Discretization of Linear Problems in Banach Spaces: Residual Minimization, Nonlinear Petrov–Galerkin, and Monotone Mixed Methods

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Abstract

This work presents a comprehensive discretization theory for abstract linear operator equations in Banach spaces. The fundamental starting point of the theory is the idea of residual minimization in dual norms, and its inexact version using discrete dual norms. It is shown that this development, in the case of strictly-convex reflexive Banach spaces with strictly-convex dual, gives rise to a class of nonlinear Petrov–Galerkin methods and, equivalently, abstract mixed methods with monotone nonlinearity. Crucial in the formulation of these methods is the (nonlinear) bijective duality map.

Under the Fortin condition, we prove discrete stability of the abstract inexact method, and subsequently carry out a complete error analysis. As part of our analysis, we prove new bounds for best-approximation projectors, which involve constants depending on the geometry of the underlying Banach space. The theory generalizes and extends the classical Petrov–Galerkin method as well as existing residual-minimization approaches, such as the discontinuous Petrov–Galerkin method.

Keywords Operators in Banach spaces · Residual minimization · Petrov–Galerkin discretization · Error analysis · Quasi-optimality · Duality mapping · Best approximation · Geometric constants

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1 Introduction

In the setting of Banach spaces, we consider the abstract problem

\[
\begin{align*}
\text{Find } u \in U : \\
Bu &= f \quad \text{in } V^*,
\end{align*}
\]

where \( U \) and \( V \) are infinite-dimensional Banach spaces and the data \( f \) is a given element in the dual space \( V^* \). The operator \( B : U \to V^* \) is a continuous, bounded-below linear operator, that is, there is a continuity constant \( M_B > 0 \) and bounded-below constant \( \gamma_B > 0 \) such that

\[
\gamma_B \|w\|_U \leq \|Bw\|_{V^*} \leq M_B \|w\|_U, \quad \forall w \in U. \tag{1.2}
\]

Problem (1.1) is equivalent to the variational statement

\[
\langle Bu, v \rangle_{V^*, V} = \langle f, v \rangle_{V^*, V} \quad \forall v \in V;
\]

commonly encountered in the weak formulation of partial differential equations (PDEs), i.e., when \( \langle Bu, v \rangle_{V^*, V} := b(u, v) \) and \( b : U \times V \to \mathbb{R} \) is a bilinear form. Note that the above Banach-space setting allows the consideration of PDEs in non-standard (non-Hilbert) settings suitable for dealing with rough data (e.g., measure-valued sources) and irregular solutions (e.g., in \( W^{1,p} \), \( p \neq 2 \), or in \( BV \)).

A central problem in numerical analysis is to devise a discretization method that, for a given family \( \{U_n\}_{n \in \mathbb{N}} \) of discrete (finite-dimensional) subspaces \( U_n \subset U \), is guaranteed to provide a near-best approximation \( u_n \in U_n \) to the solution \( u \in U \) of (1.1). This means that, for some constant \( C \geq 1 \) independent of \( n \), the approximation \( u_n \) satisfies the error bound

\[
\|u - u_n\|_U \leq C \inf_{w_n \in U_n} \|u - w_n\|_U. \tag{1.3}
\]

A discretization method for which this is true, is said to be quasi-optimal.

It is the purpose of this paper to propose and analyze a new quasi-optimal discretization method for the problem in (1.1) that generalizes and improves upon existing methods.

\footnote{The smallest possible constant \( M_B \) coincides with \( \|B\| := \sup_{w \in U \setminus \{0\}} \|Bw\|_{V^*}/\|w\|_U \), while the largest possible constant \( \gamma_B \) coincides with \( 1/\|B^{-1}\| \), where \( B^{-1} : \text{Im}(B) \to U \).}
1.1 Petrov–Galerkin discretization and residual minimization

A standard method for (1.1) is the Petrov–Galerkin discretization:

\[
\begin{align*}
\text{Find } u_n \in U_n : \\
\quad \langle Bu_n, v_n \rangle_{V^*, V} &= \langle f, v_n \rangle_{V^*, V} \quad \forall v_n \in V_n ,
\end{align*}
\]

with \( V_n \subset V \) a discrete subspace of the same dimension as \( U_n \). It is well-known however, that the fundamental difficulty of (1.4) is stability: One must come up with a test space \( V_n \) that is precisely compatible with \( U_n \) in the sense that they have the same dimension and the discrete inf–sup condition is satisfied; see, e.g., [22, Chapter 2] and [33, Chapter 10]. Incompatible pairs \((U_n, V_n)\) may lead to spurious numerical artifacts and non-convergent approximations.

An alternative method, which is not common, is residual minimization:

\[
\begin{align*}
\text{Find } u_n \in U_n : \\
\quad u_n &= \arg \min_{w_n \in U_n} \| f - Bw_n \|_{V^*} ,
\end{align*}
\]

where the dual norm is given by

\[
\| r \|_{V^*} = \sup_{v \in V \setminus \{0\}} \frac{\langle r, v \rangle_{V^*, V}}{\| v \|_V} , \quad \text{for any } r \in V^* .
\]

The residual-minimization method is appealing for its stability and quasi-optimality without requiring additional conditions. This was proven by Guermond [26], who studied residual minimization abstractly in Banach spaces, and focussed on the case where the residual is in an \( L^p \) space, for \( 1 \leq p < \infty \). If \( V \) is a Hilbert space, residual minimization corresponds to the familiar least-squares minimization method [5]; otherwise it requires the minimization of a convex (non-quadratic) functional.

1.2 Residual minimization in discrete dual norms

Although residual minimization is a quasi-optimal method, an essential complication is that the dual norm (1.6) may be non-computable in practice, because it requires a supremum over \( V \) that may be intractable. This is the case, for example, for the Sobolev space \( V = W_0^{1,p}(\Omega) \), with \( \Omega \subset \mathbb{R}^d \) a bounded \( d \)-dimensional domain, for which the dual is the negative Sobolev space \( V^* = [W_0^{1,p}(\Omega)]^* := W^{-1,q}(\Omega) \) (see, e.g., [1]), where \( p^{-1} + q^{-1} = 1 \). Situations with non-computable dual norms are very common in weak formulations of PDEs and, therefore, such complications can not be neglected.
A natural replacement that makes such dual norms computationally-tractable is obtained by restricting the supremum to discrete subspaces \( V_m \subset V \). This idea leads to the following inexact residual minimization:

\[
\begin{aligned}
\text{Find } u_n \in U_n : \\
\quad u_n = \arg \min_{w_n \in U_n} \| f - Bw_n \|_{(V_m)^*},
\end{aligned}
\]

where the discrete dual norm is now given by

\[
\| r \|_{(V_m)^*} = \sup_{v_m \in V_m \setminus \{0\}} \frac{\langle r, v_m \rangle_{V^*,V}}{\| v_m \|_V}, \quad \text{for any } r \in V^*. \tag{1.8}
\]

Note that a notation with a separate parametrization \((\cdot)_m\) is used to highlight the fact that \( V_m \) need not necessarily be related to \( U_n \).

### 1.3 Main results

The main objective of this work is to present a comprehensive abstract analysis of the inexact residual-minimization method (1.7) in the setting of Banach spaces. As part of our analysis, we also obtain new abstract results for the (exact) residual-minimization method (1.5) when specifically applied in non-Hilbert spaces. We use the remainder of the introduction to announce the main results in this work and discuss their significance.

Most of our results are valid in the case that \( V \) is a reflexive Banach space such that \( V \) and \( V^* \) are strictly convex\(^2\), which we shall refer to as the reflexive smooth setting. The reflexive smooth setting includes Hilbert spaces, but also important non-Hilbert spaces, since \( L^p(\Omega) \) (as well as \( p \)-Sobolev spaces) for \( p \in (1, \infty) \) are reflexive and strictly convex, however not so for \( p = 1 \) and \( p = \infty \) (see [14, Chapter II] and [8, Section 4.3]). We assume this special setting throughout the remainder of Section 1.

Indispensable in our analysis is the duality mapping, which is a well-studied operator in nonlinear functional analysis that can be thought of as the extension to Banach spaces of the well-known Riesz map (which is a Hilbert-space construct). In the reflexive smooth setting, the duality mapping \( J_V : V \to V^* \) is a bijective monotone operator that is nonlinear in the non-Hilbert case. To give a specific example, if \( V = W^{1,p}_0(\Omega) \) then \( J_V \) is a (normalized) \( p \)-Laplace-type operator. We refer to Section 2.1 for details and other relevant properties.

\(^2\)A normed space \( Y \) is said to be strictly convex if, for all \( y_1, y_2 \in Y \) such that \( y_1 \neq y_2 \) and \( \| y_1 \| = \| y_2 \| = 1 \), it holds that \( \| \theta y_1 + (1 - \theta) y_2 \|_Y < 1 \) for all \( \theta \in (0, 1) \), see e.g., [17, 8, 13].
Main result I. Residual minimization: Equivalences and a priori bounds

The first main result in this paper concerns equivalent characterizations of the solution to the exact residual-minimization method (1.5), see Theorem 3.B, as well as novel a priori bounds. In particular, the most important equivalence to (1.5) is given by:

\[
\begin{aligned}
\text{Find } u_n \in U_n : \\
\langle \nu_n, J^{-1}_V(f - Bu_n) \rangle_{V^*, V} = 0 & \quad \forall \nu_n \in BU_n \subset V^*. \\
\end{aligned}
\]

which we refer to as a nonlinear Petrov–Galerkin formulation. Note that statement (1.9) is equal to

\[
\begin{aligned}
\langle Bw_n, J^{-1}_V(f - Bu_n) \rangle_{V^*, V} = 0 & \quad \forall w_n \in U_n. \\
\end{aligned}
\]

In other words, the residual minimizer \( u_n \in U_n \) of (1.5) can be obtained by solving the nonlinear Petrov–Galerkin problem (1.9) for \( u_n \in U_n \), and vice versa. Owing to the equivalence, the known stability and quasi-optimality results for residual minimization (1.5) transfer to the nonlinear Petrov–Galerkin discretization (1.9).

Let us point out that the non-computable supremum norm in residual minimization translates into a non-tractable duality-map inverse \( J^{-1}_V \) in (1.9).

By introducing the auxiliary variable \( r = J^{-1}_V(f - Bu_n) \in V \) (a residual representer), one arrives at a semi-infinite mixed formulation with monotone nonlinearity, for simplicity referred to as a monotone mixed formulation:

\[
\begin{aligned}
\text{Find } (r, u_n) \in V \times U_n : \\
J_V(r) + Bu_n = f & \quad \text{in } V^*, \\
\langle Bw_n, r \rangle_{V^*, V} = 0 & \quad \forall w_n \in U_n. \\
\end{aligned}
\]

This formulation, in turn, is equivalent to a constrained-minimization formulation (i.e., a semi-infinite saddle-point problem) involving the Lagrangian \( (v, w_n) \mapsto \frac{1}{2}\|v\|_V^2 - \langle f, v \rangle_{V^*, V} + \langle Bw_n, v \rangle : V \times U_n \to \mathbb{R} \). See Section 3 for details.

In the setting of Hilbert spaces, \( J_V \) coincides with the Riesz map \( R_V : V \to V^* \), and (1.10) reduces to (recall \( R^{-1}_V \) is self-adjoint):

\[
\begin{aligned}
\langle f - Bu_n, R^{-1}_V Bw_n \rangle_{V^*, V} = 0 & \quad \forall w_n \in U_n. \\
\end{aligned}
\]

This coincides with a Petrov–Galerkin method (1.4) with the optimal, but intractable, test space \( V_n = R^{-1}_V BU_n \). Methods that aim to approximately compute this optimal \( V_n \) have received renewed interest since 2010, starting from a pioneering sequence of papers by Demkowicz & Gopalakrishnan on the so-called discontinuous Petrov–Galerkin (DPG) method; see, e.g., [18, 19] and the overviews in [20, 24]. In the Hilbert-space setting, the connection between (1.12) and residual minimization was clarified first in [19], while the connection with the mixed
formulation (1.11) (with $R_V$ instead of $J_V$) was obtained by Dahmen et al [16]. A connection with the variational multiscale framework has also been made [15, 11].

In our brief review of residual minimization, we rely on theory of best approximation in Banach spaces; see Theorem 3.A. Within this context, we prove two novel a priori bounds for abstract best approximations (hence also for residual minimizers). While the classical statement, $\|y_0\|_Y \leq \tilde{C}\|y\|_Y$ with $\tilde{C} = 2$, is valid for best approximations $y_0$ to $y$ within any Banach space $Y$, this result can be sharpened in special Banach spaces. In our first improvement (see Proposition 2.11), we prove that the constant $\tilde{C}$ can be taken as the Banach–Mazur constant $C_{BM}(Y) \in [1, 2]$ of the underlying Banach space $Y$. The Banach–Mazur constant is an example of a so-called geometrical constant that quantifies how “close” a Banach space is to being Hilbert, and this particular geometrical constant was recently introduced by Stern [34] to sharpen the a priori error estimate for the Petrov–Galerkin discretization (1.4). In our second improvement (see Proposition 2.19), we prove that $\tilde{C}$ can also be taken as $1 + C_{AO}(Y)$, where $C_{AO}(Y) \in [0, 1]$ is a newly introduced constant that we refer to as the asymmetric-orthogonality constant of $Y$.

Main result II. Inexact method: Stability and quasi-optimality

The second main result in this paper is the complete analysis of the inexact residual minimization method (1.7). To carry out the analysis, we first present equivalent characterizations of the solution of the inexact method, see Theorem 4.A. These characterizations are the fully discrete versions of the above-mentioned characterizations for exact residual minimization. In particular, the inexact nonlinear Petrov–Galerkin method corresponds to (1.9) with $J_V^{-1}$ replaced by $I_m J_V^{-1} \circ I_m^*$, where $J_V^{-1}$ is the inverse of the duality map in $V_m$, and $I_m : V_m \to V$ is the natural injection. The computationally most-insightful equivalence is the one corresponding to the mixed formulation (1.11), and we refer to the resulting discretization as a monotone mixed method:

\[
\begin{align*}
\text{Find } (r_m, u_n) &\in V_m \times U_n : \\
\left\langle J_V(r_m), v_m \right\rangle_{V^*, V} + \left\langle Bu_n, v_m \right\rangle_{V^*, V} = \left\langle f, v_m \right\rangle_{V^*, V} &\quad \forall v_m \in V_m, \quad (1.13a) \\
\left\langle B^* r_m, w_n \right\rangle_{U^*, U} = 0 &\quad \forall w_n \in U_n, \quad (1.13b)
\end{align*}
\]

where the auxiliary variable $r_m$ is a discrete residual representer.

In the analysis of the stability and quasi-optimality of the inexact method, some compatibility is demanded on the pair $(U_n, V_m)$. This compatibility is stated in terms of Fortin’s condition (involving a Fortin operator $\Pi : V \to V_m$, see Assumption 4.1), which is essentially a discrete inf–sup condition on $(U_n, V_m)$.

