REMARKS ON COUNTING NEGATIVE EIGENVALUES OF SCHRÖDINGER OPERATOR ON REGULAR METRIC TREES

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To Victor Petrovich Khavin, a friend and colleague, on his 75-th birthday

ABSTRACT. We discuss estimates on the number \( N_-(H_{\alpha V}) \) of negative eigenvalues of the Schrödinger operator \( H_{\alpha V} = -\Delta - \alpha V \) on regular metric trees, as depending on the properties of the potential \( V \geq 0 \) and on the value of the large parameter \( \alpha \). We obtain conditions on \( V \) guaranteeing the behavior \( N_-(H_{\alpha V}) = O(\alpha^p) \) for any given \( p \geq 1/2 \). For a special class of trees we show that these conditions are not only sufficient but also necessary. For \( p > 1/2 \) the order-sharp estimates involve a (quasi-)norm of \( V \) in some ‘weak’ \( L_p \)- or \( \ell_p(L_1) \)-space. We show that the results obtained can be easily derived from the ones of [8].

The results considerably improve the estimates found in the recent paper [5].

1. INTRODUCTION

1.1. Preliminaries. In this note we discuss estimates for the number \( N_-(H_V) \) of the negative eigenvalues of the Schrödinger operator \( H_V = -\Delta - V \) on regular metric trees. The classical Birman – Schwinger principle allows one to reduce such estimates to the study of the spectrum of a certain compact, self-adjoint operator acting in an appropriate Hilbert space; see Section 5 below for an explanation of this reduction. The detailed study of this spectrum was initiated by Naimark and the author [8], and some estimates for \( N_-(H_V) \) immediately follow from there.

Further results on the behavior of \( N_-(H_V) \) were obtained recently by Ekholm, Frank, and Kovařík [5]. The main novelty in [5] is an analysis of the connection between the estimates for \( N_-(H_V) \) and the global dimension of a regular tree. This is an important step in the understanding of spectral properties of the Schrödinger operator on trees.

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The proofs in [5] are based upon a scheme developed by Naimark and Solomyak [8, 9], and independently by Carlson [4]. Our main goal in this note is to show that the estimates obtained in [5] can be considerably improved by the direct reduction to the results of [8]. We also show that the estimates in terms of the ‘weak \( L_p \)-spaces' obtained by this approach, are order-sharp in the strong coupling limit. This means that when applied to the operator \( H_\alpha V \), where \( \alpha > 0 \) is a large parameter, these estimates are of the type

\[
N_-(H_\alpha V) = O(\alpha^p), \quad \alpha \to \infty,
\]

where ‘\( O \)' cannot be replaced by ‘\( o \)'. On the contrary, in the estimates obtained in [5] in terms of the standard \( L_p \), such replacement is always possible.

We also discuss (in Theorem 4.3) estimates of a different nature, where the function \( N_-(H_\alpha V) \) is evaluated in terms of a certain number sequence associated with the potential. In Theorem 4.4 we single out a sub-class of regular trees, for which the conditions guaranteeing the order \( N_-(H_\alpha V) = O(\alpha^p) \) with \( p > 1/2 \) are not only sufficient, but also necessary.

Any metric tree has local dimension one. As a consequence, for the rapidly decaying potentials \( V \) one always has \( N_-(H_\alpha V) = O(\alpha^{1/2}) \). For the slowly decaying potentials, such that (1.1) is satisfied with some \( p > 1/2 \), the estimates in \( L^p \)-terms are never order-sharp: any order-sharp estimate must involve a norm, or a quasi-norm, of some non-separable function space. In this connection, see a discussion in [11]. The results of the present paper once more substantiate this claim.

1.2. Geometry of a tree. Let \( \Gamma \) be a rooted metric tree, with the root \( o \). We suppose that the number of vertices and the number of edges are infinite, and that each edge has finite length. The distance \( \rho(x, y) \) and the partial ordering \( x \preceq y \) on \( \Gamma \) are introduced in a natural way, and we write \( x < y \) if and only if \( x \preceq y \) and \( x \neq y \). We denote \( |x| = \rho(o, x) \). On each edge \( e \subset \Gamma \) the partial ordering turns into the standard ordering defined by the inequality \( |x| \leq |y| \). With each vertex \( v \in \Gamma \) we associate its generation \( \text{Gen}(v) \), i.e. the number of vertices \( w \) such that \( o \preceq w < v \). In particular, \( \text{Gen}(o) = 0 \). The branching number \( b(v) \) of a vertex \( v \) is the number of edges emanating from \( v \). We suppose that \( b(o) = 1 \) and that \( 1 < b(v) < \infty \) for any vertex \( v \neq o \).

In this paper we consider the regular trees, see [8, 9] for more detail. A tree \( \Gamma \) is said to be regular (or symmetric, or radial), if all vertices of the same generation have equal branching numbers and lie on the equal distance from the root. Each regular tree is completely determined by
specifying two number sequences, \( \{b_n\} \) and \( \{t_n\} \), where \( b_n = b(v) \) and \( t_n = |v| \) for any vertex \( v \) of generation \( n \), \( n = 0, 1, \ldots \). We suppose that the sequence \( \{t_n\} \) is unbounded. The branching function of \( \Gamma \), defined as

\[
g_0(t) = b_0 b_1 \ldots b_n \quad \text{for} \quad t_n < t \leq t_{n+1}, \quad n = 1, 2, \ldots
\]
is an important characteristic of a regular tree.

The natural measure \( dx \) on \( \Gamma \) is induced by the Lebesgue measure on the edges. The spaces \( L^p(\Gamma) \) are understood as the Lebesgue spaces with respect to this measure. The norm in \( L^p(\Gamma) \) is denoted by \( \| \cdot \|_p \); for \( p = 2 \), we write simply \( \| \cdot \| \).

