(1) = 1$. Hence $U(\mathcal{H}_{str})$ is not strongly amenable.  

Let $A$ be a $C^*$-algebra with unit. Every $x \in A$ is a linear combination of four unitaries. Indeed, since $x = \frac{1}{2}(x + x^*) + \frac{i}{2i}(ix - ix^*)$, it is enough to check that a self-adjoint element of norm at most 1 is a linear combination of two unitaries. If $x^* = x$ and $\|x\| \leq 1$, then $u = x + i\sqrt{1 - x^2}$ and $u^* = x - i\sqrt{1 - x^2}$ are unitary, and $x = \frac{1}{2}(u + u^*)$. 

\[\square\]
Remark 4.3. (1) Let the notation be as in the proof above. Since the group \( U(\infty) \) is strongly amenable (by Claim (2) of Proposition 3.3) and dense in \( U(\mathcal{H})_{\text{str}} \), Proposition 4.2 justifies Remark 3.2(3).

(2) Proposition 4.2 has the following offspring. Let \( \mathcal{M} \) be a von Neumann algebra, realized as a weakly closed \(*\)-subalgebra of \( L(\mathcal{H}) \), for some separable Hilbert space \( \mathcal{H} \). Let \( U(\mathcal{M})_{\text{str}} \) be its unitary group, with the strong topology. Then \( U(\mathcal{M})_{\text{str}} \) is amenable if and only if \( \mathcal{M} \) is injective [Harp–79]. For a \( C^* \)-algebra \( A \), there is a similar characterization of nuclearity of \( A \) in terms of amenability of the unitary group \( U(A) \) of \( A \), with the norm topology [Pate–92].

(3) Consider the Banach algebra \( L(\mathcal{H}) \) of all bounded operators on \( \mathcal{H} \), with the usual operator norm. Let \( GL(\mathcal{H})_{\text{norm}} \) be its general linear group, for its topology as an open subset of \( L(\mathcal{H}) \). We denote by \( U(\mathcal{H})_{\text{norm}} \) the unitary group of \( \mathcal{H} \), with the topology induced by the operator norm; it is a closed subgroup of \( GL(\mathcal{H})_{\text{norm}} \). The groups \( U(\mathcal{H})_{\text{norm}} \) and \( GL(\mathcal{H})_{\text{norm}} \) are not amenable. Also, the group \( GL(p)_{\text{norm}} \) is not amenable, for \( p \) with \( 1 \leq p < \infty \). For this (and more), see [Harp–73] and [Pest–06, Examples 3.6.13-15].

From the proof of Proposition 4.2, let us extract the following observation. It appeared in [Harp–82, Page 489].

Corollary 4.4. A closed subgroup of an amenable topological group need not be amenable.

Let \( \text{Sym}(\mathbb{N}) \) denote the full symmetric group of the positive integers, with its standard Polish topology (any infinite countable set would do instead of \( \mathbb{N} \)). As a curiosity similar to that noted just before Lemma 4.1, note that \( \text{Sym}(\mathbb{N}) \) has a unique topology making it a Polish group [Gaug–67, Kall–79], indeed a unique topology making it a separable Hausdorff group [KeRo–07, Theorem 1.11]. We denote by \( \text{Sym}_f(\mathbb{N}) \) the subgroup of \( \text{Sym}(\mathbb{N}) \) of permutations with finite support; it is a locally finite group, dense in \( \text{Sym}(\mathbb{N}) \).

As in Proposition 4.2, we have:

Proposition 4.5. The topological group \( \text{Sym}(\mathbb{N}) \) is amenable and is not strongly amenable.

Proof. The proof is analogous to that of Proposition 4.2. On the one hand, since \( \text{Sym}(\mathbb{N}) \) contains a dense subgroup that is locally finite and therefore amenable, \( \text{Sym}(\mathbb{N}) \) is amenable by Proposition 3.1. On the other hand, \( \mathbb{N} \) can be identified (as a set) with a non-amenable countable group \( \Gamma \), and the metrizable group \( \text{Sym}(\Gamma) \) contains \( \Gamma \) as a
discrete closed subgroup, so that \( \text{Sym}(N) \simeq \text{Sym}(\Gamma) \) is not strongly amenable, by Corollary 3.6 \( \square \)

Let \( V \) be a vector space over a finite field \( \text{GF}(q) \). Assume that the dimension of \( V \) is infinite and countable. Observe that \( V \) is a countable infinite set, and that the general linear group \( \text{GL}(V) \) is a subgroup of \( \text{Sym}(V) \simeq \text{Sym}(N) \).

Moreover, each of the equations

\[
\begin{align*}
g(0) &= 0, \text{ with } 0 \text{ the origin of } V, \\
g(\lambda v) &= \lambda g(v), \text{ with } \lambda \in \text{GF}(q)^\times \text{ and } v \in V, \\
g(v + w) &= g(v) + g(w), \text{ with } v, w \in V,
\end{align*}
\]

defines a closed subset of \( \text{Sym}(V) \). Hence \( \text{GL}(V) \) is a closed subgroup of \( \text{Sym}(V) \); in particular \( \text{GL}(V) \) itself is a Polish group.

**Proposition 4.6.** The topological group \( \text{GL}(V) \) is amenable and is not strongly amenable.

**Proof.** This is one more variation on the same proof as for Propositions 4.2 and 4.5. Let \( (e_n)_{n \in \mathbb{N}} \) be a basis of \( V \). For every \( n \in \mathbb{N} \), denote by \( V_n \) the linear span of \( \{e_1, e_2, \ldots, e_n\} \), and by \( \text{GL}_n \) the subgroup of \( \text{GL}(V) \) consisting of those elements \( g \) such that \( g(V_n) = V_n \) and \( g(e_k) = e_k \) for all \( k > n \). We have a nested sequence

\[
\text{GL}_1 \subset \cdots \subset \text{GL}_n \subset \cdots \subset \text{GL}_\infty := \bigcup_{n=1}^\infty \text{GL}_n
\]

de of which the union \( \text{GL}_\infty \) is locally finite, in particular amenable, and dense in \( \text{GL}(V) \). Hence \( \text{GL}(V) \) is amenable.