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For an alternative approach to the analysis of residual minimization, see Guermond [26, Theorem 2.1].
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(Note that this compatibility requirement is less stringent than the one required for the Petrov–Galerkin discretization since the dimensions of $U_n$ and $V_m$ can be distinct.) Under Fortin’s condition, we prove the unique existence of the pair $(r_m, u_n) \in V_m \times U_n$ solving (1.13) and its continuous dependence on the data, see Theorem 4B. Then, we prove a corresponding a posteriori error estimate, see Theorem 4C, the result of which happens to coincide with the result in the Hilbert case [10, 24]. A straightforward consequence (see Corollary 4.4) is then the quasi-optimal error estimate (1.3) with

$$ C = \frac{(D_\Pi + C_\Pi)M_B}{\gamma_B}, \quad (1.14) $$

where $D_\Pi$ and $C_\Pi$ are boundedness constants appearing in Fortin’s condition, and $M_B$ and $\gamma_B$ are the continuity and bounded-below constants of $B$.

The importance of Fortin’s condition in the analysis of the inexact method was recognized first by Gopalakrishnan and Qiu [25], who studied the inexact optimal Petrov–Galerkin method in Hilbert spaces. In this setting, Fortin’s condition implies that $R_{V_m}^{-1}B U_n$ is a near-optimal test space that is sufficiently close to the optimal one $R_{V}^{-1}B U_n$ (cf. [9, Proposition 2.5]). The fact that near-optimal test spaces imply quasi-optimality was established by Dahmen et al.; see [16, Section 3]. Let us point out however, that the concept of an optimal test space is completely absent in our Banach-space theory, and seems to apply to Hilbert spaces only.

Although the result in (1.14) demonstrates quasi-optimality for the inexact method in Banach-space settings, the constant in (1.14) does not reduce to the known result, $C = C_\Pi M_B / \gamma_B$, when restricting to Hilbert-space settings [25, Theorem 2.1]. To resolve the discrepancy, we improve the constant in (1.14) by including the dependence on the geometry of the involved Banach spaces; see Theorem 4D. The proof of this sharper estimate is nontrivial, as it requires a suitable extension of a Hilbert-space technique due to Xu and Zikatanov [37] involving the classical identity $\|I - P\| = \|P\|$ for Hilbert-space projectors $P$, which is generally attributed to Kato [28] (cf. [35]). A key idea is the recent extension $\|I - P\| \leq C_S \|P\|$ for Banach-space projectors by Stern [34], where $C_S$ depends on the Banach–Mazur constant, however, since that extension applies to linear projectors, we generalize Stern’s result to a suitable class of nonlinear projectors (see Lemma 2.9). Combined with an a priori bound for the inexact residual minimizer involving the asymmetric-orthogonality constant, we then prove that the constant in the quasi-optimal error estimate (1.14) can be improved to

$$ C = \min \left\{ \frac{C_\Pi}{\gamma_B} (1 + C_{AO}(V)) M_B C_{BM}(U), 1 + \frac{C_\Pi}{\gamma_B} (1 + C_{AO}(V)) M_B \right\}, $$

which is consistent with the Hilbert-space result $C = C_\Pi M_B / \gamma_B$ since in that case $C_{AO}(V) = 0$ and $C_{BM}(U) = 1$. 

1.4 Discussion: Unifying aspects

Let us emphasize that the above quasi-optimality theory for the inexact method generalizes existing theories for other methods that are in some sense contained within the inexact method, and therefore it provides a unification of these theories. In particular, the theory generalizes Babuška’s theory for Petrov–Galerkin methods [3], Guermond’s theory for exact residual minimization [26], and the Hilbert-space theory for inexact residual minimization (including the DPG method) [25, 16]. For a schematic hierarchy with these connections and its detailed discussion, we refer to Section 5.

1.5 Outline of paper

The remainder of the paper is organized as follows.
– Section 2 is devoted to necessary preliminaries on the duality mapping and abstract theory of best approximation in Banach spaces.
– Section 3 is dedicated to residual minimization and its characterization via the duality mapping giving rise to several equivalences, including what is referred to as the nonlinear Petrov-Galerkin method and monotone mixed formulation.
– Section 4 analyzes the tractable inexact method. We establish the equivalence of the inexact residual minimization, inexact nonlinear Petrov–Galerkin and monotone mixed method. We then study the stability of this method and perform a comprehensive error analysis.
– Finally, Section 5 reviews the connections to other existing methods (standard Petrov–Galerkin, exact residual minimization and the inexact method in Hilbert-space settings), and points out how the presented quasi-optimality analysis applies in each situation.

2 Preliminaries: Duality mappings and best approximation

In this section we briefly review some relevant theory in the classical subject of duality mappings, and elementary results from best approximation theory in Banach spaces. These preliminaries are required for our analysis of (inexact) residual-minimization problems.

2.1 The duality mapping

An extensive treatment on duality mappings can be found in Cioranescu [14]. Other relevant treatments in the context of nonlinear functional analysis are by Brezis [8, Chapter 1], Deimling [17, Section 12], Chidume [12, Chapter 3] and
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Zeidler [38, Chapter 32.3d], while an early treatment on duality mappings is by Lions [29, Chapter 2, Section 2.2]. We recall results for duality mappings that will be useful for the characterization of best approximations and residual minimizers.

**Definition 2.1 (Duality mapping)** Let $\mathcal{Y}$ be a normed vector space. The multivalued mapping $J:\mathcal{Y}\to 2^{\mathcal{Y}^*}$ defined by
\[
J(y) := \left\{ y^* \in \mathcal{Y}^* : \langle y^*, y \rangle_{\mathcal{Y}^*, \mathcal{Y}} = \|y\|_{\mathcal{Y}}^2 = \|y^*\|_{\mathcal{Y}^*}^2 \right\},
\]
is the duality mapping on $\mathcal{Y}$.

By the Hahn-Banach extension Theorem (see, e.g., [8, Corollary 1.3]), the set $J(y) \subset \mathcal{Y}^*$ is non-empty for every $y \in \mathcal{Y}$. Some basic properties of $J$ are summarized in the following.

**Proposition 2.2 (Duality mapping)** Let $\mathcal{Y}$ be a normed vector space and $y \in \mathcal{Y}$.

(i) The set $J(y) \subset \mathcal{Y}^*$ is bounded, convex, and closed.

(ii) The duality mapping $J$ is homogeneous, and it is monotone in the sense that:
\[
\langle y^* - z^*, y - z \rangle_{\mathcal{Y}^*, \mathcal{Y}} \geq (\|y\|_{\mathcal{Y}} - \|z\|_{\mathcal{Y}})^2 \geq 0,
\]
for all $y, z \in \mathcal{Y}$, for all $y^* \in J(y)$ and for all $z^* \in J(z)$.

(iii) For any $y^* \in J(y)$, its norm supremum is achieved by $y$, i.e.,
\[
\sup_{z \in \mathcal{Y}} \frac{\langle y^*, z \rangle_{\mathcal{Y}^*, \mathcal{Y}}}{\|z\|_{\mathcal{Y}}} = \frac{\langle y^*, y \rangle_{\mathcal{Y}^*, \mathcal{Y}}}{\|y\|_{\mathcal{Y}}}.
\]  

\[\text{(2.1)}\]

**Proof** These results are classical; see, e.g., [8, Chapter 1].

We next list important properties of the duality mapping $J : \mathcal{Y} \to 2^{\mathcal{Y}^*}$ in special Banach spaces.

**2.1.1 Strict convexity of $\mathcal{Y}^*$**

The space $\mathcal{Y}^*$ is strictly convex if and only if $J : \mathcal{Y} \to 2^{\mathcal{Y}^*}$ is a single-valued map; see [17, Proposition 12.3]. In that case we use the notation:
\[
J_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}^*, \quad \text{in other words,} \quad J_{\mathcal{Y}}(y) = \{J_{\mathcal{Y}}(y)\}.
\]

Furthermore, if $\mathcal{Y}^*$ is strictly convex, then $J_{\mathcal{Y}} : \mathcal{Y} \to 2^{\mathcal{Y}^*}$ is hemi-continuous [17, Section 12.3]:
\[
J_{\mathcal{Y}}(y + \lambda z) \to J_{\mathcal{Y}}(y) \quad \text{as} \ \lambda \to 0^+, \quad \forall y, z \in \mathcal{Y}.
\]  

\[\text{(2.2)}\]
Another important property is concerned with the duality map on subspaces. We state this as the following Lemma, and we include a proof since we could not find this result in the existing literature.

**Lemma 2.3 (Duality map on a subspace)** Let \( Y \) be a Banach space, \( Y^* \) strictly convex, and \( J_Y : Y \to Y^* \) denote the duality map on \( Y \). Let \( M \subset Y \) denote a linear subspace of \( Y \), and \( J_M : M \to M^* \) denote the corresponding duality map on \( M \). Then,

\[
I^*_M J_Y \circ I_M = J_M,
\]

where \( I_M : M \to Y \) is the natural injection. \( \square \)

**Proof** Let \( z \in M \) and consider the linear and continuous functional \( J_M(z) \in M^* \). Using the Hahn–Banach extension (see [8, Corollary 1.2]), we extend this functional to an element \( \widehat{J_M(z)} \in Y^* \) such that

\[
\| \widehat{J_M(z)} \|_{Y^*} = \| J_M(z) \|_{M^*}^2.
\]

Observe that the extension satisfies

\[
\| J_M(z) \|_{Y^*} = \| I_M z \|_Y \quad \text{and} \quad \langle \widehat{J_M(z)}, I_M z \rangle_{Y^*}, Y = \langle J_M(z), z \rangle_{M^*}, M = \| I_M z \|_Y^2.
\]

So, as a matter of fact, \( \widehat{J_M(z)} = J_Y(I_M z) \). Therefore, by the extension property of \( J_M(z) \) we obtain

\[
I^*_M J_Y(I_M z) = I^*_M \widehat{J_M(z)} = J_M(z).
\]

\( \blacksquare \)

### 2.1.2 Strict convexity of \( Y \)

If \( Y \) is strictly convex, then \( J_Y \) is *strictly monotone*, that is:

\[
\langle y^* - z^*, y - z \rangle_{Y^*}, Y > 0, \quad \text{for all } y \neq z, \text{ any } y^* \in J_Y(y) \text{ and } z^* \in J_Y(z). \quad (2.3)
\]

Furthermore, \( J_Y : Y \to 2^{Y^*} \) is *injective*. In fact, if \( y \) and \( z \) are two distinct points in \( Y \), then \( J_Y(y) \cap J_Y(z) = \emptyset \) (otherwise (2.3) would be contradicted). It is known that the converse holds as well: Strict monotonicity of \( J_Y \) implies strict convexity of \( Y \), a result due to Petryshyn [31].

### 2.1.3 Reflexivity of \( Y \)

The space \( Y \) is a reflexive Banach space if and only if \( J_Y : Y \to 2^{Y^*} \) is surjective; see [17, Theorem 12.3]. This is meant in the following sense: Every \( y^* \in Y^* \) belongs to a set \( J_Y(y) \), for some \( y \in Y \).

---

\(^4\)In fact, the Hahn–Banach extension is unique on account of strict convexity of \( Y^* \).
2.1.4 Reflexive smooth setting

An important case in our study is when the Banach space $\mathbb{Y}$ has all the previously listed properties, i.e., $\mathbb{Y}$ and $\mathbb{Y}^*$ are strictly convex and reflexive Banach spaces, referred to as the reflexive smooth setting. Two important straightforward consequences need to be remarked in this situation:

(i) The duality maps $J_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^*$ and $J_{\mathbb{Y}^*} : \mathbb{Y}^* \to \mathbb{Y}^{**}$ are bijective.

(ii) $J_{\mathbb{Y}^*} = \mathcal{I}_{\mathbb{Y}} \circ J_{\mathbb{Y}}^{-1}$, where $\mathcal{I}_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^{**}$ is the canonical injection. Shortly, $J_{\mathbb{Y}^*} = J_{\mathbb{Y}}^{-1}$, by means of canonical identification.

2.1.5 Subdifferential property

A key result is that the duality mapping coincides with a subdifferential. Recall that for a Banach space $\mathbb{Y}$ and function $f : \mathbb{Y} \to \mathbb{R}$, the subdifferential $\partial f(\cdot)$ of $f$ at a point $y \in \mathbb{Y}$ is defined as the set:

$$\partial f(y) := \left\{ y^* \in \mathbb{Y}^* : f(z) - f(y) \geq \langle y^*, z - y \rangle_{\mathbb{Y}^*, \mathbb{Y}}, \forall z \in \mathbb{Y} \right\}.$$

Proposition 2.4 (Duality mapping is a subdifferential) Let $f_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{R}$ be defined by $f_{\mathbb{Y}}(\cdot) := \frac{1}{2} \| \cdot \|_{\mathbb{Y}}^2$. Then, for any $y \in \mathbb{Y}$, $J_{\mathbb{Y}}(y) = \partial f_{\mathbb{Y}}(y)$. □

Proof See, e.g., Asplund [2] or Cioranescu [14, p. 26]. ■

Remark 2.5 (Gâteaux gradient) The subdifferential of $f_{\mathbb{Y}}(\cdot) = \frac{1}{2} \| \cdot \|_{\mathbb{Y}}^2$ contains exactly one point if $\mathbb{Y}^*$ is strictly convex (recall from Section 2.1.1). In that case, $f_{\mathbb{Y}}$ is Gâteaux differentiable with Gâteaux gradient $\nabla f_{\mathbb{Y}}(\cdot)$, and for any $y \in \mathbb{Y}$ we have (see e.g. [14, Corollary 2.7]):

$$J_{\mathbb{Y}}(y) = \nabla f_{\mathbb{Y}}(y).$$

□

Example 2.6 (The $L^p$ case) We recall here an explicit formula for the duality map in the Banach space $L^p(\Omega)$ where $\Omega \subset \mathbb{R}^d$, $d \geq 1$. For $p \in (1, +\infty)$ the space $L^p$ is reflexive and strictly convex (as well as the dual space $L^q$, where $q = \frac{p}{p-1}$);

5 If $\mathbb{Y}^*$ is uniformly convex, then $f_{\mathbb{Y}}$ is Fréchet differentiable and the duality map $J_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^*$ is uniformly continuous on the unit sphere of $\mathbb{Y}$. Moreover, by the Milman–Pettis Theorem [8, Section 3.7], uniform convexity of $\mathbb{Y}^*$ implies reflexivity of $\mathbb{Y}^*$ (hence, reflexivity of $\mathbb{Y}$). Note however that a strictly convex space is not necessarily uniformly convex, see [17, Section 12.1] for an example.
see e.g. [14, Chapter II] and [8, Section 4.3]. For \( v \in L^p(\Omega) \) the duality map is defined by the action:

\[
\langle J_{L^p(\Omega)}(v), w \rangle_{L^q(\Omega), L^p(\Omega)} := \|v\|_{L^p(\Omega)}^{2-p} \int_{\Omega} |v|^{p-1} \text{sign}(v) \, w, \quad \forall w \in L^p(\Omega), \tag{2.4}
\]

which can shown by computing the Gâteaux derivative of \( v \mapsto \frac{1}{2}(\int_{\Omega} |v|^p)^{2/p} \), or by verifying the identities in Definition 2.1. In the case \( p = 1 \), the formula in the right-hand side of (2.4) also works and defines an element in the set \( J_{L^1(\Omega)}(v) \). Note however that \( L^1 \) is not a special Banach space as discussed above.

\[\Box\]

2.2 Best approximation in Banach spaces

We now consider theory of best approximation. First we recall classical results on existence, uniqueness and characterization. Then, based on geometrical constants of the underlying Banach space, we develop two novel a priori bounds for best approximations (Propositions 2.11 and 2.19), which are of independent interest.

