ON THE RELATION BETWEEN QUANTUM LIOUVILLE THEORY AND
THE QUANTIZED TEICHMÜLLER SPACES

by

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Abstract. — We review both the construction of conformal blocks in quantum Liouville theory and
the quantization of Teichmüller spaces as developed by Kashaev, Chekhov and Fock. In both cases
one assigns to a Riemann surface a Hilbert space acted on by a representation of the mapping class
group. According to a conjecture of H. Verlinde, the two are equivalent. We describe some key steps
in the verification of this conjecture.

Dedicated to A.A. Belavin on his 60th birthday

1. Introduction

Quantum Liouville theory is a crucial ingredient for a variety of models for low dimensional
quantum gravity and non-critical string theories. In the case of two dimensional quantum
gravity or non-critical string theories this is a consequence of the Weyl-anomaly [1], which
forces one to take into account the quantum dynamics of the conformal factor of the two-
dimensional metric. More recently it was proposed that Liouville theory also plays a crucial
role for three-dimensional quantum gravity in the presence of a cosmological constant in the
sense of representing a holographic dual for this theory, see e.g. [2,3] and references therein.

For all these applications it is crucial that the quantum Liouville theory has a geometric
interpretation as describing the quantization of spaces of two-dimensional metrics. Such an
interpretation is to be expected due to the close connections between classical Liouville theory
and the theory of Riemann surfaces. Having fixed a complex structure on the Riemann surface
one may represent the unique metric of negative constant curvature locally in the form
\( ds^2 = e^{2\varphi} dzd\bar{z} \) where \( \varphi \) must solve the Liouville equation \( \partial\bar{\partial}\varphi = \frac{1}{4} e^{2\varphi} \). This relation between
the Liouville equation and the uniformization problem leads to beautiful connections between
classical Liouville theory and the theory of moduli spaces of Riemann surfaces [4].

One may therefore expect that quantum Liouville theory is related to some sort of “quantization”
of the moduli spaces of Riemann surfaces. A concrete proposal in this direction was
made by H. Verlinde in [7], where it was proposed that the space of conformal blocks of the Liouville theory with its mapping class group representation is isomorphic to the space of states obtained by quantizing the Teichmüller spaces of Riemann surfaces [8, 9, 10]. We believe that understanding Verlinde’s conjecture will serve as a useful starting point for developing the geometrical interpretation of the quantized Liouville theory in general. The aim of the present paper will be to outline the definition of the objects that are involved in Verlinde’s conjecture and to describe some key steps towards the proof of it.

2. Teichmüller spaces

The Teichmüller spaces $\mathcal{I}(\Sigma)$ are the spaces of deformations of the complex structures on Riemann surfaces $\Sigma$. As there is a unique metric of constant curvature -1 associated to each complex structure one may identify the Teichmüller spaces with the spaces of deformations of the metrics with constant curvature -1. Coordinates for the Teichmüller spaces can therefore be obtained by considering the geodesics that are defined by the constant curvature metrics.

2.1. Penner coordinates. — A particularly useful set of coordinates was introduced by R. Penner in [11]. They can be defined for Riemann surfaces that have at least one puncture. One may assume having triangulated the surface by geodesics that start and end at the punctures. As an example we have drawn in Figure 1 a triangulation of the once-punctured torus. The length of these geodesics will be infinite. In order to regularize this divergence one may introduce one horocycle around each puncture and measure only the length of the segment of a geodesic that lies between the horocycles. Assigning to an edge $e$ its regularized length $l_e$ gives coordinates for the so-called decorated Teichmüller spaces. These are fiber spaces over the Teichmüller spaces which have fibers that parameterize the choices of the “cut-offs” as introduced by the horocycles.

A closely related set of coordinates for the Teichmüller spaces themselves was introduced by Fock in [8]. The coordinate $z_e$ associated to an edge $e$ of a triangulation can be expressed in terms of the Penner-coordinates via $z_e = l_a + l_c - l_b - l_d$, where $a$, $b$, $c$ and $d$ label the other edges of the triangles that have $e$ in its boundary as indicated in Figure 2.

Instead of triangulations of the Riemann surfaces it is often convenient to consider the corresponding fat graphs, which are defined by putting a trivalent vertex into each triangle and by
connecting these vertices such that the edges of the triangulation are in one-to-one correspondence to the edges of the fat-graph.

2.2. Symplectic structure. — The Teichmüller spaces carry a natural symplectic form, called Weil-Petersson symplectic form [12]. We are therefore dealing with a family of phase-spaces, one for each topological type of the Riemann surfaces. One of the crucial virtues of the Penner/Fock-coordinates is the fact that the Weil-Petersson symplectic form has a particularly simple expression in these coordinates. The corresponding Poisson-brackets are in fact constant for Fock’s variables $z_e,
\{z_e, z_{e'}\} = n_{e,e'}, \text{ where } n_{e,e'} \in \{-2, -1, 0, 1, 2\}.
\tag{1}

The value of $n_{e,e'}$ depends on how edges $e$ and $e'$ are imbedded into a given fat graph. If $e$ and $e'$ don’t have a common vertex at their ends, or if one of $e$, $e'$ starts and ends at the same vertex then $n_{e,e'} = 0$. In the case that $e$ and $e'$ meet at two vertices one has $n_{e,e'} = 2$ (resp. $n_{e,e'} = -2$) if $e'$ is the first edge to the right (1) (resp. left) of $e$ at both vertices, and $n_{e,e'} = 0$ otherwise. In all the remaining cases $n_{e,e'} = 1$ (resp. $n_{e,e'} = -1$) if $e'$ is the first edge to the right (resp. left) of $e$ at the common vertex.

If one considers a surface $\Sigma^g_s$ with genus $g$ and $s$ boundary components one will find $s$ central elements in the Poisson-algebra defined by (1). These central elements $c_k, k = 1, \ldots, s$ are constructed as $c_k = \sum_{e \in E_k} z_e$, where $E_k$ is the set of edges in the triangulation that emanates from the $k^{th}$ boundary component. The value of $c_k$ gives the geodesic length of the $k^{th}$ boundary component [8].

