Detecting Topological Superconductivity with $\varphi_0$ Josephson Junctions

Constantin Schrade, Silas Hoffman, and Daniel Loss

Department of Physics, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland

(Dated: June 26, 2018)

The interplay of superconductivity, magnetic fields, and spin-orbit interaction lies at the heart of topological superconductivity. Remarkably, the recent experimental discovery of $\varphi_0$ Josephson junctions by Szombati et al. [1], characterized by a finite phase offset in the supercurrent, requires the same ingredients as topological superconductors, which suggests a profound connection between these two distinct phenomena. Here, we theoretically show that a quantum dot $\varphi_0$ Josephson junction can serve as a new qualitative indicator for topological superconductivity: Microscopically, we find that the phase shift in a junction of $s$-wave superconductors is due to the spin-orbit induced mixing of singly occupied states on the quantum dot, while for a topological superconductor junction it is due to singlet-triplet mixing. Because of this important difference, when the spin-orbit vector of the quantum dot and the external Zeeman field are orthogonal, the $s$-wave superconductors form a $\pi$ Josephson junction while the topological superconductors have a finite offset $\varphi_0$ by which topological superconductivity can be distinguished from conventional superconductivity. Our prediction can be immediately tested in nanowire systems currently used for Majorana fermion experiments and thus offers a new and realistic approach for detecting topological bound states.

PACS numbers: 74.50.+r, 85.25.Cp, 71.10.Pm

Non-abelian anyons are the building blocks of topological quantum computers [2]. The simplest realization of a non-abelian anyon are Majorana bound states (MBSs) in topological superconductors (TSs) [3]. It has been proposed that such a TS can be induced by an $s$-wave superconductor (SC) in systems of nanowires with spin-orbit interaction (SOI) subject to a Zeeman field [4–7], in chains of magnetic atoms [8–11] and in topological insulators [12–17]. However, providing experimental evidence for the existence of this new phase of matter has remained a major challenge.

Here we present a new qualitative indicator of MBS based on $\varphi_0$ Josephson junctions ($\varphi_0$ JJJs). In $\varphi_0$ JJJs the Josephson current is offset by a finite phase, $\varphi_0$, so that a finite supercurrent flows even when the phase difference between the superconducting leads and the magnetic flux enclosed by the Josephson junction (JJ) vanishes. Such $\varphi_0$ JJJs have been discussed in systems based on unconventional superconductors [18–22], ferromagnets [24–27], quantum point contacts [28], topological insulators [29], nanowires [30, 31] and diffusive systems [32, 33]. Recently, the connection between $\varphi_0$ JJJs based on nanowires and TSs has also been discussed [34]. Most relevant for the present work, the emergence of a $\varphi_0$ JJ in $s$-wave superconducting leads and observed in recent experiments [1]. Interestingly, the ingredients for observing a $\varphi_0$ JJ in this type of system largely overlap with those required to generate MBSs. In this work, we focus on two models for $\varphi_0$ JJJs based on QDs which, compared to previous studies [35, 37], are in the singlet-triplet anticrossing regime. In the first model, two $s$-wave SCs are tunnel coupled via a two-orbital QD with SOI and subject to a Zeeman field, see Fig. 4(a), wherein we find a finite phase shift caused by the SOI-induced mixing of singly occupied states of the QD. In the second model, replacing the two $s$-wave SCs by two TSs, see Fig. 1(b), we again find a finite phase shift which results from the singlet-triplet mixing of the doubly occupied QD states. When the spin-orbit vector $S$ and the magnetic field are orthogonal, the system is invariant under a composition of time reversal and mirroring in the plane perpendicular to $S$, under which the superconducting phase goes to opposite itself; because the energy must be invariant under this symmetry, there can be no terms that are odd in the superconducting phase difference in the Hamiltonian and thus no non-trivial phase offset [25, 38]. However, unlike the ground state of the SC leads, the ground states of the TS leads transform nontrivially under the above transformations and we thus anticipate a nonzero phase shift. Indeed, we show that the phase shift $\varphi_0$ is equal to $\pi$ for the $s$-wave superconducting leads, while $\varphi_0 \neq 0, \pi$ for the TSs leads, which can, consequently, be used as a new qualitative indicator of MBSs.

Josephson junction models. Our starting point for both of the JJ models outlined above is the Hamiltonian

$$H_\nu = H_D + H_{\nu,L} + H_{\nu,t},$$

where $\nu = S,T$ corresponds to the model with $s$-wave SC leads and TS leads, respectively. The first term in this expression $H_D = H_0 + H_2 + H_{SOI}$ is the Hamiltonian of an isolated QD. Here, $H_0 = (V_g + \delta/2) n_a + (V_g - \delta/2) n_b + U/2 \sum_n n_\tau (n_{\tau} - 1) + U_{ab} n_a n_b$ describes a QD with two orbitals $\tau = a, b$ at energy difference $\delta > 0$ with respect to a gate voltage $V_g$. The particle number operator of orbital $\tau$ is $n_\tau = \sum_s d_{\tau s}^d d_{\tau s}$ with $d_{\tau s}$ the electron annihilation operator with spin $s = \uparrow, \downarrow$ quantized along the $z$-axis in orbital $\tau$. The intraorbital (interorbital) Coulomb interaction strength is $U (U_{ab})$. 

arXiv:1607.07794v1 [cond-mat.str-el] 26 Jul 2016
Furthermore, $H_z = -g\mu_B B \sum_{r} (d_{r \uparrow}^\dagger d_{r \downarrow} - d_{r \downarrow}^\dagger d_{r \uparrow})/2$ describes a Zeeman field along the z-axis of magnitude $B$ with $g$ the electron g-factor and $\mu_B$ the Bohr magneton. Lastly, $H_{SOI} = \Omega^2/2 \cdot \sum_{s,a'} (d_{s \sigma}^\dagger \sigma_{s\sigma} d_{a' \sigma} - H.c.)$ describes the SOI on the QD, where $\Omega = \Omega(\sin \theta, 0, \cos \theta)$, in which $\Omega \neq 0$, $\theta \in [0, \pi]$ is the angle of the SOI vector with respect to the Zeeman field, and $\sigma$ is the vector of Pauli matrices.

The second term in Eq. (1) describes the isolated superconducting leads. For the first model, $H_{SL} = \sum_{\eta,k_s} E_k \gamma_{\eta,k_s} \bar{\gamma}_{\eta,k_s}$, where $\gamma_{\eta,k_s}$ is the quasiparticle annihilation operator in SC $\eta = 1, 2$ with momentum $k$, pseudospin $\sigma = \uparrow, \downarrow$, and energy $E_k = \sqrt{E_k^2 + \Delta^2}$ with $\Delta$ the superconducting gap and $\xi_k$ the single-electron dispersion relation in the normal metal state. The non-degenerate ground state of the s-wave superconductors, $|\eta\rangle$, is defined so that $\gamma_{\eta,k_s}|\eta\rangle = 0$. For the second model, we assume that the localization length of the MBS wavefunctions is much smaller than the length of TSs. We also neglect contributions of bulk quasiparticles which is valid for energies much smaller than the energy gap. Consequently the MBSs are at zero energy and $H_{TS,1} = 0$. As a result, the ground state of the TS leads is four-fold degenerate which, upon choosing a fixed parity subspace, becomes two-fold degenerate. In the following, we consider the odd parity subspace; however, the results for the even parity ground state subspace are identical.

The last term in Eq. (1) describes the tunnel coupling between the superconducting leads and the QD. For the first model, it is given by

$$H_{SL} = \sum_{\eta,k_s} t_{\eta,k_s} e^{i\phi_{\eta}/2} \gamma_{\eta,k_s}^\dagger d_{\eta,k_s} + \text{H.c.},$$

with $c_{\eta,k_s}$ being the annihilation operator of an electron with momentum $k$ and spin $s$ in SC $\eta$. It is related to the quasiparticle operators by $c_{\eta,k_s} = u_k \gamma_{\eta,k_s} + v_k \gamma_{\eta,k_s}$ and $c_{\eta,-k_s} = u_k \gamma_{\eta,-k_s} + v_k \gamma_{\eta,k_s}$ with coherences $u_k = (1/\sqrt{2}) \sqrt{1 + \xi_k/\Delta_k}$ and $v_k = (1/\sqrt{2}) \sqrt{1 - \xi_k/\Delta_k}$. The tunneling Hamiltonian also contains the superconducting phase $\phi_{\eta}$ of SC $\eta$ and real, spin and momentum-independent tunneling amplitudes $t_{\eta,k_s}$. For the second model, the coupling of the TSs and the QD is given by

$$H_{TS,1} = \sum_{\eta,s} t_{\eta,s} e^{i\phi_{\eta}/2} \Gamma_{\eta} d_{\eta,s} + \text{H.c.},$$

with $\Gamma_{\eta}$ being the MBS in TS $\eta$ which is spatially closest to the QD [40]. We assume that its partner $\Gamma_{\eta}$ at the opposite end of the TS does not couple to the QD. However, they form non-local fermionic operators $C_1 = (\Gamma_{1}^\dagger + \Gamma_{1}^\uparrow)/2$ and $C_2 = (\Gamma_{2}^\dagger + \Gamma_{2}^\uparrow)/2$.

