This paper is devoted to a study of relativistic eigenstates of Dirac particles which are simultaneously bound by a static Coulomb potential and added linear confining potentials. It has recently been shown that, despite the addition of radially symmetric, linear confining potentials, some specific bound-state energies surprisingly retain their exact Dirac–Coulomb values (in the sense of an “exact symmetry”). This observation raises pertinent questions as to the generality of the cancellation mechanism. A Foldy–Wouthuysen transformation is used to find the relevant nonrelativistic physical degrees of freedom, which include additional spin-orbit couplings induced by the linear confining potentials. The matrix elements of the effective operators obtained from the scalar, and time-like confining potentials mutually cancel for specific ratios of the prefactors of the effective operators, which must be tailored to the cancellation mechanism. The result of the Foldy–Wouthuysen transformation is used to explicitly show that the cancellation is accidental and restricted (for a given Hamiltonian) to only one reference state, rather than traceable to a more general relationship among the obtained effective low-energy operators. Furthermore, we show that the cancellation mechanism does not affect anti-particle (negative-energy) states.

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I. INTRODUCTION

The Dirac-Coulomb Hamiltonian is one of the most paradigmatic bound-state Hamiltonians ever investigated [1, 4]. Its bound states form a discrete spectrum. The investigation of the spectrum of the Dirac-Coulomb Hamiltonian has led to the discovery of the fine-structure of atoms as well as the identification of the relativistic correction terms that shift the hydrogen spectrum: zitterbewegung term, and spin-orbit coupling [5, 6]. Naturally, one would assume that any added linear confining potentials (proportional to the radial coordinate) would not only shift the individual energy levels quantitatively, but even change the spectrum qualitatively, eliminating the continuous spectrum. Quite recently, in marked contrast to this expectation, Franklin and Soares de Castro [7, 8] have investigated a generalized Dirac Hamiltonian with an added linear confining potential, or, more precisely, a combination of two added linear confining potentials, and the surprising conclusion was that specific bound-state energies are not shifted at all with respect to their Dirac-Coulomb value, i.e., the added linear confining potential has no influence on the energy eigenvalue.

In Refs. [7, 8], the unaffected energies are those of the states with $\kappa = -n$ where $n$ is the principal quantum number and $\kappa$ is the Dirac angular quantum number. The papers [7, 8] leave some room for the interpretation of the generality of the cancellation mechanism. One might ask whether, given a specific ratio of the prefactors in the confining Hamiltonian, the cancellation would affect only one reference state, whose energy remains unaltered, or if the parameters could be tailored so that the cancellation occurs for more than one reference state. An exact cancellation for a general class of potentials could otherwise hint at a hidden, hitherto insufficiently discussed symmetry of the (generalized) Dirac equation. Our aim here is to analyze the problem in detail and to identify the leading relativistic corrections (effective operators) for the problem studied in Refs. [7, 8], via a Foldy–Wouthuysen transformation. Expressed differently, our goal is to investigate whether the results reported in Refs. [7, 8] constitute a coincidental relationship, crucially depending on the fine-tuning of the coupling constants multiplying the linear confining potential(s), or, if there is a more general physical pattern behind the cancellation mechanism.

For absolute clarity, we should reemphasize once more that the purpose of the paper is not to show that the accidental symmetries exist. This was done in Refs. [7, 8] and is also discussed in [A]. The purpose of this paper is to discuss the physical foundations of the cancellation mechanism discussed in [7, 8], stressing that a superficial interpretation of the cancellation could otherwise suggest that the symmetries are much more general and could be generalized to more than one reference state, and even, perhaps, as outlined in [A] to more general potentials.

Our investigation focuses on the following generalized Dirac equation,

$$H \psi(\vec{r}) = E \psi(\vec{r}), \quad H = \vec{\alpha} \cdot \vec{p} + \beta m - \frac{\lambda}{r} + \beta \mu r + \nu r,$$

(1)

where the $\vec{\alpha}$ and $\beta$ matrices are used in the standard Dirac representation [2, 4, 8, 10], while $\psi(\vec{r})$ is the bispinor.
Pauli matrices \( \vec{\sigma} \) while the corresponding quantum numbers \( \ell \) confining potential in Eq. (1). (The subscript 0 is reserved for the angular quantum numbers of the reference state, projection quantum number of the reference state because \( \kappa_0 \) and \( \mu_0 \) angular function \([9, 10]\). The magnetic projection quantum number is total angular momentum quantum number fulfill the following relations, 

\[
\psi = \left( f(\vec{r}) \chi_{\kappa_0 M_0}(\vec{r}) \bigg| g(\vec{r}) \chi_{-\kappa_0 M_0}(\vec{r}) \right), \quad f(\vec{r}) = N r^{b-1} \exp(-a r) \exp\left(-\frac{1}{2} \alpha^2 r^2\right) \chi_{\kappa_0 M_0}(\vec{r}). \tag{2a}
\]

Here, \( r = |\vec{r}| \) is the radial coordinate, and \( N \) is a normalization factor, while \( \chi_{\kappa M}(\vec{r}) = \chi_{\kappa_0 M_0}(\theta, \varphi) \) is the spin-angular function \([8, 10]\). The magnetic projection quantum number is \( \kappa_0 = (-1)^{\ell_0 + j_0 + \frac{1}{2}} \left( j_0 + \frac{1}{2} \right) \) and summarizes both the orbital angular momentum quantum number \( \ell_0 \) as well as the total angular momentum quantum number \( j_0 \) of the reference state into a single, integer-valued quantum number (we initially assume that \( \kappa_0 = -1 \), which corresponds to an \( S \) state). We use the symbol \( M_0 \) to denote the magnetic projection quantum number of the reference state because \( \mu \) is reserved for the coefficient multiplying the scalar confining potential in Eq. (1). (The subscript 0 is reserved for the angular quantum numbers of the reference state, while the corresponding quantum numbers \( \ell, j \) and \( \kappa \) denote a general state; see Sec. III below.) With the vector of Pauli matrices \( \vec{\sigma} \), one can write down the lower component \( g(\vec{r}) \) and various quantities \( \gamma, a, \) and \( b \) as follows, 

\[
g(\vec{r}) \chi_{-\kappa_0 M_0}(\vec{r}) = \gamma (\vec{\sigma} \cdot \hat{r}) f(\vec{r}) \chi_{\kappa_0 M_0}(\vec{r})
= -\gamma N r^{b-1} \exp(-a r) \exp\left(-\frac{1}{2} \alpha^2 r^2\right) \chi_{-\kappa_0 M_0}(\vec{r}). \tag{2b}
\]

