A Fractional Calculus on Arbitrary Time Scales: Fractional Differentiation and Fractional Integration

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Abstract

We introduce a general notion of fractional (noninteger) derivative for functions defined on arbitrary time scales. The basic tools for the time-scale fractional calculus (fractional differentiation and fractional integration) are then developed. As particular cases, one obtains the usual time-scale Hilger derivative when the order of differentiation is one, and a local approach to fractional calculus when the time scale is chosen to be the set of real numbers.

Keywords: fractional differentiation, fractional integration, calculus on time scales.

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1 Introduction

Fractional calculus refers to differentiation and integration of an arbitrary (noninteger) order. The theory goes back to mathematicians as Leibniz (1646–1716), Liouville (1809–1882), Riemann (1826–1866), Letnikov (1837–1888), and Grünwald (1838–1920) [21, 38]. During the last two decades, fractional calculus has increasingly attracted the attention of researchers of many different fields [1, 9, 10, 29, 31, 33, 35, 41].

Several definitions of fractional derivatives/integrals have been defined in the literature, including those of Riemann–Liouville, Grünwald–Letnikov, Hadamard, Riesz, Weyl and Caputo [21, 30, 38]. In 1996, Kolwankar and Gangal proposed a local fractional derivative operator that applies to highly irregular and nowhere differentiable Weierstrass functions [8, 26]. Here we introduce the notion of fractional derivative on an arbitrary time scale $\mathbb{T}$ (cf. Definition 6). In the particular case $\mathbb{T} = \mathbb{R}$, one gets the local Kolwankar–Gangal fractional derivative $\lim_{h \to 0} \frac{f(t+h)-f(t)}{h^\alpha}$, which has been considered in [26, 27] as the point of departure for fractional calculus. One of the

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motivations to consider such local fractional derivatives is the possibility to deal with irregular signals, so common in applications of signal processing [27].

A time scale is a model of time. The calculus on time scales was initiated by Aulbach and Hilger in 1988 [7], in order to unify and generalize continuous and discrete analysis [22, 23]. It has a tremendous potential for applications and has recently received much attention [3, 10, 17, 20, 21]. The idea to join the two subjects — the fractional calculus and the calculus on time scales — and to develop a Fractional Calculus on Time Scales, was born with the Ph.D. thesis of Bastos [12]. See also [5, 6, 13, 14, 15, 25, 37, 40] and references therein. Here we introduce a general fractional calculus on time scales and develop some of its basic properties.

Fractional calculus is of increasing importance in signal processing [35]. This can be explained by several factors, such as the presence of internal noises in the structural definition of the signals. Our fractional derivative depends on the graininess function of the time scale. We trust that this possibility can be very useful in applications of signal processing, providing a concept of coarse-graining in time that can be used to model white noise that occurs in signal processing or to obtain generalized entropies and new practical meanings in signal processing. Indeed, let $T$ be a time scale (continuous time $T = \mathbb{R}$, discrete time $T = h\mathbb{Z}$, $h > 0$, or, more generally, any closed subset of the real numbers, like the Cantor set). Our results provide a mathematical framework to deal with functions/signals $f(t)$ in signal processing that are not differentiable in the time scale, that is, signals $f(t)$ for which the equality $\Delta f(t) = f^{\Delta}(t)\Delta t$ does not hold. More precisely, we are able to model signal processes for which $\Delta f(t) = f^{(\alpha)}(t)(\Delta t)^{\alpha}$, $0 < \alpha \leq 1$.

The time-scale calculus can be used to unify discrete and continuous approaches to signal processing in one unique setting. Interesting in applications, is the possibility to deal with more complex time domains. One extreme case, covered by the theory of time scales and surprisingly relevant also for the process of signals, appears when one fix the time scale to be the Cantor set [11, 42]. The application of the local fractional derivative in a time scale different from the classical time scales $T = \mathbb{R}$ and $T = h\mathbb{Z}$ was proposed by Kolwankar and Gangal themselves: see [27, 28] where nondifferentiable signals defined on the Cantor set are considered.

The article is organized as follows. In Section 2 we recall the main concepts and tools necessary in the sequel. Our results are given in Section 3: in Section 3.1 the notion of fractional derivative for functions defined on arbitrary time scales is introduced and the respective fractional differential calculus developed; the notion of fractional integral on time scales, and some of its basic properties, is investigated in Section 3.2. We end with Section 4 of conclusions and future work.

2 Preliminaries

A time scale $T$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Here we only recall the necessary concepts of the calculus on time scales. The reader interested on the subject is referred to the books [16, 17]. For a good survey see [3].

**Definition 1.** Let $T$ be a time scale. For $t \in T$ we define the forward jump operator $\sigma : T \to T$ by $\sigma(t) := \inf\{s \in T : s > t\}$, and the backward jump operator $\rho : T \to T$ by $\rho(t) := \sup\{s \in T : s < t\}$.

**Remark 2.** In Definition 1 we put $\inf\emptyset = \sup T$ (i.e., $\sigma(t) = t$) if $T$ has a maximum $t$, and $\sup\emptyset = \inf T$ (i.e., $\rho(t) = t$) if $T$ has a minimum $t$, where $\emptyset$ denotes the empty set.

