GAME ARGUMENTS IN SOME EXISTENCE THEOREMS OF FRIEDBERG NUMBERINGS

TAKUMA IMAMURA

Abstract. We provide game-theoretic proofs of some well-known existence theorems of Friedberg numberings, including (1) the existence of two incomparable Friedberg numberings; (2) the existence of a uniformly c.e. sequence of pairwise incomparable Friedberg numberings; (3) the existence of a uniformly c.e. independent sequence of Friedberg numberings.

1. Introduction

Rogers [9] introduced the notion of computable numbering (of all partial computable functions) and the notion of reducibility. He showed that the set of all equivalence classes of computable numberings forms an upper semilattice with respect to reducibility which is called the Rogers semilattice. He asked whether this semilattice is a lattice, and if not, whether any two elements have a lower bound. Friedberg [1] constructed an injective computable numbering (called a Friedberg numbering) by a finite-injury priority argument. Pour-El [7] showed that every Friedberg numberings of is minimal. She also showed that there are two incomparable Friedberg numberings through modifying Friedberg’s construction. These numberings are non-equivalent minimal elements, and therefore they have no lower bound. Thus Rogers’ questions were negatively answered.

Shen [10] gave some examples of game-theoretic proofs of theorems in computability theory and algorithmic information theory. In particular, he gave the game-theoretic proof of the theorem of Friedberg. The game representation of Friedberg’s construction is clear and intuitive, and can be used to prove other existence theorems of Friedberg numberings as we will demonstrate in the paper.

In Section 3 we present Shen’s proof to use later. We provide game proofs of certain well-known existence criteria of Friedberg numberings for general classes of partial computable functions. In Section 4 we give two proofs of the theorem of Pour-El using two games. Also we give the proof of the existence of an infinite c.e. sequence and an independent sequence of Friedberg numberings. These are essentially modifications of Shen’s proof.

2. Notations and Definitions

We denote by \( \mathcal{P}^{(1)} \) the set of all partial computable functions from \( \mathbb{N} \) to \( \mathbb{N} \). \( \langle \cdot, \cdot \rangle \) is a computable pairing function which is a computable bijection between \( \mathbb{N}^2 \) and \( \mathbb{N} \). Let \( \mathcal{A} \) be any set. A surjective map \( \nu : \mathbb{N} \rightarrow \mathcal{A} \) is called a numbering of \( \mathcal{A} \). Let \( \nu \) and \( \mu \) be numberings of \( \mathcal{A} \). We say that \( \nu \) is reducible to \( \mu \), denoted by

2010 Mathematics Subject Classification. 03D45.
Key words and phrases. Friedberg numbering; Rogers semilattice; universal partial order; two-player infinite game.
ν ≤ μ, if there is a total computable function f : N → N such that ν = μ ∘ f. We say that ν and μ are equivalent if they are reducible to each other. We say that ν and μ are incomparable if they are not reducible to each other. In this paper, we often identify a numbering ν of a set of partial maps from X to Y with the partial map ν(i, x) = ν(i)(x) from N × X to Y. A numbering ν of a subset of \( \mathcal{P}^{(1)} \) is said to be computable if it is computable as a partial function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). A computable injective numbering is called a Friedberg numbering. A sequence \( \{ ν_i \}_{i \in \mathbb{N}} \) of numberings of a subset of \( \mathcal{P}^{(1)} \) is said to be uniformly c.e. if it is uniformly c.e. as a sequence of partial functions from \( \mathbb{N}^2 \) to \( \mathbb{N} \), or equivalently, if it is computable as a partial function from \( \mathbb{N}^3 \) to \( \mathbb{N} \). We say that a sequence \( \{ ν_i \}_{i \in \mathbb{N}} \) of numberings of a set \( A \) is independent if \( ν_i \not\in \bigoplus_{j \neq i} ν_j \) for all \( i \in \mathbb{N} \), where \( \bigoplus_{i \in \mathbb{N}} ν_j \) is the direct sum of \( \{ ν_i \}_{i \in \mathbb{N}} \) defined by \( \bigoplus_{i \in \mathbb{N}} ν_i ((j, k)) = ν_j (k) \).

