Mass - Proper Time Uncertainty Relation in a Manifestly Covariant
Relativistic Statistical Mechanics

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Abstract

We prove the uncertainty relation $T_{\Delta V} \Delta m \gtrsim 2\pi \hbar/c^2$, which is realized on a statistical mechanical level for an ensemble of events in $(1 + D)$-dimensional spacetime with motion parametrized by an invariant “proper time” $\tau$, where $T_{\Delta V}$ is the average passage interval in $\tau$ for the events which pass through a small (typical) $(1 + D)$-volume $\Delta V$, and $\Delta m$ is the dispersion of mass around its on-shell value in such an ensemble. We show that a linear mass spectrum is a completely general property of a $(1 + D)$-dimensional off-shell theory.

Key words: special relativity, mass spectrum, uncertainty relation

PACS: 03.30.+p, 05.20.Gg, 05.30.Ch, 98.20.–d

1 Introduction

In this paper we shall use a manifestly covariant form of statistical mechanics which has more general structure than the standard forms of relativistic statistical mechanics, but which reduces to those theories in a certain limit, to be described precisely below. These theories, which are characterized classically by mass-shell constraints, and the use, in quantum field theory, of fields which are constructed on the basis of on-mass-shell free fields, are associated with the statistical treatment of world lines and hence, considerable coherence (in terms of the macroscopic structure of whole world lines as the elementary objects of the theory) is implied. In nonrelativistic statistical mechanics, the elementary objects of the theory are points. The relativistic analog of this essentially structureless foundation for a statistical theory is the set of points in spacetime, i.e., the so-called events, not the world lines (Currie, Jordan and Sudarshan [1] have discussed the difficulty of constructing a relativistic mechanics on the basis of world lines).

The mass of particles in a mechanical theory of events is necessarily a dynamical variable, since the classical phase space of the relativistic set of events consists of the spacetime and energy-momentum coordinates $\{q_i, t_i; p_i, E_i\}$, with no a priori constraint on the relation between the $p_i$ and the $E_i$, and hence such theories are

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“off-shell”. It is well known from the work of Newton and Wigner \[2\] that on-shell relativistic quantum theories such as those governed by Klein-Gordon or Dirac type equations do not provide local descriptions (the wave functions corresponding to localized particles are spread out); for such theories the notion of ensembles over local initial conditions is difficult to formulate. The off-shell theory that we shall use here is, however, precisely local in both its first and second quantized forms \[3, 4\].

We finally remark that the standard formulations of quantum relativistic statistical mechanics, and quantum field theory at finite temperature, lack manifest covariance on a fundamental level. As for nonrelativistic statistical mechanics, the partition function is described by the Hamiltonian, which is not an invariant object, and hence thermodynamic mean values do not have tensor properties. [One could consider the invariant $p_{\mu}n^\mu$ in place of the Hamiltonian \[3\], where $n^\mu$ is a unit four-vector; this construction (supplemented by a spacelike vector orthogonal to $n^\mu$) implies an induced representation for spacetime. The quantity that takes the place of the parameter $t$ is then $x_\mu n^\mu$; in the corresponding quantum mechanics, the space parts of (induced form of) the momentum do not commute with this time variable. Some of the problems associated with this construction are closely related to those pointed out by Currie, Jordan and Sudarshan \[1\], for which different world lines are predicted dynamically by the change in the form of the effective Hamiltonian in different frames.] Since the form of such a theory is not constrained by covariance requirements, its dynamical structure and predictions may be different than for a theory which satisfies these requirements. For example, the canonical distribution of Pauli \[2\] for the free Boltzmann gas has a high temperature limit in which the energy is given by $3k_B T$, which does not correspond to any known equipartition rule, but for the corresponding distribution for the manifestly covariant theory, the limit is $2k_B T$, corresponding to $\frac{1}{2} k_B T$ for each of the four relativistic degrees of freedom \[1, 3\]. For the quantum field theories at finite temperature, the path integral formulation \[9\] replaces the Hamiltonian in the canonical exponent by the Lagrangian due to the infinite product of factors $\langle \phi | \pi \rangle$ (transition matrix element of the canonical field and its conjugate required to give a Weyl ordered Hamiltonian its numerical value). However, it is the $t$ variable which is analytically continued to construct the finite temperature canonical ensemble, completely removing the covariance of the theoretical framework. One may argue that some frame has to be chosen for the statistical theory to be developed, and perhaps even for temperature to have a meaning, but as we have remarked above, the requirement of relativistic covariance has dynamical consequences (note that the model Lagrangians used in the non-covariant formulations are established with the criterion of relativistic covariance in mind), and we argue that the choice of a frame, if necessary for some physical reason, such as the definition and measurement of temperature, should be made in the framework of a manifestly covariant structure.

