Topological and topological linear properties of the Sugeno-Lorentz spaces

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Abstract

The properties of spaces of Sugeno integrable functions are quite different from those of ordinary spaces of Lebesgue integrable functions. In our previous research the completeness and separability of the Sugeno-Lorentz spaces were discussed in terms of the characteristic of nonadditive measures without a detailed study of their topologies. The purpose of the paper is to further advance our study of the Sugeno-Lorentz spaces by investigating some fundamental topological and topological linear properties of the Sugeno-Lorentz spaces.

Keywords: Nonadditive measure; Sugeno integral; Sugeno-Lorentz space; Autocontinuity; Open sphere condition

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1. Introduction

The Sugeno integral was introduced by Sugeno [24] under the name of fuzzy integral and has many important and interesting applications; see [2, 9, 10, 26].

Given any nonadditive measure \( \mu \) on a measurable space \((X, \mathcal{A})\) and any Orlicz function \( \Phi \), the Orlicz space \( L_\Phi(\mu) \) and \( \Phi \)-mean convergence were defined in [28] by using the Sugeno integral. In particular, if \( \Phi(t) := t^p \), where \( 0 < p < \infty \), then the space \( L_\Phi(\mu) \) are reduced to the space \( L^p(\mu) \) consisting of all \( \mathcal{A} \)-measurable real-valued functions on \( X \) such that \( |f|^p \) are Sugeno integrable with respect to \( \mu \). Then, as a particular case of [28, Theorem 2] it follows that \( L^p(\mu) = L^1(\mu) \) and that \( p \)-th order convergence is equivalent to convergence in \( \mu \)-measure. This observation suggests that the properties of spaces of Sugeno integrable functions are quite different from those of ordinary spaces of Lebesgue integrable functions.

In [16], for any \( 0 < p < \infty \) and \( 0 < q < \infty \), the Sugeno-Lorentz prenorm \((\cdot)_{p,q}\) and the Sugeno-Lorentz space \( Su(\mu) \) were defined as the space of all \( \mathcal{A} \)-measurable functions on \( X \) such that \( |f|^q \) are Sugeno integrable with respect to the nonadditive measure \( \mu^{q/p} \). Our definition of the Sugeno-Lorentz spaces and the Sugeno-Lorentz prenorms agrees with ordinary Lorentz spaces and Lorentz norms [18, 19] if \( \mu \) is \( \sigma \)-additive and the Sugeno integral is replaced with the Lebesgue integral; see [16]. As a starting point of research, the completeness

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and separability of the Sugeno-Lorentz spaces were discussed in [16] in terms of the characteristics of nonadditive measures without a detailed study of their topologies. In this paper, we focus on the topological and topological linear properties of the Sugeno-Lorentz spaces.

This paper is organized as follows. Section 2 sets up notation and terminology. It also contains a discussion of an equivalence relation in the space of measurable functions and the topology generated by a distance function. In Section 3, given a nonadditive measure \( \mu \) on a measurable space \((X, \mathcal{A})\) the Sugeno-Lorentz space and the Sugeno-Lorentz prenorm \((\cdot)_{p,q}\) are defined for every \(0 < p < \infty\) and \(0 < q < \infty\). This section also contains a summary of results of [16] that will be used later. Topological properties of the Sugeno-Lorentz spaces are discussed in Section 4. First, the Sugeno-Lorentz topology is defined as the topology generated by the prenorm \((\cdot)_{p,q}\) and it is shown that any sphere determined by \((\cdot)_{p,q}\) is open with respect to this topology if and only if \(\mu\) is autocontinuous from above. It is also shown that the Sugeno-Lorentz topology is the relative topology induced from the Dunford-Schwartz topology introduced in [14]. This fact implies that the Sugeno-Lorentz topology does not depend on \(p\) and \(q\), and hence, it is only one regardless of \(p\) and \(q\). In Section 5 some fundamental topological linear properties of the Sugeno-Lorentz spaces are deduced from the corresponding properties of the space of all measurable functions with the Dunford-Schwartz topology. Section 6 provides a summary of our results.

2. Preliminaries

Throughout the paper, \((X, \mathcal{A})\) is a measurable space, that is, \(X\) is a nonempty set and \(\mathcal{A}\) is a \(\sigma\)-field of subsets of \(X\). Let \(\mathbb{R}\) denote the set of the real numbers and \(\mathbb{N}\) the set of the natural numbers. Let \(\mathbb{R} := [-\infty, \infty]\) be the set of the extended real numbers with usual total order and algebraic structure. Assume that \((\pm \infty) \cdot 0 = 0 \cdot (\pm \infty) = 0\) since this proves to be convenient in measure and integration theory.

For any \(a, b \in \mathbb{R}\), let \(a \lor b := \max\{a, b\}\) and \(a \land b := \min\{a, b\}\) and for any \(f, g : X \to \mathbb{R}\), let \((f \lor g)(x) := f(x) \lor g(x)\) and \((f \land g)(x) := f(x) \land g(x)\) for every \(x \in X\). Let \(\mathcal{F}_0(X)\) denote the set of all \(\mathcal{A}\)-measurable real-valued functions on \(X\). Then \(\mathcal{F}_0(X)\) is a real linear space with usual pointwise addition and scalar multiplication. For any \(f, g \in \mathcal{F}_0(X)\), the notation \(f \leq g\) means that \(f(x) \leq g(x)\) for every \(x \in X\). Let \(\mathcal{F}_0^+(X) := \{f \in \mathcal{F}_0(X) : f \geq 0\}\). A simple function is a function taking only a finite number of real numbers. Let \(\mathcal{S}(X)\) denote the set of all \(\mathcal{A}\)-measurable simple functions on \(X\).

For a sequence \(\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}\) and \(a \in \mathbb{R}\), the notation \(a_n \uparrow a\) means that \(\{a_n\}_{n \in \mathbb{N}}\) is nondecreasing and \(a_n \to a\), and \(a_n \downarrow a\) means that \(\{a_n\}_{n \in \mathbb{N}}\) is nonincreasing and \(a_n \to a\). For a sequence \(\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}\) and \(A \in \mathcal{A}\), the notation \(A_n \uparrow A\) means that \(\{A_n\}_{n \in \mathbb{N}}\) is nondecreasing and \(A = \bigcup_{n=1}^{\infty} A_n\), and \(A_n \downarrow A\) means that \(\{A_n\}_{n \in \mathbb{N}}\) is nonincreasing and \(A = \bigcap_{n=1}^{\infty} A_n\). The characteristic function of a set \(A\), denoted by \(\chi_A\), is the function on \(X\) such that \(\chi_A(x) = 1\) if \(x \in A\) and \(\chi_A(x) = 0\) otherwise. Given two sets \(A\) and \(B\), let \(A \Delta B := (A \setminus B) \cup (B \setminus A)\) and \(A^c := X \setminus A\). Let \(2^X\) denote the collection of all subsets of \(X\).
2.1. Nonadditive measures

A nonadditive measure is a set function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subseteq B$. This type of set function is also called a monotone measure [26], a capacity [3], or a fuzzy measure [21, 24] in the literature. Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu : \mathcal{A} \to [0, \infty]$.

Let $\mu \in \mathcal{M}(X)$. We say that $\mu$ is order continuous [7] if $\mu(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, continuous from above if $\mu(A_n) \to \mu(A)$ whenever $A_n \downarrow A$, continuous from below if $\mu(A_n) \to \mu(A)$ whenever $A_n \uparrow A$, and null-continuous [25] if $\mu(\bigcup_{n=1}^{\infty} N_n) = 0$ whenever $\{N_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ is nondecreasing and $\mu(N_n) = 0$ for every $n \in \mathbb{N}$. The order continuity follows from the continuity form above, while the null-continuity follows from the continuity from below.

Following the terminology used in [26], $\mu$ is called weakly null-additive if $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{A}$ and $\mu(A) = \mu(B) = 0$, null-additive if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ and $\mu(B) = 0$, autocontinuous from above if $\mu(A \cup B_n) \to \mu(A)$ whenever $A, B_n \in \mathcal{A}$ and $\mu(B_n) \to 0$, and autocontinuous from below if $\mu(A \cup B_n) \to \mu(A)$ whenever $A, B_n \in \mathcal{A}$ and $\mu(B_n) \to 0$. Furthermore, we say that $\mu$ satisfies the pseudometric generating property ((p.g.p.) for short) [6] if $\mu(A_n \cup B_n) \to 0$ whenever $A_n, B_n \in \mathcal{A}$ and $\mu(A_n) \vee \mu(B_n) \to 0$. Every nonadditive measure satisfying the (p.g.p.) is weakly null-additive. If $\mu$ is autocontinuous from above or below, then it is null-additive, hence weakly null-additive.

A nonadditive measure $\mu$ is called subadditive if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for every disjoint $A, B \in \mathcal{A}$, relaxed subadditive if there is a constant $K \geq 1$ such that $\mu(A \cup B) \leq K \{\mu(A) + \mu(B)\}$ for every disjoint $A, B \in \mathcal{A}$ (in this case $\mu$ is called $K$-relaxed subadditive). Every subadditive nonadditive measure is relaxed subadditive. If $\mu$ is relaxed subadditive, then it satisfies the (p.g.p.). See [4, 20, 26] for further information on nonadditive measures.

Remark 2.1. The relaxed subadditivity is also called the quasi-subadditivity according to the terminology used in metric space theory.

2.2. The Sugeno integral

The Sugeno integral is one of important integrals that is widely used in nonadditive measure theory and its applications. The Sugeno integral [21, 24] is defined by

$$\text{Su}(\mu, f) := \sup_{t \in [0, \infty)} t \land \mu(\{f > t\})$$

for every $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{F}_0^+(X)$. In the above definition the nonincreasing distribution function $\mu(\{f > t\})$, where $\{f > t\} := \{x \in X : f(x) > t\}$, may be replaced with $\mu(\{f \geq t\})$ and the interval of the range in which the variable $t$ moves may be replaced with $[0, \infty]$ or $(0, \infty)$ without changing the integral value.

