Cluster singularities: the unfolding of clustering behavior in globally coupled oscillatory systems

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Abstract. The ubiquitous occurrence of cluster patterns in nature still lacks a comprehensive understanding. It is known that the dynamics of many such natural systems is captured by ensembles of Stuart-Landau oscillators. Here, we investigate clustering dynamics in a mean-coupled ensemble of such limit-cycle oscillators. In particular we show how clustering occurs in minimal networks, and elaborate how the observed 2-cluster states crowd when increasing the number of oscillators. Using persistence, we discuss how this crowding leads to a continuous transition from balanced cluster states to synchronized solutions via the intermediate unbalanced 2-cluster states. These cascade-like transitions emerge from what we call cluster singularities. At those points, the bifurcations of all 2-cluster states collapse and the stable balanced cluster state bifurcates into the synchronized solution supercritically. We confirm our results using numerical simulations, and discuss how our conclusions apply to spatially extended systems.

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Clustering dynamics in ensembles of coupled oscillators, that is the splitting up of the whole ensemble into two or more groups, the members of which are behaving identically, still raise many open questions, and many aspects of it still remain unsolved. Typically, clustering occurs in systems with long-range interactions [1]. Oscillatory systems with global interactions play a crucial role in the understanding of phenomena observed in nature and technology, such as the visual perception in the mammalian brain, the circadian rhythm in the heart or the behavior of coupled Josephson junctions and electrochemical oscillators. Even phenomena such as synchronous chirping of crickets, the flashing of fireflies in unison and the synchronous clapping of an audience can be traced back to the action of a long range coupling between oscillating units (see Refs. [2, 3] and references therein).

If the coupling between such individual units is weak, each oscillator may be represented by its phase value only, and the resulting reduced models can be analyzed using powerful approaches as proposed by Okuda [1], Watanabe and Strogatz [4, 5] or Ott and Antonsen [6, 7]. In many physical systems, however, amplitude effects play a crucial role, and such a reduction is no longer possible. Nevertheless, it has been shown that oscillators close to the onset of oscillations can be reduced to normal forms such as the Stuart-Landau oscillators [8, 9], and thus one is able to draw general conclusions when investigating ensembles of such normal forms [10, 11].

Systems of globally coupled Stuart-Landau oscillators have been investigated intensively in recent years [12], revealing phenomena such as collective chaos [13], aging [14], chimera states [15, 16], oscillation death [17] and clustering [18, 19, 20, 21].

In this paper, we further analyze how clustering arises in systems of mean-coupled Stuart-Landau oscillators. In particular, we first investigate the stability of 2-cluster solutions in minimal networks of just two and then four oscillators. We subsequently elaborate how such cluster states bifurcate from the synchronous solution, and draw connections to previous works on clustering in globally coupled systems. We then show how new cluster states appear in phase space when increasing the number of oscillators in the network. Exploiting properties such as persistence yields new insights into how those states are arranged in phase space. Doing so, we find codimension-2 points which we dub cluster singularities. There, the balanced cluster states with two clusters of equal size supercritically bifurcate off the synchronized solution. We continue our considerations with an investigation of how ensembles of Stuart-Landau oscillators behave close to this point, and what this implies for spatially extended systems. We conclude with some open questions concerning clustering in coupled oscillators, and discuss a few promising directions to address them.

**Globally coupled Stuart-Landau oscillators**

It has been shown that an oscillatory system close to the onset of oscillations can be reduced to the dynamics of a so-called Stuart-Landau oscillator [8, 9]. Such analysis
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has also been extended to cases in which there is a field of oscillators which interact with each other. In particular, for \( N \) such systems which interact linearly through the common mean of some of their variables, one obtains an ensemble of the form

\[
\partial_t W_k = W_k - (1 + ic_2) |W_k|^2 W_k + (\alpha + i\beta) \left( \frac{1}{N} \sum_l W_l - W_k \right),
\]

with \( k \in \{1, \ldots, N\} \), the complex amplitude \( W_k = W_k(t) \in \mathbb{C} \), the shear \( c_2 \), and the coupling constant \( \kappa = \alpha + i\beta, \alpha, \beta \in \mathbb{R} \) [10, 11]. Hereby, the coupling is diffusive in the sense that it vanishes if \( W_k = W \forall k \) [18].

Since we take all oscillators as identical, Equation (1) is \( \Gamma \)-equivariant with respect to the symmetric group \( \Gamma = S_N \), that is, with respect to permutations of the indices \( k \). In addition, the dynamics are unchanged under a rotation in the complex plane, \( W_k \to W_k e^{i\phi} \), and under the reflection \( W_k \to \bar{W}_k, c_2 \to -c_2, \beta \to -\beta \), with the bar indicating complex conjugation.

Equation (1) can be seen as a normal form for oscillatory systems with a quickly diffusing variable or a coupling through e.g. a gas phase. There is a vast number of different dynamical states that can be observed in such an ensemble, including synchronized motion and splay states [12], cluster states [1, 19, 20], chaotic dynamics [13, 22, 23] and chimera states [15].

For the synchronized solution, in which all \( N \) oscillators move in phase, and the splay-state, in which all oscillators are frequency-synchronized but with a phase difference of \( 2\pi/N \), the respective stability boundaries can be obtained analytically, see for example Ref. [20]. For chaotic or chimera-like dynamics, however, only numerical approaches exist.

In this article, we restrict our analysis to solutions in which the ensemble splits into just two groups with each group being internally synchronized, also called a 2-cluster state [19]. Let \( \epsilon = N_1/N \) denote the fraction of oscillators in cluster 1 and \( (1 - \epsilon) = N_2/N \) the fraction of oscillators in cluster 2, then Equation (1) reduces to

\[
\begin{align*}
\partial_t W_1 &= W_1 - (1 + ic_2) |W_1|^2 W_1 + (\alpha + i\beta) (1 - \epsilon) (W_2 - W_1) \\
\partial_t W_2 &= W_2 - (1 + ic_2) |W_2|^2 W_2 + (\alpha + i\beta) \epsilon (W_1 - W_2),
\end{align*}
\]

with \( (W_1, W_2) \in \mathbb{C}^2 \cong \mathbb{R}^4 \).

