OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING, VIA CONVEXIFICATORS

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ABSTRACT. In this paper, the idea of convexificators is used to derive the Karush-Kuhn-Tucker necessary optimality conditions for local weak efficient solutions of multiobjective fractional problems involving inequality and equality constraints. In this regard, several well known constraint qualifications are generalized and relationships between them are investigated. Moreover, some examples are provided to clarify our results.

1. Introduction. The term multiobjective programming is used to denote a type of optimization problems where two or more objectives are to be minimized subject to certain constraints; see for instance [1, 3, 5]. Multiobjective fractional programming refers to a multiobjective problem where the objective functions are quotients. Fractional optimization problems arise frequently in decision making applications, including science management, portfolio selection, cutting and stock, and game theory, in the optimization of the ratio performance/cost, or profit/investment, or cost/time, see, e.g., [22]. We consider the following multiobjective fractional programming problem involving inequality and equality constraints (MFP) of the form

\[
\begin{align*}
\min & \quad \left( \frac{f_1(x)}{k_1(x)}, \ldots, \frac{f_p(x)}{k_p(x)} \right) \\
\text{s.t.} & \quad g_j(x) \leq 0 \quad j \in J := \{1, 2, \ldots, q\}, \\
& \quad h_i(x) = 0 \quad i \in I := \{1, 2, \ldots, r\},
\end{align*}
\]

(1)

(2)

(3)

where \( f_l, k_l, g_j : \mathbb{R}^m \to \mathbb{R}, \ l \in L := \{1, \ldots, p\}, \ j \in J \) are given functions and \( h_i : \mathbb{R}^m \to \mathbb{R}, \ i \in I \) are locally Lipschitz functions and \( f_l(x) \geq 0 \) and \( k_l(x) > 0 \). Many researchers have made contributions in the development of optimality conditions for MFP, see, e.g., [15, 20, 16]. Recently the idea of convexificators has been used to extend, unify and sharpen the results in various aspects of optimization. In 1994, Demyanov [6] introduced the notion of a convex and compact convexificator, which was further developed by Demyanov and Jeyakumar [7]. This notion is a
The aim of this paper is to derive the KKT conditions at weak efficient solutions of MFP, based on the convexificators. For this purpose, we generalize some well known constraint qualifications based on the Clarke generalized derivative of the equality constraints and the upper Dini derivative of the inequality constraints. We also investigate the relationships between them. One of the important features of our work is the structure of our constraint qualifications where the objective functions play no role in the definition of these CQs unlike what usually occur in multiobjective programming literature. Another major attribute is to obtain KKT conditions for weak efficient optimal solutions of MFP without any smoothness assumption on the equality constraints and without any upper semicontinuity assumption on the convexificators of the effective functions. It is worth mentioning that, the results in this study are established regardless of which convexificator is used and thus they can be applied to a large class of nonsmooth problems. Throughout the paper, several examples are provided to clarify our results.

The rest of the paper is organized as follows. Section 2 presents some basic constructions and properties of nonsmooth analysis widely used in the main body of the paper. In Section 3, we generalize various well known constraint qualifications and establish a connection link between them. Some counterexamples are provided in order to clarify that these relations are not generally true in opposite directions. Moreover, we arrive at KKT necessary optimality conditions for MFP under the weakest conditions. The paper is closed by an example to illustrate the significance of our optimality conditions.

2. Preliminaries. In this section, we recall some basic constructions and results from nonsmooth analysis needed in what follows. Our notation are basically standard. For a given subset \( S \subseteq \mathbb{R}^m \), cl \( S \), co \( S \), and cone \( S \) stand for the negative polar cone, the closure, the convex hull of \( S \) and the convex cone (including the origin) generated by \( S \), respectively. The contingent cone, the cone of feasible directions and the cone of attainable directions of \( S \) at \( x \in \text{cl} \ S \) are defined, respectively, by

\[
T(S, x) = \{ v \in \mathbb{R}^m : \exists t_n \downarrow 0 \text{ and } v_n \rightarrow v \text{ s.t. } x + t_n v_n \in S, \forall n \},
\]

\[
D(S, x) = \{ v \in \mathbb{R}^m : \exists \delta > 0 \text{ s.t. } x + tv \in S, \forall t \in (0, \delta) \},
\]

