AN ADDITIVITY THEOREM FOR COBORDISM CATEGORIES

WOLFGANG STEIMLE

Abstract. We give a new proof of the Genauer fibration sequence, relating the cobordism categories of closed manifolds with cobordism categories of manifolds with boundaries. Unlike the existing proofs, it is of combinatorial nature and generalizes to other cobordism categories of interest. Indeed we argue that the Genauer fibration sequence is an analogue, in the setting of cobordism categories, of Waldhausen’s Additivity theorem in algebraic $K$-theory.

1. Introduction

There are various reasons to investigate cobordism categories of mathematical objects which cannot be decomposed using transversality. The following examples are under current investigation:

(i) The $h$-cobordism category \cite{11},
(ii) Cobordism categories of manifolds equipped with a metric of positive scalar curvature \cite{2},
(iii) Cobordism categories of Poincaré chain complexes, and generalizations thereof \cite{5},
(iv) The cobordism category associated to a Waldhausen category \cite{10}, whose homotopy type has been shown to agree, up to degree shift, with the algebraic $K$-theory from \cite{15}.

The homotopy types of these cobordism categories are not known to be described by Thom spectra, and can hardly, if at all, be computed. For this reason, it is important to have theorems available that at least compare the homotopy types for various inputs. This program has been successfully implemented in algebraic $K$-theory, where Waldhausen’s additivity theorem \cite{15} is probably the most fundamental theorem of this kind.

In this paper we formulate and prove a “local-to-global principle” designed to produce homotopy fiber sequences of cobordism-like categories, and hence long exact sequences on their homotopy groups. Roughly, it states that under certain assumptions, pull-backs of a functor $P$ geometrically realize to homotopy pull-backs; the key assumption here is that morphisms in the target of $P$ can be universally lifted through $P$ in a specific sense, formalized by the notion of cartesian and cocartesian fibration.

Then we apply this local-to-global principle in the setting of classical cobordism categories $\text{cob}_d$ of \cite{3} and its version with boundaries $\text{cob}_{d,\partial}$ of \cite{14} (actually, possibly with higher corners and theta-structures, see section \cite{4}). It will turn out that in this case we recover the following theorem, due to Genauer \cite{4}:

**Theorem 1.1.** There is a homotopy fibration sequence of cobordism categories

\begin{equation}
\text{Bcob}_d \rightarrow \text{Bcob}_{d,\partial} \xrightarrow{\partial} \text{Bcob}_{d-1},
\end{equation}

where the functor $\partial$ takes the boundary.

In his proof of this result, Genauer first determined the homotopy type of the middle term of (1.2) in analogy to the computation for the left and right hand term.
of [3], and the fibration sequence follows from the fibration sequence
\[ MT(d) \to \Sigma^\infty_+ (BO(d)) \to MT(d-1) \]
of Thom spectra from [3 (3.3)]. In contrast, our method does not yield, nor require, knowledge of the homotopy types of any of the terms in the fibration sequence.

A virtue of the local-to-global principle is that it has interesting consequences in all the situations (i) to (iv) considered above. We will use it in [5] to relate cobordism categories of Poincaré chain complexes with the algebraic K-theory of the underlying ring; it will also be shown in that paper that in the setting of (iv) we recover, from the local-to-global principle, Waldhausen’s Additivity theorem, at least in many cases of interest. This leads us to the slogan that “Genauer’s fibration sequence is the analogue of the Additivity theorem in the setting of cobordism categories”.

In joint work with George Raptis we aim to use the local-to-global principle to continue our study of the homotopy type of the \( h \)-cobordism category and its relation to algebraic K-theory, initiated in [11]. Finally, as Johannes Ebert informs me, it will follow from [2] that the local-to-global principle also applies in the context of (ii), in which case it compares the cobordism category of p.s.c. metrics with the usual one; and that the authors of that paper have come independently to the conclusions of the local-to-global principle, in this specific situation.

I am grateful to Thomas Nikolaus for pointing out to me a generalization of the local-to-global principle, in the setting of simplicial sets, that implies Quillen’s Theorem B (see Theorem 3.5).

2. THE LOCAL-TO-GLOBAL PRINCIPLE

Let us now formulate the local-to-global principle, first in the setting of (not necessarily unital) topological categories. A more general version for semi-Segal spaces will be given afterwards. It will be derived, in the next section, from an analogous statement in the setting of simplicial sets which we formulate at the end of this section.

A morphism \( f: c \to d \) in a (non-unital) topological category \( \mathcal{C} \) is called an equivalence if for any \( t \in \text{ob} \mathcal{C} \), the maps
\[ f \circ -: \mathcal{C}(t, c) \to \mathcal{C}(t, d) \quad \text{and} \quad - \circ f: \mathcal{C}(d, t) \to \mathcal{C}(c, t) \]
are weak homotopy equivalences. A topological category is weakly unital if any object is the source or the target of an equivalence. (This is equivalent to the condition that any object \( c \) allows a weak unit, that is, an equivalence \( u: c \to c \) such that \( u \circ u \simeq u \) in \( \mathcal{C}(c, c) \), compare the proof Lemma 3.9 (iv). A (continuous) functor \( F: \mathcal{C} \to \mathcal{D} \) between weakly unital topological categories is called weakly unital provided \( F \) preserves equivalences (equivalently, sends weak units to weak units). A unital category is weakly unital, because then the identity map on an object is a weak unit; similarly a unital functor is weakly unital, because any weak unit of \( c \) is homotopic to \( \text{id}_c \) in \( \mathcal{C}(c, c) \).

**Definition 2.1.** Let \( P: \mathcal{C} \to \mathcal{D} \) a continuous functor.

(i) \( P \) is a level fibration if it is a (Serre) fibration on each level of the nerve.

(ii) \( P \) is a local fibration if both maps
\[ (P, s, t): \text{mor} \mathcal{C} \to \text{mor} \mathcal{D} \times_{\text{ob} \mathcal{D} \times \text{ob} \mathcal{D}} (\text{ob} \mathcal{C} \times \text{ob} \mathcal{C}), \]
\[ P: \text{ob} \mathcal{C} \to \text{ob} \mathcal{D} \]
are fibrations. \( \mathcal{C} \) is locally fibrant if the unique functor \( \mathcal{C} \to * \) is a local fibration, that is, if the source-target map
\[ (s, t): \text{mor}(\mathcal{C}) \to \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{C}) \]
(iii) A morphism \( f : c \rightarrow c' \) in \( C \) is \( P \)-cartesian if for all \( t \in \text{ob} \, C \), the following commutative square is a homotopy pull-back:

\[
\begin{array}{ccc}
C(t, c) & \xrightarrow{f \circ -} & C(t, c') \\
\downarrow P & & \downarrow P \\
D(P(t), P(c)) & \xrightarrow{P(f) \circ -} & D(P(t), P(c'))
\end{array}
\]

(iv) \( P \) is a cartesian fibration if for any morphism \( g : d \rightarrow d' \) and any \( c' \in \text{ob} \, C \) such that \( P(c') = d' \), there is a \( P \)-cartesian morphism \( f : c \rightarrow c' \) such that \( P(f) = g \).

(v) \( P \) is a cocartesian fibration if \( P^{\text{op}} : C^{\text{op}} \rightarrow D^{\text{op}} \) is a cartesian fibration.

Remarks. (i) A local fibration is a level fibration.

(ii) If \( P \) is a local fibration, then all induced maps \( C(c, c') \rightarrow D(P(c), P(c')) \) between morphism spaces are fibrations.

(iii) If \( C \) and \( D \) are discrete and unital categories, then the above notion of (co-)cartesian fibration agrees with the usual notion of (co-)cartesian (or Grothendieck) fibration.

**Theorem 2.2** (Local-to-global principle). Let

\[
\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{P} & \mathcal{D} \\
\downarrow F & & \downarrow P \\
\mathcal{D}' & \xrightarrow{P} & \mathcal{D}
\end{array}
\]

be a diagram of weakly unital, locally fibrant topological categories and weakly unital maps. If \( P \) a level, cartesian and cocartesian fibration, then the pull-back diagram

\[
\begin{array}{ccc}
\mathcal{D}' \times_{\mathcal{D}} C & \longrightarrow & C \\
\downarrow P' & & \downarrow P \\
\mathcal{D}' & \xrightarrow{F} & \mathcal{D}
\end{array}
\]

geometrically realizes to a homotopy pull-back diagram of spaces.

Here \( \mathcal{D}' \times_{\mathcal{D}} C \) denotes the pull-back in the category of topological categories, explicitly given by the pull-back on object and morphism spaces. The condition on weak unitality cannot be dropped, as can be seen by taking \( P \) the functor \( \partial : \text{cob}_{d, \partial} \rightarrow \text{cob}_{d-1} \) in Genauer’s sequence, and \( \mathcal{D}' \subset \text{cob}_{d-1} \) the subcategory consisting all objects and no morphisms.

It requires some care to make the composition operation in the cobordism category strictly associative, and for many perspectives it is easier and more natural to view the cobordism category as a semi-Segal space.

