SPACES OF COMMUTING ELEMENTS IN THE CLASSICAL GROUPS

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Abstract. Let $G$ be the classical group, and let $\text{Hom}(\mathbb{Z}^m, G)$ denote the space of commuting $m$-tuples in $G$. First, we refine the formula for the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)$ due to Ramras and Stafa by assigning (signed) integer partitions to (signed) permutations. Using the refined formula, we determine the top term of the Poincaré series, and apply it to prove the dependence of the topology of $\text{Hom}(\mathbb{Z}^m, G)$ on the parity of $m$ and the rational hyperbolicity of $\text{Hom}(\mathbb{Z}^m, G)$ for $m \geq 2$. Next, we give a minimal generating set of the cohomology of $\text{Hom}(\mathbb{Z}^m, G)$ and determine the cohomology in low dimensions. We apply these results to prove homological stability for $\text{Hom}(\mathbb{Z}^m, G)$ with the best possible stable range. Baird proved that the cohomology of $\text{Hom}(\mathbb{Z}^m, G)$ is identified with a certain ring of invariants of the Weyl group of $G$, and our approach is a direct calculation of this ring of invariants.

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1. Introduction

Let $G$ be a compact connected Lie group. The space of commuting elements in $G$, denoted by $\text{Hom}(\mathbb{Z}^m, G)$, is the subspace of the Cartesian product $G^m$ consisting of $(g_1, \ldots, g_m) \in G^m$ such that $g_1, \ldots, g_m$ are pairwise commutative. The

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topology of $\text{Hom}(\mathbb{Z}^m, G)$ has been studied intensely in recent years; in particular, its homological and homotopical features have been studied by a number of authors $[1, 2, 4, 5, 10, 11, 16, 24, 25, 26]$. On the other hand, as in $[6]$, $\text{Hom}(\mathbb{Z}^m, G)$ is identified with the based moduli space of flat $G$-bundles over an $m$-torus. Then it has been studied also in the context of geometry and physics, including work of Kac and Smilga $[19]$ and Witten $[29, 30]$ on supersymmetric Yang-Mills theory. We refer the reader to a comprehensive survey $[9]$ for the basics of $\text{Hom}(\mathbb{Z}^m, G)$ and a list of related work.

The purpose of this paper is two-fold. Let $G$ be the classical group. First, we refine the formula for the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)$ due to Ramras and Stafa $[24]$ in terms of integer partitions. Using the refined formula, we determine the top term of the Poincaré series, which applies to prove the dependence of the topology of $\text{Hom}(\mathbb{Z}^m, G)$ and the rational hyperbolicity of $\text{Hom}(\mathbb{Z}^m, G)$. Second, based on the description of the cohomology of $\text{Hom}(\mathbb{Z}^m, G)$ due to Baird $[4]$ in terms of a certain ring of invariants of the Weyl group of $G$, we give a minimal generating set of the cohomology and determine the cohomology in low dimensions. These results apply to prove homological stability for $\text{Hom}(\mathbb{Z}^m, G)$ with the best possible stable range.

1.1. Poincaré series. Let $\text{Hom}(\mathbb{Z}^m, G)_1$ denote the path-component of $\text{Hom}(\mathbb{Z}^m, G)$ containing $(1, \ldots, 1) \in G^m$, and let $W$ denote the Weyl group of $G$. In $[24]$, Ramras and Stafa gave a formula for the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$:

$$P(\text{Hom}(\mathbb{Z}^m, G)_1; t) = \frac{1}{|W|} \prod_{i=1}^r (1 - t^{2d_i}) \sum_{w \in W} \frac{\det(1 + tw)^m}{\det(1 - t^2w)}$$

(1)

Remarks on this formula are in order. The Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ means that of the cohomology of $\text{Hom}(\mathbb{Z}^m, G)_1$ over a field of characteristic zero or prime to the order of $W$. Then the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ does not depend on the coefficient field unless it is of characteristic zero or prime to $|W|$. The integers $d_1, \ldots, d_r$ are the characteristic degrees of the Weyl group $W$, and the determinants are taken by using the reflection group structure of $W$.

Although the formula (1) is beautiful, it is less computable. For example, we cannot determine the degree of the Poincaré series directly from the formula (1). Then in order to get information from the Poincaré series, we must refine the formula into a more computable form. Suppose $G$ is the classical group. Then the Weyl group $W$ is a symmetric group, a signed symmetric group or its subgroup. So we can assign (signed) integer partitions to elements of $W$ via the (signed) cyclic decomposition of (signed) permutations. This enables us to refine the formula (1), and we give it for unitary groups here. The formulae for other classical groups will be given in Section 4.
Theorem A. For \( \lambda = (\lambda_1^{n_1}, \ldots, \lambda_k^{n_k}) \vdash k \leq n, \) let
\[
p_{\lambda}^{m,n}(t) = \frac{t^{(m-2)(n-k)}}{\lambda_1 \cdots \lambda_k n_1! \cdots n_k!} \prod_{i=1}^{l} \left( (-1)^{m(\lambda_i-1)} t^{(m-2)\lambda_i} + \frac{(1 + (-1)^{\lambda_i+1} t^{\lambda_i})^m}{1 - t^{2\lambda_i}} \right)^{n_i}
\]
where we set \( q_{\lambda}^{m,n}(t) = t^{(m-2)n} \) if \( \lambda \) is the empty partition. Then the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, U(n))_1 \) is given by
\[
P(\text{Hom}(\mathbb{Z}^m, U(n))_1; t) = \begin{cases} 
\prod_{i=1}^{n}(1 - t^{2i}) \sum_{k=0}^{n} (-1)^k p_{\lambda}^{m,n}(t) & (m \text{ even}) \\
\prod_{i=1}^{n}(1 - t^{2i}) \sum_{k=0}^{n} (-1)^{k+1} p_{\lambda}^{m,n}(t) & (m \text{ odd}). 
\end{cases}
\]

1.2. Top terms. Using the formula in Theorem A together with enumeration of (signed) integer partitions, we can determine the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) when \( G \) is the classical group.

Theorem B. The top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, U(n))_1 \) is \( t^{n^2+(m-1)n} \) for \( m \) odd and \( \left( \frac{m+n-2}{m-1} \right) t^{n^2+(m-2)n+1} \) for \( m \) even.

The top terms for other classical groups will be also determined in Section 5. Clearly, the formula in Theorem A depends on the parity of \( m \), and Theorem B shows that the Poincaré series themselves depend on the parity of \( m \). This dependence can be stated in terms of palindromicity.

Corollary C. Let \( G \) be the classical group which is neither trivial nor \( S^1 \). Then the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is palindromic if and only if \( m \) is odd.

As mentioned above, our refinement of the formula (1) for the classical groups is based on the assignment of (signed) integer partitions to (signed) permutations. Then it does not apply to the exceptional Lie groups. However, we can calculate the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) for \( G \) exceptional directly from the formula (1) of Ramras and Stafa by using a computer. We will give the result in Appendix. In particular, we determine the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) for every simple Lie group \( G \). Using these results, we can determine the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) for every compact connected Lie group \( G \).

Theorem D. Let \( G \) be a compact connected Lie group with simple factors \( G_1, \ldots, G_k \). Then the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) is
\[
(\text{rank } G_1 + 1) \cdots (\text{rank } G_k + 1) t^{\dim G + \text{rank } \pi_1(G)}.
\]

This result has an application to the rational homotopy of \( \text{Hom}(\mathbb{Z}^m, G)_1 \). Let \( X \) be a simply-connected finite complex. As in [12, Part IV], it is well known that \( \sum_{n \geq 1} \pi_n(X) \otimes \mathbb{Q} \) is either finite or of exponential growth. In the former case, \( X \) is called rationally elliptic, and the latter, rationally hyperbolic.
Corollary E. Let $G$ be the non-trivial compact simply-connected Lie group. Then for $m \geq 2$, $\text{Hom}(\mathbb{Z}^m, G)_1$ is rationally hyperbolic.

1.3. Cohomology generators. Let $T$ denote a maximal torus of $G$, and let $F$ be a field of characteristic zero or prime to the order of $W$. In [4], Baird proved that there is an isomorphism

\[ H^*(\text{Hom}(\mathbb{Z}^m, G)_1; F) \cong (H^*(G/T; F) \otimes H^*(T; F)^\otimes m)^W. \]

Since $H^*(G/T; F)$ is identified with the ring of coinvariants of $W$, the RHS is completely determined by the Weyl group $W$. We calculate the invariant ring on the RHS to give a minimal generating set of the cohomology of $\text{Hom}(\mathbb{Z}^m, G)_1$ over $F$ for the classical group $G$, except for $SO(2n)$. Here we show the result for $U(n)$, and the results for other classical groups will be given in Section 6.

Theorem F. The cohomology of $\text{Hom}(\mathbb{Z}^m, U(n))_1$ over a field $F$ of characteristic zero or prime to $n!$ is minimally generated by

\[ S := \{ z(d, I) \mid 1 \leq d \leq n \text{ and } \emptyset \neq I \subset \{1, 2, \ldots, m\} \text{ such that } d + |I| - 1 \leq n \} \]

where $|z(d, I)| = 2d + |I| - 2$.

Let $\mathcal{H} := H^*(BT; F) \otimes H^*(T; F)^\otimes m$ and $\mathcal{K} := H^*(G/T; F) \otimes H^*(T; F)^\otimes m$. Then $\mathcal{K}$ is a quotient of $\mathcal{H}$, and by (2), we aim to give a minimal generating set of $\mathcal{K}^W$. Our calculation of the invariant ring $\mathcal{K}^W$ is quite direct and consists of two parts. First, we define a subset, say $\mathcal{S}$, of $\mathcal{H}^W$ from polynomial invariants of the Weyl group $W$, and show that $\mathcal{S}$ generate $\mathcal{K}^W$. The proof is done by an ordering on $\mathcal{H}$, which is a key ingredient. Second, we choose a subset of $\mathcal{S}$, say $S$, minimally generating $\mathcal{K}^W$. We show that every element of $\mathcal{S}$ is given in terms of $S$ by using a polynomial tensor exterior algebra analog of the Newton formula for symmetric polynomials and power sums. Then we prove minimality of $\mathcal{S}$ by describing $\mathcal{K}^W$ in low dimension. For unitary groups, we prove the following, where analogous results for other classical groups will be given in Section 6. Let $F\langle S \rangle$ denote a free graded commutative algebra generated by a graded set $S$.

Theorem G. Let $F$ be a field of characteristic zero or prime to $n!$. Then the map

\[ F\langle S \rangle \to H^*(\text{Hom}(\mathbb{Z}^m, U(n))_1; F) \]

is an isomorphism in dimension $\leq 2n - m$, where $S$ is as in Theorem F.

Applying the result for $SU(n)$, we determine the cohomology of $\text{Hom}(\mathbb{Z}^2, SU(n))_1$ for $n = 2, 3$. As is seen in this example calculation, relations among our generators are quite complicated, and we do not have a general scheme to get relations.
1.4. **Homological stability.** A sequence of spaces $X_1 \to X_2 \to X_3 \to \cdots$ is said to satisfy homological stability if for every $i \geq 0$, the induced sequence

$$H_i(X_1) \to H_i(X_2) \to H_i(X_3) \to \cdots$$

eventually consists of isomorphisms. Clearly, each sequence of the classical groups in the same series satisfies homological stability, and the classical result of Nakaoka \cite{21} shows that the sequence of (the classifying spaces of) the symmetric groups satisfy homological stability. Besides these classical results, many important series of groups \cite{3, 17, 18, 23} and spaces \cite{8, 15, 20} are proved to satisfy homological stability, and recently, the techniques for proving homological stability are rapidly developed \cite{7, 8, 14}.

Let $G_1 \to G_2 \to G_3 \to \cdots$ be one of the series of $U(n), SU(n), Sp(n), SO(2n+1)$. Then there is a sequence

$$\text{Hom}(\mathbb{Z}^m, G_1) \to \text{Hom}(\mathbb{Z}^m, G_2) \to \text{Hom}(\mathbb{Z}^m, G_3) \to \cdots$$

By applying the technique of Church and Farb \cite{8}, Ramras and Stafa \cite{25} proved that the sequence satisfies homological stability such that the map

$$H_i(\text{Hom}(\mathbb{Z}^m, G_n); \mathbb{Q}) \to H_i(\text{Hom}(\mathbb{Z}^m, G_{n+1}); \mathbb{Q})$$

is an isomorphism for $i \leq n - \lfloor \sqrt{n} \rfloor$. As an example application of Theorem A, we will calculate the Poincaré series of $\text{Hom}(\mathbb{Z}^2, SU(n))_1$ for $2 \leq n \leq 5$ in Example 4.4. As far as looking at these Poincaré series, those of $\text{Hom}(\mathbb{Z}^2, SU(n))_1$ and $\text{Hom}(\mathbb{Z}^2, SU(n+1))_1$ coincide in degree $\leq 2n - 1$. This suggest possibility of extending the stable range for $U(n)$, i.e. those degrees for which stability holds.

By construction, the generating set $S$ in Theorem F is natural with respect to the inclusion $U(n) \to U(n+1)$. Then by Theorem G together with a bit of effort, we can give an alternative proof for homological stability of $\text{Hom}(\mathbb{Z}^m, G_n)_1$, which provides the best possible stable range.

**Theorem H.** For $n \geq m$, the map

$$H_*(\text{Hom}(\mathbb{Z}^m, U(n)_1; \mathbb{Q}) \to H_*(\text{Hom}(\mathbb{Z}^m, U(n+1)_1; \mathbb{Q})$$

is an isomorphism for $* \leq 2n - m + 1$ and is not surjective for $* = 2n - m + 2$.

Homological stability for other series of the classical groups will be given in Section 6.

1.5. **Outline.** We recall in Section 2 the map giving an isomorphism of Baird (2) above, where this isomorphism is the basis of our study. In Section 3, we connect (signed) permutations to (signed) integer partitions, and show some enumerative properties of these connections. We begin Section 4 with giving a short alternative proof for the formula (1) due to Ramras and Stafa. Then we compute the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ for the classical group $G$ by applying the results in Section 3 to the formula (1). We determine in Section 5 the top term of the Poincaré series and give its applications. In Section 6, we investigate the invariant ring of
the Weyl group on the RHS of (2) to give its minimal generators and description in low dimensions. In Section 7, we apply the results in Section 6 to show that the sequence \( \text{Hom}(\mathbb{Z}^m, G_n)_1 \) satisfies homological stability and obtain the best possible stable range, where \( G_n \) is as above. We end this paper in Section 8 by posing several questions, arising in our study, connected to both topology and representation theory.

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2. **Cohomology and Weyl groups**

In this section, we recall the result of Baird [4] describing the cohomology of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) in terms of a ring of invariants of the Weyl group of \( G \). This result is the basis of our study.

