Area-Preserving Structure of Massless Matter-Gravity Fields in 1+1 Dimensions

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ABSTRACT

We derive anomalous Ward identities in two different approaches to the quantization of massless matter-gravity fields in 1+1 dimensions.
I Introduction

1+1 dimensional (“lineal”) quantum gravity is one of the areas of low-dimensional quantum field theory which has attracted more attention in recent years. The role played by symmetries in these theories is obviously very important, and here we shall be concerned with the role of symmetries in the quantization of the 1+1-dimensional matter-gravity field theory with action

$$\mathcal{I}(X, g) = \frac{1}{2} \int d^2 \xi \sqrt{-g} \ g^{\mu \nu} \partial_\mu X^A \partial_\nu X^A ,$$

where $g^{\mu \nu}$ is a metric tensor with signature (1,-1), $A = 1, 2, ..., d$, and $X$ is a d-component massless scalar field.

In the quantization of this theory one necessarily encounters anomalies that break part of the symmetry of the classical theory, which, as seen from $\mathcal{I}(X, g)$, has Weyl and diffeomorphism invariance.

In the conventional quantization approach diffeomorphism invariance is preserved, while renouncing Weyl invariance. Integrating out the matter degrees of freedom using a measure with the appropriate symmetries one obtains an effective pure gravity theory with action

$$\Gamma^D(g) = \frac{d}{96 \pi} \int d^2 \xi_1 d^2 \xi_2 \sqrt{-g(\xi_1)} R(g(\xi_1)) \Box^{-1}(\xi_1, \xi_2) \sqrt{-g(\xi_2)} R(g(\xi_2)) ,$$

where $\Box^{-1}$ is the inverse of the Laplace-Beltrami operator.

Since $\Gamma^D(g)$ is diffeomorphism-invariant but is not Weyl-invariant, the energy-momentum tensor $T^D_{\mu \nu} \equiv (2/\sqrt{-g})(\delta \Gamma^D(g)/\delta g^{\mu \nu})$ is covariantly conserved, but possesses non-vanishing trace

$$\nabla_\mu (g^{\mu \nu} T^D_{\nu \alpha}) = 0 , \quad g^{\mu \nu} T^D_{\mu \nu} = \frac{d}{24 \pi} R(g) .$$

Recently, an alternative approach to the quantization of the classical theory has been considered, in which the functional measure for the integration over the matter fields is Weyl-invariant and invariant under area-preserving diffeomorphisms (i.e. diffeomorphisms of unit Jacobian), but is not invariant under non-area-preserving diffeomorphisms. This leads to the following effective action

$$\Gamma^W(\gamma) = \frac{d}{96 \pi} \int d^2 \xi_1 d^2 \xi_2 \ R(\gamma(\xi_1)) \Box^{-1}(\xi_1, \xi_2) R(\gamma(\xi_2)) ,$$

where $\gamma^{\mu \nu} \equiv \sqrt{-g} g^{\mu \nu}$. The fact that $\Gamma^W(\gamma)$ is Weyl-invariant but is not invariant under general diffeomorphisms leads to the anomaly relations

$$\hat{\nabla}_\mu (\gamma^{\mu \nu} T^W_{\nu \alpha}) = -\frac{d}{48 \pi} \partial_\alpha R(\gamma) , \quad \gamma^{\mu \nu} T^W_{\mu \nu} = 0 .$$

Note that, for simplicity, we set the cosmological constant to zero.

Note that, by appropriate choice of measure, one can obtain more general effective actions that are invariant when $g_{\mu \nu}$ is transformed as $\delta g_{\mu \nu} = \xi^\alpha \partial_\alpha g_{\mu \nu} + g_{\nu \sigma} \partial_\mu \xi^\sigma + g_{\mu \sigma} \partial_\nu \xi^\sigma + ag_{\mu \nu} \partial_\sigma \xi^\sigma$, where $a$ is a fixed real parameter. It is seen that this combination of diffeomorphisms and Weyl transformations is equivalent to the statement that $g_{\mu \nu}$ is a tensor density of weight $a$. Such a modification of the standard formula ($a = 0$) leaves the classical action invariant because the combination $\sqrt{-g} g^{\mu \nu}$ is insensitive to the weight of $g^{\mu \nu}$. The Weyl-invariant approach considered in the present paper corresponds to the limit $a \to \infty$ with the prescription that $\partial_\mu \xi^\mu \to w/a$ for $a \to \infty$, where $w$ is an arbitrary function.
where \( \hat{\nabla} \) is the covariant derivative computed with the metric \( \gamma_{\mu\nu} \), and

\[
T^W_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma^W(\gamma)}{\delta g^{\mu\nu}} = 2 \frac{\delta \Gamma^W(\gamma)}{\delta \gamma^{\mu\nu}} - \gamma_{\mu\nu} \gamma^{\alpha\beta} \frac{\delta \Gamma^W(\gamma)}{\delta \gamma_{\alpha\beta}} ,
\]

while the invariance of \( \Gamma^W(\gamma) \) under area-preserving diffeomorphisms is encoded in the relation

\[
\hat{\nabla}_\mu \hat{\nabla}_\nu (\gamma^{\beta\nu} \epsilon^{\mu\alpha} T^W_{\alpha\beta}) = 0 ,
\]

which is consistent with (5).

