HYPERKÄHLER ARNOLD CONJECTURE AND ITS GENERALIZATIONS

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Abstract. We generalize and refine the hyperkähler Arnold conjecture, which was originally established, in the non-degenerate case, for three-dimensional time by Hohloch, Noetzel and Salamon by means of hyperkähler Floer theory. In particular, we prove the conjecture in the case where the time manifold is a multidimensional torus and also establish the degenerate version of the conjecture. Our method relies on Morse theory for generating functions and a finite-dimensional reduction along the lines of the Conley–Zehnder proof of the Arnold conjecture for the torus.

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1. INTRODUCTION

The main objective of this paper is to prove a generalization of the hyperkähler Arnold conjecture originally established via hyperkähler Floer theory by Hohloch, Noetzel and Salamon in [HNS].

The setting of the hyperkähler Arnold conjecture is similar to its standard Hamiltonian counterpart, but the time manifold is three-dimensional ($\mathbb{T}^3$ or SU(2) rather than $S^1$) and the target manifold is equipped with a hyperkähler rather than a symplectic structure. The space of maps from the time manifold to the target manifold carries a suitably defined action functional, akin to the standard action functional in Hamiltonian mechanics, provided that a version of a Hamiltonian is also furnished. In the spirit of the Arnold conjecture, the main result of hyperkähler Floer theory developed [HNS] is that the number of critical points of the action functional is bounded from below by the sum of Betti numbers of the target manifold whenever
the action functional is Morse. For technical reasons, the target manifold must be flat.

Our main goal is to show that this version of the Arnold conjecture can be further generalized and refined. We prove an analog of the conjecture (both the degenerate and non-degenerate case) for the time manifold $T^r$ and a target space equipped with $r$ flat “anti-commuting” Kähler structures. More precisely, the target space is a compact quotient of a representation of a Clifford algebra. In the degenerate case, the lower bound is given in terms of the cup-length of the target space. We also prove a version of the degenerate Arnold conjecture for the time manifold $SU(2)/D_4$ and a flat hyperkähler target space.

In contrast with [HNS], the argument we utilize to prove these results is not precisely Floer theoretic, but rather it is a finite-dimensional approximation combined with Morse or Ljusternik–Schnirelman theory for generating functions, following the line of reasoning from [CZ]. The difference is, from our perspective, rather technical and the two methods usually give the same results when they both apply, with, perhaps, the finite-dimensional approximation approach having a slight edge. (Of course, in the context of Hamiltonian dynamics, Floer theory has a much broader range.)

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2. Main Results

Let $V$ be a vector space equipped with $r$ symplectic structures $\omega_1, \ldots, \omega_r$, which are all compatible with the same inner product $\langle \cdot, \cdot \rangle$. In other words, there exist orthogonal (with respect to $\langle \cdot, \cdot \rangle$) operators $J_1, \ldots, J_r$ on $V$ such that $J_l^2 = -I$ for all $l$, i.e., these operators are complex structures, and

$$\langle X, Y \rangle = \omega_l(X, J_l Y)$$

for all $X$ and $Y$ in $V$.

Assume furthermore that the complex structures $J_l$ anti-commute:

$$J_l J_j + J_j J_l = 0 \quad \text{whenever } l \neq j.$$  \hfill (2.1)

Such a collection of complex (or equivalently symplectic) structures can exist for arbitrarily large values of $r$, depending on the dimension of $V$. It exists if and only if the unit sphere in $V$ admits $r$ linearly independent vector fields; see [Hu, Chapter 12 and 16] and, in particular, pp. 152–154 therein. More specifically, let $\dim V = 2^{d+c}b$, where $d \geq 0$ and $0 \leq c \leq 3$ are integers and $b$ is odd. Then the maximal value of $r$ for $V$ is $8d + 2^c - 1$. In fact, equipping $V$ with the structures $J_1, \ldots, J_r$ is equivalent to turning $V$ into an (orthogonal) representation of the Clifford algebra of a negative definite quadratic form on $\mathbb{R}^r$. Note also that the forms $\omega_l$ generate a “pencil” of symplectic structures, i.e., as is easy to see, any non-trivial linear combination $\omega = \sum \lambda_l \omega_l$ is symplectic. Likewise, a linear combination $J = \sum \lambda_l J_l$ is, up to a factor, a complex structure. More precisely, $J^2 = -(\sum \lambda_l^2)I$.

Example 2.1 (Hyperkähler structures). A standard example of a vector space with such structures is a hyperkähler vector space. In this case, $r = 3$ and the complex structures $J_l$ satisfy the quaternionic relations, i.e., in addition to (2.1) we also have $J_1 J_2 = J_3$. 

Let now $W$ be a smooth compact quotient of $V$ by a group of transformations preserving all of the above structures on $W$. For instance, $W$ can be the quotient of $V$ by a lattice. (There are, however, other examples; see, e.g., [HNS, p. 2548].)

Furthermore, let us fix a closed manifold $M$ equipped with a volume form $\mu$ and $r$ divergence–free vector fields $v_1, \ldots, v_r$. This manifold will take the role of "time" in Hamiltonian dynamics. More specifically, the following two examples are of interest to us.

**Example 2.2 (The torus).** In this example, $M$ is the $r$-dimensional torus $T^r = \mathbb{R}^r / \mathbb{Z}^r$ with angular coordinates $t_1, \ldots, t_r$, the vector fields $v_l$ are the coordinate vector fields $\partial_{t_l}$, and $\mu = dt_1 \wedge \ldots \wedge dt_r$. More generally, we can replace the coordinate vector fields by any basis of vector fields with constant coefficients.

