An Approach for Modelling Tachyons with Gravitation

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Abstract

This work expands previous efforts, within the classical theories of Special and General Relativity, to include tachyons (faster-than-light particles) along with ordinary (slower-than-light) particles at any energy. The objective here is to construct a Hamiltonian that includes both the particles and the gravitational field that they produce. We do this with a linear approximation for the Einstein field equations; and we also assume a time-independent gravitational metric implied by a static picture of the particles’ motion. The resulting formula will allow serious modelling to test the idea that cosmic background neutrinos may be tachyons, which can produce the observed gravitational effects now ascribed to some mysterious Dark Matter.

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1 Introduction

In earlier work \cite{1, 2, 3, 4} I have explored, theoretically, how tachyons (faster-than-light particles) would behave in Einstein’s General Relativity. That starts with the recognition that low energy tachyons would have very large velocities and thus their contribution to the energy-momentum tensor $T^{ij}$ would be very large. Previous physical ideas have been that neutrinos, which are so numerous throughout the universe, might be tachyons with a mass of around 0.1 eV and could thus produce gravitational effects that are now ascribed to mysterious sources called ”Dark Matter” or ”Dark Energy”.

The simplest calculation, based upon an unexpected minus sign in front of $T^{ij}$, gave a numerical estimate of the negative pressure that such tachyon-neutrinos would produce in the Robertson-Walker model of the universe that ”explained” Dark Energy within a factor of 2! \cite{3}

The more difficult calculation surrounds the idea that attractive gravitational forces among low energy tachyons could lead to their forming stable configurations (while all the time flowing at speeds far above the speed of light) that could attach themselves to galaxies and thus produce the local gravitational fields that are commonly ascribed to Dark Matter. This paper is about that challenging idea.

For a non-relativistic classical particle moving in a time-independent conserved force field we have the familiar equation, expressing the conservation of energy:

$$ E = \frac{1}{2}mv^2 + V(x), $$

and for a collection of such particles interacting via Newtonian gravity we write the Hamiltonian,

$$ H = \sum_{a} \frac{1}{2}m_a v_a^2 - \sum_{a<b} \frac{G m_a m_b}{r_{ab}}, \quad r_{ab} = |x_a - x_b|. $$

The main purpose of this paper is to generalize this Hamiltonian to the case of relativistic matter, including both ordinary particles ($v < c$) and tachyons ($v > c$), under Einstein’s General Theory of Relativity, with two restrictions: that we use the linearized approximation to Einstein’s equation; and that we assume the metric $g_{\mu\nu}(x)$ to be independent of the time. The result, found in Section 7, is,

$$ H = - \sum_a \omega_a m_a \gamma_a - \sum_{a,b} \frac{G(\omega_a m_a \gamma_a)(\omega_b m_b \gamma_b)}{r_{ab}} Z_{ab}, $$

$$ Z_{ab} = 2 - 4v_a \cdot v_b + (v_a^2 + v_b^2) - (\epsilon_a \gamma_a^2 + \epsilon_b \gamma_b^2 + 1) \times $$

$$ \times [(1 - v_a \cdot v_b)^2 - \frac{1}{2}(1 - v_a^2)(1 - v_b^2)]. $$

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(The factors $\omega, \epsilon$ are $\pm 1$ and will be defined later; $\gamma = 1/\sqrt{1 - v^2}$ is familiar.)

The idea of the metric being static while the particles producing that gravitational field are moving may seem contradictory. We imagine a continuous flow of particles that does not change in time, analogous to the picture of a constant electrical current used in the study of magnetostatics. One might use the term Gravitostatics for this study. (But that name has been used for a different sort of theory: see [5]).

There is also mathematical work on the static Einstein-Vlasov system [6] which uses a kinetic theory approach for many-particle systems in General Relativity; but that does not consider the possibility of tachyons. Therefore I shall begin from scratch.

With this formula (1.3) we can begin model-building, looking for potentially stable configurations of tachyon flows contained by their mutual gravity. That will be an ongoing task. That requirement of stability will be the most challenging. Even with ordinary Newtonian gravity (1.2) one sees large scale attraction that seems to lead inevitably to physical collapse; but then one brings in further physics to help us explain the observable stability of stars, solar systems, galaxies. Our new ideas are about incorporating tachyons (neutrinos?) into that cosmic modelling; and it needs to start with something better than (1.2): this leads to our new formula (1.3).

2 Beginning

Here I review previous work describing both ordinary particles ($v < c$) and tachyons ($v > c$) as classical particles in both Special and General Relativity. First, some notation and equations in common for all particles.

