Electroweakly interacting scalar and gauge bosons, and leptons, from field equations on spin 5+1 dimensional space

J. Besprosvany

Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México 01000, D. F., México

Abstract

Unification ideas motivate the formulation of field equations on an extended spin space. Demanding that the Poincaré symmetry be maintained, one derives scalar symmetries that are associated with flavor and gauge groups. Boson and fermion solutions are obtained with a fixed representation. A field theory can be equivalently written and interpreted in terms of elements of such space and is similarly constrained. At 5+1 dimensions, one obtains isospin and hypercharge $SU(2)_L \times U(1)$ symmetries, their vector carriers, two-flavor charged and chargeless leptons, and scalar particles. Mass terms produce breaking of the symmetry to an electromagnetic $U(1)$, a Weinberg’s angle with $\sin^2(\theta_W) = .25$, and additional information on the respective coupling constants. Their underlying spin symmetry gives information on the particles’ masses; one reproduces the standard-model ratio $M_Z/M_W$, and predicts a Higgs mass of $M_H \approx 114$ GeV, at tree level.

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1 Introduction

Although the accepted theory of elementary particles, the standard model (SM), is successful in describing their behavior, it is phenomenological; a justification for its input is still required at a more fundamental level, as evidenced by the large number of experimental parameters it needs. We also lack information on the origin of the particular groups $SU(3) \times SU(2)_L \times U(1)$ which define the interactions, and on why the isospin acts only on a given chirality. Neither is it clear the origin of the spectrum of fermions, the three generations, the masses and mixing angles, and the gauge-group fundamental representations in which they appear. The Higgs particle is a useful mathematical device but we lack a more basic reason for its presence.

The idea of unification has proved to be a powerful tool in physical research, finding links between originally assumed separated phenomena, and also predicting new ones. For example, by assuming the SM interactions have a shared origin in the same unified group[1] some clues can be obtained on the fermion representations in the SM and also relations among the coupling constants.

On another plane, the Kaluza-Klein idea, originally used in the framework of general relativity, has been applied to understand interactions by relating them to additional spatial dimensions. Other unification ideas may shed more light on particle-physics problems. Spin is a physical manifestation of the fundamental representation of the Lorentz group and it is more so in relation to space, which uses the vector representation. While both the spin and configuration spaces are equally necessary elements for the description of particles, the association of interactions to extended spin spaces is feasible, and gives rise to a model with new connections to the SM[2],[3].

In this paper we give a field-theory formulation of the model, providing it with additional fundament, and we further explore the 5+1 dimensional case. We review the formulation of Poincaré-invariant field equations on an extended 5+1 d spin space,
and the link of its symmetries and solutions to the SM electroweak sector. We also show that a field theory can be equivalently formulated in such a space and is similarly constrained. Thus, the demand that fields be in a 5+1 dimensional spin space determines the interactions and representations of the SM electroweak sector, and additional constraints. In particular, we extract information on the particle masses, using the underlying spin symmetry, and making a comparison with the SM.

We first consider an extended Dirac equation and develop its surrounding formalism, using the conventional relativistic quantum-mechanical framework (Section 2). We show that boson (Section 3) and fermion (Section 4) solutions are obtained that share the same spin solution space. The simplest extension of the equation at 5+1 dimensions predicts an $SU(2)_L \times U(1)$ symmetry. We obtain solutions with quantum numbers of the corresponding gauge fields, leptons, and scalars interacting electroweakly (Section 5). Typical Lagrangians are equivalently written in terms of the above degrees of freedom, with the dimension determining also the representations and the interactions (Section 6). A mass term set with an underlying spin symmetry predicts boson masses as in the SM and the other particles’ (Section 7). We also obtain information on the electroweak SM coupling constants (Section 8).

2 Generalized Dirac equation

We depart from the equation

\[ \gamma_0 (i \partial_\mu \gamma^\mu - M) \Psi = 0, \]  

where, with the aim of jointly describing bosons and fermions, we now assume $\Psi$ represents a matrix. Eq. \[ \] contains four conditions over four spinors in a $4 \times 4$ matrix. There are, then, additional possible ones to further classify $\Psi$. The transformations and symmetry operations on the Dirac operator $(i \partial_\mu \gamma^\mu - M) \rightarrow U(i \partial_\mu \gamma^\mu - M) U^{-1}$
induce the lhs of the transformation

\[ \Psi \rightarrow U \Psi U^\dagger. \]  (2)

Also, \( \Psi \) is postulated to transform as indicated on the rhs. The latter transformation is consistent with the additional equation

\[ \Psi \gamma_0 (\gamma^\mu \partial_\mu - M) = 0 \]  (3)

(the Dirac operator transforming accordingly). In fact, the conjugated fields \( \Psi^\dagger \), for which Eq. 2 is valid, satisfy this equation. It is by taking also account of this kind of fields that we can span the function space within the 32-dimensional complex \( 4 \times 4 \) matrices. Then, we shall extend our space of solutions by considering also combinations of fields \( A + B^\dagger \).

\( \Psi \) can be understood to be constructed of a tensor product of the usual configuration (or momentum) space, and a column \( |w_i\rangle \) and a row \( \langle w_j| \) state that can have a spinor interpretation, with expansion \[ \sum_{ij} a_{ij} |w_i\rangle \langle w_j| \]. The resulting tensor product of operators acting on this tensor-product space and classifying the solutions consist generators of the Poincaré algebra with spin components, (or scalars) acting as \( U \) on each side of \( \Psi \), and derivative ones acting only once, from either side; products of derivative and spin operators act from both sides. The expansion of \( \Psi \) implies it Lorentz transforms as a scalar, a vector, or an antisymmetric tensor. In fact, we will show below certain choices of symmetry operators allow for a fermion interpretation of some solutions too.

