Control of entanglement and two-qubit quantum gates with atoms crossing a detuned optical cavity

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Abstract
A scheme is proposed to generate an entangled state between two (A-type) four-level atoms that interact effectively by means of a detuned optical cavity and a laser beam that acts perpendicularly to the cavity axis. It is shown how the degree of entanglement for two atoms passing through the cavity can be controlled by manipulating their velocity and the (initial) distance between the atoms. In addition, three realistic schemes are suggested to implement the two-qubit gates within the framework of the suggested atom–cavity–laser setup, namely, the i-swap gate, controlled-Z gate and the controlled-NOT gate. For all these schemes, we analyse and discuss the atomic velocities and inter-atomic distances for which these gates are realized most reliably.

1. Introduction
In recent years, quantum entanglement has been found important not only in studying the non-classical behaviour of composite systems but also as an essential resource for the engineering and processing of quantum information. Nowadays, there are known various applications that (would) greatly benefit from having entangled quantum states available such as, for instance, super-dense coding [1], quantum cryptography [2] or the use of Grover’s quantum search algorithm [3], to name just a few. Despite recent progress in dealing with composite quantum systems, however, their manipulation and controlled interaction with the environment have remained a challenge for an experiment until the present. Apart from various other implementations of composite systems, the proof for and an excellent control of the generation of entanglement has been achieved especially with neutral atoms that are coupled to high-finesse optical cavities [4–6].

From the experimental perspective, there are two basic types of (atomic) level configurations utilized to encode and store a single qubit: apart from (i) the use of optical qubit, which simply refers to the two atomic levels separated by an optical transition frequency, one may (ii) also utilize the (so-called) hyperfine qubit that is associated with two hyperfine levels of, usually, the electronic ground state of the atom. In neutral atoms, these hyperfine levels are typically separated by a microwave frequency and are known to be robust with regard to decoherence effects and external stray fields in contrast to the optical qubits mentioned above. For the hyperfine qubits, therefore, rather long coherence times (∼1 s) have been reported in the literature [7–9]. In addition, a number of microwave techniques have been developed over the last decades in order to initialize, manipulate and detect the state of such hyperfine qubits [7–12].

Unfortunately, however, a hyperfine qubit cannot couple directly to a cavity with the resonant-mode frequency in the optical domain. Therefore, in order to manipulate the information encoded by the atom, the superposition of the two hyperfine levels must first be transferred coherently to some other two (electronically) excited states before the atom enters the cavity, and this information must be brought back in a coherent fashion after the atom exits the cavity. Instead of an atomic two-level configuration, we then need to consider a...
four-level scheme, in which the two hyperfine levels for storing quantum information are associated with two (additional) optically excited levels. In order to realize an efficient atom–cavity coupling, moreover, the energy splitting of the two electronically excited levels should be compatible with the resonant frequency of the cavity. In this way, a coupling between the hyperfine qubit and the optical cavity can be achieved, and might open a route towards the implementation of quantum gates via cavity-mediated atom–atom interactions.

The basic idea for the formation of entanglement between two four-level atoms that are coupled to an optical cavity and an external laser beam was first suggested by You and co-authors [13], which follows the novel cavity-mediated atom–atom interaction regime from [14]. This interaction regime is based on the exchange of a photon between two bi-level atoms that is stimulated by a cavity which is detuned with regard to transition frequencies of the atoms. Later in [15], moreover, it was shown experimentally how this effective atom–atom interaction leads to the generation of entanglement between two atoms that cross a detuned microwave cavity. In the theoretical analysis of You et al., however, it was assumed that both atoms couple to the same cavity mode via a constant coupling strength being independent of the atomic position inside the cavity mode. In practice, the atoms cross the cavity one after the other being separated by a macroscopic distance. This separation of the atoms implies that they have different atom–cavity couplings as given by the radiation pattern of the cavity-mode standing wave. Therefore, a more detailed description of the cavity-mediated entanglement formation has to be considered, in which the degree of entanglement between the atoms also depends on the atom–cavity coupling which depends, in turn, on the location of both atoms inside the cavity mode. Such a position-dependent coupling between the atoms and the cavity requires a revision of the previous theoretical analysis and suggests that the degree of entanglement that is finally obtained might depend substantially on the details of how the atoms cross the cavity in the course of interaction.

In the present work, we propose a scheme to generate an entangled state between the hyperfine qubits of two four-level atoms in a \( \Lambda \)-type level configuration. In this scheme, the interaction between the atoms is mediated by an optical cavity and a laser beam that acts perpendicularly to the cavity axis. In contrast to the analysis made by You et al., moreover, we assume (i) the atoms to be separated from each other by a macroscopic distance such that no direct interaction between the atoms occurs and that (ii) both atoms interact simultaneously with the same cavity mode and laser field via a position-dependent couplings while passing through the cavity–laser setup. This scheme leads to an effective atom–laser mediated interaction between the atoms and the cavity requires a revision of the previous theoretical analysis and suggests that the degree of entanglement that is finally obtained might depend substantially on the details of how the atoms cross the cavity in the course of interaction.

In this section, we propose and explain our scheme to entangle the hyperfine qubits of two four-level atoms. This includes the theoretical description of the effective atom–cavity interaction evolution that allows us to control this interaction in practice. In section 2.1, in particular, we present and explain all the steps necessary within the proposed (experimental) setup, while a more detailed view on this effective interaction is given in section 2.2 by using the adiabatic elimination procedure. In section 3, then, the schemes for the implementation of the i-swap, controlled-Z and controlled-NOT gates are presented and discussed. A few conclusions are finally given in section 4.

2. Generation of the two-atom entanglement via an optical cavity

In this section, we propose and explain our scheme to entangle the hyperfine qubits of two four-level atoms if they were initially prepared in a product state. We hereby assume that the atoms can be controlled with regard to their separation and velocity when they enter the experimental setup that is displayed in figure 1(c).