2.3. Changing the triangulation. — Changing the triangulation amounts to a change of coordinates for the (decorated) Teichmüller spaces. Any two triangulations can be related to each other by a sequence of the following elementary moves called flips:

\begin{align*}
\text{Figure 2. The labeling of the edges}
\end{align*}

\begin{align*}
\text{Figure 3. The elementary move between two triangulations}
\end{align*}

\(^{(1)}\text{w.r.t. to the orientation induced by the imbedding of the fat-graph into the surface}\)
The change of variables corresponding to the elementary move of Figure 3 is easy to describe:

\[
\begin{align*}
    z'_a &= z_a - \phi(-e^x), \\
    z'_d &= z_d + \phi(+e^x), \\
    z'_c &= -z_c, \\
    z'_b &= z_b + \phi(+e^x), \\
    z'_e &= z_e - \phi(-e^x),
\end{align*}
\]

where \(\phi(x) = \ln(e^x + 1)\), \(2\)

and all other variables are left unchanged. These transformations generate a groupoid, the Ptolemy groupoid \([11]\), that may be abstractly characterized by generators and relations: One has a generator \(\omega_{ij}\) whenever the triangles labeled by \(i\) and \(j\) have an edge in common. The relation that characterizes the Ptolemy groupoid is called the Pentagon relation. It is graphically represented in Figure 4.

2.4. The representation of the mapping class group. — The mapping class group \(\text{MC}(\Sigma)\) consists of diffeomorphisms of the Riemann surface \(\Sigma\) which are not isotopic to the identity. It is generated by the Dehn-twists, which act on an annulus \(A\) in the way indicated in Figure 5.

Elements \(\text{MC}(\Sigma)\) will map any graph drawn on the surface \(\Sigma\), in particular any triangulation of \(\Sigma\), into another one. Since any two triangulations can be connected by a sequence of elementary moves one may represent any element \(\text{MC}(\Sigma)\) by the corresponding sequence of flips. This means that the mapping class group \(\text{MC}(\Sigma)\) is a subgroup of the Ptolemy groupoid \(\text{Pt}(\Sigma)\).

It is extremely useful to think of the algebraically complicated mapping class group \(\text{MC}(\Sigma)\) as being embedded into the Ptolemy groupoid.
3. Quantization of Teichmüller spaces

3.1. Algebra of observables and Hilbert space. — The simplicity of the Poisson brackets makes part of the quantization quite simple. To each edge $e$ of a triangulation of a Riemann surface $\Sigma_g$ associate a quantum operator $z_e$. The algebra of observables $A(\Sigma_g^s)$ will be the algebra with generators $z_e$, relations

$$[z_e, z_{e'}] = 2\pi ib^2 \{z_e, z_{e'}\},$$

and hermiticity assignment $z_e^\dagger = z_e$. The algebra $A(\Sigma_g^s)$ has a center with generators $c_k$, $k = 1, \ldots, s$ defined by $c_k = \sum_{e \in E_k} z_e$, where $E_k$ is the set of edges in the triangulation that emanates from the $k$th boundary component. The representations of $A(\Sigma_g^s)$ that we are going to consider will therefore be such that the generators $c_k$ are represented as the operators of multiplication by real positive numbers $l_k$. Geometrically one may interpret $l_k$ as the geodesic length of the $k$th boundary component $[8]$. The tuple $\Lambda = (l_1, \ldots, l_s)$ of lengths of the boundary components will figure as a label of the representation $\pi(\Sigma_g^s, \Lambda)$ of the algebra $A(\Sigma_g^s)$.

To complete the definition of the representation $\pi(\Sigma_g^s, \Lambda)$ by operators on a Hilbert space $H(\Sigma_g^s)$ one just needs to find linear combinations $q_1, \ldots, q_{3g-3+s}$ and $p_1, \ldots, p_{3g-3+s}$ of the $z_e$ that satisfy $[p_m, q_n] = (2\pi i)^{-1} \delta_{mn}$. The representation of $A(\Sigma_g^s)$ on $\mathcal{H}(\Sigma_g^s) := L^2(R^{3g-3+s})$ is defined by choosing the usual Schrödinger representation for the $q_i, p_i$.

Let us discuss the example of a sphere with four holes. We shall consider the fat graph drawn in Figure 6 below. The algebra $A(\Sigma_4^0)$ has six generators $z_i = 1, \ldots, 6$ with nontrivial relations $[z_i, z_j] = 2\pi ib^2$ for

$$(i, j) \in \{(1, 2), (1, 6), (2, 3), (3, 4), (3, 5), (4, 1), (5, 1), (6, 3)\}. \tag{4}$$

The four central elements corresponding to the holes in $\Sigma_{0,4}$ are

$$c_1 = z_4 + z_6, \quad c_2 = z_1 + z_3 + z_5 + z_6, \quad c_3 = z_2 + z_5, \quad -c_4 = z_1 + z_2 + z_3 + z_4. \tag{5}$$

After fixing the lengths of the four holes one is left with two variables, say $z_4$ and $z_5$. Choosing the Schrödinger representation for $z_1, z_5$ one simply finds $\mathcal{H}(\Sigma_4^0) \simeq L^2(\mathbb{R})$. 

![Figure 6](image-url)

**Figure 6.** Fat graph for the sphere with four holes (shaded) with numbered edges.
3.2. **Representation of the mapping class group on** $A(\Sigma)$. — The first task is to find the quantum counterparts of the changes of variables corresponding to the elements of the Ptolemy groupoid. A handy formulation of the solution \[8, 10\] can be given in terms of the Fock-variables: The change of variables corresponding to the elementary move depicted in Figure 3 is given by

\[
\begin{align*}
    z_a' &= z_a - \phi_b(-z_e), \\
    z_d' &= z_d + \phi_b(+z_e), \\
    z_e' &= -z_e, \\
    z_b' &= z_b + \phi_b(+z_e), \\
    z_c' &= z_c - \phi_b(-z_e),
\end{align*}
\]  

where the special function $\phi_b(x)$ is defined as

\[
\phi_b(z) = \frac{\pi b^2}{2} \int_{i0}^{i0+\infty} dw \frac{e^{-izw}}{\sinh(\pi w) \sinh(\pi b^2 w)}. \tag{7}
\]

$\phi_b(x)$ represents the quantum deformation of the classical expression given in \[2\]. The formulae \[6\] define a representation of the Ptolemy groupoid by automorphisms of $A(\Sigma)$ \[10\].

Let us now recall that the mapping class group $\text{MC}(\Sigma)$ can be embedded into the Ptolemy groupoid. Having realized the latter therefore gives a representation of $\text{MC}(\Sigma)$ by automorphisms of $A(\Sigma)$.

3.3. **Representation of the mapping class group on** $H(\Sigma)$. — The next problem is to define unitary operators that generate the action of the mapping class group on $H(\Sigma)$ \[9, 17\].

