ON PIECEWISE AFFINE INTERVAL MAPS WITH COUNTABLY MANY LAPS

JOZEF BOBOK AND MARTIN SOUKENKA

Abstract. We study a special conjugacy class $F$ of continuous piecewise monotone interval maps: with countably many laps, which are locally eventually onto and have common topological entropy $\log 9$. We show that $F$ contains a piecewise affine map $f_\lambda$ with a constant slope $\lambda$ if and only if $\lambda \geq 9$. Our result specifies the known fact that for piecewise affine interval leo maps with countably many pieces of monotonicity and a constant slope $\pm \lambda$, the topological (measure-theoretical) entropy is not determined by $\lambda$. We also consider maps from the class $F$ preserving the Lebesgue measure. We show that some of them have a knot point (a point $x$ where $D^+ f(x) = D^- f(x) = \infty$ and $D_+ f(x) = D_- f(x) = -\infty$) in its fixed point $1/2$.

1. Introduction, main results

In their interesting article [5] the authors showed, among other results, that laws ruling piecewise monotone interval maps do not work when we admit countably many pieces of monotonicity. They showed

Theorem 1.1. [5] For $\lambda > 2$ and every $\alpha > \log 2$ there exists a continuous map $T_\lambda : [0, 1] \rightarrow [0, 1]$ with the following properties:

(i) $f$ has countably many turning points.
(ii) $f$ is locally eventually onto (leo).
(iii) $|T_\lambda'(x)| = \lambda$ for all $x \in (0, 1)$, except at the turning points of $T$.
(iv) $h_{top}(T_\lambda) \leq \alpha$.
(v) For every ergodic $T_\lambda$-invariant Borel probability measure the partition into the laps of $T_\lambda$ has finite entropy.
That is, for piecewise affine interval leo maps with countably many pieces of monotonicity and a constant slope \( \pm \lambda \), the topological (measure-theoretical) entropy is not given by \( \lambda \).

In Section 5 of the same work they also gave an example of a continuous (not piecewise monotone) locally eventually onto map satisfying

\[
\sup_{x \in [0,1]} |T'(x)| \leq r, \sup_{x \in I} |T'(x)| < r
\]

for some interval \( I \subset [0,1] \), but \( h_{top}(T) = \log r \).

In this paper we provide one specific completion of the results cited above (we do not discuss the item (v) of Theorem 1.1). We show, roughly speaking, that all maps under consideration can be taken from one conjugacy class (with common entropy value \( \log 9 \)) containing an “optimal representative”, i.e., the map \( f \) such that \( |f'(x)| = 9 \) for all \( x \in (0,1) \), except at the turning points of \( f \).

As a by-product of our construction we show that there is an element of our conjugacy class preserving the Lebesgue measure and having a knot point (a point \( x \) where \( D^+ f(x) = D^- f(x) = \infty \) and \( D^+ f(x) = D^- f(x) = -\infty \)) in its fixed point \( 1/2 \).

2. Definition of class \( \mathcal{F} \)

In what follows we introduce a conjugacy class \( \mathcal{F} \). Later on, this class will be used to demonstrate several interesting features of piecewise monotone maps with countably many laps.

We will call a pair of real increasing sequences \( V = \{v_i\}_{i \geq -1}, X = \{x_i\}_{i \geq 0} \) of points from \((0,1/2)\) admissible if

- \( V, X \) converge to \( 1/2 \)
- \( 0 = v_{-1} = x_0 = v_0 < x_1 < v_1 < x_2 < v_2 < x_3 < v_3 < \cdots \).

