MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS INVOLVING VARIABLE EXPONENTS

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Abstract. We establish the existence of multiple solutions for a non-variational elliptic systems involving $p(x)$-Laplacian operator. The approach combines the methods of sub-supersolution and Leray–Schauder topological degree.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial \Omega$. Given $p_i \in C^1(\overline{\Omega})$, $1 < p_i^- \leq p_i^+ < N$ with

$$p_i^- = \inf_{x \in \Omega} p_i(x) \quad \text{and} \quad p_i^+ = \sup_{x \in \Omega} p_i(x),$$

we consider the quasilinear elliptic system

$$\begin{cases}
-\Delta_{p_1(x)} u_1 = f_1(x, u_1, u_2) \quad \text{in } \Omega \\
-\Delta_{p_2(x)} u_2 = f_2(x, u_1, u_2) \quad \text{in } \Omega \\
u_1, u_2 = 0 \quad \text{on } \partial \Omega,
\end{cases}
$$

where $\Delta_{p_i(x)}$ stands for $p_i(x)$-Laplacian differential operator on $W^{1,p_i(x)}(\Omega)$ and the nonlinearities $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are Carathéodory functions, i.e., $f_i(\cdot, s_1, s_2)$ is measurable for all $s_1, s_2 \in \mathbb{R}$ and $f_i(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, satisfying the following conditions:

(H.1): $f_1$ and $f_2$ are bounded in bounded domain.

(H.2): There exists a constant $\eta_i > \lambda_{1,p_i} \|\phi_{1,p_i}\|_{p_i^-}^{-1}$ such that

$$\eta_i \leq \liminf_{s_i \to 0^+} \frac{f_i(x, s_1, s_2)}{s_i^{p_i^-} - 1}$$

uniformly for a.e. $x \in \Omega$, all $s_j > 0$, $i \neq j$,

$$\eta_i \leq \liminf_{s_i \to 0^-} \frac{f_i(x, s_1, s_2)}{|s_i|^{p_i^-} s_i^{p_i^- - 2}}$$

uniformly for a.e. $x \in \Omega$, all $s_j < 0$, $i \neq j$, $i = 1, 2$.

Here, $\lambda_{1,p_i}$ and $\phi_{1,p_i}$ denote the first eigenvalue and the corresponding eigenfunction of $p_i(x)$-Laplacian operator, respectively, for $i = 1, 2$.

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(H.3):
\[ \lim_{|s_i| \to \infty} \sup_{|s_j|^r - 2, j \neq i} \frac{f_i(x, s_1, s_2)}{|s_i|} = 0, \]
uniformly for a.e. \( x \in \Omega \), all \( s_j \in \mathbb{R}, i, j = 1, 2, j \neq i, i = 1, 2 \).

A solution \((u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)\) of problem (P) is understood in the weak sense, that is
\[ \int_\Omega |\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla \varphi_i \, dx = \int_\Omega f_i(x, u_1, u_2) \varphi_i \, dx, \]
for all \( \varphi_i \in W_0^{1,p_i(x)}(\Omega) \).

Throughout this paper, we assume:

\((\text{H}_p)\): One of the following condition holds:
(i) There are two vectors \( l_i \in \mathbb{R}^N \backslash \{0\} \) such that for all \( x \in \Omega \),
\( h_i(t_i) = p_i(x + t_i l_i) \) are monotone for \( t_i \in I_{i,x} = \{ t_i; x + t_i l_i \in \Omega \}, i = 1, 2 \).
(ii) There is \( x_i \notin \overline{\Omega} \) such that for all \( w_i \in \mathbb{R} \backslash \{0\} \) with \( \|w_i\| = 1 \),
the function \( h_i(t_i) = p_i(x_i + t_i w_i) \) is monotone for \( t_i \in I_{x_i,w_i} = \{ t_i \in \mathbb{R}; x_i + t_i w_i \in \Omega \}, \) for \( i = 1, 2 \).

Assumption \((\text{H}_p)\) ensures that Dirichlet problem
\begin{equation}
- \Delta_{p_i(x)} u = \lambda |u|^{p_i(x)-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\end{equation}
admits a first eigenvalue \( \lambda_{1,p_i} > 0 \) characterized by
\begin{equation}
\lambda_{1,p_i} = \inf_{u \in W_0^{1,p_i(x)}(\Omega) \backslash \{0\}} \frac{\int_\Omega |\nabla u|^{p_i(x)} \, dx}{\int_\Omega |u|^{p_i(x)} \, dx}
\end{equation}
and the corresponding eigenfunction \( \phi_{1,p_i} \) satisfies
\begin{equation}
\phi_{1,p_i} \in C^1(\overline{\Omega}), \quad \phi_{1,p_i} > 0 \text{ in } \Omega \text{ and } \frac{\partial \phi_{1,p_i}}{\partial \nu} < 0 \text{ on } \partial \Omega
\end{equation}
(see [10] [12]). Actually, assumption \((\text{H}_p)\) enables to outfit \( p_i(x)\)-Laplacian operator with an important spectral property that will be useful later on. However, this property alone does not make the study of (P) any easier because of the lack of properties such as homogeneity. This fact complicates handling \( p_i(x)\)-Laplacian operator and constitutes a serious technical difficulty to address problem (P). Moreover, notice that system (P) is not in variational form, so the variational methods are not applicable.

Problems driven by the \( p_i(x)\)-Laplacian operator are involved in various nonlinear processes related to electrorheological fluids [11] [24], and image restorations [3]. When \( p_i(\cdot) \) is reduced to be a constant, \( \Delta_{p_i(x)} \) becomes the well-known \( p^-\)-Laplacian operator. In this context, system (P) has been thoroughly investigated in the litterature (see, e.g., [6] [14] [16] [17] [20] and the references therein). However, considering that \( p(x)\)-Laplacian operator possesses more complicated nonlinearity, stretching out results of the above-mentioned papers to problems involving \( p(x)\)-Laplacian operator is
not a straightforward task. This partly explains the few existing works in the literature devoted to this topic. Actually, elliptic systems without variational structure, possibly involving singularities near the origin, are studied in [2, 3, 21, 25] while the variational case is considered in [22]. It should be noted that the systems considered in the aforementioned papers do not fit the setting of (P) under assumptions (H.1)-(H.3).

Surprisingly enough, excepting the quoted papers where existence of a positive solution is obtained, so far we were not able to find previous results providing more than one nontrivial solution for (P). Motivated by this fact, our main concern is the question of existence of multiple solutions for a system of quasilinear elliptic equations (P). We first establish the existence of opposite constant-sign solutions to system (P), which means the existence of a positive solution \((u_1^+, u_2^+), (u_1^-, u_2^-)\) and a negative solution \((u_1^-, u_2^-), (u_1^-, u_2^-)\) in the sense that both components \(u_1^+, u_2^+\) are positive, and both components \(u_1^-, u_2^-\) are negative. Our approach is chiefly based on sub-supersolutions method where a significant feature of our result lies in the obtaining of the sub- and supersolutions for (P). At this point, the choice of suitable functions as well as an adjustment of adequate constants is crucial. However, it is worth noting that the obtained sub- and supersolution are quite different from the functions considered in the quoted papers, especially those constructed in [2, 3]. Practically and contrary to preconceived ideas, the construction process of the sub- and super-solutions in the present work is broadly similar to the one used in the case of constant exponent problems (see, e.g., [7] [18] [19]), despite the loss of the homogeneity property of the operator \(\Delta_{p_i(x)}\), which constitutes in itself a major obstacle to face. The crucial aspect of the argument is the new Mean Value Theorem (cf. Lemma 2.4) which, henceforth, would become an essential tool to handle problems with variable exponents.

The first main result is formulated as follows.

**Theorem 1.1.** Assume that conditions (H.1), (H.2) and (H.3) hold. Then problem (P) possesses at least a positive solution \((u_1^+, u_2^+), (u_1^-, u_2^-)\) and a negative solution \((u_1^-, u_2^-)\) in \(C^{1, \sigma}(\Omega) \times C^{1, \sigma}(\Omega)\), for certain \(\sigma \in (0, 1)\).

Our next goal is to provide the existence of a second positive solution \((\tilde{u}_1, \tilde{u}_2)\) for system (P). To this end, we must strengthen hypothesis (H.1) by the following assumption.

\((\text{H}'.1):\) (i) \(f_i(x, s_1, s_2) \geq 0\) uniformly for a.e. \(x \in \Omega, s_i \in \mathbb{R}, i = 1, 2\).