**Example 2.3 (The special unitary group SU(2)).** Here $r = 3$ and $M = SU(2)$ is equipped with the (probability) Haar measure $\mu$. The vector fields $v_l$ are the right-invariant vector fields whose values at the unit $e$ are:

$$v_1(e) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad v_2(e) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad v_3(e) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (2.2)$$

More generally, we may replace SU(2) by the homogeneous space $M = SU(2)/G$, where $G \subset SU(2)$ is a discrete subgroup. The vector fields $v_l$ naturally descend to this quotient.

By analogy with Hamiltonian dynamics, a Hamiltonian is a smooth function $H: M \times W \to \mathbb{R}$.

The action functional $A_H$ is defined on the space $E$ of $C^\infty$-smooth (or just $C^2$), null-homotopic maps $f: M \to W$. We introduce $A_H$ in two steps. First, let $F: [0, 1] \times M \to W$ be a homotopy between $f$ and the constant map. This is an analog of a capping in the definition of the standard Hamiltonian action functional. The unperturbed action functional is

$$A(f) = \sum_i \int_{[0,1] \times M} F^* \omega_l \wedge i_{v_l} \mu.$$  

It is routine to check that $A(f)$ is well-defined, i.e., independent of $F$. (Here it would be sufficient to assume that, e.g., the universal covering of $W$ is contractible.) Finally, the total or perturbed action functional is

$$A_H(f) = A(f) - \int_M H(f) \mu. \quad (2.3)$$

For instance, when $r = 1$ and $M = T^1$, we obtain the ordinary action functional of Hamiltonian dynamics. Furthermore, it is easy to see that in the setting of Example 2.2 (with $r = 3$) or of Example 2.3 the perturbed and unperturbed action functionals coincide, up to a sign, with those defined in [HNS].

The differential of $A$ at $f \in E$ is

$$(dA)_f(w) = \sum_l \int_M \omega_l (L_{v_l} f, w) \mu,$$

where $w \in T_f E$ is a vector field along $f$. Thus, the gradient of $A$ with respect to the natural $L^2$-metric on $E$ is a Dirac type operator

$$\nabla_{L^2} A(f) = \sum_l J_l L_{v_l} f =: \hat{\phi} f.$$
Hence, we have
\[ \nabla_{L^2} A_H(f) = \delta f - \nabla H(f), \]
where \( \nabla H \) denotes the gradient of \( H \) along \( W \). As a consequence, the critical points of \( A_H \) are solutions \( f \in \mathcal{E} \) of the equation
\[ \delta f = \nabla H(f). \tag{2.4} \]
At a critical point \( f \) of \( A_H \), the Hessian \( d^2_{\delta} A_H \) is defined in the standard way as the second variation of \( A_H \). This is a quadratic form on \( T_f \mathcal{E} \) equal to the \( L^2 \)-pairing with the linearization of \( \nabla_{L^2} A_H \) at \( f \). We call \( f \) a non-degenerate critical point when this operator \( T_f \mathcal{E} \to T_f \mathcal{E} \) is one-to-one, cf. [HNS, p. 2559]. A Hamiltonian \( H \) is said to be non-degenerate when all critical points of \( A_H \) are non-degenerate. In the setting of Examples 2.2 and 2.3, non-degeneracy is a generic condition on \( H \), i.e., the set of non-degenerate Hamiltonians is residual in \( C^\infty(M \times W) \). (The proof in [HNS, p. 2574–2576] covers Example 2.3 and carries over to Example 2.2 for all \( r \) with straightforward modifications.)

Finally, denote by \( CL(W) \) the cup-length of \( W \), i.e., the maximal number of elements in \( H_{*>0}(W; F) \) such that their cup-product is not equal to zero, also maximized over all fields \( \mathbb{F} \). Likewise, let \( SB(W) \) (the sum of Betti numbers) stand for \( \sum_j \dim H_j(W; F) \), maximized again over all \( \mathbb{F} \).

In the spirit of the Arnold conjecture and of [HNS], our main result is

**Theorem 2.4.** Assume that \( M \) is as in Example 2.2, or that \( V \) is hyperkähler and \( M \) is as in Example 2.3. Then for any Hamiltonian \( H \), the action functional \( A_H \) has at least \( CL(W) + 1 \) critical points. If \( H \) is non-degenerate, the number of critical points is bounded from below by \( SB(W) \).

We emphasize that the non-degenerate case of this theorem was originally proved in [HNS] in the setting of a hyperkähler target space and the domain being either \( M = SU(2) \) or \( M = T^3 \).

Theorem 2.4 suggests that in this context a version of Hamiltonian Floer theory can be developed beyond the setting where the target space \( W \) is hyperkähler and the domain \( M \) is hypercontact as in [HNS]. It appears that more generally a collection, as above, of \( r \) symplectic and complex structures on \( W \) may be sufficient for such a theory. Note however that manifolds equipped with such structures must be extremely rare, cf. [GHJ, Chapter 21]. For instance, once \( r \geq 2 \), every such a manifold is automatically hyperkähler with the third complex structure \( J_1 J_2 \). The authors are not aware of any non-flat example where \( r > 3 \). Note also that similar, although not quite identical, types of structures (at least on the complex side of the story) are considered in [MS, Jo]. Pencils of symplectic structures also arise on the point-wise (i.e., linear algebra) level on the manifolds equipped with fat fiber bundles introduced in [We1] or fat distributions; see [Mo, Section 5.6] and references therein, and also [FZ]. It is less clear what in this setting the right structure on the time manifold \( M \) should be. We examine further generalizations of the hyperkähler Arnold conjecture elsewhere.

**Remark 2.5.** In the context of Floer theory, two properties of the operator \( \delta \), hidden in our proof, are particularly important. Namely, the operator \( \delta \) and the operator \( \partial_\ast - \delta \) on \( \mathbb{R} \times M \) must both be elliptic on the space of \( V \)-valued functions on \( M \). To see when this is the case, let us assume for the sake of simplicity that the vector fields \( v_1 \) form a basis at every point of \( M \). Then \( \delta \) is elliptic if and only
the symbol \( \sigma(\phi') = \sum \lambda_i J_i \) is invertible for all non-zero (co)vectors \( \lambda = (\lambda_1, \ldots, \lambda_r) \).

