AN APERIODIC MONOTILE THAT FORCES NONPERIODICITY THROUGH A LOCAL DENDRITIC GROWTH RULE

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Abstract. We introduce a new type of aperiodic hexagonal monotile; a prototile that admits infinitely many tilings of the plane, but any such tiling lacks any translational symmetry. Adding a copy of our monotile to a patch of tiles must satisfy two rules that apply only to adjacent tiles. The first is inspired by the Socolar–Taylor monotile, but can be realised by shape alone. The second is a local growth rule; a direct isometry of our monotile can be added to any patch of tiles provided that a tree on the monotile connects continuously with a tree on one of its neighbouring tiles. This condition forces tilings to grow along dendrites, which ultimately results in nonperiodic tilings. Our local growth rule initiates a new method to produce tilings of the plane.

1. Introduction

Almost 60 years ago, Hao Wang posed the Domino Problem [16]: is there an algorithm that determines whether a given set of square prototiles, with specified matching rules, can tile the plane? Robert Berger [4] proved the Domino Problem is undecidable by producing an aperiodic set of 20,426 prototiles, a collection of prototiles that tile the plane but only nonperiodically (lacks any translational periodicity). This remarkable discovery began the search for other (not necessarily square) aperiodic prototile sets. In the 1970s, there were two stunning results giving examples of very small aperiodic prototile sets. The first was by Raphael Robinson who found a set of six square prototiles [10]. The second was by Roger Penrose who reduced this number to two [5, 9]. Penrose’s discovery led to the planar einstein (one-stone) problem: is there a single aperiodic prototile?

In a crowning achievement of tiling theory, the existence of an aperiodic monotile was resolved almost a decade ago by Joshua Socolar and Joan Taylor [12, 15]. Several candidates had been put forth prior to their monotile, but the experts immediately recognised the importance of Socolar and Taylor’s discovery [1, 2, 3, 7, 8]. The Socolar–Taylor monotile is a hexagonal tile with two local rules that enforce aperiodicity. The first rule forces tiles to arrange themselves into collections of triangles, and the second rule ensures these triangles are nested, thereby forcing the resulting tiling to be nonperiodic. One limitation of the Socolar–Taylor monotile is that the second local rule applies to pairs of non-adjacent tiles, so aperiodicity is not enforced by adjacencies. Another limitation is that reflected copies of the monotile are required to tile the plane. The search for an aperiodic monotile with local rules that only apply to adjacent tiles or does not require reflections has been a driving force of research in tiling theory since Socolar and Taylor’s amazing discovery.

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In this paper, we put forward a new type of aperiodic monotile that does not require a reflection, and has rules that only apply to adjacent tiles. We start with a hexagonal tile satisfying Socolar and Taylor’s first local rule, and add a rule that only allows finite patches of tiles to grow along a dendrite. Our second growth rule is motivated by the proposed growth of certain quasicrystals [6, 17]. Interestingly, a consequence of not requiring a reflected copy of our monotile is that the first rule can be enforced by shape alone, while this does not hold for the Socolar–Taylor monotile [13, p.22]. Although we are comparing our monotile with Socolar and Taylor’s construction, these two monotiles are different in character. The Socolar–Taylor monotile is defined using local rules, whilst our monotile is defined by a local growth rule. Indeed, our growth rule is local in the sense of building tilings, but is not local as a rule on tilings. So the monotile we present here is not an einstein in the technical sense, but rather a variant that requires tilings to be constructed. Three representations of our monotile appear in Figure 1, and each of these representations must satisfy the local rule $R_1$ and the local growth rule $R_2$ outlined below.

Before introducing our monotile, we briefly define the terminology used in the paper. A tiling is a covering of the plane by closed topological discs, called tiles, that only intersect on their boundaries. A patch is a finite connected collection of tiles that only intersect on their boundary. The building blocks of a tiling are the prototiles: a finite set of tiles with the property that every tile is a direct isometry (an orientation preserving isometry) of a prototile. If the prototile set consists of a single tile, or a single tile and its reflection, we call it a monotile. A tiling is said to be nonperiodic if it lacks any translational periodicity, and a set of prototiles is called aperiodic if it can only form nonperiodic tilings.

