What do we learn from the CMB observations?

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Abstract

We give an account, at nonexpert and quantitative level, of physics behind the CMB temperature anisotropy and polarization and their peculiar features. We discuss, in particular, how cosmological parameters are determined from the CMB measurements and their combinations with other observations. We emphasize that CMB is the major source of information on the primordial density perturbations and, possibly, gravitational waves, and discuss the implication for our understanding of the extremely early Universe.

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1 Introduction

The existence of cosmic microwave background radiation (CMB) is one of the keystones of cosmology. It was discovered in 1964 by Perscius and Wilson, and it earned them the Nobel prize. CMB comes from the epoch when the Universe was much hotter and much younger than today. The approximate
age of the Universe at the time CMB was released equals 380 thousand years, while the current age of the Universe is about 14 billion years. Presently, the radiation has the temperature $T = 2.725 K$. Among all signals we receive directly, CMB radiation is believed to be one of the youngest\footnote{Earlier epoch of Big Bang Nucleosynthesis (starting from about 1 second after Big Bang) is probed by measuring light element abundances.}

Later, the anisotropy of CMB was discovered. This anisotropy is very small. The largest is the dipole component of order $\delta T/T \approx 10^{-3}$. It is believed to arise due to the motion of the Solar system with respect to CMB. All other multipoles are much lower, the maximal ones are of order $\delta T/T \sim 10^{-4} - 10^{-5}$ (see Fig.1).

$T = 2.725^0K$, $\frac{\delta T}{T} \sim 10^{-5}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{CMB_map.png}
\caption{The CMB temperature map, obtained by WMAP experiment \cite{1}. The brighter a region is, the hotter radiation comes from it.}
\end{figure}

It turns out that one can determine many parameters of our Universe by studying this anisotropy. CMB anisotropy is caused both by inhomogeneities in the early Universe and by the effects of propagation of CMB photons through the Universe at later time. An example of the latter is the Sunyaev–Zel’dovich effect, it is due to scattering of CMB photons off free electrons in interstellar plasma. This effect distorts CMB angular spectrum at small angular scales. There are other effects with similar consequences. For example, recombination is not an instantaneous process, the surface of last scattering has finite width, and this smooths out the CMB angular spectrum at small angular scales. Also, just before recombination the mean free path of photons drastically increases. The photons travel freely for rather long time, so they diffuse away from over-dense regions to regions of lower density, and this also smears out small inhomogeneities. This effect is called Silk damping. Because of these effects, the CMB anisotropy spectrum is not particularly informative from the cosmological point of view at angular scales of a few arc minutes and smaller.

Last scattering of the CMB photons occurs almost at the same time as the recombination does, since the main process of interaction of photons with matter in this case is the Thomson scattering off free electrons. The temperature of last scattering depends on the cosmological parameters very weakly (weaker than logarithmically), it is mainly determined by the potential of ionization of hydrogen and other world constants. The up-to-date values of the temperature and redshift of last scattering are:

$$T_r = 2970 ~ K = 0.26 ~ eV,$$

$$z_r = 1090.$$
At the time of last scattering, the Universe was in matter-dominated regime, i.e., the contribution of mass density of nonrelativistic matter to the total energy density is the largest, or, in other words, the overall pressure is much smaller than the overall energy density: $p_{\text{tot}} \ll \rho_{\text{tot}}$.

CMB gives us the photographic picture of the Universe at the time of last scattering (modulo aforementioned effects). Hence, the CMB anisotropy reflects the fact that the Universe was slightly inhomogeneous at that time. The inhomogeneities in the following components of cosmic plasma contribute to the CMB anisotropy:

1. Baryon–electron–photon plasma.
2. Dark matter.
3. Neutrinos.
4. Gravitational waves.

Rather complex structure of the CMB angular spectrum is due to the interplay between the effects caused by the first two of these components.

The purpose of this review is to discuss what information about our Universe one extracts from the CMB data. There are two classes of properties one is interested in:

1. Properties of primordial perturbations which, as we will see, were built before the hot stage of the cosmological evolution. This is a key for revealing the origin of primordial perturbations and, ultimately, understanding the epoch preceding the hot stage. The existing data are consistent with the picture that primordial density perturbations and gravitational potentials induced by them had rather simple properties, consistent with the inflationary mechanism of their generation, and that there were no primordial gravitational waves. As we discuss in this review, future CMB measurements are sensitive to primordial gravity waves at the level predicted by simple inflationary models. The observation of the effects induced in CMB by primordial gravity waves would be an outstanding discovery.

2. Properties of the Universe shortly before recombination and later, up until the present epoch. These include the energy balance (photons, neutrinos, baryons, dark matter, dark energy), spatial curvature, etc.

Studying the CMB temperature anisotropy and polarization is sufficient by itself to measure some of these properties. There are numerous cases, however, when the CMB data alone are insufficient for determining some of the cosmological parameters, because these data are approximately degenerate in the parameters (i.e., they depend on certain combinations of the parameters). The degeneracy is lifted by invoking other cosmological data, as we illustrate in this review.

There are good books and reviews on CMB, see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The reader is invited to consult them, in particular, for references to original works. In this review we mostly refer to papers containing concrete results we make use of.

## 2 Observables

The temperature variation on celestial sphere, i.e., the function $\delta T(\phi, \theta) \equiv \delta T(n)$, can be expanded in spherical harmonics:

$$\delta T(n) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi).$$  \hfill (3)

All observations today support the hypothesis that $a_{lm}$ are independent Gaussian random variables. Gaussianity means that

$$P(a_{lm}) da_{lm} = \frac{1}{\sqrt{2\pi C_l}} e^{-\frac{a_{lm}^2}{2C_l}} da_{lm},$$  \hfill (4)
where \( P(a_{lm}) \) is the probability density for the random variable \( a_{lm} \). For a hypothetical ensemble of Universes like ours, the mean values of products of the coefficients \( a_{lm} \) would obey

\[
\langle a_{lm}a_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}.
\]  

(5)

We can measure only one Universe, but this formula is still used to extract the angular spectrum \( C_l \) from the data. For given \( l \), there are \((2l + 1)\) independent coefficients \( a_{lm} \), so there exists an irreducible statistical uncertainty of order \( \delta C_l / C_l \sim 1 / \sqrt{2l + 1} \), called cosmic variance. It is particularly pronounced at small \( l \) and, indeed, it is much larger than the experimental errors in this part of the angular spectrum (as an example, error bars in the left part of Fig. 2 are precisely due to the cosmic variance).

In the case of Gaussian random variables, all odd correlators are zero and all even correlators are expressed through the two-point correlator according to the Wick theorem. Under this distribution, the mean value of the temperature fluctuation squared equals:

\[
\langle [\delta T(n)]^2 \rangle = \sum_l \frac{2l + 1}{4\pi} C_l \approx \int \frac{dl}{l} \frac{l(l + 1)}{2\pi} C_l = \int \frac{dl}{l} D_l.
\]  

(6)

Usually, graphs presenting the anisotropy spectrum show \( D_l \) (for example, in Fig. 2). It is worth noting that the approximate relationship between the multipole number \( l \) and angular scale is \( \Delta \theta \approx \pi / l \). Hence, the first peak in Fig. 2 corresponds to temperature fluctuation of angular size of about \( 10^\circ \).

Figure 2: The angular spectrum of the CMB temperature anisotropy [12]. The line is a prediction of the standard \( \Lambda \)CDM model. The quantity in vertical axis is \( D_l \) defined by (6).

Gaussianity of temperature fluctuations is a very interesting property. The linear evolution preserves Gaussianity, so this property means that the primordial density perturbations are Gaussian random field. We recall in this regard that vacuum fluctuations of free quantum fields are also Gaussian. Hence, we have an observational hint that the mechanism of the generation of primordial perturbations is the amplification of vacuum fluctuations of some linear quantum field(s). This is indeed the case in the inflationary scenario, and also in some other models.
3 Density perturbations and CMB temperature anisotropy

3.1 Background metric

To set the stage, let us begin with background. The background metric is the metric of homogeneous and isotropic space–time (Friedmann–Robertson–Walker metric):

\[
    ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

where \(a(t)\) is the scale factor, \(k\) is the spatial curvature parameter. \(k = 0, +1, -1\) correspond to the Euclidean 3–space, 3–sphere and 3–hyperboloid, respectively. In most part of this review we consider the case \(k = 0\) and use the equivalent form of the spatially flat metrics:

\[
    ds^2 = dt^2 - a(t)^2(dx_1^2 + dx_2^2 + dx_3^2).
\]

If we neglect the possible peculiar motion of cosmological observers, each observer has definite time-independent \(x\)-coordinates. Note that \(x_1^2 + x_2^2 + x_3^2\) is not a square of physical distance even in the spatially flat Universe (\(k = 0\)). The physical distance squared is \(a^2(x_1^2 + x_2^2 + x_3^2)\), and this corresponds to the fact that the physical distance between two cosmological observers grows in time as \(a(t)\) increases. This is precisely the expansion of the Universe. The coordinates \(x\) are called comoving coordinates.

