Stationary electromagnetic fields in the exterior of a slowly rotating relativistic star: a description beyond the low-frequency approximation

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29 September 2018

ABSTRACT

We investigate the electromagnetic fields in the vacuum exterior of a rotating relativistic star endowed with a magnetic dipole moment, and with the stellar surface behaving as a perfect conductor. While the stellar rotation is treated in the slow-approximation of general relativity, we do not restrict our attention to slowly rotating electromagnetic fields, and take our analysis beyond the low-frequency approximation considered so far. When the dipole moment is misaligned with the rotational axis, our approach does not yield analytic solutions as in Rezzolla et al. (2001a, b), but determines the properties of the electromagnetic fields approximately and semi-analytically, by computing the coefficients of simple expressions for the fields through the numerical solution of two partial differential equations. Because our approach provides a solution which is in principle valid throughout space, we can evaluate the accuracy and/or invalidity of previously known analytic expressions at different distances from the stellar surface. Overall, the solutions found in this way represent an efficient way of bridging in a single semi-analytic formalism the strongly relativistic (Rezzolla et al. 2001a, b) and the asymptotic regimes (Deutsch 1955) for which analytic solutions have been found.

Key words: relativity – stars: magnetic fields – stars: neutron – stars: rotation

1 INTRODUCTION

It is well known that magnetic fields often play a key role in astrophysics and an ample discussion is given, for instance, in Parker (1979); Zeldovich et al. (1983). In the limit of very high electrical conductivity, the magnetic flux is “frozen” into the fluid and increases as a result of fluid compressions. In this case the the typical value of the magnetic field $B$ can be related to the rest-mass density $\rho$ through the simple relation $B \propto \rho^{2/3}$, so that very strong magnetic field may be produced in compact stars, where the gravity is also strong. Some observations indicate that the magnetic field in young neutron stars is of order $10^{12}$ G. Recently, exceptionally stronger cases with $10^{14} - 10^{15}$ G are suggested in the phenomena with soft gamma repeaters and anomalous X-ray pulsars (Mereghetti & Stella 1992; Kouveliotou et al. 1998, 1999). Very intense magnetic fields may be produced through a dynamo action (Thompson & Duncan 1993) and the neutron stars with such magnetic fields are usually referred to as magnetars. The ideas behind the existence and formation of magnetars are gaining support from the observations and a number of authors have discussed the mechanisms generating these magnetic fields and their astrophysical relevance (see, for instance, Thompson & Duncan 1992, 1993). The combined effects of strong gravity and intense electromagnetic fields may be crucial in some astrophysical objects and it is therefore important to consider the electromagnetic field in curved spacetime produced by strong gravity. The study is important in clarifying the general relativistic effects on the electromagnetic fields, and also indispensable in constructing realistic models.

A number of works have been so far published concerning the effects of strong gravity on the electromagnetic fields. The magnetic field in principle affects the spacetime curvature through the Einstein equations, but the actual deformation at the surface is of order $10^{-5}(B/10^{15}{\text{G}})^2$ even for magnetars (Konno et al. 1994, 2000). In most astrophysical cases, therefore, the
deformation may be neglected, and general relativistic effects can be evaluated by solving the Maxwell equations in fixed but curved spacetime. Several attempts are done along this line of thought. Stationary electromagnetic fields have been considered both in Schwarzschild spacetimes (e.g. Ginzburg & Ozernoy (1964); Anderson & Cohen (1970); Petterson (1974)) and in slowly rotating spacetimes (e.g. Muslimov & Tysgan (1992); Muslimov & Harding (1997); Konno & Kojima (2000)).

More recently, Rezzolla et al. (2001a,b) and Zanotti & Rezzolla (2002) have extensively studied the interior and vacuum exterior electromagnetic fields produced by a rotating dipole moment comoving with the star. Their analysis is fully general relativistic and has provided compact and analytic expressions for both the cases in which the magnetic dipole is aligned or not with the stellar rotational axis. In these works, the star of mass $M$ and radius $R$ is assumed to be rotating at an angular velocity $\Omega_c$ much smaller than the maximum angular velocity, i.e. the Keplerian angular $\Omega_K \simeq \sqrt{GM/R^3}$, so that a slow-rotation approximation can be used satisfactorily to describe the general relativistic corrections introduced by the rotation of the spacetime. Furthermore, the electromagnetic fields are assumed to be comoving with the star at all times, thus possessing an intrinsic angular velocity $\Omega = \Omega_c \ll \Omega_K$; as a result, the expressions of Rezzolla et al. (2001a, b) are expected to be accurate in the vicinity of the star, where the general relativistic corrections are stronger and the intrinsic velocities smaller. We will refer to this as to the low-frequency approximation in the Maxwell equations.

While the slow-rotation approximation is a good one except for extremely rapidly rotating stars, the low-frequency approximation may have some limitations. The first one is that the expressions of the electromagnetic fields derived in this case cannot be accurate at larger radii, (i.e. in the vicinity and beyond the light cylinder), where they should manifest a wave-like nature. The second limitation may arise in the case in which the electromagnetic fields are not comoving with the star and, more specifically, in the case in which $\Omega \gg \Omega_c$.

