WELL-POSEDNESS OF BOUNDARY VALUE PROBLEMS FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

YUE HE

Abstract. In this paper, we study the well-posedness of boundary value problems for a special class of degenerate elliptic equations coming from geometry. Such problems is intimately tied to rigidity problem arising in infinitesimal isometric deformation. The characteristic form of this class of equations is changing its signs in the domain. Therefore the well-posedness of these above problems deserve to make a further discussion. Finally, we get the existence and uniqueness of $H^1$ solution for such boundary value problems.

1. Introduction.

In this section, we introduce briefly the history of the research to degenerate elliptic equations and some geometric backgrounds to a class of degenerate elliptic equations which we are concerned with. Next we will put forward the main question of this paper. In the last part of this section, we will summarize the main result and its trivial generalization about existence and uniqueness for solution to such a class of equations.

1.1. Historical remarks and backgrounds. In this subsection, we will introduce the brief history and the current status of investigation about the degenerate elliptic equations. We observe the equation

$$Lu \equiv a^{ij}(x)u_{x_i x_j} + b^k u_{x_k} + cu = f \text{ in } \Omega,$$

where $u_{x_k} = \partial u / \partial x_k, u_{x_i x_j} = \partial^2 u / \partial x_i \partial x_j$ etc. and the index $i, j, k$ runs from 1 to $n$ and repeated indices imply summation. (From now on we will use such summation convention throughout this paper). If for any vector $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n, a^{ij}(x)\xi_i \xi_j \geq 0$ for all $x = (x_1, ..., x_n) \in \Omega \subseteq \mathbb{R}^n$. Then $Lu = f$ is called second order PDE with nonnegative characteristic form, and also called second order degenerate elliptic equation, or second order elliptic-parabolic equation in domain $\Omega$. They contain elliptic equation, parabolic equation, one order differential equation, Brown Motion equation and some important equations introduced later.

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The study of degenerate elliptic equations can be traced back to 1910’s, first appeared in Picone’s thesis. After that, the Tricomi’s research report [1] and his research on the mixed-type partial differential equation. Besides, M.V. Keldyš [2]; Fichera; J. Kohn and L. Nirenberg [3]; O.A. Oleı̇nik [4, 5]; E.V. Radkevič etc., they all do many works for laying a foundation in this field. After half century’s development in this field, O. A. Oleı̇nik and E.V. Radkevič, published their classical monograph [6] in 1971. Their monograph summarized the theories developed before 1970’s, and established a general framework for the theories to second order degenerate linear elliptic equations. They stated the existence and uniqueness of weak solution for the general boundary value problem to such equations in $L^p$ space and some Hilbert spaces, and made a certain further contribution to the regularity theories of weak solutions for the general boundary value problems to such equations. During the past three decades, several progresses have been made in the research of second order degenerate linear elliptic equations. But we only enumerate the partial well-known and representative work at here. For instance, L. Caffarelli, L. Nirenberg and J. Spruck [8] had studied the degenerate Monge–Ampère equation; H. Brezis and P.L. Lions [10] had studied the Yang–Mills equation $-x^2 \Delta u + 2u = f(u)$ describing gauge fields; E.B. Fabes, C.E. Kenig and D. Jerison [11] had studied the general degenerate elliptic equation with the divergence form $\partial_i (a_{ij}(x) \partial_j u) = f(x)$ and so on.

A geometric background. Fanghua Lin [9] had studied the following Dirichlet problem for minimal graphs in hyperbolic space

$\begin{align*}
\begin{cases}
\Delta f - \frac{f_i f_{ij}}{1 + |df|^2} f_{ij} + \frac{n}{f} = 0 & \text{in } \Omega, \\
f > 0 & \text{in } \Omega, \\
f|_{\partial \Omega} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}

(1.1)$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open domain; $|df|^2 = \sum_{i=1}^{n} f_i^2$, $f_i = f_{x_i}$ for $0 \leq i \leq n$. Obviously, by direct computation we know that the equation (1.1) is just the Euler-Lagrange equation of the variational integral

$$A[f, K] = \int_K f^{-n} \sqrt{1 + |df|^2} dx$$

for all compact subsets $K$ of $\Omega$. Since $\text{graph}(f)$ is a $C^{1,\alpha}$-manifold with boundary in $\mathbb{R}^{n+1}$, and since the tangent planes of $\text{graph}(f)$ along the boundary $\partial \Omega$ are vertical, we view $\text{graph}(f)$ near a point at $\partial \Omega$ as a graph over such a vertical plane. This is equivalent to the hodograph transformation of (1.1). Then the corresponding P.D.E. for the function $u$ which represents the $\text{graph}(f)$ is

$\begin{align*}
\begin{cases}
y(\Delta u - \frac{u_i u_{ij}}{1 + |du|^2} u_{ij}) - nu_y = 0 & \text{in } B_1^+(0), \\
u(x, 0) = \varphi(x) & \text{for } |x| \leq 1,
\end{cases}
\end{align*}

(1.2)$
where \( B^+_1(0) = \{(x,y) \in \mathbb{R}^n_+ : |x| \leq 1, 0 \leq y \leq 1\} \). This is a degenerate quasilinear elliptic equation. Fanghua Lin shows the solutions \( u \) of (1.2) are as smooth as \( \varphi \) in \( B^+_{1/2}(0) \), and use this fact to prove the following result:

**Theorem.** If \( \partial \Omega \) is of class \( C^{k,\alpha} \), then graph(\( f \)) is a \( C^{k,\alpha} \) hypersurface with boundary for either (1) \( 1 \leq k \leq n-1 \) and \( 0 \leq \alpha \leq 1 \) or (2) \( n \leq k \leq \infty \) and \( 0 < \alpha < 1 \).

**Another geometric background.** Recently, ones come into contact with a class of second order degenerate elliptic equations when they study the rigidity problem arising in infinitesimal isometric deformation. We shall simply introduce some geometric backgrounds about the above equations in the following. The details can be found in [12], [14], [16] and [17].

Given a metric \( g \) with smooth positive curvature \( K \) on the closed unit disk \( \overline{D} \). In the sequel we always denote it by \( (\overline{D},g) \). In terms of local coordinates system \((u^1,u^2)\) on \( \overline{D} \), the metric \( g \) can be expressed as \( g_{ij} du^i du^j \).

Suppose that \( \vec{r} = (x,y,z) \) is a smooth isometric embedding of \( (\overline{D},g) \) into \( \mathbb{R}^3 \). and the boundary \( \vec{r}(\partial D) \) is a \( C^2 \) planar convex curve. By the Gauss equations we have in a local coordinate system,

\[
\vec{r}_{ij} = \Gamma^k_{ij} \vec{r}_k + \Omega_{ij} \vec{n} \quad \text{or} \quad \nabla_{ij} \vec{r} = \Omega_{ij} \vec{n}, \quad (i,j,k = 1,2),
\]

where subscripts \( i,j \) and \( \nabla_{ij} \) denote Euclidean and covariant derivatives respectively, \( \Omega_{ij} \) the coefficients of the second fundamental form, \( \Gamma^k_{ij} \) the Christoffel symbols with respect to the metric and \( \vec{n} \) the unit normal to \( \vec{r} \).

For each unit constant vector, for instance, the unit vector \( \vec{k} \) of the \( z \) axis, taking the scale product of \( \vec{k} \) with the two hind sides of (1.3) and using the Gauss equations one can get

\[
\det(\nabla_{ij} z) = K \det(g_{ij})(\vec{n},\vec{k})^2 \quad (i,j = 1,2),
\]

where \( K \) is Gaussian curvature. Notice that

\[
(\vec{n},\vec{k})^2 = 1 - \left( \frac{(\vec{r}_1 \times \vec{r}_2) \times \vec{k}}{||\vec{r}_1 \times \vec{r}_2||} \right)^2 = 1 - g^{ij}z_iz_j = 1 - |\nabla z|^2,
\]

where \( \nabla z = (g^{1l}z_l, g^{2l}z_l) \) is the gradient of \( z \). Inserting the last expression into (1.3), we deduce the Darboux equation

\[
F(z) = \det(\nabla_{ij} z) - K \det(g_{ij})(1 - |\nabla z|^2) = 0.
\]

Obviously each component of \( \vec{r} \) satisfies the Darboux equation (1.5).

Given a smooth surface \( \vec{r} \) in \( \mathbb{R}^3 \) one consider its deformation \( \vec{r}_t : (-\varepsilon,\varepsilon) \ni t \to \mathbb{R}^3 \) with \( \vec{r}_0 = \vec{r} \). If \( t = 0 \) is a critical point of the metric \( g(t) = dr_t^2 \), we say that the derivative with respect to \( t \) of \( \vec{r}_t \) at \( t = 0 \) give rise a first order infinitesimal isometric deformation of \( \vec{r} \). Denoting this infinitesimal isometric deformation by \( \vec{d} = (d\vec{r}_t/dt)(0) \), and we call it the first order infinitesimal deformation vector, or the first order deformation vector. So we have

\[
\frac{d}{dt}(dr_t^2)|_{t=0} = (d\vec{r},d\vec{d}) = 0.
\]
Obviously, any rigid body motion of \( \vec{r} \), \( \vec{r} = \vec{A} \times \vec{r} + \vec{B} \) for arbitrary two constant vectors \( \vec{A} \) and \( \vec{B} \), is always a solution of (1.6) and such solutions are called trivial ones. We say that \( \vec{r} \) is of infinitesimal rigidity for first order isometric deformation if (1.6) has no nontrivial solution. As is well known, for closed surface now we only know that closed \( C^2 \) convex surfaces are infinitesimally rigid. For a surface \( \vec{r} \) with boundary, usually it is not infinitesimally rigid if there is no restriction to the deformation on the the boundary of \( \vec{r} \). Therefore we must impose some condition, for instance, 

\[
(\vec{r}, \vec{k}) = 0 \text{ on } \partial D,
\]

where \( \vec{k} \) is the unit vector of \( z \) axis.

Let us consider an infinitesimal isometric deformation of surface \( \vec{r} \), \( \vec{r}_\epsilon = \vec{r} + \epsilon \vec{r} \) where \( \vec{r} = (\xi, \eta, \zeta) \) satisfies (1.6). Notice that \( \epsilon = 0 \) is the critical point of \( g_\epsilon = d\vec{r}^2 \) and hence, the differentiation of its Gaussian curvature \( K(\epsilon) \) and Christoffel symbols \( \Gamma^k_{ij}(\epsilon) \) (i.e. connection coefficients) in \( \epsilon \) are equal to zero at \( \epsilon = 0 \). Then differentiation of the Darboux equation (1.5) for \( z + \epsilon \zeta \) with respect to \( \epsilon \), letting \( \epsilon = 0 \), gives

\[
F^{ij}(z) \nabla_{ij} \zeta + 2K \det(g_{ij}) (\nabla z, \nabla \zeta) = 0,
\]

where \( F^{ij}(z) = \partial \det(\nabla_{kl})/\partial \nabla_{ij} z \) (i, j = 1, 2) is the algebraic cofactors of \( \nabla_{ij} z \).