In the case that $g \geq 2$ is possible to describe the action of the generators rather simply. For this case it is known \[13\] that the mapping class group is generated by the Dehn-twists along nonseparating simple closed curves. In an annular neighborhood of such a curve $c$ one may always bring the triangulation into the form depicted in Figure 7. In this case it suffices to consider the two variables $z_a, z_b$ associated to the edges $a$ and $b$ marked in Figure 7. They satisfy the algebra $[z_b, z_a] = 4\pi ib^2$. The Dehn twist can be represented by a simple flip. It can be checked that the Dehn twist around $c$ on $H(\Sigma)$ is represented by the following operator $D_c$:

\[
D_c = e^{2\pi i (p^2 - c^2)} e_b(p - x), \quad 2\pi b x = \frac{1}{2}(z_b - z_a), \quad 2\pi b p = \frac{1}{2}(z_a + z_b). \tag{8}
\]

\[\text{(2)}\] Cutting along such curves preserves connectedness of $\Sigma$, it just opens a handle.
where the special function $e_b(x)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$
\log e_b(z) \equiv \frac{1}{4} \int_{i\theta - \infty}^{i\theta + \infty} \frac{dw}{w} \frac{e^{-i2zw}}{\sinh(bw) \sinh(b^{-1}w)}.
$$

(9)

Formulae (8) and (6) suffice to completely define the action of $MC(\Sigma^s_g)$ on $H(\Sigma^s_g)$ for $g > 1$:

By means of (6) one may construct the representation of the Dehn twist along $g$ in an arbitrary triangulation from (3).

The operators that represent $MC(\Sigma^s_g)$ are of course defined by their action on $A(\Sigma^s_g)$ only up to multiplication by phase factors. Controlling the possible occurrence of phase factors in the relations of the mapping class group is a subtle task. An elegant formalism for handling this problem was given by Kashaev in [9]. It uses an enlarged set of variables, where pairs of variables are associated to the $2M = 4g - 4 + 2s$ triangles of a triangulation instead of its edges. The reduction to $(A(\Sigma^s_g), H(\Sigma^s_g))$ can be described with the help of a simple set of constraints [9]. It indeed turns out that the mapping class group is realized only projectively [17].

4. Liouville theory in the Moore-Seiberg formalism

We shall now briefly review the construction of the conformal blocks in quantum Liouville theory [5, 6] from a geometric perspective. The result will be another assignment $\Sigma \rightarrow (\mathcal{H}^L(\Sigma), \pi^L(\Sigma))$, where $\mathcal{H}^L(\Sigma)$ is the space of conformal blocks in Liouville theory and $\pi^L(\Sigma)$ is the representation of the mapping class group on $\mathcal{H}^L(\Sigma)$.

4.1. Holomorphic factorization. — The spectrum of quantum Liouville theory can be represented as follows:

$$
\mathcal{H}^L \simeq \int_{\mathbb{S}} da \ \mathcal{V}_\alpha \otimes \overline{\mathcal{V}}_\alpha, \quad \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+.
$$

(10)

In (10) we used the notation $\mathcal{V}_\alpha, \overline{\mathcal{V}}_\alpha$ for the unitary highest weight representations of the Virasoro algebras which are generated from the modes of the holomorphic and anti-holomorphic parts of the energy-momentum tensor respectively. The central charge of the representations is given by the parameter $b$ via $c = 1 + 6Q^2$, $Q = b + b^{-1}$, and the highest weight of the representations $\mathcal{V}_\alpha, \overline{\mathcal{V}}_\alpha$ is parameterized as $\Delta_\alpha = \alpha(Q - \alpha)$.

Quantum Liouville theory in genus zero is fully characterized by the set of n-point functions of the primary fields $V_\alpha(z, \bar{z})$, which are the quantized exponential functions $e^{2\alpha \phi(z, \bar{z})}$ of the Liouville field. The $V_\alpha(z, \bar{z})$ are primary fields with conformal weights $(\Delta_\alpha, \overline{\Delta}_\alpha)$. As usual one may consider $V_\alpha(z, \bar{z})$ as the generator of a family of descendant fields $V_\alpha(v \otimes w | z, \bar{z})$ parameterized by vectors $v \otimes w \in \mathcal{V}_\alpha \otimes \overline{\mathcal{V}}_\alpha$.

In principle it should always be possible to evaluate an n-point function

$$
\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_s}(z_s, \bar{z}_s) \rangle
$$

where $c_b(x)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$
\log e_b(z) \equiv \frac{1}{4} \int_{i\theta - \infty}^{i\theta + \infty} \frac{dw}{w} \frac{e^{-i2zw}}{\sinh(bw) \sinh(b^{-1}w)}.
$$

(9)
by inserting complete sets of intermediate states between each pair of fields \( V_{\alpha_i}(z_i, \bar{z}_i) \) and \( V_{\alpha_{i+1}}(z_{i+1}, \bar{z}_{i+1}) \). Due to the factorized structure of the Hilbert space \([10]\) one finds a holomorphically factorized form for the n-point functions, which may be written as

\[
\left\langle V_{\alpha_i}(v_s \otimes w_s | z_s, \bar{z}_s) \ldots V_{\alpha_1}(v_1 \otimes w_1 | z_1, \bar{z}_1) \right\rangle = \\
= \int_{\mathbb{S}_s} dS \ C(S|E) \mathcal{F}^{\Sigma}_{s,E}(v_s \otimes \ldots \otimes v_1)\bar{\mathcal{F}}^{\Sigma}_{s,E}(w_s \otimes \ldots \otimes w_1),
\]

where we have used the following notation. The tuple \( E = (\alpha_s, \ldots, \alpha_1) \) represents the “external” parameters, whereas

\[
S = (\beta_s-3, \ldots, \beta_1) \in \mathbb{S}_s \equiv \mathbb{S}^{s-3}
\]

comprises the variables of integration. The fact that \( \mathcal{F}^{\Sigma}_{s,E} \) (resp. \( \bar{\mathcal{F}}^{\Sigma}_{s,E} \)) depends holomorphically (resp. antiholomorphically) on \( z_1, \ldots, z_s \) is indicated via \( \Sigma = \mathbb{P}^1 \setminus \{z_1, \ldots, z_s\} \). The conformal blocks \( \mathcal{F}^{\Sigma}_{s,E}(v_s \otimes \ldots \otimes v_1) \) are key objects.

### 4.2. The conformal Ward identities.

It is well-known that the conformal blocks are strongly constrained by the conformal Ward-identities which express the conservation of energy-momentum on the Riemann-sphere \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_s\} \).

