A COMMUTATIVE AND COMPACT DERIVATIONS FOR W* ALGEBRAS

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ABSTRACT. In this paper, we study the compact derivations on W* algebras. Let M be W*-algebra, let LS(M) be algebra of all measurable operators with M, it is show that the results in the maximum set of orthogonal predictions. We have found that W* algebra A contains the Center of a W* algebra ß and is either a commutative operation or properly infinite. We have considered derivations from W* algebra two-sided ideals.

1. INTRODUCTION

Let M be a W*-algebra and let Z(M) be the center of M. Fix a ∈ M and consider the inner derivation δ_a on M generated by the component a, which is δ_a(·) := [a, ·].

The norm closing two sided ideal f(B) generated by the finite projections of a W* algebra B behaves somewhat similar to the idealized compact operators of B(H) (see [11],[8],[9]). Therefore, it is natural to ask about any sub-algebras d of B that is any derivation from A into f(B) implemented from an element of y(B).

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We perform two main difficulties: the presence of the center of $B$ and the fact that the main characteristic in [8] proof (that is, if $Q_n$, is a sequence of mutually orthogonal projections and $T \in B(H)$ hence $\|Q_n T Q_n\| > \alpha > 0$ for all $n$ implies that $T$ is not compact) failure to generalize to the case in which $g$ is of Type $II\infty$.

Finally, we have considered derivations from $d$ at the two-sided $C_{1+\varepsilon}(B, \tau) = B \cap L^{1+\varepsilon}(B, \tau)(1 \leq 1 + \varepsilon < \infty)$ to obtain faithful finite normal trace $\tau$ on $B$.

2. Notations Preliminary

Lemma (1). Let $B$ be a semi-finite algebra, let $Q_0 \in p(B)$ and $x_0 \in Q_0$ be such that $\omega_{x_0}$, is a faithful trace on $B_{\mathcal{Q}_0}$. Assume there are $Q_n \in p(B)$, $F_n \in p(\ell)$ and $U_n \in B$ for $n = n_1, n_2, ...$, such that the projections $Q_n$ are mutually orthogonal and $Q_n = U_n^*U_n$, $Q_n F_n = U_n^*U_n$ for all $n$ (i.e., $Q_n \sim Q_{n_0}F_n$). Let $x_n = U_nF_nx_0$. Then $x_n \rightarrow_{jRW} O$.

Proof. Assume that $\sum_{n=0}^{\infty} Q_n = n$. Let $\tau$ be a faithful semi-finite normal (fsn) trace on $B^+$ to be agreed on $B_{\mathcal{Q}_0}$ with $\omega_{x_0}$. Then for all $B \in B^+_{\mathcal{Q}_0}$ we have

$$
\tau(B) = \tau(U_n^*U_n^*BU_n^*U_n^*) = \tau(U_n^*U_n^*BU_n^*) = \tau(Q_n F_n U_n^*BU_n^* F_n) = \omega_{x_0}(F_n U_n^*BU_n^* F_n) = \omega_{x_0}(B).
$$

Let $P \in p(B)$ be any semi-finite projection. Then by [11] there is a central decomposition of the identity $\sum_{\gamma \in \Gamma} E_\gamma = 1, E_\gamma \in p(\ell), E_\gamma E_{\gamma'} = 0$ for $\gamma \neq \gamma'$ such that $\tau(PE_\gamma) < \infty$ for all $\gamma \in \Gamma$. Then

$$
\tau(PE_\gamma) = \sum_{n=1}^{\infty} \tau(Q_n PE_\gamma Q_n) = \sum_{n=1}^{\infty} \omega_{x_n}(Q_n PE_\gamma Q_n) = \sum_{n=1}^{\infty} \|PE_\gamma x_n\|^2 < \infty
$$
whence $\|PE_\gamma x\| < 0$ for all $\gamma \in \Gamma$. Let $\varepsilon > 0$ and let $\Lambda \subseteq \Gamma$ be a finite index set such that

$$\sum_{\gamma \in \Lambda} \|PE_\gamma x_0\|^2 < \varepsilon.$$  

Then for all $n$,

$$\sum_{\gamma \in \Lambda} \|PE_\gamma x_n\|^2 = \sum_{\gamma \in \Lambda} \|PE_\gamma U_n F_n x_0\|^2 = \sum_{\gamma \in \Lambda} \|PU_n F_n E_\gamma x_0\|^2 \leq \sum_{\gamma \in \Lambda} \|E_\gamma x_0\|^2 < \varepsilon$$

Hence from $\|Px_n\|^2 \leq \sum_{\gamma \in \Lambda} \|PE_\gamma x_n\|^2 + \varepsilon$, where $\|Px_n\| \to 0$, to completes the proof.

**Lemma (2).** Let $T \notin f(P)$, then there is an $\alpha > 0$ and $0 \neq E \in p(\ell)$ such that for every $0 \neq F \in p(\ell)$ with $F \leq E$ we have $\|\pi(TF)\| > \alpha$.

**Proof.** Let $\alpha = \frac{1}{2} \|\pi(T)\| > 0$ and let $G$ be the sum of a maximal family of mutually orthogonal central projections $G_\gamma$ such that $\|\pi(TG_\gamma)\| \leq \alpha$. Then

$$\|\pi(TG)\| = \sup_\gamma \|\pi(TG_\gamma)\| \leq \alpha,$$  

hence $G \neq 1$. Let $E = Z - G$ and let $0 \neq F \in p(\ell)$ with $F \leq E$.

Since $FG = 0$, by the maximally of the family we have $\|\pi(TF)\| > \alpha$.

### 3. Relatively Compact Derivation

Let $M$ be a $W^*$-algebra and let $Z(M)$ be the center of $M$. Fix $a \in M$ and consider the inner derivation $\delta_a$ on $M$ generated by the element $a$, that is $\delta_a(\cdot) := [a, \cdot]$. Obviously, $\delta_a$ there is a linear bounded operator on $(M, \| \cdot \|_M)$, where $\| \cdot \|_M$ is a $C^*$-norm on $M$. It is known that there exists $c \in Z(M)$ such that the following estimate holds: $\|\delta_a\| \geq \|a - c\|_M$. In view of this result, it is natural to ask whether there exists an element $y \in M$ with $\|y\| \leq 1$ and $c \in Z(M)$ such that $[a, y] \geq |a - c|$.