2.1. Off-resonant atom–cavity interaction

Let us start by considering an atom in the \( \Lambda \)-type four-level configuration as displayed in figure 1(a). In this level configuration, two (hyperfine) states, \( |0\rangle \) and \( |1\rangle \), of the atomic ground levels carry the qubit information and are
supplemented by two electronically excited states, |\textit{a}\rangle and |\textit{e}\rangle, that are separated from each other by an optical transition frequency. Below, we assume to have two identical atoms \textit{A}1 and \textit{A}2 in such a \textit{Λ}-type configuration that are initially prepared in the composite state |0\rangle ⊗ |1\rangle, where the bar in |\textit{a}\rangle refers to the state of atom \textit{A}1. In addition, the atoms are separated by a macroscopic distance \ell being large enough such that they do not interact directly with each other and both atoms move with the same (constant) velocity \textit{v} along the \textit{z}-axis (see figure 1(c)). Before atom \textit{A}1 enters the cavity, its electronic population is transferred from state |0\rangle to state |\textit{a}\rangle by using a pair of slightly off-resonant laser beams which are coupled to atomic transitions |0\rangle ↔ |\textit{e}\rangle and |\textit{e}\rangle ↔ |\textit{a}\rangle, respectively. Such a population transfer is known as the two-photon Raman process [16] that enables one to perform a second-order transition between the states |0\rangle and |\textit{a}\rangle. For instance, this could be done by utilizing a two phase-locked laser diode [11]. Below, we shall refer to this population transfer briefly as a Raman pulse and will distinguish between the Raman pulses (laser beams) \textit{L}1 and \textit{L}2 in front and behind the cavity (see figure 1(c)); also in the temporal diagram from figure 2, these Raman pulses are displayed as boxed circles. We assume, therefore, that the purpose of these Raman pulses is just to transfer the electronic population from the hyperfine state |0\rangle to the optical level |\textit{a}\rangle in the zone \textit{L}1, and back from |\textit{a}\rangle to |0\rangle in the zone \textit{L}2. The same Raman pulses are also applied to atom \textit{A}2, which follows \textit{A}1 subsequently with distance \ell. However, since the second atom enters the setup in the state |1\rangle, it remains unaffected by the Raman pulse \textit{L}1 and, thus, the two atoms enter the cavity in the product state |\textit{a}\rangle ⊗ |1\rangle.

Inside the cavity, both the atoms \textit{A}1 and \textit{A}2 couple via the optical transition |\textit{a}\rangle ↔ |\textit{e}\rangle (and |\textit{a}\rangle ↔ |\textit{e}\rangle, respectively) to the same cavity mode with the resonant frequency \textit{w}C (see figure 1(a)). As we discussed above, a revised description of the atom–cavity interaction evolution is based on the position-dependent atom–cavity coupling:

\[
g(\vec{r}) = g_o \exp(-|\vec{r}|^2/\textit{w}^2),
\]

where \textit{g}o denotes the vacuum Rabi frequency, and \textit{w} the (so-called) cavity-mode waist that is the minimum width of the radiation pattern given by the cavity-mode standing wave. For the two atoms which move through the cavity with the velocity \textit{v} along the \textit{z}-axis, the Gaussian profile (1) gives rise to the time-dependent atom–cavity couplings: \textit{g}_1(\textit{t}) = \textit{g}_o(\textit{z}_1 + \textit{vt}) and \textit{g}_2(\textit{t}) = \textit{g}_o(\textit{z}_2 + \textit{vt}), and where \textit{z}_1 - \textit{z}_2 = \ell > 0 denotes the initial distance between the atoms.

The cavity-mediated atom–atom interaction (i.e., without the laser beam \textit{L}) is based on the stimulated exchange of a single photon between two atoms prepared in the product state |\textit{a}\rangle ⊗ |\textit{e}\rangle. This photon exchange can be understood as the emission of a virtual photon into the cavity mode by atom \textit{A}2 and the re-absorption of the photon by atom \textit{A}1, while both atoms are coupled off-resonantly to the same cavity mode. An off-resonant atom–cavity interaction, here, refers to the case when the difference (or detuning) between the atomic |\textit{a}\rangle ↔ |\textit{e}\rangle transition frequency and the frequency of the cavity mode \textit{w}C is large enough: |\textit{w}C - (\textit{w}C - \textit{w}o)| \gg |\textit{g}_o(\textit{t})|, so that only a virtual atom–cavity energy exchange can occur [14].

In our present scheme, in contrast, the atoms enter the cavity in the composite state |\textit{a}\rangle ⊗ |1\rangle, and hence a further intermediate process, |\textit{a}\rangle ⊗ |1\rangle → |\textit{a}\rangle ⊗ |\textit{e}\rangle, is first necessary to obtain the state |\textit{a}\rangle ⊗ |\textit{e}\rangle that could evolve into |\textit{e}\rangle ⊗ |\textit{a}\rangle by means of the detuned cavity. For this reason, the atoms are exposed to a laser beam that acts transversally to the cavity axis and in addition to their interaction with the cavity mode (see figure 1(c)). The laser given by the frequency \textit{w}oC couples the atomic transitions |1\rangle ↔ |\textit{e}\rangle and |\textit{e}\rangle ↔ |\textit{a}\rangle, respectively, as shown in figure 1(a). The position-dependent atom–laser coupling, \(\bar{\Omega}(\vec{r}) = \bar{\Omega}_L \exp(-|\vec{r}|^2/\bar{\textit{w}}^2)\), hereby implies the time-dependent couplings for each atom, namely \(\bar{\Omega}_1(\textit{t}) = \Omega(z_1 + \textit{vt})\) and \(\Omega_2(\textit{t}) = \Omega(z_2 + \textit{vt})\), and where the waist of the atom–laser coupling \(\bar{\textit{w}}\) is assumed to be much larger than for the cavity mode. With the above couplings of the atoms to both, the laser and the cavity mode, the atomic composite state can be manipulated in order to create an energy exchange between |\textit{a}\rangle ⊗ |1\rangle and |\textit{e}\rangle ⊗ |\textit{a}\rangle in a similar way as has been suggested by You and coworkers. This exchange is based on the sequence of four steps:

