Finite Size Correction in a Disordered System - 
A New Divergence

Somendra M. Bhattacharjee and Sutapa Mukherji
Institute of Physics, Bhubaneswar 751 005, India
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Abstract

We show that the amplitude of the finite size correction term for the $n$th moment of the partition function, for randomly interacting directed polymers, diverges (on the high temperature side) as $(n_c - n)^{-r}$, as a critical moment $n_c$ is approached. The exponent $r$ is independent of temperature but does depend on the effective dimensionality. There is no such divergence on the low temperature side ($n > n_c$).

05.70.Fh, 64.60.Ak, 61.25.Hq, 75.10.Nr
The averages of thermodynamic quantities for quenched random systems are often done through the replica trick which requires an analytic continuation of the integer moments of the partition function \( \overline{Z^n} \) to \( n \to 0 \) (See e.g. [1]), with the overbar representing the disorder average. It is only recently that several issues related to this continuation could be debated quantitatively because of the results available for the problem of a directed polymer in a random medium [2]. This was not feasible for other models. For example, the replica prediction of the nonexistence of the higher order (> 3) cumulants of the free energy of 1+1 dimensional directed polymer problem has been challenged by direct numerical computation of those [3,4]. In large dimensions, the behaviours of the moments for \( n > 1 \) and \( n < 1 \) are found to be different [5]. The possibility of the existence of a singularity in \( n \) has also been pointed out [6,7]. All of these point finger to the analytic continuation and interchange of the \( n \to 0 \) with the thermodynamic limit. In complex situations like the spin glass, even the effective hamiltonian for the \( n \)th moment can admit of many phases adding fuel to the fire [8].

In this paper we show an even more striking possibility in a directed polymer system. It is a divergence of the finite size effect for the moments of the partition function in the high temperature phase as a critical moment is approached. This divergence is distinct from the known singularity, at the thermodynamic transition point, in the finite size correction to the free energy for disordered systems [11] like spin glasses [9], the random energy model [10], and directed polymer in a random medium [6]. In contrast, the divergence we report here is not necessarily at the bulk transition point.

Our model involves two directed polymers which interact on intersection at sites. The interaction energy is random with zero mean. The model is similar to those used for depinning transitions of line like objects [12]. In the continuum version, the random interaction model is known to exhibit a disorder induced transition from a weak disorder (high temperature) to a strong disorder (to be called the RANI phase) phase [13,14]. For detailed understanding of the phases, we study the model on a lattice using a real space renormalization group approach. Since real space renormalization group can be implemented exactly on hierarchical lattices [15], we consider the problem from the beginning on a hierarchical lattice. As usual, the procedure can be thought of as an approximate real space renormalization group on a regular lattice.

The lattice, as mentioned, is generated hierarchically by replacing each bond at the \( \mu \)th generation by a “diamond” of \( b \) branches (Fig 1). Two directed polymers start at the bottom of the lattice and meet at the upper end, without any backtracking. The model of interest is the site model where the two polymers interact whenever they meet at a site, with a random interaction energy \( \epsilon \). For simplicity, all the directed paths are taken to be identical as far as the choice of \( \epsilon \) on these is concerned. However, the energies along a particular path are independent and uncorrelated, so that the randomness is only in the longitudinal (special) direction and not in the transverse directions (see fig 1). For this paper, the only information needed about randomness is the behavior of the moments of the weight \( y = \exp(-\epsilon/T) \) (\( T \) being the temperature in units of the Boltzmann constant) which, for a gaussian distribution of \( \epsilon \), behaves as \( \overline{y^m} = \overline{y^{m^2}} \). Other distributions have been considered, though will not be elaborated upon here.

Let \( Z_\mu \) be the partition function for a given realization of randomness, and let \( S_\mu = b^{L_\mu-1} \) be the number of single chain configurations, at the \( \mu \)th generation. Here \( L_\mu = 2^\mu \) is the
length of a directed polymer. We define $Z_{\mu}(n) = \frac{Z_{\mu}}{S_{\mu}^2}$, to factor out the free chain entropy, and call $Z_{\mu}(n)$ the moments. We establish that, like the random medium problem, for a given temperature, there is a critical value $n = n_c(y)$ below which all moments are in their high temperature phase, in the thermodynamic limit of course. In this limit, $Z_{\mu}(n)$ approaches a fixed point value for $n < n_c$, whereas, for $n > n_c$, the moments diverge but with a finite “free energy” density $f_{\mu}(n) \equiv (nL_{\mu})^{-1} \ln Z_{\mu}(n)$(RANI phase). The approach to the thermodynamic limit can be written generically as

$$g_{\mu}(n) = g(n) + B_g(n) L_{\mu}^{-\psi} + ...,$$

(1)

where $g(n)$ is the thermodynamic limit ($\mu \rightarrow \infty$), and $B_g(n)$ is the amplitude of the finite size correction. We take $g_{\mu}(n)$ for $n < n_c$ to be $[Z_{\mu}(n)]^{1/n}$, and for $n > n_c$ to be $f_{\mu}(n)$. We will see that the corrections are power laws in $L$.

