The Conal representation of Quantum States and Non Trace-Preserving Quantum Operations

Pablo Arrighi¹ and Christophe Patricot²

¹Computer Laboratory, University of Cambridge, 15 JJ Thomson Avenue, Cambridge CB3 0FD, U.K.
²DAMTP, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, U.K.

We represent generalized density matrices of a d-complex dimensional quantum system as a subcone of a real pointed cone of revolution in $\mathbb{R}^d$, or indeed a Minkowskian cone in $\mathbb{E}^{1,d^2-1}$. Generalized pure states correspond to certain future-directed light-like vectors of $\mathbb{E}^{1,d^2-1}$. This extension of the Generalized Bloch Sphere enables us to cater for non-trace-preserving quantum operations, and in particular to view the per-outcome effects of generalized measurements. We show that these consist of the product of an orthogonal transform about the axis of the cone of revolution and a positive real linear transform. We give detailed formulae for the one qubit case and express the post-measurement states in terms of the initial state vectors and measurement vectors. We apply these results in order to find the information gain versus disturbance tradeoff in the case of two equiprobable pure states. Thus we recover Fuchs and Peres’ formula in an elegant manner.

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I. INTRODUCTION

The space of pure states of finite $d$-dimensional Quantum Mechanics $\mathbb{C}P^d$, set of rays in the complex Hilbert space $\mathbb{C}^d$, is, as most complex spaces, not easy to visualize. Physical motions, let alone unitary time evolutions, have no clear geometric interpretation. However, the set of hermitian operators on $\mathbb{C}^d$, $\text{Herm}_d(\mathbb{C})$, is a $(d^2-1)$-dimensional real vector space, and as such is certainly easier to represent geometrically. States, or more generally density matrices, form of course a subset of $\text{Herm}_d(\mathbb{C})$. The Generalized Bloch Sphere Representation [1]-[2] is a famous application of this fact which has proved to be popular and elucidating: a given density matrix can be represented as a real vector inside a (hyper) sphere. It turns out that this representation defined for density matrices, or unit trace positive elements of $\text{Herm}_d(\mathbb{C})$, is only good at handling unitary or trace-preserving quantum operations on density matrices: the former induce rotations of the Bloch vector, the latter affine transformations [2]. Individual outcomes of generalized measurements, for example, are not directly representable. Considering the insight the Bloch Sphere representation gave to unitary and trace-preserving operations, it seems interesting, for the mere sake of geometry at first, but mainly to give a useful picture to tackle Quantum Information problems, to extend it to cater for non-trace-preserving quantum operations. This is further motivated by the fact that the space of (semi-definite or definite) positive hermitian operators, thereafter denoted $\text{Herm}_d^+(\mathbb{C})$, is a closed convex cone, and that all admissible quantum operations should be a subset of the transformations of this cone.

In spite of being so central in Quantum Information Theory, the tradeoff between how much Shannon Information one may gain about a quantum system versus how much Disturbance the observation must necessarily cause to the system, remains extremely difficult to quantify. Quantum cryptographists tend to circumvent the problem: most of their proofs are a witty blend of the particular symmetries of the protocol in question, together with a convoluted machinery. A few attempts have been made to solve the tradeoff [1][2], but only one [6] deals with discrete ensembles - namely the case of two equiprobable pure states, and this is already something. Unfortunately the approach involves lengthy algebra and a number of assumptions. One should be able to find a method which gives a glimpse of intuition about the geometry of optimal measurements, and for this purpose, we think that our approach is useful.

In section III we consider general quantum systems of $d$ complex dimensions. We give a representation of the set of positive hermitian matrices $\text{Herm}_d^+(\mathbb{C})$ as a subcone of a real Minkowskian cone in $\mathbb{R}^{d^2}$, and analyse geometrical properties of generalized measurements in this setting.

We find that our approach is particularly useful to represent per-outcome post-measurement states, and that pure states correspond to certain light-like vectors of the cone. Unitary operators on the complex system become real orthogonal transforms, while positive operators become real positive transforms. Section III should be of special interest for quantum information theorists: we treat the $d = 2$ one qubit case in full detail. We find further geometrical relations between measurement vec-
tors, state vectors and post-measurement state vectors and give explicit formulae. In section IV we apply our results to a typical Information gain versus Disturbance tradeoff scenario in which Alice gives Eve two equiprobable pure states. Thus we recover Fuchs and Peres’ formula in an elegant and geometrical manner.

II. CONAL REPRESENTATION OF \( d \)-DIMENSIONAL QUANTUM SYSTEMS

The state of such a system is described by a \( d \times d \) density matrix. We shall express hermitian matrices as real linear combinations of Hilbert-Schmidt-orthogonal hermitian matrices, and then restrict this representation to elements of \( \text{Herm}_d^+(\mathbb{C}) \), or generalized density matrices. \( \text{Herm}_d^+(\mathbb{C}) \) turns out to be “isomorphic” to a convex subcone of a cone of revolution in \( \mathbb{R}^d^2 \), or indeed a Minkowskian future cone in \( \mathbb{R}^{1,d^2-1} \). We then analyze the effects of quantum operations on density matrices in this representation.

A. Hermitian matrices

Let \( \{\tau_i\}, i \in \{1, \ldots, d^2 - 1\} \), be a Hilbert-Schmidt orthogonal basis (as in (1)) of \( d \times d \) traceless hermitian matrices, and let \( \tau_0 \) be the identity matrix \( I \). Throughout this article latin indices will run from 1 to \( d^2 - 1 \), greek indices from 0 to \( d^2 - 2 \), and repeated indices are summed unless specified. We take the \( \tau_i \)'s to satisfy by definition:

\[
\forall \mu, \nu \quad \text{Tr}(\tau_\mu \tau_\nu) = d\delta_{\mu\nu} \tag{1}
\]

with \( \delta \) the Kronecker delta. \( \{\tau_i\}_{d^2} \) is a basis of \( \text{Herm}_d(\mathbb{C}) \), and any hermitian matrix \( A \in \text{Herm}_d(\mathbb{C}) \) decomposes on this basis as

\[
A = \frac{1}{d}(\text{Tr}(A)I + \text{Tr}(A\tau_i)\tau_i)
= \frac{1}{d}\text{Tr}(A\tau_\mu)\tau_\mu \tag{2}
\]

Letting \( A = (A_{ij}) \in \mathbb{R}^d^2 \) with \( A_{ij} = \text{Tr}(A\tau_{ij}) \) be the component vector of \( A \) in this particular basis, we have

\[
\forall A, B \in \text{Herm}_d(\mathbb{C}), \quad AB = \frac{1}{d^2}A_{ij}B_{\mu\nu}\tau_{\mu\nu}
\]

hence \( \text{Tr}AB = \frac{1}{d}A_{ij}B_{ij} \equiv \frac{1}{d}A_{ij}B_{ij} \). \( A \) is the vector in \( \mathbb{R}^{d^2} \), \( \vec{A} = (A_{ij}) \) the restricted vector in \( \mathbb{R}^{d^2-1} \), and \( \phi \) the coordinate map:

\[
\phi : \text{Herm}_d(\mathbb{C}) \to \mathbb{R}^{d^2} \quad A \mapsto \vec{A}
\]

Equation (3) says that \( \phi \) is an isometric isomorphism of \( (\text{Herm}_d(\mathbb{C}), \text{Tr}()) \) onto \( (\mathbb{R}^{d^2}, (1/d)(\cdot)) \). Therefore any linear operator \( L \) on \( \text{Herm}_d(\mathbb{C}) \) defines via \( \phi \) and \( \phi^{-1} \) an operator on \( \mathbb{R}^{d^2} \), \( M(L) = \phi \circ L \circ \phi^{-1} \). This definition yields the following “morphism” property:

Lemma 1 If \( L_1, L_2 \) are linear operators on \( \text{Herm}_d(\mathbb{C}) \), then \( M(L_i) = \phi \circ L_i \circ \phi^{-1} \) for \( i = 1, 2 \) are endomorphisms of \( \mathbb{R}^{d^2} \) and satisfy

\[
M(L_1 \circ L_2) = M(L_1)M(L_2) \tag{4}
\]

In particular, any complex \( d \times d \) matrix \( A \) defines via \( \text{Ad}_A : \rho \mapsto \rho A^\dagger \) a linear operator on \( \text{Herm}_d(\mathbb{C}) \) which corresponds to a real endomorphism \( M(\text{Ad}_A) : \rho \mapsto M(\text{Ad}_A)\rho ; \) and \( \text{Ad}_{AB} = \text{Ad}_A \circ \text{Ad}_B \) implies \( M(\text{Ad}_{AB}) = M(\text{Ad}_A)M(\text{Ad}_B) \). As a direct consequence of this and the previous definitions, calling \( \text{GL}_n(\mathbb{K}) \) the group of invertible \( n \times n \) matrices on the field \( \mathbb{K} \), we get:

Lemma 2 For any subgroup \( G \) of \( \text{GL}_d(\mathbb{C}) \), the following mapping

\[
\psi : G \to \psi(G) \subset \text{GL}_{d^2}(\mathbb{R}) \quad A \mapsto M(\text{Ad}_A) = \phi^{-1} \circ \text{Ad}_A \circ \phi \tag{5}
\]

is a group homomorphism. \( \psi(G) \) is a subgroup of \( \text{GL}_{d^2}(\mathbb{R}) \).

Note that since \( \psi(1) = \psi(-1) = I \), \( \psi \) is not necessarily injective. Moreover \( \psi \) is certainly not linear. An interesting subgroup is the Special Unitary group \( SU(d) = \{ U \in \text{GL}_d(\mathbb{C}) \mid UU^\dagger = I, \det U = 1 \} \). We call \( SO(n) = \{ O \in \text{GL}_n(\mathbb{R}) \mid OO^T = I, \det O = 1 \} \) the special orthogonal group in \( n \)-dimensions.

Lemma 3 Special Unitary transformations on \( \text{Herm}_d(\mathbb{C}) \), \( \text{Ad}_U : \rho \mapsto U\rho U^\dagger \) with \( U \in \text{SU}(d) \), induce rotations of \( \mathbb{R}^{d^2} \) about the \( 1 \)-axis. In fact the linear transforms \( \psi(U) : \rho \mapsto \psi(U)\rho \) are special orthogonal and \( \psi(SU(d^2 - 1)) \) is a subgroup of \( SO(d^2 - 1) \). It is a proper subgroup when \( d \geq 3 \). Moreover, \( \psi(U(d)) = \psi(SU(d)) \).

Proof: Let \( \rho = (1/d)(\text{Tr}(\rho)I + \rho_{ij}\tau_{ij}) \) a hermitian matrix. Using (3) and the fact that \( \text{Ad}_U \) is trace-preserving for \( U \) unitary:

\[
U\rho U^\dagger U\rho U^\dagger = (\text{Tr}(\rho))^2 + (\psi(U)\rho) \cdot (\psi(U)\rho),
= d\text{Tr}(U\rho U^\dagger U\rho U^\dagger)
= d\text{Tr}(\rho^2) = \rho^2 \rho
= (\text{Tr}(\rho)^2 + \rho^2 \rho).
\]

In addition to preserving the first component \( \rho_0 = \text{Tr}\rho \), \( \psi(U) \) preserves the \( \mathbb{R}^{d^2 - 1} \) scalar product \( \rho^2 \rho \). For all \( U \in \text{SU}(d) \), there exist \( t \in \mathbb{R} \) and \( B \in \text{su}(d) \) such that \( U = U(t) = \exp(tB) \). Since \( \det \psi(U(0)) = 1 \) and \( t \mapsto \det \psi(U(t)) \) is continuous and has values in \( \{ \pm 1 \} \), \( \det \psi(U) = 1 \). Thus \( \psi(U) \) is a special rotation about the
-axis of $\mathbb{R}^{d^2}$. By Lemma 2, $\psi(SU(d))$ is a subgroup of the special orthogonal group $SO(d^2 - 1) \subset SO(d^2)$. As for all $\theta \in \mathbb{R}$, $\text{Ad}_{\text{e}^{i\theta}} = \text{Ad}_{\text{e}^{\theta}}$, $\psi(U(d)) = \psi(SU(d))$. Since the $\{\sqrt{-1}\tau_i\}$ span the Lie algebra $su(d)$, $\psi(SU(d))$ is the Adjoint group of $SU(d)$. For $d = 2$, we get the whole of $SO(3)$, but this is not the case for $d > 2$, as is easily seen looking at the dimensions:

$$\dim SU(d) = d^2 - 1$$
$$\dim SO(d^2 - 1) = \frac{1}{2}(d^2 - 1)(d^2 - 2)$$

and $\dim SU(d) < \dim SO(d^2 - 1)$ for $d > 2$.

All the results of this section remain true of course when we just consider $\text{Herm}^+_d(\mathbb{C})$. From now on, for any $A$ complex $d \times d$ matrix we shall denote $\psi(A) = M(\text{Ad}_{\psi}A)$ the real endomorphism of $\mathbb{R}^{d^2}$.

### B. Generalized density matrices

In [2], Zanardi showed using a restriction of a mapping analogous to $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ that $d \times d$ density matrices lie in a convex subset of a ball $S \subset \mathbb{R}^{d^2 - 1}$. We shall extend this to a convex cone by considering generalized density matrices, by which we mean elements of $\text{Herm}^+_d(\mathbb{C})$. This seems more natural in the sense that we like to think of the space of states of most physical theories and indeed Quantum Mechanics as a space invariant under positive linear combinations and not just convex combinations. In addition, this bigger space allows a per-outcome representation of generalized measurements.

We define generalized pure states to be generalized density matrices which yield pure states after rescaling them to unit trace. Note that these are not the “states of partial purity” of the complex $d$-dimensional system, which are singular density matrices. In other words, generalized pure states are not the elements of the boundary of $\text{Herm}^+_d(\mathbb{C})$ in the sense of characteristic functions of cones (see [7] for example).

**Proposition 1** The cone of positive hermitian matrices $\text{Herm}^+_d(\mathbb{C})$ is isomorphic to a convex subset $C$ of the following cone of revolution in $\mathbb{R}^{d^2}$:

$$\Gamma = \{(\lambda_0, \lambda_i) \in \mathbb{R}^{d^2} / \sum_{i=1}^{d^2-1} \lambda_i^2 \leq (d-1)\lambda_0^2, \lambda_0 \geq 0\} \quad (6)$$

The set of generalized pure states verifies $\mathcal{C} = C \cap \partial \Gamma$, where $\partial \Gamma$ stands for the boundary of $\Gamma$.