The space \( V \) has a basis \( (e_\gamma)_{\gamma \in F} \) indexed by a non-abelian free group \( F \). Each \( \gamma_0 \in F \) can be viewed as an element of \( \text{GL}(V) \) mapping \( e_\gamma \) to \( e_{\gamma_0 \gamma} \) for all \( \gamma \in F \). This shows that \( \text{GL}(V) \) contains a discrete closed subgroup isomorphic to \( F \), and in particular that \( \text{GL}(V) \) is not strongly amenable. \( \square \)

**Example 4.7.** Let \( k \) be a commutative ring, with unit. Denote by \( \mathcal{J}(k) \) the substitution group of formal power series over \( k \), with elements of the form \( f(x) = x + \sum_{i \geq 2} a_i x^i \), where \( a_i \in k \), and with substitution for the group law. For what follows, and much more, about groups of this kind, see [Babe–13].

For each \( n \geq 1 \), denote by \( \mathcal{J}^{n+1}(k) \) the normal subgroup of \( \mathcal{J}(k) \) defined by the equations \( a_2 = a_3 = \cdots = a_n = 0 \) and by \( \mathcal{J}_n(k) \) the quotient \( \mathcal{J}(k)/\mathcal{J}^{n+1}(k) \). There is a natural bijection between \( \mathcal{J}_n(k) \) and \( k^n \). When \( k \) is a topological ring, we use this bijection to define a topology on \( \mathcal{J}_n(k) \); it is a group topology. Then \( \mathcal{J}(k) \) is also a
topological group, with the topology of the inverse limit \( \lim_{\leftarrow} J_n(k) \). This topology is interesting even if the ring \( k \) is discrete; note that the group \( J(k) \) is profinite when the ring \( k \) is finite.

It is known that, for every topological commutative ring \( k \), the group \( J(k) \) is amenable \([\text{BaBo–11}]\). According to \([\text{Babe–13}, \text{Page 61}]\), it is not known whether the group \( J(\mathbb{Z}) \) is strongly amenable.

5. Extreme amenability

Several of the examples we know of topological groups that are amenable and not locally compact have a property stronger than amenability: that of extreme amenability. The notion appeared in the mid 60’s. The subject became more spectacular with the article of Gromov and Milman \([\text{GrMi–83}]\), written in the late 70’s. For indications on the development of the subject, see the introduction of \([\text{Pest–06}]\).

**Definition 5.1.** A topological group \( G \) is **extremely amenable** if every continuous action of \( G \) on a compact space has a fixed point.

In the next proposition, Items (1) to (4) are reminiscent of similar items in Propositions 3.1 and 3.3.

**Proposition 5.2** (hereditary properties of extreme amenability). Let \( G \) be a topological group.

1. If \( G \) is extremely amenable, then \( G \) is amenable.
2. If \( G \) is extremely amenable, every open subgroup of \( G \) is extremely amenable.
3. If \( G \) is a directed union of a family \( (H_\alpha)_{\alpha \in A} \) of closed subgroups, and if each \( H_\alpha \) is extremely amenable, then \( G \) is extremely amenable.
4. If \( G \) has an extremely amenable closed normal subgroup \( N \) such that the quotient \( G/N \) is extremely amenable, then \( G \) is extremely amenable.
5. Let \( H \) be a topological group such that there exists a continuous homomorphism \( H \to G \) with dense image; if \( H \) is extremely amenable, then so is \( G \).
6. A closed subgroup of an extremely amenable group need not be amenable.

**Proof.** Claims (0) and (4) are straightforward. Claim (1) appears in \([\text{BoPT–11}, \text{Lemma 13}]\).

Let \( X \) be a non-empty compact \( G \)-space. For each \( \alpha \in A \), the set \( X^\alpha \) of \( H_\alpha \)-fixed points is closed and therefore compact in \( X \), and non-empty by hypothesis on \( H_\alpha \). The intersection \( \bigcap_{\alpha \in A} X^\alpha \) is non-empty,
by compactness of $X$. Since this intersection coincides with the set of $G$-fixed points in $X$, the group $G$ is extremely amenable.

For (3), see [Pest–06, Corollary 6.2.10]. A smart example confirming (5) is the group $\text{Aut}(\mathbb{Q})$ of order-preserving permutations of the set $\mathbb{Q}$ of rational numbers (see next proposition), which contains a non-abelian free discrete subgroup [Pest–98, Theorem 8.1]. Note that a non-trivial locally compact closed subgroup of an extremely amenable group is not extremely amenable, see Proposition 5.5 below.

Proposition 5.3. The following groups are extremely amenable.

(1) The unitary group $\text{U}(\mathcal{H})_{\text{str}}$ of an infinite dimensional separable complex Hilbert space $\mathcal{H}$, with the strong topology.

(2) The group $\text{Aut}(\mathbb{Q})$ of order-preserving permutations of $\mathbb{Q}$, with the topology of pointwise convergence on the discrete space $\mathbb{Q}$.

(3) The group $\mathcal{H}^+([0,1])$ of orientation-preserving homeomorphisms of the closed unit interval, with the compact-open topology.

(4) The isometry group of the Urysohn space.

(5) The group $\text{Aut}(X,\mu)$ of all measure-preserving automorphisms of a standard non-atomic finite or infinite and sigma-finite measure space, with the weak topology.

(6) The group $L^0(X,\mathcal{B},\mu;G)$ of all measurable maps from a Lebesgue space with a non-atomic probability measure $(X,\mathcal{B},\mu)$ to a second-countable locally compact group $G$, up to equality $\mu$-almost everywhere, with the topology of convergence in measure.

References for the proof. The case of $\text{U}(\mathcal{H})_{\text{str}}$ is shown in [GrMi–83]; for the translation of their result in terms of Definition 5.1, see for example [Pest–06, Section 2.2]. For $\text{Aut}(\mathbb{Q})$ and $\mathcal{H}^+([0,1])$, see [Pest–98].

For the Urysohn space and its isometry group, see [Pest–06, in particular Theorem 5.3.10 (the isometry group of the Urysohn space, with its standard Polish topology, is a Levy group) and Theorem 4.1.3 (every Levy group is extremely amenable)].

For $\text{Aut}(X,\mu)$, see [GiPe–07, Theorem 4.2]. The case of $L^0(X,\mathcal{B},\mu;G)$ is due to Eli Glasner [Glas–98] and Furstenberg-Weiss [unpublished]; see [Pest–06, Section 4.2].

