Dirac electron in a Coulomb Field in 2+1 Dimensions

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Abstract

Exact solutions of Dirac equation in two spatial dimensions in the Coulomb field are obtained. Equation which determines the so-called critical charge of the Coulomb field is derived and solved for a simple model.
I. INTRODUCTION

Planar nonrelativistic electron systems in a uniform magnetic field are fundamental quantum systems which have provided insights into many novel phenomena, such as the quantum Hall effect and the theory of anyons, particles obeying fractional statistics \[1,2\]. On the other hand, planar electron systems with energy spectrum described by the Dirac Hamiltonian have also been studied as field-theoretic models for the quantum Hall effect and anyon theory \[3,4\]. Related to these field-theoretic models are the recent interesting studies regarding the instability of the naive vacuum and spontaneous magnetization in (2+1)-dimensional quantum electrodynamics (QED), which is induced by a bare Chern-Simons term \[5\]. In view of these developments, it is essential to have a better understanding of the properties of planar Dirac particles in the presence of external electromagnetic fields.

In this paper we would like to consider solutions of Dirac equation in two spatial dimensions in the presence of a strong Coulomb field, and to discuss instability of the Dirac vacuum in a regulated strong Coulomb field. In three space dimensions the effect of positron production by strong Coulomb field was predicted in \[6\] and were studied in \[7–15\].

II. MOTION OF AN ELECTRON IN THE COULOMNB FIELD

Let us consider a relativistic electron in two spatial dimensions in a Coulomb field the vector potential of which is specified as

\[
A^0(r) = -Ze/r, \quad A^x = A^y = 0. \tag{1}
\]

In 2+1 dimensions the Dirac matrices may be represented in terms of the Pauli matrices. We choose the representation \(\vec{\alpha} = (-\sigma^2, \sigma^1)\) and \(\beta = \sigma^3\). Then the Dirac equation has the form \((c = \hbar = 1)\)

\[
(i\partial_t - H_D)\Psi = 0, \tag{2}
\]

where
\[ H_D = \vec{\alpha} \cdot \vec{P} + \beta m + eA^0 \equiv \sigma^1 P_2 - \sigma^2 P_1 + \sigma^3 m + eA^0 , \]  

is the Dirac Hamiltonian, \( P_\mu = i\partial_\mu - eA_\mu \) is the operator of generalized momentum of electron, \( m \) is the rest mass of the electron, and \( e = -e_0, e_0 > 0 \) is its electric charge. The conserved total angular momentum has only a single component, namely, \( J_z = L_z + S_z \), where \( L_z = -i\partial/\partial \varphi \) and \( S_z = \sigma^3/2 \).

We shall look for solutions of (3) in the form

\[ \Psi(t, x) = \frac{1}{\sqrt{2\pi}} \exp(-i\epsilon Et)\psi(r, \varphi) , \]  

where \( \epsilon = \pm 1 \), and \( E > 0 \) is a positive quantity. We assume the ansatz

\[ \psi(r, \varphi) = e^{il\varphi} \begin{pmatrix} f(r) \\ g(r)e^{i\varphi} \end{pmatrix} , \]  

where \( l \) is an integer number. The function \( \psi(r, \varphi) \) is an eigenfunction of the total angular momentum \( J_z \) with eigenvalue \( l + 1/2 \). Substituting (4) and (5) in (2), and taking into account of the equations

\[ P_x \pm iP_y = -ie^{\pm i\varphi} \left( \frac{\partial}{\partial r} \pm i \frac{1}{r} \frac{\partial}{\partial \varphi} \right) , \]  

we obtain

\[ \frac{df}{dr} - \frac{l}{r} f + (\epsilon E + m + \frac{Z\alpha}{r})g = 0 , \]

\[ \frac{dg}{dr} + \frac{1 + l}{r} g - (\epsilon E - m + \frac{Z\alpha}{r})f = 0 , \]  

where \( \alpha \equiv e^2 = 1/137 \) is the fine structure constant.

