Quantum Exploration Algorithms for Multi-Armed Bandits

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Abstract
Identifying the best arm of a multi-armed bandit is a central problem in bandit optimization. We study a quantum computational version of this problem with coherent oracle access to states encoding the reward probabilities of each arm as quantum amplitudes. Specifically, we show that we can find the best arm with fixed confidence using $O\left(\sqrt{n \sum_{i=2}^{n} \Delta_i^{-2}}\right)$ quantum queries, where $\Delta_i$ represents the difference between the mean reward of the best arm and the $i^{th}$-best arm. This algorithm, based on variable-time amplitude amplification and estimation, gives a quadratic speedup compared to the best possible classical result. We also prove a matching quantum lower bound (up to poly-logarithmic factors).

Introduction
The multi-armed bandit (MAB) model is one of the most fundamental settings in reinforcement learning. This simple scenario captures crucial issues such as the tradeoff between exploration and exploitation. Furthermore, it has wide applications to areas including operations research, mechanism design, and statistics.

A basic challenge about multi-armed bandits is the problem of best-arm identification, where the goal is to efficiently identify the arm with the largest expected reward. This problem captures a common difficulty in practical scenarios, where at unit cost, only partial information about the system of interest can be obtained. A real-world example is a recommendation system, where the goal is to find appealing items for users. For each recommendation, only feedback on the recommended item is obtained. In the context of machine learning, best-arm identification can be viewed as a high-level abstraction and core component of active learning, where the goal is to minimize the uncertainty of an underlying concept, and each step only reveals the label of the data point being queried.

Quantum computing is a promising technology with potential applications to diverse areas including cryptanalysis, optimization, and simulation of quantum physics. Quantum computing devices have recently been demonstrated to experimentally outperform classical computers on a specific sampling task (Arute et al. 2019). While noise limits the current practical usefulness of quantum computers, they can in principle be made fault tolerant and thus capable of executing a wide variety of algorithms. It is therefore of significant interest to understand quantum algorithms from a theoretical perspective to anticipate future applications. In particular, there has been increasing interest in quantum machine learning (see for example the surveys by Biamonte et al. 2017; Schuld, Sinayskiy, and Petruccione 2015; Arunachalam and de Wolf 2017; Dunjko and Briegel 2018). In this paper, we study best-arm identification in multi-armed bandits, establishing quantum speedup.

Problem setup.
We work in a standard multi-armed bandit setting (Even-Dar, Mannor, and Mansour 2002) in which the MAB has $n$ arms, where arm $i \in [n] := \{1, \ldots, n\}$ is a Bernoulli random variable taking value $1$ with probability $p_i$ and value $0$ with probability $1 - p_i$. Each arm can therefore be regarded as a coin with bias $p_i$. As our algorithms and lower bounds are symmetric with respect to the arms, we assume without loss of generality that $p_1 \geq \cdots \geq p_n$, and denote $\Delta_i := p_1 - p_i$ for all $i \in \{2, \ldots, n\}$. We further assume that $p_1 > p_2$, i.e., the best arm is unique. Given a parameter $\delta \in (0, 1)$, our goal is to use as few queries as possible to determine the best arm with probability $\geq 1 - \delta$. This is known as the fixed-confidence setting. We primarily characterize complexity in terms of the parameter

$$H := \sum_{i=2}^{n} \frac{1}{\Delta_i^2}$$

which arises in the analysis of classical MAB algorithms (as discussed below).

We consider a quantum version of best-arm identification in which we can access the arms coherently. This means we have access to a quantum oracle $O$ that acts as

$$O: |i\rangle_I |0\rangle_B |0\rangle_J \mapsto |i\rangle_I (\sqrt{p_i} |1\rangle_B |v_i\rangle_J + \sqrt{1-p_i} |0\rangle_B |u_i\rangle_J),$$

where $|v_i\rangle$ and $|u_i\rangle$ are arbitrary states, for all $i \in [n]$. We have used standard Dirac notation which we review in the Preliminaries section. Register $I$ is the “index” register with $n$ states that correspond to the $n$ arms. Register
Our algorithm can also be adapted to work when the reward distributions are promised to have bounded variance (for example, if they are sub-Gaussian). The adaptation essentially follows by replacing amplitude estimation (introduced in the Preliminaries section) with quantum mean estimation (Montanaro 2015), which works on any distribution with bounded variance. We remark that the situation is different for the other main type of bandits: adversarial bandits. Studies on adversarial bandits are mainly focused on regret minimization and a quantum analogue first requires a proper notion of regret which we are unsure how to even define.

Contributions. In this paper, we give a comprehensive study of best-arm identification using quantum algorithms. Specifically, we obtain the following main result:

**Theorem 1.** Given a multi-armed bandit oracle $O$ and confidence parameter $\delta \in (0, 1)$, there exists a quantum algorithm that, with probability $\geq 1 - \delta$, outputs the best arm using $O(\sqrt{1/\delta})$ queries to $O$. Moreover, this query complexity is optimal up to poly-logarithmic factors in $n, \delta$, and $\Delta_2$.

This represents a quadratic quantum speedup over what is possible classically. The speedup essentially derives from Grover’s search algorithm (Grover 1996), where a marker oracle is used to approximately “rotate” a uniform initial state to the marked state. One way to understand the quadratic speedup is to observe that each rotation step, making one query to the oracle, increases the amplitude of the marked state by $\Omega(1/\sqrt{n})$. This is possible since quantum computation linearly manipulates amplitudes, which are square roots of probabilities.

However, to establish Theorem 1 we use more sophisticated machinery that extends Grover’s algorithm, namely variable-time amplitude amplification (VTAA) (Ambainis 2010), Childs, Kothari, and Somma (2017) and estimation (VTAE) (Chakraborty, Gilván, and Jeffery 2019). We apply VTAA and VTAE on a variable-time quantum algorithm $A$ that we construct. $A$ outputs a state with labeled “good” and “bad” parts. Using that label, VTAA removes the bad part so that only the good part remains, and VTAE estimates the proportion of the good part. In our application, the good part is eventually the best-arm state.

We emphasize that our quantum algorithm, like classical ones (Even-Dar, Mannor, and Mansour 2002; Gabillon, Ghavamzadeh, and Lazarid 2012; Jamieson et al. 2014; Karnin, Koren, and Somekh 2013; Mannor and Tsitsiklis 2004), does not require any prior knowledge about the $p_i$s.

Given knowledge of $p_1$ and $p_2$, our quantum algorithm is conceptually related to the classical successive elimination (SE) algorithm (Even-Dar, Mannor, and Mansour 2002). Namely, we use that knowledge to help eliminate sub-optimal arms $i$ by checking whether $p_i < (p_1 + p_2)/2$, say. The quantum quadratic speedup arises because we can check this “in superposition” across the different arms. For intuition only, checking in superposition can be thought of as a form of checking in parallel. We stress however that while it does not make sense to compare the parallel (classical) sample complexity of best-arm identification with its usual...
(classical) sample complexity, it does makes sense to compare the latter with the quantum query complexity. We also stress that the similarity of our quantum algorithm to SE, given knowledge of $p_1$ and $p_2$, ends at the conceptual level. Technically, our algorithm makes the SE concept work by first marking all sub-optimal arms and then rotating towards the unmarked best arm in quantum state space via a careful application of VTAA. This has no classical analogue.

It is classically easy to remove any assumed knowledge of $p_1$ and $p_2$ because classical samples from a multi-armed bandit contain information about their values. Quantumly however, we cannot simply ask our quantum multi-armed bandit to supply classical samples as that would prevent interference, eliminating any quantum speedup. Therefore, we need to do something conceptually different in the quantum case. We construct another quantum algorithm whose goal is to estimate both $p_1$ and $p_2$ to precision $\Theta(\Delta^2)$ using $O(\sqrt{H})$ quantum queries. For a given test point $l$, VTAA (roughly) gives us the ability to count the number of arms $i$ with $p_i > l$, and thus allows us to perform binary search to find $p_1$ and $p_2$.

**Related work.** Classically, a naive algorithm for best-arm identification is to simply sample each arm the same number of times and output the arm with the best empirical bias (Even-Dar, Mannor, and Mansour 2002). This algorithm has complexity $O(n \log n)$ but is sub-optimal for most multi-armed bandit instances. Therefore, classical research on best-arm identification (Even-Dar, Mannor, and Mansour 2002; Gabillon, Ghavamzadeh, and Lazaric 2012; Jamieson et al. 2014; Karnin, Koren, and Somekh 2013; Mannor and Tsitsiklis 2004) has primarily focused on proving bounds of the form $O(H)$ (recall that $H := \sum_{i=2}^{n} \frac{1}{\Delta_i}$), which can be shown to be almost tight for every instance. The first work to provide an algorithm with such complexity is Even-Dar, Mannor, and Mansour (2002), giving $O(H \log n + n \Delta^{-2} \log \Delta^{-1})$. This was further improved to $O(H \log n + n \Delta^{-2} \log \Delta^{-1})$ by Gabillon, Ghavamzadeh, and Lazaric (2012; Jamieson et al. (2014; Karnin, Koren, and Somekh 2013; Mannor and Tsitsiklis 2004). More recent work (Chen and Li 2015; Chen, Li, and Qiao 2017) has focused on bringing down even this additive term by tightening both the upper and lower bounds, leaving behind a gap only of the order $\sum_{i=2}^{n} \Delta_i^{-2} \log \log n(\Delta_i^{-1})$.

Prior work on quantum machine learning has focused primarily on supervised (Lloyd, Mohseni, and Rebentrost 2014; 2013; Rehentrost, Mohseni, and Lloyd 2014; Li, Chakrabarti, and Wu 2019) and unsupervised learning (Lloyd, Mohseni, and Rebentrost 2013; Wiebe, Kapoor, and Svore 2015; Amin et al. 2018; Kerenidis et al. 2019; Dunjko, Taylor, and Briegel 2017; Dunjko et al. 2017; Jerbi et al. 2019). These algorithms for general reinforcement learning with provable guarantees, but do not consider the best-arm identification problem. The only directly comparable previous work on quantum algorithms for best-arm identification that we are aware of are Casalé et al. (2020) and Wiebe, Kapoor, and Svore (2015). By applying Grover’s algorithm, Casalé et al. (2020) shows that quantum computers can find the best arm with confidence $p_i / \sum_{i=1}^{n} p_i$ quadratically faster than classical ones. However, Casalé et al. (2020) does not show how to find the best arm with a given fixed confidence, which is the standard requirement. In fact, there is a relatively simple quantum algorithm, analogous to the naive classical algorithm, that can achieve arbitrary confidence with quadratic speedup in terms of $n/\Delta^2$. This algorithm, which appears in Fig. 3 of Wiebe, Kapoor, and Svore (2015), works by using the quantum minimum finding of Durr and Høved (1996) on top of quantum amplitude estimation (Brassard et al. 2002). As in the classical case, we show that this simple quantum algorithm is suboptimal for most multi-armed bandit instances. Specifically, we show that a quantum algorithm can achieve quadratic speedup in terms of the parameter $H$.

