Globally solving Non-Convex Quadratic Programs via Linear Integer Programming techniques

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Abstract

A quadratic program (QP) is a well-studied fundamental NP-hard optimization problem which optimizes a quadratic objective over a set of linear constraints. In this paper, we reformulate QPs as a mixed-integer linear problem (MILP). This is done via the reformulation of QP as a linear complementary problem, and the use of binary variables together with some fundamental results on the solution of perturbed linear systems, to model the complementary constraints.

Reformulating non-convex QPs as MILPs provides an advantageous way to obtain global solutions as it allows to use current state-of-the-art MILP solvers. To illustrate, we compare the performance of our solution approach with the current benchmark global QP solver quadprogBB on a large variety of QP test instances. The MATLAB code, called quadprogIP, and the instances used to perform these numerical experiments are publicly available at https://github.com/xiawei918/quadprogIP.

1 Introduction

A quadratic program (QP), is a fundamental optimization problem in which the objective is quadratic and the constraints are linear on the decision variables. In general, QPs are known to be NP-hard (see, e.g., Pardalos and Vavasis [1991], and the references therein). However, when the QP has a convex objective, it can be globally solved (within a predetermined precision $\epsilon > 0$) in polynomial time via interior-point methods (see, e.g., Renegar [2001]).

Here, the focus is on obtaining global solutions for non-convex QPs. At a fundamental level, the complexity of globally solving non-convex QPs lies in the fact that
multiple of its local optimal solutions may not necessarily be global optimal solutions (see, e.g., Bertsekas 1999). Note that a non-convex QP is the most basic instance of a non-convex, non-linear program (NLP); that is, an optimization problem in which the objective and constraints of the problem are given by non-linear functions on the decision variables (cf., Bertsekas 1999). For the purpose of brevity, in what follows we will refer to both non-convex QP and non-convex NLP as QP and NLP, respectively.

QPs commonly arise in applications in engineering, pure and social sciences, finance, and economics (see, e.g., Horst et al. 2000). As a result, there has been extensive work on studying how to obtain global solutions of QPs. In particular, Vanderbei and Shanno (1999) proposed an interior-point algorithm for NLPs (thus, applicable for QPs), which is an extension of the interior-point methods for linear and convex optimization problems (cf., Renegar 2001). Floudas and Visweswaran (1990) proposed an algorithm which globally solves certain classes of NLPs by decomposing the problem based on an appropriate partition of its decision variables. Also, the work of Belotti et al. (2009) and Tawarmalani and Sahinidis (2004) on the use of relaxation and linearization techniques (cf., Sherali and Adams 1994), in combination with spatial branching techniques (cf., Tawarmalani and Sahinidis 2004), has lead to the development of the two well-known global solvers Couenne (Belotti 2010) and BARON (Sahinidis 1996) for NLPs. For further review of numerical and theoretical results on the solution of QPs using NLP techniques, we refer the reader to Gould et al. (2003) and Gao (2004).

Besides NLP techniques, convex optimization techniques (cf., Renegar 2001, Ben-Tal and Nemirovski 2001) have also been used to address the solution of QPs. In particular, Nesterov (1998) and later Kim and Kojima (2001, 2003), explored the use of semidefinite programming (SDP) as well as second-order cone relaxations to approximately or globally solve a QP.

More recently, Burer and Vandenbussche (2009) proposed a SDP-based branch and bound approach to globally solve box-constrained QPs. Specifically, they reformulate QP by adding the QP’s corresponding Karush-Kuhn-Tucker (KKT) conditions as redundant constrains. Let us refer to this quadratically constrained quadratic program (QCQP) as $QP_{\text{KKT}}$. To solve $QP_{\text{KKT}}$ Burer and Vandenbussche (2009) construct a finite $KKT$-branching tree by branching on the resulting problem’s complementarity constraints. SDP relaxations of the $QP_{\text{KKT}}$ are used to obtain lower bounds at each node of the KKT-branching tree. On the other hand, to obtain upper bounds a (local) QP-solver based on non-linear optimization techniques is used. Chen and Burer (2012) improved the solution methodology of Burer and Vandenbussche (2009) by obtaining tighter lower bounds at each node of the KKT-branching tree. For that purpose, the double non-negative (DNN) relaxation of the completely positive reformulation (cf., Burer 2003) of $QP_{\text{KKT}}$ at each node of the KKT-branching tree is used. Chen and Burer (2012) provide a MATLAB implementation of their approach called quadprogBB. In this implementation, the MATLAB (local) QP solver quadprog is used to obtain the upper bounds while the algorithm proposed by Burer (2010) is used to obtain lower bounds. Chen and Burer (2012) show that this solution approach typically outperforms the solver Couenne and the approach proposed by Burer and Vandenbussche (2009) on a test bed of publicly available QP instances. This makes the solver quadprogBB a current benchmark for the global solution of QP problems.