We now proceed with a discussion of $H_D$ in the regime of $\delta > U > U_{AB} \gg |\Omega|$, which is common in typical experiments [1]. First, we address the case of a doubly occupied dot, $n_a + n_b = 2$. For $\Omega = 0$, the spectrum consists of three singlet (triplet) bands which are constant (split) as a function of the Zeeman field. As experimentally observed in [11], for finite $\Omega$ and $\theta$, the singlet and triplet bands anticross, see Fig. [1]c. In all following discussions, we operate the QD in the regime close to the anticrossing of the singlet $|S\rangle = |\eta\rangle |\eta\rangle_0 |0_D\rangle$ and the triplet $|T\rangle = |\eta\rangle |\eta\rangle_0 |0_D\rangle$ which occurs at the Zeeman field $B^{(2)} = (\delta - U - U_{ab})/g\mu_B$. Here, $|0_D\rangle$ is the vacuum state on the dot. The effective Hamiltonian, valid to lowest order in $\Omega$, which acts in the two-level subspace spanned by $|S\rangle$ and $|T\rangle$, is

$$H^{(2)}_{ST} = (2V_g - \delta + U)|S\rangle \langle S| + (2V_g + U_{ab} - g\mu_B B)|T\rangle \langle T| + i\Omega \sin(\theta/2)|S\rangle \langle T| \bar{\gamma}_{\eta,s} \bar{\gamma}_{\eta,s} + \text{H.c.},$$

The spectrum of $H^{(2)}_{ST}$ is given by $E_{\pm}^{(2)}$ with corresponding orthonormal eigenstates

$$|E_{\pm}^{(2)}\rangle = i S_{\pm} |S\rangle + T_{\pm} |T\rangle.$$

Here, $S_{\pm}, T_{\pm}$ are real functions of the system parameters, see [29].

Second, we discuss the case of a singly occupied dot, $n_a + n_b = 1$. For $\Omega = 0$, the energy levels for opposite spins split as a function of the Zeeman field. For finite $\Omega$ and $\theta$, an energy gap opens up at the crossing point $B^{(1)} = 2g\mu_B$ of the spin-up band in orbital $a$ and the spin-down band in orbital $b$, see Fig. [1]d. We will denote the four eigenvalues of the singly occupied sector
by \( E_\lambda^{(1)} \) for \( \lambda = 1, \ldots, 4 \). The corresponding orthonormal eigenstates are given by

\[
|E_\lambda^{(1)}⟩ = \sum_s \left( iA_{s\lambda}d^\dagger_{s\lambda} + B_{s\lambda}d^\dagger_{s\lambda} \right) |0_D⟩.
\]

(5)

Here, \( A_{s\lambda}, B_{s\lambda} \) are real functions of the system parameters, see 39. The relative imaginary unit in both Eq. 4 and Eq. 5 is due to the SOI. We adjust the filling and the gate voltage of the QD, so that its ground state is given by \( E_0^{(2)} \) while its first excited states are given by \( E_π^{(2)} \) and \( E_π^{(1)} \) for some fixed \( \lambda \). The separation between \( E_π^{(2)} \) to the states \( E_λ^{(1)} \) with \( \lambda' \neq \lambda \) is assumed to be large, \( |E_\lambda^{(1)} - E_\lambda^{(2)}| \gg E_\lambda^{(1)} - E_\lambda^{(2)} \). Finally, the remaining occupancy sectors of the QD, whose energies are much larger than the QD-lead coupling, are not relevant for our results and are hence omitted.

Detecting topological superconductivity. In order to calculate the superconducting current, we tune the chemical potential of the superconductors close to the \( E_\lambda^{(2)} \) level. We require for the SC JJ that \( πνF_\eta ττ' \ll E_\lambda^{(1)} - E_\lambda^{(2)}, \Omega \sin(θ) \), with \( νF_\eta \) the normal-state density of states of the leads at the Fermi energy and for the TS JJ that \( ττ' \ll E_\lambda^{(1)} - E_\lambda^{(2)}, \Omega \sin(θ) \), so that in both cases the states \( E_π^{(2)} \) and \( E_π^{(1)} \) on the QD serve as virtual tunneling states. Our approach is valid for angles \( θ \in (θ_s - π - θ_τ, θ_s) \) where \( θ_τ \) is a critical angle determined by the conditions above 39. Furthermore, we work in a temperature regime of \( k_B T \ll E_\lambda^{(1)} - E_\lambda^{(2)}, \Omega \sin(θ) \). The effective tunneling Hamiltonian \( H_{TS} (H_{TS,1}) \) valid up to fourth (second) order in the tunneling amplitudes acting on the ground state of the isolated dot and s-wave (odd parity) ground state of the uncoupled leads is

\[
H_{ν,±} = (E_0^ν \cos φ_ν + E_0^ν \sin φ_ν) T_ν + \tilde{E}_ν,
\]

with \( φ_S = 2φ_πTS = φ_1 − φ_2 \) and \( T_S = 1, T_G = C_1^\dagger C_2 + \text{H.c.} = iΓ †Γ \). The first term in Eq. 6 arises due to Cooper pair tunneling across the SC JJ or nonlocal fermion tunneling across the TS JJ which splits the ground states of the TS leads. The second term is an energy offset, due to processes for which there is no such transport. At zero temperature, the Josephson current, defined by \( I_ν = 2e∂_νE_{ν,GS}/h \) with \( E_{ν,GS} \) the ground state energy of the coupled system, is given by

\[
I_ν = −I_ν^0 \sin(φ_ν − φ_0^ν), \quad φ_0^ν = \arctan(E_0^ν/E_0^0),
\]

(7)

where the critical current is \( I_ν^0 = 2n_νc (∑ν(E_ν^0)^2 + (E_0^0)^2) \sin(E_0^0)/h \). Because in the TS case the ground state is a function of \( φ \), the sign of the Josephson energy also depends on the phase dependence of \( φ_νTS \). The ground state energy and \( φ_νTS \) is the ground state energy and \( φ_νTS \) is 1/2 otherwise. In the SC case the ground state is independent of \( φ \) and therefore \( s_ν = 1 \).

![FIG. 2](Color online) (a) On the left hand side of the equality: the virtual tunneling sequence which leads to \( H_{TS}^{eff} \) in terms of the eigenstates of the effective dot Hamiltonian. Because the states \( |E_π^{(2)}⟩ \) and \( |E_π^{(1)}⟩ \) are superpositions of the doubly and singly occupied eigenstates of \( H_D \) in the absence of SOI, respectively, \( H_{TS}^{eff} \) can be written as a sum of the virtual tunneling processes in that basis; two examples of which, contributing to the \( \cos φ_νTS \) and \( \sin φ_νTS \) terms, are shown on the right side of the equality. Electron spin (quasiparticle pseudospin) is denoted by \( ↑, ↓ \). Notice that it is the superposition of singly occupied dot states, e.g. in the process \( |S⟩ → d_{ττ'}^\dagger |0_D⟩ → d_{ττ'}^\dagger |0_D⟩ → |T⟩ \) (solid red box), that leads to a finite \( \sin φ_νTS \) contribution and therefore a finite \( s_νTS \).

(b) Same as (a) but for the case of \( H_{TS}^{eff} \). As compared with the SC case, it is the singlet-triplet mixing that induces a finite phase shift, e.g. in the contribution to \( \cos φ_νTS \). Here, \( 0 \) or \( 1 \) are the eigenvalues of \( C_1^\dagger C_1 \) and \( C_2^\dagger C_2 \).