Mathematically speaking a conformal block is a functional

\[
\mathcal{F}^{\Sigma}_{E}: \mathcal{V}_{\alpha_s} \otimes \ldots \otimes \mathcal{V}_{\alpha_1} \rightarrow \mathbb{C}
\]

that satisfies the following invariance condition. Let \( v(z) \) be a meromorphic vector field that is holomorphic on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_s\} \). Write the Laurent-expansion of \( v(z) \) in an annular neighborhood of \( z_k \) in the form \( v(z) = \sum_{n \in \mathbb{Z}} v_n^{(k)} (z - z_k)^{n+1} \), and define an operator \( T[v] \) on \( \mathcal{V}_{\alpha_s} \otimes \ldots \otimes \mathcal{V}_{\alpha_1} \) by

\[
T[v] = \sum_{k=1}^{s} \sum_{n \in \mathbb{Z}} v_n^{(k)} L_n^{(k)}, \quad L_n^{(k)} = \text{id} \otimes \ldots \otimes L_{n_{(k-th)}} \otimes \ldots \otimes \text{id}.
\]

The conformal Ward identities can then be formulated as the condition that

\[
\mathcal{F}^{\Sigma}_{E}(T[v]w) = 0 \quad (11)
\]

holds for all \( w \in \mathcal{V}_{\alpha_s} \otimes \ldots \otimes \mathcal{V}_{\alpha_1} \) and all meromorphic vector fields \( v \) that are holomorphic on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_s\} \).

By choosing vector fields \( v \) that are singular at a single point only one recovers the usual rules for moving Virasoro generators from one puncture to another. In this way one may convince oneself that the conformal block is uniquely determined by the value \( \mathcal{F}^{\Sigma}_{E}(v_E) \in \mathbb{C} \) that it takes on the product of highest weight states \( v_E \equiv v_{\alpha_s} \otimes \ldots \otimes v_{\alpha_1} \).

A concrete representation for the conformal blocks in genus zero can be given by means of the chiral vertex operators \( h_{\alpha_{z_{\alpha_1}}}^{\alpha}(z) \) constructed in refs. \([5, 6]\), and their descendants...
The resultant construction resembles the construction of field theoretical amplitudes by the application of a set of Feynman rules. Let us summarize the basic ingredients and their geometric counterparts.

\[ f^a_{\alpha \beta} \cdot (v | z) , \quad v \in V_a. \]

Let \( \Sigma \equiv \mathcal{P}^1 \setminus \{z_1, \ldots, z_s\}, \quad E = (\alpha_s, \ldots, \alpha_1), \quad S = (\beta_{s-3}, \ldots, \beta_1). \)

\[ \mathcal{G}^{\Sigma}_{E}: (v_0 \otimes \cdots \otimes v_1) = \langle v_0, h^0_{\alpha_1 a_1}(v_1 | z_1) h^{a_1-1}_{\alpha_1 \beta_{s-3}}(v_{s-1} | z_{s-1}) \cdots h^{a_2}_{\alpha_2 \beta_{s-3}}(v_2 | z_2) h^{a_1}_{\beta_{s-3} a_1}(v_1 | z_1) v_0 \rangle \]  

(12)

where \( v_0 \) is the highest weight vector in the vacuum representation \( V_0 \).

It is well-known that the space of solutions to the condition (11) is one-dimensional for the case of the three-punctured sphere \( s = 3 \). Invariance under global conformal transformations allows one to assume that \( \Sigma_3 = \mathbb{P}^1 \setminus \{0, 1, \infty\} \). We will adopt the normalization from [2] and denote \( \epsilon^2_{E_2} \) the unique conformal block that satisfies \( \epsilon^2_{E_2}(v_a \otimes v_{a_1}) = N(a_3, a_2, a_1) \). The function \( N(a_3, a_2, a_1) \) is defined in [2] but will not be needed in the following.

Let us furthermore note that the case of \( s = 2 \) corresponds to the invariant bilinear form \( \langle \cdot, \cdot \rangle_a : V_a \otimes V_a \rightarrow \mathbb{C} \) which is defined such that \( \langle L_n w, v \rangle_a = \langle w, L_n v \rangle_a \).

### 4.3. Sewing of conformal blocks.

For \( n > 3 \) one may generate large classes of solutions of the conformal Ward identities by the following “sewing” construction. Let \( \Sigma_i, \quad i = 1, 2 \) be Riemann surfaces with \( m_i + 1 \) punctures, and let \( \mathcal{G}^\Sigma_{E_2} \) and \( \mathcal{G}^\Sigma_{E_1} \) be conformal blocks associated to \( \Sigma_i, \quad i = 1, 2 \) and representations labeled by \( E_2 = (a_{m_2}, \ldots, a_1) \) and \( E_1 = (a, a^\prime_{m_1}, \ldots, a^\prime_1) \) respectively. Let \( p_i, \quad i = 1, 2 \) be the distinguished punctures on \( \Sigma_i \) that are associated to the representation \( V_a \). Around \( p_i \) choose local coordinates \( z_i \) such that \( z_i = 0 \) parameterizes the points \( p_i \) themselves. Let \( A_i \) be the annuli \( |z_i| < R_i \), and \( D_i \) be the disks \( |z_i| \leq r_i \). We assume \( R \) to be small enough so that the \( A_i \) contain no other punctures.

The surface \( \Sigma_2 \otimes \Sigma_1 \) that is obtained by “sewing” \( \Sigma_2 \) and \( \Sigma_1 \) will be

\[ \Sigma_2 \otimes \Sigma_1 = \left( (\Sigma_2 \setminus D_2) \cup (\Sigma_1 \setminus D_1) \right) / \sim, \]

where \( \sim \) denotes the identification of annuli \( A_2 \) and \( A_1 \) via \( z_1 z_2 = r R \). The conformal block \( \mathcal{G}^{\Sigma_2 \otimes \Sigma_1}_{A_2, \Sigma_1} \) assigned to \( \Sigma_2 \otimes \Sigma_1 \), \( E_{21} = (a_{m_2}, \ldots, a_1, a^\prime_{m_1}, \ldots, a^\prime_1) \) and \( a \in \mathcal{S} \) will then be

\[ \mathcal{G}^{\Sigma_2 \otimes \Sigma_1}_{A_2, \Sigma_1}(v_{m_2} \otimes \cdots \otimes v_1 \otimes w_{m_1} \otimes \cdots \otimes w_1) = \sum_{i,j \in I} \mathcal{G}^{\Sigma_2}_{E_2}(v_{m_2} \otimes \cdots \otimes v_1 \otimes v_i) \langle v_i^v, e^{-t L_a} v_j^\vee \rangle_{E_1} \mathcal{G}^{\Sigma_1}_{E_1}(v_j^\vee \otimes w_{m_1} \otimes \cdots \otimes w_1). \]  

(13)

The sets \( \{v_{a,i} : i \in I\} \) and \( \{v_{a,i}^\vee : i \in I\} \) are supposed to represent mutually dual bases for \( V_a \) in the sense that \( \langle v_{a,i}, v_{a,j}^\vee \rangle_a = \delta_{ij} \). In a similar way one may construct the conformal blocks associated to a surface that was obtained by sewing two punctures on a single Riemann surface.

