FLUID MODELS FOR KINETIC EFFECTS ON COHERENT NONLINEAR ALFVÉN WAVES I: FUNDAMENTAL THEORY

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Abstract

Collisionless regime kinetic models for coherent nonlinear Alfvén wave dynamics are studied using fluid moment equations with an approximate closure anzatz. Resonant particle effects are modelled by incorporating an additional term representing dissipation akin to parallel heat conduction. Unlike collisional dissipation, parallel heat conduction is presented by an integral operator. The modified derivative nonlinear Schrödinger equation thus has a spatially nonlocal nonlinear term describing the long-time evolution of the envelope of parallel-propagating Alfvén waves, as well. Coefficients in the nonlinear terms are free of the $(1 - \beta)^{-1}$ singularity usually encountered in previous analyses, and have very a simple form which clarifies the physical processes governing the large amplitude Alfvénic nonlinear dynamics. The nonlinearity appears via coupling of an Alfvénic mode to a kinetic ion-acoustic mode. Damping of the nonlinear Alfvén wave appears via strong Landau damping of the ion-acoustic wave when the electron-to-ion temperature ratio is close to unity. For a (slightly) obliquely propagating wave, there are finite Larmor radius corrections in the dynamical equation. This effect depends on the angle of wave propagation relative to $B_0$ and vanishes for the limit of strictly parallel propagation. Explicit magnetic perturbation envelope equations amenable to further analysis and numerical solution are obtained. Implications of these models for collisionless shock dynamics are discussed.

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I. Introduction

It is widely believed that Alfvén waves play an important role in interplanetary plasmas. High level of magnetohydrodynamic (MHD) wave activity and shock waves are observed in the planetary and solar wind plasmas [1-3]. The plasma environment is characterized by a quite weak mean interplanetary magnetic field and large amplitude magnetic field fluctuations caused by Alfvén and magnetosonic waves. Thus, nonlinear effects play an important role in the evolution of such waves. Interest in large-amplitude Alfvén wave dynamics arose from the attempts to understand the steepening of wave trains and shock formation in space as well as from general interest in nonlinear waves as a whole.

Previous studies have shown [4-6] that nonlinear Alfvén and fast magnetosonic waves are described by the derivative nonlinear Schrödinger equation (DNLS). In this model, a parallel ponderomotive force of a high-amplitude wave perturbs the plasma density. The force squeezes plasma out the regions of larger magnetic field, locally decreasing the plasma density, and increasing the Alfvén wave velocity. As the amplitude of a circularly polarized Alfvén wave varies on time scales much slower than the wave frequency, the section of a wave train with higher amplitude propagates faster than the part with lower one. This gives rise to nonlinear steepening of the large-amplitude Alfvén wave. Dispersion, as usual, ultimately controls steepening. The DNLS thus describes the long time-scale dynamics of the envelope of such Alfvén waves. The DNLS is an integrable equation, which describes soliton and multi-soliton solutions, shock waves, modulation instability of solitons [7-9] etc. As was shown by Longtin and Sonnerup [10] and Wong and Goldstein [11] (from analyses of the full set of MHD equations), modulation instability occurs for left-hand polarized waves if $\beta < 1$ and for right-hand polarized waves if $\beta > 1$. This DNLS modulation instability is strongly sensitive to the sign of the coefficient of the cubic nonlinearity, which is proportional to $(1-\beta)^{-1}$. Indeed, at $\beta \approx 1$ (typical of solar wind plasma), the sound speed approaches the Alfvén velocity, and resonant energy exchange between
sound and Alfvén waves is strong, in the absence of parallel dissipation. Then, the Alfvén wave is always in phase with the acoustic wave, giving rise to fast steepening of the front of the nonlinear Alfvén wave train. The rate of the steepening is roughly proportional to \((1-\beta)^{-1}\). Significant conversion of Alfvén wave energy into ion thermal energy (i.e. ion heating) occurs as well. It is interesting to note that the overwhelming preponderance of theoretical work in this field of quasi-parallel Alfvénic shocks and solitons is based upon a simple MHD plasma model. Thus, with the notable exceptions of Ref. [6,12-14], the theoretical ‘lore’ of nonlinear Alfvén waves in a collisionless, \(\beta \sim 1\) plasma is built upon a conceptual paradigm constructed for a collisional, \(\beta < 1\) system. This state of affairs is due, in part, to the intractability and unwieldiness of straightforward kinetic analysis of nonlinear Alfvén waves. In this paper, we offer a simpler approach which exploits recent developments in the theory of fluid modelling of kinetic effects.

We can expect that the influence of kinetic effects, such as Landau damping must become important when \(\beta\) approaches unity. A fully kinetic calculation was undertaken by Rogister [5] for the case of a high-beta plasma. For parallel propagation, his results coincide with equations obtained later by Mjølhus and Wyller [12, 6] and Spangler [13, 14]. The effect of Landau damping appears in the DNLS via an additional cubic term which is an integral operator over space. There is no \((1-\beta)^{-1}\) singularity in the nonlinear derivative term. The nonlocality is a consequence of the finite time history of an ion transit through an envelope modulation (i.e. the coherence time of an ion with the modulation is not infinitesimal in comparison to the modulation growth time). Thus nonlinear Alfvén wave dynamics are nonlocal, and not governed by local spatial derivatives of the perturbed field, alone. The other attempts to include Landau damping in the nonlinear Alfvén wave evolution used kinetics to calculate second-order density (or pressure) perturbation, but the DNLS-like equation (for \(\tilde{b}\)) was obtained from MHD equations. Mjølhus and Wyller [12, 6] used a guiding center formalism and Spangler [13, 14] used the full Vlasov equation. Both methods predict a nonlocal integral term, and
the coefficients of the nonlinear terms coincide with those of Rogister [5].
In these cases, the coefficient of the derivative cubic nonlinear term does
not change sign at $\beta = 1$, except for large values of $T_e/T_i$ (the electron-
to-ion temperature ratio). The coefficient of the nonlocal term is always
negative. Both coefficients of the nonlinear terms (cubic and nonlocal)
depend strongly on $\beta$ and on $T_e/T_i$.

Obviously, a simple fluid model of collisionless shocks is desirable for
reasons of tractibility. However, kinetic effects are essential for describing
interplanetary plasma dynamics. It is known that fluid models poorly
represent most kinetic effects. Several authors have suggested that kinetic
effects, such as Landau damping may be modelled in fluid equations by
adding parallel dissipation terms [15,16]. Recently, a closure method for
modelling of kinetic effects was developed [17-21] and applied to ion-
temperature gradient instabilities using gyrofluid models. This closure
anzatz for fluid moment equations i) ensures particle, momentum and
energy conservation, ii) takes a simple form in wave-number space, and
iii) has a linear response function very close to that of a collisionless,
Maxwellian plasma. In this paper we use the simplest version of the
method, namely that proposed by Hammet and Perkins [17].

In this paper we investigate effects of dissipation on the nonlinear,
parallel-propagating Alfvén wave. We first use one-fluid MHD equations
with parallel dissipation coefficients $\mu_\parallel$ and $\chi_\parallel$ which are constant and
independent of mode frequency and wave-number. Later we replace con-
stant $\chi_\parallel$ with the integral operator representation of Hammet and Perkins
[17], and obtain a modified DNLS similar to that obtained by Mjølhus
and Wyller [12, 6] and Spangler [13, 14]. However, our approach yields
expressions for coefficients of the nonlinear terms which are much simpler
than theirs and allows clear, unambiguous physical interpretation of the
results. We also derive the $T_e/T_i$ dependence using two-fluid MHD equa-
tions. In an ion-acoustic wave, ion density perturbations of the electron
background are mediated by electric field effects. In collisionless plasmas
the ion-acoustic branch, thus, replaces the acoustic branch as caused by
gas-kinetic pressure perturbations. The results indicate that the nonlinearity of Alfvén waves in collisionless plasma is controlled by the coupling of the Alfvén mode to an ion-acoustic mode, i.e. an ion-density perturbation mediated by an electron response electric field. This is in contrast to the conventional view [4,7,8] of this nonlinearity as due to a ponderomotive (i.e. a gas-kinetic/collisional) plasma density perturbation. The kinetic damping on resonant particles is nothing more than the usual strong Landau damping of an ion-acoustic wave for $T_e \simeq T_i$. This damping leads to enhanced ion heating which further raises $T_i/T_e$.