(ii) For each \(\delta > 0\), there exists \(M = M(\delta) > 0\) such that

\[|f_i(x, s_1, s_2)| \leq M, \text{ for a.e. } x \in \Omega, |s_i| \leq \delta, \text{ all } s_j \in \mathbb{R}, i, j = 1, 2, j \neq i.\]

The second main result is stated as follows.

**Theorem 1.2.** Assume that conditions (H'.1), (H.2) and (H.3) hold. Then problem (P) admits a solution \((\tilde{u}_1, \tilde{u}_2)\) in \(W^{1,p_1(x)}_0(\Omega) \times W^{1,p_2(x)}_0(\Omega)\) such
that
\[ \tilde{u}_1 \neq u_{1,+} \quad \text{and} \quad \tilde{u}_2 \neq u_{2,+}. \]

The proof is based on topological degree theory with suitable truncation as well as the Mean Value Theorem (cf. Lemma 2.4). Precisely, we prove that the degree on a ball \( B_{\tilde{R}} \) containing the obtained solutions in Theorem 1.1 is equal to 1 while the degree in a bigger ball \( B_R \supset B_{\tilde{R}} \), with \( \tilde{R} < R \), holding all potential solutions of (P) is 0. By the excision property of Leray-Schauder degree, this leads to the existence of a solution for (P) different from those obtained in Theorem 1.1.

The rest of the paper is organized as follows. Section 2 contains some technical and useful results; Section 3 deals with the existence of opposite constant-sign solutions; Section 4 establishes the existence of multiple positive solutions.

2. Preliminaries and technical results

Let \( L^{p_i(x)}(\Omega) \) be the generalized Lebesgue space that consists of all measurable real-valued functions \( u \) satisfying
\[ \rho_{p_i(x)}(u) = \int_\Omega |u(x)|^{p_i(x)} \, dx < +\infty, \]
endowed with the Luxemburg norm
\[ \|u\|_{p_i(x)} = \inf\{\tau > 0 : \rho_{p_i(x)}(\frac{u}{\tau}) \leq 1\}, \quad i = 1, 2. \]

The variable exponent Sobolev space \( W^{1,p_i(x)}(\Omega) \) is defined by
\[ W^{1,p_i(x)}(\Omega) = \{ u \in L^{p_i(x)}(\Omega) : |\nabla u| \in L^{p_i(x)}(\Omega) \}. \]
The norm \( \|u\| = \|\nabla u\|_{p_i(x)} \) makes \( W^{1,p_i(x)}(\Omega) \) a Banach space. The product space \( W^{1,p_1(x)}(\Omega) \times W^{1,p_2(x)}(\Omega) \) is endowed with the norm \( \|(u,v)\| = \|u\| + \|v\| \).

In what follows, for any constant \( C > 0 \), we denote by \( B_C \) the ball in \( W^{1,p_1(x)}(\Omega) \times W^{1,p_2(x)}(\Omega) \) defined by
\[ B_C := \left\{ (u_1, u_2) \in W^{1,p_1(x)}(\Omega) \times W^{1,p_2(x)}(\Omega) : \|(u_1, u_2)\| < C \right\}. \]
For any \( r \in \mathbb{R} \), we denote \( r^+ := \max\{r, 0\} \) and \( r^- := \max\{-r, 0\} \).

Next we formulate a serie of technical Lemmas which will be useful ater on.

**Lemma 2.1.** (i) For any \( u \in L^{p(x)}(\Omega) \) it holds
\[ \|u\|_{p(x)}^{-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{+} \quad \text{if} \quad \|u\|_{p(x)} > 1, \]
\[ \|u\|_{p(x)}^{+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{-} \quad \text{if} \quad \|u\|_{p(x)} \leq 1. \]
(ii) For \( u \in L^{p(x)}(\Omega) \setminus \{0\} \) we have

\[
\|u\|_{p(x)} = c \text{ if and only if } \rho_{p(x)} \left( \frac{u}{c} \right) = 1.
\]

**Definition 2.2.** Let \( u, v \in W^{1,p(x)}(\Omega) \). We say that \(-\Delta_{p(x)} u \leq -\Delta_{p(x)} v\) if for all \( \varphi \in W^{1,p(x)}_0(\Omega) \) with \( \varphi \geq 0 \),

\[
\int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx \leq \int_{\Omega} |\nabla v|^{p(x)} - 2 \nabla v \nabla \varphi \, dx.
\]

**Lemma 2.3.** Let \( u, v \in W^{1,p(x)}(\Omega) \). If \(-\Delta_{p(x)} u \leq -\Delta_{p(x)} v\) and \( u \leq v \) on \( \partial \Omega \), then \( u \leq v \) in \( \Omega \).

The next Lemma is crucial in our approach, which establishes a result of the Mean Value Theorem type.

**Lemma 2.4.** Let \( h \in L^{p(x)}(\Omega) \) and let \( k \in L^{\infty}(\Omega) \) be positive functions such that \( k(x) \in (m, M) \) for a.e. \( x \in \Omega \), for constants \( m, M > 0 \). Let \( u \in W^{1,p(x)}_0(\Omega) \) be the solution of the Dirichlet problem

\[
-\Delta_{p(x)} u = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

Then, for every \( \varphi \in W^{1,p(x)}_0(\Omega) \) with \( \varphi \geq 0 \) in \( \Omega \), there exists a constant \( \hat{k} \in (m, M) \), \( \hat{k} := \hat{k}(\varphi) \), such that

\[
\int_{\Omega} k(x) |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx = \hat{k} \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx.
\]

**Proof.** From the identity (2) in [4] Lemma in page 823 we get

\[
\int_{\Omega} k(x) |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx = m \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx + \int_{\Omega} (\int_{\Omega(\omega)} |\nabla u|^{p(x)} - 2 \nabla \varphi \, dx) \, dy
\]

while the identity (3) (also in [4] Lemma in page 823) implies

\[
\int_{\Omega} k(x) |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx = M \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx - \int_{\Omega} (\int_{\omega(\omega)} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx) \, dy
\]

where

\[
\Omega(y) = \{ x \in \Omega; k(x) > y \}, \quad \omega(y) = \{ x \in \Omega; \, k(x) \leq y \},
\]

for \( y \in [m, M] \). Denote by \( \chi_{\omega(y)} \) the characteristic function of the subset \( \omega(y) \). Since \( \nabla \chi_{\omega(y)}(x) = 0 \) in \( \Omega \), it follows that

\[
\int_{\omega(\omega)} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \, dx = \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \chi_{\omega(\omega)} \, dx
\]

Hence, testing with \( \varphi \cdot \chi_{\omega(y)} \in W^{1,p(x)}_0(\Omega) \) we obtain

\[
\langle -\Delta_{p(x)} u, \varphi \cdot \chi_{\omega(y)} \rangle = \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi \chi_{\omega(y)} \, dx - \int_{\partial \Omega} |\nabla u|^{p(x)} - 2 \nabla \varphi \nabla (\varphi \cdot \chi_{\omega(y)}) \, dx
\]

\[
= \int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla (\varphi \cdot \chi_{\omega(y)}) \, dx = \int_{\Omega} h (\varphi \cdot \chi_{\omega(y)}) \, dx.
\]
Lemma 2.6. Let \( h \in L^\infty(\Omega) \) a positive function in \( \Omega \) and let \( u \in W^{1,p(x)}_0(\Omega) \) be the solution of the Dirichlet problem

\[
-\Delta_{p(x)} u = h(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.
\]

Then, for every \( \varphi \in W^{1,p(x)}_0(\Omega) \) with \( \varphi \geq 0 \) in \( \Omega \), there exists \( x_0 \in \Omega \), depending on \( \varphi \), such that

\[
\int_\Omega C_{p(x)}(x) \left| \nabla u \right|^{p(x)-2} \nabla u \nabla \varphi \, dx = C_{p(x)}(x_0) \int_\Omega \left| \nabla u \right|^{p(x)-2} \nabla u \nabla \varphi \, dx,
\]

for every constant \( C > 0 \).