\[
A(S, x) = \left\{ v \in \mathbb{R}^m : \exists \delta > 0 \text{ and } \alpha : \mathbb{R} \rightarrow \mathbb{R}^m \text{ s.t. } \alpha(t) \in S, \forall t \in (0, \delta), \alpha(0) = x, \lim_{t \rightarrow 0^+} \frac{\alpha(t) - \alpha(0)}{t} = v \right\}.
\]
Let $f : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz near $x$. Taking a nonzero vector $v \in \mathbb{R}^m$, the Clarke generalized directional derivative of $f$ at $x$ in the direction $v$ and the Clarke subdifferential of $f$ at $x$ are respectively defined by

$$f^+(x; v) := \limsup_{t \downarrow 0} t^{-1}[f(x + tv) - f(x)],$$

$$\partial C f(x) := \{ \xi \in \mathbb{R}^m \mid f^+(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in \mathbb{R}^m \}. $$

It should be noted that the function $f^+(x; .)$ is finite, positively homogeneous, sublinear on $\mathbb{R}^m$ and $f^+(x; -v) = (-f)^+(x; v)$ and $\partial C f(x)$ is a compact and convex set. Now we focus on the notion of the convexificator and some of its important properties. Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a given function and $x \in \text{dom } f := \{ x \in \mathbb{R}^m : f(x) < +\infty \}$. The lower and upper Dini derivatives of $f$ at $x$ in the direction $v \in \mathbb{R}^m$ are given, respectively, as

$$f^-(x; v) := \liminf_{t \downarrow 0} t^{-1}[f(x + tv) - f(x)],$$

$$f^+(x; v) := \limsup_{t \downarrow 0} t^{-1}[f(x + tv) - f(x)].$$

If $f : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz, then the lower and upper Dini derivatives exist finitely.

On the lines of [9, 13], we recall the definition of the upper semi-regular convexificator. The function $f$ is said to have an upper semi-regular convexificator at $x \in \mathbb{R}^m$ if there is a closed set $\partial \alpha f(x) \subset \mathbb{R}^m$ such that for each $v \in \mathbb{R}^m$,

$$f^+(x; v) \leq \sup_{\xi \in \partial \alpha f(x)} \langle \xi, v \rangle.$$

**Remark 1.** If $f$ is a locally Lipschitz function, then the Clarke $\partial C f$ [4], Michel-Penot $\partial \varepsilon f$ [18], Mordukhovich $\partial M f$ [19] and Treiman $\partial T f$ [23] subdifferentials are examples of upper semi-regular convexificators of $f$. Moreover, the convex hull of an upper semi-regular convexificator of a locally Lipschitz function may be strictly contained in both Clarke and Michel-Penot subdifferentials. It was demonstrated by an example in [13].

We proceed this section by giving the following calculus rules that are useful in the sequel.

**Proposition 1.** Let $\partial \alpha f(x)$ and $\partial \alpha g(x)$ be upper semi-regular convexificators of the functions $f, g : \mathbb{R}^m \to \mathbb{R}$ at $x$. Then

(i) $\alpha \partial \alpha f(x)$ is an upper semi-regular convexificator of $\alpha f$ at $x$ for every $\alpha \geq 0$.

(ii) $\text{cl}(\partial \alpha f(x) + \partial \alpha g(x))$ is an upper semi-regular convexificator of $f + g$ at $x$.

**Proof.** The proof immediately follows from the proofs of Rules 4.1 and 4.2 in [13].

The next proposition provides a formula of the max-function.

**Proposition 2.** Let $f_i, i = 1, \ldots, k$ be continuous scalar functions on $\mathbb{R}^m$ and assume that $\partial \alpha f_i(x), \ldots, \partial \alpha f_k(x)$ are upper semi-regular convexificators of $f_1, \ldots, f_k$, respectively, at $x$. Then $\bigcup_{i \in M(x)} \partial \alpha f_i(x)$ is an upper semi-regular convexificator of the max-function $f$ defined by

$$f(x') := \max\{f_1(x'), \ldots, f_k(x')\},$$

where $M(x) := \{ i \mid f(x) = f_i(x) \}$. 

Proof. This follows directly from the proof of Rule 4.4 in [13].

