SZEGŐ KERNEL ASYMPTOTICS AND CONCENTRATION OF HUSIMI DISTRIBUTIONS OF EIGENFUNCTIONS

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ABSTRACT. We work on the boundary $\partial M_\tau$ of a Grauert tube of a closed, real analytic Riemannian manifold $M$. The Toeplitz operator $\Pi_\tau D_\sqrt{\rho} \Pi_\tau$ associated to the Reeb vector field is a positive, self-adjoint, elliptic operator on $H^2(\partial M_\tau)$. We compute $\lambda \to \infty$ asymptotics under parabolic rescaling in a neighborhood of the geodesic (Reeb) flow $G_t^\tau = \exp t \Xi_\sqrt{\rho}$ for the spectral projection kernel $\Pi_{\chi,\lambda}$ associated to $\Pi_\tau D_\sqrt{\rho} \Pi_\tau$. We also compute scaling asymptotics for tempered sums of Husimi distributions (analytic continuations) on $\partial M_\tau$ of Laplace eigenfunctions on $M$. Both asymptotic formulæ can be expressed in terms of the metaplectic representation of the linearization of the geodesic flow $G_t^\tau$ on Bargmann–Fock space. As a corollary, we obtain sharp $L^p \to L^q$ norm estimates for $\Pi_{\chi,\lambda}$ and sharp $L^p$ estimates for Husimi distributions.

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1. Statement of main results

The main purpose of this article is to study the $L^p \to L^q$ mapping norms of the spectral projections $\Pi_{\chi,\lambda}$ (3) and $\Pi_{\lambda,\lambda+1}$ (7) associated to the Szegő projector $\Pi_\tau$ (1) on the boundary $\partial M_\tau$ of a Grauert tube $M_\tau$. These norm estimates, which are sharp, are stated in Theorem 1.3 and Theorem 1.5. A key ingredient of the proof is the on-shell off-diagonal scaling asymptotics of $\Pi_{\chi,\lambda}$ on $\partial M_\tau$ (Theorem 1.1). As applications, we deduce sharp $L^p$ estimates for analytic continuations (10) of Laplace eigenfunctions (Theorem 1.3), as well as for eigenfunctions of the Toeplitz operator (5), whose principal symbol coincides with that of $\sqrt{-\Delta}$ transported to the Grauert tube boundary (Proposition 1.6). Unlike the Sogge estimates in the real domain, there is no ‘critical exponent’ $p$ separating low and high $L^p$ norms.

We now give a terse introduction to and the precise statements of our results, postponing to Section 1.1 a more detailed discussion of related works. The setup involves a closed, real analytic manifold $(M,g)$ of dimension $m \geq 2$. Its complexification $M_\C$ admits a strictly plurisubharmonic exhaustion function $\rho$ in a neighborhood of the totally real submanifold $M \subseteq M_\C$. For each $0 < \tau < \tau_{\max}$, the sublevel set $\{\sqrt{\rho} < \tau\} =: M_\tau$ is a Kähler manifold, called the Grauert tube of radius $\tau$.

Throughout, we work on the boundary of a Grauert tube with a fixed radius $\tau$. The Szegő projector associated the Grauert tube boundary

$$\Pi_\tau : L^2(\partial M_\tau) \to H^2(\partial M_\tau)$$

is the orthogonal projection onto the Hardy space of boundary values of holomorphic functions in the tube. Consider the Toeplitz operator

$$\Pi_\tau D_\sqrt{\rho} \Pi_\tau : H^2(\partial M_\tau) \to H^2(\partial M_\tau),$$

where $D_\sqrt{\rho} = \frac{1}{\sqrt{-1}} \sqrt{\rho} \partial_\sqrt{\rho}$ is a constant multiple of the Hamilton vector field of the Grauert tube function $\sqrt{\rho}$ acting as a differential operator. As in [7], we fix a positive, even Schwartz function $\chi$ whose Fourier transform is compactly supported with $\hat{\chi}(0) = 1$ and form the spectral localization

$$\Pi_{\chi,\lambda} = \int_\mathbb{R} \hat{\chi}(t)e^{-it\lambda}\Pi_\tau e^{it\Pi_\tau D_\sqrt{\rho} \Pi_\tau} dt.$$  

(3)

The classical dynamics associated to $e^{it\Pi_\tau D_\sqrt{\rho} \Pi_\tau}$ on $\partial M_\tau$ is the Hamilton flow

$$G^t_\tau : \partial M_\tau \to \partial M_\tau, \quad G^t_\tau = \exp t\sqrt{\rho}.$$  

(4)

This flow coincides with the pullback of the Riemannian geodesic flow on $S^*_\tau M$ under the diffeomorphism (17).

Our scaling asymptotics for (3) is stated in Heisenberg coordinates in the sense of [8] centered at $p \in \partial M_\tau$ and $G^s_\tau(p) \in \partial M_\tau$. In these coordinates, the derivative of the flow (4) takes the form

$$DG^s_\tau : T_p \partial M_\tau \to T_{G^s_\tau(p)} \partial M_\tau, \quad DG^s_\tau = \begin{pmatrix} 1 & 0 \\ 0 & M_s \end{pmatrix},$$

where $M_s$ is a symplectic matrix on $\mathbb{R}^{2(m-1)}$. Let $\hat{\Pi}_{\mathcal{H},M_\tau}$ denote the lift to the reduced Heisenberg group $\mathcal{H}_\text{red} = S^1 \times \mathbb{C}^{m-1}$ of the metaplectic representation of $M_\tau$ acting on the model Bargmann–Fock space $\mathcal{H}(\mathbb{C}^{m-1})$. See Section 2.5 for details. The following theorem states that under a parabolic $\lambda$-rescaling near $p$ and $G^s_\tau(p)$, the kernel of (3) behaves like $\lambda^{m-1}\hat{\Pi}_{\mathcal{H},M_\tau}$ to leading order as $\lambda \to \infty$. 

Theorem 1.1
Theorem 1.1 (On-shell scaling asymptotics for $\Pi_{\chi,\lambda}$). Let $M$ be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Let $\Pi_{\chi,\lambda}$ be as in (3) and $G^\tau_f$ be as in (4). Fix $p \in \partial M_r$ and $s \in \text{supp } \chi$. Let $(\theta, u)$ and $(\phi, v)$ be Heisenberg coordinates centered at $p$ and $G^\tau_f(p)$, respectively. Then, we have

\[
\Pi_{\chi,\lambda} \left( p + \left( \frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}} \right) ; G^\tau_f(p) + \left( \frac{\phi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \right)
= \frac{C_{m,M}}{\tau^m} e^{is\lambda^m} \sqrt{\lambda}^{m-1} \hat{\Pi}_{H,M,s} \left( \frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{v}{\sqrt{\tau}} \right) \left[ 1 + \sum_{j=1}^N \lambda^{-\frac{j}{2}} P_j(p, s, u, v, \theta, \phi) \right] + \lambda^{\frac{m+1}{2}} R_N(p, s, \theta, u, \phi, v, \lambda),
\]

where $P_j$ is a polynomial in $\theta, u, \phi, v, \lambda$, the remainder $R_N$ satisfies

\[
\|R_N(p, s, \theta, u, \phi, v, \lambda)\|_{C^1(||(\theta,u)||+(\phi,v)|| \leq \rho)} \leq C_{N,j,\rho} \text{ for } \rho > 0, \ j = 1, 2, 3, \ldots,
\]

and all quantities vary smoothly with $p$ and $s$.

Remark 1.2. When $s = 0$ so that $M_s = I$ is the identity matrix,

\[
\hat{\Pi}_{H,I}(\theta, u; \phi, v) = \frac{1}{\pi^{m-1}} e^{i(\theta-\phi)+u·v} (\frac{1}{2}|u|^2 + \frac{1}{2}|v|^2)
\]

coinsides with the Szegő kernel of level one on $H^{m-1}_{\text{red}}$, and we recover near diagonal asymptotics computed in [7].

An argument similar to that found in [23] allows us to deduce the following sharp $L^p \to L^q$ mapping norm estimate for (3).

Theorem 1.3 ($L^p \to L^q$ mapping estimate for $\Pi_{\chi,\lambda}$). Let $M$ be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Let $\Pi_{\chi,\lambda}$ be as in (3). Then we have the sharp estimate

\[
\|\Pi_{\chi,\lambda} f\|_{L^q(\partial M_r)} \leq C_{\partial M_s} \lambda^{(m-1)(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^p(\partial M_r)} \quad (2 \leq p, q \leq \infty).
\]

We now turn to estimates for eigenfunctions. The Toeplitz operator (2) is a positive, self-adjoint, elliptic operator on $H^2(\partial M_r)$ in the sense of [4], so has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with associated $L^2$-normalized eigenfunctions

\[
\Pi_r D\sqrt{\tau} \Pi_r e_{\lambda_j} = \lambda_j e_{\lambda_j}, \quad \|e_{\lambda_j}\|_{L^2(\partial M_r)} = 1.
\]

An immediate consequence of the eigenfunction expansion

\[
\Pi_{\chi,\lambda} = \sum_{j: \lambda_j \leq \lambda} \chi(\lambda - \lambda_j)e_{\lambda_j} \otimes \overline{e_{\lambda_j}},
\]

of the spectral projection (3) together with Theorem 1.3 in the case $p = 2$ is the following.

Corollary 1.4 ($L^p$ estimates for eigenfunctions of $\Pi_r D\sqrt{\tau} \Pi_r$). Let $M$ be a closed, real analytic Riemannian manifold of dimension $m \geq 2$. Let $e_{\lambda_j}$ be $L^2$-normalized eigenfunctions of $\Pi_r D\sqrt{\tau} \Pi_r$ as in (5). Then we have

\[
\|e_{\lambda_j}\|_{L^q(\partial M_r)} \leq C_{\partial M_s} \lambda_j^{(m-1)(\frac{1}{q}-\frac{1}{p})} \quad (2 \leq q \leq \infty).
\]
The conclusion of Theorem 1.3 also holds for spectral projections onto short spectral intervals
\[ \Pi_{[\lambda, \lambda+1]} = \sum_{j: \lambda \leq \lambda_j \leq \lambda+1} e_{\lambda_j} \otimes \overline{e_{\lambda_j}}, \]  
(7)
which we state below in Theorem 1.5. This result may be viewed as a Grauert tube analogue, with \( \Pi_r D_{\sqrt{\Delta}} \Pi_r \) replacing \( \sqrt{-\Delta} \), of Sogge’s \( L^p \) estimate [24] for spectral projections of the Laplacian.

**Theorem 1.5.** Let \( M \) be a closed, real analytic Riemannian manifold of dimension \( m \geq 2 \). Then we have the sharp estimate
\[ \| \Pi_{[\lambda, \lambda+1]} f \|_{L^q(\partial M_r)} \leq C_{\partial M_r} \lambda^{(m-1)(\frac{1}{2}-\frac{1}{q})} \| f \|_{L^2(\partial M_r)} \quad (2 \leq q \leq \infty) \]

Our next set of results concern analytic continuations to the Grauert tube boundary of Laplace eigenfunctions on \( M \). Let \( 0 = \mu_0 < \mu_1 < \mu_2 < \cdots \) be eigenvalues of \( \sqrt{-\Delta} \) with associated \( L^2 \)-normalized eigenfunctions
\[ -\Delta \varphi_{\mu_j} = \mu_j^2 \varphi_{\mu_j}, \quad \| \varphi_{\mu_j} \|_{L^2(M)} = 1. \]  
(8)
The analytic extensions \( \varphi_{\mu_j}^\mathbb{C} \), which are CR holomorphic functions on \( \partial M_r \), are defined by
\[ \varphi_{\mu_j}^\mathbb{C} = e^{\tau \mu_j} U(i\tau) \varphi_{\mu_j}, \]  
(9)
where \( U(i\tau) = \exp(-\tau \sqrt{-\Delta}) \) is the Poisson operator. See Section 2.4 for details. The probability amplitudes
\[ \varphi_{\mu_j}^\mathbb{C} = \frac{\varphi_{\mu_j}^\mathbb{C}}{\| \varphi_{\mu_j}^\mathbb{C} \|_{L^2(\partial M_r)}} \]  
(10)
are Husimi distributions, that is, microlocal lifts of \( \varphi_{\mu_j} \) to phase space \( \partial M_r \cong S^* M \). They are “approximate eigenfunctions” of \( \Pi_r D_{\sqrt{\Delta}} \Pi_r \) in the following sense.