**Definition 2.3.** A semi-Segal space is a semi-simplicial space \( C \) that:

(i) is locally fibrant in the sense that \( (d_1, d_0) : C_1 \rightarrow C_0 \times C_0 \) is a fibration, and

(ii) satisfies the Segal condition that for each \( n \), the Segal map

\[
C_n \rightarrow C_1 \times_{C_0} \cdots \times_{C_0} C_1
\]

into the \( n \)-fold iterated pull-back is a weak homotopy equivalence.

A map of semi-Segal spaces is, by definition, a map of the underlying semi-simplicial spaces.

We identify a topological category \( C \) with a semi-simplicial space, via the nerve. With this convention, a locally fibrant topological category is a semi-Segal space, with the Segal map being a homeomorphism.
Next we define a notion of $P$-(co-)cartesian morphism in a semi-Segal space $C$. For $\sigma \in C_k$ and $n \geq k$, we denote by $C_n/\sigma$ the homotopy fiber, over $\sigma$, of the map $C_n \to C_k$ sending an $n$-simplex to its last $k$-simplex. With this notation, we define:

**Definition 2.4.** Let $P : C \to D$ be a map of semi-Segal spaces. A 1-simplex $f$ in $C$ is called $P$-cartesian if the square

$$
\begin{array}{ccc}
C_2/f & \xrightarrow{d_1} & C_1/d \\
| & & | \\
D_2/P(f) & \xrightarrow{d_1} & D_1/P(d)
\end{array}
$$

is a homotopy pull-back square. $P$ is called cartesian fibration if for all $g \in D_1$ and all $c \in C_0$ with $P(c) = d_0(g)$, there exists $f \in C_1$ such that $P(f) = g$, which is $P$-cartesian. $P$ is called cocartesian fibration if $P^{op} : C^{op} \to D^{op}$ is a cartesian fibration.

If $C$ is a locally fibrant topological category, then $C_1/d$ is the space of all morphisms in $C$ ending at $d$; also $C_2/f$ is equivalent to the actual fiber of $C_2 \to C_1$ over $f$, which agrees with the space of morphisms ending at $c$; it follows that the upper horizontal map in (2.5) models composition by $f$ and $P(f)$, respectively. It results from this discussion that if $C$ and $D$ are locally fibrant topological categories, then our new definition of $P$-cartesian morphism is equivalent to the old one.

**Definition 2.6.** Let $C$ be a semi-Segal space and $f \in C_1$. Then $f$ is called equivalence if it is both $P$-cartesian and $P$-cocartesian, for the unique map $P : C \to *$ to the terminal objects. $C$ is called weakly unital if every $c \in C_0$ is the source or the target of an equivalence; a map of weakly unital semi-Segal spaces $F : C \to D$ is called weakly unital if it preserves equivalences.

Again if $C$ is a locally fibrant topological category, then this definition agrees with the old one. Hence the following Theorem is a generalization of the local-to-global principle for topological categories as stated above.

**Theorem 2.7** (Local-to-global principle, semi-Segal version). Let $D' \xleftarrow{C} D \xrightarrow{P} C$ be a diagram of weakly unital semi-Segal spaces and weakly unital maps. If $P$ a level, cartesian and cocartesian fibration, then the (level-wise) pull-back diagram

$$
\begin{array}{ccc}
D' \times_D C & \xrightarrow{P} & C \\
| & & | \\
D' & \xrightarrow{F} & D
\end{array}
$$

realizes to a homotopy pull-back diagram of spaces.

Finally, we give a version of the Additivity in the context of simplicial sets. Recall that a map $p : X \to Y$ between simplicial sets is called an inner fibration if for every $n > 1$ and every $0 < i < n$, every solid diagram

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\sigma} & X \\
| & & | \\
\Delta^n & \xrightarrow{\sigma} & Y
\end{array}
$$

has a dotted diagonal lift. A 1-simplex $f$ in $X$ is called $p$-cartesian if every solid diagram (2.8) has a dotted diagonal lift, provided $i = n > 1$ and the last edge of $\sigma$
is \( f \). \( p \) is a cartesian fibration if it is an inner fibration and if for every \( g \in Y_1 \) and any lift \( c \) of \( d_0(g) \) through \( p \), there exists a lift \( f \in X_1 \) of \( g \) through \( p \) such that \( d_0(f) = c \); dually \( p \) is a cocartesian fibration of \( p^{op} \) is a cartesian fibration.

**Theorem 2.9** (Local-to-global principle, simplicial set version). A cartesian and cocartesian fibration \( P : X \to Y \) between simplicial sets is a realization-fibration; that is, for any map \( Y' \to Y \) of simplicial sets, the pull-back diagram

\[
\begin{array}{ccc}
Y' \times_Y X & \longrightarrow & X \\
\downarrow p' & & \downarrow p \\
Y' & \longrightarrow & Y
\end{array}
\]

realizes to a homotopy pull-back diagram of spaces.

3. Proof of the local-to-global principle

3.1. The simplicial set case. The proof in the simplicial set version is an easy combination of two well-known results, the first of which says that simplicial (forward) homotopies may be lifted along cocartesian fibrations.

**Lemma 3.1.** Let \( p : X \to Y \) be a cocartesian fibration, and let \( J \to I \) be an injective map of simplicial sets. If in the solid commutative diagram

\[
\begin{array}{ccc}
I \times \{0\} \cup_{J \times \{0\}} J \times \Delta^1 & \longrightarrow & X \\
\downarrow F_0 & & \downarrow p \\
I \times \Delta^1 & \longrightarrow & Y
\end{array}
\]

the map \( F_0 \) sends any edge \((j,0) \to (j,1) \) (for \( j \in J_0 \)) to a \( p \)-cocartesian edge, then there exists a dotted lift \( F \) which keeps the diagram commutative, and which in addition sends any edge \((i,0) \to (i,1) \) (for \( i \in I_0 \)) to a \( p \)-cocartesian edge.

**Proof.** By induction and a colimit argument, it is enough to consider the case where \( J \to I \) is the inclusion of \( \partial \Delta^n \) into \( \Delta^n \). If \( n = 0 \), then such a lift exists by definition of a cocartesian fibration. If \( n > 0 \), then this is [S 2.4.1.8]. \( \square \)

For \( P : X \to Y \), and a simplex \( \sigma : \Delta^n \to Y \) of \( Y \), we write \( X|_{\sigma} := \Delta^n \times_Y X \).

**Corollary 3.2.** Let \( P : X \to Y \) be a cocartesian fibration of simplicial sets. For any simplex \( \sigma \) in \( Y \), with last vertex \( \ell(\sigma) \), the inclusion map

\[
X|_{\ell(\sigma)} \to X|_{\sigma}
\]

genuinely realizes to a weak equivalence.

Indeed, a simplicial homotopy inverse is obtained by lifting the standard simplicial homotopy \( \Delta^n \times \Delta^1 \to \Delta^n \) that collapses the \( n \)-simplex to its last vertex, where \( n = |\sigma| \).

**Corollary 3.3.** Let \( P : X \to Y \) be a cartesian and cocartesian fibration of simplicial sets. Then, for any simplex \( \sigma \) of \( Y \), the maps induced by inclusion of the first and last vertex respectively,

\[
X|_{i(\sigma)} \to X|_{\sigma} \leftarrow X|_{\ell(\sigma)}
\]

realize to a weak equivalence.

Thus, the simplicial version of the local-to-global principle follows from:

**Lemma 3.4.** Let \( P : X \to Y \) be a map of simplicial sets such that for each simplex \( \sigma \) of \( Y \), the inclusion maps \( Y|_{i(\sigma)} \to Y|_{\sigma} \leftarrow Y|_{\ell(\sigma)} \) realize to weak equivalences. Then, \( P \) is a realization-fibration.
This follows readily from [15, Lemma 1.4.B], which Waldhausen deduces from Quillen’s Theorem B. We find it more transparent to give a direct proof as follows (see also [12]): It is enough to show that for each \( y \in Y_0 \), the inclusion of the fiber \( X|_y \) into the homotopy fiber is an equivalence. To see this, we consider the filtration of \( Y \) by skeleta \( Y(i) \), and the induced filtration on \( X \) by \( X(i) := Y(i) \times_Y X \), so we obtain a ladder

\[
\begin{array}{cccc}
X|_y & \longrightarrow & X^{(0)} & \longrightarrow & Y^{(1)} & \longrightarrow & \cdots & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{y\} & \longrightarrow & Y^{(0)} & \longrightarrow & Y^{(1)} & \longrightarrow & \cdots & \longrightarrow & Y
\end{array}
\]

The first square is homotopy cartesian, and by a colimit argument (see [13, 1.8]) it is enough to show inductively that if the first \( n \) squares \((n \geq 1)\) are homotopy cartesian then so is the next one.