We denote the feasible set of MFP by
\[ S := \{ x | g_j(x) \leq 0, \; h_i(x) = 0 \; j \in J, i \in I \}. \]  
(4)
The index set of active constraints at \( \bar{x} \in S \) is given by
\[ J(\bar{x}) := \{ j | g_j(\bar{x}) = 0 \}. \]
Considering a feasible point \( \bar{x} \in S \), we say that:
1. \( \bar{x} \) is an efficient solution for MFP if there exists no feasible solution \( x \) such that \( \frac{f_l(x)}{k_l(x)} \leq \frac{f_l(\bar{x})}{k_l(\bar{x})} \) for each \( l \in L \) and \( \frac{f_{0_l}(x)}{k_{0_l}(x)} < \frac{f_{0_l}(\bar{x})}{k_{0_l}(\bar{x})} \) for at least one index \( l_0 \in L \).
2. \( \bar{x} \) is a weak efficient solution for MFP if there exists no feasible solution \( x \) such that for each \( l \in L \), \( \frac{f_l(x)}{k_l(x)} < \frac{f_l(\bar{x})}{k_l(\bar{x})} \).
The feasible point \( \bar{x} \) is a local efficient (local weak efficient) solution for MFP if there exists \( \delta > 0 \) such that \( \bar{x} \) is an efficient (weak efficient solution) solution in \( B_\delta(\bar{x}) \cap S \).

3. Constraint qualifications and Karush-Kuhn-Tucker necessary optimality conditions. In this section, using the upper Dini directional derivative and Clarke generalized directional derivative, we generalize several well known constraint qualifications for MFP which are extensions of classical ones. Furthermore, we establish a connection link between them and provide some examples in order to clarify that these relations are not generally true in opposite directions. Moreover, the KKT conditions at weak efficient solutions are derived under the weakest one, in the framework of convexificators. Finally, we present an example to illustrate our necessary optimality conditions.

In order to define our constraint qualifications, we use the following generalized convexity notion of functions. A function \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) is said to be pseudoconvex at \( x \in \mathbb{R}^m \) if for all \( y \in \mathbb{R}^m \),
\[ f(y) < f(x) \Rightarrow f^-(x; y - x) < 0. \]
The function \( f \) is said to be pseudoconcave at \( x \in \mathbb{R}^m \) if \( -f \) is pseudoconvex at \( x \). Since \( (-f)^-(x; v) = -f^+(x; v) \), for every \( x, v \in \mathbb{R}^m \), it follows that \( f \) is pseudoconcave at \( x \) if for all \( y \in \mathbb{R}^m \),
\[ f(y) > f(x) \Rightarrow f^+(x; y - x) > 0. \]
Considering the linearized cone \( L(S, \bar{x}) \) at \( \bar{x} \) defined as
\[ L(S, \bar{x}) := \{ v \in \mathbb{R}^m : g_j^\top(\bar{x}; v) \leq 0, h_i^\top(\bar{x}; v) \leq 0, h_i^\top(\bar{x}; -v) \leq 0 \; j \in J(\bar{x}), i \in I \}, \]
we are now in a position to introduce our constraint qualifications for MFP.

Definition 3.1. We say that:
1. the generalized Abadie CQ (GACQ) holds at \( \bar{x} \) if \( L(S, \bar{x}) \subseteq T(S, \bar{x}) \).
2. the generalized Kuhn-Tucker CQ (GKTCQ) holds at \( \bar{x} \) if \( L(S, \bar{x}) \subseteq \text{cl} \; A(S, \bar{x}) \).
3. the generalized Zangwill CQ (GZCQ) holds at \( \bar{x} \) if \( L(S, \bar{x}) \subseteq \text{cl} \; D(S, \bar{x}) \).
4. the generalized Cottle CQ (GCCQ) holds at \( \bar{x} \) if the functions \( \pm h_i \) are pseudoconcave at \( \bar{x} \), \( g_j(j \notin J(\bar{x})) \) are continuous at \( \bar{x} \), \( g_j^\top(\bar{x}; .)(j \in J(\bar{x})) \) are sublinear and the system
\[ g_j^\top(\bar{x}; v) < 0 \; j \in J(\bar{x}), h_i^\top(\bar{x}; v) \leq 0, h_i^\top(\bar{x}; -v) \leq 0 \; i \in I, \]
5. the generalized Linear CQ (GLCQ) holds at \( \bar{x} \) if the functions \( g_j(j \notin J(\bar{x})) \) are continuous at \( \bar{x} \) and \( g_j, \pm h_i \) are pseudoconcave at \( \bar{x} \) for each \( i \in I, j \in J(\bar{x}) \).

