Role of complementary correlations in the evolution of classical and quantum correlations under Markovian decoherence

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Abstract
Quantum correlation lies at the very heart of almost all of the non-classical phenomena exhibited by quantum systems composed of two or more subsystems. In recent times it has been pointed out that there is a kind of quantum correlation, namely discord, which is more general than entanglement. Some authors have investigated the phenomenon that for certain initial states the quantum correlations as well as the classical correlations exhibit sudden change under simple Markovian noise. We show that this dynamical behavior of the correlations of both types can be explained using the idea of complementary correlations. We also show that though a certain class of mixed entangled states can resist the monotonic decay of quantum correlations, this is not true for all mixed states. Moreover, pure entangled states of two qubits will never exhibit such sudden change.

Keywords: quantum correlations, Markovian noise, complementary correlations, discord

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum information processing protocols \cite{1, 2} require resources which are quantum in nature. In the past few years it has been proved that different kinds of non-classical...
correlations are necessary resources for performing various quantum information processing tasks. One of the most important non-classical correlations is quantum entanglement [3–5], which has been a central area of research in quantum information science for a long period of time. Various information-theoretic tasks such as quantum teleportation [1], quantum dense coding [2], quantum key distribution, and state merging [6] can be performed in the presence of entanglement. However, there is another kind of quantum correlation, called discord [7, 8], which is more general than quantum entanglement. Like entanglement, quantum discord has received attention during the past few years, and it has proved to be a useful non-classical resource. It provides a speedup in performing some tasks with respect to the best known classical counterparts, as shown theoretically and experimentally in a non-universal model of quantum computation [9]. Also, quantum discord finds operational significance not only in quantum computation in quantum state merging, but also in entanglement distribution, teleportation, and quantum metrology, as pointed out in [10–16]. On the other hand, quantum mutual information is an information-theoretic measure of the total correlation in a bipartite quantum state. Groisman [17] and Schumacher and Westermoreland [18] showed the significance of quantum mutual information, which can be thought of as the sum of the quantum and classical correlations [19]. It is therefore important to understand the nature of both of the correlation types, qualitatively as well as quantitatively.

In most of the ideal cases it is assumed that quantum systems are isolated from the environment, and one can use unitary evolution to illustrate the dynamics of such systems. Unfortunately, during practical applications the quantum systems interact with the environment, resulting in a loss of quantum coherence, which in turn destroys the quantum correlations. This destruction of quantum properties by the inevitable interaction with the environment is perhaps the major hindrance to the development of quantum technologies to date. Recently, several studies have revealed the dynamics of quantum and classical correlations under both Markovian [20–26] and non-Markovian [27–39] decoherence. Interestingly, contrary to the case for entanglement dynamics where sudden death may occur [40, 41], quantum correlation measured by quantum discord does not exhibit such behavior [42, 43]. More specifically, for certain classes of states, e.g. Bell diagonal states, the discord remains constant for a particular period of time and then decays, while classical correlation decays at first and then becomes constant [22].

In this paper we have focused mainly on two questions:

1. What are the underlying physical mechanisms for which classical and quantum correlations (measured in terms of discord) suffer sudden change in the decoherence process as studied by Mazzola et al [22]?
2. Does mixedness provide some advantage in preventing the loss of quantum coherence, and hence quantum correlations?

To answer the first question we use the idea of complementary correlations introduced in [44, 45] and we succeed in describing the physical mechanism underpinning the phenomena observed in [22]. More precisely, we show that under some restrictions, for certain classes of bipartite qubit states the amounts of classical correlation and quantum discord are exactly equal to the correlations between the complementary observables acting on the subsystems of the bipartite states. Consider $\rho_{AB}$ to be a bipartite system composed of subsystems $A$ and $B$. $A$ and $C$ are the two non-degenerate observables acting on subsystem $A$. Now, these observables are complementary if $|\langle a_i | c_j \rangle|^2 = \frac{1}{d^2}$ for all $i, j$. $\{|a_i\}_{i=1}^d$ and $\{|c_j\}_{j=1}^d$ denote the non-degenerate eigenstates of observables $A$ and $C$ respectively and $d$ is the dimension of the subsystems. Similarly, if $B$ and $D$ are the two complementary observables acting on
subsystem $B$, then the correlations between the measurement outcomes for $A$ and $B$ or $C$ and $D$ (figure (1)) are said to be complementary correlations.

We also find the initial class of states exhibiting sudden transition in the evolution of classical and quantum correlations. To our surprise, we find that the two-qubit pure entangled states will never show the said behavior. Rather, some particular mixture of pure states exhibits sudden change in the classical and quantum correlations in the decoherence process, which answers the second question in the affirmative.