In general, the dynamics of the scale factor is determined by the Friedmann equation, which reads:

\[
    \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2},
\]

where \(\rho\) is the total energy density in the Universe. The energy balance equation (the relation \(dE = -pdV\) applied to the comoving volume \(a^3\)) is

\[
    d\rho = -3(p + \rho) \frac{da}{a},
\]

where \(p\) is pressure. For nonrelativistic matter (dark matter, baryons and also massive neutrinos at late epoch) one has \(p \ll \rho\), and the energy density decreases as \(1/a^3\) (mass density gets diluted in inverse proportion to the comoving volume). For ultrarelativistic matter, for example, photons and neutrinos at early epoch, the equation of state is \(p = \rho/3\) and \(\rho \propto 1/a^4\) (energy density decreases also because photons get redshifted). Equation (9) then implies that if the Universe is flat (\(k = 0\)) and filled with nonrelativistic matter, then the scale factor evolves as \(a \propto r^{2/3}\) (matter domination). If the Universe is filled with ultrarelativistic matter, then scale factor increases as \(a \propto t^{1/2}\) (radiation domination). There is also dark energy in our Universe, whose density does not change in time (or almost does not change in time), \(\rho_\Lambda = \text{const}\). The Universe was radiation-dominated at early epoch, then there was long matter–dominated epoch, and recently dark energy has started to dominate the cosmological expansion.

At the epochs we consider, the Friedmann equation is specified to

\[
    H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} (\rho_{\text{rad}} + \rho_M + \rho_\Lambda) - \frac{k}{a^2} = \frac{8\pi G}{3} \rho_c \left[ \Omega_{\text{rad}} \left( \frac{a_0}{a} \right)^4 + \Omega_M \left( \frac{a_0}{a} \right)^3 + \Omega_\Lambda + \Omega_\kappa \left( \frac{a_0}{a} \right)^2 \right],
\]

where the index “0” refers to the current epoch; \(\rho_{\text{rad}}, \rho_M, \rho_\Lambda\) are energy densities of radiation, nonrelativistic matter (i. e., baryons and dark matter) and dark energy, respectively, \(\rho_c = (8\pi G)/(3H_0^2)\), and \(\Omega_{\text{rad,M,}\Lambda} = \rho_{\text{rad,M,}\Lambda}/\rho_c\) are the present energy densities in units of the critical density, while \(\Omega_\kappa\) is the relative present contribution of the spatial curvature to the current Universe expansion rate. \(\Omega\)'s are relative contributions in the sense that \(\Omega_{\text{rad}} + \Omega_M + \Omega_\Lambda + \Omega_\kappa = 1\).
The coordinate $t$ is the physical time, it is the proper time of cosmological observers. Sometimes another time coordinate is used, so-called conformal time

$$\eta(t) = \int^t \frac{dt'}{a(t')}.$$  

(12)

In coordinates $\eta, x_1, x_2, x_3$ the metric \footnote{We do not consider the effects due to structures in the relatively late Universe, such as Sunyaev–Zel’dovich effect and gravitational lensing. Yet another possible source is gravity waves, see below.} for $\kappa = 0$ reads:

$$ds^2 = a(\eta)^2 (d\eta^2 - d\mathbf{x}^2),$$  

(13)

i. e., the metric is conformally flat. Since the light propagation satisfies $ds^2 = 0$, in coordinates $\eta, x_1, x_2, x_3$ the light propagates as in the Minkowski space–time.

3.2 The angular spectrum of the CMB temperature anisotropy

3.2.1 Sachs–Wolfe, Doppler and integrated Sachs–Wolfe effects

As we have already pointed out, the major source of the CMB anisotropy are the density perturbations and gravitational potentials associated with them\footnote{This formula does not account for aforementioned smearing effects, so it is valid, strictly speaking, only at relatively large angular scales corresponding to $l \lesssim 1500$. However, smearing leaves almost unaffected some properties of the CMB angular spectrum, such as positions of the peaks. We use this formula for qualitative analysis anyway.}. In cosmological literature these are called scalar perturbations, since the relevant quantities are scalars under spatial rotations. In a certain gauge (conformal Newtonian gauge), metric with scalar perturbations is written as follows:

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 + 2\Psi)d\mathbf{x}^2,$$  

(14)

where $\Phi(\mathbf{x}, t)$ and $\Psi(\mathbf{x}, t)$ are two gravitational potentials. In the Newtonian limit, $\Phi$ coincides with the Newtonian potential.

In the case of scalar perturbations, the observed temperature fluctuation of the CMB photons coming to us along the direction $\mathbf{n}$ from the point $\mathbf{x}$ where they have last scattered is given by the general formula:\footnote{This formula does not account for aforementioned smearing effects, so it is valid, strictly speaking, only at relatively large angular scales corresponding to $l \lesssim 1500$. However, smearing leaves almost unaffected some properties of the CMB angular spectrum, such as positions of the peaks. We use this formula for qualitative analysis anyway.}

$$\frac{\delta T}{T} = \frac{1}{4} \frac{\delta \rho_\gamma}{\rho_\gamma}(\mathbf{x}) + \Phi(\mathbf{x}) + \mathbf{n} \cdot \mathbf{v}_\gamma(\mathbf{x}) + \int_{t_r}^{t_0} dt \ (\dot{\Phi} - \dot{\Psi}),$$  

(15)

where $\delta \rho_\gamma(\mathbf{x})$ is the density perturbation of photons at last scattering, $\mathbf{v}_\gamma(\mathbf{x})$ is the velocity of the baryon–photon medium at that time, and the integral is evaluated along the photon world line from last scattering to detection. All terms here have transparent interpretation. The first term is the perturbation of the temperature of the photon gas at last scattering: in general $\rho_\gamma \propto T_\gamma^4$ and we have $1/4 \delta \rho_\gamma(\mathbf{x})/\rho_\gamma = \delta T_\gamma(\mathbf{x})/T_\gamma$. The second term reflects the fact that photons get redshifted (blueshifted) when they escape from potential wells (humps). The first and second contributions together are called the Sachs–Wolfe effect. The third term is self-evident: this is the Doppler effect. Finally, the integral in (15) represents the integrated Sachs–Wolfe effect: as photons propagate through the Universe, they get red- or blueshifted due to time-dependent gravitational field. Namely, if a photon meets a potential well, it falls into it and gains energy. Suppose now that by the time the photon starts escaping from the well, the well has become shallower. Then the photon looses less energy when climbing the well, and the net effect is the blueshift of the photon. Since photons are relativistic, this effect is caused by both temporal ($\delta g_{00} \propto \dot{\Phi}$) and spatial ($\delta g_{ij} \propto \Psi \delta_{ij}$) components of metric perturbations.

For scalar perturbations, both Doppler and integrated Sachs–Wolfe effects are numerically rather small. We simplify our discussion and consider the Sachs–Wolfe effect alone (the first two terms in (15)). The main contributions here come from perturbations in the baryon–photon medium and dark
matter. It is important that to a good approximation, baryons and photons make a single fluid before recombination; they are tightly coupled due to intense scattering of photons off free electrons and intense Coulomb interaction between electrons and protons.\footnote{The tight coupling approximation is not valid for $l \gtrsim 1500$ because of the Silk damping and other effects. Our discussion, however, remains valid at qualitative level even for large $l$.}

Let us consider the effects due to baryon–photon medium and dark matter in turn.

