QUANTUM HYPERBOLIC INVARIANTS
FOR DIFFEOMORPHISMS OF SMALL SURFACES

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ABSTRACT. An earlier article [1] introduced new invariants for pseudo-Anosov diffeomorphisms of surface, based on the representation theory of the quantum Teichmüller space. We explicitly compute these quantum hyperbolic invariants in the case of the 1–puncture torus and the 4–puncture sphere.

1. INTRODUCTION

In [1], Francis Bonahon and the author constructed quantum hyperbolic invariants for pseudo-Anosov diffeomorphisms of punctured surfaces. For every odd integer $N$, these invariants associated to the pseudo-Anosov diffeomorphism $\phi : S \to S$ a square matrix $C_\phi$ of dimension $N^{3g+p-3}$ (where $g \geq 0$ is the genus of $S$ and $p > 0$ is its number of punctures), defined up to conjugation and scalar multiplication. In particular, $C_\phi$ is an $N \times N$ matrix for the 1–puncture torus and the 4–puncture sphere. The current paper provides an explicit computation for these two surfaces. It is based on the fact that the mapping class group of these surfaces is particularly simple.

For the 1–puncture torus, the isotopy class of a diffeomorphism $\phi : S \to S$ is completely determined by the matrix $A_\phi \in \text{SL}_2(\mathbb{Z})$ defined by considering the action of $\phi$ on $H_1(S) = \mathbb{Z}^2$. A diffeomorphism $\phi$ is pseudo-Anosov if and only if $|\text{Tr}(A_\phi)| > 2$. Such a matrix $A_\phi$ is conjugate to a product

$$A_\phi = A_1 A_2 \cdots A_n,$$

where $A_i = R = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ or $A_i = L = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$.

Our computation makes use of the matrices $C_R(u, v, u’, v’, h)$ and $C_L(u, v, u’, v’, h)$ defined by

$$C_R(u, v, u’, v’, h)_{ij} = \frac{1}{(1 + q^{4a-3}u)(1 + q^{4a-1}u)} \prod_{\alpha=1}^i \left( \frac{u’}{uvv’} \right)^{j-i} \left( \frac{v’}{v} \right)^i \left( \frac{u}{u’} \right)^{j-i} \left( \frac{v}{v’} \right)^i.$$

and

$$C_L(u, v, u’, v’, h)_{ij} = \frac{1}{(1 + q^{4a-3}u)(1 + q^{4a-1}u)} \prod_{\alpha=1}^k \left( \frac{u’u’}{uvv’} \right)^{j-k} \left( \frac{vv’}{v} \right)^k \left( \frac{u}{u’} \right)^{j-k} \left( \frac{v}{v’} \right)^k \prod_{\alpha=1}^k \left( \frac{1}{1 + q^{4a-3}u} \right).$$

**Theorem 1.** Let $\phi : S \to S$ be a pseudo-Anosov diffeomorphism of the 1–puncture torus $S$, and consider a matrix $A_\phi = A_1 A_2 \cdots A_n$, with $A_i = R$ or $L$, associated to $\phi$ as above. Then

$$C_\phi = C_1 C_2 \cdots C_n$$

where $C_i = C_R(u_{i-1}, v_{i-1}, u_i, v_i)$ if $A_i = R$, $C_i = C_L(u_{i-1}, v_{i-1}, u_i, v_i)$ if $A_i = L$, and where the complex numbers $u_i$, $v_i$ are explicitly defined in [7] and [8] and are determined by the complete hyperbolic metric of the mapping torus $M_\phi$ of $\phi$.

There is a similar theorem in the case of the 4–puncture sphere, using different matrix functions $C_R^*$ and $C_L^*$ defined in [8]. In this situation, the action on slopes defines a map from the mapping class group of
the 4-puncture sphere to \( \text{PSL}_2(\mathbb{Z}) \) with kernel \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). The image of a diffeomorphism \( \varphi \) can be represented by a matrix \( A_\varphi \in \text{SL}_2(\mathbb{Z}) \). Again, \( \varphi \) is pseudo-Anosov if and only if \( |\text{Tr}(A_\varphi)| > 2 \).

**Theorem 2.** Let \( \varphi : S \to S \) be a pseudo-Anosov diffeomorphism of the 4–puncture sphere \( S \), and consider a matrix \( A_\varphi = A_1A_2\cdots A_n \), with \( A_i = R \) or \( L \), associated to \( \varphi \) as above. Then

\[
C_\varphi = C_1^*C_2^*\cdots C_n^*
\]

where \( C_i^* = C_i^R(u_{i-1}, v_{i-1}, u_i, v_i, 1) \) if \( A_i = R \), \( C_i^* = C_i^L(u_{i-1}, v_{i-1}, u_i, v_i, 1) \) if \( A_i = L \), and where the complex numbers \( u_i, v_i \) are explicitly defined in \( \square \) and are determined by the complete hyperbolic metric of the mapping torus \( M_\varphi \) of \( \varphi \).