Following Ref. [24], the dynamics can be further reduced by exploiting the rotational invariance \( W_k \to W_k e^{i\phi} \) mentioned above and introducing polar coordinates \( W_1 = R_1 e^{i\phi_1}, W_2 = R_2 e^{i\phi_2} \) and the phase difference \( \Delta\phi = \phi_2 - \phi_1 \). This yields the reduced equations

\[
\begin{align*}
\partial_t R_1 &= R_1 - R_1^3 + \alpha (1 - \epsilon) (R_2 \cos(\Delta\phi) - R_1) + \beta (1 - \epsilon) R_2 \sin(\Delta\phi) \\
\partial_t R_2 &= R_2 - R_2^3 + \alpha \epsilon (R_1 \cos(\Delta\phi) - R_2) - \beta \epsilon R_1 \sin(\Delta\phi) \\
\partial_t \Delta\phi &= -c_2 (R_1^2 - R_2^2) + \beta (1 - 2\epsilon) \\
&+ \beta \cos(\Delta\phi) \left( \frac{(1 - \epsilon) R_2}{R_1} - \frac{\epsilon R_1}{R_2} \right)
\end{align*}
\]
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\[- \alpha \sin (\Delta \phi) \left( \frac{(1 - \epsilon) R_2}{R_1} + \epsilon \frac{R_1}{R_2} \right) \],

(4)
describing the dynamics in a three-dimensional phase space $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Note that the roots of the reduced model, Equations (2) to (4), correspond to sinusoidal oscillations in the full system Eq. 1.

Balanced 2-Cluster States

First, we restrict our analysis to 2-cluster states with an equal number of oscillators in each cluster, $N_1 = N_2 = N/2$, that is for $\epsilon = 1/2$. Then Equations (2) to (4) in fact describe the dynamics of just two identical oscillators. If the coupling is weak, that is if $|\kappa| << 1$, then the amplitudes relax to $R_1 = 1, R_2 = 1$ on a fast time scale, and the dynamics can be described, to a good approximation, by the phase equation only [8]. Therefore, with the adiabatic approximation $R_1 = R_2 = 1$ the system, Equations (2) to (4), reduces to

\[\frac{\partial}{\partial t} \Delta \phi = -\alpha \sin (\Delta \phi) , \]

(5)

and thus to a sine-coupled system [1]. Notice that it has two fixed points at $\Delta \phi = 0$ and $\Delta \phi = \pm \pi$, and therefore only the in-phase and anti-phase solutions exist for weak coupling. Important findings for larger ensembles of limit-cycle oscillators with weak coupling, that is for globally coupled phase oscillator systems, are summarized in [25] and [3].

The fixed points of the two-oscillator system, Equations (2) to (4) with $\epsilon = 1/2$, can be determined as

- $u_s = (R_1, R_2, \Delta \phi)^T = (1, 1, 0)$, the synchronized solution in which both oscillators have amplitude equal to one and phase difference zero,
- $u_a = (\sqrt{1-\alpha}, \sqrt{1-\alpha}, \pi)$, the anti-phase solution corresponding to a splay state for two oscillators,
- and two symmetry-broken solutions $u_{1,2}$ (see Appendix A for their derivation).

Note that the anti-phase solution $u_a$ exists only for parameter values $\alpha < 1$, and that the locations in phase space of both the synchronized and anti-phase solutions are independent of the parameters $\beta$ and $c_2$. Their stability boundaries can be determined by calculating the eigenvalues of the Jacobian evaluated at these solutions, and are given by the Benjamin-Feir condition [13, 12, 26],

\[\alpha^2 + \beta^2 + 2\alpha + 2\beta c_2 = 0, \]

(6)

for the synchronized solution and by

\[\alpha^2 + \beta^2 - 2 (1 - \alpha) (\alpha + \beta c_2) = 0 \]

(7)

for the anti-phase solutions. It is worth mentioning that these stability boundaries are independent of the number of oscillators in the ensemble [13]. The stability diagram of these two solutions for $c_2 = 0$ is depicted in Fig. 1(a). There, one can observe
that the synchronized solution is stable for positive $\alpha$ values, and either loses one stable direction at the solid blue curve and subsequently a second stable direction at the dotted blue curve, or two stable directions through a Hopf bifurcation at the horizontal blue lines. Analogously, the anti-phase solution is unstable for $\alpha > 0.5$, and gains one stable direction either at the dotted red curve and subsequently becomes stable at the solid red curve, or it becomes directly stable through a Hopf bifurcation at the horizontal red line where it gains two stable eigendirections. Note that there exist two regions in which the two solutions are bistable.

The asymmetric solutions $u_{1,2}$ (see Appendix A for their derivation) have the property that the amplitudes of the two oscillators differ and the oscillators have a non-zero phase difference, see also Ref. [21]. Note that, since our original system is $S_N$-equivariant, every

![Figure 1](image-url)

**Figure 1.** (a) Stability of the synchronized solution $u_s$ (blue) and anti-phase solution $u_a$ (red) for $c_2 = 0$. From negative to positive $\alpha$, the synchronized solution $u_s$, having two unstable directions, gains a stable direction at the dotted blue curve, and eventually becomes stable at the solid blue curve. At the horizontal blue line, the real parts of two complex conjugate eigenvalues of the Jacobian cross the imaginary axis, indicating a Hopf bifurcation. In contrast, the anti-phase solution $u_a$ loses two stable directions at the horizontal solid red line, whereas it loses one unstable direction at the solid red curve and another at the dotted red curve. This fixed point solution eventually disappears at $\alpha = 1$, indicated by the dashed red line, where the amplitudes $R_1$ and $R_2$ vanish. (b) Parameter range for $c_2 = 0$ in which the asymmetric solutions $u_{1,2}$ exist. Hereby, solution $u_1$ exists for parameter values between the saddle-node bifurcations, indicated by the dotted green lines, and between the dashed magenta curves symbolizing pitchfork bifurcations. The boundaries of the solutions $u_2$ are again the dotted green lines, and the pitchfork indicated by the dashed black lines. The gray lines indicate the one-parameter continuation cuts shown in Fig. 2.
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belong to the same group orbit.