We have used the well-known relation \((\vec{\sigma} \cdot \hat{r}) \chi_{\kappa_0 M_0}(\vec{r}) = -\chi_{-\kappa_0 M_0}(\vec{r})\). The prefactor \( \gamma \) can be shown to simultaneously fulfill the following relations, 

\[
\gamma = \frac{a}{m + E} = \frac{|\kappa_0| - b}{\lambda} = \frac{\alpha^2}{\mu - \nu}, \tag{2c}
\]

\[
\gamma = \frac{m - E}{a} = \frac{\lambda}{|\kappa_0| + b} = \frac{\mu + \nu}{\alpha^2}, \tag{2d}
\]

\[
b = \sqrt{1 - \frac{\lambda^2}{\kappa_0^2}}, \quad a = m \frac{\lambda}{|\kappa_0|}, \quad \alpha^2 = \mu \frac{\lambda}{\kappa_0}. \tag{2e}
\]

These relations hold provided we assume that 

\[
\nu = -\mu \sqrt{1 - \frac{\lambda^2}{\kappa_0^2}}. \tag{2f}
\]

The normalization constant \( N \) which ensures that \( \int d^3r \ |\psi(\vec{r})|^2 = 1 \), is found to be 

\[
N = \frac{1}{\sqrt{2^{-2b} a b \Gamma(2b) \Gamma(2 + b) \Gamma\left(1 + b, \frac{4}{2}, \frac{\alpha^2}{\sigma^2}\right)}}, \tag{2g}
\]

where \( \Gamma \) denotes the gamma function and \( U \) denotes Kummer’s confluent hypergeometric function (see Chaps. 6 and 13 of Ref. \([11]\)). It has been observed \([8, 11]\) that the ansatz \((2a)\) solves the stationary Dirac equation \((1)\) with an energy eigenvalue \( E = m \sqrt{1 - \frac{\lambda^2}{\kappa_0^2}}, \tag{2h} \)

which the informed reader will recognize as the exact Dirac–Coulomb energy for the state with the highest possible total angular momentum, for a given principle quantum number, namely the state with \( n = n_0 = -\kappa_0 \). This is easily verified based on Dirac theory \([2, 4]\). In the limit \( \nu \to 0 \) and \( \mu \to 0 \), the Hamiltonian \((1)\) becomes an exact Dirac–Coulomb Hamiltonian, and the wave function \((2a)\) tends toward the ground-state wave function of the Dirac–Coulomb problem \([2, 4]\). Apparently, the addition to two linear confining potentials has not shifted the ground-state energy at all. In (the abstract of) Ref. \([8]\), the authors state that “the method works for the ground state or for the lowest orbital
state with $\ell = j - 1$, for any $j^\dagger$. This statement may leave some room for interpretation, in particular, regarding the question of whether the cancellation affects more than one reference state, or is limited, for given parameters $n_0$ and $\kappa_0$, to one (and only one) reference state. Furthermore, we should clarify that the state with quantum numbers $n = n_0 = -\kappa_0$ actually has the highest possible total angular momentum $j_0 = |\kappa_0| - 1/2$ and the highest possible orbital angular momentum $l_0 = j_0 - 1/2$ for given principal quantum number $n_0$. One would expect that the linear confinement alters the spectrum not only quantitatively, but even qualitatively (i.e., it would be assumed to transform the continuous part of the spectrum into a discrete one, with only bound states). Further clarification is the aim of the current article. Throughout the paper, we work in units with $\hbar = c = \epsilon_0 = 1$.

II. CANCELLATION MECHANISM

We aim to investigate the physical mechanism at the roots of the intriguing observations reported in Eq. (2), which might hint at a hidden symmetry of the (generalized) Dirac equation. To this end, we carry out a Foldy–Wouthuysen transformation, which the aim of identifying the relevant physical degrees of freedom in the low-energy limit. The Foldy–Wouthuysen transformation is inherently perturbative and therefore tied to a regime where the linear confining potentials can still be treated as perturbations. This, in particular, implies that the prefactors of the linear confining potentials have to be small (parametrically suppressed). We aim to use the Schrödinger wave equations as the unperturbed wave functions.