The organization of the paper is as follows: section 2 contains a brief overview on quantum discord and mutual information; in section 3 we give a brief description of complementary correlations and discuss the correlation tensor matrix; Markovian decoherence and quantum channels are discussed briefly in section 4; we present our results in section (5); and section 6 contains discussion and concluding remarks.

2. Discord and mutual information

Entanglement is perhaps the most familiar non-classical correlation observed in quantum systems composed of two or more subsystems. But it is important to note that some states with zero entanglement can perform tasks which are not possible in the classical regime. This is due to the fact that those states have non-classical correlations even though they are unentangled. Besides entanglement, another measure of quantum correlations that has received a great deal of attention is the quantum discord ($D$), originally proposed by Ollivier and Zurek [7]. The quantum discord is defined as

$$D(\rho_{AB}) \equiv I(\rho_{AB}) - C(\rho_{AB}),$$

where $C(\rho_{AB})$ and $I(\rho_{AB})$ are respectively the classical correlations and quantum mutual information of the bipartite state $\rho_{AB}$. The quantum mutual information $I(\rho_{AB})$ measures the total correlation present in the state $\rho_{AB}$, and it is defined as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

where $\rho_A$ and $\rho_B$ are the reduced density matrices of the subsystems $A$ and $B$ respectively and $S(\rho) = -\text{Tr}\{\rho \log_2 \rho\}$ is the von Neumann entropy. From the above definition of mutual information it is clear that one can straightforwardly calculate it for a given state $\rho_{AB}$. The maximum value of $I(\rho_{AB})$ is $2 \log_{2d}$, achieved by two-qudit maximally entangled states, and its minimum value is zero, achieved by product states.
On the other hand, the classical correlations $C(\rho_{AB})$ of a composite quantum state can be quantified via the measure proposed by Henderson and Vedral [19] which is given by

$$C(\rho_{AB}) \equiv \max_{\{\Pi_j\}} \left[ S(\rho_A) - S(\rho_{A|B}) \right] ,$$

where the maximum is taken over the set of projective measurements $\{\Pi_j\}$ on subsystem $B$. $S(\rho_{A|B}) = \sum_j p_j S(\rho_A^j)$ is the entropy of subsystem $A$ conditioned on $B$, $\rho_A^j = \text{Tr}_B(\Pi_j \rho_{AB} \Pi_j) p_j$ is the density matrix of subsystem $A$ depending on the measurement outcome for $B$, and $p_j = \text{Tr}_A(\rho_{AB} \Pi_j)$ is the probability of the $j$th outcome. Note that, unlike $I(\rho_{AB})$, the classical correlation is asymmetric with respect to the subsystems involved, and so is $D(\rho_{AB})$.

As the definition of classical correlation requires optimization over all possible projective measurements (and more generally over the positive-operator-valued measurements [46]) that can be performed on one part of the composite system, it is in general very hard to find the amount of quantum discord for an arbitrary bipartite state. To avoid this complexity, various computably easy measures of quantum discord have been proposed, recently [47–49]. A few of the important properties of quantum discord are as follows:

(a) $D(\rho_{AB}) \geq 0, \forall \rho_{AB}$. The proof is straightforward. Putting the explicit forms of $I(\rho_{AB})$ and $C(\rho_{AB})$ in equation (1), we get

$$D(\rho_{AB}) = S(\rho_{A|B}) - S(\rho_{A}) ,$$

where $S(\rho_{A|B}) = S(\rho_{AB}) - S(\rho_B)$ is the quantum conditional entropy. Finding the value of $S(\rho_{A|B}) \geq S(\rho_{AB}) - S(\rho_B)$, which completes the proof. Focusing our concern on the quantum conditional density matrix $\rho_{A|B}$, then, from [50], it can be said that this conditional density matrix retains the quantum phases and coherence. So, physically, quantum discord is a measure of information which cannot be extracted without joint measurements [7].

(b) $D(\rho_{AB}) = 0$ for quantum–classical (QC) states which are of the form:

$$\rho_{QC} = \sum_i p_i \rho_A^i \otimes |i_B \rangle \langle i_B| ,$$

where $\{|i_B\rangle\}$ is an orthonormal basis for subsystem $B$, the $\rho_A^i$ are density matrices of subsystem $A$, and $\{p_i\}$ is a probability distribution. For two-qudit maximally entangled states, $D(\rho_{AB}) = \log_2 d$.

(c) Quantum discord is non-increasing under completely positive trace-preserving (CPTP) maps on the unmeasured party $A$ [51], i.e.,

$$D(\rho_{AB}) \geq D(\rho_{AB}^{A \otimes I_B}) ,$$

where $A$ is the CPTP map on $A$ [52].