Using admissible sequences \( V, X \) we define a continuous map

\[
f = f(V, X) : [0,1] \to [0,1]
\]

by (see Figure 1)

(a) \( f(v_{2i-1}) = 1 - v_{2i-1}, i \geq 1, f(v_{2i}) = v_{2i}, \ i \geq 0, \)
(b) \( f(x_{2i-1}) = 1 - v_{2i-3}, i \geq 1, f(x_{2i}) = v_{2i-2}, \ i \geq 1, \)
(c) \( f_{u,v} = \left| \frac{f(u) - f(v)}{u - v} \right| \geq 1 \) for each interval \([u, v] \subset [x_i, x_{i+1}]\),
(d) \( f(1/2) = 1/2 \) and \( f(t) = f(1 - t) \) for each \( t \in [1/2, 1] \).

The property (c) can be satisfied since for admissible \( V, X \) by (a),(b), \( f_{x_i, x_{i+1}} > 2 \) for each \( i \geq 0 \).
We denote by $F(V, X)$ the set of all continuous interval maps fulfilling (a)-(d) for a fixed admissible pair $V, X$. Finally, we put

$$F := \bigcup_{V, X \text{ admissible}} F(V, X).$$

**Proposition 2.1.** Each $f \in F$ is locally eventually onto, i.e., for every nonempty open set $U \subset [0, 1]$ there is an $n \in \mathbb{N}$ such that $f^n(U) = [0, 1]$.

**Proof.** Let $f \in F$. The points $v_{2i}, f(x_{2i+1}), i \geq 1$, resp. $1 - v_{2i-1}$, $f(1 - x_{2i}), i \geq 1$ are fixed and by (a)-(d)

$$f^{2i+1}([v_{2i}, x_{2i+1}]) = f^{2i}([1 - x_{2i}, 1 - v_{2i-1}]) = [0, 1]$$

for $i \geq 1$ and $f([0, x_1]) = [0, 1]$. Let $U$ be a nonempty open set in $[0, 1]$. From the property (c) follows that there is $n_0$ such that either $f^{n_0}(U) \supset [v_{2i}, x_{2i+1}]$ or $f^{n_0}(U) \supset [1 - x_{2i}, 1 - v_{2i-1}]$ for some $i \geq 1$. Thus $f^{n_0+2i+1}(U) = [0, 1]$. □

Let $f \in F$. For $i \geq 1$ denote $u_{2i-1}$, resp. $u_{2i}$ the unique solution (because of (c)) of the equation

1. $f(x) = 1 - x$, $x \in (x_{2i-2}, x_{2i-1})$, resp. $f(x) = x$, $x \in (x_{2i-1}, x_{2i})$.

Put $U = \{u_i\}_{i \geq 1}$. For a subset $Y \subset [0, 1]$ its symmetric extension $Y \cup (1 - Y)$ is denoted by $s(Y)$. Let

$$D(U, V) := s(U) \cup s(V) \cup \{0, 1\}.$$
An open interval \((a, b)\) with \(a, b \in D\) and \((a, b) \cap D = \emptyset\) will be called \(D\)-basic (for \(f\)).

For two maps \(f, \tilde{f} \in \mathcal{F}\) and sets \(D(U, V), X\) and \(D(\tilde{U}, \tilde{V}), \tilde{X}\) there exists the unique increasing bijection
\[
\pi: D(U, V) \cup s(X) \to D(\tilde{U}, \tilde{V}) \cup s(\tilde{X}).
\]
Two basic intervals \((a, b)\) and \((\pi(a), \pi(b))\), resp. two points \(t\) and \(\pi(t)\) will be then called corresponding.

3. \(\mathcal{F}\) IS A CONJUGACY CLASS

As we have already announced in Introduction, in Theorem 3.3 of this section we prove that any two elements of the class \(\mathcal{F}\) are topologically conjugated.