**Definition (3).** A linear subspace $I$ in the $W^*$ algebra $M$ equipped with a norm $\| \cdot \|_I$ is said to be a symmetric operator ideal if
(i) \[ \| S \|_I \geq \| S \| \text{ for all } S \in I, \]

(ii) \[ \| S^* \|_I = \| S \| \text{ for all } S \in I, \]

(iii) \[ \| A S B \| \leq \| A \| \| S \| \| B \| \text{ for all } S \in I, A, B \in M. \]

Observe, that every symmetric operator ideal \( I \) is a two-sided ideal in \( M \), and therefore by [13], it follows from \( 0 \leq S \leq T \) and \( T \in I \) that \( S \in I \) and \( \| S \|_I \leq \| T \|_I \).

**Corollary (4).** Let \( M \) be a \( W^* \)-algebra and let \( I \) be an ideal in \( M \). Let \( \delta : M \to I \) be a derivation. Then there exists an element \( a \in I \), such that \( \delta = [a,.] \).

**Proof.** Since \( \delta \) is a derivation on a \( W^* \)-algebra, it is necessarily inner [8]. Thus, there exists an element \( d \in M \), such that \( \delta(\cdot) = \delta_d(\cdot) = [d,.] \). It follows from the hypothesis that \( [d,M] \subseteq I \).

Using [22] (or [20]), we obtain \( [d^*,M] = [d,M]^* \subseteq I^* = I \) and \( [d_k,M] \subseteq I, k=1,2 \), where \( d = d_1 + i d_2, d_k = d_k^* \in M \), for \( k=1,2 \). It follows now, that there exist \( c_1, c_2 \in \mathbb{Z}(M) \) and \( u_1, u_2 \in U(M) \), such that \( \|d_k, u_k\| \geq 1/2 \|d_k - c_k\| \) for \( k=1,2 \). Again applying [20], we obtain \( d_k - c_k \in I, \) for \( k=1,2 \). Setting \( a := (d_1 - c_1) + i (d_2 - c_2) \), we deduce that \( a \in I \) and \( \delta = [a,.] \).

**Corollary (5).** Let \( M \) be a semi-finite \( W^* \)-algebra and let \( E \) be a symmetric operator space. Fix \( a = a^* \in S(M) \) and consider inner derivation \( \delta = \delta_a \) on the algebra \( LS(M) \) given by \( \delta(x) = [a,x], x \in LS(M) \). If \( \delta(M) \subseteq E \), then there exists \( d \in E \) satisfying the inequality \( \|d\|_E \leq \|\delta\|_{M \to E} \) and such that \( \delta(x) = [d,x] \).

**Proof.** The existence of \( d \in E \) such that \( \delta(x) = [d,x] \). Now, if \( u \in U(M) \), then
\[
\|\delta(u)\|_E = \|du - ud\|_E \leq \|du\|_E + \|ud\|_E = 2\|d\|_E.
\]
Hence, if \( x = \{x \in M : \|x\| \leq 1\} \), then \( x = \sum_{i=1}^4 \alpha_i u_i \), where \( u_i \in U(M) \) and \( |\alpha_i| \leq 1 \) for \( i=1,2,3,4 \), and so
\[
\|\delta(x)\|_E \leq \sum_{i=1}^4 \|\delta(\alpha_i u_i)\|_E \leq 8\|d\|_E \text{, that is } \|\delta\|_{M \to E} \leq 8\|d\|_E < \infty.
\]
4. A Commutative Operation on $W^*$ Sub-algebras

When $A$ a commutative operation is crucial because it provides the following explicit way to find an operator $T \in B$ implementing the derivation.

For the rest of this section let $A$ be any a commutative operation sub-algebras of $B$ and $\delta: A \to B$ be any derivation. Let $u$ be the unitary group of $A$ and $M$ be a given invariant mean on $u$, i.e., a linear functional on the algebra of bounded complex-valued functions on $u$ such that

(i) For all real $f$, $\inf \{ f(U) \mid U \in u \} \leq Mf \leq \sup \{ f(U) \mid U \in u \}$

(ii) For all $U \in u$, $Mf_U = MS$, where $f_U(V) = f(UV)$ for $V \in u$.

Thus $M$ is bounded and $|Mf| \leq \sup \{ |f(U)| \mid U \in u \}$ for all $f$ (see [8] for the existence and properties of $M$).

For each $\phi \in B$, the map

$$\phi \mapsto M\phi\left(U^*\delta(U)\right)$$

is linear and bounded and hence defines an element $T \in (B_u)^*$. Explicitly,

$$\phi(T) \mapsto M\phi\left(U^*\delta(U)\right) \quad \text{for all } \phi \in B_u$$

The same easy computation as in [8] shows that $\delta = aAT$. Notice that for all $A \in B$ the map

$$\phi \mapsto M\phi\left(U^*BU\right) = \phi\left(E(B)\right)$$

defines an element $E(B)$ which clearly belongs to $A \cap B$. Moreover it is easy to see that $E$ is a conditional expectation (i.e., a projection of norm one) from $B$ onto $A \cap B$ (see [6]).

**Theorem (6).** Let $A$ be a commutative operation $W^*$ sub-algebras of $B$ containing the center $\ell$ of $B$. For every derivation $\delta: A \to f(B)$ there is a $T \in f(B)$ such that $\delta = aAT$.

We have seen that given an invariant mean $M$ on $u$ there is a unique $T \in B$ such that $\delta = aAT$ and $E(T) = 0$. We are going to show that $T \in A(B)$. Reasoning by contradiction assume that $T \notin A(B)$. We proof requires several reductions to the restricted derivation
\( \delta_E : A_E \rightarrow f(B) \) for some \( 0 \neq E \in p(\ell) \). To simplify notations we shall assume each time that \( E = 1 \).