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### 2. Quantum hyperbolic invariants of surface diffeomorphisms

We briefly sketch the construction in \( \square \) of quantum hyperbolic invariants for pseudo-Anosov diffeomorphisms of a punctured surface. Details will be provided in later sections in the specific case of the 1–puncture torus and the 4–puncture sphere.

Let \( S \) be a punctured surface, that is, a closed surface with finitely many points removed. The Teichmüller space of \( S \), denoted by \( \mathcal{T}_S \), is the space of isotopy classes of complete hyperbolic metrics on \( S \). An ideal triangulation \( \lambda \) provides a certain set of global coordinates for \( \mathcal{T}_S \), called the exponential shear coordinates (these essentially are cross-ratios), which parameterize this space by a convex cell \( \mathbb{R}_+^n \) (more accurately they parametrize a finite-sheeted branched covering of \( \mathcal{T}_S \) called the enhanced Teichmüller space).

The **quantum Teichmüller space** of \( S \) is a noncommutative deformation of the algebra of rational functions on \( \mathcal{T}_S \), with a deformation parameter \( q \in \mathbb{C} \); see \([2, 4, 11, 15]\). The construction is based on a non-commutative analog of the shear coordinate parametrization for the quantum Teichmüller space, the **Chekhov-Fock algebra** \( \mathcal{T}_{\lambda}^q \) associated to an ideal triangulation \( \lambda \) of \( S \).

When we change the ideal triangulation \( \lambda \) to another one \( \lambda' \), the exponential shear coordinates transform rationally. There is a non-commutative procedure that mimics this transformation rule of cross-ratios: it is nothing but a deformation of the algebra of rational functions on \( \mathcal{T}_S \), namely such that \( \mathcal{T}_{\lambda}^q \to \mathcal{T}_{\lambda'}^q \) sends each face of \( \mathcal{T}_{\lambda}^q \) associated to the edges \( e \) of the ideal triangulation \( \lambda \) to \( \mathcal{T}_{\lambda'}^q \) associated to another one \( \lambda' \).

The Chekhov-Fock algebra \( \mathcal{T}_{\lambda}^q \) admits finite dimensional representations only when \( q \) is a root of unity. Assume that \( q \) is a primitive \( N \)-root of unity with \( N \) odd. Any finite dimensional irreducible representation of this algebra has dimension \( N^{3g+p-3} \) (where \( g \geq 0 \) is the genus of \( S \) and \( p \geq 0 \) is its number of punctures) and, up to finitely many choices, is determined by complex weights \( x_e \in \mathbb{C} - \{0\} \) associated to the edges \( e \) of the ideal triangulation \( \lambda \).

It turns out that this representation theory of the Chekhov-Fock algebras \( \mathcal{T}_{\lambda}^q \) is well-behaved with respect to the coordinate change isomorphisms \( \Phi_{\lambda, \lambda'}^{q} \). Namely, suppose that the irreducible representation \( \rho : \mathcal{T}_{\lambda}^q \to \text{End}(V) \) is associated to weights \( x_e \), as \( e \) ranges over all edges of \( \lambda \), and that these weights are in the domain of the (complexification of the) crossratio transformation rule between the exponential shear coordinates associated to \( \lambda \) and those associated to \( \lambda' \). Then it is possible to make sense of the irreducible representation \( \rho' = \rho \circ \Phi_{\lambda, \lambda'}^{q} : \mathcal{T}_{\lambda'}^q \to \text{End}(V) \), and \( \rho' \) is classified by the edge weights on \( \lambda' \) that are the image of the weights \( x_e \) under the cross-ratio transformation rule.

Now, consider a pseudo-Anosov diffeomorphism \( \varphi : S \to S \). Thurston’s Hyperbolization Theorem and Mostow’s Rigidity Theorem provide a unique finite-volume complete hyperbolic metric on the mapping torus \( M_\varphi = S \times [0,1]/\sim \), where \( \sim \) identifies each \((x,0)\) with \((\varphi(x),1)\). An arbitrary ideal triangulation \( \lambda \) determines a unique pleated surface \( f_\lambda : S \to M_\varphi \) with pleating locus \( \lambda \), namely such that \( f \) sends each face of \( \lambda \) to a totally geodesic triangle in \( M_\varphi \). This pleated surface defines a complex weight \( \varphi_\lambda \in \mathcal{T}_{\lambda}^q \) associated to the edges \( e \) of \( \lambda \).

Using the correspondence between edge weights and representations of the Chekhov-Fock algebra, this gives a representation \( \rho : \mathcal{T}_{\lambda}^q \to \text{End}(V) \) such that \( \rho' = \rho \circ \Phi_{\lambda, \lambda'}^{q} \) is isomorphic to \( \rho \circ \Phi_{\lambda} \), using the identification.
\[ \Phi : T^q_{\lambda} \cong T^q_{\lambda'} \] provided by \( \varphi \). In other words, there exists an isomorphism \( C_\varphi \in \text{GL}(V) \) such that
\[
\left( \rho \circ \Phi_{\lambda'}^q(X) \right) = C_\varphi \cdot \left( \rho \circ \Phi(X) \right) \cdot C_\varphi^{-1}
\]
for any \( X \in T^q_{\lambda'} \).