3. Complementary correlations and the correlation tensor matrix

Complementary correlations. The concept of complementary correlations was introduced recently, in [44]. Consider a quantum mechanical system described with a $d$-dimensional Hilbert space. Let $\mathcal{M}$ and $\mathcal{N}$ be two observables acting on the system, with $\{|m_i\rangle\}_{i=1}^d$ and
denoting the non-degenerate eigenstates, respectively. \( \mathcal{M} \) and \( \mathcal{N} \) are called complementary observables if \(|\langle m_i | n_j \rangle|^2 = \frac{1}{2} \) for all \( i, j \). This means that if one knows the value of one of the complementary observables, i.e., if the system is prepared in one of the eigenstates of one of the complementary observables, then all of the possible values of the other observable are equally probable.

The authors of [44] have shown that, for a composite quantum system, the correlations in the measurement of such complementary observables constitute a good signature of quantum correlations present in the state. If two quantum systems of finite dimension are considered and two observables \( \mathcal{A} \otimes \mathcal{B} \) and \( \mathcal{C} \otimes \mathcal{D} \) are taken into account, where \( \mathcal{A} \) and \( \mathcal{C} \) are complementary observables acting on one subsystem and \( \mathcal{B} \) and \( \mathcal{D} \) are complementary observables acting on the other, then the quantity \(|\chi_{AB}| + |\chi_{CD}|\) denotes the value of the complementary correlations, with \(|\chi_{AB}|\) and \(|\chi_{CD}|\) denoting the absolute values of the correlations for complementary observables. The sum not only tells us about the quantum correlations present in a composite quantum system, but also represents the overall correlations of the composite system.

Consider that \( \chi_{AB} = I_{AB} \) and \( \chi_{AB} = I_{CD} \), where \( I \) is the mutual information defined earlier and having an alternative definition as

\[
I_{AB} \equiv H(A) - H(A|B),
\]

where \( H(A) \) is the Shannon entropy of the outcome probabilities of the measurement \( A \) performed on the first system and \( H(A|B) \) denotes the conditional entropy, conditioning being done on the second system. Therefore in terms of mutual information the complementary correlation reads as \( I_{AB} + I_{CD} \). It can be easily shown that:

(i) If \( I_{AB} + I_{CD} = 2 \log_2 d \), then the bipartite quantum system is maximally entangled and there exist two complementary measurement bases, or in other words mutually unbiased bases (MUBs) [53].

(ii) If \( I_{AB} + I_{CD} > \log_2 d \), then there is entanglement in the composite system.

(iii) If \( I_{AB} + I_{CD} = \log_2 d \), then the bipartite state is a classically correlated (CC) state which belongs to the set of separable states, and the quantum correlations for such states are zero.

The correlation tensor matrix. Here we concentrate on a two-qubit quantum system defined on the Hilbert space \( \mathcal{H} = C^2 \otimes C^2 \). The collection of Hermitian operators acting on \( \mathcal{H} \) constitute an inner product space with a Hilbert–Schmidt inner product defined as \( \langle \alpha, \beta \rangle = \text{Tr}(\alpha^\dagger \beta) \), where \( \alpha \) and \( \beta \) are Hermitian operators acting on \( \mathcal{H} \). In such a Hilbert–Schmidt space, any generic state of the system can be expressed as [54]

\[
\tau = \frac{1}{4} \left( \mathbf{1}_2 \otimes \mathbf{1}_2 + a \cdot \sigma \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes b \cdot \sigma + \sum_{m,n=1}^{3} c_{nm} \sigma_n \otimes \sigma_m \right),
\]

where \( \mathbf{1}_2 \) is the identity operator, \( a \) and \( b \) denote the local Bloch vectors for each subsystem, and the \( \{\sigma_n\}_{n=1}^{3} \) are the standard Pauli spinors \( \sigma_x, \sigma_y \) and \( \sigma_z \). The \( 3 \times 3 \) matrix \( \mathcal{T} \) formed from the coefficients \( c_{nm} \) is called the correlation tensor matrix, as it is responsible for the correlations:

\[
\mathcal{E}(a, b) \equiv \text{Tr}(\rho a \cdot \sigma \otimes b \cdot \sigma) = (a, \mathcal{T} b).
\]

Note that the \( c_{nm} = \text{Tr}(\tau \sigma_n \otimes \sigma_m) \) are the expectation values of the observables \( \sigma_n \otimes \sigma_m \). The state \( \tau \), as expressed in equation (8), can always be transformed to a state \( \tilde{\tau} \) for which the matrix \( \mathcal{T} \) becomes diagonal upon acting with local unitaries \( U_1 \) and \( U_2 \). Though the unitaries transform the state \( \tau \) to \( \tilde{\tau} \), the inseparability (separability) remains invariant. The transformed state can be represented as
\[ \tau = U_1 \otimes U_2 \tau U_1^\dagger \otimes U_2^\dagger. \] (10)

The transformation of the state \( \tau \) is possible due to the fact that for any unitary transformation \( U \), there is always a unique rotation \( O \) such that \( U \hat{\mathbf{n}} \cdot \sigma \mathbf{U}^\dagger = (O \hat{\mathbf{n}}) \cdot \sigma \) and the parameters \( a, b \) and \( T \) transform themselves as \( a' = O_1 a \), \( b' = O_1 b \) and \( T' = O_1 T O_2^\dagger \) respectively, where \( a', b' \) and \( T' \) are the new parameters for the state \( \tilde{\rho} \).