### 4.4. Feynman rules for the construction of conformal blocks.

The sewing construction allows one to construct large classes of solutions to the conformal Ward identities from simple pieces. The resulting construction resembles the construction of field theoretical amplitudes by the application of a set of Feynman rules. Let us summarize the basic ingredients and their geometric counterparts.
PROPAGATOR — Invariant bilinear form:

\[ \langle v_2, e^{-tL_0}v_1 \rangle_{V_a} \sim \]

VERTEX — Invariant trilinear form:

\[ C_{\Sigma}^3(v_3, v_2, v_1) \sim \]

GLUING — Completeness:

The dashed lines have been introduced to take care of the fact that the rotation of a boundary circle by \(2\pi\) (Dehn twist) is not represented trivially. It acts by multiplication with \(e^{2\pi i \Delta_a}\). This describes a part of the action of the mapping class group on the spaces of conformal blocks.

The Riemann surfaces that are obtained by gluing cylinders and three-holed spheres as drawn will therefore carry a trivalent graph which we will call Moore-Seiberg graph. These graphs are not to be confused with the fat graphs that we had encountered in the previous subsections.

The gluing construction furnishes spaces of conformal blocks \(\mathcal{H}^L(\Sigma, \Gamma)\) associated to a Riemann surface \(\Sigma\) together with a Moore-Seiberg graph \(\Gamma\). A basis for this space is obtained by coloring the “internal” edges of the Moore-Seiberg graph \(\Gamma\) with elements of \(S\), for example

\[ \mathcal{H}^L(\Sigma, \Gamma) = \text{Span} \left\{ \text{elements of } \Gamma \right\} . \]

In order to show that the Hilbert spaces associated to each two Moore-Seiberg graphs \(\Gamma_1\) and \(\Gamma_2\) are isomorphic, \(\mathcal{H}^L(\Sigma, \Gamma_1) \simeq \mathcal{H}^L(\Sigma, \Gamma_2) \simeq \mathcal{H}^L(\Sigma)\), one needs to find unitary operators \(U_{\Gamma_2\Gamma_1} : \mathcal{H}^L(\Sigma, \Gamma_1) \to \mathcal{H}^L(\Sigma, \Gamma_2)\).

4.5. The Moore-Seiberg groupoid. — The transitions from one Moore-Seiberg graph to another generate another groupoid that will be called the Moore-Seiberg groupoid. A set of generators is pictorially represented in Figures 8-10 below.
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The Moore-Seiberg groupoid can also be characterized in terms of the generators depicted in Figures 8-10 and certain relations, but the set of relations for the Moore-Seiberg-groupoid is more complicated than the one for the Ptolemy-groupoid [15, 16].

4.6. Representation of the Moore-Seiberg groupoid on $\mathcal{H}_L(\Sigma)$. — In order to characterize a representation of the Moore-Seiberg groupoid it suffices to find the operators $U_{\Gamma_2 \Gamma_1}$ for the cases where $\Gamma_2$ and $\Gamma_1$ differ by an A-, B- or S-move. In the case of the Liouville conformal blocks in genus zero this was done in [5, 6].

**A-MOVE:** In order to describe the representation of the A-move let $\Sigma$ be the four-punctured sphere, with parameters $E = (a_4, \ldots, a_1)$ associated to the four punctures respectively. The conformal blocks corresponding to the sewing patterns indicated on the left and right halves of Figure 8 will be denoted $F_{E,a_4}^{\Sigma}$ and $G_{E,a_4}^{\Sigma}$ respectively. The A-move is then represented as an integral transformation of the following form.

$$F_{E,a_4}^{\Sigma} = \int_{\mathbb{S}} d\mu(a_t) \ F_{E}^{\Sigma}(a_s \mid a_t) \ G_{E,a_t}^{\Sigma}. \quad (14)$$
The kernel $F^u_E(a_s | a_t)$ is given by the following expression:

$$F^u_E(a_s | a_t) = \frac{s_b(u_1)}{s_b(u_2)} \frac{s_b(w_1)}{s_b(w_2)} \int dt \prod_{i=1}^4 \frac{s_b(t-r_i)}{s_b(t-s_i)},$$  \hspace{1cm} (15)

where the special function $s_b(x)$ is related to $c_b(x)$ via $s_b(x) = e^{-\frac{\pi}{2}x^2} e^{\frac{\pi}{4}(2-Q^2)} c_b(x)$. The coefficients $r_i, s_i, u_i$ and $w_i$ are

\begin{align*}
r_1 &= p_2 - p_1, & s_1 &= c_b - p_4 + p_2 - p_t, & u_1 &= p_s + p_2 - p_1, \\
r_2 &= p_2 + p_1, & s_2 &= c_b - p_4 + p_2 + p_t, & u_2 &= p_s + p_3 + p_4, \\
r_3 &= -p_4 - p_3, & s_3 &= c_b + p_s, & u_1 &= p_t + p_1 + p_4, \\
r_4 &= -p_4 + p_3, & s_4 &= c_b - p_s, & u_2 &= p_t + p_2 - p_3,
\end{align*}

where $c_b = i \frac{Q}{2}$ and $a_s = \frac{Q}{2} + ip_b$ for $b \in \{1, 2, 3, 4, s, t\}$. Setting $a_t = \frac{Q}{2} + ip_t$ one may finally write the measure $d\mu(a_t)$ in the form $d\mu(a_t) = dp_t m(p_t)$, where $m(p_t) = 4 \sinh 2\pi b p_t \sinh 2\pi b^{-1} p_t$.

**B-MOVE:** The B-move is realized simply by the multiplication with the phase factor

$$B^L(a_3, a_2, a_1) = e^{\pi i (\Delta_{a_3} - \Delta_{a_2} - \Delta_{a_1})},$$  \hspace{1cm} (17)

where $\Delta_{a_k}, k = 1, 2, 3$ are the conformal dimensions $\Delta_a = a(Q - a)$.

**S-MOVE?** It is not known yet how to construct a representation of the S-Move on the spaces of Liouville conformal blocks.

