Integrable mixing of $A_{n-1}$ type vertex models

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Abstract

Given a family of monodromy matrices $T_0, T_1, \ldots, T_{K-1}$ corresponding to integrable anisotropic vertex models of $A_{n_{\mu}-1}$ type, $\mu = 0, 1, \ldots, K - 1$, we build up a related mixed vertex model by means of glueing the lattices on which they are defined, in such a way that integrability property is preserved. The glueing process is implemented through 1-dimensional representations of rectangular quantum matrix algebras $A \left ( R_{n_{\mu}-1} : R_{n_{\mu}} \right )$, namely, the glueing matrices $\zeta_\mu$. Algebraic Bethe ansatz is applied on a pseudovacuum space with a selected basis and, for each elements of this basis, it yields a set of nested Bethe ansatz equations matching up to the ones corresponding to an $A_{m-1}$ quasi-periodic model, with $m$ equal to $\min_{\mu \in \mathbb{Z}_K} \{ \text{rank} \zeta_\mu \}$.

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I. INTRODUCTION

There exists a deep linking between solvable two-dimensional vertex models in statistical mechanics and the quantum Yang-Baxter (YB) equation, where it appears as condition for integrability related to some basic quantities of the model [1] [2]. This linking relies on the existence of an underlying symmetry, the quantum group [3] [4], which comes to provide a nice algebraic framework to study these systems. There, solutions of the YB equation are representations of some quadratic relations defining the quantum group structure and, through a process called baxterization [5], they connect with the monodromy matrices of the model. From an algebraic point of view, baxterization makes an ordinary quantum group into a YB algebra. In this way, different representations of a YB algebra lead to different integrable models.

The main aim of this work is to present a glueing process of models associated to several YB algebras preserving the integrability of the total system. We restrict ourself to those YB algebras $YB_n$ coming from baxterization of the quantum groups $A(R_n)$ [4], i.e., the duals of $U_q(su_n)$ [3], $n \in \mathbb{N}$. Beside algebras $YB_n$, the glueing process also involves their rectangular generalizations $YB_{n,m}$, defined as the spectral parameter dependent versions of the rectangular quantum matrix algebras $A(R_n : R_m)$ [4]. Here $A(R_n : R_n) = A(R_n)$ and, accordingly, $YB_{n,n} = YB_n$. The process is based on the existence of algebra homomorphisms $YB_{n,m} \to YB_{n,p} \otimes YB_{p,m}$, the cocomposition maps, that generalize the concept of coproduct in a bialgebra. Such maps can be used to build up representations of a given YB algebra $YB_n$ as a product of representations of another algebras $YB_m$, $m \neq n$, in an analogous way as the standard coproduct is used for building up usual tensor representations. More precisely, given families $T_\mu$ and $\zeta_\mu$, $\mu = 0, 1, \ldots, K - 1$, of representations of $YB_{n\mu}$ and $YB_{n\mu-1,n\mu}$, respectively, cocomposition maps ensure operator $T^{mix} = \zeta_0 \otimes T_0 \otimes \cdots \otimes \zeta_{K-1} \otimes T_{K-1}$ is a representation of $YB_{n_{K-1}}$ (symbol $\otimes$ will be defined in next section). If each $T_\mu$ defines the monodromy matrix of a given vertex model, we say $T^{mix}$ is that of the mixed model with glueing matrices $\zeta_\mu$. 
We shall see this procedure is compatible with the algebraic Bethe ansatz method for solving these models, in the sense there exists a set of pseudovacuum vectors with respect to which these technics can be applied. Moreover, we show a set of nested Bethe ansatz equations identical to the ones corresponding to an $A_{m-1}$ quasi-periodic model, with $m$ equal to $\min_{\mu \in \mathbb{Z}} \{ \text{rank } \zeta_{\mu} \}$, is related to each one of these vectors.

This work is organized as follows: in section II, we review some well known facts on the connection between YB algebras and integrable vertex models; in section III, we describe the glueing process and the glueing matrices as 1-dimensional representations of the rectangular YB algebras; finally in section IV, we prove complete integrability of mixed models, showing that diagonalization of mixed transfer matrices reduce to solve nested Bethe equations of a family of $A$-type vertex models.

II. YANG-BAXTER ALGEBRAS AND INTEGRABLE VERTEX MODELS

To start with, we describe briefly the connection between two-dimensional vertex models and $A_{n-1}$ type solutions of the YB equation.

Let us consider the class of YB operators or constant $R$-matrices

$$[R_{n}]^{kl}_{ab} = \begin{cases} q \delta_{a}^{k} \delta_{b}^{l}, & a = b; \\ \delta_{a}^{l} \delta_{b}^{k} + (q - 1/q) \delta_{a}^{l} \delta_{b}^{k}, & a < b; \\ \delta_{a}^{k} \delta_{b}^{l}, & a > b; \end{cases} \quad 1 \leq a, b, k, l \leq n;$$

where $q \in \mathbb{C} \setminus \{0, 1\}$, related to the standard Hopf algebra deformations of the simple Lie algebras $A_{n-1}$, i.e., the quantum groups $U_{q}(su_{n})$ and $A(R_{n})$, $n \in \mathbb{N}$. Baxterization process yields the spectral parameter dependent versions $R_{n}(x) = xR_{n} - PR_{n}^{-1}P/x$ of each $R_{n}$, with $P_{ij}^{kl} = \delta_{i}^{l} \delta_{j}^{k}$ the permutation matrix and $x \in \mathbb{C}$. Then, for every $N \in \mathbb{N}$ a related integrable (inhomogeneous) lattice model [7] is defined by a monodromy matrix $T \doteq T^{(n,N)}(x; \alpha)$ with entries (sum over repeated indices convention is assumed)

$$T_{a}^{b} = R_{a}^{b}(x/\alpha_{0}) \otimes R_{a_{1}}^{b_{1}}(x/\alpha_{1}) \otimes ... \otimes R_{a_{N}}^{b_{N}}(x/\alpha_{N-1}), \quad 1 \leq a, b \leq n,$$
being \( \alpha = (\alpha_0, ..., \alpha_{N-1}) \) a vector of \( \mathbb{C}^N \). Operators \( R^b_a(x) : \mathbb{C}^n \rightarrow \mathbb{C}^n \) are entries of a matrix \( R = R(x) \) such that \( \left[R^b_a(x)\right]_i^j = [R_n(x)]^b_a_{i,j} \) in the canonical basis of \( \mathbb{C}^n \). Compact notation \( T = R \otimes ... \otimes R \), where \( \otimes \) denotes matrix multiplication between consecutive factors, will be used. Note that \( T^{(n,1)} = R^n = R \). These models are the anisotropic analogous of the \( A_{n-1} \) invariant vertex models with periodic boundary conditions. Quasi-periodic versions (see \[7\] again) are given by elements \( \Upsilon \) in the symmetry group of \( R_n(x) \), i.e., \( \Upsilon \in GL(n) \) and \( [R_n(x), \Upsilon \otimes \Upsilon] = 0 \). Related monodromy matrices read \( T^\Upsilon = T \cdot \Upsilon \). Equation (2) defines operators \( T^b_a : (\mathbb{C}^n)^{\otimes N} \rightarrow (\mathbb{C}^n)^{\otimes N} \) that describe the statistical weights assigned to each vertex configuration in a given row of the lattice, graphically,

\[
\left[ T^b_a \right]_{i_0, ..., i_{N-1}}^{j_0, ..., j_{N-1}} = \frac{j_0}{a} b_1 \frac{j_1}{b_2} ... \frac{j_{N-1}}{b_{N-1}} \quad ; \quad \left[ R_n(x) \right]_{i_0, ..., i_{N-1}}^{j_0, ..., j_{N-1}} = \frac{j}{a} b \frac{i}{i_{N-1}}
\]

If lattice has \( N' \) rows, the partition function is \( Z = \text{trace} \left( t^{N'} \right) \), being \( t = \sum a T^a_a \) the transfer matrix. On the other hand, the operators \( T^b_a(x; \alpha) \), as it is well known, give a representation of the YB algebra related to \( R_n(x) \). This algebra, which we shall indicate YB\(_n\), is generated by indeterminates \( T^j_i(x), 1 \leq i, j \leq n; x \in \mathbb{C} \), subject to relations

\[
\left[ R_n(x/y) \right]_{ij}^{kl} T_k^l(x) T^r_s(y) = T_j^l(y) T^k_i(x) \left[ R_n(x/y) \right]_{kl}^{rs}; \quad 1 \leq i, j, r, s \leq n. \tag{3}
\]

These relations entail the formal integrability of the system. In fact, by taking the trace, one gets \( [t(x), t(y)] = 0 \) for all \( x, y \in \mathbb{C} \), i.e., the transfer matrix is a generating function of conserved quantities. Beside this, the model is effectively solved by means of algebraic Bethe ansatz (see \[8\] \[9\] \[10\] and references therein), where the central ingredient is the existence of an eigenstate \( \omega \in (\mathbb{C}^n)^{\otimes N} \) of each entry \( T^a_i(x) \) (and consequently of the transfer matrix \( t \)), such that \( T^b_a \omega = 0 \) for all \( a \neq b \) and \( a \geq 2 \), and \( T^b_i \omega \neq c \omega, \forall c \in \mathbb{C} \), for all \( b \geq 2 \). For latter convenience, let us express \( T \) in the block form

\[
T = \begin{bmatrix}
A & B_j \\
C_i & D_{ij}
\end{bmatrix} \quad ; \quad 1 \leq i, j \leq n - 1,
\]
i.e., define \( A \equiv T_1^1 \), \( B_j \equiv T_{i+1}^j \), \( C_i \equiv T_{i+1}^1 \) and \( D_{ij} \equiv T_{i+1}^{j+1} \). Then, \( \omega \) is an eigenstate of \( A \) and of each diagonal entry \( D_{ii} \), fulfilling \( C_i \omega = 0 \) and \( D_{ij} \omega = 0 \) for \( i \neq j \). A vector satisfying these conditions is called a pseudovacuum vector. On the other hand, since \( B_j \omega = T_{i+1}^j \omega \neq c \omega \) for all \( j \), each \( B_j (x) \) plays the role of a creation operator. Applying them repeatedly on \( \omega \) (varying \( j \) from 1 to \( n - 1 \) and \( x \) satisfying the so-called Bethe equations) we generate new eigenstates for the transfer matrix, namely the Bethe vectors, giving a priori a complete set of eigenstates for \( t \). In such a case we say the system is exactly solvable or completely integrable (see [11] and refs. therein). Nevertheless, sometimes not only a vector but a pseudovacuum subspace (cf. [12], [13]) is needed in order to insure complete integrability. This will be our case.