Remark 5.4. Propositions 4.2 and 5.3(1) show that an extremely amenable group need not be strongly amenable.

Concerning (1), recall on the one hand that the group $\text{U}(\mathcal{H})_{\text{str}}$ is known to have Kazhdan’s Property (T) [Bekk–03]. On the other hand, an amenable locally compact group which has Property (T) is compact.
Concerning (2), note that not only $\text{Aut}(\mathbb{Q})$ with the indicated topology is extremely amenable, but moreover every action by homeomorphisms of the group $\text{Aut}(\mathbb{Q})$ with the discrete topology on a compact metrizable space has a fixed point [RoSo–07, Corollary 7].

Concerning (3) recall that Thompson’s group $F$ is a dense subgroup of $\mathcal{H}^+([0,1])$. The only consequence of (3) we can state is that is does not exclude that $F$ is amenable. (This is a repetition of Remark 12 of Pest–02.)

For the next proposition, we use the following notation.

$\text{Sym}(\mathbb{N})$ is the symmetric group of $\mathbb{N}$, with its standard Polish topology, as in Proposition 4.5.

Let $p$ be a positive number with $1 \leq p < \infty$ and $p \neq 2$. Let $\ell^p$ be the Banach space of sequence $(z_n)_{n \geq 1}$ of complex numbers such that $\|z\| := (\sum_{n \geq 1} |z_n|^p)^{1/p} < \infty$. We denote by $U(\ell^p)$ the group of linear isometries of $\ell^p$, with the strong topology.

**Proposition 5.5.** The following groups are NOT extremely amenable.

1. Any locally compact group $G \neq \{1\}$.
2. The symmetric group $\text{Sym}(\mathbb{N})$.
3. The group $\mathcal{H}(C)$ of homeomorphisms of the Cantor space, with the compact-open topology.
4. The unitary group $U(\ell^p)$, for $p$ with $1 \leq p < \infty$ and $p \neq 2$.
5. The group $\text{GL}(V)$, for $V \simeq \text{GF}(q)^{\mathbb{N}}$ as in Proposition 4.6.

**References for the proof.** Every locally compact group admits a free action on a suitable compact space [Vee–77]; and (1) follows.

For (2) and $\text{Sym}(\mathbb{N})$, see Pest–98, or Pest–06, Section 2.4. As Glasner and Weiss have observed, the natural action of $\text{Sym}(\mathbb{N})$ on the compact space $\text{LO} \subset \mathbb{N}^2$ of all linear orders on $\mathbb{N}$ has no fixed point; details in Pest–06, Example 2.4.6, Page 47.

For (3) observe that the action of $\mathcal{H}(C)$ on $C$ has no fixed point. For more on actions of this group on compact spaces, see GlWe–03.

For (4) let us reproduce the argument of Pest–06 Example 3.6.15. Let $(e_n)_{n \in \mathbb{N}}$ be the canonical basis of $\ell^p$. For every sequence $(t_n)_{n \in \mathbb{N}}$ in the compact group $\prod_{n \in \mathbb{N}} T_n$, where each $T_n$ is a copy of $T = \{z \in \mathbb{C} \mid |z| = 1\}$, and every $\sigma \in \text{Sym}(\mathbb{N})$, there is an isometry of $\ell^p$ mapping $e_n$ to $t_n e_{\sigma(n)}$ for all $n \in \mathbb{N}$. Moreover, every isometry of $\ell^p$ is of this form; the proof is like that of Banach, for the $\ell^p$-space of real sequence and for $t_n \in \{-1,1\}$ for all $n$ [Bana–32, Chap. XI, § 5, Page 178]. In other terms, the unitary group $U(\ell^p)$ of $\ell^p$ is a semi-direct
product
\[ U(\ell^p) = \left( \prod_{n \in \mathbb{N}} T_n \right) \rtimes \text{Sym}(\mathbb{N}). \]

Now Claim 4 follows from 2 and Proposition 5.2(3).

For 5, see [Pest–06, Example 6.7.17]. \(\square\)

To conclude this section, we quote a result that extends Proposition 5.3(1). Compare with the way Remark 4.3(2) extends part of Proposition 4.2.

**Proposition 5.6.** A countably decomposable von Neumann algebra \(M\) is injective if and only if its unitary group \(U(M)\), with the weak topology, is the direct product of a compact group and an extremely amenable group.

In particular, an infinite dimensional factor \(M\) is injective if and only if \(U(M)\) is extremely amenable, and the same holds for \(M\) a properly infinite von Neumann algebra.

References for the proof: [GiPe–07, Theorem 3.3] and [GiNg]. \(\square\)

6. On the definition of ergodicity

Let us agree on the following terminology. Consider a Borel action of a topological group \(G\) on a Borel space \((X, \mathcal{B})\). A Borel subset \(A \subset X\) is \textit{invariant} by \(G\) if \(gA = A\) for all \(g \in G\).

Assume that we have moreover a \(G\)-invariant probability measure \(\mu\) on \((X, \mathcal{B})\). A Borel subset \(A \subset X\) is \textit{\(\mu\)-essentially invariant} by \(G\) if \(\mu(gA \Delta A) = 0\) for all \(g \in G\) (where \(\Delta\) indicates a symmetric difference). For “\(\mu\)-essentially invariant”, Varadarajan uses “\(\mu\)-invariant” [Vara–63, Page 196], Maitra “\(\mu\)-almost invariant” [Mait–77], and Phelps “invariant (mod \(\mu\))” [Phel–66].

**Definition 6.1.** Let \(G\) be a topological group acting as above on a Borel space \((X, \mathcal{B})\) with a \(G\)-invariant probability measure \(\mu\).

The action is \textbf{w-ergodic} if, for every invariant set \(A \in \mathcal{B}\), either \(\mu(A) = 0\) or \(\mu(X \setminus A) = 0\); in this situation, we say also that the invariant measure \(\mu\) is \(w\)-ergodic.

The action is \textbf{s-ergodic} if, for every \(\mu\)-essentially invariant set \(A \in \mathcal{B}\), either \(\mu(A) = 0\) or \(\mu(X \setminus A) = 0\); in this situation, we say also that the invariant measure \(\mu\) is \(s\)-ergodic.