3. FRIEDBERG’S CONSTRUCTION AND THE INFINITE GAME WITH TWO BOARDS

Theorem 1 (Friedberg [10] Corollary to Theorem 3). \( \mathcal{P}^{(1)} \) has a Friedberg numbering.

Proof (Shen [10]). First, we consider an infinite game \( G_0 \) and prove that the existence of a computable winning strategy of \( G_0 \) for one of the players implies the existence of a Friedberg numbering of \( \mathcal{P}^{(1)} \). The game \( G_0 \) is as follows:

Players: Alice, Bob.

Protocol: FOR \( s = 0, 1, 2, \ldots \):

Alice announces a finite partial function \( A_s : \mathbb{N}^2 \rightarrow \mathbb{N} \).
Bob announces a finite partial function \( B_s : \mathbb{N}^2 \rightarrow \mathbb{N} \).

Collateral duties: \( A_s \subseteq A_{s+1} \) and \( B_s \subseteq B_{s+1} \) for all \( s \in \mathbb{N} \).

Winner: Let \( A = \bigcup_{s \in \mathbb{N}} A_s \) and \( B = \bigcup_{s \in \mathbb{N}} B_s \). Bob wins if

1. for each \( i \in \mathbb{N} \), there is a \( j \in \mathbb{N} \) such that \( A(i, \cdot) = B(j, \cdot) \);
2. for any \( i, j \in \mathbb{N} \), if \( i \neq j \), then \( B(i, \cdot) \neq B(j, \cdot) \).

We consider \( A \) and \( B \) as two boards, \( A \)-table and \( B \)-table. Each board is a table with an infinite number of rows and columns. Each player plays on its board. At each move player can fill finitely many cells with any natural numbers. The collateral duties prohibit players from erasing cells.

A strategy is a map that determines the next action based on the previous actions of the opponent. Since any action in this game is a finitary object, we can define the computability of strategies via gödelization. Suppose that there is a computable winning strategy for Bob. Let Alice fill \( A \)-table with the values of some computable numbering of \( \mathcal{P}^{(1)} \) by using its finite approximation, and let Bob use some computable winning strategy. Clearly \( B \) is a Friedberg numbering of \( \mathcal{P}^{(1)} \).

Second, we consider an infinite game \( G_1 \), which is a simplified version of \( G_0 \), and describe a computable winning strategy of \( G_1 \). The game \( G_1 \) is as follows:

Players: Alice, Bob.

Protocol: FOR \( s = 0, 1, 2, \ldots \):

Alice announces a finite partial function \( A_s : \mathbb{N}^2 \rightarrow \mathbb{N} \).
Bob announces a finite partial function \( B_s : \mathbb{N}^2 \rightarrow \mathbb{N} \) and a finite set \( K_s \subseteq \mathbb{N} \).

Collateral duties: \( A_s \subseteq A_{s+1} \), \( B_s \subseteq B_{s+1} \) and \( K_s \subseteq K_{s+1} \) for all \( s \in \mathbb{N} \).

Winner: Let \( A = \bigcup_{s \in \mathbb{N}} A_s \), \( B = \bigcup_{s \in \mathbb{N}} B_s \) and \( K = \bigcup_{s \in \mathbb{N}} K_s \). Bob wins if

1. for each \( i \in \mathbb{N} \), there is a \( j \in \mathbb{N} \setminus K \) such that \( A(i, \cdot) = B(j, \cdot) \);
We consider that in this game Bob can invalidate some rows and that we ignore invalid rows when we decide the winner. Bob cannot validate invalid rows again.