In this paper we circumvent these limitations by considering the general relativistic electromagnetic fields exterior to a rotating neutron star in the slow-rotation approximation but not in the low-frequency approximation. We will not discuss here the physical conditions that may lead to electromagnetic fields with $\Omega \gg \Omega_c$, nor whether this is likely to happen for realistic astrophysical sources (indeed this condition may be difficult to occur in practice); rather we will make this our working hypothesis. As a result, we here consider the electromagnetic fields in the vacuum exterior of a relativistic star slowly rotating at frequency $\Omega_c \ll \Omega_K$ and endowed with a magnetic dipole moment rotating at a frequency (either $\Omega = \Omega_c$, or $\Omega \gg \Omega_c$). Furthermore, the stellar surface is assumed to be a perfect conductor, thus inducing a quadrupolar electric field. Our approach differs from previous ones both in the underlying assumptions (i.e. high frequency electromagnetic fields) but also in the method. In particular, we do not provide analytic expressions (as in Rezzolla et al. 2001a, b) but determine the properties of the electromagnetic fields through a semi-analytical approach in which simple expressions for the electromagnetic fields need to be completed through the calculation of coefficients coming from the numerical solution of two partial differential equations. Because the approach is based on a series expansion, the solution and its accuracy depends on the number of terms considered. On the other hand, our approach yields solutions that are valid in the whole spatial domain and this allows us to compare our results with the analytic expressions of Rezzolla et al. (2001a, b) and of Deutsch (1955), estimating the accuracy of these analytic solutions in those regimes where they are not expected to be valid.

The paper is organized as follows. In Section 2 we summarize the Maxwell equations in vacuum with stationary condition. The equations are reduced to a set of two partial differential equations among two scalar potentials. Here, we discuss how it is possible to to look for simple Deutsch-type solution which are approximate solution of the general relativistic Maxwell equations in a slowly rotating spacetime. In Section 3, we show the explicit calculations in the case in which the magnetic dipole is parallel to the rotational axis, while in Section 4, we consider the case when the dipole is perpendicular. Because of the linearity of the Maxwell equations, the general case of an oblique rotator can be constructed from the superposition of these two particular cases. Concluding remarks are finally presented in Section 4. Hereafter we will use units in which $c = G = 1$.

## 2 STATIONARY ELECTROMAGNETIC FIELD

We will consider the solution to the Maxwell equations in the fixed background spacetime of a rotating relativistic star whose generic axially symmetric line element is given by

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i_i dt)(dx^j + \beta^j_j dt)$$

$$= -\alpha^2 dt^2 + e^{\eta_\phi}(d\phi - \omega dt)^2 + e^{2\eta} dr^2 + e^{2\eta} d\theta^2.$$  \hspace{1cm} (1)

In practice and except for rapidly rotating stars, it may be sufficient to take into account the first-order rotational effect of the metric only (see, for instance Zanotti & Rezzolla (2002)) so that the metric functions exterior to the slowly rotating star are reduced to the form

$$\alpha \to \left(1 - \frac{2M}{r}\right)^{1/2}, \quad e^\eta \to \left(1 - \frac{2M}{r}\right)^{-1/2}, \quad e^\phi \to r, \quad e^\psi \to r \sin \theta, \quad \omega \to \frac{2J}{r^3},$$  \hspace{1cm} (2)

where $J$ is the dipole moment.
where $M$ and $J$ are the mass and angular momentum of the central star. In this paper, the electromagnetic fields are denoted by the values measured by “zero angular momentum observers”, i.e. ZAMOs (Bardeen et al. 1972), and their components projected to a locally orthonormal tetrad frame are expressed as $(E_r, E_\theta, E_\phi)$ and $(B_r, B_\theta, B_\phi)$. These are sometimes expressed with hatted indices to distinguish them from coordinate values (see, for instance, Rezzolla et al. 2001a, b); however, because we will always refer only to the electromagnetic fields as measured by ZAMOs, we will omit here the hatted indices.

The electric and magnetic fields in vacuum satisfy the Maxwell equations in this curved spacetime (e.g. Thorne et al. 1986):

\[
\nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{E} = 0,
\]

\[
\partial_t \vec{B} - \mathcal{L}_{\beta_\alpha} \vec{B} = -\nabla \times (\alpha \vec{E}),
\]

\[
\partial_t \vec{E} - \mathcal{L}_{\beta_\alpha} \vec{E} = \nabla \times (\alpha \vec{B}),
\]

where $\mathcal{L}_{\beta_\alpha}$ is the Lie derivative along the vector field $\vec{\beta} \equiv -\omega \partial_\theta$ and is defined for any vector $\vec{A}$ as

\[
\mathcal{L}_{\beta_\alpha} \vec{A} = (\vec{\beta}_\alpha - \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{\beta}_\alpha.
\]

Clearly, in the frame rotating with angular velocity $\vec{\beta} \equiv \Omega \partial_\theta$, the electromagnetic fields are independent of time and these are the electromagnetic fields we will consider hereafter. Note that the angular velocity $\Omega$ may differ from that of the star $\Omega_*$, which is related to the spin angular momentum $J$ and the inertial moment $I$ through the relation $\Omega_* = J/I$ in the slowly rotating spacetime. For example, the electromagnetic fields can be produced by a periodic motion with a certain angular frequency $\Omega$ on or outside the star. Furthermore, and as mentioned in the Introduction, we do not assume that the frequency $\Omega$ is small, unlike in previous work. The stationarity of a scalar function $f$ is equivalent to expressing that $f$ should depend on the time $t$ and on the azimuthal angle $\phi$ only in a combination of the type $\phi' = \phi - \Omega t$. In the locally flat four-dimensional spacetime in which the ZAMOs make their measurements, this condition of stationarity for the function $f$ can be expressed by requiring that the function $f$ is solution of the equation: $\partial_t f + \vec{\beta} \cdot \nabla f = 0$, i.e. that the function is a constant along the direction $(1, \vec{\beta})$, in the four-dimensional reference frame of the ZAMOs. In the case of the electromagnetic vector fields $\vec{E}$ and $\vec{B}$, the mathematical condition for stationarity needs to be expressed through the Lie derivative as

\[
\partial_t \vec{E} + \mathcal{L}_{\beta_\alpha} \vec{E} = 0, \quad \partial_t \vec{B} + \mathcal{L}_{\beta_\alpha} \vec{B} = 0.
\]