If \( A, B \) are solutions, the matrix product \( C = AB \), defines an algebra whose elements may or may not be solutions. The inner product of \( A, B \) is naturally defined by

\[ \langle A|B \rangle = tr(A^\dagger B), \]  (4)

\(^1\)The Bargmann-Wigner equations contain Eq. 1 but set the different second condition as \( \Psi_{BW}^\ast \gamma_0 (-i \partial_\mu \gamma^\mu - M) = 0 \). \( \Psi_{BW}^\ast \) contains charge conjugated \( |w_j| \) components (see below).
as expected for this extended tensor-product space. A trace over the coordinates is also implied.

We are interested in plane-wave solutions of the form

$$\Psi_{ki}^{(+)}(x) = u_i(k)e^{-ikx} \quad \text{(5)}$$

$$\Psi_{ki}^{(-)}(x) = v_i(k)e^{ikx}, \quad \text{(6)}$$

where $k^\mu$ is the momentum four-vector $(E, k)$, $k_0 = E$, and $u_i, v_i$ are matrices containing information on the polarization. As applied to $\Psi_{ki}^{(+)}(x)$ in Eq. (5), Eq. (4) defines the Hamiltonian $H_{lhs} = \gamma_0(k \cdot \gamma + M)$ and a projection of the Pauli-Lubansky vector on the four-vector $n_k = (1/M)(|k|, E\hat{k})$, $W \cdot n_k = \Sigma \cdot \hat{k}$, which uses

$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}, \quad \text{(7)}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and the spin operator $\Sigma = \frac{1}{2}\gamma_5\gamma_0\gamma$, which is valid both for the massless and the massive cases. However, $\Psi_{ki}^{(+)}(x)$ is classified by the energy-momentum symmetry operators corresponding to Eq. (3) (always in the relativistic quantum-mechanics framework) as a negative-energy solution. A consistent characterization of the latter can be given if we assume its rhs $\langle w_j |$ spinor component to be a hole. This interpretation is set for operators acting on $\Psi$ from the rhs, as well as on the $\Psi^\dagger$ fields. Thus, the hermitian conjugate of $\Psi_{ki}^{(-)}(x)$ in Eq. (6) has the same exponential dependence as $\Psi_{ki}^{(+)}(x)$ in Eq. (5), and is also interpreted as a positive-energy solution.

## 3 Boson solutions

On Table 1 are shown solutions of the massless equation bilinear in the $\gamma$s, and they are given together with their quantum numbers. We give them in terms of their polarizations, setting the coordinate dependence as in Eq. (5), and we assume
the spatial component of \( k^\mu \) to be along \( \hat{z} \). Here and throughout the solutions are normalized as \( tr(u_i^\dagger(k)u_i(k)) = 1 \). In the first two columns, we present the eigenvalues \( \lambda \) of the operators \( O \) in the form \( O u_i(k) = \lambda u_i(k) \).

The solutions on Table 1 are interpreted as vector bosons, for Eq. 1 alone implies they satisfy the Klein-Gordon equation and \( \Sigma \cdot \hat{k} \) classifies their different degenerate polarization components as vectors. According to the direct-product operator convention, the helicity operator is calculated in the fourth column with \( \Sigma \cdot \hat{k}_{lhs} \Psi + \Psi \Sigma \cdot \hat{k}_{rhs} \), where \( \Sigma \cdot \hat{k}_{rhs} \), and its (rhs) eigenvalue are unchanged under the hole interpretation. The use of a commutator follows from the equality \( \Sigma \cdot \hat{k}_{lhs} \Psi + \Psi \Sigma \cdot \hat{k}_{rhs} = \Sigma \cdot \hat{k}_{lhs} \Psi - \Psi \Sigma \cdot \hat{k}_{lhs} = [\Sigma \cdot \hat{k}, \Psi] \). The third column characterizes the solutions with the application of \( H_{lhs} \Psi + \Psi H_{rhs} \), with \( H_{rhs} \) coming from Eq. 3. Although the rhs spinor of \( u_{-1}(k) \) is not on-shell, as measured by \( H_{rhs} \), the latter gives the expected energy sign in the hole interpretation. Within these rules, \( u_{-1}(k), u_{-1}(\tilde{k}) \) are on-shell particles with transverse polarizations, with helicity \(-1\), propagating respectively in the \( \hat{z} \) and \(-\hat{z} \) directions; the latter is denoted through the four-vector \( \tilde{k}^\mu = k_\mu; \ u_0(k), u_0(\tilde{k}) \) are off-shell and polarized in the longitudinal-scalar directions. The \( u_i \) solutions do not represent independent polarization components as, e.g., \( u_i(\tilde{k}) \) can be obtained by rotating the \( u_i(k) \) (applying a Lorentz transformation of the form \( \mathcal{L} \)).

