Purely twistorial string with canonical twistor field quantization

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We introduce new purely twistorial scale–invariant action describing the composite bosonic $D = 4$ Nambu–Goto string with target space parametrized by the pair of $D = 4$ twistor fields. We show that by suitable gauge fixing of local scaling one gets the bilinear twistorial action and canonical quantization rules for the two–dimensional twistor–string fields. We consider the Poisson brackets of all constraints characterizing our model and we obtain four first class constraints describing two Virasoro constraints and two $U(1) \otimes U(1)$ Kac–Moody (KM) local phase transformations.

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I. INTRODUCTION

The idea, due to Penrose (see e.g. \cite{1}), of using twistors as replacing the primary space–time geometry, has been applied to many problems in physics, in particular to the description of particles and superparticles \cite{2, 3, 4, 5, 6, 7, 8, 9, 10} as well as strings and superstrings \cite{11, 12, 13, 14, 15, 16, 17}. For particle and superparticle models the equivalence of twistorial and space–time approach has been shown on classical and quantum level; for strings and superstrings the equivalence of two geometric pictures was demonstrated only by comparing the spectra of quantized excitations (see e.g. \cite{15, 16, 17}), without providing the equality of classical actions. The main results following from the models of open tensionless twistor strings presented by Witten \cite{15} and Berkovits \cite{16} was the close links with super Yang–Mills gauge theories, leading to new twistor representation of multihelicity tree and one-loop amplitudes (see e.g. \cite{20}). The tensionless closed superstring was also applied to the description of conformal supergravity sector \cite{17}; the relation with standard (Poincare) supergravity amplitudes was subsequently obtained by adding the terms breaking the conformal symmetry and introducing dimensionful parameter (Plank mass) \cite{21}. We should add also that the link between the Yang–Mills theory and twistor string framework has been further advanced by providing the Yang–Mills action (in $D = 4$) in terms of fields on twistor space \cite{22, 23}.

It should be stressed that recent twistor string models developed in \cite{15, 16, 17, 18, 19} were described by tensionless strings, which are the one-dimensional counterpart of massless point particles. It is known that the description of massive (super)particles requires in the Penrose framework the introduction of two–(super)twistor geometry (see \cite{8, 9, 24}). Following \cite{11}, using two–twistor target space purely twistorial tensorial string action was given \cite{25}, which is classically equivalent to Nambu–Goto (NG) string action with composite space–time string fields. Unfortunately, our twistorial action from \cite{25} is fourlinear, what presents a serious difficulty in performing the quantization procedure. In this paper we resolve this difficulty. Following the reduction by reparametrization gauge fixing of NG string to free string model, we show that the quadratic free twistor string can be obtained by suitable gauge fixing of a new nonlinear twistor–string model. We derive therefore the bilinear twistor–string model which leads for $D = 2$ twistor–string fields to standard Penrose quantization rules \cite{11, 26}.

In our recent paper \cite{25} we obtained purely twistorial fourlinear classical $D = 4$ string action, with target space described by the two–twistor space. The fundamental string world–sheet fields are described by the following pair of $D = 4$ twistors ($A = 1, \ldots, 4$; $i = 1, 2$; $m = 1, 2$; $\alpha, \dot{\alpha} = 1, 2$)

\begin{align}
Z_{Ai}(\xi) &= (\lambda_{\alpha i}(\xi), \mu_{\dot{\alpha} i}(\xi)), \\
\bar{Z}^{Ai}(\xi) &= (\bar{\mu}^{\alpha i}(\xi), -\bar{\lambda}^{\dot{\alpha} i}(\xi)), \quad \bar{\lambda}^{\dot{\alpha} i} = (\bar{\lambda}_{\alpha i}), \quad \bar{\mu}^{\alpha i} = (\bar{\mu}_{\dot{\alpha} i})
\end{align}