**Proof**: We begin as in [2]. Let $\mathcal{P}$ denote the space of (not generalized) pure states in $\text{Herm}_d(\mathbb{C})$. In addition to being positive, $\rho \in \mathcal{P}$ satisfies $\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1$, so we have

$$\text{Tr}(\rho^2) = \frac{1}{d} \rho \rho \rho = \frac{1}{d} \left( (\text{Tr}\rho)^2 + \rho \rho \right) = \frac{1}{d} \left( (1 + \rho \rho) \right) = 1, \quad \text{hence}$$

$$\rho \rho = d - 1 \quad (7)$$

The restricted vector $\rho \rho$ is on a $(d^2 - 2)$-sphere of radius $\sqrt{d^2 - 1}$, $\partial S^{d^2-2}$, where $S$ is the corresponding ball. In $\mathbb{R}^{d^2}$, $\rho \rho$ pure sits in the intersection of the cylinder $\mathbb{R}^d$ and the $\rho_0 = \text{Tr}(\rho) = 1$ hyperplane, in other words on $\partial S^{d^2-2}$ “centered” at $(1, 0, \ldots, 0)$.

Any density matrix can be expressed as a positive (convex) linear combination of pure states, and any positive (convex) linear combination of pure states defines a density matrix. Calling $D$ the set of (not generalized) density matrices, $D \subset \text{Hull}(\mathcal{P})$ and $\text{Hull}(\mathcal{P}) \subset D$. Since $D$ is closed, $D = \text{Hull}(\mathcal{P})$, a well-known result.

As $\phi : \text{Herm}_d(\mathbb{C}) \rightarrow \mathbb{R}^{d^2}$ is linear and bi-continuous, $\phi(D) = \phi(\text{Hull}(\mathcal{P})) = \phi(\text{Hull}(\mathcal{P})) = \text{Hull}(\phi(\mathcal{P}))$. This set is a closed convex subset of $S$ “centered” at $(1, 0, \ldots, 0)$:

$$\phi(\mathcal{P}) \subset \partial S^{d^2-2} \Rightarrow \text{Hull}(\phi(\mathcal{P})) \subset S$$

Calling $S^+ \equiv \text{Hull}(\phi(\mathcal{P}))$ the image set of density matrices as a subset of $\mathbb{R}^{d^2-1}$, we get:

$$\rho \in D \Rightarrow \text{Tr}(\rho) = \rho_0 = 1 \text{ and } (\rho \rho) \in S^+$$

Now a non-zero $\rho \in \text{Herm}_d(\mathbb{C})$ is positive if and only if $(1/\text{Tr}\rho) \rho$ is positive, that is if and only if $((1/\text{Tr}(\rho)) \rho \rho) \in S^+$. In $\mathbb{R}^{d^2}$, recalling that $\text{Tr}(\rho) \equiv \rho_0$, this reads

$$\rho \in \text{Herm}_d(\mathbb{C}) \Rightarrow \rho \in \{ (\lambda_0, \lambda_i) \in \mathbb{R}^{d^2} / \lambda_i \in \lambda_0 S^+ \} \quad (8)$$

This clearly defines a cone $C$ in $\mathbb{R}^{d^2}$. As $S^+ \subset C$, $C$ is a subcone of the cone of revolution $\Gamma$ given by [6]. $\phi$ being an isomorphism, $C$ is convex and isomorphic to $\text{Herm}_d(\mathbb{C})$. As pure states correspond to some points on the sphere $\partial S^{d^2-2}$, generalized pure states lie in the boundary of $\Gamma$. Calling $\mathcal{C}$ the set of vectors of $C$ corresponding to generalized pure states, we have $\mathcal{C} \subset C \cap \partial \Gamma$.

Moreover $C \subset C \cap \partial \Gamma$ follows from the fact that any rescaled positive matrix $\rho$ such that $\text{Tr}(\rho^2) = \text{Tr}\rho = 1$ is a pure state. Remember that $\mathcal{C}$ is not the boundary of $C$, but a cone over $\phi(\mathcal{P})$, the image set of pure states. □

As we shall see in detail in section [11] in $d = 2$ dimensions, generalized pure states correspond to future-directed light-like vectors of Minkowski space of signature $(1, 3)$. We have shown that this remains true to a certain extent in $d$-complex dimensions, $\Gamma$ being the future light-cone of Minkowski space $\mathbb{E}^{1,d^2-1}$ with metric $\eta_{\mu\nu} = \text{Diag}(d - 1, -1, \ldots, -1)$. Thus the appearance of a Minkowski product is to be expected.
As a consequence of Lemma 3 unitary transforms, since they leave \( \text{Herm}^+_{d}(\mathbb{C}) \) invariant, yield rotations which leave \( C \) (globally) invariant. This fact deserves to be analysed in detail to understand the geometry of \( C \). As Unitary transforms act transitively on pure states, \( S^+ \) is the closed convex hull of a homogeneous subspace \( \phi(\mathcal{P}) \) of \( \partial S^{d-2} \). For the moment however, we shall consider the geometric representation of general quantum operations in \( C \).

C. Generalized measurements

We call a generalized measurement \( \mathcal{E} \) a finite set \( \{M_m\}_m \) of complex \( d \times d \) matrices which satisfy:

\[
\sum_m M_m^\dagger M_m = I. \quad \text{The set of } \{E_m\}_m = \{M_m M_{m'}\}_m \text{ defines a Positive Operator Valued Measure (POVM), as } E_m \in \text{Herm}^+_{d}(\mathbb{C}) \text{ and } \sum_m E_m = I. \quad \text{Given a quantum state or density matrix } \rho \in D, \text{ the generalized measurement } \{M_m\}_m \text{ on } \rho \text{ yields outcome } m \text{ with probability } p(m) = \text{Tr}(E_m \rho), \text{ and if outcome } m \text{ occurs, the post-measurement state is } \rho'_m = (1/\text{Tr}(E_m \rho))(M_m \rho M^\dagger_m). \text{ We shall call } \rho_m = M_m \rho M^\dagger_m \in \text{Herm}^+_{d}(\mathbb{C}) \text{ the unrescaled post-measurement state.}
\]

Recall that any complex matrix can be polar-decomposed into a product of a unitary matrix and a positive matrix. For all \( m \), there exists \( U_m \in U(d) \) and \( A_m \in \text{Herm}^+_{d}(\mathbb{C}) \) such that \( M_m = U_m A_m \). As \( E_m = M_m^\dagger M_m = A_m A^\dagger_m \), \( A_m = \sqrt{E_m} \), the positive square root of \( E_m \). Using this polar decomposition, \( \rho_m \) is represented in the cone \( C \) by

\[
\rho'_m \equiv \phi(\rho'_m) = \frac{1}{\text{Tr}(E_m \rho)} \phi(U_m \sqrt{E_m \rho} U^\dagger_m) = \frac{1}{\text{Tr}(\sqrt{E_m \rho} \sqrt{E_m})} \psi(U_m)(\sqrt{E_m \rho} \sqrt{E_m})
\]

Thus when outcome \( m \) occurs, the post-measurement state \( \rho'_m \) of \( \{M_m\}_m \) is the same as that of \( \{\sqrt{E_m}\}_m \) up to a rotation \( \psi(U_m) \), and similarly for the unrescaled states. As a consequence we shall consider the geometrical effects of generalized measurements \( \{\sqrt{E_m}\}_m \) where \( E_m \) and \( \sqrt{E_m} \) are in \( \text{Herm}^+_{d}(\mathbb{C}) \) and \( \sum_m E_m = I \), bearing in mind that the most general measurements just involve rotations on the post-measurement state vectors. For example, in section \( \text{XX} \) Eve is free to perform unitary transforms on her post-measurement states, and can decide this according to the outcome \( m \). The procedure we use to find the Disturbance is to first measure with \( \{\sqrt{E_m}\}_m \) and then maximise on Unitary transforms acting upon post-measurement states. Using the conal representation, both sets of vectors \( \{E_m\}_m \) and \( \{\sqrt{E_m}\}_m \) are in \( C \), and \( \sum_m E_m = (d, 0, \ldots, 0) \). This enables us to represent elements of a measurement inside \( C \), and visualize the action of a particular non-trace-preserving operation \( \sqrt{E_m} \) on a given density matrix \( \rho \), in other words find \( \rho_m \) in terms of \( E_m \) or \( \sqrt{E_m} \).