To win this game, Bob hires a countable number of assistants who guarantee that each of the rows in A-table appears in B-table exactly once. At each move, the assistants work one by one. The i-th assistant starts working at move i. She can reserve a row in B-table exclusively, fill her reserved row, and invalidate her reserved row. The instruction for the i-th assistant: if you have no reserved row, reserve a new row. Let k be the number of rows such that you have already invalidated. If in the current state of A-table the first k positions of the i-th row are identical to the first k positions of some previous row, invalidate your reserved row. If you have a reserved row, copy the current contents of the i-th row of A-table into your reserved row. These instructions guarantee in the limit that

- if the i-th row in A-table is identical to some previous row, then the i-th assistant invalidates her reserved row infinitely many times, so she has no permanently reserved row;
- if not, the i-th assistant invalidates her reserved row only finitely many times, so she has a permanently reserved row.

In the second case, she faithfully copies the contents of the i-th row of A-table into her permanently reserved row. We can assume that each of the rows in B-table has been reserved or invalidated in the limit: when some assistant reserves a row let her select the first unused row. Then Bob wins the simplified game.

Now we prove the above properties. Suppose that the i-th row in A-table is not identical to any previous row in the limit. For each of the previous rows, select some column witnessing that this row is not identical to the i-th row. Let k be the maximum of the selected columns. Wait for convergence of the rectangular area \([0, i] \times [0, k]\) of A-table. After that, the first k positions of the i-th row in A-table are not identical to the first k positions of any previous row, and hence the i-th assistant invalidates her reserved row at most k times. Conversely, suppose that the i-th assistant invalidates her reserved row only finitely many times. Let k be the number of invalidations. After the k-th invalidation, the i-th row in A-table is not identical to any previous row, and the same is true in the limit.

Finally, we describe a computable winning strategy of \(G_0\) through modifying the winning strategy of \(G_1\) described above. We say that a row is odd if it contains a finite odd number of non-empty cells. We can assume without loss of generality that odd rows never appear in A-table: if Alice fills some cells in a row making this row odd, Bob ignores one of these cells until Alice fills other cells in this row. We replace invalidation to odd-ification: instead of invalidating a row, fill some cells in this row making it new and odd. Bob consider that odd-ified rows in B-table are invalid. This modification guarantees that each of the non-odd rows of A-table appears in B-table exactly once. Bob hires an additional assistant who guarantees that each odd row appears in B-table exactly once. At each move, the additional assistant reserves some row exclusively and fills some cells in this row making it new and odd so that all odd rows are exhausted in the limit. Thus Bob wins this game, and the theorem is proved. \(\square\)
Kummer [5] gave a priority-free proof of the existence of a Friedberg numbering of $\mathcal{P}(1)$. The key point of his proof is to split $\mathcal{P}(1)$ into $\mathcal{P}(1) \setminus \mathcal{O}$ and $\mathcal{O}$, where $\mathcal{O}$ is the set of all odd partial functions. Observe that $\mathcal{P}(1) \setminus \mathcal{O}$ has a computable numbering, $\mathcal{O}$ has a Friedberg numbering, and any finite subfunction of a partial function in $\mathcal{P}(1) \setminus \mathcal{O}$ has infinitely many extensions in $\mathcal{O}$. He provided the following useful criterion.

**Corollary 2** (Kummer [4, Extension Lemma]). Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint subsets of $\mathcal{P}(1)$. If $\mathcal{A}$ has a computable numbering, $\mathcal{B}$ has a Friedberg numbering, and every finite subfunction of a member of $\mathcal{A}$ has infinitely many extensions in $\mathcal{B}$, then $\mathcal{A} \cup \mathcal{B}$ has a Friedberg numbering.