This is clearly the case when, as above, the linear operators \( J_i \) are anti-commuting complex structures; for then \( \sigma(\phi')^2 = -\sum \lambda_i^2 I \). In a similar vein, \( \partial_a - \phi' \) is elliptic if and only if \( \sigma(\partial_a - \phi') = \lambda_0 I - \sum \lambda_i J_i \) is invertible for all \( (\lambda_0, \lambda) \neq 0 \). This is again automatically the case in our setting.

Remark 2.6. Theorem 2.4 extends to the case where the manifold \( W \) is a non-compact quotient of \( V \) without any significant changes in the proof. However, now certain restrictions must be imposed on the behavior of the Hamiltonian \( H \) at infinity and the lower bounds on the number of critical points may possibly depend on these restrictions. To be more specific, let us assume that a finite covering \( W' \) of \( W \) is a Riemannian product of a flat torus and a Euclidean space \( V' \). (For instance, \( W \) can be an iterated cotangent bundle of a flat manifold; it is not hard to see that this \( W \) carries the required structure.) Then it suffices to require the lift of \( H \) to \( M \times W' \) to coincide outside a compact set with a non-degenerate quadratic form on \( V' \) with constant coefficients. In this case, the lower bounds on the number of critical points are again \( \text{CL}(W) + 1 \) and, respectively, \( \text{SB}(W) \).

3. Proof of Theorem 2.4

As has been pointed out in the introduction, the argument follows closely the finite-dimensional reduction method of Conley and Zehnder, [CZ]. The method utilizes the Fourier expansion of \( f: M \to W \) over \( M \) to reduce the problem to a standard finite-dimensional Morse theory for generating functions. In fact, when \( M = \mathbb{T}^r \), the proof carries over essentially word-for-word with hardly more than notational changes. The case of \( M = \text{SU}(2) \) is more involved. For then we use Fourier analysis on \( \text{SU}(2) \) – the Peter–Weyl theorem – entailing somewhat lengthier calculations. In both cases, the main point of the proof is obtaining an explicit expression for \( \phi f \) in terms of the Fourier expansion of \( f \). Once this is done, we faithfully adhere to the line of reasoning from [CZ], and hence omit here some straightforward, technical details of the proof.

3.1. The \( \mathbb{T}^r \)-case. Throughout the proof, we will assume that \( v_1, \ldots, v_r \) are the coordinate vector fields on \( M = \mathbb{T}^r = \mathbb{R}^r / \mathbb{Z}^r \). The case of an arbitrary basis of constant vector fields can be handled in a similar way.

Furthermore, let us first assume that \( W \) is the quotient of a vector space \( V \) by a lattice. (As a consequence, \( W \) is a torus.) We will discuss the modifications needed to deal with the general case at the end of the proof.

In what follows, it will be convenient to view \( V \) as a complex vector space, equipped with one of the complex structures \( J_i \), say, \( J = J_r \). Since \( f \) is null-homotopic, it can be lifted to a map \( \hat{f}: M \to V \). Consider the Fourier expansion of \( \hat{f} \):

\[
\hat{f}(t) = \sum_k \exp(2\pi k \cdot t J) \hat{f}_k,
\]

where \( t = (t_1, \ldots, t_r) \in \mathbb{T}^r \) and \( k = (k_1, \ldots, k_r) \in \mathbb{Z}^r \) and the Fourier coefficients \( \hat{f}_k \) are elements of \( V \). Note that among these coefficients, the coefficients with \( k \neq 0 \) are completely determined by \( f \) and independent of the lift. (This is the point where it is essential that \( W \) is the quotient of \( V \) by a lattice.) The mean value \( \hat{f}_0 \) depends on the lift \( \hat{f} \), but its image in \( W \) is again completely determined by \( f \).
Hence we can, keeping the same notation $\hat{f}_0$ for the mean value, unambiguously express $f$ as
\[
    f(t) = \sum_k \exp(2\pi k \cdot t) \hat{f}_k,
\]
where now $\hat{f}_0 \in W$ and $\hat{f}_k \in V$ when $k \neq 0$.

In other words, here we view $E$ as an infinite-dimensional vector bundle over $W$ with projection map $f \mapsto \hat{f}_0$. This vector bundle is trivial and its fiber $F$ is canonically isomorphic to the space of smooth maps $M \to V$ with zero mean. The Fourier expansion allows us, using self-explanatory notation, to regard $E$ as a sub-bundle in $W \times L_0^2(M, V)$.

Our next goal is to obtain an explicit expression for $\hat{f}$ in terms of the Fourier expansion (3.1). As we will soon see, the operator $\hat{\phi}$ block-diagonalizes once we group together the $k$th and $(-k)$th terms in (3.1). (Note that since $\hat{\phi}$ kills constant terms we can view it as either a linear operator on $F$ or a fiberwise linear operator on $E = W \times F$ independent of the point of the base.) To be more precise, let $k^*$ stand for a pair $(-k, k)$, with $k \neq 0$. The pair is ordered lexicographically, i.e., so that the first non-zero component of $k$ is positive. Let $F_{k^*}$ be the subspace of $F$ formed by functions $\exp(-2\pi k \cdot t)X + \exp(2\pi k \cdot t)Y$ with $X$ and $Y$ in $V$. Note that $L_0^2(M, V)$ is the $L^2$-direct sum of the spaces $F_{k^*}$ for all pairs $k^*$. Below, we will use the identification $F_{k^*} = V \oplus V$, where the first term corresponds to $-k$ and the second one to $k$, and denote by $I$ the identity operator on $V$.