\[
|\textit{a}\rangle ⊗ |1\rangle → |\textit{e}\rangle ⊗ |\textit{a}\rangle → |\textit{a}\rangle ⊗ |\textit{e}\rangle → |\textit{e}\rangle ⊗ |\textit{a}\rangle → |\textit{e}\rangle ⊗ |\textit{a}\rangle → |\textit{e}\rangle ⊗ |\textit{a}\rangle → |\textit{1}\rangle ⊗ |\textit{\bar{a}}\rangle;
\]

if there were \textit{n} photons initially in the cavity mode. The sequence (2) contains in its middle part a virtual process in which a photon is emitted by the first atom and absorbed by the second atom, so that the final state of the atoms is independent of the number of cavity photons. For an initially empty cavity, we can, therefore, simplify the above sequence to

\[
|\textit{a}\rangle ⊗ |1\rangle → |\textit{e}\rangle ⊗ |\textit{a}\rangle → |\textit{1}\rangle ⊗ |\textit{\bar{a}}\rangle;
\]

if \textit{n} is assumed to be much larger than the effective \textit{w}C. An \textit{effective} atom–atom interaction evolution, \(|\textit{a}\rangle ⊗ |1\rangle → |\textit{1}\rangle ⊗ |\textit{\bar{a}}\rangle\), in which the state of the cavity field is factorized out in
the vacuum state. By exploiting this effective evolution, the maximally entangled state,
\[ |\Phi\rangle = \frac{1}{\sqrt{2}}(|a, \bar{1}\rangle + e^{i\phi}|1, \bar{0}\rangle), \]
where \( e^{i\phi} \) is a constant phase factor, can be generated by tuning the atomic velocity \( u \) and the inter-atomic distance \( \ell \) for a given set of cavity–laser parameters: \( \omega_0, \omega_L, \omega_a, g_a \) and \( \Omega_a \). In the following subsection, we shall analyse in more details how this effective atom–atom interaction depends on the velocity and distance of the atoms, while both atoms are passing through the setup.

After both the atoms \( A_1 \) and \( A_2 \) have left the cavity, the electronic population of the excited states \( |a\rangle \) and \( |\bar{a}\rangle \) is coherently transferred back to the (ground) hyperfine levels \( |0\rangle \) and \( |\bar{0}\rangle \) in order to protect them from the spontaneous decay of these levels. As before, this is achieved by applying a Raman pulse \( L_2 \) behind the cavity (see figure 1(c)). The entangled state (4) is then mapped onto the state:
\[ |\Phi'\rangle = \frac{1}{\sqrt{2}}(|0, \bar{1}\rangle + e^{i\phi}|1, \bar{0}\rangle). \]
All the manipulations with the atoms which we have just described are summarized graphically in figure 2, in which the spatio-temporal evolution of the atoms and the cavity is displayed.

2.2. Time-evolution of the effective atom–atom interaction

While the sequence (3) provides the basic idea of how an effective coupling can be achieved between the atoms, we need to analyse this sequence in more detail to understand how to control this coupling in practice. For this purpose, we shall use the adiabatic elimination procedure (see [14, 17, 18] for another derivation) which enables one to exclude all the intermediate degrees of freedom due to the action of the cavity mode and the laser field.

Formally, the time evolution of the coupled atom–cavity–laser system is driven by the Hamiltonian,
\[ H = H_1 + H_2 + H_C, \]
where \( \hbar = 1, \mu = 1, 2 \)
\[ H_\mu = \omega_\mu |1\rangle\langle 1| + \omega_\mu |e\rangle\langle e| + g_\mu |a\rangle\langle a| + \frac{1}{2} \Omega_\mu(t) e^{-i\omega_\mu t} |1\rangle\langle 1| + \Omega_\mu(t) e^{i\omega_\mu t} |e\rangle\langle e| + h.c., \]
describes the atom \( A_\mu \) and its interaction with the cavity and laser field, and where
\[ H_C = \omega_C e^{i\phi}e^{i\phi} \]
refers to the cavity–mode energy. In the atomic Hamiltonian (6), \( \hbar \omega_1, \hbar \omega_2, \) and \( \hbar \omega_a \) are the (excitation) energies of atomic states \( |1\rangle, |e\rangle \) and \( |a\rangle \) (see figure 1(a)), while \( c \) and \( e^{i\phi} \) denote the annihilation and creation operators for a photon in the cavity mode which act upon the Fock states \( |n\rangle \).

In order to simplify the evaluation of the Schrödinger equation that is associated with the Hamiltonian (6), let us switch here to the interaction picture given by [13]:
\[ U_{\text{int}}^0 = \exp \left( -i(\omega_1 + \omega_L)t \sum_{\mu} |e\rangle\langle e| - i\omega_L t \sum_{\mu} |1\rangle\langle 1| \right) \times \exp \left( -i\omega_1 t \sum_{\mu} |a\rangle\langle a| - i(\omega_L - (\omega_a - \omega_1))tc^{\dagger}c \right). \]
\[ \text{(8)} \]

In this picture, the atom–cavity–laser interaction Hamiltonian becomes
\[ H_{\text{int}} = -\delta e^{i\phi}c + \Delta \sum_{\mu} |e\rangle\langle e| + \frac{1}{2} \sum_{\mu} \Omega_\mu(t)|e\rangle\langle e| + g_\mu(t)c|e\rangle\langle e| + h.c., \]
\[ \text{(9)} \]
where \( \Delta = \omega_1 - \omega_L \) and \( \delta = \omega_L - \omega_a - \omega_0 = (\omega_1 - \omega_a) - (\omega_1 - \omega_0) \) refer to the off-resonance shifts (detuning) of the laser and cavity frequencies as depicted in figure 1(b).