**Preliminaries**

**Definitions and notations.** Quantum computing is naturally formulated in terms of linear algebra. An $n$-dimensional quantum state is a unit vector in the complex Hilbert space $\mathbb{C}^n$, i.e., $\vec{x} = (x_1, \ldots, x_n)^T$ such that $\sum_{i=1}^{n} |x_i|^2 = 1$. Such a column vector $\vec{x}$ is written in Dirac notation as $|x\rangle$ and called a "ket". The complex conjugate transpose of $|x\rangle$ is written $\langle x|$ and called a "bra", i.e., $|x\rangle := \vec{x}^\dagger$. The reason for the names is because the combination of a bra and a ket is a inner product bracket: $\langle x | y \rangle := \langle x | y \rangle = \vec{x}^\dagger \vec{y} = (x, y) \in \mathbb{C}$.

The computational basis of $\mathbb{C}^n$ is the set of vectors $\{|i\rangle\}_{i=0}^n$, where $|i\rangle := \delta_{i,0}$ and $\langle i | := \delta_{i,0}^*$. Then, for example, $|x\rangle = \sum_{i=1}^{n} x_i |i\rangle$ and $\langle x | = \sum_{i=1}^{n} x_i^* \langle i|$.

The tensor product of quantum states is their Kronecker product: if $|x\rangle \in \mathbb{C}^{n_1}$ and $|y\rangle \in \mathbb{C}^{n_2}$, then

$$
|x\rangle \otimes |y\rangle := |x\rangle \otimes |y\rangle
$$

A quantum algorithm is a sequence of unitary matrices, i.e., a linear transformation $U$ such that $U^\dagger U = I$.

For any $p \in [0, 1]$, we define the coin state in $\mathbb{C}^2$ as

$$
|\text{coin } p\rangle := \sqrt{p} |1\rangle + \sqrt{1-p} |0\rangle = (\sqrt{1-p}, \sqrt{p})^T.
$$

Measuring $|\text{coin } p\rangle$ in the computational basis gives 1 with probability $p$, hence the name.

**Quantum multi-arm bandit oracle.** Recall the quantum multi-armed bandit oracle defined in (2). The arms are accessed in superposition by applying the unitary oracle $O$ on

Wiebe, Kapoor, and Svore (2015) is not framed as solving best-arm identification, but is partly concerned with this problem.
Amplitude amplification and estimation. Our quantum speed-up can be traced back to amplitude amplification and estimation (Brassard et al. 2002). For a classical randomized algorithm for a search problem that returns a correct solution $y$ with probability $p_{\text{succ}}$, the success probability can be amplified to a constant by $O(1/p_{\text{succ}})$ repetitions. Let $A$ be a quantum procedure that outputs a quantum state $\sqrt{p_{\text{succ}}} |1\rangle |y\rangle + \sqrt{1-p_{\text{succ}}} |0\rangle |y'\rangle$ for some arbitrary quantum state $|y'\rangle$. Measuring the output state yields the solution $y$ with probability $p_{\text{succ}}$, just like a classical randomized algorithm. Brassard et al. (2002) provided an amplitude amplification procedure that amplifies the amplitude of $|1\rangle |y\rangle$ to a constant with $O(1/\sqrt{p_{\text{succ}}})$ queries to the quantum procedure $A$. This effectively provides a randomized algorithm with constant success probability with query complexity $O(t/\sqrt{p_{\text{succ}}})$ if $A$ makes $t$ queries to the oracle. The same speed-up can be achieved for the closely related task of estimating $p_{\text{succ}}$ with amplitude estimation.

Amplitude amplification and estimation originates from Grover’s search algorithm (Grover 1996). The formal statements of Grover’s algorithm and amplitude amplification and estimation are postponed to the start of the appendix. We refer the interested reader to the book Nielsen and Chuang (2000) on quantum computing for a detailed introduction to basic definitions (Section 3), Grover’s algorithm and amplitude amplification (Section 6), and related topics.

Variable-time amplitude amplification and estimation

Variable-time amplitude amplification (VTAA) and estimation (VTAE) are procedures that apply on top of so-called variable-time quantum algorithms that may stop at different (variable) time steps with certain probabilities. More precisely, for $t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$ and $w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$, a $(t, w)$-variable-time algorithm $A$ is one that can be divided into $m$ steps (i.e., $A = A_{t_1} \cdots A_{t_m}$) where $t_j$ is the query complexity of $A_j \cdots A_1$ and $w_j$ is the probability of stopping at step $j$. We have:

**Theorem 2** (Informal: Variable-time amplitude amplification and estimation—Ambainis 2010, Childs, Kothari, and Somma 2017)

Given a $(t, w)$-variable-time quantum algorithm $A = A_{t_m} \cdots A_1$ with success probability $p_{\text{succ}}$, there exists a quantum algorithm $A'$ that uses $O(Q)$ queries to output the solution with probability $\geq \frac{1}{2}$, where

$$Q := t_m \log(t_m) + t_{\text{avg}} \sqrt{\frac{\log(t_m)}{p_{\text{succ}}}}.$$  

with $t_{\text{avg}} := \sqrt{\sum_{j=1}^{m} w_j t_j^2}$ being the root-mean-square average query complexity of $A$.

There also exists a quantum algorithm that uses $O\left(\frac{t^2}{\epsilon^2} \log(t_m) \log(\frac{1}{\epsilon})\right)$ queries to estimate $p_{\text{succ}}$ with multiplicative error $\epsilon$ with probability $\geq 1 - \delta$.

For comparison, recall that applying amplitude amplification and estimation procedures on general quantum algorithms requires $O(t_m/\sqrt{p_{\text{succ}}})$ queries. See the first section of the appendix for a rigorous definition of variable-time algorithms and formal statements of the query complexities of variable-time amplitude amplification and estimation.

Fast Quantum Algorithm For Best-arm Identification

In this section, we construct a quantum algorithm for best-arm identification and analyze its performance. Specifically:

**Theorem 3.** Given a multi-armed bandit oracle $O$ and confidence parameter $\delta \in (0,1)$, there exists a quantum algorithm that outputs the best arm with probability $\geq 1 - \delta$ using $O(\sqrt{H})$ queries to $O$.

Throughout this section, the oracle $O$ is fixed, so we may omit explicit reference to it. All logs have base 2.

There are essentially two steps in our construction. In the first step, we construct two subroutines Amplify and Estimate using VTAA and VTAE, respectively, on a variable-time quantum algorithm $A$. Roughly speaking, given $l \in [0,1]$, Amplify outputs an arm index $i$ randomly chosen from those $i$ with $p_i > l$ while Estimate counts the number of such $i$. This means that if we knew the values of $p_1$ and $p_2$, we could take $l$ to be $(p_1 + p_2)/2$, then Amplify would output the best arm. But we can use Estimate in a binary search procedure to estimate $p_1$ and $p_2$. This is exactly what we do in the second step and so we are done.

We now discuss the construction more precisely. Amplify and Estimate actually use two thresholds $l_2, l_1 \in [0,1]$ with $l_2 < l_1$ instead of a single threshold $l$. In the first step, we construct a variable-time quantum algorithm denoted $\mathcal{A}$ (Algorithm 1) that is initialized in a uniform superposition state $|\psi_i\rangle := \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle$ (since initially we have no information about which arm is the best). Given an input interval $I = [l_2, l_1]$, $\mathcal{A}$ “flags” arm indices in $S'_{\text{right}} := \{ i \in [n] : p_i \geq l_1 \}$ with a bit $f = 1$ and those in $S'_{\text{left}} := \{ i \in [n] : p_i \leq l_2 \}$ with a bit $f = 0$. The flag bit $f$ is written to a separate flag register $F$, so that the state (approximately) becomes $\frac{1}{\sqrt{n}} \sum_{i \in S'_{\text{right}}} |i\rangle |1\rangle_F + \sum_{i \in S'_{\text{left}}} |i\rangle |0\rangle_F + \sum_{i \in S'_{\text{middle}}} |i\rangle |\psi_i\rangle_F$ for some states $|\psi_i\rangle \in \mathbb{C}^2$, where $S'_{\text{middle}} := [n] \setminus (S'_{\text{left}} \cup S'_{\text{right}})$.


\{i \in [n] : l_2 < p_i < l_1\}. The flag bit \( f \) stored in the \( F \) register indicates whether VTAA (resp. VTAE), when applied on \( A \), should \((f = 1)\) or should not \((f = 0)\) amplify (resp. estimate) that part of the state. We then apply VTAA and VTAE on \( A \) to construct Amplify and Estimate, respectively. Amplify produces a uniform superposition of all those indices in \( S \) with \( F \) register in \([1]\), i.e., it amplifies such indices to the others. Estimate counts the number of such indices. More precisely, Estimate (approximately) counts the number of indices in \( S'_{\text{right}} \) as their \( F \) register is in \([1]\), plus some (unknown) fraction of indices in \( S'_{\text{middle}} \) as dictated by the fraction of \([1]\) in \( S \) (the unknown) states \(|\psi_i\rangle\).

In the second step, we use Estimate as a subroutine in Locate [Algorithm 2] to find an interval \([l_2, l_1]\) such that \(p_2 < l_2 < l_1 < p_1\) and that \(|l_1 - l_2| \geq \Delta_2 / 4\). Then, running Amplify with these \(l_2, l_1\) in BestArm [Algorithm 4] gives the state \([1]\) containing the best-arm index because only \(p_1\) is to the right of \(l_2\). Locate is a type of binary search that counts the number of indices in \( S'_{\text{right}} \) using Estimate. There is a technical difficulty here because Estimate actually counts the number of indices in \( S'_{\text{right}} \) plus some fraction of indices in \( S'_{\text{middle}} \). Trying to fix this by simply setting \(l_2 = l_1\), so that \(S'_{\text{middle}} = \emptyset\), does not work as it would increase the cost of Estimate. We overcome this difficulty via the Shrink subroutine [Algorithm 3] of Locate, which employs a technique from recent work on quantum state preparation [Lin and Tong 2020]. See Figure 1 for an illustration of the overall structure of the algorithm.

**Amplify and Estimate**

We first construct a variable-time quantum algorithm [Algorithm 1] that we call \( A \) throughout. \( A \) uses the following registers: input register \( I \); bandit register \( B \); clock register \( C = (C_1, \ldots, C_{m+1}) \), where each \( C_i \) is a qubit; ancillary amplitude estimation register \( P = (P_1, \ldots, P_m) \), where each \( P_i \) has \(\Omega(m)\) qubits; and flag register \( F \). We set \( m := \lceil \log \left( \frac{1}{l_1 - l_2} \right) \rceil + 2 \) as assigned in [Algorithm 1].

\( A \) is indeed a variable-time quantum algorithm according to [Definition 1]. This is because we can write \( A = A_{m+1} A_{m} \cdots A_1 A_0 \) as a product of \(m + 2\) sub-algorithms, where \( A_0 \) is the initialization step [Line 3], \( A_j \) consists of the operations in iteration \( j \) of the for loop [Lines 6, 7] for \( j \in [n] \), and \( A_{m+1} \) is the termination step [Lines 10–11]. The state spaces \( H_C \) and \( H_A \) in [Definition 1] correspond to the state spaces of the \( C \) register and the remaining registers of \( A \), respectively. \( A_{m+1} \) ensures that Condition 4 of [Definition 1] is satisfied.