To this end, akin to the well-known so-called $Z\alpha$-expansion in atomic physics, we have to investigate the scaling of the physical quantities with the parameter $\lambda$. The following order-of-magnitude estimates are valid for Schrödinger–Pauli bound states \[12, 13\]

\[
E = m + \mathcal{O}(\lambda^2 m), \quad |\vec{p}| = \mathcal{O}(\lambda m), \quad r = \mathcal{O}((\lambda m)^{-1}), \quad \frac{\lambda}{r} = \mathcal{O}(\lambda^2 m). \tag{3}
\]

A classical analogy is obtained by considering a bound nonrelativistic particle on a classical, stationary orbit in a $(1/r)$-potential; its momentum is of the order $\lambda m$. In order for the confining potentials to represent perturbatively tractable terms, we therefore have to assume that

\[
\beta \mu r = \mathcal{O}(\lambda^3 m), \quad \nu r = \mathcal{O}(\lambda^3 m), \tag{4}
\]

and so the coupling constants of the confining potential must scale at least as

\[
\mu = \mathcal{O}(\lambda^4 m^2), \quad \nu = \mathcal{O}(\lambda^4 m^2). \tag{5}
\]

Then, Eq. (2) implies that, to order $\lambda^4$, we have

\[
\nu = -\mu + \frac{\lambda^2}{2\kappa_0^2} + \mathcal{O}(\lambda^5 m), \tag{6}
\]

which is consistent with our previous assumption that the leading-order term in both $\nu$ and $\mu$ are of order $\lambda^4 m^2$. Here, we include the correction term of relative order $\lambda^2$ which follows by expanding the square root in Eq. (2). With the radial coordinate being of order $1/(\lambda m)$, the matrix elements of the confining potentials are of order $\lambda^3 m$ and therefore parametrically suppressed with regard to the main (leading) Schrödinger energy $-\lambda^3 m/(2n^3)$, where $n$ is the principal quantum number. The perturbative hierarchy conveniently orders the terms for the application of the Foldy–Wouthuysen (FW) programs of the Hamiltonian given in Eq. (1).

For the FW transformation, one first identifies the odd part of the Hamiltonian in question, with a subsequent unitary transformation to eliminate the odd part. However, in higher orders in the perturbative parameters, this procedure often tends to introduce new odd terms in the resulting Hamiltonian, and therefore it needs to be iterated. The process is then repeated until all odd parts of the Hamiltonian are eliminated to the desired order in perturbation theory \[2, 4, 14\]. In our case we will keep all terms up to order $\lambda^3 m$ in the energy. We identify the odd part of (1) as

\[
\mathcal{O} = \vec{a} \cdot \vec{p}, \quad S = -\frac{i\beta \mathcal{O}}{2m}, \quad U = e^{-iS}, \tag{7}
\]

where we have also constructed the Hermitian operator $S$ and the unitary rotation operator $U$. The rotation is then given as \[5, 14\]

\[
H' = U H U^+ \approx H + i[S, H] + \frac{(i)^2}{2!}[S, [S, H]] + \frac{(i)^3}{3!}[S, [S, [S, H]]] + \ldots. \tag{8}
\]
We now apply the FW transformation and find
\[
H' = \beta \left( m + \frac{\bar{p}^2}{2m} - \frac{\bar{p}^4}{8m^3} \right) - \frac{\lambda}{r} + \frac{1}{8m^2} \left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] \right] \\
+ \beta \mu \left( r - \frac{\{ \vec{\alpha} \cdot \vec{\rho}, \{ \vec{\alpha} \cdot \vec{\rho}, r \} \}}{8m^2} \right) + \nu \left( r - \frac{[\vec{\alpha} \cdot \vec{\rho}, [\vec{\alpha} \cdot \vec{\rho}, r]]}{8m^2} \right) + O',
\]
where
\[
O' = -\frac{\bar{p}^2 \vec{\alpha} \cdot \vec{\rho}}{3m^2} + \frac{\bar{p}^4 \vec{\alpha} \cdot \vec{\rho}}{30m^4} - \frac{\beta}{2m} \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] - \frac{\mu}{2m} \{ \vec{\alpha} \cdot \vec{\rho}, \{ \vec{\alpha} \cdot \vec{\rho}, r \} \} + \frac{\nu}{2m} [\vec{\alpha} \cdot \vec{\rho}, r] \\
+ \frac{\beta}{48m^3} \left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] \right] \right].
\]
Note that while $O \sim \lambda m$, $O' \sim \lambda^3 m$ and higher, supporting the earlier claim that the new odd parts are of higher order than the old ones. In order to proceed, we must perform another iteration of the FW transformation, this time with
\[
S' = -i \frac{\beta O'}{2m}, \quad U' = e^{-iS'},
\]
which yields
\[
H'' = \beta \left( m + \frac{\bar{p}^2}{2m} - \frac{\bar{p}^4}{8m^3} \right) - \frac{\lambda}{r} + \frac{1}{8m^2} \left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] \right] \\
+ \beta \mu \left( r - \frac{\{ \vec{\alpha} \cdot \vec{\rho}, \{ \vec{\alpha} \cdot \vec{\rho}, r \} \}}{8m^2} \right) + \nu \left( r - \frac{[\vec{\alpha} \cdot \vec{\rho}, [\vec{\alpha} \cdot \vec{\rho}, r]]}{8m^2} \right) + O'',
\]
where
\[
O'' = \frac{\bar{p}^4 \vec{\alpha} \cdot \vec{\rho}}{6m^4} + \frac{\beta}{6m^3} \left[ \bar{p}^2 \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] + \frac{\beta}{8m^3} \{ \bar{p}^2, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] \} + \frac{1}{4m^2} \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right].
\]
The new odd part is of order $\lambda^5 m$, while to the same order, the even part remains the same. For the third transformation, we use
\[
S'' = -i \frac{\beta O''}{2m}, \quad U'' = e^{-iS''},
\]
and perform a final iteration of the FW transformation, yielding
\[
H^{(FW)} = \beta \left( m + \frac{\bar{p}^2}{2m} - \frac{\bar{p}^4}{8m^3} \right) - \frac{\lambda}{r} + \frac{1}{8m^2} \left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{\lambda}{r} \right] \right] \\
+ \beta \mu \left( r - \frac{\{ \vec{\alpha} \cdot \vec{\rho}, \{ \vec{\alpha} \cdot \vec{\rho}, r \} \}}{8m^2} \right) + \nu \left( r - \frac{[\vec{\alpha} \cdot \vec{\rho}, [\vec{\alpha} \cdot \vec{\rho}, r]]}{8m^2} \right),
\]
which will be rewritten with the help of the identities
\[
\left[ \vec{\alpha} \cdot \vec{\rho}, \left[ \vec{\alpha} \cdot \vec{\rho}, \frac{1}{r} \right] \right] = 4\pi \delta(\bar{r}) + 2 \frac{\vec{\Sigma} \cdot \vec{L}}{r^3},
\]
\[
\{ \vec{\alpha} \cdot \vec{\rho}, \{ \vec{\alpha} \cdot \vec{\rho}, r \} \} = 2\{\bar{p}^2, r\} + 2\beta \frac{K}{r},
\]
\[
[\vec{\alpha} \cdot \vec{\rho}, [\vec{\alpha} \cdot \vec{\rho}, r]] = -2\beta \frac{K}{r},
\]
\[
K \equiv \beta (\vec{\Sigma} \cdot \vec{L} + 1) = \begin{pmatrix} \vec{\sigma} \cdot \vec{L} + 1 & 0 \\ 0 & -(\vec{\sigma} \cdot \vec{L} + 1) \end{pmatrix}.
\]
While the $(4 \times 4)$-bispinor operator $K$ commutes with the Hamiltonian, $[K, H] = 0$, the $(2 \times 2)$-submatrices of the operator $K$ have the properties
\[
(\vec{\sigma} \cdot \vec{L} + 1) \chi_{\kappa_0 M_0} = -\kappa_0 \chi_{\kappa_0 M_0}, \quad \kappa_0 = (-1)^{j_0 + f_0 + 1/2} \left( j_0 + \frac{1}{2} \right).
\]
The orbital angular momentum is obtained as $\ell_0 = |\kappa_0 + 1/2| - 1/2$, and the total angular momentum reads $j_0 = |\kappa_0| - 1/2$. The eigenfunctions of the Hamiltonian given in Eq. (24) are also eigenfunctions of $K$ with eigenvalue $-\kappa_0$. The $(4 \times 4)$ FW-transformed Hamiltonian finally reads as follows,

