Simple Finite-Dimensional Modules and Monomial Bases from the Gelfand-Tsetlin Patterns

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Abstract
One of the most important classes of Lie algebras is $\mathfrak{sl}_n$, which are the $n \times n$ matrices with trace 0. The representation theory for $\mathfrak{sl}_n$ has been an interesting research area for the past hundred years and in it the simple finite-dimensional modules have become very important. They were classified and Gelfand and Tsetlin actually gave an explicit construction of a basis for every simple finite-dimensional module. This paper extends their work by providing theorems and proofs, and constructs monomial bases of the simple module.

Keywords: Finite-dimensional; Module; Irreducible; Representation; Monomial basis.

1. Introduction
Let $\mathfrak{g}$ be a Lie algebra of all matrices of order $n$. In this paper, we work with finite-dimensional modules and hence finite-dimensional representation of $\mathfrak{sl}_n$. This means for $g \in \mathfrak{sl}_n$, there exists a matrix $G$ of order $N$ defined in such a way that

if $g \to G$ and $f \to F$, then $\lambda g + \mu f \to \lambda G + \mu F$ and $[g, f] \to [G, F]$.

Choose integers $m_1, m_2, \ldots, m_n$ such that the inequality $m_1 \geq m_2 \geq \ldots \geq m_n$ is satisfied. These partitions are quite important because they appear to be the core in constructing representations. These chosen integers are used to construct some index set $\xi$ (the explicit construction of this index set will be given in the next section). For a Lie algebra with order $n$, we could construct at least $\frac{n(n-1)}{2}$ possible number of such $\xi$ with entries from a given partition. An example will be given in the next section.

Let $e_{ij}$ be a matrix of order $n$ which has 1 at the intersection of the $i^{th}$ row and the $j^{th}$ column and zeros in all other places and let $E_{ij}$ be the matrix of order $N$ from $\mathfrak{g}_n$. Note that $E_{ij}$, under our consideration corresponds to elements $e_{ij} \in \mathfrak{g}_n$. It is easy to see that each matrix $E_{ij}$ forms a linear combination of $e_{ij}$; that is $E_{ij} = \sum a_{ij} e_{ij}$ for some $a_{ij}$. Therefore, the set $E_{ij}$ distinctively defines some representation. One could find all such representations by explicitly describing all linear transformations $E_{ij}$.

The quest for irreducible representations of special linear algebra $\mathfrak{sl}_n$ was reformulated: one needs matrices $E_{ij}$ of order $N$ satisfying the following bracket relations:

$[E_{ij}, E_{kl}] = E_{il}$ when $i \neq l$,

$[E_{ij}, E_{jk}] = E_{ik} - E_{kj}$,

$[E_{ij}, E_{kl}] = 0$ when $j_1 \neq i_2$ and $i_1 \neq j_2$.

For irreducibility, the system $E_{ij}$ is required to have no invariant subspaces.

The representation theory of $\mathfrak{sl}_n$ has a unique nature in choosing a partition. For the classification of simple finite dimensional modules, one sets the last choice $m_n = 0$ in the partition. This controls differences between subsequent choices in a partition.

A comprehensive theory of infinitesimal transformations was first given by a Norwegian mathematician, Sophus Lie (1842-1899). I. M. Gelfand and M. L. Tsetlin gave an explicit construction of a basis for every simple finite-dimensional module of $\mathfrak{sl}_n$. In their work, they gave all the irreducible representations of general linear algebra ($\mathfrak{gl}_n$) but without theorems [1]. Recently, V. Futorny, D. Grantcharov and L. E. Ramirez provided a classification and explicit bases of tableaux of all irreducible generic Gelfand-Tsetlin modules for the Lie algebra $\mathfrak{gl}_n$ [2]. In 2016, V. Futorny, D. Grantcharov, and L. E. Ramirez initiated the systematic study of a large class of non-generic Gelfand-Tsetlin modules - the class of $\lambda$-singular Gelfand-Tsetlin modules. An explicit tableaux realization and the action of $\mathfrak{gl}_n$ on these modules was provided using a new construction which they call derivative tableaux. Their
construction of 1—singular modules provides a large family of new irreducible Gelfand-Tsetlin modules of \( \mathfrak{gl}_n \), and is a part of the classification of all such irreducible modules for \( n = 3 \) [3].

