Storage and retrieval of squeezing in multimode resonant quantum memories

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In this article the ability to record, store, and read out the quantum properties of light is studied. The discussion is based on high-speed and adiabatic models of quantum memory in lambda-configuration and in the limit of strong resonance. We show that in this case the equality between efficiency and squeezing ratio, predicted by the simple beamsplitter model, is broken. The requirement of the maximum squeezing in the output pulse should not be accompanied by the requirement of maximum efficiency of memory, as in the beamsplitter model. We have demonstrated a high output pulse squeezing, when the efficiency reached only about 50%.

Comprehension of this "paradox" is achieved on the basis of mode analysis. The memories eigenmodes, which have an impact on the memory process, are found numerically. Also, the spectral analysis of modes was performed to match the spectral width of the input signal to the capacities of the memories.

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I. INTRODUCTION

Quantum memory is considered as one of the major challenge for quantum communication and a key element for quantum computing/simulation schemes based on linear optics during a past decade [1–6]. The main objective of quantum memory is to write, store and read out on demand a light pulse without destroying its quantum state.

To implement the quantum memory, many different protocols were proposed (see reviews [7, 8]), based on these a number of key experimental advances have been achieved (see for instance [9–16]). Recently the interests of community shift to the protocols ensured not only high efficiency of quantum state storage and retrieval, but also fit for storage of the signals with many degrees of freedom [17–24].

Broadband or spatially multimode schemes provide this multiplexing. Addressing to quantum protocols of long-distance communication, for which memory cells are the key tool, it is clear that the information capacity of the channel strongly depends on its spectral bandwidth, so that the cell memory as a spectral filter should satisfy some requirements. On the other hand, the involvement of transverse spatial degrees of freedom of the cell and "parallelization" of information stream is also an important resource of spatial multiplexing. Assessing the benefits of quantum memories for production of multiphoton states, authors [17] have introduced an parameter \( \eta B \), where \( \eta \) and \( B \) are the efficiency and time-bandwidth product of the memories, respectively (and \( B \) is the product of the acceptance bandwidth of the memories and their coherence lifetime). Thus, in addition to the high efficiency, we have to be concerned about the spectral properties of the proposed memory protocols. Here we analyze and compare in details the efficiency along with the spectral bandwidths of two memory protocols.

Well known that if you consider off-resonant interaction between atomic ensemble (prepared in the ground state) and two fields - signal and driving, than the interaction Hamiltonian for such a system, after elimination of the excited state, can be cast in the form \( \hat{H} \sim \hat{a} \hat{b}^\dagger + h.c. \). In quantum optics this Hamiltonian is well-known as the beamsplitter Hamiltonian. The interface mixes the input atom and light states as a "beamsplitter" and the "reflection coefficient" of unity corresponds to a perfect state swapping between light and atoms.

However, this ideology not always may be applicable. If for some reason the upper excited state can not be excluded from consideration, such schemes are not beamsplitter-like, and therefore results proved for beamsplitter-like schemes can not be used to them without further justification. Moreover, even for the off-resonant models of memory, when one takes into account the spatial aspects of light-matter interaction and considers the propagation of the fields in thick atomic layer, the direct beamsplitter analogy is also inapplicable.

We consider here two quantum memory protocols based on the resonant interaction between an atomic ensemble and fields - signal and driving. The schemes differ in duration of interaction that leads to differences in formation of the atomic coherence, on which state of the signal field is mapped. One of our main issues here is to analyze the quantum efficiency. We want to find out whether this characteristic of memory is as universal for non beamsplitter-like protocols as for beamsplitter-like ones, i.e. could we predict how well a certain quantum state will be survived in the memory, if we know only the efficiency of this memory.

To demonstrate our arguments, we consider the storage of squeezing in the schemes of adiabatic and high-speed quantum memory. Thereto we analyze how the squeezed light from a particular source with the desired properties is mapped on the atomic ensemble and then read out. Squeezing storage was analyzed in [22], but as opposed to our...
consideration the authors there do not touch any spatial aspects of interaction and solved the problem for the flat input spectrum.

We will also solve the eigenfunction problem for two considered memory protocols. A similar problem was solved for Raman-type protocols [23 26], when the transformation is unitary. Based on the analysis of eigenfunctions, we compare the spectral bandwidth of the memories.

II. EFFICIENCY AND STORAGE OF SQUEEZING IN BEAMSPLITTER LIKE QUANTUM MEMORY

An ideal quantum memory (QM) maps the quantum state of a light pulse into a long lived coherence in the atomic ground state using a 2-photon transition, with the help of a strong classical control field. The signal retrieval, obtained through the interaction of the atomic coherence with the control field should provide a light pulse with the same classical and quantum properties as the input pulse. Several criteria have been proposed to measure the quality of this quantum memory. The first one is the efficiency, which measures the ratio of the input and output field intensities and uses classical quantities. The second one is the fidelity, $F$, with $F=\langle \psi | \rho | \psi \rangle$, where $\psi$ is the initial wave vector (pure state) of the light, and $\rho$ is the density matrix describing the state of light after retrieval. The fidelity is a measure of the conservation of the quantum features of the signal, and allows characterizing the memory as compared to a classical memory, for which the fidelity is limited to some maximal value. However, the fidelity depends on the stored state and cannot be considered as a characterization of the memory device itself. A third benchmark the "TV criterion" was introduced in Ref. [27] and involves a classical feature, the transmission (efficiency) but also quantum characterization with the conditional variance $\lambda$, and allows to establish quantum bounds, while evaluating the classical characteristics. In this paper, we will be interested in both the classical and the quantum benchmarks and we will consider the specific example of squeezed light storage, where several factors can impact the retrieved field.

An intuitive understanding of the connection between the storage of squeezing and the efficiency can be gained from the "beamsplitter" model of a quantum memory, where a full writing and reading cycle can be modelled by the transmission through a beamsplitter. Indeed, let us write the relation between input signal $\hat{a}_{in}$ and output signal $\hat{a}_{out}$ in the form of the beamsplitter transformation:

$$\hat{a}_{out}(t) = \sqrt{T} \hat{a}_{in}(t) - \sqrt{1-T} \hat{a}_{vac}(t).$$  \hspace{1cm} (2.1)

Here, the "transmission" coefficient $T$ models the efficiency of the quantum memory process. Let us rewrite this in the Fourier domain:

$$\hat{a}_{out,\omega} = \sqrt{T} \hat{a}_{in,\omega} - \sqrt{1-T} \hat{a}_{vac,\omega}.$$  \hspace{1cm} (2.2)

Going from the field amplitudes $\hat{a} = \hat{x} + i\hat{y}$ to the fluctuations of the quadrature components $\delta\hat{x}$ and $\delta\hat{y}$ we get

$$\langle \delta\hat{x}_{out,\omega} \delta\hat{x}_{out,\omega} \rangle = T \langle \delta\hat{x}_{in,\omega} \delta\hat{x}_{in,\omega} \rangle + (1-T) \langle \hat{x}_{vac,\omega} \hat{x}_{vac,\omega} \rangle.$$  \hspace{1cm} (2.3)

Introducing the squeezing parameters for the input field $r_{in}(\omega)$ and for the output field $r_{out}(\omega)$, we obtain

$$\langle \delta\hat{x}_{out,\omega} \delta\hat{x}_{out,\omega} \rangle = \frac{1}{4} e^{-r_{out}(\omega)}, \quad \langle \delta\hat{x}_{in,\omega} \delta\hat{x}_{in,\omega} \rangle = \frac{1}{4} e^{-r_{in}(\omega)}.$$  \hspace{1cm} (2.4)

For the vacuum field amplitude

$$\langle \delta\hat{x}_{vac,\omega} \delta\hat{x}_{vac,\omega} \rangle = \frac{1}{4}.$$  \hspace{1cm} (2.5)

As a result we get the equation

$$1 - e^{-r_{out}(\omega)} = T \left[ 1 - e^{-r_{in}(\omega)} \right].$$  \hspace{1cm} (2.6)

We can conclude that the efficiency allows a full characterization of the quantum memory for squeezing transfer if the beamsplitter model is applicable. If the efficiency is close to unity, then the light squeezing is well preserved in the read-out pulse.

In this article, we will consider two cases of resonant quantum memory, namely the adiabatic QM [28] and high-speed QM [24 29]. They are based on the ensemble of three-level atoms in A-configuration (see Fig. 1). Comparing with the non-resonant schemes, here the population of the upper level is non negligible, which leads to losses in the signal field during the light-matter interaction as well as during the storage time. As a result, the analogy between quantum memory and beamsplitter vanishes, and the ratio $\frac{2.6}{2.0}$ can be not satisfied.
III. INPUT-OUTPUT RELATION FOR SIGNAL FIELD IN THREE-LEVEL RESONANT QUANTUM MEMORIES

In this paper, we consider an ensemble of three-level atoms in a Λ-configuration (Fig. 1) that will be used to store temporal and spatial multimode quantum fields. The atoms resonantly interact with two electromagnetic fields, a signal field \( E_s \) and a driving field \( E_d \), that connect the two atomic ground states to the excited state. The driving field is a strong classical field propagating as a plane wave, while the signal field is a weak quantum field with a transverse structure. During the writing process both fields interact with the medium simultaneously.