**Proof sketch.** Let us play the game $G_0$ where Alice fills $\mathcal{A}$-table with the values of some computable numbering of $\mathcal{P}(1)$, and Bob uses the strategy which is obtained by modifying the winning strategy of $G_0$ as follows. In this strategy, we do not assume that odd rows never appear in $\mathcal{A}$-table, and Bob does not ignore cells in $\mathcal{A}$-table. Assistants use partial functions in $\mathcal{B}$ instead of odd partial functions. Replace odd-ification to $B$-ification: instead of odd-ificating a row, fill some cells in this row making it an unused member of $\mathcal{B}$ in the limit. The additional assistant guarantees that each member of $\mathcal{B}$ appears in $\mathcal{B}$-table exactly once. These actions are possible since $\mathcal{B}$ has a Friedberg numbering. Then, Bob wins, and $\mathcal{B}$ becomes a Friedberg numbering of $\mathcal{A} \cup \mathcal{B}$. $\square$

**Corollary 3** (Pour-El and Putnam [8, Theorem 1]). Let $\mathcal{A}$ be a subset of $\mathcal{P}(1)$ and $f$ be a member of $\mathcal{P}(1)$ with an infinite domain. If $\mathcal{A}$ has a computable numbering, then there is a subset $\mathcal{B}$ of $\mathcal{P}(1)$ such that

1. $\mathcal{A} \subseteq \mathcal{B}$,
2. the domain of every member of $\mathcal{B} \setminus \mathcal{A}$ is finite,
3. for any $g \in \mathcal{B} \setminus \mathcal{A}$, there is an $h \in \mathcal{A}$ with $g \subseteq f \cup h$,
4. $\mathcal{B}$ has a Friedberg numbering.

**Proof sketch.** Let us play the game $G_0$ where Alice fills $\mathcal{A}$-table with the values of some computable numbering of $\mathcal{P}(1)$, and Bob uses the strategy which is obtained by modifying the winning strategy of $G_0$ as follows. When some assistant fills a cell in the $j$-th column making this row odd, she must use $f(j)$ for filling. The instruction for the additional assistant: *if there is an odd row $i$ in $\mathcal{A}$-table such that this row is not identical to any row in $\mathcal{B}$-table, reserve a new row, copy the current contents of the $i$-th row of $\mathcal{A}$-table into your reserved row, and release your reserved row.* Released rows cannot be used forever. Note that the additional assistant exceptionally does not ignore cells in $\mathcal{A}$-table. Then, Bob wins, $\mathcal{B} = \{ B(i, \cdot) | i \in \mathbb{N} \}$ has the desired properties, and $\mathcal{B}$ becomes a Friedberg numbering of $\mathcal{B}$. $\square$

4. Modifications

**Theorem 4** (Pour-El [7, Theorem 2]). There are two incomparable Friedberg numberings of $\mathcal{P}(1)$.

The first proof is obtained from the proof of Theorem [4] through modifying in the same way done by Pour-El.

**Proof (asymmetric version).** We consider the following game $G_2$:
Players: Alice, Bob.

Protocol: FOR \( s = 0, 1, 2, \ldots \):
- Alice announces a finite partial function \( A_s : \mathbb{N}^2 \to \mathbb{N} \).
- Bob announces a finite partial function \( B_s : \mathbb{N}^2 \to \mathbb{N} \).

Collateral duties: \( A_s \subseteq A_{s+1} \) and \( B_s \subseteq B_{s+1} \) for all \( s \in \mathbb{N} \).

Winner: Let \( A = \bigcup_{s \in \mathbb{N}} A_s \) and \( B = \bigcup_{s \in \mathbb{N}} B_s \). Bob wins if
(1) for each \( i \in \mathbb{N} \), there is a \( j \in \mathbb{N} \) such that \( A(i, \cdot) = B(j, \cdot) \);
(2) for any \( i, j \in \mathbb{N} \), if \( i \neq j \), then \( B(i, \cdot) \neq B(j, \cdot) \);
(3) for each \( i \in \mathbb{N} \), if \( A(i, \cdot) \) is total, then there is a \( j \in \mathbb{N} \) such that
\( B(A(i, j), \cdot) \neq A(j, \cdot) \).