**Lemma 3.1.** The space $F_{k^*}$ is invariant under $\hat{\phi}$ and on this space, $\hat{\phi}$ acts as
\[
    A_{k^*} = 2\pi \begin{bmatrix} k_r I & -J \sum_{l=1}^{r-1} k_l J_l \\ J \sum_{l=1}^{r-1} k_l J_l & -k_r I \end{bmatrix}.
\]
Furthermore, $A_{k^*}$ is invertible and
\[
    A_{k^*}^{-1} = \frac{1}{4\pi^2 \|k\|^2} A_{k^*},
\]
where $\|k\|^2 = k_1^2 + \ldots + k_r^2$, and
\[
    \|A_{k^*}^{-1}\| = \frac{1}{2\pi \|k\|}.
\]

**Remark 3.2.** This lemma is more precise than is really necessary for the proof. In fact, explicit expressions for $A_{k^*}$, its inverse and the norm of the inverse are irrelevant. It would be sufficient to just know that $A_{k^*}$ is invertible and that $\|A_{k^*}^{-1}\| = O(1/\|k\|)$.

**Proof of the lemma.** Recall that $M$ is a torus, $v_l = \partial / \partial t_l$ and
\[
    \hat{\phi} f = \sum_l J_l \frac{\partial f}{\partial t_l}.
\]
Thus, as a straightforward calculation shows,
\[
    \hat{\phi} f = 2\pi \sum_k \exp(2\pi k \cdot t) \left( J \sum_{l=1}^{r-1} k_l J_l \hat{f}_{-k} - k_r \hat{f}_k \right).
\]
Here we use the fact that $J = J_r$ anti-commutes with $J_l$ for $l = 1, \ldots, r - 1$. This expression shows that $F_{k^*}$ is invariant under $\hat{\phi}$ and immediately implies (3.2). Now (3.3) is straightforward to check using again the fact that the complex structures $J_l$ with $l = 1, \ldots, r$ anti-commute. To finish the proof of the lemma, it remains to
establish (3.4). (The estimate $\|A_{k^*}^{-1}\| = O(1/|k|)$ mentioned in Remark 3.2 is an easy consequence of (3.3)).

The exact expression (3.4) can either be verified by a direct calculation or proved as follows. Namely, using again the fact that all complex structures $J_i$ anti-commute and are orthogonal operators, it is easy to check that $(J \sum k_i J_i)^\perp = -J(\sum k_i J_i)$. Then, from (3.2) and (3.3), we infer that $A_{k^*}$ and $A_{k^*}^{-1}$ are self-adjoint. Using again (3.3), we have

$$\langle A_{k^*}^{-1}Z, A_{k^*}^{-1}Z \rangle = \langle Z, A_{k^*}^{-1}A_{k^*}^{-1}Z \rangle = \frac{1}{4\pi^2|k|} \langle Z, A_{k^*}^{-1}A_{k^*}^{-1}Z \rangle = \frac{1}{4\pi^2|k|^2} \|Z\|^2$$

for any $Z \in F_{k^*}$. This proves (3.4) and completes the proof of the lemma. □

The rest of the argument, closely following [CZ], has become quite standard by now and is included here only for the sake of completeness. Denote by $\mathcal{F}_N$ the subspace in $\mathcal{F}$ formed by smooth maps $f$ with $\tilde{f}_k = 0$ whenever $|k| \geq N$. In other words, $\mathcal{F}_N$ consists of Fourier polynomials of degree less than $N$, where the degree is defined as $|k|$ in place of the more conventional $|k| = |k_1| + \ldots + |k_r|$. Furthermore, let $\mathcal{F}_N^\perp$ be the $L^2$-orthogonal complement of $\mathcal{F}_N$ in $\mathcal{F}$, i.e., $\mathcal{F}_N^\perp$ is the space of smooth maps $f$ with $\tilde{f}_k = 0$ whenever $|k| < N$. We can view $E_N := W \times \mathcal{F}_N$ as a subbundle in $\mathcal{E}$. It will also be useful to regard $\mathcal{E}$ as a vector bundle over $E_N$ with fiber $\mathcal{F}_N$. Denote by $P_N$ the (fiberwise) $L^2$-orthogonal projection of $\mathcal{E}$ onto $E_N$ and by $P_N^\perp$ the projection of $E = E_N \times F_N^\perp$ onto the second component $F_N^\perp$.

As is clear from Lemma 3.1, the operator $\partial|_{\mathcal{F}_N}$ is invertible. Its inverse, which we denote by $\partial^{-1}_N$, is $L^2$-bounded. Hence, $\partial^{-1}_N$ extends by continuity to the $L^2$-completion $\bar{\mathcal{F}}_N$ of $\mathcal{F}_N$. (The space $\bar{\mathcal{F}}_N$ is formed by $L^2$-maps $f: M \to V$ with zero mean such that $\tilde{f}_k = 0$ for all $k$ with $|k| < N$.) Furthermore, again by Lemma 3.1, we see that $\|\partial^{-1}_N\|_{L^2} \leq 1/2\pi N$ and $\partial^{-1}_N$ sends functions of Sobolev class $H^s$ to functions of class $H^{s+1}$. (The latter statement is, of course, also a consequence of the fact, mentioned in Remark 2.5, that $\partial$ is a first order elliptic operator; see, e.g., [LM, Chap. III].)