$$H^{(FW)} = \beta \left( m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^2}{8m^3} \right) - \frac{\lambda}{r} + \frac{\pi \lambda}{2m^2} \delta^{(3)}(\vec{r}) + \frac{\lambda}{2m^2r^3} \vec{\Sigma} \cdot \vec{L},$$

$$+ (\nu + \beta \mu)r + (\beta \nu - \mu)\frac{K}{4m^2r} - \beta \mu \left\{ \vec{p}^2, r \right\} .$$

Using Eq. (9), which we recall for convenience as

$$\nu \approx -\mu + \mu \frac{\lambda^2}{2\kappa_0^2} \quad \text{(ignoring the higher-order corrections)}$$

and expressing all terms in terms of $\mu$, we recognize that it is necessary to keep the correction term of order $\lambda^2$, in order to calculate the Hamiltonian to order $\lambda^5 m$. One obtains

$$H^{(FW)} = H_{DC} + H_C,$$

$$H_{DC} = \beta m + \frac{\beta \mu^2}{2m} r - \frac{\lambda}{r} - \beta \frac{\pi \lambda}{2m^2} \delta^{(3)}(\vec{r}) + \frac{\lambda}{2m^2r^3} \vec{\Sigma} \cdot \vec{L},$$

$$+ \frac{\lambda^2 \mu r}{2\kappa_0^2} - (\beta + 1) \mu \frac{\vec{\Sigma} \cdot \vec{L} + 1}{4m^2r} - \beta \mu \left\{ \vec{p}^2, r \right\},$$

$$H_C = \frac{\lambda^2 \mu r}{2\kappa_0^2} - (\beta + 1) \mu \frac{\vec{\Sigma} \cdot \vec{L} + 1}{4m^2r} - \beta \mu \left\{ \vec{p}^2, r \right\},$$

where $H_{DC}$ is the FW transformed Dirac-Coulomb Hamiltonian [3, 10], and $H_C$ is a perturbative term which was induced by the addition of the confining potentials. We see that the perturbation in the original Hamiltonian, under the conditions imposed in Eq. (23), with the approximation made in Eq. (6), results in the sum of three additional terms of order $\lambda^5 m$ in the FW transformed Hamiltonian, beyond the leading confining term of order $\lambda^3 m$, which is unaffected by the FW program in comparison to Eq. (1). The first of these new terms is simply a linear potential. It is followed by a spin-orbit coupling term, proportional to $\vec{\Sigma} \cdot \vec{L}/r$ instead of $\vec{\Sigma} \cdot \vec{L}/r^3$ as in $H_{DC}$. The former is the result of the FW transformation of a linear potential while the latter comes from the transformation of a $1/r$ potential. Finally we see that the linear perturbation also induces a kinetic correction term to the particle’s orbit, whose functional form is akin to the magnetic exchange term in the Breit Hamiltonian [17].

For a particle [denoted with a superscript "(+)"] as opposed to an antiparticle state, we may replace $\beta \rightarrow 1$ and isolate the upper $2 \times 2$ submatrix of the Hamiltonian,

$$H_{DC}^{(+)} = \beta \left( m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^2}{8m^3} \right) - \frac{\lambda}{r} + \frac{\pi \lambda}{2m^2} \delta^{(3)}(\vec{r}) + \frac{\lambda}{2m^2r^3} \vec{\Sigma} \cdot \vec{L},$$

$$+ \frac{\lambda^2 \mu r}{2\kappa_0^2} - (\beta + 1) \mu \frac{\vec{\Sigma} \cdot \vec{L} + 1}{4m^2r} - \beta \mu \left\{ \vec{p}^2, r \right\},$$