Notice that there is a finite phase shift only when \( E_0^ν ≠ 0 \). As such, we now turn to a more detailed comparison of the coefficients in Eq. 6. For the BCS JJ,

\[
E_0^ν = g_{ST}t_{1b}t_{2b}B_{νλ}^2 (A_λ^2t_{1b}t_{2b} + B_{νλ}^2t_{1a}t_{2a}),
\]

\[
E_0^ν = g_{ST}t_{1b}t_{2a}A_λB_{νλ}^3 (t_{1a}t_{2b} - t_{1b}t_{2a}).
\]

(8)

The prefactor \( g_{ST} > 0 \), which is not relevant for the phase shift \( φ_νTS \), includes the coherence factors and energy denominators picked up in the perturbation theory 39. Thus, the SC JJ exhibits in general a finite phase shift, when \( t_{1a}t_{2b} - t_{1b}t_{2a} ≠ 0 \). For \( φ_νTS = 0 \), the sign of the supercurrent is determined by \( \text{sgn}(t_{1a}t_{2b} - t_{1b}t_{2a}) \) and \( \text{sgn}(A_λB_{νλ}^2) \propto \text{sgn}(Ω) \). We now explain the sequence of intermediate states which leads to the contributions in Eq. 8. Our initial state on the QD is \( E_π^{(2)} \). To reach the first intermediate state, we remove one electron from the QD, whereupon its state changes to \( E_π^{(1)} \), and we create
an excitation on SC 1 (2). Next, we use the superconducting condensate to create an electron on the QD and an excitation on SC 2 (1). This changes the QD state to $E_\lambda^{(2)} [39]$. Third, we return to $E_\lambda^{(1)}$ by absorbing one of the dot electrons and the excitation on SC 1 (2) into the condensate. Finally, we go back to the initial state $E_\lambda^{(2)}$ by transferring the excitation on SC 2 (1) back on the QD. Because $E_\lambda^{(1)}$ is a superposition of different singly occupied QD orbitals, in the first and third step of this sequence the electron on the QD switches orbitals while preserving spin with amplitude $\propto iA_{\lambda\lambda'}B_{\lambda'}$ while it stays in the same orbital with amplitude $\propto (B_{\lambda\lambda'})^2$ or $\propto (A_{\lambda\lambda'})^2$. Thus, the $E_s^0$ contribution originates from processes in which the electron switches orbitals exactly once, while the remaining processes yield the $E_s^0$ contribution. The mixing of singlet and triplet states in $E_\lambda^{(2)}$ gives an overall prefactor, which due to the normalization of the states, drops out of Eq. (8). Most interestingly, for the case when the relative angle between Zeeman field and SOI axis is $\theta = \pi/2$ the phase shift $\phi_0$ vanishes, see Fig. 3(a). On a microscopic level, this is because now the SOI only mixes opposite spins in different orbitals, $A_{\lambda\lambda'} = B_{\lambda\lambda'} = 0$ for $\lambda = 1, 4$ and $A_{\lambda\lambda'} = B_{\lambda\lambda'} = 0$ for $\lambda = 2, 3$ [39]. This restricts the number of allowed virtual tunneling processes. In particular, processes which move the spin between the orbitals without flipping it are prohibited, $A_{\lambda\lambda'} = 0$ and see Fig. 2(a). However, unlike the SC JJ, the TS JJ still allows for nonzero phase shift in that case, see Fig. 3(a). At $\theta = \pi/2$, we find that the coefficients in Eq. (6) for the TS JJ when $\lambda = 1, 4$ are given by

\begin{align}
E_{TS}^0(\pi/2) &= g_{TS}B_{\lambda\lambda'}^2S_{\lambda\lambda'}(t_{1b}t_{2a} - t_{1a}t_{2b}) , \\
E_{TS}^T(\pi/2) &= -g_{TS}B_{\lambda\lambda'}^2 (S_2^2t_{1b}t_{2b} + T_2^2t_{1a}t_{2a}) ,
\end{align}

where the prefactor $g_{TS} > 0$ includes the energy denominators of the perturbation theory [39]. In comparison to the SC JJ, the sign of the supercurrent at $\varphi_{TS} = 0$ in the TS JJ is determined by parity $i\Gamma_2\Gamma_1$. If the parity fluctuates, the supercurrent exhibits fluctuations as well. So the observation of a phase shift requires sufficiently long parity life times which can be up to minutes [42]. When $\lambda = 2, 3$ we find that $E_{TS}^0 = 0$ and $E_{TS}^T \neq 0$. For $\lambda = 1, 4$ we recover the same feature when $B \gg B_2$ [2, Fig. 3 in [39]]. In both cases this is the special case of a $\varphi_{TS}^0 = \pi/2$ JJ for TS. We now focus on the case when $\lambda = 1, 4$. Recalling that $E_\lambda^{(2)}$ is a superposition of singlet and triplet states, we identify the processes that contribute to Eq. (8): $E_{TS}^0(\pi/2)$ comes from virtual tunneling sequences taking a singlet to a triplet state, with amplitude $\propto iS_{-}.T_{-}$, and the corresponding sequences taking a triplet to the singlet state, with an amplitude $\propto -iS_{-}.T_{-}$. When the order in which the non-local fermion is created or destroyed is opposite between these processes, the tunneling sequences differ in phase by $\varphi_{TS} + \pi$ and acquire the same tunneling coefficients so that their sum is proportional to $\cos(\varphi_{TS})$, see Fig. 3(b) and [39]. Distinctly, $E_{TS}^T(\pi/2)$ originates from sequences that take the singlet ($\propto S_2^T$) or triplet ($\propto T_2^T$) to itself. In both cases there exist two sequences that, again, differ in phase by $\varphi_{TS} + \pi$ but have the same tunneling coefficients, so that their sums are $\propto \sin(\varphi_{TS})$. Discussion.

We propose an experiment based on our observation that in general $\varphi_0^0(\pi/2) = 0$ but $\varphi_0^T(\pi/2) \neq 0$. We consider a nanowire setup similar to [1], see Fig. 3(b). The wire SOI axis $\Omega_W$, induced by an electric field along the $z$-direction at the SC-wire contact, is orthogonal to an external Zeeman field $B$. Via gating we create a tunnel coupled QD as a short slice in the wire. Furthermore we contact the QD to a backgate generating an electric field along the $y$-axis so that the dot SOI axis $\Omega_D$ is along the $z$-axis. We adjust the size of the QD so that the singlet-triplet anticrossing occurs for Zeeman fields close to the topological phase transition. $g\mu_BB \approx \sqrt{\Delta^2 + \mu^2}$ where $\mu$ is the chemical potential of the SCs and $B = |B|$. Also we adjust the gate voltage $V_g$ and the filling of the dot so that its ground state is $E_\lambda^{(2)}$, while its first excited states are $E_\lambda^{(2)}$ and $E_\lambda^{(1)}$. Lastly, the chemical potential of the nanowire leads is tuned to $E_\lambda^{(2)}$. We now position the Zeeman field orthogonal to both $\Omega_W$ and $\Omega_D$. When we now tune the system across the topological phase transition

![FIG. 3: (Color online) (a) Phase shift $\varphi_0(\theta)$ (left panel) and Josephson current $I_{J_0}(\theta)$ at $\varphi_0 = 0$ (right panel) for $\lambda = 4$ and $\theta \in \{0, \pi - \theta_c\}$ with $\theta_c = 0.3$. System parameters are chosen as in Fig. 1 with $B = B^{(2)}$, $V_g = -0.80$ meV, $t_{1a} = t_{2b} = 0.01$ meV, $t_{1b} = 0.05$ meV and $t_{2a} = 0.04$ meV. Compared to the SC JJ the phase shift (Josephson current at $\varphi_0 = 0$) is non-zero for the TS JJ. (b) Experimental proposal. A nanowire (dark grey) is proximity coupled to an s-wave SC (red). An electric field along the $z$-direction at the SC-wire contact induces a wire SOI axis $\Omega_W$ along the $y$-direction. An external Zeeman field $B$ is applied orthogonal to $\Omega_W$. A QD (yellow) is created by depleting the electron density via gates (light grey regions). A backgate contacted to the QD (not shown) induces an electric field along the $y$-direction and hence a SOI axis $\Omega_W$ along the $z$-direction. To measure $\varphi_0(\theta)$ and $I_{J_0}(\theta)$, $B$ is rotated in the plane orthogonal to $\Omega_W$.](image-url)
by varying $B$, we observe a change in the phase shift of the Josephson current from $\pi$ to some non-trivial $\varphi_0 \neq \pi$. Moreover, we can even determine the full dependence of the phase shift and Josephson current by rotating $B$ in the plane orthogonal to $\mathbf{W}$. Interestingly, for typical system parameters of a nanowire QD JJs we find that, at zero phase difference between the leads, $|J_S| \approx 10 \text{pA}$ while $|J_{TS}| \approx 1 \text{nA}$, which corresponds to an increase by three orders of magnitude.

