Layering in the Ising model

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Alexander, K.S., Dunlop, F., Miracle-Solé, S.: ...

Layering and wetting transitions for an SOS interface,
arXiv:0908.0321v1 [math-ph]

...

Layering in the Ising model
arXiv:0911.2105v1 [math-ph]

Warwick, November 2009.
\[ H_\Lambda(\sigma_\Lambda | \bar{\sigma}) = -2J|\Lambda_1| + J \sum_{<i,j> \cap \Lambda \neq \emptyset} (1 - \sigma_i \sigma_j) + K \sum_{i_3=1} (1 + \sigma_i). \]

\[ i, j \in \mathbb{Z}_+^3, \quad \Lambda \subset \mathbb{Z}_+^3, \quad \Lambda_1 = \Lambda \cap W, \quad 0 < K < J \]
\[ H_\Lambda(\sigma_\Lambda | \bar{\sigma}) = -2J|\Lambda_1| + J \sum_{\langle i,j \rangle \cap \Lambda \neq \emptyset} (1 - \sigma_i \sigma_j) + K \sum_{i=1}^{\Lambda} (1 + \sigma_i). \]

\[ i, j \in \mathbb{Z}_+^3, \quad \Lambda \subset \mathbb{Z}_+^3, \quad \Lambda_1 = \Lambda \cap W, \quad 0 < K < J \]

\[ t = e^{-4\beta J}, \quad u = 2\beta(J - K), \quad \text{bubble} = t^3, \quad \text{contact} = e^u \]
Ising: partial wetting

\[ K < \frac{1}{2} \tau^\pm \]

SOS: complete wetting

\[ u < -\ln(1 - t^2) \]

**Franöschlich - Pfister '87**

**Chalker '82**
Phase $n \sim n$ layers of $-$spins
Phase $n \simeq n$ layers of $-$spins

- Ising: formal power series in $t = e^{-4\beta J}$, indicating a range of stability of phase $n$ at low $T$, for $n = 0, 1, 2, 3, 4, 5, 6, 7$
- SOS: convergent 2-scale cluster expansion, proving a range of stability of phase $n$, for each $n \geq 0$
Boundary condition $\bar{\sigma} = n$, with $n = 0, 1, 2, \ldots$, defined by

$$\bar{\sigma}_i = -1 \text{ if } i_3 \leq n, \quad \bar{\sigma}_i = +1 \text{ if } i_3 > n,$$

Associated restricted ensemble partition function:

$$Z_n^\Lambda = \sum_{\sigma^\Lambda} e^{-\beta H^\Lambda(\sigma^\Lambda|\bar{\sigma})},$$

Surface free energy density $f_n$

$$f_n - f_{n+1} = \lim_{\Lambda \to Z_+^3} \frac{1}{|\Lambda_1|} \log \frac{Z_n^\Lambda}{Z_{n+1}^\Lambda}$$
Bubbles and interface excitations = contours = polymers = $\gamma$.

$\gamma$ boundary of a maximal connected set of points where the spin differs from its ground state value.

$$Z_n^\Lambda = \sum_{\{\gamma\}} \prod_{\gamma} \varphi(\gamma), \quad \varphi(\gamma) = t^{\frac{1}{2}|\gamma| - |\gamma \cap I_n|} e^{u|\gamma \cap W|}, \quad n \geq 1$$

$$\log(Z_n^\Lambda) = \sum_{\omega} \varphi^T(\omega), \quad \varphi^T(\omega) = \prod_{\gamma \in \omega} \left( \frac{1}{n_\gamma!} \varphi(\gamma)^{n_\gamma} \right) \sum_G (-1)^l$$
\[
\log(Z^\Lambda_n) = \sum_{\omega \in I_n, W} \varphi^T(\omega) + \sum_{\omega \in I_n, \omega \approx W} \varphi_0^T(\omega) + \sum_{\omega \in W, \omega \sim I_n} \varphi_1^T(\omega) + \sum_{\omega \approx W, \omega \sim I_n} \varphi_2^T(\omega)
\]

= Interface touching Wall + Interface not touching Wall + Bubble touching Wall + Bubble not touching Wall (clusters !)

\[
\varphi_0(\gamma) = t^{\frac{1}{2}}|\gamma| - |\gamma \cap I_n|, \quad \varphi_1(\gamma) = t^{\frac{1}{2}}|\gamma| e^u|\gamma \cap \{z = \frac{1}{2}\}|, \quad \varphi_2(\gamma) = t^{\frac{1}{2}}|\gamma|
\]
\[ \log(Z^n) = \sum_{\omega \in I_n, W} \varphi^T(\omega) + \sum_{\omega \in I_n, \omega \approx W} \varphi_0^T(\omega) + \sum_{\omega \in W, \omega \approx I_n} \varphi_1^T(\omega) + \sum_{\omega \approx W \omega \sim I_n} \varphi_2^T(\omega) \]

= Interface touching Wall + Interface not touching Wall + Bubble touching Wall + Bubble not touching Wall (clusters !)

\[ \varphi_0(\gamma) = t^{\frac{1}{2}}|\gamma| - |\gamma \cap I_n|, \quad \varphi_1(\gamma) = t^{\frac{1}{2}}|\gamma| e^{u|\gamma \cap \{z=\frac{1}{2}\}|}, \quad \varphi_2(\gamma) = t^{\frac{1}{2}}|\gamma| \]

