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Rotating Black Holes in Three-Dimensional Hořava Gravity Revisited: New Aspects of Asymptotically Flat and De Sitter Cases

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Abstract

I consider rotating black hole solutions in three-dimensional Hořava gravity with \( z = 2 \) as a simpler set-up of the renormalizable quantum gravity \( \text{`a la} \) Hořava, Lifshitz, and DeWitt. The solutions have a curvature singularity at the origin for a non-vanishing rotation parameter \( J \), unlike the black holes in three-dimensional Einstein gravity. For asymptotically anti-de Sitter space, there are black hole event horizons as usual and the singularity is not naked, in agreement with the cosmic censorship. For asymptotically flat or de Sitter space, there are no conventional black hole horizons as in Einstein gravity, other than the usual cosmological horizon for the latter case, so that the singularity could be naked in Hořava gravity. However, I show that, due to the Lorentz-violating higher-derivative terms, the solutions have a peculiar black hole horizon at the origin so that the singularity is not naked even without the conventional black hole horizons in asymptotically flat or de Sitter case. On the other hand, I note also that a new “cosmological” horizon exists even for the flat case, contrary to the usual wisdom, due to combined effects of the higher derivatives and the angular-momentum barrier. I study their unusual black hole thermodynamics due to lack of the absolute horizons in the Lorentz-violating gravity.

Keywords: Hořava-Lifshitz-DeWitt Gravity, Rotating Black Hole, Point-like Horizon, Cosmic Censorship, Black Hole Thermodynamics

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I. INTRODUCTION

Recently [1], I have studied an exact solution for rotating black holes in three-dimensional Hořava gravity with the dynamical critical exponent \( z = 2 \) [2] as a simpler set-up of the renormalizable quantum gravity \( \text{a la} \) Horava, Lifshitz, and DeWitt (HLD) [3–5]. The solution has a curvature singularity at the origin \( r = 0 \) for a non-vanishing rotation parameter \( J \), unlike the absence of the singularity for the black hole solution in three-dimensional Anti-de Sitter (AdS) space [6] or the three-dimensional Kerr-de Sitter solution [7], and the ring singularity for the four-dimensional Kerr black hole in Einstein gravity [8]. For asymptotically AdS space, there are two black hole event horizons \( r_{\pm} \) generally as usual where the apparent and Killing horizons coincide and the singularity is not naked, in agreement with the cosmic censorship conjecture [9]. On the other hand, for asymptotically flat or de Sitter (dS) space, there are no conventional black hole horizons as in Einstein gravity, other than the usual cosmological horizon for the latter case, so that the curvature singularity could be naked in three-dimensional Hořava gravity, contrary to the cosmic censorship.

In this paper, I revisit and resolve that issue from the recent corrections in [1]. The resolution is possible essentially due to the fact that the corrected solution in [1] shows a peculiar (non-conventional) black hole event horizon at the origin \( r = 0 \) which coincides with the curvature singularity so that the singularity is not naked even without the conventional black hole horizons in three-dimensional, asymptotically flat or dS case! I also note that a new “cosmological” horizon exists even for the flat case, contrary to the usual wisdom, due to combined effects of the higher derivatives and the angular-momentum barrier. I study their unusual black hole thermodynamics due to lack of the absolute horizons in the Lorentz-violating gravity and show that the basic results are unchanged from the earlier work in [1].

II. ROTATING BLACK HOLES IN THREE-DIMENSIONAL HOŘAVA GRAVITY REVISITED

In three-dimensional spacetime, gravity becomes enormously simplified. For example, with Einstein-Hilbert action \(^1\), there is no propagating, dynamical degrees of freedom so that there is no graviton that mediates gravity interactions between massive objects. \(^2\).

However, the system is not too simple to get trivial results only. For example, there exits the black hole solution for three-dimensional asymptotically AdS space, known as BTZ solution [6]. This suggests that the three-dimensional spacetime could be a good laboratory for studying rotating black holes in Hořava gravity. Actually, in turns out to be the case and one can get the exact solution for rotating black holes in three-dimensional Hořava gravity, contrary to those in four dimensions whose exact solutions have not been found yet. In this section, I revisit the previously obtained solution but with the recent corrections in [1].

To this ends, I start by considering the ADM decomposition of the metric
\[
ds^2 = -N^2 c_t^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \tag{1}
\]

\(^1\) However, even in three dimensions, the graviton modes may exist when higher-derivative terms are present, as in topologically massive gravity (TMG) [10] or new massive gravity (NMG) [11], for example.

