Initial bounds for analytic and bi-univalent functions by means of \((p, q)\)–Chebyshev polynomials defined by differential operator

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Abstract

In this paper, a subclass \(T^p_\lambda (m, \gamma, \lambda, p, q)\) of analytic and bi-univalent functions by means of \((p, q)\)–Chebyshev polynomials is introduced. Certain coefficient bounds for functions belong to this subclass are obtained. In addition, the Fekete-Szegő problem is solved in this subclass.

Keywords: coefficient inequalities, bi-univalent functions, Fekete-Szegő problems.

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1. Introduction and preliminaries

Let \(A\) denote the class of functions of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \(\mathbb{U} = \{z : |z| < 1\}\). Further, by \(S\) we shall denote the class of all functions in \(A\) which are univalent in \(\mathbb{U}\). It is well known that every function \(f \in S\) has an inverse \(f^{-1}\), defined by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U})
\]

and

\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4})
\]

where

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

A function \(f \in A\) is said to be in \(\Sigma\) the class of bi-univalent in \(\mathbb{U}\) if both \(f(z)\) and \(f^{-1}(z)\) are univalent in \(\mathbb{U}\). Lewin [9] showed that \(|a_2| < 1.51\) for every function \(f \in \Sigma\) given by (1.1). Posteriorly, Brannan...
and Clunie [3] improved Lewin’s result and conjectured that $|a_2| \leq \sqrt{2}$ for every function $f \in \Sigma$ given by (1.1). The coefficient estimate problem for each of the following Taylor Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N}; n \geq 4)$$

It’s still an open problem. Since then, there have been many researchers (see [2, 5, 6, 7, 11, 12, 14, 15, 13]) investigated several interesting subclasses of the class $\Sigma$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri, Jay M., and Samaneh G. Hamidi [8] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

For any integer $n \geq 2$ and $0 < q < p \leq 1$, $(p, q)$–Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$R_n(x, s, p, q) = (p^n + q^n) x R_{n-1}(x, s, p, q) + (pq)^{n-1} s R_{n-2}(x, s, p, q),$$

(1.2)

with the initial values $R_0(x, s, p, q) = 1$ and $R_1(x, s, p, q) = (p + q) x$ and $s$ is a variable.

Recently, Kızılates¸ Naim and Bayram [16] defined $(p, q)$–Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the $(p, q)$–Chebyshev polynomials of the second kind is as follows:

$$H_{p,q}(z) = \frac{1}{1 - x p z \tau_p - x q z \tau_q - s p q z^2 \tau_{p,q}} = \sum_{n=0}^{\infty} R_n(x, s, p, q) z^n \quad (z \in \mathbb{U}).$$

where the Fibonacci operator $\tau_q$ Mason and Handscomb was introduced [17], by $\tau_q f(z) = f(qz)$, similarly, $\tau_{p,q} f(z) = f(pqz)$.

First off, we present some special cases of the polynomials $H_{p,q}(z)$:

1. For $p = q = 1$ and $s = -1$, we get the Chebyshev polynomials $R_n(x)$ of the second kind.
2. For $p = q = s = 1$ and $x = \frac{1}{2}$, we get the Fibonacci polynomials $F_n(x)$.
3. For $p = q = 1$, $s = 2 y$ and $x = \frac{1}{2}$, we get the Jacobsthal polynomials $J_{n+1}(y)$.
4. If $p = q = s = 1$, then we get the Pell polynomials $P_{n+1}(x)$.

Let $w(z)$ and $v(w)$ be two analytic functions in the unit disk $\mathbb{U}$ with $w(0) = v(0) = 0$, $|w(z)| < 1$, $|v(z)| < 1$, and suppose that

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad \text{and} \quad v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (z, w \in \mathbb{U}).$$

(1.3)

Making use of the binomial series

$$(1 - \gamma)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \gamma^j \quad (m \in \mathbb{N}, \ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

recently for $f \in A$, Frasin [4] defined the differential operator $A^\gamma_{m,\lambda} f(z)$ as follows:

$$A^0 f(z) = f(z),$$

$$A^\gamma_{m,\lambda} f(z) = (1 - \gamma)^m f(z) + (1 - (1 - \gamma)^m) z f'(z) = A_{m,\lambda} f(z), \ \gamma > 0; m \in \mathbb{N},$$

$$A^\gamma_{m,\lambda} f(z) = A_{m,\lambda} \left(A^{\gamma - 1} f(z)\right)$$

$$= z + \sum_{n=2}^{\infty} \left[ 1 + (n - 1) C_f^n \gamma \right] \zetaut_n \zeta ^n; \zeta \in \mathbb{N},$$