In the case of no shear, \( c_2 = 0 \), asymmetric solutions \( u_{1,2} \) bifurcate off the anti-phase solution for large \( \alpha \) values (see the dotted magenta and black curves for \( \alpha > 0 \) in Fig. 1(b)) through pitchfork bifurcations. For smaller \( \alpha \) values, they either bifurcate in a saddle-node with each other (see the dotted green line in Fig. 1(b)) or with the synchronized solution through pitchfork bifurcations (see the dotted magenta and black curves for \( \alpha < 0 \) in Fig. 1(b)). That means, \( u_1 \) (to be more precise, the two elements of the group orbit of \( u_1 \)) exists between the dotted magenta curves (pitchforks) and the dotted green lines (saddle-nodes), and the solution \( u_2 \) exists in the two small regions between the dotted black curves (pitchforks) and the dotted green lines (saddle-nodes).

The continuation of the different solutions along one-parameter cuts (as indicated by the gray lines in Fig. 1(b)) is shown in Fig. 2, where each solution is represented by its \( \Delta \phi \) variable. This means that the synchronized solution is located at \( \Delta \phi = 0 \) and the anti-phase solution at \( \pm \pi \). In Fig. 2(a), \( \beta = 0.4 \) is held fixed and the synchronous solution is continued with increasing \( \alpha \). There, one observes a subcritical pitchfork at which the solution branches of \( u_1 \) are created and \( u_s \) gains an additional stable direction. After another pitchfork the synchronized solution becomes stable and the two branches of \( u_2 \) are born. Those two branches then reach the anti-phase solution \( u_a \) where they get destroyed in a pitchfork bifurcation, rendering \( u_a \) unstable. Finally, the branches of \( u_1 \) bifurcate with \( u_a \) in another pitchfork, adding the second unstable direction.

Fixing \( \alpha \) at \( \alpha = -0.1 \) and starting from negative \( \beta \) values, we observe two saddle-node bifurcations, each involving one branch of \( u_1 \) and \( u_2 \), respectively. The branches of \( u_2 \) are annihilated through a subcritical pitchfork at the synchronized solution. Due to the symmetry \( \beta \rightarrow -\beta \), this scenario holds also for positive \( \beta \) values. Note that there is no bifurcation at \( \beta = 0 \). The crossing of the two branches of \( u_1 \) is no real crossing but is a
consequence of the projection of the solutions onto the $\Delta \phi$ variable. The stability of the asymmetric solutions $u_{1,2}$ can further be investigated using the eigenvalues of the Jacobian. Then, we find that $u_2$ is always unstable in the regarded parameter regimes, but $u_1$ is stable for certain ranges of $\alpha$, $\beta$ and $c_2$. In particular, the number of unstable eigendirections of $u_1$ in the $\alpha - \beta$ parameter plane is shown in Fig. 3 for different values of $c_2$. Note that even for $c_2 = 0$ small regions exist in which $u_1$ is stable, but these regions grow for larger absolute values of $c_2$, see Figs. 3(b)-(e). The stable 2-cluster states lose stability through a Hopf bifurcation, leading to the yellow patches in Fig. 3 with two unstable directions. The Hopf bifurcation is supercritical for small $c_2$, creating stable periodic orbits [13]. Due to the symmetries in the equations, as discussed above, the bifurcation diagrams are symmetric under a simultaneous exchange $\beta \rightarrow -\beta$, $c_2 \rightarrow -c_2$, leading to the left-right symmetry in Fig. 3(a).

**Balanced $N$-Cluster States**

The reduced model, Equations. (2) to (4), and the eigenvalues of its Jacobian offer only limited information about the stability of 2-cluster states. This becomes obvious when...
considering the different ways in which 2-cluster solutions can bifurcate: either on or transverse to the cluster manifold. That is, either the two cluster clumps each remain together, but their relative locations change, or one of the two clusters splits up into an arbitrary number of subclusters. The stability properties of the former are fully captured by the reduced equations, but since the latter cannot happen in the reduced model, we can draw no conclusions about stability transverse to the cluster manifold.

In order to rephrase these arguments, we follow Ref. [20] and describe the stability of 2-cluster states by two kinds of Lyapunov exponents, the *cluster integrity exponents* \( \lambda_{CI}^\sigma \), and the *cluster system orbit stability exponents* \( \lambda_{SO} \). The latter describe the stability along the 2-cluster manifold and constitute 3 real numbers. The former, the cluster integrity exponents \( \lambda_{CI}^\sigma \), describe the internal stability of each cluster \( \sigma \). Due to the symmetries of the oscillator ensemble, each \( \lambda_{CI}^\sigma \) is degenerate in the sense that they consist of \( 2N^\sigma - 2 \) equal real values, with \( N^\sigma \) being the number of oscillators in cluster \( \sigma \).

We extend our considerations by regarding four instead of two equally sized clusters. By again introducing polar coordinates, we obtain the dynamics of the four amplitudes \( R_k, k = 1, \ldots, 4 \), and three phase differences \( \Delta \phi_k = \phi_k - \phi_1, k = 2, 3, 4 \). One thus has dynamics in a seven-dimensional phase space \( \mathbb{R}_+^4 \times T^3 \), which is equivalent to the dynamics of four coupled Stuart-Landau oscillators, each representing one cluster [16].

Note that the asymmetric solutions \( u_{1,2} \) now correspond to cluster solutions with two oscillators in each cluster. Due to the new dimensions in phase space, however, the stability of those 2-2 cluster solutions might differ from the stability of the \( u_{1,2} \) solutions in the two-oscillator ensemble. This can be visualized by evaluating the Jacobian of the four-oscillator system at the 2-2 cluster solutions and investigating the number of positive eigenvalues.