**Theorem 3** (Theorem 40 in [1]). The isomorphism \( C_\varphi \in \text{GL}(V) \) defined above depends only on \( q \) and \( \varphi \), up to conjugation and scalar multiplication.

This article is devoted to computing the invariant \( C_\varphi \) when the surface \( S \) is the 1–puncture torus or the 4-puncture sphere. In these cases, the irreducible representations of the Chekhov-Fock algebra have dimension \( N \), so that \( C_\varphi \) can be considered as an \( N \times N \) matrix defined up to conjugation and scalar multiplication.

### 3. Quantum Teichmüller space of the 1–puncture torus

Following the construction of the quantum enhanced Teichmüller space [5], fix an ideal triangulation \( \lambda \) of the 1-puncture torus \( T \), namely a triangulation of the unpunctured torus \( \overline{T} \) with a single vertex, located at the puncture. The Chekhov-Fock algebra \( T^q_{\lambda} \) is defined by generators \( X_1^\pm, X_2^\pm, \) and \( X_3^\pm \), respectively associated to the edges \( \lambda_1, \lambda_2, \lambda_3 \) of \( \lambda \), and by relations determined by the topology of \( \lambda \) as follows:
\[
X_i X_j = q^{2\sigma_{ij}} X_j X_i
\]
where \( \sigma_{ij} \in \{-2, -1, 0, 1, 2\} \) is the number of times one sees \( \lambda_i \) minus the number of times one sees \( \lambda_j \). In particular, the \( \sigma_{ij} \) are antisymmetric in the subscripts. When the \( \lambda_i \) occur counterclockwise around the faces of \( \lambda \) as in Figure 1, then \( \sigma_{ij} = -2 \) whenever \( j = i + 1 \mod 3 \).

![Figure 1. An ideal triangulation \( \lambda \). In this ideal triangulation \( \sigma_{12} = \sigma_{23} = \sigma_{31} = -2 \).](image)

The **diagonal exchange** \( \Delta_3 \) takes the ideal triangulation \( \lambda \) to a new ideal triangulation \( \lambda' \) obtained by replacing \( \lambda_3 \) by the other diagonal \( \lambda'_3 \) of the square formed by \( \lambda_1 \) and \( \lambda_2 \), as Figure 2. If \( \tilde{T}^q_{\lambda} \) denotes the fraction division algebra of \( T^q_{\lambda} \), consisting of rational functions in the skew-commuting variable \( X_1, X_2, X_3 \), the coordinate change isomorphism \( \Phi^q_{\lambda, \lambda'} : \tilde{T}^q_{\lambda'} \rightarrow \tilde{T}^q_{\lambda} \) introduced in [5] is such that
\[
\begin{align*}
\Phi^q_{\lambda, \lambda'}(X_1') &= (1 + qX_3)(1 + q^3X_3)X_1, \\
\Phi^q_{\lambda, \lambda'}(X_2') &= (1 + qX_3^{-1})^{-1}(1 + q^3X_3^{-1})^{-1}X_2, \\
\Phi^q_{\lambda, \lambda'}(X_3') &= X_3^{-1}.
\end{align*}
\]
in the case of Figure 2 namely if \( \sigma_{ij} = -2 \) when \( j = i + 1 \mod 3 \).

![Figure 2. The diagonal exchange \( \Delta_3 \) on edge \( \lambda_3 \).](image)

Now suppose that \( q \) is a primitive \( N \)-th root of unity with \( N \) odd. The center of \( T^q_{\lambda} \) is generated by \( X_1^N, X_2^N, \) and \( X_1X_2X_3 \).
Theorem 4 (A special case of Theorem 21 in [1]). Let \( q \) be a primitive \( N \)-th root of unity with \( N \) odd. Any irreducible representation \( \rho \) of the Chekhov-Fock algebra is of dimension \( N \), and the conjugacy class of \( \rho \) is determined by its restriction to the central elements \( X_1^N, X_2^N \) and \( H = q^{-(\sigma_{12} + \sigma_{13} + \sigma_{23})}X_1X_2X_3 \).

The coefficient introduced in the definition \( H = q^{-(\sigma_{12} + \sigma_{13} + \sigma_{23})}X_1X_2X_3 \) is designed to make \( H \) independent of the order of the \( \lambda_i \). This will also guarantee that \( H \) is invariant under coordinate change isomorphisms of the \( \rho \)-module. Let \( \Phi_{\lambda,\nu} \) be the algebra defined by generators \( \{X_1^N, X_2^N, X_3^N, X_1^{N+1}, \ldots, X_3^{N+1}\} \) and relations \( X_i^{N+1} = qX_i^N \). It is isomorphic to the Chekhov-Fock algebra \( T_\lambda^q \) for any ideal triangulation \( \lambda \) of the 1-puncture torus. The central element is \( H = q^{2UW}VW \). This will not cause any trouble because we are going to deal with automorphisms of \( \mathbb{C} \) instead of isomorphisms \( \Phi_{\lambda,\nu} \) between different algebras.

