ON THE EFFICIENCY OF FERMI ACCELERATION AT RELATIVISTIC SHOCKS

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ABSTRACT

We show that Fermi acceleration at an ultrarelativistic shock wave cannot operate on a particle for more than $1 \frac{1}{3}$ Fermi cycles (i.e., $u \rightarrow d \rightarrow u \rightarrow d$) if the particle’s Larmor radius is much smaller than the coherence length of the magnetic field on both sides of the shock, as is usually assumed. This conclusion proves to be in excellent agreement with recent numerical simulations. We thus argue that efficient Fermi acceleration at ultrarelativistic shock waves requires significant nonlinear processing of the far-upstream magnetic field with strong amplification of the small-scale magnetic power. The streaming or transverse Weibel instabilities are likely to play a key role in this respect.

Subject headings: acceleration of particles — cosmic rays — shock waves

Online material: color figures

1. INTRODUCTION

Fermi acceleration of charged particles bouncing back and forth in a collisionless shock wave is at the heart of a variety of phenomena in high-energy astrophysics. According to standard lore, this includes the acceleration of electrons at gamma-ray bursts’ internal and external relativistic shock waves, whose synchrotron light is interpreted as the prompt and afterglow radiation, respectively.

However, the inner workings of Fermi acceleration at relativistic shock waves remain the subject of intense study and debate, even in the test-particle limit. It has been argued that a universal energy spectral index $s \approx 2.2–2.3$ should be expected (e.g., Bednarz & Ostrowski 1998; Achterberg et al. 2001; Lemoine & Pelletier 2003; Ellison & Double 2004; Keshet & Waxman 2005), which would agree nicely with the index inferred from gamma-ray burst observations. Yet, recent numerical simulations that include more realistic shock-jump conditions (see, e.g., Kirk & Duffy 1999), in the test-particle limit. It has been argued that a universal energy spectral index $s \approx 2.2–2.3$ should be expected (e.g., Bednarz & Ostrowski 1998; Achterberg et al. 2001; Lemoine & Pelletier 2003; Ellison & Double 2004; Keshet & Waxman 2005), which would agree nicely with the index inferred from gamma-ray burst observations. Yet, recent numerical simulations that include more realistic shock-jump conditions (see, e.g., Kirk & Duffy 1999), so that the magnetic field can be considered as passive. In this limit, one finds $B_{\perp} = B_{\parallel} u_{B}$ and $B_{\parallel} = R_{B} B_{x}$. $B_{\perp}$ is the component of the magnetic field along the shock normal $z$, $B_{\parallel}$ is the projection of $B$ on the shock-front plane ($x$, $y$), and subscripts “$\perp$” and “$\parallel$” indicate that the quantity is measured in the downstream or upstream plasma rest frame, respectively. The quantity $R_{B} = \Gamma_{sh}^{-1} \beta_{sh}^{-1} / (\Gamma_{sh} \beta_{sh})$ is the proper shock compression ratio, expressed in terms of $\beta_{sh}$ and $\beta_{sh,\perp}$, the shock velocities measured in the respective upstream and downstream rest frames and the corresponding Lorentz factors $\Gamma_{sh,\parallel}$ and $\Gamma_{sh,\perp}$. In the ultrarelativistic limit ($\Gamma_{sh,\parallel} \gg 1$), $R_{B} \approx \Gamma_{sh,\perp} \Gamma_{sh,\parallel} \beta_{sh,\perp} \beta_{sh,\parallel}$

Hence, to an error $\sim 0(1/\Gamma_{sh,\parallel})$ in the direction of $B$, it is a good approximation to consider the magnetic field to lie in the transverse ($x, y$)-plane downstream of the shock. The magnetic field at a given point $r_{s}$ on the shock surface can be written in both downstream and upstream reference frames as follows:

$$B_{\parallel}(r_{s}) \approx B_{\parallel} \cos(\phi_{B}) x + B_{\perp} \sin(\phi_{B}) y,$$

$$B_{\perp}(r_{s}) \approx B_{\parallel} \cos(\phi_{B}) x + B_{\perp} \sin(\phi_{B}) y + B_{\perp} \parallel z.$$ (1)

The phase $\phi_{B}$ is invariant under the Lorentz transformation from downstream to upstream; this observation plays a key role in the discussion that follows.
We assume for the time being that the Larmor radius $r_\text{L}$ of the test particle is much smaller than the coherence length of the magnetic field $l_\text{coh}$; if $\alpha > 3$, then $l_\text{coh} \approx 1/k_\text{min}$, as the magnetic power is distributed on the largest spatial scales. Then, since the typical $u \rightarrow d \rightarrow u$ cycle time through the shock is of order $O(r_\text{L}/T_{\text{sh},u}) \ll l_\text{coh}$ (Achterberg et al. 2001; Lemoine & Pelletier 2003; Lemoine & Revenu 2006), to a first approximation, one can neglect the magnetic field line curvature over the trajectory of the particle.