For the parameter window shown in Fig. 4(a) (corresponding to the orange window in Fig. 3(e)), the stability of the 2–2 cluster solution in the four-oscillator system for \( c_2 = 2 \) is shown in Fig. 4(b). There, one can observe that the parameter range in which the 2-2 cluster solution is stable is smaller than the parameter regime in which the \( u_1 \) solution is stable in the two-oscillator ensemble. Using the numerical continuation software AUTO, one can further observe that the 2-2 cluster solution may either become unstable through a pitchfork (red curve in Fig. 4(b)), a saddle-node (green curve in Fig. 4(b)), or through a supercritical Hopf bifurcation (blue curve in Fig. 4(b)).

Note that for the 2-2 cluster in the four-oscillator ensemble, we have \( 2 \cdot 2 - 2 = 2 \) cluster integrity exponents of one cluster, 2 cluster integrity exponents of the other cluster and 3 cluster system orbit stability exponents, and thus 7 exponents in total. However, the degeneracy of the cluster integrity exponents implies that if a 2-2 cluster state is stable for certain parameters in the four oscillator ensemble, then any \( N/2 - N/2 \) cluster state with an even-valued \( N \geq 4 \) is stable for such parameters. Therefore, the parameter window shown in Fig. 4(b) indicates where balanced 2-cluster states (that is, cluster states with the same number of oscillators in each of the two clusters) for any even number of oscillators (larger than four) are stable.
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Figure 4. Stability of the balanced cluster solution in an ensemble of (a) two and (b) four mean-coupled Stuart-Landau oscillators for $c_2 = 2$. The parameter range corresponds to the orange window indicated in Fig. 3(e). As in Fig. 3, the color encodes the number of unstable eigendirections, and gray indicates that the balanced cluster solution does not exist for this set of parameters. The bifurcation curves plotted on the right were obtained using AUTO and symbolize a supercritical Hopf bifurcation curve (blue), a pitchfork bifurcation curve (red) and a saddle-node bifurcation curve (green). The position of the cluster singularity (as discussed in the subsequent sections of this article) is indicated by the pink x.

The question remains how this 2-2 cluster state bifurcates from or to the synchronized

Figure 5. (a) Continuation of the cluster solutions in ensemble of four coupled Stuart-Landau oscillators for $\alpha = 1.12$ and $c_2 = 2$, as indicated by the dashed orange line in Fig. 4(a) but extended to more negative $\beta$ values, using AUTO. Dotted curves indicate repellors, whereas solid lines represent attracting solutions. (b) Stability of the balanced cluster solution in an ensemble of four mean-coupled Stuart-Landau oscillators for $c_2 = 2$. The parameter range corresponds to the orange window indicated in Fig. 3(e). As in Fig. 3, the color encodes the number unstable eigendirections, and gray indicates that the balanced cluster solution does not exist for this set of parameters. The regions in which the 3-1 cluster states are stable are indicated by the white-shaded patches.
solution. In order to investigate this, we look at a cut in parameter space (indicated by the dashed orange line in Fig. 4(b)) and continue the 2-2 cluster solution by varying $\beta$ for fixed $\alpha$ and $c_2$. The amplitude of one cluster, $R_1$, from an exemplary continuation with AUTO is depicted in Fig. 5(a). There, the 2-2 cluster state is indicated by blue curves, which are dotted when this cluster solution is unstable and solid when the 2-2 cluster is stable. Note that there exist two varieties of the 2-2 cluster state, as indicated by the upper and lower blue line in Fig. 5(a). These two solutions differ in their respective value of the amplitude $R_1$, but are both stable for the same $\beta$ values. Moreover, they belong to the same group orbit (as they can be transformed into one another by interchanging the oscillators). Furthermore one can observe that the 2-2 cluster states bifurcate into 2-1-1 cluster states (green and orange curves in Fig. 5(a)), which are unstable for the parameter values regarded here. These 2-1-1 branches, however, subsequently bifurcate into a 3-1 cluster branch (red curve in Fig. 5(a)), which shows two stable regions. In contrast to the stable 2-2 cluster solutions, the stable 3 − 1 cluster solutions do not belong to the same group orbit, but correspond to a state in which the cluster with 3 oscillators has an amplitude $R_1 > 1$ (for $-1.6 \leq \beta \leq -1.2$) and to a state in which the cluster with three oscillators has an amplitude $R_1 < 1$ (for $-2.9 \leq \beta \leq -1.7$). Note that at high respective low $\beta$ values the stable 3-1 cluster states become unstable through saddle-node bifurcations. In addition, the cluster states bifurcate with the synchronized solution (black line in Fig. 5(a)) through a transcritical bifurcation, a scenario which has already been described in Refs. [28, 19]. This is in contrast to the ensemble of symmetrically related 2-2 cluster states, which bifurcate off the synchronized solution through equivariant pitchfork bifurcations at the same points. The overlapping stable 2-2 and 3-1 cluster solutions shown in Fig. 5(a) also explain the hysteretic transitions from balanced cluster states to homogeneous oscillations and vice versa. The regime in which the 3 − 1 cluster solution is stable is shown as a shaded region in Fig. 5(b).

The analysis from above can easily be extended to larger ensembles of oscillators. In particular, we now consider 16 oscillators and investigate the clustering behavior along the parameter cut indicated by the dashed orange line in Fig. 4(b). That is, we fix $c_2 = 2$ and $\alpha = 1.12$ and vary $\beta$. We do this by simulating the full model starting from the balanced 8 − 8 cluster solution, and increasing $\beta$ stepwise. Over the course of this increase, the system goes from a stable 8 − 8 solution to a stable 9 − 7 solution, 10 − 6 solution and so forth until it settles on the synchronized solution. In this way we obtain solutions for every cluster distribution, which we then use for continuation with AUTO. The continuation curves we obtain this way are depicted in Fig. 6(a), where again the amplitude of the cluster with the larger number of oscillators, $R_{C1}$, is shown as a function of the continuation parameter $\beta$. The amplitudes of the smaller cluster for the same parameter window are shown in Fig. 6(b). Note the correspondence of the 8 − 8 cluster solution, shown in blue, with the 2 − 2 cluster solution in Fig. 5(a).