Since \( \nabla_{ij} \vec{r} = \vec{r}_{ij} - \Gamma^k_{ij} \vec{r}_k = \Omega_{ij} \vec{n} \) (i, j = 1, 2), we have \( \nabla_{ij} z = z_{ij} - \Gamma^k_{ij} z_k = \Omega_{ij}(\vec{n}, \vec{k}) \) (i, j = 1, 2). So (1.7) can be written as follows

\[
(\vec{n}, \vec{k}) \Omega^{ij} \nabla_{ij} \zeta + 2(\nabla z, \nabla \zeta) = 0.
\]

where \( (\Omega^{ij}) = (\Omega_{ij})^{-1} \) is the inverse of matrix \( (\Omega_{ij}) \).

Obviously, the \( (\Omega^{ij}) \) is a positive definite matrix if Gaussian curvature \( K \) is positive. (1.8) is of nonnegative characteristic form in the subdomain \( \{ (u^1, u^2) \in D : (\vec{n}, \vec{k}) \geq 0 \} \), is of nonpositive characteristic form in the subdomain \( \{ (u^1, u^2) \in D : (\vec{n}, \vec{k}) \leq 0 \} \), and is one order PDE in the subdomain \( \{ (u^1, u^2) \in D : (\vec{n}, \vec{k}) = 0 \} \). Therefore, (1.8) is characteristic degenerate, and its characteristic form changing sign in domain \( D \).

The spherical crown is an example of this aspect (see (14)). Let us consider a spherical crown \( \Sigma_\lambda = \{ x^2 + y^2 + z^2 = 1 : z \leq 0 \} \). In spherical coordinates

\[
\Sigma_\lambda = \{ (\sin \theta \cos \phi, \sin \theta \sin \phi, -\cos \theta) \mid 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \theta_\ast = \arccos(-\lambda) \},
\]

where \( \lambda \) is a positive constant, and \( \theta = 0 \) stand for the South pole. \( \Sigma_\lambda \) is the isometric embedding of the metric \( g = d\theta^2 + \sin^2 \theta d\phi^2, 0 \leq \theta \leq \theta_\ast \). Since \( \lambda > 0 \), so \( \Sigma_\lambda \) contains the below hemisphere. In the present case, (1.8) may be written as follows

\[
\cos \theta \left[ (\sin \theta \cos \phi)_{\theta} + \left( \frac{\zeta_{\phi}}{\sin \theta} \right) \phi \right] + 2 \sin^2 \theta \zeta_{\theta} = 0, \quad \theta \in (0, \theta_\ast).
\]

with the constraint condition in here as follows:

\[
\zeta = 0 \text{ on } \theta = \theta_\ast \text{ and } \zeta \text{ is bounded near } \theta = 0.
\]
Evidently (1.9) is elliptic as \( \theta \neq \pi/2 \). We ought to show that: if \( \theta_* > 0 \), then (1.9) is of nonnegative characteristic form as \( 0 \leq \theta \leq \pi/2 \), and is of nonpositive characteristic form as \( \pi/2 \leq \theta \leq \theta_* \).

In the same way, one may consider another infinitesimal isometric deformation of surface \( \vec{r} \) in the following:

\[
\vec{r}_\epsilon = \vec{r} + \epsilon \vec{\tau}_1 + \epsilon^2 \vec{\tau}_2 + \cdots.
\]

Denote \( \vec{\tau}_1 = (\xi_1, \eta_1, \zeta_1) \), \( \vec{\tau}_2 = (\xi_2, \eta_2, \zeta_2) \), and so on. We respectively call \( \vec{\tau}_1 \), \( \vec{\tau}_2 \) the first order deformation vector, the second order deformation vector, etc.

Obviously, \( g_\epsilon = (d\vec{r}_\epsilon, d\vec{r}_\epsilon) = (d\vec{r}, d\vec{r}) + 2\epsilon (d\vec{r}, d\vec{\tau}_1) + \epsilon^2 [2(d\vec{r}, d\vec{\tau}_2) + (d\tau_1, d\tau_1)] + O(\epsilon^3) \). If \( g_\epsilon = (d\vec{r}, d\vec{r}) + O(\epsilon^3) \), then \( \vec{\tau}_1 \) and \( \vec{\tau}_2 \) should satisfy the following systems:

\[
\begin{align*}
(d\vec{r}, d\vec{\tau}_1) &= 0, \\
(d\vec{r}, d\vec{\tau}_2) &= -\frac{1}{2}(d\tau_1, d\tau_1).
\end{align*}
\]

We may analogously definite and discuss the rigidity of the second order, even higher order infinitesimal isometric deformation of surface \( \vec{r} \). But in here we only show that \( g_\epsilon \) gives rise to a second order infinitesimal isometric deformation of \( g \) if \( g_\epsilon \) is equal to \( g \) up to second order, i.e.

\[
(dg_\epsilon/de)|_{\epsilon=0} = (d^2g_\epsilon/de^2)|_{\epsilon=0} = 0.
\]

Consequently,

\[
\frac{d\Gamma^k_{ij}(\epsilon)}{d\epsilon}|_{\epsilon=0} = \frac{d^2\Gamma^k_{ij}(\epsilon)}{d\epsilon^2}|_{\epsilon=0} = \frac{dK(\epsilon)}{d\epsilon}|_{\epsilon=0} = \frac{d^2K(\epsilon)}{d\epsilon^2}|_{\epsilon=0} = 0.
\]

And then, Then two order derivative of the Darboux equation (1.5) for \( z + \epsilon \zeta \) with respect to \( \epsilon \), letting \( \epsilon = 0 \), gives

\[
(\vec{n}, \vec{k}) \Omega_{ij} \nabla_{ij} \zeta_2 + 2(\nabla z, \nabla \zeta_2) = -\frac{\det(\nabla_{ij} \zeta_1)}{\det(\Omega_{ij})} - |\nabla \zeta_1|^2,
\]

where \( \zeta_1 \) and \( \zeta_2 \) respectively is the third component of \( \vec{\tau}_1 \) and \( \vec{\tau}_2 \).

**Remark 1.1.** (1.8) is the linearization with respect to (1.5). In addition, (1.8), (1.10) and the linearization with respect to (1.2) all are belong to the same kind of degenerate elliptic equation.

The degenerate elliptic equations we shall study is very closely related to rigidity problems arising from infinitesimal isometric deformation, as well as other geometry problem, such as minimal surface in hyperbolic space, etc. In particular, the existence of solutions with high order regularity is very important to investigate many geometry problems. One would like to know under what conditions the solution of such equations are as smooth as the given data. The theory on well-posedness and regularity of solutions to such equations, plays a crucial role in the above fields. Anyway, such equations are deserved to be investigated vastly. However, so far such equations might not be able to be treated by any standard methods. Therefore maybe they will stimulate a general study of linear, semilinear, quasilinear, and fully nonlinear degenerate elliptic equations.
1.2. **The main question.** The present paper is devoted to investigate the well-posedness of boundary value problems for a special class of degenerate linear elliptic equations with previous geometric backgrounds. The aim of this subsection is to bring up the main question of this paper. We start with a few definitions and introduce notation and terminology that is consistent throughout this paper.

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with smooth boundary $\partial \Omega$. And $\Omega$ is divided into two subdomains by a smooth closed curve $\Gamma$. One of the subdomains is called interior subdomain which is a simply-connected one of $\Omega$. We denote it by $\Omega^+$, i.e. $\Omega^+ \subset \subset \Omega$. Another is denoted by $\Omega^- := \Omega \setminus \overline{\Omega^+}$, which is connected. The boundary of $\Omega^+$ is denoted by $\partial \Omega^+$. In addition, let $\varphi$ be a function in $\Omega$, and set $\Gamma := \partial \Omega^+ = \{ (\xi_1, \xi_2) \in \Omega \mid \varphi(\xi_1, \xi_2) = 0 \}$, $\Omega^+ := \{ (\xi_1, \xi_2) \in \Omega \mid \varphi(\xi_1, \xi_2) > 0 \}$, $\Omega^- := \Omega \setminus \overline{\Omega^+} = \{ (\xi_1, \xi_2) \in \Omega \mid \varphi(\xi_1, \xi_2) < 0 \}$, where $\varphi$ is called the definition function of $\Gamma$. Moreover, we suppose $\nabla \varphi \neq 0$ on $\Gamma$, and denote the inward normal direction to the boundary $\partial \Omega^+$ by $\bar{n}$. Obviously,

$$\bar{n} = (n_1, n_2) = \left( \frac{\varphi_{\xi_1}}{|\nabla \varphi|}, \frac{\varphi_{\xi_2}}{|\nabla \varphi|} \right) \big|_{\Gamma} \neq 0,$$

Consider

$$Lu \equiv \varphi (A^{ij} u_{\xi_i \xi_j} + Cu) + B^l u_{\xi_l} \quad \text{in } \Omega \subset \mathbb{R}^2_+,$$

where

$$\varphi, A^{ij}, B^l, C \in C^\infty(\overline{\Omega}) \quad \text{for } i, j = 1, 2.$$

Assume that

$$A^{ij} \eta_i \eta_j \geq \lambda_0 |\eta|^2 \quad \text{for all } \eta = (\eta_1, \eta_2) \in \mathbb{R}^2, \quad (i, j = 1, 2),$$

where $\lambda_0$ is a positive constant;

$$B^l \varphi_{\xi_l} < 0 \quad \text{on } \Gamma;$$

and

$$C \leq 0 \quad \text{for all } (\xi_1, \xi_2) \in \overline{\Omega}.$$  

Obviously, from the above assumptions it is easy to see that

$$\left. (A^{ij} \varphi_{\xi_i} \varphi_{\xi_j}) \right|_{\Gamma} \geq \lambda_0 (|\nabla \varphi|^2) \big|_{\Gamma} > 0.$$

Assume that $F \in L^2(\Omega)$, $g \in H^2(\Omega)$, we shall discuss the well-posedness of the following boundary value problem (Abbreviation: BVP)

\[
\begin{cases}
Lu = F & \text{in } \Omega, \\
u = g & \text{on } \Gamma, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Now we state the definition of weak solution in the following
Definition 1.1. Assume that $F \in L^2(\Omega), g \in H^2(\Omega)$. $A : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ a continuous bilinear form, is defined by

$$A(u, v) = \int_\Omega \left[ -u_{ij}(A_{ij} \varphi v)_{ij} + \varphi Cu v + B^l u^l v \right] d\xi$$

for all $u, v \in H^1_0(\Omega)$.

If $u \in H^1_0(\Omega)$ satisfies that

$$\begin{cases}
A(u, v) = (F, v) & \text{for all } v \in H^1_0(\Omega), \\
u = g & \text{on } \Gamma,
\end{cases}$$

where the boundary value is to be interpreted in the sense of traces. Then $u$ is called $H^1$ weak solution of BVP (1.16).