**Conclusions.** We have introduced a new qualitative indicator for the detection of topological superconductivity based on a QD $\varphi_0$JJ. We found that for this setup the trivial SCs always form a $\pi$JJ while the TSs can form a $\varphi_0$JJ with $\varphi_0 \neq 0, \pi$. We have also seen that this change in phase shift is accompanied by a significant increase in the magnitude of the critical current. These observations can be probed by simple modifications of recent experimental setups in nanowire QD JJs.

**Acknowledgments.** We acknowledge support from the Swiss NSF and NCCR QSIT. We are grateful to J. Klinovaja for useful comments.

[1] D. B. Szombati, S. Nadji-Perge, D. Car, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Nat. Phys. **12**, 568 (2016).
[2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[3] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
[4] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
[5] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett., **105**, 177002 (2010).
[6] J. Klinovaja, and D. Loss, Phys. Rev. B **86**, 085408 (2012).
[7] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012).
[8] S. Nadji-Perge, I. K. Drozdov, B. A. Bernevig, and A. Yazdani, Phys. Rev. B **88**, 020407 (2013).
[9] F. Pientka, F.J. Glazman, F. von Oppen, Phys. Rev. B **88**, 155420 (2013).
[10] S. Nadji-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, Ju. Seo, A. H. MacDonald, B. A. Bernevig, A. Yazdani, Science **346**, 602 (2014).
[11] R. Pawlak, M. Kisiel, J. Klinovaja, T. Meier, S. Kawai, T. Glatzel, D. Loss, and E. Meyer, arXiv:1505.06078.
[12] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[13] S. Hart, H. Ren, T. Wagner, P. Leubner, M. Muhlbauer, C. Brune, H. Buhmann, L. W. Molenkamp, and A. Yacoby, Nat. Phys. **10**, 638 (2014).
[14] V. S. Pribiag, A. J. A. Beukman, F. Qu, M. C. Cassidy, C. Charpentier, W. Wegscheider, and L. P. Kouwenhoven, Nat. Nanotechnology **10**, 593 (2015).
[15] J. Wiedenmann, E. Bocquillon, R. S. Deacon, S. Hartinger, O. Herrmann, T. M. Klapwijk, L. Maier, C. Ames, C. Brune, C. Gould, A. Oiwa, K. Ishibashi, S. Tarucha, H. Buhmann, and L. W. Molenkamp, Nat. Commun. **7**, 10303 (2016).
[16] E. Bocquillon, R. S. Deacon, J. Wiedenmann, P. Leubner, T. M. Klapwijk, C. Brune, K. Ishibashi, H. Buhmann, and L. W. Molenkamp, arXiv:1601.08055.
[17] R. S. Deacon, J. Wiedenmann, E. Bocquillon, T. M. Klapwijk, P. Leubner, C. Brune, S. Tarucha, K. Ishibashi, H. Buhmann, L. W. Molenkamp, arXiv:1603.09611 (2016).
[18] V.B. Geshkenbein and A.I. Larkin, Pisma Zh. Eksp. Teor. Fiz. **43**, 306 (1986) [JETP Lett. **43**, 395 (1986)].
[19] S. Yip, Phys. Rev. B **52**, 3087 (1995).
[20] M. Sigrist, Prog. Theor. Phys. **99**, 899 (1998).
[21] S. Kashiwaya, and Y. Tanaka, Rep. Prog. Phys. **63**, 1641 (2000).
[22] Y. Asano, Y. Tanaka, M. Sigrist, and S. Kashiwaya, Phys. Rev. B **71**, 214501 (2005).
[23] Y. Avishai and T.K. Ng, Phys. Rev. B **81**, 104501 (2010).
[24] A. Buzdin, Phys. Rev. Lett. **101**, 107005 (2008).
[25] J.-F. Liu and K. S. Chan, Phys. Rev. B **82**, 184533 (2010).
[26] E. Goldobin, D. Koelle, R. Kleiner, and R. G. Mints, Phys. Rev. Lett. **107**, 227001 (2011).
[27] H. Sickinger, A. Lipman, M. Weides, R.G. Mints, H. Kohlstedt, D. Koelle, R. Kleiner, and E. Goldobin, Phys. Rev. Lett. **109**, 107002 (2012).
[28] A. A. Reynoso, G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. Lett. **101**, 107001 (2008).
[29] F. Dolcini, M. Houzet, and J. S. Meyer, Phys. Rev. B **92**, 035428 (2015).
[30] T. Yokoyama, M. Eto, and Y. V. Nazarov, Phys. Rev. B **89**, 195407 (2014).
[31] G. Campagnano, P. Lucignano, D. Giuliani, and A. Tagliacozzo, J. Phys. Condens. Matter **27**, 205301 (2015).
[32] M. Alidoust and J. Linder, Phys. Rev. B **87**, 060503 (2013).
[33] F. S. Bergeret and I. V. Tokatly, Europhys. Lett. **110**, 57005 (2015).
[34] K. N. Nesterov, M. Houzet, and J. S. Meyer, Phys. Rev. B **93**, 174502 (2016).
[35] L. Dell’Anna, A. Zazunov, R. Egger, and T. Martin, Phys. Rev. B **75**, 085305 (2007).
[36] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Phys. Rev. Lett. **103**, 147004 (2009).
[37] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, Phys. Rev. B **88**, 144515 (2013).
[38] A. Rasmussen, J. Danon, H. Suominen, F. Nichele, M. Kjaergaard, and K. Flensberg, Phys. Rev. B **93**, 155406 (2106).
[39] See Supplemental Material for a more detailed discussion of the bare QD Hamiltonian and the Josephson junction models.
[40] M. Lee, J. Soo Lim, and R. Lopez, Phys. Rev. B **87**, 241402(R) (2013).
[41] C. Fasth, A. Fuhrer, L. Samuelson, V.N. Golovach, and D. Loss, Phys. Rev. Lett. **98**, 266801 (2007).
[42] D. J. van Woerkom, A. Geresdi, and L. P. Kouwenhoven, Nat. Phys. **11**, 547 (2015).
Supplemental Material to ‘Detecting Topological Superconductivity with $\varphi_0$ Josephson junctions’

Constantin Schrade$^1$, Silas Hoffman$^1$, and Daniel Loss$^1$

$^1$Department of Physics, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland

A QUANTUM DOT WITH SPIN ORBIT INTERACTION IN A ZEEMAN FIELD

This first section of the supplemental material provides a more detailed discussion of the model for an isolated QD with SOI subject to an external Zeeman field as given by $H_D$ in the main text. The Hilbert space of the system is spanned by the occupation number states

$$|n_{a\uparrow}, n_{a\downarrow}, n_{b\uparrow}, n_{b\downarrow}\rangle = (d_{a\uparrow}^{\dagger})^{n_{a\uparrow}}(d_{a\downarrow}^{\dagger})^{n_{a\downarrow}}(d_{b\uparrow}^{\dagger})^{n_{b\uparrow}}(d_{b\downarrow}^{\dagger})^{n_{b\downarrow}}|0_D\rangle,$$

where $n_{\tau s} \in \{0, 1\}$ is the occupation number of an electron with spin $s$ in orbital $\tau$. Since the total number of electrons on the QD is conserved, we can address each sector with fixed total occupation number separately.