The dynamics of such a system is described by a set of equations for connecting the signal field amplitude \( \hat{a} \) that is going to be recorded in the atomic medium, operator \( \hat{b} \), associated with the atomic coherence on the transition \(|1\rangle \rightarrow |2\rangle\), which arises in the writing process, and operator \( \hat{c} \), associated with an atomic coherence on the transition \(|1\rangle \rightarrow |3\rangle\) [30]:

\[
\frac{\partial}{\partial z} \hat{a}(z,t;\tilde{q}) = -g\sqrt{N} \hat{c}(z,t;\tilde{q}),
\]

\[
\frac{\partial}{\partial t} \hat{c}(z,t;\tilde{q}) = -\gamma \hat{c}(z,t;\tilde{q}) + g\sqrt{N} \hat{a}(z,t;\tilde{q}) + \Omega \hat{b}(z,t;\tilde{q}) + \sqrt{2\gamma} \hat{\xi}(z,t;\tilde{q}),
\]

\[
\frac{\partial}{\partial t} \hat{b}(z,t;\tilde{q}) = -\Omega^* \hat{c}(z,t;\tilde{q}),
\]

Here \( N \) is a mean density of atoms, \( g \) is the coupling constant between atoms and field on the \(|1\rangle \rightarrow |3\rangle\) transition, \( \Omega \) is a Rabi frequency of the driving field pulse with a rectangular profile, \( \gamma \) is the spontaneous relaxation rate of upper level. These equations are written in the Fourier domain relative to the transverse co-ordinates \((\tilde{r} = (\tilde{\rho}, \tilde{z})\) and \(\tilde{\rho} \rightarrow \tilde{q}\)). The Langevin source \( \hat{\xi}(z,t;\tilde{q}) \) is a due to the decay of optical atomic coherence and is assumed to fulfill the following conditions (derived in the same approximations as the equations (3.1)-(3.3)):

\[
\langle \hat{\xi}(z,t;\tilde{q}) \rangle = 0, \quad \langle \hat{\xi}(z,t;\tilde{q})\hat{\xi}^\dagger(z',t';\tilde{q}') \rangle = \delta(t-t')\delta(z-z')\delta^2(\tilde{q}-\tilde{q}'), \quad \langle \hat{\xi}^\dagger(z,t;\tilde{q})\hat{\xi}(z',t';\tilde{q}') \rangle = 0.
\]

Amplitudes \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \) have to obey the canonical commutation relations:

\[
[\hat{a}(z,t;\tilde{q}), \hat{a}^\dagger(z,t';\tilde{q}')] = \delta(t-t')\delta^2(\tilde{q}-\tilde{q}'),
\]

\[
[\hat{b}(z,t;\tilde{q}), \hat{b}^\dagger(z',t';\tilde{q}')] = [\hat{c}(z,t;\tilde{q}), \hat{c}^\dagger(z',t';\tilde{q}')] = \delta(z-z')\delta^2(\tilde{q}-\tilde{q}').
\]

We will compare the dynamics of the atomic and field variables in two limits: first, when the signal and driving field assumed to be very short pulses, much shorter than the excited state lifetime \( T \ll \gamma^{-1} \), so that we can neglect the spontaneous emission during the writing process (high-speed quantum memory), and, vice versa, when \( T \gg \gamma^{-1} \) and we can neglect the time derivative of the optical coherence \( \hat{c} \) (adiabatic quantum memory).

A. Adiabatic Quantum Memory

The adiabatic approximation can be applied if the evolution time of the system greatly exceeds the life-time of atomic state \(|3\rangle\). In this case we can set \( \partial \hat{c}(z,t;\tilde{q})/\partial t = 0 \) in (3.2). Then equations (3.1)-(3.3) can be written as
$\frac{\partial}{\partial z} \xi(z, t; \bar{q}) = -C_1 \dot{a}(z, t; \bar{q}) - C \dot{b}(z, t; \bar{q}) - 2g\sqrt{N/\gamma} \xi(z, t; \bar{q}), \quad (3.7)$

$\frac{\partial}{\partial t} \dot{b}(z, t; \bar{q}) = -C_2 \dot{b}(z, t; \bar{q}) - C^* \dot{a}(z, t; \bar{q}) - 2\Omega^*/\sqrt{\gamma} \xi(z, t; \bar{q}), \quad (3.8)$

In Eqs. (3.7)-(3.8) we have introduced the following notation for the constants:

$C_1 = \frac{2g^2 N}{\gamma}, \quad C_2 = \frac{2|\Omega|^2}{\gamma}, \quad C = \frac{2\sqrt{N/\gamma}}{\gamma} \quad (C_1 C_2 = |C|^2), \quad \Omega = |\Omega| e^{i\varphi}. \quad (3.9)$

For simplicity, in the following we will assume $\varphi = 0$.

Let us now introduce dimensionless variables for space and time:

$\bar{z} = C_1 z, \quad \bar{t} = C_2 t. \quad (3.10)$

As shown in [28] using Laplace transformation, one can get a general solution of this system in the form

$\dot{a}(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{t} d\bar{t}' \dot{a}(0, \bar{t}'; \bar{q}) G_{aa}(\bar{z}, \bar{t} - \bar{t}') - p \int_0^\bar{z} d\bar{z}' \dot{b}(\bar{z}', 0; \bar{q}) G_{ba}(\bar{z} - \bar{z}', \bar{t}) + \dot{D}_a(\bar{z}, \bar{t}; \bar{q}), \quad (3.11)$

$\dot{b}(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{z} d\bar{z}' \dot{b}(\bar{z}', 0; \bar{q}) G_{bb}(\bar{z} - \bar{z}', \bar{t}) - \frac{1}{p} \int_0^\bar{t} d\bar{t}' \dot{a}(0, \bar{t}'; \bar{q}) G_{ab}(\bar{z}, \bar{t} - \bar{t}') + \dot{D}_b(\bar{z}, \bar{t}; \bar{q}). \quad (3.12)$

$p = \sqrt{\frac{C_2}{C_1}} = \frac{|\Omega|}{g\sqrt{N}}. \quad (3.13)$

The kernels are expressed with the modified Bessel functions $I_0$ and $I_1$ as

$G_{aa}(\bar{z}, \bar{t}) = \left[ e^{-\bar{z}} \delta(\bar{t}) + e^{-\bar{t}} - \bar{z} \sqrt{\bar{t}/\bar{z}} I_1 \left( 2 \sqrt{\bar{t}/\bar{z}} \right) \right] \Theta(\bar{t}), \quad (3.14)$

$G_{bb}(\bar{z}, \bar{t}) = \left[ e^{-\bar{t}} \delta(\bar{z}) + e^{-\bar{t}} - \bar{z} \sqrt{\bar{t}/\bar{z}} I_1 \left( 2 \sqrt{\bar{t}/\bar{z}} \right) \right] \Theta(\bar{t}), \quad (3.15)$

$G_{ba}(\bar{z}, \bar{t}) = G_{ab}(\bar{z}, \bar{t}) = e^{-\bar{t}} - \bar{z} I_0 \left( 2 \sqrt{\bar{t}/\bar{z}} \right) \Theta(\bar{t}). \quad (3.16)$

Here $\Theta(\bar{t})$ is a window function: $\Theta(\bar{t}) = 1$ for $0 < \bar{t} < \bar{T}$ and equals zero otherwise, where $\bar{T}$ is the interaction time ($\bar{T} = \bar{T}_W$ for the writing stage and $\bar{T} = \bar{T}_R$ for the read out). The operators $\dot{D}_a(\bar{z}, \bar{t}; \bar{q})$ and $\dot{D}_b(\bar{z}, \bar{t}; \bar{q})$ are functions of the Langevin source operator $\xi(\bar{z}, \bar{t}; \bar{q})$:

$\dot{D}_a(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{t} \int_0^\bar{z} d\bar{t}' d\bar{z}' \xi(\bar{z}', \bar{t}'; \bar{q}) \left( -\sqrt{2C_1} G_{aa}(\bar{z} - \bar{z}', \bar{t} - \bar{t}') + p \sqrt{2C_2} G_{ba}(\bar{z} - \bar{z}', \bar{t} - \bar{t}') \right), \quad (3.17)$

$\dot{D}_b(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{t} \int_0^\bar{z} d\bar{t}' d\bar{z}' \xi(\bar{z}', \bar{t}'; \bar{q}) \left( -\sqrt{2C_2} G_{ab}(\bar{z} - \bar{z}', \bar{t} - \bar{t}') + \frac{\sqrt{2C_1}}{p} G_{ab}(\bar{z} - \bar{z}', \bar{t} - \bar{t}') \right). \quad (3.18)$

It is important to note that the noise operators $\dot{D}_a$ and $\dot{D}_b$ are independent of the initial conditions.

The solutions given by Eqs. (3.11) and (3.12) can be first used to model the writing process. For this we should take into account the initial conditions for operator $\dot{a}$: $\dot{a}^W(0, \bar{t}; \bar{q}) = \dot{a}_{in}(\bar{t}; \bar{q})$, which means that a signal field in some specific quantum state enters the cell at $\bar{z} = 0$. Before the writing process, all the atoms are in the lower level $|1\rangle$, and the initial state of the coherence operator $\dot{b}^W(\bar{z}, 0; \bar{q})$ is the vacuum state. Equations (3.11) and (3.12) can then be written as

$\dot{a}^W(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{t} d\bar{t}' \dot{a}_{in}(\bar{t}'; \bar{q}) G_{aa}(\bar{z}, \bar{t} - \bar{t}') - p \int_0^\bar{z} d\bar{z}' \dot{b}^W(\bar{z}', 0; \bar{q}) G_{ba}(\bar{z} - \bar{z}', \bar{t}) + \dot{D}_a(\bar{z}, \bar{t}; \bar{q}), \quad (3.19)$

$\dot{b}^W(\bar{z}, \bar{t}; \bar{q}) = \int_0^\bar{z} d\bar{z}' \dot{b}^W(\bar{z}', 0; \bar{q}) G_{bb}(\bar{z} - \bar{z}', \bar{t}) - \frac{1}{p} \int_0^\bar{t} d\bar{t}' \dot{a}_{in}(\bar{t}'; \bar{q}) G_{ab}(\bar{z}, \bar{t} - \bar{t}') + \dot{D}_b(\bar{z}, \bar{t}; \bar{q}). \quad (3.20)$
In Eq. (3.20), the second term is the memory term; it shows the creation of the atomic coherence \( \hat{b}^W \) due to the interaction with the input signal \( \hat{a}_{in} \) and driving field \( \Omega \), and corresponds to the storage of the input field in the atomic coherence, while the first term corresponds to the evolution of the initial atomic coherence \( \hat{b}^W(z, t = 0; \vec{q}) \) interacting with the driving field and the third term to added noise. For \( z = L \), equation (3.19) gives the output signal field, the first term corresponds to the leakage of the signal field, the second one to the generation of a field in the mode of the signal field from the interaction of the initial atomic coherence \( \hat{b}^W(z, t = 0; \vec{q}) \) with the driving field and the third one to added noise.

