Abstract—This paper studies the stability of some reconstruction algorithms for compressed sensing in terms of the bit precision. Considering the fact that practical digital systems deal with discretized signals, we motivate the importance of the total number of accurate bits needed from the measurement outcomes in addition to the number of measurements. It is shown that if one uses a $2k \times n$ Vandermonde matrix with roots on the unit circle as the measurement matrix, $O(\ell + k \log \frac{n}{\ell})$ bits of precision per measurement are sufficient to reconstruct a $k$-sparse signal $x \in \mathbb{R}^n$ with dynamic range (i.e., the absolute ratio between the largest and the smallest nonzero coefficients) at most $2^\ell$ within $\ell$ bits of precision, hence identifying its correct support. Finally, we obtain an upper bound on the total number of required bits when the measurement matrix satisfies a restricted isometry property, which is in particular the case for random Fourier and Gaussian matrices. For very sparse signals, the upper bound on the number of required bits for Vandermonde matrices is shown to be better than this general upper bound.

I. INTRODUCTION

Compressed sensing is an emerging field which deals with new sampling techniques for sparse signals. The goal is to exploit the sparsity of the signal and try to reconstruct the signal using a number of linear measurements far below the signal’s dimension. Formally, let $x \in \mathbb{R}^n$ be a $k$-sparse vector ($k \ll n$) and $y = Ax$, where $A$ is an $m \times n$ measurement matrix with possibly complex entries, and $y \in \mathbb{C}^m$ is the observed vector. The main problem is to design a measurement matrix with $m \ll n$ for which there exists an efficient reconstruction algorithm that is able to reconstruct any $k$-sparse signal from the measurement $y$. It is easy to check that at least $2k$ measurements are required if all $k$-sparse signals have to be distinguishable. However, it can be shown that $k + 1$ linear measurements with random coefficients are enough to reconstruct almost all signals almost surely [1], [2]. To reconstruct the signal we need to solve the following $\ell_0$ minimization problem

$$(P_0) \quad \min \|\hat{x}\|_0, \quad A\hat{x} = y,$$

which is known to be NP-hard for a generic matrix $A$.

A key observation by Candès et al. [3], [4], [5] and Donoho [6] shows that if matrix $A$ obeys a so-called restricted isometric property (RIP\footnote{This property was originally called uniform uncertainty principle (UUP) by Candès and Tao .}) which essentially requires that any set of up to $k$ columns of $A$ behaves close to an orthonormal system, then the signal can be exactly reconstructed using the following $\ell_1$ minimization program

$$(P_1) \quad \min \|\hat{x}\|_1, \quad A\hat{x} = y.$$

This is an easier problem compared to $\ell_0$ minimization and in particular can be solved in polynomial time using linear programming (LP) techniques. There are families of random matrices which satisfy the RIP with high probability if $m$, the number of rows, is large enough. Two examples of such families are given by random Gaussian measurements and random Fourier measurements. If $A$ is a random matrix with i.i.d. Gaussian entries and $m = O(k \log \frac{n}{\ell})$, or if $A$ is constructed from $m = O(k \log n)$ random rows of the $n \times n$ discrete Fourier transform matrix, then the matrix can be shown to satisfy the RIP with high probability [3]. Even though the results for random matrices hold with high probability, there is no known efficient way to verify if a random matrix satisfies the RIP. This motivates the problem of finding an explicit construction of a measurement matrix $A$ with small number of measurements for which we can solve $(P_0)$ efficiently. It is shown that explicit matrices can be constructed based on group testing techniques [7] as well as expander graphs and randomness extractors [8], [9], [10], [11], [12].

A closer inspection reveals that $(P_0)$ is an analog of the so-called syndrome decoding problem over real or complex numbers, and hence it is natural to expect that known techniques from coding theory might be applicable in compressed sensing. In particular, Akcakaya and Tarokh [13] show that several results known for the Reed-Solomon codes over finite fields can be extended to the field of complex numbers. Therefore, similar coding and decoding algorithms can be used for sensing sparse vectors over the real or complex field. Specifically, they show that it is possible to reconstruct any $k$-sparse vector from only $2k$ measurements, which is the minimum number of measurements one can hope for, using $O(n^2)$ arithmetic operations.