In this paper, we reformulate QPs as a mixed-integer linear problem (MILP). This provides an advantageous way to obtain global solutions as it allows to use current state-
of-the-art MILP solvers. Moreover, the numerical experiments of Section 3 show that a basic implementation of the proposed algorithm, which we refer to as quadprogIP, typically outperforms quadprogBB on most test instances considered in the related literature. The MATLAB code and the instances used to perform these numerical experiments are available at https://github.com/xiawei918/quadprogIP.

In order to introduce the proposed MILP-reformulation, we begin by using the fact that the QP’s KKT conditions can be used to reformulate the QP as a linear complementarity problem (LCP); that is, as an equivalent problem with linear objective, and linear and complementary constraints (Giannessi and Tomasin 1973, Thm. 2.4). It is not useful to apply the KKT-branching approach on this reformulation of the problem, as the underlying linear relaxations at the root node of the KKT-branching tree is (under mild assumptions) unbounded (Burer and Vandenbussche 2009, Corollary 2.3). Alternatively, the complementarity constraints can be reformulated using binary variables. However, this requires the knowledge of bounds on the problem’s KKT multipliers, which may be unbounded, in order to directly use MILP solvers for the solution of the QP (cf. Hu et al. 2012, Section 6.1 and 6.2). Here, we overcome this requirement by making novel use of fundamental results on the approximate solution of systems of linear equations (e.g., Güler et al. 1995, Mangasarian 1981). One of the advantages the proposed methodology is that unlike previous related work, the convergence of the MILP-based approach to the the QP’s global optimal solution in finite time follows in straightforward fashion (see Section 3.1.1), and it can be applied to QPs without the need for assumptions on the relative interior of its feasible set (see Section 2.3 for details).

The rest of the paper is organized as follows. In Section 2, we formally introduce the QP problem and present the theoretical results that serve as the foundation for the proposed solution approach. In Section 3, we illustrate the effectiveness of this approach by presenting relevant numerical results on test instances of the QP problem. To conclude, in Section 4, we provide conclusions and directions for future work.

2 Solution Approach for non-convex QPs

A quadratic program (QP) is a linearly constrained optimization problem with quadratic objective. Formally, a QP can be formulated (w.l.o.g.) as follows:

\[
\begin{align*}
\min & \quad \frac{1}{2} x^T H x + f^T x \\
\text{s.t.} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\] (QP)

where \(f \in \mathbb{R}^n\), \(A \in \mathbb{R}^{m \times n}\), \(b \in \mathbb{R}^m\), and \(H\) is a \(n \times n\) symmetric matrix. Note that there is no assumption on the matrix \(H\) being positive semidefinite; that is, (QP) is in general a non-convex optimization problem (cf., Bertsekas 1999). Similar to Burer and Vandenbussche (2009) and Chen and Burer (2012), we assume that (QP) is feasible, and that its feasible set is bounded. However, in what follows (unless stated otherwise), no assumption is made about the relative interior of the feasible set of (QP).
2.1 Mixed-integer linear programming reformulation

After introducing the Lagrange multipliers $\mu \in \mathbb{R}^m$ for its equality constraints and $\lambda \in \mathbb{R}^n$ for its non-negativity constraints, it follows that the KKT conditions for problem (QP) are given by

$$
\begin{align*}
Hx + f + A^T \mu - \lambda &= 0 \quad (1a) \\
x^T \lambda &= 0 \quad (1b) \\
Ax &= b \quad (1c) \\
x \geq 0, \lambda \geq 0. \quad (1d)
\end{align*}
$$

In what follows, we will refer to the set

$$
\Lambda_{KKT} = \{(x, \mu, \lambda) \in \mathbb{R}^{2n+m} : (x, \mu, \lambda) \text{ satisfy (1a)–(1d)}\}
$$

as the KKT points of (QP).

The KKT conditions (1) are first order necessary conditions for the optimal solutions of (QP). Thus, one can add these KKT conditions as constraints to (QP) in order to obtain the following equivalent formulation for (QP),

$$
\begin{align*}
\min & \quad \frac{1}{2} x^T H x + f^T x \\
\text{s.t.} & \quad Hx + f + A^T \mu - \lambda = 0 \\
& \quad x^T \lambda = 0 \\
& \quad Ax = b \\
& \quad x \geq 0, \lambda \geq 0.
\end{align*}
$$

(2)