Since we use the Heisenberg formalism, we deal with operators evolving in time and space and acting on fixed state vector. The state vector of the full system is determined by the states of subsystems - the signal field, the atomic coherence, and Langevin noise source \( \xi \). At the initial time, i.e. before the beginning of the writing procedure, all of these subsystems are independent, and the state vector \( |\psi\rangle \) is factorized:

\[
|\psi^{W}_{a}\rangle = |\psi^{W}_{a}\rangle_{a}|\psi^{W}_{b}\rangle_{b}|\psi^{W}_{\xi}\rangle_{\xi},
\]

where \( |\psi^{W}_{a}\rangle_{a} \) is the initially prepared signal field state, \( |\psi^{W}_{b}\rangle_{b} \) is the initial state of the atomic coherence (vacuum state), and \( |\psi^{W}_{\xi}\rangle_{\xi} \) is the state of the Langevin noise source \( \xi \) (vacuum state).

At the end of the writing stage \( (\tilde{t} = \tilde{T}_W) \), the signal field is vacuum, the driving field is zero, and we assume that the coherence \( \hat{b}^W(\tilde{z}, \tilde{t} = \tilde{T}_W; \vec{q}) \) does not decay with time.

Turning to the analysis of the reading stage, we use again the Heisenberg-Langevin equations (3.11) and (3.12), with different initial conditions and different initial state vector. We assume that there is no input field at the entrance of the cell and the coherence is \( \hat{b}^R(\tilde{z}, 0; \vec{q}) = \hat{b}^W(\tilde{z}, \tilde{T}_W; \vec{q}) \). The state vector of the full system before reading is also factorized:

\[
|\psi^{R}_{a}\rangle = |\psi^{R}_{a}\rangle_{a}|\psi^{R}_{b}\rangle_{b}|\psi^{R}_{\xi}\rangle_{\xi},
\]

Let us derive the signal field operator during the reading process:

\[
\hat{a}^R(\tilde{z}, \tilde{t}; \vec{q}) = \int_{0}^{\tilde{t}} d\tilde{\tau} \hat{a}^R(0, \tilde{\tau}; \vec{q}) G_{aa}(\tilde{z}, \tilde{t} - \tilde{\tau}) - p \int_{0}^{\tilde{T}_W} d\tilde{z} \hat{b}^W(\tilde{z}, \tilde{T}_W; \vec{q}) G_{ba}(\tilde{z} - \tilde{z}', \tilde{t}) + \hat{D}_a(\tilde{z}, \tilde{t}; \vec{q}).
\]

In this equation, the second term shows to the generation of a signal field through the interaction of the atomic coherence \( \hat{b}^W(\tilde{z}, \tilde{T}_W; \vec{q}) \) with the control field and corresponds to the memory read-out. The first term gives the interaction of the input (vacuum) field with the atoms and the control field. The last term corresponds to added noise from the Langevin reservoir.

Substitute \( \hat{a}_{in}^R \) into \( \hat{a}^R(\tilde{z}, \tilde{t}; \vec{q}) \) we obtain:

\[
\hat{a}^R(\tilde{z}, \tilde{t}; \vec{q}) = \int_{0}^{\tilde{t}} d\tilde{\tau} \hat{a}^R(0, \tilde{\tau}; \vec{q}) G_{aa}(\tilde{z}, \tilde{t} - \tilde{\tau}) + \int_{0}^{\tilde{T}_W} d\tilde{z} \hat{a}_{in}(\tilde{\tau}; \vec{q}) G_{ab}(\tilde{z}', \tilde{T}_W - \tilde{\tau}) G_{ba}(\tilde{z} - \tilde{z}', \tilde{t}) - p \int_{0}^{\tilde{z}} d\tilde{z}' \int_{0}^{\tilde{T}_W} d\tilde{z}'' \hat{b}^W(\tilde{z}'', 0; \vec{q}) G_{bb}(\tilde{z}' - \tilde{z}'', \tilde{T}_W) G_{ba}(\tilde{z} - \tilde{z}'', \tilde{t}) - p \int_{0}^{\tilde{z}} d\tilde{z}' \hat{D}_b(\tilde{z}', \tilde{T}_W; \vec{q}) G_{ba}(\tilde{z} - \tilde{z}', \tilde{t}) + \hat{D}_a(\tilde{z}, \tilde{t}; \vec{q}).
\]

Now let us make averaging of this expression over the vacuum subsystems. Then instead of the canonical amplitude \( \hat{a}^R(\tilde{z}, \tilde{t}; \vec{q}) \) the new one arises:

\[
\hat{A}(\tilde{z}, \tilde{t}; \vec{q}) \equiv \langle vac^W|\langle vac^W|\hat{a}^R(\tilde{z}, \tilde{t}; \vec{q})|vac^W\rangle|vac^R\rangle,
\]

where \( |vac^W\rangle = |vac\rangle_{a}|vac\rangle_{\xi} \) are vacuum state vectors before writing and \( |vac^R\rangle = |vac\rangle_{a}|vac\rangle_{\xi} \) are vacuum state vectors before reading. Further we shall call operator \( \hat{A}(\tilde{z}, \tilde{t}; \vec{q}) \) the non-canonical amplitude (NCA), because the corresponding commutation relations have acquired the non-canonical form.

Let us emphasize that we do not average over the complete wave function, but only over those subsystems that are in the vacuum state. In this case, the quantum nature of the input signal is taken into account, and we can keep some of the quantum properties of the output signal field. As a result, all the terms except the second one in (3.24) turn out equal to 0:

\[
\hat{A}^R(\tilde{L}, \tilde{t}; \vec{q}) = \int_{0}^{\tilde{T}_W} d\tilde{\tau} \hat{a}_{in}(\tilde{T}_W - \tilde{\tau}; \vec{q}) G(\tilde{t}, \tilde{\tau}).
\]
We thus cast an integral transformation of the input field operator into the operator of the output field retrieved in the reading process.

For the forward read-out (when the reading driving field propagates in the same direction as the writing driving field), the kernel $G(\tilde{t}, \tilde{t}')$ reads:

$$G(\tilde{t}, \tilde{t}') = \int_0^L d\tilde{z}' G_{ab}(\tilde{z}', \tilde{t}') G_{ba}(\tilde{L} - \tilde{z}', \tilde{t}).$$  \hspace{1cm} (3.27)

For the backward read-out (when these two driving fields propagate in opposite directions) the transformation keeps the same form \eqref{3.20} with the kernel

$$G(\tilde{t}, \tilde{t}') = \int_0^L d\tilde{z}' G_{ab}(\tilde{z}', \tilde{t}') G_{ba}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (3.28)

where $G_{ab}$ and $G_{ba}$ are given by Eq. \eqref{3.10}. Although the introduced partially averaged operators do not convey all the quantum properties of the field (because, for example, they do not obey canonical commutation relations that can be easily seen from comparison of the canonical amplitude \eqref{3.24} with non-canonical one \eqref{3.26}), they will be very useful in the following. For example, for the $x$-quadrature of the retrieved field $\hat{a}_R$,

$$\langle \hat{x}_a(\tilde{z}, \tilde{t}; \tilde{q}) \hat{x}_a(\tilde{z}', \tilde{t}'; \tilde{q}) \rangle = \langle \hat{X}_A(\tilde{z}, \tilde{t}; \tilde{q}) \hat{X}_A(\tilde{z}', \tilde{t}'; \tilde{q}) \rangle; \hspace{0.5cm} \hat{A}_R = \hat{X}_A + i \hat{Y}_A, \hspace{0.5cm} \hat{a}_R = \hat{x}_a + i \hat{y}_a,$$  \hspace{1cm} (3.29)

where notation $\langle \ldots \rangle$ stands for normal ordering of the operators.

### B. High-speed quantum memory

In the high-speed model of quantum memory the signal and driving pulses are assumed to be much shorter than the excited state lifetime $\gamma^{-1}$, so that we can neglect spontaneous emission during the interaction processes, which eliminates a source of fluctuations and dissipation in \eqref{3.2}. Then the equations for signal field amplitude $a(z,t;\tilde{q})$, atomic coherence between low levels $b(z,t;\tilde{q})$, and optical coherence between levels $|1\rangle$ and $|3\rangle$ $c(z,t;\tilde{q})$ read \cite{24}:

$$\frac{\partial}{\partial z} \hat{a}(z,t;\tilde{q}) = -g\sqrt{N} \hat{c}(z,t;\tilde{q}),$$  \hspace{1cm} (3.30)

$$\frac{\partial}{\partial t} \hat{c}(z,t;\tilde{q}) = g\sqrt{N} \hat{a}(z,t;\tilde{q}) + \Omega \hat{b}(z,t;\tilde{q}),$$  \hspace{1cm} (3.31)

$$\frac{\partial}{\partial t} \hat{b}(z,t;\tilde{q}) = -\Omega^* \hat{c}(z,t;\tilde{q}),$$  \hspace{1cm} (3.32)

The notation here is the same as in \eqref{3.1}-\eqref{3.3}.

