ELECTROMAGNETIC AND WEAK CURRENT OPERATORS FOR INTERACTING SYSTEMS WITHIN THE FRONT-FORM DYNAMICS

F.M. Lev\textsuperscript{a}, E. Pace\textsuperscript{b} and G. Salmè\textsuperscript{c}

\textsuperscript{a}Laboratory of Nuclear Problems, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia
\textsuperscript{b}Dipartimento di Fisica, Università di Roma ”Tor Vergata”, and Istituto Nazionale di Fisica Nucleare, Sezione Tor Vergata, Via della Ricerca Scientifica 1, I-00133, Rome, Italy
\textsuperscript{c}Istituto Nazionale di Fisica Nucleare, Sezione di Roma, P.le A. Moro 2, I-00185 Rome, Italy

Abstract

Electromagnetic and weak current operators for interacting systems should properly commute with the Poincaré generators and satisfy Hermiticity. The electromagnetic current should also satisfy $\mathcal{P}$ and $\mathcal{T}$ covariance and continuity equation. In front-form dynamics the current can be constructed from auxiliary operators, defined in a Breit frame where initial and final three-momenta of the system are directed along the $z$ axis. Poincaré covariance constraints reduce for auxiliary operators to the ones imposed only by kinematical rotations around the $z$ axis; while Hermiticity requires a suitable behaviour of the auxiliary operators under rotations by $\pi$ around the $x$ or $y$ axes. Applications to deep inelastic structure functions and electromagnetic form factors are discussed. Elastic and transition form factors can be extracted without any ambiguity and in the elastic case the continuity equation is automatically satisfied, once Poincaré, $\mathcal{P}$ and $\mathcal{T}$ covariance, together with Hermiticity, are imposed.

Pacs: 11.40-q 11.40.Dw 13.40.Gp 13.60.Hb 25.30.Bf
1 Introduction

Experiments on modern accelerators make it possible to investigate a variety of electromagnetic (em) and weak properties of hadrons. A comprehensive theoretical analysis of these properties encounters serious difficulties, since perturbative QCD does not apply to the bound state problem. In view of these difficulties, effective models have been developed, but both the theory and models use the current operator as a fundamental input for evaluating elastic and inelastic form factors of relativistic interacting systems. For this reason, it is important to understand which constraints on the current operators can be imposed taking into account only general properties, e.g., Poincaré covariance and Hermiticity.

For instance, in the relevant case of deep inelastic scattering (DIS) the cross-section is fully defined by the hadronic tensor

\[ W_{\mu\nu} = \frac{1}{4\pi} \int e^{iqx} \langle P', \chi' | J^\mu(x) J^\nu(0) | P', \chi' \rangle d^4x \]  

where \( q \) is the momentum transfer and \( | P', \chi' \rangle \) is the initial state of the nucleon with four-momentum \( P' \) and internal wave function \( \chi' \). This tensor will have correct transformation properties relative to the Poincaré group (i.e., \( W_{\mu\nu} \) will be a true tensor) only if both the state \( | P', \chi' \rangle \) and the operator \( J^\mu(x) \) have correct transformation properties with respect to the same representation of the Poincaré group. In the parton model proposed by Bjorken [1] and Feynman [2] the nucleon is described as a bound system, while the (em or weak) current operator \( J^\mu(x) \) (\( x \) is a point in Minkowski space) is taken in impulse approximation (IA), i.e., it is the same as for noninteracting particles. According to the present theory based on the operator product expansion [3] and factorization theorem [4], the parton model (even in the Bjorken limit) is accurate up to anomalous dimensions and perturbative QCD corrections; therefore the corrections to the parton model can be considered in the framework of perturbation theory. However the nucleon is a bound state of quarks and gluons and cannot be described perturbatively. Therefore, in principle, the problem arises whether the operator \( J^\mu(x) \), treated perturbatively at large \( Q = |q^2|^{1/2} \), is compatible with the correct transformation properties of bound states.
In the case of model approaches, another example of the necessity of compatibility between the generators of the Poincaré group and the current operators is clearly met in the investigation of elastic and inelastic hadron form factors within the front-form Hamiltonian dynamics [3]. In this framework hadron form factors have generally been calculated assuming that, in the reference frame where \( q^+ = 0 \), the component \( J^+(0) \) can be taken in IA (the \( \pm \) components of four-vectors are defined as \( p^\pm = (p^0 + p^z)/\sqrt{2} \)). The main argument in favor of this assumption (see, e.g., Refs. [6, 7, 8, 9]) is that in the reference frame where \( q^+ = 0 \) the production of pairs from the vacuum is forbidden by momentum conservation and the operator \( J^+(0) \) gives a contribution only for positive-energy components of Dirac spinors. However, the hadron form factors are determined by matrix elements of \( J^\mu(0) \) between initial and final bound states and, by analogy with the case of DIS, the problem arises whether the assumption that \( J^+(0) \) is free is compatible with the correct transformation properties of its matrix elements. Indeed, consider for example the elastic electron-deuteron scattering in the Breit frame of the deuteron, i.e., in the reference frame where the initial and final three-momenta \( \mathbf{P}' \) and \( \mathbf{P}'' \) satisfy the condition \( \mathbf{P}' + \mathbf{P}'' = 0 \). If \( \lambda' \) and \( \lambda'' \) are the deuteron helicities in the initial and final states, respectively, and \( I_{\lambda',\lambda''} = \langle \lambda''|J^+(0)|\lambda' \rangle \) then, as follows from \( \mathcal{P} \) and \( \mathcal{T} \) covariance, all the matrix elements \( I_{\lambda',\lambda''} \) can be expressed in terms of \( I_{11}, I_{00}, I_{10} \) and \( I_{1,−1} \). As follows from Poincaré covariance, current conservation and Hermiticity, the elastic electron-deuteron scattering is described by three independent real form factors and therefore the above matrix elements are not independent. As shown in Refs. [10, 11] and others, if \( \eta = Q^2/4m_d^2 \), with \( m_d \) the deuteron mass, then the following constraint, called ”angular condition” must be fulfilled in the \( q^+ = 0 \) frame, viz.

\[
(1 + 2\eta)I_{11} - I_{00} - (8\eta)^{1/2}I_{10} + I_{1,−1} = 0.
\]  

(2)

However this relation is not satisfied if the matrix elements \( I_{\lambda',\lambda''} \) are calculated with the free operator, \( J^+_{\text{free}}(0) \), and therefore interactions term are needed.

A way to avoid this difficulty has been proposed, e.g., in [12], where it is noted that the three form factors can be determined by using the free operator \( J^+_{\text{free}}(0) \) for calculating only three matrix elements, while the fourth one can be determined (if necessary) from Eq. (2). It is clear that such a
procedure contains a large extent of freedom. In absence of any dynamical scheme, only the comparison of the results with the data can yield insight, if any, on the choice of the three matrix elements to be preferred. Another approach has been proposed in [12] within the covariant formulation of the front-form dynamics. In this approach the matrix elements \( \langle \lambda'' | J^\mu(0) | \lambda' \rangle \) with \( J^\mu(0) \equiv J^\mu_{\text{free}}(0) \) are given by the sum of eleven contributions. Only three of them depend upon the physical form factors (as they must do if the operator fulfills the Poincaré covariance and the current conservation), while the other contributions contain the null vector \( \omega^\mu \), which determines the direction of the null plane in Minkowski space, and are unphysical. The physical form factors can be formally obtained if, following Ref. [13], two matrix elements are calculated by using \( J^\mu_{\text{free}}(0) \) and the third matrix element is calculated by using \( J^\mu_{\text{free}}(0) \) with \( j = 1 \) or \( j = 2 \). However, it remains unclear whether the contribution of interaction terms in the current operator to the physical form factors is important.

In view of the above discussion it is important to know the constraints imposed by Poincaré covariance on the exact current operator \( J^\mu(x) \), and, as far as the em current is concerned the further constraints imposed by current conservation, parity and time reversal. These constraints were investigated in detail in Refs. [14, 15, 16]. As shown in Ref. [14], if the mass and spin operators for the system as a whole are diagonalized, the matrix elements of the current operator can be expressed in terms of reduced matrix elements which are not constrained by Poincaré covariance. However for practical calculations it is desirable to know all the constraints directly in terms of operators (see, e.g., [16]), and furthermore the Hermiticity condition (discussed in detail in this paper) and the cluster separability (see, e.g., Refs. [17, 18, 19, 20]) are to be satisfied. For systems with a fixed number of interacting relativistic particles a current operator satisfying Poincaré covariance, Hermiticity and cluster separability to order \((v/c)^4\) has been constructed in Ref. [15]. An exact solution in the point form of dynamics has been considered in Ref. [16], where the current operator is expressed in terms of auxiliary operators defined in the equal-velocity frame, with the \( z \) axis directed along the three-momentum transfer. In particular, it has been shown that the free-current operator represents a possible solution for the auxiliary operators.
Finally, we mention that the requirement of locality, in the sense that the commutator \([J^\mu(x), J^\mu(y)]\) should vanish when \(x - y\) is a space-like vector, could be used for choosing between different solutions, if they exist, satisfying Poincaré covariance and the other general properties.

Aim of this paper is the investigation of the constraints imposed on the current operator by extended Poincaré covariance (continuous + discrete transformations), Hermiticity and current conservation within the front-form dynamics \([5]\). This dynamics is widely adopted and exhibits many interesting features, such as the largest set of kinematical Poincaré generators and the boundedness from below of the \(P^+\) component (see, e.g., \([9, 21]\)). We have followed an approach analogous to the one of Ref. \([16]\), but applied directly in the front form. In Ref. \([16]\), by using the unitary equivalence of the different forms of relativistic dynamics \([22]\), the current operators in the front and the instant form corresponding to the particular solution found in the point form, have been constructed. It should be pointed out that the point-form auxiliary operators obtained from the free current generate both non interacting and interacting terms in the corresponding front-form (instant) operators. Therefore it is interesting to investigate if, also in the front form, auxiliary operators obtained directly from a free current can produce an exact solution. In Ref. \([16]\) the direct construction of the current in the front-form was not considered, arguing the presence of extra difficulties with respect to the point form. In this paper such problems will be solved expressing the current operator in terms of auxiliary operators, which act only through internal variables and depend upon given masses for the initial and final system (namely considering a spectral decomposition of the current operator). A particular attention has to be devoted to the choice of the reference frame where the auxiliary operators are defined. As a matter of fact, in the conventional (instant form) approach to elastic scattering the form factors are generally evaluated in the reference frame where \(q^0 = 0\) and then, as follows from rotational invariance of the ordinary three-dimensional space, the \(z\) axis can be chosen along the common direction of the initial and final three-momenta of the system. Hence the problem becomes symmetric with respect to rotations around the \(z\) axis (≡ the spin quantization axis). In the front form, the reference frame analogous to the one adopted in the
conventional approach is the frame where $q^+ = 0$ (for both elastic and inelastic scattering). In this case, it is again possible to find a reference frame where both initial and final three-momenta of the system are directed along the same vector $n$, but $n$ obviously cannot coincide with the $z$ axis. Therefore the rotational invariance around the $z$ axis is lost. It is worth noting that in the front form the rotation around the $z$ axis are kinematical, while those around the $x$ and $y$ axes are dynamical. In order to take advantage of this peculiarity of the $z$ axis and to restore the symmetry of the physical problem, we perform our analysis in the Breit frame where the initial and final three-momenta of the system are directed along the $z$ axis, and as a consequence $q^+ \neq 0$.

One of the main results of our investigation is that, in order to satisfy the Poincaré covariance, the auxiliary operators in our Breit frame have to be covariant only with respect to rotations around the $z$ axis.

The paper is organized as follows: in Sects. 2-6 the general formalism relevant for relativistic interacting systems is presented; in Sects. 7 and 8 the constraints imposed on the current operators are explicitly given; in Sect. 9 the matrix elements of the current are discussed; in Sect. 10 the application to DIS is investigated; in Sect. 11 the cases of elastic and inelastic scattering are considered. Finally in Sect. 12 conclusions are drawn.