**Proposition 1.6.** Let \( \{ \lambda_j \} \) and \( \{ \mu_j \} \) be the eigenvalues of \( \Pi_r D_{\sqrt{\Delta}} \Pi_r \) and \( \sqrt{-\Delta} \), respectively. Let \( \Pi_{\lambda, \lambda} \) be as in (3) and \( \varphi_{\mu_j}^\mathbb{C} \) be as in (10). Then, \( \mu_j = \lambda_j + O(1) \) as \( j \to \infty \) and
\[ \| \Pi_{\lambda, \lambda} \varphi_{\mu_j}^\mathbb{C} - \varphi_{\mu_j}^\mathbb{C} \|_{L^2(\partial M_r)} = O(1), \]
\[ \| \Pi_r D_{\sqrt{\Delta}} \Pi_r \varphi_{\mu_j}^\mathbb{C} - \lambda_j \varphi_{\mu_j}^\mathbb{C} \|_{L^2(\partial M_r)} = O(1). \]
Furthermore, we have the sharp estimate
\[ \| \varphi_{\mu_j}^\mathbb{C} \|_{L^p(\partial M_r)} \leq C_{\partial M_r} \lambda_j^{(m-1)(\frac{1}{2}-\frac{1}{q})} \quad (2 \leq p \leq \infty). \]  
(11)
The \( L^p \) bound (11) may also be deduced from the sup norm bound of Zelditch [31] and log-convexity of \( L^p \) norms. In Section 5.1, we show that the bound is saturated by complexified Gaussian beams. This result is yet another Grauert tube analogue, with the analytically continued \( \varphi_{\lambda_j}^\mathbb{C} \) replacing \( \varphi_{\lambda_j} \), of Sogge’s \( L^p \) estimate for eigenfunctions. Note that, unlike in the real domain, there are no separate estimates for “high” versus “low” \( L^p \); see Section 1.1.3 for further discussion.

**Remark 1.7.** In [7], we showed that analytically continued eigenfunctions on a torus are approximate eigenfunctions of \( \Pi_r D_{\sqrt{\Delta}} \Pi_r \) by an explicit computation, and that analytically continued spherical harmonics are in fact exact eigenfunctions of \( \Pi_r D_{\sqrt{\Delta}} \Pi_r \).
To state the last result, we fix, as before, a positive, even Schwartz function \( \chi \) whose Fourier transform is compactly supported with \( \hat{\chi}(0) = 1 \) to construct the tempered partial sums

\[
P_{\chi,\mu} = \sum_{j: \mu_j \leq \mu} \chi(\mu - \mu_j)e^{-2\tau \mu_j}\varphi^C_{\mu_j} \otimes \hat{\varphi}^C_{\mu_j},
\]

using (9). The prefactor \( e^{-2\tau \mu_j} \) is introduced to “temper” the exponential growth estimate ([34, Corollary 3]) for complexified eigenfunctions:

\[
C_1\mu_j^{m-\frac{1}{2}} e^{\tau \mu_j} \leq \| \varphi^C_{\mu_j}(\zeta) \|_{L^\infty(\partial M_\tau)} \leq C_2\mu_j^{m-\frac{1}{2}} e^{\tau \mu_j}.
\]

**Remark 1.8.** It is shown in [16,34] that \( e^{-\tau \mu_j}\varphi^C_{\mu_j} \) is a Riesz basis (but not an orthonormal basis) in general, so they fail to be reproduced by \( P_{\chi,\mu} \).

The proof of Theorem 1.1 is easily adapted to prove scaling asymptotics for (12). Comparing the statements of Theorem 1.1 and Theorem 1.9 below, we see the leading order asymptotics of \( \Pi_{\chi,\lambda} \) and \( P_{\chi,\mu} \) differ only in the powers of the frequency parameters \( \lambda \) or \( \mu \).

**Theorem 1.9** (On-shell asymptotics for \( P_{\chi,\mu} \)). Let \( P_{\chi,\mu} \) be as in (3). Under the same hypotheses as Theorem 1.1, we have

\[
P_{\chi,\mu} \left( p + \left( \frac{\theta}{\mu}, \frac{u}{\sqrt{\mu}} \right); G^s_\tau(p) + \left( \frac{\phi}{\mu}, \frac{\nu}{\sqrt{\mu}} \right) \right)
\]

\[
= C_{m,M} \tau^m e^{i\nu \mu} \mu^{m-\frac{1}{2}} \tilde{\Pi}_{\mu,M_\tau} \left( \frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{\nu}{\sqrt{\tau}} \right) \left[ 1 + \sum_{j=1}^N \mu^{-\frac{1}{2}} P_j(p, s, u, v, \theta, \phi) \right]
\]

\[
+ \mu^{-\frac{N+1}{2}} R_N(p, s, \theta, u, \phi, v, \mu),
\]

where \( P_j \) is a polynomial in \( \theta, u, \phi, v \), the remainder \( R_N \) satisfies

\[
\| R_N(p, s, \theta, u, \phi, v, \mu) \|_{C^j((\|\theta, u\|+\|\phi, v\|) \leq \rho)} \leq C_{N,j,\rho} \quad \text{for } \rho > 0, \ j = 1, 2, 3, \ldots,
\]

and all quantities vary smoothly with \( p \) and \( s \).

**Remark 1.10.** Our techniques for proving Theorem 1.1 and Theorem 1.3 hold in the more general setting of a compact, strictly pseudoconvex CR manifold \( X \) for which \( \Box_b \) has closed range. In particular, the Boutet de Monvel–Sjöstrand description of the Szegö projector remains valid and quantization of the geodesic flow (4) can be replaced by that of the Reeb flow. The main interest in the Grauert tube setting is the manifestation of the underlying Riemannian geometry as well as the connection between analytic extensions and microlocal lifts of eigenfunctions.

1.1. **Comparison to prior results.** In Section 1.1.1, we briefly justify the operator (3) as a Grauert tube analogue of the Bergman projections in the line bundle setting; a more detailed discussion is contained in [7, Section 1.1]. Section 1.1.2 recalls some results on \( L^p \) estimates on Bergman kernels associated to line bundles, as well as asymptotic expansions of quantized Hamiltonian symplectomorphisms. We return to the real domain in Section 1.1.3 with a comparison of \( L^p \) norms of eigenfunctions on the manifold \( M \) versus those of analytically continued eigenfunctions on the tube \( \partial M_\tau \).
1.1.1. The operators $\Pi_{\chi,\lambda}$ as Fourier components of the Grauert tube Szegö kernel.

Let $(L, h) \to (X, \omega)$ be a positive Hermitian line bundle over a closed Kähler manifold. Let $\partial D = \{ \ell \in L^* : \ell|_{h^*} = 1 \}$ be the unit co-circle bundle. The orthogonal projections

$$
\Pi_{h^k} : L^2(X, L^k) \to H^0(X, L^k) \quad \text{and} \quad \Pi_h : L^2(\partial D) \to H^2(\partial D) \quad \text{(13)}
$$

are the Bergman and Szegö projections, respectively. The space $H^0(X, L^k)$ of holomorphic sections is unitarily equivalent to the set of equivariant CR functions $f \in H^2(\partial D)$ satisfying $f(r_\theta x) = e^{ik\theta} f(x)$, where $r_\theta$ denotes the circle action on $\partial D$. Under this identification, operators (13) are related by

$$
\Pi_{h^k} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} (r_\theta)^* \Pi_h \, d\theta. \quad \text{(14)}
$$

Note that the Fourier decomposition of $\Pi_h$ coincides with the spectral decomposition of $D_\theta = \frac{1}{2\pi} \partial_\theta$ on $\partial D$.

Turning to the Grauert tube setting of (1) and (4), the naive approach of replacing $(r_\theta)^* \Pi_h$ by $(G^\ast_{\theta})^* \Pi_r$ is insufficient. Instead, the former is replaced by $\Pi_r \hat{\sigma} (G^\ast_{\theta})^* \Pi_r$, where $\hat{\sigma}$ is a polyhomogeneous pseudodifferential operator on $\partial M_r$ that makes the resulting composition unitary. Thus, we are led to the CR holomorphic analogue of (14) that is

$$
\Pi_{\chi,\lambda} := \int_{\mathbb{R}} \hat{\chi}(t) e^{-i\lambda t} \Pi_r \hat{\sigma} (G^\ast_{t})^* \Pi_r \, dt \sim \int_{\mathbb{R}} \hat{\chi}(t) e^{-i\lambda t} \Pi_r e^{itD} e^{\sigma D} \Pi_r \, dt. \quad \text{(15)}
$$

Note that (15) is essentially the spectral decomposition of the elliptic Toeplitz operator $\Pi_r D_\sigma \Pi_r$ introduced in (2). The same operator is also studied in [31, Theorem 0.12].

1.1.2. L^p estimates and quantized Hamiltonians on line bundles. In the line bundle setting $(L, h) \to (X, \omega)$, Shiffman–Zelditch [23, Lemma 4.1] proved $\|\Pi_{h^k}\|_{L^p \to L^q} \leq C k^{m(1/p-1/q)}$ by combining the Shur–Young inequality with a near-diagonal Gaussian estimate [22, Lemma 5.2]. Consequently, $\|s\|_{L^p} = O(k^{-m(1/2-1/p)})$ for all $L^2$-normalized holomorphic sections $s \in H^0(X, L^k)$.

The proof techniques of our Grauert tube analogue, Theorem 1.3, are similar. But, in place of a near-diagonal scaling asymptotics, we need the full strength of Theorem 1.1, which is an asymptotic expansion in a $\lambda^{-\frac{1}{2}}$-neighborhood of the orbit $p \mapsto G^\ast_{\theta}(p)$. This finer control of the time evolution under the Reeb flow (i.e., $G^\ast_{\theta}$ on $\partial M_r$ or $r_\theta$ on the circle bundle) is unnecessary in the line bundle setting because rotations of the fiber introduce only an overall phase factor to the near-diagonal scaling asymptotics.

There has also been prior work on quantized Hamiltonian flows on line bundles. More precisely, let $f \in C^\infty(X)$ be a Hamiltonian on the classical phase space (Kähler manifold) $X$ that induces a 1-parameter group of symplectomorphisms $\varphi_t : X \to X$ which lifts to a family of contactomorphisms $\tilde{\varphi}_t : \partial D \to \partial D$. As shown by Zelditch [32], these contactomorphisms may be quantized as unitary maps

$$
\Phi_t : L^2(\partial D) \to L^2(\partial D), \quad \Phi_t = R_t \Pi_h (\tilde{\varphi}_t)^* \Pi_h,
$$

in which $R_{-t}$ is a zeroth order Toeplitz operator chosen to ensure the unitarity of $\Phi_t$. In a series of papers, Paoletti [19, 20, 21] computed scaling asymptotics for the Fourier coefficients (with respect to the $S^1$ action) of $\Phi_t$ near points on the graph of $\tilde{\varphi}_t$. When $t = 0$, this specializes to the scaling asymptotics of [3, 22]. We
emphasize that in contrast, $G_r^c$ is simultaneously playing the role of the $S^1$ action and the Hamiltonian flow in our set up. Nevertheless, our main argument borrows heavily from that of [20]. We also mention the works of Zelditch–Zhou [36, 37, 38], which treat other types of asymptotics for partial Bergman kernels of quantized Hamiltonian flows on line bundles.

1.1.3. $L^p$ norms of eigenfunctions in the real domain. In this section we discuss how the $L^p$ estimates of Theorem 1.3 and Corollary 1.4 compare with those of Sogge eigenfunctions in the real domain. Let $P_{\lambda, \lambda+1}$ denote the orthogonal projection onto the span of Laplace eigenfunctions $\varphi_{\lambda_j}$ with frequencies $\lambda \leq \lambda_j < \lambda + 1$.

