On the ambiguity of the interfering resonances parameters determination.

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Abstract

In the paper the interfering resonances parameters determination ambiguity is considered. It is shown that there are two solutions for two fixed width resonances. Analytical relation between different solutions is derived. Numeric experiments for fixed width three and four resonances, and for model energy-dependant width two resonances confirm ambiguity of the resonances parameters determination.

I. INTRODUCTION

One of the typical tasks during experimental data processing is determination of the parameters of several resonances from the experimental cross-section measurements taking into account their interference with arbitrary phases. Often it occurs that for the resonances with arbitrary phases several almost equally good solutions can be found. In the paper an attempt to analyze this problem is performed using simple examples.

The preliminary version of this paper has been published as a preprint of Budker Institute of Nuclear Physics [1].

II. TWO RESONANCES INTERFERENCE WITH SIMPLE NON-RELATIVISTIC BREIT-WIGNER AMPLITUDE

Let us consider a model cross section

\[
\sigma(E) = \left| \frac{A}{E - m_1 + i\frac{\Gamma_1}{2}} + \frac{B}{E - m_2 + i\frac{\Gamma_2}{2}} \right|^2,
\]  

(1)
where $A, B$ are some complex numbers (coupling constants), $m_1, m_2, \Gamma_1, \Gamma_2$ are real numbers (masses and widths of resonances).

First let us derive the conditions which lead to the identical cross section as a function of energy with different set of parameters.

Two identical continuous functions should have identical Fourier images.

For the function of interest the Fourier image \( \phi(t) \) is easily calculated:

$$
\phi(t) = \int_{-\infty}^{+\infty} \sigma(E) e^{itE} \, dE =
\left\{
\begin{array}{ll}
\frac{A^* A}{\Gamma_1} + \frac{iA^* B}{m_1-m_2+i\frac{1}{2}(\Gamma_1+\Gamma_2)} e^{it\left(m_1-\frac{i\Gamma_1}{2}\right)} + \\
\frac{-iB^* A}{m_2-m_1+i\frac{1}{2}(\Gamma_1+\Gamma_2)} + \frac{B^* B}{\Gamma_2} e^{it\left(m_2+i\frac{\Gamma_2}{2}\right)}, & t > 0,
\end{array}
\right.
$$

\hspace{1cm} (2)

$$
= 2\pi \left\{
\begin{array}{ll}
\frac{A^* A}{\Gamma_1} - \frac{iAB^*}{m_1-m_2-\frac{1}{2}(\Gamma_1+\Gamma_2)} e^{it\left(m_1+\frac{i\Gamma_1}{2}\right)} + \\
\frac{-iBA^*}{m_2-m_1-\frac{1}{2}(\Gamma_1+\Gamma_2)} + \frac{B^* B}{\Gamma_2} e^{it\left(m_2-i\frac{\Gamma_2}{2}\right)}, & t < 0.
\end{array}
\right.
$$

In order that function

$$
\sigma_x(E) = \left| \frac{A_x}{E-m_{1x}+i\frac{\Gamma_{1x}}{2}} + \frac{B_x}{E-m_{2x}+i\frac{\Gamma_{2x}}{2}} \right|^2
$$

be equal to $\sigma(E)$ at every point $E$, evidently the following equalities should be valid

$$
m_{1x} = m_1, \quad \Gamma_{1x} = \Gamma_1;
$$

$$
\frac{A_x^* A_x}{\Gamma_{1x}} + \frac{iA_x^* B_x}{m_{1x}-m_{2x}+\frac{1}{2}(\Gamma_{1x}+\Gamma_{2x})} = \frac{A^* A}{\Gamma_1} + \frac{iA^* B}{m_1-m_2+i\frac{1}{2}(\Gamma_1+\Gamma_2)};
$$

$$
m_{2x} = m_2, \quad \Gamma_{2x} = \Gamma_2;
$$

$$
\frac{-iB_x^* A_x}{m_{2x}-m_{1x}+\frac{1}{2}(\Gamma_{1x}+\Gamma_{2x})} + \frac{B_x^* B_x}{\Gamma_{2x}} = \frac{iB^* A}{m_2-m_1+i\frac{1}{2}(\Gamma_1+\Gamma_2)} + \frac{B^* B}{\Gamma_2}.
$$

(4)

Apparently the resonance masses should be ordered here otherwise additional trivial solutions would appear due to parameter sets exchange.

For the amplitudes $A_x, B_x$ we have four equations with four variables (separate equations for the real and imaginary parts). Because the equations are non-linear there could be more than one solution.

Since only amplitude absolute value squared has a physical sense there is a freedom in absolute phases with definite relative phase value. So we can take, for example, that $A_x$ is a real number and $B_x$ defines their relative phase, or equivalent:

$$
A_x = |A_x| e^{i\psi}, \quad B_x = |B_x| e^{-i\psi},
$$
If the latter definition is admitted then

\[ A_x = a_x e^{i\psi_x}, \quad B_x = b_x e^{-i\psi_x}, \quad A = ae^{i\psi}, \quad B = be^{-i\psi}, \]  

(5)

where \( a, b, \psi, a_x, b_x, \psi_x \) are real numbers.

Now one can write down system of equations:

\[
\begin{align*}
\frac{a^2}{\Gamma_1} + \frac{a b_x \left[ \cos(2\psi_x) - (m_1 - m_2) \sin(2\psi_x) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2} &= \frac{a^2}{\Gamma_1} + \frac{ab \left[ \cos(2\psi) - (m_1 - m_2) \sin(2\psi) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2} \\
\frac{b^2}{\Gamma_2} + \frac{a b_x \left[ \cos(2\psi_x) + (m_1 - m_2) \sin(2\psi_x) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2} &= \frac{b^2}{\Gamma_2} + \frac{ab \left[ \cos(2\psi) + (m_1 - m_2) \sin(2\psi) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2} \\
\frac{a x b_x \left[ \cos(2\psi_x) - (m_1 - m_2) \sin(2\psi_x) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2} &= \frac{a x b_x \left[ \cos(2\psi) - (m_1 - m_2) \sin(2\psi) \right]}{(m_1 - m_2)^2 + (\Gamma_1 + \Gamma_2)^2}
\end{align*}
\]  

(6)

Evidently the last two equations are identical, so for three unknown variables \( a_x, b_x, \psi_x \) we have three independent equations.

Trivial solution: \( a_x = a, \ b_x = b, \ \psi_x = \psi \). Let us check whether there are some other solutions. First exclude \( \psi_x \). One equation without \( \psi_x \) can be derived by subtracting the second equation in (6) from the first one:

\[
\frac{a^2}{\Gamma_1} - \frac{b^2}{\Gamma_2} = \frac{a^2}{\Gamma_1} - \frac{b^2}{\Gamma_2}.
\]  

(7)

Let us introduce a new variable

\[ y = \frac{a^2}{\Gamma_1} - \frac{b^2}{\Gamma_2} \Leftrightarrow a_x = \sqrt{a^2 + y\Gamma_1}, \quad b_x = \sqrt{b^2 + y\Gamma_2} \]  

(8)
Now for two variables $\psi_x$ and $y$ we have two equations:

\[
\begin{align*}
\begin{cases}
axb_x \cdot \left[ \frac{\cos(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi_x) \right] = \\
= ab \cdot \left[ \frac{\cos(2\psi)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi) \right] - \\
- y \cdot \left[ (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 \right],
\end{cases}
\end{align*}
\]

\begin{align*}
axb_x \cdot \left[ (m_1 - m_2) \cos(2\psi_x) - \frac{\sin(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) \right] = \\
= ab \cdot \left[ (m_1 - m_2) \cos(2\psi) - \frac{\sin(2\psi)}{2} (\Gamma_1 + \Gamma_2) \right].
\end{align*}

(9)

Linear equation for $\tan(2\psi_x)$ can be obtained dividing the first equation in (9) on the second one:

\[
\frac{\Gamma_1 + \Gamma_2 + (m_1 - m_2) \tan(2\psi_x)}{(m_1 - m_2) - \tan(2\psi_x) / (\Gamma_1 + \Gamma_2)} =
\]

\[
= \frac{ab \left( \frac{\cos(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi_x) \right) - y \left( (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 \right)}{ab \left( (m_1 - m_2) \cos(2\psi) - \frac{\sin(2\psi)}{2} (\Gamma_1 + \Gamma_2) \right)}.
\]

(10)

The solution is amazingly simple:

\[
\tan^{2}\psi_x = \frac{ab \sin(2\psi) + (m_2 - m_1) y}{ab \cos(2\psi) - \frac{y}{2} (\Gamma_1 + \Gamma_2)}.
\]

(11)

Now we can check whether it is an actual solution of the system (9). Let us check the following values:

\[
S_1 = \frac{axb_x \cdot \left[ \frac{\cos(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi_x) \right]}{ab \left( \frac{\cos(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi_x) \right) - y \left( (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 \right)},
\]

(12)

\[
S_2 = \frac{axb_x \cdot \left[ (m_1 - m_2) \cos(2\psi_x) - \frac{\sin(2\psi_x)}{2} (\Gamma_1 + \Gamma_2) \right]}{ab \left( (m_1 - m_2) \cos(2\psi) - \frac{\sin(2\psi)}{2} (\Gamma_1 + \Gamma_2) \right)}.
\]

For an actual solution there should be $S_1 = 1$, $S_2 = 1$. After substitution $\psi_x$ we obtain:

\[
S_1 = S_2 = \frac{axb_x \cos(2\psi_x)}{ab \cos(2\psi) - \frac{y}{2} (\Gamma_1 + \Gamma_2)}
\]

(13)

From $S_{1,2} = 1$:

\[
\frac{\cos(2\psi_x)}{ab \cos(2\psi) - \frac{y}{2} (\Gamma_1 + \Gamma_2)} = \frac{1}{axb_x} = \frac{1}{\sqrt{(a^2 + y\Gamma_1)(b^2 + y\Gamma_2)}},
\]

(14)

and from expression (11) we get:

\[
\frac{\cos(2\psi_x)}{ab \cos(2\psi) - \frac{y}{2} (\Gamma_1 + \Gamma_2)} =
\]

\[
= \frac{1}{\sqrt{a^2b^2 + y^2 \left( (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 \right) + 2ab \cdot y \cdot \left[ (m_2 - m_1) \sin(2\psi) - \frac{\Gamma_1 + \Gamma_2}{2} \cos(2\psi) \right]}},
\]

(15)
So the equation for $y$:

$$
y \cdot \left\{ y \cdot \left[ (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 - \Gamma_1 \Gamma_2 \right] + 2a b \cdot \left[ (m_2 - m_1) \sin(2\psi) - \frac{\Gamma_1 + \Gamma_2}{2} \cos(2\psi) \right] - a^2 \Gamma_2 - b^2 \Gamma_1 \right\} = 0
$$

(16)

This equation has two solutions for $y$:

$$
y = 0
$$

(17)

$$
a_x = a, \ b_x = b, \ \tan(2\psi_x) = \tan(2\psi), \ S_1 = S_2 = \frac{\cos(2\psi_x)}{\cos(2\psi)}.
$$

(18)

It is obvious that the values $\sin(2\psi_x)$ and $\cos(2\psi_x)$ must match exactly with $\sin(2\psi)$ and $\cos(2\psi)$, correspondingly.

$$
y = \frac{a^2 \Gamma_2 + b^2 \Gamma_1 + 2 a b \Gamma_1 \frac{\cos(2\psi)}{2} (\Gamma_1 + \Gamma_2) + (m_2 - m_1) \sin(2\psi)}{(m_1 - m_2)^2 + \left( \frac{\Gamma_1 - \Gamma_2}{2} \right)^2}
$$

(19)

If one formally substitutes this solution, then two cross section curves become identical.

However in order to this solution be admissible some conditions must be satisfied:

1. $y \geq -\frac{a^2}{\Gamma_1} \iff a_x^2 \geq 0$,
2. $y \geq -\frac{b^2}{\Gamma_2} \iff b_x^2 \geq 0$,
3. $|\cos(2\psi_x)| \leq 1$,
4. $|\sin(2\psi_x)| \leq 1$.