To see this, we write the map \( X^{(n)} \to Y^{(n)} \) as the induced map on horizontal push-outs in the following diagram

\[
\begin{array}{cccc}
\prod_i \Delta^n \times_Y X & \leftarrow & \prod_i \partial \Delta^n \times_Y X & \longrightarrow & Y^{(n-1)} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i \Delta^n & \leftarrow & \prod_i \partial \Delta^n & \longrightarrow & X^{(n-1)}
\end{array}
\]

It is then enough to show that both squares are homotopy cartesian (see [13, 1.7]). By the inductive assumption, we have already shown that the right vertical map is a realization-fibration for pulling back to simplicial sets of dimension at most \( n-1 \), so that the right square is a homotopy pull-back. For the left square, we argue in each summand separately and denote by \( \Delta^0 \to \Delta^n \) the inclusion of either the left or the last vertex. In the diagram

\[
\begin{array}{cccc}
X \times_Y \Delta^0 & \longrightarrow & X \times_Y \partial \Delta^n & \longrightarrow & X \times_Y \Delta^n \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^0 & \longrightarrow & \partial \Delta^n & \longrightarrow & \Delta^n
\end{array}
\]

the total square is homotopy cartesian by assumption and the left one again by induction hypothesis. Since every component of \( \partial \Delta^n \) contains either the first or the last vertex, this proves that the right square is also homotopy cartesian. \( \square \)

We close this section with a remark on the relation of this simplicial version of the local-to-global principle to Quillen’s Theorem B. I am grateful to Thomas Nikolaus for pointing this out to me.

By Corollary 3.2 we may define, for each 1-simplex \( f : c \to c' \) in \( Y \), a fiber transport

\[
f_* : X|_c \to X|_f \overset{\sim}{\longrightarrow} X|_{c'}
\]

if \( p \) is a cocartesian fibration; dually a fiber transport

\[
f^* : X|_{c'} \to X|_f \overset{\sim}{\longrightarrow} X|_{c}
\]

if \( p \) is a cartesian fibration.

**Theorem 3.5.** Let \( P : X \to Y \) be a map of simplicial sets. Suppose that \( P \) a cocartesian fibration, such that for each 1-simplex \( f \) in \( Y \), the fiber transport \( f_* \) realizes to an equivalence. Then, \( P \) is a realization-fibration.
Of course, the same conclusion then holds if $P$ is a cartesian fibration, such that all fiber transports $f^*$ realize to weak equivalences; for the dual of a realization-fibration is again a realization-fibration.

The proof of this Theorem consists in noting that the inclusions $Y|_{f(\sigma)} \to Y|_{\sigma}$ are equivalences (by Corollary 3.2), but also for each 1-simplex $\sigma$ of $Y$, the inclusion $Y|_{f(\sigma)} \to Y|_{\sigma}$ is an equivalence (by the hypothesis on fiber transport). But this still implies the statement of Corollary 3.3 so that we may apply Lemma 3.4 just as above.

Theorem 3.5 implies Quillen’s Theorem B [9] as follows: For an arbitrary functor $F: \mathcal{C} \to \mathcal{D}$ between ordinary categories, we form a new category $\mathcal{F}/\mathcal{D}$ as the pull-back category

$$\begin{array}{ccc}
\mathcal{F}/\mathcal{D} & \to & \mathcal{D}^{[1]} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

Then the composite projection

$$\mathcal{F}/\mathcal{D} \to \mathcal{D}^{[1]} \xrightarrow{eval_1} \mathcal{D}$$

is a cocartesian fibration, with fiber over $d$ the translation category $\mathcal{F}/d$; the fiber transport along $f: d \to d'$ is homotopic to the canonical map $\mathcal{F}/d \to \mathcal{F}/d'$ induced by postcomposition with $f$. If we assume that these maps all realize to weak equivalences, then Theorem 3.5 implies that the canonical sequence

$$\mathcal{F}/d \to \mathcal{F}/\mathcal{D} \to \mathcal{D}$$

realizes to a homotopy fiber sequence. On the other hand the projection

$$\mathcal{F}/\mathcal{D} \to \mathcal{C}$$

is a cartesian fibration over $\mathcal{C}$, with contractible fibers $f(c)/\mathcal{D}$, so that it realizes to a weak equivalence again by Theorem 3.5. The projection $\mathcal{F}/\mathcal{D} \to \mathcal{D}^{[1]}$ defines natural transformation in the triangle

$$\begin{array}{ccc}
\mathcal{F}/\mathcal{D} & \to & \mathcal{D}^{[1]} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

so that it becomes homotopy-commutative after realization. Therefore, for each object $d$ of $\mathcal{D}$ there is a fibration sequence

$$B(\mathcal{F}/d) \to BC \xrightarrow{BF} BD,$$

which is the conclusion of Quillen’s Theorem B.

3.2. The semi-Segal case. We deduce the semi-Segal, and hence the topological version of the local-to-global principle from the simplicial one. The main input is the existence of simplicial structures from [14]. (Note that the free addition of identity morphism, while preserving the homotopy type of the classifying space, usually does not preserve (co-)cartesian morphisms.) We start by recalling the concept of a Reedy fibration.

For a semi-simplicial space $X$, and a semi-simplicial set $A$, we denote

$$X_A := \lim_{[n] \to A} X_n,$$

where the limit runs over the category of simplices of $A$, in the semi-simplicial sense. An equivalent description of $X_A$ is the mapping space from $A$ to $X$, that is, the set of natural transformations $A \to X$, equipped with the subspace topology
of $\prod_n \text{map}(A_n, X_n)$. With this notation, the canonical map \( X_n \rightarrow X_{[n]} \) is an isomorphism, which we will view as an identification. (Here, and throughout this section, we denote by \([n]\) the semi-simplicial \(n\)-simplex (i.e., presheaf represented by \([n]\)).

**Definition 3.6.** A map \( P : X \rightarrow Y \) between semi-simplicial spaces is a Reedy fibration if for all \( n \), the map

$$X_n = X_{[n]} \rightarrow Y_{n} \times_{Y_{[n]}} X_{[n]}$$

is a fibration. \( X \) is Reedy fibrant if the map to the terminal object \( * \) is a Reedy fibration.

It is easily seen that the class of Reedy fibrations is closed under composition and pull-back. Also, the standard techniques show that if \( P \) is a Reedy fibration, then for all inclusions of semi-simplicial sets \( A \rightarrow B \), the induced map \( X_B \rightarrow Y_B \times_{Y_A} X_A \) is a fibration.

**Lemma 3.7.** Any map \( P : X \rightarrow Y \) can be factored into a level equivalence \( X \rightarrow X' \), followed by a Reedy fibration \( X' \rightarrow Y \).

**Proof.** If \( P : X \rightarrow Y \) satisfies the Reedy condition up to simplicial degree \( n - 1 \), then we define a new semi-simplicial space \( X' \) as follows: Choose a factorization

$$X_n \xrightarrow{i} X'_n \xrightarrow{p} Y_{n} \times_{Y_{[n]}} X_{[n]}$$

of the Reedy map where \( i \) is the a weak equivalence, and \( p \) is a fibration. Letting \( X'_k := X_k \) for \( k \neq n \) we obtain a new semi-simplicial space \( X' \); \( i \) and \( p \) induce semi-simplicial maps \( I : X \rightarrow X' \) and \( P' : X' \rightarrow Y \) whose composite is \( P \). By construction, \( P' \) satisfies the Reedy condition up to degree \( n \).

Now the claim follows by doing this construction iteratively, starting at \( n = 0 \), and taking the colimit. \( \square \)

The following result states the applying Reedy fibrant replacement does not destroy the hypotheses in the local-to-global principle.

**Lemma 3.8.** Suppose that \( P : C \rightarrow D \) and \( P' : C' \rightarrow D' \) are maps between semi-Segal spaces that are related by a compatible zigzag of level weak equivalences.

(i) If \( P \) is a weakly unital map of weakly unital semi-Segal spaces, then so is \( P' \).

(ii) Suppose that \( P' \) is a Reedy fibration. If \( P \) is a cartesian (resp. cocartesian) fibration, then so is \( P' \).

**Proof.** For a map of semi-Segal spaces \( P : C \rightarrow D \), denote by \( \alpha^{P-\text{cart}}_1 \subset C_1 \) the subspace of \( P \)-cartesian morphisms, and by \( C^\sim_1 \subset C_1 \) the subspace of equivalences. From the definitions, it follows:

(a) The subspaces \( C^{P-\text{cart}}_1 \) and \( C^\sim_1 \) of \( C_1 \) consist of entire path components.

(b) If \( P \) is level equivalent to \( P' : C' \rightarrow D' \) then \( C^{P-\text{cart}}_1 \simeq (C')^{P-\text{cart}}_1 \) and \( C^\sim_1 \simeq (C')^\sim_1 \).

Let us now show (i). If \( C \) is weakly unital, then \( d_1 : C^\sim_1 \subset C_1 \rightarrow C_0 \) is surjective and it follows that \( d_1 : (C')^\sim_1 \subset C'_1 \rightarrow C'_0 \) is at least surjective on \( \pi_0 \). As \( C' \) is locally fibrant, this map is also a fibration and we conclude that the map is actually surjective, that is, every object is the target of an equivalence. So \( C' \) is weakly unital.