**Theorem 1.11** (Sogge [24], see also [25, 15]). Let $(M, g)$ be a closed Riemannian manifold of dimension $n$, then the following estimates are sharp.

$$\|P_{\lambda, \lambda+1}f\|_{L^q(M)} \leq \left\{ \begin{array}{ll} C\lambda^{(n-1)\left(\frac{1}{2} - \frac{1}{q}\right)}\|f\|_{L^2(M)} & \text{for } 2 \leq q \leq \frac{2(n+1)}{n-1}, \\
C\lambda^{n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{q}}\|f\|_{L^2(M)} & \text{for } \frac{2(n+1)}{n-1} \leq q < \infty. \end{array} \right.$$  

Consequently, for $L^2$-normalized Laplace eigenfunctions $\varphi_{\lambda_j}$ with frequencies $\lambda_j$, we have

$$\|\varphi_{\lambda_j}\|_{L^q(M)} \leq \left\{ \begin{array}{ll} C\lambda_j^{\left(\frac{2(n+1)}{n-1} - \frac{1}{q}\right)} & \text{for } 2 \leq q \leq \frac{2(n+1)}{n-1}, \\
C\lambda_j^{n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{q}} & \text{for } \frac{2(n+1)}{n-1} \leq q < \infty. \end{array} \right.$$

Note the presence of a critical exponent $q_n = \frac{2(n+1)}{n-1}$ at which the sharp estimates change. Roughly speaking, high $L^q$ norms measure concentration around single points, whereas low $L^q$ norms measure concentration around larger sets such as geodesics and hypersurfaces. It is well known on the round sphere $S^n$ that the sequence of zonal spherical harmonics at a pole saturate the estimate for $q > q_n$. On the other hand, the sequence of highest weight spherical harmonics, that is Gaussian beams along a stable elliptic geodesic, saturate the estimate for $q < q_n$. However, these bounds are rarely sharp on other manifolds. For example, on the flat torus all eigenfunctions have $L^q$ norms bounded by $O(1)$. An interesting question in this direction is which manifolds admit sequences of eigenfunctions that saturate these bounds. We point the readers to [26, 28, 29, 27] for research in this topic.

Although we project onto the orthonormal basis consisting of eigenfunctions of $\Pi_r D\sqrt{\tau} \Pi_r$ rather than onto the span $C_{\lambda_j}$, thanks to Proposition 1.6 we can interpret Theorem 1.5 as a complexified version of the theorem above. In the complex setting there is no critical exponent $q_n$ differentiating the behavior between the low and high $L^q$ norms. Indeed, the exponent in our sharp estimate is analogous to that of Sogge’s in the low $L^q$ regime, and we show in Section 5.1 that complexified Gaussian beams are extremals for all $p$.

Our main result has an interpretation as measuring concentration in phase space. As discussed in [31], the squares of $C_{\lambda_j}$ are microlocal lifts of $\varphi_{\lambda_j}$ to $\partial M_\tau \cong S^*_\tau M$, so they may be viewed as probability densities of finding a quantum particle at a phase space point in $\partial M_\tau$. Their marginals are given by the pushforward $\pi_*(C_{\lambda_j})^2$ under the natural projection $\pi: S^*_\tau M \to M$. It is natural to ask how the marginal densities of these Husimi distributions relate to eigenfunction concentration on $M$. Other types of phase space norms of eigenfunctions have been studied by Blair–Sogge [1, 2]. We also mention the work of Galkowski [10], which uses defect measures to
study eigenfunction concentration. It would be interesting to compare the results and techniques with those of complexification.

1.2. Organization of the paper. Section 2 collects the relevant facts pertaining to Grauert tubes. We recall the Szegő projector as a complex Fourier integral operator (FIO) with a positive complex canonical relation as well as the Boutet de Monvel–Sjöstrand parametrix for the kernel. We also recall the relevant microlocal properties of the Poisson wave operator (31) and the tempered spectral projection (32) studied in [33, 34, 31]. Also in Section 2 is a brief review the metaplectic representation on the Bargmann–Fock space of \( C^n \).

The proofs of the two scaling asymptotics are contained in Section 3, while the \( L^p \) estimates for the projector and for eigenfunctions are found in Section 4. In section 6 we prove the Theorem 1.3, Theorem 1.5, and Proposition 1.6. Finally in section 7 we demonstrate that Gaussian beams on the sphere saturate \( L^p \) bounds and give a geometric explanation.

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2. Background

We assume throughout that \( (M, g) \) is a closed, real analytic Riemannian manifold of dimension \( m \geq 2 \). Readers may consult [12, 13, 17, 18, 11] for geometry of and analysis on Grauert tubes (particularly in relation to the complex Monge–Ampère equation and complexified geodesics), as well as a paper [7] of the authors with a more detailed discussion.

2.1. Kähler geometry on Grauert tubes. A real analytic Riemannian manifold \( M \) admits a complexification \( M_C \) into which \( M \) embeds as a totally real submanifold. The Grauert tube function is defined by

\[
\sqrt{\rho} : U \subseteq M_C \to \mathbb{R}, \quad \sqrt{\rho}(z) = \frac{1}{2} i \sqrt{r^2_C(z, \bar{z})},
\]

where \( r^2_C(z, \bar{w}) \) is the analytic extension of the square of the Riemannian distance function \( r : M \times M \to \mathbb{R} \) to a neighborhood of the diagonal in \( M_C \times \overline{M_C} \). In a neighborhood of \( M \) in \( M_C \), the square \( \rho \) of (16) is the unique strictly plurisubharmonic function such that the metric induced by the Kähler form \( i\partial \bar{\partial} \rho \) restricts to the Riemannian metric \( g \) on \( M \).

For each \( 0 < \tau \leq \tau_{\max} \), the sublevel set

\[
M_\tau = \{ z \in M_C : \sqrt{\rho}(z) < \tau \}
\]

is called the Grauert tube of radius \( \tau \). It is diffeomorphic [11, Theorem 1.5] to the co-ball bundle \( B^*_\tau M = \{(x, \xi) \in T^*M : \|\xi\|_x < \tau \} \) of radius \( \tau \) under the imaginary-time exponential map

\[
E : B^*_\tau M \to M_\tau, \quad E(x, \xi) = \exp^C_x i\xi.
\]

Let \( G^t \) denote the homogeneous geodesic flow, that is, the Hamilton flow of \( |\xi|_x \), on the cotangent bundle. Then, for each \( 0 < \tau \leq \tau_{\max} \), the \( C^\omega \) diffeomorphism
(17) conjugates the geodesic flow on the sphere bundle $S^*_cM$ to the Hamilton flow $G^t = \exp t\Xi_{\sqrt{\rho}}$ of the Grauert tube function on $\partial M_r$:

$$G^t_r : \partial M_r \to \partial M_r, \quad G^t_r = E \circ G^t \circ E^{-1}|_{\partial M_r}. \tag{18}$$

With this identification in mind, we will henceforth refer to $G^t_r : \partial M_r \to \partial M_r$ as the "geodesic flow."

2.2. **Contact and CR structure on the Grauert tube boundary.** The Grauert tube $M_r$ is a strongly pseudoconvex domain thanks to the existence of the strictly plurisubharmonic exhaustion function $\sqrt{\rho}$. The pullbacks of the canonical 1-form $\alpha_{T\cdot M} = \xi dx$ and the symplectic form $\omega_{T\cdot M} = d\xi \wedge dx$ on the cotangent bundle under the diffeomorphism (17) are

$$\alpha := (E^{-1})^* \alpha_{T\cdot M} = d^c \sqrt{\rho} \quad \text{and} \quad (E^{-1})^* \omega_{T\cdot M} = dd^c \rho. \tag{19}$$

We endow the Grauert tube boundary $\partial M_r$ with the volume form

$$d\mu_r = (E^{-1})^*(\alpha_{T\cdot M} \wedge \omega_{T\cdot M}^{-1}) \big|_{\partial M_r}, \tag{20}$$

which is the pullback of the standard Liouville volume form on $S^*_cM$. Since $\partial M_r$ is a real hypersurface in the Kähler manifold $M_{max} \subseteq (M_C, J)$, we see that $H = JT\partial M_r \cap T\partial M_r$ is a real $J$-invariant hyperplane bundle. The restriction $\alpha|_{\partial M_r}$ of the 1-form in (19) is a contact form on $\partial M_r$ with ker $\alpha = H$.

The characteristic vector field given by $T = \Xi_{\sqrt{\rho}}$ is the unique one on $\partial M_r$ satisfying $\alpha(T) = 1$ and $d\alpha(T, \cdot) = 0$. The complexification of the decomposition $T\partial M_r = H \oplus \mathbb{R}T$ yields a CR structure:

$$T^C\partial M_r = H^{(1,0)} \partial M_r \oplus H^{(0,1)} \partial M_r \oplus \mathbb{C}T,$$

where $H^{(1,0)}$ and $H^{(0,1)}$ are the $J$-holomorphic and $J$-antiholomorphic subspaces, respectively.

2.3. **The Szegő projector and the Boutet de Movel–Sjöstrand parametrix.** The Szegő projector $\Pi_r$ associated to the boundary of a Grauert tube is the orthogonal projection

$$\Pi_r : L^2(\partial M_r, d\mu_r) \to H^2(\partial M_r, d\mu_r)$$

onto the Hardy space consisting of boundary values of holomorphic functions in $M_r$ that are square integrable with respect to the volume form (20). This is a Fourier integral operator with a positive complex canonical relation whose real points are the graph of the identity map on the symplectic cone

$$\Sigma_r = \{ (\zeta, r\alpha_\zeta) : r \in \mathbb{R}_+ \} \subseteq T^*(\partial M_r) \tag{21}$$

spanned by the contact form (19). Using (17), we can construct a symplectic equivalence

$$\iota_r : T^*M = 0 \to \Sigma_r, \quad \iota_r(x, \xi) = \left( E(x, \tau \frac{\xi}{|\xi|}), |\xi| E(x, \tau \frac{\xi}{|\xi|}) \right). \tag{22}$$

Details of the symbol of the Szegő projector can be found in [4, Theorem 11.2]. Briefly, let $\Sigma_r^+ \otimes \mathbb{C}$ be the complexified normal bundle of (21). The symbol $\sigma(\Pi_r)$ of $\Pi_r$ is a rank one projection onto a ground state $\epsilon_{\Lambda_r}$, which is annihilated by a Lagrangian system of Cauchy–Riemann equations corresponding to a Lagrangian subspace $\Lambda_r \subseteq \Sigma_r^+ \otimes \mathbb{C}$. 
The time evolution $\Pi_\tau \mapsto G^{-1}_\tau \Pi_\tau G^1_\tau$ under the Hamilton flow (18) yields another a rank one projection onto some time-dependent ground state $e_{\Lambda^\tau}$, where $\Lambda^\tau$ is the pushforward of $\Lambda_\tau$ under the flow. The quantity

$$
\sigma_{t,\tau,0} = \langle e_{\Lambda^\tau}, e_{\Lambda_\tau} \rangle^{-1}
$$

appears in (30) and (34). See [35, Section 4.3] for details.

The Szegő kernel $\Pi_\tau(z, w)$ is defined by the relation

$$
\Pi_\tau f(z) = \int_{\partial M_\tau} \Pi_\tau(z, w)f(w) \, d\mu_\tau(w) \quad \text{for all } f \in L^2(\partial M_\tau).
$$

To describe an oscillatory integral representation for the kernel, we introduce the defining function

$$
\varphi_\tau : M_{\text{max}} \rightarrow [0, \infty), \quad \varphi_\tau(z) = \rho(z) - \tau^2,
$$

so that $\varphi_\tau < 0$ in $M_\tau$ and $\varphi_\tau = 0$ on $\partial M_\tau$. Let $\varphi_\tau(z, \bar{w})$ be the analytic extension of $\varphi_\tau(z) = \varphi_\tau(z, \bar{\tau})$ to $M_\tau \times \bar{M}_\tau$ obtained by polarization.

$$
\psi_\tau(z, w) = \frac{1}{i} \varphi_\tau(z, \bar{w}) = \frac{1}{i} \left( -\frac{1}{4} \bar{\tau}^2(z, \bar{w}) - \tau^2 \right).
$$

By construction, $\psi$ is holomorphic in $z$, antiholomorphic in $w$, and satisfies $\psi(z, w) = -\bar{\psi}(z, w)$. It appears as the phase function of the following parametrix for $\Pi_\tau$ due to Boutet de Monvel and Sjöstrand.