This paper will show that the Gelfand-Tsetlin constructions given in the year 1950 [1] forms all the irreducible representations of special linear algebra \( SL_n \) by providing proofs to results. It will also show that \( SL_n \) —module is simple and also construct monomial basis from these modules. Section 2 discusses some previous work and gives some notations and Section 3 presents proofs to results and shows that \( SL_n \) —module is simple. Then a conclusion is drawn in Section 4.

2. Notations and Preliminaries

Definition 1 (Upper Triangular Matrix). This is a matrix with entries \((i, j)\) where \(i \geq j\) are zeros. Let \( U^+ \) be the set of all upper triangular matrices. \( E_{ij}, E_{ij} \in U^+, \) then \([E_{ij}, E_{kj}] \in U^+\). Therefore, \( U^+ \) is a Lie algebra and \( E_{ij}, i < j \) is a basis of \( U^+\). So \( E_{ij} \) acts by zero. Hence \([E_{ij},\xi]\) are generators of \( U^+\). We will denote a sequence of upper triangular matrices by \( E^\xi \) and a sequence of upper triangular matrices in relation to \( \xi \) by \( E^\xi(\xi)\).

Definition 2 (Lower Triangular Matrix). This is a matrix with entries \((i, j)\) where \(i \leq j\) are zeros. Similarly, from now on, we will denote a lower triangular matrix by \( F_{ij} \) such that entry \((i, j)\) has a 1 and all others are zeros. Let \( U^- \) be the set of all lower triangular matrices. \( F_{ij}, F_{ij} \in U^-\), then \([F_{ij}, F_{kj}] \in U^-\). Therefore, \( U^-\) is a Lie algebra and \( F_{ij}, i > j \) is a basis of \( U^-\), with \( F_{ij} \) acting by zero, so \([F_{ij},\xi]\) are generators of \( U^-\). Similarly, we will denote a sequence of lower triangular matrices by \( F^\xi \) and a sequence of lower triangular matrices in relation to \( \xi \) by \( F^\xi(\xi)\).

Definition 3 (Diagonal Matrix). This is a matrix with some non-zero entries on its diagonal while all other entries away from the diagonal are zero.

It is well known that the entries of the diagonals of such a square matrix are the eigenvalues. Let \( h \) be the set of all diagonal matrices with trace zero. For \( H_{ij}, H_{ij} \in h, \) \([H_{ij}, H_{ij}] = 0 \in h, \) so \([h, h] = \{0\}\) is a basis of \( h\). Suppose \( h^* = \{\varphi: h \to \mathbb{C} | \varphi \) is linear\}. The map \( \varphi \) is defined by giving the image of \( H_{ij} \), for all \( i \). By definition, \( H_{ij} \) generates all of \( h \), then \( \{H_{ij}\} \) is a basis of \( h^*\).

Definition 4 (Representation \([4]\)). Suppose \( L \) is a Lie algebra and let \( x, y \in L \). The operation \( \rho: L \to \text{End}(K_n) \)

\[ \rho([x,y]) = [\rho(x),\rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x). \]

is a Lie algebra representation. The vector space \( K_n \) is the representation space. The bracket \([\cdot,\cdot]\) is bilinear and also an endomorphism. That means \([\cdot,\cdot]: \text{End}(K_n) \times \text{End}(K_n) \to \text{End}(K_n)\).