$$H_C^{(+)} = \frac{\lambda^2 \mu r}{2\kappa_0^2} - (\beta + 1) \mu \frac{\vec{\Sigma} \cdot \vec{L} + 1}{4m^2r} - \beta \mu \left\{ \vec{p}^2, r \right\}.$$
The relation $\kappa_0(\kappa_0 + 1) = \ell_0(\ell_0 + 1)$ holds regardless of the relative orientation of the orbital angular momentum $\ell_0$ and spin projection $\pm \frac{1}{2}$. Using the quantum numbers $n_0$ and $\kappa_0$, as well as the magnetic momentum projection $M$, one may describe a state which otherwise needs the quantum numbers $n_0$, $\ell_0$, and $j_0$. In the usual $n_0\ell_j$ notation, examples are as follows. For the $1S_{1/2}$ ground state, we have $n_0 = 1$, $\kappa_0 = -1$, while the $2P_{3/2}$ states are given as follows. Namely, for $2P_{1/2}$, we have $n_0 = 2$, $\kappa_0 = 1$, while for $2P_{3/2}$, we have $n_0 = 2$, $\kappa_0 = -2$. The $(n_0 = 3)$ state are $3D_{3/2}$, which $n_0 = 3$ and $\kappa_0 = 2$, and $3D_{5/2}$, with $n_0 = 3$ and $\kappa_0 = -3$. The $P$ states with $n_0 = 2$ are $3P_{1/2}$ with $\kappa_0 = 1$, and $3P_{3/2}$ with $\kappa_0 = -2$. For given $n_0$, $\kappa_0$ attains all integer values in the interval $(-n_0, -n_0 + 1, \ldots, n_0 - 1)$, excluding zero.

Let us now look at the expectation value of the perturbation for a $|n_0\kappa_0\rangle$ reference state. First we note that in such a state, the expectation of $\hat{\sigma} \cdot \hat{L}$ is $\kappa_0 - 1$. We can use the Schrödinger-Pauli states [8] [10] as basis states when evaluating the energy perturbation. This means that if the wave function is given as $\phi(\vec{r}) = R_{n_0\ell_0}(\vec{r})\chi_{\kappa_0} M_{\ell_0}(\vec{r})$, then the radial part $R_{n_0\ell_0}(r)$ is given by the radial solution to the Coulomb–Schrödinger Hamiltonian $\left(\frac{\ell^2}{2m} - \frac{1}{r}\right)$. In leading order, using formulas from [18], we then find that

$$E_{n_0\kappa_0} \approx \langle n_0\kappa_0 | m + \frac{\hat{r}^2}{2m} - \frac{\lambda}{r} | n_0\kappa_0 \rangle = m - \frac{\lambda^2 m}{2n_0},$$

(23a)

$$\langle r \rangle_0 = \langle n_0\kappa_0 | r | n_0\kappa_0 \rangle = \frac{3n_0^2 - \kappa_0(\kappa_0 + 1)}{2\lambda m},$$

(23b)

$$\langle r^{-1} \rangle_0 = \langle n_0\kappa_0 | r^{-1} | n_0\kappa_0 \rangle = \frac{\lambda m}{n_0^2}.$$  

(23c)

Independent of the magnetic projection $M_{\ell_0}$ of the reference state, and with the help of Eq. (17), one then finds that

$$\langle n_0\kappa_0 | H^{(+)}_C | n_0\kappa_0 \rangle = \frac{\mu \lambda}{4m} \left(\frac{(n_0 - \kappa_0) (n_0 + \kappa_0)}{\kappa_0^2} \frac{3n_0^2 - \kappa(\kappa + 1)}{n_0^2 + \kappa(\kappa - 1)} \right) = 0,$$

(24)

because we explicitly assume that $n_0 = -\kappa_0$. As anticipated, due to the fact that the energy is equivalent to that of the corresponding state of the DC Hamiltonian, we see that the perturbation disappears, but only because we have adjusted the relationship [8] among the prefactors of the two confining potentials. One might ask if this cancellation is a more general phenomenon, or specific to the particular reference state at hand.

III. THE GENERAL EXPECTATION VALUE OF $H^{(+)}_C$

Let us consider the perturbed particle Hamiltonian, $H^{(+)}_C$, given in Eq. (21b), and its expectation value for a general reference state $|n\kappa \rangle \neq |n_0\kappa_0 \rangle$. Again, using formulas from Ref. [18], we find that the first-order energy shift $\Delta E_{n\kappa}$ reads as

$$\Delta E_{n\kappa} = \langle n\kappa | H^{(+)}_C | n\kappa \rangle = \frac{\mu \lambda}{4m} \left(\frac{3n^2 - \kappa(\kappa + 1)}{\kappa_0^2} \frac{n^2 + \kappa(\kappa - 1)}{n^2} \right).$$

(25)

The question then is whether at least the leading correction to the energy due to the confining potential, given by Eq. (25), vanishes for states other than $|n\kappa \rangle = |n_0\kappa_0 \rangle$. We recall that $\Delta E_{n\kappa}$ is of order $\lambda^3$ as $\mu$ is of order $\lambda^4$. Now, $\Delta E_{n\kappa}$ vanishes provided

$$\kappa_0^2 = n^2 \Xi^2, \quad \Xi^2 = \frac{3n^2 - \kappa(\kappa + 1)}{n^2 + \kappa(\kappa - 1)}.$$  

(26)

If we set $\kappa = \pm n$, then $\Xi$ evaluates to unity, so that $\kappa_0 = \pm n$. The only sign which can be realized for bound states pertains to $n = -\kappa$ [see the text following Eq. (23)], in which case $\kappa = \kappa_0 = -n = -n_0$, and we reproduce the case already discussed in Eq. (24).