A ”world line” $\xi^\mu(\tau) = (t(\tau), x(\tau))$ maps the trajectory of the particle in space and time with some as yet undefined scalar parameter $\tau$. I use notation $\mu = (0, i)$, $i = 1, 2, 3$ and set the velocity of light $c = 1$; and use the overhead dot notation to represent $d/d\tau$. The argument $x$ is meant to stand for all four spacetime coordinates $x^\mu = (t, \mathbf{x})$, where $\mathbf{x} = (x^1, x^2, x^3)$; and partial derivatives are written as $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

We write a conserved current density as

$$j^\mu(x) = \int d\tau \, \dot{\xi}^\mu(\tau) \, \delta^4(x - \xi(\tau)), \quad (2.1)$$

$$j^\mu_\nu(x) = \partial_\nu j^\mu(x) = \int d\tau \, \dot{\xi}^\mu(\tau) \partial_\nu \delta^4(x - \xi(\tau)) = \int d\tau \, \frac{d\xi^\mu}{d\tau} \frac{d}{d\xi^\nu} \delta^4(x - \xi(\tau)) = \int d\tau \, \frac{d}{d\tau} \delta^4(x - \xi(\tau)) = 0. \quad (2.3)$$

All that we required for that last step was to take the end points of the $\tau$ integral.
far away from the place where the particle is at this location \( x \).

We can also write an energy-momentum tensor,

\[
T^\mu_\nu (x) = m \int d\tau \dot{\xi}^\mu(\tau) \dot{\xi}^\nu(\tau) \delta^4(x - \xi(\tau)).
\]

(2.4)

When we take the divergence on one index we follow the above calculation but get something left over from the final partial integration.

\[
T^\mu_\nu ,\mu (x) = -m \int d\tau \dot{\xi}^\nu d\tau \delta^4(x - \xi(\tau)) = m \int d\tau \ddot{\xi}^\nu \delta^4(x - \xi(\tau)).
\]

(2.5)

If the only forces acting upon the particle are those due to gravity, then we also have the geodesic equation (equally correct for ordinary particles or tachyons):

\[
\dddot{\xi}^\nu + \Gamma^\nu_\alpha\beta \dot{\xi}^\alpha \dot{\xi}^\beta = 0,
\]

(2.6)

involving the Christoffel symbols \( \Gamma \), defined in terms of derivatives of the metric tensor \( g_{\mu\nu}(x) \), evaluated at the point where the particle is at any given value of \( \tau \). This lets us write the result of the ordinary divergence calculation as,

\[
T^\mu_\nu ,\mu + \Gamma^\nu_\alpha\beta T^\alpha\beta = 0.
\]

(2.7)

From this result we can construct a modified tensor, multiplied by the square root of the determinant of the metric, which will have zero as its covariant divergence.

\[
\mathcal{T}^\mu_\nu = \sqrt{|\det(g)|} T^\mu_\nu, \quad \mathcal{T}^\nu_\alpha = \mathcal{T}^\mu_\nu + \Gamma^\nu_\alpha\beta T^\alpha\beta = 0.
\]

(2.8)

That is proper for the full Einstein equation; but here we will be satisfied with the linear approximation.

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad h = \eta^\mu\nu h_{\mu\nu}, \quad \partial^\mu h_{\mu\nu} = 0,
\]

(2.9)

\[
\partial^\alpha \partial_\alpha h_{\mu\nu}(x) = [\partial^2 - \nabla \cdot \nabla] h_{\mu\nu}(x) = -16\pi G T_{\mu\nu}(x),
\]

(2.10)

with the Minkowski metric \( \eta_{\mu\nu} = \delta_{\mu\nu}(+1, -1, -1, -1) \) used to raise and lower indices.

### 3 Free particles

First, we sit in a flat spacetime. There should be nothing new here; we want to practice for later. At any point in the particle’s trajectory, where it happens to have a velocity \( v \) in the original reference frame we can make a Lorentz transformation
to a local frame where the particle is seen momentarily at rest (in the rest frame: \( v' = 0 \)). This Lorentz transformation has a velocity \( v_{LT} = v \) and the gamma factor \( \gamma_{LT} = 1/\sqrt{1 - v^2} \). We take the scalar parameter for this particle to be the time in that local rest frame: \( d\tau = dt' = dt/\gamma_{LT} \). This \( \gamma_{LY} \) is exactly the \( \gamma \) of the particle at that point of its trajectory in the original frame. Thus we can write,

\[
\dot{\xi}^\mu = (\gamma, \gamma v), \quad v = \frac{dx}{dt}.
\]

This leads to, with the Minkowski metric,

\[
\dot{\xi}^\mu \dot{\xi}^\nu \eta_{\mu\nu} = \gamma^2 (1 - v^2) = +1,
\]

which is fine for a free particle; but in a gravitational field it should be the entire metric \( g_{\mu\nu} \) that fills this role. The geodesic equation implies that \( \dot{\xi}^\mu \dot{\xi}^\nu g_{\mu\nu} \) is a constant along the particle’s trajectory.