D. Quantum operations represented in \( C \)

One might wonder here why not just rescale all the post-measurement states and only consider the density matrices \( \rho'_m \). The reason for not doing so is that the unrescaled states encode extra information: their “height” in the cone, the first component \( \rho_{m0} = \text{Tr}(E_m \rho) \), is simply the probability of their outcomes. Under a given generalized measurement, post-measurement vectors with identical first components are equiprobable. Thus the sections of \( C \) of constant \( \lambda_0 \) have a clear physical interpretation. We shall need the following simple properties:

Lemma 4 For \( A \in \text{Herm}_d(\mathbb{C}) \) and \( B, C \in \text{Herm}^+_{d}(\mathbb{C}) \),

\[
\text{Tr}(BC) \geq 0 \quad \text{Tr}(BABA) \geq 0
\]

Proof: Let \( B = \sqrt{B} \sqrt{B} \), then \( \text{Tr}(BC) = \text{Tr}(\sqrt{B} C \sqrt{B}) \geq 0 \) since \( \sqrt{B} C \sqrt{B} \in \text{Herm}^+_{d}(\mathbb{C}) \). Then polar decompose \( A \) into \( A = U |A| \), with \( U \) unitary and \( |A| \in \text{Herm}^+_{d}(\mathbb{C}) \). As \( A \in \text{Herm}_d(\mathbb{C}) \), \( A = |A| U^\dagger = A^\dagger \), and

\[
\text{Tr}(BABA) = \text{Tr}(BU |A| B |A| U^\dagger) = \text{Tr}(U^\dagger BU |A| B |A|)
\]

This is non-negative by the previous result since \( U^\dagger BU, |A| B |A| \in \text{Herm}^+_{d}(\mathbb{C}) \). ◻

Unitary transforms induce rotations in \( C \), and generalised measurements have the following geometric properties:

Proposition 2 The linear transforms \( \psi(\sqrt{E_m}) : \rho \mapsto \rho_m \) associated to a generalized measurement \( \{\sqrt{E_m}\}_m \) correspond to real symmetric matrices which are positive. They individually map \( \mathcal{C} \) into itself. In addition, for any generalised pure state \( \theta, \psi(\theta) \) maps \( C \) into \( \mathcal{C} \).

The probability of outcome \( m \) for a quantum system in state \( \rho \) is given by

\[
p(m) = \frac{1}{d} E_m \rho
\]

(9)

Proof: By using (2) successively, we have

\[
\rho_{m \mu} = \text{Tr}(\sqrt{E_m \rho} \sqrt{E_m \tau_\mu}) = \frac{1}{d} \text{Tr}(\sqrt{E_m \tau_\nu} \sqrt{E_m \tau_\mu}) \psi_\nu = M_{m \nu}^\dagger \psi_\nu
\]

(10)

Clearly \( M_{m \nu}^\dagger \) is real symmetric by cyclicity of the trace and the fact that \( \sqrt{E_m \tau_\nu} \sqrt{E_m \tau_\mu} \) and \( \tau_\mu \) are hermitian. (Actually \( \psi(A) \) is real for any complex \( d \times d \) matrix \( A \), and real symmetric for any \( A \) hermitian). Let \( v = (v_\nu) \in \mathbb{R}^{d^2} \).

Using (10) we get

\[
v^T \psi(\sqrt{E_m} \rho) v = v_\mu M_{m \nu}^\dagger \psi_\nu = \frac{1}{d} \text{Tr}(\sqrt{E_m \tau_\nu} \sqrt{E_m \tau_\mu}) \psi_\nu
\]

\[
= \frac{1}{d} \text{Tr}(\sqrt{E_m \tau_\nu} \sqrt{E_m \tau_\mu} \sqrt{E_m \tau_\nu} \sqrt{E_m \tau_\mu}) \geq 0
\]
This follows from Lemma 1 since \( \varphi_{\mu} \tau_{\mu} \in \text{Herm}_{d}(\mathbb{C}) \). Hence \( M_{\mu \nu}^{m} \) is a positive real (symmetric) matrix. The properties on purity simply follow from general facts on quantum operations on density matrices which remain true for generalized density matrices:

For any kernel \( A \) complex \( d \times d \) matrix and any generalized density matrix \( \rho \):

\[
A|u><v|A^\dagger = |Au><Au| \quad \text{and} \quad |v><v|\rho|v><v| = |v><v|\rho|v><v| \quad |v><v| \quad (11)
\]

are generalized pure states. Relation (9) follows from Lemma 4 since \( \rho \equiv \rho_{m} \). The index \( m \)

sequence, if \( M \) is such an eigenvector, then the rescaled density matrix \( \rho \) is such that \( \rho = \rho_{m} \), i.e. \( \rho \) is unchanged if outcome \( m \) occurs.

We now give the general expressions for \( \rho_{m} \) in terms of \( \rho \equiv (1/d)\rho_{\mu} \tau_{\mu} \) and \( \sqrt{E_{m}} \equiv (1/d)\sqrt{\varphi_{\mu} \tau_{\mu}} \), where we drop the index \( m \) and do not underline the components of the vectors \( \rho \) and \( \sqrt{E_{m}} \) for convenience. By definition:

\[
\rho_{m} = \frac{1}{d^{2}}\sqrt{\varepsilon_{\mu} \rho_{\nu} \sqrt{\varepsilon_{\tau} \tau_{\mu} \tau_{\nu} \tau_{\sigma}}}
\]

Expanding this using \( \tau_{0} = 1 \) and grouping the products of the \( \tau_{i} \)'s in hermitian terms, we easily derive:

\[
\rho_{m} = \frac{1}{d^{3}}\left\{ \sqrt{\varepsilon_{0} \rho_{0} \sqrt{\varepsilon_{1} \tau_{1}}} + (2\sqrt{\varepsilon_{0} \rho_{0} \sqrt{\varepsilon_{1}} + \varepsilon_{0} \rho_{1} \sqrt{\varepsilon_{0}}})\tau_{1} + \frac{1}{2}\sqrt{\varepsilon_{1} \rho_{1} \sqrt{\varepsilon_{0} \tau_{0}}} \right\}
\]

To push the general \( d \)-dimensional analysis further, we need a particular choice of \( \tau_{i} \)'s whose anti-commutation relations are convenient. This is subject to current work. We now treat in full detail the \( d = 2 \) (one qubit) case and apply our geometric approach to a challenging quantum information theoretical problem.

III. THE QUBIT CASE PUSHED FURTHER

Applied to qubit states the representation yields two of the most familiar objects in fundamental physics: the

2 \times 2 density matrices yield a Minkowskian future-light-cone in \( \mathbb{E}^{1,3} \) whose vertical sections are nothing but Bloch spheres. The correspondence between light-like vectors and fully determined spins is puzzling, but it requires a little more than one qubit to be investigated further. Meanwhile in this simple case we are able to give explicit coordinates for states posterior to non trace-preserving quantum operations. These formulae remain simple provided Minkowskian products are introduced alongside the Euclideans. They constitute a sufficient armoury to deal, using only four-vectors, with the most general evolutions to happen on a qubit.