As shown by Giannessi and Tomasin (1973, Thm. 2.4), one can use the KKT conditions (1a)–(1c) to linearize the objective of (2). Namely, for any feasible solution $x \in \mathbb{R}^n$ of (2), we have

$$
\frac{1}{2} x^T H x + f^T x = \frac{1}{2} (f^T x - x^T A^T \mu + x^T \lambda) = \frac{1}{2} (f^T x - b^T \mu).
$$

(3)

As a result, problem (2) is equivalent to the following problem with a linear (instead of quadratic) objective.

$$
\begin{align*}
\frac{1}{2} \min & \quad f^T x - b^T \mu \\
\text{s.t.} & \quad Hx + f + A^T \mu - \lambda = 0 \\
& \quad x^T \lambda = 0 \\
& \quad Ax = b \\
& \quad x \geq 0, \lambda \geq 0.
\end{align*}
$$

(4)

Notice that in (4), the “complexity” of (QP) is captured in the complementary constraints $x^T \lambda = 0$. Next, we address the complementary constraints in (1) by using Big-M constraints. For that purpose, in Section 2.2, we derive upper bounds $U, V \in \mathbb{R}^n$ on the decision variables $x, \lambda \in \mathbb{R}^n$ of (4) such that there are (globally) optimal KKT points $(x, \mu, \lambda) \in \mathbb{R}^{2n+m}$ of (QP) satisfying $x \leq U, \lambda \leq V$. Using these upper bounds, one can show (see, Theorem 2) that a global optimal solution of (QP) can be obtained.
by solving the following optimization problem,

$$
\frac{1}{2} \min \ f^T x - b^T \mu \\
\text{s.t.} \quad H x + f + A^T \mu - \lambda = 0 \\
Ax = b \\
0 \leq x_j \leq z_j U_j \quad j = 1, \ldots, m \\
0 \leq \lambda_j \leq (1 - z_j) V_j \quad j = 1, \ldots, m \\
z_j \in \{0, 1\} \quad j = 1, \ldots, m.
$$

(IQP)

Specifically, problem (IQP) is a MILP with the same optimal value as (QP) whose optimal solutions are optimal solutions of (QP).

### 2.2 Bounding the primal and dual variables

As mentioned earlier, the first step in obtaining problem (IQP) is to derive explicit upper bounds $U, V \in \mathbb{R}^n$ such that there are optimal KKT points $(x, \mu, \lambda) \in \mathbb{R}^{2n+m}$ of (QP) satisfying $x \leq U$, $\lambda \leq V$.

Similar to Chen and Burer (2012), using the assumption that the feasible set of (QP) is non-empty and bounded, one can compute the following upper bounds on the primal variables $x \in \mathbb{R}^n$ of the problem:

$$
U_j := \max \{x_j : Ax = b, x \geq 0\}.
$$

In Chen and Burer (2012) it is shown, under stronger assumptions than ours, that the set of KKT points $\Lambda_{KKT}$ is bounded. As the following example illustrates, under our weaker assumptions, the set $\Lambda_{KKT}$ could be unbounded.

**Example 1.** Consider the instance of (QP) defined by setting

$$
H = \begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
f = \begin{bmatrix}
2 \\
4 \\
3
\end{bmatrix} \\
A = \begin{bmatrix}
1 & 2 & 2 \\
1 & 1 & 1
\end{bmatrix} \\
b = \begin{bmatrix}
2 \\
1
\end{bmatrix}.
$$

Note that in this case the feasible region of (QP) is $\{[0, t, 1-t]^T : 0 \leq t \leq 1\}$, which is bounded and non-empty. However, the set of KKT points $\Lambda_{KKT}$ is unbounded. Specifically, notice that for any $t \geq 0$ the following is a KKT point for (QP):

$$
x = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \\
\mu = \begin{bmatrix}
t \\
0
\end{bmatrix} \\
\lambda = \begin{bmatrix}
t + 2 \\
0 \\
0
\end{bmatrix}.
$$

Thus, in order to handle the complementarity constraints in (4) using Big-M constraints, we do not try to obtain a bound for the value of the entries of $\lambda \in \mathbb{R}^m$ for all KKT points. Instead, in Theorem 2, we prove that there exist a bound that we can impose in the dual variables, without discarding all optimal points of (QP). For this purpose, we make use of fundamental results on the approximate solution of systems of linear equations (e.g., G"uler et al. 1995, Mangasarian 1981). Let us first define a particular instance of the well-known Hoffman bound (Hoffman 1952).