Throughout this section, let \( G \) be a compact connected simple Lie group, let \( T \) be its maximal torus, and let \( W \) denote the Weyl group of \( G \). Consider the action of \( W \) on \( G/T \times T^m \) given by

\[
w \cdot (gT, t_1, \ldots, t_m) = (gwT, w^{-1}t_1w, \ldots, w^{-1}t_mw)
\]

for \( w \in W, g \in G, t_1, \ldots, t_m \in T \). Then the naive map

\[
G \times T^m \to \text{Hom}(\mathbb{Z}^m, G)_1, \quad (g, t_1, \ldots, t_m) \mapsto (gt_1g^{-1}, \ldots, gt_mg^{-1})
\]

for \( g \in G, t_1, \ldots, t_m \) defines a map

\[
\phi: G/T \times_W T^m \to \text{Hom}(\mathbb{Z}^m, G)_1.
\]

Let \( \mathbb{F} \) be a field of characteristic zero or prime to the order of \( W \). In [4], Baird proved that the map \( \phi \) is an isomorphism in cohomology over a field \( \mathbb{F} \). Since the action of \( W \) on \( G/T \times T^m \) is free, we have an isomorphism

\[
H^*(G/T \times_W T^m; \mathbb{F}) \cong (H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F})^\otimes m)^W.
\]

Then we can restate the result of Baird as follows.

**Theorem 2.1.** Let \( \mathbb{F} \) be a field of characteristic zero or prime to the order of \( W \). Then there is an isomorphism

\[
H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \cong (H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F})^\otimes m)^W.
\]

We recall the characteristic degrees of the Weyl group \( W \). Suppose \( G \) is of rank \( n \), and let

\[
\mathcal{P}(n) := \mathbb{F}[x_1, \ldots, x_n], \quad |x_i| = 2
\]

where \( \mathbb{F} \) is a field as above. Then \( \mathcal{P}(n) \) is identified with the cohomology of the classifying space \( BT \), so that \( W \) acts on \( \mathcal{P}(n) \). By the Shephard-Todd theorem, we have

\[
\mathcal{P}(n)^W = \mathbb{F}[a_1, \ldots, a_n], \quad |a_i| = 2d_i
\] (3)
for some integers \(d_1, \ldots, d_n\). We call these integers \(d_1, \ldots, d_n\) the characteristic degrees of \(W\). Notice that the number of the characteristic degrees of \(W\) coincides with the rank of \(G\). We give a table including information on the Weyl groups of simple Lie groups, where \(\Sigma_n\) and \(B_n\) denote the symmetric group and the signed symmetric group, respectively, and \(B_n^+\) denotes the subgroup of \(B_n\) consisting of signed permutation with total sign one.

| Type | Lie group | Rank | \(W\) | \(|W|\) | Characteristic degrees |
|------|-----------|------|--------|--------|------------------------|
| \(A_n\) | \(SU(n+1)\) | \(n\) | \(\Sigma_{n+1}\) | \((n+1)!\) | \(2, 3, \ldots, n+1\) |
| \(B_n\) | \(SO(2n+1)\) | \(n\) | \(B_n\) | \(2^n n!\) | \(2, 4, \ldots, 2n\) |
| \(C_n\) | \(Sp(n)\) | \(n\) | \(B_n\) | \(2^n n!\) | \(2, 4, \ldots, 2n\) |
| \(D_n\) | \(SO(2n)\) | \(n\) | \(B_n^+\) | \(2^{n-1} n!\) | \(2, 4, \ldots, 2n - 2, n\) |
| \(G_2\) | \(G_2\) | \(2\) | —— | 12 | 2, 6 |
| \(F_4\) | \(F_4\) | \(4\) | —— | 1152 | 2, 6, 8, 12 |
| \(E_6\) | \(E_6\) | \(6\) | —— | 51840 | 2, 5, 6, 8, 9, 12 |
| \(E_7\) | \(E_7\) | \(7\) | —— | 2903040 | 2, 6, 8, 10, 12, 14, 18 |
| \(E_8\) | \(E_8\) | \(8\) | —— | 696729600 | 2, 8, 12, 14, 16, 20, 24, 30 |

Now we give a short alternative proof of the following theorem due to Ramras and Stafa [24], the formula (1) in Section 1, using Theorem 2.1.

**Theorem 2.2.** The Poincaré series of \(\text{Hom}(\mathbb{Z}^m, G)_{1}\) is given by

\[
P(\text{Hom}(\mathbb{Z}^m, G)_{1}; t) = \frac{1}{|W|} \prod_{i=1}^{r} (1 - t^{2d_i}) \sum_{w \in W} \frac{\det(1+tw)^m}{\det(1-t^2w)}
\]

where \(d_1, \ldots, d_n\) are the characteristic degrees of \(W\).

**Proof.** Let us work over a field \(F\) of characteristic zero or prime to \(|W|\). By the Shephard Todd theorem, there is an isomorphism of \(W\)-modules

\[
\mathcal{P}(n) \cong \mathcal{P}(n)^W \otimes \mathcal{P}(n)_{W}
\]

where \(\mathcal{P}(n)_{W}\) denotes the ring of coinvariants of \(W\). Then since \(H^*(G/T; \mathbb{F})\) is identified with \(\mathcal{P}(n)_{W}\), there is an isomorphism

\[
(H^*(G/T; \mathbb{F}) \otimes H^*(T)^{\otimes m})^W \cong (\mathcal{P}(n) \otimes H^*(T)^{\otimes m})^W / (a_{d_1}, \ldots, a_{d_n})
\]

where \(a_1, \ldots, a_n\) are as in (3). Since the sequence \(a_1, \ldots, a_n\) is regular in \(\mathcal{P}(n)\), it is also regular in \((\mathcal{P}(n) \otimes H^*(T)^{\otimes m})^W\). Then it follows from Theorem 2.1 that

\[
P(\text{Hom}(\mathbb{Z}^m, G)_{1}; t) = P((\mathcal{P}(n) \otimes H^*(T)^{\otimes m})^W; t) \prod_{i=1}^{n} (1 - t^{2d_i}).
\]
By the standard argument in representation theory of finite groups,

\[ P((\mathcal{P}(n) \otimes H^*(T)^{\otimes m})^W; t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{i=0}^\infty \text{tr}(w|_{\mathcal{P}(n)})t^{2i} \right) \left( \sum_{i=0}^\infty \text{tr}(w|_{H^*(T)})t^i \right)^m \]

and it is easy to see that

\[ \sum_{i=0}^\infty \text{tr}(w|_{\mathcal{P}(n)})t^{2i} = \frac{1}{\det(1 - t^2w)} \quad \text{and} \quad \sum_{i=0}^\infty \text{tr}(w|_{H^*(T)})t^i = \det(1 + tw) \]

where \( A^i \) denote the \( i \)-dimensional part of a graded algebra \( A \). Thus the proof is complete.

3. Integer partitions and permutations

In this section, we assign (signed) integer partitions to (signed) permutations, and show enumerative properties of this assignment. The result in this section will be used to compute the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) for the classical group \( G \) in the next section.

3.1. Integer partitions. A partition of a positive integer \( n \) is a sequence of positive integers \( \lambda = (\lambda_1, \ldots, \lambda_k) \) such that \( \lambda_1 \leq \cdots \leq \lambda_k \) and \( \lambda_1 + \cdots + \lambda_k = n \). For this integer partition \( \lambda \), let

\[ \ell(\lambda) := k. \]

We denote an integer partition \( \mu = (\mu_1, \ldots, \mu_{l_1}, \ldots, \mu_{l_n}) \) by \( (\mu_1^{n_1}, \ldots, \mu_l^{n_l}) \) alternatively. For this integer partition \( \mu \), let

\[ \theta(\mu) := \mu_1 \cdots \mu_{l_1}! \cdots n_l!. \]

If \( \lambda \) is the empty partition, then we set \( \theta(\lambda) = 1 \). If \( \lambda \) is a partition of \( n \), we write \( \lambda \vdash n \).

We mean by \( \lambda \vdash 0 \) that \( \lambda \) is the empty partition. For \( \lambda \vdash n \) and \( \mu \vdash k \) with \( n \geq k \), let \( \text{Emb}(\mu, \lambda) \) denote the set of subsequences of \( \lambda \) which are equal to \( \mu \).

**Example 3.1.** If \( \lambda = (1^2, 2^2, 3^2, 4^2) \) and \( \mu = (1, 2, 4) \), then \( |\text{Emb}(\mu, \lambda)| = 2^3 = 8 \).

For a cyclic permutation \( c \in \Sigma_n \), let \(|c|\) denote the order of \( c \). Let \( w \in \Sigma_n \). Then there are disjoint cyclic permutations \( c_1, \ldots, c_k \in \Sigma_n \) such that

\[ w = c_1 \cdots c_k \quad \text{and} \quad |c_1| + \cdots + |c_k| = n \]

which is called a cycle decomposition. Note that we do not omit 1-cycles in a cycle decomposition. A cycle decomposition \( w = c_1 \cdots c_k \) is called standard if \(|c_1| \leq \cdots \leq |c_k|\) and the least element of \( c_i \) is less than the least element of \( c_{i+1} \).
whenever \(|c_i| = |c_{i+1}|\). Clearly, every permutation has a unique standard cycle decomposition.

**Example 3.2.** A cycle decomposition \((1\ 6)(2\ 5)(3\ 4\ 7)\) is standard, but \((2\ 5)(1\ 6)(3\ 4\ 7)\) is not standard.

Let \(w \in \Sigma_n\) with the standard cycle decomposition \(w = c_1 \cdots c_k\). Then we assign to \(w\) a partition of \(n\)
\[
c(w) := (|c_1|, \ldots, |c_k|).
\]
We show a property of \(c(w)\) that we are going to use. Let \([a\ b]\) denote the Stirling number of the first kind.

**Lemma 3.3.** For \(i \geq 0\) and \(\lambda \vdash k\) with \(k \leq n\),
\[
\sum_{w \in \Sigma_n} \ell(c(w)) - \ell(\lambda) = i \left\lfloor \frac{n! \theta(\lambda)(n-k)!}{\theta(\lambda)(n-k)!} \right\rfloor.
\]

**Proof.** Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\). Every subsequence of \(c(w)\) which is equal to \(\lambda\) is obtained by the following constructions:

1. We choose \(k\) disjoint cyclic permutations of order \(\lambda_1, \ldots, \lambda_k\), and
2. We divide the remaining subset of \([n]\) by disjoint \(i\) cycles.

Since the number of cyclic permutations of order \(d\) in \(\Sigma_n\) is \(\frac{n!}{\theta(d)(n-d)!}\), the number of the first choices is \(\frac{n!}{\theta(\lambda)(n-k)!}\). Since the Stirling number \([a\ b]\) counts the number of permutations on \(a\) letters having a cycle decomposition by \(b\) cyclic permutations, the number of the second divisions is \([n-k]\). Thus the identity in the statement holds. \(\square\)

We record an identity involving Stirling numbers that we will use later. See [27, Proposition 1.3.4] for the proof.

**Lemma 3.4.** There is an equality
\[
\sum_{k=1}^{n} \binom{n}{k} x^k = x(x+1) \cdots (x+n-1).
\]

### 3.2. Signed integer partitions

A *signed partition* of a positive integer \(n\) is an ordered sequence of non-zero integers \(\lambda = (\lambda_1, \ldots, \lambda_k)\) such that \((|\lambda_1|, \ldots, |\lambda_k|)\) is a partition of \(n\).

**Example 3.5.** Sequences \((1, -1, 2)\) and \((-1, 1, 2)\) are distinct signed partitions of 4, because signed partitions are ordered sequences.

If \(\lambda\) is a signed partition of \(n\), then we write
\[
\lambda \vdash \pm n.
\]
For $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash \pm n$, we define

$$\lambda^+ = (|\lambda_1|, \ldots, |\lambda_k|) \quad \text{and} \quad \text{sgn}(\lambda) = \text{sgn}(\lambda_1) \cdots \text{sgn}(\lambda_k).$$

For $\lambda \vdash \pm n$ and $\mu \vdash \pm k$ with $n \geq k$, let $\text{Emb}(\mu, \lambda)$ denote the set of ordered subseuqences of $\lambda$ which are equal to $\mu$.

**Example 3.6.** If $\lambda = (1, -1, 1, 2, 3, -3, 4)$ and $\mu = (-1, 1, 2, 3)$, then $|\text{Emb}(\mu, \lambda)| = 2$.

If $|\text{Emb}(\mu, \lambda)| > 0$, then by choosing a specific ordered subsequence $\bar{\mu}$ of $\lambda$ which is equal to $\mu$, we define $\text{sgn}(\mu, \lambda)$ to be the product of signs of elements of $\lambda$ which are not in $\mu$. Note that $\text{sgn}(\mu, \lambda)$ does not depend on the choice of $\bar{\mu}$.

**Example 3.7.** If $\lambda = (1, -1, 1, -1, 1, 2, -2, 3, -3, 4)$ and $\mu = (-1, 1, 2, 3)$, then $\text{sgn}(\mu, \lambda) = (-1)^3 = -1$.

A signed permutation on $n$ letters is a permutation $w$ on $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $w(-i) = -w(i)$. Then signed permutations on $n$ letters form a group, which we denote by $B_n$. Clearly, $B_n$ is isomorphic with a semidirect product $(\mathbb{Z}/2)^n \rtimes \Sigma_n$. Then to each $w \in B_n$ one can assign unique $s \in (\mathbb{Z}/2)^n$ and $v \in \Sigma_n$ such that $w = sv$. In this case, let

$$w^+ := v \quad \text{and} \quad \text{sgn}(w) := s_1 \cdots s_n$$

where $s = (s_1, \ldots, s_n) \in (\mathbb{Z}/2)^n = \{-1, +1\}^n$. Let $B_n^+$ denote the subgroup of $B_n$ consisting of $w \in B_n$ with $\text{sgn}(w) = 1$.

It is obvious that every signed permutation $w \in B_n$ admits a decomposition $w = c_1 \cdots c_k$ by disjoint signed cyclic permutations. A signed cycle decomposition $w = c_1, \ldots, c_k$ is said to be standard if $w^+ = c_1^+ \cdots c_k^+$ is the standard cycle decomposition. Then every signed permutation admits a unique standard signed cycle decomposition.

To every signed permutation $w \in B_n$ with the standard signed cycle decomposition $w = c_1 \cdots c_k$, we assign a signed partition of $n$

$$c(w) := (\text{sgn}(c_1)|c_1^+|, \ldots, \text{sgn}(c_k)|c_k^+|).$$

**Lemma 3.8.**

1. For $i \geq 0$ and $\lambda \vdash \pm k$ with $k \leq n$,

$$\sum_{w \in B_n} |\text{Emb}(\lambda, c(w))| = \frac{2^{n-\ell(\lambda^+)} n!}{\theta(\lambda^+)(n-k)!} \left[ \begin{array}{c} n-k \end{array} \right].$$

2. Let $\lambda, \lambda'$ be signed partitions of $n-1$ such that all but one elements are the same. Then

$$\sum_{w \in B_n^+} |\text{Emb}(\lambda, c(w))| = \sum_{w \in B_n^+} |\text{Emb}(\lambda', c(w))|. $$
Proof. The first identity is proved by the same argument as in the proof of Lemma 3.3 with an easy modification about signs. Let \( \lambda, \lambda' \) be as in the second statement. Then each element of \( \text{Emb}(\lambda, c(w)) \) and \( \text{Emb}(\lambda', c(w)) \) are obtained by adding \( \pm 1 \) to \( \lambda \) and \( \lambda' \), respectively. Since all but one elements of \( \lambda, \lambda' \) are the same, the difference is by sign. Thus the second identity is obtained. \( \square \)

4. Poincaré series

In this section, we refine the formula in Theorem 2.2 for the classical groups by applying the results in the previous section, where the refined formula is given in terms of (signed) integer partitions.