### 3.2.2 Oscillating part of the angular spectrum: baryon–photon medium

The oscillating part of the CMB temperature angular spectrum comes from sound waves in the baryon–photon medium before recombination. So, we have to understand the behavior of these waves in the expanding Universe. Instead of writing and then solving the equations for sound waves, let us consider a toy example, the case of inhomogeneities in massless scalar field. In fact, example is useful for demonstrating the main notions and ideas of treatment of various kinds of perturbations. The action for the scalar field is:

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] = \int d^3 x dt a^3 \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2a^2} (\partial_i \phi)^2 \right].$$

(16)

The field equation thus reads:

$$- \frac{d}{dt} (a^3 \phi) + a \partial_i \partial_i \phi = 0,$$

(17)

i. e.,

$$\ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \partial_i^2 \phi = 0.$$

(18)

Here, $H$ denotes the Hubble parameter: $H \equiv \dot{a}/a$. This equation is linear in $\phi$ and homogeneous in space, so it is natural to represent $\phi$ in terms of the Fourier expansion:

$$\phi(x) = \int e^{ikx} \hat{\phi}(k) d^3 k.$$

(19)

Clearly, the value of $k$ for a given Fourier mode is a constant. However, $k$ is not a physical wavenumber (physical momentum), since $x$ is not a physical distance. $k$ is called conformal momentum, while physical momentum equals $q \equiv 2\pi/\lambda = 2\pi/(a(t)\Delta x) = k/a(t)$. $\lambda$ here is the physical wavelength of perturbation, and it grows together with the expansion of the Universe. Accordingly, as the Universe expands, the physical momentum of a given mode decreases (gets redshifted), $q(t) \propto 1/a(t)$. For a mode of given conformal momentum $k$, Eq.(18) gives:

$$\ddot{\phi} + 3H \dot{\phi} + \frac{k^2}{a^2} \phi = 0.$$

(20)

This equation has two time–dependent parameters of the same dimension: $k/a$ and $H$. One can consider two limiting cases: $k/a \ll H$ and $k/a \gg H$. In the cosmological models with conventional equation of state of the dominant component (e. g., matter–dominated or radiation–dominated Universe), one has $H \approx 1/L$, where $L$ is the size of cosmological horizon. So, the regime $k/a \ll H$ is the regime when the physical wavelength is greater than the horizon size (super-horizon mode), while for $k/a \gg H$ the physical wavelength is much smaller than the horizon size (sub-horizon mode). The time when the wavelength of the mode coincides with the horizon size is called horizon crossing. This time is denoted by the symbol $\times$. It is straightforward to see that at both radiation–dominated and matter–dominated epochs, the ratio $k/(aH)$ grows. This means that every mode was at some early time super-horizon, and later it becomes sub-horizon.
For a super-horizon mode, we can neglect the term $\phi \cdot k^2/a^2$ in comparison with other terms. Then the field equation, e.g., in the radiation-dominated Universe ($a \propto t^{1/2}, H = 1/2t$), becomes

$$\ddot{\phi} + \frac{3}{2t} \dot{\phi} = 0. \quad (21)$$

The solution of this equation is

$$\phi(t) = A + \frac{B}{\sqrt{t}}. \quad (22)$$

This behavior is generic for all cosmological perturbations: there is a constant mode ($A$ in our case) and a mode that decays in time. If we extrapolate the decaying mode $B/\sqrt{t}$ back in time, we get very strong (infinite in the limit $t \to 0$) perturbation. For density perturbations this means that this mode corresponds to strongly inhomogeneous early Universe. Therefore, this mode is usually assumed to be absent for actual cosmological perturbations. Hence, for given $k$, the solution is determined by a single parameter, the initial amplitude $A$ of the mode $\phi_k$.

After entering the sub-horizon regime, the modes begin to oscillate. The equation in the sub-horizon regime is:

$$\ddot{\phi} + \frac{3}{2t} \dot{\phi} + \frac{k^2}{t} \phi = 0, \quad (23)$$

and its general solution in the WKB-approximation reads:

$$\phi(t) = \frac{A'}{a(t)} \cos \left( \int_0^t \frac{k}{a(t')} dt' + \psi_0 \right), \quad (24)$$

with the two constants being the amplitude $A'$ and the phase $\psi_0$. The amplitude $A'$ of these oscillations is determined by the amplitude of the super-horizon initial perturbation, while the phase $\psi_0$ of these oscillations is uniquely determined by the condition of the absence of the decaying mode, $B = 0$. Imposing these conditions yields

$$\phi(t) = D \frac{a \propto}{a(t)} \sin \left( \int_0^t \frac{k}{a(t')} dt' \right). \quad (25)$$

The decreasing amplitude of oscillations $\phi(t) \propto 1/a(t)$ and the particular phase $\psi_0 = -\pi/2$ in $[24]$ are peculiar properties of the wave equation $[18]$, as well as the radiation-dominated cosmological expansion. However, the fact that the phase of oscillations is uniquely determined by the requirement of the absence of the super-horizon decaying mode is generic.

The perturbations in the baryon–photon medium before recombination — sound waves — behave in a rather similar way. Their evolution is as follows:

$$\frac{\delta \rho_\gamma}{\rho_\gamma} = \begin{cases} \text{const}, & \text{outside horizon,} \\ \text{const} \cdot \cos \left( \int_0^t v_s \frac{k}{a(t')} dt' \right), & \text{inside horizon,} \end{cases} \quad (26)$$

where $v_s \equiv \sqrt{dp/d\rho}$ is the sound speed. The baryon–photon medium before recombination is almost relativistic [7] since $\rho_B < \rho_\gamma$. Therefore, $v_s \approx 1/\sqrt{3}$. At the time of last scattering $t_r$ we have:

$$\frac{\delta \rho_\gamma}{\rho_\gamma} = \text{const} \cdot \cos \left( \int_0^{t_r} v_s \frac{k}{a(t')} dt' \right) = \text{const} \cdot \cos kr_s, \quad (27)$$

This does not contradict the statement that the Universe is in matter-dominated regime at recombination. The dominating component at this stage is dark matter.
where

\[ r_s = \int_0^t v_s \frac{dt'}{a(t')} \]  

(28)

is the comoving size of the sound horizon, while the physical size equals \( a(t_r) r_s \). So, we see that the value of the density perturbation at recombination oscillates as a function of wavelength. The period of this oscillation is determined by \( r_s \). These oscillations are shown in Fig. 2. Indeed, one can show that gravitational potentials induced by the perturbations in the baryon–photon medium are negligible at recombination, while, as we mentioned above, the Doppler and integrated Sachs–Wolfe effects are also rather small. Hence, the main effect of the baryon–photon perturbations on CMB comes from the oscillatory first term in (15). The absolute values of perturbations\(^6\) in the baryon–photon plasma are maximal at conformal momenta \( k_n \) obeying \( k_n r_s = \pi n \) for integer \( n \). Hence, their coordinate half wavelengths (distances between maxima of the absolute value) are equal to \( \Delta x_n = \pi / k_n \). Now, according to (13), photons propagate in conformal coordinates \((\eta, x)\) precisely like in the Minkowski space–time. Hence, an interval of coordinate length \( \Delta x \) at the time of last scattering is seen today at an angle \( \Delta x / \eta_0 \), where \( \eta_0 \) is the conformal time interval between the present and last scattering epochs, i.e., the comoving distance to the sphere of last scattering.\(^7\)

So, the strongest CMB temperature fluctuations have angular scales \( \Delta \theta_n \approx r_s / (n \eta_0) \). We recall the approximate correspondence \( \Delta \theta \leftrightarrow \pi / l \) and conclude that the peaks in the CMB angular spectrum are expected to be at the positions

\[ l_n \approx \pi n \frac{\eta_0}{r_s} \]  

(29)

These peaks are indeed seen in Fig. 2.

Let us pause here to record the approximate correspondence between the conformal momenta of perturbations and the multipole numbers of the CMB anisotropy they generate:

\[ l \leftrightarrow k \eta_0 \]  

(30)

This correspondence follows from the argument just presented and will be used later on.

The very existence of the peaks in the CMB angular spectrum strongly suggests that the hot Big Bang stage of the cosmological expansion was preceded by another epoch with entirely different properties. That was the epoch when the cosmological perturbations were generated. Indeed, we have seen that the definite phases of sound waves which are behind the peaks are determined by the properties of the scalar perturbations in the early, super-horizon regime. So, the modes we observe were indeed super-horizon, their wavelengths exceeded considerably the size of the cosmological horizon. In the hot Big Bang theory, there is no causal connection between horizon-size regions. Hence, within this theory it is impossible to invent a mechanism which would generate these perturbations in a causal manner. We conclude that the generation of the cosmological perturbations must have occurred at some epoch preceding the hot Big Bang stage. Furthermore, the cosmological evolution at that epoch was such that the combination \( a(t) H(t) \) was small early on, and then it increased enormously towards the beginning of the hot Big Bang stage. Only in that case the physical momentum \( k / a(t) \) could exceed the Hubble parameter \( H \) at early times, the modes could be sub-horizon “in the beginning”, and the perturbations could be generated by some causal mechanism. This is precisely what happens in the inflationary scenario.

An alternative would be the generation of the density perturbations at the hot Big Bang stage in a causal way. This can only happen when a mode of perturbations is in the sub-horizon regime. In that

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\(^6\) According to (5), the relevant quantity is the temperature fluctuation squared. This is the reason for considering absolute values of density perturbations.

\(^7\) This is true for intervals normal to the line of sight. One can show that the sound waves in the baryon–photon component, whose wave vectors are normal to the line of sight, indeed give dominant contribution to the oscillations in the CMB temperature angular spectrum. Contributions to the oscillating part of the angular spectrum coming from other waves get smeared out upon integration over the directions of the wave vectors.
Figure 3: The real CMB angular spectrum and inflationary prediction compared with the spectra obtained in models with casual mechanisms of the generation of density perturbations [13]. In all casually connected models (all models above except inflation), there are no oscillations.