Double occupancy sector

We start with an analysis of the double occupancy sector. A basis is given by the singlet states

$$|1, 1, 0, 0\rangle, \quad |S\rangle = |0, 0, 1, 1\rangle, \quad (|1, 0, 0, 1\rangle - |0, 1, 1, 0\rangle)/\sqrt{2},$$

and the triplet states

$$|T\rangle = |1, 0, 1, 0\rangle, \quad (|1, 0, 0, 1\rangle + |0, 1, 1, 0\rangle)/\sqrt{2}, \quad |0, 1, 0, 1\rangle.$$

Representing $H_D$ in terms of these basis states we find that

$$H_D^{(2)} = \begin{pmatrix}
2V_g + \delta + U & 0 & 0 & -i\Omega \sin(\theta)/2 & i\Omega \cos(\theta)/\sqrt{2} & i\Omega \sin(\theta)/2 \\
0 & 2V_g - \delta + U & 0 & -i\Omega \sin(\theta)/2 & i\Omega \cos(\theta)/\sqrt{2} & i\Omega \sin(\theta)/2 \\
i\Omega \sin(\theta)/2 & i\Omega \sin(\theta)/2 & 2V_g + U_{ab} & 0 & 0 & 0 \\
-\Omega \cos(\theta)/\sqrt{2} & -i\Omega \cos(\theta)/\sqrt{2} & 0 & 2V_g + U_{ab} - g\mu_B B & 0 & 0 \\
-i\Omega \sin(\theta)/2 & -i\Omega \sin(\theta)/2 & 0 & 0 & 2V_g + U_{ab} & 0 \\
-\Omega \cos(\theta)/\sqrt{2} & -i\Omega \cos(\theta)/\sqrt{2} & 0 & 0 & 0 & 2V_g + U_{ab} + g\mu_B B
\end{pmatrix}.$$

Here, the top left $3 \times 3$ block acts on the singlet subspace, while the bottom right $3 \times 3$ block acts on the triplet subspace and the off-diagonal blocks contain the SOI which couples the singlet to the triplet subspace. The spectrum of $H_D^{(2)}$ is depicted in Fig. 1(c) of the main text. The effective Hamiltonian, valid to lowest order in $\Omega$, which acts in $|S\rangle$ and $|T\rangle$ is

$$H_{ST}^{(2)} = \begin{pmatrix}
2V_g - \delta + U & -i\Omega \sin(\theta)/2 \\
-i\Omega \sin(\theta)/2 & 2V_g + U_{ab} - g\mu_B B
\end{pmatrix}.$$

It contains the bare energies of the singlet $|S\rangle$ and the triplet $|T\rangle$ on its diagonal. The SOI interaction then couples these levels via the off-diagonal terms. The spectrum of $H_{ST}^{(2)}$ is given by

$$E_{\pm}^{(2)} = 2V_g + |(U + U_{ab} - g\mu_B B - \delta)/2| \pm \sqrt{[(U - U_{ab} + g\mu_B B - \delta)/2]^2 + (\Omega \sin(\theta)/2)^2}.$$

We see that the effect of the SOI is the opening of an energy gap at the crossing point of the bare singlet and triplet energy levels. In terms of the angle between the Zeeman field and the SOI axis, the gap is maximal when $\theta = \pi/2$ and vanishes when $\theta = 0$. The eigenstates of $H_{ST}^{(2)}$ are

$$\left| E_{\pm}^{(2)} \right\rangle = \left( iS_{\pm} \left| T_{\pm} \right\rangle \right) \Leftrightarrow \left| E_{\pm}^{(2)} \right\rangle = iS_{\pm} \left| S \right\rangle + T_{\pm} \left| T \right\rangle,$$

where the coefficients are given by

$$T_{\pm} = \pm \frac{1}{\sqrt{2}} \sqrt{1 \mp \frac{U - U_{ab} + g\mu_B B - \delta}{\sqrt{(U - U_{ab} + g\mu_B B - \delta)^2 + (\Omega \sin(\theta))^2}}}, \quad S_- = -\text{sgn}(\Omega) T_+ \quad S_+ = \text{sgn}(\Omega) T_-.$$

The mixing of the singlet and the triplet is minimal when $\Omega = 0$ or $\theta = 0$ and it is maximal when $\theta = \pi/2$. 
Single occupancy sector

We next discuss the single occupancy sector of the QD which is spanned by the basis states

\[ |1, 0, 0, 0\rangle, |0, 1, 0, 0\rangle, |0, 0, 1, 0\rangle, |0, 0, 0, 1\rangle. \]  

(18)

The matrix representation of \( H_D \) in terms of these basis states is given by

\[
H^{(1)}_D = \frac{1}{2} \begin{pmatrix}
2V_g - \delta - g\mu_B B & 2V_g - \delta + g\mu_B B & i\Omega \cos \theta & i\Omega \sin \theta \\
0 & 2V_g - \delta + g\mu_B B & i\Omega \sin \theta & -i\Omega \cos \theta \\
-i\Omega \cos \theta & -i\Omega \sin \theta & 2V_g + \delta - g\mu_B B & 0 \\
-i\Omega \sin \theta & i\Omega \cos \theta & 0 & 2V_g + \delta + g\mu_B B
\end{pmatrix}.
\]  

(19)

Here, the top left \( 2 \times 2 \) block acts on the subspace of orbital \( a \), while the bottom right \( 2 \times 2 \) block acts on the subspace of orbital \( a \). The off-diagonal blocks contain the SOI which couples the \( a \) orbital to the \( b \) orbital. The spectrum of \( H^{(1)}_D \) is depicted in Fig. 1(d) of the main text and is given by

\[
E^{(1)}_\lambda = V_g + \frac{1}{2} \left( \delta_{\lambda 1} + \delta_{\lambda 2} - \delta_{\lambda 3} - \delta_{\lambda 4} \right) \sqrt{\left( \Omega \sin \theta \right)^2 + \left( g\mu_B B + \left( \delta_{\lambda 1} - \delta_{\lambda 2} - \delta_{\lambda 3} + \delta_{\lambda 4} \right) \sqrt{\delta^2 + \left( \Omega \cos \theta \right)^2} \right)^2}.
\]  

(20)

Here, \( \delta_{\lambda \lambda'} \) for \( \lambda, \lambda' = 1, ..., 4 \), is the Kronecker delta. The eigenstates of \( H^{(1)}_D \) are of the form

\[
|E^{(1)}_\lambda \rangle = \begin{pmatrix} B_{\lambda \uparrow} \\ B_{\lambda \downarrow} \\ iA_{\lambda \uparrow} \\ iA_{\lambda \downarrow} \end{pmatrix} \quad \Rightarrow \quad |E^{(1)}_\lambda \rangle = \sum_s \left( iA_{\lambda s} a^\dagger_{\alpha s} + B_{\lambda s} a^\dagger_{\beta s} \right) |0_D\rangle.
\]  

(21)

We now determine the coefficients \( A_{\lambda s} \) and \( B_{\lambda s} \) for the different relative angles \( \theta \) between Zeeman field and SOI axis.

Zeeman field and SOI axis are orthogonal \( (\theta = \pi/2) \)

For \( \theta = \pi/2 \), the SOI is proportional to \( \sigma^z \) so that we expect the eigenstates of \( H^{(1)}_D \) to be linear combinations of opposite spins in different orbitals. Indeed, we find that the only coefficients which are non-zero are given by

\[
B_{1\uparrow} = A_{4\downarrow} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{g\mu_B B + \delta}{\sqrt{(g\mu_B B + \delta)^2 + \Omega^2}}}, \quad B_{4\uparrow} = -A_{1\downarrow} = \frac{\text{sgn}(\Omega)}{\sqrt{2}} \sqrt{1 + \frac{g\mu_B B + \delta}{\sqrt{(g\mu_B B + \delta)^2 + \Omega^2}}},
\]

\[
A_{3\uparrow} = -B_{2\downarrow} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{g\mu_B B - \delta}{\sqrt{(g\mu_B B - \delta)^2 + \Omega^2}}}, \quad A_{2\uparrow} = B_{3\downarrow} = \frac{\text{sgn}(\Omega)}{\sqrt{2}} \sqrt{1 - \frac{g\mu_B B - \delta}{\sqrt{(g\mu_B B - \delta)^2 + \Omega^2}}}.
\]  

(22)

The remaining coefficients are vanishing, \( B_{4\downarrow} = B_{3\uparrow} = A_{2\downarrow} = A_{3\uparrow} = A_{4\uparrow} = 0 \).