**Algorithm 1:** \( \mathcal{A}(O, l_2, l_1, \alpha) \)

**Input:** Oracle \( O \) as in \([2]\); \( 0 < l_2 < l_1 < 1; \) approximation parameter \( 0 < \alpha < 1 \).

1. \( \Delta \leftarrow l_1 - l_2 \)
2. \( m \leftarrow \lceil \log \frac{1}{\alpha} \rceil + 2 \)
3. \( a \leftarrow \frac{2m+2}{\alpha} \)
4. Initialize state to \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle |\text{coin } p_i\rangle_B |0\rangle_C |0\rangle_F \)
5. for \( j = 1, \ldots, m \) do
6. \( \epsilon_j \leftarrow 2^{-j} \)
7. if register \( I \) is in state \( |i\rangle \) and registers \( C_1, \ldots, C_{j-1} \) are in state \( |0\rangle \) then
8. \( \text{Apply GAE}(\epsilon_j, a; l_1) \) with \( O_{p_i} \) on registers \( B, C_j, \) and \( P_j \)
9. \( \text{Apply controlled-NOT gate with control on register } C_j \) and target on register \( F \)
10. if registers \( C_1, \ldots, C_m \) are in state \( |0\rangle \) then
11. Flip the bit stored in register \( C_{m+1} \)

With \( \Delta := l_1 - l_2 \) being the length of \([l_2, l_1]\), we define the following three sets that partition \([n]\):

\[ S_{\text{left}} := \{i \in [n] : p_i < l_1 - \Delta / 2\}, \quad (9) \]

\[ S_{\text{middle}} := \{i \in [n] : l_1 - \Delta / 2 \leq p_i < l_1 - \Delta / 8\}, \quad (10) \]

\[ S_{\text{right}} := \{i \in [n] : p_i \geq l_1 - \Delta / 8\}. \quad (11) \]

These sets play the roles of aforementioned \( S'_{\text{left}}, S'_{\text{middle}}, \) and \( S'_{\text{right}} \). They can be regarded as functions of (the input to) \( A \). For later convenience, we also define \( S_{\text{left}} := S_{\text{left}} \cup S_{\text{middle}} \) and \( S_{\text{right}} := S_{\text{middle}} \cup S_{\text{right}} \).

**Lemma 1** (Correctness of \( A \)). Let \( p_{\text{succ}} \) denote the success probability \( A \). Then \( |p_{\text{succ}} - p'_{\text{succ}}| \leq \frac{2m}{n} \) where \( p'_{\text{succ}} = \frac{1}{n} \sum_{i \in S_{\text{right}}} |\beta_{i,1}|^2 \) for some \( |\beta_{i,1}|^2 \in [0, 1] \).

At a high level, at iteration \( j \) [Line 8] approximately identifies those \( i \in S_{\text{left}} \) with \( p_i \in [l_1 - 2\epsilon_1 j, l_1 - \epsilon_j) \) and stops computation on these \( i \) by setting their associated \( C \) registers to \([1]\). [Line 9] then flips these \( i \) by setting their associated \( F \) registers to \([0]\), indicating failure. We defer the detailed proof to the supplementary material which is mainly concerned with bounding the error in the aforementioned approximation, as well as the lemma as follows.

**Lemma 2** (Complexity of \( A \)). With \( \Delta := l_1 - l_2 \) being the length of the interval, we have:

1. The \( j^{th} \) stopping time \( t_j \) of \( A_1 A_1 \cdots A_0 \) is of order \( \sum_{k=1}^{j} \frac{1}{n} \log \frac{1}{\alpha} \leq 2^{j+1} \log \frac{1}{\alpha} \). In particular, \( t_{m+1} = O(\frac{\Delta^2}{\log \frac{1}{\alpha}}) \).

2. The average stopping time squared, \( t_{\text{avg}}^2 \), is of order

\[ \frac{1}{n} \left( \frac{|S_{\text{right}}|}{\Delta^2} + \sum_{i \in S_{\text{left}}} \frac{1}{(l_1 - p_i)^2} \right) \log^2 \left( \frac{1}{\alpha} \right). \quad (12) \]

Now we fix algorithm \( A \) and its input parameters. We always assume that \(|S_{\text{right}}| \geq 1 \), which we need for some of
the following results to hold. This is without loss of generality as we can always add an artificial arm 0 with bias \( p_0 = 1 \) to the bandit oracle \( \mathcal{O} \), as we do in \[\text{Line 3 of Algorithm 3}\].

We apply VTAA and VTAE (Theorem 2) on our variable-time quantum algorithm \( \mathcal{A} \) to prepare the state \( |\psi_{\text{succ}}\rangle \) and to estimate the probability \( p_{\text{succ}} \), respectively. This gives two new algorithms Amplify and Estimate with the following performance guarantees.

**Lemma 3** (Correctness and complexity of Amplify, \( \mathcal{A}, \delta \), Estimate, \( \mathcal{A}, \epsilon, \delta \)). Let \( \mathcal{A} = \mathcal{A}(\mathcal{O}, I, l_1, l_2, \delta) \). Then Amplify, \( \mathcal{A}, \delta \), uses \( O(Q) \) queries to output an index \( i \in S_{\mathcal{O}} \) with probability \( \geq 1 - \delta \), and Estimate, \( \mathcal{A}, \epsilon, \delta \), uses \( O(Q/\epsilon) \) queries to output an estimate \( r \) of \( p_{\text{succ}} \) (defined in Lemma 1) such that

\[
(1 - \epsilon) \left( p_{\text{succ}} - \frac{0.1}{n} \right) < r < (1 + \epsilon) \left( p_{\text{succ}} + \frac{0.1}{n} \right)
\]

with probability \( \geq 1 - \delta \), where \( Q \) is

\[
\left( \frac{1}{n^2} + \left| S_{\text{right}} \right| \sum_{i=1}^{n} \frac{1}{(l_1 - p_i)^2} \right) \log \left( \frac{1}{\delta} \right).
\]

where \( \Delta = l_1 - l_2 \).

This lemma follows by applying Lemma 1 and Lemma 2 to Theorem 2. The proof detail is given in the appendices.

**Quantum algorithm for best-arm identification**

In this subsection, we use Amplify and Estimate to construct three algorithms (Algorithms 2, 3, 4) that work together to identify the best arm following the outline that we described at the beginning of this section.

**Algorithm 2**: Locate, \( \mathcal{O}, \delta \)

**Input**: Oracle \( \mathcal{O} \) as in (2); confidence parameter \( 0 < \delta < 1 \).

1. \( I_1, I_2 \leftarrow [0, 1] \)
2. \( \delta \leftarrow \delta/8 \)
3. while \( \min I_1 - \max I_2 < 2 |I_1| \) do
   4. \( I_1 \leftarrow \text{Shrink}(\mathcal{O}, 1, I_1, \delta) \)
   5. \( I_2 \leftarrow \text{Shrink}(\mathcal{O}, 2, I_2, \delta) \)
   6. \( \delta \leftarrow \delta/2 \)
7. return \( I_1, I_2 \)

We state the correctness and complexities of Amplify and Estimate as follows:

**Lemma 4** (Correctness and complexity of Algorithm 1). Fix a confidence parameter \( 0 < \delta < 1 \). Then the event \( E = \{ p_1 \in I_1 \text{ and } p_2 \in I_2 \text{ in all iterations of the while loop} \} \) holds with probability \( \geq 1 - \delta \). When \( E \) holds, Algorithm 1 also satisfies the following for both \( k \in \{1, 2\} \):

1. Its while loop (Line 3) breaks at or before the end of iteration \( \left( \log_{5/3} \left( \frac{1}{\Delta} \right) \right) + 3 \) and then returns \( I_k \) with \( p_k \in I_k \) and \( \min I_1 - \max I_2 \geq 2 |I_1| \); during the while loop, we always have \( |I_1| = |I_2| \geq \Delta/2 \); and

\footnote{The state spaces \( \mathcal{H}_C, \mathcal{H}_F, \text{ and } \mathcal{H}_W \) correspond to the state spaces of the \( C, F, \text{ and } \mathcal{A} \), respectively.}

**Algorithm 3**: Shrink, \( \mathcal{O}, k, I, \delta \)

**Input**: Oracle \( \mathcal{O} \) as in (2); interval \( I = [a, b] \); confidence parameter \( 0 < \delta < 1 \).

1. \( \epsilon \leftarrow (b - a)/5 \)
2. \( \delta \leftarrow \delta/2 \)
3. Append arm \( i = 0 \) with bias \( p_0 = 1 \) to \( \mathcal{O} \); call the resulting oracle \( \mathcal{O}' \)
4. Construct variable-time quantum algorithms \( \mathcal{A}_1, \mathcal{A}_2 \):
   5. \( \mathcal{A}_1 \leftarrow \mathcal{A}(\mathcal{O}', l_2 = a + \epsilon, l_1 = a + 3\epsilon, 0.01\delta) \)
   6. \( \mathcal{A}_2 \leftarrow \mathcal{A}(\mathcal{O}', l_2 = a + 2\epsilon, l_1 = a + 4\epsilon, 0.01\delta) \)
7. \( r_1 \leftarrow \text{Estimate}(\mathcal{A}_1, \epsilon = 0.1) \)
8. \( r_2 \leftarrow \text{Estimate}(\mathcal{A}_2, \epsilon = 0.1) \)
9. \( B_1 \leftarrow \mathbb{1}(r_1 > \frac{k+0.5}{n+1}); B_2 \leftarrow \mathbb{1}(r_2 > \frac{k+0.5}{n+1}) \)
10. switch \( (B_1, B_2) \) do
   11. case \( (0, 0) \) : \( I \leftarrow [a, a + 3\epsilon] \)
   12. case \( (0, 1) \) : \( I \leftarrow [a + \epsilon, a + 4\epsilon] \)
   13. case \( (1, 0) \) : \( I \leftarrow [a + \epsilon, a + 4\epsilon] \)
   14. case \( (1, 1) \) : \( I \leftarrow [a + 2\epsilon, a + 5\epsilon = b] \)
15. return \( I \)

2. it uses \( O\left( \sqrt{H} \log \left( \frac{n}{\delta^2} \right) \right) \) queries.

**Lemma 5** (Correctness and complexity of Algorithm 3). Fix \( k \in \{1, 2\} \), an interval \( I = [a, b] \), and a confidence parameter \( 0 < \delta < 1 \). Suppose that \( p_k \in I \) and \( |I| \geq \Delta/2 \). Then Algorithm 3
   1. outputs an interval \( J \) with \( |J| = \frac{k}{3} |I| \) such that \( p_k \in J \) with probability \( \geq 1 - \delta \), and
   2. uses \( O\left( \sqrt{H} \log \left( \frac{n}{\delta^2} \right) \right) \) queries.

The proofs of Lemma 4 and Lemma 5 appear in the supplementary material.

The following theorem is equivalent to Theorem 3:

**Theorem 4** (Correctness and complexity of Algorithm 4). Fix a confidence parameter \( 0 < \delta < 1 \). Then, with probability \( \geq 1 - \delta \), Algorithm 4
   1. outputs the best arm, and
   2. uses \( O\left( \sqrt{H} \log \left( \frac{n}{\delta^2} \right) \right) \) queries.

Proof. Note that \( \delta \) is halved at the beginning, on Line 1. For the first claim, we know from the first claim of Lemma 4 that, with probability \( \geq 1 - \delta/2 \), the two intervals \( I_k \) assigned in Line 3 have min \( I_1 - \max I_2 \geq 2 |I_1| \geq \Delta/4 \).
Corollary 1. In this setting as well. More precisely, we can modify the algorithm settings. In the $\text{PAC}$ (Probably Approximately Correct) and fixed-budget settings. In the $O(\sqrt{H})$ queries for outputting the best arm is at least $1 - \delta$. The second claim follows immediately from adding the complexity of $\text{Locate}(O, \delta/2)$ (Lemma 4) and $\text{Amplify}(A, \delta/2)$ (Lemma 3) using $l_1 - l_2 \geq \Delta_2/4$.