Since each YB\(_n\) is a bialgebra, with coalgebra structure

\[
\Delta : T_i^j (x) \mapsto T_i^k (x) \otimes T_j^l (x), \quad \varepsilon : T_i^j (x) \mapsto \delta_i^j,
\]

for every couple of monodromy matrices \( T^{(n,N)} \) and \( T^{(n,P)} \) as above we have another one,

\[
T^{(n,N+P)} = T^{(n,N)} \otimes T^{(n,P)}, \quad \text{with entries} \quad T_a^b (n,N+P) = T_a^c (n,N) \otimes T_c^b (n,P),
\]

giving again a representation of YB\(_n\). Furthermore, if \( \omega \) and \( \phi \) are the pseudovacuums of \( T^{(n,N)} \) and \( T^{(n,P)} \), then \( \omega \otimes \phi \) defines a pseudovacuum for \( T^{(n,N+P)} \). Consequently the enlarged model, or the glueing of \( T^{(n,N)} \) and \( T^{(n,P)} \), is also integrable. In particular, thermodynamic limit \( N \to \infty \) preserves integrability. But, can we glue models which give representations of different YB algebras, e.g., YB\(_n\) and YB\(_m\) with \( n \neq m \), and such that a pseudovacuum exists for the resulting model? The aim of this paper is to answer last question. More precisely, we build up from a family \( \{ T_\mu : \mu \in \mathbb{Z}_K \} \) of pure models, i.e., \( T_\mu = T^{(n_\mu,N_\mu)} \), \( n_\mu, N_\mu \in \mathbb{N} \), a mixing of them by means of glueing the lattices on which they are defined, in such a way that resulting mixed model can be solved by means of algebraic Bethe ansatz technics.
III. THE GLUEING PROCESS

For any pair \((R_n, R_m)\) of matrices (1), there exist an associated quadratic algebra \(A (R_n : R_m)\). They are called rectangular quantum matrix algebras [3]. There are also parameter dependent versions, the algebras \(YB_{n,m}\), generated by indeterminates \(T^i_j(x), 1 \leq i \leq n, 1 \leq j \leq m\) and \(x \in \mathbb{C}\), and defined by the quadratic relations

\[
[R_n (x/y)]^{kl}_{ij} T^r_k(x) T^s_l(y) = T^l_j(y) T^k_i(x) [R_m (x/y)]^{rs}_{kl},
\]

(5)

\(1 \leq i, j \leq n, 1 \leq r, s \leq m\). Obviously, \(YB_{n,n} = YB_n\). In the same way as for the constant case [3], [14], there exist homomorphisms

\[
\Delta_p : YB_{n,m} \to YB_{n,p} \otimes YB_{p,m}; \quad n, m, p \in \mathbb{N};
\]

(6)

inherited from the cocomposition notion of the internal coHom objects, enjoying the coassociativity property \((\Delta_p \otimes id) \Delta_r = (id \otimes \Delta_r) \Delta_p\) [15]. In the \(n = m = p\) cases, these reduce to the usual comultiplication maps [see Eq. (4)]. In particular, we have morphisms

\[
YB_m \to YB_{m,n} \otimes YB_n \otimes YB_{n,m} \otimes YB_m
\]

for all \(n, m\). Now, consider pure monodromy matrices \(T^{(n,N)}\) and \(T^{(m,P)}\) related to \(YB_n\) and \(YB_m\), and representations \(\lambda\) and \(\beta\) of \(YB_{n,m}\) and \(YB_{n,m}\), respectively, where \(\lambda\) and \(\beta\) denote rectangular matrices whose coefficients are representative of the corresponding generator algebra elements. Mentioned morphism implies \(\lambda \otimes T^{(n,N)} \otimes \beta \otimes T^{(m,P)}\) gives a representation of \(YB_m\). As we do not want to add new degrees of freedom others than the related to the original models \(T^{(n,N)}\) and \(T^{(m,P)}\), we ask \(\lambda\) and \(\beta\) to be constant (i.e., spectral parameter independent) 1-dimensional representations. In this case \(\lambda \otimes T^{(n,N)} \otimes \beta \otimes T^{(m,P)}\) gives an operator on \((\mathbb{C}^n)^{\otimes N} \otimes (\mathbb{C}^m)^{\otimes P}\), which we shall call the glueing of \(T^{(n,N)}\) and \(T^{(m,P)}\) through matrices \(\lambda\) and \(\beta\). It is worth remarking that this is not the glueing operation defined in [3]. Physically, \(\lambda\) and \(\beta\) define vertices with statistical weights

\[
\lambda^b_a \quad \text{and} \quad \beta^d_c, \quad 1 \leq a, d \leq m \quad \text{and} \quad 1 \leq b, c \leq n.
\]
In general, for a family of pure monodromy matrices as described above, we can define a \textit{mixing} of them, namely

\[ T^{\text{mix}} = \lambda_0 \otimes T_0 \otimes \lambda_1 \otimes T_1 \otimes \ldots \otimes \lambda_{K-1} \otimes T_{K-1}, \]  

(7)

where each \( \lambda_\mu \) is a constant 1-dimensional representation of the rectangular YB algebra \( YB_{n_{\mu-1},n_\mu} \pmod{K} \). Graphically,

\[ [T^b_{a \text{ mix}}]_{J_0,\ldots,J_{K-1}}^{I_0,\ldots,I_{K-1}} = \]

\[ \begin{array}{c}
\text{a} \\
\text{b}
\end{array} \begin{array}{c}
\text{b}_0 \\
\text{b}_1 \\
\vdots \\
\text{b}_K
\end{array} \begin{array}{c}
\text{J}_0 \\
\text{J}_1 \\
\vdots \\
\text{J}_{K-1}
\end{array} \begin{array}{c}
\text{I}_0 \\
\text{I}_1 \\
\vdots \\
\text{I}_{K-1}
\end{array}; \]

\[ [T^b_\mu]_{a J}^{b I} = \]

\[ \begin{array}{c}
\text{a} \\
\text{b}
\end{array} \begin{array}{c}
\text{I}
\end{array} \begin{array}{c}
\text{J}
\end{array} ; \]

\[ [\lambda^b_\mu]_{a} = \]

\[ \begin{array}{c}
\text{a} \\
\text{b}
\end{array} , \]

being \( I_\mu \) and \( J_\mu \) multi-indices for spaces \((\mathbb{C}^{n_\mu})^{\otimes N_\mu}\) on which each \( T_\mu \) acts. Since the quadratic relations (5) and the cocomposition maps (6), one may see that \( T^{\text{mix}} \) provides a representation of \( YB_{n_{K-1}} \). This is a direct consequence of the algebra map

\[ YB_{n_{K-1}} \rightarrow YB_{n_{K-1},n_0} \otimes YB_{n_0} \otimes YB_{n_0,n_1} \otimes \ldots \otimes YB_{n_{K-2},n_{K-1}} \otimes YB_{n_{K-1}}. \]

(8)

Of course, these representations are highly reducible in general, as we shall see later.

\textbf{A. Constant one dimensional representations of } YB_{n,m}

Representations \( \lambda_\mu \) appearing in (7) match exactly with 1-dimensional representations of \( A (R_n : R_m) \), i.e., rectangular matrices \( \lambda \in \text{Mat} [n \times m] \) in \( \mathbb{C} \) such that

\[ [R_n]^{kl}_{ij} \lambda^k_i \lambda^s_l = \lambda^l_i \lambda^k_j [R_m]^{rs}_{kl}; \quad 1 \leq i, j \leq n, \ 1 \leq r, s \leq m. \]

(9)

We are considering the same parameter \( q \neq 0,1 \) for all involved \( R\)-matrices. Otherwise, the only solution to (9) is the trivial one. Using explicit form of \( R_n \) given in (4), last equation is equivalent to
\[ \lambda_i^r \lambda_j^s = 0, \quad 1 \leq r \leq m, \ 1 \leq i \leq j \leq n, \]
\[ \lambda_i^r \lambda_i^s = 0, \quad 1 \leq r < s \leq m, \ 1 \leq i \leq n, \]
\[ \lambda_i^r \lambda_j^s = 0, \quad 1 \leq r < s \leq m, \ 1 \leq j < i \leq n. \]

First and second lines imply coefficients of \( \lambda \) in a given column and row, respectively, are null except for almost one of them. Last line says, if \( \lambda_i^j \neq 0 \) then all coefficients \( \lambda_a^b \) with \( i < a, \ b < j, \) and with \( a < i, \ j < b, \) are null. Thus, each solution \( \lambda \) of (1) is a diagonal matrix to which columns and rows of zeros were added. From that it follows immediately the set of solutions for all \( m, n \) form a semigroupoid, or a category, generated by the abelian groups \( D_n \) of invertible \( n \times n \) diagonal matrices, and also by matrices \( \sigma_i^n \in Mat \ [n \times (n + 1)] \) and \( \partial_i^n \in Mat \ [(n + 1) \times n], \ i = 1, \ldots, n + 1, \ n \in \mathbb{N}, \) given by

\[
\sigma_i^n = \begin{bmatrix}
\begin{array}{c}
Id_{(i-1) \times (i-1)} & O_{(i-1) \times (n-i+2)} \\
O_{1 \times (i-1)} & O_{1 \times (n-i+2)} \\
O_{(n-i) \times (i-1)} & Id_{(n-i) \times (n-i+2)}
\end{array}
\end{bmatrix}, \quad \partial_i^n = \begin{bmatrix}
\begin{array}{ccc}
Id_{(i-1) \times (i-1)} & O_{(i-1) \times 1} & O_{(i-1) \times (n-i)} \\
O_{(n-i+2) \times (i-1)} & O_{(n-i+2) \times 1} & Id_{(n-i+2) \times (n-i)}
\end{array}
\end{bmatrix},
\]

being \( O_{n \times m} \) the \( n \times m \) null matrix. In fact, a general solution of Eq. (9) has the form

\[
\lambda = \partial_{j_1}^{a_1} \ldots \partial_{j_b}^{a_b} D \sigma_{i_1}^{m_1} \ldots \sigma_{i_a}^{m_a-1} \in Mat \ [n \times m]; \\
a, b \geq 0, \ m - a = n - b = k \geq 0,
\]

with \( i_1 \leq \ldots \leq i_a \leq m, \ j_1 \leq \ldots \leq j_b \leq n, \) and \( D \in D_k. \) If \( a \) (resp. \( b \)) is equal to zero, then factors of type \( \sigma \) (resp. \( \partial \)) do not appear. Such a solution has \( k \) non null entries equal to the diagonal elements of \( D, \) a number \( a \) of null columns in positions \( i_1, \ldots, i_a, \) and \( b \) null rows in positions \( j_1, \ldots, j_b. \) Note that rank \( \lambda = k. \)