Zeeman field and SOI axis are parallel \( (\theta = 0, \pi) \)

In the case of \( \theta = 0, \pi \), the SOI is proportional to \( \sigma^z \). Consequently, we expect the eigenstates of \( H^{(2)}_D \) to be mixtures of same spins in different orbitals. For \( \theta = 0 \), we find that the non-vanishing coefficients are given by

\[
B_{1\downarrow} = A_{2\uparrow} = -B_{3\uparrow} = A_{4\downarrow} = \frac{\text{sgn}(\Omega)}{\sqrt{2}} \sqrt{1 - \frac{\delta}{\sqrt{\Omega^2 + \delta^2}}}, \quad A_{1\downarrow} = -B_{2\uparrow} = A_{3\uparrow} = B_{4\downarrow} = B_{4\uparrow} = 0 = A_{4\uparrow}.
\]  

(23)

The remaining coefficients are all zero, \( B_{1\uparrow} = A_{1\uparrow} = A_{2\uparrow} = B_{2\uparrow} = A_{3\downarrow} = B_{3\downarrow} = B_{4\downarrow} = A_{4\downarrow} = 0 \). For \( \theta = \pi \), we find that

\[
B_{1\downarrow} = A_{2\uparrow} = -B_{3\uparrow} = A_{4\downarrow} = -\frac{\text{sgn}(\Omega)}{\sqrt{2}} \sqrt{1 - \frac{\delta}{\sqrt{\Omega^2 + \delta^2}}}, \quad A_{1\downarrow} = -B_{2\uparrow} = A_{3\uparrow} = B_{4\downarrow} = B_{4\uparrow} = 1 \sqrt{2} \sqrt{1 + \frac{\delta}{\sqrt{\Omega^2 + \delta^2}}}.
\]  

(24)

As before, the remaining coefficients vanish, \( B_{1\uparrow} = A_{1\uparrow} = A_{2\uparrow} = B_{2\uparrow} = A_{3\downarrow} = B_{3\downarrow} = B_{4\downarrow} = A_{4\downarrow} = 0 \).
We assume that $\Omega \neq 0$; for $\Omega = 0$ we note that $H^{(1)}_D$ is already diagonal. When $\theta \neq 0, \pi/2, \pi$, the SOI is proportional to both $\sigma^x$ and $\sigma^z$. This means that the SOI mixes states of all spin species in all orbitals. We find that the components of the respective eigenstates are given by

\[ B_{1\uparrow} = \frac{1}{N_1} \frac{g \mu_B + \sqrt{\delta^2 + (\Omega \cos \theta)^2}}{\Omega \sin \theta}, \]
\[ B_{2\uparrow} = \frac{1}{N_2} \frac{g \mu_B - \sqrt{\delta^2 - (\Omega \cos \theta)^2}}{\Omega \sin \theta}, \]
\[ B_{3\uparrow} = \frac{1}{N_3} \frac{g \mu_B - \sqrt{\delta^2 - (\Omega \cos \theta)^2}}{\Omega \sin \theta}, \]
\[ B_{4\uparrow} = \frac{1}{N_4} \frac{g \mu_B + \sqrt{\delta^2 + (\Omega \cos \theta)^2}}{\Omega \sin \theta}, \]
\[ B_{1\downarrow} = \frac{1}{N_1} \frac{\Omega \cos \theta}{\delta + \sqrt{\delta^2 + (\Omega \cos \theta)^2}}, \quad B_{4\downarrow} = \frac{1}{N_4} \frac{\Omega \cos \theta}{\delta + \sqrt{\delta^2 + (\Omega \cos \theta)^2}}, \]
\[ B_{2\downarrow} = \frac{1}{N_2} \frac{\Omega \cos \theta}{\delta - \sqrt{\delta^2 + (\Omega \cos \theta)^2}}, \quad B_{3\downarrow} = \frac{1}{N_3} \frac{\Omega \cos \theta}{\delta - \sqrt{\delta^2 + (\Omega \cos \theta)^2}}, \]
\[ A_{\lambda\uparrow} = \frac{1}{N_\lambda} B_{\lambda\uparrow} B_{\lambda\downarrow}, \quad A_{\lambda\downarrow} = \frac{1}{N_\lambda}, \]

where $N_\lambda$ is a normalization factor which we choose so that $A_{\lambda\uparrow}^2 + A_{\lambda\downarrow}^2 + B_{\lambda\uparrow}^2 + B_{\lambda\downarrow}^2 = 1$. The normalization also ensures that when $\theta \to 0, \pi/2, \pi$ the expressions above reproduce the corresponding limiting cases.

**AN S-WAVE SUPERCONDUCTOR $\varphi_0$ JOSEPHSON JUNCTION**

This second section of the supplemental material gives a more detailed discussion of the SC JJ described by $H_S$ in the main text.

**Effective tunneling Hamiltonian**

We begin with a derivation of the effective tunneling Hamiltonian $H_{S,t}^{\text{eff}}$. Compared to the main text, we allow for a slightly more general tunneling Hamiltonian with spin-dependent tunneling amplitudes,

\[ H_{S,t} = \sum_{\sigma} \sum_{k} t_{\eta'\eta} e^{i\varphi_{\eta'/\eta}/2} c^\dagger_{\eta'k} d_{\sigma k} + \text{H.c.} \]  

(26)

Because it is only the relative phase between the two superconductors which is a physical quantity, we assume that $\varphi_2 = 0$ while $\varphi_1 \equiv \varphi$. We now briefly discuss the different tunneling processes which can occur in the system. Therefore, we rewrite $H_{S,t}$ in terms of the quasiparticle operators,

\[ H_{S,t} = \sum_{\tau} \sum_{k} t_{1\tau} e^{i\varphi/2} u_k \gamma_1 d_{\tau k} + t_{1\tau} e^{i\varphi/2} u_k \gamma_1 d_{\tau k} + t_{2\tau} u_k \gamma_2 d_{\tau k} + t_{2\tau} u_k \gamma_2 d_{\tau k} + t_{2\tau} u_k \gamma_2 d_{\tau k} + t_{2\tau} u_k \gamma_2 d_{\tau k} + H.c., \]  

(27)
where we have assumed that $\xi_k = -\xi_{-k}$. We see that there are two types of tunneling processes: On the one hand, there are processes in which we destroy an electron on the dot and create a quasiparticle on one of the SC leads (or vice versa). Here, electrons and quasiparticles carry the same type of spin or pseudospin. On the other hand, there are processes in which we use the superconducting condensate to simultaneously create (or destroy) an electron on the dot and a quasiparticle on the SC leads. In this case, electron and quasiparticle always carry the opposite type of spin or pseudospin. Because of our convention for the superconducting phases, whenever we destroy (create) an electron on the dot and create a quasiparticle on one of the SC leads or destroy (create) an electron on the dot and destroy or create a quasiparticle on SC $\eta = 1$ we pick up a phase of $e^{i\varphi/2}$ ($e^{-i\varphi/2}$) during the tunneling process.

We now derive the effective tunneling Hamiltonian $H_{S,t}^{\text{eff}}$ using the projection method [1]. Up to fourth order in the tunneling amplitudes we find that

$$H_{S,t}^{\text{eff}} = P_S H_{S,t}^{(E(2))} - H_{D} - H_{S,L}^{-1}(1 - P_S)H_{S,t}^{(E(2))}P_S + P_S H_{S,t}^{(E(2))} \left[(E(2) - H_D - H_{S,L})^{-1}(1 - P_S)H_{S,t}^{(E(2))} \right]^3 P_S,$$

where $P_S = |0_1, E_-(^{(2)}, 0_2)\rangle\langle 0_1, E_-(^{(2)}, 0_2)|$ is the projector on the $E(2)$ state on the dot and the ground states of the SC leads. It acts within the reduced Hilbert space of the states $E_\pm^{(2)}, E^{(1)}_\lambda$ on the dot and the full Hilbert space of the SC

![Tunneling sequences (up to hermitian conjugation) of the SC JJ for contributions $\propto \cos \varphi S$. We use the basis $|n_{1k}\uparrow, n_{1k}\downarrow, n_{n\uparrow}, n_{n\downarrow}, n_{b\uparrow}, n_{b\downarrow}, n_{2q\uparrow}, n_{2q\downarrow} \rangle = (\gamma_{1k}\uparrow)_{+1/2}^{(n_{1k}\downarrow)}(d_{1k}\uparrow)^{n_{n\uparrow}}(d_{1k}\downarrow)^{n_{n\downarrow}}(d_{b\downarrow})^{n_{b\downarrow}}(d_{b\uparrow})^{n_{b\uparrow}}(\gamma_{2q\uparrow})^{n_{2q\uparrow}}(\gamma_{2q\downarrow})^{n_{2q\downarrow}}|0_1, 0_D, 0_2 \rangle$. Filled (empty) dots are used to visually represent a filled (an empty) level.](image-url)
The coupling constant is given by \( \varphi \) transport in Fig. 4 and Fig. 5. Here, we note that the sum of the processes in each row of Fig. 4 and Fig. 5 is

\[
\nu \cdots \text{sum of processes in each row of Fig. 4 and Fig. 5}
\]

This factor is unity because the states \( S \sum_k (\lambda_k t_{1\alpha} t_{2\alpha} + B_{\lambda\alpha}^2 (t_{1\alpha} t_{2\alpha} - t_{1\beta} t_{2\beta})) \).