**1.3. Sobolev spaces on a tree.** Differentiation, in the direction compatible with the ordering on \( \Gamma \), is well-defined at any point \( x \in \Gamma \) except for the vertices. The Sobolev space \( H^1(\Gamma) \) consists of all continuous functions \( u \) on \( \Gamma \), such that \( u \mid e \in H^1(e) \) on each edge \( e \subset \Gamma \), and the condition

\[
\|u\|_{H^1(\Gamma)}^2 := \int_\Gamma (|u'(x)|^2 + |u(x)|^2) dx < \infty
\]
is satisfied. It is easy to show that any function \( u \in H^1(\Gamma) \) vanishes as \( |x| \to \infty \), see, e.g., Lemma 3.1 in [12]. It immediately follows that the functions \( u \in H^1(\Gamma) \) with bounded support are dense in \( H^1(\Gamma) \). The boundary condition \( u(o) = 0 \) selects the subspace

\[
H^{1,0}(\Gamma) \subset H^1(\Gamma).
\]
The functions \( u \in H^{1,0}(\Gamma) \) with bounded support are dense in this subspace.

Along with \( H^{1,0}(\Gamma) \), we need also the homogeneous Sobolev space \( \mathcal{H}^{1,0}(\Gamma) \) formed by all continuous functions \( u \) on \( \Gamma \), such that \( u(o) = 0 \), \( u \mid e \in H^1(e) \) on each edge \( e \subset \Gamma \), and \( u' \in L^2(\Gamma) \). By definition,

\[
\|u\|_{\mathcal{H}^{1,0}(\Gamma)} = \|u'\|.
\]

In contrast with the space \( H^{1,0}(\Gamma) \), the functions \( u \in \mathcal{H}^{1,0}(\Gamma) \) with bounded support are not always dense in \( \mathcal{H}^{1,0}(\Gamma) \). As a consequence, the set \( H^{1,0}(\Gamma) \), which is evidently contained in \( \mathcal{H}^{1,0}(\Gamma) \), is not always dense in the latter space. This property depends on geometry of the tree. A tree such that \( H^{1,0}(\Gamma) \) is dense in \( \mathcal{H}^{1,0}(\Gamma) \) is called recurrent, otherwise it is called transient. We shall denote the closure of \( H^{1,0}(\Gamma) \) in \( \mathcal{H}^{1,0}(\Gamma) \) by \( \mathcal{H}^c(\Gamma) \). So, \( \mathcal{H}^c(\Gamma) = \mathcal{H}^{1,0}(\Gamma) \) if and only if the tree \( \Gamma \) is recurrent.

A simple necessary and sufficient condition of transiency is known for the regular trees, see, e.g., [9], Theorem 3.2. Namely, such tree is
transient if and only if its reduced height

\[ l(\Gamma) := \int_{\Gamma} \frac{dt}{g_0(t)} \]
is finite.

2. Schrödinger operator on \( \Gamma \).

Our main object in this paper is the Schrödinger operator on \( \Gamma \),

\[ H_{\alpha V} = -\Delta_D - \alpha V, \]

with the non-negative potential \( V(x) \) and a large parameter (the coupling constant) \( \alpha > 0 \). In (2.1) the symbol \( \Delta_D \) stands for the Dirichlet Laplacian on \( \Gamma \). In Section 8 we discuss also an analogue of (2.1) for the Neumann Laplacian \( \Delta_N \).

The operator (2.1) is defined via its quadratic form which is

\[ h_{\alpha V}[u] = \int_{\Gamma} (|u'|^2 - \alpha V|u|^2)dx, \quad u \in H^{1,0}(\Gamma). \]

We are interested in estimating the number of the negative eigenvalues of the operator \( H_{\alpha V} \) in terms of the potential \( V \) and the coupling constant \( \alpha \).

Assume that the potential \( V \geq 0 \) is such that the inequality

\[ \int_{\Gamma} V|u|^2dx \leq C_V \int_{\Gamma} |u'|^2dx, \quad \forall u \in H^{1,0}(\Gamma) \]
is satisfied. By the continuity, it extends to all \( u \in \mathcal{H}^0(\Gamma) \). We assume that \( C_V \) is the least possible constant in (2.3). We call any such potential \( V \) a Hardy weight on \( \mathcal{H}^0(\Gamma) \), and we say that the Hardy weight is normalized, if \( C_V = 1 \). If \( V \) is a Hardy weight, the operator (2.1) is well-defined at least for small values of \( \alpha \). Some further assumptions about \( V \), that we impose later, imply finiteness of the negative spectrum of this operator.

In general, for a self-adjoint, bounded from below operator \( H \) in a Hilbert space \( \mathcal{H} \), whose negative spectrum is finite, we denote by \( N_-(H) \) the number of its negative eigenvalues counted according to their multiplicities.

For estimating the quantity \( N_-(H_{\alpha V}) \) it is important to have a description of the Hardy weights on \( \mathcal{H}^0(\Gamma) \). Such description, for general trees and the Hardy weights on a wider class \( \mathcal{H}^{1,0}(\Gamma) \), was obtained in [6]. A similar result for \( \mathcal{H}^0(\Gamma) \) seems to be unknown so far. The
situation changes if we restrict ourselves to the regular trees and to the symmetric weights

\[ V(x) = v(|x|), \quad \forall x \in \Gamma \]

where \( v \geq 0 \) is a measurable function on \([0, \infty)\). For the regular trees an exhaustive description of all symmetric Hardy weights can be easily derived from the classical Hardy-type inequalities, see, e.g., [10], or Section 1.3 in the book [7]. The result of such reduction was presented in [9], Theorem 5.2, but only for the transient regular trees. We reproduce it below, including also the recurrent case. The proof for this case remains the same.

**Proposition 2.1.** Let \( \Gamma \) be a regular tree, and let \( g_0(t) \) be its branching function (1.2). Suppose \( V(x) = v(|x|) \geq 0 \) is a symmetric weight function on \( \Gamma \).

