Nodal Structure of Toroidal Standing Alfvén Waves and Its Implication for Field Line Mass Density Distribution

Kazue Takahashi and Richard E. Denton

Abstract We have conducted a statistical study of toroidal mode standing Alfvén waves detected by the Van Allen Probes spacecraft in the dayside inner magnetosphere, with an emphasis on the nodal structure of the fundamental through fifth harmonics. We developed a technique to accurately assign harmonic mode numbers to peaks in the power spectra of the electric ($E_r$) and magnetic ($B_\phi$) field components of toroidal waves and then determine the spectral intensities of $E_r$ and $B_\phi$ and the coherence and cross-phase between these field components for each harmonic. The magnetic latitude (MLAT) dependence of these quantities was statistically examined to determine the location of the nodes. In addition to the equatorial nodes located close to the equator (MLAT = 0), we identified several nodes away from the equator within the MLAT range from −20° to +20°. We found that the $E_r$−$B_\phi$ cross-phase is very close to ±90° except near the nodes, indicating that the fixed-end approximation is appropriate in modeling dayside toroidal waves. Noting that the node latitudes depend on the distribution of the mass density ($\rho$) along the background magnetic field, we inferred the distribution from the nodes observed at $L \approx 4$–6. If we adopt a model field line mass density ($\rho$) distribution of the form $\rho \propto (1/r)^{\alpha}$, where $r$ is geocentric distance to the field line and $\alpha$ is a free parameter, the statistically determined node latitudes indicate that $\alpha \sim 1.5$ is appropriate for both the plasmasphere and the plasmatrough.

Plain Language Summary Interaction of the solar wind plasma with the outer boundary of Earth’s magnetosphere results in compressional magnetohydrodynamic waves in the 1–100 mHz band that propagate into the magnetosphere. Each geomagnetic field line is stimulated by the waves and executes its own discrete eigenmode oscillations because the field line is tied to the rigid ionosphere. The oscillations contain valuable information on how the plasma mass is distributed along the field line. Unlike the mass distribution on a guitar string, which is uniform, the mass distribution of geomagnetic field lines changes with distance from the ionosphere. The mass distribution affects various magnetospheric phenomena but is difficult to directly determine using particle experiments on spacecraft. Fortunately, we can estimate the mass distribution from the location of the nodes of the field line eigenmode oscillations, which can be determined using spacecraft observations of the electric and magnetic fields of the oscillations. In this study, we use data from the Van Allen Probes spacecraft to demonstrate that the node location can be determined for the fundamental through fifth harmonics and describe how this information is used to estimate the field line mass density distribution.

1. Introduction

Standing Alfvén waves in the magnetosphere are a well-known phenomenon with a long history of research in theory (e.g., Dungey, 1954) and observation (e.g., Sugira & Wilson, 1964). The waves have been extensively studied with regard to their basic physical properties (e.g., Radoski & Carovillano, 1966), excitation mechanisms (e.g., Chen & Hasegawa, 1974), interaction with particles (e.g., Southwood & Kivelson, 1981), and relation to the plasma mass density (e.g., Obayashi & Jacobs, 1958). In this study, we address a relatively unexplored property of the waves: the mode structure, that is, the variation of the amplitude and phase of the electric and magnetic field perturbations along the background magnetic field. We pay special attention to the location of the nodes. We focus on toroidal standing Alfvén waves (hereinafter referred to as toroidal waves), which are the most frequently observed class of magnetospheric ultralow frequency (ULF) waves. In space, toroidal waves are recognized as narrowband oscillations in the radial component ($E_r$) of the electric field ($E$) (e.g., Cahill et al., 1986) and the azimuthal component ($B_\phi$) of the magnetic field ($B$) (e.g.,...
Arthur et al., 1977). The plasma bulk velocity (V) also oscillates because it is related to $E$ and $B$ through the magnetohydrodynamics equation $E = -V \times B$ (e.g., Junginger et al., 1984).

Toroidal waves are excited by driver fast mode waves through steady field line resonance (Chen & Hasegawa, 1974; Southwood, 1974) or impulsive triggering (Takahashi & Heilig, 2019). The driver waves may originate from solar wind dynamic pressure pulses (Sarris et al., 2010), foreshock ULF waves (Claussen et al., 2009), or nightside disturbances associated with substorms (Takahashi et al., 1996). Because these source disturbances often have a broad frequency spectrum, they are capable of exciting toroidal waves at multiple harmonics (Cahill et al., 1986; Engebretson et al., 1986; Takahashi & McPherron, 1982; Waters et al., 1994). The presence of harmonics is central to the present study. We refer to the $n$th harmonic of a toroidal wave as the $T_n$ wave and denote its frequency as $f_{T_n}$, with $n = 1$ for the fundamental mode.

The motivation for our mode structure analysis is twofold. First, we are interested in confirming theoretical predictions of the mode structure of toroidal waves in the dayside magnetosphere. Although dayside toroidal waves are usually assumed to be close to the idealized fixed-end modes (perfect reflection at the ionosphere) because of the high ionospheric conductivity (Newton et al., 1978), few studies have examined the structure in a quantitative manner. We compare spacecraft observations with theoretical models.

Second, we are interested in using the mode structure in constraining mass density models. Observed frequencies of toroidal waves have been used to estimate the mass density using a technique known as magnetoseismology (Menk & Waters, 2013). In using this technique, one needs to assume a functional form of the mass density ($\rho$) variation along the background magnetic field line because the frequencies are related to the integral of the Alfvén velocity over the field line. The field line distribution has important implications for the way the background plasma establishes an equilibrium and how various plasma waves behave (Denton et al., 2006). Previous studies used the frequency of toroidal harmonics to constrain the model (Takahashi & Denton, 2007; Takahashi et al., 2004). Observationally determined nodal structure of the waves is a new approach to the field line mass density distribution. We discuss the results of the mode structure analysis from this point of view.

The remainder of this paper is as follows. Section 2 presents theoretical models of toroidal waves. Section 3 describes experiments. Section 4 presents data analysis in the frequency domain. Section 5 presents statistical results on the frequency and mode structure of toroidal harmonics. Section 6 presents discussion, and section 7 presents the conclusions.

### 2. Model Toroidal Waves

To guide the data analysis, we review key properties of theoretical fixed-end toroidal waves. We obtain the frequencies and mode structures of the waves using the toroidal wave equation of Cummings et al. (1969) with the mass density specified by the power law model (Radoski & Carovillano, 1966)

$$\rho = \rho_{eq}(LR_E / r)^\alpha,$$

where $\rho_{eq}$ is the equatorial mass density, $L$ is the magnetic shell parameter, $R_E$ is the Earth’s radius, $r$ is geocentric distance to the field line, and the power law index $\alpha$ specifies how $\rho$ varies along the field line. If $\alpha = 0$, $\rho$ is constant along the field line. A positive (negative) $\alpha$ means that $\rho$ increases (decreases) from the equator toward the ionosphere. This mass density model has been used widely (Dent et al., 2006; Kabbin et al., 2007; Menk et al., 1999; Orr & Matthew, 1971; Price et al., 1999; Waters et al., 1996) not only for its simplicity but also for its relevance to theory (Angerami & Carpenter, 1966). We solve the equation for $L = 5$, which is representative of observations by Van Allen Probes. Although $\alpha$ is typically assumed to be in the range 0–4 (Orr & Matthew, 1971; Poulter et al., 1984; Takahashi et al., 2004; Vellante & Förster, 2006), we vary $\alpha$ from $-6$ to $+6$. We compute the mode frequencies and node latitudes for integer values of $\alpha$. For non-integer $\alpha$ values, we obtain the frequencies and node latitudes by linear interpolation. One could adopt other field line density models with more free parameters (Dent et al., 2004), but we do not consider such models in this study.
2.1. Mode Frequencies

Figure 1 shows the $\alpha$ dependence of $f_{T1}$ through $f_{T5}$. In Figure 1a, the frequencies are normalized to $f_{T1}$ for $\alpha = 0$. The normalized frequencies decrease as $\alpha$ increases. This occurs because the integral of $\rho$ over the field line is larger for larger $\alpha$ when $\rho_{eq}$ is held constant, that is, the frequencies are lower when the field lines are “heavier.” Although Figure 1a is helpful in understanding how $\alpha$ is physically related to the frequencies, normalization by $f_{T1}$ is not a good choice in practice when we try to constrain $\alpha$ using the observed frequencies. The reason is that $T1$ waves are not the easiest to detect, as indicated in Figure 9, which is presented in Section 2. This leads us to use $f_{T3}$ as the reference for frequency normalization, as was done previously (Takahashi & Denton, 2007).

Figure 1b shows the frequencies normalized to $f_{T3}$ for each value of $\alpha$. As $\alpha$ increases, $f_{T1}/f_{T3}$ and $f_{T2}/f_{T3}$ increase while $f_{T4}/f_{T3}$ and $f_{T5}/f_{T3}$ decrease. The rate of the change of the ratio with $\alpha$ is greatest for $T1$ waves: a 13.0% change from 0.238 for $\alpha = 0$ to 0.269 for $\alpha = 3$. For $T2$, $T4$, and $T5$ waves, the changes are much smaller: 2.4% (from 0.620 to 0.635), −1.1% (from 1.378 to 1.363), and −1.7% (1.756–1.726). This means that $f_{Tn}/f_{T3}$ for $n \geq 2$ is less sensitive to changes in $\alpha$ than for $n = 1$.

For a fixed value of $\alpha$, the variation of the frequency ratios with $L$ is small. If we take $f_{T2}/f_{T3}$ as an example, the ratio decreases from $L = 4$ to $L = 6$ by 2.4% for $\alpha = 0$ and by 2.0% for $\alpha = 3$. This justifies the comparison made in Section 5.5 between the theoretical frequency ratios for $L = 5$ and the frequency ratios statistically determined using satellite observations made at $L = 4–6$.

