Concentration of measure for classical Lie groups

Sergio L. Cacciatori\textsuperscript{1,2} · Pietro Ursino\textsuperscript{3}

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Abstract
We study the concentration of measure in metric-measurable (mm)-spaces. We define the notion of concentration locus of a flag sequence of metric-measurable (mm)-spaces. Some examples of infinite group action on an infinite dimensional compact and non-compact manifold show the role played by the trajectory of concentration locus. We also provide some applications in physics, which emphasize the role of concentration of measure in gravitational effects.

Keywords Concentration of measure · Compact Lie groups · Riemann geometry · Metric measurable spaces · Topological dynamics

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1 Introduction
Our work has two primary goals. The first consists of introducing a tool, the \textit{concentration loci}, which allows determining whether an infinite sequence of open sets yields a concentration of measure into a significantly small object, then in Proposition 2.7...

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\begin{itemize}
  \item Pietro Ursino \textsuperscript{3}
pietro.ursino@uninsubria.it
  \item Sergio L. Cacciatori
sergio.cacciatori@uninsubria.it
\end{itemize}

\textsuperscript{1} Department of Science and High Technology, Università dell’Insubria, Via Valleggio 11, 22100 Como, Italy
\textsuperscript{2} INFN sezione di Milano, via Celoria 16, 20133 Milano, Italy
\textsuperscript{3} Department of Mathematics and Informatics, Università degli Studi di Catania, Viale Andrea Doria 6, 95125 Catania, Italy
we establish a connection between the notion of concentration locus and the notion of concentration in the sense of Gromov [8]. Next we study the concentration inherited by Lie group actions on compact and non-compact infinite dimensional spaces.

The second goal consists of investigating the geometric trajectory of a concentration locus while concentrating the measure by increasing the dimension. The concentration of measure, especially in relation to an increase in dimensions, has an interesting physical counterpart, which we shall discuss in two physical applications.

In physics, after Einstein, spacetime is described by a four-dimensional manifold with a dynamical Lorentzian geometry. However, at present, there is no reason to think that such a description should continue to work at sub-Planckian scales. On the contrary, quantum effects are expected to become dominant at these scales, even though any theory proposed so far is mainly at a speculative level. Quantum corrections are expected, for example, to solve the problem of curvature singularities in black hole physics, as predicted by singularity theorems. There is no proof indicating exactly what happens below such scales. For example, regarding curvature singularities of black holes, we may conjecture that they consist of an infinite energy density concentrating into a point. Remaining at a speculative level, it may be useful to think of the dynamical process of such a collapse: for instance, one cannot exclude that as soon as in a very small region the energy density grows excessively, the number of spatial dimensions increases more and more. This would in turn lead to a “spreading of directions”, yield a lowering of the energy density, and, maybe, smooth out the singularity, since, as it is well known, the Lebesgue measure concentrates on the boundaries when dimensions increase.

Further, the concentration of energy in higher dimensions could lead to new kinds of forces induced from the “infinite” dimensional spacetime to the usually visible finite-dimensional world, and could then appear as a mysterious force in lower dimensions. We conjecture that one of these forces is indeed due to the phenomenon of measure concentration. Of course, all our arguments are merely speculative, and to substantiate them we need to improve our understanding of the phenomenon of measure concentration, which does not depend only on the varying of dimensions but also on topology.

In mathematics, the first illustration of the concentration of measure is an elementary calculus performed on spheres by Lévy [15], where it is shown that by increasing the dimensions of spheres the measure concentrates along the equator, an \((n - 1)\)-sphere. The work of Gromov and Milman [9] has given a strong impetus to deepen this area of research. The reason must be sought in the fact that, by isoperimetric inequalities, they showed that this phenomenon is uniform for compact manifolds and depends only on the rate of curvature. They also exhibit, using fixed point arguments, examples of some natural fibrations, which have continuous sections but have no uniformly continuous sections (see also [20] for connections between extreme amenability and the existence of global continuous sections).

One of the most important applications of concentration of measure is the existence of fixed points on compacta. A topological group \(G\) has a very strong fixed point property (i.e., extreme amenability), if any continuous \(G\)-action on a compact set has a fixed point (i.e., there exists a point \(x\) such that \(g \cdot x = x\) for all \(g \in G\)).
Extreme amenability of automorphism groups of countable structures has been related to Ramsey property, a pure combinatorial phenomenon, in the exhaustive treatise of Kechris, Pestov and Todorcević [12]. Concentration of measure and Milman–Gromov approach are instead, in a sense, a Ramsey property counterpart for groups that are smooth manifolds [9].

The Gromov and Milman results allow to single out a fixed point starting from any part of the manifold where the group acts. Compactness guarantees that fixed points exist but does not localize them and does not show how they become fixed along the incremental dimension process.

We have decided to give a notion (Definition 2.2) of concentration which does not require uniformity as that one proposed in [9]. Namely, we have emphasized the phenomenon of concentration of a specific sequence of sets, which contains a concentration locus, instead of saying that all sequences of sets concentrate. However, we show that, under suitable conditions, our definition implies uniform concentration.

In Sect. 3, two examples show how the measure concentrates from a purely geometrical point of view. In particular, we single out that the direction of concentration is essential in the non-compact case (see Proposition 3.6). Remarkably, we find two different directions. Following the first one, the measure concentrates on a circle, while following the second one, it shrinks to a point. Therefore, even in non-compact cases, we may have fixed points but we do not have any uniformity. Indeed, following the wrong direction, we may not find any fixed point. In this last case, we have a concentration locus, where the orbits are confined, without fixed point property. This puts the accent on the fact that extreme amenability and concentration are different phenomena. Moreover, we show, under an asymmetric action, how the concentration phenomenon could appear.

In all cases, it is clear that the concentration on the compact manifold depends on how the concentration locus of the group is mapped to the compact set under the action of the group.

For our considerations, the relevant extreme amenable groups are only those obtained as limits of Lévy families [5, 6].

With regard to the connection between the concentration of measure and extreme amenability, the starting point is [9, Theorem 7.1] which proves extreme amenability of the infinite unitary groups (observe that originally this proof requires an equicontinuous action, later this additional hypothesis has been shown to be unnecessary by Eli Glasner).

It is known that, if one works with the Hilbert–Schmidt topology and considers canonical embeddings, then \( U(\infty) = SU(\infty) \) is extremely amenable. Nevertheless, if one considers finer topologies, as the inductive limit topology (notice that it is not at all obvious that the limit of a sequence of Lie groups produces Lie groups, see [7, 18]), then \( SU(\infty) \subset U(\infty) \), \( U(\infty)/SU(\infty) \equiv S^1 \) and, obviously, the action of \( U(\infty) \) on such \( U(1) \) cannot have fixed points, this implies that \( U(\infty) \) cannot be extremely amenable (with the finer topology).

By the cited Theorem 7.1 with the inductive limit topology \( (U(\infty), U(\infty)) \) cannot be Lévy since \( U(\infty) \) is not extremely amenable. Observe that, strictly following Gromov–Milman approach, the notion of Lévy family does not apply to \( U(\infty) \) since it is not a metric space. However, direct limits of (paracompact) finite-dimensional
manifolds with closed embeddings between them are paracompact, hence normal and, therefore completely regular (see [10]). Indeed, arbitrary direct limits of paracompact finite-dimensional smooth manifolds with smooth injective immersions are smoothly paracompact (have smooth partitions of unity subordinate to any open cover) and hence normal (see [7]), which in turn implies that they are uniform spaces. Pestov generalizes the notion of Lévy family to uniform spaces [19], so our example perfectly makes sense. So, \((U(\infty), U(\infty))\) is an example of a group which has a concentration locus \((S^1)\) which is not a point. Again, this stresses the difference between the fixed point property and measure concentration. The paper is organized as follows.

- In Sect. 2 we outline basic definitions and give the first results.
- In Sect. 3 we show examples of infinite group action on compact and non-compact infinite dimensional manifolds. In the compact case, we prove the uniqueness of the fixed point and the exact position.
- In Sect. 4 we make two physical applications of the above results.

