The Minimum Feasible Tileset problem

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We consider the Minimum Feasible Tileset problem: Given a set of symbols and subsets of these symbols (scenarios), find a smallest possible number of pairs of symbols (tiles) such that each scenario can be formed by selecting at most one symbol from each tile. We show that this problem is NP-complete even if each scenario contains at most three symbols. Our main result is a $4/3$-approximation algorithm for the general case. In addition, we show that the Minimum Feasible Tileset problem is fixed-parameter tractable both when parameterized with the number of scenarios and with the number of symbols.

1 Introduction

Consider the general assignment problem where several devices (e.g., workers, robots, microchips, ...) each can be used in one of $k$ functions/modes (e.g., employing different skills, tools, instruction sets, ...) at a time. Given a set of scenarios, the goal is to assign $k$ different functions to each device, such that, for each scenario, all functions requested by the scenario are available simultaneously. In this paper, we initiate the study of this problem for $k = 2$ and the case that each function is requested at most once by each scenario. Formally, we study the following problem (we use “tile” instead of “device” to intuitively capture the fact that a device/tile has two modes/sides).

**Minimum Feasible Tileset**

**Input:** A universe of symbols $F$, scenarios $S \subseteq 2^F \setminus \{F\}$, and $\ell \in \mathbb{N}$.

**Problem:** Is there a tileset $T$ of at most $\ell$ tiles $T \in \binom{F}{2}$ that is feasible for all scenarios in $S$?

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In the above, we refer to (multi-)sets of tiles as tilesets. A tileset $T$ is feasible for scenario $S$, if we can produce all symbols in $S$ by taking at most one symbol from each tile in $T$. Formally, a tileset $T$ is feasible for a scenario $S \subseteq F$ if there is a mapping $\phi: T \rightarrow F$, such that $\phi(T) \in T$ for all $T \in T$, and $S \subseteq \phi(T) := \{\phi(T) \mid T \in T\}$. By definition, no scenario contains all symbols of $F$. Note that such a scenario would require $|F|$ tiles, making the problem trivial. Similarly, we may assume that all symbols in $F$ appear in at least one scenario, otherwise we can simply remove each symbol that does not occur in any scenario. Finally, the requirement that tiles contain no less than two symbols can be met by arbitrarily assigning a second symbol to all tiles of cardinality one.

Apart from practical motivations Minimum Feasible Tileset is appealing from a structural point of view. In this work we exhibit equivalent definitions for the problem which are interesting in their own right. At first glance, Minimum Feasible Tileset is a covering problem since we must cover all scenarios using tiles that can each cover one of their two symbols in each scenario. It turns out that the problem can also be phrased as a packing/partitioning problem, but with an objective function different from the classical one in terms of number of packed objects or sets (see Section 3). In addition, having tiles be symbol sets of size two suggests a graph interpretation where we are asked to find a minimum set of edges such that for each scenario there is an orientation where each vertex has indegree at least one. We favor the tileset formulation, since it most naturally generalizes to the original assignment problem with tiles of larger sizes and scenarios which contain multiple copies of the same symbols. Also, the Minimum Feasible Tileset interpretation appears suitable for studying the effect of parameters, such as the number of symbols/scenarios, on the complexity.

Results and Outline. We analyze the structure of the graph that has the tiles of a minimum cardinality tileset as its edges, and show that this graph is always (wlog.) a forest. In fact, only the component structure of this forest matters: We may replace trees by arbitrary trees spanning the same components without affecting the feasibility of the corresponding tileset (Section 2). This lets us view Minimum Feasible Tileset as a partitioning problem, which in turn allows us to prove NP-completeness even when scenarios have size at most three (Section 3). As our main result, we complement the hardness with a 4/3-approximation algorithm (for scenarios of arbitrary sizes) inspired by the component structure of the optimum solution (Section 4). Finally, we show that the problem is fixed-parameter tractable with respect to the number of scenarios (Section 5) and the number of symbols (Section 6), respectively.

Related Work. The problem most closely related to Minimum Feasible Tileset is arguably Set Packing, as 3-Set Packing appears as a subproblem in our approximation algorithm and also as the source problem for our NP-hardness reduction. Set Packing has been extensively studied for both approximability and parameterized complexity (see, e.g., [1, 3, 19] and [6, 17] for some recent results). The main difference between the two problems is that Set Packing is a maximization problem whereas Minimum Feasible Tileset seeks to minimize the size of a feasible tileset—a measure that is only indirectly related to the number of sets (scenarios). In particular, Set Packing becomes trivial for a bounded number of sets, whereas for Minimum Feasible Tileset we get a nontrivial polynomial-time algorithm via integer linear programming.

As mentioned above, the Minimum Feasible Tileset problem can equivalently be seen as designing an edge-minimal graph on the set of symbols such that, for each scenario, the edges (tiles) can be oriented in such a way that all symbols in the scenario have indegree at least one. The question whether a given graph admits an orientation with certain properties has been
studied in various settings. For example, Biedl et al. [2] proposed an approximation algorithm for finding a balanced acyclic orientation. Another natural constraint on an orientation that has been studied is to prescribe degrees for each vertex [8, 10, 14].

More abstractly, we are looking for a graph on the set of symbols that fulfills a certain constraint for each scenario. The case where the subgraph induced by each scenario has to be connected is well-studied [3, 4, 9, 13, 15]. In particular, it is NP-hard to find the minimum number of edges needed [9] and to decide whether a planar solution [3, 15] or a solution of treewidth at most three [13] exists.

2 Graph structure of tilesets

The tiles in a tileset $T$ over a universe of symbols $F$ can be viewed as the edges of the undirected (multi-) graph $G(T) := (F, T)$. In this section, we establish that there always exist optimal tilesets with a simple graph structure. This is made formal in the following lemma which will be useful in later sections.

**Lemma 1.** Let $F$ be a universe of symbols, $S$ a family of scenarios over $F$, and $T$ a tileset feasible for $S$. There is a tileset $T' \subseteq \binom{F}{2}$ feasible for $S$ such that $|T'| \leq |T|$ and $G(T')$ is a forest.

Note that each connected component of $G(T')$ has size at least two because each symbol occurs in at least one scenario and hence is incident with at least one edge.

In the proof of [Lemma 1] it is convenient to think of feasibility of $T$ via orientations of the graph $G(T)$. Let us say that an orientation of $G(T)$ is feasible for the scenario $S$ if each vertex in $S$ has indegree at least one. It is easy to see that deciding whether $T$ is feasible for some scenario $S \subset F$ is equivalent to deciding whether there is a feasible orientation of the edges of $G(T)$ for $S$. We obtain the following lemma.

**Lemma 2.** For every tileset $T$ and scenario $S$ over a universe of symbols $F$ the following are equivalent.

(i) $T$ is feasible for $S$,

(ii) there is a feasible orientation of $G(T)$ for $S$,

(iii) for every connected component $C$ of $G(T)$, there is a feasible orientation of $G(T)$ for $S \cap C$, (iv) for every connected component $C$ of $G(T)$, tileset $T$ is feasible for $S \cap C$.

**Proof.** Note that it suffices to prove the equivalence of the first three statements, since equivalence of (i) and (ii) implies equivalence of (iii) and (iv).

(i) $\Rightarrow$ (ii): Assume that $T$ is feasible for $S$ and let $\phi : T \rightarrow F$ be the corresponding mapping with $\phi(T) \in T$ for all $T \in T$, and $S \subseteq \phi(T)$. We obtain an orientation of $G(T)$ by orienting each edge $T \in T$ towards $\phi(T)$. Since $S \subseteq \phi(T)$, each symbol in $S$ has indegree at least one and we have a feasible orientation of $G(T)$ for $S$.

(ii) $\Rightarrow$ (iii): Clearly, a feasible orientation of $G(T)$ for $S$ is, in particular, a feasible orientation for $S \cap C$ for every connected component $C$ of $G(T)$.