Our monotile has two distinct features: a disconnected set of three curves that meet tile edges off-centre, and a connected tree that meets itself whenever two edges with a tree intersect. Once a single tile has been placed, a direct isometry of our monotile can be added to the plane provided the resulting collection of tiles is a patch, and

- **$R_1$:** the black off-centre lines and curves must be continuous across tiles (c.f. [12, $R_1$]) and
- **$R_2$:** the new tile’s red tree continuously connects with at least one tree of an adjacent tile.

We note that $R_1$ can be realised by shape alone, represented as puzzle like edge contours, while $R_2$ can be represented by magnetic dipole-dipole coupling. These representations appear in Figure 1.

In what follows, we will refer to the off-centre decorations that determine $R_1$ as R1-curves. These combine to form R1-triangles. Similarly, we will refer to the decorations that determine $R_2$ as R2-trees. We say that a tiling $T$ satisfies local growth rule $R$ if every patch in $T$ is contained in a patch that can be constructed following local growth rule $R$. Therefore, a tiling $T$ satisfies $R_2$ if and only if the union of R2-trees in $T$ is connected. Two legal
patches satisfying $R1$ and $R2$ appear in Figure 2. From this point forward we will use the representation from the left hand side of Figure 1, since we use R1-triangles heavily in the arguments that follow.

![Figure 2](image)

**Figure 2.** Two sets of equivalent legal patches. The patches on the top can be constructed directly using $R1$ and $R2$, while the patches on the bottom cannot, but are subsets of legal patches.

The goal of the paper is to prove the following theorem.

**Theorem A.** The monotile in Figure 1 is aperiodic. That is, there is a tiling of the plane using only direct isometries of the monotile that satisfies $R1$ and $R2$, and any such tiling $T$ is nonperiodic.

The proof of Theorem A is essentially the contents of the rest of this paper. Since the proof is quite involved, we now provide a brief sketch. In Section 2, we identify two classes, $C_0$ and $C_1$, of tilings satisfying $R1$ and $R2$. We prove that both classes are non-empty and only contain nonperiodic tilings. To prove $C_0$ contains only nonperiodic tilings, we use a clever construction of Socolar and Taylor to build tilings satisfying $R1$, but not necessarily $R2$. We then build a tiling in $C_0$ by recursively constructing a spiral fixed point of R2-trees about the origin. We prove that $C_1$ is non-empty and that every tiling is nonperiodic by building all possible tilings in the class. The key to this construction is Lemma 2.4, which shows that no...
legal tiling can contain an infinite R1-triangle. In Section 3, we show that R2 rules out two patterns of R2-trees, which we call R2-cycles and R2-anticycles. These patterns are exactly those that are formed between R1-triangles when they are arranged into a periodic lattice, as illustrated in Figure 16. A nice consequence of ruling these patterns out is that the union of R2-trees in any tiling is always a connected tree. We are left to show that every tiling satisfying R1 and R2 must be in C₀ or C₁. This is achieved by proving that the absence of R2-cycles and R2-anticycles implies that every tiling fits into one of these two classes.

After proving Theorem A we discuss the continuous hull of tilings arising from our monotile. We show that all tilings in the continuous hull satisfy R1 and a weak version of R2.

Our use of dendrites in constructing the local growth rule for our monotile was motivated by growth in quasicrystals. Several papers hypothesise that dendritic structures in molecules, particularly in soft-matter quasicrystals, are the mechanism that force nonperiodicity, see [6, 17]. In [18], magnetic dipole-dipole coupling of quasicrystals is mentioned, which can also be modelled by our monotile as indicated in Figure 1. According to the recent survey paper of Steurer [14], one of the most pressing questions in quasicrystal theory is understanding how they form, and when they grow periodically and quasiperiodically. The local growth rules in this paper show that dendritic growth can lead to aperiodic tile sets.

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2. Classes of tilings arising from the monotile

In this section, we consider two classes of tilings satisfying R1 and R2. We show that each class is non-empty and only contains nonperiodic tilings. In the following section, we show that these classes exhaust the possible tilings that can be constructed from our monotile.

We begin with a closer look at the R1-triangles. Notice that the small R1-curves only occur at angle \( \pi/3 \), so any R1-triangle must be equilateral. We also observe that the R1-curves can only give rise to either nested R1-triangles, or a bi-infinite R1-line. Let us denote the length of an R1-triangle by the number of tiles comprising the straight R1-line segment of any given side, as in Figure 3.