Let us check

$$
a_x^2 = a^2 + \Gamma_1 y =
$$

$$
= \frac{b^2 \Gamma_1 + a^2 \left[ (m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2 \right] + 2 a b \Gamma_1 \frac{\cos(2\psi)}{2} \left( \Gamma_1 + \Gamma_2 \right) + (m_2 - m_1) \sin(2\psi)}{(m_1 - m_2)^2 + \left( \frac{\Gamma_1 - \Gamma_2}{2} \right)^2}
$$

$$
= \left[ a \sqrt{(m_1 - m_2)^2 + \left( \frac{\Gamma_1 + \Gamma_2}{2} \right)^2} + b \Gamma_1 \sin \left( 2\psi + \arctg \frac{\Gamma_1 + \Gamma_2}{2(m_1 - m_2)} \right) \right]^2
$$

$$
+ \frac{b^2 \Gamma_1^2 \cos^2 \left( 2\psi + \arctg \frac{\Gamma_1 + \Gamma_2}{2(m_1 - m_2)} \right)}{(m_1 - m_2)^2 + \left( \frac{\Gamma_1 - \Gamma_2}{2} \right)^2} \geq 0.
$$

(20)
Similarly the condition $b_x^2 \geq 0$ is checked. The last two conditions are easily confirmed by the check that derived by their own formulae $\sin(2\psi_x)$ and $\cos(2\psi_x)$ satisfy the Pythagorean theorem $\cos^2(2\psi_x) + \sin^2(2\psi_x) = 1$.

So we have found non-trivial solution that means: any pair of resonances can be replaced by another pair of resonances with the same masses and widths but different amplitudes and phases so that the resonance cross section does not change.

Using these formulae one can get the solution for the case of interference of Breit-Wigner amplitude with complex constant, substituting

$$b \rightarrow c \cdot i \frac{\Gamma_2}{2}$$

and setting $\Gamma_2 \rightarrow \infty$. Corresponding cross section reads

$$\sigma(E) = \left| \frac{ae^{i\psi}}{E - m_1 - i\frac{\Gamma_1}{2}} + ce^{-i\psi} \right|^2.$$  \hspace{1cm} (21)

Finally we get

$$y = \lim_{\Gamma_2 \to \infty} \frac{a^2\Gamma_2 + (c \cdot i \frac{\Gamma_2}{2})^2 \Gamma_1 + a \cdot c \cdot i \frac{\Gamma_2}{2} \left( \frac{\cos(2\psi)}{2} (\Gamma_1 + \Gamma_2) + (m_1 - m_2) \sin(2\psi) \right)}{(m_1 - m_2)^2 + \left( \frac{\Gamma_1 - \Gamma_2}{2} \right)^2} = 2iac \cos 2\psi - c^2 \Gamma_1, \hspace{1cm} (22)$$

$$a_x = \sqrt{a^2 + 2iac\Gamma_1 \cos 2\psi - c^2 \Gamma_1^2}, \quad c_x = c,$$

$$\tan 2\psi_x = \frac{a \sin 2\psi}{a \cos 2\psi + i c_x}.$$  \hspace{1cm} (23)

Here the variable $y$, which was real in previous consideration became complex as well as variable $a_x$. If reassemble imaginary and real parts of these variables or solve this problem from the beginning with cross section (21), then one gets

$$a_x = \sqrt{a^2 + 2ac\Gamma_1 \sin 2\psi + c^2 \Gamma_1^2} = \sqrt{(c\Gamma_1 + a \sin 2\psi)^2 + a^2 \cos^2 2\psi},$$

$$\sin 2\psi_x = -\frac{a \sin 2\psi + c\Gamma_1}{a_x}, \quad \cos 2\psi_x = \frac{a \cos 2\psi}{a_x}. \hspace{1cm} (23)$$

Let us look at the numerical example of $\omega$ and $\phi$ mesons interference in the channel $e^+e^- \to \pi^+\pi^-\pi^0$. Approximate values of resonance parameters: $m_1 = m_\omega = 782.6, \Gamma_\omega = 8.4, a = 1, m_2 = m_\phi = 1019.4, \Gamma_\phi = 4.46, b = 0.1, \psi = -\frac{155^\circ}{2} = -77.5^\circ$.

Substituting to the formulae one gets

$$y = 0.00042; \quad a_x = 1.0018; \quad b_x = 0.109; \quad \psi_x = 74.4^\circ.$$  \hspace{1cm} (24)
So we get quite different phase while amplitudes changed just a little bit.

At this point the most urgent question is: whether this ambiguity is a unique property of just this simple resonance description, or in a more sophisticated and realistic parameterization of resonance cross section such resonance phase ambiguity will take place? The matter is that for actual experimental data processing the much more complicated resonance cross section formulae are used.

### III. RELATIVISTIC BREIT-WIGNER RESONANCE AMPLITUDE

A little more complicated variant of Breit-Wigner formula (relativistic) is:

\[
\sigma(E) = \left| \frac{A}{s - m_1^2 + i\Gamma_1 m_1} + \frac{B}{s - m_2^2 + i\Gamma_2 m_2} \right|^2,
\]

where \( s = E^2 \), evidently has the same property, just some redefinition of variables is necessary.

It is also evident that additional general factor, even strongly dependent on energy, does not change the solution.

However in the most accurate variant of this formula instead of constants \( \Gamma_1, \Gamma_2 \) there are used more complicated expressions containing phase space of final states, transition to which are probable for these resonances.

### IV. ENERGY DEPENDENCE OF RESONANCE WIDTH

For multiparticle final states, such as \( \pi^+\pi^-\pi^0 \), there are no simple formulae for the final state phase space, so for our exercise let us choose a simple model formula:

\[
\Gamma_i \rightarrow \Gamma_i \cdot \left( \frac{s - 4\mu^2}{m_i^2 - 4\mu^2} \right)^{\frac{3}{2}},
\]

where the effective mass \( \mu \) defines the reaction threshold, so at \( s < 4\mu^2 \) cross section becomes equal zero. New cross section can be rewritten as follows

\[
\sigma(s) = \left( \frac{s-4\mu^2}{s^2} \right)^{\frac{3}{2}} \times
\]

\[
\times \left| \frac{A}{s - m_1^2 + i\Gamma_1 m_1 \left( \frac{s - 4\mu^2}{m_1^2 - 4\mu^2} \right)^{\frac{3}{2}}} + \frac{B}{s - m_2^2 + i\Gamma_2 m_2 \left( \frac{s - 4\mu^2}{m_2^2 - 4\mu^2} \right)^{\frac{3}{2}}} \right|^2
\]

(27)
For this function it is hard to make a Fourier transform in order to apply the same trick we have done for the approximate resonance curve. Substitute

\[ s = 4\mu^2 + \rho^4, \quad \rho = (s - 4\mu^2)^{\frac{1}{4}}, \]

\[ \rho_1 = (m_1^2 - 4\mu^2)^{\frac{1}{4}}, \quad \rho_2 = (m_2^2 - 4\mu^2)^{\frac{1}{4}}. \]

Cross section dependence on the new variable \( \rho \) is the following

\[ \sigma(\rho) = \frac{\rho^6}{(\rho^4 + 4\mu^2)^2} \times \]

\[ \frac{A}{\rho^4 + 4\mu^2 - m_1^2 + i\Gamma_1 m_1 \left( \frac{\rho}{\rho_1} \right)^4} + \frac{B}{\rho^4 + 4\mu^2 - m_2^2 + i\Gamma_2 m_2 \left( \frac{\rho}{\rho_2} \right)^4} \]  

The task of Fourier transform of this function is already not so hard if we can find all its irregular points. Just let us simplify the function first so as the general factor does not change the problem solution.

Thus investigated function of \( \rho \) is

\[ f(\rho) = \left| \frac{A}{\rho^4 + 4\mu^2 - m_1^2 + i\Gamma_1 m_1 \left( \frac{\rho}{\rho_1} \right)^4} + \frac{B}{\rho^4 + 4\mu^2 - m_2^2 + i\Gamma_2 m_2 \left( \frac{\rho}{\rho_2} \right)^4} \right|^2 \]

16 irregular points are determined by the equations

\[ \rho^4 + 4\mu^2 - m_j^2 \pm i\Gamma_j m_j \cdot \left( \frac{\rho}{\rho_j} \right)^3 = 0, \quad j = 1, 2 \]

Rewrite this equation in the form

\[ \rho^4 + R_1 \rho^3 + R_2 = 0, \]

where \( R_1 = \pm i \frac{\Gamma_j m_j}{\rho_j^4}, \quad R_2 = 4\mu^2 - m_j^2 = -\rho_j^4. \)

If we solve this equation in a standard way, then the solution can be excessive complicated. Let us try to apply the idea of Ferrari solution directly to the equation \[ \text{(32)} \].

\[ \rho^4 + R_1 \rho^3 + R_2 = (\rho^2 + \lambda_1 \rho + \lambda_2) \cdot (\rho^2 + \lambda_3 \rho + \lambda_4) \]

This equality is valid for any \( \rho \) if

\[ \begin{align*}
\lambda_1 + \lambda_3 &= R_1 \\
\lambda_2 + \lambda_4 + \lambda_1 \lambda_3 &= 0 \\
\lambda_2 \lambda_3 + \lambda_1 \lambda_4 &= 0 \\
\lambda_2 \lambda_4 &= R_2
\end{align*} \]
Exclude $\lambda_1$, $\lambda_3$:

$$\lambda_1 = \frac{\lambda_2}{\lambda_2 - \lambda_4} R_1, \quad \lambda_3 = \frac{\lambda_4}{\lambda_4 - \lambda_2} R_1.$$  

(35)

Now for the variables $\lambda_2$ and $\lambda_4$ we have two equations:

$$\begin{cases}
(\lambda_2 + \lambda_4) \cdot (\lambda_2^2 + \lambda_4^2 - 2R_2) - R_1^2 R_2 = 0, \\
\lambda_2 \lambda_4 = R_2.
\end{cases}$$  

(36)

Substituting $\lambda_2 + \lambda_4 = y$, obtain

$$\begin{align*}
\lambda_1 &= \frac{y + \sqrt{y^2 + 4\rho_j^4}}{2\sqrt{y^2 + 4\rho_j^4}} R_1, \\
\lambda_2 &= \frac{1}{2} \left( y + \sqrt{y^2 + 4\rho_j^4} \right), \\
\lambda_3 &= -\frac{y - \sqrt{y^2 + 4\rho_j^4}}{2\sqrt{y^2 + 4\rho_j^4}} R_1, \\
\lambda_4 &= \frac{1}{2} \left( y - \sqrt{y^2 + 4\rho_j^4} \right),
\end{align*}$$  

(37)

and $y$ must satisfy the equation

$$y^3 - 4R_2 y - R_1^2 R_2 = 0.$$  

(38)

Applying Cardano solution [2]:

$$y = \alpha + \beta \implies \alpha^3 + \beta^3 + (\alpha + \beta) \cdot (3\alpha\beta - 4R_2) - R_1^2 R_2 = 0.$$  

(39)

Setting $\beta = \frac{4R_2}{3\alpha}$, we get

$$\alpha^6 - R_1^2 R_2 \alpha^3 + \left( \frac{4R_2}{3} \right)^3 = 0,$$  

(40)

$$\alpha = \sqrt[3]{\frac{1}{2} R_1^2 R_2 + \sqrt{\left( \frac{R_1^2 R_2}{2} \right)^2 - \left( \frac{4R_2}{3} \right)^3}},$$  

(41)

$$\beta = \sqrt[3]{\frac{1}{2} R_1^2 R_2 - \sqrt{\left( \frac{R_1^2 R_2}{2} \right)^2 - \left( \frac{4R_2}{3} \right)^3}}.$$  

Substituting $R_1$, $R_2$, we get

$$\alpha = \sqrt[3]{\frac{1}{2} \frac{\Gamma^2 m^2}{2\rho_j^2} + \sqrt{\left( \frac{\Gamma^2 m^2}{2\rho_j^2} \right)^2 + \left( \frac{4\rho_j^4}{3} \right)^3}},$$  

(42)

$$= \rho_j^2 \sqrt[3]{\frac{1}{2} \frac{\Gamma^2 m^2}{2\rho_j^2} + \sqrt{\left( \frac{\Gamma^2 m^2}{2\rho_j^2} \right)^2 + \left( \frac{4\rho_j^4}{3} \right)^3}}.$$  

