Broadcasting Spanning Forests on a Multiple-Access Channel

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Abstract

The problem of finding a spanning forest of a graph in a distributed-processing environment is studied. If an input graph is weighted, then the goal is to find a minimum-weight spanning forest. The processors communicate by broadcasting. The output consists of the edges that make a spanning forest and have been broadcast on the network. Input edges are distributed among the processors, with each edge held by one processor.

The underlying broadcast network is implemented as a multiple-access channel. If exactly one processor attempts to perform a broadcast, then the broadcast is successful. A message broadcast successfully is delivered to all the processors in one step. If more than one processors broadcast simultaneously, then the messages interfere with each other and no processor can receive any of them.

Optimality of algorithmic solutions is investigated, by way of comparing deterministic with randomized algorithms, and adaptive with oblivious ones. Lower bounds are proved that either justify the optimality of specific algorithms or show that the optimal performance depends on a class of algorithms.

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*This work was published as [5]. The results of this paper were presented in a preliminary form in [6].
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Work supported by the National Science Foundation under Grant No. 0310503.
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1 Introduction

We consider a distributed system in which processing units communicate by broadcasting. The underlying broadcast network is implemented as a multiple-access channel. The processing units are called stations. A message sent at a step is successfully received by all the stations only if there is exactly one station that performs a broadcast at this step. We work with a model in which a collision, of the messages sent simultaneously by different stations at a step, results in a feedback that allows all the stations to detect the collision.

We study the problem of finding a spanning forest of a graph. The input may be either a simple graph, with no weights assigned to the edges, or it may have a weight given for each edge, then the goal is to find a minimum-weight spanning forest. The input is distributed among the stations: each edge is held by one station. Some edges are heard on the channel during an execution of an algorithm, we require that all the edges of a spanning forest we seek are revealed explicitly by having been heard at some step.

We consider two possible ways to obtain an input. In a static case, the input is provided to the participating stations at the start of an execution. In a dynamic case, there is an adversary that controls the timing of when the station holding a given edge wakes up and joins the process of computation.

The questions we address in this paper concern the time efficiency of finding a spanning forest. We compare adaptive and oblivious algorithms, and deterministic with randomized ones, and prove lower bounds. For a given class of algorithms, the goal is to identify the optimum time performance of an algorithm in the class. When comparing classes of algorithms, the question is whether the optimum performance differs among the classes.

Now we give a detailed overview of the results of this paper. Let $n$ be the number of vertices and let $m$ be the number of edges of an input graph, which also means that there are $m$ stations participating in the computation. The size $|T|$ of a spanning forest $T$ is defined to be equal to the number of edges in $T$. A deterministic algorithm is said to be oblivious if its actions are defined in advance for any input of a given size, otherwise an algorithm is adaptive.

I. Simple graphs and deterministic algorithms. We give a deterministic adaptive algorithm that finds a spanning forest $T$ in time $O\left(\min\left[ m, |T| \log m \right]\right)$. We prove that, for any number of edges $m$ and any oblivious deterministic algorithm $A$, there exists some graph $G_A$ with $\Theta(m)$ edges that makes $A$ perform $\Omega(m)$ steps on $G_A$. This shows that adaptive and oblivious algorithms have different optimum-time performances, for sufficiently many edges in the output.

II. Simple graphs and randomized algorithms. We give a randomized algorithm that finds a spanning forest $T$ in expected time proportional to its size $|T|$, which is optimal.

III. Graphs with weights and deterministic algorithms. We prove that any deterministic adaptive algorithm requires time $\Omega(m)$ to find a minimum-weight spanning forest. This shows, if $m = \omega(n \log n)$, that deterministic adaptive algorithms can solve the problem of finding any spanning forest of a simple graph faster than the problem of finding a minimum-weight spanning tree of a graph with weights assigned to its edges.

IV. Graphs with weights and randomized algorithms. We give a randomized algorithm that finds a
minimum-weight spanning forest $T$ in expected time $O(\min\{m, |T| + W \log m\})$, where $W$ is the number of different weights on the edges in $T$. This shows that, for the problem of finding minimum-weight spanning forests, randomized algorithms are provably more efficient than deterministic ones when the number $m$ of edges is $\omega(n \log n)$.

V. Graphs specified dynamically by adversaries. We develop a deterministic algorithm that finds a spanning forest of a simple graph in time $O(|T| \log m)$, if an adversary can control the time when stations holding edges are activated and join the computation. We also prove that for any deterministic adaptive algorithm $A$ and an input graph containing a forest with $m$ edges there is a strategy of an adversary that forces $A$ to perform $\Omega(m \log m)$ steps on $G$. This shows that the time performance $\Theta(m \log m)$, in terms of $m$, is optimal among deterministic adaptive algorithms in the dynamic adversarial model.

Related work. Early work on multiple-access channels has concentrated on distributed protocols to handle bursty traffic of packets carrying dynamically generated messages. It included the development of protocols like Aloha [1] and Ethernet [31]. A survey by Gallager [12] covers the results up to 1985, recent papers include those by Goldberg et al. [16], Håstad et al. [21], and by Raghavan and Upfal [34]. A multiple-access channel can be viewed as a special case of a multi-hop radio network. It is actually single-hop, since one step is sufficient to have a message delivered between any two stations. For an overview of work on communication in radio networks, including the single-hop ones, see a survey by Chlebus [4].

In static problems on multiple-access channels, we assume that the input has been given to the stations at the start of a protocol. One of the most natural such problems is that of selection, where some among the $N$ stations are given messages and the goal is to have any of them heard on the channel. Willard [36] developed a randomized protocol for this problem that operates in expected time $O(\log \log N)$, for a channel with collision detection. Solving this problem requires expected time $\Omega(\log N)$, if detection of collision is not available, as was shown by Kushilevitz and Mansour [25], hence there is an exponential gap between these two models. In the all-broadcast problem we are asked to have all the messages, given to some stations selected among the $N$ ones, heard on the channel. If collision detection is available then this can be done deterministically with a logarithmic overhead per message by an algorithm of Komlós and Greenberg [24], which is optimal as shown by Greenberg and Winograd [18].

Gašieniec et al. [15] studied the wakeup problem, which is a dynamic version of the selection problem. The goal again is to have a successful transmission as soon as possible, but the timing of stations joining the protocol is controlled by an adversary. Paper [15] shows how efficiency depends on various levels of synchrony, and it compares randomized and deterministic solutions. Other related work on radio networks has been done by Jurdziński et al. [22] and by Jurdziński and Stachowiak [23].