Let us start by noticing that if \( Q_i \in p(A) \) for \( i = n, n + 1 \), \( Q_n, Q_{n+1} = 0 \) and \( P = Q_n + Q_{n+1} \), then

\[
PTP = \sum_{i=n}^{n+1} Q_i T Q_i + \delta(Q_{n+1})Q_n + \delta(Q_n)Q_{n+1}
\]

hence

\[
\|\pi(PTP)\| = \left\| \sum_{i=n}^{n+1} \pi(Q_i T Q_i) \right\| + \max_i \pi(Q_i T Q_i)
\]

**Definition (7).** For every \( Q \in p(A) \) define \([Q] = [Q, \varepsilon]\) to be the central projection. Set

\[
P = \{ P \in p(A) \mid \|P\| = 1 \}.
\]

Thus \( P \in p \) iff \( \|\pi(PTP)\| = \|\pi(TG)\| \) for all \( G \in p(\ell) \). We collect several properties of \([Q]\).

**Corollary (8).** Let \( B \) be a semi-finite \( W^* \) algebra with a trace \( \tau \), let \( A \) be a properly infinite \( W^* \) sub-algebras of \( B \) and let \( 1 \leq 1 + \varepsilon < \infty \). Then for every derivation \( \delta : A \rightarrow C_{1+\varepsilon}(B, \tau) \) there is \( a T \in C_{1+\varepsilon}(B, \tau) \) such that \( \tilde{\delta} = a \Lambda T \).

In the notations introduced there, it is easy to see that \( \phi(C_{1+\varepsilon}(B, \tau)) = C_{1+\varepsilon}(\tilde{B}, \tilde{\tau}) \), where \( \tau = \tau \oplus \tau_0 \) and \( \tau_0 \) is the usual trace on \( B(H_0) \). We can actually simplify the proof by choosing \( \tilde{A}_n = I \otimes \ell \) since the condition \( \ell \subset A \) is no longer required.

**Corollary (9).** Let \( P = Q_n + Q_{n+1} \). Then there is a largest central projection \([Q_n, Q_{n+1}]\) such that for every \( G \in p(\ell) \) with \( G \leq [Q_n, Q_{n+1}] \), we have \( \|\pi(Q_i T Q_i G)\| = \|\pi(PTP)\| \).

**Proof.** Let \( G_i = \{ G \in p(\ell) \mid \|\pi(Q_i T Q_i G)\| = \|\pi(PTP)\| \} \) and \( \Xi = \{ G + \varepsilon \in p(\ell) \mid \text{if} \ G \in p(\ell) \text{ and } \varepsilon \geq 0 \text{ then } G \in G_n \} \). Since \( \|\pi(PTP)\| = \max_i \|\pi(Q_i T Q_i G)\| \) for all \( G \in p(\ell) \), we see that \( G_n \cup G_n+1 = p(\ell) \). Notice that \( \Xi \) is hereditary (i.e., \( G - \varepsilon \in \Xi \) and \( F \in p(\ell), F \leq G + \varepsilon \) imply \( F \in \Xi \)).
Let \( [Q_n, Q_{n+1}] = \sup \Xi \). We have only to show that \( [Q_n, Q_{n+1}] \in \Xi \). Let \( G + \varepsilon = \sum \gamma (G + \varepsilon)_\gamma \) be the sum of a maximal collection of mutually orthogonal projections \( (G + \varepsilon)_\gamma \in \Xi \). Then for every \( F \in \Xi \) we have \( ([Q_n, Q_{n+1}] - (G + \varepsilon))F = 0 \) because of the maximal of the collection of \( \Xi \).

Then \( [Q_n, Q_{n+1}] = G + \varepsilon \). Consider now any \( G \in p(\ell), \varepsilon \geq 0 \), then \( G = \sum \gamma G (G + \varepsilon)_\gamma \) and since \( G (G + \varepsilon)_\gamma \leq (G + \varepsilon)_\gamma \in \Xi \), we have \( \| \pi \left( Q_n TQ_n G (G + \varepsilon)_\gamma \right) \| = \| \pi \left( PTPG (G + \varepsilon)_\gamma \right) \| \) for all \( \gamma \).

Since \( \pi \left( Q_n TQ_n G \right) \) (resp. \( \pi \left( PTPG \right) \)) is the direct sum of then

\[
\| \pi \left( Q_n TQ_n G (G + \varepsilon)_\gamma \right) \| = \sup \| \pi \left( Q_n TQ_n G (G + \varepsilon)_\gamma \right) \|
\]

\[
= \sup \| \pi \left( PTPG (G + \varepsilon)_\gamma \right) \|
\]

\[
= \| \pi \left( PTPG \right) \|
\]

whence \( G \in G_n \). Since \( \varepsilon \geq 0 \) is arbitrary, we have \( G + \varepsilon = [Q_n, Q_{n+1}] \in \Xi \) which completes the proof.

**Corollary (10).** (i) If \( Q_n Q_{n+1} = 0 \) with \( Q_j \in p(A) \) then \( 1 - [Q_n, Q_{n+1}] \leq [Q_n, Q_{n+1}] \).

(ii) If \( Q_n \leq Q_{n+1} \) with \( Q_j \in p(A) \) then \( [Q_n] \leq [Q_{n+1}] \).

(iii) If \( \varepsilon \geq 0 \) with \( Q \in p(A), Q + \varepsilon \in p \) then \( [Q] = [Q, \varepsilon] \) and \( 1 - [Q] \leq [\varepsilon] \)

If \( \pi (TG) \neq 0 \) for all \( 0 \neq E \in p(\ell) \) then the following hold:

(iv) If \( E \in p(\ell) \) then \( E = [E] \).

(v) If \( Q \in p(A) \) then \( [Q] \leq c(Q) \), where \( c(Q) \) is the central support of \( Q \).