**Definition 3.1.** Fix \(f(V, X) \in \mathcal{F}\) and \(D = D(U, V)\), let \(\mathcal{J}\) denote the set of all \(D\)-basic intervals. An interval \(K \in \mathcal{J}\) is in preimage \(\mathcal{P}(J)\) of an interval \(J \in \mathcal{J}\) if \(I = \pi(J, K) = f^{-1}(J) \cap K \neq \emptyset\). If \(I\) consists of one connected component that is a subset of an interval of monotony \((x_i, x_{i+1})\) of \(f\), then \(K\) will be called increasing, resp. decreasing for \(\mathcal{P}(J)\) in accordance with the type of monotony of \(f|I\). Otherwise, if \(I\) consists of two connected components that are subsets of interval of monotony \((x_{i-1}, x_i), (x_i, x_{i+1})\) of \(f\), then \(K\) will be called nonmonotone for \(\mathcal{P}(J)\).

One can see that by our construction
\[
(\forall f \in \mathcal{F})(\forall t \in [0, 1] \setminus \{1/2\}): 1 \leq \#f^{-1}(t) < \infty.
\]

In what follows, for \(f \in \mathcal{F}\) we will need the complete backward orbit \(\mathcal{O}(f, t)\) of a point \(t \in [0, 1] \setminus \{1/2\} \cup D(U, V)\). Since \(f(D(U, V)) \subseteq D(U, V)\) we get \(\mathcal{O}(f, t) \cap D(U, V) = \emptyset\). For \(i_0 = 1\) we formally put \(t = t(i_0) \in f^{-0}(\{t\})\) and
\[
\mathcal{O}(f, t) = \{t(i_0, i_1, \ldots, i_n)\}_{n \geq 0},
\]
where \(t(i_0, i_1, \ldots, i_n) \in f^{-n}(\{t\})\) satisfies
\[
f(t(i_0, i_1, \ldots, i_n)) = t(i_0, i_1, \ldots, i_{n-1}), \ n \geq 1;
\]
if \(k = \#f^{-1}(\{t(i_0, i_1, \ldots, i_{n-1})\})\) then
\[
t(i_0, i_1, \ldots, i_{n-1}, 1) < t(i_0, i_1, \ldots, i_{n-1}, 2) < \ldots < t(i_0, i_1, \ldots, i_{n-1}, k).
\]

By \(J(t) \in \mathcal{J}\) we denote the \(D\)-basic interval that contains a point \(t\).

**Lemma 3.2.** Let \(f, \tilde{f} \in \mathcal{F}\). The following is true.
(i) If $D$-basic intervals $J, \tilde{J}$, resp. $K \in \mathcal{P}(J), \tilde{K} \in \mathcal{P}(\tilde{J})$ are corresponding then $K$ is increasing, resp. decreasing for $\mathcal{P}(J)$ if and only if $\tilde{K}$ is increasing, resp. decreasing for $\mathcal{P}(\tilde{J})$.

(ii) For every $J \in \mathcal{J}$ and corresponding $\tilde{J} \in \mathcal{J}$, the preimages $\mathcal{P}(J), \mathcal{P}(\tilde{J})$ contain corresponding intervals.

(iii) For every $J \in \mathcal{J}$, the preimage $\mathcal{P}(J)$ contains either 0 or 2 nonmonotone $D$-basic intervals.

(iv) For any two corresponding intervals $J, \tilde{J}$ and points $t \in J, \tilde{t} \in \tilde{J}$ and $t_m = t(i_0, i_1, \ldots, i_m)$, $\tilde{t}_m = \tilde{t}(i_0, i_1, \ldots, i_m)$, the intervals $J(t_m), \tilde{J}(\tilde{t}_m)$ are corresponding.

(v) For every points $u, v \in s(X)$ and corresponding $\tilde{u}, \tilde{v} \in s(\tilde{X})$, $\text{sgn}(u_m - v_n) = \text{sgn}(\tilde{u}_m - \tilde{v}_n)$.

Proof. The properties (i), (ii) directly follow from the definition of functions $f, \tilde{f}$ and the corresponding intervals.