In the massless case, we have also negative-energy solutions \( v_i(k) = u_i(k) \) (and \( \tilde{k} \) terms) from Eq. 3, that is, with opposite helicities. The positive-energy solutions containing \( v^\dagger(k), v^\dagger(\tilde{k}) \) have also opposite helicities. By applying the parity operator \( \gamma_0 \varphi \), with \( \varphi x_\mu = \tilde{x}_\mu \), in the form \( \mathcal{L} \) one classifies the solutions according to the weight of the vector \( V \) and axial \( A \) components. The solutions in Table 1 are \( V - A \), while \( V + A \) solutions are also obtained. Their combination,

\[
A_\mu(x) = g_{\mu\nu}(x) \frac{i}{2} \gamma_0 \gamma^\nu,
\]

with \( g_{\mu\nu}(x) \) a polarization tensor, transforms into \( A^\mu(\tilde{x}) \) under parity \( P \), that is,
as a vector. An axial term can be also obtained. A combination of $A_\mu(x)$ and its hermitian conjugate can be shown to transform as a vector under the charge conjugation operator $C = i\gamma_2 K$, with $K_i = -iK_i$; given its quantum numbers, it becomes then possible to relate $A_\mu(x)$ to the vector potential of an electromagnetic field satisfying Maxwell’s equations within the Lorentz gauge. Actually, we may also view $\frac{i}{2}\gamma_0\gamma_\mu$ as an orthonormal polarization basis, $A_\mu = \text{tr} \frac{i}{2}\gamma_\mu A^\nu \frac{i}{2}\gamma_\nu$ just as $n_\mu$ in $A_\mu = g_{\mu\nu} A^\nu = n_\mu \cdot A^\nu n_\nu$. In fact, the sum of Eqs. of 1 and 3 implies for a $\Psi$ containing $\gamma_0 A = A^\mu \gamma_0 \gamma_\mu$ that $A_\mu$ satisfies the free Maxwell’s equations[4]. Both interpretations for the polarization components as basis for solutions in an extended Dirac equation, and for standard fields shall be considered in this work.

The remaining eight degrees of freedom in the massless case contain six forming an antisymmetric tensor $A_{\mu\nu} = \frac{1}{4}\gamma_0[\gamma_\mu, \gamma_\nu]$, and scalar, and pseudoscalar terms $\phi = \frac{1}{2}\gamma_0$, $\phi_5 = \frac{1}{2}\gamma_5\gamma_0$.

4 Fermion solutions

The chirality invariance allows for more choices of Lorentz generators. Using $J_{\mu\nu}^- = \frac{1}{2}(1 - \gamma_5)J_{\mu\nu}$ in $\Sigma$ to classify the solutions in Table 1, they remain $V - A$. However, when applying $J_{\mu\nu}^-$ to the other solutions of the form $(1 \pm \gamma_5)\gamma$ it leads to one of the sides (either $|w_i\rangle$ or $\langle w_j|)$ to transform trivially, and therefore, to spin-1/2 objects transforming as the $(1/2, 0)$ or $(0, 1/2)$ representations of the Lorentz group. Having obtained both fermion and boson solutions constitutes progress in the task of giving a unified description of these fields. Clearly, their nature depends on the Hamiltonian and on the set of valid symmetry transformations, chosen among limited options. But once the choice is made, there is no ambiguity.

As for $\bar{\psi} = \psi^1 \gamma_0$, a unitary transformation can be applied to the fields and operators to convert them to a covariant form.
The equation
\[ i(1 - \gamma_5)\gamma_0 \partial_{\mu} \gamma^\mu \Psi = 0 \] (9)
has all these fields as solutions. If we stick to \( J_{\mu\nu} \) to classify them, the invariance algebra of the equation has as additional symmetry the group of linear complex transformations \( G(2, C) \) with eight components, generated by \((1/2)(1 + \gamma_5)\) and \( f_{\mu\nu} = -(i/2)(1 + \gamma_5)\sigma_{\mu\nu} \). The unitary subgroups \( SU(2) \times U(1) \) of \( G(2, C) \) imply two additional quantum numbers we can assign to the solutions. In consideration that this symmetry does not act on the vector solution part, and taking account of the known quantum numbers of fermions in nature, we shall associate these operators with the flavor and spin-1/2 number operators, respectively. The \( SU(2) \) set of operators leads to a flavor doublet. The \( U(1) \) is in this case not independent from the chirality. Choosing \( f_{30} \) to classify the solutions of Eq. 9, these are given in Table 2 (as \( H \) and \( \Sigma \cdot \hat{p} \) act trivially from the rhs, commutators are not needed.) Overall, the matrix objects we use allow for a fermion interpretation, for one of its components behaves as a spinor, while the other carries the flavor quantum number.
Left-handed spin 1/2 particles  $\frac{1}{2}(1 - \gamma_5)\gamma_0\gamma^3$  $\frac{i}{4}(1 - \gamma_5)\gamma_1\gamma_2$  [f_{30}]

\[
\begin{align*}
w_{-1/2}(k) &= \frac{1}{4}(1 - \gamma_5)(\gamma_0 + \gamma_3) & 1 & -1/2 & 1/2 \\
w_{-1/2}(\tilde{k}) &= \frac{1}{4}(1 - \gamma_5)(\gamma_1 + i\gamma_2) & -1 & 1/2 & 1/2 \\
\hat{w}_{-1/2}(k) &= \frac{1}{4}(1 - \gamma_5)(\gamma_1 - i\gamma_2) & 1 & -1/2 & -1/2 \\
\hat{w}_{-1/2}(\tilde{k}) &= \frac{1}{4}(1 - \gamma_5)(\gamma_0 - \gamma_3) & -1 & 1/2 & -1/2 
\end{align*}
\]

Table 2: Massless fermions.