![R1-triangles length](image)

**Figure 3.** R1-triangles have length \( 2^n - 1 \) for \( n = 0, 1, 2, \ldots \).
The following lemma is easily deduced from R1 and the geometry of the R1-curves.

**Lemma 2.1.** Any nested R1-triangle has length $2^n - 1$ for some $n = 0, 1, 2, \ldots$.

**Definition 2.2.** Let $\mathcal{C}$ denote the collection of tilings whose prototile set is the monotile from Figure 1 satisfying R1 and R2. Consider the subcollections of $\mathcal{C}$ defined by the properties:

- $\mathcal{C}_0$: if the corners of a pair of R1-triangles meet at a common tile in $T$, then these R1-triangles have the same length;
- $\mathcal{C}_1$: $T$ contains a bi-infinite R1-line.

We first consider the collection $\mathcal{C}_0$. Let us introduce a convention that will be used in the proof of the following proposition. Define $R_\theta$ to be the rotation operator that rotates a patch counterclockwise around the origin by $\theta$.

**Proposition 2.3.** The collection $\mathcal{C}_0$ is non-empty and only contains nonperiodic tilings.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The interlaced honeycombs of R1-triangles in the Socolar–Taylor construction}
\end{figure}

**Proof.** To see that $\mathcal{C}_0$ contains only nonperiodic tilings, we appeal to a construction of Socolar and Taylor [12, Theorem 1], which we now summarise. Tilings satisfying R1 are produced by adding markings to tiles that form successively larger hexagonal grids of interlaced R1-triangles, see Figure 4 for a pictorial representation of their construction. To force these honeycomb lattices of length $2^n - 1$ R1-triangles, Socolar and Taylor use their second local rule to deduce the condition that all R1-triangles whose corners meet at a common tile have the same size. Since we have restricted ourselves to tilings in $\mathcal{C}_0$, this condition is one of our hypotheses. The honeycomb lattices of R1-triangles have no largest translational periodicity constant, so that all of the infinite tilings produced must be nonperiodic.

We are left to show that $\mathcal{C}_0$ is non-empty. To construct a tiling $T_0 \in \mathcal{C}_0$, we recursively define patches $P_n$ with three key properties:

1. $P_n$ satisfies R1 and R2,
2. the patch $P_n$ is a strict subset of $P_{n+1}$, and
3. as $n$ increases, the patch around the origin in $P_n$ increases exponentially.

Such a collection of patches naturally leads to a tiling of the plane, see (2.2) below.

To define the recursive algorithm, we carefully look at how patches grow with respect to R2. The first few steps of our algorithm appear in Figure 5, which should help decipher the recursive definition below. Essentially, the patch $P_n$ is constructed from $P_{n-1}$ by gluing $P_{n-1}$
Figure 5. Constructing an infinite tiling by piecing together R2-trees and three direct isometries of $P_{n−1}$ together by a single connecting tile in the centre of the new patch $P_n$. These central tiles have centre at a point $x_n$ (explicitly described below), and each such $x_n$ is marked by a dot in Figure 5, which helps to see $P_n$ in $P_{n+1}$.

Let $P_0$ be our monotile in exactly the orientation appearing in Figure 1, placed with its centre on the origin. Using polar coordinates $(r, θ)$, the recursive formula for patch $P_n$ is defined by points

$$x_0 := (0, 0) \quad \text{and} \quad x_n := \sum_{i=1}^{n} (2^{i-1}, 4i\pi/3),$$

along with patches

$$P_n := R_{\frac{2\pi}{3}} (P_0 + x_n) \bigsqcup P_{n-1} \bigsqcup R_{\frac{\pi}{3}} (P_{n-1} - x_{n-1}) + x_n + (2^{n-1}, 4n\pi/3 - 2\pi/3)$$
$$\bigsqcup R_{\pi} (P_{n-1} - x_{n-1}) + x_n + (2^{n-1}, 4n\pi/3)$$
$$\bigsqcup R_{\frac{4\pi}{3}} (P_{n-1} - x_{n-1}) + x_n + (2^{n-1}, 4n\pi/3 + \pi/3).$$

Note that the points $x_n$ appear at each corner of the superimposed spiral in Figure 6.
The method we have used to build $P_n$ ensures that both $\textbf{R1}$ and $\textbf{R2}$ are satisfied. Moreover, the patches $P_n$ overlap where they intersect, and are space filling in a spiral pattern that successively connects the points $x_n$ around the origin, see the Figure 6. Thus, the union

$$T_0 := \bigcup_{n=0} P_n$$

is a tiling of the plane satisfying both $\textbf{R1}$ and $\textbf{R2}$.