It is easy to see that the Lie algebra \( sl_n = U^- \bigoplus h \bigoplus U^+ \). If \( K_n \) is a finite dimensional module, then \( H \in h \) (where \( H = \bigoplus^m_i H_i \)) acts on \( K_n \) such that

\( K_n = H \cdot \xi_1 + \cdots + H_n \cdot \xi_n = \bigoplus^{m}_{i} K_n(h_{\lambda}). \)

where \( \lambda \) runs over \( h^*\) (a dual) and \( (K_n)_{\lambda} = \{ \xi \in (K_n) | H \cdot \xi = \lambda(\xi), \forall H \in h \} \).

The weight spaces \( (K_n)_{\lambda} \) are infinitely many and different from zero when \( K_n \) is infinite dimensional. \( (K_n)_{\lambda} \) is called a weight space, \( \xi \) a weight vector and we called \( \lambda \) a weight of \( K_n \). A highest weight vector (maximal vectors) in \( sl_n \) —module is a non-zero weight vector \( \xi \) annihilated by the action of all upper triangular matrices. We will prove in this paper that a highest weight vector is indeed maximal and hence a generator.

The index set, \( \xi \) is an interesting construction and we will show how it is built. \( K_n \) is a vector space with bases \( \xi \) [1] . These bases depend on the choice of integer partition \( m_1, m_2, \ldots, m_n \) with \((m_1 \geq m_2 \geq \cdots \geq m_n)\).

\[ \xi = \left( \begin{array}{c} p_{1,1} \\ p_{2,1} \\ \vdots \\ p_{j,i} \\ \vdots \\ p_{n,j} \\ \end{array} \right) \quad (1) \]

In order to understand the construction of this basis vector quite well, let us consider rows \((i - 1), i \) and \((i + 1)\) and entry \( p_{i,j} \) in \( \xi \). For all \( p_{i,j} \), if \( j < 1 \) or \( j > n - 1 \), then \( p_{i,j} \) is not an entry in the index set. Otherwise, the relations of the three rows and specifically the entry \( p_{i,j} \) are

\[ \begin{cases} p_{i,j} \geq p_{i+1,j} \geq p_{i-1,j+1} \\ p_{i+1,j-1} \geq p_{i,j} \geq p_{i+1,j}, \\ p_{i,j} = m_j. \end{cases} \]

Below is a pictorial representation of \( p_{i,j} \).
Let $m_1 = 1, m_2 = 1$, and $m_3 = 0$. All possible bases from this partition, as given by the construction of Figure 1, are
\[
\left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).
\]

Here, we discuss the module structure on $K_n$. Our representation space $K_n$ is a $\mathfrak{sl}_n$-module. Although this is true, we will not prove it. It is a $\mathfrak{sl}_n$-module via actions of upper triangular matrices, lower triangular matrices and the diagonal matrices on $\xi$ [1].

In Gelfand [1], a comprehensive construction was presented for the action of upper triangular, lower triangular and diagonal matrices on basis vector $\xi$. For upper triangular matrices in general, suppose $\xi^l_{k-1,k}$ is the pattern obtained from $\xi$ by replacing $m_{i,k-1}$ with $m_{i,k-1} + 1$, the upper triangular matrix $E_{k-1,k}$ acts on $\xi$ as
\[
E_{k-1,k}(\xi) = \sum_j a^j_{k-1,k}(\xi^l_{k-1,k}).
\]  
(2)

For a $3 \times 3$ matrix, the action of $E_{ij}$ on $\xi$ raises the $i^{th}$ row in the basis $\xi$ by 1 on every entry in that row accordingly. For the case $n = 3$, the formulas for computing the action can be found in Gelfand [1]. In general, $E_{ij}$ is generated by $E_{i+1,j}$.

The action of $F_{ij}$ on $\xi$ reduces the entries of the $i^{th}$ row in $\xi$ by 1 accordingly. This is done in such a way that rules governing the size of entries are observed. Suppose $\xi^l_{k,k-1}$ is the pattern obtained from $\xi$ by replacing $m_{i,k-1}$ with $m_{i,k-1} - 1$. The lower triangular matrix $F_{k,k-1}$ acts on $\xi$ as
\[
F_{k,k-1}(\xi) = \sum_j b^j_{k,k-1}(\xi^l_{k,k-1}).
\]  
(3)

The formulas for $n = 3$ can be found in Gelfand [1]. In general, $F_{i+1,j}$ generates all $F_{ij}$ and other actions can be computed using the Lie bracket operation.