The question

Is there a $H^1$ solutions of BVP (1.16) and unique is such solution?

is unknown as the well-posedness of the boundary value problem for degenerate elliptic equations.

Throughout this paper we will utilize such Convention: (1) The $C$ that appearing in paper, all express positive bounded constant. But they are possibly different when they are appearing in different rows. (2) We often use "$\rightharpoonup"$ and "$\to"$ expressing respectively weak convergence and strong convergence in the corresponding function spaces.

1.3. The main result and its trivial generalization. The main result of this paper is the following

Theorem 1.1. Suppose $\varphi, A_{ij}, B^l (i, j, l = 1, 2)$, and $C$ satisfy the conditions (1.11), (1.12), (1.13), and (1.14). Let $F \in L^2(\Omega)$ and $g \in H^2(\Omega)$. Then there exists an unique $H^1$ weak solution $u$ of the BVP (1.16), and $u$ satisfies

$$\|u\|_{H^1(\Omega)} \leq C\left[ \|F\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)} \right],$$

where $C$ is a constant depending only on $\Gamma$, $\|\varphi\|_{C^2(\Omega)}$, $\|A_{ij}\|_{C^2(\Omega)}$, $\|B^l\|_{C^1(\Omega)} (i, j, l = 1, 2)$ and $\|C\|_{C^1(\Omega)}$.

Remark 1.2. The BVP (1.16) can be discussed in $\mathbb{R}^{n+1}$ under the same conditions. And the similar result also can be obtained by the same methods in the case $\mathbb{R}^{n+1}$. Since all the generalization is trivial, we omit to state this result and the details of its proof at here.

Remark 1.3. We explain briefly some known facts about the works of Oleinik and Radkevič (for details, to see [6]). Their theory requires that the characteristic form of the equation is non-negative in the global domain. But the characteristic form of equations in the problems which we deal with is changing its signs in the domain. Next the theory of Oleinik and Radkevič requires that the coefficient of unknown function term for the equation is negative enough. It is usually not provided with this condition in the practical problems. Hence we could not get the $L^2$ solution from the direct
applications of their conclusion to our problems. This is the difficulties in our problems.

Of course the regularity of solutions plays an important role in the study of geometry problems. So we need to make a further discussion on the corresponding regularity of solutions to such problems. The further results on regularity will be given in our preprint paper [25]. In spite of many relevant progress, up to now there has been no standard way to deal with such kind of problems, and some crucial problems remain unsolved. Therefore, maybe our methods are helpful in studying the general degenerate elliptic equations.

2. Preliminaries.

2.1. Homogenization of the BVP (1.16). Suppose $g$ may be extend to domain $\Omega$, still be denoted by $g$, such that $g$ satisfies $g = 0$ on $\partial \Omega$. Without loss of generality we may assume that $g \equiv 0$. In fact, if $u$ is a solution of BVP (1.16), let $v = u - g$, then $v$ is a solution of the following BVP:

$$\begin{cases}
Lv = F - Lg & \text{in } \Omega, \\
v = 0 & \text{on } \Gamma \cup \partial \Omega.
\end{cases}$$

Hence, instead of the primary BVP (1.16) we may discuss the well-posedness of the following BVP:

$$\begin{cases}
Lu = F & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma \cup \partial \Omega.
\end{cases}$$

2.2. Simplification of the form to $Lu = F$. Firstly, we will simplify the form of the equation $Lu = F$ in some neighborhood of $\Gamma$. By a appropriate transformations of the variable in a neighborhood of $\Gamma$, we get the following result:

**Lemma 2.1.** By a appropriate transformation $\Phi$ of the variable in some neighborhood $N_0(\Gamma)$ of $\Gamma$, $Lu = F$ can be translated to the following form:

$$Lu \equiv y(\omega u_{xx} + u_{yy} + cu) + au_x + bu_y = f, \quad \text{on } D_{d_0},$$

where

$$D_{d_0} = \Phi(N_0(\Gamma)) = \{(x, y) \mid -\pi \leq x \leq \pi, -d_0 < y < d_0\},$$

and $\omega, a, b, c, f$ are all periodic functions with period $2\pi$ on $x$. $\omega, a, b, c \in C^\infty(D_{d_0})$.

**Proof.** We might as well suppose that $\Gamma$ may be expressed as follow

$$\Gamma = \{(\nu_1(s_1), \nu_2(s_1)) \mid 0 \leq s_1 \leq l\},$$

where $s_1$ is the parameter of arc length, and $l$ is the length of $\Gamma$. Obviously,

$$\dot{\nu}_1^2 + \dot{\nu}_2^2 = 1 \quad \text{on } \Gamma,$$
where "\(\cdot\)" means the derivative with respect to \(s_1\). Then there exists a neighborhood \(N_0(\Gamma)\) of \(\Gamma\), such that any \(\xi = (\xi_1, \xi_2) \in N(\Gamma)\) may be express as

\[
\begin{align*}
\xi_1 &= \nu_1(s_1) + n_1(s_1)s_2, \\
\xi_2 &= \nu_2(s_1) + n_2(s_1)s_2,
\end{align*}
\]

where \(\bar{n} = (n_1, n_2)\) is the inward normal direction of \(\Omega_+\). Meanwhile it is also the exterior normal direction of \(\Omega_-\). In \(N_0(\Gamma)\), we know easily that \(\xi = (\xi_1, \xi_2) \in \Gamma\) for \(s_2 = 0\). On the other hand, since \((\nu_1, \nu_2)\) is the unit tangent direction of \(\Gamma\), so \((\nu_2, -\nu_1)\) is the inward normal direction of \(\Gamma\). Hence \((\nu_2, -\nu_1) = (n_1, n_2)\).

Clearly,

\[
\frac{D(\xi_1, \xi_2)}{D(s_1, s_2)}|_{s_2=0} = \det \left( \begin{array}{c}
\nu_1 + \tilde{n}_1s_2 \\
\nu_2 + \tilde{n}_2s_2
\end{array} \right)_{s_2=0} = [\nu_1n_2 - \nu_2n_1 + (\nu_1\nu_2 - \nu_1\nu_2)s_2]_{s_2=0} = n_1^2 + n_2^2 = 1.
\]

Thus by the inverse function theorem, for \(|s_2| \leq \delta\), there exists a sufficient small constant \(\delta > 0\), such that \(s_1, s_2\) are smooth functions on \(\xi_1, \xi_2\). We express \(s_1, s_2\) as follow

\[
\Xi : \begin{cases}
s_1 = s_1(\xi_1, \xi_2), \\
s_2 = s_2(\xi_1, \xi_2).
\end{cases}
\]

Applying (2.3) again, we have

\[
\begin{align*}
\frac{\partial s_1}{\partial \xi_1} &= n_2/\triangle, \\
\frac{\partial s_2}{\partial \xi_1} &= -(\nu_2 + n_2s_2)/\triangle, \\
\frac{\partial s_1}{\partial \xi_2} &= -n_1/\triangle, \\
\frac{\partial s_2}{\partial \xi_2} &= (\nu_1 + n_1s_2)/\triangle,
\end{align*}
\]

where \(\triangle = \nu_1n_2 - \nu_2n_1 + (\tilde{n}_1n_2 - \tilde{n}_1\tilde{n}_2)s_2\).

In the sequel, by the transform \(\Xi\) and direct calculation, the equation \(Lu = F\) become

\[
\varphi(\tilde{A}^{ij}u_{s_i,s_j} + Cu) + \tilde{B}^ku_{s_k} = F,
\]

where

\[
\tilde{A}^{ij} = A^{ij} = A^{lr} \frac{\partial s_l}{\partial \xi_i} \frac{\partial s_r}{\partial \xi_j}, \quad \tilde{B}^k = \varphi A^{ij} \frac{\partial^2 s_k}{\partial \xi_i \partial \xi_j} + B^l \frac{\partial s_k}{\partial \xi_l}.
\]

Obviously, the transform \(\Xi : N(\Gamma) \rightarrow \Xi(N(\Gamma))\), and makes \(\Gamma\) to become

\[
\Xi(\Gamma) = \{ (s_1, s_2) \mid 0 \leq s_1 \leq l, s_2 = 0 \}.
\]

So, we have

\[
\varphi(s_1, s_2) = s_2\bar{\varphi},
\]

where

\[
\bar{\varphi}(s_1, s_2) = \int_0^1 \varphi_{s_2}(s_1, \varsigma s_2) d\varsigma,
\]
and
\[ \bar{\varphi}(s_1, 0) = \varphi_{s_2}(s_1, 0) \]
\[ = \varphi_{\xi_1}\Big|_\Gamma \frac{\partial \xi_1}{\partial s_1} \big|_{s_2=0} + \varphi_{\xi_2}\Big|_\Gamma \frac{\partial \xi_2}{\partial s_1} \big|_{s_2=0} \]
\[ = (\varphi_{\xi_1} n_1 + \varphi_{\xi_2} n_2)\big|_\Gamma \]
\[ = \left\{ \sqrt{\varphi_{\xi_1}^2 + \varphi_{\xi_2}^2} \left( \frac{\varphi_{\xi_1}}{\sqrt{\varphi_{\xi_1}^2 + \varphi_{\xi_2}^2}} n_1 + \frac{\varphi_{\xi_2}}{\sqrt{\varphi_{\xi_1}^2 + \varphi_{\xi_2}^2}} n_2 \right) \right\}\big|_\Gamma \]
\[ = \left( \frac{\varphi_{\xi_1}^2 + \varphi_{\xi_2}^2}{\sqrt{\varphi_{\xi_1}^2 + \varphi_{\xi_2}^2}} \right)\big|_\Gamma (n_1^2 + n_2^2) \]
\[ = (|\nabla \varphi|)\big|_\Gamma \neq 0. \]

Then there exists some neighborhood \( U \subseteq \Xi(\mathcal{N}(\Gamma)) \), such that \( \bar{\varphi} \neq 0 \) in \( U \).