As the unitaries \( U_1 \) and \( U_2 \) diagonalize the correlation tensor matrix, we hence have

\[ T' = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}, \] (11)

where \( c_1 = \text{Tr}(\sigma_x \otimes \sigma_x) \), \( c_2 = \text{Tr}(\sigma_y \otimes \sigma_y) \) and \( c_3 = \text{Tr}(\sigma_z \otimes \sigma_z) \) are the expectation values of the observables \( \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y \) and \( \sigma_z \otimes \sigma_z \), respectively.

4. Markovian decoherence and quantum channels

Markovian decoherence. Decoherence is a physical process of gradual loss of the coherence present in any quantum system [55]. The process can be represented by some family of linear maps \( \{A_{t_{0,1}}, t_2 \geq t_1 \geq t_0 \} \), where \( t_0, t_1 \) and \( t_2 \) denote times [56]. If the linear map \( A_{t_{0,1}} \) satisfies the following three properties:

(i) preservation of the trace, i.e., \( \text{Tr}(\rho) = \text{Tr}(A[\rho]) \),
(ii) being completely positive, i.e., \( A \otimes I_n \) is positive for all \( n \), where \( I_n \) denotes the identity map acting on \( n \)-dimensional Hilbert space, and
(iii) having \( A_{t_{0,1}} = A_{t_{1,2}} A_{t_{0,1}} \), then the decoherence process is called Markovian decoherence [56]. The linear map \( A(\cdot) \) basically describes the time evolution of a quantum system that interacts with its environment. The above-mentioned conditions, which a linear map should meet to represent a Markovian process, come from a very useful theorem, namely the Kraus representation theorem [57], which provides the following operator sum representation for any CPTP map:

\[ A(A) = \sum_{j=1}^{r} \Gamma_j A \Gamma_j^\dagger, \] (12)

where the \( \Gamma_j \) are the Kraus operators and \( r \) is the Kraus rank, which determines the number of Kraus operators in the operator sum representation of the linear map \( A(\cdot) \). The normalization principle leads to the fact that \( A(\cdot) \) is trace preserving if \( \sum_{j=1}^{r} \Gamma_j \Gamma_j^\dagger = 1 \).

Quantum channels. In any communication protocol, one has to send information through channels. If the information is quantum in nature, then the time evolution of the quantum system carrying the information can be modeled using quantum channels. Mathematically, quantum channels are superoperators or CPTP linear maps having the operator sum representation. For our purposes, we briefly discuss the following three quantum channels: the bit flip channel, the bit–phase flip channel and the phase flip/phase damping channel:

1. The bit flip channel. The Kraus operators which represent the bit flip channel are

\[ \Gamma_0 = \sqrt{1 - p} I, \quad \Gamma_i = \sqrt{p} \frac{1}{\sqrt{2}} \sigma_i, \]

where \( p \) represents the probability and \( \sigma_i \) is the standard Pauli matrix. For the bit flip
channel, any general density matrix $\rho$ evolves as

$$\rho \rightarrow \rho' = \left(1 - \frac{P}{2}\right)\rho + \frac{P}{2}(\sigma_i \rho \sigma_i).$$

The bit flip channel basically flips $|0\rangle$ to $|1\rangle$ (and vice versa). So any single-qubit state $|\psi\rangle$ is mapped to $\sigma_i |\psi\rangle$ under the action of the bit flip channel.

(II) **The bit–phase flip channel.** The name of this channel signifies that it flips both the bit and the phase of a qubit. The Kraus operators for representing the bit–phase flip channel are as follows:

$$I_0 = \sqrt{1 - \frac{P}{2}} I, \quad I_1 = \frac{P}{\sqrt{2}} \sigma_2.$$

The evolution of a density matrix $\rho$ for the bit–phase flip channel is given by

$$\rho \rightarrow \Lambda(\rho) = I_0 \rho I_0^\dagger + I_1 \rho I_1^\dagger = \left(1 - \frac{P}{2}\right)\rho + \frac{P}{2}(\sigma_2 \rho \sigma_2).$$

If one considers a single-qubit pure state $|\psi\rangle$, then the evolution of the state for the bit–phase flip channel can be given by the mapping $|\psi\rangle \rightarrow \sigma_2 |\psi\rangle$.