Using these conditions in eqs. (4) and (5), some combinations of $\vec{E}$ and $\vec{B}$ are expressed by gradient of two scalar potentials, $H$ and $G$. After solving for $\vec{E}$ and $\vec{B}$, we have

\[
\vec{E} = -\frac{1}{\alpha \Lambda} \left[ e^{-\mu} G_{,r} + \varpi e^{-\eta} \alpha^{-1} H_{,\theta} , \ e^{-\eta} G_{,\theta} - \varpi e^{-\mu} \alpha^{-1} H_{,r} , \ e^{-\eta} \Lambda G_{,\phi} \right],
\]

\[
\vec{B} = -\frac{1}{\alpha \Lambda} \left[ e^{-\mu} H_{,r} - \varpi e^{-\eta} \alpha^{-1} G_{,\theta} , \ e^{-\eta} H_{,\theta} + \varpi e^{-\mu} \alpha^{-1} G_{,r} , \ e^{-\eta} \Lambda H_{,\phi} \right],
\]

where a comma denotes a partial derivative with respect to the coordinate $j(= r, \theta, \phi)$ and

\[
\varpi = \Omega - \omega.
\]

From eqs. (4) and (5), we have a pair of second-order partial differential equations

\[
\Lambda \left[ \alpha \left( e^{-\mu+\eta+\psi} \alpha^{-1} G_{,r} \right)_{,r} + \alpha \left( e^{\mu-\eta+\psi} \alpha^{-1} G_{,\theta} \right)_{,\theta} + e^{\mu+\eta-\psi} \Lambda G_{,\phi} \right] - e^{-\mu+\eta+\psi} \Lambda_{,r} G_{,r} - e^{\mu+\eta+\psi} \Lambda_{,\theta} G_{,\theta} + K_r G_{,\theta} - K_\theta H_{,r} = 0,
\]

\[
\Lambda \left[ \alpha \left( e^{-\mu+\eta+\psi} \alpha^{-1} H_{,r} \right)_{,r} + \alpha \left( e^{\mu-\eta+\psi} \alpha^{-1} H_{,\theta} \right)_{,\theta} + e^{\mu+\eta-\psi} \Lambda H_{,\phi} \right] - e^{-\mu+\eta+\psi} \Lambda_{,r} H_{,r} - e^{\mu+\eta+\psi} \Lambda_{,\theta} H_{,\theta} - K_r G_{,\theta} + K_\theta G_{,r} = 0,
\]

where

\[
K_j = \alpha \left( \frac{\varpi E_{,r}}{\alpha^2} \right)_{,j} + \frac{\varpi^2 e^{4\psi}}{\alpha^3} \left( \varpi \right)_{,j} \quad \text{(for } j = r, \theta).\]

These equations (14) and (15) are symmetric under the transformation between $\vec{E}$ and $\vec{B}$, i.e. $\vec{E} \to \vec{B}$, by changing $G \to H$ and $H \to -G$. Note that this is a generic property of the source-free Maxwell equations.
3 AXIALLY SYMMETRIC CASE

We now look for solutions of eqs. (14) and (15) in the slowly rotating spacetime. Note that these equations have a singularity at the light cylinder $\Lambda = 0$, so that a special care should be paid when performing a numerical integration. We now expand the electromagnetic fields using a series expansion with respect to the angular part which is particularly advantageous for the imposition of regularity conditions. Since the electric and magnetic fields are respectively polar and axial vectors with respect to parity transformation, we expand them through two scalar potentials $G$ and $H$, which have parity of $+1$ and $-1$, respectively. Therefore, for an axially symmetric system, these functions are respectively expanded in terms of the spherical harmonics $Y_{2l}^0$ and $Y_{2l+1}^0$. Note that the following forms are equivalent to the spherical harmonics expansion, with the difference that the regularity condition can be imposed more easily.

$$G(r, \theta) = \sum_{n=0}^{\infty} g_n(r) (\sin \theta)^{2n},$$

(17)

$$H(r, \theta) = \cos \theta \left( \sum_{n=0}^{\infty} h_n(r) (\sin \theta)^{2n} \right).$$

(18)

The electromagnetic fields are described as

$$B_r = -\frac{\cos \theta}{\Lambda} \left[ \sum_{n=0}^{\infty} \left( h_n' - \frac{2n\pi}{\alpha^2} g_n \right) (\sin \theta)^{2n} \right],$$

(19)

$$B_\theta = -\frac{\sin \theta}{\alpha \Lambda} \left[ \sum_{n=0}^{\infty} \left( \frac{\pi}{\alpha} r^2 g_n' + 2(n+1)h_{n+1} - (2n+1)h_n \right) (\sin \theta)^{2n} \right],$$

(20)

$$E_r = \frac{-1}{\Lambda} \left[ g_0' + \sum_{n=1}^{\infty} \left( g_n' + \frac{\pi}{\alpha^2} (2nh_n - (2n-1)h_{n-1}) \right) (\sin \theta)^{2n} \right],$$