In the framework of a manifestly covariant relativistic statistical mechanics, the dynamical evolution of a system of $N$ particles, for the classical case, is governed by
equations of motion that are of the form of Hamilton equations for the motion of \( N \) events which generate the space-time trajectories (particle world lines) as functions of a continuous Poincaré-invariant parameter \( \tau \) [10, 11], usually referred to as a “proper time”. These events are characterized by their positions \( q^\mu = (t, \mathbf{q}) \) and energy-momenta \( p^\mu = (E, \mathbf{p}) \) in an \( 8N \)-dimensional phase-space. For the quantum case, the system is characterized by the wave function \( \psi_\tau(q_1, q_2, \ldots, q_N) \in L^2(\mathbb{R}^{4N}) \), with the measure \( d^4q_1d^4q_2\cdots d^4q_N \equiv d^{4N}q \), \( q_i \equiv q_i^\mu; \mu = 0, 1, 2, 3; \ i = 1, 2, \ldots, N \), describing the distribution of events, which evolves with a generalized Schrödinger equation [11]. The collection of events (called “concatenation” [12]) along each world line corresponds to a particle, and hence, the evolution of the state of the \( N \)-event system describes, \textit{a posteriori}, the history in space and time of an \( N \)-particle system.

For a system of \( N \) interacting events (and hence, particles) one takes [11]

\[
K = \sum_i \frac{p_i^\mu p_i^\mu}{2M} + V(q_1, q_2, \ldots, q_N),
\]

where \( M \) is a given fixed parameter (an intrinsic property of the particles), with the dimension of mass, taken to be the same for all the particles of the system. The Hamilton equations are

\[
\begin{align*}
\frac{dq_i^\mu}{d\tau} &= \frac{\partial K}{\partial p_i^\mu} = \frac{p_i^\mu}{M}, \\
\frac{dp_i^\mu}{d\tau} &= -\frac{\partial K}{\partial q_i^\mu} = -\frac{\partial V}{\partial q_i^\mu}.
\end{align*}
\]

In the quantum theory, the generalized Schrödinger equation

\[
i \frac{\partial}{\partial \tau} \psi_\tau(q_1, q_2, \ldots, q_N) = K \psi_\tau(q_1, q_2, \ldots, q_N)
\]

describes the evolution of the \( N \)-body wave function \( \psi_\tau(q_1, q_2, \ldots, q_N) \). To illustrate the meaning of this wave function, consider the case of a single free event. In this case (1.3) has the formal solution

\[
\psi_\tau(q) = (e^{-iK_0\tau} \psi_0)(q)
\]

for the evolution of the free wave packet. Let us represent \( \psi_\tau(q) \) by its Fourier transform, in energy-momentum space:

\[
\psi_\tau(q) = \frac{1}{(2\pi)^2} \int d^4pe^{-i\frac{q^2}{2M}\tau}e^{ip\cdot q} \psi_0(p),
\]

where \( p^2 \equiv p^\mu p_\mu, p\cdot q \equiv p^\mu q_\mu \), and \( \psi_0(p) \) corresponds to the initial state. Applying the Ehrenfest arguments of stationary phase to obtain the principal contribution to \( \psi_\tau(q) \) for a wave packet at \( p_c^\mu \), one finds (\( p_c^\mu \) is the peak value in the distribution \( \psi_0(p) \))

\[
q_c^\mu \approx \frac{p_c^\mu}{M\tau},
\]
consistent with the classical equations (1.2). Therefore, the central peak of the wave packet moves along the classical trajectory of an event, i.e., the classical world line.