In case, there is no reason for very definite phases of the sound waves in the baryon–photon component, and hence for oscillations in the CMB temperature angular spectrum. For example, Fig.3 shows the angular spectrum in a model where density perturbations are generated by accretion on cosmic strings, and the spectra created by other casual mechanisms at the hot Big Bang epoch. All these spectra do not exhibit the oscillatory behavior. So, the generation of density perturbations at the hot Big Bang stage is ruled out by the CMB observations.

The positions and strengths of the peaks are sensitive to various cosmological parameters, so from observations one can determine these parameters or at least their combinations. In particular, the peak positions depend on:

1. The ratio of baryons to photons.
2. The rate of the cosmological expansion, thus the density of dark energy and other forms of matter.
3. The curvature of space.

The size of the sound horizon entering (29) depends on the sound speed in the baryon–photon component. It is given by

$$v_s^2 = \frac{dp}{d\rho} = \frac{3}{\rho_B} \frac{d\rho_\gamma/\rho_\gamma}{\rho_B/\rho_\gamma} + \frac{d\rho_B/\rho_B}{\rho_\gamma} = \frac{1}{3} \frac{4dT/T}{\rho_B/\rho_\gamma} = \frac{1}{3} \left[ 1 + \frac{3}{4} \frac{dT}{T} \right].$$

The dependence of $v_s^2$ and hence $r_s$ on $\rho_B$ is rather weak, since $\rho_B/\rho_\gamma \ll 1$ up to recombination. More sensitive to the baryon fraction are the heights of the peaks, see below. For given baryon abundance, precisely determined basically from the peak positions, the sound horizon serves as a standard ruler at recombination. The angle at which it is seen today depends on the expansion history after recombination and spatial curvature.

The dependence on the expansion history of the Universe comes about in the following way. We know precisely the factor $a(t_0)/a(t_r)$ determining how much the Universe has expanded from the time of last scattering (since $a(t_0)/a(t_r) = T_r/T_0$ and we know the present CMB temperature $T_0$ and the temperature of last scattering $T_r$). The Universe could expand fast, and then the last scattering surface
would be close to us, or it could expand slowly, and the last scattering surface would be further away, so the angular scale at the peaks would be smaller. Notably, the expansion rate depends strongly on the value of the cosmological constant $\Lambda$ (or, generally, on dark energy density); we discuss this point shortly in some detail.

The dependence on curvature of space comes from the fact that the same interval viewed from the same distance is seen at different angles depending on curvature. For example, in the space of positive curvature (3-sphere) the angle is larger than the one in the Euclidean space, and the angle in the flat space is larger that the angle in the space of negative curvature (Lobachevsky geometry). So, both the curvature and the cosmological constant lead to similar shift of the peaks in the CMB angular spectrum, see Fig. 4. Therefore, there is a degeneracy in the parameters of $\Omega_\Lambda$ and $\Omega_{\kappa}$, see [14]. Actually, the degeneracy is lifted by making use of the results of cosmological observations unrelated to CMB. We give an example of the latter in Section 4.

Figure 4: The effect of curvature (left panel) and of $\Lambda$ (right panel) on the CMB temperature angular spectrum [11].

Let us illustrate the dependence of peak positions on the dark energy density by considering the spatially flat Universe. The Friedmann equation [12] in the flat Universe can be written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_0 \left(\Omega_{\text{rad}}(1+z)^4 + \Omega_M(1+z)^3 + \Omega_{\Lambda}\right),$$

(32)

where $z = a_0/a(t) - 1$ is redshift. As we have already noted, the light propagates according to the equation $ds^2 = 0$, i.e., $dt^2 = a^2 dx^2$, so $\eta = \int (1/a) dt$ is the comoving distance traveled by light. To calculate the comoving distance which light has traveled since recombination, one uses Eq. (32) and writes

$$dz = -\frac{a_0}{a^2} da = -\frac{a_0}{a} \frac{\dot{a}}{a} dt = -a_0 H \frac{dt}{a}.$$  

(33)

Therefore,

$$\eta_0 = \int_0^{\eta_r} \frac{dt}{a} = \int_0^{z_r} \frac{dz}{a_0 H(z)} = \frac{1}{a_0 H_0} \int_0^{z_r} \frac{dz}{\sqrt{\Omega_{\text{rad}}(1+z)^4 + \Omega_M(1+z)^3 + \Omega_{\Lambda}}}.$$  

(34)

The effect on $\eta_0$ due to $\Omega_{\Lambda}$ is seen from this formula. Note that by definition $\Omega_{\text{rad}} + \Omega_M + \Omega_{\Lambda} = 1$ in the spatially flat Universe, so the larger $\Omega_{\Lambda}$ is, the larger is the integrand at $z > 0$. Since the positions of the peaks are approximately given by Eq. (29), these positions are shifted to the left at larger $\Omega_{\Lambda}$. 


3.2.3 Nonoscillating part of the spectrum: the effect of dark matter

As we saw above, matter with finite sound speed produces oscillating angular spectrum of the CMB temperature anisotropy. But there exists another form of matter, which has zero (or nearly zero) sound speed. This is dark matter. Due to the fact that \( p = 0 \) for dark matter, and hence \( v_s^2 = 0 \), there are no waves in this medium, and thus by itself it does not produce oscillations in the CMB angular spectrum. Nevertheless, dark matter density perturbations affect the CMB angular spectrum quite strongly, via the gravitational potentials they create. These gravitational potentials induce the CMB temperature anisotropy both directly, through the second term in (15), and through the induced perturbations in the baryon–photon medium, which falls into the potential wells.

In the first place, we have to understand the properties of the gravitational potential \( \Phi_{\text{DM}} \) generated by dark matter perturbation. Let us consider a perturbation whose wavelength is much smaller than the horizon size at recombination. Then the Newtonian approximation is valid, and the gravitational potential is related to the mass density perturbation by the Poisson equation. We take into account the fact that the physical Laplacian written in terms of comoving coordinates is \( \Delta = a^{-2} \frac{\partial^2}{\partial x^2} \) and write the Poisson equation in the expanding Universe in the following way (we sometimes omit subscript \( \text{DM} \) in \( \Phi_{\text{DM}} \) in this Section):

\[
- \frac{k^2}{a^2(t)} \Phi = 4\pi G \delta \rho_{\text{DM}}. \tag{35}
\]

In matter–dominated regime \( \delta \rho_{\text{DM}}/\rho_{\text{DM}} \propto a(t) \) for sub-horizon modes, while \( \rho_{\text{DM}} \propto 1/a^3 \). Therefore, \( \delta \rho_{\text{DM}} \propto 1/a^2 \) and \( \Phi \) is independent of time. Using the assumption of the flat spectrum of primordial perturbations, for which \( \delta \rho_{\text{DM}}/\rho_{\text{DM}} \) depends on \( k \) weakly (see below), we obtain:

\[
\Phi_{\text{DM}}(k) \propto \frac{1}{k^2}. \tag{36}
\]

In fact, this relation is valid modulo logarithm, whose origin is the evolution of the dark matter perturbations at the radiation–dominated epoch; we are not going to elaborate on this point.

To evaluate the effect of dark matter on CMB, we have to obtain the photon density perturbation \( \delta \rho_\gamma/\rho_\gamma \), which is induced by the gravitational potential \( \Phi_{\text{DM}} \). In this way we are going to find the first term in (15); as we have already pointed out, the third and fourth terms are rather small, and we neglect them for the sake of argument. To this end, let us consider the baryon–photon medium in static background gravitational potential. Baryon-photon plasma is then in the state of hydrostatic equilibrium. In Newtonian mechanics, the equation of hydrostatic equilibrium is:

\[
\nabla p = -\rho \nabla \Phi. \tag{37}
\]

In the case of relativistic medium, when \( p \) is of order \( \rho \) (but \( \Phi \) is small, \( \Phi \ll 1 \)), this equation is generalized to

\[
\nabla p = -(p + \rho) \nabla \Phi, \tag{38}
\]

or, in the Fourier space:

\[
\delta p = -(p + \rho) \Phi. \tag{39}
\]

Here \( p \) and \( \rho \) are pressure and energy density of the baryon–photon medium before recombination. Using the equation of state \( p_\gamma = \rho_\gamma/3 \), \( p_B = 0 \) we obtain:

\[
\frac{1}{3} \delta \rho_\gamma = - \left( \frac{4}{3} \rho_\gamma + \rho_B \right) \Phi, \tag{40}
\]

and, finally,

\[
\frac{1}{4} \frac{\delta \rho_\gamma}{\rho_\gamma} = - \left( 1 + \frac{3}{4} \frac{\rho_B}{\rho_\gamma} \right) \Phi. \tag{41}
\]
Let us pause here to point out that there is a subtlety in this result. Namely, suppose for the moment that baryon density is negligible, and consider static flat background space–time. Let there be dark matter which does not interact with photons, but creates the gravitational potential. Then in thermodynamic equilibrium the temperature of photons should be the same everywhere in space. This is in apparent contradiction with the result (41), since the left hand side is precisely the perturbation of the photon temperature: $\frac{\delta \rho_\gamma}{\rho_\gamma} = 4 \frac{\delta T_\gamma}{T_\gamma}$ in view of the general property $\rho_\gamma \propto T_\gamma^4$. The reader is invited to reconcile thermal equilibrium with the result (41).

