The Hausdorff Dimension of Two-Dimensional Quantum Gravity

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We argue that the Hausdorff dimension of a quantum gravity random surface is always \( D_H = 4 \), irrespective of the conformal central charge \( c \), \( -2 \leq c \leq 1 \), of a critical statistical model possibly borne by it. The Knizhnik-Polyakov-Zamolodchikov (KPZ) relation allows us to determine \( D_H \) from the exact Hausdorff dimension of random maps with large faces, recently rigorously studied by Le Gall and Miermont, and reformulated by Borot, Bouttier and Guitter as a loop model on a random quadrangulation. This contradicts several earlier conjectured formulae for \( D_H \) in terms of \( c \).

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Introduction.—Understanding and describing the quantum nature of gravity is one of the most critical issues in physics. Thirty years ago, Polyakov introduced Liouville quantum gravity [1] to describe the random geometry of world sheets of gauge fields and strings. This seminal work was followed by a flurry of studies of this novel “quantum geometry” [2]. Such 2D random surfaces are also expected to arise as continuum limits of the random-planar-graph surfaces developed in random matrix theory, as became clear after KPZ discovered their famous relation between critical exponents on a random surface and in the Euclidean plane [3-6]. The fractal properties of 2D gravity are thereby understood from the so-called Schramm-Loewner evolution (SLE) is also being developed [7].

More recently, a combinatorial approach to this subject has emerged. It uses bijections from random planar graphs, or “maps”, to labelled random trees. It allows a refined probabilistic description, in particular of metric or geodesic properties [8, 9] not yet accessible in Liouville or matrix approaches, and of universal scaling limits [10]. A major effort now seeks to prove that all these approaches converge to a same universal continuous object, possibly coupled to a statistical system or CFT.

One of the most enduring questions concerns the intrinsic Hausdorff dimension \( D_H \), that describes how the area of a quantum ball, \( A \propto l^{D_H} \), scales with its geodesic radius \( l \). For the so-called pure gravity case of large planar maps, one has \( D_H = 4 \) [11, 12] resulting from the so-called Schramm-Loewner evolution (SLE) is also being developed [7].

When “matter” is present, that is, when the surface bears a critical system represented by a CFT of central charge \( c \leq 1 \), the situation is much less clear. A first formula, \( D_H^{(1)} = \frac{25 - c}{8} - 1 \), was proposed in the so-called “string field theory” framework [13]. An improper representation of the geodesic distance there was pinpointed in [14], and a value \( D_H(c = 1/2) = 4 \) derived for the Ising model case. Another formula, \( D_H^{(2)} = 2\sqrt{25 - c} + \sqrt{49 - c} \), was then proposed [15] and claimed to fit well results from numerical simulations, in particular for \( c \in [-2, 0] \). However, it has been consistently observed that, at least for \( c \geq 0 \), a value \( D_H \approx 4 \) is also plausible [19].

In this Letter, we argue that one has \( D_H = 4 \) independent of the value of \( c \), for \( c \in [-2, 1] \). We focus on the case of a random surface bearing a critical loop model, and use recent combinatorial results [20, 21].

Le Gall and Miermont [20] recently introduced a model of random planar maps with (macroscopically) large faces in the scaling limit. They proved rigorously that these maps have, with probability one, a parameter dependent Hausdorff dimension \( D_G \) continuously varying between 2 and 4. As anticipated by them, and shown recently by Borot, Bouttier and Guitter [21], these maps can also be derived from maps bearing an \( O(n) \) loop model. The so-called gasket, defined by erasing the interiors of all the outermost loops, is such a planar map with large faces. In this work we argue that the underlying dimension \( D_H \) can be derived from \( D_G \) by using the KPZ relation for the \( O(n) \) model gasket. We first describe the essentials of the constructions in [21].

