Elastic wave diffraction by infinite wedges

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Abstract. We compare two recently developed semi-analytical approaches to the classical problem of diffraction by an elastic two dimensional wedge, one based on the reciprocity principle and Fourier Transform and another, on the representations of the elastodynamic potentials in the form of Sommerfeld Integrals. At present, in their common region of validity, the approaches are complementary, one working better than the other at some isolated angles of incidence.

1. Introduction

In the nuclear industry one often needs to characterise a smooth planar defect at or near the back-wall of the component. This is easily detected when the defect is normal to the wall because of a specular “cat’s-eye” reflection. However, the defect may still be detectable even if it is tilted, through diffracted edge waves from the crack tips. If the defect breaks the back-wall, one of these edge waves comes from the corner or “wedge” formed by the defect and back-wall (Fig. 1). The inspection configuration is normally two dimensional. In order to model it using GTD (Geometrical Theory of Diffraction), it is necessary to know the diffraction coefficients for both the embedded crack tip and the surface-breaking corner. The diffraction coefficients for the embedded crack have been described in [1]. As far as the wedge diffraction coefficients are concerned, for horizontally polarised transversal (TH) waves, this is an acoustic problem for which an analytic solution has been available for a long time [2]. For longitudinal (L) and vertically polarised transverse (TV) waves the problem is elastic and reliable codes have been developed only recently [17] - [18]. It appears that a purely analytical solution is impossible, and instead there have been three major semi-analytical approaches developed so far:

1. In the first approach, the displacement is represented in the integral form form using a form of Green’s theorem, otherwise known as the reciprocity principle—the superposition of the
fields radiated by imaginary point sources situated on the faces of the wedge. Their FT (Fourier Transforms) satisfy certain analyticity requirements and the functional equations that are reformulated as Fredholm integral equations of the second kind. The latter are solved numerically (see the above references by Gautesen, Fujii, and Croisille and Lebeau).

2. In the second approach, each elastodynamic potential is represented in the form of a SI (Sommerfeld Integral)—the superposition of plane waves propagating in all (including complex) directions. The amplitudes of the plane waves belong to a certain class of analytical functions and satisfy a system of functional equations that are reformulated as singular integral equations. Budaev and Bogy [7]–[10] have offered a numerical schedule for solving the problem, and for the incident Rayleigh wave they calculated the Rayleigh reflection and transmission coefficients. The schedule has been clarified by Kamotski et al. [13]. It has been extended to finding the diffraction coefficients in [18].

3. In the third approach, the functional equations are solved using the Padé approximants, without recourse to any integral equations [15].

The paper is organised as follows: in the next Section we present the mathematical statement of the problem. We then briefly compare and cross-validate the first approach which we call FT with the second approach which we call SI.

2. Problem Statement

Let us consider a two dimensional wedge made of an isotropic solid, with the wedge angle $\alpha$ which is less than $\pi$ and introduce the Cartesian wedge coordinate system based on the orthonormal basis vectors $\{e_1, e_2\}$, with the origin at the wedge tip and the axis $e_1$ running along the lower edge. Then an arbitrary point in space $x$ can be described (see Fig. 1) using the Cartesian coordinates $(x_1,x_2)$ or associated polar coordinates $(r,\theta)$,

$$(x_1,x_2) = r(\cos(\theta + \alpha/2), \sin(\theta + \alpha/2)), \quad -\pi - \alpha/2 < \theta < \pi - \alpha/2.$$  

(1)

The polar angle is chosen in an unorthodox way to allow for an easy comparison with the results presented in [17]. Let the wedge be irradiated by an incident harmonic plane wave. This means that the incident field and all the resulting scattered fields contain the common factor $\exp(-i2\pi ft)$, where $f$ denotes the frequency and $t$ time. For simplicity of presentation, we suppress this factor throughout.

If the incident wave has transverse or longitudinal motions, then the incident displacements are respectively,

$$u^{inc}(x) = \frac{1}{ik_0} \nabla_\perp (e^{-ik_0p^{inc} \cdot x} e_3),$$  

(2)

$$u^{inc}(x) = \frac{1}{ik_1} \nabla e^{-ik_1p^{inc} \cdot x},$$  

(3)
where \( \nabla = (\partial_1, \partial_2) \), \( \nabla_\perp = (\partial_2, -\partial_1) \), \( \partial_i \) denotes the partial derivative with respect to \( x_i \); the wave numbers are

\[
k_j = \frac{2\pi f}{c_j} = \frac{2\pi}{\lambda_j}, \quad j = 0, 1,
\]

with \( c_0, c_1, \lambda_0 \) and \( \lambda_1 \) denoting the speeds and wave lengths of the transverse and longitudinal waves, respectively, and the symbol \( i \) when neither a superscript nor a subscript always denotes \( \sqrt{-1} \). The direction of incidence \(-p_{inc}\) is shown in Fig. 1 and the unit vector

\[
p_{inc} = (\cos(\theta_{inc} + \alpha/2), \sin(\theta_{inc} + \alpha/2)), \quad -\frac{\alpha}{2} < \theta_{inc} < \frac{\alpha}{2}.
\]