This approximation may be justified as follows: Consider a particle moving over a length scale $l \sim r_\text{L}/T_{\text{sh},u} \ll l_\text{coh}$. The radius of curvature $R_{\text{sh}}$ of the magnetic field on scales larger than $l$ can be calculated as

$$R_{\text{sh}}^{-1} = \left| \frac{(B \cdot \nabla)B}{B^2} \right|^2 \frac{1}{2}.$$  

Assuming that $\alpha > 3$ (or $l_\text{coh} \sim 1/k_\text{min}$) and decomposing $\delta B$ in a Fourier series, one finds that, indeed, $1/R_{\text{sh}} \sim (\delta B^2)^{-1} \times (l/l_\text{coh})^{\alpha-2}/2$. Hence the large-scale component is approximately uniform over a length scale $l$. It is safe to neglect the magnetic power on scales smaller than $l$, since $\delta B^2 \sim \delta B^2 \times (l/l_\text{coh})^{\alpha-2} \ll \delta B^2$, the latter being comparable to the large-scale component. If $\alpha < 3$ similar conclusions apply, since the assumption $r_\text{L} \ll l_\text{coh}$ translates into $r_\text{L} \ll k_\text{min}$, which means that the particle only experiences a smooth large-scale magnetic field.

Now, if the magnetic field is approximately regular over the path of the particle in a $u \rightarrow d \rightarrow u$ cycle, Fermi acceleration in the ultrarelativistic regime becomes similar to superluminal acceleration in a fully regular magnetic field, which is known to be inefficient (Begelman & Kirk 1990). In the following, we extend that discussion and compare the predictions with numerical simulations of particle propagation in realistic turbulence.

2.2. Analytical Trajectories

Upstream.—The equation of motion reads

$$\frac{d\mathbf{\dot{B}}}{dt} = \Omega_\text{L} \times \mathbf{B}$$

with $\mathbf{\dot{B}}$ the velocity of the particle and $\Omega_\text{L} = e\mathbf{r}_\text{L}$ the Larmor frequency. Shock crossing from downstream to upstream requires $\beta_{\text{sh}}^0 \geq \beta_{\text{sh},u}$ with $\beta_{\text{sh},u}$ the ingressing component of the velocity along the shock normal. Hence $\beta_{\text{sh},u} \sim O(1/T_{\text{sh},u})$. By working to first order in $1/T_{\text{sh},u}$, Achterberg et al. (2001) were able to obtain analytically the particle trajectory and its direction at shock recrossing $u \rightarrow d$.

Assuming that $\phi_\text{p} = 0$ so that the transverse component of $\mathbf{B}$ lies along $x$, one obtains the outgoing velocity vector as

$$\beta_{\text{sh},u}^0 = \beta_{\text{sh},u}^0,$$

$$\beta_{\text{sh},u}^0 = -\frac{1}{2} \beta_{\text{sh},u}^0 + \left[ \frac{3}{\Gamma_{\text{sh},u}} - 3\beta_{\text{sh},u}^0 - \frac{3}{4} \beta_{\text{sh},u}^0 \right]^{1/2},$$

$$\beta_{\text{sh},u}^0 = 1 - \beta_{\text{sh},u}^0 - \beta_{\text{sh},u}^0.$$  

(4)

Downstream.—Here we must proceed differently, as the return timescale is $\sim O(r_\text{L}/c)$ and can no longer be treated as a small quantity ($r_\text{L}$ is evaluated in the downstream rest frame; Lemoine & Revenu 2006).

We may assume that $\phi_\text{p} = 0$, since the phase is preserved by the Lorentz transformation; furthermore, shock compression results in a magnetic field essentially oriented transversely to the shock normal. To this order of approximation, the trajectory along the shock normal reads

$$\Omega_\text{L}(t) = \beta_{\text{sh},u}^0 \sin \phi, (\cos \Omega_\text{L} t - 1) + \beta_{\text{sh},u}^0 \sin \Omega_\text{L} t, \quad (5)$$

where $\phi$ denotes the phase of the ingressing velocity vector in the $(x, y)$-plane: $\beta_{\text{sh},u}^0 \equiv \beta_{\text{sh},u}^0 \cos \phi$, and $\beta_{\text{sh},u}^0 \equiv \beta_{\text{sh},u}^0 \sin \phi$. The shock front follows the trajectory $z_{\text{sh},u}(t) = \beta_{\text{sh},u} t$, and return to the shock will occur if and when