where $\xi^m = (\xi^0, \xi^1)$ denote the world–sheet coordinates; the indices $\alpha, \dot{\alpha}$ and $i$ are lifted by $(2 \times 2)$ antisymmetric tensor $\epsilon, \epsilon^{12} = \epsilon_{21} = 1$ ($\lambda^a = \epsilon^{a\beta} \lambda_{\beta}$, $\lambda_{\alpha} = \epsilon_{\alpha \beta} \lambda^\beta$ etc.). In twistorial string model the generalized Penrose relations \cite{1}, expressing the string phase space fields $X_{\mu}(\xi)$, $P_{\mu}(\xi)$\cite{27, 28} ($\mu = 1, \ldots, 4$) by the twistor fields $Z_{AI}(\xi)$, $\bar{Z}^{AI}(\xi)$ are the following ($P_{m}^{m} = \sigma_{\alpha \dot{\alpha}} P_{m}^{\alpha \dot{\alpha}}$)

- string extension of Cartan–Penrose formula

\begin{align}
P_{\alpha \dot{\beta}}(\xi) &= \epsilon(\xi) \bar{\lambda}^{\beta}_{\dot{\alpha}}(\rho^{\alpha}) i^j \lambda_{\alpha j}(\xi).
\end{align}

where $\epsilon = \det \epsilon_{m}^{\alpha}$, $(\epsilon_{m}^{\alpha}(\xi)$ is the zweibein), $(\rho^{\alpha})_{i}^{j} = \epsilon_{\alpha}^{m}(\rho^{\mu})_{i}^{j}$ ($\rho^{\mu}$ are $2 \times 2$ Dirac matrices; $\mu = 1, 2$), $\bar{\lambda}_{\dot{\alpha}}^{\alpha} = \bar{\lambda}_{\dot{\alpha}}^{\beta} (\rho^{\beta})_{j}^{i}$. 

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• string extension of Penrose incidence relations

\[ \mu^{\hat{a}}(\xi) = X^{\alpha\beta}(\xi) \lambda_{\beta i}(\xi), \quad \bar{\mu}^{\hat{a}}(\xi) = \bar{X}^{\hat{a}}{\beta}_i(\xi) \]

In [25] it was shown that if we insert (1.2) into the first order phase space formulation of Nambu–Goto string due to Siegel (see [27])

\[ S = \int d^2\xi \left[ P^m \partial_m X + \frac{1}{27} (-h)^{-1/2} h_{mn} P^m P^n \right] \]

one obtains the bosonic string model proposed by Soroka, Sorokin, Tkach and Volkov (SSTV) [12]

\[ S = \int d^2\xi \left[ \hat{\lambda}_a \rho^m \lambda_\alpha \partial_m X^{\hat{a}} + \frac{1}{27} (\lambda^{\alpha i} \lambda_{ai}) (\hat{\lambda}_a \hat{\lambda}_a) \right] \]

where we use Weyl representation for \( \tilde{D} \), \( c_\rho \)

\[ \rho \]

where we use Weyl representation for D = 2 Dirac matrices \( \rho_\alpha \), with the products \( \rho_\alpha \rho_\alpha \) diagonal. In [25] we suitably fixed one local scale transformation with \( c = c_1 \) and one local phase transformation with \( \varphi = \varphi_1 + \varphi_2 \) and imposed the constraints

\[ \lambda^{\alpha i} \lambda_\alpha = \bar{T} = \hat{\lambda}_a \hat{\lambda}_a. \]

Subsequently after employing in (1.5) the relations (1.3) we obtained the fourlinear purely twistorial Lagrangian [25] which is induced on the world-sheet by the following canonical 2-form

\[ \Theta^{(2)} = \Theta_i^{(1)} \wedge \Theta_2^{(1)} \]

where \( (i = 1, 2) \) [30]

\[ \Theta_i^{(1)} = (\partial Z_{A\hat{A}} + d\tilde{Z}^A Z_{A\hat{A}}) \]

(1.9) describes the standard twistorial one-form, defining the twistorial particle actions [3, 8].

In this paper we shall transform the action (1.5) into 2-twistor action before any gauge fixing. Further we shall fix suitably the local gauges (1.6), in different way than in [25]. In such a way we shall cancel the fourlinear terms in the action and we obtain the solvable gauge-fixed bilinear twistor–string action.