It is clear from the form of (1.3) that one can construct relativistic transport theory in a form analogous to that of the nonrelativistic theory; a relativistic Boltzmann equation and its consequences, for example, was studied in ref. [13].

Since, in the Hilbert space \( L^2(\mathbb{R}^4) \) the operators \( x^\mu, p^\nu \) obey the canonical commutation relations \((g^{\mu\nu} = \text{diag}(-, +, +, +))\)

\[
[x^\mu, p^\nu] = i\hbar g^{\mu\nu},
\]  

(1.7)
the uncertainty relations

\[
\Delta x \Delta p \geq \frac{\hbar}{2}, \quad \Delta t \Delta E \geq \frac{\hbar}{2}
\]  

(1.8)
follow directly from the mathematical structure of the theory, and are on the same footing (with the usual statistical interpretation [14]). The dispersion \( \Delta t \) is a property of the wave function \( \psi_\tau(q) \) at a given \( \tau \).

Arshansky and Horwitz [3] have studied thought experiments analogous to those discussed by Landau and Peierls [15], within the framework of the manifestly covariant relativistic quantum theory which we are using here, and derived the (causal) Landau-Peierls relation

\[
\Delta t \Delta p \sim \frac{\hbar}{c},
\]  

(1.9)
concerning the time interval \( \Delta t \) during which the momentum of a particle is measured, and the momentum dispersion of the state.

In this paper we shall discuss another uncertainty relation following directly from the mathematical structure of the theory which is realized on a statistical mechanical level (for an ensemble of events),

\[
T_{\Delta V} \Delta m \sim \frac{2\pi \hbar}{c^2},
\]  

(1.10)
where \( \Delta m \) is the mass dispersion around the on-shell value due to the off-shellness of the events making up the ensemble, and \( T_{\Delta V} \) is the average passage interval in \( \tau \) for the events which pass through the small (typical) four-volume \( \Delta V \) in the neighborhood of the \( R^4 \)-point \([4]\). The four-volume \( \Delta V \) is the smallest that can be considered a macrovolume in representing the ensemble. \( T_{\Delta V} \) is related to the (average) extent of

\footnote{This result is analogous to the nonrelativistic \( \Delta t \Delta E \) relation, which, as for (1.10), does not follow from commutation relations and the Schwartz inequality. The nonrelativistic time-energy uncertainty relation, in fact, follows from (1.10) and (1.11) in the nonrelativistic limit for which \( \tau \rightarrow t, \langle E \rangle/M \rightarrow 1, \) and \( \Delta m c^2 \rightarrow \Delta E \). This result implies the existence, in the non-relativistic limit, of a residual ensemble over \( t \), consistently with the treatment of the non-relativistic limit given in ref. [7].}
the ensemble along the time axis, through the Hamilton equation (1.2) for \( \mu = 0 \) (in the sense of a statistical average),
\[
\frac{\Delta t}{T_{\Delta V}} = \frac{\langle E \rangle}{M},
\]
if the ensemble is constructed with the minimum time span to characterize the physical system.