2.2.1 Existence, uniqueness and characterization

Best approximation in Banach spaces is founded on the following classical result.

**Theorem 2.A (Best approximation)** Let \( \mathbb{Y} \) be a Banach space, and \( y \in \mathbb{Y} \).

(i) Suppose \( \mathbb{M} \subset \mathbb{Y} \) is a finite-dimensional subspace, then there exists a best approximation \( y_0 \in \mathbb{M} \) to \( y \) such that

\[
\|y - y_0\|_{\mathbb{Y}} = \min_{z_0 \in \mathbb{M}} \|y - z_0\|_{\mathbb{Y}}.
\]

(ii) Suppose \( \mathbb{M} \subset \mathbb{Y} \) is any subspace and \( \mathbb{Y} \) is strictly convex, then a best approximation \( y_0 \in \mathbb{M} \) to \( y \) is unique.

(iii) Suppose \( \mathbb{M} \subset \mathbb{Y} \) is a closed subspace, then the following statements are equivalent:

- \( y_0 = \arg \min_{z_0 \in \mathbb{M}} \|y - z_0\|_{\mathbb{Y}}. \)
- There exists a functional \( y^* \in J_{\mathbb{Y}}(y - y_0) \) which annihilates \( \mathbb{M} \), i.e.,

\[
\langle y^*, z_0 \rangle_{\mathbb{Y}^*, \mathbb{Y}} = 0, \quad \forall z_0 \in \mathbb{M}, \quad \text{where} \quad J_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^* \quad \text{is the duality mapping defined in Definition 2.1}. \]

\[\Box\]

**Proof** For parts (i) and (ii) see, e.g., Stakgold & Holst [33, Section 10.2] or DeVore & Lorentz [21, Chapter 3]. For part (iii) in case of \( y \in \mathbb{Y} \setminus \mathbb{M} \) see, e.g., Singer [32] or Braess [7]. The case of \( y \in \mathbb{M} \) is trivial, because in that case \( y_0 = y \) and one can choose \( y^* = 0 \). \[\blacksquare\]
2.2.2 Banach–Mazur constant and nonlinear projector estimate

To describe the first of two novel a priori bounds for best approximations, we recall the Banach–Mazur constant [34, Definition 2]. The Banach–Mazur constant is based on the classical Banach–Mazur distance, whose motivation is best described by Banach himself [4, p. 189]:

“A Banach space $X$ is isometrically isomorphic to a Hilbert space if and only if every two-dimensional subspace of $X$ is isometric to a Hilbert space.”

Definition 2.7 (Banach–Mazur constant) Let $Y$ be a normed vector space with $\dim Y \geq 2$, and let $\ell_2(\mathbb{R}^2)$ be the 2-D Euclidean space endowed with the 2-norm. The Banach–Mazur constant of $Y$ is defined by

$$C_{BM}(Y) := \sup \left\{ \left( d_{BM}(W, \ell_2(\mathbb{R}^2)) \right)^2 : W \subset Y, \dim W = 2 \right\},$$

where $d_{BM}(\cdot, \cdot)$ is the (multiplicative) Banach–Mazur distance:

$$d_{BM}(W, \ell_2(\mathbb{R}^2)) := \inf \left\{ \|T\| \|T^{-1}\| : T \text{ is a linear isomorphism}^{6} W \to \ell_2(\mathbb{R}^2) \right\}. \square$$

Since the definition only makes sense when $\dim Y \geq 2$, henceforth, whenever $C_{BM}(\cdot)$ is written, we assume this to be the case. (Note that $\dim Y = 1$ is often an uninteresting trivial situation.)

Remark 2.8 (Elementary properties of $C_{BM}$) It is known that $1 \leq C_{BM}(Y) \leq 2$, $C_{BM}(Y) = 1$ if and only if $Y$ is a Hilbert space, and $C_{BM}(Y) = 2$ if $Y$ is non-reflexive; see [34, Section 3]. In particular, for $Y = \ell_p(\mathbb{R}^2)$, $C_{BM}(Y) = 2^{\frac{2}{p}-1}$; cf. [36, Section II.E.8] and [27, Section 8]. This result is also true for $L^p$ and Sobolev spaces $W^{k,p}$ ($k \in \mathbb{N}$), see [34, Section 5]. \square

The Banach–Mazur constant is used in the Lemma below to state a fundamental estimate for an abstract nonlinear projector. This nonlinear projector estimate, which is an extension of Kato’s identity $\|I - P\| = \|P\|$ for Hilbert-space projectors [28], and a generalization of the estimate obtained by Stern [34, Theorem 3] for linear Banach-space projectors, will be used to prove the a priori bound in Proposition 2.11 and also Corollary 4.9 in Section 4.

Lemma 2.9 (Nonlinear projector estimate) Let $Y$ be a normed space, $I : Y \to Y$ the identity and $Q : Y \to Y$ a nonlinear operator such that:

(i) $Q$ is a nontrivial projector: $0 \neq Q = Q \circ Q \neq I$.

(ii) $Q$ is homogeneous: $Q(\lambda y) = \lambda Q(y), \quad \forall y \in Y$ and $\forall \lambda \in \mathbb{R}$.\[6^{i.e.}, T \text{ is a linear bounded bijective operator.}
(iii) $Q$ is bounded in the sense that $\|Q\| := \sup_{y \in Y \setminus \{0\}} \frac{\|Q(y)\|_Y}{\|y\|_Y} < +\infty$.

(iv) $Q$ is a generalized orthogonal projector in the sense that

$$Q(y) = Q(Q(y) + \eta(I - Q)(y)),$$

for any $\eta \in \mathbb{R}$ and any $y \in Y$.

Then the nonlinear operator $I - Q$ is also bounded and satisfies

$$\|I - Q\| \leq C_S \|Q\|,$$

where $C_S$ is the constant introduced by Stern [34]:

$$C_S = \min \left\{ 1 + \|Q\|^{-1}, C_{BM}(Y) \right\}. \quad (2.5)$$

Proof The proof of this result follows closely Stern [34, Proof of Theorem 3]. Although Stern considers linear projectors, his result generalizes to projectors with the properties in (i)–(iv). See Section 2.3 for the complete proof.

Remark 2.10 (Generalized orthogonal projectors) Requirement (iv) in Lemma 2.9 is a key nonlinear property. We point out that it is satisfied by linear projectors, by best-approximation projectors, by $I$ minus best-approximation projectors (as in the proof of Proposition 2.11), and by (inexact) nonlinear Petrov-Galerkin projectors $P_n$ of Definition 4.7 (see Corollary 4.9).

2.2.3 A priori bound I

The first a priori bound for best approximations is obtained by applying Lemma 2.9.

Proposition 2.11 (Best approximation: A priori bound I) Let $Y$ be a Banach space and $M \subset Y$ a closed subspace. Suppose $y_0 \in M$ is a best approximation in $M$ of a given $y \in Y$ (i.e., $\|y - y_0\|_Y \leq \|y - z_0\|_Y$, for all $z_0 \in M$), then $y_0$ satisfies the a priori bound:

$$\|y_0\|_Y \leq C_{BM}(Y)\|y\|_Y,$$

where $C_{BM}(Y)$ is the Banach-Mazur constant of the space $Y$ (see Definition 2.7).

Proof We assume that $M \neq \{0\}$ and $M \neq Y$ (otherwise the result is trivial). Consider a (nonlinear) map $P^\perp : Y \to Y$ such that $P^\perp(y) = y - y_0$, where $y_0 \in M$ is a best approximation to $y \in Y$. The map $P^\perp$ can be chosen in a homogeneous way, i.e., satisfying $\lambda P^\perp(y) = P^\perp(\lambda y)$ for any $\lambda \in \mathbb{R}$. Observe that

$$\|P^\perp(y)\|_Y = \|y - y_0\|_Y \leq \|y - 0\|_Y = \|y\|_Y.$$
Hence, $P^\perp$ is bounded with $\|P^\perp\| \leq 1$. Additionally, it can be verified that $P^\perp(P^\perp(y)) = y - y_0 - 0 = P^\perp(y)$. Thus, $Q = P^\perp$ satisfies the requirements (i), (ii) and (iii) of Lemma 2.9 with $\|P^\perp\| = 1$. To verify requirement (iv), notice that for any $\eta \in \mathbb{R}$,

$$P^\perp \left( P^\perp(y) + \eta (I - P^\perp)(y) \right) = P^\perp \left( y - y_0 + \eta y_0 \right) = y - y_0,$$

since $\eta y_0$ is a best approximation in $\mathbb{M}$ to $y - y_0 + \eta y_0$. Therefore, by Lemma 2.9 we get:

$$\|y_0\|_Y = \|(I - P^\perp)y\|_Y \leq \min \left\{ 1 + \|P^\perp\|^{-1}, C_{\text{BM}}(Y) \right\} \|P^\perp\| \|y\|_Y,$$

and (2.6) follows since $\|P^\perp\| = 1$ and $C_{\text{BM}}(Y) \leq 2$. □

**Remark 2.12 (Sharpness of (2.6))** Bound (2.6) improves the classical bound $\|y_0\|_Y \leq 2 \|y\|_Y$ (see, e.g., [33, Sec. 10.2]), in the sense that it shows an explicit dependence on the geometry of the underlying Banach space. In particular, (2.6) contains the standard result $\|y_0\|_Y \leq \|y\|_Y$ for a Hilbert space, as well as the classical bound $\|y_0\|_Y \leq 2 \|y\|_Y$ for non-reflexive spaces such as $\ell_1(\mathbb{R}^2)$ and $\ell_\infty(\mathbb{R}^2)$ (for which the bound is indeed sharp; see Example 2.13). However, (2.6) need not be sharp for intermediate spaces; see Example 2.16. □

**Example 2.13 ($\ell_1(\mathbb{R}^2)$)** In $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, i.e. $Y = \ell_1(\mathbb{R}^2)$, the best approximation of the point $(1,0)$ over the line $\{(t,t) : t \in \mathbb{R}\}$ is the whole segment $\{(t,t) : t \in [0,1]\}$. Moreover, the point $(1,1)$ is a best approximation and $\|(1,1)\|_1 = 2 = 2 \|(0,1)\|_1$. Since the Banach–Mazur constant for this case equals 2, Eq. (2.6) is sharp for this example. □

### 2.2.4 Asymmetric-orthogonality constant

We now construct an alternative a priori bound for best approximations (compare with Proposition 2.11). This bound is also a novel result, which is of independent interest. The describe the bound, we introduce the following new geometric constant.

**Definition 2.14 (Asymmetric-orthogonality constant)** Let $Y$ be a normed vector space with $\dim Y \geq 2$. The asymmetric-orthogonality constant is defined by:

$$C_{\text{AO}}(Y) := \sup_{(z_0,z) \in O_Y \atop z_0 \in J_Y(z)} \frac{\langle z_0, z \rangle_{Y^*,Y}}{\|z\|_Y \|z_0\|_Y},$$

where the above supremum is taken over the set $O_Y$ consisting of all pairs $(z_0, z)$ which are orthogonal in the following sense:

$$O_Y := \left\{ (z_0, z) \in Y \times Y : \exists z^* \in J_Y(z) \text{ satisfying } \langle z^*, z_0 \rangle_{Y^*,Y} = 0 \right\}.$$
As in the case of $\text{BM}_\mathbb{R}(\mathbb{Y})$, $\text{AO}_\mathbb{R}(\mathbb{Y})$ only makes sense when $\dim \mathbb{Y} \geq 2$. Therefore as before, whenever $\text{AO}_\mathbb{R}()$ is written, we assume this to be the case.

**Remark 2.15 (Elementary properties of $\text{CJ}$)** The constant $\text{CJ}(\mathbb{Y})$ is a geometric constant since, it measures the degree to which the orthogonality relation (2.8) fails to be symmetric. Using the Cauchy–Schwartz inequality it is easy to see that $0 \leq \text{CJ}(\mathbb{Y}) \leq 1$. If $\mathbb{Y}$ is a Hilbert space, then $\text{CJ}(\mathbb{Y}) = 0$, since the single-valued duality map $J_\mathbb{Y}(\cdot)$ coincides with the self-adjoint Riesz map, and $\langle J_\mathbb{Y}(\cdot), \cdot \rangle_{\mathbb{Y}^*}$ coincides with the (symmetric) inner product in $\mathbb{Y}$. On the other hand, the maximal value $\text{CJ}(\mathbb{Y}) = 1$ holds for example for $\mathbb{Y} = \ell_1(\mathbb{R}^2)$. Indeed taking $z_0 = (1, -1)$ and $z = (\alpha, 1)$, with $\alpha > 0$, then $(2, -2) \in J_\mathbb{Y}(z_0)$ and $(1 + \alpha, 1 + \alpha) \in J_\mathbb{Y}(z)$, so that upon taking $\alpha \to +\infty$ one obtains $\langle z_0^*, z \rangle_{\mathbb{Y}^*}/\|z_0\|_{\mathbb{Y}} \|z\|_{\mathbb{Y}} \to 1$.

**Example 2.16 ($\text{CJ}(\ell_p)$)** Consider the Banach space $\ell_p \equiv \ell_p(\mathbb{R}^2)$ with $1 < p < +\infty$ (i.e., $\mathbb{R}^2$ endowed with the $p$-norm). In this case the duality map is given by:

$$\langle J_\ell(x_1, x_2), (y_1, y_2) \rangle_{\ell_p} = \|x_1, x_2\|_{\ell_p}^{2-p} \sum_{i=1}^{2} |x_i|^{p-1} \text{sign}(x_i) y_i,$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. By homogeneity of the duality map, the supremum in definition (2.7) can be taken over a normalized set (with unitary elements), in which case the computation of $\text{CJ}(\ell_p)$ is derived from the following constrained maximization problem:

$$\max x_1 |x_1|^{p-2} x_1 y_1 + |x_2|^{p-2} x_2 y_2,$$
subject to $\begin{cases} |x_1|^p + |x_2|^p = 1, \\ |y_1|^p + |y_2|^p = 1, \\ |x_1|^{p-2} x_1 y_1 + |x_2|^{p-2} x_2 y_2 = 0. \end{cases}$

Using polar coordinates, the above constraints, and some symmetries, it is possible to reduce the above problem to the following one dimensional maximization:

$$\text{CJ}(\ell_p) = \max_{\theta \in [0, \frac{\pi}{2}]} \left| (\cos \theta)^{\frac{q}{p}} (\sin \theta)^{\frac{q}{p}} - (\sin \theta)^{\frac{q}{p}} (\cos \theta)^{\frac{q}{p}} \right| \| \cos \theta, \sin \theta \|_{\ell_p}^{\frac{q}{p}} \| \cos \theta, \sin \theta \|_{\ell_p}^{\frac{q}{p}},$$
where $q = \frac{p}{p-1}$. Observe that $\text{CJ}(\ell_p) = \text{CJ}(\ell_q)$ since the formula remains the same by switching the roles of $p$ and $q$ (cf. Lemma 2.17). One can also show that $\text{CJ}(\ell_1) = \text{CJ}(\ell_\infty) = 1$. Figure 1 shows the dependence of $\text{CJ}(\ell_p)$ versus $p - 1$. It also illustrates the Banach–Mazur constant $\text{BM}(\ell_p)$ and the best-approximation projection constant $\text{best}(\ell_p) := \max_{u \in \ell_p(\mathbb{R}^2)} \|u\|/\|u\|$, with $u_n$ the best approximation to $u$ on the worst 1-dimensional subspace of $\ell_p(\mathbb{R}^2)$. The figure shows that

$$\text{best}(\ell_p) < \text{BM}(\ell_p) < 1 + \text{CJ}(\ell_p)$$
except for $p = 1, 2$ and $+\infty$, for which they coincide. \qed
We conclude our discussion of $C_{\text{AO}}$ with a Lemma describing three important properties that are going to be used later in Section 4.4.