All the above results hold under the assumption that measurements and arithmetic over real numbers are carried out precisely. However, in digital systems we generally cannot deal
with real numbers simply because we would need infinitely many bits to represent a real number. So it is inevitable to resort to truncated representations of real vectors. Thus a natural question to ask is how precise the measurement outcomes need to be so as to be able to reconstruct the original data within a target precision. In Section II, we will show how it becomes important to not only take the total number of measurements into account, but also the precision required from individual measurements. Together, these two quantities give a suitable measure of the amount of information (in bits) that needs to be extracted from the measurements in order to approximate the sparse signal. We will use a simple example to justify the point that if the precision of the measurements is allowed to be sufficiently high, even one measurement is sufficient to reconstruct discrete signals.

The main result of this paper is a bit precision analysis of the syndrome decoding algorithm for Reed-Solomon codes when applied in the context of compressed sensing as a reconstruction algorithm for Vandermonde measurement matrices [13]. The analysis is based on the assumption that the sparse signal is to be reconstructed within a certain chosen precision in the fixed-point model and the additional requirement that the support of the reconstructed vector is the same as that of the original signal. In particular, we show that if the dynamic range of $x$ is at most $2^\ell$ then having each measurement available within $O(\ell + k \log \frac{\ell}{k})$ bits of precision is sufficient to identify $x$ within $\ell$ bits, which is the minimum precision needed to ensure that the smallest nonzero entry of $x$ is not confused with zero. Since we have a total of $2k$ measurements, the total number of bits required from the measurement outcomes is upper bounded by $O(\ell k + k^2 \log \frac{\ell}{k})$.

The rest of the paper is organized as follows. First, in Section II we motivate the total bit precision as a practical measure for assessing the quality of compressed sensing algorithms. Then, in Section III we give a more rigorous definition of the problem that we consider and state our stability theorem for Vandermonde measurements and the syndrome decoding algorithm. Section IV gives the sketch of the proof for the stability theorem and in Section V we will upper bound the total number of bits required from the measurements obtained from matrices satisfying certain restricted isometry properties.

II. THE IMPORTANCE OF BIT PRECISION

The main purpose of this work is to show that while the number of measurements is an important criterion for assessing the quality of a compressed sensing scheme, it is by itself insufficient without considering the precision needed for the reconstruction algorithm to work properly. Indeed, a more favorable approach than simply bounding the number of measurements would be to quantify the total amount of information (in bits) that needs to be extracted from the measurement outcomes so as to enable a reliable reconstruction of the original signal within a pre-specified precision. Intuitively, a single real number can pack an infinite amount of information and for virtually all real world applications, either the signal to be measured is a priori known to be discrete (for example, the output of a sensor measuring the temperature over a long period of time), or is only needed within a certain pre-specified number of accurate bits. For all such cases, a single measurement is in principle capable to carry all the needed information. The following example illustrates this point.

Example 1: Suppose that $A$ is an $m \times n$ binary matrix that allows recovery of $k$-sparse vectors over $\mathbb{F}_2$. Such a matrix can be obtained from a parity check matrix of a binary code with minimum distance at least $k+1$. We now “compress” the matrix $A$ into a vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that any $k$-sparse $x \in \{0,1\}^n$ can be exactly reconstructed from $a \cdot x \in \mathbb{R}$. We define

$$a_i := \sum_{j=1}^{m} A(j, i) \cdot 2^{j(\lfloor \log (k+1) \rfloor)},$$

where $A(j, i)$ denotes the the entry of $A$ at the $j$th row and the $i$th column. This vector simply encodes all the rows of the matrix $A$ by shifting each row by a sufficient amount to prevent any confusion, and a moment’s thought reveals that indeed $x$ can be uniquely reconstructed from $a \cdot x$. However, by a simple counting, the number of rows of $A$ has to be at least $\log \binom{n}{k} = \Omega(k \log(n/k))$, and we need at least $m \times \lceil \log(k+1) \rceil$ bits from $a \cdot x$ to be able to reconstruct $x$. Hence, although the number of measurements is extremely low, the total number of bits that we need to extract from the measurement has to be at least $\Omega(k \log k \log(n/k))$.