\(^2\) For a comoving frame on collapsing matters, the (negative) binding energy still exits for a positive Newton’s constant, \( G_3 > 0 \), as in the conventional four and higher-dimensional black holes [12].
and the three-dimensional renormalizable action with \( z = 2 \), \( \text{à la Horã-ra, Lifshitz, and DeWitt} \), which reads

\[
I = \frac{1}{\kappa} \int dt d^2 x \sqrt{g} N \left( K_{ij} K_{ij} - \lambda K^2 + \xi R + \alpha R^2 - 2\Lambda \right),
\]

up to surface terms, where \( \kappa \equiv 16\pi G_3 \),

\[
K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i)
\]

is the extrinsic curvature, \( R \) is the Ricci scalar of the Euclidean two-geometry, \( \lambda, \xi \) are the IR Lorentz-violating parameters, and \( \Lambda \) is the cosmological constant. Note that the action (2) is general enough since in two-spatial dimensions all curvature invariants can be expressed by the Ricci scalar \( R \) due to the identities, \( R_{ijkl} = (g_{ik} g_{jl} - g_{il} g_{jk}) R/2 \), \( R_{ij} = g_{ij} R/2 \). However, I do not consider the terms like \( \nabla^2 R \) since the qualitative structure of the solutions is expected to be similar, as in the four dimensions \([14]\).

Let me now consider an axially symmetric solution with the metric ansatz (I adopt the convention of \( c_\ell \equiv 1 \), hereafter)

\[
ds^2 = -N^2(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 \left( d\phi + N\phi(r) dt \right)^2.
\]

Substituting the metric ansatz into the action (2) gives the reduced Lagrangian, after angular integration,

\[
\mathcal{L} = \frac{2\pi N}{\kappa \sqrt{f}} \left[ \frac{f r^3 (N\phi')^2}{2N^2} - \xi f' + \alpha \frac{f'^2}{r} - 2\Lambda r \right],
\]

where the prime (’) denotes the derivative with respect to \( r \). Note that there is no dependance on \( \lambda \) but only on \( \xi \) in the Lagrangian, due to the peculiar property \( K \equiv g_{ij} K^{ij} = 0 \) for the ansatz \([14]\).

Varying the metric functions \( N \), \( N\phi \), and \( f \) give the equations of motions as follows, respectively:

\[
- \frac{f r^3 (N\phi')^2}{2N^2} - \xi f' + \alpha \frac{f'^2}{r} - 2\Lambda r = 0,
\]

\[
\left( \frac{\sqrt{f}}{N} r^3 N\phi' \right)' = 0,
\]

\[
\left( \frac{N}{\sqrt{f}} \right)' \left( 2\alpha \frac{f'}{r} - \xi \right) + 2\alpha \frac{N}{\sqrt{f}} \left( \frac{f''}{r} - \frac{f'}{r^2} \right) = 0.
\]

\[3\] Here, I do not consider the terms which depend on \( a_i = \partial_i N/N \) and \( \nabla_j a_i \) either since those will change the dynamical degrees of freedom and also the IR as well as UV behaviors a lot from those of the standard action \([2]\) \([15]\).

\[4\] This gives a peculiar constraint algebra in three dimensions. For the fully non-linear constraint analysis, see \([17]\).

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4 This gives a peculiar constraint algebra in three dimensions. For the fully non-linear constraint analysis, see \([17]\).
For arbitrary $\alpha$, $\Lambda$, and $\xi$, the general solution is obtained as \(^5\) \(^6\)

\[
\begin{align*}
f &= -\mathcal{M} + \frac{br^2}{2} \left[ 1 - \sqrt{a + \frac{c}{r^4}} + \sqrt{\frac{c}{ar^4}} \ln \left( \sqrt{\frac{c}{ar^4}} + \sqrt{1 + \frac{c}{ar^4}} \right) \right], \\
N \equiv W &= 1/\sqrt{1 + \frac{c}{ar^4}}, \\
N^\phi &= -\frac{\mathcal{J}}{2r^2} \sqrt{\frac{a}{c}} \ln \left[ \sqrt{\frac{c}{ar^4}} + \sqrt{1 + \frac{c}{ar^4}} \right],
\end{align*}
\]

(9)

where

\[
\begin{align*}
a &= 1 + \frac{8\alpha \Lambda}{\xi^2}, \\
b &= \frac{\xi}{2\alpha}, \\
c &= \frac{2\alpha \mathcal{J}^2}{\xi^2},
\end{align*}
\]

(10)

and I have set $W(\infty) \equiv 1$, $N^\phi(\infty) \equiv 0$ so that the solution approaches to the three-dimensional (A)dS or flat asymptotically, depending on the sign of $\Lambda$ \([6, 7, 12]\). Here, I assume $a > 0$, $c \geq 0$, or equivalently, $8\alpha \Lambda / \xi^2 > -1$ and $\alpha \geq 0$ so that the metric functions $f$, $N$, and $N^\phi$ are all real-valued.