All the problems mentioned above concern the communication itself on a multiple-access channel. Some research has also been done concerning distributed algorithmics for specific combinatorial or optimization problems when the underlying communication is implemented by a multiple-access channel. Martel and Vayda [29, 30] studied the problem of finding the maximum value among those stored at a subset of stations. Chlebus et al. [17] and Clementi et al. [8] considered the problem of performing a set of independent unit tasks, when the stations may fail by crashing. This problem is called Do-All, it was first studied in a message-passing model by Dwork et al. [10].
Distributed algorithms finding minimum-weight spanning trees have been studied before in other models. The most popular among them assumes that the processors are vertices and that the communication links are edges of an input graph. Such a setting provides a unique combination in which the underlying communication network is also an input. The problem of finding a minimum-weight spanning tree in such a model was first proposed by Gallager et al. [13]. Awerbuch [2] developed an algorithm working in time $O(n)$, where $n$ is the number of vertices. Garay et al. [14] found a solution with performance proportional to the diameter, if the diameter is sufficiently large. Faloutsos and Molle [11] studied tradeoffs between the time and the number of messages. Lower bounds on the time have been given by Peleg and Rubinovich [33] and by Lotker et al. [28]. Other graph problems studied for such a distributed setting include finding maximal matchings, a deterministic algorithm has been developed by Hančkowiak et al. [20], and edge coloring, a randomized algorithm has been given by Grable and Panconesi [17]. An algorithm finding $k$-dominating sets was given by Kutten and Peleg [26]. The issues of locality in distributed graph algorithms have been studied by Linial [27], see the book by Peleg [32] for a comprehensive coverage.

2 Technical Preliminaries

We consider distributed algorithms performed by stations that communicate over a broadcast network. See the books by Bertsekas and Gallager [3] and Tanenbaum and Wetherall [35] for systematic overviews of communication networks. The computations are synchronous, all the stations have access to a global clock.

Adversaries. The stations are categorized at each step as either active or passive, and only active stations participate in the computation. A passive station may be activated at any step, then it changes its status and becomes active. Such decisions, concerning which stations to activate and when to do this, are made by an adversary. In a static scenario the active stations are stipulated at the start of an execution, and are not changed later by the adversary. In a static setting we do not mention the adversary at all. A dynamic scenario involves an adversary who can decide on the timing when each passive station is activated in the course of an execution.

Multiple-access channel. The broadcast operation is implemented on a multiple-access channel. All the stations receive the same information from the channel at each step, unless they are passive, this information is said to be heard on the channel. The basic property of the channel is that if only one station attempts the broadcast operation at a step, then its message is delivered to all the active stations by the end of the step. We assume that the size of a message that can be heard in one step is as large as required by the algorithm. In particular, in Section 5 the algorithm relies on the property that a single message can carry a list of an arbitrary subset of the set of all the stations.

Multiple-access channels come in two variants: either with or without collision detection. In the former case, if more than one stations broadcast at a step then all the stations can hear the collision noise on the channel. This signal is distinct from the background noise, which is heard when no station performs a broadcast at a step. Both kinds of noise signals are received as indistinguishable if the channel is without collision detection. A multiple-access channel is a single-hop radio network, in the terminology of radio networks (see [4]).

In this paper we work with the collision-detection variant. This simplifies algorithms, and we do
not lose much generality, at least in the static case. Collision detection can be implemented without affecting the asymptotic performance, provided that the number $m$ of edges is large enough. To this end, it is sufficient first to elect a leader among the active stations. Then the consecutive steps of an execution are used depending on their parity: the algorithm uses the even steps $2i$, and the stations scheduled by it to broadcast at step $2i$ repeat this at step $2i + 1$, but the leader always broadcasts a dummy message at odd steps. Hence if no signal is received at two consecutive steps $2i$ and $2i + 1$, then this means that there was a collision at step $2i$, since the leader’s attempt to broadcast failed. Otherwise there is no collision, because the dummy message of the leader is heard at step $2i + 1$, while nothing was heard at step $2i$. Selecting a leader, in the static case, can be achieved in time $O(\log n)$ by a deterministic tree-like algorithm, see [4] for a survey of solutions of related problems in radio networks, and [22] for a recent work. In the dynamic case the problem of selecting a leader is essentially equivalent to the wakeup problem, and requires time $\Omega(m)$, as shown by Gąsieniec et al. [15].

**Graphs and algorithms.** We consider graph problems, the inputs are simple graphs, possibly with weights assigned to the edges. The goal is to find a maximal spanning forest of the input graph, of the smallest weight in the case of weighted graphs. A spanning forest $T$ is maximal in graph $G$ if adding a new edge from $G$ to $T$ creates a cycle. Throughout the rest of this paper, when we refer to a spanning forest, we always mean a maximal one. The size $|T|$ of a spanning forest $T$ is defined to be the number of edges in it.

Let $n$ be the number of vertices of the input graph. This number $n$ is assumed to be known by all the stations. A vertex of an input graph is identified by a number in the interval $[1..n]$, and an edge is identified as a pair of such numbers. Each station holds a single edge. Additionally, each station is assigned a unique identifier, referred to as its ID, which is a positive integer. We assume that the IDs of stations form a contiguous segment of integers $[1..m]$, but the number $m$ is not assumed to be known by the stations at the start of an execution. The assumption of contiguity is relied upon when we consider the case of sparse graphs in a static scenario, it makes an exhaustive enumeration efficient.

We require that all the edges of the forest sought are revealed by being heard on the channel. After the edges of a spanning subgraph have been revealed, and this subgraph has as many connected components as the input graph, then its spanning forest can be determined by some direct rule. For instance, if there are no weights on edges, the first spanning forest in a lexicographic ordering of sets of edges may be designated as the output, and if there are weights on edges, only spanning forests of the smallest weight are considered. We abstract from such specific rules when presenting algorithms, since our main concern is the communication involved in revealing a sufficiently large subgraph with as few transmission attempts as possible.

Typically, when a station performs a broadcast, the message contains only its input edge. An algorithm proceeds as a sequence of queries, each specifies the stations that are to attempt to broadcast at the given step. More precisely, a query is a list including IDs of stations and/or edges. A station with its ID equal to $p$ attempts to broadcast its edge on the channel at step $i$ if it is specified by the query $Q_i$. This means that either the ID $p$ is in $Q_i$ or the station holds an edge that is in $Q_i$. We often refer to a station with its ID equal to $p$ as “the station $p$,” and to a station holding some edge $e$ as “the station $e$.”

An algorithm is adaptive if each query $Q_{i+1}$ depends on the feedback heard on the channel when the preceding queries $Q_1, Q_2, \ldots, Q_i$ were executed. An algorithm is oblivious if the queries
$Q_1, Q_2, \ldots, Q_i, \ldots$ are all known in advance, and depend only on the number $n$ of vertices of the input graph. More precisely, a query of an oblivious algorithm may be given in terms of properties of edges of a graph on $n$ vertices, for instance by specifying some of the endpoints, or refer to specific IDs of stations. Randomized algorithms, that we develop, are also described by queries, but each station specified by a query additionally first performs a random experiment, and then actually broadcasts only if the result of the experiment was a success. Such experiments are Poisson trials, that is, they return either success or failure, which are independent over the steps and the stations, the probability of success at a step is the same for all the stations, these probabilities may vary over the steps.

3 Simple Graphs

Graphs considered in this section are simple and they do not have weights assigned to the edges. The input is given at the start of an execution, and no adversaries are involved.