2 Background and first results

In the first instance, we give the definition of Lévy family as it was given by Gromov and Milman in [9].

**Definition 2.1** For a set \(A\) in a metric space \(X\) we denote by \(N^\varepsilon(A)\), \(\varepsilon > 0\), its \(\varepsilon\)-neighborhood. Consider a family \((X_n, \mu_n)\) with \(n = 1, 2, \ldots\) of metric spaces \(X_n\) with normalized Borel measures \(\mu_n\). We call such a family Lévy if for any sequence of Borel sets \(A_n \subset X_n, n = 1, 2, \ldots\), such that \(\lim \inf_{n \to \infty} \mu_n(A_n) > 0\), and for every \(\varepsilon > 0\), we have \(\lim_{n \to \infty} \mu_n(N^\varepsilon(A)) = 1\).

Following the considerations in the introduction we will adopt the following definitions:

**Definition 2.2** Let \(\{X_n, \mu_n\}_{n \in \mathbb{N}}\) be a family of metric spaces with metrics \(g_n\), and \(\mu_n\) measures w.r.t. which open sets are measurable of non-vanishing measure. Assume the measures to be normalized, \(\mu_n(X_n) = 1\). Let \(\{S_n\}_{n \in \mathbb{N}}\) be a family of proper closed subsets, \(S_n \subset X_n\). Fix a sequence \(\{\varepsilon_n\}_{n \in \mathbb{N}}\) such that \(\varepsilon_n > 0\), \(\lim_{n \to \infty} \varepsilon_n = 0\), and let \(\{U_{\varepsilon_n}^n\}_{n \in \mathbb{N}}\) be the sequence of tubular neighbourhoods of \(S_n\) of radius \(\varepsilon_n\). We say that the family \(\{S_n\}\) is a concentration locus if

\[
\lim_{n \to \infty} \mu_n(X_n - U_{\varepsilon_n}^n) = 0.
\]

Moreover, if such a sequence \(\varepsilon_n\) converges to 0 at rate \(k\) (so that \(\lim_{n \to \infty} n^k \varepsilon_n = c\) for some constant \(c\)), we say that the family \(\{S_n\}\) is a concentration locus at least at rate \(k\).

Notice that in general, we may have \(\mu_n(S_n) = 0\). Moreover, with these definitions we do not need any notion of convergence of \(S_n\) to a final subset. \(S_n\) just gives a “direction of concentration”. Also, our definitions do not pretend to provide any optimality in concentration: it can happen that for a given sequence of \(S_n\) there exists a sequence
of proper subsets \( S'_n \subset S_n \) on which we still have concentration. A special example of Definition 2.2 consists of the case \( X_n = X \) and \( S_n = S \) where \( X \) is a metric space, \( \{\mu_n\}_{n \in \mathbb{N}} \) a sequence of normalized measures on \( X \), compatible with the metric of \( X \) and \( S \subset X \) a proper closed subset of \( X \).

**Remark 2.3** It is apparent that the above definition has some similarities with the usual definition of concentration. However, they are different, since our definition is centered on a specific sequence of sets, the concentration locus, while the standard one is on generic open sets of positive measure. Even the usual notion of concentration has a rate of concentration, represented by the concentration function (see for example [14] for definition and an extensive discussion). Again, this notion is based on generic open sets of positive measure instead of a specific sequence of closed sets. Roughly speaking the standard definition checks if a sequence concentrates and, the concentration function, how it concentrates, our definition says the same thing but specifies where it concentrates.

An interesting rate of concentration is determined by all sequences \( \varepsilon_n \) which run to 0 with order less than \( 1/\sqrt{n} \) (the corresponding concentration function is called “normal concentration” [14]).

We want now to provide a point of contact between the notion of concentration locus and the notion of a sequence of mm-spaces \( (X_n)_{n \in \mathbb{N}} \) that concentrates to an mm-space \( X \). As far as we know, the latter notion has been originally described in Gromov’s Green Book [8, Chapter 3 12]. Actually, for the definition of observable distance between two mm-spaces \( d_{\text{conc}}(X, Y) \) and for all notations and definitions needed for the definition of \( d_{\text{conc}} \), we refer to a more concise presentation in [22] or [21].

In any case, for the sake of completeness of the present paper, we outline a brief resume of all useful definitions and notations.

Given a set \( X \), we denote by \( l^\infty(X) \) the unital Banach algebra of all bounded real-valued functions on \( X \) equipped with the supremum norm. Let \( X \) be a topological space. If the topology of \( X \) is generated by a metric \( d \), then we call \( d \) a compatible metric on \( X \).

We will denote by \( C(X) \) the set of all continuous real-valued functions on \( X \), and we let \( CB(X) = l^\infty(X) \cap C(X) \). Let us denote by \( \mathcal{B}(X) \) the Borel \( \sigma \)-algebra of \( X \) and by \( P(X) \) the set of all Borel probability measures on \( X \). The weak topology on \( P(X) \) is defined to be the initial topology on \( P(X) \) generated by the maps of the form \( P(X) \to \mathbb{R}, \mu \to \int f d\mu \) where \( f \in CB(X) \).

The support of a measure \( \mu \in P(X) \) is defined as

\[
\text{spt } \mu = \{ x \in X \mid \forall U \subseteq X \text{ open: } x \in U \Rightarrow \mu(U) > 0 \}.
\]

Given \( \mu \in P(X) \) and a Borel subset \( B \subseteq X \) with \( \mu(B) = 1 \), we let \( \mu|_B := \mu|_{\mathcal{B}(B) \in P(B)} \). The push-forward of a measure \( \mu \in P(X) \) along a Borel map \( f : X \to Y \) into another topological space \( Y \) is defined to be \# \( f \mu : \mathcal{B}(Y) \to [0, 1], B \mapsto \mu(f^{-1}(B)) \).
Furthermore, let us note that each \( \mu \in P(X) \) gives rise to a pseudo-metric \( \text{me}_\mu \) on the set of all Borel measurable real-valued functions \( X \) defined by

\[
\text{me}_\mu(f, g) := \inf \{ \epsilon > 0 \mid \mu(\{ x \in X \mid |f(x) - g(x)| > \epsilon \}) \leq \epsilon \}
\]

for any two Borel functions \( f, g : X \to \mathbb{R} \).

Let \((X, d)\) be a pseudo-metric space. Given a subset \( A \subseteq X \), we abbreviate \( d \mid A := d \mid A \times A \) and define \( \text{diam}(A, d) := \sup\{d(x, y) \mid x, y \in A\} \). For \( x \in A \subseteq X \) and \( \epsilon > 0 \) we let

\[
B_d(x, \epsilon) := \{ y \in X \mid d(x, y) < \epsilon \}, \quad B_d(A, \epsilon) := \{ y \in X \mid \exists a \in A; d(x, y) < \epsilon \}.
\]

Then the Hausdorff distance between any two subsets \( A, B \subseteq X \) is given by

\[
d_H(A, B) := \inf \{ \epsilon > 0 \mid B \subseteq B_d(A, \epsilon), A \subseteq B_d(B, \epsilon) \}.
\]

For \( l, r \geq 0 \), we denote by \( \text{Lip}_l(X, d) \) the set of all \( l \)-Lipschitz real-valued functions on \((X, d)\), and we define

\[
\text{Lip}_l^\infty(X, d) := \text{Lip}_l(X, d) \cap l^\infty(X),
\]

\[
\text{Lip}_l^r(X, d) := \{ f \in \text{Lip}_l(X, d) \mid \| f \|_\infty \leq r \}.
\]

Moreover, we let \( \text{Lip}(X, d) := \bigcup \{ \text{Lip}_l(X, d) \mid l \geq 0 \} \) and \( \text{Lip}(X, d)^\infty := \text{Lip}(X, d) \cap l^\infty(X) \).