There are two $\beta$ values, $\beta_1$ and $\beta_2$, between which the synchronized solution is unstable, see Fig. 6. At the bifurcation points, there are also transcritical bifurcations of all unbalanced cluster states and pitchfork bifurcations of the balanced state. The two
solution branches of each transcritical and of the pitchfork bifurcations, respectively, connect the two bifurcation points at $\beta_1$ and $\beta_2$. Each of the unbalanced clusters is born respectively destroyed in a saddle-node bifurcation, the most outer one corresponding to the $(N-1)$-1 cluster state being the only one that posses a stable branch. The other cluster states, as well as the balanced one, are stabilized through further (equivariant) pitchfork bifurcations (not shown), compare Fig. 5(a). In this way, two staircases of overlapping stable cluster states are generated, whereby the cluster distribution of the cluster states in subsequent steps differ by just one oscillator. This leads to two cascades of transitions between the two synchronized regions.
Figure 7. The amplitude of one of 16 oscillators as a function of the parameter $\beta$ starting from the stable synchronized solution and increasing $\beta$ (blue curve) and when subsequently reducing $\beta$ (orange curve). Due to the addition of finite noise in the numerical simulations when increasing $\beta$, not all of the densely located cluster states close to $\beta_2$ are resolved.

If we start from the stable synchronized solution for $\beta < \beta_1$ and slowly increase $\beta$, the system goes from the synchronized solution to a $15 - 1$ cluster state, and then two a $14 - 2$ state and so forth, traversing a cascade up to the balanced 8-8 cluster and back, until it settles again on the synchronized solution, see the blue curve in Fig. 7. Thereby, the originally larger cluster with amplitude $R_{C_1}$ (cf. Fig. 6(a)) becomes the smaller one with amplitude $R_{C_2}$ (cf. Fig. 6(b)) beyond the balanced cluster state. The second staircase, and with it the hysteretic behavior, can be seen when we subsequently reduce $\beta$ again, cf. the orange curve in Fig. 7.

Furthermore, from Fig. 6(a) it becomes obvious that all unbalanced cluster solutions, that is all cluster solutions except the $8 - 8$ cluster, bifurcate with the synchronized solution in a transcritical bifurcation. In particular, the unbalanced cluster states exist on both sides of both bifurcation points where the synchronized solution changes stability. However, all cluster solutions lose stability through equivariant pitchfork bifurcations (the resulting branches are not shown in Fig. 6(a)), in which either the cluster with the larger or smaller amplitude breaks up. This is true except for the most unbalanced, the $15 - 1$, cluster, for which the smallest cluster cannot break up. There we find that this stable state is destroyed in a saddle-node bifurcation instead. In addition, each kind of unbalanced cluster solution is stable in two different regions in parameter
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Each of these stable regions lies close but slightly shifted to the stable regions of neighboring cluster states.

Persistence and Cluster Singularities

The observed phenomena of slightly shifted bifurcations can be explained with the concept of persistence. Loosely speaking, if two attractors are hyperbolic and close in phase space, then bifurcations of those attractors are also close in parameter space [19]. In addition, one can infer that between the cluster states shown in Fig. 6, which must also exist for larger ensembles, there lie many more cluster states in larger networks, and using persistence, their stability must be similar to that of the ones seen in Fig. 6. This explains the cascade-like transition from balanced cluster states to the homogeneous solution in large ensembles, where, when changing a parameter, one oscillator after another joins the other cluster until the synchronized solution is reached. We conjecture that for infinitely large ensembles, the cluster attractors are infinitesimal close, and thus this process becomes continuous.

Turning back to Fig. 4(b), we observe that there is a codimension-2 point (pink point in Fig. 4(b)) where the stable 2-2 cluster bifurcates into the synchronized solution in a pitchfork bifurcation. This is in contrast to the phenomena observed in the literature, where the transition to the synchronized solution occurs via the unbalanced cluster solutions [20]. For the Stuart-Landau ensemble, Equation (1), this point can be found analytically as

\[ \alpha_{\text{CS}} = -\frac{1 \pm \sqrt{3}c_2}{2}, \]

\[ \beta_{\text{CS}} = -c_2 \pm \frac{\sqrt{3}}{2}, \]

with the derivation shown in Appendix B.

The characteristics of such a point is that, when starting from a balanced 2-cluster solution and changing the parameters over this bifurcation point, the two clusters approach each other and finally merge and form the synchronized solution. This is what we call a cluster singularity. However, when varying the parameters such that one turns around the codimension-2 point either clock- or anticlockwise (that is, changing \( \alpha \) and \( \beta \) along a path which circumvents the singularity on the left or on the right, cf. pink arrows in Fig. 4(b)), then either of the clusters shrinks and single oscillators join the other cluster until all oscillators finally form the synchronized solution. This scenario can be verified using numerical simulations and is shown in Fig. 8. There, simulations of \( N = 20 \) Stuart-Landau oscillators are shown when avoiding the cluster singularity clockwise (Fig. 8(a)), when directly crossing over the cluster singularity (Fig. 8(b)) and when avoiding the cluster singularity in an anticlockwise manner (Fig. 8(c)).

The cluster singularity serves as an organizing center for nearby unbalanced cluster solutions. Recall that all unbalanced 2-cluster solutions get destroyed in saddle-node bifurcations, cf. Fig. 6. In the cluster singularity, all these saddle-node bifurcations
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as well as the pitchfork bifurcations that alter the stability properties of the cluster states collapse to a single point in phase space, suggesting the name cluster singularity. Note that when crossing the singularity, the stable balanced cluster solution directly bifurcates into the synchronized solution.