The following properties of the Sugeno integral are easy to prove and used without explicitly mentioning; see also [26]. Let $f, g \in \mathcal{F}_0^+(X), A \in \mathcal{A}, \alpha \geq 0$, and $0 < p < \infty$.

- Monotonicity: If $f \leq g$ then $\text{Su}(\mu, f) \leq \text{Su}(\mu, g)$.
- Generativity: $\text{Su}(\mu, \alpha \chi_A) = \alpha \land \mu(A)$. 

3
• Truncated subhomogeneity: \( \text{Su}(\mu, \alpha f) \leq \max\{1, \alpha\} \text{Su}(\mu, f) \).

• Exponentiation: \( \text{Su}(\mu, f^p) = \text{Su}(\mu^{1/p}, f)^p \).

• Integrability: \( \text{Su}(\mu, f) \leq t \lor \mu(\{|f| > t\}) \) for every \( t \in [0, \infty] \), so that \( \text{Su}(\mu, f) < \infty \) if and only if there is \( t_0 \in [0, \infty) \) such that \( \mu(\{|f| > t_0\}) < \infty \).

Note that the Sugeno integral is neither additive nor positively homogeneous in general. The relaxed subadditivity of the Sugeno integral can be characterized in terms of nonadditive measures; see [16, Proposition 2.2].

Proposition 2.2. Let \( \mu \in \mathcal{M}(X) \). The following conditions are equivalent.

(i) \( \mu \) is \( K \)-relaxed subadditive for some \( K \geq 1 \).

(ii) The Sugeno integral is \( K \)-relaxed subadditive, that is, for any \( f, g \in \mathcal{F}_0(X) \) it follows that

\[
\text{Su}(\mu, f + g) \leq K \{\text{Su}(\mu, f) + \text{Su}(\mu, g)\}.
\]

In particular, the Sugeno integral is subadditive if and only if \( \mu \) is subadditive.

Remark 2.3. The subadditivity of the Sugeno integral was proved in [11, Proposition 5] for nonadditive measures on the discrete space \((\mathbb{N}, 2^\mathbb{N})\).

2.3. Convergence in measure of measurable functions

The concept of convergence in measure is not quite intuitive, but it has some advantages in analysis. Let \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0(X) \) and \( f \in \mathcal{F}_0(X) \). We say that \( \{f_n\}_{n \in \mathbb{N}} \) converges in \( \mu \)-measure to \( f \), denoted by \( f_n \xrightarrow{\mu} f \), if \( \mu(\{|f_n - f| > \varepsilon\}) \to 0 \) for every \( \varepsilon > 0 \). This mode of convergence requires that the differences between the elements \( f_n \) of the sequence and the limit function \( f \) should become small in some sense as \( n \) increases. The following definition involves only the elements of the sequence. We say that \( \{f_n\}_{n \in \mathbb{N}} \) is Cauchy in \( \mu \)-measure if for any \( \varepsilon > 0 \) and \( \delta > 0 \) there is \( n_0 \in \mathbb{N} \) such that \( \mu(\{|f_m - f_n| > \varepsilon\}) < \delta \) whenever \( m, n \in \mathbb{N} \) and \( m, n \geq n_0 \).

See a survey paper [17] for further information on various modes of convergence of measurable functions in nonadditive measure theory.

2.4. Equivalence relation and quotient space

The quotient space of \( \mathcal{F}_0(X) \) is constructed by an equivalence relation determined by a nonadditive measure \( \mu \). The proof of the following statements is routine and left it to the reader.

• Assume that \( \mu \) is weakly null-additive. Given \( f, g \in \mathcal{F}_0(X) \), define the binary relation \( f \sim g \) on \( \mathcal{F}_0(X) \) by \( \mu(\{|f - g| > c\}) = 0 \) for every \( c > 0 \) so as to become an equivalence relation on \( \mathcal{F}_0(X) \). For every \( f \in \mathcal{F}_0(X) \) the equivalence class of \( f \) is the set of the form \( \{g \in \mathcal{F}_0(X): f \sim g\} \) and denoted by \([f]\). Then the quotient space of \( \mathcal{F}_0(X) \) is defined by \( \mathcal{F}_0(X) := \{[f]: f \in \mathcal{F}_0(X)\} \) and the mapping \( \kappa: \mathcal{F}_0(X) \to \mathcal{F}_0(X) \) defined by \( \kappa(f) := [f] \) for every \( f \in \mathcal{F}_0(X) \) is known as the quotient map.
• Assume that μ is weakly null-additive. Given equivalence classes \([f], [g] \in F_0(X)\) and \(\alpha \in \mathbb{R}\), define addition and scalar multiplication on \(F_0(X)\) by 
\([f] + [g] := [f + g]\) and \(\alpha [f] := [\alpha f]\). They are well-defined, that is, they are independent of which member of an equivalence class we choose to define them. Then \(F_0(X)\) is a real linear space.

The space \(F_0(X)\) is exactly the same as the quotient space \(F_0(X)/N\), where 
\(N := \{f \in F_0(X) : \mu(\{|f| > c\}) = 0 \text{ for every } c > 0\}\). The binary relation on \(F_0(X)\) defined above may not be transitive unless \(\mu\) is weakly null-additive; see [14, Example 5.1]. In what follows, let \(S(X) := \{[h] : h \in S(X)\}\).

2.5. Prenorms

Let \(V\) be a real linear space. A prenorm on \(V\) is a nonnegative real-valued function \(\| \cdot \|\) defined on \(V\) such that \(\|0\| = 0\) and \(\|−x\| = \|x\|\) for every \(x \in V\). Then the pair \((V, \| \cdot \|)\) is called a prenormed space. A prenorm \(\| \cdot \|\) is called homogeneous if it follows that \(\|αx\| = |α|\|x\|\) for every \(x \in V\) and \(\alpha \in \mathbb{R}\) and truncated subhomogeneous if it follows that \(\|αx\| \leq \max(1, |α|)\|x\|\) for every \(x \in V\) and \(\alpha \in \mathbb{R}\). Following [5], a prenorm \(\| \cdot \|\) is called relaxed if it satisfies a relaxed triangle inequality, that is, there is a constant \(K \geq 1\) such that \(\|x + y\| \leq K(\|x\| + \|y\|)\) for every \(x, y \in V\) (in this case, we say that \(\| \cdot \|\) satisfies \(K\)-relaxed triangle inequality).

To associate with similar characteristics of nonadditive measures, a prenorm \(\| \cdot \|\) is called weakly null-additive if \(\|x + y\| = 0\) whenever \(x, y \in V\) and \(\|x\| = \|y\| = 0\) and null-additive if \(\|x + y\| = \|x\|\) whenever \(x, y \in V\) and \(\|y\| = 0\).

Let \((V, \| \cdot \|)\) be a prenormed space. Let \(\{x_n\}_{n \in \mathbb{N}} \subset V\) and \(x \in V\). We say that \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x\), denoted by \(x_n \to x\), if \(\|x_n - x\| \to 0\). We also say that \(\{x_n\}_{n \in \mathbb{N}}\) is Cauchy if for any \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(\|x_m - x_n\| < \varepsilon\) whenever \(m, n \in \mathbb{N}\) and \(m, n \geq n_0\). Not every converging sequence is Cauchy since prenorms satisfy neither the triangle inequality nor its relaxed ones in general. A subset \(B\) of \(V\) is called bounded if \(\sup_{x \in B} \|x\| < \infty\). If the prenorm is relaxed, then every converging sequence is Cauchy and every Cauchy sequence is bounded.

A prenormed space \((V, \| \cdot \|)\) is called complete if every Cauchy sequence in \(V\) converges to an element in \(V\). It is called quasi-complete if every bounded Cauchy sequence in \(V\) converges to an element in \(V\). The denseness and the separability can be defined in the same way as in ordinary normed spaces. We say that \(V\) is separable if there is a countable subset \(D\) of \(V\) such that \(D\) is dense in \(V\), that is, for any \(x \in V\) and \(\varepsilon > 0\) there is \(y \in D\) for which \(\|x - y\| < \varepsilon\).

In the above terms, if we want to emphasize that we are thinking of \(\| \cdot \|\) as a prenorm, then the phrase “with respect to \(\| \cdot \|\)” is added to each term.

2.6. Topological concepts

The topological concepts used in this paper are standard and can be found in [27]. It is the custom of a great many mathematicians to use “neighborhood of \(x\)” to mean “open neighborhood of \(x\)”. In this paper, neighborhoods are not necessarily open, however, unless so described. Given a sequence \(\{x_n\}_{n \in \mathbb{N}} \subset T\) and \(x \in T\) in a topological space \((T, T)\), the notation \(x_n \to x\) means that \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x\) with respect to the topology \(T\).
2.7. Distance functions and distance topologies

A distance function (distance for short) on \( T \) is a nonnegative real-valued function \( d \) defined on \( T \times T \) such that \( d(x, x) = 0 \) and \( d(x, y) = d(y, x) \) for every \( x, y \in T \); see for instance [5, p. 3] and [27, p. 16]. It is called a pseudometric if it satisfies the triangle inequality, that is, \( d(x, y) \leq d(x, z) + d(z, y) \) for every \( x, y, z \in T \). A pseudometric is a metric if it separates points of \( T \), that is, for any pair of points \( x, y \in T \), if \( d(x, y) = 0 \) then \( x = y \).