Transforming the equation \( Lu = F \) again, by \( \Theta : \)
\[ \begin{align*}
  x_1 &= \psi(s_1, s_2), \\
  x_2 &= s_2,
\end{align*} \]
where \( \psi \) is a undetermined function. Then the form of the equation \( Lu = F \) is translated from (2.5) into the following form:
\[ x_2 \bar{\varphi} \left[ \left( A^{ij} \frac{\partial u_{x_{1x_2}}}{\partial x_i} + \frac{\partial^2 u_{x_1}}{\partial s_1 s_j} + C^i \frac{\partial^2 u_{x_2}}{\partial s_i s_k} + \bar{B}^i \frac{\partial u_{x_2}}{\partial s_k} \right) u_{x_i} = F. \]

By simplification, we obtain
\[ x_2 \bar{\varphi} \left[ \left( A^{ij} \psi_{s_i} \psi_{s_j} \right) u_{x_1} + \left( A^{12} \psi_{s_i} \right) u_{x_1x_2} + \bar{A}^{22} u_{x_2x_2} + C u \right] \]
\[ + (x_2 \bar{A}^{ij} \psi_{s_i} \psi_{s_j} + \bar{B}^i \psi_{s_i}) u_{x_1} + \bar{B}^2 u_{x_2} = F. \]

In order to delete the mixed derivative term \( (\bar{A}^{12} \psi_{s_1}) u_{x_1x_2} \), we consider the Cauchy problem
\[ (2.9) \]
\[ \begin{align*}
  &\bar{A}^{12} \psi_{s_1} + \bar{A}^{22} \psi_{s_2} = 0, \\
  &\psi(s_1, 0) = s_1.
\end{align*} \]

Together with (2.4), we have
\[ \bar{A}^{22} = \begin{pmatrix} n_2, -n_1 \end{pmatrix} \begin{pmatrix} A^{22} & -A^{12} \\ -A^{12} & A^{11} \end{pmatrix} \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} > 0, \quad \text{for } s_2 = 0. \]

Therefore,
\[ \begin{pmatrix} \bar{A}^{12}, \bar{A}^{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{A}^{22} > 0 \quad \text{on } \Gamma. \]

On one hand, according to the theory of one order PDE, there exists an unique solution \( \psi^* \) of the Cauchy problem (2.9) in a neighborhood \( V (\subseteq U) \).
of \( \{0 \leq s_1 \leq l, s_2 = 0\} \). Obviously, \( \psi^* (s_1, s_2) + l \) is a solution of the Cauchy problem

\[
\begin{aligned}
\tilde{A}^{12} \psi_{s_1} + \tilde{A}^{22} \psi_{s_2} &= 0, \\
\psi(s_1, 0) &= s_1 + l.
\end{aligned}
\]

(2.10)

On the other hand, since \( \tilde{A}^{12} \) and \( \tilde{A}^{22} \) are both periodic function with period \( l \) on \( s \), thus \( \psi^* (s_1 + l, s_2) \) is also a solution of the Cauchy problem (2.10). Consequently, by the uniqueness of solution of the Cauchy problem, \( \psi^* (s_1 + l, s_2) = \psi^* (s_1, s_2) + l \). By choosing \( \psi = \psi^* \), it follows that \( \det J|_{s_2=0} = 1 > 0 \), where

\[
J = \begin{pmatrix}
\psi_{s_1}^* & \psi_{s_2}^* \\
0 & 1
\end{pmatrix}
\]

is the Jacobi matrix of the transform \( \Theta \). Additionally, by the theory of one order PDE, we know that the solution depends on continuously the initial data and the equation’s coefficients. So there exists a sufficient small neighborhood \( W (\subseteq V) \) of \( \{0 \leq s \leq l, s_2 = 0\} \), such that \( \psi^* \in C^\infty(W) \) and \( \det J > 0 \) for all \( (s_1, s_2) \in W \). Therefore, the equation (2.8) can be simplified as follow

\[
y \tilde{\varphi} [(\tilde{A}^{12} \psi_{s_i}^* \psi_{s_j}^*) u_{x_1 x_1} + \tilde{A}^{22} u_{x_2 x_2} + C u] + (y \tilde{A}^{12} \psi_{s_i}^* \psi_{s_j}^*) u_{x_1} + \tilde{B}^2 u_{x_2} = F
\]

in the neighborhood \( W \) of \( \{0 \leq s \leq l, s_2 = 0\} \).

Without loss of the generality, we may assume that

\[
\Theta \circ \Xi (\Gamma) = \{ (x_1, x_2) \mid -\pi \leq x_1 \leq \pi, x_2 = 0 \},
\]

and

\[
\Theta : W \rightarrow \Theta(W) \subseteq \Theta \circ \Xi (\mathcal{N}(\Gamma)) \subseteq \{ (x_1, x_2) \mid -\pi \leq x_1 \leq \pi \}.
\]

According to all the above analyses, we know that \( D_{d_0} \subseteq \Theta(W) \) for an adequate small \( d_0 \), where

\[
D_{d_0} = \{ (x_1, x_2) \mid -\pi \leq x_1 \leq \pi, -d_0 < x_2 < d_0 \}.
\]

Define \( \Phi = \Theta \circ \Xi \), and denote \( \mathcal{N}_0(\Gamma) = \Phi^{-1}(D_{d_0}) \). Obviously, \( \mathcal{N}_0(\Gamma) \subseteq \mathcal{N}(\Gamma) \).

Since \( \tilde{\varphi}|_{\mathcal{N}_0(\Gamma)} > 0, \tilde{A}^{22}|_{\mathcal{N}_0(\Gamma)} > 0 \), therefore (2.8) can be simplified as follow

\[
Lu \equiv x_2(\omega u_{x_1 x_1} + u_{x_2 x_2} + cu) + au_{x_1} + bu_{x_2} = f \quad \text{in} \ D_{d_0},
\]

where

\[
\begin{aligned}
\omega &= \frac{\tilde{A}^{ij} \psi_{s_i}^* \psi_{s_j}^*}{\tilde{A}^{22}}, & a &= \frac{x_2 \tilde{A}^{ij} \psi_{s_i}^* \psi_{s_j}^* + \tilde{B}^k \psi_{s_k}^*}{\tilde{A}^{22} \tilde{\varphi}}, \\
b &= \frac{\tilde{B}^2}{\tilde{A}^{22} \tilde{\varphi}}, & c &= \frac{C}{\tilde{A}^{22}}, & f &= \frac{F}{\tilde{A}^{22} \tilde{\varphi}}.
\end{aligned}
\]

(2.11)

(2.12)

For the simplicity, we still denote respectively \( x_1, x_2 \) as \( x, y \) from now on. Thus we simply rewrite (2.11) as (2.2). Obviously, from (2.12) it follows that \( \omega, a, b, c, f \) all are periodic function with period \( 2\pi \) on \( x \). In addition, from (1.11) it follows that \( \omega, a, b, c \in C^\infty(D_{d_0}) \).
For the sake of simplicity, denoting \( \omega_0 = \omega(x, 0) \), \( a_0 = a(x, 0) \), \( b_0 = b(x, 0) \):

\[
L^\varepsilon u \equiv (\varphi + \varepsilon)(A^{ij}u_{\xi_i \xi_j} + Cu) + B^l u_{\xi_l} \quad (i, j, l = 1, 2);
\]

\[
L^{(-\varepsilon)} u \equiv (\varphi - \varepsilon)(A^{ij}u_{\xi_i \xi_j} + Cu) + B^l u_{\xi_l} \quad (i, j, l = 1, 2);
\]

\[
L^\varepsilon u \equiv (y + \varepsilon)(\omega u_{xx} + u_{yy} + cu) + au_x + bu_y;
\]

\[
L^{(-\varepsilon)} u \equiv (y - \varepsilon)(\omega u_{xx} + u_{yy} + cu) + au_x + bu_y;
\]

\[
D^+_d = \{(x, y) \mid -\pi \leq x \leq \pi, 0 < y < d\};
\]

\[
D^-_d = \{(x, y) \mid -\pi \leq x \leq \pi, -d < y < 0\}.
\]

In the sequel, we shall always use such convention: we sometimes identify \( u \circ \Phi^{-1} \) as \( u \), and still denote \( u \circ \Phi^{-1} \) by \( u \). Anyway, no confusion of ideas will rise in this paper if only one keep concretely close touch with the context.

**Proposition 2.2.**

\[
(2.13) \quad b_0 = \left( \frac{B^l \varphi_{\xi_l}}{A^{ij} \varphi_{\xi_i} \varphi_{\xi_j}} \right) \bigg|_{\Gamma}.
\]

**Proof.** The result follows from (2.6), (2.12) and (2.7). \( \square \)

**Remark 2.1.** By (2.13) and (2.12), it is easily verify that

\[
b_0 < 0 \quad \text{is equivalent to} \quad (1.13) \quad \text{i.e.} \quad (B^l \varphi_{\xi_l}) < 0 \quad \text{on} \quad \Gamma;
\]

\[
c \leq 0 \quad \text{is equivalent to} \quad (1.14) \quad \text{i.e.} \quad C \leq 0 \quad \text{for all} \quad (\xi_1, \xi_2) \in \Omega.
\]

2.3. **Elliptic regularization of the BVP (2.1).** Under the conditions (1.12), (1.13) and (1.14), we will employ elliptic regularization to discuss the well-posedness of the following BVP:

\[
(2.14) \quad \left\{ \begin{array}{l}
Lu = F \quad \text{in} \quad \Omega_+,
\text{u} = 0 \quad \text{on} \quad \Gamma,
\end{array} \right.
\]

and

\[
(2.15) \quad \left\{ \begin{array}{l}
Lu = F \quad \text{in} \quad \Omega_-,
\text{u} = 0 \quad \text{on} \quad \Gamma \cup \partial \Omega.
\end{array} \right.
\]

Since \( \varphi > 0 \), on \( \Omega_+ \); \( \varphi < 0 \) on \( \Omega_- \), then we thus may construct the following subsidiary BVP:

\[
(2.16) \quad \left\{ \begin{array}{l}
L^\varepsilon u = F^\varepsilon \quad \text{in} \quad \Omega_+,
\text{u} = 0 \quad \text{on} \quad \Gamma,
\end{array} \right.
\]

and

\[
(2.17) \quad \left\{ \begin{array}{l}
L^{(-\varepsilon)} u = F^\varepsilon \quad \text{in} \quad \Omega_-,
\text{u} = 0 \quad \text{on} \quad \Gamma \cup \partial \Omega,
\end{array} \right.
\]

where \( F^\varepsilon \in C^\infty(\Omega) \). In fact, we may choose \( F^\varepsilon \) as the mollification of \( F \). Moreover, if \( F \in H^k(\Omega) \), then

\[
(2.18) \quad \|F^\varepsilon\|_{H^k(\Omega)} \leq C_k \|F\|_{H^k(\Omega)},
\]
where $C_k$ is a constant depending only on $k$; Furthermore,
\[
\|F^\varepsilon - F\|_{H^k(\Omega)} \to 0, \quad \varepsilon \to 0.
\]

3. $H^1$ estimates for solutions of the BVP (2.16).