### A. The Cone and the Bloch Sphere

A suitable Hilbert-Schmidt orthogonal basis for \( 2 \times 2 \) traceless hermitian matrices is given by the set of Pauli matrices:

\[
\tau_{1} = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_{2} = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_{3} = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Together with the identity

\[
\tau_{0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

one may express any \( 2 \times 2 \) hermitian matrix as a sum \( A = \frac{1}{2}A_{\mu} \tau_{\mu} \) with the \( A_{\mu} \)'s real. The positivity conditions for those matrices turns out simple.

#### Lemma 5

The cone of positive hermitian matrices \( \text{Herm}_{2}^{+}(\mathbb{C}) \) is isomorphic to the following cone of revolution in \( \mathbb{R}^{4} \):

\[
\Gamma = \{(\lambda_{\mu}) \in \mathbb{R}^{4} / \lambda_{0}^{2} - \sum_{i=1}^{3} \lambda_{i}^{2} \geq 0, \lambda_{0} \geq 0\}
\]

Generalized pure states lie on the boundary of \( \Gamma \).

#### Proof:

The eigenvalues of \( A \) are given by \( \lambda_{\pm} = \frac{1}{2}(A_{0} \pm \sqrt{A_{0}^{2} - A}) \). \( A \) is positive if and only if \( \lambda_{+} \lambda_{-} \geq 0 \) and \( \lambda_{+} + \lambda_{-} \geq 0 \). This is equivalent to:

\[
\eta_{\mu\nu}A_{\mu}A_{\nu} \geq 0 \quad \land \quad A_{0}\geq 0 \quad (13)
\]

with \( \eta_{\mu\nu} = \text{Diag}(1, -1, -1, -1) \). The purity condition is an obvious consequence of Proposition 1.

Thus the generalized (not necessarily normalized) density matrices of a qubit cover the whole Minkowskian future-light-cone in \( \mathbb{E}^{1,3} \). Taking a vertical cross-section of the cone is equivalent to fixing the trace \( A_{0} \) of the density matrix, which might be thought of physically as the overall probability of occurrence for the state. By doing
so we are left with only the spin degrees of freedom along $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, therefore each vertical cross-section is a Bloch sphere with radius $a = A_0$.

The ability to represent states with different traces is convenient when dealing with quantum ensembles $\{(\rho_x, \rho_y)\}$. When we seek to represent non trace-preserving quantum operations the feature becomes absolutely crucial.

**B. The Post-measurement State**

As we have seen in subsection III C the most general quantum operation can be described as $\{M_m\}_m = \{U_m \sqrt{E_m}\}_m$ with $U_m$ unitary and $\sqrt{E_m}$ positive (the only extra feature Kraus operators allow is the possibility to ignore one’s knowledge of some measurement outcomes, but in our setting this is easily catered for by adding up the undistinguished non-normalized post-measurement states). While the action of $U_m$ is well understood in terms of four-vectors (as a mere rotation in the Bloch Sphere, see Lemma 6), the authors of this paper are not aware of a solid geometrical framework for representing the effects of $\sqrt{E_m}$ - other than the one presented here. In Lemma 6 if $A \equiv \sqrt{E_m}$ while $\rho$ is the initial state, then $A\rho A$ stands for the (not renormalized) ‘post-measurement’ state when outcome $m$ has occurred (up to a unitary evolution $U_m$).

**Lemma 6** Let $A$ and $\rho$ be two matrices in $\text{Herm}_2^+(\mathbb{C})$. Then:

$$A\rho A = \frac{1}{8} \left[ -\rho_0 (\eta_{\mu\nu} A_{\mu\nu} A_{\mu\nu}^\dagger) + 2 A_0 (A \rho A) \right] \tau_0 + \frac{1}{8} \left[ \rho_1 (\eta_{\mu\nu} A_{\mu\nu} A_{\mu\nu}^\dagger) + 2 A_1 (A \rho A) \right] \tau_1 + \frac{1}{8} \left[ \rho_2 (\eta_{\mu\nu} A_{\mu\nu} A_{\mu\nu}^\dagger) + 2 A_2 (A \rho A) \right] \tau_2 + \frac{1}{8} \left[ \rho_3 (\eta_{\mu\nu} A_{\mu\nu} A_{\mu\nu}^\dagger) + 2 A_3 (A \rho A) \right] \tau_3 + \frac{1}{8} \left[ (\eta_{\mu\nu} \rho_1 A_{\mu\nu} A_{\mu\nu}^\dagger) + 2 A_{\mu\nu} (A \rho A) \right] \tau_{\mu\nu}$$  \hspace{1cm} (14)

**Proof: Consider**

$$\mathcal{A} = \begin{bmatrix} a & \beta & \gamma & \delta \\ x & y & z & 0 \end{bmatrix}$$

We have:

$$A\rho A = \frac{1}{8} \left[ a(a^2 + \beta^2 + \gamma^2 + \delta^2) + 2\alpha(\beta x + \gamma y + \delta z) \right] \tau_0 + \frac{1}{8} \left[ x(\alpha^2 + \beta^2 - \gamma^2 - \delta^2) + 2\beta(\alpha a + \gamma y + \delta z) \right] \tau_1 + \frac{1}{8} \left[ y(\alpha^2 - \beta^2 + \gamma^2 - \delta^2) + 2\gamma(\alpha a + \beta x + \delta z) \right] \tau_2 + \frac{1}{8} \left[ z(\alpha^2 - \beta^2 - \gamma^2 + \delta^2) + 2\delta(\alpha a + \beta x + \gamma y) \right] \tau_3$$  \hspace{1cm} (15)

This formula can be be obtained either by brute force calculation using the Pauli multiplication relations, or by exploiting the fact that Pauli matrices form a Clifford Algebra i.e. $\{\tau_\mu, \tau_\nu\} = 2\delta_{\mu\nu} \tau_0$ together with equation (12). Regrouping the terms gives formula (14).

**Corollary 1** Let $A$ and $\rho$ be two matrices in $\text{Herm}_2^+(\mathbb{C})$. $A\rho A$ can be expressed as a linear combination of $\rho$, $A$ and the Identity:

$$A\rho A = \frac{1}{2} \left( \eta_{\mu\nu} A_{\mu\nu} A_{\mu\nu}^\dagger \right) (\rho - \rho_0 \tau_0)$$

This last corollary provides much geometrical insight on non trace-preserving quantum operations. We find that the effect of $\sqrt{E_m}$ is not that difficult to visualize: the resulting state is a weighted sum of $\sqrt{E_m}$, the initial state and the identity, with real coefficients.

It is a somewhat strange fact that the structure equation (13) does not become apparent until one brings the Minkowskian product to the rescue. The spurious appearance of special relativistic products in quantum mechanics bears some explanation in this setting however, since the Minkowski metric is intrinsically related to the characteristic function of pointed cones of revolution.

Finally it is important to notice that the results expressed in these two last subsections are invariant under any orthogonal change of basis $\{\tau_\mu\}$. This is because rotations about the vertical axis leave the Minkowskian product invariant. The Pauli matrices have been helpful in computing those results, but from now and in the rest of the paper we may consider ourselves in the more general setting of section II.

**C. Square and Square Root**

In our quest towards representing non trace-preserving quantum operations in the cone we have managed to obtain the probability of occurrence $p(m)$ in terms of $E_m$ (Proposition 2). In the previous subsection we have also worked out the evolved state $\rho_{\text{evolved}}$ but unfortunately this was done in terms of $\sqrt{E_m}$. In order to deal fully with these operations in the Cone formalism we need to understand ways of switching back and forth from $E_m$ to $\sqrt{E_m}$. The next Lemma is a direct consequence of equation (12) when $\rho = I$.