If \( P \) is weakly unital, then it sends \( \pi_0 C^\sim_1 \) into \( \pi_0 D^\sim_1 \), and hence \( P' \) sends \( \pi_0 (C')^\sim_1 \) into \( \pi_0 (D')^\sim_1 \). Therefore \( P' \) is weakly unital.

Next we prove (ii). If \( P \) is a cartesian fibration, then the map

\[(P, d_0) : (C_1)^{P-\text{cart}} \subset C_1 \rightarrow D_1 \times_{D_0} C_0 \]
is surjective. We note that the pull-back in the target is a homotopy pull-back. Under the zigzag of weak equivalences, this map corresponds to the map

\[(P', d_0): (C'_0)_{\text{P'\text{-cart}}} \rightarrow D'_1 \times_{D'_0} C'_0\]

where the pull-back in the target is also a homotopy pull-back. We conclude that the map \((P', d_0)\) is surjective on \(\pi_0\). But the map is a Serre fibration because \(P'\) is a Reedy fibration, so it is even surjective. This translates back to saying that \(P'\) is a cartesian fibration. The statement for cocartesian fibrations holds by the dual argument. □

Now, the following allows to translate the weakly unital semi-Segal setting into the setting of simplicial sets.

**Lemma 3.9.** Let \(P: C \rightarrow D\) be Reedy fibration of Reedy fibrant semi-Segal spaces. Then,

(i) \(P^\delta\) is an inner fibration between inner fibrant semi-simplicial sets.

(ii) If \(f \in C_1\) is \(P\)-cartesian (in the Segal sense), then it is also \(P^\delta\)-cartesian (in the quasicategorical sense).

(iii) If \(P\) is a (co-)cartesian fibration then so is \(P^\delta\).

If, furthermore, \(C, D,\) and \(P\) are weakly unital, then

(iv) \(C^\delta\) and \(D^\delta\) admit simplicial structures such that \(P^\delta\) is simplicial.

(v) The canonical maps \(C^\delta \rightarrow C\) and \(D^\delta \rightarrow D\) realize to weak equivalences.

**Proof.** It follows from [14, 3.4] that the map \(C_n \rightarrow D_n \times_{D_A^n} C_{A^n}\) is an acyclic fibration of spaces for each \(0 < i < n\); therefore it is surjective. But this means that \(P_i\) and equally \(P^\delta_i\), have the right lifting property against all inner horn inclusions, so \(P^\delta\) is an inner fibration. Applying this to the projection \(D \rightarrow \ast\) shows that \(D^\delta\) is inner fibrant. This shows (i). Part (ii) follows by the same reasoning from [14, 3.6]; (iii) is a direct consequence of (ii).

Let us prove (iv). We first note that every object \(c\) admits a homotopy idempotent self-equivalence, in other words, a two-simplex \(\tau\), all whose vertices are \(c\), and all whose faces agree and are equivalences. Indeed, if \(f: d \rightarrow c\) is an equivalence, then we can use the cocartesian property of \(f\) to find a 2-simplex \(\sigma\)

\[
\begin{array}{ccc}
d & \rightarrow & c \\
\downarrow f & & \downarrow u \\
c & \leftarrow & u
\end{array}
\]

such that \(d_1 \sigma = d_2 \sigma = f\), and \(u := d_0 f\) is still an equivalence, for equivalences in semi-Segal spaces are easily seen to satisfy the 2-out-of-3 property. Applying the cocartesian property once more, we obtain a 3-simplex

\[
\begin{array}{ccc}
d & \rightarrow & c \\
\downarrow f & & \downarrow u \\
c & \leftarrow & u
\end{array}
\]

whose first three faces are \(\sigma\); its last boundary is then a simplex \(\tau\) as required. The dual argument works if \(c\) is the domain of an equivalence.

By part (ii) a self-equivalence of \(D\) (in the Segal sense) is a self-equivalence of \(D^\delta\) (in the quasicategorical sense). Then it follows from the [14, Theorem 1.2] that \(D^\delta\) admits a simplicial structure.
Now, since $P$ is weakly unital, any equivalence of $C$ is both $P$-cartesian and $P$-cocartesian. Applying the same argument as above in the relative situation, we then see that any object of $C$ admits a 2-simplex, all whose vertices are $c$, and all whose faces agree and are $P$-cartesian and $P$-cocartesian, and which maps under $P$ to the degenerate simplex on $c$, in the newly constructed simplicial structure. Then it follows from [14, Theorem 2.1] that $C^\delta$ admits a simplicial structure such that $P^\delta$ is simplicial.

It remains to prove (v). By definition, $D$ being Reedy fibrant means that each diagram $0 \leq i \leq k$

\[
\begin{array}{ccc}
\Lambda^i_k \times \Delta^\bullet \cup_{\Lambda^i_k \times \partial \Delta^\bullet} \Delta^k \times \partial \Delta^\bullet & \longrightarrow & D \\
\Delta^k \times \Delta^\bullet & \downarrow & \\
\end{array}
\]

has a dotted extension, which again means that the restriction map

\[
\text{map}(\Delta^k, D) \to \text{map}(\Lambda^i_k, D)
\]

has the right lifting property against all inclusions $\partial \Delta^\bullet \to \Delta^\bullet$. Since each inclusion $\Delta^k \to \Delta^{k+1}$ may be obtained by filling in horns, it follows that evaluation at each vertex

\[
\text{map}(\Delta^k, D) \to \text{map}(\Delta^0, D)
\]

also has the right lifting property against the same set of inclusions. Now it follows from Lemma 3.8 that (3.10) is a weakly unital map of weakly unital semi-Segal spaces; by part (iv) it follows that the induced map

\[
\text{map}(\Delta^k, D)^\delta \to \text{map}(\Delta^0, D)^\delta
\]

may be given a simplicial structure, and still has the right lifting property against the same set of inclusions; therefore realizes to a weak equivalence. In other words, the degree-wise singular construction $S_\ast D$ is homotopically constant in the $\ast$-direction, after realizing the semi-simplicial direction of $D$. It follows that

\[
|D^\delta| = |S_0 D| \to |S_\ast D| \simeq |D|
\]

is an equivalence. □

**Proof of Theorem 2.7.** Applying Lemma 3.7 multiple times, we obtain a level equivalent diagram where all semi-simplicial spaces are Reedy fibrant and all maps are Reedy fibrations. It still satisfies the hypotheses in the statement of Theorem 2.7 by Lemma 3.8. Now, applying $(-)^\delta$ level-wise at each entry, by Lemma 3.9 we obtain diagram of semi-simplicial sets and inner fibrations, where the right vertical map is a cartesian and cocartesian fibration, and where all terms can be given compatible simplicial structures so that all maps become simplicial; and which becomes equivalent to the old diagram after geometric realization. Therefore the claim follows from Theorem 2.9. □

4. Cobordism categories of $(k)$-manifolds

In this section we use the concept of $(k)$-manifolds, see Appendix 5. We view always the interval $([a,b], \{a\}, \{b\})$ as a $(2)$-manifold and denote $\partial_- [a,b] = a$, $\partial_+ [a,b] = b$. Accordingly, if $M$ is a $(k)$-manifold, the product $[a,b] \times M$ inherits the structure of a $(k+2)$-manifold, whose faces we denote by

\[
\partial_- ([a,b] \times M) = \{a\} \times M, \quad \partial_+ ([a,b] \times M) = \{b\} \times M
\]

and

\[
\partial_i ([a,b] \times M) = \partial_i [a,b] \times M, \quad 0 \leq i \leq k.
\]
Loosely, we define the $d$-dimensional cobordism category of $\langle k \rangle$-manifolds as follows: An object is a compact $M \subset \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$ which is a smooth, neat submanifold of dimension $(d-1)$. A morphism is a pair $(W, a)$ of $a > 0$ and a compact $W \subset [0, a] \times \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$ which is a smooth neat submanifold of dimension $d$. Such a $(W, a)$ is viewed as a morphism from $\partial_- W$ to $\partial_+ W$. Composition is given by stacking and adding the real parameters.

Instead of giving this (non-unital) category a topology, we view this as the value at $[0]$ of a simplicial category $\text{cob}_{d,\langle k \rangle, \bullet}$. In simplicial level $n$, an object is a fiber bundle $E \to \Delta^n$ of smooth $(k)$-manifolds, which is fiberwise $\delta$-neatly embedded into $\Delta^n \times \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$, for some $\delta > 0$. A morphism is a continuous map $\bar{a}: \Delta^n \to (0, \infty)$ and a fiber bundle over $\Delta^n$ of compact smooth $(k+2)$-manifolds, fiberwise $\delta$-neatly embedded into $\Delta^n \times [0, a] \times \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$ for some $\delta > 0$. Here we use the notation

$$[0, a] \times X := \{(t, x) \in \mathbb{R} \times X \mid 0 \leq t \leq a(\pi(x))\}$$

for a space $X$ with a map $\pi: X \to \Delta^n$.