(21)

$$E_\theta = \frac{\sin \theta \cos \theta}{\alpha \Lambda} \left[ \sum_{n=0}^{\infty} \left( \frac{\pi}{\alpha} r^2 h_n' - 2(n+1)g_{n+1} \right) (\sin \theta)^{2n} \right],$$

(22)

where a prime $'$ denotes a radial derivative and

$$\Lambda = 1 - b^2 \sin^2 \theta = 1 - \left( \frac{\pi r}{\alpha} \right)^2 \sin^2 \theta.$$ 

(23)

Note also that the expansions (17) and (18) are not necessarily valid in whole space. The denominators of eqs. (19) - (22) become zero at the light cylinder, which is located at $b(r) \sin \theta = 1$. Taking the limit of $\sin \theta \to 1/b(r)$ in the numerators, we have two regularity conditions among the radial functions, $g_n$ and $h_n$:

$$\sum_{n=0}^{\infty} \left( g_n' + \frac{\pi}{\alpha^2} \left( 2n - \frac{2n+1}{b^2} \right) h_n \right) b^{-2n} = 0,$$

(24)

and

$$\sum_{n=0}^{\infty} \left( h_n' - \frac{2n\pi}{\alpha^2} g_n \right) b^{-2n} = 0.$$ 

(25)

These relations should be satisfied for the region $1/b \leq 1$, which is equivalent to $r \geq r_c$, where the radius $r_c$ is defined by $b(r_c) = 1$ and, of course, $r_c$ is reduced to $1/\Omega$ in the flat spacetime. Note that it is not necessary for the radial functions to satisfy the conditions (24), (25) in the inner region $r < r_c$. It may be possible to construct a global solution by smoothly matching the radial functions across the radius $r_c$, which are solved with and without the constraints (24), (25) on either side of $r_c$. We however look for the solutions that satisfy (at least approximately) the constraints (24), (25) in the whole space exterior to the stellar surface. This method may restrict the class of solutions, but avoids to locate and handle a matching at $r_c$. If no solution can be found satisfying the constraints (24), (25) in entire region outside the star, we will then look for solutions under weaker constraints.

We consider the electromagnetic field described or approximated by a finite number of functions $g_n(r)$ and $h_n(r)$. For example, the field by rotating magnetic dipole in a flat spacetime can be described by $g_n, h_n (n = 0, 1)$. See Appendix for the results in the flat case for comparison. It is natural to approximate the electromagnetic fields in a similar fashion also in the case of a slowly rotating spacetime.

1 The function $H$ is not a scalar one, but rather a "pseudo-scalar".
The regularity conditions for the non-vanishing components \( g_n, h_n (n = 0, 1) \) are given by

\[
0 = \mathcal{X} = g_1' + b^2 g_0' - \frac{\omega}{\alpha^2} h_0 - \frac{1}{c^2} (3 - 2b^2) h_1
\]

(26)

and

\[
0 = \mathcal{Y} = h_1' + b^2 h_0' - \frac{2\omega}{\alpha^2} g_1.
\]

(27)

Using these conditions in eqs. (19)–(22), we have

\[
B_r = -h_0' \cos \theta,
\]

(28)

\[
B_\theta = \frac{1}{\alpha r} \left( h_0 - 2h_1 - \frac{\omega}{\alpha^2} g_0' \right) \sin \theta,
\]

(29)

\[
E_r = -g_0' - \frac{3}{\alpha^2} h_0 \sin^2 \theta = - \left( g_0' + \frac{2}{\alpha^2} h_1 \right) + \frac{2}{c^2} h_1 P_2(\cos \theta),
\]

(30)

\[
E_\theta = \frac{1}{\alpha r} \left( \omega^2 h_0' - 2g_1 \right) \sin \theta \cos \theta.
\]

(31)

The electric field is given by a mixture of monopolar and quadrupolar terms. In particular, when the total electric charge is zero, the monopolar part vanishes, with the quadrupolar one being the one induced by rotation. This condition can be written as

\[
g_0 + \frac{2}{\alpha^2} h_1 = 0.
\]

(32)

The divergence of the magnetic field yields

\[
\mathbf{\nabla} \cdot \mathbf{B} = -\alpha \mathcal{H} \cos \theta,
\]

(33)

where

\[
\mathcal{H} = h_0'' + \frac{2}{r} h_0' + \frac{2\omega}{\alpha^2} g_0 + \frac{2}{\alpha^2} (2h_1 - h_0).
\]

(34)

Using the condition (32) in eq. (34), we have a second order differential equation of \( h_0 \) only. Enforcing that the magnetic field is divergence-free is equivalent to setting \( \mathcal{H} = 0 \) so that equation (31) has solution

\[
\begin{align*}
\mathcal{H} &= \frac{3\mu}{4M^2} \left[ \alpha^2 \ln \alpha^2 - \frac{2M^2}{r} + 2M \right],
\end{align*}
\]

(35)

where \( \mu \) denotes the magnetic dipole moment. This relativistic solution for the magnetic field is well known in the literature

\cite{GinzburgOzernoy1964, AndersonCohen1970, Petterson1974}.