**Proof.** We have to show that for every \( G \in p(\ell), G \leq 1 - [Q_n, Q_{n+1}] \) we have \( G \in G_{n+1} \). Let \( E + \varepsilon \) be the sum \( \sum E_\gamma \) of a maximal collection of mutually orthogonal projections of \( G_{n+1} \) that are majored by \( G \). Then
\[ \| \pi(Q_n TQ_n F) \| = \sup_{\gamma} \| \pi(Q_n TQ_n F_\gamma) \| \]
\[ = \sup_{\gamma} \| \pi(Q_{n+1} + Q_{n+1}) T (Q_{n+1} + Q_{n+1}) F_\gamma \| \]
\[ = \| \pi(Q_{n+1} + Q_{n+1}) T (Q_{n+1} + Q_{n+1}) F \| \]

whence \( E + \varepsilon \in G_{n+1} \). By the maximalist of the collection, \( 0 \leq G - (E + \varepsilon) \) does not majority any nonzero projection of \( G_{n+1} \) and since \( p(\ell) = G_n \cup G_{n+1} \), any central projection \( G' \leq G - (E + \varepsilon) \) must be in \( G_n \). By definition of \( \Xi \), this implies that \( G - (E + \varepsilon) \in \Xi \) whence \( G - (E + \varepsilon) \leq [Q_n, Q_{n+1}] \). So, \( G - (E + \varepsilon) \leq G \leq 1 - [Q_n, Q_{n+1}] \) and hence \( G = E + \varepsilon \in G_{n+1} \) which completes the proof.

(ii) Let \( G \in p(\ell) \) and \( G \leq [Q_n] \). Then \( \| \pi(TG) \| = \| \pi(Q_n TQ_n G) \| \leq \| \pi(Q_{n+1} TQ_{n+1} G) \| \leq \| \pi(TG) \| \) whence equality holds and \( [Q_n] \leq [Q_{n+1}] \) by the maximalist of \( Q_{n+1} \).

(iii) \([Q, \varepsilon] \) is maximal under the condition: if \( G \in p(\ell) \) and \( G \leq [Q, \varepsilon] \) then
\[ \| \pi(QTQG) \| \leq \| \pi((Q + \varepsilon) T (Q + \varepsilon) G) \| = \| \pi(TG) \| \]
which is the same condition defining \([Q, I - Q] = [Q]\). Thus \([Q] = [Q, \varepsilon]\). Applying this to \( \varepsilon \) we have \([\varepsilon] = [\varepsilon, Q] \) and thus by (i) we have \([\varepsilon] \geq 1 - [Q, \varepsilon] = 1 - [Q] \).

(ii) Let \( E + \varepsilon, E \in p(\ell) \) then \( \| \pi(ETE(E + \varepsilon)) \| = \| \pi(T(E(E + \varepsilon))) \| \). This implies that if \( \varepsilon \geq 0 \), then \( E + \varepsilon \leq [E] \) so \( E \leq [E] \) and if \( E + \varepsilon = [E] - E \leq [E] \) then
\[ 0 = \| \pi(ETE(E + \varepsilon)) \| = \| \pi(T(E(E + \varepsilon))) \| \] whence \( E = [E] \).

(v) Follows at once from (ii) and (iv).

The condition that \( \| \pi(TE) \| \neq 0 \) for all \( 0 \neq E \in p(\ell) \) is of course meaningless unless \( B \) is properly infinite. Hence, we may assume without loss of generality that:
\( B \) is properly infinite and semi-finite.

There is an \( \alpha > 0 \) such that \( \| \pi(TE) \| > \alpha \) for all \( 0 \neq E \in p(\ell) \).
Lemma (11). Let \( P \in p \) and \( R_n = X_{\text{PTP}} ([\alpha, \infty), \), \( R_{n+1} = X_{\text{PTP}} (-\infty, -\alpha]) \), where \( X_{\text{PTP}} (\ ) \) denotes the spectral measure of the self-adjoint operator \( \text{PTP} \). Then there is an \( E_n \in p (\ell) \), with \( E_n = I - E \) such that \( R_i E_i \) are properly infinite and \( c (R_i E_i) = E_j \) for \( i = n, n+1 \).

Proof. Let \( R = R_n + R_{n+1} = X_{\text{PTP}} ([\alpha, \alpha]) \) and let \( F \not= 0 \) be any central projection. If \( RF \) were finite, we would have

\[
\| \pi(TF) \| = \| \pi(\text{PTP}F) \| \\
= \| \pi(\text{PTP}(1-R)F) \| \\
= \| \pi(|\text{PTP}(1-R)F|) \| \\
\leq \alpha
\]

Thus \( RF \) is infinite and nonzero. Hence \( R \) is properly infinite and \( c (R) = n \). Now let \( E_i \) be the maximal central projection majorized by \( c (R_n) \), such that \( R_n F_n \) is properly infinite. Then \( c (R_n, E_n) = E_n \) and \( R_n (n-E_n) \) is finite, hence \( R_n+ (n-E_n) = R_{n+1} E_{n+1} \) is properly infinite and \( c (R_{n+1}, E_{n+1}) = E_{n+1} \).

End of the Proof of Theorem (6). Take any \( 0 \not= Q_0 \in p (B) \) such that \( B_{Q_0} \) has a faithful trace \( \omega_{x_0} \) with \( x_0 \in Q_0 H \) and assume \( \| x_0 \| = 1 \). Let \( P_\gamma \in p, \gamma \in \Gamma \) be the not decreasing to zero. We are going to construct inductively a sequence \( \gamma_n \in \Gamma, F_n \in p (\ell), Q_n \in p (B), U_n \) partial isometrics in \( B, x_n \in H \) such that

(a) \( U_n^* U_n = Q_n, U_n^* U_n = Q_0 F_n \), i.e., \( Q_n \sim Q_0 F_n \)

(b) \( x_n = U_n F_n x_0 \in Q_n H \)

(c) \( Q_n Q_m = 0 \) for \( n \not= m \)

(d) \( \gamma_n > \gamma_m \) (hence \( P_{\gamma_n} < P_{\gamma_m} \)) for \( n > m \)

(e) \( Q_n \leq p_{\gamma_n} \)

(f) \( \| p_{\gamma_n} x_n \| < \gamma_n \)