One of the main results of the paper is the blowing up of $Z_c(n)$ as $(n_c - n)^{-r}$, for $n \rightarrow n_c-$, a feature whose existence was not even dreamed of before. No such divergence occurs for $n > n_c$. We study the variation of $r$ with the various parameters of the problem and establish its universality for a given distribution. We also characterize the behavior of $f_{\mu}(n)$ for $n \rightarrow n_c+$.

For a given realization of disorder, the partition function can be written as (see Fig [1])

$$Z_{\mu+1} = bZ_{\mu}^{(A)}yZ_{\mu}^{(B)} + b(b - 1)S_{\mu}^4,$$

(2)

The first term originates from the configurations that require the two directed polymers to meet at $C$, while the second term counts the “no encounter” cases. There are no energy costs at the two end points. The moments of the partition function, from Eq. [3], are

$$Z_{\mu+1}(n) = b^{-n} \sum_{m=0}^{n} P_{nm} Z_{\mu}^2(m),$$

(3)

where $P_{nm} = \binom{n}{m}(b - 1)^{n-m}\overline{y}^m$, with the initial condition $Z_0(n) = 1$ for all the moments because there is no interaction in the zeroth generation (one single bond).

The behaviour of the moments are determined by the stable fixed points of the recursion relation as $\mu \rightarrow \infty$. The first moment has no fixed point for $\overline{y} > y_1 \equiv b^2/[4(b - 1)]$. For $y_1 \equiv \overline{y}/y_1 < 1$, $Z_{\mu}(1)$ reaches a fixed point value $Z^*(1)$ for large $\mu$. For $y_1 > 1$, $f_{\mu}(1)$ approaches a definite limit. For higher moments, the stable fixed points in extended spaces can be determined with a high precision on a computer by just solving a quadratic equation. It is easy to see that, for the stable fixed points of the first $n - 1$ moments, there is again a critical value $y_n$,

$$y_n^{-1} = 4b^{-2n} \sum_{m=0}^{n-1} P_{nm} [Z^*(m)]^2,$$

so that for $y_n \equiv \overline{y}/y_n > 1$ there is no fixed point for the $n$th moment eventhough the lower moments do have so. The stable fixed point, if it is real, can be written as

$$Z^*(n) = b^n(2\overline{y}^{n-1})^{-1} \left[ 1 - (1 - y_n)^{1/2} \right].$$

(4)

The property to be used later is that $2Z^*(n)\overline{y}^nb^{-n} < 1$. 


Numerical analysis: By iterating the recursion relations for given $b$ and $\bar{y}$, we computed the moments of the partition function with large (70 digit) accuracy using Mathematica. Iterations up to 100 generations are done, and for checking, several cases with 300 generations are also considered. For a given $\bar{y}$, there is a critical value $n_c$ so that for $n < n_c$ the moments reach their fixed point value, as shown in the lower inset of Fig 2. This $n_c$ depends on $\bar{y}$ (i.e. temperature), $b$, and the distribution. For $n < n_c$, the moments are in the high temperature phase, as per Eq. 4, because for these $y_n > \bar{y}^{\mu}$. By the same token, for $n > n_c$, the moments are in the low temperature phase.

To analyse the finite size data (finite generations) for $n < n_c$, we adopt the following numerical procedure. First, construct the differences $\Delta_{\mu}(n) \equiv Z_{\mu+1}^{1/n}(n) - Z_\mu^{1/n}(n)$, from Eq. 4, as

$$\Delta_{\mu}(n) = B(n)(1 - 2^{-\psi}) L_\mu^{-\psi},$$

omitting, for simplicity, the subscript $z$. Hence, a log-log plot would give $\psi$ and $B(n)$, provided $\mu$ is sufficiently large. Such an analysis has been done for all the moments for various $\bar{y}$ and $b$. The sample plots of Fig 2 clearly show that the exponent $\psi$ is independent of $n$. It, however, depends on $\bar{y}$. (The upper inset of Fig 2 shows a few sample plots for $[Z_\mu(n)]^{1/n} - Z_\mu^{1/n}(n)$ vs $\mu$ against $B(n)L_\mu^{-\psi}$ with the estimated values.) Fig 3 shows the growth of the amplitude with the moment indicating a divergence as the critical $n_c$ is approached from below. The location of $n_c$ and the exponent $r$, as defined after Eq. 4, can be determined by choosing $r$ such that $B^{-1/r}(n)$ is a straight line with $n$. The intercept gives $n_c$.