**Theorem 2.1** (The Boutet de Monvel–Sjöstrand parametrix, [6, Theorem 1.5]). With $\psi$ as in (26), there exists a classical symbol

$$
s \in S^{m-1}(\partial M_\tau \times \partial M_\tau \times \mathbb{R}^+) \quad \text{with} \quad s(z, w, \sigma) \sim \sum_{k=0}^{\infty} \sigma^{m-1-k}s_k(z, w)
$$

so that the Szegő kernel (24) has the oscillatory integral representation

$$
\Pi_\tau(z, w) = \int_{\mathbb{R}^+} e^{i \sigma \psi_\tau(z, w)}s(z, w, \sigma) \, d\sigma \quad \text{modulo a smoothing kernel.}
$$

A key estimate for $\varphi_\tau$ (or equivalently for the phase function $\psi$) can be stated in terms of the Calabi diastatis function, which is defined by

$$
D(z, w) = \varphi_\tau(z, \bar{z}) + \varphi_\tau(w, \bar{w}) - \varphi_\tau(z, \bar{w}) - \varphi_\tau(w, \bar{z}).
$$

In the closure of the Grauert tube, [6, Corollary 1.3] gives the lower bound

$$
D(z, w) \geq C (d(z, \partial M_\tau) + d(w, \partial M_\tau) + d(z, w)^2) \quad \text{for } z, w \in \overline{M_\tau}.
$$

2.4. The Toeplitz operator $\Pi, D_{\sqrt{\nu}}\Pi_\tau$, and tempered eigenfunction sums.

In this section, we introduce the two operators for which we compute the scaling asymptotics in Theorem 1.1 and Theorem 1.9.

The operator $\Pi, D_{\sqrt{\nu}}\Pi_\tau$ is a generalized Toeplitz operator in the sense of Boutet de Monvel–Guillemin [4]. Here, $D_{\sqrt{\nu}} = \frac{1}{i} \bar{\Xi} \sqrt{\nu}$ is differentiation along the Hamilton vector field of the Grauert tube function. The symbol of $D_{\sqrt{\nu}}$ is nowhere vanishing on $\Sigma_\tau = 0$, where $\Sigma_\tau$ is the symplectic cone (21). Thus, $\Pi_\tau D_{\sqrt{\nu}}\Pi_\tau$ is elliptic and its spectrum discrete.
As discussed in Section 1.1.1, the Grauert tube analogue of Fourier coefficients of the Szegő kernel are given by the spectral localizations

\[ \Pi_{\chi,\lambda} = \Pi_{\tau}(\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} - \lambda) = \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda} \Pi_{\tau} e^{it\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau}} dt. \]

Here, \( \hat{\chi} \) is a Schwartz function whose Fourier transform is compactly supported in some small neighborhood \([-\varepsilon, \varepsilon]\) of the origin, and \( \hat{\chi}(0) = 1 \).

If we denote by the pullback by the Hamilton flow (18) of \( \Xi_{\sqrt{\rho}} \) on \( \partial M_{\tau} \), then due to [31, Proposition 5.3], there exists a classical polyhomogeneous pseudodifferential operator \( \tilde{\sigma}_{t,\tau}(w, D_{\sqrt{\rho}}) \) on \( \partial M_{\tau} \) so that

\[ \Pi_{\tau} e^{it\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau}} \sim \Pi_{\tau} \tilde{\sigma}_{t,\tau}(G_{\tau}^t)^* \Pi_{\tau} \pmod{\text{smooth Toeplitz operator}}. \]

The symbol \( \sigma_{t,\tau} \) of \( \tilde{\sigma}_{t,\tau} \) admits a complete asymptotic expansion

\[ \sigma_{t,\tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t,\tau,j}(w) r^{-j}, \]

in which \( \sigma_{t,\tau,0} = (\varepsilon_{A_1}, \varepsilon_{A_\rho})^{-1} \) is to leading order the reciprocal of the overlap of two Gaussians, as in (23).

**Remark 2.2.** It follows that

\[ \Pi_{\chi,\lambda}(x, y) \sim \left( \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda} \Pi_{\tau} \tilde{\sigma}_{t,\tau}(G_{\tau}^t)^* \Pi_{\tau} dt \right) (x, y) \]

\[ = \int_{\mathbb{R}} \int_{\partial M_{\tau}} \hat{\chi}(t) e^{-it\lambda} \Pi_{\tau}(x, w) \sigma_{t,\tau}(w) \Pi_{\tau}(G_{\tau}^t(w), y) dt dy. \]

In the proof of Theorem 1.1, we replace the two Szegő kernels in the expression above by the parametrices (27) and directly compute the resulting oscillatory integral in parameter \( \sqrt{\lambda} \) using stationary phase.

We now introduce the tempered spectral projections kernel \( P_{\chi,\lambda}(z, w) \) constructed using analytically continued eigenfunctions. Recall the eigenvalue equation (8) for the Laplacian on \( M \). The eigenfunction expansion of the Schwartz kernel of the half-wave operator \( U(t) = e^{it\sqrt{-\Delta}} \) is given by

\[ U(t, x, y) = \sum_{j} e^{it\lambda_j} \varphi_{\lambda_j}(x) \overline{\varphi_{\lambda_j}(y)}. \]

As shown in [5, 13, 11, 34], for \( 0 < \tau \leq \tau_{\text{max}} \), the Schwartz kernel \( U(t, x, y) \) admits an analytic extension \( U(t + \i\tau, x, y) \) in the time variable \( t \mapsto t + \i\tau \in \mathbb{C} \), and then in the spacial variable \( x \mapsto z \in M_{\tau} \). Let \( O^s(\partial M_{\tau}) \) denote the order \( s \) Sobolev space of CR holomorphic functions on the Grauert tube boundary. Then the Poisson operator

\[ U(\i\tau) = e^{-\tau\sqrt{-\Delta}}: L^2(M) \to O^{\frac{s}{4} - \frac{1}{4}}(\partial M_{\tau}) \]

with kernel \( U(\i\tau, z, y) \) is a Fourier integral operator of order \( -(m - 1)/4 \) with complex phase associated to the canonical relation \( \{(y, \eta, \i\tau, (y, \eta)) \subseteq T^* M \times \Sigma_{\tau} \). Here, the quantities \( \i\tau \) and \( \Sigma_{\tau} \) are defined in (22) and (21). We recall the following lemma.

**Lemma 2.3 ([34, Lemma 8.2]).** Let \( \Psi^s \) denote the class of pseudodifferential operators of order \( s \). Then,
\[ U(\iota r) = U(\iota r) * U(\iota r) \in \mathcal{Ψ}^{1/2} (M) \] with principal symbol \(|\xi|^\frac{m-1}{2}\).

(2) \( U(\iota r)U(\iota r)^* = \Pi_{r} A_{r} \Pi_{r} \) where \( A_{r} \in \mathcal{Ψ}^{1/2} (\partial M_{r}) \) has principal symbol \(|\sigma|^{\frac{m-1}{2}}\) as a function on \( \Sigma_{r} \).

To introduce the complexified spectral projection kernels, we need to further continue \( U(\iota r, z, y) \) anti-holomorphically in the \( y \) variable. Consider the operator
\[
U_{C} (t + 2i\iota r) = U(\iota r)U(t)U(\iota r)^* : \mathcal{O}(\partial M_{r}) \to \mathcal{O}(\partial M_{r})
\]
with Schwartz kernel
\[
U_{C} (t + 2i\iota r, z, w) = \sum_{j=1}^{\infty} e^{i(t+2i\iota)r_{j} n_{j} \sqrt{2}} g_{C}^{j}(z) g_{C}^{j}(w).
\]
Set \( t = 0 \), then the partial sums of the expression above becomes
\[
P_{\lambda}(z, w) = \sum_{j: \lambda_{j} \leq \lambda} e^{-2i\iota r_{j} n_{j} \sqrt{2}} g_{C}^{j}(z) g_{C}^{j}(w).
\] (32)

To smooth out the kernel, we fix \( \varepsilon > 0 \) and fix \( \chi \) a positive even Schwartz function such that \( \hat{\chi}(0) = 1 \) and \( \text{supp} \hat{\chi} \subseteq [-\varepsilon, \varepsilon] \). Define
\[
P_{\chi, \lambda}(z, w) = \chi * d_{\lambda} P_{\lambda}(z, w) \sim \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda \sqrt{2} U_{C}(t + 2i\iota r, z, w)} \, dt.
\] (33)

As before, if we denote by \((G^{\iota r}_{\iota r})^{*}\) the pullback by the Hamilton flow (18) of \( \Xi_{\sqrt{2}} \) on \( \partial M_{r} \), then [31, Proposition 7.1] establishes the existence a classical polyhomogeneous pseudodifferential operator \( \tilde{\sigma}_{t, \iota r} (w, D_{\sqrt{2}}) \) on \( \partial M_{r} \) so that
\[
U_{C} (t + 2i\iota r) \sim \Pi_{r} \tilde{\sigma}_{t, \iota r} (G^{\iota r}_{\iota r})^{*} \Pi_{r} \text{ modulo a smoothing Toeplitz operator. (34)}
\]
The symbol \( \sigma_{t, \iota r} \) of \( \tilde{\sigma}_{t, \iota r} \) admits a complete asymptotic expansion
\[
\sigma_{t, \iota r} (w, r) \sim \sum_{j=0}^{\infty} \sigma_{t, \iota r, j} (w) r^{-\frac{m}{2} - j},
\]
in which \( \sigma_{t, \iota r, 0} = \{ e_{\mathcal{A}^{t, \iota r}}, e_{\mathcal{A}^{t, \iota r}} \}^{-1} \) is to leading order the reciprocal of the overlap of two Gaussians, as in (23).

2.5. Quantization of linear symplectic maps on Bargmann–Fock space.

The proofs of our theorems involve Taylor expansions in appropriate coordinates to reduce the geometry to the model linear space, so we briefly review the metaplectic representation on Bargmann–Fock space used to quantize symplectic linear mappings. Details can be found [30, 9].

The Bargmann–Fock space on \( \mathbb{C}^{m} \) is
\[
\mathcal{H}(\mathbb{C}^{m}) = \left\{ f(z)e^{-\frac{|z|^{2}}{4}} \in L^{2}(\mathbb{C}^{m}, dz) \mid f \in \mathcal{O}(\mathbb{C}^{m}) \right\}.
\]
The reproducing Bergman kernel has the exact formula
\[
\Pi_{\mathcal{H}}(z, w) = (2\pi)^{-m} e^{-\frac{|z|^{2}}{4} - \frac{|w|^{2}}{4}} |z - w|^{2m}.
\]
Let \( Sp(m, \mathbb{R}) \) denote the space of real symplectic matrices on \( \mathbb{R}^{2m} = \mathbb{R}^{m} \times \mathbb{R}^{m} \) with respect to the standard symplectic form. Then matrix multiplication \( M \in Sp(m, \mathbb{R}) \) in real coordinates takes the form
\[
M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}.
\] (35)
We map $\mathbb{R}^{2m}$ into $\mathbb{C}^m$ via $(x, y) \mapsto (x+iy, x-iy) =$: $(z, \bar{z})$. Under this mapping, (35) becomes
\[
\mathcal{M} \left( \frac{z}{\bar{z}} \right) = \begin{pmatrix} P & Q \\ Q & \bar{P} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix},
\]
where the holomorphic component $P$ and antiholomorphic component $Q$ of the symplectic mapping are given by
\[
\begin{pmatrix} P & Q \\ Q & \bar{P} \end{pmatrix} = W^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} W, \quad W = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ iI & iI \end{pmatrix}.
\]
(The choice of normalization is taken so that $W$ is unitary.) The explicit formula for the holomorphic component is
\[
P = \frac{1}{2} (A + D + i(C - B)).
\]

The metaplectic representation on $\mathcal{H}(\mathbb{C}^m)$ is defined by $M \mapsto \Pi_{\mathcal{H},M}$, the latter being a unitary operator with kernel
\[
\Pi_{\mathcal{H},M}(z, w) = (\det P)^{-\frac{1}{2}} \int_{\mathbb{C}^m} \Pi_{\mathcal{H}}(z, \mathcal{M}v) \Pi_{\mathcal{H}}(v, w) \, dv,
\]
in which we set $\mathcal{M}v := P v + Q \bar{v}$. (The ambiguity of the sign of $(\det P)^{-\frac{1}{2}}$ is determined by the lift to the double cover.) Explicit computations involving standard Gaussian integrals show
\[
\Pi_{\mathcal{H},M}(z, w) := \mathcal{K}_M(z, w) e^{-\frac{|z|^2}{4} - \frac{1}{4} \frac{|w|^2}{4}}, \quad \mathcal{K}_M(z, w) := (2\pi)^{-m} (\det P)^{-\frac{1}{2}} \exp \left\{ \frac{1}{2} \left( z\bar{Q}P^{-1}z + 2\bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w} \right) \right\}.
\]

The principal term of Theorem 1.1 contains the lift of $\Pi_{\mathcal{H},M}$ to the reduced Heisenberg group $\mathbb{H}^m_{\text{red}} \cong S^1 \times \mathbb{C}^m$, which is given by
\[
\hat{\Pi}_{\mathcal{H},M}(\theta, z, \phi, w) = e^{i(\theta - \phi)} \Pi_{\mathcal{H},M}(z, w).
\]

3. PROOFS OF NEAR GRAPH SCALING ASYMPTOTICS

This section is focused on proving the scaling asymptotics Theorem 1.1 and Theorem 1.9, with the two proofs being identical. The techniques are similar to those of [7].