It can be easily confirmed that $\alpha > 0$, $\beta < 0$, $y > 0$, $\lambda_2 > 0$, $\lambda_4 < 0$ are the real numbers, $\lambda_1$, $\lambda_3$ are complex numbers.
Two roots are determined by the equation

\[ \rho^2 + \lambda_1 \rho + \lambda_2 = 0, \]  

and two more roots by the equation

\[ \rho^2 + \lambda_3 \rho + \lambda_4 = 0. \]

None of these roots can be a real number that is evident from the initial form of quartic equation: \( \rho = 0 \) is not root, and for any real \( \rho \neq 0 \) polynomial has non-zero imaginary part for non-zero width \( \Gamma_j > 0 \) and mass \( m_j > 0 \). In order to calculate integrals with infinite limits by residue method we are interested to know the sign of imaginary part of roots. Introduce notation for all roots.

\[
\begin{align*}
\lambda_{11} &= \frac{y_1 + \sqrt{y_1^2 + 4 \rho_1^4 \Gamma_1 m_1}}{2 \sqrt{y_1^2 + 4 \rho_1^4}} \frac{\Gamma_1 m_1}{\rho_1^2}, \\
\lambda_{21} &= \frac{1}{2} \left( y_1 + \sqrt{y_1^2 + 4 \rho_1^4} \right), \\
\lambda_{31} &= -\frac{y_1 - \sqrt{y_1^2 + 4 \rho_1^4}}{2 \sqrt{y_1^2 + 4 \rho_1^4}} \frac{\Gamma_1 m_1}{\rho_1^2}, \\
\lambda_{41} &= \frac{1}{2} \left( y_1 - \sqrt{y_1^2 + 4 \rho_1^4} \right), \\
y_1 &= \rho_1^2 \left( \frac{3 \Gamma_1^2 m_1^2}{2 \rho_1^2} + \sqrt{\left( \frac{3 \Gamma_1^2 m_1^2}{2 \rho_1^2} \right)^2 + \left( \frac{4}{3} \right)^3} \right) + \\
&\quad \quad + \frac{3 \Gamma_1^2 m_1^2}{2 \rho_1^2} - \sqrt{\left( \frac{3 \Gamma_1^2 m_1^2}{2 \rho_1^2} \right)^2 + \left( \frac{4}{3} \right)^3}, \\
z_{11} &= -\frac{\lambda_{11}}{2} + \sqrt{\frac{\lambda_{11}^2}{4} - \lambda_{21}}, \\
z_{21} &= -\frac{\lambda_{11}}{2} - \sqrt{\frac{\lambda_{11}^2}{4} - \lambda_{21}}, \\
z_{31} &= -\frac{\lambda_{31}}{2} + \sqrt{\frac{\lambda_{31}^2}{4} - \lambda_{41}}, \\
z_{41} &= -\frac{\lambda_{31}}{2} - \sqrt{\frac{\lambda_{31}^2}{4} - \lambda_{41}}.
\end{align*}
\]

One can see that \( \Re(z_{11}) = \Re(z_{21}) = 0, \Im(z_{11}) > 0, \Im(z_{21}) < 0, \Im(z_{31}) < 0, \Im(z_{41}) < 0. \)
Four roots correspond better to the equation with index \( j = 2 \):

\[
\begin{align*}
\lambda_{12} &= \frac{y_2 + \sqrt{y_2^2 + 4\rho_2^2}}{2\sqrt{y_2^2 + 4\rho_2^2}} \Gamma m_2, \\
\lambda_{22} &= \frac{1}{2} \left( y_2 + \sqrt{y_2^2 + 4\rho_2^2} \right), \\
\lambda_{32} &= -i \frac{y_2 - \sqrt{y_2^2 + 4\rho_2^2}}{2\sqrt{y_2^2 + 4\rho_2^2}} \Gamma m_2, \\
\lambda_{42} &= \frac{1}{2} \left( y_2 - \sqrt{y_2^2 + 4\rho_2^2} \right), \\
y_2 &= \rho_2^2 \cdot \left( \sqrt{\frac{\Gamma^2 m_2^2}{2\rho_2^2}} \left( \frac{\Gamma^2 m_2^2}{2\rho_2^2} \right)^2 + \left( \frac{4}{3} \right)^3 \right) + \\
&\quad + \frac{3}{\sqrt{\frac{\Gamma^2 m_2^2}{2\rho_2^2}} - \sqrt{\frac{\Gamma^2 m_2^2}{2\rho_2^2}} \left( \frac{4}{3} \right)^3}, \\
z_{12} &= -\frac{\lambda_{12}}{2} + \sqrt{\frac{\lambda_{12}^2}{4} - \lambda_{22}}, \\
z_{22} &= -\frac{\lambda_{22}}{2} - \sqrt{\frac{\lambda_{22}^2}{4} - \lambda_{22}}, \\
z_{32} &= -\frac{\lambda_{32}}{2} + \sqrt{\frac{\lambda_{32}^2}{4} - \lambda_{42}}, \\
z_{42} &= -\frac{\lambda_{42}}{2} - \sqrt{\frac{\lambda_{42}^2}{4} - \lambda_{42}}.
\end{align*}
\]

Here \( \Re(z_{12}) = \Re(z_{22}) = 0, \Im(z_{12}) > 0, \Im(z_{22}) < 0, \Im(z_{32}) < 0, \Im(z_{42}) < 0 \). Now the function \( f(\rho) \) can be presented in the form

\[
f(\rho) = \left( \frac{A}{(\rho - z_{11})(\rho - z_{21})(\rho - z_{31})(\rho - z_{41})} + \frac{B}{(\rho - z_{12})(\rho - z_{22})(\rho - z_{32})(\rho - z_{42})} \right) \times \\
\times \left( \frac{A^*}{(\rho - z_{11}^*)(\rho - z_{21}^*)(\rho - z_{31}^*)(\rho - z_{41}^*)} + \frac{B^*}{(\rho - z_{12}^*)(\rho - z_{22}^*)(\rho - z_{32}^*)(\rho - z_{42}^*)} \right),
\]

where symbol * designates complex conjugate number. Fourrier image \( F(t) = \int_{-\infty}^{+\infty} f(\rho) e^{it\rho} d\rho \) of the real function has a property that \( F(-t) = F^*(t) \), so it is enough to calculate Fourrier transform only for positive value of \( t \), that is determined by the sum of residues on the
irregular points above the abscissa axis, that is \( z_{11}, z_{21}, z_{31}, z_{41}, z_{12}, z_{22}, z_{32}, z_{42} \).

\[ F(t) = 2\pi i \times \]

\[
\left\{ A_{11} e^{i\pi z_{11}} \left( \frac{z_{21}-z_{11}}{z_{21}-z_{21}}(z_{21}-z_{41}) \right) + \right.
\left. A_{12} e^{i\pi z_{12}} \left( \frac{z_{22}-z_{12}}{z_{22}-z_{22}}(z_{22}-z_{42}) \right) + \right.
\left. A_{13} e^{i\pi z_{13}} \left( \frac{z_{23}-z_{13}}{z_{23}-z_{23}}(z_{23}-z_{43}) \right) + \right.
\left. A_{14} e^{i\pi z_{14}} \left( \frac{z_{24}-z_{14}}{z_{24}-z_{24}}(z_{24}-z_{44}) \right) \right\} 
\]

Because the coordinates of irregular points determine the functional dependence of Fourier image on the parameter \( t \), so they must be unchanged for any solution. This condition produces the equations:

\[
\rho^4 + 4\mu^2 - m_{jx}^2 \pm i\Gamma_{jx}m_{jx} \left( \frac{\rho}{\rho_{jx}} \right)^3 \equiv \rho^4 + 4\mu^2 - m_j^2 \pm i\Gamma_jm_j \left( \frac{\rho}{\rho_j} \right)^3 
\]
for \( j = 1, 2 \). Thus we get the system of equations:

\[
\begin{align*}
4\mu_x^2 - m_{1x}^2 &= 4\mu^2 - m_1^2, \\
\Gamma_{1x}m_{1x} \rho_{1x}^{-2} &= \Gamma_1m_1 \rho_1^2, \\
4\mu_x^2 - m_{2x}^2 &= 4\mu^2 - m_2^2, \\
\Gamma_{2x}m_{2x} \rho_{2x}^{-2} &= \Gamma_2m_2 \rho_2^2,
\end{align*}
\]

where \( \rho_1 = (m_1^2 - 4\mu^2)^{1/4} \), \( \rho_2 = (m_2^2 - 4\mu^2)^{1/4} \). For five variables \( \mu, \Gamma_1, \Gamma_2, m_1, m_2 \) we got the system of the four equations. However the factor \( \rho^3 \) in the original function determined the threshold behaviour, and from \( \rho = (s - 4\mu^2)^{1/4} \) is evident that \( \mu \) must not change, so

\[
\mu_x = \mu \quad (51)
\]

and we have four equations for the four variables. Taking into account that \( m_{jx} > 0 \), we derive \( m_{jx} = m_j \) and can immediately conclude that \( \rho_{jx} = \rho_j \), and \( \Gamma_{jx} = \Gamma_j \).

Let us try to simplify the denominators in the formula \([48]\). Apparently

\[
2z_{11}(z_{11} - z_{21}^*) (z_{11} - z_{31}^*) (z_{11} - z_{41}^*) =
\]

\[
= z_{11}^4 + 4\mu^2 - m_1^2 - i\Gamma_1m_1 \left( \frac{z_{11}}{\rho_1} \right)^3 =
\]

\[
= z_{11}^4 + 4\mu^2 - m_1^2 - i\Gamma_1m_1 \left( \frac{z_{11}}{\rho_1} \right)^3 -
\]

\[
- \left[ z_{11}^4 + 4\mu^2 - m_1^2 + i\Gamma_1m_1 \left( \frac{z_{11}}{\rho_1} \right)^3 \right] = -2i\Gamma_1m_1 \left( \frac{z_{11}}{\rho_1} \right)^3
\]

Similarly

\[
(z_{21}^* - z_{11})(2z_{21}^* - z_{21})(z_{21}^* - z_{41}) = 2i\Gamma_1m_1 \left( \frac{z_{21}^*}{\rho_1} \right)^3
\]

\[
(z_{31}^* - z_{11})(z_{31}^* - z_{21})(z_{31}^* - z_{31})(z_{31}^* - z_{41}) = 2i\Gamma_1m_1 \left( \frac{z_{31}^*}{\rho_1} \right)^3
\]

\[
(z_{41}^* - z_{11})(z_{41}^* - z_{21})(z_{41}^* - z_{31})(z_{41}^* - z_{41}) = 2i\Gamma_1m_1 \left( \frac{z_{41}^*}{\rho_1} \right)^3
\]

\[
2z_{12}(z_{12} - z_{22}^*)(z_{12} - z_{32}^*)(z_{12} - z_{42}) = -2i\Gamma_2m_2 \left( \frac{z_{12}}{\rho_2} \right)^3
\]

\[
(z_{22}^* - z_{12})(2z_{22}^* - z_{22})(z_{22}^* - z_{32}) = 2i\Gamma_2m_2 \left( \frac{z_{22}^*}{\rho_2} \right)^3
\]

\[
(z_{32}^* - z_{12})(z_{32}^* - z_{22})(z_{32}^* - z_{32})(z_{32}^* - z_{42}) = 2i\Gamma_2m_2 \left( \frac{z_{32}^*}{\rho_2} \right)^3
\]
\[(z_{12}^* - z_{12}) (z_{22}^* - z_{22}) (z_{14}^* - z_{32}) (z_{42}^* - z_{42}) = 2i\Gamma_2 m_2 \left( \frac{z_{12}^*}{\rho_2} \right)^3 \] (59)

