Black Holes with Zero Mass

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abstract

We consider the spacetimes corresponding to static Global Monopoles with interior boundaries corresponding to a Black Hole Horizon and analyze the behavior of the appropriate ADM mass as a function of the horizon radius \( r_H \). We find that for small enough \( r_H \), this mass is negative as in the case of the regular global monopoles, but that for large enough \( r_H \) the mass becomes positive encountering an intermediate value for which we have a Black Hole with zero ADM mass.

I. INTRODUCTION

Global monopoles are topological defects that arise in certain theories where a global symmetry is spontaneously broken. The simplest and most studied example is the \( O(3) \) model, in which one finds that the static, spherically symmetric solutions, have in general an energy density that decreases for large distances as \( 1/r^2 \) \cite{1}. This would lead in a Newtonian analysis to a divergent expression for the total mass. When we turn to the general relativistic analysis, this problem translates in the fact that the resulting spacetime is not asymptotically flat, and thus the standard ADM mass is not well defined. The effect of the \( 1/r^2 \) behavior of the density on the spacetime is that the latter develops a deficit angle at large distances. The fact that, at small distances, the behavior deviates from that, results in the appearance of a small phenomenological ”core mass” which turns out to be negative in all cases considered \cite{2}. Moreover, an analysis of the behavior of geodesics in the large distance regime does indeed support such interpretation of this core mass because its effect turns out to be repulsive \cite{2}. This “core mass” is then evidently not the standard ADM mass. The question of what exactly one is talking about when referring to this core mass has been resolved in \cite{3} through the application of the standard type of Hamiltonian analysis to the class of spacetimes that are Asymptotically-flat-but-for-a-deficit-angle (AFDA\( \alpha \)) \cite{3}.

For these (AFDA\( \alpha \)) spacetimes one can also define future and past conformal null infinity, and thus the notion of a black hole and of its horizon. In fact solutions corresponding to global monopoles with such interior horizons have been found in \cite{4}, \cite{5}.

In this paper we study the behavior of the ADM mass of these AFDA\( \alpha \) spacetimes as a function of the horizon area, concentrating in particular on its sign, which we find changes in the regime where one would interpret as going from a situation that would be naturally described as a “black hole

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inside a monopole core” to that which would be naturally described as a “black hole with a global monopole inside”.

We shall adhere to the following conventions on index notation in this paper: Greek indices \((\alpha, \beta, \mu, \nu,...)\) range from 0 to 3, and denote tensors on (four-dimensional) spacetime. Latin indices, alphabetically located after the letter \(i\) (i,j,k,...) range from 1 to 3, and denote tensors on a spatial hipersurface \(\Sigma\); whereas Latin indices, from the beginning of the alphabet (a,b,c,d,...) range from 1 to 3, and denote indices in the internal space of the scalar fields. The metric for the internal space is just the flat Euclidean metric \(\delta_{ab}\). The signature of the spacetime metric \(g\) is \((- , +, +, +)\).

Geometrized units, for which \(G_N = c = 1\) are used in this paper.

II. THE GLOBAL MONOPOLE SPACETIME

The theory of a scalar field with spontaneously broken internal \(O(3)\) symmetry, minimally coupled to gravitation, is described by the action:

\[
S = \int \sqrt{\left(-g\right)} \left[\frac{1}{16\pi}R - \frac{1}{2}(\nabla^\mu \phi_a)(\nabla_\mu \phi^a) - V(\phi)\right] d^4x.
\]  

(1)

where \(R\) is the scalar curvature of the spacetime metric, \(\phi_a\) is a triplet of scalar fields, and \(V(\phi)\), is potential depending only on the magnitude \(\phi = (\phi_a \phi^a)^{1/2}\), which we will take to be the “Mexican Hat” \(V(\phi) = (\lambda/4)(\phi^2 - \eta^2)^2\).