If $\sigma(t) > t$, then we say that $t$ is right-scattered; if $\rho(t) < t$, then $t$ is said to be left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $t < \sup T$ and $\sigma(t) = t$, then $t$ is called right-dense; if $t > \inf T$ and $\rho(t) = t$, then $t$ is called left-dense. The graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

We make use of the set $T^\kappa$, which is derived from the time scale $T$ as follows: if $T$ has a left-scattered maximum $M$, then $T^\kappa = T \setminus \{M\}$; otherwise, $T^\kappa = T$.

**Definition 3** (Delta derivative [2]). Assume $f : T \to \mathbb{R}$ and let $t \in T^\kappa$. We define

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(s)) - f(t)}{\sigma(s) - t}, \quad t \neq \sigma(s),$$
provided the limit exists. We call \( f^{\Delta}(t) \) the delta derivative (or Hilger derivative) of \( f \) at \( t \). Moreover, we say that \( f \) is delta differentiable on \( \mathbb{T}^\sigma \) provided \( f^{\Delta}(t) \) exists for all \( t \in \mathbb{T}^\sigma \). The function \( f^{\Delta} : \mathbb{T}^\sigma \to \mathbb{R} \) is then called the delta derivative of \( f \) on \( \mathbb{T}^\sigma \).

Delta derivatives of higher-order are defined in the usual way. Let \( r \in \mathbb{N} \), \( \mathbb{T}^0 : = \mathbb{T} \), and \( \mathbb{T}^i : = \left( \mathbb{T}^{i-1} \right)^{\kappa} \), \( i = 1, \ldots, r \). For convenience we also put \( f^{\Delta_0} = f \) and \( f^{\Delta_1} = f^{\Delta} \). The \( r \)-th delta derivative \( f^{\Delta_r} \) is given by \( f^{\Delta_r} = \left( f^{\Delta_{r-1}} \right)^{\Delta} : \mathbb{T}^\sigma \to \mathbb{R} \) provided \( f^{\Delta_{r-1}} \) is delta differentiable.

The following notions will be useful in connection with the fractional integral (Section 3.2).

**Definition 4.** A function \( f : \mathbb{T} \to \mathbb{R} \) is called regulated provided its right-sided limit exist (finite) at all right-dense points in \( \mathbb{T} \) and its left-sided limits exist (finite) at all left-dense points in \( \mathbb{T} \).

**Definition 5.** A function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( \mathbb{T} \) and its left-sided limits exist (finite) at left-dense points in \( \mathbb{T} \). The set of rd-continuous functions \( f : \mathbb{T} \to \mathbb{R} \) is denoted by \( \mathcal{C}_{rd} \).

## 3 Main Results

We develop the basic tools of any fractional calculus: fractional differentiation (Section 3.1) and fractional integration (Section 3.2). Let \( \mathbb{T} \) be a time scale, \( t \in \mathbb{T} \), and \( \delta > 0 \). We define the left \( \delta \)-neighborhood of \( t \) as \( \mathcal{U}^- : = [t - \delta, t] \cap \mathbb{T} \).

### 3.1 Fractional Differentiation

We begin by introducing a new notion: the fractional derivative of order \( \alpha \in [0,1] \) for functions defined on arbitrary time scales. For \( \alpha = 1 \) we obtain the usual delta derivative of the time-scale calculus.

**Definition 6.** Let \( f : \mathbb{T} \to \mathbb{R} \), \( t \in \mathbb{T}^\sigma \), and \( \alpha \in [0,1] \). For \( \alpha \in [0,1] \cap \{1/q : q \text{ is a odd number} \} \) (resp. \( \alpha \in [0,1] \setminus \{1/q : q \text{ is a odd number} \} \)) we define \( f^{(\alpha)}(t) \) to be the number (provided it exists) with the property that, given any \( \epsilon > 0 \), there is a \( \delta \)-neighborhood \( \mathcal{U}^- \subset \mathbb{T} \) of \( t \) (resp. left \( \delta \)-neighborhood \( \mathcal{U}^- \subset \mathbb{T} \) of \( t \)), \( \delta > 0 \), such that

\[
\left| f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^\alpha \right| \leq \epsilon |\sigma(t) - s|^\alpha
\]

for all \( s \in \mathcal{U} \) (resp. \( s \in \mathcal{U}^- \)). We call \( f^{(\alpha)}(t) \) the fractional derivative of \( f \) of order \( \alpha \) at \( t \).

Along the text \( \alpha \) is a real number in the interval [0,1]. The next theorem provides some useful relationships concerning the fractional derivative on time scales introduced in Definition 6.

**Theorem 7.** Assume \( f : \mathbb{T} \to \mathbb{R} \) and let \( t \in \mathbb{T}^\sigma \). The following properties hold:

(i) Let \( \alpha \in [0,1] \cap \left\{ \frac{1}{q} : q \text{ is a odd number} \right\} \). If \( t \) is right-dense and if \( f \) is fractional differentiable of order \( \alpha \) at \( t \), then \( f \) is continuous at \( t \).

(ii) Let \( \alpha \in [0,1] \setminus \left\{ \frac{1}{q} : q \text{ is a odd number} \right\} \). If \( t \) is right-dense and if \( f \) is fractional differentiable of order \( \alpha \) at \( t \), then \( f \) is left-continuous at \( t \).