**Lemma 2.17 (C_{\text{AO}} in reflexive smooth setting)** Assume the reflexive smooth setting where $Y$ and $Y^*$ are strictly convex and reflexive Banach spaces (see Section 2.1.4). The following properties hold true:

(i) $C_{\text{AO}}(Y) = \sup_{(z_0, z) \in \mathcal{O}_Y} \frac{\langle J_Y(z_0), z \rangle_{Y^*, Y}}{\|z\|_Y \|z_0\|_{Y^*}}$, where $\mathcal{O}_Y = \{(z_0, z) \in Y \times Y : \langle J_Y(z), z_0 \rangle_{Y^*, Y} = 0\}$.

(ii) $C_{\text{AO}}(Y^*) = C_{\text{AO}}(Y)$.

(iii) $C_{\text{AO}}(M) \leq C_{\text{AO}}(Y)$, for any closed subspace $M \subset Y$ with $\dim M \geq 2$ endowed with the norm $\|\cdot\|_Y$.

**Proof** See Section 2.4.

Note that result (ii) and (iii) in Lemma 2.17 actually imply that

$$C_{\text{AO}}(\ell_p) = C_{\text{AO}}(M) = C_{\text{AO}}(Y),$$

because $C_{\text{AO}}(Y) = C_{\text{AO}}(Y^*) \leq C_{\text{AO}}(M^*) = C_{\text{AO}}(M) \leq C_{\text{AO}}(Y)$.

**Example 2.18 (C_{\text{AO}}(L^p))** Let $\Omega \subset \mathbb{R}^d$ be an open set and consider the Banach space $Y := L^p(\Omega)$, $1 < p < +\infty$. Let $\Omega_1$ and $\Omega_2$ be two open bounded disjoint subsets. Define the functions $f_i \in L^p(\Omega)$ ($i = 1, 2$) by $f_i := |\Omega_i|^{-\frac{1}{p}} 1_{\Omega_i}$ and let $M := \text{span}\{f_1, f_2\} \subset Y$. It is easy to see that $M$ is isometrically isomorphic to $\ell_p(\mathbb{R}^2)$ and thus, using (2.9), we have

$$C_{\text{AO}}(\ell_p) = C_{\text{AO}}(M) = C_{\text{AO}}(L^p).$$

![Figure 1: Three different geometric constants and its dependence on $p - 1$.](image)
2.2.5 A priori bound II

The second a priori bound for best approximations is based on the equivalent characterization given in Theorem 2.1 and the asymmetric-orthogonality constant.

**Proposition 2.19 (Best approximation: A priori bound II)** Let \( Y \) be a Banach space, \( y \in Y \) and \( M \subseteq Y \) a closed subspace. Let \( y_0 \in M \) be such that \( \| y - y_0 \|_Y \leq \| y - z_0 \|_Y \), for all \( z_0 \in M \). Then \( y_0 \) satisfies the a priori bound:

\[
\| y_0 \|_Y \leq (1 + C_{AO}(Y)) \| y \|_Y ,
\]

where \( C_{AO}(Y) \in [0, 1] \) is the asymmetric-orthogonality constant of \( Y \) (see Definition 2.14).

**Proof** If \( y_0 = 0 \) or \( y_0 = y \), then the result is obvious. Hence, let us assume that \( \| y_0 \|_Y > 0 \) and \( \| y - y_0 \|_Y > 0 \). First of all, we estimate the error using the minimizing property of \( y_0 \in M \):

\[
\| y - y_0 \|_Y \leq \| y - 0 \|_Y = \| y \|_Y .
\]

Next, by Theorem 2.1, there exists a \( z^* \in J_Y(y - y_0) \) which annihilates \( M \). Therefore, \( (y_0, y - y_0) \in \mathcal{O}_Y \), and we thus obtain for any \( z^*_0 \in J_Y(y_0) \):

\[
\| y_0 \|_Y = \frac{\langle z^*_0, y_0 \rangle_{Y^*, Y}}{\| y_0 \|_Y} \\
= \frac{\langle z^*_0, y \rangle_{Y^*, Y} - \langle z^*_0, y - y_0 \rangle_{Y^*, Y}}{\| y_0 \|_Y} \\
\leq \| y \|_Y - \frac{\langle z^*_0, y - y_0 \rangle_{Y^*, Y}}{\| y_0 \|_Y} \| y - y_0 \|_Y \\
\leq \| y \|_Y + C_{AO}(Y) \| y - y_0 \|_Y .
\]

Conclude by using the estimate in (2.11). \( \square \)

2.3 Proof of Lemma 2.9

In this section, we prove Lemma 2.9.

The inequality \( \| I - Q \| \leq 1 + \| Q \| = (1 + \| Q \|^{-1}) \| Q \| \) is trivial, so we focus our attention in showing that

\[
\| y - Q(y) \|_Y \leq C_{nm}(Y) \| Q \| \| y \|_Y , \quad \forall y \in Y .
\]

If \( y - Q(y) = 0 \) the result holds true immediately. On the other hand, by requirement (i), since \( 0 \neq Q = Q \circ Q \), we have that \( \| Q \| \geq 1 \). Moreover, \( C_{nm}(Y) \geq 1 \) (see Remark 2.8). Thus, if \( Q(y) = 0 \), then

\[
\| y - Q(y) \|_Y = \| y \|_Y \leq C_{nm}(Y) \| Q \| \| y \|_Y .
\]
Hence, we assume now on that $y - Q(y) \neq 0$ and $Q(y) \neq 0$.

First of all observe that $y - Q(y)$ and $Q(y)$ are linearly independent. Indeed, suppose on the contrary that there exists $t \in \mathbb{R} \setminus \{0\}$ such that $y - Q(y) = tQ(y)$, then $y = (1 + t)Q(y)$, hence applying $Q$ and using homogeneity (requirement (ii)), we get $t = 0$ (a contradiction).

The proof follows next using a two-dimensional geometrical argument. Let us define $\mathbb{W} := \text{span}\{Q(y), y - Q(y)\}$ and note that $\dim \mathbb{W} = 2$. Let $T : \mathbb{W} \to \ell_2(\mathbb{R}^2)$ be any linear isomorphism between $\mathbb{W}$ and $\ell_2(\mathbb{R}^2)$ (the two-dimensional Euclidean vector space endowed with the norm $\| \cdot \|_2$). Define

$$0 \neq \alpha := \| T(y - Q(y)) \|_2, \quad \text{(2.12a)}$$
$$0 \neq \beta := \| TQ(y) \|_2, \quad \text{(2.12b)}$$

and, subsequently, let $\tilde{y} \in \mathbb{W}$ be defined by

$$\tilde{y} := \frac{\alpha}{\beta}Q(y) + \frac{\beta}{\alpha}(y - Q(y)). \quad \text{(2.13)}$$

The proof will next be divided into four steps:

(S1) To show that $\| y - Q(y) \|_Y \leq (\| T \| \| T^{-1} \| \) \| \frac{\alpha}{\beta}Q(y) \|_Y$.

(S2) To show that $\frac{\alpha}{\beta}Q(y) \|_Y \leq \| Q \| \\| \tilde{y} \|_Y$.

(S3) To show that $\| \tilde{y} \|_Y \leq (\| T \| \| T^{-1} \| \) \| y \|_Y$.

(S4) To conclude that $\| y - Q(y) \|_Y \leq C_{\text{BM}}(Y) \| Q \| \| y \|_Y$.

Proof of (S1): This follows from elementary arguments since $\beta \neq 0$:

$$\| y - Q(y) \|_Y \leq \| T^{-1} \| \| T(y - Q(y)) \|_2$$
$$= \| T^{-1} \| \frac{\alpha}{\beta} \| TQ(y) \|_2$$
$$\leq \| T^{-1} \| \| T \| \| \frac{\alpha}{\beta}Q(y) \|_Y.$$

Proof of (S2): Use requirement (iv) with $\eta = \frac{\beta^2}{\alpha^2}$, and subsequently requirements (ii) and (iii), to obtain:

$$\| \frac{\alpha}{\beta}Q(y) \|_Y = \| \frac{\alpha}{\beta}Q\left(Q(y) + \frac{\beta^2}{\alpha^2}(I - Q)(y)\right) \|_Y = \| Q(\tilde{y}) \|_Y \leq \| Q \| \| \tilde{y} \|_Y. \quad \text{(2.14)}$$
Proof of (S3): The key point here is to observe that \( \|T \hat{y}\|_2 = \|Ty\|_2 \), indeed,

\[
\|T \hat{y}\|_2^2 = \left\| \frac{\alpha}{\beta} T Q(y) + \frac{\beta}{\alpha} T(y - Q(y)) \right\|_2^2 \\
= \alpha^2 + 2 T Q(y) \cdot T(y - Q(y)) + \beta^2 \\
= \|T(y - Q(y)) + T Q(y)\|_2^2 \\
= \|Ty\|_2^2.
\]

Therefore,

\[
\|\hat{y}\|_Y \leq \|T^{-1}\| \|T \hat{y}\|_2 = \|T^{-1}\| \|Ty\|_2 \leq \|T^{-1}\| \|T\| \|y\|_Y.
\]

Proof of (S4): Combining (S1)–(S3) we get

\[
\|y - Q(y)\|_Y \leq (\|T\| \|T^{-1}\|)^2 \|Q\| \|y\|_Y.
\]

Finally, taking the infimum over all linear isomorphisms \( T : W \to \ell_2(\mathbb{R}^2) \) we obtain

\[
\|y - Q(y)\|_Y \leq (d_{\text{nm}}(W, \ell_2(\mathbb{R}^2)))^2 \|Q\| \|y\|_Y \leq C_{\text{nm}}(Y) \|Q\| \|y\|_Y.
\]

\[\blacksquare\]

### 2.4 Proof of Lemma 2.17

In this section we prove Lemma 2.17.

The reflexive smooth setting ensures that the duality mappings are single-valued bijections \( J_Y : Y \to Y^* \) and \( J_{Y^*} : Y^* \to Y^{**} \). Moreover, \( J_{Y^*} = J_Y^{-1} \) by canonical identification.

Property (i) is a direct consequence of the definition of the constant \( C_{\text{AO}}(Y) \) (see (2.7)) and the fact that the duality mapping is single-valued, i.e., \( J_Y(y) = \{J_Y(y)\} \), for all \( y \in Y \).

To prove property (ii), we make use of property (i) replacing \( Y \) by \( Y^* \). We get

\[
C_{\text{AO}}(Y^*) = \sup_{(z^*, z^*_0) \in \mathcal{O}_{Y^*}} \frac{\langle J_{Y^*}(z^*), z^*_0 \rangle_{Y^{**}, Y^*}}{\|z^*_0\|_{Y^*} \|z^*\|_{Y^*}} = \sup_{(z^*, z^*_0) \in \mathcal{O}_{Y^*}} \frac{\langle z^*_0, J_Y^{-1}(z^*) \rangle_{Y^*, Y}}{\|z^*_0\|_{Y^*} \|z^*\|_{Y^*}}.
\]

Defining \( z = J_Y^{-1}(z^*) \) and \( z_0 = J_Y^{-1}(z_0^*) \) we obtain

\[
C_{\text{AO}}(Y^*) = \sup_{(z^*, z_0^*) \in \mathcal{O}_{Y^*}} \frac{\langle J_Y(z_0^*), z \rangle_{Y^*, Y}}{\|z_0\|_{Y^*} \|z\|_{Y^*}}.
\]

Now observe that

\[
\mathcal{O}_{Y^*} = \{(z^*, z_0^*) \in Y^* \times Y^*: \langle J_Y(z_0^*), z^* \rangle_{Y^{**}, Y^*} = 0\} = \{(J_Y(z), J_Y(z_0)) \in Y^* \times Y^*: \langle J_Y(z), z_0 \rangle_{Y^*, Y} = 0\} = \{(J_Y(z), J_Y(z_0)) \in Y^* \times Y^*: (z_0, z) \in \mathcal{O}_Y\}.
\]
Hence the supremum in (2.15) can be taken over all \((z_0, z) \in \mathcal{O}_Y\) and thus \(C_{AO}(Y^*) = C_{AO}(Y)\).

For the last property (iii) we make use of Lemma 2.3 to show that

\[
C_{AO}(M) = \sup_{(z_0, z) \in \mathcal{O}_M} \frac{\langle J_M(z_0), z \rangle_{M^*, M}}{\|z\|_Y \|z_0\|_Y} = \sup_{(z_0, z) \in \mathcal{O}_M} \frac{\langle J_Y(I_M z_0), I_M z \rangle_{Y^*, Y}}{\|z\|_Y \|z_0\|_Y}.
\]

To conclude, it remains to show that \(C_{AO}(Y)\) takes the supremum over a larger set (i.e., \(I_M \mathcal{O}_M \subset \mathcal{O}_Y\)). Indeed, if \((z_0, z) \in \mathcal{O}_M\), then \((I_M z_0, I_M z) \in \mathcal{Y} \times \mathcal{Y}\) and

\[
\langle J_Y(I_M z), I_M z_0 \rangle_{Y^*, Y} = \langle J_M(z), z_0 \rangle_{M^*, M} = 0,
\]

by Lemma 2.3. Hence \((I_M z_0, I_M z) \in \mathcal{O}_Y\). \(\square\)

3 Residual minimization, nonlinear Petrov–Galerkin and monotone mixed formulation

In this section, we analyze the residual minimization method (1.5) and characterize its solution by means of the duality mapping. The characterization will give rise to a nonlinear Petrov–Galerkin discretization and corresponding mixed formulation. The inexact version of this method is the subject of Section 4.

3.1 Equivalent best-approximation problem

To carry out the analysis, we reformulate (1.5) as an equivalent best-approximation problem and apply the classical theory of Section 2.2. Let us introduce the norm

\[
\|\cdot\|_E := \|B(\cdot)\|_{Y^*},
\]

which, in some applications, is referred to as the energy norm on \(U\). Since \(B\) is continuous and bounded below, it is clear that \(\|\cdot\|_E\) is an equivalent norm on \(U\); see (1.2).

Let us recall that existence of a unique solution to (1.1) is guaranteed for continuous and bounded-below \(B\), if \(f \in \text{Im} B\) or if \(\text{Ker} B^* = \{0\}\) (in which case \(B\) is surjective); see, e.g., [22, Appendix A.2] or [30, Section 5.17]. Therefore, supposing \(f \in \text{Im} B\), then upon substituting \(f = Bu\), (1.5) is equivalent to finding a best approximation \(u_n \in U_n\) to \(u\) measured by the energy norm:

\[
\begin{align*}
\text{Find } u_n &\in U_n : \\
&u_n = \arg \min_{w_n \in U_n} \|u - w_n\|_E.
\end{align*}
\] (3.1)
3.2 Analysis of residual minimization

The main result for the residual-minimization method (1.5) now follows from the classical Theorem 2.A for best approximations, while novel a priori bounds follow from Propositions 2.11 and 2.19.

