ON THE FREENESS OF RATIONAL CUSPIDAL PLANE CURVES

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Abstract. We bring additional support to the conjecture saying that a rational cuspidal plane curve is either free or nearly free. This conjecture was confirmed for curves of even degree, and in this note we prove it for many odd degrees. In particular, we show that this conjecture holds for the curves of degree at most 34.

1. Introduction

A plane rational cuspidal curve is a rational curve $C : f = 0$ in the complex projective plane $\mathbb{P}^2$, having only unibranch singularities. The study of these curves has a long and fascinating history, some long standing conjectures, as the Coolidge-Nagata conjecture being proved only recently, see [20], other conjectures, as the one on the number of singularities of such a curve being bounded by 4, see [24], are still open. The classification of such curves is not easy, there are a wealth of examples even when additional strong restrictions are imposed, see [17, 18, 19, 22, 23, 27].

Free divisors, defined by a homological property of their Jacobian ideals, have been introduced in a local analytic setting by K. Saito in [25], and then extended to projective hypersurfaces, see [3, 29] and the references there. We have remarked in [13] that many plane rational cuspidal curves are free. The remaining examples of plane rational cuspidal curves in the available classification lists turned out to satisfy a weaker homological property, which was chosen as the definition of a nearly free curve, see [14]. Subsequently, a number of authors have establish interesting properties of this class of curves, see [2, 21].

In view of the above remark, we have conjectured in [14] Conjecture 1.1 that any plane rational cuspidal curve $C$ is either free or nearly free. This conjecture was proved in [14] Theorem 3.1] for curves $C$ whose degree $d$ is even, as well as for some cases when $d$ is odd, e.g. when $d = p^k$, for a prime number $p > 2$. In this note we take a closer look at the case $d$ odd.

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables $x, y, z$ with complex coefficients, $f \in S$ a reduced homogeneous polynomial of degree $d \geq 2$, and let $f_x, f_y$ and $f_z$ be the partial derivatives of $f$ with respect to $x, y$ and $z$ respectively. Consider the graded $S$–submodule $AR(f) \subset S^3$ of all relations involving the derivatives of $f$, namely

$$\rho = (a, b, c) \in AR(f)$$

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if and only if \( af_x + bf_y + cf_z = 0 \) and \( a, b, c \) are in \( S_q \), the space of homogeneous polynomials of degree \( q \). The minimal degree of a Jacobian relation for the polynomial \( f \in S_d \) is the integer \( \text{mdr}(f) \) defined to be the smallest integer \( m \geq 0 \) such that \( AR(f)^m \neq 0 \). When \( \text{mdr}(f) = 0 \), then \( C : f = 0 \) is a union of lines passing through one point, and hence \( C \) is cuspidal only for \( d = 1 \).

We assume from now on in this note that \( \text{mdr}(f) \geq 1 \). It turns out that a rational cuspidal curve \( C : f = 0 \) with \( \text{mdr}(f) = 1 \) is nearly free. Indeed, this follows from [8, Proposition 4.1]. To see this, note that the implication (1) \( \Rightarrow \) (2) there holds for any \( d \geq 2 \).

Assume from now on that \( d \) is odd, and let

\[
d = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}
\]

be the prime decomposition of \( d \). We assume also that \( m \geq 2 \), the case \( m = 1 \) of our conjecture being settled in [14, Corollary 3.2]. By changing the order of the \( p_j \)'s if necessary, we can and do assume that \( p_1^{k_1} \geq p_2^{k_2} \) for any \( 2 \leq j \leq m \). Set \( e_1 = d/p_1^{k_1} \). With these assumptions and notations, the main results of this note are the following.

**Theorem 1.1.** Let \( C : f = 0 \) be a rational cuspidal curve of degree \( d = 2d' + 1 \) an odd number. Then \( \text{mdr}(f) \leq d' \) and if equality holds, then \( C \) is either free or nearly free.

**Theorem 1.2.** Let \( C : f = 0 \) be a rational cuspidal curve of degree \( d = 2d' + 1 \), an odd number as in (1.1). Then, if

\[
\text{mdr}(f) \leq r_0 := \frac{d - e_1}{2},
\]

then \( C \) is either free or nearly free. In particular, the following hold.

i) If \( d = 3p^k \), with \( p \) a prime number, then \( C \) is either free or nearly free.

ii) \( d = 5p^k \), with \( p \) a prime number, \( p^k > 3 \), then \( C \) is either free or nearly free, unless \( \text{mdr}(f) = d' - 1 \).