$$\sin \phi_t = g(t) \equiv \frac{\beta_{\text{sh},u} t - \beta_{\text{sh},u} \sin \hat{t}}{\beta_{\text{sh},u}^0 (\cos \hat{t} - 1)}, \quad (6)$$

with $\hat{t} = \Omega_\text{L} t$. The function $g(t)$ diverges toward $-\infty$ for $t \rightarrow 0$ or $2\pi$, and its derivative is monotonic in the interval $0 \leq \hat{t} \leq 2\pi$. Hence, return to the shock can occur if and only if the maximum of $g(t)$ exceeds the value $\sin \phi$. Note also that $g(t)$ is always negative, since by assumption $\beta_{\text{sh},u}^0 \leq \beta_{\text{sh},u}^0$. Therefore, a necessary condition for return to the shock front is $\phi_t \in [\pi, 0]$, or equivalently $\beta_{\text{sh},u}^0 \leq 0$.

Once the time $t_\text{d}$ of shock return has been determined (numerically), the outgoing velocity vector can be derived from the solutions to the equations of motion:

$$\beta_{\text{sh},u}^0 = \beta_{\text{sh},u}^0,$$

$$\beta_{\text{sh},u}^0 = \beta_{\text{sh},u}^0 \cos \Omega_\text{L} t_\text{d} + \beta_{\text{sh},u}^0 \sin \Omega_\text{L} t_\text{d},$$

$$\beta_{\text{sh},u}^0 = \beta_{\text{sh},u}^0 \cos \Omega_\text{L} t_\text{d} - \beta_{\text{sh},u}^0 \sin \Omega_\text{L} t_\text{d}.$$  

(7)

2.3. Mappings: Downstream to Upstream and Vice Versa

Equations (4) and (7) define mappings from the ingress angles on either side of the shock. The ingress angles in one rest frame are related to the egress angles in the other rest frame by the Lorentz transformations

$$\phi_t^0 = \phi_t^0, \quad \phi_y^0 = \phi_y^0,$$

$$\beta_{\text{sh},u}^0 = \frac{\beta_{\text{sh},u}^0 + \beta_{\text{sh},u}^0}{1 + \beta_{\text{sh},d}^0 \beta_{\text{sh},u}^0}, \quad \beta_{\text{sh},u}^0 = \frac{\beta_{\text{sh},u}^0 - \beta_{\text{sh},d}^0}{1 - \beta_{\text{sh},d}^0 \beta_{\text{sh},u}^0}.$$  

(8)

where $\beta_{\text{sh},d}^0 = (\beta_{\text{sh},u}^0 - \beta_{\text{sh},d}^0)(1 - \beta_{\text{sh},d}^0 \beta_{\text{sh},u}^0)$ is the relative velocity between the upstream and downstream rest frames. Using these mappings and transformations, one can follow the trajectory of a particle. Since the cycle time is of order $r_\text{L}/T_{\text{sh}} \ll l_\text{coh}$, it is reasonable to assume that $\phi_\text{p} = 0$ remains constant from one Fermi cycle to the next.

Now, in § 2.2 we argued that $\beta_{\text{sh},u}^0 \leq 0$ is a necessary condition for the particle to be able to return to the shock. However, as the particle travels upstream and exits back toward downstream, its outgoing velocity is given by equation (4), and it can be shown that $\beta_{\text{sh},u}^0 \geq 0$ irrespective of the upstream ingress angle. In effect, for a given $\beta_{\text{sh},u}^0$, the quantity $\beta_{\text{sh},u}^0$ is minimal when $\beta_{\text{sh},u}^0$ is maximal, that is, when $(\beta_{\text{sh},u}^0)^2 = 1 - \beta_{\text{sh},d}^0(\beta_{\text{sh},u}^0)^2$. Then the final $\beta_{\text{sh},u}^0 = (-\beta_{\text{sh},u}^0 + 3|\beta_{\text{sh},u}^0|)/2 \geq 0$. The minimum is then
zero, which corresponds to a particle entering upstream along $x$ (tangentially to the shock surface), that is, $\beta_{x,i} = \beta_{x,i}$ and $\beta_{y,i} = 0$. Hence, if a particle that travels downstream is able once to return to the shock, it will not do so in the subsequent cycle.