The gravitational field equations following from the Lagrangian (1) can be written as

\[
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi T^{\mu\nu}
\]  

(2)

where

\[
T^{\mu\nu} = \nabla^\mu \phi^a \nabla_\nu \phi_a - g^{\mu\nu} \left[ \frac{1}{2}(\nabla \phi^a)(\nabla \phi_a) + V(\phi) \right].
\]  

(3)

The equation of motion for the scalar fields become

\[
\Box \phi^a = \frac{\partial V(\phi)}{\partial \phi_a}.
\]  

(4)

We are interested in spacetimes with topology \(\Sigma \times R\), where \(\Sigma\) has the topology of \((R^3 - B) \cup C\), with \(B\) a 3-ball, and \(C\) a compact manifold with \(S^2\) boundary.

We will focus on the sector corresponding to the asymptotic behavior characteristic of the Hedgehog ansatz:

\[
\phi^a \approx \eta x^a / r.
\]  

(5)

where the \(x^a\) are asymptotic Cartesian coordinates. Within this sector, there is a static, spherically
symmetric solution \[1\] with metric given by:

\[ds^2 = -B(r)dt^2 + S(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2),\] (6)

and scalar field

\[\phi^a = \eta f(r)x^a/r\] (7)

and with the following asymptotic behavior of,

\[B \approx S^{-1} \approx 1 - \alpha - 2M/r + O(1/r^2), \quad f \approx 1 + O(1/r^2)\] (8)

where \(\alpha = 8\pi\eta^2\). Redefining the \(r\) and \(t\) coordinates as \(r \rightarrow (1 - \alpha)^{1/2}r\) and \(t \rightarrow (1 - \alpha)^{-1/2}t\), respectively, and defining \(\tilde{M} = M(1 - \alpha)^{-3/2}\), we obtain the asymptotic form for the metric:

\[ds^2 = -(1 - 2\tilde{M}/r)dt^2 + (1 - 2\tilde{M}/r)^{-1}dr^2 + (1 - \alpha)r^2(d\theta^2 + \sin^2(\theta)d\varphi^2).\] (9)

As we mentioned it is natural to associate the parameter \(\tilde{M}\) with the mass of the configuration because it can be seen that the proper acceleration of the \((\theta, \varphi, r) = \text{constant}\) world lines is \(a = -\tilde{M}/(r(r - 2\tilde{M}))\). However, as we explained before it is not the standard ADM mass. This is also evident from the fact that in the specific solution \(\tilde{M}\) turns out to be negative \[2\], while the matter certainly satisfy the dominant energy condition, under which the ADM mass of a regular solution would be positive \[3\], \[4\].

These issues are clarified by the introduction of concept of asymptotically-flat-but-for-a-deficit-angle spacetimes (A.F.D.A \(\alpha\)) and the standard asymptotically-flat-but-for-a-deficit-angle spacetime (S.A.F.D.A \(\alpha\)). The ADM mass of any spatial hypersurface of the former being defined in terms of its spatial metric and the metric of a particular slice of the S.A.F.D.A \(\alpha\) spacetime (see \[3\] for details).

In fact we take as the S.A.F.D.A \(\alpha\) spacetime the metric,

\[ds^2 = -dt^2 + dr^2 + (1 - \alpha)r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)\] . (10)

The natural generalization of the ADM mass for the A.F.D.A \(\alpha\) spacetimes is

\[16\pi(1 - \alpha)M_{ADM\alpha} = \int_{\partial\Sigma} dS_i(h_{ij}h^{0ij} - h_{ij}h^{0ij})D_j^0(h_{jk}).\] (11)

where \(h_{ij}\) is the spatial metric of the slice of the spacetime for which the ADM\(\alpha\) mass will be evaluated, \(h_{ij}^{0}\) is the spatial metric of a static slice of the S.A.F.D.A \(\alpha\) spacetime, and \(D_j^0\) the covariant derivative associated with the latter (in fact, it looks just like the usual ADM formula, but with the quantities associated with the flat metric replaced by the S.A.F.D.A.\(\alpha\) metric), and just like this, it
is the numerical value of the true Hamiltonian (a true generator of “time translations”); so, it is natural to interpret this as the mass (or energy) of the A.F.D.A. spacetimes.