Eight other denominators are not so compact:

\[(z_{11} - z_{11}^*) (z_{11} - z_{22}^*) (z_{11} - z_{32}^*) (z_{11} - z_{42}^*) =
\]
\[= -i\Gamma_2 m_2 \left( \frac{z_{11}^*}{\rho_2} \right)^3 - i\Gamma_1 m_1 \left( \frac{z_{11}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (60)

\[(z_{21}^* - z_{21}) (z_{21}^* - z_{32}) (z_{21}^* - z_{42}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{21}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{21}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (61)

\[(z_{31} - z_{31}^*) (z_{31} - z_{32}) (z_{31} - z_{42}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{31}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{31}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (62)

\[(z_{41} - z_{41}^*) (z_{41} - z_{32}) (z_{41} - z_{42}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{41}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{41}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (63)

\[(z_{12} - z_{12}^*) (z_{12} - z_{31}^*) (z_{12} - z_{31}) (z_{12} - z_{41}) =
\]
\[= -i\Gamma_2 m_2 \left( \frac{z_{12}^*}{\rho_2} \right)^3 - i\Gamma_1 m_1 \left( \frac{z_{12}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (64)

\[(z_{22} - z_{11}^*) (z_{22}^* - z_{21}) (z_{22}^* - z_{31}) (z_{22} - z_{41}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{22}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{22}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (65)

\[(z_{32} - z_{32}^*) (z_{32} - z_{31}) (z_{32} - z_{41}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{32}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{32}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (66)

\[(z_{42} - z_{11}^*) (z_{42}^* - z_{21}) (z_{42}^* - z_{31}) (z_{42} - z_{41}) =
\]
\[= i\Gamma_2 m_2 \left( \frac{z_{42}^*}{\rho_2} \right)^3 + i\Gamma_1 m_1 \left( \frac{z_{42}^*}{\rho_1} \right)^3 - m_2^2 + m_1^2 \] (67)

Here we can shorten the expressions using notations:

\[G = \frac{\Gamma_1 m_1}{\rho_1^2} + \frac{\Gamma_2 m_2}{\rho_2^2}, \quad D_m^2 = m_2^2 - m_1^2. \] (68)

For the rest of free parameters \( A_x = a_x e^{i\psi_x}, B_x = b_x e^{-i\psi_x} \) there is a system of eight complex equations:
\[
\frac{a_x e^{i\psi_x}}{-2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{b_x e^{i\psi_x}}{-iG_{z_{11}} - D_m^2} =
\]
\[
= ae^{i\psi} \left( \frac{ae^{-i\psi}}{-2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{be^{i\psi}}{-iG_{z_{11}} - D_m^2} \right),
\]
\[
\frac{a_x e^{-i\psi_x}}{2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{b_x e^{-i\psi_x}}{iG_{z_{21}} - D_m^2} =
\]
\[
= ae^{-i\psi} \left( \frac{ae^{i\psi}}{2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{be^{-i\psi}}{iG_{z_{21}} - D_m^2} \right)
\]
\[
\frac{a_x e^{-i\psi_x}}{2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{b_x e^{-i\psi_x}}{iG_{z_{31}} - D_m^2} =
\]
\[
= ae^{-i\psi} \left( \frac{ae^{i\psi}}{2i\Gamma_1 m_1 \left( \frac{11}{r_1} \right)^3} + \frac{be^{-i\psi}}{iG_{z_{31}} - D_m^2} \right)
\]
\[
\frac{b_x e^{-i\psi_x}}{-iG_{z_{12}} + D_m^2} + \frac{b_x e^{i\psi_x}}{-2i\Gamma_2 m_2 \left( \frac{12}{r_2} \right)^3} =
\]
\[
= be^{-i\psi} \left( \frac{ae^{-i\psi}}{-iG_{z_{12}} + D_m^2} + \frac{be^{i\psi}}{-2i\Gamma_2 m_2 \left( \frac{12}{r_2} \right)^3} \right)
\]
\[
\frac{b_x e^{i\psi_x}}{iG_{z_{22}} + D_m^2} =
\]
\[
= be^{i\psi} \left( \frac{ae^{i\psi}}{iG_{z_{22}} + D_m^2} + \frac{be^{i\psi}}{2i\Gamma_2 m_2 \left( \frac{12}{r_2} \right)^3} \right)
\]
\[
\frac{b_x e^{i\psi_x}}{iG_{z_{32}} + D_m^2} =
\]
\[
= be^{i\psi} \left( \frac{ae^{i\psi}}{iG_{z_{32}} + D_m^2} + \frac{be^{i\psi}}{2i\Gamma_2 m_2 \left( \frac{12}{r_2} \right)^3} \right)
\]
Let us introduce new variables:

\[ \xi = \frac{a_x b_x}{ab}, \quad \nu_x = \frac{a_x}{b_x}, \quad \nu = \frac{a}{b} \]  

(77)

Of course the trivial solution of the system of equations is valid: \( \psi_x = \psi, \ a_x = a, \ b_x = b, \ \xi = 1, \ \nu_x = \nu \).

The new set of unknown variables \( \psi_x, \nu_x, \xi \). Old variables are connected with new ones with the following relations: \( a_x = \sqrt{ab\xi \nu_x}, \ b_x = \sqrt{\frac{abK}{\nu_x}} \).

The system of equations with new unknowns \( \xi, \nu_x \) is:

\[
\begin{align*}
e^{2i\psi_x} &= \frac{e^{2i\psi}}{\xi} + \frac{iGz_{21}^3 + D_m^2}{2i\Gamma m_1 \left( \frac{z_{21}^3}{\rho} \right)^3} \cdot \left( \frac{\nu}{\xi} - \nu_x \right), \\
e^{-2i\psi_x} &= \frac{e^{-2i\psi}}{\xi} + \frac{iGz_{21}^3 - D_m^2}{2i\Gamma m_1 \left( \frac{z_{21}^3}{\rho} \right)^3} \cdot \left( \frac{\nu}{\xi} - \nu_x \right), \\
e^{-2i\psi_x} &= \frac{e^{-2i\psi}}{\xi} + \frac{iGz_{31}^3 - D_m^2}{2i\Gamma m_1 \left( \frac{z_{31}^3}{\rho} \right)^3} \cdot \left( \frac{\nu}{\xi} - \nu_x \right), \\
e^{-2i\psi_x} &= \frac{e^{-2i\psi}}{\xi} + \frac{iGz_{12}^3 - D_m^2}{2i\Gamma m_1 \left( \frac{z_{12}^3}{\rho} \right)^3} \cdot \left( \frac{\nu}{\xi} - \nu_x \right), \\
e^{2i\psi_x} &= \frac{e^{2i\psi}}{\xi} + \frac{iGz_{12}^3 + D_m^2}{2i\Gamma m_1 \left( \frac{z_{12}^3}{\rho} \right)^3} \cdot \left( \frac{1}{\xi} - \nu_x \right), \\
e^{2i\psi_x} &= \frac{e^{2i\psi}}{\xi} + \frac{iGz_{31}^3 + D_m^2}{2i\Gamma m_1 \left( \frac{z_{31}^3}{\rho} \right)^3} \cdot \left( \frac{1}{\xi} - \nu_x \right), \\
e^{2i\psi_x} &= \frac{e^{2i\psi}}{\xi} + \frac{iGz_{21}^3 + D_m^2}{2i\Gamma m_1 \left( \frac{z_{21}^3}{\rho} \right)^3} \cdot \left( \frac{1}{\xi} - \nu_x \right). \\
\end{align*}
\]

(78)

Subtracting the third equation from the second one, we get

\[
\left( \frac{iGz_{21}^3 - D_m^2}{2i\Gamma m_1 \left( \frac{z_{21}^3}{\rho} \right)^3} - \frac{iGz_{31}^3 - D_m^2}{2i\Gamma m_1 \left( \frac{z_{31}^3}{\rho} \right)^3} \right) \cdot \left( \frac{\nu}{\xi} - \nu_x \right) = 0,
\]

(79)

which can be only valid if one of the multipliers equals zero. So one of the solution is

\[ \nu_x = \frac{\nu}{\xi} \]  

(80)
It can be checked easily that the first multiplier cannot be equal to zero for any resonance parameters. If it were so then

\[ iG - \frac{D_m^2}{z_{21}^3} = iG - \frac{D_m^2}{z_{31}^3}, \]  

(81)

and this is equivalent to the equality \( z_{21} = z_{31} \). Similarly using the equation pairs “second – fourth”, “third – fourth”, and supposing \( \nu_x \neq \frac{\nu}{\xi} \), one gets \( z_{21} = z_{31} = z_{41} \). So the three roots of quatric equation are equal to the same value.

\[
\rho^4 + 4\mu^2 - m_1^2 + i\Gamma_1 m_1 \left( \frac{\rho}{m} \right)^3 = (\rho - z_{11})(\rho - z_{21})^3 = \\
= \rho^4 - (z_{11} + 3z_{21})\rho^3 + \\
+ 3z_{21}(z_{11} + z_{21})\rho^2 - z_{21}^2(3z_{11} + z_{21})\rho + z_{11}z_{21}^3.
\]  

(82)

So as none of the roots is equal to zero, then comparing to representations of the same equation one concludes that

\[
\begin{cases}
  z_{11} + z_{21} = 0, \\
  3z_{11} + z_{21} = 0,
\end{cases}
\]  

(83)

and hence \( z_{11} = z_{21} = 0 \), which is impossible. So our guess \( \nu_x \neq \frac{\nu}{\xi} \) is invalid and the solution of (79) is

\[ \nu_x = \frac{\nu}{\xi}. \]

(84)

Similar analysis of equations 6–8 in (78) gives

\[ \nu_x = \xi \nu. \]

(85)

So one can immediately conclude that

\[ \xi = 1, \quad \nu_x = \nu. \]

(86)

Now from any equation from the system (78) one derives

\[ e^{2i\psi_x} = e^{2i\psi} \implies \psi_x = \psi. \]

(87)

As a result we conclude that for the case of energy dependent resonance width the degeneration disappears and there is the only set of resonance parameters presenting the given energy dependence of cross section.
V. NUMERIC EXPERIMENTS

For additional check of these conclusions it is useful to carry out numeric experiments simulating some actual experiment in high energy physics. Such experiments can show the role of experimental statistics.

In all cases we should suppose some true process cross section function \( \sigma(E) \), where \( E \) is the energy of colliding beams, then for the finite number of points \( E_k \) we generate experimental number of events, according to the Poisson probability distribution (integrated luminosity at every point is equal to the same value \( L \)).

In order to get the parameters of resonances, consisting the given model of process, we minimize the likelihood function as follows

\[
L = \sum_{k=1}^{N} 2 \left( p_k - n_k + n_k \ln \frac{n_k}{p_k} \right),
\]

which is equal to the doubled logarithmic likelihood function with opposite sign. Here \( p_k = \sigma(E_k) \cdot L \). For greater statistics this function limits to \( \chi^2 \) value, that can be used for the check of statistical confidence level. Minimization will be performed with well-known MINUIT package [3].

For presentation of experimental cross section on the plots, the experimental number of events will be ascribed asymmetric statistical errors:

\[
\Delta n_i^{(+)} = \sqrt{n_i + 1}, \quad \Delta n_i^{(-)} = \sqrt{n_i}.
\]

A. Approximate expression for the resonance amplitude

\[
\sigma(E) = \left| \frac{a}{E - m_1 + i\frac{\Gamma_1}{2}} + \frac{be^{i\psi}}{E - m_2 + i\frac{\Gamma_2}{2}} \right|^2
\]

with “true” values of parameters

\[
m_1 = 782.6, \quad \Gamma_1 = 8.4, \quad a = 1, \quad m_2 = 1019.4, \quad \Gamma_2 = 4.5, \quad b = 0.3, \quad \psi = 155^\circ.
\]

This function is equal to \( \sigma(m_1) \approx \frac{1}{4 \pi^2} = 0.057, \sigma(m_2) \approx \frac{0.09}{2 \pi^2} = 0.018 \). In order to an accuracy at the resonance maxima to be at least at the 5% level let us appoint \( L = 2 \cdot 10^4 \).

Fig. I shows the “experimental” set of points.
Figure 1: Result of fit to “experimental” points. Cross section model is \[90\]. \(\chi^2/n_D = 64.3/(74-7)\).

