Compact Quantum Metric Spaces

Marc A. Rieffel

Abstract. We give a brief survey of many of the highlights of our present understanding of the young subject of quantum metric spaces, and of quantum Gromov-Hausdorff distance between them. We include examples.

My interest in developing the theory of compact quantum metric spaces was stimulated by certain statements in the high-energy physics and string-theory literature, concerning non-commutative spaces that converged to other (possibly non-commutative) spaces. These statements appeared to me to deserve a more precise formulation. (See the references in the introductions of [33] [34].) Here I will just give a brief survey of some of the main developments in this very young subject. I will also indicate some of the main classes of examples which have been explored so far. I include a few related arguments which are not quite in place in the existing literature.

1. The definition of compact quantum metric spaces

The concept of a quantum metric space has its origins in Connes’ paper [8] of 1989 in which he first proposes using Dirac operators as the vehicle for metric data in non-commutative geometry. He was motivated by his observation that for a compact spin Riemannian manifold one can recover its smooth structure, its Riemannian metric, and much else, directly from its standard Dirac operator. This led him to the concept of a spectral triple, \((A, \mathcal{H}, D)\), consisting of a \(\ast\)-algebra \(A\) represented by bounded operators on a Hilbert space \(\mathcal{H}\), and of a (usually unbounded) self-adjoint operator, \(D\), on \(\mathcal{H}\) such that the commutator \([D, a]\) is a bounded operator for each \(a \in A\). Connes also requires that \(D\) have compact resolvant. (Spectral triples are very closely related to “unbounded K-cycles” or “unbounded Fredholm modules”, the difference being that for spectral triples the representation of \(A\) must be faithful.) Connes pointed out that from a spectral triple one obtains a metric, \(\rho_D\), on the state space, \(S(A)\), of \(A\) by means of the formula

\[
\rho_D(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : \|[D, a]\| \leq 1\},
\]

2000 Mathematics Subject Classification. Primary 46L87, 14E20; Secondary 53C23, 58B34. Key words and phrases. quantum metric spaces, \(C^\ast\)-algebra, state, Lipschitz, Dirac operator. The author’s research was supported in part by NSF Grant DMS-0200591.

©2003 American Mathematical Society
where the value $+\infty$ may occur. But he did not explore this metric much for non-commutative algebras.

In [31] [32] I pointed out that, motivated by what happens for ordinary compact metric spaces, it is natural to desire that the topology on $S(A)$ determined by $\rho_{D}$ coincide with the weak-∗ topology. To be more specific about this motivation, let $(X, \rho)$ be an ordinary compact metric space. It is common to define a seminorm, $L_{\rho}$, the Lipschitz seminorm, on the algebra $A = C(X)$ of continuous functions on $A$, by

$$L_{\rho}(f) = \sup\{|f(x) - f(y)|/\rho(x, y) : x \neq y\},$$

where the value $+\infty$ is permitted. The metric $\rho$ can be recovered from $L_{\rho}$ by

$$\rho(x, y) = \sup\{|f(x) - f(y)| : L_{\rho}(f) \leq 1\}.$$ 

Even more, Kantorovich [18] [19] used $L_{\rho}$ to define a metric on $S(A)$, which now is identified with the space of all probability measures on $X$, by

$$\rho_{L}(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : L_{\rho}(f) \leq 1\}.$$ 

It is clear that $\rho_{L}$ extends $\rho$ from the set of point-measures to the set of all probability measures. Kantorovich showed, among other properties, that the topology on $S(A)$ from $\rho_{L}$ coincides with the weak-∗ topology.

Since the data $(C(X), L_{\rho})$ is equivalent to the data $(X, \rho)$, we see that we have reformulated the notion of a metric in terms of just the commutative $C^{\ast}$-algebra $C(X)$, by means of $L_{\rho}$. It is then natural to try to formulate metric data for a non-commutative unital $C^{\ast}$-algebra, $A$, by means of a suitable seminorm on $A$ which should play the role of $L_{\rho}$. For Connes’ spectral triples the role of $L_{\rho}$ is played, in effect, by $L(a) = \|[D, a]\|$. 

However, for technical flexibility it is useful to consider the more general situation in which $A$ is just an order-unit space. The definition of an order-unit space is due to Kadison [16], influenced by work of Stone. For our present purposes it is sufficient to know that any order-unit space can be realized concretely (in many ways) as a real linear space of self-adjoint operators on a Hilbert space, containing the identity operator (the order unit), and equipped with the usual partial ordering and norm on operators. An order-unit space $A$ has a state-space, $S(A)$, defined just as for $C^{\ast}$-algebras. We can consider a seminorm, $L$, on an order-unit space, and use it to define a metric, $\rho_{L}$, on $S(A)$, much as above, by

$$\rho_{L}(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}.$$ 

To be useful we need some conditions on $L$. As first condition, we require for convenience that $L(1) = 0$ where 1 is the order-unit. This is well-motivated by the case of ordinary metric spaces. But the main requirement which we make is that the topology on $S(A)$ from $\rho_{L}$ should coincide with the weak-∗ topology. This is motivated by the result of Kantorovich for ordinary metric spaces which we mentioned above. We view the role of this requirement as follows. Given a compact topological space $X$, there are plenty of metrics on $X$ as a set, but when we say “a metric on $X$” we nearly always also have in mind “whose topology coincides with the given topology on $X$”. We consider the requirement that $\rho_{L}$ give $S(A)$ the weak-∗ topology to be the analog of this idea for seminorms on order-unit spaces.

We can always restrict to the subspace on which $L$ takes finite values. This does not change $\rho_{L}$. (We do not require our order-unit spaces to be complete.) Thus we make [33]:
DEFINITION 1.1. Let $A$ be an order-unit space, and let $L$ be a seminorm on $A$ taking finite values. We say that $L$ is a Lip-norm if

1) $L(1) = 0$.
2) The topology on $S(A)$ from $\rho_L$ coincides with the weak-$*$ topology.

By a compact quantum metric space we mean a pair $(A, L)$ where $A$ is an order-unit space and $L$ is a Lip-norm on $A$.

It is easily seen that condition 2 implies that if $L(a) = 0$ then $a \in \mathbb{R}1$. If $L$ is actually a seminorm on a dense subalgebra of a unital $C^*$-algebra, $A$, or more generally on a (complex) self-adjoint linear subspace of bounded operators, then we require that $L(a) = L(a^*)$ for all $a \in A$. Then a simple argument (in section 2 of [33]) shows that $L$, and the restriction of $L$ to the order-unit space $A_{sa}$ of self-adjoint elements of $A$, determine the same metric on $S(A)$. We can then require that either one gives $S(A)$ the weak-$*$ topology.

Aside from the motivation given by the case of ordinary metric spaces, it is reasonable to ask why we care that $\rho_L$ give $S(A)$ the weak-$*$ topology. The topic of quantum metric spaces is still in its infancy, and it is far too soon to know what demands new important examples will bring. At the moment the main answer to this question which I have is that this requirement permits an effective notion of Gromov–Hausdorff distance for compact quantum metric spaces. We will discuss this in Section 5, followed by examples in Section 6.