II. PURELY TWISTORIAL PICTURE OF SSTV STRING MODEL

Let us apply the generalized incidence relations (1.3) in order to remove from (1.6) the string space–time field \( X_\mu(\xi) \). It follows from (1.3) that the twistor fields \( Z_{Ai}(\xi) \), \( \tilde{Z}^A(\xi) \) should satisfy four constraints (\( [V^i_I] = -V^i_I \))

\[ V^i_I \equiv Z_{Ai} \tilde{Z}^{AI} = \lambda_{ai} \bar{\mu}^{\hat{a}} - \mu^{\hat{a}} \lambda_{ai} \approx 0. \]

Further, taking into account the incidence relations (1.3) we obtain that

\[ \hat{\lambda}_a \rho^m \lambda_\alpha \partial_m X^{\hat{a}} = \frac{1}{2} \partial_m \tilde{Z}^A \rho^m Z_A - \tilde{Z}^A \rho^m \partial_m Z_A. \]

Inserting (2.2) into (1.5) and taking into consideration the constraints (2.1) we obtain the action

\[ S = \frac{1}{2} \int d^2 \xi \left( \partial_m \tilde{Z}^A \rho^m Z_A - \tilde{Z}^A \rho^m \partial_m Z_A \right) - \int d^2 \xi \left( e^{\tilde{M}M} + \Lambda^I V^i_I \right) \]

where

\[ M \equiv e_{ij} I_{AB} Z_{Ai} Z_{ Bj} = \lambda^{\alpha i} \lambda_{ai}, \]

\[ \tilde{M} \equiv -e_{ij} I_{AB} \tilde{Z}^A_j \tilde{Z}^B_i = \bar{\lambda}_{\alpha i} \bar{\lambda}_a i \]

and \( I_{AB} \) are the asymptotic twistors [11], described by singular \( 4 \times 4 \) matrices \( I_{AB} = \begin{pmatrix} e^{\alpha \beta} & 0 \\ 0 & 0 \end{pmatrix} \)

\[ I_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

Let us indicate here that Berkovits [16] and Siegel [18] string extension of Penrose incidence relations [25] models can be obtained from the action (2.3) in particular tensionless limit (see also [19]). In Weyl representation for \( d = 2 \) gamma-matrices the action can be rewritten in the form

\[ S = \int d^2 \xi e^{-m} (\tilde{Z}^A \partial_m Z_A - \partial_m \tilde{Z}^A Z_A) + \]

\[ + \int d^2 \xi e^{-m} (\tilde{Z}^A \partial_m Z_{A\hat{A}} - \partial_m \tilde{Z}^A Z_{A\hat{A}}) \]

\[ - \int d^2 \xi \left( e^{\tilde{M}M} + \Lambda^I V^i_I \right) \]

(2.5)

where \( e^{m} = \frac{1}{2} (e^{m} + e^{m}) \). Then, making rescaling \( Z_{Ai} \rightarrow T^{1/2} Z_{Ai}, e^{-m} \rightarrow T^{-1} e^{-m}, e^{-m} \rightarrow T e^{-m}, A^I \rightarrow T^{-1} A^I \) and taking tensionless limit \( T \rightarrow 0 \) we obtain Siegel twistor string action [18, 29]

\[ S_{T \rightarrow 0} = \int d^2 \xi \left( \tilde{Z}^A \nabla_+ Z_{A\hat{A}} - \nabla_- \tilde{Z}^A Z_{A\hat{A}} \right) \]

(2.6)

where \( \nabla_+ = e^{m} \partial_m + iA \) is the world-sheet covariant derivative with U(1)–connection \( A = \tilde{A} = \frac{1}{2\pi} \Lambda^I A^I \). One can add that by putting in (2.5) \( Z_{A\hat{A}} = 0 \) one obtains as well the action (2.6). The Berkovits model [16] can
be considered as double variant of the Siegel model \(^\text{18}\) with sum of two actions (2.6), one for left moving, and second for right moving twistor string fields (1.1).

