Abstract

The stable chromomagnetic vacuum for SU(2) Yang-Mills theory found earlier is shown to give a model for confinement in QCD, using Wilson loop and a linear potential (in the leading order) for quark-antiquark interaction. The coefficient $k$ in this potential is found to be $\sim 0.25 \text{ GeV}^2$, in satisfactory agreement with non-relativistic potential model calculations for charmonium. At finite temperature, the real effective energy density found earlier is used to obtain estimates of the deconfining temperature agreeing reasonably with lattice study for SU(2).

The economical definition of confinement of quarks in QCD is the 'area law' for the Wilson loop. The gauge invariant Wilson loop is

$$W(C) = \text{Tr} \ P e^{ig \oint dx^\mu A_\mu^a(x) t^a},$$

where $P$ denotes the path ordering and $t^a$ are the generators of the gauge group. We shall consider SU(2) Yang-Mills theory and choose the Savvidy [1] classical background

$$A_0^a = 0 \ ; \ A_i^a = \delta^{a3} \left( -\frac{Hy}{2}, \frac{Hx}{2}, 0 \right).$$

$^1$e-mail address: sarathy@cmi.ac.in; sarathy@imsc.res.in
This choice solves the classical equation of motion \( \bar{D}_\mu^{ab} F^{\mu\nu b} = 0 \), where \( \bar{D}_\mu^{ab} = \partial_\mu \delta^{ab} + g e^{abc} \bar{A}^c_\mu \). The classical background corresponds to constant chromomagnetic field in the third color direction \( \bar{F}^3_{12} = H \) and this comes from the derivative terms in \( \bar{F}^a_{\mu\nu} = \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu + g e^{abc} \bar{A}^b_\mu \bar{A}^c_\nu \). For this reason, the Savvidy ansatz (2) is called 'Abelian like'. So the classical background (2) is essentially Abelian-like, taking values in the Cartan subgroup of \( SU(2) \).

The use of Abelian-like field strength can be understood from the idea of 't Hooft [2] who proposed 'Abelian Projection'. This is a particular gauge fixing, breaking the gauge group \( SU(N) \) to the maximal torus subgroup \( H = U(1)^{N-1} \). For \( SU(2) \), \( H = U(1) \). This is realized in a specific gauge called the 'Maximal Abelian Gauge'. In the continuum formulation, this has the form

\[
(\partial_\mu \delta^{ab} + g e^{a3b} \bar{A}^3_\mu) \bar{A}^b_\mu = 0, \tag{3}
\]

and the classical Savvidy background (2) satisfies (3). Numerical simulations on the lattice have found that the Abelian projected Wilson loop defined by \( \bar{A}^3_\mu \) exhibits the 'area law' [3]. So (1) becomes

\[
W(C) = \langle e^{-ig \oint dx^\mu \bar{A}^3_\mu} \rangle, \\
= \langle e^{-i\frac{g}{2} \int d^4S F^{\mu\nu}_{3\mu}} \rangle, \\
= \langle e^{-i\frac{g}{2} H \times \text{area}} \rangle, \tag{4}
\]

where in (4), \( H \) should correspond to the minimum value of the energy density as \( W(C) \) involves vacuum expectation value.

The classical energy density for the background (2), in the Euclidean formulation, is \( \mathcal{E} = \frac{H^2}{2} \). This energy density has a minimum \( \mathcal{E} = 0 \) at \( H = 0 \) and so \( W(C) \) in (4) does not give the area law. In order to realize the area law from (4), the minimum energy density should correspond to \( H \neq 0 \).

Savvidy [1] has studied the quantum 1-loop effective energy density which has a minimum lower than the above classical minimum and for which \( H \neq 0 \). However, Nielsen and Olesen [4] pointed out that the 1-loop effective energy density in the background (2) had an imaginary part, stemming from the lowest Landau level and so the vacuum (ground state) of such a model is
unstable. Various attempts were made to circumvent this sensitive issue which inhibited the progress. Instead of using the background (2), constant non-Abelian background $\bar{A}_a^0 = 0; \bar{A}_i^a = K\delta_i^a$ was tried [5] and the said instability persisted. All these calculations were performed in the Gaussian (keeping only the terms quadratic in quantum fluctuations) approximation.