To finish the proof, we show that $T_0$ satisfies the defining condition of $C_0$. Observe that each patch $P_n$ has two R2-tree arms of length $2^n - 1$ extending from the tile containing the point $x_n$ at angles $\frac{n\pi}{3}$ and $\frac{(n+3)\pi}{3}$. The recursive definition extending $P_n$ into $P_{n+1}$ ensures that these R2-tree arms terminate at length $2^n - 1$, which implies that the lengths of the R1-triangles (realised in $P_{n+3}$) along those arms also have length $2^n - 1$, and occur on opposite sides of an R1-line segment. These R1-triangles force all R1-triangles of smaller length to have the same length if their corners meet in a common tile. Thus, the tiling $T_0$ is in $C_0$. □

We now consider the collection $C_1$, but first a lemma.

**Lemma 2.4.** Any tiling satisfying $\textbf{R1}$ and containing an infinite R1-triangle (two R1-rays connected by an R1-corner) does not satisfy $\textbf{R2}$. 

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**Figure 6.** Part of the patch $P_6$ with the spiral branch of the R2-tree highlighted.
Figure 7. The pictorial argument for Lemma 2.4

Proof. Suppose $T$ is a tiling satisfying $R_1$ and containing an infinite $R_1$-triangle. Let $A$ and $B$ be the respective connected components of the union of $R_2$-trees associated with each side of the infinite $R_1$-triangle, see Figure 7. We will argue that $A$ and $B$ cannot be connected. Notice that the branches of $A$ and $B$ into the interior of the infinite $R_1$-triangle never reach the opposite side of the $R_1$-triangle, and are disjoint. At the infinite $R_1$-triangle corner, one of the trees extends into the corner tile, while the other does not. Let us assume $A$ does not extend. The only remaining connection possible between $A$ and $B$ is along $R_1$-branches of $A$ growing outside the infinite $R_1$-triangle. Every such $R_2$-branch extends along the side of an $R_1$-triangle, and terminates at the corner of the $R_1$-triangle, if the $R_1$-triangle is finite, or extends infinitely if the $R_1$-triangle is infinite. However, as noted above, $R_2$-branches into the interior of an $R_1$-triangle never reach the opposite side of the $R_1$-triangle and are disjoint from any $R_2$-branches extending from the opposite side. It follows that $A$ never meets $B$, and so $T$ cannot satisfy $R_2$. □

Proposition 2.5. The collection $C_1$ is non-empty and only contains nonperiodic tilings.

Proof. We will construct a tiling in $C_1$, starting with a bi-infinite string of tiles forming an $R_1$-line. Note that the union of $R_2$-trees along this string is connected. In order to simplify the argument, we fix the orientation of this string of tiles, as in Figure 8, and will refer to the top, bottom, left and right as per the orientation depicted. Our construction will produce every possible tiling in $C_1$ up to direct isometry.
We begin by adding tiles above the R1-line. Lemma 2.4 implies that we can never add an infinite R1-triangle with a corner meeting the R1-line. So every R1-triangle meeting the R1-line must have length $2^n - 1$ for some $n \in \mathbb{N}$ by Lemma 2.1.

Suppose we add an R1-triangle of length $2^n - 1$ whose bottom corner is the R1-corner of a tile in the R1-line, as in Figure 9 with $n = 2$. We note that the union of R2-trees is no longer connected, but this will be rectified shortly. Between the length $2^n - 1$ R1-triangle and the R1-line, R1-triangles of all shorter legal lengths are forced, as depicted in Figure 10.