**Lemma 7** The square of a matrix $A$ in $\text{Herm}_2^+(\mathbb{C})$ is given by:

$$A^2 = \mathcal{A}_0 A - \frac{1}{4} (\eta_{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu) \tau_0$$

Inversely the square root operation obeys:

$$\sqrt{A} = \frac{1}{r} (A + \frac{1}{2} \eta_{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu) \tau_0$$

with: $r = \sqrt{\mathcal{A}_0} + \sqrt{\eta_{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu}$
Note that $A$ is proportional to $\sqrt{A}$ if and only if $A$ is
generalized pure or $A \propto 1$.
But when we seek to express a function of $E_m$ in terms
of $\sqrt{E_m}$ (or the reverse) the next formulae become con-
venient.

Lemma 8 Let $A$ and $\rho$ be two matrices in Herm$_2^+$($\mathbb{C}$).
The following relations hold:

$$
\eta_{\mu\nu}\sqrt{A} \sqrt{A} = 2\sqrt{\eta_{\mu\nu}A\cdot A}
$$

$$
A^\top \rho = A_0(A\cdot \rho) - \frac{1}{2}\rho_0(\eta_{\mu\nu}A_\mu A_\nu)
$$

$$
\sqrt{A} \rho = \frac{1}{r}(A\cdot \rho + \rho_0\sqrt{\eta_{\mu\nu}A_\mu A_\nu})
$$

with: $r = \sqrt{A_0 + \sqrt{\eta_{\mu\nu}A_\mu A_\nu}}$

On the whole taking the square root of $E_m$ is not so easy.
It would be much more convenient if we could make all
calculations in terms of $E_m$, with the added advantage condition:

$$
\sum_m E_m = 2r_0
$$

(16)
is easily visualized. Results in the following subsection are
most useful for this purpose.

D. Inner Products Through Quantum Operations

Consider two states $\rho^0$, $\rho^1$. Suppose they undergo a quantum operation $\{M_m\}_m = \{U_m\sqrt{E_m}\}_m$ and outcome
$m$ occurs. Rather than seeking the coordinates of the
rescaled post-measurement states $\rho^0_m$ and $\rho^1_m$, we may be
interested in their positions relative to one another.
Note this subsection reuses a number of notational con-
veniences introduced in section IV.

Lemma 9 Let $\rho^0$, $\rho^1$ be two initial states in Herm$_2^+$($\mathbb{C}$)
and $\sqrt{E_m}$ a measurement element in Herm$_2^+$($\mathbb{C}$).
The inner products of the post-measurement states satisfy:

$$
\rho^0_m \cdot \rho^0_m = \frac{1}{4}[(E_m\cdot \rho^0)^{(E_m\cdot \rho^0)} - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho^0 \cdot \rho^0)]
$$

$$
\rho^0_m \cdot \rho^1_m = \frac{1}{4}[(E_m\cdot \rho^0)^{(E_m\cdot \rho^1)} - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho^0 \cdot \rho^1)]
$$

$$
\rho^1_m \cdot \rho^0_m = \frac{1}{4}[(E_m\cdot \rho^0)^{(E_m\cdot \rho^1)} - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho^1 \cdot \rho^0)]
$$

(17)

$$
\rho^0_m \cdot \rho^1_m = \frac{1}{4}[(E_m\cdot \rho^0)^{(E_m\cdot \rho^1)} - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho^0 \cdot \rho^1)]
$$

(18)

Proof: By using (3) we have:

$$
\rho^0_m \rho^1_m = \sqrt{E_m}\rho^0\sqrt{E_m}\rho^1\sqrt{E_m}
$$

$$
= 2Tr(\sqrt{E_m}\rho^0\sqrt{E_m}\rho^1\sqrt{E_m})
$$

$$
= 2Tr(\sqrt{E_m}\rho^0\rho^1)
$$

From there we readily obtain equation (17) by applying
equation (14) once. □

By letting $\rho^0 = \rho^1 = \rho$ in the above Lemma we get:

$$
||\rho_m||^2 = \frac{1}{4}[(E_m\cdot \rho)^2 - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho \cdot \rho)]
$$

$$
||\rho'_m||^2 = 2 - \frac{(\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho \cdot \rho)}{(E_m\cdot \rho)^2}
$$

$$
||\rho''_m||^2 = \frac{1}{4}[(E_m\cdot \rho)^2 - (\eta_{\mu\nu}E_m\cdot E_m)(\eta_{\mu\nu}\rho \cdot \rho)]
$$

(19)

Equation (19) clearly exhibits the general property we
stated in Proposition 2 that if the initial state is
generalized pure ($\eta_{\mu\nu}E_m\cdot E_m = 0$) or the measurement
is generalized pure ($\eta_{\mu\nu}E_m\cdot E_m = 0$) then we have
$||\rho_m|| = 1$ (pure), which implies that $\rho_m$ is generalized
pure.

The above lemma enables us to determine all the repre-
tative positions (angles and norms) of quantum states using
relatively compact formulae which do not involve $\sqrt{E_m}$.
It is only when the coordinates of each post-measurement
state are required that one needs to take the impractical
square root of $E_m$. But remember we are allowed an
arbitrary rotation $U_m$ in order to complete the quan-
tum operation. This means we have full freedom to fix
the absolute coordinates at will (so long as the relative
positions are respected).

Most Quantum Information Theoretical problems seek
to evaluate the limits of quantum operations, e.g. quan-
tum cloning [3], distinguishability [10], Information Gain
versus Disturbance tradeoff [4]. In these situations the
precise individual coordinates of the states after $Ad\sqrt{E_m}$
tend not to matter; usually they will need to be rotated
anyhow into a position which optimizes the fidelity mea-
ure in question. What counts is the relative position of
the post-measurement states. Therefore these problems
can be treated comfortably in our framework. Section XIV
provides a good example of such an application.

There are, however, some rare situations where we
would like to see quantum operations act step by step,
yielding precise coordinates - instead of just fixing the
coordinates of the final state as we would do in order to
avoid taking the square root of $E_m$. This is the case for
instance in quantum complexity, where one needs an ap-
preciation of how many basic computational operations
it takes to accomplish some calculation. Yet in this type
of problems it turns out that the basic operations can be
taken to be unitary operators, with measurements only performed at the end (principle of delayed measurement \[\text{[8]}\]). Therefore these scenarios may still be analyzed comfortably within our conal representation: the basic unitary operators will just be a set of chosen real orthogonal rotations, and the final measurement statistics will be evaluated straight from \(E_m\).

IV. APPLICATION: INFORMATION GAIN VERSUS DISTURBANCE TRADEOFF

The following idealized scenario captures a key situation for any quantum cryptographic protocol:

Alice owns a random variable \(X = \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}\). According to the outcome \(x\) she prepares either \(|\psi_0\rangle\) or \(|\psi_1\rangle\), i.e. she runs \(|x\rangle\langle 0| \xrightarrow{U} |x\rangle\langle x|\psi_x\rangle\). Eve knows \(U\) and the distribution \(X\), but not the particular outcome Alice has drawn. Later Eve gains access to \(|\psi_x\rangle\) and may use of this opportunity to try and learn about \(x\). How much she learns is quantified using Information theoretical notions. Even though Alice has had to expose \(|\psi_x\rangle\), still she really wanted to keep \(x\) secret. But now she gets a chance of checking upon Eve’s honesty - by asking her to return \(|\psi_x\rangle\). Suppose Eve’s measurement and further manipulations have modified \(|\psi_x\rangle\) into \(\rho_x\). Alice then measures \(|\langle \psi_x\rangle\langle x|\psi_x\rangle|,\ I_x = |\langle \psi_x\rangle|\rangle\langle x|\psi_x\rangle|\rangle and has a probability \(1 - |\langle \psi_x|\rho_x|\psi_x\rangle|^2\) of detecting the felony.