The diagonal matrices can also be generated by $E_{i,i+1}$ and $F_{i+1,i}$. Some coefficients from the action of $H_{ij}$ can be zero but not all coefficients. In general
\[
H_{ij}(\xi) = \left( \sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} \right)(\xi) \quad \text{where} \quad \left( \sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} \right)
\]  
(4)

is the coefficient of $\xi$. The formulas for computing the action of diagonal matrices $(H_{ij})$ when $n = 3$ can be found in Gelfand [1].

Theorem 5 (sln_module). The representation space $K_n$ is a $\mathfrak{sl}_n$-module.

A highest weight vector is the weight vector that is annihilated by every $(n \times n)$ upper triangular matrix (that is $E_{ij}$ with $i < j$). We fixed $\xi$ as our basis vector in $K_n$, the representation space where $q$ is any integer depending on some conditions [1]. The nature of each basis vector depends on the dimension $n$ of operator $E_{ij}$ acting on it and the partition. For $n = 2$, we choose some integers $m_1, m_2$, such that $m_1 \geq q \geq m_2$ is satisfied. When $n = 3$, we choose three integers $m_1, m_2, m_3$ such that $m_1 \geq q \geq m_2 \geq m_3$. The bases vectors in the representation space are now numbered by triples, $p_1, p_2, q$. The representation is given by $m_1 \geq p_1 \geq m_2 \geq p_2 \geq m_3$ and $p_1 \geq q \geq p_2$.

We have our bases vectors of the form
\[
\xi = \left( \begin{array}{c} p_1 \\ q \\ p_2 \end{array} \right).
\]

Every weight vector has a corresponding weight. The bases vectors are the weight vectors. Constructing these bases depends on the choices of $m_1 \geq m_2 \geq \cdots \geq m_n$ as defined above.

Suppose $H_{ij}$ is a square diagonal matrix. The action $H_{ij}(\xi) = \kappa_i(\xi)$, where $\kappa_i$ is the eigenvalue of corresponding weight vector $\xi$. There is a map $\omega$ such that for $h \in H$, $\omega(h) \in \mathbb{C}$ such that $H_{ij} \mapsto \kappa$. The map $\omega$ is the weight. Now,
for arbitrary partition $m_1 \geq m_2 \geq \cdots \geq m_n$, $H_{i,k}(\xi) = \kappa_k(\xi)$ has weight $\omega = \kappa_1 \epsilon_1 + \kappa_2 \epsilon_2 + \cdots + \kappa_n \epsilon_n$ where $\epsilon_i$ is the weight for $\epsilon_1, \epsilon_1 + \epsilon_2$ the weight of $\epsilon_2$ and so on. Since $s_{\lambda_i}$ is trace free, $\epsilon_1 + \cdots + \epsilon_n = 0$. In general, 

$$H_{i,k}(\xi) = \left( \sum_{k=1}^{i} m_{i,k} - \sum_{k=i}^{i-1} m_{i,k-1} \right) (\xi).$$

Suppose we let $m_1 = 3, m_2 = 2$ and $m_3 = 0$. Then $E_{1,2}^1 E_{2,3}^2 E_{1,2}^1 \xi = \beta$ where $\beta$ is a highest weight vector and $l_i$ is the maximum times each operator can act on $\xi$ while all conditions are observed to either raise the first row or the second row of $\xi$. Due to the nature of transitions as a consequence of the action of the sequence, the result is unique (proved later).

The representation space $K_n$ is simple if for all $v \in K_n$, there exist upper triangular square matrices such that 

$$E^\alpha(\xi) \cdot v = \beta,$$

a highest weight vector. The weight vectors could be of the form $v = \sum a_i(\text{patterns}) \alpha_i \alpha$ where $\alpha$ is a weight vector. We will show that there exists a sequence of upper triangular matrices $E^\alpha(\xi)$ such that its action on any sum of weight vectors annihilates all but one. That resulting weight vector is a highest weight vector.

