Conditional Entanglement

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Based on the ideas of quantum extension and quantum conditioning, we propose a generic approach to construct a new kind of entanglement measures called conditional entanglement. The new measures, built from the known entanglement measures, are convex, automatically super-additive, and even smaller than the regularized versions of the generating measures. More importantly, new measures can also be built directly from measures of correlations, enabling us to introduce an additive measure and generalize it to a multipartite entanglement measure.

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Entanglement, as a key resource and ingredient in quantum information and computation as well as communication, plays a crucial role in quantum information theory. It is necessary to quantify entanglement from different standpoints. A number of entanglement measures have been formulated, and their properties have been explored extensively (see, e.g., Ref. 1, 2 and references therein). Nevertheless, little is known on how to systematically introduce new entanglement measures. It is likely accepted that an appropriate entanglement measure is necessarily non-increasing under local operations and classical communication (LOCC), while this requirement makes the definition of entanglement measure notoriously difficult and challenging. So far, most of existing methods to construct entanglement measures are based on the “convex roof” 3 and the concept “distance” 4 — the distance from the entangled state to its closest separable state. The well-known entanglement of formation $E_f$ 5 is established for a mixed state $\rho_{AB}$ of a bipartite AB-system via the technique of convex roof. On the other hand, the relative entropy of entanglement $E_r$ was based on a concept of “distance” 4, and squashed entanglement $E_{sq}$ was built from conditional quantum mutual entropy 6—a quantum analog to intrinsic information 6 known from classical cryptography, as well as the logarithmic negativity $E_N$ was suggested 7, 8 on the basis of the well-known separability criterion—partial transposition 9. Among the known measures, additivity holds for $E_{sq}$ and $E_N$ and is conjectured to hold for $E_f$, but $E_r$ is nonadditive 10. $E_N$ is computable for a generic mixed state, while it does not reduce to the von Neumann entropy of subsystem for pure states. $E_r$ can be generalized to a measure for multipartite states, but still it is nonadditive. Very recently, $E_{sq}$ was extended to multipartite cases 11.

In this paper, we introduce a generic approach to construct a new kind of entanglement measures, which is defined in analogy to the conditional entropy 12 and thus referred to as conditional entanglement. The key ideas are quantum extension and quantum conditioning 12. New entanglement measures can be built from old ones and the order between them is known. Of particular importance, conditional entanglement can be formulated by quantum conditioning of functions that describe correlations rather than entanglement. Taking the quantum mutual information as as exemplary measure of correlations, we show that a new entanglement measure can be established by quantum conditioning. Remarkably, it is additive and can straightforwardly be generalized to multipartite states for two different choices of multipartite mutual information.

**Definition 1** Let $\rho_{AB}$ be a mixed state on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A conditional entanglement of $\rho_{AB}$ is defined as

$$CE(\rho_{AB}) = \inf \{E(\rho_{AA':BB'}') - E(\rho_{A':B'})\},$$

where the infimum is taken over all extensions of $\rho_{AB}$, i.e., over all states satisfying the equation $\text{Tr}_{A'B'}\rho_{AA':BB'} = \rho_{AB}$, and $E(\cdot)$ is an entanglement measure. Note that the above definition is similar to that of conditional entropy $S(A|B) = S(AB) - S(B)$ with $S(\rho)$ as the von Neumann entropy $S(\rho) = -\text{Tr}\rho \log \rho$.

To show that conditional entanglement is a good entanglement measure, we now elaborate that it does satisfy two essential axioms that an entanglement measure should obey 1.

1. Entanglement does not increase under local operations and classical communication (LOCC) i.e. $E(\Lambda(\rho)) \leq E(\rho)$, for any LOCC operation $\Lambda$. The reason that $CE$ inherits the monotonicity of $E$ is straightforward,

$$E(\Lambda^{AB}(\rho_{AA':BB'}')) - E(\rho_{A':B'}) \leq E(\rho_{AA':BB'}) - E(\rho_{A':B'}).$$

2. Entanglement is not negative and is zero for separable states. The inequality $CE(\rho_{AB}) \geq 0$ comes from the fact that any entanglement measure is non-increasing by tracing subsystems, while the equality $CE = 0$ for separable states lies in that separable extensions can be found for separable states.
The monotonicity under LOCC implies that entanglement remains invariant under local unitary transformations. This comes from the fact local unitary transformations are reversible LOCC. The convexity of entanglement used to be considered as a mandatory ingredient of the mathematical formulation of monotonicity \[ \text{CE} \]

Now the convexity is merely a convenient mathematical property. Also there is a common agreement that the strong monotonicity—monotonicity on average under LOCC is unnecessary but useful \[ \text{CE} \]. Many known existing entanglement measures are convex and satisfy the strong monotonicity. We will show that \( CE \) naturally inherits these properties.