We now consider the possible tiles we may add above the tiles occurring $2^n$ tiles to the left or right of the bottom corner of the $2^n - 1$ R1-triangle along the R1-line. The geometry of the situation forces one of these tiles to be the corner of a length $2^{n+1} - 1$ R1-triangle. As above, this triangle forces the corner of a length $2^{n+2} - 1$ triangle to the left or right of the bottom corner of the $2^{n+1} - 1$ R1-triangle along the R1-line. This process of adding successively larger R1-triangles whose corners occur at distance $2^{n+j}$ tiles along the R1-line from its successor carries on ad infinitum. Lemma 2.4 implies that tilings in $\mathcal{C}$ cannot contain infinite triangles, so we must change the direction of our choice an infinite number of times so that every tile on the R1-line contains the corner of some finite length R1-triangle. Moreover, these triangles are forced to occur periodically. That is, placing a length $2^m - 1$ R1-triangle of tiles at tile position $l$ on the R1-line yields a length $2^m - 1$ R1-triangle of tiles at positions $l + k2^{m+1}$ for
all $k \in \mathbb{Z}$, and at least one R1-triangle of length $2^m - 1$ appears in any string of tiles along the R1-line of length $2^{m+1}$. This construction leads to a half-plane of tiles that satisfies $\mathbf{R1}$.

We now argue that the half-plane of tiles constructed above also satisfies $\mathbf{R2}$. Indeed, the union of R2-trees along the bi-infinite string of tiles is connected. Given a length $2^m - 1$ R1-triangle whose corner meets the R1-line, the union of R2-trees in a triangular arrangement of tiles between its right side and the R1-line is connected to the union of R2-trees along the R1-line, the shaded region in Figure 12 is an example of such a patch. Since there are no infinite R1-triangles, every tile in the upper half-plane is to the right of some R1-triangle whose corner meets the R1-line. Thus, the union of R2-trees in the upper half-plane is connected, so the upper half-plane is a patch satisfying $\mathbf{R1}$ and $\mathbf{R2}$.

An analogous argument implies that all tiles in the lower half-plane satisfy $\mathbf{R1}$ and $\mathbf{R2}$. Since the upper and lower half-planes intersect along the bi-infinite R1-line, the resulting tiling is in $\mathcal{C}_1$. Since there are arbitrarily large R1-triangles arranged in interlaced periodic
patterns whose corners meet the R1-line, the resulting tiling is nonperiodic, giving the desired result.

We note that the classes $C_0$ and $C_1$ have non-trivial intersection. Indeed, if the R1-triangles on opposite sides of the bi-infinite R1-line of a tiling in $C_1$ have the same length, then it is also in $C_0$.

3. Proof of Theorem A

We have shown in Propositions 2.3 and 2.5, that the classes $C_0$ and $C_1$ from Definition 2.2 are non-empty and contain only nonperiodic tilings. In this section, we will prove that $C = C_0 \cup C_1$, which will prove Theorem A. The key is to prove that any tiling in $C$ that does not belong to $C_0$ must have an infinite R1-line through it, so it belongs to $C_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{Monotiles with dashed lines represent legal patches of length $2^n - 1$}
\end{figure}

In order to provide general arguments, we introduce pictorial notation. For $n \in \mathbb{N}$, a monotile with dashed lines represents a patch of tiles depicted in Figure 13, where the main diameter of R2-trees has length $2^n - 1$. We note that these are the patches $P_n$ that appeared when we constructed a tiling in Proposition 2.3. Moreover, notice that these patches fit together in the manner depicted in Figure 14.

The fundamental tool of this section is to prove that R2 rules out three R1-triangles meeting corners to sides in the cyclic fashion appearing in Figure 15, where the solid lines have length one, and the dotted lines have length $2^n - 1$ for $n \in \{0, 1, 2, \ldots\}$. Due to the behaviour of the R2-trees in the centre of the cyclic R1-triangles, we will refer to these configurations as R2-cycles and R2-anticycles, respectively. We note that Lemma 2.1 implies that R2-cycles and R2-anticycles only occur with side length $2^n - 1$.

An immediate consequence of ruling out R2-cycles and R2-anticycles is that periodic lattices of R1-triangles, as depicted in Figure 16, is no longer possible. Of course, it is clear that such lattices do not satisfy R2. However, a growth rule that disallows these periodic lattices was the key to the results of this paper.
Figure 14. For $n = 2$ and $n = 3$, the patches in Figure 13 fit together as above

an R2-cycle of length $2^n - 1$

an R2-anticycle of length $2^n - 1$

Figure 15. The illegal patches appearing in Lemmas 3.1 and 3.3

Lemma 3.1. Suppose $T$ is a tiling in $\mathcal{C}$, then $T$ does not contain an R2-cycle.