Some authors use “ergodic” for our “\(w\)-ergodic” (see for example [Zimm–84, beginning of Chapter 2]), others use “ergodic” for our “\(s\)-ergodic” (see for example [Phel–66, Chapter 10]). Proposition 6.7 shows that, in a standard setting, the two notions coincide.
Remark 6.2. Consider a Borel action of a topological group $G$ on a Borel space $(X, \mathcal{B})$, and a $G$-invariant probability measure $\mu$ on $X$. Assume that $\mu$ is s-ergodic. Then every Borel subset $A$ in $X$ such that $\mu(A) = 1$ is $\mu$-essentially invariant by $G$.

Indeed, let $g \in G$. The subset $g(A) \setminus (A \cap g(A))$ is contained in $X \setminus A$, hence is negligible for $\mu$. Similarly, $A \setminus (A \cap g(A)) = g(g^{-1}(A) \setminus (A \cap g^{-1}(A)))$ is negligible for $\mu$, because $\mu$ is $G$-invariant. Hence $\mu(A \Delta g(A)) = 0$ for all $g \in G$.

Suppose moreover that $X$ is a compact space, and that $\mathcal{B}$ is the $\sigma$-algebra of Borel subsets of $X$, i.e. the $\sigma$-algebra of subsets of $X$ generated by the open subsets of $X$. The space $\mathcal{P}(X)$ of probability measures on $X$ is a compact convex subspace of the dual space of $C(X)$, with the weak-$*$-topology, and the space $\mathcal{P}^G(X)$ of $G$-invariant probability measures is a compact convex subspace of $\mathcal{P}(X)$.

Definition 6.3. With the notation above, a $G$-invariant measure $\mu \in \mathcal{P}^G(X)$ is indecomposable if it is in the subset of extreme points $\mathcal{E}^G(X)$ of $\mathcal{P}^G(X)$.

In other words, $\mu$ is decomposable if there exist two distinct measures $\mu_1, \mu_2 \in \mathcal{P}^G(X)$ and a constant $c$ with $0 < c < 1$ such that $\mu = c\mu_1 + (1 - c)\mu_2$.

The following proposition appears in [Bogo–39].

Proposition 6.4. Let $G$ be a topological group acting continuously by homeomorphisms on a compact space $X$, and let $\mu$ be a $G$-invariant probability measure on $X$. The following two properties are equivalent:

(i) $\mu$ is indecomposable,
(ii) $\mu$ is s-ergodic;

and imply the following third property

(iii) $\mu$ is w-ergodic.

Note. We will add another property equivalent to (i) and (ii) in Proposition 8.3.

It is shown below that (iii) does imply (ii) when $G$ is second countable locally compact (Proposition 6.7), but not in general (Proposition 7.1(3)).

Proof. For (i) $\iff$ (ii) we reproduce the proof of [Bogo–39]. This proof has remained the standard one: see for example [Walt–82, Theorem 610]. Walters writes his proof for one continuous map $T : X \rightarrow X$, but it works without change in the present situation. For (ii) $\Rightarrow$ (i) see also [Phel–66, Proposition 10.4] (Proposition 12.4 of the Second Edition).
We prove the contraposition. Assume that there exists a \(\mu\)-essentially \(G\)-invariant subset \(A\) of \(X\) with \(0 < \mu(A) < 1\). Then
\[
\mu = \mu(A)\mu_1 + (1 - \mu(A))\mu_2,
\]
with
\[
\mu_1(B) = \frac{1}{\mu(A)}\mu(B \cap A), \quad \mu_2(B) = \frac{1}{1 - \mu(A)}\mu(B \cap (X \setminus A)),
\]
for all Borel subsets \(B\) of \(X\).

Assume (again by contraposition) that there exist two distinct \(G\)-invariant probability measures \(\mu_1, \mu_2\) of which
\[
\mu = c_1\mu_1 + c_2\mu_2
\]
is a convex combination. For every Borel subset \(B\) of \(X\) with \(\mu(B) = 0\), we have \(\mu_1(B) = 0 = \mu_2(B)\), i.e. \(\mu_1, \mu_2\) are absolutely continuous with respect to \(\mu\). By the Radon-Nikodym theorem (one of many convenient references is [Rudi–66, Theorem 6.9]), there exist well-defined and unique positive-valued functions \(f_1, f_2 \in L^1(X, \mu)\) such that \(\mu_1 = f_1\mu\) and \(\mu_2 = f_2\mu\). By unicity, \(f_1\) and \(f_2\) are \(G\)-invariant.

Set \(A = \{x \in X \mid f_1(x) > f_2(x)\}\) and \(B = \{x \in X \mid f_1(x) \leq f_2(x)\}\).

Then \(A, B\) are \(\mu\)-essentially \(G\)-invariant Borel subsets of \(X\) of positive measure, and constitute a partition of \(X\).

The implication \((\mathbf{ii}) \Rightarrow (\mathbf{iii})\) is trivial. \(\square\)

**Definition 6.5.** Let \((\Omega, \mathcal{B})\) be a Borel space; assume that it is a standard Borel space, i.e. that there exist an isomorphism of \((\Omega, \mathcal{B})\) with a Borel subset of a complete separable metric space. Let \(G\) be a topological group acting in a Borel way on \((\Omega, \mathcal{B})\); assume that there exists a probability measure \(\mu\) on \((\Omega, \mathcal{B})\) which is a \(G\)-quasi-invariant, i.e. such that, for all \(A \subset \mathcal{B}\) and \(g \in G\), we have \(\mu(A) = 0\) if and only if \(\mu(gA) = 0\).

Let \((Y, \mathcal{C})\) be another standard Borel space (the most important case here is \(Y = \mathbb{R}\), with the usual Borel \(\sigma\)-algebra). A Borel function \(f : \Omega \to Y\) is \(\mu\)-essentially \(G\)-invariant if, for each \(g \in G\), we have \(f(g\omega) = f(\omega)\) for \(\mu\)-almost all \(\omega \in \Omega\); it is \(G\)-invariant if, for each \(g \in G\), we have \(f(g\omega) = f(\omega)\) for all \(\omega \in \Omega\).

**Lemma 6.6** (Lemma 3.3 in [Vara–63], or Lemma 2.2.16 in [Zimm–84]). Let \(G\) be a second countable locally compact group acting in a Borel way on a standard Borel space \(\Omega\). Assume that \(\Omega\) has a \(G\)-quasi-invariant probability measure \(\mu\).