(iii) $\Rightarrow$ (i): Let $C_1, \ldots, C_k$ denote the connected components of $G(T)$ and assume that there are feasible orientations $\tilde{G}_1, \ldots, \tilde{G}_k$ for $S \cap C_1, \ldots, S \cap C_k$, respectively. Since $S \subseteq \bigcup_i C_i$, we can obtain a feasible orientation $\tilde{G}$ of $G(T)$ for $S$ as $\tilde{G}_1[C_1] \cup \cdots \cup \tilde{G}_k[C_k]$. We define the mapping $\phi : T \rightarrow F$ by setting $\phi(T) = s$, where $s$ is the symbol towards which the edge $T$ is oriented in $\tilde{G}$. By definition, $\phi(T) \in T$ for all $T \in T$, and, since $\tilde{G}$ is feasible for $S$, we have $S \subseteq \phi(T)$. By existence of such a mapping $T$ is feasible for $S$. \qed
Using the notion of feasible orientations we can observe that connected components in $G(T)$ yield feasibility for each of their strict subsets.

**Lemma 3.** Let $T$ be a tileset, $C$ a connected component of $G(T)$ and $C' \subseteq C$. Then, $T$ is feasible for $C'$.

*Proof.* The proof is by induction over the size of $C'$. If $C' = \{s\}$ contains a single symbol, then we obtain a feasible orientation by orienting an arbitrary edge towards $s$; such an edge exists because $C'$ is part of the (larger) connected component $C$. Consider the case $|C'| > 1$. First assume that $G(T)[C']$ contains no edges, i.e., $C'$ is an independent set. Then, there is an edge in $G(T)[C]$ for each symbol $s \in C'$, connecting $s$ to $C \setminus C'$. A feasible orientation for $C'$ can simply be obtained by orienting these edges towards $C'$. Now, assume there is an edge $\{s, s'\}$ in $G(T)[C']$ and consider the graph $G'$ obtained by contracting $\{s, s'\}$. By induction, there is a feasible orientation of $G'$ for $C''$, where $C''$ is obtained from $C'$ by identifying $s$ and $s'$. Hence, there is an orientation of $G(T)$ such that all vertices in $C'$ except one of $\{s, s'\}$ have indegree at least one. We orient the edge $\{s, s'\}$ towards the vertex of smaller indegree to obtain the desired feasible orientation of $G(T)$. \qed

We are ready for a proof of **Lemma 1.** Intuitively, we observe that cycle components in $G(T)$ yield feasibility for *any* of their subsets and hence are a safe replacement for every component with a large number of edges. Then we show how to break cycle components into trees.

*Proof of **Lemma 1.*** We replace connected components in $G(T)$, maintaining feasibility of $T$ and without increasing the cardinality of $T$.

Let $C_1, \ldots, C_k$ be the connected components of $G(T)$ that contain a cycle, and let $C = \bigcup_{i=1}^k C_i$. We obtain a new tileset $T'$ by replacing the edges in $G(T)[C]$ with a cycle on $C$. (If $|C| = 1$, we introduce a self-loop and if $|C| = 2$, we introduce two parallel edges.) Since each component $C_i, i \in [k]$, originally contained at least $|C_i|$ edges, and we only introduced $|C|$ new edges, the cardinality of $T'$ is not larger than the cardinality of $T$. We observe that $T'$ is still feasible. Consider any scenario $S \in S$. Clearly, if $S \cap C = \emptyset$ then $T'$ is feasible for $S$. Hence, assume $S \cap C \neq \emptyset$. By **Lemma 2** there is a feasible orientation of $G(T)$ for $S \setminus C$. This implies that there is still a feasible orientation of $G(T')$ for $S \setminus C$. By **Lemma 2** it suffices to prove that there is a feasible orientation of $G(T')$ for $S \cap C$. Indeed, since $G(T')[C]$ is a cycle, orienting the edges in one direction along the cycle yields a feasible orientation for any subset of $C$, and, in particular, for $S \cap C$. Hence, $T'$ is still feasible for every $S \in S$.

By definition, every connected component $C'$ of $G(T)$ outside of $C$ is a tree. Hence, the connected components of $G(T')$ are trees, with the exception of at most one component $C$ that is a cycle. We now modify $C$ in order to obtain our final feasible tileset $T''$ with the desired structure.

First, consider the case that $C$ is the only connected component of $G(T')$. Then, we can remove an arbitrary tile from $T'$ to obtain $T''$: Since, by definition, $S$ does not contain the set of all vertices $F$, by **Lemma 3** $T''$ is feasible for all scenarios $S \in S$. Now, assume there is at least one tree component $C'$ in $G(T')$ along with $C$. Consider an arbitrary edge $\{s, s'\}$ in $C$ and an arbitrary vertex $s'' \in C'$. We remove $\{s, s'\}$ from $T'$ and instead add the edge $\{s, s''\}$ to obtain the tileset $T''$. Clearly, $G(T'')$ is a forest. It remains to prove that $T''$ is feasible for every scenario $S \in S$. By **Lemma 2** $T'$ is feasible for $S \setminus C$ and, hence, so is $T''$. Because $C \cup C'$ is a connected component in $G(T')$, **Lemma 3** guarantees that $T''$ is feasible for every $S' \subseteq (C \cup C')$ and, in particular, $T''$ is feasible for $S \cap C$. Hence, as $T''$ is feasible for $S \cap C$ and $S \setminus C$, with **Lemma 2** we obtain that $T''$ is feasible for $S$, as required. \qed
Intuitively, Lemmas 2 and 3 imply that only the partition of the symbols induced by the component structure of the graph of a tileset matters, but not the exact topology of each of the trees. This leads to the following.

**Theorem 1.** Let $S$ be a family of scenarios and $T$ be a tileset over symbols $F$. If $G(T)$ is a forest, then $T$ is feasible for $S$ if and only if no connected component $C$ of $G(T)$ is fully contained in any scenario $S \in S$, i.e., $C \not\subseteq S$ for all scenarios $S \in S$ and all connected components $C$ of $G(T)$.

**Proof.** “⇒”: Assume towards a contradiction that $T$ is feasible, $G(T)$ is a forest, and there is a scenario $S \in S$ and a component $C$ of $G(T)$ such that $C \subseteq S$. By [Lemma 2] there is a feasible orientation of $G(T)$ for $S \cap C = C$. But this is absurd, because $G(T)$ is a forest and hence $G(T)[C]$ contains only $|C| - 1$ edges.

“⇐”: For each component $C$ of $G(T)$ and every scenario $S \in S$, we have that $C \cap S \subseteq C$, since $C \not\subseteq S$. By [Lemma 3] we get that $T$ is feasible for $C \cap S$. Since this is true for all choices of $C$ and $S$, [Lemma 2] implies that $T$ is feasible for $S$.

**3 NP-Completeness of Minimum Feasible Tileset**

In this section we establish the following completeness result.

**Theorem 2.** Minimum Feasible Tileset is NP-complete, even if each scenario has size at most three.

Let us check that Minimum Feasible Tileset is contained in NP: A feasible tileset can be encoded using polynomially many bits with respect to $|F|$. Verifying feasibility comes down to solving one bipartite matching problem for each scenario on an auxiliary graph that has an edge between each symbol in the scenario and every tile containing that symbol, which is possible in polynomial time.

It remains to prove NP-hardness. For this, we first give a reduction from the following partition problem, and later prove this problem to be NP-hard.

**Fine Constrained Partition**

**Input:** A universe $U$, constraints $V \subseteq 2^U \setminus U$, and $p \in \mathbb{N}$.

**Problem:** Is there a partition $P$ of $U$, $|P| \geq p$, such that $P \not\subseteq V$ for all parts $P \in P$ and all $V \in V$?