We also consider the case of a slightly obliquely propagating wave. There, one can expect that other kinetic effects may be relevant to the dynamics. It is shown that the $\mathbf{E} \times \mathbf{B}$ drift in the electric field produced by charge separation in an ion-acoustic wave with an ambient magnetic field is not significant. Another effect which is important, however, is caused by gyro-averaged electric and magnetic fields acting on a particle over the scale of a Larmor orbit. (There are no $(k_\perp \rho_i)^2$ effects on dispersion in the DNLS approximation. Such effects may appear in higher-order in $\tilde{b}/B_0$ calculations.) This effect enters the modified DNLS in a way similar to collisional dissipation. The finite Larmor radius correction depends on the angle of propagation of a wave and disappears for a strictly parallel-propagating waves. We explore modulational stability for the general case of a dissipative nonlinear Alfvén wave. This instability is important when one considers the origin and evolution of solitons, wave packets, shock waves, etc.

The rest of this paper is organized as follows. In Section II we derive the evolution equation for dissipative nonlinear parallel-propagating Alfvén waves. In Section III the modified DNLS with the resonant particle effect is obtained. In Section IV we consider the influence of finite Larmor radius corrections to the dynamics of a slightly oblique, nonlinear Alfvén wave. In Section V we investigate the modulation instability of dissipative nonlinear Alfvén waves. Section VI is a discussion of the results obtained.
II. Dissipative Nonlinear Alfvén Waves

We obtain the equation governing nonlinear wave dynamics from a multiple time scale expansion of the dissipative MHD equations. Dissipation is included via the parallel viscosity $\mu_{\parallel}$ or the parallel heat conductivity $\chi_{\parallel}$. We considered both these cases for the following reason. The first case, $\mu \neq 0$, corresponds to the three-moment fluid model of Landau damping. The four-moment fluid model should contain both $\chi_{\parallel}$ and $\mu_{\parallel}$ to close the equations. As shown in Ref. [17], the best fit of the linear-response function of this model to the linear-response function of a Maxwellian plasma is achieved when $\mu_{\parallel} = 0$ and $\chi_{\parallel} \neq 0$. This is the second case we consider here. Note that any closure of this sort which tacitly assumes a Maxwellian plasma intrinsically fails to capture the strong local modification of the distribution function associated with large-amplitude turbulence, trapping, etc.

In the derivation, we follow Ref. [4]. We assume a plane wave propagating in the $z$ direction. Thus, all quantities are functions of $z$ and $t$. The MHD equations are written as:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial z} + u \frac{\partial \rho}{\partial z} = 0,$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + \frac{\partial p}{\partial z} + \frac{\partial S}{\partial z} + \frac{\partial b^2}{8\pi} = 0,$$

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial z}\right) \left(\frac{p}{\rho \gamma}\right) + \frac{\partial q}{\partial z} = 0,$$

$$\frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial z} - \frac{B_0}{4\pi} \frac{\partial b}{\partial z} = 0,$$

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial z} + b \frac{\partial u}{\partial z} - B_0 \frac{\partial v}{\partial z} = 0,$$

where the velocity components are $v = V_{\perp}, u = V_{\parallel}$, the magnetic field components are $b = B_{\perp}, B_0 = B_{\parallel}, \rho$ is the mass density, $p$ is the pressure, $\gamma$ is the polytropic constant. Also, $S$ and $q$ are the parallel momentum and heat fluxes respectively, i.e. $S = -\rho \mu_{\parallel} (\partial u / \partial z)$ and $q = -n \chi_{\parallel} (\partial T / \partial z)$, where $n = \rho / m_i$ and $T$ is the temperature. There is no longitudinal perturbation of the magnetic field, as $\nabla \cdot B = 0$. Eqs. (1)-(5) are the equations of
continuity, longitudinal momentum, energy conservation (with $q = 0$ being the adiabatic equation), transverse momentum and transverse flux, respectively. Dissipation, as stated above, appears only via either momentum flux ($\mu \neq 0, \chi = 0$) or heat flux ($\mu = 0, \chi \neq 0$). We expand the system of Eqs. (1)-(5) in powers of $\epsilon = b/B_0$. To avoid secularities at third order, we assume multiple time-scale dependence, i.e. $t_{(2n)} \sim t e^{2n}$ are the independent variables.

The zeroth-order solution of the system (1)-(5) is the equilibrium: $\rho_0 = \text{const}, V_0 = 0, B_0 = B_0 \hat{e}_z$, where $\hat{e}_z$ is the unit vector in $z$ direction. The first-order solution of Eqs. (4) and (5) is the linearized Alfvén wave, i.e. $b_1 = b_1(z \pm v_A t_{(0)}, t_{(2)}), v_1 = (v_A/B_0) b_1 + \tilde{v}_1(t_{(2)})$, where $v_A^2 = B_0^2/4\pi \rho_0$ is the Alfvén velocity. Eqs. (1)-(3) also have dissipative acoustic waves as a solution. As we consider travelling hydromagnetic waves propagating to the right, we set $\rho_1 = u_1 = 0$ and and choose a minus sign in the argument of $b_1$.

At second order, we consider $\mu_\parallel \neq 0$ and $\chi_\parallel \neq 0$ models, separately.

a) $\mu_\parallel \neq 0$ model.

Eqs. (1)-(3) become, in second order:

\begin{align*}
\frac{\partial \rho_2}{\partial t_{(0)}} + \rho_0 \frac{\partial u_2}{\partial z} &= 0, \\
\rho_0 \frac{\partial u_2}{\partial t_{(0)}} + \frac{\partial p_2}{\partial z} - \rho_0 \mu_\parallel \frac{\partial^2 u_2}{\partial z^2} &= -\frac{\partial}{\partial z} \frac{b_1^2}{8\pi}, \\
\frac{\partial}{\partial t_{(0)}} (p_2 - c_s^2 \rho_2) &= 0,
\end{align*}

where $c_s^2 = \gamma p_0/\rho_0$ is the sound speed. From this system, excluding the free sound wave solution (as in first order), we have:

\begin{align*}
u_2 &= \frac{v_A}{\rho_0} \rho_2 + \tilde{u}_2(t_{(2)}), \\
p_2 &= \frac{v_A^2}{\rho_0} \rho_2 + \mu_\parallel v_A \frac{\partial \rho_2}{\partial z} - \frac{b_1^2}{8\pi}, \\
p_2 &= c_s^2 \rho_2 + \tilde{p}_2(t_{(2)}).
\end{align*}
Solving this system of equations, we, finally, arrive at the equation for the second-order velocity perturbation:

\[ u_{2(\mu)} = \frac{v_A^2}{2\mu_{||}} e^{-z/L_\mu} \int^z e^{z'/L_\mu} \left[ \frac{b_1^2(z') - \langle b_1^2(z') \rangle}{B_0^2} \right] dz', \]  

(12)

where \( \beta \equiv c_s^2/v_A^2 \) and \( L_\mu = (\mu_{||}/v_A)(1 - \beta)^{-1} \). This is the inhomogeneous solution of Eq. (11), finite for all \(-\infty < z < +\infty\). The term \( \langle b_1^2 \rangle \) appears in Eq. (12) from density conservation: \( d\langle \rho \rangle/dt = 0 \). Note that it is the deviation from the mean ponderomotive force which causes density bunching and thus wave steepening. Thus \( \langle \rho \rangle = \rho_0 \), so that \( \langle u_2 \rangle = (v_A/\rho_0) \langle \rho_2 \rangle = 0 \).

b) \( \chi_{||} \neq 0 \) model.