4.1. Unitary groups. We start with unitary groups. For a positive integer \( k \), let

\[
q_k^m(t) = (-1)^{m(k-1)}t^{(m-2)k} + \frac{(1 + (-1)^{k+1}t^k)^m}{1 - t^{2k}}.
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_l) \vdash k \leq n \), let

\[
q_{\lambda}^{m,n}(t) = t^{(m-2)(n-k)}q_{\lambda_1}^m(t) \cdots q_{\lambda_l}^m(t)
\]

where we set \( q_{\lambda}^{m,n}(t) = t^{(m-2)n} \) if \( \lambda \) is the empty partition.

Theorem 4.1 (Theorem A). The Poincaré series of \( \text{Hom}(\mathbb{Z}^m, U(n))_1 \) is given by

\[
P(\text{Hom}(\mathbb{Z}^m, U(n))_1; t) = \begin{cases}
\prod_{i=1}^{n} (1 - t^{2i}) \sum_{k=0}^{n} \sum_{\lambda \vdash k} (-1)^{n+k} \frac{\theta(\lambda)}{\theta(\lambda)} q_{\lambda}^{m,n}(t) & \text{if } m \text{ even} \\
\prod_{i=1}^{n} (1 - t^{2i}) \sum_{k=0}^{n} \sum_{\lambda \vdash k} (-1)^k \frac{1}{\theta(\lambda)} q_{\lambda}^{m,n}(t) & \text{if } m \text{ odd}.
\end{cases}
\]

Proof. The Weyl group of \( U(n) \) is \( \Sigma_n \) acting canonically on \( \mathbb{F}^n \). Then we apply the formula of Theorem 2.2 to the canonical representation of \( \Sigma_n \). Let \( w \in \Sigma_n \) with the standard cycle decomposition \( w = c_1 \cdots c_k \). Since \( \det(1 + tc) = 1 + (-1)^{|c|+1}t^{|c|} \) for a cyclic permutation \( c \in \Sigma_n \) and \( c_1, \ldots, c_k \) are disjoint,

\[
\frac{\det(1 + tw)^m}{\det(1 - t^2w)} = \prod_{i=1}^{k} \frac{(1 + (-1)^{|c_i|+1}t^{|c_i|})^m}{1 - t^{2|c_i|}} = \prod_{i=1}^{k} (-1)^{|c_i|+1}t^{(m-2)|c_i|} + q_{|c_i|}^m(t).
\]

Then it follows from Lemma 3.3 that

\[
\sum_{w \in \Sigma_n} \frac{\det(1 + tw)^m}{\det(1 - t^2w)} = \sum_{k=0}^{n} \sum_{\lambda \vdash k} \sum_{i=0}^{n-k} (-1)^{mk+(m+1)i} |\text{Emb}(\lambda, c(w))| q_{\lambda}^{m,n}(t)
\]

\[
= \sum_{k=0}^{n} \sum_{\lambda \vdash k} \sum_{i=0}^{n-k} (-1)^{mk+(m+1)i} \frac{n!}{\theta(\lambda)(n-k)!} \binom{n-k}{i} q_{\lambda}^{m,n}(t)
\]
By Lemma 3.4, \( \sum_{i=1}^{n-k} (-1)^i \binom{n-k}{i} = 0 \) for \( n-k \geq 2 \) and \( \sum_{i=1}^{n-k} \binom{n-k}{i} = (n-k)! \). Thus by Theorem 2.2 the proof is complete. \( \square \)

Applying Theorem 4.1, we give a formula for the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, SU(n))_1 \).

**Lemma 4.2.** Let \( G, H \) be compact connected Lie groups, and let \( \mathbb{F} \) be a field of characteristic zero or prime to the Weyl group of \( G \). If there is a covering \( G \to H \), then

\[
H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \cong H^*(\text{Hom}(\mathbb{Z}^m, H)_1; \mathbb{F}).
\]

**Proof.** Since there is a covering \( G \to H \), the Weyl groups of \( G \) and \( H \) have the same order. Then the characteristic of \( \mathbb{F} \) is also prime to the order of the Weyl group of \( H \), and so the Weyl groups of \( G \) and \( H \) are isomorphic as reflection groups over \( \mathbb{F} \). Then by Theorem 2.1 below, the proof is done. \( \square \)

**Corollary 4.3.** The Poincaré series of \( \text{Hom}(\mathbb{Z}^m, SU(n))_1 \) is given by

\[
P(\text{Hom}(\mathbb{Z}^m, SU(n))_1; t) = \begin{cases} 
\prod_{i=1}^{n} (1 - t^{2^i}) \sum_{k=0}^{n} \frac{(-1)^{n-k}}{\theta(\lambda)(1 + t)^m} q_{\lambda}^{m,n}(t) & \text{ (m even)} \\
\prod_{i=1}^{n} (1 - t^{2^i}) \sum_{k=0}^{n} \frac{(-1)^k}{\theta(\lambda)(1 + t)^m} q_{\lambda}^{m,n}(t) & \text{ (m odd)}
\end{cases}
\]

**Proof.** Since \( U(n) = (SU(n) \times S^1)/(\mathbb{Z}/n) \), it follows from Lemma 4.2 that

\[
H^*(\text{Hom}(\mathbb{Z}^m, U(n))_1; \mathbb{F}) \cong H^*(\text{Hom}(\mathbb{Z}^m, SU(n))_1; \mathbb{F}) \otimes H^*((S^1)^m; \mathbb{F})
\]

where \( \text{Hom}(\mathbb{Z}^m, S^1)_1 = (S^1)^m \). Then by Theorem 4.1 the proof is complete. \( \square \)

**Example 4.4.** We give example calculations of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, SU(n))_1 \) by using Theorem 4.3.

\[
P(\text{Hom}(\mathbb{Z}^2, SU(2))_1; t) = 1 + t^2 + 2t^3
\]
\[
P(\text{Hom}(\mathbb{Z}^2, SU(3))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + t^6 + 2t^7 + 3t^8
\]
\[
P(\text{Hom}(\mathbb{Z}^2, SU(4))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 6t^8 + 6t^9 + 8t^{10}
+ 6t^{11} + 7t^{12} + 2t^{13} + 3t^{14} + 4t^{15}
\]
\[
P(\text{Hom}(\mathbb{Z}^2, SU(5))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 10t^8 + 14t^9
+ 13t^{10} + 16t^{11} + 22t^{12} + 18t^{13} + 21t^{14} + 20t^{15} + 22t^{16}
+ 18t^{17} + 14t^{18} + 14t^{19} + 10t^{20} + 10t^{21} + 3t^{22} + 4t^{23} + 5t^{24}
\]
The Poincaré series of symplectic and special orthogonal groups. For a nonzero integer implying that the homology dimension of \( \text{Hom}(\mathbb{Z}, \text{Symplectic and special orthogonal groups}) \).

4.2. The homology dimension of \( \text{Hom}(\mathbb{Z}, \text{Symplectic and special orthogonal groups}) \) is three. This occurs because we are taking the ground field of characteristic zero or greater than two for the Poincaré series.

The first observation will be proved to be true for every \( n \) in Section 5, and the second will be justified by homological stability proved in Section 7.

Remark 4.5. The homology dimension of \( \text{Hom}(\mathbb{Z}, G)_1 \) can be greater than the degree of its Poincaré series. For example, as is observed in [11], there is a stable homotopy decomposition

\[
\text{Hom}(\mathbb{Z}, SU(2))_1 \simeq S^2 \vee S^3 \vee S^3 \vee S^2 \mathbb{RP}^2,
\]

implying that the homology dimension of \( \text{Hom}(\mathbb{Z}, SU(2))_1 \) is four. But the degree of its Poincaré series is three. This occurs because we are taking the ground field of characteristic zero or greater than two for the Poincaré series.

4.2. Symplectic and special orthogonal groups. We refine the formula for symplectic and special orthogonal groups. For a nonzero integer \( k \), let

\[
q_k(t) = (-1)^{m(k+1)} \text{sgn}(k)^{m+1} t^{(m-2)|k|} + \frac{(1 + (-1)^{k+1} \text{sgn}(k)t^{|k|})^m}{1 - \text{sgn}(k)t^{|k|}}
\]

and for \( \lambda = (\lambda_1, \ldots, \lambda_t) \vdash \pm k \) with \( k \leq n \), let

\[
q_{\lambda_1}^{m,n} = t^{(m-2)(n-|k|)} q_{\lambda_1}^{m}(t) \cdots q_{\lambda_t}^{m}(t).
\]

We set \( q_{\lambda_1}^{m,n} = t^{(m-2)n} \) for the empty partition \( \lambda \) as in the case of \( U(n) \).

Theorem 4.6. The Poincaré series of \( \text{Hom}(\mathbb{Z}^m, Sp(n))_1 \) is given by

\[
P(\text{Hom}(\mathbb{Z}^m, Sp(n))_1; t) = \begin{cases} 
\prod_{i=1}^{n} (1 - t^{4i}) \sum_{\lambda_1^i \vdash n} \frac{1}{2^{(\lambda_1^i)^+} \theta(\lambda_1^i)} q_{\lambda_1}^{m,n}(t) & \text{ (m even)} \\
\prod_{i=1}^{n} (1 - t^{4i}) \sum_{k=0}^{ \lambda_1^i \vdash n \lambda_1^i \vdash n} \frac{(-1)^{n-k}}{2^{(\lambda_1^i)^+} \theta(\lambda_1^i)} q_{\lambda_1}^{m,n}(t) & \text{ (m odd)}.
\end{cases}
\]
Proof. The Weyl group of $Sp(n)$ is $B_n$ acting canonically on $\mathbb{F}^n$. Then we apply the formula of Theorem 2.2 to the canonical representation of $B_n$. Then since $\det(1 + tc) = 1 + (-1)^{|c|+1} \sgn(c)t^{|c|}$ for a signed cyclic permutation $c \in B_n$, for $w \in B_n$ with the standard cycle decomposition $w = c_1 \cdots c_k$, we have

$$
\frac{\det(1 + tw)^m}{\det(1 - t^2 w)} = \prod_{i=1}^k ((-1)^{m(|c_i|)+1} \sgn(c_i)^{m+1} t^{m-2|c_i|} + q_{\sgn(c_i)|c_i|}^m(t))
$$

$$
= \sum_{k=0}^n \sum_{\ell(n-k) + 1} (-1)^{m(n-k)+1} \sgn(\lambda, c(w))^{m+1} \left| \text{Emb}(\lambda, c(w)) \right| q_\lambda^m(t).
$$

It is obvious that

$$
\sum_{s \in (\mathbb{Z}/2)^n} \sum_{\ell(n-k) + 1} \sgn(\lambda, c(sw)) \left| \text{Emb}(\lambda, c(sw)) \right| = 0
$$

unless $i = 0$. Then it follows from Lemma 3.8 that for $m$ even

$$
\sum_{w \in B_n} \frac{\det(1 + tw)^m}{\det(1 - t^2 w)} = \sum_{\lambda, \pm n} \frac{2n-\ell(\lambda)}{\theta(\lambda)} q_\lambda^m(t).
$$

For $m$ odd, by Lemmas 3.4 and 3.8,

$$
\sum_{w \in B_n} \frac{\det(1 + tw)^m}{\det(1 - t^2 w)} = \sum_{k=0}^n \sum_{\ell(n-k) + 1} \frac{(-1)^{n-k} 2^{n-\ell(\lambda)} n!}{\theta(\lambda^+)} \left[ {n-k \atop i} \right] q_\lambda^m(t)
$$

$$
= \sum_{k=0}^n \sum_{\lambda, \pm k} \frac{(-1)^{n-k} 2^{n-\ell(\lambda)} n!}{\theta(\lambda^+)} q_\lambda^m(t).
$$

Thus by Theorem 2.2 the proof is complete. □

**Theorem 4.7.** Let $\mathbb{F}$ be a field of characteristic zero or prime to $2(n!)$. Then

$$
H^*(\text{Hom}(\mathbb{Z}^m, Sp(n)); \mathbb{F}) \cong H^*(\text{Hom}(\mathbb{Z}^m, SO(2n+1)); \mathbb{F}).
$$

**Proof.** The Weyl groups of $Sp(n)$ and $SO(2n + 1)$ are isomorphic as reflection groups over a field of characteristic $> 2$. Then by Theorem 2.1, we obtain the isomorphism in the statement. □

The following corollary is immediate from Theorem 4.7

**Corollary 4.8.** $P(\text{Hom}(\mathbb{Z}^m, SO(2n+1)); t) = P(\text{Hom}(\mathbb{Z}^m, Sp(n)); t)$.

Then by Theorem 4.6, we obtain a formula for the Poincaré series of $\text{Hom}(\mathbb{Z}^m, SO(2n+1))$. 
Theorem 4.9. The Poincaré series of $\text{Hom}(\mathbb{Z}^m, SO(2n))$, is given by

$$(1 - t^{2n}) \prod_{i=1}^{n-1} (1 - t^{4i}) \left( \sum_{\lambda \vdash \pm(n-1)} \frac{-\text{sgn}(\lambda)}{2^{(\lambda^+)/\theta(\lambda^+)}q_{\lambda}^{m,n} + \sum_{\lambda \vdash n} \frac{1}{2^{(\lambda^+)/\theta(\lambda^+)}q_{\lambda}^{m,n}} \right)$$

for $m$ even and

$$(1 - t^{2n}) \prod_{i=1}^{n-1} (1 - t^{4i}) \sum_{k=0}^{n} \sum_{\lambda \vdash \pm k} \frac{(-1)^{n-k}}{2^{(\lambda^+)/\theta(\lambda^+)}q_{\lambda}^{m,n}}$$

for $m$ odd.

Proof. The Weyl group of $SO(2n)$ is $B_n^+$ with the canonical representation. Then we apply the formula of Theorem 2.2 to the canonical representation of $B_n^+$. As in the case of $Sp(n)$, we can show the identity

$$\frac{\det(1 + tw)^m}{\det(1 - t^{2w})} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{\ell(c(w)^+ - \ell(\lambda^+)) = i} (-1)^{m(n-k)+(m+1)i} \text{sgn}(\lambda, c(w))^{m+1} |\text{Emb}(\lambda, c(w))| q_{\lambda}^{m,n}(t)$$

for $w \in B_n^+$. By Lemma 3.8,

$$\sum_{w \in B_n^+} |\text{Emb}(\lambda, c(w))| = \frac{2^{n-\ell(\lambda^+)-1}n!}{\theta(\lambda^+)(n-k)!} \binom{n-k}{i}.$$ 

Since $\text{sgn}(c(w)) = \text{sgn}(w) = 1$ for $w \in B_n^+$, $\text{sgn}(\lambda, c(w)) = \text{sgn}(\lambda)$. Then it follows that

$$\sum_{w \in B_n^+} \frac{\det(1 + tw)^m}{\det(1 - t^{2w})} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{\ell(c(w)^+ - \ell(\lambda^+)) = i} (-1)^{m(n-k)+(m+1)i} \text{sgn}(\lambda)^{m+1} \theta(\lambda^+)(n-k)! \binom{n-k}{i} q_{\lambda}^{m,n}.$$ 

By Lemma 3.4, $\sum_{i=1}^{n-k} \binom{n-k}{i} = 0$ for $n - k \geq 2$. Hence for $m$ even,

$$\sum_{w \in B_n^+} \frac{\det(1 + tw)^m}{\det(1 - t^{2w})} = -\sum_{\lambda \vdash n-1} \frac{\text{sgn}(\lambda)2^{n-\ell(\lambda^+)-1}n!}{\theta(\lambda^+)} q_{\lambda}^{m,n} + \sum_{\lambda \vdash n} \frac{2^{n-\ell(\lambda^+)-1}n!}{\theta(\lambda^+)} q_{\lambda}^{m,n}.$$ 

For $m$ odd, by Lemmas 3.4 and 3.8,

$$\sum_{w \in B_n^+} \frac{\det(1 + tw)^m}{\det(1 - t^{2w})} = \sum_{k=0}^{n} \sum_{\lambda \vdash \pm k} \frac{(-1)^{n-k}2^{n-\ell(\lambda^+)-1}n!}{\theta(\lambda^+)} q_{\lambda}^{m,n}.$$
Thus by Theorem 2.2 the proof is complete.