Matrices \( \sigma_i^n \) and \( \partial_i^n, \) which give solutions to (9) for \( m = n + 1 \) and \( n = m + 1, \) respectively, are related each other by matrix transposition, i.e., \( \partial_i^n = (\sigma_i^n)^t, \) and enjoy relations

\[
\sigma_{j}^{i-1} \sigma_{i}^{n} = \sigma_{i}^{n-1} \sigma_{j+1}^{n}, \quad i \leq j; \\
\partial_{i}^{n+1} \partial_{j}^{n} = \partial_{j+1}^{n+1} \partial_{i}^{n}, \quad i \leq j; \\
\sigma_{j}^{n+1} \partial_{i}^{n+1} = \partial_{i}^{n} \sigma_{j-1}^{n}, \quad i < j; \\
\sigma_{i}^{n} \partial_{i}^{n} = Id, \quad i = j; \\
\sigma_{j}^{n+1} \partial_{i}^{n+1} = \partial_{i-1}^{n} \sigma_{j}^{n}, \quad i \geq j + 1.
\]
In spite of these relations, they do not define the simplicial category. Note that, for instance, \( \sigma_j^n \partial_{j+1}^n \neq Id \). Nevertheless we name \( \Delta = \bigvee_{n,m \in \mathbb{N}} \Delta_{n,m} \) the category formed out by them.

On the other hand, as it is well known, the group of diagonal matrices \( \mathcal{D}_n \) defines precisely the symmetry group of \( R_n \), given by matrices \( D \in GL(n) \) such that \( [R_n, D \otimes D] = 0 \). Moreover, they are also the symmetry group of \( R_n(x) \) or \( R_n(x, y) = R_n(x/y) \). Let us mention that, when \( R_n(x, y) \) is changed by a similarity transformation \( Q(x) \otimes Q(y) \) such that \( Q_l(x) = \delta_l^i x^{2l/n} \), the group enlarges to \( \mathcal{D}_n \times \mathbb{Z}_n \). This is why systems related to such \( R \)-matrices were called \( \mathbb{Z}_n \)-symmetric vertex models [16].

Elements \( D \in \mathcal{D}_n \) give rise to multiparametric solutions \( (id \otimes D)^{-1} R_n (D \otimes id) \) of the YB equation [14], and related twist transformations of original quantum groups [14] [18]. Associated integrable models, which differ from the original ones by a twisting of the boundary conditions, were described in [19]. We shall see latter that also in mixed models the role of matrices \( D \) is to make a twist on the boundary conditions.

The commutation relations between elements of \( \mathcal{D} = \bigvee_{n \in \mathbb{N}} \mathcal{D}_n \) and \( \Delta \) can be written

\[
D \sigma_i^n = \sigma_i^n D^+_i, \quad \partial_i^n D = D^+_i \partial_i^n,
\]

\[
\sigma_i^{n-1} D = D_i^- \sigma_i^{n-1}, \quad D \partial_i^{n-1} = \partial_i^{n-1} D_i^-,
\]

being \( D_i^+ = diag (d_1, ..., d_{i-1}, 1, d_i, ..., d_n) \) and \( D_i^- = diag (d_1, ..., d_{i-1}, d_{i+1}, ..., d_n) \) whenever \( D = diag (d_1, ..., d_n) \in \mathcal{D}_n \). It is worth mentioning that \( \lambda \) can also be expressed as a product \( \lambda = \zeta D' \), where \( \zeta \in \Delta \) is obtained from \( \lambda \) by taking \( D = Id \), and \( D' \in \mathcal{D}_n \) is the result of passing \( D \) to the right, through matrices \( \sigma \)'s, using commutation rules (11).

\section*{B. Equivalent forms for a mixed monodromy matrix}

From results of last section, it is clear that any mixed model has a monodromy matrix

\[
\lambda_0 \cdot R_0 \hat{\otimes} \lambda_1 \cdot R_1 \hat{\otimes} ... \hat{\otimes} \lambda_{N-1} \cdot R_{N-1},
\]

(12)

with \( R_\nu = R_{n_\nu} \) for some related dimension \( n_\nu \), and \( \lambda_\nu = \zeta_\nu D_\nu \), where \( \zeta_\nu \) is in \( \Delta_{n_{\nu-1}, n_\nu} \) and \( D_\nu \) in \( \mathcal{D}_{n_\nu} \). Here \( \nu \in \mathbb{Z}_N \). Equation (7) corresponds to the case in which there exist \( K \) numbers \( N_\mu, \mu \in \mathbb{Z}_K \), giving a partition of \( N \) and such that
\[ R_{M_\mu} = \ldots = R_{M_\mu + N_\mu - 1} = R^{nM_\mu} \quad \text{and} \quad \lambda_\nu = \text{Id}_{n_\nu} \quad \text{for} \ \nu \neq M_0, \ldots, M_{K-1}, \]

being \( M_0 = 0 \) and \( M_\mu = \sum_{\sigma=0}^{\mu-1} N_\sigma \) for \( 1 \leq \mu \leq K - 1 \). Furthermore, Eq. (12) can be brought to an equivalent form

\[ T^{mix} = \zeta_0 \cdot R_0 \otimes \zeta_1 \cdot R_1 \otimes \ldots \otimes \zeta_{N-1} \cdot R_{N-1} \ \Upsilon, \]

where \( \Upsilon \) is an element of \( D_{nN-1} \). To see that, let us define matrices \( D^{(k)}_\nu \in D_{n_k}, k \in \mathbb{Z}_N \), by

\[ D^{(k)}_\nu = \begin{cases} \text{Id}_{n_k}, & 0 \leq k < \nu, \\ D_\nu, & k = \nu, \end{cases} \]

and for each \( k > \nu \) by the solution of \( D_\nu \zeta_{\nu+1} \zeta_{\nu+2} \ldots \zeta_k = \zeta_{\nu+1} \zeta_{\nu+2} \ldots \zeta_k D^{(k)}_\nu \). We mean by ‘the solution’ of last equation the invertible diagonal matrix \( D^{(k)}_\nu \) that arises when passing, in the first member, the matrix \( D_\nu \) to the right using (11). Then (12) and (14) are similar through \( \Pi_{\nu\in\mathbb{Z}_N} \otimes_{k\in\mathbb{Z}_N} D^{(k)}_\nu \), and \( \Upsilon = \Pi_{\nu\in\mathbb{Z}_N} D^{(N-1)}_\nu \).

In other words, every mixed model is physically equivalent to a twisted version of another one whose corresponding matrices \( \lambda_\nu \) are in the category \( \Delta \). The role of matrices \( D \)'s is to implement a twisting of the boundary conditions. Accordingly, we can describe each mixing \( T^{mix} \) in terms of a family of elements \( \zeta_\nu \in \Delta_{n_{\nu-1}n_\nu} \), the glueing matrices, and a diagonal matrix \( \Upsilon = \text{diag} \left( \tau_1, \ldots, \tau_{nN-1} \right) \) of \( D_{nN-1} \), the boundary matrix.

Another useful expression for the monodromy matrices of these models can be given from the following observation. Any matrix \( \lambda \in \Delta_{n,m} \) of rank \( k \) (note that \( k \leq m, n \)) may be written \( \lambda = P \hat{\lambda} P' \), where \( \hat{\lambda} = \partial^{n-1}_a \ldots \partial^k_{i_k+1} \sigma^k_{i_k+1} \ldots \sigma^{m-1}_m \in \text{Mat} \ [n \times m] \), that is,

\[ \hat{\lambda} = \begin{pmatrix} \text{Id}_k & O_{k \times (m-k)} \\ O_{(n-k) \times k} & O_{(n-k) \times (m-k)} \end{pmatrix}, \]

and \( P, P' \) are appropriate permutations. More precisely, if

\[ \lambda = \partial^{n-1}_{j_b} \ldots \partial^k_{i_1} \sigma^k_{i_1} \ldots \sigma^{m-1}_{i_a}, \ i_1 \leq \ldots \leq i_a \leq m, \ j_1 \leq \ldots \leq j_b \leq n \]

[see Eq. (10)], then we can choose, for instance, \( P \in \text{Mat} \ [n] \) and \( P' \in \text{Mat} \ [m] \) to be
\[ P = C_{j_b,n} C_{j_{b-1},n} \ldots C_{j_1,n} \quad \text{and} \quad P' = C_{i_1,m} \ldots C_{i_{a-1},m} C_{i_a,m}, \]

respectively, being \( C_{r,s} \), \( r \leq s \), the matrix that acting on the right (resp. left) makes a cyclic permutation sending \( s \)-th column (resp. row) to \( r \)-th one, and acts as an identity for the rest of columns (resp. rows). Hence, for a given family of glueing matrices we have \( \zeta_\nu = P_\nu \hat{\zeta}_\nu P'_\nu \). Introducing last expression for \( \zeta_\nu \) into Equation (14), and making a similarity transformation \( \otimes_\nu \in \mathbb{Z}_N P'_\nu \), an equivalent system

\[ \tilde{T}^{mix} = P_0 \hat{\zeta}_0 \cdot \tilde{R}_0 \cdot Q_0 \otimes \hat{\zeta}_1 \cdot \tilde{R}_1 \cdot Q_1 \otimes \ldots \]

\[ \ldots \otimes \hat{\zeta}_{N-2} \cdot \tilde{R}_{N-2} \cdot Q_{N-2} \otimes \hat{\zeta}_{N-1} \cdot \tilde{R}_{N-1} P'_{N-1} \Upsilon, \]

where \( Q_\nu = P'_\nu P_{\nu+1} \) and \( \tilde{R}_\nu = (P'_\nu \otimes P'_\nu) R_\nu (P'_\nu \otimes P'_\nu)^{-1} \), follows. That is,

\[ \tilde{T}^{mix} = (\otimes_{\nu \in \mathbb{Z}_N} P'_\nu) \quad T^{mix} \quad (\otimes_{\nu \in \mathbb{Z}_N} P'_\nu)^{-1}. \]