In order to check whether the perturbation vanishes for any other state, one investigates if the case $\kappa_0^2 = n^2 N^2$ may occur, where $N$ is an integer greater than unity, i.e., if the case $\Xi^2 = N^2$ with $N \geq 2$ can be realized. We set

$$\frac{3n^2 - \kappa(\kappa + 1)}{n^2 + \kappa(\kappa - 1)} = N^2.$$

(27)
Again, if we set \( N = 1 \), then we find \( \kappa^2 = n^2 \), which is the physically relevant solution already discussed. For \( N > 1 \), we rearrange Eq. (27) to find
\[
(N^2 - 3)n^2 = \kappa((N^2 - 1) - (N^2 + 1)\kappa).
\] (28)

Under the assumption that \( N \geq 2 \), one ascertains that
\[
(N^2 - 3)n^2 > 0, \quad \kappa((N^2 - 1) - (N^2 + 1)\kappa) < 0,
\] (29)
regardless of whether \( \kappa \) is negative or positive. Thus Eq. (28) cannot be fulfilled, and the only possible integer value for \( N^2 \) is 1. It then follows that \( \Delta E_{n\kappa} = 0 \) if and only if \( \kappa = \kappa_0 = -n = -n_0 \).

We thus find an important restriction to the cancellation mechanism. Indeed, we can tailor the prefactors of the two confining potentials, according to the modified relation [21],
\[
\nu = -\mu \sqrt{1 - \frac{\lambda^2}{\kappa_0}}.
\] (30)

and obtain the cancellation for the state \(|n_0\kappa_0\rangle\) with \( n_0 = -\kappa_0 \), which happens to be the state of maximum total angular momentum for given principal quantum number. However, once \( \kappa_0 \) is fixed, no other reference states are affected by the cancellation, and in fact, are shifted with regard to their unperturbed (Dirac–Coulomb) values.

### IV. EFFECTIVE CONFINING POTENTIALS FOR ANTI-PARTICLE STATES

Let us recall Eq. (20) and discuss how it needs to be applied to antiparticle states,
\[
H^{(FW)} = H_{DC} + H_C,
\] (31a)
\[
H_{DC} = \beta m + \frac{\beta \vec{p}^2}{2m} - \frac{\lambda}{r} - \beta \frac{\lambda^4}{8m^3} + \frac{\pi \lambda}{2m^2} \bar{S} \cdot \bar{L} + \frac{\lambda}{2m^2} \bar{S} \cdot \bar{\sigma} \cdot \bar{L} \cdot \bar{\sigma} + \frac{\lambda}{4m^2} \bar{\sigma} \cdot \bar{L} + \frac{1}{4m^2} - \beta \mu \{\vec{p}^2, r\}.
\] (31b)
\[
H_C = \mu (\beta - 1) r + \frac{\lambda^2}{2} \frac{\mu r}{\kappa_0} - (\beta + 1) \mu \frac{\vec{S} \cdot \vec{L} + 1}{4m^2} - \beta \mu \{\vec{p}^2, r\}.
\] (31c)

The FW transformation has disentangled the particle from the antiparticle degrees of freedom, as described by the Dirac \( \beta \) matrix,
\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (32)

Formally, the Hamiltonian \( H^{(FW)} \), being the time evolution operator, has negative energy eigenvalues corresponding to the antiparticle states; however, the reinterpretation principle [1] dictates that the physical energy operator for antiparticles equals the negative of the lower \( (2 \times 2) \) submatrix of the Hamiltonian (31). We endow this physical Hamilton operator for the antiparticle states with the superscript "\((-)" and write
\[
H^{(-)}_{DC} = m + \frac{\vec{p}^2}{2m} + \frac{\lambda}{r} - \frac{\lambda^4}{8m^3} - \frac{\pi \lambda}{2m^2} \bar{S} \cdot \bar{L} - \frac{\lambda}{2m^2} \bar{S} \cdot \bar{\sigma} \cdot \bar{L} \cdot \bar{\sigma} + \frac{\lambda}{4m^2} \bar{\sigma} \cdot \bar{L} + \frac{1}{4m^2} - \mu \{\vec{p}^2, r\}.
\] (33a)
\[
H^{(-)}_C = \frac{2\mu r}{\kappa_0} - \mu \frac{\lambda^2}{2\kappa_0^2} r - \mu \frac{\{\vec{p}^2, r\}}{4m^2}.
\] (33b)

Here, the order-of-magnitude estimates in regard to \( \lambda \) pertain to the expectation values that would otherwise be obtained for bound Schrödinger–Pauli reference states. However, as is well known [9, 10] and manifest in Eq. (33), for antiparticles, the terms of order \( O(\lambda^2 m) \) in Eq. (33) actually describe a repulsive ("positron") Schrödinger Hamiltonian
\[
H^{(-)}_S = m + \frac{\vec{p}^2}{2m} + \frac{\lambda}{r}.
\] (34)
where the first term is just the rest mass. The spectrum of $H_S^{(-)}$ consists only of continuum states. The Coulomb potential $\lambda/r$ has “changed sign” for the antiparticles, and the spectrum consists only of a continuum. This means that the analysis outlined above for bound particle states is not applicable to the antiparticle states in the FW transformed Hamiltonian. Furthermore, the two confining potentials, in sharp contrast to the cancellations observed for the particle Hamiltonian, add up for antiparticles to a confining linear term $2\mu r$ which, perturbatively, is of order $\lambda^3 m$.