The Hamiltonian (9) drives the state of the composite atom–cavity–laser system due to the Schrödinger equation:
\[ i\frac{d|\Psi(t)\rangle}{dt} = H_{\text{int}}|\Psi(t)\rangle, \]
\[ \text{(10)} \]
where the (composite) wavefunction, \( |\Psi(t)\rangle \), is defined in the product space of three (sub)systems: \( A_1(|1\rangle, |e\rangle, |a\rangle), A_2(|\bar{1}\rangle, |\bar{e}\rangle, |\bar{a}\rangle) \) and the cavity Fock states \( C(|0\rangle, |1\rangle) \). Moreover, by taking into account the composite states that occur in sequence (3), we may restrict this wavefunction to the subspace:
\[ |\Psi(t)\rangle = C_1(t)|a, \bar{1}; 0\rangle + C_2(t)|a, \bar{e}; 0\rangle + C_3(t)|a, \bar{a}; 1\rangle + C_4(t)|e, \bar{a}; 0\rangle + C_5(t)|1, \bar{a}; 0\rangle, \]
\[ \text{(11)} \]
for which the Schrödinger equation (10) gives rise to a set of closed equations:
\[ i\dot{C}_1(t) = \frac{1}{2} \Omega_2(t)C_2(t), \]
\[ i\dot{C}_2(t) = \Delta C_2(t) + g_2(t)C_3(t) + \frac{1}{2} \Omega_2(t)C_1(t), \]
\[ i\dot{C}_3(t) = -\delta C_3(t) + g_1(t)C_4(t) + g_2(t)C_2(t), \]
\[ i\dot{C}_4(t) = \Delta C_4(t) + g_1(t)C_3(t) + \frac{1}{2} \Omega_1(t)C_2(t), \]
\[ i\dot{C}_5(t) = \frac{1}{2} \Omega_1(t)C_4(t), \]
\[ \text{(12a)} \]
\[ \text{(12b)} \]
\[ \text{(12c)} \]
\[ \text{(12d)} \]
\[ \text{(12e)} \]
and where the dot denotes the time derivative.

The off-resonant regime of the atom–cavity and atom–laser interactions, we assume, implies
\[ |\delta| \gg |g_\mu(t)|, \] \[ |\Delta| \gg |\Omega_\mu(t)|, \] \[ |\delta\Delta| \gg |g_\mu^2(t)|. \]
\[ \text{(13)} \]
These conditions, therefore, justify the adiabatic elimination procedure for a sufficiently slow-varying time-dependent atom–cavity, \( g_\mu(t) \), and atom–laser coupling, \( \Omega_\mu(t) \). The adiabatic elimination procedure implies the vanishing of the time derivatives \( \dot{C}_2(t), \dot{C}_3(t) \), and \( \dot{C}_4(t) \), which together with conditions (13) lead to the exclusion of equations (12b)–(12f) that account for the evolution of the state vectors \( |a, \bar{e}; 0\rangle, |a, \bar{a}; 1\rangle \) and \( |e, \bar{a}; 0\rangle \), respectively. Here, we shall omit the details of the derivation for which the reader is referred to the literature [13, 14, 17, 18]. The remaining equations (12a) and (12e) for the functions \( C_1(t) \) and \( C_2(t) \) take the closed form:
\[ i\dot{C}_1(t) = -\frac{\Omega_2^2(t)}{4\Delta}C_1(t) + \lambda(t)C_3(t), \]
\[ i\dot{C}_3(t) = \lambda(t)C_1(t) - \frac{\Omega_2^2(t)}{4\Delta}C_5(t), \]
\[ \text{(14a)} \]
\[ \text{(14b)} \]
Figure 3. (a) Atomic velocities $\nu$ and inter-atomic distances $\ell$ for which the initial product state $|a, \bar{1}\rangle$ becomes maximally entangled due to the cavity–laser mediated atom–atom interaction. The velocity $\nu$ is displayed in units of $\Omega^2 g_0^2 w / \delta \Delta^2$ and the inter-atomic distance in units of $w$. Along the lines, the condition, $\theta(\nu, \ell) = (2n + 1)\pi / 4$, is satisfied for the asymptotic couplings angle with $n = 0, 1, 2, 3, 4$. The straight dashed line corresponds to a vanishing inter-atomic distance, which is obtained in the formal limit $\ell \to 0$ for $n = 0$. (b) von Neumann entropy, $E(\nu, \ell)$, as a function of the atomic velocity $\nu$ and inter-atomic distance $\ell$ (using the same units).

where

$$
\lambda(t) = \frac{\Omega_1(t)\Omega_2(t)g_1(t)g_2(t)}{4\delta \Delta^2} \tag{15}
$$

is the effective coupling between the initial and final composite states $|a, \bar{1}\rangle$ and $|1, \bar{a}\rangle$, respectively.

The atom–laser coupling, $\Omega_n(t)$, is determined by the interaction of the electric dipole of the atom with the electric field of the laser. We have assumed that the waist of the laser beam is much larger than that of the cavity mode, and therefore we may take $\Omega_n(t) = \Omega = \text{const}$. and include the time variation only due to the atom–cavity coupling $g_n(t)$. With this simplification in mind, an analytical solution of equations \((14a) \) and \((14b)\) can be obtained in the form

$$
|\Psi(t)\rangle = e^{i\frac{\lambda t}{\Delta \omega} |\Phi(t)\rangle} \tag{16}
$$

with

$$
|\Phi(t)\rangle = \cos \xi(t) |a, \bar{1}\rangle - i \sin \xi(t) |1, \bar{a}\rangle, \tag{17}
$$

if the wavefunction, $|\Psi(t)\rangle$, was prepared initially in the product state $|a, \bar{1}\rangle$. In expression \((17)\), moreover, the cavity field state is not shown as being factorized out in the vacuum state, and the effective atom–atom coupling angle is given by

$$
\xi(t) = \int_0^t \lambda(s) \, ds. \tag{18}
$$

The wavefunction \((17)\) describes an entangled state for the atoms $A_1$ and $A_2$, whose time evolution can also be obtained from the effective Hamiltonian,

$$
H_{\text{eff}} = \lambda(t) (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+). \tag{19}
$$