Inclusion/Exclusion \[ \Rightarrow \]

\[ \log(Z_{n+1}^\Lambda / Z_{n+1}^\Lambda) = \sum_{\omega \in I_n, W} \varphi^T(\omega) - \sum_{\omega \in I_n, \omega \approx W_0} \varphi_0^T(\omega) - \sum_{\omega \in I_{n+1}, W} \varphi^T(\omega) \]

\[ - \left( \sum_{\omega \in W, \omega \approx I_n} \varphi_1^T(\omega) - \sum_{\omega \approx I_{n+1}} \varphi_2^T(\omega) \right) + \left( \sum_{\omega \approx W, \omega \approx I_n} \varphi_2^T(\omega) - \sum_{\omega \approx W, \omega \approx I_{n+1}} \varphi_2^T(\omega) \right) \]
\[ t^{-2n}(f_{n+1} - f_n) = 1 + \ldots \]
\[ t^{-2n}(f_{n+1} - f_n) = 1 + \ldots \]
\[-64 \left( \frac{n-1}{2} \right) + 10 \left( \frac{n-1}{2} \right) + 10 \left( \frac{n-1}{2} \right) + 4(n-2) - 16 \left( \frac{n-2}{2} \right) - 6 \left( \frac{n-1}{2} \right)\]
\[-64 \left( \frac{n-1}{2} \right) + 10 \left( \frac{n-1}{2} \right) + 10 \left( \frac{n-1}{2} \right) + 4(n-2) - 16 \left( \frac{n-2}{2} \right) - 6 \left( \frac{n-1}{2} \right) \]

\[+8(n-4) + 5(n-1) - 1 + 1 \]

\[-\left( \frac{n-1}{2} \right) - 8(n-2)\]
\[ Q_{n+1}^2 = -5t^3 P_n + 5t^4 P_n - 10t^3 Q_n^1 + 6t^4 Q_n^1 + 2t^4 Q_n^1 + 2t^4 Q_n^1 \]
\[ Q_{n+1}^2 = -5t^3 P_n + 5t^4 P_n - 10t^3 Q_n^1 + 6t^4 Q_n^1 + 2t^4 Q_n^1 + 2t^4 Q_n^1 \]

\[ + t^2 Q_n^2 - 8t^3 Q_n^{2a} - 4t^3 Q_n^{2a} - 8t^3 Q_n^{2b} - 4t^3 Q_n^{2b} + \ldots \]
Let $t = e^{-4\beta J} \ll 1$ and $u = 2\beta (J - K) = O(t^2)$. We find the following approximation to the coexistence (first order transition) lines starting from $(t = 0, u = 0)$:

0/1 : $u = -\ln(1 - t^2) + t^3 + O(t^4)$

1/2 : $u = -\ln(1 - t^2) - t^3 + 5t^4 + O(t^5)$

2/3 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 4t^5 + O(t^6)$

3/4 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{51}{2} t^6 + O(t^7)$

4/5 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2} t^6 - 51t^7 + O(t^8)$

5/6 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2} t^6 - 53t^7 + 144t^8 + O(t^9)$

6/7 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2} t^6 - 53t^7 + 142t^8$

$+ (B_9 - 2)t^9 + O(t^{10})$

7/8 : $u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2} t^6 - 53t^7 + 142t^8$

$+ B_9 t^9 + O(t^{10})$
Theorem

\[ \forall n \geq 0, \forall \epsilon > 0, \exists t_0(n, \epsilon) > 0 \text{ such that, if } 0 < t < t_0(n, \epsilon) \text{ and } \]
\[ -\ln(1 - t^2) + (2 + \epsilon)t^{n+3} < u < -\ln(1 - t^2) + (2 - \epsilon)t^{n+2}, \quad n \geq 1, \]
\[ -\ln(1 - t^2) + (2 + \epsilon)t^3 < u < \sqrt{t}, \quad n = 0, \]

then there is a unique translation invariant Gibbs state \( \mu_n \), a pure phase associated to the level \( n \), that satisfies
\[ \mu_n(\{\phi_x \neq n\}) = O(t^2) \text{ for any given } x. \]
\[ \beta^{-1} \]

(a): triple points accumulate at the roughening temperature from below

(b): triple points accumulate at the roughening temperature from above

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(d): multiphase point at the roughening temperature

(c): critical points accumulate at the roughening temperature from above
(e): triple points accumulate at zero temperature
Lemma (SOS): let \( k \geq 8 \) and define the \( k \)-restricted ensemble by:

\[
\text{diameter of contours} \leq 3k + 3
\]

Let \( t \leq (3k + 3)^{-4} \) and \( u \leq t^{1/2} \). Then the expansion of the restricted free energy is an absolutely convergent power series in \( t \). Choosing \( s = te^{2t^{1/4}} \) and \( \mu(\gamma) = \varphi_{s,0}(\gamma) \), we have

\[
|\varphi(\gamma)| \leq \mu(\gamma) \exp \left( - \sum_{\gamma' : \gamma' \sim \gamma} \mu(\gamma') \right),
\]

\[
\sum_{X : \gamma \in X} |\varphi^T_u(X)| \leq \mu(\gamma),
\]

\[
\sum_{X : \gamma \in X} X(\gamma)|\varphi^T_u(X)| \leq \varphi(\gamma)e^{\sum_{\gamma' : \gamma' \sim \gamma} \mu(\gamma')} \leq e^\mu(\gamma) - 1.
\]
Convergence of 2-scale cluster expansion requires

\[(16t)^{3k+4} < \frac{\varepsilon}{4} t^{3n+3},\]

which is satisfied as soon as \(k \geq \max(8, 2n)\) and \(t \leq \varepsilon^{1/12}/2000\).

And we still need, from the restricted ensemble expansion,

\[t \leq (3k + 3)^{-4}\]
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