Eqs. (1)-(3) in this case are replaced by:

\[ \frac{\partial \rho_2}{\partial t} + \rho_0 \frac{\partial u_2}{\partial z} = 0, \]  

(6)

\[ \rho_0 \frac{\partial u_2}{\partial t} + \frac{\partial p_2}{\partial z} = -\frac{\partial}{\partial z} \frac{b_1^2}{8\pi}, \]  

(7')

\[ \frac{\partial}{\partial t}(p_2 - c_s^2 \rho_2) = -\frac{\partial q_2}{\partial z}, \]  

(8')

The conductive heat flux can be represented as follows:

\[ q_2 = -n_0 \chi_{||} \frac{\partial}{\partial z} T_2 - n_1 \chi_1 \frac{\partial}{\partial z} T_1 - n_2 \chi_2 \frac{\partial}{\partial z} T_0. \]  

(13)

The second term here vanishes because \( n_1 = \rho_1/m_i = 0 \), as we set in the first-order equations. The third term vanishes also because we consider only a homogeneous plasma temperature \( T_0 = \text{const} \). The second-order temperature perturbation is, in turn, given by:

\[ T_2 = \frac{p_2 - T_0 \rho_2}{n_0} = \frac{p_2 - v_t^2 \rho_2}{n_0}, \]  

(14)

where \( v_t^2 = T_0/m_i \) is the thermal velocity of the particles. Now, from Eqs. (6), (7'), (8'), and using (13) and (14) we obtain:

\[ u_2 = \frac{v_A}{\rho_0} \dot{u}_2(t_2), \]  

(9)

\[ p_2 = v_A^2 \rho_2 - \frac{b_1^2}{8\pi} + \tilde{p}_2(t_2), \]  

(10')

\[ (p_2 - c_s^2 \rho_2) + \frac{\chi_{||}}{v_A} \frac{\partial}{\partial z} (p_2 - c_s^2 \rho_2) = 0. \]  

(11')
From these equations we obtain the second-order velocity perturbation for the $\chi_\parallel \neq 0$ model as:

$$u_2(\chi) = \frac{v_A}{2(1 - \beta/\gamma)} \left\{ \frac{b_1^2}{B_0^2} - \langle b_1^2 \rangle + \frac{v_A}{\chi_\parallel} \left( 1 - \frac{1 - \beta}{1 - \beta/\gamma} \right) e^{-z/L_\chi} \int e^{z'/L_\chi} \left[ \frac{b_1^2(z') - \langle b_1^2(z') \rangle}{B_0^2} \right] dz' \right\}. \quad (12')$$

Here $L_\chi = (\chi_\parallel/v_A)(1 - \beta/\gamma)/(1 - \beta)$. The velocities $u_{2(\mu)}$ and $u_{2(\chi)}$ are expressed in the frame co-moving with a wave (i.e. $z \to z - v_At(0)$). The coefficient $\gamma = 3$ is chosen to ensure energy conservation [17].

Eqs. (4) and (5) are, at second order, equations for linear Alfvén waves. One may easily show that for all $t(2)$ one may set $b_2$ and $v_2$ to zero [4]. At third order, upon substituting first- and second-order solutions into Eqs. (4) and (5) one obtains the equation for $\partial b_1/\partial t(2)$:

$$\frac{\partial b_1}{\partial t(2)} = -\frac{1}{2} \frac{\partial}{\partial z} (u_2 b_1). \quad (15)$$

For a dispersive term [6, 24] one must invoke finite Larmor radius effects. Since Eq. (15) describes the transverse components of $b$, there is no influence of parallel dissipation effects on the linear dispersive term. We thus add a general form of this term to obtain a DNLS-like magnetic perturbation envelope equation. Introducing $\tau = t(2)$ and $\phi = (b_{1x} + ib_{1y})/B_0$ we have:

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial z} (u_2 \phi) \mp i \frac{v_A^2}{2\Omega_i} \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (16)$$

where $\Omega_i = eB_0/m_i c$ is the ion-cyclotron frequency, upper ($-$) and lower ($+$) signs on the third term refer to right and left elliptically (circularly) polarized waves. Eq. (16) with (12) or (12') inserted for $u_1$ describes the coherent nonlinear dynamics of an Alfvén wave train subject to damping. Damping results in the appearance of a new nonlocal integral contribution to the envelope equation. We define the integral dissipation operator as:

$$J_L[F](x) = e^{-x/L} \int_x^x e^{x'/L} F(x') dx', \quad (17)$$
where $F$ is the arbitrary function the operator acts on and $L$ is a characteristic length proportional to the dissipation coefficient (i.e. $L \sim \mu_{\parallel}/v_A(1 - \beta)$ or $L \sim \chi_{\parallel}/v_A(1 - \beta)$).

From comparison of Eqs. (12), (12') to the result of Ref. [13] we conclude that three-moment approximation ($\chi_{\parallel} \neq 0$ model) of Landau damping recovers the correct functional dependence $u_2 \simeq c_1 b_1^2 + c_2 \mathcal{L}[b_1^2]$, where $\mathcal{L}$ is a nonlocal operator, and $c_1$ and $c_2$ are some coefficients. However, the integral operator in this case (Eq. (17)) is different from the resonant particle operator of Ref. [13] (see Eq. (21) in the next section). The two-moment approximation ($\mu_{\parallel} \neq 0$ model) describes the nonlinear physics incorrectly. Indeed, it lacks a free cubic nonlinear contribution to the expression for $u_2$. This is in agreement with the conclusions of Ref. [17], which, stated simply, are the higher the moment approximation used, the better the description of Landau damping in a fluid model which results. In particular, the linear response function of the three-moment fluid model is much closer to the exact Maxwellian linear response function then that of the two-moment fluid model. From now on we use the three-moment fluid model ($\chi_{\parallel} \neq 0$).

III. Coherent Nonlinear Alfvén Waves with Landau Damping

Using the $\chi \neq 0$ model, kinetic effects are modeled by a longitudinal heat flux. In the linear closure approximation, this flux is (in wave-number space [17]):

$$q_{2k} = -n_0 \chi_1 \sqrt{2v_t} \frac{\sqrt{2v_t}}{|k|} ikT_{2k}.$$  

(18)

Here the temperature perturbation is given by Eq. (14), and $\chi_1$ is a dimensionless fit coefficient for the model. The choice $\chi_1 = 2/\sqrt{\pi}$ gives the
best fit to linear Landau damping. By performing the inverse Fourier transform of $q_{2k}$, we obtain the real-space representation of $q_2$ as:

$$q_2(z) = \frac{1}{\sqrt{2\pi}} \lim_{\delta \to 0} \int_{-\infty}^{\infty} dk e^{-|k|\delta} e^{ikz} q_{2k}$$

$$= -\frac{n_0\chi_1\sqrt{2}v_t}{\pi} \int_{0}^{\infty} \frac{T_2(z+z') - T_2(z-z')}{z'} dz'$$

$$= -\frac{n_0\chi_1\sqrt{2}v_t}{\pi} \int_{-\infty}^{\infty} \mathcal{P} \frac{T_2(z')}{z'-z} dz'.$$

(19)

Here $\mathcal{P}$ stands for the Cauchy principal value integral. We have also added the factor of $\exp(-|k|\delta)$ to control the otherwise infinite integral. From comparison of this equation with Eq. (13), we conclude that the effect of resonant particles in our theory is reduced to the following replacement:

$$\frac{\chi||}{v_A} \frac{\partial}{\partial z} \rightarrow \hat{\chi}|| \mathcal{L}.$$  

(20)

We thus define here the resonant particle (nonlocal) integral operator:

$$\mathcal{L}[F](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{P}}{x'-x} F(x') dx'$$  

(21)

with the coefficient $\hat{\chi}|| = \chi_1\sqrt{2}v_t/v_A$.