![FIG. 1. (Courtesy of E. Guitter [21].)](arXiV:1108.3327v1 [math-ph] 16 Aug 2011) A random planar quadrangulation with a boundary of length \( 2p = 8 \), and an \( O(n) \) loop model on the dual lattice. The configuration here of 28 quadrangles (quads) decomposes into \( N_0 = 9 \) empty quads, \( N_1 = 8 \) trans-quads, and \( N_2 = 11 \) cis-quads, building up 3 loops, and resulting in a weight \( W_G(a) = n^3(h_0)^5(h_1)^5(h_2)^3 \). The gasket \( G \) results from erasing the interior (shaded area) of all outermost loops, and retaining only those vertices which lay outside of the loops. This yields a bipartite planar map \( \tilde{G} \) with the same boundary, and made here of five of the original empty quads and of two larger faces of degrees \( 2k = 6 \) and 10, with a total weight \( W_\mathcal{Q}(\tilde{G}) = (q_2)^3(q_3)^3(q_5)^4 \).
Planar maps with large faces.—A planar map is a proper embedding of a finite connected graph $G$ on the two-dimensional sphere, and is said to be bipartite when all its faces have even degrees (number of edges). Given a sequence of nonnegative real numbers $q := (q_k)_{k \in \mathbb{N}}$, one defines the Boltzmann weights $W_q(G) := \prod_{f \in G} q_{\deg(f)/2}$, where $\delta(G)$ is the set of faces of $G$ and $\deg(f)$ is the degree of face $f$. More precisely, one considers sequences of the form

$$q_k = c_0 \beta^k q_0, \quad q_k \sim k^{-a}, \quad a \in (3/2, 5/2),$$

where $(q_k)_{k \in \mathbb{N}}$ is a sequence of nonnegative real numbers with a heavy tail, and $\sim$ means the ratio of the two sides has limit 1. The constants $c_0$ and $\beta$ are fine tuned to this sequence, via the auxiliary series $f_0(z) := \sum_{k \geq 1} \binom{2k-1}{k} q_k \frac{z^k}{k!}$. Using Stirling’s asymptotics for the binomial coefficient, its radius of convergence is 1 with finite values $f_0(1)$ and $f_0'(1)$, because $a > 3/2$. One takes $c_0 := 4/f_0'(1)$ and $(\beta^{-1}) := 1 + f_0(1)/f_0'(1)$.

Scaling limit and Hausdorff dimension.—For the particular choice of face statistics with a heavy tail, the authors are able to prove rigorously that a scaling limit of these random maps exists, where “macroscopic” faces remain present. A continuous so-called distance process of these random maps exists, where “macroscopic” faces of these random maps exist, with external (empty) face has a fixed degree 2 sphere, rooted along some (boundary) edge, whose external (empty) face has a fixed degree 2 sphere, rooted along some (boundary) edge, whose existence is in the so-called disk partition function has the distinctive aspect of the $O(n)$ model on random trivalent graphs, as well as by that of the model at hand in the scaling limit. (See below.)

Asymptotics.—On the critical surface $\tilde{h} = 2a - 1 \in (2, 4)$, the so-called disk partition function has the distinctive asymptotic behavior

$${\mathcal{Z}}^{(p)}_{O(n)}(\tilde{h}) \sim c_0 \beta^{-p} \tilde{h}^{-a}, \quad \beta = h_1 + 2h_2,$$

where $c_0$ is a factor independent of $p$. The universal critical exponent $a$ in (4) is given by

$$n = 2 \sin \pi(a - 3/2) \in (0, 2),$$

$$a \in (3/2, 2), \quad h = C+, \quad a \in (2, 5/2), \quad \tilde{h} \in C_+,$$

the two ranges of $a$ corresponding to the dense and dilute phases, respectively, and $a = 2$ to the limit case $n = 2$ where these phases coincide. In the pure gravity phase $h \in C_+$, one has $a = 5/2$.

Relation to bipartite maps with large faces.—From [21, 23] and [8], one observes that both partition functions share the same asymptotic behavior. This strongly suggests that the two models can in fact be put in bijection one to another, this resulting in a complete identity of their partition functions $Z^{(p)}_{\phi(n)} = Z^{(p)}_q \forall p \geq 1$. This relation was introduced in [20] and established combinatorially in [21] for the present model.

A quadrangulation carrying $\mathcal{L}$ loops, made of $N_0$ empty quads, $N_1$ trans-quads and $N_2$ cis-quads, receives a weight $W_{O(n)} := n^C h_0^{N_0} h_1^{N_1} h_2^{N_2}$.