We now express the divergence of the electric field as

\[
\mathbf{\nabla} \cdot \mathbf{E} = -\alpha \mathcal{G} + Q \sin^2 \theta,
\]

(36)

where

\[
\mathcal{G} = g_0'' + \frac{2}{r} g_0' - \frac{2\omega}{\alpha^2} h_0' + \frac{4}{\alpha^2 r^2} g_1,
\]

(37)

and

\[
Q = \frac{3\alpha}{\omega^2} \left( \frac{\omega}{\alpha} h_1 - \mathcal{Y} \right).
\]

(38)

In vacuum and in the absence of a net electrical charge, the solution to Maxwell equations is determined by the following set of differential equations

\[
\mathcal{H} = \mathcal{G} = Q = 0,
\]

(39)

To check the regularity conditions \( \mathcal{X} = \mathcal{Y} = 0 \). It is apparent that this is an overdetermined system if \( \omega h_1/\omega \neq 0 \), for in this case the two equations \( Q = 0 \) and \( \mathcal{Y} = 0 \) cannot be satisfied in the presence of spacetime dragging.

An additional constraint \( h_1 = 0 \) leads to a trivial solution, \( g_n = h_n = 0 \) and it is therefore impossible to impose \( h_1 = 0 \) everywhere. As a result, the solutions found with this method are only approximate solutions to the Maxwell equations in vacuum, with the deviation being measured through the term \( \omega h_1/\omega \). If the value of this quantity, which clearly depends on position, is small enough, then the level of approximation can be very good. Hereafter we will monitor the term \( \omega h_1/\omega \) as a consistency check to the solutions found.
The functions $g_1$ and $h_1$ are obtained from $G = \mathcal{X} = 0$.

$$
g_1 = \frac{\pi r^2}{2} h_0 + \frac{3\mu}{2\pi^3} \left[ c_1 \left\{ 3r^3(r^2 - 3Mr + 2M^2) \ln \alpha^2 + 2Mr^2(3r^2 - 6Mr + M^2) \right\} + \frac{J}{M} \right],
$$

$$
h_1 = -\frac{\pi r^2}{2} g_0 - \frac{\mu r}{4r} \left[ c_1 \left\{ 6r^3(2r - 3M) \ln \alpha^2 + 4Mr(6r^2 - 3Mr - M^2) \right\} + \frac{3J}{M} (r \ln \alpha^2 + 2M) \right],
$$

where $c_1$ is given by the values at surface ($r = R$) such as $a_R = a(R)$,

$$
c_1 = -\frac{1}{4\pi^3} \left( R^2 \ln \alpha_R^2 + 2MR + 2M^2 \right) R^2 \Omega - 2J(R \ln \alpha_R^2 + 2M) - 6MR^2 - 12M^2 R + 2M^3
$$

We have assumed that the stellar surface behaves as a perfect conductor so that the following boundary condition can be imposed at the stellar surface: $E = 0$. Note that these analytic solutions have already been derived elsewhere (e.g. Konno & Kojima 2000; Rezzolla et al. 2001a,b).

By direct calculations, we can check that these solutions satisfy $\mathcal{Q} = 0$, but never $\mathcal{Y} = 0$. Indeed, these solutions are such that $\mathcal{Y} = \mathcal{Y}^r h_1/\pi \neq 0$ and for $r \gg M$ this term can be easily estimated to be

$$
\frac{\pi^r}{\pi} h_1 \sim -\frac{9}{10} (4M^6 c_1 + 5J) \frac{\mu \Omega}{M^3} \propto r^{-6}.
$$

Since this term rapidly decreases with distance, the approximate solutions will be rather accurate at a sufficient distance from the stellar surface and could therefore be used in practical applications.

The analytic expressions (35), (40), (41) and (42) are the same ones as obtained in previous works, although with some slight differences in notations. Note that the “error term” (44) is effectively a second-order term in $\Omega$ and is therefore absent in previous analyses which were restricted to the low-frequency approximation. Because in our treatment such a term is not neglected a priori, we can explicitly check the accuracy of the expressions (35), (40), (41) and (42) at all radial positions. Indeed, these solutions represent a rather good approximation and can be used in practice in a slowly rotating spacetime, if the “error term” (which is usually quite small) is within a tolerable range.

## 4 NON-AXIALLY SYMMETRIC CASE

In the previous Section, the axis of the magnetic dipole is aligned with the rotational axis. Here, instead, we consider what happens when the dipole is misaligned. In particular, we focus on the case in which the axis of the magnetic dipole is perpendicular to the rotational axis.

The method used here to solve the Maxwell equations is almost identical to the one used in the previous Section for the axially symmetric configuration. In order to point out the similarities, we will adopt the same notation for the radial functions $g_n, h_n$. While this may induce some confusion, it is sufficient to bear in mind that the basis functions used in the expansion are different in the two cases [cf. eqs. (11) and (12)].

Since the potentials $G$ and $H$ would depend on the azimuthal wave number $m = 1$, they are expanded by the spherical harmonics $Y_{2l}^1$ and $Y_{2l+1}^1$, respectively. As a result, the angular expansion can be written as

$$
G(r, \theta, \phi, t) = e^{i\phi'} \sin \theta \cos \theta \left( \sum_{n=0}^{\infty} g_n(r)(\sin \theta)^{2n} \right),
$$

$$
H(r, \theta, \phi, t) = e^{i\phi'} \sin \theta \left( \sum_{n=0}^{\infty} h_n(r)(\sin \theta)^{2n} \right),
$$

where

$$
\phi' = \phi - \Omega t.
$$

The functions should depend only on $r, \theta, \phi'$ from the stationary condition as mentioned in Section 2. As shown in eqs. (10) and (11), the denominators in the poloidal electric and magnetic fields become zero at the light cylinder. Using the expansion forms, we have two regularity conditions, which should be satisfied for $r \geq r_c$. The conditions for the radial functions are written as