We now consider properties of the resonant particle operator in more detail. We need to know the inverse operator, $\mathcal{L}^{-1}$, so that $\mathcal{L}^{-1}[\mathcal{L}] = 1$. It is easy to find in $k$-space representation, i.e. $\mathcal{L}^{-1}\mathcal{L} = \mathcal{L}_k^{-1} i k/|k| = \mathcal{L}_k^{-1}$ $\text{sign}(k) = 1$, thus, $\mathcal{L}_k^{-1} = -i \text{sign}(k)$ and, returning to real space:

$$\mathcal{L}^{-1} = -\mathcal{L}.$$  

(22)

Formula (22) reflects the fact that Landau damping is time reversible, i.e. that a system returns to its initial state when time is reversed. There is no microscopic information loss in a Vlasov system, unlike in the case of collisional damping. Writing an evolution equation in the form $(\partial/\partial t + \mathcal{L})\psi = 0$, we see that after time inversion ($t \rightarrow -t$), $\mathcal{L}$ must satisfy Eq. (22)
to represent damping. It can also be shown that the operator $L$ commutes with any standard differential or integral operator, i.e.

$$\frac{\partial}{\partial z}(L[F(z)]) = L\left[\frac{\partial}{\partial z}F(z)\right],$$

$$\int (L[F(z)])dz = L\left[\int F(z)dz\right],$$

(23)

$L[c] = 0$,

where $c = \text{const}$.

Making a replacement, Eq. (20), in the energy conservation equation (11'), we have:

$$\left((1 - \beta) + \left(1 - \frac{\beta}{\gamma}\right)\hat{\chi}_\parallel L\right)\rho_2 = \frac{\rho_0}{2} \left(1 + \hat{\chi}_\parallel L\right)\frac{b_1^2 - \langle b_1^2 \rangle}{B_0^2}.$$

(24)

This equation can be easily solved for $\rho_2$ using Eq. (22). The second-order velocity perturbation is then:

$$u_2 = v_A \left\{ M_1 |\Phi|^2 + M_2 L\left[|\Phi|^2\right] \right\},$$

(25)

where $|\Phi|^2 \equiv |\phi|^2 - \langle |\phi|^2 \rangle = (b_1^2 - \langle b_1^2 \rangle)/B_0^2$, and the factors $M_1$ and $M_2$ are:

$$M_1^{(1)} = \frac{1}{2} \frac{(1 - \beta) + \hat{\chi}_\parallel^2(1 - \beta/\gamma)}{(1 - \beta)^2 + \hat{\chi}_\parallel^2(1 - \beta/\gamma)^2},$$

(26a)

$$M_2^{(1)} = \frac{1}{\hat{\chi}_\parallel} - \frac{1}{\gamma} \frac{1}{(1 - \beta)^2 + \hat{\chi}_\parallel^2(1 - \beta/\gamma)^2},$$

(26b)

Here $\hat{\chi}_\parallel = \chi_1 \sqrt{2\beta/\gamma}$, $\gamma = 3$ and $\chi_1 = 2/\sqrt{\pi}$ according to Ref. [17]. The superscript (1) means that the result refers to the one-fluid model.

Eq. (16) together with Eq. (25) recovers the modified derivative nonlinear Schrödinger equation governing the envelope of nonlinear Alfvén modes subject to Landau damping. Fig. 1 represents the dependence of the factors $M_1^{(1)}$ and $M_2^{(1)}$ on $1/\beta$ for comparison with the work by Spangler [13]. This case corresponds to the $T_e/T_i = 1$ case of Spangler. The coefficient $M_1$ of Spangler is positive, always. The coefficient $M_1^{(1)}$ in our analysis changes sign at $\beta = 1$, although where negative, its absolute value is small. The disagreement is related to the lack of higher moments in
the closure scheme we use. The coefficients $M_2^{(1)}$ and $M_2$ of Spangler are negative and look alike. Analytical expressions for the coefficients in our approach are much simpler than those obtained from kinetic calculations. The phenomenon which is described here by Eqs. (25), (26) is typical for other resonance phenomena. The sharp resonance without damping (here the resonance between Alfvén and sound waves) is, of course, smoothed by increasing damping. In other words, there is resonance broadening caused by the interaction of particles with a wave. In this respect, the coupling to dissipation calculated here is the coherent analogue of nonlinear ion Landau damping.

Until now we have considered only one-fluid magnetohydrodynamics. Based on this model, we derived the Alfvén-sound wave resonance broadening in a DNLS-like equation. Astrohysical plasmas are essentially collisionless, i.e. the wave-length is much larger than mean free path, $\lambda \ll \ell_{mf}$. Thus, an ion-acoustic mode replaces the sound wave in the collisionless regime. Hence, it is natural to adopt the perspective that the nonlinearity of finite-amplitude Alfvén waves arises from an ion density perturbation, that is from coupling of the ion-acoustic and Alfvén branches of the plasma oscillation. In other words, we must consider electrostatic density perturbations instead of pure neutral density perturbation as in a sound wave. The electric fields of the ion-acoustic wave play a role in the dynamics of oblique nonlinear Alfvén waves, as will be shown later. To include the ion-acoustic wave coupling effects instead of the gas-kinetic pressure perturbation, we must use two-fluid dynamics for both electron and ion fluids. The longitudinal momentum equation, Eq. (2), for ions in a collisionless plasma is:

$$\frac{\rho}{\partial t} \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + e \frac{\partial \varphi}{\partial z} + \frac{\partial b^2}{\partial z} \frac{b^2}{8\pi} = 0,$$

where $\varphi$ is the electric potential. We neglect wave damping by electrons, which is always much smaller than damping on ions, i.e. $\gamma_{elect}/\gamma_{ion} \simeq$
Due to the quasineutrality of ion-acoustic waves, the first nonvanishing term of the expansion of the Boltzmann distribution for electrons is:

\[ \tilde{n}_i \simeq \tilde{n}_e = n_0 \frac{e \varphi}{T_e}. \]  

(28)

Upon substituting this into Eq. (27), and formally defining (for ions)

\[ p_2^* = n_2 T_0 e = (T_e / T_i) p_2, \]

we can write Eq. (7') as:

\[ \rho_0 \frac{\partial u_2}{\partial t(0)} + \frac{T_e}{T_i} \frac{\partial p_2}{\partial z} = - \frac{\partial}{\partial z} \frac{b_1^2}{8\pi}. \]  

(29)

Multiplying Eq. (8') by the factor \( T_e / T_i \) and rewriting it with \( p_2^* \) we have for this new pressure the same equation, with the \( c_{s_i}^2 = (T_e / T_i) c_s^2 \), - the speed of an ion-acoustic wave. We do the same with the closure equation (14). Thus, all equations still apply for the new pressure \( p_2^* \) when we substitute

\[ \beta \rightarrow \frac{T_e}{T_i} \beta \]  

(30)

everywhere except in the \( \tilde{\chi}_{||} \) coefficient. There \( \beta \) appears as a combination of the ion thermal velocity and the Alfvén velocity.

When \( T_e / T_i \gg 1 \) an ion-acoustic wave experiences almost no damping (neglecting the feeble damping by electrons). In the opposite case, \( T_e / T_i \simeq 1 \), there is strong ion Landau damping. This is the case of the one-fluid calculation. So, we must redefine the fit coefficient:

\[ \chi_1 \rightarrow \chi_1 \, f \left( \frac{T_e}{T_i} \right) \]  

(31)

with some correction function \( f(T_e / T_i) \) that reflects the behavior in two asymptotic regimes. This function must reflect the temperature dependence of the damping rate, and must equal unity at \( T_e / T_i = 1 \). The damping rate of ion-acoustic waves in a Maxwellian plasma is well known [22]:

\[ \frac{\gamma}{\omega} \simeq \left( \frac{\pi}{8} \right)^{1/2} \left( \frac{T_e}{T_i} \right)^{3/2} \exp \left\{ - \frac{T_e}{2T_i} \right\}. \]  

(32)

Then, for the correction function \( f \) we obtain:

\[ f \left( \frac{T_e}{T_i} \right) = \left( \frac{T_e}{T_i} \right)^{3/2} \exp \left\{ - \frac{T_e}{2T_i} + \frac{1}{2} \right\}. \]  

(33)
Upon substitutions given by Eqs. (30) and (31) in Eqs. (26), we arrive at:

\[ M_1 = \frac{1}{2} \left( 1 - \frac{T_e}{T_i} \beta \right) + \frac{\hat{\chi}_\parallel^2}{2 \left( 1 - \frac{T_e}{T_i} \beta \right)^2} \left( 1 - \frac{T_e}{T_i} \beta \right)^2 \]  
\[ M_2 = -\frac{1}{2} \frac{\chi_\parallel T_e}{T_i} \beta \frac{\gamma - 1}{\gamma} \frac{1}{1 - \frac{T_e}{T_i} \beta} \left( 1 - \frac{T_e}{T_i} \beta \right)^2 + \frac{\hat{\chi}_\parallel^2}{2} \left( 1 - \frac{T_e}{T_i} \beta \right)^2 \]  

