0. Introduction

In some recent papers ([2], [3]) a broad class of differential equations, the \textit{relaxed Dirichlet problems}, was introduced by G. Dal Maso and U. Mosco. The solutions of these problems describe the asymptotic behaviour of sequences of solutions to perturbed Dirichlet problems with homogeneous boundary conditions on varying domains as well as of Schrödinger equations with varying nonnegative potentials. These equations have the following formal expression

$$Lu + \mu u = \nu \quad \text{in } \Omega$$

(0.1)

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $L$ is a uniformly elliptic operator with bounded (Lebesgue) measurable coefficients in $\mathbb{R}^N$, $\mu$ belongs to the space $\mathcal{M}_0(\Omega)$ of all non-negative Borel measures on $\Omega$, which vanish on sets of zero capacity and $\nu$ is a Radon measure belonging to a suitable subspace $K_N(\Omega)$ of $H^{-1}(\Omega)$.

A variational Wiener criterion for these problems has been formulated in [2]; this criterion is inspired by the classical one of potential theory ([18]). The main result in [2] is the characterization of the \textit{regular Dirichlet points} of $\mu$ (i.e., the points $x_0$ of $\Omega$ such that every local weak solution $u$ of (0.1) is continuous at $x_0$ with value $u(x_0) = 0$) as the points where the \textit{Wiener modulus} of $\mu$, defined by

$$\omega(r, R) \equiv \exp \left( - \int_r^R \frac{\operatorname{Cap}_\mu(B_\rho(x_0), B_{2\rho}(x_0))}{\operatorname{Cap}(B_\rho(x_0), B_{2\rho}(x_0))} d\rho \right),$$

(0.2)

vanishes as $r \to 0^+$, for some fixed positive $R$.

In the same paper the necessity of the Wiener condition is proved when the dimension $N$ of the space is greater or equal to 2.

The proof of the sufficient condition is given by means of a joint estimate of the energy and the continuity modulus of local weak solutions for problem (0.1), in terms of the Wiener modulus, by making use of tools which require the hypothesis $N \geq 3$. Indeed, the proof of this estimate needs the equivalence between the Wiener criterion given in terms of annuli and in terms of balls.

This equivalence can be obtained directly when $N \geq 3$ by using the fact that the function $\gamma(\rho) = \operatorname{Cap}(B_\rho, B_{2\rho})$ is homogeneous of degree $N - 2$. The purpose of this paper is to give a proof of the previous estimate when $N \geq 2$, having in mind some techniques already used by M. Biroli and U. Mosco ([1]) in the case of obstacle problems for degenerate elliptic operators.

As a first step we will define the function

$$V(r) \equiv \sup_{B_r(x_0)} u^2 + \int_{B_r} |Du|^2 G_{x_0}^{\frac{2q}{q-1}} dx + \int_{B_r} u^2 G_{x_0}^{\frac{2q}{q-1}} d\mu,$$

(0.3)

where $0 < q < 1$ and $G_{x_0}^{\frac{2q}{q-1}}$ is the Green function, with singularity in $x_0$, of the Dirichlet problem for the operator $L$ in the ball $B_{\frac{2r}{q}}(x_0)$, and then we will establish the following estimate

$$V(r) \leq k\omega(r, R)^\beta V(R) + k\|\nu\|^2_{K_N(B_R)},$$

(0.4)
for any $0 < r < R$ ($R$ such that $B_{\frac{2R}{q}}(x_0) \subset \Omega$), where $k$ and $\beta$ are two positive constants and the norm of $\nu$ is taken in the Kato space $K_N(B_R)$.

We want to point out that the difference between our definition of $V(r)$ and the definition given in [2] is that in (0.3) we use the Green function relative to the ball $B_{\frac{2r}{q}}(x_0)$ instead of the fundamental solution in $\mathbb{R}^N$ for the Laplace operator. It is the presence of the Green function, together with the estimates connected with the maximum principle, that will allow us to obtain estimate (0.4) avoiding the comparison between the capacity of the balls and that of the annulus.

Then, as in [2], we obtain from (0.4) not only a proof of the sufficient Wiener condition, but also an estimate of the continuity modulus of the local weak solution of (0.1) in terms of the Wiener modulus. This estimate extends that one given by Maz'ja ([13] and [14]) in the case of regular boundary points for Dirichlet problems. In addition, we obtain also an estimate for the decay of the $\mu$–energy

$$\mathcal{E}_\mu(r) \overset{\text{def}}{=} \int_{B_r} |Du|^2 \, dx + \int_{B_r} u^2 \, d\mu,$$

in the ball $B_r$ as $r \to 0^+$.

Finally we specialize our result to the classical case, obtaining the continuity modulus estimate proved by Maz'ja and finding an energy decay estimate, at a point at the boundary, valid in dimension $N \geq 2$.

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1. Notation and preliminary results

In this paper $\Omega$ will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\bar{\Omega}$ its closure and $\partial \Omega$ its boundary.

1.1. Sobolev spaces

We denote by $H^{1,p}(\Omega)$, $1 \leq p < +\infty$, the Sobolev space of all functions $u \in L^p(\Omega)$ with distribution derivatives $D_i u \in L^p(\Omega)$, $i = 1, \ldots, N$. The space $H^{1,p}(\Omega)$ is endowed with the norm

$$\| u \|_{H^{1,p}(\Omega)} = \left( \| u \|_{L^p(\Omega)}^p + \| Du \|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

where $Du = (D_1 u, \ldots, D_N u)$ is the gradient of $u$. By $H^{1,p}_{\text{loc}}(\Omega)$ we denote the set of functions belonging to $H^{1,p}(\Omega')$ for every open set $\Omega' \subset \subset \Omega$. By $H^{1,p}_0(\Omega)$ we denote the closure of $C_0^1(\Omega)$ in $H^{1,p}(\Omega)$. As usual, for the space $H^{1,2}(\Omega)$, $H^{1,2}_{\text{loc}}(\Omega)$ and $H^{1,2}_0(\Omega)$ we use the notations $H^{1}(\Omega)$, $H^{1}_{\text{loc}}(\Omega)$ and $H^{1}_0(\Omega)$. Moreover by $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$ and by $\langle \cdot, \cdot \rangle$ the dual pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. 

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Finally, for every \( u \in H^1(\Omega) \) and for every \( E \) open subset of \( \Omega \), we denote by \( \text{osc}_E u = \sup_E u - \inf_E u \) the (essential) oscillation of \( u \) in \( E \) (i.e. the difference between the essential sup and the essential inf of \( u \) on \( E \)).

We will say that \( u \) is (essentially) continuous at \( x_0 \in \Omega \) if

\[
\lim_{\rho \to 0^+} \left( \text{osc}_{B_\rho(x_0)} u \right) = 0,
\]

where \( B_\rho(x_0) \) (or \( B(\rho, x_0) \)) is the ball of radius \( \rho \) and center \( x_0 \).

### 1.2. The harmonic capacity

Let \( A \) be an open subset of \( \Omega \). The harmonic capacity of \( A \) with respect to \( \Omega \) is defined by

\[
\text{Cap}(A, \Omega) \overset{\text{def}}{=} \inf \left\{ \int_\Omega |Du|^2 \, dx : u \in H^1_0(\Omega), u \geq \chi_A \text{ a.e. on } \Omega \right\},
\]

where \( \chi_A \) is the characteristic function of \( A \). If the set \( \{ u \in H^1_0(\Omega), u \geq \chi_A \text{ a.e. on } \Omega \} \) is empty, we define \( \text{Cap}(A, \Omega) = +\infty \). This definition can be extended to any subset \( E \) of \( \Omega \) in the following way:

\[
\text{Cap}(E, \Omega) \overset{\text{def}}{=} \inf \{ \text{Cap}(A, \Omega) : A \text{ open; } A \subseteq E \}.
\]

We say that a set \( E \) of \( \mathbb{R}^N \) has zero capacity if \( \text{Cap}(E \cap \Omega, \Omega) = 0 \) for every bounded open set \( \Omega \) of \( \mathbb{R}^N \). Then, we say that a property holds quasi-everywhere in a set \( S \) (q.e. in \( S \)), if it holds in \( S - E_0 \), where \( E_0 \) is subset of \( S \) with capacity zero.