For large $r$ and small $\alpha$, one can expand the solution (9) as \(^7\)

\[
\begin{align*}
f &= -\frac{\Lambda}{\xi} r^2 \left( 1 - \frac{2\alpha \Lambda}{\xi^2} \right) - \mathcal{M} + \frac{\mathcal{J}^2}{4r^2 \xi} \left( 1 - \frac{4\alpha \Lambda}{\xi^2} \right) - \frac{\alpha \mathcal{J}^4}{24 \xi^2 r^6} + \mathcal{O}(\alpha^2, r^{-10}), \\
W &= 1 - \frac{\alpha \mathcal{J}^2}{\xi^2} \frac{1}{r^4} + \mathcal{O}(\alpha^2, r^{-8}), \\
N^\phi &= -\frac{\mathcal{J}}{2r^2} + \frac{\alpha \mathcal{J}^3}{6 \xi^2 r^6} + \mathcal{O}(\alpha^2, r^{-10}).
\end{align*}
\]

(11)

In the Einstein gravity limit of $\alpha \to 0$, the solution reduces to

\[
\begin{align*}
N^2 &= f = -\frac{\Lambda}{\xi} r^2 - \mathcal{M} + \frac{\mathcal{J}^2}{4r^2 \xi}, \\
N^\phi &= -\frac{\mathcal{J}}{2r^2},
\end{align*}
\]

(12)

which corresponds (with $\xi = 1$) to the BTZ black hole solution for $\Lambda < 0, \mathcal{M} > 0$ \([6]\); the three-dimensional Kerr-de Sitter solution ($KdS_3$) for $\Lambda > 0, \mathcal{M} < 0$ \([7]\); the three-dimensional conical space-time outside of spinning point masses for $\Lambda = 0, \mathcal{M} < 0$ \([12]\).

The non-vanishing curvature invariants are given by

\[
R = -\frac{f'}{r} = -b \left( 1 - \sqrt{a + \frac{c}{r^4}} \right),
\]

\(^5\) In the earlier published work \([1]\), there was an error in identifying the solution for $N$ and, as the results, $N^\phi$ also. These corrections do not affect the main conclusion of the work but they are crucial for resolving the naked singularity issue for asymptotically flat or dS space.

\(^6\) The solution for $W$ can also be obtained rather easily by observing that (8) can be written as a total derivative form. I thank D. O. Devecioglu for pointing out this.

\(^7\) Note that the (two) leading terms of $W$ and $N^\phi$ are the same as in the earlier work \([1]\) so that the conserved quantities and their thermodynamics relations in \([1]\), which are defined at the asymptotic infinity, are unchanged.
FIG. 1: Plots of \( f(r) \) (green, bright solid), \( N^2(r) = W^2 f(r) \) (red, dash-dotted), \( N^\phi(r) \) (blue, dark solid) curves for asymptotically AdS space with \( \mathcal{M} > 0 \) (left) and \( \mathcal{M} \leq 0 \) (right). Here, I have plotted \( \mathcal{M} = 5, -5 \) with \( \xi = 1, \Lambda = -0.5, \mathcal{J} = 1, \alpha = 0.1 \). For \( \mathcal{M} > 0 \), there are two horizons, \( r_+, r_- \), which are solutions of \( f = 0, N^2 = 0 \), simultaneously, as usual (left). There is another new horizon at the origin \( r = 0 \), where \( N^2 = 0 \) but \( f \neq 0 \). For \( \mathcal{M} \leq 0 \), the solution does not have the usual black hole horizons and corresponds to a point particle solution (right).

\[
K^{ij}K_{ij} = -\frac{\xi}{2\alpha} \left( 1 - \sqrt{1 + \frac{8\alpha \Lambda}{\xi^2}} \right) + \frac{\mathcal{J}^2}{2\xi \sqrt{1 + \frac{8\alpha \Lambda}{\xi^2}}} \frac{1}{r^4} + O(\mathcal{J}^4 r^{-8}),
\]

and these show the curvature singularities at \( r = 0 \) for \( \mathcal{J} \neq 0 \). This is in contrast to Einstein gravity case \( (\alpha = 0, \xi = 1) \), where the (covariant) three-curvature scalar \( R^{(3)} \) becomes finite, \( R^{(3)} = R + K^{ij}K_{ij} - K^2 - f'/r - f'' = 6\Lambda \), due to exact concealment of the unphysical singularities in \( R, K^{ij}K_{ij} \), and the remainders.

**III. THE HORIZON STRUCTURE**

**A. The asymptotically AdS case**

For the asymptotically AdS space, \( \text{i.e., } \Lambda < 0 \) and \( \mathcal{M} > 0 \), the solution (9) has two black hole horizons, \( r_+, r_- \), generally where \( f \) and \( N \) vanish simultaneously so that the apparent and Killing horizons coincide (Fig. 1 (left)) and the curvature singularity at the origin is not naked, in agreement with the cosmic censorship [9].