**Lemma 4.** Minimum Feasible Tileset and Fine Constrained Partition are equivalent if we identify scenarios and constraints.

**Proof.** We claim that an instance $(F, S, \ell)$ of Minimum Feasible Tileset admits a solution if and only if the instance $(F, S, |F| - \ell)$ of Fine Constrained Partition admits a solution.

“⇒”: By [Lemma 1] there is a feasible tileset $T'$ for $S$ of cardinality at most $\ell$ such that $G(T')$ is a forest. The connected components $C_1, \ldots, C_k$ of $G(T')$ induce a partition that is a solution for the Fine Constrained Partition instance: By [Theorem 1] we indeed have $C_i \not\subseteq S$ for all connected components $C_i$, $i \in [k]$, and scenarios $S \in S$. Furthermore, since there are at most $\ell$ edges in $G(T')$ and each connected component is a tree, we have $\ell \geq \sum_{i=1}^{k} |C_i| - 1 = |F| - k$. Hence, our partition has at least $k \geq |F| - \ell$ parts.

“⇐”: Let $P = \{P_1, \ldots, P_p\}$ be a solution for the Fine Constrained Partition instance. We construct a tileset $T$ by setting $G(T)[P_i]$ to an arbitrary spanning tree for each $i \in [p]$. Since $P_i \not\subseteq S$ for each $S \in S$ and each $i \in [p]$, by [Theorem 1] $T$ is feasible for $S$. The number of tiles in $T$ is $\sum_{i=1}^{p} |P_i| - 1 = |F| - p \leq |F| - (|F| - \ell) = \ell$, as required.
Note that the corresponding optimization problems are dual to each other in the sense that one is to minimize $\ell$ and the other to maximize $|F| - \ell$. We are now ready to give a reduction to Fine Constrained Partition from Exact Cover by 3-Sets, which is well known to be NP-hard \[12\], hence, completing the proof of Theorem 2.

**Exact Cover by 3-Sets**

**Input:** A universe $X$ and a family $C$ of three-element sets $C \in \binom{X}{3}$.

**Problem:** Is there an exact cover for $X$, i.e., a partition of $X$ into a family $C' \subseteq C$ of disjoint sets?

**Lemma 5.** There is a polynomial-time reduction from Exact Cover by 3-Sets to Fine Constrained Partition with constraints of size at most three.

**Proof.** Let an instance $(X, C)$ of Exact Cover by 3-Sets be given. Without loss of generality, we may assume that $|X| = 3q$ for some integer $q$, as otherwise no exact cover exists. We construct an instance of Fine Constrained Partition with universe $X$ asking for a partition of size at least $q$. First, we add constraints $V_2 = \binom{X}{2}$ that exclude every two-element subset of $X$ from all solution partitions. Since every solution partition needs to contain at least $q$ parts and $|X| = 3q$, each such partition consists of sets of size exactly three. Next, we exclude partitions that contain sets outside of $C$ by simply adding the constraints $V_C^\bar{=} = \binom{X}{3} \setminus C$. This concludes the construction of the Fine Constrained Partition instance $(X, V_2 \cup V_C, q)$. Clearly, this takes polynomial time.

Now, if there is a partition $P$ with at least $q$ parts for the Fine Constrained Partition instance, by the above, we know that each of its parts is a set in $C$. Hence $P$ is an exact cover of $X$ for family $C$. Conversely, let $C' \subseteq C$ be an exact cover for $X$. Then $|C'| \geq q$ and for all $C \in C'$ and $V \in V_2 \cup V_C$ we have $C \not\subseteq V$, because $C$ has size three and is not from $V_C$. Hence also the Fine Constrained Partition instance has a solution.

4 A 4/3-approximation for Minimum Feasible Tileset

In this section, we propose an approximation algorithm for Minimum Feasible Tileset with unbounded scenario size. Motivated by the structural insights of Section 2, we construct a tileset that induces a forest in the corresponding graph, with the property that none of its components are contained in a single scenario. Since a component of size $k$ requires $k - 1$ tiles, we additionally aim for small components in order to keep the resulting tileset small.

We first take as many components of size two as possible among all disjoint sets of two symbols that are not both contained in the same scenario. This can easily be achieved by computing a maximum matching in the graph that has an edge for each candidate component. Similarly, among all remaining symbols, we try to form many (disjoint) components of size three, without creating components that are contained in a single scenario. For this, we employ a simple greedy strategy, that repeatedly takes any possible component until no possible candidates remain. (While there are better packing strategies available for sets of size three, we will see that improving the packing strategy alone does not improve our approximation ratio.) Finally, for each leftover symbol we add an individual tile (pairing that symbol in such a way as to prevent cycles).

We give a more formal listing in Algorithm A. We use $\tilde{F}_i(F') = \{C \in \binom{F'}{i} \mid \forall S \in \mathcal{S}: C \not\subseteq S\}$ to denote the family of all sets of symbols in $F'$ that are of size $i$ and not fully contained in a single scenario. In the following, we identify connected components with their sets of vertices.
\begin{algorithm}
\textbf{Input:} A set $F$ of symbols and a set $\mathcal{S}$ of scenarios, where $\mathcal{S} \subseteq 2^F \setminus \{F\}$.
\textbf{Output:} A set of tiles $\mathcal{T}$.
\begin{itemize}
  \item $T_2 \leftarrow$ maximum matching in graph $G(\hat{F}_2(F))$.
  \item $\mathcal{P} \leftarrow$ greedy set packing of $\hat{F}_3(F \setminus \bigcup_{t \in T_2} t)$.
  \item $T_3 \leftarrow \bigcup_{(f_1, f_2, f_3) \in \mathcal{P}} \{\{f_1, f_2\}, \{f_2, f_3\}\}$.
  \item if $T_2 \cup T_3 \neq \emptyset$ then take $f_{\text{root}} \in \bigcup_{t \in T_2 \cup T_3} t$ else take $f_{\text{root}} \in F$.
  \item $T_1 \leftarrow \{\{f, f_{\text{root}}\} \mid f \in F \setminus \bigcup_{t \in T_2 \cup T_3} t, f \neq f_{\text{root}}\}$.
  \item return $T = T_1 \cup T_2 \cup T_3$.
\end{itemize}
\end{algorithm}

\textbf{Theorem 3.} \textbf{[Algorithm A] computes a $4/3$-approximation for Minimum Feasible Tileset.}

\textbf{Proof.} We first argue that the set of tiles $T = T_1 \cup T_2 \cup T_3$ computed by Algorithm A is feasible for $\mathcal{S}$. First observe that $G(T)$ is a forest. This is true, because $G(T_2 \cup T_3)$ consists of trees of sizes 2 and 3, $G(T_1)$ is a star, and $T_1 \cap (T_2 \cup T_3)$ contains at most one node ($f_{\text{root}}$). Using Theorem 1, it only remains to show that no connected component $C$ of $G(T)$ is contained in any scenario $S \in \mathcal{S}$, i.e., $C \cap S \subseteq C$. By definition of Algorithm A, this is true for all connected components of the graph $G(T_2 \cup T_3)$. If $T_2 \cup T_3 \neq \emptyset$, then each component of $G(T)$ is a superset of a component of $G(T_2 \cup T_3)$, and is thus not contained in any scenario. If $T_2 \cup T_3$ is empty, then $G(T) = G(T_1)$ consists of a single component that is not contained in any scenario, since, by definition, $F \notin \mathcal{S}$. Thus $T$ is feasible for $\mathcal{S}$.

We now bound the size of $T$ with respect to a minimum cardinality tileset $T^*$. To do this we distribute virtual currency (gold) to the symbols in $F$, such that the total gold distributed is $4/3$ times the size of $T^*$. We later use this gold to pay one unit of gold to certain symbols that can in turn use to provide for (at most) one tile of $T$ that involves this symbol. To complete the proof, we establish that each tile of $T$ is provided for by one of its two symbols.