Coming back to CMB, we find from Eq. (41) that the Sachs–Wolfe effect on the observed CMB temperature — the first two terms in (15) — is

$$\frac{\delta T}{T} = -3 \frac{\rho_B}{4 \rho_\gamma} \Phi_{DM}. \quad (42)$$

Notably, the net effect of dark matter is proportional to $\rho_B$: if there were no baryons, the gravitational potential would not affect CMB.

According to (30) and (36), the dark matter contribution to the CMB angular spectrum is a smooth function of $l$ that decays as $1/l^2$. To large extent this contribution determines the overall decline of the spectrum, with oscillations superimposed on it, which is clearly seen in Fig. 2 (our Newtonian analysis is not valid for perturbations that remain super-horizon at recombination, so our discussion applies to the region $l \gtrsim 100$ only).

The effect of dark matter on CMB is so strong that the existence of dark matter could be deduced from CMB observations even in the absence of astrophysical evidence. As a digression, we point out that the presence of dark matter is crucial also for structure formation in the Universe. Indeed, at matter–dominated epoch, sub-horizon density perturbations in nonrelativistic matter (e. g., in baryons) grow as $\delta \rho/\rho \propto a$. At recombination, inhomogeneities in the baryon component were of order $10^{-5}$, the Universe since then has the expansion factor $a(t_0)/a(t_r) = z_r + 1 \approx 1000$, so the perturbations at present would be of order $10^{-2}$. They would not have left the linear regime of evolution, and thus would not form galaxies, stars, planets, men, etc. What saves the day is the property that the perturbations in dark matter grow well before recombination, and soon after recombination baryons catch up with them, as shown in Fig. 5.

### 3.2.4 Interference

The sum of the contributions of the baryon–photon and dark matter perturbations to the CMB temperature fluctuation has the following form:

$$\frac{\delta T}{T} = A \cdot \cos(kr_s) - 3 \frac{\rho_B}{4 \rho_\gamma} \Phi_{DM}. \quad (43)$$

The first term comes from the baryon–photon acoustic oscillations, and the second one from dark matter perturbations. According to (5), the observables are quadratic in $\delta T/T$, so an important issue is whether these two terms add up coherently. If they are coherent, we have:

$$\left(\frac{\delta T}{T}\right)^2 = A^2 \cos^2(kr_s) - 3 \frac{\rho_B}{2 \rho_\gamma} \cos(kr_s) A \Phi_{DM} + \left(3 \frac{\rho_B}{4 \rho_\gamma} \Phi_{DM}\right)^2. \quad (44)$$

Suppose now that the relative sign of $A$ and $\Phi_{DM}$ is positive. Then the peaks corresponding to $\cos kr_s = -1$ (odd $n$ in (29)) are enhanced, and the peaks corresponding to $\cos kr_s = 1$ (even $n$) are suppressed due to the second, interference term. Since $\Phi \propto 1/k^2$, the effect is more pronounced for the first peaks. And this is exactly what is observed.

Let us recall that the density perturbations are random fields. Hence, the interference is possible only if the baryon–photon perturbations (the coefficient $A$ in (43)) and dark matter perturbations ($\Phi_{DM}$ in (43)) are proportional to one and the same random field, call it $R(k)$. This means that the baryon–photon and dark matter perturbations are of one and the same origin, with $R$ being the common amplitude of primordial scalar perturbation. In fact, CMB data are consistent with the property that
primordial scalar perturbations are in the adiabatic mode. The definition of the adiabatic mode is
that the particle content is one and the same throughout the Universe. In other words, adiabatic
mode would appear if one contracts or expands some regions of the Universe without changing the
chemical composition of matter in these regions. The invariant and time independent characteristic
of the baryon abundance is the ratio $n_B/s$ of the baryon number density to the entropy density. Hence,
in the adiabatic mode $n_B/s$ is constant in time and space. Likewise, the ratio of the number density
of dark matter particles to the entropy density $n_{DM}/s$ is a universal constant. Since $s \propto T^3$ and
$\rho_i \propto T^4$, for the adiabatic mode in super-horizon regime we have\footnote{The relations (45) are
valid in the conformal Newtonian gauge. A convenient gauge-invariant definition of $R$ is the
spatial curvature of comoving hypersurfaces.}
\begin{equation}
\frac{\delta \rho_{DM}}{\rho_{DM}} = \frac{3 \delta \rho_\gamma}{4 \rho_\gamma} = \frac{\delta \rho_B}{\rho_B} = \frac{3 \delta \rho_\nu}{4 \rho_\nu} = R(k).
\end{equation}

After the horizon entry, the perturbations in the baryon–photon and dark matter components start
to evolve. This evolution is linear (perturbations are small) up until recombination, so we have at
recombination
\begin{align}
\Phi_{DM}(k) &= T_{DM}(k)R(k), \\
A(k) &= T_{B\gamma}(k)R(k).
\end{align}

The functions $T_{DM}$ and $T_{B\gamma}$ are called transfer functions. They describe how the perturbations in
different media evolve. Yet $\Phi_{DM}$ and $A$ at recombination are proportional to one and the same random
field $R$, and they add up coherently.
In principle, besides the adiabatic mode, there could exist so-called CDM entropy mode and baryon entropy mode. In the CDM entropy mode, there are no primordial perturbations in the baryon–photon medium, perturbations exist only in dark matter. Such a situation persists until the horizon crossing, due to causality. The inhomogeneities in the baryon–photon medium are generated at the time of horizon crossing, and afterwards they oscillate as follows:

$$\frac{\delta \rho}{\rho_\gamma} \propto \sin \left( \int_{t_X}^{t} \frac{k}{a(t')} dt' \right).$$  (48)

We emphasize that the phase of oscillations is different for adiabatic perturbations (cosine rather than sine, see (27)). So, if primordial perturbations were in the CDM entropy mode, the peaks in the CMB angular spectrum would be shifted, see Fig. 6. Exactly the same result holds for the baryon entropy mode. Presently, the admixture of the CDM entropy perturbations is ruled out by the CMB observations at the level of about 10%.

![CMB Angular Spectrum](image)

**Figure 6:** The effects of adiabatic and entropy perturbations on the CMB angular spectrum [11].

Purely adiabatic mode of perturbations is very natural. If our Universe was in complete thermal equilibrium (including chemical equilibrium) at some early stage of the hot Big Bang epoch, the only mode of perturbations is adiabatic. Physical processes that generated dark matter and baryon asymmetry, if that happened at the hot Big Bang epoch, were the same everywhere in space, so the values $n_{DM}/s$ and $n_B/s$ are the same everywhere [6]. Reversing the argument, if there exists an entropy mode in our Universe, the generation of the baryon asymmetry and/or dark matter must have happened before the hot Big Bang epoch. Hence, the detection of any of the entropy modes would be a strong signal for rather unconventional cosmology.

To end up with the interference, we note that the effect is proportional to $\rho_B$ and hence $\Omega_B$. The same holds for the third, nonoscillating term in (44), which gives particularly strong contribution in the region of the first peak.

In this way the CMB temperature anisotropy is very sensitive to $\rho_B/\rho_\gamma$ and, therefore, to the baryon asymmetry parameter $\eta_B = n_B/n_\gamma$ and $\Omega_B$. This is shown in Fig. 7. The CMB result for the latter is:

$$\Omega_B = 0.045 \pm 0.003.$$  (49)

Notably, it is consistent with the determination of $\eta_B$ from Big Bang Nucleosynthesis.

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9The argument is, in fact, even more general: the adiabatic mode is the only mode also in the Universe where particles of all types are decay products of one and the same field, like inflaton in the inflationary scenario.
4 Baryon acoustic oscillations

We now digress from the discussion of CMB and introduce one of the independent methods of measuring combinations of cosmological parameters. This method has to do with a peculiar property of matter density distribution in the Universe.