where $\sigma_1^+ = |1\rangle_{1} \langle a|$ and $\sigma_1^- = |a\rangle_{1} \langle 1|$ denote the two-photon atomic excitation and de-excitation operators. Owing to its obvious simplicity, this Hamiltonian provides a much better understanding of the effective two-atom evolution \((17)\) that is mediated by the cavity–laser fields and by using the ansatz \((11)\) within the adiabatic regime. Below, we shall restrict ourselves to the evolution of the function $|\Phi(t)\rangle$, since the Hamiltonian that drives the wavefunction $|\Psi(t)\rangle$ differs from \((19)\) by just the constant term $H_0 = h \Omega_1^2 (|1\rangle \langle 1| + |\bar{1}\rangle \langle \bar{1}|)$. This factor need not be considered since we could utilize a modified interaction picture given by the unitary transformation, $U_{\text{int}}^1 = \exp(\imath H_0 t)$, for which the (original) wavefunction \((16)\) would coincide with \((17)\).

When both atoms have left the cavity (which is formally obtained in the limit $t \to +\infty$), the state \((17)\) becomes

$$
|\Phi_{+\infty}\rangle = \cos \theta(\nu, \ell) |a, \bar{1}\rangle - i \sin \theta(\nu, \ell) |1, \bar{a}\rangle, \tag{20}
$$

and where the asymptotic coupling angle is given by

$$
\theta(\nu, \ell) \equiv \xi(+\infty) = \sqrt{\frac{\pi}{32}} \frac{\Omega^2 g_0^2 w}{\delta \Delta^2 \nu} \exp \left( -\frac{\ell^2}{2w^2} \right). \tag{21}
$$

Note that, according to our scheme in figure 2, the atomic states $|a\rangle$ and $|\bar{a}\rangle$ are mapped onto the hyperfine states $|0\rangle$ and $|\bar{0}\rangle$ by applying a Raman pulse $L_2$ shortly after the atoms have crossed the cavity. Therefore, the wavefunction \((20)\) becomes

$$
|\Phi'_{+\infty}\rangle = \cos \theta(\nu, \ell) |0, \bar{1}\rangle - i \sin \theta(\nu, \ell) |1, \bar{0}\rangle. \tag{22}
$$

From equation \((22)\), we can easily read off the condition, $\theta(\nu, \ell) = (2n + 1)\pi / 4$, with $n$ being an integer, for which the two atoms become maximally entangled with each other initially being prepared in the product state $|a, \bar{1}\rangle$. For fixed cavity–laser parameters ($\delta, \Delta, w, g_0$, and $\Omega$), this condition implies that the values of atomic velocity $\nu$ and inter-atomic distances $\ell$ cannot be chosen arbitrarily but must follow the (so-called) \emph{lines of maximal entanglement} displayed in figure 3(a) for $n = 0, 1, 2, 3, 4$. According to this figure, the change between the (maximally) entangled and disentangled state occurs more and more rapidly as the velocity is decreased (or $n$ increases). In figure 3(a), all velocities are given in units of $\Omega^2 g_0^2 w / \delta \Delta^2$ and all distances in units of the cavity waist $w$. For typical atom–cavity–laser parameters: $\delta = 360 \text{ MHz}$, $\Delta = 380 \text{ MHz}$, $g_0 = 27 \text{ MHz}$, $\Omega = 50 \text{ MHz}$ and $w = 13 \mu\text{m}$, these velocity and distance units take the values of 0.46 m s$^{-1}$ and 13 $\mu\text{m}$, respectively. These values are compatible with the velocities in the range 0.01–1 m s$^{-1}$ which were utilized in the recent cavity QED experiments [19–21], in which atoms are coherently transported inside the cavity by means of an optical lattice trap (see below).

Next, let us analyse how the degree of entanglement depends on the velocity $\nu$ and distance $\ell$ of the atoms. For this
reason, we display in figure 3(b) the von Neumann entropy \[22\]:

\[
E(\nu, \ell) \equiv -\text{Tr}[\rho(\nu, \ell) \log_2 \rho(\nu, \ell)]
\]

\[
= -\cos^2 \theta(\nu, \ell) \log_2[\cos^2 \theta(\nu, \ell)]
-\sin^2 \theta(\nu, \ell) \log_2[\sin^2 \theta(\nu, \ell)],
\]

where \(\rho(\nu, \ell) = \text{Tr}_2(\ket{\Phi'_{\infty}}\bra{\Phi'_{\infty}})\) denotes the reduced density operator of the first hyperfine qubit (see equation \[22\]). The maximal values of the von Neumann entropy, i.e., \(E(\nu, \ell) = 1\), are obtained for the velocities and distances as displayed in figure 3(a). Moreover, as seen from figure 3(b), the velocities and distances along the \(n = 0\) line from figure 3(a) appear to be the most appropriate for any practical implementation of this scheme, since for these values of \(\nu\) and \(\ell\), the obtained entanglement is less sensitive with regard to small uncertainties. This leads us to the conclusion that the \(\nu\) and \(\ell\) combinations along this line \((n = 0)\) might be relevant for experimental attempts to generate the atom–atom entanglement by means of the suggested setup.

Since the atom–atom interaction sequence \[3\] can easily be time reversed to

\[
|1; \bar{a}; 0\rangle \rightarrow |e; \bar{a}; 0\rangle \rightarrow |a, \bar{a}; 1\rangle \rightarrow |a, \bar{e}; 0\rangle \rightarrow |a, \bar{I}; 0\rangle,
\]

(24)

we can also generate the state \((t \rightarrow +\infty)\),

\[
|\bar{\Phi}_{1\infty} \rangle = \cos \theta(\nu, \ell) |1, \bar{a}\rangle - i \sin \theta(\nu, \ell) |a, \bar{I}\rangle,
\]

(25)

from the atoms initially being prepared in the product state \(|1, \bar{a}\rangle\). Together with the Raman pulse \(L_2\) that maps back the atomic states \(|a\rangle \rightarrow |0\rangle\) and \(|\bar{a}\rangle \rightarrow |\bar{0}\rangle\), we then obtain the state

\[
|\bar{\Phi}'_{1\infty} \rangle = \cos \theta(\nu, \ell) |1, 0\rangle - i \sin \theta(\nu, \ell) |0, \bar{I}\rangle.
\]

(26)

For the other two initial (product) states \(|a, \bar{a}\rangle\) and \(|1, \bar{I}\rangle\), in contrast, no effective interaction occurs on the atoms when they pass through the cavity–laser system. From this fact and equations \[20\], \[25\], we conclude that the effective Hamiltonian \[19\] gives a complete description of the (effective) atom–atom interaction for all four possible initial product states of the two atoms being mediated by the cavity–laser fields in the adiabatic regime.