But let’s stick in Minkowski space, no gravity, for a while and look at the comparable calculation for tachyons. Now we have no rest frame to give us a nice definition of the scalar parameter \( \tau \). So we find another special frame of reference: one where the tachyon has infinite velocity. (This is the one used in quantum group theory to find the ”Little Group” for tachyons.) This involves a velocity of the Lorentz transformation, \( v_{LT} = 1/v \), where this \( v \) is the tachyon’s velocity \( (v > 1) \) in the original reference frame. (To see this, recall the velocity addition formula \( v' = (v + v_{LT})/(1 - vv_{LT}) \).) The gamma for this Lorentz transformation is \( \gamma_{LT} = 1/\sqrt{1 - v^2} \). We now make the definition of the scalar parameter as \( d\tau = \hat{v} \cdot dx' \), marking the path of the infinitely fast particle. Here \( dx' \) is the differential of the spatial coordinate in this special reference frame, which is related to that in the original frame by \( \hat{v} \cdot dx' = \gamma_{LT}^{-1} \hat{v} \cdot dx \). So we have, for tachyons, in the original reference frame,

\[
d\tau = (v\gamma)^{-1} \hat{v} \cdot dx, \quad \gamma = 1/\sqrt{(v^2 - 1)}, \quad \dot{\xi}^\mu = \gamma (1, v),
\]

where that last equation is what we expected.

For ordinary particle of extremely low velocities, we had \( d\tau \approx dt \); for tachyons of extremely high velocities we have \( d\tau \approx \hat{v} \cdot dx \). This is nice.

Now let’s look at how these results influence our physical interpretation of the conserved currents. For ordinary particles,

\[
j^\mu = \int d\tau (\gamma, \gamma v) \delta(t - \gamma \tau) \delta^3(x - \gamma v \tau) = (\gamma, \gamma v) \gamma^{-1} \delta^3(x - vt),
\]

where I have used the first delta-function to do the integral over \( \tau \). If we now do the standard integral over all space, we get a very simple answer,

\[
\int d^4 x j^0 = 1.
\]
This says we have one particle there (somewhere in space) at any time.

The analogous calculation for a tachyon goes like this.

\[ j^\mu = \int d\tau \gamma(1, v) \delta(t - \gamma \tau) \delta^2(x_\perp) \delta(x_\parallel - \gamma v \tau), \quad (3.6) \]

where the parallel and perpendicular subscripts refer to the direction of the velocity. Now I do the integration over \( \tau \) using the last delta-function to get,

\[ j^\mu = (1, v) \frac{1}{v} \delta(t - x_\parallel / v) \delta^2(x_\perp). \quad (3.7) \]

To get the count of "one particle" from this I integrate the component of \( j \) parallel to the velocity over the transverse plane and integrate over time.

\[ \int dt \int d^2 x_\perp \hat{v} \cdot j = 1. \quad (3.8) \]

We recite this conservation law as: We have one particle passing through a transverse plane at some time - and this could be any transverse plane. This is completely consonant with my earlier writings about tachyon kinematics.

Suppose I try to treat the tachyon as I did the ordinary particle. I still go to the frame where the tachyon is at \( v' = \infty \) but I choose to define \( \tau = dt' \) in that frame. I again write \( dt' = dt / \gamma_{LT} \) but remember that \( \gamma_{LT} = 1 / \sqrt{(1 - 1/v^2)} = v\gamma \), where this last is the usual gamma for the tachyon in the original reference frame. This gives us \( \dot{\xi}^\mu = \gamma v(1, v) \). We now calculate the current, as before,

\[ j^\mu = \int d\tau v \gamma(1, v) \delta(t - v\gamma \tau) \delta^3(x - v\gamma v \tau) = (1, v) \delta^3(x - vt), \quad (3.9) \]

integrating over \( \tau \) by using the first delta-function. This looks just like the case with ordinary particles. We are tempted to integrate over \( d^3x \) and say that we have one particle in a large box at any time - just as we did for ordinary particles. However, this is really not acceptable for tachyons: the velocity \( v \) occurs in that delta-function \( \delta(x - vt) \) and that velocity can be arbitrarily large. Thus, given any finite "box" over which we do the \( \int d^3x \) at time \( t_1 \) there may be tachyons that will be located out of that box at time \( t_2 \). (This cannot happen for ordinary particles.) We conclude, as in earlier writings, that the first method of treating tachyons - using space displacement to define the parameter \( \tau \) - is the correct one for them.