In the following we shall derive the anomalous Ward identities both for the Weyl-invariant approach and the conventional diffeomorphism invariant approach, and observe that, although the difference in the symmetries leads to several differences at intermediate steps of the derivation, the final results are equivalent.

II Anomalous Ward Identities

We start by considering the functional integrals

\[
Z^D[J] = \int \frac{Dg}{\Omega_{diff}} \exp \left( i \Gamma^D(g) + i \int \sqrt{-g} g^{\mu\nu} J^D_{\mu\nu} \right) ,
\]

\[
Z^W[J] = \int \frac{D\gamma}{\Omega_{Sdiff}} \exp \left( i \Gamma^W(\gamma) + i \int \gamma^{\mu\nu} J^W_{\mu\nu} \right) ,
\]

where \( J^D_{\mu\nu} \) and \( J^W_{\mu\nu} \) are sources, \( \Omega_{diff} \) is the volume of the diffeomorphism group, and \( \Omega_{Sdiff} \) is the volume of the group of the area-preserving diffeomorphisms. The volume of the Weyl group does not appear in (10) because the functional integral is already written in terms of the Weyl-invariant field \( \gamma \).

\( Z^D[J] \) and \( Z^W[J] \) are the generating functionals for the Green’s functions of the diffeomorphism-invariant approach and the Weyl-invariant approach respectively.

In order to factorize out the gauge volume one can fix the gauge and introduce the corresponding action for the ghost fields. We choose to work in the light-cone gauge, and, after integrating out the ghost fields, \( Z^D \) and \( Z^W \) take the following form

\[
Z^D[J] = \int Dg_{++} \exp \left( i \Gamma^D(g) + i \int g^{++} J^D_{++} \right) ,
\]

\[
Z^W[J] = \int D\gamma_{++} \exp \left( i \Gamma^W(\gamma) + i \int \gamma^{++} J^W_{++} \right) ,
\]

where \( \Gamma^D + \Gamma^D_{gh} \) and \( \Gamma^W + \Gamma^W_{gh} \) are gauge-fixed actions for gravity.

Our choice of gauge is motivated by the fact that in the light-cone gauge \( \Gamma^W + \Gamma^W_{gh} \) takes the same form of \( \Gamma^D + \Gamma^D_{gh} \), and we intend to exploit this correspondence in the investigation of the anomalous Ward identities. Still, in the analysis we shall need to take into account the fact that the measure \( D\gamma \), which is Weyl-invariant but not diffeomorphism-invariant, is different from the diffeomorphism-invariant but not Weyl-invariant measure \( Dg \). Moreover,
since \( \gamma \) and \( g \) have different transformation properties, \( \Gamma^W + \Gamma^D_{gh} \) and \( \Gamma^D + \Gamma^D_{gh} \) satisfy different anomaly relations; specifically [7]

\[
\nabla_\mu (g^{\mu \nu} \Theta^D_{\nu \alpha}) = 0, \quad g^{\mu \nu} \Theta^D_{\mu \nu} = \frac{d - 28}{24\pi} R(g),
\]

(13)

\[
\nabla_\mu (\gamma^{\mu \nu} \Theta^W_{\nu \alpha}) = -\frac{d - 28}{48\pi} \partial_\alpha R(\gamma), \quad \gamma^{\mu \nu} \Theta^W_{\mu \nu} = 0,
\]

(14)

where

\[
\Theta^D_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta (\Gamma^D_{W} + \Gamma^D_{gh})}{\delta g^{\mu \nu}}.
\]

(15)

We now consider the following infinitesimal shifts in the functional variables of integration

\[
\delta_f g_{++} = (2\partial_+ - g_{++} \partial_-) \delta f + \delta f \partial_- g_{++}, \quad \delta_f \gamma_{++} = (2\partial_+ - \gamma_{++} \partial_-) \delta f + \delta f \partial_- \gamma_{++},
\]

(16)

(17)

and observe that

\[
\int \frac{\delta [\Gamma^D(g) + \Gamma^D_{gh}(g)]}{\delta g_{++}} \delta_f g_{++} = \int [\nabla_\mu (g^{\mu \nu} \Theta^D_{\nu \alpha}) - \frac{1}{2} \nabla_- (g^{\mu \nu} \Theta^D_{\mu \nu})] \delta f = \int \frac{28 - d}{48\pi} \partial_+^2 g_{++} \delta f
\]

(18)

\[
\int \frac{\delta [\Gamma^W(\gamma) + \Gamma^W_{gh}(\gamma)]}{\delta \gamma_{++}} \delta_f \gamma_{++} = \int [\nabla_\mu (\gamma^{\mu \nu} \Theta^W_{\nu \alpha}) - \frac{1}{2} \nabla_- (\gamma^{\mu \nu} \Theta^W_{\mu \nu})] \delta f = \int \frac{28 - d}{48\pi} \partial_+^3 \gamma_{++} \delta f,
\]

(19)

where we used the anomaly relations [13] and [14].