We recall that for every function \( u \) of \( H^1_{\text{loc}}(\Omega) \), it is possible to find a quasi-continuous representative. Then the limit

\[
\lim_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(x_0)} u(x) \, dx
\]

exists and is finite quasi-everywhere in \( \Omega \) (\( |B_\rho| \) is the Lebesgue measure of \( B_\rho(x_0) \)). Therefore we can determine the pointwise value of \( u \in H^1(\Omega) \) using, for every \( x_0 \in \Omega \), the following convention:

\[
\liminf_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(x_0)} u(x) \, dx \leq u(x_0) \leq \limsup_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(x_0)} u(x) \, dx.
\] (1.1)

If \( \Omega \) is bounded it is possible to prove that

\[
\text{Cap}(E, \Omega) = \inf \left\{ \int_\Omega |Du|^2 \, dx : u \in H^1_0(\Omega), u \geq 1 \text{ q.e. on } E \right\},
\]
for any set $E \subset \Omega$. The function $u \in H^1_0(\Omega)$ that realizes the minimum is said the **capacitary potential** of $E$ in $\Omega$. It is easy to prove that $u = 1$ q.e. in $E$ and $-\Delta u = 0$ in $\Omega - E$.

1.3. The Green function

Let us consider a second order elliptic differential operator in divergence form

$$Lu = -\sum_{i,j=1}^{N} D_j (a_{ij} D_i u),$$

(1.2)

where $a_{ij}$, $i, j = 1, \ldots, N$, are measurable, real valued functions such that

$$\exists \Lambda > 0 : |a_{ij}(x)| \leq \Lambda \text{ a.e. in } \Omega,$$

(1.3)

and that the following uniformly elliptic condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^N,$$

(1.4)

holds for some $\lambda > 0$. The bilinear form on $H^1(\Omega)$ associated to $L$ is denoted by

$$a(u, v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) D_i u D_j v dx.$$

We define the **Green function** $G(x, y)$ (or $G^y(x)$) for the Dirichlet problem in $\Omega$ relative to the operator $L$ as the unique solution in $H^1_0(\Omega)$, for the equation

$$a(\phi, G^y) = \phi(y), \quad \forall \phi \in H^1_0(\Omega) \cap C(\Omega) \text{ with } L\phi \in C(\Omega).$$

It is well known that $G(x, y) \in H^1(\Omega - B_r(y))$ for every $r > 0$, that it is Hölder continuous on every compact subset of $\Omega \times \Omega - \{(y, y) : y \in \Omega\}$ and that vanishes q.e. on $\partial \Omega$. Moreover, for every measure $\mu \in H^{-1}(\Omega)$, the function

$$u(y) = \int_{\Omega} G(x, y) d\mu(x)$$

is the unique solution in $H^1_0(\Omega)$ of the equation

$$a(u, \phi) = \int_{\Omega} \phi d\mu, \quad \forall \phi \in C^1_0(\Omega).$$
We also recall that, if \( \Omega \) is a ball, say \( \Omega = B_R(x_0) \), for every \( 0 < q < 1 \) there exists a constant \( K > 0 \), depending only on \( q \) and \( N \), such that for every \( y \in B_R(x_0) \) and \( r > 0 \), with \( B_{q^r}(y) \subset B_R(x_0) \), and for every \( x \in \partial B_r(y) \) the following estimate holds

\[
\frac{\Lambda^{-1}K^{-1}}{\text{Cap}(B_r(y), B_R(x_0))} \leq G(x, y) \leq \frac{\lambda^{-1}K}{\text{Cap}(B_r(y), B_R(x_0))},
\]

where \( \lambda \) and \( \Lambda \) are the ellipticity constants of \( L \). Moreover there exists a positive constant \( \alpha \), depending only on \( \frac{\Lambda}{\lambda} \) and \( N \), such that, for every \( x, y \in B_R(x_0) \),

\[
G(x, y) \leq \frac{\alpha}{\lambda} |x - y|^{2-N}
\]

if \( N \geq 3 \), and

\[
G(x, y) \leq \frac{\alpha}{\lambda} \log \frac{4R}{|x - y|}
\]

if \( N = 2 \). For the main properties of the Green function and for a proof of estimate (1.5) see \cite{16}, \cite{12} and \cite{5}.

Let us return to an arbitrary bounded open set \( \Omega \). For every \( y \in \Omega \) and \( \rho > 0 \) such that \( B(x, \rho) \subset \Omega \), we define the approximate Green function \( G^\rho(x, y) \) (or \( G^y_\rho(x) \)) as the unique solution in \( H^{1}_0(\Omega) \) of the equation

\[
a(v, G^\rho_\rho) = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} v(x) \, dx, \quad \forall v \in H^1_0(\Omega).
\]

Thanks to De Giorgi-Nash theorem, \( G^y_\rho \) is Hölder continuous for every \( \rho > 0 \) and \( G^y_\rho \) converges uniformly to \( G^y \), as \( \rho \) tends to zero, on every compact subset of \( \Omega - \{y\} \).

### 1.4. Kato measures

The Kato space \( K_N(\Omega) \) is the set of all Radon measures \( \nu \) on \( \Omega \) such that

\[
\lim_{r \to 0^+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |y - x|^{2-N} \, d|\nu|(y) = 0
\]

if \( N \geq 3 \), and

\[
\lim_{r \to 0^+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} \log \frac{1}{|y - x|} \, d|\nu|(y) = 0
\]

if \( N = 2 \), where \( |\nu| \) is the total variation of \( \nu \). With \( K^{\text{loc}}_N(\Omega) \) we denote the set of Radon measures \( \nu \) on \( \Omega \) such that \( \nu \in K_N(\Omega') \) for every open set \( \Omega' \subset \subset \Omega \). In \( K_N(\Omega) \) we can define the following norms

\[
\| \nu \|_{K_N(\Omega)} \overset{\text{def}}{=} \sup_{x \in \Omega} \int_{\Omega} |y - x|^{2-N} \, d|\nu|(y),
\]

\[
\| \nu \|_{K^{\text{loc}}_N(\Omega)} \overset{\text{def}}{=} \sup_{\Omega' \subset \subset \Omega} \int_{\Omega'} |y - x|^{2-N} \, d|\nu|(y).
\]
\[ \| \nu \|_{K^2(\Omega)} = \sup_{x \in \Omega} \int_{\Omega} \frac{1}{|y - x|} d|\nu|(y), \]

the former when \( N \geq 3 \) and the latter when \( N = 2 \). With this norm \( K_N(\Omega) \) is a Banach space (see [2], Proposition 4.6). From the definition of \( K_N(\Omega) \) it follows that

\[ \lim_{r \to 0^+} \| \nu \|_{K_N(B_r(x))} = 0, \]

for every \( \nu \in K_N(\Omega) \) and \( x \in \Omega \). Moreover if \( \nu \in K_N(\Omega) \), then \( \nu \in H^{-1}(\Omega) \) and

\[ \| \nu \|_{H^{-1}(\Omega)} \leq k \| \nu \|_{K_N(\Omega)}, \]

where \( k \) is a positive constant depending only on the dimension of the space, i.e. \( K_N(\Omega) \subset H^{-1}(\Omega) \) with continuous imbedding.