Lemma 5 (Lemma 27 in [1]). Let \( q \) be a primitive \( N \)-th root of unity with \( N \) odd, and let \( \rho : T_\lambda^q \to \text{End}(V) \) be an irreducible representation with
\[
\rho(X_1^N) = x_1 \text{id}_V, \quad \rho(X_2^N) = x_2 \text{id}_V, \quad \rho(X_3^N) = x_3 \text{id}_V, \quad \rho(H) = h \text{id}_V
\]
then \( \rho' := \rho \circ \Phi_{\lambda,\nu}^q : T_\lambda^q \to \text{End}(V) \) is an irreducible representation with
\[
\rho'(X_1^N) = x_1' \text{id}_V, \quad \rho'(X_2^N) = x_2' \text{id}_V, \quad \rho'(X_3^N) = x_3' \text{id}_V, \quad \rho'(H') = h' \text{id}_V
\]
where \((x_1', x_2', x_3') = \Phi_{\lambda,\nu}^{-1}(x_1, x_2, x_3)\) and \(h' = h\).

Note: Since \( \Phi_{\lambda,\nu}^q(T_\lambda^q) \not\subseteq T_\lambda^q \), one needs Lemma 25 and Lemma 26 in [1] to rigorously make sense of \( \rho \circ \Phi_{\lambda,\nu}^q \) as a representation of \( T_\lambda^q \). However, this turns out to coincide with our intuitive understanding.

4. Algebraic Lemmas

Given a primitive \( N \)-th root of unity with \( N \) odd, let \( W_q \) be the algebra defined by generators \( U^{\pm 1}, V^{\pm 1}, W^{\pm 1} \) and relations \( UV = qVU, VW = qVW, UW = qWU \). It is isomorphic to the Chekhov-Fock algebra \( T_\lambda^q \) for any ideal triangulation \( \lambda \) of the 1-puncture torus. The central element is \( H = q^{2UW}VW \). This will not cause any trouble because we are going to deal with automorphisms of \( W \) instead of isomorphisms \( \Phi_{\lambda,\nu} \) between different algebras.

Lemma 6 (See for instance [1 Sect. 4]). Every finite-dimensional irreducible representation of \( W_q \) is conjugate to \( \chi_{u,v,h} : W_q \to \text{End}(\mathbb{C}^N) \), defined by
\[
\chi_{u,v,h}(U) = u \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & q^4 & 0 & \cdots & 0 \\
0 & 0 & q^8 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q^{4(N-1)}
\end{pmatrix}, \quad \chi_{u,v,h}(V) = v \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]
and
\[
\chi_{u,v,h}(W) = \frac{q^{-2h}}{uv} \begin{pmatrix}
0 & 0 & 0 & \cdots & q^{-4(N-1)} \\
1 & 0 & 0 & \cdots & 0 \\
0 & q^{-4} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & q^{-4(N-2)} & 0
\end{pmatrix}.
\]
Note that in this case \( \chi_{u,v,h}(H) = h \text{id}_{\mathbb{C}^N} \).
Let $\hat{\mathcal{W}}_q$ be the fraction division algebra of $\mathcal{W}_q$. Consider the automorphism $\mathcal{R} : \hat{\mathcal{W}}_q \rightarrow \hat{\mathcal{W}}_q$ defined by the property that

$$
\mathcal{R}(U) = (1 + qU^{-1})^{-1}(1 + q^3U^{-1})^{-1}W
$$

(4.1)

$$
\mathcal{R}(V) = (1 + qU)(1 + q^3U)V
$$

$$
\mathcal{R}(W) = U^{-1}.
$$

One easily see that $\mathcal{R}$ does define an algebra isomorphism, namely that $\mathcal{R}(U)$, $\mathcal{R}(V)$ and $\mathcal{R}(W)$ satisfy the same relations as $U$, $V$ and $W$ (compare Proposition 5 in [5]), and that $\mathcal{R}$ is invertible.

**Lemma 7.** If $\chi_{u,v,h}$ is a standard representation as in Lemma 4 then

$$
\chi_{u,v,h} \circ \mathcal{R}(X) = C_R \cdot \chi_{u',v',h'}(X) \cdot C_R^{-1}, \quad \forall X \in \mathcal{W}_q,
$$

where

$$
(u')^N = \frac{h^N}{u^Nv^N(1 + u^{-N})^2}, \quad (v')^N = (1 + u^N)^2v^N, \quad h' = h
$$

and where the matrix $C_R(u,v,u',v',h) \in \text{GL}_N(\mathbb{C})$ is defined by its entries

$$
(C_R)_{ij} = q^{2(j-1)^2} \left( \frac{u'v'}{uh} \right)^{j-i} \left( \frac{v'}{v} \right)^i \prod_{\alpha=1}^{i} \frac{1}{(1 + q^{4\alpha-3}u)(1 + q^{4\alpha-1}u)}.
$$