Let us write a static spherically symmetric metric in the form

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) e^{-2\tilde{r}} dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + (1 - \alpha) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .$$

(12)

where the functions $m(r)$ and $\delta(r)$ depend only on the radial coordinate $r$.

We also introduce the following dimensionless quantities

$$\tilde{r} := r \cdot \eta \lambda^{1/2} ,$$

(13)

$$\tilde{m} := m \cdot \eta \lambda^{1/2} ,$$

(14)

$$\tilde{E} := E \cdot \frac{1}{\eta^2 \lambda} ,$$

(15)

$$\alpha := 8\pi \eta^2 ,$$

(16)

Introducing the metric (12) and the ansatz (1) in the expressions (2) and (4), we obtain the final form of the equations of motion:

$$\partial_{\tilde{t}} \tilde{m} = 4\pi \tilde{r}^2 \tilde{E} - \frac{\alpha}{2(1 - \alpha)} ,$$

(17)

$$\partial_{\tilde{r}} \delta = - \frac{\alpha \tilde{r}}{2} (\partial_{\tilde{r}} \tilde{f})^2 ,$$

(18)

$$\partial_{\tilde{r}} \tilde{f} = - \left[ \frac{2}{\tilde{r}} - \partial_{\tilde{r}} \delta - 2A^2 \left( 4\pi \tilde{r} \tilde{E} - \frac{\alpha}{2(1 - \alpha) \tilde{r}} - \frac{\tilde{m}}{\tilde{r}^2} \right) \right] (\partial_{\tilde{r}} \tilde{f})$$

$$+ A^2 \left[ f (f^2 - 1) + \frac{2f}{(1 - \alpha) \tilde{r}^2} \right] ,$$

(19)

where

$$A^2 = \left( 1 - \frac{2\tilde{m}}{\tilde{r}} \right)^{-1} ,$$

(20)

and

$$\tilde{E} = \frac{\alpha}{8\pi} \left[ \frac{(\partial_{\tilde{r}} \tilde{f})^2}{2A^2} + \frac{f^2}{(1 - \alpha) \tilde{r}^2} + \frac{(f^2 - 1)^2}{4} \right] .$$

(21)

In this paper, we will make use of the fact that when the metric is written as (12), the formula (11) for the ADM $\alpha$ mass of the spacetime can be expressed as:

$$M_{\text{ADM} \alpha} = M = \lim_{r \to \infty} m(r) .$$

(22)

Regular solutions require the following boundary conditions at the origin $f(0) = 0$, $\tilde{m}(0) = 0$, $\tilde{m}(0)\tilde{r} = -\alpha/(2(1 - \alpha))$ and using a standard shooting method to compute $f(0)\tilde{r}$. The solutions
are found by the standard one parameter shooting method [8]. This is possible because the equation for \( \delta \) decouples, and can thus be solved independently, after the rest of the functions are solved, and because, once the function \( f \) approaches 1 (sufficiently fast) at \( \infty \), eq.(17) ensures that \( \tilde{m} \) converges to a finite value.

A very useful tool in the analysis of this kind of configurations is provided by the consideration of the limit in which \( \lambda \to \infty \) which can formally be taken by replacing the potential term by the constraint \( \phi^2 = \eta^2 \).