We point out that unlike Eq. (8) in the main text, this results holds also for spin-dependent tunneling amplitudes. The coupling constant is given by

\[
g_{S} = 2 \sum_{k,q} \frac{u_k u_q v_k v_q}{E_{\lambda}^1 + E_{\alpha} - E_{\alpha}^2 (E_{\alpha}^2 + E_{k} + E_{q} - E_{\alpha}^2)} > 0.
\]

We give a complete table of the tunneling sequences (up to hermitian conjugation) contributing to the Cooper pair transport in Fig. 4 and Fig. 5. Here, we note that the sum of the processes in each row of Fig. 4 and Fig. 5 is \( \propto \sum_k t_{1\beta} t_{2\beta} \). This factor is unity because the states \( E^2_{\alpha} \) are orthonormal, see Eq. (17). This explains why the singlet-triplet mixing does not enter the effective tunneling Hamiltonian. We omit the presentation of \( E^1_{\alpha} \) since it is not relevant to compute the Josephson current. The phases shifts \( \varphi_{S}(\theta) \) and Josephson currents \( I_{S}(\theta) \) at \( \varphi_{S} = 0 \) are plotted in Fig. 10.
We devote this third part of the supplemental material to the derivation and discussion of the effective tunneling Hamiltonian $H_{TS,t}^{\text{eff}}$ for the TS JJ. Similar to the SC JJ, we also allow for spin-dependent tunneling amplitudes in the tunneling Hamiltonian up to second order in the tunneling amplitudes is given by,

$$H_{TS,t} = \sum_{\eta T} \sum_{s} t_{\eta T s} e^{i \varphi_{\eta}} \Gamma_{\eta} d_{\eta s}^\dagger d_{\eta s} + \text{H.c.}$$  \hspace{1cm} (31)$$

For our derivation we adopt the same assumptions as in the main text. Compared to the SC JJ the lowest order processes which contribute to the Josephson current are of second order in the tunneling amplitudes. In particular these processes do not mix the total fermion parity of the TS leads. Because of that, we focus on the odd parity subspace of the TSs. The results for the even parity subspace of the TSs are identical. The effective tunneling Hamiltonian up to second order in the tunneling amplitudes is given by,

$$H_{TS,t}^{\text{eff}} = P_{TS} H_{TS,t} (E^{(2)}_{-} - H_D - H_{TS,L})^{-1}(1 - P_{TS}) H_{TS,t} P_{TS},$$  \hspace{1cm} (32)$$

where $P_{TS} = |1_1, E^{(2)}_{-}, 0_2\rangle \langle 1_1, E^{(2)}_{-}, 0_2| + |0_1, E^{(2)}_{-}, 1_2\rangle \langle 0_1, E^{(2)}_{-}, 1_2|$ is the projector on the $E^{(2)}_{-}$ state on the dot and the ground states of the TS leads. It acts within the reduced Hilbert space of the states $E^{(2)}_{-}, E^{(1)}_{+}$ on the dot and the odd parity ground state subspace of the TS leads. In particular, $0_{\eta}$ ($1_{\eta}$) denotes the ground state in which the non-local fermionic mode in TS $\eta$ is unoccupied (occupied). When evaluating Eq. (32) we find that the result is of the form as given in the main text by Eq. (6) with $\nu = \text{TS}$ and

$$E_{TS}^0 = g_{TS} \left[ B_{\lambda_1 T_+} (A_{\lambda_1 T_-} + B_{\lambda_1 S_-}) (t_{1a\uparrow} t_{2b\uparrow} - t_{1b\uparrow} t_{2a\uparrow}) + B_{\lambda_1 S_-} (A_{\lambda_1 T_-} + B_{\lambda_1 S_-}) (t_{1b\downarrow} t_{2a\uparrow} - t_{1a\downarrow} t_{2b\uparrow}) \right]$$  \hspace{1cm} (33)$$

$$E_{TS}^\nu = - g_{TS} \left[ (A_{\lambda_1 T_-} + B_{\lambda_1 S_-})^2 t_{1b\downarrow} t_{2b\downarrow} + B_{\lambda_1 S_-}^2 (S^2 t_{1b\downarrow} t_{2b\downarrow} + T^2 t_{1a\uparrow} t_{2a\uparrow}) - B_{\lambda_1 S_-} (A_{\lambda_1 T_-} + B_{\lambda_1 S_-}) (t_{1b\downarrow} t_{2b\downarrow} + t_{1a\downarrow} t_{2a\uparrow}) \right],$$

A TOPOLOGICAL SUPERCONDUCTOR $\varphi_0$ JOSEPHSON JUNCTION

Effective tunneling Hamiltonian

Figure 6: Tunneling sequences of the TS JJ for $\theta = \pi/2$. We use the basis $|n_1, n_{a\uparrow}, n_{a\downarrow}, n_{b\uparrow}, n_{b\downarrow}, n_2\rangle = (C_{a\uparrow}^n)^{n_1} (d_{a\uparrow}^\dagger)^{n_{a\uparrow}} (d_{a\downarrow}^\dagger)^{n_{a\downarrow}} (d_{b\uparrow}^\dagger)^{n_{b\uparrow}} (C_{b\downarrow}^n)^{n_{b\downarrow}} |0_1, 0_2, 0_2\rangle$. Filled (empty) dots are used to visually represent a filled (an empty) level. (a) Tunneling sequences that give contributions $\propto \cos(\varphi_{TS})$. (b) Tunneling sequences that give contributions $\propto \sin(\varphi_{TS})$. 
FIG. 7: Phase shift $\varphi_{TS}^0$ as a function of the magnitude of the external magnetic field $B$ at $\theta = \pi/2$ for $\lambda = 1, 4$. For $\lambda = 2, 3$ the phase shift is independent of $B$ and given by $\varphi_{TS}^0 = \pi/2$. For the SC JJ we do not observe a phase shift when $\theta = \pi/2$, $\varphi_S^0 = 0$. We see that the phase shift is peaked at $B = B^{(2)}$ when the singlet triplet mixing is maximal and it saturates at $\pi/2$ when $B \gg B^{(2)}$. Note however that our perturbative approach is not valid when $B \ll B^{(2)}$, because additional energy levels would have to be taken into account.

FIG. 8: Estimate of the critical angle $\theta_c$ when $\lambda = 4$ by analyzing the conditions for the weak coupling limit as a function of $\theta$. The system parameters are chosen as in the main text and supplemental material. In the left panel we plot $\pi \nu F t^2/(\Omega \sin \theta)$ (red dashed) and $\pi \nu F t^2/|E_4^{(1)} - E_-^{(2)}|$ (red solid). In the right panel we plot $t/(\Omega \sin \theta)$ (blue dashed) and $t/|E_4^{(1)} - E_-^{(2)}|$ (blue solid). We find that $\theta_c = 0.3$. This choice of critical angle also works for $\lambda = 1, 2, 3$.

where we have introduced the coefficient

$$g_{TS} = \frac{2}{E_4^{(1)} - E_-^{(2)}} > 0. \tag{34}$$

There are also processes which do not transport a non-local fermion across the JJ and thus lead to a contribution $\tilde{E}_{TS}$ which is independent of the superconducting phase difference. In these processes each TS interacts separately with the QD. In particular this means that the action of the effective tunneling Hamiltonian on the two odd parity ground states of the TS is identical. Consequently, this contribution is proportional to the identity operator and is not relevant when computing the zero-temperature Josephson current. For the case when $\theta = \pi/2$ we have listed all the intermediate tunneling sequences which contribute to the Josephson current in Fig. 6. The phase shift $\varphi_{TS}^0(\theta = \pi/2)$ for $\lambda = 1, 4$ is plotted as a function of the external Zeeman field in Fig. 7. Lastly, the phase shifts $\varphi_{TS}^0(\theta)$ and Josephson currents $I_{TS}(\theta)$ at $\varphi_{TS} = 0$ are plotted in Fig. 10.