Details. By “fiber bundle $E \to B$ of smooth $\langle k \rangle$-manifolds” we mean a fiber bundle with a reduction of the structure group to $\text{Diff}(M)$, the diffeomorphism group of the typical fiber, a $(k)$-manifold $M$ (see Appendix 5). A fiberwise embedding from one bundle into another is fiberwise smooth and neat if, in local charts, it is given by a continuous map to $\text{Emb}(M, N)$ (again defined as in Appendix 5). Of course, the reduction of the structure group for the bundle $\Delta^n \times [0, a] \times \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$ is given by the unique chart $\Delta^n \times [0, 1] \times \mathbb{R}^{\infty+d-1}_{\langle k \rangle}$ which rescales fiberwise by $\bar{a}$.

The simplicial category $\text{cob}_{d, \bullet} := \text{cob}_{d, (0), \bullet}$ is a simplicial version of the usual $d$-dimensional cobordism category, and $\text{cob}_{d, 1, \bullet} := \text{cob}_{d, (1), \bullet}$ is a simplicial version of the $d$-dimensional cobordism category of manifolds with boundaries. We denote by $\text{cob}_{d, \langle k \rangle}$ the topological category obtained from $\text{cob}_{d, \langle k \rangle, \bullet}$ by geometric realization of objects and morphisms and by $B\text{cob}_{d, \langle k \rangle}$ the its classifying space.

There is an obvious commutative square of functors

$$\begin{array}{ccc}
\text{cob}_{d, \langle k \rangle} & \to & \text{cob}_{d, (k+1)} \\
\downarrow \text{id} & & \downarrow \partial_1 \\
\text{cob}^\emptyset & \to & \text{cob}_{d-1, \langle k \rangle}
\end{array}$$

where the upper horizontal functor is the inclusion

$$(M; M_1, \ldots, M_k) \mapsto (M; \emptyset, M_1, \ldots, M_k)$$

(using a standard inclusion $\mathbb{R} \to [0, \infty)$ in the relevant coordinate) on objects and morphisms, and the functor $\partial_1$ takes the first boundary on objects and morphisms. The lower left entry denotes the subcategory of $\text{cob}_{d-1, \langle k \rangle}$ on the empty set and empty morphisms. Actually this is isomorphic to the topological semi-group $\text{S}([0, \infty))$ (the geometric realization on the singular construction on the space $(0, \infty)$, semi-group structure by addition). The lower horizontal functor is the inclusion and the left vertical functor sends $(W, \bar{a})$ to $\bar{a}$.

Theorem 4.1. The above square realizes to a homotopy pull-back square. Since $B\text{cob}^\emptyset$ is contractible, we obtain a fibration sequence

$$B\text{cob}_{d, \langle k \rangle} \xrightarrow{\partial_1} B\text{cob}_{d, (k+1)} \xrightarrow{\partial_1} B\text{cob}_{d-1, \langle k \rangle}.$$ 

In the case $k = 0$ this yields the sequence from Theorem 1.1. — To prove this theorem, we will verify the criterion of the local-to-global principle for the functor $\partial_1$. First both $\text{cob}_{d, \langle k \rangle}$ and $\text{cob}_{d-1, (k-1)}$ are weakly unital and $\partial$ is a weakly unital
Lemma 4.3. Let $S_G$ be a Kan fibration. Then the projection map

$$\pi: X/G \rightarrow X$$

is a Kan fibration.

Proof. Let $S_G$ be a simplicial group acting simplicially on a simplicial set $X$, with level-wise quotient simplicial set $(X/G)_\bullet$. Suppose that the action is free in each simplicial degree. Then the projection map $X_\bullet \rightarrow (X/G)_\bullet$ is a Kan fibration.

Now any simplicial group is Kan so $g'$ extends to a map $g: \Delta^n \rightarrow G_\bullet$. Then $g \cdot y$ is a lift of $x$ restricting to $y'$.\qed

Moreover the vertical maps in (4.2) are surjective by definition. It is a formal consequence that the lower horizontal map is a Kan fibration as well. This proves that $\text{cob}_{d-1}(k)$ is locally fibrant. A very similar argument shows that the map $\partial_1$ is a local fibration, and hence a level fibration.

Next we show that $\text{cob}_{d-1}(k)$ is locally fibrant, that is, that the combined source-target map of $\text{cob}_{d-1}(k)$ is a Serre fibration. Let $W$ be a morphism of $\text{cob}_{d-1}(k)$ and denote by $\text{Emb}(W, \mathbb{R}^n_{(k)} \times [0,1])$ the space of neat embeddings of $(k+2)$-manifolds as defined in Appendix 5. By Theorem 5.1, the restriction map

$$\text{Emb}(W, \mathbb{R}^n_{(k)} \times [0,1]) \rightarrow \text{Emb}(M, \mathbb{R}^n_{(k)}) \times \text{Emb}(N, \mathbb{R}^n_{(k)})$$

is a Kan fibration.

With $S_n$ the singular construction, we have a commutative square

$$\begin{array}{ccc}
\prod_{[W]} |S_n(\text{Emb}(W, \mathbb{R}^n_{(k)} \times [0,1]) \times (0, \infty))| & \longrightarrow & \prod_{[M,N]} |S_n(\text{Emb}(M, \mathbb{R}^n_{(k)})| \times |S_n(\text{Emb}(N, \mathbb{R}^n_{(k)})|) \\
\downarrow & & \downarrow \\
\text{mor} \text{cob}_{d-1}(k) & \longrightarrow & \text{ob} \text{cob}_{d-1}(k) \times \text{ob} \text{cob}_{d-1}(k)
\end{array}$$

where the upper horizontal map is still a Kan fibration because both $S_n$ and $| - |$ preserve Kan fibrations. The vertical maps are Kan fibrations by letting $G_\bullet = S_n \text{Diff}(W)$ in the following criterion:

Lemma 4.3. Let $G_\bullet$ be a simplicial group acting simplicially on a simplicial set $X_\bullet$, with level-wise quotient simplicial set $(X/G)_\bullet$. Suppose that the action is free in each simplicial degree. Then the projection map $X_\bullet \rightarrow (X/G)_\bullet$ is a Kan fibration.

Proof. Let $x: \Delta^n \rightarrow (X/G)_\bullet$ be a simplicial map and $y': (A^n_\bullet)_\bullet \rightarrow X_\bullet$ be a partial lift. As $X_n \rightarrow (X/G)_n$ is surjective, there is a lift $y: \Delta^n \rightarrow X_\bullet$ of $x$. As $G_\bullet$ acts freely in each degree, there is a unique map $g': (A^n_\bullet)_\bullet \rightarrow G_\bullet$ such that $y' = g' \cdot y(A^n_\bullet)_\bullet$.

Now any simplicial group is Kan so $g'$ extends to a map $g: \Delta^n \rightarrow G_\bullet$. Then $g \cdot y$ is a lift of $x$ restricting to $y'$.\qed

Moreover the vertical maps in (4.2) are surjective by definition. It is a formal consequence that the lower horizontal map is a Kan fibration as well. This proves that $\text{cob}_{d-1}(k)$ is locally fibrant. A very similar argument shows that the map $\partial_1$ is a local fibration, and hence a level fibration.

Next we show that $\partial_1$ is a cocartesian fibration. Let $(W, a): M \rightarrow N$ be a cobordism between $(d-1)$-dimensional $(k)$-manifolds, and $X$ a $d$-dimensional $(k+1)$-manifold $\partial_1 X = N$; we will construct a $\partial_1$-cocartesian lift of the homotopy morphism $W' := (N \times [0,a]) \circ W$ starting at $N$. (By local fibrancy, it follows easily the general existence of $\partial_1$-cocartesian lifts.) Basically the cocartesian lift is obtained by rearranging the boundary parts of the manifold $[0,a] \times (X \cup_M W)$ as to obtain a morphism from $X$ to $X \cup_M W$ with vertical boundary $W'$.

More precisely, we let

$$B: \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$$

be a homeomorphism, which is a diffeomorphism except at $(0,a)$ and $(0,0)$, such that (see figure 4):

(i) $B$ maps the different pieces of the boundary isometrically as displayed in the picture.

(ii) $B$ is cylindrical in a neighborhood of $\partial \mathbb{R}^2_+ - \{(0,a), (0,0)\}$
(iii) In a neighborhood of \((0, a)\), \(B\) preserves the radial coordinate and scales the angle by a smooth function

\[
\lambda : [0, \pi] \rightarrow [0, \pi/2]
\]

which is id on \([0, \pi/3]\) and \(\text{id} - \pi/2\) on \([2\pi/3, \pi]\). Similarly, a neighborhood of \((0, 0)\), \(B\) scales the angle by the inverse of \(\lambda\).

(iv) The restriction of \(B\) to \([a/2, \infty) \times \mathbb{R}\) is translation by \(a\) to the right.

We call \(B\) the bending homeomorphism. It induces a bending homeomorphism (denoted by the same letter)

\[B : \mathbb{R}_{(k)}^\infty \rightarrow \mathbb{R}_{(k)}^\infty\]

by applying \(B\) in the first coordinates and the identity in the other ones.