A $p$-quadrangulation with a boundary of length $2p \geq 1$ is then defined as a planar map whose external face has degree $2p$, all other faces being quadrangles. The partition function $Z^{(p)}_{\phi(n)}$ is then obtained by summing the weights $W_{O(n)}$ over all such (rooted) $p$-quadrangulations equipped with all possible collections of loops on the dual graph.

For a given $n \in [0, 2]$, there exists a critical surface $C(n)$ in the parameter space $\tilde{h} := (h_0, h_1, h_2) \in \mathbb{R}^3_+$. It is encoded by a positive decreasing function $h_0^0(h_1, h_2, n)$ s.t. the model is well-defined in the bounded domain $0 \leq h_0 < h_0^0$, critical on the surface $h_0 = h_0^0$, ill-defined elsewhere. On $C(n)$, there exists a doubly critical line $C_+(n) : h_1 = h_1^*(h_2, n), h_0 = h_0^*(h_1^*, \cdot, \cdot)$, where the model is in the so-called dense critical phase; in the open sets $C_+(n) : h_1 = h_1^*(h_1^*, h_2, \cdot, \cdot)$ is the model in the dense critical phase $C_+$ or pure gravity phase $C_-$. This description of critical loci is corroborated by exact solutions of the $O(n)$ model on random trivalent graphs [22], as well as by that of the model at hand in the $(h_1 \geq 0, h_2 = 0, h_0 \geq 0)$ sector [19].

In the scaling limit, the dense phase $C_+$ is characterized by macroscopic fractal loops that bounce off themselves and off each other infinitely often; in the dilute phase $C_*$, these loops have no self nor mutual intersections; in the pure gravity phase $C_-$, they stay microscopic, vanishing in the scaling limit. (See below.)

Relation to bipartite maps with large faces.—In this model [21], self- and mutually-avoiding loops are drawn on a trivalent graph dual of a quadrangulation, each loop carrying a non local weight $n$ (Fig. 1). They delineate three types of quadrangles (quads): empty ones not traversed by any loop, of local weight $h_0$, trans-quads crossed by a loop at opposite edges (local weight $h_1$), and cis-quads with loop crossings at adjacent edges (local weight $h_2$) (Fig. 1). A quadrangulation carrying $\mathcal{L}$ loops, made of $N_0$ empty quads, $N_1$ trans-quads and $N_2$ cis-quads, receives a weight $W_{O(n)} := n^C h_0^{N_0} h_1^{N_1} h_2^{N_2}$.
As can be observed geometrically, faces of arbitrary degree $2k$ can be drawn as contours of outermost loops of length $2k$ in the $O(n)$ model (Fig. 1). When $k$ is large, the contour loop, of weight $n$, of a face of length $2k$ crosses a long thin quads’ rim of width essentially one and length $2k$. The result is a weight $(h_1 + 2h_2)2k$, each factor corresponding to one choice of trans-quad and two choices of cis-quads. Inside the rim lies a face of large degree of order $2k$ and containing an $O(n)$ $2k$-quadrangulation of total weight $Z_{O(n)}^{(k)}$. The result is an overall face weight $[3, 4]$, rigorously proven in [5]:

$$q_k \sim n^{-\beta} n^{\Delta(\gamma)} k^2 Z_{O(n)}^{(k)} \left( \frac{h}{k} \right),$$

with $\beta = h_1 + 2h_2$. Then [3] exactly coincides with [3] in the face model, in agreement with the relation [3] [21]. Lastly, comparing [1], [3] and [3] gives $c_0 = n c_0$.

To summarize, erasing the content of the outermost loops of an $O(n)$ model defines its gasket $G$. For a critical model on a random quadrangulation, parameterized as in [3], this yields the large face planar map model with weights [1]. Therefore, the gasket $G$ of a critical $O(n)$ model on a random surface $S$, defined as the scaling limit of the random quadrangulation, has a Hausdorff dimension rigorously given by [2].

Coulomb gas and SLE.—In the so-called Coulomb gas (CG) representation of the $O(n)$ model in the Euclidean plane [24], one has the parameterization $n = -2 \cos \pi g$ with, for $n \in [0, 2]$, a coupling constant $g \in [1/2, 1)$ in the dense phase and $g \in [1, 3/2]$ in the dilute phase. The critical scaling limit is known to correspond to Schramm-Loewner evolutions SLE$_{\kappa}$, or more precisely to conformal loop ensembles CLE$_{\kappa}$ [23], with a parameter $\kappa = 4/g$.