The Wasserstein distance \( W_1(\mu, \nu) \) whenever \( X, d \) is a separable metric space, is a compatible metric for the weak topology on \( P(X) \) defined by

\[
W_1(\mu, \nu) := \sup_{f \in \text{Lip}_1(X, d)} \left| \int f \, d\mu - \int f \, d\nu \right| \quad (\mu, \nu \in P(X)).
\]

**Definition 2.4** (Gromov–Milman [8]) A space with a metric \( g \) and a measure \( \mu \), or an **mm-space**, is a triple \((X, \mu, g)\), consisting of a set \( X \), a metric \( g \) on \( X \) and a probability Borel measure such that \((X, g)\) is a separable complete metric space.

Moreover, an mm-space \((X, \mu, d)\) is called **compact** if \((X, d)\) is compact, and **fully supported** if \( \text{spt} \mu = X \). Henceforth, we will denote by \( \lambda \) the Lebesgue measure on \([0, 1]\).

A parametrization of an mm-space \((X, \mu, d)\) is a Borel measurable map \( \phi : [0, 1) \to X \) such that \( \# \phi \lambda = \mu \). It is well known that any mm-space admits a parametrization (see, e.g. [22]).

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1 Different names appearing in the literature include Monge–Kontorovich distance, bounded Lipschitz distance, mass transportation distance, and Fortet–Mourier distance [23].
Definition 2.5 (Gromov–Milman [8, 22]) In the set of isomorphism classes of mm-spaces we can define the following distance:

$$d_{\text{conc}}(X, Y) := \inf \left\{ (\text{me}_{\lambda})_H(\text{Lip}_1(X) \circ \phi, \text{Lip}_1(Y) \circ \psi) \right\} \left| \phi \text{ is a parametrization of } X \right. \left. \right| \psi \text{ is a parametrization of } Y \}.$$

Two mm-spaces $X$ and $Y$ are isomorphic if there exists an isomorphism between mm-spaces $(X, \mu, d)$, $(Y, v, d')$, i.e an isometry

$$f : (\text{spt } \mu, d \mid X) \rightarrow (\text{spt } v, d' \mid Y)$$

such that $\#f(\mu \mid \text{spt } \mu) = v \mid \text{spt } v$.

A sequence of mm-spaces $(X_n, \mu_n, g_n)$ is said to concentrate to an mm-space $(X, \mu, g)$ if

$$\lim_{n \to \infty} d_{\text{conc}}(X_n, X) = 0.$$

In the following, we shall use optimal transports $p$ from $(X, \mu, d)$ to $S$ such that $d(x, p(x)) = d(x, S)$, we denote such transports as “transports to the border”.

For a natural example of optimal transports to the border $p$ from $(X, \mu, d)$ to $(S, v)$, the following remark will be useful.

Remark 2.6 A natural example of optimal transport to the border is the projection function. Let $(M, \mu, g)$ be a Riemann manifold and $C \subseteq M$ the projection map $\text{proj}_C : M \rightarrow C$ defined as a map such that $g(x, \text{proj}_C(x)) = g(x, C)$. In this case $\text{proj}_C$ is a transport to the border from $\text{spt } \mu$ to $\text{spt } \# \text{proj}_C \mu$. For the existence and, more in general, a theory of distance functions in $\mathbb{R}^n$ context, see e.g. [4]; for an extension of this theory to a Riemann context, see e.g. [17].

Let $(X_n, \mu_n, g_n)$ and $(X, \mu, g)$ be mm-spaces, $\bigcup X_n$ dense in $X$, $g_n = g \mid X_n$, $W_1(\mu_n, \mu)$ converge to 0 in $P(X)$, then we say that $(X_n, \mu_n, g_n)$ converges to $(X, \mu, g)$ as mm-spaces.

An example for $(X_n, \mu_n, g_n)$, $(X, \mu, g)$ mm-spaces, taken from [19], is given by the sequence of unitary groups $U_i$, with the natural embeddings. The limit metric is determined by the Hilbert–Schmidt norm $\| \cdot \|_2$, defined in a coordinate-free fashion by $\| A \|_2 = \sqrt{\text{tr}(A^*A)}$ for $A \in \text{Mat}(n \times n)$. This closes to the set of compact operators $K$ such that the eigenvalues of $\sqrt{K^*K}$ are the coordinates. If this sequence is in $\ell^2$, then $K$ is said to be of Schatten 2-class. The operators of Schatten 2-class are exactly those for which the Hilbert–Schmidt norm, as given by the above equation, makes sense.

We denote by $U(\infty)_2$ the completion of all unitary operators of the form $I + K$, where $I$ is the identity and $K$ as above. The Hilbert–Schmidt metric on the group $U(\infty)_2$, given by the rule $d_{\text{HS}}(u, v) = \| u - v \|_2$, is (well-defined and) bi-invariant. With this metric, $U(\infty)_2$ is a Polish group. For every $n \in N$, the unitary group $U(n)$ of rank $n$ belongs into $U(\infty)_2$ in the usual way: by enlarging the matrix with 1 along the diagonal. The union $\bigcup U(n)$ is dense in $U(\infty)_2$. $U(\infty)_2$ is the completion of the
abstract group $\bigcup U(n)$ with regard to the Hilbert–Schmidt metric. It is well known that the limit space $U(\infty)_2$ is extremely amenable since the support of the measures collapses to one point (see e.g. [9]).

**Proposition 2.7** Let $(X_n, \mu_n, g_n)$ be a sequence of manifolds with bounded curvature which converges to $(X, \mu, g)$, $(S_n, \nu_n)$ a concentration locus, with measures $\nu_n$, for $(X_n, \mu_n, g_n)$ and, for each $n$, $p_n'$ be a continuous optimal transport to the border from $\mu_n$ to $\nu_n$ such that $W_1(\nu_n, \mu_n)$, defined in $P(X)$, converges to 0. If $(S_n, \nu_n, g_n|_{S_n})$ converges to a manifold $(S, \nu, g)$ with bounded curvature, then $(X_n, \mu_n', g_n)$ concentrates to $(S, g, \nu)$ (i.e. $g = g | S$).

**Proof** This proof does not make any use of Lévy’s family or either condition on the first eigenvalue [9].

We use a characterization for concentration from [22, Corollary 5.36] or [21, Theorem 2.2].

Let $p_n'$ be the continuous (in particular Borel) optimal transport of $X_n$ on $S_n$, $i_n$ the isometric injection of $S_n$ in $S$ and $p_n = i_n \circ p_n'$. By our hypothesis # $p_n \mu_n = \nu_n$.

$$W_2(\mu_n, \nu) \leq W_1(\mu_n, \nu_n) + W_1(\nu_n, \nu) \leq \epsilon$$
(by hypothesis for large enough $n$).

By [22, Lemma 9.12], $\mu_n$ weakly converges to $\nu$ and this verifies condition 1 of Corollary 5.36 of Shioya.

Regarding condition 2 of Corollary 5.36 of Shioya [22], it is sufficient to prove that for a given $f' \in \text{Lip}_1(S, g)$ there exists for a sufficiently large $n$ and $f''_n \in \text{Lip}_1(X_n)$ such that $\text{me}_{\mu_n}(f''_n, f \circ p_n) < \epsilon$. On the other hand, for a given $f''_n \in \text{Lip}_1(X_n, g)$, we need to prove that there exists an $f \in \text{Lip}_1(S, g)$ such that

$$\text{me}_{\mu_n}(f''_n, f \circ p_n) < \epsilon.$$

First we consider $\tilde{f}'_n = f|_{S_n}$ which is in $\text{Lip}_1(S_n)$. By [13] we can extend $\tilde{f}'_n$ to an $f''_n \in \text{Lip}_1(X_n)$. Let us consider open sets $U^{\epsilon}_n$ around $S^n$. Choose an $n_1$ such that for all $n > n_1$, $\mu(X_n \setminus U^{\epsilon}_n) < \epsilon$. Observe that $f \circ p_n$ inside $S^n$ coincides with $f''_n$ since optimal transport does not move points inside the destination support. Therefore we have to check the measure of $\{x \text{ such that } |f''_n(x) - f \circ p_n| \geq \epsilon\}$ just outside of $S_n$. In $X_n \setminus U^{\epsilon}_n$ the measure is less than $\epsilon$. While inside $U^{\epsilon}_n \setminus S_n$ for each point $x$, $g_n(x, p_n'(x)) = g_n(x, S_n)$ which is less than $\epsilon_n$. We can choose $n > n_1$ in such a way $\epsilon_n < \epsilon$, since $f''_n \in \text{Lip}_1(X_n)$ we get $|f''_n(x) - f \circ p_n(x)| < \epsilon$.