In fact in this case the field equations become:

\[
R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R = 8\pi T^{\mu \nu}_{\text{const}}
\]  

(23)

where

\[
T^{\mu \nu}_{\text{const}} = \nabla^{\mu} \phi^a \nabla_{\nu} \phi_a - g^{\mu \nu} \left[ \frac{1}{2} (\nabla \phi^a)(\nabla \phi_a) \right].
\]  

(24)

while the equation of motion for the scalar fields become

\[
\Box \phi^a = \frac{1}{\eta^2} \phi^b (\Box \phi_b) \phi^a .
\]  

(25)

Using the metric (12) we can verify that the anzats \( \phi_a = \eta x_a / r \), satisfies identically eq. (23) and the eqs. (23) reduce to

\[
\partial_r \tilde{m} = 0 , \\
\partial_r \delta = 0 ,
\]  

(26)

(27)

and

\[
\tilde{E} = \frac{1}{8\pi} \left( \frac{\alpha}{(1 - \alpha) r^2} \right) .
\]  

(28)

Which results in the solution given by the metric

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + (1 - \alpha)(r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) .
\]  

(29)

which for the choice \( M = 0 \), is in fact what is taken as the S.A.F.D.A \( \alpha \) spacetime (10).

III. SOLUTIONS WITH BLACK HOLE HORIZONS

We note that eq. (29) with \( M > 0 \) corresponds to the case of a solution with a regular event horizon. In this case, as in the standard asymptotically flat case, the mass is in fact positive and given by \( M = \left( \frac{A_H}{16\pi(1-\alpha)} \right)^{1/2} \) where \( A_H \) is the area of the horizon.
The issue is then, whether in the general situation, i.e. without taking the limit \( \lambda \to \infty \), we will find also positive masses, or whether the presence of the monopole will in some instances dominate and make the mass negative?. Intuition of course suggests that the latter will be the case, and that the two regimes (positive and negative mass) are possible, with the interplay between the two scales, the monopole core scale given by \( r_c \equiv (\eta \lambda^{1/2})^{-1} \), and the black hole radius given by \( r_H \), defining which one prevails. In the case \( r_H > r_c \), which we will consider as a “monopole within a black hole’”, one expects the black hole features to dominate, and that therefore the \( M_{\text{ADM}} \) will be positive. In the case \( r_H < r_c \), which we will consider as a “black hole inside a monopole”, one expects the monopole features to dominate, and thus that the \( M_{\text{ADM}} \) will be negative. Furthermore there should exist a specific regime where \( r_H \approx r_c \) and for which the two tendencies will exactly compensate each other so that the mass should be zero.

We will investigate these issues numerically and will find that the above picture is in fact confirmed by the results.

Configurations corresponding to static spherically symmetric regular black hole horizons with area \( A = 4\pi r_H^2 = \frac{4\eta f_H^2}{\lambda^2} \) are those that satisfy the standard boundary conditions at infinity, i.e., \( \lim_{r \to \infty} f'(r) = 1 \), \( \lim_{r \to \infty} m'(r) = M \), where \( M \) is a constant, and that at \( \tilde{r} = \tilde{r}_H \) satisfy:

\[
2\tilde{m}(\tilde{r}_H) = \tilde{r}_H \, .
\]  

We ensure that the time-translational Killing field of the metric is normalized to unit at infinity. This is done by fixing the constant that appears in the integration of the equation for \( \delta \) in such a way that \( \lim_{r \to \infty} \delta(r) = 0 \).

The value of the monopole field at \( \tilde{r} = \tilde{r}_H \), \( f(\tilde{r}_H) = f_H \), is taken as the shooting parameter, and is thus fixed by the boundary conditions at spatial infinity. The derivatives at \( r = r_H \) of the functions \( f(r) \) and \( m(r) \) are given by:

\[
\tilde{m}(\tilde{r}_H) = \frac{\alpha (f_H^2 - 1)}{2(1 - \alpha)} \left[ 1 + \frac{(1 - \alpha)\tilde{r}_H^2 (f_H^2 - 1)}{4} \right] \, ,
\]

\[
f(\tilde{r}_H) = \frac{f_H(2 + \tilde{r}_H^2(1 - \alpha)(f_H^2 - 1))}{\tilde{r}_H(1 - \alpha)(1 - 2m(\tilde{r}_H, \tilde{r}))} \, .
\]

In the same way as in the regular case the system is analyzed as a standard one dimensional shooting problem.