(iii) There are no nonmonotone $D$-basic intervals in $\mathcal{P}(J)$ for any $J$ of type $(u, v), (1 - v, 1 - u)$. All other types $(0, u), (1 - u, 1), (v, u), (1 - u, 1 - v)$ have 2 nonmonotone $D$-basic intervals in their preimage.

(iv) Number from the left to the right all intervals $K_1, K_2, \ldots, K_k \in \mathcal{J}$ that are in preimage $\mathcal{P}(J)$ of $J$, i.e. $\mathcal{P}(J) = \{K_1 < K_2 < \ldots < K_k\}$. Using (iii) assume that $\ell \in \{0, 2\}$ of them are nonmonotone. Then for every $t \in J$ we have

$$f^{-1}\{\{t\}\} = \{t_1 < t_2 < \ldots < t_{k+\ell}\},$$

where each monotone $K$, resp. nonmonotone $K$ corresponds to one $t_i \in K$, resp. to two consequent $t_i, t_{i+1} \in K$ with $(t_i, t_{i+1}) \cap s(X) = \{x\}$. Thus, the coordinate $i_1 \in \{1, 2, \ldots, k + \ell\}$ of $t(i_0, i_1)$ uniquely determines an interval $K$ from $\mathcal{P}(J)$ with $t(i_0, i_1) \in K$ and, if $K$ is nonmonotone, also the position of $t(i_0, i_1)$ with respect to $\{x\} = s(X) \cap K$. Using this fact repeatedly for $f, t$, resp. $\tilde{f}, \tilde{t}$, we get the conclusion.

Let us show (v). We will proceed by induction.

Arguing as in (iv) we can see that the conclusion is correct when $m = 0$ (or by the symmetry, $n = 0$), since then $u_0 \in s(X)$ and $\tilde{u}_0 \in s(\tilde{X})$ is corresponding.

Let $m, n \geq 1$ and $m \geq n$, consider points $u_m = u(i_0, \ldots, i_m)$, $\tilde{u}_m = \tilde{u}(i_0, \ldots, i_m)$, $v_n = v(j_0, \ldots, j_n)$, $\tilde{v}_n = \tilde{v}(j_0, \ldots, j_n)$.

Assume that for some $k \in \{1, 2, \ldots, n\}$ the equality

$$\text{sgn}(u_{m-k} - v_{n-k}) = \text{sgn}(\tilde{u}_{m-k} - \tilde{v}_{n-k})$$

(7)
holds true. By (iv), if $J(u_{m-k+1}) \neq J(v_{n-k+1})$ then
\[ \text{sgn}(u_{m-k+1} - v_{n-k+1}) = \text{sgn}(\tilde{u}_{m-k+1} - \tilde{v}_{n-k+1}). \]

Let $K = J(u_{m-k+1}) = J(v_{n-k+1})$.

If $K$ is monotone for $\mathcal{P}(J(u_{m-k}))$ and also for $\mathcal{P}(J(v_{n-k}))$ then by (i)–(iii) the corresponding $\tilde{K}$ has the same type of monotony for $\mathcal{P}(\tilde{J}(\tilde{u}_{m-k}))$ and $\mathcal{P}(\tilde{J}(\tilde{v}_{n-k}))$, hence we get $\text{sgn}(u_{m-k+1} - v_{n-k+1}) = \text{sgn}(\tilde{u}_{m-k+1} - \tilde{v}_{n-k+1})$.

If $K$ is nonmonotone for $\mathcal{P}(J(u_{m-k}))$, resp. $\mathcal{P}(J(v_{n-k}))$ ($K$ can be nonmonotone for one of them only), the last coordinates $i_{m-k+1}, j_{n-k+1}$ determine the effective connected components of $f^{-1}(J(u_{m-k})) \cap K$ (containing $u_{m-k+1}$), resp. $f^{-1}(J(v_{n-k})) \cap K$ (containing $v_{n-k+1}$) and for $\tilde{f}$ the order of effective component is the same. Thus,
\[ \text{sgn}(u_{m-k+1} - v_{n-k+1}) = \text{sgn}(\tilde{u}_{m-k+1} - \tilde{v}_{n-k+1}). \]