1° Let \( \Gamma \) be recurrent. Then \( V(x) \) is a Hardy weight on \( \mathcal{H}^\circ(\Gamma) \) if and only if

\[
B_0(v) := \sup_{t > 0} \left( \int_t^{\infty} v(s)g_0(s)ds \cdot \int_0^t \frac{ds}{g_0(s)} \right) < \infty.
\]

Moreover, the least possible constant \( C_V \) in (2.3) satisfies

\[
B_0(v) \leq C_V \leq 4B_0(v).
\]

2° Let \( \Gamma \) be transient. Then \( V(x) \) is a Hardy weight on \( \mathcal{H}^\circ(\Gamma) \) if and only if the following two conditions are satisfied:

\[
B_1(v) := \sup_{t > t_1} \left( \int_{t_1}^{t} v(s)g_0(s)ds \cdot \int_t^{\infty} \frac{ds}{g_0(s)} \right) < \infty;
\]

\[
B_2(v) := \sup_{t < t_1} \left( t \cdot \int_{t_1}^{t} v(s)ds \right) < \infty.
\]

Moreover,

\[
C_V \leq 4(B_1(v) + B_2(v)); \quad B_1(v) \leq C_V; \quad B_2(v) \leq \left( 1 + \frac{b_1t_1}{t_2 - t_1} \right) C_V.
\]

### 3. Classes of functions, of number sequences, and of compact operators

As we shall see, the sharp estimates of the function \( N_-(\mathcal{H}_{aV}) \) involve the weak \( L_p \)-spaces on the tree, rather than the classical Lebesgue spaces \( L_p \). Here we recall their definitions. See, e.g., [4], Section 1.3, for more detail.
Let \((\mathcal{X}, \mu)\) be a measure space with \(\sigma\)-finite measure. A measurable function \(f\) on \(\mathcal{X}\) belongs to the class \(L_{p,w}(\mathcal{X}, \mu)\), \(0 < p < \infty\), if and only if
\[
\|f\|_{L_{p,w}(\mathcal{X}, \mu)} := \sup_{t>0} \left( t \cdot (\mu\{x \in \mathcal{X} : |f(x)| > t\})^{1/p} \right) < \infty.
\]
The spaces \(L_{p,w}\) are linear. The functional (3.1) defines a quasi-norm on \(L_{p,w}\), and the space is complete with respect to this quasi-norm. If \(p > 1\), and only in this case, a norm equivalent to the quasi-norm (3.1) does exist. However, in various estimates it is more convenient to use the quasi-norm (3.1).

If the measure \(\mu\) does not reduce to the sum of a finite number of atoms, the space \(L_{p,w}(\mathcal{X}, \mu)\) is non-separable. The condition
\[
\mu\{x \in \mathcal{X} : |f(x)| > t\} = o(t^{-p}) \quad \text{as } t + t^{-1} \to \infty
\]
singles out a separable subspace in \(L_{p,w}(\mathcal{X}, \mu)\), which we denote by \(L_{p,w}^{\circ}(\mathcal{X}, \mu)\). It is well-known (and easy to check) that
\[
\|f\|_{L_{p,w}(\mathcal{X}, \mu)}^p \leq \int_{\mathcal{X}} |f|^p d\mu; \quad L_p(\mathcal{X}, \mu) \subset L_{p,w}^{\circ}(\mathcal{X}, \mu).
\]

In particular, let \(\mathcal{X} = \Gamma\) be a metric tree, and let \(\mu\) be the measure generated by a weight function \(\Phi(x) \geq 0\) on \(\Gamma\):
\[
d\mu = \Phi(x)dx.
\]
We denote the corresponding space by \(L_{p,w}(\Gamma, \Phi)\) and the functional (3.1) by \(\|f\|_{p,w;\Phi}\). So,
\[
\|f\|_{p,w;\Phi}^p = \sup_{t>0} \left( \int_{|f(x)| > t} \Phi(x) dx \right).
\]

We also need the weak \(\ell_p\)-spaces of number sequences. These are a particular case of the spaces \(L_{p,w}(\mathcal{X}, \mu)\), where \(\mathcal{X} = \mathbb{N}\), or \(\mathcal{X} = \mathbb{N}_0 := \{0, 1, \ldots\}\). The measure \(\mu\) is either the standard counting measure, or it is given by a sequence of positive weights \(\Phi_n = \mu(\{n\})\). For a sequence \(f = \{f_n\}_{n \in \mathcal{X}}\) we have
\[
\|f\|_{p,w;\Phi_n}^p = \sup_{t>0} \left( \int_{|f(n)| > t} \Phi_n dx \right).
\]

The case of counting measure is especially simple; here we drop the symbol \(\Phi_n\) in the notation. Consider the sequence \(\{f_n^*\}\) whose terms are the numbers \(|f_n|\), rearranged in the non-increasing order. Then
\[ f \in \ell_{p,w} \text{ if and only if } f^*_n = O(n^{-1/p}), \text{ and } f \in \ell^o_{p,w} \text{ if and only if } f^*_n = o(n^{-1/p}). \] Moreover,
\[ \|f\|_{p,w} = \sup_n (n^{1/p} f^*_n). \]

Now, let \( A \) be a compact operator in a Hilbert space \( \mathcal{H} \), and let \( \{s_n(A)\} \) be the sequence of its singular numbers, counted according to their multiplicities. Recall that for a non-negative self-adjoint operator its singular numbers coincide with the eigenvalues. By definition, \( A \) belongs to the weak Schatten class \( C_{p,w} \), if \( \{s_n(A)\} \in \ell_{p,w} \), and
\[ \|A\|_{p,w} := \|\{s_n(A)\}\|_{p,w} = \sup_n (n^{1/p} s_n(A)). \]
The class \( C^o_{p,w} \) is defined as the set of all compact operators such that \( s_n(A) = o(n^{-1/p}). \)

For a compact operator \( A \) we denote
\[ n(s; A) = \#\{n : s_n(A) > s\}, \quad s > 0. \]
Evidently, the inclusion \( A \in C^o_{p,w} \) is equivalent to the inequality
\[ n(s; A) \leq C n^{-p}, \quad \forall n \in \mathbb{N}, \]
and moreover, the best possible constant here is \( C = \|A\|_{p,w}^p \). Also, \( A \in C^o_{p,w} \) if and only if \( n(s; A) = o(n^{-p}). \)

4. Main results on the behavior of \( N_-(H_{\alpha V}) \)

Here we present the basic estimates for the number of negative eigenvalues of the operator \( H_{\alpha V} \). Their proofs are given in Section 5.