2. Sources of examples

Of course, it is condition 2 in the definition of a Lip-norm which may be difficult to verify in naturally occurring examples. It can be reformulated in terms of $A$ itself in the following way [31]. Let $\hat{A} = A/\mathbb{C}1$, equipped with the corresponding quotient norm $\| \cdot \|\sim$.

THEOREM 2.1. Let $L$ be a seminorm on an order-unit space $A$ such that $L(1) = 0$. Set $B_1 = \{a \in A : L(a) \leq 1\}$.

a) Then $\rho_L$ gives $S(A)$ finite diameter iff the image of $B_1$ in $\hat{A}$ is bounded.
b) And $\rho_L$ gives $S(A)$ the weak-$*$ topology iff the image of $B_1$ in $\hat{A}$ is totally bounded (for $\| \cdot \|\sim$).

It can be shown by a somewhat unnatural construction [35] that every separable order-unit space has an abundance of Lip-norms (finite on dense subspaces). Certain other constructions are also known [35] [5] [11]. But most interesting constructions so far have come from actions of groups. In fact, we will eventually see that in a certain sense all Lip-norms come from actions of Lie groups, and in fact actions of $\mathbb{R}$.

Let $A$ be a unital $C^*$-algebra, let $G$ be a locally compact group, and let $\alpha$ be a strongly continuous action of $G$ on $A$ by automorphisms. To define a Lip-norm we must somewhere put in metric data. We do this by choosing a continuous length-function $\ell$ on $G$. This means that $\ell$ has values in $\mathbb{R}^+$, that $\ell(xy) \leq \ell(x) + \ell(y)$, that $\ell(x^{-1}) = \ell(x)$, and that $\ell(x) = 0$ iff $x = e$. Then we can define a seminorm, $L$, on $A$ by

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e\}.$$  

It can be seen by a “smoothing argument” that $\{a : L(a) < \infty\}$ is a dense $*$-subalgebra of $A$ if $\ell$ is a proper function. A proof of this for compact $G$ is given in
proposition 2.2 of [31]. But that proof can be easily modified to apply also to locally compact groups if \( \ell \) is proper, as follows. For each \( n \) let \( g_n = (n^{-1} - \ell)^+ \in C_c(G) \). Let \( f_n = c_n g_n \) where \( c_n \) is the positive constant such that \( \|f_n\|_1 = 1 \). Then \( \{f_n\} \) is an approximate identity for \( L^1(G) \), consisting of functions which are Lipschitz for the right-invariant metric on \( G \) defined by \( \rho(x,y) = \ell(xy^{-1}) \). Simple calculations then show that \( L(\alpha f_n(a)) < \infty \) for any \( a \in A \), where \( \alpha f_n(a) = \int f_n(x) \alpha_x(a) dx \).

If we are to have that \( L(a) = 0 \) only when \( a \in C1 \), it is clear that \( \alpha \) must be ergodic, in the sense that the only elements of \( A \) which are invariant under \( \alpha \) are those in \( C1 \). It is shown in [31] that:

**Theorem 2.2.** If \( G \) is compact and \( \alpha \) is ergodic, then \( L \) (restricted to \( A_{sa} \)) is a Lip-norm.

This easily fails if \( G \) is not compact.

If \( G \) is a Lie group, then we can consider the set \( A^{\infty} \) of smooth elements of \( A \) for the action \( \alpha \), that is, elements \( a \in A \) such that the function \( x \mapsto \alpha_x(a) \) on \( G \) is infinitely differentiable. Then \( A^{\infty} \) is a dense \( * \)-subalgebra of \( A \) [4], and \( \alpha \) gives a representation (also denoted by \( \alpha \)) of the Lie algebra, \( g \), of \( G \) by derivations on \( A^{\infty} \). Given \( a \in A^{\infty} \), we can define its differential, \( da \), to be the operator from \( g \) into \( A^{\infty} \) defined by \( da(X) = \alpha_X(a) \) for \( X \in g \). If we bring metric data into the situation by choosing a norm on \( g \), then the norm of \( da \) as an operator from \( g \) to \( A \) is defined, and we can define a seminorm \( L \) on \( A^{\infty} \) by \( L(a) = ||da|| \). If \( G \) is compact then we can again show [31] that \( L \) is a Lip-norm, by reducing this situation to that of Theorem 2.2.

In general, compact groups have many ergodic actions on unital \( C^* \)-algebras. Here are three constructions.

1) If \( U \) is an irreducible unitary representation of a compact group \( G \) on a Hilbert space \( \mathcal{H} \), then we can define an action, \( \alpha \), of \( G \) on \( A = \mathcal{B}(\mathcal{H}) \) by \( \alpha_x(T) = U_x T U_x^* \). This action is ergodic.

2) If \( G \) is a compact Abelian group and if \( c \) is a 2-cocycle with values in \( T \) on its (discrete) dual group \( \hat{G} \), then we can define [43] the twisted group \( C^* \)-algebra \( A = C^*(\hat{G}, c) \). There is a natural action, \( \alpha \), of \( G \) on \( A \) given by \( \alpha_x(f)(\gamma) = \langle x, \gamma \rangle f(\gamma) \) for \( f \in \ell^1(\hat{G}) \) and \( \gamma \in \hat{G} \). This action (called the “dual action”) is ergodic.

3) If \( H \) is a closed subgroup of \( G \) and if \( \beta \) is an action of \( H \) on a unital \( C^* \)-algebra \( B \), then we can form the induced \( C^* \)-algebra \( A \), consisting of the continuous \( B \)-valued functions \( F \) on \( G \) which satisfy the condition that \( F(xs) = \beta_s(F(x)) \) for \( x \in G \) and \( s \in H \). The action \( \alpha \) of left translation by elements of \( G \) carries \( A \) into itself, and it is easily verified that if \( \beta \) is ergodic then so is \( \alpha \).

We can then combine this inducing construction with the two previous constructions to obtain many ergodic actions. But I do not know of other constructions of ergodic actions of compact groups. For \( G \) Abelian the second and third constructions give all ergodic actions [24]. But for non-Abelian compact groups there is no known classification of the possible ergodic actions. For example, it seems still to be unknown whether \( SU(4) \) has any ergodic actions other than those given by the above constructions [41] [39] [40], and in particular, any ergodic actions on infinite-dimensional unital \( C^* \)-algebras which are simple.
Anyway, Theorem 2.2 combined with the above three constructions gives many examples of compact quantum metric spaces. We note in particular that for the non-commutative tori, $A_0$, which come from the second construction described above, the dual action is an action of $T^d$. Thus any continuous length function on $T^d$ (of which there is an abundance) gives Lip-norms on the $A_0$’s. Here $\theta$ is a real $d \times d$ skew-symmetric matrix, $c_0$ is the bicharacter on $\mathbb{Z}^d$ defined by $c_0(x, y) = e^{2\pi i (x \cdot y)}$, and $A_0$ is a twisted group $C^*$-algebra $C^*(\mathbb{Z}^d, c_0)$.