$$
\sum_{n=0}^{\infty} \left\{ g_n' + (2n + 1) \frac{\pi}{\alpha^2} h_n \right\} b^{-2n} = 0,
$$

where $b = \frac{r}{r_c}$. The method used here to solve the Maxwell equations is almost identical to the one used in the previous Section for the axially symmetric configuration. In order to point out the similarities, we will adopt the same notation for the radial functions $g_n, h_n$. While this may induce some confusion, it is sufficient to bear in mind that the basis functions used in the expansion are different in the two cases [cf. eqs. (11) and (12)].
\[
\sum_{n=0}^{\infty} \left\{ h_n' - \frac{\omega}{\alpha^2} \left( 2n + 1 - \frac{2n+2}{b^2} \right) g_n \right\} b^{-2n} = 0.
\]

In a spherically symmetric and static spacetime the electromagnetic fields are described completely in terms of four radial functions \( g_n, h_n (n = 0, 1) \). This set of functions can also provide approximate expressions in the case of a slowly rotating spacetime. The regularity conditions for the non-vanishing components are given by

\[
0 = X \equiv g_1' + b^2 g_0 + \frac{\omega^2}{\alpha^2} h_0 + \frac{3\omega}{\alpha^2} h_1, \\
0 = Y \equiv h_1' + b^2 h_0 - \frac{\omega}{\alpha^2} (b^2 - 2) g_0 - \frac{\omega}{\alpha^2} \left( 3 - \frac{4}{b^2} \right) g_1.
\]

Using these conditions, the expressions for the electromagnetic fields regular at the light cylinder are

\[
B_r = - \left[ h_0' - \frac{\omega}{\alpha^2} g_0 - \frac{4}{\omega r^2} g_1 \sin^2 \theta \right] e^{i\omega r} \sin \theta, \\
B_\theta = - \frac{1}{\omega} \left[ h_0 + (b^2 h_0 + 3h_1 + \omega r^2 g_0') \sin^2 \theta \right] e^{i\omega r} \cos \theta, \\
B_\phi = - \frac{i}{\omega} \left[ h_0 + h_1 \sin^2 \theta \right] e^{i\omega r}, \\
E_r = - \left[ g_0' + \frac{\omega}{\alpha^2} h_0 \right] e^{i\omega r} \sin \theta \cos \theta, \\
E_\theta = - \frac{1}{\omega} \left[ g_0 - \left\{ (2 - b^2) g_0 - 3g_1 + \omega r^2 h_0' \right\} \sin^2 \theta \right] e^{i\omega r}, \\
E_\phi = - \frac{i}{\omega} \left[ g_0 + g_1 \sin^2 \theta \right] e^{i\omega r} \cos \theta.
\]

The divergence of the electric and magnetic fields yields

\[
\vec{\nabla} \cdot \vec{E} = - \alpha \mathcal{G} \sin \theta \cos \theta, \\
\vec{\nabla} \cdot \vec{B} = - \alpha \mathcal{H} \sin \theta - \mathcal{Q} \sin^3 \theta,
\]

where

\[
\mathcal{G} = g_0'' + \frac{2}{r} g_0' - \frac{2\omega}{\alpha^2} h_0' + \frac{8}{(\alpha r)^2} g_1 + \frac{1}{(\alpha r)^2} (3b^2 - 6) g_0 + \frac{1}{r^2} \left( \frac{\omega r^2}{\alpha^2} \right)' h_0, \\
\mathcal{H} = h_0'' + \frac{2}{r} h_0' + \frac{2\omega}{\alpha^2} g_0' - \frac{1}{r^2} \left( \frac{\omega r^2}{\alpha^2} \right)' g_0 + \frac{8}{(\alpha r)^2} h_1 + \frac{1}{(\alpha r)^2} (3b^2 - 2) h_0, \\
\mathcal{Q} = \frac{4\alpha}{\omega r^2} \left( \frac{\omega'}{\alpha} g_1 - X \right).
\]

The solutions \( g_n, h_n (n = 0, 1) \) of the Maxwell equations in vacuum are determined by \( \mathcal{G} = \mathcal{H} = \mathcal{Q} = 0 \) and the regularity conditions \( X = Y = 0 \), but it is clear that the system of differential equations is overdetermined in a slowly rotating spacetime because of the presence of a redundant equation. In this respect, and as mentioned in the previous Section, the solutions to these equations are not exact.

In order to obtain the Deutsch-type solution, we impose the following conditions (see the Appendix for details)

\[
g_1 = 0, \quad h_1 = \frac{1}{3} \left( \frac{\omega r^2}{\alpha} g_0' + b^2 h_0 \right).
\]

Using these conditions, we have \( X = 0 \) and \( \mathcal{Q} = 0 \). The two functions \( g_0, h_0 \) are determined by solving \( \mathcal{G} = \mathcal{H} = 0 \). We introduce the functions \( F_l (l = 1, 2) \) so as to express \( h_0 \) and \( g_0 \) as

\[
g_0 = \frac{\alpha^2}{6} F_2' - \frac{\omega}{2} F_1, \\
h_0 = \frac{\alpha^2}{2} F_1' + \frac{\omega}{6} F_2,
\]

where \( F_1 \) and \( F_2 \) satisfy the coupled equations

\[
\alpha^2 (\alpha^2 F_1')' + \left( \frac{\omega^2 - \frac{2\alpha^4}{r^2}}{\alpha^2} \right) F_1 + \frac{1}{3} \alpha^2 \omega^2 F_2 = 0, \\
\alpha^2 (\alpha^2 F_2')' + \left( \frac{\omega^2 - \frac{6\alpha^4}{r^2}}{\alpha^2} \right) F_2 - 3\alpha^2 \omega^2 F_1 = 0.
\]