2.2. Mode Structures

Figure 2 shows the mode structure of $T1$–$T5$ waves obtained for $\alpha = 1$. Each panel shows the normalized amplitudes of the $E_r$ and $B_\phi$ components as a function of magnetic latitude (MLAT). These components oscillate with a cross-phase (denoted $\theta_{E_r, B_\phi}$) of ±90°, which means that the time-averaged Poynting flux along the background magnetic field vanishes. We define $\theta_{E_r, B_\phi}$ in the range from −180° to +180° and give it a positive sign when $E_r$ leads $B_\phi$. The sign of $\theta_{E_r, B_\phi}$ changes each time a node of either field component is crossed. In the MLAT domain shaded orange, $\theta_{E_r, B_\phi} = 90^\circ$. In the MLAT domain shaded blue, $\theta_{E_r, B_\phi} = -90^\circ$.

For the $n$th harmonic, the number of nodes, excluding those at the northern and southern field line foot points, is $n – 1$ for $E_r$ and $n$ for $B_\phi$. The $T1$ wave (Figure 2a) has only one node (in the $B_\phi$ component), which is located at the equator. Therefore, for this wave, a spacecraft sees $\theta_{E_r, B_\phi} = -90^\circ$ in the southern hemisphere and $\theta_{E_r, B_\phi} = 90^\circ$ in the northern hemisphere regardless of the magnitude of MLAT. As the harmonic number increases, the number of nodes increases, and the MLAT spacing between the nodes decreases. As a consequence, even a spacecraft with a low orbital inclination encounters off-equatorial nodes. Consider Van Allen Probes as an example. The spacecraft had an orbital inclination of 10° and covered the MLAT range from −20° to +20°, as indicated by the black horizontal bar at the bottom of Figure 2. If we take the $T5$ wave (Figure 2e) as an example, the spacecraft have the possibility of encountering up to five nodes, excluding those located at MLAT = ±20°.

Figure 3 shows the $\alpha$ dependence of the theoretical node latitude (denoted MLAT$_{node}$). For both $E_r$ and $B_\phi$, the non-equatorial nodes move away from the equator as $\alpha$ increases. For example, Figure 3b indicates that the $E_r$ node moves from 10.0° for $\alpha = 0$ to 17.3° for $\alpha = 6$. The implication is that we can infer the $\alpha$ value with an accuracy of ∼1°, if we can determine MLAT$_{node}$ with an accuracy of ∼1°. Of course, this is contingent upon having a good model for the background magnetic field, which also controls the mode structure. The real magnetic field can be very different from a dipole field, depending on the location and geomagnetic...
activity. Nevertheless, Figure 3 gives us good motivation to search for the node locations as a means to determine $\alpha$.

To evaluate whether the mode structures derived using a dipole field are good enough for comparison with observations, we examine the mode structures for realistic model magnetic fields. Figure 4 shows a dipole and T89c (Tsyganenko, 1989) comparison of the mode structures of T3 waves excited on field lines that pass the dipole equator at geocentric distance 5 $R_E$ and at 12 h magnetic local time (MLT). The structures were obtained by numerically solving the wave equation derived by Singer et al. (1981). For simplicity, we considered the case of zero dipole tilt angle. Two T89c fields are considered, one for Kp = 3 (IOPT = 4) and the other for Kp $\geq$ 6 (IOPT = 7), where IOPT is the input parameter required for the T89c model. We refer to the dipole and the two T89c fields as model 1, model 2, and model 3, respectively. The mass density is given by Equation 1 with $\alpha = 1$. The figure shows the absolute value of the $E_\nu$ and $B_\phi$ components of the wave, with
the amplitudes normalized as in Figure 2. The mode structures are symmetric about the magnetic equator, so the figure covers only the northern hemisphere (MLAT ≥ 0).

The mode structure differs very little between model 1 and model 2. With model 1, the nodes are found at 10.6° for $E_v$ and at 26.3° for $B_\phi$. With model 2, the nodes are found at 10.4° and 25.8°. Within the MLAT range covered by Van Allen Probes, only the $E_v$ node will be detected, and the node latitudes differ only by 0.2° between model 1 and model 2. We also examined the mode structures on other field lines (not shown) and found very small changes from those shown. Specifically, on field lines passing the equator at 4 $R_E$, the $E_v$ node is located at 10.5° with model 1 and at 10.3° with model 2. On field lines passing the equator at 6 $R_E$, which is close to the apogee of the spacecraft, the $E_v$ node is located at 10.6° with model 1 and at 10.9° with model 2.

We need to be careful when we consider the mode structures during geomagnetically active periods. The model 3 results shown in Figure 4 indicate a notable departure of the node latitudes from those obtained using model 1 or model 2. With model 3, we find the nodes at lower latitudes, 9.1° ($E_v$) and 22.8° ($B_\phi$). We also need to pay attention to the local time dependence of the magnetic field, which requires use of realistic models such as T89c. However, we believe that the dipole model is adequate in the present study. The reason is that we discuss the mode structures by statistically processing satellite observations that were made mostly in the noon sector, at $L = 4–6$, and during quiet to moderately disturbed times (Kp < 3).

We assume that the node locations do not change unless the background magnetic field or plasma mass distribution changes. Compressional Pc5 waves are known to exhibit frequency doubling and waveform deformation near the magnetic equator, which can be explained by a latitudinal oscillation of the equatorial node that is phase-locked to the waves (Takahashi et al., 1987). Frequency doubling and waveform deformation have also been reported for fundamental poloidal waves (Takahashi et al., 2011). We have not seen frequency doubling for toroidal waves and believe that these waves do not induce detectable node oscillations.

3. Experiments and Data

Data used in this study were acquired by Van Allen Probes A and B. The twin spacecraft made scientific measurements from 2012 to 2019 on elliptical orbits with an inclination of ~10°, apogee of ~5.8 $R_E$ (maximum dipole $L$ value of 6.6), and orbital period of ~9 h. The spacecraft were spin stabilized with the spin periods maintained at ~11 s. The spin axes approximately pointed to the Sun with offset angles lower than 27°.

The Van Allen Probe experiments relevant to the present study are the electric field spectrum analyzers (for electron density) and the fluxgate magnetometers included in the Electric and Magnetic Field Instrument Suite and Integrated Science (EMFISIS) experiments (Kletzing et al., 2013) and DC and low-frequency electric field experiments included in the electric field and waves experiments (Wygant et al., 2013). The E-field data used in the present study were obtained by the spinfit method applied to the two orthogonal components measured in the spacecraft spin plane and rotated into modified geocentric solar ecliptic (MGSE) coordinates (Wygant et al., 2013). The data have a time resolution of the spacecraft spin period (~11 s), placing the upper (Nyquist) limit of our wave frequency analysis ($f_{Nyq}$) at ~45 mHz. When the angle $\lambda_B$ between the measured $B$-field and the spin plane is sufficiently large (>10°), we can use the $\mathbf{B} \cdot \mathbf{E} = 0$ assumption to get a reliable estimate of the electric field component parallel to the spacecraft spin axis ($E_{MGSE}$). According to Ali et al. (2016), the error in $E_{MGSE}$ is 0.5 mV/m when $\lambda_B = 5.75°$ and decreases quickly as $\lambda_B$ increases. Our

![Figure 3. (a–d) Theoretical dependence of the magnetic latitudes (MLATs) of the nodes of toroidal waves on the mass density power law index $\alpha$, plotted for the second through fifth harmonics. The solid (open) circles indicate the $E_v (B_\phi)$ nodes. The node locations are for $L = 5$ in a dipole field.](image)
threshold $\lambda_b$ of 10° is lower than the 15° that is normally recommended (e.g., Dai et al., 2013). We use this relaxed threshold because we are mainly interested in finding wave frequencies and need not be very strict about the accuracy of the amplitude of $\mathbf{E}$.

To extract perturbations associated with ULF waves and examine their spectral and polarization properties, we rotate the measured $\mathbf{B}$-field vectors into locally defined magnetic field aligned (MFA) coordinates. The MFA coordinate axes are defined using a reference magnetic field $\mathbf{B}_{\text{ref}}$ and the spacecraft position vector $\mathbf{R}$ relative to the center of the Earth. We obtain $\mathbf{B}_{\text{ref}}$ by fitting a function to the three components of the observed $\mathbf{B}$-field vector time series given in geocentric solar ecliptic (GSE) coordinates. The function is a polynomial of the form $c_0 + c_1 \tau + c_2 \tau^2 + c_3 \tau^3 + c_4 \tau^4$, where $\tau$ is UT rescaled to the range from −0.5 to 0.5 for the data segment selected for analysis. The segment length is 15 min in the routine moving data window analysis of toroidal waves described below. The coefficients $c_0$ to $c_4$ are determined using the least-squares method. In the MFA coordinate system, the parallel or compressional ($\mu$) component is in the direction of $\mathbf{B}_{\text{ref}}$, the azimuthal ($\phi$) component (positive eastward) is in the direction of $\mathbf{B}_{\text{ref}} \times \mathbf{R}$, and the radial ($\nu$) component is given by $\mathbf{e}_\nu = \mathbf{e}_\phi \times \mathbf{e}_\mu$. The transverse components $B_\phi$ and $B_\nu$ are perturbations about $\mathbf{B}_{\text{ref}}$ by definition, and the compressional perturbation is given by $B_\mu = |\mathbf{B}_{\text{ref}}|$. By using this method, we can effectively remove slow variations of the magnetic field, either spatial or temporal, without using magnetic field models.