(16)

It can be seen for an arbitrary permutation that

\[ \tilde{R}_a^a(x) = R_a^a(x), \quad \text{and} \quad \tilde{R}_a^b(x) = x^{2 \varepsilon_{ab}} R_a^b(x) \quad \text{for} \quad a \neq b, \]

(18)

where coefficients \( \varepsilon_{ab} \) takes values \(-1, 0, 1\) depending on the considered permutation. We shall show in the next section that mixed models with

\[ P_0 = P'_{N-1} = Id_{n_{N-1}}, \quad \text{and} \quad Q_\nu = Id_{n_\nu}, \forall \nu \in \mathbb{Z}_{N-1}, \]

(19)

are solvable by means of algebraic Bethe ansatz technics [actually, (19) can be slightly relaxed and ask \( Q_\nu = Id_{n_\nu}, \forall \nu \in \mathbb{Z}_N \), instead]. Furthermore, we shall see complete integrability implies transfer matrix obtained from (16) is similar to the trace of

\[ T^{mix} = \hat{\zeta}_0 \cdot R_0 \otimes \hat{\zeta}_1 \cdot R_1 \otimes \ldots \otimes \hat{\zeta}_{N-1} \cdot R_{N-1} \Upsilon. \]

(20)

Thus, we can solve all vertex models with glueing matrices \( \zeta_\nu = P_\nu \hat{\zeta}_\nu P'_\nu \) satisfying (19) by solving those with \( T^{mix} \) given in Equation (20). In addition, all \( \hat{\zeta}_\nu \) can be supposed to have the same rank.
In order to show exact solvability of these models (or unless of a subclass of them), since
needed commutation rules follow from map (8), we must prove there exists a suitable set of
pseudovacuum vectors for $T_{\text{mix}}$ from which all its eigenstates and corresponding eigenvalues
can be constructed. In other terms, using block form

$$T_{\text{mix}} = \begin{bmatrix} A_{\text{mix}} & B_{\text{mix}}^j \\ C_i^{\text{mix}} & D_{ij}^{\text{mix}} \end{bmatrix}; \quad 1 \leq i, j \leq n_{N-1} - 1,$$

we look for elements $\Phi \in H_{\text{mix}} = \otimes_{\nu \in \mathbb{Z}_N} \mathbb{C}^{n_{\nu}}$ which are eigenvectors of $A_{\text{mix}}$ and of each
diagonal entry $D_{ii}^{\text{mix}}$, such that $C_i^{\text{mix}} \Phi = 0$, and $D_{ij}^{\text{mix}} \Phi = 0$ for $i \neq j$. In this way, we build
up recursively all eigenvalues and eigenstates by applying repeatedly operators $B_{mj}^{\text{mix}}$ to the
mentioned vectors. Completeness problem will be studied separately. Of course, the smaller
the rank of involved glueing matrices, the smaller the set of monomials in $B_{mj}^{\text{mix}}$ and the bigger
the number of pseudovacuum vectors we need to construct the complete set of eigenstates.
In the singular case for which rank $\zeta_\nu = 0$ for some $\nu$, we have $T_{\text{mix}} = 0$ and accordingly
every vector of $H_{\text{mix}}$ is trivially a pseudovacuum vector, and no creation operator is needed
in order to diagonalize the transfer matrix $t_{\text{mix}}$. Note in this case, operators $B_{mj}^{\text{mix}}$ are null.
Thus we can have pseudovacuum vectors which are annihilated by operators $B_{mj}^{\text{mix}}$ and still
be able to build up an eigenstate basis for $t_{\text{mix}}$.

We actually show exact solvability for a particular class of mixed models. Concretely,
we concentrate ourself in monodromy matrices whose related glueings $\zeta_\nu$ satisfy Eq. (19).

**A. The pseudovacuum subspace**

Let us first consider the mixed models with monodromy matrices $T_{\text{mix}}$ defined by Equation (20), that is, each $\zeta_\nu = \hat{\zeta}_\nu$ is of the form (13). They are a particular case of those
with glueing matrices satisfying (19). At the end of this section the general case will be
addressed. In order to simplify our calculations, we shall suppose
\[ m \doteq \text{rank } \zeta_0 \leq \text{rank } \zeta_\nu \quad \text{for all } \nu \in \mathbb{Z}_N. \quad (21) \]

This can be reached by a similarity transformation cyclically permuting tensor factors of the linear space \( \mathcal{H}^{\text{mix}} \). Of course, \( m \leq \text{rank } \zeta_\nu \leq \min\{n_{\nu-1}, n_\nu\} \pmod{N} \). Also, we suppose \( m > 0 \), since for \( m = 0 \) diagonalization of \( T^{\text{mix}} \) is immediate. Note that (21) implies

\[ T^b_a^{\text{mix}} = 0 \quad \text{for } a > m, \quad (22) \]

and in particular

\[ t^{\text{mix}} = \sum_{a=1}^{n-1} T^a_a^{\text{mix}} = \sum_{a=1}^{m} T^a_a^{\text{mix}} = A^{\text{mix}} + \sum_{i=1}^{m-1} D^{\text{mix}}_{ii}. \quad (23) \]

For \( a \leq m \) we have (non sum over \( b \))

\[ T^b_a^{\text{mix}} = \tau_b [R^{a1}]_{la} \otimes [R^{a1}]_{lc1} \ldots [R^{a1}]_{cN-2} \otimes [R^{a1}]_{cN-1}, \quad (24) \]

where sum over each \( c_\nu \) is in the interval \( 1 \leq c_\nu \leq \text{rank } \zeta_\nu \).

Let us indicate by \( e_1, \ldots, e_n \) the elements of the canonical basis of \( \mathbb{C}^n \). Then \( \mathcal{H}^{\text{mix}} \) is spanned by vectors of the form \( e_{f_0} \otimes \ldots \otimes e_{f_{N-1}} \), which can be identified with an obvious subset \( \mathcal{F} \) of functions \( f : \mathbb{Z}_N \to \mathbb{N} : \nu \mapsto f_\nu \). In particular, given \( f \in \mathcal{F} \), we denote \( \Omega^f \) the corresponding vector of \( \mathcal{H}^{\text{mix}} \). We shall show there exists a set of pseudovacuum vectors, on which algebraic Bethe ansatz will be applied, labeled by the subset \( \mathcal{F}_0 \) of functions

\[ f \in \mathcal{F} \quad / \quad \text{Image } f \subset \{1\} \cup \{ n \in \mathbb{N} : n > m \}. \quad (25) \]

More precisely, there exist vectors \( \Phi^f \in \mathcal{H}^{\text{mix}} : f \in \mathcal{F}_0 \), expanding a space

\[ \mathcal{H}_0 = \text{span} \left\{ \Omega^f \in \mathcal{H}^{\text{mix}} : f \in \mathcal{F}_0 \right\} \subset \mathcal{H}^{\text{mix}}, \quad (26) \]

namely the pseudovacuum subspace, and fulfilling

\[ A^{\text{mix}} \Phi^f = \tau_1 d \prod_{\nu \in f^{-1}(1)} G(x/\alpha_\nu) \Phi^f, \]

\[ D^{\text{mix}}_{ii} \Phi^f = \tau_{i+1} d \Phi^f \quad (i < m), \quad D^{\text{mix}}_{i \neq j} \Phi^f = C^{\text{mix}}_i \Phi^f = 0, \quad (27) \]

being \( d = \prod_{\nu \in \mathbb{Z}_N} 1/G(x/\alpha_\nu) \) and \( G(x) = (xq-1/qx)/(x-1/x) \). In particular for the equal rank case, i.e., if \( \text{rank } \zeta_\nu = m \) for all \( \nu \), then \( \Phi^f = \Omega^f \). Note that \( \mathcal{H}_0 \subset \ker C^{\text{mix}}_i \forall i. \)
We also show
\[ B_j^{\text{mix}} \Phi^f \neq c \Phi^f \] if \( 1 \in \text{Image } f \); otherwise, \( B_j^{\text{mix}} \Phi^f = 0; \) (28)
i.e., each \( B_j^{\text{mix}} \) creates new states when \( f^{-1} (1) \neq \emptyset \). Using that we construct a set of Bethe vectors from each \( \Phi^f \), with \( j \) from 1 to \( m \) and \( f^{-1} (1) \neq \emptyset \), and generate in this way all eigenstates of the transfer matrix.

To find the vectors \( \Phi^f \) we need some previous results.