Whenever necessary below, the subscript \( n \) is used to denote the nature of motions on the incident wave, with \( n = 0 \) corresponding to transverse motions and \( n = 1 \) corresponding to longitudinal motions. The geometry of the problem and direction of incidence are shown in Fig. 1. The scattered field consists of the GE (geometrico-elastodynamic) reflected and maybe, multiple reflected waves as well as the field diffracted by the wedge tip. It satisfies the standard elastodynamic equations, the zero traction boundary conditions as well as the standard radiation conditions at infinity and the tip conditions of bounded energy. The mathematical description of these equations and conditions can be found in many books and papers, see e.g. [13].

The GE displacement is well known and its description is given in many sources, see e.g. [13] and [18]. According to the Geometrical Diffraction Theory, as \( k_j r \to \infty \) the diffracted displacement can represented as:

\[
H\left(\frac{\alpha}{2} - \theta\right)u^{diff}(x) \approx D^T_n(\theta; \theta_{inc}) e^{ik_0 r} \frac{e^{ik_0 r}}{\sqrt{r/\lambda_0}} e_{\theta} + D^L_n(\theta; \theta_{inc}) e^{ik_1 r} \frac{e^{ik_1 r}}{\sqrt{r/\lambda_1}} e_r,
\]

where \( H \) is the Heaviside function and \( e_r \) and \( e_\theta \) are the unit vectors corresponding to the polar coordinate system \((r, \theta)\)—see Fig. 1.

![Figure 1. Geometry of the traction free elastic wedge.](image-url)
3. A comparative description of two semi-analytical approaches

In this paper we aim to compare two different semi-analytical approaches to the solution of the above wedge diffraction problem, one based on FT (Fourier Transforms) and reported in [18] and another, based on SI (the Sommerfeld Integrals) reported in [13]. Both methods involve going through eight stages:

1. The first stage of the FT approach involves combining the elastodynamic equation and boundary conditions to produce an integral equation known as a generalised reciprocity principle or else Extinction Theorem for displacements and then applying to it the operations of dilatation and rotation. The first stage of the SI approach involves rewriting the elastodynamic equation and boundary conditions in terms of the elastodynamic potentials $\psi_0(x)$ and $\psi_1(x)$, using the usual two-dimensional decomposition

$$u(x) = \nabla_\perp[\psi_0(x)e_3] + \nabla\phi_1(x).$$  \hspace{1cm} (7)

It is well known that the elastodynamic equation decouples into one Helmholtz equation for $\psi_0$ and another Helmholtz equation for $\psi_1$. In view of that, $\psi_0$ and $\psi_1$ are simply related to the rotation and dilatation of the displacement, respectively.

2. At the second stage, whatever the approach, one employs symmetrisation with respect to the wedge bisectrix. The symmetric and antisymmetric problems decouple and their solutions have to be added up to give the solution of the original problem.

3. As the next stage, the FT method requires application of the Fourier transform in $x_1$ to both the symmetric and anti-symmetric versions of the Extinction Theorem for elastodynamic potentials. The Fourier variable is called $\xi$. Starting with the Extinction Theorem rather than the reciprocity relationship, the FT method allows one to take the full Fourier transform on a line just below the $x_1$ axis. The new unknowns are the one sided Fourier transforms of the displacement components

$$\hat{u}_i(\xi) = k_1 \int_0^\infty u_i(x_1,0)e^{ik_1\xi x_1}dx_1, \ i = 1, 2,$$  \hspace{1cm} (8)

which are analytic in the upper half of the complex $\xi$-plane. Let us describe their singularities: $\hat{u}_j(\xi)$ possess branch points $-\kappa_0$ and $-1$ and can be rendered single-valued by a branch cut along the line $\{\xi : \text{Re} (\xi) < -1, \text{Im} (\xi) = 0^- \}$. Also, on the real axis, each $\hat{u}_j(\xi)$ has a simple Rayleigh pole at $\xi = -\kappa_R$, where $\kappa_R = c_1/c_R$ and $c_R$ is the Rayleigh speed as well as two other poles, $\kappa_nP_{inc}$ and $\kappa_nP_{sym}$, where $P_{sym} = (\cos(\alpha/2 - \theta_{inc}), \sin(\alpha/2 - \theta_{inc}))$. We remind the reader that the subscript $n$ denotes the nature of the motions on the incident wave, with $n = 0$ for shear and $1$ for compressional. Given the above two poles, the functional equations can be used to “propagate” them, that is, evaluate successive reflection angles and find the relationships between their residues, which are the corresponding
reflection coefficients. All the poles corresponding to the body waves have to be located on the real axis in such a way that when the inverse Fourier transform is taken, they are passed from above, so that their contributions satisfy the radiation condition and represent waves outgoing to infinity. The above branch cut as well as the Rayleigh and incident pole arising in the FT approach are all shown in Fig. 2, all moved down from the real axis by infinitesimal amounts.