Following a standard procedure [4], the relations (18) and (19) lead to the following anomalous Ward identities

\[
\sum_i^n \langle g_{++}(\xi_1) \ldots \delta g_{++}(\xi_i) \ldots g_{++}(\xi_n) \rangle + \frac{d-28+\lambda^D}{i48\pi} \int d\xi^2 \delta f(g) \left\langle \partial_+^2 g_{++}(\xi) g_{++}(\xi_1) \ldots g_{++}(\xi_n) \right\rangle = 0.
\]

(20)

\[
\sum_i^n \langle \gamma_{++}(\xi_1) \ldots \delta \gamma_{++}(\xi_i) \ldots \gamma_{++}(\xi_n) \rangle + \frac{d-28+\lambda^W}{i48\pi} \int d\xi^2 \delta f(g) \left\langle \partial_+^3 \gamma_{++}(\xi) \gamma_{++}(\xi_1) \ldots \gamma_{++}(\xi_n) \right\rangle = 0.
\]

(21)

Here \( \lambda^D \) is the additional contribution to the anomaly which is due to the fact that \( \delta_f g_{++} \) is a composition of a diffeomorphism and a Weyl transformation on \( g_{++} \), and therefore the diffeomorphism-invariant but not Weyl-invariant measure \( D g_{++} \) is not invariant under \( g_{++} \rightarrow g_{++} + \delta_f g_{++} \). Analogously, the presence of \( \lambda^W \) is due to the fact that \( \delta_f \gamma_{++} \) is an infinitesimal (not area-preserving) diffeomorphism transformation on \( \gamma_{++} \), and therefore the measure \( D \gamma_{++} \) is not invariant under \( \gamma_{++} \rightarrow \gamma_{++} + \delta_f \gamma_{++} \). The values of \( \lambda^D \) and \( \lambda^W \) can be fixed by requiring that the theory be independent of the choice of gauge. In Ref. [2] the class of gauges \( g_{--} = g^B_{--}, g_{++} = 1 \) is considered, and it is found that the independence of the partition function on the choice of \( g^B_{--} \) requires that

\[
d - 28 + \lambda^D = \frac{d - 13 - \sqrt{(d - 1)(d - 25)}}{2}.
\]

(22)

Following the corresponding procedure for the Weyl invariant approach one finds that also \( \lambda^W \) must satisfy Eq. (22), i.e. \( \lambda^W = \lambda^D \). This observation together with the results (24) and (21) indicates that the anomalous Ward identities satisfied by \( \gamma_{++} \) in the Weyl-invariant approach are identical to the ones satisfied by \( g_{++} \) in the diffeomorphism-invariant approach. Since these Ward identities completely determine [2] the Green’s functions, also the Green’s functions are identical.
III Conclusion

The investigation of the anomalous Ward identities indicates that the two approaches are equivalent, and this is consistent with the results of the (classical) Dirac Hamiltonian analysis. It appears that the physics described by the model is independent of the local term that one needs to add to the action in order to convert the Weyl anomaly into a diffeomorphism anomaly. This does not always happen in anomalous quantum field theories, for example in the chiral Schwinger model the mass emergent at the quantum level as a result of the anomaly does depend on the coefficient of one such local term.

It is also interesting to notice that in deriving the equivalence of the two approaches at the level of the anomalous Ward identities a key role is played by the component of the combination
\[ \nabla_\mu (g^{\mu\nu} \Theta_{\nu\alpha}) - \nabla_\alpha (g^{\mu\nu} \Theta_{\mu\nu})/2 \]
[see Eqs. (18) and (19)], which (in the chosen gauges) takes the same form in both approaches. Clearly this combination of the anomaly relations encodes some essential feature of the model, but its physical interpretation is not yet clear to us.

Finally, we want to point out that 1+1-dimensional quantum gravities of the type here considered and their supersymmetric extensions are related to some low-dimensional models in statistical physics, such as the Ising model, random surfaces, percolation, tree-like polymers, and self-avoiding polymers. In several occasions results first obtained in the study of the quantum field theories have been useful also in the context of the statistical models and vice versa. Since only recently there has been increased interest in the Weyl invariant approach, the possibility of application of this new viewpoint to the study of statistical models has not yet been investigated.

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