Proof. We begin with a patch containing a R2-cycle of length $2^n - 1$, and show that any tiling that extends the patch fails to satisfy R2. Fix $n \in \{0, 1, 2, \ldots\}$, and suppose we start with the patch on the left-hand side of Figure 15.

Since R2 implies that the R2-tree must be infinite, at least one branch of the R2-tree leaving the central R2-cycle must be infinite. We will refer to the R1-triangle associated with the infinite tree as triangle $A$. Lemma 2.4 implies that triangle $A$ cannot be infinite. We will show that triangle $A$ cannot be finite either. Lemma 2.1 implies that triangle $A$ must have length $2^n + m + 1 - 1$ for some $m \in \{1, 2, \ldots\}$. Since the union of R2-trees terminates at the R1-corner of $A$, there must be an infinite branch leaving the main tree. All R2-branches towards the interior of $A$ are finite, so any infinite branch must turn away from triangle $A$. A straightforward, but geometrically technical, induction proves that if triangle $A$ has length...
An R1-triangle with length $2^n(2^2 - (2^1 - 1)) - 1$ (i.e. $m = k = 1$), which is impossible by Lemma 2.1

$2^{m+n+1} - 1$, then the next R1-triangle clockwise in the R2-cycle of length $(2^n - 1)$ (labelled B in Figures 17 and 18) forces an R1-triangle of length $2^n(2^{m+1} - (2^k - 1)) - 1$ for some
Lemma 2.1 implies that such triangles cannot exist in a tiling, giving us the desired contradiction. So $T$ cannot belong to $C$. \hfill $\Box$

Remark 3.2. A comment on the omitted geometric induction argument from the proof of Lemma 3.1 is in order. Figure 17 shows the argument for arbitrary $n \in \mathbb{N}$ and $m = 1$. Figure 18 shows the geometric argument for $n = 1$ and $m = 1, 2$. Using Figure 18, the general argument for $m = 1, 2$ follows by using dashed tiles of length $2^n - 1$, as depicted in Figures 13 and 14.

Lemma 3.3. Suppose $T$ is a tiling in $C$, then $T$ does not contain an R2-anticycle.

Proof. We begin with a patch containing a R2-anticycle of length $2^n - 1$, and show that any tiling that extends the patch does not belong to $C$. Fix $n \in \{0, 1, 2, \ldots\}$, and suppose we start with the patch on the right-hand side of Figure 15.

Rule R2 implies that all three branches of the R2-anticycle must be infinite and must all be connected. Let us concentrate on just one of these branches. Lemma 2.4 implies that the R1-triangle associated with this branch cannot be infinite, and then Lemma 2.1 implies it must have length $2^{n+m} - 1$ for some $m \in \{1, 2, \ldots\}$. However, if this R1-triangle has length $2^{n+m} - 1$, then an R2-cycle of length $2^n - 1$ is forced where the R1-triangle associated with this branch has an R1-corner. Lemma 3.1 implies that $T$ is not in $C$. See Figure 19 for a pictorial representation, where $x$ is the location of the R2-cycle in the case $m = 2$. \hfill $\Box$

We are now able to tackle the proof of Theorem A. The reader is encouraged to consider Figures 20 and 21 while reading through the proof.
Proof of Theorem A. Propositions 2.3 and 2.5 prove that $C_0$ and $C_1$ are both non-empty and only contain nonperiodic tilings. We will prove that $C = C_0 \cup C_1$, thereby proving the result.
By definition, $C_0$ and $C_1$ satisfy $\textbf{R1}$ and $\textbf{R2}$ so that $C_0 \cup C_1 \subseteq C$. We are left to prove the reverse inclusion. Suppose $T$ is in $C$. If all pairs of R1-triangles in $T$ that meet at a common tile have the same length, then $T$ is in $C_0$. If there exists a pair of R1-triangles which meet at a common tile, but do not have the same length, we claim that $T$ is in $C_1$, which would imply that $C \subseteq C_0 \cup C_1$. To do this, we must show that such a tiling $T$ has a bi-infinite R1-line through it.