3.1 Deterministic Adaptive Algorithm

We use a routine $\text{Resolve}(S)$ to resolve conflicts among a set of stations $S$ that want to reveal one among their edges or one among their IDs. We discuss the case of edges in detail, the case of IDs is similar. The procedure is based on a binary-search paradigm. Let us fix a linear ordering $I$ of all the possible edges of a graph with vertices in $[1..n]$, for instance the lexicographic one. First all the stations (whose edges are) in $S$ broadcast simultaneously. If this results in silence then $S$ is empty and $\text{Resolve}(S) = \emptyset$ is completed. If an edge $e$ is heard then the set $S$ is a singleton and the conflict is resolved, with $\text{Resolve}(S) = \{e\}$. Otherwise a collision is detected. Let $I_1$ and $I_2$ be a partition of $I$ into left and right subintervals, respectively, determined by the median of $I$. In the next step all the stations in $S \cap I_1$ broadcast. If this results in a single edge $e$ heard on the channel then $\text{Resolve}(S) = \{e\}$. If there is a collision then $I_1 \cap S$ is searched recursively, with set $I_1$ replacing $I$. If there is silence then $I_2 \cap S$ is searched recursively, with $I_2$ replacing $I$. It takes $O(\log m)$ steps to resolve a conflict and hear the smallest edge held by a station in $S$.

**Basic algorithm.** We start with the algorithm $\text{DetSimple}$. It operates by iterating phases in a loop. At all times the input edges are partitioned into three subsets:

- **Revealed**: the edges already revealed on the channel, each of them is called revealed;
- **Cycle**: the edges that would make a cycle if added to those in Revealed, each of them is called cycle;
- **Waiting**: the remaining edges, called waiting.

Each of the waiting edges could be added to Revealed and still the property that Revealed is a forest would be maintained. A station holding a revealed, cycle or waiting edge is called a revealed, cycle or waiting station, respectively. Initially the sets Revealed and Cycle are empty, and the set Waiting consists of all the input edges. During one iteration, the procedure $\text{Resolve(Waiting)}$ is called, and the edge eventually heard on the channel is added to Revealed. Each of the remaining waiting stations checks to see if it is now in Cycle, and if this is the case, then it will never attempt to perform a broadcast in this execution. A pseudocode of the algorithm is given in Figure 1.
INPUT: the number $n$ of graph vertices; edge $e_p$.

INITIALIZATION: Revealed := $\emptyset$, Cycle := $\emptyset$, $e_p$ is waiting;

repeat
  if $e_p$ is waiting then broadcast a dummy message;
  if silence was heard in the previous step then terminate;
  \{e\} = Resolve(Waiting);
  move edge $e$ from Waiting to Revealed;
  if there is a cycle in the graph induced by $e_p$ and the edges in Revealed
  then move $e_p$ from Waiting to Cycle
until termination;

OUTPUT: all the revealed edges.

Figure 1: Algorithm DetSimple. Code for the station $p$ storing the edge $e_p$.

Correctness. The correctness of algorithm DetSimple is guaranteed by the following invariant maintained in each iteration: the edges in Revealed make a forest on the set of vertices $[1..n]$.

Performance. The overall cost of algorithm DetSimple to find a spanning forest $T$ is $O(|T| \log m)$ since Resolve is called $|T|$ times.

General algorithm. We give an algorithm that works in time $O(m)$ if $m = o(n \log n)$, it operates as follows. The algorithm DetSimple is run during the odd-numbered steps. During the even-numbered steps, the stations broadcast their edges on the channel one by one, in the order of their IDs. If there is a silence heard during an even-numbered step, then this is interpreted as a termination signal, since all the stations have revealed their edges by this step. The two processes, run during the odd-numbered and the even-numbered steps, do not affect one another, in particular the edges broadcast in the even-numbered steps are not treated as revealed in algorithm DetSimple. The following theorem follows directly from the design of this general algorithm:

**Theorem 1** There is a deterministic adaptive algorithm for the static model that finds a spanning forest $T$ of a simple graph in time $O(\min\{m, |T| \log m\})$.

3.2 Randomized Algorithm

We present a randomized algorithm RandSimple that finds a spanning forest in a graph in expected time proportional to its size. The algorithm is similar in structure to the deterministic one, and we use the same terminology. The main difference and advantage is that the expected time to reveal a waiting edge is constant.

A pseudo code of the algorithm is in Figure 2. The algorithm uses a variable denoted $a$, which is maintained by all the stations, to approximate the number of waiting stations during an execution. The stations broadcast with the probability $1/a$. They update the estimate $a$ if silence or noise is heard. If an edge is heard, then all the sets Revealed, Waiting and Cycle are updated as in the algorithm DetSimple.
INPUT: edge $e_p$;

INITIALIZATION: $a := 1$; edge $e_p$ is waiting; $\text{counter} := 0$;

repeat

A: if $e_p$ is waiting then broadcast it with the probability $1/a$;

B: begin case

(a) the edge $e_p$ was heard: change the status of $e_p$ to revealed;
(b) an edge distinct from $e_p$ was heard: check if $e_p$ is now cycle,
   and if so then change its status to cycle;
(c) collision was heard: set $a := 3a$;
(d) silence was heard: set $a := \min\left\{a/3, 1\right\}$;

end case;

$\text{counter} := \text{counter} + 1$;

C: if $p = \text{counter}$ then broadcast $e_p$; begin case

(a) the edge $e_p$ was heard: if $e_p$ is waiting
   then change its status to revealed;
(b) an edge distinct from $e_p$ was heard: check if $e_p$ became cycle,
   and if so then change its status to cycle;

end case;

D: if $e_p$ is waiting then broadcast a dummy message;

until silence was heard in step D;

OUTPUT: the set of the revealed edges.

Figure 2: Randomized algorithm $\text{RANDSIMPLE}$ for simple graphs without weights.
Code for the station $p$ that stores the input edge $e_p$. 

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Correctness. The correctness of algorithm RandSimple follows from the following two facts. First, the output is a superset of a spanning forest. This follows from the observation that as long as a spanning forest has not been found, then there are still some waiting edges. Secondly, the algorithm terminates in the worst-case time $O(m)$. This is because in each iteration a specific edge is verified directly if it is waiting, exhausting all of them in a systematic way, except possibly for the last iteration. When the counter variable attains the value equal to the maximum ID of a station and its edge is broadcast in step C, then it was the last waiting edge, hence silence is heard in step D and the algorithm terminates.

Performance. We will show that expected time of the algorithm is proportional to the number of edges in the obtained forest.

Let a round denote one iteration of the main loop of the algorithm. An execution of algorithm RandSimple can be partitioned into phases, each comprising a sequence of consecutive full rounds, and finished by an iteration in which (at least one) edge in a forest has been broadcast. An additional closing round, in which the algorithm terminates, is also possible. For the purpose of the probabilistic analysis we assume conservatively that no edges are ever broadcast successfully during step C.

Let $\mathcal{W}_i$ denote the set of waiting edges in the beginning of the $i$th phase. In particular, the set $\mathcal{W}_1$ consists of all the $m$ input edges. Let $a_i$ denote the value of variable $a$ at the start of the $i$th phase; this number is interpreted as an approximation of the size $|\mathcal{W}_i|$ of the set $\mathcal{W}_i$. The value stored in the variable $a$ at the start of the $j$th round of phase $i$ is denoted as $a_{i,j}$. We say that the algorithm has a good approximation if the inequalities

$$\frac{1}{3} |\mathcal{W}_i| \leq a_{i,j} \leq 3 |\mathcal{W}_i|$$

hold. We first estimate the probabilities of silence, noise and a successful broadcast, respectively, heard on the channel just after step A in a single round of a phase.