Eqs. \eqref{3.30}-\eqref{3.32} can be solved in the general form by using the Laplace transform. Details of the procedures are discussed in Appendix A. As a result, applying the same technique of averaging over the vacuum subsystems, we can derive the solution for retrieval field formally in the same form as for adiabatic case:

$$\hat{A}_R(\tilde{L}, \tilde{t}; \tilde{q}) = \int_0^{\tilde{T}_W} d\tilde{t}' \hat{a}_{in}(\tilde{T}_W - \tilde{t}'; \tilde{q}) G(\tilde{t}, \tilde{t}'),$$  \hspace{1cm} (3.33)

Here the kernel $G(\tilde{t}, \tilde{t}')$ is given by

$$G(\tilde{t}, \tilde{t}') = \frac{1}{2} \int_0^L d\tilde{z}' G_{ab}(\tilde{z}', \tilde{t}') G_{ba}(\tilde{L} - \tilde{z}', \tilde{t})$$  \hspace{1cm} (3.34) \hspace{0.5cm} - for the forward retrieval

$$G(\tilde{t}, \tilde{t}') = \frac{1}{2} \int_0^L d\tilde{z}' G_{ab}(\tilde{z}', \tilde{t}') G_{ba}(\tilde{z}', \tilde{t})$$  \hspace{1cm} (3.35) \hspace{0.5cm} - for the backward retrieval,

and functions $G_{ab}$ and $G_{ba}$ now read

$$G_{ab}(\tilde{z}, \tilde{t}) = G_{ba}(\tilde{z}, \tilde{t}) = \int_0^{\tilde{t}} d\tilde{t}' g_{ab}(\tilde{z}, \tilde{t} - \tilde{t}') g_{ab}(\tilde{z}, \tilde{t}'),$$  \hspace{1cm} (3.36)

$$g_{ab}(\tilde{z}, \tilde{t}) = e^{-\tilde{t}^2} J_0 \left( \sqrt{\tilde{t}^2} \right) \Theta(\tilde{t}),$$  \hspace{1cm} (3.37)

where $J_0$ is the Bessel functions of the first kind of the zero order. Here and below when we discuss high-speed QM we use dimensionless variables $\tilde{t}$ and $\tilde{z}$ given by

$$\tilde{t} = \Omega t, \hspace{1cm} \tilde{z} = 2g^2 N z/\Omega.$$  \hspace{1cm} (3.38)
IV. COMPARISON OF THE QUANTUM MEMORY PERFORMANCE FOR PULSE ENERGY AND FOR SQUEEZING

In this section we want to compare the results of two mental experiments obtained for both adiabatic and high-speed memory models: an experiment to study the efficiency of QM and experiment for the storage of squeezed light. As shown in Section III the results of such experiments are uniquely related for the case of the beamsplitter-type interaction. We will consider whether they related in some way in our models.

The study of the quantum efficiency of the QM implies the following scheme: signal and control pulses with some defined profiles both are incident on the input face of the atomic cell. Propagating through the cell, these fields generate a coherence between the two lower levels with a certain spatial distribution, which depends both on the nature of light-matter interaction and on the duration of this interaction. After some storage time, the retrieved control pulse is sent to the input or output cell face and triggers the emission at the signal frequency, which is collected by a photodetector. One compares the total photon numbers in reconstructed signal and in the input pulse.

Analytically, the quantum efficiency can be read as the following ratio of the averages:

\[ \mathcal{E}_q = \int_0^{T_R} d\tilde{t} \langle \hat{a}_{out}^\dagger(\tilde{t}; \tilde{q}) \hat{a}_{out}(\tilde{t}; \tilde{q}) \rangle^{-1} \int_0^{T_W} d\tilde{t} \langle \hat{a}_{in}^\dagger(\tilde{t}; \tilde{q}) \hat{a}_{in}(\tilde{t}; \tilde{q}) \rangle. \] (4.1)

Then taking into account (3.26), one can obtain for the arbitrary input pulse

\[ \mathcal{E}_q = \left( \int_0^{T_R} d\tilde{t} \int_0^{T_W} d\tilde{t}' \int_0^{T_W} d\tilde{t}'' \langle \hat{a}_{in}^\dagger(\tilde{T}_W - \tilde{t}'\tilde{q}) \hat{a}_{in}(\tilde{T}_W - \tilde{t}''\tilde{q}) \rangle G(\tilde{t}, \tilde{t}') G(\tilde{t}, \tilde{t}'') \right)^{-1}. \] (4.2)

Choosing a rectangular signal profile, we have to neglect the time dependencies of the input correlation functions

\[ \langle \hat{a}_{in}^\dagger(\tilde{T}_W - \tilde{t}'\tilde{q}) \hat{a}_{in}(\tilde{T}_W - \tilde{t}''\tilde{q}) \rangle = \langle \hat{a}_{in}^\dagger(\tilde{q}) \hat{a}_{in}(\tilde{q}) \rangle = \langle \hat{a}_{in}^\dagger(\tilde{q}) \hat{a}_{in}(\tilde{q}) \rangle. \] (4.3)

Then the formula for the efficiency turns out to be essentially simpler

\[ \mathcal{E}_q = \frac{1}{T_W} \int_0^{T_R} d\tilde{t} \left( \int_0^{T_W} d\tilde{t}' G(\tilde{t}, \tilde{t}') \right)^2. \] (4.4)

One can see, for different components of \( \tilde{q} \) the efficiencies are equal.

In the second case, we assume that the input pulse is squeezed. Let us assume that pulsed light with given quantum properties is obtained from a steady source of squeezed light in which a pulse has been cut. It is this kind of light source that was used in the experiments on quantum memory for squeezed states [31].

Here, we consider two examples of pulsed quantum light generation when the single pulse is obtained by cutting the time of radiation 1) of single-mode phase-locked sub-Poissonian laser, which emits a squeezed light in a close to coherent state [32], and 2) of the degenerate optical parametric oscillator (DOPO), operating in the subthreshold regime [33, 34]. To assess the conservation of the quantum light properties, we will compare the quantum properties of the read-out pulse with the original one.

Let us introduce the degree of light squeezing \( S_{\text{in}, \omega, \tilde{q}} \) and \( S_{\text{out}, \omega, \tilde{q}} \) for the input and output pulses, respectively:

\[ S_{\text{out}, \omega, \tilde{q}} = e^{-r_{\text{out}}(\tilde{\omega}, \tilde{q})}, \quad S_{\text{in}, \omega, \tilde{q}} = e^{-r_{\text{in}}(\tilde{\omega}, \tilde{q})}, \quad \tilde{\omega} = \omega/\Omega. \] (4.5)

We assess the ratio of these parameters to compare squeezing in the output pulse with squeezing in the input one. Taking into account Eq. (2.4) this ratio can be derived in the form

\[ \frac{1 - S_{\text{out}, \omega, \tilde{q}}}{1 - S_{\text{in}, \omega, \tilde{q}}} = \frac{\langle \hat{x}_{\text{out}}(\omega, \tilde{q}) \hat{x}_{\text{in}}(\omega, \tilde{q}) \rangle}{\langle \hat{x}_{\text{in}}^2(\omega, \tilde{q}) \rangle}. \] (4.6)

According to (3.33) input and output correlation functions in time domain are connected with the expression:

\[ \langle \hat{x}_{\text{in}}^R(\tilde{T}, \tilde{t}', \tilde{q}') \hat{x}_{\text{in}}(\tilde{T}, \tilde{t}, \tilde{q}) \rangle = \int_0^{T_W} d\tilde{t}_1 \int_0^{T_W} d\tilde{t}' \langle \hat{x}_{\text{in}}(\tilde{T}_W - \tilde{t}_1, \tilde{q}) \hat{x}_{\text{in}}(\tilde{T}_W - \tilde{t}_2, \tilde{q}) \rangle G(\tilde{t}_1, \tilde{t}_2) G(\tilde{t}', \tilde{t}_2). \] (4.7)
Passing to the Fourier domain one can obtain

\[
\langle : \delta \hat{x}_{\text{out},\bar{\omega},\bar{q}} \delta \hat{x}_{\text{out},\tilde{\omega}',\tilde{q}'} : \rangle = \int_{0}^{\tilde{T}_{W}} d\tilde{t}_{1} \int_{0}^{\tilde{T}_{W}} d\tilde{t}_{2} \langle : \delta \hat{x}_{\text{in}}(\tilde{T}_{W} - \tilde{t}_{1}, \tilde{q}') \delta \hat{x}_{\text{in}}(\tilde{T}_{W} - \tilde{t}_{2}, \tilde{q}'): \rangle G(\tilde{\omega}, \tilde{t}_{1})G(\tilde{\omega}', \tilde{t}_{2}),
\]

where the Fourier transformation is given by

\[
\delta \hat{x}_{\text{out},\bar{\omega},\bar{q}} = \frac{1}{\sqrt{R}} \int_{0}^{\tilde{T}_{R}} d\tilde{t} \delta \hat{x}(\tilde{L}, \tilde{t}, \tilde{q}) e^{i\tilde{\omega}\tilde{t}}.
\]

As for the input correlation function \(\langle : \delta \hat{x}_{\text{in},\bar{\omega},\bar{q}} \delta \hat{x}_{\text{in},\tilde{\omega}',\tilde{q}'} : \rangle\) in the denominator of (4.8), one can find it in the explicit form in Appendix B for two squeezed light sources - the sub-Poissonian phase-locked laser and degenerated optical parametric oscillator.