**Proof.** Lemma 4 shows that $\chi_{u,v,h} \circ \mathcal{R}$ is isomorphic to a standard representation $\chi_{u',v',h'}$ with $(u')^N = \frac{h^N}{u^Nv^N(1 + u^{-N})^2}$, $(v')^N = (1 + u^N)^2v^N$ and $h' = h$. To compute the conjugation matrix $C_R$, the last two equalities of (4.1) provide

$$
(C_R)_{ij} = \frac{v'}{v} \frac{1}{(1 + q^{4(i-1)}u)(1 + q^{4i-1}u)} (C_R)_{i-1,j-1}
$$

$$
(C_R)_{ij} = q^{-4i}q^{4(j-1)} \frac{u'v'}{uq^{-2h}} (C_R)_{i,j-1}
$$

and these inductive relations immediately give the entries of $C_R$. \hfill \square

Similarly, let the isomorphism $\mathcal{L} : \hat{\mathcal{W}}_q \rightarrow \hat{\mathcal{W}}_q$ be defined by

$$
\mathcal{L}(U) = (1 + qV^{-1})^{-1}(1 + q^3V^{-1})^{-1}U
$$

(4.2)

$$
\mathcal{L}(V) = (1 + qV)(1 + q^3V)W
$$

$$
\mathcal{L}(W) = V^{-1},
$$

**Lemma 8.** If $\chi_{u,v,h}$ is a standard representation as in Lemma 4 then

$$
\chi_{u,v,h} \circ \mathcal{L}(X) = C_L \cdot \chi_{u'',v'',h''}(X) \cdot C_L^{-1}, \quad \forall X \in \mathcal{W}_q,
$$

where

$$
(u'')^N = \frac{u^N}{(1 + v^{-N})^2}, \quad (v'')^N = \frac{h^N(1 + v^N)^2}{u^Nv^N}, \quad h'' = h
$$

and where $C_L = G\hat{C}_L$ is the product of matrices $G$, $\hat{C}_L(u,v,u',v'',h) \in \text{GL}_N(\mathbb{C})$ with entries $G_{ij} = q^{4ij}$ and

$$
(\hat{C}_L)_{ij} = q^{2(j-1)^2 + 2i^2} \left( \frac{u''v''}{vh} \right)^{j-i} \left( \frac{u''v''}{v} \right)^i \prod_{\alpha=1}^{i} \frac{1}{(1 + q^{4\alpha-3}v)(1 + q^{4\alpha-1}v)}.
$$

**Proof.** The proof is similar to the proof of the previous lemma, except that $\chi_{u,v,h}(V)$ is not diagonal. This inconvenience is bypassed in the following way. Define another irreducible representation $\mu_{u,v,h}$ by

$$
\mu_{u,v,h}(U) = u \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \mu_{u,v,h}(V) = v \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q^4 & 0 & \cdots & 0 \\ 0 & 0 & q^8 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^{4(N-1)} \end{pmatrix},
$$
\[ \mu_{u,v,h}(W) = \frac{q^{-2}h}{uv} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & q^{-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^{-4(N-2)} \\ q^{-4(N-1)} & 0 & 0 & \cdots & 0 \end{pmatrix}. \]

One easily computes that \( G \cdot \mu_{u,v,h}(X) \cdot G^{-1} = \chi_{u,v,h}(X) \), with \( G_{ij} = q^{4ij} \). Let \( \tilde{C}_L \) be the matrix such that \( \mu_{u,v,h} \circ \mathcal{L}(X) = \tilde{C}_L \cdot \chi_{u',v',h'}(X) \cdot \tilde{C}_L^{-1} \). Obviously \( C_L = G\tilde{C}_L \). As in the previous lemma, we obtain the inductive equations
\[
(\tilde{C}_L)_{ij} = q^{4(i-1)} \frac{\left(u'v''\right)}{q^{2h}} \frac{1}{(1 + q^{4i-3}v')(1 + q^{4i-1}v)} \tilde{C}_{L,i-1,j-1}^i \quad (\tilde{C}_L)_{ij} = q^{4(j-1)} \frac{u''v''}{vq^{2h}} (\tilde{C}_L)_{i,j-1}^j
\]
which provide the entries of \( \tilde{C}_L \).

\[ \square \]

**Remark 9.** Note that, although the quantities \( u, v, h, u', v' \), involved in the definition of the matrix \( C_R \) are related by equations involving only their \( N \)-th powers, the matrix itself depends on the choice of these \( N \)-roots. The same holds for \( C_L \). In the next section, the \( N \)-powers will be determined by geometric information but the actual computation will depend on actual choices of \( N \)-roots for this geometric data.

### 5. The invariant in the case of the 1-punctured torus

We now compute the invariant \( C_\varphi \) of Theorem\[3\] in the case of the 1-puncture torus \( T \). The computation is here greatly simplified by the fact that the mapping class group of \( T \) is isomorphic to \( \text{SL}_2(\mathbb{Z}) \). In addition, recall that a mapping class is pseudo-Anosov exactly when the corresponding element of \( \text{SL}_2(\mathbb{Z}) \) has a trace of absolute value greater than 2.