## 2 General Formalism

Let $P$ be the operator of the four-momentum for the system under consideration and $M^{\mu\nu} \ (M^{\mu\nu} = -M^{\nu\mu})$ be the representation generators of the Lorentz group. We shall always assume that the commutation relations for the representation generators of the Poincaré group are realized in the form

$$[P^\mu, P^\nu] = 0, \ [M^{\mu\nu}, P^\rho] = -i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu),$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}) \tag{3}$$

where $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$, the metric tensor in Minkowski space has the nonzero components $\eta^{00} = -1 = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$, and we use the system of units with $\hbar = c = 1$. 

6
As explained in well-known textbooks and monographs (see, e.g., Refs. [23, 24]), matrix elements of field operators have the correct transformation properties relative to transformations of the Poincaré group, only if the transformation of the operators are compatible with the transformations of the states. This implies that the current operator should satisfy the conditions

\begin{equation}
\exp(iPx)J^\mu(0)\exp(-iPx) = J^\mu(x),
\end{equation}

\begin{equation}
U(l)^{-1}J^\mu(x)U(l) = L(l)^\mu_\nu J^\nu(L(l)^{-1}x)
\end{equation}

where \(L(l)\) is the element of the Lorentz group corresponding to \(l \in SL(2, \mathbb{C})\) and \(U(l)\) is the representation operator corresponding to \(l\).

In particular

\begin{equation}
U(l)^{-1}J^\mu(0)U(l) = L(l)^\mu_\nu J^\nu(0)
\end{equation}

Therefore, as a consequence of Lorentz covariance, one has

\begin{equation}
[M^{\mu\nu}, J^\rho(0)] = -i(\eta^{\mu\rho} J^\nu(0) - \eta^{\nu\rho} J^\mu(0))
\end{equation}

Since some of the Poincaré group generators describing Poincaré transformations of the bound states are necessarily interaction dependent, the above expressions show that \(J^\mu(x)\) is a relativistic vector operator only if it depends on the interaction present in the system under consideration. In general, it is not clear whether this condition can be compatible with the usual assumption that \(J^+(0)\) is free. Moreover, as already explained, there exist cases where this assumption is definitely incompatible with Poincaré covariance and current conservation (see, e.g., Refs. [10, 25]).

3 Irreducible representations of the Poincaré group with positive mass

In order to describe a relativistic system of interacting particles it is necessary to choose first an explicit form of the unitary irreducible representation (UIR) of the Poincaré group describing an elementary particle of mass \(m > 0\) and spin \(s\). There are many equivalent ways to construct an explicit realization of such a representation [20, 27]. We choose the realization which is convenient in the front-form dynamics [2].
Let \( p \) be the particle 4-momentum, \( g = p/m \) be the particle 4-velocity and \( s \) be the spin operator. Since \( p^2 = m^2 \), only three components of \( p \) are independent. We choose \( p_\perp \) and \( p^+ \) as such components, where \( p_\perp \equiv (p_x, p_y) \). Let \( \sigma \) be the projection of the spin on the \( z \) axis. The one-particle space can be chosen as the space of functions \( \phi(p, \sigma) = \phi(p_\perp, p^+, \sigma) \) with the norm

\[
\langle \varphi|\varphi \rangle = \sum_\sigma \int |\phi(p_\perp, p^+, \sigma)|^2 d\rho(p_\perp, p^+) \tag{8}
\]

where

\[
d\rho(p_\perp, p^+) = \frac{d^2 p_\perp dp^+}{2(2\pi)^3 p^+} \tag{9}
\]

If an element of the Poincaré group \((a, l)\) is defined by the four-vector \(a\) and by the matrix \(l \in SL(2, C)\), then the corresponding representation operator acts as

\[
\langle p, \sigma|U(a, l)|\phi \rangle = U(a, l)\phi(p, \sigma) = \exp(ip'a) \sum_{\sigma'} D^s_{\sigma\sigma'}[W(l, g')][\phi(p', \sigma')] \tag{10}
\]

where \( p' = L(l)^{-1}p \) and \( W(l, g') \) is the front-form Wigner rotation defined as

\[
W(l, g') = \beta(g)^{-1}l\beta(g') \tag{11}
\]

The matrices \( \beta(g) \in SL(2,C) \) represent the front-form boosts and their components are given by

\[
\beta_{11} = \beta_{22}^{-1} = 2^{1/4}(g^+)^{1/2}, \quad \beta_{12} = 0, \quad \beta_{21} = (g_x + ig_y)\beta_{22} \tag{12}
\]

In Eq. (10), \( D^s(u) \) is the matrix of the UIR of the group \( SU(2) \) with the spin \( s \), corresponding to \( u \in SU(2) \) (it is easy to verify that \( W(l, g') \in SU(2) \)).

A direct calculation shows that, for the UIR defined by Eqs. (10) and (11), the generators have the well-known form (see, for example, Refs. [28, 29])

\[
P^+ = p^+, \quad P_\perp = p_\perp, \quad P^- = p^- = \frac{m^2 + p^2_\perp}{2p^+}, \tag{13}
\]

\[
M^{++} = ip^+ \frac{\partial}{\partial p^+}, \quad M^{+j} = -ip^+ \frac{\partial}{\partial p^j}, \quad M^{xy} = \ell^z(p_\perp) + s^z,
\]

\[
M^{-j} = -i(p_j \frac{\partial}{\partial p^+} + p^- \frac{\partial}{\partial p^j}) - \frac{\epsilon_{jl}}{p^+}(ms^l + p^l s^z)
\]
where a sum over $j, l = x, y$ is assumed, $\epsilon_{jl}$ has the components $\epsilon_{xy} = -\epsilon_{yx} = 1$, $\epsilon_{xx} = \epsilon_{yy} = 0$ and $\ell(p) = -i p \times (\partial / \partial p)$ is the orbital angular-momentum operator.

The presence of the matrices $\beta(g)$ in Eq. (11) is very relevant. Let $B$ be a subgroup of $\text{SL}(2, \mathbb{C})$ such that $b \in B$ if $b_{11} = b_{22}^{-1} > 0$, $b_{12} = 0$ and $b_{21}$ is an arbitrary complex number. Then, it is clear from Eq. (12) that $B$ is the set of the front-form boosts and one can verify by a direct calculation that

$$b\beta(g) = \beta(L(b)g) \quad \text{if} \quad b \in B \quad (14)$$

Therefore, as follows from Eqs. (11) and (14), the Wigner rotations corresponding to $b \in B$ are equal to 1 and, as follows from Eq. (10), the action of the representation operators corresponding to $b \in B$ and $a = 0$ is especially simple, viz.

$$U(b)p, \sigma = \phi(L(b)^{-1} p, \sigma) \quad (15)$$

The representation generators of the group $B$ are $M^+^−$ and $M^+_j$ (see, e.g., [21, 30]) and it is clear from Eq. (15) why they do not depend on $s$ (see Eq. (13)). The important role of the group property (14) has been pointed out in Ref. [31].

Each element of the group $\text{SL}(2, \mathbb{C})$ can be uniquely written as $l = \beta(g)u$, where $u \in SU(2)$ (see, e.g., Ref. [27]). Another possible representation is $l = \alpha(g)u'$ [27], where

$$\alpha(g) = \frac{g^0 + 1 + \tau g}{[2(g^0 + 1)]^{1/2}} \quad (16)$$

and $\tau$ are the Pauli matrices. The matrices $\alpha(g)$ represent the instant-form boosts and do not form a group. The choice of $\alpha(g)$ instead of $\beta(g)$ is convenient for investigating discrete symmetries and conventional three-dimensional rotations. In particular one has

$$u\alpha(g) = \alpha(L(u)g)u \quad (17)$$

The relation between the matrices $\alpha(g)$ and $\beta(g)$ is

$$\beta(g) = \alpha(g)v(g), \quad v(g) = \frac{1 + g^0 + g^3 + \nu \epsilon_{jl} \tau^j g^l}{[2(1 + g^0)(g^0 + g^3)]^{1/2}} \quad (18)$$
where $v(g) \in SU(2)$ is the so called Melosh matrix \cite{32}, which in the given context was first considered in Ref. \cite{29}. In particular, note that if $g^l = 0$ ($l = 1, 2$) then $v = 1$. Such a property will be used in the following sections.

Let us now consider the discrete symmetries, space reflection and time reversal. For the reasons which will be clear later, it is convenient to consider not the full space reflection $P$, but only the reflection relative the $y$ axis, $P_y = P R_y (\pi)$ (where $R_y (\pi)$ is a rotation around the $y$ axis by $\pi$). The action of the corresponding operator is given by

$$\langle p | U_y | \varphi \rangle = U_y \varphi (p) = \eta_P \exp (-i \pi s_y) \varphi (\tilde{p})$$

(19)

where $\eta_P$ is the $P$ parity of the particle under consideration, $\tilde{p}$ differs from $p$ by the sign of the $y$ component: $\tilde{p} = (p_x, -p_y, p^+, p^-)$ and the action of $R_y (\pi)$ can be obtained from Eq. (10).

Instead of $T$ covariance we will consider $\theta$ covariance, where $\theta = PT$. The action of the corresponding antiunitary operator is given by

$$\langle p, \sigma | U_\theta | \varphi \rangle = U_\theta \varphi (p, \sigma) = \eta_\theta \overline{\langle p, \sigma | \exp (-i \pi s_y) | \varphi \rangle}$$

(20)

where $\eta_\theta$ is the $\theta$ parity and the bar means the complex conjugation.

4 Representations of the extended Poincaré group for systems of noninteracting particles

The space $\mathcal{H}$ for the representation of the Poincaré group describing a system of $N$ free particles with the masses $m_i$ and spins $s_i$ ($i = 1, 2, ..., N$) can be realized as the space of functions $\phi(p_{1\perp}, p_{1}^{+}, \sigma_1, ..., p_{N\perp}, p_{N}^{+}, \sigma_N)$ with the norm

$$\langle \varphi | \varphi \rangle = \sum_{\sigma_1...\sigma_N} \int |\phi(p_{1\perp}, p_{1}^{+}, \sigma_1, ..., p_{N\perp}, p_{N}^{+}, \sigma_N)|^2 \prod_{i=1}^{N} d\rho(p_{i\perp}, p_{i}^{+})$$

(21)

Instead of the variables $p_{1\perp}, p_{1}^{+}, ..., p_{N\perp}, p_{N}^{+}$, we introduce the variables $P_{\perp}, P^{+}, k_1, ..., k_N$, where $P = p_1 + ... + p_N$ is the total four-momentum, and $k_i$ is the spatial part of the four-vector

$$k_i = L[\beta(G)]^{-1} p_i,$$

(22)
with \( G = P/M_0 \) and \( M_0 = |P| \equiv |P^2|^{1/2} \). The action of the boost \( L[\beta(G)]^{-1} \) is such that \( P' = L[\beta(G)]^{-1}P \equiv (P'_\perp = 0, P'^+ = M_0/\sqrt{2}, P'^- = M_0/\sqrt{2}) \).

As follows from Eqs. (12) and (22), it is also possible to use the following internal variables:

\[
\xi_i = \frac{p_i^+}{P^+} = \sqrt{2}k_i^+ \quad k_i^\perp = p_i^\perp - \xi_i P^\perp
\]

The four-vectors \( p_i \) have canonical components \((\omega_i(p_i), p_i)\), and the four-vectors \( k_i \) have the components \((\omega_i(k_i), k_i)\), where \( \omega_i(k) = (m_i^2 + k^2)^{1/2} \) and \( k_z = (\xi - 1/2)M_0 \). In turn, only \( N - 1 \) vectors \( k_i \) are independent since, as follows from Eqs. (22) and (23), \( k_1 + ... + k_N = 0 \), i.e., \( k_i \) are intrinsic three-momenta. It is easy to show that \( M_0 = \omega_1(k_1) + ... + \omega_N(k_N) \).