In general, a counting argument shows that matrices with entries from a small domain cannot be used to bring down the number of measurements below a certain level. This is captured in the proposition below:

**Proposition 1:** Let $A$ be an $m \times n$ matrix whose entries are integers in range $[-2^\ell, 2^\ell]$. Assume that $A$ can be used for reconstruction of $k$-sparse signals in $\mathbb{R}^n$. Then $m = \Omega \left( \frac{k \log \log(n/k)}{\ell + \log k} \right)$.

**Proof:** The matrix $A$ must be in particular able to distinguish binary $k$-sparse vectors. The number of such vectors is $\binom{n}{k}$. Let $x$ be a $k$-sparse binary vector and $y := Ax$. Each entry of $y$ must be an integer in range $[-2^\ell, 2^\ell]$, and the number of vectors in $\mathbb{R}^n$ satisfying this property is $(2^{2\ell+1}+1)^m$, and this number must be lower bounded by the number of $k$-sparse binary vectors. This gives the desired bound.

The above result explains why the entries of our single-measurement matrix had long binary representations. However, as shown in [13], one can “break” this lower bound using Vandermonde matrices and achieve a total of $2k$ measurements. This special property of Vandermonde matrices is due to the fact that the entries of the matrix cannot be represented by bounded precision numbers and the amount of required precision must necessarily grow to infinity as $n$ gets large. Hence, Vandermonde matrices use large precision in an essential way and it becomes a crucial task to quantitatively analyze
III. Problem Definition and Main Result

We consider the problem of recovering a $k$-sparse signal $x \in \mathbb{R}^n$ from $m \ll n$ discretized observations. Let $A$ be an $m \times n$ matrix with possibly complex entries and $\hat{y} := y + e \in \mathbb{C}^m$, be our observation vector, where $y := Ax$ and $e \in \mathbb{C}^m$ is the truncation noise in the observation. Throughout the paper we consider the fixed-point binary representation of real numbers and define precision as follows.

Definition 1: We say that a vector $\hat{z} \in \mathbb{C}^n$ is an approximation of $z \in \mathbb{C}^n$ within $\ell$ bits of precision (or $\ell$ accurate bits) if $\|\hat{z} - z\|_\infty/\|z\|_\infty < 2^{-\ell}$.

Definition 2: The dynamic range of a nonzero vector $x$ is defined as the ratio $|x_{\text{max}}/x_{\text{min}}|$, where $x_{\text{max}}$ and $x_{\text{min}}$ are the largest and the smallest nonzero entries of $x$ in absolute value, respectively.

The problem is to find the sufficient precision for $y$ such that we can ensure that a signal $x$ with known dynamic range can be recovered with the correct support. This course depends on the matrix $A$ and the reconstruction algorithm. We pick as $A$ a Vandermonde matrix with roots on the unit circle; namely,

$$A := \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{m-1} & \alpha_2^{m-1} & \ldots & \alpha_n^{m-1}
\end{pmatrix},$$

where $\alpha_i := \exp\left(j \frac{2\pi \sqrt{i}}{n}\right)$. This choice of $A$ is motivated by Reed-Solomon codes over finite fields. By the properties of Reed-Solomon codes if the “error pattern” (i.e., the vector $x$) is $k$-sparse and we pick $m = 2k$ then it can be uniquely identified from the measurement outcomes. As noted in [13] this property holds over the complex field as well and therefore if we use a Vandermonde matrix with distinct roots as the sensing matrix, we can exploit an analog of Reed-Solomon decoding algorithm over the complex field to reconstruct the signal. The Reed-Solomon decoding algorithm we use is the so-called syndrome decoding algorithm, where (for the case $e = 0$) the measurements $y_0, y_1, \ldots, y_{m-1}$ are considered as syndromes from which we wish to find the corresponding error pattern (i.e., the vector $x$). The decoding algorithm is as follows. First, we solve the following Toeplitz linear system

$$\begin{pmatrix}
y_0 & y_1 & \ldots & y_k \\
y_1 & y_2 & \ldots & y_{k+1} \\
y_2 & \ldots & \vdots & \vdots \\
y_{k-1} & y_k & \ldots & y_{2k-1}
\end{pmatrix}\begin{pmatrix}h_0 \\ h_1 \\ \vdots \\ h_{k-1}\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ \vdots \\ 0\end{pmatrix},$$

for a nonzero solution, and let $h(x) := h_0 + h_1 x + \ldots + h_k x^k$.

