LOCAL AND NONLOCAL COMPLEX DISCRETE AND SEMI-DISCRETE SINE-GORDON EQUATIONS AND SOLUTIONS

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Abstract. In this paper, local and nonlocal complex reduction of a discrete and a semi-discrete negative order Ablowitz-Kaup-Newell-Segur equations is studied. Cauchy matrix type solutions, including soliton solutions and Jordan-block solutions, for the resulting local and nonlocal complex discrete and semi-discrete sine-Gordon equations are constructed. Dynamics of 1-soliton solution are analyzed and illustrated.

1. Introduction

The nonlocal integrable systems have been quite widely studied in recent years. The first example of nonlocal integrable systems is the reverse-space nonlinear Schrödinger equation

\[ iw_t(x,t) + w_{xx}(x,t) \pm w^2(x,t)w^*(−x,t) = 0, \]

(1.1)

proposed by Ablowitz and Musslimani in [1], where asterisk means the complex conjugate. This equation is parity-time symmetric because the potential \( V(x,t) = w(x,t)w^*(−x,t) \). Since then, the nonlocal integrable systems have attracted much interest in view of their wide range of applications in various fields of mathematics and physics. In mathematics, this type of equation possesses Lax integrability and admits infinite number of conservation laws [2]. Physically speaking, this type of equation can be usually applied to describing the two-place (Alice-Bob) physics or multi-place physics [3-4]. Besides the model (1.1), many other nonlocal integrable systems have been proposed, including nonlocal Korteweg-de Vries model, nonlocal modified Korteweg-de Vries model, nonlocal sine-Gordon (sG) model, nonlocal Davey-Stewartson model, (e.g. [5-6]). Up to now, many traditional methods have been used to search for exact solutions to the nonlocal integrable systems [7-18], such as the inverse scattering transformation, the Riemann-Hilbert approach, the Hirota’s bilinear method, the Darboux transformation, etc.

Although great progress has been got in the nonlocal systems within the field of the continuous or semi-discrete integrable systems. There was a little work on the nonlocal discrete integrable systems. In [19], Zhang et al. investigated two types of nonlocal discrete integrable equations, which were constructed as reductions of 2-component Adler-Bobenko-Suris systems [20]. In particular, they showed that the 2[0,1] Adler-Bobenko-Suris system allows reverse-\( n \) nonlocal reduction and the 2[1,1] Adler-Bobenko-Suris system admits reverse-(\( n, m \)) nonlocal reduction. Moreover, they used the nonlocal H1 equations as examples to demonstrate their idea. Up to now, there was no open literature on the solutions to the nonlocal discrete integrable systems to the best of the author’s knowledge.

Key words and phrases. local and nonlocal complex discrete and semi-discrete sine-Gordon equations, Cauchy matrix solutions, dynamics.
In this paper, a discrete Ablowitz-Kaup-Newell-Segur (AKNS) type equation that we want to investigate is of form

\[
q(\tilde{u} - u)[p^2 - (u + \tilde{u})(v + \tilde{v})]^{1/2} + (u + \tilde{u})[1 - q^2(\tilde{u} - u)(\tilde{v} - v)]^{1/2}
\]

\[
+ (\tilde{u} + \tilde{v})[1 - q^2(\tilde{u} - u)(\tilde{v} - v)]^{1/2} + q(\tilde{u} - u)[p^2 - (\tilde{u} + \tilde{v})(\tilde{v} + v)]^{1/2} = 0, \tag{1.2a}
\]

\[
q(\tilde{v} - v)[p^2 - (u + \tilde{u})(v + \tilde{v})]^{1/2} + (v + \tilde{v})[1 - q^2(\tilde{u} - u)(\tilde{v} - v)]^{1/2}
\]

\[
+ (\tilde{u} + \tilde{v})[1 - q^2(\tilde{u} - u)(\tilde{v} - v)]^{1/2} + q(\tilde{v} - v)[p^2 - (\tilde{u} + \tilde{v})(\tilde{v} + v)]^{1/2} = 0, \tag{1.2b}
\]