A similar analysis is done for $f_\mu(n)$ for $n > n_c$. The exponent $\psi = 1$, and there is no divergence of the amplitude near $n_c$. The finite size correction, from Eq. 3, is $(\ln \bar{y} - n \ln b)L^{-1}$. In this case, the “free energy”, $f(n)$ in the thermodynamic limit vanishes in a singular fashion as $n_c$ is approached. Assuming $f(n) \sim (n - n_c)^\sigma$, $\sigma$ and $n_c$ can be estimated by linearizing the $f^{1/\sigma}(n)$ vs $n$ plot.

The estimates of $n_c$ from the two sides are consistent with each other. For $n < n_c$, see Fig 3, fits over a wide range in $n$ give the exponent $r$ as $0.71 \pm 0.02$ and this value is independent of temperature. We have checked this insensitivity to temperature up to $\ln \bar{y} = 0.005$ or $n_c = 223.53$. There is definitely a curvature near $n_c$, and if only the last three or four points are taken, a lower value for $r$ is obtained. We, however, believe that this or any other more elaborate procedure to estimate $r$ is not warranted because the last few points are still far off from $n_c$. We, therefore, tend to believe the obtained value of $r$ to be an upper limit like a mean field estimate. The free energy exponent $\sigma$ is found to be temperature dependent, i.e., nonuniversal. For example, for $b = 4$, $\sigma$ changes from $1.78 \pm 0.03$ at log $\bar{y} = 0.04$ to $\sigma = 1.9 \pm 0.05$ at log $\bar{y} = 0.08$. What really happens at $n_c$ eludes us because it is not possible to hit an integer $n_c$ numerically. It is plausible that $n \to n_c \pm$, and $n = n_c$ are to be treated separately.

There is a strong dependence of the exponents on $b$ and the distribution. For the gaussian distribution, the variation of $r$ with $b$ is shown in Fig 4. This dependence might be expected because as $b$ is changed the effective dimensionality $2b = 2^d$ changes. Interestingly enough, the exponent reaches a saturation as $b \to 2$, the minimum value of $b$ for a transition.

Linearized renormalization group analysis: To rationalize the results, we now analyze the linearized renormalization group transformation.
For a given $\mathbf{y}$, we linearize the recursion relation, Eq. (3), up to the moment for which the
stable fixed point is reached (i.e., $n < n_c$). Let us start with the situation where the moments
are close to the fixed points, and define a column vector $\mathbf{z}$ of size $n_0$, the integer part of $n_c$,
with $\mathbf{z}(n) = Z_\mu(n) - Z^*(n)$ as the $n$th element. The transformation matrix $\mathbf{R}$ that takes the
vector to a new one $\mathbf{z}' = \mathbf{R}\mathbf{z}$, is lower triangular with elements $2b^{-n}P_{nm}Z^*(m)$, for $m \leq n$, 
and zero otherwise. $\mathbf{R}$ is of size $n_0 \times n_0$.

The eigenvalues of this matrix $\mathbf{R}$ are just the diagonal elements $\lambda_n = 2Z^*(n)y^n b^{-n}$, of
which, in all cases, we find $\lambda_1$ to be the largest. By Eq. (4) $\lambda_n < 1$ for all $n < n_c$. (See Fig 4
for the spectrum for a particular case.) If $\mathbf{e}_n$ is the $n$th eigenvector, then, after $\mu$ iterations,
$\mathbf{z}^{(\mu)} = \sum_k A_k \lambda_k^n \mathbf{e}_k$, where $A_k$ is the projection of the starting vector along $\mathbf{e}_k$. For large $\mu$, 
it follows that the convergence to the fixed point for the $n$th moment is as $\mathcal{B}_n L^{-\psi}_n$, with

$$\psi = -\log_2 \lambda_1 = -\log_2 \left[ 1 - (1 - y_1)^{1/2} \right],$$

for all $n$, and amplitude $\mathcal{B}_n = A_1 e_{1,n}$, where $e_{1,n}$ is the $n$th element of $\mathbf{e}_1$.

There is a universality in the exponent $\psi$ when considered as a function of $y_1 = \mathbf{y}/y_1$. 
The numerical values for various temperatures, $b$, and distributions can be made to collapse 
on this curve. Incidentally, this is the same exponent that a pure system would have, with
attractive interaction, if expressed as $y/y_c$ where $y_c$ is the binding-unbinding transition.

The ratio $K_n$ of $n[Z^*(n)]^{(n-1)/n} B_n/e_{1,n}$ is plotted against $n$ in Fig 5 for the same situation. 
This shows the proportionality of the two. We have ensured that this proportionality is maintained 
for all the cases. Hence, we infer that the divergence of the finite size amplitude is really a consequence of the divergence of the elements of the eigenvector for the largest
eigenvalue.