At the time right after recombination, the total matter density contrast (dark matter and baryons) is

$$\frac{\delta \rho}{\rho} = \frac{\delta \rho_B + \delta \rho_{DM}}{\rho_B + \rho_{DM}} = \frac{\Omega_B}{\Omega_M} \frac{\delta \rho_B}{\rho_B} + \frac{\Omega_{DM}}{\Omega_M} \frac{\delta \rho_{DM}}{\rho_{DM}}.$$  \hfill (50)

As we have seen above, right after recombination $\delta \rho_B$ oscillates as a function of $k$, while $\delta \rho_{DM}$ is a smooth function of $k$. After recombination, the matter density distribution is essentially constant in time (in comoving coordinates), because baryons have decoupled from photons, and the sound speed in the baryon component is essentially zero. Therefore, the power spectrum of the total matter density perturbations today has an oscillating in $k$ part. This phenomenon is called baryon acoustic oscillations (BAO). In the adiabatic mode, the two terms in (50) add up coherently, so there is an interference between them in $(\delta \rho/\rho)^2$. The interference is important, since if it did not exist, the oscillations would be the second order effect in the small parameter $\Omega_B/\Omega_M$. Due to the interference, the oscillations are of the first order in $\Omega_B/\Omega_M$:

$$\left( \frac{\delta \rho}{\rho} \right)^2 = \frac{\Omega_{DM}}{\Omega_M} B(k) + \frac{\Omega_B}{\Omega_M} C(k) \cos(kr_s).$$  \hfill (51)

This effect has been detected in the analysis of the distribution of galaxies in the Universe at rather small redshifts, $z \simeq 0.2$ and $z \simeq 0.35$, see [15].

There is a simple interpretation of the effect. In the adiabatic mode, the overdensities in the baryon–photon medium and in the dark matter are at the same place before horizon crossing. But before recombination the sound speed in baryon–photon plasma is of the order of the speed of light, while the sound speed in dark matter is basically zero. So, the overdensity in baryons generates the outgoing density wave after horizon crossing. This wave propagates until recombination, and then freezes out. On the other hand, the overdensity in the dark matter remains in its original place. The
current distance from the overdensity in dark matter to the front of the baryon density wave equals \(10^{150} \text{ Mpc}\). Hence, there is an enhanced correlation between matter perturbations at this distance scale, which shows up as a feature in the correlation function, see Fig. 8. In the Fourier space, this feature produces oscillations \([15]\).

Since the sound speed in the baryon–photon medium before recombination is well known, these \(150 \text{ Mpc}\) is a robust prediction, so BAO provides a standard ruler at low redshifts. In principle, it can be used to determine the Hubble parameter as function of \(z\) at low \(z\) (if the ruler is directed along the line of sight, its absolute length \(l\) is related to its measured extension in redshift by \(\Delta z = H(z)l\)) and also its angular diameter distance \(D_a(z)\) (if the ruler is normal to the line of sight, the angle at which it is seen is \(\Delta \theta = l/D_a(z)\)). The latter again depends on the expansion history (now at low \(z\)) and spatial curvature. In practice, one determines from observations a combination \([D_a^2(z)/H(z)]^{1/3}\), which is a rather complicated function of \(\Omega_\Lambda\) and \(\Omega_\kappa\). Importantly, this is a different combination as compared to the one best measured in the CMB observations. Hence, BAO are used to lift the degeneracy in parameters inherent in the CMB data, see \([14]\).

5 Primordial perturbations

5.1 Scalar power spectrum and scalar tilt

We have seen in Section 3.2.2 that the oscillations in the CMB angular spectrum can be naturally explained if we assume that the scalar perturbations existed already at the beginning of the hot Big Bang epoch of the cosmological expansion. Also we mentioned that the scalar perturbations are observed to be Gaussian, and thus all information about them is contained in the two-point correlation

\[\xi(z)\]

Figure 8: The correlation function of matter density versus distance \([16]\). (1) - The correlation function in the model with baryons and \(CDM\), (2) - the correlation function in pure \(CDM\)-model without baryons.

\[^{10}\text{We remind that the separation at Fig. 8 is given in }h^{-1}\text{ Mpc, where }h = H/100\text{ km/(s \cdot Mpc)}, H\text{ is a current Hubble parameter. Currently }h \approx 0.7, \text{ so }100\ h^{-1}\text{ Mpc roughly corresponds to }150\ \text{Mpc.}\]
function:

$$\langle R(k)R^*(k') \rangle = \frac{P(k)}{(2\pi)^3} \delta(k - k'), \quad (52)$$

where $R(k)$ parametrizes the value of the initial perturbation, and $P(k)$ is called power spectrum. In the isotropic situation, $P(k)$ depends only on the magnitude of vector $k$, but not on its direction. The mean square of fluctuation is

$$\langle R^2(x) \rangle = \langle \int e^{ikx} R(k) d^3k \int e^{-ik'x} R^*(k') d^3k' \rangle = \int d^3k \frac{P(k)}{(2\pi)^3} = \int_0^\infty \frac{dk}{k} P(k), \quad (53)$$

where $P(k) = k^3 P(k)/(2\pi^2)$ is also called power spectrum. The quantity $P(k)$ measures the contribution of a logarithmic interval of $k$ to the total power of fluctuations. Usually, $P(k)$ is approximated as follows,

$$P_s(k) = A_s \left( \frac{k}{k_s} \right)^{n_s - 1}, \quad (54)$$

where $k_s$ is some fiducial momentum, $A_s$ is the amplitude of the power spectrum at this momentum, and $n_s$ is called the spectral tilt. Subscript $s$ refers to scalar perturbations. In 1960’s, Zel’dovich and Harrison predicted the flat spectrum of perturbations, that is the spectrum with $n_s = 1$. Approximately flat, Harrison–Zel’dovich spectrum is consistent with the inflationary scenario.

CMB data (and also data on the distribution of galaxies in space) are obviously sensitive to both the amplitude $A_s$ and the tilt $n_s$. The amplitude determines the overall magnitude of the CMB temperature anisotropy, while the tilt gives the relative power at high and low multipoles. Both parameters can be measured rather precisely. The current values with $k_s = (500 \text{ Mpc})^{-1}$ are [14]:

$$A_s = (2.46 \pm 0.09) \cdot 10^{-9}, \quad (55)$$
$$n_s = 0.960 \pm 0.014. \quad (56)$$

Note that the Harrison–Zel’dovich value $n_s = 1$ is somewhat disfavored. The quoted values are obtained, however, under the assumption of the absence of tensor perturbations (see below) and exact form [54] of the primordial power spectrum. Relaxing these assumptions makes the allowed intervals of $A_s$ and $n_s$ wider. Note also that $A_s$ characterizes the perturbation squared; the amplitude of perturbation itself is of order

$$\sqrt{A_s} \simeq 5 \cdot 10^{-5}. \quad (57)$$

According to the formula [45], $R(k)$ is a relative amplitude of initial perturbations. Also, $\sqrt{A_s}$ measures $R(k_s)$, so $\sqrt{A_s}$ is approximately equal to the relative CMB temperature fluctuation $\delta T/T$.

### 5.2 Inflation and generation of perturbations

Inflation is fast, nearly exponential expansion of the Universe:

$$a(t) \propto \exp \left( \int H dt \right), \quad H \approx \text{const}. \quad (58)$$

Inflation occurs if the Universe is filled with a scalar field $\varphi$ (inflaton) which has nonvanishing scalar potential $V(\varphi)$. The homogeneous field $\varphi$ satisfies the equation

$$\ddot{\varphi} + 3H \dot{\varphi} = -\frac{dV}{d\varphi}. \quad (59)$$
If $H$ is large (the Universe expands rapidly) and $dV/d\varphi$ is small, the field $\varphi$ varies slowly in time. The Friedmann equation in this case is $H^2 = 8\pi G V(\varphi)/3$, so if $\varphi$ varies slowly, then $V(\varphi)$ and thus $H$ also vary slowly. This gives a self-consistent mechanism of inflation.