5. Top terms

In this section, we determine the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ for the classical group $G$ by applying the formulae obtained in the previous section. We also determine the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^2, G)_1$ for any compact connected Lie group $G$ by combining the results for the classical group and a computer calculation for the exceptional groups. This has an application to the rational homotopy of $\text{Hom}(\mathbb{Z}^m, G)_1$.

5.1. Classical groups. We determine the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ for the classical group $G$. Define

$$c(m, G) = \begin{cases} m + n - 2 & (G = U(n), SU(n)) \\ m + n - 1 & (G = Sp(n), SO(2n + 1), SO(2n)) \end{cases}$$

and

$$d(m, G) = \begin{cases} n^2 + (m - 1)n & (G = U(n)) \\ n^2 + (m - 1)(n - 1) - 1 & (G = SU(n)) \\ 2n^2 + mn & (G = Sp(n), SO(2n + 1)) \\ 2n^2 + (m - 2)n & (G = SO(2n)) \end{cases}$$

We also set $\epsilon(U(n)) = 1$ and $\epsilon(G) = 0$ for other classical groups $G$.

**Theorem 5.1** (Theorem B). Let $G$ be the classical group, where we assume $n > 2$ for $G = SO(2n)$. The top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ is

$$\left\{ \begin{array}{ll} t^{d(m, G)} & (m \text{ odd}) \\ \frac{t^{c(m, G)}}{m-1} t^{d(m-1, G) + \epsilon(G)} & (m \text{ even}) \end{array} \right.$$  

**Proof.** We define the degree of a rational function $\frac{P(t)}{Q(t)}$ by $\deg P(t) - \deg Q(t)$, where $P(t), Q(t)$ are polynomials.

**Unitary groups.** Suppose $m$ is odd. Since $q^{m,n}_\lambda$ is of degree $(m - 2)n - k$ for $\lambda \vdash k$, it follows from Theorem 4.1 that the Poincaré series of $\text{Hom}(\mathbb{Z}^m, U(n))_1$ is of degree at most $n^2 + (m - 1)n$ such that the degree $n^2 + (m - 1)n$ term is included in $\frac{1}{\sigma(\lambda)} q^{m,n}_\lambda(t)$ for the empty partition $\lambda$. Then its coefficient is 1, and so the top term of the Poincaré series is $t^{n^2 + (m - 1)n}$.

Suppose $m$ is even. Then by Theorem 4.1 the Poincaré series is of degree at most $n^2 + (m - 2)n + 1$. Going back to the calculation of the Poincaré series, one can see that the coefficient of the degree $n^2 + (m - 2)n + 1$ term is

$$\frac{1}{n!} \sum_{\lambda \vdash n-1} m^{(\lambda)} \sum_{w \in \Sigma_n} |\text{Emb}(\lambda, c(w))| = \frac{1}{n!} \sum_{i=1}^{n-1} m^i \sum_{\ell(\lambda) = i} \sum_{w \in \Sigma_n} |\text{Emb}(\lambda, c(w))|.$$
By Lemma 3.3 for $k = 1$, 
\[
\sum_{\lambda \vdash n - 1, \ell(\lambda) = i} \sum_{w \in S_n} |\text{Emb}(\lambda, c(w))| = n \binom{n - 1}{i}
\]
and by Lemma 3.4, 
\[
\sum_{i=1}^{n-1} m^i \binom{n-1}{i} = \frac{(m+n-2)!}{(m-1)!}.
\]
Thus the coefficient of the degree $n^2 + (m - 2)n + 1$ term is $\binom{m+n-2}{m-1}$.

**Special unitary groups.** This follows from (4) and the case of $U(n)$ above.

**Symplectic groups.** For $m$ odd, it is immediate from Theorem 4.6 that the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, \text{Sp}(n))_1$ is $t^{2n^2 + mn}$. Suppose that $m$ is even. Then by Theorem 4.6, the Poincaré series is of degree $\leq 2n^2 + (m - 1)n$.

Going back to the calculation of the Poincaré series, we can see that the coefficient of the degree $2n^2 + (m - 1)n$ term is
\[
\frac{1}{2^{m!} n!} \sum_{\lambda \vdash n, \ell(\lambda) = n} m^{\ell(\lambda)} \sum_{w \in B_n} |\text{Emb}(\lambda, c(w))| = \frac{1}{2^{m!} n!} \sum_{i=1}^{n} \sum_{\lambda \vdash n, \ell(\lambda) = i} m^i \sum_{w \in B_n} |\text{Emb}(\lambda, c(w))|.
\]

By Lemma 3.8 (1) for $k = 0$, 
\[
\sum_{\lambda \vdash n, \ell(\lambda) = i} \sum_{w \in B_n} |\text{Emb}(\lambda, c(w))| = 2^n \binom{n}{i}.
\]
Thus by Lemma 3.4, the coefficient is $\binom{m+n-1}{m-1}$.

**Special orthogonal groups.** By Theorem 4.7, the $\text{SO}(2n + 1)$ case follows from the $\text{Sp}(n)$ case above. Then we consider the $\text{SO}(2n)$ case. For $m$ odd, it is immediate to see that the top term is $t^{2n^2 + (m-2)n}$. For a nonzero integer $k$, $q_k(t) - q_{-k}(t)$ is of degree $(m-4)k$. Then $\sum_{\lambda \vdash n \text{ even}, \ell(\lambda) = i} \text{sgn}(\lambda) q_{\lambda}^{m,n}$ is of degree at most $(m-4)n + 2$. On the other hand, since $q_{\lambda}^{m,n}(t)$ is of degree $(m-3)k$ for $\lambda \vdash k$, $\sum_{\lambda \vdash n} q_{\lambda}^{m,n}$ is of degree at most $(m-3)n$. Then the top term is included in
\[
(1 - t^{2n}) \prod_{i=1}^{n-1} (1 - t^{4i}) \sum_{\lambda \vdash n, \ell(\lambda) = i} \frac{1}{2^{\ell(\lambda)} \theta(\lambda)} q_{\lambda}^{m,n}
\]
because we are assuming $n > 2$. Thus by the same calculation as in the $\text{Sp}(n)$ case above, one gets that the top term is $(\frac{m+n-1}{m-1})t^{2n^2 + (m-3)n}$. 

**Remark 5.2.** The top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, \text{SO}(2n))_1$ for $n = 1, 2$ is easily obtained as follows. Since $\text{Hom}(\mathbb{Z}^m, \text{SO}(2))_1 = (S^1)^m$, its Poincaré series is $(1 + t)^m$. Then its top term is $t^m$. Since $\text{SO}(4) = (\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}/2)$, $P(\text{Hom}(\mathbb{Z}^m, \text{SO}(4))_1; t) = P(\text{Hom}(\mathbb{Z}^m, \text{SU}(2))_1; t)^2$ by Lemma 4.2. Then the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^m, \text{SO}(4))_1$ is $m^2 t^{2m+2}$. 

**Proof.**
Observe that Theorem 5.1 implies that the topology of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) depends on the parity of \( m \). We give an alternative expression for this dependence. Antolín, Gritschacher and Villarreal (see [24, Remark 6.5]) observed that the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is palindromic for \( m \) odd. On the other hand, it follows from Theorems 5.1 and Remark 5.2 that when \( G \) is the classical group which is neither trivial nor \( S^1 \), the coefficient of the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is greater than one for \( m \) even. Thus we obtain the following.

**Corollary 5.3** (Corollary C). Let \( G \) be the classical group which is neither trivial nor \( S^1 \). Then the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is palindromic if and only if \( m \) is odd.

5.2. **General Lie groups.** We further determine the top term of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) for any compact connected Lie group \( G \). We first observe the case of simple Lie groups, and then consider the general case.

**Corollary 5.4.** Let \( G \) be a compact connected simple Lie group. Then the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) is \((\text{rank } G + 1)t^{\text{dim } G}\).

**Proof.** Combine Theorem 5.1 and Appendix. \( \Box \)

**Theorem 5.5** (Theorem D). Let \( G \) be a compact connected Lie group with simple factors \( G_1, \ldots, G_k \). Then the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) is

\[
(\text{rank } G_1 + 1) \cdots (\text{rank } G_k + 1)t^{\text{dim } G + \text{rank } \pi_1(G)}.
\]

**Proof.** There are compact simply-connected simple Lie groups \( G_1, \ldots, G_k \) (simple factors in the statement), a torus \( T \), and a discrete subgroup \( K \) of the center of \( G_1 \times \cdots \times G_k \times T \) such that

\[
G \cong (G_1 \times \cdots \times G_k \times T)/K.
\]

Then by Lemma 4.2,

\[
P(\text{Hom}(\mathbb{Z}^2, G)_1; t) = \prod_{i=1}^{k} P(\text{Hom}(\mathbb{Z}^2, G_i)_1; t).
\]

Since \( \text{Hom}(\mathbb{Z}^2, T)_1 = T^2 \), the top term of \( P(\text{Hom}(\mathbb{Z}^2, T)_1; t) \) is \( t^{2\text{dim } T} \). Then by Corollary 5.4, the top term of \( P(\text{Hom}(\mathbb{Z}^2, G)_1; t) \) is

\[
(\text{rank } G_1 + 1) \cdots (\text{rank } G_k + 1)t^{\text{dim } G_1 + \cdots + \text{dim } G_k + 2\text{dim } T}.
\]

Since \( \text{dim } G = \text{dim } G_1 + \cdots + \text{dim } G_k + \text{dim } T \) and \( \text{rank } \pi_1(G) = \text{dim } T \), the proof is complete. \( \Box \)

We consider an application of Theorem 5.5 to rational homotopy. Let \( X \) be a simply connected finite complex. Then by dichotomy in rational homotopy theory [12], \( \sum_{n \geq 0} \pi_n(X) \otimes \mathbb{Q} \) is either finite or of exponential growth. In the former case, \( X \) is called rationally elliptic, and the latter, rationally hyperbolic.
It is shown by Gómez, Pettet, and Souto [16] that there is an isomorphism
\[ \pi_1(\text{Hom}(\mathbb{Z}^m, G)_1) \cong \pi_1(G)^m. \]

In particular, if \( G \) is simply-connected, then so is \( \text{Hom}(\mathbb{Z}^m, G)_1 \). On the other hand, since \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is a real algebraic variety, it is a finite complex. Then \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is either rationally elliptic or hyperbolic when \( G \) is simply-connected. For \( m = 1 \), \( \text{Hom}(\mathbb{Z}, G)_1 = G \) which is rationally elliptic. For \( m \geq 2 \), one has:

**Corollary 5.6 (Corollary E).** If \( G \) is a non-trivial compact simply-connected Lie group and \( m \geq 2 \), then \( \text{Hom}(\mathbb{Z}^m, G)_1 \) is rationally hyperbolic.

**Proof.** Since \( G \) is non-trivial and \( m \geq 2 \), the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) has coefficient greater than one by Theorem 5.5. On the other hand, the rational cohomology of a rationally elliptic space satisfies the Poincaré duality. In particular, the top term of the Poincaré series has coefficient one. Thus \( \text{Hom}(\mathbb{Z}^2, G)_1 \) must be rationally hyperbolic. By definition, if \( X \) is a retract of \( Y \) and is rationally hyperbolic, then \( Y \) is rationally hyperbolic too. Since \( \text{Hom}(\mathbb{Z}^2, G)_1 \) is a retract of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) for \( m \geq 2 \), the proof is complete. \( \square \)

### 6. Cohomology generators

In this section we give a minimal generating set of the cohomology of \( \text{Hom}(\mathbb{Z}^m, G)_1 \) for the classical group \( G \), except for \( \text{SO}(2n) \). We set notation that are used throughout this section. Let \( G \) be a compact connected Lie group of rank \( n \), and let \( W \) denote its Weyl group. We set

\[ \tilde{H}(m, G) := \mathcal{P}(n) \otimes \bigotimes_{i=1}^m \Lambda(y_1^i, \ldots, y_n^i) \quad \text{and} \quad H(m, G) := \mathcal{P}(n)_W \otimes \bigotimes_{i=1}^m \Lambda(y_1^i, \ldots, y_n^i) \]

where \( \mathcal{P}(n) \) is as in Section 2. By Theorem 2.1,

\[ H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \cong \tilde{H}(m, G)_W \]

where the ground field \( \mathbb{F} \) is of characteristic zero or prime to \( |W| \). Then we understand the results on \( H^*(m, G)_W \) below as those on \( H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \).

#### 6.1. Generators for unitary groups.

We assume that the ground field \( \mathbb{F} \) is of characteristic prime to \( n! \). We give generators of \( \tilde{H}(m, U(n))^{\Sigma_n} \). Let \( e_i \) denote the \( i \)-th symmetric polynomial in \( \mathcal{P}(n) \) for \( 1 \leq i \leq n \). Then

\[ \mathcal{P}(n)^{\Sigma_n} = \mathbb{F}[e_1, \ldots, e_n]. \]

Let \( p_i \) denote the \( i \)-th power sum in \( \mathcal{P}(n) \) for \( i \geq 1 \), i.e. \( p_i = x_1^i + \cdots + x_n^i \). By the Newton formula

\[ ke_k = \sum_{i=0}^{k} (-1)^{i-1} e_{k-i} p_i \quad (5) \]
where we set \( e_0 = p_0 = 1 \). Then since the ground field \( \mathbb{F} \) is of characteristic zero or prime to \( n \), \( \mathcal{P}(n)^{\Sigma_n} \) is alternatively expressed as

\[
\mathcal{P}(n)^{\Sigma_n} = \mathbb{F}[p_1, \ldots, p_n].
\]

This simple observation is the core idea to construct generators of \( \mathcal{H}(m, U(n))^\Sigma_n \).

Let \( q_i := x_1^i y_1 + \cdots + x_n^i y_n \in \mathcal{P}(n) \otimes \Lambda(y_1, \ldots, y_n) \) for \( i \geq 0 \). We observe a property of these element. For non-negative integers \( k_1, \ldots, k_n \), let

\[
q_{k_1, \ldots, k_n} := \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) x_{\sigma(1)}^{k_1} \cdots x_{\sigma(n)}^{k_n}.
\]

Then \( q_{k_1, \ldots, k_n} \) is anti-symmetric. Note that \( q_{k_1, \ldots, k_n} \) can be trivial for some \( k_1, \ldots, k_n \). For instance,

\[
q_{1,0,0} = \operatorname{sgn}(123)x_1 + \operatorname{sgn}(213)x_2 + \operatorname{sgn}(321)x_3
\]
\[
+ \operatorname{sgn}(132)x_1 + \operatorname{sgn}(231)x_2 + \operatorname{sgn}(312)x_3
\]
\[
= 0
\]

where \( n = 3 \). Clearly, each anti-symmetric polynomial in \( \mathcal{P}(n) \) is a linear combination of \( q_{k_1, \ldots, k_n} \). By a straightforward calculation, we can prove the following.