From the theory of Reed-Solomon codes (cf. [14]) we know that $h(x)$ is a multiple of the error locator polynomial $L(x) := \prod_{e \in E} (1 - xa_e)$, where $E \subseteq \{1, 2, \ldots, n\}$ is a set of size at most $k$ containing the error positions (i.e., the support of $x$). Therefore the set of the zeros of $h(x)$ determines a superset of the error positions (and the exact set if $h$ is a nonzero solution with the smallest degree). Having found a superset of error locations with size $k$, we can solve a $k \times k$ system of linear equations to find the actual error values.

In the presence of truncation noise, it is natural to consider the same reconstruction method using the truncated syndrome vector $\hat{y}$, and ask how stable the method is in this situation. The following theorem quantifies the amount of precision of the measurements needed to ensure correct recovery of the support:

**Theorem 1:** If we use a $2k \times n$ Vandermonde matrix with roots on the unit circle and observe the syndromes within $O(\ell + k \log \frac{2}{\ell})$ accurate bits, then we can stably reconstruct any $k$-sparse signal with dynamic range at most $2^\ell$. The reconstructed signal has the same support as the original signal and approximates its nonzero elements within $\ell$ bits of precision.

The proof of this theorem is sketched in Section [IV].

Remark 1: For the fixed point model that we are considering in this work, if the signal has dynamic range $2^\ell$, then we obviously need at least $\ell$ bits of precision in the reconstructed signal to make sure that all the nonzero entries are being recovered with nonzero magnitudes.

Theorem states that if we want to recover the correct support of a signal with dynamic range $2^\ell$, $O(\ell + k \log \frac{2}{\ell})$ bits of precision per measurement is sufficient. Since we have $2k$ measurements, the total number of measurement bits add up to $O(k\ell + k^2 \log \frac{2}{\ell})$. In particular, if we consider binary signals for which $\ell = 0$, then we will require $O(k^2 \log \frac{2}{\ell})$ bits in total. Nevertheless, this bound is $k$ times larger than $O(k \log \frac{2}{\ell})$, the information theoretical lower bound on the number of required bits to identify $k$-sparse binary signals.

IV. Sketch of the Proof of Theorem

In this section we outline the proof of Theorem and omit certain details due to space restrictions. In particular, we focus on the case where the support size of the sparse signal is exactly $k$ or known to the decoder. The proof is done in three steps. First, we find sufficient precision for the $h_i$ so as to be able to detect the positions of nonzero elements, namely the roots of the error locator polynomial. Next, an upper bound is derived on the required precision for the $y_i$ so as to guarantee that the $h_i$ can be solved, with desired precision, from the set of linear equations given in (1). This upper bound depends on the condition number of a $k \times k$ matrix with entries given by the $y_i$, for which an upper bound is derived in the last step.

Step 1: In the presence of noise, $h_0, h_1, \ldots, h_k \in \mathbb{C}$ are noisy and $h(x)$, assuming that $h_0 = 1$, is not necessarily equal to the error locator polynomial $L(x)$. Thus, we need to make sure that the error in the $h_i$ is small enough to allow us to...
reliably find the roots of \( L(x) \). From the choice of \( L(x) = \prod_{e \in E} (1 - xa_e) \), the minimum nonzero magnitude of \( L(x) \) can be bounded as
\[
|L(x)| \geq k^4 \left( \frac{2\pi}{n} \right)^k,
\]
since each \( a_e \) and the evaluation point \( x \) is an \( n \)th root of unity and the quantity \( |1 - xa_e| \) is the length of a chord on the unit circle whose corresponding angle is a distinct multiple of \( 2\pi/n \) (as the \( a_e \) are distinct).