On the other side, consider a restriction $\tilde{f}'_n$ of $f''_n \in \text{Lip}_1(X_n, g)$ to $S_n$, then again by [13] extend $\tilde{f}'_n$ by an $f \in \text{Lip}_1(S)$. Using a similar argument as above, we get $|f''_n(x) - f \circ p_n(x)| < \epsilon$ inside $U^{\epsilon}_n \setminus S_n$, $\mu(X_n \setminus U^{\epsilon}_n) < \epsilon$ and $x \in S_n$ left fixed by the optimal transport.

In an on-going paper, we proved that a large class of Riemann manifolds\(^2\) satisfies the conditions for applying Proposition 2.7. In these cases, concentration locus property strongly depends on curvatures conditions of $X_n$ and $S_n$.

\(^2\) For example, $SU(n)$ belongs to this class.
Remark 2.8 Schneider proved the following Pestov’s conjecture [19, Conjecture 7.4.26]:

If $G$ is a metrizable topological group, equipped with a compatible right-invariant metric $d$, and $(K_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subgroups such that

- The union $\bigcup_{n \in \mathbb{N}} K_n$ is everywhere dense in $G$, and
- $(K_n, d |_{K_n}, \mu_n)_{n \in \mathbb{N}}$ concentrates to a fully supported, compact mm-space $(X, d_X, \mu_X)$, where $\mu_n$ denotes the normalized Haar measure on $(K_n)_{n \in \mathbb{N}}$,

then the topological space $X$ supports the structure of a $G$-flow, with respect to which it admits a morphism to every $G$-flow.

As a consequence, whenever $X_n$ are compact topological groups and the limit of a concentration locus is minimal, by Schneider’s result, the limit space $S$ is the Universal Minimal flow (provided that they satisfy the further property A of [21, Theorem 1], since they are topological groups). In case $S$ should be a point, the sequence is a Lévy family, hence $X$ is extremely amenable (see e.g. [21, 22]). Moreover, under conditions of Theorem 2.7, if a concentration locus concentrates to a point (e.g. in the case of Hilbert–Schmidt distance) then $X_n$ is a Lévy family.

In view of the observations made above, the concentration loci could be a valuable tool in order to explicitly describe new infinite universal minimal flows.

3 Concentration of measure on infinite dimensional compact sets

The extreme amenability of a group is related to the concentration of the measure induced by the Lévy family on any compact Lie group. In order to improve our intuition on the phenomenon, we will consider two simple examples of infinite dimensional compact sets and the concentration of measure on them associated to the continuous action of certain infinite dimensional groups.

3.1 An elementary example

In this example, we construct an infinite compact space on which an infinite-dimensional group acts. In this case, it will happen that there is no well-defined limit of the concentration locus in the group, but, instead, there is a convergence of the concentration locus in the compact set, given by the fixed point. In other words, the concentration locus on the group induces a concentration locus on the compact set that converges to the fixed point.

Let $\mathcal{H}$ be a real infinite-dimensional Hilbert space, with scalar product $(\cdot | \cdot)$ and norm $\| \cdot \|$. We define the spheres and the closed balls as

$$S_r^\infty := \{ x \in \mathcal{H} \mid \|x\| = 1 \}, \quad B_r^\infty := \{ x \in \mathcal{H} \mid \|x\| \leq 1 \}.$$

Both are closed and bounded but none of them is compact in $\mathbb{H}$ with the strong topology. Therefore, let us consider the weak topology, the weakest topology which makes the maps of the topological dual $\mathcal{H}'$ of $\mathcal{H}$ continuous. Its open sets are generated by open sets of the form
\[ U_{v, \varepsilon}(x_0) = \{ x \in \mathcal{H} \mid (v \mid x - x_0) < \varepsilon \} , \]

under finite intersections and arbitrary unions, where \( v \in \mathcal{H} \).

If \( \mathcal{H}_v \equiv v^\perp \), \( v \neq 0 \), then

\[ U_{v, \varepsilon}(x_0) = \bigcup_{\alpha \in \mathbb{R}, |\alpha| < \varepsilon} \left\{ x_0 + \frac{\alpha v}{\|v\|_2} + \mathcal{H}_v \right\} . \]

Of course, \( S_r^\infty \) is not closed in the weak topology, and \( \overline{S_r^\infty} = \overline{B_r^\infty} \). Indeed, let us first consider \( y \in B_r^\infty \), and fix an orthonormal complete system \( \{ e_j \}_{j=1}^\infty \) for \( \mathcal{H} \) such that \( y = \|y\| e_1 \). Therefore, the sequence

\[ x_n := y + \sqrt{r^2 - |y|^2} e_n, \quad n > 1, \]

is in \( S_r^\infty \) and satisfies

\[ \lim_{n \to \infty} (v \mid y - x_n) = 0 \]

for any \( v \in \mathcal{H} \), which means that \( y \) is an accumulation point for \( S_r^\infty \) in the weak topology. On the opposite, if \( y \notin B_r^\infty \), then \( \|y\| > r \) and, for any \( x \in S_r^\infty \),

\[ |(y \mid y - x)| = ||y||^2 - (y \mid x) \geq q ||y||^2 - |(y \mid x)| \]

\[ \geq q ||y||(||y|| - ||x||) = ||y||(||y|| - r) > 0 , \]

so that \( y \) is isolated from the ball.

Let us now fix a complete orthonormal system \( \{ e_j \}_{j=1}^\infty \) for \( \mathcal{H} \). Set

\[ \mathbb{R}^N := \mathbb{R} e_1 \oplus \cdots \oplus \mathbb{R} e_N . \]

Notice that its orthogonal complement in \( \mathbb{H} \) is

\[ \mathbb{R}^{N\perp} = \mathcal{H}_{e_1} \cap \cdots \cap \mathcal{H}_{e_N} . \]

We can consider the linear action of the group SO\((N)\) on \( \mathcal{H} \) defined by its fundamental action on \( \mathbb{R}^N \) and by its trivial action on \( \mathbb{R}^{N\perp} \). Let us consider the sequence of canonical embeddings

\[ \cdots \subset \text{SO}(N) \subset \text{SO}(N + 1) \subset \text{SO}(N + 2) \subset \cdots \]

and its limit SO\((\infty)\) w.r.t. to any topology which makes the embeddings SO\((N) \hookrightarrow \text{SO}(\infty)\) continuous for all \( N \). Let \( K = B_r^\infty \) be endowed with the weak topology, so

\[ \circlearrowleft \text{Springer} \]
that $K$ is a compact set. For any given point $x \in B_r^\infty$, we define the measures $\mu^x_N$ over $K$ induced by the action of $SO(N)$ over $K$ as follows. If

$$f : SO(N) \times \mathcal{H} \to \mathcal{H}$$

is the action of $SO(N)$ on $\mathcal{H}$ defined above, it is clear that its restriction over $SO(N) \times B_r^\infty$ defines an action of $SO(N)$ on $B_r^\infty$. Given a subset $A \subseteq B_r^\infty$, let us consider the subset $SO(N)^x_A$ of $SO(N)$ defined by

$$SO(N)^x_A := \{ g \in SO(N) \mid f(g, x) \in A \}. $$

We say that $A$ is measurable if $SO(N)^x_A$ is measurable with respect to the normalized Haar measure $\mu_{SO(N)}$ of $SO(N)$, and put

$$\mu^x_N(A) := \mu_{SO(N)}(SO(N)^x_A).$$

**Remark 3.1** In [3] we calculated explicitly a concentration locus for classical compact Lie groups, in particular for $SO(n)$.