Before we start to compare the input and output squeezing, it should be noted that the squeezing degree in the input pulse is already less than in the steady beam \[33\]. This fact is important, because the mechanism of QM depends essentially on the pulsed nature of the recording light, and we cannot neglect by this feature. However, in \[33\] it has been shown that if the condition \(\kappa T \gg 1\) for the pulse duration \(T\) and the spectral width of the light \(\kappa\) is fulfilled then the pulse squeezing degree does not depend on its duration and equals to one in stationary radiation. We will assume that this condition is verified. All the necessary details specifying the input radiation parameters one can find in Appendix B. Graphically, the input pulse spectrum is shown in Fig. 2.

Bellow we are interested in squeezing of the pulse as a whole that corresponds to the zero frequency component in the spectrum of squeezing \(S_{\text{out},\bar{\omega},\bar{q}}\), i.e., in value \(S_{\text{out},0,0}\), it characterizes the maximum achievable degree of squeezing.

It should be noted that the properties of input radiation in two above-described mental experiments are quite different. In fact, studying the efficiency we are restricted by the analysis of the essentially narrow-band light, because for given profile of the input it can be characterized by a single time-independent amplitude. That is, one is discussing the storage for single mode (rectangular in our description). Input-output transformation converts the profile, resulting in the other field profile at the output, but input and output profiles are connected by unique integral transformation and the single mode at the input is converted into a single (different from the original) mode at the output. When we study the degree of squeezing we are dealing with a broadband signal: the input field can not be characterized by the single amplitude \(\hat{a}_{\text{in}}\), but is a function of time. This is the time dependence is the subject of the study in this experiment. The question of the legitimacy for comparing the results of two such a different mental experiments, we will discuss below.

Fig. 3 depicts the results of the numerical calculations of the efficiency \(E\) and pulse squeezing \(S_{\text{out},0,0}\) (more precisely, \(1 - S_{\text{out},0,0}\)) as a function of read-out time \(\tilde{T}_{R}\) for forward retrieval in two models: high-speed QM (Fig. 3a) and adiabatic QM (Fig. 3b).

The parameters for numerical calculation have been chosen as follows. In the scheme of high-speed QM for given dimensionless length of the medium \(\tilde{L} = 10\), we found the duration of the input pulse, resulting in the lowest signal losses in writing (see \[24\]), \(\tilde{T}_{W} = 5.5\) meets this requirement. The efficiency in Fig. 3a is plotted for these parameters. To calculate the pulse squeezing the same values of \(\tilde{T}_{W}\) and \(\tilde{L}\) have been chosen, and in addition, we used the input field parameters \(\tilde{\kappa} = 100/5.5, \quad p = 1\), to provide knowingly a good squeezing of the input pulse (below we will demonstrate how to optimize the choice of the spectral width of the input signal).
Figure 3: Efficiency (dashed curve) and pulse squeezing (solid curve) as a function of reading time $\tilde{T}_R$ in two models: high-speed QM (a) and adiabatic QM (b) for forward retrieval.

Figure 4: Efficiency (dashed curve) and pulse squeezing (solid curve) as a function of reading time $\tilde{T}_R$ in two models: high-speed QM (a) and adiabatic QM (b) for backward retrieval.

For the calculation in the adiabatic limit we have chosen $\tilde{T}_W = \tilde{L} = 55$, providing approximately the same efficiency as for the high-speed QM. The value of $\tilde{\kappa}$ is chosen to satisfy the condition $\tilde{\kappa}\tilde{T}_W \gg 1$. Plot in Fig. 3b corresponds to $\tilde{\kappa} = 100/55$, but it should be noted that the increase of $\kappa$ has no effect on the behavior of the curves.

First, let us note that the efficiency and pulse squeezing in Fig. 3 do not always coincide. The efficiency increases monotonically and comes to saturation, while the pulse squeezing decreases after a certain time $\tilde{T}_R$. Such a behavior of the curves is a clear demonstration of the non-beamsplitter type of interaction in these models. This result is in line with our knowledge of the role of noises in these models. It is not hard to believe that we have a high efficiency of QM, but lose the squeezing of radiation due to the contribution of vacuum noise when the read-out go on too long time.

On the other hand, the figure shows the range of values $\tilde{T}_R$ where pulse squeezing higher than the efficiency. The same situation takes place for curves in the case of backward retrieval (see Fig. 4). On the face of it, these curves seem paradoxical: they appeared a range of values $\tilde{T}_R$ such that under sufficiently low efficiency (50% only), we stored a good squeezing in the output. Could it be? A clue to this paradox lies in the multimode nature of the interaction. In the next section we discuss the mode properties in both adiabatic and high-speed QM models.

V. DISCUSSION

A. Eigenmodes of the memories

It is not difficult to see that for both considered here models of memory the kernels $G(t, t')$ in Eqs. (3.26) and (3.33), which couple the input and output signals, are symmetrical with respect to permutation of the arguments $t \leftrightarrow t'$. This means we have a right to derive the equation for the eigenfunctions $\psi_i(t)$ and eigenvalues $\sqrt{\lambda_i}$ of these
kernels in the form
\[
\sqrt{\lambda_i} \psi_i(t) = \int_0^{T_W} dt' \psi_i(t') G(t, t').
\] (5.1)

Here the functions \(\psi_i(t)\) form the complete orthonormal set:
\[
\int_0^{T_W} dt \psi_i^*(t) \psi_j(t) = \delta_{ij}, \quad \sum_i \psi_i^*(t) \psi_i(t') = \delta(t - t').
\] (5.2)

The equation (5.1) can be solved for each of the four kernels of the integral transformations (3.26) and (3.33). In this article we will analyze the eigenfunctions and eigenvalues only for backward read-out, as this is the case for both models when the “abnormally” high pulse squeezing at the relatively low efficiency appears.

We will carry out the numerical calculations at the same values of parameters as before (in Fig. 3 and 4). First, let us note that the eigenvalues in both cases decrease rapidly (see Fig. 5). Especially sharp decrease of eigenvalues is observed for the model of high-speed QM, where, in fact, only the first two eigenvalues differ from zero. This behavior shows that this scheme is a good filter for the first two eigenmodes. Figures 6 and 7 shows the first three eigenmodes (the first row) for the high-speed QM and adiabatic QM models, correspondingly, as well as the squares of these functions (the second row).

The distinguishing feature of the high-speed QM model is not only the filtering in this scheme of only two modes, but also the localization of these modes in different intervals of the timeline (the first mode is actually localized in the range of \(\tilde{T}_R \in [0, 2.75]\), and the second one - in \(\tilde{T}_R \in [2.75, 5.5]\)). Draw attention that the boundaries of intervals coincide with the locations of two peaks in the curve in fig. 4a. For the adiabatic memory such a mode “separation” is not observed.

Now let us see how squeezing is distributed over the modes, that means we want to follow whether all eigenmodes of the cell are squeezed. For this aim we plotted the spectra of the first three modes (see the third row in Figs. 6 and 7). Recall that the input field has the wide spectrum of squeezing (see Fig. 2). One can see that all of our spectra of eigenmodes are concentrated in the area of good squeezing, that means, they are well squeezed. However, not all of them contribute to the squeezing of pulse as whole (squeezing at the zero frequency).

Going back to the curves in Fig. 4b, we see that at time \(\tilde{T}_R = 2.75\) the first mode of the field is mostly read-out, the second mode is localized on the other time interval, and the rest is filtered out by the cell. The zero spectral component of the first mode is high and squeezed well (that is the first mode as whole is squeezed well). Thus, although the half part of the photons still is not read-out - they will be retrieved from the second mode, one can obtain a high squeezing in the retrieved pulse.

B. Comparing the actual parameters for two schemes

It should be noted that the two QM models are very different spectrally. To estimate the typical spectral widths for eigenmodes in both cases, we should revert from the dimensionless variables to dimensional ones. Recall that the dimensionless procedures for the models are different. For time we got the relations (3.38) and (3.10), respectively:
\[
\Omega_{HS} t \rightarrow \tilde{t}_{HS}, \quad \frac{2\Omega_{AD}^2}{\gamma} t \rightarrow \tilde{t}_{AD},
\] (5.3)
Figure 6: The first three eigenmodes of the integral transformation of the field operator from the input to the output of memory cell (upper row), squared eigenmodes (middle row) and Fourier spectra of eigenmodes (lower row) for high-speed QM model. Dimensionless time is $\Omega t$, dimensionless frequency is $\omega/\Omega$.

Figure 7: The first three eigenmodes of the integral transformation of the field operator from the input to the output of memory cell (upper row), squared eigenmodes (middle row) and Fourier spectra of eigenmodes (lower row) for adiabatic QM model. Dimensionless time is $2\Omega^2 t/\gamma$, dimensionless frequency is $\omega\gamma/(2\Omega^2)$. 
where we now added the below indexes "HS" and "AD" to indicate the high-speed or adiabatic model, respectively. Besides, the requirements imposed on the Rabi frequency, are opposite:

\[ \gamma \ll \Omega_{HS}, \quad \gamma \gg \Omega_{AD}, \]  

whence it follows \( \Omega_{AD} \ll \Omega_{HS} \), and the ratio of this frequencies has to be at least 2 orders. Based on these inequalities and noting that the ratio of dimensionless spectral widths in Figs. 6 and 7 is \( \Delta \omega_{HS}/\Delta \omega_{AD} \approx 10 \), we find that the width of eigenmodes in the adiabatic model is at least 4 orders of magnitude smaller than the corresponding width in the high-speed model. At the same time, one can see that although the dimensionless values of \( L \) are different, we can assume that both calculations were carried out at the same optical depth \( d = 2g^2NL/\gamma \sim 55 \) (supposing the relation \( \Omega_{HS}/\gamma \approx 10 \) to be fulfilled). Let us note that the relation between the parameters typical for the high-speed QM, can be easily reached in experiments with resonant \( \Lambda \)-atoms, especially for cold atomic ensemble, than for the adiabatic condition.