**Proposition 10.** Let \( A \) be an element of \( \text{SL}_2(\mathbb{Z}) \) with \( |\text{Tr}(A)| > 2 \). Then the conjugacy class of \( A \) in \( \text{SL}_2(\mathbb{Z}) \) contains an element of the form

\[
\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_n & 1 \end{pmatrix}
\]

where \( n > 0 \) and the \( a_i, b_i \) are positive integers. Moreover, the right hand side is unique up to cyclic permutation of the factors \( \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ b_i & 1 \end{pmatrix} \).

\[ \square \]

In other words, after conjugation, every mapping class of \( T \) can be represented by a matrix \( A = A_1 A_2 \cdots A_n \) with \( A_i = R \) or \( L \), where \( L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \).

The identification between \( H_1(T) \cong \mathbb{Z}^2 \) assigns a slope in \( \mathbb{Q} \cup \{\infty\} \) to each edge of an ideal triangulation \( \lambda \) of \( T \). Consider the ideal triangulation \( \lambda_{(0)} \) whose edges have respective slopes 0, \( \infty \) and 1. Then define ideal triangulations \( \lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(n)} = \varphi(\lambda_{(0)}) \) by the property that \( \lambda_{(i)} = A_1 A_2 \ldots A_i(\lambda_{(0)}) \).

It is immediate that the edges \( \lambda_{(i),1}, \lambda_{(i),2} \) and \( \lambda_{(i),3} \) of \( \lambda_{(i)} \) all have non-negative slope. Choose the indexing of these edges so that \( \lambda_{(i),1} \) has the lowest slope and \( \lambda_{(i),2} \) has the highest slope of the three. Note that with this convention the edges \( \lambda_{(i),1}, \lambda_{(i),2} \) and \( \lambda_{(i),3} \) occur counterclockwise in this order around the two triangles of \( \lambda_{(i)} \).

The key observation is the following:

- if \( A_i = R \), then \( \lambda_{(i)} \) is obtained from \( \lambda_{(i-1),1} \) by a diagonal exchange along the edge \( \lambda_{(i-1),1} \), followed by a reindexing exchanging \( \lambda_{(i),1} \) and \( \lambda_{(i),3} \);
- if \( A_i = L \), then \( \lambda_{(i)} \) is obtained from \( \lambda_{(i-1)} \) by a diagonal exchange along the edge \( \lambda_{(i-1),2} \), followed by a reindexing exchanging \( \lambda_{(i),2} \) and \( \lambda_{(i),3} \).

This property is illustrated in Figure\[4\] for the case where \( i = 1 \), and the other cases follow from this one by observing that

\[ \lambda_{(i)} = (A_1 A_2 \cdots A_{i-1})A_i(A_1 A_2 \cdots A_{i-1})^{-1}(\lambda_{(i-1)}). \]

Incidentally, this accounts for the somewhat unnatural order of the \( A_k \) in the definition of \( \lambda_{(i)} \).
For the complete hyperbolic metric on the mapping torus $M_\varphi = S \times [0, 1]/\sim$ with its complete hyperbolic metric, each ideal triangulation $\lambda_{(i)}$ determines a unique pleated surface $f_{\lambda_{(i)}} : T \to M_\varphi$ with pleating locus $\lambda_{(i)}$. The geometry of this pleated surface associates complex weights $x_{(i)1}, x_{(i)2}, x_{(i)3} \in \mathbb{C} - \{0\}$ to the edges of $\lambda_{(i)}$, corresponding to the exponential shear-bend coordinates of $f_{\lambda_{(i)}}$ along the components of its pleating locus.

**Remark 11.** In [13], Guéritaud proves the remarkable fact that the pleated surfaces $f_{\lambda_{(i)}}$ are embeddings. In addition, one passes from $f_{\lambda_{(i-1)}}$ to $f_{\lambda_{(i)}}$ by a diagonal exchange across a positively oriented hyperbolic ideal tetrahedra with dihedral angles strictly between 0 and $\pi$.

**Theorem 12.** Given any pseudo-Anosov diffeomorphism $\varphi : T \to T$, if $\varphi$ has the canonical decomposition $A_1 A_2 \cdots A_n$ with $A_i = R$ or $L$ as in Proposition [17], then the conjugacy class of $C_\varphi$ has a matrix representative

$$C_\varphi = C_1 C_2 \cdots C_n$$

where

- if $A_i = R$, then $C_i = C_R(u_{i-1}, v_{i-1}, u_i, v_i, 1)$, defined in Lemma [7] with

  $$(u_i)^N = \frac{1}{u_{i-1}^N v_{i-1}^N (1 + u_{i-1}^N)^2}, \quad (v_i)^N = (1 + u_{i-1}^N)^2 v_{i-1}^N,$$