Since \( D(S, \bar{x}) \subseteq A(S, \bar{x}) \subseteq T(S, \bar{x}) \) and \( T(S, \bar{x}) \) is a closed set, we have the following implications:

\[
GZCQ \rightarrow GKTCQ \rightarrow GACQ.
\]

The following theorems relate GCCQ and GLCQ to GZCQ.

**Theorem 3.2.** If GCCQ holds at \( \bar{x} \), then GZCQ is satisfied at this point.

**Proof.** Let \( v \in L(S, \bar{x}) \), then

\[
g_j^+(\bar{x}; v) \leq 0 \quad j \in J(\bar{x}), \quad h_i^+(\bar{x}; v) \leq 0, \quad h_i^-(\bar{x}; -v) \leq 0 \quad i \in I. \tag{5}
\]

On the other hand, GCCQ gives us a vector \( d \in \mathbb{R}^m \) such that

\[
g_j^+(\bar{x}; d) < 0 \quad j \in J(\bar{x}), \quad h_i^+(\bar{x}; d) \leq 0, \quad h_i^-(\bar{x}; -d) \leq 0 \quad i \in I. \tag{6}
\]

It follows from the sublinearity of the functions \( g_j^+(\bar{x}; .) \) and \( h_i^+(\bar{x}; .) \) together with the relations in (5) and (6), for a given \( r > 0 \) that

\[
g_j^+(\bar{x}; v + rd) < 0 \quad j \in J(\bar{x}), \quad h_i^+(\bar{x}; v + rd) \leq 0, \quad h_i^-(\bar{x}; -v - rd) \leq 0 \quad i \in I. \tag{7}
\]

Therefore, there exists \( \delta > 0 \) such that for all \( t \in (0, \delta) \) we have \( g_j(\bar{x} + t(v + rd)) < 0 \) \((j \in J(\bar{x}))\). Also the continuity of \( g_j(j \notin J(\bar{x})) \) yields \( g_j(\bar{x} + t(v + rd)) < 0 \) for sufficiently small \( t \). Considering for each \( v, h_i^+(\bar{x}; v) \leq h_i^+(\bar{x}; v) \) and \( h_i^-(\bar{x}; -v) = (-h_i)^+(\bar{x}; v) \), it follows from the relations in (7) that for each \( t > 0 \),

\[
h_i^+(\bar{x}; t(v + rd)) \leq 0 \quad i \in I.
\]

Now, the pseudoconcavity of the functions \( h_i - h_i \) implies that \( h_i(\bar{x} + t(v + rd)) = 0 \). Putting these facts together, one has some positive scalar \( \delta > 0 \) such that \( \bar{x} + t(v + rd) \in S \) for all \( t \in (0, \delta) \) which means \( v + rd \in D(S, \bar{x}) \). Since \( r > 0 \) was chosen arbitrarily, we conclude that \( v \in cl\ D(S, \bar{x}) \) which finishes the proof.

**Theorem 3.3.** If GLCQ holds at \( \bar{x} \), then GZCQ is fulfilled at this point.

**Proof.** Assume that \( v \in L(S, \bar{x}) \) then, similar to the proof of Theorem 3.2, the inequalities in (5) and the pseudoconcavity property of the functions \( g_j, h_i \) and \( -h_i \) give us for any \( t > 0 \),

\[
g_j(\bar{x} + tv) \leq 0 \quad j \in J(\bar{x}) \quad \text{and} \quad h_i(\bar{x} + tv) = 0 \quad i \in I.
\]

Since the functions \( g_j, j \notin J(\bar{x}) \) are continuous, there exists \( \delta > 0 \) such that \( g_j(\bar{x} + tv) < 0 \) for all \( t \in (0, \delta) \). Therefore, we get \( \bar{x} + tv \in S \) for all \( t \in (0, \delta) \). It means \( v \in D(S, \bar{x}) \), which completes the proof.

The following diagram summarises the relationships between the above constraint qualifications.

\[
\begin{align*}
\text{GCCQ} & \quad \downarrow \\
\text{GZCQ} & \quad \rightarrow \quad \text{GKTCQ} \quad \rightarrow \quad \text{GACQ} \\
\text{GLCQ} & \quad \uparrow
\end{align*}
\]
It is worth mentioning that, in general, the reverse arrows in this diagram may fail to hold, as the following examples indicate. The first example shows that GACQ does not imply GKTCQ.