We [6] have reexamined this important issue by retaining all the terms in the quantum fluctuations. Besides quadratic terms, there are terms cubic and quartic in quantum fluctuations. The detail of these calculations are given in Ref.6, in which the effective energy density has been shown to be real. Briefly, the Euclidean functional integral for SU(2) pure Yang-Mills theory

$$ Z = \int [dA^a_{\mu}] e^S; $$

$$ S = \int d^4 x \{-\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}\}; $$

$$ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu, $$

(5)

is expanded around the classical background $\bar{A}_a^a$ in (2) as

$$ A^a_{\mu} = \bar{A}^a_{\mu} + a^a_{\mu}; $$

(6)

and the quantum fluctuations $a^a_{\mu}$ are taken to satisfy the 'background gauge'

$$ \bar{D}^{ab}_{\mu} a^b_{\mu} = 0. $$

(7)

This gauge choice is important. First of all, there is no Gribov ambiguity [7] in using this background gauge. It has been shown by Amati and Rouet [8] that the multiplicity of classical solutions satisfying the gauge condition is an irrelevant issue for quantizing non-Abelian Yang-Mills theories in the background gauge and an unambiguous generating functional is now possible. The correct treatment of the zero modes of the 1-loop operator gives the background gauge relative to the classical solution. See also [9]. The crucial point is that under a gauge transformation $U$, the quantum fluctuations $a^a_{\mu}$ in (6) transform homogeneously, namely, $a_\mu \equiv a^a_{\mu} t^a; a_\mu \rightarrow U a_\mu U^{-1}$ [9]. Second, with the background gauge (7), and using (3), we have $\bar{D}^{ab}_{\mu} (\bar{A}^b_{\mu} + a^b_{\mu}) = 0$ and so the 'Maximal Abelian Gauge' or Abelian projection is realized for the full gauge field $\bar{A}^a_{\mu} + a^a_{\mu}$. 
Now using (6) and (7) in (5), the unambiguous Euclidean generating functional $Z$ becomes,

$$Z = \int [da_\mu^a] e^{S'}, \quad (8)$$

with

$$S' = \int d^4x \left\{ -\frac{1}{4} \bar{F}_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} a_\mu^a \Theta^{ac}_{\mu\nu} a_\nu^c + g e^{acd} (\bar{D}_\nu a^e_{\mu}) a^c_{\mu} a^d_{\nu} 
- \frac{g^2}{4} \left( (a_\mu^a a_\mu^a)^2 - a_\mu^a a_\mu^e a^e_{\mu} a^c_{\nu} \right) \right\} - \log \det (-\bar{D}_\mu^a \bar{D}^{bc}) , \quad (9)$$

where

$$\Theta^{ac}_{\mu\nu} = (\bar{D}^a_{\lambda} \bar{D}^{bc}) \delta_{\mu\nu} + 2 g e^{ace} \bar{F}_{\mu\nu}^c. \quad (10)$$

In arriving at (9), we have introduced the gauge fixing and the Faddeev-Popov ghost Lagrangian for the background gauge (7) and integrated the ghost fields, resulting in the last term in (9). The expansion in (9) is exact. The purpose of writing $S'$ in the form above is to isolate the stable and unstable modes of $\Theta^{ac}_{\mu\nu}$.

For the Savvidy background, $\Theta^{ac}_{44} = \Theta^{ac}_{33} = \bar{D}^a_{\lambda} \bar{D}^{bc}$, so that their contributions to $\Gamma$ cancel the ghost contribution. Further the non-vanishing $\Theta$’s are $\Theta^{ec}_{ij}$ for $i, j = 1, 2$. Their eigenmodes and eigenvalues are:

$$a_1^1 \pm ia_2^1 : \quad k_1^2 + k_2^2 + k_3^2 + k_4^2 \quad (plane \ waves),$$

$$(a_1^1 \pm ia_2^1) - i(a_1^2 + ia_2^2) : \quad (2n + 1)gH + 2gH + k_2^2 + k_4^2 \quad (stable),$$

$$(a_1^1 - ia_2^1) + i(a_1^2 - ia_2^2) : \quad (2n + 1)gH + 2gH + k_3^2 + k_4^2 \quad (stable),$$

$$(a_1^1 + ia_2^1) + i(a_1^2 + ia_2^2) : \quad (2n + 1)gH - 2gH + k_3^2 + k_4^2 \quad (unstable),$$

$$(a_1^1 - ia_2^1) - i(a_1^2 - ia_2^2) : \quad (2n + 1)gH - 2gH + k_3^2 + k_4^2 \quad (unstable).$$

The last two eigenvalues become negative when $n = 0$ and for low momenta. These are called the ‘unstable modes’. As we encounter logarithm of the eigenvalues, in the quadratic approximation, negative eigenvalues make it imaginary and hence the effective energy density becomes complex indicating vacuum instability. This in the Gaussian approximation.
The stable modes (the first two eigenvalues and the last two with \(n \neq 0\)) can be safely treated in the quadratic approximation. The contribution from the stable modes (see Ref.6 for details) are found to be

\[
\frac{\log^2 \mu^2}{96\pi^2} \{\log \left( \frac{gH}{\mu^2} \right) + C \},
\]

where \(C\) is a real (infinite) constant and \(\mu^2\) is a dimensionful constant introduced to render the argument of the logarithm dimensionless.