**Lemma 6.1.** For non-negative integers \( k_1, \ldots, k_n \),

\[
q_{k_1} \cdots q_{k_n} = q_{k_1, \ldots, k_n} y_1 \cdots y_n.
\]

In particular, every anti-symmetric polynomial in \( \mathcal{P}(n) \) is generated by \( q_i \).

Let \( [m] := \{1, 2, \ldots, m\} \). We consider an ordering on the power set \( 2^{[m]} \) which sorts by cardinality first and by the lexicographic ordering with \( 1 < 2 < \cdots < m \) next.

**Example 6.2.** The power set \( 2^{[3]} \) is ordered as

\[
\emptyset < \{1\} < \{2\} < \{3\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}.
\]

We give a basis of \( \mathcal{H}(m, U(n))^\Sigma_n \). For \( I = \{i_1 < \cdots < i_k\} \subset [m] \), let

\[
y^I_j := y_{i_1}^{j_1} \cdots y_{i_k}^{j_k}
\]

and for \( d_1, \ldots, d_{2^m-1} \geq 0 \), let

\[
z(d_1, \ldots, d_{2^m-1}) := \prod_{k=1}^{2^m-1} \prod_{i=d_1+\cdots+d_{k-1}+1}^{d_1+\cdots+d_k} y^I_k
\]

where \( S_k \) is in \( 2^{[m]} = \{\emptyset \cup S_1 \cup \cdots \cup S_{2^m-1}\} \) by the above ordering.

**Example 6.3.** For \( m = 3 \), \( z(d_1, \ldots, d_7) \) of degree 2 are:

\[
z(0, 1, 0, 0, 0, 0, 0) = y_1^2 y_2^3 \quad z(1, 0, 1, 0, 0, 0, 0) = y_1^1 y_2^3 \quad z(1, 1, 0, 0, 0, 0, 0) = y_1^1 y_2^2
\]
\[
z(0, 0, 0, 0, 0, 1, 0) = y_1^2 y_3^3 \quad z(0, 0, 0, 0, 1, 0, 0) = y_1^3 y_2^3 \quad z(0, 0, 0, 1, 0, 0, 0) = y_1^3 y_3^2
\]
\[
z(0, 0, 2, 0, 0, 0, 0) = y_1^2 y_2^3 \quad z(0, 2, 0, 0, 0, 0, 0) = y_1^2 y_3^2 \quad z(2, 0, 0, 0, 0, 0, 0) = y_1^2 y_3^1
\]
Let $M(m, n)$ be a subset of $\widetilde{\mathcal{H}}(m, U(n))$ consisting of monic monomials of the form

$$M(x_1, \ldots, x_n)z(d_1, \ldots, d_{2^m-1}).$$

Clearly, for every monic monomial $w$ of $\widetilde{\mathcal{H}}(m, U(n))$, there is a unique $v \in M(m, n)$ such that

$$w = \sigma(v)$$

for some $\sigma \in \Sigma_n$. Since the ground field $F$ is of characteristic zero or prime to $n!$, we get the following.

**Lemma 6.4.** The set

$$\left\{ \sum_{\sigma \in \Sigma_n} \sigma(v) \left| v \in M(m, n) \right. \right\}$$

is a basis of $\widetilde{\mathcal{H}}(m, U(n))^{\Sigma_n}$.

Now we are ready to give a basis of $\mathcal{H}(m, U(n))^{\Sigma_n}$.

**Lemma 6.5.** $\mathcal{H}(m, U(n))^{\Sigma_n}$ is spanned by elements represented by

$$\sum_{\sigma \in \Sigma_n} \sigma(M(x_1, \ldots, x_n)z(d_1, \ldots, d_{2^m-1})) \in \mathcal{H}(m, U(n))^{\Sigma_n}$$

where $M(x_1, \ldots, x_n)$ are monomials, satisfying the following conditions:

1. $M(x_1, \ldots, x_n) = M(x_1, \ldots, x_r, 0, \ldots, 0)$ for $r = d_1 + \cdots + d_{2^m-1}$;
2. if the above representative includes the term
   $$P(x_{d_1+\cdots+d_{i-1}+1}, \ldots, x_{d_1+\cdots+d_{i-1}+d_i})Q$$
   where $Q$ does not include $x_{d_1+\cdots+d_{i-1}+1}, \ldots, x_{d_1+\cdots+d_{i-1}+d_i}$, then $P$ is symmetric for $|S_i|$ even and anti-symmetric for $|S_i|$ odd.

**Proof.** (1) As in the proof of Theorem 2.2, the natural map

$$\mathcal{H}(m, U(n))^{\Sigma_n} \rightarrow \mathcal{H}(m, U(n))^{\Sigma_n}$$

is surjective. Then by Lemma 6.4, each element of $\mathcal{H}(m, U(n))^{\Sigma_n}$ is a linear combination of elements of $\widetilde{\mathcal{H}}(m, U(n))^{\Sigma_n}$ represented by elements of Lemma 6.4. By definition, $M(x_1, \ldots, x_n)$ is invariant under permutations on $\{r + 1, r + 2, \ldots, n\}$. Then by (6)

$$M(x_1, \ldots, x_n) = \sum_{i \geq 0} M'_i(x_1, \ldots, x_r)p_i(x_{r+1}, \ldots, x_n)$$

where $M'_i$ are monomials and $p_i$ denotes the $i$-th power sum as above. Since

$$p_i(x_{r+1}, \ldots, x_n) = -p_i(x_1, \ldots, x_r) \text{ in } P(n)^{\Sigma_n}$$

by (6), we can choose the monomials $M(x_1, \ldots, x_n)$ as stated.

(2) By definition, the representative in the statement is invariant under permutations on $\{d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_{i-1} + 2 \ldots, d_1 + \cdots + d_{i-1} + d_i\}$, so that
$z(d_1, \ldots, d_{2^{m-1}})$ is invariant for $|S_i|$ even and anti-invariant for $|S_i|$ odd. Then the second statement follows.

We define an ordering on $M(m, n)$. Note that those elements are bigraded by the degrees in the polynomial and the exterior algebra part.

(1) We sort by total degree.
(2) After (1), we sort by degree with respect to the exterior algebra part.
(3) After (2), $M(x_1, \ldots, x_n)z(d_1, \ldots, d_{2^{m-1}}) > M'(x_1, \ldots, x_n)z(d_1', \ldots, d_{2^{m-1}}')$ whenever $d_1 + \cdots + d_{2^{m-1}} < d_1' + \cdots + d_{2^{m-1}}'$.
(4) After (3), $M(x_1, \ldots, x_n)z(d_1, \ldots, d_{2^{m-1}}) > M'(x_1, \ldots, x_n)z(d_1', \ldots, d_{2^{m-1}}')$ whenever $d_i = d_i'$ for $i < k$ and $d_k > d_k'$ for some $k$.
(5) Finally, we sort by the lexicographic ordering on $P(n)$ with $x_1 < \cdots < x_n$.

Given any $w \in \mathcal{H}(m, U(n))^\Sigma$, it follows from Lemma 6.4 that

$$w = \sum_{v \in M(m, n)} a_v \sum_{\sigma \in \Sigma_n} \sigma(v)$$

where $a_v \in \mathbb{F}$. We define the least term of $w$ to be the least $v$ such that $a_v \neq 0$, which is well defined by the observation on (7).

Example 6.6. By Examples 6.2 and 6.3, elements of $M(3, 2)$ of degree 2 are ordered as

$$y_1^3 y_2^3 < y_1^2 y_2 < y_1 y_2^2 < y_1^2 y_2 < y_1 y_2 y_3^2 < y_1^2 y_2 < y_1 y_2 y_3 < y_1 y_2 < y_1 x_1 < x_1 < x_2.$$ 

By analogy to the power sums in $P(n)$, we define for $\emptyset \neq I \subset [m]$,

$$z(d, I) := x_1^{d-1} y_1^I + \cdots + x_n^{d-1} y_n^I.$$ 

Clearly, $z(d, I) \in \mathcal{H}(m, U(n))^\Sigma$. Clearly, if $k > n$, then for any $d_1, \ldots, d_k \geq 1$ and $\emptyset \neq I_1, \ldots, I_k \subset [m]$,

$$z(d_1, I_1) \cdots z(d_k, I_k) = 0.$$

Lemma 6.7. The least term of $z(d_1, I_1) \cdots z(d_k, I_k)$ for $I_1 \leq \cdots \leq I_k$ and $k \leq n$ is

$$x_1^{d_1-1} \cdots x_k^{d_k-1} y_{i_1}^{I_1} \cdots y_{i_k}^{I_k}.$$ 

Proof. The exterior algebra part of any monomial in $z := z(d_1, I_1) \cdots z(d_k, I_k)$ is

$$y_{i_1}^{I_1} \cdots y_{i_k}^{I_k}$$

where $i_1 < \cdots < i_k$ and $J_i = I_{j_1} \cup \cdots \cup I_{j_k}$ for disjoint $I_{j_1}, \ldots, I_{j_k}$ such that distinct $J_i$ include no common $I_j$. Then by the condition (3) in the definition of the ordering on $M$, the exterior algebra part of the least term is $y_{i_1}^{I_1} \cdots y_{i_k}^{I_k}$. Thus the least term of $z(d_1, I_1) \cdots z(d_k, I_k)$ is $x_1^{d_1-1} \cdots x_k^{d_k-1} y_{i_1}^{I_1} \cdots y_{i_k}^{I_k}$, as stated. □
We set 
\[ \tilde{\mathcal{S}}(m, U(n)) := \{ z(d, I) \mid d \geq 1 \text{ and } \emptyset \neq I \subset [m] \}. \]
If \( A \) is a subset of \( \tilde{\mathcal{H}}(m, U(n)) \), then we will use the same symbol \( A \) for the image of \( A \) under the projection \( \tilde{\mathcal{H}}(m, U(n)) \to \mathcal{H}(m, U(n)) \).

**Theorem 6.8.** \( \mathcal{H}(m, U(n))^{\Sigma_n} \) is generated by \( \tilde{\mathcal{S}}(m, U(n)) \).

**Proof.** Let \( R \) denote the subring of \( \mathcal{H}(m, U(n)) \) generated by \( \tilde{\mathcal{S}}(m, U(n)) \). Take any element \( w \in \mathcal{H}(m, U(n))^{\Sigma_n} \) represented by an element \( \tilde{w} \in \tilde{\mathcal{H}}(m, U(n))^{\Sigma_n} \) in Lemma 6.5. By (6) and Lemmas 6.1 and 6.7, the least term of \( \tilde{w} \) is included in some element \( v \in R \). Since \( v \) and \( \tilde{w} \) are \( \Sigma_n \)-invariant, their least terms coincide, and so by eliminating the least terms inductively, we obtain \( w \in R \). Thus the statement holds. \( \square \)

### 6.2. Low dimensional cohomology for unitary groups

We continue to assume that the ground field \( \mathbb{F} \) is of characteristic zero or prime to \( n! \). We determine \( H^*(\text{Hom}(\mathbb{Z}^m, U(n))_1; \mathbb{F}) \cong \mathcal{H}(m, U(n))^{\Sigma_n} \) in low dimensions, and show that a certain subset of \( \tilde{\mathcal{S}}(m, U(n)) \) must be contained in every subset of \( \tilde{\mathcal{S}}(m, U(n)) \) that generates \( \mathcal{H}(m, U(n))^{\Sigma_n} \).

**Lemma 6.9.** For any \( \sigma \in \Sigma_n \),

\[ \mathcal{B}(\sigma) := \{ x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \cdots x_{\sigma(n-1)}^{i_{n-1}} \in \mathcal{P}(n)^{\Sigma_n} \mid i_k \leq n - k \text{ for } 1 \leq k \leq n - 1 \} \]

is a basis of \( \mathcal{P}(n)^{\Sigma_n} \).

**Proof.** As in [13, Section 10, Proposition 3], \( \mathcal{B}(1) \) is a basis of \( \mathcal{P}(n)^{\Sigma_n} \). Then the statement holds by symmetry of \( \mathcal{P}(n)^{\Sigma_n} \). \( \square \)

Let \( \mathcal{E}(m, n) \) denote the set of all monic monomials of \( \bigotimes_{i=1}^n \Lambda(y_1, y_2, \ldots, y_n) \). Then for any map \( \alpha: \mathcal{E}(m, n) \to \Sigma_n \), the set

\[ \mathcal{B}(\alpha) := \{ XY \mid X \in \mathcal{B}(\alpha(Y)), Y \in \mathcal{E}(m, n) \} \]

is a basis of \( \mathcal{H}(m, U(n)) \).

Let \( \mathbb{F}(S) \) denote the free commutative graded algebra over \( \mathbb{F} \) generated by a graded set \( S \), and define a subspace of \( \mathbb{F}(\tilde{\mathcal{S}}(m, U(n))) \)

\[ V(m, n) := \{ z(d_1, I_1) \cdots z(d_k, I_k) \mid I_1 < \cdots < I_k \text{ and } k + \max\{d_1, \ldots, d_k\} - 1 \leq n \}. \]

**Lemma 6.10.** The projection

\[ \mathbb{F}(\tilde{\mathcal{S}}(m, U(n))) \to \mathcal{H}(m, U(n))^{\Sigma_n} \]

is injective on \( V(m, n) \)

**Proof.** By Lemma 6.7, the least term of \( z(d_1, I_1) \cdots z(d_k, I_k) \in V(m, n) \) for \( I_1 < \cdots < I_k \) is

\[ x_1^{d_1-1} \cdots x_k^{d_k-1} y_1^{I_1} \cdots y_k^{I_k}. \]
Note that \( d_i - 1 \leq \max\{d_1, \ldots, d_k\} - 1 \leq n - k \leq n - i \) for \( 1 \leq i \leq k \). Then the least terms of elements of \( V(m, n) \) are mutually distinct elements of \( \overline{E}(\alpha) \) for any map \( \alpha: E(m, n) \to \Sigma_n \) satisfying \( \alpha(y_1^{I_1} \cdots y_k^{I_k}) \in \Sigma_k \). Thus elements of \( V(m, n) \) are linearly independent in \( \mathcal{H}(m, U(n))^\Sigma_n \), completing the proof. \( \square \)

We set
\[
S(m, U(n)) := \{ z(d, I) \in \overline{S}(m, U(n)) \mid d + |I| - 1 \leq n \}.
\]

Now we are ready to describe \( \mathcal{H}(m, \Sigma_n)^\Sigma_n \) in low dimensions.

**Theorem 6.11 (Theorem G).** The map \( \mathcal{F}(S(m, U(n))) \to \mathcal{H}(m, U(n))^\Sigma_n \) is an isomorphism in dimension \( \leq 2n - m \).