Applying \(B^{-1}\) to the subspace \(X \cup W \subset \mathbb{R}_{(k+1)}^{\infty + d\langle k+2 \rangle}\) which happens to be a neatly embedded submanifold of \(0 \times \mathbb{R}_{(k+1)}^{\infty + d-1}\).

Then \(C := B([0, a] \times Y)\) is a neat submanifold of \([0, 2a] \times \mathbb{R}_{(k+1)}^{\infty + d-1}\). Hence \(C\) defines a morphism in \(\text{cob}_{d,(k+1)}\) and we claim that is indeed cocartesian. In other words, we claim that for any object \(Z\) of \(\text{cob}_{d,(k+1)}\), with \(P := \partial_1 Z\), the diagram

\[
\begin{array}{ccc}
\text{cob}_{d,(k+1)}(Y, Z) & \rightarrow & \text{cob}_{d,(k+1)}(X, Z) \\
\downarrow \alpha & & \downarrow \beta \\
\text{cob}_{d-1,(k)}(N, P) & \rightarrow & \text{cob}_{d-1,(k)}(M, P)
\end{array}
\]

is a homotopy pull-back. We prove the equivalent assertion that the induced map on all vertical fibers

\[
\text{cob}_{d,(k+1)}(Y, Z)/V \xrightarrow{-\circ C} \text{cob}_{d,(k+1)}(X, Z)/(V \circ W')
\]

is an equivalence.

By the special form of \(Y = X \cup_M W\), applying \(B\) induces a homeomorphism

\[B : \text{cob}_{d,(k+1)}(Y, Z)/V \rightarrow \text{cob}_{d,(k+1)}(X, Z)/(V \circ W').\]

The triangle

\[
\begin{array}{ccc}
\text{cob}_{d,(k+1)}(Y, Z)/V & \rightarrow & \text{cob}_{d,(k+1)}(X, Z)/(V \circ W') \\
\downarrow \circ \partial([0, a] \times Y) & & \downarrow \circ \partial([0, a] \times Y) \\
\text{cob}_{d,(k+1)}(Y, Z)/(V \circ (N \times [0, a])) & \cong & B
\end{array}
\]

is commutative by property (iv) of the map \(B\). As \([0, a] \times Y\) is a weak unit, the diagonal map is an equivalence, from which we conclude that the horizontal map is an equivalence as well.
This finishes the construction of the cocartesian lifts. It follows that our functor $\partial_1$ is also a cartesian fibration, because $\text{cob}_{d,\langle k \rangle} \cong \text{cob}_{d,\langle k \rangle}^{\text{op}}$ by reflecting everything in the last coordinate.

**Tangential structures.** Our proof of Theorem 4.1 also has a generalization to cobordism categories with tangential structures. We give the relevant definitions and results and show how to modify the proof. Recall from [4] that a $\langle k \rangle$-space is a $P(k)$-shaped diagram of spaces, and a $\langle k \rangle$-vector bundle is $P(k)$-shaped diagram of vector bundles and fiberwise linear maps.

**Definition 4.4.** A **collar** on a $\langle k \rangle$-vector bundle $\xi$ is a collection of bundle maps (isomorphisms in each fiber)

$$c_{AB} : \varepsilon^{B-A} \oplus \xi(A) \to \xi(B), \quad A \subset B \subset k$$

such that

(i) each $c_{AB}$ extends to the structure map $\xi(A) \to \xi(B)$, and

(ii) for any $A \subset B \subset k$, the following triangle commutes:

$$\begin{array}{ccc}
\varepsilon^{C-A} \oplus \xi(A) & \xrightarrow{id \oplus c_{AB}} & \varepsilon^{C-B} \oplus \xi(B) \\
\downarrow c_{AC} & & \downarrow c_{BC} \\
\xi(C) & & \xi(C)
\end{array}$$

A $\langle k \rangle$-bundle map $f : \xi \to \eta$ between collared $\langle k \rangle$-bundles is called **collared** if for each $A \subset B \subset k$, the resulting square commutes:

$$\begin{array}{ccc}
\varepsilon^{B-A} \oplus \xi(A) & \xrightarrow{id \oplus f(A)} & \xi(B) \\
\downarrow \varepsilon^{B-A} \oplus \eta(A) & & \downarrow \varepsilon^{B-A} \oplus \eta(A) \\
\eta(B) & & \eta(B)
\end{array}$$

The category of collared $\langle k \rangle$-vector bundles and collared maps is denoted by $\langle k \rangle\text{-Bun}^{\text{col}}$. The **rank** of a collared $\langle k \rangle$-vector bundle $\xi$ is defined to be the rank of the ordinary vector bundle $\xi(k)$.

**Examples.**

(i) The tangent bundle of a neatly embedded $\langle k \rangle$-manifold $M \subset \mathbb{R}^n_{\langle k \rangle}$.

(ii) For a vector bundle $\xi$ over $X$, there is a canonical extension to a collared $\langle k \rangle$-vector bundle, where $\xi(A)$ is universally characterized by requiring a bijection

$$\text{map}(\eta, \xi(A)) \cong \text{map}(\eta \oplus \varepsilon^{k-A}, \xi)$$

which is natural in the vector bundle $\eta$.

Now let $\theta$ be a collared $\langle k \rangle$-vector bundle of rank $d$. We define $\text{cob}_{\theta}$, the $\theta$-cobordism category of $\langle k \rangle$-manifolds as follows, for simplicity only in simplicial degree 0: A morphism is a morphism $W$ of $\text{cob}_{d,\langle k \rangle}$ together with a collared bundle map $l_W : TW \to \theta$. An object is an object $M$ of $\text{cob}_{d,\langle k \rangle}$ together with a collared bundle map $l_M : \varepsilon \oplus TM \to \theta$. The source/target of $(W, l_W)$ is $(\partial_1 W, l_{\partial_1 W})$, where we identify the tangent bundle of an interval with the trivial bundle $\varepsilon$ by sending $\partial/\partial x$ to 1.

For a (collared) $\langle k + 1 \rangle$-vector bundle $\theta$, we define:

(i) $\partial_1 \theta$ as the first face, viewed as a (collared) $\langle k \rangle$-bundle – that is, restriction along the embedding

$$j_1 : \mathcal{P}(k) \subset \mathcal{P}(k+1), \quad A \mapsto A + 1.$$
(ii) \( i_1^* \theta \) to be the (collared) \( \langle k \rangle \)-vector bundle where the first face is omitted – that is, restriction of the diagram along the embedding
\[
i_1: P(k) \to P(k+1), \quad A \mapsto A + 1 \cup \{1\}.
\]
Note that the dimension of \( \theta \) (that is, the dimension of the bundle at the terminal object), does not change under \( i_1^* \), but drops under \( \partial_1 \) by one.

**Theorem 4.5.** For any collared \( \langle k \rangle \)-bundle, the square
\[
\begin{array}{ccc}
\text{cob}_{i_1^* \theta} & \xrightarrow{i_1} & \text{cob}_\theta \\
\downarrow & & \downarrow_{\partial_1} \\
\text{cob} \theta & \xrightarrow{\partial_1} & \text{cob}_{\partial_1 \theta}
\end{array}
\]
realizes to a homotopy pull-back square. Since \( B\text{cob}_{\emptyset} \) is contractible, we obtain a fibration sequence
\[
B\text{cob}_{i_1^* \theta} \xrightarrow{i_1} B\text{cob}_\theta \xrightarrow{\partial_1} B\text{cob}_{\partial_1 \theta}.
\]

The proof is a modification of the proof of Theorem 4.1, and we will only show how put a \( \theta \)-structure on the \( \partial_1 \)-cocartesian morphism \( C \) constructed above from the datum of \( W \) and \( X \), provided that \( W \) comes equipped with a \( \partial_1 \theta \)-structure, and \( X \) with a \( \theta \)-structure. To this end, we first note:

**Remark 4.6.** There are compatible 1-to-1 correspondences:

(i) Between collared \( \langle k+1 \rangle \)-bundles \( \xi \), and maps of collared \( \langle k \rangle \)-bundles of the form \( \varepsilon \oplus \xi' \to \xi'' \), and

(ii) Between collared \( \langle k+1 \rangle \)-maps \( f: \xi \to \eta \) and commutative squares of collared \( \langle k \rangle \)-bundle maps
\[
\begin{array}{ccc}
\varepsilon \oplus \xi' & \xrightarrow{id \oplus f'} & \xi'' \\
\downarrow & & \downarrow_{f''} \\
\varepsilon \oplus \eta' & \xrightarrow{f} & \eta''
\end{array}
\]

In technical, and more precise terms, the Remark states that the category \( (k+1)\text{-Bun}_{\text{col}} \) is isomorphic to the category \( (\varepsilon \oplus -)/\langle k \rangle \text{-Bun}_{\text{col}} \).