Since we have shown at the beginning of this part that a $k \in \{1, 2, \ldots, n\}$ satisfying (7) has to exist, our proof is finished. \(\square\)

**Theorem 3.3.** The set $\mathcal{F}$ is a conjugacy class, i.e., for any two functions $f, \tilde{f} \in \mathcal{F}$ there exists a homeomorphism $H$ of the unit interval such that $f = H^{-1} \circ \tilde{f} \circ H$.

**Proof.** Put
\[ \mathcal{O}(f) = \bigcup_{x \in s(X)} \mathcal{O}(f, x). \]

From Proposition 2.1 follows that both the sets $\mathcal{O}(f), \mathcal{O}(\tilde{f})$ are dense in $[0, 1]$. Moreover, using Lemma 3.2 (v) we get that there is an increasing bijection $h : \mathcal{O}(f) \to \mathcal{O}(\tilde{f})$ coinciding with the $\pi$ from (3) on the set $s(X)$ and defined by $h(x(i_0, i_1, \ldots, i_n)) = \tilde{x}(i_0, i_1, \ldots, i_n)$ for $x(i_0, i_1, \ldots, i_n) \in \mathcal{O}(f)$. Since by the property (b) the $f, \tilde{f}$ images of corresponding points from $s(X), s(\tilde{X})$ are corresponding and (6) holds true, we get for each $x \in \mathcal{O}(f),$
\[ (h \circ f)(x) = (\tilde{f} \circ h)(x). \]

Finally, we know that both the sets $\mathcal{O}(f), \mathcal{O}(\tilde{f})$ are dense in $[0, 1]$. In such a case $h$ can be (uniquely) extended to the homeomorphism $H$ of the unit interval such that $f = H^{-1} \circ \tilde{f} \circ H$. \(\square\)
4. Maps with constant slope in \( \mathcal{F} \)

In this section we will check if the class \( \mathcal{F} \) contains piecewise affine maps with constant slopes, i.e., to a given \( \lambda > 1 \) a map \( f = f_\lambda \) such that \( |f_\lambda'(x)| = \lambda \) for all \( x \in (0, 1) \), except at the turning points of \( f_\lambda \) (clearly, if it exists, then it is unique). The reader can easily verify by standard computation that such a map \( f_\lambda \) would be uniquely determined by a sequence \( V = \{v_i\}_{i \geq 1} \) fulfilling the two-dimensional difference equation

\[
\begin{pmatrix}
  v_{n+2} \\
  v_{n+3}
\end{pmatrix}
= \begin{pmatrix}
  \frac{-2}{\lambda-1} & 1 \\
  \frac{\lambda-3}{\lambda-1} & \frac{\lambda-1}{\lambda-1}
\end{pmatrix}
\begin{pmatrix}
  v_n \\
  v_{n+1}
\end{pmatrix}
+ \begin{pmatrix}
  \frac{1}{2} \\
  \frac{1}{2}
\end{pmatrix}, \quad n \geq 1,
\]

with the initial condition \((v_1, v_2) = (\frac{1}{\lambda-1}, \frac{2}{\lambda-1})\).

Figure 2. (a) The map \( f_9 \); (b) The map \( f_{20} \).

**Proposition 4.1.** The map \( f_\lambda \) exists if and only if \( \lambda \geq 9 \).

**Proof.** By our construction we are interested in increasing solutions \( \{w_n = (v_{2n-1}, v_{2n})\}_{n \geq 1} \) of (8) \( (w_{n+1} > w_n \text{ for each } n \in \mathbb{N}) \) and such that \( \lim_{n \to \infty} w_n = (1/2, 1/2) \).

Denote by \( A(\lambda) \) the matrix from the equation (8).