## 5. Length operators

We have described how to assign to a Riemann surface $\Sigma$ two vector spaces $\mathcal{H}^L(\Sigma)$ (for genus 0) and $\mathcal{H}^T(\Sigma)$ (for arbitrary genus, but $s > 0$), each equipped with a (projective) representation of the mapping class group. According to the conjecture of H. Verlinde the two are equivalent. One possible strategy to prove this conjecture is to construct a representation of the Moore-Seiberg groupoid on the quantized Teichmüller spaces, allowing one to compare the respective representations of its generators.

### 5.1. Length-Twist coordinates

We find it instructive to discuss the corresponding classical story first. Regarding the Teichmüller spaces as a collection of phase-spaces naturally leads one to look for suitable Hamiltonians. A natural choice is furnished by the lengths of geodesics around simple closed curves, considered as functions on $\mathcal{T}(\Sigma)$. It is known that the length-functions $l_c, l_{c'}$ associated to *non-intersecting* curves $c$ and $c'$ Poisson-commute w.r.t. the Weil-Petersson symplectic form [21].

In the case of a Riemann surface $\Sigma_g$ of genus $g$ with $s$ boundary components it is well-known that there will be a collection of $3g - 3 + s$ closed geodesics $c_i$ such that cutting $\Sigma_g$ along $c_i, i = 1, \ldots, 3g - 3 + s$ decomposes it into a collection of pants (three-holed spheres). The collection of lengths $l_i$ of the geodesics $c_i$ furnishes a set of functions on the Teichmüller space
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$T(\Sigma^g)$ that Poisson-commute with each other, $\{l_i, l_j\} = 0$. Having half as many commuting “Hamiltonians” as $\dim(T(\Sigma^g)) = 6g - 6 + 2s$ we thereby recognize the Teichmüller space as a completely integrable system.

A nice feature of the Fock coordinates is that they lead to a particularly simple way to express the lengths $l_\gamma$ of simple closed curves $\gamma$ in terms of the variables $z_\gamma$. Assume given a closed path $P_\gamma$ on the fat graph homotopic to a simple closed curve $\gamma$. Let the edges $e_i$, $i = 1, \ldots, r$ be labeled according to the order in which they appear on the path $P_\gamma$, and define $\sigma_i$ to be 1 if the path turns left\(^{(3)}\) at the vertex that connects edges $e_i$ and $e_{i+1}$, and to be equal to $-1$ otherwise. The length $l_\gamma$ is then computed as follows [8].

$$2 \cosh \left( \frac{1}{2}l_\gamma \right) = |\text{tr}(X_\gamma)|, \quad X(\gamma) = V^{\sigma_r}E(z_{e_r}) \cdots V^{\sigma_1}E(z_{e_1}),$$

where the matrices $E(z)$ and $V$ are defined respectively by

$$E(z) = \begin{pmatrix} 0 & e^{\frac{z}{2}} \\ -e^{-\frac{z}{2}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

Having chosen the $\{l_1, \ldots, l_{3g-3+s}\}$ as the set of “action”-variables, it is amusing to note that the corresponding “angle”-variables are nothing but the twist-angles corresponding to the deformation of cutting $\Sigma^g$ along $c_i$ and twisting by some angle $\varphi_i$ before gluing back along $c_i$ [21]. The set $\{l_1, \ldots, l_{3g-3+s}; \varphi_1, \ldots, \varphi_{3g-3+s}\}$ gives another set of coordinates on the Teichmüller space $T(\Sigma^g)$, the classical Fenchel-Nielsen coordinates.

Dehn twists along one of the curves $c_i$, $i = 1, \ldots, 3g - 3 + s$ do not change the pants decomposition and are described by $\varphi_i \rightarrow \varphi_i + 2\pi$. It seems natural to associate length-twist coordinates $\{l_1, \ldots, l_{3g-3+s}; \varphi_1, \ldots, \varphi_{3g-3+s}\}$ to each Moore-Seiberg graph $\Gamma$. Changing the Moore-Seiberg graph $\Gamma$ by the Dehn-twist along one of the curves $c_i$ corresponds to a change of the twist coordinate $\varphi_i \rightarrow \varphi_i' = \varphi_i + 2\pi$. The Moore-Seiberg groupoid thereby gets the interpretation as the natural groupoid of canonical transformations of the length-twist coordinates.

5.2. The length operators. — Our aim is to make contact with the Liouville conformal field theory. In the Moore-Seiberg formalism for conformal field theories one considers bases for the space of conformal blocks that are associated to pants decompositions of Riemann surfaces. Our task may be seen as a quantum counterpart of the task to construct the change of variables from the Penner- to the Fenchel-Nielsen coordinates.

Let us consider a simple closed curve $c$ in $\Sigma$ such that the triangulation in an annular neighborhood of $c$ looks as depicted in Figure 7. In this case one finds according to (18) an expression for the hyperbolic cosine of the geodesic length function that is easy to quantize,

$$L_c = 2 \cosh 2\pi b x + e^{2\pi b x}, \quad 2\pi b x = \frac{1}{2}(z_b - z_a), \quad 2\pi b x = \frac{1}{2}(z_a + z_b). \tag{20}$$

\(^{(3)}\)w.r.t. to the orientation induced by the imbedding of the fat-graph into the surface
We shall work in the representation where \( p \) is diagonal. It is easy to check that the functions
\[
\Psi(s \mid p) = s_b^{-1}(p - s - c_b)s_b^{-1}(p + s - c_b)
\]
are eigenfunctions of \( L_c \) with eigenvalue being \( 2 \cosh 2\pi b s \). Let \( \langle d_p \mid \) be the distribution represented by \( \Psi(s \mid p + i0) \). The following important result was proven by R. Kashaev in \([19]\).

**Theorem 1.** — \([19]\) The following set of distributions
\[
\mathcal{B}_R^b \equiv \{ \mid d_p \rangle ; \ p_s \in \mathbb{R}^+ \}
\]
does a basis for \( L^2(\mathbb{R}) \) in the sense of generalized functions. We have the relations
\[
\langle d_p \mid d_q \rangle = m^{-1}(p) \delta(p - q) \quad \text{(Orthogonality),}
\]
\[
\int_{\mathbb{R}^+} dp \ m(p) \langle d_p \rangle \langle d_p \rangle = \text{id} \quad \text{(Completeness),}
\]
where the measure \( m(p) \) is defined as
\[
m(p) = 4 \sinh 2\pi b p \sinh 2\pi b^{-1}p.
\]

It follows that there exists a self-adjoint operator \( l_c \) with spectrum \( \mathbb{R}^+ \) such that \( L_c = 2 \cosh \frac{1}{2} l_c \). \( l_c \) is the quantum operator corresponding to the hyperbolic length around the geodesic \( c \).