### 2.1. Main Results

Theorem 6. The representation space $K_n$ is a simple $s_{\lambda_i}$ module.

This theorem requires a proof for many parts so we break it down into two propositions and two lemmas.

Proposition 7. For every given partition there is a highest weight vector, $\beta$.

Proof. Suppose for integers $m_1, m_2, \ldots, m_n$ with $(m_1, m_2, \ldots, m_n)$ that 

$$\beta = \left( \begin{array}{cccc} m_1 & m_2 & m_3 & \cdots & m_n \end{array} \right).$$

Suppose there exists some $\xi_i$ such that (we have a total ordering) 

$$\xi_1 < \xi_2 < \cdots < \xi_s$$

and $E_{i,i+1} \cdot \xi_1 \neq 0$ and $E_{i,i+1} \cdot \xi_2 \neq 0$ and so on. Also, suppose that entries in both basis vectors $\xi_1, \xi_2$ are equal at the bottom, except for a certain row such that in that row, the sum of the entries for $\xi_1$, denote the first entry of the row by $^{1}\xi_1$ and the second entry by $^{2}\xi_1$ and so on) 

$$^{1}\xi_1 + ^2\xi_1 + \cdots + ^s\xi_1 < ^{a_1}\xi_2 + ^{a_2}\xi_2 + \cdots + ^{a_s}\xi_2.$$ 

We can write $\xi_1 = (v - \xi_1)$, where $\xi_1$ is some weight vector and $v = \sum_{i=1}^{s} c_i \xi_i, c_i \neq 0$ is a complex number. The action 

$$E_{i,i+1} \cdot v = E_{i,i+1} \cdot \xi_1 + E_{i,i+1} \cdot (v - \xi_1) = \xi_1 + \sum \Psi_{i,j},$$

where $\Psi_{i,j}$ is the set of all resulting weight vectors the sum of whose entries in the $n$th row are greater than that of $^{1}\xi_1$. Now, with a sequence of upper triangular matrices which raises the entries of $^{1}\xi_1$, 

$$E^\alpha(\xi_1) \cdot v = E^\alpha(\xi_1) \cdot \sum c_i \xi_i$$

The sequence is actually raising the weight vectors by the series of actions and the supposedly the smallest basis vector becomes a highest weight vector as a consequence. So 

$$E^\alpha(\xi_1) \cdot \sum c_i \xi_i = \lambda_\beta \beta + \sum \Psi_{i,j} \beta$$

Therefore, $\beta$ is a highest weight vector.

The weight for $\beta$ is such that 

$$H_{i,k} \cdot \beta = \left( \sum_{k=1}^{i} m_{i,k} - \sum_{k=i}^{i-1} m_{i,k-1} \right) \cdot \beta$$

$$= [q + (p_1 + p_2 - q) + (m_1 + m_2 + m_3 - p_1 - p_2) + \cdots + q + (p_1 + p_2 - q) +$$

$$+ (m_1 + m_2 + \cdots + m_n - p_{i-1} - p_{i-2} - \cdots - p_{n-1})] \cdot \beta$$

So $\beta$ has weight $c_1 \epsilon_1 + c_2 \epsilon_2 + \cdots + c_n \epsilon_n$, which is a highest weight. Q.E.D.