Lemma 1 There is a constant $c_1 > \frac{1}{2}$ such that if $a > 3|\mathcal{W}_i|$, then the probability of silence in a round of phase $i$ is at least $c_1$.

Proof: If $\mathcal{W}_i \neq \emptyset$, then $a > 3$. Interpreting actions of stations as Bernoulli trials, we obtain that the probability of silence is at least

$$(1 - \frac{1}{a})^{|\mathcal{W}_i|} \geq (1 - \frac{1}{a})^{a/3} \geq \left(1 - \frac{1}{4}\right)^{4/3},$$

which is larger than $\frac{1}{2}$. \qed

Lemma 2 There is a constant $c_2 > \frac{1}{2}$ such that if $a < |\mathcal{W}_i|/3$, then the probability of noise during a round in phase $i$ is at least $c_2$.

Proof: We start by estimating the probabilities of silence and a successful broadcast. The probability of silence is at most

$$(1 - \frac{1}{a})^{|\mathcal{W}_i|} \leq \left(1 - \frac{1}{a}\right)^{3a} \leq e^{-3}.$$
The probability of a successful broadcast is that of an exactly one success in a sequence of $|W_i|$ Bernoulli trials, and is at most

$$|W_i| \frac{1}{a} \left(1 - \frac{1}{a}\right)^{|W_i| - 1} = \frac{|W_i|}{a} \frac{a}{a-1} \left(1 - \frac{1}{a}\right)^{|W_i|} \leq 2 \frac{|W_i|}{a} \left(1 - \frac{1}{a}\right)^{|W_i|/a} \leq 6 e^{-3}.$$ 

Since the probability of noise is 1 minus the probabilities of silence and of a successful broadcast, we obtain the following estimate as a lower bound:

$$1 - (e^{-3} + 6e^{-3}) = 1 - 7e^{-3},$$

and it is larger than $\frac{1}{2}$. □

**Lemma 3** There is a constant $c_3 > 0$ such that if both the inequalities $|W_i|/3 \leq a \leq 3|W_i|$ hold then the probability of a successful broadcast during a round in phase $i$ is at least $c_3$.

**Proof:** We estimate the probability of exactly one success in a sequence of $|W_i|$ Bernoulli trials:

$$\frac{|W_i|}{a} \left(1 - \frac{1}{a}\right)^{|W_i| - 1} = \frac{|W_i|}{a} \frac{a}{a-1} \left(1 - \frac{1}{a}\right)^{|W_i|} \geq \frac{1}{3} \left(1 - \frac{1}{a}\right)^{3a},$$

which is at least $1/(3e^3)$. □

We model an execution of the algorithm as a combination of two random processes. One is a discrete-time random walk with a retaining barrier: the nonnegative integers are the possible coordinates of a particle, the barrier is at the origin with the coordinate equal to zero. Each time the particle is at the origin then the second process is started which is just a single Bernoulli trial. The random walk terminates after the first success and is restarted at some positive integer coordinate. Details are as follows.

If the algorithm has a good approximation, then this is interpreted as the particle being at the barrier. If either the inequalities

$$\frac{|W_i|}{3k+1} \leq a_{i,j} < \frac{|W_i|}{3k}$$

or the inequalities

$$3^k |W_i| < a_{i,j} \leq 3^{k+1} |W_i|$$

hold, for $k \geq 1$, then this is interpreted as the particle being at the distance $k$ from the barrier. If the particle is at the origin then a Bernoulli trial is performed with the probability of success equal to the number $c_3$ of Lemma 3. A success is interpreted as a successful broadcast, which starts a new phase $i$. After a success, the particle is moved to the location determined by the current $a_i$ and $W_i$, and after a failure, the particle is moved to location 1. If the particle is at the location with a coordinate $k > 0$, then it moves either to $k - 1$ or to $k + 1$ in the next step, according to the following rules. It moves to $k - 1$ with the probability $c_0 = \min[c_1, c_2] > \frac{1}{2}$, where $c_1$ and $c_2$ are
as in Lemmas 1 and 2. It moves to \( k + 1 \) with the probability \( 1 - c_0 < \frac{1}{2} \). All the random moves are independent of each other.

This underlying random-walk model captures the behavior of the algorithm for the following two reasons. First, the moves of the particle correspond to the modifications of the variable \( a \) in steps \( B(c) \) and \( B(d) \) in Figure 2. Second, any positive integer is always between two consecutive powers of 3; take the number \( k \) such that \( 3^k \leq |W_i| < 3^{k+1} \), for integer \( k \geq 0 \), and if \( a = 3^k \) or \( a = 3^{k+1} \), then this is reflected in the model by the particle located at the barrier. Our probabilistic analysis is based on the following property of such a random walk on a discrete axis with a barrier: if a particle starts at location \( \ell > 0 \), then the expected time needed to reach the barrier is \( O(\ell) \), where the constant hidden in the notation \( O \) depends on the probability \( c_0 \), see [19].

**Theorem 2** Algorithm RandSimple works in expected time proportional to the size \(|T|\) of a spanning forest \( T \) it finds, and in the worst-case time \( O(m) \).

**Proof:** Let \( L_i \) be the distance from the origin in the beginning of phase \( i \). Then \( \sum L_i = O(\log m) \) because the sizes of the set of waiting edges are monotonically decreasing. It follows that the expected total time spent by the particle between the beginnings of phases and then reaching the origin for the first time is \( O(\log m) \). This amount of time is not asymptotically more than the size of the spanning forest, which is \( \Omega(\sqrt{m}) \).

Let \( X \) be a random variable equal to the number of steps it takes the particle to reach the barrier at the origin after a start at position 1. Let \( Y \) be the number of attempts in a sequence of Bernoulli trials, each with the probability \( c_3 \) of success, before a success occurs. Then the expected length of a phase is at most

\[
\sum_{k \geq 1} (k \cdot E[X]) \cdot \Pr(Y = k) = E[X] \cdot \sum_{k \geq 1} k \cdot \Pr(Y = k) = \frac{1}{c_3} \cdot E[X],
\]

which is \( O(1) \). This completes the proof for the expected performance, since a phase contributes an edge in a spanning forest. The worst-case upper bound is straightforward. \( \square \)

A spanning forest of a graph with \( n \) nodes an \( m \) edges has at most \( \min \{ m, n - 1 \} \) edges, which gives another possible form of a performance bound following directly from Theorem 2.

### 3.3 Lower Bound for Simple Graphs

In this subsection we prove a lower bound \( \Omega(m) \) on the number of queries required by a deterministic oblivious algorithm, when the input graph has no weights assigned to its edges. This shows that there is a gap in the optimum performances between adaptive and oblivious algorithms, among deterministic ones.