\[ \xi_{inc} = k_n p_{inc} \]

**Figure 2.** The main singularities in the FT method: the incident pole \( \xi_{inc} = k_n p_{inc} \), the Rayleigh pole, the branch points and a branch cut. Poles corresponding to multiply reflected waves are not indicated.

In the SI the solutions of the Helmholtz equations are represented in the form of the Sommerfeld Integrals

\[ \psi_{0,1}(kr, \theta) = \int_{C \cup \tilde{C}} \Psi_{0,1}(\omega \theta) e^{-ik_0,1 r \cos \omega} d\omega, \]

where the contour C runs from \(-\pi/2 + i\infty \) to \(3\pi/2 + i\infty\); and the contour \( \tilde{C} \) is the reflection of C with respect to the origin (see in Fig. 3). The contour of integration \( C \cup \tilde{C} \) is normally transformed into \( C_1 \cup C_2 \), where \( C_1 \) and \( C_2 \) are the steepest descent paths indicated in Fig. 3. The singularities lying inside the region bounded by all four contours give rise to reflected and head waves. The integrals along the contours \( C_1 \) and \( C_2 \) describe the tip diffracted waves. Only the plane waves with the complex angles of propagation lying in the shaded regions satisfy the radiation conditions. The Sommerfeld Integrals are substituted into the boundary conditions. The new unknown are \( \Psi_0(\omega) \) and \( \Psi_1(\omega) \), the so-called called Sommerfeld amplitudes.

4. At the fourth stage, assuming that the tip condition and radiation conditions are satisfied, each approach leads to a system of functional equations in the unknown functions of one complex variable, the horizontal component \( \xi \) of the wave vector and an angle \( \omega \) it forms with the observation direction, respectively. The functional equations can be cast in different forms, with the arguments in different terms related as in the Snell’s law. In the FT approach this means rotation of the wave vector through \( \alpha \), so that its horizontal
Figure 3. The contours used in the SI approach. The shaded areas represent physical regions.

component $\xi$ transforms to $\zeta_1 = \xi \cos \alpha + \sqrt{\kappa_0^2 - \xi^2} \sin \alpha$ for the transverse waves and to $\zeta_2 = \xi \cos \alpha + \sqrt{1 - \xi^2} \sin \alpha$ for the longitudinal waves. In the SI approach, when no mode conversion takes place the underlying transformation is a simple $\alpha$-shift in $\omega$. Otherwise, it is more complicated: prior to the $\alpha$-shift the angles are transformed from the longitudinal to transverse via a multi-valued function $g(\omega) = \cos^{-1}(\kappa_0^{-1} \cos \omega)$ and from transverse to longitudinal via its inverse $g^{-1}(\omega) = \cos^{-1}(\kappa_0 \cos \omega)$. The branch cuts of $g(\omega)$ are chosen so that the deformed contour integration $\{\tilde{\omega} = g(\omega) : \omega \in C \cup \tilde{C}\}$ may be transformed back to $C \cup \tilde{C}$ without touching them. They are the segments

$$[-\theta_h + \pi m, \theta_h + \pi m], \quad \theta_h = \cos^{-1} \kappa_0^{-1}, \quad m \text{ - integer}.$$  

The branch of $g(\omega)$ is chosen so that it has the properties

$$g\left(\frac{\pi}{2}\right) = \frac{\pi}{2},$$

$$g(-\omega) = -g(\omega), \quad g(\omega + \pi m) = g(\omega) + \pi m,$$

$$g(\omega) \simeq \omega - i \ln \kappa_0^{-1} + O\left(e^{-2|\Im \omega|}\right), \quad \text{as Im } \omega \to \infty. \quad (11)$$

The branches of the function $g(\omega)$ lie on the vertical lines $\Re \omega = m\pi$.