Suppose that $T$ contains a tile where a pair of R1-triangles meet that have lengths $2^m - 1$ and $2^n - 1$ for $m \neq n$. Since these two R1-triangles meet at their respective R1-corners on a common tile, they are separated by the R1-line segment in this tile. We will argue that this line segment must extend indefinitely in both directions. For the sake of contradiction, suppose the extended R1-line segment has an R1-corner, which must occur at length $2^{m+j}$ along the straight R1-line segment from the corner of the $2^m-1$ R1-triangle, for $j \in \{1, 2, \ldots\}$.

At any such R1-corner, there is either an R2-cycle or an R2-anticycle of length $(2^k - 1)$ for $k \in \{0,\ldots,\min\{m,n\}\}$. Since Lemma 3.1 and Lemma 3.3 imply that R2-cycles and R2-anticycles cannot exist in $T$, the R1-line segment must extend infinitely in both directions. Thus, $T \in C_1$ so that $C \subseteq C_0 \cup C_1$, as required.

4. The continuous hull of our aperiodic monotile

We conclude the paper with a brief discussion of the tiling space, or continuous hull, of the tilings in the class $C$. Recall that the continuous hull of a collection of tilings $\Lambda$ is the completion of $\{T + \mathbb{R}^d : T \in \Lambda\}$ in the tiling metric, typically denoted by $\Omega$. Under mild assumptions, the continuous hull is a compact topological space endowed with a continuous $\mathbb{R}^d$ action, making $(\Omega, \mathbb{R}^d)$ a dynamical system. For further details see [11, Section 1.2]. We are interested in the continuous hull of $C = C_0 \cup C_1$.

**Theorem 4.1.** All tilings in the continuous hull of $C$ are nonperiodic and satisfy $\textbf{R1}$. Moreover, in any such tiling, there are at most three connected components of R2-trees, and each component crosses an infinite number of tiles.

**Proof.** We begin by considering the class $C_1$. As described in Proposition 2.5, a tiling is in $C_1$ if it contains an bi-infinite R1-line with intertwined $2^n$-periodic patterns of R1-triangles of length $2^n - 1$ meeting either side of the R1-line, which typically have no relation with one another. All such tilings satisfy $\textbf{R2}$.
Figure 22. Elements in the completion of $C_1$ have at most three connected components of R2-trees.

Since we can have arbitrarily large R1-triangles on either side of the bi-infinite R1-line, the completion of $C_1$ in the tiling metric contains an infinite R1-triangle whose corner meets the R1-line. An infinite R1-triangle meeting a bi-infinite R1-line forces a half plane of R1-triangles with exactly two connected R2-components, and both of these components cross an infinite number of tiles. Moreover, arbitrarily large R1-triangles with corners meeting opposite sides of the R1-line can occur an arbitrary distance apart. Thus, in addition to tilings in $C_1$, the continuous hull of $C_1$ contains tilings with an infinite R1-triangle on one or both sides of the bi-infinite R1-line. Such tilings have either two or three connected components of R2-trees; see Figure 22. This exhausts all possible tilings in $C_1$ decorated only with R1-curves, so we have found all possible tilings in the continuous hull of $C_1$.

Figure 23. All possible patches of $P_n$ connected by a single tile in $C_0$ are represented on the left, and a representation of the partial tiling $P$ with infinitely small tiles appears on the right.
Figure 24. Cauchy sequences of the above configurations of patches $P_n$, with the central dots on the origin, converge to tilings $R$ and $S$ in the continuous hull of $C_0$. Each of these tilings has three connected components of $\mathbb{R}^2$-trees.