Direct computations show that:

- An increasing solution of (8) that converges to the \((1/2, 1/2)\) exists if and only if \( \lambda \geq 9 \) and it is if and only if all eigenvalues of \( A(\lambda) \) are real positive.
- For \( \lambda = 9 \) the matrix \( A(9) \) has the unique eigenvalue \( 1/4 \) of multiplicity two; the solution of (8) is then given by the explicit
formula

\[(v_{2n-1}, v_{2n}) = \left(\frac{1}{2} - \frac{4n + 2}{4^{n+1}}, \frac{1}{2} - \frac{2n + 2}{4^{n+1}}\right), \quad n \geq 2, \]

as one can easily check substituting (9) into (8).

□

Since the topological entropy is a conjugacy invariant (see [8]) and Theorem 3.3 holds true, we can speak about entropy value \(h_{\text{top}}(\mathcal{F})\) of the class \(\mathcal{F}\). We know that \(f_9 \in \mathcal{F}\) and \(f_9\) is 9-Lipschitz hence by [3, Theorem 3.2.9], \(h_{\text{top}}(\mathcal{F}) \leq \log 9\). In order to show that it equals to \(\log 9\) we will use standard tools developed for interval maps.

Let \(f: [0, 1] \to [0, 1]\) be a continuous interval map, and \(Q = \{q_1 < q_2 < \cdots < q_n\}\) be a finite subset of \([0, 1]\) \((Q\) need not be \(f\)-invariant\). The matrix of \(Q\) (with respect to \(f\)) is the \((n - 1) \times (n - 1)\) matrix \(A_Q\), indexed by \(Q\)-basic intervals and defined by \(A_{JK}\) is the largest non-negative integer \(l\) such that there are \(l\) subintervals \(J_1, \ldots, J_l\) of \(J\) with pairwise disjoint interiors such that \(f(J_i) = K, i = 1, 2, \ldots, l\). An interval map \(f: [0, 1] \to [0, 1]\) is called transitive if for some point \(x \in [0, 1]\) its orbit is dense in \([0, 1]\).

The following lemma is needed in the proof of Theorem 4.4.

**Lemma 4.2.** [2] Let \(f: [0, 1] \to [0, 1]\) be transitive, \(Q\) be a finite subset of the ambient interval, and let \(A_Q\) be the matrix of \(Q\) with respect to \(f\). Then \(h_{\text{top}}(f) \geq \log r(A_Q)\), with equality if \(Q\) is \(f\)-invariant and contains the endpoints of the ambient interval, and \(f\) is monotone (but not necessarily strictly monotone) on each \(Q\)-basic interval.

One of well-known results from one-dimensional dynamics is the following.

**Theorem 4.3.** [7],[6] Any continuous, transitive, piecewise monotone map \(G: [a, b] \to [a, b]\) is topologically conjugate to a piecewise affine map \(\pi: [0, 1] \to [0, 1]\) which has slope \(\pm \beta\) (\(\log \beta\) is the topological entropy of \(G\)) on each affine piece.

**Theorem 4.4.** The entropy value \(h_{\text{top}}(\mathcal{F})\) is equal to \(\log 9\).

**Proof.** By the previous, \(h_{\text{top}}(f_9) = h_{\text{top}}(\mathcal{F}) \leq \log 9\). In what follows we will use Lemma 4.2 and Theorem 4.3 to show that even \(h_{\text{top}}(f_9) = h_{\text{top}}(\mathcal{F}) = \log 9\).