(1.4)
where

\[ C_j^m(\gamma) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \gamma^j. \]

Using the relation (1.4), it is easily verified that

\[ C_j^m(\gamma)z(A_{m,\gamma}^f(z))' = A_{m,\gamma}^{\xi+1}f(z) - (1 - C_j^m(\gamma))A_{m,\gamma}^\xi f(z). \]  \hfill (1.5)

By specializing the parameters we observe that, for \( m = 1, A_{1,\lambda}^\xi \) defined by Al-Oboudi [1] and for \( m = \gamma = 1, A_{1,1}^\xi \) defined by Salagean [10].

2. The function class \( T_\xi^\gamma (m, \gamma, \lambda, p, q) \)

**Definition 2.1.** A function \( f(z) \in \Sigma \) is said to be in the class \( T_\xi^\gamma (m, \gamma, \lambda, p, q) \) if and only if

\[
(1 - \lambda) \frac{A_{m,\gamma}^\xi f(z)}{z} + \lambda (A_{m,\gamma}^\xi f(z))' < H_{p,q}(z) = \frac{1}{1 - xp\tau_p - xq\tau_q - spqz^2\tau_{p,q}}
\]

and

\[
(1 - \lambda) \frac{A_{m,\gamma}^\xi g(w)}{w} + \lambda (A_{m,\gamma}^\xi g(w))' < H_{p,q}(w) = \frac{1}{1 - xp\omega\tau_p - xq\omega\tau_q - spqw^2\tau_{p,q}}
\]

where \( 0 \leq \lambda \leq 1 \), \( z, w \) in \( \mathbb{U} \) and \( g(w) = f^{-1}(w) \).

3. Coefficient bounds for the function class \( T_\xi^\gamma (m, \gamma, \lambda, p, q) \)

We begin with the following result involving initial coefficient bounds for the function class \( T_\xi^\gamma (m, \gamma, \lambda, p, q) \).

**Theorem 3.1.** If \( f(z) \) given by (1.1) is in the class \( T_\xi^\gamma (m, \gamma, \lambda, p, q) \). Then

\[
|a_2| \leq \frac{(p + q)x\sqrt{(p + q)x}}{\chi} \]  \hfill (3.1)

and

\[
|a_3| \leq \frac{(p + q)^2x^2}{\chi} + \frac{(p + q)x}{(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\xi}. \]  \hfill (3.2)

where

\[
\chi = \left| (1 + 2C_j^m(\gamma))^\xi (1 + 2\lambda)(p + q)^2x^2 - [(p^2 + q^2)(p + q)x^2 + pqs] (1 + C_j^m(\gamma))^\xi (1 + \lambda) \right|^2.
\]

**Proof.** Let \( f(z) \in T_\xi^\gamma (m, \gamma, \lambda, p, q) \). Then there are analytic functions \( u \) and \( v \), with \( u(0) = v(0) = 0 \), \( |u(z)| < 1 \), \( |v(z)| < 1 \), given by (1.3) and satisfying the following conditions:

\[
(1 - \lambda) \frac{A_{m,\gamma}^\xi f(z)}{z} + \lambda (A_{m,\gamma}^\xi f(z))' = H_{p,q}(w(z)) \]  \hfill (3.3)

and

\[
(1 - \lambda) \frac{A_{m,\gamma}^\xi g(w)}{w} + \lambda (A_{m,\gamma}^\xi g(w))' = H_{p,q}(v(w)). \]  \hfill (3.4)

where \( g(w) = f^{-1}(w) \).
A short calculation shows that

\[ w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}), \]

and

\[ v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (w \in \mathbb{U}), \]

such that \( w(0) = v(0) = 0, \) \( |w(z)| < 1 \) \( (z \in \mathbb{U}) \) and \( |v(w)| < 1 \) \( (w \in \mathbb{U}). \)

It follows from (3.3) and (3.4) that

\[
(1 - \lambda) \frac{\mathcal{A}_{m, \gamma} f(z) - \mathcal{A}_{m, \gamma} f(z)'}{z} + \lambda (\mathcal{A}_{m, \gamma} f(z))' = 1 + R_1(x, s, p, q) c_1 z + [R_1(x, s, p, q) c_2 + R_2(x, s, p, q) c_1^2] z^2 + \cdots
\]