A direct calculation shows that

\[
\prod_{i=1}^{N} d\rho(p_i^\perp, p_i^+) = d\rho(P_\perp, P^+)d\rho(int),
\]

\[
d\rho(int) = 2(2\pi)^3 M_0 \delta^{(3)}(k_1 + ... + k_N) \prod_{i=1}^{N} d\rho_i(k_i^\perp, k_i^+)
\]

Therefore the space \( \mathcal{H} \) can be realized as the space of functions \( \phi(P_\perp, P^+; k_1, \sigma_1, ..., k_N, \sigma_N) \) such that

\[
\langle \phi|\phi \rangle = \sum_{\sigma_1,...,\sigma_N} \int |\phi(P_\perp, P^+; k_1, \sigma_1, ..., k_N, \sigma_N)|^2 d\rho(P_\perp, P^+)d\rho(int)
\]

Let us also define the ”internal” space \( \mathcal{H}_{int} \) as the space of functions \( \chi(k_1, \sigma_1, ..., k_N, \sigma_N) \) such that the norm is equal to

\[
\langle \chi|\chi \rangle = \sum_{\sigma_1,...,\sigma_N} \int |\chi(k_1, \sigma_1, ..., k_N, \sigma_N)|^2 d\rho(int)
\]

Note that in front-form dynamics the operators \( P_\perp \) and \( P^+ \) are always equal to the operators of multiplication by the corresponding variables. Therefore the use of the same notations \((P_\perp, P^+)\) for both the variables and the operators should not lead to misunderstanding. Since the structure of the operators \((P_\perp, P^+)\) is clear, in the following we will consider only the structure of the remaining seven generators of the Poincaré group. For non-interacting particles they are equal to sums of the corresponding one-particle...
generators given by Eq. (13). A direct calculation of these sums shows that in the variables $P_\perp, P^+, k_1, ..., k_N$ one has

$$P^- = \frac{M_0^2 + P_\perp^2}{2P^+}, \quad M^{+-} = iP^+ \frac{\partial}{\partial P^+},$$

$$M^{+j} = -iP^+ \frac{\partial}{\partial P^j}, \quad M^{xy} = \ell^z(P_\perp) + S_0^z,$$

$$M^{-j} = -i(P^j \frac{\partial}{\partial P^-} + P^- \frac{\partial}{\partial P^j}) - \frac{\epsilon_{jl}}{P^+}(M_0S_0^l + P^lS_0^z)$$

(27)

where $\ell(P) = -iP \times (\partial/\partial P)$.

The operator $S_0$ in Eq. (27) is the spin operator for the system as a whole. It acts only through the variables of the space $H_{int}$ and is unitarily equivalent to the spin operator in the conventional form (see, e.g., Refs. [28, 29, 33, 30]):

$$S_0 = \{\prod_{i=1}^N D^{s_i}[v(\frac{k_i}{m_i})]\}^{-1}(\mathcal{L} + s_1 + ... + s_N)\{\prod_{i=1}^N D^{s_i}[v(\frac{k_i}{m_i})]\}^{-1}$$

(28)

where $\mathcal{L}$ is the total internal orbital angular momentum operator.

We see that the many-particle generators have the same form as the free ones, if in Eq. (13) $p$ is replaced by $P$, $m$ by $M_0$ and $s$ by $S_0$.

It is possible to show that the same is valid for the many-particle operator $U_y$. However, for practical purposes it is sufficient to represent the action of this operator in the form

$$\langle P_x, P_y, P^+|U_y|\varphi\rangle = U_{y, int}\langle P_x, -P_y, P^+|\varphi\rangle$$

(29)

where, as follows from Eqs. (19) and (23), the action of the operator $U_{y, int}$ in the space of internal variables is given by

$$\langle k_1, ..., k_N|U_{y, int}|\chi\rangle = \{\prod_{i=1}^N \eta_\mathcal{P}_i exp(-\imath\pi s_iy)\}\chi(k_1, ..., k_N)$$

(30)

where $\eta_\mathcal{P}_i$ is the internal $\mathcal{P}$ parity of particle $i$, and $\tilde{k}_i \equiv (k_{ix}, -k_{iy}, k_{iz})$.

Analogously, as follows from Eq. (20), the action of the operator $U_\theta$ for the system as a whole can be written as

$$\langle P_\perp, P^+; SS_z|U_\theta|\varphi\rangle = \eta_\theta \langle P_\perp, P^+; SS_z|exp(-\imath\pi S_0y)|\varphi\rangle$$

(31)
5 Systems of interacting particles in the front form of dynamics

If the particles interact with each other, then the representation space remains the same as in the case of free particles, but the representation generators of the Poincaré group differ from the corresponding free generators. One of the simplest way to preserve the relativistic commutation relations is to replace $M_0$ in Eq. (27) by a mass operator $\mathcal{M}$ which acts only through the variables of the space $\mathcal{H}_{int}$ and commutes with $S_0$. Then

$$P^- = \frac{\mathcal{M}^2 + P_+^2}{2P_+}, \quad M^{+j} = iP_+ \frac{\partial}{\partial P_+}, \quad M^{xy} = l^z (P_\perp) + S_0^z,$$

$$M^{-j} = -i(P_+ \frac{\partial}{\partial P_+} - P^- \frac{\partial}{\partial P_-}) - \frac{\epsilon_{ij}}{P_+} (\mathcal{M}S_0^i + P^i S_0^z) \quad (32)$$

Such a procedure was first proposed by Bakamjian and Thomas [34]. According to the Dirac classification [5], the generators in Eq. (32) are given in the front form of dynamics, since this form is characterized by the condition that only the operators $P^-$ and $M^{-j}$ are interaction dependent, while all the other seven generators are free.

In this procedure, however, cluster separability [17, 18, 19, 20] is not implemented. In order to satisfy cluster separability the spin operator in the general case ($N \geq 3$) has to be interaction dependent and the generators can be obtained from Eq. (32) by replacing $\mathcal{M}$ by $M = A \mathcal{M} A^{-1}$ and $S_0$ by the operator $S$, such that $S^z = S_0^z$ and $AS_0 A^{-1} = S$ [33, 24, 30], where the unitary operator $A$ acts only through the internal variables. In this case the operators $M$ and $S$ must commute with each other. The operator $A$ is the front-form analog of the Sokolov packing operator [18]; an explicit expression for $A$ can be found, for example, in Refs. [19, 24, 30, 35]. The choice $A = 1$ is possible when in a system of $N$ particles there exists only the $N$-particle interaction or there is a confining interaction (see, e.g., [20, 21]).

It is important to note that the form of the operators in Eq. (32) (even if $S_0$ is replaced by $S$) does not explicitly depend on the number of particles. Therefore one could argue that such a form can also be valid when the number
of particles is infinite and even when this number is not a conserved physical quantity. Indeed, according to our intuition, there should always exist a representation in which the external motion is purely kinematical, while all the information about dynamics is contained in the mass operator which acts only through internal variables. The representation (32) has just such properties and therefore one might expect that any other representation in the front form is unitarily equivalent to that given in Eq. (32).

A difficulty in front-form dynamics is that the operators $U_P$ and $U_T$ corresponding to space reflection and time reversal should necessarily be interaction dependent. This follows in particular from the relations

$$U_P P^+ U_P^{-1} = U_T P^+ U_T^{-1} = P^- \quad (33)$$

However, as noted by Coester [36], the discrete transformation $P_y$ such that $P_y x := \{x^0, x_1, -x_2, x_3\}$ leaves the light cone $x^+ = 0$ invariant, and therefore it is kinematical. The full space reflection $P$ is the product of $P_y$ and a dynamical rotation around the $y$ axis by $\pi$. Thus $P$ is not an independent dynamical transformation to be considered besides the rotations around transverse axes. Similarly the transformation $\theta$ leaves $x^+ = 0$ invariant and $T = \theta P_y R_y(\pi)$. Therefore the interaction dependence of the operators $U_P$ and $U_T$ in the front form does not mean that there are discrete dynamical symmetries in addition to the rotations around transverse axes. We conclude that the operators $U_y$ and $U_\theta$ are interaction independent and can be chosen to be the same as for a system of free particles (see the preceding section). Then the generators given by Eq. (32) will satisfy extended Poincaré covariance if

$$U_{y,int} M U_{y,int}^{-1} = M, \quad U_{y,int} A U_{y,int}^{-1} = A,$$

$$U_{\theta} M U_{\theta}^{-1} = M, \quad U_{\theta} A U_{\theta}^{-1} = A \quad (34)$$

Let $\Pi_i$ be the orthogonal projector onto the subspace $H_i \equiv \Pi_i \mathcal{H}$ corresponding to the eigenvalue of the operator $M$ equal to $M_i$ and to the eigenvalue of the spin operator equal to $S_i$. Therefore by analogy with Ref. [12] we work in the representation where the mass and spin operators are diagonalized. In constituent quark models the spectrum of the mass operator is discrete, but in the general case one has also to consider the continuous
spectrum (e.g., in the parton model). For this reason we will not specify whether the index enumerating the eigenstates of the mass operator is discrete or continuous. In the latter case a sum over $i$ should be understood as an integration.

If $\phi(\mathbf{P}_\perp, P^\perp; k_1, \sigma_1, ..., k_N, \sigma_N) \in \mathcal{H}_i$ it will be convenient to use the notation $\phi_i(P; k_1, \sigma_1, ..., k_N, \sigma_N)$ having in mind that the four-vector $P$ has the components $(\mathbf{P}_\perp, P^+, P^-)$, where $P^- = (M^2_i + \mathbf{P}_\perp^2)/2P^+$. Then, as follows from the comparison of Eqs. (10), (13) and (32), the action of the representation operators of the Lorentz group can be written as

$$
\langle P; S_i, S_{iz} | U(l) | \phi_i \rangle = \sum_{S'_{iz}} D_{S_i S'_{iz}}^S [W(l, P'/M_i)] \langle P'; S_i, S'_{iz} | \phi_i \rangle \quad (35)
$$

where $P' = L(l)^{-1}P$.

6 Current operators in the front form of dynamics

The translational covariance of the current operator implies that Eq. (4) is satisfied and we can consider this expression as the definition of $J^\mu(x)$ in terms of $J^\mu(0)$. Adopting this definition it is easy to show that Poincaré covariance of $J^\mu(x)$ takes place if i) $J^\mu(0)$ satisfies the Lorentz covariance condition (6) and ii) the Poincaré group generators satisfy the condition (3). Therefore the problem of constructing $J^\mu(x)$ can be reduced to that of constructing an operator $J^\mu(0)$ which satisfies the condition (8) and therefore Eq. (7).

As follows from Eq. (4), the continuity equation $\partial J^\mu(x)/\partial x^\mu = 0$ in terms of $J^\mu(0)$ reads

$$
[P_\mu, J^\mu(0)] = 0 \quad (36)
$$

Eqs. (7) and (36) show that in the general case the operator $J^\mu(0)$ cannot be chosen the same as for free particles. Indeed, the free operator $J^\mu_{\text{free}}(0)$ obviously satisfies the conditions (7) and (36) when the corresponding representation generators of the Poincaré group are free, but in general does not properly commute with $M^{\mu\nu}$ and $P^\mu$, when these operators are interaction dependent.

In the front form the set $M^{\mu\nu}$ contains both free and interaction dependent operators and therefore the problem of constructing the operator $J^\rho(0)$ satisfying Eq. (7) appears to be more complicated than in the point form,
where only $P^\mu$ contains the interaction \[16\]. Osborn \[37\] used this equation to express the operators $J^j(0)$ and $J^-(0)$ in terms of $J^+(0)$ and then he obtained a restriction on the kernel of the latter operator. This restriction, which was called the angular condition (as well as the condition (2)), involves triple commutators and therefore it is difficult to solve. As shown in Ref. \[25\], the Osborn angular condition is automatically satisfied to first order in $Q$, but it is not satisfied to second order if $J^+(0)$ is free. In what follows we will consider the constraints on the whole set of components of the current, adopting a spectral decomposition of the current operator. Such a procedure allows one to overcome the difficulties related to the presence of the interaction in $M^{\mu\nu}$, since the dependence upon the interacting mass operator becomes a dependence upon its eigenvalues.

