On Bosonic Wightman Quantum Field Theories

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Abstract

We prove that field operators in a Wightman quantum field theory generally have self-adjoint extensions. If the theory is bosonic and the field operators also obey canonical commutation relations (CCRs), then the Weyl form of the CCRs exits. This entails that the field operators emerge from the corresponding CCR algebra, which is a unique C*-algebra and which is determined by the two-point Wightman function.

Keywords: Wightman quantum field theory, canonical commutation relations.

1 Introduction

The canonical approach to quantum field theory (QFT) combined with path integral methods has produced very successful physical theories [1]. At the outset, one postulates canonical commutation relations (CCRs) for Boson fields and canonical anti-commutation relations (CARs) for Fermion fields. Surprisingly, neither CCRs nor CARs are included in the Gårding-Wightman axioms, which are the basis of Wightman QFTs and which mark a mathematically rigorous approach to relativistic QFTs [2,3]. If one wishes to recast QFTs, which are based on the canonical approach, as Wightman QFTs, then I would expect CCRs or CARs to be present in the latter theories as well. This is at least the case for Wightman QFTs of free fields and for Wightman QFTs of interacting scalar fields in two and three space-time dimensions.
We discuss bosonic Wightman QFTs in this paper. Starting point is the general setting used in the proof of the Wightman reconstruction theorem. We extend the setting in Sec. 2 so that we can proof the existence of certain operators in Sec. 3. In particular, we prove that field operators in Wightman QFTs generally have self-adjoint extensions. In Sec. 4, we assume that the field operators obey CCRs as they are present in free field theories. We prove that this entails the Weyl form of the CCRs, and that the field operators therefore emerge from the corresponding CCR algebra. We finally discuss the consequences of our results for the existence of bosonic Wightman QFTs.

2 Preliminaries

Let us assume a Wightman QFT with test-function space, $S$, which is a space of smooth vector-valued functions. In particular, for fixed $x \in \mathbb{R}^4$, a test function, $f \in S$, is a vector, $(f_a(x))$, in some finite-dimensional vector space, $V$, and the coordinate functions, $f_a(x)$, typically belong to Schwarz space. The latter usually is a convenient choice, but other types of smooth coordinate functions are possible [5].

Field operators, $\Phi(f)$ ($f \in S$), are defined on a dense domain, $D \subset \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the Wightman QFT, and $S$ possesses a conjugation, which is compatible with the operator adjoint:

$$\langle \Phi(f)u, v \rangle = \langle u, \Phi(f)^*v \rangle = \langle u, \Phi(f^*)v \rangle \quad (u, v \in D).$$

The set of hermitian test-functions is $S_r = \{ f \in S : f^* = f \}$, and the corresponding field operators, $\{ \Phi(f) : f \in S_r \}$ are symmetric. We note that we suppress indices of the test functions for the sake of notational convenience. Explicit examples of test-function spaces, their index structure, and the index structure of Wightman functions are given in appendix A. Without loss of generality, we further assume that the one-point Wightman functions vanish, i.e. we assume zero vacuum expectation values of the field operators [3].

For the general discussion of Wightman QFTs, let us consider the setting, which is used in the proof of the Wightman reconstruction theorem. The Hilbert space of the QFT is constructed from the tensor algebra of $S$,

$$T(S) = \bigoplus_{n=0}^{\infty} S^\otimes n \quad (S^\otimes 0 = \mathbb{C}).$$
As a set, the tensor algebra is a subset of the direct product,

\[ T(S) \subset Z(S), \quad Z(S) = \prod_{n=0}^{\infty} S^\otimes n = \{(u_n) : u_n \in S^\otimes n (n \geq 0)\}, \]

and \((u_n) \in T(S)\) if only finitely-many \(u_n \neq 0\). We note that both, \(T(S)\) and \(Z(S)\), are algebras with respect to the multiplication

\[(u_n) \otimes (v_n) = (w_n), \quad w_n = \sum_{k=0}^{n} u_k \otimes v_{n-k},\]

where \((u_n), (v_n)\) are both in \(T(S)\) or both in \(Z(S)\), respectively.