In this section, we will discuss the $H^1$ estimates for solutions of the BVP (2.16). According to the $L^2$ theory of second order elliptic type equation, there exists a solution $u^\varepsilon \in C^\infty(\overline{\Omega}_+)$ of the BVP (2.16), and also a solution $u^{(-\varepsilon)} \in C^\infty(\overline{\Omega}_-)$ of the BVP (2.17). Obviously, the interior $H^1$ estimates for solutions of above the BVPs, can be derived from the standard interior $H^2$ estimates of second order elliptic type equation. Therefore, we only need to give the local estimates in a neighborhood of $\Gamma$. In addition, from the Lemma 2.1, it follow that if we want to estimate the solution $u^\varepsilon$ (or $u^{(-\varepsilon)}$) of the BVP (2.16) (or (2.17)) in a neighborhood of $\Gamma$, then only need under the conditions $b_0 < 0$, $c \leq 0$, to make the local $H^1$ estimates for the solution of $\mathcal{L}^\varepsilon u = f^\varepsilon$ in $D_{d_0}^+$ with $u(x, 0) = 0$ (or $\mathcal{L}^{(-\varepsilon)} u = f^\varepsilon$ in $D_{d_0}^-$, with $u(x, 0) = 0$) in a neighborhood of $\{y = 0\}$, where $f^\varepsilon$ is the mollification of $f$, satisfies that $\|f^\varepsilon\|_{L^2} \leq C\|f\|_{L^2}$, and constant $C$ is independent of $\varepsilon$.

3.1. $H^1$ estimates for the solution of $\mathcal{L}^\varepsilon u = f^\varepsilon$ with $u(x, 0) = 0$. Under the condition $b_0 < 0$, $c \leq 0$, we will give the $H^1$ estimates for the solution $u^\varepsilon$ of $\mathcal{L}^\varepsilon u = f^\varepsilon$ with $u(x, 0) = 0$ in the following

**Proposition 3.1.** Suppose $\omega \geq \omega_* > 0$; $b_0 < 0$, $c \leq 0$. Then there exists a constant $\sigma = \sigma(\omega, a, b) \in (0, d_0/2]$, such that for any $d \in (0, \sigma]$, the solution $u^\varepsilon$ of the equation $\mathcal{L}^\varepsilon u = f^\varepsilon$ with $u(x, 0) = 0$ has the following estimates
\[
\|\nabla u^\varepsilon\|_{L^2(D_{d}^+)} \leq C\|f^\varepsilon\|_{L^2(D_{d_0}^+)} + \|u^\varepsilon\|_{L^2(D_{d_0}^+)},
\]
where $C$ is a constant depending only on $\omega$, $a$, $b$ and $d$.

**Proof.** First, we construct the cutoff function $\vartheta \in C^\infty(\mathbb{R}_+)$, such that $0 \leq \vartheta \leq 1$, and satisfies
\[
\vartheta = \vartheta(y) = \begin{cases} 1, & 0 \leq y \leq d, \\ 0, & y \geq 2d, \end{cases} \quad \text{and} \quad \left| \frac{\partial^k \vartheta}{\partial y^k} \right| \leq \frac{C}{d^k}, \quad k \in \mathbb{N}.
\]
Define $v^\varepsilon = \vartheta u^\varepsilon$. Thus, from $\mathcal{L}^\varepsilon u^\varepsilon = f^\varepsilon$ it follows
\[
\mathcal{L}^\varepsilon v^\varepsilon = \vartheta f^\varepsilon + 2(y + \varepsilon)\vartheta_y u^\varepsilon_y + [(y + \varepsilon)\vartheta_{yy} + b\vartheta_y]u^\varepsilon \quad \text{in} \quad D_{d_0}^+.
\]
Clearly, $u^\varepsilon(x, 0) = 0$ implies $v^\varepsilon(x, 0) = 0$.

Now we make both sides of (3.3) inner product with $v^\varepsilon/(y + \varepsilon)$, i.e.,
\[
-\left(\mathcal{L}^\varepsilon v^\varepsilon, \frac{v^\varepsilon}{y + \varepsilon}\right) = -\left(\vartheta f^\varepsilon + 2(y + \varepsilon)\vartheta_y u^\varepsilon_y + [(y + \varepsilon)\vartheta_{yy} + b\vartheta_y]u^\varepsilon, \frac{v^\varepsilon}{y + \varepsilon}\right).
\]
Moreover, denoting
\[
\iint := \int_{D_{d}^+} := \int_{-\pi}^{\pi} \int_{0}^{d}.
\]
For the left-hand side of (3.4), we get

\[-(L^\varepsilon v^\varepsilon, \frac{v^\varepsilon}{y + \varepsilon}) = \iint [- (\omega v^\varepsilon_{xx} + v^\varepsilon_{yy}) v^\varepsilon - c|v^\varepsilon|^2 - \frac{a}{y + \varepsilon} v^\varepsilon_x v^\varepsilon - \frac{b}{y + \varepsilon} v^\varepsilon_y v^\varepsilon]
\]

\[= \iint [- (\omega v^\varepsilon_{x} v^\varepsilon_{x}) - (v^\varepsilon_y v^\varepsilon_{y}) - c|v^\varepsilon|^2 + \omega v^\varepsilon_x v^\varepsilon + \omega |v^\varepsilon|^2 + |v^\varepsilon|^2
\]

\[+ \frac{\alpha_x}{2(y + 2\varepsilon)} |v^\varepsilon|^2 + (\frac{b}{2(y + 2\varepsilon)}) y |v^\varepsilon|^2]
\]

\[- \int_0^{2d} \frac{1}{2y + 2\varepsilon} (a |v^\varepsilon|^2)_{2y + 2d} dy - \int_{-2d}^{2d} (\frac{b |v^\varepsilon|^2}{2y + 2\varepsilon})_{2d} dx \]

\[= \iint \{\omega v^\varepsilon_x |v^\varepsilon|^2 + |v^\varepsilon_y|^2 + \omega v^\varepsilon_y v^\varepsilon + \left[\frac{\alpha_x}{2(y + 2\varepsilon)} + (\frac{b}{2(y + 2\varepsilon)}) y - c\right]|v^\varepsilon|^2\}
\]

\[= \iint \{|v^\varepsilon_x|^2 + |v^\varepsilon_y|^2 + \omega v^\varepsilon x v^\varepsilon + \left[\frac{(y + \varepsilon)(a_x + b_y) - b}{2(y + \varepsilon)^2} - c\right]|v^\varepsilon|^2\},
\]

and

\[\iint \omega v^\varepsilon x v^\varepsilon \leq \frac{1}{4} \iint \omega |v^\varepsilon_x|^2 + \iint \frac{\partial^2 \omega^2}{\omega} |u^\varepsilon|^2.
\]

In addition, the terms in the right-hand side of (3.4) have the following estimates:

\[\iint \partial f^\varepsilon \frac{v^\varepsilon}{y + \varepsilon} \leq \frac{4}{\delta} \iint \partial^2 |f^\varepsilon|^2 + \frac{16}{\delta} \iint \frac{|v^\varepsilon|^2}{(y + \varepsilon)^2};
\]

\[\iint 2(y + \varepsilon) \partial_y u^\varepsilon \frac{v^\varepsilon}{y + \varepsilon} = \iint 2\partial_y (\partial u^\varepsilon_y) u^\varepsilon
\]

\[= \iint 2\partial_y [(\partial u^\varepsilon)_y - \partial_y u^\varepsilon] u^\varepsilon
\]

\[= \iint 2(\partial_y u^\varepsilon y u^\varepsilon - \partial_y^2 |u^\varepsilon|^2),
\]

and

\[\iint \partial_y v^\varepsilon_y u^\varepsilon \leq \iint \partial_y^2 |u^\varepsilon|^2 + \frac{1}{4} \iint |v^\varepsilon|^2,
\]

Thus we have

\[\iint 2(y + \varepsilon) \partial_y u^\varepsilon \frac{v^\varepsilon}{y + \varepsilon} \leq \frac{1}{2} \iint |v^\varepsilon|^2;
\]

\[\iint b\partial_y u^\varepsilon \frac{v^\varepsilon}{y + \varepsilon} \leq \frac{4}{\delta} \iint |b|_2^2 \partial_y^2 |u^\varepsilon|^2 + \frac{16}{\delta} \iint \frac{|v^\varepsilon|^2}{(y + \varepsilon)^2};
\]
and
\[ \int \int (y + \varepsilon) \partial_y u^\varepsilon \frac{v^\varepsilon}{y + \varepsilon} \leq \int \int \partial \partial_y u^\varepsilon |u^\varepsilon|^2. \]

Together with (3.4) and all above estimates, it implies
\[ \int \int \left\{ \omega |v^\varepsilon_x|^2 + |v^\varepsilon_y|^2 + \left[ \frac{(y + \varepsilon)(a_x + b_y) - b}{2} - c \right] |v^\varepsilon|^2 \right\} \]
\[ \leq \frac{4}{\delta} \int \int \partial^2 |f^\varepsilon|^2 + \int \int \left( \frac{\delta}{\delta} |b|_\infty \partial_y^2 + 2 \partial^2 \frac{\omega^2}{\omega} + 2 \partial |\partial_y u| \right) |u^\varepsilon|^2 \]
\[ + \frac{1}{4} \int \int \omega |v^\varepsilon_x|^2 + \frac{1}{2} \int \int |v^\varepsilon_y|^2 + \frac{\delta}{8} \int \int |v^\varepsilon|^2 \frac{\omega^2}{(y + \varepsilon)^2}. \]

Hence, by a simplification procedure we obtain
\[ \text{(3.5)} \quad \int \int \left\{ \frac{3\omega}{2} |v^\varepsilon_x|^2 + |v^\varepsilon_y|^2 + \left[ \frac{(y + \varepsilon)(a_x + b_y) - b - \frac{\delta}{4}}{2} - c \right] |v^\varepsilon|^2 \right\} \]
\[ \leq \frac{8}{\delta_0} \int \int \partial^2 |f^\varepsilon|^2 + \int \int \left( \frac{8}{\delta_0} |b|_\infty \partial_y^2 + 2 \partial^2 \frac{\omega^2}{\omega} + 2 \partial |\partial_y u| \right) |u^\varepsilon|^2. \]

Since \( b_0 < 0 \). By the continuity of \( b_0 \), there exists a constant \( \delta_0 > 0 \) such that \( -b_0 > \delta_0 \). Subsequently, by the continuity of \( b \), there exists an adequate small constant \( \sigma_1 \) such that \( -b \geq \frac{\delta_0}{2} \) for \( 0 \leq y \leq \sigma_1 \). So, we have \( -b - \frac{\delta_0}{4} > \frac{\delta_0}{4} \) for \( 0 \leq y \leq \sigma_1 \). Choose \( \delta = \delta_0 \), from (3.5), we get
\[ \text{(3.6)} \quad \int \int \left\{ \frac{3\omega}{2} |v^\varepsilon_x|^2 + |v^\varepsilon_y|^2 + \left[ \frac{(y + \varepsilon)(a_x + b_y) + \frac{\delta_0}{4}}{2} - c \right] |v^\varepsilon|^2 \right\} \]
\[ \leq \frac{8}{\delta_0} \int \int \partial^2 |f^\varepsilon|^2 + \int \int \left( \frac{8}{\delta_0} |b|_\infty \partial_y^2 + 2 \partial^2 \frac{\omega^2}{\omega} + 2 \partial |\partial_y u| \right) |u^\varepsilon|^2. \]

Obviously, we may choose a sufficient small constant \( \sigma_2 > 0 \), which only depends on \( \delta_0, a, b \); such that \( (y + \varepsilon)(a_x + b_y) + \frac{\delta_0}{4} \geq 0 \), for \( 0 \leq y \leq \sigma_2 \).