Suppose that there is a computable winning strategy of \( G_2 \) for Bob. Let Alice fill \( A \)-table with the values of some Friedberg numbering of \( P^{(1)} \), and let Bob use some computable winning strategy. Then, \( A \) is a Friedberg numbering of \( P^{(1)} \), and \( B \) is a Friedberg numbering of \( P^{(1)} \) to which \( A \) is not reducible. Since \( A \) is minimal, \( B \) is also not reducible to \( A \).

We describe a computable winning strategy of \( G_2 \). Bob uses the winning strategy of \( G_0 \) described in the proof of Theorem [1] which guarantees that the first two winning conditions are satisfied. For the third winning condition, Bob adds the following instruction for the \( i \)-th assistant: if the \( i \)-th row \( i \)-th column in \( A \)-table has been filled, and you have reserved the \( A(i, i) \)-th row, then odd-ify the \( A(i, i) \)-th row. Each of these instructions is done at most once because after doing this the corresponding requirement is permanently satisfied. Hence they do not interrupt satisfying the first two winning conditions. It remains to show that the third winning condition is also satisfied. Suppose that \( A(i, \cdot) \) is total. We can assume without loss of generality that \( i \) is the least index of \( A(i, \cdot) \), i.e., there is no \( j < i \) with \( A(j, \cdot) = A(i, \cdot) \). Since \( A(i, \cdot) \) is not either odd or even, the \( i \)-th assistant has a permanently reserved row \( j \). The additional instruction guarantees that \( j \neq A(i, \cdot) \).

By the second winning condition, we have that \( B(A(i, i), \cdot) \neq B(j, \cdot) = A(i, \cdot) \). Thus Bob wins this game.

\( \square \)

The second proof is more symmetric and parameterized. In the second proof, we consider an infinite game where Alice constructs two partial functions \( A \) and \( R \), and Bob constructs two partial functions \( B \) and \( C \). The aim of Bob is to guarantee that each of \( B \) and \( C \) contains all the partial functions from \( A \) exactly once, and \( B \) and \( C \) are not reducible to each other by any total function from \( R \). A sketch of the winning strategy is as follows: let Bob fix a computable injective numbering \( \varphi \) of non-odd partial functions which will be used as a witness of incomparability of \( B \) and \( C \) relative to \( R \). The assistant responsible for \( \varphi_{2i} \) guarantees that \( R(i, \cdot) \) is not a reduction function from \( C \) to \( B \). The assistant responsible for \( \varphi_{2i+1} \) guarantees that \( R(i, \cdot) \) is not a reduction function from \( B \) to \( C \). We now describe the details.

Proof (symmetric version). We consider the following game \( G_3 \):

Players: Alice, Bob.

Protocol: FOR \( s = 0, 1, 2, \ldots \):
- Alice announces two finite partial functions \( A_s, R_s : \mathbb{N}^2 \to \mathbb{N} \).
- Bob announces two finite partial functions \( B_s, C_s : \mathbb{N}^2 \to \mathbb{N} \).

Collateral duties: \( A_s \subseteq A_{s+1} \), \( R_s \subseteq R_{s+1} \), \( B_s \subseteq B_{s+1} \), and \( C_s \subseteq C_{s+1} \) for all \( s \in \mathbb{N} \).
Winner: Let $A = \bigcup_{s \in \mathbb{N}} A_s$, $R = \bigcup_{s \in \mathbb{N}} R_s$, $B = \bigcup_{s \in \mathbb{N}} B_s$, and $C = \bigcup_{s \in \mathbb{N}} C_s$.
Bob wins if