(g) \( |Tx_n, x_n| \geq \gamma_n \).
The induction can be started with an arbitrary \( P \); assume we have the construction for \( n - 1 \). Let us apply Lemma(11) to \( P = P_{\gamma_n} \) and obtain \( E_i \in p(\ell) \), \( R_i \in p(B) \) for \( i = n, n + 1 \) as defined there. Then

\[
1 = \left\| x_n \right\|^2 = \left\| E_n x_0 \right\|^2 + \left\| E_{n+1} x_0 \right\|^2
\]

Let \( F_n \) be (any of) the projection \( E_n \) or \( E_{n+1} \) for which \( \left\| E_n x_0 \right\|^2 \geq \frac{1}{2} \) and let \( i \) be the corresponding index. Then \( RF_n \) is properly infinite and has central support \( F_n \). Now \( Q_0 \) is finite having a finite faithful trace \( \omega_0 \), hence so is \( Q_j \sim F_jQ_0 \leq Q_0 \) for \( 1 \leq j \leq n - 1 \) and \( \left( \sum_{j=1}^{n-1} Q_j \right) F_n \). Let \( S_n = \inf \left\{ RF_n, \left( 1 - \sum_{j=1}^{n-1} Q_j \right) F_n \right\} \). By the parallelogram law (see [2]) applied to \( F_n \) we have that

\[
RF_n - S_n \sim \left( \sum_{j=1}^{n-1} Q_j \right) F_n - \inf \left\{ \left( \sum_{j=1}^{n-1} Q_j \right) F_n, (1-R_i)F_n \right\}
\]

whence \( RF_n - S_n \) is finite and hence \( S_n \) is properly infinite and \( c(S_n) = F_n \). Since \( Q_0F_n \) is finite and \( c(Q_0F_n) \leq F_n \) we have \( Q_0F_n \prec S_n \), i.e., there is a partial isometry \( U_n \in B \) and a \( Q_n \in p(B) \), \( Q_n \leq S_n \) such that (a) holds. Let \( x_n \) be defined by (b) and choose \( \gamma_{n+1} > \gamma_n \) so that (d) and (f) hold. Since \( Q_nR \leq P_{\gamma_n} \), we have (e), since \( Q_n \leq \left( 1 - \sum_{j=1}^{n-1} Q_j \right) F_n \) we have (c). Finally \( x_n = R_{\gamma_n}x_n = P_{\gamma_n}x_n \), hence (g) follows from

\[
\left\| (Tx_n, x_n) \right\| = \left\| \left( P_{\gamma_n}TP_{\gamma_n}x_n, x_n \right) \right\|
\]

\[
= \left\| \left( P_{\gamma_n}TP_{\gamma_n}Rx_n, R_ix_n \right) \right\|
\]

\[
\geq \alpha \left\| (Rx_n, R_ix_n) \right\|
\]

\[
= \alpha \left\| x_n \right\|^2
\]

\[
= \alpha \left\| F_n x_0 \right\|^2
\]

\[
\geq \frac{1}{2} \alpha.
\]

Let now \( y_n = x_n - P_{\gamma_{n+1}}x_n \). \( B \) is semi-finite, hence we can apply Lemma (1) to obtain that \( x_n \to_{BEW} 0 \). Since \( \left\| P_{\gamma_{n+1}}x_n \right\| \to 0 \) we thus have \( y_n \to_{BEW} 0 \) and \( y_n \in P_nH \), where
\[ P_n = P_{x_n} - P_{x_{n+1}} \in p(d) \] and are mutually orthogonal by (d). Clearly for \( n \) large enough, 
\[ \|(T y_n, y_n)\| = |\omega_{x_n}(T)| > \frac{1}{2} \alpha. \] Since \( \omega_{x_n}(T) = M \omega_{x_n}(U^* \delta(U)) \), by the properties of the invariant mean mentioned, we have that \( \sup \{ |\omega_{x_n}(U^* \delta(U))| \|U \in u\} > \frac{1}{4} \alpha \). Thus we can find for every \( n \), a unitary \( V_n \in u \) such that \( \|V_n^* \delta(V_n) y_n, y_n\| > \frac{1}{4} \alpha \). Let \( A = \sum_{n=1}^{\alpha} V_n P_n \), then \( A \in d \) and
\[
A^* \delta(A) y_n, y_n = \left( P_n A^* \delta(A) P_n y_n, y_n \right) \\
= \left( P_n \left( A^* AT - A^2 A \right) P_n y_n, y_n \right) \\
= \left( P_n V_n^* \delta(V_n) P_n y_n, y_n \right) \\
= \left( V_n^* \delta(V_n) y_n, y_n \right) \\
= \frac{1}{4} \alpha
\]
for all \( n \). Therefore \( \|\delta(A) y_n\| \to 0 \). But because of (\Pi), we have \( \delta(A) \notin f(B) \), which completes the proof.

5. The Property of Infinite W* Sub-algebra

**Lemma (12)**. Let \( 0 < b \in Z(M) \), \( s(b) = 1 \); \( e_z^a(0,\infty) \) be a properly infinite projection and \( c \left( e_z^a(0,\infty) \right) = 1 \). Let projection \( q \in P(M) \) be finite or properly infinite, \( c(q) = 1 \) and \( q \ll e_z^a(0,\infty) \). Let \( \mathbb{R} \ni \mu_n \downarrow 0 \). For every \( n \in \mathbb{N} \) we denote by \( z_n \) such a projection that \( 1 - z_n \) is the largest central projection, for which \( (1-z_n)q \geq (1-z_n) e_z^a(\mu_n b, +\infty) \) holds. We have \( z_n \uparrow_n 1 \) and for
\[
d = \left[ \mu_1 z_1 + \sum_{n=1}^{\infty} \mu_{n+1} (z_{n+1} - z_n) \right] b
\]
the following relations hold: \( q \ll e_z^a(d, +\infty) \), \( 0 < d \leq \mu b \) and \( s(d) = 1 \). Moreover, if all projections \( e_z^a(\mu_n b, +\infty), n \geq 1 \) are finite then \( e_z^a(d, +\infty) \) is a finite projection as well.