Proposition 8. For any basis vector $\xi$, there exists a set of upper triangular matrices, $E^\alpha(\xi)$, such that 

$$E^\alpha(\xi) \cdot \xi = \lambda_\beta \beta$$

for $\lambda_\beta \neq 0$.

where 

$$E^\alpha(\xi) = E_{1,2}^{a_1} E_{2,3}^{a_2} \cdots E_{n-2,n-1}^{a_{n-2}} E_{n-1,n}^{a_{n-1}} E_{n,n+1}^{a_n}$$

Proof. From the order $^{1}\xi_1 < ^2\xi_2 < \cdots < ^s\xi_s$ introduced in Proposition 7, we see that $^{1}\xi_1$ is smaller than all other basis vectors. The action
\[ E^\xi(\xi) \cdot \xi = E^\xi(\xi_1) \cdot \xi_1^1 + E^\xi(\xi_1) \left( \sum_{\psi_{i,j} > \xi_1} \psi_{i,j} \right). \]

But \( (\sum_{\psi_{i,j} > \xi_1} \psi_{i,j}) \) will be annihilated by the action since its elements are bigger and \( E^\xi(\xi) \) will be the sequence that raises \( \xi_1 \) to \( \beta \), which is a highest weight. Q.E.D.

Lemma 9. Suppose \( v \) is a non-zero element in \( K_n \),
\[ v = \sum_{i=1}^s c_i \xi_i, \text{ with } c_i \neq 0 \in \mathbb{C}. \]

Then there exists a sequence of upper triangular matrices such that
\[ E^{\xi(v)} \cdot v = \lambda_\beta \beta \text{ where } \lambda_\beta \neq 0. \]

Proof. From Proposition 7, for \( \xi_1 < \xi_2 \), we established that
\[ \xi_1^1 < (\xi_1^2 + \cdots + \xi_1^s) < (a_1 \xi_2 + \cdots + a_s). \]

Then, for all \( c_i \neq 0 \),
\[ E^{\xi(\xi_i)} \cdot v = E^{\xi(\xi_i)} \cdot \left( \sum_{i=1}^s c_i \xi_i \right) = E^{\xi(\xi_i)} \cdot \left( c_1 \xi_1 + \sum_{\xi_i > \xi_1} c_i \xi_i \right). \]

Since \( \xi_1^1 \) is the smallest basis, the action will be
\[ E^{\xi(\xi)} \cdot v = \lambda_\beta \beta + \sum_{\xi_i > \xi_1} \lambda_i \xi_i = \lambda_\beta \beta. \]

Therefore, \( E^{\xi(\xi)} \cdot v = \lambda_\beta \beta \). Q.E.D.

This implies

Corollary 10. If \( S \subset K_n \) is a non-zero submodule, then \( \beta \in S \).

We proved from Proposition 7 that there is a highest weight vector \( \beta \in K_n \). So if \( S \subset K_n \) is a non-zero submodule, then \( \beta \in S \).

Let \( M \) be a simple finite-dimensional module and \( \nu \) be a highest weight vector, the following result claims that \( M \) is generated by \( \nu \) through applying iterative lower triangular matrices on \( \nu \). We can view this iterated applying as being a product in some algebra (namely the universal enveloping algebra).

Definition 11 (Monomial Basis). For \( M \) a finite-dimensional module and \( \nu \) a highest weight vector, consider the fixed basis \( \{ \xi_i \} \) and the monomials in these only. A given set \( \{ \nu_{i,j} \} \) of monomials is called a monomial basis of \( M \) if
\[ \nu_{i,j} = \prod_{k=i}^N \xi_k \]

where \( \xi_i = x_i \) and \( \xi_2 = \sum y_{i,j} \). Suppose
\[ \sum_{i=1}^N c_i \nu_{i,j} = 0, \]
where \( N \) is the size of the basis \( \xi_i \), and all \( c_i \neq 0 \). Then
\[ \sum_{i=1}^N c_i \left( x_i + \sum_{y_{i,j} > x_i} y_{i,j} \right) = 0. \]

We fix \( x_i \) such that \( x_i < x_2 < \cdots < x_N \). So
\[ \sum_{i=1}^N c_i \left( x_i + \sum_{y_{i,j} > x_i} y_{i,j} \right) = c_1 x_1 + \sum_{i=2}^N c_i \left( x_i + \sum_{y_{i,j} > x_i} y_{i,j} \right) + y_1 = 0. \]