5 5+1 Dimensional Extension

The simplest generalization of the above model is to consider the six-dimensional Clifford algebra, (the $d = 5$ lives also in a $4 \times 4$ space), composed of 64 $8 \times 8$ matrices. To describe it we use the quaternion-like objects $1$, $I$, $J$, $K$ and now generalized $8 \times 8$ matrices $\gamma_\mu$ (and $\gamma_5$). A 6-d Clifford algebra $\{\gamma'_\mu, \gamma'_\nu\} = g_{\mu\nu}$ can be formed with the 4-d elements $\gamma'_\mu = \gamma_\mu$, $\mu = 0, 1, 3$, $\gamma'_2 = I\gamma_2$, and the 4-d scalars $\gamma'_5 = J\gamma_2$, $\gamma'_6 = K\gamma_2$. Altogether, the scalars (and pseudo-) are $1$, $I$, $J\gamma_2$, $\gamma_5$, $I\gamma_5$, $J\gamma_2\gamma_5$, $K\gamma_2\gamma_5$. From these, $I\gamma_5$ commutes with the rest. Excluding it and the identity, the remaining six elements generate an $SO(4)$ algebra, or equivalently, an $SU(2) \times SU(2)$ algebra. The eight scalars have a Cartan algebra of dimension four, for which we can take the basis $1$, $I$, $\gamma_5$, $I\gamma_5$.

We obtain the useful prediction of a limited number of scalar symmetries and representations consistent with the 4-d Lorentz symmetry. We may choose among them the closest to reproduce aspects of the SM. The only projector operator $L$ that can be constructed with the latter operators, describes both fermions and bosons through the Lorentz generator $J_L^{\mu\nu} = LJ'_{\mu\nu}$, with $J'_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma'_{\mu\nu}$, $\sigma'_{\mu\nu} = \frac{i}{2}[\gamma'_\mu, \gamma'_\nu]$, allows for a non-abelian group symmetry, and permits parity to be a good
quantum number for some solutions is \( L = \frac{3}{4} - \frac{1}{4}(I + \gamma_5 + I \gamma_5) = 1 - \frac{1}{4}(1 + I)(1 + I \gamma_5) \); the latter can be interpreted as the lepton number. The equation

\[ iL\gamma_0 \partial^\mu \gamma'_\mu \Psi = 0, \quad \mu = 0, \ldots, 3, \quad (10) \]

is invariant under \( J^L_{\mu \nu} \), and allows for a flavor symmetry as defined previously. It is also invariant under the \( L \)-commuting scalars: the hypercharge \( Y = -1 + 1/2(I + \gamma_5) \), and the isospin \( SU(2)_L \) with generators

\begin{align*}
I_1 &= \frac{i}{4}(1 - I\gamma_5)J_2, \\
I_2 &= -\frac{i}{4}(1 - I\gamma_5)K_2, \\
I_3 &= -\frac{1}{4}(1 - I\gamma_5)I;
\end{align*}

(11)

(12)

(13)

this identification of \( Y \) and \( I_i \) comes from their commutation relations, and from the correct vertices and quantum numbers they determine on the bosons and leptons, as argued below. \( Y \) can also be deduced from other requirements related to gauge symmetry.

These scalars lead, as expected, to a restricted set of solutions. The massless positive solutions are shown schematically on Table 3 (see Ref. 2 for details) and they include \( V - A \) and \( V + A \) vectors, with components \( B_\mu = \frac{1}{4y_2} (1 - I)(1 + I \gamma_5)\gamma_0 \gamma_\mu \), \( \tilde{B}_\mu = \frac{1}{4}(1 - I\gamma_5)\gamma_0 \gamma_\mu \), which amount to eight degrees of freedom (df); isospin \( V - A \) vectors \( W^i_\mu = I_i \tilde{B}_\mu \) with twelve df; also left and right-handed antisymmetric tensors and scalars \( n_{L,R}, v_{L,R} \) in an isospin doublet, amounting to eight df, and with their antiparticles sixteen df. These add up to thirty-six bosons. We also obtain spin-1/2 left-handed particles in an isospin doublet \((\nu, l^-)_L\) with \( Y = -1 \) and a right-handed isospin singlet \( l_R \) with \( Y = -2 \); for example

\[ l_R^k = \frac{1}{8}(1 + I\gamma_5)(J_2 \gamma_2 - iK_2)\gamma_0 (\gamma_1 + iI\gamma_2) \]

(14)

is a right-handed spin component with these quantum numbers and given flavor. Taking account of antiparticles and the two flavors, we have twenty-four fermion df.
The reason for not having altogether sixty-four active df is the four inert df, defining the flavor, projected by \( 1 - L \), and which are not connected to the Hamiltonian (top-left matrix on Table 3).

\[
\begin{array}{cccc}
 e^+ & \bar{\nu}_R & l^+_R \\
 e^- & \tilde{B}_\mu & \tilde{n}_R & \tilde{\nu}_R \\
 \nu_L & n_L & B_\mu, W^0_\mu & W^1_\mu, W^2_\mu \\
 l^-_L & \nu_L & W^1_\mu, W^2_\mu & B_\mu, W^0_\mu \\
\end{array}
\]

Table 3: Arrangement of solutions in a \( 6 - d \) \( 8 \times 8 \) matrix model, with each box occupying a \( 2 \times 2 \) matrix.