The variation of (2.3) with respect to the zweibein \(e^a_m\) gives the relation

\[
e^a_m = \frac{T_{MM}}{\rho^a} \left( \tilde{Z}^A \partial_m Z - \partial_m \tilde{Z}^A \rho^a Z_A \right). \tag{2.7}
\]

Inserting (2.7) into (2.3) we obtain our new nonlinear twistor string model

\[
S = \int d^2 \xi \left[ \frac{T_{MM}}{\rho^A} \varepsilon^{mn} \left( \partial_m Z A_1 \tilde{Z}^{A1} - Z A_1 \partial_m \tilde{Z}^{A1} \right) \times \left[ \partial_m \tilde{Z}^{B2} - Z B_2 \partial_m \tilde{Z}^{B2} \right) - \Lambda^j V_i \right], \tag{2.8}
\]

where

\[
\Lambda^i = - (\Lambda^j). \tag{2.9}
\]

It should be observed that the twistor string fields have different mass dimensions in comparison with the twistor coordinates in particle mechanics. For consistency we should assume in (1.2), (1.3) that

\[
[\lambda_a] = m^1, \quad [\mu^a] = m^0.
\]

Lagrangian density in the action (2.8) can be represented in the following form (we use the usual notations: \( \hat{X} \equiv \frac{\partial L}{\partial \dot{X}}, \hat{\dot{X}} \equiv \frac{\partial L}{\partial \ddot{X}} \) where \( \tau \equiv \xi^0, \sigma \equiv \xi^1 \))

\[
\mathcal{L} = \frac{T_{MM}}{\rho^A} Q_1 \left( \tilde{Z}_1 \tilde{Z}^1 - Z_1 \tilde{Z}^1 \right) - \frac{T_{MM}}{\rho^A} Q_1 \left( \tilde{Z}_2 \tilde{Z}^2 - Z_2 \tilde{Z}^2 \right) - \Lambda^i V_i \tag{2.9}
\]

where

\[
\Lambda^i = \tilde{Z}_1 \tilde{Z}^1 - Z_1 \tilde{Z}_1, \quad \Lambda^2 = \tilde{Z}_2 \tilde{Z}^2 - Z_2 \tilde{Z}^2
\]

We observe that the nonlinearity in (2.10) is not invariant under the two scale gauge transformations (2.12). One gets

\[
\delta (Q_1 / N M) = - 2c_2 \left( Q_1 / N M \right), \quad \delta (Q_2 / N M) = - 2c_1 \left( Q_2 / N M \right). \tag{2.13}
\]

If we supplement (2.12) by suitable variation of Lagrange multipliers (\( \delta \lambda_1 = - (c_1 + c_2) \lambda_1 \)) we obtain that the complete twistor–string Lagrangian (2.9) is invariant under the local scaling described by (2.12).

### III. SCALE GAUGE FIXING AND CANONICAL QUANTIZATION

Let us put the following gauge fixing conditions for the gauge transformations (2.12), (2.13)

\[
(T Q_1) / (M M) = - 1/2, \quad (T Q_2) / (M M) = 1/2. \tag{3.1}
\]

As a result, in this gauge we obtain from (2.9) the bilinear gauge–fixed twistor–string action \( S_{gf} \):

\[
S_{gf} = \int d^2 \xi \left[ \frac{1}{2} \left( \tilde{Z}_1 \tilde{Z}^1 - Z_1 \tilde{Z}^1 \right) - \Lambda^i V_i \right] \tag{3.2}
\]

where \( Z_{A1}, \tilde{Z}^{A1} \) are restricted by the constraints (2.1) and the gauge fixing (3.1)

\[
\Phi_1 = \frac{1}{2} (\tilde{Z}_1 \tilde{Z}^1 - Z_1 \tilde{Z}^1) + \frac{M \tilde{M}}{4T} \approx 0, \tag{3.3}
\]

\[
\Phi_2 = \frac{1}{2} (\tilde{Z}_2 \tilde{Z}^2 - Z_2 \tilde{Z}^2) - \frac{M \tilde{M}}{4T} \approx 0.
\]

Let us observe that similarly like the mass parameter in phase space formulation of relativistic particle action, the tension parameter \( T \) enters into the constraints (see (3.3)).