CRITICAL ANGLE

The effective Hamiltonians for the SC JJ and the TS JJ are valid in the weak tunnel coupling limit. For the SC JJ this limit is defined by

$$\pi \nu F t_{\eta \tau} t_{\eta' \tau'} \ll E_\lambda^{(1)} - E_-^{(2)}, \Omega \sin(\theta), \Delta \tag{35}$$

and for the TS JJ by

$$t_{\eta \tau} \ll E_4^{(1)} - E_-^{(2)}, \Omega \sin(\theta). \tag{36}$$
These conditions fix a critical angle $\theta_c > 0$ so that our perturbative approach is valid when $\theta \in [\theta_c, \pi - \theta_c]$. In this section we want to determine this critical angle for the system parameters which we have chosen in Fig. 3 of the main text. To get a sense of scales, we consider an InAs nanowire QD JJ with SC leads of length $L = 1 \mu m$. We assume that the effective mass of the electrons in the wire is given by $m = 0.05 m_e$, where $m_e$ is the bare electron mass. Furthermore, we expect that the Fermi energy of the leads is given by $E_F = 20 \text{ meV}$ and the induced superconducting gap by $\Delta = 0.1 \text{ meV}$. The density of states at the Fermi level of the nanowires in the normal metal state is given by $\nu_F = \frac{L}{\pi} \frac{\sqrt{2m}}{\sqrt{\epsilon_F}}$. For the order of magnitude of the tunnel coupling between dot and leads we assume that $t = 0.01 \text{ meV}$. Furthermore, we fix $V_g$ so that $E^{(1)}_\lambda (\pi/2) - E^{(2)}_\lambda (\pi/2) \approx 0.1 \text{ meV}$. This means that depending on the choice of $\lambda$ we have $(V_g|_{\lambda=1}, V_g|_{\lambda=2}, V_g|_{\lambda=3}, V_g|_{\lambda=4}) = (0.89 \text{ meV}, 0.20 \text{ meV}, -0.12 \text{ meV}, -0.80 \text{ meV})$. We can now graphically find an estimate for $\theta_c$, see Fig. 8. A choice of critical angle that works for all $\lambda$ is given by $\theta_c = 0.3$.

**CRITICAL CURRENTS**

**Critical current of the SC JJ**

In this section of supplemental material we compute the critical current $I_{c,SC}$. First, we need to find an approximate value for the coefficient $g_S$. To this end, we notice that it can be rewritten as

$$g_S = \frac{\Delta^2}{2} \int_{-h\omega_c}^{h\omega_c} \nu(E_1) \, dE_1 \int_{-h\omega_c}^{h\omega_c} \nu(E_2) \, dE_2 \frac{1}{\sqrt{E_1^2 + \Delta^2}} \frac{1}{\sqrt{E_2^2 + \Delta^2}}$$

$$\times \left[ \left( E^{(1)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_1^2 + \Delta^2} \right] \left[ \left( E^{(1)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_2^2 + \Delta^2} \right] \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_1^2 + \Delta^2} \right] \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_2^2 + \Delta^2} \right]$$

where $\nu(E) = \sum_k \delta(E - E_k)$ is the density of state of the leads in the normal state at energy $E$ and $\omega_c$ is a cut-off frequency which is typically of the order of the Debye frequency of the crystal. For simplicity, we now assume that $\nu(E) \approx \nu_F$ for $|E| > \Delta$ and $\nu(E) = 0$ for $|E| < \Delta$. This yields

$$g_S \approx \frac{(\Delta \nu_F)^2}{2} \left( \int_{-h\omega_c}^{h\omega_c} \nu(E_1) \, dE_1 \right) \left( \int_{-h\omega_c}^{h\omega_c} \nu(E_2) \, dE_2 \right) \frac{1}{\sqrt{E_1^2 + \Delta^2}} \frac{1}{\sqrt{E_2^2 + \Delta^2}}$$

$$\times \left[ \left( E^{(1)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_1^2 + \Delta^2} \right] \left[ \left( E^{(1)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_2^2 + \Delta^2} \right]$$

$$\times \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_1^2 + \Delta^2} \right] \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_2^2 + \Delta^2} \right] \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_1^2 + \Delta^2} \right] \left[ \left( E^{(2)}_\lambda - E^{(2)}_\lambda \right) + \sqrt{E_2^2 + \Delta^2} \right]$$

Defining $\xi_\pm = (E^{(1)}_\lambda - E^{(2)}_\lambda)/\Delta$ allows us to rewrite this expression as

$$g_S \approx \frac{40 \mu m L^2}{\pi^2 \hbar^2 \Delta E_F}$$

where we have introduced the dimensionless factor

$$\alpha = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{1 + x^2} \sqrt{1 + y^2} \left( \sqrt{1 + x^2} + \sqrt{1 + y^2} + \xi_- - \xi_+ \right) \left( \sqrt{1 + x^2} + \xi_+ \right) \left( \sqrt{1 + y^2} + \xi_- \right)}$$

FIG. 9: Magnitude of the critical current $|I_c(\theta)|$ for different choices of $\lambda$. The system parameters are chosen as in the main text and supplemental material.
FIG. 10: Phase shift $\varphi^0_{\lambda}(\theta)$ (top row) and Josephson current $I_{\lambda}(\theta)$ at $\varphi_S = 0$ (bottom row) for $\lambda = 1, 2, 3$. The system parameters are chosen as in the main text. The jumps in the Josephson current $I_{TS}(\theta)$ correspond to a change of the ground state of the junction.

and we have assumed that $\hbar \omega_c \gg \Delta$ which ensures that the Cooper potential of the BCS theory is a good approximation to the actual electron pairing potential. We note that $\alpha$ is a function of the relative orientation of SOI axis and Zeeman field, $\alpha = \alpha(\theta)$. For the system parameters chosen in the main text we find that $\alpha \approx 10^{-1}$. In total the critical current is then given by

$$I^c_S \approx \frac{8\alpha}{\pi^2} \frac{m e L^2}{\hbar^2 \Delta E_F} \sqrt{(E^{(1)}_S)^2 + (E^{(2)}_S)^2 \text{sgn}(E^{(0)}_S)}.$$ (41)

We have plotted $I^c_S(\theta)$ in Fig. 9. For the case when $\theta = \pi/2$ and $\lambda = 2, 3$ we have $I^c_S = 0$ because $B_{2(3)\uparrow} = 0$. Moreover, there exists a significant difference in magnitude of the critical currents for the cases when $\lambda = 1, 4$ which are most relevant for our experimental proposal in the main text. We can understand this because $I_{S,c}\mid_{\lambda=1} / I_{S,c}\mid_{\lambda=4} \propto (B_{1\uparrow}/B_{4\uparrow})^4 \approx 10^{-6}$. The virtual state $E^{(1)}_1$ only contains a small amount of $B_{1\uparrow}$ due to the SOI, while $E^{(1)}_4$ consists mostly of $B_{4\uparrow}$, hence $B_{4\uparrow} \gg B_{1\uparrow}$. The conclusion is that the absence or presence of a phase shift can most easily be measured when virtual tunneling occurs via the $E^{(1)}_4$ state.

Critical current of the TS JJ

For the TS JJ we find that the critical current is given by

$$I^c_{TS} = \frac{4\kappa_{TS} e}{\hbar(E^{(1)}_{\lambda_0} - E^{(2)}_{-\lambda})} \sqrt{(E^{(1)}_{TS})^2 + (E^{(2)}_{TS})^2 \text{sgn}(E^{(0)}_{TS})}. (42)$$

We plot $I^c_{TS}(\theta)$ in Fig. 9. Again we see a significant difference in magnitude when comparing the most relevant cases of $\lambda = 1$ and $\lambda = 4$. This can be explained in the same way as for the SC JJ. However, this time we have for example at $\theta = \pi/2$, $I_{TS,c}\mid_{\lambda=1} / I_{TS,c}\mid_{\lambda=4} \propto (B_{1\uparrow}/B_{4\uparrow})^2 \approx 10^{-3}$.

[1] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer-Verlag, Berlin, 1994).