We know \( x_i \) is the smallest and \( \{ x_i \} \) are linearly independent for \( 1 \leq i \leq N \), then \( c_i = 0 \). Therefore, the set \( \{ \nu_{i,j} \mid \nu_{i,j} \in B \} \)
is linearly independent.
We are given that \( \xi_i \) is a basis of \( K_n \) implying \( \xi (\beta \text{ in particular}) \) is a basis element. The cardinality of \( \xi \) is \( D \) (that is \( \dim K_n \), in other words the number of basis vectors one can make from a given partition). Since \( \xi \) has \( N \) linearly independent elements, then \( \dim \{ F^{a(\beta)} \cdot \beta \} = \dim K_n = D \). So \( \{ F^{a(\beta)} \cdot \beta \mid F^{a(\beta)} \in B \} \) spans and is a basis in \( K_n \). Since \( \{ F^{a(\beta)} \cdot \beta \mid F^{a(\beta)} \in B \} \) spans \( K_n \) and all its elements are linearly independent, then it is all of \( K_n \). Therefore, the weight vector \( \beta \) generates all of \( K_n \). Q.E.D.

From the above proofs, we can make out that if \( \beta \) is a highest weight vector, \( S \) a submodule of \( K_n \) (i.e \( \beta \in S \) and \( S \) is all of \( K_n \)) implies \( \beta \) generates all of \( K_n \). Therefore, there is no invariant subspace of \( K_n \).

Corollary 13. The representation space \( K_n \) is generated by \( \beta \), and moreover if \( S \subset K_n \) is a non-zero submodule, then \( S = K_n \).

This completes the proof for Theorem 6. So, the representation space \( K_n \) is a simple \( sl_n \) –module. Already, a monomial basis is constructed in Lemma 12.

3. Conclusion

In this paper, our representation is actually \( \rho: sl_n \rightarrow \text{End}(K_n) \) where \( x \mapsto \rho(x) \). The map \( \rho \) is linear and also the identity. Suppose \( v \in K_n \) and \( v = \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n \), where \( \xi_1, \cdots, \xi_n \) are basis vectors and \( \lambda_1, \cdots, \lambda_n \) are non-zero coefficients. Let \( \lambda_1 \neq 0 \) and \( \lambda_2 = \cdots = \lambda_n \). Then \( F_{ij} \cdot v = \lambda_1 F_{ij} \cdot \xi_1 \) and \( E_{ij} \cdot (F_{ij} \cdot v) = \lambda_1 E_{ij} \cdot (F_{ij} \cdot \xi_1) \) are both well defined operations in our representation. Now, let \( \lambda_1 = \cdots = \lambda_n \) and \( \lambda_2 \neq 0 \). Then \( E_{ij} \cdot v = \lambda_2 E_{ij} \cdot \xi_2 \) and \( F_{ij} \cdot (E_{ij} \cdot v) = \lambda_2 F_{ij} \cdot (E_{ij} \cdot \xi_2) \) again are both well defined operations in our representation. The diagonal matrices act by a scalar; that is \( H_{ij} \cdot \xi = \kappa \xi \). In all the actions above, the results are all accounted for in formulas of Equations (2), (3) and (4). If \( \xi_1, \cdots, \xi_n \in S \), then \( S \) is all of \( K_n \). So, \( \rho \) has no invariant subspace. Therefore, \( \rho \) is an irreducible representation of the special linear algebra, \( sl_n \).

For any partition, we can construct all possible basis vectors and modules as discussed above. We apply total ordering on basis vectors to identify the smallest basis vector. A sequence of upper triangular matrices that acts maximally on the smallest bases vector will eventually act on a set of bases vectors resulting in a total annihilation of all bases vectors but raising the smallest basis vector maximally, to a highest weight vector which has weight \( \omega_1 = c_1 \xi_1 + \cdots + c_n \xi_n \). We also proved that every basis vector has a sequence of upper triangular matrices that acts on it maximally to yield a highest weight vector. We proved the existence of monomial basis and gave a construction. Each of these results contributes in proving our main result, that \( sl_n \) –module is simple, and has monomial basis.

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