**Proof.** Viewing the cube \( P(k+1) \) as a product \( P(k) \times [1] \), a \( \langle k+1 \rangle \)-bundle \( \xi \) may be viewed as a map of \( \langle k \rangle \)-bundle \( \partial_1 \xi \to i_1^* \xi \). Now we observe that a collar \( c \) on \( \xi \) determines and is determined by a compatible collection of maps
\[
c_{A - \{i\}, A}: \varepsilon \oplus \xi(A - \{i\}) \to \xi(A)
\]
for \( i \in A \subset k \). Such a collection may be viewed, in turn, as the datum of a collar both on \( \partial_1 \xi \) and on \( i_1^* \xi \), and a map
\[
\varepsilon \oplus \partial_1 \xi \to i_1^* \xi
\]
of collared \( \langle k \rangle \)-bundles. From this description it is clear that a map \( f: \xi \to \eta \) is collared if and only if the maps \( \partial_1 f \) and \( i_1^* f \) are collared, and the above square commutes. \( \square \)

Then, using Remark 4.6, the compatible tangential structures on \( X \) and \( W \) amount to the datum of two collared \( \langle k \rangle \)-bundle maps
\[
l'_X: (\partial/\partial x) \oplus i_1^* TX \to i_1^* \theta, \quad l_W: TW \to \partial_1 \theta,
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
\langle \partial / \partial x \rangle \oplus i^*_X TX & \xrightarrow{r'_X} & i^*_X \theta \\
\langle \partial / \partial x, \partial / \partial y \rangle \oplus TM & \xrightarrow{(\partial / \partial y) \oplus TW} & \langle \partial / \partial y \rangle \oplus TW \oplus \varepsilon \oplus \partial_1 \theta.
\end{array}
\]

Also, to endow \( C := B([0, a] \times Y) \) (notation from above) with a \( \theta \)-structure compatible with the given structures on \( X \) and \( W \), it is enough to construct a collared bundle map \( l^*_C : i^*_1 TC \rightarrow i^*_1 \theta \), extending \( l^*_X \) and such that the following square is commutative:

\[
\begin{array}{ccc}
i^*_1 TC & \xrightarrow{l^*_C} & i^*_1 \theta \\
\langle \partial / \partial y \rangle \oplus TW & \xrightarrow{id \oplus l_W} & \varepsilon \oplus \partial_1 \theta.
\end{array}
\]

We form the pushout

\[
\xi := \langle \partial / \partial x \rangle \oplus i^*_1 TX \cup \langle \partial / \partial y \rangle \oplus TW
\]

(union of subbundles of \( TR^{x+d}_{(k+2)} \)), which is a collared \( \langle k \rangle \)-bundle over the \( \langle k \rangle \)-space \( i^*_1(X) \cup W \), and comes with a collared \( \langle k \rangle \)-map \( L : \xi \rightarrow i^*_1 \theta \). Using this, we would like to define a \( i^*_1 \theta \)-structure on \( Y = B^{-1}(X \cup W) \) by precomposing \( L \) with the derivative of \( B \). This does not make sense however, as the derivative of the smooth map \( B \) away from \( p_0 = (0, 0) \) and \( p_1 = (0, a) \) does not extend to a bundle map on all of \( TR^{2}_{(2)} \). (Indeed \( DB \) is the identity on the \( x \)-axis, but rotation by \( \pi / 2 \) on the \( y \)-axis.)

To rectify this, choose \( \varepsilon > 0 \) so small that both \( W \) and \( X \) are \( \varepsilon \)-neatly embedded, and that the \( \varepsilon \)-neighborhoods around \( p_1 \) and \( p_2 \) in \( R^2_{(2)} \) are contained in the region where \( B \) is specified by property (iii) above. Then choose a continuous map

\[
R^2_{(2)} \rightarrow \mathbb{R}^2_{(2)} - \{p_0, p_1\}, \quad p \mapsto \tilde{p}
\]

which is the identity except in the \( \varepsilon \)-neighborhoods of \( p_0 \) and \( p_1 \), and define a map of ordinary vector bundles

\[
\tilde{D}B : TR^2_{(2)} \rightarrow TR^2_{(2)}, \quad (p, v) \mapsto (B(p), Dp(v)).
\]

Then

\[
\tilde{D}B := \tilde{D}B \times id : TR^{x+d}_{(k+2)} \rightarrow TR^{x+d}_{(k+2)}
\]

is a collared \( \langle k \rangle \)-bundle map (after forgetting the faces given by the equations \( x = 0 \) and \( y = 0 \)) and restricts to a collared \( \langle k \rangle \)-bundle map

\[
\tilde{D}B : \langle \partial / \partial x \rangle \oplus i^*_1 TY \rightarrow \xi.
\]

Then, we define \( l^*_C \) as the composite of the following collared bundle maps:

\[
i^*_1 TC \xrightarrow{DB^{-1}} i^*_1 T[a, b] \times Y \xrightarrow{proj} \langle \partial / \partial x \rangle \oplus i^*_1 TY \xrightarrow{\tilde{D}B} \xi \xrightarrow{L} i^*_1 \theta.
\]

This finishes the construction of the \( \theta \)-structure on \( C \). Applying \( (B, \tilde{D}B) \) induces a homeomorphism

\[
B : \text{cob}_\varepsilon^\theta(Y, Z)/V \rightarrow \text{cob}_\varepsilon^\theta(X, Z)/(V \circ W)
\]

where the addition upper index \( \varepsilon \) indicates that we only consider those morphisms that are \( \varepsilon \)-neatly embedded at the non-smooth points of \( \Phi \) – this change does not affect the homotopy type, provided \( \varepsilon \) is small enough. Then, the argument continues as above.
appendix

5. Isotopy Extension on \((k)\)-Manifolds

Recall that an \(n\)-dimensional (smooth) manifold is a second countable paracompact topological space equipped with an atlas of local homeomorphisms to open subsets of \(\mathbb{R}^n = [0, \infty)^n\), such that the change of charts is smooth. (This means that our definition of manifold allows corners.) A submanifold of \(M\) is a subset \(N \subset M\) which, locally in a suitable chart, looks like \(\mathbb{R}^n \times \{1\} \subset \mathbb{R}^n\).

All manifolds in this appendix will be smooth. One major example of a manifold with corners is \(\mathbb{R}^n_{(k)} := \mathbb{R}^{n-k} \times \mathbb{R}^k_+\). We note that this comes with the extra structure of a \((k+1)\)-ad in the sense of Wall: That is, its boundary (as a topological manifold) is partitioned into the \(k\) many subspaces
\[
\partial_i \mathbb{R}^n_{(k)} := \{ x \in \mathbb{R}^n_{(k)} \mid x_{n-k+i} = 0 \} \quad (i \in \{1, \ldots, k\}).
\]
(This \((k+1)\)-ad structure is compatible with the manifold structure in a suitable way, giving \(\mathbb{R}^n_{(k)}\) the structure of a \((k)\)-manifold, see [6] or [4]). We denote as usual, for \(A \subset k = \{1, \ldots, k\}\), the value of the \((k+1)\)-ad \(\mathbb{R}^n_{(k)}\) at \(A\), given by
\[
\mathbb{R}^n_{(k)}(A) := \bigcap_{i \notin A} \partial_i \mathbb{R}^n_{(k)} = \{ x \in \mathbb{R}^n_{(k)} \mid \forall i \notin A x_{n-k+i} = 0 \}.
\]
We always identify \(\mathbb{R}^n_{(k)}\) with \(\mathbb{R}^n_{(k)}(A) \times [0, \infty)^{k-A}\) by writing the coordinates \(x_{n-k+i}, i \notin A\), into the second factor. With this identification, we have canonical collar embeddings
\[
c_{AB} : \mathbb{R}^n_{(k)}(A) \times [0, \varepsilon)^{k-A} \to \mathbb{R}^n_{(k)}(B) \times [0, \varepsilon)^{k-B},
\]
for \(A \subset B \subset k\), and any \(\varepsilon > 0\).

The only manifolds that we will consider are submanifolds \(M \subset \mathbb{R}^n_{(k)}\). These inherit the structure of a \((k+1)\)-ad by means of
\[
M(A) := M \cap \mathbb{R}^n_{(k)}(A).
\]
(In fact, they inherit the structure of a \((k)\)-manifold; conversely any \((k)\)-manifold is isomorphic, as manifold and as \((k+1)\)-ad, to a submanifold \(M \subset \mathbb{R}^n_{(k)}\).)

Furthermore we will restrict our attention to submanifolds which are neat: For all \(A \subset B \subset k\), we require that there is an \(\varepsilon > 0\) such that
\[
M(B) \cap (\mathbb{R}^n_{(k)}(A) \times [0, \varepsilon)^{k-A}) = M(A) \times [0, \varepsilon)^{k-A}.
\]
If we want to specify the value of \(\varepsilon\), we also speak of an \(\varepsilon\)-neat submanifold. An \(\varepsilon\)-neat smooth submanifold \(M \subset \mathbb{R}^n_{(k)}\) inherits extra structure of collar embeddings
\[
c_{AB} : M(A) \times [0, \varepsilon)^{k-A} \to M(B) \times [0, \varepsilon)^{k-B},
\]
by restricting the collar embeddings on \(\mathbb{R}^n_{(k)}\).