For the unstable modes, we [6] considered the full action in (9). The unstable modes involve the Lorentz indices 1 and 2 and the \(SU(2)\) indices 1 and 2, because the classical background (2) is in the third color direction and so the cubic term in (9), namely, \(\epsilon^{acd}(\hat{D}_a^\nu a^\nu_b) a^c_{\mu} a^d_{\nu}\) vanishes. The quartic term in (9) for the unstable modes is found to be

\[
\frac{1}{8} \left( |a_u|^2 \right)^2,
\]

where \(a_u\) is the unstable mode. The functional integral \(Z\) for the unstable modes is evaluated in Ref.6 and from this the finite part of the unstable mode contribution to the energy density is found to be

\[
\frac{g^2 H^2}{8\pi^2} \log \left( \frac{gH}{\mu^2} \right) - \frac{g^2 H^2}{4\pi^2} \log I,
\]

where

\[
I = \int dc' e^{-\{c'^2(k_3'^2+k_4'^2-1)+\frac{g^2}{256\pi^2}c'^4\}}.
\]

The integral \(I\) is convergent irrespective of whether \(k_3'^2 + k_4'^2\) is < or > 1. Further \(I\) is real, finite and independent of \(H\). Adding (12) to (11) and including the classical energy density, the effective energy density is found to be

\[
E = \frac{H^2}{2} + \frac{11g^2 H^2}{48\pi^2} \{\log \left( \frac{gH}{\mu^2} \right) + C' \}.
\]

where \(C'\) includes the second term in (12) along with \(C\) in (11). The real constant \(C'\) is then fixed by Coleman-Weinberg normalization \(\frac{\partial}{\partial H^2}E|_{gH=\mu^2} = \frac{1}{2}\) as \(-\frac{1}{2}\). Thus the effective energy density for \(SU(2)\) Yang-Mills theory in Savvidy background becomes

\[
E = \frac{H^2}{2} + \frac{11g^2 H^2}{48\pi^2} \{\log \left( \frac{gH}{\mu^2} \right) - \frac{1}{2} \}.
\]
This is real. The above result is non-Abelian gauge theory effect.

Quarks (fermions) can be added by minimally coupling them with the background (2) and functionally integrating $\psi$ and $\bar{\psi}$ in $Z$. The only change is the replacement of 11 in (15) by $(11 - N_f)$ for $N_f$ quark flavors. The prefactor $\frac{11 - N_f}{48\pi^2}$ can be obtained from group theory considerations as well. Extending to $SU(3)$, this factor becomes $\frac{33 - 2N_f}{96\pi^2}$.

In contrast to the classical energy density, the effective energy density in (15) has a minimum at non-zero $H$. From (15), we have

$$\frac{\partial \mathcal{E}}{\partial H^2} = \frac{1}{2} + \frac{11g^2}{48\pi^2} \log\left(\frac{gH}{\mu^2}\right),$$

$$\frac{\partial^2 \mathcal{E}}{\partial (H^2)^2} = \frac{11g^2}{96\pi^2 H^2} > 0. \tag{16}$$

The energy density has a minimum. When quarks are included, in order to have a minimum energy density, $N_f < 11$ for $SU(2)$ or $N_f \leq 16$ for $SU(3)$. The minimum occurs when

$$H = \frac{\mu^2}{g} e^{-\frac{4g^2}{11\pi^2}}, \tag{17}$$

with 11 appropriately replaced when quarks are included. Thus vacuum expectation value $H$ (corresponding to the minimum of $\mathcal{E}$) is not zero which gives the Wilson loop the 'area law' and hence confinement.\(^2\)

The minimum energy density using (16) in (15) is

$$\mathcal{E}_{\text{min}} = -\frac{11g^2H^2}{96\pi^2}, \tag{18}$$

\(^2\)In the case of QED, as photons do not have self-interactions, the electrons alone contribute to the effective energy density in the constant magnetic field background. In this case, the effective energy density will be $\mathcal{E} = \frac{\mu^2}{2} - A(e^2H^2)\left\{\log\left(\frac{eH}{\mu}\right) - \frac{1}{2}\right\}$ where $A$ is a positive constant. This energy density has a maximum at non-zero value of $H$. Interpreting the maximum of the energy density to correspond to excited state, the area-law gives confinement of electrons by a linear effective potential, in a constant magnetic field when the electrons are in the excited state, akin to magnetic confinement of plasma state.
which is lower than the classical minimum. This is the energy of the vacuum in the pure $SU(2)$ Yang-Mills theory.