(III) **The phase damping channel.** Characterization of the decoherence process in realistic physical situations is possible by means of the phase damping channel. Although there are phenomenological models leading to decoherence, none of the models represent the real physical situations. Nevertheless, using the operator sum representation, the evolution of a quantum state for the phase damping channel can be easily understood.

The Kraus operators required to represent the phase damping channel are

$$I_0 = \sqrt{1 - \frac{P}{2}} I, \quad I_1 = \frac{P}{\sqrt{2}} \sigma_3.$$

The most general single-qubit density matrix can be written as

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix},$$

where the diagonal real elements represent the probabilities of finding the qubit in the states $|0\rangle$ and $|1\rangle$ respectively, if the measurement is done in the $\sigma_z$ basis. The off-diagonal elements (quantum coherences) have no classical analogue, and the phase damping channel induces a decay of those elements resulting in decoherence. The single-qubit density matrix evolves as

$$\Lambda(\rho) = I_0 \rho I_0^\dagger + I_1 \rho I_1^\dagger = \left(1 - \frac{P}{2}\right)\rho + \frac{P}{2}(\sigma_3 \rho \sigma_3) = \begin{pmatrix} \rho_{00} & \rho_{01}(1 - p) \\ \rho_{10}(1 - p) & \rho_{11} \end{pmatrix}. \quad (13)$$

It is clear from the above equation that the off-diagonal terms will gradually decay as the time elapses, and the initial coherent superposition will turn into an incoherent superposition or mixture, i.e.,

$$\rho \rightarrow \rho' = |\rho_{00}|^2 |0\rangle \langle 0| + |\rho_{11}|^2 |1\rangle \langle 1|.$$

The phase damping channel plays the central role in the transition from the quantum to
the classical world. The decay of the off-diagonal terms and hence decoherence can be well understood if we consider the interaction of the qubit with the environment as a rotation (or phase kick) about the $z$-axis of the Bloch sphere through an angle $\theta$, due to which the axes transform as $x' = e^{-\gamma t} x$, $y' = e^{-\gamma t} y$ and $z' = z$, where $\lambda$ is the damping parameter. In other words, it can be said that the channel picks out a preferred basis for the qubit, which is $\{|0\rangle, |1\rangle\}$, as the $z$ basis is the only one in which a bit flip never occurs.

5. Results

5.1. Complementary correlations and decoherence

Consider a generic two-qubit bipartite state $\tau$ as expressed in equation (8). Let us apply local unitaries, and diagonalize the correlation tensor matrix and transform the state $\tau$ to $\tilde{\tau}$, such that

$$\tilde{\tau} = \frac{1}{4} \left( \mathbf{1}_2 \otimes \mathbf{1}_2 + a \cdot \mathbf{\sigma} \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes b \cdot \mathbf{\sigma} + \sum_{n=1}^{3} c_n \mathbf{\sigma}_n \otimes \mathbf{\sigma}_n \right).$$

(14)

For simplicity, we consider the states with maximally mixed marginals, i.e., $a = 0$ and $b = 0$. Thus we have finally

$$\rho_{AB} = \frac{1}{4} \left( \mathbf{1}_2 \otimes \mathbf{1}_2 + \sum_{n=1}^{3} c_n \mathbf{\sigma}_n \otimes \mathbf{\sigma}_n \right).$$

(15)

where the $c_n$ are the diagonal elements of correlation tensor matrix $\mathbf{\tau}$ and $0 \leq |c_n| \leq 1$. The class of states represented by $\rho_{AB}$ in equation (15) are called the Bell diagonal states, which include the pure Bell states $|c_1| = |c_2| = |c_3| = 1$ and the Werner class of states $|c_1| = |c_2| = |c_3| = c$ [22].

When both of the subsystems of the composite state of equation (15) are subjected to local Markovian noise, the time evolution of the composite state is given by

$$\rho_{AB}(t) = \lambda_{\Psi}^+(t) |\Psi^+\rangle \langle \Psi^+| + \lambda_{\Phi}^+(t) |\Phi^+\rangle \langle \Phi^+| + \lambda_{\Phi}^-(t) |\Phi^-\rangle \langle \Phi^-| + \lambda_{\Psi}^-(t) |\Psi^-\rangle \langle \Psi^-|.$$  

(16)

where

$$\lambda_{\Psi}^+(t) = \frac{1}{4} \left[ 1 \pm c_1(t) \mp c_2(t) + c_3(t) \right],$$

(17)

$$\lambda_{\Phi}^+(t) = \frac{1}{4} \left[ 1 \pm c_1(t) \mp c_2(t) - c_3(t) \right],$$

(18)

and $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)$, $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle)$ are Bell states [46].