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It is easy to check that the functions (65) and (66) satisfy the coupled equations \( G = H = 0 \) with the conditions (63). Indeed, in the limit of a non-rotating spacetime, equations (65) and (66) become the Regge-Wheeler wave-like equations with frequency \( \Omega \) for \( l = 1,2 \) (Wheeler 1955). If the spacetime is rotating, however, the two equations are coupled because of the spacetime dragging through the coefficient \( \varpi' = -\omega' = 6J/r^3 \). From this choice of functions, we have

\[
\mathcal{Y} = -\frac{2J}{r^3} F_2.
\]  

(68)

This does not vanish in general, but rapidly decreases to zero with the radius because of the factor \( 2J/r^3 \). In fact, we can find \( \mathcal{Y} \sim -6\mu J\Omega^2 r^6/\varpi^2 \) near the light cylinder and \( \mathcal{Y} \sim 2\mu J\Omega^2 R^2 \exp\{\Omega(r + 2M \ln(r/2M - 1))\}/r^4 \) beyond the light cylinder.

Since the function \( F_2 \) describes the induced electromagnetic fields by a rotating perfect conductor, it is a first-order quantity in \( \Omega \). Consequently, the right-hand-side of (65) is second-order in \( \Omega \) and was thus absent in previous works performed in the slow-frequency approximation. Furthermore, eqs. (65) and (66) are significantly simplified when the terms \( \mathcal{O}(\Omega^2) \) can be neglected and, in fact, analytic expressions have been found in this case by Rezzolla et al. (2001a,b). It should be stressed, however, that such analytic solutions can be valid only in the vacuum exterior close to the stellar surface, i.e. for \( r < r_c \). Beyond the light cylinder, in fact, the wave-like nature of the electromagnetic fields becomes important and the term \( \mathcal{O}(\Omega^2) \) in eqs. (65) and (66) can no longer be neglected.

Expressing the electromagnetic fields in terms of functions \( F_l \) we have

\[
\vec{B} = \left[ -\frac{1}{r} F_1 \sin \theta e^{i\varphi}, \quad -\left( \frac{\alpha}{2r^2} F_1 + \frac{\varpi}{6\alpha r} F_2 \right) \cos \theta e^{i\varphi}, \quad -i \left( \frac{\alpha}{2r} F_1 + \frac{\varpi}{6\alpha r} F_2 \cos 2\theta \right) e^{i\varphi} \right].
\]  

(69)

\[
\vec{E} = \left[ -\frac{1}{2r^2} F_2 \sin 2\theta e^{i\varphi}, \quad -\left( \frac{\varpi}{2\alpha r} F_2 - \frac{\varpi}{6r} F_2 \cos 2\theta \right) e^{i\varphi}, \quad i \left( \frac{\varpi}{2\alpha r} F_2 - \frac{\varpi}{6r} F_2 \cos 2\theta \right) e^{i\varphi} \right].
\]  

(70)

and we can use the perfect conductor condition at the surface, i.e. \( E_\theta + B_\varphi \alpha^{-1} \varpi r \sin \theta = 0 \), which is reduced to \( F_2^2 - 3\varpi F_1/\alpha^2 = 0 \) at \( r = R \).

Using this boundary condition, the numerical solution of the coupled equations (65) and (66) for the magnetic field \( B_r \) is shown in Fig.1 for a typical choice of the stellar parameters. In the same figure, we also show the approximate solution in curved spacetime by Rezzolla et al. (2001a) [indicated by (a)] and the asymptotic one in flat spacetime by Deutsch (1953) [indicated by (f)]. The function \( B_r \) decreases with radius and the difference between our numerical result and analytic expressions is not so clear when shown in absolute magnitude. To highlight the differences we have therefore multiplied the magnetic field intensity \( B_r \) in Fig.1 by the factor \( r^3/(\mu \sin \theta e^{i\varphi}) \). Note that our numerical solution agrees rather well with that of Rezzolla et al. (2001a) near the surface. Around \( r \sim 0.2\Omega^{-1} \), our result deviates from it, approaching however the solution for flat spacetime by Deutsch (1953) as the distance from the stellar surface increases.

The explanation for this behavior is rather simple since the analytic expression for curved spacetime in the low-frequency approximation is valid only well within the light cylinder region \( r \sim 0.2\Omega^{-1} < r_c \), where the relativistic corrections are large and the electromagnetic frequencies small. Indeed, in the low-frequency approximation the point \( r_c \sim 1/\Omega \) is regarded as “infinity” and, as expected, the results of the low-frequency approximation cannot be accurate near this limit. As the distance from the star increases, a qualitative difference appears from the solutions of Rezzolla et al. (2001a) and this is represented by the sinusoidal, wave-like nature which is clear beyond the light cylinder. In this region, the gravitational corrections are not important and, in fact, our numerical result agrees well with that of flat spacetime. Fig.1 therefore serves to illustrate how our approach is very useful in connecting two different regimes, i.e. the relativistic one and the and the asymptotically flat one within a single, semi-analytic approach.