Suppose \( \mathcal{A} \) is an algorithm and \( m \) is the number of edges we would like to be the size of an input. To simplify exposition, we assume that the queries in \( \mathcal{A} \) contain only edges and no IDs of stations. Our goal is to construct a connected graph \( G = (V, E) \), with the set of nodes \( V = \{1, \ldots, n\} \), for a suitable number \( n \), and the set \( E \) of \( m \) edges, such that there is a vertex \( v \in V \) with the property that, for each vertex \( w \in V \) different from \( v \), and any query \( Q_i \), for \( 1 \leq i \leq k \), we have \( Q_i \cap E \neq \{(v, w)\} \). Such a graph would make the algorithm \( \mathcal{A} \) perform more than \( k \) steps, because after these many steps no edge with \( v \) as its endpoint would have been heard.
The following specification of an input graph for $A$ is referred to as the five-phase construction. Take a number $n$ such that the inequalities $2n \leq m \leq n(n-1)/4$ hold. Consider the first $k$ queries $Q_1, \ldots, Q_k$ of algorithm $A$, where $k = \lceil \frac{m-n}{2} \rceil$. Notice that $k = \Omega(m)$. We start from a configuration such that, for each edge in the complete graph on $n$ vertices, there is a station holding this edge. We specify which of these stations, and hence also edges, should be removed, so that the number of the remaining edges is exactly $m$. Formally, we proceed through a sequence of five phases and maintain three sets of edges: $Q$, $F$ and $T$. The set $Q$ is initialized to $Q := \bigcup_i Q_i$, it contains all the edges that can appear in any of the queries. The set $F$ is initialized to all the possible edges between the $n$ vertices. The set $E$ of edges of the input graph $G$, that we will identify in the course of the construction, will be stored in the set $T$.

Phase 1  \hspace{1cm} (preventing edges to be heard for as long as possible)

\begin{verbatim}
stop := 0 ;
while stop = 0 do begin
  stop := 1 ;
  if there is a $Q_i$ such that
      ($|Q_i \cap F| = 1$) and ($\text{there is no isolated vertex in } F \setminus Q_i$)
  then begin $F := F \setminus Q_i$ ; stop := 0 end ;
end
\end{verbatim}

Comment: We remove from $F$ those edges that do not isolate any vertex but would be heard if the algorithm asked about them and they were in the graph. The purpose is to delay the step when the algorithm learns about a pair of vertices being connected. If it keeps asking about specific neighbors of a vertex, then it hears nothing except for the last possible moment. We keep passing through all the queries until we are sure that the last modification has not created new singletons $Q_i \cap F$ that would be heard. It follows that the graph spanned by the set of edges $F$ is connected.

Phase 2  \hspace{1cm} (initialization)

Set $T$ equal to a spanning tree of the graph spanned by the set $F$ of edges.

Comment: This is just an initialization of the set $T$. We will keep increasing it during the next phases as long as its size is not large enough.

Phase 3  \hspace{1cm} (adding collisions)

for $i := 1$ to $k$ do
  if $Q_i \cap T = \{e_1\}$ and $|Q_i \cap F| > 1$ then
    begin choose $e_2 \in Q_i \cap F$ such that $e_2 \neq e_1$ ; $T := T \cup \{e_2\}$ end ;
  if $|Q_i \cap T| = 0$ and $|Q_i \cap F| > 1$ then
    begin choose $e_1, e_2 \in Q_i \cap F$ such that $e_1 \neq e_2$ ; $T := T \cup \{e_1, e_2\}$ end ;
endfor

Comment: Now the set $T$ makes the algorithm hear collisions at all the steps when the set $F$ does.
Phase 4  (increasing size)

for $i := 1$ to $k$ do

if $|T| < m$ and $Q_i \cap T \neq \emptyset$ then begin

choose $X \subseteq Q_i \cap F$ of a maximal size such that $|T \cup X| \leq m$; $T := T \cup X$

end

Comment: In this phase we increase the size of $T$, by adding edges included in queries, to be as close as possible to $m$. This phase is productive if the set $Q$ is large enough, otherwise we still have Phase 5 as the last resort.

Phase 5  (padding with edges not mentioned)

If $|T| < m$ then add $m - |T|$ edges to $T$ that are not in $Q$.

Comment: This phase is needed in case algorithm $A$ has the queries involving only a small set of edges.

This completes the five-phase construction of the graph $G$. We say that a vertex $v$ is a $Y$-witness, for a set $Y$ of edges, if none of the edges $(v, w)$ satisfies $Q_i \cap Y = \{(v, w)\}$, for any number $i \leq k$ and for any vertex $w$ in $V$ distinct from $v$. Correctness of the five-phase construction is formulated as Lemma 4.

Lemma 4  The five-phase construction results in obtaining a set $T$ of size $m$ and with a vertex that is a $T$-witness.

Proof: We examine the five phases one by one.

After Phase 1. Suppose to the contrary that there are no $F$-witness vertices. We have $Q_i \cap F = \{e\}$ holds only if $F \setminus \{e\}$ has some isolated vertex. An edge $(v, w)$, such that $Q_i \cap F = \{(v, w)\}$, is not removed only if otherwise one among the vertices $v$ and $w$ would become isolated in $F$. Notice also that at least

$$(n - 1) + (n - 2) + \cdots + (n - l) = l \cdot (2n - l - 1)/2$$

edges need to be removed in order to isolate $l$ or more vertices, which follows by induction on $l$. Removing one edge results in the intersection of at least one query with $F$ being empty. We can remove at most $k$ edges. Should we like to restrain all the $n$ vertices from being $F$-witnesses we would need to isolate at least $n/2$ vertices, thus needing to remove at least $n/2(2n - n/2 - 1)/2 = n(\frac{3}{8}n - 1)/4 = \frac{3}{8}n^2 - n/4$ edges from $F$. Because of the inequality $k < n^2/8 - n/2$, at least one vertex $v$ has not been prevented from being an $F$-witness.

It follows that all the singleton intersections that contain an edge adjacent to vertex $v$ have been removed from $F$. The inequality $|F| \geq |Q| - k$ follows by the fact that an edge is removed from the set $F$ only if at least one query $Q_i$ becomes disjoint with the updated $F$.

After Phase 2. The tree $T$ contains each edge $(v, w)$ such that $Q_i \cap F = \{(v, w)\}$, for some $i \leq k$. This follows from the observation that such an edge $(v, w)$ has not been removed in Phase 1, hence one among the vertices $v$ and $w$ is of degree 1 in $F$, and the edge $(v, w)$ was put into set $T$. Clearly, $|T| \leq n - 1$, and hence there is a $T$-witness at this point.
After Phase 3. Set $T$ has at most $n - 1 + 2k < m$ edges. This is because at most two edges can be added to the set $T$ in each step $i \leq k$. The vertex $v$ that is a $T$-witness is the same as in the analysis of the previous phases. More precisely, there was no edge $(v, w)$ such that $Q_i \cap T = \{(v, w)\}$ in the beginning of Phase 3, the set $T$ was not decreased during Phase 3, and no edge such that $Q_i \cap F = \emptyset$ was added in step $i$.

After Phase 4. We obtain set $T$ of a size that is between $|Q| - k$ and $m$. This is because we can add to $T$ any number of edges from $F \setminus T$, as long as the inequality $|F| \geq |Q| - k$ holds. There is a $T$-witness because during Phase 4 we add only edges in such queries that have at least one element in $T$.