**Definition 1.** Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, let $F := \{x \in \mathbb{R}^n_+ : Ax = b\}$. Let $\mathcal{H}_{A,b} \in \mathbb{R}$ be the smallest constant satisfying that for all $y \in \mathbb{R}^n$ such that $Ay = b$

$$
\min_{x \in F} \|x - y\|_\infty \leq \mathcal{H}_{A,b}\|y\|_\infty.
$$
Above, for any \( y \in \mathbb{R}^n \) and \( y^- \in \mathbb{R}^n \) is the vector defined by \( y_i^- = \min \{0, y_i\} \), \( i = 1, \ldots, n \). That is, Definition 1 corresponds to the Hoffman bound obtained when looking at perturbations of only the non-negative constraints of the polyhedron \( \mathcal{P} := \{ x \in \mathbb{R}_+^n : Ax = b \} \).

For any \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), the constant \( \mathcal{H}_{A,b} \) is well defined, and depends only on \( A \) (Hoffman 1952). This is formally stated in Theorem 1 which follows directly from (Güler et al. 1995, Theorem 2.1).

**Theorem 1 (Güler et al. (1995)).** Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) such that the set \( F := \{ x \in \mathbb{R}_+^n : Ax = b \} \neq \emptyset \). Also, let

\[
\sigma(A) = \left\{ (\mu, \lambda) \in \mathbb{R}^{m+n} : \| A^T \mu - \lambda \|_1 \leq 1, \mu \in \mathbb{R}^m, \lambda \in \mathbb{R}_+^n \right\}.
\]

Then

\[
\mathcal{H}_{A,b} = \max\{\|(\mu, \lambda)\|_1 : (\mu, \lambda) \text{ is an extreme point of } \sigma(A)\}.
\]

Using Definition 1 and Theorem 1 we can now provide the desired bound in Theorem 2 below.

**Theorem 2.** Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) such that the set \( F := \{ x \in \mathbb{R}_+^n : Ax = b \} \) is non-empty and bounded. Let \( \mathcal{H}_{A,b} \) be defined by (7) and \( M = \left( \frac{1}{2} \| H \|_{1,\infty} (2n + \mathcal{H}_{A,b}) + \| f \|_1 \right) \mathcal{H}_{A,b} \), where \( \| x \|_\infty \leq u \) for any \( x \in F \). Then, there exists an optimal KKT point \((x^*, \mu^*, \lambda^*)\) for \( (QP) \) such that \( e^T \lambda^* \leq M \), and \( \mu^* \in \mathbb{R}^m \).

**Proof.** Consider the following perturbed version of \( (QP) \):

\[
\begin{align*}
\min & \quad \frac{1}{2} x^T H x + f^T x + M t \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq -te \\
& \quad 0 \leq t \leq 1,
\end{align*}
\]

where \( e \) is the vector of all ones. Notice that the feasible set of \( (\mathcal{P}) \) is a closed subset of \( \{ x \in \mathbb{R}^n : Ax = b, x \geq -e \} \times [0, 1] \) which is non-empty and bounded as \( \{ x \in \mathbb{R}_+^n : Ax = b, x \geq 0 \} \) is non-empty and bounded. Thus, the optimal value of \( (\mathcal{P}) \) exists and it is attained. Let \((x^*, u^*)\) be an optimal solution of \( (\mathcal{P}) \). Then, there exists \((\mu^*, \lambda^*, \alpha^*, \beta^*) \in \mathbb{R}^{n+m+2} \) such that \((x^*, u^*, \mu^*, \lambda^*, \alpha^*, \beta^*)\) satisfies the KKT conditions associated with problem \( (\mathcal{P}) \) (see, eq. (9) below), where \( \mu^* \in \mathbb{R}^m \), \( \lambda^* \in \mathbb{R}_+^n \), \( \alpha^* \in \mathbb{R} \), \( \beta^* \in \mathbb{R} \), are respectively the Lagrangean multipliers of problem \( (\mathcal{P}) \) associated with the linear constraints, lower bounds in the decision variables \( x \in \mathbb{R}^n \), and lower and upper bounds on the decision variable \( t \in \mathbb{R} \). Specifically, there exists \((x^*, t^*, \mu^*, \lambda^*, \alpha^*, \beta^*) \in \mathbb{R}^{2n+m+3} \) satisfying:

\[
\begin{align*}
H x^* + f + A^T \mu^* - \lambda^* &= 0 \\
M - e^T \lambda^* - \alpha^* + \beta^* &= 0 \\
(x^* - t^* e) \lambda^* &= 0 \\
t^* \alpha^* &= 0 \\
(1 - t^*) \beta^* &= 0 \\
Ax^* &= b \\
x^* + t^* e &\geq 0 \\
0 &\leq t^* \leq 1 \\
\lambda^*, \alpha^*, \beta^* &\geq 0.
\end{align*}
\]
Now we claim that \( t^* = 0 \). In that case, notice that the complementarity constraint \((1 - t^*)\beta^* = 0\) in (9) implies \( \beta^* = 0 \) and thus from (9), it follows that \((x^*, \mu^*, \lambda^*)\) satisfies the statement of the theorem.