When $G$ is not compact, one will need stronger conditions on $\alpha$ in order to obtain a Lip-norm, but this direction has barely been explored. I know of only two relevant papers, both of which deal with the action of the Heisenberg Lie group on non-commutative Heisenberg manifolds [27]. In the first, by Weaver [42], the metric structures come from sub-Riemannian metrics, and so it does not quite fit into the above framework. The second, by Chakraborty [6], uses a Lip-norm which is not defined in terms of the operator norm, so again does not fit exactly into the above framework. Thus it is still unclear what happens for non-commutative Heisenberg manifolds within our present framework.

We find further examples of compact quantum metric spaces by a simple application of Theorem 2.1b to obtain:

**Proposition 2.3.** Let $L$ be a Lip-norm on an order-unit space $A$, and let $B$ be a subspace of $A$ which contains the order unit (so that $B$ is an order-unit space). Then the restriction of $L$ to $B$ is a Lip-norm.

In particular, if $A$ is (the self-adjoint part of) a dense $*$-subalgebra of a unital $C^*$-algebra, and if $B$ is a unital $*$-subalgebra of $A$, then any Lip-norm on $A$ restricts to a Lip-norm on $B$. When used in conjunction with Theorem 2.2 this gives many more examples of compact quantum metric spaces. We will only describe one specific class of examples here. Let $A_0$ be the non-commutative 2-torus with unitary generators $U$ and $V$ satisfying the relation $VU = e^{2\pi i \theta} UV$ for some real number $\theta$. As mentioned above, there is a natural ergodic action $\alpha$ of $T^2$ on $A_0$, the dual action. For any continuous length function on $T^2$ we obtain a Lip-norm $L$ on $A_0$ by using $\alpha$. The algebraic $*$-subalgebra, $A_0^\alpha$, generated by $U$ and $V$ is carried into itself by $\alpha$, and $L$ is finite on $A_0^\alpha$. There is a (unique) involutory automorphism, $\beta$, of $A_0$ determined by $\beta(U) = U^{-1}$ and $\beta(V) = V^{-1}$. Thus the 2-element group acts on $A_0$, and carries $A_0^\alpha$ into itself. Let $B_0$ be the fixed point algebra of this automorphism. Then $B_0$ is not carried into itself by $\alpha$, so that we can not apply Theorem 2.2 directly to $B_0$. But $B_0 \cap A_0^\alpha$ is easily seen to be a dense $*$-subalgebra of $B_0$, and the restriction of $L$ to $B_0 \cap A_0^\alpha$ will be a Lip-norm by Proposition 2.3. Now the $B_0$’s were extensively studied by Bratteli, Elliott, Evans and Kishimoto, under the name “non-commutative spheres”. Bratteli and Kishimoto showed [8] that $B_0$ is actually an AF $C^*$-algebra when $\theta$ is irrational. So we have obtained in this way some fairly natural and interesting examples of Lip-norms on AF $C^*$-algebras.

Another interesting class of examples of Lip-norms on AF $C^*$-algebras is studied in [21], in connection with the development of a notion of “dimension” for compact quantum metric spaces, and a notion of entropy for automorphisms of ($C^*$-algebraic) compact quantum metric spaces. Also among the examples to which this notion of entropy is applied are automorphisms of non-commutative tori.
3. Dirac operators

Let $G$ be a Lie group, not necessarily compact, and let $\beta$ be an action of $G$ on a unital $C^*$-algebra $B$. For a given continuous length function on $G$ we can define a seminorm, $L$, on $B$ as before. But now we will not assume that $\beta$ is ergodic, and so we may have a large subalgebra of $B$ on which $L$ takes value 0. It may nevertheless happen that for suitable unital $\ast$-subalgebras $A$ (or even order-unit subspaces) of $B$, which are not carried into themselves by $\beta$, the restriction of $L$ to $A$ is a Lip-norm.

Actually, up to now the only situation that I know of in which this possibility has been used is that in which we have a (strongly-continuous) unitary representation $U$ of $G$ on a possibly infinite-dimensional Hilbert space $\mathcal{H}$. Then, much as in the first construction of Section 2 we define an action, $\beta$, of $G$ on $B(\mathcal{H})$ by $\beta_x(T) = U_x T U_x^*$. In general this action will not be strongly continuous. We let $B (= B_\beta)$ be the largest subalgebra of $B(\mathcal{H})$ on which $\beta$ is strongly continuous. (Then $B$ is weak-operator dense in $B(\mathcal{H})$ by lemma 7.5.1 of [26].) Furthermore, $B$ is carried into itself by $\beta$. Thus we can use $(B, \beta)$ as in the previous paragraph, and ask, for any given unital $C^*$-subalgebra $A$ of $B$ (or any order-unit subspace of $B$) whether the restriction to $A$ of a seminorm on $B$ coming from a continuous length function on $G$ is a Lip-norm.

But actually, up to now, the only situation that I know of in which this possibility has been used is that in which $G = \mathbb{R}$, with its usual length function. This is the very important case of Connes’ Dirac operators. Any strongly continuous unitary representation $U$ of $\mathbb{R}$ has an infinitesimal generator, which we will denote here by $D$, and which is a (usually unbounded) self-adjoint operator on $\mathcal{H}$. By means of the functional calculus for unbounded self-adjoint operators we have $U_t = e^{itD}$ for $t \in \mathbb{R}$. Let $\beta$ be defined in terms of $U$ as above, and let $B_\beta$ be the $C^*$-subalgebra of $B(\mathcal{H})$ on which $\beta$ is strongly continuous. Let $T \in B_\beta$, and assume further that $t \mapsto \beta_t(T)$ is once differentiable for the operator norm. Just from the definition of $D$ being the infinitesimal generator of $U$ it follows easily that $T$ carries the domain of $D$ into itself and that $[D, T]$ is bounded on that domain and so extends to a bounded operator on $\mathcal{H}$. Conversely, if $T$ carries the domain of $D$ into itself and if $[D, T]$ is bounded, then $T$ is in $B_\beta$. (See the first line of the proof of corollary 10.16 of [14].

If $t \mapsto \beta_t(T)$ is twice differentiable, so that $t \mapsto \beta_t([D, T])$ is differentiable, then $[D, T] \in B_\beta$, and the main calculation in the proof of corollary 10.16 of [14] shows that the derivative of $t \mapsto \beta_t(T)$ at $t = 0$ is $[D, T]$. (We remark that it can be useful, more generally, to use the fact that $B(\mathcal{H})$ is a von Neumann algebra so that the weak operator topology is available. We can then just ask that derivatives exist for the weak operator topology. See in particular proposition 3.2.53 of [4]. But we will not pursue this aspect here.)