**Remark 1.3.** Note that, for \( d \neq 15 \), we have \( e_1 \leq d/7 \) and hence

\[
r_0 = \frac{d - e_1}{2} \geq \left\lceil \frac{d(1 - \frac{1}{7})}{2} \right\rceil = \left\lceil \frac{3d}{7} \right\rceil.
\]

Therefore, the only cases not covered by our results correspond to curves of odd degree \( d \), such that \( r = \text{mdr}(f) \) satisfies

\[
\left\lceil \frac{3d}{7} \right\rceil + 1 \leq r_0 + 1 \leq r \leq d' - 1 = \frac{d - 3}{2}.
\]

**Corollary 1.4.** A rational cuspidal curve \( C : f = 0 \) of degree \( d \) is either free or nearly free, if one of the following holds.

(1) \( \text{mdr}(f) \leq 15 \), or

(2) \( d \leq 90 \), unless we are in one of the following situations.

i) \( d = 35 \) and \( \text{mdr}(f) = 16 \);
ii) \(d = 45\) and \(\text{mdr}(f) = 21\);  
iii) \(d = 55\) and \(\text{mdr}(f) = 26\);  
iv) \(d = 63\) and \(\text{mdr}(f) \in \{29, 30\}\);  
v) \(d = 65\) and \(\text{mdr}(f) = 31\);  
vi) \(d = 77\) and \(\text{mdr}(f) \in \{36, 37\}\).  

In the excluded situations, our results do not allow us to conclude.

The proof of our main results are based on a deep result by U. Walther, see [31, Theorem 4.3] bringing into the picture the monodromy of the Milnor fiber \(F : f = 1\) associated to the curve \(C : f = 0\). A second ingredient is our results on the relations between the Hodge filtration and pole order filtration on the cohomology group \(H^1(F, \mathbb{C})\), see [12, Theorem 1.2] and [16, Proposition 2.2].

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2. Some facts about free and nearly free curves

Here we recall some basic notions on free and nearly free curves. We denote by \(J_f\) the Jacobian ideal of \(f\), i.e. the homogeneous ideal of \(S\) spanned by the partial derivatives \(f_x, f_y, f_z\) and let \(M(f) = S/J_f\) be the corresponding graded ring, called the Jacobian (or Milnor) algebra of \(f\). Let \(I_f\) denote the saturation of the ideal \(J_f\) with respect to the maximal ideal \(\mathfrak{m} = (x, y, z)\) in \(S\) and recall the relation with the 0-degree local cohomology

\[ N(f) := I_f/J_f = H^0_\mathfrak{m}(M(f)). \]

It was shown in [9, Corollary 4.3] that the graded \(S\)-module \(N(f)\) satisfies a Lefschetz type property with respect to multiplication by generic linear forms. This implies in particular the inequalities

\[ 0 \leq n(f)_0 \leq n(f)_1 \leq \ldots \leq n(f)_{\lceil T/2 \rceil} \geq n(f)_{\lfloor T/2 \rfloor + 1} \geq \ldots \geq n(f)_T \geq 0, \]

where \(T = 3d - 6\) and \(n(f)_k = \dim N(f)_k\) for any integer \(k\). If we set \(\nu(f) = \dim N(f)_{\lfloor T/2 \rfloor}\), then \(C : f = 0\) is a free curve if \(\nu(f) = 0\). We say that \(C : f = 0\) is a nearly free curve if \(\nu(f) = 1\), see [2, 7, 8, 13, 14] for more details, equivalent definitions and many examples.

Note that the curve \(C : f = 0\) is free if and only if the graded \(S\)-module \(AR(f)\) is free of rank 2, i.e. there is an isomorphism of graded \(S\)-modules

\[ AR(f) = S(-d_1) \oplus S(-d_2) \]

for some positive integers \(d_1 \leq d_2\). When \(C\) is free, the integers \(d_1 \leq d_2\) are called the exponents of \(C\). They satisfy the relations

\[ d_1 + d_2 = d - 1 \text{ and } \tau(C) = (d - 1)^2 - d_1d_2, \]

where \(\tau(C)\) is the total Tjurina number of \(C\), that is \(\tau(C) = \sum_{i=1}^p \tau(C, x_i)\), the \(x_i\)'s being the singular points of \(C\), and \(\tau(C, x_i)\) denotes the Tjurina number of the
isolated plane curve singularity \((C, x_i)\), see for instance \([10, 13]\). In the case of a nearly free curve, there are also the exponents \(d_1 \leq d_2\), and this time they verify

\begin{equation}
 d_1 + d_2 = d \quad \text{and} \quad \tau(C) = (d - 1)^2 - d_1(d_2 - 1) - 1.
\end{equation}

Both for a free and a nearly free curve \(C : f = 0\), one has \(mdr(f) = d_1\), and hence \(mdr(f) \leq (d - 1)/2\) for a free curve \(C\), and \(mdr(f) \leq d/2\) for a nearly free curve \(C\). It follows that Theorem 1.1 gives a similar inequality for any rational cuspidal curve. Our examples of rational cuspidal curves given in \([11]\), which are also free or nearly free, show that all the possible values of \(mdr(f)\) do actually occur for any fixed degree \(d\). It follows that

If we set \(r = mdr(f)\), then the curve \(C : f = 0\) is free (resp. nearly free) if and only if

\begin{equation}
 (2.4) \quad \tau(C) = \tau(d, r) := (d - 1)^2 - r(d - 1 - r)
\end{equation}

(resp. \(\tau(C) = \tau(d, r) - 1\)), see \([7]\).