What if we take this second version of \( j^\mu \) for tachyons and integrate it as we did for the first version.

\[ \int dt \int d^2 x_\perp \hat{v} \cdot j = v \int dt \delta(x_\parallel - vt) = 1, \quad (3.10) \]
which looks nice.

We can apply the same analysis to the energy-momentum tensor: just add the factor \( m \dot{\xi}^\nu \) to the results above for \( j^\mu \).

\[
T^{\mu\nu} = m \gamma(1, \mathbf{v})(1, \mathbf{v}) \delta^3(\mathbf{x} - \mathbf{v}t),
\]

or

\[
T^{\mu\nu} = m \gamma(1, \mathbf{v})(1, \mathbf{v}) v^{-1} \delta(t - x_\parallel/v)\delta^2(x_\perp),
\]

for the ordinary particle or tachyon, respectively.

4 Lagrangian/Hamiltonian formalism

Textbooks show how to write a Lagrangian and a Hamiltonian for a single relativistic (ordinary) particle; but then say that one cannot make this “manifestly covariant” for many-particle systems because each individual particle’s \( d\tau \) is independent of the others’. Traditional Lagrangian formalism involves particles and fields but all described on a common space-time manifold. When we come to add gravitation, there is the familiar caveat that “Energy” is not well defined in Einstein’s theory, at least because there are possible gravitational waves that need to be attended to; but that need not bother us here. There is also a sophisticated literature about the “positive energy theorem” in general relativity [7]; but that work explicitly requires \( T_{00} \geq 0 \) everywhere in each local Lorentz frame and that would prohibit our inclusion of tachyons.

Our objective here is to study a “static” physical system of particles - both ordinary and tachyons - with gravitational interaction, derived from Einstein’s equation. By the word “static” we mean that the particles are moving, but their pattern of flow does no change with time. This should imply that the gravitational field they produce - via the metric \( g_{\mu\nu}(x) \) - is independent of the time. But this must mean that we are restricting ourselves to one (or a particular set of) Lorentz frames. If any field is independent of time (but varying with spatial position) in one reference frame, a Lorentz transformation that takes us to a frame moving relative to the original frame will show the field (at any place) as varying with time. So, our final analysis will be done in a particular Lorentz frame: and this is ok. Nevertheless we want to start with a generally invariant/covariant formalism, and specialize to the static case later.

I want to be especially careful about minus signs here. Start with one particle:

\[
L = \int d^3x \mathcal{L}, \quad \mathcal{L} = \zeta m \int d\tau \sqrt{\dot{\xi}^\mu(\tau)\dot{\xi}^\nu(\tau)\epsilon_{\mu\nu}} \delta^4(x - \xi(\tau)),
\]

6
\[ L = \zeta m \int d\tau \sqrt{\dot{\xi}^\mu(\tau)\dot{\xi}^\nu(\tau)} \epsilon_{\mu\nu} \delta(t - \xi^0(\tau)). \] (4.2)

Here the dot means derivative with respect to \( \tau \), \( \eta_{\mu\nu} \) is the Minkowski metric; \( \epsilon = \pm 1 \) distinguishes ordinary particles from tachyons; and \( \zeta \) is another \( \pm 1 \) factor that we will have to argue about later on. We now use the remaining delta-function to eliminate the integral over \( \tau \) - and this leaves us with a factor \( |\dot{\xi}^0|^{-1} \). We now write,

\[ \dot{\xi}^\mu = (dt/d\tau, d\mathbf{x}/d\tau) = \dot{\xi}^0(1, \mathbf{v}), \quad \mathbf{v} = d\mathbf{x}/dt. \] (4.3)

and this yields,

\[ L = \zeta m \sqrt{\epsilon(1 - v^2)}. \] (4.4)

We then proceed with the "canonical" formalism,

\[ p_i = \frac{\partial L}{\partial v^i} = \zeta \epsilon m \gamma, \quad \gamma = 1/\sqrt{\epsilon(1 - v^2)}, \quad H = p_i v^i - L = -\zeta \epsilon m \gamma. \] (4.5)

For ordinary particles (\( \epsilon = +1 \)) at low velocities this gives,

\[ H = -\zeta (mc^2 + 1/2mv^2 + ...) \] (4.6)

Thus it is conventional to choose \( \zeta = -1 \). For tachyons (\( \epsilon = -1 \)) we have,

\[ H = +\zeta m \gamma. \] (4.7)

It is tempting to choose \( \zeta = +1 \) but maybe we should wait to see about this sign.