Also, the magnitude of the error in evaluation of \( h(x) \) can be upper bounded by \( k^2 2^{-\ell_h} \) if the \( h_i \) are available within \( \ell_h \) bits of precision. This is because \( L(x) \) has \( k \) monomials with coefficients of magnitude at most \( k \).

Hence, to find the roots of \( L(x) \) correctly, it suffices to have
\[
k^2 2^{-\ell_h} < k! \left( \frac{2\pi}{n} \right)^k,
\]
which can be satisfied by having the coefficient vector of \( h(x) \) within \( \ell_h = O(k \log \frac{\pi}{\epsilon}) \) bits of precision.

**Step 2:** Now that we have a bound on the precision that we need for the \( h_i \), we have to calculate the precision we need for \( y_i \). We will use the following theorem.

**Theorem 2:** [15, Theorem 7.2] Let \( Ax = b \), where \( A \) is a square invertible matrix and \( x \) and \( b \) are vectors, and \( (A + \Delta A)y = b + \Delta b \), where \( \|\Delta A\| \leq \epsilon \| E \| \) and \( \|\Delta b\| \leq \epsilon \| f \| \).

The matrix \( E \) and the vector \( f \) are arbitrary and \( \|\cdot\| \) is any absolute norm. Furthermore, assume that \( \epsilon \| A^{-1}\| \| E \| \leq 1 \).

Then
\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon}{1 - \epsilon \| A^{-1}\| \| E \|} \left( \frac{\|A^{-1}\| \| f \|}{\| x \|} + \| A^{-1}\| \| E \| \right).
\]

By picking \( E := A, f := b \) and \( \|\cdot\| \) as the norm function, we obtain the following corollary:

**Corollary 1:** For a square matrix \( A \) let \( Ax = b \) and \( (A + \Delta A)y = b + \Delta b \), where \( \|\Delta A\| \leq \epsilon \| A \| \) and \( \|\Delta b\| \leq \epsilon \| \Delta b \| \leq \epsilon \|b\| \), and assume that \( \epsilon \kappa_\infty(A) \leq \frac{1}{2} \).

Then
\[
\frac{\|x - y\|}{\|x\|} \leq 4\epsilon \kappa_\infty(A),
\]
where \( \kappa_\infty(A) \) denotes the condition number of the matrix \( A \) with respect to the \( \infty\)-norm.

The corollary states that if we wish to obtain the solution \( \hat{x} \) of a linear system \( Ax = b \) up to \( r \) accurate bits, i.e., \( \|\hat{x} - x\|/\|x\| < 2^{-r} \), it suffices to have \( \|\Delta A\| \leq \epsilon \| A \| \) and \( \|\Delta b\| \leq \epsilon \|b\| \) with \( \epsilon = O(2^{-(r+\log \kappa_\infty(A)))}) \).

From Step 1, we know that we need \( O(k \log \frac{\pi}{\epsilon}) \) bits of precision for the \( h_i \). Now suppose that the decoder receives a perturbed version of the syndrome vector and finds a nonzero solution for the (perturbed) system of linear equations \( (1) \), namely, \( h := (h_0, \ldots, h_k) \). We show that there is a solution \( \hat{h} := (\hat{h}_0, \ldots, \hat{h}_k) \) for the original system \( (1) \) that is sufficiently close to the perturbed solution, i.e., \( \|\hat{h} - h\|/\|h\| < 2^{-\Omega(k \log \frac{\pi}{\epsilon})} \), as required by Step 1.