In the \( \lambda \to \infty \) limit we can see from \cite{29} that the ADM mass is always positive, and this can be easily understood if we note that this limit can be thought to represent the extreme case of a monopole inside a black hole .

The Fig.1 shows the behavior of the ADM\( \alpha \) mass as a function of the horizon radius. Note that as expected there exists a radius where the mass ADM\( \alpha \) vanishes. These are therefore zero mass black holes.

We must emphasize that in this class of spacetimes the mass is not positive definite and therefore there is nothing really paradoxical about the fact that there are black holes that have zero mass.
However we must also point out that the mass in this case is not just a definition as it really reflects the effects on the test particles at large distances from the "body" which in the cases treated here are the monopole core and/or the black hole horizon. In this case a zero value for the mass means that at large distances, the proper acceleration of the static bodies (i.e., those following integral curves of the static Killing field) falls off faster than $1/r^2$. Moreover, as pointed out in [3] these black holes satisfy the standard laws of black hole dynamics and are nondegenerate as can be seen from the evaluation of the surface gravity $\kappa$. This is obtained from the expression $t^\mu \nabla_\mu t^{\nu} = \kappa t^{\nu}$, i.e., the surface gravity is defined in terms of the acceleration at the horizon of the time-translational Killing field $t^\mu$ which is unit at infinity. In general for a spherically symmetric system the surface gravity is [3],

$$\kappa = (\eta \lambda^{1/2}) \frac{1}{2 \tilde{r}_H} e^{-\delta(\tilde{r}_H)} [1 - 2 \tilde{m}(\tilde{r}_H), s]$$

(33)

here the derivative of the metric function evaluated at the horizon can be evaluated from formula (31). The surface gravity is positive definite as it is shown in Figure 2.

Previous works on this system (e.g., [5]) were unable to study the main point of this work because they lacked an appropriate definition of the mass for the class of A.F.D.A. $\alpha$ spacetimes.

These type of models with unusual asymptotic are a very interesting ground to investigate the robustness of the standard results of the physics of black holes and to understand which of those results are specific to the asymptotic flatness assumption. This point might seem to be of purely academic interest, but we must remember that our universe is not asymptotically flat, and that the latter is just an approximation that is introduced in order to simplify the treatment of regions of spacetime that might be regarded as “isolated” from the rest of the universe, thus the issue of the degree to which the standard results are independent of the precise form of the asymptotics is indeed of practical importance because it will tell us which of them might have to be taken as really pertinent to our universe. In this work we have learned for example that the mass of a black hole need not be positive and might indeed be negative or even zero. Here it is worth recalling that, as a result of the fact that the spacetimes in question are not asymptotically flat, the notion of ADM mass used throughout this work is different from the usual one, and thus, in particular, the standard theorems concerning the positivity of the standard ADM mass do not apply (for details see [3]). Other results that we feel should be studied in this context include, for example, the black hole uniqueness theorems, the black hole entropy results arising from the various proposals for a quantum theory of gravity, etc. Some of these issues are currently under investigation and will be the subject of forthcoming articles.
FIG. 1. The ADM mass vs the horizon radius. Here $\alpha = 0.5$ and $M_c = r_c \equiv (\eta \lambda^{1/2})^{-1}$. 
FIG. 2. The surface gravity vs the horizon radius. Here $\alpha = 0.5$ and $\kappa_c = r_c^{-1} \equiv (\eta^{1/2})$. 
ACKNOWLEDGMENTS

This work was in part supported by DGAPA-UNAM grant No IN121298 and by CONACyT grant 32272-E. U.N. is supported by a CONACyT Postdoctoral Fellowship Grant 990490.

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