To show that \( t^* = 0 \), let \( K = \mathcal{H}_{A,b} \) and \( x' \in \mathbb{R}_+^n \) be such that \( Ax' = b \), and

\[
\|x^* - x'\|_\infty \leq Kt^*.
\]

Notice that the existence of \( x' \) is ensured from Theorem 1 (cf., (7)). By construction, \((x', 0)\) is a feasible solution of problem (8). Also, by optimality, the objective value of \((x', 0)\) is larger than or equal to the objective value of \((x^*, t^*)\). Thus,

\[
\frac{1}{2} x^T H x^* + f^T x^* + M t^* \leq \frac{1}{2} x'^T H x' + f^T x'.
\]

Therefore,

\[
M t^* \leq \frac{1}{2} (x'^T H x' - x^T H x^*) + f^T (x' - x^*)
\]

\[
= \frac{1}{2} (x' + x^*)^T H (x' - x^*) + f^T (x' - x^*)
\]

\[
\leq \frac{1}{2} \|H\|_{1,\infty} \|x' + x^*\|_\infty \|x' - x^*\|_\infty + \|f\|_1 \|x' - x^*\|_\infty
\]

\[
\leq \left( \frac{1}{2} \|H\|_{1,\infty} (2\|x^*\|_\infty + \|x' - x^*\|_\infty) + \|f\|_1 \right) \|x' - x^*\|_\infty
\]

\[
\leq \left( \frac{1}{2} \|H\|_{1,\infty} (2u^* + K) + \|f\|_1 \right) K t^*.
\]

As \( M > \left( \frac{1}{2} \|H\|_{1,\infty} (2u^* + K) + \|f\|_1 \right) K \) and \( t^* \geq 0 \), we have \( t^* = 0 \). \( \square \)

2.3 Computation of the dual bounds

In general, Theorem 1 provides the desired constant \( K \) used in Theorem 2. However, obtaining an efficient way to compute this type of condition bounds is still an open question (see, e.g., [Zheng and Ng 2004, Güler et al. 1995]). Thus, we show efficient ways to compute the desired dual bounds \( V \in \mathbb{R}^n \), provided in general by Theorem 2 for specific instances of the \((\text{QP})\).

Following the article’s notation, let us start by considering a standard quadratic program (SQP):

\[
\max_{x \in \Delta} \frac{1}{2} x^T H x + f^T x
\]

(SQP)

where

\[
\Delta = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x \geq 0 \right\}
\]

is the standard simplex. The (SQP) problem is fundamental in optimization and arises in many applications (cf., [Bomze 1998]). Next, we show that the Hoffman-type bound \( \mathcal{H}_{A,b} \) used in Theorem 2 can be computed explicitly in this case.

Proposition 1. Let \( A = e^T \) and \( b = 1 \). Then \( \mathcal{H}_{A,b} = n - 1 \).
Thus, the following (QP) problem:

\[ \text{minimize } c^T x + \frac{1}{2} x^T Q x \]
\[ \text{subject to } A x = b, \quad x \geq 0, \quad x \leq u \]

\[ \text{for some } A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad l \leq u \in \mathbb{R}^n. \]

Next, we show that the Hoffman-type bound \( H_{A,b} \) used in Theorem 2 can be computed explicitly for the specific case of box-constraints, where \( I \) denotes the identity matrix in \( \mathbb{R}^{n \times n} \).

**Proposition 2.** Let \( u \in \mathbb{R}_+^n \), \( A = [I, I] \), and \( b = u \). Then \( H_{A,b} \leq 1 \).

**Proof.** Let \( (y, z) \in \mathbb{R}^{2n} \) such that \( y + z = u \) be given. Define \( x = y^+ - z^- \) and \( s = z^+ - y^- \). We claim \( (x, s) \in F = \{ (x, s) \in \mathbb{R}^{2n}_+ : x + s = u \} \). To show this notice first that \( x + s = y^+ - z^- + z^+ - y^- = y + z = u \). Now let \( i \in \{1, \ldots, n\} \). If \( z_i^- = 0 \) then \( x_i = y_i^+ \geq 0 \). Thus assume \( z_i^- > 0 \). Then \( z_i^+ = 0 \) and \( x_i = u_i - s_i = u_i + y_i^- \geq 0 \). Thus \( x \geq 0 \). Similarly \( s \geq 0 \). To finish notice that \( \|(y, z) - (x, s)\|_\infty = \|(-y^- - z^-, -z^- - y^-)\|_\infty \leq \|(y^-, z^-)\|_\infty = \|(y, z^-)\|_\infty \) because for each \( i \in \{1, \ldots, n\} \), at least one of \( y_i^- \) and \( z_i^- \) is zero, as \( y_i + z_i = u_i \geq 0 \). This shows that \( H_{A,b} \leq 1 \). To show \( H_{A,b} \geq 1 \), let \( y = -e \) and \( z = u + e \). For any \( (x, s) \in F \), it follows that \( \|(x, s) - (y, z)\|_\infty \geq |x_1 + 1| \geq 1 = \|(y, z^-)\|_\infty \).
Finally, for more general instances of (QP), here we use the bounds on the dual variables proposed by Chen and Burer (2012, Proposition 3.1) which are valid for (QP) instances having a strictly non-negative feasible solution (i.e., a feasible solution satisfying \( x > 0 \)), and can be computed by solving a LP (cf., Chen and Burer 2012, eq. (19)). Specifically, notice that after obtaining the primal dual bounds \( U \in \mathbb{R}^n \) using (5), problem (QP) is equivalent to