For convex \( E \), convexity of \( CE \) can be obtained by noticing that for any extension states \( \rho_{AA'BB'} \) and \( \sigma_{AA'BB'} \), a new extension state can be constructed as \( \tau_{AA'E:BB'} = \lambda \rho_{AA'BB'} \otimes (|0\rangle \langle 0|)_E + (1 - \lambda) \sigma_{AA'BB'} \otimes (|1\rangle \langle 1|)_E \), and therefore

\[
E(\tau_{AA'E:BB'}) - E(\tau_{A'E:B'}) = E(\rho_{AA'BB'}) - E(\rho_{A'B'}) + (1 - \lambda)[E(\sigma_{AA'BB'}) - E(\sigma_{A'B'})] \quad (2)
\]

Now, let us show that \( CE(\cdot) \) satisfies the monotonicity on average under LOCC if the convex \( E(\cdot) \) does. It is sufficient to prove that \( CE \) is non-increasing under measurement on one party. For any extension \( \rho_{AA'BB'} \), a measurement on party \( A \) reduces the extension state to an ensemble \( \{p_k, \hat{\rho}_{AA'BB'}^k\} \)

\[
E(\rho_{AA'BB'}) - E(\rho_{A'B'}) \geq \sum_k p_k E(\hat{\rho}_{AA'BB'}^k) - E(\rho_{A'B'}) = \sum_k p_k E(\hat{\rho}_{AA'BB'}^k) - \sum_k p_k E(\hat{\rho}_{A'B'}^k) + \sum_k p_k E(\hat{\rho}_{A'B'}^k) - E(\rho_{A'B'}) \geq \sum_k p_k [E(\hat{\rho}_{AA'BB'}^k) - E(\hat{\rho}_{A'B'}^k)]. \quad (3)
\]

The first inequality comes from the fact that \( E \) is non-increasing on average under local measurement, while the second one is due to the convexity of \( E \). As a result, we have \( CE(\rho_{AB}) \geq \sum_k p_k CE(\hat{\rho}_{AB}^k) \).

Remarkably, while most of the known entanglement measures are sub-additive, \( CE \) is super-additive.

**Proposition 1** \( CE(\rho \otimes \sigma) \geq CE(\rho) + CE(\sigma) \).

**Proof** For any extension state \( \tau_{A_1A_2A':B_1B_2B'} \) of \( \rho_{A_1B_1} \otimes \sigma_{A_2B_2} \)

\[
E(\tau_{A_1A_2A':B_1B_2B'}) - E(\tau_{A':B'}) = E(\tau_{A_1A_2A':B_1B_2B'}) - E(\tau_{A_2A':B_2B'}) + E(\tau_{A_2A':B_2B'}) - E(\tau_{A':B'}) \geq CE(\rho) + CE(\sigma). \quad (4)
\]

Some entanglement measures are upper bounds for distillable entanglement. Their so-called regularizations provide stronger bounds. Here \( CE \) is even smaller than the regularized entanglement measure:

\[
CE(\rho) \leq E^\infty(\rho) \quad \text{for all states } \rho, \quad (5)
\]

where \( E^\infty(\rho) = \lim_{n \to \infty} E(\rho^\otimes n)/n \) is the regularized version of the generating entanglement measure \( E \). Indeed, it is explicit that \( CE(\rho) \leq E(\rho \otimes |00\rangle \langle 00|) - E(|00\rangle \langle 00|) = E(\rho) \). From the super-additivity of \( CE \), we know \( nCE(\rho) \leq CE(\rho^\otimes n) \leq E(\rho^\otimes n) \), which leads to \( 5 \).

One also finds that \( E_f(\rho_{A:B:C:D}) - E_f(\rho_{C:D}) \geq G(\rho_{A:B}) \), where \( G(\rho_{A:B}) > 0 \) iff \( \rho_{A:B} \) is entangled \( 14 \). We then get \( G(\rho_{A:B}) \leq CE_f(\rho_{A:B}) \leq E_c \), where \( E_c = E^\infty_f \) is so-called entanglement cost \( 15 \). Thus for any entangled state, \( CE_f > 0 \). It is an open question, whether \( CE \) is nonzero for entangled states.