**Theorem 4.1.** Let \( \Gamma \) be a regular metric tree and let a normalized and symmetric Hardy weight \( \Psi(x) = \psi(|x|) > 0 \) on \( \mathcal{H}^o(\Gamma) \) be chosen. Define the weight function \( \Phi(x) = |x|\Psi(x) \). Suppose \( V \) is a non-negative potential on \( \Gamma \) such that \( V\Psi^{-1} \in L_{p,w}(\Gamma, \Phi) \) for some \( p > 1 \). Then

\[ N_-(H_{\alpha V}) \leq C\alpha^p \sup_{t>0} \left( \int_{V(x) > t\Psi(x)} t^p \Phi(x) dx \right), \quad \forall \alpha > 0, \]

and

\[ \limsup_{\alpha \to \infty} \alpha^{-p} N_-(H_{\alpha V}) \leq C \limsup_{t+t^{-1} \to \infty} \left( \int_{V(x) > t\Psi(x)} t^p \Phi(x) dx \right). \]
In particular, if
\[
\int_{V(x)>t\Psi(x)} \Phi(x)dx = o(t^{-p}) \text{ as } t + t^{-1} \to \infty,
\]
then
\[
N_-(H_{\alpha V}) = o(\alpha^p).
\]

If \( V\Psi^{-1} \) belongs to the narrower space \( L_p(\Gamma, \Phi) \), then
\[
N_-(H_{\alpha V}) \leq C\alpha^p \int_{\Gamma} |x|^p \Psi^{-1-p}dx,
\]
and \((4.3)\) is satisfied.

**4.1. Global dimension of a regular tree.** Suppose that the branching function \( g_0(t) \) satisfies the two-sided inequality
\[
c_1(1 + t)^{d-1} \leq g_0(t) \leq c_2(1 + t)^{d-1}, \quad 0 < c_1 < c_2, \quad \forall t > 0,
\]
with some \( d \geq 1 \). Following [5], we shall say that the number \( d \) is the global dimension of the regular tree \( \Gamma \). This is an important invariant of a tree. The existence of a global dimension is itself a serious restriction on the structure of the tree.

It is more convenient for us to use an equivalent definition of global dimension:
\[
c'_1 t^{d-1} \leq g_0(t) \leq c'_2 t^{d-1}, \quad 0 < c'_1 \leq c'_2, \quad \forall t > t_1.
\]
A direct inspection shows that the function \( |x|^{-2} \) is a symmetric Hardy weight on \( H^\circ(\Gamma) \), provided that \( \Gamma \) has global dimension \( d \neq 2 \). Taking \( \Psi(x) = c|x|^{-2} \) with an appropriate \( c \) (and hence, \( \Phi(x) = c|x|^{-1} \)) and applying Theorem 4.1, we come to the following result.

**Theorem 4.2.** Let \( \Gamma \) be a regular metric tree of global dimension \( d \neq 2 \). Suppose the potential \( V \geq 0 \) is such that \( V|x|^2 \in L_{p,w}(\Gamma, |x|^{-1}) \) for some \( p > 1 \). Then
\[
N_-(H_{\alpha V}) \leq C\alpha^p \sup_{t>0} \left( t^p \int_{V(x)|x|^2>t} |x|^{-1}dx \right)
\]
and
\[
\limsup_{\alpha \to \infty} \alpha^{-p} N_-(H_{\alpha V}) \leq C \limsup_{t+t^{-1} \to \infty} \left( t^p \int_{V(x)|x|^2>t} |x|^{-1}dx \right).
\]
In particular, if \( \int_{V(x)|x|^2>t} |x|^{-1}dx = o(t^{-p}) \) as \( t + t^{-1} \to \infty \), then \((4.3)\) is satisfied. If \( V|x|^2 \in L_p(\Gamma, |x|^{-1}) \), then along with \((4.3)\) we have
\[
N_-(H_{\alpha V}) \leq C\alpha^p \int_{\Gamma} V^p|x|^{2p-1}dx.
\]
Any metric tree has local dimension one, and hence, for the fast decaying potentials one must have $N_-(H_{aV}) = O(\alpha^{1/2})$. At the same time, both Theorems 4.1 and 4.2 describe only the situation where $N_-(H_{aV}) = O(\alpha^p)$ with $p > 1$, which corresponds to the potentials that decay rather slowly. The next result concerns the case $1/2 < p \leq 1$. On the other hand, it applies only to the symmetric potentials $V(x) = v(|x|)$. We also suppose $V \in L_1(\Gamma)$. Actually, the latter is a restriction on the behavior of $V$ only near the root. It is necessary in case of the Neumann boundary condition, see Section 6, but not in case of the Dirichlet condition.

The properties of the potential in the next two theorems are expressed in terms different from those in Theorems 4.1 and 4.2. With any symmetric potential $V(x) = v(|x|) \geq 0$ from $L_1(\Gamma)$ we associate the sequence $\eta(V) = \{\eta_n(V)\}$, where

$$\eta_n(V) = t_{n+1}^{-1} \int_{t_n}^{t_{n+1}} v(t) dt, \quad n = 0, 1, \ldots$$

**Theorem 4.3.** Let $V \in L_1(\Gamma)$ be a symmetric, non-negative potential on a regular tree $\Gamma$.

1° Suppose the sequence $\eta(V) = \{\eta_n(V)\}$ belongs to the space $\ell_{1/2}$ with respect to the weight sequence $\{g_0(t_{n+1})\}$. Then the estimate

\[ N_-(H_{aV}) \leq C \alpha^{1/2} \sum_n \eta_n^{1/2}(V)g_0(t_{n+1}) \tag{4.9} \]

and the Weyl type asymptotic formula

\[ \lim_{\alpha \to \infty} \alpha^{-1/2} N_-(H_{aV}) = \frac{1}{\pi} \int_{\Gamma} V^{1/2}(x) dx = \frac{1}{\pi} \int_0^\infty v^{1/2}(t)g_0(t) dt. \tag{4.10} \]

are satisfied.