We can define $\mathbf{B}_{\text{ref}}$ by taking running averages of the measured $\mathbf{B}$-field, but we prefer the polynomial fitting technique. In taking running averages one usually selects a data window with a fixed length ($T_w$). To define perturbations in a time segment $[t_0, t_1]$, the running average must be calculated using $\mathbf{B}$-field data starting at $t_0 - T_w/2$ and ending at $t_1 + T_w/2$. This means that perturbations near $t_0$ and $t_1$ are affected by $\mathbf{B}$-field data outside $[t_0, t_1]$. This can be a problem when we analyze spacecraft data taken near perigee, where the background $\mathbf{B}$-field changes rapidly in a nonlinear manner. By contrast, polynomial fitting does not require data outside $[t_0, t_1]$.

We rotated the spinfit $\mathbf{E}$-field vector samples into an MFA system that uses the spin ($\sim 11$ s) averages of the observed $\mathbf{B}$-field to define the field-aligned component. This MFA system was used to produce $\mathbf{E}$-field data files for the entire Van Allen Probes mission period, before the start of this study. As far as wave $\mathbf{E}$-fields are concerned, the difference between this system and the MFA system defined above for wave $\mathbf{B}$-fields is negligible because the $E_\mu$, $E_\phi$ or $E_\nu$ values are essentially the same regardless of whether we use the observed $\mathbf{B}$ or $\mathbf{B}_{\text{ref}}$ in applying the $\mathbf{B} \cdot \mathbf{E} = 0$ assumption in coordinate transformation. To make the spectral properties of the $\mathbf{E}$-field vectors consistent with those of the $\mathbf{B}$-field vectors in the MFA system, we remove the trend, also a polynomial of fourth degree, from $E_\mu$ and $E_\phi$.

To compare observations with the theoretical mode structures (see Figure 2), we use dipole coordinates for the spacecraft position. We define $L$, MLAT, and MLT using the Gauss coefficients relevant to the centered dipole that are specified by Thébault et al. (2015) for the International Geomagnetic Reference Field model.

### 4. Frequency-Domain Analysis

To conduct mode structure analysis, we first need to separate harmonics of toroidal waves in the frequency domain. This section describes how we determine $f_{\text{m}}$. With the exception of geostationary satellites, spacecraft move in $L$ and MLAT. $L$ shell crossing means continuous changes in $f_{\text{m}}$ (Engebretson et al., 1986), and changing MLAT means that spacecraft cross $E_\phi$ and $B_\mu$ nodes (Cahill et al., 1986). In this situation, correctly assigning harmonic mode numbers to multiple spectral peaks is often challenging. In addition, ULF waves
other than toroidal waves are present in the magnetosphere. Spectral peaks caused by these waves need to be excluded.

In previous studies using spacecraft data, toroidal wave frequencies were determined by visually inspecting the dynamic spectra of $B_y$ (Min et al., 2013; Nosé et al., 2011, 2015, 2018; Takahashi et al., 2010), $E_x$ (Takahashi et al., 2004), or both (Takahashi et al., 2018). In some studies, the plasma bulk velocity (Takahashi et al., 2014) or ion flux anisotropy (Takahashi et al., 2002) in the azimuthal direction was used as a proxy to $E_x$. Our approach is similar. However, to improve the accuracy of mode identification, we also compute the cross-phase between $E_x$ and $B_y$ and compare it with a theoretical model.

4.1. Dynamic Spectra

Figure 5 illustrates toroidal waves detected by Van Allen Probe B on orbit 1596. Figures 5a–5e are time-frequency spectrograms of the $E_x$ autopower $S_{E_x, E_x}$, $B_y$ autopower $S_{B_y, B_y}$, $E_x$-$B_y$ cross-power $S_{E_x, B_y}$, $E'_y$-$B'_y$ coherence $\gamma_{E'_y, B'_y}$, and $E'_y$-$B'_y$ cross-phase $\theta_{E'_y, B'_y}$. These are derived from the elements of the $2 \times 2$ spectral matrix constructed from the Fourier transform of input $E_x$ and $B_y$ time series as described by Bendat and Piersol (1971). We compute these spectral parameters in a 15 min data window, which is shifted in 5 min steps. The window size gives a frequency resolution of 1.1 mHz for the raw Fourier components. A three-point (3.3 mHz) boxcar smoothing in frequency is applied to the elements of the spectral matrix before deriving the spectral parameters displayed. Figures 5f and 5g show the $\lambda_E$ angle defined in section 3 and the electron density $n_e$ derived using plasma wave spectra (Kurth et al., 2015). The condition $\lambda_E > 10^°$ is satisfied except after 1700 UT, which means that the $E_x$ data are mostly reliable. The rapid changes of $n_e$ occurring at $\sim$1000 UT ($L \sim 3.8$) and $\sim$1530 UT ($L \sim 4.6$) indicate plasmapause crossings. The $n_e$ variation is very smooth both inside and outside the plasmapause.

Multiharmonic toroidal waves are evident in all spectrograms. The waves produce multiple spectral lines, which change smoothly in frequency but vary significantly in intensity. The most persistent spectral line starts at $\sim$1040 UT ($L \sim 4.6$) at 30 mHz, reaches a minimum frequency of 18 mHz at $\sim$1300 UT ($L \sim 5.9$), and lasts until $\sim$1520 UT ($L \sim 4.9$). This spectral line is attributed to T3 waves, as described below. T3 waves reappear after $\sim$1600 UT and indicate a rapid frequency change as the spacecraft moves inward within the plasmasphere ($L < 4.1$). T1 and T2 waves follow the same trend. The toroidal waves are similar to those observed by other elliptically orbiting spacecraft (Cahill et al., 1986; Clausen et al., 2009; Sarris et al., 2009; Takahashi et al., 2004). Features to be ignored in this example are the elevated $E''_y$-$B''_y$ coherence $\gamma_{E''_y, B''_y}$ and $S_{E''_y, B''_y}$ autopower $S_{E''_y, E''_y}$ and $S_{E''_y, B''_y}$ panel. Multiharmonic toroidal waves intensities near perigee and the instrumental noise lines in $S_{E_x, E_x}$, labeled “Noise.” According to Takahashi et al. (2015), the most persistent noise appears between 20 and 25 mHz, and the intensity of the noise varies from orbit to orbit and also during an orbit. This noise limits our ability to determine toroidal wave frequencies using $E_x$. Fortunately, this is not a very serious problem in our statistical analyses of toroidal waves because the noise occupies a small fraction of the frequency range surveyed. Also, there are many orbits on which the noise is very weak or nonexistent, and the $B_y$ component is free from this noise line.

We find the $\theta_{E'_y, B'_y}$ spectrogram (Figure 5e) to be particularly useful in identifying the harmonic mode of toroidal waves. The $\theta_{E'_y, B'_y}$ value is color coded and displayed only when $\gamma_{E'_y, B'_y} > 0.5$. The spectrogram shows several bands of either blue or orange shade, with their frequency-versus-time patterns matching those found in the $S_{E_x, E_x}$, $S_{B_y, B_y}$ and $S_{E_x, B_y}$ panels. The shade of $\theta_{E'_y, B'_y}$ depends on both the harmonic mode and the position of the spacecraft relative to the $E_x$ and $B_y$ nodes. Because we know the spacecraft position and can estimate the node positions from theoretical models, we can place a constraint on the possible harmonic modes using information available from the $\theta_{E'_y, B'_y}$ spectrogram. Details are presented below.

4.2. Sample Data Segment

We explain how we determine the harmonic modes using a sample 20 min data segment. The segment is marked by the black bar at the bottom of Figure 5e. The spacecraft was located at $L \sim 5.7$ and MLAT$\sim10^°$. Figures 6a and 6b show that $E_x$ and $B_y$ oscillate with different waveforms. The $E_x$ oscillation consists of a
strong (2 mV/m peak-to-peak) long period (∼3 min) component and weaker shorter-period components. The \( B_\phi \) oscillation is dominated by a 1 min component with peak-to-peak amplitudes of ∼1 nT, but the oscillation is not purely monochromatic.

Figures 6c–6g show that the oscillations consist of multiple harmonics. The five vertical dashed lines indicate the frequencies of T1, T2, T3, T5, and T6 waves, identified based on the following spectral features. First, these harmonics produce peaks in \( S_{E_\nu E_\nu} \) (Figure 6c), \( S_{R_\phi R_\phi} \) (Figure 6d), and \( S_{E_\nu R_\phi} \) (Figure 6e). The \( S_{E_\nu E_\nu} \) peak appearing at 25 mHz is one of the known noise lines, not to be attributed to a T4 wave. Second, \( \gamma_{E_\nu R_\phi} \) is elevated at the T1, T2, T3, and T5 frequencies (Figure 6f) to higher than 0.75, which we selected as the threshold to obtain reliable \( \theta_{E_\nu R_\phi} \). The low \( \gamma_{E_\nu R_\phi} \) value (0.3) at the T6 frequency is explained by the

**Figure 5.** (a–e) Dynamic display of spectral parameters computed from the toroidal field components \( E_\nu \) and \( B_\phi \) measured by Van Allen Probe B on orbit 1596. The labels “T1,” “T2,” and “T3” indicate the fundamental, second, and third harmonics of toroidal waves, respectively. The label “Noise” in the \( E_\nu \) spectrogram indicates known noise lines. (f) Magnetic field elevation angle from the spacecraft spin plane, defined in the range 0°–90°. The shading indicates the domain 0°–10° in which the \( \mathbf{B} \cdot \mathbf{E} = 0 \) assumption needs to be used with caution in deriving the third component of \( \mathbf{E} \). (g) Electron number density.
Figure 6. A 20 min snapshot of toroidal waves detected by Van Allen Probe B on the orbit shown in Figure 5. (a) $E_v$ waveform. (b) $B_y$ waveform. (c) $E_v$ power spectrum $S_{E_v,E_v}$. (d) $B_y$ power spectrum $S_{B_y,B_y}$. (e) $B_y-E_v$ cross-power spectrum $S_{B_y,E_v}$. (f) $E_v-B_y$ coherence spectrum $\gamma_{E_v, B_y}$. A horizontal dashed line is drawn at $\gamma_{E_v, B_y} = 0.75$. (g) $E_v-B_y$ cross-phase spectrum $\theta_{E_v, B_y}$, shown if $\gamma_{E_v, B_y} > 0.75$. The sign of $\theta_{E_v, B_y}$ is positive if $E_v$ leads $B_y$.

