New Multiply Nutty Spacetimes

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Abstract

We construct new solutions of the vacuum Einstein field equations with multiple NUT parameters, with and without cosmological constant. These solutions describe spacetimes with non-trivial topology that are asymptotically $dS$, $AdS$ or flat. We also find the the multiple nut parameter extension of the inhomogeneous Einstein metrics on complex line bundles found recently by Lü, Page and Pope. We also provide a more general form of the Eguchi-Hanson solitons found by Clarkson and Mann. We discuss the global structure of such solutions and possible applications in string theory.
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1 Introduction

Despite their unusual properties, Taub-NUT spacetimes have received increased attention in recent years. Part of their attractiveness issues from the reputation NUT-charged spacetimes have of being a ‘counterexample to almost anything’ in General Relativity [1]. For example such spacetimes typically (but not always) have closed time-like curves (CTCs) in their Lorentzian section. This apparently less-than-desirable feature actually makes them more interesting. They have served as testbeds for the AdS/CFT conjecture [2, 3, 4, 5]. Even more recently a very interesting result was obtained via a non-trivial embedding of the Taub-NUT geometry in heterotic string theory, with a full conformal field theory definition (CFT) [6]. It was found that nutty effects were still present even in the exact geometry, computed by including all the effects of the infinite tower of massive string states that propagate in it. This might be a sign that string theory can very well live even in the presence of nonzero NUT charge, and that the possibility of having CTCs in the background can still be an acceptable physical situation.

Furthermore, in asymptotically dS settings, regions near past/future infinity do not have CTCs, and NUT-charged asymptotically dS spacetimes have been shown to yield counterexamples to some of the conjectures advanced in the still elusive dS/CFT paradigm [7]- such as the maximal mass conjecture and Bousso’s entropic N-bound conjecture [8, 9, 10]. Moreover, they have been used to uncover deep results regarding gravitational entropy [2, 3, 11, 12] and, in particular, they exhibit breakdowns of the usual relation between entropy and area [13] (even in the absence of Misner string singularities). Furthermore, it has been recently noted that the boundary metric Lorentzian sector of these spaces in AdS-backgrounds is in fact similar with the Gödel metric [14, 15].

The first solution in four dimensions describing such an object was presented in [16, 17]. Intuitively a NUT charge corresponds to a magnetic type of mass. Its presence induces a so-called Misner singularity in the metric, analogous to a ‘Dirac string’ in electromagnetism [1]. This singularity is only a coordinate singularity and can be removed by choosing appropriate coordinate patches. However, expunging this singularity comes at a price: in general we must make coordinate identifications in the spacetime that yield CTCs in certain regions.

There are known extensions of the Taub-NUT solutions to the case when a cosmological constant is present and also in the presence of rotation [18, 19, 20, 21]. In these cosmological settings, the asymptotic structure is only locally de Sitter (for a positive cosmological constant) or anti-de Sitter (for a negative cosmological constant) and we speak about Taub-NUT-(a)dS solutions.

Generalizations to higher dimensions follow closely the four-dimensional case [22, 23, 24, 12, 25, 13, 26, 27, 28, 29]. In constructing these metrics the idea is to regard Taub-NUT spacetimes as radial extensions of $U(1)$ fibrations over a $2k$-dimensional base space endowed with an Einstein-Kähler metric $g_B$. Then the $(2k + 2)$-dimensional Taub-NUT spacetime has the metric:

$$ds^2 = F^{-1}(r) dr^2 + (r^2 + N^2) g_B - F(r)(dt + NA)^2$$

where $t$ is the coordinate on the fibre $S^1$ and the one-form $A$ has curvature $J = dA$, which is proportional to some covariantly constant 2-form. Here $N$ is the NUT charge and $F(r)$ is a function of $r$. 


Recently NUT-charged spacetimes with more than one nut parameter have been obtained as exact solutions to the higher-dimensional Einstein equations [26]. However explicit solutions were provided only up to seven-dimensions. These higher-dimensional spaces are constructed as radial extensions of circle fibrations over even-dimensional base spaces that can be factored in the form \( B = M_1 \times \cdots \times M_{d-2} \), where \( M_i \) are two-dimensional spaces of constant curvature. One can then associate a NUT charge \( N_i \) for every such two-dimensional factor. The above metric ansatz would be then modified by replacing \((r^2 + N^2)g_B\) with the sum \(\sum_i (r^2 + N^2_i)g_{M_i}\), while \(2NA\) would be replaced by \(\sum_i 2N_iA_i\). For example we can use the sphere \(S^2\), the torus \(T^2\) or the hyperboloid \(H^2\) as factor spaces. These solutions represent the generalizations of the spacetimes studied in refs. [22, 12, 25, 5].