The exact solutions and the energy eigenvalues with \( \epsilon E < m \) corresponding to stationary states of the Dirac equation may be found in full analogy with the case of three space dimensions. We shall follow Ref. [16]. Let us look for functions \( f \) and \( g \) in the form

\[ f = \sqrt{m + Ee^{-\rho/2} \rho^{\gamma-1}(Q_1 + Q_2)} , \]

\[ g = \sqrt{m - Ee^{-\rho/2} \rho^{\gamma-1}(Q_1 - Q_2)} , \]  

(8)
where
\[ \rho = 2\lambda r, \quad \lambda = \sqrt{m^2 - E^2}, \quad \gamma = 1/2 + \sqrt{(l + 1/2)^2 - (Z\alpha)^2}. \quad (9) \]

The value of \( \gamma \) is to be found by studying the behavior of the wave function at small \( r \). From (4) and (8), together with the equality \( (\gamma - 1/2)^2 - (Z\alpha E/\lambda)^2 = (l + 1/2)^2 - (Z\alpha m/\lambda)^2 \), one can derive the differential equations satisfied by \( Q_1 \) and \( Q_2 \). It turns out that the functions \( Q_1 \) and \( Q_2 \) which rendered the solutions of (7) finite at \( \rho = 0 \) are given in terms of the confluent hypergeometric function \( F(a, b; z) \) as
\[ Q_1 = AF(\gamma - 1/2 - (Z\alpha E/\lambda), 2\gamma; \rho), \]
\[ Q_2 = BF(\gamma + 1/2 - (Z\alpha E/\lambda), 2\gamma; \rho). \quad (10) \]

The constants \( A \) and \( B \) are related by
\[ B = \frac{\gamma - 1/2 - Z\alpha E/\lambda}{l + 1/2 + Z\alpha m/\lambda}A. \quad (11) \]

The energy eigenvalues are defined by
\[ \gamma - \frac{1}{2} - \frac{Z\alpha E}{\lambda} = -n_r. \quad (12) \]

It is easy to show that the following values of the quantum number \( n_r \) are allowed: \( n_r = 0, 1, 2, \ldots \), if \( l \geq 0 \), and \( n_r = 1, 2, 3, \ldots \) if \( l < 0 \). Therefore, the electron energy spectrum in the Coulomb field (1) has the form
\[ E = m \left[ 1 + \frac{(Z\alpha)^2}{(n_r + \sqrt{(l + 1/2)^2 - (Z\alpha)^2})^2} \right]^{-1/2}. \quad (13) \]

It is seen that
\[ E_0 = m\sqrt{1 - (2Z\alpha)^2} \]
for \( l = n_r = 0 \), and \( E_0 \) becomes zero at \( Z\alpha = 1/2 \), whereas in three spatial dimensions \( E_0 \) equals zero at \( Z\alpha = 1 \). Thus, in two space dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge \( Z|e| \) no longer has a physical meaning at a much lower value of \( Z\alpha = 1/2 \), and the corresponding solution of the Dirac equation oscillates near the point \( r \to 0 \).
III. CRITICAL CHARGE

It is known \[16,14\] that in three spatial dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge \(Z|e|\) becomes purely imaginary when \(Z > 137\), and that its interpretation as electron energy no longer has a physical meaning. To determine the electron energy spectrum in the Coulomb field with such a charge we need to eliminate the singularity of the Coulomb potential of a point-charge at \(r = 0\) by cutting off the Coulomb potential at small distances. This is equivalent to taking into account of the nucleus size. In three space dimensions the electron energy spectrum in the Coulomb field regulated at small distances was first considered in \[13\]. With increasing \(Z\) in the region \(Z > 137\), the electron energy levels in such a field were found to decrease, become negative, and may cross the boundary of the lower energy continuum, \(E = -m\). The value of \(Z|e| = Z_{\text{cr}}|e|\) at which the lowest electron energy level cross the boundary of the lower energy continuum is called the critical charge for the electron ground state \[12–14\]. If \(Z\) continues to grow and enters the transcritical region with \(Z > Z_{\text{cr}}\), the lowest electron energy level “sinks” into the lower energy continuum, which result in a rearrangement of the vacuum of the QED. This rearrangement is constrained by Pauli’s exclusion principle. If the electron ground state at \(Z < Z_{\text{cr}}\) is vacant, two electron-positron pairs are created; if it is half-occupied, one pair is created; and if it is occupied, no pairs are created. The Coulomb potential is repulsive for the created positrons, so they go to infinity. Hence at \(Z > Z_{\text{cr}}\) a quasistationary state appears in the lower energy continuum and the new vacuum of QED, which corresponds to the filling of all the electron states with \(E < -m\), has the total electric charge \(2e\) \[12, 14\]. Indeed, all the electron states with \(E < -m\) (the Dirac sea) were filled at \(Z < Z_{\text{cr}}\), so electrons created by the strong Coulomb field with \(Z > Z_{\text{cr}}\) cannot be described by means of a convenient wave function, and the notion of charged vacuum was introduced to describe these states \[6, 3, 12, 13\]. In terms of the new vacuum, the density of electric charge \(\rho(r)\) is classical. It is a function characterizing the spatial distribution of the real electric charge appearing in the new (charged) vacuum, while in terms of the old
(uncharged) vacuum this function should be interpreted as the probability of two electrons
(with charge $2e$) being present at a given point in space.