The action of the field operator, \(\phi(h)\) \((h \in S)\), which eventually leads to the operator \(\Phi(h)\) on \(D\), is defined on \(T(S)\) as

\[\phi(h)(u_n) = (0, hu_0, h \otimes u_1, h \otimes u_2, ... ) = h \otimes (u_n),\]

For the sake of simplicity we use the notation \(h^\otimes = \phi(h)\) in the following.

However, the action of \(h^\otimes\) extends to every \(u \in Z(S)\). We note that the advantage of considering \(Z(S)\) is that while on \(T(S)\), we can define polynomials of field operators, we can even define functions of field operators on \(Z(S)\) with the help of expansions as series. This is the main idea of this paper. For example, let

\[e^{h^\otimes} = \left(\sum_{n=0}^{\infty} \frac{h^\otimes^n}{n!}\right)^\otimes \quad (h^\otimes^0 = 1),\]

then

\[e^{h^\otimes}(u_n) = \left(u_0, hu_0 + u_1, \frac{1}{2} h \otimes hu_0 + h \otimes u_1 + u_2, ...ight) \quad ((u_n) \in Z(S))\]

\[= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} h^\otimes^k \otimes u_{n-k}\right).\]

In general, for a formal power series,

\[F(x) = \sum_{k} a_k x^k,\]

we define

\[F(h^\otimes)(u_n) = \sum_{k} (a_k h^\otimes^k) \otimes (u_n) \quad ((u_n) \in Z(S))\]

\[= (w_n),\]

\[w_n = \sum_{k=0}^{n} a_k h^\otimes^k \otimes u_{n-k}.\]
The definition can further be extended to elements \((u_n) \in Z(S)\), for which \(u_0 = 0\):

\[
F((u_n) \otimes (v_n)) = \sum_k (a_k (u_n) \otimes (v_n)) \quad ((v_n) \in Z(S)),
\]

\[
(u_n)^{\otimes k} = (u_n) \otimes ... \otimes (u_n) \quad ((u_n)^{\otimes 0} = 1),
\]

and the definition makes also sense for general \((u_n) \in Z(S)\) if \(F\) is only a polynomial.

**Proposition 1:** Let \(f, g \in S\), then

\[
e^{f\otimes} e^{g\otimes} = e^{w\otimes} \quad (w = (w_n) \in Z(S)),
\]

and \(w\) is given by the Baker-Campbell-Hausdorff formula.

**Proof:**

\[
w = \log \left( e^{f\otimes} e^{g\otimes} \right)
\]

is algebraically given by the Baker-Campbell-Hausdorff formula [6], which shows that each \(w_n\) is the sum of finitely many monomials of order \(n\) and that \(w_0 = 0\). Therefore, \((w_n), e^{w\otimes} \in Z(S)\).

We introduce further a conjugation on \(Z(S)\). Let

\[
u_n^*(x_1, ..., x_n) = u_n(x_n, ..., x_1)^* \quad (u_n \in S^{\otimes n}),
\]

\[
(iu_n)^* = -iu_n^*,
\]

\[
(u_n)^* = (u_n^*)^* \quad ((u_n) \in Z(S)).
\]

We obtain, for example,

\[
((u_n) \otimes (v_n))^* = (v_n)^* \otimes (u_n)^*
\]

\[
(h^{\otimes}(u_n))^* = (u_n)^* \otimes (0, h^*, 0, 0, ...)
\]

\[
e^{ih^{\otimes}}(u_n)^* = (u_n)^* e^{-ih^{\otimes}}.
\]

### 3 Self-adjointness of field operators

With the help of the conjugation on \(Z(S)\) we can define the sesquilinear map

\[(u_n)^* \otimes (v_n) \quad ((u_n), (v_n) \in Z(S)),\]

so that for \(h \in S_r\),

\[
((e^{ih\otimes}) \otimes (u_n))^* \otimes ((e^{ih\otimes}) \otimes (v_n)) = (u_n)^* \otimes (e^{-ih^{\otimes}}) \otimes (e^{ih\otimes}) \otimes (v_n)
\]

\[
= (u_n)^* \otimes (v_n)
\]
holds.