which was introduced in [21] by using the generalized Cauchy matrix approach [22]. This is one of the discrete versions of the first negative order AKNS equation [23]. In the rest part of the paper, we denote system (1.2) by dAKNS(-1) for short. The notation adopted in (1.2) is as follows: both of dependent variables \(u\) and \(v\) are functions defined on the two-dimensional lattice with discrete coordinates \((n, m) \in \mathbb{Z}^2\), e.g., \(u = u(n, m) =: u_{n,m}\) and the operations \(u \mapsto \tilde{u}\), \(u \mapsto \hat{u}\) denote elementary shifts in the two directions of the lattice, i.e., \(\tilde{u} = u_{n+1,m}\), \(\hat{u} = u_{n,m+1}\), while for the combined shift we have: \(\hat{\tilde{u}} = u_{n+1,m+1}\), \(p\) and \(q\) are continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables \(n\) and \(m\). Reinterpreting the variables \(u\) and \(v\) as \(u_{n,m} =: U_n(t)\) and \(v_{n,m} =: V_n(t)\) with \(t = -\frac{\mu}{q} \sim O(1)\), then under the continuum limit \(m \to \infty\), \(q \to \infty\), a semi-discrete version of equation (1.2) was also revealed, which reads

\[
(U' - \tilde{U}')(p^2 - (U + \tilde{U})(V + \tilde{V}))^{1/2} + (U + \tilde{U})((1 - U'V')^{1/2} + (1 - \tilde{U}'\tilde{V}')^{1/2}) = 0, \tag{1.3a}
\]

\[
(V' - \tilde{V}')(p^2 - (U + \tilde{U})(V + \tilde{V}))^{1/2} + (V + \tilde{V})((1 - U'V')^{1/2} + (1 - \tilde{U}'\tilde{V}')^{1/2}) = 0, \tag{1.3b}
\]

where the prime means the derivative of \(t\). We name the equation (1.3) as sdAKNS(-1). Following the reduction technique developed in recent papers [24], we would like to consider local and nonlocal complex reduction of the systems (1.2) and (1.3). We call the resulting local and nonlocal complex discrete \(G\) equation as cnd-s\(G\), as well as the resulting local and nonlocal complex semi-discrete \(G\) equation as cnsd-s\(G\). We will construct Cauchy matrix solutions, including soliton solutions and Jordan-block solutions, for these \(G\) type equations.

The paper is organized as follows. In Sec. 2, we briefly recall Cauchy matrix type solutions for the dAKNS(-1) equation (1.2). Local and nonlocal complex reduction for the dAKNS(-1) equation (1.2) is investigated. We construct some exact solutions for the cnd-s\(G\) equation. 1-soliton solution, 2-soliton solutions and the simplest Jordan-block solution are listed. The dynamic behaviors of 1-soliton solution are analyzed and illustrated. In Sec. 3, we consider local and nonlocal complex reduction for the sdAKNS(-1) equation (1.2) and discuss the exact solutions to the resulting local and nonlocal complex semi-discrete \(G\) equation. Section 4 is for the conclusions.

2. LOCAL AND NONLOCAL COMPLEX REDUCTION FOR THE DAKNS(-1) EQUATION (1.2)

In this section, we investigate the local and nonlocal complex reduction for the dAKNS(-1) equation (1.2). Moreover, we discuss two types of Cauchy matrix solutions for the resulting cnd-s\(G\) equation. For the sake of simplicity, in what follows we omit the index of each unit matrix \(I\) to indicate its size.

We start by recalling Cauchy matrix type solutions for the dAKNS(-1) equation (1.2), which were expressed as

\[
u = \epsilon_2(I - M_2M_1)^{-1}r_2, \quad v = \epsilon_1(I - M_1M_2)^{-1}r_1. \tag{2.1}
\]
where \(M_1 \in \mathbb{C}_{N_1 \times N_2}, M_2 \in \mathbb{C}_{N_2 \times N_1}, r_j \in \mathbb{C}_{N_j \times 1}, \) \(c_j \in \mathbb{C}_{1 \times N_j}\) with \(N_1 + N_2 = 2N\) satisfy the following determining equation set (DES)

\[
\begin{align*}
K_1M_1 - M_1K_2 &= r_1 \cdot c_2, \quad K_2M_2 - M_2K_1 = r_2 \cdot c_1, \\
(pI - K_1)\tilde{r}_1 &= (pI + K_1)r_1, \quad (pI + K_2)\tilde{r}_2 = (pI - K_2)r_2, \\
(qI - K_1^{-1})\tilde{r}_1 &= (qI + K_1^{-1})r_1, \quad (qI + K_2^{-1})\tilde{r}_2 = (qI - K_2^{-1})r_2.
\end{align*}
\]