**Proof**. Since \( e_z^a(\mu_{n+1} b, +\infty) \geq e_z^a(\mu_n b, +\infty) \) we have
\( e^\alpha_z (1 - z_{n+1}) q \geq (1 - z_{n+1}) e^\alpha_z (\mu_{n+1} b, +\infty) \geq (1 - z_{n+1}) e^\alpha_z (\mu_n b, +\infty) \). Hence, \( z_{n+1} \geq z_n \) for every \( n \in \mathbb{N} \).

In addition, \( e^\alpha_n (\mu_n b, +\infty) \uparrow_n e^\alpha_z (0, +\infty) \) and \( e^\alpha_z (0, +\infty) \) is properly infinite projection. Hence, in the case when \( q \) is finite projection, it follows that \( z_n \uparrow_n 1 \). Let us consider the case when \( q \) is a properly infinite projection with \( c(q) = 1 \) and such that \( q \prec \prec e^\alpha_z (0, \infty) \). In this case, with \( p = q, q = e^\alpha_z (0, +\infty), q_n = e^\alpha_z (\mu_n b, +\infty) \) and deduce \( \bigvee_{n=1}^\infty z_n \geq c(q) = 1 \).

All other statements follow from the form of element \( d \). Since, \( z_q d = \mu_z z_q b \), \( (z_{n+1} - z_n) = \mu_{n+1} (z_{n+1} - z_n) b \) and \( z_n q \prec \prec z_n e^\alpha_z (\mu_n b, +\infty) \) for every \( n \in \mathbb{N} \). Observe also that \( s(d) = s(b) \left( z_q + \sum_{n=1}^\infty (z_{n+1} - z_n) \right) = 1 \).

Finally, let all projections \( e^\alpha_z (\mu_n b, +\infty), n \geq 1 \) be finite. Since 
\[
dz = \mu_i b, d(z_{n+1} - z_n) = \mu_{n+1} b(z_{n+1} - z_n),
\]
we have 
\[
e^\alpha_z (d, +\infty) z_1 = e^\alpha_z (\mu_b, +\infty) z_1,
\]
\[
e^\alpha_z (d, +\infty) (z_{n+1} - z_n) = e^\alpha_z (\mu_{n+1} b, +\infty) (z_{n+1} - z_n)
\]
for every \( n \in \mathbb{N} \). There projections standing on the right-hand sides are finite. Hence, \( e^\alpha_z (d, +\infty) \) is finite projection as a sum of the left-hand sides [22].

We shall use a following well-known implication 
\[
p \prec \prec q \Rightarrow zp \prec \prec zq, \forall z \in P\left( Z(M) \right), 0 < z \leq c(p) \vee c(q).
\]
We supply here a straightforward argument. Let \( z' \in z \in Z(M) \) be such that 
\[0 < z' \leq c(pz) \vee c(qz) z(c(p) \vee c(q)) \]. Then \( z' \leq c(p) \vee c(q) \) and therefore 
\[z'(zp) = z'p \prec z'q = z'(zq) \]. This means \( zp \prec \prec zq \).

As in [6] we can use Theorem (6) to extend the result to the properly infinite case.

**Theorem (13).** Let \( A \) be a properly infinite \( W^* \) sub-algebra of \( B \) containing the center \( \ell \) of \( B \). For every derivation \( \delta : A \to f(B) \) there is \( aT \in f(B) \) such that \( \delta = aA T \).

Before we start the proof let us recall that if \( A \) is properly infinite there is an infinite countable decomposition of the identity into mutually orthogonal projections of \( A \), all
equivalent in $A$ to $I$, and thus a fortify equivalent in $B$ to $1$ [8]. Therefore there is a spatial isomorphism

$$\phi: B \rightarrow \tilde{B} = B \otimes B(H_0)$$

with $H_0 = l^{n+1}(\mathbb{Z})$ and

$$\phi(A) = \tilde{A} = A \otimes B(H_0)$$

[5]. Recall also that the elements $B$ of $\tilde{B}$ (or $\tilde{A}$) are represented by bounded matrices $[B_{ij}], i, j \in \mathbb{Z}$ with entries in $B$ (or $A$) by the formula

$$\left(I \otimes E_j\right)T\left(I \otimes E_i\right) = T_{ik} \otimes E_{il}$$

where $E_j$ is the canonical matrix unit of $B(H_0)$. In particular if $\ell, \psi$ are the maximal a commutative operation subalgebras of $B(H_0)$ of Laurent (resp. diagonal) matrices, then $B \in B \otimes \ell$ (resp. $B \in B \otimes \psi$) iff $[B_{ij}]$ is a Laurent matrix with entries in $B$, i.e., $B_{ij} = B_{i-j}$, where $B_{ij}$ denotes the entry along the $k$th diagonal (resp. $B_{ij} = \delta_{ij}B_{ii}$) for all $i, j \in \mathbb{Z}$.

**Proof.** Let $\tilde{\delta} = \phi \circ \delta \circ \phi^{-1}$ then

$$\tilde{\delta}: d \rightarrow \phi\left(f\left(B\right)\right) = f\left(\tilde{B}\right)$$

is a relative compact derivation. Let us define the following $W^*$ algebras:

$$\tilde{\ell} = \tilde{B} \cap \tilde{\ell}, \quad \tilde{A}_n = \ell \otimes \ell, \quad A_n = \phi^{-1}\left(\tilde{A}_n\right), \quad \tilde{\ell}_{n+1} = \ell \otimes \ell, \quad \tilde{A}_{n+1} = A \otimes B(H_0), \quad \tilde{A}_{n-1} = A \otimes \psi, \quad \tilde{A}_{n+2} = A_n \otimes \psi.$$

First, let us notice that

$$\tilde{A}_n \cap f\left(\tilde{B}\right) = \left(\ell \otimes \ell \cap B \otimes B(H_0)\right) \cap f\left(\tilde{B}\right)$$

$$= \left(B \otimes \ell\right) \cap f\left(\tilde{B}\right)$$

$$= \{0\}$$

by [22]. Therefore

$$A_n' \cap f\left(B\right) = \phi^{-1}\left(\tilde{A}_n\right) \cap f\left(B\right) = \phi^{-1}\left(\tilde{A}_n \cap f\left(\tilde{B}\right)\right) = \{0\}$$

because $\phi$ is spatial. Now

$$\tilde{\ell} = \left(B \otimes B(H_0)\right) \cap \left(B' \otimes I\right)$$

$$= \ell \otimes I \subset \tilde{A}_n \subset \tilde{A}.$$
Thus we can apply Theorem (6) to the derivation \( \delta \) restricted to the a commutative operation sub-algebra \( \tilde{A}_n \) of \( \tilde{B} \) and we obtain a \( T_n \in f(\tilde{B}) \) such that \( \delta_n = \delta - a AT_n \) vanishes on \( \tilde{A}_n \).