**1.5. Relaxed Dirichlet problems**

By \( \mathcal{M}_0(\Omega) \) we denote the set of non-negative Borel measures \( \mu \) on \( \Omega \) such that \( \mu(E) = 0 \) for every Borel subset \( E \) of \( \Omega \) of capacity zero.

The problem we consider have the following formal expression

\[ Lu + \mu u = f \quad \text{in } \Omega, \]

where \( \mu \in \mathcal{M}_0(\Omega) \), \( f \in H^{-1}_{loc}(\Omega) \) and \( L \) is the operator defined in 1.3. These are called relaxed Dirichlet problems.

**Definition 1.1.** We say that \( u \) is a local weak solution of the relaxed problem

\[ Lu + \mu u = f \quad \text{in } \Omega \]  

\[(1.8)\]

if

\[ u \in H^1_{loc}(\Omega) \cap L^2_{loc}(\Omega, \mu) \]

and it satisfies the following variational equation

\[ a(u, v) + \int_{\Omega} uv \, d\mu = \langle v, f \rangle \]  

\[(1.9)\]

for every \( v \in H^1(\Omega) \cap L^2(\Omega, \mu) \) with compact support in \( \Omega \).

In (1.9) and in all other expressions of this kind we always choose the quasi-continuous representatives for \( u \) and \( v \). So, since \( \mu \in \mathcal{M}_0(\Omega) \) (i.e. it vanishes on set of zero capacity), the integral with respect to \( \mu \) is well defined and does not depend on the choice of such representative.

**Definition 1.2.** Given \( g \in H^1(\Omega) \) and \( f \in H^{-1}_{loc}(\Omega) \), we will say that \( u \) is a weak solution of the relaxed Dirichlet problem

\[
\begin{cases}
Lu + \mu u = f & \text{in } \Omega \\
u = g & \text{on } \partial\Omega
\end{cases}
\]

\[(1.10)\]
if \( u \) is a local weak solution of (1.8) and \( u - g \in H^1_0(\Omega) \).

**Theorem 1.1.** Suppose that \( f \in H^{-1}(\Omega) \) and that there exists a function \( w \in H^1(\Omega) \cap L^2(\Omega, \mu) \) such that \( w - g \in H^1_0(\Omega) \). Then problem (1.10) has one and only one weak solution \( u \). Moreover we have \( u \in H^1(\Omega) \cap L^2(\Omega, \mu) \) and

\[
a(u, v) + \int_{\Omega} uv \, d\mu = \langle v, f \rangle
\]

for every \( v \in H^1_0(\Omega) \cap L^2(\Omega, \mu) \).

If \( L \) is a symmetric operator, the solution \( u \) is the unique minimum point of the functional

\[
F(v) = a(v, v) + \int_{\Omega} v^2 \, d\mu - 2 \langle f, v \rangle
\]

in the set \( V(g) = \{ v \in H^1(\Omega) : v - g \in H^1_0(\Omega) \} \).

**Proof.** See [2], Theorem 2.4 and Proposition 2.5. \( \square \)

### 1.6. The \( \mu \)-capacity

Let \( \mu \) belong \( \mathcal{M}_0(\Omega) \) and let \( L \) be the elliptic operator defined in 1.3. Let \( E \subseteq \Omega \) be a Borel set and let \( \mu_E \) be a Borel measure in \( \Omega \) such that, for every Borel set \( B \subseteq \Omega \), \( \mu_E(B) = \mu(B \cap E) \). If \( \mu \in \mathcal{M}_0(\Omega) \), then \( \mu_E \in \mathcal{M}_0(\Omega) \) for every Borel subset \( E \) of \( \Omega \).

**Definition 1.3.** For every Borel set \( E \subseteq \Omega \), the \( \mu \)-capacity of \( E \) in \( \Omega \) is defined by

\[
\text{Cap}_\mu(E, \Omega) \overset{\text{def}}{=} \min \left\{ \int_{\Omega} |Du|^2 \, dx + \int_{\Omega} u^2 \, d\mu_E : u - 1 \in H^1_0(\Omega) \right\}.
\]

We recall the main properties of the \( \mu \)-capacity.

**Proposition 1.1.** Let \( u, v \in \mathcal{M}_0(\Omega) \), let \( E \) and \( F \) be Borel subsets of \( \Omega \) and let \( \Omega' \) be an open subset of \( \Omega \). Then

(a) \( 0 = \text{Cap}_\mu(\emptyset, \Omega) \leq \text{Cap}_\mu(E, \Omega) \leq \text{Cap}(E, \Omega) \);  
(b) if \( E \subseteq F \) then \( \text{Cap}_\mu(E, \Omega) \leq \text{Cap}_\mu(F, \Omega) \);  
(c) \( \text{Cap}_\mu(E \cup F, \Omega) + \text{Cap}_\mu(E \cap F, \Omega) \leq \text{Cap}_\mu(E, \Omega) + \text{Cap}_\mu(F, \Omega) \);  
(d) if \( E \subseteq \Omega' \subseteq \Omega \) then \( \text{Cap}_\mu(E, \Omega) \leq \text{Cap}_\mu(E, \Omega') \);  
(e) if \( \mu \leq \nu \) then \( \text{Cap}_\mu(E, \Omega) \leq \text{Cap}_\nu(E, \Omega) \).

Finally we recall a Poincaré inequality, involving \( \mu \)-capacity, that will be useful in the following.

**Theorem 1.2.** There exists a constant \( k > 0 \), depending only on \( N \), such that, given \( x_0 \in \mathbb{R}^N \) and \( r > 0 \), the following inequality holds for every \( u \in H^1(B_r(x_0)) \)

\[
\int_{B_r} u^2 \, dx \leq \frac{kr^N}{\text{Cap}_\mu(B_{2r}, B_{2r})} \left[ \int_{B_r} |Du|^2 \, dx + \int_{B_r} u^2 \, d\mu \right].
\]
For a complete treatment of the arguments in 1.4, 1.5 and 1.6 see [2] and [3].

2. Wiener Criterion for relaxed problem

We are going to study the behaviour of the local weak solution of a given relaxed problem in some special points: the regular Dirichlet points.

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^N \). Let \( L \) be the second order elliptic operator in divergence form defined by (1.2) with bounded coefficients satisfying conditions (1.3) and (1.4).

Let \( \mu \in \mathcal{M}_0(\Omega) \), \( x_0 \in \Omega \) and \( R_0 > 0 \) such that \( B_{R_0}(x_0) \subset \Omega \).

**Definition 2.1.** We say that \( x_0 \in \Omega \) is a regular Dirichlet point for the measure \( \mu \) in \( \Omega \) if every local weak solution \( u \), in an arbitrary neighbourhood of \( x_0 \), of the equation

\[
Lu + \mu u = 0,
\]

is continuous in \( x_0 \) and \( u(x_0) = 0 \).

We recall that for the definition of the pointwise value of \( u \) we use the convention (1.1).

**Definition 2.2.** For every \( 0 < r \leq R \leq R_0 \), we define the Wiener modulus of \( \mu \) in \( x_0 \) by

\[
\omega(r, R) = \exp \left( - \int_r^R \delta(\rho) \frac{d\rho}{\rho} \right),
\]

where

\[
\delta(\rho) = \frac{\text{Cap}_{\mu}(B_\rho, B_{2\rho})}{\text{Cap}(B_\rho, B_{2\rho})}
\]

for every \( 0 < \rho < R_0 \).