**Lemma 3.1.** Suppose that $t \mapsto \beta_t(T)$ is twice differentiable as above. Then

$$\|[D, T]\| = \sup\{\|\beta_t(T) - T\|/|t| : t \neq 0\}.$$ 

**Proof.** From the remarks above it follows that the derivative of $t \mapsto \beta_t(T)$ is $t \mapsto \beta_t([D, T])$, and that this derivative is norm-continuous. Thus

$$\beta_t(T) - T = \int_0^t \beta_s([D, T])ds,$$
and so \( \| \beta_t(T) - T \| \leq |t| \| [D, T] \| \). This gives inequality in one direction. But for any \( \epsilon > 0 \) we can find \( t > 0 \) close enough to 0 that \( \| \beta_s([D, T]) - [D, T] \| < \epsilon \) for \( 0 \leq s \leq t \). Then from the above integral we obtain
\[
\| \beta_t(T) - T \| \geq t(\| [D, T] \| - \epsilon),
\]
which yields the reverse inequality. \( \square \)

Again, one can extend the above lemma by using the weak operator topology. But for many purposes one wants to deal just with elements which are at least twice differentiable—see the discussion of regular tuples in section 10.3 of [13].

We can now relate our earlier construction of seminorms by means of length functions to that in terms of “Dirac” operators as follows, where the length function on \( \mathbb{R} \) is the usual one, \( \ell(t) = |t| \).

**Proposition 3.2.** Let \( D \) and \( B_\beta \) be as above. Let \( A \) be a unital \( \ast \)-subalgebra of \( B_\beta \) such that for any \( a \in A \) both \( [D, a] \) and \( [D, [D, a]] \) are bounded. Then the seminorms on \( A \) defined by
\[
L(a) = \| [D, a] \|
\]
and
\[
L(a) = \sup \{ \| \beta_t(a) - a \|/|t| : t \neq 0 \}
\]
coincide.

In this setting we can thus again ask whether the seminorm \( L \) gives \( S(A) \) the weak-\( \ast \) topology. Usually it will not. For example, if all of the elements of \( A \) commute with \( D \) then \( L \equiv 0 \). But there are some quite interesting situations for which it is known that the answer is affirmative. We give four classes of examples.

**Example 3.3.** This first example is the source of the whole topic, namely the ordinary Dirac operator for a compact spin Riemannian manifold \( M \), and Connes’ observation that one recovers the ordinary metric on \( M \) from the Dirac operator. More precisely, if \( f \in C^\infty(M) \) and if \( f \) is viewed as an operator by “pointwise multiplication” on the spinor bundle with its Hilbert space structure, then Connes shows that \( \| [D, f] \| \) coincides with the usual Lipschitz seminorm of \( f \) from the ordinary metric. We remark that the Dirac operator is usually defined for a Spin\(^c\) manifold, because these are the ones which have a spinor bundle. But if one is only interested in the smooth structures and ordinary metric of a Riemannian manifold (from which the Riemannian metric can be recovered), and if the homological information which the Dirac operator contains is not so important, then one can treat any compact Riemannian manifold. One simply replaces the spinor bundle (which won’t exist if the manifold is not Spin\(^c\)) with the Clifford algebra bundle itself equipped with a continuous choice of faithful tracial states so as to give a Hilbert space structure to the space of cross sections. Then one uses the corresponding left regular representation of the Clifford algebra bundle. The corresponding Dirac-like operator will again give the smooth structure and the usual Lipschitz norm for each \( f \in C^\infty(M) \).

**Example 3.4.** The type of construction used above is also used in our second class of examples, which involves the situation considered right after Theorem 2.2 consisting of a compact Lie group \( G \) acting ergodically on a unital \( C^\ast \)-algebra \( A \). If we put an arbitrary inner-product on the Lie algebra \( \mathfrak{g} \) of \( G \), and form the corresponding Clifford algebra, then the usual construction of a Dirac operator
can be imitated to give a Hilbert space $\mathcal{H}$, an operator $D$ on it, and a faithful representation of $A$ on $\mathcal{H}$. It is shown in \cite{31} that the corresponding seminorm on $A$ is a Lip-norm.

**Example 3.5.** This class of examples concerns the $\theta$-deformed spheres and manifolds of Connes, Landi and Dubois–Violette \cite{12,10,9,11,22}. These can be constructed whenever one has an action of the $d$-dimensional torus $T^d$, $d \geq 2$, on any compact manifold $M$. For any skew-symmetric real $d \times d$ matrix $\theta$ one considers the corresponding non-commutative torus $A_\theta$ with its dual action of $T^d$ which we mentioned earlier. Then $T^d$ has a diagonal action on $C(M) \otimes A_\theta$, and we let $M_\theta$ be the fixed-point subalgebra for this diagonal action. This construction is a reformulation, for the special case of actions of $T^d$, of the general deformation quantization construction for actions of $\mathbb{R}^d$ described in \cite{30}. This reformulation is also discussed in \cite{37,36}, and the relation with the quantum groups of \cite{29} is discussed in \cite{37}.

Connes and Landi \cite{12,11} show that when $M$ is a spin Riemannian manifold, and when the action $\alpha$ is smooth, leaves the Riemannian metric invariant, and lifts to the spin bundle, then there is a natural Dirac operator for $M_\theta$. Hanfeng Li shows in his doctoral thesis \cite{22} that the seminorm on $M_\theta$ obtained from this Dirac operator is a Lip-norm.

**Example 3.6.** This class of examples returns to the main example in Connes’ first paper on this subject \cite{8}. We now let $G$ be a discrete group, and consider its reduced $C^*$-algebra $C^*_r(G)$ acting on $\ell^2(G)$. Let $\ell$ be a length function on $G$. We take as our Dirac operator the operator $D = M_\ell$ of pointwise multiplication by $\ell$ on $\ell^2(G)$. It is easily seen \cite{8} that for any $f \in C_c(G)$, viewed as an operator in $C^*_r(G)$, the operator $[D,f]$ is bounded. Thus again we can ask if the seminorm $L(f) = \| [D,f] \|$ is a Lip-norm. An affirmative answer is now known for two classes of groups. In \cite{35} it is shown that if $G = \mathbb{Z}^d$ and if $\ell$ is either a word-length function or the restriction to $G$ of a norm on $\mathbb{R}^d$, then the answer is affirmative. (The proof of this requires a substantial and interesting development, involving, among things, boundaries for non-compact metric spaces.) In \cite{25} (and see also \cite{11}) it is shown, by means of techniques which are entirely different from those used in \cite{35}, that the answer is affirmative if $G$ is a hyperbolic group and $\ell$ is a word-length function, as well as for certain reduced free-product $C^*$-algebras. The techniques involve filtered $C^*$-algebras and Dirac operators determined by filtrations. They do not apply to the case of $G = \mathbb{Z}^d$. It would be interesting to find a unified proof for the two cases.