Choose a constant \( \sigma = \min\{\sigma_1, \sigma_2, \delta_0/2 \} \) and together \( c \leq 0 \) with (3.6), we therefore obtain that
\[ \int \int \left( \frac{3\omega}{2} |v^\varepsilon_x|^2 + |v^\varepsilon_y|^2 \right) \leq \frac{8}{\delta_0} \int \int \partial^2 |f^\varepsilon|^2 + \int \int \left( \frac{8}{\delta_0} |b|_\infty \partial_y^2 + 2 \partial^2 \frac{\omega^2}{\omega} + 2 \partial |\partial_y u| \right) |u^\varepsilon|^2. \]

always holds for any \( d \leq \sigma \). Finally, by utilizing \( \omega \geq \omega_s > 0 \) and (3.24), we have
\[ \int_{D_1^+} |\nabla u^\varepsilon|^2 \leq C(\omega, b) \int_{D_{2d}^+} |f^\varepsilon|^2 + \frac{C(\omega, a, b)}{d^2} \int_{D_{2d}^+} |u^\varepsilon|^2. \]
This implies (3.11). \( \square \)
Proposition 3.2 (Boundary $H^1$ estimates). Assume the coefficients $\varphi, A^{ij}, B^l (i, j, l = 1, 2)$ and $C$ of the operator $L$ satisfy the conditions (1.11), (1.12), (1.13) and (1.14). Then the solution $u^\varepsilon$ of the BVP (2.16), satisfies that
\[
\|u^\varepsilon\|_{H^1(\Phi^{-1}(\partial^+ D^+_{\sigma^2}))} \leq C \left[ \|u^\varepsilon\|_{L^2(\Omega_+)} + \|F^\varepsilon\|_{L^2(\Omega_+)} \right],
\]
where $C$ is a constant depending only on $\|\varphi\|_{C^3(\partial^+ \Omega_+)}$, $\|A^{ij}\|_{C^2(\partial^+ \Omega_+)}$, and $\|B^l\|_{C^1(\partial^+ \Omega_+)}$.

Proof. Rewriting the inequality (3.1) in term of the variable $\xi = (\xi_1, \xi_2)$, we have
\[
\|\nabla u^\varepsilon\|_{L^2[\Phi^{-1}(\partial^+ D^+_{\sigma^2})]} \leq C \left\{ \|u^\varepsilon\|_{L^2(\Phi^{-1}(\partial^+ D^+_{\sigma^2}))} + \|F^\varepsilon\|_{L^2(\Phi^{-1}(\partial^+ D^+_{\sigma^2}))} \right\}.
\]
Let the integral domain $\Phi^{-1}(\partial^+ D^+_{\sigma^2})$ of the right-hand side of the above inequality extend to $\Omega_+$. The proof is complete. \qed

Lemma 3.3 (Interior $H^1$ estimates). Under the conditions of Proposition 3.2. Then for arbitrary $\Omega' \subset \subset \Omega_+$, the solution $u^\varepsilon$ of the equation $L^\varepsilon u = F^\varepsilon$ has the following interior estimates
\[
\|u^\varepsilon\|_{H^1(\Omega')} \leq C \left[ \|u^\varepsilon\|_{L^2(\Omega_+)} + \|F^\varepsilon\|_{L^2(\Omega_+)} \right],
\]
where $C$ is a constant depending only on $\text{dist}\{\Omega', \Gamma\}$, $\|\varphi\|_{C^2(\partial^+ \Omega_+)}$, $\|A^{ij}\|_{C^2(\partial^+ \Omega_+)}$, $\|B^l\|_{C^1(\partial^+ \Omega_+)}$, and $\|C\|_{C^1(\partial^+ \Omega_+)}$.

Proof. From the interior $H^2$ estimates of second order elliptic type equation directly follows the conclusion. \qed

Proposition 3.4 (Global $H^1$ estimates). Under the conditions of Proposition 3.2. Then the solution $u^\varepsilon$ of BVP (2.16), has the following global estimates
\[
\|u^\varepsilon\|_{H^1(\Omega_+)} \leq C \left[ \|u^\varepsilon\|_{L^2(\Omega_+)} + \|F^\varepsilon\|_{L^2(\Omega_+)} \right],
\]
where $C$ is a constant depending only on $\|\varphi\|_{C^3(\partial^+ \Omega_+)}$, $\|A^{ij}\|_{C^2(\partial^+ \Omega_+)}$, $\|B^l\|_{C^1(\partial^+ \Omega_+)}$, and $\|C\|_{C^1(\partial^+ \Omega_+)}$.

Proof. Choose $\Omega' = \Omega_+ \setminus \Phi^{-1}(\partial^+ D^+_{\sigma^2})$. (3.9) follows from (3.7) and (3.8). \qed

3.2. $H^1$ estimates of solutions of the BVP (2.14).

Lemma 3.5 (Lemma 3.1 of [24]). Let $f \in H^1(D^+_{d_0})$, if $v \in H^1(D^+_{d_0})$ is a weak solution of the following BVP:
\[
\begin{cases}
(L - \lambda I)v = f & \text{in } D^+_{d_0}, \\
v \text{ is a periodic function with period } 2\pi \text{ on } x, \\
v(x, 0) = v(x, d_0) = 0.
\end{cases}
\]
Then there exists a $\lambda_0 > 0$, such that when $\lambda \geq \lambda_0$, we have $v_x \in H^1(D^+_{d_0})$, and
\[
\|v_x\|_1 \leq C \|f\|_1,
\]
where $C$ is a constant.
Lemma 3.6. Let \( u, u_x \in H^1(\Omega) \). Then \( u \in C(\overline{\Omega}) \).

Proof. In fact, by the localization technique, we only need prove one conclusion:

Firstly, that \( u, u_x \in H^1(\mathbb{R}^2) \), then \( u \in C(\mathbb{R}^2) \).

Secondly, it is easy to verify

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} |\hat{u}|^2 \frac{1}{1 + \xi^2 + \eta^2 + \xi^2(\xi^2 + \eta^2)} d\xi d\eta < +\infty.
\]

Finally, we define \( G \) only discuss the continuity of \( u \).

So we have

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} |\hat{u}| d\xi d\eta = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\hat{u}| \left( 1 + \xi^2 + \eta^2 + \xi^2(\xi^2 + \eta^2) \right)^{-\frac{1}{2}} \frac{1}{\left[ 1 + \xi^2 + \eta^2 + \xi^2(\xi^2 + \eta^2) \right]^2} d\xi d\eta
\]

\[
\leq \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} |\hat{u}|^2 \left( 1 + \xi^2 + \eta^2 + \xi^2(\xi^2 + \eta^2) \right) d\xi d\eta \right)^{\frac{1}{2}} \times \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{1 + \xi^2 + \eta^2 + \xi^2(\xi^2 + \eta^2)} d\xi d\eta \right)^{\frac{1}{2}} < +\infty.
\]

Thirdly, since \( u(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x\xi + y\eta)} \hat{u}(\xi, \eta) d\xi d\eta \), thus we have

\[
|u(x, y) - u(x_0, y_0)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |e^{i(x\xi + y\eta)} - e^{i(x_0\xi + y_0\eta)}| \cdot |\hat{u}(\xi, \eta)| d\xi d\eta.
\]

Finally, we define \( G(\xi, \eta; x, y; x_0, y_0) = |e^{i(x\xi + y\eta)} - e^{i(x_0\xi + y_0\eta)}| \cdot |\hat{u}(\xi, \eta)| \).

Obviously,

\[
G(\xi, \eta; x, y; x_0, y_0) \to 0 \quad \text{a.e. in } \mathbb{R}^2 \quad \text{as} \quad (x, y) \to (x_0, y_0)
\]

In addition, \( |G(\xi, \eta; x, y; x_0, y_0)| \leq 2|\hat{u}(\xi, \eta)| \in L^1(\mathbb{R}^2) \). Hence the Lebesgue Dominated Convergence Theorem implies

\[
|u(x, y) - u(x_0, y_0)| \to 0 \quad \text{as} \quad (x, y) \to (x_0, y_0).
\]

i.e., \( u \in C(\mathbb{R}^2) \).

Lemma 3.7. Let \( u^* \) be a \( H^1 \) weak solution of the following BVP:

\[
(3.10) \quad \begin{cases}
Lu = 0 & \text{in } \Omega_+,
\vspace{0.5cm}
u = 0 & \text{on } \Gamma.
\end{cases}
\]

Then \( u^* \equiv 0 \).

Proof. Firstly, we prove \( u^* \in C(\overline{\Omega}_+) \cap C^\infty(\Omega_+) \). According to the interior regularity of second order elliptic type equation, we know that \( u^* \in C^\infty(\Omega) \), for all \( \Omega' \subset \subset \Omega_+ \). Because of \( L \) only degenerates on \( \Gamma \). Thus we only discuss the continuity of \( u^* \) in some neighborhood of \( \Gamma \).
From the Lemma 3.4, it is easy to know that $Lu^* = 0$ may be simplified as $\mathcal{L}u^* = 0$ by the transform $\Phi : \mathcal{N}_0(\Gamma) \rightarrow D_{d_0}$. Define a cutoff function $\zeta \in C^\infty(\mathbb{R}_+^1)$ as follows

$$
\zeta = \zeta(y) = \begin{cases} 
1 & \text{for } 0 \leq y \leq \delta, \\
0 & \text{for } y \geq 2\delta.
\end{cases}
$$

From $\mathcal{L}u^* = 0$, it follows easily that $(\mathcal{L} - \lambda)(\zeta u^*) = f^*$, where $f^* = 2y\zeta_y u^* + (y\zeta_{yy} + b\zeta_y - \lambda\zeta)u^*$. Obviously, $\zeta u^*$ is a $H^1$ weak solution of problem

$$
\begin{cases} 
(\mathcal{L} - \lambda)w = f^* & \text{in } D_{d_0}^+ \\
w \text{ is a periodic function with period } 2\pi & \text{on } x, \\
w(x,0) = w(x,d_0) = 0.
\end{cases}
$$

Obviously, $u^* \in H^2(D_{d_0}^+ \backslash \overline{D}_{\delta}^+)$. Direct calculation shows that $f^* \in H^1(D_{d_0}^+)$. Hence by applying Lemma 3.5, we have $(\zeta u^*)_x \in H^1(D_{d_0}^+)$. Thus Lemma 3.6 implies $\zeta u^* \in C(D_{d_0}^+)$. Consequently, $u^* \in C(\overline{D}_{d_0}^+)$. Set $N_{d_0}^+ = \Phi^{-1}(D_{d_0}^+)$. Then $u^* \in C(N_{d_0}^+(\Gamma))$. Using the interior regularity of elliptic equation, we deduce $u^* \in C(\overline{\Omega}_+^\times) \cap C^\infty(\Omega_+)$. 