Figure 2: Plot of the likelihood function on the phase of the second resonance amplitude \(\psi_{2x}\). Two equivalent minima with \(\chi^2/n_D = 56.703/(74-7)\) are obtained at \(\psi_{2x} = -157.14^\circ\) and \(\psi_{2x} = 157.51^\circ\). Although the separating maximum is not high (\(\chi^2 = 56.907\) at \(\psi_{2x} = \pm180^\circ\)), these are the different solutions. Values of all parameters at these minimum points are cited at the TableI.

This numeric experiment confirmed the analytical conclusion — fitting the data with the approximate resonance formula produce the ambiguity of resonance phases and amplitudes.
Table I: Parameters of resonances at the minimum points of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|---------|--------------|------|---------|--------------|--------------|--------|
| 1.0125 | 782.62  | 8.5371       | 0.30284 | 1019.4   | 4.5554       | 157.815      | 56.703 |
| 1.0167 | 782.62  | 8.5371       | 0.31033 | 1019.4   | 4.5554       | -157.065     | 56.703 |

Table II: Resonances parameters at the two minimum points of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|---------|--------------|------|---------|--------------|--------------|--------|
| 1.0020 | 782.60  | 8.4116       | 0.30007 | 1019.4   | 4.5093       | 155.283      | 74.58624 |
| 1.0071 | 782.60  | 8.4116       | 0.30708 | 1019.4   | 4.5093       | -154.807     | 74.58624 |

B. Relativistic form of resonance amplitude

Let us consider more accurate dependence of the resonance amplitude on energy:

$$\sigma(E) = \frac{m_1^4 \sqrt{E^2 - 4\mu^2}}{E^4 \sqrt{m_1^2 - 4\mu^2}} \left| \frac{2m_1 a}{E^2 - m_1^2 + i\Gamma_1 m_1} + \frac{2m_2 b e^{i\psi}}{E^2 - m_2^2 + i\Gamma_2 m_2} \right|^2$$  \hspace{1cm} (92)

with the same “true” parameters values

$$m_1 = 782.6, \quad \Gamma_1 = 8.4, \quad a = 1,$$

$$m_2 = 1019.4, \quad \Gamma_2 = 4.5, \quad b = 0.3, \quad \psi = 155^\circ,$$

where the factors $2m_i$ are introduced in order to keep the values of amplitudes at the resonance masses, and general factor imitates the threshold behaviour of cross section with

$$\mu \approx \frac{3m_\pi}{2} \approx \frac{140+140+135}{2} \approx 208.$$  \hspace{1cm} (93)

In order that minima of likelihood function to be more demonstrative let us increase the integrated luminosity to the value of $L = 10^6$.

In Fig. 3 the set of points and optimal cross section of the type (92) are presented.

The plot of the likelihood function on the second resonance phase $\psi_{2x}$ is presented in Fig. 4.

As we expected, we get two equivalent minima again. Parameters values at the minimum points of $\mathcal{L}$ are shown in Table II.
Figure 3: Result of fit of “experimental” points. Cross section model is described with the formula \( \chi^2/n_D = 74.6/(74 - 7) \).

Figure 4: Plot of likelihood function on the second resonance phase \( \psi_{2x} \).

C. Resonance width dependent on energy

Finally let us consider the case where the degeneracy is expected to disappear and only one global minimum will be found:
Figure 5: Result of fit of “experimental” points. Cross section is described by the formula (94). \[ \chi^2/n_D = 74.8/(74 - 7). \]

Table III: Resonance parameters at the two points of minimum of likelihood function.

| \(a_x\) | \(m_{1x}\) | \(\Gamma_{1x}\) | \(b_x\) | \(m_{2x}\) | \(\Gamma_{2x}\) | \(\psi_{2x}\) | \(\chi^2\) |
|---|---|---|---|---|---|---|---|
| 1.0020 | 782.60 | 8.4117 | 0.30007 | 1019.4 | 4.5093 | 155.289 | 74.82254 |
| 1.0071 | 782.60 | 8.4117 | 0.30699 | 1019.4 | 4.5093 | -155.529 | 74.81663 |

\[
\sigma(E) = \left. \left( m_1^4 \frac{E^2-4\mu^2}{m_1^2-4\mu^2} \right) \frac{2m_1a}{E^2-m_1^2+i\Gamma_1 m_1 \left( \frac{E^2-4\mu^2}{m_1^2-4\mu^2} \right)^{\frac{3}{2}}} + \frac{2m_2 b e^{i\psi}}{E^2-m_2^2+i\Gamma_2 m_2 \left( \frac{E^2-4\mu^2}{m_2^2-4\mu^2} \right)^{\frac{3}{2}}} \right| ^2 \tag{94}
\]

Fig. 5 shows the set of energy points and optimal cross section of the type (94).

Likelihood function plot on the phase of the second resonance \(\psi_{2x}\) is shown in Fig. 6.

Indeed, the minimum values are not equal now (see Table III), however the difference is very small and, what was totally unexpected, the “better minimum” corresponds to the “wrong” minimum.

Let us look what will change if the experimental statistics will increase by factor of 100 (\(L = 10^8\)). In Table IV the parameters of resonances are shown for two minimum points.
Figure 6: Likelihood function plot on the phase of the second resonance $\psi_{2x}$

Table IV: Resonance parameters at the two minima of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|----------|----------------|------|----------|----------------|---------------|---------|
| 0.99982 | 782.60  | 8.3977        | 0.30015 | 1019.4 | 4.5025        | 154.985 | 70.085 |
| 1.0050 | 782.60  | 8.3977        | 0.30711 | 1019.4 | 4.5025        | -155.237 | 70.060 |

of likelihood function. Again the difference at the minimum points is very small and again “wrong” minimum is a little preferable although the total statistics is extremely high and practically unreachable in real experiments.

Let us return to the previous level of statistics ($L = 10^6$), but change the threshold factor $\mu = 350$ instead of 208. Again two minima (Table IV) difference is statistically unreliable.

Evidently the more narrow are the resonances, the less is the influence of width dependence on energy to the form of cross section. Let us set both widths large — $\Gamma_{1,2} = 100$. In

Table V: Resonance parameters at the minimum points of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|----------|----------------|------|----------|----------------|---------------|---------|
| 0.99946 | 782.61  | 8.3931        | 0.30013 | 1019.4 | 4.5202        | 151.335 | 63.799 |
| 1.0051 | 782.61  | 8.3935        | 0.30777 | 1019.4 | 4.5203        | -153.908 | 63.711 |
Thus, we have an expression for the cross section model related to the resonance maxima. The fit result of the data points is shown in Fig. 7. The resonances are described by the formula

\[ \chi^2 / n_D = 46.8 / (50 - 7). \]

Likelihood function plot on the second resonance phase $\psi_{2x}$ is presented in Fig. 8. This time the minimum values are essentially unequal and the “better” minimum has “correct” phase. Table VI shows the parameters of resonances at these minimum points. $\chi^2$ confidence level of the first minimum is $P_{50-7}(46.844) = 0.318$, the second one — $P_{50-7}(64.428) =$
Table VI: Resonance parameters at the minimum points of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|----------|----------------|-------|----------|----------------|---------------|---------|
| 10.050 | 782.71 | 100.61 | 5.0111 | 1019.4 | 100.22 | 154.985 | 46.844 |
| 10.848 | 783.39 | 102.03 | 5.9767 | 1019.1 | 99.816 | -173.990 | 64.428 |

Figure 9: Result of fit of “data points”. Model of the cross section is described by the formula (94). $\chi^2/n_D = 52.05/(53 - 7)$. $0.0188$.

For completeness let us consider the intermediate case: $\Gamma_{1,2} = 30$, $a = 3$, $b = 1$. Fig. 9 demonstrates the “data points” and fit result.

Again two minima (Table VII) are statistically equivalent (the difference of $\chi^2$ values much less than unit).

After this numerical experiments we can conclude: if resonance width depends on energy,

Table VII: Resonance parameters at the two likelihood minimum points.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $\chi^2$ |
|-------|----------|----------------|-------|----------|----------------|---------------|---------|
| 3.0103 | 782.62 | 30.128 | 0.9986 | 1019.5 | 30.013 | 155.128 | 52.054 |
| 3.0745 | 782.63 | 30.172 | 1.1311 | 1019.4 | 30.019 | -158.262 | 51.714 |
Table VIII: Resonances parameters at the two likelihood function minimum points.

| aₓ  | m₁ₓ | Γ₁ₓ | bₓ  | m₂ₓ | Γ₂ₓ | ψ₂ₓ | cₓ  | m₃ₓ | Γ₃ₓ | ψ₃ₓ | χ²  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.99805 | 782.62 | 8.3768 | 0.30038 | 1019.4 | 4.5055 | 153.925 | 3.0128 | 1199.7 | 100.75 | 28.634 | 125.969 |
| 1.0535 | 782.62 | 8.3768 | 0.30323 | 1019.4 | 4.5055 | 30.931 | 3.1673 | 1199.7 | 100.75 | -72.318 | 125.969 |

then the two minimum points of likelihood function describe not the same cross section function of energy. However the difference can be used for cutting off the false minimum only under the favourable conditions: high statistics, coverage of wide energy interval with both resonances within, and the widths of the resonances must be compatible with the mass difference. The final result can be obtained only after comparison the likelihood function values χ² at minima: if the difference of levels is much greater than unit, then one can choose better set of phases and amplitudes, otherwise one should involve additional considerations for the choice of interference phase.

D. Three resonances

The case of three resonances with constant widths:

$$
\sigma(E) = \frac{m_4 \sqrt{E^2 - 4\mu^2}}{E^4 \sqrt{m_1^2 - 4\mu^2}} \left| \frac{2m_1 a}{E^2 - m_1^2 + i\Gamma_1 m_1} + \frac{2m_2 b e^{i\psi_2}}{E^2 - m_2^2 + i\Gamma_2 m_2} + \frac{2m_3 c e^{i\psi_3}}{E^2 - m_3^2 + i\Gamma_3 m_3} \right|^2
$$

with “true” values of parameters

$$
m_1 = 782.6, \quad \Gamma_1 = 8.4, \quad a = 1,
\quad m_2 = 1019.4, \quad \Gamma_2 = 4.5, \quad b = 0.3, \quad \psi_2 = 155^\circ,
\quad m_3 = 1200, \quad \Gamma_3 = 100, \quad c = 3, \quad \psi_3 = 30^\circ,
\quad \mu = 350, \quad \text{and integrated luminosity equals } L = 10^6.
$$

Fig. [10] demonstrates “experimental” data and fit result.

Likelihood function plot vs the second resonance phase ψ₂ₓ is shown in Fig. [11].

One can see two minimum points on this plot (Table [VIII]).

Despite that we could not derive explicit analytical solutions for the case of three resonances it seems that there are at least two equivalent solutions. Let us check whether
Figure 10: Result of the fit to “data” points. Process cross section model is described by the formula (95). \( \chi^2/n_D = 126.0/(123 - 7) \).

Figure 11: Likelihood function plot on the phase of the second resonance phase \( \psi_{2x} \). There are some more solutions scanning the space of two parameters: \( \psi_{2x} \) and \( \psi_{3x} \). All local minima are presented in Table IX. There are four minimum points with the same values of mass, width and likelihood function value. It is quite a surprize that the second resonance phase value are different for all points. It means that we should see the four minima at the plot of likelihood function, but we have only two of them. In principle it can be. For
Table IX: Resonance parameters at the likelihood function local minimum points.

|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |   |   |
| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}$ | $c_x$ | $m_{3x}$ | $\Gamma_{3x}$ | $\psi_{3x}$ | $\chi^2$ |
| 1.0535 | 782.62 | 8.3768 | 0.30323 | 1019.4 | 4.5055 | 30.931 | 3.1673 | 1199.7 | 100.75 | -72.319 | 125.969 |
| 1.0508 | 782.62 | 8.3768 | 0.23505 | 1019.4 | 4.5055 | 144.655 | 3.0147 | 1199.7 | 100.75 | -53.143 | 125.969 |
| 0.99805 | 782.62 | 8.3768 | 0.30038 | 1019.4 | 4.5055 | 153.925 | 3.0128 | 1199.7 | 100.75 | 28.634 | 125.969 |
| 0.99545 | 782.62 | 8.3768 | 0.23284 | 1019.4 | 4.5055 | -21.661 | 2.8676 | 1199.7 | 100.75 | 47.811 | 125.969 |

Figure 12: Likelihood function plot on the phase of the second resonance phase $\psi_{2x}$

every new minimization run we take as a starting point the final point of the previous minimization. Thus the minimization could converge to “bad” local minimum. Let us try to get another plot of likelihood function, starting minimization at every point $\psi_2$ closer to the known “good” minima (Fig. 12).