Let $G^* := G(T^*)$ be the graph induced by $T^*$ and $F_i^*$ be the set of connected components of size $i \in \{2, \ldots, |F|\}$ in $G^*$. By Lemma 1, we may assume that $G^*$ is a forest. Furthermore, because each symbol appears in at least one scenario, graph $G^*$ does not contain components of size 1. Since the symbols in a component of size $i \geq 1$ are part of exactly $i - 1$ tiles in $T^*$, we may distribute all available gold by giving $4/3 \cdot \frac{1}{i} |F|$ gold to each symbol in a component of $F_i^*$, for all $i \in \{2, \ldots, |F|\}$. This gold is used to pay symbols in what follows. We call a symbol $s \in F$ sufficiently paid if one of the following holds: (i) $s$ is paid, (ii) $s$ appears in a tile $T \in T_2$ and the other symbol of $T$ is paid, or (iii) $s$ appears in a tile $T \in T_3$ and the other two symbols in the same component of $G(T_3)$ are paid. Below, we show how to sufficiently pay all symbols. This completes the proof, since then all tiles in $T_1 \cup T_2 \cup T_3$ can be provided for (note that then each tile in $T_1$ contains its own paid symbol). We call a component of $G^*$ sufficiently paid, if all its symbols are sufficiently paid. Let $F_{i,4}^* := F \setminus \bigcup_{C \subseteq F_i \cup F_3^*} C$ be the set of all symbols not in components of size two or three in $G^*$. In paying the symbols we will maintain the invariant that each element of $F_2^* \cup F_3^* \cup F_{i,4}^*$ is either sufficiently paid, or it still holds its gold (all its symbols still hold their gold, respectively).

We define a graph $H = (V, E)$ that has the components in $F_2^* \cup F_3^*$ as its vertices, as well as the symbols that are not part of these components, i.e., $V = F_2^* \cup F_3^* \cup F_{i,4}^*$ (cf. Figure 1). In this way, each vertex of $H$ represents up to three symbols. For each tile $T \in T_2$ we introduce an edge connecting the vertices of $H$ representing the two symbols of $T$, possibly introducing self-loops. Since $T_2$ is a matching, and since the vertices in $H$ represent at most three symbols
each, all vertices in $H$ have degree at most 3. We partition the edges of $H$ into paths, cycles, and self-loops, and show for each how to use the gold remaining at its vertices to pay all symbols in the components of $G^*$ that are intersected by the path/cycle/self-loop. We will ensure that every symbol (except possibly $f_{\text{root}}$) on a tile in $T_1$ is paid. Since each symbol on a tile of $T_2$ appears only exactly on this and no other tile of $T_2 \cup T_3$, it is thus sufficient to pay only one of the two symbols on each tile of $T_2$.

Let $\mathcal{P}$ be the set of all paths in $H$ connecting (different) vertices of degree 1 or 3 with internal nodes of degree 2. Consider the paths in $\mathcal{P}$ one by one. We use the gold available along path $P \in \mathcal{P}$ of length $k$ as follows (cf. Figure 2). Let $N_2, N_3$ be the number of internal nodes of $P$ that represent 2 and 3 symbols, respectively. Note that $P$ has no inner nodes that represent a single symbol, since $T_2$ is a matching, and hence $k = 1 + N_2 + N_3$. Also, $P$ is the only path visiting these inner nodes and hence they all still hold their gold. Let $N_{\text{end}}^1, N_{\text{end}}^2, N_{\text{end}}^3 \leq 2$ be the number of endpoints of $P$ that still hold gold and represent 1, 2, and 3 symbols, respectively. Similarly, let $N_{\text{end}}^0$ be the number of endpoints without gold. By our invariant, the symbols or components represented by the endpoints without gold left have already been sufficiently paid before. We make sure that all other nodes along $P$ are sufficiently paid. We do this by, for all tiles that form the path $P$, paying one of the two corresponding symbols, and, in addition, paying every further symbol represented by nodes along $P$. Note that this preserves the invariant. The total cost is

$$C^- = k + N_{\text{end}}^1 + 2N_{\text{end}}^2 + N_3 - N_{\text{end}}^0 = 1 + N_{\text{end}}^1 + 2N_{\text{end}}^2 + N_2 + 2N_3 - N_{\text{end}}^0. \quad (1)$$

Using that each endpoint of $P$ that contributes to $N_{\text{end}}^1$ represents a symbol that is part of a component in $G^*$ of size $i \geq 4$, we get that the gold available at this symbol is at least \( \frac{4}{3} \cdot \frac{N_{\text{end}}^1}{i} \geq 1 \). Hence, the gold available to us is at least

$$C^+ = \frac{4}{3}(N_{\text{end}}^1 + 2N_{\text{end}}^2 + N_2 + 2N_3 + \frac{3}{4}N_{\text{end}}^0). \quad (2)$$
components in $\bar{\Sigma}$ symbols and have not yet been sufficiently paid (i.e., still hold their gold). Further, let $N_0^{\text{end}} = N_2^{\text{end}} = 1$, $N_2 = 3$, $N_0^{\text{end}} = N_2^{\text{end}} = 3 = 0$. For the bottom path we have $N_0^{\text{end}} = N_2^{\text{end}} = 1$, $N_2 = 2$, $N_3 = 1$, $N_1^{\text{end}} = N_3^{\text{end}} = 0$.

Since $N_0^{\text{end}} + N_1^{\text{end}} + N_2^{\text{end}} + N_3^{\text{end}} = 2$, we get

$$C^+ − C^- = 1 − \frac{2}{3}N_2^{\text{end}} − \frac{1}{3}N_3^{\text{end}} + \frac{1}{3}N_2 + \frac{2}{3}N_3.$$  

Hence, we have $C^+ \geq C^−$, unless $N_2^{\text{end}} = 2$ and $N_0^{\text{end}} = N_1^{\text{end}} = N_3^{\text{end}} = N_2 = N_3 = 0$, i.e. $P$ is of length one, connecting two tiles $p_1, p_2 \in F_2^*$ by an edge which corresponds to a tile $t \in T_2$. To see that this case cannot occur, observe first, that $p_1$ and $p_2$ are of degree 1 in $H$. Second, since $T^*$ is feasible, no component of $G^*$ is contained in a single scenario (Theorem 1), and thus $p_1, p_2 \in F_2^* \subseteq F_2(F)$. This is a contradiction to $T_2$ being a maximum matching in graph $G(F_2(F))$, as the matching can be augmented by removing $t$ and adding $p_1$ and $p_2$.

Similarly to the above, we can consider all cycles in $H$ with at most one node of degree 3 one by one. (Note that cycles with at least two nodes of degree 3 contain a path as before.) If a cycle of length $k$ does not contain a node of degree 3, or the node of degree 3 is not yet sufficiently paid (and thus still holds its gold), the cost for the cycle and its available gold are

$$C^- = k + N_3 = N_2 + 2N_3 = \frac{3}{4}C^+ < C^+,$$

where $N_2, N_3$ are the numbers of nodes of $P$ that represent 2 and 3 symbols, respectively. If the node of degree 3 has no gold left, then it has already been sufficiently paid and $C^- = N_2 + 2N_3 − 3 < 4C^+$. In either case, the available gold allows to sufficiently pay all nodes along the cycle. Finally, each self-loop in $H$ connects two symbols in the same component $C$ of size 2 or 3 in $G^*$. If $|C| = 2$, the gold available among the two symbols is $C^+ = \frac{4}{3}$, while we require only $C^- = 1$ unit of gold. If $|C| = 3$, we have $C^+ = \frac{5}{7}$ and $C^- = 2$.