**Proof.** By a degree reason, the inclusion \( \mathcal{F}(S(m, U(n))) \to \mathcal{F}(\overline{S}(m, U(n))) \) is an isomorphism in dimension \( \leq 2n - m \), and the image of this inclusion is contained in \( V(m, n) \) in dimension \( \leq 2n - m \). Then by Lemmas 6.8 and 6.10, the proof is complete. \( \square \)

We end this subsection by proving the following lemma.

**Lemma 6.12.** \( S(m, U(n)) \) is included in every subset of \( \overline{S}(m, U(n)) \) that generates \( \mathcal{H}(m, U(n))^\Sigma_n \).

**Proof.** Let \( F_i \) denote the subspace of \( \mathcal{H}(m, U(n))^\Sigma_n \) generated by \( z(d_1, I_1) \cdots z(d_k, I_k) \) such that \( d_1 + \cdots + d_k + |I_1| + \cdots + |I_k| - k \leq i \). Then we get a filtration
\[
\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset \mathcal{H}(m, U(n))^\Sigma_n.
\]
Note that this filtration is induced from that of \( \overline{\mathcal{H}}(m, U(n))^\Sigma_n \) defined by word-length with respect to \( x_i \) and \( y_i \). Clearly,
\[
F_i \cdot F_j \subset F_{i+j} \quad \text{and} \quad S(m, U(n)) \subset V(m, n) \cap F_n.
\]
Suppose that an element \( z(d, I) \) of \( S(m, U(n)) \) with \( d + |I| - 1 = i \) is expressed by the remaining elements of \( S(m, U(n)) \) Then \( z(d, I) \) belongs to \( F_{i-1} \cap V(m, n) \). This is a contradiction to Lemma 6.10, completing the proof. \( \square \)

6.3. Minimal generating set for unitary groups. We continue to assume that the ground field \( \mathbb{F} \) is of characteristic zero or prime to \( n! \). We give a minimal generating set of \( \mathcal{H}(m, U(n))^\Sigma_n \) contained in \( \overline{S}(m, U(n)) \). We begin with a simple observation. In \( \mathcal{H}(3, U(2))^\Sigma_2 \),
\[
z(1, \{1\})z(1, \{2\})z(1, \{3\})
\]
\[
= (y_1^1 + y_2^1)(y_2^2 + y_3^2)(y_1^3 + y_3^3)
\]
\[
= (y_1^1y_2^2 + y_2^1y_3^2)(y_3^3 + y_1^3) - (y_1^1y_1^3 + y_2^1y_2^3)(y_1^2 + y_2^2)
\]
\[
+ (y_2^2y_3^3 + y_3^2y_3^3)(y_1^1 + y_1^2) - 2(y_1^1y_2^3y_3^1 + y_2^1y_2^3y_3^2)
\]
\[
= z(1, \{1, 2\})z(1, \{3\}) - z(1, \{1, 3\})z(1, \{2\})
\]
\[
+ z(1, \{2, 3\})z(1, \{1\}) - 2z(1, \{1, 2, 3\})
\]
Then we get a relation among \( S(3, \mathbb{U}(2)) \). We can think that this relation comes out from the Newton formula (5), hence we prove the following generalized Newton formula to deduce relations among \( S(m, \mathbb{U}(n)) \).

For \( d \geq 0 \) and \( I = \{i_1 < \cdots < i_k\} \subset [m] \), we define an element of \( \tilde{\mathbb{H}}(m, \mathbb{U}(n)) \)

\[
e(d, I) := \sum_{\sigma \in \text{Emb}(k,n)} x_{\sigma(1)}^{i_1} y_{\sigma(2)}^{i_2} \cdots y_{\sigma(k)}^{i_k}.
\]

**Lemma 6.13.** For \( k \leq n \) and \( I = \{i_1 < \cdots < i_k\} \subset [m] \),

\[
\sum_{l=0}^{k-2} \sum_{i_1 \in J \subset I \atop |J|=l+1} (-1)^{l+d(J)}! z(d, J) e(0, I - J) + (-1)^{k+1} (k-1)! z(d, I)
\]

\[
= \begin{cases} 
  e(d - 1, I) & (k \leq n) \\
  0 & (k > n)
\end{cases}
\]

in \( \tilde{\mathbb{H}}(m, \mathbb{U}(n)) \), where \( d(J) = (\sum_{i \in J} j) - \frac{|J|(|J|+1)}{2} \).

**Proof.** Suppose that the identities

\[
\sum_{l=0}^{k-2} \sum_{i_1 \in J \subset I \atop |J|=l+1} (-1)^{l+d(J)}! x_i^{d-1} y_i^{l} e(0, I - J) + (-1)^{k+1} (k-1)! x_i^{d-1} y_i^{l}
\]

\[
= \begin{cases} 
  \sum_{\sigma} x_i^{d-1} y_i^{j_1} y_{\sigma(2)}^{i_2} \cdots y_{\sigma(k)}^{i_k} & (k \leq n) \\
  0 & (k > n)
\end{cases}
\]

in \( \tilde{\mathbb{H}}(m, \mathbb{U}(n)) \) hold, where \( \sigma \) ranges over all injections \([k-1] \to [n]-i\). Then the identities in the statement are obtained by summing up these identities. Hence we prove these identities. Let

\[
f(i, I) := \sum_{\sigma \in \text{Emb}(k,n)} y_{\sigma(1)}^{i_1} \cdots y_{\sigma(k)}^{i_k}.
\]

Then for \( 1 \leq l \leq k-2 \),

\[
\sum_{i_1 \in J \subset I \atop |J|=l} (-1)^{d(J)} x_i^{d-1} y_i^{l} f(i, I - J)
\]

\[
= \sum_{i_1 \in J \subset I \atop |J|=l} (-1)^{d(J)} x_i^{d-1} y_i^{l} \sum_{\sigma \in \text{Emb}(k-l,n)} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(k-l-1)}^{j_{k-l-1}}.
\]
Moreover,
\[
\sum_{\sigma \in \text{Emb}(k-l,n)} \sum_{i \in \sigma([k-l])} y_{\sigma(i(1))} \cdots y_{\sigma(k-l)} = \sum_{1 \leq s \leq k-l} (-1)^{s-1} y_i^s \sum_{\sigma \in \text{Emb}(k-l-1,n)} \sum_{i \in \sigma([k-l-1])} y_{\sigma(i(1))} \cdots y_{\sigma(s-1)} y_{\sigma(s)} \cdots y_{\sigma(k-l-1)}.
\]

Then it follows that
\[
\sum_{i \in J \subset I \mid |J| = l} (i, I - J) = \sum_{i \in J \subset I \mid |J| = l+1} (i, I - J).
\]

By a similar calculation, one also gets
\[
\sum_{i \in J \subset I \mid |J| = k-l} (i, I - J) = (k-l)x_i^{d-1}y_i^k(e(0, I - J) - f(i, I - J)).
\]

By a similar calculation, one also gets
\[
\sum_{i \in J \subset I \mid |J| = k-l} (i, I - J) = (k-l)x_i^{d-1}y_i^k(e(0, I - J) - f(i, I - J)).
\]

and for \(k \leq n\),
\[
\sum_{i \in J \subset I \mid |J| = 1} (i, I - J) = \sum_{\sigma} x_i^{d-1}y_i^1y_{\sigma(2)} \cdots y_{\sigma(k)},
\]

where \(\sigma\) ranges over all injections \([k] - 1 \to [n] - i\). Thus by combining these three identities, we obtain the desired identities. \(\square\)

We need the following three lemmas to deduce relations among \(S(m, U(n))\) from Lemma 6.13.

**Lemma 6.14.** For \(k \leq n\),
\[
\sum_{i_1 + \cdots + i_k = n-k+1} x_1^{i_1} \cdots x_k^{i_k} = 0 \quad \text{in} \quad \mathcal{P}(n)_{\Sigma_n}.
\]
Proof. Define \( h_{a,k}, e_{a,b,k} \in \mathcal{P}(n) \) by the identities of formal power series in \( t \) over \( \mathcal{P}(n) \)

\[
\sum_{a \geq 0} h_{a,k} t^a = \prod_{1 \leq j \leq k} \frac{1}{1 - x_j t} \quad \text{and} \quad \sum_{0 \leq a \leq k} (-1)^a e_{a,b} t^a = \prod_{k \leq j \leq k+b} (1 - x_j t).
\]

By a straightforward calculation, we can see that

\[
h_{a,k} = \sum_{i_1 + \cdots + i_k = a} x_1^{i_1} \cdots x_k^{i_k} \quad \text{and} \quad e_{a,b,k} = \sum_{k \leq i_1 < \cdots < i_a \leq k+b} x_1^{i_1} \cdots x_a^{i_a}.
\]

Note that

\[
\prod_{1 \leq j \leq k} \frac{1}{1 - x_j t} - \prod_{k+1 \leq j \leq n} (1 - x_j t) = \frac{1 - \prod_{1 \leq j \leq n} (1 - x_j t)}{\prod_{1 \leq j \leq k} (1 - x_j t)} = \left( \sum_{1 \leq a \leq n} (-1)^{l+1} e_{a,n-1,l} t^a \right) \prod_{k \leq j \leq k+n} \frac{1}{1 - x_j t}.
\]

Since \( e_{a,n-1,1} = 0 \) in \( \mathcal{P}(n)_{\Sigma_n} \), we get

\[
\prod_{1 \leq i \leq k} \frac{1}{1 - x_j t} = \prod_{1 \leq j \leq n-k} (1 - x_{k+j} t)
\]

as formal power series over \( \mathcal{P}(n)_{\Sigma_n} \). Note that the coefficient of \( t^{n-k+1} \) in the LHS is \( h_{n-k+1,k} \) and the RHS is 0. Thus \( h_{n-k+1,k} = 0 \) in \( \mathcal{P}(n)_{\Sigma_n} \), which is the identity in the statement. \( \square \)

Let \( \left\{ \binom{n}{k} \right\} \) denote the Stirling number of the second count. Then \( \left\{ \binom{n}{k} \right\} \) counts the number of partitions of \( [n] \) into \( k \) non-empty subsets. Recall from [27, (1.94d)] that the following identity holds.

**Lemma 6.15.** There is a equation

\[
\sum_{k=1}^{n} (-1)^k (k - 1)! \left\{ \binom{n}{k} \right\} = 0.
\]

**Lemma 6.16.** For \( k \geq 2 \), let \( I_1, \ldots I_k \) be pairwise disjoint subsets of \( [n] \), and let \( d_1, \ldots d_k \) be non-negative integers. Then the element

\[
w := \sum_{\sigma \in \text{Emb}(k,n)} x_{\sigma(1)}^{d_1} \cdots x_{\sigma(k)}^{d_k} y_{\sigma(1)}^{I_1} \cdots y_{\sigma(k)}^{I_k}
\]

of \( \mathcal{H}(m, U(n)) \) includes the term

\[
(-1)^{k+1+d(I_1, \ldots I_k)} (k - 1)! z (1 + d_1 + \cdots + d_k, I_1 \cup \cdots \cup I_k)
\]
where $d(I_1, \ldots, I_k) \in \mathbb{Z}/2$ is defined by
\[ (-1)^{d(I_1, \ldots, I_k)} y_1^{I_1} \cdots y_k^{I_k} = y_1^{I_1} \cdots y_k^{I_k}. \]

**Proof.** We prove the statement by induction on $k$. For $k = 2$, it is easy to see that
\[
\sum_{\sigma \in \text{Emb}(2, n)} x_{\sigma(1)}^{d_1} x_{\sigma(2)}^{d_2} y_{\sigma(1)}^{I_1} y_{\sigma(2)}^{I_2} = (-1)^{d(I_1, I_2)+1} z(d_1 + d_2 + 1, I_1 \cup I_2) + z(d_1 + 1, I_1)z(d_2 + 1, I_2).
\]
Then the statement holds.

Assume that the statement holds for $< k$. Let
\[ r_i = \sum_{j \in I_i} d_j \quad \text{and} \quad K_i = \bigcup_{j \in I_i} J_i. \]
Then by a straightforward calculation, we can see that
\[
z(1 + d_1, I_1) \cdots z(1 + d_k, I_k) - \sum_{\sigma \in \text{Emb}(k, n)} x_{\sigma(1)}^{d_1} \cdots x_{\sigma(k)}^{d_k} y_{\sigma(1)}^{I_1} \cdots y_{\sigma(k)}^{I_k}
\]
\[= \sum_{s=1}^{k-1} \sum_{J_1 \sqcup \cdots \sqcup J_s = [k]} (-1)^{d'(J_1, \ldots, J_s)} x_{\sigma(1)}^{r_1} \cdots x_{\sigma(s)}^{r_s} y_{\sigma(1)}^{K_1} \cdots y_{\sigma(s)}^{K_s}, \]
where $d'(J_1, \ldots, J_s) \in \mathbb{Z}/2$ is defined by
\[ (-1)^{d'(J_1, \ldots, J_s)} y_1^{K_1} \cdots y_1^{K_s} = y_1^{I_1} \cdots y_1^{I_k}. \]

By assumption, the coefficient of $z(1 + d_1 + \cdots + d_k, I_1 \cup \cdots \cup I_k)$ in $w$ is
\[ \sum_{s=1}^{k-1} (-1)^{d(I_1, \ldots, I_k)+s+1}(s-1)! \binom{k}{s}. \]

Thus the induction is complete by Lemma 6.15. \qed

Now we are ready to prove:

**Theorem 6.17 (Theorem F).** $\mathcal{H}(m, U(n))^{\Sigma_n}$ is minimally generated by $S(m, U(n))$.

**Proof.** By Theorem 6.8 and Lemma 6.12, it is sufficient to show that $z(d, I) \in S(m, U(n))$ for $d + |I| \geq n + 2$ belongs to a subring of $\mathcal{H}(m, U(n))^{\Sigma_n}$ generated by $S(m, U(n))$. First, we consider the case $|I| > n$. Applying Lemma 6.13 for $k = 1$, one gets that for any $J \subset [m]$, $e(0, J)$ belongs to the subring of $\mathcal{H}(m, U(n))^{\Sigma_n}$ generated by $z(d, I) \in S(m, U(n))$ for $|I| \leq n$. One also gets that $z(d, I) \in S(m, n)$ for $|I| > n$ belongs to the subring of $\mathcal{H}(m, U(n))^{\Sigma_n}$ generated by $e(0, J)$ for $J \subset [m]$ and $z(d, J) \in S(m, n)$ for $|J| < |I|$. Thus $z(d, I) \in S(m, U(n))$ for $|I| > n$ turn out to belong to the subring of $\mathcal{H}(m, U(n))^{\Sigma_n}$ generated by $z(d, I) \in S(m, U(n))$ for $|I| \leq n$. 