3.1. Identities in Heisenberg coordinates. We briefly recall the notion of Heisenberg coordinates. We point the reader to [8] for detailed construction of these coordinates on any strongly pseudoconvex CR manifold, and to [7] for the Grauert tube setting. Roughly speaking, in Heisenberg coordinates, the strongly pseudoconvex boundary $\partial M_\tau \subseteq M_C$ is, to a first approximation, the Heisenberg group viewed as a Seigel domain in complex Euclidean space. More precisely, given a point $p \in \partial M_\tau \subseteq M_C$ one may use the Levi procedure as in [8, Section 18] to construct holomorphic coordinates $(z_0, z_1, \ldots, z_{n-1}) = (z_0, z')$ on an open neighborhood $U \subseteq M_C$ such that for $w \in U$
\[
\rho(z_0, z') = -\text{Im} z_0 + |z'|^2 + O \left( |z_0||z'| + |z'|^3 \right)
\]
(38)

We note that $(t, z') := (\text{Re} z_0, z')$ constrained by $\rho(z_0, z') = 0$ provides a coordinate system on the open neighborhood $V = U \cap \partial M_\tau$ in $\partial M_\tau$. We will refer to both the coordinates on $M_C$ as well as the coordinates on $\partial M_\tau$ as Heisenberg coordinates.
coordinates. Furthermore, let \( Z_0 = \frac{T}{\partial t} \) where \( T \) is the characteristic vector field and \( Z_1, \ldots, Z_{m-1} \) denote an orthonormal frame of \( T^{1,0} \partial M_r \). Then in Heisenberg coordinates centered at \( p \) we have
\[
Z_0|_p = \frac{\partial}{\partial t}|_p, \quad Z_j|_p = \frac{\partial}{\partial z_j}|_p
\] (39)

We now record several Taylor expansions in Heisenberg coordinates established by the authors in [7] that will be useful in subsequent sections. In the following \( \lambda \in \mathbb{R}^+ \) is a parameter tending to \( \infty \). We state all of the identities in rescaled form as they appear in the main argument.

**Lemma 3.1 (Expansion of the rescaled phase function).** Let \( \psi_r \) be as in (26). In Heisenberg coordinates centered at \( p \in \partial M_r \), we have
\[
i\lambda\psi_r\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{w}{\sqrt{\lambda}}\right)\right) = -\sqrt{\lambda} \frac{i}{2} \text{Re} w_0 + \tilde{R}\left(\frac{\theta}{\lambda}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right),
\]
where
\[
\tilde{R} = \frac{i}{2} \theta - \frac{|u|^2}{2} - \frac{|w|^2}{2} + u \cdot \overline{w} + \lambda Q\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right)
\] (40)
and \( Q \) takes the form
\[
Q\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) = O\left(\frac{|\text{Re} w_0| |u|}{\lambda} + \frac{|\text{Re} w_0| |w'|}{\lambda}\right) + O\left(\lambda^{-\frac{2}{3}}\right)
\]

**Proof.** This is a special case of the computation immediately following [7, Remark 4.6]. Briefly, let \( \varphi_r = i\psi_r \) be the defining function (25) obtained by polarizing (38). Then, by [7, Lemma 3.4], in Heisenberg coordinates we may write
\[
i\psi_r(z, \overline{w}) = \varphi_r(z, \overline{w}) = \frac{i}{2} \left( z_0 - \overline{w}_0 \right) + \sum_{j=1}^{m-1} z_j \overline{w}_j + R(z, \overline{w}).
\]

Here, the remainder term \( R \) may be written as
\[
R(z, \overline{w}) = R_2(z_0, \overline{w}_0, z', \overline{w}') + R_2(z_0, \overline{w}_0) + R_3(z', \overline{w}'),
\]
where \( R_2(z_0, \overline{w}_0, z', \overline{w}') \) only contains terms of the form \( z_0^\alpha \overline{w}_0^\beta \) and \( \overline{w}_0^\alpha z'^\beta \) with \( |\alpha| + |\beta| \geq 2 \). Similarly, \( R_2(z_0, \overline{w}_0) \) (resp. \( R_3(z', \overline{w}') \)) only contains terms of the form \( z_0^\alpha \overline{w}_0^\beta \) with \( |\alpha| + |\beta| \geq 2 \) (resp. terms of the form \( z'^\alpha \overline{w}'^\beta \) with \( |\alpha| + |\beta| \geq 3 \)).

Keeping track of the powers of \( \sqrt{\lambda} \) under parabolic rescaling results in the statement of the lemma. \( \square \)

To prove our scaling asymptotics we will simultaneously be working with two sets of Heisenberg coordinate systems, one centered at \( p \) and another centered at \( G^*_r(p) \). We recall that \( G^*_r \) is the Hamiltonian flow of the characteristic vector field which also preserves \( T^{1,0} \partial M_r \oplus T^{0,1} \partial M_r \). Its derivative \( DG^*_r \) is a linear map \( T^*_p \partial M_r \rightarrow T^*_r(G^*_r(p)) \partial M_r \). With respect to Heisenberg coordinates (39) at \( p \) and \( G^*_r(p) \), we have
\[
DG^*_r = \begin{pmatrix} 1 & 0 \\ 0 & M_s \end{pmatrix} \quad \text{with} \quad M_s \in Sp(m-1, \mathbb{R}).
\]
We denote its complexification by \( \mathcal{M}_s \) as in (36) and use the same notation \( P, Q \) for its holomorphic and anti-holomorphic components. We have the following identities for Heisenberg coordinates centered at \( G^*_r(p) \).
Lemma 3.2 (Expansion of the rescaled geodesic flow). Let $w = (w_0, w')$ be a point in a Heisenberg coordinate chart centered at $p \in \partial M_r$. Then, in Heisenberg coordinates centered at $G_r^s(p) \in \partial M_r$, we have

$$G_r^s \left( \frac{w}{\sqrt{\lambda}} \right) = \left( \frac{\text{Re} w_0}{\sqrt{\lambda}} + \frac{2\tau r}{\sqrt{\lambda}} + O \left( \frac{|r|}{\lambda} \right) + O \left( \lambda^{-3/2} \right) \right),$$

$$\mathcal{M}w' + O \left( \frac{|r|}{\lambda} \right) + O \left( \lambda^{-3/2} \right).$$

Proof. This follows from the Taylor expansion $G_r^s(z_0, z') = (z_0 + 2\tau t + t \cdot O^1 + O(t^2), z' + t \cdot O^1 + O(t^2))$ of [7, Lemma 3.6].

Lemma 3.3 (Combined expansion of the rescaled phase and flow). In Heisenberg coordinates centered at $G_r^s(p) \in \partial M_r$, we have

$$i\lambda \Psi_r \left( G_r^s \left( \frac{w}{\sqrt{\lambda}} \right), \left( \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) = \sqrt{\frac{\lambda}{2}} (\text{Re} w_0 + 2\tau r)$$

$$+ \tilde{S} \left( \frac{\phi}{\lambda}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right),$$

where

$$\tilde{S} = -\frac{i}{2} \phi - \frac{|v|^2}{2} - \frac{|\mathcal{M}w|^2}{2} + \mathcal{T} (\mathcal{M}w) + \lambda T \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right)$$

and $T$ takes the form

$$T \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \frac{\text{Re} w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right) = O \left( \frac{|r||v|}{\lambda} + \frac{|r||w'|}{\lambda} + \frac{|\text{Re} w_0||v|}{\lambda} + \frac{|\text{Re} w_0||w'|}{\lambda} \right) + O \left( \lambda^{-2} \right).$$

Proof. This follows from Lemma 3.1 and Lemma 3.2.

3.2. Proof of Theorem 1.1: asymptotic expansion for $\Pi_{\chi,\lambda}$. Fix $p \in \partial M_r$ and let $(\theta, u), (\phi, v) \in \partial M_r$ be two points in Heisenberg coordinates centered at $p$ and $G_r^s(p)$ respectively. Then, as discussed in Remark 2.2, substituting the parametrix (27) for each instance of $\Pi_\tau$ above and composing the resulting kernels, we arrive at the oscillatory integral representation

$$\Pi_{\chi,\lambda} \left( p + \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), G_r^s(p) + \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right)$$

$$\sim \int_{\mathbb{R} \times \partial M_r \times \mathbb{R}^+ \times \mathbb{R}^+} e^{i\lambda \Psi} A d\sigma_1 d\sigma_2 d\mu_r(w) dt,$$

in which the phase $\Psi$ and the amplitude $A$ are given by

$$\Psi = -t + \frac{1}{\lambda} \sigma_2 \psi_r \left( p + \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), w \right) + \frac{1}{\lambda} \sigma_1 \psi_r \left( G_r^s(w), G_r^s(p) + \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right),$$

$$A = \tilde{\chi}(t) s \left( p + \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), w, \sigma_2 \right) s \left( G_r^s(w), G_r^s(p) + \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right), \sigma_1 \right) \sigma_t \tau (w, \sigma_1).$$

From now on, we suppress $p$ and $G_r^s(p)$ from the notation, keeping in mind that they are the origin in each of their respective coordinates. Make the change-of-variables...
σ_j \mapsto \lambda \sigma_j$. Homogeneity of the symbols implies

$$
\Pi_{\lambda, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim \lambda^{2m} \int_{\mathbb{R} \times \partial M_r \times \mathbb{R}^+} e^{i \lambda \Psi} A \, d\sigma_1 \, d\sigma_2 \, d\mu_\tau (w) \, dt, \quad (44)
$$

in which the phase $\Psi$ and the amplitude $\tilde{A}$ are given by

$$
\tilde{A} = \lambda^{-2m} A.
$$

We begin by localizing in $(w, t) \in \partial M_r \times \mathbb{R}$. Fix $C > 0$ and $0 < \delta < \frac{1}{2}$. Set

$$
V_\lambda = \left\{ (w, t) : \max \left\{ d(w, (\theta, \frac{u}{\sqrt{\lambda}})), d(G_r^1(w), (\phi, \frac{v}{\sqrt{\lambda}})) \right\} < \frac{2C}{\delta} \lambda^{\delta - \frac{1}{2}} \right\},
$$

$$
W_\lambda = \left\{ (w, t) : \max \left\{ d(w, (\theta, \frac{u}{\sqrt{\lambda}})), d(G_r^1(w), (\phi, \frac{v}{\sqrt{\lambda}})) \right\} > \frac{2C}{\delta} \lambda^{\delta - \frac{1}{2}} \right\}.
$$

Let $\{ \varrho_\lambda, 1 - \varrho_\lambda \}$ be a partition of unity subordinate to the cover $\{ V_\lambda, W_\lambda \}$ and decompose the integral (44) into

$$
\Pi_{\lambda, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim I_1 + I_2,
$$

$$
I_1 = \lambda^{2m} \int e^{i \lambda \Psi} \varrho_\lambda (t, w) \tilde{A} \, d\sigma_1 \, d\sigma_2 \, d\mu_\tau (w) \, dt,
$$

$$
I_2 = \lambda^{2m} \int e^{i \lambda \Psi} (1 - \varrho_\lambda (t, w)) \tilde{A} \, d\sigma_1 \, d\sigma_2 \, d\mu_\tau (w) \, dt.
$$