After processing all paths, cycles, and self-loops all nodes of $H$ intersecting a tile of $T_2$ are sufficiently paid. In particular, since $T_2$ is a maximum matching, all components in $F_2^*$ are sufficiently paid. In the next step we ensure that all components of $F_3^*$ are sufficiently paid. By construction, every element of $F_3^*$, that is not sufficiently paid yet, intersects at least one tile of $T_3$. We can thus consider the components of $G(T_3)$ one by one and make sure to sufficiently pay each element of $F_3^*$ that intersects the considered component of $G(T_3)$.

Consider a component of $G(T_3)$ involving the three symbols $f_1, f_2, f_3$ (cf. Figure 3 in the following). Let $C_3 \subseteq F_3^*$ be the set of components of size 3 in $G^*$ that involve at least one of these symbols and have not yet been sufficiently paid (i.e., still hold their gold). Further, let $N_n$ be the number of symbols among $\{f_1, f_2, f_3\} \cap F_3^*$ that are not yet sufficiently paid. Since all components in $F_2^*$ are sufficiently paid, the gold we have available is at least $C^+ \geq \frac{4}{3}(2|C_3| + \frac{3}{4}N_n)$. We ensure that (at least) two symbols among $f_1, f_2, f_3$ are paid, as well as all other symbols appearing in $C_3$. In this way, each component in $C_3$ is sufficiently paid. Note that this preserves our
invariant that each element of $\bar{F}_2^* \cup \bar{F}_3^* \cup F_{\geq 4}^*$ is either sufficiently paid, or still holds its gold. The cost for paying the symbols $f_1, f_2, f_3$ is at most 2. Since in addition to $f_1, f_2, f_3$ there are $3|C_3| + N_n - 3$ symbols needing pay in $\bigcup_{C \in C_3} C \cup \{f_1, f_2, f_3\}$, and because $|C_3| \leq 3$, the total cost is

$$C^- \leq 3|C_3| + N_n - 1 \leq \frac{8}{3}|C_3| + N_n \leq C^+.$$ 

At this point, we have sufficiently paid all components in $\bar{F}_2^* \cup \bar{F}_3^*$ using gold only from these components. This means that all remaining symbols that are not sufficiently paid yet have at least $\frac{4}{3} \cdot \frac{4-1}{3} = 1$ gold available, which we can use to pay these symbols themselves. Now all elements of $\bar{F}_2^* \cup \bar{F}_3^* \cup F_{\geq 4}^*$ have been sufficiently paid and the proof is complete.

Our analysis of Algorithm $A$ is tight in three different spots: (i) A path of length 1 in the graph $H$ defined above that visits a component of size 2 and a component of size 3 of the optimum solution $T$ may lead to 4 tiles in our solution compared to the 3 tiles required in the optimum solution, i.e., Equations (1) and (2) coincide if $N_{end}^3 = N_{end}^2 = 1$ and all other terms vanish. (ii) The first intersection of a component of $G(T_3)$ with components of $G^*$ illustrated in Figure 3 may lead to 8 tiles in our solution compared to the 6 tiles required in the optimum solution. (iii) Each symbol of a component of size 4 in $G^*$ might result in a single tile for this symbol only, in which case the optimum solution requires 3 tiles for the symbols of the component, while our solution requires 4 tiles. To improve Algorithm $A$ we have to address each of these three bottlenecks. For (i), we either would have to alter the matching $T_2$ to prevent the described situation, or combine the analysis to account for the loss in other places. The aspect (ii) can easily be prevented by employing a more sophisticated set packing algorithm (e.g., the $(4/3 + \varepsilon)$-approximation of Cygan [5]). Finally, to avoid (iii), we would need to pack sets of size 4 similarly to our packing of sets of size 3. In addition to requiring one more level of analysis, this would also complicate the other levels, as we would have to include sets of size 4 in our reasoning there.
5 Bounded number of scenarios

We prove that Minimum Feasible Tileset can be solved in polynomial time when the number \(|S|\) of scenarios is bounded. More precisely, we provide an algorithm that solves any instance \((F, S, k)\) in time \(f(|S|)|F, S, k|^c\), i.e., in time \(O(|F, S, k|^c)\) for bounded values of \(|S|\). In other words, Minimum Feasible Tileset is fixed-parameter tractable with respect to the number of scenarios.

Our algorithm works by first translating the input instance \((F, S, \ell)\) into an integer linear program (ILP) in such a way that the ILP is feasible (i.e., contains at least one integer point) if and only if \((F, S, \ell)\) admits a feasible tileset with at most \(\ell\) tiles. The ILP uses \(O(|S|^{|S|})\) variables. Lenstra [18] proved that deciding feasibility of any ILP is fixed-parameter tractable with respect to the number of variables; the currently fastest algorithm was obtained by Frank and Tardos [11], modifying an algorithm by Kannan [16].

**Theorem 4** (Frank and Tardos [11]). Feasibility of an ILP with \(p\) variables can be decided in time \(O^*(p^{O(p)})\).

Using this, we can prove the following result.

**Theorem 5.** Minimum Feasible Tileset on instances with at most \(k\) scenarios can be solved in time \(O^*(k^{O(k+k+1)})\).

Intuitively, a bounded number of scenarios also implies a bound on the number of different subsets of scenarios in which a tile can appear. Thus, one would like to forget the actual identities of the symbols and only remember how many symbols appear, say, exactly in scenarios \(S_1, S_5, \text{ and } S_6\). It appears, however, that grouping symbols in this way is insufficient since symbols from the same group can nevertheless have different patterns for how they are provided by tiles: E.g., one tile could provide such a symbol in all three scenarios \(S_1, S_5, \text{ and } S_6\), whereas other symbols of the same group might need three separate tiles for \(S_1, S_5, \text{ and } S_6\). To cope with this, the constructed ILP has separate variables for all partitions of scenario subsets as well as variables for all ways of using a tile (recall that a tile has two symbols, meaning that it has two disjoint subsets of the scenario that express when either symbol is provided by the tile).

**Proof of Theorem 5.** We formulate Minimum Feasible Tileset as an ILP and employ Kannan’s algorithm. Intuitively, each tile contributes both of its symbols to different (disjoint) subsets of the scenarios. For example, if we have 5 scenarios, a tile might contribute one of its symbols to scenarios 1 and 4, the other to scenarios 3 and 5, and neither to scenario 2. Each tile is associated with such a pattern of how it contributes to scenarios, and one part of the variables of our ILP track the number of tiles having each of the possible patterns. On the other hand, each symbol has a pattern associated with it, depending on which occurrences of the symbol are provided by the same tile. In our example, a symbol appearing in scenarios 1, 2, and 4 might be provided by the same tile in scenarios 1 and 4, and by a different tile in scenario 2. The remaining variables of the ILP track the number of symbols having each of the possible patterns. We provide exchange arguments to show that enforcing correct totals for these variables by linear constraints is sufficient to ensure that a feasible assignment of tiles to symbols exists for each scenario.

**ILP formulation.** To make our description precise, let an instance \((F, S, \ell)\) with \(k\) scenarios \(S = \{S_1, \ldots, S_k\}\) be given. For brevity, we refer to a subset of \(S\) by the corresponding index set. For every subset \(I \subseteq [k]\) of scenarios we count the number of symbols that occur exactly in these scenarios and denote this number by \(c_I = |\bigcap_{i \in I} S_i \setminus \bigcup_{i \notin I} S_i|\). The family of all partitions of \(I\) is denoted by \(\Pi(I)\). The ILP is constructed as follows.

\[
\text{Variables: } x_{I,S} \in \mathbb{Z}^{|I|}, \quad y_{I,S} \in \mathbb{Z}^{|I|}, \quad z_I \in \mathbb{Z}^{|I|}.
\]

\[
\text{Objective: } \min \sum_{I \subseteq [k]} \sum_{S \subseteq I} (c_I - |S|)x_{I,S} + \sum_{I \subseteq [k]} \sum_{S \subseteq I} \ell y_{I,S}.
\]

\[
\text{Constraints: }\sum_{I \subseteq [k]} \sum_{S \subseteq I} x_{I,S} = \sum_{I \subseteq [k]} |S| y_{I,S},
\]

\[
\sum_{I \subseteq [k]} \sum_{S \subseteq I} x_{I,S} = c_I, \quad \sum_{I \subseteq [k]} \sum_{S \subseteq I} y_{I,S} = \ell, \quad \sum_{I \subseteq [k]} \sum_{S \subseteq I} z_I = |S|,
\]

\[
\text{for all } I \subseteq [k], S \subseteq I.
\]
1. For each set \( I \subseteq [k] \) and each partition \( \mathcal{I} = \{I_1, \ldots, I_s\} \in \Pi(I) \) we introduce a variable \( y_{\mathcal{I}} \).