(iii) If \( f \) is continuous at \( t \) and \( t \) is right-scattered, then \( f \) is fractional differentiable of order \( \alpha \) at \( t \) with

\[
f^{(\alpha)}(t) = \frac{f^\sigma(t) - f(t)}{(\mu(t))^\alpha}.
\]
(iv) Let $\alpha ]0, 1] \cap \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$. If $t$ is right-dense, then $f$ is fractional differentiable of order $\alpha$ at $t$ if, and only if, the limit

$$
\lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}
$$

exists as a finite number. In this case,

$$
f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}.
$$

(v) Let $\alpha ]0, 1] \setminus \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$. If $t$ is right-dense, then $f$ is fractional differentiable of order $\alpha$ at $t$ if, and only if, the limit

$$
\lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}
$$

exists as a finite number. In this case,

$$
f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}.
$$

(vi) If $f$ is fractional differentiable of order $\alpha$ at $t$, then $f(\sigma(t)) = f(t) + (\mu(t))^{\alpha} f^{(\alpha)}(t)$.

Proof. (i) Assume that $f$ is fractional differentiable at $t$. Then, there exists a neighborhood $U$ of $t$ such that

$$
\left| f(\sigma(t)) - f(s) - f^{(\alpha)}(t) [\sigma(t) - s]^\alpha \right| \leq \epsilon |\sigma(t) - s|^\alpha
$$

for $s \in U$. Therefore, for all $s \in U \cap ]t - \epsilon, t + \epsilon[$,

$$
|f(t) - f(s)| \leq \left| f^{(\alpha)}(t) [\sigma(t) - s]^\alpha \right|
$$

and, since $t$ is a right-dense point,

$$
|f(t) - f(s)| \leq \left| f^{(\alpha)}(t) [\sigma(t) - s]^\alpha \right| + \left| f^{(\alpha)}(t) [\sigma(t) - t]^\alpha \right|
$$

It follows the continuity of $f$ at $t$.

(ii) The proof is similar to the proof of (i), where instead of considering the neighborhood $U$ of $t$ we consider a left neighborhood $U^-$ of $t$.

(iii) Assume that $f$ is continuous at $t$ and $t$ is right-scattered. By continuity,

$$
\lim_{s \to t} \frac{f^{(\alpha)}(t) - f(s)}{(\sigma(t) - s)^\alpha} = \frac{f^{(\alpha)}(t) - f(t)}{(\sigma(t) - t)^\alpha} = \frac{f^{(\alpha)}(t) - f(t)}{(\mu(t))^{\alpha}}.
$$

Hence, given $\epsilon > 0$ and $\alpha ]0, 1] \cap \{ 1/q : q \text{ is a odd number} \}$, there is a neighborhood $U$ of $t$ (or $U^-$ if $\alpha ]0, 1] \setminus \{ 1/q : q \text{ is a odd number} \}$) such that

$$
\left| \frac{f^{(\alpha)}(t) - f(s)}{(\sigma(t) - s)^\alpha} - \frac{f^{(\alpha)}(t) - f(t)}{(\mu(t))^{\alpha}} \right| \leq \epsilon.
$$
for all \( s \in \mathcal{U} \) (resp. \( \mathcal{U}^- \)). It follows that
\[
|f^\sigma(s) - f(s)| - \frac{f^\sigma(t) - f(t)}{(\mu(t))^\alpha} (\sigma(t) - s)^\alpha \leq \epsilon |\sigma(t) - s|^\alpha
\]
for all \( s \in \mathcal{U} \) (resp. \( \mathcal{U}^- \)). Hence, we get the desired result:
\[
f^{(\alpha)}(t) = \frac{f^\sigma(t) - f(t)}{(\mu(t))^\alpha}.
\]

(iv) Assume that \( f \) is fractional differentiable of order \( \alpha \) at \( t \) and \( t \) is right-dense. Let \( \epsilon > 0 \) be given. Since \( f \) is fractional differentiable of order \( \alpha \) at \( t \), there is a neighborhood \( \mathcal{U} \) of \( t \) such that
\[
|f^\sigma(s) - f(s)| - f^{(\alpha)}(t)(\sigma(t) - s)^\alpha \leq \epsilon |\sigma(t) - s|^\alpha
\]
for all \( s \in \mathcal{U} \). Since \( \sigma(t) = t \),
\[
|f(t) - f(s)| - f^{(\alpha)}(t)(t - s)^\alpha \leq \epsilon |t - s|^\alpha
\]
for all \( s \in \mathcal{U} \). It follows that
\[
\left| \frac{f(t) - f(s)}{(t - s)^\alpha} - f^{(\alpha)}(t) \right| \leq \epsilon
\]
for all \( s \in \mathcal{U} \), \( s \neq t \). Therefore, we get the desired result:
\[
f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}.
\]

Now assume that
\[
\lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}
\]
exists and is equal to \( L \) and \( t \) is right-dense. Then, there exists \( \mathcal{U} \) such that
\[
\left| \frac{f(t) - f(s)}{(t - s)^\alpha} - L \right| \leq \epsilon
\]
for all \( s \in \mathcal{U} \). Because \( t \) is right-dense,
\[
\left| \frac{f^\sigma(t) - f(s)}{(\sigma(t) - s)^\alpha} - L \right| \leq \epsilon.
\]

Therefore,
\[
|f^\sigma(t) - f(s)| - L (\sigma(t) - s)^\alpha \leq \epsilon |\sigma(t) - s|^\alpha,
\]
which lead us to the conclusion that \( f \) is fractional differentiable of order \( \alpha \) at \( t \) and \( f^{(\alpha)}(t) = L \).