**Remark 2.1.** It is easy to verify that \( 0 \leq \delta(\rho) \leq 1 \) for every \( \rho > 0 \) and that \( \frac{r}{R} \leq \omega(r, R) \leq 1 \) for every \( 0 < r < R \).

**Definition 2.3.** We say that \( x_0 \in \Omega \) is a Wiener point for the measure \( \mu \) if

\[
\lim_{r \to 0^+} \omega(r, R) = 0,
\]

for some \( R > 0 \) or, equivalently, if the following Wiener condition for the measure \( \mu \) in \( x_0 \) holds

\[
\int_0^R \delta(\rho) \frac{d\rho}{\rho} = +\infty.
\]
As mentioned in the introduction, the proof of Theorem 2.1 was given in [2]. The proof of the necessity of the Wiener condition is valid for $N \geq 2$, but the proof of the sufficiency given in [2] holds only for $N \geq 3$. In the next section we shall prove an energy estimate (Thm 3.1) which is valid in the general case $N \geq 2$, and from which the sufficiency of the Wiener condition can be obtained immediately (Thm 3.2).

We want to point out that the notion of Wiener point for the measure $\mu$ does not depend on the operator $L$. Therefore, Theorem 2.1 implies that the notion of regular Dirichlet point is independent of $L$.

3. Energy estimate

In this section an energy estimate, similar to that one given in [2], is proved under the general hypothesis $N \geq 2$.

Let $u$ be a local weak solution of the problem

$$Lu + \mu u = \nu \quad \text{in } \Omega,$$

where $\Omega$ is a bounded open set of $\mathbb{R}^N$, $L$ is the elliptic operator defined by (1.2), (1.3) and (1.4), $\mu \in \mathcal{M}_0(\Omega)$ and $\nu \in K_{loc}^{N}(\Omega)$. We fix a point $x_0 \in \Omega$ and a radius $R_0 > 0$ such that $B_{R_0} \subseteq \Omega$. For every $\rho > 0$ we denote $B_{\rho}(x_0)$ by $B_\rho$; the Green function for the Dirichlet problem relative to the operator $L$ in the ball $B_\rho(x_0)$ with singularity at $x$ will be denoted by $G_{B_\rho}^x(y)$ (or $G_{B(x_0,\rho)}^x$).

**Definition 3.1.** Let $q \in (0, \frac{1}{5m})$ be fixed with $m \geq 1$. For every $r$ such that $0 < \frac{2r}{q} < R_0$, we define the function

$$V(r) \overset{\text{def}}{=} \sup_{B_r(x_0)} u^2 + \int_{B_r} |Du|^2 G_{B_\rho}^x dx + \int_{B_r} u^2 G_{B_\rho}^x d\mu.$$

**Theorem 3.1.** There exist two constants $k > 0$ and $\beta > 0$, depending only on $q$, $\lambda$, $\Lambda$ and $N$, such that

$$V(r) \leq k\omega(r,R)^{\beta}V(R) + k \| \nu \|_{K_N(B_R)}^2,$$

for every $0 < r < R < \frac{2R}{q} \leq R_0$.

Before proving this theorem we note that as a consequence of Theorem 3.1 and of Lemma 3.2 we obtain the following result, with the same proof as in [2].

**Theorem 3.2.** If $x_0$ is a Wiener point for the measure $\mu$, then

$$\lim_{r \to 0^+} V(r) = \lim_{x \to x_0} u(x) = u(x_0) = 0.$$

If $x_0$ is a Wiener point for the measure $\mu$, then $u$ is continuous at $x_0$ and $u(x_0) = 0$. This result holds in particular for solutions of (3.1) with $\nu = 0$. Then Theorem 3.1 proves also the sufficient condition in the Wiener Criterion.
In this section $k$ will denote a positive constant, independent of $r$ and $R$, that can assume different values. We state some lemmas that will be useful in the proof of Theorem 3.1.

**Lemma 3.1.** For every $0 < q < 1$ there exists a constant $k > 0$, depending only on $q$, $\lambda$, $\Lambda$ and $N$, such that

$$
\sup_{x \in B_{qR}(x_0)} |u| \leq k \left( \frac{1}{R^2} \int_{B_R - B_qR} u^2 \, dx \right)^{\frac{1}{2}} + k \| \nu \|_{K_N(B_R)},
$$

for every $0 < R \leq R_0$.

**Lemma 3.2.** For every fixed $0 < R < 2R_0$ and for every $q$ such that $0 < q < 1$ there exists a constant $k > 0$, depending only on $q$, $\lambda$, $\Lambda$ and $N$, such that

$$
V(qR) \leq k \frac{1}{R^N} \int_{B_R - B_qR} u_2^2 \, dx + k \| \nu \|_{K_N(B_R)}^2.
$$

For the proofs of Lemmas 3.1 and 3.2 see [2]. We give here, for the sake of completeness, the proof of the following lemma given in [1] for the case of obstacle problems with elliptic degenerate operators.

**Lemma 3.3.** For quasi every $z$ in $\Omega$ and $R > 0$ such that $B_R(z) \subseteq \Omega$, for every $\gamma > 0$ we have

$$
2\lambda \int_{B_{\rho R}(z)} |Du|^2 G^{\rho^z}_{B(tR,z)} \, dx + (u(z))^2 \leq (2 + \gamma) \sup_{B_{tR}(z)} u^2 + \frac{A}{\gamma} \int_{B_{tR}(z) - B_{\rho R}(z)} |Du|^2 G^{\rho^z}_{B(tR,z)} \, dx + \frac{\alpha^2}{\lambda^2} \| \nu \|_{K_N(B_R(z))}^2,
$$

with $t \in (1, \frac{1}{2})$, $p < \frac{2}{3}t$, $A$ is a positive constant depending only on $\lambda$, $\Lambda$ and $N$ and $\alpha$ is the constant appearing in (1.6) and (1.7).

**Proof.** Let $G^\rho_{\rho^z}$ be the approximate Green function for $G^\rho_{B(tR,z)}$. Consider $v = uG^\rho_{\rho^z}^z\varphi$ with $\rho < \frac{1}{2}R$ and $\varphi$ the capacitary potential of $B(pR, z)$ in $B(tR, z)$ for the operator $L$, i. e., $\varphi \in H^1_0(B_{tR}(z))$, $\varphi \geq 1$ q.e. on $B_{pR}(z)$, and

$$
a(\varphi, \psi - \varphi) \geq 0 \quad \forall \psi \in H^1_0(B_{tR}(z)) \text{ with } \psi \geq 1 \text{ q.e. on } B_{pR}(z).
$$

It turns out that $\varphi = 1$ q.e. on $B_{pR}(z)$ and that $\varphi \geq 0$ q.e. on $B_{tR}(z)$ (see [16]). Since $u \in H^1_0(B_{tR}(z)) \cap L^2(B_{tR}(z), \mu) \cap L^\infty(B_{tR}(z), \mu)$ (Lemma 3.1) and $G^\rho_{\rho^z} \in H^1_0(B_{tR}(z)) \cap L^\infty(B_{tR}(z), \mu)$, then $v \in H^1_0(B_{tR}(z)) \cap L^2(B_{tR}(z), \mu)$ and $v$ has compact support in $\Omega$ provided that we extend it to all $\mathbb{R}^N$ in the trivial way. We can use $v$ as test function in the variational equation verified by $u$, obtaining

$$
\sum_{i,j=1}^N \int_{B_{tR}(z)} a_{ij} D_i u D_j u \varphi G^\rho_{\rho^z} \, dx + \sum_{i,j=1}^N \int_{B_{tR}(z)} a_{ij} D_i u D_j G^\rho_{\rho^z} \varphi u \, dx =
$$
= \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu - \int_{B_{tR}(z)} u^2 G_{\rho}^z \varphi \, d\mu - \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i u D_j \varphi uG_{\rho}^z \, dx \
\leq \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu - \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i u D_j \varphi uG_{\rho}^z \, dx.