![Figure 8.](image)

Figure 8. Simulations of the globally coupled Stuart-Landau ensemble close to the cluster singularity for \( N = 20 \) oscillators and \( c_2 = 2 \), (a) \( \beta = \beta_{CS} + 0.3 > \beta_{CS} \), (b) \( \beta = \beta_{CS} \) and (c) \( \beta = \beta_{CS} - 0.05 < \beta_{CS} \). The direction in which \( \alpha \) is changed is from small to large values.

Clustering in Spatially Extended Systems

Adding a diffusive coupling in addition to the global coupling, one obtains a globally coupled version of the complex Ginzburg-Landau equation [29],

\[
\partial_t W = W - (1 + i c_2) |W|^2 W + d \nabla^2 W + (\alpha + i \beta) \left( \int W \, dx - W \right),
\]

with \( W = W(x, t) \) and \( d \in \mathbb{C} \). In a sense, such a system can be viewed as an ensemble of infinitely many oscillators, coupled locally and globally. If the local coupling is weak (\(|d| \ll 1\)), then we expect the solutions of the Stuart-Landau ensemble to exist also in the spatially extended system. For infinitely many oscillators, however, the 2-cluster solutions become infinitesimally close in phase space (cf. Fig. 6 for 16 oscillators) and thus infinitesimally small perturbations are sufficient to drive the solution from one cluster state to another. This is also what we observe in numerical simulations: the diffusive coupling leads to the selection of a particular cluster distribution, and the multistability of different cluster solutions, as apparent in Fig. 6 for Stuart-Landau oscillators, seems no longer to exist. What is special about the then globally stable 2-cluster solution, however, still remains unknown.
Conclusion

To summarize our results, we have shown ways how clustering can occur in globally coupled ensembles of Stuart-Landau oscillators. In particular, starting from small ensembles, we described how 2-cluster branches bifurcate, and extended this analysis to larger ensembles of oscillators. Doing so, we found a codimension-2 point which we dubbed a cluster singularity: at this point, the stable balanced cluster solution bifurcates directly into the synchronized solution. In addition, all saddle-node bifurcations generating unbalanced cluster solutions collapse in this point. Using numerical simulations, we showed how ensembles of Stuart-Landau oscillators behave close to this cluster singularity. Since any oscillatory system close to the onset of oscillations can be mapped onto the dynamics of the Stuart-Landau oscillator, we believe that cluster singularities are common in oscillatory systems with global coupling, and that an experimental observation of these should be possible. Concludingly, we discussed how our results extend to spatially extended systems, where the diffusive coupling seems to destroy the multistability.

We believe that our considerations may serve as a further step towards a better understanding of clustering behavior in coupled oscillators. In addition, 2-cluster solutions in the regarded parameter windows may also become unstable through supercritical Hopf bifurcations for smaller $\alpha$ values, and even bifurcate into chimera states [13, 16]. How this transition occurs for different cluster distributions is still an open question.

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Appendix A: Balanced cluster solution

In order to find the solutions of balanced 2-cluster states, it is sufficient to find the fixed points of the reduced two-oscillator system