**Corollary 2.5.** Let \( w_1 \geq 0 \) and \( w_2 > 0 \) be two nonconstant differentiable functions in \( \Omega \). For all \( x \in \Omega \) define

\[
\mathcal{L}_1(w_1, w_2) = \left| \nabla w_1 \right|^{p(x)} + (p(x) - 1) \left| \nabla w_2 \right|^{p(x)} \left( \frac{w_1(x)}{w_2(x)} \right)^{p(x)} - p(x) \left| \nabla w_2 \right|^{p(x)-2} \nabla w_2 \nabla \left( \frac{w_1(x)}{w_2(x)} \right)^{p(x)-1},
\]

\[
\mathcal{L}_2(w_1, w_2) = \left| \nabla w_1 \right|^{p(x)} - p(x) \left| \nabla w_2 \right|^{p(x)-2} \nabla w_2 \nabla \left( \frac{w_1(x)}{w_2(x)} \right)^{p(x)-1}.
\]

Then \( \mathcal{L}_1(w_1, w_2) = \mathcal{L}_2(w_1, w_2) \geq 0 \).
Lemma 2.7. Assume \((H_p)\) holds true and let
\[
0 < J < \lambda_1, p (p^- - 1).
\]
Then, the Dirichlet problem
\[
(2.9) \quad \begin{cases}
-\Delta_{p(x)} u = J\left(\frac{u^+}{\max\{1, u\}}\right)^{p(x)-1} + \delta \lambda_{1, p} \phi_{1, p}^{p(-1)} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
does not admit solutions \(u \in W^{1, p(x)}_0(\Omega)\) for every \(\delta > 0\) small.

**Proof.** Arguing by contradiction, let \(u \in W^{1, p(x)}_0(\Omega)\) be a solution of \((2.9)\).

According to \([11, \text{Theorem 4.1}]\), \(u\) is bounded in \(L^\infty(\Omega)\) and therefore, owing to \([10, \text{Theorem 1.1}]\), \(u\) is bounded in \(C^{1, \sigma}(\overline{\Omega})\) for a certain \(\sigma \in (0, 1)\).

Furthermore, by strong maximum principle in \([12]\) one can write
\[
(2.10) \quad u \geq \delta \phi_{1, p(x)} \quad \text{in } \Omega, \text{ for } \delta > 0 \text{ small}.
\]
Applying Picone’s Identity in Lemma 2.6 to functions \(u\) and \(\phi_{1, p} + \varepsilon\) for \(\varepsilon > 0\), and by Lemma 2.7, there is \(k \in (p^-, p^+)\) such that
\[
0 \leq \int_{\Omega} L_2(u, \phi_{1, p} + \varepsilon) \, dx \\
= \int_{\Omega} |\nabla u|^{p(x)} \, dx - \int_{\Omega} p(x)|\nabla \phi_{1, p}|^{p(x)} - 2 |\nabla \phi_{1, p}|^{p(x)} \frac{u^{p(x)}}{\left(\phi_{1, p}^{p(x)} + \varepsilon\right)} \, dx \\
= \int_{\Omega} |\nabla u|^{p(x)} \, dx - k \int_{\Omega} |\nabla \phi_{1, p}|^{p(x)} - 2 |\nabla \phi_{1, p}|^{p(x)} \frac{u^{p(x)}}{\left(\phi_{1, p}^{p(x)} + \varepsilon\right)} \, dx \\
= \int_{\Omega} |\nabla u|^{p(x)} \, dx - \lambda_{1, p} k \int_{\Omega} \left(\frac{\phi_{1, p}}{\phi_{1, p}^{p(x)} + \varepsilon}\right)^{p(x)-1} u^{p(x)} \, dx \\
\leq \int_{\Omega} |\nabla u|^{p(x)} \, dx - \lambda_{1, p} p^{-} \int_{\Omega} \left(\frac{\phi_{1, p}}{\phi_{1, p}^{p(x)} + \varepsilon}\right)^{p(x)-1} u^{p(x)} \, dx.
\]
Passing to the limit as \(\varepsilon \to 0\), by means of the Lebesgue dominated convergence theorem, we obtain
\[
(2.11) \quad 0 \leq \int_{\Omega} |\nabla u|^{p(x)} \, dx - \lambda_{1, p} p^{-} \int_{\Omega} u^{p(x)} \, dx.
\]
Acting with \(u\) in \((2.9)\) and using \((2.10)\) lead to
\[
(2.12) \quad \int_{\Omega} |\nabla u|^{p(x)} \, dx = \int_{\Omega} \left(J u^{p(x)} + \left(\lambda_{1, p} \phi_{1, p}^{p(-1)} \right) u \right) \, dx \\
\leq \int_{\Omega} \left(J u^{p(x)} + (\delta \lambda_{1, p} \phi_{1, p})^{p(x)-1} u \right) \, dx \leq \int_{\Omega} \left(J + \lambda_{1, p}\right) u^{p(x)} \, dx.
\]
Gathering \((2.11)-(2.12)\) together we get
\[
0 \leq (J - \lambda_{1, p} (p^- - 1)) \int_{\Omega} u^{p(x)} \, dx < 0,
\]
a contradiction due to \((2.8)\). Consequently, problem \((2.9)\) has no solutions.

\[\square\]

3. **Proof of Theorem 1.1: Opposite constant-sign solutions**

We establish the existence of two opposite constant-sign solutions to system \((P)\). Our approach relies on sub-supersolutions method (see \([3, \text{Theorem 3.1}]\)). We recall that a sub-supersolution for \((P)\) consists of two pairs
\((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2)\) \(\in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) such that there hold \(\overline{u}_i \geq \underline{u}_i\) in \(\Omega\), and
\[
\int_{\Omega} |\nabla \underline{u}_i|^{p_i(x)-2} \nabla \underline{u}_i \nabla \phi_i \, dx - \int_{\Omega} f_i(x, u_1, u_2) \phi_i \, dx \leq 0,
\]
\[
\int_{\Omega} |\nabla \overline{u}_i|^{p_i(x)-2} \nabla \overline{u}_i \nabla \phi_i \, dx - \int_{\Omega} f_i(x, u_1, u_2) \phi_i \, dx \geq 0,
\]
for all \(\phi_i \in W^{1,p_i}_0(\Omega)\) with \(\phi_i \geq 0\) a.e. in \(\Omega\) and for all \(u_i \in W^{1,p_i}_0(\Omega)\) satisfying \(u_i \in [\underline{u}_i, \overline{u}_i]\) a.e. in \(\Omega\), for \(i = 1, 2\).

**Existence of supersolution:**

Let \(\widetilde{\Omega}\) be a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \widetilde{\Omega}\), such that \(\overline{\Omega} \subset \widetilde{\Omega}\). We denote by \(\lambda_{1,p_i}\) the first eigenvalue of \(-\Delta_{p_i}(x)\) on \(W^{1,p_i}_0(\Omega)\) and by \(\tilde{\phi}_{1,p_i}\) the positive eigenfunction corresponding to \(\lambda_{1,p_i}\), that is
\[
-\Delta_{p_i}(x) \tilde{\phi}_{1,p_i} = \lambda_{1,p_i} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \text{ in } \widetilde{\Omega}, \quad \tilde{\phi}_{1,p_i} = 0 \text{ on } \partial \widetilde{\Omega}.
\]
By the definition of \(\tilde{\Omega}\) and the strong maximum principle, there exists a constant \(\tau > 0\) sufficiently small such that
\[
\tilde{\phi}_{1,p_i}(x) > \tau \text{ in } \overline{\Omega}.
\]
Moreover, one can find a constant \(\bar{\eta} > 0\) such that
\[
\bar{\eta} < \min_{i=1,2} \left\{ \frac{\lambda_{1,p_i}}{2} \tau^{p_i^{\ast} - 1} \left\| \tilde{\phi}_{1,p_i} \right\|_{\infty}^{-(p_i^{\ast} - 1)} \right\}.
\]
For a constant \(\varepsilon \in (0,1)\) set
\[
(\overline{u}_1, \overline{u}_2) = \varepsilon^{-1}(\tilde{\phi}_{1,p_1}, \tilde{\phi}_{1,p_2}).
\]
It follows that
\[
\int_{\Omega} |\nabla \overline{u}_i|^{p_i(x)-2} \nabla \overline{u}_i \nabla \phi_i \, dx = \int_{\Omega} \varepsilon^{-1} |\nabla \tilde{\phi}_{1,p_i}|^{p_i(x)-2} \nabla \tilde{\phi}_{1,p_i} \nabla \phi_i \, dx.
\]
Using (3.2) and Corollary 2.5, there is \(\tilde{x}_i \in \Omega\) such that
\[
\int_{\Omega} \varepsilon^{-1} |\nabla \tilde{\phi}_{1,p_i}|^{p_i(x)-2} \nabla \tilde{\phi}_{1,p_i} \nabla \phi_i \, dx
\]
\[
= \varepsilon^{-1} \lambda_{1,p_i} \int_{\Omega} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \phi_i \, dx
\]
\[
\geq \varepsilon^{-1} \lambda_{1,p_i} \int_{\Omega} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \phi_i \, dx
\]
\[
= \varepsilon^{-1} \lambda_{1,p_i} \int_{\Omega} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \phi_i \, dx
\]
\[
\geq \varepsilon^{-1} \lambda_{1,p_i} \int_{\Omega} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \phi_i \, dx,
\]
provided \(\varepsilon > 0\) small enough. Since, from (3.3), we have
\[
\frac{1}{2} \lambda_{1,p_i} \tilde{\phi}_{1,p_i}^{p_i(x)-1} \geq \frac{1}{2} \lambda_{1,p_i} \left\{ \begin{array}{ll}
\tilde{\phi}_{1,p_i}^{p_i^{\ast} - 1}(x) & \text{if } \tilde{\phi}_{1,p_i}(x) \geq 1 \\
\tilde{\phi}_{1,p_i}^{p_i^{\ast} - 1}(x) & \text{if } \tilde{\phi}_{1,p_i}(x) < 1
\end{array} \right.
\]
\(\geq \bar{\eta} \tilde{\phi}_{1,p_i}^{p_i^{\ast} - 1}(x)\) in \(\Omega\),
then it follows that