We would like to see how the same system behaves in two dimensions. Let us therefore
consider the solutions and the energy eigenvalues corresponding to stationary states of the
Dirac equation in the Coulomb field with $2Z > 137$ and find the corresponding value of
$Z_{cr}$. To find $Z_{cr}$ it is enough to study the energy region near the boundary of the lower
energy continuum, $-m$. We shall rewrite the Dirac equation, taking account of the fact that
$\epsilon E \approx -m$. Introducing functions $F(r) = rf(r)$ and $G(r) = rg(r)$, and eliminating $G(r)$
from (7), we arrive at the equation for the function $F$ near the boundary of the lower energy
continuum $-m$ in the form

$$\frac{d^2 F(r)}{dr^2} + \left( E^2 - m^2 + \frac{2\epsilon E Z \alpha}{r} + \frac{(Z \alpha)^2 - l(l+1)}{r^2} \right) F(r) = 0 .$$

We note that near the boundary of the upper energy continuum for $\epsilon E \approx m$, the function
$G(r)$ obeys the equation (15) with $F(r)$ replaced by $G(r)$.

Solution of (15), which tends to zero at $r \to \infty$, may be expressed by means of the
Whittaker function (see, also [17])

$$F(r) \sim W_{\beta, i \nu/2} (2\lambda r) ,$$

where

$$\beta = \epsilon E Z \alpha / \lambda , \quad \nu = 2 \sqrt{(Z \alpha)^2 - (l + 1/2)^2} , \quad \lambda = \sqrt{m^2 - E^2} .$$

From (7), the function $G(r)$ at $\epsilon E = -m$ can be obtained as

$$G(r) = \frac{1}{Z \alpha} \left( (1 + l) F - r \frac{dF}{dr} \right) .$$

Near the boundary of the upper energy continuum, the function $G(r)$ is given by the Whittaker function in eq. (16) with $\epsilon E = m$, while the function $F(r)$ can be found from the
relation

$$F(r) = \frac{1}{Z \alpha} \left( lG + r \frac{dG}{dr} \right) .$$
Using the asymptotic representation for Whittaker function at large \(|z|\) in the form

\[ W_{\beta,\mu}(z) \sim e^{-z/2}(z)^\beta, \quad (20) \]

it is seen that the bound electron state (with \(|E| < m\) or \(\lambda > 0\)) is localized in the plane.

Such a behavior of the wave function of the bound electron state may be easily understood if we treat (13) as a one-dimensional Schrödinger-type equation which describes a particle with “particle energy” \( E' = (E^2 - m^2)/2m \) in the field of the effective potential (in particular, for \( l = 0 \))

\[ U_{\text{eff}}(r) = -\epsilon E Z\alpha/mr - (Z\alpha)^2/2mr^2. \]

We note that the effective potential is wide enough near the boundary of the lower energy continuum (for behavior of the effective potential in three space dimensions see, for eg. [14]), and that the effective potential in two space dimensions does not contain the spin electron term \(-s(s+1) \equiv -3/4\).

The solution of (15) at \( \epsilon E = -m \) can be written in terms of the MacDonald function of imaginary order

\[ F(r) = \sqrt{r}K_{|l|}(\sqrt{8mZ\alpha}r). \quad (21) \]

The function \( G(r) \) at \( \epsilon E = -m \) is determined by (18). In the following we shall determine the critical value \( Z_{\text{cr}} \) for a simple model in which the potential \( A_0(r) \) is regulated at small distances as follows:

\[ A_0^Z(r) = -Ze/r, \quad r \geq R; \quad A_0^Z(r) = -Ze/R, \quad r \leq R. \quad (22) \]

In the region \( r \leq R \) the function \( F(r) \) obeys the equation

\[ \frac{d^2F}{dr^2} - \frac{dF}{dr} + \left( \epsilon E + \frac{Z\alpha}{R} \right)^2 - m^2 + \frac{1 - l^2}{r^2} \right) F(r) = 0. \quad (23) \]

The solution of (23) is

\[ F(r) = r(A_1 J_{|l|}(kR) + B_1 Y_{|l|}(kR)), \quad (24) \]
where
\[ \kappa = \sqrt{(\epsilon E + \frac{Z\alpha}{R})^2 - m^2}, \] (25)
and \( J_n(z) \) and \( Y_n(z) \) are the Bessel and the Neumann functions of integer order \( n \).