If we consider the phase damping channel as the local Markovian noise, then the coefficients in equations (17) and (18) will be

$$c_1(t) = c_1(0)e^{-2\gamma t},$$

$$c_2(t) = c_2(0)e^{-2\gamma t},$$

$$c_3(t) = c_3(0) \equiv c_3,$$

(19)

where $\gamma$ is the phase damping rate. For our analysis, we consider the initial states as $c_1(0) = \pm 1$ and $c_2(0) = \mp c_3(0)$, with the condition $|c_3| \leq 1$, as considered by other authors [22] also. Thus the states read as
\[ \rho_{AB} = \frac{(1 + c_1)}{2} |\Psi^z\rangle \langle \Psi^z| + \frac{(1 - c_1)}{2} |\Phi^z\rangle \langle \Phi^z|. \]  

(20)

The subsystems of the above state are qubits, and \( \sigma_x \) and \( \sigma_z \) are complementary observables for a qubit quantum system. From the definition of complementary correlations, the total complementary correlation is therefore

\[ I^c[\rho_{AB}(t)] = I(\sigma_x^A; \sigma_x^B) + I(\sigma_z^A; \sigma_z^B). \]  

(21)

where the superscript c signifies complementarity. The first term on the right-hand side of equation (21) denotes the correlation between the outcomes of \( \sigma_x \) measurement performed on the two sides; similarly, the second term denotes the same, but for \( \sigma_z \). Using equation (7), we have

\[ I(\sigma_x^A; \sigma_x^B) = P[c_1(t)] + P[-c_1(t)], \]  

(22)

\[ I(\sigma_z^A; \sigma_z^B) = P[c_3] + P[-c_3], \]  

(23)

where \( P[\alpha] = \frac{1+\alpha}{2} \log_2(1 + \alpha) \). Inserting equations (22) and (23) in equation (21), we get

\[ I^c[\rho_{AB}(t)] = P[c_1(t)] + P[-c_1(t)] + P[c_3] + P[-c_3]. \]  

(24)

Interestingly, for the class of states concerned, \( I^c[\rho_{AB}(t)] \) is exactly equal to the mutual information \( I \) of the state \( \rho_{AB} \), i.e.,

\[ I^c[\rho_{AB}(t)] = I(\rho_{AB}(t)). \]  

(25)

On the other hand, the classical correlation \( C(\rho_{AB}(t)) \) in this case turns out to be

\[ C(\rho_{AB}(t)) = P[K(t)] + P[-K(t)], \]  

(26)

where \( K(t) = \max\{ |c_1(t)|, |c_2(t)|, |c_3(t)| \} \). It is to be noted that the coefficients \( c_1(t), c_2(t) \) and \( c_3(t) \) are the expectation values of \( \sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2 \) and \( \sigma_3 \otimes \sigma_3 \) respectively. So, during the calculation of classical correlations of the specific states considered, the conditional entropy in equation (3) reaches its minimum when the projective measurements are performed on the eigenstate of that complementary observable \( \sigma_{\alpha}^{(B)} \) for which \( \text{Tr}(\rho_{AB} \sigma^{(A)}_{\alpha} \otimes \sigma^{(B)}_{\alpha}) \) is at its maximum. Hence, we conclude that, for the class of states taken into consideration, \( \sigma_x, \sigma_y, \sigma_z \) form a set of complementary observables, and the classical correlation \( C(\rho_{AB}(t)) \) is

\[ C(\rho_{AB}(t)) = \max\{ I(\sigma_x^A; \sigma_x^B) \}. \]  

(27)

For our purposes, we assume that

\[ I(\rho_{AB}) = Q(\rho_{AB}) + C(\rho_{AB}), \]  

(28)

which in turn yields

\[ I^c[\rho_{AB}(t)] = Q(\rho_{AB}) + C(\rho_{AB}). \]  

(29)

We are now in a position to explain the sudden transition in classical and quantum decoherence for the states considered in equation (20)

(i) At time \( t = 0 \), \( \text{Tr}(\rho_{AB} \sigma_1^{(A)} \otimes \sigma_1^{(B)}) = c_1(0) = 1 \). So projective measurements on eigenstates of \( \sigma_1 \) will yield the minimum conditional entropy \( S(A|B) \), and hence the amount of classical correlation, which is equal to the correlation between the measurement outcomes for \( \sigma_1 \) on either side of the bipartite state, reads as
From equations (1), (23) and (28), the value of the quantum correlations or discord comes out as
\[
D(\rho_{AB}(t_0)) = \mathbb{I}(\sigma^A : \sigma^B) = 2P[c_1(t_0)],
\]
(30)

(ii) In the time interval \(0 < t < t' = -\ln(|c_1|)/2\gamma\), we have \(c_1(t) > c_3\). So for the same reasons as were detailed above, the classical correlation \(C\) and discord \(D\) of the initial state will be
\[
C(\rho_{AB}(t)) = \mathbb{I}(\sigma^A : \sigma^B) = P[c_1(t)] + P[-c_1(t)],
\]
(32)
\[
D(\rho_{AB}(t)) = \mathbb{I}(\sigma^A : \sigma^B) = P[c_1] + P[-c_3].
\]
(33)