\[
\int_{\Omega} \varepsilon^{-(p_1^{-1})} \frac{1}{2} \lambda_{1,p_i} \varphi_i^{p_i(x)-1} \varphi_i \, dx \geq \int_{\Omega} \varepsilon^{-(p_i^{-1})} \bar{\varphi}_i^{p_i^{-1}} \varphi_i \, dx \\
= \int_{\Omega} \bar{\eta}(\varepsilon^{-1} \bar{\varphi}_i^{p_i^{-1}}) \varphi_i \, dx = \int_{\Omega} \varphi_i^{p_i^{-1}} \varphi_i \, dx,
\]

for all \( \varphi_i \in W_0^{1,p_i(x)}(\Omega) \) with \( \varphi_i \geq 0 \). On the other hand, assumption (H.3) yields \( \rho = \rho(\bar{\eta}) > 0 \) such that

\[
\frac{f_i(x, s_1, s_2)}{|s_i|^{p_i} - 2 s_i} \leq \bar{\eta}, \text{ for a.e. } x \in \Omega, \text{ for all } |s_i| > \rho, s_j \in \mathbb{R},
\]

while assumption (H.1) ensures the existence of a constant \( c_\rho > 0 \) for which we have

\[
|f_i(x, s_1, s_2)| \leq c_\rho, \text{ for a.e. } x \in \Omega, \text{ for all } |s_1|, |s_2| \leq \rho, i = 1, 2.
\]

Thus, it turns out that

\[
\varepsilon^{-(p_i^{-1})} \frac{1}{2} \lambda_{1,p_i} \varphi_i^{p_i^{-1}} \geq c_\rho.
\]

Then, gathering (3.5) - (3.9) together yields

\[
\int_{\Omega} |\nabla \bar{u}_1|^{p_1(x)-2} \nabla \bar{u}_1 \nabla \varphi_1 \, dx \geq \int_{\Omega} (c_\rho + \bar{\eta} \bar{\varphi}_1^{p_1^{-1}}) \varphi_1 \, dx \\
\geq \int_{\Omega} f_1(x, \bar{u}_1, s_2) \varphi_1 \, dx
\]

and

\[
\int_{\Omega} |\nabla \bar{u}_2|^{p_2(x)-2} \nabla \bar{u}_2 \nabla \varphi_2 \, dx \geq \int_{\Omega} (c_\rho + \bar{\eta} \bar{\varphi}_2^{p_2^{-1}}) \varphi_2 \, dx \\
\geq \int_{\Omega} f_2(x, s_1, \bar{u}_2) \varphi_2 \, dx,
\]

for all \( \varphi_i \in W_0^{1,p_i(x)}(\Omega) \) with \( \varphi_i \geq 0 \), for all \( (s_1, s_2) \in [0, \bar{u}_1] \times [0, \bar{u}_2] \). This proves that \((\bar{u}_1, \bar{u}_2)\) is a supersolution for system (P).

**Existence of subsolution:**

Next, we show that

\[
(\bar{u}_1, \bar{u}_2) = \varepsilon(\phi_{1,p_1}, \phi_{1,p_2})
\]

is a subsolution for (P) for \( \varepsilon \in (0, 1) \). We claim that \( \bar{u}_i \geq \bar{u}_i \) in \( \Omega \). Indeed, from (1.1), (1.3) and Corollary 2.5 there is \( \bar{x}_i \in \Omega \) such that

\[
\int_{\Omega} \varepsilon^{p_i(x)-1} |\nabla \phi_{1,p_i}|^{p_i(x)-2} \nabla \phi_{1,p_i} \nabla \varphi_i \, dx \\
= \varepsilon^{p_i(\bar{x}_i)-1} \lambda_{1,p_i} \int_{\Omega} \phi_{1,p_i}^{p_i(x)-1} \varphi_i \, dx \\
\leq \varepsilon^{p_i^{-1}} \lambda_{1,p_i} \int_{\Omega} \phi_{1,p_i}^{p_i(x)-1} \varphi_i \, dx,
\]

for \( \varepsilon > 0 \) sufficiently small, for all \( \varphi_i \in W_0^{1,p_i(x)}(\Omega) \) with \( \varphi_i \geq 0 \). Then, on account of (3.4), (3.10), (3.11) and the first equality in (3.10), it holds

\[
\int_{\Omega} |\nabla \bar{u}_i|^{p_i(x)-2} \nabla \bar{u}_i \nabla \varphi_i \, dx \leq \int_{\Omega} |\nabla \bar{u}_i|^{p_i(x)-2} \nabla \bar{u}_i \nabla \varphi_i \, dx,
\]
for all $\varphi \in W_{0}^{1,p_{i}}(\Omega)$ with $\varphi \geq 0$. This proves the claim.

In view of assumption (H.2) there exists $\hat{\rho} = \hat{\rho}(\eta_{i}) > 0$ such that

\[
\frac{f_i(x, s_1, s_2)}{s_{i}^{p_{i}-1}} \geq \eta_{i}, \text{ for a.e. } x \in \Omega, \text{ for all } 0 < s_{i}, s_{j} < \hat{\rho}.
\]

Thus

\[
f_i(x, s_1, s_2) \geq \eta_{i} s_{i}^{p_{i}-1}, \text{ for all } 0 < s_1, s_2 < \hat{\rho}.
\]

For $\phi_{1,p_{i}}(x) > 1$, in view of (H.2), one has

\[
\lambda_{1,p_{i}} \phi_{1,p_{i}}^{p_{i}(x)-1}(x) \leq \lambda_{1,p_{i}} \phi_{1,p_{i}}^{p_{i}-1}(x) \leq \lambda_{1,p_{i}} \|\phi_{1,p_{i}}\|_{\infty}^{p_{i}-1} \leq \eta_{i} \phi_{1,p_{i}}^{p_{i}-1}(x) \text{ in } \Omega,
\]

while, if $\phi_{1,p_{i}}(x) \leq 1$, we have

\[
\lambda_{1,p_{i}} \phi_{1,p_{i}}^{p_{i}(x)-1}(x) \leq \lambda_{1,p_{i}} \phi_{1,p_{i}}^{p_{i}-1}(x) \leq \eta_{i} \phi_{1,p_{i}}^{p_{i}-1}(x) \text{ in } \Omega.
\]

Hence, it turns out that

\[
\varepsilon^{p_{i}-1} \lambda_{1,p_{i}} \int_{\Omega} \phi_{1,p_{i}}^{p_{i}(x)-1} \varphi_{i} \, dx \leq \varepsilon^{p_{i}-1} \eta_{i} \int_{\Omega} \phi_{1,p_{i}}^{p_{i}-1} \varphi_{i} \, dx
\]

\[
= \eta_{i} \int_{\Omega} (\varepsilon \phi_{1,p_{i}})^{p_{i}-1} \varphi_{i} \, dx,
\]

for all $\varphi_{i} \in W_{0}^{1,p_{i}}(\Omega)$ with $\varphi_{i} \geq 0$. Then, assuming $\varepsilon > 0$ so small that $\varepsilon \phi_{1,p_{i}}(x) \leq \hat{\rho}, \forall x \in \Omega, i = 1, 2$, gathering (3.10), (3.11), (3.12) and (3.13) together yield

\[
\int_{\Omega} |\nabla \underline{u}_{i}|^{p_{i}(x)-2} \nabla \underline{u}_{i} \nabla \varphi_{1} \, dx
= \int_{\Omega} \varepsilon^{p_{i}(x)-1} |\nabla \phi_{1,p_{i}}|^{p_{i}(x)-2} \nabla \phi_{1,p_{i}} \nabla \varphi_{1} \, dx
\]

