Single-Step Distillation Protocol with Generalized Beam Splitters

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We develop a distillation protocol for multilevel qubits (qudits) using generalized beam splitters like in the proposal of Pan et al. for ordinary qubits. We find an acceleration with respect to the scheme of Bennet et al. when extended to qudits. It is also possible to distill entangled pairs of photons carrying orbital angular momenta (OAM) states that conserves the total angular momenta as those produced in recent experiments.

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I. INTRODUCTION

The controlled manipulation of orbital angular momenta (OAM) degrees of freedom of light photon’s has opened a new field of potential applications in quantum information, like the experimental realization of entanglement with OAM states of photons using parametric down conversion (PDC) that conserves the angular momentum [1], the preparation of photons with OAM states [2], measurements of the OAM in a single photon [3] and the construction of OAM sorters [2], [4], that discriminates single photons with different angular momentum by sorting them out, etc. These developments point towards a feasible employment of the orbital angular momentum of photons to achieve quantum cryptography with higher alphabets increasing the information flux through the communication channels, checking Bell’s inequalities in qudit states, spintronics, capacity-increased quantum communication channels, checking Bell’s inequalities while the OAM from the azimutal phase of the complex electric field. This is a quantum case, qubits were considered as the two independent polarization of photons (|0⟩ = |V⟩, |1⟩ = |H⟩). This is a very remarkable proposal since there is an enormous difference in the experimental effort required between implementing the CNOT operation an overlapping two photons on a polarization beam splitter.

It would be interesting to have a distillation protocol for qudits with similar properties as the one of Pan et al. [2], using some sort of generalized beam splitters. Here we develop one such a scheme. Although we are clearly inspired by the existence of qudits realized as OAM states of photons, we want to stress that our constructions are independent of this concrete realization and we shall describe them from an abstract viewpoint.

II. GENERALIZED BEAM SPLITTERS

We shall use the standard notation for denoting the computational basis states for qudits, |0⟩, |1⟩,..., |D − 1⟩ instead of the physical notation |−M⟩, |−M + 1⟩,..., |M − 1⟩, |M⟩. We will also consider even and odd values of D. The following result is central in our constructions.

Proposition 1. There is an exact mapping (one-to-one) between the Polarization Beam Splitter and a certain realization of the CNOT gate for qudits.

Proof: By construction. We need to construct a generalization of the PBS for qudits such that it faithfully implements the CNOT gate $U_{\text{CNOT}}$ defined as [3].

$$U_{\text{CNOT}}|i⟩|j⟩ := |i⟩|i \oplus j⟩, \ i, j = 0, 1, \ldots, D − 1.$$ (1)

We shall refer to the generalized beam splitter for qudits as the Orbital Angular Momentum Beam-Splitter (OAM-BS). This is a black-box that takes D directions, or channels, as input and convert them into D output directions. In doing so, when the light is in each of the OAM states |l⟩, l = 0, 1,..., D − 1 entering one of the input directions, then the OAM-BS converts that state into another outgoing state according to precise rules. We construct these rules with an operator $T_D$ that implements the action of this OAM-BS. In order to reproduce the action of the CNOT gate as a beam splitter, we define the action of $T_D$ on input states |l⟩, along the direcion $i$ as follows

$$T_D|l⟩_i := |l⟩_{OAM}, \ l, i = 0, 1, \ldots, D − 1.$$ (2)
The action of the generalized beam splitter changes the outgoing direction with respect to the incoming direction while leaving the qudit state unchanged. As an example, the action of $T_3$ for qutrits is depicted in Fig. 1: a) A black-box $T_D$ representing an OAM Beam Splitter, with $D$ incoming and outgoing directions (channels) labeled from top to bottom. b) An example of $T_3$ acting on qutrits.