**Example 1.** Consider \( g(x_1, x_2) := d_K(x_1, x_2) \), where \( d_K \) is the distance function and \( K := \{ (\frac{1}{n}, 0) : n \in \mathbb{N} \} \cup \{ (0, 0) \} \) and let \( S := \{ (x_1, x_2) : g(x_1, x_2) \leq 0 \} \). A simple calculation confirms that \( S = K, \ T(S, (0, 0)) = \{ (x_1, x_2) : x_2 = 0, x_1 \geq 0 \} \),

and \( d_K^+(0, 0; (v_1, v_2)) = \begin{cases} |v_1| + |v_2| & v_1 \leq 0 \\ |v_2| & v_1 > 0 \end{cases} \), and \( L(S, (0, 0)) \subseteq T(S, (0, 0)) \).

As shown in [2], \( cl \ A(S, (0, 0)) = \{ (0, 0) \} \) thus \( L(S, (0, 0)) \) is not contained in \( cl \ A(S, (0, 0)) \) which means that GKTCQ does not hold at \( (0, 0) \). Whereas GACQ is satisfied at this point.

The second example illustrates that GKTCQ does not imply GZCQ.

**Example 2.** Consider \( h(x_1, x_2) := x_2 - x_1^2 \) and let \( S := \{ (x_1, x_2) : h(x_1, x_2) = 0 \} \). Since \( h^+(0, 0; (v_1, v_2)) = v_2 \), then \( L(S, (0, 0)) = \{ (v_1, v_2) : v_2 = 0 \} \). Furthermore, \( cl \ A(S, (0, 0)) = \{ (\lambda \pm 1, 0) : \lambda \geq 0 \} \) and \( cl \ D(S, (0, 0)) = \{ (0, 0) \} \) which imply that at the point \( (0, 0) \), GKTCQ is fulfilled whereas GZCQ does not hold.

The following two examples clarify that GCCQ and GLCQ may not hold simultaneously.

**Example 3.** Consider \( g(x) := \begin{cases} x & x \in Q \\ -x & x \notin Q \end{cases} \), where \( Q \) denotes the set of rational numbers. Clearly, \( g^+(0; v) = |v| \), thus for each \( v \neq 0 \), \( g^+(0; v) > 0 \) which implies that the function \( g \) is pseudoconcave at \( 0 \). On the other hand, there is no \( v \in \mathbb{R} \) satisfying \( g^+(0; v) < 0 \). Therefore, GLCQ is satisfied but GCCQ does not hold at \( 0 \).

**Example 4.** Consider \( g(x_1, x_2) := x_1^2 + x_2 \) and \( h(x_1, x_2) := |x_1| \). Since \( g^+(0, 0); (v_1, v_2)) = v_2 \) and \( h^+(0, 0; (v_1, v_2)) = |v_1| \), hence the conditions in GCCQ hold true at \( (0, 0) \). While at the point \( x = (2, -1) \), we have \( g(2, -1) > g(0, 0) \) and \( g^+(0, 0; (2, -1)) = -1 \), namely the function \( g \) is not pseudoconcave at \( 0 \), which implies that GLCQ does not hold.

The next lemma is used in the proof of the optimality conditions and is proved immediately from [21, Theorem 3.3].

**Lemma 3.4.** Let \( S \) and \( T \) be two nonempty subsets of \( \mathbb{R}^m \). Then 
\[
\text{co} \ (S \cup T) = \bigcup_{0 \leq \lambda \leq 1} \lambda \text{co} \ S + (1 - \lambda) \text{co} \ T.
\]

Now let us present the KKT conditions under GACQ based on the convexificators.

**Theorem 3.5.** Let all the functions \( f_l, -k_l \) and \( g_j \) admit upper semi-regular convexificators \( \partial^* f_l(\bar{x}) \), \( \partial^* (-k_l)(\bar{x}) \) and \( \partial^* g_j(\bar{x}) \), respectively, at a local weak efficient solution \( \bar{x} \) of MFP. Assume also that for each \( l \in L \), the functions \( f_l, k_l \) are locally Lipschitz and \( \partial^* f_l(\bar{x}), \partial^* (-k_l)(\bar{x}) \) are bounded. If GACQ is satisfied at \( \bar{x} \), then there exists a nonzero nonnegative vector \( \alpha \in \mathbb{R}^m \) such that
\[ (i) \quad 0 \in \sum_{l \in L} \alpha_l \co \left[ \partial^* f_l(\bar{x}) + \frac{f_l(\bar{x})}{k_l(\bar{x})} \partial^*(-k_l)(\bar{x}) \right] + \cl \Delta, \text{ where } \Delta = \sum_{j \in J(\bar{x})} \text{cone} \partial g_j(\bar{x}) + \sum_{i \in I} \text{cone} \partial C h_i(\bar{x}) + \sum_{i \in I} \text{cone} - \partial C h_i(\bar{x}). \]