The intention is that variable \( y_{\mathcal{I}} \) counts the number of symbols that occur (exactly) in scenarios \( I := I_1 \cup \ldots \cup I_s \) and have pattern \( \mathcal{I} \) associated with them, in the following way:

Exactly \( s \) tiles, say, \( T_1, \ldots, T_s \), are used for such a symbol and the symbol is provided by tile \( T_i \) in the scenarios \( I_i \).

For each \( I \) we add a constraint that enforces the total number of patterns to equal the number \( c_I \) of symbols that occur in the scenarios \( I \):

\[
c_I = \sum_{\mathcal{I} \in \Pi(I)} y_{\mathcal{I}} \quad \forall I \subseteq [k].
\]

For example, if \( I = \{1, 2, 3\} \), the following variables are created:

\[
y_{\{1,2,3\}}, y_{\{1,2\}}, y_{\{1,3\}}, y_{\{2,3\}}, y_{\{1\}}, y_{\{2\}}, y_{\{3\}}.
\]

The number of \( y \)-variables equals the number of subpartitions of the set \([k]\). This is upper bounded by \( k^k + 1 \): We can \( k \)-color all subpartitions other than the partition into singletons by using color \( k \) for all unused elements and colors \( 1, \ldots, k-1 \) for the elements of each set in the partition (only the partition into singletons has \( k \) sets). Thus, we get an injective mapping of all but one subpartition into the \( k \) colorings of \([k]\); this gives a total of \( k^k + 1 \).

2. For the tiles, we introduce variables \( x_{I,J} \) for all \( I, J \subseteq [k] \) with \( I \cap J = \emptyset \) and \( I \cup J \neq \emptyset \); for convenience we identify \( x_{I,J} = x_{J,I} \). Intuitively, the variable \( x_{I,J} \) stands for the number of tiles that provide one of their symbols for scenarios \( I \) and the other symbol for scenarios \( J \).

For example, for \( k = 3 \) we create the following variables:

\[
x_{0,\{1\}}, x_{0,\{2\}}, x_{0,\{3\}}, x_{0,\{1,2\}}, x_{0,\{1,3\}}, x_{0,\{2,3\}}, x_{0,\{1,2,3\}},
x_{\{1\},\{2\}}, x_{\{1\},\{3\}}, x_{\{1\},\{2,3\}}, x_{\{2\},\{3\}}, x_{\{2\},\{1,3\}}, x_{\{3\},\{1,2\}}.
\]

The number of \( x \)-variables is \( \frac{3^k-1}{2} \) corresponding to all partitions of \([k]\) into three sets (i.e., \( I, J, \) and \([k] \setminus (I \cup J)\)), without \( I = J = \emptyset \), and identifying \( x_{I,J} \) with \( x_{J,I} \).

We add constraints that enforce that the number of tiles of each pattern match the sum of the corresponding \( y \)-variables. Concretely, we add

\[
\sum_{\substack{I \subseteq J \subseteq [k] \quad J \in \Pi(J) \quad I \in J}} y_{J} = \sum_{\substack{I \subseteq J \subseteq [k] \quad J \in \Pi(J) \quad I \in J}} x_{I,J} \quad \forall I \subseteq [k], I \neq \emptyset.
\]

We compare the number of tiles that provide one of their symbols for scenarios in \( I \) with the number of symbols that have \( I \) in their pattern. For the set of scenarios \( J \) such symbols appear in we must have \( I \subseteq J \subseteq [k] \), and we need partitions \( J \in \Pi(J) \) that contain \( I \).

3. As a final constraint we enforce that the total number of used tiles is no more than \( \ell \). To this end, we simply sum over all \( x \)-variables and add

\[
\frac{1}{2} \sum_{\substack{I,J \subseteq [k] \quad I \cap J = \emptyset \quad I \cup J \neq \emptyset}} x_{I,J} \leq \ell.
\]
This completes our construction. We use \( p \leq k^2 + 1 + \frac{3^k - 1}{2} = \mathcal{O}(k^2) \) variables and, thus, Kannan’s algorithm decides feasibility of our ILP in time \( \mathcal{O}^*(p^{\mathcal{O}(p)}) = \mathcal{O}^*((k^2)^{\mathcal{O}(k^2)}) = \mathcal{O}^*((k^{\mathcal{O}(k^2)})).

Correctness. First assume that the given instance \((F, S, \ell)\) of MINIMUM FEASIBLE TILES
SET admits a feasible tileset \(T\) of minimum cardinality \(|T| \leq \ell\). Since \(T\) is feasible for each scenario \(S_i \in S\), we may let \(\varphi_i : S_i \rightarrow T\) be an injective function that assigns each symbol in \(S_i\) a unique tile in \(T\) that can provide it. We specify feasible values for the \(x\)- and \(y\)-variables.

1. \(x\)-variables. Each tile \(T \in T\) has two symbols, say, \(T = \{s, s'\}\), and, hence, for each \(i \in [k]\) it is the image of at most one of \(s\) and \(s'\). Formally, let

\[
I := \{ i \in [k] \mid \varphi_i(s) = T \}, \\
J := \{ j \in [k] \mid \varphi_j(s') = T \}.
\]

That is, the set \(I\) contains all scenarios for which tile \(T\) provides symbol \(s\), and \(J\) is the analogue for symbol \(s'\). Since the functions \(\varphi_i\) are injective, we must have that \(I \cap J = \emptyset\).

We have \(I \cup J \neq \emptyset\) as otherwise \(T\) would not be used for any scenario, contradicting the minimality of \(T\). We say that tile \(T\) has pattern \(\{I, J\}\).

For each \(I, J \subseteq [k]\) with \(I \cap J = \emptyset\) and \(I \cup J \neq \emptyset\), we set \(x_{I, J}\) to the number of tiles with pattern \(\{I, J\}\). Clearly, the constraint forcing the total value of the \(x\)-variables to be at most \(\ell\) is fulfilled since \(|T| \leq \ell\).

2. \(y\)-variables. Similarly to the tiles in \(T\) we determine a pattern for each symbol \(s \in F\). We let \(T(s) := \{ T \in T \mid \exists i \in [k] : \varphi_i(s) = T \} = \{ T_1, \ldots, T_r \}\), i.e., the set of tiles that provide \(s\) in at least one scenario. Let \(I \subseteq [k]\) be the set of scenarios containing \(s\). We define a partition \(\{I_1, \ldots, I_r\}\) of \(I\) by

\[
I_p := \{ i \in [k] \mid \varphi_i(s) = T_p \},
\]

for all \(p \in [r]\). We say that symbol \(s\) has pattern \(\{I_1, \ldots, I_r\} \in \Pi(I)\).

For each \(I \in [k]\) and each partition \(\mathcal{I} \in \Pi(I)\) we set \(y_{\mathcal{I}}\) to the number of symbols in \(F\) with pattern \(\mathcal{I}\). Clearly, this fulfills the constraint that all \(y\)-variables whose pattern is a partition of some set \(I \subseteq [k]\) equals the total number \(c_I\) of symbols that occur exactly among the scenarios in \(I\).