In the gauge (3.1) the constraints (2.10) have the form

\[
\mathcal{D} A^1 \rightarrow D A^1 \equiv P A^1 - \frac{1}{2} Z A^1 \approx 0, \quad D A^2 \rightarrow \tilde{D} A^1 \equiv \tilde{P} A^1 + \frac{1}{2} Z A^1 \approx 0. \tag{3.4}
\]

If we use canonical equal time Poisson brackets

\[
\{ Z A^1(\sigma), P B^j(\sigma') \} = \delta^B_A \delta^j_i \delta^2(\sigma - \sigma'), \quad \{ \tilde{Z} A^1(\sigma), \tilde{P} B^j(\sigma') \} = \delta^B_A \delta^j_i \delta^2(\sigma - \sigma')
\]

we obtain the brackets

\[
\{ D A^1(\sigma), \tilde{D} B^j(\sigma') \} = - \delta^B_A \delta^j_i \delta^2(\sigma - \sigma'). \tag{3.5}
\]

Inserting into Dirac brackets (we denote it by “\( \mathcal{T} \)” as twistor brackets):

\[
\{ A(\sigma), B(\sigma') \} = \{ A(\sigma), B(\sigma') \} - \int d\sigma \{ A(\sigma), D A^i(\sigma) \} \{ \tilde{D} A^i(\sigma), B(\sigma') \} + \int d\sigma \{ A(\sigma), \tilde{D} A^i(\sigma) \} \{ D A^i(\sigma), B(\sigma') \}
\]
we arrive at the standard twistor canonical relations for free twistorial string
\[
{\{ZA_i(\sigma), Z^B_j(\sigma')\}_T} = \delta^B_A \delta^j_i \delta(\sigma - \sigma')
\]
(3.7)
which were assumed e.g. a priori in \[26\], but not derived from the twistorial string action.

In such a way we obtained free two dimensional twistor model which corresponds e.g. to the twistor string formulation given in \[20\]. It should be added that in the free twistor string action \[3.2\] the constraints are derived, in similar way as Virasoro conditions in Nambu-Goto string framework, by gauge-fixing procedure which leads uniquely to bilinear action.

Using the relations \(2.1\), \(3.3\) one can calculate the PB of primary constraints \(2.1\), \(3.3\). One obtains
\[
\{Q_i^1, Q_j^1\}_T = -\left( \delta^1_i \delta^1_j - \delta^1_i \delta^1_j \right) \delta(\sigma - \sigma') \quad (3.8)
\]
\[
\{\Phi_{+}(\sigma), \Phi_{+}(\sigma')\}_T = \left( \Phi_{+}(\sigma) + \Phi_{+}(\sigma') \right) \delta(\sigma - \sigma') \quad (3.9)
\]
\[
\{\Phi_{-}(\sigma), \Phi_{-}(\sigma')\}_T = \left( \Phi_{-}(\sigma) + \Phi_{-}(\sigma') \right) \delta(\sigma - \sigma') \quad (3.10)
\]
where \(\Phi_{\pm} = \Phi_{1} \pm \Phi_{2}\) and
\[
Q_i^1 \equiv \dot{Z}_{A_i} \dot{Z}^{A_2} - Z_{A_1} \dot{Z}^{A_2},
Q_i^2 \equiv \dot{Z}_{A_2} \dot{Z}^{A_1} - Z_{A_1} \dot{Z}^{A_1}.
\quad (3.12)
\]

Interestingly enough we see that the constraints \(3.3\) describe Virasoro algebra, and the constraints \(V_i^1\) \(2.1\) form the \(U(2)\) Kac–Moody algebra. The cross relations between these algebras are however not closed due to the last two PB in relations \(3.11\) where do appear the bilinears \(Q_i^1, Q_i^2\). We shall show in Sect. IV that the bilinears \(3.12\) define secondary constraints in our model.

IV. PRIMARY AND SECONDARY CONSTRAINTS

Further we shall consider Hamiltonian formulation of our twistorial action \(2.3\) in the gauge \(3.1\), i.e. with the constraints \(D^{A_i}, D_{A_i}\) replaced by \(D^{A_1}, D_{A_1}\). Remaining primary contraints \(\Phi_i, V_i^1\) are described by the relations \(2.1, 3.3\). The Hamiltonian corresponding to the action \(3.2\) looks as follows
\[
H_1 = \int d\sigma \left( A_{A_1} D^{A_1} + \bar{A}_{A_1} D_{A_1} + V_i^1 (V_i^1 + N\Phi_i) \right) \quad (4.1)
\]
Because the twistor–string momenta \(P^{A_1}, \bar{P}_{A_1}\) are entering only into constraints \(2.1\), \(3.3\), the nonvanishing canonical PB of the constraints are only those with \(D^{A_1}, D_{A_1}\).