Now there are all four minimum points on the plot. However the curve is not smooth, so probably not at every point of $\psi_{2x}$ the global minimum was achieved, although after convergence MINUIT executed command *IMPROVE*, which tries to seach better minimum.
Table X: Resonance parameters at the local minima of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $c_x$ | $m_{3x}$ | $\Gamma_{3x}$ | $\psi_{3x}^0$ | $\chi^2$ |
|-------|----------|----------------|-------|----------|----------------|--------------|-------|----------|----------------|--------------|--------|
| 1.0497 | 782.62   | 8.3787         | 0.30316 | 1019.4   | 4.5061        | 39.985       | 3.1635 | 1199.7   | 100.79        | -64.366      | 127.005 |
| 1.0470 | 782.62   | 8.3790         | 0.23583 | 1019.4   | 4.5058        | -148.442     | 3.0063 | 1199.7   | 100.71        | 28.687       | 126.529 |
| 0.99824 | 782.62  | 8.3792         | 0.30038 | 1019.4   | 4.5057        | 153.914      | 3.1635 | 1199.7   | 100.65        | 38.983       | 37.682 |
| 0.99559 | 782.62  | 8.3795         | 0.23365 | 1019.4   | 4.5054        | -34.675      | 2.8630 | 1199.7   | 100.71        | -45.063      | 126.218 |

Table XI: Resonance parameters at the minimum points of likelihood function.

| $a_x$ | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$ | $m_{2x}$ | $\Gamma_{2x}$ | $\psi_{2x}^0$ | $c_x$ | $m_{3x}$ | $\Gamma_{3x}$ | $\psi_{3x}^0$ | $\chi^2$ |
|-------|----------|----------------|-------|----------|----------------|--------------|-------|----------|----------------|--------------|--------|
| 11.685 | 783.30   | 99.681         | 10.503 | 1019.2   | 100.61        | -166.020     | 0.75199 | 1199.5   | 104.28        | 157.901      | 38.983 |
| 9.9572  | 782.86   | 100.31         | 10.013 | 1019.0   | 100.10        | 152.298      | 3.138  | 1195.7   | 102.44        | -25.172      | 37.682 |

Let us look which set of minima we can obtain if the resonance width depends on energy:

$$
\sigma(E) = \frac{m_4 \sqrt{E^2 - 4\mu^2}}{E^4 \sqrt{m_1^2 - 4\mu^2}} \left[ \frac{2m_1 a}{E^2 - m_1^2 + i\Gamma_1} \left( \frac{E^2 - 4\mu^2}{m_1^2 - 4\mu^2} \right)^{\frac{3}{4}} \right] + \frac{2m_2 b e^{i\psi_2}}{E^2 - m_2^2 + i\Gamma_2} \left( \frac{E^2 - 4\mu^2}{m_2^2 - 4\mu^2} \right)^{\frac{3}{4}} + \frac{2m_3 c e^{i\psi_3}}{E^2 - m_3^2 + i\Gamma_3} \left( \frac{E^2 - 4\mu^2}{m_3^2 - 4\mu^2} \right)^{\frac{3}{4}}
$$

$$
(97)
$$

Again we get the result that for energy dependent resonance width degeneration disappears (Table X).

Again for narrow resonances this difference is negligible from statistical point of view.

Let us change the following parameters of two resonances:

$$
\Gamma_1 = 100, \quad \Gamma_2 = 100, \quad a = 10, \quad b = 10.
$$

“Experimental” data and fit result are shown in Fig. 13.

On the likelihood function plot vs $\psi_{2x}$ (Fig. 14) one can see the minima, listed in Table XI.

Table XII lists the local minimum points found by scanning angles $\psi_{2x}$ and $\psi_{3x}$.

During this scan five local minima were found. The difference between lowest minimum and “highest” one is significant — $\Delta \chi^2 = 4.27$. However the difference between global
Figure 13: Result of fit the “experimental” points. Process cross section is described by the formula \( \chi^2/n_D = 37.7/(43 - 11) \).

Figure 14: Likelihood function plot on the second resonance phase \( \psi_{2x} \)

minimum and closest one is not so big — \( \Delta\chi^2 = 0.28 \). Statistically these two minima are almost equivalent. Nevertheless the global minimum has the resonance parameters closer to the “true” ones.
Table XII: Resonance parameters at the minimum points of likelihood function.

| $a_x$  | $m_{1x}$ | $\Gamma_{1x}$ | $b_x$  | $m_{2x}$ | $\Gamma_{2x}$ | $\psi^0_{2x}$ | $c_x$  | $m_{3x}$ | $\Gamma_{3x}$ | $\psi^0_{3x}$ | $\chi^2$ |
|--------|----------|----------|--------|----------|----------|------------|--------|----------|----------|------------|--------|
| 9.9572 | 782.86   | 100.31   | 10.013 | 1019.0   | 100.10   | 152.298    | 3.138  | 1195.7   | 102.45   | 25.172     | 37.682 |
| 9.769  | 782.84   | 100.34   | 8.9003 | 1019.0   | 100.08   | 138.747    | 0.64948| 1195.5   | 102.39   | 68.199     | 37.964 |
| 9.7644 | 782.48   | 99.576   | 8.9027 | 1019.2   | 99.970   | 141.698    | 0.62815| 1198.7   | 100.53   | 77.349     | 41.951 |
| 11.881 | 783.31   | 99.608   | 11.810 | 1019.2   | 100.63   | -153.260   | 3.5041 | 1199.8   | 104.31   | 108.19     | 39.608 |
| 11.685 | 783.30   | 99.681   | 10.503 | 1019.2   | 100.61   | -166.026   | 0.75197| 1199.5   | 104.28   | 157.901    | 38.983 |

VI. CONCLUSION

As a result of analytical solution of the problem of parameter definition of the two interfering resonances by experimental data there was demonstrated that for cross section parameterization with constant widths there are always two different solutions (for different sets of resonance parameters one gets the same cross section function of energy). If the dependence of resonance width on energy is taken into account, then the degeneration disappears, but quantitatively two solutions usually differ very little, and this difference is determined by many factors.

For illustration of analytical conclusions a series of numerical experiments was carried out. Above conclusions for the case of two resonances are confirmed, although the statistical difference of two solutions is not large even if energy dependence of resonance width is taken into account. In every particular case this problem should be investigated separately.

In the case of three resonances for constant widths there occurred already four equivalent solutions with the same likelihood function minimum. Analytical solution of this problem appeared too hard due to technical difficulties. For one numeric example the system of equations was solved, and four different solutions were derived (see appendix A). One can guess that the number of different solutions equals $2^{n-1}$, where $n$ is the number of resonances. In case of any number of resonances the degeneration disappears when the dependence of resonance width on energy is taken into account, however for narrow resonances the statistical difference between different solutions is usually not significant.

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Appendix A: NUMBER OF SOLUTIONS FOR \( n \) RESONANCES

Let us consider a case of \( n \) interfering resonances:

\[
\sigma(s) = \left| \sum_{k=1}^{n} \frac{A_k}{s - m_k^2 + i \Gamma_k m_k} \right|^2, \tag{A1}
\]

where \( m_k, \Gamma_k \) are mass and width of \( k \)-th resonance, \( m_k < m_{k+1} \), and \( A_k \) are some complex numbers.

This function is entirely defined by the location and residues of its irregular points, so some other function of the form

\[
\sigma_x(s) = \left| \sum_{k=1}^{n} \frac{A_{kx}}{s - m_k^2 + i \Gamma_k m_k} \right|^2 \tag{A2}
\]

can be equal to the first function over all region of \( s \) only if the system of equations is satisfied

\[
\sum_{k=1}^{n} \frac{A_{jx} \cdot A_{kx}}{m_j^2 + i \Gamma_j m_j - m_k^2 + i \Gamma_k m_k} = \sum_{k=1}^{n} \frac{A_j^* \cdot A_k}{m_j^2 + i \Gamma_j m_j - m_k^2 + i \Gamma_k m_k}, \quad j = 1, \ldots, n \tag{A3}
\]

If we have only two resonances then the system of equations looks like

\[
\begin{align*}
A_{1x}^* A_{1x} + \frac{2i \Gamma_1 m_1 A_{1x}^* A_{2x}}{G} & = A_1^* A_1 + \frac{2i \Gamma_1 m_1 A_1^* A_2}, \\
A_{2x}^* A_{2x} - \frac{2i \Gamma_2 m_2 A_{2x}^* A_{1x}}{G} & = A_2^* A_2 - \frac{2i \Gamma_2 m_2 A_2^* A_1},
\end{align*} \tag{A4}
\]

where \( G = m_1^2 + i \Gamma_1 m_1 - m_2^2 + i \Gamma_2 m_2 \).

Let \( A_{1x} = A_1 z_1, R_1^2 = z_1^* z_1, A_{2x} = A_2 z_2 z_1 \):

\[
\begin{align*}
A_1^* \cdot \left[ (R_1^2 z_2 - 1) A_2 \frac{2i \Gamma_1 m_1}{G} + (R_1^2 - 1) A_1 \right] & = 0, \\
A_2^* \cdot \left[ (R_1^2 |z_2|^2 - 1) A_2 - \frac{2i \Gamma_2 m_2}{G^*} \cdot (R_1^2 z_2^* - 1) A_1 \right] & = 0. \tag{A5}
\end{align*}
\]

Now we can derive \( z_2 \) value from the first equation and substitute to the second one:

\[
\begin{align*}
& \left( R_1^2 - 1 \right) \left( R_1^2 - \frac{1 + \frac{2i \Gamma_1 m_1 A_{1x}^* A_2}{A_1^* G} - \frac{2i m_1 \Gamma_1 A_2^*}{A_1 G} + \frac{4m_1^2 \Gamma_1^2 |A_2|^2}{|A_1|^2 G G^*}}{1 - \frac{4m_1 m_2 \Gamma_1 \Gamma_2}{G G^*}} \right) = 0. \tag{A6}
\end{align*}
\]

One can see that there are two solutions: the first one is trivial \( R_1^2 = 1, z_2 = 1, A_{1x} = A_1, A_{2x} = A_2 \), and another solution is

\[
R_1^2 = \frac{1 + \frac{2i m_1 \Gamma_1 A_2}{A_1 G} - \frac{2i m_1 \Gamma_1 A_2^*}{A_1 G^*} + \frac{4m_1^2 \Gamma_1^2 |A_2|^2}{|A_1|^2 G G^*}}{1 - \frac{4m_1 m_2 \Gamma_1 \Gamma_2}{G G^*}} \tag{A7}
\]
If we take the parameters of resonances from the first line of Table II
\[ m_1 = 782.60, \quad \Gamma_1 = 8.4116, \quad A_1 = 2m_1a = 1568.3304, \]
\[ m_2 = 1019.4, \quad \Gamma_2 = 4.5093, \quad A_2 = 2m_2be^{i\psi_2} = -555.7337 + 255.8088i, \quad (A8) \]
then the second solution should correspond to the second line in this Table: \[ R_1^2 = 1.010302, \]
\[ A_{1x} = A_1R_1 = 1576.3879, \quad z_2 = 0.6557 + 0.77886i, \quad A_{2x} = A_2R_1z_2 = -563.6463 - 265.09954i, \]
\[ a_x = |A_{1x}|/(2m_1) = 1.00715, \quad b_x = |A_{2x}|/(2m_2) = 0.3055, \quad \psi_{2x} = \arg(A_{2x}) = -154.81^\circ. \]
Analytical solution matches numerical one within the accuracy defined by rounding errors.