Given a distance \( d \) on \( T \), one can construct a topology on \( T \) in the following way. For each \( x \in T \) and \( r > 0 \), let the sphere centered at \( x \) with radius \( r \) be the subset of \( T \) defined by

\[
S(x, r) := \{ y \in T : d(x, y) < r \}.
\]

For each \( x \in T \), let \( U(x) \) be the collection of all subsets \( U \) of \( X \) such that \( S(x, r) \subseteq U \) for some \( r > 0 \). Let \( T \) be the collection of all subsets \( G \) of \( T \) such that \( G \in U(x) \) for every \( x \in G \). It is obvious that \( T \) is the collection of all subsets \( G \) of \( T \) with the property that for any \( x \in G \) there is \( r > 0 \) such that \( S(x, r) \subseteq G \).

**Theorem 2.4.** The collection \( T \) is a topology on \( T \) and has the following properties.

1. For any \( x \in T \), the neighborhood system of \( x \) is contained in \( U(x) \).
2. For any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset T \) and \( x \in T \), if \( d(x_n, x) \to 0 \) then \( x_n \to x \).
3. The following conditions are equivalent.
   1. For any \( x \in T \) and \( r > 0 \), the sphere \( S(x, r) \) is a neighborhood of \( x \).
   2. For any \( x \in T \), the neighborhood system of \( x \) coincides with \( U(x) \).
   3. The topology \( T \) is first countable and for any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset T \) and \( x \in T \), it follows that \( x_n \to x \) if and only if \( d(x_n, x) \to 0 \).

**Proof.** The proof of (1), (2) and implication (i)\( \Rightarrow \) (iii) of (3) is the same as [14, Theorem 4.1].

(ii)\( \Rightarrow \) (iii) The first countability of \( T \) follows from the fact that for each \( x \in T \) the collection \( B(x) := \{ S(x, 1/n) : n \in \mathbb{N} \} \) is a neighborhood base of \( x \).

The proof of the rest is easy.

(iii)\( \Rightarrow \) (i) Let \( x \in T \) and \( r > 0 \). Let \( \{U_n\}_{n \in \mathbb{N}} \) be a neighborhood base of \( x \). Without loss of generality we may assume that \( \{U_n\}_{n \in \mathbb{N}} \) is nonincreasing. Suppose, contrary to our claim, that \( S(x, r) \) is not a neighborhood of \( x \). Then, for any \( n \in \mathbb{N} \), we have \( U_n \not\subseteq S(x, r) \), for if not, \( U_n \subseteq S(x, r) \), which is impossible since \( S(x, r) \) is not a neighborhood of \( x \). Form this, for each \( n \in \mathbb{N} \), we can take \( x_n \in U_n \) such that \( x_n \not\in S(x, r) \). Then \( x_n \to x \), hence \( d(x_n, x) \to 0 \) by assumption. This leads to a contradiction since \( d(x_n, x) \geq r \) for every \( n \in \mathbb{N} \). 

This way of constructing a topology is the same as that of constructing the usual metric topology, but it should be noted that \( U(x) \) is not always the neighborhood system of \( x \). The topology constructed in the way described above is called the distance topology generated by a distance \( d \). A topological space \((T, \mathcal{T})\) is called pseudometrizable if the space \( T \) can be given a pseudometric \( d \)
such that the distance topology generated by $d$ coincides with the given topology $\mathcal{T}$. In this case, $\mathcal{T}$ is called pseudometrizable by the pseudometric $d$. A metrizable topological space can be defined in the same way.

Finally, a distance $d$ on a real linear space $V$ is called translation-invariant if $d(x,y) = d(x+z, y+z)$ for every $x, y, z \in V$ and truncated subhomogeneous if $d(\alpha x, \alpha y) \leq \max\{1,|\alpha|\}d(x,y)$ for every $x, y \in V$ and $\alpha \in \mathbb{R}$.

3. The Sugeno-Lorentz spaces

Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{S}\mu(\mu)$ denote the set of all Sugeno integrable functions on $X$, that is,

$$\mathcal{S}\mu(\mu) := \{f \in \mathcal{F}_0(X) : \mathcal{S}\mu(|f|) < \infty\}.$$

In this section we define a counterpart of ordinary Lorentz spaces by using the Sugeno integral. Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. When $\mu$ is $\sigma$-additive, the Lorentz space is defined by

$$L^{p,q}(\mu) := \{f \in \mathcal{F}_0(X) : \|f\|_{p,q} < \infty\},$$

where $\|f\|_{p,q}$ is the Lorentz quasi-seminorm on $L^{p,q}(\mu)$ defined by the Lebesgue integral as

$$\|f\|_{p,q} := p^{1/q}\left(\int_0^\infty \left[\mu(\{|f| > t\})^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} \quad (1)$$

for every $f \in \mathcal{F}_0(X)$ [1, Theorem 6.6]. The right side of (1) can be expressed as

$$\text{Ch}(\mu^{q/p}, p|f|^{q/q})^{1/q}$$

in terms of the Choquet integral defined by

$$\text{Ch}(\mu, f) := \int_0^\infty \mu(\{|f| > t\})dt$$

for every $f \in \mathcal{F}_0^+(X)$ and $\mu \in \mathcal{M}(X)$. This rewriting of equation follows from the fact that

$$\text{Ch}(\mu, |f|^q) = \int_0^\infty qt^{q-1}\mu(\{|f| > t\})dt$$

and the positive homogeneity of the Choquet integral, that is,

$$\text{Ch}(\mu, \alpha|f|) = \alpha \text{Ch}(\mu, |f|) \quad (\alpha > 0).$$

This observation leads to the definition of the Sugeno-Lorentz prenorm.

**Definition 3.1.** Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Define the function $(\cdot)_{p,q} : \mathcal{F}_0(X) \to [0, \infty]$ by

$$(f)_{p,q} := \mathcal{S}\mu^{q/p}, p|f|^{q/q})^{1/q} \quad (2)$$

for every $f \in \mathcal{F}_0(X)$. 

7
Since $\text{Su}(\mu, |f|) < \infty$ if and only if $\text{Su}(\mu^{q/p}, p|f|^{q/q}) < \infty$ by [26, Lemma 9.4], it follows that
$$\text{Su}(\mu) = \{ f \in \mathcal{F}_0(X) : (f)_{p,q} < \infty \}.$$ Furthermore, if $\mu$ is finite, then $\text{Su}(\mu, |f|) \leq \mu(X) < \infty$ for every $f \in \mathcal{F}_0(X)$, hence we have
$$\text{Su}(\mu) = \mathcal{F}_0(X).$$ The functions $(\cdot)_{p,q}$ are called the Sugeno-Lorentz prenorms on $\text{Su}(\mu)$ and the spaces $\text{Su}(\mu)$ equipped with them are called the Sugeno-Lorentz spaces. In particular, if $p = q = 1$ then $(\cdot)_1$ is called the Sugeno prenorm and written by $(\cdot)_1$, that is, $(f)_1 = (f)_{1,1}$ for every $f \in \mathcal{F}_0(X)$. The space $\text{Su}(\mu)$ equipped with the Sugeno prenorm is called the Sugeno space. In Section 4 we will show that the pronorms $(\cdot)_{p,q}$ and $(\cdot)_1$ generate the same topology on $\text{Su}(\mu)$ and that those topologies coincide with the relative topology induced from the topology generated by the Dunford-Schwartz prenorm $\| \cdot \|_0$ on $\mathcal{F}_0(X)$ introduced in [14].

**Remark 3.2.** The Sugeno-Lorentz prenorm was defined in [16] by
$$ (f)_{p,q} = \left( \frac{p}{q} \right)^{1/q} \text{Su}(\mu^{q/p}, |f|^q)^{1/q}, $$
which is not equivalent to formula (2) in Definition 3.1 since the Sugeno integral is not positively homogeneous. However, formula (2) seems to be a more appropriate definition of the Sugeno-Lorentz prenorm since it can reflect the feature that the Sugeno integral does not enjoy the positive homogeneity. Therefore, in this paper, we adopt formula (2) as the definition of the Sugeno-Lorentz prenorm. As a matter of fact, it turns out that this change in definition does not have an essential effect on the results obtained in [16] or the results of this paper, that is, with obvious modifications they can be shown for the Sugeno-Lorentz prenorm (2) in the same way.

The properties of the Sugeno-Lorentz prenorm are collected in the following proposition, all of which follow from easy calculation; see also [14, 15, 16].

**Proposition 3.3.** Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$.

1. For any $f \in \text{Su}(\mu)$ it follows that
   $$ (f)_{p,q} = \text{Su}(\mu^{1/p}, (p/q)^{1/q}|f|). $$

2. For any $A \in \mathcal{A}$ and $\alpha \in \mathbb{R}$ it follows that
   $$ (\alpha \chi_A)_{p,q} = \min \left\{ \alpha, \left( \frac{p}{q} \right)^{1/2}, \mu(A)^{1/2} \right\}. $$

3. For any $f \in \text{Su}(\mu)$ it follows that $(f)_{p,q} = 0$ if and only if $\mu(\{|f| > c\}) = 0$ for every $c > 0$; they are equivalent to the condition that $\mu(\{|f| > 0\}) = 0$ if $\mu$ is null-continuous.

4. For any $f \in \text{Su}(\mu)$ and $\alpha \in \mathbb{R}$ it follows that
   $$ (\alpha f)_{p,q} \leq \max \{ 1, |\alpha| \} (f)_{p,q}. $$
   Hence the prenorm $(\cdot)_{p,q}$ is truncated subhomogeneous.
(5) For any \( f \in \mathcal{S}u(\mu) \) and \( \varepsilon > 0 \) it follows that
\[
\min \left\{ \varepsilon \left( \frac{p}{q} \right)^{\frac{1}{q}}, \mu(|f| > \varepsilon) \right\} \leq (f)_{p,q}^p.
\]

(6) For any \( f, g \in \mathcal{S}u(\mu) \), if \(|f| \leq |g|\) then \((f)_{p,q} \leq (g)_{p,q}\).

(7) \( \mu \) is weakly null-additive if and only if \((\cdot)_{p,q}\) is weakly null-additive.

(8) \( \mu \) is null-additive if and only if \((\cdot)_{p,q}\) is null-additive.