**Lemma 3.4.** We have $I_2 = O(\lambda^{-\infty})$.

**Proof.** By definition of $W_\lambda$, on the support of $1 - \varrho_\lambda$ either

$$
|d_{\sigma_2} \tilde{\Psi}| = \left| \psi_\tau \left( \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), w \right) \right| \geq 2D \left( \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), w \right) \geq C' \lambda^{2\delta - 1}
$$
or

$$
|d_{\sigma_1} \tilde{\Psi}| = \left| \psi_\tau \left( \left( G_r^1 (w), \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \right| \geq 2D \left( \left( G_r^1 (w), \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \geq C' \lambda^{2\delta - 1}
$$

where $D$ is the Calabi diastasis (28) and the inequalities follow from (29). Repeated integration by parts in $\sigma_1$ or $\sigma_2$ as appropriate completes the proof. \[ \square \]

In preparation for stationary phase we make the following change of variables

$$
t \mapsto s + \frac{r}{\sqrt{\lambda}} \quad \text{and} \quad (\text{Re} \, w_0, w') \mapsto \left( \frac{\text{Re} \, w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right).
$$

Substituting in our formulas from lemmas 3.2 and 3.4 we obtain the following oscillatory integral with parameter $\sqrt{\lambda}$.

$$
\Pi_{\lambda, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim e^{-i \lambda \lambda m} \int e^{i \lambda \Psi} \tilde{A} \, d\sigma_1 \, d\sigma_2 \, dwdr \quad (46)
$$

where

$$
\tilde{\Psi} = -r - \frac{\sigma_2}{2} \text{Re} \, w_0 + \frac{\sigma_1}{2} (\text{Re} \, w_0 + 2 \tau r),
$$

$$
\tilde{A} = e^{\sigma_2 \tilde{r} + \sigma_1 \tilde{s}} \varrho_\lambda \tilde{A} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \frac{\text{Re} \, w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}, \frac{r}{\sqrt{\lambda}}, \sigma_1, \sigma_2 \right) J \left( \frac{w'}{\sqrt{\lambda}} \right). \quad (47)
$$
with $J(\cdot)$ the volume density in Heisenberg coordinates.

We may further localize this integral in the $\sigma_1, \sigma_2$ variables. Let $\{\eta, 1-\eta\}$ be a partition of unity subordinate to the cover

$$\left\{ (\sigma_1, \sigma_2) : 0 < \sigma_1, \sigma_2 < \frac{2}{\tau} \right\}$$

and

$$\left\{ (\sigma_1, \sigma_2) : \sigma_1, \sigma_2 > \frac{3}{2\tau} \right\}.$$

Decompose (46) into two integrals:

$$\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim I'_1 + I'_2,$$

$$I'_1 = e^{-i \sigma_1 \lambda} \int e^{i \sigma_2 \lambda} \eta(\sigma_1, \sigma_2) \hat{A} d\sigma_1 d\sigma_2 d\mu_t(w) dr$$

$$I'_2 = e^{-i \sigma_1 \lambda} \int e^{i \sigma_2 \lambda} (1-\eta(\sigma_1, \sigma_2)) \hat{A} d\sigma_1 d\sigma_2 d\mu_t(w) dr$$

with $\hat{A}$ and $\hat{\Psi}$ as in (47).

**Lemma 3.5.** We have $I'_2 = O(\lambda^{-\infty})$.

**Proof.** Notice that

$$\left| \nabla_{\text{Re } w_0} \hat{\Psi} \right|^2 \geq \left( \frac{\sigma_1}{2} - \frac{\sigma_2}{2} \right)^2 + (\tau \sigma_2 - 1)^2 \geq \frac{1}{4}$$

on the support of $1-\eta$. Thus, the lemma follows from repeated integration by parts in $(\text{Re } w_0, t)$. $\square$

We have reduced the spectral localization kernel to the oscillatory integral

$$\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim e^{-i \sigma_1 \lambda} \int e^{i \sigma_2 \lambda} \hat{\Psi} \hat{A} dw'(\text{Re } w_0) d\sigma_1 d\sigma_2 dr, \quad (48)$$

with phase and amplitude

$$\hat{\Psi} = -r - \frac{\sigma_2}{2} \text{Re } w_0 + \frac{\sigma_1}{2} (\text{Re } w_0 + 2\tau r),$$

$$\hat{A} = e^{\sigma_1 R + \sigma_2 \hat{S}_t} \eta \hat{\chi} \hat{A} J.$$

Since the exponential of the terms of order $\lambda^{-\frac{1}{2}}$ appearing in $\hat{R}, \hat{S}$ is bounded it may be absorbed into the main amplitude. We will now reduce (48) to a Gaussian integral over $\mathbb{C}^{m-1}$ by integrating out the variables $\text{Re } w_0, \sigma_1, \sigma_2, r$ using the method of stationary phase. We note the following derivatives:

$$\partial_{\sigma_2} \hat{\Psi} = -\frac{1}{2} \text{Re } w_0,$$

$$\partial_{\sigma_1} \hat{\Psi} = \frac{1}{2} (\text{Re } w_0 + 2\tau r),$$

$$\partial_t \hat{\Psi} = -1 + \tau \sigma_1,$$

$$\partial_{\text{Re } w_0} \hat{\Psi} = -\frac{\sigma_2}{2} + \frac{\sigma_1}{2}.$$

The critical set of the phase is the point $C = \{\text{Re } w_0 = 0, r = 0, \sigma_1 = \sigma_2 = \frac{1}{2}\}$. The Hessian matrix and its inverse at the critical point are

$$\hat{\Psi}''_C = \begin{pmatrix} r & \sigma_1 & \sigma_2 & \text{Re } w_0 \\ r & 0 & 0 & 0 \\ \sigma_1 & 0 & 0 & \frac{1}{2} \\ \sigma_2 & 0 & 0 & -\frac{1}{2} \\ \text{Re } w_0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad (\hat{\Psi}''_C)^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -2 \end{pmatrix}.$$
Set
\[ L_{\Psi} = \left\langle \left( \frac{s''}{s} \right)^{-1} D, D \right\rangle = \frac{2}{\tau} \partial_{\sigma} \partial_{\tau} + \frac{2}{\tau} \partial_{\sigma} \partial_{s} - 4 \partial_{\sigma} \partial_{\text{Re} w_0} \]
By the method of stationary phase ([14, Theorem 7.75]), we have
\[
e^{-i s \lambda^m} \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}} e^{i \sqrt{\lambda} \Psi_{\psi}} \hat{A} du' ds \partial_{s} ds \partial_{\tau} dr
\]
\[
= \frac{8 \pi^2}{\lambda \tau} e^{-i s \lambda^m} \sum_{j=0}^{N-1} \lambda^{-j} \frac{1}{2^{j \nu} \nu!} \sum_{\nu=0}^{\nu} \sum_{\mu=2}^{\nu} \frac{1}{\nu!} \partial_{\sigma} \partial_{\tau} \partial^{\nu-\mu} \partial^{\mu} \int_{\mathbb{R} \times \mathbb{R}} \left[ e^{i \mu (\sigma_2 R + \sigma_1 \tilde{S})} \eta \lambda \hat{A} J_{C} + \hat{R}_{N} \right]
\]
with the remainder term satisfying
\[
\int_{C_{m}} |\hat{R}_{N}| dw' \leq \lambda^{-\frac{N}{2}} \int_{C_{m}} \sum_{|\alpha| \leq 2 N} \sup |D^\alpha (\eta \rho \lambda \hat{A} J_{C})| dw' \leq C N \lambda^{-\frac{N}{2}}.
\]
(Here, the supremum and the derivative $D^\alpha$ are taken over $t, \sigma_1, \sigma_2, \text{Re} w_0$ and the integral is with respect to the remaining variable $w'$. Note that $\hat{A}$, defined in (45), is a symbol of order zero.)

Thanks to the remainder estimate, we may integrate the asymptotic expansion (49) term-by-term in $w'$ to obtain (48). Upon substituting expressions (40) and (41) the leading term is given by the following Gaussian integral
\[
C_{m} \frac{\lambda^{m-1}}{\tau} e^{-i s \lambda^m} \sigma_{s, \tau, 0}(p)e^{\frac{1}{\tau}(\theta - \phi)}
\]
\[
\times \int_{C_{m}} \exp \left\{ \frac{1}{\tau} \left( -\frac{|u|^2}{2} - \frac{|w|^2}{2} + u \cdot \tilde{w} - \frac{|v|^2}{2} - \frac{|M_s w'^2|}{2} + \tilde{v} \cdot M_s w' \right) \right\} dw'
\]
(50)
The symbol $\sigma_{s, \tau, 0}(p)$ can be computed as in [31] to be $(\det P_s)^{-\frac{1}{2}}$. This is precisely the same integral as (37) and so we obtain the leading term
\[
\frac{C_{m}}{\tau} \left( \frac{\lambda}{\tau} \right)^{m-1} e^{-i s \lambda^m} \hat{A}_{H, M_s} \left( \frac{\theta}{2 \tau}, \frac{\phi}{2 \tau}, \frac{v}{2 \tau}, \frac{\tilde{v}}{2 \tau} \right)
\]
The lower order terms have the form
\[
\frac{C_{m}}{\tau^m} \lambda^{m-1} e^{-i s \lambda^m} e^{\frac{1}{\tau}(\theta - \phi)}
\]
\[
\times \int_{C_{m}} P_j(u, v, w, s, \theta, \phi)e^{\frac{1}{\tau} \left( -\frac{|w|^2}{2} - \frac{|v|^2}{2} + u \cdot \tilde{w} - \frac{|M_s w|^2}{2} + \tilde{v} \cdot M_s w' \right)} dw',
\]
with $j$ a positive integer and $P_j(u, v, w, s, \theta, \phi)$ a polynomial. This can be rewritten as
\[
\frac{C_{m}}{\tau^m} \lambda^{m-1} e^{-i s \lambda^m} e^{\frac{1}{\tau}(\theta - \phi)}
\]
\[
\times \int_{C_{m}} e^{\frac{1}{\tau} \left( -\frac{|w|^2}{2} - \frac{|v|^2}{2} + u \cdot \tilde{w} + \tilde{v} \cdot M_s w' \right)} \tilde{P}_j(u, v, s, \theta, \phi, D)e^{\frac{1}{\tau} \left( -\frac{|w|^2}{2} - \frac{|M_s w|^2}{2} \right)} dw',
\]
where $\tilde{P}_j$ is a differential operator with polynomial coefficients. We integrate by parts with the $\tilde{P}_j$ operator from which we obtain the same Gaussian integral as
Theorem 1.9: asymptotic expansion for $P_{\chi,\lambda}$. We can also study the on-shell scaling asymptotics for the tempered spectral projection kernel (33) under Heisenberg-type rescaling. The proof is nearly identical to that of Theorem 1.1. We first write out the kernel using Equation 34 and (27):

$$P_{\chi,\lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \sim \int_{\mathbb{R} \times \partial M_{\tau} \times \mathbb{R}^{+} \times \mathbb{R}^{+}} e^{i\lambda \Psi} B \, d\sigma_{1} d\sigma_{2} d\mu_{\tau}(w) \, dt, \quad (51)$$

in which the phase $\Psi$ and the amplitude $B$ are given by

$$\Psi = -t + \frac{1}{\lambda} \sigma_{2} \psi_{\tau} \left( \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right), w \right) + \frac{1}{\lambda} \sigma_{1} \psi_{\tau} \left( G_{r}^{t}(w),\left( \frac{\phi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \right),$$

$$B = \hat{\chi}(t) s \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{w}{\sigma_{2}}, \frac{\phi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{\sigma_{1}}{\lambda} \right) \sigma_{t,\tau}(w,\sigma_{1}).$$

The only modification is that despite identical notation, the unitarization symbol $\sigma_{t,\tau}$ for $B$ is now of order $-(m-1)/2$, whereas $\sigma_{t,\tau}$ in the expression for $A$ in (43) is of order zero. Hence, the oscillatory integral expression (51) is exactly $\lambda^{-(m-1)/2}$ times the expression (42). The rest of the computations proceed in the same manner.