Let \(Y\) be a standard Borel space and \(f : \Omega \to Y\) be a Borel function. Assume that \(f\) is essentially \(G\)-invariant. Then there exists a Borel function \(\tilde{f} : \Omega \to Y\) which is \(G\)-invariant, and \(\tilde{f}(\omega) = f(\omega)\) for \(\mu\)-almost all \(\omega \in \Omega\).
In particular, if \( A \subset \Omega \) is a \( \mu \)-essentially \( G \)-invariant Borel subset, there exists a \( G \)-invariant Borel set \( \tilde{A} \subset \Omega \) such that \( \mu(\tilde{A} \Delta A) = 0 \).

On the proof. Note that the particular case follows from the result on functions, with \( Y = \mathbb{R} \) and \( f \) the characteristic function of \( A \).

It is crucial here that \( G \) is locally compact (hence \( G \) has a Haar measure) and “not too large” (more precisely “second countable” in \([\text{Zimm–84}]\) and \([\text{Vara–63}]\), because the proof relies very strongly on Fubini’s theorem, and Fubini’s theorem can be used for Haar measure with appropriate conditions only. \( \square \)

**Proposition 6.7.** Let \( G \) be a second countable locally compact group acting in a Borel way on a standard Borel space \( \Omega \). Assume that \( \Omega \) has a \( G \)-quasi-invariant probability measure \( \mu \).

Then the action is s-ergodic if and only if it is w-ergodic.

**Proof.** The proposition follows from the lemma and the definitions. \( \square \)

We quote now the following decomposition theorem. It can be seen as an elaboration of results going back to von Neumann, in the early 30’s. Two convenient references are \([\text{Vara–63}]\), for invariant measures, and \([\text{GrSc–00}]\), for quasi-invariant measures.

**Theorem 6.8 (Ergodic Decomposition Theorem).** Let \( G \) be a second countable locally compact group acting in a Borel way on a standard Borel space \( (X, \mathcal{B}) \); assume that \( \mathcal{P}(X)^G \) is non-empty (equivalently that \( \mathcal{E}(X)^G \) is non-empty). Denote by \( (Y, \mathcal{C}) \) the standard Borel space with \( Y = \mathcal{E}(X)^G \) and \( \mathcal{C} \) its Borel \( \sigma \)-algebra.

Then there exist a family \((p_y)_{y \in Y}\) of probability measures on \( X \), with the following properties:

1. for every Borel subset \( B \) in \( X \), the map \( y \mapsto p_y(B) \) from \( Y \) to \([0, 1]\) is Borel;
2. for every \( y \in Y \), the measure \( p_y \) is \( G \)-invariant and ergodic;
3. for \( y, y' \in Y \) with \( y \neq y' \), the measures \( p_y \) and \( p_{y'} \) are mutually singular;
4. for every \( \mu \in \mathcal{P}(X)^G \), there exists a probability measure \( \nu \) on \((Y, \mathcal{C})\) such that \( \mu(B) = \int_Y p_y(B) d\nu(y) \) for every \( B \in \mathcal{B} \).

Note: a fortiori, the theorem holds for a continuous action of \( G \) on a metrizable compact space.

**Corollary 6.9.** Let \( G \) be a second-countable locally compact group acting continuously by homeomorphisms on a metrizable compact space \( X \).
For every $\mu \in \mathcal{E}^G(X)$, there exists a Borel subset $A_\mu$ of $X$ such that

$\mu(A_\mu) = 1$ and $\mu'(A_\mu) = 0$ for all $\mu' \in \mathcal{E}^G(X)$ with $\mu' \neq \mu$.

On the proof. This follows from a version of the Ergodic Decomposition Theorem more comprehensive than that of Theorem 6.8, see [GrSc–00]. Theorem 1.4 of [GrSc–00] is such a theorem for a countable discrete group, say $\Gamma$. It is shown there that

(a) the sub-$\sigma$-algebra $\mathcal{T}$ of $\mathcal{B}$ consisting of $\Gamma$-invariant sets is countably generated, say generated by a countable sub-algebra $\mathcal{T}'$;

(b) for every $x \in X$, the set $[x]_\mathcal{T} = \bigcap C$, where the intersection is over all $C \in \mathcal{T}'$ with $x \in C$, is a Borel subset of $X$ (and it coincides with the intersection $\bigcap C$ over all $C \in \mathcal{T}$ with $x \in C$);

(c) $[x]_\mathcal{T} \in \mathcal{T}$, i.e. $[g(x)]_\mathcal{T} = [x]_\mathcal{T}$ for all $x \in X$ and $g \in \Gamma$;

(d) there exist a $\Gamma$-invariant Borel subset $X_0 \subset X$, with $\eta(X \setminus X_0) = 0$ for all $\eta \in \mathcal{P}^\Gamma(X)$, and a surjective map $p : X_0 \rightarrow Y_\Gamma$, $x \mapsto p_x$, measurable with respect to $\mathcal{T}$ and $\mathcal{C}$, such that

\[
p_x([x]_\mathcal{T}) = 1,
\]

\[
p_{x'}([x]_\mathcal{T}) = 0,
\]

for all $x, x' \in X_0$ with $p(x') \neq p(x)$.

Then, for $\mu \in \mathcal{E}^\Gamma(X)$, it suffices to set:

(e) $A_\mu = [x]_\mathcal{T}$, with $x \in X_0$ such that $p(x) = \mu$.

[Note that, in [GrSc–00], the set $[x]_\mathcal{T}$ is defined as an intersection over $C \in \mathcal{T}'$ with $x \in C$; but $[x]_\mathcal{T}$ is Borel because this is also an intersection over $C$ in the countable subalgebra $\mathcal{T}'$. We are grateful to K. Schmidt for an e-mail clarifying this point for us.]