Now

\[
\tilde{A}_{n+1} \subset B \otimes \ell' \subset \ell' \otimes \ell = \tilde{A}'_n.
\]

Therefore, for all \( A_n \in \tilde{A}_n \) and \( A_{n+1} \in \tilde{A}_{n+1} \) we have

\[
\delta_n (A_n A_{n+1}) = A_n \delta_n (A_{n+1}) = \delta_n (A_{n+1} A_n) = \delta_n (A_{n+1}) A_n
\]

i.e., \( \delta_n (A_{n+1}) \) and \( A_n \) commute and hence

\[
\tilde{\delta}_n (\tilde{A}_{n+1}) \subset \tilde{A}'_n \cap f(\tilde{B}) = \{0\}
\]

Thus \( \tilde{\delta}_n \) also vanishes on \( \tilde{A}_{n+1} \). Now \( \tilde{A}_n \) is a commutative operation and hence so are \( A_n \) and \( \tilde{A}_{n+2} \). Moreover,

\[
\tilde{\ell} \subset \tilde{A}_n \subset \tilde{A} \subset \tilde{B}
\]

Implies

\[
\ell = \phi^{-1} (\tilde{\ell}) \subset A_n \subset A \subset B
\]

and hence

\[
\tilde{\ell} = \ell \otimes I \subset A_n \otimes I \subset \tilde{A}_{n+2} \subset \tilde{A} \subset \tilde{B}
\]

Thus we can apply again Theorem (6) to the relative compact derivation \( \tilde{\delta}_n \) restricted to \( \tilde{A}_{n+2} \).

Let \( T_{n+1} \in f(\tilde{B}) \) be such that \( \tilde{\delta}_n \) agrees with \( \text{ad} T_{n+1} \) on \( \tilde{A}_{n+2} \). Since

\[
A_n \otimes I \subset A \otimes I \subset A \otimes \ell = \tilde{A}_{n+1}
\]

and \( \tilde{\delta}_n \) vanishes on \( \tilde{A}_{n+1} \), we see that \( \text{ad} T_{n+1} \) vanishes on \( A_n \otimes I \), i.e.,

\[
T_{n+1} \in (A_n \otimes I)' \cap f(\tilde{B}) = (A'_n \otimes B(H_0)) \cap f(\tilde{B})
\]

Then for all \( i, j \in \mathbb{Z}, (T_{n+1})_{ij} \in A'_n \) and

\[
(T_2)_{ij} \otimes E_{mn} = (I \otimes E_{ni}) T_{n+1} (I \otimes E_{jm}) \in f(\tilde{B})
\]
whence by Lemma(12)(a) \( (T_{n+1})_j \in f(B) \). But we saw that \( d'_n \cap f(B) = \{0\} \), hence \( (T_{n+1})_j = 0 \) for all \( i, j \in \mathcal{J} \), so \( T_{n+1} = 0 \). Therefore \( \delta_n \) vanishes also on \( \tilde{A}_{n+2} \) and hence on \( I \otimes \varnothing \). Now \( \ell \) and \( \varnothing \) generate \( B(H_0) \), whence \( \tilde{A}_{n+1} = A \otimes \ell \) and \( I \otimes \varnothing \) generate \( \tilde{A} \). Thus by the \( \sigma \)-weak continuity of \( \delta_n \) (see [6]) we see that

\[
\delta_n = \delta - aAT_n = 0, \text{ i.e., } \delta = aAT_n. \text{ Clearly } \delta = ad\phi^{-1}(T_n) \text{ and } \phi^{-1}(T_n) \in A(B).
\]

Let us assume in this part that \( B \) is semi-finite and let \( \tau \) be a fsn trace on it. Beside the closed ideal \( f(B) \) we can also consider the (non closed) two-sided norm-ideals \( C_{1+\varepsilon}(B,\tau) \) for \( 1 \leq 1 + \varepsilon < \infty \) defined by

\[
C_{1+\varepsilon}(B,\tau) = \left\{ B \in B \mid \tau \left( |B|^{1+\varepsilon} \right) < \infty \right\}
\]

\[
\|B\|_{1+\varepsilon} = \tau \left( |B|^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \text{ for } B \in C_{1+\varepsilon}(B,\tau).
\]

Obviously,

\[
C_{1+\varepsilon}(B,\tau) = B \cap L^{1+\varepsilon}(B,\tau),
\]

where the latter is the non commutative \( L^{1+\varepsilon} \)-space of \( B \) relative to \( \tau \) (see [14]).

Recall the following facts about \( L^{1+\varepsilon}(M) \) spaces in the case of a general W* algebra \( M \) and \( 1 \leq 1 + \varepsilon < \infty \) (\( L^\varepsilon(M) \) is identified with \( M \)): \( L^{1+\varepsilon}(M) \) is a Banach space, its dual is isomorphic to \( L^{\varepsilon/(1+\varepsilon)}(M) \) (with \( x^{1+\varepsilon} + \varepsilon x^{1+\varepsilon} = 1 \)), and the duality is established by the functional \( \text{tr} \) on \( L(M) \),

where if \( A \in L^{1+\varepsilon}(M), B \in L^{\varepsilon/(1+\varepsilon)}(M) \) we have \( AB, BA \in L(M) \) and

\[
\text{tr}(AB) = \text{tr}(BA), \quad |\text{tr}(AB)| \leq \|A\|_{1+\varepsilon} \|B\|_{\varepsilon/(1+\varepsilon)}, \quad \|A\|_{1+\varepsilon} = \left( |\text{tr} A^{1+\varepsilon} |^{\varepsilon/(1+\varepsilon)} \right)^{1/(\varepsilon/(1+\varepsilon))} = \max \left\{ |\text{tr} AB| \mid B \in L^{\varepsilon/(1+\varepsilon)}(M), \|B\|_{\varepsilon/(1+\varepsilon)} \leq 1 \right\}
\]