(9) \( \mu \) is null-additive if and only if it follows that \((f)_{p,q} = (g)_{p,q}\) whenever \( f, g \in \mathcal{S}u(\mu) \) and \( f \sim g \).

(10) If \( \mu \) is \( K \)-relaxed subadditive for some \( K \geq 1 \), then \((\cdot)_{p,q}\) satisfies \((2K)^{1/p}\)-relaxed triangle inequality. Furthermore, \( \mu \) is subadditive if and only if the Sugeno prenorm \((\cdot)_1\) satisfies the triangle inequality.

From (4) and (10) of Proposition 3.3 it follows that \( \mathcal{S}u(\mu) \) is a real linear subspace of \( \mathcal{F}_0(X) \) if \( \mu \) is relaxed subadditive.

There is a close relationship between convergence in measure and convergence with respect to \((\cdot)_{p,q}\). The conclusion of the following proposition can be found in [26, 28], where \( \mu \) is assumed to satisfy a kind of continuity. The same proof works for any nonadditive measures; see also [12, 13].

**Proposition 3.4.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). For any \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0(X) \) and \( f \in \mathcal{F}_0(X) \), it follows that \((f_n - f)_{p,q} \rightarrow 0\) if and only if \( f_n \xrightarrow{\mu} f \).

The quotient space
\[
\mathcal{S}u(\mu) := \{ [f] : f \in \mathcal{S}u(\mu) \}
\]
is defined by the equivalence relation introduced in Subsection 2.4. Given an equivalence class \([f] \in \mathcal{S}u(\mu)\), define the prenorm on \( \mathcal{S}u(\mu) \) by \((|f|)_{p,q} := (f)_{p,q}\), which is well-defined by (9) of Proposition 3.3, provided that \( \mu \) is null-additive. This prenorm has the same properties as the prenorm on \( \mathcal{S}u(\mu) \) and separates points of \( \mathcal{S}u(\mu) \), that is, for any \([f] \in \mathcal{S}u(\mu)\), if \((|f|)_{p,q} = 0\) then \([f] = 0\). If \( \mu \) is finite, then \( \mathcal{S}u(\mu) = \mathcal{F}_0(X) \).

For any \( a, b \in [0, \infty] \) and \( 0 < r < \infty \) the following inequalities
\[
a \land b^r \leq (a \land b)^r + a \land b, \quad (a \land b)^r \leq (a \land b^r)^r + a \land b^r
\]
holds [12]. The first inequality yields
\[
(f)_{p,q} \leq \left[ \max \left\{ 1, \left( \frac{p}{q} \right)^{\frac{1}{q}} \right\} (f)_1 \right]^\frac{1}{p} + \max \left\{ 1, \left( \frac{p}{q} \right)^{\frac{1}{q}} \right\} (f)_1 \quad (3)
\]
and the second one yields
\[
(f)_1^\frac{1}{p} \leq \left[ \max \left\{ 1, \left( \frac{p}{q} \right)^{\frac{1}{q}} \right\} (f)_{p,q} \right]^\frac{1}{p} + \max \left\{ 1, \left( \frac{p}{q} \right)^{\frac{1}{q}} \right\} (f)_{p,q} \quad (4)
\]
both of which hold for every \( f \in \mathcal{F}_0(X) \). From these inequalities we see that for any sequence \( \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_0(X) \), it converges with respect to \( (\cdot)_{p,q} \) if and only if it converges with respect to \( (\cdot)_1 \) and that \( \{\xi_n\}_{n \in \mathbb{N}} \) is Cauchy with respect to \( (\cdot)_{p,q} \) if and only if it is Cauchy with respect to \( (\cdot)_1 \).

The completeness and separability of the Sugeno-Lorentz spaces are already investigated in [16]. For later use, in the following some results are extracted from [16] together with related terms. Let \( \mu \in \mathcal{M}(X) \). We say that \( \mu \) satisfies property (C) if for any sequence \( \{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A} \), it follows that \( \mu \left( \bigcup_{n=k}^{\infty} E_n \right) \to 0 \) whenever \( \sup_{n \in \mathbb{N}} \mu \left( \bigcup_{n=k}^{\infty} E_n \right) \to 0 \). Furthermore, we say that \( \mu \) is monotone autocontinuous from below [22] if \( \mu(A \setminus B_n) \to \mu(A) \) whenever \( A, B_n \in \mathcal{A} \), \( \mu(B_n) \to 0 \), and \( \{B_n\}_{n \in \mathbb{N}} \) is nonincreasing. Every nonadditive measure that is continuous from below satisfies property (C). Other examples of nonadditive measures satisfying property (C) can be found in [15, Proposition 3.3]. The monotone autocontinuity from below obviously follows from the autocontinuity from below. If \( \mu \) is monotone autocontinuous from below, then it is null-additive.

**Theorem 3.5.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). Assume that \( \mu \) is monotone autocontinuous from below and satisfies property (C) and the \((p,g,p)\). Then \( \mathbb{S}(\mu) \) and \( \mathbb{S}(\mu) \) are quasi-complete with respect to \( (\cdot)_{p,q} \).

**Corollary 3.6.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). Assume that \( \mu \) is relaxed subadditive, monotone autocontinuous from below, and satisfies property (C). Then \( \mathbb{S}(\mu) \) and \( \mathbb{S}(\mu) \) are complete with respect to \( (\cdot)_{p,q} \). Furthermore, the prenorms \( (\cdot)_{p,q} \) satisfy relaxed triangle inequalities.

**Remark 3.7.** Example 4.6 of [16] shows that property (C) cannot be dropped in Theorem 3.5 and Corollary 3.6.

The denseness and separability of the Sugeno-Lorentz spaces can also be characterized in terms of some properties of nonadditive measures. Recall that \( S(X) \) is the set of all \( \mathcal{A} \)-measurable simple functions on \( X \) and \( S(X) := \{[f] : f \in S(X) \} \).

**Theorem 3.8.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). Assume that \( \mu \) is order continuous. Then, \( S(X) \) is a dense subset of \( \mathbb{S}(\mu) \) with respect to \( (\cdot)_{p,q} \). If \( \mu \) is additionally assumed to be null-additive, then \( S(X) \) is a dense subset of \( \mathbb{S}(\mu) \).

We say that \( \mu \) has a countable base if there is a countable subset \( \mathcal{D} \) of \( \mathcal{A} \) such that for any \( A \in \mathcal{A} \) and \( \varepsilon > 0 \) there is \( D \in \mathcal{D} \) for which \( \mu(A \Delta D) < \varepsilon \).

**Theorem 3.9.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). Assume that \( \mu \) is order continuous and satisfies the \((p,g,p)\). Assume that \( \mu \) has a countable base. Then there is a countable subset \( \mathcal{E} \) of \( \mathbb{S}(\mu) \) such that for any \( f \in \mathbb{S}(\mu) \) and \( \varepsilon > 0 \) there is \( h \in \mathcal{E} \) for which \( (f - h)_{p,q} \). Hence \( \mathbb{S}(\mu) \) is separable with respect to \( (\cdot)_{p,q} \). If \( \mu \) is additionally assumed to be null-additive, then \( \mathbb{S}(\mu) \) is separable with respect to \( (\cdot)_{p,q} \).

4. **Topological properties of the Sugeno-Lorentz spaces**

Let \( \mu \in \mathcal{M}(X) \). For any \( 0 < p < \infty \) and \( 0 < q < \infty \), let \( (\cdot)_{p,q} \) be the Sugeno-Lorentz prenorm on \( \mathbb{S}(\mu) \). The Sugeno-Lorentz topology on \( \mathbb{S}(\mu) \) is
defined as the distance topology generated by the distance
\[ d(f, g) := (f - g)_{p,q} \]
for every \( f, g \in \mathcal{S}_\mu \). Let us recall the way of constructing the topology. For each \( f \in \mathcal{S}_\mu \) and \( r > 0 \), let
\[ S(f, r) := \{ g \in \mathcal{S}_\mu : (f - g)_{p,q} < r \}. \]
For each \( f \in \mathcal{S}_\mu \), let \( \mathcal{U}(f) \) be the collection of all subsets \( U \) of \( \mathcal{S}_\mu \) such that \( S(f, r) \subset U \) for some \( r > 0 \). Let \( \mathcal{T} \) be the collection of all subsets \( G \) of \( \mathcal{S}_\mu \) such that \( G \in \mathcal{U}(f) \) for every \( f \in G \). Then Theorem 2.4 takes the following form.

**Theorem 4.1.** The collection \( \mathcal{T} \) is a topology on \( \mathcal{S}_\mu \) and has the following properties.

1. For any \( f \in \mathcal{S}_\mu \), the neighborhood system of \( f \) is contained in \( \mathcal{U}(f) \).
2. For any sequence \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\mu \) and \( f \in \mathcal{S}_\mu \), if \( (f_n - f)_{p,q} \to 0 \) then \( f_n \to f \).
3. The following conditions are equivalent.
   - (i) For any \( f \in \mathcal{S}_\mu \) and \( r > 0 \), the sphere \( S(f, r) \) is a neighborhood of \( f \).
   - (ii) For any \( f \in \mathcal{S}_\mu \), the neighborhood system of \( f \) coincides with \( \mathcal{U}(f) \).
   - (iii) The topology \( \mathcal{T} \) is first countable and for any sequence \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\mu \) and \( f \in \mathcal{S}_\mu \), it follows that \( f_n \to f \) if and only if \( (f_n - f)_{p,q} \to 0 \).

**Definition 4.2.** The topology \( \mathcal{T} \) on \( \mathcal{S}_\mu \) constructed in the way described above is called the **Sugeno-Lorentz topology generated by** \((\cdot)_{p,q}\). The topological space \((\mathcal{S}_\mu, \mathcal{T})\) or its topology \( \mathcal{T} \) is called compatible with \((\cdot)_{p,q}\)-convergence if one of equivalent conditions (i)–(iii) of (3) in Theorem 4.1 is satisfied. In terms of Proposition 3.4 it may be called compatible with convergence in \( \mu \)-measure.