However, we can give a more general reason as to why this extension does not work. Let us notice that having $D$ input channels, the number of possible output states with non-empty outgoing directions is $D!$, but corresponding to this number of outputs we have that the number of all equal input states of the form $|l⟩_0|l⟩_1⋯|l⟩_{D-1}$ is only $D$. Thus, in order to guarantee that whenever the output channels are occupied then the input channels are occupied by all equal states we must demand that $D! = D$, whose only solution is $D = 2$ (qubits). □

To overcome this difficulty, we have found that the following generalization is good enough for the purpose of qudit distillation with an OAM-BS instead of a CNOT gate.

Property. The “$2D$-Extended Mode Case” property, denoted by “$2D$-EMC” for short, is defined as follows: when the $2D$ output and input channels are occupied then we have some means to be sure that the input and output states are all equal. □

This means that for $D > 2$ we need extra information not contained in $T_D$ in order to perform the qudit distillation based on a OAM-BS. Lemma 2 guarantees us that this is the unique extension of the “4-Mode Case” property to qudits. The graphical content of the “$2D$-EMC” property is shown in Fig. 2 which rules out a situation like that in (3). Physically, this extended property implies that we must have some way of distinguishing when all channels are occupied by the same states $|l⟩$ on input and output, from the rest of situations. This is like if we had some sort of “color property” associated to $|l⟩$ in order to discriminate them.

III. SINGLE-STEP DISTILLATION PROTOCOLS

With the ingredients introduced so far, we are able to set up a purification protocol for qudits that implements the scheme of Bennet et al. following the prescription introduced by Pan et al. of substituting the CNOT gates by OAM-BS. Indeed, we shall need some...
extra ingredients that will show up along the way. The schematic representation of this implementation is illustrated in Fig. 3.

To be specific, we shall consider first a simple type of mixed entangled state made up with qudits of arbitrary dimension $D$ that are shared by Alice and Bob, namely,

$$\rho := \sum_{i=0}^{M} q_i |\Psi_{0i}\rangle\langle\Psi_{0i}|, \quad M \leq D - 1,$$

$$1 := \sum_{i=0}^{M} q_i, \quad q_i \geq 0,$$  \hspace{1cm} (4)

where $q_i$ is the probability weight of the generalized Bell state $|\Psi_{0i}\rangle$ for appearing in the mixture $\rho$. The generalized Bell states that we consider here are

$$|\Psi_{kj}\rangle := U_{\text{CNOT}} [(U_F |k\rangle \otimes |j\rangle)]$$

$$= \frac{1}{\sqrt{D}} \sum_{y=0}^{D-1} e^{2\pi i y/k} |y\rangle|y \oplus j\rangle, \quad k, j = 0, \ldots, D - 1,$$  \hspace{1cm} (5)

where $U_F$ is the quantum Fourier transform (QFT). Thus, we shall be working with the subset of all possible generalized Bell states of the form $\{|\Psi_{0i}\rangle\}_{i=0}^{D-1}$. We have shown that this kind of states can be successfully distilled with the BBPSSW protocol based on CNOT gates with the following result for the new weights after the purification

$$q'_i = \frac{q_i^2}{\sum_{j=0}^{M} q_j^2}.$$  \hspace{1cm} (6)

For $M = 2$, i.e., considering a mixed state formed of just two Bell states of the form $|\Psi_{0i}\rangle$, the protocol has the following recursion relation

$$q'_i = \frac{q_i^2}{q_i^2 + (1 - q_i)^2}.$$  \hspace{1cm} (7)

In other words, we have found a direct $D$-dimensional generalization of the distillation protocols for qubits, with $q_i := F$. For $M = D - 1$ and taking $q_0 := F$ and $q_i = \frac{1-F}{D-1}$ and $i = 1, \ldots, D - 1$ we can find a more advantageous protocol than the previous one. In fact,

$$q'_0 := F' = \frac{F^2}{F^2 + (1-F)^2}.$$  \hspace{1cm} (8)

Our purpose now is to purify states of this form with the help of OAM-BS and the “2D-EMC” property. The steps of this OAM Distillation Protocol (OAM-DP) are as follows.