(see [14]). Of course, if \( M = B \) we can identify \( L^{1+\varepsilon}(M) \) with \( L^{1+\varepsilon}(B,\tau) \) and \( \text{tr} \) with \( \tau \). The following inequality will be used here only in the semi-finite case and in the context of \( C_{1+\varepsilon} \)-ideals, but since the same proof holds for \( L^{1+\varepsilon} \)-spaces, we shall consider the general case.
Corollary (14). Let $M$ be a $W^*$ algebra, $\varepsilon \geq 0, A \in L^{1+\varepsilon}(M)$ and $Q_n, Q_{n+1} \in p(M)$. Let $Q_n Q_{n+1} = 0, Q_n + Q_{n+1} = 1$. Then
\[ \|A\|^{1+\varepsilon} \geq \|Q_n AQ_n\|^{1+\varepsilon} + \|Q_{n+1} AQ_{n+1}\|^{1+\varepsilon} \]

Proof. Let us first note that
\[ \sum_{i=n}^{n+1} Q_i A Q_i^{1+\varepsilon} = \sum_{i=n}^{n+1} |Q_i A Q_i|^{1+\varepsilon} \]

And
\[ \left\| \sum_{i=n}^{n+1} Q_i A Q_i \right\|^{1+\varepsilon} = \sum_{i=n}^{n+1} \|Q_i A Q_i\|^{1+\varepsilon} \]

Consider first $1 + \varepsilon = n$ and take the polar decomposition's
\[ Q_i A Q_i = U_i |Q_i A Q_i|, \quad i = n, n+1. \]

Then $U_i U_i^*$ and $U_i^* U_i$ are majorized by $Q_i$ and hence $U_i$ commutes with $Q_i$. Therefore
\[ B = (U_n + U_{n+1})^* \] commutes with $Q_i$ and $\|B\| = 1$. Then
\[ \|A\| \geq |trAB| = |tr\left( \sum_{i=n}^{n+1} Q_i BAQ_i \right)| = tr\left( \sum_{i=n}^{n+1} Q_i A Q_i \right) = \sum_{i=n}^{n+1} \|Q_i A Q_i\|_n. \]

Consider now $\varepsilon > 0$. Let $B \in L^{\frac{\varepsilon}{1+\varepsilon}}(M)$ be such that $\|B\|_{\frac{\varepsilon}{1+\varepsilon}} \leq 1$ and
\[ \left\| \sum_{i=n}^{n+1} Q_i A Q_i \right\|^{1+\varepsilon} = tr\left( \left( \sum_{i=n}^{n+1} Q_i A Q_i \right) B \right). \]

Take the polar decomposition's $A = U |A|$ and $B = V |B|$, then $VU$ are in $M$ and $|A|, |B|$ are in $L^{\frac{\varepsilon}{1+\varepsilon}}(M), L^{\frac{\varepsilon}{1+\varepsilon}}(M)$, respectively. Let
\[ f(z) = \text{tr} \left( \sum_{i=n}^{n+1} Q_i U^{|(1+\varepsilon)t|} Q_i V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right) . \]

Then by standard arguments, it is easy to see that \( f \) is analytic on \( 0 < Re \ z < n \) and continuous and bounded on \( 0 \leq Re \ z \leq n \). Then by the three-line theorem (see [4]) we have

\[
 f \left( \frac{1}{1+\varepsilon} \right) \leq \text{Max}_{t \in \mathbb{R}} f \left( it \right)^{\frac{\varepsilon}{1+\varepsilon}} \text{Max}_{t \in \mathbb{R}} f \left( 1+it \right)^{\frac{\varepsilon}{1+\varepsilon}}
\]

Now \( f \left( \frac{1}{1+\varepsilon} \right) = \left\| \sum_{i=n}^{n+1} Q_i A \right\|_{1+\varepsilon} \) and by Holder’s inequality

\[
 |f(it)| = \text{tr} \left( \sum_{j=n}^{n+1} Q_j U^{|(1+\varepsilon)t|} Q_j V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right)
\]

\[
 \leq \left\| \sum_{j=n}^{n+1} Q_j U^{|(1+\varepsilon)t|} Q_j V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right\|
\]

\[
 \leq \left( \text{max}_{j} \left\| Q_j U^{|(1+\varepsilon)t|} Q_j V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right\| \right)
\]

\[
 \leq n.
\]

Again by Holder’s inequality applied twice and by the result already obtained in the \( \varepsilon = 0 \) case,

\[
 |f(1+it)| = \text{tr} \left( \sum_{j=n}^{n+1} Q_j U^{|(1+\varepsilon)t|} |A|^{1+\varepsilon} Q_j V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right)
\]

\[
 \leq \left\| \sum_{j=n}^{n+1} Q_j U^{|(1+\varepsilon)t|} |A|^{1+\varepsilon} Q_j V^{|(1+\varepsilon)t|} |B| \frac{e^{\frac{\varepsilon}{1+\varepsilon}}}{1+\varepsilon} \right\|
\]

\[
 \leq \left\| U^{|(1+\varepsilon)t|} |A|^{1+\varepsilon} \right\|
\]

\[
 \leq \left\| U^{|(1+\varepsilon)t|} \right\| \left\| A^{1+\varepsilon} \right\|
\]

\[
 \leq \left\| A^{1+\varepsilon} \right\|
\]

Thus \( f \left( \frac{1}{1+\varepsilon} \right) \leq \left\| A \right\|_{1+\varepsilon} \) whence by the second equality in this proof,

\[
 \left\| A \right\|_{1+\varepsilon} \leq \left\| \sum_{i=n}^{n+1} Q_i A Q_i \right\|_{1+\varepsilon} = \sum_{i=n}^{n+1} \left\| Q_i A Q_i \right\|_{1+\varepsilon}
\]

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No data were used to support this study.
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