(3.3)

By the definition of $G_{\rho}^z$, we have

$$\frac{1}{|B_{\rho}|} \int_{B_{\rho}(z)} u^2 \, dx = \frac{1}{|B_{\rho}|} \int_{B_{\rho}(z)} u^2 \varphi \, dx = \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i u D_j \varphi uG_{\rho}^z \, dx =$$

$$= 2 \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i u D_j \varphi uG_{\rho}^z \, dx + \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i \varphi D_j (G_{\rho}^z u^2) \, dx. \quad (3.4)$$

From (1.4), (3.3), and (3.4) it follows

$$\lambda \int_{B_{tR}(z)} |Du|^2 \varphi G_{\rho}^z \, dx + \frac{1}{|B_{\rho}|} \int_{B_{\rho}(z)} u^2 \, dx \leq \frac{1}{2} \sum_{i,j=1}^{N} \int_{B_{tR}(z)} a_{ij} D_i \varphi D_j (G_{\rho}^z u^2) \, dx +$$

$$- \sum_{i,j=1}^{N} \int_{B_{tR}(z)} (a_{ij} + a_{ji}) D_i u D_j \varphi uG_{\rho}^z \, dx + \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu =$$

$$= \frac{1}{2} a(\varphi, u^2 G_{\rho}^z) + \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu - \sum_{i,j=1}^{N} \int_{B_{tR}(z)} (a_{ij} + a_{ji}) D_i u D_j \varphi uG_{\rho}^z \, dx. \quad (3.5)$$

Now we know that $u^2 G_{\rho}^z \in H_0^1(B_{tR}(z))$, $L \varphi \geq 0$, $G_{\rho}^z \geq 0$, $\varphi = 1$ q.e. in $B_{pR}(z)$; thus, using the definition of $G_{\rho}^z$, we obtain

$$a(\varphi, u^2 G_{\rho}^z) \leq a(\varphi, G_{\rho}^z) \sup_{B_{tR}(z)} u^2 = \sup_{B_{tR}(z)} u^2 \frac{1}{|B_{\rho}|} \int_{B_{\rho}} \varphi \, dx = \sup_{B_{tR}(z)} u^2. \quad (3.6)$$

Therefore from (3.5) and (3.6) we have

$$2\lambda \int_{B_{tR}(z)} |Du|^2 \varphi G_{\rho}^z \, dx + \frac{1}{|B_{\rho}|} \int_{B_{\rho}(z)} u^2 \, dx \leq$$

$$\leq \sup_{B_{tR}(z)} u^2 + 2 \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu - 2 \sum_{i,j=1}^{N} \int_{B_{tR}(z)} (a_{ij} + a_{ji}) D_i u D_j \varphi uG_{\rho}^z \, dx. \quad (3.7)$$

We can estimate from above the absolute value of the second term on the right-hand side of (3.7) as follows

$$\left| \int_{B_{tR}(z)} uG_{\rho}^z \varphi \, d\nu \right| \leq \sup_{B_{tR}(z)} |u| \int_{B_{tR}(z)} G_{\rho}^z d|\nu|. \quad (3.8)$$
Now we define
\[ w(y) = \int_{B_tR(z)} G_R(x, y) d|\nu|(x), \]
where \( G_R \) is the Green function for the Dirichlet problem in \( B_R(z) \) with the operator \( L \). The function \( w \) is the solution in \( B_R(z) \) of the equation \( Lw = |\nu|_t \), where \( \nu_t = \nu(E \cap B_tR(z)) \) for every Borel set \( E \subseteq B_tR(z) \). Then, using (1.6) and (1.7), we get
\[ 0 \leq w(x) \leq \frac{\alpha}{\lambda} \| \nu \|_{K_N(B_R(z))} \quad \text{q.e. in } B_tR(z). \]

From (3.8) we obtain
\[
\left| \int_{B_tR(z)} uG^z_{\rho} \varphi \, d\nu \right| \leq \sup_{B_tR(z)} |u| \sum_{i,j=1}^N \int_{B_R} a_{ij} D_i w D_j G^z_{\rho} \, dx =
\]
\[ = \sup_{B_tR(z)} |u| \frac{1}{|B_{\rho}|} \int_{B_{\rho}(z)} w \, dx \leq \sup_{B_tR(z)} |u| \frac{\alpha}{\lambda} \| \nu \|_{K_N(B_R(z))}. \]

Finally, we estimate the last term of (3.7) using the Young inequality, the boundedness of the coefficients of \( L \), and the fact that \( |D\varphi| = 0 \) q.e. in \( B_{pR}(z) \):
\[
-2 \sum_{i,j=1}^N \int_{B_tR(z)} (a_{ij} + a_{ji}) D_i u D_j \varphi G^z_{\rho} \, dx \leq
\]
\[ \leq 4N\lambda \int_{B_tR(z) - B_{pR}(z)} |Du||D\varphi||u|G^z_{\rho} \, dx \leq
\]
\[ \leq 2N\lambda \eta \int_{B_tR(z) - B_{pR}(z)} |D\varphi|^2 u^2 G^z_{\rho} \, dx + \frac{2N\lambda}{\eta} \int_{B_tR(z) - B_{pR}(z)} |Du|^2 G^z_{\rho} \, dx, \]
where \( \eta > 0 \) is an arbitrary positive constant. As \( \varphi \) is the \( L \)-capacitary potential of \( B_{pR}(z) \) in \( B_tR(z) \), there exists a constant \( k \), depending only on \( N, \lambda, \Lambda \) such that
\[
\int_{B_tR(z) - B_{pR}(z)} |D\varphi|^2 \, dx \leq \frac{1}{\lambda} \sum_{i,j=1}^N \int_{B_tR(z)} a_{ij} D_i \varphi D_j \varphi \, dx \leq k\text{Cap}(B_{pR}, B_tR)
\]
(see [16]). In the estimates obtained up to now we can pass to the limit as \( \rho \to 0^+ \). From (3.7), applying the maximum principle to \( G^z_{B(tR,z)} \), that is \( L \)-harmonic in the annulus \( B_tR(z) - B_{pR}(z) \), we obtain
\[
2\lambda \int_{B_{pR}(z)} |Du|^2 G^z_{B(tR,z)} \, dx + u^2(z) \leq
\]
\[ \leq \sup_{B_tR(z)} u^2 + 2N\lambda \eta \sup_{B_tR(z)} u^2 \sup_{\partial B_{pR}(z)} G^z_{B(tR,z)} \text{Cap}(B_{pR}, B_tR) + \]

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\[
+ \frac{2N\Lambda}{\eta} \int_{B_{tR}(z) - B_{pR}(z)} |Du|^2 G^\gamma_{B(tR,z)} \, dx + 2 \sup_{B_{tR}(z)} \left| u \frac{\alpha}{\lambda} \right| \nu \|K_N(B_{R}(z)) \), \quad (3.9)
\]
where for the pointwise value of \( u \) we use convention (1.1). Then it follows by (1.5)

\[
2\lambda \int_{B_{pR}(z)} |Du|^2 G^\gamma_{B(tR,z)} \, dx + u^2(z) \leq (2 + \gamma) \sup_{B_{tR}(z)} u^2 + \frac{A}{\gamma} \int_{B_{tR}(z) - B_{pR}(z)} |Du|^2 G^\gamma_{B(tR,z)} \, dx + \alpha^2 \frac{\lambda}{\lambda^2} \nu \|K_N(B_{R}(z)) \),
\]
where \( \gamma = 2N\Lambda \lambda^{-1} \alpha \eta \) and \( A = 4N^2 \lambda^2 \lambda^{-1} \alpha \).