\textbf{Remark 2.1.} If the equation \(f = 0\) of the curve \(C\) is given explicitly, then one can use a computer algebra software, for instance Singular \([4]\), in order to compute the integer \(mdr(f)\). Such a computer algebra software can of course decide whether the curve \(C\) is free or nearly free, see for instance the corresponding code on our website \texttt{http://math.unice.fr/~dimca/singular.html}.

However, for large degrees \(d\), it is much quicker to determine the integer \(mdr(f)\).

\section{The proofs}

First we recall the setting used in the proof of \([14, \text{Theorem 3.1}]\). The key results of U. Walther in \([31, \text{Theorem 4.3}]\) yield the inequality

\begin{equation}
 (3.1) \quad \dim N(f)_{2d-2-j} \leq \dim H^2(F, \mathbb{C})_{\lambda},
\end{equation}

for \(j = 1, 2, \ldots, d\), where \(F : f(x, y, z) - 1 = 0\) is the Milnor fiber in \(\mathbb{C}^3\) associated to the plane curve \(C\), and the subscript \(\lambda\) indicates the eigenspace of the monodromy action corresponding to the eigenvalue \(\lambda = \exp(2\pi i(d+1-j)/d) = \exp(-2\pi i(j-1)/d)\).

Assume that \(C\) is a rational cuspidal curve of degree \(d\). Denote by \(U\) the complement \(\mathbb{P}^2 \setminus C\), and note that its topological Euler characteristic is given by \(E(U) = E(\mathbb{P}^2) - E(C) = 1\). Since \(F\) is a cyclic \(d\)-fold covering of the complement \(U\), it follows that \(H^m(F, \mathbb{C})_1 = H^m(U, \mathbb{C}) = 0\) for \(m = 1, 2\). We have also

\[
 \dim H^2(F, \mathbb{C})_{\lambda} - \dim H^1(F, \mathbb{C})_{\lambda} + \dim H^0(F, \mathbb{C})_{\lambda} = E(U) = 1,
\]

see for instance \([5, \text{Prop. 1.21, Chapter 4}]\) or \([6, \text{Cor. 5.1 and Remark 5.1}]\). For any \(\lambda \neq 1\), since clearly \(H^0(F, \mathbb{C})_{\lambda} = 0\), we get

\begin{equation}
 (3.2) \quad \dim H^2(F, \mathbb{C})_{\lambda} = \dim H^1(F, \mathbb{C})_{\lambda} + 1.
\end{equation}
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3.1. Proof of Theorem 1.1. Suppose now that \( d \) is odd, say \( d = 2d' + 1 \). In order to prove \( \nu(f) \leq 1 \), in view of the inequality (3.1), it is enough to show that \( \dim H^2(F, \mathbb{C})_\lambda = 1 \), for \( \lambda = \exp(2\pi id'/2d' + 1) \) which corresponds to \( j = d' + 2 \). The equation (3.2) tells us that this is equivalent to \( \dim H^1(F, \mathbb{C})_\lambda = 0 \). Using [16, Proposition 2.2], see also [15, Remark 4.4], it follows that

\[
\dim H^1(F, \mathbb{C})_\lambda = \dim E_2^{1,0}(f)_k + \dim E_2^{1,0}(f)_{d-k},
\]

where \( k = j - 1 = d' + 1 \). Here \( E_2^{1,0}(f)_k \) and \( E_2^{1,0}(f)_{d-k} \) denote some terms of the second page of spectral sequences used to compute the monodromy action on the Milnor fiber \( F \), see [5, 12, 15, 26] for details. Note also that the weaker result in [12, Theorem 1.2] is enough for this proof. By the construction of these spectral sequences, it follows that, for \( q \leq d \), one has an identification

\[
E_2^{1,0}(f)_q = \{(a, b, c) \in AR(f)_{q-2} : a_x + b_y + c_z = 0\},
\]

where \( a_x \) is the partial derivative of \( a \) with respect to \( x \) and so on. It follows that

\[
\dim E_2^{1,0}(f)_k + \dim E_2^{1,0}(f)_{d-k} \leq \dim AR(f)_{d-1} + \dim AR(f)_{d-2}.
\]

If \( mdr(f) \geq d' \), it follows that \( AR(f)_{d-1} = AR(f)_{d-2} = 0 \), and hence the curve \( C \) is either free or nearly free. But this implies that \( mdr(f) \leq d' \), as explained in the previous section.