(v) The proof is similar to the proof of (iv), where instead of considering the neighborhood \( \mathcal{U} \) of \( t \) we consider a left-neighborhood \( \mathcal{U}^- \) of \( t \).

(vi) If \( \sigma(t) = t \), then \( \mu(t) = 0 \) and
\[
f^\sigma(t) = f(t) = f(t) + (\mu(t))^\alpha f^{(\alpha)}(t).
\]

On the other hand, if \( \sigma(t) > t \), then by (iii)
\[
f^\sigma(t) = \frac{f(t) - f(s)}{(\mu(t))^\alpha} = f(t) + (\mu(t))^\alpha f^{(\alpha)}(t).
\]
The proof is complete.
Remark 8. In a time scale $\mathbb{T}$, due to the inherited topology of the real numbers, a function $f$ is always continuous at any isolated point $t$.

Proposition 9. If $f: \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then $f^{(\alpha)}(t) \equiv 0$.

Proof. If $t$ is right-scattered, then, by Theorem 7 (iii), one has
\[
f^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{(\mu(t))^{\alpha}} = \frac{c - c}{(\mu(t))^{\alpha}} = 0.
\]
Assume $t$ is right-dense. Then, by Theorem 7 (iv) and (v), it follows that
\[
f^{(\alpha)}(t) = \lim_{s \to t} \frac{c - c}{(t-s)^{\alpha}} = 0.
\]
This concludes the proof.

Proposition 10. If $f: \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then
\[
f^{(\alpha)}(t) = \begin{cases} (\mu(t))^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}
\]

Proof. From Theorem 7 (vi) it follows that $\sigma(t) = t + (\mu(t))^{\alpha} f^{(\alpha)}(t)$, that is, $\mu(t) = (\mu(t))^{\alpha} f^{(\alpha)}(t)$. If $\mu(t) \neq 0$, then $f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}$ and the desired relation is proved. Assume now that $\mu(t) = 0$, that is, $\sigma(t) = t$. In this case $t$ is right-dense and by Theorem 7 (iv) and (v) it follows that
\[
f^{(\alpha)}(t) = \lim_{s \to t} \frac{t-s}{(t-s)^{\alpha}}.
\]
Therefore, if $\alpha = 1$, then $f^{(\alpha)}(t) = 1$; if $0 < \alpha < 1$, then $f^{(\alpha)}(t) = 0$. The proof is complete.

Let us consider now the two classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, $h > 0$.

Corollary 11. Function $f: \mathbb{R} \to \mathbb{R}$ is fractional differentiable of order $\alpha$ at point $t \in \mathbb{R}$ if, and only if, the limit
\[
\lim_{s \to t} \frac{f(t) - f(s)}{(t-s)^{\alpha}}
\]
exists as a finite number. In this case,
\[
f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t-s)^{\alpha}}. \tag{1}
\]

Proof. Here $\mathbb{T} = \mathbb{R}$ and all points are right-dense. The result follows from Theorem 7 (iv) and (v). Note that if $\alpha \in [0,1] \setminus \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$, then the limit only makes sense as a left-side limit.

Remark 12. The definition (1) corresponds to the well-known Kolwankar–Gangal approach to fractional calculus [26, 39].

Corollary 13. Let $h > 0$. If $f: h\mathbb{Z} \to \mathbb{R}$, then $f$ is fractional differentiable of order $\alpha$ at $t \in h\mathbb{Z}$ with
\[
f^{(\alpha)}(t) = \frac{f(t+h) - f(t)}{h^{\alpha}}.
\]

Proof. Here $\mathbb{T} = h\mathbb{Z}$ and all points are right-scattered. The result follows from Theorem 7 (iii).

We now give an example using a more sophisticated time scale: the Cantor set.
Example 14. Let $\mathbb{T}$ be the Cantor set. It is known (see Example 1.47 of [10]) that $\mathbb{T}$ does not contain any isolated point, and that

$$
\sigma(t) = \begin{cases} 
t + \frac{1}{3^{m+1}} & \text{if } t \in L, \\
t & \text{if } t \in \mathbb{T} \setminus L,
\end{cases}
$$

where

$$
L = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N} \text{ and } a_k \in \{0, 2\} \text{ for all } 1 \leq k \leq m \right\}.
$$

Thus,

$$
\mu(t) = \begin{cases} 
\frac{1}{3^{m+1}} & \text{if } t \in L, \\
0 & \text{if } t \in \mathbb{T} \setminus L.
\end{cases}
$$

Let $f : \mathbb{T} \to \mathbb{R}$ be continuous and $\alpha \in [0, 1]$. It follows from Theorem 7 that the fractional derivative of order $\alpha$ of a function $f$ defined on the Cantor set is given by

$$
f^{(\alpha)}(t) = \begin{cases} 
[f \left(t + \frac{1}{3^{m+1}}\right) - f(t)] 3^{(m+1)\alpha} & \text{if } t \in L, \\
\lim_{s \to t} \frac{f(t) - f(s)}{(t-s)^\alpha} & \text{if } t \in \mathbb{T} \setminus L,
\end{cases}
$$

where $\lim_{s \to t} = \lim_{s \to t}$ if $\alpha = \frac{1}{q}$ with $q$ an odd number, and $\lim_{s \to t} = \lim_{s \to t}$ otherwise.