Combining that map with the Wightman functions on \( T(S) \) yields the scalar product used in the proof of the Wightman reconstruction theorem:

\[
\langle (u_n), (v_n) \rangle_W = W((u_n)^* \otimes (v_n)) \quad ((u_n), (v_n) \in T(S)),
\]

\[
W((u_n)^* \otimes (v_n)) = \sum_{j,k} W_{j+k}(u_j^* \otimes v_k).
\]

Note that in a scalar Wightman QFT,

\[
\sum_{j,k} W_{j+k}(u_j^* \otimes v_k) = \sum_{j,k} \int dx_1 ... dx_j dy_1 ... dy_k u_j(x_1, ..., x_j)^* W_{j+k}(x_j, ..., x_1, y_1, ..., y_k)v(y_1, ..., y_k),
\]

and that we give examples of the form of the Wightman functions for more complex theories in appendix A.

However, let us extend the definition of that scalar product to be able to utilise the calculus developed above. Let \( D_{1,a} \) be the set of \((u_n) \in Z(S)\), for which the sum

\[
\|u\|_W^2 = W((u_n)^* \otimes (u_n)) = \sum_{j,k} W_{j+k}(u_j^* \otimes u_k) \quad (u = (u_n))
\]

converges absolutely, and let \( D_1 \) be the linear hull of \( D_{1,a} \). We note that for \( h \in S_r \), the operator \( e^{ih} \otimes \) leaves both \( D_{1,a} \) and \( D_1 \) invariant.

**Lemma 2:** \( \langle \cdot, \cdot \rangle_W \) is a scalar product on \( D_1 \).

**Proof:** For \( u = (u_n) \in Z(S) \) let \( u^{(n)} = (u_0, u_1, ..., u_n, 0, 0, ...) \), i.e. \( u_k^{(n)} = u_k \) for \( k \leq n \) and \( u_k^{(n)} = 0 \) for \( k > n \). If \( u \in D_{1,a} \), then \( \{u^{(n)}\} \) is a Cauchy sequence with respect to \( \| \cdot \|_W \), since

\[
\|u^{(n)} - u^{(m)}\|_W \leq \sum_{j,k=m}^n |W_{j+k}(u_j^* \otimes u_k)| \quad (m \leq n).
\]

Therefore, \( \{\langle u^{(n)}, v^{(n)} \rangle_W \} \) is a Cauchy sequence for \((u_n), (v_n) \in D_{1,a}\), and

\[
\langle u, v \rangle_W = \lim_n \langle u^{(n)}, v^{(n)} \rangle_W \quad (u = (u_n), v = (v_n)).
\]

This induces the scalar product on \( D_1 \): Let

\[
u = \sum_{j=1}^m u_{[j]}, \quad v = \sum_{k=1}^n v_{[k]} \quad (u_{[j]}, v_{[k]} \in D_{1,a})
\]
then
\[ \langle u, v \rangle_W = \sum_{j,k} \langle u[j], v[k] \rangle_W = \lim_{n} \sum_{j,k} \langle u^{(n)}[j], v^{(n)}[k] \rangle_W. \]

Let \( D_0 = \{ u \in D_1 : \|u\|_W = 0 \} \), then \( D_1/D_0 \) is a pre-Hilbert space. Since for each \( u \in D_1 \), \( \lim_n \|u - u^{(n)}\|_W = 0 \), \( T(S)/D_0 \) is dense in \( D_1/D_0 \), so that the completion of \( D_1/D_0 \) yields the Hilbert space of the Wightman QFT, \( \mathcal{H} \).