In the more general case of a second-countable locally compact group $G$, we refer to [GrSc–00, Theorem 5.2]. Choose a countable dense subgroup $\Gamma$ of $G$. Define $X_0$ as above, in terms of $\Gamma$ and, for $x \in X_0$, define also $p_x$ and $[x]_\mathcal{T}$ in terms of $\Gamma$. Then we have:

(f) $[g(x)]_\mathcal{T} = [x]_\mathcal{T}$ for all $g \in G$ and $x \in X$ (compare with (c));

(g) $\mathcal{E}^G(X) = \mathcal{E}^\Gamma(X)$;

(h) $p_x \in \mathcal{Y} = \mathcal{E}^G(X)$ (compare with (d));

(i) $p_x([x]_\mathcal{T}) = 1$ and $p_{x'}([x]_\mathcal{T}) = 0$ for all $x, x' \in X_0$ with $p(x') \neq p(x)$ (as in (d)).

[The sets $[x]_\mathcal{T}$ may depend on the choice of $\Gamma$, but we will not discuss this further here.]

The article [GrSc–00] is written for the case of quasi-invariant measures. For the particular case of invariant measures, as discussed in our review, we could equally have quoted an earlier article by Varadarajan. Specifically, we refer to [Vara–63, Theorem 4.2] for the fact that our
Borel space \((Y, \mathcal{C})\) is standard. Our sets \(A_{\mu}\), with \(\mu \in \mathcal{E}^G(X)\), correspond to the sets \(X_{e}\), with \(e\) in the space \(\mathcal{J}\) of ergodic \(G\)-invariant measures, in [Vara–63]. □

We end this section with a simpler version of the previous corollary, for comparison with Propositions 7.2 and 7.3 below.

**Corollary 6.10.** Let \(G\) be a second-countable locally compact group acting continuously by homeomorphisms on a metrizable compact space \(X\). Assume that there exist two distinct ergodic probability measures \(\mu_1, \mu_2 \in \mathcal{E}^G(X)\).

Then there exists two \(G\)-invariant Borel subsets \(A_1, A_2 \subset X\) such that

\[
\begin{align*}
\mu_1(A_1) &= 1, & \mu_1(A_2) &= 0 \\
\mu_2(A_1) &= 0, & \mu_2(A_2) &= 1.
\end{align*}
\]

**Proof.** This is a consequence of Theorem 6.8 since, on the one hand, two distinct measures in \(\mathcal{E}^G(X)\) are mutually singular, and, on the other hand, two measures \(\mu_1\) and \(\mu_2\) are mutually singular precisely when there exist two Borel subsets \(A_1, A_2\) in \(X\) for which the equalities of (1) hold. □

### 7. Kolmogorov’s example

The example of this section is due to Kolmogorov. It appears in [Fomi–50], and is revisited in [Vara–63], with reference to Kolmogorov, and in [Mait–77], without. Note that Fomin was a student of Kolmogorov [Ale+–76].

Consider a positive number \(p\) with \(0 \leq p \leq 1\) and the measure \(\lambda_p\) on \(Y := \{0, 1\}\) defined by \(\lambda_p(0) = p\) and \(\lambda_p(1) = 1 - p\). For \(i \in \mathbb{Z}\), let \((Y_i, \lambda_{p,i})\) be a copy of \((Y, \lambda_p)\). Let \(X = \prod_{i \in \mathbb{Z}} X_i = \{0, 1\}^\mathbb{Z}\) be the product of the \(Y_i\)'s, and \(\mathcal{B}\) the usual \(\sigma\)-algebra; \((X, \mathcal{B})\) is a standard Borel space. Let \(\mu_p\) be the product probability measure \(\prod_{i \in \mathbb{Z}} \lambda_{p,i}\), which is called a Bernoulli measure. We denote by \(S : X \rightarrow X\) the corresponding **Bernoulli shift**, defined by \((Sx)_i = x_{i+1}\) for all \(i \in \mathbb{Z}\); observe that the measure \(\mu_p\) is preserved by \(S\). Recall that the transformation \(S\) of \((X, \mu_p)\) is s-ergodic, indeed strongly mixing; see e.g. [Walt–82, Theorems 1.12 and 1.30].

Denote as in Proposition 4.5 by \(\text{Sym}(\mathbb{Z})\) the full symmetric group of \(\mathbb{Z}\), with its standard Polish topology, and by \(\text{Sym}_f(\mathbb{Z})\) the subgroup of \(\text{Sym}(\mathbb{Z})\) of permutations with finite support. Recall that \(\text{Sym}_f(\mathbb{Z})\) is countable, locally finite, and dense in \(\text{Sym}(\mathbb{Z})\). The natural action of \(\text{Sym}(\mathbb{Z})\) on \(X\) is continuous, and preserves \(\mu_p\).
Denote as in Definition 6.3 by \( P_{\text{Sym}_f(Z)}(X) \) the compact convex set of \( \text{Sym}_f(Z) \)-invariant probability measures on \( X \); it is a compact convex set in the dual space of \( \mathcal{C}(X) \), with the weak-*-topology, and \( \mathcal{E}_{\text{Sym}_f(Z)}(X) \) is the set of its extreme points. Similarly for \( \mathcal{E}_{\text{Sym}(Z)}(X) \). Since \( \text{Sym}_f(Z) \) is a subgroup of \( \text{Sym}(Z) \), the space \( P_{\text{Sym}(Z)}(X) \) is contained in \( P_{\text{Sym}_f(Z)}(X) \). Since \( \text{Sym}_f(Z) \) is dense in \( \text{Sym}(Z) \), and the latter acts continuously on \( P(X) \), we have indeed
\[
P_{\text{Sym}(Z)}(X) = P_{\text{Sym}_f(Z)}(X).
\]
In terms of extreme points of compact convex sets, we have consequently
\[
(2) \quad \mathcal{E}_{\text{Sym}(Z)}(X) = \mathcal{E}_{\text{Sym}_f(Z)}(X).
\]

**Proposition 7.1.** Consider as above the natural actions of the Polish group \( \text{Sym}(Z) \) and of its subgroup \( \text{Sym}_f(Z) \) on the compact space \( X = \{0, 1\}^\mathbb{Z} \).