Theorem 3.A (Residual minimization) Let $U$ and $V$ be two Banach spaces and let $B : U \to V^*$ be a linear, continuous and bounded-below operator with continuity constant $M_B > 0$ and bounded-below constant $\gamma_B > 0$. Given $f \in V^*$ and a finite-dimensional subspace $U_n \subset U$, the following statements hold:

(i) There exists a residual minimizer $u_n \in U_n$ such that:

$$u_n = \arg \min_{w_n \in U_n} \| f - Bw_n \|_{V^*}.$$  \hspace{1cm} (3.2)

(ii) Any residual minimizer $u_n$ of (3.2) satisfies the a priori bounds

$$\|u_n\|_U \leq \frac{C_{BM}(V^*)}{\gamma_B} \| f \|_{V^*},$$  \hspace{1cm} (3.3a)

$$\|u_n\|_U \leq \frac{(1 + C_{AO}(V))}{\gamma_B} \| f \|_{V^*}.$$  \hspace{1cm} (3.3b)

where $C_{BM}(V^*) \in [1, 2]$ is the Banach-Mazur constant of $V^*$ and $C_{AO}(V) \in [0, 1]$ is the asymmetric-orthogonality constant of $V$ (see Definitions 2.7 and 2.14).

(iii) If $V^*$ is a strictly-convex Banach space, then the residual minimizer $u_n$ of (3.2) is unique.

(iv) If $f \in \text{Im}(B)$ and $u \in U$ is the solution of the problem $Bu = f$, then we have the a posteriori and a priori error estimates:

$$\|u - u_n\|_U \leq \frac{1}{\gamma_B} \| f - Bu_n \|_{V^*} \leq \frac{M_B}{\gamma_B} \inf_{w_n \in U_n} \|u - w_n\|_U.$$  \hspace{1cm} (3.4)

Proof We first consider the case that $f \in \text{Im}(B)$, in which case $Bu = f$.

The proof of parts (i), (iii) and (iv) can be found in Guermond [26], but we present an alternative based on Theorem 2.A. Since $U$ endowed with the energy norm is a Banach space, the first statement is a direct application of Theorem 2.A(i) by using the energy norm topology in $U$ and the equivalence between (1.5) and (3.1). If $V^*$ is strictly convex, then $U$ endowed with the energy norm is also strictly convex. Hence, by 2.A(ii) the minimizer $u_n \in U_n$ is unique, which
proves the third statement. Finally, using the norm equivalence (1.2), together with the minimizing property of \( u_n \) in the energy norm, we get

\[
\| u - u_n \|_U \leq \frac{1}{\gamma_B} \| u - u_n \|_E \leq \frac{1}{\gamma_B} \| u - w_n \|_E \leq \frac{M_B}{\gamma_B} \| u - w_n \|_U,
\]

for all \( w_n \in U_n \), which proves the last statement.

We now prove part (ii). The bound provided by Proposition 2.11 shows that

\[
\| u_n \|_U \leq \frac{1}{\gamma_B} \| u_n \|_E \leq C \beta M (V^\ast) \frac{\| u \|_E}{\gamma_B} = C \beta M (V^\ast) \frac{\| f \|_{V^\ast}}{\gamma_B},
\]

which proves (3.3a). A similar argument based on Proposition 2.19 proves (3.3b) (using also Lemma 2.17(ii)).

The proof for general \( f \in V^\ast \), including the case that \( f \notin \text{Im}(B) \), follows similarly by considering the best approximation of \( f \) in the space \( BU_n \), and using that any \( g_n \in BU_n \) has a unique \( w_n \in U_n \) such that \( Bw_n = g_n \).

\[\Box\]

**Remark 3.1 (Finite element methods)** In the context of finite elements, there is a sequence \( \{U_h\}_{h>0} \) of finite-dimensional subspaces, \( U_h \subset U \), having the approximation property

\[
\inf_{w_h \in U_h} \| w - w_h \|_U \leq \varepsilon(h) \| w \|_Z, \quad \forall w \in Z,
\]

where \( Z \subset U \) is a more regular subspace and \( \varepsilon(h) \) is a function that is continuous at zero and \( \varepsilon(0) = 0 \). This last statement, together with (3.4), gives a guarantee that minimizers \( u_n \in U_n \equiv U_h \) of (3.2) converge to \( u = B^{-1}f \) upon \( h \to 0^+ \).

\[\Box\]

**Remark 3.2 (Optimal test-space norm)** As proposed in [39] (cf. [16]), if \( B \) is a linear bounded bijective operator and \( V \) reflexive (hence \( B^\ast : V \to U^\ast \) is bijective), one can endow the space \( V \) with the equivalent optimal norm

\[
\| \cdot \|_{V_{opt}} = \| B^\ast(\cdot) \|_{U^\ast}.
\]

Then, residual minimization in \( (V_{opt})^\ast \) reduces precisely to best approximation of \( u \) measured in \( \| \cdot \|_U \). In particular, instead of (3.4), one then obtains

\[
\| u - u_n \|_U = \| f - Bu_n \|_{(V_{opt})^\ast} = \inf_{w_n \in U_n} \| u - w_n \|_U.
\]

\[\Box\]

### 3.3 Characterization of residual minimization

The first characterization for residual minimizers is given in general Banach spaces:
Proposition 3.3 (Characterization of residual minimization) Let \( U \) and \( V \) be two Banach spaces and let \( B : U \to V^* \) be a linear, continuous and bounded-below operator. Given \( f \in V^* \) and a finite-dimensional subspace \( U_n \subset U \), an element \( u_n \in U_n \) is a solution of the residual minimization problem (1.5), if and only if there is an \( r^{**} \in J_{V^*}(f - Bu_n) \subset V^{**} \) satisfying:

\[
\langle r^{**}, Bu_n \rangle_{V^{**}, V^*} = 0, \quad \forall w_n \in U_n.
\]  

(3.5)

**Proof** Apply Theorem 2.A(iii) to the minimization problem (1.5), using \( Y = V^* \) and \( M = BU_n \).

Defining the discrete space \( V_n^* := BU_n \), we see that (3.5) can be interpreted as the nonlinear Petrov–Galerkin discretization:

\[
\begin{align*}
\text{Find } u_n \in U_n \text{ such that for some } r^{**} \in J_{V^*}(f - Bu_n), \\
\langle r^{**}, \nu_n \rangle_{V^{**}, V^*} &= 0, \quad \forall \nu_n \in V_n^*.
\end{align*}
\]  

(3.6)

Observe that \( r^{**} \in J_{V^*}(f - Bu_n) \) must fulfill the set of nonlinear equations:

\[
\|r^{**}\|^2_{V^{**}} = \langle r^{**}, f - Bu_n \rangle_{V^{**}, V^*} = \|f - Bu_n\|^2_{V^*}.
\]

In special Banach spaces, because of specific properties of \( J_{V^*} \) depending on the geometry of \( V \), the above characterizations can be reduced to other forms. For example, if \( V \) is reflexive, then \( f - Bu_n \in J_{V^*}(r) \) for some \( r \in V \) such that \( \langle Bw_n, r \rangle_{V^*, V} = 0 \), for all \( w_n \in U_n \). To have more useful characterizations, we shall restrict to the reflexive smooth setting. Recall from Section 2.1.4 that in this setting the duality mapping in \( V \) is a single-valued and bijective map, denoted by \( J_V : V \to V^* \).

**Theorem 3.B (Equivalent characterizations)** Let \( U \) and \( V \) be two Banach spaces and let \( B : U \to V^* \) be a linear, continuous and bounded-below operator. Assume additionally that \( V \) and \( V^* \) are strictly convex and reflexive. Given \( f \in V^* \) and a finite-dimensional subspace \( U_n \subset U \). The following statements are equivalent:

(i) \( u_n \in U_n \) is the unique residual minimizer such that

\[
u_n = \arg\min_{w_n \in U_n} \|f - Bw_n\|_{V^*}.
\]

(ii) \( u_n \in U_n \) is the solution of the nonlinear Petrov–Galerkin formulation:

\[
\langle \nu_n, J_V^{-1}(f - Bu_n) \rangle_{V^*, V} = 0, \quad \forall \nu_n \in BU_n.
\]
(iii) There is a unique residual representation \( r \in V \) such that \( u_n \in U_n \) together with \( r \) satisfy the semi-infinite monotone mixed formulation:

\[
\begin{aligned}
\{ \langle J_V(r), v \rangle_{V^*, V} + \langle Bu_n, v \rangle_{V^*, V} = \langle f, v \rangle_{V^*, V}, \quad \forall v \in V, \quad (3.7a) \\
\langle B^*r, w_n \rangle_{U^*, U} = 0, \quad \forall w_n \in U_n. \quad (3.7b) \}
\end{aligned}
\]

(iv) \( u_n \in U_n \) is the Lagrange multiplier of the constrained minimization:

\[
\min_{v \in (B^*)^{-1}} \frac{1}{2} \|v\|_V^2 - \langle f, v \rangle_{V^*, V}. \quad (3.8)
\]

**Proof** We proceed by proving consecutively: (i) \(\iff\) (ii) \(\iff\) (iii) \(\Rightarrow\) (iv) \(\Rightarrow\) (iii).

(i) \(\iff\) (ii) : By Proposition 3.3, \( u_n \in U_n \) is the unique minimizer of (3.2) if and only if \( J_V^{-1}(f - Bu_n) \in V^{**} \) annihilates the discrete space \( BU_n \subset V^* \). In other words, because of the identification \( J_V = J_V^{-1} \) (see Section 2.1.4),

\[
\langle Bw_n, J_V^{-1}(f - Bu_n) \rangle_{V^*, V} = \langle J_V^{-1}(f - Bu_n), Bw_n \rangle_{V^{**}, V^*} = 0,
\]

for all \( w_n \in U_n \).

(ii) \(\iff\) (iii) : Note that \( r = J_V^{-1}(f - Bu_n) \).

(iii) \(\Rightarrow\) (iv) : The Lagrangian \( \mathcal{L} : V \times U_n \rightarrow \mathbb{R} \) associated with the constrained minimization (3.8) is:

\[
\mathcal{L}(v, w_n) := \frac{1}{2} \|v\|_V^2 - \langle f, v \rangle_{V^*, V} + \langle B^*v, w_n \rangle_{U^*, U}.
\]

Let \( (r, u_n) \) denote the solution to the mixed formulation (3.7). Firstly, since \( r \in (BU_n)^{-1} \), it is straightforward to see that \( \mathcal{L}(r, w_n) = \mathcal{L}(r, u_n) \). Secondly,

\[
0 = \langle J_V(r), v - r \rangle_{V^*, V} + \langle Bu_n, v - r \rangle_{V^*, V} - \langle f, v - r \rangle_{V^*, V} \quad \text{(by (3.7a))}
\]
\[
\leq \frac{1}{2} \|v\|_V^2 - \frac{1}{2} \|r\|_V^2 + \langle Bu_n, v - r \rangle_{V^*, V} - \langle f, v - r \rangle_{V^*, V} \quad \text{(by Prop. (2.4))}
\]
\[
= \mathcal{L}(v, u_n) - \mathcal{L}(r, u_n).
\]

Therefore \( (r, u_n) \) is a saddle-point of the Lagrangian, i.e.,

\[
\mathcal{L}(r, w_n) \leq \mathcal{L}(r, u_n) \leq \mathcal{L}(v, u_n) \quad \forall (v, w_n) \in V \times U_n, \quad (3.9)
\]

which is equivalent to (3.8).

(iv) \(\Rightarrow\) (iii) : Let \( (r, u_n) \) be a solution of (3.8), i.e., (3.9) holds. The first inequality in (3.9) implies

\[
\langle Bw_n, r \rangle_{V^*, V} \leq \langle Bu_n, r \rangle_{V^*, V} \quad \forall w_n \in U_n,
\]
which implies (3.7b) by a vector-space argument. Next, considering the second inequality in (3.9) with \( v \) equal to \( r + \lambda v \), and \( \lambda > 0 \), it follows that
\[
0 \leq \lambda^{-1} \left( \mathcal{L}(r + \lambda v, u_n) - \mathcal{L}(r, u_n) \right) = \lambda^{-1} \left( \frac{1}{2} \| r + \lambda v \|^2 - \frac{1}{2} \| r \|^2 \right) - (f, v)_{\mathcal{V}^*, \mathcal{V}} + (Bu_n, v)_{\mathcal{V}^*, \mathcal{V}} \\
\leq \langle J_{\mathcal{V}}(r + \lambda v), v \rangle_{\mathcal{V}^*, \mathcal{V}} + (Bu_n, v)_{\mathcal{V}^*, \mathcal{V}} - (f, v)_{\mathcal{V}^*, \mathcal{V}} \quad \text{(by Prop. 2.4)}
\]
Therefore, upon \( \lambda \to 0^+ \), invoking hemi-continuity of \( J_{\mathcal{V}} \) (see (2.2)) and repeating the above with \(-v\) instead of \( v \), one recovers (3.7a).

Remark 3.4 (Mixed form for optimal test-space norm) If one assumes \( B \) is a linear bounded bijective operator and, instead of \( \| \cdot \|_{\mathcal{V}} \), one uses the norm \( \| \cdot \|_{\mathcal{V}_{\text{opt}}} \) on \( \mathcal{V} \) (recall from Remark 3.2), then one can show that (3.7a) holds with \( (J_{\mathcal{V}}(r), v)_{\mathcal{V}^*, \mathcal{V}} \) replaced by \( \langle B^* v, J_{\mathcal{U}}^{-1}(B^* r) \rangle_{\mathcal{U}^*, \mathcal{U}} \).

4 Analysis of the inexact method

We now consider the tractable approximation. The reflexive smooth setting guarantees that the semi-infinite mixed formulation (3.7) introduced in Theorem 3.B is well posed. For convenience, this formulation will be the starting point for the inexact method.

In addition to \( \mathcal{U}_n \subset \mathcal{U} \), let \( \mathcal{V}_m \subset \mathcal{V} \) be a finite-dimensional subspace. We shall then consider:

\[
\begin{bmatrix}
\langle J_{\mathcal{V}}(r_m), v_m \rangle_{\mathcal{V}^*, \mathcal{V}} + (Bu_n, v_m)_{\mathcal{V}^*, \mathcal{V}} = (f, v_m)_{\mathcal{V}^*, \mathcal{V}} & \forall v_m \in \mathcal{V}_m, \\
\langle B^* r_m, w_n \rangle_{\mathcal{U}^*, \mathcal{U}} = 0 & \forall w_n \in \mathcal{U}_n.
\end{bmatrix}
\]

(4.1a) (4.1b)

Because the nonlinear operator \( J_{\mathcal{V}} \) is monotone, we refer to the above as a monotone mixed method.