4. PROOFS OF $L^{p}$ ESTIMATES

In this section we prove $L^{p}$ estimates of the Szegő kernel, namely Theorem 1.3 and Theorem 1.5. Corollary 1.4, an $L^{p}$ upper bound for normalized eigenfunctions of $\Pi_{\tau} \hat{D}_{g} \Pi_{\tau}$, is then deduced.

We begin by establishing the following Gaussian decay estimate for $\Pi_{\chi,\lambda}(z, w)$ away from a small neighborhood of the graph $(z, w) = (p, G_{r}^{\tau}(p))$.

**Lemma 4.1** (Gaussian decay estimate). Fix $z \in \partial M_{\tau}$. Set $\delta = |\text{supp} \chi| = 2\varepsilon$ and $T_{\delta}(z) = \{ G_{r}^{t}(z) : |t| < \delta \}$. Then, after possibly shrinking $\text{supp} \chi$, there exists $C > 0$ such that whenever $d(T_{\delta}(z), w) < C \lambda^{-\frac{1}{2}}$ we have

$$|\Pi_{\chi,\lambda}(z, w)| \leq C(1+o(1))\lambda^{m-1} e^{-\frac{1}{4\varepsilon} d(T_{\delta}(z),w)^{2}} + O(\lambda^{-\infty}).$$

**Proof.** Let $|s| < \varepsilon = \delta/2$. In Heisenberg coordinates centered at $G_{r}^{\tau}(z)$, consider points of the form $w = G_{r}^{\tau}(z) + (\frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}})$ with $|\phi, u| < \lambda^{\frac{1}{2}}$. We repeat the stationary phase computation in the proof of Theorem 1.1:

$$\Pi_{\chi,\lambda} \left( z, G_{r}^{\tau}(z) + \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \sim e^{-i\lambda \psi} \int e^{i\lambda \hat{\chi} \hat{\Psi}} e^{i\sqrt{\lambda} \hat{\Psi}} dw' \, d(Re w_{0}) d\sigma_{1} d\sigma_{2} dt,$$

with phase and amplitude defined in the same way as (48). Keeping track of the first remainder term, we find

$$\left| \Pi_{\chi,\lambda} \left( z, G_{r}^{\tau}(z) + \left( \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \right| \leq \lambda^{m-1} e^{-\frac{1}{4\varepsilon} \left| u \right|^{2} - \frac{\lambda}{4\varepsilon} P_{s}^{-1} Q_{s} s} + \lambda^{m-1} \varepsilon e^{-\frac{1}{2\varepsilon} (|u|^{2} + \tilde{u} P_{s}^{-1} Q_{s} \tilde{u})} R(p, s, u, \lambda),$$

where $P_{s}, Q_{s}$ are matrices defined in (36) and $R(p, s, u, N) \leq C(s) |u|$ for $|u| < \lambda^{\frac{1}{2}}$. 


Since \( P^{-1}Q_s = o(s) \) and \(|u| \approx \sqrt{\lambda}d(G_s^r(z), G_s^r(z) + \frac{\rho}{\sqrt{\lambda}})\), we get uniformly for \(|s| < \varepsilon\) and \(|u| < \lambda^{\frac{1}{2}}\) that

\[
\left| \Pi_{\chi,\lambda} \left(z, G_s^r(z) + \left(\frac{\phi}{\lambda}, \sqrt{\frac{r}{\lambda}}\right)\right) \right| \leq C\lambda^{m-1} \left(1 + \frac{|u|}{\sqrt{\lambda}}\right) e^{-\frac{1}{\lambda^{\frac{1}{2}}}d(G_s^r(z), G_s^r(z) + \frac{\rho}{\sqrt{\lambda}})},
\]

as desired. \(\square\)

To establish sharpness we will also need the following lower bound on \(\Pi_{\chi,\lambda}\) in a \(\lambda^{-\frac{1}{2}}\) of the graph.

**Lemma 4.2.** Fix \(z \in \partial M_r\) and \(D > 0\). Set \(\delta = |\text{supp } \tilde{\chi}| = 2\varepsilon\) and \(T_\delta(z)\) as in Lemma 4.1. Then, after possibly shrinking \(\text{supp } \tilde{\chi}\), there exists \(C > 0\) such that whenever \(d(T_\delta(z), w) < D\lambda^{-\frac{1}{2}}\) we have

\[
|\Pi_{\chi,\lambda}(z, w)| \geq C(1 - o(1))\lambda^{m-1}e^{-\frac{\lambda d(T_\delta(z), w)^2}{\lambda^{\frac{1}{2}}}}.
\]

**Proof.** This is an immediate corollary of Theorem 1.1 when we take \(w = G_s^r(z) + (\frac{\phi}{\lambda}, \sqrt{\frac{r}{\lambda}})\) with \(|\phi, u| < D\). \(\square\)

### 4.1. Proof of Theorem 1.3: sharp norm estimates for \(\Pi_{\chi,\lambda}\)

We invoke the Shur-Young inequality

\[
\|\Pi_{\chi,\lambda}\|_{L^r \to L^q} \leq C_p \left[ \sup_z \int_{\partial M_r} |\Pi_{\chi,\lambda}(z, w)|^r \, dw \right]^\frac{1}{r}, \quad \frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}.
\]

With \(T_\delta\) as in Lemma 4.1, we break up the integral

\[
\int_{\partial M_r} |\Pi_{\chi,\lambda}(z, w)|^r \, dw = \int_{d(T_\delta(z), w) \leq \lambda^{-\frac{1}{2}}} |\Pi_{\chi,\lambda}(z, w)|^r \, dw \quad \text{(52)}
\]

\[
+ \int_{d(T_\delta(z), w) \geq \lambda^{-\frac{1}{2}}} |\Pi_{\chi,\lambda}(z, w)|^r \, dw. \quad \text{(53)}
\]

The integration by parts argument for Lemma 3.4 can be adapted to show that (53) is \(O(\lambda^{-\infty})\). We use Lemma 4.1 to see that (52) is to leading order

\[
C\lambda^{r(m-1)} \int_{\mathbb{R}^{2(m-1)}} e^{-r\lambda |\phi|^2} \, du \leq C\lambda^{(r-1)(m-1)}. \quad \text{(54)}
\]

Combining these estimates establishes the desired upper bound.

We now show that this upper bound is sharp. Set

\[
\Phi_{\chi,\lambda}^w(z) := \frac{\Pi_{\chi,\lambda}(z, w)}{\|\Pi_{\chi,\lambda}(\cdot, w)\|_{L^r(\partial M_r)}}.
\]

Note that \(\|\Phi_{\chi,\lambda}^w\|_{L^p(\partial M_r)} = 1\) and by (6),

\[
\Pi_{\chi,\lambda}(\Phi_{\chi,\lambda}^w)(z) = \frac{\Pi_{\chi,\lambda}(z, w)}{\|\Pi_{\chi,\lambda}(\cdot, w)\|_{L^r(\partial M_r)}}. \quad \text{(55)}
\]

To estimate the numerator of (55), we observe

\[
\int_{\partial M_r} |\Pi_{\chi,\lambda}(z, w)|^q \, dw \geq \int_{d(T_\delta(z), w) \leq D\lambda^{-\frac{1}{2}}} |\Pi_{\chi,\lambda}(z, w)|^q \, dw,
\]

so by applying Lemma 4.2 to the integrand and a using similar argument used to show (54), we may conclude \(\|\Pi_{\chi,\lambda}(\cdot, w)\|_{L^q(skr_M)} \geq C\lambda^{(m-1)(1-\frac{1}{2})}\). Therefore, \(\|\Pi_{\chi,\lambda}\|_{L^p(\partial M_r)} \sim \lambda^{(m-1)(1-\frac{1}{2})}\).
Similarly, the denominator of (55) is asymptotically $\|\Pi_{X,\lambda}(\cdot, w)\|_{L^p(\partial M_r)} \sim \lambda^{(m-1)(1-\frac{1}{p})}$. Together we have

$$\|\Pi_{X,\lambda}(\Phi_{X,\lambda}^w(z))\|_{L^p(\partial M_r)} \sim \lambda^{(m-1)(1-\frac{1}{p})-(m-1)(1-\frac{1}{q})} = \lambda^{(m-1)(\frac{1}{q} - \frac{1}{p})},$$

which shows $\Phi_{X,\lambda}^w$ saturates the upper bound.

4.2. **Proof of Theorem 1.5: norm estimates for $\Pi_{[\lambda, \lambda+1]}$.** A standard argument [25, Chapter 5] converts the $L^p \to L^q$ estimate for $\Pi_{X,\lambda}$ to that for the projection $\Pi_{[\lambda, \lambda+1]}$ onto a short spectral interval $[\lambda, \lambda + 1]$ as defined in (7). We include a proof here for the readers’ convenience.

**Theorem 1.5** is equivalent to sharpness of the dual inequality

$$\|\Pi_{[\lambda, \lambda+1]}f\|_{L^2(\partial M_r)} \leq C\lambda^{(m-1)(\frac{1}{q} - \frac{1}{p})}\|f\|_{L^p(\partial M_r)},$$

which we now establish. For the upper bound, we compute

$$\|\Pi_{[\lambda, \lambda+1]}f\|_{L^2}^2 = \sum_{\lambda \leq \lambda_j < \lambda + 1} |\langle f, e_j \rangle|^2 \leq C \sum_{j=1}^{\infty} \chi(\lambda - \lambda_j)^2 |\langle f, e_j \rangle|^2 = C \|\Pi_{X,\lambda}f\|_{L^2}^2 \leq \lambda^{2(m-1)(\frac{1}{q} - \frac{1}{p})}\|f\|_{L^p}^2.$$

To show this upper bound is saturated, fix $w \in \partial M_r$ and set $f_{X,\lambda}(z) = \Pi_{X,\lambda}(z, w)$. We compute

$$\|\Pi_{[\lambda, \lambda+1]}f_{X,\lambda}\|_{L^2}^2 = \sum_{\lambda \leq \lambda_j < \lambda + 1} |\chi(\lambda - \lambda_j)^2| e_{\lambda_j}(w)|^2 \|e_{\lambda_j}\|_{L^2}^2 \geq C \sum_{\lambda \leq \lambda_j < \lambda + 1} |e_j(w)|^2 \geq \frac{1}{\text{vol}(\partial M_r)} \int_{\partial M_r} \sum_{\lambda \leq \lambda_j < \lambda + 1} |e_j(z)|^2 dz \geq C(N(\lambda + 1) - N(\lambda)) \sim C\lambda^{m-1},$$

where $N(\lambda) = \#\{j : \lambda_j < \lambda\}$ is the eigenvalue counting function.

It follows from the proof of the sharpness of **Theorem 1.3** that $\|f_{X,\lambda}\|_{L^p} \sim \lambda^{(m-1)(1-\frac{1}{p})}$, so

$$\sup_{f \in L^p(\partial M_r)} \lambda^{-(m-1)(\frac{1}{q} - \frac{1}{p})} \frac{\|\Pi_{[\lambda, \lambda+1]}f\|_{L^2}}{\|f\|_{L^p}} \geq C \lambda^{-\frac{(m-1)}{2}}(N(\lambda + 1) - N(\lambda))^{\frac{1}{2}} \geq C + o(1).$$

Taking the lim sup of both sides we get

$$\limsup_{\lambda \to \infty} \sup_{f \in L^p(\partial M_r)} \lambda^{-(m-1)(\frac{1}{q} - \frac{1}{p})} \frac{\|\Pi_{[\lambda, \lambda+1]}f\|_{L^2}}{\|f\|_{L^p}} > 0$$

which shows that upper bound is sharp.
5. Proof of Proposition 1.6: complexified Laplace eigenfunctions and eigenfunctions of $\Pi_r D_{\sqrt{r}} \Pi_r$

Here we give a proof of Proposition 1.6. In the following we use the parameter $\mu$ for the frequencies of $-\Delta$ to distinguish it from the spectral parameter $\lambda$ used for $\Pi_r D_{\sqrt{r}} \Pi_r$. Set

$$\tilde{\varphi}_{\mu}^C(z) = \frac{\varphi_{\mu}^C(z)}{\|\varphi_{\mu}^C\|_{L^2(\partial M_r)}}$$

to be the $L^2(\partial M_r)$ normalized complexified Laplace eigenfunction.