Let \( q : D_1 \to D_1/D_0 \) denote the quotient map. Field operators are compatible with \( q \), since
\[ (h^\otimes u)^* \otimes (h^\otimes u) = u^* \otimes (h^\otimes h^\otimes u) \quad (u \in T(S)), \]
so that
\[ \|h^\otimes u\|_W^2 \leq \|h^\otimes h^\otimes u\|_W \|u\|_W = 0 \]
for \( u \in T(S) \cap D_0 \). The analogous statement holds for operators \( e^{ith^\otimes} \) with \( h \in S_r \). Therefore, there exist unique operators, so that
\[ (q \circ h^\otimes) u = (\Phi(h) \circ q) u \quad (u \in T(S)), \]
\[ q \circ e^{ith^\otimes} = U_h \circ q \quad (h = h^*). \]

In particular, \( U_h \) is a unitary operator.

**Theorem 3:** For \( h \in S_r \), \( \{U_{th}\}_{t \in \mathbb{R}} \) is a strongly-continuous one-parameter group of unitary operators, whose generator, \( \Psi(h) \), is a self-adjoint extension of \( \Phi(h) \).

**Proof:** Since
\[ ((e^{ith^\otimes}) \otimes u)_n = \sum_{k=0}^{n} \frac{(it)^k}{k!} h^\otimes h^\otimes u_{n-k} \quad (u \in D_1) \]
we conclude that, for \( u, v \in T(S) \),
\[ \langle v, ((e^{ith^\otimes}) \otimes u)^{(n)} \rangle_W \]
is a continuous function with respect to \( t \in \mathbb{R} \). Hence,
\[ \lim_n \langle v, ((e^{ith^\otimes}) \otimes u)^{(n)} \rangle_W = \langle v, ((e^{ith^\otimes}) \otimes u) \rangle_W \]
is measurable. Moreover, \( T(S)/D_0 \) is dense in \( \mathcal{H} \), so that for any pair \( u, v \in \mathcal{H} \) there exists Cauchy sequences, \( (u_{[n]}), (v_{[n]}) \) in \( T(S)/D_0 \), which converge to \( u, v \), respectively. The function
\[ \langle v, U(th)u \rangle = \lim_n \langle v_{[n]} U(th)u_{[n]} \rangle \]
is also measurable with respect to $t$, since it is the point-wise limit of measurable functions. $H$ is further separable, so that we can apply theorem VIII.9 in Ref. [7], i.e., $\{U_{th}\}$ is a strongly continuous one-parameter group of unitary operators. Moreover, for $u \in T(S)$,

$$\frac{d}{dt}(e^{i\theta u}) \otimes u_{|t=0} = i\theta \otimes u,$$

and

$$q \circ \frac{d}{dt}e^{i\theta u} = \frac{d}{dt}(q \circ e^{i\theta u}) = \frac{d}{dt}U_{th} \circ q = i\Psi(h) \circ q.$$

4 Weyl form of the CCRs

In bosonic Wightman QFTs, which are formulated along the formalism of canonical quantisation, field operators obey canonical commutation relations (CCRs),

$$[\Phi(f), \Phi(g)] = i\sigma(f, g),$$

where $f, g \in S_r$, and where $\sigma$ is a symplectic bilinear form. Taking the vacuum expectation value, we see that $\sigma$ is related to the two-point Wightman function,

$$\sigma(f, g) = 2\text{Im} W_2(f, g) \quad (f, g \in S_r).$$

Let us consider a corresponding operator on $Z(S)$:

$$C(f, g) = [f \otimes, g \otimes] - i\sigma(f, g) \quad (f, g \in S_r).$$

We note that $C(f, g)^* = C(g, f)$. Since

$$(q \circ C(f, g)) u = ([\Phi(f), \Phi(g)] - i\sigma(f, g))q(u) = 0$$

for $u \in T(S)$, we obtain

$$W(u^* \otimes (C(f, g)v)) = W((C(g, f)u)^* \otimes v) = 0 \quad (u \in T(S), v \in D_1),$$

i.e. $C(f, g)(D_1) \subset D_0$ and $(q \circ C(f, g)) v = 0$ for all $v \in D_1$. Analogously, we obtain

$$q \circ P(f_1 \otimes, ..., f_n \otimes) C(f, g) = 0 \quad (f_1, ..., f_n \in S),$$

where $P$ is a polynomial of $n$ variables. This entails the Weyl form of the CCRs.