2 Ideal relativistic gas of events

To describe an ideal gas of events in the grand canonical ensemble, we use the expression for the number of events given in [7] (in the following we shall use the system of units in which \( \hbar = c = k_B = 1 \), unless otherwise specified),
\[
N = \sum_{p^\mu} n_{p^\mu} = \sum_{p^\mu} \frac{1}{e^{(E-\mu-\mu_K m^2)/2M} \mp 1},
\]
where \( E \equiv p^0, \ m^2 \equiv -p^2 = -p^\mu p_\mu \), and the sign in the denominator of (2.1) is determined by the event statistics in the usual way; \( \mu_K \) is an additional mass potential \( [7] \), which arises in the grand canonical ensemble as the derivative of the free energy with respect to the value of the dynamical evolution function \( K \), interpreted as the invariant mass of the system. In the kinetic theory \( [7] \), \( \mu_K \) enters as a Lagrange multiplier for the equilibrium distribution for \( K \), as \( \mu \) is for \( N \) and \( 1/T \) for \( E \). In order to simplify subsequent considerations, we shall take it to be a fixed parameter.

We restrict ourselves, in the following, to the case of the events obeying Bose-Einstein statistics and use, therefore, the relation (2.1) with the minus sign in the denominator. To ensure a positive-definite value for \( n_{p^\mu} \), the number density of bosons with four-momentum \( p^\mu \), we require that
\[
m - \mu - \mu_K \frac{m^2}{2M} \geq 0.
\]
The discriminant for the l.h.s. of the inequality must be nonnegative, i.e.,
\[
\mu \leq \frac{M}{2\mu_K}.
\]
For such \( \mu \), (2.2) has the solution
\[
m_1 \equiv \frac{M}{\mu_K} \left( 1 - \sqrt{1 - \frac{2\mu\mu_K}{M}} \right) \leq m \leq \frac{M}{\mu_K} \left( 1 + \sqrt{1 - \frac{2\mu\mu_K}{M}} \right) \equiv m_2.
\]
For small $\mu \mu_K / M$, the region (2.4) may be approximated by

$$\mu \leq m \leq \frac{2M}{\mu_K}. \quad (2.5)$$

One sees that $\mu_K$ plays a fundamental role in determining an upper bound of the mass spectrum, in addition to the usual lower bound $m \geq \mu$. In fact, small $\mu_K$ admits a very large range of off-shell mass, and hence can be associated with the presence of strong interactions [16]. For our present purposes it will be sufficient to assume that the mass distribution has a finite range $m_1 \leq m \leq m_2$ around the on-shell value $m_c = M / \mu_K$ corresponding to the limiting value for which the inequality (2.3) becomes an equality.

In order to show that our results hold independent of the dimensionality of space-time, we shall consider our ensemble in one temporal and $D$ spatial dimensions, $D \geq 1$.

Replacing the sum over $p^\mu$ (2.1) by an integral,

$$\sum_{p^\mu} \Rightarrow \frac{V^{(1+D)}}{(2\pi)^{1+D}} \int d^{1+D}p,$$

where $V^{(1+D)}$ is the system’s $(1 + D)$-volume, and using the relation $(p^\mu = (p^0, \mathbf{p}))$

$$d^{(1+D)}p = \frac{d^Dp}{2p^0} dm^2, \quad m^2 \equiv -p^\mu p_\mu, \quad \mu = 0, 1, \ldots, D,$$

one obtains for the density of events per unit $(1 + D)$-volume, $n \equiv N/V^{(1+D)}$,

$$n = \int_{m_1}^{m_2} \frac{dm}{2\pi} \int \frac{d^Dp}{(2\pi)^Dp^0} f(p), \quad (2.6)$$

with $f(p) \equiv n_{p^\mu}$, as given in Eq. (2.1). Typical average values are given by the relations

$$\langle p^\mu \rangle = \frac{1}{n} \int_{m_1}^{m_2} \frac{dm}{2\pi} \int \frac{d^Dp}{(2\pi)^Dp^0} p^\mu f(p), \quad (2.7)$$

$$\langle p^\mu p^\nu \rangle = \frac{1}{n} \int_{m_1}^{m_2} \frac{dm}{2\pi} \int \frac{d^Dp}{(2\pi)^Dp^0} p^\mu p^\nu f(p), \quad \text{etc.} \quad (2.8)$$