**Theorem 3.2** The following holds:

$$\lim_{N \to \infty} \mu^e_1^N = D_0$$

where $D_0$ is the Dirac measure with support in 0.

**Proof** The image of $e_1$ under the action of $SO(N + 1)$ is

$$f(SO(N + 1), e_1) = S_1^N = S_1^\infty \cap \mathbb{R}^N = \left\{ y = \sum_{j=1}^{N+1} y_j e_j \mid \| y \| = 1 \right\}. $$

It follows immediately that

$$\mu^e_1^{N+1} = D_{S_1^N}$$

the Dirac measure uniformly supported on $S_1^N$ (so that it is the normalized Lebesgue measure when restricted to the support). Choosing for any $N$ the spherical polar coordinates with azimuthal axes defined by $e_1$, it is a standard argument (see [15]) to show that when $N \to \infty$ the measure concentrates on $S_1^\infty \cap \mathcal{H}_{e_1}$. This is exactly the image of a concentration locus of the $SO(N + 1)$ family: after identifying $SO(N + 1)$ with the $SO(N)$ Hopf fibration over $S^N$ and $SO(N)$ with the isotropy group of $e_1$ under the action of $SO(N)$, following the results in [3], it is easy to see that a concentration locus on the group can be identified with the restriction of the Hopf fibration to an equator of $SO(N + 1)/SO(N) \simeq S^N$. This family of fibrations is mapped to the family of the equators of the orbits of $SO(N + 1)$ through $e_1$.

---

3 The normalization is $\mu_{SO(N)} = 1$. 
Iterating the procedure w.r.t. $e_2 \in S_1^\infty \cap \mathcal{H}_{e_1}$ and so forth, we see that for any finite $k$ the measure concentrates on $S_1^\infty \cap \mathbb{R}^k \perp$. Taking the closure of these sets and noting that $\bigcap_{k \in \mathbb{N}} \mathcal{H}_k = 0 \in B_1^\infty$, we get the assertion. \hfill \Box

**Remark 3.3** The proof not only gives the assertion, but also shows that the concentration locus has growing codimension in the family of $\text{SO}(N + 1)$, and when we take the limit its image on the spheres collapses to the fixed point 0. So, while the concentration locus has no limit on the groups, it has a well-defined limit on the compact spheres on which the groups act.

This phenomenon of concentration of measure on a compact space can be also interpreted as an optimal transport of mass along geodesics of measures, which converge to $D_0$, see e.g. [16]. Such flow of measures can be geometrically visualized in our case as a geodesic flow transporting the equators toward the north pole $e_1$ along meridian geodesics. In particular, a recent work by Schneider [21] uses this approach to prove a conjecture by Pestov [19].

**Proposition 3.4** If $x \in B_r^\infty$, then

$$\lim_{N \to \infty} \mu_N^x = D_0.$$  

**Proof** If $x \neq 0$ then the proof follows exactly the same line as above. The case $x = 0$ follows from $\mu_N^0 = D_0$ for any $N$. \hfill \Box

Notice that we can get a completely geometrical picture of this phenomenon: acting on $x \in K = B_r^\infty$, $\text{SO}(\infty)$ generates $K$ as a union of (infinitely many) layers like an onion when $\|x\|$ varies in $[0, r]$. In fact, $\text{SO}(\infty)$ acts transitively on each layer, but all layers have a common center, the origin 0, which is left fixed by each $\text{SO}(N)$. Since the normalised measure induced on each $S_N^N|_x$ is always the same as for $S_1^N$ if $x \neq 0$, it is natural to expect that the measure will concentrate on the common accumulation point, which is just 0.

The same result can be extended by moving to a complex Hilbert space $\mathcal{H}_C$ and acting with the groups $\text{SU}(N)$, where projective spaces replace the spheres. More precisely, the reference compact for $\text{SU}(N + 1)$ is $\mathbb{C}P^N$. Again, for each $N$ and $x \in K$ (the weak closure of the $\mathbb{C}P^\infty$) we can define the measure $\mu_N^x$ and get

**Proposition 3.5** If $x \in K$, then

$$\lim_{N \to \infty} \mu_N^x = D_0.$$  

We do not know whether $\text{SO}(\infty)$ and $\text{SU}(\infty)$ are extremely amenable or not, if endowed with the inductive topology. Nevertheless, we know that $\text{U}(\infty)$ is not extremely amenable with such a topology. Despite this, probably as a consequence of the weakness of the weak topology, its action on $K$ results in the concentration of the measure in a point. This concentration is “non-essential” in the sense that it is due to the property of the compact and not of the Lévy sequence of groups. This is easily seen
by changing the compact set. If $K = B^\infty$ with the weak topology and we identify $S^1$ endowed with the metric topology with $U(\infty)/SU(\infty)$, then $K' = K \times S^1$ with the product topology is a compact set. Now, after fixing a point $x' = (x, \theta) \in K'$, we see that, because of Proposition 3.5, necessarily the measures generated by the sequence of $SU(N)'s$ concentrate on the point $x_0' = (0, \theta)$. But since $U(\infty)$ has no fixed points on $S^1 = U(\infty)/SU(\infty)$, then the measures induced by the sequence of $U(N)'s$ will obviously concentrate uniformly on $0 \times S^1$.

We can now state a simple but interesting result, giving us some information when working on a non-compact infinite dimensional set.

**Proposition 3.6** Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of sets $K_n \subset K_{n+1}$, over which acts the family of groups $\{G_n\}$, endowed with an invariant measure. Let $G_\infty$ be the limit group defined by the sequence $G_n \subset G_{n+1}$. Assume that there exists a concentration locus $c_n \subset K_n$, such that cutting $K_n$ along $c_n$, it is divided into disconnected parts $K^n_+$ and $K^n_-$, having the following properties:

- $c_n \subset K^n_+$ and $K_n = K^n_+ \cup K^n_-$;
- For $n$ large enough all $K^n_+$ are contained in a compact subset $K^+$ of $K$;
- There exists a sequence of subgroups $F_n \subseteq G_n$ acting on $K^n_+$ and having the same limit $G_\infty$.

Then, if $G_\infty$ is extremely amenable, $K = \bigcup_n K_n$ admits a fixed point.

**Proof** Since $K^+$ is compact and $F_n$ act on $K^+$, we get that $G_\infty$ has a fixed point in $K^+$, and therefore in $K$. \qed

This shows how eventually fixed points could be individuated even on a non-compact set $K$. It is clear that to do it, it is essential to have a well-posed definition of concentration locus.

### 3.2 Linear non-symmetric action

In the second example, we want to consider a non-compact sphere on which, however, the action of an infinite dimensional group has a fixed point. This fixed point is again individuated exactly by the concentration locus induced by the action of the group.

Let $a \in \mathbb{R}^{N+2}$, and let $B_{N+2}$ be the closed ball

$$
B_{N+2} = \{x \in \mathbb{R}^{N+2} \mid \|x\| \leq 1\},
$$

while

$$
B_{N+2} = \{x \in \mathbb{R}^{N+2} \mid \|x\| < 1\}
$$

is the set of its internal points. We call $\alpha = \|a\|$ and assume $0 < \alpha < 1$. Finally, we call

$$
S_{N+1} = \partial B_{N+2}
$$
that is an \((N + 1)\)-sphere of radius 1, centered in \(o \equiv 0\). We now define a linear action of \(\text{SO}(N + 2)\) on \(S_{N+1}\), which we simply call the non-symmetric action. This is a left action

\[
\nu_N : \text{SO}(N + 2) \times S_{N+1} \rightarrow S_{N+1}
\]

defined as follows. Let \(p \in S_{N+1}\), and consider the right-half line \(r\) starting from the origin \(a\) through the point \(p\). If \(R \in \text{SO}(N + 2)\), the right-half line \(R(r)\) will meet \(S_{N+1}\) at a single point \(q\). We define

\[
\nu_N(R, p) := q.
\]

Since \(0 < \alpha < 1\), it is well defined and gives rise to a transitive action of \(\text{SO}(N + 2)\) on \(S_{N+1}\).