2° Suppose $\eta(V) \in \ell_{p,w}(\mathbb{N}_0; g_0(t_{n+1}))$ for some $p \in (1/2, 1)$. Then

\[ N_-(H_{aV}) \leq C \alpha^p \sup_{t>0} \left( t^p \sum_{\eta_n(V)>t} g_0(t_{n+1}) \right) \tag{4.11} \]

and

\[ \limsup_{\alpha \to \infty} \alpha^{-p} N_-(H_{aV}) \leq C \limsup_{t \to 0} \left( t^p \sum_{\eta_n(V)>t} g_0(t_{n+1}) \right). \tag{4.12} \]

In particular, if $\sum_{\eta_n(V)>t} g_0(t_{n+1}) = o(t^{-p})$ as $t \to 0$, then $\text{[4.3]}$ is satisfied.
If $\eta(V) \in \ell_p(N_0; g_0(t_{n+1}))$ for some $p \in (1/2, 1]$, then along with (4.3) we have

\[ N_-(H_{\alpha V}) \leq C\alpha^p \sum_{n=0}^\infty \eta_n^p(V) g_0(t_{n+1}). \]

4.2. Estimates for the $b$-regular trees. More advanced results can be obtained if we suppose that for all the vertices $v \neq o$ the branching number is the same integer $b > 1$. In other words, we suppose that

\[ b_n = b, \quad \forall n > 0. \]

In the paper [5], Section 7.2, such trees were called $b$-regular. For a $b$-regular tree one has

\[ g_0(t) = b^n, \quad t_n < t \leq t_{n+1}. \]

If, besides, $\Gamma$ has global dimension $d$, then (4.14), together with (4.5), implies (as $t \to t_n+$) that

\[ c'_1 t_n^{d-1} < b^n < c'_2 t_n^{d-1}. \]

Hence, the sequence $\{t_n\}$ grows as a geometric progression.

For the $b$-regular trees the result of Theorem 4.3 can be considerably strengthened: it extends to arbitrary values of $p > 1/2$ and, most importantly, the condition

\[ \eta(V) \in \ell_p,w(N_0; b_n) \]

turns out to be not only sufficient, but also necessary for $N_-(H_{\alpha V}) = O(\alpha^p)$.

Theorem 4.4. Let $V \in L_1(\Gamma)$ be a symmetric, non-negative potential on a $b$-regular tree $\Gamma$. Suppose also that $\Gamma$ has global dimension $d \neq 2$.

1° Suppose $\eta(V) \in \ell_p,w(N_0; b^n)$ for some $p > 1/2$. Then

\[ N_-(H_{\alpha V}) \leq C\alpha^p \sup_{t>0} \left( t^p \sum_{\eta_n(V) > t} b^n \right) \]

and

\[ \limsup_{\alpha \to \infty} \alpha^{-p} N_-(H_{\alpha V}) \leq C \limsup_{t \to 0} \left( t^p \sum_{\eta_n(V) > t} b^n \right). \]

In particular, if $\eta(V) \in \ell_p,w(N_0, b^n)$, then (4.3) is satisfied.

If $\eta(V) \in \ell_p(N_0, b^n)$, then along with (4.3) we have

\[ N_-(H_{\alpha V}) \leq C\alpha^p \sum_{n=0}^\infty \eta_n^p(V) b^n. \]
Conversely, if \( N_-(H_\alpha V) = O(\alpha^p) \) with some \( p \geq 1/2 \), then \( \eta(V) \in \ell_{p,w}(N_0; b^n) \). If \( N_-(H_\alpha V) = o(\alpha^p) \) with some \( p > 1/2 \), then \( \eta(V) \in \ell_{p,w}(N_0; b^n) \).

5. Proofs

5.1. Using the Birman – Schwinger principle. The proofs of all theorems 4.1–4.4 are based upon the Birman – Schwinger principle, see [2, 3] for its exposition most convenient for our purposes. In order to formulate it for the particular case we need, let us consider the quadratic form

\[
a_V[u] = \int_\Gamma V(x)|u(x)|^2\,dx
\]

where \( V \geq 0 \) is a Hardy weight on \( \mathcal{H}^c(\Gamma) \). Then the quadratic form \( a_V \) is bounded in this space, and therefore, it generates in \( \mathcal{H}^c(\Gamma) \) a bounded, self-adjoint and non-negative operator which we denote \( A^c_V \).

The notation \( A^c_{V,V} \) was used for this operator in [8], see Eq. (3.7) there.

The following statement expresses the Birman – Schwinger principle for the case of operators on trees.

**Proposition 5.1.** Let a Hardy weight \( V \) on \( \mathcal{H}^c(\Gamma) \) be such that the operator \( A^c_V \) is compact. Then for any \( \alpha > 0 \) the quadratic form \( (2.2) \) is bounded below and closed in \( L^2(\Gamma) \), the negative spectrum of the corresponding operator \( H_\alpha V \) is finite, and moreover,

\[
N_-(H_\alpha V) = n(\alpha^{-1}; A^c_V), \quad \forall \alpha > 0.
\]

This proposition reduces estimates of \( N_-(0; H_\alpha V) \) to the spectral analysis of the compact operator \( A^c_V \). Such analysis was carried out in [8].

The equality (5.2) implies that for any \( p > 0 \)

\[
\sup_{\alpha > 0} \alpha^{-p} N_-(H_\alpha V) = \sup_{s > 0} s^p n(s; A^c_V) = \|A^c_V\|^p_{p,w}
\]

and

\[
\limsup_{\alpha \to \infty} \alpha^{-p} N_-(H_\alpha V) = \limsup_{s \to 0} s^p n(s; A^c_V) = \inf_{T \in \mathcal{C}_{p,w}} \|A^c_V - T\|^p_{p,w}.
\]

5.2. Proof of Theorems 4.1–4.3. Due to (5.3), the inequality (4.1) turns into the estimate (3.14) in Theorem 3.4 of the paper [8]. By the inequality (3.13) in the same theorem, finiteness of the integral in (4.4) implies \( A^c_V \in \mathcal{C}_p \). Since \( \mathcal{C}_p \subset \mathcal{C}_{p,w} \), this immediately yields (4.4).