The result that the minimum of the energy density occurs when $H \neq 0$ \((17)\) and $F_{12}^3 = H$, imply that the vacuum expectation value $\langle F_{12}^3 \rangle \neq 0$.

This indicates that $\langle g^2 F^a_{\mu\nu} F^a_{\mu\nu} \rangle \neq 0$, the occurrence of 'gluon condensate'. In this case

$$\langle g^2 F^a_{\mu\nu} F^a_{\mu\nu} \rangle = 2 g^2 H^2_{\text{min}} = 2 \mu^4 e^{-\frac{24\pi^2}{\alpha g^2}}. \quad (19)$$

Instead of using the strong coupling $g$ which runs, we use the result from the Charmonium decay analysis, $\langle g^2 F^a_{\mu\nu} F^a_{\mu\nu} \rangle \sim 0.5 GeV^4$. Then, $gH \sim 0.5 GeV^2$.

With this estimate

$$W(C) \sim e^{\frac{9H}{2} \text{area}} = e^{\sigma \text{area}}, \quad (20)$$

where $\sigma = \frac{aH}{2} = 0.25 GeV^2$. It is well known that the 'area law' corresponds to a linear potential and in the leading order $V = \sigma R$ where $R$ is the separation of static quark and anti-quark.\(^3\)

For a linear potential $V = kr$, the non-relativistic potential model calculations give the $c\bar{c}$ bound states for $k = 0.272 GeV^2$ which agrees with our estimate of $\sigma$ as $0.25 GeV^2$.

Thus, the stable vacuum in the chromomagnetic background is very much indicative of confinement, giving in the leading order the linear potential whose parameter $k$ is satisfactorily obtained.

\(^3\)The Wilson loop in (4) involves the area in the $(x-y)$ plane for the Savvidy background (2). Following M.Baker, J.S.Ball, N.Brambilla, G.M.Prosperi and F.Zachariasen, Phys.Rev. D54 (1996) 2829, the closed loop is defined by the quark (antiquark) trajectories $\vec{x}_1(t)(\vec{x}_2(t))$ running from $\vec{y}_1$ to $\vec{x}_1$ ($\vec{x}_2$ to $\vec{y}_2$) as $t$ varies from $t_i$ to $t_f$. The quark (antiquark) trajectories are the world lines $C_1$ (and $C_2$) running from $t_i$ to $t_f$ ($t_f$ to $t_i$). The world lines $C_1$ and $C_2$ along with two straight lines at fixed time connecting $\vec{y}_1$ to $\vec{y}_2$ and $\vec{x}_1$ to $\vec{x}_2$ then make up the contour. Parameterising the quark (antiquark) trajectories as: $y_1(-\frac{R}{2}, -\frac{T}{2}); x_1(-\frac{R}{2}, \frac{T}{2}); x_2(\frac{R}{2}, \frac{T}{2}); y_2(\frac{R}{2}, -\frac{T}{2})$, where $t_i = -\frac{T}{2}, t_f = \frac{T}{2}$ and $R$ is the separation of the quark from antiquark, the area of the loop is $RT$. So, $W(C) = e^{-\frac{aH}{2} RT}$. The potential is $V = \frac{1}{T} \log W(C)$ in the limit $T \to \infty$ and so $V = \frac{aH}{2} R$. 

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Now we consider the $SU(2)$ Yang-Mills theory at finite temperature. In the studies of the Savvidy vacuum at finite temperature in the Gaussian Approximation, the effective energy density involved a temperature dependent imaginary part [10]. This inhibited the progress of examining the phase transition. We [11] have extended our zero-temperature studies (including the cubic and quartic terms) to finite temperature with chemical potential for the $SU(2)$ gauge bosons.