It remains to verify that the constraint relating \(x\)- and \(y\)-variables is satisfied. To this end, let us fix some \(I \subseteq [k]\), \(I \neq \emptyset\), and consider the constraint

\[
\sum_{\substack{I \subseteq J \subseteq [k] \\
J \in \Pi(I) \\
I \in J}} y_J = \sum_{\substack{J \subseteq [k] \setminus I}} x_{I, J}.
\]

For each tile \(T \in T\) that contributes to the right-hand-side, there must be a unique symbol \(s\) in \(F\), such that \(\varphi_i(s) = T\) if and only if \(i \in I\). For this symbol, we have \(T \in T(s)\), the set of scenarios \(J\) containing \(s\) satisfies \(I \subseteq J \subseteq [k]\), and \(I\) is part of the pattern of \(s\). Hence, \(s\) contributes to the left-hand-side. Conversely, if \(s\) is a symbol contributing to the left-hand-side, then \(I\) must be part of the pattern of \(s\). This means that there is a unique tile \(T \in T\), such that \(\varphi_i(s) = T\) if and only if \(i \in I\). This tile has \(I\) in its pattern and thus contributes to the right-hand-side. Overall, the contribution to both sides is equal, and our assignment to \(x\) and \(y\)-variables is feasible, as claimed.
Now, assume that the ILP constructed from \((F, S, \ell)\) is feasible and fix a feasible assignment to the \(x\)- and \(y\)-variables. We derive a feasible tileset for all scenarios in \(S\). The set of all symbols can be partitioned according to the scenarios \(I \subseteq [k]\) that each symbol appears in. The total count \(c_I\) of symbols in \(I\) is matched by the sum of \(y\)-variables that are indexed by the partitions \(I \in \Pi(I)\). We arbitrarily assign to each symbol with scenario set \(I\) a pattern \(I \in \Pi(I)\) under the sole constraint that the total number of symbols with pattern \(I\) matches the corresponding variable \(y_I\). For a symbol with assigned pattern \(I = \{I_1, \ldots, I_r\}\) the intention is to use \(r\) tiles \(T_1, \ldots, T_r\) that are each responsible for one set \(I_p \in \mathcal{I}\).

We will use a number of tiles that exactly matches the sum of \(x\)-variables, and thereby ensure that the final tileset has cardinality at most \(\ell\). We do not pick symbols for each tile but, according to the \(x\)-variables, we pick for each tile two disjoint sets of scenarios in which its two symbols will be used. Concretely, exactly \(x_{I,J}\) tiles will be used in \(I\)-scenarios for one symbol and in \(J\) scenarios for their other symbol, i.e., we use \(x_{I,J}\) tiles of pattern \(\{I, J\}\). Recall that \(I \cap J = \emptyset\) and that the sum of these variables does not exceed the maximum number of allowed tiles \(\ell\).

Finally, we assign symbols to tiles according to symbol and tile patterns in a canonical way. Specifically, symbols whose pattern contains some fixed \(I \subseteq [k]\) are assigned to tiles that contain \(I\) in their pattern. By constraint (3) the number of symbols and the number of tiles are equal. Note that each tile is used for two disjoint sets \(I, J \subseteq [k]\) and each variable \(x_{I,J}\) appears in two (3)-constraints (for \(I\) and for \(J\)). Thus, each tile with pattern \(\{I, J\}\) is assigned two symbols, one requiring the tile for the scenarios in \(I\) and the other requiring it the ones in \(J\). Similarly, a symbol with pattern \(I = \{I_1, \ldots, I_r\}\) contributes to \(r\) constraints (3), one for each \(I_1, \ldots, I_r\). Accordingly, these constraints enforce the correct sum of the corresponding variables \(x_{I_1, \ldots, I_r}\). (Recall that we identified \(x_{I, J}\) with \(x_{I, J}\).

We argue that the constructed tileset is indeed feasible for all scenarios \(S_i \in S\). Consider any symbol \(s \in S_i\) with pattern \(J\). Since \(s\) appears in \(S_i\), we have \(i \in J\) for some set \(I\). By the above, we know that there is a tile \(T\) containing \(s\) that has \(I\) as a part of its pattern \(\{I, J\}\). Since, by definition, \(I \cap J = \emptyset\), we have \(i \notin J\) and may safely use \(T\) for symbol \(s\) in scenario \(S_i\). □

6 Bounded number of symbols

We analyze the influence of the number of symbols \(|F|\) on the complexity of solving an instance \((F, S, \ell)\) of Minimum Feasible Tileset. It is easy to see that the problem becomes solvable in polynomial time when \(F\) is bounded: The instance is trivial if \(\ell \geq |F|\) since, in that case, we can afford to dedicate a separate tile for each symbol. Otherwise, there are only \(O(|F|^{2\ell}) \subseteq O(|F|^{2|F|})\) ways to fix \(\ell\) tiles. As mentioned in Section 3, each candidate tileset can be verified by solving a bipartite matching problem for each scenario, on a graph that has an edge between each symbol in the scenario and every tile containing that symbol. This yields an overall runtime of \(O^*(|F|^{2|F|})\), and, hence, fixed-parameter tractability in \(|F|\). Using structural insights of Section 2 we are able to improve on this naive running time.

Theorem 6. Instances \((F, S, \ell)\) of Minimum Feasible Tileset can be solved in time \(O^*(3^{|F|})\).

Note that, as every symbol occurs in a scenario, \(\ell \geq |F|/2\). Hence, Theorem 6 gives a fixed-parameter algorithm also for parameter \(\ell\).

Proof of Theorem 6. We describe a dynamic program for solving an instance \((F, S, \ell)\). Recall that we may assume \(\ell < |F|\); otherwise the instance is trivial. Our algorithm uses a table \(M\) of size \(2^{|F|}\) that is indexed by subsets \(D \subseteq F\), with each entry taking integer values from \([|F|] \cup 2^{|F|}\)
\{−∞\}. At the end of the computation, each entry \(M(D)\) will be set to \(−∞\) if \(D \subseteq S\) for some scenario \(S \in S\), and otherwise to the maximum integer \(i \in [|F|]\) for which there is a partition of \(D\) into \(i\) sets \(D_1, \ldots, D_i\) such that no scenario contains any set in \(\{D_1, \ldots, D_i\}\) as a subset.

In the end, by Theorem 1, the entry \(M(F)\) contains the maximum number of components in the graph corresponding to a feasible tileset. Accordingly, every corresponding tileset \(T\) has minimum cardinality. Hence, and since each connected component \(C\) in the graph \((F, T)\) is composed of \(|C| − 1\) tiles, the instance \((F, S, ℓ)\) admits a tileset of size \(ℓ\) if and only if \(M(F) \geq |F| − ℓ\).

We fill out the entries of the table in order of increasing subset sizes. Each entry is computed via the following recurrence. (Note that the 1 in the maximum taken over subsets into at least two sets can be found such that both sets are not subsets of scenarios.)

\[
M(D) = \begin{cases} 
−∞, & \text{if } D \subseteq S \text{ for some } S \in S, \\
\max_{2 \leq |D'| \leq |D|/2} \{1, M(D') + M(D \setminus D')\}, & \text{otherwise.}
\end{cases}
\]

Thus, for each \(D \subseteq F\) that is not a subset of a scenario we need to compute the maximum of \(M(D') + M(D \setminus D')\) over less than \(2^{|D|}\) subsets \(D'\) of \(D\). By the well-known binomial theorem the total number of evaluations taken over all \(D \subseteq F\) can be upper bounded by \(3^{|F|}\) giving us the claimed runtime.