The inequality (4.2) follows from (4.1) automatically, see the proof of a similar statement in [2], Section 4.3. Note also that below we give
the detailed proof of the inequality (4.12) in Theorem 4.3. The proof of (4.12) can be reconstructed by analogy. Theorem 4.2 is a particular case of Theorem 4.1 for a special choice of the weight function, and it does not need a separate proof.

Theorem 4.3, except for the estimate (4.12), is a direct consequence (actually, just a reformulation in the equivalent terms) of Theorem 4.1 in [8], for the regular trees and the symmetric weights. One only has to take into account that in [8] the operators $A_V$ were considered, rather than $A_V^\circ$. The operator $A_V^\circ$ corresponds to the quadratic form (5.1) on the space $H^1(\Gamma)$. If the tree is transient, this space is wider than $H^0(\Gamma)$. For this reason, 

$$\|A_V^\circ\|_c \leq \|A_V\|_c,$$

where $\|\cdot\|_c$ stands for the (quasi-)norm in any class of operators we are dealing with. It immediately follows that all the estimates derived in [8] for $A_V$, hold also for the operator $A_V^\circ$.

To justify (4.12), we note that for any $V \in L_1(\Gamma)$ with compact support the estimate (4.9) holds. For the corresponding operator $A_V^\circ$, this implies

$$n(s; A_V^\circ) = O(s^{-1/2}) = o(s^{-p})$$

for any $p > 1/2$. Given an $\epsilon > 0$, we can find a finite subset $E_\epsilon \subset \mathbb{N}$ such that

$$\sup_{t > 0} \left( t^p \sum_{n \in E_\epsilon; \eta_n(V) > t} g_0(t_{n+1}) \right) \leq \limsup_{t > 0} \left( t^p \sum_{n \in \mathbb{N}_0; \eta_n(V) > t} g_0(t_{n+1}) \right) + \epsilon.$$

Let $\chi_\epsilon$ stand for the characteristic function of the set $\bigcup_{n \in E_\epsilon} (t_n, t_{n+1})$. Take $V_\epsilon(x) = V(x) \chi_\epsilon(|x|)$ and $V'(x) = V(x) - V_\epsilon(x)$, then $A_V^\circ = A_{V_\epsilon} + A_{V'}^\circ$ and $n(s; A_V^\circ) = o(s^{-p})$. It follows from (4.11) that

$$\limsup_{s \to 0} s^p n(s; A_V^\circ) = \limsup_{s \to 0} s^p n(s; A_{V_\epsilon}^\circ) \leq \sup_{t > 0} \left( t^p \sum_{\eta_n(V') > t} g_0(t_{n+1}) \right) \leq \limsup_{t > 0} \left( t^p \sum_{n \in \mathbb{N}_0; \eta_n(V) > t} g_0(t_{n+1}) \right) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (4.12) is justified.

5.3. Proof of Theorem 4.4. The estimate (4.16) for $1/2 < p < 1$ is a part of Theorem 4.3 above. Now we prove that if $\eta(V) \in \ell_\infty$, then the operator $A_V^\circ$ is bounded, with the estimate

$$\|A_V^\circ\| \leq C \sup_n \eta_n(v).$$
To check this, it is sufficient to show that $V(x)$ is a Hardy weight on $H^0(\Gamma)$. Suppose first that $d < 2$, so that the tree is recurrent. The condition (2.4) is equivalent to
\[
\int_t^\infty v(s)s^{d-1}ds \leq Ct^{d-2}, \quad \forall t > 0.
\]
Since the numbers $t_n$ grow as a geometric progression, it is sufficient to check this inequality for the values $t = t_n$, $n \in \mathbb{N}$. We have
\[
\int_{t_n}^\infty v(s)s^{d-1}ds = \sum_{j=n}^{\infty} \int_{t_j}^{t_{j+1}} v(s)s^{d-1}ds
\]
\[\leq \sum_{j=n}^{\infty} t_{j+1}^{d-2} \eta_j(V) \leq \sup_j \left( \eta_j(V) \sum_{j=n}^{\infty} t_{j+1}^{d-2} \right).
\]
By (4.15), the series in the last expression converges and its sum is controlled by $t_n^{d-2}$. This gives the desired result in the recurrent case.

For the transient case $d > 2$ the argument is similar.

The estimates (4.16) and (4.18) follow from the estimates for $p = 1/2$ and $p = \infty$ by the same interpolation argument as the one used in [8] for the proof of Theorem 4.1. For interpolation, it is necessary to extend the definition of the operator $A_V^\circ$ to the case where $V(x)$ is an arbitrary measurable function on $\Gamma$, such that $|V(x)|$ is a Hardy weight. This extension is evident and we give no detail; cf. Section 3.1 in [8].

We interpolate the mapping $\Pi : v \mapsto A_V^\circ$, $V(x) = v(|x|)$. In both cases, one of the basic results for interpolation is the one for $p = 1/2$, that is the the estimate
\[
\|A_V^\circ\|_{1/2,w} \leq C\|\eta(V)\|_{\ell_{1/2}}.
\]
Recall that this is equivalent to (4.9). Under the assumptions of Theorem 4.4, due to the special character of the tree and the function $V$, another basic result concerns the case $p = \infty$, while for the general trees and arbitrary potentials we could use only Theorem 3.3 in [8], which corresponds to the case $p = 1$. This argument justifies the statement 1° of Theorem 4.4.

For justifying the statement 2°, we use Theorem 4.6 and Remark to Corollary 4.7 in [8]. Note that in this theorem the properties of the operators $A_V$ on $\mathcal{H}^{1,0}(\Gamma)$ were discussed. However, the construction in the proof of Theorem 4.6, suggested in [8], uses only functions with bounded support, and hence the result applies also to the operators $A_V^\circ$. Note also that Corollary 4.7 in [8] summarizes the results of Theorems 4.1 and 4.6 of this paper. For the $b$-regular trees the first
of these results extends to all $p > 1/2$, and this automatically leads to extension of the Corollary 4.7 to all such $p$.