\[
\leq \int_{\Omega} \eta_{i} \phi_{1,p_{i}}^{p_{i}-1} \varphi_{1} \, dx
\leq \int_{\Omega} f_{i}(x, \underline{u}_{1}, s_{2}) \varphi_{1} \, dx,
\]

and

\[
\int_{\Omega} |\nabla \underline{u}_{2}|^{p_{2}(x)-2} \nabla \underline{u}_{2} \nabla \varphi_{2} \, dx
= \int_{\Omega} \varepsilon^{p_{2}(x)-1} |\nabla \phi_{1,p_{2}}|^{p_{2}(x)-2} \nabla \phi_{1,p_{2}} \nabla \varphi_{2} \, dx
\]

\[
\leq \int_{\Omega} \eta_{i} \phi_{1,p_{2}}^{p_{2}-1} \varphi_{2} \, dx
\leq \int_{\Omega} f_{2}(x, s_{1}, \underline{u}_{2}) \varphi_{2} \, dx,
\]

for all $\varphi_{i} \in W_{0}^{1,p_{i}}(\Omega)$ with $\varphi_{i} \geq 0$, for all $(s_{1}, s_{2}) \in [\underline{u}_{1}, \bar{u}_{1}] \times [\underline{u}_{2}, \bar{u}_{2}]$, showing that $(\underline{u}_{1}, \underline{u}_{2})$ is a subsolution for (P).

**Proof of Theorem 4.1**

Now we are in position to apply [3 Theorem 3.1] which guarantees the existence of a positive solution $(u_{1,+}, u_{2,+})$ satisfying $\underline{u}_{i} \leq u_{i,+} \leq \bar{u}_{i}$. By an analogous approach as before, on the basis of assumptions (H.1), (H.2) and (H.3), we can show that the pair of functions $(-\bar{u}_{1}, -u_{1})$ and $(-\bar{u}_{2}, -u_{2})$ constitute a pair of negative sub- and supersolution for problem (P). Consequently, we obtain a negative solution $(u_{1,-}, u_{2,-})$ within $[-\bar{u}_{1}, -\underline{u}_{1}] \times [-\bar{u}_{2}, -\underline{u}_{2}]$. Furthermore, the nonlinear regularity theory up to the boundary (see [10 Theorem 1.2]) implies that the solutions $(u_{1,+}, u_{2,+})$ and $(u_{1,-}, u_{2,-})$ belong to $C^{1,\sigma}(\overline{\Omega}) \times C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0, 1)$. This completes the proof.
4. Proof of Theorem 1.2: Positive solutions

In this section we show that problem (P) admits a second positive solution different from \((u_{1,+}, u_{2,+})\). The proof is based on topological degree theory. Precisely, we prove that the degree of an operator corresponding to system (P) is equal to 0 on a ball \(B_R\), while the degree is 1 in a smaller ball \(B_{\hat{R}} \subset B_R\), with \(\hat{R} < R\). By the excision property of Leray-Schauder degree, we find a positive solution \((\hat{u}_1, \hat{u}_2)\) in \(B_{\hat{R}} \backslash \overline{B_R}\) such that \(\hat{u}_1 \neq u_{1,+}\) and \(\hat{u}_2 \neq u_{2,+}\).

4.1. Topological degree on \(B_R\). For every \(t \in [0,1]\), we consider the problem

\[
\begin{aligned}
(P_t) \quad & \begin{cases} 
-\Delta_{p_i(x)} u_i = f_{i,t}(x, u_1, u_2) & \text{in } \Omega, \\
u_i = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

with

\[
f_{i,t}(x, u_1, u_2) = tf_i(x, u_1, u_2) + (1-t) \left[ J_i \frac{(u_i^+)^{p_i(x)-1}}{(\max(1,\|u_i\|))^{p_i(x)-1}} + \delta \lambda_1 \phi_1^\prime(p_i(x)-1) \right],
\]

where \(\delta > 0\) is a small constant and

\[
0 < J_i < \lambda_1 \phi_1 \min \{1, p_i^- - 1\}, \quad i = 1, 2.
\]

With a constant \(R > 0\), let define the homotopy

\[
\mathcal{H} : [0,1] \times \overline{B_R} \rightarrow W^{1,p_1(x)}(\Omega) \times W^{1,p_2(x)}(\Omega)
\]

\[
(t, u_1, u_2) \mapsto (\mathcal{H}_1(t, u_1, u_2), \mathcal{H}_2(t, u_1, u_2))
\]

where \(\mathcal{H}_i\) are given by

\[
\langle \mathcal{H}_i(t, u_1, u_2), \varphi_i \rangle = \int_{\Omega} |\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla \varphi_i \, dx - \int_{\Omega} f_{i,t}(x, u_1, u_2) \varphi_i \, dx,
\]

for \(\varphi_i \in W^{1,p_i(x)}_0(\Omega)\) and \(\overline{B_R}\) is the closure of \(B_R\) in \(W^{1,p_1(x)}_0(\Omega) \times W^{1,p_2(x)}_0(\Omega)\) with

\[
B_R := \left\{ (u_1, u_2) \in W^{1,p_1(x)}_0(\Omega) \times W^{1,p_2(x)}_0(\Omega) : \|(u_1, u_2)\| < R \right\}.
\]

Lemma 4.1. The homotopies \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are continuous and compact.

Proof. We prove only the continuity of \(\mathcal{H}_1\) because that of \(\mathcal{H}_2\) can be justified similarly. Let \((t_n, u_{1,n}, u_{2,n}) \in [0,1] \times \overline{B_R}\) with

\[
(t_n, u_{1,n}, u_{2,n}) \rightarrow (t, u_1, u_2) \quad \text{in } [0,1] \times W^{1,p_1(x)}_0(\Omega) \times W^{1,p_2(x)}_0(\Omega).
\]

Passing to relabeled subsequences, there holds the convergence

\[
u_{i,n} \rightarrow u_i \quad \text{a.e. in } \Omega
\]

and there exists a function \(h_i \in L^{p_i(x)}(\Omega)\) such that

\[
|u_{i,n}(x)| \leq h_i(x) \quad \text{a.e. in } \Omega, \quad \text{for } i = 1, 2.
\]

Noticing that

\[
(t_n f_1(x, u_{1,n}, u_{2,n}) - t f_1(x, u_1, u_2)
= (t_n - t)f_1(x, u_{1,n}, u_{2,n}) + t [f_1(x, u_{1,n}, u_{2,n}) - f_1(x, u_1, u_2)],
\]

we conclude...
it suffices to prove that

\[(4.6) \quad \{f_{1,t_n}(x,u_1,n,u_2,n)\} \to \{f_{1,t}(x,u_1,u_2)\} \text{ in } L^{p_1(x)\gamma}(\Omega).\]

From (3.8) we have that \(f_1(x,u_1,n,u_2,n) \in L^{p_1(x)/p_1(x)-1}(\Omega)\) while the fact that \(f_1\) is a Carathéodory function implies

\[f_1(x,u_1,n(x),u_2,n(x)) \to f_1(x,u_1(x),u_2(x)) \text{ a.e. in } \Omega.\]

Using (3.8), (4.5) and the embedding \(W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)\), it follows that

\[|f_1(x,u_1,n_k,u_2,n_k) - f_1(x,u_1,u_2)|^{p_1(x)/p_1(x)-1} \leq \left[2C_p + \alpha \left(|h|^{p_1(x)} + |u_1|^{p_1(x)}\right)\right]^{p_1(x)/p_1(x)-1}.

Then, the dominated convergence result in [8, Lemma 2.3.16] implies that (4.6) holds true.

The next step in the proof is to show that

\[(1 - t_n)\frac{(u_{1,n})^{p_1(x)-1}}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} \to (1 - t)\frac{(u_{1})^{p_1(x)-1}}{(\max\{1,|u_1|\})^{p_1(x)-1}} \text{ in } L^{p_1(x)\gamma}(\Omega).

As above one can write

\[(4.7) \quad (1 - t_n)\frac{(u_{1,n})^{p_1(x)-1}}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} - (1 - t)\frac{(u_{1})^{p_1(x)-1}}{(\max\{1,|u_1|\})^{p_1(x)-1}}
\]

\[= (t - t_n)\frac{(u_{1,n}^+)^{p_1(x)-1}}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} + (1 - t)\left(\frac{(u_{1,n}^+)^{p_1(x)-1}}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} - \frac{(u_{1})^{p_1(x)-1}}{(\max\{1,|u_1|\})^{p_1(x)-1}}\right)
\]

\[= (t - t_n)\frac{(u_{1,n}^+)^{p_1(x)-1}}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} + (1 - t)\left(\frac{1}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} - \frac{1}{(\max\{1,|u_1|\})^{p_1(x)-1}}\right).