In seeking a massive extension of Eq. \([10]\), we expect all the hermitian combinations of the scalar terms \( M_i \gamma_0 \) to be scalars with respect to \( J'_{\mu\nu} \). However, if we also demand that they be scalars with respect to \( J^L_{\mu\nu} \), then the choices are reduced to \( M_1 = (M/2)(1 - I) \), \( M_2 = i(M/2)(\gamma_5 - I\gamma_5) \), \( M_3 = -(M/2)J\gamma_2(1 + \gamma_5) \), \( M_4 = (M/2)K\gamma_2(1 + \gamma_5) \), where \( M \) is the mass constant. Now, the only non-trivial scalar that commutes with all \( M_i \gamma_0 \) terms is \( L \). Nevertheless, if we relax this condition we obtain in addition that only

\[
Q = I_3 + \frac{1}{2}Y^\prime
\]

commutes with \( M_3 \gamma_0 \) and \( M_4 \gamma_0 \). As \( Q \) is the electric charge we deduce the electromagnetic \( U(1)_{em} \) remains a symmetry while the hypercharge and isospin are broken. We stress that \( Q \) is deduced, rather than being imposed, as the only additional symmetry consistent with massive terms.
6 Interactive field theory and physical fields

We argue below that an interactive field theory can be constructed for fields belonging to an extended spin space. Although we refer to 5+1 d, this applies to other dimensions as well. As in Section 5, the spin-space dimension determines the representations, and we shall also find constraints on their possible interactions.

The expression for the kinetic-component Lagrangian density of a non-abelian gauge-invariant vector field

$$\mathcal{L}_V = -\frac{1}{4} F^a_{\mu \lambda} \gamma^\mu \delta_{ab} F^{b \eta}_{\eta} = -\frac{1}{4 N_o} \text{tr} \mathcal{P}_D F^a_{\mu \lambda} \gamma^\mu \gamma^\lambda G_a F^{b \mu}_{\eta} \gamma^\mu \gamma^\eta G_b$$

(16)

shows $\mathcal{L}_V$ is equivalent to a trace over combinations over normalized components $\frac{1}{\sqrt{N_o}} \gamma^\mu \gamma^\mu G_a$, $\mu = 0, ...3$, with coefficients $F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g A^b_\mu A^c_\nu C_{bc}^a$, $g$ the coupling constant, $C_{bc}^a$ the structure constants, and $\mathcal{P}_D$ a Lorentz-scalar projection operator, defining the solution space and Poincaré algebra in 5+1 d. Limited choices exist for $\mathcal{P}_D$, and the restrictions in Section 5 imply $\mathcal{P}_D = L$, e.g., $G_a = I_a$ are 8 $\times$ 8 matrices, and $N_o g^\mu_\nu \delta_{ab} = \text{tr} \gamma^\mu \gamma^\nu G_a \gamma^\mu \gamma^\nu G_b = g^\mu_\nu \text{tr} G_a G_b$, where for non-abelian irreducible representations we use $\text{tr} G_i G_j = 2 \delta_{ij}$.

Similarly, the interactive part of the fermion gauge-invariant Lagrangian

$$\mathcal{L}_f = \frac{1}{2} \psi^\dagger \gamma^\mu (i \partial_\mu - g A^a_\mu G_a) \gamma^\mu \psi^\alpha,$$

(17)

with $\psi^\alpha$ a massless spinor with flavor $\alpha$, can be written

$$\mathcal{L}_{int} = -g \frac{1}{2 N_o} \text{tr} L A^a_\mu \gamma^\mu G_a j^{b \nu}_\lambda \gamma^\mu \gamma^\nu G_b,$$

(18)

with $j^{a \alpha}_\mu = \text{tr} \Psi^\dagger \gamma^\mu \gamma^\mu G_a \Psi^\alpha$ containing $\Psi^\alpha = \psi^\alpha \langle \alpha |$, and $\langle \alpha |$ is a row state accounting for the flavor. Similar matches between standard Lagrangians and their expressions in terms of fields in an extended spin space for components with scalar fields, and their interaction with the vector and spin-1/2 fields, are considered below. $\mathcal{L}_{int}$ presents $A^a_\mu$ and $j^a_\mu$ as components over $\frac{1}{\sqrt{N_o}} \gamma^\mu \gamma^\mu G_a$, that is, the vector field and the current occupy...
the same spin space. This connection and the quantum field theory (QFT) understanding of this vertex as the transition operator between fermion states, exerted by a vector particle, with the coupling constant as a measure of the transition probability, justifies the interpretation for it.\[ \frac{1}{2}gA^{a\mu}j^{a\alpha} = A^{a\mu} \frac{1}{\sqrt{N_o}}\text{tr}\Psi^\alpha\gamma^\mu G_\alpha \Psi^\alpha, \]

leading to the identification \[ g \rightarrow 2\sqrt{\frac{K}{N_o}}, \]

\( K \) correcting for over-counted reducible representations, which need not be considered at 5+1 d. The theoretical assignment of \( g \) complements QFT, in which the coupling constant is set experimentally. It should be also understood as tree-level information, while the values are modified by the presence of a virtual cloud of fields, at given energy. Although in QFT the coupling constant is obtained perturbatively in terms of powers of the bare, which takes infinite values absorbed through renormalization, we may take the view that renormalization is a calculational device and that its physical value is a manifestation of the bare one; this is feasible for small coupling constants, which can give small corrections. Energy corrections are also necessary for a more detailed calculation.

We found in Section 5 that among a limited number of choices, at 5+1, \( L \) reproduces SM input. In this section’s approach, it restricts the possible gauge symmetries and representations, for \( \gamma^0\gamma^\mu G_i \) needs to be contained in the space it projects. Thus, it determines the symmetries, which are global, and in turn, the allowed gauge interactions. Furthermore, it fixes the representations. In the same vein, it restricts the allowed Lagrangians. Fixing \( L \), the physically feasible \( (\text{tr}G_iG_j = 2\delta_{ij}) \) gauge groups are comprised of the \( U(1) \) and compact simple algebras generated in it. With the conditions of renormalizability (this excludes antisymmetric fields), and \( L \) conservation, the allowed Lagrangian reduces basically to that of the electroweak sector of the SM, with representations as on Table 3.
Information on SM fields and their masses is next derived from the polarization and generator components of the 5+1 d spin solutions and their expectation values under mass operators and then, in relation to the SM. These values’ scalar nature must be ensured in order to describe mass.