Next, we consider the case $|I| \leq n$ and $d + |I| \geq n + 2$. Let $I = \{i_1 < \cdots < i_k\}$ be a subset of $[n]$, and let $d$ be a positive integer with $d + k \geq n + 2$. By Lemma 6.14, there is an identity in $\mathcal{H}(m, U(n))^\Sigma_n$

$$\sum_{j_1 + \cdots + j_k = n - k + 1} \sum_{\sigma \in \text{Emb}(k, m)} x_{\sigma(1)}^{d-n+k-2} x_{\sigma(1)}^{j_1} \cdots x_{\sigma(k)}^{j_k} y_{\sigma(1)}^{i_1} \cdots y_{\sigma(k)}^{i_k} = 0.$$ 

On the other hand, it follows from Lemma 6.16 that the LHS includes a non-zero multiple of $z(d, I)$. Thus $z(d, I)$ turns out to belong to the subring of $\mathcal{H}(m, U(n))^\Sigma_n$ generated by $z(d', I') \in \mathcal{S}(m, U(n))$ with $d' + |I'| < d + |I|$. Therefore the proof is complete by induction on $d + |I|$. $\square$

Let $S(m, SU(n)) := \{z(d, I) \in S(m, U(n)) \mid d > 1 \text{ or } |I| > 1\}$. Then $S(m, SU(n))$ is a subset of $S(m, U(n))$ consisting of elements of degree 1. Thus by (4) and Theorem 6.17, we obtain the following corollary.

**Corollary 6.18.** $\mathcal{H}(m, SU(n))^\Sigma_n$ is minimally generated by $S(m, SU(n))$.

We give example calculations of the cohomology of $\text{Hom}(\mathbb{Z}^2, SU(n))_1$ for $n = 2, 3$.

**Example 6.19.** By Example 4.4, $\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(2)); \mathbb{F})$ is non-trivial only for $s = 0, 2, 3$, where $\mathbb{F}$ is a field of characteristic zero or prime to 2. Then all products in $\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(2))_1; \mathbb{F})$ are trivial. By Theorem 6.18, $\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(2))_1; \mathbb{F})$ is generated by

$$a := z(2, \{1\}), \quad b := z(2, \{1\}), \quad c := z(1, \{1, 2\}).$$

Thus we obtain

$$\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(2))_1; \mathbb{F}) = \mathbb{F} \langle a, b, c \rangle / (a, b, c)^2$$

where $|a| = |b| = 3$ and $|c| = 2$.

**Example 6.20.** We assume the ground field $\mathbb{F}$ is of characteristic zero or prime to 6. By Theorem 6.18, $\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(3))_1; \mathbb{F})$ is generated by

$$a_i := z(i + 1, \{1\}), \quad b_i := z(i + 1, \{1\}), \quad c_i := z(i, \{1, 2\})$$

for $i = 1, 2$. Then $(a_1, a_2, b_1, b_2, c_1, c_2)^3 = 0$. Moreover, by a direct computation, we can verify that the following are trivial in $\mathcal{H}(2, SU(n))$.

$$c_1 c_2, \quad a_2 c_2, \quad b_2 c_2, \quad a_2 c_1 - a_1 c_2, \quad b_2 c_1 - b_1 c_2, \quad a_2 b_1 - b_2 a_1.$$ 

Let $I$ be the ideal of $\mathbb{F} \langle a_1, a_2, b_1, b_2, c_1, c_2 \rangle$ generated by these elements. It is easy to see that the Poincaré series of

$$\mathbb{F} \langle a_1, a_2, b_1, b_2, c_1, c_2 \rangle / (a_1, a_2, b_1, b_2, c_1, c_2)^3 + I$$

coincides with that of $\text{Hom}(\mathbb{Z}^2, SU(3))_1$ in Example 4.4. Thus we obtain

$$\mathcal{H}^*(\text{Hom}(\mathbb{Z}^2, SU(3))_1; \mathbb{F}) \cong \mathbb{F} \langle a_1, a_2, b_1, b_2, c_1, c_2 \rangle / (a_1, a_2, b_1, b_2, c_1, c_2)^3 + I.$$
6.4. **Results for symplectic groups.** By calculations similar to the case of unitary groups, we give a minimal generating set of $\mathcal{H}(m, Sp(n))^{B_n}$. Since the calculations are quite similar, we only state results without proofs.

We assume that the ground field $F$ is of characteristic zero or prime to $2(n!)$. Then analogously to (6), we have

$$\mathcal{P}(n)^{B_n} = F[p_2, p_4, \ldots, p_{2n}].$$

For $d_1, \ldots, d_{2m-1} \geq 1$, let

$$w(d_1, \ldots, d_{2m-1}) := \prod_{k=1}^{2m-1} d_k \prod_{i=d_1+\cdots+d_{k-1}+1}^1 x_i^{(|I_k|)} y_i^k$$

where

$$\epsilon(k) = \begin{cases} 0 & (k \text{ even}) \\ 1 & (k \text{ odd}) \end{cases}$$

and $\emptyset \neq I_1 < \cdots < I_{2m-1}$ is the ordering of $2^m$ defined above. Let $N(m, n)$ be the subset of $\mathcal{H}(m, Sp(n))$ consisting of monic monomials of the form

$$M(x_1, \ldots, x_n)w(d_1, \ldots, d_{2m-1}).$$

Analogously to Lemmas 6.4 and 6.5, one can prove the following two lemmas.

**Lemma 6.21.** The set

$$\left\{ \sum_{\sigma \in B_n} \sigma(v) \mid v \in N(m, n) \right\}$$

is a basis of $\mathcal{H}(m, Sp(n))^{B_n}$.

**Lemma 6.22.** $\mathcal{H}(m, Sp(n))^{B_n}$ is spanned by elements represented by

$$\sum_{\sigma \in B_n} \sigma(M(x_1^2, \ldots, x_n^2)w(d_1, \ldots, d_{2m-1}))$$

satisfying the following conditions:

1. $M(x_1^2, \ldots, x_r^2) = M(x_1^2, \ldots, x_r^2, 0, \ldots, 0)$ for $r = d_1 + \cdots + d_{2m-1}$;
2. if the above representative includes the term

$$P(x_{d_1+\cdots+d_{i-1}+1}^2, \ldots, x_{d_1+\cdots+d_{i-1}+d_i}^2)Q$$

where $Q$ does not include $x_{d_1+\cdots+d_{i-1}+1}^2, \ldots, x_{d_1+\cdots+d_{i-1}+d_i}^2$, then $P$ is symmetric for $|S_i|$ even and anti-symmetric for $|S_i|$ odd.

For $\emptyset \neq I \subset [m]$ and $d \geq 1$, let

$$w(d, I) := x_1^{2d+\epsilon(|I|)-2} y_1^I + \cdots + x_n^{2d+\epsilon(|I|)-2} y_n^I$$

and let

$$\mathcal{S}(m, Sp(n)) := \{w(d, I) \mid d \geq 1 \text{ and } \emptyset \neq I \subset [m]\}.$$
We can define an ordering on \( N(m, n) \) analogously to that on \( M(m, n) \). Then by arguing as in the proof of Theorem 6.8, one gets:

**Theorem 6.23.** \( \mathcal{H}(m, Sp)^{B_n} \) is generated by \( \overline{S}(m, Sp(n)) \).

Then it remains to find a minimal generating set of \( \mathcal{H}(m, Sp(n))^{B_n} \) contained in \( \overline{S}(m, Sp(n)) \). Quite similarly to Lemma 6.9 (cf. [13, Section 10, Proposition 3]), we can prove the following.

**Lemma 6.24.** \( P(n)^{B_n} \) has a base

\[ \{x_1^{2i_1+\epsilon_1} \cdots x_n^{2i_n-1+\epsilon_n-1} x_n^{\epsilon_n} \mid i_k \leq n - k \text{ and } \epsilon_k = 0, 1 \} \]

We define a subset of \( \overline{S}(m, Sp(n)) \) by

\[ S(m, Sp(n)) := \{ w(d, I) \in \overline{S}(m, Sp(n)) \mid 2d + |I| + \epsilon(|I|) - 2 \leq 2n \} \]

Similarly to Theorem 6.11, we can prove the following.

**Theorem 6.25.** The map \( \mathbb{F}(S(m, Sp(n))) \to \mathcal{H}(m, Sp(n))^{B_n} \) is an isomorphism in dimension \( \leq 2n + 1 \).

Let \( S(m, SO(2n+1)) := S(m, Sp(n)) \). Then by Theorems 4.7 and 6.25, we get:

**Corollary 6.26.** The map \( \mathbb{F}(S(m, SO(2n+1))) \to \mathcal{H}(m, SO(2n+1))^{B_n} \) is an isomorphism in dimension \( \leq 2n + 1 \).

We define

\[ W(m, n) := \{ w(d_1, I_1) \cdots w(d_k, I_k) \mid I_1 < \cdots < I_k \text{ and } k + \max\{d_1, \ldots, d_k\} - 1 \leq n \} \]

By the same argument as Lemma 6.12, we can prove:

**Lemma 6.27.** \( S(m, Sp(n)) \) is included in every subset of \( \overline{S}(m, Sp(n)) \) that generates \( \mathcal{H}(m, Sp(n))^{B_n} \).

Thus by using Lemmas 6.13, similarly to Theorem 6.17, we finally obtain the following.

**Theorem 6.28.** \( \mathcal{H}(m, Sp(n))^{B_n} \) is minimally generated by \( S(m, Sp(n)) \).

The following is immediate from Theorems 4.7 and 6.28.

**Corollary 6.29.** \( \mathcal{H}(m, SO(2n+1))^{B_n} \) is minimally generated by \( S(m, SO(2n+1)) \).

### 7. Homological stability

Throughout this section, let \( G_n \) be either \( U(n), SU(n), Sp(n) \) or \( SO(2n+1) \). Then there is a sequence of spaces

\[ \text{Hom}(\mathbb{Z}^m, G_1)_1 \to \text{Hom}(\mathbb{Z}^m, G_2)_1 \to \text{Hom}(\mathbb{Z}^m, G_3)_1 \to \cdots \]
In this section, we prove that this sequence satisfies homological stability with rational coefficients and give the best possible stable range. For \( d \geq 1 \) and \( 1 \leq k \leq n \), we define a set

\[
U(d, k) := \{ z(d_1, I_1) \cdots z(d_k, I_k) \mid \text{max}\{d_1, \ldots, d_k\} = d \}.
\]

**Lemma 7.1.** If \( m + 1 \leq k \leq n \) and \( d \geq 2 \), then least degree elements of \( U(d, k) \) are

\[
a_{d,k}(i, I_1, \ldots, I_k) := z(d, \{i\}) \prod_{j=1}^{m} z(1, \{j\}) \prod_{l=1}^{k-m-1} z(1, I_l)
\]

where \( i \in [m] \) and \( |I_l| = 2 \) for all \( l \). Otherwise, \( U(d, k) = \{0\} \) or least degree elements of \( U(d, k) \) are of degree \( > 2d + 2k - m - 3 \).

**Proof.** Let \( z := z(d_1, I_1) \cdots z(d_k, I_k) \) be a least degree element of \( U(d, k) \). We may assume that \( d_1 = d \). Since \( z \) is of degree \( 2d_1 + \cdots + 2d_k - 2k + |I_1| + \cdots + |I_k| \), it follows from minimality of the degree that \( d_2 = \cdots = d_k = 1 \) and \( |I_1| = 1 \). Thus the proof is done by an easy degree counting. \( \square \)

By Lemma 7.1, elements of \( \bigoplus_{d+k=n+2} U(d, k) \) are of degree \( \geq 2n - m + 1 \). Let \( U \) denote the vector space spanned by elements of \( \bigoplus_{d+k=n+2} U(d, k) \) of degree \( 2n - m + 1 \). Then \( U = \{0\} \) for \( n = m \), and \( U \) is spanned by \( a_{d,k}(i, I_1, \ldots, I_k) \) in Lemma 7.1 for \( n > m \).

For a graded vector space \( V \), let \( V_d \) denote the \( d \)-dimensional part of \( V \).

**Proposition 7.2.** For \( n \geq m \), there is an isomorphism

\[
\mathcal{H}(m, U(n))_{2n-m+1} \cong V(m, n)_{2n-m+1} \oplus U.
\]

**Proof.** By Lemma 7.1, there is a natural surjection

\[
V(m, n)_{2n-m+1} \oplus U \rightarrow \mathcal{H}(m, U(n))_{2n-m+1}.
\]

By Lemma 6.10, if \( a_{d,k}(i, I_1, \ldots, I_k) \) and the canonical basis of \( V(m, n)_{2n-m+1} \) are linearly independent in \( \mathcal{H}(m, U(n))_{2n-m+1} \) for \( d+k = n+2 \) and \( d \geq 2 \), then this map is injective.

Let \( \alpha: \mathcal{E}(m, n) \rightarrow \Sigma_n \) be a map sending \( y_1^1 \cdots y_m^m y_{m+1}^{l_1} \cdots y_{m+k+1}^{l_k} \) to the transposition of \((1, m+1)\) and other monomials to 1, where \( |I_1| = \cdots = |I_k| = 2 \). We may assume \( I_1 < \cdots < I_k \). Then by Lemma 6.7, the least term of \( a_{d,k}(i, I_1, \ldots, I_k) \) is given by

\[
x^{n-m-k}_i y^i_1 y^i_2 \cdots y^i_{m+1} y^i_{m+2} \cdots y^i_{m+k+1}.
\]

On the other hand, the least terms of the canonical basis are given in the proof of Lemma 6.10. Then since the exponents of \( x_i \) in the least terms of the canonical basis of \( V(m, n)_{2n-m+1} \) are less than \( n - m \), the least terms of \( a_{d,k}(i, I_1, \ldots, I_k) \) and the canonical basis of \( V(m, n)_{2n-m+1} \) are mutually distinct elements of \( \mathcal{B}(\alpha) \), where we consider the map \( \alpha \) because \( x^i_{m+1} \not\in \mathcal{B}(1) \) and \( x^{n-m}_{m+1} \in \mathcal{B}((1 m+1)) \).
Corollary 7.3. The map \( p_n : \mathcal{H}(m, U(n + 1))^{\Sigma_n + 1} \to \mathcal{H}(m, U(n))^{\Sigma_n} \) is an isomorphism in dimension \( 2n - m + 1 \).

Proof. By Lemma 7.1, the map \( p_n \) induces an isomorphism \( V(m, n + 1)_{2n-m+1} \cong V(m, n)_{2n-m+1} \oplus U \). Then the proof is done by Lemma 6.10, Theorem 6.11 and Proposition 7.2.

We define

\[
d_{m,n} := \begin{cases} 
2n - m + 1 & (G_n = U(n), SU(n)) \\
2n + 1 & (G_n = Sp(n), SO(2n + 1)).
\end{cases}
\]

Theorem 7.4 (Theorem H). Suppose that \( n \geq m \) for \( G_n = U(n), SU(n) \) and \( m \geq 2 \) for \( G_n = Sp(n), SO(2n + 1) \). Then the map

\[
H_* (\text{Hom}(\mathbb{Z}^m, G_{n+1}; \mathbb{Q})) \to H_* (\text{Hom}(\mathbb{Z}^m, G_n; \mathbb{Q}))
\]

is an isomorphism for \( * \leq d_{m,n} \) and not surjective for \( * = d_{m,n} + 1 \).

Proof. We assume that the ground field is \( \mathbb{Q} \). Let \( W_n \) denote the Weyl group of \( G_n \). There is a natural projection

\[
p_n : \mathcal{H}(m, G_{n+1})^{W_{n+1}} \to \mathcal{H}(m, G_n)^{W_n}.
\]

By the construction of the isomorphism in Theorem 2.1, this projection satisfies a commutative square

\[
\begin{array}{ccc}
H^* (\text{Hom}(\mathbb{Z}^m, G_{n+1}; \mathbb{Q})) & \xrightarrow{\cong} & \mathcal{H}(m, G_{n+1})^{W_{n+1}} \\
\downarrow & & \downarrow p_n \\
H^* (\text{Hom}(\mathbb{Z}^m, G_n; \mathbb{Q})) & \xrightarrow{\cong} & \mathcal{H}(m, G_n)^{W_n}.
\end{array}
\]

Then it suffices to show that \( p_n \) is an isomorphism in dimension \( \leq d_{m,n} \) and not injective in dimension \( d_{m,n} + 1 \).