1. The action of \( T^{\text{mix}} \) on vectors \( \Omega^f \)

Let us evaluate the entries \( T^{\text{mix}} b \) on each vector \( \Omega^f \). From Eq. (22) it follows that \( T^{\text{mix}} b \Omega^f = 0 \) for all \( a > m \). So we only consider \( a \leq m \). As usual [7], we normalize operators \( R^b = R^{\nu \nu} (x/\alpha_\nu) : \mathbb{C}^{\nu \nu} \rightarrow \mathbb{C}^{\nu \nu} \) in such a way that on the canonical basis of \( \mathbb{C}^{\nu \nu} \)

\[ R^b e_k = \begin{cases} \delta^b_a / G \left( x/\alpha_\nu \right) e_k, & k \neq a, \\ \left( \delta^b_a + \left( 1 - \delta^b_a \right) c_{a b - a} \left( x/\alpha_\nu \right) \right) e_b, & k = a, \end{cases} \] (29)
being \( c_\pm (x) = (q - 1/q) \, x^{\pm 1} / (xq - 1/qx) \). Then, from (24) and the first part of (29), it follows that

\[ T^{\text{mix}} b \Omega^f = \tau_b \delta^b_a \prod_{\nu \in \mathbb{Z}_N} 1/G \left( x/\alpha_\nu \right) \Omega^f = \tau_a \delta^b_a d \Omega^f \] (30)
if \( a \notin \text{Image } f \), i.e., if \( f (\nu) \neq a \) for all \( \nu \in \mathbb{Z}_N \). In particular

\[ T^{\text{mix}} b \Omega^f = 0 \] if \( a \notin \text{Image } f \cup \{ b \} \). (31)
Also, if \( f \in F_0 \) and \( 1 < a \leq m \), since in this case \( a \notin \text{Image } f \) [see (23)], we have that

\[ D^{\text{mix}} i \Omega^f = T^{i+1 \text{mix}} i \Omega^f = \tau_{i+1} d \Omega^f, \quad \text{for } 1 \leq i < m, \]
\[ D^{\text{mix}} j \Omega^f = T^{j+1 \text{mix}} j \Omega^f = 0, \quad \text{for } 1 \leq i, j < m, \quad i \neq j, \]
\[ C^{\text{mix}} i \Omega^f = T^{i+1 \text{mix}} i \Omega^f = 0, \quad \text{for } 1 \leq i < m, \] (32)
putting \( i = a - 1 \) in (30). Otherwise, let \( \sigma_a \) be the first integer such that \( f (\sigma_a) = a \), that is, \( f (\nu) \neq a \) for all \( \nu < \sigma_a \) and \( f (\sigma_a) = a \). Let us write
\[ \Omega^f = \Omega g^a \otimes e_a \otimes \Omega f^a, \quad \text{with} \quad g^a, f^a : \mathbb{Z}_{\sigma_a}, \mathbb{Z}_{N-\sigma_a-1} \to \mathbb{N}. \] (33)

If \( \sigma_a = N - 1 \), we take \( \Omega^{f^a} \) equal to 1. Then, using (24) and (29) again (note that \( a \) does not belong to \( \text{Image} \ g^a \)) we have

\[
\hat{T}_a^{\text{mix}} \Omega^f = \sum_{i=1}^{\text{rank} \zeta_{\sigma_a}} C_{a,i} \Omega g^a \otimes e_i \otimes \hat{T}_i^{\text{mix}} \Omega f^a, \quad (34)
\]

\[
C_{a,i} = D_{a,i} \prod_{\nu < \sigma_a} 1/G (x/\alpha_\nu); \quad D_{a,i} = \delta_i + (1 - \delta_i) c_{sg(i-a)} (x/\alpha_{\sigma_a}).
\]

Here, operators \( \hat{T}_i^{\text{mix}} \) are given by the last \( N - \sigma_a - 1 \) factors of \( T^{\text{mix}} \). From Eq. (34) applied to \( \hat{T}_i^{\text{mix}} \Omega^{f^a} \), since \( C_{a,i} \neq 0 \) for all \( 1 \leq i \leq \text{rank} \zeta_{\sigma_a} \), the non zero terms of (34) are those with \( i \) inside \( I_a = \{ i \in \text{Image} \ f^a \cup \{ b \} : i \leq \text{rank} \zeta_{\sigma_a} \} \). If \( b \) is the unique element of \( I_a \) and \( b \notin \text{Image} \ f^a \), then

\[
\hat{T}_a^{\text{mix}} \Omega^f = \tau_b C_{a,b} \left( \prod_{\nu \notin \sigma_a} 1/G (x/\alpha_\nu) \right) \Omega g^a \otimes e_b \otimes \Omega f^a
\]

\[
= \tau_b C_{a,b} d G (x/\alpha_{\sigma_a}) \Omega g^a \otimes e_b \otimes \Omega f^a. \quad (35)
\]

Otherwise, suppose there exists \( c_1 \in I_a \cap \text{Image} \ f^a \), and let \( \sigma_{ac_1} \) be the first integer such that \( f^a (\sigma_{ac_1} - \sigma_a) = c_1 \). Then \( \hat{T}_c^{\text{mix}} \Omega^{f^a} = \sum_{i \in I_{ac_1}} C_{ac_1,i} \Omega g^{ac_1} \otimes e_i \otimes \hat{T}_i^{\text{mix}} \Omega^{f^{ac_1}} \) with

\[
C_{ac_1,i} = D_{c_1,i} \prod_{\nu < \sigma_{ac_1}} 1/G (x/\alpha_\nu), \quad \Omega f^{ac_1} = \Omega g^{ac_1} \otimes e_{c_1} \otimes \Omega f^{ac_1},
\]

\[
I_{ac_1} = \{ i \in \text{Image} f^{ac_1} \cup \{ b \} : i \leq \text{rank} \zeta_{\sigma_{ac_1}} \}.
\]

A recursive process easily follows and the generic term reads

\[
\hat{T}_{c_{k+1}}^{\text{mix}} \Omega^{f^{ac_1 \ldots c_{k-1}}} = \sum_{i \in I_{ac_1 \ldots c_{k}}} C_{ac_1 \ldots c_{k+1},i} \Omega g^{ac_1 \ldots c_k} \otimes e_i \otimes \hat{T}_i^{\text{mix}} \Omega^{f^{ac_1 \ldots c_k}},
\]

with \( C_{ac_1 \ldots c_{k+1},i} = D_{c_{k+1},i} \prod_{\sigma_{ac_1 \ldots c_{k+1}} < \nu < \sigma_{ac_1 \ldots c_k}} 1/G (x/\alpha_\nu) \). Of course, \( c_1 \in I_a \cap \text{Image} f^a \),

\[
I_{ac_1 c_2 \ldots c_{j-1}} = \{ i \in \text{Image} f^{ac_1 c_2 \ldots c_{j-1}} \cup \{ b \} : i \leq \text{rank} \zeta_{\sigma_{ac_1 \ldots c_{j-1}}} \} \quad (36)
\]

and \( c_j \in I_{ac_1 c_2 \ldots c_{j-1}} \cap \text{Image} f^{ac_1 c_2 \ldots c_{j-1}} \) for \( 2 \leq j \leq k \). The process ends when \( b \in I_{ac_1 \ldots c_k} \), \( b \notin \text{Image} f^{ac_1 \ldots c_k} \), and we choose \( c_{k+1} = b \). In this case

\[
\hat{T}_{c_k}^{\text{mix}} \Omega^{f^{ac_1 \ldots c_{k-1}}} = \tau_b C_{ac_1 \ldots c_k,b} \prod_{\nu > \sigma_{ac_1 \ldots c_k}} 1/G (x/\alpha_\nu) \Omega g^{ac_1 \ldots c_k} \otimes e_b \otimes \Omega f^{ac_1 \ldots c_k}.
\]

In particular, writing \( T_a^{\text{mix}} \Omega^f = \sum_{g \in f} t_{ab}^g \Omega^g \) we have the given sequence of numbers \( c_1, \ldots, c_k \in \text{Image} f \) defines a function \( g \) such that \( t_{ab}^g \neq 0 \), being
\[ \Omega^g = \Omega^g \otimes e_{c_1} \otimes \Omega^{g_{ac_1}} \otimes e_{c_2} \otimes \Omega^{g_{ac_1c_2}} \otimes ... \]
\[ ... \otimes \Omega^{g_{ac_1\cdots c_{k-1}}} \otimes e_{c_k} \otimes \Omega^{g_{ac_1\cdots c_k}} \otimes e_b \otimes \Omega^{f_{ac_1\cdots c_k}}. \]

(37)

Note that \( \Omega^f \) can be written
\[ \Omega^f = \Omega^g \otimes e_a \otimes \Omega^{g_{ac_1}} \otimes e_{c_1} \otimes \Omega^{g_{ac_1c_2}} \otimes ... \]
\[ ... \otimes \Omega^{g_{ac_1\cdots c_{k-1}}} \otimes e_{c_k} \otimes \Omega^{g_{ac_1\cdots c_k}} \otimes e_{c_k} \otimes \Omega^{f_{ac_1\cdots c_k}}. \]

(38)

Furthermore, defining \( J_{fg} = \{ \sigma_a, \sigma_{ac_1}, \ldots, \sigma_{ac_1\cdots c_k} \} \) and \( c_0 = a \), and recalling \( c_{k+1} = b \), we have [compare with Eq. (35)]
\[ t_{fg}^{ab} = \tau_b d \prod_{j=1}^{k+1} D_{c_j-1,c_j} \prod_{\nu \in J_{fg}} G(x/\alpha_\nu). \]

(39)

If \( \# [f^{-1}(a)] = k + 1 \) and \( a = b \), the sequence of numbers \( c_i = a, i = 1, \ldots, k \), corresponds to the vector \( \Omega^g = \Omega^f \). Also, \( J_{ff} = f^{-1}(a) \) and accordingly, since \( D_{a,a} = 1 \),
\[ t_{ff}^{aa} = \tau_a d \prod_{\nu \in f^{-1}(a)} G(x/\alpha_\nu). \]

(40)

Comparing (37) and (38), we see that functions \( g \) such that \( t_{fg}^{ab} \neq 0 \) necessarily satisfy
\[ \text{Image } g \cup \{ a \} = \text{Image } f \cup \{ b \}. \]

(41)

In addition, for each element \( \mu \in \text{Image } f \), \( \mu \neq a, b \), function \( g \) must hold
\[ \# [g^{-1}(\mu)] = \# [f^{-1}(\mu)], \]

(42)

and
\[ \# [g^{-1}(a)] = \# [f^{-1}(a)] - (1 - \delta_{ab}), \]
\[ \# [g^{-1}(b)] = \# [f^{-1}(b)] + (1 - \delta_{ab}). \]

(43)

Now, defining the classes of functions \( C \subset F \), in such a way that \( f, g \in C \) iff
\[ \text{Image } f = \text{Image } g \quad \text{and} \quad \# [f^{-1}(\mu)] = \# [g^{-1}(\mu)] \]

for all \( \mu \) contained in their respective images, we can write
\[ T_{a}^{b \text{mix}} \Omega^f = \sum_{g \in C_{a}^{b}} t_{fg}^{ab} \Omega^g, \quad \text{if } f \in C, \]