1. for each $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that $A(i, \cdot) = B(j, \cdot)$;
2. for each $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that $A(i, \cdot) = C(j, \cdot)$;
3. for any $i, j \in \mathbb{N}$, if $i \neq j$, then $B(i, \cdot) \neq B(j, \cdot)$;
4. for any $i, j \in \mathbb{N}$, if $i \neq j$, then $C(i, \cdot) \neq C(j, \cdot)$;
5. for each $i \in \mathbb{N}$, if $R(i, \cdot)$ is total, then there is a $j \in \mathbb{N}$ such that $B(R(i, j), \cdot) \neq C(j, \cdot)$;
6. for each $i \in \mathbb{N}$, if $R(i, \cdot)$ is total, then there is a $j \in \mathbb{N}$ such that $C(R(i, j), \cdot) \neq B(j, \cdot)$.

First, to consider the constant functions separately, we modify the winning strategy of $G_0$ described in the proof of Theorem 1. Bob adds the following instruction for the $i$-th assistant: let $k$ be the number of rows such that you have already odd-ified. If in the current state of $A$-table the first $k$ positions of the $i$-th row are constant, odd-ify your reserved row. Next, Bob hires a countable number of additional assistants who guarantee that each of the constant functions appears in $B$-table exactly once. At each move, all assistants work one by one. The $i$-th additional assistant starts working at move $i$. She can reserve a row in $B$-table exclusively, fill her reserved row, and odd-ify her reserved row. The instruction for the $i$-th additional assistant: if you have no reserved row, reserve a new row. If you have a reserved row, fill your reserved row so that this row becomes the constant function $i$ in the limit. The modified strategy is still a computable winning strategy of $G_0$.

Second, we describe a computable winning strategy of $G_3$. To guarantee that the first four winning conditions are satisfied, Bob uses two copies of the winning strategy of $G_0$ described in the previous paragraph. At each move, the strategies work one by one. For the fifth winning condition, Bob adds the following instruction for the $2i$-th additional assistant: if you have reserved the $j$-th row in $C$-table, the $i$-th row $j$-th column in $R$-table has been filled, and you have reserved the $R(i, j)$-th row in $B$-table, then odd-ify the $R(i, j)$-th row in $B$-table. Symmetrically, for the sixth condition, Bob adds the following instruction for the $2i + 1$-st additional assistant: if you have reserved the $j$-th row in $B$-table, the $i$-th row $j$-th column in $R$-table has been filled, and you have reserved the $R(i, j)$-th row in $C$-table, then odd-ify the $R(i, j)$-th row in $C$-table. Since each of these instructions is done at most once, the first four winning conditions are still satisfied. We now prove that the last two winning conditions are also satisfied. Suppose that $R(i, \cdot)$ is total. The $2i$-th additional assistant has a permanently reserved row $c$ in $C$-table. The additional instruction guarantees that she does not permanently reserve the $R(i, c)$-th row in $B$-table. Hence $B(R(i, c), \cdot)$ is not identical to the constant function $2i$. It follows that $B(R(i, c), \cdot) \neq C(c, \cdot)$. Similarly, we have that $C(R(i, b), \cdot) \neq B(b, \cdot)$ for some $b$. Thus Bob wins this game.

The following can be proved in a way similar to the second proof of Theorem 1. It suffices to consider an infinite game where Bob constructs a countable number of partial functions $B^0, B^1, B^2, \ldots$ simultaneously. To ensure finiteness of actions, Bob shall start constructing the $i$-th numbering $B^i$ at move $i$.

Corollary 5 (Khutoretskii Corollary 2). There is a uniformly c.e. sequence of pairwise incomparable Friedberg numberings of $\mathcal{P}^{(1)}$. 


Corollary 6. Let $A$, $B$ and $C$ be disjoint subsets of $\mathcal{P}^{(1)}$. If $A$ has a computable numbering, $B$ and $C$ have Friedberg numberings, and every finite subfunction of a member of $A \cup C$ has infinitely many extensions in $B$, then there is a uniformly c.e. sequence of pairwise incomparable Friedberg numberings of $A \cup B \cup C$.