After Phase 5. The property of an existence of a $T$-witness is maintained because Phase 5 does not interfere with intersections of $F$ and $Q_i$. Since the size $|T|$ was between $|Q| - k$ and $m$, we can add sufficiently many elements from the set $\{(v, w) : v, w \in V, v \neq w\} \setminus Q$ of size $n(n - 1)/2 - |Q| \geq m + k - |Q|$ to obtain $|T| = m$.

Lemma 4 contains all the essential ingredients of a lower bound in terms of the number $m$ of edges alone. We show a more general fact formulated in terms of both the numbers of nodes and edges.

**Theorem 3** For any deterministic oblivious algorithm $A$ finding spanning forests, and numbers $n$ and $m$, where $m = O(n^2)$, there exists a connected graph $G_A$ with $n$ vertices and $\Theta(m)$ edges, such that algorithm $A$ requires time $\Omega(m)$ to find a spanning tree of $G_A$.

**Proof:** If the inequality $2n \leq m \leq n(n - 1)/4$ holds, then apply the five-phase construction. It follows from Lemma 4 that if we take the final $T$, obtained by completing the five-phase construction, as the set of edges $E$, then the oblivious algorithm $A$ needs more than $k = \Omega(m)$ steps to broadcast a spanning tree of the graph $G = (V, E)$. This is because a $T$-witness has not been heard by the $k$th step as an endpoint of an edge.

In the remaining cases of dependencies between the numbers $m$ and $n$, we can proceed as follows. If $n(n - 1)/4 < m$, then we can use the same construction as if the number $m$ were equal to $n(n - 1)/4$. If $m < 2n$, then a simple path of length $\ell = \min\{n - 1, m\}$ does the job, because any algorithm needs to broadcast at least $\ell$ edges to reveal the whole path.

There is a simple oblivious algorithm that operates in time $m + 1$: the station $i$ broadcasts its edge at step $i$. Theorem 3 may be interpreted as follows: listing all the edges systematically is asymptotically optimal among oblivious algorithms.

**4 Weighted Graphs**

In this section we consider graphs that have positive weights assigned to their edges. The input is specified at the start of an execution, no adversaries are involved.
INPUT: edge $e_p$ of weight $w_p$;
INITIALIZATION: edge $e_p$ is waiting;
repeat
    $a := 1$ ; $weight := +\infty$ ;
A: repeat
    if ( $e_p$ is waiting) and ( $w_p < weight$) then broadcast $w_p$ with the probability $1/a$ ;
    begin case
        (a) some weight $w$ was heard : set weight := $w$ ;
        (b) collision was heard : set $a := 3a$ ;
        (c) silence was heard : set $a := \min[a/3, 1]$ ;
    end case ;
    if ( $e_p$ is waiting) and ( $w_p < weight$) then broadcast a dummy message ;
until silence was heard in the previous step ;
B: repeat
    if ( $e_p$ is waiting) and ( $w_p = weight$) then broadcast $w_p$ with the probability $1/a$ ;
    begin case
        (a) the edge $e_p$ was heard : change the status of $e_p$ to revealed ;
        (b) an edge different from $e_p$ was heard : check if $e_p$ became cycle, and if so then change its status to cycle ;
        (c) collision was heard : set $a := 3a$ ;
        (d) silence was heard : set $a := \min[a/3, 1]$ ;
    end case ;
    if ( $e_p$ is waiting) and ( $w_p = weight$) then broadcast a dummy message ;
until silence was heard in the previous step ;
if $e_p$ is waiting then broadcast a dummy message ;
until silence was heard in the previous step ;
OUTPUT: the set of the revealed edges.

Figure 3: Randomized algorithm RANDWEIGHTED for simple graphs with weights.
Code for the station $p$ that stores the input edge $e_p$ with weight $w_p$. 
4.1 Randomized Algorithm

We present a randomized algorithm that finds a minimum-weight spanning forest of a graph. It operates in expected time $O(|T| + W \log m)$, where $W$ is the number of different weights assigned to the edges in $T$. The algorithm is called RandWeighted, its pseudocode is given in Figure 3.

The algorithm is a generalization of RandSimple, it uses the same categories of edges, and the same variable $a$ serving as a stochastic estimate of a set of edges. There is an additional variable weight interpreted as an edge weight. Algorithm RandWeighted is a loop in which two inner loops A and B are executed. The purpose of the first loop A is to find the smallest weight of a waiting edge. This loop terminates with the variable weight storing this value. Then the next loop B follows in which a maximal set of edges is found such that each of these edges is of a weight equal to that stored in the variable weight, and all can be added to the current set of the revealed edges with the property that it is a forest being preserved.

Similarly as before, an edge is waiting if it would not create a cycle if added to the revealed part of a minimum-weight forest. We depart from the previous terminology in this section, in that an edge is said to be revealed only if it was broadcast as a minimum-weight one among those that were waiting. The remaining edges, broadcast during the selection in the inner loop A, are not categorized as revealed. This is in contrast with algorithms from the preceding sections, where every waiting edge broadcast successfully became revealed immediately.

Correctness. The algorithm RandWeighted is essentially an implementation on a multiple-access channel of a greedy minimum-spanning-forest algorithm (see [9]), hence a set of the revealed edges is a minimum-weight spanning forest of the input graph. The algorithm terminates with probability 1 because its expected time is finite, as we show next.

Performance. Our analysis of the behavior of algorithm RandWeighted is an extension of that for RandSimple.

Lemma 5 The randomized algorithm RandWeighted finds a minimum-weight spanning forest $T$ in expected time $O(|T| + W \log m)$, where $W$ is the number of distinct weights on the edges of $T$.

Proof: Consider an iteration of the main loop. The purpose of the first inner loop A is to find the minimum weight among all the waiting edges. There are two phenomena here that occur concurrently. One is a randomized binary search that governs the selection of weights. The expected number of selections made is $O(\log m)$, which follows from the fact that the expected height of a random binary-search tree is logarithmic in the number of its leaves, see for instance [9]. The other phenomenon is similar to the behavior of algorithm RandSimple, as modeled by a discrete random walk in the proof of Theorem 2. After each new weight has been broadcast, there is an adjustment needed to the variable $a$ to catch up with the decreasing size of the set of the edges of smaller weights, if there are still any. This corresponds to placing the particle at a place possibly distant from the origin in the terminology of the proof of Theorem 2. The expected total time spent on such catching up is $O(\log m)$, since the subsets keep decreasing. Except for that, the expected time spent on producing a new edge is $O(1)$. These two $O(\log m)$ bounds add up, and this is the cost of producing the first revealed edge, of the minimum weight among the waiting ones, when the second inner loop B is executed for the first time in the given iteration of the main loop. Loop B may need to be repeated more times as long as there are waiting edges of the same weight, but each of them...
is produced with expected time \( O(1) \) per edge within the same iteration of the main loop.

**Theorem 4** There is a randomized algorithm that finds a minimum-weight spanning forest \( T \) in expected time \( O(|T| + W \log m) \), where \( W \) is the number of distinct weights on the edges in \( T \), and in worst-case time \( O(m) \).