After this fixed-parameter tractability result, and taking into account the trivial bound of \(2^{|F|}\) for the number of scenarios (giving a worst-case size of instances of \(O(2^{|F|}|F|)\)), it is natural to ask whether polynomial-time preprocessing can simplify input instances to size polynomial in \(|F|\). We show that this is impossible unless \(NP \subseteq coNP/poly\) (and the polynomial hierarchy collapses). More generally, we prove that for the restricted case \(d\)-MINIMUM FEASIBLE TILES\(\text{SET}\), where scenarios have size at most \(d\), no polynomial-time algorithm can achieve a size of \(O(k^{d−ε})\). Note that this restricted case has an essentially matching upper bound of \(|S| < (|F| + 1)^d = O(|F|^d)\)\(^1\). As a consequence there is no reduction to size polynomial in \(|F|\) for the general \(MINIMUM\ FEASIBLE\ TILES\ SET\) problem: Any size \(O(k^ε)\) preprocessing for \(MINIMUM\ FEASIBLE\ TILES\ SET\) could be used for \(d\)-MINIMUM FEASIBLE TILES\(\text{SET}\), for any \(d > c\), and violate the lower bound.

**Theorem 7.** Let \(d \geq 3\) and \(ε\) be a positive real. There is no polynomial-time algorithm that reduces every instance of \(d\)-MINIMUM FEASIBLE TILES\(\text{SET}\) to an equivalent instance (possibly of a different problem) of size \(O(|F|^{d−ε})\), unless \(NP \subseteq \text{coNP/poly}\).

To prove Theorem 7 we employ a similar result by Dell and Marx \(^6\) for **Exact Cover by \(d\)-Sets**, which is defined as follows\(^2\).

**Exact Cover by \(d\)-Sets**

**Input:** A universe \(X\) and a family \(C\) of \(d\)-element sets \(C \in \binom{X}{d}\).

**Problem:** Is there an **exact \(d\)-set cover** for \(X\), i.e., a partition of \(X\) into a family \(C' \subseteq C\) of disjoint sets?

---

\(^1\)A compression to \(O(|F|^d)\) size can be achieved by specifying one bit for each possible scenario in \(S\) and setting it to one if the scenario is present and zero otherwise.

\(^2\)Dell and Marx called this problem **Perfect \(d\)-Set Matching**.
Note that the original result by Dell and Marx \cite{DellMarx} is given in terms of the size $k$ of an exact $d$-set cover. Clearly, $k = \frac{|U|}{d}$ and, thus, we have $O(k^{d-\varepsilon}) = O(|U|^{d-\varepsilon})$ and may instead phrase the result in terms of $|U|$. Furthermore, their result builds on work by Dell and van Melkebeek \cite{DellMelkebeek} and, thus, extends to any polynomial time algorithms (rather than just kernels) whose output instances can be with respect to a different problem. We give the following paraphrased version of the result.

**Theorem 8** (Dell and Marx \cite{DellMarx}). Let $d \geq 3$ and $\varepsilon$ be a positive real. There is no polynomial-time algorithm that reduces every instance $(U, \mathcal{H})$ of Exact Cover by $d$-Sets to an equivalent instance of size $O(|U|^{d-\varepsilon})$ (possibly with respect to a different problem), unless \text{NP} \subseteq \text{coNP}/\text{poly}.

The following lemma, together with Theorem 8, directly implies Theorem 7.

**Lemma 6.** There is a polynomial-time reduction from Exact Cover by $d$-Sets to Minimum Feasible Tileset such that instances $(X, \mathcal{C})$ are mapped to instances $(F, S, \ell)$ with $F = X$ and scenario size at most $d$.

**Proof sketch for Lemma 6.** The proof is similar to the proof of Theorem 2. Given an instance of Exact Cover by $d$-Sets with universe $X$ and a family $\mathcal{C}$ we construct an instance $(F, S, p)$ of Fine Constrained Partition with $F = X$ and $p = |F| - \ell$. Applying the equivalence of Fine Constrained Partition and Minimum Feasible Tileset (Lemma 4) then gives Lemma 6.

To do this, we simply set $S = \binom{X}{d-1} \cup (\binom{X}{d}) \setminus \mathcal{C}$. Similarly to the reduction used for Lemma 5, the constraints $\binom{X}{d-1}$ enforce that every feasible partition contains only parts of size exactly $d$. The constraints $\binom{X}{d} \setminus \mathcal{C}$ enforce that only sets of $\mathcal{C}$ occur in a feasible partition. Hence, each feasible partition is also an exact $d$-set cover and vice-versa.

We now consider a more general setting: In the Generalized Minimum Feasible Tileset problem we are also given a set of symbols and a set of scenarios, but here each scenario may be a multi-set of symbols (or, equivalently, each scenario is a function $S : F \to \mathbb{N}$ indicating the number of copies of each symbol $f$ needed for $S$). We prove that Generalized Minimum Feasible Tileset can be solved in time $O^*(|F|^O(|F|^2))$. Note that for this problem the solution size $\ell$ may be much larger than $|F|$ and similarly the number of scenarios cannot in general be bounded in $|F|$.

**Theorem 9.** Generalized Minimum Feasible Tileset can be solved in time $O^*(|F|^O(|F|^2))$, i.e., it is fixed-parameter tractable with respect to $|F|$.

**Proof.** Let $(F, S, \ell)$ be an instance of Generalized Minimum Feasible Tileset and let $k := |F|$. We will construct an integer linear program (ILP) with $\binom{k}{2}$ variables and $O^*(2^k)$ constraints that is feasible if and only if $(F, S, \ell)$ admits a feasible tileset with at most $\ell$ tiles. Using Kannan’s algorithm (Theorem 4) then completes the proof.

We introduce one variable $x_{s,s'} \geq 0$ for each possible tile type, i.e., for each pair of symbols $s, s' \in \binom{f}{k}$. We interpret $x_{s,s'}$ as the number of tiles of type $s, s'$ that the solution will contain. We begin with the constraint ensuring that we do not use more than $\ell$ tiles overall:

$$\sum_{\{s,s'\} \in \binom{f}{2}} x_{s,s'} \leq \ell$$
We need to add constraints to the ILP to ensure that the resulting assignment to the $x_{s,s'}$-variables corresponds to a feasible tileset, i.e., that each scenario $S$ can be implemented using the corresponding numbers of tiles of each type. This is the case if and only if there is a matching from the symbols in $S$ to the tiles that cover all symbols in $S$. Clearly, in order not to use too many variables, we do not want to compute a (one-sided perfect) matching for each scenario $S$. By Hall’s Theorem, it is instead sufficient to ensure that for each subset $I \subset F$ of symbols appearing at least once in scenario $S$ there are at least that many tiles involving these symbols. If $c_{s,S}$ denotes the number of occurrences of symbol $s$ in scenario $S$, we obtain the following constraints:

$$\sum_{(s,s') \cap I \neq \emptyset} x_{s,s'} \geq \sum_{s \in I} c_{s,S} \quad \forall S \in S, \forall I \subseteq F$$

In total we use $\binom{k}{2} = O(k^2)$ variables and $1 + m \cdot 2^k + \binom{k}{2}$ constraints. Using Kannan’s algorithm for testing feasibility of an ILP with $p$ variables in time $O^*(p^{O(p)})$ (Theorem 4) we get a total running time of $O^*(k^{O(k^2)})$.

7 Conclusion

We initiated the study of the Minimum Feasible Tileset problem and exposed an interesting combinatorial structure. We proved the problem to be NP-complete even in the restricted case with scenarios of size at most three. On the positive side, we showed that the Minimum Feasible Tileset problem admits a 4/3-approximation algorithm and that it is fixed-parameter tractable with respect to the number of scenarios and number of symbols. The latter algorithm works also for the Generalized Minimum Feasible Tileset problem where each scenario can contain multiple copies of a symbol and we believe that it can be further generalized to work also for the original assignment problem where also tiles of larger (but constant) size are allowed. It would be interesting to see whether our other positive results transfer to this more general setting. We note that our approximation algorithm relies heavily on the structural observations from Section 2 which do not seem to generalize well. Our integer linear program for a fixed number of scenarios does not seem easily adaptable either.

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