Let \(M, N \subset \mathbb{R}^n_{(k)}\) be neat submanifolds. By definition, an \(\varepsilon\)-neat embedding, resp. \(\varepsilon\)-neat diffeomorphism \(\epsilon : M \to N\) is a map which is both an allowable map of \((k+1)\)-ads (that is, for all \(A \subset k\), we have \(\epsilon^{-1}(N(A)) = M(A)\)) and an embedding (resp. diffeomorphism) of manifolds, which is cylindrical at the collars in the sense that for each \(A \subset B \subset k\), the following square commutes:
\[
\begin{array}{ccc}
M(A) \times [0, \varepsilon)^{k-A} & \xrightarrow{c_{AB}} & M(B) \times [0, \varepsilon)^{k-B} \\
\downarrow{\epsilon \times \text{id}} & & \downarrow{\epsilon \times \text{id}} \\
N(A) \times [0, \varepsilon)^{k-A} & \xrightarrow{c_{AB}} & N(B) \times [0, \varepsilon)^{k-A}
\end{array}
\]
We will now study family versions of isotopy extension theorems for neat submanifolds of \( \mathbb{R}^n_{(k)} \). To this end, we let

\[
\text{Emb}^\varepsilon(M, \mathbb{R}^n_{(k)}) \quad \text{and} \quad \text{Aut}^\varepsilon(M)
\]
denote the spaces of \( \varepsilon \)-neat embeddings \( M \to \mathbb{R}^n_{(k)} \) and \( \varepsilon \)-neat automorphisms \( M \to M \), equipped with the strong \( C^r \)-topology \( \| \cdot \| \leq 1, r \geq 1 \). We then define

\[
\text{Emb}(M, \mathbb{R}^n_{(k)}) = \lim_{\varepsilon} \text{Emb}^\varepsilon(M, \mathbb{R}^n_{(k)}) \quad \text{and} \quad \text{Aut}(M) = \lim_{\varepsilon} \text{Aut}^\varepsilon(M)
\]
equipped with the colimit topology.

We are interested in restricting an embedding \( M \to \mathbb{R}^n_{(k)} \) to an embedding \( M(A) \to \mathbb{R}^n_{(k)}(A) \) for \( A \subset \mathbb{R}^n_{(k)} \) or to an embedding \( \partial M \to \partial \mathbb{R}^n_{(k)} \). As \( \partial M = \bigcup_{\varepsilon \in \mathbb{R}} M(A) \), we define

\[
\text{Emb}^\varepsilon(\partial M, \partial \mathbb{R}^n_{(k)}) := \lim_{\varepsilon} \text{Emb}^\varepsilon(M(A), \mathbb{R}^n_{(k)}(A)),
\]
and, more generally, for a subcomplex \( X \subset \Delta^k \) (that is, a subset \( X \subset P(k) \) which is closed under taking subsets),

\[
\text{Emb}^\varepsilon(M(X), \mathbb{R}^n_{(k)}(X)) := \lim_{\varepsilon} \text{Emb}^\varepsilon(M(A), \mathbb{R}^n_{(k)}(A)).
\]

Note that

\[
\text{Emb}^\varepsilon(M(A), \mathbb{R}^n_{(k)}(A)) = \text{Emb}^\varepsilon(M(\sigma A), \mathbb{R}^n_{(k)}(\sigma A))
\]
with \( \sigma A := \{ B \subset \mathbb{R}^n_{(k)} | B \subset A \} \) the simplex spanned by \( A \), so the first case is also included in this notation. Again we let

\[
\text{Emb}(M(X), \mathbb{R}^n_{(k)}(X)) := \lim_{\varepsilon} \text{Emb}^\varepsilon(M(X), \mathbb{R}^n_{(k)}(X)), \quad \text{Aut}(M(X)) := \lim_{\varepsilon} \text{Aut}^\varepsilon(M(X)).
\]

**Theorem 5.1.** Let \( M \subset \mathbb{R}^n_{(k)} \) be a compact neat submanifold. Let further \( Y \subset X \subset \Delta^k \) be subcomplexes. In the square formed by restriction maps

\[
\begin{array}{ccc}
\text{Aut}(\mathbb{R}^n_{(k)}(X)) & \longrightarrow & \text{Emb}(M(X), \mathbb{R}^n_{(k)}(X)) \\
\downarrow & & \downarrow \\
\text{Aut}(\mathbb{R}^n_{(k)}(Y)) & \longrightarrow & \text{Emb}(M(Y), \mathbb{R}^n_{(k)}(Y))
\end{array}
\]

all maps are Serre fibrations, as well as the map from the left corner to the pull-back of the remaining diagram.

**Proof.** Step 1. We show that the left vertical map admits a local section

\[
s : \text{Aut}(\mathbb{R}^n_{(k)}(Y)) \supset U \to \text{Aut}(\mathbb{R}^n_{(k)}(X))
\]
on an open neighborhood \( U \) of the neutral element, so that the map under consideration is locally trivial. We first construct, for any \( i \in \mathbb{R}^n_{(k)} \), a local section of the map \( \text{Aut}(\mathbb{R}^n_{(k)}(Y)) \to \text{Aut}(\partial_i \mathbb{R}^n_{(k)}) \).

Choose some smooth map

\[
\alpha : [0, \infty) \to [0, 1]
\]
which is identically 0 near 0 and identically 1 on \([1, \infty)\), and consider the map

\[
\hat{s}_i : \text{Aut}(\partial_i \mathbb{R}^n_{(k)}) \to C^r(\mathbb{R}^n_{(k)}, \mathbb{R}^n_{(k)})
\]
defined by

\[
\hat{s}_i(\phi)(x, t) := \alpha(t) \cdot x + (1 - \alpha(t)) \cdot \phi(x)
\]
where we identify \( \mathbb{R}^n_{(k)} \) with \( \partial_i \mathbb{R}^n_{(k)} \times [0, \infty) \). Since \( \text{Aut}(\mathbb{R}^n_{(k)}) \) is open in the space of all allowable \( C^r \)-selfmaps of \( \mathbb{R}^n_{(k)} \) \([1, 1.1.4.2]\), the map \( \hat{s}_i \) restricts to a local section \( s_i \) as required.
Now we show that the map $\alpha: \text{Aut}(\mathbb{R}_n^k) \to \text{Aut}(\partial \mathbb{R}_n^k)$ also has a local section. To this end, we note that if some $\phi \in \text{Aut}(\partial \mathbb{R}_n^k)$ had the property that it restricts to the identity on some other face $\partial_i \mathbb{R}_n^k$, then so does the map $s_i(\phi)$ constructed above. Therefore, if we locally define maps

$$\sigma_i: \text{Aut}(\partial \mathbb{R}_n^k) \supset U_i \to \text{Aut}(\mathbb{R}_n^k), \quad i \in \mathbb{Z}$$

iteratively by

$$\sigma_1 := s_1, \quad \sigma_{i+1}(\phi) := \sigma_i(\phi) \cdot s_{i+1}(\sigma_i(\phi)^{-1} \cdot \phi),$$

then we iteratively see that $\sigma_i(\phi)$ restricts to $\phi$ over the first $i$ faces, so that $\sigma_k$ defines a local section as required.

Since, for any $A \subset \mathbb{Z}$, we have $\mathbb{R}_n^k(A) = \mathbb{R}_n^{n-k(|A|)}$, the above shows that the restriction map $\text{Aut}(\mathbb{R}_n^k(A)) = \text{Aut}(\mathbb{R}_n^k(1 \cdot A)) \to \text{Aut}(\mathbb{R}_n^k(\partial A))$ has a local section. But now, in the general case, the inclusion $X \to Y$ may be factored into inclusions each of which fills in precisely one simplex, and therefore has a local section. Composing these local sections defines a local section $s$ as required in the general case.

**Step 2.** We show that the map from the left upper corner to the pull-back of the remaining diagram admits a local section. As in step 1, one reduces to the case where $X$ is the simplex spanned by $\mathbb{Z}$, and $Y$ its boundary; so we need to show that the map

$$\text{Aut}(\mathbb{R}_n^k) \to \text{Emb}(M, \mathbb{R}_n^k) \times_{\text{Emb}(\partial M, \partial \mathbb{R}_n^k)} \text{Aut}(\partial \mathbb{R}_n^k)$$

has a local section. Using the local section from step 1, one reduces to showing that the restriction map

$$\text{Aut}(\mathbb{R}_n^k; \partial) \to \text{Emb}(M, \mathbb{R}_n^k; \partial)$$

on diffeomorphism group and embedding space relative boundary has a local section. As the behaviour near the boundary is standard, we can smooth corners out. But then the argument from [7] guarantees the existence of a local section.

**Step 3.** Taking $Y = \emptyset$ in Step 2, we conclude that the upper (hence also the lower) horizontal map is a Serre fibration; as well as the diagonal map. But then it easily follows that the right vertical map is also a Serre fibration. $\square$

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