### 3. Two-qubit quantum logic gates

In the previous section, we have shown how the atomic hyperfine qubits of the two atoms \(A_1\) and \(A_2\) can be manipulated adiabatically by means of the cavity–laser setup from figure 1(c) and the sequence of steps from figure 2. Independent of the initial state of the qubits, the evolution of the two-qubit hyperfine input state \(|\psi_\text{in} \rangle = \sum_i c_i |v_i \rangle\) into the output state \(|\psi_\text{out} \rangle = \sum_i c_i(\nu, \ell)|v_i \rangle\) \((i, j = 1, \ldots, 4)\) is given by the unitary matrix

\[
U_{ij}(\nu, \ell) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta(\nu, \ell) & -i \sin \theta(\nu, \ell) & 0 \\
0 & -i \sin \theta(\nu, \ell) & \cos \theta(\nu, \ell) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(27)

expressed in the two-qubit hyperfine basis

\[
|v_1 \rangle = |0, 0\rangle, \quad |v_2 \rangle = |0, 1\rangle, \\
|v_3 \rangle = |1, 0\rangle, \quad |v_4 \rangle = |1, 1\rangle,
\]

(28)

and where \(c_i(\nu, \ell) = \sum_j U_{ij}(\nu, \ell)c'_j\). For different values of the atomic velocity \(\nu\) and inter-atomic distance \(\ell\), different transformations are therefore realized including, for instance, the generation of maximally entangled state \[5\] if one starts from the initial product state \(|0, \bar{1}\rangle\). Moreover, we can analyse the atom–atom coupling angle \(\theta(\nu, \ell)\) for different combinations of \(\nu\) and \(\ell\), and for its capability to realize non-trivial two-qubit quantum gates. In fact, the suggested setup is suitable for realizing the i-swap, controlled-Z and the controlled-NOT quantum gates for different choices of the velocity and distance, together with some minor modifications in the steps that are necessary to prepare the atoms before (afterwards) they enter (leave) the cavity–laser system (see below). In the following, we consider these gates in more details and display their temporal diagrams and possible values \((\nu, \ell)\) for which these gates are realized.

#### 3.1. i-swap gate

Perhaps the simplest quantum gate is the i-swap gate \[23\] which is expressed in the atomic basis \[28\] as

\[
U_{ij}^{\text{i-swap}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(29)

By comparing equations \[27\] and \[29\], we see that this gate can be generated whenever the effective coupling angle fulfils the condition

\[
\theta(\nu, \ell) = 3\pi/2 + 2\pi n.
\]

(30)

Figure 4(a) displays the combinations of the atomic velocity \(\nu\) and the inter-atomic distance \(\ell\) which satisfy this condition for \(n = 0, 1, 2\). For the i-swap gate, moreover, the sequence of steps that needs to be carried out before and after the atoms have crossed the cavity is the same as shown in figure 2, and no additional manipulations are required in order to implement this gate.

From the viewpoint of an experiment, as we have discussed, it is important to know how stable a gate operation can be performed for small deviations in the \((\nu, \ell)\) parameters. This stability can be seen from the fidelity (distance) between the i-swap gate \[29\] and the unitary matrix \[27\] obtained for different values of \((\nu, \ell)\). Figure 4(b) displays such a fidelity that we have defined as

\[
F_{\text{i-swap}}(\nu, \ell) \equiv 1 - \mathcal{N}(\|U(\nu, \ell) - U_{\text{i-swap}}^{\text{i-swap}}\|)
\]

\[
= 1 - \sqrt{1 + \sin \theta(\nu, \ell)}^2,
\]

(31)

where \(\|M\| = \text{Tr}(M^\dagger M)^{1/2}\) is the Frobenius norm \[24\], and \(\mathcal{N}(f_{\theta, \ell}) \equiv f_{\theta, \ell} \cdot (\text{Max}(f_{\theta, \ell}))^{-1}\) is used for its normalization upon the interval \(0 \leq F_{\text{i-swap}} \leq 1\).

By construction, this fidelity is a continuous function for which the realization of the i-swap gate occurs when \(F_{\text{i-swap}}(\nu, \ell) = 1\), which corresponds to the values \((\nu, \ell)\) displayed in figure 4(a).
3.2. Controlled-Z gate

For two interacting qubits $A$ and $B$, the controlled-Z gate is defined by the transformation [22]

$$U_{CZ}|\alpha A, \beta B\rangle = (-1)^{\alpha \beta}|\alpha A, \beta B\rangle,$$  \hspace{1cm} (32)

where $\alpha, \beta = 0, 1$ are the basis states. This gate is a simple example of the conditional quantum dynamics which introduces an additional phase, $e^{i\pi} = -1$, whenever both qubits are in the state $|1_A, 1_B\rangle$.