It will be shown \([20]\) that the definition of the length operators \( l_c \) can be extended to arbitrary closed geodesics \( \gamma \) such that (i) \( l_c \) is self-adjoint with spectrum \( \mathbb{R}^+ \), and (ii) \( [l_\gamma, l_{\gamma'}] = 0 \) if \( \gamma \cap \gamma' = \emptyset \). Moreover, diagonalization of the length operator \( l_\gamma \) for a closed geodesic \( \gamma \subset \Sigma \) leads to a factorization of \( \pi(\Sigma, \Lambda) \) in the following sense. Let \( \Sigma' \equiv \Sigma \setminus \gamma \) be the possibly disconnected Riemann surface obtained by cutting along \( \gamma \). The coloring \( \Lambda \) of the boundary components of \( \Sigma \) can be naturally extended to a coloring \( \Lambda'_{\gamma,l} \) for \( \Sigma' \) by assigning the number \( l \in \mathbb{R}^+ \) to the two new boundary components that were created by cutting along \( \gamma \). The spectral representation for \( l_\gamma \) then yields the following representation for \( \pi(\Sigma, \Lambda) \).
\[
\pi(\Sigma, \Lambda) \simeq \int_{\mathbb{R}^+} dl \ \pi(\Sigma'_{\gamma,l}, \Lambda'_{\gamma,l}).
\]
The corresponding representations of the mapping class group factorize/restrict accordingly \([20]\). This allows one to construct bases (in the sense of generalized functions) for \( \mathcal{H}(\Sigma) \) labeled by the assignments of lengths to the closed geodesics \( c_1, \ldots, c_{3g-3+s} \) that define a pants decomposition.

6. **Realization of the Moore-Seiberg groupoid on** \( \mathcal{H}(\Sigma) \)

Thanks to the factorization properties of the quantized Teichmüller spaces one may indeed construct a realization of the Moore-Seiberg groupoid by associating unitary operators to the elementary moves depicted in Figures 8-10 \([20]\).
6.1. A-Move. — In order to construct the representation of the A-move let us consider the four-holed sphere with the fat graph introduced in Figure 6. The operators $L_s$ and $L_t$ that represent the lengths of the geodesics isotopic to the $s$- and $t$-cycles drawn in Figure 6 respectively are then given by the following expressions.

$$L_s = 2 \cosh 2\pi b p_s + e^{\pi bx} 2 \cosh \pi b (p_s + s_1 - s_2) 2 \cosh \pi b (p_s + s_3 + s_4) e^{\pi bx},$$

$$L_t = 2 \cosh 2\pi b p_t + e^{-\pi bx} 2 \cosh \pi b (p_t + s_2 - s_3) 2 \cosh \pi b (p_t - s_4 - s_1) e^{-\pi bx},$$

(24)

where we have used the notation $p_s = p - s_3 - s_4, p_t = p - s_3 - s_2$ with

$$p = (2\pi b)^{-1} z_5, \quad x = (2\pi b)^{-1} z_1, \quad s_k = (4\pi b)^{-1} c_k, \quad k = 1, \ldots, 4.$$  

(25)

In order to find eigenfunctions of the operators $L_s$ and $L_t$ let us observe that the ansatz

$$| d_p^b \rangle = \tilde{U}_D p U_{d_t}^b | d_p \rangle, \quad b = s, t, \quad p \in \mathbb{R}^+.$$  

(26)

allows one to construct eigenstates of $L_s$ and $L_t$ from the eigenstates $| d_p \rangle$ of the operator $L_c$ that were defined in (21), provided that the unitary operators $U_p^b$ are chosen as

$$U_p^s = e^{2\pi i(s_3 + s_4)} s_b (p + s_1 - s_2) s_b (p + s_3 + s_4),$$

$$U_p^t = e^{2\pi i(s_2 + s_3)} s_b^{-1} (p + s_2 - s_3) s_b^{-1} (p - s_1 - s_4).$$  

(27)

The tuple $D = (s_4, \ldots, s_1)$ parameterizes the fixed lengths of the four boundary holes via $l_k = 4\pi b s_k, k = 1, \ldots, 4$, and the $\rho_{b,p}^p$ are pure phases that will be fixed shortly. We have

Theorem 2. — [22] The following two sets of distributions

$$\mathfrak{g}^b_E \equiv \{ | d_p^b \rangle : p \in \mathbb{R}^+ \}, \quad b \in \{ s, t \}$$

form bases for $L^2(\mathbb{R})$ in the sense of generalized functions. We have the relations

$$\langle d_q^b | d_p^b \rangle = m^{-1}(p) \delta(p - q) \quad \text{(Orthogonality)},$$

$$\int_{\mathbb{R}^+} dp \, m(p) \langle d_q^b | d_p^b \rangle \langle d_p^b | = \text{id} \quad \text{(Completeness).}$$  

(28)

This theorem was first obtained by another method [22], but thanks to [26] it also follows easily from Kashaev’s Theorem [1]. Another simple implication of Theorem 2 is the relation

$$\int_{\mathbb{R}^+} dq \, m(q) \langle d_q^t | F_E^{T} (p | q) = | d_p^s \rangle, \quad F_E^{T} (p | q) \equiv \langle d_q^t | d_p^s \rangle.$$  

(29)

The unitary operator that represents the A-move will simply be the one that describes the change of basis between the eigenfunctions of the length operators $l_s$ and $l_t$ respectively. It is represented by the kernel $F_D^{T} (p | q) = \langle d_q^t | d_p^s \rangle$. It is natural to choose the phase factors $\rho_{b,p}^p$ in such a way that the b-6-j symbols $\{ \ldots \}_b$ defined by

$$F_D^{T} (q | p) = \{ s_1 \, s_2 \, s_4 \, q \}$$  

(30)
satisfy the following tetrahedral symmetries
\[
\begin{align*}
\{ s_1, s_2, s_3 \}_b &= \{ s_4 - s_5 - s_6 \}_b = \{ s_4 - s_5 - s_6 \}_b = \{ s_4 - s_5 - s_6 \}_b.
\end{align*}
\] (31)

Let us choose the \( \rho_{O,p}^b \) as follows:
\[
\begin{align*}
\rho_{O,p}^s &= \frac{s_b(p + s_2 - s_1)}{s_b(p + s_3 + s_4)}, \quad \rho_{O,q}^s &= \frac{s_b(q + s_2 - s_3)}{s_b(p + s_1 + s_4)}.
\end{align*}
\] (32)

The kernel \( F^q_0(q \mid p) \) will then differ from the b-Racah-Wigner coefficients for the quantum group \( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})) \) studied and calculated in [22] only by multiplication with the measure \( m(q) \). It follows from the results of [23] (Appendix B.2) that the corresponding b-6-j symbols indeed satisfy \( (31) \).