Every sphere of any metric space is open with respect to the metric topology. This is not the case for the Sugeno-Lorentz topology \( \mathcal{T} \) on \( \mathcal{S}_\mu \) as the following example shows.

**Example 4.3.** Let \( X = \mathbb{N} \) and \( \mathcal{A} := 2^X \). Let \( \mu : \mathcal{A} \to [0, \infty] \) be the nonadditive measure defined by
\[
\mu(A) := \begin{cases} 
0 & \text{if } A = \emptyset, \\
|A| \cdot \sum_{i \in A} \frac{1}{2^i} & \text{otherwise}
\end{cases}
\]
for every \( A \in \mathcal{A} \), where \( |A| \) stands for the number of elements of \( A \). Let \( A := \{1\} \) and \( B_n := \{n + 1\} \) for every \( n \in \mathbb{N} \). For simplicity of notation, let \( M_0 := (q/p)^{1/q} \). Let \( f := M_0 \chi_A \) and \( f_n := M_0 \chi_{A \cup B_n} \) for every \( n \in \mathbb{N} \). Then
\[
(f_n - f)_{p,q} = (M_0 \chi_{B_n})_{p,q} = \min \left\{ 1, \mu(B_n)^{\frac{1}{p}} \right\} = \min \left\{ 1, \frac{1}{2n+1} \right\} \to 0,
\]
which implies $f_n \to f$ by (2) of Theorem 4.1. In addition, we have

$$(fn)_{p,q} = \min \left\{ 1, \left(1 + \frac{1}{2^n}\right)^\frac{1}{p} \right\} = 1$$

and

$$(f)_{p,q} = \min \left\{ 1, \left(\frac{1}{2}\right)^\frac{1}{p} \right\} < 1,$$

so that $f_n \notin S(0,1)$ and $f \in S(0,1)$. Therefore, the sphere $S(0,1)$ is not open with respect to the Sugeno-Lorentz topology $T$ on $\mathcal{SU}(\mu)$.

The following theorem gives a necessary and sufficient condition that every sphere be open.

**Theorem 4.4.** Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Let $T$ be the Sugeno-Lorentz topology on $\mathcal{SU}(\mu)$ generated by $(\cdot)_{p,q}$.

1. The following conditions are equivalent.
   
   (i) $\mu$ is autocontinuous from above.
   
   (ii) For every $f \in \mathcal{SU}(\mu)$ and $r > 0$ the sphere $S(f, r)$ is open with respect to $T$.

   Furthermore, if $\mu$ is autocontinuous from above, then $U(f)$ is the neighborhood system of $f$ for every $f \in \mathcal{SU}(\mu)$ and the topology $T$ is first countable.

2. The following conditions are equivalent.

   (i) $\mu$ is autocontinuous from below.

   (ii) For every $f \in \mathcal{SU}(\mu)$ and $r > 0$, the set $\{g \in \mathcal{SU}(\mu) : (f - g)_{p,q} \leq r\}$ is closed with respect to $T$.

To prove Theorem 4.4 the following Fatou and reverse Fatou type lemmas of the Sugeno integral are needed; see [26, Theorem 9.9] for the proof.

**Proposition 4.5.** Let $\mu \in \mathcal{M}(X)$.

1. The following conditions are equivalent.

   (i) $\mu$ is autocontinuous from below.

   (ii) The Fatou convergence in measure lemma holds for $\mu$, that is, for any sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0^+(X)$ and $f \in \mathcal{F}_0^+(X)$, if $f_n \mathcal{M} \to f$, then it follows that

   $$\operatorname{Su}(\mu, f) \leq \liminf_{n \to \infty} \operatorname{Su}(\mu, f_n).$$

2. The following conditions are equivalent.

   (i) $\mu$ is autocontinuous from above.

   (ii) The reverse Fatou convergence in measure lemma holds for $\mu$, that is, for any sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0^+(X)$ and $f \in \mathcal{F}_0^+(X)$, if $f_n \mathcal{M} \to f$, then it follows that

   $$\limsup_{n \to \infty} \operatorname{Su}(\mu, f_n) \leq \operatorname{Su}(\mu, f).$$
Proof of Theorem 4.4. For simplicity of notation, let $L_0 := (p/q)^{1/q}$ and $M_0 := (q/p)^{1/q}$.

(1) (i)\Rightarrow(ii) Suppose, contrary to our claim, that $S(f_0, r_0) \not\subseteq \mathcal{T}$ for some $f_0 \in \mathcal{S}(\mu)$ and $r_0 > 0$. Then $S(f_0, r_0) \not\subseteq U(g_0)$ for some $g_0 \in S(f_0, r_0)$. Hence, for each $n \in \mathbb{N}$ there is $g_n \in S(g_0, 1/n)$ such that $g_n \not\in S(f_0, r_0)$. Since $(g_n - g_0)_{p,q} < 1/n$ for every $n \in \mathbb{N}$, we have $g_n \xrightarrow{\mu} g_0$ by Proposition 3.4, hence $L_0|g_0| \xrightarrow{\mu} L_0|f_0 - g_0|$. Since $\mu^{1/p}$ is autocontinuous from above, it follows from Proposition 4.5 that

$$
\limsup_{n \to \infty}(f_0 - g_n)_{p,q} = \limsup_{n \to \infty} \su(\mu^{1/p}, L_0|f_0 - g_n|) \\
\leq \su(\mu^{1/p}, L_0|f_0 - g_0|) \\
= (f_0 - g_0)_{p,q} < r_0
$$

and this contradicts to the fact that $g_n \not\in S(f_0, r_0)$ for every $n \in \mathbb{N}$.

(ii)\Rightarrow(i) Suppose, contrary to our claim, that there are $A_0 \in \mathcal{A}$ and a sequence $\{B_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\mu(B_n) \to 0$ and $\mu(A_0 \cup B_n) \neq \mu(A_0)$. If $\mu(A_0) = \infty$ we would see that $\mu(A_0 \cup B_n) = \infty$ for every $n \in \mathbb{N}$, which contradicts to the fact that $\mu(A_0 \cup B_n) \neq \mu(A_0)$. We thus assume that $\mu(A_0) < \infty$. Then there are $\varepsilon_0 > 0$ and a subsequence $\{B_{n_k}\}_{k \in \mathbb{N}}$ of $\{B_n\}_{n \in \mathbb{N}}$ such that

$$
\mu(A_0 \cup B_{n_k}) > \mu(A_0) + \varepsilon_0
$$

for every $k \in \mathbb{N}$. Let $r_0 := (\mu(A_0) + \varepsilon_0)^{\frac{1}{p}} > 0$. Let $f_0 := M_0 r_0 \chi_{A_0}$ and $f_k := M_0 r_0 \chi_{A_0 \cup B_{n_k}}$ for every $k \in \mathbb{N}$. First calculate $(f_0)_{p,q}$ and have

$$(f_0)_{p,q} = \min \left\{ r_0, \mu(A_0)^{\frac{1}{p}} \right\} = \mu(A_0)^{\frac{1}{p}} < r_0.$$  

Hence $f_0 \in S(0, r_0)$. Next calculate $(f_k)_{p,q}$ and by (5) we have

$$(f_k)_{p,q} = \min \left\{ r_0, (\mu(A_0 \cup B_{n_k})^{\frac{1}{p}} \right\} \\
\geq \min \left\{ r_0, (\mu(A_0) + \varepsilon_0)^{\frac{1}{p}} \right\} = r_0.$$  

Hence $f_k \not\in S(0, r_0)$ for every $k \in \mathbb{N}$. Finally calculate $(f_k - f_0)_{p,q}$ and have

$$(f_k - f_0)_{p,q} = \min \left\{ r_0, \mu(B_{n_k} \setminus A_0)^{\frac{1}{p}} \right\} \leq \mu(B_{n_k})^{\frac{1}{p}}.$$  

Therefore $(f_k - f_0)_{p,q} \to 0$ since $\mu(B_{n_k}) \to 0$.

Now, let us get a contradiction. By assumption we have $S(0, r_0) \subseteq \mathcal{T}$, hence it follows from $f_0 \in S(0, r_0)$ that $S(0, r_0) \subseteq U(f_0)$. Therefore, $S(f_0, s_0) \subseteq S(0, r_0)$ for some $s_0 > 0$. For this $s_0$, since $(f_k - f_0)_{p,q} \to 0$, we can find $k_0 \in \mathbb{N}$ such that $(f_{k_0} - f_0)_{p,q} < s_0$. Hence $f_{k_0} \in S(0, r_0)$. This is impossible since $f_k \not\in S(0, r_0)$ for every $k \in \mathbb{N}$. The last statement follows from Theorem 4.1.

(2) (i)\Rightarrow(ii) For each $f \in \mathcal{S}(\mu)$ and $r > 0$, let

$$D(f, r) := \{ g \in \mathcal{S}(\mu) \mid (f - g)_{p,q} \leq r \}$$

and show that $D(f, r)^c \not\subseteq \mathcal{T}$. Suppose, contrary to our claim, that $D(f_0, r_0)^c \not\subseteq \mathcal{T}$ for some $f_0 \in \mathcal{S}(\mu)$ and $r_0 > 0$. Then, by the same argument as the proof
of (1) one can find a sequence \( \{g_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\mu \) and \( g_0 \in \mathcal{S}_\mu \) such that 

\[ L_0 |f_0 - g_0| - \mu \rightarrow L_0 |f_0 - g_0| \quad \text{and} \quad \forall n \in \mathbb{N} \quad \text{such that} \quad |\mu| \leq \pi \]

and \( g_n \in D(f_0, r_0) \) for every \( n \in \mathbb{N} \). Since \( \mu^{1/p} \) is autocontinuous from below, it follows from Proposition 4.5 that

\[
\lim_{n \to \infty} (f_0 - g_n)_{p,q} = \lim_{n \to \infty} \mathcal{S}_\mu(\mu^{1/p}, L_0 |f_0 - g_n|) \\
\geq \mathcal{S}_\mu(\mu^{1/p}, L_0 |f_0 - g_0|) \\
= (f_0 - g_0)_{p,q} > r_0
\]

and this contradicts to the fact that \( g_n \in D(f_0, r_0) \) for every \( n \in \mathbb{N} \).