4.3. Interactive Identification of Mode Frequencies

We have developed a routine procedure to identify toroidal wave frequencies and harmonic mode numbers using Van Allen Probes data. Automated frequency detection procedures have been developed for ground magnetometer data (Berube et al., 2003; Chi et al., 2013; Sandhu et al., 2018; Wharton et al., 2018), incorporating the cross-spectral analysis technique (Waters et al., 1991). This approach cannot be used for spacecraft data because no spacecraft pair maintains the short radial distance that is required for the technique to work properly. Fortunately, we can determine toroidal wave frequencies using data from individual spacecraft. As Figure 5 shows, dynamic spectra generated from spacecraft data exhibit clear spectral peaks arising from toroidal waves. A major difficulty is assigning a harmonic mode number to each peak. To reduce this difficulty, we pay attention to the nodal structures of toroidal waves.

Figure 7 illustrates our approach using Van Allen Probe B data for orbit 1596. Figure 7a shows that the spacecraft was mostly in the prenoon sector, where toroidal waves in the Pc3–4 band (7–100 mHz) are routinely detected (Anderson et al., 1990). Our first step is to find spectral peaks that are potentially associated with toroidal waves. We do this by simply searching for peaks in $S_j$ computed using the same 15 min moving data window as that used in generating Figure 5, where $S_j$ represents $S_{E_v, E_v}$, $S_{B_y, B_y}$, or $S_{E_v, B_y}$. If a spectral peak is found at frequency $f_p$, we examine whether we can define the full-width-at-half-maximum (FWHM, delimited by $f_0$ and $f_1$) around the peak, within the entire band covered by the spinfit data, 0–$f_{\text{NHF}}$. If we can, and there are no other spectral peaks between $f_0$ and $f_1$, we compute the weighted frequency $f_w$, given by

$$f_w = \frac{\int_{f_0}^{f_1} S_j(f) df}{\int_{f_0}^{f_1} S_j(f) df}.$$

This frequency is a good representation of the wave frequency when a wave has a bandwidth spanning several Fourier components (Takahashi & McPherron, 1982). We then integrate $S_{E_v, E_v}$, $S_{B_y, B_y}$, $S_{B_y, B_y}$, and $S_{B_y, B_y}$ from $f_0$ to $f_1$ and take the square root of the integral to obtain the root-mean-square amplitudes $\langle S_{E_v} \rangle$, $\langle S_{B_y} \rangle$, $\langle S_{B_y} \rangle$, and $\langle S_{B_y} \rangle$. We similarly define the band integral of the $E_v-B_y$ coherence and cross-phase, denoted $\langle \gamma_{E_v, B_y} \rangle$ and $\langle \theta_{E_v, B_y} \rangle$, respectively. If there are multiple peaks between $f_0$ and $f_1$, we discard the spectral peak at $f_0$ to avoid confusion in the subsequent analyses. The large colored dots in 7b are the $f_w$ data points that resulted from the peak search in $S_{E_v, E_v}$, $S_{B_y, B_y}$, and $S_{E_v, B_y}$. The color indicates whether the associated $\langle \theta_{E_v, B_y} \rangle$ is positive (orange) or negative (blue).
In the second step, we narrow down the $f_w$ samples by examining whether they are reliable and consistent with toroidal waves. We reject an $f_w$ sample if it is:

1. derived from $S_{E_x E_y}$ or $S_{E_x B_y}$ and $\lambda < 10^\circ$
2. derived from any of $S_{E_x E_y}$, $S_{E_x B_y}$, or $S_{B_y B_x}$, and $(f_1 - f_0)/f_w > 1$
3. derived from $S_{E_x B_y}$, and $\langle f_{E_x B_y} \rangle < 0.5$
4. derived from $S_{E_x E_y}$, and $\langle \delta E_x \rangle < \langle \delta E_y \rangle$
5. derived from $S_{B_y B_x}$, and $\langle \delta B_x \rangle < \langle \delta B_y \rangle$ or $\langle \delta B_y \rangle < \langle \delta B_x \rangle$
6. derived from $S_{E_x B_y}$, and $\langle \delta E_x \rangle < \langle \delta E_y \rangle$ or $\langle \delta B_y \rangle < \langle \delta B_x \rangle$

**Figure 7.** Example of data display used in interactive frequency and harmonic mode identification. (a) Spacecraft location in $L$ (radius) versus magnetic local time (MLT) (azimuth). (b) Frequencies identified in the spectra computed from the observed $E_x$ and $B_y$ time series. See text for symbols and colors of the data points. (c) Model frequencies for the T1–T5 waves. The color indicates whether the $E_x$-$B_y$ cross-phase is positive (orange) or negative (blue). (d) Electron number density derived from plasma wave spectra. (e) MLAT of the spacecraft.
The small black dots in Figure 7b indicate the $f_w$ samples that survived the downselection procedure. In the third step, we generate a plot of model wave frequencies with information on the $E_\nu - B_\phi$ cross-phase included to guide manual determination of the harmonic mode numbers. In the example shown in Figure 7c, the frequencies and cross-phase are obtained by solving the dipole field toroidal wave equation (Cummings et al., 1969). Here we adopt $\alpha = 1$ for the mass density model and an average ion mass ($M$, given by $\rho/n_e$) of 4 amu. Once this setup is done, the $n_e$ data shown in Figure 7d give the Alfvén velocity along the field line that is necessary to solve the wave equation. To reduce computation, the equation is solved at discrete $L$ shells (2, 3, ..., 7) for a fixed equatorial mass density (1 amu cm$^{-3}$), and the wave frequencies and the node latitudes are saved in a lookup table. The theoretical frequencies at the spacecraft at any UT epoch are obtained by interpolation of the frequencies saved in the lookup table and then adjusting the frequencies by the mass density at the spacecraft given by $Mn_e$. The node latitudes are also interpolated. From the MLAT of the spacecraft (Figure 7e) and the theoretical node latitudes, we can predict the sign of $\theta_{E_\nu - R_y}$ at the spacecraft, which is color coded in Figure 7b. We caution not to take the model frequencies too literally, because $M$ varies with solar activity, geomagnetic activity, and $L$ (Denton et al., 2011; Nosé et al., 2015; Takahashi et al., 2004). Also, the real magnetic field will be very different from the dipole model when the geomagnetic activity level is high, resulting in large differences between the observed and model frequencies.

In the fourth step, we visually compare Figures 7b and 7c and use an interactive computer tool to select $f_w$ samples that belong to a target harmonic mode. The tool allows one to use a computer mouse to draw a rectangular area called “marquee” in an orbital $f_w$ plot displayed on a computer screen in the format of Figure 7b and to save all data points in the area into a temporary data array by choosing the “save” option from a pulldown menu. The saved data points appear on the screen with a different symbol. If one selects wrong data points by mistake, they can be removed from the temporary array by invoking the “remove” option. We typically repeat the marquee procedure several times to cover all possible $f_w$ samples belonging to a target harmonic mode. When the selected data points appear satisfactory for the entire orbit, the data in the temporary array are written out into an output file with the file name specifying the spacecraft, orbit number, and the harmonic mode.

In the present example, the target is the third harmonic (T3), which is highlighted by the thick line in Figure 7c. The model predicts a switch of the sign of $\theta_{E_\nu - R_y}$ at 1418 UT from positive to negative, in association with the satellite crossing of an $E_\nu$ node MLAT = 10.6°. In Figure 7b, we find a sequence of $f_w$ samples that fit the model, in terms of both the frequency trend and the sign of the cross-phase. Because the real field line mass density distribution is unknown, the model prediction of the node latitudes based on a single $\alpha$ value should have some error relative to the observation. However, we expect the error to be of the order of a few degrees at most and find that the predicted cross-phase plotted in the format of Figure 7c is quite helpful to determine harmonic modes. The procedure is repeated for the T1 and T2 modes. The downselected frequencies are denoted $f_wT1$, $f_wT2$, and $f_wT3$.

Our frequency selection procedure involves human judgments, and the results certainly depend on the individual (operator) who runs the procedure. To make proper judgments, the operator needs to be familiar with the properties (e.g., instrument noise) of the spacecraft experiments and magnetospheric ULF waves, and the outcome depends on the experience of the operator. However, we believe that our results on the mode frequencies and node latitudes depend little on the operator because the judgment errors are likely random and averaged out in statistics. In case studies, we can afford paying attention to far more details of the wave properties to determine toroidal wave frequencies with high confidence (e.g., Takahashi et al., 2015). A possible approach in the future to remove the operator dependency is to generate high-confidence frequency data files for a subset of Van Allen Probes orbits and use the files to develop a neural network technique to automatically determine the frequencies. This approach has been taken in the Neural-network-based Upper hybrid Resonance Determination technique (Zhelavskaya et al., 2016) developed for automated determination of electron density using plasma wave spectra.
5. Statistics

We determined $f_{wT1}$, $f_{wT2}$, and $f_{wT3}$, using Van Allen Probe A and B data for January–June 2014. We use the resulting data set to statistically evaluate the frequencies and mode structures of toroidal harmonics.

5.1. Overview

Figures 8a–8c show the locations of wave detection. Each data point is the spacecraft position at the center of the 15 min data window mapped to the equatorial plane of solar magnetic (SM) coordinates along the dipole field line. The samples for each harmonic mode are distributed in the noon sector and in the $L$ range from $\sim 3$ to 6.6 (geostationary orbit). Figures 8d–8f show time series plots of all $f_{wT1}$, $f_{wT2}$, and $f_{wT3}$ samples. The vertical spread of the data points results mostly from satellite motion in $L$. On each orbit, the frequencies change with $L$ as shown in Figure 7.