By establishing Theorem 4, we have established our main claim. As discussed previously, the main complexity measure of interest in the classical case is $H$, and we see that we get a quadratic speedup in terms of this parameter.

We can see that the poly-logarithmic factor has degree about 6 from (38), (40), and (42). It would be interesting to reduce this degree. A more fundamental challenge is to remove the variable $n$ that appears in our log factors. In the classical case, $n$ was already removed from log factors in early work by a procedure called “median elimination”. However, quantizing the median elimination framework is nontrivial, as the query complexity for outputting the $n/2$ smallest items among $n$ elements is $\Theta(n)$ [Ambainis 2010a, Theorem 1], exceeding our budget of $O(\sqrt{H})$.

As corollaries of our main results in the fixed-confidence setting, we provide results on best-arm identification in the $\text{PAC}$ (Probably Approximately Correct) and fixed-budget settings. In the $(\epsilon, \delta)$-PAC setting, the goal is to identify an arm $i$ with $p_i \geq p_1 - \epsilon$ with probability $\geq 1 - \delta$. Our best-arm identification algorithm can be modified to work in this setting as well. More precisely, we can modify Locate (Algorithm 2) by adding a breaking condition to the while loop when $|I_1|$ (or equivalently $|I_2|$) is smaller than $\epsilon$. This gives the following result:

Corollary 1. There is a quantum algorithm that finds an $\epsilon$-optimal arm with query complexity $O\left(\sqrt{\min\{\frac{\Delta_1}{\epsilon^2}, H\}} \cdot \text{poly}(\log(\frac{1}{\delta}))\right)$.

Note that our modification means that the $\text{Amplify}$ step in [Algorithm 4] takes an input interval $I$ with $|I| = l_1 - l_2 \leq \frac{1}{\epsilon^2}$. The correctness and complexity follow directly from Lemma 1 and Lemma 3. For comparison, Even-Dar, Mannor, and Mansour (2002) gave a classical PAC algorithm with complexity $O\left(\frac{1}{\epsilon^2} \log\left(\frac{1}{\delta}\right)\right)$, which was later improved to $O\left(\sum_{i=1}^{n} \min\{\epsilon^{-2}, \Delta^{-2}\} \cdot \log\left(\frac{1}{\delta} \cdot \frac{1}{\epsilon^2}\right)\right)$ by Gabillon, Ghavamzadeh, and Lazaric (2012).

In the supplementary material, we also show how to identify the best arm with high probability for a fixed number of total queries (the fixed-budget setting) given knowledge of $H$.

Quantum lower bound

In this section, we describe a lower bound for the quantum best-arm identification problem. Our lower bound shows that the algorithm of Theorem 3 is optimal up to poly-logarithmic factors.

Theorem 5. Let $p \in (0, 1/2)$. For any biases $p_i \in [p, 1-p]$, any quantum algorithm that identifies the best arm requires $\Omega(\sqrt{H})$ queries to the multi-armed bandit oracle $\mathcal{O}$.

To prove this lower bound, we use the quantum adversary method to show quantum hardness of distinguishing $n$ oracles $\mathcal{O}_x$, $x \in [n]$, corresponding to the following $n$ bandits. In the 1st bandit, we assign bias $p_1$ to arm $i$ for all $i$. In the $x$th bandit for $x \in \{2, \ldots, n\}$, we assign bias $p_1 + \eta$ to arm $x$ and $p_1$ to arm $i$ for all $i \neq x$, where $\eta$ is an appropriately chosen parameter. This hard set of bandits is inspired by the proof of a corresponding classical lower bound [Mannor and Tsitsiklis 2004, Theorem 5].

More precisely, for a positive integer $T$, consider an arbitrary $T$-query quantum algorithm that distinguishes the oracles $\mathcal{O}_x$. The main idea of the adversary method is to keep track of certain quantities $s_k \in \mathbb{R}$ where $k \in \{0, 1, \ldots, T\}$. For each $k$, $s_k$ quantifies how close the states of the quantum algorithm are when it operates using $k$ queries to the different $\mathcal{O}_x$. At the start, when $k = 0$, $s_0$ must be large because when no queries have been made, the states must be close. At the end, when $k = T$, $s_T$ must be small because the states are distinguishable by assumption.

The key point is that we can also bound how much $s_k$ can change in one query, that is we can bound the quantities $|s_{k+1} - s_k|$ for each $k$. Of course, this bound immediately gives a lower bound on $T$, the number of queries it takes to go from $s_0$ (large) to $s_T$ (small). To bound $|s_{k+1} - s_k|$, the key point is to bound the distance between oracles, i.e., matrices, $\mathcal{O}_x$ and $\mathcal{O}_y$ for different $x, y \in [n]$.

We defer the full proof and full description of the quantum adversary method to the supplementary material.

Conclusions

In this paper, we propose a quantum algorithm for identifying the best arm of a multi-armed bandit, which gives a quadratic speedup compared to the best possible classical result. We also prove a matching quantum lower bound (up to poly-logarithmic factors).

This work leaves several natural open questions:

- Can we give fast quantum algorithms for the exploitation of multi-armed bandits? In particular, can we give online algorithms with favorable regret? The quantum hedging algorithm [Hamoudi et al. 2020] and the quantum boosting algorithm [Aruna el al. 2020] might be relevant to this challenge.

- Can we give quantum algorithms for other types of multi-armed bandits, such as contextual bandits or adversarial bandits (e.g., Beygelzimer et al. 2011, Agarwal et al. 2014, Auer et al. 2002)?

- Can we give fast quantum algorithms for finding a near-optimal policy of a Markov decision process (MDP)? MDPs are a natural generalization of MABs, where the goal is to maximize the expected reward over sequences of decisions. [Even-Dar, Mannor, and Mansour 2002] gave a reduction from this problem to best-arm identification by viewing the Q-function of each state as a multi-armed bandit.
Ethics Statement
This work is purely theoretical. Researchers working on theoretical aspects of bandits and quantum computing may immediately benefit from our results. In the long term, once fault-tolerant quantum computers have been built, our results may find practical applications in multi-armed bandit scenarios arising in the real world. As far as we are aware, our work does not have negative ethical impact.

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Preliminaries on Quantum Algorithms

Grover’s search and amplitude amplification and estimation

Our quantum speedup conceptually originates from Grover’s search algorithm (Grover 1996). Consider a function \( f_w : [n] \rightarrow \{-1, 1\} \) such that \( f_w(i) = 1 \) if and only if \( i \neq w \), so that \( w \) can be viewed as a (unique) marked item. To search for \( w \), classically we need \( \Omega(n) \) queries to \( f_w \). Quantumly, we can use one call of \( f_w \) to create an oracle \( U_w \) such that \( U_w |i\rangle = |i\rangle \) for all \( i \neq w \) and \( U_w |w\rangle = -|w\rangle \). Now consider the uniform superposition \( |u\rangle := \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle \) as well as the state \( |r\rangle := \frac{1}{\sqrt{n-1}} \sum_{i \in [n]/\{w\}} |i\rangle \).

The angle between \( U_w |u\rangle \) and \( |u\rangle \) is \( \theta := \arccos(1/n) = \Theta(1/\sqrt{n}) \). Note that the unitary \( U_w \) reflects about \( |r\rangle \), and the unitary \( U_w |w\rangle = 2 |w\rangle \langle w| - I \) reflects about \( |w\rangle \). If we start with \( |w\rangle \), the angle between \( U_w |w\rangle \) and \( U_w |w\rangle U_w |w\rangle \) is amplified to 20\(^\circ\), and in general the angle between \( U_w |u\rangle \) and \( (U_w U_w)^k |u\rangle \) is \( 2k\theta \). It thus suffices to take \( k = \Theta(\sqrt{n}) \) to find \( w \).

This method of alternatively applying two reflections to boost the amplitude for success can be generalized to a technique called amplitude amplification. For the case with some unknown number \( k \in [n] \) of marked items, there is also a quadratic quantum speedup for estimating \( \theta := \arccos(k/n) \) via a technique called amplitude estimation (Brassard et al. 2002).

In the context of searching, consider a quantum procedure \( \mathcal{A} \) that returns a state \( |\psi\rangle \) with \( t \) oracle queries, such that the overlap between the target state \( |w\rangle \) and output state \( |\psi\rangle \) is \( p_{\text{succ}} := (\langle w|\psi\rangle)^2 \). By amplitude amplification and estimation (Brassard et al. 2002), \( O(t/\sqrt{p_{\text{succ}}}) \) oracle queries suffice to amplify the overlap to constant order and to estimate \( p_{\text{succ}} \) respectively. We describe amplitude estimation more formally:

**Theorem 6** (Amplitude estimation). Suppose \( \mathcal{O}_p \) is a unitary with \( \mathcal{O}_p |0\rangle_B = |\text{coin } p\rangle_B \). Then there is a unitary procedure \( \mathcal{AE}(\epsilon, \delta) \), making \( O(\frac{1}{\epsilon} \log \frac{1}{\delta}) \) queries to \( \mathcal{O}_p \) and \( \mathcal{O}_p^\dagger \), that on input \( |\text{coin } p\rangle_B |0\rangle_C \) prepares a state of the form

\[
|\text{coin } p\rangle_B \left( \sum_{p'} \alpha_{p'} |p'\rangle \right) \left( |\text{coin } p\rangle_B |0\rangle_C \right) + \alpha |\bot\rangle \left( |\text{coin } p\rangle_B |0\rangle_C \right),
\]

where \( |\alpha| \geq \sqrt{1 - \sum_{p'} |\alpha_{p'}|^2} \leq \delta \), \( \langle p' | p, \bot \rangle = 0 \) for all \( p' \), and \( |p' - \bot| \leq \epsilon \) for all \( p' \).

Strictly speaking, the parts of Theorem 6 involving \( \delta \) come from measuring the output state of the original amplitude estimation procedure (Brassard et al. 2002) \( O(\log \frac{1}{\epsilon}) \) times and taking the median. This can be made coherent by the principle of deferred measurement.

**Variable-time amplitude amplification and estimation**

In this section we review variable-time amplitude amplification (VTAA) and estimation (VTAE), which are essential components of our algorithm. VTAA and VTAE are procedures applied on top of so-called “variable-time” quantum algorithms, which can be formally defined as follows:

**Definition 1** (Variable-time quantum algorithm, cf. Ambainis 2010b, Section 3.3 and Childs, Kothari, and Somma 2017, Section 5.1). Let \( \mathcal{A} \) be a quantum algorithm in a space \( \mathcal{H} \) that starts in the state \( |0\rangle_H \), the all-zeros state in \( \mathcal{H} \). We say \( \mathcal{A} \) is a variable-time quantum algorithm if the following conditions hold:

1. \( \mathcal{A} \) is the product of \( m \) sub-algorithms, \( \mathcal{A} = \mathcal{A}_m \cdot \mathcal{A}_{m-1} \cdot \ldots \cdot \mathcal{A}_1 \).
2. \( \mathcal{H} \) is a tensor product \( \mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_A \), where \( \mathcal{H}_C \) is a tensor product of \( m \) single-qubit registers denoted \( \mathcal{H}_{C_1}, \mathcal{H}_{C_2}, \ldots, \mathcal{H}_{C_m} \).
3. Each \( \mathcal{A}_j \) is a controlled unitary that acts on the registers \( \mathcal{H}_{C_j} \otimes \mathcal{H}_A \) controlled on the first \( j \) – 1 qubits of \( \mathcal{H}_C \) being set to \( |0\rangle \).
4. The final state of the algorithm, \( \mathcal{A} |0\rangle_H \), is perpendicular to \( |0\rangle_C := |0\rangle_{C_1} |0\rangle_{C_2} \ldots |0\rangle_{C_m} \).