Now let us carry out similar procedure in case of three resonances with parameters (the first row in Table IX):
\[ m_1 = 782.62, \quad \Gamma_1 = 8.3768, \quad A_1 = 1.0535 \cdot 2m_1 = 1648.98, \]
\[ m_2 = 1019.4, \quad \Gamma_2 = 4.5055, \quad A_2 = 0.30323 \cdot 2m_2e^{30.931^\circ} = 530.3056 + 317.771i, \quad (A9) \]
\[ m_3 = 1199.7, \quad \Gamma_3 = 100.75, \quad A_3 = 3.1673 \cdot 2m_3e^{-172.319^\circ} = 2308.1347 - 7240.6307i \]
The system of equations:
\[
\begin{align*}
(21.4572 - 33.7750i) R_1^2 z_2 - (224.715 - 111.758i) R_1^2 z_3 &= 3271.14 - 145.533i - \\
&
\quad - 3474.402R_1^2, \\
374.929R_1^2 |z_2|^2 - (100.699 + 7.32447i) R_1^2 z_2^* z_3 + (11.5409 + 18.1606i) R_1^2 z_3^* = \\
&
\quad = 285.770 + 10.8416i, \\
48140.6R_1^2 |z_3|^2 - (2251.80 - 163.787i) R_1^2 z_3^* z_2 - (2702.72 - 1344.14i) R_1^2 z_2^* = \\
&
\quad = 43186.0 + 1507.93i, \\
\end{align*}
\]
where \[ R_1^2 = |z_1|^2. \] We can derive variable \( z_2 \) from the first equation:
\[
\begin{align*}
(0.653988 + 6.23783i) z_3 - 46.5605 - 73.2892i + \frac{46.9065 + 67.0514i}{R_1^2}
(14637.5 + 623.355i) R_1^2 |z_3|^2 - (1827002 - 90862.5i) R_1^2 z_3^* - \\
&
\quad - (177596 + 97961.8i) R_1^2 z_3 + (168317 - 93261.3i) z_3^* + \\
&
\quad + (163103 + 99669.8i) z_3 =
\end{align*}
\]
\[ = 5321121 - 67.4339i - 2824793R_1^2 - \frac{2510567}{R_1^2}, \]
\[ (45646.2 - 13939.2i) R_1^2 |z_3|^2 + (114146 + 158750i) R_1^2 z_3^* - (116606 + 143303i) z_3^* = \\
\quad = 43186.0 + 1507.93i \]
We got the system of two equations for \( z_3, z_3^* \), but both equations are quadratic. Let us introduce \( R_3^2 = |z_3|^2 \). From the last equation \( z_3^* \) can be derived as a linear expression of \( R_3^2 \),
then we substitute $z_3$ and $z_3^*$ to another equation and derive the only root of $R_3^2$:

\[
\begin{align*}
R_3^2 &= \frac{3705.61 R_0^5 - 13991.2 R_0^4 + 19810.0 R_0^3 - 12464.7 R_0^2 + 2940.37}{R_0^5 (R_0^2 - 0.892805)}, \\
z_3 &= \frac{-(290.536 + 856.587) R_0^9 + (871.330 + 2410.08) R_0^8 - (867.126 + 2257.00) R_0^7 + 286.440 + 703.515 i}{R_0^5 (R_0^2 - 0.892805)}.
\end{align*}
\] (A12)

Now we can use expression $R_3^2 = z_3 z_3^*$ as a final equation for $R_1^2$:

\[
(R_1^2 - 0.94482) (R_1^2 - 1) (R_1^2 - 0.89281) (R_1^2 - 0.89747) (R_1^2 - 0.99480) \times
\left[ (R_1^2 - 0.94320)^2 + 0.05633^2 \right] \left[ (R_1^2 - 0.94447)^2 + 0.02783^2 \right] = 0
\] (A13)

This is a polynomial of degree 20 in variable $R_1$, or that of degree 10 in variable $R_1^2$, so there are 20 formal solutions for $R_1$ or 10 different solutions for $R_1^2$. But there are only five real roots $R_1^2 > 0$:

| $R_1$     | $z_2$     | $z_3$     | $a_x = \frac{R_1}{2 m_1^2}$ | $b_x = \frac{1}{2 m_1^2}$ | $\psi_{2x} = \arctan \frac{\psi_{2x}}{m_{k}}$ | $\psi_{3x} = \arctan \frac{\psi_{3x}}{m_{k}}$ |
|----------|-----------|-----------|-----------------------------|-----------------------------|---------------------------------|---------------------------------|
| 1        | 1         | 1         | 1.0535                      | 0.30323                     | 30.931°                         | 3.1673°                         |
| 0.97205  | (2.6 + 5.2 i) \cdot 10^{11} | (8.8 - 3.2 i) \cdot 10^{10} | 1.0241                      | 1.7 \cdot 10^{11}           | 94.5°                           | 2.9 \cdot 10^{11}               |
| 0.9449   | 0.4937 - 0.6455 i | -0.4810 + 0.8288 i | 0.99543                     | 0.23284                     | -21.663°                        | 2.8677                         |
| 0.9473   | -0.5694 + 0.8771 i | -0.1907 + 0.9855 i | 0.99803                     | 0.30038                     | 153.92°                         | 3.0128                         |
| 0.9974   | -0.7749 - 0.0598 i | 0.9014 + 0.3135 i | 1.05076                     | 0.23505                     | -144.65°                        | 3.0147                         |

Four of these solutions match with the parameters of resonances in Table IX and one is very strange (second row). If we substitute the found solutions to the initial system of equations, then the four “legal” solutions satisfy the equations within rounding errors, and “illegal” second solution does not satisfy neither second equation nor the third one. Obviously this false solution corresponds to the case $z_2 = z_3 = 0$, which should be denied.

Let us consider the case of three resonances where one of them has infinite width:

\[
\sigma_x(s) = \left| A_0 + \sum_{k=1}^{2} \frac{A_{kx}}{s - m_k^2 + i \Gamma_k m_k} \right|^2
\] (A14)

The system of equations reads:

\[
\begin{align*}
|A_0|^2 &= |A_0|^2, \\
A_{1x}^* \cdot \left[ A_{0x} + \sum_{k=1}^{2} \frac{A_{kx}}{m_1^2 - m_k^2 + i \Gamma_k m_k + i \Gamma_1 m_1} \right] &= A_{1x} \cdot \left[ A_0 + \sum_{k=1}^{2} \frac{A_k}{m_1^2 - m_k^2 + i \Gamma_k m_k + i \Gamma_1 m_1} \right], \\
A_{2x}^* \cdot \left[ A_{0x} + \sum_{k=1}^{2} \frac{A_{kx}}{m_2^2 - m_k^2 + i \Gamma_k m_k + i \Gamma_2 m_2} \right] &= A_{2x} \cdot \left[ A_0 + \sum_{k=1}^{2} \frac{A_k}{m_2^2 - m_k^2 + i \Gamma_k m_k + i \Gamma_2 m_2} \right].
\end{align*}
\] (A15)
Here we can choose $A_{0x} = A_0$ and $A_{0^*} = A_{0^*}$, so we get the system of two equations for two complex variables $A_{1x}, A_{2x}$. Introduce new notations:

\[ A_{1x} = A_1 z_1, \quad A_{2x} = A_2 z_1 z_2, \quad \rho_1 = |z_1|^2. \] (A16)

It is very hard job to derive the solution for general case. So let us try to solve this problem for the following numerical example:

\[
A_0 = 50, \quad m_1 = 728, \quad \Gamma_1 = 9, \quad A_1 = \cos 15^\circ - i \sin 15^\circ, \\
m_2 = 1019, \quad \Gamma_2 = 4, \quad A_2 = 39 \cdot (\cos 155^\circ - i \sin 155^\circ). \] (A17)

$z_1, z_2$ can be re-written as functions of $\rho_1$, and for $\rho_1$ we get the algebraic equation of 12-th degree:

\[
(\rho_1 - 1) \cdot (-3.743751 \cdot 10^{67} + (\rho_1 - 1) \cdot (9.058522 \cdot 10^{70} + \\
+ (\rho_1 - 1) \cdot (1.511509 \cdot 10^{65} + (\rho_1 - 1) \cdot (-1.333590 \cdot 10^{59} + \\
+ (\rho_1 - 1) \cdot (-5.218886 \cdot 10^{53} + (\rho_1 - 1) \cdot (-8.183609 \cdot 10^{47} + \\
+ (\rho_1 - 1) \cdot (-2.813253 \cdot 10^{40} + (\rho_1 - 1) \cdot (1.405807 \cdot 10^{36} + \\
+ (\rho_1 - 1) \cdot (9.868724 \cdot 10^{29} + (\rho_1 - 1) \cdot (1.843597 \cdot 10^{23} + \\
+ (\rho_1 - 1) \cdot (-8.587479 \cdot 10^{11} + (\rho_1 - 1))))))))) = 0. \] (A18)

All roots (both real and complex) are located within circle $|\rho_1| = 10^{13}$. One root is trivial: $\rho_1 = 1, z_1 = z_2 = 1$. Using Sturm method \cite{2} for the remaining polynomial of 11-th order, there was found, that some roots are doubled, and the polynomial

\[
(\rho_1 - 1)^3 + 2.49 \cdot 10^6 (\rho_1 - 1)^2 - 4.29 \cdot 10^{11} (\rho_1 - 1) - 1.07 \cdot 10^{18} \] (A19)

was a common divisor for the original polynomial and its derivative. Repeating the Sturm procedure for the remaining polynomial of 8-th order, it is possible to check that there are 6 real roots of this equation. Preliminary localization defined exactly one root between every two of the following points:

\[-4.9 \cdot 10^6, -2.5 \cdot 10^6, 0, 4.1 \cdot 10^5, 8.2 \cdot 10^5, 4.295 \cdot 10^{11}, 4.297 \cdot 10^{11}.\] (A20)

Negative roots should be rejected, because $\rho_1$ can be only positive. Table XIII presents the values of positive roots and some additional information. The third row in this table has
Table XIII: Real and positive roots of the equation (A18).

| $\rho_1$ | $A_{1x}$ | $|A_{1x}|$ | $\psi_1$, deg. | $A_{2x}$ | $|A_{2x}|$ | $\psi_2$, deg. |
|----------|----------|------------|----------------|----------|------------|----------------|
| 1.000000 | 0.9659 + 0.2588i | 1.0000 | -15.00 | -35.346 − 16.482i | 39.000 | -155.00 |
| 1.000413 | 0.9702 − 0.2434i | 1.0002 | -14.08 | -35.350 − 4.08 \cdot 10^5i | 4.076 \cdot 10^5 | -90.005 |
| 655336.0 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 4.292879 \cdot 10^{11} | 1.3863 − 6.552 \cdot 10^5i | 6.552 \cdot 10^5 | -90.000 | -35.766 − 15.569i | 39.008 | -156.48 |
| 4.294653 \cdot 10^{11} | 10507 − 6.552 \cdot 10^5i | 6.553 \cdot 10^5 | -89.081 | -10541 − 4.075 \cdot 10^5i | 4.077 \cdot 10^5 | -91.482 |

Figure 15: Plot of the cross section (A14) with amplitude parameters from the first row in Table XIII.

inappropriate solution, because $z_1$ for it goes to infinity. $z_1$ is the ratio of two polynomials and for $\rho_1 = 655336$ polynomial in the denominator equals zero. The last two solution are quite unexpected because of high value of amplitudes. Original cross section is shown in Fig.15. Within the same interval of $s$ the ratio of cross section of alternative solution and the original cross section was evaluated and occurred to be equal to 1 with high accuracy. In order to avoid some digital surprises all these calculations were carried out with high accuracy of 150 decimal digits, using REDUCE system [5]. In order to illustrate the strange
Figure 16: Trajectories $S_{12}(s)$ for different solutions (marked by the number of row in Table XIII) for the parameter $s \in (0.36, 1.21)$ GeV$^2$. Left picture for the solution in the first row, right picture is for all the rest solutions. Step for the trajectory plot equals $\Delta \sqrt{s} = 1$ MeV.

two last solutions, in Fig[16] the trajectories of the complex function

$$S_{12}(s) = \frac{A_{1x}}{s - m_1^2 + i\Gamma_1 m_1} + \frac{A_{2x}}{s - m_2^2 + i\Gamma_2 m_2} \quad (A21)$$

are presented on the complex plane.