In the SM, quadratic terms of the form \( \langle \bar{\phi} F^\dagger F \bar{\phi} \rangle \), where \( \bar{\phi} \) is a scalar-particle field (and eventually, a vacuum expectation value), give masses to the boson fields \( F \). A similar scalar-expectation value for bosons is implemented using Eq. 4 by

\[
M_F^2 = \text{tr} \left[ \tilde{H}, F \right]^\dagger_{\pm} \left[ \tilde{H}, F \right]_{\pm},
\]

where \( \tilde{H} \) is constructed with the scalar-chargeless solutions of \( n_{L,R} \), which are the counterpart of the Higgs particle in the SM. Indeed, these terms’ components \( \tilde{n}_0 = \frac{1}{4\sqrt{2}}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0 = \frac{i}{\sqrt{2}}\gamma_0 I_+ \), \( n_0 = \frac{i}{\sqrt{2}}\gamma_0 I_- \), \( I_+ = I_1 + iI_2 \), from Eqs. [1]-[3]

form the parity-conserving scalar combinations \( M_i\gamma_0, i = 3, 4 \). Specifically,

\[
\tilde{H} = v' \frac{1}{\sqrt{2}}(n_0 + i\tilde{n}_0)
\]

with \( v' \) an energy scale, is symmetric in particle and antiparticle \( n_0, \tilde{n}_0 \) components and we justify the coefficients’ choice below. Covariance requires the use of anti-commutators for the spatial vector components, leading to the same scalar quantity, hence the \( \pm \) index in Eq. [19]. When applying this equation to the scalar fields both methods give the same result.

We note that \( M_i\gamma_0, i = 3, 4 \), by construction, immediately give \( [n_0, Q] = [\tilde{n}_0, Q] = 0 \). Thus, the term \( A_\mu = \frac{1}{2}Q\gamma_0\gamma_\mu \) has quantum numbers of a massless vector and induces the fermion vertex that identifies it with a photon.

There are two additional neutral combinations of vector bosons orthogonal to \( Q \). One, the axial combination \( Z_\mu \) can be related to its namesake in the SM. Indeed, \( A_\mu \)
can be represented as a mixture of two chargeless and massless components. On the one hand, the $B_\mu$ and $\tilde{B}_\mu$ terms form $B_{Y\mu} = \frac{1}{2\sqrt{3}} Y_0 \gamma_\mu$ $(B_{Y0} = g'\frac{\sqrt{3}}{2} Y)$, that is, the hypercharge carriers. Thus, we obtain another argument to set $Y$ whose origin is in the way we arrive at the expression for $Q$ in Eq. 13. With hindsight, we write in parenthesis the $B_{Y0}$ component, and other fields' below, more generally in terms of a hypercharge coupling constant $g'$ which encompasses both the arbitrary parameter and normalized-linked interpretations. On the other hand, we can use the chargeless vector isospin triplet component $W^{3\mu}_\mu = I_3 \gamma_0 \gamma_\mu$ $(W^{3}_0 = g I_3)$, $g$ the $SU(2)$ coupling. From the expressions for $Q$, $A_\mu$, and $W^{3\mu}_\mu$ we easily obtain

$$A_\mu = \frac{1}{2} W^{3\mu}_\mu + \frac{\sqrt{3}}{2} B_{Y\mu}. \quad (21)$$

The value of Weinberg’s angle $\theta_W$ is derived immediately from this equation by making an analogy with the combination of fields obtained in the SM, after application of the Higgs mechanism\[7\]. The photon there has the form

$$\bar{A}_\mu = \frac{1}{\sqrt{g'^2 + g^2}} (g \bar{B}_{Y\mu} + g' \bar{W}^0_\mu), \quad (22)$$

where the bar denotes here and below SM fields. We obtain $\frac{g'}{g} = \frac{1}{\sqrt{3}}$. As in the SM $\tan(\theta_W) = \frac{g'}{g}$, we find $\sin^2(\theta_W) = .25$. We derive the other possible combination constructed with this form:

$$Z_\mu = \frac{\sqrt{3}}{2} W^{3\mu}_\mu - \frac{1}{2} B_{Y\mu} \quad (23)$$

$$(Z_0 = \frac{1}{\sqrt{g'^2 + g^2}} (g'^2 \frac{\sqrt{3}}{2} Y - g^2 I_3)).$$

To calculate $M_F$ in Eq. 19 for the vector bosons we may use the underlying isospin and hypercharge $SU(2)_L \times U(1)_Y$ classification of the generators. It gives $[\tilde{n}_0, Z_0] = \frac{1}{2} \sqrt{g^2 + g'^2} \tilde{n}_0$, $[n_0, Z_0] = -\frac{1}{2} \sqrt{g^2 + g'^2} n_0$. For the charged vectors, $[n_0, W^+_\mu] = \frac{g}{\sqrt{2}} v^+_0$, $[\tilde{n}_0, W^{+\mu}_0] = 0$, where $\tilde{v}^-_0 = \frac{1}{4\sqrt{2}} (1 + I\gamma_5)(1 + I)\gamma_0$, $v^+_0 = \frac{1}{4\sqrt{2}} (1 - I\gamma_5)(1 + I)\gamma_0$ are charged-scalar components. The resulting values are presented on Table 4 in terms of the $W^{\pm\mu}$ mass. They are compared with SM expressions and values. Remarkably,
the calculation reproduces the SM relation \( M_Z/M_W = \sqrt{1 + tan^2(\theta_W)} \), for the derived and general values of \( g \) and \( g' \). The additional neutral orthogonal vector and pseudo-vector combination is