As already stated, the order-of-magnitude estimates given in powers of $\lambda$ in Eq. (25) are relevant to bound states and otherwise are consistent with the classical analogue of bound particles orbiting the binding $(-\lambda/r)$-potential with velocities of order $\lambda c$. For antiparticles, in the repulsive $(+\lambda/r)$-potential, the physics changes. Here, one can typically assume that the energy of the particle with regard to the continuum threshold in the repulsive potential is large compared to the binding energy scale of the sign-reversed potential. This means, in particular, that the wave functions of the continuum states are typically spread further out than the scale $\langle r \rangle \sim 1/(\lambda m)$, which is typical of bound states. For states with $\langle r \rangle \gg 1/(\lambda m)$, the Coulomb potential is suppressed in comparison to the effective confining term $2\mu r$ given in Eq. (33). In this regime, we can thus approximate

$$H^{(-)} = H^{(-)}_{DC} + H^{(-)}_{C} \approx m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} + 2\mu r,$$

and treat the repulsive part of the Coulomb potential, given by the term $\lambda/r$, as a perturbation. As compared to the particle states, the effective Hamiltonian for antiparticle states, obtained after the FW transformation, leads up to no cancellation between the two confining potentials, which add up rather than cancel. This observation illustrates the different physics that result from one and the same generalized Dirac Hamiltonian, given in Eq. (20), in the case where the Dirac Hamiltonian is not charge conjugation invariant and therefore may result in fundamentally different physics for particles and corresponding antiparticles.

The confining potential $2\mu r$ leads to a transformation of the spectrum in terms of a discrete spectrum. For not-too-large value of the coupling parameter $\mu$, this spectrum will contain a series of relatively dense eigenvalues.

V. CONCLUSIONS

We have studied the generalized Dirac Hamiltonian with two linear radial confining potentials given in Eq. (11). Given the general complexity of the Dirac equation, one would intuitively assume that the addition of the confining potentials shifts the energy values of bound states, no matter what the relation between the prefactors of the two confining potentials in Eq. (11) is, and irrespective of the quantum numbers of the “unperturbed” Dirac–Coulomb eigenstates. However, it can be verified, both on the basis of an explicit solution of the radial Dirac equation as well as on the basis of a perturbative calculation (see Sec. II), that an interesting accidental cancellation occurs provided the prefactor of the two confining potentials $(\mu$ and $\nu$), and the “Coulomb” coupling parameter $\lambda$, fulfill the relationship given in Eq. (21). This cancellation may occur for the reference 1S ground state (if we set $n_0 = -\kappa_0 = 1$), which retains its precise energy eigenvalue under the addition of the confining potentials, but only, if the relation (2) is fulfilled.

As revealed by the FW transformed Hamiltonian given in Eq. (20a), the cancellation is not fully accidental: Namely, the leading-order effective operators [of order $O(\lambda^3 m)$] derived from the two confining potentials cancel for particle states [see Eq. (21b)], while they add up for antiparticle states [see Eq. (33b)]. At order $O(\lambda^5 m)$, the cancellation is more subtle and involves both the kinetic corrections in the confining potential as well as the additional spin-orbit coupling terms which are manifest in Eq. (21b). Such spin-orbit terms are known to affect bound Dirac particles in comparable situations, such as in nuclei where the hadrons are confined by the short-range meson exchange potentials, and yet, spin-orbit terms have a tremendous influence on the spectrum [17, 20]. The cancellation is explicitly verified in Eq. (24).

Yet, in Sec. III we verify that the cancellation can be tailored to only one reference state $|n_0 \kappa_0 \rangle$ with $\kappa_0 = -n_0$; other states are shifted by the added potential. This result, expressed in Eq. (29), would be impossible to obtain without the explicit Foldy–Wouthuysen transformed Hamiltonian being available (its matrix elements can be expressed analytically). Furthermore, the analysis of corresponding antiparticle states, carried out in Sec. IV, reveals that the physical degrees of freedom are drastically different as compared to particle states [for the nonrelativistic approximation relevant to antiparticles, see Eq. (35)].
In summary, we hope to have accomplished three goals in this paper: (i) First, to show that the considerations reported in Refs. [7, 8] are restricted to one particular reference state of the “double-confining” potential given in Eq. (1). This restriction may not be completely obvious from the papers [7, 8] (see also [A]). Furthermore, while the cancellation mechanism can be tailored to any reference state, it can be tailored to work for one and only one reference state, not more than one (see Sec. III). Moreover, our considerations suggest that the existence of the cancellation mechanism actually is tied to a certain proportionality of the upper and lower radial wave functions of the “unperturbed” Dirac-Coulomb problem, which persists only for \( n = n_0 = -\kappa_0 \). (ii) Second, for the model problem studied in Refs. [7, 8], we identify, via a Foldy–Wouthuysen transformation, the nonrelativistic effective operators which pertain to an intuitive physical interpretation of the Dirac equation in the nonrelativistic limit. These considerations (or slight generalizations thereof) might become important if and when the effective Dirac equations with linear confining potentials are used for particles bound in linear confining wells, such as excited states of quarks in a nucleon, or appropriately designed traps. This calculation confirms that the cancellation is rather accidental and does not imply a further, hitherto insufficiently appreciated symmetry of effective operators pertaining to the (nonrelativistic limit of the) Dirac Hamiltonian. (iii) Third, we identify the effective operators that pertain to antiparticles as opposed to particles, in the double-confining potential. We show that the cancellation mechanism relevant for particles is reversed for the corresponding antiparticles which are described by the same (generalized) Dirac equation. In passing, we obtain the effective form of the confining potential for antiparticles and the effective (fully confining) operators relevant in the nonrelativistic limit (for antiparticles). This latter conclusion illustrates the intricacies, and perhaps, also the power, of the Dirac formalism: namely, to describe two different particle “species” (particles and corresponding antiparticles) at the same time, and with the same equation, predicting a different, and in our case, opposite influence (addition versus cancellation) of the two confining potentials for the antiparticle as opposed to the particle states.