1. The set \( \mathcal{E}_{\text{Sym}_f(Z)}(X) \) coincides with the set \( \{\mu_p\}_{0 \leq p \leq 1} \) of Bernoulli measures.
2. Similarly, \( \mathcal{E}_{\text{Sym}(Z)}(X) = \{\mu_p\}_{0 \leq p \leq 1} \); in particular, for every \( p \in [0, 1] \), the Bernoulli measure \( \mu_p \) is invariant and s-ergodic for the action of \( \text{Sym}(Z) \).
3. Consider an integer \( k \geq 1 \), a finite sequence \( (p_j)_{j=1,\ldots,k} \) with \( 0 < p_1 < \cdots < p_k < 1 \), positive constants \( c_1, \ldots, c_k \) with \( c_1 + \cdots + c_k = 1 \), and the \( \text{Sym}(Z) \)-invariant probability measure \( \mu = c_1\mu_{p_1} + \cdots + c_k\mu_{p_k} \in P_{\text{Sym}(Z)}(X) \). Then \( \mu \) is \( w \)-ergodic.

Proof. Claim [1] is known as a result of de Finetti. The original article seems to be [dFin–37], but we rather refer to [Fell–71, Section VII.4, Pages 228–229].

Claim [2] follows by Equality [2]. It is the way de Finetti’s result is quoted in [Fomi–50, Section 2.4].

For [3] observe that the \( \text{Sym}(Z) \)-orbits on \( X \) are easily described:

(a) Two one-point orbits, one with 0’s only, the other with 1’s only,
(b) For each \( k \geq 1 \) two countable infinite orbits \( \{(x_i)_{i \in \mathbb{Z}} \in X \mid \sum_{i \in \mathbb{Z}} x_i = k\} \) and \( \{(x_i)_{i \in \mathbb{Z}} \in X \mid \sum_{i \in \mathbb{Z}} (1-x_i) = k\} \),
(c) And the uncountable orbit, that we denote by \( X' \), of sequences that have infinitely many 0’s and infinitely many 1’s.

In particular, the complement \( N \) of \( X' \) in \( X \) is countable, the partition \( X = X' \cup N \) is \( \text{Sym}(Z) \)-invariant, and \( \text{Sym}(Z) \) is transitive on \( X' \). It follows that every \( \text{Sym}(Z) \)-invariant subset of \( X \) is either inside \( N \) or contains \( X' \).
Since \( N \) is countable and the measure \( \mu \) of \([3]\) without atoms, \( \mu(N) = 0 \). Hence the action of \( \text{Sym}(\mathbb{Z}) \) on \((X, \mu)\) is \( w \)-ergodic.

If \( k \geq 2 \), the measure \( \mu \) is by definition decomposable, i.e. not \( s \)-ergodic (Proposition 6.4). \( \square \)

**Proposition 7.2.** Corollary 6.10 does not extend to the situation of the group \( \text{Sym}(\mathbb{Z}) \) acting on the compact space \( X \).

More precisely, there exists a \( \text{Sym}(\mathbb{Z}) \)-invariant Borel subset \( X' \) of \( X \) such that, for every \( p \in ]0, 1[ \), we have \( \mu_p \in \mathcal{E}^{\text{Sym}(\mathbb{Z})}(X) \) and \( \mu_p(X') = 1 \).

**Proof.** Let \( A \subset X \) be a \( \text{Sym}(\mathbb{Z}) \)-invariant Borel subset. Suppose that there exists some \( p \in ]0, 1[ \) such that \( \mu_p(A) \neq 0 \). In the previous proof, we have checked that \( X \setminus A \) is countable. Hence, for every \( p \in ]0, 1[ \), we have \( \mu_p(A) = 1 \). \( \square \)

Despite the failure of Corollary 6.10 in situations like that of the previous proposition, we have the following result, that is Theorem 6 of [Fomi–50]:

**Proposition 7.3** (Fomin). Let \( G \) be a topological group acting continuously by homeomorphisms on a compact space \( \Omega \). Assume that there exist two distinct \( s \)-ergodic \( G \)-invariant probability measures on \( \Omega \), say \( \nu_1 \) and \( \nu_2 \).

Then there exists a Borel partition of \( \Omega \) in two subsets \( A_1, A_2 \) that are \( \nu_1 \)-essentially invariant and \( \nu_2 \)-essentially invariant by \( G \), and such that

\[
\begin{align*}
(1) & \quad \nu_1(A_1) = 1 \quad \text{and} \quad \nu_2(A_1) = 0, \\
(2) & \quad \nu_1(A_2) = 0 \quad \text{and} \quad \nu_2(A_2) = 1.
\end{align*}
\]

When \( G \) is a second countable locally compact group, recall we gave a much stronger conclusion in Corollary 6.10 in particular, the conclusion of Proposition 7.3 holds with \( A_1, A_2 \) actually \( G \)-invariant, a condition stronger than the above conditions of essential invariance.

Before giving an illustration of Proposition 7.3 with Kolmogorov’s example, we recall the following well-known fact. For \( p \in ]0, 1[ \), let \( E_p \) denote the Borel subset of \( X \) consisting of sequences \((x_i)_{i \in \mathbb{Z}}\) such that \( \lim_{k \to \infty} \frac{1}{2k+1} \sum_{i=-k}^{k} x_i = p \), i.e. of sequences in which the 1’s have density \( p \). Note that \( E_p \) is invariant by \( \text{Sym}_f(\mathbb{Z}) \), but not by \( \text{Sym}(\mathbb{Z}) \).

**Proposition 7.4.** Let \( p, q \in ]0, 1[ \) with \( p \neq q \), and let \( E_p, E_q \subset X \) be as above. Then \( \mu_p(E_p) = 1 \) and \( \mu_p(E_q) = 0 \).

**Proof.** Let \( \varphi_0 : X \to \mathbb{R} \) be defined by \( \varphi_0(x) = x_0 \). Then

\[
\int_X \varphi_0(x) d\mu_p(x) = p,
\]
by definition of $\mu_p$. By Birkhoff’s ergodic theorem, the limit

$$\varphi_0^*(x) := \lim_{k \to \infty} \frac{1}{2k + 1} \sum_{i=-k}^{k} \varphi_0(S^i x)$$

exist for $\mu_p$-almost all $x \in X$, and defines a $\mu_p$-almost everywhere constant function $\varphi_0^*$ of essential value $p$ (because the shift $S$ is s-ergodic on $(X, \mu_p)$). Observe that, for $x \in X$, we have $\varphi_0^*(x) = p$ if and only if $x \in E_p$. Hence $E_p = X$, up to $\mu_p$-negligible sets; otherwise said: $\mu_p(E_p) = 1$.