Let us formally denote \(A_n = \{a_0, a_1, \ldots, a_n\}\). Using two admissible sequences \(V, X\) (introduced in Section 2) corresponding to the map \(f_9\), put for \(k \in \mathbb{N}\)

\[Q_k = s(X_{2k} \cup V_{2k-1}) \cup \{1 - x_{2k+1}\}, \quad g_k = f_9|Q_k,\]
where as before, \( s(Y) \) denotes the symmetric extension \( Y \cup (1 - Y) \) of \( Y \). Let \( G_k \) be “connect-the-dots” map given by the pair \((Q_k, g_k)\). One can use a similar way as in Proposition 2.1 to show that \( G_k \) is locally eventually onto hence also transitive. Since the set \( Q_k \) is \( G_k \)-invariant and \( G_k \) is affine on each \( Q_k \)-basic interval, Lemma 4.2 applies. From that lemma we get

\[
\log r(A_{Q_k}) = h_{\text{top}}(G_k) \leq h_{\text{top}}(f_9) = \log \beta; \tag{10}
\]

the topological entropy is lower semicontinuous on the space of all continuous interval maps equipped with the supremum norm (see [4]) and \( f_9 = \lim_k G_k \). This fact together with (10) imply

\[
\lim_k h_{\text{top}}(G_k) = \log \beta. \tag{11}
\]

Let \( \alpha_k \) be a map guaranteed by Theorem 4.3, i.e., a piecewise affine map from \([0, 1]\) into itself which has slope \( \pm e^{h_{\text{top}}(G_k)} \) on each affine piece. From above we know that \( \lim_k e^{h_{\text{top}}(G_k)} = e^{h_{\text{top}}(f_9)} = \beta \leq 9 \); by our definitions of \( \alpha_k \)'s and the class \( \mathcal{F} \),

\[
\lim_k \alpha_k = f_\beta \in \mathcal{F},
\]

where \( f_\beta \) is piecewise affine which has slope \( \pm \beta \). Using Proposition 4.1 we get \( \beta = 9 \), i.e., \( h_{\text{top}}(f_9) = \log 9 = h_{\text{top}}(\mathcal{F}) \). \( \Box \)

![Figure 3. The maps \( g \) and \( f \).](image)

**Theorem 4.5.** There is a map \( f \in \mathcal{F} \) and a union of two intervals \( I \subset (0, 1) \) such that if \( f'(x) \) exists then either \( |f'_\lambda(x)| = 9 \) when \( x \notin I \) or \( |f'_\lambda(x)| = 23/7 \) for \( x \in I \).
Proof. To the map $f_9$ correspond the sequences $V$ and $U$ introduced in Section 2. In particular, from (8),(9) we get $u_3 = 3/10$ and $v_3 = 11/32$. Define a continuous map $g: [1/4, 3/4] \to [1/4, 3/4]$ by

$$
g(x) = \begin{cases} 
  f_9(x), & x \in [1/4, 1/2] \setminus (u_3, v_3), \\
  3/4, & x = 29/92, \\
  \text{affine on laps } [u_3, 29/92], [29/92, v_3], \\
  g(1-x), & x \in [1/2, 3/4].
\end{cases}
$$

Put $f = r^{-1} \circ g \circ r$, where $r$ is affine, preserves orientation and maps the unit interval onto $[1/4, 3/4]$ - see Figure 3. The reader can verify by direct computations that $f$ and $I = r^{-1}(s([u_3, v_3]))$ satisfy the conclusion. \hfill \Box

5. Maps preserving the Lebesgue measure in $F$

The class $F$ contains also maps preserving the Lebesgue measure. One possible way how to see it follows from Figure 4. It shows a piecewise affine map with countably many laps which is uniquely determined by a sequence $\{k_i\}_{i \geq 1}$ of reals from the interval $(2, \infty)$. One can easily verify that such a map preserves the Lebesgue measure if and only if

$$
\sum_{i=1}^{\infty} \frac{1}{|k_{2i-1}|} = \frac{1}{2}, \quad \frac{1}{|k_{2j}|} + \sum_{i=1}^{j} \frac{1}{|k_{2i-1}|} = \frac{1}{2}, \quad j = 1, 2, \ldots.
$$

Figure 4. A map from $F$ preserving the Lebesgue measure.
We recall that by a knot point of a function $f$ we mean a point $x$ where $D^+ f(x) = D^- f(x) = \infty$ and $D^+ f(x) = D^- f(x) = -\infty$. It was discussed elsewhere [1] that for the problem of understanding of relationship of two characteristics of an interval (or tree) map - its topological entropy and cardinalities of level sets - it could be useful to understand the role of knot points of Lebesgue measure preserving maps, for example, to evaluate the topological entropy of such a map having a knot point at its fixed point. The best estimate is not clear at all, but using elements of the class $\mathcal{F}$ we obtain the following.