The $f_{wT1}$ samples are found mostly between 4 and 20 mHz (Figure 8d). The lower cutoff occurs because we use a 15-min data window for spectral analysis. The frequency resolution of the Fourier transform with this data window is 1.1 mHz. To define a spectral peak after taking three-point smoothing, we need to have a few Fourier components on either side of the peak, which is the reason we do not have $f_w$ values lower than $\sim 4$ mHz. The absence of $f_{wT1}$ samples above $\sim 30$ mHz results from our inability to detect T1 waves in the 15 min data window when a spacecraft is located at $L < 3$ where the frequency changes rapidly as the spacecraft moves in $L$.

The $f_{wT2}$ samples are more evenly spread out in frequency (Figure 8e). There is no obvious low frequency cutoff originating from the finite data window length. The variation of the lower cutoff of $f_{wT2}$, seen most clearly in April and May, is caused by changes in the space environment. As the geomagnetic activity increases (see Dst in Figure 8g), the plasmapause shrinks, and the spacecraft are in the plasmatrough when located outside of $L \sim 4$. At a given $L$, the frequencies are higher in the plasmatrough than in the plasmasphere because the mass density is lower in the plasmatrough.

The $f_{wT3}$ samples are distributed similarly to $f_{wT2}$ (Figure 8f). A notable difference is that $f_{wT3}$ reaches the upper cutoff ($f_{Nyq}$, 45 mHz) of the 11 s data. An example of the cutoff is found in Figure 5b at 1610 UT, when the spacecraft was in the plasmasphere and moving inward.

5.2. Plasma Domains

We are interested in separating the plasmasphere and plasmatrough because ions in these domains come from different source regions and go through different transport processes. Toroidal waves could provide valuable information on the difference between the domains, for example, regarding the ion composition and field line mass density distribution. We could use $n_e$ data to identify the domain. However, the two plasma domains can be distinguished using a statistical approach without relying on the $n_e$ data.

Figure 9 shows the number (given in color code) of the $f_{wT1}$ (left column), $f_{wT2}$ (middle column), and $f_{wT3}$ (right column) samples in $L$ versus frequency bins, sorted by the source spectrum (rows). The $f_{wT1}$ distributions form a group that resides in the $L$ range 2.0–4.5 with the frequency falling from $\sim 20$ to $\sim 6$ mHz and another group that resides in the $L$ range 4.0–6.5 with the frequency falling from $\sim 15$ to $\sim 6$ mHz. The $f_{wT2}$ and $f_{wT3}$ distributions show similar patterns. The distinction between the two groups is most clearly seen in Figure 9f ($f_{wT3}$ samples derived from $S_{\theta_B}$).

The interpretation of the two groups is easy. The low-$L$ group comes from measurements made in the plasmasphere, and the high-$L$ group comes from measurements made in the plasmatrough. The two groups overlap in $L$ because the plasmapause distance depends on the solar wind and geomagnetic conditions. The large vertical (frequency) spread of the samples in the second group can be explained by larger variability of the mass density outside the plasmasphere than inside.

To distinguish the two $f_{wT3}$ groups, we introduce a model boundary frequency $f_{bT3}$.
which is shown by the dotted line in Figure 9f. We assumed a power-law $L$ dependence following previous studies of toroidal waves detected by spacecraft (e.g., Takahashi et al., 2014). This particular equation was obtained from a least-squares fit of a power-law function to several data points taken along the trough in the 2-D distribution shown in Figure 9f. In the following analyses, measurements are judged to be made in the plasmasphere (plasmatrough) if $f_{wT3} < f_{BT3}$ ($f_{wT3} \geq f_{BT3}$).

We find that the largest number of frequency samples is obtained using $S_{B_0}$ for T3 waves ($N = 15,565$). This was also the case in studies that used magnetic field data from geostationary satellites (Takahashi &
This leads us to use $f_{wT3}$ as the key parameter in our statistical data analysis.

5.3. Frequency Ratios and Field Line Mass Density Distribution

Frequency ratios among toroidal harmonics depend on the field line mass density distribution, as illustrated in Figure 1. This dependence has been used to infer the distribution from observed toroidal wave frequencies (Takahashi & Denton, 2007; Takahashi et al., 2004). We took the same approach to the Van Allen Probes data and show the results in Figure 10. To generate the figure, we used $f_{wT3}$ derived from $S_{E_{E}E_{v}}$ to normalize all $f_{wTn}$ values that are determined using $S_{E_{E}E_{v}}$, $S_{R_{v}R_{v}}$, and $S_{E_{v}R_{v}}$ at a common time step (see section 4.3). If no $f_{wT3}$ value is determined from $S_{R_{v}R_{v}}$ at the given time step, we skip that time step. The normalized

Denton, 2007; Takahashi & McPherron, 1982). This leads us to use $f_{wT3}$ as the key parameter in our statistical data analysis.

Figure 9. (a–c) $L$-frequency joint distributions of the $f_{wT1}$, $f_{wT2}$, and $f_{wT3}$ samples obtained using $S_{E_{E}E_{v}}$. The total number of samples is shown in the upper left corner of each panel (d–f) Same as (a–c) but for $S_{R_{v}R_{v}}$. The curve shown in panel (f) separates plasmasphere and plasmatrough samples. (g–i) Same as (a–c) but for $S_{E_{v}R_{v}}$. 

5.3. Frequency Ratios and Field Line Mass Density Distribution

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frequencies are shown for the plasmasphere and plasmatrough as time series (Figures 10a and 10c) and as histograms (Figures 10b and 10d).

We defined the \( f_{wT3}/f_{wT3} \) domains (bands) occupied by T1–T5 waves by paying attention to notable features of the total \( f_{wT3}/f_{wT3} \) histogram for the plasmatrough (Figure 10d). The bands are 0.10–0.36, 0.36–0.82, 0.82–1.19, 1.19–1.55, and 1.55–1.92. The T1 band was defined based on the lower cutoff (0.10) and the first minimum (0.36) of the histogram. The limits for the remaining bands are defined to be the midpoints between the neighboring peaks of the histogram. We apply the same band definition to the plasmasphere because the total \( f_{wT3}/f_{wT3} \) histogram for the plasmasphere are peaked very near the center of the bands. In each \( f_{wT3}/f_{wT3} \) band, we calculated the mean (denoted \( \langle f_{wT3}/f_{wT3} \rangle \)) and standard deviation of the mean (denoted \( \sigma_{\text{mean}} \)). The results are summarized in Table 1 along with the number of samples (\( N \)). The mean values are also shown to the right of Figures 10b and 10d. We do not define \( f_{wT4}/f_{wT3} \) in the plasmasphere because the total histogram does not exhibit a clear peak.

The frequency ratios are similar to those obtained in a previous study (Takahashi et al., 2004) that used data from Combined Release and Radiation Effects Satellite (CRRES), which had orbits similar to those of Van Allen Probes. The CRRES study used \( f_{T1} \) detected in the \( E_{i} \) component as the key frequency and found \( f_{T2}/f_{T1} \sim 2.5 \) and \( f_{T3}/f_{T1} \sim 4.0 \) for \( L = 4–6 \) and MLT = 12–18 h. These ratios translate to \( f_{T1}/f_{T3} \sim 0.25 \) and \( f_{T2}/f_{T3} \sim 0.63 \), which are close to our present results.
Figures 11a and 11b illustrate estimation of $\alpha$ using $\langle f_{wTn}/f_{wT3}\rangle$ for the plasmasphere and the plasmatrough. The black curves in these figures are the theoretical relationship between $\alpha$ and $f_{wT}/f_{wT3}$ repeated from Figure 1b with the horizontal and vertical axes switched. The red vertical dashed lines indicate the $\langle f_{wTn}/f_{wT3}\rangle$ values taken from Table 1. The filled red circles mark the intersections of the red lines with the black curves, which give $\alpha$ estimates labeled “$\alpha_{freq\_obs}$.” As an indicator of the error, we evaluated the $\alpha$ values corresponding to $\langle f_{wTn}/f_{wT3}\rangle \pm \sigma_{mean}$ and show them using short red horizontal bars. The errors are small compared with the difference in $\alpha_{freq\_obs}$ between different harmonics. In both plasma domains, the estimated $\alpha_{freq\_obs}$ varies among the harmonic modes, which means that the mass density model given by Equation 1 should be taken only as first approximation.

Given the harmonic dependence of $\alpha_{freq\_obs}$, we define an optimal $\alpha$ value by minimizing

$$e_f(\alpha) = \sum_n \left( \frac{\langle f_{wTn} / f_{wT3}\rangle}{\langle f_{wT}/f_{wT3}\rangle_\alpha} - \frac{\langle f_{wTn} / f_{wT3}\rangle}{\langle f_{wT}/f_{wT3}\rangle} \right)^2,$$

where $\langle f_{wTn} / f_{wT3}\rangle_\alpha$ is the theoretical ratio given as a function of $\alpha$. With this approach, which follows Taka-
hashi and McPherron (1982), we obtain $\alpha = 1.8$ for the plasmasphere (Figure 11a) and a nearly identical result, $\alpha = 1.9$, for the plasmatrough (Figure 11b).