Let us now determine the invariant measure on the sphere induced by the action \(\nu_N\). First notice that, like for the symmetric action, the isotropic subgroup of a point \(p \in S_{N+1}\) is the \(\text{SO}(N + 1)\) fixing the segment \(op\). The complementary generators originate the displacements defining the measure. We can easily compare the resulting measure \(d\mu_a\) with the standard measure \(d\mu_0\). Let \(\phi\) be the angle between the segments \(oa\) and \(op\) with orientation from \(oa\) to \(op\). Then

**Proposition 3.7**  With the above notations, we have

\[
d\mu_a = \frac{1 + \alpha \cos \phi}{(1 + \alpha^2 + 2\alpha \cos \phi)^{N/2+1}} d\mu_0,
\]

where \(\mu_0\) is the Lebesgue measure over \(S_{N+1}\) normalised so that \(\mu_0(S_{N+1}) = 1\).

In particular, \(\mu_a(S_{N+1}) = 1\).

**Proof** The normalised measure \(d\mu_0\) is nothing but the element of solid angle as seen by \(o\), divided by the total solid angle. By construction, \(d\mu_a\) is the element of solid angle as seen by \(a\), again divided by the total solid angle, so that we have of course \(\mu_a(S_{N+1}) = 1\). Now fix a point \(p\) on \(S_{N+1}\). By definition, the relation between the two measures at \(p\) is

\[
d\mu_a(p) = \frac{\cos \psi}{R^{N+1}} d\mu_0,
\]

where \(R = \|p - a\|\) and \(\psi\) is the angle \(\angle opa\). The cosines theorem says us that

\[
R^2 = 1 + \alpha^2 + 2\alpha \cos \phi.
\]

As for \(\cos \psi\), we can again apply the cosines theorem to the triangle \(apo\) in the form

\[
\|oa\|^2 = \|op\|^2 + \|pa\|^2 - 2\|pa\|\|op\| \cos \psi.
\]
Using that \( \|oa\| = \alpha, \|op\| = 1 \), and \( \|pa\| = R \), we get

\[
\cos \psi = \frac{1 + \alpha \cos \phi}{(1 + \alpha^2 + 2\alpha \cos \phi)^{1/2}},
\]

and the proposition is proved. \( \square \)

### 3.2.1 The extremal infinite sphere and the concentration of measure

Let \( \mathcal{H} \) be a separable real Hilbert space, and \( \{e_n\}_{n=0}^{\infty} \) be a complete orthonormal system (CONS) for it. Let \( a \in \mathcal{H} \), with \( \alpha := \|a\| = 1 \), we define the extremal sphere

\[
S_a := \{ x \in \mathcal{H} \mid \|x\| = 1 \}.
\]

Here the \( a \) means only that the sphere has the point \( a \) as a marked point. We assume to choose the CONS in such a way that \( a_n = (e_n | a) \neq 0 \) for all \( n \).

For any given \( N = 0, 1, 2, \ldots \), we identify

\[
\mathbb{R}^{N+2} \equiv H_N = \{ x \in \mathcal{H} \mid (e_n | x) = 0, \forall n > N + 1 \},
\]

\[
S_{a_N} = H_N \cap S_a, \quad a_N = (a_0, \ldots, a_{N+1}).
\]

By hypothesis, \( 0 < \|a_N\| \equiv \alpha_N < 1 \), so that \( a_N \) is internal to \( S_{a_N} \). Let \( O(H) \) be the unitary group of \( H \). The subgroup fixing \( \mathcal{H} \perp_N \) is \( O(N + 2) \), the orthogonal group acting on \( \mathbb{R}^{N+2} \). \( SO(N + 2) \) is the subgroup fixing the orientation of \( (e_0, \ldots, e_{N+1}) \). Let us consider the sequence

\[
SO(\infty) := (SO(2) \subset SO(3) \subset \cdots \subset SO(N) \subset SO(N + 1) \subset \cdots)
\]

defined by the identifications above. We see that on each \( S_{a_N} \) we can define a left action \( \nu_{a_N} \) of \( G_N := SO(N + 2) \) on \( S_{a_N} \), defined as in the previous section. Notice that the action of \( G_N \) over \( S_{a_N} \) is just the restriction to \( S_{a_N} \) of the action on \( S_{a_{N+1}} \) of the subgroup of \( G_{N+1} \) leaving fixed the segment \( a_{N+1}a_N \).

The measure induced on \( S_{a_N} \) by \( G_N \) is

\[
d\mu_{a_N}(p) = \frac{1 + \alpha_N \cos \phi_N}{(1 + \alpha_N^2 + 2\alpha_N \cos \phi_N)^{N/2+1}} \, d\mu_{N,0}(p),
\]

where \( \phi_N \) is the azimuthal angle of \( p \) w.r.t. \( ao \), and \( d\mu_{N,0}(p) \) is the Lebesgue measure on \( S_{a_N} \) normalised to 1. We get:

**Proposition 3.8** With the above notation we have

\[
\lim_{N \to \infty} \mu_{a_N} = D_a,
\]
where $D_a$ is the Dirac measure with support in $a$. The action of $\text{SO}(\infty)$ on $S_a$ has a fixed point in $a$.

**Proof** Using standard polar coordinates on the sphere, w.r.t. the azimuthal angle $\phi_N$, we can write

$$d\mu_{a_N}(p) = \frac{1 + \alpha_N \cos \phi_N}{(1 + \alpha_N^2 + 2\alpha_N \cos \phi_N)^{N/2+1}} \sin^N \phi_N \, d\phi_N \, d\mu_{N-1,0}(p).$$

On the other hand, if $\theta = \angle apo$ and $R(\phi_N) = \|ap\| = \sqrt{1 + \alpha^2 + 2\alpha \cos \phi_N}$, we have

$$\sin \phi_N = R(\phi_N) \sin \theta.$$

From this we get

$$\frac{1 + \alpha_N \cos \phi_N}{(1 + \alpha_N^2 + 2\alpha_N \cos \phi_N)^{N/2+1}} \sin^N \phi_N \, d\phi_N = \sin^N \theta \, d\theta.$$

Therefore, in these coordinates

$$d\mu_{a_N}(p) = \sin^N \theta \, d\theta \, d\mu_{N-1,0}(p).$$

This is the standard spherical measure, which is well known to concentrate on the equatorial sphere $\theta = \pi/2$. On $S_{N+1}$ this is an $S^N$ sphere with center in $a_N$ and radius $\sqrt{1 - \|a_N\|^2}$. Since $a_N \rightarrow a$, and $\|a\| = 1$, we get the assertion. \(\Box\)

**Remark 3.9** $S_a$ is not a compact space in the topology of $H$. Notice that this result is nothing but a consequence of Proposition 3.6: here the equators play the role of $c_n$ of the proposition. Again, the equators are just the image of the concentration locus of $\text{SU}(N+1)$ mapped on the sphere now through the non-symmetric action. Under the action, the locus is shrunk down to $a$.

### 3.3 Pictorial idea

We want now to give a pictorial vision of the construction done above. This is because it will help us to better understand what happens and to proceed further to the compact case. The idea of the asymmetric action of rotation is depicted in Figs. 1 and 2. They show the relation between the rotation angle $\theta$ and the azimuthal angle $\phi$ of spherical coordinates. When the segment $ap$ belongs to the plane of the rotation (assuming a plane rotation), the main relation is $\sin \phi = \|ap\| \sin \theta$.

When the rotation is on a plane orthogonal to $ao$, then the relation is just $\|op\| \, d\psi = \|ap\| \, d\chi$. Notice that in this case the length of $op$ is preserved.