The triangle inequalities

\[\|u_1,n\| \leq \|u_1,n - u_1\| + \|u_1\| \text{ and } \|u_1\| \leq \|u_1,n - u_1\| + \|u_1,n\|\]

ensure that \(\|u_1\| > 1\) (resp. \(\leq 1\)) whenever \(\|u_1,n\| > 1\) (resp. \(\leq 1\)) and therefore, due to (4.3), one has

\[\max\{1,|u_1,n|\} \to \|u_1\| = \max\{1,|u_1|\}.

Hence, for all \(x \in \Omega\), we have

\[\frac{1}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} \to 0,
\]

which implies that

\[\frac{1}{(\max\{1,|u_1,n|\})^{p_1(x)-1}} - \frac{1}{(\max\{1,|u_1|\})^{p_1(x)-1}} \to 0.
\]
Moreover, thanks to the estimate

\[
\left| \frac{1}{(\max\{1,||u_{1,n}||\})^{p_1(x)-1}} - \frac{1}{(\max\{1,||u_1||\})^{p_1(x)-1}} \right| \leq 2,
\]
we conclude, from the dominated convergence theorem, that

\[
\left( \frac{1}{(\max\{1,||u_{1,n}||\})^{p_1(x)-1}} \right) \rightarrow \left( \frac{1}{(\max\{1,||u_1||\})^{p_1(x)-1}} \right) \text{ in } L_{\frac{p_1(x)}{p_1(x)-1}}(\Omega).
\]

Now, we focus on the last term in (4.7). Observe that

\[
(4.10) \quad \left( u_{1,n}^{+} \right)^{p_1(x)-1} - \left( u_{1}^{+} \right)^{p_1(x)-1} = \chi_{\{u_{1,n} \geq 0\}} |u_{1,n}|^{p_1(x)-1} - \chi_{\{u_{1} \geq 0\}} |u_1|^{p_1(x)-1}
\]

Due to (4.4) and the estimate $|\chi_{\{u_{1,n} \geq 0\}} (x) - \chi_{\{u_1 \geq 0\}} (x)| \leq 2$, it follows that

\[
\chi_{\{u_{1,n} \geq 0\}} - \chi_{\{u_1 \geq 0\}} \rightarrow 0 \text{ in } L_{\frac{p_1(x)}{p_1(x)-1}}(\Omega).
\]

Moreover, since by (4.4) and (4.5) we have

\[
|u_{1,n}|^{p_1(x)-1} - |u_1|^{p_1(x)-1} \rightarrow 0 \text{ a.e } x \in \Omega
\]

and

\[
\left| |u_{1,n}|^{p_1(x)-1} - |u_1|^{p_1(x)-1} \right| \leq h^{p_1(x)-1} + |u_1|^{p_1(x)-1},
\]

where the positive function $h^{p_1(x)-1} + |u_1|^{p_1(x)-1}$ belongs to $L^{p_1(x)/p_1(x)-1}(\Omega)$.

The dominated convergence theorem implies that

\[
\lim_{n \rightarrow +\infty} \rho_{\frac{p_1(x)}{p_1(x)-1}} \left( |u_{1,n}|^{p_1(x)-1} - |u_1|^{p_1(x)-1} \right) = 0.
\]

which by [12] Theorem 1.4] shows that

\[
|u_{1,n}|^{p_1(x)-1} \rightarrow |u_1|^{p_1(x)-1} \text{ in } L_{\frac{p_1(x)}{p_1(x)-1}}(\Omega).
\]

Hence, bearing in mind (4.8) and (4.9), we derive that

\[
(4.10) \quad \left( u_{1,n}^{+} \right)^{p_1(x)-1} \rightarrow \left( u_1^{+} \right)^{p_1(x)-1} \text{ in } L_{\frac{p_1(x)}{p_1(x)-1}}(\Omega).
\]

Gathering (4.6) and (4.10) together, we conclude that the homotopy $\mathcal{H}_1$ is continuous from $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ to $L_{\frac{p_1(x)}{p_1(x)-1}}(\Omega)$. We proceed analogously to prove that the homotopy $\mathcal{H}_2$ is continuous from $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ to $L_{\frac{p_2(x)}{p_2(x)-1}}(\Omega)$.

Finally, from the estimate (3.8) and the compactness of the embedding $W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, it is readily seen that homotopies $\mathcal{H}_1$ and $\mathcal{H}_2$ are compact. This completes the proof.

\[\square\]

**Proposition 4.1.** Assume (H.1) and (H.3) hold. If $R > 0$ is sufficiently large, then the Leray-Schauder topological degree

\[
\deg(\mathcal{H}(t, \cdot, \cdot), B_R, 0)
\]
is well defined for every \( t \in [0, 1] \). Moreover, it holds
\[
\deg(H(1, \cdot, \cdot), B_R, 0) = \deg(H(0, \cdot, \cdot), B_R, 0) = 0.
\]