For many particles, labelled with the subscript "a", we now write the Lagrangian density for all this matter, in the presence of a gravitational field as follows.

\[ \mathcal{L}_M(x) = \sum_a \zeta_a m_a \int d\tau_a \sqrt{\dot{\xi}_a^\mu(\tau_a)\dot{\xi}_a^\nu(\tau_a)} \epsilon_a g_{\mu\nu}(x) \delta^4(x - \xi_a(\tau_a)), \] (4.8)

and, following the method used above for each particle’s coordinates,

\[ L_M = \sum_a \zeta_a m_a \sqrt{\epsilon_a g_{\mu\nu}(x_a) v_a^\mu v_a^\nu}, \] (4.9)

where \( v_a^\mu = (1, \mathbf{v}_a) = (1, d\mathbf{x}_a/dt) \). What we have here, for the physical problem posed in Section 1, is an expression where the metric \( g_{\mu\nu} \) does not depend explicitly on the time \( t \); it does depend on the coordinates of the particle \( \mathbf{x}_a \) in each term, and those coordinates do depend on the time \( t \). The particle velocities \( \mathbf{v}_a \) also depend
implicitly on the time \( t \). So we can do conventional steps of Lagrangian analysis, as follows.

\[
p_a \mu = \frac{\partial L_M}{\partial \dot{v}_a^\mu} = \zeta_a m_a \epsilon_a g_{\mu\nu}(x_a) v_a^\nu / \sqrt{\epsilon_a g_{\mu\nu}(x_a) v_a^\mu v_a^\nu}.
\] (4.10)

Since we have defined \( v^0 = 1 \) this equation should be read only for \( \mu = i = 1, 2, 3 \) in term of Lagrangian formalism. However, as we shall see below, this may be read as a generally covariant definition of momentum. We also have the geodesic equation for each individual particle,

\[
\ddot{\xi}_a^\mu + \Gamma_{\alpha\beta}^{\mu} \dot{\xi}_a^\alpha \dot{\xi}_a^\beta = 0,
\] (4.11)

which comes from varying each worldline \( \xi_a(\tau_a) \) in the action made with this Lagrangian density (4.8) in the most general case. (This geodesic equation does not involve the factors \( \zeta, \epsilon \).) From this geodesic equation we have, in the general case, the integral,

\[
g_{\mu\nu}(x = \xi_a(\tau_a)) \dot{\xi}_a^\mu \dot{\xi}_a^\nu = \text{constant} = \epsilon_a \kappa_a^2
\] (4.12)

Here I put in the factor \( \epsilon_a = \pm 1 \) to distinguish the two species of particles we study; and I also put in some constants \( \kappa_a \). As usual, we make these constants \( \kappa_a \) equal to 1 by scaling the previously arbitrary parameters \( \tau_a \).

With the above information, we now calculate the Hamiltonian for these particles,

\[
H_M = \sum_a p_a v_a^i - L_M = - \sum_a p_a^0.
\] (4.13)

We can also write,

\[
p_a \mu = \zeta_a \epsilon_a g_{\mu\nu}(x_a) p_a^\nu
\] (4.14)

where I have \( p_a^\nu = m_a \dot{\xi}_a^\nu \), following the usual definition of particle 4-momentum (contravariant). Except for the factors \( \zeta, \epsilon \), this relation seems to conform with the standard relation between "covariant" and "contravariant" vectors. Furthermore, in the situation where the metric \( g \) is independent of the time, there is a textbook proof \(^{[8]}\) that the time component of this covariant momentum \( p_a^0 \) is constant along the particle’s worldline. We shall continue this analysis of particle dynamics in Section 6.

## 5 Gravitational field

Now we write that part of the Lagrangian that describes the gravitational field. I am now limiting this part to the Linear approximation to Einstein’s full theory of General Relativity.
\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) - \frac{1}{2} \eta_{\mu\nu} h + O(G^2) = \eta_{\mu\nu} + \lambda_{\mu\nu}(x) + O(G^2), \quad (5.1) \]

where \( h_{\mu\nu} \) is first order in \( G \) and \( h = \eta^{\mu\nu} h_{\mu\nu} \); and henceforward we use the Minkowski metric \( \eta_{\mu\nu} \) to raise and lower indices.

The equation of motion (Einstein’s theory in the linear approximation) is,

\[ \partial^\alpha \partial_\alpha h_{\mu\nu}(x) = -16\pi G T_{\mu\nu}(x), \quad (5.2) \]

\[ \partial^\alpha \partial_\alpha \lambda_{\mu\nu}(x) = -16\pi G [T_{\mu\nu}(x) - \frac{1}{2} \eta_{\mu\nu} T(x)], \quad (5.3) \]

with the gauge condition \( \partial^\mu h_{\mu\nu} = 0 \).