- if $A_i = L$, then $C_i = C_L(u_{i-1}, v_{i-1}, u_i, v_i, 1)$, defined in Lemma [5] with

  $$(u_i)^N = \frac{u_{i-1}^N}{(1 + v_{i-1}^N)^2}, \quad (v_i)^N = (1 + v_{i-1}^N)^2 u_{i-1}^N,$$

and, the initial values $u_0$, $v_0$, $h$ are chosen so that

$$u_n = u_0, \quad v_n = v_0, \quad h = 1.$$

**Proof.** The action of the word $A_1 A_2 \cdots A_n$ produces a sequence of ideal triangulations

$$\lambda = \lambda_{(0)} \xrightarrow{A_1} \lambda_{(1)} \xrightarrow{A_2} \lambda_{(2)} \cdots \xrightarrow{A_n} \lambda_{(n)} = \varphi(\lambda),$$

which in turn gives a composition of isomorphisms

$$\widehat{T}_\varphi(\lambda) \xrightarrow{\Phi^q_{\lambda_{(n-1)}}(\lambda_{(n)})} \widehat{T}_\lambda(\lambda_{(n-1)}) \xrightarrow{\cdots} \widehat{T}_\lambda(\lambda_{(1)}) \xrightarrow{\Phi^q_{\lambda_{(0)}}(\lambda_{(1)})} \widehat{T}_\lambda(\lambda).$$

By sending $X_{\lambda_{(i)1}}, X_{\lambda_{(i)2}}, X_{\lambda_{(i)3}}$ respectively to $U, V, W$ we identify each $\widehat{T}_\lambda(\lambda)$ to the algebra $\widehat{W}_q$ of [4].

Then we get an automorphism $A_1 A_2 \cdots A_n : \widehat{W}_q \to \widehat{W}_q$ with $A_i \in \{R, L\}$. This automorphism is conjugate to the automorphism $\Phi^{-1} \circ \Phi^q_{\lambda(\varphi)} : \widehat{T}_\varphi(\lambda) \to \widehat{T}_\lambda(\lambda)$, $(\Phi : \widehat{T}_\varphi(\lambda) \to \widehat{T}_\lambda(\lambda)$ is the natural identification). If we

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**Figure 3.**

![Figure 3](image-url)
choose a standard representation $\chi_{u,v,h}$, then Lemma 7 and Lemma 8 tell us that for any $i$, $1 \leq i \leq n$, we have

$$\chi_{u,v,h} \circ A_1A_2\cdots A_i(X) = C_i(u_0, v_0, u_1, v_1, h) \cdot \left( \chi_{u_1,v_1,h} \circ A_2\cdots A_i(X) \right) \cdot C_i(u_0, v_0, u_1, v_1, h)^{-1} = \cdots = (C_1\cdots C_i)\left( \chi_{u_k,v_k,h}(X) \right) (C_1\cdots C_i)^{-1}$$

for all $X \in W_q$. In particular, the hyperbolic metric of the mapping torus $M_\varphi$ provides an interesting initial standard representation $\chi_{u_0,v_0,h_0}$ such that $u_n^N = u_0^N$, $v_n^N = v_0^N$ (we have interpreted $u_i^N$, $v_i^N$ to be the shear-bend coordinates $x(i)_1$, $x(i)_2$). We choose the standard representations from their conjugacy classes so that $u_n = u_0$, $v_n = v_0$, and we choose $h = 1$ because it is natural to work on the “cusped quantum Teichmüller space” (see [1, 5]) in which we should put the “cusp condition” $H^2 = 1$, and a geometric consideration fixes $H = 1$. The above equation applied to $i = n$ gives

$$\chi_{u_0,v_0,h_0} \circ A_1A_2\cdots A_n(X) = (C_1\cdots C_n)\left( \chi_{u_0,v_0,h_0}(X) \right) (C_1\cdots C_n)^{-1}$$

Comparing with equation 2.1, the product of matrices $C_1\cdots C_n$ turns out to be a representative of the invariant $C_\varphi$.

6. The Invariant in the Case of the 4-Puncture Sphere

We identify the 4-puncture sphere to be the quotient $S \cong (\mathbb{R}^2 - \mathbb{Z}^2)/G$ where $G$ is the group generated by four elements $(x, y) \mapsto (-x, y)$, $(x, y) \mapsto (x + 2, y)$, $(x, y) \mapsto (x, -y)$ and $(x, y) \mapsto (x, y + 2)$. The mapping class group is the semi-directed product of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The image of a diffeomorphism $\varphi$ can be represented by a matrix $A_\varphi \in \text{SL}_2(\mathbb{Z})$.

![Figure 4. An ideal triangulation $\lambda$ of $S$.](image)

We will restrict attention to ideal triangulation of $S$ which are isomorphic to the 2-skeleton of an ideal tetrahedron, as in the figure above. This is equivalent to the property that each puncture is adjacent to exactly 3 edges.