Proof. We claim that the solution set of problem \((P_t)\) is uniformly bounded in \( W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \) with respect to \( t \in [0, 1] \). To do so, suppose by contradiction that for every positive integer \( n \) there exist \( t_n \in [0, 1] \) and a solution \((u_{1,n}, u_{2,n})\) of \((P_{t_n})\) such that \( t_n \to t \in [0,1] \) and \(||(u_{1,n}, u_{2,n})||_{M_{p}} \to \infty \) as \( n \to \infty \). We have
\[
\left\{ \begin{array}{l}
\int_{\Omega} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla \varphi_1 \, dx = \int_{\Omega} f_{1,t_n}(x, u_{1,n}, u_{2,n}) \varphi_1 \, dx \\
\int_{\Omega} |\nabla u_{2,n}|^{p_2(x)-2}\nabla u_{2,n} \nabla \varphi_2 \, dx = \int_{\Omega} f_{2,t_n}(x, u_{1,n}, u_{2,n}) \varphi_2 \, dx,
\end{array} \right.
\]
for all \( \varphi_i \in W^{1,p_i}(\Omega) \). Without loss of generality we may admit that
\[
\theta_n := \|u_{1,n}\| \to \infty \text{ as } n \to \infty.
\]
Denote
\[
\hat{u}_{1,n} := \frac{1}{\theta_n} u_{1,n} \in W^{1,p_1}(\Omega).
\]
Then, there exists \( \hat{u}_1 \in W^{1,p_1}(\Omega) \) such that \( \hat{u}_{1,n} \to \hat{u}_1 \) weakly in \( W^{1,p_1}(\Omega) \), strongly in \( L^{p_1}(\Omega) \) and a.e. in \( \Omega \). Putting \( \varphi_1 = \hat{u}_{1,n} - \hat{u}_1 \), we have
\[
\int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2}\nabla \hat{u}_{1,n} \nabla \varphi_1 \, dx = \int_{\Omega} \frac{1}{\theta_n^{p_1(x)-1}} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla \varphi_1 \, dx
\]
Noticing that
\[
\int_{\Omega} \frac{1}{\theta_n^{p_1(x)-1}} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla \varphi_1 \chi_{\{\varphi_1 \geq 0\}} \, dx = \int_{\Omega} \frac{1}{\theta_n^{p_1(x)-1}} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla \varphi_1 \chi_{\{\varphi_1 \geq 0\}} \chi_{\{\varphi_1 > 0\}} \, dx, 
\]
by \((H'.1)(i)\), the successive application of Corollary \([2,5]\) for \( \varphi = \varphi_1 \chi_{\{\varphi_1 > 0\}} \) and \( \varphi = -\varphi_1 \chi_{\{\varphi_1 < 0\}} \) guarantee the existence of \( x_0, x_0 \in \Omega \) such that
\[
\int_{\Omega} \frac{1}{\theta_n^{p_1(x)-1}} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla \varphi_1 \chi_{\{\varphi_1 \geq 0\}} \, dx
\]
and
\[
-\int_{\Omega} \frac{1}{\theta_n^{p_1(x)-1}} |\nabla u_{1,n}|^{p_1(x)-2}\nabla u_{1,n} \nabla (-\varphi_1 \chi_{\{\varphi_1 < 0\}}) 
\]
and
Thence
\[
\int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2} \nabla \hat{u}_{1,n} \nabla \varphi_1 \, dx
\]
\[
= \left( \frac{1}{\theta_n^{p_1(x_0)-1}} + \frac{1}{\theta_n^{p_1(x_0)-1}} \right) \int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2} \nabla \hat{u}_{1,n} \nabla \varphi_1 \chi_{\{\varphi_1 \geq 0\}}
\]
\[
+ \left( \frac{1}{\theta_n^{p_1(x_0)-1}} + \frac{1}{\theta_n^{p_1(x_0)-1}} \right) \int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2} \nabla \hat{u}_{1,n} \nabla \varphi_1 \chi_{\{\varphi_1 < 0\}}
\]
\[
= \left( \frac{1}{\theta_n^{p_1(x_0)-1}} + \frac{1}{\theta_n^{p_1(x_0)-1}} \right) \int_{\Omega} J_{1,t_n}(x, u_{1,n}, u_{2,n}) \varphi_1 \, dx
\]
which, by (4.11), is equivalent to
(4.15)
\[
\int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2} \nabla \hat{u}_{1,n} \nabla (\hat{u}_{1,n} - \hat{u}_1) \, dx
\]
\[
= \left( \frac{1}{\theta_n^{p_1(x_0)-1}} + \frac{1}{\theta_n^{p_1(x_0)-1}} \right) \left[ \int_{\Omega} t_n f_1(x, u_{1,n}, u_{2,n})(\hat{u}_{1,n} - \hat{u}_1) \, dx
\]
\[
+ (1 - t_n) \int_{\Omega} \left( J_1 \left( \frac{u_{1,n}^+}{\max(1,|u_{1,n}|)} \right)^{p_1(x)-1} + \delta \lambda_{1,p_1} \phi_1^{p_1(x)-1} \right) (\hat{u}_{1,n} - \hat{u}_1) \, dx \right]
\]
Thus, bearing in mind (H'.1)(i) and (4.13), one gets
\[
\left| \int_{\Omega} |\nabla \hat{u}_{1,n}|^{p_1(x)-2} \nabla \hat{u}_{1,n} \nabla (\hat{u}_{1,n} - \hat{u}_1) \, dx \right|
\]
\[
\leq \frac{2}{\theta_n^{p_1(x_0)-1}} \left[ \int_{\Omega} t_n f_1(x, u_{1,n}, u_{2,n})(\hat{u}_{1,n} - \hat{u}_1) \, dx
\]
\[
+ (1 - t_n) \int_{\Omega} \left( J_1 \left( \frac{u_{1,n}^+}{\max(1,|u_{1,n}|)} \right)^{p_1(x)-1} + \delta \lambda_{1,p_1} \phi_1^{p_1(x)-1} \right) |\hat{u}_{1,n} - \hat{u}_1| \, dx \right]
\]
as well as
(4.16)
\[
\frac{2J_1}{\theta_n^{p_1(x_0)-1}} \int_{\Omega} \left( \frac{u_{1,n}^+}{\max(1,|u_{1,n}|)} \right)^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
\[
= \frac{2J_1}{\theta_n^{n^{-\theta_{1,n}}}} \int_{\Omega} \left( \frac{u_{1,n}^+}{\max(1,|u_{1,n}|)} \right)^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
\[
= \frac{2J_1}{\theta_n^{n^{-\theta_{1,n}}}} \int_{\Omega} (\hat{u}_{1,n}^{+})^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx \leq \int_{\Omega} (\hat{u}_{1,n}^{+})^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
and
\[
\frac{2\delta \lambda_{1,p_1}}{\theta_n^{n^{-\theta_{1,n}}}} \int_{\Omega} \phi_1^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx \leq \frac{2\delta \lambda_{1,p_1}}{\theta_n^{n^{-\theta_{1,n}}}} \int_{\Omega} \phi_1^{p_1(x)-1} |\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
\[
\leq \frac{2\delta \lambda_{1,p_1}}{\theta_n^{n^{-\theta_{1,n}}}} \max(1, \|\phi_1,p_1\|_{\infty})^{p_1(x)-1} \int_{\Omega} |\hat{u}_{1,n} - \hat{u}_1| \, dx \leq \int_{\Omega} |\hat{u}_{1,n} - \hat{u}_1| \, dx.
\]
On the other hand, assumption (H.3) yields \( \tilde{\eta} > 0 \) and \( \kappa = \kappa(\tilde{\eta}) > 0 \) fulfilling
(4.17)
\[ |s_1| > \kappa \Leftrightarrow |f_1(x, s_1, s_2)| < \tilde{\eta}|s_1|^{p_1(x)-1} \text{ for } x \in \Omega, \ s_2 \in \mathbb{R}.\]
Given \( n \in \mathbb{N} \) observe that
\[
\frac{2J_1}{\theta_n^{n^{-\theta_{1,n}}}} \int_{\Omega} f_1(x, u_{1,n}, u_{2,n})|\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
\[
= \frac{2J_1}{\theta_n^{n^{-\theta_{1,n}}}} \left[ \int_{|u_n| > \kappa} f_1(x, u_{1,n}, u_{2,n})|\hat{u}_{1,n} - \hat{u}_1| \, dx
\]
\[
+ \int_{|u_n| \leq \kappa} f_1(x, u_{1,n}, u_{2,n})|\hat{u}_{1,n} - \hat{u}_1| \, dx \right].
\]
Thus, (H.3) and (4.17) entail
\[
\int_{|u_n| > \kappa} \frac{2t_n}{\theta_n^{p_1}-1} f_1(x, u_1, u_2) |\hat{u}_1 - \hat{u}_1| \, dx
\]
(4.18)
\[
= \int_{|u_n| > \kappa} 2t_n |\hat{u}_1|^{p_1-1} f_1(x, \beta u_n, u_n) |\hat{u}_1 - \hat{u}_1| \, dx
\]
\[
\leq 2\eta \int_{|u_n| > \kappa} |\hat{u}_1|^{p_1-1} |\hat{u}_1 - \hat{u}_1| \, dx,
\]
while, by (H'.1)(ii) and (4.13), we have
\[
\int_{|u_n| \leq \kappa} \frac{2t_n}{\theta_n^{p_1}-1} f_1(x, u_1, u_2) |\hat{u}_1 - \hat{u}_1| \, dx \leq M \int_{|u_n| \leq \kappa} |\hat{u}_1 - \hat{u}_1| \, dx\]
(4.19)
\[
\leq M \int_{\Omega} |\hat{u}_1 - \hat{u}_1| \, dx,
\]
Thus, passing to the limit as \( n \to \infty \), Lebesgue dominate convergence theorem implies
\[
\lim_{n \to \infty} \langle -\Delta_{p_1(x)} \hat{u}_1, \hat{u}_1 - \hat{u}_1 \rangle = 0.
\]
Consequently, the \( S_\pm \) property of the operator \(-\Delta_{p_1(x)}\) shows that
\[
\hat{u}_1 \to \hat{u}_1 \text{ strongly in } W_0^{1,p_1(x)}(\Omega) \text{ with } ||\hat{u}_1|| = 1.
\]
Acting in (4.15) with \( \varphi_1 = \hat{u}_1 \) instead of \( \varphi_1 = \tilde{u}_1 - \hat{u}_1 \) and passing to the limit as \( n \to \infty \) one gets
\[
\int_{\Omega} |\nabla \hat{u}_1|^{p_1(x)} \, dx \leq (1 - t) J_1 \int_{\Omega} (\hat{u}_1^+)^{p_1(x)} \, dx, \quad \text{for } t \in [0, 1].
\]
Testing with \(-\tilde{u}_1^-\) in (4.15), using (H'.1)(i) and passing to the limit leads to
\[
\tilde{u}_1 = \hat{u}_1^+ \text{, which is nonzero because } ||\hat{u}_1|| = 1. \text{ Thus}
\]
\[
\int_{\Omega} |\nabla \hat{u}_1|^{p_1(x)} \, dx \leq (1 - t) J_1 \int_{\Omega} \hat{u}_1^{p_1(x)} \, dx, \quad \text{for } t \in [0, 1].
\]
If \( t = 1 \) then \( \tilde{u}_1 = 0 \) which contradicts the fact that \( \tilde{u} \neq 0 \). Assume \( t \in [0, 1) \).

By (1.2) and (4.22) it follows that
\[
(\lambda_{1,p_1} - (1 - t) J_1) \int_{\Omega} \hat{u}_1^{p_1(x)} \, dx \leq 0,
\]
which is a contradiction because \((1 - t) J_1 < \lambda_{1,p_1}\) for \( t \in [0, 1] \) (see (4.2)) and \( \hat{u}_1 > 0 \). The claim is thus proved.

As a consequence of the previous claim, the Leray-Schauder topological degree \( \deg(\mathcal{H}(t,\cdot,\cdot), \mathcal{B}_R, 0) \) is well defined for every \( t \in [0, 1] \).