4.1 Equivalent discrete settings

Analogous to the semi-infinite mixed formulation, which is equivalent to residual minimization (see Theorem 3.B), the monotone mixed method is related to residual minimization in the discrete dual norm
\[
\| \cdot \|_{(\mathcal{V}_m)^*} = \sup_{v_m \in \mathcal{V}_m} \frac{\langle \cdot, v_m \rangle_{(\mathcal{V}_m)^*, \mathcal{V}_m}}{\| v_m \|_{\mathcal{V}}}. 
\]

The next theorem summarizes this equivalence and, additionally, shows the equivalence with an inexact version of the nonlinear Petrov–Galerkin discretization and a discrete constrained minimization.
Theorem 4.A (Discrete equivalent characterizations) Let $U$ and $V$ be two Banach spaces and let $B : U \to V^*$ be a linear, continuous and bounded-below operator. Assume that $V$ and $V^*$ are reflexive and strictly convex. Given $f \in V^*$ and finite-dimensional subspaces $U_n \subset U$ and $V_m \subset V$, the following statements are equivalent:

(i) $(r_m, u_n) \in V_m \times U_n$ is a solution of the discrete mixed problem:
$$
\begin{align*}
\langle J_{V_m}(r_m), v_m \rangle_{V^*, V} + \langle Bu_n, v_m \rangle_{V^*, V} &= \langle f, v_m \rangle_{V^*, V}, & \forall v_m \in V_m, \\
\langle B^* r_m, w_n \rangle_{U^*, U} &= 0, & \forall w_n \in U_n.
\end{align*}
$$

(ii) $u_n \in U_n$ is a solution of the inexact non-linear Petrov-Galerkin discretization:
$$
\left\langle \nu_n, I_m J^{-1}_{V_m} \circ I_m^*(f - Bu_n) \right\rangle_{V^*, V} = 0, \quad \forall \nu_n \in BU_n.
$$

and $r_m = J^{-1}_{V_m} \circ I_m^*(f - Bu_n)$, where $I_m : V_m \to V$ is the natural injection.

(iii) $u_n \in U_n$ is a minimizer of the discrete residual minimization problem:
$$
\min_{w_n \in U_n} \| I_m^*(f - Bw_n) \|_{(V_m)^*},
$$

and $r_m = J^{-1}_{V_m} \circ I_m^*(f - Bu_n)$, where $I_m^* : (V_m)^* \to V^*$ is the natural injection.

(iii) $u_n \in U_n$ is the Lagrange multiplier of the discrete constrained minimization problem:
$$
\min_{v_m \in V_m \cap (BU_n)^*} \left\{ \frac{1}{2} \| v_m \|^2_{V^*} - \langle f, v_m \rangle_{V^*, V} \right\},
$$
while $r_m \in V_m$ is the minimizer of it. \hfill \Box

Proof First notice the following direct equivalences:
$$
r_m = J^{-1}_{V_m} \circ I_m^*(f - Bu_n)
\iff J_{V_m}(r_m) = I_m^*(f - Bu_n)
\iff I_m^* J_{V_m}(I_m r_m) = I_m^*(f - Bu_n),
$$

(by Lemma 2.3)

where the last statement is equivalent to (4.1a).

(i) $\Rightarrow$ (ii). If $(u_n, r_m) \in U_n \times V_m$ is a solution of (4.1), then $r_m = J^{-1}_{V_m} \circ I_m^*(f - Bu_n)$ and (4.1b) is nothing but (4.2).
(ii) ⇒ (iii). Observe that for any \( w_n \in U_n \) we have:

\[
\begin{align*}
\| I_m^* (f - B u_n) \|_{(V_m)^*} &= \sup_{v_m \in V_m} \frac{\langle I_m^* (f - B u_n), v_m \rangle_{(V_m)^*, V_m}}{\| v_m \|_V} \\
&= \sup_{v_m \in V_m} \frac{\langle J_{V_m}(r_m), v_m \rangle_{(V_m)^*, V_m}}{\| v_m \|_V} \\
&= \frac{\langle J_{V_m}(r_m), r_m \rangle_{(V_m)^*, V_m}}{\| r_m \|_V} \quad \text{(by (2.1))} \\
&= \frac{\langle I_m^* (f - B w_n), r_m \rangle_{(V_m)^*, V_m}}{\| r_m \|_V} \quad \text{(by (4.2))} \\
&\leq \| I_m^* (f - B w_n) \|_{(V_m)^*}.
\end{align*}
\]

Thus, \( u_n \) is a minimizer of (4.3).

(iii) ⇒ (i). If \( u_n \in U_n \) is a minimizer of (4.3) and \( r_m = J_{V_m}^{-1} \circ I_m^* (f - B u_n) = J_{(V_m)^*} \circ I_m^* (f - B u_n) \), then by Theorem 2.A, with \( M = I_m^* B U_n \subset (V_m)^* = \mathbb{Y} \), \( r_m \) satisfies:

\[
0 = \langle I_m^* B w_n, r_m \rangle_{(V_m)^*, V_m} = \langle B w_n, I_m r_m \rangle_{V^*, V} = \langle B^* r_m, w_n \rangle_{U^*, U}, \quad \forall w_n \in U_n,
\]

which verifies (4.1b).

(i) ⇔ (iv). The proof of this equivalence follows exactly the same reasoning as in the semi-infinite setting; see the proof of Theorem 3.B, part (iii) ⇔ (iv). □

### 4.2 Well-posedness of the inexact method

We now study the existence and uniqueness of solutions to the inexact method (4.1). A critical ingredient for the uniqueness analysis is the following condition:

**Assumption 4.1 (Fortin condition)** Let \( \{ (U_n, V_m) \} \) be a family of discrete subspace pairs, where \( U_n \subset U \) and \( V_m \subset V \). For each pair \( (U_n, V_m) \) in this family, there exists an operator \( \Pi_{n,m} : V \rightarrow V_m \) and constants \( C_\Pi > 0 \) and \( D_\Pi > 0 \) (independent of \( n \) and \( m \)) such that the following conditions are satisfied:

\[
\begin{align*}
\| \Pi_{n,m} v \|_V &\leq C_\Pi \| v \|_V, \quad \forall v \in V, \quad (4.5a) \\
\| (I - \Pi_{n,m}) v \|_V &\leq D_\Pi \| v \|_V, \quad \forall v \in V, \quad (4.5b) \\
\langle B w_n, v - \Pi_{n,m} v \rangle_{V^*, V} &\leq 0, \quad \forall w_n \in U_n, \forall v \in V, \quad (4.5c)
\end{align*}
\]

where \( I : V \rightarrow V \) is the identity map in \( V \). For simplicity, we write \( \Pi \) instead of \( \Pi_{n,m} \). □
Such an operator is referred to as a Fortin operator after Fortin’s trick in mixed finite element methods [6, Section 5.4]. For the existence of $\Pi$, note that the last identity (4.5c) requires that $\dim V_m \geq \dim \text{Im}(B|_{U_n}) = \dim U_n$ (for a bounded-below operator $B$). Note that (4.5a) implies (4.5b) with $D_{\Pi} = 1 + C_{\Pi}$, but to allow for sharper estimates, we prefer to retain the independent constant $D_{\Pi}$.

**Theorem 4.B (Inexact method: Discrete well-posedness)** Let $U$ and $V$ be two Banach spaces and let $B : U \to V^*$ be a linear, continuous and bounded-below operator, with continuity constant $M_B > 0$ and bounded-below constant $\gamma_B > 0$. Assume that $V$ and $V^*$ are reflexive and strictly convex. Let $U_n \subset U$ and $V_m \subset V$ be finite-dimensional subspaces such that the (Fortin) Assumption 4.1 holds true. Given $f \in V^*$, there exists a unique solution $(r_m, u_n) \in V_m \times U_n$ of the inexact method:

\[
\begin{cases}
    \langle J_V(r_m), v_m \rangle_{V^*,V} + \langle Bu_n, v_m \rangle_{V^*,V} = \langle f, v_m \rangle_{V^*,V}, & \forall v_m \in V_m, \quad (4.6a) \\
    B^* r_m, w_n \rangle_{U^*,U} = 0, & \forall w_n \in U_n. \quad (4.6b)
\end{cases}
\]

Moreover, let $u \in U$ be such that $Bu = f$, then we have the a priori bounds:

\[
\begin{cases}
    \|r_m\|_V \leq \|f\|_{V^*} \leq M_B \|u\|_U \quad \text{and} \\
    \|u_n\|_U \leq \frac{C_{\Pi}}{\gamma_B} (1 + C_{\lambda_0}(V)) \|f\|_{V^*} \leq \frac{C_{\Pi}}{\gamma_B} (1 + C_{\lambda_0}(V)) M_B \|u\|_U, \quad (4.7a)
\end{cases}
\]

where $C_{\Pi} > 0$ is the boundedness constant of the Fortin operator (see Assumption 4.1) and $C_{\lambda_0}(V) \in [0,1]$ is the asymmetric-orthogonality geometrical constant related to the space $V$ (see Definition 2.14).

**Proof** To prove existence, we consider the equivalent discrete constrained minimization problem (4.4). The existence of a minimizer $r_m \in V_m \cap (B U_n)^\perp$ is guaranteed since the functional $v_m \mapsto \frac{1}{2} \|v_m\|^2_V - \langle f, v_m \rangle_{V^*,V}$ is convex and continuous, and $V_m \cap (B U_n)^\perp$ is a closed subspace.

Next, we claim that there exist a $u_n \in U_n$ such that

\[\langle Bu_n, v_m \rangle_{V^*,V} = \langle f - J_V(r_m), v_m \rangle_{V^*,V}, \quad \forall v_m \in V_m.\]

To see this, consider the restricted operator $B_n : U_n \to V^*$, such that $B_n w_n = B w_n$ for all $w_n \in U_n$, and recall the natural injection $I_m : V_m \to V$. Then, the above translates into

\[I_m^* B_n u_n = I_m^* (f - J_V(r_m)) \quad \text{in} \quad (V_m)^*.\]

Thus, to prove existence, we show that $I_m^* (f - J_V(r_m))$ is in the (closed) range of the finite-dimensional operator $I_m^* B_n : U_n \to (V_m)^*$. Since $r_m$ is the minimizer
of (4.4), we have
\[ 0 = \langle J_Y(r_m) - f, I_m v_m \rangle_{\mathcal{Y}', \mathcal{Y}} = \langle I_m^* (J_Y(r_m) - f), v_m \rangle_{(\mathcal{Y}^s)^*, \mathcal{Y}}, \]
\[ \forall v_m \in \mathcal{Y}_m \cap (B \mathcal{U}_n)^{\perp} = \text{Ker}(B^*_n I_m). \]
Hence, \( I_m^*(f - J_Y(r_m)) \in (\text{Ker}(B^*_n I_m))^\perp = \text{Im}(I_m^* B_n). \)

To prove uniqueness assume that \((u_n, r_m)\) and \((\tilde{u}_n, \tilde{r}_m)\) are two solutions of problem (4.1). Then, by subtraction, it is immediate to see that:
\[ \langle J_Y(r_m) - J_Y(\tilde{r}_m), r_m - \tilde{r}_m \rangle_{\mathcal{Y}', \mathcal{Y}} = 0, \]
which implies that \(\tilde{r}_m = r_m\) by strict monotonicity of \(J_Y\) (see (2.3)). Going back to (4.6a) we now obtain \(\langle B(u_n - \tilde{u}_n), v_m \rangle_{\mathcal{Y}', \mathcal{Y}} = 0, \) for all \(v_m \in \mathcal{Y}_m\). Therefore, by the Fortin-operator property (4.5c),
\[ \langle B(u_n - \tilde{u}_n), v \rangle_{\mathcal{Y}', \mathcal{Y}} = \langle B(u_n - \tilde{u}_n), \Pi v \rangle_{\mathcal{Y}', \mathcal{Y}} = 0, \quad \forall v \in \mathcal{Y}. \]
Thus, \(B(u_n - \tilde{u}_n) = 0\) which implies \(u_n - \tilde{u}_n = 0\) since \(B\) is bounded below.

The a priori bound (4.7a) is straightforwardly obtained by replacing \(v_m = r_m\) in (4.6a) and using (4.6b) together with the Cauchy–Schwartz inequality.

For the a priori bound (4.7b), we refer to Proposition 4.8 in Section 4.4. □

Although \(\mathcal{Y}_m\) should be sufficiently large for stability, there is no need for it to be close to the entire \(\mathcal{Y}\). The following proposition essentially shows that the goal of \(\mathcal{Y}_m\) is to resolve the residual \(r\) in the semi-discrete formulation (3.7).

**Proposition 4.2 (Optimal \(\mathcal{Y}_m\))** Assuming the same conditions of Theorem 4.B, let \((r, u_n) \in \mathcal{Y} \times \mathcal{U}_n\) be the solution of the semi-discrete formulation (3.7). If \(r \in \mathcal{Y}_m\), then the pair \((r, u_n)\) is also the unique solution of the fully-discrete formulation (4.6).

**Proof** Let \((\tilde{r}_m, \tilde{u}_n) \in \mathcal{Y}_m \times \mathcal{U}_n\) be the unique solution of (4.6) which is guaranteed by Theorem 4.B. The aim is to prove that \((\tilde{r}_m, \tilde{u}_n) = (r, u_n)\). So testing (3.7a) with \(v_m \in \mathcal{Y}_m\) and subtracting (4.6a) (satisfied by \((\tilde{r}_m, \tilde{u}_n)\)) we get:
\[ \langle B(u_n - \tilde{u}_n), v_m \rangle_{\mathcal{Y}', \mathcal{Y}} = -\langle J_Y(r) - J_Y(\tilde{r}_m), v_m \rangle_{\mathcal{Y}', \mathcal{Y}}, \quad \forall v_m \in \mathcal{Y}_m. \quad (4.8) \]
In particular for \(v_m = r - \tilde{r}_m \in B(\mathcal{U}_n)^\perp\) we obtain:
\[ \langle J_Y(r) - J_Y(\tilde{r}_m), r - \tilde{r}_m \rangle_{\mathcal{Y}', \mathcal{Y}} = 0, \]
which implies \(r = \tilde{r}_m\) by strict monotonicity of \(J_Y\). Going back to (4.8) we get:
\[ \langle B(u_n - \tilde{u}_n), v_m \rangle_{\mathcal{Y}', \mathcal{Y}} = 0, \quad \forall v_m \in \mathcal{Y}_m, \]
which implies \(u_n = \tilde{u}_n\) by the Fortin condition (4.5c) and the injectivity of \(B\). □
4.3 Error analysis of the inexact method

We next present an error analysis for the inexact method. Since the method is fundamentally related to (discrete) residual minimization, the most straightforward error estimate is of a posteriori type. Immediately after, an a priori error estimate follows naturally from the a posteriori estimate (compare with the error estimates for the exact residual-minimization method in (3.4)). The constant in the resulting a priori estimate can however be improved by resorting to an alternative analysis technique, which we present in Section 4.4.