In section 2, we concluded that the initial product state $|1, 1\rangle$ of the two atoms does not undergo any evolution mediated by the cavity–laser fields. Therefore, the direct identification of the atomic hyperfine states $|0\rangle, |1\rangle$ and $|0\rangle, |1\rangle$ with the (logical) qubit states $|0_A, 1_A\rangle, |1_A\rangle$ and $|0_B, 1_B\rangle, |1_B\rangle$ in (32) will not allow us to realize the controlled-Z gate, while the reversed assignment for the qubit $A$,

$$|0\rangle = |1_A\rangle, \quad |1\rangle = |0_A\rangle, \quad |\bar{0}\rangle = |0_B\rangle, \quad |\bar{1}\rangle = |1_B\rangle,$$  \hspace{1cm} (33)

would do so. With this assignment of the basis (28), the transformation matrix for the requested controlled-Z gate becomes

$$U_{ij}^{CZ} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}. \hspace{1cm} (34)$$

In contrast to the $i$-swap gate (29), however, the matrix (34) cannot be obtained from the evolutionary matrix (27) by just imposing a condition of the type (30) on the coupling angle $\theta(\nu, \ell)$. Instead, we must consider here a new temporal diagram as displayed in figure 5(c). The difference between this diagram and the sequence from figure 2 is that the second atom $A_2$ is not subjected to the Raman pulses $L_1$ and $L_2$, implying that its (hyperfine) state $|0\rangle$ is not mapped upon the (optical) state $|\bar{a}\rangle$ or back. Due to the suggested setup from figure 1(c) this modification, for instance, is realized simply...
by switching off the pairs of Raman laser beams while atom \( A_2 \) crosses the zones \( L_1 \) and \( L_2 \).

Following the temporal sequence in figure 5(c) and by making use of equation (20), we see that the four input states will evolve (after mapping \( |a| \rightarrow |0\rangle \)) into

\[
\begin{align*}
|0, 0\rangle & \rightarrow |0, 0\rangle, \\
|0, 1\rangle & \rightarrow \cos \theta(\upsilon, \ell)|0, 0\rangle - i \sin \theta(\upsilon, \ell)|1, \bar{a}\rangle, \\
|1, 0\rangle & \rightarrow |1, 0\rangle, \\
|1, 1\rangle & \rightarrow |1, 1\rangle,
\end{align*}
\]  

(35)

when both atoms passed through the setup. Although the output state in the second line does not belong to the basis set (28), the transformation matrix (34) is obtained whenever the condition,

\[
\theta(\upsilon, \ell) = \pi + 2\pi n,
\]  

(36)
is fulfilled. In this case, the unwanted part, \(|1, \bar{a}\rangle\), in the second line vanishes. Figure 5(a) displays the values of \( \upsilon \) and \( \ell \) for which condition (36) is satisfied. Moreover, by applying the fidelity we introduced in section 3.1, the fidelity between the state of the hyperfine qubits to evolve (up to a global phase factor)

\[
\begin{align*}
|\bar{a}\rangle & \rightarrow \cos(\eta/2)|\bar{a}\rangle - \sin(\eta/2)|\bar{1}\rangle, \\
|\bar{1}\rangle & \rightarrow \sin(\eta/2)|\bar{a}\rangle + \cos(\eta/2)|\bar{1}\rangle,
\end{align*}
\]  

(41a)

(41b)

where the rotation angle \( \eta \) is proportional to the microwave pulse duration. In the literature, such an atom–field interaction is often called a Ramsey pulse and is denoted in figure 5(d) by grey circles. These circles contain the interaction time in units of Ramsey rotations \( \eta \), and the letters \( R_1 \) and \( R_2 \) are associated with the Ramsey zones in front of and behind the cavity (see figure 1(c)).

According to figure 5(d) and equations (41a) and (41b), the state of \( A_2 \) is first transformed:

\[
\begin{align*}
|\bar{0}\rangle & \rightarrow \frac{\pi/2}{\ell_1}(1 - |\bar{a}\rangle - |\bar{1}\rangle) \quad \text{or} \quad |\bar{1}\rangle \rightarrow \frac{\pi/2}{\ell_1}(1 - |\bar{a}\rangle + |\bar{1}\rangle),
\end{align*}
\]  

(42)

by using a \( \pi/2 \) Ramsey pulse in the zone \( R_1 \), and immediately afterwards, the atomic hyperfine state \(|\bar{0}\rangle\) is mapped upon the optical state \(|\bar{a}\rangle\) by means of the Raman pulse \( L_1 \), which overall gives rise to the superposition

\[
\begin{align*}
|\bar{0}\rangle \rightarrow \frac{\pi/2}{\ell_1}(1 - |\bar{a}\rangle - |\bar{1}\rangle) \quad \text{or} \quad |\bar{1}\rangle \rightarrow \frac{\pi/2}{\ell_1}(1 - |\bar{a}\rangle + |\bar{1}\rangle).
\end{align*}
\]  

(43)

Inside the cavity, as mentioned above, only the product state \(|1, \bar{a}\rangle\) of the two atoms evolves according to equation (25). This makes the target qubit \( A_2 \) to remain unchanged if the control qubit \( A_1 \) was set initially to \(|0\rangle\). If the control qubit was set to \(|1\rangle\), then the effective atom–atom evolution (25) applies and gives rise to a swap of the target qubit \( A_2 \) for a proper choice of the velocity \( \upsilon \) and the inter-atomic distance \( \ell \). When both atoms have passed through the cavity, the state \(|\bar{a}\rangle\) is mapped back to \(|\bar{0}\rangle\) by the Raman pulse \( L_2 \) and, finally, atom \( A_2 \) is subjected to a \( \pi/2 \) Ramsey pulse in the zone \( R_2 \).