In each iteration of the variable-time algorithm we shall construct, we use a subroutine that we call **gapped amplitude estimation** (GAE). Standard amplitude estimation (Brassard et al. 2002) performs phase estimation on a particular unitary, and GAE is essentially the same as “gapped phase estimation” (Childs, Kothari, and Somma 2017, Lemma 22) of that unitary. We recall the standard technique of amplitude estimation (Brassard et al. 2002), which we have stated in Theorem 6. It implies the following:

**Corollary 2** (Gapped amplitude estimation). Suppose \( \mathcal{O}_p \) is a unitary with \( \mathcal{O}_p |0\rangle = |\text{coin } p\rangle \). Then there is a unitary procedure \( \mathcal{GAE}(\epsilon, \delta; l) \), making \( O(\frac{1}{\epsilon} \log \frac{1}{\delta}) \) queries to \( \mathcal{O}_p \) and \( \mathcal{O}_p^\dagger \), that on input \( |\text{coin } p\rangle_B |0\rangle_C \) prepares a state of the form

\[
|\text{coin } p\rangle_B \left( \beta_0 |0\rangle_C |\gamma_0\rangle \right) \left( |\text{coin } p\rangle_B |0\rangle_C \right) + \beta_1 |1\rangle_C |\gamma_1\rangle \left( |\text{coin } p\rangle_B |0\rangle_C \right),
\]

where \( \beta_0, \beta_1 \in [0, 1] \) satisfy \( \beta_0^2 + \beta_1^2 = 1 \) with \( \beta_1 \leq \delta \) if \( p \geq l - \epsilon \) and \( \beta_0 \leq \delta \) if \( p < l - 2\epsilon \).

**Proof.** We first run \( \mathcal{AE}(\epsilon/4, \delta) \) on registers \( B, P \). Then, in register \( C \), we output 1 if the value stored in register \( P \) is closer to \( l - \epsilon \), and output 0 if it is closer to \( l - 2\epsilon \). This gives the desired unitary procedure. For convenience, we put any phase factors on the \( \beta_i \) into the \( |\gamma_i\rangle \).
Theorem 7 (Variable-time amplitude amplification and estimation [Ambainis 2010b; Childs, Kothari, and Somma 2017; Chakraborty, Gilvén, and Jeffery 2019]. Let $A = A_{m} \cdots A_{1}$ be a variable-time quantum algorithm on the space $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_W$. Let $|0\rangle_{\mathcal{H}}$ be the all-zeros state in $\mathcal{H}$ and let $t_j$ be the query complexity of the algorithm $A_{j} \cdots A_{1}$. We define

$$w_{j} := \|\Pi_{C_{j}}A_{j} \cdots A_{1}|0\rangle_{\mathcal{H}}\|^2 \quad \text{and} \quad t_{\text{avg}} := \sqrt{\sum_{j=1}^{m} w_{j} t_{j}^2}$$

(17)

to be the probability of halting at step $j$ and the root-mean-square average query complexity of the algorithm, respectively, where $\Pi_{C_{j}}$ denotes the projector onto $|1\rangle$ in $\mathcal{H}_{C_{j}}$. We also define

$$p_{\text{succ}} := \|\Pi_{F}A_{m} \cdots A_{1}|0\rangle_{\mathcal{H}}\|^2 \quad \text{and} \quad |\psi_{\text{succ}}\rangle := \frac{\Pi_{F}A_{m} \cdots A_{1}|0\rangle_{\mathcal{H}}}{\|\Pi_{F}A_{m} \cdots A_{1}|0\rangle_{\mathcal{H}}\|}$$

(18)

to be the success probability of the algorithm and the corresponding output state, respectively, where $\Pi_{F}$ projects onto $|1\rangle$ in $\mathcal{H}_{F}$. Then there exists a quantum algorithm that uses $O(Q)$ queries to output the state $|\psi_{\text{succ}}\rangle$ with probability $\geq 1/2$ and a bit indicating whether it succeeds, where

$$Q := t_{m} \log(t_{m}) + \frac{t_{\text{avg}}}{p_{\text{succ}}} \log(t_{m}).$$

(19)

There also exists a quantum algorithm that uses $O\left(\frac{Q}{\epsilon^2} \log^2(t_{m}) \log\left(\frac{t_{m}}{\epsilon^2}\right)\right)$ queries to estimate $p_{\text{succ}}$ with multiplicative error $\epsilon$ with probability $\geq 1 - \delta$.

Quantum lower bounds by the adversary method

Suppose we have $n$ multi-armed bandit oracles $\mathcal{O}_{x}$, $x \in [n]$, corresponding to $n$ multi-armed bandits where the best arm is located at a different index in each. Suppose that we also have a best-arm identification algorithm $A$ that uses no more than $T$ queries to identify the best arm with probability $\geq 1 - \delta$.

The basic quantum adversary method [Ambainis 2002; Høyer and Spalek 2005] considers a quantity of the form

$$s_{k} := \sum_{x \neq y} w_{x,y} \langle \psi_{x}^{(k)} | \psi_{y}^{(k)} \rangle,$$

(20)

where $k \in \{0, 1, \ldots, T\}$, $x, y \in [n]$, $w_{x,y} \geq 0$, and $|\psi_{x}^{(k)}\rangle$ is the state of $A$ after the $k$th query to the oracle $\mathcal{O}_{x}$.

At step $k = 0$, $A$ has made no queries to the oracle, so $|\psi_{x}^{(0)}\rangle$ must be the same for all $x$. Therefore $s_{0} = \sum_{x \neq y} w_{x,y}$ as $\langle \psi_{x}^{(0)} | \psi_{y}^{(0)} \rangle = 1$.

At step $k = T$, $A$ must output the index of the best arm with probability $\geq 1 - \delta$. Since the location of the best arm is different for each $\mathcal{O}_{x}$, the states $|\psi_{x}^{(T)}\rangle$ must be distinguishable by a quantum measurement with probability $\geq 1 - \delta$. This means that $|\langle \psi_{x}^{(T)} | \psi_{y}^{(T)} \rangle| \leq 2\sqrt{\delta(1 - \delta)}$. Therefore $|s_{T}| \leq 2\sqrt{\delta(1 - \delta)} \cdot \sum_{x \neq y} w_{x,y}$.

Combining the above observations, we have

$$|s_{0} - s_{T}| \geq |s_{0}| - |s_{T}| \geq (1 - 2\sqrt{\delta(1 - \delta)}) \cdot \sum_{x \neq y} w_{x,y}.$$ 

(21)

Hence, if we can upper bound $|s_{k+1} - s_{k}|$ by $B$ for some constant $B$, we can deduce that

$$T \geq \frac{1 - 2\sqrt{\delta(1 - \delta)}}{B} \cdot \sum_{x \neq y} w_{x,y},$$

(22)

giving a lower bound on the query complexity.

Note that we apply the quantum adversary method to multi-armed bandit oracles of the form given in [2], whereas most results from the literature on quantum lower bounds assume a different form of oracle. We remark that [Belson 2015] treats a more general class of oracles, so it should be possible to prove Theorem 5 using its results. However, we give a self-contained proof using the formulation described above as this approach is straightforward in our case.

Proof Details of the Quantum Upper Bound

Proof of [Lemma 1]

We first state a more detailed version of [Lemma 1]. We say that states $|\psi\rangle$ and $|\phi\rangle$ are $\epsilon$-close if $\| |\psi\rangle - |\phi\rangle \| \leq \epsilon$. 


Lemma 6 (Full version of Lemma 1, correctness of \( A \)). The output state \(|\phi(A)\rangle\) of \( A \) is \((\alpha/n)\)-close to

\[
|\psi(A)\rangle := \frac{1}{\sqrt{n}} \sum_{i,j} |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F 
+ \frac{1}{\sqrt{n}} \sum_{i,j} |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |0\rangle_F 
+ \frac{1}{\sqrt{n}} \sum_{i,j} |i\rangle_j \langle \text{coin}_i| B (\beta_{i,1} |\psi_{i,1}\rangle_{C,P} |1\rangle_F + \beta_{i,0} |\psi_{i,0}\rangle_{C,P} |0\rangle_F )
\]

for some \( \beta_{i,1}, \beta_{i,0} \in \mathbb{C} \) and states \(|\psi_i\rangle, |\psi_{i,j}\rangle\). In particular, we have \(|p_{\text{succ}} - p'_{\text{succ}}| \leq \frac{2\alpha}{n} \) where \( p_{\text{succ}} := \|\Pi_F |\phi(A)\rangle\|^2 \) and \( p'_{\text{succ}} := \|\Pi_F |\psi(A)\rangle\|^2 = \frac{1}{n} (|S_{\text{right}}| + \sum_{i,j} |S_{\text{middle}}| \beta_{i,1}^2) \).

As our proof is similar to that presented in Section 5.3 of Childs, Kothari, and Somma (2017), we only sketch it in a way that highlights the differences. For comparison, it may be helpful to note that our states \(|i\rangle_j \langle \text{coin}_i| B\rangle\) are analogous to the matrix eigenstates \(|\lambda\rangle\) in Childs, Kothari, and Somma (2017). The controlled-NOT operation in Line 9 of our Algorithm 1 takes the place of the simulation subroutine called “\( W \)” in Lemma 23 of Childs, Kothari, and Somma (2017), which is much more elaborate.

We proceed with the proof sketch. Let \( A_{\text{main}} := A_{m+1} \cdots A_1 \) denote the part of \( A \) after initialization. We show that, for each fixed \( i, A_{\text{main}} |i\rangle_j \langle \text{coin}_i| B\rangle_{C,P,F} \) is \((\alpha/\sqrt{n})\)-close to

Case \( i \in S_{\text{middle}}\): \(|i\rangle_j \langle \text{coin}_i| B (\beta_{i,1} |\psi_{i,1}\rangle_{C,P} |1\rangle_F + \beta_{i,0} |\psi_{i,0}\rangle_{C,P} |0\rangle_F )\) for some \( \beta_{i,1}, \beta_{i,0} \in \mathbb{C} \) and states \(|\psi_i\rangle, |\psi_{i,j}\rangle\);

Case \( i \in S_{\text{right}}\): \(|i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F\);

Case \( i \in S_{\text{left}}\): \(|i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |0\rangle_F\).

Then \(|\psi(A)\rangle\) = \( A_0 |1\rangle_{B,C,P,F} = A_{\text{main}} \sum_{i=1}^n |i\rangle_j \langle \text{coin}_i| B\rangle_{C,P,F} \) is \((\alpha/\sqrt{n} \cdot \frac{2\alpha}{n}) = \frac{\alpha}{n}\)-close to \(|\psi(A)\rangle\) as claimed.