The point is that most quantum cryptographic protocols rely upon the fact that Eve cannot eavesdrop a state without causing it an irreversible, detectable damage. In spite of their central role, Information Gain versus Disturbance tradeoffs upon discrete ensemble states remain largely unknown, due to the mathematical difficulties they raise. In 1995 Fuchs and Peres accomplished the mathematical feat of obtaining an analytic formula for the above case of two non-orthogonal states. But the method they used relies upon a number of “plausible” assumptions - and does not provide a geometrical intuition of what the family of optimal measurements looks like.

The Cone, by enabling a per outcome geometrical representation of generalized measurements, permits us to overcome some of these shortcomings and greatly facilitate the derivation of Fuchs and Peres’ formula. We hope this illustrates the power of the geometrical framework developed in this paper.

A. Information Contribution, Disturbance Contribution

Suppose the \(|\psi_x\rangle\}_{x=0,1}\) states Alice prepares verify the following basic relations:

\[
\begin{align*}
\psi^x &= \phi(|\psi_x\rangle\langle \psi_x|) \\
\psi^0 \cdot \psi^1 &= d = \sqrt{1 - c^2}
\end{align*}
\]

By choosing a suitable basis in the Bloch Sphere and since the \(|\psi_x\rangle\}_{x=0,1}\) are pure we may we fix:

\[
\begin{align*}
\psi^0 &= \begin{bmatrix} 1 & c & d & 0 \end{bmatrix} \\
\psi^1 &= \begin{bmatrix} 1 & -c & d & 0 \end{bmatrix}
\end{align*}
\]

The most general thing Eve can ever do is to attack the states with a measurement \(\{M_m\}_m\). This procedure is equivalent to first measuring \(\sqrt{E_m}\), and then, conditional to \(m\), applying the unitary transformation \(U_m\), with \(E_m\) and \(U_m\) defined as in subsection II C. It is rather interesting to observe that the second step has no other use but to “repair” the post-measurement states as much as is possible. The first step on the other hand may partially destroy the initial states so as to collect the Information Eve seeks. This is the step we now study in order to quantify her Information Gain.

Let \(Y\) be the random variable arising from the measurement outcomes, i.e. \(Y = \{(p(m), m)\}_m\). Eve’s Information Gain is given by:

\[
I = H(X : Y) = H(Y) - H(Y|X)
\]

\[
= \sum_{x,m} p(m) \log(p(m)) - \sum_{x,m} p(x,m) \log(p(m|x))
\]

\[
= \sum_m I_m \quad \text{with}
\]

\[
I_m = p(m) \log(p(m)) - \sum_x p(x,m) \log(p(m|x))
\]

\(I_m\) must be understood as the Information Contribution brought by the measurement element:

\[
\varepsilon_m = [\alpha \quad \beta \quad \gamma \quad \delta] = \phi(E_m)
\]

By making use of the relations \([20], [21]\) and \([9]\) one can express \(I_m\) geometrically in terms of scalar products in the cone:

\[
I_m = -(p_m + q_m) \log(p_m + q_m)
\]

\[
+ p_m \log(2p_m) + q_m \log(2q_m)
\]

with

\[
p_m = \frac{\alpha + \beta c + \gamma d}{4} = \frac{\varepsilon_m^0}{4} \equiv p(0, m)
\]

\[
q_m = \frac{\alpha - \beta c + \gamma d}{4} = \frac{\varepsilon_m^1}{4} \equiv p(1, m)
\]

Notice that if \(\varepsilon_m\) is orthogonal to \(\psi^0\) (resp. \(\psi^1\)) then \(I_m = p_m\) (resp. \(q_m\)). Such a measurement element may
be said to be “all or nothing”: it brings a whole bit of information when it occurs, but does so only with probability \(p_m\) (resp. \(q_m\)). Taken individually these measurement elements seem ideal: they fully identify \(|ψ_x⟩\) and thus they let you reconstruct the initial state perfectly, with no disturbance at all. The downside is that failure to occur comes at a high price. In order to verify the condition \(\xi_m\) the other measurement elements generally become rather inefficient with respect to the tradeoff. The family of the optimizing \(\{M_m\}_m\) is not constructed in such simple ways.

Next we seek an expression of the Disturbance Contribution brought by each measurement element. For this purpose we must first assume outcome \(m\) has occurred. Eve knows it, and now she will try to maximize her chances of fooling Alice by applying a carefully tailored unitary evolution \(U_m\). First we will give \(D_m\) as a function of \(U_m\), and next proceed to the maximization which determines \(U_m\). Remember that upper indices \(x\) distinguish initial states, while lower indices \(m\) specify the measurement outcome.

\[
p(\text{fool}|m) = \sum_x p(x|m)\text{Tr}(|ψ_x⟩⟨ψ_x|U_mρ_m^xU_m^∗)
\]

\[
= \frac{1}{2} \sum_x p(x|m)\|v^x\| \|r^x_m\| \cos(\theta^x, r^x_m)
\]

Negating back to the Disturbance we obtain:

\[
D = \sum_mD_m \quad \text{with} \quad D_m = p(\text{fool}, m)
\]

\[
= \frac{p(m) - \sum_x p(x, m)\|v^x\| \|r^x_m\| \cos(\theta^x, r^x_m)}{2}
\]

In our scenario the \(|ψ_x⟩\) are pure. Thus by Lemma 2 or equation \(\xi_m\) we have \(\|v^x\| \|r^x_m\| = 1\). Now let us deal with \(\cos(\theta^x, r^x_m)\) by making the following definitions:

\[
\theta = (v^0, v^1) \quad \delta = (0, 0)
\]

\[
\theta_m = (r^0_m, r^1_m) \quad \Delta_m = \theta - \theta_m
\]

\[
\omega_m = (r^0_m + r^1_m, v^0 + v^1)
\]

\(\omega_m\) is the angle between the bisector of \((r^0_m, r^1_m)\) and that of \((v^0, v^1)\). Given that we want to minimize \(D_m\) in terms of \(U_m\) we can safely assume \(r^0_m, r^1_m, v^0, v^1\) to be coplanar. Thus \(D_m\) may now be rewritten in terms of those angles as well as \(p_m\) and \(q_m\):

\[
D_m = \frac{p_m + q_m - \cos(\Delta_m - \omega_m) - q_m \cos(\Delta_m + \omega_m)}{2}
\]

In this equation the values of \(p_m, q_m\), and \(\Delta_m\) are fully determined by \(\xi_m\), as described in \(\xi_m\). \(\omega_m\) on the other hand solely depends on \(U_m\): it can be chosen at will by rotation in the Bloch Sphere. We now show how Eve must tune \(\omega_m\) so as to minimize \(D_m\).

\[
\frac{\partial D_m}{\partial \omega_m} = 0 \Rightarrow p_m \sin(\Delta_m - \omega_m) - q_m \sin(\Delta_m + \omega_m) = 0
\]

The minimum occurs at:

\[
\omega_m = \arcsin \left( \frac{p_m - q_m}{\sqrt{p_m^2 + q_m^2 + 2p_mq_m\cos(2\Delta_m)}} \right)
\]

which yields, after simplification:

\[
D_m = \frac{p_m + q_m - \sqrt{p_m^2 + q_m^2 + 2p_mq_m\cos(2\Delta_m)}}{2}
\]