The mentioned Ramsey and Raman pulses together with the cavity and laser field make, therefore, the four input states of the hyperfine qubits to evolve (up to a global phase factor)

\[
\begin{align*}
|0, 0\rangle & \rightarrow |0, \bar{0}\rangle, \\
|0, 1\rangle & \rightarrow |0, \bar{1}\rangle, \\
|1, 0\rangle & \rightarrow \frac{1}{2}(1 + \cos \theta(\upsilon, \ell))|1, \bar{0}\rangle - \frac{1}{2}(1 - \cos \theta(\upsilon, \ell))|1, \bar{1}\rangle + i \sin \theta(\upsilon, \ell)(|\bar{a}\rangle - |\bar{1}\rangle), \\
|1, 1\rangle & \rightarrow \frac{1}{2}(1 + \cos \theta(\upsilon, \ell))|1, \bar{0}\rangle - \frac{1}{2}(1 - \cos \theta(\upsilon, \ell))|1, \bar{1}\rangle + i \sin \theta(\upsilon, \ell)(|\bar{a}\rangle - |\bar{1}\rangle).
\end{align*}
\]  

(44a)

(44b)

(44c)

(44d)

Although, again, the output states in the last two lines do not belong entirely to the basis set (28), the matrix (39) can be

\[
U_{\bar{a}}^{C\bar{N}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]  

(39)

where we utilized the assignment

\[
|\bar{0}\rangle = |0\rangle, \quad |\bar{1}\rangle = |1\rangle, \quad |\bar{0}\rangle = |0\rangle, \quad |\bar{1}\rangle = |1\rangle.
\]  

(40)

Obviously, the matrix (39) cannot be obtained from the evolutionary matrix (27) by just imposing a single restriction on the coupling angle \( \theta(\upsilon, \ell) \). Hence, we consider the modified temporal diagram displayed in figure 5(d). According to this diagram, the control atom \( A_1 \) is not subjected to the Raman pulses \( L_1 \) and \( L_2 \); however, the target atom \( A_2 \) passes through two additional classical microwave fields, in which it undergoes the (coherent) rotation of atomic hyperfine states:

\[
\begin{align*}
|\bar{0}\rangle & \rightarrow \cos(\eta/2)|\bar{0}\rangle - \sin(\eta/2)|\bar{1}\rangle, \\
|\bar{1}\rangle & \rightarrow \sin(\eta/2)|\bar{0}\rangle + \cos(\eta/2)|\bar{1}\rangle,
\end{align*}
\]  

(41a)

(41b)

In contrast to the controlled-\( \overline{\text{NOT}} \) gate we have defined in equation (38) the conventional controlled-\( \overline{\text{NOT}} \) gate [22]. \( U = |0\rangle\langle 0| + |1\rangle\langle 1| \times U_{\bar{a}}^{C\bar{N}} \) implies only the flipping of the target qubit (without accumulating a phase) when the control qubit is set to \(|1\rangle\).
realized if we impose the condition,
\[ \theta(\nu, \ell) = \pi + 2\pi n, \]
for the effective coupling angle. Since it is the same condition as equation (36) for the controlled-Z gate, the combinations of \( \nu \) and \( \ell \) that are appropriate for the controlled-NOT gate are displayed already in figure 5(a) for \( n = 0, 1, 2 \). The same also applies to the fidelity that is displayed in figure 5(b).

4. Summary and outlook

In summary, a scheme is proposed to generate an entangled state between the hyperfine qubits of two non-interacting four-level atoms being separated by a macroscopical distance. An effective interaction between the atoms is mediated by a detuned optical cavity and a laser beam. The purpose of our work is to analyse how the position-dependent coupling of each atom to the same cavity mode and a laser beam affects the effective interaction among the atoms, and whether it is possible to create a (maximally) entangled state between the atoms. In particular, the analytical expressions for the (asymptotic) coupling angle (21) and the evolutionary matrix (27) tell us explicitly how the degree of entanglement depends on both, the atomic velocity and the (initial) inter-atomic distance. For a position-dependent atom–cavity coupling (1), these expression have been derived for the first time for a four-level scheme as described above. In figure 3(a), for instance, we have shown that the atom–cavity interaction with a position-independent atom–cavity coupling, which was suggested in [13], leads to the set of constant atomic velocities for which the maximally entangled state can be generated (see the dashed line). Under more realistic assumptions of position-dependent atom–cavity coupling, however, these velocities are not constant but depend on the values of inter-atomic distance according to expression (21). From figure 3(b), moreover, it can be seen how sensitive the entanglement depends on variations in these parameters, an important requisite for any experimental realization. Finally, a few realistic schemes are suggested to implement some basic two-qubit quantum gates, such as i-swap gate, controlled-Z and the controlled-NOT gate in the framework of the given cavity–laser setup. For all these schemes, we displayed the atomic velocities and inter-atomic distances for which these gates are realized, i.e., the gate fidelities become maximal.

Following the recent experiments [20, 21, 25] and the theoretical works of [18, 26–28], the position-dependent effects on the effective atom–atom interaction and entanglement formation mediated by a (detuned) optical cavity are acknowledged today as a notable step in obtaining the control over the entanglement of atoms within the framework of cavity QED. In particular, Li and coworkers [18] suggested that the distance between the atoms is an important parameter that can be utilized to control the (position-dependent) atom–cavity coupling which also implies the control over the atomic entanglement. Instead of using a two-level atomic configuration, however, for the cavities in the optical domain it appears more suitable to consider a four-level Λ-type level configuration, in which the quantum information is stored in the hyperfine levels of the atomic ground state. Such a configuration appears to be essential for the recent experimental attempts [19–21] in which atoms are transported coherently inside the cavity by means of an optical lattice trap (conveyor belt). For such a belt, the inter-atomic distance is given by the wavelength of the (standing) optical lattice, while the velocity of the atoms is set by a shift in the frequencies of the counter-propagating laser beams.

A further extension of the effective atom–atom evolution, as described in section 2, might be a chain of \( N \) four-level atoms that cross the experimental setup in figure 1(c) and interacts simultaneously with the same cavity mode while passing through the cavity. This extension would lead naturally to the generation of various \( N \)-partite entangled states depending on the \( (\nu, \ell) \) regime and the succession of Raman and Ramsey zones (see, for instance, [29], where we discussed the formation of genuine entangled states for a chain of \( N \) bi-level atoms, which cross an analogous experimental setup we considered in this paper). Finally, we remark that a realistic atom–cavity interaction evolution should also include decoherence effects, which have been avoided in this paper so far. We note that in order to analyse the time evolution of such quantum systems embedded into a reservoir or under the external noise and to analyse different entanglement or separability measures, including those in equations (23) and (31), a quantum simulator has been developed recently in our group [30] that can be utilized for such studies in future.

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