(34a)  

(34b)

with \( \hat{\chi}_\parallel = \chi_1 \sqrt{2\beta/\gamma} f(T_e/T_i) \). Figs. 2a and 2b show the functions \( M_1 \) and \( M_2 \), respectively, plotted vs. \( 1/\beta \) for different electron-to-ion temperature ratios \( (T_e/T_i = 1, 3, 5, 10) \). These figures closely resemble the graphs of Spangler [14] and Mjølhus and Wyller [6], however the expressions Eqs. (34) are much simpler than their counterparts in Refs. [6,14], and easily amenable to detailed calculation and evaluation, etc.. The coefficient \( M_2 \) is always negative and has a peak near \( \beta_0 = T_i/T_e \). There is slight difference when \( T_e \approx T_i \). The coefficient \( M_1 \) should be positive in this case, for almost all values of \( \beta \). This difference can result from insufficient accuracy of the three-moment approximation \( (\chi_\parallel \neq 0) \) when Landau damping is very strong. To get a better fit one should take higher-moment approximations. Of course, for large amplitude Alfvén waves the assumption of a Maxwellian plasma fails, as well.

IV. Slightly Oblique Nonlinear Alfvén Waves with Landau Damping and Finite Larmor Radius Effects

In the previous sections we considered Alfvén waves propagating strictly parallel to the magnetic field lines. Here we try to generalize our approach to waves propagating at a small angle to the magnetic field. It has been shown [12, 23-25] that dissipationless oblique nonlinear Alfvén waves are governed by the DNLS equation, as well. We investigate here the influence of kinetic effects on wave dynamics. To include kinetic effects in the fluid model of the DNLS we use a two-fluid gyrofluid
model [18,19]. Here we construct the simplest model which includes kinematic effects such as finite gyro-frequency, Landau damping, etc. We omit gyro-viscous corrections to the Reynolds stress. These corrections are small for the slightly oblique propagation case in straight field geometry, and can be incorporated (approximately) into the viscous model considered in Section II.

Typically, the gyrofluid equations are written in components projected onto the magnetic field direction, \( z' \), and perpendicular to it. It is more convenient to us to work in the frame of a wave propagating in the \( z \) direction, which makes some angle \( \Theta \ll 1 \) relative to the \( z' \) direction. We choose the \( y \) direction coincident with \( y' \). We leave variables ‘unprimed’ when measured in the frame of a wave, and use primes for variables measured in the frame of the ambient magnetic field. We consider contributions of kinetic effects separately for every equation of the system (1)-(5). The equation for the transverse flux is obtained from the Maxwell equations \( \nabla \times \mathbf{E} = (-1/c)\partial \mathbf{B}/\partial t, \nabla \times \mathbf{B} = (4\pi/c)\mathbf{J} \) and the Ohm’s law. We write the generalized Ohm’s law as follows:

\[
\mathbf{E} = -\frac{1}{c} [\mathbf{v} \times \mathbf{B}] + \frac{m_i}{e\rho} [\mathbf{J} \times \mathbf{B}] - \frac{m_i}{2e\rho} \nabla p + \nabla \varphi_{el}. \quad (35)
\]

There are no contributions from the last two terms. There is no contribution from the term \( \nabla \varphi \times \nabla \rho \) because \( \varphi \) is the first-order perturbation, and we have set \( \rho_0 = const, \rho_1 = 0 \). The first term has already been accounted for. The second term leads to an additional term on the right-hand-side of Eq. (5). We write it in components as follows:

\[
\frac{\partial b_x}{\partial t} + \frac{\partial}{\partial z}(ub_x) - B_{0z} \frac{\partial v_x}{\partial z} + B_{0x} \frac{\partial u}{\partial z} = -\frac{v_A^2}{\Omega_i} \left( B_{0z} \frac{\partial^2 b_x}{\partial z^2} \right), \quad (36a)
\]

\[
\frac{\partial b_y}{\partial t} + \frac{\partial}{\partial z}(ub_y) - B_{0z} \frac{\partial v_y}{\partial z} = \frac{v_A^2}{\Omega_i} \left( B_{0z} \frac{\partial^2 b_y}{\partial z^2} \right). \quad (36b)
\]

The terms on the right-hand-side give rise to a linear dispersion term in the DNLS. This term describes dispersion due to finite ion Larmor radius. In gyrofluid models there is no correction to the transverse momentum
equation, Eq. (4). We write it with the substitution of $B_0$ by $B_{0z}$, as is obvious for oblique propagation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho u \frac{\partial \mathbf{v}}{\partial z} = \frac{B_{0z}}{4\pi} \frac{\partial \mathbf{b}}{\partial z}. \tag{37}$$

In the gyrofluid equations [18,19] there are corrections from both $\mathbf{E} \times \mathbf{B}$ velocity and an effect of gyro-averaging of fields over the Larmor orbit. The $\mathbf{E} \times \mathbf{B}$ drift velocity $\mathbf{v}_E = c[\mathbf{E} \times \mathbf{B}]/B^2$ enters the gyrofluid analogs of Eqs. (1)-(3) in the combination $\nabla \cdot (u'\hat{\mathbf{e}}_{z'} + \mathbf{v}_E')$, where $\hat{\mathbf{e}}_{z'}$ - is the unit sector in $z'$ direction. Upon substituting the electric field from Eq. (35) we obtain the following. The first term contributes to the velocity $u'$, giving rise to $\nabla \cdot (u'\hat{\mathbf{e}}_{z'} + \mathbf{v}_\perp') = \nabla \cdot \mathbf{v} = \partial u/\partial z$, i.e. recovering the usual velocity divergence term. The second term does not make a contribution because $(4\pi/c)\mathbf{J} = \nabla \times \mathbf{B} = \nabla \times \mathbf{b}_\perp$ and, thus, $\mathbf{J}_\perp \equiv 0$. The last two terms are both gradient terms. For a plane wave the gradient is reduced to $\nabla = \hat{\mathbf{e}}_z(\partial/\partial z)$. So, both these terms contribute to $\mathbf{v}_E$ like $[\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_{z'}] \sim \hat{\mathbf{e}}_y$. Upon substituting into the divergence term of Eqs. (1)-(3), it vanishes. Thus, we conclude that for a plane single coherent wave, the $\mathbf{E} \times \mathbf{B}$ drift does not make a significant contribution to the nonlinear Alfvén wave dynamics.

Another effect we should consider is the gyro-averaging of electric and magnetic fields over the Larmor orbits of particles. This effect enters the equations via a differential operator which approaches unity when the ion Larmor radius vanishes. Since it acts on fields only and because we construct the model which must reduce to MHD in the case of zero ion Larmor radius, finite gyro-radius corrections appear only in Eq. (2), leaving Eqs. (1) and (3) unchanged. In the case of oblique propagation, Eqs. (1)-(3) are:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) &= 0, \\
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} &= -\langle J_0 \rangle \left\{ \frac{\partial p^*}{\partial z} + \frac{\partial b_x^2 + b_y^2}{8\pi} + \frac{B_{0z} \partial b_x}{4\pi} \right\} = 0, \\
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial z} \right) \left( \frac{p^*}{\rho^*} \right) + \frac{T_e}{T_i} \frac{\partial q}{\partial z} &= 0.
\end{align*}
\]
Here we have already used the quasineutrality condition together with Eq. (28) for the electric potential perturbation. This appears on the right-hand-side of Eq. (39) as a gradient of $p^*$. This accounts for the charge separation in an ion-acoustic wave, and the subsequent gyro-averaging of the electric field associated with the electron response. Eq. (40) is a trivial rewriting of Eq. (3) for the quantity $p^*$.