Since \( \eta > 0 \) is arbitrary this concludes the proof of the lemma. \( \square \)

Finally we recall the following integration lemma. For the proof see e.g. [15].

**Lemma 3.4.** Let \( V(\rho) \) be a non-decreasing function of \( \rho \in (0, R], R > 0 \) and \( \delta(\rho) \) be a function such that \( 0 \leq \delta(\rho) \leq 1 \). Let \( q \) and \( k \) be two constants such that \( 0 < q < 1, k > 0 \) and let \( 0 < r < qR \). Suppose that

\[
V(q \rho) \leq \frac{1}{1 + k \delta(\rho)} V(\rho), \quad (3.10)
\]

for every \( \rho \in \left[ \frac{r}{q}, R \right] \). Then we have

\[
V(r) \leq k_0 \exp \left( -\beta |\log q|^{-1} \int_r^R \delta(\rho) \frac{d\rho}{\rho} \right) V(R),
\]

where \( \beta = \frac{k}{1+k} \) and \( k_0 = \exp(\beta) \).

**Proof of Theorem 3.1.** We choose in (3.2) \( t = \frac{1}{m} - q, p = 2q, \) with \( q \in (0, \frac{1}{sm}) \) and \( m \geq 1 \) so that \( q < p < t \) and \( t + q \leq 1 \). Then for every \( z \in B_{qR}(x_0) \) we have \( B_{tR}(z) \subset B_R(x_0) \) and

\[
\sup_{z \in B_{qR}(x_0)} \sup_{B_{tR}(z)} u^2 \leq \sup_{B_R(x_0)} u^2.
\]

We take in (3.2) the supremum for \( z \in B_{qR}(x_0) \) and obtain:

\[
\sup_{z \in B_{qR}(x_0)} u^2 \leq (2 + \gamma) \sup_{B_R(x_0)} u^2 +
\]

\[
+ \frac{A}{\gamma} \sup_{z \in B_{qR}(x_0)} \sup_{\partial B_{pR}(z)} G^\gamma_{B_{tR}(z)} \int_{B_{tR}(z) - B_{pR}(z)} |Du|^2 dx + \frac{\alpha^2}{\lambda^2} \nu \|K_N(B_{R}(z)) \leq
\]

\[
\leq (2 + \gamma) \sup_{B_R(x_0)} u^2 + \frac{A \lambda^{-1} K R^{2-N}}{\gamma \text{Cap}(B_p, B_t)} \int_{B_R(x_0) - B_{qR}(x_0)} |Du|^2 dx +
\]

\[
\frac{2N\Lambda}{\eta} \int_{B_{tR}(z) - B_{pR}(z)} |Du|^2 G^\gamma_{B(tR,z)} \, dx + 2 \sup_{B_{tR}(z)} \left| u \frac{\alpha}{\lambda} \right| \nu \|K_N(B_{R}(z)) \),
\]
\[ + \frac{\alpha^2}{\lambda^2} \| \nu \|^2_{KN(B_R)} . \] (3.11)

For the last inequality we used the estimate (1.5) for the Green function and the fact that \( B_{pR}(z) \supset B_{qR}(x_0) \) for every \( z \in B_{qR}(x_0) \). Moreover, we have by (1.5)

\[
G^{x_0}_{B_{2R}(x_0)} \geq \frac{\Lambda^{-1} K^{-1} R^{2-N}}{\text{Cap}(B_1, B_2)},
\]

for every \( x \in B_R(x_0) - B_{qR}(x_0) \); from (3.11) it follows that

\[
\sup_{B_{qR}(x_0)} u^2 \leq (2 + \gamma) \sup_{B_R(x_0)} u^2 + \frac{C_1}{\gamma} \int_{B_R(x_0) - B_{qR}(x_0)} |Du|^2 G^{x_0}_{B_{2R}(x_0)} \, dx + \frac{A}{N^4 \Lambda^2} \| \nu \|^2_{KN(B_R)},
\] (3.12)

where \( C_1 = 4AK^2 \Lambda^2 \gamma^{-1} \text{Cap}(B_1, B_2) / \text{Cap}(B_p, B_t) \) and in the right-hand side we added the integral with respect to the non-negative measure \( \mu \). In the sequel we will use the notation \( G_\rho \) for \( G^{x_0}_{B_{2R}(x_0)} \).

By Lemma 3.2 we have

\[
\int_{B_{qR}} |Du|^2 G_{2R} \, dx + \int_{B_{qR}} u^2 G_{2R} \, d\mu \leq V(qR) \leq k \frac{1}{R^N} \int_{B_R - B_{qR}} u^2 \, dx + k \| \nu \|^2_{KN(B_R)} \leq k' \sup_{B_R} u^2 + k' \| \nu \|^2_{KN(B_R)},
\]

where we can choose \( k' > 1 \) arbitrarily large. Therefore

\[
\sup_{B_R} u^2 + \| \nu \|^2_{KN(B_R)} \geq C_2 \left[ \int_{B_{qR}} |Du|^2 G_{2R} \, dx + \int_{B_{qR}} u^2 G_{2R} \, d\mu \right],
\] (3.13)

where \( C_2 = \frac{1}{k'} \) can be fixed arbitrarily small; this fact will be useful later. Now from (3.12) and (3.13) we obtain

\[
C_2 \left[ \int_{B_{qR}} |Du|^2 G_{2R} \, dx + \int_{B_{qR}} u^2 G_{2R} \, d\mu \right] + \sup_{B_{qR}} u^2 \leq
\]

\[
\leq (3 + \gamma) \sup_{B_R} u^2 + \frac{C_1}{\gamma} \left[ \int_{B_R - B_{qR}} |Du|^2 G_{2R} \, dx + \int_{B_R - B_{qR}} u^2 G_{2R} \, d\mu \right] + \left( 1 + \frac{\alpha^2}{\lambda^2} \right) \| \nu \|^2_{KN(B_R)} .
\] (3.14)
All the relations we established up to now hold for every $R$ such that $0 < R \leq \frac{qR_0}{2}$. In particular if we fix $R \leq \frac{qR_0}{2}$, then they hold for every $\rho$ such that $0 < \rho \leq R$. Fixed $0 < r < qR$, we want to start from (3.14), in order to arrive to a relation like $V(q\rho) \leq \frac{V(\rho)}{1+k\delta(\rho)}$ for every $\frac{r}{q} < \rho < R$ and then apply the integration lemma. To obtain this we have to distinguish different cases.

Consider first the case of fixed $r$ and $R$ such that $r \leq qR$ and

$$C_2 \left[ \int_{B_r} |Du|^2 G_{\frac{2q}{q}} \, dx + \int_{B_r} u^2 G_{\frac{2q}{q}} \, d\mu \right] + \sup_{B_r} u^2 \geq 2M \| \nu \|_{K_N(B_R)^2}, \quad (3.15)$$

with $M$ a positive constant greater than all the constants appearing as factors of $\| \nu \|_{K_N(B_R)^2}$ in Lemma 3.1, in Lemma 3.2 and in (3.14). Therefore, for every $\frac{r}{q} < \rho < R$, we have

$$M \| \nu \|_{K_N(B_R)^2} \leq \frac{C_2}{2} \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right] + \frac{1}{2} \sup_{B_{q\rho}} u^2, \quad (3.16)$$

where we took into account that, thanks to the maximum principle, $G_{\frac{2q}{q}} \leq G_{2\rho}$.