6. Neumann boundary condition at the root

Here we discuss the changes in the above results, appearing if in (2.1) we replace the Dirichlet Laplacian $\Delta_D$ by the Neumann Laplacian $\Delta_N$. In order to indicate the difference between these two cases, below we denote the corresponding operators by $H_{\alpha V;D}$ and $H_{\alpha V;N}$. It immediately follows from the variational principle that

$$N_-(H_{\alpha V;D}) \leq N_-(H_{\alpha V;N}) \leq N_-(H_{\alpha V;D}) + 1,$$

so that both quantities behave in a similar way as $\alpha \to \infty$. The behavior in the weak coupling limit (i.e., as $\alpha \to 0$) may differ. Indeed, if the tree $\Gamma$ is recurrent, then for any non-trivial potential $V \geq 0$ the operator $H_{\alpha V;N}$ has at least one negative eigenvalue. Hence, no estimate homogeneous in $\alpha$ is possible for $N_-(H_{\alpha V;N})$ in the recurrent case.

Such estimates of $N_-(H_{\alpha V;N})$ are possible for the transient trees. They are based upon an analogue of Proposition 5.1 for the $N$-case. In order to define an appropriate substitute for the operator $A_{\phi} \circ V$, we need one more version of the homogeneous Sobolev space on $\Gamma$.

Let $\Gamma$ be a regular transient tree, and let $u \in H^1(\Gamma)$ be a function with bounded support. Suppose also that $u$ is symmetric, that is, $u(x) = \phi(|x|)$ where $\phi(t)$ is some function on $[0, \infty)$. Then

$$u(o) = -\int_0^{\infty} \phi'(t) \frac{dt}{\sqrt{g_0(t)}},$$

whence

$$u(o) \leq l(\Gamma) \int_0^{\infty} |\phi'(t)|^2 g_0(t) dt = l(\Gamma) \int_{\Gamma} |u'|^2 dx.$$

Here $l(\Gamma)$ is the reduced height of the tree $\Gamma$, see (1.3). The resulting inequality extends to all $u \in H^1(\Gamma)$ with bounded support, since for the symmetric component $u_{sym}$ of $u$ we always have $\|u_{sym}'\| \leq \|u'\|$ by [12], Theorem 3.2 (applied to the function $u'$).

Define the space $\mathcal{H}(\Gamma)$ as the completion of $H^1(\Gamma)$ in the norm

$$\|u\|_\mathcal{H} := \|u'\|_{L_2(\Gamma)}.$$

The inequality (6.1) extends by continuity to all $u \in \mathcal{H}(\Gamma)$, and it shows that $\mathcal{H}(\Gamma)$ can be considered as a function space on $\Gamma$. This is no more true if the tree is recurrent: then the elements of such completion are the classes of functions, differing by an arbitrary constant.
Below we assume that $\Gamma$ is a transient regular tree. We have shown that then the linear functional $\gamma : u \mapsto u(o)$ on the space $\mathcal{H}(\Gamma)$ is continuous, and

\[ \|\gamma\| \leq \sqrt{l(\Gamma)}. \]  

(6.2)

The space $\mathcal{H}^o(\Gamma)$, introduced in Section 1.3, can be realized as the subspace $\{u \in \mathcal{H}(\Gamma) : \gamma u = 0\}$. Clearly,

$$\dim \mathcal{H}(\Gamma)/\mathcal{H}^o(\Gamma) = 1.$$  

By the Riesz theorem, there is a unique function $h \in \mathcal{H}(\Gamma)$, such that

$$u(o) = (u, h)_{\mathcal{H}(\Gamma)} = \int_{\Gamma} u' h' \, dx, \quad \forall u \in \mathcal{H}(\Gamma).$$

It is easy to see that this function is symmetric, and is given by

$$h(x) = h_0(|x|) \quad \text{where} \quad h_0(t) = \int_{t}^{\infty} \frac{ds}{g_0(s)}.$$  

Actually, $h(t)$ is the harmonic function on $\Gamma$, vanishing at infinity and such that $h(o) = h_0(0) = l(\Gamma)$. Its norm in $\mathcal{H}(\Gamma)$ is given by

$$\|h\|_{\mathcal{H}(\Gamma)}^2 = \int_{\Gamma} |h'(x)|^2 \, dx = \int_{0}^{\infty} h_0'(t)^2 g_0(t) \, dt = l(\Gamma).$$

This shows that in (6.2) we actually have the equality sign.

Let a function $V(x) \geq 0$ on a transient regular tree $\Gamma$ be such that the quadratic form (5.1) is bounded in the space $\mathcal{H}(\Gamma)$. Then it defines a bounded operator in this space, say, $\tilde{A}_V$. It also defines the bounded operator $A^o_V$ in its subspace $\mathcal{H}^o(\mathcal{F})$, see Section 5.1. Our next goal is to compare their (quasi-)norms in various spaces of operators.

**Theorem 6.1.** Let $\Gamma$ be a transient regular tree, and let $V \in L_1(\Gamma)$, $V \geq 0$. Suppose that the operator $A^o_V$ belongs to some space $\mathcal{C}$, where $\mathcal{C} = \mathcal{C}_p, \mathcal{C}_{p,w}$, or $\mathcal{C}_{p,w}$. Then the operator $\tilde{A}_V$ belongs to the same space, and, moreover,

\[ \|\tilde{A}_V\|_{\mathcal{C}} \leq C \left( \|A^o_V\|_{\mathcal{C}} + t_1 \int_{e_0} \int V \, dx \right). \]

(6.3)

where $e_0$ stands for the edge of $\Gamma$ emanating from the root, and the constant $C$ depends only on the reduced height $l(\Gamma)$ and on the length of $e_0$ (that is, on the number $t_1$).