The Hawking temperature for the black hole horizon, \( r_H = r_+, r_- \), is given by

\[
T_H = \frac{\hbar (W f')}{{\mathcal{E}}} \bigg|_{r_H} \\
= \frac{\hbar}{4\pi} b r_H \left( 1 - \sqrt{a + \frac{c}{r_H^4}} \right) \sqrt{1 + \frac{c}{ar_H^4}}
\]

(15)
from the regularity condition of the horizon in the Euclidean space-time, as usual. For a non-vanishing \( c \) or \( \mathcal{J} \), the temperature vanishes in the usual extremal black hole limit, where the inner horizon \( r_{-} \) meets the outer horizon \( r_{+} \), i.e., a degenerate horizon exists at

\[
r_{H}^{*} = \left( \frac{c}{1-a} \right)^{1/4} = \left( \frac{\mathcal{J}^2}{4\Lambda} \right)^{1/4}
\]

(Fig. 2 (left)) and the integration constant,

\[
\mathcal{M} = \frac{b y_{H}^2}{2} \left[ 1 - \sqrt{a + \frac{c}{r_h^4}} + \sqrt{\frac{c}{r_h^4}} \ln \left( \frac{\sqrt{\frac{c}{ar_h^4}} + 1}{\sqrt{\frac{c}{ar_h^4}} - 1} \right) \right]
\]

gets the minimum \( \mathcal{M}^{*} \) (Fig. 2 (right)),

\[
\mathcal{M}^{*} = \frac{b \sqrt{c}}{2} \ln \left( \frac{\sqrt{1-a} + 1}{\sqrt{a}} \right)
\]

\[
= \frac{\xi}{4\alpha} \sqrt{\frac{2\alpha J^2}{\xi^2}} \ln \left( \frac{\sqrt{\frac{8\alpha\Lambda}{\xi^2}} + 1}{\sqrt{\frac{8\alpha\Lambda}{\xi^2}} - 1} \right).
\]

In addition to these conventional black holes, \( r_{+}, r_{-} \), remarkably, one can also find that there is another peculiar horizon at the origin \( r_{---} = 0 \), if \( c \neq 0 \), i.e., \( \alpha \neq 0 \) and \( \mathcal{J} \neq 0 \), where \( N^2 = W^2 f = 0 \) but \( f \neq 0 \), which corresponds to an apparent horizon but not a Killing horizon, from the behaviors of the metric near the origin. It is interesting to note that \( r_{H} \) has no explicit \( \alpha \)-dependence and has the same expression as in Einstein gravity, in contrast to the explicit \( \alpha \)-dependence in \( \mathcal{M} \) and its minimum value \( \mathcal{M}^{*} \), as can be seen also in Fig. 2.

8 Here, the very meaning of the horizon and its Hawking temperature in Hořava gravity are not the same as those of Einstein gravity. However, we have used the same mathematical definition of Killing/apparent horizon and the Hawking temperature which should be understood as what the IR, i.e. lower energy, test particles with the Lorentz symmetry probe. Of course, this would be justified only if the Lorentz symmetry (for both gravity and matter sectors) is recovered in IR. For the gravity sector, this issue has been a long-standing debate but recently it seems to be clear that the non-perturbative effect would be important in resolving the debate. For the matter sectors, on the other hand, the issue, which is usually a naturalness problem in IR, also seems to be related to the non-perturbative effect.

9 Here, the apparent horizon is defined by the null hypersurface \( g_{\mu\nu}(\partial_{\mu}r)(\partial_{\nu}r) = N^2 = 0 \), whereas the Killing horizon by the surface where the norm of the Killing vector \( \chi = \partial_t + \Omega_H \partial_{\phi} \) vanishes, i.e., \( \chi^2 = g_{tt} - (g_{\phi\phi})^2/g_{\phi\phi} = -f^2 = 0 \) with the angular velocity of the horizon \( \Omega_H = -(g_{t\phi}/g_{\phi\phi})|_H \). In the conventional stationary black holes, these two horizons coincide. For the black holes in (three-dimensional) non-commutative space, there is a splitting of the two horizons due to the non-commutativity. But for the new horizon in our three-dimensional Hořava gravity, there exists only the apparent horizon without its paired Killing horizon as in IR.

10 Note that the limiting behavior of \( x^n \ln x \to 0^{-} \) as \( x \to 0^{+} \), or equivalently \( \ln x/x^n \to 0^{+} \) as \( x \to \infty \), for \( n > 0 \). The latter limit is usually summarized as “logarithms grow more slowly than any power or root of \( x \).
FIG. 2: Plots of $T_H$ (left) and $M$ (right) vs. $r_H$ for asymptotically AdS space. The two solid curves represent the three-dimensional rotating Hořava black holes for different Lorentz-violating higher-derivative coupling $\alpha = 0.24$, 0.1 for the dark and bright curves, respectively, in comparison with the BTZ case ($\alpha = 0$) in the dotted curve. Here, I have considered $\xi = 1, \Lambda = -0.5, J = 1$, and $\hbar \equiv 1$.