Using these relations and decomposing arbitrary rotations in terms of rotation on planes containing $ao$ and rotations in hyperplanes orthogonal to $ao$, one easily computes the measure induced by the action $\nu_a$ on a sphere. This is shown in Fig. 3.
Concentration of measure for classical…

One can then inductively generate a sequence of actions $v_{a_N}$ on a sequence of spheres $S_{a_{N+1}}, S_{a_N} \subset S_{a_{N+1}} \subset S_{a_{N+2}}$, as done in the previous section and depicted in Fig. 4.

The invariant measure induced by this action is nothing but the solid angle on which the point $a_N$ “sees” the portions of $S_{N+1}$. The hyperplane through $a_N$ perpendicular to $ao$ intersects $S_{N+1}$ in a sphere $S_N$, which, by construction, separates $S_{N+1}$ in two portions of equal measure. Therefore, we call it equator. If one excludes to each sphere a small tubular neighbourhood of the equator, with fixed radius $\epsilon$, then the resulting sets have measures that tend to zero when $N$ diverges. Indeed, this is exactly the same thing which happens to the usual construction for spheres of radius 1 and increasing dimension. The phenomenon is obviously exactly the same, so we can state that the concentration collapses on the equators. In Fig. 5 a particular case is shown, with two portions that have equal measure with respect to the asymmetric action.
Fig. 3 Rotations decompose on the ones along planes containing $oa$ (like $R_1$) and the ones on hyperplanes through $o$ perpendicular to $oa$ (like $R_2$ and $R_3$).

Fig. 4 Asymmetric actions in successive dimensions. The picture represents the 1- and 2-dimensional cases, but one can imagine to consider the general case after replacing 0, 1, 2 with $N$, $N + 1$, $N + 2$ everywhere. The points $a_0 = (a_0, a_1)$ and $a_1 = (a_0, a_1, a_2)$ are the centers of the rotations action on $S_1$ and $S_2$ respectively.
Fig. 5 The two dark spherical cups have the same volume since are seen by \( a \) with the same solid angle. The asymmetric equator separates the sphere in two parts having equal volumes. The measure tends to concentrate on the asymmetric equator for the same reason and the same sense the Lebesgue measure concentrates on the equator.

There is, however, an important difference between the asymmetric case and the usual one. The point is that, while the equators remain perfectly symmetric from the point of view of the measures, in the asymmetric case it is not so from the geometrical perspective. On \( S_{N+1} \) we can individuate the south pole as the intersection of the sphere with the half-right line from \( o \) passing through \( a_N \). We see that \( a_N \) is the closest to the south pole, with a radius smaller than that of the equator. Since our sequence of points \( a_N \) by construction converges to \( a \), with \( \|a\| = 1 \), so that \( a \in S_a \), we see that the radii of the equators tend to zero and collapse to \( a \). This is shown in Fig. 6. This difference is quite important in our opinion for the following reason.

In the symmetric case, as we said, the Lebesgue measure concentrates on equators. Which equator depends on choices. Any equator we choose is good, after fixing polar coordinates with azimuth orthogonal to that equator, everything goes as predicted. The point is that, however, the concentration does not localize, in the sense that it does not concentrate on a specific point of a limit equator. The same phenomenon happens in the asymmetric case if we consider different equators, see Fig. 7.

If in place of asymmetric equators we choose meridian equators, as shown in Fig. 7, we will not see a collapse on a point, but a delocalised concentration on the meridians. This is due to the fact that these equators do not collapse into a single point. The concentration on a defined point is due to a “compactification” of the sequence of equators as given by Proposition 3.6: our deformation is such that the equators relative to the south poles tend to enter a compact region around the south pole \( a \). They are
Fig. 6 Nearest the point $a$ is to the sphere, smallest is the equator. Since the sequence $a_N$ converges to a point $a$ on $S_d$, the equators of concentration collapse to $a$.

Fig. 7 Foliation of the infinite-dimensional ball

flattened to the south pole. In the other direction this does not happen, since the infinite-dimensional sphere is not compact. This (probably) shows the role of compactness in determining the fixed points of the action. In any case, the equators where the concentration happens are in all cases the images of the concentration locus in the sequence of groups.

We will now proceed by further investigating the compact case more deeply.
3.4 Non-symmetric action on a compact space

In the previous example we have seen that in a non-compact space one can use the concentration locus of the group as a probe in order to look for a fixed point under its action. In the next example, we want to show that in the compact case the convergence is uniform around the fixed point.

Up to now, we worked with non-compact spaces. On them, we can of course have actions without fixed points. For example, it is sufficient to take $\|a\| < 1$ to get an action of $\text{SO}(\infty)$ on $S_\alpha$, which has no fixed points. Let us keep in mind such an action. We want now to extend it to an action on a compact space. To this end, let us first realise the sphere in a well-specified space.

Let us consider the real space $V = H^1_0((0, 2\pi)) = \left\{ f \in W^{1,2}([0, 2\pi]) \mid f(0) = f(2\pi) = 0, f \text{ is absolutely continuous} \right\}$.

Here, $W^{1,2}([0, 2\pi])$ is the Sobolev space of all square-Lebesgue-summable functions over $[0, 2\pi]$ whose weak first derivatives are still square-Lebesgue-summable over $[0, 2\pi]$; $H^1_0((0, 2\pi))$ is defined as the closure in $W^{1,2}([0, 2\pi])$ of the set $C_c^\infty([0, 2\pi])$ of smooth functions with compact support in $[0, 2\pi]$.

The functions $u_n(x) = \frac{\sin(nx)}{\sqrt{\pi(n^2 + 1)}}$, $n = 1, 2, 3, \ldots$, form an orthonormal complete set (CONS) for $V$. The unit sphere $S$ in $V$ can then be described, w.r.t. the given (CONS), as

$$ S = \left\{ \sum_{n=1}^{\infty} x_n u_n \mid \sum_{n=1}^{\infty} x_n^2 = 1 \right\}. $$

Now, consider the embedding $J : H^1_0((0, 2\pi)) \hookrightarrow L^2([0, 2\pi])$. $J$ is a compact operator, which means that the image of a closed bounded set has compact closure. Let us now consider the set $J(S)$. In $L^2([0, 2\pi])$, we consider the real Hilbert subspace $H$ generated by the orthonormal system

$$ v_n = \frac{\sin nx}{\sqrt{\pi}}, \quad n = 1, 2, \ldots. $$

Then $J : V \rightarrow H$ is again compact and if $y = J(x)$, we see that its components w.r.t. $\{v_n\}$ are

$$ y_n = \frac{x_n}{\sqrt{n^2 + 1}}. $$
Therefore, the image \( J(S) \) of the unit sphere is
\[
J(S) = \left\{ \sum_{n=1}^{\infty} y_n v_n \mid \sum_{n=1}^{\infty} (n^2 + 1)y_n^2 = 1 \right\}.
\]

Thus, \( J(S) \) looks like an infinite-dimensional ellipsoid whose principal form has length
\[
L_n = \frac{1}{\sqrt{n^2 + 1}}.
\]

In particular, it is squashed more and more in the increasing directions, since \( L_n \to 0 \).
This implies that all the points satisfying \( \sum_{n=1}^{\infty} (n^2 + 1)y_n^2 < 1 \) are accumulation points for \( J(S) \), and we conclude that if
\[
B := \{ x \in V \mid \| x \|_V \leq 1 \},
\]
so that \( S = \partial B \), then the compact set generated by \( J(S) \) is
\[
K_S := \overline{J(S)} = J(B).
\]

So, in order to get an action of \( SO(N) \) on \( K_S \) we have to extend the action on \( S \) to a continuous action on the whole \( B \). To this end, we can foliate \( B \) in spheres:
\[
B = \bigcup_{\xi \in [0,1]} S_\xi, \quad S_\xi = \{ x \in V \mid \| x - a(1 - \xi) \| = \xi \}.
\]

If \( B_\xi \) is the ball having \( S_\xi \) as a boundary, then notice that if \( \xi_1 < \xi_2 \) then \( S_{\xi_1} \subset B_{\xi_2} - S_{\xi_2} \).
Indeed, for \( x \in S_{\xi_1} \) we have
\[
\| x - a(1 - \xi_2) \| \leq \| x - a(1 - \xi_1) \| + (\xi_2 - \xi_1)\| a \|
= \xi_1 + (\xi_2 - \xi_1)\| a \| = \xi_2 - (\xi_2 - \xi_1)(1 - \| a \|) < \xi_2,
\]
since we assumed \( \| a \| < 1 \).
Therefore, \( x \in B_{\xi_2} - S_{\xi_2} \). This allows us to extend the action of the rotations to the whole \( B \) as shown in Fig. 8.