To find the expressions for the pressure and energy density in our ensemble, we study the particle energy-momentum tensor defined by the relation

$$T^{\mu\nu}(q) = \sum_i \int d\tau \frac{p_i^\mu p_i^\nu}{m_c^{1+D}} \delta^{1+D}(q - q_i(\tau)), \quad (2.9)$$

\[\text{2The corresponding relation of ref. [13] is given in four-dimensional spacetime.}\]
in which $m_c$ is the value around which the mass of the events making up the ensemble is distributed. Upon integrating over a small $(1 + D)$-volume $\Delta V$ and taking the ensemble average, (2.9) reduces to

$$
\langle T^{\mu\nu} \rangle = \frac{T_{\Delta V}}{m_c} n \langle p^\mu p^\nu \rangle. \tag{2.10}
$$

In this formula, $n = N/V$, and $T_{\Delta V}$ is the average passage interval in $\tau$ for the events which pass through $\Delta V$, which we discussed above. The formula (2.10) reduces, through Eq. (2.8), to

$$
\langle T^{\mu\nu} \rangle = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \int \frac{d^Dp}{(2\pi)^D p^0} p^\mu p^\nu f(p). \tag{2.11}
$$

Using the standard expression

$$
\langle T^{\mu\nu} \rangle = pg^{\mu\nu} - (p + \rho) u^\mu u^\nu, \tag{2.12}
$$

where $p$ and $\rho$ are the particle pressure and energy density, respectively, we obtain

$$
\rho = \langle T^{00} \rangle, \quad p = \frac{1}{D} g^{ii} \langle T_{ii} \rangle, \quad i = 1, 2, \ldots, D,
$$

and therefore, through (2.11),

$$
p = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \int \frac{d^Dp}{(2\pi)^D D p^0} p^\mu f(p), \tag{2.13}
$$

$$
\rho = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \int \frac{d^Dp}{(2\pi)^D} p^0 f(p). \tag{2.14}
$$

We now calculate the particle number density per unit $D$-volume. The particle $D+1$-current is given by the formula

$$
J^\mu(q) = \sum_i \int d\tau \frac{p_i^\mu}{m_c} \delta^{1+D}(q - q_i(\tau)), \tag{2.15}
$$

which upon integrating over a small $(1 + D)$-volume and taking the average reduces to

$$
\langle J^\mu \rangle = \frac{T_{\Delta V}}{m_c} n \langle p^\mu \rangle, \tag{2.16}
$$

then the particle number density is

$$
N_0 \equiv \langle J^0 \rangle = \frac{T_{\Delta V}}{m_c} n \langle E \rangle. \tag{2.17}
$$
so that, with the help of Eq. (2.7),

\[ N_0 = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \int \frac{d^Dp}{(2\pi)^D} f(p). \quad (2.18) \]

Since

\[ p = \int \frac{d^Dp}{(2\pi)^D} \frac{p^2}{m^0} f(p) \equiv p(m), \quad (2.19) \]
\[ \rho = \int \frac{d^Dp}{(2\pi)^D} m^0 f(p) \equiv \rho(m) \quad (2.20) \]

and

\[ N_0 = \int \frac{d^Dp}{(2\pi)^D} f(p) \equiv N_0(m) \quad (2.21) \]

are the standard expressions for the pressure, energy density and particle number density in 1 + \( D \) dimensions, respectively \[18, 19, 20\], we have the following relations:

\[ p = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \ m \ p(m), \quad (2.22) \]
\[ \rho = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \ \rho(m), \quad (2.23) \]
\[ N_0 = \frac{T_{\Delta V}}{2\pi m_c} \int_{m_1}^{m_2} dm \ N_0(m). \quad (2.24) \]

It is seen in these relations that the manifestly covariant framework provides a linear mass spectrum, independent of the dimensionality of spacetime. In order to obtain the expressions for the basic thermodynamic quantities, one has to integrate the standard (on-shell) results over this spectrum within the range of the mass distribution.