If extended Poincaré covariance is required (as it is the case for the em current operator), the operator $J^\mu(0)$ should also properly commute with the operators $U_P$ and $U_T$ which, as explained in the preceding section, are interaction dependent. However, as explained in that section, the usual Poincaré covariance and the proper commutation relations with $U_y$ and $U_\theta$ guarantee that extended Poincaré covariance takes place. Therefore the operator $J^\mu(0)$ should satisfy the conditions

\[ U_y J^\mu(0) U_y^{-1} = (\Lambda_y)^\mu_\nu J^\nu(0), \]

\[ U_\theta J^\mu(0) U_\theta^{-1} = J^\mu(0) \]

where the only nonzero components of the matrix $\Lambda_y$ are

\[ (\Lambda_y)_0^0 = (\Lambda_y)_1^1 = - (\Lambda_y)_2^2 = (\Lambda_y)_3^3 = 1. \]

The action of the operator $J^\mu(0)$ can be written in the form

\[ \langle P_\perp, P^+ | J^\mu(0) | \varphi \rangle = \int J^\mu(P_\perp, P^+; P'_\perp, P'^+) \langle P'_\perp, P'^+ | \varphi \rangle d\rho(P'_\perp, P'^+) \]

where the kernel $J^\mu(P_\perp, P^+; P'_\perp, P'^+)$ is an operator in $\mathcal{H}_{int}$ at any fixed value of its arguments and the projection $\langle P_\perp, P^+ | \varphi \rangle$ is a state belonging to $\mathcal{H}_{int}$. As follows from this expression, the operator $J^\mu(0)$ will be selfadjoint if

\[ J^\mu(P_\perp, P^+; P'_\perp, P'^+) = J^\mu(P'_\perp, P'^+; P_\perp, P^+) \]
where * means the Hermitian conjugation in $\mathcal{H}_{int}$ (in the general case the property of an operator to be selfadjoint is stronger than to be Hermitian, but we shall not discuss this question).

As mentioned above, the key property that allows one to generalize to the front form (and also to the instant form) the approach of [16] is the following spectral decomposition of the current operator, viz.

$$J^\mu(0) = \sum_{ij} \Pi_i J^\mu(0) \Pi_j = \sum_{ij} J^\mu(M_i, M_j)$$

where

$$J^\mu(M_i, M_j) \equiv \Pi_i J^\mu(0) \Pi_j$$

is the part of $J^\mu(0)$ describing the transition from $\mathcal{H}_j$ to $\mathcal{H}_i$ (for the sake of brevity we do not write the arguments $S_j$ and $S_i$). If $\langle P_\perp, P_+|\varphi_j\rangle \in \mathcal{H}_j$, then we can reexpress Eq. (39) in the form

$$\langle P_\perp, P_+|J^\mu(M_i, M_j)|\varphi_j\rangle = \int J^\mu(P_i; P'_j)$$

$$\langle P'_\perp, P'_+|\varphi_j\rangle d\rho(P'_\perp, P'_+)$$

where

$$J^\mu(P_i; P'_j) \equiv \langle P_\perp, P_+|\Pi_i J^\mu(0) \Pi_j|P'_\perp, P'_+\rangle =$$

$$J^\mu(P_\perp, P_+, M_i; P'_\perp, P'_+, M_j) = \Pi_i J^\mu(P_\perp, P_+; P'_\perp, P'_+) \Pi_j$$

Therefore the operator $J^\mu(0)$ is fully defined by the set of the operators $J^\mu(P_i, P'_j)$, with definite values of the masses. At any fixed values of $(P'_\perp, P'_+)$ and $(P_\perp, P_+)$ these operators act from $\mathcal{H}_{j,int}$ to $\mathcal{H}_{i,int}$, where $\mathcal{H}_{i,int} = \Pi_i \mathcal{H}_{int}$. Since the spaces $\mathcal{H}_i$ are invariant under the action of the representation operators of the extended Poincaré group, the restrictions imposed on the operator $J^\mu(0)$ by extended Poincaré covariance and current conservation (see Eq. (36)) can be formulated in terms of $J^\mu(P_i; P'_j)$. As follows from Eqs. (40), (42), (43) and (44), the operator $J^\mu(0)$ will be selfadjoint if

$$J^\mu(P_i, P'_j)^* = J^\mu(P'_j, P_i)$$
7 Extended Lorentz covariance of the current operator

As explained in the preceding section, the problem of constructing a Poincaré covariant operator $J^\mu(x)$ can be reduced to that of constructing a Lorentz covariant operator $J^\mu(0)$. This operator is fully defined by the set of operators $J^\mu(P_\perp, P^+; P'_\perp, P'^+)$ (which in turn are defined by the set of operators $J^\mu(P_i; P'_j)$) acting through the internal variables.

First of all, let us consider the covariance with respect to continuous Lorentz transformations. From relativistic invariance of $d\rho(P'_\perp, P'^+)$ and from Eqs. (6), (11), (35), (42) and (43), the operator $J^\mu(0)$ will be Lorentz covariant if $J^\mu(P_i; P'_j)$ fulfills the following relation

$$L(l)_\mu^\nu J^\nu(L(l)^{-1}P_i, L(l)^{-1}P'_j) = D^S_i[W(l, L(l)^{-1}P_i/M_i)]^{-1}. $$

$$J^\mu(P_i, P'_j) D^S_j[W(l, L(l)^{-1}P'_j/M_j)] $$

(46)

for (almost) all values of $(P'_\perp, P'^+)$ and $(P_\perp, P^+)$. If $l$ is a front-form boost, i.e. $l = b \in B$ (see Sect. 3), the Wigner rotation becomes the identity and then, as follows from Eq. (46), one has

$$L(b)_\mu^\nu J^\nu(L(b)^{-1}P_i, L(b)^{-1}P'_j) = J^\mu(P_i, P'_j) $$

(47)

In order to investigate in detail the constraints imposed on $J^\mu(P_i; P'_j)$ by Lorentz covariance, it is convenient to consider the current in a general Breit frame, and then the current in a particular Breit frame where the three-momentum is directed along the $z$ axis. In the latter frame, as will be clear in what follows, one can take advantage of the rotational symmetry, differently from the case where the frame $q^+ = 0$ is chosen. It is worth noting that the definition of a Breit frame is possible because the masses are well defined in both the initial and final states. (In the point form, cf. [16], the construction of the covariant current was carried out in the equal-velocity frame, and therefore it was not necessary to fix the masses). The Breit frame is defined as the reference frame where the initial and final momenta are

$$K_i = B(H_{ij})^{-1}P_i, \quad K'_j = B(H_{ij})^{-1}P'_j $$

(48)
In Eq. (48) $H_{ij} \equiv (P_i + P_j')/|P_i + P_j'|$ and $B(H_{ij})$ denotes the Lorentz transformation $L[\beta(H_{ij})]$. The four-vectors $K_i$ and $K_j'$ in Eq. (48) are such that

$$K_i^2 = M_i^2, \quad K_j'^2 = M_j'^2, \quad K_i + K_j' = 0 \quad (49)$$

Therefore the four-vectors $K_i$ and $K_j'$ are fully determined by one three-dimensional vector $K_{ij} \equiv K_i$. The relations (49) can also be directly obtained from Eqs. (12) and (48), since, as follows from these expressions (compare with Eq. (23)) one has

$$K_{i\perp} = P_{i\perp} - \sqrt{2}K_i^+ H_{ij\perp}$$

and $K_j'$ is given by the same expressions with $P_i$ replaced by $P_j'$.

As follows from Eqs. (47) and (48),

$$J^\mu(P_i, P_j') = B(H_{ij})^\mu_{\nu} j'^\nu(K_{ij}; M_i, M_j) \quad (51)$$

where we use $j'^\nu(K_{ij}; M_i, M_j)$ to denote $J^\mu(K_i, K_j')$, i.e. the current in the Breit frame. From Eqs. (45) and (51), the condition for the Hermiticity of the operator $J^\mu(P_i, P_j')$ defined by Eq. (51) satisfies if and only if

$$j'^\nu(K_{ij}; M_i, M_j)^* = j'^\nu(-K_{ij}; M_j, M_i) \quad (52)$$

Since the operator $J^\mu(P_i, P_j')$ can be expressed in terms of $j'^\nu(K_{ij}; M_i, M_j)$, we will look for the properties of $j'^\nu(K_{ij}; M_i, M_j)$ such that the operator $J^\mu(P_i, P_j')$ defined by Eq. (51) satisfies Eq. (46). This latter becomes in terms of $j'^\nu(K_{ij}; M_i, M_j)$

$$j'^\nu(L[W^{-1}(l, L(l)^{-1}H_{ij})]K_{ij}; M_i, M_j) = L[W^{-1}(l, L(l)^{-1}H_{ij})]_{\nu}^\mu \cdot$$

$$D^{S_i}[W^{-1}(l, L(l)^{-1}P_i/M_i)]j'^\nu(K_{ij}; M_i, M_j)D^{S_j}[W^{-1}(l, L(l)^{-1}P_j'/M_j)]^{-1} \quad (53)$$

Let us define $u \in SU(2)$ as follows

$$u = W^{-1}(l, L(l)^{-1}H_{ij}) \quad (54)$$

As shown in Appendix A, from Eq. (54) one has

$$W^{-1}(l, L(l)^{-1}P_i/M_i) = W(u, K_i^i_{/M_i}), \quad W^{-1}(l, L(l)^{-1}P_j'/M_j) = W(u, K_j'^i_{/M_j}) \quad (55)$$
Therefore Eq. (53) becomes
\[ j^\mu(L(u)K_{ij}; M_i, M_j) = L(u)_\nu^\mu D^{S_i}[W(u, \frac{K_i}{M_i})] \cdot j^\nu(K_{ij}; M_i, M_j)D^{S_j}[W(u, \frac{K'_j}{M_j})]^{-1} \] (56)

It is clear that if Eq. (56) is satisfied for any \( u \in SU(2) \), then Eq. (53) is satisfied too. Therefore we can investigate Eq. (56) only.

As follows from Eqs. (11), (17) and (18), Eq. (56) can be written in the form
\[ j^\mu(L(u)K_{ij}; M_i, M_j) = L(u)_\nu^\mu D^{S_i}[\nu(L(u)\frac{K_i}{M_i})^{-1}uv(\frac{K_i}{M_i})] \cdot j^\nu(K_{ij}; M_i, M_j)D^{S_j}[\nu(L(u)\frac{K'_j}{M_j})^{-1}u^{-1}v(L(u)\frac{K'_j}{M_j})] \] (57)

where the Melosh rotations \( v \) appear instead of the boosts \( \beta \) (see Eq.(18)). This replacement will be useful in the following. Note that when \( K_{ij} = 0 \) Eq. (57) becomes
\[ j^\mu(0; M_i, M_j) = L(u)_\nu^\mu D^{S_i}(u) j^\nu(0; M_i, M_j)D^{S_j}(u)^{-1} \] (58)

We conclude that the operator \( J^\mu(P_i, P'_j) \) will satisfy the Lorentz covariance condition (16) if the operator \( j^\nu(K_{ij}; M_i, M_j) \) satisfies the rotational covariance condition (57) for any \( u \). In the point form, the equation analogous to Eq. (57) does not contain Melosh matrices, but the relation between \( J^\mu \) and \( j^\nu \) is more complicated than Eq. (51) [16].

Equation (57) can be used for expressing \( j^\nu(K; M_i, M_j) \), corresponding to an arbitrary three-momentum \( K \), in terms of auxiliary operators \( j^\mu(Ke_z; M_i, M_j) \), which are constrained only by covariance relative to rotations around the \( z \) axis, \( u_z \), and not by the full SU(2) covariance. The auxiliary operators \( j^\mu(Ke_z; M_i, M_j) \) represent the current in a Breit frame where \( K \) is along the \( z \) axis. If \( K = Ke_z \), where \( K = |K| \) and \( e_z \) is the unit vector along the positive direction of the \( z \) axis, then from (57) one has for rotations around the \( z \) axis
\[ j^\mu(Ke_z; M_i, M_j) = L(u_z)_\nu^\mu \exp(-\nu c S^z) j^\nu(Ke_z; M_i, M_j) \exp(\nu c S^z) \] (59)
where \( \exp(-\nu \varphi S^z_{i(j)}) = D^{S_{i(j)}}(u_z) \) and the relation \( v(g) = 1 \) for \( g_{\perp} = 0 \) has been used (see Eq. (18)). From Eq. (59), it is clear that both \( j^+ \) and \( j^- \) must be diagonal with respect to the third component of the spin, while \( j_{\perp} \) should properly transform with respect to the rotations \( u_z \).