\[ (ii) \quad \text{If the set } \Delta \text{ is closed, then there exist nonnegative multipliers } \mu_j, \ j \in J(\bar{x}), \gamma_i, \nu_i, \ i \in I \text{ such that} \]

\[ 0 \in \sum_{l \in L} \alpha_l \co \left[ \partial^* f_l(\bar{x}) + \frac{f_l(\bar{x})}{k_l(\bar{x})} \partial^*(-k_l)(\bar{x}) \right] + \sum_{j \in J(\bar{x})} \mu_j \co \partial^* g_j(\bar{x}) + \sum_{i \in I} \gamma_i \partial C h_i(\bar{x}) - \sum_{i \in I} \nu_i \partial C h_i(\bar{x}). \]

**Proof.** First, we consider the following scalar problem

\[
\min \quad \theta(x) \\
\text{s.t. } x \in S,
\]

where \( \theta(x) := \max \left\{ f_l(x) - \frac{f_l(x)}{k_l(x)} k_l(x) : l \in L \right\} \) and \( S \) is the feasible set defined in (4). It is easy to confirm that \( \bar{x} \) is a local minimum of \( \theta \) on \( S \). Furthermore, Propositions 1 and 2 imply that the set

\[
\partial^* \theta(\bar{x}) := \bigcup_{l \in L} \cl \left[ \partial^* f_l(\bar{x}) + \frac{f_l(\bar{x})}{k_l(\bar{x})} \partial^*(-k_l)(\bar{x}) \right] = \bigcup_{l \in L} \partial^* f_l(\bar{x}) + \frac{f_l(\bar{x})}{k_l(\bar{x})} \partial^*(-k_l)(\bar{x}) \quad (9)
\]

is a bounded upper semi-regular convexificator of \( \theta \) at \( \bar{x} \).

Next, we assert that \( 0 \in \co \partial^* \theta(\bar{x}) + \cl \Delta \). Assume by contradiction that \( 0 \notin \co \partial^* \theta(\bar{x}) + \cl \Delta \). Applying the separation theorem, we can choose the scalars \( c_1, c_2 \in \mathbb{R} \) together with the nonzero vector \( v \in \mathbb{R}^m \) such that

\[
\langle \xi, v \rangle < c_1 < c_2 \leq \langle \eta, v \rangle \quad \forall \xi \in \co \partial^* \theta(\bar{x}), \forall \eta \in -\cl \Delta.
\]

Since \( \cl \Delta \) is a closed convex cone, on the one hand we get

\[
\langle \xi, v \rangle < c_1 < 0, \quad \forall \xi \in \co \partial^* \theta(\bar{x}),
\]

which implies that

\[
\theta^+(\bar{x}; v) \leq c_1 < 0. \quad (10)
\]

On the other hand, we have \( \langle \eta, v \rangle \leq 0 \) for all \( \eta \in \cl \Delta \), consequently,

\[
\langle \eta_j, v \rangle \leq 0, \quad \forall \eta_j \in \partial^* g_j(\bar{x}), \forall j \in J, \\
\langle \eta_i, v \rangle \leq 0, \quad \forall \eta_i \in \partial C h_i(\bar{x}), \forall i \in I, \\
\langle \eta_i, -v \rangle \leq 0, \quad \forall \eta_i \in \partial C h_i(\bar{x}), \forall i \in I,
\]

which imply that \( v \in L(S, \bar{x}) \). Thus, by GACQ, we have \( v \in T(S, \bar{x}) \) which entails the existence of sequences \( t_n \downarrow 0 \) and \( v_n \to v \) such that \( \bar{x} + t_n v_n \in S \). Now, according to Lipschitz condition of the function \( \theta \) near \( \bar{x} \) and inequality in (10), we obtain for sufficiently large \( n \),

\[
\theta(\bar{x} + t_n v_n) < \theta(\bar{x}),
\]

which impossible by \( \bar{x} \) is a local minimum of Problem (8). Hence the assertion is true and we get

\[
0 \in \co \partial^* \theta(\bar{x}) + \cl \Delta. \quad (11)
\]