(3.5)

and

\[
(1 - \lambda) \frac{\mathcal{A}_{m, \gamma} g(w) - \mathcal{A}_{m, \gamma} g(w)'}{w} + \lambda (\mathcal{A}_{m, \gamma} g(w))' = 1 + R_1(x, s, p, q) d_1 w + [R_1(x, s, p, q) d_2 + R_2(x, s, p, q) d_1^2] w^2 + \cdots.
\]

(3.6)

A short calculation shows that

\[
(1 + \lambda)(1 + C_{j}^{m}(\gamma)) \xi a_2 = R_1(x, s, p, q) c_1,
\]

(3.7)

\[
(1 + 2\lambda)(1 + 2C_{j}^{m}(\gamma)) \xi a_3 = R_1(x, s, p, q) c_2 + R_2(x, s, p, q) c_1^2,
\]

(3.8)

\[-(1 + \lambda)(1 + C_{j}^{m}(\gamma)) \xi a_2 = R_1(x, s, p, q) d_1,
\]

(3.9)

and

\[
(1 + 2\lambda)(1 + 2C_{j}^{m}(\gamma)) \xi (2a_2^2 - a_3) = R_1(x, s, p, q) d_2 + R_2(x, s, p, q) d_1^2.
\]

(3.10)

From (3.7) and (3.9), we get

\[
c_1 = -d_1,
\]

(3.11)

and

\[
2[(1 + C_{j}^{m}(\gamma)) \xi (1 + \lambda)]^2 a_2^2 = R_1^2(x, s, p, q) (c_1^2 + d_1^2).
\]

(3.12)

By adding (3.8) to (3.10), we have

\[
2(1 + 2C_{j}^{m}(\gamma)) \xi (1 + 2\lambda) a_2^2 = R_1(x, s, p, q) (c_2 + d_2) + R_2(x, s, p, q) (c_1^2 + d_1^2).
\]

(3.13)

Therefore, from equalities (3.12) and (3.13) we find that

\[
[2(1 + 2C_{j}^{m}(\gamma)) \xi (1 + 2\lambda) R_1^2(x, s, p, q) - 2R_2(x, s, p, q) \left( (1 + C_{j}^{m}(\gamma)) \xi (1 + \lambda) \right) ^2] a_2^2
\]

\[= R_1^3(x, s, p, q) (c_2 + d_2). \]

(3.14)

Then

\[
|a_2| \leq \frac{(p + q)x \sqrt{(p + q)x}}{\sqrt{\chi}},
\]

where

\[
\chi = \left| (1 + 2C_{j}^{m}(\gamma)) \xi (1 + 2\lambda)(p + q)^2 x^2 - [(p^2 + q^2)(p + q)x^2 + pq] \left( (1 + C_{j}^{m}(\gamma)) \xi (1 + \lambda) \right) ^2 \right|.
\]

Next, in order to find the bound on \( |a_3|, \) subtracting (3.10) from (3.8) and using (3.11), we get

\[
2(1 + 2\lambda)(1 + 2C_{j}^{m}(\gamma)) \xi a_3 = 2(1 + 2\lambda)(1 + 2C_{j}^{m}(\gamma)) \xi a_2^2 + R_1(x, s, p, q)(c_2 - d_2).
\]

(3.15)
Then in view of (3.15) and (3.11), we have
\[ 2(1 + 2\lambda)(1 + 2C_j^m(\gamma))\zeta |a_3| \leq 2(1 + 2\lambda)(1 + 2C_j^m(\gamma))\zeta |a_2|^2 + 2R_1(x, s, p, q) \]

From (3.7), we immediately have
\[ |a_3| \leq |a_2|^2 + \frac{(p + q)x}{(1 + 2\lambda)(1 + 2C_j^m(\gamma))\zeta}. \]

Now the assertion (3.2) follows from (3.1). This evidently completes the proof of Theorem 4.1. \( \square \)

**Corollary 3.2.** If \( f(z) \) given by (1.1) is in the class \( T_2^\zeta(m, \gamma, p, q) \). Then
\[
|a_2| \leq \frac{(p + q)x\sqrt{(p + q)x}}{\sqrt{3(1 + 2C_j^m(\gamma))\zeta(p + q)^2x^2 - 4[(p^2 + q^2)(p + q)x^2 + pqs]\left(1 + C_j^m(\gamma)\zeta\right)^2}} \tag{3.16}
\]
and
\[
|a_3| \leq \frac{(p + q)^2x^2}{\tau} + \frac{(p + q)x}{3(1 + 2C_j^m(\gamma))\zeta}, \tag{3.17}
\]
where
\[
\tau = \left|3(1 + 2C_j^m(\gamma))\zeta(p + q)^2x^2 - 4[(p^2 + q^2)(p + q)x^2 + pqs]\left(1 + C_j^m(\gamma)\zeta\right)^2\right|.
\]

4. Fekete-Szegő inequalities for the function class \( T_2^\zeta(m, \gamma, \lambda, p, q) \)

Now, we are ready to find the sharp bounds of Fekete-Szegő functional \( a_3 - \delta a_2^2 \) defined for \( f \in T_2^\zeta(m, \gamma, \lambda, p, q) \) given by (1.1).