(ii) \( \Rightarrow \) (i) Suppose, contrary to our claim, that there are \( A_0 \in \mathcal{A} \) and a sequence \( \{B_n\} \subset \mathcal{A} \) such that \( \mu(B_n) \rightarrow 0 \) and \( \mu(A_0 \setminus B_n) \neq \mu(A_0) \). If \( \mu(A_0) = 0 \) then we would see that \( \mu(A_0 \setminus B_n) = 0 \) for every \( n \in \mathbb{N} \), which contradicts to the fact that \( \mu(A_0 \setminus B_n) \neq \mu(A_0) \). We thus assume that \( \mu(A_0) > 0 \). Then the proof falls naturally into two cases.

First consider the case where \( \mu(A_0) = \infty \). Then there are \( r_0 > 0 \) and a sequence \( \{B_{n_k}\}_{k \in \mathbb{N}} \) of \( \{B_n\}_{n \in \mathbb{N}} \) such that

\[
\mu(A_0 \setminus B_{n_k}) \leq r_0^p
\]

for every \( k \in \mathbb{N} \). Let \( f_0 := M_0(r_0 + 1) \chi_{A_0} \) and \( f_k := M_0(r_0 + 1) \chi_{A_0 \setminus B_{n_k}} \) for every \( k \in \mathbb{N} \). Then \( (f_0)_{p,q} = r_0 + 1 \), hence \( f_0 \notin D(0, r_0) \). Next, by (6) we have

\[
(f_k)_{p,q} = \mu(A_0 \setminus B_{n_k})^\ast \leq r_0,
\]

so that \( f_k \in D(0, r_0) \) for every \( k \in \mathbb{N} \). Finally we have

\[
(f_k - f_0)_{p,q} = \min \left\{ r_0 + 1, \mu(A_0 \cap B_{n_k})^\ast \right\} \leq \mu(B_{n_k})^\ast,
\]

so that \( (f_k - f_0)_{p,q} \rightarrow 0 \) since \( \mu(B_{n_k}) \rightarrow 0 \). These observations lead to a contradiction by the same argument as the proof of (1).

Next consider the case where \( 0 < \mu(A_0) < \infty \). Then there are \( \varepsilon_0 > 0 \) and a subsequence \( \{B_{n_k}\}_{k \in \mathbb{N}} \) of \( \{B_n\}_{n \in \mathbb{N}} \) such that

\[
\mu(A_0) - \varepsilon_0 > \mu(A_0 \setminus B_{n_k})
\]

for every \( k \in \mathbb{N} \). Let \( r_0 := (\mu(A_0) - \varepsilon_0)^\ast > 0 \). Let \( f_0 := M_0\mu(A_0)^\ast \chi_{A_0} \) and \( f_k := M_0\mu(A_0)^\ast \chi_{A_0 \setminus B_{n_k}} \) for every \( k \in \mathbb{N} \). Then, the same calculation as above shows that \( f_0 \notin D(0, r_0) \), \( f_k \in D(0, r_0) \) for every \( k \in \mathbb{N} \), and \( (f_k - f_0)_{p,q} \rightarrow 0 \). We thus get a contradiction.

**Definition 4.6.** We say that the topology \( T \) satisfies the *open sphere condition* if for any \( f \in \mathcal{S}_\mu \) and \( r > 0 \) the sphere \( S(f, r) \) is open with respect to \( T \).

At first glance, the Sugeno-Lorentz topology \( T \) looks like depending on constants \( p \) and \( q \), but actually it is only one regardless of \( p \) and \( q \). This is the reason why this topology is written \( T \) instead of \( T_{p,q} \). To verify this fact, first recall the Dunford-Schwartz topology on \( F_0(X) \) introduced in [14].

Let \( \mu \in M(X) \). Following [8, Definition III.2.1], define the distance \( \rho \) on \( F_0(X) \) by

\[
\rho(f,g) := \inf_{\varepsilon > 0} \varphi \left( c + \mu(\{|f-g| > \varepsilon\}) \right)
\]

14
for every $f, g \in \mathcal{F}_0(X)$, where the function $\varphi: [0, \infty] \to [0, \pi/2]$ is given by

$$\varphi(t) := \begin{cases} \arctan t & \text{if } t \neq \infty, \\ \pi/2 & \text{if } t = \infty. \end{cases}$$

Let

$$\|f\|_0 := \rho(f, 0)$$

for every $f \in \mathcal{F}_0(X)$, which is called the Dunford-Schwartz prenorm on $\mathcal{F}_0(X)$.

For each $f \in \mathcal{F}_0(X)$ and $r > 0$, let

$$S_0(f, r) := \{g \in \mathcal{F}_0(X) : \|f - g\|_0 < r\}.$$

For each $f \in \mathcal{F}_0(X)$, let $U_0(f)$ be the collection of all subsets $U$ of $\mathcal{F}_0(X)$ such that $S_0(f, r) \subset U$ for some $r > 0$. Let $T_0$ be the collection of all subsets $G$ of $\mathcal{F}_0(X)$ such that $G \in U_0(f)$ for every $f \in G$. Then, by [14, Theorem 4.1] the collection $T_0$ is a topology on $\mathcal{F}_0(X)$ satisfying properties (1)–(3) of Theorem 2.4. The topology $T_0$ is referred to as the Dunford-Schwartz topology on $\mathcal{F}_0(X)$ generated by $\mu$.

The Sugeno-Lorentz prenorm is closely related to the Dunford-Schwartz prenorm. This relation is crucial to the proof of the fact that the Sugeno-Lorentz topology is independent of $p$ and $q$.

**Proposition 4.7.** Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$.

1. For any $f \in \mathcal{F}_0(X)$ it follows that

$$\|f\|_0 \geq \min \left\{ \varphi \left( (f)^{p}_{p,q} \right), \varphi \left( \left( \frac{q}{p} \right)^{1/q} (f)^{p}_{p,q} \right) \right\}.$$

2. For any $f \in \mathcal{F}_0(X)$ it follows that

$$\|f\|_0 \leq \varphi \left( (f)^{p}_{p,q} + \left( \frac{q}{p} \right)^{1/q} (f)^{p}_{p,q} \right).$$

**Proof.** For simplicity of notation, let $L_0 := (p/q)^{1/q}$ and $M_0 := (q/p)^{1/q}$.

(1) The conclusion is obvious if $(f)^{p}_{p,q} = 0$. We thus assume that $(f)^{p}_{p,q} > 0$. Then, for any $c \in (0, M_0(f)^{p}_{p,q})$, we have

$$(f)^{p}_{p,q} = \text{Su}(\mu^{1/p}, L_0 |f|) \leq \min \left\{ L_0 c, \mu(\{|f| > c\})^{1/p} \right\},$$

hence

$$\mu(\{|f| > c\}) \geq (f)^{p}_{p,q},$$

and finally

$$\inf_{0 < c < M_0(f)^{p}_{p,q}} \varphi(c + \mu(\{|f| > c\})) \geq \varphi \left( (f)^{p}_{p,q} \right).$$

Meanwhile, we have

$$\inf_{c \geq M_0(f)^{p}_{p,q}} \varphi(c + \mu(\{|f| > c\})) \geq \varphi \left( M_0(f)^{p}_{p,q} \right).$$

15
Since
\[ \|f\|_0 = \min \left\{ \inf_{0 < c < M_0(f)_{p,q}} \varphi(c + \mu(\{|f| > c\})), \inf_{c \geq M_0(f)_{p,q}} \varphi(c + \mu(\{|f| > c\})) \right\}, \]
the desired inequality follows from (9) and (10).

(2) The conclusion is obvious if \( \|f\|_0 = 0 \). Thus the proof naturally falls into two cases.

First consider the case where \( \|f\|_0 = \pi/2 \). Then \( \mu(\{|f| > c\}) = \infty \) for every \( c > 0 \), so that \( (f)_{p,q} = \infty \). This leads to the conclusion.

Next consider the case where \( 0 < \|f\|_0 < \pi/2 \). Since the function \( \xi(t) := t^n + M_0t \) is a continuous and increasing function on \((0, \infty)\) satisfying \( \lim_{t \to 0} \xi(t) = 0 \) and \( \lim_{t \to \infty} \xi(t) = \infty \), by the intermediate value theorem there is \( c_0 \in (0, \infty) \) such that \( \xi(c_0) = c_0^n + M_0c_0 = \varphi^{-1}(\|f\|_0) \). For this \( c_0 > 0 \), since
\[ \|f\|_0 \leq \varphi(M_0c_0 + \mu(\{|f| > M_0c_0\])), \]
it follows that
\[ \mu(\{L_0|f| > c_0\})^{\frac{1}{p}} \geq \left( \varphi^{-1}(\|f\|_0) - M_0c_0 \right)^{\frac{1}{p}} = c_0. \]
Hence
\[ (f)_{p,q} = Su(\mu^{1/p}, L_0|f|) \geq c_0 \wedge \mu(\{L_0|f| > c_0\})^{\frac{1}{p}} = c_0 = \left( \varphi^{-1}(\|f\|_0) - M_0c_0 \right)^{\frac{1}{p}}, \]
which yields
\[ \varphi^{-1}(\|f\|_0) \leq (f)_{p,q}^p + M_0c_0, \tag{9} \]
and finally
\[ c_0 = \left( \varphi^{-1}(\|f\|_0) - M_0c_0 \right)^{\frac{1}{p}} \leq (f)_{p,q}. \tag{10} \]
Hence, the desired inequality follows from (9) and (10). \( \square \)

**Theorem 4.8.** Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \). The Sugeno-Lorentz topology \( \mathcal{T} \) on \( \mathfrak{Su}(\mu) \) is the relative topology induced from the Dunford-Schwartz topology \( T_0 \) on \( F_0(X) \). Hence, the Sugeno-Lorentz topology \( \mathcal{T} \) does not depend on \( p \) and \( q \).