Corollary 7 is an instance of Corollary 6 when $B$ is the set of all odd partial functions, $C$ is the set of all constant functions, and $A$ is the complement of $B \cup C$.

Corollary 7 (Kummer [10, Lemma]). Let $A$ and $B$ be disjoint subsets of $\mathcal{P}^{(1)}$. If $A$ and $B$ have Friedberg numberings, and every finite subfunction of a member of $A$ has infinitely many extensions in $B$, then there is a uniformly c.e. sequence of pairwise incomparable Friedberg numberings of $A \cup B$.

Proof. Fix a Friedberg numbering $\nu$ of $A$. Apply Corollary 6 to $A'$ = \{ $\nu(2i)$ | $i \in \mathbb{N}$ \}, $B$, and $C$ = \{ $\nu(2i+1)$ | $i \in \mathbb{N}$ \}. \qed

Conversely, we can prove Corollary 6 by using Corollary 7 as follows. Applying Corollary 2 to $A$ and $C$, we have a Friedberg numbering of $A \cup C$. By Corollary 7 ($A \cup C \cup B$) has a uniformly c.e. sequence of pairwise incomparable Friedberg numberings.

Corollary 8 (Kummer [10, Theorem 1]). Let $A$ be a subset of $\mathcal{P}^{(1)}$. If $A$ has a Friedberg numbering, and every finite subfunction of a member of $A$ has infinitely many extensions in $A$, then there is a uniformly c.e. sequence of pairwise incomparable Friedberg numberings of $A$.

Proof. See [10]. \qed

It is immediate from Corollary 6 that the Rogers semilattice of $\mathcal{P}^{(1)}$ has a countable antichain. Naturally, we can ask whether the Rogers semilattice of $\mathcal{P}^{(1)}$ has a countable chain. This, in fact, is true. This was proved by Khutoretskii [3, Proof of Corollary 1]. Moreover, there is a uniformly c.e. independent sequence of Friedberg numberings of $\mathcal{P}^{(1)}$. Consequently, the Rogers semilattice of $\mathcal{P}^{(1)}$ is a universal countable partial order. We provide a game-theoretic proof of this theorem.

Theorem 9. There is a uniformly c.e. independent sequence of Friedberg numberings of $\mathcal{P}^{(1)}$.

Proof. Let us consider the following game $G_4$:

Players: Alice, Bob.

Protocol: FOR $s = 0, 1, 2, \ldots$ :

Alice announces two finite partial functions $A_s, R_s : \mathbb{N}^2 \to \mathbb{N}$.

Bob announces $s+1$ finite partial functions $B^k_s : \mathbb{N}^2 \to \mathbb{N}$ ($k \leq s$).

Collateral duties: $A_s \subseteq A_{s+1}$, $R_s \subseteq R_{s+1}$ and $B^k_s \subseteq B^k_{s+1}$ ($k \leq s$) for all $s \in \mathbb{N}$.

Winner: Let $A = \bigcup_{s \in \mathbb{N}} A_s$, $R = \bigcup_{s \in \mathbb{N}} R_s$ and $B^k = \bigcup_{k \leq s} B^k_s$ ($k \in \mathbb{N}$). Bob wins if

1. for any $k \in \mathbb{N}$ and for each $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that $A(i, \cdot) = B^k(j, \cdot)$;
2. for any $i, j, k \in \mathbb{N}$, if $i \neq j$, then $B^k(i, \cdot) \neq B^k(j, \cdot)$;
3. for any $k \in \mathbb{N}$ and for each $i \in \mathbb{N}$, if $R(i, \cdot)$ is total, then there is a $j \in \mathbb{N}$ such that $\bigoplus_{l \neq k} B^l(R(i,j), \cdot) \neq B^k(j, \cdot)$.