\[
\min \quad \frac{1}{2} x^T H x + f^T x \\
\text{s.t.} \quad Ax = b \\
0 \leq x \leq U.
\] (16)

Following the notation used thus far and letting \( \rho \in \mathbb{R}^n \) be the dual variables associated with the upper bound constraints on the variables \( x \in \mathbb{R}^n \) in (16), it follows from the KKT conditions of (16) that any of its optimal solutions must satisfy:

\[
H x + f + A^T \mu - \lambda + \rho = 0 \tag{17a}
\]

\[
x^T \lambda = 0, (U - x)^T \rho = 0 \tag{17b}
\]

\[
Ax = b \tag{17c}
\]

\[
x \geq 0, \lambda \geq 0, \rho \geq 0.
\]

Also, after multiplying (17a) by a feasible solution \( x \in \mathbb{R}^n \) of (16) and using (17b), (17c), it follows that any optimal solution of (16) also satisfies:

\[
x^T H x + f^T x + b^T \mu + U^T \rho = 0.
\] (18)

Then, if (QP) has a feasible solution \( x \in \mathbb{R}^n \) satisfying \( x_i > 0, i = 1, \ldots, m \), it follows from Chen and Burer (2012, Proposition 3.1) that the bounds on the dual variables \( V \in \mathbb{R}^n \) required for the MILP reformulation (IQP) of (QP) can be computed by solving the following LP:

\[
V_j = \max \left\{ \lambda_j : \begin{array}{l}
H x + f + A^T \mu - \lambda + \rho = 0 \\
H \bullet X + f^T x + b^T \mu + U^T \rho = 0 \\
0 \leq X_{ij} \leq U_i, U_j, i, j = 1, \ldots, n \\
0 \leq x \leq U, \lambda, \rho \geq 0, X \in S^n
\end{array} \right\},
\] (19)

where \( H \bullet X \) indicates the trace of the matrix \( HX \), the matrix \( X \in S^n \) represents the linearization of the matrix \( xx^T \in S^n \), and \( S^n \) is the set of \( n \times n \) real symmetric matrices. It is worth to mention that in Chen and Burer (2012), eq. (18) is used to refine the dual variable bounds after scaling the problem so that its variables are between zero and one. However, this refinement of the bounds is not necessary to obtain their result in Proposition 3.1.

As we illustrate in the next section, using these bounds in the MILP reformulation (IQP) of (QP), together with state-of-the-art MILP solvers, results in a simple and global solution approach for (QP)s that performs well against Quadprogbb, a benchmark solver for this type of problems.

### 3 Computational results

In this section, we provide a detailed description of the implementation of the solution approach for (QP) problems described in the previous sections. Also, we illustrate the
performance of the solution approach by presenting the results of numerical experiments on a diverse set of (QP) test problems.

3.1 Problem instances

To test the performance of the proposed solution approach for (QP), we use the set of BoxQP (cf., Globallib (cf., http://www.gamsworld.org/global/globallib.htm), CUTEr (Gould et al. 2003), and RandQP test problems used in (Chen and Burer 2012, Section 4.2 and Table 1). In addition to these test problems, we consider the following (QP) test instances:

- **SQP.** Standard quadratic programming instances (cf., (SQP)) are created by replacing the constraints of each of the BoxQPs considered in (Chen and Burer 2012, Section 4.2 and Table 1) by the constraint that the decision variables belong to the standard simplex of appropriate dimension (cf., (12)).

- **StableQP.** These instances are particular SQPs resulting from the problem of computing the stability number of a graph (see, e.g., Motzkin and Straus 1965). Here, we use instances of this type arising from a class of graphs that have been used for testing purposes in the literature (Dobre and Vera 2015, Section 4.2.2). A more detailed description of these instances is presented in Section 3.1.1.