6.2. B-move. — In order to discuss the B-move we find it convenient to use the Fock-variables associated to the edges of the fat graph depicted in Figure 11. Let us furthermore introduce the following linear combinations of the variables \( z_e, e \in \{1, 2, 3, 5, 6\} \).
\[
\begin{align*}
p_2 &= (4\pi b)^{-1}(z_6 - z_3), \quad s_2 = (4\pi b)^{-1}(z_6 + z_3), \quad q_2 = -(2\pi b)^{-1}z_5, \\
p_1 &= (4\pi b)^{-1}(z_1 - z_2), \quad s_1 = (4\pi b)^{-1}(z_1 + z_2), \quad q_1 = +(2\pi b)^{-1}z_5.
\end{align*}
\] (33)

The representation of the braiding of two punctures in terms of the generators of the Ptolemy groupoid was first discussed in [18]. Considering punctures (holes of zero size) corresponds to setting \( s_2 = 0 = s_1 \). The expression for the generator of the B-move that was found in [18] may be written as
\[
R = e^{\pi ip_2p_1} \frac{s_b(p_1)}{s_b(p_2)} e_b^{-1}(q_1 - q_2) \frac{s_b(p_2)}{s_b(p_1)} e^{\pi ip_2p_1}.
\] (34)

As already observed in [18], \( R \) is identical to the R-operator for the modular double \( \mathcal{D}\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})) \) of \( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})) \) that was first proposed in [24] and further studied in [25]. By working out the operator \( L_{21} \) that represents the length of the geodesic surrounding the two punctures via \( L_{21} = 2 \cosh \frac{1}{2}l_{21} \), one finds that \( L_{21} \) coincides with the realization of the Casimir \( C_{21} \) of \( \mathcal{D}\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})) \) in the tensor product of two of its representations.

These observations may then be used in order to find the operator \( R_{s_2,s_1} \) that represents the braiding of two holes with finite circumferences \( l_k = (4\pi b)^{-1}s_k \). In order to do this one may “fill” the holes by gluing disks with two punctures each to the boundaries of the holes.
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The braiding of the two holes can be decomposed into a sequence of braidings of the four punctures. By using (Theorem 3) it is then possible to show that

$$R_{s_2 s_1} = e^{\pi i p_2 p_1} \frac{s_b(p_1 - s_1)}{s_b(p_2 + s_2)} e_b^{-1} (q_1 - q_2) \frac{s_b(p_2 + s_2)}{s_b(p_1 - s_1)} e^{\pi i p_2 p_1}. \quad (35)$$

It remains to observe that according to (Theorem 6) the operator $R_{s_2 s_1}$ becomes diagonal on eigenstates of the length operator $L_{21}$. The eigenvalue of $R_{s_2 s_1}$ is

$$B^T(l_3, l_2, l_1) = e^{\pi i (\Delta_{l_3} - \Delta_{l_2} - \Delta_{l_1})}, \quad \Delta_l = \frac{Q^2}{4} + \left(\frac{l}{4\pi b}\right)^2. \quad (36)$$

6.3. S-move. — One possibility to introduce the operator that represents the S-move is to consider the change of basis between bases that diagonalize the length operators of a- and b-cycles on the once-holed torus respectively. The proof that the operators that represent the A-, B- and S-moves satisfy the relations of the Moore-Seiberg groupoid will be given in [20]. Concerning the relations coming from genus zero surfaces we may observe that their validity immediately follows if one combines the above observations concerning relations with $\mathcal{D}U_q(sl(2, R))$ with the results of [22, 25].

It furthermore turns out that the kernel that represents the S-move can be expressed in terms of $(F^T, B^T)$ by means of an expression which is similar to the one that was found for rational conformal field theories in [26]. The data $(F^T, B^T)$ are therefore sufficient to characterize the projective representation of the Moore-Seiberg groupoid on the quantized Teichmüller spaces completely.

6.4. The Verlinde conjecture. — The main observation to be made is the following. One has

$$(F^T, B^T) \equiv (F^L, B^L) \quad \text{provided that} \quad a_k = \frac{Q}{2} + i \frac{J_k}{4\pi b}. \quad (37)$$

It follows that the spaces of conformal blocks of Liouville theory and the Hilbert spaces from the quantization of the Teichmüller spaces are indeed isomorphic as representations of the mapping class group.

7. Final remarks

First let us note that so far the conformal blocks of Liouville theory were only constructed in genus zero, but our results imply that the corresponding mapping class group representation has a consistent and essentially unique extension to higher genus.

One may wonder if there is a more direct relation between the conformal blocks of Liouville theory as discussed in Section 4 and the eigenfunctions of the length operators from Sections 5 and 6. In this regard one may observe that so far the quantization of the Teichmüller spaces $\mathcal{T}(\Sigma)$ was based on the choice of a real polarization for $\mathcal{T}(\Sigma)$. However, the complex structure on the Teichmüller spaces $\mathcal{T}(\Sigma)$ should allow one to introduce an alternative (“coherent state”) representation where the states are represented by wave-functions that are analytic on $\mathcal{T}(\Sigma)$. Such a representation should have the property that the action of the mapping class group
The symmetry group of our phase-space is represented by monodromy transformations. The field theoretical conformal blocks have exactly this property. This leads us to suggest that the Liouville conformal blocks are in fact just representing the change of basis from a coherent state basis for $\mathcal{H}(\Sigma)$ to a basis in which the length operators are diagonal.

Our results finally establish a precise correspondence \((\ref{37})\) between the labels $a_k = Q^2 + iP$ of a Liouville vertex operator $V_a$ and the length $l_k$ of the hole to which $V_a$ is associated. The fact that all the basic objects are meromorphic w.r.t. the variables $a_k$ or $l_k$ allows one to analytically continue from the case considered above to cases where $a_k \in \mathbb{R}$. The geometrical interpretation of the cases $a_k \in \mathbb{R}$ was previously inferred from the semi-classical relation between $a$ and the type of singularity that the metric $ds = e^{2\phi} dz d\bar{z}$ develops near the insertion point of $V_a$, see e.g. \([27]\). Combining these two observations gives the diagram drawn in Figure 12, which relates the parameter $a$ to the “shape” of the boundary component to which $V_a$ is associated. It is remarkable and nontrivial that the point $a = 0$ indeed corresponds to the complete disappearance of a boundary component, as follows from \([23]\) (Appendix B.1).

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