Theorem 4.3 (Inexact method: A posteriori error estimate) Let $U$ and $V$ be two Banach spaces and let $B : U \to V^*$ be a linear, continuous and bounded-below operator, with continuity constant $M_B > 0$ and bounded-below constant $\gamma_B > 0$. Assume that $V$ and $V^*$ are reflexive and strictly convex. Let $U_n \subset U$ and $V_m \subset V$ be finite-dimensional subspaces such that the (Fortin) Assumption 4.1 holds true. Given $f = Bu \in V^*$, let $(r_m, u_n) \in V_m \times U_n$ be the unique solution of the discrete mixed problem:

$$
\begin{cases}
\langle J_V(r_m), v_m \rangle_{V^*,V} + \langle Bu_n, v_m \rangle_{V^*,V} = \langle f, v_m \rangle_{V^*,V}, & \forall v_m \in V_m, \\
\langle B^* r_m, w_n \rangle_{U^*_n,U} = 0, & \forall w_n \in U_n.
\end{cases}
$$

Then $u_n$ satisfies the following a posteriori error estimate:

$$
\|u - u_n\|_U \leq \frac{1}{\gamma_B} \text{osc}(f) + \frac{C_\Pi}{\gamma_B} \|r_m\|_V, \tag{4.9}
$$

where the data-oscillation term $\text{osc}(f)$ satisfies

$$
\text{osc}(f) := \sup_{v \in V} \frac{\langle f, v - \Pi v \rangle}{\|v\|_V} \leq M_B D_{\Pi} \inf_{w_n \in U_n} \|u - w_n\|_U, \tag{4.10}
$$

and $\|r_m\|_V$ satisfies

$$
\|r_m\|_V \leq M_B \inf_{w_n \in U_n} \|u - w_n\|_U. \tag{4.11}
$$

The constants $C_\Pi$ and $D_{\Pi}$ correspond to the boundedness constants related to the Fortin operator $\Pi : V \to V_m$ (see eq. (4.5a)-(4.5b)).

Remark 4.3 (Lower bounds) Observe that (4.11) and (4.10) say that:

$$
\|r_m\|_V \leq M_B \|u - u_n\|_U \quad \text{and} \quad \text{osc}(f) \leq M_B D_{\Pi} \|u - u_n\|_U.
$$

Hence, the a posteriori error estimate in Theorem 4.3 is reliable and efficient. This extends the result by Carstensen, Demkowicz & Gopalakrishnan [10] and Cohen, Dahmen & Welper [15, Proposition 3.2] for the Hilbert-space version of the method.
Proof Using that $B$ is bounded from below, and that $Bu = f$, we get:

$$
\|u - u_n\| \leq \frac{1}{\gamma_B} \|Bu - Bu_n\| = \frac{1}{\gamma_B} \sup_{v \in V} \frac{(f - Bu_n, v - \Pi v + \Pi v)_{V^*,V}}{\|v\|_V}.
$$

Next, by definition of the $\Pi$ operator (eq. (4.5)), $Bu_n \in V^*$ annihilates $v - \Pi v$, for all $v \in V$. Hence, splitting the supremum we obtain:

$$
\|u - u_n\| \leq \frac{1}{\gamma_B} \sup_{v \in V} \frac{(f, v - \Pi v)_{V^*,V}}{\|v\|_V} + \frac{1}{\gamma_B} \sup_{v \in V} \frac{(f - Bu_n, \Pi v)_{V^*,V}}{\|v\|_V}
$$

where we used boundedness of $\Pi$. To obtain (4.9), we observe that:

$$
\sup_{v \in V} \frac{(f - Bu_n, \Pi v)_{V^*,V}}{\|\Pi v\|_V} = \sup_{v \in V} \frac{(f, v - \Pi v)_{V^*,V}}{\|\Pi v\|_V} \leq \|J(V(r_m), \Pi v)_{V^*,V} \leq \|J(V(r_m))\|_{V^*} = \|r_m\|_{V^*}.
$$

Next, observe that for all $w_n \in U_n$ we have

$$
osc(f) = \sup_{v \in V} \frac{(f, v - \Pi v)_{V^*,V}}{\|v\|_V} = \sup_{v \in V} \frac{(f - Bw_n, v - \Pi v)_{V^*,V}}{\|v\|_V} \leq MB\|u - w_n\|_U.
$$

Finally, by the proof of Theorem 4.A, part (i) $\iff$ (ii),

$$
\|I^*_m(f - Bu_n)\|_{(V_m)^*} = \frac{(J(V(r_m), r_m)_{V^*,V}}{\|r_m\|_V} \leq \|r_m\|_V \leq \|I^*_m(f - Bw_n)\|_{(V_m)^*},
$$

and

$$
\|I^*_m(f - Bw_n)\|_{(V_m)^*} \leq \|Bu - Bw_n\|_{V^*} \leq MB\|u - w_n\|_U.
$$

A straightforward a priori error estimate follows naturally from the results in Theorem 4.C.

Corollary 4.4 (Inexact method: A priori error estimate I) Under the same assumptions of Theorem 4.C, we have the following a priori error estimate:

$$
\|u - u_n\| \leq \frac{1}{\gamma_B} osc(f) + \frac{C\Pi M_B}{\gamma_B} \inf_{w_n \in U_n} \|u - w_n\|_U
$$

(4.12a)

$$
\leq \frac{(D\Pi + C\Pi)M_B}{\gamma_B} \inf_{w_n \in U_n} \|u - w_n\|_U.
$$

(4.12b)

Remark 4.5 (Oscillation) In the context of finite-element approximations, the data-oscillation term in (4.12a) can generally be expected to be of higher order than indicated by the upper bound in (4.12b); see discussion in [10].

Remark 4.6 ($V_m = V$) Note that if $V_m = V$, then $osc(f) = 0$, $D\Pi = 0$ and $C\Pi = 1$ (choose $\Pi = I$), so that the estimates in (4.9) and (4.12) reduce to those in the semi-infinite case (3.4).
4.4 Direct a priori error analysis of the inexact method

A direct a priori error analysis is possible for the inexact method, without going through an a posteriori error estimate. The benefit of the direct analysis is that the resulting estimate is sharper than the worst-case upper bound given in (4.12b).

The main idea of the direct analysis is based on the sequence of inequalities (formalized below):

\[
\|u - u_n\|_U \leq \|I - P_n\| \|u - w_n\|_U \leq C \|P_n\| \|u - w_n\|_U \quad \forall w_n \in U_n ,
\]

(4.13)

where \(I\) is the identity, \(P_n\) is the projector defined below in Definition 4.7 and the norm \(\|\cdot\|\) corresponds to the standard operator norm.

To define our projector \(P_n\), consider any \(u \in U\). Next, let \((r_m, u_n) \in V_m \times U_n\) be the solution of the inexact monotone mixed method (4.1) with \(f = Bu \in V^*\), i.e.,

\[
\begin{cases}
\langle J_V(r_m), v_m \rangle_{V^*, V} + \langle Bu_n, v_m \rangle_{V^*, V} = \langle Bu, v_m \rangle_{V^*, V} & \forall v_m \in V_m , \\
\langle B^* r_m, w_n \rangle_{U^*, U} = 0 & \forall w_n \in U_n .
\end{cases}
\]

(4.14a)

(4.14b)

**Definition 4.7 (Nonlinear PG projector)** Under the same conditions of Theorem 4.B, we define the (inexact) **nonlinear Petrov–Galerkin projector** to be the well-defined map

\(P_n : U \to U_n\) such that \(P_n(u) := u_n\), with \(u_n\) the second argument of the solution \((r_m, u_n)\) of (4.14).

The next result establishes important properties of \(P_n\), including a fundamental bound that depends on the geometric constant \(C_{AO}(V) \in [0, 1]\) (see Definition 2.14).

**Proposition 4.8 (Nonlinear PG projector properties)** Under the conditions of Theorem 4.B, let \(P_n : U \to U_n\) denote the nonlinear Petrov–Galerkin projector of Definition 4.7. Then the following properties hold true:

(i) \(P_n\) is a nontrivial projector: \(0 \neq P_n = P_n \circ P_n \neq I\).

(ii) \(P_n\) is homogeneous: \(P_n(\lambda u) = \lambda P_n(u), \forall u \in U\) and \(\forall \lambda \in \mathbb{R}\).

(iii) \(P_n\) is bounded and

\[
\|P_n\| = \sup_{u \in U} \frac{\|P_n(u)\|_U}{\|u\|_U} \leq \frac{C_B^U}{\gamma_B} (1 + C_{AO}(V)) M_B .
\]

(4.15)

(iv) \(P_n\) is distributive in the following sense:

\[
P_n(u - P_n(w)) = P_n(u) - P_n(w), \quad \forall u, w \in U .
\]

(4.16)
(v) \( P_n \) is a generalized orthogonal projector in the sense that
\[
P_n(u) = P_n \left( P_n (u) + \eta (I - P_n)(u) \right), \quad \text{for any } \eta \in \mathbb{R} \text{ and any } u \in U.
\]

**Proof** See Section 4.5. ■

Property (iv) in Proposition 4.8 is key to establishing the first inequality in (4.13), indeed, for any \( w_n \in U_n \),
\[
\| u - P_n(u) \|_U = \| u - w_n - P_n(u - w_n) \|_U \leq \| I - P_n \| \| u - w_n \|_U.
\]

(4.17)

On the other hand, the second inequality in (4.13) can be established through properties (i)–(iii) and (v) in Proposition 4.8. Indeed, these properties correspond to the four requirements for the abstract nonlinear projector \( Q \) of Lemma 2.9. Hence we immediately obtain the following key estimate:

**Corollary 4.9 (Nonlinear PG projector estimate)** Under the conditions of Proposition 4.8, it holds that
\[
\| I - P_n \| \leq C_S \| P_n \|,
\]
with \( C_S := \min \left\{ 1 + \| P_n \|^{-1}, C_{bm}(U) \right\} \).

**Proof** Apply Lemma 2.9. ■

In conclusion, by combining (4.17) with Corollary 4.9 and the bound in (4.15), we have established the following main result.

**Theorem 4.4 (Inexact method: A priori error estimate II)** Let \( U \) and \( V \) be two Banach spaces and let \( B : U \to V^* \) be a linear, continuous and bounded-below operator, with continuity constant \( M_B > 0 \) and bounded-below constant \( \gamma_B > 0 \). Assume that \( V \) and \( V^* \) are reflexive and strictly convex. Let \( U_n \subset U \) and \( V_m \subset V \) be finite-dimensional subspaces such that the (Fortin) Assumption 4.1 holds true. Given \( f = Bu \in V^* \), let \((r_m, u_n) \in V_m \times U_n\) be the unique solution of the discrete mixed problem:
\[
\begin{aligned}
\langle J_V(r_m), v_m \rangle_{V^*, V} + \langle Bu_n, v_m \rangle_{V^*, V} &= \langle f, v_m \rangle_{V^*, V}, \quad \forall v_m \in V_m, \\
\langle B^* r_m, w_n \rangle_{U^*, U} &= 0, \quad \forall w_n \in U_n.
\end{aligned}
\]

Then \( u_n \) satisfies the a priori error estimate:
\[
\| u - u_n \|_U \leq C \inf_{w_n \in U_n} \| u - w_n \|_U
\]
with
\[
C = \min \left\{ \frac{C_H}{\gamma_B} \left( 1 + C_{\alpha}(V) \right) M_B C_{bm}(U), 1 + \frac{C_H}{\gamma_B} \left( 1 + C_{\alpha}(V) \right) M_B \right\}.
\]
(See Assumption 4.1 for the definition of \( C_H \); Definition 2.7 for \( C_{bm}(U) \); and Definition 2.14 for \( C_{\alpha}(V) \).)
Remark 4.10 (Hilbert-space case) If $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, then $C_{\text{BM}} = 1$ and $C_{\text{AO}}(\mathcal{V}) = 0$, hence $C = C_{\Pi}M_B/\gamma_B$ in the a priori error estimate of Theorem 4.D. This coincides with the known result in the Hilbert-space setting [25]; see Section 5.3 for further details on the connection to the method in Hilbert spaces. □

Corollary 4.11 (Vanishing discrete residual) If $r_m = 0$, the a priori error estimate in Theorem 4.D reduces to:

$$\|u - u_n\|_{\mathcal{U}} \leq \min \left\{ \frac{C_{\Pi}M_B}{\gamma_B} C_{\text{BM}}(\mathcal{U}), 1 + \frac{C_{\Pi}M_B}{\gamma_B} \right\} \inf_{w_n \in \mathcal{U}_n} \|u - w_n\|_{\mathcal{U}}.$$

□

Proof If $r_m = 0$, Eq. (4.19) in the proof of Proposition 4.8 implies the simpler bound:

$$\|P_n(u)\|_{\mathcal{U}} \leq \frac{C_{\Pi}}{\gamma_B} M_B \|u\|_{\mathcal{U}}.$$

Combining (4.17) with Corollary 4.9 and this bound, gives the desired result. □

One particular situation for which $r_m = 0$ occurs, is when discrete dimensions are matched: $\text{dim } \mathcal{V}_m = \text{dim } \mathcal{U}_n$. In that case, one recovers actually the standard Petrov–Galerkin method; see Section 5.1 for an elaboration on this connection.

4.5 Proof of Proposition 4.8

In this section we proof Proposition 4.8. We proceed item by item.

(i) Take $u \in \mathcal{U}$ and plug $u_n = P_n(u)$ in the right-hand side of (4.14a). Then the unique solution of the mixed system (4.14) will be $(0, u_n)$. Therefore $P_n(P_n(u)) = P_n(u_n) = u_n$. The fact that $P_n \neq 0$ and $P_n \neq I$ is easy to verify whenever $\mathcal{U}_n \neq \{0\}$ and $\mathcal{U}_n \neq \mathcal{U}$.

(ii) The result follows by multiplying both equations of the mixed system (4.14) by $\lambda \in \mathbb{R}$ and using the homogeneity of the duality map (see Proposition 2.2).

(iii) Consider any $u \in \mathcal{U}$ and let $(r_m, u_n) \in \mathcal{V}_m \times \mathcal{U}_n$ denote the solution to (4.14). Observe that

$$\|P_n(u)\|_{\mathcal{U}} = \|u_n\|_{\mathcal{U}} \leq \frac{1}{\gamma_B} \sup_{v \in \mathcal{V}} \frac{\langle Bu_n, v \rangle_{\mathcal{V}^*, \mathcal{V}}}{\|v\|_{\mathcal{V}}} \leq \frac{C_{\Pi}}{\gamma_B} \sup_{v \in \mathcal{V}} \frac{\langle Bu_n, \Pi v \rangle_{\mathcal{V}^*, \mathcal{V}}}{\|\Pi v\|_{\mathcal{V}}}.$$

Let $y_m = J_{\Pi}^{-1} V_m(I_m B u_n)$ and note that $y_m \in \mathcal{V}_m \subset \mathcal{V}$ is the supremizer of the last expression in (4.18). Hence, using (4.14a) we get

$$\|P_n(u)\|_{\mathcal{U}} \leq \frac{C_{\Pi}}{\gamma_B} \frac{\langle Bu_n, y_m \rangle_{\mathcal{V}^*, \mathcal{V}}}{\|y_m\|_{\mathcal{V}}} \leq \frac{C_{\Pi}}{\gamma_B} \left( \frac{\langle Bu_n, y_m \rangle_{\mathcal{V}^*, \mathcal{V}}}{\|y_m\|_{\mathcal{V}}} - \frac{\langle J_{\Pi}(r_m), y_m \rangle_{\mathcal{V}^*, \mathcal{V}}}{\|r_m\|_{\mathcal{V}}} \right).$$

(4.19)
The first term in the parentheses above is clearly bounded by \( \| Bu \|_{\mathcal{V}^*} \). To bound the second term first observe that

\[
\langle J_{\mathcal{V}}(y_m), r_m \rangle_{\mathcal{V}^*, \mathcal{V}} = \langle Bu_n, r_m \rangle_{\mathcal{V}^*, \mathcal{V}} = 0,
\]

where we used (4.14b). Thus, \((r_m, y_m) \in \mathcal{O}_\mathcal{V}\) (see Lemma 2.17 (i)) which implies that the second term is bounded by \( C_{AO}(\mathcal{V}) \| r_m \|_{\mathcal{V}} \). Using (4.14), note that

\[
\| r_m \|_{\mathcal{V}} = \| Bu - Bu_n, r_m \|_{\mathcal{V}^*, \mathcal{V}} \leq \| Bu \|_{\mathcal{V}^*}.
\]

In conclusion,

\[
\| P_n(u) \|_{\mathcal{U}} \leq \frac{C_{\Pi}}{\gamma_B} (1 + C_{AO}(\mathcal{V})) \| Bu \|_{\mathcal{V}^*}, \quad \forall u \in \mathcal{U}
\]

and we get the desired result upon using \( \| Bu \|_{\mathcal{V}^*} \leq M_B \| u \|_{\mathcal{U}} \).