5. In both approaches, the fifth stage involves recasting the functional equations by inverting a matrix whose determinant is, or is proportional to, the Rayleigh function. In the FT approach, this is matrix $\{A_{ij}\}$ given by

$$A_{11}(\xi) = 1 - 2\xi^2, A_{12}(\xi) = -2\xi \sqrt{\kappa_0^2 - \xi^2}, A_{21}(\xi) = 2\xi \sqrt{1 - \xi^2}, A_{22}(\xi) = 1 - 2\xi^2,$$

where we use the notation $\kappa_0 = c_1/c_0$. In [13], it is the matrix $\{t_{ij}\}$, where

$$t_{11} = \cos 2\omega \sin \omega / \sqrt{\kappa_0^2 - \cos^2 \omega}, \quad t_{12} = t_{21} = \sin 2\omega, \quad t_{22} = -\cos 2\omega.$$

6. At the sixth stage it is assumed that the unknown functions can be decomposed so that the first (known) terms possesses poles corresponding to the incident, reflected and—provided they exist—multiply reflected body waves. The decomposition

$$\hat{u}(\xi) = \hat{v}^{ge}(\xi) + \hat{\phi}^{ge(E)}(\xi) + D^R_n \hat{d}^R(\xi) + D^{R(E)}_n \hat{d}^{R(E)}(\xi) + \hat{v}(\xi). \quad (12)$$
used in the FT approach contains three unknown vector terms, two associated with the Rayleigh pole and one, $\hat{v}(\xi)$, which is analytic everywhere outside the branch cuts. Above, the superscript $E$ stands for extraneous poles that appear on the lower non-physical side of the branch cut and $R$ stands for the Rayleigh wave. Note that the unit vectors $\hat{d}_R$ and $\hat{d}_{R(E)}$ are known. In the SI approach the decomposition contains two types of terms,

$$\Psi_{0,1}(\omega) = \hat{\Psi}_{0,1}(\omega) + \tilde{\Psi}_{0,1}(\omega), \quad \ell = 1, 2.$$  \hspace{1cm} (13)

Here and everywhere below $\ell = 1$ for the symmetric problem and $\ell = 2$ for the anti-symmetric one; the unknown terms $\hat{\Psi}_{0,1}(\omega)$ are regular in the strip $\pi/2 - \alpha/2 \leq \text{Re} \omega \leq \pi/2 + \alpha/2$, and the known terms $\hat{\Psi}_{0,1}$ with poles at $\theta_m^{(\ell)}$ are

$$\hat{\Psi}_{0,1}(\omega) = \sum_m \text{Res}(\Psi_{0,1}; \theta_m^{(\ell)}) \sigma(\omega - \theta_m^{(\ell)}), \quad \frac{\pi}{2} - \frac{\alpha}{2} \leq \text{Re} \theta_m^{(\ell)} \leq \frac{\pi}{2} + \frac{\alpha}{2}. \hspace{1cm} (14)$$

Above, an otherwise arbitrary function $\sigma(\omega)$ should be chosen to be analytic everywhere inside the strip $\pi/2 - \alpha/2 \leq \text{Re} \omega \leq \pi/2 + \alpha/2$, except for a simple pole at zero, where it has the residue one. The position of the analyticity strip is indicated on Fig. 4.

7. At the seventh stage, the above decompositions are substituted into the respective original functional equations to arrive at the functional equations for the new unknowns with prescribed analytic properties.

8. At the eighth stage, the FT approach relies on the Hilbert transform

$$\hat{v}(\xi) = \frac{1}{2\pi i} \int_{1}^{\infty} \frac{w(\zeta)}{\zeta + \xi} d\zeta \hspace{1cm} (15)$$
where the vector unknown, $w(\xi)$ is the jump of $\hat{v}(\xi)$ across the branch cut,

$$w(\xi) = \hat{v}(\xi + i0) - \hat{v}(\xi - i0). \quad \xi > 1.$$  \hspace{1cm} (16)

The SI approach relies on two further Hilbert type transforms $H$ and $\overline{H}$,

$$(Hf)(\omega) = \frac{1}{2\pi i} V.P. \int_{\pi/2-i\infty}^{\pi/2+i\infty} \frac{f(\omega')d\omega'}{\sin\left[\frac{\pi}{2\omega}(\omega' - \omega)\right]}, \quad \text{Re } \omega = \frac{\pi}{2} \hspace{1cm} (17)$$

and

$$\overline{H}f(\omega) = \frac{1}{2\pi i} V.P. \int_{\pi/2-i\infty}^{\pi/2+i\infty} \frac{(f(\omega')g'(\omega')d\omega')}{\sin\left[\frac{\pi}{2\omega}(g'(\omega') - g(\omega))\right]}, \quad \text{Re } \omega = \frac{\pi}{2}, \hspace{1cm} (18)$$

(see Eqs. (3.21) and (3.22), [13]. Here $g'(\omega') = dg(\omega')/d\omega'$. The above transforms allow us to reformulate each system of functional equations that appears in the FT approach as a system involving integral equations with a non-singular kernel and each system of functional equations that appears in the SI approach as an algebraic equation and integral equation with a singular kernel.