Figures 6 and 7 shows that the spectral bandwidths of memories are much less than the chosen spectral width of the input signal. We can say that this choice was not optimal, and it is necessary to match the input signal to the memory capacity. In this sense, the model of high-speed QM looks more attractive because it provides a much wider spectral bandwidth of the signal as mentioned above.

The decrease in the value of the spectral width \( \kappa \), while maintaining the duration of the input pulse results in a loss of input pulse squeezing (see Fig. 8). However, it is seen that for \( \tilde{\kappa} \approx 2 \) for the high-speed QM and for \( \tilde{\kappa} \approx 0.2 \) for the adiabatic QM these losses are relatively small (about 10%), and the ratio \( \tilde{\kappa} \tilde{T} \gg 1 \) is still satisfied.

In Fig. 8 the input and retrieval spectra of squeezing are plotted for the case when spectral bandwidth of squeezing of the input signal is matched to the memory bandwidth. Since considered memory cells are the mode filters, i.e. they are sensitive to only a few the first modes of input signal, then the presence of other modes at the input does not affect to the pulse squeezing. From the Fig. 8 one can see that the zero spectral component is large enough only in the spectra of the first and second eigenmodes. At the same time, the square input pulse is well approximated by the superposition of the first two modes. That is why the comparison of the two mental experiments (estimation of the efficiency and of the storage of squeezing) with the different input signals seems to us justified. For example, in the scheme of high-speed QM if knowing the properties of our cell as a mode filter, we form an input signal as a superposition of the first two modes, then the curves in Fig. 4 remain almost unchanged.

**VI. IS THE EFFICIENCY OF EXHAUSTIVE CHARACTERISTIC OF QUANTUM MEMORY?**

In Raman memory models based on three-level atoms, two quantum sub-systems (the signal field and atomic coherence) are interacted with each other. The writing and read-out processes are formally realized by means of the interaction Hamiltonian of the kind of \( g(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \). This Hamiltonian ensures the beamsplitter-like interaction for the thin atomic layer and as a result one can derive the input-output equation (2.1) and conclude that squeezing and efficiency are coupled by the formula (2.6). This means that the efficiency turns out to be sufficient to assess the quality of memory [30, 36].

In this article we consider more complicated situation, which takes place in the cases of adiabatical and high-speed memories in the strongly resonant limit. In each of these models there are three quantum sub-systems: the signal
field and two atomic coherences for the high-speed memory and the signal field, atomic coherence, and the vacuum reservoir, forming the Langevin source, for the adiabatic memory. The presence of more than two active subsystems makes impossible direct interpretation of interaction as a beamsplitter-like one. However maybe we still can apply such an interpretation in some sense? Let’s consider again the equation (4.2) coupling the signal field at the output of the memory cell after the read-out with the signal at the front of the cell before the writing process. Let’s rewrite it in the form of the non-averaged over vacuum sub-systems

\[ \hat{a}_{\text{out}}(\tilde{t}, \tilde{q}) = \int_0^{\tilde{T}_W} G(\tilde{t}, \tilde{t}') \hat{a}_{\text{in}}(\tilde{T}_W - \tilde{t}', \tilde{q}) d\tilde{t}' + \hat{F}_{\text{vac}}(\tilde{t}, \tilde{q}). \]  

(6.1)

Remind the indexes "in" and "out" mean \( \tilde{z} = 0 \) and \( \tilde{z} = \tilde{L} \), respectively. The second term on the right \( \hat{F}_{\text{vac}} \) concentrates contributions of all sub-systems, which initially were in the vacuum state. It is not difficult to see that the input-output amplitudes obey the correct commutation relations

\[ [\hat{a}_{\text{out}}(\tilde{t}_1, \tilde{q}_1), \hat{a}_{\text{out}}^\dagger(\tilde{t}_2, \tilde{q}_2)] = [\hat{a}_{\text{in}}(\tilde{t}_1, \tilde{q}_1), \hat{a}_{\text{in}}^\dagger(\tilde{t}_2, \tilde{q}_2)] = \delta(\tilde{t}_1 - \tilde{t}_2) \delta^2(\tilde{q}_1 - \tilde{q}_2). \]  

(6.2)

provided that

\[ [\hat{F}_{\text{vac}}(\tilde{t}_1, \tilde{q}_1), \hat{F}_{\text{vac}}(\tilde{t}_2, \tilde{q}_1)] = \left( \delta(\tilde{t}_1 - \tilde{t}_2) - \int_0^{\tilde{T}_W} G(\tilde{t}_1, \tilde{t}') G(\tilde{t}_2, \tilde{t}') d\tilde{t}' \right) \delta^2(\tilde{q}_1 - \tilde{q}_2). \]  

(6.3)

As mentioned above the kernel \( G(\tilde{t}, \tilde{t}') \) is symmetrical and we have a right to derive Eq. (6.1) for its eigenfunctions, which obey the requirements (6.2).

Let us decompose input and output amplitudes in the form

\[ \hat{a}_{\text{in}}(\tilde{T}_W - \tilde{t}, \tilde{q}) = \sum_i \hat{e}_{\text{in};i,\tilde{q}} \psi_i(\tilde{t}), \quad \hat{a}_{\text{out}}(\tilde{t}, \tilde{q}) = \sum_i \hat{e}_{\text{out};i,\tilde{q}} \psi_i(\tilde{t}), \]  

(6.4)

where

\[ [\hat{e}_{\text{in};i,\tilde{q}}, \hat{e}_{\text{in};i',\tilde{q}}^\dagger] = [\hat{e}_{\text{out};i,\tilde{q}}, \hat{e}_{\text{out};i',\tilde{q}}^\dagger] = \delta_{ii'} \delta^2(\tilde{q} - \tilde{q}'). \]  

(6.5)

The similar decomposition can be used for function \( \hat{F}_{\text{vac}}(\tilde{t}, \tilde{q}) \):

\[ \hat{F}_{\text{vac}}(\tilde{t}, \tilde{q}) = \sum_i \hat{f}_{\text{vac};i,\tilde{q}} \psi_i(\tilde{t}), \quad \text{where} \quad \hat{f}_{\text{vac};i,\tilde{q}} = \int_0^{\tilde{T}_W} d\tilde{t} \psi_i(\tilde{t}) \hat{F}_{\text{vac}}(\tilde{t}, \tilde{q}). \]  

(6.6)

Taking into account the equation (6.3) one can obtain

\[ [\hat{f}_{\text{vac};i,\tilde{q}}, \hat{f}_{\text{vac};i',\tilde{q}}^\dagger] = (1 - \lambda_i) \delta_{ii'} \delta^2(\tilde{q} - \tilde{q}'). \]  

(6.7)

or making normalization

\[ [\hat{e}_{\text{vac};i,\tilde{q}}, \hat{e}_{\text{vac};i',\tilde{q}}^\dagger] = \delta_{ii'} \delta^2(\tilde{q} - \tilde{q}'), \quad \text{where} \quad \hat{f}_{\text{vac};i,\tilde{q}} = \sqrt{1 - \lambda_i} \hat{e}_{\text{vac};i,\tilde{q}}. \]  

(6.8)

Substituting (6.4), (6.6) and (6.8) into (6.1) one can obtain for each of modes

\[ \hat{e}_{\text{out};i,\tilde{q}} = \sqrt{\lambda_i} \hat{e}_{\text{in};i,\tilde{q}} - \sqrt{1 - \lambda_i} \hat{e}_{\text{vac};i,\tilde{q}}. \]  

(6.9)

Thus, in this sense, we can consider our model as a beamsplitter-like ones. However such a classification associated with choice of the input signal as one (and only one) of the memory eigenfunction.