Let me try the following construction, which is first order in \( G \):

\[ \mathcal{L}_G = \frac{1}{64\pi G} [(\partial^\alpha \lambda^{\mu\nu})(\partial_\alpha \lambda_{\mu\nu}) - \frac{1}{2} (\partial^\alpha \lambda)(\partial_\alpha \lambda)], \quad (5.4) \]

where \( \lambda = \eta^{\mu\nu} \lambda_{\mu\nu} \). When we go through variation of the action it will involve partial integrations over time and space. We assume that the deviations of the metric from the Minkowski form are contained in space, so there should be no surface terms from the partial integration over space. Regarding integration over time, the usual action rules say that there is no variation at the time endpoints, whatever they may be.

\[ \frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu}(x)} = \frac{-1}{32\pi G} [\partial^\alpha \partial_\alpha \lambda^{\mu\nu}(x) - \frac{1}{2} \eta^{\mu\nu} \partial^\alpha \partial_\alpha \lambda]; \quad (5.5) \]

Using the matter Lagrangian density from the previous section we have,

\[ \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}(x)} = \sum_a \frac{\xi_a \xi_a}{2} m_a \int d\tau_a \dot{\mathbf{r}}_a \dot{\mathbf{r}}_a \delta^4(x - \xi_a(\tau_a)) = \frac{-1}{2} T^{\mu\nu}(x). \quad (5.6) \]

This defines the energy-momentum tensor \( T^{\mu\nu}(x) \). Thus we have the complete Lagrangian density, \( \mathcal{L} = \mathcal{L}_M + \mathcal{L}_G \), giving us the correct equations of motion (5.3) upon variation of the metric. Now we calculate the Hamiltonian for this entire system of particles and (linearized) gravitational field, adding \[1.13\]

\[ H = H_M + H_G = - \sum_a p_a \cdot \mathbf{v}_a + \frac{1}{64\pi G} \int d^3 x [(\partial_\alpha \lambda^{\mu\nu})(\partial_\alpha \lambda_{\mu\nu}) - \frac{1}{2} (\partial_\alpha \lambda)(\partial_\alpha \lambda)], \quad (5.7) \]

where that second term has no longer the Lorentz invariant \( \partial^\alpha \ldots \partial_\alpha \) but what looks like a Euclidean sum.
Let’s explore this result. For any set of fields $\varphi_b(x)$ in 3-dimensional Euclidean space that are produced by localized source densities $\rho_b(x)$, we have

$$\triangle \varphi_b = \partial_i \partial_i \varphi_b = -4\pi \rho_b, \quad \varphi_b(x) = \int d^3x' \frac{\rho_b(x')}{|x - x'|}, \quad \sim M_b/r,$$  \hspace{1cm} (5.8)

where the $\sim$ means at a large distance $r$ from the source. Thus we have for the gravitational field part of the Hamiltonian, setting $\partial_t \lambda^\mu_{\nu} = 0$,

$$H_G = G \int d^3x \int d^3x' \frac{1}{|x - x'|} [T^\mu_{\nu}(x)T_{\mu\nu}(x') - \frac{1}{2} T(x)T(x')].$$ \hspace{1cm} (5.10)

Putting in the earlier formula for the source $T$:

$$H_G = \sum_{a,b} G(\zeta_a \xi_a \mu_a \gamma_a) \frac{G(\zeta_b \xi_b \mu_b \gamma_b)}{r_{ab}} [(1 - v_a \cdot v_b)^2 - \frac{1}{2}(1 - v_a^2)(1 - v_b^2)].$$ \hspace{1cm} (5.11)

If I look at the non-relativistic limit, $v_a \to 0$, this term becomes

$$H_G \to \frac{1}{2} \sum a \sum b \frac{G \mu_a \mu_b}{r_{ab}},$$ \hspace{1cm} (5.12)

which looks familiar; and for the other limit, $v_a \to \infty$

$$H_G \to \frac{1}{2} \sum a \sum b \frac{G \mu_a \mu_b}{r_{ab}} \zeta_a \zeta_b v_a v_b (\cos^2 \theta_{ab} - \frac{1}{2}),$$ \hspace{1cm} (5.13)

where $\theta_{ab}$ is the angle between the two velocity vectors. This formula is not familiar; but one may want to compare it to the expression for the energy of the magnetic field produce by a spatial distribution of static electric currents. Note that the velocities may vary with spatial position of the particles, so this calculation is not as simple as it may look.