There is ample opportunity to discover additional natural examples of compact quantum metric spaces. And there are many further aspects to explore. For example, what are the isometry groups \cite{39} of the above examples? What about quantum isometry groups \cite{38,2}? What is the analog of a continuous length function for a compact quantum group, such that it defines Lip-norms on quantum spaces on which the quantum group acts?

### 4. Universality of the Dirac approach

Although the Dirac operator construction appears fairly special, we now show that in fact every compact quantum metric space can be obtained from the Dirac operator approach (though this may not be the most useful presentation). We
should make clear now that by this point the term “Dirac operator” does not refer to any special kind of self-adjoint operator, but rather to how a self-adjoint operator is being used, namely to provide metric data by means of its commutant with the elements of the algebra (or order-unit space) which specifies the “space” being metrized.

We begin by considering ordinary compact metric spaces \((X, ρ)\). Let

\[
Z = \{(x, y) \in X \times X : x \neq y\}
\]

Choose any positive measure on \(X\) whose support is all of \(X\), and let \(ω\) be the restriction to \(Z\) of the square of this measure on \(X \times X\). Let

\[
A = C(X), \text{ acting on } L^2(Z, ω) \text{ by } (fξ)(x, y) = f(x)ξ(x, y).
\]

This is a faithful representation. Let \(D\) be the operator on \(L^2(Z, ω)\) defined by

\[
(Dξ)(x, y) = ξ(y, x)/ρ(y, x),
\]

for those \(ξ\) for which \(Dξ \in L^2(Z, ω)\). Some simple calculations \[33\] show that \([D, f]\) is a bounded operator exactly if \(f\) is a Lipschitz function for \(ρ\), and that in this case \(\|[D, f]\|\) is exactly the Lipschitz constant of \(f\). Thus we can recover \(ρ\) from \(D\).

For the ordinary Dirac operator of a compact Riemannian manifold it is an important fact that it has compact resolvant, that is, the Hilbert space has a basis consisting of eigenvectors, the eigensubspaces have finite dimensions, and any bounded interval of \(R\) contains only a finite number of eigenvalues. This led Connes to require this property of a Dirac operator in his definition of a spectral triple. Connes shows that many wonderful properties come from the hypothesis of compact resolvant (such as the existence of a “volume” if the spectral triple is “\(p\)-summable”). In other examples the compact resolvant property seems tied to having some kind of differential structure on \(X\). The operator \(D\) constructed earlier for a general compact metric space will usually not have compact resolvant. Thus we are led to:

**Problem 4.1.** Characterize those compact metric spaces for which there is a spectral triple (for which \(D\) has compact resolvant) which gives their metric. When can the spectral triple be chosen to be \(p\)-summable for some \(p\)? How does one characterize the minimal \(p\)?

We now turn to the non-commutative case, or more generally the order-unit case. Let \(L\) be a Lip-norm on an order-unit space \(A\). Then \((S(A), ρ_L)\) is an ordinary compact metric space, and so, as above, there is a representation of \(C(S(A))\) on a Hilbert space \(H\) and a self-adjoint operator \(D\) on \(H\) which gives \(ρ_L\). But there is a canonical inclusion of \(A\) into \(C(S(A))\) given by \(a(μ) = μ(a)\) for \(a ∈ A\) and \(μ ∈ S(A)\).

When \(A\) is a \(C^*\)-algebra, this inclusion is not an algebra homomorphism.) Under this inclusion elements of \(a\) are carried to functions which are Lipschitz for \(ρ_L\) with Lipschitz constant \(L(a)\). On composing this inclusion with the representation of \(C(S(A))\) on \(H\) we obtain a faithful representation of \(A\) preserving the order unit structure. Furthermore, \(L(a) = \|[D, a]\|\) for every \(a ∈ A\).

**Problem 4.2.** For the case of \(C^*\)-algebras \(A\) characterize those Lip-norms which come from triples \((A, H, D)\), where we do not require that \(D\) have compact resolvant, but we do require that the representation of \(A\) on \(H\) is a unital \(*\)-algebra homomorphism.
5. Gromov–Hausdorff distance

Let \((Z, \rho)\) be an ordinary compact metric space. For a subset \(Y\) of \(Z\) and a positive real number \(r\) define the open \(r\)-neighborhood, \(\mathcal{N}_r^\rho(Y)\), of \(Y\) by
\[
\mathcal{N}_r^\rho(Y) = \{z \in Z: \text{there is a } y \in Y \text{ with } \rho(z, y) < r\}.
\]
Hausdorff defined the distance, \(\text{dist}^\rho_H(X, Y)\), between two closed subsets \(X\) and \(Y\) of \(Z\) by
\[
\text{dist}^\rho_H(X, Y) = \inf \{r : Y \subseteq \mathcal{N}_r^\rho(X) \text{ and } X \subseteq \mathcal{N}_r^\rho(Y)\}.
\]
This defines an ordinary metric on the set of closed subsets of \(Z\), for which this set is compact.

Gromov generalized this notion of distance to one between any two compact metric spaces. (See \([15]\).) His notion is now called Gromov–Hausdorff distance. Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be two compact metric spaces. The Gromov–Hausdorff distance between them, denoted by \(\text{dist}^\rho_G(X, Y)\), is defined as follows. Let \(X \cup Y\) denote the disjoint union of \(X\) and \(Y\). Let \(\mathcal{M}(\rho_X, \rho_Y)\) be the set of all metrics on the compact set \(X \cup Y\) whose restrictions to \(X\) and \(Y\) are \(\rho_X\) and \(\rho_Y\) respectively. Then set
\[
\text{dist}^\rho_G(X, Y) = \inf \{\text{dist}^\rho_H(X, Y) : \rho \in \mathcal{M}(\rho_X, \rho_Y)\}.
\]
For simplicity of notation we are not explicitly indicating the metrics on the left-hand side.

There is by now a large and rich literature concerning the Gromov–Hausdorff limits of compact Riemannian manifolds and related spaces. As just one example, but one in which the role of Dirac operators is prominent, and which might suggest phenomena worth investigating in the non-commutative setting, we mention \([33]\).

We want to define a corresponding notion of Gromov–Hausdorff distance between compact quantum metric spaces \([33]\). Thus let \((A, L_A)\) and \((B, L_B)\) be compact quantum metric spaces. We form the order-unit space \(A \oplus B\), and consider its canonical projections \(\pi_A\) and \(\pi_B\) onto \(A\) and \(B\), respectively. We let \(\mathcal{M}(L_A, L_B)\) be the set of all Lip-norms \(L\) on \(A \oplus B\) whose quotient seminorms on \(A\) and \(B\) are \(L_A\) and \(L_B\), respectively. This means that for each \(a \in A\) we have
\[
L_A(a) = \inf \{L((a, b)) : b \in B\},
\]
and similarly for \(L_B\). There are evident canonical injections of \(S(A)\) and \(S(B)\) into \(S(A \oplus B)\). Through these injections we will simply view \(S(A)\) and \(S(B)\) as closed subsets of \(S(A \oplus B)\). From the requirement that \(L_A\) is the quotient of \(L\) it follows (proposition 3.1 of \([33]\)) that the restriction of \(\rho_L\) to \(S(A)\) coincides with \(\rho_{L_A}\), and similarly for \(B\). Thus it is reasonable to define the distance between \((A, L_A)\) and \((B, L_B)\), which we denote by \(\text{dist}_q(A, B)\), by
\[
\text{dist}_q(A, B) = \inf \{\text{dist}^\rho_H(S(A), S(B)) : L \in \mathcal{M}(L_A, L_B)\}.
\]
For this to be well-defined and to work smoothly it is essential that the metrics from our Lip-norms give the state spaces the weak-* topology, for which the state spaces are compact. This is an important reason for our emphasis in the earlier sections of this paper on proving that the seminorms we consider do give metrics whose topology is the weak-* topology.