Next we will prove $u^* \equiv 0$. Since $u^*$ on $\Gamma = 0$ and $u^* \in C(\overline{\Omega}_+^\times)$. For any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon)$ such that

$$
|u^*(\xi,\eta)| < \epsilon \text{ for } \text{dist}(\xi,\eta,\Gamma) < \delta.
$$

Choose simple connected domain $\Omega' \subset \subset \Omega_+$, such that $\text{dist}(\partial\Omega', \Gamma) < \frac{\delta}{2}$. By the maximum principle of elliptic equation, then $u^*$ must attains its maximum and minimum of $\overline{\Omega'}$ on $\partial\Omega'$, i.e., $\max_{\overline{\Omega'}} |u^*| \leq \max_{\partial\Omega'} |u^*|$. 

Therefore,

$$
\max_{\overline{\Omega}_+^\times} |u^*| \leq \max_{\Omega'} \{ \max_{\overline{\Omega'}} |u^*|, \max_{\overline{\Omega'}} |u^*| \} \leq \max_{\partial\Omega'} \{ \max_{\overline{\Omega'}} |u^*|, \max_{\overline{\Omega'}} |u^*| \} < \epsilon.
$$

By the arbitrariness of $\epsilon$, we deduce that $\max_{\overline{\Omega}_+^\times} |u^*| \leq 0$. This implies $u^* \equiv 0$. 

Now we consider a family of elliptic operators

$$
\Psi = \{ L^\varepsilon \mid L^\varepsilon = (\varphi + \varepsilon)(A^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} + C) + B^l \frac{\partial}{\partial \xi_l}; \varphi, A^{ij}, B^l, \text{ and } C \text{ satisfy } (1.11), (1.12), (1.14); i, j, l = 1, 2, 0 < \varepsilon \leq 1. \}.
$$

**Lemma 3.8.** Suppose that for any $L^\varepsilon \in \Psi$, $\Gamma \in C^{1,1}$. Let $u \in H^1(\Omega_+)$ with $u = 0$ on $\Gamma$, and satisfies the estimates

$$
\|u\|_{H^1(\Omega_+)} \leq C[\|u\|_{L^2(\Omega_+)} + \|L^\varepsilon u\|_{L^2(\Omega_+)}],
$$

Then we have

$$
\|u\|_{H^1(\Omega_+)} \leq C\|L^\varepsilon u\|_{L^2(\Omega_+)}.
$$

where constant $C$ is independent of $u$ and $\varepsilon$. 

Proof. If the conclusion does not hold, then for any \( n \in \mathbb{N} \), there exist sequences \( \{u_n\} \) satisfying the assumed conditions and \( \{\varepsilon_n\} \), such that
\[
\|u_n\|_{H^1} \geq n \|L^{\varepsilon_n} u_n\|_{L^2} \quad \text{and} \quad \|u_n\|_{L^2} = 1.
\]
By (3.13), it follows that
\[
\|u_n\|_{H^1} \leq C \left( \frac{\|u_n\|_{H^1}}{n} + 1 \right).
\]
Thus we have
\[
\|u_n\|_{H^1} \leq C \quad \text{for} \quad n \geq 2C.
\]
Therefore a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) converges weakly in \( H^1(\Omega_+) \), i.e.,
\[
u_{n_k} \rightharpoonup u^* \quad \text{weakly in} \quad H^1(\Omega_+) \quad \text{as} \quad k \to \infty.
\]
According to the Banach-Saks Theorem (cf. [19]), we may choose a subsequence of \( \{u_{n_k}\} \), might as well still denote by \( \{u_{n_k}\} \), such that \( \tilde{u}_k \) which is consist of the arithmetic mean of \( \{u_{n_k}\} \) as follow
\[
\tilde{u}_k = \frac{1}{k}(u_{n_1} + \cdots + u_{n_k})
\]
converges strongly in \( H^1(\Omega_+) \) to \( u^* \in H^1(\Omega_+) \), i.e.,
\[
\tilde{u}_k \rightharpoonup u^* \quad \text{strongly in} \quad H^1(\Omega_+) \quad \text{as} \quad k \to \infty.
\]
Obviously,
\[
\tilde{u}_k = \frac{1}{k}(u_{n_1} + \cdots + u_{n_k}) = 0 \quad \text{on} \quad \Gamma.
\]
By the Trace Theorem, it follows that \( u^* = 0 \quad \text{on} \quad \Gamma, \) in the sense of traces.

Since \( \{\varepsilon_{n_k}\} \) is bounded, there exists a subsequence of \( \{\varepsilon_{n_k}\} \), might as well still denote by \( \{\varepsilon_{n_k}\} \), such that
\[
\varepsilon_{n_k} \rightharpoonup \varepsilon_0 \quad \text{as} \quad k \to \infty.
\]
In addition, by the Sobolev embedding theorem,
\[
u_{n_k} \rightharpoonup u^* \quad \text{strongly in} \quad L^2(\Omega_+) \quad \text{as} \quad k \to \infty.
\]
So we have
\[
L^{\varepsilon_{n_k}} u_{n_k} \rightharpoonup L^{\varepsilon_0} u^* \quad \text{as} \quad k \to \infty.
\]
in the sense of distribution. Together (3.13) with (3.14), we deduce
\[
\|L^{\varepsilon_{n_k}} u_{n_k}\|_{L^2} \leq \frac{C}{n_k},
\]
Letting \( k \to \infty \), we conclude \( \|L^{\varepsilon_0} u^*\|_{L^2} = 0 \). Hence in the sense of distribution, we obtain \( L^{\varepsilon_0} u^* = 0 \).

Since \( u^* = 0 \quad \text{on} \quad \Gamma \) and \( L^{\varepsilon_0} u^* = 0 \), thus \( u^* \) is a \( H^1 \) weak solution of the following BVP:
\[
(3.15) \quad \left\{ \begin{array}{l}
L^{\varepsilon_0} u = 0 \quad \text{in} \quad \Omega_+, \\
u = 0, \quad \text{on} \quad \Gamma.
\end{array} \right.
\]
Then we have $u^* \equiv 0$.

We will respectively prove $u^* \equiv 0$ in the following two cases:

1) $\varepsilon_0 > 0$, then by the Maximal Principle of second order elliptic type equation, we know that the BVP (3.15) only has null solution. So, $u^* = 0$.

2) $\varepsilon_0 = 0$, i.e., $u^*$ is a $H^1$ weak solution of the BVP (3.10). Then Lemma 3.7 implies $u^* \equiv 0$.

On the other hand,

$$1 = \|u_{n_k}\|_{L^2} \to \|u^*\|_{L^2} \text{ as } k \to \infty,$$

yields $\|u^*\|_{L^2} = 1$. This contradicts $u^* = 0$. The proof for (3.12) is complete.

\[\square\]

**Theorem 3.9 (Global $H^1$ estimates).** Under the conditions of Proposition 3.4. Then the solution $u^\varepsilon$ of the BVP (2.16), has the following global estimates

\[(3.16)\]

$$\|u^\varepsilon\|_{H^1(\Omega_+)} \leq C\|F\|_{L^2(\Omega_+)},$$

where $C$ is a constant depending only on $\|\varphi\|_{C^3(\overline{\Omega}_+)}$, $\|A^{ij}\|_{C^2(\overline{\Omega}_+)}$, $\|B^i\|_{C^1(\overline{\Omega}_+)}$ $(i,j,l = 1,2)$ and $\|C\|_{C^1(\overline{\Omega}_+)}$.

**Proof.** By Proposition 3.4 and Lemma 3.8, we immediately obtain

\[(3.17)\]

$$\|u^\varepsilon\|_{H^1(\Omega_+)} \leq C\|F^\varepsilon\|_{L^2(\Omega_+)},$$

Inequalities (2.18) and (3.17) yield (3.16).

\[\square\]

**Lemma 3.10.** There exist a subsequence $\{u^\varepsilon\} \subset \{u^\varepsilon\}$ of solution of the BVP (2.14), such that

(i) $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_+)$, as $\varepsilon \to 0$;

(ii) $u^\varepsilon \to u$ strongly in $L^2(\Omega_+)$, as $\varepsilon \to 0$;

Furthermore,

(iii) we have the estimates

\[(3.18)\]

$$\|u\|_{H^1(\Omega_+)} \leq C\|F\|_{L^2(\Omega_+)},$$

where $C$ is a constant;

(iv) $u = 0$ on $\Gamma$ in the sense of traces, denoted by $\gamma(u) = 0$.

**Proof.** Together (3.16) with the fact $H^1(\Omega_+) \hookrightarrow L^2(\Omega_+)$ compactly, it follows the conclusions of (i), (ii) and (iii).

In addition, from $u^\varepsilon = 0$ on $\Gamma$, the conclusion of (i), and the fact $\gamma : H^1(\Omega_+) \hookrightarrow L^2(\Gamma)$ is compact map, it follows easily the conclusions of (iv).

\[\square\]

3.3. Some properties of solutions of the BVP (2.16).

**Lemma 3.11.** $\|(\varphi + \varepsilon)u^\varepsilon\|_{H^2(\Omega_+)} \leq C\|F\|_{L^2(\Omega_+)}$. 

Proof. Consider the following BVP:
\[
\begin{cases}
  A^{ij}[(\varphi + \varepsilon)u^\varepsilon]_{\xi_i \xi_j} = F^\varepsilon + (2A^{ij} \varphi_{\xi_i} - B^j)u^\varepsilon_{\xi_j} - cu^\varepsilon & \text{in } \Omega_+,
  \\
  (\varphi + \varepsilon)u^\varepsilon = 0 & \text{on } \Gamma.
\end{cases}
\]

In the sequel, observe that \(\|F^\varepsilon + (2A^{ij} \varphi_{\xi_i} - B^j)u^\varepsilon_{\xi_j} - cu^\varepsilon\|_{L^2(\Omega_+)} \leq C\|F\|_{L^2(\Omega_+)}\). Thus, by employing the regularity theory of second order elliptic type equations, we obtain the claim. \(\square\)

Without loss of generality, we may assume \(d_0 = 1\). So \(D^+_0 = D^+_1 = [-\pi, \pi] \times [0, 1]\). \(\eta \in C^\infty([0, 1])\), is defined by
\[
\eta = \eta(y) = \begin{cases}
  1 & \text{for } 0 \leq y \leq \frac{\pi}{2}, \\
  0 & \text{for } \frac{\pi}{2} \leq y \leq 1,
\end{cases}
\text{ and } 0 \leq \eta \leq 1.
\]

Lemma 3.12. The solution \(u^\varepsilon\) of \(\mathcal{L}^\varepsilon u = f^\varepsilon\) with \(u(x, 0) = 0\), satisfies that
\[
\lim_{\varepsilon \to 0} \|\varepsilon u^\varepsilon(\cdot, 0)\|_{L^2([-\pi, \pi])} = 0.
\]