On one hand, by the first part of Lemma 2.3, we may write $U(i\tau)^* U(i\tau) = (-\Delta)^{-m+1} + R$ for some $R \in \Psi^{-m+1}(M)$. It follows that

$$U(i\tau)\sqrt{-\Delta}^{m+1} U(i\tau)^* \varphi_{\mu}^C = \frac{U(i\tau)\sqrt{-\Delta}^{m+1} U(i\tau)^* U(i\tau) \varphi_{\mu}^C}{\sqrt{U(i\tau)^* U(i\tau) \varphi_{\mu}^C, \varphi_{\mu}^C}}$$

$$\quad = \frac{U(i\tau)\sqrt{-\Delta} \varphi_{\mu}^C + U(i\tau)\sqrt{-\Delta}^{m+1} R \varphi_{\mu}^C}{\sqrt{U(i\tau)^* U(i\tau) \varphi_{\mu}^C, \varphi_{\mu}^C}}$$

$$\quad = \mu \varphi_{\mu}^C + O_{L^2(\partial M_r)}(1).$$

In the last equality, the first term follows from the definition of complexification and the eigenvalue equation; the second term follows from $L^2$ boundedness of $(-\Delta)^{m+1} R$ as a zeroth order $\Psi DO$ and $U(i\tau)$ being a continuous isomorphism $L^2(M) \to O^{-m+1}(\partial M_r)$.

On the other hand, by the second part of Lemma 2.3, there exists $A \in \Psi^1(\partial M_r)$ such that that

$$U(i\tau)\sqrt{-\Delta}^{m+1} U(i\tau)^* = \Pi_r A \Pi_r \quad \text{and} \quad \sigma(A)|_{\Sigma_r} = \sigma(D_{\sqrt{r}}).$$

Therefore, we may write $\Pi_r A \Pi_r = \Pi_r D_{\sqrt{r}} \Pi_r + \Pi_r B \Pi_r$ for some $B \in \Psi^0(\partial M_r)$. It follows from $L^2$ boundedness of $B$ that

$$\Pi_r D_{\sqrt{r}} \Pi_r \varphi_{\mu}^C = \Pi_r A \Pi_r \varphi_{\mu}^C - \Pi_r B \Pi_r \varphi_{\mu}^C = \mu \varphi_{\mu}^C + O_{L^2(\partial M_r)}(1).$$

By a standard theorem giving the distance to the spectrum (see for example [39, Theorem C.11]), if $\lambda_j \in \text{spec}(\Pi_r D_{\sqrt{r}} \Pi_r)$ and $\mu_j \in \text{spec}(\Pi_r A \Pi_r)$ then there exists $M > 0$ such that for $|\mu_j - \lambda_j| < M$ for all $j$ sufficiently large. Therefore, we can view the $\varphi_{\mu}^C$ as approximate eigenfunctions for $\Pi_r D_{\sqrt{r}} \Pi_r$.

Additionally, we know $\|\varphi_{\lambda_j}^C - e_{\lambda_j}\|_{L^\infty(\partial M_r)} = O(\lambda_j^{\frac{m-1}{2}})$ thanks to [31, Theorem 0.1]. Since $\|\varphi_{\lambda_j}^C - e_{\lambda_j}\|_{L^2(\partial M_r)} = O(1)$, by the log convexity of $L^p$ norms we get $\|\varphi_{\lambda_j}^C - e_{\lambda_j}\|_{L^p(\partial M_r)} = O(\lambda^{\frac{m-1}{2} - \frac{1}{2}})$.

5.1. Complexified Gaussian beams as extremals: direct computation. In this section, we show that the $L^p$ estimate of Proposition 1.6 on complexified Laplace eigenfunctions is saturated by analytic continuations of Gaussian beams on the round $S^2$. We use spherical coordinates

$$x = \sin \varphi \cos \theta, \quad y = \sin \varphi \sin \theta, \quad z = \cos \varphi,$$
where $0 \leq \varphi \leq \pi$ and $0 \leq \theta < 2\pi$. The standard spherical harmonics are the joint eigenfunctions

$$-\Delta_{S^2} Y_N^m = N(N+1)Y_N^m, \quad \frac{1}{i} \frac{\partial}{\partial \theta} Y_N^m = mY_N^m, \quad -N \leq m \leq N.$$ 

of the spherical Laplacian and the angular momentum operator. The highest weight spherical harmonic (Gaussian beam) is of the form

$$Y_N^N(\theta, \varphi) = c_N \sin^N(\varphi) e^{i N \theta}, \quad c_N = (-1)^N 2^{-N/2} (2N)!^{1/2} N! \sim N^N.$$

It is convenient to transfer the computations of Guillemin–Stenzel [12] from Cartesian coordinates to spherical coordinates. In terms of complexified Cartesian coordinates, the Grauert tube $S^2_\tau$ of the sphere is the set

$$\{(x + i\xi_x, y + i\xi_y, z + i\xi_z) : (x + i\xi_x)^2 + (y + i\xi_y)^2 + (z + i\xi_z)^2 = 1\},$$

and the Grauert tube function is

$$\sqrt{\rho}(x + i\xi_x, y + i\xi_y, z + i\xi_z) = \sinh^{-1}\left[\left(\xi_x^2 + \xi_y^2 + \xi_z^2\right)^{1/2}\right]. \quad (56)$$

In terms of complexified spherical coordinates, we have

$$x + i\xi_x = \sin(\varphi + i\xi_\varphi) \cos(\theta + i\xi_\theta)$$

$$= \cos(\varphi) \sinh(\xi_\varphi) \sin(\theta) \sinh(\xi_\theta) + \sin(\varphi) \cosh(\xi_\varphi) \cos(\theta) \cosh(\xi_\theta)$$

$$+ i \left[\cos(\varphi) \sinh(\xi_\varphi) \cos(\theta) \cosh(\xi_\theta) - \sin(\varphi) \cosh(\xi_\varphi) \sin(\theta) \sinh(\xi_\theta)\right]$$

$$y + i\xi_y = \sin(\varphi + i\xi_\varphi) \sin(\theta + i\xi_\theta)$$

$$= -\cos(\varphi) \sinh(\xi_\varphi) \cos(\theta) \sinh(\xi_\theta) + \sin(\varphi) \cosh(\xi_\varphi) \sin(\theta) \cosh(\xi_\theta)$$

$$+ i \left[\cos(\varphi) \sinh(\xi_\varphi) \sin(\theta) \cosh(\xi_\theta) + \sin(\varphi) \cosh(\xi_\varphi) \cos(\theta) \sinh(\xi_\theta)\right]$$

$$z + i\xi_z = \cos(\varphi) \cosh(\xi_\varphi) - i \sin(\varphi) \sinh(\xi_\varphi).$$

The formula for $\sqrt{\rho}$ in spherical coordinates is complicated. Since we will be simplifying our expressions by picking special values, it suffices to note

$$\xi_x^2 + \xi_y^2 + \xi_z^2 = \cos^2(\varphi) \sin^2(\xi_\varphi) \cosh^2(\xi_\theta) + \sin^2(\varphi) \cosh^2(\xi_\varphi) \sinh^2(\xi_\theta) + \sin^2(\xi_\varphi). \quad (57)$$

We also note that the analytically continued highest weight spherical harmonic is of the form

$$(Y_N^N)^C(\theta + i\xi_\theta, \varphi + i\xi_\varphi) = c_N \left[\sin(\varphi) \cosh(\xi_\varphi) + i \cos(\varphi) \sinh(\xi_\varphi)\right]^N e^{i N \theta} e^{-N \xi_\theta} \quad (58)$$

To simplify (57) and (58), we fix $\varphi = \pi/2$ so that

$$\xi_x^2 + \xi_y^2 + \xi_z^2 = \cosh^2(\xi_\varphi) \sinh^2(\xi_\theta) + \sinh^2(\xi_\varphi), \quad (59)$$

$$(Y_N^N)^C(\theta + i\xi_\theta, i\xi_\varphi) = c_N \cosh^N(\xi_\varphi) e^{i N \theta} e^{-N \xi_\theta} \quad (c_N \sim N^N).$$

We additionally set $\tau = 1$, so that (56) and (59) imply the Grauert tube boundary is given by $\sinh^2(1) = \cosh^2(\xi_\varphi) \sinh^2(\xi_\theta) + \sinh^2(\xi_\varphi)$. Direct computation shows the equality is satisfied whenever

$$-1 < \xi_\varphi < 1 \quad \text{and} \quad \xi_\theta = -\sinh^{-1}\left[\text{sech}(\xi_\varphi) \left(\sinh^2(1) - \sinh^2(\xi_\varphi)\right)^{1/2}\right].$$
Note that $\xi_\theta < 0$, so
\[
\| (Y_N^N)^\mathcal{C} \|_{L^p} = c_N \int_{\partial S^2_1} |\cosh^N(\xi_\phi)|^p e^{-Np\xi_\phi} d\xi_\theta d\xi_\phi \\
\geq c_N \int_{-1}^1 |\cosh^N(\xi_\phi)|^p e^{-Np\sinh^{-1}\left[\text{sech}(\xi_\phi)\left(\sinh^2(1) - \sinh^2(\xi_\phi)\right)\right]^\frac{1}{2}}\ d\xi_\phi \\
\geq c_N \int_{-1}^1 |\cosh^N(\xi_\phi)|^p d\xi_\phi \\
\geq c_N \int_{0}^1 e^{-Np\xi_\phi} d\xi_\phi \\
\sim N^\frac{p}{4} e^{N^p}.
\]
In the last line we used $c_N \sim N^\frac{1}{4}$. Combined with the universal asymptotics \[
\| (Y_N^N)^\mathcal{C} \|_{L^2(\partial M_\tau)} = N^{-\frac{1}{2}} e^N(1 + O(N^{-\frac{1}{2}}))
\] proved in [31, Lemma 0.2], we conclude
\[
\frac{\| (Y_N^N)^\mathcal{C} \|_{L^p(\partial S^2_1)}}{\| (Y_N^N)^\mathcal{C} \|_{L^2(\partial S^2_1)}} \sim \frac{N^\frac{1}{2} N^{-\frac{1}{2}} e^N}{N^{-\frac{1}{2}} e^N} = N^{-\frac{1}{2}} e^{N^p},
\]
showing that Proposition 1.6 is sharp.

5.2. **Complexified Gaussian beams as extremals: geometric explanation.**

As mentioned earlier, in the real domain Gaussian beams only saturate the low $L^p$ norms whereas zonal spherical harmonics saturate the high $L^p$ norms. As in [31], we give a heuristic symplectic geometry explanation for why complexifications of Gaussian beams are also extremals for high $L^p$ norms on $\partial S^2_1$.

Let $P$ be the north pole and let $\frac{\partial}{\partial \theta}$ denote the generator of rotation about the $z$-axis. The zonal spherical harmonics denoted $Y_N^0$ are semiclassical Lagrangian distributions associated to
\[
\Lambda_P = \{ g^t(S^*_P S^2) : t \in \mathbb{R} \}
\]
Under the natural projection $S^* S^2 \rightarrow S^2$ there is a blowdown singularity at $P$, which leads to peaking of sup norms. This is a heuristic explanation for the zonal harmonics saturating high $L^p$ norms in the real domain.

Now let $E$ denote the equator. The Gaussian beams $Y_N^N$ associated to $E$ have wavefront set
\[
\Gamma = \{ g^t(p,0) : p \in E, \ t \in \mathbb{R} \}
\]
Th sympelctic cone $\Sigma = (\partial M_\tau, \mathbb{R}^+ d\alpha) \cong (S^*_P M, \mathbb{R}^+ d\xi)$ from (21) is the phase space of the Grauert tube boundary. Under the identification (22), $\Lambda_P$ is a Lagrangian submanifold embedded in $\partial M_\tau$ and no blowdown singularities occur, suggesting that zonal harmonics are longer extremals in the complex domain. Instead, the geodesic flow and the lift of rotations to $S^* S^2$ coincide on $\Gamma$, and $\Gamma$ is a singular leaf of the foliations of $\partial M_\tau$ generated by the geodesic flow together with rotations. This singularity suggests that Gaussian extremizes $L^p$ norms in the complex domain.

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