Our goal is to show that equation (2.4) has at least the desired number of solutions. Let $f = g + h$ with $g \in E_N$ and $h \in \bar{\mathcal{F}}_N$. Clearly, $f$ satisfies (2.4) if and only if we have

$$\partial g = P_N \nabla H(g + h) \quad (3.5)$$

and

$$\partial h = P_N^\perp \nabla H(g + h). \quad (3.6)$$

Let us focus on the second of these equations with $g$ fixed and both sides viewed as functions of $h$, cf. [CZ]. Clearly, (3.6) is equivalent to

$$h = \partial^{-1}_N P_N^\perp \nabla H(g + h). \quad (3.7)$$

Note that the right hand side is now defined for all $h \in \bar{\mathcal{F}}_N$ without any smoothness requirement. We claim that when $N$ is large enough, for any $g \in E_N$, equation (3.7) (and hence (3.6)) has a unique solution $h = h(g)$ and this solution is smooth.

To show this, note first that, when $N$ is sufficiently large, $h \mapsto \partial^{-1}_N P_N^\perp \nabla H(g + h)$ is a contraction operator on $\bar{\mathcal{F}}_N$ with respect to the $L^2$-norm. Indeed,

$$\|\partial^{-1}_N P_N^\perp \nabla H(g + h_1) - \partial^{-1}_N P_N^\perp \nabla H(g + h_0)\|_{L^2} \leq \frac{1}{2\pi N} \|\nabla H(g + h_1) - \nabla H(g + h_0)\|_{L^2}$$
and, in obvious notation,

\[ \| \nabla H(g + h_1) - \nabla H(g + h_0) \|_{L^2} = \left\| \int_0^1 \frac{d}{ds} \nabla H(g + sh_1 + (1 - s)h_0) \, ds \right\|_{L^2} \leq \| \nabla^2 H \|_{L^\infty} |h_1 - h_0|_{L^2}. \]

Hence,

\[ \| \phi_N^{-1} P_N^+ \nabla H(g + h_1) - \phi_N^{-1} P_N^+ \nabla H(g + h_0) \|_{L^2} \leq O(1/N) |h_1 - h_0|_{L^2}, \]

which shows that we can indeed choose \( N \) such that the map \( h \mapsto \phi_N^{-1} P_N^+ \nabla H(g + h) \) is a contraction. The fact that the fixed point \( h = h(g) \) of this operator is a smooth function is established by the standard bootstrapping argument. Namely, we have \( \phi h = P_N^+ \nabla H(g + h) \in L^2 = H^0 \), and therefore \( h \in H^1 \). Now, since \( H \) and \( g \) are smooth, we also have \( P_N^+ \nabla H(g + h) \in H^1 \), and hence \( h \in H^2 \), etc.

From a more geometrical perspective, \( h(g) \) is the unique critical point of the action functional \( A_H \) on the fiber over \( g \) of the vector bundle \( E \to \mathcal{E}_N \). Set \( \Phi(g) := A_H(g + h(g)) \). In other words, \( \Phi \) is obtained from \( A_H \) by restricting the action functional to the section \( g \mapsto h(g) \) of this vector bundle, formed by the fiber-wise critical points. Therefore, \( g \) is a critical point of \( \Phi \) if and only if \( f = g + h(g) \) is a critical point of \( A_H \), i.e., a solution of (2.4), and every critical point of \( A_H \) is captured in this way. It remains to show that the generating function \( \Phi \) on \( \mathcal{E}_N \) has the required number of critical points.

The key feature of this function is that it is asymptotically (i.e., at infinity in the fibers of \( \mathcal{E}_N \)) a non-degenerate quadratic form. To be more precise, set

\[ \Phi_0(g) = A(g) = \langle \phi g, g \rangle_{L^2} \quad \text{and} \quad R = \Phi - \Phi_0. \]

The unperturbed action \( \Phi_0 \) is a fiberwise non-degenerate quadratic form. By definition, \( \nabla \Phi_0(g) = \phi g \). (The quadratic form \( \Phi_0 \) has zero signature, but this is not essential for what follows.) Furthermore, the perturbation \( R \) is small compared to \( \Phi_0 \), when \( N \) is sufficiently large. Namely, for our purposes it is sufficient to show that fiberwise

\[ |R| + |\nabla R| < \| \nabla \Phi_0 \| \quad \text{outside a compact set.} \quad (3.8) \]

Here and throughout the rest of the proof, the metric on \( \mathcal{E}_N = W \times \mathcal{F}_N \) is the product of the fiberwise \( L^2 \)-metric and the metric on \( W \).

To establish (3.8), note first that \( H \) and \( \nabla H \) are bounded; for \( H \) is a function on a compact manifold. Therefore, the integral of \( H \) makes a bounded contribution to \( R \) and \( \nabla R \), while the right hand side of (3.8) grows linearly as \( g \to \infty \) in the fiber. Thus, we can ignore \( H \) in (3.8) and only need to estimate the growth of the difference

\[ R_0 := A(g + h(g)) - A(g) = 2 \langle \phi g, h(g) \rangle + \langle \phi h(g), h(g) \rangle, \]

or to be more precise of \( |R_0| \) together with \( |\nabla R_0| \). First observe that \( |R_0(g)| \) is bounded by \( O(1/N)(\| \nabla \Phi_0(g) \| + 1) \). (Here and below all the bounds are in the \( L^2 \)-norm.) This follows from the facts that the function \( g \mapsto h(g) \) is uniformly bounded by a constant \( O(1/N) \), due to (3.7), and that the function \( g \mapsto \phi h(g) \) is uniformly bounded, due to (3.6). In a similar vein, it is not hard to show that \( \| \nabla R_0(g) \| \) is bounded from above by \( O(1) + O(1/N)\| \nabla \Phi_0(g) \| \). (To this end, one also uses the fact that the derivative of the function \( g \mapsto h(g) \) is uniformly bounded by a constant \( O(1/N) \), as can be seen by differentiating (3.7) with respect to \( g \).) Together, these upper bounds prove (3.8).
A similar argument shows that a critical point \( g \) of \( \Phi \) is non-degenerate when \( f = g + h(g) \) is a non-degenerate critical point of \( A_H \).