5 CONCLUDING REMARKS

We have studied the stationary solutions of the Maxwell equations in slowly rotating spacetime. The electromagnetic fields exterior to rotating magnetic dipole moment have been derived with the perfect conductor condition at the surface. This problem is not new, but it has been here considered through a different type of approach and with a different level of accuracy. In particular, we have considered the general relativistic electromagnetic fields exterior to a rotating neutron star in the slow-rotation approximation but not in the low-frequency approximation. The solutions derived here reduce to the exact analytic ones if the spacetime is spherically symmetric and no frame dragging effect is present. Furthermore, in this same limit, they can be expressed in terms of analytic functions in the limit of flat Minkowski space (Deutsch 1953).

Using these results as a guide, we have looked for solutions that have similar angular dependence but more complicated radial functional behavior. Adopting a semi-analytical approach in which simple expressions for the electromagnetic fields need to be completed through the calculation of coefficients coming from the numerical solution of two partial differential equations, we have found approximate solutions to the Maxwell equations. The error terms present in our solutions are \( \mathcal{O}(\Omega^2) \) and were therefore absent in previous approaches using the low-frequency approximation. The numerical solutions found in this way agree well with the known analytic results both in the strong-field region (Rezzolla et al. 2001a) and in the
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Figure 1. Magnetic field $B_r \sim (\mu r^{2}\sin \theta e^{i\phi})$ as a function of radius $r/r_c$. The left and right panels show the behavior near the surface ($R \leq r \leq r_c/2$) and that for wave region ($R \leq r \leq 10r_c$). The solid line with symbol ‘n’ represents our result solved for $2M/r_c = 0.01$, $R/r_c = 0.05$, $J = 10^{-3}r_c^2$. The dotted lines with symbols ‘a’ and ‘f’ respectively represent the analytic result for curved spacetime (Rezzolla et al. 2001a,b) and that for flat spacetime (Deutsch 1955).

Our approach is fundamentally based on a series expansion for which we discuss here only the lowest-order terms. However, it is in principle possible to increase the accuracy of our results by increasing the number of expansion series. For example, by including the higher-order functions $g_2, h_2$, the solutions can be calculated for a larger system of partial differential equations. Also in this case, there would be a redundant equation in the set of equations signalling terms which are incompatible within a set of differential equations. While we will not show any explicit calculations here, it is not difficult to realize that in the lowest extension to the results presented here the new correction would come through a term $\sim \omega' h_2/\omega$ (or $\omega' g_2/\omega$) in the system of the functions $g_n, h_n (n = 0, 1, 2)$. The function $h_2$ (or $g_2$) is expected to be small, because it should be zero both at the surface and infinity and the overall accuracy is therefore improved in this enlarged system of equations. By increasing further the number of functions, the difference between the incompatible equations becomes small. Of course, the limit of $n \gg 1$ may be preferable because of its increase in accuracy, but it also has clear limitations in practical applications.

ACKNOWLEDGMENTS

We thank the referee, Luciano Rezzolla, for many useful comments that improved the manuscript. This work was supported in part by a Grant-in-Aid for Scientific Research (No. 14047215) from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

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APPENDIX A: DEUTSCH SOLUTION

We here summarize the Deutsch solution and offer it for comparison. The solution describes the electromagnetic fields produced by a dipole magnet rotating with angular frequency $\Omega$. The quadrupolar electric field is induced by a perfect-conductor condition at the surface, written as $E_\theta + \Omega R \sin \theta B_r = 0$ at $r = R$. We here assume that the angle between the dipole moment and rotational axis is $\chi$ so that the components of the electromagnetic fields are given by

\begin{align*}
B_r &= \frac{2\mu}{r^3} \cos \chi \cos \theta + \frac{2\mu}{r^3} s_1 e^{i\lambda} \sin \chi \sin \theta, \\
B_\theta &= \frac{i\mu}{r^3} \cos \chi \sin \theta - \frac{i\mu}{r^3} (s_2 - d_1 s_3) e^{i\lambda} \sin \chi \cos \theta, \\
B_\phi &= \frac{i\mu}{r^3} (s_2 - d_1 s_3 \cos 2\theta) e^{i\lambda} \sin \chi, \\
E_r &= \frac{2\mu \Omega R^2}{r^4} \cos \chi P_2(\theta) + \frac{3\mu d_1}{r^4} s_3 e^{i\lambda} \sin \chi \sin 2\theta, \\
E_\theta &= \frac{2\mu \Omega R^2}{r^4} \cos \chi \sin 2\theta - \left( \frac{\mu \Omega}{r^2} s_1 + \frac{2\mu d_1}{r^4} s_4 \cos 2\theta \right) \sin \chi e^{i\lambda}, \\
E_\phi &= -i \left( \frac{\mu \Omega}{r^2} s_1 + \frac{2\mu d_1}{r^4} s_4 \right) \sin \chi \cos \theta e^{i\lambda},
\end{align*}

where

\begin{align*}
s_1 &= 1 - i\Omega r, \\
s_2 &= 1 - i\Omega r - \Omega^2 r^2, \\
s_3 &= 1 - i\Omega r - \frac{1}{3} \Omega^2 r^2, \\
s_4 &= 1 - i\Omega r - \frac{1}{2} \Omega^2 r^2 + \frac{7}{6} \Omega^3 r^3, \\
\lambda &= \phi - \Omega(t - r), \\
d_1 &= -\frac{3\Omega^2 R^2 (1 - i\Omega R)}{6 - i6\Omega R - 3\Omega^2 R^2 + i\Omega^3 R^2}. 
\end{align*}