The task is now to prove (4.11). Thanks to the homotopy invariance property of the Leray-Schauder topological degree, the first equality in (4.11) is fulfilled. For \( t = 0 \), \( (P_0) \) is expressed as a decoupled system:
\[
(P_0) \quad \begin{cases}
-\Delta_{p_1(x)} u_i = J_1 \left( \frac{(u_i^+)^{p_1(x)-1}}{(\max\{1,|u_i|\})^{p_1(x)-1}} \right) + \delta \lambda_{1,p_1} \phi_{1,p_1(x)}^{-1} \quad \text{in } \Omega, \\
\quad u_i = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
which, by Lemma 2.7, has no solutions. Thus, the second equality in (4.11) holds true. This completes the proof. \( \square \)
4.2. Topological degree on $\mathcal{B}_{\tilde{R}}$. We slightly modify the homotopy $\mathcal{H}$ related to problem $(P_t)$. Specifically, let us consider for every $t \in [0,1]$ the Dirichlet problem:

\[
(P_t) \quad \begin{cases}
-\Delta_{p_i(x)} u_i = \tilde{f}_{i,t}(x, u, v) & \text{in } \Omega, \\
u_i = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with

\[
(4.23) \quad \tilde{f}_{i,t}(x, u_1, u_2) = tf_i(x, u_1, u_2) + (1 - t)J_i\frac{(u_i^+)^{p_i(x)-1} - (u_i^-)^{p_i(x)-1}}{(\max\{1,\|u_i\|\})^{p_i(x)-1}},
\]

where $J_i$ satisfies (4.2).

For a constant $\tilde{R} > 0$, let define the homotopy

\[
\tilde{\mathcal{H}} : \ [0,1] \times \overline{\mathcal{B}_{\tilde{R}}} \to W^{-1,p_1'(x)}(\Omega) \times W^{-1,p_2'(x)}(\Omega)
\]

then $\tilde{\mathcal{H}}(t, u_1, u_2)$ is given by

\[
\tilde{\mathcal{H}}_i(t, u_1, u_2, \varphi_i) = \int_{\Omega} |\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla \varphi_i \, dx - \int_{\Omega} \tilde{f}_{i,t}(x, u_1, u_2) \varphi_i \, dx,
\]

for $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$, and $\overline{\mathcal{B}_{\tilde{R}}}$ is the closure of $\mathcal{B}_{\tilde{R}}$ in $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ with

\[
\mathcal{B}_{\tilde{R}} := \left\{ (u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega) : \|(u_1, u_2)\| < \tilde{R} \right\}
\]

Proposition 4.2. Assume that condition (H.1) and (H.3) are satisfied. If $\tilde{R} > 0$ is sufficiently large, then the Leray-Schauder topological degree

\[
\deg(\tilde{\mathcal{H}}(t, \cdot, \cdot), \mathcal{B}_{\tilde{R}}, 0)
\]

is well defined for every $t \in [0,1]$. Moreover, it holds

\[
(4.24) \quad \deg(\tilde{\mathcal{H}}(1, \cdot, \cdot), \mathcal{B}_{\tilde{R}}, 0) = \deg(\tilde{\mathcal{H}}(0, \cdot, \cdot), \mathcal{B}_{\tilde{R}}, 0) = 1.
\]

Proof. Arguing as in the proof of Proposition 4.1 we show that the solution set of problem $(\tilde{P}_t)$ is bounded in $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ uniformly with respect to $t \in [0,1]$. Thus, for $\tilde{R} > 0$ is sufficiently large the Leray-Schauder topological degree $\deg(\tilde{\mathcal{H}}(t, \cdot, \cdot), \mathcal{B}_{\tilde{R}}, 0)$ is well defined for every $t \in [0,1]$. Moreover, the first equality in (4.24) is true thanks to the homotopy invariance property of Leray-Schauder topological degree.

On the other hand, for $t = 0$, $(\tilde{P}_0)$ is expressed as a decoupled system:

\[
(\tilde{P}_0) \quad \begin{cases}
-\Delta_{p_1(x)} u = J_1\frac{(u^+)^{p_1(x)-1} - (u^-)^{p_1(x)-1}}{(\max\{1,\|u\|\})^{p_1(x)-1}} & \text{in } \Omega, \\
-\Delta_{p_2(x)} v = J_2\frac{(v^+)^{p_2(x)-1} - (v^-)^{p_2(x)-1}}{(\max\{1,\|v\|\})^{p_2(x)-1}} & \text{in } \Omega, \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
which, since \( J_i \in (0, \lambda_{1,i}(p^- - 1)) \), admits only the trivial solution \((u, v) = (0, 0)\). Then, from the definition of Leray-Schauder topological degree together with its homotopy invariance property, the equalities in (4.24) hold true. This completes the proof.

4.3. Topological degree on \( B_R \setminus \bar{B}_R \). Fix \( \hat{R} > 0 \) in Proposition 4.2 so large that every element \((u_1, u_2)\) in \([-\bar{u}_1, \bar{u}_1] \times [-\bar{u}_2, \bar{u}_2]\) belongs to \( B_R \). Take \( R > \hat{R} \), with \( R \) so large to fulfill the conclusion of Proposition 4.1. For this construction, it is essential to observe that \( \hat{R} > 0 \) in Proposition 4.2 and \( R > 0 \) in Proposition 4.1 must necessarily verify \( \hat{R} < R \). This is the consequence of the weak comparison principle in Lemma 2.3 applied to problems \((P_t)\) and \((\bar{P}_t)\) making use of the inequality \( f_{i,t}(x, s_1, s_2) < f_{i,t}(x, s_1, s_2) \), for a.e. \( x \in \Omega \), all \( s_1, s_2 \in \mathbb{R}, t \in [0, 1] \). Hence, the strict inclusion \( \overline{B_R} \subset B_R \) is fulfilled.

In view of the expressions of the homotopies \( \mathcal{H} \) and \( \hat{\mathcal{H}} \) used in Propositions 4.1 and 4.2 it is seen that

\[
\mathcal{H}(1, \cdot, \cdot) = \hat{\mathcal{H}}(1, \cdot, \cdot) \text{ in } \overline{B_R}.
\]

The Leray-Schauder degree \( \deg(\mathcal{H}(1, \cdot, \cdot), B_R \setminus \partial B_R, 0) \) of \( \mathcal{H}(1, \cdot, \cdot) \) on \( B_R \setminus \overline{B_R} \) makes sense according to (4.25) because it was shown in Propositions 4.1 and 4.2 that \( \mathcal{H}(1, \cdot, \cdot) \) and \( \hat{\mathcal{H}}(1, \cdot, \cdot) \) do not vanish on \( \partial B_R \) and \( \partial B_R \), respectively. Then the excision property of Leray-Schauder degree (see, e.g., [15, p. 72]) yields

\[
\deg(\mathcal{H}(1, \cdot, \cdot), B_R, 0) = \deg(\mathcal{H}(1, \cdot, \cdot), B_R \setminus \partial B_R, 0),
\]

whereas by virtue of the domain additivity property of Leray-Schauder degree it turns out that

\[
\deg(\mathcal{H}(1, \cdot, \cdot), B_R, 0) = \deg(\mathcal{H}(1, \cdot, \cdot), B_R, 0) + \deg(\mathcal{H}(1, \cdot, \cdot), B_R \setminus \overline{B_R}, 0).
\]

Combining the preceding equalities with (4.11) and (4.24), we infer that

\[
\deg(\mathcal{H}(1, \cdot, \cdot), B_R \setminus \overline{B_R}, 0) = -1.
\]

Therefore, there exists \((\bar{u}_1, \bar{u}_2) \in B_R \setminus \overline{B_R}\) satisfying \( \mathcal{H}(1, \bar{u}_1, \bar{u}_2) = 0 \). This implies that the pair \((\bar{u}_1, \bar{u}_2)\) is a solution of system \((P)\) belonging to the set \( B_R \setminus \overline{B_R} \).

4.4. Proof of Theorem 1.2. Since \((\bar{u}_1, \bar{u}_2) \in B_R \setminus \overline{B_R}\) and the ordered rectangle \([-\bar{u}_1, \bar{u}_1] \times [-\bar{u}_2, \bar{u}_2]\) is contained in the ball \( B_R \), we have that \((\bar{u}_1, \bar{u}_2) \notin [-\bar{u}_1, \bar{u}_1] \times [-\bar{u}_2, \bar{u}_2]\). In particular, we note that \((\bar{u}_1, \bar{u}_2) \neq (u_{1,\cdot}, u_{2,\cdot})\), so \((\bar{u}_1, \bar{u}_2)\) is a second nontrivial positive solution of system \((P)\). This completes the proof.

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