The relations in (11) and (9) together with Lemma 3.4 give us nonnegative scalars \( \alpha_l \) with \( \sum_{l \in L} \alpha_l = 1 \) satisfying

\[
0 \in \sum_{l \in L} \alpha_l \co \left[ \partial^* f_l(\bar{x}) + \frac{f_l(\bar{x})}{k_l(\bar{x})} \partial^*(-k_l)(\bar{x}) \right] + \cl \Delta,
\]

which ends the proof of part (i).
(ii) According to part (i) and hypothesis in part (ii), we have
\[ 0 \in \sum_{i \in L} \alpha_i \co \left[ \partial^* f_i(\bar{x}) + \frac{f_i(\bar{x})}{k_i(\bar{x})} \partial^* (-k_i)(\bar{x}) \right] + \Delta, \]
thus there exist \( a_i \in \alpha_i \co \left[ \partial^* f_i(\bar{x}) + \frac{f_i(\bar{x})}{k_i(\bar{x})} \partial^* (-k_i)(\bar{x}) \right], b_j \in \text{cone } \partial^* g_j(\bar{x}) (j \in J(\bar{x})), c_i \in \text{cone } \partial C h_i(\bar{x}) \) and \( d_i \in \text{cone } - \partial C h_i(\bar{x}) (i \in I) \) such that
\[ 0 = \sum_{i \in L} a_i + \sum_{j \in J(\bar{x})} b_j + \sum_{i \in I} c_i + \sum_{i \in I} d_i. \]
Therefore, there exist nonnegative multipliers \( \mu_j (j \in J(\bar{x})) \) and \( \gamma_i, \nu_i (i \in I) \) such that \( b_j \in \mu_j \co \partial^* g_j(\bar{x}), c_i \in \gamma_i \partial C h_i(\bar{x}) \) and \( d_i \in -\nu_i \partial C h_i(\bar{x}) \). Putting these facts together, (ii) is proved. \( \square \)

The final example illustrates the KKT conditions.

**Example 5.** Consider the following MFP:
\[
\begin{align*}
  f_1(x_1, x_2) := \max & \{0, x_1 + x_2\}, \\
  f_2(x_1, x_2) := & \|x_1\| + \|x_2\| + 1, \\
  k_1(x_1, x_2) := & x_1^2 + 2, \\
  k_2(x_1, x_2) := & 3|x_2| + 5, \\
  g_1(x_1, x_2) := & \max \{x_1, x_2\} \text{ and let } S := \{(x_1, x_2) : g(x_1, x_2) \leq 0\}.
\end{align*}
\]
One can easily check that \( \bar{x} = (0, 0) \) is a local weak efficient solution for this problem and \( g^+(\bar{x}, v) = \max \{v_1, v_2\} \). Therefore,
\[
L(S, \bar{x}) = T(S, \bar{x}) = S = \{(x_1, x_2) : x_1 \leq 0, x_2 \leq 0\},
\]
which implies that GACQ holds at \( \bar{x} \). On the other hand, it is not hard to see that the sets
\[
\begin{align*}
  \partial^* f_1(\bar{x}) &= \{(0, 0), (1, 1)\}, \\
  \partial^* f_2(\bar{x}) &= \{(0, 0)\}, \\
  \partial^* (-k_1)(\bar{x}) &= \{-1, 1\} \times \{-1, 1\}, \\
  \partial^* (-k_2)(\bar{x}) &= \{0\} \times \{-3, 3\} \\
  \text{and } \partial^* g(\bar{x}) &= \{(0, 1), (1, 0)\},
\end{align*}
\]
are upper semi-regular convexificators of the functions \( f_1, f_2, -k_1, -k_2 \) and \( g \), respectively, at \( \bar{x} \). Taking \( \alpha_1 = \alpha_2 = 1 \) and \( \mu = 6/5 \), we get
\[
0 \in \sum_{i=1}^{2} \alpha_i \co \left[ \partial^* f_i(\bar{x}) + \frac{f_i(\bar{x})}{k_i(\bar{x})} \partial^* (-k_i)(\bar{x}) \right] + \mu \co \partial^* g(\bar{x}).
\]

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