For the fractional derivative on time scales to be useful, we would like to know formulas for the derivatives of sums, products and quotients of fractional differentiable functions. This is done according to the following theorem.

**Theorem 15.** Assume $f, g : \mathbb{T} \to \mathbb{R}$ are fractional differentiable of order $\alpha$ at $t \in \mathbb{T}^\circ$. Then,

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at $t$ with $(f + g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$;

(ii) for any constant $\lambda, \lambda f : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at $t$ with $(\lambda f)^{(\alpha)}(t) = \lambda f^{(\alpha)}(t)$;

(iii) if $f$ and $g$ are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at $t$ with

$$
(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(t)g^{(\alpha)}(t) = f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t);
$$

(iv) if $f$ is continuous and $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is fractional differentiable at $t$ with

$$
\left(\frac{1}{f}\right)^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))};
$$

(v) if $f$ and $g$ are continuous and $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is fractional differentiable at $t$ with

$$
\left(\frac{f}{g}\right)^{(\alpha)}(t) = \frac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}.
$$

**Proof.** Let us consider that $\alpha \in [0, 1] \cap \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$. The proofs for the case $\alpha \in [0, 1] \setminus \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$ are similar: one just needs to choose the proper left-sided neighborhoods. Assume that $f$ and $g$ are fractional differentiable at $t \in \mathbb{T}^\circ$. (i) Let $\epsilon > 0$. Then there exist neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of $t$ for which

$$
|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^\alpha| \leq \frac{\epsilon}{2} |\sigma(t) - s|^\alpha \text{ for all } s \in \mathcal{U}_1
$$
and 
\[ |g(\sigma(t)) - g(s) - g^{(\alpha)}(t)[\sigma(t) - s]| \leq \frac{\epsilon}{2}|\sigma(t) - s|^\alpha \text{ for all } s \in U_2. \]

Let \( U = U_1 \cap U_2 \). Then 
\[
\begin{align*}
|f + g)(\sigma(t)) - (f + g)(s) - \left[ f^{(\alpha)}(t) + g^{(\alpha)}(t) \right] (\sigma(t) - s)|^\alpha \\
= |f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s] + g(\sigma(t)) - g(s) - g^{(\alpha)}(t)[\sigma(t) - s]|^\alpha \\
\leq \frac{\epsilon}{2}|\sigma(t) - s| + \frac{\epsilon}{2}|\sigma(t) - s|\alpha = \epsilon|\sigma(t) - s|\alpha
\end{align*}
\]
for all \( s \in U \). Therefore, \( f + g \) is fractional differentiable at \( t \) and \( (f + g)^{\alpha}(t) = f^{\alpha}(t) + g^{\alpha}(t) \).

(ii) Let \( \epsilon > 0 \). Then there exists a neighborhood \( U \) of \( t \) with 
\[
|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|\alpha \text{ for all } s \in U.
\]
It follows that 
\[
|(\lambda f)(\sigma(t)) - (\lambda f)(s) - \lambda f^{(\alpha)}(t)[\sigma(t) - s]| \leq \epsilon|\lambda| |\sigma(t) - s|\alpha \text{ for all } s \in U.
\]
Therefore, \( \lambda f \) is fractional differentiable at \( t \) and \( (\lambda f)^{\alpha} = \lambda f^{(\alpha)} \) holds at \( t \). (iii) If \( t \) is right-dense, then 
\[
(fg)^{\alpha}(t) = \lim_{s \to t} \frac{(fg)(t) - (fg)(s)}{(t-s)^\alpha} = \lim_{s \to t} \frac{f(t) - f(s)}{(t-s)^\alpha} g(t) + \lim_{s \to t} \frac{g(t) - g(s)}{(t-s)^\alpha} f(s)
\]
\[
= f^{(\alpha)}(t)g(t) + g^{(\alpha)}(t)f(t)
\]
\[
= f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t).
\]
If \( t \) is right-scattered, then 
\[
(fg)^{\alpha}(t) = \frac{(fg)^{\alpha}(t) - (fg)(t)}{(\mu(t))^{\alpha}}
\]
\[
= \frac{f^{\alpha}(t) - f(t)}{(\mu(t))^{\alpha}} g(t) + \frac{g^{\alpha}(t) - g(t)}{(\mu(t))^{\alpha}} f(\sigma(t))
\]
\[
= f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t).
\]
The other product rule formula follows by interchanging in \( (fg)^{\alpha}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) \) the functions \( f \) and \( g \). (iv) We use the fractional derivative of a constant (Proposition 9 and Theorem 15 (iii) just proved: from Proposition 9 we know that 
\[
\left(f \cdot \frac{1}{f}\right)^{\alpha}(t) = (1)^{(\alpha)}(t) = 0
\]
and, therefore, by (iii) 
\[
\left(\frac{1}{f}\right)^{(\alpha)}(t) = \frac{f^{(\alpha)}(t)}{f(t)}
\]
Since we are assuming \( f(\sigma(t)) \neq 0 \), 
\[
\left(\frac{1}{f}\right)^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}.
\]
For the quotient formula (v), we use (ii) and (iv) to calculate
\[
\left( \frac{f}{g} \right)^{(\alpha)}(t) = \left( f \cdot \frac{1}{g} \right)^{(\alpha)}(t) \\
= f(t) \left( \frac{1}{g} \right)^{(\alpha)}(t) + f^{(\alpha)}(t) \frac{1}{g(\sigma(t))} \\
= -f(t) \frac{g^{(\alpha)}(t)}{g(t)g(\sigma(t))} + f^{(\alpha)}(t) \frac{1}{g(\sigma(t))} \\
= \frac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}.
\]
This concludes the proof.