**Proof.** For simplicity of notation, let \( M_0 := (q/p)^{1/q} \). We first show that \( \mathcal{T} \subset T_0 \cap \mathfrak{Su}(\mu) \). Let \( H \in \mathcal{T} \) and \( f \in H \). Then \( S(f, r) \subset H \) for some \( r > 0 \). Let
\[ r_0 := \min \{ \varphi(r^p), \varphi(M_0r) \}. \]
Then we must have \( S_0(f, r_0) \subset S(f, r) \), for otherwise there is \( g_0 \in S_0(f, r_0) \) but \( g_0 \notin S(f, r) \). Since \( (f - g_0)_{p,q} \geq r \), by (1) of Proposition 4.7 we have
\[ \|f - g_0\|_0 \geq \{ \varphi(r^p), \varphi(M_0r) \} = r_0, \]
a contradiction. Consequently, \( H \in T_0 \). Since \( H \) is a subset of \( \mathfrak{Su}(\mu) \), we conclude that \( H \in T_0 \cap \mathfrak{Su}(\mu) \).
Next we show that \( T_0 \cap \mathfrak{Su}(\mu) \subset T \). To this end, let \( H \in T_0 \cap \mathfrak{Su}(\mu) \). Then \( H = G \cap \mathfrak{Su}(\mu) \) for some \( G \in T_0 \). Let \( f \in H \). Since \( f \in G \), it follows that \( S_0(f, r) \subset G \) for some \( r > 0 \). Since the function \( \xi(t) := t^{p} + M_{0}t \) is a continuous and increasing function on \((0, \infty)\) satisfying \( \lim_{t \to +0} \xi(t) = 0 \) and \( \lim_{t \to \infty} \xi(t) = \infty \), by the intermediate value theorem there is \( r_{0} \in (0, \infty) \) such that \( \xi(r_{0}) = \varphi^{-1}(r) \), hence
\[
\varphi(r_{0}^{p} + M_{0}r_{0}) = r. \tag{11}
\]

Then we must have \( S(f, r_{0}) \subset S_0(f, r) \), for otherwise there is \( g_{0} \in S(f, r_{0}) \) but \( g_{0} \notin S_0(f, r) \). Since \((f - g_{0})_{p,q} < r_{0} \), by (2) of Proposition 4.7 we have
\[
\|f - g_{0}\|_{0} < \varphi(r_{0}^{p} + M_{0}r_{0}) = r,
\]
a contradiction. It thus follows that \( S(f, r_{0}) \subset H \), hence \( H \in T \).

From Theorem 4.8 we see that the topologies on \( \mathfrak{Su}(\mu) \) generated by the Sugeno-Lorentz prenorm \((\cdot)_{p,q}\) and by the Sugeno prenorm \((\cdot)_{1}\) are equal to each other and that they coincide with the relative topology induced from the Dunford-Schwartz topology on \( F_{0}(X) \).

In the rest of this section, assume that \( \mu \) is null-additive in such a way that the quotient space \( \mathfrak{Su}(\mu) \) and the prenorm \((\cdot)_{p,q}\) on \( \mathfrak{Su}(\mu) \) are well-defined. Let \( \tilde{T} \) be the distance topology generated by the distance
\[
\tilde{d}([f], [g]) := ([f] - [g])_{p,q}
\]
for every \([f], [g] \in \mathfrak{Su}(\mu)\). In other words, \( \tilde{T} \) is defined in the following way: For each \([f] \in \mathfrak{Su}(\mu)\) and \( r > 0 \), let
\[
\tilde{S}([f], r) := \{[g] \in \mathfrak{Su}(\mu) : ([f] - [g])_{p,q} < r\}.
\]
For each \([f] \in \mathfrak{Su}(\mu)\), let \( \tilde{U}([f]) \) be the collection of all subsets \( \tilde{U} \) of \( \mathfrak{Su}(\mu) \) such that \( \tilde{S}([f], r) \subset \tilde{U} \) for some \( r > 0 \). Let \( \tilde{T} \) be the collection of all subsets \( \tilde{G} \) of \( \mathfrak{Su}(\mu) \) such that \( \tilde{G} \in \tilde{U}([f]) \) for every \([f] \in \tilde{G} \). Then Theorems 2.4 and 4.4 take the following form.

**Theorem 4.9.** The collection \( \tilde{T} \) is a topology on \( \mathfrak{Su}(\mu) \) and has the following properties.

1. For any \([f] \in \mathfrak{Su}(\mu)\), the neighborhood system of \([f] \) is contained in \( \tilde{U}([f]) \).
2. For any sequence \( \{[f_{n}]\}_{n \in \mathbb{N}} \subset \mathfrak{Su}(\mu) \) and \([f] \in \mathfrak{Su}(\mu)\), if \( ([f_{n}] - [f])_{p,q} \to 0 \) then \([f_{n}] \to [f] \).
3. The following conditions are equivalent.
   (i) For any \([f] \in \mathfrak{Su}(\mu)\) and \( r > 0 \), the sphere \( \tilde{S}([f], r) \) is a neighborhood of \([f] \).
   (ii) For any \([f] \in \mathfrak{Su}(\mu)\), the neighborhood system of \([f] \) coincides with \( \tilde{U}([f]) \).
   (iii) The topology \( \tilde{T} \) is first countable and for any sequence \( \{[f_{n}]\}_{n \in \mathbb{N}} \subset \mathfrak{Su}(\mu) \) and \([f] \in \mathfrak{Su}(\mu)\), it follows that \([f_{n}] \to [f] \) if and only if \( ([f_{n}] - [f])_{p,q} \to 0 \).
(4) The following conditions are equivalent.

(i) \( \mu \) is autocontinuous from above.

(ii) For every \([f] \in Su(\mu)\) and \( r > 0 \) the sphere \( \tilde{S}([f], r) \) is open with respect to \( \mathcal{T} \).

Furthermore, if \( \mu \) is autocontinuous from above, then \( \tilde{U}([f]) \) is the neighborhood system of \([f]\) for every \([f] \in Su(\mu)\) and the topology \( \mathcal{T} \) is first countable.

(5) The following conditions are equivalent.

(i) \( \mu \) is autocontinuous from below.

(ii) For every \([f] \in Su(\mu)\) and \( r > 0 \) the set \( \{[g] \in Su(\mu)\}: ([f] - [g])_{p,q} \leq r \) is open with respect to \( \mathcal{T} \).

Definition 4.10. The topological space \((Su(\mu), \mathcal{T})\) or its topology \( \mathcal{T} \) is called compatible with \((\cdot)_{p,q}\)-convergence if one of equivalent conditions (i)–(iii) of (3) in Theorem 4.9 is satisfied. In terms of Proposition 3.4 it may be called compatible with convergence in \( \mu\)-measure. Furthermore, if for any \([f] \in Su(\mu)\) and \( r > 0 \) the sphere \( \tilde{S}([f], r) \) is open with respect to \( \mathcal{T} \), then we say that the topology \( \mathcal{T} \) satisfies the open sphere condition.

Theorem 4.11. Let \( \kappa: Su(\mu) \to Su(\mu) \) be the quotient map defined by \( \kappa(f) := [f] \) for every \( f \in Su(\mu) \). Then \( \mathcal{T} \) is the collection of all subsets \( \tilde{G} \) of \( Su(\mu) \) such that \( \kappa^{-1}(\tilde{G}) \in \mathcal{T} \). Therefore, \( \mathcal{T} \) is the quotient topology induced on \( Su(\mu) \) by \( \mathcal{T} \) and \( \kappa \). Furthermore, \( \kappa \) is continuous.

Proof. Let \( \tilde{G} \in \mathcal{T} \). For any \( f \in \kappa^{-1}(\tilde{G}) \), we have \([f] = \kappa(f) \in \tilde{G} \), so that \( \tilde{S}([f], r) \subset \tilde{G} \) for some \( r > 0 \). Hence it follows that \( S(f, r) = \kappa^{-1}(\tilde{S}([f], r)) \subset \kappa^{-1}(\tilde{G}) \), and hence, \( \kappa^{-1}(\tilde{G}) \in \mathcal{T} \). Conversely, let \( \tilde{G} \) be a subset of \( Su(\mu) \) such that \( \kappa^{-1}(\tilde{G}) \in \mathcal{T} \). For any \([f] \in \tilde{G} \), we have \( f \in \kappa^{-1}(\tilde{G}) \), so that \( S(f, r) \subset \kappa^{-1}(\tilde{G}) \) for some \( r > 0 \). Then it follows that \( \tilde{S}([f], r) = \kappa(S(f, r)) \subset \tilde{G} \), and hence, \( \tilde{G} \in \mathcal{T} \). The fact that \( \kappa \) is continuous is easy to show.

From Theorems 4.8 and 4.11 it is also seen that the Sugeno-Lorentz topology \( \mathcal{T} \) on \( Su(\mu) \) does not depend on \( p \) and \( q \) and that it is only one regardless of \( p \) and \( q \).

It is desirable that the topologies on \( Su(\mu) \) and \( Su(\mu) \) contain enough open sets to distinguish their points in some way. This is accomplished when the topologies satisfy the Hausdorff separation axiom.

Theorem 4.12. Let \( \mu \in \mathcal{M}(X) \). Let \( 0 < p < \infty \) and \( 0 < q < \infty \).