In order to demonstrate that the auxiliary operators are constrained only by covariance relative to rotations around the \( z \) axis, let \( r(K) \in SU(2) \) be such that \( L[r(K)]K e_z = K \). In particular, if \( \varphi \) and \( \theta \) are the polar angles characterizing the vector \( K \), then \( r(K) \) can be chosen in the form

\[
\exp(-\frac{i}{2} \varphi \tau_3) \exp(-\frac{i}{2} \theta \tau_2).
\]

Then, from Eq. (57), replacing \( K_{ij} \) with \( K_{ij} e_z \) and \( u \) with \( r(K) \) one has

\[

\begin{align*}
     j^\mu(K_{ij}; M_i, M_j) &= L[r(K_{ij})]\mu D^S_i[v(M_i)]^{-1} r(K_{ij})] \\
     j^\nu(K_{ij} e_z; M_i, M_j) &= D^S_j[r(K_{ij})^{-1} v(M_j)]
\end{align*}
\]

(61)

where the property of the Melosh rotations that \( v(g) = 1 \) for \( g_{\perp} = 0 \) has been used once more. Now we consider this expression as the definition of \( j^\mu(K_{ij} e_z; M_i, M_j) \) in terms of \( j^\mu(K_{ij}; M_i, M_j) \). Note that this definition is meaningful only if \( K_{ij} \neq 0 \). Using Eq. (61), it is easy to show that Eq. (57) in terms of \( j^\mu(K_{ij} e_z; M_i, M_j) \) becomes

\[

\begin{align*}
     j^\mu(K_{ij} e_z; M_i, M_j) &= L[r(L(u)K_{ij})^{-1} ur(K_{ij})]\mu \\
     D^S_i[r(L(u)K_{ij})^{-1} ur(K_{ij})]j^\nu(K_{ij} e_z; M_i, M_j) \\
     D^S_j[r(L(u)K_{ij})^{-1} ur(K_{ij})]^{-1}
\end{align*}
\]

(62)

From the properties of the products of rotation operators (see, e.g., [38]), one obtains \( r(L(u)K_{ij})^{-1} ur(K_{ij}) = u_z \), where \( u_z \) is a well-defined rotation around the \( z \) axis. Therefore, if \( j^\mu(K e_z; M_i, M_j) \) satisfies Eq. (59) for any \( u_z \), then \( j^\mu(K; M_i, M_j) \), defined by Eq. (61), fulfills Eq. (57). This means that, in order to fulfill Poincaré covariance, \( J^\mu(P_i, P_j) \) can be expressed in terms of the auxiliary operators \( j^\mu(K e_z; M_i, M_j) \) constrained only by rotations around the \( z \) axis.
The property of Hermiticity (cf. Eq. (52)) for $j^\mu(0; M_i, M_j)$ reads

$$j^\mu(0; M_i, M_j)^* = j^\mu(0; M_j, M_i)$$

while for $|K| \neq 0$, from Eq. (61), the property of Hermiticity for $j^\mu(Ke_z; M_i, M_j)$ becomes

$$j^\mu(Ke_z; M_i, M_j)^* = L(r(K)^{-1}r(-K))j^\nu D^S_j[r(K)^{-1}r(-K)] = L[r x(-\pi)]j^\nu D^S_j[r x(-\pi)]j^\nu(Ke_z; M_i, M_j)D^S_i[r x(-\pi)]^{-1},$$

since from Eq. (60) one has

$$r(K)^{-1}r(-K) = exp(i\frac{\pi}{2}\tau_1) = i\tau_1 \equiv r x(-\pi),$$

where $r x(-\pi) \in SU(2)$ yields the rotation by $-\pi$ around the $x$ axis. It is worth noting that Eq. (64) represents a non-trivial constraint when $M_i = M_j$ (i.e., for elastic form factors), because in this case the rhs and the lhs contain the same operator. In the inelastic case, one could construct the current defining the operator $j^\mu(Ke_z; M_i, M_j)$ for $M_i < M_j$ and use Eq. (64) in order to extend the definition of the current also for $M_i > M_j$. Another possible choice is to define the current for any value of the masses; in this case Eq. (64) becomes a non-trivial constraint also in inelastic processes. We will call Hermiticity condition the relation given by Eq. (64).

Let us now summarize the above results. The current operator $J^\mu(0)$ satisfies Lorentz covariance if the operator $J^\mu(P_i, P_j')$ satisfies the condition (40). This condition is satisfied if i) $J^\mu(P_i, P_j')$ is defined by Eq. (51), ii) $j^\mu(K; M_i, M_j)$ is defined by Eq. (61), if $K \neq 0$, and satisfies Eq. (58) if $K = 0$, and iii) the auxiliary operators $j^\mu(Ke_z; M_i, M_j)$ satisfy Eq. (59). In addition, Eqs. (63) and (64) guarantee that the operator $J^\mu(0)$ constructed in such a way is Hermitian.

Of course it is possible to choose another set of minimally constrained operators by choosing $K$ along any other axis. However the choice of the reference frame where $K = Ke_z$ is the most convenient, since the rotations around the $z$ axis are interaction independent and furthermore there are no Melosh matrices in Eqs. (59) and (64) (this follows from the fact that $v(g) = 1$.
if \( g \) is directed along the \( z \) axis). It is worth noting that the continuous Lorentz transformations constrain the current \( j^\mu(Ke_z; M_i, M_j) \) for a non-interacting system in the same way as in the interacting case, namely Eq. (59) holds for both non-interacting and interacting systems, since rotations around the \( z \) axis are interaction free.

For the em current, also \( \mathcal{P} \) and \( \mathcal{T} \) covariance is required, i.e. extended Lorentz covariance is needed. As explained in Sects. 3 and 4, the current operator satisfies \( \mathcal{P} \) covariance if it satisfies Poincaré covariance and Eq. (57). As follows from Eqs. (29), (30), (39-44), (50) and (51), the condition (37) is satisfied if and only if

\[
(\Lambda_y)_{\mu}^{\nu} j^\nu(\tilde{K}, M_i, M_j) = U_{y,\text{int}} j^\mu(K, M_i, M_j) U_{y,\text{int}}^{-1} \tag{66}
\]

where \( \tilde{K} \equiv (K_x, -K_y, K_z) \)

As follows from Eq. (61), this condition in terms of \( j^\mu(Ke_z; M_i, M_j) \) reads

\[
(\Lambda_y)_{\mu}^{\nu} j^\nu(Ke_z, M_i, M_j) = U_{y,\text{int}} j^\mu(Ke_z; M_i, M_j) U_{y,\text{int}}^{-1} \tag{67}
\]

where the properties \( \Lambda_y L[r(\tilde{K})] \Lambda_y^{-1} = L[r(K)] \) and

\[
U_{y,\text{int}} D\Sigma_i[v(K_i/M_i)^{-1}r(\tilde{K})] U_{y,\text{int}}^{-1} = D\Sigma_i[v(\tilde{K}/M_i)^{-1}r(\tilde{K})]
\]

have been used.

Analogously, as follows from Eqs. (31), (39-44), and (51), the condition (38), which guarantees \( \theta \) covariance, is satisfied if and only if

\[
j^\mu(Ke_z; M_i, M_j) = U_\theta j^\mu(Ke_z; M_i, M_j) U_\theta^{-1} \tag{68}
\]

The constraints imposed on the current for an interacting system by extended Lorentz covariance can be fulfilled by a current composed in our Breit frame by the sum of only one-body currents (e.g. \( \sum_{i=1}^N j^\mu_{\text{free},i} \), where \( N \) is the number of constituents in the interacting system), while some additional care must be adopted for the Hermiticity (cf. Eq. (64) and Sects. 10 - 14). The extended Lorentz covariance is clearly satisfied by the one-body free current, since the constraints are the same for a non-interacting and an interacting system (cf. Sect. 3). It is worth noting that the same analysis performed in our Breit frame can be carried out in any other reference frame obtained by a boost along the \( z \) axis, since the symmetry around the \( z \) axis is preserved.

23
If the cluster separability is important, i.e. $A \neq 1$, one can adopt $A j^\mu_{\text{free}} A^{-1}$ (as discussed in Sects. 10 - 11 and in [16, 39]) in order to construct a current which fulfills extended Lorentz covariance and Hermiticity.

8 Current conservation and charge operator

The results of the preceding section give the full solution of the problem of constructing the current in the front form as far as Poincaré covariance and Hermiticity are concerned. However the em current operator (differently from the weak one) should also satisfy current conservation and a proper normalization condition in terms of the electric charge of the system.

As follows from Eqs. (36) and (41-44), the continuity equation will be satisfied if

$$\left( P - P' \right)_\mu j^\mu(P, P') = 0, \quad (69)$$

In the Breit frame Eq. (69) (cf. Eq. (51)) becomes

$$\left( K_\perp - K_\text{h} \right) j^{+}(K; M_i, M_j) + \left( K_i^+ - K_j^+ \right) j^{-}(K; M_i, M_j) - 2K_\perp \cdot j_\perp(K; M_i, M_j) = 0 \quad (70)$$

where $K_i^+ = (\sqrt{M_i^2 + |K|^2} + K_z)/\sqrt{2}$ and $K_j^+ = (\sqrt{M_j^2 + |K|^2} - K_z)/\sqrt{2}$.

If $K_\perp = 0$ and $K_z \neq 0$, then Eq. (70) yields

$$\left( \frac{M_i^2}{2K_i^+} - \frac{M_j^2}{2K_j^+} \right) j^{+}(K e_z; M_i, M_j) + (K_i^+ - K_j^+) j^{-}(K e_z; M_i, M_j) = 0 \quad (71)$$

while, if $K = 0$ and $M_i \neq M_j$, then from Eq. (71) we have

$$j^-(0; M_i, M_j) = - j^+(0; M_i, M_j). \quad (72)$$

By taking the derivatives of Eq. (70) at $K = 0$, one has

$$j_\perp(0; M_i, M_j) = \frac{1}{2\sqrt{2}}(M_i - M_j) \frac{\partial}{\partial K_\perp} \left[ j^+(0; M_i, M_j) + j^-(0; M_i, M_j) \right],$$

$$\left[ j^+(0; M_i, M_j) - j^-(0; M_i, M_j) \right] = \frac{1}{2}(M_i - M_j) \frac{\partial}{\partial K_z} \left[ j^+(0; M_i, M_j) + j^-(0; M_i, M_j) \right] \quad (73)$$
In particular if $M_i = M_j$ one has from Eq. (73)

$$\begin{align*}
\mathbf{j}_\perp(0; M_i, M_i) &= 0 \\
\mathbf{j}^+(0; M_i, M_i) &= \mathbf{j}^-(0; M_i, M_i)
\end{align*}$$

(74)

Note that the signs in Eq. (72) (inelastic case) and in the second line of (74) (elastic case) differ each other.

As follows from Eq. (71), if $K \neq 0$ then only $j^+(K \mathbf{e}_z; M_i, M_j)$ and $j^-(K \mathbf{e}_z; M_i, M_j)$ are constrained by the continuity equation, viz.

$$j^-(K \mathbf{e}_z; M_i, M_j) = \frac{-M_i^2/(2K^+_i) - M_j^2/(2K^+_j)}{(K^+_i - K^+_j)} j^+(K \mathbf{e}_z; M_i, M_j).$$

(75)

while $j_\perp(K \mathbf{e}_z; M_i, M_j)$ remains unconstrained. If we choose in our Breit frame $j^+(K \mathbf{e}_z; M_i, M_j)$ and $j_\perp(K \mathbf{e}_z; M_i, M_j)$ free, then, as follows from Eq. (75), $j^-(K \mathbf{e}_z; M_i, M_j)$ must be interaction dependent, because of current conservation. However, in the actual calculations of any elastic and inelastic form factor only three components of the current are necessary (cf. Sects. 10 and 11), and these components can be chosen free.