Case \( i \in S_{\text{middle}}\). This is trivially true because \( \beta_{i,1} |\psi_{i,1}\rangle_{C,P} |1\rangle_F + \beta_{i,0} |\psi_{i,0}\rangle_{C,P} |0\rangle_F \) can represent any state on registers \( C, P, F \).

Case \( i \in S_{\text{left}}\). Let \( j \in [m-1] \) be such that \( l_1 - 2\epsilon_j \leq \epsilon_i < l_1 - \epsilon_j \). Note that this \( j \) uniquely exists by the definition of \( S_{\text{left}}, m, \) and \( \epsilon_j \). Then the state of the algorithm after the \((j-1)\)st iteration of the for-loop in Line 5 is \((2(j-1)\alpha)-close to

\[
|\psi(\text{Line 5})\rangle := |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F 
+ \sum_{j' \leq j} \alpha \sum_{i=1}^n |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F ,
\]

where, for each \( i \), the state \(|0\rangle_C |\gamma_0\rangle_P\) corresponds to the state \(|0\rangle_C |\gamma_0\rangle_P\) in \( \text{GAE}(\epsilon_j, a; l_1) \). Note that we incur an error of at most \( 2\alpha \) at each iteration which comes from running \( \text{GAE}(\epsilon_j, a; l_1) \) (cf. the case where \( \beta_1 \leq \alpha \) in Corollary 2). This error accumulates additively.

The state after the \( j \)th iteration is \((2j\alpha)-close to

\[
|\psi(\text{Line 5})\rangle := |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F 
+ \sum_{j' \leq j} \alpha \sum_{i=1}^n |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F ,
\]

where \( j := 0^1 \cdots 10^{m-1} \) denotes a unary representation of the integer \( j \).

At the \((j + 1)\)st iteration, the part of the state in the second line of Eq. (24) is unchanged because its register \( C \) indicates “stop”, but the part in the first line of Eq. (24) changes to being \((2(j + 1)\alpha)\)-close to

\[
|\psi(\text{Line 5})\rangle := |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F 
+ \sum_{j' \leq j} \alpha \sum_{i=1}^n |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |1\rangle_F ,
\]

where \( j := 0^1 \cdots 10^{m-1} \) denotes a unary representation of the integer \( j \).

Since the \( C \) register of all parts of the state in Eq. (25) indicates “stop”, the remaining iterations \( j + 2, \ldots, m \) of \( A \) do not alter it. Hence the final state of \( A \) is \((2ma)\)-close to the state in Eq. (26), which is of the form

\[
|\psi(\text{Line 5})\rangle := |i\rangle_j \langle \text{coin}_i| B |\psi_i\rangle_{C,P} |0\rangle_F .
\]

Note that \( 2ma = \alpha/m^2 \), so the closeness of approximation is as claimed.
Case \( i \in S_{\text{right}} \). In this case, there does not exist a \( j \in [m-1] \) such that \( l_1 - 2\epsilon_j \leq p_i < l_1 - \epsilon_j \). Thus a simplified version of the argument above, in which we do not have to consider different cases according to the iteration number, shows that the resulting state is \((2ma)\)-close to a state of the same form as Eq. (27) but with the \( F \) register remaining in state 1.

Lastly, we show that \( p_{\text{suc}} \) is close to \( p'_{\text{suc}} \) as claimed:

\[
|p_{\text{suc}} - p'_{\text{suc}}| = \left| \left( \sqrt{p_{\text{suc}}} + \sqrt{p'_{\text{suc}}} \right) \cdot \left( \sqrt{p_{\text{suc}}} - \sqrt{p'_{\text{suc}}} \right) \right|
\]

\[
= \left( \sqrt{p_{\text{suc}}} + \sqrt{p'_{\text{suc}}} \right) \cdot \left| \|\Pi_F |\phi(A)\| - \|\Pi_F |\psi(A)\| \right|
\]

\[
\leq 2 \left| \|\Pi_F |\phi(A)\| - \|\psi(A)\| \right|
\]

\[
\leq 2 \frac{\alpha}{n}.
\]

**Proof of Lemma 2**

The proof is similar to that presented in Section 5.4 of [Childs, Kothari, and Somma (2017)]. For the first claim, note first that \( A_0 \) and \( A_{m+1} \) use a constant number of queries (1 and 0, respectively), so we can ignore them. For \( k \in [m] \), \( A_k \) only uses queries to perform \( \text{GAE}(\epsilon_k, d; l_1) \), which takes \( O\left(\frac{1}{\epsilon_k} \log \frac{1}{\epsilon_k}\right) \) queries. Therefore, \( \tau_j \), the number of queries in \( A_j A_{j-1} \cdots A_1 \), is of order

\[
\sum_{k=1}^{j} \frac{1}{\epsilon_k} \log \left( \frac{1}{\alpha} \right) = \sum_{k=1}^{j} 2^k \log \left( \frac{1}{\alpha} \right) \leq 2^{i+1} \log \left( \frac{1}{\alpha} \right)
\]

(29)

because \( \epsilon_k = 2^{-k} \) by definition. In addition, we have \( t_m = O(\log \frac{1}{\epsilon_m}) \) because \( m = \lceil \log \frac{1}{\epsilon_m} \rceil + 2 \) by definition. The first claim follows.

For the second claim, we have

\[
t_{\text{avg}}^2 = \sum_{j=1}^{m} w_j t_j^2 = \sum_{j=1}^{m} \|\Pi_{C_j} A_j \cdots A_1 |i\rangle |\text{coin} p_i \rangle_B |0\rangle_C |0\rangle_P |1\rangle_F \|^2 t_j^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{i,j} t_j^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \tau_i^2,
\]

(30)

(31)

(32)

where \( w_{i,j} := \|\Pi_{C_j} A_j \cdots A_1 |i\rangle |\text{coin} p_i \rangle_B |0\rangle_C |0\rangle_P |1\rangle_F \|^2 \in [0,1] \) and \( \tau_i := \sum_{j=1}^{m} w_{i,j} t_j^2 \).

Note that \( w_{i,j} \) can be thought of as the probability that \( A \) stops at the end of iteration \( j \) if initialized with arm \( i \); \( \tau_i^2 \) can be thought of as the squared average stopping time of \( A \) if initialized with arm \( i \).

For each fixed \( i \), we consider \( \tau_i^2 \) according to the following three cases.

Case \( i \in S_{\text{right}} \). We have \( \sum_{j=1}^{m} w_{i,j} = 1 \), so \( \tau_i^2 \leq t_m^2 = O(2^{2m} \log^2 (\frac{1}{\alpha})) = O\left(\frac{1}{\alpha} \log^2 (\frac{1}{\alpha})\right) \) because \( m = \lceil \log \frac{1}{\alpha} \rceil + 2 \) by definition.

Case \( i \in S_{\text{middle}} \). We still have \( \tau_i^2 = O\left(\frac{1}{\alpha} \log^2 (\frac{1}{\alpha})\right) \) as in the case \( i \in S_{\text{right}} \), by exactly the same argument. But by the definition of \( S_{\text{middle}} \), we have \( l_1 - p_i \leq \Delta/2 \), so we can also write \( \tau_i^2 = O\left(\frac{1}{(l_1 - p_i)^2} \log^2 (\frac{1}{\alpha})\right) \).

Case \( i \in S_{\text{left}} \). For \( i \in S_{\text{left}} \), let \( j \in [m-1] \) be such that \( l_1 - 2\epsilon_j \leq p_i < l_1 - \epsilon_j \) as in the proof of Lemma 1.

We know that after the \( (j+1) \)st iteration, the state is \( (ma = \alpha/n) \)-close to the state in (26) on which the algorithm terminates. Therefore, the probability \( w_{i,j+1} \) of terminating after the \( (j+1) \)th iteration is \( 1 - O((\alpha/n)^2) \). It can also be seen that the probability \( w_{i,j+r} \) of terminating after the \( (j+r) \)th iteration is \( 1 - O((\alpha/n)^2) \cdot O((\alpha/n)^2(r-1)) \). Hence

\[
\tau_i^2 \leq t_{j+1}^2 + O\left( \sum_{r=2}^{m-j} \left( \frac{\alpha}{n} \right)^{2(r-1)} t_j^{2+r} \right) = O(t_{j+1}^2) = O\left( \frac{\log^2 (\frac{1}{\epsilon_j})}{\epsilon_j+1} \right) = O\left( \frac{\log^2 (\frac{1}{\epsilon_j})}{(l_1 - p_i)^2} \right),
\]

(33)

where we used \( \epsilon_{j+1} = \epsilon_j/2 \geq (l_1 - p_i)/4 \) for the last inequality.

Substituting the above results into (32) tells us that \( t_{\text{avg}}^2 \) is of order

\[
\frac{1}{n} \left( \frac{|S_{\text{right}}|}{\Delta^2} + \sum_{i \in S_{\text{left}} \cup S_{\text{middle}}} \frac{1}{(l_1 - p_i)^2} \right) \cdot \log^2 (\frac{1}{\alpha})
\]

as desired.
Proof of Lemma 4

We set the approximation parameter in $A$ to be $\alpha = c\delta$ for some constant $c < 0.05$ to be determined later. Then $\alpha < 0.05$.

We apply VTAA (Theorem 7) on $A$. This gives an algorithm that outputs a state $|\psi_{\text{suc}}\rangle$ that is $(\frac{n}{n} = \frac{c\delta}{n})$-close to the (normalized) state proportional to

$$
\Pi_F |\psi(A)\rangle = \frac{1}{\sqrt{n}} \left( \sum_{i \in S_{\text{right}}} |i\rangle_I |\text{coin } p_i\rangle_B |\psi_i\rangle_{C,P} |1\rangle_F + \sum_{i \in S_{\text{middle}}} \alpha_{i,1} |i\rangle_I |\text{coin } p_{i,1}\rangle_B |\psi_{i,1}\rangle_{C,P} |1\rangle_F \right)
$$

with success probability at least $1/2$ and a bit indicating success or failure. Now, we repeat the entire procedure $O(\log \frac{1}{\delta})$ times to prepare $|\psi_{\text{suc}}\rangle$ at least once with probability $\geq 1 - \delta/2$. Once $|\psi_{\text{suc}}\rangle$ has been successfully prepared, as indicated by the algorithm, we measure its index register $J$. This procedure outputs an arm index in $S_{\text{right}} \cup S_{\text{middle}}$ with probability $\geq (1 - \delta/2) \cdot (1 - 2c\delta/n)$ which is $\geq 1 - \delta$ for $c \leq 1/4$ sufficiently small. So, as we also need $c < 0.05$, we choose $c = 0.01$.

We call this procedure Amplify$(A, \delta)$.