A chemical potential for massless non-Abelian bosons has been introduced in [12,13], by observing that there are conserved color charges $Q^a = \int d^3x j^a_0; \ j^a_\mu = f^{abc} A^b_\nu F^c_{\nu\mu}$. For $SU(2)$, one chooses $Q^3$. The grand canonical partition function will now have $\mu Q^3$ in the hamiltonian. This leads to the result of using $A^a_0 = -i\mu \delta^a_3$. This is not possible for Abelian gauge bosons.

The role of chemical potential as a constant term in $A^a_0$ is similar to the use of Polyakov loop specified by a constant $A^a_0$ field in the third color direction. Now the Savvidy background becomes

$$\tilde{A}^a_\mu = \delta^a_3 \left\{ \frac{i\mu}{g}, -\frac{Hy_2}{2}, \frac{Hx_2}{2}, 0 \right\};$$

which gives $\tilde{F}^3_{12} = H$ and which solves the classical equation of motion. The background covariant derivative (Euclidean) now is

$$\tilde{D}^{ab}_\lambda = \partial_\lambda \delta^{ab} + g\epsilon^{abc} \tilde{A}^c_\lambda + \mu \epsilon^{a3b} v_\lambda,$$

where $v_\lambda = (1, 0, 0, 0)$.

Once again we have isolated the unstable modes and treated them including the cubic and quartic terms in the fluctuations. The involved calculation (see Ref.11 for details) leads to the result that the effective energy density is real. The result for the effective energy density, including the zero temperature contribution is

$$\mathcal{E} = \frac{H^2}{2} + \frac{11(gH)^2}{48\pi^2} \left( \log \left( \frac{gH}{\Lambda^2} \right) - \frac{1}{2} \right) + \frac{\pi^2}{45\beta^4} + \frac{(gH)^2}{\beta\pi^2} \sum_{\ell=1}^{\infty} \frac{\cos(\mu\beta\ell)}{\ell} \left\{ -\frac{\pi}{2} Y_1(\beta\ell\sqrt{gH}) + K_1(\beta\ell\sqrt{gH}) \right\}$$
where \( Y_1, K_1 \) are modified Bessel functions. Setting \( \beta = \frac{b}{\sqrt{gH}} \) and \( \mu = b\sqrt{gH} \), the temperature dependence is written as
\[
\frac{\mathcal{E}_T}{(gH)^2} = \frac{\pi^2}{45a^4} + \frac{1}{\pi^2a} \sum_{\ell=1}^{\infty} \frac{\cos(ab\ell)}{\ell} \left\{ -\frac{\pi}{2} Y_1(a\ell) + K_1(a\ell) \right\}
+ 2 \sum_{n=1}^{\infty} \sqrt{2n+1} K_1(\ell\sqrt{a\ell}) \}.
\]

In [11], we have plotted \( \frac{\mathcal{E}_T}{(gH)^2} \) with \( T \) in units of \( \sqrt{gH} \). The parameter \( b \) involves the chemical potential. For \( b = 0 \), zero chemical potential, the variation is smooth. At high temperatures, the behaviour is like that of non-interacting relativistic gas. For \( b = 1, 2, 3 \) the variation shows a minimum and then smooth rise. A non-zero chemical potential thus triggers deconfinement phase transition.

Deconfinement occurs for \( b = 1 \) around \( T/\sqrt{gH} \sim 0.4 \) and for \( b = 2 \) around \( T/\sqrt{gH} \sim 0.7 \). We have earlier identified the string tension \( \sigma \) as \( \frac{gH}{2} \) and so our results give for the deconfining temperature \( T/\sqrt{\sigma} \sim 0.5656 \) for \( b = 1 \) and 0.9899 for \( b = 2 \). It is interesting to compare with the lattice studies. For \( SU(2) \), Lucini, Teper and Wenger [14] find the deconfining temperature \( T/\sqrt{\sigma} \sim 0.709 \); the agreement is satisfactory for \( 1 > b < 2 \). As the chemical potential \( \mu = b\sqrt{gH} \), using \( gH = 0.5 \, GeV^2 \), the lowest value for the chemical potential triggering deconfinement is 0.7 \( GeV < \mu < 1.41 \, GeV \).

To summarize, the stable chromomagnetic vacuum for \( SU(2) \) Yang-Mills theory found in [6] gives a model for confinement using Wilson loop and hence a linear potential (in the leading order) for the quark-antiquark interaction. The coefficient \( k \) in this potential is \( \sim 0.25 \, GeV^2 \), in satisfactory agreement with non-relativistic potential model for charmonium. At finite temperature, the real effective energy density found in [11] is used to obtain estimates of the deconfining temperature and this reasonably agrees with the lattice study for \( SU(2) \).
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