Since $E_p \cap E_q = \emptyset$, we have $\mu_p(E_q) = 0$. \qed

If we particularize the pair $(G, \Omega)$ to the pair $(G = \text{Sym}(\mathbb{Z}), X = \{0, 1\}^\mathbb{Z})$ of Kolmogorov’s example, and $\nu_1, \nu_2$ to the Bernoulli measures $\mu_{1/3}, \mu_{2/3}$, then the conclusion of Proposition 7.3 holds for the subsets $E_{1/3}, E_{2/3}$ of Proposition 7.4; for the essential invariance of these two subsets, see Remark 6.2.

Let us finally mention a generalization of Equality (2), from just before Proposition 7.1, and of Part (1) of the same proposition. Consider a measure space $(Y, \mathcal{C})$ and the product space $(X, \mathcal{B})$, with $X = Y^\mathbb{Z}$, with the natural action of the groups $\text{Sym}(\mathbb{Z})$ and $\text{Sym}_f(\mathbb{Z})$. For every probability measure $\lambda$ on $(Y, \mathcal{C})$, let $\tilde{\lambda}$ be the probability measure on $(X, \mathcal{B})$ that is the product of copies of $\lambda$ indexed by $\mathbb{Z}$; we denote by $\text{Bern}(X)$ the set of measures of the form $\tilde{\lambda}$; observe that $\text{Bern}(X) \subseteq \mathcal{P}^{\text{Sym}(\mathbb{Z})}(X)$. Then:

1. $\mathcal{P}^{\text{Sym}(\mathbb{Z})}(X) = \mathcal{P}^{\text{Sym}_f(\mathbb{Z})}(X) \quad \text{HeSa–63, Theorem 3.2}$, and therefore $\mathcal{E}^{\text{Sym}(\mathbb{Z})}(X) = \mathcal{E}^{\text{Sym}_f(\mathbb{Z})}(X)$

2. $\mathcal{E}^{\text{Sym}_f(\mathbb{Z})}(X) = \text{Bern}(X) \quad \text{HeSa–63, Theorem 5.3}$. 

8. Fomin’s representations

In [Fomi–50, § 1], Fomin proves the equivalence $(\text{i}) \Leftrightarrow (\text{ii})$ of Proposition 6.4 by adding one more equivalent condition, of independent interest, in terms of unitary representations. The object of this section is to describe this condition.

Let $G$ be a topological group acting continuously by homeomorphisms on a compact space $X$. Let $\mathcal{C}(X, \mathbb{T})$ denote the group of all continuous functions from $X$ to the compact group $\mathbb{T}$ of complex numbers of modulus one. We consider the natural action of $G$ on $\mathcal{C}(X, \mathbb{T})$, defined by $(g(\varphi))(x) = \varphi(g^{-1}(x))$ for all $g \in G$, $\varphi \in \mathcal{C}(X, \mathbb{T})$, and
Consider a $G$-invariant probability measure $\mu$ on $X$, and the complex Hilbert space $L^2_\mathbb{C}(X, \mu)$. For $(\varphi, g) \in \mathcal{F}$, define a linear operator $\rho_\mu(\varphi, g)$ on $L^2_\mathbb{C}(X, \mu)$ by

$$(\rho_\mu(\varphi, g)\xi)(x) = \varphi(x)\xi(g^{-1}(x))$$

for all $\xi \in L^2_\mathbb{C}(X, \mu)$ and $x \in X$. The following proposition, which is now straightforward to check, is Theorem 1 in [Fomi–50].

**Proposition 8.1.** Let $G, X, \mathcal{F}$, and $\mu$ be as above. Then $(\varphi, g) \mapsto \rho_\mu(\varphi, g)$ defines a unitary representation of the group $\mathcal{F}$ on the Hilbert space $L^2_\mathbb{C}(X, \mu)$.

For every essentially bounded function $\psi \in L^\infty_\mathbb{C}(X, \mu)$, we denote by $M_\psi$ the multiplication operator $\xi \mapsto \varphi\xi$ on $L^2_\mathbb{C}(X, \mu)$.

**Lemma 8.2.** With the notation above, let $S$ be a continuous linear operator on $L^2_\mathbb{C}(X, \mu)$ that commutes with $\rho_\mu(\varphi, g)$ for all $(\varphi, g) \in \mathcal{F}$.

Then $S = M_\psi$ for some $\psi \in L^\infty_\mathbb{C}(X, \mu)$; moreover $\psi$ is $\mu$-essentially $G$-invariant (in the sense of Definition 6.5).

**Proof.** By hypothesis, $S$ commutes with $\rho_\mu(\varphi, 1) = M_\varphi$ for every $\varphi \in \mathcal{C}(X, \mathbb{T})$, and therefore with sums of products of such multiplication operators. By the Stone-Weierstrass theorem, $S$ commutes also with $M_\varphi$ for every continuous function $\varphi : X \rightarrow \mathbb{C}$. By a standard argument, it follows that there exists $\psi \in L^\infty_\mathbb{C}(X, \mu)$ such that $S = M_\psi$; see [BourTS] Chap. II, § 3, no 3, Lemma 3].

Since $S = M_\psi$ commutes with $\rho_\mu(1, g)$ for all $g \in G$, the function $\psi$ is $\mu$-essentially $G$-invariant. \qed

The following proposition is Theorem 2 in [Fomi–50]. Note that Property \[\text{[ii]}\] below coincides with Property \[\text{[ii]}\] of Proposition 6.4.

**Proposition 8.3.** Let $G, X, \mathcal{F}$, and $\mu$ be as above. The following properties are equivalent:

(i) the representation $\rho_\mu$ is irreducible,

(ii) the measure $\mu$ is s-ergodic.

**Proof.** For \[\text{[i]} \Rightarrow \text{[ii]}\] we prove the contraposition. If $\mu$ is not s-ergodic, there exists a $\mu$-essentially invariant Borel subset $A \subset X$ with $0 < \mu(A) < 1$. The subspace of $L^2_\mathbb{C}(X, \mu)$ of functions which vanish outside $A$ is invariant by $\rho_\mu$, and therefore the representation is reducible.

The converse implication \[\text{[ii]} \Rightarrow \text{[i]}\] follows from Lemma 8.2 and Schur’s lemma (for which we refer to [BeHV–08 Theorem A.2.2]). \qed
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