**Step 1.** Alice and Bob share $D$ pairs of qudit states $\rho$ as shown in Fig. 3. This is an unavoidable consequence of using the OAM-BS. This is a clear difference with respect to the standard purification scheme like the BBPSSW protocol that employ only 2 pairs $\rho \otimes \rho$. We use the following notation

$$\rho^{(D)}_{ab} := \rho_{ab0} \otimes \rho_{ab1} \otimes \cdots \otimes \rho_{abD-1bd},$$

$$\rho_{ab} := F|\Psi_{00}\rangle_{a,b} \langle\Psi_{00}| + \frac{1-F}{D-1} \sum_{i=1}^{D-1} |\Psi_{0i}\rangle_{a,b} \langle\Psi_{0i}|,$$  \hspace{1cm} (9)

where the subscripts have the following important meaning: $a_i$ ($b_i$) denotes Alice’s (Bob’s) qudit entering the channel $i$ of the OAM-BS.

**Step 2.** Alice and Bob apply the “2D-EMC” property to the state $\rho^{(D)}_{ab}$. This state is a probabilistic mixture of the following types of states: non-crossed terms $|\Psi_{01}\rangle_{a,b_0}\langle\Psi_{01}|_{a,b_1} \cdots |\Psi_{01}\rangle_{a,b_{D-1}}$, $i = 0, 1, \ldots, D - 1$, and the remaining crossed terms $|\Psi_{0i}\rangle_{a,b_0}|\Psi_{0j}\rangle_{a,b_i} \cdots |\Psi_{0k}\rangle_{a,b_{D-1}}$, $i \neq j \neq \cdots \neq k$. Now, the key point is that based on this property we can discard the crossed terms, which can be seen by expanding them in the computational basis using (5). As for the non-crossed terms, we also expand them in the computational basis and then the “2D-EMC” property projects each of these states labeled by $i$ onto the non-normalized states

$$\frac{(q_i)^2}{\sqrt{D}} \sum_{y=0}^{D-1} \sum_{d=0}^{D-1} |y\rangle_{a,d} \langle y \oplus i|_{b,d}, \quad i = 0, 1, \ldots, D - 1.$$  \hspace{1cm} (10)

It is important to notice that now the subscripts in (10) denote outgoing channels in the OAM-BS (see Fig. 3).

**Step 3.** Alice and Bob measure only $D - 1$ outgoing channels $a'_1, b'_1, a'_2, b'_2, \ldots, a'_{D-1}, b'_{D-1}$ (see Fig. 3) in a rotated basis defined through the QFT

$$|y\rangle_{a'} := U_F^{-1} |y\rangle_{d'}, \quad d' = a'_1, b'_1, \ldots, a'_{D-1}, b'_{D-1}.$$  \hspace{1cm} (11)

Then, if we substitute (11) into (10), and Alice and Bob measure the channels $a'_1, b'_1, a'_2, b'_2, \ldots, a'_{D-1}, b'_{D-1}$ in this rotated basis and compare their results via classical communication, we can check that when they find coincident results (channel by channel) then the remaining pair of shared photons in the channel $a'_0b'_0$ is left in the state $|\Psi_{01}\rangle_{a'_0b'_0}$. Therefore, the final outcome is a mixed state at output channels $a'_0b'_0$ as the original one

$$\rho_{a'b'} := F' |\Psi_{00}\rangle_{a'_0b'_0} \langle\Psi_{00}| + \frac{1-F'}{D'} \sum_{i=1}^{D'} |\Psi_{0i}\rangle_{a'_0b'_0} \langle\Psi_{0i}|,$$

with

$$F' = \frac{F^D}{F^D + (D-1) \left[\frac{1-F}{D-1}\right]^D}.$$  \hspace{1cm} (12)