We now consider the continuous hull of $C_0$. As described in Proposition 2.3, such tilings have successively larger hexagonal grids of interlaced $\mathbb{R}^1$-triangles. For any $n \in \mathbb{N}$, a tiling of interlaced $\mathbb{R}^1$-triangles decomposes into patches $P_n$ connected by single tiles, where $P_n$ is defined in Proposition 2.3 (also see Figure 14). The image on the left-hand side of Figure 23 shows all possible arrangements of $P_3$ connected by a single tile. Up to direct isometry, there are exactly two configurations. These are shown for $P_3$ in Figure 24. For $n \in \mathbb{N}$, let us call these two patches $R_n$ and $S_n$. Placing the origin at the centre of each patch, as shown in Figure 24, we see that $R_n \subset R_{n+1}$ and $S_n \subset S_{n+1}$. For example, the reader can compare the patches in Figure 23, where $n = 2$, with the patches in Figure 24, where $n = 3$. Therefore,

$$R := \bigcup_{n=0}^{\infty} R_n \quad \text{and} \quad S := \bigcup_{n=0}^{\infty} S_n$$

are tilings satisfying $\mathbf{R1}$. Thus, in addition to tilings in $C_0$, the continuous hull of $C_0$ contains direct isometries of the tilings $R$ and $S$. As $n$ tends to infinity, the patches $P_n$ converge to a partial tiling, which can be scaled down as each step so as to be depicted as the fractal on the right-hand side of Figure 23. The fractal nature of this partial tiling implies that, up to direct isometry, $R$ and $S$ are the only two additional elements in the completion of $C_0$. Notice that the tilings $R$ and $S$ have exactly three connected components of $\mathbb{R}^2$-trees and each component crosses an infinite number of tiles, as desired.

The astute reader will notice that the hull of $C$ contains all possible tilings satisfying $\mathbf{R1}$ with the condition that either $C_0$ or $C_1$ holds. That is, the local growth rule $\mathbf{R2}$ ensures that there are no periodic tilings, but does not factor into the final description of the hull.
A DENDRITIC MONOTILE

REFERENCES

[1] S. Akiyama and J-Y Lee, The computation of overlap coincidence in Taylor–Socolar substitution tiling, Osaka J. Math. 51 (2014), 597–609.
[2] M. Baake, F. Gähler and U. Grimm, Hexagonal inflation tilings and planar monotiles, Symmetry 4 (2012), 581–602.
[3] M. Baake and U. Grimm, Aperiodic Order. Volume 1: A Mathematical Invitation, Cambridge University Press, Cambridge, 2013.
[4] R. Berger, The Undecidability of the Domino Problem, Memoirs AMS 66, Providence, 1966.
[5] B. Grunbaum and G.C. Shephard, Tilings and Patterns, W.H. Freeman, New York, 1987.
[6] C. R. Iacovella, A. S. Keysa and S. C. Glotzera, Self-assembly of soft-matter quasicrystals and their approximants, Proc. Nat. Acad. Sci. 108 (2011), 20935–20940.
[7] J-Y Lee, Tiling spaces of Taylor-Socolar tilings, Acta Phys. Polon. A 126 (2014), 508–511.
[8] J-Y Lee and R.V. Moody, Taylor-Socolar hexagonal tilings as model sets, Symmetry 5 (2013), 1–46.
[9] R. Penrose, Pentaplexity: A class of non-periodic tilings of the plane, Math. Intel. 2 (1979), 32–37.
[10] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane, Invent. Math. 12 (1971), 177–209.
[11] L. Sadun, Topology of Tiling Spaces, University Lecture Series 46, American Mathematical Society, Providence, 2008.
[12] J.E.S Socolar and J.M. Taylor, An aperiodic hexagonal tile, J. Comb. Th. A 118 (2011), 2207–2231.
[13] J.E.S Socolar and J.M. Taylor, Forcing nonperiodicity with a single tile, Math Intel. 34 (2012), 18–28.
[14] W. Steurer, Quasicrystals: What do we know? What do we want to know? What can we know, Acta Cryst. A 74 (2018), 1–11.
[15] J.M. Taylor, Aperiodicity of a functional monotile, preprint (2010); available from http://www.math.uni-bielefeld.de/sfb701/preprints/view/420.
[16] H. Wang, Proving theorems by pattern recognition, II, Bell Sys. Tech. J. 40 (1961), 1–41.
[17] X. Zeng, G. Ungar, Y. Liu, V. Percec, A. E. Dulcey and J. K. Hobbs, Supramolecular dendritic liquid quasicrystals, Nature 428 (2004), 157–160.
[18] M. Zu, P. Tan and N. Xu, Forming quasicrystals by monodisperse soft core particles, Nature Comm. 8 (2017), 2089–2098.

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