**Theorem 4**: Assume a bosonic Wightman QFT, in which the field operators satisfy CCRs, then the Weyl form of the CCRs holds,

$$U_f U_g = e^{i\sigma(f,g)/2} U_{f+g} \quad (f, g \in S_r).$$
Proof: Let $f, g \in S_r$. As stated in proposition 1,
$$e^{if} e^{ig} = e^i w \quad (w = (w_n) \in Z(S)),$$
and $w$ is given by the Baker-Campbell-Hausdorff formula. In particular, $e^{iw}$
maps $D_{1,a}^1$ onto $D_{1,a}^1$. Let $u \in T(S)$. There exists a $k > 0$ so that $u_n = 0$ for
$n > k$. Let further $m \in \mathbb{N}$ and assume $n > 2m + k$. Using the arguments
presented before this theorem, we conclude that terms in
$$q (w^{\otimes m} u)^{(n)}$$
vanish, if they involve more than one Lie bracket. Therefore,
$$q (w^{\otimes m} u)^{(n)} = q \left( \left( f + g + \frac{\sigma(f, g)}{2} \right)^{\otimes m} u \right)^{(n)}.$$
The right-hand side is a Cauchy sequence in $\mathcal{H}$, so that
$$q (w^{\otimes m} u) = q \left( \left( f + g + \frac{\sigma(f, g)}{2} \right)^{\otimes m} u \right) = \left( \Psi(f + g) + \frac{\sigma(f, g)}{2} \right)^n q(u).$$
Let $v \in \mathcal{H}$ be a vector, for which
$$\sum_{n=0}^{\infty} \frac{\|\Psi(f + g)^n v\|}{n!} < \infty$$
holds. Such vectors are dense in $\mathcal{H}$, which can be inferred from the spectral
theorem. However,
$$\sum_{m=1}^{n} \frac{\langle v, q (i^{m} w^{\otimes m} u) \rangle}{m!} = \sum_{m=1}^{n} \frac{\langle v, (i^m \Psi(f + g) + i \sigma(f, g) / 2)^m q(u) \rangle}{m!}$$
is a Cauchy sequence, and we therefore obtain
$$\langle v, q (e^{iw} u) \rangle = e^{i \sigma(f, g)/2} \langle v, U_{f+g} q(u) \rangle.$$ This yields the Weyl-form of the CCRs. 

The $C^*$-algebra generated by the unitary operators $U_f \ (f \in S_r)$ is called CCR
algebra $\mathbb{C}$. We show now that it is uniquely determined by the real linear
space $S_r/N_r(\Phi)$ ($N_r(\Phi) = \{ f \in S_r : \Phi(f) = 0 \}$), and by the symplectic
bilinear form, $\sigma$. Let
$$N_\sigma = \{ f \in S_r : (\forall g \in S_r) \sigma(g, f) = 0 \}.$$
We note that \( N_r(\Phi) \subset N_\sigma \).

**Corollary 5:** \( \sigma \) is non-degenerate on \( S_r/N_r(\Phi) \).

**Proof:** Let \( f \in S_r \), and assume that \( \sigma(f, g) = 0 \) for all \( g \in S_r \). Due to theorem 4, \( U_f \) commutes with all \( U_g \ (g \in S_r) \). Let \( u, v \in q(T(S)) \), then

\[
\langle u, U_f U_g v \rangle = \langle u, U_g U_f v \rangle \quad (t \in \mathbb{R}),
\]

and differentiating both sides at \( t = 0 \) with respect to \( t \) yields

\[
\langle u, U_f \Phi(g)v \rangle = \langle u, \Phi(g)U_f v \rangle.
\]

Since the vacuum state of the Wightman QFT is cyclic and since the set of field operators therefore is irreducible \([2]\), we obtain \( U_f = c_f \in \mathbb{C} \) and \( \Psi(f) = a_f \in \mathbb{R} \). Moreover, the vacuum expectation value vanishes, \( W_1(f) = 0 \), so that \( \Phi(f) = a_f = 0 \). Therefore \( N_r(\Phi) = N_\sigma \). \( \blacksquare \)