### 6 Particles in a gravitational field

Now we pick up and continue the work of Section 4 with particles in a gravitational field. We have a different scalar $\tau_a$ for each particle, labeled with "a". We want to define them all in relation to the common time $t$ in this chosen reference frame:
\[ d\tau_a = c_a^{-1}dt, \] for some variables \( c_a \) assigned to each particle at each point of its trajectory. We identify \( c_a = \dot{\xi}_a^0 \). This lets us go from Eq. (4.8) to Eq. (4.9) above but we have yet no formula for \( c_a \). The condition from the geodesic equation is

\[ c^2[g_{00} + 2g_{0i}v^i + g_{ij}v^iv^j] = \epsilon \]  

(6.1)

Here \( \epsilon = +1 \) means an ordinary particle and \( \epsilon = -1 \) designates a tachyon. We should put the subscript "a" on all parts of this equation since it should hold true for each particle: this means that the metric \( g \) is evaluated at the point \( x^i_a = \xi^i_a(\tau_a) \) and the same is true for \( v^i_a \) and thus for \( c_a \).

With the metric independent of time, we also have, for each particle [8]:

\[ \dot{\xi}_a = [g_{00} + g_{0i}v^i]\dot{\xi}_0^a = [g_{00} + g_{0i}v^i]c = \text{constant} \]  

(6.2)

or,

\[ c_a[g_{00} + g_{0i}v^i] = -E_a/(\zeta_a \epsilon \gamma m_a). \]  

(6.3)

For a free particle, \( g_{\mu\nu} = \eta_{\mu\nu} \), the first (6.1) gives the familiar \( c = \gamma = 1/\sqrt{\epsilon(1 - v^2)} \) and the second (6.3) gives \( E = -\zeta \epsilon \gamma m_a \), agreeing with the earlier formula for a free particle. For convenience we set \( \omega_a = \zeta_a \epsilon \gamma m_a \).

For the general case, we combine (6.1) and (6.3), eliminating \( c \), to get

\[ E_a/(\omega_a m_a) = -A/\sqrt{\epsilon B}, \quad A = [g_{00} + g_{0i}v^i], \quad B = [g_{00} + 2g_{0i}v^i + g_{ij}v^iv^j]. \]  

(6.4)

where, again, everything on the right deserves the label "a", since this relation is true for each particle.

Expanding this to first order in the gravitational constant,

\[ E_a = -\omega_a m_a \gamma a[1 + W], \quad W = (\lambda_{00} + \lambda_{0i}v^i) - \frac{1}{2} \epsilon_a \gamma_a^2(\lambda_{00} + 2\lambda_{0i}v^i + \lambda_{ij}v^iv^j). \]  

(6.5)

From (5.3) and (5.6) we have,

\[ \lambda^{\mu\nu}(x) = 4G \sum_b \omega_b m_b \gamma_b [v_b^\mu v_b^\nu - \frac{1}{2} \eta^{\mu\nu}(1 - v_b^2)], \]  

(6.6)

and this leads us to,

\[ H_M = \sum_a E_a = -\sum_a \omega_a m_a \gamma_a - \sum_{a,b} G(\omega_a m_a \gamma_a)(\omega_b m_b \gamma_b) W_{ab}, \]  

(6.7)

\[ W_{ab} = 2 - 4v_a \cdot v_b + (v_a^2 + v_b^2) - (\epsilon_a \gamma_a^2 + \epsilon_b \gamma_b^2)[(1 - v_a \cdot v_b)^2 - \frac{1}{2}(1 - v_a^2)(1 - v_b^2)]. \]  

(6.8)
In the low energy limit for ordinary particle we have $W_{ab} = 1$ and so,

$$\text{Newton} : \quad H_M = \sum_a E_a = \sum_a m_a \gamma_a - \sum_a \sum_b G m_a m_b / \lvert x_a - x_b \rvert. \quad (6.9)$$

The potential energy term is too big by a factor of two. We know, in non-relativistic Newtonian physics, that the total potential energy of such a system counts each pair of interacting particles once. This formula counts each pair twice. But we have to add this particle energy to the field energy calculated in the previous section: and now we get the correct answer:

$$\text{Newton} : \quad H = \sum_a m_a \gamma_a - \frac{1}{2} \sum_a \sum_b \frac{G m_a m_b}{r_{ab}}. \quad (6.10)$$

7 Final formulas

For the general case we have the final formula for Gravito-Statics, allowing tachyons and ordinary particles at any energy:

$$H = - \sum_a \omega_a m_a \gamma_a - \sum_{a,b} \frac{G(\omega_a m_a \gamma_a)(\omega_b m_b \gamma_b)}{r_{ab}} Z_{ab},$$

$$Z_{ab} = 2 - 4 \mathbf{v}_a \cdot \mathbf{v}_b + (v_a^2 + v_b^2) - (\epsilon_a \gamma_a^2 + \epsilon_b \gamma_b^2 + 1) \times$$

$$\times \left[ (1 - \mathbf{v}_a \cdot \mathbf{v}_b)^2 - \frac{1}{2}(1 - v_a^2)(1 - v_b^2) \right]. \quad (7.1)$$

One may wish to drop the self-energy terms $a = b$: $-(G m_a^2 \epsilon_0)^2 (1 + v^2) / (r_{ab} = 0)$.