(1) Assume that \( \mu \) is autocontinuous from below or satisfies the \((p.g.p.)\). Then, for any \( f, g \in Su(\mu) \), if \((f - g)_{p,q} > 0\) then \( S(f, r) \cap S(g, r) = \emptyset \) for some \( r > 0 \).

(2) Assume that \( \mu \) is autocontinuous from above. If \( \mu \) is autocontinuous from below or satisfies the \((p.g.p.)\), then the topological space \((Su(\mu), \mathcal{T})\) is Hausdorff.

18
Proof. (1) Let \( f, g \in \mathcal{Su}(\mu) \) and assume that \( (f - g)_{p,q} > 0 \). Suppose, contrary to our claim, that \( S(f,r) \cap S(g,r) \neq \emptyset \) for every \( r > 0 \). Then there is \( \{h_n\}_{n \in \mathbb{N}} \subset \mathcal{Su}(\mu) \) such that \( (h_n - f)_{p,q} \to 0 \) and \( (h_n - g)_{p,q} \to 0 \). Hence \( h_n \xrightarrow{\mu} f \) and \( h_n \xrightarrow{\mu} g \) by Proposition 3.4. Let \( c > 0 \). Let \( A_n := \{|h_n - f| > c/2\} \) and \( B_n := \{|h_n - g| > c/2\} \) for every \( n \in \mathbb{N} \). Then \( \mu(A_n) \to 0 \) and \( \mu(B_n) \to 0 \). In addition, for any \( n \in \mathbb{N} \),

\[
\{|f - g| > c\} \cap A_n \subset B_n
\]

and

\[
\{|f - g| > c\} \subset A_n \cup B_n.
\]

If \( \mu \) is autocontinuous from below, then

\[
\mu(\{|f - g| > c\} \setminus A_n) = \mu(\{|f - g| > c\})
\]

so that

\[
\mu(\{|f - g| > c\}) = 0
\]

follows from (12). If \( \mu \) satisfies the (p.g.p.), then (14) follows from (13). Since \( c > 0 \) is arbitrary, by (14) we have \( (f - g)_{p,q} = 0 \), a contradiction.

(2) Let \([f],[g] \in \mathcal{Su}(\mu)\) and assume that \([f] \neq [g]\). Then \((f - g)_{p,q} > 0\), hence assertion (1) of this theorem implies that \( S(f,r) \cap S(g,r) = \emptyset \) for some \( r > 0 \). Let \( U := \tilde{S}([f],r) \) and \( V := \tilde{S}([g],r) \). Since \( \mu \) is autocontinuous from above, it follows from Theorem 4.9 that \( U \) and \( V \) are open neighborhoods of \([f]\) and \([g]\), respectively. The disjointness of \( U \) and \( V \) follows from the fact that \( \tilde{S}(f,r) \cap \tilde{S}(g,r) = \emptyset \).

Metrizability of a topology is also one of important issues in the study of topological spaces. Recall that a topological space is (pseudo)metrizable if the space can be given a (pseudo)metric such that the distance topology generated by the (pseudo)metric coincides with the given topology on the space. In this case, the given topology is called (pseudo)metrizable by the (pseudo)metric.

**Theorem 4.13.** Let \( \mu \in \mathcal{M}(X) \). Assume that \( \mu \) is subadditive. Then the topology \( \mathcal{T} \) is pseudometrizable by the pseudometric \( d \) on \( \mathcal{Su}(\mu) \) defined by \( d(f,g) := (f - g)_1 \) for every \( f, g \in \mathcal{Su}(\mu) \), while \( \tilde{\mathcal{T}} \) is metrizable by the metric \( \tilde{d} \) on \( \mathcal{Su}(\mu) \) defined by \( \tilde{d}([f],[g]) := ([f] - [g])_1 \) for every \([f],[g] \in \mathcal{Su}(\mu)\).

**Proof.** It follows from Proposition 2.2 that \( d \) is a pseudometric on \( \mathcal{Su}(\mu) \), while \( \tilde{d} \) is a metric on \( \mathcal{Su}(\mu) \). Furthermore, by Theorems 4.8 and 4.11 the topologies \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \) are generated by \( d \) and \( \tilde{d} \), respectively.

**Remark 4.14.** If \( \mu \) is \( K \)-relaxed subadditive for some \( K \geq 1 \), then \( d \) and \( \tilde{d} \) satisfy the \( K \)-relaxed triangle inequality and the topologies \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \) are generated by \( d \) and \( \tilde{d} \), respectively.

5. **Topological linear properties of the Sugeno-Lorentz spaces**

In this section, basic topological linear properties of the Sugeno-Lorentz spaces are deduced from the corresponding properties of the space \( F_0(X) \) developed in [14].
Theorem 5.1. Let $\mu \in \mathcal{M}(X)$. Assume that $\mu$ is relaxed subadditive. If $\mu$ is order continuous and autocontinuous from above, then $(\text{Su}(\mu), \mathcal{T})$ and $(\text{Su}(\mu), \tilde{\mathcal{T}})$ are topological linear spaces satisfying the open sphere condition. Furthermore, $\tilde{\mathcal{T}}$ is completely regular.

Proof. By assumption, $\mu$ is even null-additive and satisfies the (p.g.p.). It thus follows from [14, Theorem 9.3] that $(\mathcal{F}_0(X), \mathcal{T}_0)$ is a topological linear space. Note that the relaxed subadditivity of $\mu$ implies that $\text{Su}(\mu)$ is a linear subspace of $\mathcal{F}_0(X)$. Therefore, it follows from [23, page 17] that $(\text{Su}(\mu), \mathcal{T})$ is a topological linear space since $\mathcal{T}$ is the relative topology induced from $\mathcal{T}_0$ by Theorem 4.8. The open sphere condition of $\mathcal{T}$ follows from Theorem 4.4.

Next, since $\tilde{\mathcal{T}}$ is the quotient topology induced by $\mathcal{T}$ and $\kappa$ by Theorem 4.11, it follows from [23, page 20] that $(\text{Su}(\mu), \tilde{\mathcal{T}})$ is a topological linear space. The open sphere condition of $\tilde{\mathcal{T}}$ follows from Theorem 4.9.

Finally, the linear topology $\tilde{\mathcal{T}}$ is Hausdorff by Theorem 4.12. It thus follows from [23, page 16] that $\tilde{\mathcal{T}}$ is completely regular. $\blacksquare$

Remark 5.2. Theorem 5.1 contains [13, Corollary 2] since every finite nonadditive measure that is weakly subadditive and continuous from above in the sense of [13] is relaxed subadditive, order continuous, and autocontinuous from above.

The following corollary is an immediate consequence of (4) of Proposition 3.3 and Theorem 4.13.

Corollary 5.3. Let $\mu \in \mathcal{M}(X)$. Assume that $\mu$ is order continuous and subadditive. Then the topological linear space $(\text{Su}(\mu), \mathcal{T})$ is pseudometrizable by the pseudometric $d$ on $\text{Su}(\mu)$ defined by $d(f, g) := (f - g)_1$ for every $f, g \in \text{Su}(\mu)$, while $(\text{Su}(\mu), \tilde{\mathcal{T}})$ is metrizable by the metric $\tilde{d}$ on $\text{Su}(\mu)$ defined by $\tilde{d}(\lfloor f \rfloor, \lfloor g \rfloor) := (\lfloor f \rfloor - \lfloor g \rfloor)_1$ for every $\lfloor f \rfloor, \lfloor g \rfloor \in \text{Su}(\mu)$. Furthermore, $d$ and $\tilde{d}$ are translation-invariant and truncated subhomogeneous.

6. Summary of results

In this paper, given a nonadditive measure $\mu$ and constants $0 < p < \infty$ and $0 < q < \infty$, the Sugeno-Lorentz space $\text{Su}(\mu)$ and its quotient space $\text{Su}(\mu)$ are defined by using the Sugeno integral, together with the Sugeno-Lorentz topologies $\mathcal{T}$ on $\text{Su}(\mu)$ and $\tilde{\mathcal{T}}$ on $\text{Su}(\mu)$ generated by $(\cdot)_{p,q}$. As a continuation of our research [16] on the completeness and separability of the spaces, this paper focuses on their topological and topological linear properties. Some of our results are as follows.

- Any sphere defined by the Sugeno-Lorentz prenorm $(\cdot)_{p,q}$ is open with respect to the Sugeno-Lorentz topology if and only if $\mu$ is autocontinuous from above.

- The Sugeno-Lorentz topology on $\text{Su}(\mu)$ is the relative topology induced from the Dunford-Schwartz topology on $\mathcal{F}_0(X)$. Consequently, the Sugeno-Lorentz topology does not depend on $p$ and $q$, and hence, it is only one regardless of $p$ and $q$.

- Assume that $\mu$ is autocontinuous from above. Then, the Sugeno-Lorentz topology on $\text{Su}(\mu)$ is Hausdorff if $\mu$ is autocontinuous form below or satisfies the (p.g.p.).
Assume that \( \mu \) is subadditive. Then the Sugeno-Lorentz topology on \( Su(\mu) \) is pseudometrizable, while the Sugeno-Lorentz topology on \( Su(\mu) \) is metrizable.

As the basic topological linear properties of the Sugeno-Lorentz spaces, the following results are shown.

- Assume that \( \mu \) is relaxed subadditive. If \( \mu \) is order continuous and autocontinuous from above, then the Sugeno-Lorentz spaces \( Su(\mu) \) and \( Su(\mu) \) are topological linear spaces satisfying the open sphere condition.

- If \( \mu \) is order continuous and subadditive, then \( (Su(\mu), T) \) is a pseudometrizable topological linear space, while \( (Su(\mu), \tilde{T}) \) is a metrizable topological linear space.

Finally it should be remarked that some of the foregoing results are peculiar to the Sugeno integrable function spaces. A proper nonadditive counterpart of ordinary Lorentz spaces is certainly the Choquet-Lorentz spaces defined by the Choquet integral; see [15].

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