As $1 + \frac{q^2}{4\pi} \leq M$, by (3.14) and (3.16) it follows that

$$\frac{C_2}{2} \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right] + \frac{1}{2} \sup_{B_{q\rho}} u^2 \leq$$

$$\leq (3 + \gamma) \sup_{B_{\rho}} u^2 + \frac{C_1}{\gamma} \left[ \int_{B_{\rho} - B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{\rho} - B_{q\rho}} u^2 G_{2\rho} \, d\mu \right],$$

for every $\frac{r}{q} < \rho < R$. Now, after multiplication by $\gamma$, we ‘fill the hole” of the annulus adding the term

$$C_1 \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right]$$

to both sides and we obtain

$$\frac{1}{2}(C_2\gamma + 2C_1) \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right] + \frac{\gamma}{2} \sup_{B_{q\rho}} u^2 \leq$$

$$\leq \gamma(3 + \gamma) \sup_{B_{\rho}} u^2 + C_1 \left[ \int_{B_{\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{\rho}} u^2 G_{2\rho} \, d\mu \right]. \quad (3.17)$$

Now we want to replace $G_{2\rho}$ with $G_{\frac{2q}{q}}$ in the right-hand side of (3.17). We consider $F = G_{\frac{2q}{q}} - G_{2\rho}$ and thus $G_{2\rho} = G_{\frac{2q}{q}} - F$. From the definition of the Green function, $F$ is $L$-harmonic in $B_{2\rho}$ and $F = G_{\frac{2q}{q}}$ q.e. on $\partial B_{2\rho}$. It follows that

$$\min_{B_{\rho}} F \geq \min_{B_{2\rho}} F \geq \min_{\partial B_{2\rho}} F = \min_{\partial B_{2\rho}} G_{\frac{2q}{q}} \geq \frac{\Lambda^{-1}K^{-1}}{\text{Cap}(B_{\rho}, B_{\frac{2q}{q}})} \rho^{2-N}. \quad (3.18)$$
Then by (3.17) and (3.18) we obtain

\[
\frac{1}{2} (C_2 \gamma + 2C_1) \left[ \int_{B_{\rho \gamma}} |Du|^2 G_{2 \rho} \, dx + \int_{B_{\rho \gamma}} u^2 G_{2 \rho} \, d\mu \right] + \frac{\gamma}{2} \sup_{B_{\rho \gamma}} u^2 \leq \gamma (3 + \gamma) \sup_{B_\rho} u^2 + C_1 \left[ \int_{B_\rho} |Du|^2 G_{\frac{2 \rho}{q}} \, dx + \int_{B_\rho} u^2 G_{\frac{2 \rho}{q}} \, d\mu \right]
\]

\[+ \frac{C_1 \Lambda^{-1} K^{-1}}{\text{Cap}(B_2, B_{\frac{2 \rho}{q}})} \rho^{2-N} \left[ \int_{B_\rho} |Du|^2 \, dx + \int_{B_\rho} u^2 \, d\mu \right]. \tag{3.19}
\]

Therefore, applying Poincaré inequality (see Theorem 1.2) to the last term of (3.19), we have

\[- \frac{C_1 \Lambda^{-1} K^{-1}}{\text{Cap}(B_2, B_{\frac{2 \rho}{q}})} \rho^{2-N} \left[ \int_{B_\rho} |Du|^2 \, dx + \int_{B_\rho} u^2 \, d\mu \right] \leq -\kappa \delta(\rho) \frac{1}{\rho^N} \int_{B_\rho} u^2 \, dx, \tag{3.20}
\]

where \(\kappa\) is a positive constant depending only on \(q, \lambda, \Lambda\) and \(N\). Choosing \(C_2 < 1\), by (3.16) it follows that

\[M \parallel \nu \parallel_{K_N(B_\rho)}^2 \leq \frac{1}{2} V(q \rho),\]

for every \(\frac{\rho}{q} < \rho < R\). Since the constant \(k\) which appears in Lemma 3.2 satisfies \(k \leq M\), from Lemma 3.2 we obtain

\[\frac{1}{2} \sup_{B_{\rho \gamma}} u^2 \leq \frac{1}{2} V(q \rho) \leq k \frac{1}{\rho^N} \int_{B_\rho} u^2 \, dx,
\]

with \(k > 1\) arbitrarily large. Then from (3.20) we get

\[- \frac{C_1 \Lambda^{-1} K^{-1}}{\text{Cap}(B_2, B_{\frac{2 \rho}{q}})} \rho^{2-N} \left[ \int_{B_\rho} |Du|^2 \, dx + \int_{B_\rho} u^2 \, d\mu \right] \leq -C_3 \delta(\rho) \sup_{B_{\rho \gamma}} u^2, \tag{3.21}
\]

with \(\frac{\rho}{q} < \rho < R\), where \(C_3\) (as well as \(C_2\)) is a constant that can be chosen arbitrarily small. Then we can take, without loss of generality, \(C_3 = \frac{15}{2} C_2\). Therefore, by (3.19) and (3.21) it follows that

\[\frac{1}{2} (C_2 \gamma + 2C_1) \left[ \int_{B_{\rho \gamma}} |Du|^2 G_{2 \rho} \, dx + \int_{B_{\rho \gamma}} u^2 G_{2 \rho} \, d\mu \right] + \frac{1}{2} [\gamma + 15C_2 \delta(\rho)] \sup_{B_{\rho \gamma}} u^2 \leq \gamma (3 + \gamma) \sup_{B_\rho} u^2 + C_1 \left[ \int_{B_\rho} |Du|^2 G_{\frac{2 \rho}{q}} \, dx + \int_{B_\rho} u^2 G_{\frac{2 \rho}{q}} \, d\mu \right]. \tag{3.22}
\]
By adding \( \frac{16C_1}{C_2} \sup_{B_{q\rho}} u^2 \) to both sides of (3.22) we obtain

\[
(C_2\gamma + 2C_1) \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right] + 
\]

\[
+ \frac{16}{C_2} \left[ 2C_1 + C_2 \left( \frac{\gamma + 15C_2\delta(\rho)}{16} \right) \right] \sup_{B_{q\rho}} u^2 \leq 
\]

\[
\leq 2 \frac{16}{C_2} \left[ C_1 + \frac{C_2}{16} \gamma(3 + \gamma) \right] \sup_{B_{\rho}} u^2 + 2C_1 \left[ \int_{B_{\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{\rho}} u^2 G_{2\rho} \, d\mu \right].
\]

Then, since \( \gamma \) is an arbitrary constant, we can choose \( \gamma = C_2\delta(\rho) < 1 \) and we get

\[
(2C_1 + C_2^2\delta(\rho)) \left[ \int_{B_{q\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{q\rho}} u^2 G_{2\rho} \, d\mu \right] + 
\]

\[
+ \frac{16}{C_2} (2C_1 + C_2^2\delta(\rho)) \sup_{B_{q\rho}} u^2 \leq 
\]

\[
\frac{16}{C_2} \left( 2C_1 + \frac{1}{2} C_2^2\delta(\rho) \right) \sup_{B_{\rho}} u^2 + 2C_1 \left[ \int_{B_{\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{\rho}} u^2 G_{2\rho} \, d\mu \right].
\] (3.23)

We now introduce the non-decreasing function \( U(\rho) \) defined by

\[
U(\rho) \overset{\text{def}}{=} \int_{B_{\rho}} |Du|^2 G_{2\rho} \, dx + \int_{B_{\rho}} u^2 G_{2\rho} \, d\mu + \frac{16}{C_2} \sup_{B_{\rho}} u^2.
\]

From (3.23) we have

\[
U(q\rho) \leq \frac{2C_1 + \frac{1}{2} C_2^2\delta(\rho)}{2C_1 + C_2^2\delta(\rho)} U(\rho),
\] (3.24)

for every \( \frac{r}{q} < \rho < R \). Since we can choose \( C_2 \) such that \( \frac{C_2^2}{2C_1} < 1 \) holds, then from (3.24) we obtain

\[
U(q\rho) \leq \frac{1}{1 + k\delta(\rho)} U(\rho),
\]

for every \( \frac{r}{q} < \rho < R \), with \( k = \frac{C_2^2}{6C_1} \). Therefore by Lemma 3.4 we have

\[
U(r) \leq k_0 \exp \left( -\beta \int_{r}^{R} \delta(\rho) \frac{d\rho}{\rho} \right) U(R).
\]

Then, choosing \( C_2 < 16 \), we get

\[
V(r) \leq \frac{16}{C_2} k_0 \exp \left( -\beta \int_{r}^{R} \delta(\rho) \frac{d\rho}{\rho} \right) V(R),
\] (3.25)
for every \( r \) and \( R \) with \( r < qR \) such that (3.15) holds.