The preservation of the constraints \(D^{A_1}, D_{A_1}\) in time \((\dot{D}^{A_1} = \{D^{A_1}, H_1\}_T \equiv 0, \dot{D}_{A_1} = \{D_{A_1}, H_1\}_T \equiv 0\) leads to the expressions for \(A_{A_1}, \bar{A}_{A_1}\) as suitable linear combinations of the Lagrange multipliers \(L_{A, i}, L^i\). The time independence of the remaining constraints \(2.1\), \(3.3\)
\[(\dot{F}_M = \{F_M, H_1\}_T \equiv 0; F_M = \{F_i, V_i^1\})\] leads after long but simple algebraic calculation to the following conditions \((i = 1, 2)\)
\[
\dot{\Phi}_1 \equiv 0 \Rightarrow \Lambda^1 Q^1_2 - \Lambda^2 Q^2_1 \equiv 0, \quad (4.2)
\]
\[
\dot{V}_i^1 \equiv 0 \Rightarrow \Lambda^i Q^2_2 \equiv 0, \quad i \neq j. \quad (4.3)
\]
where \(\Lambda^i \equiv \Lambda^i - \Lambda^2\). Vanishing of time derivatives of the constraints \(V_i^1, V_i^2\) do not require additional relations.

After substitution of \(\Lambda_{A_1}, \bar{A}_{A_1}\) in terms of remaining Lagrange multipliers the Hamiltonian \(4.1\) takes the form
\[
H_1 = \int d\sigma \left( \Lambda^i \dot{V}_i^1 + \Lambda^2 \dot{\Phi}_1 \right) \quad (4.4)
\]
where
\[
\dot{V}_i^1 \equiv Z_i P^i - \bar{P}_i \dot{Z}_i, \quad (4.5)
\]
\[
\dot{\Phi}_1 \equiv \frac{1}{2} (\dot{Z}_1 P^1 - Z_1 \dot{P}^1 + \dot{P}_1 \dot{Z}_1 - \dot{P}_1 \dot{Z}_1) + R, \quad (4.6)
\]
\[
\dot{\Phi}_2 \equiv \frac{1}{2} (\dot{Z}_2 P^2 - Z_2 \dot{P}^2 + P_2 \dot{Z}_2 - P_2 \dot{Z}_2) + R, \quad R \equiv \frac{M}{4T} (P^1 \dot{Z}_1^2 + \dot{Z}_1^1 P^2) - \frac{M}{4T} (P_1 I \dot{Z}_2 + Z_1 I \dot{P}_2) - \frac{M}{4T}.
\]

One can check that the constraints \(\dot{\Phi}_1, \dot{V}_i^1\) differ from the constraints \(\Phi_i, V_i^1\) by terms linear in constraints \(D^{A_1}, D_{A_1}\).

The equations \(4.2-4.3\) describe the additional restrictions. There are two possible choices:

i) We choose \(\Lambda^1 \neq 0\) and the secondary constraints
\[
Q_i^1 \equiv 0, \quad Q_i^2 \equiv 0. \quad (4.7)
\]
In such a case one should add to \(4.1\) the secondary constraints \(Q_i^1 \equiv 0, Q_i^2 \equiv 0\), and check the closure for arbitrary time.

ii) One can choose alternatively \(Q_i^1 = (Q_i^2) \neq 0\) and \(\Lambda^1 = 0\), \(\Lambda^1 - \Lambda^2 = (\Lambda^2 + \Lambda^2)(Q_i^2 - Q_i^2)/(Q_i^2 + Q_i^2)\).
In such a case the closure of the constraints algebra at arbitrary time implies the change of the nature of two primary constraints from first class to second class. One can show that in such a case the number of degrees of freedom of tensor string will not coincide with the number of physical degrees of freedom of bosonic string. Further we shall study only the case i).