Similar to Chen and Burer (2012), Table 1 provides a summary of the basic information of all the test instances. In Table 1 “n” denotes the range of the number of decision variables required to formulate the corresponding problem instance using $m_{ineq}$ inequality constraints, and $m_{eq}$ equality constraints. Also, “density” denotes the corresponding range of the density of the matrix defining the quadratic problem’s objective.

| Type     | # Instances | n     | $m_{ineq} + m_{eq}$ | density   |
|----------|-------------|-------|---------------------|-----------|
| StableQP | 8           | [5, 26]| [0,1]               | [0.30, 0.60] |
| SQP      | 90          | [20, 100] | [0, 90]            | [0.19, 0.99] |
| BoxQP    | 90          | [20, 100] | [0, 0]              | [0.19, 0.99] |
| Globallib| 83          | [2, 100] | [1,52]              | [0.01, 1]  |
| CUTEr    | 6           | [4, 12]  | [0, 13]             | [0.08, 1]  |
| RandQP   | 64          | [20, 50] | [14, 35]            | [0.23, 1]  |

Table 1: Statistics of the test (QP) instances.

3.1.1 StableQP instances

Given a graph $G(V, E)$, it is well-known (see, e.g. Motzkin and Straus 1965) that the inverse of the stability number $\alpha(G)$ of a graph can be computed by solving the following (SQP).

$$\frac{1}{\alpha(G)} = \min_{x \in \Delta} x^T(A + I)x$$

(20)
where $A$ is the adjacency matrix of the graph $G$ and $I$ is the identity matrix.

The StableQP instances mentioned in the previous section are obtained by solving (20) for a class of graphs $G_k$, $k = 1, \ldots$ introduced in Dobre and Vera (2015) that have proven to be hard instances for approximation methods for $\alpha(G)$ proposed in Bomze et al. (2010), Bundfuss and Dür (2009), Dong and Anstreicher (2013), Dobre and Vera (2015). As discussed in Dobre and Vera (2015, Section 4.2.2), these graphs can be constructed as follows for any given $k = 1, \ldots$. Start with the complete bipartite graph with $2k$ number of vertices $V = \{(i, -1), (i, 1) : i = 0, \ldots, k\}$. For each edge of the form $\{(i, -1), (i, 1)\}$ where $i = 1, \ldots, k$, add a vertex $j$ on the edge, which then transform the original edge $\{(i, -1), (i, 1)\}$ into two edges, $\{(i, -1), (j, 0)\}$ and $\{(j, 0), (j, 1)\}$. The resulting graph is the desired $G_k$. For an illustration of the graph $G_2$, we refer the reader to Dobre and Vera (2015, Fig. 2).

3.2 Implementation details

The solution approach for (QP) proposed here is implemented as follows. First, explicit upper and lower bounds for the instance’s decision variables are obtained. Then, the problem instance is reformulated as (QP) by linearly shifting its decision variables, and adding slack variables to the problem as necessary (see, e.g., (14)). The upper bounds on the added slack variables are computed using (5) to obtain the primal variable upper bounds $U \in \mathbb{R}^n$. Next, we calculate the dual variable upper bounds $V \in \mathbb{R}^n$ following the discussion on Section 2.3 (cf. equations (13), (15), (19)). Finally, we invoke CPLEX (cf., http://www-eio.upc.edu/lceio/manuals/CPLEX-11/html/) to solve (IQP) with the following parameter settings for its CPLEX MILP solver:

- Max time: This is the user specified maximum running time of the algorithm and is set to $10^4$ seconds. Any problem taking longer than this value to be solved will be deemed as “out of time”.

- Tol: The solver will stop when

$$\frac{|\text{bestnode} - \text{bestinteger}|}{1 - 10^{-5} + |\text{bestinteger}|} \leq 10^{-6}.$$ 

To be consistent with quadprogBB stopping criteria (cf., Chen and Burer 2012), which is

$$\frac{\text{Greatest upper bound} - \text{current lower bound}}{\max\{1, |\text{Greatest upper bound}|\}} \leq 10^{-6}.$$ 

- Other parameters of the CPLEX MILP solver such as TolXInteger, MaxIter, BranchStrategy, NodeSelect, that are set to their default values.

We refer to the procedure above to solve (QP) as quadprogIP, which is coded using Matlab 2014a, and is publicly available at https://github.com/xiawei918/quadprogIP.