Now let us pass to constructing entanglement measures by conditioning correlation measures \( 25 \). Most intriguingly, we illustrate below that a new additive measure can indeed be constructed based on quantum conditioning and can be generalized to multipartite states.

For a function \( f \) quantifying correlations we have two candidates for its conditioned version

\[
C_f^+ (\rho_{AB}) = \inf [f(\rho_{AA':BB'}) - f(\rho_{A'B'})], \quad (6a)
\]

\[
C_f^- (\rho_{AB}) = \inf [f(\rho_{A:BE}) - f(\rho_{AE})], \quad (6b)
\]

where \( S_f(\cdot) \) is taken over all extensions \( \rho_{AA'BB'} \) \( \rho_{AB} \). \( C^+ \) the symmetric conditioned version of \( f \) while \( C^- \) the asymmetric one.

Taking \( f \) to be quantum mutual information \( I(X:Y) = S(X) + S(Y) - S(XY) \), we obtain conditional entanglement of mutual information given by \( C_f^+ \). We add a factor \( 1/2 \) and will denote it by \( C_f \), Explicitly

\[
C_f(\rho_{AB}) = \inf \left\{ I(AA':BB') - I(A':B') \right\}, \quad (7)
\]

where the infimum is taken over all the extension states \( \rho_{AA'BB'} \) of \( \rho_{AB} \). Now we justify that \( C_f \) is an appropriate entanglement measure.

1. We prove that \( C_f \) satisfies the strong monotonicity. From a symmetry consideration, it is sufficient to prove that \( C_f \) is non-increasing under a measurement on subsystem \( A \), namely, \( C_f(\rho_{AB}) \geq \sum_k p_k C_f(\hat{\rho}_{AB}^k) \), where \( \hat{\rho}_{AB}^k = A_k \rho_{AB} A_k^\dagger/\rho_k \), \( p_k = tr A_k \rho_{AB} A_k^\dagger \), and \( \sum_k A_k^\dagger A_k = I_A \). Another way to describe the measurement process is as following. First, one attatches two ancillary systems \( A_0 \) and \( A_1 \) in states \( |0\rangle_{A_0} \) and \( |0\rangle_{A_1} \) to system \( AB \). Secondly, a unitary operation \( U_{A_0A_1} \) on \( A_0A_1 \) is performed. Thirdly, the system \( A_1 \) is traced out to get the state as \( \hat{\rho}_{A_0AB} = \sum_k A_k \rho_{AB} A_k^\dagger \otimes |k\rangle\langle k| \). Now for any extension state \( \rho_{AA'BB'} \), we get the state after the measurement on \( A_0 \), \( \hat{\rho}_{A_0AA'BB'} = \sum_k A_k \rho_{AA'BB'} A_k^\dagger \otimes |k\rangle\langle k| \). \( 27 \)
always be decomposed into a separable form: non-increasing on average under LOCC operation.

For a separable state \(|\rho\rangle\langle\rho|\) of the ensemble \(\{\rho_k, p_k\}\), the quantum mutual information is invariant under local unitary operation, while the inequalities of (S3) and (S4) stem from, respectively, the facts that quantum mutual information and the Holevo quantity are non-increasing by tracing subsystem. Consequently, we prove that \(C_t\) is non-increasing on average under LOCC operation.

2. \(C_t \geq 0\) comes from the fact that the quantum mutual information is non-increasing under tracing subsystems of both sides. For a separable state \(\rho_{AB}\), it can always be decomposed into a separable form: \(\rho_{AB} = \sum_{i,j} p_{ij} \phi_A^i \otimes \phi_B^j\). An extension state may be chosen to be \(\rho_{AA'B'B'} = \sum_{i,j} p_{ij} \phi_A^i \otimes (|i\rangle\langle i|)_{A'} \otimes \phi_B^j \otimes (|j\rangle\langle j|)_{B'}\). It is obvious that \(I(AA' : BB') = I(A' : B')\), and thus \(C_t = 0\) for separable states.

Continuity. The conditional entanglement of quantum mutual information is asymptotically continuous, i.e. if \(|\rho_{AB} - \sigma_{AB}| \leq \epsilon\), then \(|C_t(\rho) - C_t(\sigma)| \leq K\epsilon \log d + O(\epsilon)\), where \(|\cdot|\) is the trace norm for matrix, \(K\) is a constant, \(d = \dim \mathcal{H}_{AB}\), and \(O(\epsilon)\) is any function that depends only on \(\epsilon\) (in particular, it does not depend on dimension) and satisfies \(\lim_{\epsilon \to 0} O(\epsilon) = 0\).