**Proof.** Define a function $\phi$ on $\Gamma$ in the following way. Let $\phi(x) = 0$ everywhere outside the edge $e_0$. We identify $e_0$ with the segment $[0, t_1]$.
and take $\phi(x) = t_1^{-1}(t_1 - x), \ x \in e_0$. Given a function $u \in H^1(\Gamma)$, we decompose it as
\[ u(x) = u_0(x) + u(o)\phi(x), \]
then $u_0 \in H^0(\Gamma)$. We have
\[
\int_\Gamma V(x)|u(x)|^2dx \leq 2 \left( \int_\Gamma V(x)|u_0(x)|^2dx + |u(o)|^2 \int_{e_0} V(x)|\phi(x)|^2dx \right) 
\]
(6.4) \[ \leq 2 \left( \int_\Gamma V(x)|u_0(x)|^2dx + |u(o)|^2 \int_{e_0} V(x)dx \right). \]
Further, we have $||\phi'||^2 = t_1^{-1}$, and hence,
\[
||u'|| \leq ||u'|| + |u(o)||||\phi'|| \leq ||u'|| \left( 1 + \sqrt{l(G)t_1^{-1}} \right).
\]
It follows that
\[
||u'_0||^2 + |u(o)|^2||\phi'||^2 \leq ||u'||^2(2 + 3l(\Gamma)t_1^{-1}). \]
(6.5)

The inequalities (6.4) and (6.5) show that the (quasi-)norm $||\tilde{A}_V||_C$ is controlled by the sum of two terms. The first is $||A^c_V||_C$ and the second is the (quasi-)norm in $C$ of the operator of multiplication by the number $M = \int_{e_0} Vdx||\phi'||^{-2} = t_1 \int_{e_0} Vdx = \eta_0(V)$, acting in the one-dimensional space, generated by the function $\phi(x)$. This results in the estimate (6.3).

An analogue of Proposition 5.1 for the $N$-case shows that
\[
N_+(H^{\alpha V};N) = n(\alpha^{-1}; \tilde{A}_V).
\]

As a result of this equality and Theorem 6.1, we come to the following general conclusions.

**Theorem 6.2.** Let $\Gamma$ be a regular, transient metric tree and let $V \in L_1(\Gamma), \ V \geq 0$. Then
1° Each estimate (4.1), (4.4), (4.6), (4.8), (4.11), (4.13), (4.16), (4.18), with an additional term $C\alpha t_1 \int_{e_0} Vdx$ in the right-hand side, holds for the operator $H^{\alpha V};N$. If $p = 1/2$, the same is true for the estimate (4.9), and the asymptotic formula (4.10) remains valid.
2° the inequalities (4.2), (4.7), (4.12), and (4.17), remain valid for the operator $H^{\alpha V};N$. In particular, the finiteness of the right-hand sides in (4.4), (4.8), (4.13), and (4.18) implies (4.3).
3° The statement 2° of Theorem 4.4 remains valid for the operator $H^{\alpha V};N$.

We also present a simple but useful result which follows from Theorem 6.1 and Theorem 3.3 in [8].
Theorem 6.3. Let $Γ$ be a transient regular tree, and let $V ≥ 0$ be a measurable function on $Γ$. The operator $\tilde{A}_V$ is trace class if and only if $\int_Γ |x|V(x)dx < \infty$, and

$$\|\tilde{A}_V\|_1 = \text{Tr} \tilde{A}_V ≤ C \left( \int_Γ |x|V(x)dx + t_1 \int_{x_0} Vdx \right)$$

where $C$ is the constant from (6.3).

It follows from this theorem that under its assumptions we have

$$N_-(H_αV;N) ≤ Cα \left( \int_Γ |x|V(x)dx + t_1 \int_{x_0} Vdx \right),$$

and also $N_-(H_αV;N) = o(α)$.

7. Concluding remarks

7.1. On the sharpness of Theorem 4.2. Below we present an example in which both Theorems 4.2 and 4.4 apply. Then, based upon the statement 2° of the latter, we derive the sharpness of the first theorem.

Example 7.1. Let $Γ$ be a $b$-regular tree with some $b > 1$, and let $t_n = b^{\frac{d}{d-2}}$. Then $Γ$ has global dimension $d$, and we assume $d > 2$. On $Γ$ we consider the symmetric potential $V(x) = v(|x|)$ where

$$v(t) = t^{-2}b^{-\frac{n}{p}}, \quad t_n < t ≤ t_{n+1}.$$  

Then

$$η_n(V) = t_{n+1}^{-2}(t_{n+1} - t_n)b^{-\frac{n}{p}} = cb^{-\frac{n}{p}}, \quad c = 1 - b^{-\frac{2}{d-2}}.$$  

Let us check the assumptions of Theorem 4.4 1°. For small $t > 0$ the sum

$$\sum_{cb^{-\frac{n}{p}} > t} b^n = \sum_{b^n < (t/c)^{-p}} b^n$$

has the order $O(t^{-p})$ but not $o(t^{-p})$. It follows from Theorem 4.4 that $N_-(H_αV) = O(α^p)$ where $O$ cannot be replaced by $o$.

Now, assuming $p > 1$, we apply Theorem 4.2 to the same potential. A simple calculation shows that the supremum in the right-hand side of (4.6) is finite, and hence, $N_-(H_αV) = O(α^p)$. The reference to Theorem 4.3 shows that the estimate is sharp. As a consequence, we see that the estimate (4.7) in Theorem 4.2 is order-sharp.
7.2. **Comparison with the results of** [5]. We do not discuss here the general Theorem 2.4 in [5] and restrict ourselves to the comparison between its Corollary 2.6 and our Theorem 4.2 – more exactly, its analogue for the operator $H_{\alpha V; N}$. For the symmetric potentials and $p > 1$ the corollary (where one should take $a = 2p - 1$) is equivalent to our estimate (4.3), within the value of the constant factor. The authors of [5] overlook the possibility to extend their result to the general (non-symmetric) potentials. For $p = 1$ the result is covered by our Theorem 6.3. They also do not discuss the important problem of the sharpness of the estimate in the strong coupling limit, and they have nothing similar to our inequalities (4.6) and (4.7), where the weak $L_p$-spaces are involved.

The results of [5] concern only the case $p \geq 1$, while our Theorem 4.3 covers the case $1/2 \leq p < 1$, and for the $b$-regular trees Theorem 4.4 gives the necessary and sufficient conditions for $N_{-}(H_{\alpha V}) = O(\alpha^p)$ with an arbitrary $p > 1/2$.

We would like to emphasize that this critics does not concern other parts of the paper [5], including a material on the so-called Lieb – Thirring estimates on trees.

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