The existence of a new horizon at the origin can also be checked by the vanishing of temperature $T_H$ (Fig. 2 (left)) as

$$N^2 = \left( \frac{a}{c} \right) \left[ -M + \frac{b\sqrt{c}}{2} \left( \ln \left( \frac{2\sqrt{c}}{a} \right) - 1 \right) \right] r^4 - \frac{ab}{\sqrt{c}} r^4 \ln r + \frac{ab}{2c} r^6 + \mathcal{O}(r^8 \ln r).$$

The existence of a new horizon at the origin can also be checked by the vanishing of temperature $T_H$ (Fig. 2 (left)) as

$$T_H = -b\sqrt{a} r_H + b\sqrt{\frac{a}{c}} r_H^3 + \mathcal{O}(r_H^5)$$

near $r_H = 0$, which implying another “degenerate” horizon at the origin, though the “negative” Hawking temperature for the inner horizon $r_-$ would not make sense along the way to the origin $r_- = 0$, due to the lower bound of the mass spectrum $[22]$. Actually, as $M$ increases, the inner horizon $r_-$ approaches to the origin where the peculiar horizon $r_{-} = 0$ is located and finally $r_-$ coincides with $r_{-} = 0$ at $M \to \infty$ limit (Fig. 2 (right)), which can be seen analytically as

$$M = \frac{b\sqrt{c}}{2} \left( \ln \left( \frac{2\sqrt{c}}{a} \right) - 1 \right) + \frac{b}{2} r_H^2 - b\sqrt{c} \ln r_H + \mathcal{O}(r_H^4).$$

However, the new horizon at the origin would not have any important effect for the outsider observer since the usual black hole horizons are already formed and hides the new horizon at the origin. It could have an important effect when there is no conventional black hole horizons at finite radius, like the case of $M \leq 0$ (Fig. 1 (right)), which corresponds to a point particle solution $[22]$. However, even in this case, even though there a curvature singularity at the origin, it is not naked due to the new point-like (apparent) horizon which coincides with the location of the singularity!

**B. The asymptotically dS case**

For the asymptotically dS space, \textit{i.e.}, $\Lambda > 0$, the solution $[9]$ has no conventional black hole horizon but the usual cosmological horizon, $r_{++}$, where $N^2$ and $f$ vanish simultaneously,
FIG. 3: Plots of $f(r)$ (green, bright solid), $N^2(r) = W^2 f(r)$ (red, dash-dotted), $N^\phi(r)$ (blue, dark solid) curves for dS space with $\mathcal{M} > 0$ (left) and $\mathcal{M} \leq 0$ (right). Here, I have plotted $\mathcal{M} = 5, -5$, respectively, with $\xi = 1, \Lambda = 0.5, \mathcal{J} = 1, \alpha = 0.1$. There is no conventional black hole horizons but just the usual cosmological horizon $r_C = r_{++}$, which is the solution of $f = 0$, $N^2 = 0$, simultaneously. However, there is a peculiar (apparent) black hole horizon at the origin $r_{--} = 0$, where $N^2 = 0$ but $f \neq 0$.

regardless of the sign of $\mathcal{M}$ (Fig. 3): In the case of $\mathcal{J} \neq 0$, there is no constraint on $\mathcal{M}$ for the existence of the cosmological horizon, ranging from $-\infty$ to $+\infty$, whereas in the case of $\mathcal{J} = 0$, there is no conventional cosmological horizon for $\mathcal{M} > 0$. However, as in the asymptotically AdS case, a peculiar (apparent) horizon $r_{--} = 0$, where $N^2 = 0$ but $f \neq 0$ generally, exists but now it becomes the unique “black hole” horizon which hides the singularity at $r = 0$. The curvature singularity at the origin $r = 0$, for the non-vanishing $\alpha$ and the rotation parameter $\mathcal{J}$, is not naked by the point-like horizon which is degenerated at the same location so that the cosmic censorship is satisfied in an interesting way. This is in contrast to the non-rotating case in the Hořava gravity solution [7] or $KdS_3$ in Einstein gravity [7], where there is neither $r = 0$ singularity in (13), (14) nor black hole horizons so that the cosmic censorship conjecture is trivially satisfied.