How could the amplitude diverge? The eigenvalues are shown in Fig 3 for one temperature. 
In all cases studied, we find the rise of the last few eigenvalues approaching from 
below the first (largest) eigenvalue ($\lambda_1$). This leads to an increase in the components of the 
first eigenvector $\mathbf{e}$, whose $i$th component $e_{1,i}$ is of the form

$$(\lambda_1 - \lambda_i)^{-1} \left( 1 + \sum_{j=2}^{i-1} \frac{R_{ij} R_{j1}}{\Lambda_j} + \sum_{j,k} \frac{R_{ij} R_{jk} R_{kl}}{\Lambda_j \Lambda_k} + \ldots \right),$$

where $\Lambda_p = \lambda_1 - \lambda_p$. We, therefore, make the following hypothesis. In an analytic continuation
in $n$, the largest eigenvalue of the transformation operator (not necessarily diagonal anymore) will be degenerate; the eigenvalues of the modes at the end, near $n_c$, will come up and merge with $\lambda_1$. It is this degeneracy that leads to the singular behaviour and produces
a nontrivial exponent $r$.

It is interesting to compare this situation with the problem of a directed polymer in a 
random medium [3,4,7] and the similar randomly interacting directed polymers but now 
interacting on the bonds [8]. In these cases, unlike Eq. (4) the temperature does not enter the recursion relations explicitly but only through the initial condition. For this reason, linearizing around the fixed point, with the associated eigenvalues and vectors, will not produce any strong $n$ dependence, and hence no divergence in the finite size correction. In 
fact, an attempt (not serendipity) to formulate a problem, where the renormalization group 
transformations would contain all the important information, led us to this particular site 
version of the random interaction model.
In the context of the random medium directed polymer problem, in 1+1 dimensions, a scaling form has been proposed for the moments, \( \log Z^n = nf + g(nL^\omega) \), where \( \omega \) is the free energy fluctuation exponent, and \( f \) is the thermodynamic free energy [7]. In this case, the whole phase is the low temperature phase. In our case, for \( b > 2 \), there is a phase transition which is reflected through a nonzero \( n_c \). The power law growth of the amplitude suggests a different scaling form for \( n < n_c \), namely \( \log Z^n = nf + g((n_c - n)L^\Omega) \), where \( \Omega = \psi/r \). It is tempting to speculate that this \( \Omega \) is also the free energy fluctuation exponent. The question of scaling on the other side however remains an open question.

To summarize, we have shown the existence of a diverging finite size effect with a weakly universal exponent, at a critical moment from the high temperature side - a scar left by the disorder. The thermodynamic limit on the high temperature side is identical to the pure system. The growth of the amplitude observed numerically can be understood through a renormalization group argument. However, what controls these exponents remains a puzzling issue. A theory to understand the universality (or its absence) of the exponent \( r \) is lacking. Full significance of the divergence is yet to be elucidated but it cautions that an analytic continuation in \( n \) has to be done with proper care.

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* electronic address: sb@iopb.ernet.in
** electronic address: sutapa@iopb.ernet.in

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FIGURES

FIG. 1. Construction of a hierarchical lattice. (a) This is for b=2. Three generations ($\mu=0,1,2$) are shown. For this paper, identically marked points are taken to have the same interaction energy $\epsilon$. But, $\epsilon$ is random for points with different markers. (b) A more general motif with $2b$ bonds.

FIG. 2. Plot of log $\Delta_\mu(n)$ vs log $L_\mu$ for $n=5$ (curve a), $n=20$ (curve b), and $n=26$ (curve c). These are for $b=4$ and log $\overline{y} = 0.04$. The straight lines are the fits to these log-log plots. The lower inset shows the plot of $f(n)$ vs $n$. The transition is at $n_c = 26.6$. The lower moments, $n < n_c$, are in the high temperature phase while the higher ones are in the low temperature RANI phase. The upper inset compares Eq. 1 with data points for $n = 5$ (curve d) and $n = 26$ (curve e) on a log scale.

FIG. 3. Plot of $[B(n)]^{-1/r}$ vs $n$ for $b = 4$ and (a) log $\overline{y} = 0.065$, and $r = 0.73$, (b) log $\overline{y} = 0.04$, and $r = 0.73$ (c) log $\overline{y} = 0.03$, and $r = 0.72$ (d) log $\overline{y} = 0.02$, and $r = 0.72$. The straight lines are the best fits through the points.

FIG. 4. Dependence of the exponent $r$ on $b$. The line through the points is just a guide to the eye.

FIG. 5. (a) The 26 eigenvalues of $R$ for log $\overline{y} = 0.04$ for which $n_c = 26.6$. Note the log scale along y axis. (b) The horizontal straight line is the ratio $K_n/K_1$. This plot is on a linear scale, shown on the right.