*Here for the sake of clarity we are neglecting the lower order term \( O(1/n^2) \) in the approximation on the length of the chord, but it should be clear that this will not affect the analysis.*

Denote by \( M \) the coefficient matrix of \( (1) \) (prior to the perturbation of syndromes), and by \( M_i \) the \( k \times k \) minor of \( M \) obtained by removing the column corresponding to \( h_i \). Moreover, define \( y_{\text{max}} \) as the largest syndrome in absolute value so that \( |y_{\text{max}}| = \|y\| \), and note that each \( M_i \) contains all the syndromes but one. It is always possible to set \( h_i = \hat{h}_i \), for some choice of \( i \) such that \( M_i \) contains an entry with magnitude \( |y_{\text{max}}| \), and we can rewrite \( (1) \)

\[
M_i(h_k, \ldots, h_i+1, h_i-1, \ldots, h_0)^\top = -\hat{h}_i(\hat{y}_k, \hat{y}_{k-1}, \ldots, \hat{y}_{k-I-1})^\top. \tag{2}
\]

Here we mention a technicality that the precision of the \( y_i \) is bounded relative to the largest coefficient \( y_{\text{max}} \), which is not necessarily present on the right hand side of the above system. However, we can add an additional "dummy" equation \( y_{\text{max}} \hat{h}_i = \hat{h}_i y_{\text{max}} \) to the system and ensure that the requirements of Corollary \( \Pi \) on the error bounds are fulfilled. It is easy to see that the new system will have condition number at most \( \max\{\kappa_\infty(M_i), k\} \). We can now apply Corollary \( \Pi \) on the system given by \( (2) \) and find a sufficient precision for the \( y_i \). In particular, we conclude that

\[
O(k \log (n/k)) + \log \kappa_\infty(M_i) \tag{3}
\]
bits of precision for the \( y_i \) would be sufficient for finding the \( h_i \) within the precision required by Step 1, and thus, the correct support of the sparse vector.

**Step 3:** For the last step, we find a good upper bound on \( \kappa_\infty(M_i) \). We know that \( \kappa_\infty(M_i) \leq \sqrt{k} \cdot \kappa(M_i) \), where \( \kappa(\cdot) \) denotes the condition number with respect to the \( \ell_2 \)-norm. Thus we equivalently upper bound \( \kappa(M_i) \). It is straightforward to see that \( M_i \) can be decomposed as \( M_i = DV_kX_kV_k^\top \), where \( X_k \) is a \( k \times k \) diagonal matrix containing the nonzero coefficients of the sparse vector \( x \) on its diagonal, \( V_k \) is a \( k \times k \) Vandermonde matrix with roots on the unit circle, and \( D \) is a diagonal (and unitary) \( k \times k \) matrix containing appropriate powers of the \( a_i \). Obviously, \( \kappa(D) = 1 \). Moreover, as the dynamic range of \( x \) is bounded by \( 2^\ell \), we have that \( \kappa(V_k) = |x_{\text{max}}|/|x_{\text{min}}| \leq 2^\ell \). Moreover, we use the following lemmas:

**Lemma 1:** [16] The condition number of any complex \( k \times k \) Vandermonde matrix \( V \) with roots on the unit circle is at most \( \sqrt{2k} \).

**Lemma 2:** If \( Q \) and \( R \) are square matrices with complex entries, then \( \kappa(QR) \leq \kappa(Q)\kappa(R) \).

The second lemma is easy to derive and we omit its proof. Altogether, we conclude that \( \kappa(M_i) \leq k2^{\ell+1} \), which, combined with Step 2, implies that it suffices to have the \( y_i \) within \( O(k \log \frac{\pi}{\epsilon} + \ell) \) bits to correctly reconstruct the support of \( x \).

After we find the correct support, the reconstruction problem is reduced to a \( k \times k \) system of linear equations defined by \( k \) columns of the Vandermonde measurement matrix and the corresponding measurement outcomes. As \( k \) columns of a Vandermonde matrix also form a Vandermonde matrix, by Lemma \( L \) the condition number of the matrix defining the equations is at most \( \sqrt{2k} \). Hence, again using Corollary \( \Pi \).
knowing the measurement outcomes within $O(\log k + \ell)$ bits would be sufficient for obtaining $\ell$ bits of precision in the reconstruction of $x$. However, this number is less than the bound that we derived before for finding the correct support of $x$. This concludes the proof.