3.2. Proof of Theorem 1.2. For the reader’s convenience, we divide this proof into two steps.

Proposition 3.3. With the above notation, we have \( \dim N(f)_j \leq 1 \), for any integer \( j \leq d - 3 + r_0 \) and for any integer \( j \geq 2d - 3 - r_0 \).

Proof. Let \( t_1 = (p_1^{k_1} - 1)/2 \) and note that

\[
d - 3 + r_0 = d - 3 + e_1 t_1 < d - 3 + \frac{d}{2} = \frac{T}{2}.
\]

We apply the inequality (3.1) with \( j = d + 1 - e_1 t_1 = d + 1 - r_0 \). It follows that

\[
(3.3) \quad \dim N(f)_{d-3-r_0} \leq \dim H^2(F, \mathbb{C})_\lambda,
\]

with

\[
\lambda = \exp(2\pi i r_0/d) = \exp\left(\frac{2\pi i t_1}{p_1^{k_1}}\right).
\]

Since this eigenvalue has order a prime power, it follows from Zariski’s Theorem, see [11, Proposition 2.1], that \( H^1(F, \mathbb{C})_\lambda = 0 \). Using (3.2), we get \( \dim N(f)_j \leq 1 \) for \( j = d - 3 + r_0 \). The claim for \( j = 2d - 3 - r_0 \) follows from the fact that the graded module \( N(f) \) enjoys a duality property: \( \dim N(f)_j = \dim N(f)_{T-j} \), for any integer \( j \), see [28, 30]. The Lefschetz type property of the graded module \( N(f) \), see (2.1), completes the proof of this Proposition.

Proposition 3.4. A rational cuspidal curve \( C : f = 0 \) of degree \( d \) as in (1.1), and such that \( r = mdr(f) \leq r_0 \), is either free or nearly free.
Proof. We use the formulas (2.2) and (2.3) from [7] and get the following equality
\[
\dim AR(f)_{d-r-2} - \dim N(f)_{2d-r-3} + \dim AR(f)_{r-2} = \\
3\left(\frac{d-r}{2}\right) - \left(\frac{2d-r-1}{2}\right) + \tau(C).
\]
For any curve \( C : f = 0 \), it is known that, if \( \rho_1 \in AR(f)_r \), then the first relation \( \rho \in AR(f)_m \) which is not a multiple of \( \rho_1 \) occurs in a degree \( m \geq d-r-1 \), see [29, Lemma 1.1]. It follows that
\[
\dim AR(f)_{d-r-2} = \dim S_{d-2r-2} : \rho_1 = \left(\frac{d-2r}{2}\right) = \frac{(d-2r)(d-2r-1)}{2},
\]
for any \( r \) such that \( 2r \leq d \). Using the obvious fact that \( AR(f)_{r-2} = 0 \), a direct computation shows that
\[
\tau(C) = \tau(d, r) - \dim N(f)_{2d-r-3}.
\]
Since \( r \leq r_0 \), it follows that \( 2d - r - 3 \geq 2d - r_0 - 3 \), and hence \( \dim N(f)_{2d-r-3} \leq 1 \) by Proposition 3.3. The claim follows now using the characterization of free (resp. nearly free) curves given above in (2.4).

It remains to prove the last claim in Theorem 1.2. If \( d = 3p^k \), we can assume \( p^k = 2p' + 1 > 3 \) and then
\[
r_0 = \frac{d - e_1}{2} = 3p'.
\]
On the other hand, \( d = 3p^k = 6p' + 3 = 2(3p' + 1) + 1 \), and hence \( d' = 3p' + 1 \). Since \( mdr(f) \leq d' \) by Theorem 1.2, we get either \( mdr(f) = d' = 3p' + 1 \), and then we conclude by Theorem 1.2 or \( mdr(f) \leq d' - 1 = 3p' = r_0 \), and then we conclude using Theorem 1.2.

3.5. Proof of Corollary 1.4. To prove the first claim, we have to consider the minimal possible value of
\[
r_0 = \frac{d - e_1}{2} = e_1 \frac{p_{k_1} - 1}{2},
\]
when \( d \) is odd, but neither a prime power, nor of the form \( 3p \), with \( p > 3 \) prime. First, if \( p_1 = 3 \), then \( k_1 \geq 2 \) and \( e_1 \geq 5 \), hence \( r_0 \geq 20 \). Otherwise, \( p_1 \geq 5, e_1 \geq 3, \) but both cannot be equalities. It follows that the minimal values are obtained for \( p_1 = 5^2 \) and \( e_1 = 3 \), or \( p_1 = 7 \) and \( e_1 = 5 \). In the first case we get \( r_0 = 36 \), in the second we get \( r_0 = 15 \). To prove the second claim, just use Remark 1.3.

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