(44)
where $C^\pm_a$ is the class given by functions $g$ satisfying (41), (42) and (43). Let us note that $C = C^\pm_a$ iff $a = b$. Then, denoting

$$C = \sum_{a < b} \text{span} \{ g \in H^\text{mix} : f \in X \} \text{ for each } X \subset F,$$

(45)

spaces $\mathcal{H}_C$ are $T^a_{\text{mix}}$-invariant for all $a$. On the other hand, when $a \neq b$ (since $C \neq C^\pm_a$), vector $\Omega^f$ can not be written as a linear combination of vectors $\Omega^{g}$’s appearing in (44) (they form a linearly independent set of vectors). That is, $T^b_{\text{mix}} \Omega^f$ is not proportional to $\Omega^f$. Also, if $a \notin \text{Image } f$, then $C^\pm_a = \emptyset$ and consequently $T^b_{\text{mix}} \Omega^f = 0$, such as follows from Eq. (31) for $a \neq b$. Last observations translate for operators $B^\text{mix}_j$ into equations

$$B^\text{mix}_j \Omega^f \neq c \Omega^f, \text{ if } f^{-1}(1) \neq \emptyset; \quad B^\text{mix}_j \Omega^f = 0 \text{ otherwise.}$$

(46)

Let us briefly study the reducibility of the action on $\mathcal{H}^\text{mix}$ of the algebra generated by operators $T^\text{mix}$. It follows from Eq. (36) that, if $M = \max_{\nu} \{ \text{rank } \zeta_{\nu} \}$, numbers $c_1, \ldots, c_k$ and $c_{k+1} = b$ must be smaller than or equal to $M$. This implies $T^b_{\text{mix}} = 0$ for $b > M$, and we can restrict ourself to the $a, b \leq M$ case. Also, comparing (37) and (38), if $f(\nu) > M$ then $g(\nu) = f(\nu)$. As a consequence, beside (41), (42) and (43), condition

$$g(\nu) = f(\nu) \forall \nu \in \mathbb{Z}_N \text{ such that } g(\nu), f(\nu) > M$$

(47)

is necessary in order to have $\iota_{ab}^{fg} \neq 0$. Thus, defining the classes $E \subset F$ as those whose functions satisfy (47), it is clear that spaces $\mathcal{H}_E$ are invariant under the action of $T^\text{mix}$. It actually can be found smaller invariant spaces inside $\mathcal{H}_E$, depending locally on the ranks of glueing matrices, but we will not discuss it here.

For the equal rank case we have $m = M$, and accordingly the classes $E$ are in bijection with elements of $F_0$. Thus, we can decompose $\mathcal{H}^\text{mix}$ into $T^\text{mix}$-invariant subspaces $\mathcal{H}_{E(f)}$ labeled by elements of $F_0$. In addition, by a simple inspection of coefficients (39), it can be shown the actions on $\mathcal{H}_{E(f)}$ and $\mathcal{H}_{E(g)}$ are equivalent provided $f^{-1}(1) = g^{-1}(1)$. Moreover, in the homogeneous case, namely, $\alpha_{\nu} = 1$ for all $\nu$, above equivalence still holds when $\# [f^{-1}(1)] = \# [g^{-1}(1)]$. 

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In the following subsection we diagonalize (when possible) the operator $A_{mix}$ restricted to each $\mathcal{H}_C$, and show its eigenvectors, when $C \subset F_0$, are precisely the pseudovacuum vectors we are looking for.

2. Diagonalization of $A_{mix}$ and vectors $\Phi^f$

Let us consider a class of functions $C$. Using Eqs. $\text{(40)}$ and $\text{(44)}$ for $a = b = 1$, and defining $a_{fg} \equiv t_{11}^{fg}$ for $f \neq g$, we have that

$$A_{mix} \Omega^f = a_f \Omega^f + \sum_{g \in C, g \neq f} a_{fg} \Omega^g, \quad a_f = \tau_1 d \prod_{\nu \in f^{-1}(1)} G(x/\alpha_{\nu}). \quad (48)$$

Now, we are going to show there exists a total order relation between the functions of $C$, such that w.r.t. this order we can write

$$A_{mix} \Omega^f = a_f \Omega^f + \sum_{g < f} a_{fg} \Omega^g. \quad (49)$$

In other words, operator $A_{mix}$ restricted $\mathcal{H}_C$ is represented by a triangular matrix w.r.t. the resulting ordered basis [recall Eq. $\text{(45)}$ for $X = C$].

To see that, let us consider a function $f \in C$. Assign to $f_\nu$ the number 1 if $f_\nu = 1$ or 0 if $f_\nu \neq 1$. Denote $b^f$ the binary expression related to the sequence $f_0, ..., f_{N-1}$. From Eqs. $\text{(37)}$ and $\text{(38)}$ for $a = b = 1$, we see that $b^f > b^g$ (as real numbers) for $f \neq g$, since unless one $f_\nu = 1$ were moved to the right. This implies $a_{fg} = 0$ if $b^f \leq b^g$, that is,

$$A_{mix} \Omega^f = a_f \Omega^f + \sum_{b^g < b^f} a_{fg} \Omega^g. \quad (50)$$

So let us define an order $<$ between the elements of $C$ by saying $g < f$ if $b^g < b^f$, and when $b^g = b^f$ we choose an arbitrary order. Using that and equation above, Eq. $\text{(49)}$ follows immediately.

Since eigenvalues of $A_{mix}$ are given by the numbers $a_f$, in order to insure its diagonalizability we can ask the considered model to be completely inhomogeneous, i.e., $\alpha_\nu \neq \alpha_\mu$ for all $\nu, \mu \in \mathbb{Z}_N$. Then $a_f \neq a_g$ provided $f^{-1}(1) \neq g^{-1}(1)$. Thus, eigenvalues are distinct, unless those related to $f$-th and $g$-th rows for which $f^{-1}(1) = g^{-1}(1)$. But $f^{-1}(1) \neq g^{-1}(1)$.
if \( b^f > b^g \). Therefore [see Eq. (50)], \( A^{mix} \) does not mix vectors related to rows with the same diagonal entries, and accordingly \( A^{mix} \) is diagonalizable. Actually, we just can insure \( A^{mix} = A^{mix}(x) \) is diagonalizable for almost all values \( x \) of the spectral parameter. Note that for some isolated points \( x_o \in \mathbb{C} \), we can have \( a_f(x_o) = a_g(x_o) \), in spite of condition \( f^{-1}(1) \neq g^{-1}(1) \) holds.

Using usual recursion formulae for diagonalizing triangular matrices, we can define for each subspace \( \mathcal{H}_C \) the basis \( \Phi^f, f \in \mathcal{C} \), given by

\[
\Phi^f = \begin{cases} 
\Omega^{min_C}, & f = min_C, \\
\Omega^f + \sum_{g < f} \chi_{fg} \Phi^g, & f > min_C,
\end{cases}
\]

with

\[
\chi_{fg} = \begin{cases} 
a_{g^+} / (a_{g^+} - a_g), & f = g^+, \\
(a_{fg} - \sum_{g < h < f} a_{fh} \chi_{hg}) / (a_f - a_g), & f > g^+.
\end{cases}
\]

Here \( min_C \) is the minimal \( f \in \mathcal{C} \) w.r.t. the defined order, and \( g^+ \) is the first element in \( \mathcal{C} \) bigger than \( g \). Equation (52) must be understood as a recursive formula on \( f \) for each \( g \). Because \([A^{mix}(x), A^{mix}(x')] = 0\) for all \( x, x' \in \mathbb{C} \) [that follows from commutation relations given in (3)], operators \( A^{mix}(x) \) can be diagonalized simultaneously. Thus numbers \( \chi_{fg} \) and vectors \( \Phi^f \) do not depend on the spectral parameter.

Let us note diagonal entries \( D^{mix}_{ii} \) can be diagonalized as above. But this is not enough to diagonalize \( t^{mix} \), since operators \( A^{mix} \) and \( D^{mix}_{ii} \) do not commute among themselves. Nevertheless, last operators restricted to \( \mathcal{H}_0 \) do commute, and accordingly can be simultaneously diagonalized. This follows from the facts that \( \mathcal{H}_0 \subset \ker C^{mix}_i \) and that Eq. (3) implies

\[
[D^{mix}_{ii}(x), A^{mix}(y)] = -B^{mix}_{i}(x) C^{mix}_i(y) c_-(x/y) + B^{mix}_{i}(y) C^{mix}_i(x) c_+(x/y).
\]

Now, let us see that vectors \( \Phi^f \) for \( f \in \mathcal{I}_0 \), given by (51) and (52), satisfy Equations (27) and (28). Since they are eigenvectors of \( A^{mix} \) with eigenvalues \( a_f \), the first part of (27) follows immediately. For the second part, note \( \Phi^f \) is a linear combination of vectors \( \Omega^g \) with
g inside C. Also note, if \( f \in F_0 \), then the class defined by \( f \) is inside \( F_0 \) too. Hence, using Eq. (24) we arrive at the wanted result. The same happens for (28) using Eq. (46).

For the equal rank case, it can be shown that \( a_{fg} = 0 \) for all \( f \in F_0 \). In fact, sets \( I_{c_1...c_i} \) defined by (26) (putting \( a = b = 1 \)) has 1 as the unique element, and consequently the only possible sequence is \( c_i = 1 \) for \( i = 1, ..., \# [f^{-1}(1)] - 1 \). Such sequence correspond to the vector \( \Omega_f \). Then, the latter is an eigenvector of \( A^{\text{mix}} \) (without any inhomogeneity condition).

In other words, \( A^{\text{mix}} \) restricted to \( \mathcal{H}_0 \) is represented by a diagonal matrix for the basis \( \Omega_f \), \( f \in F_0 \), and accordingly \( \Phi_f = \Omega_f \).