\[
\begin{align*}
\partial_t R_1 &= R_1 - R_1^3 + \alpha (R_2 \cos (\Delta \theta) - R_1) + \beta R_2 \sin (\Delta \theta) \\
\partial_t R_2 &= R_2 - R_2^3 + \alpha (R_1 \cos (\Delta \theta) - R_2) - \beta R_1 \sin (\Delta \theta) \\
\partial_t \Delta \theta &= -c_2 \left( R_1^2 - R_2^2 \right) + \beta \cos (\Delta \theta) \left( \frac{R_2}{R_1} - \frac{R_1}{R_2} \right) \\
&\quad - \alpha \sin (\Delta \theta) \left( \frac{R_2}{R_1} + \frac{R_1}{R_2} \right).
\end{align*}
\]
In addition, these equations can be simplified by introducing the sum and the difference of the squared amplitudes, \( \gamma = R_1^2 + R_2^2 \) and \( \rho = R_1^2 - R_2^2 \), with
\[
\begin{align*}
\partial_t \gamma &= 2R_1 \partial_t R_1 + 2R_2 \partial_t R_2 \\
\partial_t \rho &= 2R_1 \partial_t R_1 - 2R_2 \partial_t R_2.
\end{align*}
\]
This transforms Equations (1) to (2) into
\[
\begin{align*}
\partial_t \gamma &= 2 (1 - \alpha - \gamma) \gamma + \gamma^2 - \rho^2 + 2\alpha \sqrt{\gamma^2 - \rho^2} \cos (\Delta \theta) \\
\partial_t \rho &= 2 (1 - \alpha - \gamma) \rho + 2\beta \sqrt{\gamma^2 - \rho^2} \sin (\Delta \theta) \\
\partial_t \Delta \theta &= -c_2 (R_1^2 - R_2^2) - \beta \cos (\Delta \theta) \frac{R_1^2 - R_2^2}{R_1 R_2} - \alpha \sin (\Delta \theta) \frac{R_1^2 + R_2^2}{R_1 R_2}
\end{align*}
\]
and using \( R_1 R_2 = \sqrt{\gamma^2 - \rho^2}/2 \),
\[
\begin{align*}
\partial_t \gamma &= 2 (1 - \alpha - \gamma) \gamma + \gamma^2 - \rho^2 + 2\alpha \sqrt{\gamma^2 - \rho^2} \cos (\Delta \theta) \\
\partial_t \rho &= 2 (1 - \alpha - \gamma) \rho + 2\beta \sqrt{\gamma^2 - \rho^2} \sin (\Delta \theta) \\
\partial_t \Delta \theta &= -c_2 \rho - 2\beta \cos (\Delta \theta) \frac{\rho}{\sqrt{\gamma^2 - \rho^2}} - 2\alpha \sin (\Delta \theta) \frac{\gamma}{\sqrt{\gamma^2 - \rho^2}}.
\end{align*}
\]
At a fixed point solution, this system of equations must satisfy
\[
\begin{align*}
0 &= (1 - \alpha - \gamma) \gamma + \frac{\gamma^2 - \rho^2}{2} + \alpha \sqrt{\gamma^2 - \rho^2} \cos (\Delta \theta) \\
0 &= (1 - \alpha - \gamma) \rho + \beta \sqrt{\gamma^2 - \rho^2} \sin (\Delta \theta) \\
0 &= -c_2 \rho - 2\beta \cos (\Delta \theta) \frac{\rho}{\sqrt{\gamma^2 - \rho^2}} - 2\alpha \sin (\Delta \theta) \frac{\gamma}{\sqrt{\gamma^2 - \rho^2}}.
\end{align*}
\]
Solving the first two equations for \( \cos (\Delta \theta) \) and \( \sin (\Delta \theta) \) yields
\[
\begin{align*}
\sqrt{\gamma^2 - \rho^2} \cos (\Delta \theta) &= - (1 - \alpha - \gamma) \frac{\gamma}{\alpha} - \frac{\gamma^2 - \rho^2}{2\alpha} \quad (4)\\
\sqrt{\gamma^2 - \rho^2} \sin (\Delta \theta) &= - (1 - \alpha - \gamma) \frac{\rho}{\beta} \quad (5)
\end{align*}
\]
and inserted into the last equation,
\[
\begin{align*}
0 &= -c_2 \rho + \frac{\beta}{\alpha} \frac{\rho}{\gamma^2 - \rho^2} \left[2 (1 - \alpha - \gamma) \gamma + \gamma^2 - \rho^2\right] + 2\alpha \frac{\gamma}{\beta} \frac{1}{\gamma^2 - \rho^2} (1 - \alpha - \gamma) \rho \\
\Rightarrow 0 &= -c_2 (\gamma^2 - \rho^2) + \frac{\beta}{\alpha} \left[2 (1 - \alpha - \gamma) \gamma + \gamma^2 - \rho^2\right] + 2\alpha \frac{1}{\beta} (1 - \alpha - \gamma) \gamma \\
\Rightarrow 0 &= -c_2 \alpha \beta (\gamma^2 - \rho^2) + 2\beta^2 (1 - \alpha - \gamma) \gamma + \beta^2 (\gamma^2 - \rho^2) + 2\alpha^2 (1 - \alpha - \gamma) \gamma \\
\Rightarrow 0 &= \left(\beta^2 - c_2 \alpha \beta\right) (\gamma^2 - \rho^2) + 2 \left(\alpha^2 + \beta^2\right) (1 - \alpha - \gamma) \gamma.
\end{align*}
\]
Solving for \( \rho^2 \)
\[
\rho^2 = 2 \frac{\alpha^2 + \beta^2}{\beta^2 - c_2 \alpha \beta} (1 - \alpha - \gamma) \gamma + \gamma^2. \quad (6)
\]
Using the identity \( 1 = \sin^2 (\Delta \theta) + \cos^2 (\Delta \theta) \), we can write Equations (4) and (5), yielding
\[
\begin{align*}
\gamma^2 - \rho^2 &= (1 - \alpha - \gamma)^2 \frac{\rho^2}{\beta^2} + \left(- (1 - \alpha - \gamma) \frac{\gamma}{\alpha} - \frac{\gamma^2 - \rho^2}{2\alpha}\right)^2 \\
\Rightarrow \gamma^2 - \rho^2 &= (1 - \alpha - \gamma)^2 \frac{\rho^2}{\beta^2} + \frac{1}{4\alpha^2} (\gamma^2 - \rho^2 + 2 (1 - \alpha - \gamma) \gamma)^2. \quad (7)
\end{align*}
\]
By inserting Equation (6) into Equation (7), we can solve it for $\gamma$ and obtain

$$\gamma = \frac{(1 - \alpha)(3\beta - 4\alpha c_2 - \beta c_2^2)}{2\beta - 4\alpha c_2 - 2\beta c_2^2} \pm \frac{\beta \sqrt{(1 - \alpha)^2(1 + c_2^2)^2 - 8\beta^2(1 - c_2^2) + 8\alpha c_2(3\beta - 2\alpha c_2 - \beta c_2^2)}}{2\beta - 4\alpha c_2 - 2\beta c_2^2}. \quad (8)$$

Together with Equation (6), this can be used to calculate $R_1$, $R_2$ and $\Delta \theta$.

**Appendix B: Cluster Singularities**

The idea is that at the cluster singularity, the saddle-node bifurcations of all unbalanced 2-cluster solutions hit the pitchfork at which the synchronous solution becomes unstable. This must be true for any $\epsilon = N_1/N$. Therefore, for simplicity, we take the limit $\epsilon \to 0$ in system Equations (2) to (4), yielding

$$0 = R_1 - R_1^3$$
$$0 = R_2 - R_2^3 + \alpha (R_1 \cos(\Delta \phi) - R_2) - \beta R_1 \sin(\Delta \phi)$$
$$0 = -c_2(R_1^2 - R_2^2) + \beta - \beta \cos(\Delta \phi) \left(\frac{R_1}{R_2}\right) - \alpha \sin(\Delta \phi) \left(\frac{R_1}{R_2}\right).$$

This means $R_1 = 1$ and thus leaves

$$0 = R_2 - R_2^3 - \alpha R_2 + \alpha \cos(\Delta \phi) - \beta \sin(\Delta \phi)$$
$$0 = -c_2(R_2 - R_2^3) + \beta R_2 - \beta \cos(\Delta \phi) - \alpha \sin(\Delta \phi).$$

$$R_2^2 = \frac{-2\alpha + 2\beta c_2 - 1 - c_2^2}{2(1 + c_2^2)} \pm \frac{\sqrt{(2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4(\alpha^2 + \beta^2)(1 + c_2^2)}}{2(1 + c_2^2)}$$