The question arises whether we can correctly determine the efficiency for an arbitrary input signal and whether it will be as comprehensive characteristic of the memory as the efficiency associated with the transfer of eigenmode. If we consider the arbitrary input signal as a single mode from some complete orthonormal set of functions \( \hat{A}_{\text{in}}(\tilde{T}_W - \tilde{t}) = h_1(\tilde{t}) \hat{E}_1 \), then the efficiency can be expressed analytically. In this case, one can obtain from the formula (4.2)

\[ \mathcal{E} = \int_0^{\tilde{T}_R} d\tilde{t} \left( \int_0^{\tilde{T}_W} d\tilde{t}' h_1(\tilde{t}') G(\tilde{t}, \tilde{t}') \right)^2. \]  

(6.10)
To assess how squeezing is changed under the full memory process, we can use ratio (6.10) putting there $\omega = 0$ and $\tilde{q} = 0$

$$\mathcal{E}_{\text{squeezing}} = \frac{\langle \delta \tilde{x}_{\text{out}}, \omega = 0, \tilde{q} = 0 | \delta \tilde{x}_{\text{out}}, \omega = 0, -\tilde{q} = 0 \rangle}{\langle \delta \tilde{x}_{\text{in}}, \omega = 0, \tilde{q} = 0 | \delta \tilde{x}_{\text{in}}, \omega = 0, -\tilde{q} = 0 \rangle}.$$

(6.11)

According (4.8) the expression in numerator is given by

$$\langle \delta \tilde{x}_{\text{out}}, \omega = 0, \tilde{q} = 0 | \delta \tilde{x}_{\text{out}}, \omega = 0, -\tilde{q} = 0 \rangle =$$

$$= \int_0^{\tilde{T}_W} d\tilde{t}_1 \int_0^{\tilde{T}_W} d\tilde{t}_2 \langle \delta \tilde{x}_{\text{in}}(\tilde{T}_W - \tilde{t}_1, \tilde{q} = 0) \delta \tilde{x}_{\text{in}}(\tilde{T}_W - \tilde{t}_2, -\tilde{q} = 0) : G(\omega = 0, \tilde{t}_1)G(-\omega = 0, \tilde{t}_2).$$

(6.12)

Now taking into account that

$$\langle : \delta \tilde{x}_{\text{in}}, \omega = 0, \tilde{q} = 0 | \delta \tilde{x}_{\text{in}}, \omega = 0, -\tilde{q} = 0 : \rangle =$$

$$= \frac{1}{2\pi} \int_0^{\tilde{T}_W} d\tilde{t}_1 \int_0^{\tilde{T}_W} d\tilde{t}_2 \langle \delta \tilde{x}_{\text{in}}(\tilde{t}_1, \tilde{q} = 0) \delta \tilde{x}_{\text{in}}(\tilde{t}_2, -\tilde{q} = 0) : \rangle.$$

(6.13)

and $\hat{A}_\text{in}(\tilde{t}) = h(\tilde{t})\hat{E}_1$ one can obtain

$$\mathcal{E}_{\text{squeezing}} = \left[ \int_0^{\tilde{T}_W} d\tilde{t} \int_0^{\tilde{T}_W} d\tilde{t}' h_1(\tilde{t}') G(\tilde{t}, \tilde{t}') / \int_0^{\tilde{T}_W} h_1(\tilde{t}) d\tilde{t} \right]^2.$$

(6.14)

It is clear that the expressions (6.10) and (6.14) coincide only if the profile $h_1(\tilde{t})$ is an eigenfunction of the kernel $G(\tilde{t}, \tilde{t}')$. Thus, the efficiency determined for a single non-eigenmode provides an answer to the question about the ability of the memory to store the total number of photons, but does not characterize the preservation of other quantum properties of light, such as squeezing.

Beamsplitter relation can be generalized in the form [37]:

$$\tilde{r} = \sqrt{\lambda} \hat{s} - \sqrt{1 - \lambda} \hat{e}_\text{vac},$$

(6.15)

where operator $\hat{s}$ is a quantum amplitude of single input mode of the signal, and $\hat{r}$ is an amplitude of different mode of output signal, which can be defined as a projection of input mode on the full set of the eigenmodes $\psi_i(t)$ of the kernel of integral transformation (6.1). It is easy to see that this equation turns into Eq. (6.1) when the input field is chosen in the form of one of the modes $\psi_i(t)$. Eq. (6.14) can be used for comparison of the degree of squeezing in given (for example, zero) frequency that is the most important parameter for multimode squeezing. We should note that Eq. (6.14) is valid only when writing time is equal to reading time. Moreover, it links two single modes - one mode at the input and one mode at the output - and is inapplicable for multimode stochastic fields considered here.

Such a generalization can be a subject for further study.

VII. CONCLUSION

Addressing once again to the comparison of two memory models presented in this paper, we note that different factors lead to a vanishing of the role of the Langevin noises in these cases. In high-speed QM model the noises can be excluded from consideration in very beginning, when we construct equations, since on the typical interaction time the spontaneous decay of the excited state is negligible, and hence the noise associated with this decay is also small. The situation is different in adiabatic QM model. Here, the rate of the spontaneous relaxation is large that yields to the presence of Langevin noise sources, which associated with population decay of the excited state as well as with decay of the coherences on the transitions coupled with the excited state. However, due to the initial assumption that the population of level $|3\rangle$ is small (and almost all atoms remain on the level $|1\rangle$) the additional noise terms turn out to be small.

In this study, we demonstrated that both memory models considered here allow us to achieve high efficiency for writing and restore of the light pulse, and are suitable for storage of the squeezed states.

We have shown that the quantum efficiency of whole writing-reading cycle for a pulse with arbitrary time profile is not a universal characteristic of the memory, through which the other properties of the memory (for example, the ability to store a squeezing) can be expressed. Only the efficiency defined in relation to a single eigenmode of the
memory is such a comprehensive parameter. Thus for a pulse with an arbitrary shape it is not enough to provide the high efficiency of the memory to conclude about preserving the quantum state of light.

We have analyzed the eigenmodes of the two memory models discussed here and demonstrated interesting feature connected with the ability to manipulate separately by two first (high-efficient) eigenmodes in the high-speed QM scheme.

Since the quantum memory operates as a mode filter, its important characteristic is a spectral bandwidth. Estimating the spectral bandwidth of the adiabatic and high-speed QM schemes, we have shown that it is significantly different for these schemes. The spectral bandwidth of the high-speed QM is at least 4 orders of magnitude higher than of the adiabatic QM. This demonstrate the benefits of the high-speed memory for storage of wideband signals.

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Appendix A: General solutions for high-speed memory model

The general solutions of the Eqs. $[5,30]-[5,32]$ under the arbitrary initial and boundary conditions can be written over the non-canonical amplitudes, introduced analogously to our consideration in Section III A:

$$\hat{A}^W(\tilde{z}; \tilde{t}; \tilde{q}) = \int_0^\tilde{t} d\tilde{t}' \hat{a}_{in}(\tilde{t}' - \tilde{t}; \tilde{q}) G_{aa}(\tilde{z}, \tilde{t}),$$  \hspace{1cm} (A1)

$$\hat{B}^W(\tilde{z}; \tilde{t}; \tilde{q}) = -\frac{p}{2} \int_0^\tilde{z} d\tilde{z}' \hat{B}^R(\tilde{z} - \tilde{z}'; 0; \tilde{q}) G_{ba}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (A2)

$$\hat{C}^W(\tilde{z}; \tilde{t}; \tilde{q}) = \frac{1}{2} \int_0^\tilde{z} d\tilde{z}' \hat{B}^R(\tilde{z} - \tilde{z}'; 0; \tilde{q}) G_{ac}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (A3)

where $p = \Omega/(g\sqrt{N})$.

$$\hat{A}^R(\tilde{z}; \tilde{t}; \tilde{q}) = -\frac{p}{2} \int_0^\tilde{z} d\tilde{z}' \hat{B}^R(\tilde{z} - \tilde{z}'; 0; \tilde{q}) G_{ba}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (A4)

$$\hat{B}^R(\tilde{z}; \tilde{t}; \tilde{q}) = \frac{1}{2} \int_0^\tilde{z} d\tilde{z}' \hat{B}^R(\tilde{z} - \tilde{z}'; 0; \tilde{q}) G_{bb}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (A5)

$$\hat{C}^R(\tilde{z}; \tilde{t}; \tilde{q}) = \frac{1}{2} \int_0^\tilde{z} d\tilde{z}' \hat{B}^R(\tilde{z} - \tilde{z}'; 0; \tilde{q}) G_{bc}(\tilde{z}', \tilde{t}),$$  \hspace{1cm} (A6)

where upper indexes "W" and "R" indicate the writing and read-out stage, respectively.

Here kernels $G_{ik}(\tilde{z}, \tilde{t})$ are bilinear combinations of the expressions depending on the n-th Bessel functions of the first kind denoted by $J_n$:

$$g_{aa}(\tilde{z}, \tilde{t}) = \delta(\tilde{t}) - e^{-i\tilde{t}} \sqrt{\tilde{z}/(4\tilde{t})} J_1(\sqrt{i\tilde{t}}) \Theta(\tilde{t}),$$  \hspace{1cm} (A7)

$$g_{bb}(\tilde{z}, \tilde{t}) = e^{-i\tilde{t}} \sqrt{4\tilde{t}/\tilde{z}} J_1(\sqrt{i\tilde{t}}) \Theta(\tilde{t}),$$  \hspace{1cm} (A8)

$$g_{ab}(\tilde{z}, \tilde{t}) = e^{-i\tilde{t}} J_0(\sqrt{i\tilde{t}}) \Theta(\tilde{t}).$$  \hspace{1cm} (A9)
The kernels in Eqs. (A1)–(A6) read

\[
G_{aa}(\tilde{z}, \tilde{t}) = \int_0^\tilde{t} d\tilde{t}' g_{aa}(\tilde{z}, \tilde{t} - \tilde{t}', g_{aa}^*(\tilde{z}, \tilde{t}'), \quad (A10)
\]

\[
G_{ab}(\tilde{z}, \tilde{t}) = \int_0^\tilde{t} d\tilde{t}' g_{ab}(\tilde{z}, \tilde{t} - \tilde{t}', g_{ab}^*(\tilde{z}, \tilde{t}'), \quad (A11)
\]

\[
G_{ac}(\tilde{z}, \tilde{t}) = \frac{1}{2} \int_0^\tilde{t} d\tilde{t}' g_{ac}(\tilde{z}, \tilde{t} - \tilde{t}', g_{ac}^*(\tilde{z}, \tilde{t}')) + c.c., \quad (A12)
\]

\[
G_{ba}(\tilde{z}, \tilde{t}) = G_{ab}(\tilde{z}, \tilde{t}), \quad (A13)
\]

\[
G_{bb}(\tilde{z}, \tilde{t}) = 2 \delta(\tilde{z}) \cos \tilde{t} + \int_0^\tilde{t} d\tilde{t}' g_{bb}(\tilde{z}, \tilde{t} - \tilde{t}', g_{bb}^*(\tilde{z}, \tilde{t}')), \quad (A14)
\]

\[
G_{bc}(\tilde{z}, \tilde{t}) = \delta(\tilde{z}) \sin \tilde{t} - \frac{1}{2} \int_0^\tilde{t} d\tilde{t}' g_{ac}(\tilde{z}, \tilde{t} - \tilde{t}', g_{ac}^*(\tilde{z}, \tilde{t}')) + c.c. \quad (A15)
\]