To end this subsection let us say last results, valid for monodromy matrices \( T^{\text{mix}} \) of the form (24), also holds for those given by Eq. (16) and satisfying condition (19). In fact, on the canonical basis \( e_1, ..., e_{n_\nu} \) of \( \mathbb{C}^{n_\nu} \), using Eq. (18) and (29), we have that

\[
\tilde{R}_{a}^{b} e_k = \begin{cases} 
\delta_a^b / G(x/\alpha_\nu) e_k, & k \neq a, \\
(\delta_a^b + 1 - \delta_a^b) (x/\alpha_\nu)^{2a_b} c_{sg(b-a)}(x/\alpha_\nu) e_b, & k = a.
\end{cases}
\]

Then, applying \( \tilde{T}^{\text{mix}}_a \) to a vector \( \Omega_f \) we arrive at Eqs. (30) or (34), depending on Image \( f \), where the second term of coefficients \( C_i \) [see Eq. (34)] must be just changed by a factor \( (x/\alpha_\nu)^{2a_b} \). Therefore, all above results follows. In particular, all we have said for \( A^{\text{mix}} \) is also true for \( \tilde{A}^{\text{mix}} \), and the former is diagonalizable iff so is the latter. There is a minor change in coefficients \( a_{fg} \), and consequently in the linear combinations (51) that define eigenvectors of \( \tilde{A}^{\text{mix}} \). Denoting the latter by \( \tilde{\Phi}_f \), and recalling Eq. (17), we conclude

**Theorem 1.** Given a mixed vertex model \( T^{\text{mix}} = \zeta_0 \cdot R_0 \otimes \zeta_1 \cdot R_1 \otimes ... \otimes \zeta_{N-1} \cdot R_{N-1} \cdot Y \), with glueing matrices \( \zeta_\nu = P_\nu \hat{\zeta}_\nu P'_\nu \) satisfying Eq. (19) and (21), and assuming \( A^{\text{mix}} \) is diagonalizable (e.g., \( T^{\text{mix}} \) is completely inhomogeneous), it follows that vectors

\[
\Phi_f \doteq (\otimes_{\nu \in \mathbb{Z}_N} P'_\nu)^{-1} \tilde{\Phi}_f, \quad f \in F_0,
\]

are pseudovacuum states for \( T^{\text{mix}} \) satisfying Eqs. (27) and (28). When rank \( \zeta_\nu = m \forall \nu \), \( A^{\text{mix}} \) is diagonalizable and \( \Phi_f \doteq (\otimes_{\nu \in \mathbb{Z}_N} P'_\nu)^{-1} \Omega_f \). \( \square \)
All that can be rephrased in terms of our original mixed monodromy matrices, i.e., in the form (7). We just must regard them as particular cases of (12) subject to (13).

B. Nested Bethe equations

Let \( T^{\text{mix}} \) be a monodromy matrix as that given in theorem above. Thanks to the algebra embeddings \( \text{YB}_{n-1} \hookrightarrow \text{YB}_n, n > 1 \), which are a direct consequence of equations

\[
[R_{n-1}]^{kl}_{ab} = [R_n]^{kl}_{ab}, \quad \text{for } 1 \leq a, b, k, l \leq n - 1,
\]

it follows that \( T^{b, \text{mix}}_a \) for \( a, b \leq m \) satisfy relations corresponding to the YB algebra \( \text{YB}_m \).

Then, following for each \( \Phi^f \in \mathcal{H}_0 \) analogous technics to the ones developed in refs. [7] [20], that is, proposing as eigenstates for \( t^{\text{mix}} \) [see (23)] the Bethe vectors

\[
\Psi^f = \Psi_{j_1} \ldots j_r \, \mathcal{B}_{j_1}^{\text{mix}} (x_1; \alpha) \ldots \mathcal{B}_{j_r}^{\text{mix}} (x_r; \alpha) \, \Phi^f, \quad j_1, \ldots, j_r < m,
\]

and separating in the so-called wanted and unwanted terms, we arrive at a set of nested Bethe ansatz equations which in its recursive form are given by

\[
\frac{\prod_{p=1}^{r_l} G \left( \frac{x_k^{(l)}}{x_p^{(l)}} \right)}{\prod_{\nu \in f^{-1}(1)} G \left( \frac{x_k^{(1)}}{\alpha_{\nu}} \right)} \Lambda_1 \left( \frac{x_k^{(1)}}{x_p^{(1)}} \right) + \tau_1 \prod_{p=1}^{r_1} G \left( \frac{x_k^{(1)}}{x_p^{(1)}} \right) = 0,
\]

\[
\frac{\prod_{p=1}^{r_l} G \left( \frac{x_k^{(l)}}{x_p^{(l)}} \right)}{\prod_{p=1}^{r_{l-1}} G \left( \frac{x_k^{(l-1)}}{x_p^{(l-1)}} \right)} \Lambda_l \left( \frac{x_k^{(l)}}{x_p^{(l)}} \right) + \tau_l \prod_{p=1}^{r_l} G \left( \frac{x_k^{(l)}}{x_p^{(l)}} \right) = 0, \quad (l > 1)
\]

and

\[
\Lambda_{m-1} (x) = \tau_m,
\]

\[
\Lambda_l (x) = \frac{\prod_{p=1}^{r_l} G \left( \frac{x^{(l+1)}}{x^{(l)}} \right)}{\prod_{u=1}^{r_{l+1}} G \left( \frac{x^{(l+1)}}{x^{(l)}} \right)} \Lambda_{l+1} (x) + \tau_{l+1} \prod_{p=1}^{r_{l+1}} G \left( \frac{x^{(l+1)}}{x^{(l)}} \right), \quad (l < m - 2)
\]

where \( l = 1, \ldots, m - 1, \ k = 1, \ldots, r_l \), and \( 0 \leq r_l \leq r_{l-1} \leq \# [f^{-1}(1)] \). Thus, Bethe equations related to a given \( \Phi^f \in \mathcal{H}_0 \), are the ones corresponding to an \( A_{m-1} \) type quasi-periodic vertex model with \( n_f = \# [f^{-1}(1)] \) sites per row and inhomogeneity vector \( \alpha_f = (\alpha_{\nu_0}, \alpha_{\nu_1}, \ldots, \alpha_{\nu_{n_f}}) \), such that \( \nu_i \in f^{-1}(1) \) and \( \nu_i < \nu_{i+1} \) for all \( i \in \mathbb{Z}_{n_f} \). When
For each solution
\[ x = \{ x^{(l)} = (x_1^{(l)}, ..., x_r^{(l)}): l = 1, ..., m - 1 \} \]
of Equations (53) and (54),
\[ \Lambda^f(x; x) = d \prod_{k=1}^{r_1} G \left( x/x_1^{(1)} \right) \Lambda_1(x) + \tau_1 d \prod_{\nu \in \Phi^0} G \left( x/\alpha_\nu \right) \prod_{k=1}^{r_1} G \left( x_1^{(1)}/x \right) \]  
(55)
gives an eigenvalue of \( t^{\text{mix}} \). Note that \( \Lambda^f(x; x) = \Lambda^g(x; x) \) if \( f^{-1}(1) = g^{-1}(1) \). This is the main source of degeneracy for the transfer matrix. It can be seen each \( \Lambda^f(x; x) \) differs by a factor \( \prod_{\nu \notin f^{-1}(1)} 1/G \left( x/\alpha_\nu \right) \) from the corresponding eigenvalue related to the mentioned \( A_{m-1} \) model. Eigenvectors \( \Psi^f(x) \), i.e., the Bethe vectors, can also be given recursively, but now through vectors \( \Psi_l \in (\mathbb{C}^{m-l})^{\otimes r_l} \) with coordinates \( (\Psi_l)^{j_1, ..., j_{r_l}} \) (w.r.t. the canonical basis of \( \mathbb{C}^{m-l} \)) such that
\[ \Psi^f(x) = (\Psi_1)^{j_1, ..., j_{r_1}} \ B_{j_1}^{\text{mix}} \left( x_1^{(1)}; \alpha \right) \ ... \ B_{j_{r_l}}^{\text{mix}} \left( x_{r_l}^{(1)}; \alpha \right) \Phi^f, \]  
(56)
for \( 1 \leq l \leq m - 2 \)
\[ \Psi_l = (\Psi_{l+1})^{j_1, ..., j_{r_l+1}} \ B_{j_1}^{(m-l, r_l)} \left( x_1^{(l+1)}; x^{(l)} \right) \ ... \ B_{j_{r_{l+1}}}^{(m-l, r_l)} \left( x_{r_{l+1}}^{(l+1)}; x^{(l)} \right) \omega_l, \]
and \( \Psi_{m-1} = 1 \). Here \( j_1, ..., j_{r_{l+1}} < m \). We are denoting by \( \omega_l \) the pseudovacuum for the pure monodromy matrix \( T^{(m-l, r_l)} \). Let us mention, in the \( l \)-th level of nesting process the involved monodromy matrix actually is the twisting
\[ T^{(m-l, r_l)} \cdot \Upsilon_l, \text{ being } \Upsilon_l = \text{diag} (\tau_1, ..., \tau_{m-l}), \]
which also has \( \omega_l \) as pseudovacuum vector.

Summing up, we have constructed a set of eigenvectors for \( t^{\text{mix}} \) by applying creation operators \( B_{j}^{\text{mix}} \)'s over all \( \Phi^f, f \in \mathcal{F}_0 \). In the following section we address the combinatorial completeness of that set of states.