Let us explain how Bob wins this game by an effective way. To guarantee that the first two winning conditions are satisfied, Bob uses countable copies of the winning strategy of $G_0$ described in the second proof of Theorem 4. At each move, the strategies work one by one. The $i$-th strategy starts running at move $i$. For the third winning condition, Bob adds the following instruction for the $(i,k)$-th additional assistant: if you have reserved the $j$-th row in $B^k$-table, and the $i$-th row $j$-th column in $R$-table has been filled, and, in addition, you have reserved the $R(i,j)$-th row in $\bigoplus_{l \neq k} B^l$-table, odd-ify the $R(i,j)$-th row in $\bigoplus_{l \neq k} B^l$-table. Each instruction is done at most once. The first two winning conditions are satisfied.

Finally, we prove that the third winning condition is also satisfied. Suppose that $R(i,\cdot)$ is total. The $(i,k)$-th additional assistant has a permanently reserved row $j$ in $B^k$-table. Then, by the additional instruction, she does not permanently reserve the $R(i,j)$-th row in $\bigoplus_{l \neq k} B^l$-table, so $\bigoplus_{l \neq k} B^l(R(i,j),\cdot) \neq B^k(j,\cdot)$. Thus Bob wins this game.

\[ \square \]

**Corollary 10.** If $A$, $B$ and $C$ are subsets of $\mathcal{P}(1)$ satisfying the hypotheses of Corollary 6, then there is a uniformly c.e. independent sequence of Friedberg numberings of $A \cup B \cup C$.

**Corollary 11.** If $A$ and $B$ are subsets of $\mathcal{P}(1)$ satisfying the hypotheses of Corollary 7, then there is a uniformly c.e. independent sequence of Friedberg numberings of $A \cup B$.

**Corollary 12.** If $A$ is a subset of $\mathcal{P}(1)$ satisfying the hypotheses of Corollary 8, then there is a uniformly c.e. independent sequence of Friedberg numberings of $A$.

**Acknowledgement**

The author is grateful to Alexander Shen for discussion and helpful comments. He recommended that we consider the symmetric game $G_3$ instead of the asymmetric game $G_2$, and he gave the sketch of the winning strategy of $G_3$.

**References**

[1] Richard M. Friedberg, *Three Theorems on Recursive Enumeration. I. Decomposition. II. Maximal Set. III. Enumeration without Duplication*, The Journal of Symbolic Logic 23 (1958), no. 3, 309–316.

[2] A. B. Khutoretskii, *On the reducibility of computable numerations*, Algebra and Logic 8 (1969), no. 2, 145–151.

[3] ________, *On the cardinality of the upper semilattice of computable numerations*, Algebra and Logic 10 (1971), no. 5, 348–352.

[4] Martin Kummer, *Numberings of $R_1 \cup F$*, Lecture Notes in Computer Science 385 (1989), 166–186.

[5] ________, *An easy priority-free proof of a theorem of Friedberg*, Theoretical Computer Science 74 (1990), no. 2, 249–251.

[6] ________, *Some applications of computable one-one numberings*, Archive for Mathematical Logic 30 (1990), no. 4, 219–230.

[7] Marian Boykan Pour-El, *Gödel Numberings Versus Friedberg Numberings*, Proceedings of the American Mathematical Society 15 (1964), no. 2, 252–256.

[8] Marian Boykan Pour-El and Hilary Putnam, *Recursively enumerable classes and their application to recursive sequences of formal theories*, Archiv für mathematische Logik und Grundlagenforschung 8 (1965), no. 3, 104–121.

[9] Hartley Rogers, Jr., *Gödel Numberings of Partial Recursive Functions*, The Journal of Symbolic Logic 23 (1958), 331–341.
[10] Alexander Shen, *Game Arguments in Computability Theory and Algorithmic Information Theory*, Lecture Notes in Computer Science **7318** (2012), 655–666.

Department of Mathematics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan

E-mail address: s1240008@ems.u-toyama.ac.jp