In DES \([2.2]\), \(\{r_j\}\) and \(\{M_j\}\) are functions of \((n, m)\) while \(\{c_j\}\) and \(\{K_j\}\) are non-trivial constant matrices; \(K_1^{-1}\) and \(K_2^{-1}\) are the inversion of matrices \(K_1\) and \(K_2\), respectively. We assume that matrices \(\{pI + (-1)^jK_j\}\) and \(\{qI + (-1)^jK_j^{-1}\}\) are inverse.

Functions \(u\) and \(v\) are invariant and DES \((2.2)\) is covariant under similarity transformations

\[
K_j = \Gamma_j^{-1}K_j\Gamma_j, \quad r_j = \Gamma_j^{-1}r_j, \quad s_j = \Gamma_j^{T}s_j, \quad M_1 = \Gamma_1^{-1}M_1\Gamma_2, \quad M_2 = \Gamma_2^{-1}M_2\Gamma_1,
\]

where \(\{\Gamma_j\}\) are transform matrices. Thus solutions to the dAKNS(-1) equation \((1.2)\) can be still given by \((2.1)\), where the entities satisfy canonical DES

\[
\begin{align*}
\Omega_1M_1 - M_1\Omega_2 &= r_1 \cdot c_2, \quad \Omega_2M_2 - M_2\Omega_1 = r_2 \cdot c_1, \\
(pI - \Omega_1)\tilde{r}_1 &= (pI + \Omega_1)r_1, \quad (pI + \Omega_2)\tilde{r}_2 = (pI - \Omega_2)r_2, \\
(qI - \Omega_1^{-1})\tilde{r}_1 &= (qI + \Omega_1^{-1})r_1, \quad (qI + \Omega_2^{-1})\tilde{r}_2 = (qI - \Omega_2^{-1})r_2,
\end{align*}
\]

where \(\Omega_1\) and \(\Omega_2\) are the canonical forms of the matrices \(K_1\) and \(K_2\), respectively.

Equations \((2.4)\) are linear, where equations \((2.4a)\) and \((2.4c)\) are used to determine plane-wave factor vector \(r\) and the equations \((2.4b)\) are used to give matrices \(M_1\) and \(M_2\). The two equations in \((2.4a)\) are nothing but the famous Sylvester equations, which have a unique solution \(\{M_1, M_2\}\) if and only if \(E(\Omega_1) \cap E(\Omega_2) = \emptyset\), where \(E(\Omega_1)\) and \(E(\Omega_2)\) represent the eigenvalue sets of \(\Omega_1\) and \(\Omega_2\), respectively. From \((2.4b)\) and \((2.4c)\), we know

\[
r_j = (pI + (-1)^j\Omega_j)^n(pI + (-1)^j\Omega_j)^{-n}(qI + (-1)^jI)^m(qI + (-1)^jI)^{-m}C_j,
\]

where constant column vectors \(\{C_j\}\) are phase terms of \(\{r_j\}\). The key point of the solving procedure of \((2.4a)\) is to factorize \(M_1\) and \(M_2\) into triplets, i.e. \(M_1 = F_1G_1H_1\) and \(M_2 = F_2G_2H_1\), where \(\{F_j, H_j\} \subset \mathbb{C}_{N_j \times N_j}, G_1 \in \mathbb{C}_{N_1 \times N_2}\) and \(G_2 \in \mathbb{C}_{N_2 \times N_1}\). When \(\{\Omega_j\}\) being diagonal matrices, one can get the soliton solutions. When \(\{\Omega_j\}\) being Jordan-block matrices, multiple-pole solutions can be derived. For the detailed calculations, one can refer to \([21]\).