In terms of the critical exponent $a$ defined in [2], we thus have the parameter relations

$$g = a - 1 = 4/\kappa \in (1/2, 3/2).$$

Quantum scaling exponents and KPZ.—Given a fractal $X$ on a random surface $S$, its quantum fractal measure $Q$ (i.e., the mass of $X$) in a $S$-subset of area $A$ scales as

$$Q \sim A^{1-\Delta},$$

where $\Delta$ is the quantum scaling (KPZ) exponent of $X$. It is related to the Euclidean scaling exponent $x$ of $X$, of fractal dimension $d = 2-2x$, via the celebrated KPZ relation [3]:

$$x = \gamma^2 \Delta^2 / 4 + (1 - \gamma^2 / 4) \Delta,$$

where $\gamma$ is the so-called quantum Liouville parameter of the random surface.

When the fractal $X$ is part of a geometrical critical model, such as the $O(n)$ loop model considered here, $\gamma$ depends on the model’s central charge $c$ [3, 4] and in the CG-SLE$_{\kappa}$ framework (where $c = (6-\kappa)(6-16/\kappa)/4$) [3]:

$$\gamma = (4/g) \wedge 4g = \kappa \wedge (16/\kappa) \leq 4.$$  

Quantum Hausdorff dimensions in the dilute phase.—

We now claim the following relation between the Hausdorff dimension $D_G$ of the gasket $G$ of a dilute critical $O(n)$ model in quantum gravity, and that $D_H$ of $S$:

$$D_G = D_H(1 - \Delta_G),$$

where $\Delta_G$ is the gasket quantum scaling exponent. We explain this relation as follows. Denote by $l_G$ the length of an element of geodesic in $G$. The gasket quantum measure $Q_G$ in a ball of radius $l_G$ in $G$ is, with [2],

$$Q_G \propto l_G^{D_G}.$$  

A natural but crucial question is then whether an element of geodesic on the gasket $G$ is also (locally) a geodesic of the whole fractal surface $S$. We expect this to be true for the dilute phase of the $O(n)$ model, as explained now.

In the Euclidean plane, the fractal dimension of the loop contact points is $d_G = 2 - 2x_4$, with a “four-leg” exponent $x_4 = (3g + 2 - g^{-1})/4$. Thus $d_G \leq 0$ in the dilute phase $g \geq 1$. In quantum gravity, the corresponding exponent is obtained from $x_4$ by using KPZ [9], [10] for $g \geq 1$: $\Delta_4 = (1 + g)/2 \geq 1$ [22]. This implies a negative quantum fractal dimension $D_4 = (1 - \Delta_4)$ for loop contact points. In the scaling limit, they disappear and locally the quantum gasket should look like a patch of random surface, with matching geodesics.

The quantum area $A$ in the $S$-ball of shared geodesic radius $l = l_G$ is then, by definition of $D_H$,

$$A \propto l^{D_H},$$

The Hausdorff dimension identity [11] then simply follows from comparing [12] and [13] to [8]. (See also [27].)

The gasket Euclidean fractal dimension $d_G$ was first determined by CG techniques [28], and recently rigorously derived [23] as $d_G = 2 - 2x_4$, with $x_4 = (8 - 4g - 3g^{-1})/16$. The gasket quantum exponent is then obtained from KPZ [9], [10] for $g \geq 1$: $\Delta_G = (3 - 2g)/4$. By combining this with [2], [7] and [11] we get

$$D_H = 4,$$

independent of the value of $g \in [1/3, 2/3]$, i.e., of the universality class of the dilute critical $O(n)$ model. Note that in that phase $c \in [0, 1]$ for $n \in [0, 2]$, after inclusion of the pure gravity case $c = 0, n = 0$.