**Proof:** To guarantee the claimed worst-case performance, we apply a similar stratagem as in algorithm RandSimple for simple graphs. We have all the edges broadcast on the channel in a systematic way, until a forest of minimum weight has been found or all the edges have been exhausted, whatever comes first. More precisely, let the algorithm RandWeighted be run in odd-numbered steps, while in even-numbered steps the stations broadcast their edges and weights in order of their IDs. Unlike algorithm RandSimple, the processes should be independent of each other, in the sense that the edges broadcast in the odd-numbered steps are not categorized as revealed, cycle or waiting, until all have been exhausted and the algorithm stops with all the edges broadcasted. This allows us to combine the expected performance of RandWeighted given in Lemma 5 with a worst-case upper bound proportional to the number \( m \) of all the edges.

## 4.2 Lower Bound for Weighted Graphs

In this section we prove a lower bound \( \Omega(m) \) for deterministic adaptive algorithms for weighted graphs. The lower bound is of the same form as in Section 3.3, the difference is that the algorithms are adaptive rather than oblivious, while the graphs are weighted rather than simple.

For each deterministic adaptive algorithm \( A \), we construct a certain weighted graph \( G_A \) of \( n \) vertices and \( m \) edges. We start with any assignment of edges to the stations at the very beginning of computation, so that the graph \( G = (V, E) \) has no isolated vertices. Let \( \mathbb{N} \) be the set of positive integers. In the construction we use only weights from set \( \{1/j : j \in \mathbb{N}\} \). Our goal is to assign weights to all the edges. Each station \( i \) will have a set \( A_i(t) \) of numbers in \( \{1/j : j \in \mathbb{N}\} \) assigned to it after step \( t \) of the construction. Initially we set \( A_i(0) = \{1/j : j \in \mathbb{N}\} \), for every station \( i \).

Since there is a one-to-one correspondence between the edges and the stations, we treat \( E \) also as a set of stations. Denote by \( E(t) \) the set of stations which have an edge with some weight assigned to it by step \( t \) of algorithm \( A \). Each station \( i \) from \( E(t) \) has just one element in \( A_i(t) \). We assume the following invariant after step \( t \): for each station \( i \in E \setminus E(t) \), the set \( A_i(t) \) is infinite.

Consider step \( t+1 \) of algorithm \( A \), as determined by step \( t \) of the construction. Let \( S_0(t+1) \) denote the set of these stations \( i \) that do not broadcast during step \( t+1 \) of algorithm \( A \), for an infinite number of possible weights from \( A_i(t) \). Similarly, let \( S_1(t+1) \) denote the set of stations that broadcast in step \( t+1 \), for infinitely many possible initial values.

Step \( t+1 \) of the construction is broken into the following cases:

**Case 1:** \( S_1(t+1) \setminus S_0(t+1) = \emptyset \).

Make set \( A_i(t+1) \) contain all the weights from \( A_i(t) \), for which station \( i \) does not broadcast during step \( t+1 \) of algorithm \( A \), for each station \( i \in S_0(t+1) \). Hence \( E(t+1) = E(t) \) and the invariant holds after step \( t+1 \). Moreover, there are no new stations broadcasting on the channel, hence we can determine what happens in step \( t+1 \) by considering only stations in \( E(t) \).

**Case 2:** \( S_1(t+1) \setminus S_0(t+1) = \{i\} \).
We set $A_i(t + 1) = \{1/\ell\}$, where $1/\ell$ is the maximum weight in $A_i(t)$ for which station $i$ broadcasts in step $t + 1$ of algorithm $\mathcal{A}$. For the other stations $j$ in $E\setminus E(t)$, we set $A_j(t + 1) = A_j(t) \cap (0, 1/\ell)$. In this case $E(t + 1) = E(t) \cup \{i\}$ and the invariant holds after step $t + 1$. In step $t + 1$ only station $i$ broadcasts, among stations in $E\setminus E(t)$, hence by an argument similar to that used in Case 1, we have control over what happens in step $t + 1$.

**Case 3:** $|S_1(t + 1) \setminus S_0(t + 1)| > 1$.

We choose two different stations $i_1, i_2 \in S_1(t + 1) \setminus S_0(t + 1)$ and decide on the sets to be $A_{i_1}(t + 1) = \{1/\ell_1\}$ and $A_{i_2}(t + 1) = \{1/\ell_2\}$, similarly as for vertex $i$ in Case 2. For the other stations $j$ in $E\setminus E(t)$ we set $A_j(t + 1) = A_j(t) \cap (0, 1/\max(\ell_1, \ell_2))$. In step $t + 1$ only stations $i_1$ and $i_2$ broadcast, among all the stations in $E\setminus E(t)$, hence, by a similar argument as in Case 1, we can decide on the events in step $t + 1$.

This construction gives the following result:

**Theorem 5** For each adaptive deterministic algorithm $\mathcal{A}$ that finds a minimum spanning forest, and a possible number of edges $m$, there is a graph $G_\mathcal{A}$ such that algorithm $\mathcal{A}$ terminates after $\Omega(m)$ steps when given the graph $G_\mathcal{A}$ as input.

**Proof:** Consider the first $m/2$ steps and graph $G_\mathcal{A}$ constructed as described above. Our construction has the following properties. First, in one step of construction at most two stations (edges) can have assigned weights; each among such stations $i$ is moved to $E(t)$ and has $A_i(t)$ of unit size. Second, for any station $j$ not in $E(t)$ we have $|A_j(t)| = \infty$, hence in the next steps some sufficiently small weight could be assigned to $j$. It follows, by induction on the number of steps $t$ of algorithm $\mathcal{A}$ on graph $G_\mathcal{A}$, that if $t < m/2$ then an edge with the minimum weight has not been broadcast successfully on the channel. \hfill \square

## 5 Adversarial Environment

In this section we consider dynamic graphs with no weights assigned to their edges. There is an adversary who is able to decide on timing when the station holding a particular edge is activated. An activated station is aware of being activated at the given step, and of the number of the step, as counted by a global clock. Multiple stations may be activated at a step, or none. We assume that the global clock is started exactly at the first step of the algorithm, and that at least one station is active then.

The stations are activated by an adversary but they halt on their own. The issue of termination and correctness needs to be clarified precisely, since an adversary might activate a number of stations just before the stations already active have decided to terminate. We say that the algorithm terminates at the step when some station successfully broadcasts a special termination signal. Our approach to correctness is based on disregarding the edges held by the stations activated too late. To make this precise, we call the period of the last $c$ steps by termination the closing $c$ steps, for any fixed integer $c > 0$. An algorithm is said to be $c$-correct, for a positive integer $c$, if the output is a spanning forest for the input graph having the edges that are held by the stations activated before the closing $c$ steps. An algorithm is correct if it is $c$-correct for some integer $c > 0$ and a sufficiently large number $n$ of vertices. We consider only the $m$ edges held by the stations activated before
the last $c$ closing steps, where parameter $c$ is some fixed constant, as required in the definition of correctness.

First we prove a lower bound for deterministic adaptive algorithms in this model. This lower bound is strong in the sense that it holds for arbitrary graphs containing a forest of a given size.

**Theorem 6** For any positive constant $c$ and an adaptive deterministic $c$-correct algorithm $A$ that is able to find spanning forests in simple graphs, and for any positive integer $m$ that is sufficiently large depending on $c$, and for any simple forest $G$ with $m$ edges, there is such a strategy of an adversary to activate stations that forces algorithm $A$ to perform $\Omega(m \log m)$ steps on the input $G$ if run against the strategy.