The integrable symmetry reduction for dAKNS(-1) \((1.2)\) is of \((\sigma n, \sigma m)\) type:

\[
v = \delta u^\sigma, \quad \delta, \sigma = \pm 1,
\]

where for the function \(f = f(x_1, x_2)\) we have used notation \(f_\sigma = f(\sigma x_1, \sigma x_2)\), which makes equations \((1.2a)\) and \((1.2b)\) self-consistent leading to a single integrable equation

\[
q(\widehat{u} - u)[p^2 - \delta(u + \widehat{u})(u^\sigma + \widehat{u}^\sigma)]^{1/2} + (u + \widehat{u})[1 - \delta q^2(\widehat{u} - u)(\widehat{u}^\sigma - u^\sigma)]^{1/2} \\
+ (\widehat{u} + u)[1 - \delta q^2(\widehat{u} - u)(\widehat{u}^\sigma - u^\sigma)]^{1/2} + q(\widehat{u} - u)[p^2 - \delta(u + \widehat{u})(u^\sigma + \widehat{u}^\sigma)]^{1/2} = 0.
\]

When \(\sigma = 1\), equation \((2.7)\) is nothing but exactly the complex discrete sG equation. When \(\sigma = -1\), equation \((2.7)\) is referred to as a complex reverse-\((n, m)\) discrete sG equation. We observe that equation \((2.7)\) is preserved under transformation \(u \to -u\). Besides, equation \((2.7)\) with \((\sigma, \delta) = (\pm 1, 1)\) and with \((\sigma, \delta) = (\pm 1, -1)\) can be transformed into each other by taking \(u \to iu\).

To derive solutions of the cnd-sG equation \((2.7)\), we take \(N_1 = N_2 = N\). For its solution, we have the following result.
Theorem 1. The function
\[ u = t^c_2(I - M_2M_1)^{-1}r_2 \]  \hspace{1cm} (2.8)
solves the cnd-sG equation (2.7), provided that the entities satisfy canonical DES (2.4) and simultaneously obey the constraints
\[ r_1 = \varepsilon T r_{2,\sigma}, \quad t^c_1 = \varepsilon t^c_2 T^{-1}, \quad M_1 = -\delta \sigma T M^*_{2,\sigma} T^*, \]  \hspace{1cm} (2.9)
in which \( T \in \mathbb{C}^{N \times N} \) is a constant matrix satisfying
\[ \Omega_1 T + \sigma T \Omega_2^* = 0, \quad C_1 = \varepsilon T C_2^*, \quad \varepsilon^2 = \varepsilon^* = \delta. \]  \hspace{1cm} (2.10)

Proof. According to the assumption (2.10), we have
\[ r_1 = (pI + \Omega_1)^n(pI - \Omega_1)^{-n}(q\Omega_1 + I)^m(q\Omega_1 - I)^{-m}C_1 \]
\[ = T(pI - \sigma \Omega^*_2)^n(pI + \sigma \Omega^*_2)^{-n}(q\sigma \Omega^*_2 - I)^m(q\sigma \Omega^*_2 + I)^{-m}T^{-1}C_1 \]
\[ = T(pI - \Omega^*_2)\sigma^n(pI + \Omega^*_2)^{-\sigma^n}(q\Omega^*_2 - I)^\sigma_m(q\Omega^*_2 + I)^{-\sigma_m}T^{-1}C_1 \]
\[ = \varepsilon T r_{2,\sigma}, \]  \hspace{1cm} (2.11)
where we have used the identity \((aI + \sigma L)(aI - \sigma L)^{-1} = (aI + L)^\sigma(aI - L)^{-\sigma}\) with \(a \in \mathbb{C}\). Substituting \( \Omega_1 = -\sigma T \Omega_2^* T^{-1} \) into the first equation in (2.4a) and through a straightforward calculation, we find
\[ \Omega_2^* (\sigma T^{-1} M_1 T^{*-1} + \delta M_{2,\sigma}^*) - (\sigma T^{-1} M_1 T^{*-1} + \delta M_{2,\sigma}^*) \Omega_1^* = 0, \]  \hspace{1cm} (2.12)
which yields the third relation in (2.9). With (2.9) and (2.10) at hand, we immediately have
\[ u = t^c_1(I - M_1 M_2)^{-1} r_1 = \varepsilon^2 t^c_2 (I - M_{2,\sigma}^* M^*_{1,\sigma})^{-1} r_{2,\sigma} = \delta u^*_\sigma, \]
which coincides with the reduction (2.6) for the cnd-sG equation (2.7).