The proof of the asymptotic continuity is similar to that for the squashed entanglement and is presented in the Appendix.

Convexity. \(C_t\) is convex, i.e., \(C_t(\lambda \rho + (1 - \lambda) \sigma) \leq \lambda C_t(\rho) + (1 - \lambda) C_t(\sigma)\) for \(0 \leq \lambda \leq 1\).

Proof. For any extension states \(\rho_{AA'B'B'}\) and \(\sigma_{AA'B'B'}\), we consider the extension state \(\tau_{AA'\cdot\cdot'BB'B'} = \lambda \rho_{AA'B'B'} \otimes (|0\rangle 0)_{A'} \otimes (|0\rangle 0)_{B'} + (1 - \lambda) \sigma_{AA'B'B'} \otimes (|1\rangle 1)_{A'} \otimes (|1\rangle 1)_{B'}\), and have \(I(\tau_{AA'\cdot\cdot'BB'B'}) - I(\rho_{AA'B'B'}) = \lambda[I(\rho_{AA'B'B'}) - I(\rho_{A'B'})] + (1 - \lambda)[I(\sigma_{AA'B'B'}) - I(\sigma_{A'B'})].\) This implies \(C_t\) is convex.

An immediate corollary of convexity is that \(C_t \leq E_f\) and furthermore \(C_t \leq E_c\) due to the following additivity.

**Proposition 2** \(C_t(\rho_{AB} \otimes \sigma_{CD}) = C_t(\rho_{AB}) + C_t(\sigma_{CD})\).

**Proof.** On the one hand, for any extension states \(\rho_{AA'B'B'}\) and \(\sigma_{CC'D'D'}\), \(\rho_{AA'B'B'} \otimes \sigma_{CC'D'D'}\) is an extension state of \(\rho_{AB} \otimes \sigma_{CD}\).

\[
I(AA'CC' : BB'DD') - I(A'A'C' : B'D') = I(AA' : BB') - I(A' : B') + I(CC' : DD') - I(C' : D').
\]

So \(C_t(\rho_{AB} \otimes \sigma_{CD}) \leq C_t(\rho_{AB}) + C_t(\sigma_{CD})\) holds.

On the other hand, for extension states \(\tau_{ACE' : BDF'}\) of \(\rho_{AB} \otimes \sigma_{CD}\), \(\tau_{ACE' : BDF'}\) is an extension state of \(\rho_{AB}\) and \(\tau_{CE' : DF'}\) is an extension state of \(\sigma_{CD}\). Therefore we have

\[
I(ACE' : BDF') - I(E' : F') = I(ACE' : BDF') - I(CE' : DF') + I(CE' : DF') - I(E' : F') = 0.
\]

This means that \(C_t(\rho_{AB} \otimes \sigma_{CD}) \geq C_t(\rho_{AB}) + C_t(\sigma_{CD})\).

So we have finally the additivity equality.

It is quite remarkable that the property of additivity is rather easy to prove for conditional entanglement while it is extremely tough for other candidates. The reason lies in that the conditional entanglement is naturally super-additive while others are usually sub-additive. Also the proof for the conditional entanglement shares a similarity with that of squashed entanglement. As a matter of fact, squashed entanglement can be constructed in the same spirit: it is based on asymmetric conditioning of mutual information

\[
E_{sq}(\rho_{AB}) = \frac{1}{2} \inf \{I(A : BE) - I(A : E)\} \equiv \frac{1}{2} C_t^q(\rho_{AB}),
\]

where the infimum is taken all extensions \(\rho_{ABE}\) of \(\rho_{AB}\). It is notable that \(I(A : BE) - I(A : E) = I(AB : B) - I(E : B)\) is symmetric w.r.t. systems \(AB\) though each term in the formula is asymmetric w.r.t. both parties. This gives the possibility to build symmetric entanglement measures by asymmetric conditioning.

In [11], we call the squashed entanglement q-squashed entanglement \(E_{sq}^q\) because the extension is generic and the system \(E\) is required to be quantum memory. If we restrict \(E\) to classical memory, another proper entanglement measure—c-squashed entanglement \(E_{sq}^c\) can be obtained [11]. Here we show the order relation among these three measures.