Using the formulas (2.22)-(2.24), one can establish the uncertainty relation (1.10) for a narrow mass width around the on-shell value: as \( m \to m_c \),

\[ \int_{m_1}^{m_2} dm \ m \ f(m) \to \Delta m \ m_c \ f(m_c), \quad (2.25) \]

where \( f(m) \) stands for each of the \( p(m), \rho(m), N_0(m) \), and \( \Delta m \) is the (infinitesimal) width of the mass distribution around \( m_c \). The requirement that the results for \( p, \rho \) and \( N_0 \) coincide with those of the usual on-shell theories implies \( p = p(m_c), \rho = \rho(m_c), N_0 = N_0(m_c) \) in Eqs. (2.22)-(2.24), which leads, with Eq. (2.25), to the relation\(^3\)

\[ T_{\Delta V} \Delta m = 2\pi, \quad (2.26) \]

the case of equality in the relation (1.10), in agreement with the results obtained earlier in ref. \[21\]. One can understand this relation, up to a numerical factor, in

\(^3\)In c.g.s. units, this relation has a factor \( \hbar/c^2 \) on the right hand side.
terms of the uncertainty principle (rigorous in the $L^2(R^4)$ quantum theory) \( \Delta E \cdot \Delta t \geq 1/2 \). Since the time interval for the particle to pass the volume \( \Delta V \) (this smallest macroscopic volume is bounded from below by the size of the wave packets) \( \Delta t \simeq E/M \Delta \tau \), and the dispersion of \( E \) due to the mass distribution is \( \Delta E \sim m\Delta m/E \), one obtains a lower bound for \( T_{\Delta V \Delta m} \) of order unity.

3 Mass-proper time uncertainty relation

We now wish to prove the relation (1.10) for the general case, not only for the case of a narrow mass width as done in the previous section. First, we note that we previously considered a relativistic ensemble without degeneracy; therefore no degeneracy factor appeared in the expressions for the basic thermodynamic quantities. Now suppose that we have \( \nu \) internal degrees of freedom in our ensemble which correspond to degeneracy. In this case considerations made previously will remain valid and lead to the formulas (2.22)-(2.24) with the extra factor of \( \nu \) on their right hand side:

\[
\begin{align*}
p &= \frac{T_{\Delta V \nu}}{2\pi m_c} \int_{m_1}^{m_2} dm \ m \ p(m), \\
\rho &= \frac{T_{\Delta V \nu}}{2\pi m_c} \int_{m_1}^{m_2} dm \ m \ \rho(m), \\
N_0 &= \frac{T_{\Delta V \nu}}{2\pi m_c} \int_{m_1}^{m_2} dm \ m \ N_0(m).
\end{align*}
\]

On the other hand, according to our previous arguments, one can consider the \( \nu \) degrees of freedom as being distributed in the mass interval \( m_1 \leq m \leq m_2 \) with a linear mass spectrum,

\[\tau(m) = Cm,\]

which leads to the formula

\[
p = \int_{m_1}^{m_2} dm \ \tau(m) \ p(m),
\]

and analogous relations for \( \rho \) and \( N_0 \) (similar to the treatment of a strongly interacting system by means of a particle resonance spectrum [22]). In fact, the linear mass spectrum finds its confirmation in the experimental hadronic resonance spectrum: if one calculates, for example, the pressure in the hadronic resonance gas by summing up the individual contributions of a finite number of the different hadronic species with the corresponding degeneracies,

\[
p = \sum_i g_i \ p(m_i), \quad p(m_i) = \frac{T^2 m_i^2}{2\pi^2} K_2(m_i/T),
\]
and by using the formula (3.5), in which \( m_1 \) and \( m_2 \) are the masses of the lightest and the heaviest species, respectively, the results coincide \cite{23}.