The perturbations about the homogeneous field $\varphi$ evolve according to the same equation as in Section 3.2.2 (one can show that the effect of $V(\varphi)$ on $\delta \varphi$ is negligible):

$$\ddot{\delta \varphi} + 3H \dot{\delta \varphi} + \frac{k^2}{a^2} \delta \varphi = 0. \tag{60}$$

Recall that if $k/(aH) \gg 1$, the mode is sub-horizon, and if $k/(aH) \ll 1$ the mode is super-horizon. In the inflationary regime one has $a(t) \propto \exp(HT)$, $H$ is almost constant, and $k/(aH)$ decreases in time. A mode of a fixed $k$ is sub-horizon at the beginning of inflation. Then it exits the horizon, and the mode’s evolution terminates. Note that this sequence of events is reverse as compared to what happens in the matter-dominated or radiation-dominated Universe. So, to find the perturbations in the Universe towards the end of inflation, one needs to calculate the inflaton perturbation at the moment of the horizon crossing\footnote{Note that the meaning of the moments denoted by the symbol $\times$ is different in this Section and in Section 3.2.2. A given mode crosses the horizon at least twice, for the first time at inflation (it turns from sub-horizon to super-horizon), and for the second time after inflation (it becomes sub-horizon again). In this Section $\times$ denotes the first crossing, and in Section 3.2.2 the second one.}. There are two quantities of the right dimension: $k/a$ and $H$. But at the time of the horizon exit one has $k/a = H$. So, by dimensional argument, one obtains that at the time of the horizon exit

$$\delta \varphi_x \sim H, \tag{61}$$

and $\delta \varphi_x$ is almost independent of $k$. This value remains intact until the end of inflation.

After inflation ends, the energy stored in the inflaton field gets converted into heat, and the Universe enters the hot Big Bang epoch. The inflaton perturbations get reprocessed into scalar perturbations (energy density perturbations of the hot gas of particles and gravitational potentials). These are automatically in the adiabatic mode, since the perturbations in all forms of matter have one and the same origin everywhere in space\footnote{In inflationary models with more than one scalar field, the generation of an admixture of entropy modes is also possible.}. Since the parameters in the inflating Universe are almost time-independent in the time interval when all interesting modes exit the horizon, inflation produces nearly Harrison–Zel’dovich, flat spectrum (we have already noticed that $\delta \varphi_x$ is almost independent of $k$). Yet the parameters are not exactly time-independent at inflation, so the predicted spectral tilt ($n_s - 1$) is small but nonzero. It can be positive or negative, depending on the shape of the scalar potential $V(\varphi)$. In particular, it is negative for the simplest power-law potentials like

$$V(\varphi) = \frac{m^2}{2} \varphi^2 \quad \text{or} \quad V(\varphi) = \frac{\lambda}{4} \varphi^4. \tag{62}$$

Finally, the primordial perturbations in the field $\varphi$ have to be small in amplitude to generate small amplitude of the scalar perturbations\footnote{The meaning of the moments denoted by the symbol $\times$ is different in this Section and in Section 3.2.2. A given mode crosses the horizon at least twice, for the first time at inflation (it turns from sub-horizon to super-horizon), and for the second time after inflation (it becomes sub-horizon again). In this Section $\times$ denotes the first crossing, and in Section 3.2.2 the second one.}. Thus, $\delta \varphi_x$ can be treated in the linear approximation. Linear quantum fluctuations of scalar fields are known to be Gaussian random variables, so inflation naturally explains this property of density perturbations. We conclude that the inflationary mechanism of the generation of scalar perturbations is consistent with everything we know about them.

Actually, the derivation of approximately flat spectrum does not depend on whether we deal with scalar or tensor field. So, inflation generates also tensor perturbations (transverse traceless perturbations of spatial metric $h_{ij}$, i.e., gravitational waves). We obtain the same picture for them: primordial tensor perturbations are Gaussian random field with almost flat power spectrum. The standard approximation for primordial tensor perturbations spectrum (omitting the details concerning polarization) is:

$$P_T(k) = A_T \left( \frac{k}{k_*} \right)^{n_T}. \tag{63}$$
Note the flat spectrum corresponds to $n_T = 0$, unlike in (54). It is convenient to introduce the parameter $r = P_T/P_s$ which measures the ratio of tensor to scalar perturbations. The simplest theories of inflation like (62) predict $r \sim 0.1 - 0.3$. The effect of tensor perturbations has not yet been reliably observed, but there is great effort to discover it. The allowed regions in the plane $(n_s, r)$ are shown in Fig. 9. The detection of tensor perturbations is an extremely important test for the simplest inflationary models. The flat primordial spectrum of scalar perturbations is predicted by some theories other than inflation, but at the moment only inflation predicts the flat spectrum of primordial gravitational waves.

The tensor perturbations behave at the hot Big Bang epoch exactly in the same way as massless scalar field studied in Section 3.2.2. These are purely metric perturbations (essentially no perturbations in matter):

$$ds^2 = dt^2 - a^2(t)[\delta_{ij} - h_{ij}(x, t)]dx^idx^j,$$

where $h_{ij}$ is a transverse traceless tensor, $\partial_i h_{ij} = 0; h_{ii} = 0$. Before the horizon re-entry, metric perturbation $h_{ij}$ stays constant in time; after the horizon entry it oscillates and its amplitude decreases as $1/a$. For modes that cross the horizon at radiation domination one has (compare with (25); we omit indices $i, j$)

$$h(k, t) = h_{(i)}(k) \cdot \frac{a(t_x)}{a(t)} \cdot \sin \left( k \int_0^t \frac{dt'}{a(t')} \right),$$

where $h_{(i)}$ is the primordial amplitude (Gaussian random field). The phase of oscillations is different for modes that cross the horizon at matter domination.

For nearly flat primordial power spectrum, the shorter the mode, the earlier it crosses the horizon, the smaller $a(t_x)$, the lower the present amplitude. For this reason it is almost hopeless to detect relic gravity waves directly by existing and planned detectors of gravitational waves. Presently, the best constraints on tensor perturbations come from the analysis of the CMB temperature anisotropy.

Figure 9: The allowed regions of $n_s$ and $r$ [14]. The contours show 68% and 95% CL from WMAP+BAO+SN. The label “HZ” corresponds to the flat, Harrison–Zel’dovich scalar spectrum in the absence of tensor perturbations. Also shown are the predictions of various inflationary models, whose potentials are given on the right. N-flation is a model with numerous inflaton fields.
effect of gravitational waves on the CMB temperature anisotropy is given by:

\[
\frac{\delta T}{T} = \int_{t_r}^{t_0} dt \; \dot{h}_{ij} n^i n^j, \tag{66}
\]

where \(n^i\) is the unit vector along the photon path and the integration is performed along the photon world line. This is nothing but the integrated Sachs–Wolfe effect; other effects we considered in Section 3.2.1 are absent for tensor perturbations. The modes which are super-horizon at recombination \((l \lesssim 50)\) have not changed since inflation, as they have never been sub-horizon after inflation but before recombination. So, their spectrum at recombination is flat, and they generate flat contribution to the CMB temperature angular spectrum at low \(l\). At higher multipoles, shorter gravity waves contribute (recall the relationship (30)), their amplitudes decrease with \(k\), and hence their effect on the CMB temperature decreases with \(l\). This is shown in Fig.10. The oscillations in the angular spectrum are due to the oscillatory behavior in (65).

Clearly, it is hard to disentangle the contributions of scalar and tensor perturbations to the CMB temperature anisotropy. In particular, tensor perturbations and red scalar tilt \((n_s < 1)\) have similar effect of the enhancement of low multipoles (this is the basic reason for the shape of the allowed region in Fig.9). Potentially more powerful way to detect primordial gravity waves is the measurement of the CMB polarization.

![Image](image.png)

Figure 10: The effect of gravity waves on CMB [11].

6 CMB polarization and primordial gravitational waves

6.1 The physical mechanism of polarization

CMB polarization is generated by a simple mechanism: light gets polarized when it scatters off free electrons. Indeed, when the light is scattered off an electric charge, the electromagnetic wave induces the oscillations of the charge in the direction of electric field. Electric field is perpendicular to the direction of the wave propagation. The oscillating charge emits radiation only in directions perpendicular to the direction of its oscillations. So, if the initial light is polarized in the plane formed by the direction of initial propagation and direction of scattered light (scattering plane), smaller amount of radiation is emitted as compared with incoming light polarized normal to the scattering plane. A photon with polarization normal to the scattering plane scatters at the cross-section

\[
\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi}, \tag{67}
\]
and its polarization remains unchanged. If the initial polarization is in the scattering plane, it remains in that plane, and the cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} \cos^2 \theta,$$

(68)

where $\theta$ is the scattering angle. If at some position the flux of photons is anisotropic, the radiation scattered there towards an observer is linearly polarized. The local anisotropy before the last scattering can be caused by both temperature inhomogeneities and gravitational waves. The effect involves the factor $\lambda_\gamma/\lambda(k)$, where $\lambda_\gamma$ is the photon mean free path just before last scattering, and $\lambda(k)$ is the physical wavelength of the perturbation which is responsible for the local anisotropy. So, the estimate for the degree of polarization $P$ induced by temperature inhomogeneities is

$$P = \frac{\lambda_\gamma}{\lambda(k)} \delta T,$$

(69)

For $l \sim 100$ (this roughly corresponds to the horizon at recombination) the degree of polarization is approximately $10^{-6}$.