Dilute critical lines have a “two-leg” quantum exponent $\Delta_2 = 1/2$, independent of $n$, which governs the scaling of their quantum length $L \propto A^{1-\Delta_2} \propto A^{1/2}$ [22]. The gasket boundary exponent is given by [20]:

$$\Delta_G = 2\Delta_2 - (1 - g) = 1/2 - \Delta_2,$$

viz., the gasket’s boundaries indeed are loops. From [13], we thus conclude that the quantum fractal dimension of dilute loops or gasket boundaries is $D_2 = D_H(1 - \Delta_2) = 2, \forall n \in [0, 2]$.

Quantum Hausdorff dimensions in the dense phase.—

This phase corresponds to $g \in (1/2, 1)$, hence $c \in (-2, 1).
The fractal dimension of the surface $\mathcal{S}$ is still $D_H = 4$, but its derivation requires a modified geometrical argument.

From KPZ (9), (10) applied to $x_0$ above for $g < 1$, we get a quantum contact exponent $\Delta_D = (3-g^{-1})/2 < 1$ in the dense phase 22, 26. Hence, like the Euclidean one $d_4$, the quantum dimension $D_4 \propto (1 - \Delta_D^2)$ for contact points is now positive and we have to take into account the non-simple, infinitely bouncing, nature of dense loops. The relation between the loop quantum length and the area is known to be modified to $\mathbb{L} \propto \mathcal{A}^{1-2\Delta_D^2} \propto \mathcal{A}_\mathbb{H}^{\nu}$, with an exponent $\Delta_D^2 = 1 - (2g)^{-1}$ and $\nu = g$ 22, 26.

We now distinguish the length $l_G$ of an element of geodesic on $\mathcal{G}$ from the (shorter) length $l$ of the corresponding direct geodesic on $\mathcal{S}$. In the scaling limit, the $\mathcal{G}$-geodesic, continually pinched at all scales by dense loops, is forced to pass through all their contact points. It thus seems natural to assume that the same scaling relation holds between $l_G$ and $l$ as holds between $\mathbb{L}$ and the typical length $\mathcal{A}^{1/2}$:

$$l_G \propto 1^{1/\nu} \propto 1^{1/g} \gg 1.$$  (15)

Eqs. (12) and (13) inserted into relation (8) then give

$$D_Q = \nu D_H(1 - \Delta_D^2)$$  (16)

instead of (11), and with $\Delta_D$ the dense gasket quantum exponent. The latter is obtained from KPZ (9), (10) for $g < 1$, and from the same $x_0$ as above: $\Delta_D = (2-g^{-1})/4$. Using (2) in (10) again gives $D_H = 4$, $\forall g \in (1/2, 1)$, thus $\forall c \in (-2, 1)$. By continuity, this should also hold at $c = -2$ and (1). The gasket has a boundary exponent 29

$$\Delta_D^2 = 2\Delta_Q^2 = \Delta_D^2, \text{ viz., is made of loops with a quantum fractal dimension in } \mathcal{G}: D_2 = 2\nu D_H(1 - \Delta_Q^2) = 2.$$

Concluding remarks.—Recall that the Fortuin-Kasteleyn representation of the critical $Q$-state Potts model yields a dense loop model with $n^2 = Q \in [0, 4]$, so that our result should also apply to the Potts model in quantum gravity. The dilute and dense loop-model phases share the interval $c \in [0, 1]$, and have the same dimension there. In particular, $c = 0$ corresponds to dilute $n = 0$ self-avoiding walk or dense $n = 1$ critical percolation models. Their Euclidean partition functions are both 1, which leads to pure gravity with $D_H = 4$.

For (non-unitary) $c < 0$ models, numerical values less than 4 have been reported for $D_H$ in 12, 18, in particular for $c = -2$ (spanning trees), while recent simulations, by contrast, suggest there $1/D_H \gtrsim 0.26$ 29. Admittedly, our result rests on assumption (14), but for $g = 1/2, c = -2$, this just agrees with the further anomalous boundary scaling found in 17. A direct combinatorial study of the geodesic scaling and of the superuniversality of the Hausdorff dimension would thus be especially interesting.

Lastly, the so-called $m$-multicritical models ($m \in \mathbb{N}$) have fractal dimensions $D_H = 2(m+1) 3$, with $c \leq -22/5$ for $m \geq 2$, hence $c$ alone does not determine $D_H$.

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