Now I want to look at this general formula in several special cases, of possible physical interest. In this we will want to specify two species of particles: ordinary $\epsilon = +1$ and tachyon $\epsilon = -1$. Also we may have two types of tachyons: $\zeta = \pm 1$. This last dichotomy was predicted in [4] on the basis of quantum theory for a spin 1/2 field obeying a Dirac equation for tachyons.

I. For low energy ordinary particles ($v \rightarrow 0$) we have already noted that (7.1) becomes the Newtonian formula (1.2).

II. In the case of all low energy tachyons $v \rightarrow \infty$,

$$H \approx \sum_a m_a / v_a + \sum_{a,b} \frac{G m_a m_b}{r_{ab}} \zeta_a \zeta_b v_a v_b (\cos^2 \theta_{ab} - 1/2). \quad (7.2)$$

This is a new formula; and it will be subjected to further study elsewhere. However there is one feature to be noted now. If all the $\zeta$ factors are of the same sign, then
this says that colinear motion of the tachyons (θ near 0 or π) is not a stable bound state. This is contrary to what I had anticipated in earlier study [1]; and this is because (7.1) comes from the field term \(H_G\), which I had not previously considered.

III For all high energy particles \(v \to 1\) and thus \(\gamma\) very large,

\[Z_{ab} \approx -(\epsilon_a \gamma_a^2 + \epsilon_b \gamma_b^2)(1 - \cos \theta_{ab})^2.\] (7.3)

Thus, except for strictly colinear motion \(\theta_{ab} = 0\), this says: a strong repulsive force among ordinary particles; a strong attractive force among same-type tachyons; a strong repulsive force between opposite-type tachyons. For the interaction between the two species, it is more complicated. That may be interesting in thinking about the early universe.

IV. For low energy ordinary particles \(v_a \to 0\) and high energy tachyons \(v_b \to 1\) we find,

\[H_{int} \approx -\sum_{a,b} \frac{G m_a \zeta_b \gamma_b^2}{r_{ab}}.\] (7.4)

This appears to be large and attractive for one type of tachyon but repulsive for the other type. This may be interesting in thinking about intermediate times in the evolution of the universe when ordinary matter has cooled but tachyons (e.g. neutrinos) still have energies far above their mass.

V. For the case of \(v_a \to 0\) and \(v_b \to \infty\) we have the interaction between these two species to be,

\[H_{int} \approx -3 \sum_{a,b} \frac{G m_a \zeta_b \gamma_b^2}{r_{ab}}[1 + O(v_a^2 v_b^2)],\] (7.5)

which looks like a weak Newtonian force between low energy tachyons and ordinary matter: it can be attractive or repulsive, depending on the sign of \(\zeta\) for the tachyons. This may be interesting in looking at the most recent evolution of the universe, with temperatures well below the mass of ordinary matter as well as tachyon-neutrinos. We note that this force is a lot weaker than that seen in (7.4).

One more general remark about this result. According to some authorities [8] the word "static" should be used to imply that the metric is not only independent of the time but also invariant under time-reversal. That would mean excluding components \(T_{0i}\) in the source of the gravitational field and thus eliminating the components \(\lambda_{0i}\) in the metric. While I have not made those restrictions, we see that the final formula (7.1) is invariant under time-reversal: \(v \to -v\) for all particles.
8 Conclusion

With the general formula and its various specialized forms, presented in the last Section, one may start the hard work of trying to build models of tachyon flows that are stable and confined and may contribute effectively to the proposition that neutrinos-as-tachyons may explain the observed phenomena now ascribed to Dark Matter.

The one sharp conclusion from this work is the rejection of my original [1] model of low energy tachyons being attracted to one another in a rope-like structure.

I will save my own further modelling for a separate paper, while offering the above mathematical framework for others to explore independently.

As a bit of self-criticism, I offer the thought that the Static model, upon which this paper is based, may be very inappropriate for the high energy situations considered in Cosmology, although I have blithely ventured into that domain just above.

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