Regarding the differences of $\alpha_{freq\_obs}$ among the harmonics, we examine the accuracy of $\langle f_{wTn}/f_{wT3}\rangle$. In Figures 9a, 9d, and 9g, we find a 4-mHz lower cutoff to the lower-frequency (plasmasphere) group of $f_{wTn}$ data points for $L > 4$. This is an artifact of the finite length of the data window for Fourier transform, as described in Section 5.1. The cutoff results in a sharp lower cutoff of the distribution of the normalized frequency for the plasmasphere at 0.16 (Figure 10b). A cutoff occurs in the plasmatrough distribution as well but at a lower value of 0.10 (Figure 10d). This difference can be explained by the fact that toroidal wave frequencies are higher in the plasmatrough; most of them are higher than 4 mHz. As a consequence of this skewing, $\langle f_{wTn}/f_{wT3}\rangle$ is higher for the plasmasphere. To avoid this skewing, we applied Equation 4 only to $\langle f_{wT3}/f_{wT3}\rangle$ and obtained $\alpha = 2.0$ (green horizontal dashed line in Figure 11a). We also replaced the $\langle f_{wT3}/f_{wT3}\rangle$ value with that for the plasmasphere and obtained $\alpha = 1.2$ (blue horizontal dashed line in Figure 11a). This exercise indicates that the skewing of the $f_{wT3}/f_{wT3}$ distribution does not produce a large error in estimating plasmaspheric $\alpha$.

### 5.4. Example of a Node Crossing

Figure 12 illustrates how we derive the spectral parameters and use them to detect the nodes of toroidal waves. We focus on T3 waves using data from Van Allen Probe B acquired on orbit 1596 (Figures 5 and 7) as an example. Our desire is to continuously track the behavior of the spectral parameters at the T3 frequency. If the waves were continuously detected, this would be easy. The reality is that toroidal waves often become undetectable either from node crossings or from disappearance of driver fast mode waves. In Figure 12a, $f_{wT3}$ samples are found from 1055 to 1520 UT, with a gap at ~1315 UT. By comparison, $f_{wT1}$ and $f_{wT2}$ samples are sparse but are present outside the time span of the $f_{wT3}$ samples.

When $f_{wT3}$ is not directly determined from power spectra, we can still define a T3 frequency using the T1 or T2 frequencies, which are included in the frequency survey illustrated in Figure 8. This is done first by

---

**Table 1**

| $n$, Mode number | $f_{wT1}/f_{wT3}$ band | $\langle f_{wTn}/f_{wT3}\rangle \pm \sigma_{mean}$ (N) | $\langle f_{wTn}/f_{wT3}\rangle \pm \sigma_{mean}$ (N) |
|------------------|-------------------------|---------------------------------|---------------------------------|
|                  | Plasmasphere            |                                 | Plasmatrough                    |
| 1                | 0.10–0.36               | 0.2509 ± 0.0007 (2,906)         | 0.2395 ± 0.0004 (13,981)        |
| 2                | 0.36–0.82               | 0.6233 ± 0.0011 (7,189)         | 0.6244 ± 0.0006 (24,765)        |
| 3                | 0.82–1.19               | 1.0036 ± 0.0004 (15,643)        | 0.9970 ± 0.0004 (27,142)        |
| 4                | 1.19–1.55               | 1.3594 ± 0.0008 (11,675)        |                                 |
| 5                | 1.55–1.92               | 1.7345 ± 0.0010 (8,335)         | 1.7249 ± 0.0012 (5,308)        |
estimating \( f_{wT3} \) from \( f_{wT1} \) or \( f_{wT2} \) using the frequency ratios found in Figure 10d for the plasmatrough:

\[
\frac{f_{wT3}}{f_{wT1}} = \frac{0.24}{0.2} = 0.62; \quad \frac{f_{wT3}}{f_{wT2}} = \frac{0.62}{0.2} = 3.1.
\]

We use the ratios for the plasmatrough because we have a larger number of \( f_{wTn} \) samples from that region than from the plasmasphere. We have confirmed that adopting the plasmaspheric ratios does not change the outcome of the mode structure analysis described in Section 5.5. The plus symbols in Figure 12b indicate both the \( f_{wT3} \) values directly derived and those estimated using \( f_{wT1} \) or \( f_{wT2} \). Given the possibility of multiple \( f_{wT3} \) estimates at a time step, we obtain a unique value simply by calculating the mean of the \( f_{wT3} \) values defined above. The mean is denoted \( f_{T3\_mean} \) and is plotted by black filled circles in Figure 12b. We use \( f_{T3\_mean} \) as the key to define the mean frequencies of other harmonics. Adopting the plasmatrough \( (f_{wTn}/f_{wT3}) \) values listed in Table 1, we get the following mean frequencies:

\[
\begin{align*}
    f_{T1\_mean} &= 0.24 f_{T3\_mean}, \\
    f_{T2\_mean} &= 0.62 f_{T3\_mean}, \\
    f_{T4\_mean} &= 1.36 f_{T3\_mean}, \\
    f_{T5\_mean} &= 1.73 f_{T3\_mean}.
\end{align*}
\]

These frequencies are used in Section 5.5 to statistically evaluate the mode structure of toroidal waves.

Figures 12c–12f show spectral parameters related to the nodal structure of T3 waves. Each parameter is evaluated at \( f_{T3\_mean} \) after smoothing the elements of the spectral matrix by three-point (3.3 mHz) averaging. From 1030 to 1300 UT, \( S_{E_xE_y} \) and \( S_{\theta_R\theta_R} \) track each other (Figure 12c). At ~1300 UT, however, they start to separate, with \( S_{E_xE_y} \) reaching a minimum at 1445 UT while \( S_{\theta_R\theta_R} \) remains at a nearly constant level. As a
consequence, $S_{E_vE_v} / S_{B\phi B\phi}$ decreases monotonically until 1445 UT and then increases (Figure 12d). This behavior is indicative of crossing of an $E_v$ node, given that the spacecraft was steadily moving northward away from the magnetic equator (Figure 12g). The low $S_{E_vE_v}$ intensity at 1445 UT causes $\gamma_{E_vB\phi}$ to drop to $\sim 0$ (Figure 12e). Most importantly, $\theta_{E_vB\phi}$ changes from $\sim 90^\circ$ to $\sim -90^\circ$ (Figure 12f) in association with the $S_{E_vE_v} / S_{B\phi B\phi}$ minimum. Taken together, it is clear that the spacecraft crossed an $E_v$ node at 1445 UT, when the spacecraft was at MLAT = 11.7°. According to Figure 3b, this corresponds to $\alpha = 2.4$, provided the background magnetic field and mass distribution are symmetric about the dipole equator. We believe that we can determine the latitude of this particular node with an accuracy of 0.2°, which corresponds to one-time step (=5 min) in the moving data window analysis. This translates to an accuracy of ±0.2 for $\alpha$. 

Figure 12. Time series plots of parameters relevant to the mode structure analysis of toroidal waves, shown for Van Allen Probe B orbit 1596. (a) Frequencies of T1 (circles), T2 (crosses), and T3 (diamonds) waves. The symbol color indicates the source spectrum inspected: red for $S_{E_vE_v}$, blue for $S_{B\phi B\phi}$, and green for $S_{E_vB\phi}$. (b) Frequency of the T3 waves. The crosses indicate the values given by $f_{T3} / (f_{T2}^2 - f_{T1}^2)$, where $f_{Tn}$ represents the frequencies shown in panel (a). The filled circles, labeled $f_{T3, mean}$, indicate the T3 wave frequencies that are obtained at each time step by averaging $f_{T3} / (f_{T2}^2 - f_{T1}^2)$ for $n = 1$–3. (c) Power spectral density of $E_v$ and $B\phi$ evaluated at $f_{T3, mean}$. (d) $E_v$ to $B\phi$ power spectral density ratio evaluated at $f_{T3, mean}$. The green shading indicates the minimum of the ratio, which is attributed to a node of $E_v$. (e) $E_v$-$B\phi$ coherence evaluated at $f_{T3, mean}$. (f) $E_v$-$B\phi$ cross-phase evaluated at $f_{T3, mean}$, shown when the coherence is > 0.5. (g) MLAT of the spacecraft.
This example indicates that node crossings can be used to gain information on field line mass distribution. The node latitude may be a superior variable to look at than the frequency ratios (Takahashi & McPherron, 1982) for the distribution because wave frequencies cannot be very accurately determined at each time step. In the Takahashi and McPherron (1982) study, the length of the data window was 1 h, 4 times that of the present study, but the estimated $\alpha$ still varied between 0 and 4, very likely because of the uncertainly in the wave frequencies. On the other hand, detection of a node is dictated by the spacecraft orbit. A clean node crossing like the one illustrated in Figure 12 occurs only a few times per orbit at most. This make a statistical approach essential to understand the mode structure of toroidal waves.

5.5. Mode Structure Statistical Analysis

We have applied the spectral analysis method illustrated in Figure 12 to T1–T5 waves at Van Allen Probes A and B during January–June 2014. This resulted in orbital data files containing $S_{E_\nu E_\nu}$, $S_{B_\phi B_\phi}$, $\gamma_{E_\nu B_\phi}$, and $\theta_{E_\nu B_\phi}$ that are evaluated at the five frequencies $f_{T1, \text{mean}}$, …, $f_{T5, \text{mean}}$ defined in Section 5.4. We demonstrate in Figure 13 that we can routinely determine the most important parameter for the mode structure analysis, $\theta_{E_\nu B_\phi}$, using the T3 waves observed in a 4 day period in January 2014. Figure 13a shows all $f_{T3, \text{mean}}$ samples from both spacecraft. Figure 13b shows $\theta_{E_\nu B_\phi}$ for cases with high coherence ($\gamma_{E_\nu B_\phi} > 0.5$). The data points in this figure are clustered around ±90°, with the sign being consistent with the MLAT of the spacecraft shown in Figure 13c. That is, $\theta_{E_\nu B_\phi}$ is ~90° (~90°) when the spacecraft is in the MLAT domain shaded blue (orange).