\[
Z'_\mu = \left[ \sqrt{3}/2(L + 2Q) - \frac{1}{\sqrt{2}} Z_0 \right] \gamma_0 \gamma_\mu,
\]

and its expectation value \( M_F \) in Eq. 19 gives \( M_{Z'} = \sqrt{2} M_Z \). \( M_{Z'} \) is unstable under changes of some of the \( Z'_\mu \) components (\( Z_\mu, W_\mu \) are stable) which casts doubt on whether it represents a physical particle.

| \( \{ tr[\bar{H}, F]_\pm \} \) | \( fer \) | \( \nu \) | \( A_\mu \) | \( W^\pm_\mu \) | \( Z_\mu \) | \( H \) |
|-----------------|-----|-----|-----|-----|-----|-----|
| \( M_W \) | 0 | 0 | \( M_W \) | \( \sqrt{4/3} M_W \) | \( \sqrt{2} M_W \) | \( [ GeV ] \) |
| [ GeV ] | 80.4 | 0 | 0 | 80.4 | 92.8 | 113.7 |
| \( \{ tr[\bar{H}, F]_\pm \} \) | \( - \) | 0 | 0 | \( M_W \) | \( \sqrt{1 + tan^2(\theta_W)} M_W \) | \( \frac{2\sqrt{25}}{g} M_W \) |
| \( SM \) | \( - \) | 0 | 0 | 80.4 | 91.2 | \( - \) |
| [ GeV ] | \( - \) | 0 | 0 | 80.4 | 91.2 | \( - \) |

Table 4: Predictions from \( M_F \) in Eq. 19, based on the \( sin^2(\theta_W) = .25 \) result, for masses of massive fermion (\( fer \)), neutrino (\( \nu \)), photon (\( A_\mu \)), \( W, Z \), and Higgs (\( H \)) particles, in terms of \( M_W \), as compared to the SM, with numerical values for both cases.

\[ ^3 M_F \text{ is not a constant or minimum upon variation of these parameters.} \]
Application of Eq. 19 to the state with scalar polarization $H = \frac{\theta}{\sqrt{2}} (n_0 + \tilde{n}_0) = -\frac{1}{2} M_\gamma \gamma_0$ leads to $[\bar{H}, H] = \frac{\theta^u}{2} (-1 + i) [\tilde{n}_0, n_0] = \frac{\theta^u}{64} (-1 + i) [\gamma_0 I_+, \gamma_0 I_-] = i \frac{\theta^u}{8} (-1 + i)(1 - I) \gamma_5$, implying $M_H = \sqrt{2} M_W$. The choice $\tilde{H}$ in Eq. 20 gives mass to the hermitian scalar combinations $M_i \gamma_0$, $i = 3, 4$ in Eq. 10, and is stable ($n_0 \pm \tilde{n}_0$ makes them either massless or unstable). It is pertinent to compare this value with recent experiments that indicate a Higgs particle exists with a 114-114.5 GeV mass[8].

It is illustrative to derive the same results within the SM, by using its equivalence to a formulation in terms of the fields’ underlying spin structure. We consider the Lagrangian component that gives masses to the electroweak vector fields after spontaneous symmetry breaking[7]:

$$L_M = \bar{v}^\dagger (i \frac{g'}{2} B_\mu + i \frac{g}{2} \tau^i W^\mu_i) \frac{1}{2} (1 + \tau^3) + \frac{1}{2 \sqrt{g^2 + g'^2}} \bar{Z}_\mu (g'^2 - g^2 \tau^3) + i \frac{g}{2 \sqrt{2}} W^\mu_+ \tau^+ \bar{v},$$

where $\tau^i$ are the Pauli matrices, we use Eq. 22, $\bar{Z}_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' B_\mu - g W^0_\mu)$, $W^\pm_\mu = \frac{1}{\sqrt{2}} (W^1_\mu \mp i W^2_\mu)$, $\tau^\pm = \tau^1 \pm i \tau^2$, $e = \frac{g'}{\sqrt{g^2 + g'^2}}$ is the electric charge, $\bar{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$, and $v$ the Higgs vacuum expectation value. $g$ and $g'$ can again either represent coupling parameters or specific numbers linked to the polarization normalization. $L_M$ can be equivalently written in terms of the 5+1 $d$ generators

$$L_M = \frac{v'^2}{2} tr[\ldots]^\dagger [ie \bar{A}_\mu (I_3 + \frac{1}{2} Y) + \frac{i}{\sqrt{g^2 + g'^2}} \bar{Z}_\mu (g'^2 - g^2 I_3)] + \frac{g}{\sqrt{2}} W^\mu_+ I_+ \gamma_0 \gamma^\mu \bar{H}$$

$$= \frac{1}{2} tr[\ldots]^\dagger (\bar{Z}_\mu [\bar{H}, Z^\mu]_\pm + \sqrt{2} W^\mu_+ [\bar{H}, W^\mu_+]_\pm),$$