The following theorem is proved in \([16]\) for \(\alpha = 1\). Here we show its validity for \(\alpha \in ]0, 1[\).

**Theorem 16.** Let \(c\) be a constant, \(m \in \mathbb{N}\), and \(\alpha \in ]0, 1[\).

(i) If \(f(t) = (t - c)^m\), then
\[
f^{(\alpha)}(t) = (\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m - 1} (\sigma(t) - c)^\nu (t - c)^{m - 1 - \nu}.
\]

(ii) If \(g(t) = \frac{1}{(t - c)^m}\), then
\[
g^{(\alpha)}(t) = -(\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m - 1} \frac{1}{(\sigma(t) - c)^{m - \nu} (t - c)^{\nu + 1}},
\]
provided \((t - c) (\sigma(t) - c) \neq 0\).

**Proof.** We prove the first formula by induction. If \(m = 1\), then \(f(t) = t - c\) and \(f^{(\alpha)}(t) = (\mu(t))^{1 - \alpha}\) holds from Propositions \([9]\) and \([10]\) and Theorem \([15]\) (i). Now assume that
\[
f^{(\alpha)}(t) = (\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m - 1} (\sigma(t) - c)^\nu (t - c)^{m - 1 - \nu}
\]
holds for \(f(t) = (t - c)^m\) and let \(F(t) = (t - c)^{m+1} = (t - c)f(t)\). We use the product rule (Theorem \([15]\) (iii)) to obtain
\[
F^{(\alpha)}(t) = (t - c)^{\alpha} f(\sigma(t)) + f^{(\alpha)}(t)(t - c) = (\mu(t))^{1 - \alpha} f(\sigma(t)) + f^{(\alpha)}(t)(t - c)
\]
\[
= (\mu(t))^{1 - \alpha} (\sigma(t) - c)^m + (\mu(t))^{1 - \alpha} (t - c) \sum_{\nu = 0}^{m - 1} (\sigma(t) - c)^\nu (t - c)^{m - 1 - \nu}
\]
\[
= (\mu(t))^{1 - \alpha} \left[ (\sigma(t) - c)^m + \sum_{\nu = 0}^{m - 1} (\sigma(t) - c)^\nu (t - c)^{m - 1 - \nu} \right]
\]
\[
= (\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m} (\sigma(t) - c)^\nu (t - c)^{m - \nu}.
\]
Hence, by mathematical induction, part (i) holds. For \(g(t) = \frac{1}{(t - c)^m} = \frac{1}{f(t)}\), we apply Theorem \([15]\) (iv) to obtain
\[
g^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))} = -(\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m - 1} \frac{(\sigma(t) - c)^\nu (t - c)^{m - 1 - \nu}}{(t - c)^m (\sigma(t) - c)^m}
\]
\[
= -(\mu(t))^{1 - \alpha} \sum_{\nu = 0}^{m - 1} \frac{1}{(t - c)^{\nu + 1} (\sigma(t) - c)^{m - \nu}},
\]
provided \((t - c) (\sigma(t) - c) \neq 0\). \(\square\)
Let us illustrate Theorem 19 in special cases.

**Example 17.** Let \( \alpha \in [0,1] \).

(i) If \( f(t) = t^2 \), then \( f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}[\sigma(t) + t] \).

(ii) If \( f(t) = t^3 \), then \( f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}[t^2 + t\sigma(t) + (\sigma(t))^2] \).

(iii) If \( f(t) = \frac{1}{t} \), then \( f^{(\alpha)}(t) = -(\mu(t))^{1-\alpha} \).

From the results already obtained, it is not difficult to see that the fractional derivative does not satisfy a chain rule like \((f \circ g)^{\alpha}(t) = f^{\alpha}(g(t))g^{\alpha}(t)\):

**Example 18.** Let \( \alpha \in [0,1] \). Consider \( f(t) = t^2 \) and \( g(t) = 2t \). Then,

\[
(f \circ g)^{\alpha}(t) = (4t^2)^{\alpha} = 4(\mu(t))^{1-\alpha}(\sigma(t) + t) \tag{2}
\]

while

\[
f^{\alpha}(g(t))g^{\alpha}(t) = (\mu(2t))^{1-\alpha}(\sigma(2t) + 2t)(\mu(t))^{1-\alpha} \tag{3}
\]

and, for example for \( T = \mathbb{Z} \), it is easy to see that \((f \circ g)^{\alpha}(t) \neq f^{\alpha}(g(t))g^{\alpha}(t)\).

Note that when \( \alpha = 1 \) and \( T = \mathbb{R} \) our derivative \( f^{(\alpha)} \) reduces to the standard derivative \( f' \) and, in this case, both expressions (2) and (3) give \( 8t \), as expected. In the fractional case \( \alpha \in [0,1] \) we are able to prove the following result, valid for an arbitrary time scale \( T \).