We used the full 6 months data to generate plots of the MLAT dependence of the spectral parameters, which are shown in Figure 14 for the plasmasphere and in Figure 15 for the plasmatrough. Note that we limited the $L$ range to 4–6 for comparison with the theoretical models for $L = 5$ (Figure 3). The spectral parameters are sorted into 1° MLAT bins, and their median, upper, and lower quartile values are plotted. The $\theta_{E_\nu B_\phi}$ samples are included in the statistics only when $\gamma_{E_\nu B_\phi} > 0.5$ and are plotted only when the number of samples in a bin is > 15. The vertical lines passing the lower three panels indicate the $E_\nu$ (red) and $B_\phi$ (blue) nodes determined in 1° resolution by visually examining the MLAT dependence of the spectral parameters. We refer to the MLAT value of these nodes as $\text{MLAT}_{\text{node, obs}}$.

We pay attention to the median values of the spectral parameters and examine the plasmasphere results first. For T1 waves (Figures 14a–14e), we find that $S_{E_\nu E_\nu}$ has a broad maximum at the equator (MLAT = 0), whereas $S_{B_\phi B_\phi}$ has a minimum there. Not suprisingly, $S_{E_\nu E_\nu}$ / $S_{B_\phi B_\phi}$ peaks at MLAT = 0. The coherence $\gamma_{E_\nu B_\phi}$ exhibits a clear minimum at the equator, and the cross-phase ($\theta_{E_\nu B_\phi}$) is ~90° at MLAT < 0 and ~90° at MLAT > 0. These are signatures of the equatorial $B_\phi$ node predicted by theory (Figure 2a). Accordingly, we judge that a $B_\phi$ node is on average located within the MLAT bin centered on 0 (MLAT$_{\text{node, obs}}$ = 0) and indicate the judgment by a blue vertical line.

A cautionary statement is warranted here regarding the behavior of $S_{E_\nu E_\nu}$. Going back to the theoretical T1 mode structure shown in Figure 2a, we find that the $E_\nu$ amplitude is actually peaked at MLAT = ±42° and has a minimum value (83% of the peak value) at the equator. This is inconsistent with the equatorial peak of $S_{E_\nu E_\nu}$ appearing in Figure 14a. This discrepancy could be explained by broadband equatorial $E_\nu$ oscillations that are not associated with toroidal waves. We speculate that fast mode waves generated by irregular solar wind dynamic pressure variations are the source of such $E_\nu$ oscillations. Mixture of toroidal and non-toroidal waves is inevitable in our analysis because we evaluate $S_{E_\nu E_\nu}$ at $f_{T1, \text{mean}}$ even when we do not detect an $E_\nu$ spectral peak at that frequency. By comparison, $S_{B_\phi B_\phi}$ exhibits MLAT dependence that is consistent with the model.

In the results for T2 waves (Figures 14f–14j), we also find a generally good agreement with theory (Figure 2b). There is an equatorial minimum of $S_{E_\nu E_\nu}$ / $S_{B_\phi B_\phi}$ (Figure 14h), which is colocated with the minimum of $\gamma_{E_\nu B_\phi}$ (Figure 14i) and a positive-to-negative change of $\theta_{E_\nu B_\phi}$ (Figure 14j). Accordingly, we judge that an $E_\nu$ node is located at the equator, that is, MLAT$_{\text{node, obs}}$ = 0. However, we note that $S_{E_\nu E_\nu}$ does not
exhibit an equatorial minimum expected from the \( E_\nu \) node predicted by theory (Figure 2b), possibly because of the non-toroidal waves we speculated in discussing the T1 wave results. The non-toroidal waves are also likely the cause of the gradual change of \( \theta_{E_\nu B_\phi} \) at the equator.

The equatorial nodes found for T1 and T2 waves are valuable confirmation of the basic concept of standing Alfvén waves and their symmetry about the magnetic equator (Sugiura & Wilson, 1964). Equatorial nodes are expected regardless of the harmonic modes provided the field line mass distribution and the background magnetic field are symmetric about the equator. Therefore, the equatorial nodes do not provide information on how the mass density changes with distance from the equator. The mode structure analysis yields more interesting results as the mode number increases because we begin to detect nodes away from the magnetic equator that can be used to infer the mass density distribution.

For T3 waves (Figures 14k–14o), we confirm signatures of the equatorial \( B_\phi \) node: an \( \theta_{E_\nu B_\phi} \) minimum and an \( \sim \)180° shift of \( \theta_{E_\nu B_\phi} \). The sharp \( \theta_{E_\nu B_\phi} \) transition implies that equatorial \( E_\nu \) of T3 waves is stronger than that of non-toroidal waves. More importantly, we identify two \( E_\nu \) nodes, MLAT\(_{\text{node,obs}} = -10^\circ\) and MLAT\(_{\text{node,obs}} = 11^\circ\), based on the minima of \( S_{E_\nu B_\phi} \) and rapid changes of \( \theta_{E_\nu B_\phi} \) associated with the minima. These node latitudes agree with those of the theoretical model (\( \pm 10.6^\circ \); see Figure 2c) within the MLAT resolution (1°) of the bin statistics.

The results for T4 waves (Figures 14p–14t) are puzzling. For example, \( \theta_{E_\nu B_\phi} \) changes from \( \sim -90^\circ \) to \( \sim 90^\circ \) across MLAT = 0, which is opposite to what we expect from the theoretical model (Figure 2d). In addition, we find no evidence for the \( B_\phi \) nodes that theory predicts at MLAT \( \sim \)8°. We can explain these findings by low intensity of T4 waves. Figure 10b indicates that the spectral peak detection rate is low and that a clear histogram peak is missing in the T4 band. Weak T4 waves can be masked by non-toroidal waves. Also, it is
possible that signals from T3 and T5 waves leak into the T4 band masking T4 waves. This latter explanation is plausible because the negative-to-positive change across MLAT = 0 seen in Figure 14t is qualitatively the same as those found for T3 (Figure 14o) and T5 (Figure 14y) waves.

The results for T5 waves (Figures 14u–14y) are more in line with the theoretical model. Specifically, $S_{B\phi}$ exhibits an equatorial minimum indicating the equatorial $B\phi$ node, $S_{E\nu}$ exhibits maxima at MLAT = −12°, 0, and 14° indicating $B\phi$ nodes, and minima at MLAT = −6° and 7° indicating $E\nu$ nodes. Rapid sign changes occur in $\theta_{E\nu,B\phi}$ across these latitudes as expected for the nodes. The inferred node latitudes agree with or are close to the theoretical nodes shown in Figure 2e at ±6.0° for $E\nu$ and at 0 and ±12.6° for $B\phi$. To summarize, the T5 wave statistics give five MLAT node data points −12°, −6°, 0, 7°, and 14°.

In the plasmatrough (Figure 15), T1, T2, T3, and T5 waves show node latitudes that are generally similar to those found in the plasmasphere. The $\theta_{E\nu,B\phi}$ plots for these waves indicate an equatorial node (Figures 15e, 15i, 15o, and 15y). In addition, we find $E\nu$ nodes of T3 waves, MLAT node = −11° and MLAT node = 12°, the latter of which matches the example shown in Figure 12, and $B\phi$ nodes of T5 waves, MLAT node = ±13°. A difference from the plasmasphere results is that $B\phi$ nodes of T4 waves are visible at MLAT node = ±8° (Figures 15r–15t), which is not surprising because Figure 10d shows clear presence of T4 waves in the plasmatrough. Another difference is that no $E\nu$ nodes are selected for T5 waves.
Figures 14 and 15 both indicate small N-S asymmetries in the mode structure. In the plasmasphere, the detected $E_{\nu}$ nodes of T3 and T5 waves are located at latitudes that are 1° or 2° higher in the northern hemisphere than in the southern hemisphere. The same is true in the plasmatrough for T3 waves. This asymmetry could be explained by assuming that the effective magnetic equator is located slightly (∼0.5°) north of the dipole equator (MLAT = 0). Note that the equatorial nodes identified in Figures 14 and 15 come with an uncertainty of 0.5° because the MLAT bins for the statistics have a width of 1°. Alternatively, the asymmetry could be attributed to an asymmetry in the background magnetic field, mass density distribution, or the ionospheric boundary condition. It is also possible that the asymmetry is related to the MLAT bias in sampling the northern and southern hemispheres, combined with the MLT dependence of the background magnetic field or field line mass distribution. The dipole tilt angle might also induce a small north-south (N-S) asymmetry to the background magnetic field. In a study of compressional Pc5 waves, mostly detected at $L\sim 8$ in the dawn and dusk sectors, Takahashi et al. (1990) suggested that the equatorial node of the waves is shifted from the dipole equator by as much as 9° when the tilt angle is large. A similar tilt angle effect might cause a node shift of dayside ULF waves. We hope to conduct a detailed analysis of the asymmetry in the future when we have processed Van Allen Probes data for more orbits.

### 5.6. Field Line Mass Density Distribution Inferred From Node Latitudes

Figure 16 shows how we use MLAT$_{\text{node,obs}}$ to infer the field line mass density distribution within the context of the model Equation 1. The red and blue curves in this figure show the theoretical MLAT$\text{node,ct}$ rela-
tionship for T3, T4, and T5 waves repeated from Figure 3 in a modified format. The filled circles indicate MLAT_{node,obs} and the corresponding α, which is denoted α_{node,obs}. The crosses indicate MLAT_{node,obs} ± 1°, in consideration of the MLAT bin width used in the statistics, and the corresponding α, which serve as a measure of the uncertainties in α_{node,obs}. In this case, the uncertainty of α_{node,obs} is ∼±1.5.

Ignoring the uncertainties, we find that α_{node,obs} varies from one node to another and also between the plasmasphere and the plasmatrough. This situation is similar to α estimation from the frequency ratios (Figure 11) and implies that Equation 1 does not completely describe the variation of density along the field line. Nevertheless, we can define an optimal α value by minimizing

$$
\varepsilon_\alpha(\alpha) = \sum_n (\text{MLAT}_{node,obs} - \text{MLAT}_{node}(\alpha))^2.
$$

(5)

This equation gives α = 1.4 when applied to data for the plasmasphere (Figures 16a–16c, thick black horizontal dashed lines) and α = 1.7 for the plasmatrough (Figures 16g–16i). Noting the N-S difference of the node latitudes, we shifted the magnetic equator northward by 0.5° to generate additional figures for the plasmasphere (Figures 16d–16f) and the plasmatrough (Figures 16j–16l). This adjustment does not change the α values from Equation 5 for either plasma domain.