Proof. Note that \(\mathcal{L}^\varepsilon u^\varepsilon = f^\varepsilon\) implies
\[
(y + \varepsilon)u^\varepsilon_{yy} = f^\varepsilon - \omega(y + \varepsilon)u^\varepsilon_{xx} - au^\varepsilon_x - bu^\varepsilon_y + cu^\varepsilon.
\]
Therefore, we obtain easily
\[
(y + \varepsilon)(\eta u^\varepsilon)_y = (y + \varepsilon)\eta_y u^\varepsilon_y + \eta(y + \varepsilon)u^\varepsilon_{yy} = h^\varepsilon,
\]
where \(h^\varepsilon := \eta[f^\varepsilon - \omega\eta(y + \varepsilon)u^\varepsilon_{xx} - au^\varepsilon_x - bu^\varepsilon_y + cu^\varepsilon] + [(y + \varepsilon)\eta_y - \eta b]u^\varepsilon_y\). So
\[
(\eta u^\varepsilon)_y = \frac{1}{y + \varepsilon} h^\varepsilon.
\]

It follows that
\[
\eta u^\varepsilon_y = -\int_y^1 \frac{1}{\tau + \varepsilon} h^\varepsilon(x, \tau) d\tau
\]
and
\[
\eta(y + \varepsilon)u^\varepsilon_y = -(y + \varepsilon) \int_y^1 \frac{1}{\tau + \varepsilon} h^\varepsilon(x, \tau) d\tau
\]
From this inequality we get
\[
\|\eta(y + \varepsilon)u^\varepsilon_y\|_{L^2([-\pi, \pi])} = \int_\pi \int_0^1 (y + \varepsilon)^2 \left( \int_y^1 \frac{1}{\tau + \varepsilon} h^\varepsilon(x, \tau) d\tau \right)^2 dx
\]
\[
\leq \int_\pi \int_0^1 (y + \varepsilon)^2 \left( \int_y^1 \frac{1}{(\tau + \varepsilon)^2} d\tau \right) \left( \int_y^1 |h^\varepsilon(x, \tau)|^2 d\tau dx \right)
\]
\[
= (y + \varepsilon)^2 \left( \frac{1}{y + \varepsilon} - \frac{1}{1 + \varepsilon} \right) \int_\pi \int_y^1 |h^\varepsilon(x, \tau)|^2 d\tau dx
\]
\[
\leq (y + \varepsilon) \int_\pi \int_y^1 |h^\varepsilon(x, \tau)|^2 d\tau dx
\]
\[
\leq (y + \varepsilon) \|h^\varepsilon\|_{L^2(D^+_1)}^2
\]
On the other hand, by a direct calculation and together with Lemma 3.11, we get
\[ \| h^\varepsilon \|_{L^2(D_+^1)} \leq C \| F \|_{L^2(\Omega_+)} . \]

From the previous two results we deduce
\[ \| \eta(y + \varepsilon) u_\varepsilon \|_{L^2(\Omega)} \leq C (y + \varepsilon)^{\frac{1}{2}} \| F \|_{L^2(\Omega_+)} . \]

In particular, for \( y = 0 \), with \( \eta(0) = 1 \), the above inequality yields
\[ \| \varepsilon u_\varepsilon \|_{L^2(\Omega)} \leq C \varepsilon \| F \|_{L^2(\Omega_+)} \to 0, \quad \text{as} \quad \varepsilon \to 0. \]

\[ \square \]

**Corollary 3.13.** The solution \( u_\varepsilon \) of the BVP (2.16) satisfies that
\[ \lim_{\varepsilon \to 0} \| \varepsilon u_\varepsilon \|_{L^2(\Gamma)} = 0. \]

**Proof.** The solution \( u_\varepsilon \) of the equation \( L^\varepsilon u = f^\varepsilon \) with \( u(x,0) = 0 \), satisfies that \( u_\varepsilon = 0 \). So it is apparent that \( \| \varepsilon u_\varepsilon \|_{L^2(\Omega)} = 0 \). By Lemma 3.12 and the transformation of variable, we easily prove the conclusion of this Corollary. \( \square \)

**Remark 3.1.** Under the same condition, we may estimate the solution of the BVP (2.15) in the case \( \Omega_- \). By the similar procedure and argument, it will lead to the completely same results as the case \( \Omega_+ \). But we do not state these corresponding results, as well as omit the proof of these results at here.

**4. The well-posedness of the BVP (2.1).**

In this section, we will discuss the well-posedness of BVP (1.16), i.e. to prove Theorem 1.1. For this purpose, we shall also need the following lemma:

**Lemma 4.1** (see [20]). Suppose that \( u_+ \in H^1(\Omega_+) \), \( u_- \in H^1(\Omega_-) \), and their traces on \( \Gamma \) is equal, i.e. \( \gamma u_+ = \gamma u_- \). A function \( u \in L^2(\Omega) \) is defined by
\[ u = \begin{cases} u_+ & \text{in } \Omega_+ , \\ u_- & \text{in } \Omega_- . \end{cases} \]

Then \( u \in H^1(\Omega) \).

**Theorem 4.2.** Under the conditions of Theorem 1.1. Then there exists a unique \( H^1 \) weak solution \( u \) of the BVP (2.1) in domain \( \Omega \). And \( u \) satisfies
\[ \| u \|_{H^1(\Omega)} \leq C \| F \|_{L^2(\Omega)} , \]

where \( C \) is a constant depending only on \( \Gamma, \| \varphi \|_{C^3(\Omega)}, \| A^{ij} \|_{C^2(\Omega)}, \| B^l \|_{C^2(\Omega)} \) (\( i,j,l = 1,2 \)) and \( \| C \|_{C^1(\Omega)} \).
Proof. Recall that \( u^{(\varepsilon') \l} \) and \( u^{(-\varepsilon')} \) are the solutions of the BVP (2.16) and (2.17) respectively. For the convenience, we use \( \overline{u} \) and \( u \) to express respectively the limits of \( u^{(\varepsilon')} \) and \( u^{(-\varepsilon')} \), i.e.

(i) \( u^{(\varepsilon')} \to \overline{u} \) weakly in \( H^1(\Omega^+) \), as \( \varepsilon' \to 0 \);
(ii) \( u^{(-\varepsilon')} \to u \) weakly in \( H^1(\Omega^+) \), as \( \varepsilon' \to 0 \).

A function \( u \in L^2(\Omega) \) is defined by

\[
\begin{align*}
  u &= \begin{cases} 
    \overline{u} & \text{in } \Omega^+, \\
    u & \text{in } \Omega^-.
  \end{cases}
\end{align*}
\]

By Theorem 3.9, Remark 3.1, Lemma 4.1, and \( \gamma(\overline{u}) = \gamma(u) = \mathbf{0}_{\partial \Omega} = 0 \), we obtain that \( u \in H^1(\Omega) \), and \( u = 0 \) on \( \Gamma \cup \Omega \), in the sense of traces.

On the other hand, from \( L^\varepsilon u^\varepsilon = F^\varepsilon \) and \( L^{(-\varepsilon)} u^{(-\varepsilon)} = F^{\varepsilon}_\varepsilon \), we get

\[
( L^\varepsilon u^\varepsilon, v ) = ( F^\varepsilon, v ), \quad \text{for all } v \in C_0^1(\Omega),
\]

and

\[
( L^{(-\varepsilon)} u^{(-\varepsilon)}, v ) = ( F^\varepsilon, v ), \quad \text{for all } v \in C_0^1(\Omega).
\]

Thus, by Green formula, \( u^\varepsilon = 0 \) on \( \Gamma \), and \( u^{(-\varepsilon)} = 0 \) on \( \Gamma \cup \partial \Omega \), we obtain that the following identities

\[
\int_{\Omega^+} \left[ -u^\varepsilon_{,i} (A^{ij} \varphi_{,j}) + (\varphi C u^\varepsilon + B^i u^\varepsilon_{,i}) v \right] d\xi = \int_{\Gamma} \varepsilon u^\varepsilon_{,i} A^{ij} \varphi_{,j} ds = \int_{\Omega^+} F v d\xi
\]

and

\[
\int_{\Omega^-} \left[ -u^{(-\varepsilon)}_{,i} (A^{ij} \varphi_{,j}) + (\varphi C u^{(-\varepsilon)} + B^i u^{(-\varepsilon)}_{,i}) v \right] d\xi = \int_{\Gamma} \varepsilon u^{(-\varepsilon)}_{,i} A^{ij} \varphi_{,j} ds = \int_{\Omega^-} F v d\xi
\]

hold for all \( v \in C_0^1(\Omega) \), where

\( \overline{n} = (n_1, n_2) = \left( \frac{\varphi_{,1}}{|\nabla \varphi|}, \frac{\varphi_{,2}}{|\nabla \varphi|} \right) |_{\Gamma} \neq 0 \)

is the unit inward normal direction of \( \partial \Omega^+ \), Meanwhile it is also the unit outward normal direction of \( \partial \Omega^- \).

Letting \( \varepsilon \) tends to 0, by the above (i), (ii), Corollary 3.13 and Remark 3.1, we get

\[
\int_{\Omega^+} \left[ -u_{,i} (A^{ij} \varphi_{,j}) + \varphi C u + B^i u_{,i} \right] d\xi = \int_{\Omega^+} F v d\xi, \quad \text{for all } v \in C_0^1(\Omega);
\]

and

\[
\int_{\Omega^-} \left[ -u_{,i} (A^{ij} \varphi_{,j}) + \varphi C u + B^i u_{,i} \right] d\xi = \int_{\Omega^-} F v d\xi, \quad \text{for all } v \in C_0^1(\Omega).
\]

Consequently, \( u \) satisfies that

\[
\int_{\Omega} \left[ -u_{,i} (A^{ij} \varphi_{,j}) + \varphi C u + B^i u_{,i} \right] d\xi = \int_{\Omega} F v d\xi, \quad \text{for all } v \in C_0^1(\Omega).
\]

Since \( C_0^1(\Omega) \) is dense in \( H_0^1(\Omega) \). So, we have thus the existence of \( H^1 \) weak solution of the BVP (2.1) on \( \Omega \) in a very simple manner. In addition, \( (\varepsilon, 1) \) will be followed by applying (3.15) and Remark 3.1.
Finally, together Lemma 3.7 with Remark 3.1, we obtain easily the uniqueness of $H^1$ solution to the BVP (2.1) in $\Omega$. 

According to subsection 2.1, we see easily that Theorem 1.1 is a trivial Corollary of Theorem 4.2. So far, we finish the proof for the main result of this paper.

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**Address 1:** Department of Mathematics, School of Mathematics & Computing, Nanjing Normal University, Nanjing 210097, P. R. China

**Current address:** Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China

**E-mail address:** heyueyn@163.com & heyue@njnu.edu.cn.