An important and non-obvious theorem concerning ordinary Gromov–Hausdorff distance is that if the Gromov–Hausdorff distance between two compact metric spaces is 0 then the two metric spaces are isometric \([16]\). Thus Gromov–Hausdorff
distance is actually a metric on the set of isometry-classes of compact metric spaces. A similar result holds for our quantum Gromov–Hausdorff distance (theorem 7.8 of [33]), but for this purpose the order-unit spaces of our compact quantum metric spaces must be suitably completed with respect to their Lip-norm, so that isometries which should exist will have a place to land. (The isometries form a compact group in a natural way [33].)

However, a defect of the present theory is that two \( C^* \)-algebras can have self-adjoint parts which are isomorphic as order-unit spaces, while the algebras themselves are not isomorphic. This will happen, for example, for those \( C^* \)-algebras which are not isomorphic to their opposite algebra. Thus two \( C^* \)-algebras equipped with Lip-norms can be of distance 0 from each other without the algebras being isomorphic.

One way of avoiding this defect has been developed by Li [22]. He defines for \( C^* \)-algebras a variant of our quantum Gromov–Hausdorff distance which explicitly uses the product in the \( C^* \)-algebras. (Thus it has no counterpart for order-unit spaces.) Li shows that his variant has many favorable properties, and that it applies to examples of the types which we discuss in the next section. Underlying Li’s variant is another variant which does apply to order-unit spaces, and which he calls “order-unit space quantum Gromov-Hausdorff distance”. It involves looking at the Gromov-Hausdorff distance between sets like \( L_1 = \{ a \in A : L(a) \leq 1 \text{ and } ||a|| \leq 1 \} \) (which are pre-compact, as follows from Theorem 2.1) in the order-unit space itself, rather than using the state spaces. Li shows that this variant has some substantial technical advantages, for example in understanding when a continuous field of \( C^* \)-algebras, equipped with Lip-norms in a continuous way, will be continuous for Gromov-Hausdorff distance.

In section 7 we will discuss another way, due to Kerr [20], of avoiding the defect. It involves using operator systems and their matricial norms.

The set of isometry classes of compact quantum metric spaces, equipped with \( \text{dist}_q \), is a complete metric space (theorem 12.11 of [33]). There is also a natural analog of a useful theorem of Gromov describing the totally bounded subsets of this metric space, in terms the number of \( \epsilon \)-balls needed to cover \( S(A) \) for the various \( A \)’s in such a subset (theorem 13.5 of [33]).

If \((X, \rho)\) is an ordinary compact metric space, and if \( A_X \) is the algebra of real-valued Lipschitz functions on \( X \), then \((A_X, L_{\rho})\) is a compact quantum metric space. If \((X, \rho_X)\) and \((Y, \rho_Y)\) are compact metric spaces, then one can show that \( \text{dist}_q (A_X, A_Y) \leq \text{dist}_{GH} (X, Y) \). But equality may fail. A simple explicit example of this failure was found by Hanfeng Li. (See appendix 1 of [33].) This can be viewed as a defect in the theory. But its origin is not difficult to understand. For ordinary Gromov–Hausdorff distance one is asking, in effect, that \( X \) and \( Y \) be combined into a metric space in such a way that the pure states of \( C(X) \) are close to the pure states of \( C(Y) \) (and the other way around); whereas for \( \text{dist}_q \) one is asking only that each pure state of \( X \) be close to some state of \( Y \), not necessarily a pure state. For non-commutative \( C^* \)-algebras it is known that the set of pure states need not be closed, and can be a somewhat bad set, often fairly inaccessible. Thus it is not clear how to develop a useful theory in which one requires pure states of one algebra to be close to pure states of the other. On the other hand, for any faithful representation of a \( C^* \)-algebra the finite convex combinations of vector states will be dense in the state space, and so provide an accessible set of states which can...
serve well for the theory that we have developed. (Note also that for a simple unital infinite dimensional $C^\ast$-algebra $A$ the closure of the set of pure states is all of $S(A).$)

6. Examples of quantum Gromov-Hausdorff convergence

For two given ordinary compact metric spaces it is seldom possible to calculate the Gromov–Hausdorff distance between them precisely. What has been found useful in a number of situations is to calculate upper bounds for the distance. (Lower bounds seem much harder to obtain.) Then one can try to verify that certain sequences of compact metric spaces converge to a given space (or form Cauchy sequences, which must have limit spaces).

The above comments apply equally well for quantum Gromov–Hausdorff distance. There are now three interesting classes of examples where convergence has been established.

Example 6.1. For a fixed integer $d$ consider the non-commutative tori $A_\theta$ discussed in Section 2, where $\theta$ ranges over real $d \times d$ skew-symmetric matrices. We have the dual action of $T^d$ on each $A_\theta$ as discussed earlier. Fix a continuous length function on $T^d$, and for each $\theta$ let $L_\theta$ be the corresponding Lip-norm on $A_\theta$ according to Theorem 2.2. Then in [33] it is shown that if a sequence $\{\theta_n\}$ of matrices converges to a matrix $\theta$, then the sequence $(A_{\theta_n},L_{\theta_n})$ converges to $(A_\theta,L_\theta)$ for quantum Gromov–Hausdorff distance. In other words, the mapping from matrices $\theta$ to quantum metric spaces $(A_\theta,L_\theta)$ is continuous for the usual topology on the set of matrices and for quantum Gromov–Hausdorff distance.

Example 6.2. This example works for any compact semisimple Lie group when suitably rephrased [34], but for simplicity of exposition we describe it only for $G = SU(2)$. Fix a continuous length function on $SU(2)$. Let $A = C(S^2)$, for $S^2$ the two-sphere, and consider the usual action of $SU(2)$ on $A$ through the homomorphism from $SU(2)$ to $SO(3)$ and the action of $SO(3)$ on $S^2$. This action on $A$ is ergodic. Equip $A$ with the corresponding Lip-norm as in Theorem 2.2. For each $n$ let $(U_n,\mathcal{H}_n)$ be the irreducible representation of $SU(2)$ of dimension $n$, and let $A_n = \mathcal{B}(\mathcal{H}_n)$, equipped with the ergodic action consisting of conjugating by $U_n$, as described in Section 2. (Thus $A_n$ is a full matrix algebra.) Then equip each $A_n$ with the corresponding Lip-norm, $L_n$. In [34] it is shown that the sequence $\{(A_n,L_n)\}$ converges to $\{(A,L)\}$ for quantum Gromov–Hausdorff distance. The proof involves Berezin symbols, which are closely related to coherent states. This example gives a possible precise meaning to statements occurring in places in the theoretical physics literature to the effect that a sequence of matrix algebras converges to the sphere (or some related space). See [34] for references.