**Proposition 5.1.** The class $\mathcal{F}$ contains a map $f$ with the following properties.

(i) $f$ preserves the Lebesgue measure.

(ii) $h_{\text{top}}(f) = \log 9$.

(iii) $f$ has a knot point at $1/2$.

**Proof.** By Theorem 4.4 it is sufficient to find a sequence $\{k_i\}_{i \geq 1}$ fulfilling (13) and (iii). Denote $f_{u,v} = \left| \frac{f(u) - f(v)}{u - v} \right|$. With the help of Figure 4, we briefly explain how to ensure

$$\lim_{i \to \infty} f_{x_i,1/2} = \infty,$$

which implies (iii).

Choose $k_1 > 2$ to satisfy $f_{x_1,1/2} > 1$; then by (13), $k_2 = \left( \frac{1}{2} - \frac{1}{k_1} \right)^{-1}$. One can choose $k_3 > k_2$ (but close to $k_2$) such that $f_{x_2,1/2} > 2$; similarly as before, by (13), $k_4 = \left( \frac{1}{2} - \frac{1}{k_1} - \frac{1}{k_3} \right)^{-1}$. We choose $k_5 > k_4$ (close to $k_4$) such that $f_{x_3,1/2} > 3$; if have already defined numbers $k_1, \ldots, k_{2n-1}$, the number $k_{2n}$ satisfies $k_{2n} = \left( \frac{1}{2} - \sum_{i=1}^{2n} \frac{1}{k_{2i-1}} \right)^{-1}$. We can choose $k_{2n+1} > k_{2n}$ (close to $k_{2n}$) such that $f_{x_{n+1},1/2} > n + 1$; etc. \hfill $\square$

**References**

[1] Bobok J., Soukenka M., Irreducibility, infinite level sets and small entropy, submitted, 12 pp., 2009.

[2] Coven E.M., Hidalgo M.C., On the topological entropy of transitive maps of the interval, Bull. Aust. Math. Soc. 44(1991), 207-213.

[3] Katok A., Hasselblatt B. Introduction to the modern theory of dynamical systems. Encyclopedia of Mathematics and its Applications 54, Cambridge University Press, Cambridge, 1995.

[4] Misiurewicz M., Horseshoes for mappings of an interval, Bull. Acad. Pol. Sci., Sér. Sci. Math. 27(1979), 167-169.

[5] Misiurewicz M., Raith P. Strict Inequalities for the Entropy of Transitive Piecewise Monotone Maps, Discrete and Continuous Dynamical Systems, 13(2005), 451-468.
[6] Milnor J., Thurston W., *On iterated maps of the interval*, Dynamical systems, Lecture Notes in Math. **1342**, pp. 465-563, Springer Verlag, Berlin, 1988.

[7] Parry W., *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. **122**(1966), 368-378.

[8] Walters P., *An Introduction to Ergodic Theory*, Springer-Verlag, New Yourk, 1982.

E-mail address: bobok@mat.fsv.cvut.cz
E-mail address: soukenkam@mat.fsv.cvut.cz

JOZEF BOBOK*, MARTIN SOUKENKA†

* KM FSV ČVUT, Thákurova 7, 166 29 Praha 6, Czech Republic
† ÚT AV ČR, Dolejškova 1402/5, 182 00 Praha 8, Czech Republic