In order to show that after adding \((4.7)\) we obtained complete set of constraints we should consider the second stage Hamiltonian

\[
H_2 = H_1 + \int d\sigma \left( L^2_1 Q^2_1 + L^2_2 Q^2_2 \right) \tag{4.8}
\]

where \(H_1\) is defined in \((4.11)\).

The preservation of the constraints \(D^{A_i}, \bar{D}_{A_i}\) in time (\(\bar{D}^{A_i} = \{ D^{A_i}, H_2 \}_\sigma \approx 0, \bar{D}_{A_i} = \{ \bar{D}_{A_i}, H_2 \}_\sigma \approx 0\)) leads again to the formulae, expressing \(\Lambda_{A_i}, \Lambda^{A_i}\) by means of the linear combination of the Lagrange multipliers \(\Lambda_j, \Lambda^i\), \(L^1_2, L^2_1\). After substitution of these formulae in \((4.8)\) we can check that the constraints \(\hat{V}_i^j, \hat{\Phi}_i^j\) remain the same, i.e. are given by \((4.5), (4.6)\), but the secondary constraints \((4.7)\) are modified and have the form

\[
\begin{align*}
\hat{Q}^1_2 &\equiv Z_1 P^2 + \bar{P}_1 \hat{Z}^2 - Z_1 \hat{P}^2 - \hat{P}_1 Z^2, \\
\hat{Q}^1_2 &\equiv Z_2 P^1 + P_2 \hat{Z}^1 - \bar{Z}_2 \hat{P}^1 - \bar{P}_2 Z^1.
\end{align*}
\tag{4.9}
\]

Time independence of other constraints \((F_M = \{ V^i_j, \hat{\Phi}_i^j, Q^2_1, Q^2_2 \}; \bar{F}_M = \{ F_M, H_2 \}_\sigma \approx 0\) leads to the following new four conditions:

\[
\begin{align*}
\hat{V}^i_j &\approx 0 \Rightarrow M \bar{M} \lambda_j^i \approx 0, \quad i \neq j, \tag{4.10} \\
\hat{Q}^j_i &\approx 0 \Rightarrow M \bar{M} \lambda^i_j \approx 0, \quad i \neq j. \tag{4.11}
\end{align*}
\]

From \((4.10)\)–\((4.11)\) follows the vanishing of \(L^2_1, L^1_2, \lambda^2_1\) and \(\lambda^1_2\) in the Hamiltonian \((4.13)\). As a result, in comparison with the Hamiltonian \((4.14)\), the constraints \(\hat{V}^2_2, \hat{V}^2_1\) change their nature (from first to second class) and in final Hamiltonian \((4.13)\) remain only \(\hat{V}^1_1, \hat{V}^2_2, \hat{\Phi}^i_j\) as first class constraints. These four constraints have the following canonical nonvanishing equal time PB brackets \((30)\):

\[
\begin{align*}
\{ \hat{\Phi}_i(\sigma), \hat{\Phi}_j(\sigma') \}_\sigma &\equiv \delta_{ij} \left( \hat{\Phi}_i(\sigma) + \hat{\phi}_i(\sigma') \right) \delta'(\sigma - \sigma'), \tag{4.12} \\
\{ \hat{\Phi}_i(\sigma), \hat{V}^j_i(\sigma') \}_\sigma &\equiv \delta_{ij} \hat{V}^i_j(\sigma) \delta'(\sigma - \sigma'). \tag{4.13}
\end{align*}
\]

Calculating at a given time \(\tau\) the Poisson brackets of \(Z_{A_i}, Z^{A_i}\) with the four–parameter generator of local symmetry transformations in our model

\[
\begin{align*}
\sum_{k=1}^2 \int d\sigma \left( \varepsilon_k(\sigma, \tau) \hat{\Phi}_k(\sigma, \tau) + i \varphi_k(\sigma, \tau) \hat{V}^k(\sigma, \tau) \right)
\end{align*}
\tag{4.14}
\]

one obtains using e.g. the considerations in \((28)\) (see Sect. 12.2.2), that the functions \(\varepsilon, \varphi\) describe infinitesimal local world–sheet transformations, and \(\varphi_i, \varepsilon_i\) lead to the Abelian phase transformations \((4.9)\).