Let us now consider the charge operator. In front-form dynamics it is defined as

$$Q = \int J^+(x) \delta(x^+) d^4x$$

(76)

Therefore, from Eqs. (4) and (39), one has

$$\langle \mathbf{P}_\perp, P^+ | Q | \varphi \rangle = \int \int \int J^+(\mathbf{P}_\perp, P^+; \mathbf{P}'_\perp, P'^+) \cdot \exp \left[ i ((P^+ - P'^+) x^- - (\mathbf{P}_\perp - \mathbf{P}'_\perp) \cdot \mathbf{x}_\perp) \right] \langle \mathbf{P}'_\perp, P'^+ | \varphi \rangle \cdot d\mathbf{x}_\perp dx^- d\rho(\mathbf{P}'_\perp, P'^+) = \sum_{ij} \frac{1}{2P^+} J^+(P_i, P_j) \langle \mathbf{P}_\perp, P^+ | \varphi \rangle$$

(77)

where Eqs. (44) and (44) have been used. From Eq. (52), for $P'^+ = P^+$ and $\mathbf{P}'_\perp = \mathbf{P}_\perp$, one has $(P_i^+ - P_j^-) J^+(P_i, P_j) = 0$, and then $J^+(P_i, P_j) = 0$, if $M_i \neq M_j$. Therefore only the terms with $i = j$ contribute to the sum in Eq. (77). Since from Eqs. (48) and (50), $B^+_{ij}(H_{ii}) = \sqrt{2} H^+_{ii}$, and all the other components of $B^+_{ij}(H_{ii})$ are equal to zero, using Eq. (51), $J^+(P_i, P_i)$ can be expressed in terms of $j^+(K_{ij}; M_i, M_i)$, where in our case $K_{ij} = 0$. Therefore,
Eq. (77) becomes
\[ \langle P_{\perp}, P_{\perp}^+ | Q | \varphi \rangle = \sum_{i} \frac{1}{\sqrt{2M_i}} j^+(0; M_i, M_i) \langle P_{\perp}, P_{\perp}^+ | \varphi \rangle \] (78)

We conclude that for each subspace \( \mathcal{H}_i \) one must have (cf. Eq. (74))
\[ j^+(0; M_i, M_i) = j^-(0; M_i, M_i) = \sqrt{2} e M_i \Pi_i \] (79)
where \( e \) is the total electric charge of the system under consideration. This normalization condition is trivially fulfilled by \( \sum_{i=1}^{N} j^+_{\text{free},i} \).

All the above results show that the operator \( J^\mu(0) \) satisfies i) Lorentz covariance, ii) \( \mathcal{P} \) and \( \mathcal{T} \) covariance, iii) Hermiticity, iv) continuity equation and v) charge conservation, if the operator \( j^\mu(K e_z; M_i, M_j) \) satisfies Eqs. (58), (59), (67), (68), (63), (64), (71) and (79). However, even if all these conditions are satisfied, this does not guarantee that the current operator fulfills locality and cluster separability (cf. Sect. 1 and Ref. [39]).

9 Matrix elements of the current operator

In the scattering theory, one-particle states with four-momentum \( p' \) and spin projection \( \sigma' \) are usually normalized as
\[ \langle p'', \sigma'' | p', \sigma' \rangle = 2(2\pi)^3 p'^+ \delta^{(2)}(\mathbf{p}'_{\perp} - \mathbf{p}_{\perp}) \delta (p''^+ - p'^+) \delta_{\sigma'' \sigma'} \] (80)
where \( \delta_{\sigma\sigma'} \) is the Kronecker symbol.

Since the form of the generators (32) is analogous to the form of the one-particle generators (13), the wave function of the state with four-momentum \( P' \) and internal wave function \( \chi' \) can be written in the form
\[ \langle P_{\perp}, P_{\perp}^+; \mathbf{k}_1, \sigma_1, \ldots, \mathbf{k}_N, \sigma_N | P', \chi' \rangle = 2(2\pi)^3 P'^+ \delta^{(2)}(\mathbf{P}_{\perp} - \mathbf{P}'_{\perp}) \cdot \delta (P^+ - P'^+) \chi'(\mathbf{k}_1, \sigma_1, \ldots, \mathbf{k}_N, \sigma_N) \] (81)
Indeed, as follows from Eqs. (25) and (26), in this case the normalization is analogous to that in Eq. (80):
\[ \langle P'', \sigma'' | P', \chi' \rangle = 2(2\pi)^3 P'^+ \delta^{(2)}(\mathbf{P}_{\perp}'' - \mathbf{P}'_{\perp}) \delta (P''^+ - P'^+) \langle \chi'' | \chi' \rangle \] (82)
where the scalar product on the right-hand side is understood only in the space \( H_{\text{int}} \).

Let us now consider the em or weak transition of the state with mass \( M_j \), four-momentum \( P'_j \) (such that \( P'^2_j = M_j^2 \)) and internal wave function \( \chi_j \) to the state with mass \( M_i \), four-momentum \( P_i \) and internal wave function \( \chi_i \). Then, as follows from Eqs. (4), (39-44) and (51),

\[
\langle P_i, \chi_i | J^\mu(x) | P'_j, \chi_j \rangle = \exp[i(P_i - P'_j)x] B(H_{ij})_{\mu}^{\nu} \langle \chi_i | j''(K_{ij}, M_i, M_j) | \chi_j \rangle
\]

(83)

where the matrix element on the right-hand side must be calculated only in the space \( H_{\text{int}} \).

From Eq. (83), it is clear that a process in an arbitrary frame can be investigated in terms of the current in the Breit frame. This observation has been essentially used in the method of Ref. [14].

In turn, using Eq. (61), we can express matrix elements of the current operator in terms of the matrix elements of the operator \( j^\mu(K_{ij} e_z; M_i, M_j) \):

\[
\langle P_i, \chi_i | J^\mu(x) | P'_j, \chi_j \rangle = \exp[i(P_i - P'_j)x] \cdot L[\beta(H_{ij}) r(K_{ij})] \langle \chi_i | D^{S_i} [v(K_i / M_i)^{-1} r(K_{ij})] \cdot j''(K_{ij} e_z; M_i, M_j) D^{S_j} [r(K_{ij})^{-1} v(K'_j / M_j)] | \chi_j \rangle
\]

(84)

As follows from Eqs. (48), (84) and from the definition of \( K_{ij} \), the matrix elements of the current operator in terms of the matrix elements of the operator \( j^\mu(K_{ij} e_z; M_i, M_j) \) have the simplest form in the reference frame where

\[
P_{i\perp} = P'_{j\perp} = 0, \quad P_i^z + P_j'^z = 0, \quad P_i^z \neq 0
\]

(85)

Indeed, in this case Eq. (84) obviously yields

\[
\langle P_i, \chi_i | J^\mu(0) | P'_j, \chi_j \rangle = \langle \chi_i | j^\mu(\pm K_{ij} e_z; M_i, M_j) | \chi_j \rangle
\]

(86)

where \( K_{ij} = |P_j^z| \) and \( \pm = \text{sign}(P_i^z) \).

It is useful to investigate the matrix elements of the current \( j^\mu(K e_z; M_i, M_j) \) between different internal states \( |\chi_i\rangle \) and \( |\chi_j\rangle \), and the constraints imposed by the covariance for rotations around the \( z \) axis. In the em case, one has to consider also the constraints imposed by parity and time
reversal covariance (the current conservation will be furtherly discussed in Sect. [1]). From Eq. (59) one immediately obtains

\[ j^+(K e_z; M_i, M_j) = \exp(-\nu S_z) j^+(K e_z; M_i, M_j) \exp(\nu S_z) \]
\[ j^-(K e_z; M_i, M_j) = \exp(-\nu S_z) j^-(K e_z; M_i, M_j) \exp(\nu S_z) \]  

(87)

and therefore only the diagonal matrix elements of \( j^\pm(K e_z; M_i, M_j) \) are different from zero. As for the \( \perp \) components of the current, one has

\[ j_x(K e_z; M_i, M_j) = \exp(-\nu S_z) \]
\[ [\cos \varphi j_x(K e_z; M_i, M_j) + \sin \varphi j_y(K e_z; M_i, M_j)] \exp(\nu S_z) \]

\[ j_y(K e_z; M_i, M_j) = \exp(-\nu S_z) \]
\[ [-\sin \varphi j_x(K e_z; M_i, M_j) + \cos \varphi j_y(K e_z; M_i, M_j)] \exp(\nu S_z) \]  

(88)

Let \( |\chi_{i(j)}\rangle = |M_{i(j)} S_{i(j)} S_{iz(jz)}\rangle \in \mathcal{H}_{int} \) be an eigenstate of mass, \( M_{i(j)} \), intrinsic angular momentum, \( S_{i(j)} \), and third component of angular momentum, \( S_{iz(jz)} \). From Eq. (88) it is straightforward to obtain the matrix elements of \( j_y \) from the ones of \( j_x \), namely

\[ \langle S_{iz} S_i M_i | j_y(K e_z; M_i, M_j) | M_j S_j S_{jz} \rangle = -\exp[-\frac{i \pi}{2} (S_{iz} - S_{jz})] \]
\[ \langle S_{iz} S_i M_i | j_x(K e_z; M_i, M_j) | M_j S_j S_{jz} \rangle \]  

(89)

Furthermore, after substituting Eq. (89) in Eq. (88) one finds that the matrix elements are vanishing unless \( (S_{iz} - S_{jz})^2 = 1 \).

The \( U_y \)-parity covariance, (Eq. (67)), yields

\[ \langle S_{iz} S_i M_i | j^\pm(K e_z; M_i, M_j) | M_j S_j S_{jz} \rangle = \eta_{p_i} \eta_{p_j} (-1)^{(S_{iz} - S_{jz})} \]
\[ \langle -S_{iz} S_i M_i | j^\pm(K e_z; M_i, M_j) | M_j S_j - S_{jz} \rangle \]  

(90)

\[ \langle S_{iz} S_i M_i | j_x(K e_z; M_i, M_j) | M_j S_j S_{jz} \rangle = -\eta_{p_i} \eta_{p_j} (-1)^{(S_{iz} - S_{jz})} \]
\[ \langle -S_{iz} S_i M_i | j_x(K e_z; M_i, M_j) | M_j S_j - S_{jz} \rangle \]  

(91)

\[ \langle S_{iz} S_i M_i | j_y(K e_z; M_i, M_j) | M_j S_j S_{jz} \rangle = \eta_{p_i} \eta_{p_j} (-1)^{(S_{iz} - S_{jz})} \]
\[ \langle -S_{iz} S_i M_i | j_y(K e_z; M_i, M_j) | M_j S_j - S_{jz} \rangle \]  

(92)
and therefore only the matrix elements with $S_{iz} \geq 0$ are independent.

Finally the $U_{y}$-parity (‘time reversal’) covariance, (Eq. (63)), gives

$$\langle S_{iz}S_{i}M_{i}\jmath^{\mu}(Ke_{z};M_{i},M_{j})|M_{j}S_{j}S_{jz}\rangle = \eta_{\theta_{i}}\eta_{\theta_{j}}(-1)^{(S_{i}+S_{j}-S_{iz})}(93)$$

As is well known, combining parity, Eqs. (90)-(91), and time reversal, Eq. (93), one obtains that the matrix elements of $j_{x}$ and $j_{\pm}$ are real, while $j_{y}$ is imaginary, as also follows from Eq. (89).

### 10 Applications to deep inelastic scattering

Let us first consider the problem of calculating the tensor (4). As follows from Eq. (4)

$$W_{\mu \nu} = \frac{1}{4 \pi} \sum(2\pi)^{4}\delta^{(4)}(P' + q - P'\mu)\langle P', \chi' | J^{\mu}(0) | P', \chi \rangle \cdot \langle P', \chi' | J^{\nu}(0) | P', \chi \rangle (94)$$

where the sum is taken over all possible final states with four-momentum $P''$ and internal wave functions $\chi''$.