The α values obtained from the node latitudes are close to those obtained from the frequency ratios (Figure 11). The various α values from the two methods are in the narrow range 1.2–2.0 and do not show any significant differences between the plasmasphere and plasmatrough. Based on these results, it appears that α~1.5 describes the field line mass distribution for both plasma domains.

While the good agreement between the two methods implies that either method can be used for mass density modeling, we note that there are advantages in using the node latitude method. First, a clear node crossing such as that shown in Figure 12 can be used to model the density more accurately than is possible using the frequency ratios. Second, the node latitudes provide information on the possible N-S asymmetry of the mass density distribution. Third, the number of parameters for constraining mass density models can be larger in the node latitude method. Using Van Allen Probes data, we are able to identify six off-equatorial nodes, equal to the number of harmonic frequencies identified using Geostationary Operational Environmental Satellite (GOES) magnetometer data (Takahashi & Denton, 2007). If we use data from satellites reaching higher MLAT than Van Allen Probes, we will be able to determine more node latitudes and use that information to construct mass density models that are more structured than Equation 1.

6. Discussion

6.1. Significance of Mode Structure Analysis

Numerous previous studies have shown the validity of the concept of standing Alfvén waves in terms of the N-S conjugacy on the ground (Sugiura & Wilson, 1964), a±90° phase delay between \( \mathbf{E} \) and \( \mathbf{B} \) fields measured in space (Cahill et al., 1986), and the diminishing amplitude of a field component near the magnetic equator (Singer & Kivelson, 1979).

Our results are new in that they include more quantitative details on the mode structure for multiple harmonics. For example, the equatorial nodes are confirmed to be very close (within 1°) to the dipole equator. Nodes away from the equator are detected, and their locations indicate high degree of symmetry (also within 1°) about the magnetic equator. Another important finding is that the cross-phase between \( E_\phi \) and \( B_\phi \) is close to ±90° except near the nodes. This means that the time-averaged Poynting flux along the background magnetic field is small and confirms that the fixed-end approximation (Newton et al., 1978) is valid for describing the mode structures observed by spacecraft. Therefore, our statistical analysis confirms that the existing theoretical models of toroidal waves in a dipole magnetosphere (Cummings et al., 1969; Radoski & Carovillano, 1966) fairly accurately describe the mode structure of toroidal waves excited at \( L \sim 5 \) under average geomagnetic conditions.
6.2. Uncertainty of $\alpha$ Estimated Using Node Latitudes

The information we gain on the mode structure allows us to take a new approach to infer the field line mass density distribution. Nodes located off the equator but at MLATs accessible by spacecraft are of particular importance. In this regard, T1 waves are irrelevant because the waves have only one node in space (at the magnetic equator) regardless of $\alpha$. The same is true for T2 waves because Van Allen Probes did not.

Figure 16. Relationship between the node latitude (MLAT$_{node}$) of T3, T4, and T5 waves and the mass density power law index $\alpha$. The color distinguishes between variables associated with the $E_\nu$ (red) and $B_\phi$ (blue) components. Each filled circle indicates the intersection of the vertical dashed line drawn at an observationally determined node latitude and the theoretical MLAT$_{node}$-$\alpha$ curve. The numbers shown next to the filled circles are the $\alpha$ values corresponding to the node latitudes, which are given at the top of each panel. The crosses indicate the range of $\alpha$ corresponding to the change of the node latitude from $-1^\circ$ to $+1^\circ$ about the value marked by the vertical dashed line. The horizontal dashed line indicates the value of $\alpha$ determined using Equation 5. (a–c) Result for the plasmasphere. No nodes are shown for T4 waves. (d–f) Result for the plasmasphere obtained assuming a 0.5° northward shift of the magnetic equator. (g–i) Result for the plasmatrough. (j–l) Result for the plasmasphere obtained assuming a 0.5° northward shift of the magnetic equator.
go beyond $\text{MLAT} = \pm 20^\circ$ and thus were not able to fully straddle the off-equatorial $B_{\phi}$ nodes expected at $|\text{MLAT}| > 19^\circ$ (for $\alpha \geq 0$) according to Figure 3a. The third or higher harmonics are valuable because they have off-equatorial nodes within the MLAT range of Van Allen Probes. Not all off-equatorial nodes carry equal significance for estimating $\alpha$. This has an impact on our ability to estimate $\alpha$ from observed node latitudes. To understand this, let us take a look at the $\text{MLAT}_{\text{node}}$--$\alpha$ relationship for T5 waves shown in Figure 16c. In the case of the lower-latitude $E_{\nu}$ node located in the northern hemisphere (node number 6 in Figure 2e), $\text{MLAT}_{\text{node}}$ changes from $5.6^\circ$ for $\alpha = 0$ to $10.3^\circ$ for $\alpha = 6$, an increase of $4.7^\circ$. In the case of the $B_{\phi}$ node located in the northern hemisphere (node number 7 in Figure 2e), $\text{MLAT}_{\text{node}}$ changes from $11.8^\circ$ for $\alpha = 0$ to $20.9^\circ$ for $\alpha = 6$, an increase of $9.1^\circ$. Accordingly, for a given error in $\text{MLAT}_{\text{node, obs}}$, the corresponding $\alpha_{\text{node, obs}}$ has a smaller error with nodes located at the higher latitudes. Therefore, it will be advantageous to use spacecraft that reach high MLATs to expand on the $\alpha$ estimation technique introduced in this study. For example, Figure 2 indicates that with a hypothetical satellite reaching $\text{MLAT} = \pm 35^\circ$ (orbital inclination of $25^\circ$), we would be able to detect nodes 1 and 3 of T2 waves, nodes 1 and 5 of T3 waves, nodes 1 and 7 of T4 waves, and nodes 1, 2, 8, and 9 of T5 waves, in addition to those detected by Van Allen Probes.

On the other hand, it is also true that the wave amplitude becomes smaller as the harmonic mode number increases. This makes it difficult to accurately determine the node latitudes for higher-order harmonics. Therefore, there is a trade-off between using higher-order harmonics for $\alpha$ sensitivity and using lower-harmonics for better statistics of node latitudes. We have not discussed rigorous statistical approaches to address this issue. Nonetheless, the simple approaches described above demonstrate that we can obtain valuable information on the field line mass density distribution using the node latitudes detected by spacecraft.

### 6.3. Comparison with Previous $\alpha$ Estimation

Because our use of node latitudes for mass density modeling is new, we compare our results with previous results on $\alpha$ obtained using frequency analysis. Denton et al. (2006) statistically analyzed toroidal wave frequencies observed by CRRES and found that $\alpha \sim 2$ is appropriate for $L = 4–5$ and $\alpha \sim 1$ is appropriate for $L = 5–6$. This is in agreement with our results within a margin of $\pm 0.5$.

In the $L > 6$ region, the GOES satellites ($L \sim 7$, usually the plasmatrough) have been a major data source for magnetoseismic studies (Denton et al., 2015; Takahashi & Denton, 2007; Takahashi et al., 2010). Because the satellites provide a large volume of frequency data extending even to the nightside, it is possible to examine the MLT dependence of $\alpha$. Denton et al. (2015) found that $\alpha$ changes from $\sim 0$ at noon to $\sim 2$ at dawn. Such MLT dependence in the $L > 6$ can be addressed by expanding the Van Allen Probes $f_T$ survey to locations outside the noon local time sector.

The power-law function (Equation 1) is also used to model field line distributions of electron density $n_e$. A series of papers used $n_e$ data from the Polar spacecraft to estimate $\alpha$ for electrons (Denton, Goldstein, Menietti, 2002; Denton, Goldstein, Menietti, Young, 2002; Goldstein et al., 2001). Because the spacecraft was on a polar orbit, $n_e$ measurements were made over a wide range of MLAT, and the field line distribution was determined by binning data in $L$ and MLAT. These studies reported $\alpha$ in the range 0–1 in the plasmasphere and in the range 1.6–2.1 in the plasmatrough. These results are remarkably close to our results for $\rho$ derived using the node latitudes. Because different ion species (e.g., H$^+$ and O$^+$) could have different density variation along field lines, $\alpha$ does not need to be the same for $\rho$ and $n_e$. Thus, the Polar results might indicate that H$^+$ and O$^+$ do have similar field line dependence for $L < 6$.

### 7. Conclusions

Van Allen Probes routinely detected several harmonics of toroidal waves within the frequency range covered by spinfit data, which are well suited for studying magnetohydrodynamic waves. We developed a technique to identify the frequency of the harmonics and to derive spectral parameters for each harmonic that are related to the latitudinal mode structure of the harmonics. We conclude the following:
1. Wave frequencies can be classified into plasmasphere and plasmatrough groups
2. Spectral parameters for the harmonics exhibit clear MLAT dependence, from which we can determine the location of the nodes (node latitudes) of the harmonics
3. If we assume a $r^{-m}$ mass density variation along field lines, the node latitudes indicate that $\alpha \sim 1.5$ is appropriate for both the plasmasphere and the plasmatrough in the $L = 4-6$ region where the majority of toroidal waves were detected

Data Availability Statement

Data used in this study are available from the following sources: NASA Goddard Space Flight Center Space Physics Data Facility Coordinated Data Analysis Web (https://cdaweb.gsfc.nasa.gov) for Van Allen Probes; World Data Center for Geomagnetism, Kyoto (http://wdc.kugi.kyoto-u.ac.jp) for the Dst index.

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