On the other hand, the Hawking temperature for the cosmological horizon, $r_C = r_{++}$, in asymptotically dS space is given by

$$T_C = \frac{\hbar |W^2 f'|_{r_C}}{4\pi} = \frac{\hbar}{4\pi} \left| r_C \left( 1 - \sqrt{a + \frac{c}{r_C^4}} \right) \right| \sqrt{1 + \frac{c}{a r_C^4}} ,$$

following the usual convention of the positive Hawking temperature [7, 23, 24] (Fig. 4 (left)) and the integration constant $\mathcal{M}$ is given by the same formula as in (17) by replacing $r_H$ with $r_C$ (Fig. 4 (right)). The existence of the two extremal points in the Hawking temperature

\[12\] Near the horizon $r_{--} = 0$, the escape time for the radial null signals from the horizon is $t = \int_0^\infty dr / \sqrt{N^2 f} \sim \int_0^\infty dr / r^2 \ln r = Ei(\ln r)$ which becomes infinite as $r \to 0$ so that any signal from the singularity can not be observed. Here, the exponential integral $Ei(x)$ is defined as $Ei(x) = - \int_{-x}^\infty dt \ e^{-t} t^{-1}$, for real non-zero values of $x$.  


FIG. 4: Plots of $T_C$ (left) and $M$ (right) vs. $r_c$ for asymptotically dS space. The three solid curves represent the rotating Hořava-de-Sitter solutions for different Lorentz-violating higher-derivative coupling $\alpha = 0.5, 0.25, 0.1$ (bottom to top (the left part for $M$)) in comparison with the $KdS_3$ case ($\alpha = 0$) in the dotted curve. Here, I have considered $\xi = 1, \Lambda = 0.5, \mathcal{J} = 1$, and $\bar{h} \equiv 1$. The two extremal points at $r_{c1}$ and $r_{c2}$ ($r_{c1} > r_{c2}$) in the temperature imply two phase transitions between the stable horizons and the evaporating (unstable) horizons.

implies two phase transitions between the stable horizons and the evaporating horizons at

$$r_{c1} = \left\{ \frac{c}{a(a-1)} \left[ 2 - a + \Delta + \Delta^{-1} \right] \right\}^{1/2}$$

$$r_{c2} = \left\{ \frac{c}{a(a-1)} \left[ 2 - a - \frac{1}{2}(\Delta + \Delta^{-1}) - \frac{\sqrt{3}}{2}(\Delta - \Delta^{-1}) \right] \right\}^{1/4}$$

where

$$\Delta \equiv \left[ 1 - 2(a-1)^2 + a(a-1)\sqrt{2(a-2)} \right].$$

As the observers in the interior region of the rotating dS space, $r \leq r_c$, detect an isotropic background of thermal radiations with the Gibbons-Hawking temperature $T_C$ at the expense of the horizon area, $T_C$ goes down for the case of $r_c \geq r_{c1}$, but $T_C$ goes up for the case of $r_c < r_{c1}$, up to the radius $r_{c2}$ of the local maximum in $T_C$; there is a phase transition between the large stable horizons and the small evaporating (unstable) event horizons at the critical point $r_c = r_{c1}$. This is similar to the case of $KdS_3$ solution in Einstein gravity but different from the previously studied asymptotically AdS space black holes in Hořava gravity as well as the BTZ black hole in Einstein gravity, which are always stable. As the cosmological horizon $r_c$ shrinks smaller than $r_{c2}$ by the further thermal radiation, $T_C$ goes down again up to the point of vanishing cosmological horizon $r_c = 0$ so that there is another stable phase for $0 \leq r_c < r_{c2}$. The existence of a new stable phase is the genuine effect of the higher-derivative terms in Hořava gravity which persist until a critical coupling $\alpha_c = \xi^2/8\Lambda$, i.e., $a = 2$ so that $\Delta = -1$ ($\alpha = 0.25$ in Fig. 4 (left)) is being reached.

Moreover, the vanishing Hawking temperature $T_C$ in Fig. 4 (left) as

$$T_C = \left| -b\sqrt{a} r_c + b\sqrt{\frac{1}{c}} r_c^3 + \mathcal{O}(r_c^5) \right|$$

(26)
FIG. 5: Plots of $f(r)$ (green, bright solid), $N^2(r) = W^2 f$ (red, dash-dotted), $N^\phi(r)$ (blue, dark solid) curves for flat space with $\mathcal{M} > 0$ (left, center), $\mathcal{M} < 0$ (right). (The plotted parameters, except $\Lambda = 0$, are the same as in Fig. 3) For $\mathcal{M} > 0$, there is one cosmological horizon as well as a peculiar black hole horizon at the origin, as in the dS case. For $\mathcal{M} < 0$, there is no conventional cosmological/black hole horizon but just the point particle solution remains.

when the cosmological horizon $r_\mathcal{C} = r_{++}$ approaches the origin $r_\mathcal{C} = 0$ implies the merging of the two horizons $r_{++}$ and $r_{--} = 0$, similarly to asymptotically AdS case 13.