Depending on the method and decomposition used, apart from two unknown functions with prescribed analyticity, the equations for the symmetric and antisymmetric problems contain up to two unknown constants each. In the FT approach presented in [18] these are $D_{n}^{R(\ell)}$ and $D_{n}^{R(E)(\ell)}$ and in the form of SI approach described in [13] we have $c_{1}^{(\ell)}$. The $c_{1}^{(\ell)}$ constants appear as the result of application of the Sommerfeld Integral Nullification Theorem and the fact that the tip asymptotics of the displacement contain a non-zero constant term (see [13]). At the eight stage, in each approach the resulting integral equations are solved by employing solvability conditions that eliminate whatever constants are present. This is achieved by expressing them as integrals of unknown functions. In the FT approach described in [18] the solvability condition takes the form

$$D_{n}^{R(\ell)} = \int_{1}^{\infty} M_{R(\ell)}^{(1)}(z)w(z)dz + H_{R(\ell)}, \hspace{1cm} (19)$$

where we use the notation

$$M_{j}^{R(\ell)}(z) = [\hat{A}^{(\ell)}]_{11}^{-1}M_{j\ell}^{(1)}(-\kappa_{R}, z) + [\hat{A}^{(\ell)}]_{21}^{-1}M_{1j}^{(1)}(-\kappa_{R} - i0, z),$$

$$M_{j}^{R(E)(\ell)}(z) = [\hat{A}^{(\ell)}]_{12}^{-1}M_{j\ell}^{(1)}(-\kappa_{R}, z) + [\hat{A}^{(\ell)}]_{22}^{-1}M_{1j}^{(1)}(-\kappa_{R} - i0, z),$$

$$H_{R(\ell)} = [\hat{A}^{(\ell)}]_{11}^{-1}H_{1\ell}^{(1)}(-\kappa_{R}, \hat{u}^{ge}(-\kappa_{R})) + [\hat{A}^{(\ell)}]_{21}^{-1}H_{1\ell}^{(1)}(-\kappa_{R} - i0, \hat{u}^{ge}(-\kappa_{R} - i0)),$$

$$H_{R(E)(\ell)} = [\hat{A}^{(\ell)}]_{12}^{-1}H_{1\ell}^{(1)}(-\kappa_{R}, \hat{u}^{ge}(-\kappa_{R})) + [\hat{A}^{(\ell)}]_{22}^{-1}H_{1\ell}^{(1)}(-\kappa_{R} - i0, \hat{u}^{ge}(-\kappa_{R} - i0)). \hspace{1cm} (20)$$
Above, $[A^{(l)}]_{ij}^{-1}$ denote the components of the inverse of the matrix $A^{(l)}$ and matrices $A^{(l)}$ and $M^{(l)}$ are

\[
\begin{align*}
A^{(l)}_{11} & = \hat{R}(\kappa_R) - H_1^{(l)}(-\kappa_R, \hat{d}^R(-\kappa_R)), \\
A^{(l)}_{12} & = -H_1^{(l)}(-\kappa_R, \hat{d}^R(-\kappa_R)), \\
A^{(l)}_{21} & = -H_1^{(l)}(-\kappa_R - i0, \hat{d}^R(-\kappa_R - i0)), \\
A^{(l)}_{22} & = \hat{R}(\kappa_R) - H_1^{(l)}(-\kappa_R - i0, \hat{d}^R(-\kappa_R - i0)) \\
M^{(l)}_{11}(\xi, z) & = -\frac{1}{2\pi i}(-1)^l \left( \frac{a(\xi)a(\zeta)}{\zeta_1 + z} + \frac{\bar{b}_2(\xi)b_1(\xi)}{\zeta_2 + z} \right), \\
M^{(l)}_{12}(\xi, z) & = \frac{1}{2\pi i}(-1)^l \left( \frac{a(\xi)b_1(\xi)}{\zeta_1 + z} + \frac{b_1(\xi)a(\zeta)}{\zeta_2 + z} \right), \\
M^{(l)}_{21}(\xi, z) & = \frac{1}{2\pi i}(-1)^l \left( \frac{b_2(\xi)a(\zeta_1)}{\zeta_1 + z} + \frac{\bar{b}_2(\xi)a(\zeta)}{\zeta_2 + z} \right), \\
M^{(l)}_{22}(\xi, z) & = -\frac{1}{2\pi i}(-1)^l \left( \frac{\bar{b}_1(\xi)b_2(\xi)}{\zeta_1 + z} + \frac{a(\xi)a(\zeta_2)}{\zeta_2 + z} \right)
\end{align*}
\]