We will consider the process in the reference frame where $P_{\perp} = q_{\perp} = 0$, $P_{z} = -P''_{z} = K > 0$. Then, as follows from Eqs. (86),

$$W_{\mu \nu} = \frac{1}{4 \pi} \sum(2\pi)^{4}\delta^{(4)}(P' + q - P''\mu)\langle \chi' | j^{\mu}(Ke_{z};m,M'') | \chi'' \rangle \cdot \langle \chi'' | j^{\nu}(-Ke_{z};M'',m) | \chi' \rangle (95)$$

where $m$ is the mass of the nucleon and $M''$ is the mass of the final state. As follows from the delta function in Eq. (93) and the definition of the Bjorken variable $x = Q^{2}/2(P'q)$, in the Bjorken limit (when $Q \gg m$ and $x$ is not too close to 0 or 1)

$$M''^{2} = \frac{Q^{2}(1-x)}{x}, \quad K = \frac{Q}{2[(2-x)x]^{1/2}} (96)$$
Using these expressions it is easy to show that in the reference frame under consideration

\[ P' + = \sqrt{2} K, \quad P'' + = \sqrt{2} K (1 - x), \quad P'' - = \sqrt{2} K (2 - x) \quad (97) \]

Because of Eq. (52), we have to choose the operators \( j^\mu (-K e_z; M'', m) \) and \( j^\mu (K e_z; m, M'') \) in such a way that

\[ j^\mu (-K e_z; M'', m)^* = j^\mu (K e_z; m, M'') \quad (98) \]

This expression makes it possible to rewrite Eq. (93) in the form

\[ W_{\mu \nu} = \frac{1}{4 \pi} \sum (2\pi)^4 \delta^{(4)}(P' + q - P'') \langle \chi' | j^\nu (K e_z; m, M'') | \chi'' \rangle^2 \quad (99) \]

From Eq. (104) one has

\[ j^\mu (K e_z; m, M'') = \Pi J^\mu (0, P'^{+}; 0, P''^+) \Pi'' \quad (100) \]

where \( \Pi' \) and \( \Pi'' \) are the orthogonal projectors onto the states with the masses \( m \) and \( M'' \) respectively, and \( P' + \) and \( P'' + \) are given by Eq. (97).

A usual assumption in the parton model (where the final state interaction is neglected) is that the current operator can be taken in IA, viz.

\[ j^\mu (K e_z; m, M'') = \Pi J^\mu_{\text{free}} (0, P'^{+}; 0, P''^+) \Pi'' \quad (101) \]

where \( J^\mu (0)_{\text{free}} = \sum_{i=1}^{N} J^\mu_{\text{free}, i} \). In Sect. 7, we have already shown that, in general, the free current fulfills the extended Lorentz covariance; moreover, as already noted, the Hermiticity property, Eq. (94), does not impose any further constraint in DIS, since \( m \neq M'' \). In the actual calculations of the structure functions, for any value of the momentum transfer, only three components of the current are needed and can be chosen unconstrained with respect to the current conservation, while the fourth component can be determined through the current conservation, see Eq. (75). Therefore the structure functions could be calculated by using the + and \( \perp \) components of the free current operator in the Breit frame, even in the case where the final state interaction is present (cf., e.g., [40]). However, in the parton model all
components of the free current are compatible with current conservation in the Bjorken limit. As a matter of fact, the $+\,\text{and}\,\perp$ components of the vector $P$ are the same as for the system of free quarks and gluons, and we have to discuss only the value of $P^-$, that differs from the free one, for demonstrating that the free current satisfies the current conservation in the parton model.

For the initial state, given our choice of the reference frame, the value of $P'-$ is negligible and therefore the difference with respect to the value in the free case is vanishing in the Bjorken limit. For the final state, if the interaction is disregarded, as in the parton model, $P''-$ is the same as for the free system. Therefore Eq. (75) is satisfied by the free current in the Bjorken limit.

The above discussion indicates that a consistent calculation of the tensor (1) can be achieved in the parton model by replacing the full current operator $J^\mu(x)$ with the IA one. Obviously this assumption does not imply that the IA current operator can be adopted for calculating matrix elements in any other reference frames, apart the ones reachable by front-form boosts.

Summarizing, the parton model does not contradict Poincaré covariance and current conservation, although the nucleon is described as a bound state of quarks and gluons (see the discussion in Sect. [4]). However, it is clear that the conditions (59), (67), (68), (71), and (79) are not too restrictive and they allow many choices of $j^\mu$.

If the cluster separability is important, then in order to recover the parton model results (cf. [4]) one can choose the em current as follows

$$j^\mu(Ke_\perp; m, M') = \Pi'AJ^\mu_{\text{free}}(0, P'; 0, P'^+; 0)A^{-1}\Pi'$$  \hspace{1cm} (102)

where $A$ is the packing operator.

11 Elastic and inelastic scattering

As shown in Sect. [4], in the case of elastic scattering the Hermiticity condition represents a constraint to be imposed on $j^\mu(Ke_\perp; M_i, M_i)$, besides the extended Lorentz covariance and current conservation. Indeed, in this case the operators $j^\mu(Ke_\perp; M_i, M_i)$ and $j^\mu(Ke_\perp; M_i, M_i)^*$ must be connected by $\pi$ rotations around the $x$ or $y$ axes (see, e.g., Eq. (6)). However, as already
noted, Hermiticity can represent a non trivial constraint also in the inelastic case, if the current operator is defined for any value of the masses.

Let $M_{i(j)}$ be the mass of a bound state $|\chi_{i(j)}\rangle$ of spin $S_{i(j)}$ and third component $S_{i z(j)}$, $\Pi_{i(j)}$ be the projector onto the corresponding subspace and $\mathcal{J}^\mu(Ke_z; M_i, M_j)$ be a current which fulfills Eq. (59) for any rotation $u_z$ around the $z$ axis (in the em case we assume that $\mathcal{J}^\mu(Ke_z; M_i, M_j)$ satisfies also Eqs. (67) and (68)). As we have already shown, a possible choice is the following one

$$\mathcal{J}^\mu(Ke_z; M_i, M_j) = \Pi_i J^\mu_{\text{free}}(0, P^+; 0, P^{\prime}+) \Pi_j$$

where

$$P^+ = \frac{1}{\sqrt{2}}[(m^2 + K^2)^{1/2} + K], \quad P^{\prime}+ = \frac{1}{\sqrt{2}}[(m^2 + K^2)^{1/2} - K]$$

and $K = \frac{Q}{2}$. Then a choice for the current compatible with the Hermiticity condition, Eq. (64), and with the extended Lorentz covariance is

$$j^\mu(Ke_z; M_i, M_j) = \frac{1}{2}\{\mathcal{J}^\mu(Ke_z; M_i, M_j) + \mathcal{J}^\mu(-Ke_z; M_j, M_i)\}^*$$

where $\mathcal{J}^\mu(-Ke_z; M_j, M_i)$ is given by a $\pi$ rotation of $\mathcal{J}^\mu(Ke_z; M_j, M_i)$ around the $x$ axis, in agreement with Eq. (61), viz.

$$\mathcal{J}^\mu(-Ke_z; M_j, M_i) = L^\mu[\rho_x(\pi)] \exp(i\pi S_x)$$

$$\mathcal{J}^\mu(Ke_z; M_j, M_i)\exp(-i\pi S_x)$$

It is easy to show that the Hermiticity condition, Eq. (64), is satisfied by the current defined by Eqs. (103) and (106) by noting that

$$\exp(i2\pi S_x) j^\mu(Ke_z; M_i, M_j)\exp(-i2\pi S_x) = j^\mu(Ke_z; M_i, M_j).$$

It is also straightforward to check that such a current fulfills extended Lorentz covariance (i.e., Eqs. (59), (67) and (68)). In particular, Eq. (59) holds since $L[\rho_x(\pi)]L[u_z]L[\rho_x(-\pi)] = L[-u_z]$ and $\exp(-i\pi S_x)S_x\exp(i\pi S_x) = -S_z$. Parity covariance, Eq. (67), holds since $L[\rho_x(\pi)]A_y L[\rho_x(-\pi)] = A_y$ and $U_{y \text{ int}} \exp(i\pi S_x) = \exp(-i2\pi S_z) \exp(i\pi S_x) U_{y \text{ int}}$. For the time reversal one can follow analogous arguments.
The Hermiticity condition, Eq. (64), imposes the following constraints on the matrix elements

\[ \langle S_j z S_j M_j | j^\pm (Ke_z; M_i, M_j)^* | M_i S_i S_{iz} \rangle = \delta_{S_j, S_{iz}} \left( -1 \right)^{(S_i - S_j)} \]
\[ \langle -S_j z S_j M_j | j^\pm (Ke_z; M_j, M_i) | M_i S_i - S_{iz} \rangle, \]

(107)

\[ \langle S_j z S_j M_j | j_x (Ke_z; M_i, M_j)^* | M_i S_i S_{iz} \rangle = (-1)^{(S_i - S_j)} \]
\[ \langle -S_j z S_j M_j | j_x (Ke_z; M_j, M_i) | M_i S_i - S_{iz} \rangle, \]

(108)

\[ \langle S_j z S_j M_j | j_y (Ke_z; M_i, M_j)^* | M_i S_i S_{iz} \rangle = \left( -1 \right)^{(S_i - S_j)} \]
\[ \langle -S_j z S_j M_j | j_y (Ke_z; M_j, M_i) | M_i S_i - S_{iz} \rangle \]

(109)

since \( \langle S_z S_m | D^S[r_x(-\pi)]|mSS'z \rangle = \delta_{S_z, S'_z} \left( -1 \right)^{-S_z + S_{s} \exp(i\pi S_z)} \). Equations (108)-(109) are obviously compatible with Eq. (39).

Let us discuss now the current conservation in the elastic case (\( M_i = M_j = m; S_i = S_j = S \)). If \( j^+(Ke_z; m, m) \) and \( j_\perp(Ke_z; m, m) \) are chosen as independent components of the operator \( j^\mu(Ke_z; m, m) \), then, in order to satisfy the continuity equation, \( j^-(Ke_z; m, m) \) has to be defined through Eq. (75), which reads for the elastic scattering

\[ j^-(Ke_z; m, m) = j^+(Ke_z; m, m) \]

(110)

This condition implies that \( j_z(Ke_z; m, m) = 0 \).

It is important to notice that Eq. (110) can be obtained as a consequence of extended Lorentz covariance and Hermiticity, or in other words that, in the elastic case, the extended Lorentz covariance together with Hermiticity imposes current conservation. Indeed from Lorentz covariance (Eq. (87)) only the diagonal matrix elements of \( j^\pm(Ke_z; m, m) \) are different from zero, while from parity and time reversal covariance we know that they are real. Then from Hermiticity (Eq. (107))

\[ \langle mSS_z | j^-(Ke_z; m, m)^* | mSS_z \rangle = \langle mSS_z | j^-(Ke_z; m, m) | mSS_z \rangle = \langle mS - S_z | j^+(Ke_z; m, m) | mS - S_z \rangle. \]

(111)
Therefore using parity covariance (Eq. (90)) we obtain
\[
\langle mS_S z | j^- (K e_z; m, m) | mS_S z \rangle = \langle mS_S z | j^+ (K e_z; m, m) | mS_S z \rangle.
\] (112)
i.e., Eq. (110).

In particular the current defined by Eqs. (103) and (105) represents a possible choice. In this case it can be immediately seen that
\[
\langle mS_S z | j^- (K e_z; m, m) | mS_S z \rangle = \langle mS_S z | J^- (K e_z; m, m) | mS_S z \rangle,
\] (113)
\[
\langle mS_S z | j_x (K e_z; m, m) | mS'_S z \rangle = \frac{1}{2} [\langle mS_S z | J_x (K e_z; m, m) | mS'_S z \rangle - \langle mS'_S z | J_x (K e_z; m, m) | mS_S z \rangle],
\] (114)

An elementary application is represented by the elastic scattering from a target with \( S = 0 \). In this case only the matrix elements of \( j^+ \) are relevant; they are diagonal and real, and therefore the terms in Eq. (103) are equal each other (note that: i) \( L_+^+[r_x(-\pi)] = 1 \), and ii) the matrix elements of \( \exp(-i\pi S_x) \) are equal to one).