C. The asymptotically flat case

For the asymptotically flat space, i.e., $\Lambda = 0$, the solution (9) shows the most dramatic case. The peculiar property of this solution is that, regardless of the absence of the cosmological constant $\Lambda$, there is also another “cosmological” horizon for $\mathcal{M} > 0$, contrary to the usual wisdom 14, as well as the point-like horizon at $r = 0$, if $c$ does not vanish, i.e., $\alpha \neq 0$ and $J \neq 0$ (Fig. 5). This would be due to the combined effects of the (repulsive) nature of the higher derivatives and the angular-momentum barrier, as can be checked analytically in (19) also. Due to the point-like horizon at $r = 0$, the curvature singularity at the origin is not naked as in the asymptotically dS case. But, the difference from the asymptotically dS case is that the new cosmological horizon moves to the spatial infinity as $\mathcal{M}$ approaches to zero and disappears as $\mathcal{M}$ becomes negative in the flat case. This is a generic behavior of three-dimensional gravity, without much dependence on the higher curvature terms in Hořava gravity (Fig. 6). On the other hand, the case of $\mathcal{M} < 0$ corresponds to the point

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13 For an observer in asymptotically AdS space sitting at the interior region of $0 < r < r_-$, the inner horizon $r_-$ would be considered as a cosmological horizon so that the negative Hawking temperature $T_-$ for the inner horizon $r_-$ may be interpreted as the positive Hawking $T_C \equiv |T_-|$ for the “inner cosmological horizon” $r_C \equiv r_-$ (cf. 23), for the consistency with the convention of the outer cosmological horizon $r_{++}$. Then, the merging process of the inner cosmological horizon $r_-$ and the point-like horizon $r_{--} = 0$ is exactly corresponding to that of the outer cosmological horizon $r_{++}$.

14 It seems that the existence of a cosmological horizon even for the three-dimensional flat Einstein gravity ($\alpha = 0$) has not been well recognized until recently 24; actually, it could have been recognized earlier since the solution is just at the border of the BTZ solution for $\Lambda < 0, \mathcal{M} > 0$ 4 and $KdS_3$ solution for $\Lambda > 0, \mathcal{M} > 0$ 2. But in this case, there is neither the point horizon nor the curvature singularity at $r = 0$. 

FIG. 6: Numerical plots of \( r_h \equiv (r_H, r_C) \) vs. \( \Lambda \) for different masses, \( M = 5, 2, 0, -2 \) (left to right). Depending on the values of \( M \) and \( \Lambda \), there are two branches of solutions. For the first branch (left two curves), the solutions have two (black hole) horizons (dotted lines for outer horizons and solid lines for inner horizon) for asymptotically AdS (\( \Lambda < 0 \)) and their inner horizons (only) remain for asymptotically flat or dS (\( \Lambda \geq 0 \)). In the latter case, the outer horizons in AdS case moves to infinity and the inner horizons become the cosmological horizons in flat or dS case. For the second branch (right two curves), there is no corresponding solution in AdS or flat but exists only in dS case. Here, I have considered \( \xi = 1, J = 5, \alpha = 0.1 \).

FIG. 7: Plots of \( T_C \) (left) and \( \mathcal{M} \) (right) vs. \( r_C \) for asymptotically flat space. (The plotted parameters are the same as in Fig. 4). There is only one extremal point in the temperature which implies one phase transition between the (large) unstable cosmological horizon and the (small) stable horizon.

The integration parameter \( \mathcal{M} \) and the Hawking temperature \( T_C \) can be obtained by setting \( a = 1 \), i.e., \( \Lambda = 0 \) in (17) and (22), respectively (Fig. 7). In contrast to the particle solution in Hořava gravity \(^\text{15}\).

\(^{15}\) In the Einstein gravity limit \( \alpha \to 0 \) \(^\text{12, 27}\), the mass and angular momentum are given by \( m = \frac{1 - \sqrt{-M^4}}{4G}, j = \frac{J}{8G\sqrt{-M}} \), respectively. It would be an interesting problem to identify those for \( \alpha \neq 0 \) also (cf \(^\text{28}\)).
dS case, there is only one extremal point of the temperature with the maximum $T_{\text{max}} = \sqrt{2(2\sqrt{3} - 3)^{3/4}b^{1/4}/8\pi}$ and this persists for any coupling as far as the cosmological horizon exists, i.e., $\mathcal{M} > 0$. So, there is only one phase transition between the large evaporating and the small stable (cosmological) horizons at $r = r_{c2} = [(2/\sqrt{3} - 1)c]^{1/4}/2$ with $\Delta = 1$.