Finally, recall that whenever \( \Phi = \Phi_0 + R \) is a function on the total space of a vector bundle over an arbitrary closed manifold \( W \), such that \( \Phi_0 \) is a fiberwise non-degenerate quadratic form and (3.8) holds, the function \( \Phi \) has at least \( CL(W) + 1 \) critical points. Moreover, when \( \Phi \) is Morse, the number of critical points is bounded from below by \( SB(W) \). This is a standard fact and we refer the reader to [CZ] for the original proof and to, e.g., [We2] for a different argument. (Here we only mention that the requirement (3.8) enables one to modify \( \Phi \) outside a sufficiently large compact set, without creating new critical points, to turn it into a function identically equal to \( \Phi_0 \) at infinity.)

Turning to the general case where \( W \) is the quotient of \( V \) by a group \( \Gamma \), we argue as follows. First recall that \( \Gamma \) contains a finite-index subgroup \( \Gamma' \) consisting of only parallel transports, [Wo, p. 110]. Thus \( W' = V/\Gamma' \) is a torus and the projection \( W' \to W \) is a covering map with the finite group \( \Pi = \Gamma/\Gamma' \) acting as the group of deck transformations. The previous argument applies to the natural lift of the problem to \( W' \) and the entire construction is \( \Pi \)-equivariant. As a result, we obtain a vector bundle \( E'_N \to W' \) equipped with a \( \Pi \)-action covering the \( \Pi \)-action on \( W' \) and a \( \Pi \)-invariant function \( \Phi' \) on \( E'_N \), which is asymptotically quadratic at infinity. The critical points of \( A_H \) for the original problem correspond to the \( \Pi \)-orbits of the critical points of \( \Phi' \). Passing to the quotient by \( \Pi \), we arrive at a vector bundle over \( W \) and a smooth function \( \Phi \) on its total space \( E'_N/\Pi \). (The total space is smooth; for the \( \Pi \)-action on \( E'_N \), it is free as an action covering a free action on \( W' \).) The function \( \Phi \) is asymptotically quadratic and its critical points are in one-to-one correspondence with the critical points of \( A_H \) for the original problem.

The theorem now follows as before from the lower bounds on the number of critical points of \( \Phi \).

### 3.2. The SU(2)-case.

Let us now consider the setting where \( M = SU(2) \) and \( r = 3 \) and \( W \) is the quotient, by a lattice, of a hyperkähler vector space \( V \) with complex structures \( J_1, J_2 \) and \( J_3 \). (In particular, \( W \) is a torus.) The case of a more general quotient \( W = V/\Gamma \) can be reduced to this one exactly as in Section 3.1; see the previous paragraph. Furthermore, the case where \( M = SU(2)/G \) does not present any new difficulties and in fact follows from the argument below. Throughout the rest of the proof, we will treat \( V \) as a real vector space or as a complex vector space with complex structure \( J = J_3 \). Let us also fix a Hermitian inner product on \( V \), which, when necessary, we can also view as a real inner product by discarding the imaginary part.

The space \( L^2(SU(2), V) \) is a unitary representation of \( SU(2) \), which, by the Peter–Weyl theorem, decomposes into an \( L^2 \)-sum of irreducible representations \( P_k \), \( k = 0, 1, 2, \ldots \), of \( SU(2) \) with \( P_k \) entering the sum with multiplicity \( \dim_C (P_k \otimes V) \); see, e.g., [Bo].

The irreducible representation \( P_k \) is the natural representation of \( SU(2) \) on the space of homogeneous polynomials of degree \( k \) in two complex variables \( z_1 \) and \( z_2 \). The \( SU(2) \)-action on \( P_k \) is given by \( x \cdot p = p \circ x^{-1} \) for \( p \in P_k \) and \( x \in SU(2) \). Let us turn \( P_k \) into a unitary representation by fixing a Hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( P_k \) which is invariant under the group action. (Note that that such an inner product is unique up to a factor; the normalization of the inner product is immaterial for what follows.) Set \( e_a^{(k)} = z_1^a z_2^k - z_2^a z_1^k \) for \( a = 0, \ldots, k \). This is an orthogonal basis of \( P_k \).
with respect to $\langle \cdot, \cdot \rangle$. The matrix coefficients $c^{(k)}_{a,b}: \text{SU}(2) \to \mathbb{C} \text{ for } a, b \in \{0, \ldots, k\}$ are defined as

$$c^{(k)}_{a,b}(x) = \langle x \cdot e^a, e^b \rangle.$$

These are complex-valued functions on $\text{SU}(2)$. With $i$ acting as $J = J_3$, we will view matrix coefficients as $\text{GL}(V)$-valued functions.

As in the torus case, the domain $\mathcal{E}$ of the action functional $\mathcal{A}_H$ consists of smooth null-homotopic functions $f: \text{SU}(2) \to W$. Such a function $f$ lifts to an $L^2$-map $\tilde{f}: \text{SU}(2) \to V$. Using the Peter–Weyl theorem, we can decompose $\tilde{f}$ as

$$\tilde{f}(x) = \sum_{k \geq 0} \sum_{a, b = 0}^k c^{(k)}_{a,b}(x) \hat{f}^{(k)}_{a,b}.$$ 

Here the sum converges in $L^2(\text{SU}(2), V)$, the terms are mutually $L^2$-orthogonal, and the Fourier coefficients $\hat{f}^{(k)}_{a,b} \in V$ are uniquely determined by $\tilde{f}$. (The same of course holds for any $V$-valued $L^2$-function on $\text{SU}(2)$.)