Let us consider the query complexity of Amplify$(A, \delta)$. We have

$$
t_{m+1} = O\left( \frac{1}{\Delta} \log \left( \frac{1}{\alpha} \right) \right) = O\left( \frac{1}{\Delta} \log \left( n \log \left( \frac{1}{\Delta} \right) \right) \right)
$$

because $\alpha = \frac{c}{2(\log(1/\Delta)) + 2n^{1/2}}$ by definition. We also have

$$
p_{\text{suc}} \geq p'_{\text{suc}} - 2\alpha n \geq \frac{|S_{\text{right}}|}{n} - \frac{0.1}{n} > \frac{|S_{\text{right}}|}{2n},
$$

where we used the assumption $|S_{\text{right}}| > 0$ for the last inequality. Lastly, $t_{\text{avg}}$ is of order given in (12) (reproduced in (34) above). Therefore, substituting all these bounds into (19) of Theorem 7 we see that Amplify$(A, \delta)$ has query complexity of order

$$
\left( \frac{1}{\Delta^2} + \frac{1}{|S_{\text{right}}|} \sum_{i \in S_{\text{right}}} \frac{1}{(1 - p_i)^2} \right) \cdot \log \left( \frac{n}{\delta} \log \left( \frac{1}{\Delta} \right) \right) \cdot \log \left( \frac{n}{\Delta} \right) \cdot \log \left( \frac{n}{\Delta} \right) \cdot \log \left( \frac{1}{\delta} \right).
$$

We also apply VTAE (Theorem 7) with multiplicative accuracy $\epsilon$ and confidence $\delta$ on $A$. This gives an algorithm, Estimate$(A, \epsilon, \delta)$, that outputs an estimate $r$ of $p_{\text{suc}}$ with multiplicative accuracy $\epsilon$ (i.e., $|r - p_{\text{suc}}| < \epsilon p_{\text{suc}}$) with probability $\geq 1 - \delta$. Combining $|r - p_{\text{suc}}| < \epsilon p_{\text{suc}}$ with $|p_{\text{suc}} - p'_{\text{suc}}| \leq 2\alpha n < \frac{0.1}{n}$ gives

$$
(1 - \epsilon)(p'_{\text{suc}} - \frac{0.1}{n}) < r < (1 + \epsilon)(p'_{\text{suc}} + \frac{0.1}{n})
$$

as claimed.

The query complexity of Estimate$(A, \epsilon, \delta)$ is given by (38) times

$$
\frac{1}{\epsilon} \log^2(t_{m+1}) \log \left( \log \left( \frac{n^{1/2}}{\delta} \right) \right) = O\left( \frac{1}{\epsilon} \text{poly} \left( \log \left( \frac{n}{\delta} \Delta^2 \right) \right) \right)
$$

according to Theorem 7 and (36).

Proof of Lemma 5

From the first claim of Lemma 5 we see that the probability of $E^c$ is at most $\sum_{i=0}^{\infty} 2^{-i} = \delta/2$, where the geometric series arises because of Line 6. Henceforth, we assume $E$.

Consider the first claim. For given intervals $I_2, I_1$, let us write

$$
gap(I_2, I_1) := \min I_1 - \max I_2.
$$

At the end of iteration $i \geq 1$ (i.e., after Line 6), we have $|I_k| = (3/5)^i$ by the first claim of Lemma 5. At the end of iteration $\left[ \log_{5/3} \left( \frac{1}{\Delta_2} \right) \right] + 3$, we have $|I_k| < \Delta_2/4$, so gap$(I_2, I_1) > \Delta_2 - 2\Delta_2/4 = \Delta_2/2 > 2 |I_1|$ because $p_k \in I_k$. Therefore the while loop must break at this point if it has not done so earlier. For the returned $I_k$, we clearly have $p_k \in I_k$ because $E$ holds, and gap$(I_2, I_1) > 2 |I_1|$ because the while loop has broken. During the while loop, because $|I_k|$ decreases from iteration to iteration, we always have $|I_k| \geq (3/5)^i \log_{5/3} \left( \frac{1}{\Delta_2} \right) + 3 \geq \Delta_2/8$. Note that $|I_1| = |I_2|$ because, at each iteration of the while loop, the Shrink subroutine always shrinks intervals by the same factor of 3/5 and $|I_1| = |I_2| = 1$ initially.

Now, consider the second claim. From the first claim, we know that the while loop breaks at or before the end of iteration $\left[ \log_{5/3} \left( \frac{1}{\Delta_2} \right) \right] + 3$, and we always have $1/\delta_i = O(2^{\log_{5/3} \left( \frac{1}{\Delta_2} \right) + 3}) = O(\Delta_2^2/\delta)$, where $\delta_i = \delta/2^{i+1}$ is the confidence parameter in Shrink at iteration $i$. Therefore, using the second claim of Lemma 5, the total number of queries used is at most

$$
O(\log(\Delta_2^2)) \cdot O\left( \sqrt{H} \cdot \text{poly} \left( \log \left( \frac{n}{\Delta_2} \cdot \frac{\Delta_2^2}{\delta} \right) \right) \right),
$$

which is $O(\sqrt{H} \cdot \text{poly} \left( \log \left( \frac{n}{\Delta_2^2} \right) \right))$ as desired.
Proof of \textbf{Lemma 5} 

Throughout, we fix $k \in \{0, 1\}$.

For the first claim, it is clear that $|J| = 3 |I| / 5$ because all the intervals appearing in Lines (11) and (14) have length $3 \varepsilon$. Our proof that $p_k \in J$ with high probability is similar to that in Section 4 of \cite{Lin and Tong (2020)}, so we only present a brief sketch below.

Let us write $x_j = a + j \varepsilon$ for $j = 0, \ldots, 5$, so that $x_0 = a$ and $x_5 = b$. Let $E$ be the event that both Estimates in Lines 7 and 8 return the correct result. The probability of $E^c$ is at most $\delta$ so we restrict to the case of $E$ in the following paragraph.

For $j \in \{1, 2\}$, we can use (13) in Lemma 3 to see that if $p_k \leq x_j$, then $B_j = 0$ because $r_j \leq (1+0.1)(\frac{3}{n+1} + \frac{3}{n+1}) < \frac{60.5}{n+1}$, whereas if $p_k \geq x_{j+2}$, then $B_j = 1$ because $r_j \geq (1-0.1)(\frac{3}{n+1} - \frac{3}{n+1}) > \frac{0.5}{n+1}$. Here we use the fact $k \in \{1, 2\}$. By considering the contrapositive of the previous two if-then statements, we establish the first claim.

For more details, we refer the reader to Section 4 of \cite{Lin and Tong (2020)}, in particular its Table 2 and Algorithm 1. Note that in the case of $(B_1, B_2) = (0, 1)$, we could have shrunk the interval to $[a+2 \varepsilon, a+3 \varepsilon]$ and still maintained $p_k \in J$, as is done in \cite{Lin and Tong (2020)}. However, it is important for us to keep the shrinkage factor $(3/5)$ the same in all cases because we use this to prove correctness in \textbf{Lemma 4}.

We now prove the second claim. Since we run Estimate with constant multiplicative error $\varepsilon = 0.1$, its query complexity is of order (45), which is

$$\frac{1}{\Delta^2} + \frac{1}{S_{\text{right}}} \sum_{i \in S_{\text{left}} \cup S_{\text{middle}}} \frac{1}{(l_1 - p_i)^2}$$

up to polylog factors, where we recall that $\Delta = l_1 - l_2$. In addition, we recall

$$S_{\text{left}} \cup S_{\text{middle}} = \{i : p_i < l_1 - \Delta/8\}$$

from (10) and (11). Note that $|S_{\text{right}}| > 0$ because we appended an arm with bias $p_0 = 1$.

By assumption, $|I| \geq \Delta_2/8$. So, in view of Lines 5 and 6 we have $\Delta = 2 \varepsilon = 2 |I| / 5 \geq \Delta_2 / 20$. Therefore $1/\Delta^2 = O(1/\Delta^2_2)$.

We also need to compare $p_1 - p_i$ with $l_1 - p_i$ for $i \in S_{\text{left}} \cup S_{\text{middle}}$. By definition, we have $p_i < l_1 - \Delta/8$, so $l_1 - p_i > \Delta/8$.

Note that we also have $|p_k - l_1| \leq |I| = 5 \Delta/2$ because $p_k \in I$ by assumption and $l_1 \in I$ by definition. If $k = 1$, this says $|p_1 - l_1| \leq 5 \Delta/2$. If $k = 2$, this says $|p_2 - l_1| \leq 5 \Delta/2$, but we can still bound

$$|p_1 - l_1| \leq \Delta_2 + |p_2 - l_1| \leq 20 \Delta + 5 \Delta/2 < 25 \Delta.$$

So regardless of whether $k = 1$ or $k = 2$, we have that $|p_1 - l_1| < 25 \Delta$. Therefore

$$\frac{p_1 - p_i}{l_1 - p_i} = 1 + \frac{p_1 - l_1}{l_1 - p_i} < 1 + \frac{25 \Delta}{\Delta/8} = 201,$$

and so $1/(l_1 - p_i)^2 = O(1/(p_1 - p_i)^2)$. Hence we have established the second claim.

**Corollaries for the Fixed-budget Setting**

As mentioned near the end of the main body, by using a reduction similar to that from Monte Carlo to Las Vegas algorithms, we can construct a fixed-budget algorithm from our fixed-confidence one. For completeness, we state and prove the following result:

\textbf{Lemma 7} (Reduction to fixed confidence). Let $O$ be a multi-armed bandit oracle. Suppose that for any $\delta \in (0, 1)$, we have an algorithm $A_c(\delta)$ that with probability $\geq 1 - \delta$, terminates before using $T_c(\delta)$ queries to $O$ and returns the best-arm index $i^* = 1$. Suppose that we also know $T_c(\delta)$. Then, for any positive integer $T$, we can construct an algorithm $A_c(T)$ that returns

$$i^* = 1$$

with probability $\geq \min_{\delta \in (0, 1)} \exp(-|T/T_c(\delta)| D(\frac{1}{2} || \delta))$ using at most $T$ queries to $O$, where $D(p||q)$ is the relative entropy between Bernoulli random variables with bias $p$ and $q$.

**Proof.** Since $T_c(\delta)$ is known, consider the modified version of the fixed-confidence algorithm where the algorithm is forced to halt and return some blank symbol “⊥” if the running time exceeds $T_c(\delta)$. We refer to the modified algorithm as $A'_c(\delta)$. $A'_c(\delta)$ returns the best-arm index $i^* = 1$ with probability $\geq 1 - \delta$ and returns some symbol in $\{2, \ldots, n, \bot\}$ with probability $\leq \delta$.

For any $T$, we construct $A_c(T)$ as follows. Pick some $\delta \in (0, 1)$, run $A'_c(\delta)$ $m := |T/T_c(\delta)|$ times, and take a majority vote over the outcomes. The failure probability can be upper bounded by the probability that $i^*$ is observed fewer than $m/2$ times. The Chernoff bound upper bounds the latter probability by $\exp(-mD(\frac{1}{2} || \delta)) = \exp(-|T/T_c(\delta)| D(\frac{1}{2} || \delta))$. But $\delta$ was arbitrary, so we can take the $\delta$ that minimizes this upper bound.

As a direct corollary of \textbf{Theorem 3} and \textbf{Lemma 7}, we see that when $H$ (therefore $T_c$) is known in advance, for sufficiently large $T$, there is a quantum algorithm using at most $T$ queries that returns the best arm with probability $\geq 1 - \exp(-\Omega(T/\sqrt{H}))$. 


Proof Details of the Quantum Lower Bound

Proof of Theorem 5

For convenience, we reproduce the statement of the result:

**Theorem 5.** Let \( p \in (0, 1/2) \). For any biases \( p_i \in [p, 1-p] \), any quantum algorithm that identifies the best arm requires \( \Omega(\sqrt{n}) \) queries to the multi-armed bandit oracle \( \mathcal{O} \).