B. The Tradeoff

How many elements should Eve’s measurement contain? Levitin has proved that there exists a two-element measurement \(\{M_m\}_{m=0,1}\) which maximizes Eve’s Information Gain \(\xi_m\). While this was never formally shown to be the case for the measurements which optimize the Information Gain versus Disturbance Tradeoff, there is strong numerical evidence in support of this assumption \(\xi_m\). Suppose this is the case and let \(\hat{\xi}_m\) denote the \(\mu\)-th coordinate of \(\xi_m\). Using the constraint equation \(\xi_m\) we have:

\[
\delta\hat{\xi}_\mu = -\delta\overline{\xi}_\mu, \quad \text{and thus:}
\]

\[
\forall \mu \quad \frac{\partial f}{\partial \hat{\xi}_\mu} = -\frac{\partial f}{\partial \overline{\xi}_\mu}
\]

Optimizing the Tradeoff implies finding a stationary point for the Disturbance while keeping the Information Gain fixed. We need to find \(\xi_\mu\) such that

\[
\sum_\mu \frac{\partial D}{\partial \xi_\mu} \delta\xi_\mu = 0
\]

where the variations \(\delta\xi_\mu\) are subject to the additional constraint:

\[
\sum_\mu \frac{\partial I}{\partial \xi_\mu} \delta\xi_\mu = 0
\]
Using equation (24) and \( D = D_0 + D_1 \) and \( I = I_0 + I_1 \) this gives:

\[
\sum_{\mu} \frac{\partial D_0}{\partial \varepsilon_{0,\mu}} \delta \varepsilon_{0,\mu} = \sum_{\mu} \frac{\partial D_1}{\partial \varepsilon_{1,\mu}} \delta \varepsilon_{1,\mu}, \quad (25)
\]

subject to

\[
\sum_{\mu} \frac{\partial I_0}{\partial \varepsilon_{0,\mu}} \delta \varepsilon_{0,\mu} = \sum_{\mu} \frac{\partial I_1}{\partial \varepsilon_{1,\mu}} \delta \varepsilon_{1,\mu}, \quad (26)
\]

Guided by the geometrical picture of the scenario one may consider the following attack:

\[
\varepsilon_0 = [1 \quad \beta \quad 0 \quad 0], \quad \varepsilon_1 = [1 \quad -\beta \quad 0 \quad 0].
\]

The fact that this is indeed a solution follows from its obvious symmetries:

\[
\frac{\partial D_0}{\partial \varepsilon_{0,\mu}} = \frac{\partial D_1}{\partial \varepsilon_{1,\mu}} \quad \text{and for } \mu = 2 \quad \frac{\partial D_0}{\partial \varepsilon_{0,\mu}} = \frac{\partial D_1}{\partial \varepsilon_{1,\mu}}, \quad (27)
\]

\[
\frac{\partial I_0}{\partial \varepsilon_{0,\mu}} = \frac{\partial I_1}{\partial \varepsilon_{1,\mu}} \quad \text{and for } \mu = 2 \quad \frac{\partial I_0}{\partial \varepsilon_{0,\mu}} = \frac{\partial I_1}{\partial \varepsilon_{1,\mu}}, \quad (28)
\]

Substituting (28) in the constant Information constraint (26), we get \( \varepsilon_{0,\mu} = 0 \). Using this fact together with equation (27) it becomes clear that condition (26) is fulfilled. Thus \( \varepsilon_0 \) is a stationary point. We may now proceed to compute the values of the Disturbance and the Information Gain under this family of optimal attacks. First by making a few additional observations:

\[
p_0 = q_1 = p \quad p_1 = q_0 = q \quad D_0 = D_1 = D/2 \quad I_0 = I_1 = I/2
\]

\[
D = \frac{1}{2} - \sqrt{p^2 + q^2 + 2pq + \cos(2\Delta_m)} \quad I = 1 + 2p \log(2p) + 2q \log(2q)
\]

and second by plugging in the relations (18), (20), (23), we reproduce the exact content of Fuchs and Peres’ formulae:

\[
D = \frac{1}{2} - \frac{1}{2} \sqrt{1 + (c^2 - c^4)(\beta^2 - 2 + 2\sqrt{1 - \beta^2})} \\
I = \frac{1}{2}((1 + \beta c) \log(1 + \beta c) + (1 - \beta c) \log(1 - \beta c))
\]

V. CONCLUSION

In this paper we considered a linear embedding taking \( d \times d \) positive hermitian matrices into vectors of \( d^2 \) real entries, \( \phi : \rho \mapsto \rho = (\text{Tr}(\rho \eta_m))_m \). It is a well-known fact that the most general evolution a density matrix \( \rho \) may undergo is a generalised measurement \( \{M_m\}_m = \{U_m \sqrt{E_m} \}_m \), where the polar decomposition was applied. In order to represent \( M_m \)’s per-outcome effect upon the real vectors we defined \( \psi : A \rightarrow \phi \circ \text{Ad}_A \circ \phi^{-1} \) and showed that \( \psi(U_m) \) is a real orthogonal transform while \( \psi(\sqrt{E_m}) \) turns out to be a real positive matrix. Thus the geometrical effect of a generalized measurement can be viewed in terms of real transformations only.

Such a nice correspondence suggests quantum mechanics could be expressed elegantly over the real numbers in this manner, quite differently from its formulation in terms of real Jordan algebras [12]. However we first need to gain more geometrical intuition about the set of real vectors \( \phi(\text{Herm}^+_d(C)) \), and the sets of allowed orthogonal and positive transforms. For now we know that \( \phi(\text{Herm}^+_d(C)) \) is a subcone of the future-light-cone \( \Gamma = \{(\lambda_{\mu}) \in \mathbb{R}^{d^2} / \sum_{i=1}^{d^2} \lambda_i^2 \leq (d - 1)\lambda_0^2, \lambda_0 \geq 0 \} \).

One of the advantages of defining \( \phi \) upon \( \text{Herm}^+_d(C) \) instead of the restricted set of density matrices is that \( E_m = M_m M_m^\dagger \) can be visualized. In order to characterize its effects we derived rather compact and powerful formulae for the qubit case, such as the one giving the scalar product of the post-measurement states:

\[
\frac{1}{4}(2(E_m, \rho_0^0)(E_m, \rho_1^1) - (\eta_{\mu\nu}, E_m, E_m^\mu\nu)(\eta_{\mu\nu}, \rho_0^0, \rho_1^1))
\]

By looking at such expressions it becomes apparent that Minkowskian products have a crucial role to play in our framework, and even more so as we showed that pure quantum states correspond to light-like vectors (i.e. they sit on the boundary of \( \Gamma \)), even in dimensions greater than 2. It seems interesting to notice that BPS states of supersymmetric theories can also be thought of as lying on the boundary of a cone of positive operators [3], and that their stability is related to that fact. Somehow our setting seems to single out generalized pure states in a more natural way than merely characterizing them as unit rank elements of the boundary of \( \text{Herm}^+_d(C) \). The reminiscence of special relativity must be investigated further; this will be a subject for future work.

Pauli matrices together with special relativistic considerations have already brought some fruitful results to quantum information theory. This is the case for instance in [13], where some limits of quantum cloning are derived by using the no-signalling condition. Armed with the present representation one should be able tackle more of these difficult quantum information theoretical problems. Already in this paper we recovered Fuchs and Peres’ information gain versus disturbance formula simply and geometrically. In the future we should be able to extend our analysis to the case of two non-equiprobable states. Some highly symmetric \( n \)-states scenarios may well cease to be out of reach.
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