At this point we are ready to discuss how a measure on \( K_S \) is induced by an “observer”. Let us fix any point of \( K_S \), say \( s \) in Fig. 8. This is our “observer”. If we consider the orbit of \( SO(\infty) \) through \( s \), we get a measure on \( K_S \) as defined above. In particular, we consider the family of measures \( \mu^s_N \), defined as the normalized angular measure under the action of \( SO(N + 1) \). It is defined as follows. Suppose \( M \) is the lower integer such that \( (u_M \mid s - a) \neq 0 \). Then, for \( N < M \), \( s \) is left fixed by the action of \( SO(N + 1) \), so that \( \mu^s_N = D_s \) is the Dirac measure with support \( J(s) \). If \( N \geq qM \), let \( O^s_N \) be the orbit of \( SO(N + 1) \) through \( s \). Then, for any measurable subset of \( K_S \) (a subset \( E \subset K_S \) such that \( E \cap \mathbb{R}^n \) is Lebesgue measurable for any \( n \)), we set
\[
\mu^s_N(E) := \Omega_N(O^s_N \cap E),
\]
where $\Omega_N(R)$ is the solid angle in $\mathbb{R}^{N+1}$ under which $J_a$ sees $R \in \mathbb{R}^{N+1}$. Here we mean

$$\mathbb{R}^{N+1} = J(a) + \mathbb{R}v_1 + \cdots + \mathbb{R}v_{N+1} \subset L^2([0, 2\pi]).$$

Also, we mean the angular measure to be normalized so that $\Omega_N(\Omega^x_N) = 1$. We now show by hand that the measures so defined on the compact concentrate on $a$.

**Proposition 3.10** Fix arbitrarily $\varepsilon > 0$ and consider $U_\varepsilon(a)$ the open ball neighbourhood of radius $\varepsilon$ of $J(a)$ in $L^2([0, 2\pi])$. If $E$ is a measurable set such that $E \subseteq K_S \cap U_\varepsilon(a)^c$, then

$$\lim_{N \to \infty} \mu_N(E) = 0.$$

**Proof** It is sufficient to prove that $\lim_{N \to \infty} \mu_N(U_\varepsilon(a)^c) = 0$. Since $K_S$ is an ellipsoid whose half diameter in the $(N + 1)$-th direction is $L_{N+1} = ((N + 1)^2 + 1)^{1/2}$, we see that for $N$ large enough we have $L_{N+1} < \varepsilon$. If $\theta_{N+1}$ is the angle in $\mathbb{R}^{N+1}$ w.r.t. the axis fixed by $v_{N+1}$, we get that $U_\varepsilon(a)^c$ is seen by $J(a)$ under an angle range $[\pi_2 - \tilde{\theta}, \pi_2 + \tilde{\theta}]$, with

$$\sin \theta \leq \frac{L_{N+1}}{\varepsilon}.$$
Thus, it is seen to be around an equator by an angle $\bar{\theta}$ that goes to zero as

$$\frac{L_{N+1}}{\varepsilon} \sim \frac{1}{\varepsilon N}$$

as $N$ diverges. Since it decreases faster than $N^{-1/2}$, the thesis follows immediately from Lévy’s argument. 

\[ \square \]

**Remark 3.11** It follows that the measures concentrate uniformly on the fixed point $J(a)$, contrarily to the non-compact case. Once again this is a consequence of the fact that as $N$ grows, the concentration locus of $\text{SO}(N)$ is mapped into a smaller neighbourhood of the point $a$.

From our attempts it seems that even more interesting examples could arise from nonlinear actions of Lie groups. Unfortunately, analytical nonlinear Lie group actions are very difficult to determine. The most notable examples can be found in algebraic topology whenever the orbit space is investigated. Regrettably, in these cases only the nature of the action is determined rather than their actual analytic form. Indeed, they set algebraic criteria to detect whether an action is equivalent to a linear one or not (see e.g. [1, 2, 11]). This does not serve our scopes. However our results in this area are still not satisfactory enough, so we have decided to postpone this topical discussion to future research.

### 4 Applications in physics

Finally, we apply our approach to some examples from physics. The phenomenon of concentration of measure is of course relevant in describing statistical ensembles. The measure on the phase space determines the equilibrium configuration of the statistical system. The eventual concentration of measure is therefore characterizing the system. As an application of what we have seen up to now, let us consider a statistical ensemble of free particles on $\text{SU}(N)$ manifolds. The phase space is given by the cotangent bundle

$$T^*\text{SU}(N) \simeq \text{SU}(N) \times \mathfrak{su}(N)^*.$$ 

Since $\text{SU}(N)$ is compact, it is endowed with a negative definite Killing product that naturally induces a negative definite scalar product over $\mathfrak{su}(N)^*$:

$$\langle \cdot, \cdot \rangle : \mathfrak{su}(N)^* \times \mathfrak{su}(N)^* \longrightarrow \mathbb{R}.$$ 

It defines the kinetic energy of the particles of mass $m$ by

$$E(\vec{p}) = -\frac{\langle \vec{p}, \vec{p} \rangle}{2m}, \quad \vec{p} \in \mathfrak{su}(N)^*.$$
If $κ$ is the Boltzmann constant and $dm_N$ the Lebesgue measure over $\mathfrak{su}(N)^*$, the Gibbs measure at temperature $T$ is

$$dμ(ξ, \vec{p}) = \frac{1}{Z_N} e^{-\frac{⟨\vec{p}, \vec{p}⟩}{2mκT}} dμ_{\mathfrak{SU}(N)}(\xi) dμ_{\mathfrak{SU}(N)}(\vec{p}), \quad (ξ, \vec{p}) ∈ T^*\mathfrak{SU}(N),$$

where $Z$ is the normalization factor. We, therefore, get an easy result for the concentration of the measure here, since the Gaussian measure concentration is discussed in [15, p. 2]. This distribution is uniform, which means that the concentration will not appear to be on an “a priori defined locus”, but that the locus depends on the observer. This is the same as expected in a homogeneous distribution on a uniform sphere of large dimension: any given observer will conclude that the largest probability is in finding a particle near the equator with respect to the observer placed at the north pole.

In the example, we have just considered the measure concentrates without fixed points. It is interesting to consider the same situation in an example where concentration determines a fixed point. An example is the case of the spheres when the measure is determined by the asymmetric action, as presented in Sect. 3. Again, we can consider a free gas of particles of mass $m$ on $S_{N+1}$, so that the Gibbs measure over $T^*S_{N+1}$ is

$$dμ_{\text{Gibbs}}(x, p) = dμ_{\mathfrak{a}_N}(x) dm_{N+1}(p) e^{-\frac{p^2}{2mκT}},$$

where $m_{N+1}$ is the Lebesgue measure on $\mathbb{R}^{N+1}$. The consequence of our analysis in Sects. 3.2.1 and 3.3 is that the measure concentrates on neighbourhoods of about $1/\sqrt{N}$ of radius around asymmetric equators collapsing over $a$, while momenta behave exactly as in the previous example. Therefore, this looks as if the equators and the fixed point behave as attractors for the particles. This could be how the concentration seems to contribute to the gravitational effects.

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