V. A GENERAL BOUND ON PRECISION

In the preceding section we obtained a bound on the amount of precision required for the measurements obtained from a Vandermonde measurement matrix for ensuring reliable recovery of the sparse signal using a particular reconstruction algorithm, namely, syndrome decoding. In this section, we consider a similar problem, but for a general class of measurement matrices satisfying a suitable restricted isometry property (RIP) and considering convex optimization as the recovery method. We remark that, as in the case of Vandermonde measurements, our main focus here is on the amount of information that needs to be extracted from the measurement outcomes, and we do not take the imprecision of numerical computations into account. In particular, we assume that the reconstruction algorithm uses an idealized computation model, but receives truncated measurement outcomes on its input.

The bound that we obtain in this section is a direct corollary of a result by Candes et al. on robust recovery of sparse signals from inaccurate measurements [17]. We begin by recalling this result and the required notation. For a complex $m \times n$ matrix $A$ (where $m \leq n$) and positive integer $s$, denote by the $s$-isometry constant $\delta_s$ the infimum over all choices of $\delta$ that satisfy $(1 - \delta)||e||_2^2 \leq ||A'e||_2^2 \leq (1 + \delta)||e||_2^2$, for every $m \times s$ submatrix $A'$ of $A$ and every $e \in \mathbb{C}^s$. The following is the main result proved in [17]:

**Theorem 3:** Suppose that the measurement matrix $A$ satisfies the restricted isometry property that $\delta_{3s} + 3\delta_4 < 2$, for some positive integer $s$. Then for every $\epsilon > 0$, every $s$-sparse signal $x \in \mathbb{C}^n$, and $y := Ax + \epsilon$, $||e||_2 \leq \epsilon$, the solution $\hat{x}$ to the convex program

\[
(P_2) \quad \min ||x'||_1, \quad ||Ax' - y||_2 \leq \epsilon,
\]

satisfies $||\hat{x} - x|| \leq C\epsilon$, where $C$ is a positive constant only depending on $\delta_{3s}$.

As pointed out in [17], the program $(P_2)$ can be solved efficiently using known techniques from convex optimization. Now suppose in the sequel that $A$ is an $m \times n$ matrix satisfying the isometry property needed above, and that we wish to reconstruct an approximation $\hat{x}$ of a $k$-sparse signal $x \in \mathbb{C}^n$ within at least $\ell$ significant bits (in the fixed-point model) from $y := Ax + \epsilon$, where $\epsilon$ is the rounding error, such that $\hat{x}$ has the same support as that of $x$. Obviously, for that to be possible the dynamic range of $x$ must be at most $2^\ell$, as otherwise a nonzero but small coefficient of $x$ might be confused with zero. The following straightforward corollary of Theorem 3 quantifies the amount of precision needed for $y$:

**Corollary 2:** Let $e \in \mathbb{C}^m$ denote the quantization error in $y$. In order to ensure that $||\hat{x} - x||/||x|| < 2^{-\ell}$, it suffices to have

\[
\frac{||e||_{\infty}}{||y||_{\infty}} < \frac{2^{-\ell}}{Ck\sqrt{m}}.
\]

The result states that if we have the measurements within $O(\ell + \log k + \log m) = O(\ell + \log m)$ bits of precision, we can ensure that the program $(P_2)$ obtains a reconstruction that approximates $x$ within $\ell$ bits of precision, which in particular implies correct support recovery of $x$. Hence, the total number of bits needed from the measurements can be upper bounded by $O(m(\ell + \log m))$.

As a concrete example, consider a measurement matrix $A$ that outputs a set of $m$ random Fourier coefficients of the signal. It is shown in [17] that, in order for $A$ to satisfy the RIP needed by Corollary 2 with overwhelming probability, it is sufficient to take $m = O(k(\log n)^4)$. Thus in this case, $O(k(\log n)^4(\ell + \log k + \log \log n))$ bits from the measurement vector $y$ would be sufficient to reconstruct $x$ within precision $\ell$. On the other hand, the upper bound that we obtained for the Vandermonde matrix with syndrome decoding is a total of $O(\ell k + k^2(\log(n/k)))$ bits from $y$. The two bounds are incomparable, but the latter is better for very sparse signals (e.g., $k = O(\log n)$).

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