However, in this time interval the classical correlation \(C(\rho_{AB}(t))\) decays and the discord \(D(\rho_{AB}(t))\) remains constant. In other words, we can say that the correlations \(\mathbb{I}(\sigma^A : \sigma^B)\) decay and correlations \(\mathbb{I}(\sigma^A : \sigma^B)\) remain constant. The physical origin of this is that at \(t > 0\) the phase damping channel induces a decay in the quantum coherence (phase) of the state \(\rho_{AB}(t)\), which results in a decay in the expectation value \(c_1(t) = \text{Tr}[\rho_{AB}(t)\sigma^A \otimes \sigma^B]\), whereas the expectation value \(c_3(t) = \text{Tr}[\rho_{AB}(t)\sigma^B \otimes \sigma^B]\) remains constant. The phase damping channel picks the \(\{|0\rangle, |1\rangle\}\) basis as the preferred basis for each qubit and destroys all other superpositions of \(|0\rangle\) and \(|1\rangle\), resulting in the decay of \(c_1(t)\) and, hence, \(C(\rho_{AB}(t))\). Here the discord remains constant.

(iii) For \(t > t'\), we have \(c_1(t) < c_3\). So, in this case the conditional entropy \(S(A|B)\) will be a minimum if projective measurements are performed on eigenstates of \(\sigma_z\). Thus the classical correlation will be
\[
C(\rho_{AB}(t)) = \mathbb{I}(\sigma^A : \sigma^B) = P[c_1] + P[-c_3].
\]
(34)

From equations (1), (22) and (28), the discord of \(\rho_{AB}(t)\) will be
\[
D(\rho_{AB}(t)) = \mathbb{I}(\sigma^A : \sigma^B) = P[c_1(t)] + P[-c_1(t)].
\]
(35)

Hence, for \(t > t'\), the classical correlation is constant in time, whereas the discord starts to decay. To understand this sudden change in the evolution of classical and quantum correlations for \(t > t'\), we focus on the definition of classical correlations and notice that during this time, projective measurement on the eigenstates of \(\sigma_z\) will yield the classical correlations, i.e., only the term \(\mathbb{I}(\sigma^A : \sigma^B)\) contributes in \(C(\rho_{AB}(t))\). Now, as the channel is the phase damping channel, the expectation value \(c_3\) does not change with time and, as a result, the classical correlation becomes constant after time \(t'\).
On the other hand, during this time the correlation between measurement outcomes for another complementary observable, $\sigma_x$, represents the amount of $\rho_{AB}(t)$ present in the state $\rho_{AB}(t)$. However, it was mentioned before that the expectation value $c_1(t)$ decays due to the effect of phase damping noise. Therefore, $D(\rho_{AB}(t))$ decays after time $t'$.

If we consider, instead of phase damping noise, the bit–phase flip noise, then the coefficients $c_i$ change as follows:

\[
c_1(t) = c_1(0)e^{-2\rho t}, \quad c_2(t) = c_2(0) \equiv c_2, \quad c_3(t) = c_3(0)e^{-2\rho t}.
\]

Hence, rewriting the state $\rho_{AB}$ of equation (20) as

\[
\rho_{AB} = \frac{1-c_1(t)}{2}|\Psi^+\rangle\langle\Psi^+| + \frac{1+c_1(t)}{2}|\Phi^+\rangle\langle\Phi^+|,
\]

it is very easy to show that under bit–phase flip noise the classical and quantum correlations of this state will exhibit the same sudden transition in their evolution. Moreover, such sudden transitions can also be explained through complementary correlations. Similar arguments hold for bit flip noise with $\rho_{AB} = \frac{1-c_1(t)}{2}|\Psi^\pm\rangle\langle\Psi^\pm| + \frac{1+c_1(t)}{2}|\Phi^\pm\rangle\langle\Phi^\pm|$ (considering $c_2(0) = \pm 1$ and $c_1(0) = \mp c_3(0)$ with the condition $|c_3| \leq 1$). The coefficients $c_i$ evolve under bit flip noise as

\[
c_1(t) = c_1(0) \equiv c_1, \quad c_2(t) = c_2(0)e^{-2\rho t}, \quad c_3(t) = c_3(0)e^{-2\rho t}.
\]