First, we consider the case \( G_n = Sp(n) \). Clearly, \( p_n(w(d, I)) = w(d, I) \) for \( 2d + |I| + e(|I|) - 2 \leq 2n \). Then it follows from Theorem 6.25 that \( p_n \) is an isomorphism in dimension \( \leq d_{m,n} \). Since we are supposing \( m \geq 2 \), there is an element

\[
w(1, \{1, 2\}) = y_1^1 y_1^2 + \cdots + y_k^1 y_k^2 \in \mathcal{H}(m, Sp(k))^{B_k}
\]

for \( k = n, n + 1 \). It is obvious that \( w(1, \{1, 2\})^{n+1} = 0 \) in \( \mathcal{H}(m, Sp(n))^{B_n} \), and by Theorem 6.25, \( w(1, \{1, 2\})^{n+1} \neq 0 \) in \( \mathcal{H}(m, Sp(n+1))^{B_{n+1}} \). Thus the statement is proved. By Theorem 4.7, the case \( G_n = SO(2n + 1) \) is also proved.

Next, we consider the case \( G_n = U(n) \). By definition, \( p_n(z(d, I)) = z(d, I) \) for \( d + |I| = 1 \leq n \). Then by Theorem 6.11, \( p_n \) is an isomorphism in dimension \( \leq d_{m,n} - 1 \). The element \( z(n - m + 2, [m]) \) belongs to \( \mathcal{H}(m, U(n + 1))^{\Sigma_{n+1}} \), but it
does not belong to \( S(m, U(n))^{\Sigma n} \). Since \( z(n - m + 2, [m]) \) is of degree \( 2n - m + 2 \), this implies that \( p_n \) is not injective in dimension \( d_{m,n} + 1 \). Then it remains to show that \( p_n \) is an isomorphism in dimension \( d_{m,n} \), and this is already proved by Corollary 7.3. Thus the proof for \( G_n = U(n) \) is complete. The case \( G_n = SU(n) \) follows from this result and (4). 

\[ \square \]

Remark 7.5. (1) Let \( m = 1 \) and \( G_n = Sp(n), SO(2n+1) \). Since \( \text{Hom}(\mathbb{Z}, G_n)^1 = G_n \), the map (8) is an isomorphism for \( * \leq 4n + 2 \) and not surjective for \( * = 4n + 3 \).

(2) Let \( n < m \). Then quite similarly to the above proof, we can show that the map (8) is an isomorphism for \( * \leq n \) and not surjective for \( * = n + 1 \).

8. Questions

In this section, we pose several questions arising in our work.

In Section 4, we gave explicit formulae for the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)^1 \) by investigating combinatorics of (signed) permutation. So the method does not directly apply to the exceptional Lie groups. On the other hand, by a direct calculation of the formula in Theorem 2.2 using an explicit matrices for the canonical representation of the Weyl group of \( G_2 \), Ramras and Stafa [24] gave an explicit formula for the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G_2)^1 \). However, this formula does not show a connection to combinatorics of the Weyl group. Then we ask the following question.

**Question 8.1.** Is there a formula for the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)^1 \) in terms of combinatorics of the Weyl group of \( G \) when \( G \) is exceptional?

A formula for the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)^1 \) when \( G \) is exceptional must be more computable than the one in Theorem 2.2. We determined the top term of the Poincaré series by using our formula for the classical groups.

**Question 8.2.** Can we determine the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^m, G)^1 \) when \( G \) is exceptional?

As we can see in Theorem 5.5, the top term of the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)^1 \) is given in terms of simple factors of \( G \). This was obtained by a formal calculation using our formula and the computer calculation in Appendix. Then we do not understand its topological meaning.

**Question 8.3.** Is there a topological interpretation of the identity in Theorem 5.5?

In Section 6, we gave a minimal generating set of \( \mathcal{H}(m, G)^W \) for the classical group \( G \), where there is an isomorphism

\[ H^*(\text{Hom}(\mathbb{Z}^m, G)^1; \mathbb{F}) \cong \mathcal{H}(m, G)^W \]
for a field \( \mathbb{F} \) of characteristic zero or prime to \(|W|\). Our method is a direct calculation of invariants of (signed) symmetric groups, which does not apply to other Lie groups. However, for \( m = 1 \) we have the theorem of Solomon, from which we can determine \( \mathcal{H}(1, G)^W \). By the Shephard-Todd theorem, we have

\[
\mathcal{P}(n)^W = \mathbb{F}[f_1, \ldots, f_n]
\]

where \( G \) is of rank \( n \). Define a map \( d: \mathcal{P}(n) \to \mathcal{P}(n) \otimes \Lambda(y_1, \ldots, y_n) \) by \( dx_i = y_i \) together with the Leibniz rule. Then the theorem of Solomon states that

\[
\tilde{\mathcal{H}}(1, G)^W \cong \mathcal{P}(n)^W \otimes \Lambda(df_1, \ldots, df_n).
\]

As in the proof of Theorem 2.2,

\[
\mathcal{H}(1, G)^W = \tilde{\mathcal{H}}(1, G)^W / (f_1, \ldots, f_n).
\]

Then we obtain

\[
\mathcal{H}(1, G)^W = \Lambda(df_1, \ldots, df_n)
\]

which is a well known result in topology. Applying this observation to (6), we obtain Theorem 6.17. Note that our generators \( z(d, I) \) can be obtained by "differentiating" the power sums \( p_1, \ldots, p_n \). Then Theorems 6.17, 6.28 and Corollary 6.18, 6.29 may be alternatively proved by a representation theoretic way.

**Question 8.4.** Is there a generalization of the theorem of Solomon, which gives a minimal generating set of \( \mathcal{H}(m, G)^W \)?

By the theorem of Solomon, if Lie groups \( G_1 \) and \( G_2 \) are simple of the same rank, then the cohomology of \( \text{Hom}(\mathbb{Z}, G_1)_1 \) and \( \text{Hom}(\mathbb{Z}, G_2)_1 \) are isomorphic as ungraded rings. In [28], the second author determines the cohomology of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) when \( G \) is simple Lie group of rank two, including the exceptional group \( G_2 \). In particular, he finds that cohomology of these spaces are isomorphic to each other as ungraded rings. Considering the case of \( \text{Hom}(\mathbb{Z}, G)_1 \) above, this supports the affirmative answer to this question.

As we can see in Examples 6.19 and 6.20, we do not have a general scheme to get relations among our minimal generating set of \( \mathcal{H}(m, G)^W \).

**Question 8.5.** Is there a general scheme to get relations among our minimal generating set of \( \mathcal{H}(m, G)^W \)?

The best answer to this question is an answer to Question 8.4, which completely describes \( \mathcal{H}(m, G)^W \) in terms of \( W \).

**Appendix A. Exceptional Lie groups**

We can calculate the Poincaré series of \( \text{Hom}(\mathbb{Z}^2, G)_1 \) for the exceptional Lie group \( G \) by Theorem 2.2 with an assistance of computer. In this section, we only record the result.
$P(\text{Hom}(\mathbb{Z}^2, G_2); t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + t^{10} + 2t^{11} + 2t^{13} + 3t^{14}$

$P(\text{Hom}(\mathbb{Z}^2, F_4); t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + t^{12} + 4t^{13} + 6t^{14} + 6t^{15} + 6t^{16} + 8t^{17} + 9t^{18} + 6t^{19} + 6t^{20} + 6t^{21} + 6t^{22} + 6t^{23} + 5t^{24} + 8t^{25} + 9t^{26} + 8t^{27} + 14t^{28} + 12t^{29} + 8t^{30} + 10t^{31} + 7t^{32} + 4t^{33} + 4t^{44} + 4t^{35} + 8t^{36} + 8t^{37} + 6t^{38} + 8t^{39} + 10t^{40} + 6t^{41} + 3t^{42} + 6t^{43} + 3t^{44} + 3t^{48} + 4t^{49} + 4t^{51} + 5t^{52}$

$P(\text{Hom}(\mathbb{Z}^2, E_6); t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 3t^8 + 4t^9 + 4t^{10} + 8t^{11}$

$P(\text{Hom}(\mathbb{Z}^2, E_7); t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + 2t^9 + 3t^{10} + 4t^{11} + 3t^{12} + 6t^{13} + 8t^{14} + 8t^{15} + 8t^{16} + 10t^{17} + 12t^{18} + 12t^{19} + 14t^{20} + 18t^{21} + 19t^{22} + 20t^{23} + 24t^{24} + 28t^{25} + 29t^{26} + 30t^{27} + 39t^{28} + 44t^{29} + 44t^{30} + 48t^{31} + 58t^{32} + 64t^{33} + 66t^{34} + 70t^{35} + 82t^{36} + 90t^{37} + 91t^{38} + 98t^{39} + 112t^{40} + 118t^{41} + 123t^{42} + 134t^{43} + 146t^{44} + 152t^{45} + 158t^{46} + 174t^{47} + 183t^{48} + 186t^{49} + 198t^{50} + 214t^{51} + 221t^{52} + 222t^{53} + 235t^{54} + 256t^{55} + 252t^{56} + 252t^{57} + 274t^{58} + 288t^{59} + 282t^{60} + 282t^{61} + 300t^{62} + 314t^{63} + 303t^{64} + 300t^{65} + 319t^{66} + 328t^{67} + 313t^{68} + 306t^{69} + 326t^{70} + 330t^{71} + 313t^{72} + 314t^{73} + 324t^{74} + 320t^{75} + 305t^{76} + 296t^{77} + 300t^{78} + 292t^{79} + 277t^{80} + 278t^{81} + 278t^{82} + 260t^{83} + 256t^{84} + 256t^{85} + 241t^{86} + 222t^{87} + 219t^{88} + 220t^{89} + 200t^{90} + 178t^{91} + 181t^{92} + 180t^{93} + 160t^{94} + 140t^{95} + 145t^{96} + 158t^{97} + 129t^{98} + 104t^{99} + 118t^{100} + 112t^{101} + 84t^{102} + 70t^{103} + 76t^{104} + 88t^{105} + 68t^{106} + 50t^{107} + 63t^{108} + 66t^{109} + 43t^{110} + 30t^{111} + 38t^{112} + 42t^{113} + 25t^{114} + 16t^{115} + 25t^{116} + 26t^{117} + 17t^{118} + 12t^{119} + 17t^{120} + 22t^{121} + 12t^{122} + 6t^{123} + 12t^{124} + 6t^{125} + 6t^{129} + 7t^{130} + 7t^{132} + 8t^{133}$
\( P(\text{Hom}(\mathbb{Z}^2, E_6)_1; t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + 2t^{12} + 2t^{13} + 3t^{14} + 4t^{15} + 3t^{16} + 6t^{17} + 6t^{18} + 4t^{19} + 6t^{20} + 6t^{21} + 7t^{22} + 8t^{23} + 7t^{24} + 10t^{25} + 12t^{26} + 12t^{27} + 13t^{28} + 16t^{29} + 17t^{30} + 16t^{31} + 18t^{32} + 18t^{33} + 20t^{34} + 22t^{35} + 21t^{36} + 60t^{48} + 64t^{49} + 64t^{50} + 64t^{51} + 75t^{52} + 74t^{53} + 76t^{54} + 90t^{55} + 97t^{56} + 100t^{57} + 107t^{58} + 116t^{59} + 122t^{60} + 126t^{61} + 124t^{62} + 128t^{63} + 145t^{64} + 144t^{65} + 151t^{66} + 180t^{67} + 191t^{68} + 198t^{69} + 210t^{70} + 218t^{71} + 225t^{72} + 224t^{73} + 218t^{74} + 228t^{75} + 245t^{76} + 242t^{77} + 258t^{78} + 298t^{79} + 310t^{80} + 328t^{81} + 345t^{82} + 348t^{83} + 361t^{84} + 354t^{85} + 347t^{86} + 364t^{87} + 374t^{88} + 382t^{89} + 413t^{90} + 452t^{91} + 464t^{92} + 488t^{93} + 509t^{94} + 494t^{95} + 493t^{96} + 486t^{97} + 478t^{98} + 494t^{99} + 496t^{100} + 520t^{101} + 570t^{102} + 598t^{103} + 615t^{104} + 650t^{105} + 668t^{106} + 640t^{107} + 625t^{108} + 616t^{109} + 607t^{110} + 614t^{111} + 606t^{112} + 646t^{113} + 707t^{114} + 708t^{115} + 721t^{116} + 758t^{117} + 760t^{118} + 720t^{119} + 687t^{120} + 670t^{121} + 671t^{122} + 670t^{123} + 663t^{124} + 716t^{125} + 782t^{126} + 768t^{127} + 785t^{128} + 810t^{129} + 784t^{130} + 744t^{131} + 693t^{132} + 672t^{133} + 668t^{134} + 650t^{135} + 652t^{136} + 704t^{137} + 753t^{138} + 736t^{139} + 745t^{140} + 756t^{141} + 710t^{142} + 678t^{143} + 627t^{144} + 594t^{145} + 600t^{146} + 578t^{147} + 591t^{148} + 640t^{149} + 658t^{150} + 652t^{151} + 661t^{152} + 638t^{153} + 571t^{154} + 546t^{155} + 504t^{156} + 456t^{157} + 460t^{158} + 448t^{159} + 468t^{160} + 506t^{161} + 498t^{162} + 512t^{163} + 521t^{164} + 474t^{165} + 417t^{166} + 400t^{167} + 369t^{168} + 322t^{169} + 323t^{170} + 322t^{171} + 348t^{172} + 372t^{173} + 334t^{174} + 360t^{175} + 376t^{176} + 302t^{177} + 254t^{178} + 248t^{179} + 226t^{180} + 190t^{181} + 186t^{182} + 202t^{183} + 228t^{184} + 240t^{185} + 208t^{186} + 230t^{187} + 247t^{188} + 178t^{189} + 147t^{190} + 146t^{191} + 124t^{192} + 104t^{193} + 93t^{194} + 110t^{195} + 136t^{196} + 132t^{197} + 104t^{198} + 122t^{199} + 140t^{200} + 82t^{201} + 62t^{202} + 72t^{203} + 53t^{204} + 48t^{205} + 42t^{206} + 54t^{207} + 83t^{208} + 74t^{209} + 56t^{210} + 68t^{211} + 71t^{212} + 36t^{213} + 24t^{214} + 30t^{215} + 17t^{216} + 16t^{217} + 12t^{218} + 16t^{219} + 39t^{220} + 30t^{221} + 19t^{222} + 30t^{223} + 31t^{224} + 14t^{225} + 7t^{226} + 14t^{227} + 7t^{228} + 6t^{229} + 7t^{230} + 6t^{231} + 19t^{232} + 14t^{233} + 7t^{234} + 14t^{235} + 7t^{236} + 7t^{241} + 8t^{245} + 8t^{247} + 9t^{248}

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