Using formulae \((4.5)\)–\((4.9)\) one can also calculate the canonical PB matrix of second class constraints \(D^{A_i}, \bar{D}_{A_i}, \hat{V}^2_1, \hat{V}^1_2, \bar{Q}^2_1, \bar{Q}^1_2\). If we observe that the canonical PB of the constraints \(D^{A_i}, \bar{D}_{A_i}\) with all other constraints \(\hat{\Phi}_i, \hat{V}^i_j, \bar{Q}^i_j, \bar{Q}^i_2\) are proportional to \(D^{A_i}, \bar{D}_{A_i}\), we can show that the Dirac bracket eliminating the constraints \(D^{A_i}, \bar{D}_{A_i}\) (twistor brackets) provide the same algebra of all the constraints as the canonical PB.

Let us calculate finally in our model the number of physical degrees of freedom. The unconstrained phase space field variables \(Z_{A_i}, Z^{A_i}\) contains sixteen field variables (after introducing twistor brackets which eliminate \(P^{A_i}, \bar{P}_{A_i}\)). The second class constraints \(\hat{V}^2_1, \hat{V}^2_2, \bar{Q}^2_1, \bar{Q}^2_2\) remove four, and the first class constraints \(\hat{V}^1_1, \hat{V}^2_2, \hat{\Phi}_i, \varepsilon_i\) remove eight degrees of freedom. In conclusion, we have four fields describing physical real degrees of freedom, as in the case of Nambu–Goto string.

V. FINAL REMARKS

One of important problems of twistorial formulation of string theory is its relation with standard string theory. In this paper we propose to relate these two descriptions of reparametrization–invariant two–dimensional elementary objects in very close way – it appears that already on classical level one can relate the twistor and space–time actions by suitable nonlinear change of variables. In this paper we find new formulation of the twistor string model, which permits to obtain by suitable gauge fixing the bilinear twistorial Lagrangian and the standard twistorial commutation relations (see \((3.6)\)). In such a way we proceed in analogous way as in Nambu–Goto formulation of string model in order to get the solvable bilinear action.

In this paper we performed the complete constraint analysis of our twistor string model and derived the set of local symmetries. After suitable gauge fixing and performing the constraints analysis we are left with a pair of Virasoro algebras, describing the reparametrization of the local world–sheet parameters; besides there are present two KM generators \(U(1) \otimes U(1)\), producing the change of local phases of two twistors.

There are several problems which could be further studied:

1) In our case we consider the linear Lie–algebraic closure of four first class constraints. It is interesting to study if there exists a closure of coupled two Virasoro and four \(U(2)\) KM generators \(\varepsilon_i, \varphi_i\) (six constraints!) in the framework with nonlinearly extended Poisson structures. In such framework one can look for the comparison with interesting considerations in \((11)\), where the twistorial string was constructed without action, by postulating six constraints in the twistorial string phase space.
ii) In [26] it was assumed that the twistorial model analogous to (3.2) can be useful for the description of string–like quarks, as two–dimensional fundamental \( SU(2,2|4) \) fields. It should be added that the twistor-string models in [15, 16, 17, 18, 19, 20, 21] are all supersymmetric in twistorial target space, mostly with \( N = 4 \) \( D = 4 \) supersymmetry. Because of known theoretical advantages of superstring models it is interesting to extend our scheme, in particular to \( D = 10 \) \( N = 1 \) superstring, and possibly perform the dimensional reduction \( D = 10 \rightarrow D = 4 \) [31].

iii) In this paper we discuss the nonchiral bosonic strings. One can also develop our scheme for the twistor strings constructed from two left- (right-) handed twistor fields \( Z_{Ai}(\xi_+) \) \( (Z_{Ai}(\xi_-)) \), where \( \xi_\pm = \tau \pm \sigma \), and consider their supersymmetric extensions. In such a way we would achieve closer link with the twistor string models, proposed in [16, 18].

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