with $\hat{R}(\xi)$ - the derivative of $R(\xi)$ and

\[
\begin{align*}
H_1^{(l)}(\xi, u(\xi)) &= (-1)^l \{ a(\xi)[-a(\zeta_1)\hat{u}_1(\zeta_1) + \bar{b}_1\hat{u}_2(\zeta_1)] + b_1(\xi)[\bar{b}_2\hat{u}_1(\zeta_2) + a(\zeta_2)\hat{u}_2(\zeta_2)] \}, \\
\zeta_1 & = \xi \cos \alpha + \sqrt{\kappa_0^2 - \xi^2} \sin \alpha, \quad \zeta_2 = \xi \cos \alpha + \sqrt{\kappa_0^2 - \xi^2} \sin \alpha, \\
\eta_1 & = \xi \sin \alpha - \sqrt{\kappa_0^2 - \xi^2} \cos \alpha, \quad \eta_2 = \xi \sin \alpha - \sqrt{\kappa_0^2 - \xi^2} \cos \alpha, \\
\bar{b}_j(\xi) &= 2\zeta_j \eta_j, \quad j = 1, 2.
\end{align*}
\]

In the SI approach the solvability conditions are

\[
\begin{align*}
c^{(1)}_1 & = \int_{-\infty}^{\infty} \left[ (K y^{(1)})(\eta) - q_0^{(1)}(\eta) \right] d\eta \int_{-\infty}^{\infty} q_1^{(1)}(\eta) d\eta, \\
c^{(2)}_1 & = \int_{-\infty}^{\infty} \left[ (K y^{(2)})(\eta) - q_0^{(2)}(\eta) \right] d\eta \int_{-\infty}^{\infty} q_1^{(2)}(\eta) d\eta
\end{align*}
\]

- see Eq. (4.4) in [13]. Above, the new unknown functions are

\[
\begin{align*}
x^{(l)}(\frac{\pi}{2} + i\eta) & = \tilde{\Psi}_0^{(l)} [g(\frac{\pi}{2} + i\eta) + \frac{\alpha}{2}] + \tilde{\Psi}_0^{(l)} [g(\frac{\pi}{2} + i\eta) - \frac{\alpha}{2}], \\
y^{(l)}(\frac{\pi}{2} + i\eta) & = \tilde{\Psi}_1^{(l)} (\frac{\pi}{2} + \frac{\alpha}{2} + i\eta) + \tilde{\Psi}_1^{(l)} (\frac{\pi}{2} - \frac{\alpha}{2} + i\eta)
\end{align*}
\]
and we use the notation

\[ K f(\eta) = \frac{1}{\alpha i} \int_{-\infty}^{\infty} \left\{ \frac{\tanh^2 2t}{\chi'(t) \sinh \frac{\alpha}{2} (t-\eta)} - \frac{\tanh 2t \tanh 2\eta}{\sinh \frac{\alpha}{2} (\chi(t) - \chi(\eta))} \right\} f(t) \, dt, \]

\[ q_0^{(1)}(\eta) = r_2^{(1)}(\eta) - a'(\eta) \tilde{\mathcal{H}}r_1^{(1)}(\eta), \]

\[ q_1^{(1)}(\eta) = -\tan \frac{\alpha}{2} \frac{\cosh \eta}{\chi'(\eta) \cosh 2\eta} - \frac{\tanh 2\eta}{\alpha \chi'(\eta)} V.P. \int_{-\infty}^{\infty} \frac{\cosh \tau \, d\tau}{\cosh 2\tau \sinh \frac{\alpha}{2} (\tau - \eta)}, \]

\[ q_0^{(2)}(\eta) = r_1^{(2)}(\eta) - b'(\eta) \tilde{\mathcal{H}}r_2^{(2)}(\eta), \]

\[ q_1^{(2)}(\eta) = -\tan \frac{\alpha}{2} \frac{\cosh \eta}{\chi'(\eta) \cosh 2\eta} - \frac{\tanh 2\eta}{\alpha \chi'(\eta)} V.P. \int_{-\infty}^{\infty} \frac{\cosh \tau \, d\tau}{\cosh 2\tau \sinh \frac{\alpha}{2} (\tau - \eta)}, \]

in which we have defined

\[ a'(\eta) = -i \tanh 2\eta, \]

\[ b'(\eta) = \frac{2i \sinh \eta \sqrt{\kappa_0^{-2} + \sinh^2 \eta}}{\cosh 2\eta}, \]