Trivially, if \( r < qR \) and (3.15) does not hold, i.e.,

\[
C_2 \left[ \int_{B_r} |Du|^2 G_{\frac{2}{\alpha}} \, dx + \int_{B_r} u^2 G_{\frac{2}{\alpha}} \, d\mu \right] + \sup_{B_r} u^2 < 2M \left\| \nu \right\|^2_{K_N(B_R)},
\]

then we have

\[
V(r) \leq \frac{2M}{C_2} \left\| \nu \right\|^2_{K_N(B_R)}. \tag{3.26}
\]

If \( qR \leq r \leq R \), then

\[
\int_r^R \delta(\rho) \frac{d\rho}{\rho} \leq \int_{qR}^R \delta(\rho) \frac{d\rho}{\rho} \leq \log \frac{1}{q};
\]

hence

\[
\exp\left( -\beta \int_r^R \delta(\rho) \frac{d\rho}{\rho} \right) \geq q^\beta.
\]

Therefore from \( V(r) \leq V(R) \) it follows that

\[
V(r) \leq q^{-\beta} \exp\left( -\beta \int_r^R \delta(\rho) \frac{d\rho}{\rho} \right) V(R). \tag{3.27}
\]

Finally from (3.25), (3.26) and (3.27) it follows that, for every \( 0 < r \leq R \leq \frac{2R_0}{q} \), we have

\[
V(r) \leq k \exp\left( -\beta \int_r^R \delta(\rho) \frac{d\rho}{\rho} \right) V(R) + k \left\| \nu \right\|^2_{K_N(B_R)},
\]

where \( k = \max \left\{ \frac{16}{C_2} k_0, \frac{2M}{C_2}, q^{-\beta} \right\} \).

As a consequence of Theorem 3.1 we have the following estimate of the \( \mu \)-energy

\[
E_\mu(r) \overset{\text{def}}{=} \int_{B_r} |Du|^2 \, dx + \int_{B_r} u^2 \, d\mu, \quad 0 < r \leq R_0
\]

**Theorem 3.3.** There exist two constants \( k > 0 \) and \( \beta > 0 \), depending only on \( \lambda, \Lambda \) and \( N \), such that

\[
E_\mu(r) \leq k \omega(r, R)^{\beta} \frac{r^{N-2}}{\text{Cap}_\mu(B_{2R}; B_{4R})} E_\mu(2R) + kr^{-N-2} \left\| \nu \right\|_{K_N(B_{2R})}
\]

for every \( 0 < r \leq R \leq \frac{qR_0}{2} \).

**Proof.** We proceed as in Theorem 6.5 of [2], having in mind that when in [2] it is used the estimate of the fundamental solution for the Laplace operator, we must use the estimate of the Green function. \( \square \)
4. Classical case

Choosing a suitable $\mu$ in $M_0(\Omega)$ it is possible to obtain from a relaxed problem of the type (3.1) a problem equivalent to the following variational Dirichlet problem

$$\begin{cases}
Lu = f & \text{in } \Omega \\
u \in H^1_0(\Omega)
\end{cases} \tag{4.1}$$

where $f \in H^{-1}(\Omega)$.

Let $E$ be a subset of $\mathbb{R}^N$. We denote with $\infty_E$ the measure of $M_0(\Omega)$ defined by

$$\infty_E(B) \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } \text{Cap} (E \cap B) = 0 \\
+\infty & \text{otherwise}
\end{cases}$$

and we consider the equation

$$Lu + \infty_E u = f \quad \text{in } \Omega. \tag{4.2}$$

First of all we remark that if $v \in L^2_{\text{loc}}(\Omega, \infty_E)$, then $v = 0$ q.e. in $\Omega \cap E$. Thus $u$ is a local weak solution of (4.2) if and only if

$$u \in H^1(\Omega)$$

$$u = 0 \text{ q.e. in } \Omega \cap E$$

$$\int_\Omega Du Dv \, dx = \int_\Omega fv \, dx$$

for every $v \in H^1_0(\Omega)$ with compact support in $\Omega$ and such that $v = 0$ q.e. in $\Omega \cap E$.

In particular if $E$ is a closed set, $u$ is a weak solution of problem

$$\begin{cases}
Lu + \infty_E u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

if and only if $u = 0$ q.e. in $\Omega \cap E$ and $u|_{\Omega \cap E}$ is a solution of

$$\begin{cases}
Lu = f & \text{in } \Omega - E \\
u \in H^1_0(\Omega - E).
\end{cases}$$

Let $\Omega'$ be a bounded open set such that $\Omega' \supset \supset \Omega$. Consider the equation (4.2) in $\Omega'$, choosing $E = \Omega' - \Omega$. In this case $u$ is a local weak solution of (4.2) in $\Omega'$ if and only if it is a solution of (4.1).

If we consider the Wiener Criterion (Theorem 2.1) for this special case, we obtain exactly the classical Wiener Criterion for the variational Dirichlet problem (4.1). Actually it is easy to see that $\text{Cap}_{\infty_{\Omega' - \Omega}}(B_\rho, B_{2\rho}) = \text{Cap}(B_\rho \cap C\Omega, B_{2\rho})$, for $\rho$ small enough, and then the Wiener modulus at a point $x_0$ on the boundary of $\Omega$ is given by

$$\omega(r, R) = \exp \left( - \int_r^R \frac{\text{Cap}(B_\rho(x_0) \cap C\Omega, B_{2\rho}(x_0))}{\text{Cap}(B_\rho(x_0), B_{2\rho}(x_0))} \frac{d\rho}{\rho} \right)$$

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for every $0 < \rho < R_0$ (with $R_0$ such that $\overline{B_{R_0}(x_0)} \subset \Omega'$), where $C\Omega = \mathbb{R}^N - \Omega$.

Moreover, if we consider problem (4.1) with $f = \nu \in K_N(\Omega)$, by the estimate of Theorem 3.1 we obtain a continuity modulus estimate already proved by Maz’ja in [13]; by Theorem 3.2 we have the following estimate of the energy decay in terms of the Wiener modulus

$$\int_{B_r} |Du|^2 \, dx \leq k \frac{r^{N-2}}{\text{Cap}(B_{2R} \cap C\Omega, B_{4R})} \exp \left( -\beta \int_r^R \frac{\text{Cap}(B_\rho \cap C\Omega, B_{2\rho})}{\rho^{N-1}} d\rho \right)$$

$$\times \int_{B_{2R}} |Du|^2 \, dx + kr^{N-2} \| \nu \|_{K_N(B_{2R})}$$

($\nu$ is extended out of $\Omega$ in the trivial way), that holds for every $0 < r \leq R$ and in dimension $N \geq 2$.

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