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Appendix A: The “Bear Trap” Consideration

The purpose of this Appendix is to show how easy it is to fall into a “bear trap” and conclude that the cancellation mechanism outlined in Refs. [7, 8] might be more general and applicable to more than one reference state in a given Hamiltonian. Moreover, one could almost assume that any two potentials can be added to a known Dirac–Coulomb Hamiltonian and still allow for an easy computation of at least one eigenvalue. We start from a know Dirac equation of the form

\[
(\bar{\alpha} \cdot \bar{p} + \beta m + V_0(r)) \psi(\bar{r}) = E \psi(\bar{r}), \quad \psi(\bar{r}) = \left( \frac{f_0(r)x_{\kappa,M}}{ig_0(r)x_{-\kappa,M}} \right),
\]

where \( \psi(\bar{r}) \) is known and \( V_0(r) \) is a kind of “Coulomb potential”. We can add a linear combination of radial potentials to the equation that defines the energy eigenvalue \( E \),

\[
(\bar{\alpha} \cdot \bar{p} + \beta m + V_0(r) + \beta V_1(r) + V_2(r)) \psi(\bar{r}) = E \psi(\bar{r}), \quad \psi(\bar{r}) = \left( \frac{f(r)x_{\kappa,M}}{ig(r)x_{-\kappa,M}} \right),
\]

where we have the same energy \( E \) in Eqs. \( \text{(A1)} \) and \( \text{(A2)} \). The radial equations for the original “unperturbed” Dirac–Coulomb Hamiltonian are then

\[
\left( \frac{\partial}{\partial r} + \frac{\kappa + 1}{r} \right) f_0(r) = (E + m + V_0(r)) g_0(r), \quad \text{(A3a)}
\]

\[
\left( -\frac{\partial}{\partial r} + \frac{\kappa - 1}{r} \right) g_0(r) = (E - m + V_0(r)) f_0(r), \quad \text{(A3b)}
\]
while the radial equations for the perturbed Hamiltonian read as

\[
\left( \frac{\partial}{\partial r} + \frac{\kappa + 1}{r} \right) f(r) = (E + m + V_0(r) + V_1(r) - V_2(r)) g(r),
\]

(A4a)

\[
\left( - \frac{\partial}{\partial r} + \frac{\kappa - 1}{r} \right) g(r) = (E - m + V_0(r) - V_1(r) - V_2(r)) f(r).
\]

(A4b)

It is then straightforward to show that for a given energy eigenvalue \(E\) there exists a solution for the perturbed system which is obtained by a rescaling of the original radial wave functions, specifically

\[
f(r) = e^{h(r)} f_0(r), \quad g(r) = e^{h(r)} g_0(r), \quad \psi(\vec{r}) = e^{h(\vec{r})} \psi_0(\vec{r}).
\]

(A5)

This would imply that

\[
\frac{\partial h(r)}{\partial r} f(r) = (V_1(r) - V_2(r)) g(r),
\]

(A6a)

\[
\frac{\partial h(r)}{\partial r} g(r) = (V_1(r) + V_2(r)) f(r).
\]

(A6b)

By multiplying these equations together we find

\[
\frac{\partial h(r)}{\partial r} = \pm \sqrt{(V_1(r))^2 - (V_2(r))^2}.
\]

(A7)

Based on these considerations, one might conclude that it is possible to add any combinations of radial potentials \(V_1\) and \(V_2\) to a known Dirac–Hamiltonian, provided that the potentials fulfill the condition that \((V_1(r))^2 \geq (V_2(r))^2\), and quickly find a solution using Eq. (A5). The state \(\psi_0(\vec{r})\) has retained its energy \(E\) even after the addition of \(V_1\) and \(V_2\), and remains an exact eigenstate of the perturbed problem (with nonvanishing \(V_1\) and \(V_2\)). If this argument were universally applicable, then the Dirac equation should have at least one exact solution for very wide classes of potentials \(V_0, V_1\) and \(V_2\), which can be expressed in closed analytic form [provided \(f_0(r)\) and \(g_0(r)\) admit such a form]. However, there is another relation that we can obtain from Eq. (A6) which comes about through division, giving us

\[
\frac{g(r)}{f(r)} = \frac{g_0(r)}{f_0(r)} = \frac{\sqrt{V_1(r) + V_2(r)}}{\sqrt{V_1(r) - V_2(r)}}.
\]

(A8)

where we used Eq. (A5) to relate the ratio of the original wave functions to the ratio of the wave functions of the perturbed Hamiltonian. This relation implies that the “unperturbed” radial wave functions must be related by a global prefactor. For convenience, we denote the proportionality factor as \(-\gamma\), so that \(f_0(r) = -\gamma g_0(r)\), with reference to Sec. II. Taking into account Eq. (A5), this also implies that \(f(r) = -\gamma g(r)\). We then find that

\[
V_2(r) = \frac{1 - \gamma^2}{1 + \gamma^2} V_1(r).
\]

(A9)

Since the value of \(\gamma\) is determined by the relationship of the original radial wave function, this turns out to be an extremely restrictive condition. In the case where the original Hamiltonian is the Dirac–Coulomb Hamiltonian, for example, the relationship between the two potentials is found to be expressible in a simpler form (see Sec. III)

\[
1 - \gamma^2 = \frac{E}{m} = \sqrt{1 - \frac{\lambda^2}{\kappa_0^2}}.
\]

(A10)

For the classes of potentials discussed in Sec. III and the states with \(n_0 = -\kappa_0\), all these relations (A8), (A9) and (A10) are fulfilled. Otherwise, the relations (A8), (A9) and (A10) represent important restrictions which can be fulfilled only for particular solutions of the “unperturbed” problem (the one with \(V_0\)) and also limit the occurrence of the exact cancellation to potentials \(V_1\) and \(V_2\) which have to be proportionally related according to Eq. (A9). Furthermore, as shown in Sec. III the cancellation cannot be parametrically adjusted for more than one reference state.

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