### 6.2 $E$- and $B$-modes

Consider a polarimeter oriented along a vector $n$ normal to the line of sight. The intensity it measures is given by

$$I_n = \langle (\mathbf{E}n)^2 \rangle = n_a n_b \langle E_a E_b \rangle = \langle E^2 \rangle P_{ab} n_a n_b,$$

(70)

where the polarization tensor is defined as follows:

$$P_{ab} = \frac{\langle E_a E_b \rangle}{\langle E^2 \rangle}.$$

(71)

All averages here are the time averages over the period of the wave. For linear polarization, the tensor $P_{ab}$ is real. Additionally, it is symmetric and satisfies the constraint $\text{Tr}(P) = P_{aa} = 1$. A symmetric 2-tensor has three parameters, and since the polarization tensor satisfies one additional trace constraint, it is characterized by only two parameters. The eigenvectors of this tensor are the directions of maximal and minimal intensity. As is clear from the above discussion, the direction of maximal intensity near a hot spot is perpendicular to the direction towards the spot. Near a cold spot the direction of maximal intensity is the direction towards the spot.

CMB polarization can be visualized by showing the direction of maximal polarization and the degree of polarization $P = \sqrt{1 - 4\det P}$. The actual CMB polarization map is shown in Fig.11.

It is convenient to parametrize the polarization tensor as follows:

$$P_{ab} - \frac{1}{2} \delta_{ab} = -\{\nabla_a \nabla_b \} P_E - \{\varepsilon^c_a \nabla_b \nabla_c \} P_B,$$

(72)

where $\{A_{ab}\} = (A_{ab} + A_{ba} - \delta_{ab} A_{cc})/2$. The modes generated by the potentials $P_E$ and $P_B$ are called $E$-mode and $B$-mode, respectively, in distant analogy with electric and magnetic fields. The $B$-mode is obviously parity-odd, while the $E$-mode is parity-even. Scalar perturbations generate only $E$-mode\textsuperscript{13} while tensor perturbations generate both modes. This is because that parity-odd $B$-mode can not generate parity-even scalar perturbations. Similarly to the expansion of temperature, one expands potentials of CMB polarization in spherical harmonics; omitting $l$-dependent factors, we write

$$P_{E,B}(n) = \sum_{l,m} a_{lm}^{E,B} Y_{lm}(n).$$

(73)

\textsuperscript{13}Some admixture of $B$-mode is generated at small angular scales from $E$-mode via gravitational lensing by structures (clusters of galaxies, etc.) in the late Universe.
Figure 11: CMB polarization map in WMAP K band.

Equivalently, $a^{E,B}_{lm}$ are the expansion coefficients of $P_{ab}$ in parity-even and parity-odd tensor spherical harmonics. Polarization observables are the correlators

$$C^{EE}(l) \propto \langle a^E_{lm}a^E_{lm} \rangle,$$

$$C^{BB}(l) \propto \langle a^B_{lm}a^B_{lm} \rangle.$$  \hspace{1cm} (74)

Another possible correlator, $\langle a^E_{lm}a^B_{lm} \rangle$, vanishes because of its odd parity ($a^E$ are parity-even and $a^B$ are parity-odd). Since CMB polarization is caused predominantly by local anisotropies in the medium, which are generated by the temperature inhomogeneities, one naturally expects that polarization will correlate with the CMB temperature fluctuations. Hence, one introduces the cross-correlator

$$C^{TE}(l) \propto \langle a^E_{lm}a^T_{lm} \rangle.$$  \hspace{1cm} (75)

The cross-correlator $C^{TB}(l) \propto \langle a^B_{lm}a^T_{lm} \rangle$ also vanishes because of the parity argument. In observations, only $C^{TE}$ and $C^{EE}$ correlators were discovered so far, and there exist upper limits on $C^{BB}$, see Fig.12.

Scalar perturbations contribute to polarization mainly because of the motion of the baryon–photon medium. Photons coming along the direction of motion are hotter than photons coming from the opposite direction. This is precisely the Doppler effect that produces local anisotropy before the very last scattering, and hence CMB polarization. It is clear from the discussion in Section 6.1 that this local anisotropy by itself is insufficient for producing CMB polarization (effects due to photons coming from opposite directions would cancel out), so there is still a suppression factor $\lambda_\gamma/\lambda(k)$. For the same reason, the effect of density inhomogeneities on CMB polarization is of the second order in $\lambda_\gamma/\lambda(k)$. In acoustic waves, the phase of oscillations of velocity is shifted by $\pi/2$ from the phase of oscillations of density. The effect of this shift is that the oscillations of $C^{TE}_l$ are shifted by half-period with respect to oscillations in $C^{TT}_l$ (recall that the oscillating part of the CMB temperature anisotropy spectrum $C^{TT}_l$ comes mainly from $\delta\rho_\gamma$, recall also the correspondence \[\text{superscript } TT \text{ in } C_l\]); zeroes of $C^{TE}_l$ should coincide with maxima and minima of $C^{TT}_l$. This can be seen in Fig.13.

The effect of gravitational waves on CMB is much more interesting. At relatively low multipoles, primordial gravitational waves (tensor perturbations) are the only possible source of considerable B-mode of CMB polarization. Thus, the detection of the B-mode will be the detection of primordial gravity waves. Measuring their power spectrum will be of paramount importance for understanding the origin of cosmological perturbations.
Figure 12: Correlators of various polarization potentials and temperature as functions of $l$ [17]. $C^{EE}$ and $C^{BB}_l$ are multiplied by $T_0^2$, and $C^{TE}_l$ is multiplied by $T_0$, so all these quantities have the same dimension as $C^{TT}$. The $BB$-mode has not yet been detected; the lower right panel shows the existing limits and predicted spectrum generated via gravitational lensing by structures in the late Universe in a model without primordial gravitational waves.

As we have already mentioned, the smaller the wavelength of a gravitational wave, the earlier it becomes sub-horizon and the smaller its amplitude at recombination. So, the effect of gravitational waves is small for small wavelengths, and hence for large $l$. On the other hand, the effect of gravitational waves at recombination contains the factor $(\lambda_\gamma/\lambda(k))^2$, so it is small for very large wavelengths and thus for very small $l$. The maximal effect is at $l \sim 50$. We note, however, that the interaction of gravitational waves with light, which leads to CMB polarization, occurs not only at the time of last scattering. Approximately at $z \sim 10$, intergalactic gas was reionized by the light of first very massive stars with $M \sim 100 \, M_\odot$. Tensor modes of wavelengths approximately equal to the size of the horizon at this reionization epoch are most important in this situation. This means that the effect is most pronounced at $l \lesssim 10$. This is seen in Fig. 13, a wide peak at $l \sim 2 - 7$ is precisely due to reionization. The optical depth at reionization (i. e., the probability for a photon to be scattered) is about 0.1, this is the reason for the smallness of the effect.

Special experiments intended to measure the CMB polarization, such as CMBPole, are being prepared. The sensitivity of CMBPole to $r = A_T/A_s$ is approximately 10%, and it is planned to increase the sensitivity to the level of 1% in the next generation experiments.
Summary

To summarize, measurements of the CMB temperature anisotropy and polarization is a powerful tool in cosmology. Combined with other methods, they are capable of determining with high precision the cosmological parameters characterizing the recent Universe and revealing the properties of the primordial perturbations. The current picture of the Universe is quite simple. As to the recent Universe, the data are consistent with the spatially flat $\Lambda$CDM model whose ingredients are time-independent dark energy density, cold dark matter, baryons, electrons, photons and fairly light neutrinos ($m_\nu < 0.2$ eV for each of the neutrino species). Known properties of scalar perturbations are also simple: they were generated before the hot Big Bang epoch, have no decaying super-horizon modes, are adiabatic,
Gaussian and have flat or nearly flat power spectrum. Tensor perturbations have not been discovered, though the current bound on their amplitude is not particularly strong. All these properties are consistent with the inflationary mechanism of the generation of cosmological perturbations, but inflation is not the only option for the moment.

What can one expect to be discovered in future? On the recent Universe side, the major issues are whether the dark energy density is exactly constant in time or not, and what are dark matter particles. CMB alone will not shed much light on these issues, but its analysis will be instrumental in combination with other methods. Concerning the cosmological perturbations, one intrigue is whether the scalar power spectrum is exactly flat or not. Even more interesting are tensor perturbations whose primordial amplitude is predicted by the simplest inflationary models to be quite large; furthermore, the smoking-gun prediction of inflation is nearly flat tensor spectrum. Unexpected discoveries cannot be excluded either, like sizeable non-Gaussianity or admixture of entropy modes in scalar perturbations. All these are the tasks for future CMB experiments, which will thus serve as windows to the extremely early cosmological epoch preceding the hot Big Bang stage.

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