In terms of the Theorem 1, we know that solution to the cnd-sG equation (2.7) reads
\[ u = t^c_2(I + \delta \sigma M_2 T M^*_{2,\sigma} T^*)^{-1} r_2, \]  \hspace{1cm} (2.13)
where \( r_2 \) is given by (2.5) and \( M_2 \) and \( T \) are determined by
\[ \Omega_2 M_2 T + \sigma M_2 T \Omega_2^* = \varepsilon r_2 \ t^c_2. \]  \hspace{1cm} (2.14)
We denote \( M_2 T \rightarrow \tilde{M}_2 \) and simplify solution (2.13) together with Sylvester equation (2.14) as
\[ u = t^c_2(I + \delta \sigma \tilde{M}_2 \tilde{M}^*_{2,\sigma})^{-1} r_2, \]  \hspace{1cm} (2.15a)
\[ \Omega_2 \tilde{M}_2 + \sigma \tilde{M}_2 \Omega_2^* = \varepsilon r_2 \ t^c_2. \]  \hspace{1cm} (2.15b)

As three examples, we just list 1-soliton solution, 2-soliton solutions and the simplest Jordan-block solution. For the sake of brevity, we introduce two notations
\[ \alpha_{ij} = k_i + \sigma k^*_j, \quad \rho_j = \rho_j(n, m) = \left( \frac{p - k_j}{p + k_j} \right)^n \left( \frac{q k_j - 1}{q k_j + 1} \right)^m \rho^0_j, \]  \hspace{1cm} (2.16)
as well as \( \alpha = \alpha_{11} |_{k_1 \rightarrow k} \) and \( \rho = \rho_1 |_{k_1 \rightarrow k, \rho^0_j \rightarrow \rho} \), where \( \{ k_j, \rho^0_j, k, \rho^0 \} \) are complex constants. \( \rho_j \) and \( \rho \) play the role of discrete plane-wave factors.

When \( N = 1 \), we denote
\[ \Omega_2 = k_1, \quad t^c_2 = c_1, \]  \hspace{1cm} (2.17)
and write down 1-soliton solution

\[ u = \frac{c_1 \alpha_1^2 \rho_1}{\alpha_{11}^2 + \delta |c_1|^2 \rho_1 \rho_{1,\sigma}} \] (2.18)

with module | · |. When \( N = 2 \), we take

\[ \Omega_2 = \text{diag}(k_1, k_2), \quad \mathbf{c}_2 = (c_1, c_2). \] (2.19)

In this case, the 2-soliton solutions are expressed as \( u = \frac{g_2}{g_1} \), where

\[ g_1 = 1 + \delta \sum_{i=1}^{2} \sum_{j=1}^{2} \left( \frac{c_i c_j^* \rho_i \rho_j^*}{\alpha_{ij}} \right) + \frac{|c_1 c_2|^2 |k_1 - k_2|^4 \rho_1 \rho_2 \rho_{1,\sigma}^* \rho_{2,\sigma}^*}{\alpha_{11}^2 \alpha_{12}^2 \alpha_{21}^2 \alpha_{22}^2}, \] (2.20a)

\[ g_2 = c_1 \rho_1 + c_2 \rho_2 + \delta c_1 c_2 (k_1 - k_2)^2 \rho_1 \rho_2 \left( \frac{c_1^* \rho_{1,\sigma}^*}{\alpha_{11}^2 \alpha_{21}^2} + \frac{c_2^* \rho_{2,\sigma}^*}{\alpha_{12}^2 \alpha_{22}^2} \right). \] (2.20b)

For presenting the simplest Jordan-block solution, we set \( N = 2 \) together with

\[ \Omega_2 = \begin{pmatrix} k & 0 \\ 1 & k \end{pmatrix}, \quad \mathbf{c}_2 = (c_1, c_2). \] (2.21)