The operator $J_0$, which carries out the gyro-averaging operation, is a linear operator. It is simply a Bessel function represented in Fourier space:

$$J_0\left(\frac{k'_i v'_i}{\Omega_i}\right) = \int_0^{2\pi} d\vartheta \exp\left\{i \frac{k'_i v'_i}{\Omega_i} \cos \vartheta \right\} = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{v'^2}{2\Omega^2_i}\right)^n \nabla'^2_{\perp}, \quad (41)$$

where $\nabla'^2_{\perp}$ is the gradient in the plane perpendicular to $z'$. Going to the frame of the wave, it contains a longitudinal projection equal to $\sin \Theta (\partial/\partial z)$. All other components vanish for a plane wave in a homogeneous plasma. The operator $\langle J_0 \rangle$ is the operator $J_0$ averaged over the (Maxwellian) particle distribution function. There are different ways [19] to approximate $\langle J_0 \rangle$. We choose one of the more simple approximations:

$$\langle J_0 \rangle \cong \Gamma_0^{1/2} \simeq 1 - \frac{\rho_i^2}{2} \nabla'^2_{\perp} + ... \simeq 1 - \frac{\rho_i^2}{2} \sin^2 \Theta \frac{\partial^2}{\partial z^2}, \quad (42)$$

where $\rho_i = v_i/\Omega_i$ is the ion Larmor radius. We neglect, as usual, terms of higher order in $\rho_i^2$.

Equations (36)-(40) replace Eqs. (1)-(5) for the case of an obliquely propagating wave subject to Landau damping and including finite Larmor radius effects. The procedure for derivation of the DNLS-like equation describing nonlinear, obliquely propagating Alfvén waves is similar to that explained in Section II for parallel-propagating waves, but requires a bit more algebra. From Eqs. (36) and (37) we finally arrive at the DNLS-like equation, Eq. (16), with a new field:

$$\psi = \frac{b_{1x} + ib_{1y} + B_{0x}}{B_{0z}} \quad (43)$$
instead of $\phi = (b_{1x} + ib_{1y})/B_0$. One can mention that in terms of the field $\phi$ the DNLS for oblique waves is more complicated and contains Korteweg-de Vries nonlinearities. These terms are proportional to $B_0x$ and, thus, disappear in the case of strictly parallel propagation.

The velocity perturbation $u_2$ is found from the second-order expansion of Eqs. (38)-(40). After one integration these equations can be written in the frame moving with the wave as:

$$v_A\rho_0 u_2 = \Gamma_0^{1/2} p_2^* + \Gamma_0^{1/2} \frac{b_{1x}^2 + b_{1y}^2}{8\pi} + \Gamma_0^{1/2} \frac{B_0 b_{1x}^2}{4\pi},$$ (44)

$$\left( p_2^* - \frac{c_2^2 \rho_0}{v_A} u_2 \right) + \hat{\chi}_\parallel L \left[ p_2^* - \frac{c_2^2 \rho_0}{\gamma v_A} u_2 \right] = 0.$$ (45)

Upon acting on the second equation with the operator $\Gamma_0^{1/2}$ and substituting $\Gamma_0^{1/2} p_2^*$ from the first, we obtain the equation for $u_2$ as follows:

$$\left\{ \left( 1 - \beta^* \Gamma_0^{1/2} \right) + \hat{\chi}_\parallel L \left( 1 - \frac{\beta^* \Gamma_0^{1/2}}{\gamma} \right) \right\} [u_2] = \frac{v_A}{2} \left\{ 1 + \hat{\chi}_\parallel L \right\} \Gamma_0^{1/2} \left[ |\Psi|^2 \right],$$ (46)

where $\beta^* = (T_e/T_i)\beta$ and $|\Psi|^2 = |\psi|^2 - \langle |\psi|^2 \rangle$. We solve this equation using the inverse operator (22) and the commutation relations (23). Using approximation for the operator $\Gamma_0^{1/2}$ given by Eq. (42), we finally obtain the following equation for $u_2$:

$$\left( 1 - \Lambda^2 \frac{\partial^2}{\partial z^2} \right) u_2 = v_A \left\{ \left( M_1 - N_1 \frac{\partial^2}{\partial z^2} \right) + L \left( M_2 - N_2 \frac{\partial^2}{\partial z^2} \right) \right\} \left[ |\Psi|^2 \right],$$ (47)

where

$$\Lambda^2 = -2\eta_{||}^2 \beta^* \frac{(1 - \beta^*) + \hat{\chi}_\parallel^2 (1 - \beta^*/\gamma)}{(1 - \beta^*)^2 + \hat{\chi}_\parallel^2 (1 - \beta^*/\gamma)^2},$$ (48a)

$$N_1 = \frac{1}{2} \eta_{||}^2 \frac{(1 - 2\beta^*) + \hat{\chi}_\parallel^2 (1 - 2\beta^*/\gamma)}{(1 - \beta^*)^2 + \hat{\chi}_\parallel^2 (1 - \beta^*/\gamma)^2},$$ (48b)

$$N_2 = -\eta_{||}^2 \hat{\chi}_\parallel \beta^* \frac{\gamma - 1}{\gamma} \frac{1}{(1 - \beta^*)^2 + \hat{\chi}_\parallel^2 (1 - \beta^*/\gamma)^2}. $$ (48c)
Here we used the notation $\eta_2^2 \equiv (\rho_2^2/2) \sin^2 \Theta$, and $M_1$ and $M_2$ are given by Eqs. (34). The solution of this equation can be easily obtained in terms of the integral dissipation operator $J$ given by Eq. (17). We thus have:

$$u_2 = v_A \left( \left( M_1 + M_2 \mathcal{L} \right) \frac{1}{2\Lambda} (J_\Lambda - J_{-\Lambda}) + \left( N_1 + N_2 \mathcal{L} \right) \frac{1}{\Lambda^2} \right) \left[ |\Psi|^2 \right]. \quad (49)$$

This is the inhomogeneous solution valid for all values of $\Lambda^2$. There is the finite homogenous solution when $\Lambda^2$ is negative, i.e. $\beta^* < 1$:

$$u_{2\text{hom}} = u_{2\text{in}} \sin(z/|\Lambda|), \quad (49')$$

where $u_{2\text{in}}$ is the constant that can be obtained from initial conditions. This solution represents, in a sense, a free modulation wave, i.e. a traveling modulation of the envelope of the nonlinear Alfvén wave train with amplitude set by initial conditions. When $\Lambda^2$ becomes positive, this solution of the homogenous equation diverges at infinity like $\sinh(z/\Lambda)$.

The dissipation operator combination can be written as:

$$\frac{1}{2\Lambda} (J_\Lambda - J_{-\Lambda}) [F] = \frac{1}{2\Lambda} \left[ \int z' e^{(z'-z)/\Lambda} F(z') dz' - \int z' e^{-(z'-z)/\Lambda} F(z') dz' \right]$$

$$= \begin{cases} 
\frac{1}{\Lambda} \int z' \sin \left[ \frac{(z' - z)}{\Lambda} \right] F(z') dz', & \text{if } \Lambda^2 > 0; \\
-\frac{1}{|\Lambda|} \int z' \sin \left[ \frac{(z' - z)}{|\Lambda|} \right] F(z') dz', & \text{if } \Lambda^2 < 0.
\end{cases} \quad (50)$$

It is interesting to note that the kernel of this integral is an antisymmetric function of $z' - z$, as in the resonant particle integral, but is not singular at $z' = z$. Thus, the main contribution to this integral is not from the instantaneous position $z$, unlike the resonant particle operator. It is also interesting that this operator does not correspond to pure dissipation, because the kernel of the operator $J_L$ itself consists of both symmetric and antisymmetric parts.