Setting $R_2^2 = 1$ means we are at the point where the cluster solution meets the synchronous solution ($\Delta \phi = 0$ follows from $R_2 = R_1 = 1$), and from this the previous expression turns into

$$1 = \frac{-2\alpha + 2\beta c_2 - 1 - c_2^2}{2(1 + c_2^2)} \pm \frac{\sqrt{(2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4(\alpha^2 + \beta^2)(1 + c_2^2)}}{2(1 + c_2^2)}$$

$$\rightarrow 2(1 + c_2^2) = -(2\alpha + 2\beta c_2 - 1 - c_2^2) \pm \sqrt{(2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4(\alpha^2 + \beta^2)(1 + c_2^2)}$$

$$\rightarrow (2\alpha + 2\beta c_2 + 1 + c_2^2) = \pm \sqrt{(2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4(\alpha^2 + \beta^2)(1 + c_2^2)}$$
$$\rightarrow (2\alpha + 2\beta c_2 + 1 + c_2^2)^2 = (2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4(\alpha^2 + \beta^2)(1 + c_2^2)$$
$$\rightarrow (2\alpha + 2\beta c_2)^2 + (1 + c_2^2)^2 + 2(2\alpha + 2\beta c_2)(1 + c_2^2) = (2\alpha + 2\beta c_2)^2$$
$$+ (1 + c_2^2)^2 - 2(2\alpha + 2\beta c_2)(1 + c_2^2)$$
$$- 4(\alpha^2 + \beta^2)(1 + c_2^2)$$
Cluster singularities

\[ \rightarrow 4 (2\alpha + 2\beta c_2) (1 + c_2^2) = -4 (\alpha^2 + \beta^2) (1 + c_2^2) \]
\[ \rightarrow \alpha^2 + \beta^2 + 2 (\alpha + \beta c_2) = 0 \] (9)

which coincides with the curve at which the homogeneous solution becomes unstable. For the saddle-node curve of the cluster solutions, the two solutions of \( R_2^2 \) must equal, and thus the discriminant must equal zero

\[
0 = (2\alpha + 2\beta c_2 - 1 - c_2^2)^2 - 4 (\alpha^2 + \beta^2) (1 + c_2^2) \\
0 = 4 (\alpha^2 + \beta^2) (1 + c_2^2) = (2\alpha + 2\beta c_2 - 1 - c_2^2)^2 \\
0 = 4 (\alpha^2 + \beta^2) (1 + c_2^2) = (2\alpha + 2\beta c_2)^2 + (1 + c_2^2)^2 - 2 (2\alpha + 2\beta c_2) (1 + c_2^2) \\
0 = 4 (\alpha^2 + \beta^2) (1 + c_2^2) = 4 (\alpha + \beta c_2)^2 + (1 + c_2^2)^2 - 4 (\alpha + \beta c_2) (1 + c_2^2)
\]

Now use that \(-2 (\alpha + \beta c_2) = \alpha^2 + \beta^2\) from Equation (9) above,

\[
4 (\alpha^2 + \beta^2) (1 + c_2^2) = 4 (\alpha + \beta c_2)^2 + (1 + c_2^2)^2 - 4 (\alpha + \beta c_2) (1 + c_2^2) \\
4 (\alpha^2 + \beta^2) (1 + c_2^2) = (\alpha^2 + \beta^2)^2 + (1 + c_2^2)^2 + 2 (\alpha^2 + \beta^2) (1 + c_2^2) \\
0 = (\alpha^2 + \beta^2)^2 + (1 + c_2^2)^2 - 2 (\alpha^2 + \beta^2) (1 + c_2^2) \\
0 = \alpha^2 + \beta^2 - 1 - c_2^2
\] (10)

Equation (10) gives the saddle-node curve of the cluster with \( \epsilon = 0 \). So for the cluster singularity, this saddle-node bifurcation coincides with the point at which the homogeneous solution becomes unstable, as given by Equation (9), which yields

\[
0 = \alpha^2 + \beta^2 + 2 (\alpha + \beta c_2) \\
0 = \alpha^2 + \beta^2 - 1 - c_2^2.
\]

Subtraction of these two equations yields

\[
0 = 2 (\alpha + \beta c_2) + 1 + c_2^2 \\
\alpha = -\beta c_2 - \frac{1}{2} (1 + c_2^2)
\] (11)

and thus

\[
0 = \left( \beta c_2 + \frac{1 + c_2^2}{2} \right)^2 + \beta^2 - (1 + c_2^2) \\
0 = (2\beta c_2 + 1 + c_2^2)^2 + 4\beta^2 - 4 (1 + c_2^2) \\
0 = 4\beta^2 c_2^2 + (1 + c_2^2)^2 + 4\beta c_2 (1 + c_2^2) + 4\beta^2 - 4 (1 + c_2^2) \\
0 = 4\beta^2 (1 + c_2^2) + (1 + c_2^2)^2 + 4\beta c_2 (1 + c_2^2) - 4 (1 + c_2^2) \\
0 = 4\beta^2 + (1 + c_2^2) + 4\beta c_2 - 4 \\
0 = \beta^2 + \beta c_2 + \frac{-3 + c_2^2}{4} \\
\beta = \frac{-c_2 \pm \sqrt{3}}{2}.
\]
This solution plugged into Equation (11) yields

\[
\alpha = -\frac{c_2 \pm \sqrt{3}}{2} c_2 - \frac{1}{2} (1 + c_2^2) \\
= \frac{c_2 \mp \sqrt{3}}{2} c_2 - \frac{1}{2} (1 + c_2^2) \\
= -\frac{1 \mp \sqrt{3} c_2}{2}.
\]

So, in total, we have at the cluster singularity

\[
\alpha = -\frac{1 \pm \sqrt{3} c_2}{2} \quad \text{(12)} \\
\beta = -\frac{c_2 \pm \sqrt{3}}{2} \quad \text{(13)}
\]

This indicates two possible solutions for the cluster singularity. Furthermore, it is worth mentioning that it seems to exist for all \(c_2\) values.

**Appendix C: Numerical Methods**

For the integration of the Stuart-Landau ensemble, an implicit Adams method with a fixed time step of \(dt = 0.01\) is used. All figures are generated using matplotlib [30].

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