3.3 Numerical performance

In order to test the performance of the quadprogIP methodology proposed here, the (QP) test instances discussed in Section 3.1 are solved using both quadprogIP and the quadprogBB solver introduced by Chen and Burer (2012). Both methodologies are tested using Matlab 2014a, together with CPLEX 12.5.1, on an AMD Opteron 2.0 GHz
similar to Chen and Burer (2012), the performance between quadprogBB and quadprogIP is compared by plotting the CPU time it takes to solve a particular (QP) instance with both solvers as a square in a 2D plane, where the y-axis denotes quadprogBB’s CPU time and the x-axis denotes quadprogIP’s CPU time. The dashed line in the plots indicates the $y = x$ line in the plane, that represents equal CPU solution times. Thus, a square that is above the diagonal line indicates an instance for which it takes quadprogBB more CPU time to solve than quadprogIP. Furthermore, the size of the square illustrates the size (number of decision variables) of the instance. That is, smaller squares represent smaller size instances while bigger squares represent bigger size instances. In the figures below, only instances in which at least one of the methodologies solves the problem within the maximum allowed time are displayed.

The results for the SQP test instances are shown in Figures 1 and 2. To make the illustration of the results more clear, the SQP instances are divided into two groups, according to quadprogBB running time. That is, the smaller SQP instances, those that are solved in less than 150 seconds by quadprogBB (see Figure 1), and the larger SQP instances those that are solved in more than 150 seconds by quadprogBB (see Figure 2). Also, a different time scale is used in each of the Figure’s axis. Note that in these SQP test instances, quadprogIP clearly outperforms quadprogBB by solving all instances in less than 10 seconds. As Figures 1 and 2 illustrate, the performance of quadprogIP against quadprogBB improves as the SQP instance becomes larger.

![Figure 1: Solution time in seconds for smaller size SQP instances.](image)

In line with the performance of quadprogIP on SQP instances, it is interesting to see in Table 2 that quadprogIP clearly outperforms quadprogBB in the StableQP instances (cf., Section 3.1). In fact, while quadprogIP solves each of the instances in less than a second, quadprogBB is unable to solve the instances beyond $k \geq 4$ within the maximum allowed solution time of $10^4$ seconds.
Next, in Figure 3, we compare the performance on quadprogBB and quadprogIP on the BoxQP instances. It is clear from Figure 3 that while quadprogIP outperforms quadprogBB in the smaller BoxQP instances (ranging between 20–60 decision variables in Table 1), quadprogBB outperforms quadprogIP for larger BoxQP instances (ranging between 60–100 decision variables in Table 1), where quadprogIP is typically unable to solve the instance within the $10^4$ maximum solution time.

Although quadprogIP does not in general outperforms quadprogBB on BoxQP instances, it does on more general CUTEr, globallib, and randQP instances of (QP). As Figure 4 illustrates, except for a few instances, quadprogIP has shorter solution times than quadprogBB. Moreover, quadprogIP typically solves these problems about 10 times faster than quadprogBB. For these CUTEr, Globallib, and randQP we find nine (9) instances that are successfully solved by quadprogIP but not by quadprogBB.

| $k$ | quadprogIP | quadprogBB |
|-----|------------|------------|
| 1   | 0.44       | 3.67       |
| 2   | 0.51       | 6.28       |
| 3   | 0.42       | 12.56      |
| 4   | 0.50       | (*)        |
| 5   | 0.51       | (*)        |
| 6   | 0.60       | (*)        |
| 7   | 0.64       | (*)        |
| 8   | 0.58       | (*)        |

Table 2: Solution time in seconds of StableQP instances, where (*) indicates that maximum allowed solution time was reached.
Figure 3: Solution time in seconds of BoxQP instances.

within the maximum allowed solution time of $10^4$ seconds.

Figure 4: Solution time in seconds of CUTEr, Globallib and RandQP instances

4 Conclusion

In this paper, we present a new simple and effective approach for the global solution of a (non-convex) linearly constrained quadratic problem (QP) by combining the use
of the problem’s necessary KKT conditions together with state-of-the-art integer programming solvers. This is done via a reformulation of the QP as a mixed-integer linear program (MLIP). We show that in general, this MILP reformulation can be done for QPs with a bounded feasible set via fundamental results related to the solution of perturbed linear systems of equations (see, e.g., Güler et al. 1995). In practice, this approach is shown to typically outperform the benchmark solver quadprogBB on a test set including instances considered in the literature of global solution of QPs, together with additional standard quadratic programming (SQP) problems. The performance of this methodology in the latter class of problems allows for the potential use of this solution approach as a basis for the solution of copositive programs (cf., Dür 2010). This will be an interesting direction of future research work.

Recall that the proposed IP formulation of general QPs requires the computation of Hoffman-like bound’s (see, e.g., Hoffman 1952) on the system of linear equations defining the problem’s feasible set. Thus, obtaining general and effectively computable Hoffman-like bounds is another interesting open question.

We finish by mentioning that a basic implementation of the proposed solution approach referred as quadprogIP is publicly available at https://github.com/xiawei918/quadprogIP, together with pointers to the test instances used in the article for the numerical experiments.

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