In the last equality we use a commutator or anticommutator for $\mu = 0$ or $\mu = 1, 2, 3$, respectively. Their expectation values give the $M_Z$ and $M_W$ masses as in Eq. 19. The SM sets $v' = v$ in $\bar{H}$. Thus, the vector-boson masses can also be interpreted as stemming from a single mass term defined on a spin space.
On the other hand, the Higgs $\bar{\rho}$ Lagrangian can be similarly written

$$\mathcal{L}_H = \frac{1}{2} \partial_\mu \bar{\rho} \partial^\mu \bar{\rho} - V\left[\frac{\bar{\rho} + v}{2}\right]^2,$$

with $V[\bar{\phi}^\dagger \bar{\phi}] = \mu^2 \bar{\phi}^\dagger \phi + \lambda (\bar{\phi}^\dagger \phi)^2$ the potential of the Higgs doublet, $\mu^2 < 0$. In particular, after spontaneous symmetry breaking, the mass term can be written

$$\mathcal{L}_{MH} = -\frac{1}{2} 2\mu^2 \bar{\rho}^2 = -\frac{1}{2} \text{tr}[...][\tilde{H}, H]\bar{\rho}^2,$$

with $v^2 = -\mu^2/\lambda$. The second equality has been set so that $\mathcal{L}_M$ in Eq. 24 and $\mathcal{L}_{MH}$ in Eq. 27 exhibit the same underlying operator, with a symmetry assumed under the exchange of fields occupying the same (spin) space. Thus, a unique term is understood to produce masses at tree level, both among the vector and the scalar fields, setting the Higgs parameter $\mu$.

8 Vector fermion-current vertices

The spin 1/2 components form a charge $q = 1$ (from $Q$) massive Dirac particle $l$ (left-handed part $l_L$) and a $q = 0$ particle $\nu$, which remains massless. This is as occurs in the SM for charged leptons and neutrinos, to be identified respectively with $l^-$ and $\nu$. Assuming the same spin symmetry for the fermion-mass term implies taking $Hl^- = \frac{1}{\sqrt{2}}[n_0 + \bar{n}_0, l^-] = \frac{1}{2} l^-$, where, e.g., $l^- = \frac{1}{\sqrt{2}}(l^- + \bar{l}_L)$, $\bar{l}_R$ is given in Eq. [4], $l^-_L = \frac{1}{2}(1 - I\gamma_5)(1 + I)(\gamma_1 + iI\gamma_2)$, with equivalent results with $\tilde{H}$ in $tr[...][\tilde{H}, l^-]$. Although an $M_W$ mass is obtained for the massive leptons, extended models accounting for quarks[3] will presumably give a similar prediction, so that a connection to the top-quark mass scale may be at hand.

SM vertices describing the fermions to the vector-fermion couplings are obtained from the argument after Eq. [18]. Following it, the interaction of the $l$, $\nu$, and $W^\pm$ particles can be described through the Lagrangian

$$\mathcal{L}_{W\text{int}} = W^\pm_\mu \frac{g}{2\sqrt{2}}[\bar{l}_L^\dagger (1 - \gamma_5)\gamma_0 \gamma^\mu \bar{l}_L + hc]$$

(28)
\[
\begin{align*}
= & \ W_{\mu}^+ tr \frac{g}{2\sqrt{2}} [\nu^\dagger (1 - I\gamma_5) \gamma_0 \gamma^\mu l_L^+ + hc] \\
= & \ W_{\mu}^+ tr [\nu^\dagger W_{\mu}^+ l_L^+ + hc],
\end{align*}
\]

where in the second equality 5+1 spin polarizations matrices are used, as in Eq. \ref{eq:18}. Similar expressions as in the SM can be obtained for the lepton interactions with the \( Z_\mu \) particle.

In addition, the vertices give information on the coupling constants \( g' \) and \( g \), which can also be extracted from the normalization constants of the vector bosons in Section 5. \( g \) can be obtained from the coupling of the massive charged vectors \( W_{\mu}^+ \) and the charged current, defined by the neutrino and the charged leptons wave functions, represented in Eq. \ref{eq:28}. These are \( g = 1 \), \( g' = 1/\sqrt{3} \). The next Clifford algebra model at 7+1 \( d \), reproduces the same results and, under the assumption that \( K = 1 \), one finds a \( \frac{1}{\sqrt{2}} \) factor in the coupling obtained from the trace in Eq. \ref{eq:4}, with respect to the lower-dimensional algebra, with predictions \( g = 1/\sqrt{2} \approx .707 \) \( g' = 1/\sqrt{6} \approx .408 \). It is a consistency feature of the theory that any these values are in accordance with \( \theta_W \). In addition, these values are to be compared with the experimentally measured ones at energies of the mass of the \( Z_\mu \) particle, which is where the breakdown of the \( SU(2)_L \times U(1)_Y \) symmetry occurs. These are \( g'_{\text{exp}} \approx .36 \), \( g_{\text{exp}} \approx .65 \), and \( \sin^2(\theta_{W,\text{exp}}) \approx .23 \). Closer numbers are obtained when considering the SM strong sector \ref{3}.

The theory thus presented succeeds in reproducing many aspects of the electroweak sector of the standard model. We conclude that the latter allows for a consistent spin interpretation that enriches it with relevant information on the couplings, the representations, and the masses. The close connection between the results hence derived and the physical particles’ phenomenology makes plausible the idea that, as with the spin, the gauge vector and matter fields, and their interactions originate in the structure of space-time.

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Vitae. J. Besprosvany, B. Sc. in Physics, Universidad Nacional Autónoma de México (UNAM), Mexico City (1985); M. Sc., Weizmann Institute of Science, Rehovot (1988); Ph. D., Weizmann Institute of Science, Rehovot (1993); researcher at UNAM from 1995.