The fixed points of this distillation recursion relation are $F_c = 0$, $F_c = 1$, with 0,1 stable and $\frac{1}{D}$ unstable. Thus, for an initial fidelity $F > \frac{1}{D}$, we can successfully distill the initial state. □

Comparing this recursion relation with the one from the BBPSSW protocol that used pairs of mixed
Our goal now is to distill a conserving state of photons carrying OAM states, the available entangled states are of the form (9). Instead, the pairs of photons are produced by the PDC mechanism based on the conservation of angular momentum [1]. This fact constraints the entanglement of the photons pairs which are of the form in the computational basis:

\[ |\Phi_C\rangle := \frac{1}{\sqrt{D}} \sum_{i=0}^{D-1} |i\rangle|D - 1 - i\rangle, \]  

(13)

since this corresponds to \(|\Phi_C\rangle = \frac{1}{\sqrt{2M+1}} \sum_{l=-M}^{+M} |l\rangle|l\rangle - l\rangle\) in the physical basis of the OAM states, so that the pairs of photons have zero total angular momentum. Thus, the appropriate entangled mixed state is now of the form

\[
\rho_{ab} := F |\Phi_C\rangle_{ab} \langle \Phi_C| + (1 - F)|\Psi_{NC}\rangle_{ab} \langle \Phi_{NC}|.
\]

(14)

Our goal now is to distill a conserving state \(|\Phi_C\rangle\) at the expense of a non-conserving state \(|\Psi_{NC}\rangle\). We have found that this state (14) can also be successfully distilled. The proof relies on the following results:

a/ the action of \(U_{BCNOT}\) on \(|\Phi_C\rangle\langle \Phi_C|\) plus the process of measurement coincidences in the target states yields as the only possibility

\[ \frac{1}{\sqrt{D}} |\Phi_C\rangle|00\rangle; \]  

(15)

b/ the same process of L.O.C.C. operations on the crossed states \(|\Phi_C\rangle\langle \Psi_{NC}|, |\Psi_{NC}\rangle\langle \Phi_C|\) yields no coincidences in the target states in the form \(|00\rangle\) as in a/;

c/ similarly, for the other direct states \(|\Psi_{NC}\rangle\langle \Psi_{NC}|\) this process yields

\[ \frac{1}{\sqrt{D(D-1)}} |\Psi_{NC}\rangle|00\rangle. \]  

(16)

These results imply that distilling a state \(\rho_{ab} \otimes \rho_{ab}\) as in (14) yields the following unnormalized state

\[ \rho_{ab}' \sim \frac{F^2}{D} |\Phi_C\rangle\langle \Phi_C| + \frac{(1 - F)^2}{D(D-1)} |\Psi_{NC}\rangle\langle \Psi_{NC}|. \]  

(17)

Therefore, we recover precisely the distillation recursion relation (12) for diagonal states. Likewise, we can adapt this distillation of OAM-preserving states into a OAM-BS protocol described above, to yield the distillation relation (12).

IV. CONCLUSIONS

There is a current interest and high activity concerning the distillation or purification of entanglement from quantum states since these methods are necessary for any realistic implementation of the useful properties implied by entanglement. More concretely, in recent experiments the manipulation of photons carrying Orbital Angular
Momenta (OAM) states has been achieved. This represent a very important experimental realization of the concept of qudits, i.e., multilevel quantum systems in quantum information.

In this paper the notion of a generalized beam splitter is introduced motivated by the existence of entangled OAM states of photons. We call these beam splitters as OAM-BS and we show how they can be used to distill qudit states. In particular, we have proposed a distillation protocol for qudits with the following properties:

i/ it is based on a non-trivial generalization of the method by Pan et al. [1] based on polarization Beam Splitters (PBS);

ii/ it amounts to an acceleration w.r.t. the standard BBPSSW protocol [5] and for suitable cases the distillation is achieved in a single step;

iii/ it allows the distillation a variety of qudit states including those of photons carrying Orbital Angular Momenta states as those employed in recent experiments [1].

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