Example 6.3. Let the $M_\theta$'s be as in Section 4, for a fixed spin Riemannian manifold $M$ with action of $T^d$, and varying $\theta$'s. For each $\theta$ let $L_\theta$ be the Lip-norm on $M_\theta$ coming from the Dirac operator. Hanfeng Li shows in [22] that, much as in Example 6.1 above, if $\{\theta_n\}$ is a sequence of matrices which converges to a matrix $\theta$, then the sequence $\{(M_{\theta_n},L_{\theta_n})\}$ converges to $(M_{\theta},L_{\theta})$ for quantum Gromov–Hausdorff distance.

Clearly we are still near the beginning in producing examples of quantum Gromov–Hausdorff convergence. But the literature of high-energy physics and
string theory suggests a variety of possible examples. See the introduction of \[33\] for a number of references. Furthermore, the consequences of convergence are essentially unexplored in the non-commutative case, but the discussion of “degeneration of Riemannian manifolds” in the literature on classical Riemannian manifolds (see references in \[33\], and, in particular, \[23\]) suggests many interesting questions concerning the non-commutative case.

7. Matricial quantum Gromov–Hausdorff distance

A matricial version of quantum Gromov–Hausdorff distance has been developed by David Kerr \[20\]. For a variety of reasons it is natural to seek such a version, but one benefit of it is that it provides one way to repair the defect of quantum Gromov–Hausdorff distance mentioned in Section 5, namely that two unital C*-algebras equipped with Lip-norms can have distance 0 yet not be isomorphic.

The setting is that of operator systems \[13\]. An operator system is a self-adjoint subspace of bounded operators on a Hilbert space which contains the identity operator. (Equivalently, it can be a corresponding subspace of a unital C*-algebra.) The self-adjoint part of an operator system is an order-unit space. But the crucial difference is that now one is dealing with an order-unit space which has a specific choice of representation as operators on a Hilbert space (or in a unital C*-algebra).

The essential feature of this situation is that if \(A\) is an operator system, and if \(M_m(A)\) denotes the linear space of \(m \times m\) matrices with entries in \(A\) for each \(m\), then there is a canonical operator norm and partial order on \(M_m(A)\) coming from viewing the matrices as operators on the sum of \(m\) copies of the Hilbert space. Different (isometric) representations of an order-unit space can give quite different operator norms and partial orders on \(M_m(A)\) for \(m \geq 2\), and thus give distinct operator systems.

If \(A\) and \(B\) are operator systems and if \(\varphi\) is a linear map from \(A\) to \(B\), then we define the corresponding map \(M_m(\varphi)\) from \(M_m(A)\) to \(M_m(B)\) by \((M_m(\varphi))(a_{ij}) = (\varphi(a_{ij}))\). We say that \(\varphi\) is \(m\)-positive if \(M_m(\varphi)\) is positive, and that \(\varphi\) is completely positive if \(M_m(\varphi)\) is positive for all \(m\).

Now let \(M_n = M_n(\mathbb{C})\) have its canonical matricial structure from its representation on \(\mathbb{C}^n\). We let \(UCP_n(A)\) denote the set of unital completely positive maps from \(A\) into \(M_n\). This is the \(n\)-th matricial state space of \(A\). It has the evident point-norm topology, for which it is compact. Notice that \(S(A) = UCP_1(A)\).

Now let \(L\) be a Lip-norm on (the self-adjoint part of) \(A\). Then for each \(n\) we can define a metric, \(\rho_{L,n}\), on \(UCP_n(A)\) by

\[
\rho_{L,n}(\varphi, \psi) = \sup\{||\varphi(a) - \psi(a)|| : L(a) \leq 1\}.
\]

Kerr shows that because \(L\) is a Lip-norm the topology from \(\rho_{L,n}\) on \(UCP_n(A)\) coincides with the point-norm topology. If \(\Phi : A \to B\) is a unital completely positive map from \(A\) onto an operator system \(B\), then \(UCP_n(B)\) embeds into \(UCP_n(A)\) for each \(n\) by composition with \(\Phi\). Kerr shows that each of these embeddings is isometric when \(UCP_n(B)\) is equipped with the metric from the quotient on \(B\) of \(L\).

The pieces are then in place to imitate the definition of quantum Gromov–Hausdorff distance.

**Definition 7.1.** (Definition 3.2 of \[20\]) Let \((A, L_A)\) and \((B, L_B)\) be Lip-normed operator systems. For each \(n\) we define the \(n\)-distance, \(dist^n_n(A, B)\), between
(A, L_A) and (B, L_B) by
\[ \text{dist}_n^m(A, B) = \inf \{ \text{dist}_H^{\mu,n}(UCP_n(A), UCP_n(B)) : L \in M(L_A, L_B) \}. \]

If \( m > n \) then \( \text{dist}_n^m(A, B) \geq \text{dist}_n^m(A, B) \) (by lemma 4.10 of [20]), and we define the complete distance, \( \text{dist}_s(A, B) \) by
\[ \text{dist}_s(A, B) = \sup_n \{ \text{dist}_n^m(A, B) \}. \]

Kerr shows that these distances satisfy the triangle inequality. Furthermore, he shows that if \( \text{dist}_n^m(A, B) = 0 \) for some \( n \), then there is an isometric \( n \)-order isomorphism between \( A \) and \( B \). For \( C^* \)-algebras \( A \) and \( B \) a \( 2 \)-order isomorphism will be a \( * \)-algebra isomorphism (because by corollary 5 of [17] a unital order isomorphism \( \Psi \) will satisfy \( \Psi(a^2) = (\Psi(a))^2 \) for self-adjoint \( a \in A \), and Choi shows (corollary 3.2 of [7]) that if \( \Psi \) has this latter property and is also \( 2 \)-positive, then \( \Psi \) is a \( * \)-algebra homomorphism). Thus:

**Theorem 7.2.** Let \( A \) and \( B \) be unital \( C^* \)-algebras equipped with Lip-norms \( L_A \) and \( L_B \). If \( \text{dist}_s^2(A, B) = 0 \) then there is an isometric \( * \)-algebra isomorphism of \( A \) upon \( B \) (which carries \( L_A \) to \( L_B \)).

Kerr shows that the continuity of quantum Gromov–Hausdorff distance for noncommutative tori as described in Example 6.1, and for matrix algebras converging to the sphere (and other coadjoint orbits) as described in Example 6.2, carries over to complete quantum Gromov–Hausdorff distance.