**Appendix B: Squeezed pulse radiation from sub-Poissonian laser and DOPO**

1. **Stationary radiation**

According to [32], the mean-square intracavity quadrature fluctuations of a single-mode laser are determined as

\[
\langle : \delta x_\omega^2 : \rangle = -\frac{p(1 - \mu)}{4} \frac{\kappa}{\kappa^2(1 - \mu/2)^2 + \omega^2}, \quad (\langle : \delta y_\omega^2 : \rangle = -\frac{1 - \mu}{2} \frac{\kappa}{\kappa^2\mu^2/4 + \omega^2}, \quad \mu = \sqrt{\frac{n_0}{n}} \ll 1. \quad (B1)
\]

Here, the parameter $0 < p < 1$ reflects the degree of ordering of the upper laser level excitation: $p = 0$ for a completely random Poissonian pumping, and $p = 1$ for a completely regular one. Here, $\kappa$ is the spectral width of the laser mode, $n$ is the average number of photons in the oscillation mode in the steady-state regime of oscillation and $n_0$ is the average number of photons appearing in an empty cavity under the influence of an external locking field. This locking is necessary for suppression of phase diffusion in the oscillation, which prohibits efficient quadrature squeezing. Parameter $\mu$ determining laser locking to an external field is assumed to be small compared to unity. However, despite small value of $\mu$, ignoring this quantity would violate the Heisenberg uncertainty relation at small frequencies. Therefore, this quantity should be handled carefully.

Similarly, we can write the corresponding expressions for the case of a degenerate optical parametric generation [33]:

\[
\langle : \delta x_\omega^2 : \rangle = \frac{1}{4} \frac{\kappa s}{\kappa^2/4(1 - s)^2 + \omega^2}, \quad (\langle : \delta y_\omega^2 : \rangle = -\frac{1}{4} \frac{\kappa s}{\kappa^2/4(1 + s)^2 + \omega^2}, \quad s = \sqrt{\frac{n_p}{n_0}} < 1. \quad (B2)
\]

Here, $n_p$ is the average steady-state number of photons stored in the pump mode, $n_0$ is the threshold number of photons in the pump mode, and $\kappa$ is the spectral width of the signal mode. Here, we analyze the operation of a DOPO in the below-threshold regime and the parameter $s$ characterizes the degree of approaching the generation threshold.

2. **Spectrum of squeezing in isolated pulse of light**

Applying inverse Fourier transformation to (B1), we have

\[
\langle : \delta x(t) \delta x(t') : \rangle = -\frac{p}{8} \frac{1 - \mu}{1 - \mu/2} e^{-\kappa(1 - \mu/2)|t - t'|}, \quad (B3)
\]

\[
\langle : \delta y(t) \delta y(t') : \rangle = -\frac{1 - \mu}{\mu} e^{-\kappa\mu|t - t'|}. \quad (B4)
\]

When describing a pulse of light in the quantum theory, it should be remembered that the field outside of the pulse is in a vacuum state, which can be explicitly expressed by writing the extracavity field amplitude in the form

\[
\hat{A}(t) \rightarrow \Theta^T(t)\hat{A}(t) + (1 - \Theta^T(t))\hat{A}_{vac}(t). \quad (B5)
\]
Here, the index ”vac” in front of the Heisenberg operator means that the operator must be averaged over the vacuum state when calculating measurable quantities. The function $\Theta^T(t)$ is zero outside of the $0 < t < T$ interval and is equal to unity within this interval.

To characterize squeezing, along with the field quadratures for the intracavity field $\hat{x}(t)$ and $\hat{y}(t)$, the quadratures for the field leaving the cavity for free-space propagation

$$\hat{X}(t) = \frac{1}{2} \left( \hat{A}^\dagger(t) + \hat{A}(t) \right), \quad \hat{Y}(t) = \frac{i}{2} \left( \hat{A}^\dagger(t) - \hat{A}(t) \right). \tag{B6}$$

are introduced. Quadrature squeezing for stationary generation is determined based on the equalities relating the correlator of the quadrature operators with the corresponding average of the normal-ordered operators:

$$\langle \hat{X}(t)\hat{X}(t') \rangle = \frac{1}{4} \delta(t - t') + \kappa \langle : \hat{x}(t)\hat{x}(t') : \rangle, \tag{B7}$$

$$\langle \hat{Y}(t)\hat{Y}(t') \rangle = \frac{1}{4} \delta(t - t') + \kappa \langle : \hat{y}(t)\hat{y}(t') : \rangle. \tag{B8}$$

For the isolated pulse equations (B7) and (B8) can be written in the form

$$\langle \Theta^T(t) \delta \hat{X}(t) \Theta^T(t') \delta \hat{X}(t') \rangle = \frac{1}{4} \Theta^T(t) \Theta^T(t') \delta(t - t') + \kappa \langle : \Theta^T(t) \delta \hat{x}(t) \Theta^T(t') \delta \hat{x}(t') : \rangle, \tag{B9}$$

$$\langle \Theta^T(t) \delta \hat{Y}(t) \Theta^T(t') \delta \hat{Y}(t') \rangle = \frac{1}{4} \Theta^T(t) \Theta^T(t') \delta(t - t') + \kappa \langle : \Theta^T(t) \delta \hat{y}(t) \Theta^T(t') \delta \hat{y}(t') : \rangle. \tag{B10}$$

Let us define the Fourier transformation for function $F(t)$ in limited time interval $[0, T]$ as

$$F^T_\omega = \frac{1}{\sqrt{T}} \int_0^T dt F(t) e^{i\omega t}. \tag{B11}$$

Application of this transformation to (B9) and (B10) yields

$$\langle \delta \hat{X}_\omega^T \delta \hat{X}_{\omega'}^T \rangle = \frac{1}{4} \delta^T(\omega + \omega') + \kappa \langle : \delta \hat{x}_\omega^T \delta \hat{x}_{\omega'}^T : \rangle, \tag{B12}$$

$$\langle \delta \hat{Y}_\omega^T \delta \hat{Y}_{\omega'}^T \rangle = \frac{1}{4} \delta^T(\omega + \omega') + \kappa \langle : \delta \hat{y}_\omega^T \delta \hat{y}_{\omega'}^T : \rangle, \tag{B13}$$

where we introduced the notation

$$\delta^T(\omega + \omega') = \frac{\sin(\omega + \omega')T/2}{(\omega + \omega')T/2} e^{i(\omega + \omega')T/2}. \tag{B14}$$

The second terms in (B12)-(B13), which are related to calculation of the normally ordered average, are different for different sources. In the case of single-mode phase-locked sub-Poissonian laser, we readily find that

$$\langle : \delta \hat{x}_\omega^T \delta \hat{x}_{\omega'}^T : \rangle = \frac{p}{8} \frac{1 - \mu}{1 - \mu/2} \left[ - \left( \frac{1}{\kappa_x + i\omega} + \frac{1}{\kappa_x + i\omega} \right) \delta^T(\omega + \omega') + \frac{1}{T(\kappa_x - i\omega)(\kappa_x + i\omega)} \left( 1 - e^{-\kappa_x T + i\omega T} \right) + \frac{1}{T(\kappa_x - i\omega') (\kappa_x + i\omega)} \left( 1 - e^{-\kappa_x T + i\omega' T} \right) \right], \tag{B15}$$

where

$$\kappa_x = \kappa (1 - \mu/2)$$

Using definition (2.4) and substituting it into (B12) and (B15), we can express the squeezing parameter for the case of a sub-Poissonian laser in the form

$$e^{-\nu_{in}(\omega)} = 1 - \frac{p\kappa^2(1 - \mu)}{\kappa_x^2 + \omega^2} + \frac{p\kappa^2(1 - \mu)}{2\kappa_x T} \left[ \frac{1}{(\kappa_x - i\omega)^2} \left( 1 - e^{-\kappa_x T + i\omega T} \right) + c.c. \right]. \tag{B16}$$
For parametric oscillation, the most interesting feature is the phase quadrature. Formally acting similar to the laser case, i.e., limiting analysis to time interval $0 < t < T$ and applying Fourier transformation, we have

\begin{equation}
\langle \delta \hat{Y}_\omega \delta \hat{Y}_{\omega'} \rangle = \frac{1}{4} \left[ 1 - \frac{\kappa s}{1 + s} \left( \frac{1}{\kappa/2(1 + s) + i\omega'} + \frac{1}{\kappa/2(1 + s) + i\omega} \right) \right] \delta^T(\omega + \omega') +
\frac{\kappa s}{4T(1 + s)(\kappa/2(1 + s) - i\omega)(\kappa/2(1 + s) + i\omega')} \left( 1 - e^{-\kappa/2(1 + s)T + i\omega'T} \right) \left( 1 - e^{-\kappa/2(1 + s)T + i\omega'T} \right).
\end{equation}

(B17)

The obtained expression allows estimating the degree of squeezing of a pulse of light of arbitrary duration $T$ at arbitrary frequency.

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