Then we have

\[ \mathbf{r}_2 = (\rho, \check{\rho})^T, \quad \check{M}_2 = FGH, \] (2.22a)

in which

\[ F = \begin{pmatrix} \rho & 0 \\ \check{\rho} & \rho \end{pmatrix}, \quad G = \begin{pmatrix} \alpha^{-1} & \alpha^{-2} \\ -\alpha^{-2} & -2\alpha^{-3} \end{pmatrix}, \quad H = \begin{pmatrix} c_1 & c_2 \\ c_2 & 0 \end{pmatrix}, \] (2.22b)

where \( \check{\rho} =: \partial_k \rho \). Substituting (2.21) and (2.22) into (2.15a), we get the simplest Jordan-block solution \( u = \frac{h_2}{h_1} \) with

\[ h_1 = \alpha^8 + |c_2|^4 \rho_2 \rho_{2,\sigma}^*|^2 + \sigma \delta \left[ c_2^2^\alpha^5 \check{\rho}(c_2 \sigma \alpha \check{\rho}_{\sigma} + (c_1 \sigma \alpha + c_2 \beta) \rho_{\sigma}^* \right. \]
\[ \left. + \alpha^4 \rho_2 \rho_{\sigma}(c_1 \alpha \zeta + c_2 (c_1 \beta - 2c_2^2 \sigma + \beta)) + c_2 \alpha \check{\rho}_{\sigma}^* \right], \] (2.23a)

\[ h_2 = \alpha^3 \left[ \alpha^5 (c_1 \rho_2 + c_2 \check{\rho}) + \sigma \delta \rho_2 (c_2 (|c_2|^2 - \sigma \alpha \gamma) \check{\rho}_{\sigma}^* \right. \]
\[ \left. - (\alpha (c_1 \sigma \alpha \gamma + c_2 \beta (c_1 \rho_2^* + \gamma)) + c_2^2 (c_2^2 \beta - c_1^* \sigma \alpha) \rho_2^* \right], \] (2.23b)

where \( \beta = \sigma - 1, \gamma = c_1 c_2^* - c_1^* c_2, \zeta = c_1^* \sigma \alpha - c_2^2 \beta \).

To understand the dynamic behavior of soliton solution (2.18), we set

\[ k_1 = \mu + i \nu, \text{ with } \mu, \nu \in \mathbb{R}. \] (2.24)

For \( \sigma = 1 \), i.e., local case, the carrier wave is expressed as

\[ |u|^2 = \begin{cases} \mu^2 \text{sech}^2 \left( \frac{1}{2} \ln |A| + \ln |c_1 \rho_1^\delta| \right), & \text{with } \delta = 1, \\ \mu^2 \text{csch}^2 \left( \frac{1}{2} \ln |A| + \ln |c_1 \rho_1^\delta| \right), & \text{with } \delta = -1, \end{cases} \] (2.25)

where \( A = \left( \frac{(p - \mu)^2 + \nu^2}{(p + \mu)^2 + \nu^2} \right)^n \left( \frac{q \mu - 1)^2 + q^2 \nu^2}{(q \mu + 1)^2 + q^2 \nu^2} \right)^m \). As \( \delta = 1 \), the solution (2.25) is nonsingular and provides a bell-type traveling wave which propagates with initial phase \( \ln |c_1 \rho_1^\delta| \). The
amplitude is approximately equal to $\mu^2$. As $\delta = -1$, the solution (2.25) has large value in the neighborhood of straight line

$$\frac{y}{2} \ln \frac{(p-\mu)^2 + \nu^2}{(p+\mu)^2 + \nu^2} + \frac{z}{2} \ln \frac{(q\mu - 1)^2 + (q\nu)^2}{(q\mu + 1)^2 + (q\nu)^2} + \ln \left| \frac{c_1 \rho_1^0}{2\mu} \right| = 0, \quad y, z \in \mathbb{R}. \quad (2.26)$$

We illustrate these two solitons in Figure 1.

![Figure 1](image1)

**Fig. 1** shape and motion with $|u|^2$ given by (2.25) for $k_1 = 1 + 3i, p = 2, q = 1.5, \rho_1^2 = 1$ and $c_1 = 1 + i$.

(a) 3D-plot for $\delta = 1$. (b) waves in blue and yellow stand for plot (a) at $m = -1$ and $m = 3$, respectively.

(c) waves in blue and yellow stand for plot (a) at $n = -1$ and $n = 3$, respectively. (d) 3D-plot for $\delta = -1$.