It is essential for what follows that the coefficients of the non-constant matrix elements, i.e., the vectors $\hat{f}^{(k)}_{a,b} \in V$ for $k \neq 0$, depend only on $f$ and are independent of the lift. As in the torus case, we can therefore write, slightly abusing notation,

$$f(x) = \sum_{k \geq 0} \sum_{a, b = 0}^k c^{(k)}_{a,b}(x) \hat{f}^{(k)}_{a,b},$$

where $\hat{f}^{(k)}_{a,b} \in V$ when $k \neq 0$ and the mean value $\hat{f}^{(0)}_{0,0}$ is an element of $W$. Thus, the space $\mathcal{E}$ can be viewed as an infinite-dimensional vector bundle over $W$ with projection map $f \mapsto \hat{f}^{(0)}_{0,0}$. This vector bundle is trivial and its fiber $\mathcal{F}$ is canonically isomorphic to the space of smooth maps $\text{SU}(2) \to V$ with zero mean.

For a fixed $k > 0$, we denote by $F_k$ the subspace of $L^2(\text{SU}(2), V)$ which is spanned by all functions $c^{(k)}_{a,b}(x)w$ for $w \in V$ and $a, b \in \{0, \ldots, k\}$. This is the subspace formed by the functions $f$ such that $\hat{f}^{(l)}_{a,b} = 0$ for $l \neq k$.

We are now in a position to find an explicit representation of the operator $\hat{\delta}f$ in terms of the Fourier expansion of $f$. The image of a function $f$ under $\hat{\delta}$ is independent of the mean value $\hat{f}^{(0)}_{0,0}$, since the constant term is killed by the derivatives in $\hat{\delta}$. Therefore, we can view $\hat{\delta}$ as a fiberwise linear map on $\mathcal{E} = W \times \mathcal{F}$, which is independent of the point in the base $W$. Our goal is to block-diagonalize $\hat{\delta}$. In what follows, it is useful to keep in mind that this operator is not complex linear.

In order to identify the invariant subspaces of $\hat{\delta}$, we utilize the decomposition of $\mathcal{F}$ over irreducible representations along with the quaternionic structure on $V$. To be more precise, recall that $V$ is not just a complex vector space, but also a quaternionic vector space: for the complex structures $J_1$, $J_2$, $J_3$ satisfy the quaternionic relations. Thus, we can decompose $V$ as the sum of four real vector spaces intertwined by the operators $J_m$, i.e., $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 = V_0^2$, where $V_m = J_m V_0$ for $m = 1, 2, 3$. Let us denote by $I$ the identity map on $V$ or $V_0$.

**Lemma 3.3.** The operator $\hat{\delta}$ preserves the subspaces $F_k$ and on each of this subspaces block-diagonalizes as the sum of the following operators:

(i) **the scalar operator** $k \cdot I$ on the space $c^{(k)}_{a,0} V$,

(ii) **the operator**
Using this, we can rewrite (3.10) on the subspace $e_{a,b}^{(k)} V_0 \oplus e_{k-a,k-b+1}^{(k)} V_2 \cong V_0 \oplus V_0$ for $b \in \{1, \ldots, k\}$.

(iii) the operator

\[
\begin{bmatrix}
(k - 2b)I & (-1)^{a+b+2}(k - b + 1)I \\
(-1)^{a+b+2}bI & (2b - k - 2)I
\end{bmatrix}
\]

on the space $e_{a,b}^{(k)} V_1 \oplus e_{k-a,k-b+1}^{(k)} V_3 \cong V_0 \oplus V_0$ for $b \in \{1, \ldots, k\}$.

Furthermore, on each of the subspaces $F_k$ with $k > 0$, the operator $\phi$ is invertible and its inverse has norm $1/k$.

Remark 3.4. As in the torus case, this lemma is more precise than is really necessary for the proof. It would be sufficient to know that $\phi|_{F_k}$ is invertible for $k > 0$ and that its inverse has norm $O(1/k)$.

Proof. First let us determine the matrix representation of the operator $\phi$ and show that the subspaces $F_k$ are invariant. Recall that the operator $\phi$ is given by

\[
\phi = J_1 L_{v_1} + J_2 L_{v_2} + J_3 L_{v_3},
\]

where the right-invariant vector fields $v_l$ are defined by their values at the identity as in (2.2). Computing the Lie derivatives yields

\[
J_1 L_{v_1} e_{a,b}^{(k)} = J_1 J_3 \left( b e_{a,b-1}^{(k)} + (k - b) e_{a,b+1}^{(k)} \right),
\]

\[
J_2 L_{v_2} e_{a,b}^{(k)} = J_2 \left( -b e_{a,b-1}^{(k)} + (k - b) e_{a,b+1}^{(k)} \right)
\]

and

\[
J_3 L_{v_3} e_{a,b}^{(k)} = J_3 \left( -J_3 (k - 2b) e_{a,b}^{(k)} \right).
\]

Here we set $e_{a,-1}^{(k)} = 0 = e_{a,k+1}^{(k)}$. (In fact, the actual definition of these functions is immaterial since they enter the formulas with zero coefficients.) Taking into account the quaternionic relations between the complex structures, we obtain

\[
\phi e_{a,b}^{(k)} = (k - 2b) e_{a,b}^{(k)} - J_2 2b e_{a,b-1}^{(k)}.
\]

(3.12)

When $b = 0$, this is the result of part (i) of the lemma.