By last, let us say that equations (53) and (54) do not depend neither on permutations \( P_\nu, P'_\nu \) defining the glueing matrices of \( T^{\text{mix}} \) (recall conditions of theorem above), nor on
the set of ranks of the latter. They only depends on the minimum \( m = \min_{\nu} \{ \text{rank} \zeta_{\nu} \} \) of that set, on the boundary matrix \( \Upsilon \), and on the inhomogeneity vector \( \alpha \). Hence, assuming complete integrability, the spectrum of the related transfer matrix \( t^{\text{mix}} \), which would be given by the numbers \( \Lambda^{f}(x;\mathbf{x}) \) defined in (52), only depends on \( m, \Upsilon \) and \( \alpha \). Accordingly,

**Theorem 2.** Assuming complete integrability, every mixed model with glueing matrices satisfying Equation (19) is physically equivalent to one with monodromy matrix of the form (20) and satisfying the equal rank condition: \( \text{rank} \zeta_{\nu} = m \) for all \( \nu \).

\[ \square \]

**C. Combinatorial completeness**

In this section we are going to show that Eq. (56) (varying indices \( j \) from 1 to \( m \), functions \( f \) in \( F_{0} \), and \( \mathbf{x} \) along solutions of (53) and (54)) defines unless \( \dim \mathcal{H}^{\text{mix}} = \prod_{\nu \in \mathbb{Z}_{N}} n_{\nu} \) different vectors. That is to say, we have a set of Bethe vectors from which, *a priori*, a basis of eigenstates for the related transfer matrix can be extracted. To see that, we shall assume combinatorial completeness of Bethe ansatz equations related to the \( A_{n-1} \) vertex models, i.e., for a model with \( N \) sites in a row we suppose there is unless \((n-1)^{r} \binom{N}{r} \) different solutions for the Bethe equations corresponding to \( r \) creation operators. This has been shown for \( n = 2 \) (see for instance [21]), but we do not know about any similar result for bigger \( n \). In our case, we would be saying for each vector \( \Phi^{f} \) with \( f \in F_{0} \), there exists unless a number \((m-1)^{r} \binom{n_{f}}{r} \) of different solutions of (53) and (54) corresponding to \( r_{1} = r \) creation operators. Recall that \( n_{f} = \# \{ f^{-1}(1) \} \). Let us first see why this assumption is useful for our purposes.

It is enough to analyze the case of monodromy matrices given by (20). The other cases, i.e., those given by (14) and satisfying (19), follow analogously. So let us come back to §IV.A.1 and consider the action of operators \( \mathcal{B}_{j}^{\text{mix}} \) with \( j < m \), on vectors \( \Omega^{f} \) with \( f \in F_{0} \). Suppose first that \( \text{rank} \zeta_{\nu} = m \) for all \( \nu \). For \( a = 1 \) and \( b = j + 1 \), sequences \( c_{i} = 1, i = 1, \ldots, k \), with \( 1 \leq k < n_{f} \) define terms proportional to vectors \( \Omega^{g} = \Omega^{f_{\nu,j}} \), with
\( \mu \in f^{-1}(1), f_{\mu,j}(\nu) = f(\nu) \) for all \( \nu \neq \mu \) and \( f_{\mu,j}(\mu) = j + 1 \). That is, we change a vector \( e_1 \) by a vector \( e_{j+1} \) in position \( \mu \in f^{-1}(1) \). They are the only possible sequences. Thus, the action of each \( B_j^{mix} \), \( j = 1, ..., m - 1 \), on a vector \( \Omega^f \) gives rise to a linear combination of \( n_f \) linearly independent vectors. Existence of \( n_f \) different solutions to Eqs. (53) and (54) for \( r_1 = 1 \) and for each \( j \), is a necessary condition to obtain \( n_f \) l.i. eigenstates from the set of Bethe vectors. Then, varying \( j \) from 1 to \( m - 1 \), we shall have, a priori, \( (m - 1) n_f \) l.i. eigenstates. Applying \( B_i^{mix} \) and \( B_j^{mix} \) we have \( (m - 1)^2 \) vectors, each one of them having \( n_f (n_f - 1)/2 \) l.i. terms. In general, if we apply \( r \) creation operators to \( \Omega^f \), we have \( (m - 1)^r \) vectors with related \( \binom{n_f}{r} \) terms. Now it becomes clear why our assumption is needed. The same argument can be given for the general rank case. There, when an operator \( B_j^{mix} \) acts on \( \Omega^f \) we have as above the terms proportional to \( \Omega^{f_{\mu,j}}, \mu \in f^{-1}(1) \), together with additional terms given by vectors \( \Omega^{h_{\sigma,j}} \) with \( h \) belonging to the same class of \( f \). Thus, the latter appears as terms when \( B_j^{mix} \) is applied to \( \Omega^h \). Accordingly, in order to avoid overcounting, we do not have to take them into account.

Let us come back to our original problem. If combinatorial completeness holds there exists unless a number \( \sum_{r=0}^{n_f} (m - 1)^r \binom{n_f}{r} = ((m - 1) + 1)^{n_f} = m^{n_f} \) of Bethe vectors for each function \( f \in F_0 \). Thus, since \( 0 \leq n_f \leq N \) for every \( f \in F \), the total number of Bethe vectors is \( \sum_{f \in F_0} m^{n_f} = \sum_{k=0}^{N} m^k p_k \), being \( p_k \) the number of functions \( f \in F_0 \) such that \( n_f = k \). Let us calculate \( p_k \). It is clear that the number of functions \( f \) in \( F_0 \) with the same pre-image \( f^{-1}(1) \) is

\[
\prod_{\nu \in \mathbb{Z}_N} (n_\nu - m)^{\varepsilon_\nu}, \quad \varepsilon_\nu = \begin{cases} 
0, & \nu \in f^{-1}(1), \\
1, & \text{otherwise}.
\end{cases}
\] (57)

In terms of numbers \( \varepsilon_0, ..., \varepsilon_{N-1} \), the condition \( n_f = \# [f^{-1}(1)] = k \) can be characterized by equality \( \varepsilon_0 + ... + \varepsilon_{N-1} = N - k \). Then, in order to obtain \( p_k \) we must sum over all configurations of \( \varepsilon_0, ..., \varepsilon_{N-1} \) (\( \varepsilon_\nu \) equal to 0 or 1), such that last condition holds, i.e.,

\[
p_k = \sum_{\varepsilon_0, ..., \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} (n_\nu - m)^{\varepsilon_\nu} \delta_{\varepsilon_0 + ... + \varepsilon_{N-1} = N - k}.
\] (58)
Accordingly,

\[
\sum_{k=0}^{N} m^k p_k = \sum_{k=0}^{N} m^k \left( \sum_{\varepsilon_0, \ldots, \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} (n_\nu - m)^{\varepsilon_\nu} \delta_{\varepsilon_0 + \ldots + \varepsilon_{N-1}, N-k} \right)
\]

\[
= \sum_{\varepsilon_0, \ldots, \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} (n_\nu - m)^{\varepsilon_\nu} \sum_{k=0}^{N} m^k \delta_{\varepsilon_0 + \ldots + \varepsilon_{N-1}, N-k}
\]

\[
= \sum_{\varepsilon_0, \ldots, \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} (n_\nu - m)^{\varepsilon_\nu} m^{N-(\varepsilon_0 + \ldots + \varepsilon_{N-1})}
\]

\[
= m^N \sum_{\varepsilon_0, \ldots, \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} \left( \frac{n_\nu}{m} - 1 \right)^{\varepsilon_\nu}
\]

But

\[
\sum_{\varepsilon_0, \ldots, \varepsilon_{N-1}} \prod_{\nu \in \mathbb{Z}_N} \left( \frac{n_\nu}{m} - 1 \right)^{\varepsilon_\nu} = \prod_{\nu \in \mathbb{Z}_N} \left( \sum_{\varepsilon_\nu} \left( \frac{n_\nu}{m} - 1 \right)^{\varepsilon_\nu} \right)
\]

\[
= \prod_{\nu \in \mathbb{Z}_N} \left( \left( \frac{n_\nu}{m} - 1 \right)^0 + \left( \frac{n_\nu}{m} - 1 \right) \right) = \prod_{\nu \in \mathbb{Z}_N} \frac{n_\nu}{m} = m^{-N} \prod_{\nu \in \mathbb{Z}_N} n_\nu,
\]

and consequently \( \sum_{f \in F_0} m^{n_f} = \prod_{\nu \in \mathbb{Z}_N} n_\nu \), as we wanted to see.

**CONCLUSIONS**

From last equation we see that, under conditions of Theor. 1 and assuming complete integrability, \( \mathcal{H}^{mix} \) can be decomposed into a direct sum of \( m^{n_f} \)-dimensional spaces \( \mathcal{H}_f \), each one of them generated by the Bethe vectors related with some \( f \) inside \( F_0 \). Note this sum, in general, is not orthogonal w.r.t. the usual scalar product in \( \bigotimes_{\nu \in \mathbb{Z}_N} \mathbb{C}^{n_\nu} \). Thinking of the quantum spin ring related to our vertex model, whose Hamiltonian \( H \) is constructed from the logarithmic derivative (if there exists) of the transfer matrix, states of \( \mathcal{H}_f \) can be interpreted as those of an anisotropic \( A_{m-1} \) type spin chain with \( n_f \) sites, which are localized on the subring \( \mathbb{Z}_{n_f} \prec f^{-1}(1) \subset \mathbb{Z}_N \). In other words, we have decomposed a mixed spin model as a direct sum of \( A_{m-1} \) type ones with different numbers of sites and generically different inhomogeneities. Multiplicity of these models is given by \( (57) \) [recall eigenvalues \( (55) \) only depend on \( f \) through \( f^{-1}(1) \)]. In connection with Theor. 2 let us say that for the equal rank case, since we have \( \mathcal{H}_f = \mathcal{H}_{E(f)} \) (see at the end of §IV.A.1), described decomposition (which results orthogonal) and mentioned multiplicity are direct consequences of the facts that last spaces are \( T^{mix} \)-invariant, and that corresponding actions on spaces \( \mathcal{H}_f \) and \( \mathcal{H}_g \) are equivalent when \( f^{-1}(1) = g^{-1}(1) \).
In the homogeneous case we have in addition actions on $\mathcal{H}_f$ and $\mathcal{H}_g$ are equivalent still when $n_f = n_g$. In other terms, for the homogeneous equal rank case we can write $\mathcal{H}^{mix}$ as the orthogonal direct sum $\mathcal{H}^{mix} = \bigoplus_{k=0}^N \mathbb{C}^{p_k} \otimes \mathcal{H}_{f_k}$ [see (58) for numbers $p_k$], being $f_k$ some function with $n_{f_k} = k$.

Concluding, we have presented a procedure for glueing different integrable vertex models in such a way that the integrability of the whole system is preserved. This procedure relies on some generalization of the coalgebra structure to the case of rectangular quantum matrices and their representations, enhancing the deep linking between these algebraic structures and integrability.

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