**Theorem 4.1.** Let \( f(z) \) given by (1.1), be in the class \( T_2^\zeta(m, \gamma, \lambda, p, q) \). Then
\[
|a_3 - \delta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{(p + q)x}{2(p + q)|h(\delta)|x} & \text{for } 0 \leq |h(\delta)| < \frac{1}{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))\zeta}, \\
\frac{(p + q)^2x^2(1 - \delta)}{2[(1 + 2C_j^m(\gamma))\zeta(1 + 2\lambda)(p + q)]x^2 - [(p^2 + q^2)(p + q)x^2 + pqs]\left(1 + C_j^m(\gamma)\zeta(1 + \lambda)\right)^2} & \text{for } |h(\delta)| \geq \frac{1}{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))\zeta},
\end{array} \right. \tag{4.1}
\]
where
\[
h(\delta) = \frac{(p + q)^2x^2(1 - \delta)}{2[(1 + 2C_j^m(\gamma))\zeta(1 + 2\lambda)(p + q)]x^2 - [(p^2 + q^2)(p + q)x^2 + pqs]\left(1 + C_j^m(\gamma)\zeta(1 + \lambda)\right)^2}.
\]

**Proof.** From (3.14) and (3.15), we get
\[
a_2^2 = \frac{R_3^2(x, s, p, q)(c_2 + d_2)}{2[(1 + 2C_j^m(\gamma))\zeta(1 + 2\lambda)R_2^2(x, s, p, q) - R_2(x, s, p, q)\left(1 + C_j^m(\gamma)\zeta(1 + \lambda)\right)^2]} \tag{4.2}
\]
and
Then, we easily conclude that

\[ a_3 = \frac{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\delta a_2^2 + R_1(x, s, p, q)(c_2 - d_2)}{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\xi}. \]  

(4.3)

From the equations (4.2) and (4.3), it follows that

\[ a_3 - \delta a_2^2 = R_1(x, s, p, q)\left[ \left( h(\delta) + \frac{1}{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\xi} \right) c_2 + \left( h(\delta) - \frac{1}{2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\xi} \right) d_2 \right], \]

where

\[ h(\delta) = \frac{R_1^2(x, s, p, q)(1 - \delta)}{2[(1 + 2C_j^m(\gamma))^\xi(1 + 2\lambda)R_1^2(x, s, p, q) - R_2(x, s, p, q) (1 + C_j^m(\gamma))^\xi(1 + \lambda)]}. \]

Then, we easily conclude that

\[
|a_3 - \delta a_2^2| \leq \begin{cases} 
\frac{(p + q)x}{(1 + 2\lambda)(1 + 2C_j^m(\gamma))^\delta}, & |h(\delta)| \leq \frac{1}{2(1 + 2\lambda + 6\mu)} \\
2(p + q)|h(\delta)|x, & |h(\delta)| \geq \frac{1}{2(1 + 2\lambda + 6\mu)}
\end{cases}
\]

This proves Theorem 4.1. \qed

By taking \( \lambda = 1 \) in Theorem 4.1, we have

Corollary 4.2. Let \( f(z) \) given by (1.1), be in the class \( T^\xi_2(m, \gamma, p, q) \). Then

\[
|a_3 - \delta a_2^2| \leq \begin{cases} 
\frac{(p + q)x}{3(1 + 2C_j^m(\gamma))^\delta}, & \text{for } 0 \leq |h(\delta)| < \frac{1}{6(1 + 2C_j^m(\gamma))^\xi} \\
2(p + q)|h(\delta)|x, & \text{for } |h(\delta)| \geq \frac{1}{6(1 + 2C_j^m(\gamma))^\xi}
\end{cases}
\]

(4.4)

where

\[ h(\delta) = \frac{(p + q)^2 x^2(1 - \delta)}{2[3(1 + 2C_j^m(\gamma))^\xi(p + q)^2x^2 - 4((p^2 + q^2)(p + q)x^2 + pqs)^2(1 + C_j^m(\gamma))^\xi]}. \]

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