**IV. THE MASS, ANGULAR MOMENTUM, AND THEIR UNUSUAL THERMODYNAMICAL LAW**

There are some subtleties in defining the conserved quantities when there is the cosmological horizon, partially due to the absence of an asymptotic observer. However, due to some recent developments on the entanglements [29], the extension of the usual methodology to the hypothetical observer in the causally-disconnected regions could be more than just a mathematical apparatus [1, 24, 30]. Here, I consider and extend the canonical methodology for computing the conserved mass and angular momentum of the rotating solution (9) for the black hole as well as the cosmology solution, based on the boundary variation of the action. To this ends, I start by considering the variation of the total action $I_{\text{total}} = I + B$ with boundary terms $B$ at the infinity $r = \infty$, which may be space-like or time-like, such that the boundary variation $(\delta I)(\infty)$ is canceled by $\delta B$ and there remain only the bulk terms in $\delta I_{\text{total}}$, which vanish when the equations of motions hold. Then, for the class of fields that approach our solution (9), or more generally the asymptotic form (11) al least, at $r = \infty$, one finds

$$B = \pm(t_2 - t_1)\left[-W(\infty)M + N^\phi(\infty)J\right],$$

which defines the canonical mass and angular momentum

$$M = \pm \frac{2\pi\xi}{\kappa}a^{1/2}\mathcal{M}, \ J = \pm \frac{2\pi\xi}{\kappa}J,$$

as the conjugates to the asymptotic displacements $N^\phi(\infty)$ and $N\phi(\infty)$, respectively, when kept as independent parameters. Here, the upper (+) and lower (−) signs correspond to the asymptotically AdS and dS/flat cases, respectively, which may be traced back their origins to the opposite role of space and time coordinates on the space-like (AdS case) and the time-like (dS/flat case) boundaries [31]. With the above choice of signs, one can now get the same mass and angular momentum as in the literatures for the dS and flat cases [24, 30], as well as the AdS case [1].

Even though the conserved mass and angular momentum can be obtained, their thermodynamical law for Lorentz-violating black holes has not been well established yet. Actually, previously for the AdS case, it has been shown that the conventional first law of black hole thermodynamics with the usual Hawking temperature and chemical potential does not work [1], and this is fundamentally unchanged even for the dS and flat cases also. To see this, let me consider the variation of the mass $M$ as a function of $J$ and $r_h \equiv (r_H, r_C)$, which can be written as

$$dM = Adr_h + \Omega_h dJ$$

with the chemical potential $\Omega_h \equiv -N^\phi|_h$ and

$$A \equiv \pm \frac{\pi\xi^2}{\kappa\alpha}r_h\sqrt{a}\left(1 - \sqrt{a + \frac{c}{r_h^4}}\right)$$
Then, in order to see whether the black hole entropy can be defined through the first law of thermodynamics in the conventional form

\[ dM = T_H dS + \Omega_h dJ \]  

with the usual Hawking temperature \( T_H \) of (15) for the AdS case, or \( T_C \) of (22) for the dS/flat case, let me define the black hole entropy function \( S \equiv S(r_h, J) \), as a function of \( r_h \) and \( J \), with

\[ dS \equiv \partial_{r_h} S \, dr_h + \partial_J S \, dJ. \]  

Then, from (29), (30), and (31), one can find

\[ \partial_{r_h} S = \frac{A}{T_h}, \quad \partial_J S = 0 \]  

but

\[ \partial_J \partial_{r_h} S - \partial_{r_h} \partial_J S = \pm \frac{8\pi\alpha J}{\sqrt{ar_h^4/1 + c/ar_h^4}}, \]  

which shows that the (horizon) entropy is not integrable with the non-relativistic higher-curvature corrections (\( \alpha \neq 0 \)) for rotating black holes. This proves that the entropy cannot be defined in the conventional form of the first law of thermodynamics with the usual Hawking temperature and chemical potential, for the generic black holes or cosmology solutions with a rotation.

V. CONCLUDING REMARKS

In conclusion, I have studied the (corrected) rotating black hole solutions in the three-dimensional Hořava gravity for asymptotically flat or (A)dS case [1], and found that the curvature singularity at the origin is not naked due to a peculiar point-like horizon at the origin, even without the conventional black hole horizons. This resolves the naked singularity problems in the earlier work so that there exists the Hořava gravity generalization of the known solutions for the \( KdS_3 \) and flat solutions in Einstein gravity. I have also found that a new “cosmological” horizon exists even for the flat case, contrary to the usual wisdom, due to combined effect of the higher derivatives and the angular momentum barrier. I have studied their unusual black hole thermodynamics due to lack of the absolute horizons in the Lorentz-violating gravity [16] and shown that the basic results are unchanged from the earlier work.

The existence of the point-like horizon at the origin resembles the case of black “plane” solutions in four-dimensional Hořava gravity, though the details of horizon structures are different [33]. The point-like horizon in three-dimensional Hořava gravity might be due the

16 Recently, it was shown that, for a low energy limit, there still exists a region bounded by an universal horizon where even an infinite-speed particle can not escape from [32]. It would be interesting to investigate the universal horizon and its thermodynamical law for the full (three-dimensional) Hořava gravity.
lower-derivative equations of motion for $N$ in (8). It would be interesting to see whether the introduction of higher-derivative terms, like the $\nabla^2 R$ term, which has not been considered in this paper, could make the point horizon at $r = 0$ to be expanded so that the more sizable black hole may be obtained [34, 35].

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