So we got two solutions for the case of two resonances and four solutions for the case of three resonances. It is not enough to choose the rule for the number $N_s$ of solutions for $n$ resonances: it can be $N_s = 2^{n-1}$, or $N_s = 2(n - 1)$, or something else.

In order to check whether it is possible to solve the system of equations in every case (at least numerically), let us solve a similar problem with four resonances, but choose the most simple input data making easier all calculations:

$$m_1 = 1, \quad \Gamma_1 = 1, \quad A_1 = 1,$$
$$m_2 = 2, \quad \Gamma_2 = 1, \quad A_2 = 1 + i,$$
$$m_3 = 3, \quad \Gamma_3 = 1, \quad A_3 = 1 - i,$$
$$m_4 = 4, \quad \Gamma_4 = 1, \quad A_4 = -1 + i. \quad (A22)$$
The system of equations looks like

\[
\begin{align*}
A_{x1}^* \cdot (A_{x1} + \frac{1-i}{3} A_{x2} + \frac{1-2i}{10} A_{x3} + \frac{1-3i}{25} A_{x4}) &= \frac{247-21i}{150}, \\
A_{x2}^* \cdot (\frac{2+2i}{3} A_{x1} + A_{x2} + \frac{2-2i}{5} A_{x3} + \frac{2+4i}{15} A_{x4}) &= \frac{46-8i}{15}, \\
A_{x3}^* \cdot (\frac{3+6i}{10} A_{x1} + \frac{3+3i}{5} A_{x2} + A_{x3} + \frac{3-3i}{7} A_{x4}) &= \frac{2071-25}{70}, \\
A_{x4}^* \cdot \left(\frac{4+12i}{25} A_{x1} + \frac{4+8i}{15} A_{x2} + \frac{4+4i}{7} A_{x3} + A_{x4}\right) &= \frac{1178-1216i}{525}.
\end{align*}
\] (A23)

If we describe the j-th equation in the form

\[
A_{xj}^* \cdot \sum_{k=1}^{4} G_{jk} A_{xk} = R_j,
\] (A24)

then the solution of every equation can be written as follows

\[
A_{xj} = -\frac{1}{2} S_j - \frac{i\Im(R_j)}{S_j^*} + S_j Q_j = S_j \cdot \left(Q_j - \frac{1}{2} \frac{i\Im(R_j)}{|S_j|^2}\right),
\] (A25)

where

\[
S_j = \sum_{k \neq j} G_{jk} A_{xk}, \quad Q_j^2 = \frac{1}{4} + \frac{\Re(R_j)}{|S_j|^2} - \frac{\Im(R_j)^2}{|S_j|^4}.
\] (A26)

These four solutions together can be considered as a system of linear equations:

\[
\sum_{k=1}^{n} B_{jk} A_{xk} = 0, \quad B_{jk} = \begin{cases} \frac{1}{C_j}, & k = j, \\ G_{jk}, & k \neq j, \end{cases} \quad j = 1, \ldots, n,
\] (A27)

where

\[
C_j = \frac{1}{2} + \frac{i\Im(R_j)}{|S_j|^2} \pm \sqrt{\frac{1}{4} + \frac{\Re(R_j)}{|S_j|^2} - \frac{\Im(R_j)^2}{|S_j|^4}}.
\] (A28)

This system can have non-zero solution only if the determinant is equal to zero. If it is, we can use the last three equations to express all amplitudes $A_{xj}$ through the amplitude $A_{x1}$:

\[
A_{x2} = \frac{3}{5} \left(\frac{876+4908i}{4032}C_2C_3C_4+(6615+2205i)C_3+(2744+392i)C_4-12250(1+i)\right) C_2 A_{x1},
\]
\[
A_{x3} = \frac{21}{10} \left(\frac{-(4616+7888i)C_2C_4+21000C_2+3600(i+2)C_4-7875(1+2i)}{4032}C_3 A_{x1},
\] (A29)
\[
A_{x4} = \frac{14}{25} \left(\frac{(8882-8874i)C_2C_3+(3i-1)(3500C_2-3375C_3)-3150(1+3i)}{4032}C_4 A_{x1}.
\]

If the variables $C_j$ were the predefined constants, then these expressions would be the set of infinite number of solutions with arbitrary $A_{x1}$. But here $C_j$ depend on $A_{xk}$ via the relation (A28). And even more, instead of constraint on the determinant of the system (A27) we can use the first equation of the system [A23], which can be presented in the form:

\[
R_1^2 = F(C_2, C_3, C_4) = \frac{2(247-21i)}{3} \times
\]
\[
\times \frac{4032C_2C_3C_4-5292C_2C_3-1960C_2C_4-5400C_3C_4+11025}{39576C_2C_3C_4-176400C_2C_3-70560C_2C_4-490000C_2-419040C_3C_4-165375C_3-70560C_4+110250},
\] (A30)
Table XIV: Results of the search for the solutions in case of four resonances

| $Q_2$  | $Q_3$  | $Q_4$  | $R_1$         | $A_{x2}$         | $A_{x3}$         | $A_{x4}$         | min $\Phi$ |
|--------|--------|--------|--------------|-----------------|-----------------|-----------------|------------|
| -2.000 | -0.1704 | -0.5899 | 1.0000       | 1.0000$ + 1.0000i$ | 0.9999$ - 1.0002i$ | -1.0000$ + 1.0000i$ | 7.5$ \cdot 10^{-9}$ |
| -2.8900 | -4$ \cdot 10^{-8}$ | 0.6880 | 1.0492       | 0.9378$ + 1.2978i$ | 1.9073$ - 0.3002i$ | -2.0751$ - 2.2721i$ | 0.009      |
| -7$ \cdot 10^{-6}$ | 0.3203 | -0.3965 | 1.0459       | 1.0731$ + 1.3992i$ | 0.5103$ - 2.5446i$ | -0.4493$ + 1.5012i$ | 4.2$ \cdot 10^{-4}$ |
| -2.9180 | 0.1103 | 0.6893 | 1.0536       | 0.9359$ + 1.3216i$ | 1.9127$ - 0.4163i$ | -2.0985$ - 2.2274i$ | 8.1$ \cdot 10^{-8}$ |
| 0.9510 | -0.0139 | -0.5047 | 2.2669       | -1.8270$ - 2.5487i$ | -1.7284$ + 0.0352i$ | 1.4877$ + 0.2031i$ | 6.5$ \cdot 10^{-4}$ |
| 0.8089 | -5$ \cdot 10^{-8}$ | 0.6846 | 2.4065       | -1.8727$ - 3.2613i$ | -1.4980$ - 1.0233i$ | -0.6968$ + 3.0950i$ | 0.018      |
| 0.7616 | 0.3213 | -0.4026 | 1.0472       | 1.0754$ + 1.4242i$ | 0.4838$ - 2.6127i$ | -0.3946$ + 1.5174i$ | 4.6$ \cdot 10^{-11}$ |
| 2.5546 | 0.4321 | 0.6691 | 1.1033       | 0.9392$ + 1.8158i$ | 2.5179$ - 2.6845i$ | -3.2574$ - 0.9483i$ | 1.3$ \cdot 10^{-7}$ |

where $R_1 = |A_{x1}|$.

The problem looks very much complicated. Let us try to solve it using numeric minimization procedure. The free parameters are complex variables $C_2$, $C_3$, $C_4$. Minimized function

$$\Phi = (\Im (F(C_2, C_3, C_4)))^2 + \sum_{j=2}^{4} \left| \frac{1}{2} \left( \Im(R_j) \right) + \frac{\Im(R_j)}{|S_j|^2} \pm \sqrt{\frac{1}{4} + \frac{\Re(R_j)}{|S_j|^2} - \frac{\Im(R_j)^2}{|S_j|^4} - C_j^2} \right|^2$$

(A31)

The true solution is found if the minimum value of $\Phi$ equals zero. Eight different combinations of signs of square roots provide possible eight solutions. Attempt to use the code MINUIT [3] failed because of very complicated function profile. Use of [4] brought more success. Table XIV presents the results of minimization.

Despite minimization problems this algorithm allows to localize the solutions, exactly $2^{n-1}$ of them, where $n$ is the number of resonances. To improve the amplitudes values and make sure that localization is good enough, one can minimize the function

$$\Psi = \sum_{j=1}^{4} \left| A_{xj}^* \sum_{k=1}^{4} G_{jk} A_{xk} - R_j \right|^2,$$

(A32)

starting minimization from the found points. There are 7 free parameters: $R_1 = \Re(A_{x1})$ and real and imaginary parts of $A_{x2}$, $A_{x3}$, $A_{x4}$ ($\Im(A_{x1}) = 0$). The result of this operation is shown in Table XV. One can see that for those cases, where all $Q_j$ had non-zero values, the
Table XV: Improved parameters of solutions. Minimization of $\Psi$ started from the approximation from the Table XIV.

| $A_{x1}$ | $A_{x2}$ | $A_{x3}$ | $A_{x4}$ | min $\Psi$ |
|----------|----------|----------|----------|------------|
| 1.0000 + 0i$^a$ | 1.0000 + 1.0000i | 1.0000 − 1.0000i | −1.0000 + 1.0000i | 1.2 · 10$^{-13}$ |
| 1.0536 + 0i$^b$ | 0.9361 + 1.3216i | 1.9126 − 0.4174i | −2.0986 − 2.2262i | 5.8 · 10$^{-8}$ |
| 1.0471 + 0i$^c$ | 1.0762 + 1.4232i | 0.4831 − 2.6116i | −0.3947 + 1.5171i | 6.2 · 10$^{-6}$ |
| 1.0536 + 0i$^d$ | 0.9360 + 1.3217i | 1.9126 − 0.4172i | −2.0985 − 2.2262i | 3.9 · 10$^{-11}$ |
| 2.2528 + 0i$^e$ | −1.8210 − 2.5396i | −1.7049 + 0.5456i | 1.4949 + 0.2602i | 1.9 · 10$^{-6}$ |
| 2.3744 + 0i$^f$ | −1.5650 − 3.2183i | −2.0232 − 1.2148i | −0.6648 + 3.2157i | 1.3 · 10$^{-5}$ |
| 1.0472 + 0i$^g$ | 1.0754 + 1.4242i | 0.4838 − 2.6127i | −0.3946 + 1.5174i | 9.6 · 10$^{-15}$ |
| 1.1033 + 0i$^h$ | 0.9393 + 1.8152i | 2.5170 − 2.6818i | −3.2560 − 0.9502i | 7.5 · 10$^{-10}$ |

Additionally found solutions

| $A_{x1}$ | $A_{x2}$ | $A_{x3}$ | $A_{x4}$ | min $\Psi$ |
|----------|----------|----------|----------|------------|
| 2.4858 + 0i$^i$ | −1.3958 − 4.2950i | −4.4264 + 0.2797i | 1.1745 + 3.4443i | 4.8 · 10$^{-10}$ |
| 2.3593 + 0i$^j$ | −1.8330 − 3.4916i | −2.5821 + 1.8974i | 1.5754 − 0.5896i | 4.9 · 10$^{-9}$ |

$^a$ Improved solution point matched the initial approximation from the Table XIV
$^b$ The minimum point moved away essentially from the approximation in Table XIV and matched the solution in the fourth row
$^c$ The minimum point moved away essentially from the approximation in Table XIV and matched the solution in the seventh row
$^d$ The minimum point moved away essentially from the approximation in Table XIV, $Q_2 = 0.811$, $Q_3 = 0.253$, $Q_4 = 0.676$
$^e$ Reconstructed values $Q_2 = 0.676$, $Q_3 = 0.461$, $Q_4 = 0.653$
$^f$ Reconstructed values $Q_2 = 0.743$, $Q_3 = 0.397$, $Q_4 = −0.210$

Improved points practically match the approximate values of $A_{xj}$. On contrary, for “bad” points (rows 2, 3 and 6), the improved values of $A_{xj}$ are rather far from approximation in Table XIV. Furthermore, the found solutions in rows 2 and 3 match exactly other solutions, so the approximations in Table XIV were not close to some new solutions. Starting randomly from different points, one can find additional two solutions, presented at the bottom of Table XV.

This exercise shows that the suggested algorithm cannot localize reliably all solutions of this problem. But it supports the rule $2^{n-1}$ for the number of solutions for $n$ resonances.
problem.

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