**Proof:** For the number $m$ to be sufficiently large, the property $c < \lfloor \log(m/4) \rfloor$ suffices, as will be seen in the proof, where $\log x$ is the logarithm of $x$ to base 2. We partition an execution of algorithm $A$ into $\lceil m/8 \rceil$ stages, each lasting for $\lfloor \log(m/4) \rfloor$ consecutive steps. The adversary activates at least two, and up to four, stations holding specific edges precisely at the start of each stage. The chosen stations and edges are to have the following property: none of these at least two edges is broadcast successfully during the stage when they are activated.

Consider the beginning of the $k$th stage, where $k \leq \lfloor m/8 \rfloor$, and suppose that during the previous stages the execution has proceeded as required. In particular, a total of at most $4(k - 1)$ stations have been activated, and none of the edges activated $\lfloor \log(m/4) \rfloor$ steps ago, when the previous stage $k - 1$ started, has been heard by this step. The algorithm has not terminated yet because it is $c$-correct, the inequality $c < \lfloor \log(m/4) \rfloor$ holds and the edges recently activated have to be in a spanning forest of the input.

Prior to the beginning of a stage, we need a pool of passive stations holding edges to choose the ones to activate from. The specific fractions of $m$ we use serve the purpose of the proof for the following reason: each stage has at most four new edges activated, for a total of $m/2$ edges after $m/8$ stages, and before each stage there are at least $m - m/2 = m/2$ edges to choose from.

Let the next consecutive queries be $Q_i$, for $1 \leq i \leq \lfloor \log(m/4) \rfloor$. They are determined uniquely if we assume that there are no successful broadcasts of edges that have not been activated prior to this point. This property will be guaranteed by the construction. We proceed by considering sets $S_i$ and $E_i$, for $0 \leq i \leq \lfloor \log(m/4) \rfloor$. Initialize the set $S_0$ to the IDs of the stations still passive at this point, and the set $E_0$ to the edges held by these stations. The sets $S_i$ and $E_i$ are determined inductively. Suppose $S_i$ and $E_i$ have already been determined, then $S_{i+1}$ and $E_{i+1}$ are defined as follows. If $|Q_{i+1} \cap S_i| \geq |S_i|/2$, then set $S_{i+1} := Q_{i+1} \cap S_i$, otherwise set $S_{i+1} := Q_{i+1} - S_i$. Similarly, if $|Q_{i+1} \cap E_i| \geq |E_i|/2$, then set $E_{i+1} := Q_{i+1} \cap E_i$, otherwise set $E_{i+1} := Q_{i+1} - E_i$. The sets $S_i$ and $E_i$, for $i = \lfloor \log(m/4) \rfloor$, contain at least two elements each. Choose any two stations in the final set $S_i$ and any two stations in the final set $E_i$ to activate in the next stage.

When the last stage has been completed, there are still at least two stations that have been activated prior to the last $c$ steps and they hold edges that have to be in a spanning forest of the graph with edges held by all the stations active by this step. This means that the algorithm still needs to perform at least two more steps.

Theorem [6] can be strengthened to hold for any graph that contains a forest of $m$ edges, a proof is a straightforward modification, details are omitted.

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**Input:** the number $n$ of graph vertices; edge $e_p$;

**Initialization:** $\text{Revealed} := \emptyset, \text{Cycle} := \emptyset, e_p$ is waiting;

if this is the start of an execution then participate in $\text{Resolve}$ to elect a leader

else begin
    wait for the first update message;
    update $\text{Revealed}$;
    adjust the status of $e_p$
end;

repeat
    if $e_p$ is waiting then broadcast a dummy message;
    if ( silence was heard in the previous step ) and ( $p$ is a leader ) then broadcast a termination signal;
    \( \{ e \} := \text{Resolve}(\text{Waiting}) \);
    move edge $e$ from $\text{Waiting}$ to $\text{Revealed}$;
    if ( $e_p$ is waiting ) and ( there is a cycle in the graph induced by $e_p$ and the edges in $\text{Revealed}$ ) then move $e_p$ from $\text{Waiting}$ to $\text{Cycle}$;
    if $p$ is a leader then broadcast an update message;
until a termination signal was heard

**Output:** all the revealed edges.

Figure 4: Algorithm DETADVERSARIAL. Code for the station $p$ that holds the input edge $e_p$. 

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Next we give a deterministic adaptive algorithm \textsc{DetAdversarial}, whose performance matches the lower bound of Theorem 6. The design principle under which it operates is similar to that of the adaptive algorithm for the static case. The main difference is that now new stations can wake up at arbitrary steps and they need to be incorporated into an execution. When a station is activated after the start of an execution, it pauses and listens to the channel until it hears a special update message. This message carries a list of all the edges that have been revealed in the course of the execution so far. Such update messages are sent by one designated station, called a leader. The algorithm starts by having those stations that are active from the very beginning select a leader among themselves. This is achieved by running the procedure \textsc{Resolve}, and using IDs in it rather than edges. If a station \( p \) joins the execution at some point, after having been activated, then it waits for the first update message, then sets \textsc{Revealed} to the list obtained, and if there is a cycle in the graph induced by \( e_p \) and the edges in \textsc{Revealed}, then \( e_p \) becomes a cycle. A pseudocode of the algorithm is presented in Figure 4.

\begin{theorem}
\text{Algorithm \textsc{DetAdversarial} finds a spanning forest \( |T| \) in time \( \mathcal{O}(|T| \log m) \) and is 2-correct against any adversary.}
\end{theorem}

\begin{proof}
It is checked whether there is any waiting edge just before a possible termination signal is to be broadcast. A station that has been activated at least two steps before has a chance to receive the update message that was broadcast as the last action performed in the previous iteration of the main loop. It takes \( |T| \) calls of the procedure \textsc{Resolve} to contribute all the edges, each call takes the time \( \mathcal{O}(\log m) \).
\end{proof}

\section{Discussion}

This paper presents a study of the problem of finding a minimum-weight spanning forest in a distributed setting, for the model when single edges are held by stations that communicate by broadcasting on a multiple-access channel.

We show that adaptive deterministic algorithms are more efficient than oblivious ones, even for simple graphs without weights. Finding the optimum performance of a deterministic adaptive algorithm for simple graphs is an open problem. We claim that Theorem 1 actually gives the best possible bound.

We develop an optimal randomized algorithm for simple graphs without weights. It is an open problem if the performance of this algorithm can be matched by that of a deterministic one. We conjecture that this is not the case.

We also develop a randomized algorithm finding a minimum-weight spanning forest \( T \) of a graph in expected time \( \mathcal{O}(|T| + W \log m) \), where \( W \) is the number of distinctive weights on the edges of \( T \), and show that any deterministic one requires time \( \Omega(m) \). This shows that randomization helps for this problem, for sufficiently many edges. The optimality of finding a minimum-weight spanning forest by a randomized algorithm is an open problem.

We develop a deterministic algorithm for an adversarial environment that is time optimal, but its properties rely on a possibly large size of a message broadcast in a single step. An interesting problem is what is the optimum-time complexity of the problem in an adversarial model with the
size of messages restricted so that each can carry up to a constant number of edges or IDs of stations?

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