To deal with the case $b \neq 0$, recall first that $J_2$ is not a complex linear operator on $V$: it does not commute with $J = J_3$. However, it anti-commutes with $J$, i.e., $J_2 J = -J J_2$, and hence

\[
J_2 e_{a,b}^{(k)} = e_{a,b}^{(k)} J_2,
\]

since the matrix coefficients are C-valued functions. A direct calculation of the matrix coefficients or an argument using the conjugate representation of $SU(2)$ yields that

\[
\overline{e_{a,b}^{(k)} J_2} = (-1)^{a+b} e_{k-a,k-b}^{(k)} J_2.
\]

Using this, we can rewrite (3.12) for $b \neq 0$ as

\[
\phi e_{a,b}^{(k)} = (k - 2b) e_{a,b}^{(k)} + (-1)^{a+b+2}b e_{k-a,k-b+1}^{(k)} J_2.
\]

With the identifications $V_0 \oplus J_2 V_0 = V_0 \oplus V_2 \cong V_0 \oplus V_0$ and $V_3 \oplus J_2 V_3 \cong V_3 \oplus V_3$, this formula immediately implies the matrix representations given in parts (ii) and (iii).
Let us now turn to the “moreover” part of the lemma and prove the bounds on the inverse of $\hat{\phi}$. It is not hard to check that the operator $\hat{\phi}$ is invertible on $F_k$ for $k > 0$ by computing the eigenvalues of the matrices. In the case $b = 0$ in part (i), it is clear that $k$ is the only eigenvalue. On the subspaces considered in parts (ii) and (iii), one easily computes the eigenvalues to be $k$ and $-k - 2$. Thus, zero is not an eigenvalue for $k > 0$ and the operator is invertible and, moreover, the inverse of $\hat{\phi}|_{F_k}$, has eigenvalues $1/k$ and $-1/(k + 2)$. Furthermore, the eigenvectors are mutually orthogonal since $\hat{\phi}$ is self-adjoint. This shows that the norm of the inverse of $\hat{\phi}|_{F_k}$ is indeed $1/k$ as stated in the lemma. $\square$

The remaining part of the proof of Theorem 2.4 goes through almost word-for-word as in the torus case and in [CZ]. Recall that we view the space of all null-homotopic functions as a trivial vector bundle $E = W \times F$ and that the fiber $F$ is the direct sum of the subspaces $F_k$ for $k > 0$. Closely following the reasoning in the torus case (see Section 3.1), we denote the direct sum of $F_k$ for $0 < k < N$ by $F_N$ and its $L^2$-orthogonal complement by $F_N^\perp$. Thus, $F_N$ consists of all functions with $f_a^{(k)} = 0$ for $k \geq N$. For $f \in F_N^\perp$, we have $f_a^{(k)} = 0$ for $k < N$.

Set $\mathcal{E}_N = W \times F_N$ and denote the fiberwise orthogonal projection of $\mathcal{E}$ onto $\mathcal{E}_N$ by $P_N$. Let $\mathcal{P}_N$ again denote the projection of $\mathcal{E} = \mathcal{E}_N \times F_N^\perp$ onto the second component. By Lemma 3.3, the restriction of the operator $\hat{\phi}$ to $F_N^\perp$ is invertible and, on $F_N^\perp$, the $L^2$-norm of the inverse $\hat{\phi}_N^{-1} := (\hat{\phi}|_{F_N^\perp})^{-1}$ is bounded by $O(1/N)$. As a consequence, this operator extends by continuity to the $L^2$-completion $\bar{F}_N^\perp$ of $F_N^\perp$.

We need to show that equation (2.4) has at least the desired number of solutions. As in Section 3.1, we write $f = g + h$ with $g \in \mathcal{E}_N$ and $h \in F_N^\perp$ and break the equation (2.4) into equations (3.5) and (3.6). For a fixed $g \in \mathcal{E}_N$, equation (3.6) gives rise to the fixed point problem

$$h = \hat{\phi}_N^{-1} P_N \nabla H (g + h)$$

(3.13)

for $h \in F_N^\perp$, where the right hand side is defined for all $h \in F_N^\perp$ without any smoothness requirement. We claim that for any sufficiently large $N$ and any $g \in \mathcal{E}_N$, equation (3.13) has a unique solution $h = h(g)$ and that this solution is smooth.

The existence of the solution $h(g)$ is established by the same argument as in the torus case. Namely, the Hamiltonian $H$ is smooth and compactly supported. Thus, $H$ and $\nabla H$ are uniformly bounded by a constant. The norm of $\hat{\phi}_N^{-1}$ is bounded by $O(1/N)$, due to Lemma 3.3. For fixed $g$ and $H$, we can therefore choose $N$ sufficiently large so that the operator $h \mapsto \hat{\phi}_N^{-1} P_N \nabla H (g + h)$ is a contraction. This proves the existence and uniqueness of the fixed point $h(g)$.

To show that $h(g)$ is smooth, we invoke elliptic regularity. Namely, recall that, since $\hat{\phi}$ is a first order elliptic operator (see Remark 2.5), a solution $h$ of the equation $\hat{\phi}h = y$ is of Sobolev class $H^{s+1}$ whenever $h$ and $y$ are $H^s$; see, e.g., [LM, Chap. III]. Applying this to $y = P_N \nabla H (g + h)$ and using the standard bootstrapping argument as in Section 3.1, we conclude that $h$ is $C^{\infty}$-smooth.

From here on, the argument from the torus case applies without any modifications. The calculations in Section 3.1 are independent of the specific setting of the torus case, relying only on the definition of the action $A_H$ by (2.3). The function $\Phi$ is asymptotically a non-degenerate quadratic form in the fibers of the bundle $E \to \mathcal{E}_N$. (Note that, in contrast to the torus case, the quadratic form $\Phi$ does not have zero signature on the subspaces $F_k$. However, this is not relevant for the proof
of the theorem.) A critical point \( g \) of \( \Phi \) is non-degenerate if and only if \( f = g + h(g) \) is a non-degenerate critical point of \( \mathcal{A}_H \).

Finally, recall that, as was already mentioned in Section 3.1, a function \( \Phi \) on the total space of a vector bundle over a closed manifold \( W \) has at least \( \text{CL}(W) + 1 \) critical points, whenever \( \Phi \) is asymptotically a non-degenerate quadratic form and \( \Phi \) satisfies (3.8). Moreover, when \( \Phi \) is Morse, the number of critical points is bounded from below by \( \text{SB}(W) \). This completes the proof of the theorem.

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