In order for the function \( F(r) \) to be finite at the point \( r = 0 \) we need to set \( B_1 = 0 \). To determine the energy spectrum we need to match the solutions at the point \( r = R \):
\[ \left( \frac{G(r)}{F(r)} \right)_{r=R-0} = \left( \frac{G(r)}{F(r)} \right)_{r=R+0}. \] (26)
Taking into account of the fact that \( R \) is much less than \( 1/m \), so that \( \kappa \approx Z\alpha/R \), we obtain, for the state with \( l = 0 \) and \( \epsilon E = -m \), the following equation that determine (at fixed \( R \)) the critical charge:
\[ \frac{J_1(X)}{J_0(X)} = \frac{1}{2X} \left( 1 - \sqrt{z} \frac{K_{i\nu}^\prime(z)}{K_{i\nu}(z)} \right). \] (27)
Here \( X = Z_{\text{cr}}\alpha, \ \nu = \sqrt{4X^2 - 1}, \ z = \sqrt{8mRX} \), and \( K_{i\nu}^\prime(z) = dK_{i\nu}(z)/dz \). Eq. (27) may be solved numerically. As we are interested only in the critical charge corresponding to the ground state, we can consider small values of \( z \). In this case, the Macdonald function with imaginary order \( K_{i\nu}(z) \) has the following expansion:
\[ K_{i\nu}(z) \to \sqrt{\frac{\pi}{\nu \sinh \pi \nu}} \left[ \sin \left( \nu \ln \frac{2}{z} + \arg \Gamma(1 + i\nu) \right) \right. \]
\[ \left. + \frac{z^2}{4\sqrt{1 + \nu^2}} \sin \left( \nu \ln \frac{2}{z} + \arg \Gamma(1 + i\nu) + \tan^{-1}(1) \right) + \ldots \right]. \] (28)
(29)
Numerical solutions of eq.(27) give \( Z_{\text{cr}} \approx 84, 89 \) at \( Rm = 0.02 \) and 0.03, respectively. For comparison purpose, we recall that \( Z_{\text{cr}} \approx 170 \) at \( Rm = 0.03 \) for the analogical model in three space dimensions [12, 14].

Thus, the Dirac vacuum in two space dimensions in the presence of a strong Coulomb field is unstable against electron-positron production at significantly smaller values of the critical charge than in the case of three spatial dimensions. Another difference between these two cases results from the fact that electrons confined to a plane behave like a spinless fermion. So if the ground electron state at \( Z < Z_{\text{cr}} \) is vacant, one pair is created; if it is occupied, no pairs are created.
IV. SUMMARY

In this paper we present the exact solutions of the 2 + 1-dimensional Dirac equation with a Coulomb field, and determine the critical charge $Z_{cr}$ of a regulated Coulomb source for which the Dirac vacuum of the system become unstable. At $Z > Z_{cr}$ the lowest electron state of discrete spectrum is the state with $n_r \neq 0$. So if the electron ground state at $Z < Z_{cr}$ was vacant, then at $Z > Z_{cr}$ an electron is created, together with a hole in the lower energy continuum. According to Dirac this hole is to behave as a real positive charged particle far from the Coulomb center. Thus, phenomena that may occur at $Z > Z_{cr}$ are many-particle, and to describe them it is necessary to apply the quantum field theory. From the point of view of QED, the strong Coulomb field with $Z > Z_{cr}$ creates a positron and changes the vacuum in such a way that it gains the electric charge which is exactly equal to the electron charge $e$. The spatial distribution of the electric charge appearing in the vacuum looks like the spatial distribution of the electron charge in the level with $n_r = 0$ in an atom with $Z < Z_{cr}$. However, the density of the vacuum electric charge is a function characterizing the spatial distribution of the real electric charge appearing in the vacuum, while in the atom this function gives the probability density that the electron (with charge $e$) may be found at a given point in space.

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