If $\varphi : S \to S$ is associated to the matrix $A_\varphi \in \text{SL}_2(\mathbb{Z})$, then $\varphi$ is pseudo-Anosov exactly when $|\text{Tr} \varphi| > 2$. Similar to the 1-puncture torus, such a matrix has an $LR$ decomposition. The difference now is that, to stay within tetrahedral ideal triangulations, the $L$ and $R$ are now associated to the product of two diagonal exchanges along opposite edges. More precisely, with the indexing convention of the figure above, $L$ corresponds to $\Delta_2\Delta_4$ and $R$ corresponds to $\Delta_1\Delta_3$. By [5], the Chekhov-Fock algebra $T^\lambda_\alpha$ ($\lambda$ as in the figure) is generated by $X_1$, $X_2$ and the central elements

$$P_1 = qX_1X_2X_5, \quad P_2 = q^{-1}X_2X_3X_6, \quad P_3 = qX_3X_4X_5, \quad P_4 = qX_1X_4X_6,$$

$$H = q^2X_1X_2X_3X_4X_5X_6.$$ 

There exist automorphisms $\mathcal{R}$ and $\mathcal{L}$ of the Chekhov-Fock algebra $T^\lambda_\alpha$ which fix the central elements and map $X_1$, $X_2$ as follows,

$$\mathcal{R}(X_1) = (1 + qX_1^{-1})^{-1}(1 + qX_3^{-1})^{-1}X_6,$$

$$\mathcal{R}(X_2) = (1 + qX_1)(1 + qX_3)X_2.$$
and
\[ \mathcal{L}(X_1) = (1 + qX_2^{-1})^{-1}(1 + qX_1^{-1})^{-1}X_1, \]
\[ \mathcal{L}(X_2) = (1 + qX_2)(1 + qX_4)X_5. \]

If we use the standard representation
\[
\chi_{u,v,h,p}(X_1) = u \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q^2 & 0 & \cdots & 0 \\ 0 & 0 & q^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^{2(N-1)} \end{pmatrix}, \quad \chi_{u,v,h,p}(X_2) = v \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},
\]
and
\[ \chi_{u,v,h,p}(H) = h \text{id}_{C^N}, \quad \chi_{u,v,h,p}(P_j) = p_j \text{id}_{C^N}, \quad j = 1, 2, 3, 4, \]
then we have similar algebraic Lemmas for the 4-puncture sphere, with the new matrix functions,

\[
(C_R^*)_{ij} = q^{(i-j)(i-j-2)} \left( \frac{u'}{v} \right)^{j-i} (uvp^3 u')^{j} \prod_{\alpha=1}^{i-j} \frac{1}{(1 + q^{1-2\alpha} u^{-1}) \left(1 + \frac{q^{1-2\alpha} h}{up^2 p_3} \right)} \times 
\prod_{\beta=i-j+1}^{i} \frac{1}{(1 + q^{2\beta-1} u)} \left(1 + \frac{q^{2\beta+1} up^2 p_3}{h} \right)
\]

\[
(C_L^*)_{ij} = q^{2j} \left( \frac{u'}{u} \right)^{j-i} \left( \frac{uv'}{p_1} \right)^j \prod_{\alpha=1}^{i-j} \frac{1}{(1 + q^{1-2\alpha} v^{-1}) \left(1 + \frac{q^{1-2\alpha} h}{vp^3 p_4} \right)} \times 
\prod_{\beta=i-j+1}^{i} \frac{1}{(1 + q^{2\beta-1} v)} \left(1 + \frac{q^{2\beta+1} vp^3 p_4}{h} \right)
\]

The same arguments as in [?5] then give:

**Theorem 13.** Given any pseudo-Anosov diffeomorphism \( \varphi : S \to S \), if \( \varphi \) has the canonical decomposition \( A_1 A_2 \cdots A_n \) with \( A_i = R \) or \( L \) as in Proposition 13, then the conjugacy class of \( C_\varphi \) has a matrix representative
\[ C_\varphi^* = C_1^* C_2^* \cdots C_n^* \]

where

- if \( A_i = R \), then \( C_i^* = C_R^*(u_{i-1}, v_{i-1}, u_i, v_i) \), defined above, with
  \[ (u_i)^N = -\frac{1}{u_{i-1}^N v_{i-1}^N (1 + u_{i-1}^{-N})^2}, \quad (v_i)^N = (1 + u_{i-1}^{-N})^2 v_{i-1}^N, \]

- if \( A_i = L \), then \( C_i^* = C_L^*(u_{i-1}, v_{i-1}, u_i, v_i, 1) = G^* \cdot \tilde{C}_L^*(u_{i-1}, v_{i-1}, u_i, v_i, 1) \), defined above, with
  \[ (u_i)^N = \frac{u_{i-1}^N}{(1 + v_{i-1}^{-N})^2}, \quad (v_i)^N = -\frac{(1 + v_{i-1}^{-N})^2}{u_{i-1}^N v_{i-1}^N}, \]

and, the initial values \( u_0, v_0, h, p \) are chosen so that
\[ u_n = u_0, \quad v_n = v_0, \quad h = 1, \quad p_j = 1, \quad \forall \quad j. \]

\[ \square \]
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