\[ \mathcal{H} f(\eta) = \frac{1}{\alpha i} V.P. \int_{-\infty}^{\infty} \frac{f(t) \, dt}{\sinh \left\{ \frac{\alpha}{2} (t-\eta) \right\}}, \]

\[ \tilde{\mathcal{H}} f(\eta) = \frac{1}{\alpha i} V.P. \int_{-\infty}^{\infty} \frac{f(t) \chi'(t) \, dt}{\sinh \left\{ \frac{\alpha}{2} (\chi(t) - \chi(\eta)) \right\}}, \]

\[ r_1^{(\ell)}(\eta) = -\left[ \hat{\Psi}_0^{(1)} \left( g \frac{\pi}{2} + \eta \right) + \frac{\alpha}{2} + (-1)\ell+1 \hat{\Psi}_0^{(1)} \left( g \frac{\pi}{2} - \eta \right) - \frac{\alpha}{2} \right] - \]

\[ - \left[ \hat{\Psi}_1^{(1)} \left( g \frac{\pi}{2} + \eta \right) + \frac{\alpha}{2} + (-1)\ell+1 \hat{\Psi}_1^{(1)} \left( g \frac{\pi}{2} - \eta \right) - \frac{\alpha}{2} \right], \]

\[ r_2^{(\ell)}(\eta) = -a'(\eta) \left[ \hat{\Psi}_0^{(1)} \left( g \frac{\pi}{2} + \eta \right) + \frac{\alpha}{2} + (-1)\ell \hat{\Psi}_0^{(1)} \left( g \frac{\pi}{2} + \eta \right) - \frac{\alpha}{2} \right] - \]

\[ - \left[ \hat{\Psi}_1^{(1)} \left( g \frac{\pi}{2} + \eta \right) + \frac{\alpha}{2} + (-1)\ell \hat{\Psi}_1^{(1)} \left( g \frac{\pi}{2} + \eta \right) - \frac{\alpha}{2} \right], \]

\[ \chi(\eta) = \cosh^{-1} \left( \kappa_0^{-1} \sinh(\eta) \right), \]

\[ \chi'(\eta) = \frac{\cosh \eta}{\sqrt{\kappa_0^{-2} + \sinh^2 \eta}}. \]

This summary shows that the use of the SI approach involves extra technical complications, such as various integral transforms and their inverses as well as the function \( \cos^{-1}(\kappa_0^{-1} \cos \omega) \) with multiple cuts in its domain.

### 4. The testing and cross-validation of codes

As explained in [18], the FT code has been tested in two different ways:
1. The diffraction coefficients defined by Eq. (6) have been computed for $\theta < -\alpha/2$ and $\theta > \alpha/2$. These values of $\theta$ lie outside the wedge. As no energy is diffracted into this region, the diffraction coefficients should vanish. We have found that as a rule, the magnitude of the coefficients for this range of $\theta$ varies from $10^{-8}$ for the larger wedge angles $\alpha$ to around $10^{-3}$ for the smaller wedge angles.

2. When calculating the backscatter diffraction coefficients $D_{T0}^L(\theta; \theta)$ and $D_{L1}^T(\theta; \theta)$, simultaneously the following reciprocity relationship

$$D_{L0}^L(\theta; \theta) = \frac{k_1^2}{k_0^2} D_{T1}^T(\theta; \theta),$$

has been checked. This has been done by calculating the reciprocity discrepancy

$$\Delta(\theta) = |D_{L0}^L(\theta; \theta) - \frac{k_1^2}{k_0^2} D_{T1}^T(\theta; \theta)|,$$

The SI code has been also tested in two ways:

1. Checking that the symmetry properties of the computed Sommerfeld amplitudes are the same as predicted for the corresponding theoretical amplitudes.

2. Using the same reciprocity principle as above.

Note that the above tests involve conditions that are necessary but not sufficient for the diffraction coefficients to be correct.

In addition to results presented in [17] and [18], Fig. 5 shows that in their common region of validity, the outputs of both codes usually coincide. It is argued in [18] that whenever experimental data are available, the computed results are well within the experimental error. Note that due to various recent improvements in the SI code, results in Fig. 5 are in an even better agreement with experimental data than reported in [17]. Note too that the results displayed in Figs. 5 have been obtained using 800 nodes to solve the integral equation in the FT code and 200 nodes to solve the singular integral equation in the SI code.

When the FT code fails the reciprocity test in its region of applicability, the situation can be remedied by changing the incident angle by a fraction of a degree. When the SI code fails the test in its region of applicability, the situation can be remedied by increasing the accuracy - either by allowing for “a safe and slow mode” in which all poles are treated in a careful manner or else by increasing the number of nodes required to solve the underlying singular integral equation or both. At present, the FT code has been optimised to run very fast (with 800 nodes for each particular couple of incident and diffracted angles the run time is under one minute on a modern Pentium), but no such optimisation has been performed in the SI code. The latter takes about a minute to solve the underlying singular integral equations when the number of nodes is 200 and but over 15 minutes when this number is 800.