5.2. Mixedness and decoherence

We now focus on two questions. (i) Will the pure entangled two-qubit states exhibit sudden change in the evolution of $\rho_{AB}(t)$ and $\rho_{\sigma}(t)$? (ii) Does mixedness always ensure sudden change in decoherence processes? To answer the first question, we consider a pure two-qubit entangled state:

\[
|\Psi_{AB}\rangle = a|00\rangle + b|11\rangle,
\]

where $|a|^2$ and $|b|^2$ are probabilities. This state belongs to the class of states for which the correlation tensor matrix $(T)$ is diagonal. The three diagonal elements, or the expectation values of the observables $\sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2$ and $\sigma_3 \otimes \sigma_3$, are found to be

\[
c_1 = (|a|^2 + |b|^2) = 1,
\]
\[
c_2 = -(|a|^2 + |b|^2) = -1,
\]
\[
c_3 = (|a|^2 - |b|^2) = 1.
\]

If both of the qubits of the state are subjected to a local phase damping channel, then all of the $c_i$ change as in equation (19). The classical correlation $C(\rho_{AB}(t))$ and discord $D(\rho_{AB}(t))$ of the state for $t > 0$ read as

\[
C(\rho_{AB}(t)) = I(\sigma^A_1: \sigma^B_1) = P[c_1] + P[-c_3] = 1,
\]
\[
D(\rho_{AB}(t)) = I(\sigma^A_1: \sigma^B_1) = P[c_1(t)] + P[-c_3(t)]
\]
\[
= P[c_1(t)] + P[e^{-2\rho t}]
\]
\[
= P[e^{-2\rho t}].
\]

From equations (38) and (39), it is clear that $C(\rho_{AB}(t))$ will always remain constant, while $D(\rho_{AB}(t))$ will decay from the beginning, which means that there will be no such sudden transitions in the evolution of the classical correlations and discord. So, we can conclude that pure two-qubit entangled states will never show sudden transitions.

To address our second question, we first consider the two-qubit Werner class of states, which is represented as
where $\beta$ represents the singlet fraction and $(1 - \beta)$ represents the random fraction. Simple calculations show that for such states, the diagonal elements of the correlation tensor matrix are all equal, i.e., $|c_1| = |c_2| = |c_3| = \text{constant} = k$ (say). Therefore, for such states, $C(\rho_{AB}(t))$ and $D(\rho_{AB}(t))$ are found to be

$$C(\rho_{AB}(t)) = I(\sigma^A: \sigma^B) = P[c_3] + P[-c_3] = P[k] + P[-k],$$

$$D(\rho_{AB}(t)) = \frac{1}{2} \bigg( P[k + 2k \cdot e^{-2\eta}] + P[k - 2k \cdot e^{-2\eta}] - 2P[k] \bigg).$$

Hence, for phase damping noise, classical correlations for this class of states will remain constant, while the discord will gradually decay without showing any kind of sudden transition. Figure 2 clearly illustrates the change of the discord. The dynamical behavior of the discord of Werner states shown here is in agreement with the findings of [42], where the authors have elaborately depicted the aforesaid dynamics considering different kinds of noises. It is thus confirmed that mixedness is not the only factor in sudden transitions in classical and quantum decoherence. It is important that, along with the bipartite state $\rho_{AB}$ having a mixed nature, the state should also have asymmetry in the correlations between the measurement outcomes for different complementary observables, or, stated more specifically, $I(\sigma^A: \sigma^B) > I(\sigma^A_1: \sigma^B_1) = I(\sigma^A_2: \sigma^B_2)$.

6. Conclusions

We have shown that in the case of certain Bell diagonal states, the sudden transition in the evolution of classical and quantum correlations under Markovian noise (the phase damping channel) can be well understood in terms of complementary correlations. For two-qubit Bell diagonal states, $\sigma_x$ and $\sigma_z$ are complementary observables, and for those specific Bell diagonal states, the overall complementary correlations are exactly equal to the mutual information ($I$) of the states. Interestingly, the sudden transition behavior of classical correlations finds a good interpretation in the quantum measurement problem. The whole measurement process can be modeled as if an isolated system is initially correlated with its measurement apparatus, which in turn locally interacts with an environment. In such a scenario, the transition of system–apparatus classical correlations from a decay regime to a plateau has been interpreted as the emergence of a pointer basis, while the system and the apparatus can still be quantum correlated [58]. Analogous considerations are expected to be valid for the situation presented in our work, because both qubits interact locally with their environment, and also the qubits act as both the system and the apparatus for each other. However, we explain here the sudden transition behaviors of not only classical but also quantum correlations in terms of complementary correlations.

We have also confirmed that two-qubit pure entangled states will never show such sharp transitions in the classical and quantum decoherence. Taking the example of the Werner class of states, we have reconfirmed that the freezing property of the discord ($D$), which is in contrast with the case for the entanglement sudden death, is not inherent to all mixed entangled states of
the Bell diagonal class [24]. Our analysis put forward an interesting question: using complementary correlations, is it possible to find the general class of two-qubit composite states exhibiting sudden change in the evolutions of the classical and quantum correlations? We hope that our findings will provide better insight into the evolutions of classical and quantum correlations of composite quantum systems, when subjected to noises.

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