The normalization constant \( C \) is determined by the condition

\[
\int_{m_1}^{m_2} \tau(m) \, dm = \nu; \tag{3.6}
\]

therefore

\[
C = \frac{2\nu}{m_2^2 - m_1^2} = \frac{\nu}{\Delta m \,(m_1 + m_2)/2}, \tag{3.7}
\]

where \( \Delta m = m_2 - m_1 \) is the width of the mass distribution in a general case. Since \( m_c \) should be associated with one of the averages \( \langle m \rangle \) or \( \langle m^2 \rangle \) which both are closer to \( m_2 \) than to \( m_1 \) for a linear spectrum \([ (m_1 + m_2)/2 \lesssim m_c] \),

\[
C \gtrsim \frac{\nu}{m_c \, \Delta m}. \tag{3.8}
\]

Direct comparison of the formulas (3.1)-(3.3) with the relation (3.5) and analogous formulas for \( \rho \) and \( N_0 \), with \( \tau(m) \) given by (3.4),

\[
p = C \int_{m_1}^{m_2} dm \, m \, p(m), \tag{3.9}
\]

\[
\rho = C \int_{m_1}^{m_2} dm \, m \, \rho(m), \tag{3.10}
\]

\[
N_0 = C \int_{m_1}^{m_2} dm \, m \, N_0(m), \tag{3.11}
\]

leads to the relation

\[
C = \frac{T_{\Delta V} \nu}{2\pi m_c},
\]

which reduces, through (3.8), to

\[
T_{\Delta V} \Delta m \gtrsim 2\pi, \tag{3.12}
\]

the relation (1.10) for the general case of the finite range mass distribution.

In order that our considerations be valid, the effective degeneracy in a relativistic ensemble should be large, \( \nu >> 1 \), so that one could speak of the distribution of \( \nu \) degrees of freedom in a finite mass range. This is the case for realistic physical systems such as high temperature strongly interacting hadronic matter \cite{22}.
4 Concluding remarks

In this paper we have proved the uncertainty relation (1.10) for the general case of a finite range of mass distribution in a relativistic ensemble. This relation allows one to admit the following picture of a strongly interacting system: one can consider a strongly interacting system as a distribution of free particles which temporarily go off-shell while undergoing an interaction. Then $T_{ΔV}$ may be associated with the time for the particle to remain very close to its mass shell. So, for weakly interacting systems, $Δm ∼ 0, T_{ΔV} ∼ ∞$, i.e., the particle remains on-shell almost all the time. In contrast, for a strongly interacting system, $μ_K ∼ 0$, then, in view of (2.5), $Δm ∼ ∞$, and $T_{ΔV} ∼ 0$, i.e., the particle is off-shell almost always (because it undergoes interaction almost continuously).

We have remarked that the non-relativistic limit of the mass-“proper time” uncertainty relation provides a new derivation of the $ΔE Δt < \hbar$ relation of the non-relativistic theory. This result implies, as we pointed out, the existence of a residual ensemble over $t$, even in the non-relativistic limit. Such an ensemble has been introduced recently [24] in order to achieve an exact semigroup (exponential decay) law of evolution for the reduced motion of an unstable system. The usual derivation [25] of the non-relativistic energy-time uncertainty relation studies the motion of the system under the action of the full Hamiltonian relative to the eigenstates of unperturbed energy; this procedure corresponds precisely to that of Weisskopf and Wigner [26] for the description of the decay of unstable systems. It is argued in refs. [24] that the introduction of an ensemble over $t$ is necessary for achieving the semigroup property as well as for the consistency of the interpretation.

The uncertainty relation (1.10) may be useful in practical calculations concerning realistic strongly interacting systems in which the particles are necessarily off-shell. For example, it may allow one to estimate the relaxation times for the quark-gluon plasma created in ultrarelativistic heavy-ion collisions.

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