A quantitative assessment of this discussion is shown in Figure 1, which presents the loci of ingress and egress velocity vectors in the $(x, y)$-plane as seen in the upstream rest frame. The gray area shows the region of egress, $\beta_{x,e}$ and $\beta_{y,e}$ (equivalently ingress as seen from downstream), for which the particle is bound to return to the shock. The circles show the ingress $\beta_{x,i}$ and $\beta_{y,i}$ of a particle that crosses toward upstream. The various circles correspond to different values of $\beta_{y,i}$ upon entry; the radii of these circles are bounded by the shock-crossing condition $\beta_{x,i} \geq \beta_{ab,i}$. Finally, the kidney-shaped contours map these ingress upstream velocities into the egress velocities, according to equation (4). The fact that these do not overlap anywhere with the blue area confirms that at most $1 \frac{1}{2}$ $u \rightarrow d \rightarrow u \rightarrow d$ cycles are permitted.

2.4. Comparison with Numerical Work

The previous discussion relies on several approximations, most notably that the field lines can be considered as straight over the trajectory of the particle. Comparison of the previous results with numerical simulations of particle propagation in refined descriptions of the magnetic field are best suited to assess the error that results from these approximations. Figure 2, which shows a contour plot of the return probability defined as a function of $\beta_{x,i}$ and $\beta_{y,i}$, can be directly compared to the gray area of Figure 1. Indeed, the agreement is excellent. The parameters of the simulations whose results are shown in Figure 2 are as follows: $r_s/L_{\text{rms}} = 7 \times 10^{-4}$, $\alpha = 11/3$ (Kolmogorov turbulence), $B_0 = 0$ (pure turbulence), and $\Gamma_{ab,i} =$ 38. The numerical procedure used to follow the particle trajectory has been described in Lemoine & Pelletier (2003) and Lemoine & Revenu (2006).

This discussion also explains the results of recent Monte Carlo simulations. For instance, Niemiec & Ostrowski (2006) report that Fermi acceleration is inefficient in the ultrarelativistic regime for upstream Kolmogorov turbulence; their simulations indicate very steep spectra if any, in good agreement with the present discussion. In contrast, other studies of Fermi acceleration obtain power-law spectra of various spectral indices. However, one can check that these latter studies have, one way or another, either assumed an isotropic downstream turbulence or implicitly marginalized over the angle between the particle trajectory and $B_1$ at shock crossing, which amounts to picking $\phi_0$ at random in each half-cycle. In the light of the above discussion, it is then easy to understand why Fermi acceleration seemed efficient in these studies.

Strictly speaking, our results do not apply to scale-invariant turbulence, that is, if $\alpha = 3$. However, results of numerical simulations for this particular case are similar to those shown in Figure 2; the return probability is nonzero everywhere, but it is 10 times lower in the kidney-shaped region than in the area of negative $\beta_{y,i}$. This suggests that quite steep power-law spectra should emerge from Fermi acceleration in such turbulence.

3. DISCUSSION

Fermi acceleration is thus inefficient at ultrarelativistic shock waves if the Larmor radius $r_s$ of injected particles is much smaller than the coherence length $l_{\text{coh}}$ of the turbulent magnetic field on both sides of the shock. This does not mean that Fermi acceleration is bound to fail. In particular, if $r_s > l_{\text{coh}}$, power-law spectra must emerge, as the memory of the magnetic field direction (in the transverse plane) at shock crossing is erased.
recently, it has been suggested that a generalization of the streaming instability to the relativistic regime could amplify the magnetic field to the values required by gamma-ray burst observations (Milosavljević & Nakar 2005). Because of the very short upstream return timescale, \( \sim r_s/T_{\text{sh}} \), the particle can never stream too far ahead of the shock, so that the turbulence is generated on small scales, \( \sim 10^{11} \, \text{cm} \ll r_s \) (Milosavljević & Nakar 2005). Therefore, one naturally expects in this case too that Fermi acceleration would be efficient; here as well, one needs to understand the turbulence properties before conclusions can be drawn about the index \( s \).

To summarize, we have shown in § 2 that Fermi acceleration cannot operate successfully at ultrarelativistic shock waves if one (somewhat naively) assumes large-scale turbulence on both sides of the shock wave. The conclusions of the present discussion are thus more optimistic and open a wealth of new possibilities; in particular, they suggest that the success of Fermi acceleration is intimately connected with the mechanism of magnetic field amplification in the shock vicinity. The comprehension of Fermi acceleration will eventually require understanding the generation of the magnetic field and deriving the properties of the turbulence, as well as characterizing the transport of accelerated particles in this possibly anisotropic turbulence.

**Note added in manuscript.**—While this work was being completed, a recent preprint by Niemiec et al. (2006) appeared, reporting on Fermi acceleration with small-scale turbulence. Although their simulations are limited to \( \Gamma_{\text{sh}} = 10 \), these authors observe that the inclusion of small-scale turbulence allows power-law spectra to emerge through Fermi acceleration, in good agreement with the above discussion.

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