We now can apply theorem 5.2.8. in Ref. \([8]\), which states the uniqueness of the CCR algebra. The standard representation of the CCR algebra can be obtained if we use the two-point Wightman function as a scalar product on \( S/N(\Phi) \),

\[
\langle f, g \rangle = 2W_2(f, g) \quad (f, g \in S).
\]

\( S/N(\Phi) \) becomes a pre-Hilbert space, whose completion is \( \mathcal{H}_1 \), and the standard Fock-space construction (Segal quantization) yields the representation of the CCR algebra \([5]\). We note that the Fock-space vacuum vector leads to a Wightman QFT, which is a generalised free field theory. An example of the latter construction is given in Ref. \([5]\).

However, examples for this \( C^* \)-algebraic situation are scalar Wightman QFTs of self-interacting fields in two and three space-time dimensions. One starts with a free-field theory and introduces self-interaction, which enforces a different representation of the CCR algebra \([2, 4, 5]\). As shown in this paper, this is not a coincident. A bosonic Wightman QFT is generally algebraically determined by the CCR algebra related to the two-point Wightman function. However, finding the correct representation, i.e. the correct state that is defined by the Wightman functions, remains the hard problem.

### A Example Test-Function Spaces

Probably the most well-known test-function space is Schwartz space, \( S = S(\mathbb{R}^4) \), which is commonly used for scalar fields. In particular, for the field
operators in scalar Wightman QFTs, complex conjugation on $S(\mathbb{R}^4)$ yields,
\[
\Phi(f^*)u = \Phi(f)^*u \quad (u \in D).
\]
For general vector fields, we would use Schwartz-space functions with a finite-dimensional vector space, $V$, as target space, $S = S(\mathbb{R}^4, V)$. For fixed $x \in \mathbb{R}^4$, $f(x)$ is a vector in $V$, i.e. $f(x) = (f_j(x)) \ (1 \leq j \leq \dim(V))$, and each coordinate function, $f_j$, is a scalar Schwartz function. We have only written $\Phi(f)$ in the preceding sections, suppressing coordinate indices for field operators for the sake of notational convenience. A more explicit representation can be retrieved as follows:
\[
(f^{(j)})_k = \delta_{j,k}f_k \quad (f \in S), \quad \text{i.e.} \quad f = \sum_j f^{(j)},
\]
\[
\Phi^{(j)}(f) = \Phi(f^{(j)}), \quad \Phi = \sum_j \Phi^{(j)}.
\]
The simplest example of a vector field has four components, i.e. $V$ is four-dimensional. Complex conjugation on $S(\mathbb{R}^4, V)$ yields
\[
\Phi(f^*)u = \Phi(f^*^*)u \quad (u \in D),
\]
\[
\Phi^{(j)}(f^*)u = \Phi(f^{(j)^*})u = \Phi(f^{(j)*})u = \Phi^{(j)}(f^*)u.
\]
More complicated vector spaces are however possible. For example, in the case of non-commutative gauge fields, $V$ could consist of matrix-valued vectors,
\[
f(x) = (f_{\mu,(l,k)}(x)) \quad (1 \leq \mu \leq 4, \ 1 \leq k, l \leq m),
\]
where the coordinate functions, $f_{\mu,(k,l)}(x)$ are scalar Schwartz functions and $\dim(V) = 4m^2$. Conjugation is given by
\[
f^*(x) = (f_{\mu,(k,l)}(x)^*),
\]
i.e. for fixed $\mu, x$, $f_{\mu}^*(x)$ is the matrix adjoint of $f_{\mu}(x)$.

The indices of Wightman functions have a structure corresponding to the structure of the indices of the test functions. For example, in the case of vector fields, the $n$-point function is a sum of functions:
\[
W_n(f^{[1]}, \ldots, f^{[n]}) = \sum_{j_1 \ldots j_n} W^{(n)}_{j_1 \ldots j_n}(f^{[1]}_{j_1}, \ldots, f^{[n]}_{j_n}) \quad (f^{[1]}, \ldots, f^{[n]} \in S).
\]
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