**Theorem 19 (Chain rule).** Let \( \alpha \in [0,1] \). Assume \( g : \mathbb{R} \to \mathbb{R} \) is continuous, \( g : T \to \mathbb{R} \) is fractional differentiable of order \( \alpha \) at \( t \in T^\omega \), and \( f : \mathbb{R} \to \mathbb{R} \) is continuously differentiable. Then there exists \( c \) in the real interval \([t, \sigma(t)]\) with

\[
(f \circ g)^{\alpha}(t) = f'(g(c))g^{\alpha}(t). \tag{4}
\]

**Proof.** Let \( t \in T^\omega \). First we consider \( t \) to be right-scattered. In this case

\[
(f \circ g)^{\alpha}(t) = \frac{f(g(\sigma(t))) - f(g(t))}{(\mu(t))^{\alpha}}.
\]

If \( g(\sigma(t)) = g(t) \), then we get \((f \circ g)^{\alpha}(t) = 0 \) and \( g^{\alpha}(t) = 0 \). Therefore, (4) holds for any \( c \) in the real interval \([t, \sigma(t)]\) and we can assume \( g(\sigma(t)) \neq g(t) \). By the mean value theorem,

\[
(f \circ g)^{\alpha}(t) = \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \cdot \frac{g(\sigma(t)) - g(t)}{(\mu(t))^{\alpha}} = f'(\xi)g^{\alpha}(t),
\]

where \( \xi \) is between \( g(t) \) and \( g(\sigma(t)) \). Since \( g : \mathbb{R} \to \mathbb{R} \) is continuous, there is a \( c \in [t, \sigma(t)] \) such that \( g(c) = \xi \), which gives us the desired result. Now consider the case when \( t \) is right-dense. In this case

\[
(f \circ g)^{\alpha}(t) = \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \cdot \frac{g(t) - g(s)}{(t-s)^{\alpha}} = \lim_{s \to t} \left\{ f'(\xi_s) \cdot \frac{g(t) - g(s)}{(t-s)^{\alpha}} \right\}
\]

by the mean value theorem, where \( \xi_s \) is between \( g(s) \) and \( g(t) \). By the continuity of \( g \) we get that \( \lim_{s \to t} \xi_s = g(t) \), which gives us the desired result. \( \Box \)
Example 20. Let $\mathbb{T} = \mathbb{Z}$, for which $\sigma(t) = t + 1$ and $\mu(t) \equiv 1$, and consider the same functions of Example 18: $f(t) = t^2$ and $g(t) = 2t$. We can find directly the value $c$, guaranteed by Theorem 19 in the interval $[4, \sigma(4)] = [4, 5]$, so that

$$(f \circ g)^{(\alpha)}(4) = f'(g(c))g^{(\alpha)}(4). \quad (5)$$

From (2), it follows that $(f \circ g)^{(\alpha)}(4) = 36$. Because $g^{(\alpha)}(4) = 2$ and $f'(g(c)) = 4c$, equality (5) simplifies to $36 = 8c$, and so $c = \frac{9}{2}$.

We end Section 3.1 explaining how to compute fractional derivatives of higher-order. As usual, we define the derivative of order zero as the identity operator: $f^{(0)} = f$.

**Definition 21.** Let $\beta$ be a nonnegative real number. We define the fractional derivative of $f$ of order $\beta$ by

$$f^{(\beta)} := (f^{\Delta N})^{(\alpha)},$$

where $N := \lfloor \beta \rfloor$ (that is, $N$ is the integer part of $\beta$) and $\alpha := \beta - N$.

Note that the $\alpha$ of Definition 21 is in the interval $[0, 1]$. We illustrate Definition 21 with some examples.

**Example 22.** If $f(t) = c$ for all $t \in \mathbb{T}$, $c$ a constant, then $f^{(\beta)} \equiv 0$ for any $\beta \in \mathbb{R}_+^*$.

**Example 23.** Let $f(t) = t^2$, $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $\beta = 1.3$. Then, by Definition 21, we have $f^{(1.3)} = (f^{\Delta})^{(0.3)}$. It follows from $\sigma(t) = t + h$ that $f^{(1.3)}(t) = (2t + h)^{0.3}$. Proposition 4 and Theorem 13 (i) and (ii) allow us to write that $f^{(1.3)}(t) = 2(t)^{0.3}$. We conclude from Proposition 10 with $\mu(t) \equiv h$ that $f^{(1.3)}(t) = 2h^{0.7}$.

### 3.2 Fractional Integration

The two major ingredients of any calculus are differentiation and integration. Now we introduce the fractional integral on time scales.

**Definition 24.** Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite fractional integral of $f$ of order $\beta$, $0 \leq \beta \leq 1$, by

$$\int f(t)\Delta^\beta t := \left(\int f(t)\Delta t\right)^{(1-\beta)},$$

where $\int f(t)\Delta t$ is the usual indefinite integral of time scales [7].

**Remark 25.** It follows from Definition 24 that $\int f(t)\Delta^1 t = \int f(t)\Delta t$ and $\int f(t)\Delta^0 t = f(t)$.