When $\Lambda^2 \simeq 0$ we can write the approximate solution of Eq. (47). Using an iterative method, we obtain:

$$u_2 = v_A \left\{ \left( M_1 + M_2 \mathcal{L} \right) + \left[ (\Lambda^2 M_1 - N_1) + (\Lambda^2 M_2 - N_2) \mathcal{L} \right] \frac{\partial^2}{\partial z^2} \right\} \left[ |\Psi|^2 \right]. \quad (51)$$
Fig. 3 represents the dependence of $\Lambda^2/\eta^2_\parallel$ vs. $1/\beta$ for the same $(T_e/T_i)$ values as in Figs. 2a and 2b. This quantity changes sign at $\beta \simeq \beta_0$. It defines the particular structure of the finite Larmor radius integral operator given by Eq. (50). The analogous plots of $N_1/\eta^2_\parallel$ and $N_2/\eta^2_\parallel$ are shown in Figs. 4a and 4b respectively. These coefficients are negative (there is a region where $N_1$ is positive, but very small), peaked near $\beta_0$ and look similar to graphs of the coefficient $M_2$.

V. Modulation Instability

It has been demonstrated [9,26,27] that modulation instability leads to the destruction of wave trains and the production of solitons, transfer of energy from large-scale wave-modes to (damped) small-scale modes, the appearance of shock waves and shocklets in the vicinity of bow-shocks of planets, etc.. As was already mentioned, waves described by the DNLS can be modulationly unstable depending on the sense of their polarization. Left-hand polarized waves are unstable when the cubic nonlinearity coefficient is positive, and right-hand polarized waves are unstable when it is negative. In this Section we investigate the modulation instability of dissipative wave packets described by the modified DNLS, Eqs. (16) and (12’), and compare the result with that of the DNLS with the resonant particle term. We write the general equation as follows:

$$\frac{\partial \phi}{\partial \tau} + \frac{\partial}{\partial z} \left[ \phi(Q_1(|\phi|^2 - \langle |\phi|^2 \rangle) + Q_2J_L [|\phi|^2 - \langle |\phi|^2 \rangle]) \right] - i\mu \frac{\partial^2}{\partial z^2} \phi = 0, \quad (52)$$

where

$$Q_1 = \frac{v_A}{4} \frac{1}{1 - \beta/\gamma}, \quad Q_2 = -\beta \frac{v_A^2}{4\chi_\parallel} \frac{\gamma - 1}{\gamma} \frac{1}{(1 - \beta/\gamma)^2} \quad (53)$$

and $\mu = v_A^2/2\Omega_i$. We consider only left-hand polarized waves (i.e. with a minus sign on the third term) which are modulationally stable when $Q_1 > 0$ in the dissipationless limit, $Q_2 = 0$. 

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Our approach follows that of Mio et al. [28] and Spangler [14]. Expressing the solution of Eq. (52) in the form:

$$\phi = ue^{i\vartheta},$$

(54)

where \(u(\tau, z)\) and \(\vartheta(\tau, z)\) are real functions and substituting this into Eq. (52), we obtain two equations for the real and imaginary parts, respectively:

$$\dot{u} + Q_13u^2u_z - Q_1\langle u^2 \rangle u_z + Q_2u_zJ_L[u^2 - \langle u^2 \rangle]$$

$$+ Q_2u(J_L[u^2 - \langle u^2 \rangle])_z + 2\mu \vartheta_z u_z + \mu \vartheta_{zz} u = 0,$$

(55a)

$$\dot{\vartheta} + Q_1\vartheta u^3 - Q_1\vartheta \langle u^2 \rangle u + Q_2\vartheta u_z J_L[u^2 - \langle u^2 \rangle] - \mu u_{zz} + \mu \vartheta_z u = 0,$$

(55b)

where \(\dot{u} \equiv \partial u/\partial t,\ u_z \equiv \partial u/\partial z,\) and similarly for \(\vartheta\) and \(J_L\). Using the solutions \(U_0\) and \(\Theta_0\), we superimpose small-amplitude and phase modulation such as:

$$u = U_0 + \epsilon \tilde{u},\ \vartheta = \Theta_0 + \epsilon \tilde{\vartheta},$$

(56)

where \(\epsilon \ll 1, U_0 = \text{const}, \dot{\Theta}_0 = -\omega_0, (\Theta_0)_z = k_0,\) where, in turn, \(\omega_0\) and \(k_0\) are the frequency and wavenumber of high-amplitude Alfvén wave. Eqs. (55) with \(u\) and \(\vartheta\) of the form (56) yields (in zeroth order of \(\epsilon\)) the dispersion relation:

$$\omega_0 = \mu k_0^2.$$  

(57)

Note that no nonlinearity enters the dispersion relation when obtained in the frame moving with a wave.

In first order the linearized Eqs. (55) become:

$$\dot{\tilde{u}} + \tilde{u}_z(2Q_1U_0^2 + 2\mu k_0) + 2Q_2U_0^2(J_L[\tilde{u}])_z + \tilde{\vartheta}_{zz} \mu U_0 = 0,$$

(58a)

$$\dot{\tilde{\vartheta}} + 2\tilde{\vartheta}_z \mu k_0 + 2\tilde{\vartheta}Q_1k_0U_0 - \tilde{u}_{zz} \frac{\mu}{U_0} + 2Q_2k_0U_0J_L[\tilde{u}] = 0,$$

(58b)

To go further, we need to know the Fourier representation of the operator \(J_L\). We write:

$$J_L[\tilde{u}] = \int_{-\infty}^{0} e^{\zeta/L} \tilde{u}(z + \zeta) d\zeta = \int_{-\infty}^{0} e^{\zeta/L} \left( \int_{-\infty}^{\infty} dk \ u_k e^{i(k(z + \zeta))} \right) d\zeta$$

$$= \int_{-\infty}^{\infty} dk \ e^{ikz} \frac{u_k}{ik + 1/L}.$$  

(59)
Taking the Fourier transformation of Eqs. (58), with $\tilde{u}, \tilde{\vartheta} \sim \exp(ikz - i\omega \tau)$, we obtain:

\begin{align*}
(-i\omega + A_r + iA_i)u_{k,\omega} - C_r \vartheta_{k,\omega} &= 0, \\
(B_r - iB_i)u_{k,\omega} + (-i\omega + iC_i)\vartheta_{k,\omega} &= 0.
\end{align*}

(60a)

(60b)

The coefficients are:

\begin{align*}
A_r &= 2Q_2U_0^2L^2k^2, \quad A_i = k \left(2Q_1U^2 + 2\mu k_0 + \frac{2Q_2U_0^2L}{1 + k^2L^2}\right), \\
B_r &= 2Q_1U_0k_0 + \frac{2Q_2U_0Lk_0}{1 + k^2L^2} + k^2\frac{\mu}{U_0}, \quad B_i = \frac{2Q_2U_0L^2k_0k}{1 + k^2L^2}, \\
C_r &= k^2\mu U_0, \quad C_i = 2\mu kk_0.
\end{align*}

(61)

Instability appears when the imaginary part of the complex frequency $\omega$ is positive. When $Q_2$ vanishes there is no instability, so typically $\gamma \ll \omega_r$. Substituting the frequency $\omega = \omega_r + i\gamma$, we may write an expression for the growth rate $\gamma$ as:

\begin{equation}
\gamma \simeq \mu k_0 \frac{|Q_2|}{|Q_1|} \frac{k^2L^2}{1 + k^2L^2}.
\end{equation}

(62)

It is notable that the growth rate in this case does not depend on the amplitude of the nonlinear Alfvén wave, unlike the resonant particle case. In astrophysical plasma $\beta \simeq 1$, so that $k^2L^2 \gg 1$, i.e. the modulation length is much smaller the characteristic length of dissipation $L$. In this case:

\begin{equation}
\gamma \sim \frac{v_A^3k_0\beta}{\Omega_i\chi_\parallel}.
\end{equation}

(62’)

In the opposite case $k^2L^2 \ll 1$:

\begin{equation}
\gamma \sim \frac{v_A\chi_\parallel k_0}{l_m^2\Omega_i} \beta \frac{(1 - \beta/\gamma)^2}{(1 - \beta)^2},
\end{equation}

(62’’)

where $l_m = 1/k$ is the modulation length. One can see that those waves which are modulationally stable in the dissipationless limit become unstable when collisional dissipation increases. The growth rate of modulation instability is proportional to the ratio of the coefficients of the nonlocal and the cubic nonlinearity terms. When the nonlocal term coefficient is
zero, we obtain from the system (60) the known criterion of modulation stability which depends on the sense of polarization of the wave [10,14].