Several issues with the FT code still remain unresolved:
• sometimes it fails for unknown reasons and the reciprocity test indicates the failure, e.g. when the T wave is incident at 20° on the 80° wedge (and there exists a secondary diffracted T wave that propagates at 180° along a wedge face) - see Fig. 5;

• sometimes it fails for unknown reasons but the reciprocity test does not indicate the failure, e.g. when the T wave is incident at 10° on the 100° wedge (so that the incidence angle is close to critical).

The main issue with the SI code is that when the accuracy is increased it takes a long time to run. Also, when 800 nodes are used for wedge angles smaller than 45° the singular integral equations to be solved by the SI codes become ill-posed.

![Figure 5. Comparison of corrected computations based on the FT approach (dashed line) and corrected computations based on the SI approach (solid line) for the wedge angle 80° and \( \kappa_0 = 1.82222222157 \). The magnitudes and phases of the backscatter diffraction coefficients \( D_L^T \) - (a) and (b) and \( D_{T0}^T \) - (c) and (d), respectively. The reciprocity discrepancy for the corrected FT results (dashed line) and corrected SI results (solid line) - (e).]

5. The uniform GTD

As is to be expected, the magnitudes of the diffraction coefficients blow up at the shadow boundaries of multiply reflected waves, where the standard (non-uniform) GTD is inapplicable. For any given scattered mode, longitudinal or transverse, the non-uniform asymptotics covering the penumbral region are

\[
\mathbf{u}^{\text{non-uniform}} = \mathbf{H} \mathbf{C}_{\text{plane}} e^{ik_{\text{plane}}} + \mathbf{C}_{\text{diff}} \frac{e^{ik_{\text{diff}}}}{\sqrt{k_{\text{diff}}}},
\]  

(28)
Figure 6. The amplitude of displacement (in dimensionless units) versus the observation angle $\theta$ (in degrees) for $80^0$ wedge. Solid line - uniform asymptotics, dashed line - non-uniform asymptotics. (a), (b) and (c) the incident wave - L, with $\theta^{inc} = -5^0$, $10^0$ and $25^0$; (d), (e) and (f) the incident wave - T, with $\theta^{inc} = -5^0$, $10^0$ and $25^0$, respectively.

where $H$ is the Heaviside function, 0 in the shadow and 1 in the illuminated regions and $k$ is the corresponding wave number. This expansion is applicable far from the shadow boundaries, i.e. when $|s_{plane} - s_{diff}| \gg 1$. As a shadow boundary is approached, $|s_{plane} - s_{diff}| \to 0$, the diffraction coefficient $C_{diff}$ tends to infinity.

The uniform asymptotics of (28) have in the form

$$u_{\text{uniform}} = F \left( \sqrt{k|s_{diff} - s_{plane}|} \right) C_{plane} e^{ik s_{plane}} + \tilde{C}_{diff} e^{ik s_{diff}},$$

(29)

where $F$ is the Fresnel integral

$$F(x) = \frac{1}{\sqrt{12\pi}} \int_{-\infty}^{x} e^{i\xi^2} d\xi,$$

(30)

if $s_{diff} > s_{plane}$. The sign of the root $\sqrt{k|s_{diff} - s_{plane}|}$ is taken positive in the illuminated region, and negative in the shadow region. The unknown coefficient $\tilde{C}_{diff}$ is regular in the vicinity of the shadow boundary.

The simplest way to find $\tilde{C}_{diff}$ is by using the method of matched expansions. As $|s_{diff} - s_{plane}| \gg 1$ we expand the Fresnel integral in (29) into the asymptotic series and compare the resulting expression with (28). The higher order terms cancel out and matching the second order terms gives

$$\tilde{C}_{diff} = C_{diff} \frac{1}{\sqrt{k s_{diff}}} + C_{plane} \frac{e^{i\pi/4}}{2\sqrt{\pi} \sqrt{k(s_{diff} - s_{plane})}}.$$

(31)
5.1. Testing subroutines for calculating the uniform asymptotics

To test our code we studied the pulse train generated by the interaction of one cycle of sine pulse with an infinite wedge at the observation point which is inside a penumbra. All the arrivals have a bounded amplitude and do not blow up as they would if we used the non-uniform asymptotics (28) - see Fig. 6. This confirms the correct behaviour of the diffraction coefficients in the penumbrae, allowing for the mutual cancelation of the singularities in (31).

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