**Proof.** We use the adversary method and consider the following \( n \) different multi-armed bandit oracles.

In the 1st bandit, we assign bias \( p_1 \) to arm \( i \). Let \( \eta > 0 \) be a constant to be determined later. In the \( x \)-th bandit, \( x \in \{2, \ldots, n\} \), we assign bias \( p_i' := p_1 + \eta \) to arm \( x \) and \( p_i \) to arm \( i \) for all \( i \neq x \). A best-arm identification algorithm must output arm \( x \) on assignment \( x \) for all \( x \in [n] \) with probability \( \geq 1 - \delta \).

Following the adversary method, we consider the sum

\[
s_k := \sum_{x > 1} \frac{1}{\Delta_x^2} \langle \psi_x^{(k)} | \psi_1^{(k)} \rangle
\]

for \( x \in [n] \), where \( \Delta_x' := p_i' - p_x \). Clearly

\[
s_0 = \sum_{x > 1} \frac{1}{\Delta_x^2}.
\]

We also have

\[
s_T \leq \sum_{x > 1} \frac{1}{\Delta_x^2} \cdot 2\sqrt{\delta(1 - \delta)}.
\]

Next, we bound the difference \( |s_{k+1} - s_k| \). For \( i > 1 \), we let

\[
A_i := \begin{pmatrix}
\sqrt{1-p_i} & \sqrt{p_i} \\
\sqrt{p_i} & -\sqrt{1-p_i}
\end{pmatrix},
\]

while

\[
A_1 := \begin{pmatrix}
\sqrt{1-p_1} & \sqrt{p_1} \\
\sqrt{p_1} & -\sqrt{1-p_1}
\end{pmatrix},
\]

where we recall \( p_i' = p_i + \eta \) by definition.

Now, let us write

\[
|\psi_x^{(k)}\rangle = \sum_{z,i,b} \alpha_{x,z,i,b} |z,i,b\rangle, \quad |\psi_1^{(k)}\rangle = \sum_{z,i,b} \alpha_{1,z,i,b} |z,i,b\rangle.
\]

Then

\[
|\psi_x^{(k+1)}\rangle = \mathcal{O}_x |\psi_x^{(k)}\rangle = \sum_{z,b} \alpha_{x,z,x,b} |z,x\rangle A_1 |b\rangle + \sum_{i \neq x} \sum_{z,b} \alpha_{x,z,i,b} |z,i\rangle A_1 |b\rangle
\]

and similarly

\[
|\psi_1^{(k+1)}\rangle = \mathcal{O}_1 |\psi_1^{(k)}\rangle = \sum_{z,b} \alpha_{1,z,x,b} |z,x\rangle A_x |b\rangle + \sum_{i \neq x} \sum_{z,b} \alpha_{1,z,i,b} |z,i\rangle A_1 |b\rangle.
\]

Then

\[
|s_{k+1} - s_k| \leq \sum_{x > 1} \frac{1}{\Delta_x^2} \left| \langle \psi_x^{(k)} | \mathcal{O}_x^\dagger \mathcal{O}_1 |\psi_1^{(k)}\rangle - \langle \psi_x^{(k)} | \psi_1^{(k)} \rangle \right|.
\]

Using (53) and (54), and after cancellations, we find that

\[
\langle \psi_x^{(k)} | \mathcal{O}_x^\dagger \mathcal{O}_1 |\psi_1^{(k)}\rangle - \langle \psi_x^{(k)} | \psi_1^{(k)} \rangle = \sum_{z,b,b'} \alpha^*_{x,z,x,b} \alpha_{1,z,x,b'} \langle b| (A_1^\dagger A_x - \mathbb{I}) |b'\rangle.
\]

With

\[
\begin{pmatrix}
u_x & u_x \\ -v_x & u_x
\end{pmatrix} := A_1^\dagger A_x - \mathbb{I}
\]

\[
= \begin{pmatrix}
\sqrt{(1-p_1')(1-p_x)} + \sqrt{p_1'(1-p_x)} & \sqrt{(1-p_1')(1-p_x)} - \sqrt{p_1'(1-p_x)} \\
-\sqrt{(1-p_1')(1-p_x)} + \sqrt{p_1'(1-p_x)} & \sqrt{(1-p_1')(1-p_x)} + \sqrt{p_1'(1-p_x)} - 1
\end{pmatrix},
\]

Using (53) and (54), and after cancellations, we find that

\[
\langle \psi_x^{(k)} | \mathcal{O}_x^\dagger \mathcal{O}_1 |\psi_1^{(k)}\rangle - \langle \psi_x^{(k)} | \psi_1^{(k)} \rangle = \sum_{z,b,b'} \alpha^*_{x,z,x,b} \alpha_{1,z,x,b'} \langle b| (A_1^\dagger A_x - \mathbb{I}) |b'\rangle.
\]
we have

$$|s_{k+1} - s_k| \leq \sum_{x > 1} \sum_{z, b} \frac{|u_x|}{\Delta_x^2} |\alpha_{x, z, x, b}| |\alpha_{1, z, x, b}| + \sum_{x > 1} \sum_{z, b \neq b'} \frac{|v_x|}{\Delta_x^2} |\alpha_{x, z, x, b}| |\alpha_{1, z, x, b}|. \quad(58)$$

Clearly, $|u_x| = 1 - \sqrt{(1 - p_1)(1 - p_x)} - \sqrt{p_1 p_x} \leq 1 - (1 - p_1') - p_x = p_1' - p_x = \Delta_x'$. It can also be seen that $|v_x| \leq \Delta_x' / c(p - \eta)$, where $c(x) := 2\sqrt{x(1 - x)}$ is a monotone increasing function when $x \in [0, 1/2]$. For completeness, we prove the latter inequality as an auxiliary [Lemma 8] immediately after this proof.

We can establish the following bounds using the Cauchy-Schwarz inequality:

$$\sum_{x > 1} \sum_{z, b} \frac{|u_x|}{\Delta_x^2} |\alpha_{x, z, x, b}| |\alpha_{1, z, x, b}| \leq \sqrt{\sum_{x > 1} \sum_{z, b} \frac{|u_x|^2}{\Delta_x^2} |\alpha_{x, z, x, b}|^2} \cdot \sqrt{\sum_{x > 1} \sum_{z, b} \frac{|\alpha_{1, z, x, b}|^2}{\Delta_x^2}} \quad(59)$$

and

$$\sum_{x > 1} \sum_{z, b \neq b'} \frac{|v_x|}{\Delta_x^2} |\alpha_{x, z, x, b}| |\alpha_{1, z, x, b}| = \sum_{b \neq b'} \sum_{x > 1} \sum_{z, b} \frac{|v_x|}{\Delta_x^2} |\alpha_{x, z, x, b}| |\alpha_{1, z, x, b}| \leq \sum_{b \neq b'} \sqrt{\sum_{x > 1} \sum_{z, b} \frac{|v_x|^2}{\Delta_x^2} |\alpha_{x, z, x, b}|^2} \cdot \sqrt{\sum_{x > 1} \sum_{z, b} \frac{|\alpha_{1, z, x, b}|^2}{\Delta_x^2}} \quad(60)$$

Therefore, we find that

$$|s_{k+1} - s_k| \leq \left(1 + \frac{2}{c(p - \eta)}\right) \sqrt{\sum_{x > 1} \frac{1}{\Delta_x^2}}. \quad(61)$$

Hence, from Eqs. (58), (59), and (61), we find that

$$T \geq 1 - 2\sqrt{\delta(1 - \delta)} \frac{1 + 2/c(p - \eta)}{1 + 2/c(p - \eta)} \sqrt{\sum_{x > 1} \frac{1}{\Delta_x^2}}. \quad(62)$$

We then set $\eta = p(p_1 - p_2)/2$. Now, it can be seen that

$$c(p - \eta) = c\left(1 + \frac{p_1 - p_2}{2}\right) \geq c(p/2) \quad(63)$$

because $p \leq 1/2$ and $p_1 - p_2 \leq 1$. Moreover, for $x > 1$,

$$\Delta_x' = p_1 + \eta - p_x = \frac{p_1}{2}(p_1 - p_2) + (p_1 - p_x) \leq \left(1 + \frac{p}{2}\right)(p_1 - p_x) \leq \frac{5}{4} \Delta_x \quad(64)$$

because $p_x \leq p_2$ and $p \leq 1/2$. Therefore, we find that

$$T \geq \frac{4}{5} \frac{1 - 2\sqrt{\delta(1 - \delta)}}{1 + 2/c(p/2)} \sqrt{\sum_{x > 1} \frac{1}{\Delta_x^2}}. \quad(65)$$

and hence $T = \Omega\left(\sqrt{\sum_{i=1}^n \frac{1}{\Delta_i^2}}\right)$. □

**Lemma 8.** Suppose that $p_1, p_2 \in [p, 1 - p]$ where $0 < p \leq 1/2$. Then

$$|\sqrt{(1 - p_1)p_2} - \sqrt{(1 - p_2)p_1}| \leq \frac{|p_1 - p_2|}{2\sqrt{p(1 - p)}}, \quad(66)$$

and the term in the denominator is optimal.
Proof. Note that

\[ \sqrt{(1-p_1)p_2} - \sqrt{(1-p_2)p_1} = \frac{(1-p_1)p_2 - (1-p_2)p_1}{\sqrt{(1-p_1)p_2} + \sqrt{(1-p_2)p_1}} \tag{67} \]

\[ = \frac{-(p_1 - p_2)}{\sqrt{(1-p_1)p_2} + \sqrt{(1-p_2)p_1}} \tag{68} \]

Therefore, it suffices to prove

\[ \sqrt{(1-p_1)p_2} + \sqrt{(1-p_2)p_1} \geq 2\sqrt{p(1-p)}. \tag{69} \]

Since \( p_1, p_2 \in [p, 1-p] \), we have

\[ (p_1-p)(p_1-(1-p)) \leq 0 \tag{70} \]
\[ (p_2-p)(p_2-(1-p)) \leq 0 \tag{71} \]
\[ |2p_1 - 1| \leq 1 - 2p \tag{72} \]
\[ |2p_2 - 1| \leq 1 - 2p. \tag{73} \]

Eqs. (70) and (71) are equivalent to

\[ p_1 - p_1^2 \geq p(1-p), \quad p_2 - p_2^2 \geq p(1-p). \tag{74} \]

Eqs. (72) and (73) imply

\[ 4p_1p_2 - 2p_1 - 2p_2 + 1 = (2p_1 - 1)(2p_2 - 1) \leq (2p - 1)^2 = 4p^2 - 4p + 1, \tag{75} \]

which gives

\[ p_1 + p_2 - 2p_1p_2 \geq 2p - 2p^2. \tag{76} \]

Now, we have

\[ \left( \sqrt{(1-p_1)p_2} + \sqrt{(1-p_2)p_1} \right)^2 = (1-p_1)p_2 + (1-p_2)p_1 + 2\sqrt{(1-p_1)p_2(1-p_2)p_1} \tag{77} \]
\[ = p_1 + p_2 - 2p_1p_2 + 2\sqrt{p_1(1-p_1)\sqrt{p_2(1-p_2)}} \tag{78} \]
\[ \geq 2p - 2p^2 + 2p(1-p) = (2\sqrt{p(1-p)})^2, \tag{79} \]

where the inequality comes from (74) and (76). Therefore, we have established (69). Note that this is optimal as taking \( p_1 = p_2 = p \) makes the two sides in (69) equal.