For $\sigma = -1$, i.e., nonlocal case, the wave package reads

$$|u|^2 = \frac{4\nu^2 A}{B + B^{-1} - 2\delta CA^{-1} \cos(2n \arctan \theta_1 + 2m \arctan \theta_2)}, \quad (2.27)$$

where $A$ is defined by (2.25) and

$$B = 4\nu^2 |c_1 \rho_1^0|^2, \quad \theta_1 = \frac{2p\nu}{\mu^2 + \nu^2 - p^2}, \quad \theta_2 = \frac{2q\nu}{q^2(\mu^2 + \nu^2) - 1}, \quad (2.28a)$$

$$C = \frac{(p^2 - \mu^2 - \nu^2)^2 + 4p^2 \nu^2}{(p + \mu)^2 + \nu^2} (q^2(\mu^2 + \nu^2) - 1)^2 + 4q^2 \nu^2)^m. \quad (2.28b)$$

The solution (2.27) has oscillatory phenomenon since the involvement of cosine function in denominator. We illustrate soliton (2.27) in Figure 2.
3. LOCAL AND NONLOCAL COMPLEX REDUCTION FOR THE SDAKNS(-1) EQUATION

In this section, we shall use a similar strategy to consider the local and nonlocal complex reduction for the sdAKNS(-1) equation (1.3).

We notice that
\[ U = \text{tc}_2(I - M_2M_1)^{-1}r_2, \quad V = \text{tc}_1(I - M_1M_2)^{-1}r_1, \]  
(3.1)
solve the sdAKNS(-1) equation (1.3) [21], where the components \{M_1, M_2, r_j, \text{tc}_j\} satisfy the equations (2.4a), (2.4b) together with
\[ r_1' = -2\Omega_1^{-1}r_1, \quad r_2' = 2\Omega_2^{-1}r_2. \]  
(3.2)

Under the complex integrable symmetry reduction
\[ V = \delta U_\sigma^*, \quad \delta, \sigma = \pm 1, \]  
(3.3)
the coupled equations (1.3a) and (1.3b) are compatible and give rise to the cnsd-sG equation
\[ (U' - \tilde{U}')(p^2 - \delta(U + \tilde{U})(U_\sigma^* + \tilde{U}_\sigma^*))^{\frac{1}{2}} + (U + \tilde{U})((1 - \delta U'(U_\sigma^*)')^{\frac{1}{2}} + (1 - \delta \tilde{U}'(\tilde{U}_\sigma^*)')^{\frac{1}{2}}) = 0, \]  
(3.4)
which is the complex semi-discrete sG equation as \(\sigma = 1\), respectively, the complex reverse-(n,t) semi-discrete sG equation as \(\sigma = -1\). Similar to the equation (2.7), the cnsd-sG equation (3.4) is preserved under transformation \(u \rightarrow -u\). And equation (3.4) with \((\sigma, \delta) = (\pm 1, 1)\) and with \((\sigma, \delta) = (\pm 1, -1)\) can be transformed into each other by taking \(u \rightarrow iu\).

Under the assumption \(N_1 = N_2 = N\), solution to the equation (3.4) can be summarized by the following theorem, where we skip the proof because it is very similar to the cnd-sG case.
Theorem 2. The function
\[ U = \mathcal{c}_2(I - M_2M_1)^{-1}r_2, \]  
(3.5)
solves the cnsd-sG equation \((3.4)\), provided that the entities satisfy DES \((2.4a)\), \((2.4b)\), \((3.2)\) and simultaneously obey the constraints \((2.9)\) and \((2.10)\).

We find that solution for the cnsd-sG equation \((3.4)\) is expressed by
\[ U = \mathcal{c}_2(I + \delta \sigma M_2M_2^\sigma,\sigma)^{-1}r_2, \]  
(3.6)
in which the entities still satisfy the Sylvester equation \((2.15b)\) but with
\[ r_2 = (pI - \Omega_2)^n(pI + \Omega_2)^{-n}\exp(2\Omega_2^{-1}t)D_2, \]  
(3.7)
where \(D_2\) is a \(N\)-th order constant column vector.