**Proposition 3** \(E_{sq}^c \leq C_t \leq E_{sq}^q\).

**Proof.** \(E_{sq}^c \leq C_t\) comes from the chain rule for quantum mutual information.

\[
I(AA' : BB') - I(A' : B') \geq I(A' : BB') + I(A : BB'|A') - I(A' : B') = I(A' : B)|B'\rangle + I(A : B|A'B') \geq I(A : B|A'B').
\]

(12)
The conditional entanglement for multipartite entanglement. Moreover, the known additive measures built from quantum conditioning are additive, which means that quantum conditioning leads to additive entanglement. Conditional entanglement measures from other candidates and further properties will be addressed elsewhere.

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Proof of the asymptotic continuity of $C_1$

The proof is similar to the continuity of the squashed entanglement [5] that is based on a basic result in [22] asserting that for any two states $\rho_{AB}$ and $\sigma_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, if $|\rho_{AB} - \sigma_{AB}| = \epsilon$, then

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq 4\epsilon \log d_A + 2H(\epsilon), \quad (13)$$

where $d_A$ is the dimension of $\mathcal{H}_A$ and $H(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log (1-\epsilon)$. Note that the righthand side of Eq (13) does not explicitly depend on the dimension of $\mathcal{H}_B$. Iteratively using the relations between fidelity and trace norm [23], if $|\rho_{AB} - \sigma_{AB}| \leq \epsilon$, then the fidelity $F(\rho_{AB},\sigma_{AB}) \geq 1 - \epsilon$, then there exist purifications $\Phi_{ABC}$ and $\Psi_{ABC}$ of $\rho_{AB}$ and $\sigma_{AB}$ respectively such that $F(\Phi_{ABC},\Psi_{ABC}) \geq 1 - \epsilon$, and then $|\Phi_{ABC} - \Psi_{ABC}| \leq 2\sqrt{\epsilon}$. For any quantum operation $\mathcal{E}$ acting on $C$ into $A'B'$, it creates the extensions $\rho_{AA'BB'}$ and $\sigma_{AA'BB'}$ of $\rho_{AB}$ and $\sigma_{AB}$ satisfying $|\rho_{AA'BB'} - \sigma_{AA'BB'}| \leq 2\sqrt{\epsilon}$. Notice that $I(AA' : BB') - I(A' : B') = S(A|A') + S(B|B') - S(AB|A'B')$, we get

$$\begin{align*}
|I(\rho_{AA'BB'}) - I(\sigma_{AA'BB'})| &\leq \left| [S(A|A')_\rho - S(A|A')_\sigma] + [S(B|B')_\rho - S(B|B')_\sigma] \right| \\
&\leq \sum_i |S(\rho_{AA'B'_i})_\rho - S(\rho_{AA'B'_i})_\sigma| + |S(\sigma_{AA'B'_i})_\rho - S(\sigma_{AA'B'_i})_\sigma| \\
&\leq 16\sqrt{\epsilon} \log (d_Ad_B) + 6H(2\sqrt{\epsilon}) = \epsilon' \quad (14)
\end{align*}$$

\[\text{For a sequence of operation } \mathcal{E}_i \text{ that creates a sequence of extensions such that } I(AA' : BB')_\rho - I(A' : B')_\rho \to E_I(\rho_{AB}), \text{ we have } |C_i(\rho_{AB}) - [I(AA' : BB')_\sigma - I(A' : B')_\sigma]| \leq \epsilon', \text{ then } C_i(\sigma_{AB}) \leq I(AA' : BB')_\sigma - I(A' : B')_\sigma \leq C_i(\rho_{AB}) + \epsilon'. \text{ Similarly } C_i(\rho_{AB}) \leq C_i(\sigma_{AB}) + \epsilon', \text{ so } |C_i(\rho_{AB}) - C_i(\sigma_{AB})| \leq \epsilon'.\]

Notice that we have $\sqrt{\epsilon}$ instead of $\epsilon$, but it does not change the essence of condition referring asymptotic continuity [24].

Definition of $E_{sq}^c$ [11] The c-squared entanglement $E_{sq}^c$ is defined as

$$E_{sq}^c(\rho_{AB}) = \inf \frac{1}{2} I(A : B|E) \quad (15)$$

where infimum is taken over the extension states of the form $\sum_i p_i \rho_{AB} \otimes (i\langle i|)_E$.

In deed, it is equivalent to the mixed convex roof of the quantum mutual information, i.e.

$$E_{sq}^c(\rho_{AB}) = \min \frac{1}{2} \sum_i p_i I(\rho_{AB}), \quad (16)$$

where $\rho_{AB} = \sum_i p_i \rho_{AB}$. 

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