**Definition 26.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Let

$$F^\beta(t) = \int f(t)\Delta^\beta t$$

denote the indefinite fractional integral of $f$ of order $\beta$ with $0 \leq \beta \leq 1$. We define the Cauchy fractional integral by

$$\int_a^b f(t)\Delta^\beta t := F^\beta(t)|_a^b = F^\beta(b) - F^\beta(a), \quad a, b \in \mathbb{T}.$$
(ii) \( \int_a^b (\xi f)(t) \Delta^\beta t = \xi \int_a^b f(t) \Delta^\beta t; \)

(iii) \( \int_a^b f(t) \Delta^\beta t = - \int_b^a f(t) \Delta^\beta t; \)

(iv) \( \int_a^b f(t) \Delta^\beta t = \int_a^c f(t) \Delta^\beta t + \int_c^b f(t) \Delta^\beta t; \)

(v) \( \int_a^b f(t) \Delta^\beta t = 0. \)

Proof. The equalities follow from Definition 24 and Definition 26 analogous properties of the delta integral of time scales, and the properties of Section 3.1 for the fractional derivative on time scales.

(i) From Definition 26

\[
\int_a^b (f + g)(t) \Delta^\beta t = \int_a^b (f(t) + g(t)) \Delta^\beta t
\]

and, from Definition 24

\[
\int_a^b (f + g)(t) \Delta^\beta t = \left( \int_a^b (f(t) + g(t)) \Delta t \right) \Delta^\beta t
\]

It follows from the properties of the delta integral and Theorem 15 (i) that

\[
\int_a^b (f + g)(t) \Delta^\beta t = \left( \int_a^b f(t) \Delta t \right)^{(1-\beta)} + \left( \int_a^b g(t) \Delta t \right)^{(1-\beta)}
\]

Using again Definition 24 and Definition 26 we arrive to the intended relation:

\[
\int_a^b (f + g)(t) \Delta^\beta t = \int_a^b f(t) \Delta^\beta t + \int_a^b g(t) \Delta^\beta t.
\]

(ii) From Definition 26 and Definition 24 one has

\[
\int_a^b (\xi f)(t) \Delta^\beta t = \int_a^b (\xi f)(t) \Delta^\beta t = \left( \int_a^b (\xi f)(t) \Delta t \right)^{(1-\beta)}
\]

It follows from the properties of the delta integral and Theorem 15 (ii) that

\[
\int_a^b (\xi f)(t) \Delta^\beta t = \xi \left( \int_a^b f(t) \Delta t \right)^{(1-\beta)}
\]

We conclude the proof of (ii) by using again Definition 24 and Definition 26

\[
\int_a^b (\xi f)(t) \Delta^\beta t = \xi \int_a^b f(t) \Delta^\beta t = \xi \int_a^b f(t) \Delta^\beta t = \xi (F^\beta(b) - F^\beta(a))
\]

\[
= \xi \int_a^b f(t) \Delta^\beta t.
\]

The last three properties are direct consequences of Definition 26

(iii)

\[
\int_a^b f(t) \Delta^\beta t = F^\beta(b) - F^\beta(a) = - (F^\beta(a) - F^\beta(b))
\]

\[
= - \int_b^a f(t) \Delta^\beta t.
\]
\[(iv)\]
\[
\int_a^b f(t) \Delta^\beta t = F^\beta(b) - F^\beta(a) = F^\beta(c) - F^\beta(a) + F^\beta(b) - F^\beta(c)
\]
\[
= \int_a^c f(t) \Delta^\beta t + \int_c^b f(t) \Delta^\beta t.
\]
\[(v)\]
\[
\int_a^b f(t) \Delta^\beta t = F^\beta(a) - F^\beta(b) = 0.
\]

The proof is complete.

We end with a simple example of a discrete fractional integral.

**Example 28.** Let \(\mathbb{T} = \mathbb{Z}, 0 \leq \beta \leq 1\), and \(f(t) = t\). Using the fact that in this case
\[
\int t \Delta t = \frac{t^2}{2} + C
\]
with \(C\) a constant, we have
\[
\int_1^{10} t \Delta^\beta t = \left( \int t \Delta t \right)^{(1-\beta)} \bigg|_1^{10} = \left( \frac{t^2}{2} + C \right)^{(1-\beta)} \bigg|_1^{10}.
\]

It follows from Example 17 (i) with \(\mu(t) \equiv 1\), Theorem 15 (i) and (ii) and Proposition 7 that
\[
\int_1^{10} t \Delta^\beta t = \frac{1}{2} (2t + 1)^{(1-\beta)} \bigg|_1^{10} = \frac{21}{2} - \frac{3}{2} = 9.
\]

4 Conclusion

Fractional calculus, that is, the study of differentiation and integration of noninteger order, is here extended, via the recent and powerful calculus on time scales, to include, in a single theory, the discrete fractional difference calculus and the local continuous fractional differential calculus.

We have only introduced some fundamental concepts and proved some basic properties, and much remains to be done in order to develop the theory here initiated: to prove concatenation properties of derivatives and integrals, to consider partial fractional operators on time scales, to introduce a suitable fractional exponential on time scales, to study boundary value problems for fractional differential equations on time scales, to investigate the usefulness of the new fractional calculus in applications to real world problems where the time scale is partially continuous and partially discrete with a time-varying graininess function, etc. We would like also to mention that it is possible to develop fractional calculi on time scales in other different directions than the one considered here. For instance, instead of following the delta approach we have adopted, one can develop a nabla [4, 32], a diamond [30, 34], or a symmetric [18, 19] time scale fractional calculus. These and other questions will be subject of future research.

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