With \((2.17)\) we identify the 1-soliton solution to equation \((3.4)\)
\[ U = \mathcal{c}_1\alpha_{11}^2\sigma_{1,\sigma}, \]  
(3.8)
where the semi-discrete plane-wave factor \(\sigma_{1,\sigma}\) is given by
\[ \sigma_{1,\sigma} = \left(\frac{p - k_1}{p + k_1}\right)^n\exp\left(\frac{2n}{\mu_1}\right). \]  
(3.9)
As \(\delta = 1\), the solution \((3.9)\) is nonsingular and the wave propagates with initial phase \(\ln \frac{|c_1\sigma_{1,\sigma}|}{2n\mu_1}\), amplitude \(\mu^2\), top trajectory
\[ \frac{n}{2} \ln \frac{(p - \mu)^2 + \nu^2}{(p + \mu)^2 + \nu^2} + \frac{2\mu t}{\mu^2 + \nu^2} + \ln \frac{|c_1\sigma_{1,\sigma}|}{2n\mu} = 0, \]  
(3.10)
and velocity \(n'(t) = \frac{4\mu}{\mu^2 + \nu^2} \ln^{-1} \frac{(p + \mu)^2 + \nu^2}{(p - \mu)^2 + \nu^2}\). As \(\delta = -1\), the solution \((3.9)\) has singularities along with point trace \((3.10)\). We illustrate these two solitons in Figure 3.
Fig. 3 shape and motion with $|U|^2$ given by (3.9) for $k_1 = 1 + 3i$, $p = 2$, $\rho_0^2 = 1$ and $c_1 = 1 + i$. (a) 3D-plot for $\delta = 1$. (b) waves in blue and yellow stand for plot (a) at $t = -1$ and $t = 3$, respectively. (c) waves in blue and yellow stand for plot (a) at $n = -1$ and $n = 3$, respectively. (d) 3D-plot for $\delta = -1$.

For $\sigma = -1$, the wave package reads

$$|U|^2 = \frac{4\nu^2D}{B + B^{-1} - 2\delta ED^{-1}\cos(2n\arctan\theta_1 - 4\nu t)},$$

where $B$ is the same as (2.28a) and $E = \frac{(p - \mu^2 - \nu^2)^2 + 4p^2\nu^2}{((p + \mu)^2 + \nu^2)^2n}\exp\left(\frac{4\mu t}{\mu^2 + \nu^2}\right)$. We illustrate this wave in Figure 4.

Fig. 4 shape and motion with $|U|^2$ given by (3.11) for $k_1 = -0.1 + 5i$, $p = 2$, $\rho_0^2 = 1$ and $c_1 = 1 + i$. (a) 3D-plot for $\delta = 1$. (b) wave of plot (a) at $t = 1$. (c) wave of plot (a) at $n = -20$. (d) 3D-plot for $\delta = -1$.

4. Conclusions

In this paper, we have investigated local and nonlocal complex reduction for the dAKNS(-1) equation (1.2). A local complex discrete sG equation (equation (2.7) with $\sigma = 1$) and a nonlocal complex discrete sG equation (equation (2.7) with $\sigma = -1$) have been revealed. Based on the canonical DES (2.4), by imposing suitable constraints (2.9) on the elements $(r_1, c_1, M_1)$ and $(r_2, c_2, M_2)$ in the Cauchy matrix solution of the dAKNS(-1) equation (1.2), we have
obtained formal solution to the cnd-sG equation (2.7). 1-soliton solution, 2-soliton solutions and the simplest Jordan-block solution were given as three examples of the exact solutions. Dynamics for 1-soliton solution were analyzed with graphical illustration. For the local complex discrete sG equation, its 1-soliton solution exhibited the usual bell-type structure. For the nonlocal complex discrete sG equation, its 1-soliton solution showed quasi-periodic phenomenon. Besides the dAKNS(-1) equation (1.2), we also discussed local and nonlocal complex reduction for the sdAKNS(-1) equation (1.3). Soliton solutions and Jordan-block for the resulting local and nonlocal complex semi-discrete sine-Gordon equation are constructed. Some discrete models of positive order AKNS-type equations, admitting Cauchy matrix solutions, have been proposed [27,28]. How to consider their local and nonlocal complex reduction and derive the solutions of the resulting local and nonlocal complex equations are interesting questions worth consideration.

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