It is clear from the above discussion that in the elastic case one has (as usual) only \( 2S + 1 \) independent matrix elements for the em current defined in Eqs. (103) and (106), corresponding to \( 2S + 1 \) elastic form factors. The matrix elements of \( j_y \) and \( j^- \) can be obtained from the matrix elements of \( j_x \) and \( j^+ \), respectively, because of Eqs. (89) and (110). As follows from Eqs. (90) or (93), there are \( [S + 1] \) non-zero independent matrix elements of \( j^+ \) ([\( S + 1 \)] is the integer part of \( S + 1 \)) and they can be chosen as the diagonal ones with \( S_z \geq 0 \). The independent matrix elements of \( j_x \) are \( [S + 1/2] \) and can be chosen to be \( \langle mS_S z | j_x (K e_z; m, m) | mS_S z - 1 \rangle \) with \( S_z \geq +1/2 \), as follows from Eqs. (108), and (91) or (23).

Also in the inelastic case, the matrix elements of \( j_y \) and \( j^- \) can be obtained from the matrix elements of \( j_x \) and \( j^+ \), respectively, because of Eqs. (89) and (75). As follows from Eqs. (90) or (93), there are \( [S + 1] \) non-zero independent matrix elements of \( j^+ \) (with \( S = \min(S_i, S_j) \)). The matrix elements of \( j_x \) are constrained by the rules \( |S_i z - S_j z| = 1 \) and \( S_i z \geq 0 \) (see Eq. (94)). For instance, in the case of the transition from the nucleon to a resonance with \( S_j = 1/2 \) one has only two independent matrix elements,
while for a transition to any resonance with $S_j > 1/2$ one remains with only three independent matrix elements, one for $j^+ \langle 1/2 | j^+ | 1/2 \rangle$ and two for $j_x \langle 1/2 | j_x | 1/2 \rangle$ and $\langle 1/2 | j_x | 3/2 \rangle$, corresponding to the three helicity amplitudes.

Summarizing, if one adopts the current defined by the Eqs. (103), (104) and (106) the extraction of em form factors in both the elastic and inelastic scattering is no more plagued by the ambiguities discussed in the introduction.

12 Conclusion

The results of the present paper show that the constraints imposed by (extended) Poincaré covariance and current conservation allows one to determine the current operator through some auxiliary operators $j^\mu (K e_z; M_i, M_j)$, which act only through internal variables and are covariant for rotations around the $z$ axis. Unfortunately the latter operators are not unique. This is in agreement with the results of Ref. [14] where it has been shown that all matrix elements of the current operator can be expressed in terms of some set of fully unconstrained matrix elements. We have demonstrated that it is possible to choose explicit models for $j^\mu (K e_z; M_i, M_j)$ such that all the necessary requirements are satisfied. In particular, as noted in Sects. 7, 8, 11 and 12 (see especially Eqs. (101), (102), (103) and (105)), the operator $j^\mu (K e_z; M_i, M_j)$ can be obtained by projecting the free current operator onto the subspaces corresponding to definite eigenvalues of the mass and spin operators.

It is also worth noting that, although the choice of the Breit frame where the initial and final momenta are directed along the $z$ axis is rather convenient, analogous results for constructing the full current operator from auxiliary ones can be derived by choosing any frame obtained from the Breit one by Lorentz transformations corresponding to the group $\tilde{B}$, where $\tilde{B}$ is a subgroup of $SL(2,\mathbb{C})$ such that $\tilde{b} \in \tilde{B}$ if $\tilde{b}_{12} = 0$ (it is easy to see that $\tilde{B}$ is obtained from $B$ by adding rotations around the $z$ axis). This follows from the fact that in the front-form dynamics the Lorentz transformations corresponding to the group $\tilde{B}$ are kinematical. In particular, we can use
the plus and $\perp$ components of the free current operator for constructing the same components of the auxiliary operators in all such reference frames and then the matrix elements of the minus components can be determined (if necessary) from the continuity equation.

In Sect. 10 we have considered the application of our results to DIS. It has been shown that for calculating the matrix elements of the current operator in the infinite momentum frame where the initial and final momenta are directed along the $z$ axis we indeed can use the free current operator and this does not contradict the fact that the nucleon is a bound state of quarks and gluons. At the same time it has been briefly mentioned that problems with locality and cluster separability exist. These problems will be considered elsewhere [39].

In Sect. 11 we have applied our results to the elastic and inelastic scattering for particles with arbitrary spin. In contrast with the approaches discussed in the Introduction, we have no problem with the angular condition, since our model current is in agreement with extended Poincaré covariance and current conservation, by construction. Therefore the number of independent matrix elements of the current is equal to the number of physical form factors.

Finally, it should be pointed out that our approach, based on the reduction of the whole complexity of the Poincaré covariance to the SU(2) symmetry can represent a simple framework where to investigate the possible many-body terms to be added to the free current, since they must obviously fulfill the rotational covariance condition of Eq. (59).

Numerical calculations for the deuteron elastic form factors and applications to the hadron elastic and transition form factors are in progress.

Acknowledgements

The authors are grateful to F. Coester, W. Klink and H. J. Weber for useful discussion. The work of one of the authors (FML) was supported in part by grant 96-02-16126a from the Russian Foundation for Basic Research.

Appendix A
In this appendix it will be shown that

\[ W^{-1}(l, L(l)^{-1} \frac{P_i}{M_i}) = W(u, \frac{K_i}{M_i}) \]  \hspace{1cm} (A.1) 

where \( K_i = B(H_{ij})^{-1}P_i \) and \( u \in SU(2) \) is given by Eq. (54), viz.

\[ u = \beta(g')^{-1}l^{-1} \beta(g) \]  \hspace{1cm} (A.2) 

with \( g' = L(l)^{-1}H_{ij} \) and \( g = H_{ij} \). Using Eqs. (11) and (A.2), one has

\[ W^{-1}(l, L(l)^{-1} \frac{P_i}{M_i}) = \beta(L_l^{-1} \frac{P_i}{M_i})^{-1}l^{-1} \beta(\frac{P_i}{M_i}) \]

\[ W(u, \frac{K_i}{M_i}) = \beta(L(u) \frac{K_i}{M_i})^{-1}l^{-1} \beta(g')^{-1}l^{-1} \beta(H_{ij}) \beta(\frac{K_i}{M_i}) \]  \hspace{1cm} (A.3) 

Using the group property of the boosts (Eq. (14)) one has

\[ \beta(H_{ij}) \beta(\frac{K_i}{M_i}) = \beta(L(\beta(H_{ij})) \frac{K_i}{M_i}) = \beta(\frac{P_i}{M_i}) \]  \hspace{1cm} (A.4) 

and

\[ \beta(L(u) \frac{K_i}{M_i})^{-1}l^{-1} \beta(g')^{-1} = \beta(L(\beta(g'))L(u) \frac{K_i}{M_i})^{-1} \]  \hspace{1cm} (A.5) 

Finally the multiplicative rule of the Poincaré group yields

\[ L(\beta(g'))L(u) \frac{K_i}{M_i} = L(\beta(g'))L(\beta(g'))^{-1}L(l)^{-1}L(\beta(H_{ij})) \frac{K_i}{M_i} = \]

\[ L(l)^{-1} \frac{P_i}{M_i} \]  \hspace{1cm} (A.6) 

Then, collecting the results from Eqs. (A.3)-(A.6) we find that Eq. (A.1) holds. The same it is true for \( W^{-1}(l, L(l)^{-1} P_j/M_j) = W(u, K_j/M_j) \)
References

[1] J.D. Bjorken, Phys. Rev. 179 (1969) 1547; J.D. Bjorken and E.A. Paschos, Phys. Rev. 185 (1969) 1975.

[2] R.P. Feynman, Phys. Rev. Lett. 23 (1969) 1415; Photon-Hadron Interaction (Benjamin, Reading, Mass., 1972).

[3] K.G. Wilson, Phys. Rev. 179 (1969) 1499; Phys. Rev. D 3 (1971) 1818.

[4] G. Sterman et. al., Rev. Mod. Phys. 67 (1995) 157.

[5] P.A.M. Dirac, Rev. Mod. Phys. 21 (1949) 392.

[6] S. Weinberg, Phys. Rev. 150 (1966) 1313.

[7] F. Close, Introduction to Quarks and Partons (Academic Press, London - New York 1979).

[8] G.P. Lepage and S.J. Brodsky, Phys. Rev. D22 (1980) 2157.

[9] J.M. Namyslowski, Prog. Part. Nucl. Phys. 14 (1984) 49.

[10] I.L. Grach and L.A. Kondratyuk, Yad. Fiz. 39 (1984) 316.

[11] P.L. Chung, F. Coester, B.D. Keister and W.N. Polyzou, Phys. Rev. C37 (1988) 2000.

[12] V.A. Karmanov and A.V. Smirnov, Nucl. Phys. A546 (1992) 691; Nucl. Phys. A575 (1994) 520.

[13] S. Glazek, Acta Phys. Pol. B14 (1983) 893.

[14] W.N. Polyzou and W. Klink, Annals of Physics, 185 (369) 1988.

[15] F.M. Lev, Nucl. Phys. A 567 (1994) 797.

[16] F.M. Lev, Annals of Physics, 237 (1995) 355.

[17] S.N. Sokolov, Teor. Mat. Fiz. 23 (1975) 355.
[18] S.N. Sokolov, Teor. Mat. Fiz. 36 (1978) 193; Doklady Akademii Nauk SSSR 233 (1977) 575.

[19] F. Coester and W.N. Polyzou, Phys. Rev. D26 (1982) 1348.

[20] U. Mutze, Jou. Math. Phys. 19 (1978) 231; Phys. Rev. D29 (1984) 2255.

[21] B.D. Keister and W. Polyzou, Adv. Nucl. Phys. 21 (1991) 225.

[22] S.N. Sokolov and A.M. Shatny, Teor. Mat. Fiz. 37 (1978) 291.

[23] A.I. Akhiezer and V.B. Berestetsky, Quantum Electrodynamics (Nauka, Moscow 1969); N.N. Bogolubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Nauka, Moscow, 1976); J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields (McGraw-Hill Book Company, New York, 1976).

[24] N.N. Bogolubov, A.A. Logunov, A.I. Oksak and I.T. Todorov, General Principles of Quantum Field Theory (Nauka, Moscow, 1987).

[25] L.A. Kondratyuk and M.I. Strikman, Nucl. Phys. A426 (1984) 575.

[26] E.P. Wigner, Ann. of Math. 40 (1939) 1; Proc. Amer. Phil. Soc. 93 (1949) 521; Rev. Mod. Phys. 29 (1957) 255.

[27] I.M. Gelfand, R.A. Minlos and Z. Ya Shapiro, ”Representations of the Rotation and Lorentz Groups and their Applications”, Pergamon Press, London (1963); M.A. Naimark, ”Linear Representations of the Lorentz Groups”, Pergamon Press, London (1964).

[28] K. Bardakci and M.B. Halpern, Phys. Rev. 176 (1968) 1686.

[29] M.V. Terentiev, Yad. Fiz. 24 (1976) 207.

[30] F.M. Lev, Rivista Nuovo Cimento 16 (1993) 1; Fortschr. Phys. 31 (1983) 75.

[31] L.A. Kondratyuk and M.V. Terentiev, Yad. Fiz. 31 (1980) 1087.

[32] H. Melosh, Phys. Rev. D9 (1974) 1095.
[33] B.L.G. Bakker, L.A. Kondratyuk and M.V. Terentiev, Nucl. Phys. B158 (1979) 497.

[34] L.H. Bakamjian and L.H. Thomas, Phys. Rev. 92 (1953) 1300.

[35] F.M. Lev, J. Phys. 17 (1984) 2047; Nucl. Phys. A433 (1985) 605.

[36] F. Coester, private communication of May 23, (1995).

[37] H. Osborn, Nucl. Phys. B 38 (1972) 429.

[38] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, ”Quantum Theory of Angular Momentum”, World Scientific (Singapore), 1988, p. 32.

[39] F. Lev, E. Pace, G. Salmè, to be published.

[40] E. Pace, G. Salmè, F. Lev, Phys. Rev. C 58 (1998) 2655.

[41] F.M. Lev, Rus. Jou. of Nucl. Phys. (1998) in press, hep-ph/9803254.