We note that an alternative proof can be given based on the second a priori bound for the best approximation (Proposition 2.19) and Lemma 2.17(ii).

(iv) Let \((r_m, u_n)\) be the solution of the mixed system (4.14) and for some \( \tilde{w} \in \mathcal{U} \), let \( \tilde{w}_n = P_n(\tilde{w}) \in \mathcal{U}_n \). By subtracting \( \langle Bu_n, v_m \rangle_{\mathcal{V}^*, \mathcal{V}} \) on both sides of the identity in (4.14a), we get that \((r_m, u_n - \tilde{w}_n)\) is the unique solution of (4.14) with right-hand side \( \langle B(u - \tilde{w}_n), v_m \rangle_{\mathcal{V}^*, \mathcal{V}} \). Therefore \( P(u - \tilde{w}_n) = u_n - \tilde{w}_n \).

(v) Statement (v) follows from statements (ii) and (iv), indeed, for any \( \eta \in \mathbb{R} \),

\[
P_n\left(P_n(u) + \eta \left(u - P_n(u)\right)\right) = P_n\left(\eta u + P_n\left((1 - \eta)u\right)\right) \quad \text{(by (ii))}
= P_n(\eta u) + P_n\left((1 - \eta)u\right) \quad \text{(by (iv))}
= P_n(u). \quad \text{(by (ii))}
\]

5 Connection to other theories

In this last section, we elaborate on how the presented quasi-optimality analysis in Section 4 generalizes existing theories for other methods. In doing so, we collect some of our earlier observations and provide a coherent summary of the connections.

Figure 2 presents a schematic hierarchy with the connections among the methods and the constants \( C \) in their respective quasi-optimality bound:

\[
\| u - u_n \|_{\mathcal{U}} \leq C \inf_{w_n \in \mathcal{U}_n} \| u - w_n \|_{\mathcal{U}}.
\]

(5.1)
Figure 2: Hierarchy of discretization methods and their quasi-optimality result; see Sections 5.1–5.3 for a detailed explanation. To lighten the notation, \( \gamma \equiv \gamma_B, \ M \equiv M_B \) and \( C_{BM} \equiv C_{BM}(U) \).

At the top of the figure is the inexact residual minimization (iRM) method, or equivalently, the inexact nonlinear Petrov–Galerkin (iNPG) or monotone mixed method (MMM). By considering certain special cases, quasi-optimality constants are recovered for Petrov-Galerkin (PG) methods, exact residual minimization (RM), and inexact residual minimization in Hilbert spaces (iRM-H), which includes the DPG method. Naturally, in these connections, the conditions of Theorem 4.4 (discrete well-posedness of the inexact method) are assumed to hold.

### 5.1 Petrov–Galerkin methods

The standard Petrov–Galerkin method (PG in Figure 2) is obtained when \( \dim V_m = \dim U_n \). Indeed, under this stipulation, the (Fortin) Assumption 4.1 implies the well-known discrete inf-sup condition (see, e.g., [22, 23]) with discrete inf-sup constant \( \zeta_B = \gamma_B/C_{\Pi} \), where \( \gamma_B \) is the bounded-below constant of \( B \) and \( C_{\Pi} \) the boundedness constant of the Fortin operator \( \Pi \). Therefore, \( (r_m, w_n) \mapsto \langle Bu_n, v_m \rangle_{V, V} \) corresponds to an invertible square system, so that (4.1b) implies \( r_m = 0 \), while (4.1a) reduces to the standard (linear) Petrov–Galerkin form:

\[
\langle Bu_n, v_m \rangle_{V, V} = \langle f, v_m \rangle_{V, V}, \quad \forall v_m \in V_m.
\] (5.2)

The quasi-optimality result of Corollary 4.11 applies in this situation
(since \( r_m = 0 \)), resulting in the constant
\[
C = \min \left\{ \frac{M_B}{\tilde{\gamma}} C_{nm}(U), \ 1 + \frac{M_B}{\tilde{\gamma}} \right\}.
\]

This coincides with the recent result obtained by Stern [34]. Historically, the first quasi-optimality analysis for the PG method was carried out in the pioneering work of Babuška [3] who obtained the classical result \( C = 1 + M_B/\tilde{\gamma} \).

Furthermore, if \( U \) is a Hilbert space, then \( C_{nm}(U) = 1 \). Hence, when \( U \) and \( V \) are Hilbert spaces, the constant reduces to \( C = M_B/\tilde{\gamma} \), which is the established result for PG methods in Hilbert spaces (PG-H); see Xu & Zikatanov [37]. Furthermore, when the discrete test space \( V_m \) equals \( \mathbb{R}^{-1} V_B U_n \), one obtains the optimal PG method in Hilbert spaces (oPG-H). In that case \( \tilde{\gamma} = \gamma \), and one recovers the result \( C = M_B/\gamma_B \) as obtained by Demkowicz & Gopalakrishnan [19]. We note that \( M_B/\gamma_B \) can be made equal to 1, by suitably re-norming \( U \) or \( V \) [39, 16].

5.2 Residual minimization

The exact residual minimization method (1.5) (RM in Figure 2) is obviously recovered when \( V_m = V \) in (1.7). To demonstrate that the corresponding quasi-optimality result is also recovered, note that when \( V_m = V \), the (Fortin) Assumption 4.1 is straightforwardly satisfied by taking \( \Pi = I \) (the identity). Then, \( C_{\Pi} = 1 \) and \( \text{osc}(f) \) as defined in (4.10) vanishes, which reduces the a priori error estimate in Corollary 4.4 to (5.1) with \( C = M_B/\gamma_B \). This result indeed coincides with the one for exact residual minimization (see (3.4)) as originally obtained by Guermond [26]. Furthermore, the a priori bound (3.3b) can also be recovered, upon substituting \( C_{\Pi} = 1 \) in (4.20) and taking into account that \( C_{AO}(V^*) = C_{AO}(V) \) (see Lemma 2.17).

As an alternative to the case \( V_m = V \), the exact residual minimizer \( u_n = \arg \min_{w_n \in U_n} \| f - B w_n \|_{V^*} \) is also obtained when the (continuous) residual representer happens to be in \( V_m \), i.e.,
\[
r := J_V^{-1}(f - B u_n) \in V_m.
\]
(5.3)

See Proposition 4.2 for the equivalence in this special situation.

Interestingly, when \( U \) and \( V \) are Hilbert spaces, the quasi-optimality constant \( C = M_B/\gamma_B \) for exact residual minimization remains the same as in the Banach-space case. This is consistent with the fact that the resulting method is equivalent to the optimal Petrov–Galerkin method in Hilbert spaces (oPG-H), as discussed in Section 1.3; see (1.12).
5.3 Inexact method in Hilbert spaces

The most important fact of the inexact residual minimization method in Hilbert spaces (iRM-H in Figure 2) is that it is a linear method. Indeed, in this case the duality map $J_V$ is the Riesz map $R_V$, hence

$$\langle J_V(r_m), v_m \rangle_V = \langle R_V r_m, v_m \rangle_{V^*} = \langle r_m, v_m \rangle_V,$$

where $\langle \cdot, \cdot \rangle_V$ denotes the inner product in $V$. Therefore, the monotone mixed method (4.1) reduces to:

$$\begin{cases}
(r_m, v_m)_V + \langle Bu_n, v_m \rangle_{V^*} = \langle f, v_m \rangle_{V^*}, & \forall v_m \in V_m, \\
\langle B^* r_m, w_n \rangle_{U^*} = 0, & \forall w_n \in U_n,
\end{cases}$$

which is equal to the mixed form of the DPG method [20] as well as the inexact optimal Petrov–Galerkin method in Hilbert spaces [15]. In the Hilbert case, $C_{\text{BM}}(U) = 1$ and $C_{\text{AO}}(V) = 0$, so that the quasi-optimality constant in Theorem 4.D reduces to $C = C_{\Pi} M_B / \gamma_B$. This coincides with the Hilbert-space result due to Gopalakrishnan and Qiu [25].

Furthermore, if $\dim V_m = \dim U_n$, by the same reasoning as in Section 5.1, one obtains a Petrov–Galerkin method in Hilbert spaces (PG-H). The quasi-optimality constant reduces to $C = M_B / \hat{\gamma}$, since the discrete inf-sup constant $\hat{\gamma}$ can be taken as $C_{\Pi} / \gamma$. On the other hand, if $V_m = V$ (or (5.3) is valid), by the same reasoning as in Section 5.2, one obtains exact residual minimization in Hilbert spaces, which in turn is equivalent to the optimal Petrov–Galerkin method in Hilbert spaces (oPG-H). In that case $C_{\Pi} = 1$, so that the quasi-optimality constant reduces to the expected result $C = M_B / \gamma_B$.

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References

[1] R. A. Adams and J. F. Fournier, Sobolev Spaces, vol. 140 of Pure and Applied Mathematics, Academic Press, Oxford, 2nd ed., 2003.

[2] E. Asplund, Positivity of duality mappings, Bull. Amer. Math. Soc., 73 (1967), pp. 200–203.

[3] I. Babuška, Error-bounds for finite element method, Numer. Math., 16 (1971), pp. 322–333.

[4] S. Banach, Theory of Linear Operations, Dover Books on Mathematics, Dover Publications, 2009. Reprint of the Elsevier Science Publishers, 1987 edition. Translation of original French version 1931.

[5] P. B. Bochev and M. D. Gunzburger, Least-Squares Finite Element Methods, vol. 166 of Applied Mathematical Sciences, Springer Science & Business Media, 2009.

[6] D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications, vol. 44 of Springer Series in Computational Mathematics, Springer, Berlin, 2013.

[7] D. Braess, Nonlinear Approximation Theory, vol. 7 of Springer Series in Computational Mathematics, Springer, Berlin, 1986.

[8] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.

[9] D. Broersen and R. Stevenson, A robust Petrov–Galerkin discretisation of convection–diffusion equations, Comput. Math. Appl., 68 (2014), pp. 1605–1618.

[10] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan, A posteriori error control for DPG methods, SIAM J. Numer. Anal., 52 (2014), pp. 1335–1353.

[11] J. Chan, J. A. Evans, and W. Qiu, A dual Petrov–Galerkin finite element method for the convection–diffusion equation, Comput. Math. Appl., 68 (2014), pp. 1513–1529.

[12] C. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, vol. 1965 of Lecture Notes in Mathematics, Springer, London, 2009.

[13] P. G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM, Philadelphia, 2013.

[14] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, vol. 62 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[15] A. Cohen, W. Dahmen, and G. Welper, Adaptivity and variational stabilization for convection-diffusion equations, M2AN Math. Model. Numer. Anal., 46 (2012), pp. 1247–1273.
[16] W. Dahmen, C. Huang, C. Schwab, and G. Welper, *Adaptive Petrov–Galerkin methods for first order transport equations*, SIAM J. Numer. Anal., 50 (2012), pp. 2420–2445.

[17] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[18] L. Demkowicz and J. Gopalakrishnan, *A class of discontinuous Petrov–Galerkin methods. Part I. The transport equation*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 1558–1572.

[19] ———, *A class of discontinuous Petrov–Galerkin methods. II. Optimal test functions*, Numer. Methods Partial Differential Equations, 27 (2011), pp. 70–105.

[20] ———, *An overview of the discontinuous Petrov Galerkin method*, in Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations: 2012 John H Barrett Memorial Lectures, X. Feng, O. Karakashian, and Y. Xing, eds., vol. 157 of The IMA Volumes in Mathematics and its Applications, Springer, Cham, 2014, pp. 149–180.

[21] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, vol. 303 of Grundlehren der Mathematischen Wissenschaften, Springer, 1993.

[22] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Element Methods*, vol. 159 of Applied Mathematical Sciences, Springer-Verlag, New York, 2004.

[23] ———, *A converse to Fortin's Lemma in Banach spaces*, C. R. Math. Acad. Sci. Paris, 354 (2016), pp. 1092–1095.

[24] J. Gopalakrishnan, *Five lectures on DPG methods*. arXiv:1306.0557v2 [math.NA], Aug 2014.

[25] J. Gopalakrishnan and W. Qiu, *An analysis of the practical DPG method*, Math. Comp., 83 (2014), pp. 537–552.

[26] J. L. Guermond, *A finite element technique for solving first-order PDEs in $L^p$*, SIAM J. Numer. Anal., 42 (2004), pp. 714–737.

[27] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in Handbook of the Geometry of Banach Spaces, W. B. Johnson and J. Lindenstrauss, eds., vol. 1, Elsevier Science B. V., 2001, ch. 1, pp. 1–84.

[28] T. Kato, *Estimation of iterated matrices with application to von Neumann condition*, Numer. Math., 2 (1960), pp. 22–29.

[29] J. Lions, *Quelques Méthodes de Réolution des Problèmes aux Limites Non Linéaires*, Études Mathématiques, Dunod, 1969.

[30] J. T. Oden and L. F. Demkowicz, *Applied Functional Analysis*, CRC Press, 2nd ed., 2010.

[31] W. V. Petryshyn, *A characterization of strict convexity of Banach spaces and other uses of duality mappings*, J. Funct. Anal., 6 (1970), pp. 282–291.
[32] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, vol. 171 of Die Grundlehren der mathematischen Wissenschaften, Springer, Berlin, 1970.

[33] I. Stakgold and M. Holst, *Green’s Functions and Boundary Value Problems*, vol. 99 of Pure and Applied Mathematics, John Wiley & Sons, Hoboken, New Jersey, 3rd ed., 2011.

[34] A. Stern, *Banach space projections and Petrov–Galerkin estimates*, Numer. Math., 130 (2015), pp. 125–133.

[35] D. Szyld, *The many proofs of an identity on the norm of oblique projections*, Numer. Algorithms, 42 (2006), pp. 309–323.

[36] P. Wojtazczyk, *Banach Spaces for Analysts*, no. 25 in Cambridge studies for advanced mathematics, Cambridge University Press, Cambridge, 1991.

[37] J. Xu and L. Zikatanov, *Some observations on Babuška and Brezzi theories*, Numer. Math., 94 (2003), pp. 195–202.

[38] E. Zeidler, *Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

[39] J. Zitelli, I. Muga, L. Demkowicz, J. Gopalakrishnan, D. Pardo, and V. M. Calo, *A class of discontinuous Petrov–Galerkin methods. Part IV: The optimal test norm and time-harmonic wave propagation in 1D*, J. Comput. Phys., 230 (2011), pp. 2406–2432.