Kerr also provides interesting matricial versions of the theorem on the completeness of the space of isometry classes of compact quantum metric spaces, and of the characterization of totally bounded subsets of that space.

**References**

[1] C. Antonescu and E. Christensen, Group \( C^* \)-algebras, metrics and an operator theoretic inequality, arXiv:math.OA/0211312.
[2] T. Banica, Quantum automorphism groups of small metric spaces, arXiv:math.QA/0304025.
[3] O. Bratteli and A. Kishimoto, Noncommutative spheres. III. Irrational rotations, Comm. Math. Phys. 147 (1992), no. 3, 605–624. MR 93g:58008
[4] O. Bratteli and D. W. Robinson, Operator algebras and quantum statistical mechanics. Vol. I, Springer–Verlag, New York, 1979. MR 81a:46070
[5] P. S. Chakraborty, From \( C^* \)-algebra extensions to CQMS, \( SU_q(2) \), Podles sphere and other examples, arXiv:math.OA/0210155.
[6] , Metrics On The Quantum Heisenberg Manifold, arXiv:math.OA/0112309.
[7] M. D. Choi, A Schwarz inequality for positive linear maps on \( C^* \)-algebras, Illinois J. Math. 18 (1974), 565–574. MR 50 #8089
[8] , Noncommutative geometry—year 2000, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 481–559, GAFA 2000 (Tel Aviv, 1999). MR 2003g:58010
[9] , A short survey of noncommutative geometry, J. Math. Phys. 41 (2000), no. 6, 3832–3866. MR 2001m:58016
[10] A. Connes and M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I. spherical manifolds and related examples, Comm. Math. Phys. 230 (2002), no. 3, 539–579. arXiv:math.QA/0107070.
[11] A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, Commun. Math. Phys. 221 (2001), 141–159. arXiv:math.QA/0011194.
[12] E. G. Effros and Z.-J. Ruan, Operator spaces, The Clarendon Press Oxford University Press, New York, 2000. MR 1 793 753.
[14] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, Elements of noncommutative geometry, Birkhäuser Boston Inc., Boston, MA, 2001. MR 1 789 831
[15] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser Boston Inc., Boston, MA, 1999. MR 2000d:53065
[16] R. V. Kadison, A representation theory for commutative topological algebra, Mem. Amer. Math. Soc. 1951 (1951), no. 7, 39. MR 13,3606
[17] , , , A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. of Math. (2) 56 (1952), 494–503. MR 14,481c
[18] L. V. Kantorovič, On the translocation of masses, C. R. (Doklady) Acad. Sci. URSS (N.S.) 37 (1942), 199–201. MR 5,174d
[19] L. V. Kantorovič and G. Š. Rubinštein, On a functional space and certain extremum problems, Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 1058–1061. MR 20 #1219
[20] D. Kerr, Matricial quantum Gromov-Hausdorff distance, arXiv:math.OA/0207282
[21] , , Dimension and dynamical entropy for metrized C*-algebras, Comm. Math. Phys. 232 (2003), no. 3, 501–534, arXiv:math.OA/0211043. MR 1 952 475
[22] Hanfeng Li, Quantum Gromov-Hausdorff distance and continuity of Connes and Dubois-Violette’s θ-deformations, 2002, Doctoral dissertation, University of California, Berkeley.
[23] John Lott, Collapsing and Dirac-type operators. Geom. Dedicata 91 (2002), 175–196, arXiv:math.DG/0005009. MR 2003h:58042
[24] D. Olesen, G. K. Pedersen, and M. Takesaki, Ergodic actions of compact abelian groups, J. Operator Theory 3 (1979), no. 4, 531–562. MR 81e:46092
[25] N. Ozawa and M. A. Rieffel, Hyperbolic group C*-algebras and free-product C*-algebras as compact quantum metric spaces, Canadian J. Math, to appear, arXiv:math.OA/0302210
[26] G. K. Pedersen, C*-algebras and their automorphism groups, Academic Press Inc., London, 1979. MR 81c:46037
[27] M. A. Rieffel, Deformation quantization of Heisenberg manifolds, Comm. Math. Phys. 122 (1989), no. 4, 531–562. MR 90e:46060
[28] , Noncommutative tori — a case study of noncommutative differentiable manifolds, Geometric and topological invariants of elliptic operators, Amer. Math. Soc., Providence, RI, 1990, pp. 191–211. MR 91d:58012
[29] , Compact quantum groups associated with toral subgroups, Representation theory of groups and algebras, Amer. Math. Soc., Providence, RI, 1993, pp. 465–491. MR 94i:22022
[30] , Deformation quantization for actions of R^n, Mem. Amer. Math. Soc. 106 (1993), no. 506, x+93. MR 94d:46072
[31] , Metrics on states from actions of compact groups, Doc. Math. 3 (1998), 215–229, arXiv:math.OA/9807084
[32] , Metrics on state spaces, Doc. Math. 4 (1999), 559–600, arXiv:math.OA/9906151. MR 1 727 499
[33] , Gromov-Hausdorff distance for quantum metric spaces, (2000), Memoirs A. M. S. to appear, arXiv:math.OA/0011063
[34] , Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance, (2001), Memoirs A. M. S. to appear, arXiv:math.OA/0108005
[35] , Group C*-algebras as compact quantum metric spaces, Documenta Mathematica 7 (2002), 605–651, arXiv:math.OA/0205195
[36] A. Sitarz, Rieffel's deformation quantization and isospectral deformations, Internat. J. Theoret. Phys. 40 (2001), no. 10, 1693–1696, arXiv:math.QA/0102075. MR 2002i:53125
[37] J. C. Várilly, Quantum symmetry groups of noncommutative spheres, Commun. Math. Phys. 221 (2001), 511–523, arXiv:math.QA/0102065. MR 2002i:58004
[38] Shuzhou Wang, Quantum symmetry groups of finite spaces, Comm. Math. Phys. 195 (1998), no. 1, 195–211, math.OA/9807091. MR 99h:58014
[39] A. Wassermann, Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions, Canad. J. Math. 40 (1988), no. 6, 1482–1527. MR 92d:46168
[40] , Ergodic actions of compact groups on operator algebras. III. Classification for SU(2), Invent. Math. 93 (1988), no. 2, 309–354. MR 91e:46093
[41] , Ergodic actions of compact groups on operator algebras. I. General theory, Ann. of Math. (2) 130 (1989), no. 2, 273–319. MR 91e:46092
[42] N. Weaver, Sub-Riemannian metrics for quantum Heisenberg manifolds, J. Operator Theory 43 (2000), no. 2, 223–242, arXiv:math.OA/9801014. MR 2001b:46114
[43] G. Zeller-Meier, Produits croisés d’une C*-algèbre par un groupe d’automorphismes, J. Math. Pures Appl. (9) 47 (1968), 101–239. MR 39 #3329

Department of Mathematics, University of California Berkeley, CA 94720-3840
E-mail address: rieffel@math.berkeley.edu