Modulation stability analysis of the modified DNLS with the resonant particle integral operator has been implemented in Refs. [14,29]. The growth rate for this case is:

\[
\gamma \sim \frac{\mu^2 k_0^2 |k|}{B_0^2 U_0^2 M_1} \left( \frac{|M_2|}{M_1} \right).
\]

Using Eqs. (34) for the case of low beta, \( \beta \ll 1 \), we obtain the scaling:

\[
\gamma \sim \frac{\hat{\chi}_{||} T_e}{T_i} \left( \frac{v_A^4 k_0^2}{\Omega_i^2 |b|^2 l_m} \right).
\]

Modulation instability now appears for waves which are stable in the absence of resonant particle effects. The growth rate again is proportional to the nonlocal-to-nonlinear term coefficient ratio. Roughly, this growth rate is \( M_2/M_1 \) times smaller than the modulation instability growth rate predicted by the simple DNLS.

VI. Conclusions

In this paper we have considered the influence of kinetic effects on nonlinear Alfvén wave dynamics. The effects under consideration are i) the dissipative (collisional) longitudinal viscous and thermal fluxes, ii) resonant particle effects, i.e. Landau damping, iii) the \( \mathbf{E} \times \mathbf{B} \) drift and finite Larmor radius effects such as those associated with the gyro-averaging of fields over the Larmor orbits of particles. All calculations have been based on fluid moment and gyrofluid models with Landau damping modelled by additional dissipation-like terms. The results obtained are given below.

a. Dissipative (collisional) longitudinal viscosity, \( \mu_{||} \), and thermoconductivity, \( \chi_{||} \), give rise to an additional, new integral nonlinear term in
the evolution equation for the nonlinear Alfvén waves, and also resolve the \((1 - \beta)^{-1}\) singularity of the derivative nonlinear term in the MHD model. The dissipative integral operator is different from the integral operator representing resonant particle effects, and is given by Eq. (17).

b. Modulationally stable dissipationless Alfvén waves become unstable when dissipation is included. The growth rate is proportional to the ratio of the coefficients of the integral and nonlinear terms. This is similar to the case of the modified DNLS with Landau damping, studied in Ref. [14].

c. Using the three-moment fluid equations [17], we derived the modified DNLS including Landau damping effects. Since in this model dissipation is no longer an algebraic constant, but an integral operator given by Eq. (21), we obtained exactly the same functional form of the equation as that obtained from lengthy, full kinetic calculations [6,14]. However, the coefficient of both the nonlinear and resonant particle operator (nonlocal) terms obtained here are much simpler, thus facilitating clear physical interpretation and further analysis. Particularly, the phenomenon under study is just the broadening of the resonance between acoustic and Alfvénic branches due to the resonant particle interaction with a sound wave.

d. In the collisionless regime, there is coupling of Alfvén and ion-acoustic waves, instead of coupling of Alfvén and sound waves. The resonant particle effect is then strong Landau damping of the ion-acoustic wave when \(T_e/T_i \simeq 1\). We obtained the temperature ratio dependent coefficients \(M_1\) and \(M_2\) of the nonlinear and integral terms, respectively. These coefficients are much simpler than those obtained full Vlasov model [14]. However, they display similar qualitative dependence on the \(\beta\) and \(T_e/T_i\) parameters, Figs. 2a, 2b.

e. In the case of obliquely propagating waves, other kinetic effects may be relevant. It was shown that even though the \(\mathbf{E} \times \mathbf{B}\) drift velocity is not zero (electric field is generated from charge separation in
an ion-acoustic wave), it does not contribute to the dynamic wave equation. Another effect which does contribute is a gyro-averaging of electric and magnetic fields acting on a particle over a Larmor orbit of this particle. This effect depends on the angle between the wave propagation direction and the ambient magnetic field direction as \( \sin^2 \Theta \), and vanishes for a strictly parallel-propagating wave. We studied this effect using a gyrofluid model constructed from gyrofluid equations [18,19]. This effect results in additional terms in the amplitude evolution equation. These terms are expressed in terms of the dissipation integral operator, given in Eq. (17). Hence, this effect can be viewed as some specific nonlinear dissipation (similar to collisional dissipation) of an obliquely propagating wave due to finite Larmor radius.

We have shown in this paper that MHD models with linear kinetic corrections provide correct quantitative description of effects such as the dynamics of high-amplitude Alfvén waves. It is significant that the expressions obtained are, nevertheless, much simpler than those obtained from kinetic (Vlasov) calculations, thus facilitating further analysis and numerical calculations, and allowing very clear physical interpretation. There are some points unresolved in this work. First, when the amplitude of a wave is high, there are particles trapped in the wave. These particles traverse the regions of lower field and bounce between the regions of higher field. This trapped motion in a beat wave results in amplitude oscillation of this wave at the particle bounce frequency, similar to the nonlinear Landau damping process. Obviously, such phenomena cannot be represented by a theory based on an assumption that the distribution function deviates slightly from a local Maxwellian. Obviously, the physical process of trapping will lead to plateau formation and other modification of Maxwellian structure. Indeed, waves with \( b_\perp/B_0 \sim 1 \) will have large trapping width and undoubtedly lead to significant distortion of the distribution function. It is interesting to note that such distortions could potentially mitigate the effects of Landau damping, via local flattening of \( \langle f \rangle \) at the resonant velocity (for \( \beta \sim 1 \)). Second, the gyrofluid
model of this paper is the simplest possible. It is important to consider full set of nonlinear gyrofluid equations to include gyro-viscosity, finite Larmor radius corrections to the Reynolds stress, etc., which are relevant for the case of oblique and near-perpendicular propagation.

As noted above, the kinetic effects which govern the dynamics of a collisionless plasma may significantly alter the picture of nonlinear Alfvén wave dynamics built upon the MHD plasma model. Inclusion of wave-particle resonance effects can significantly alter predictions for modulational instability (i.e. left vs. right circular polarization dependence of growth rate upon parameters such as $T_e/T_i$, $\beta$, etc.). Also, the nonlocal structure of the envelope equation which arises from the effects of parallel streaming, will likely result in departure from the traditional paradigm of collisionless shocks as solitons formed by the competition between nonlinear steepening and dispersion. In particular, a new time scale, namely the ion transit time through the envelope modulation, enters along with the steepening and dispersion rates. Strong ion heating will occur, as well. Thus, collisionless shock structure may be smoothed or exhibit secondary temporal oscillations. These speculations may be easily addressed by studies of numerical solution of the (tractable) envelope equations derived in this paper. The result of these numerical studies will be published in Part II of this series.

Finally, it should be mentioned that an improved understanding of nonlinear Alfvén dynamics in a compressible plasma may have application in contexts other than collisionless shocks in the solar wind plasma. First, a significant fraction of the interstellar medium [30] is hot, collisionless compressible plasma. Thus, the problems of the galactic dynamo and interstellar turbulence should be approached in the context of a collisionless, compressible plasma model. In particular, collisionless dissipation via ion heating is a natural mechanism for controlling the growth of small scale magnetic energy which has been shown to inhibit the mean field dynamo in purely incompressible MHD theories [31,32]. Similarly, the processes of wave steepening and shock or soliton formation can strongly affect the parallel dynamics of Alfvénic turbulence in
the interstellar medium. Such turbulence is thought to be related to interstellar scintillations. These issues will be addressed in future publications.

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Fig. 1. The coefficients of the nonlinear and integral terms, $M_1^{(1)}$ and $M_2^{(1)}$, in the one-fluid model vs. $1/\beta$. 
Figs. 2a, 2b. The same coefficients as on Fig. 1 in the two-fluid model vs. $1/\beta$ for four $T_e/T_i$ temperature ratios ($T_e/T_i = 1, 3, 5, 10$).
Fig. 3. Same dependence as on Figs. 2a, 2b for the normalized quantity $\Lambda^2/\eta_\parallel^2$. 

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Figs. 4a, 4b. Same as on Figs. 2a, 2b for the normalized coefficients $N_1/\eta_\parallel$ and $N_2/\eta_\parallel$. 