Another class of solutions introduced in [26] used a generalised ansatz:

\[
d s^2 = F^{-1}(r)dr^2 + (r^2 + N^2)g_M + \alpha r^2 g_Y - F(r)(dt + 2NA)^2
\]

in which one constructs the higher dimensional Taub-NUT space as a generalised fibration over an Einstein-Kähler manifold \(M\). The non-trivial feature of this ansatz is that now the fibre contains besides the \((r,t)\)-sector a general Einstein space \(Y\), endowed with an Einstein metric \(g_Y\). This type of solutions was later generalised to arbitrary dimensions by Lü, Page and Pope in [27].

In this letter we generalise both of these types of solutions to include multiple NUT parameters in arbitrary dimensions. We first describe the generalisation of the ansatz (1) for an arbitrary number of factors \(M_i\) in the factored form of the base space \(B\). To analyse the possible singularities of these metrics we switch over to their Euclidian sections by performing analytic continuations of the time coordinate \(t\) and of the nut parameters. Following [23] we go on to analyse the regularity constraints to be imposed on these Euclidian sections in order to obtain regular metrics that can be extended globally to cover the whole manifold. We find that for generic values of the parameters these metrics are singular: it is only for a sagacious choice of the parameters that they become regular. As an example of this general analysis we focus on the six-dimensional case and we explicitly consider the cases of a Taub-NUT-like fibration over the base spaces \(S^2 \times S^2\) and \(CP^2\).

In Section 3 we present a more general form of the solution (1) in which we replace the 2-dimensional factors \(M_i\) by arbitrary even-dimensional Einstein-Kähler manifolds. We use here the normalisation that the Ricci tensor for each manifold \(M_i\) can be written as \(Ricci(M_i) = \delta_i g(M_i)\). For each factor \(M_i\) we associate a nut parameter \(N_i\). Consistent with what was conjectured in [26], we find that generically there are constraints to be imposed on the possible values of the cosmological constant \(\lambda\), the nut parameters \(N_i\) and the values of the various \(\delta\)’s. These solutions represent the multiple nut parameter generalisation of the inhomogeneous Einstein metrics on complex line-bundles described in [23]. We find that we can cast these solutions in another form that explicitly encodes the constraint conditions into the metric.

In Section 4 we present the multiple nut parameter extension of the metrics (2) constructed by Lü, Page and Pope [27]. In this case we replace the Einstein-Kähler manifold \(M\) by a product of Einstein-Kähler manifolds \(M_i\) with arbitrary even-dimensions and to each such factor we associate a nut parameter \(N_i\). The case in which \(Y\) is one-dimensional is particularly interesting to us since it will provide us with the general form of the odd-dimensional
Eguchi-Hanson-type solitons found recently by Clarkson and Mann [30].

The final section is dedicated to conclusions and possible applications.

Our conventions are: \((-, +, ..., +)\) for the (Lorentzian) signature of the metric; in even \(d\) dimensions our metrics will be solutions of the vacuum Einstein field equations with cosmological constant \(\Lambda = \pm \frac{(d-1)(d-2)}{2l^2}\), which can be expressed in the form \(G_{ij} + \Lambda g_{ij} = 0\) or in the equivalent form \(R_{ij} = \lambda g_{ij}\), where \(\lambda = \frac{2\Lambda}{d-1} = \pm \frac{d-1}{l^2}\). By an abuse of terminology we will still call \(\lambda\) cosmological constant.

2 The general solution

We assume that the \((d - 2)\)-dimensional base space in our construction can be factored as a product of \(p\) factors, \(B = M_1 \times \cdots \times M_p\) where \(M_i\) are 2-dimensional spaces of constant curvature normalised such that \(\text{Ricci}(M_i) = \delta_i g(M_i)\). The metric ansatz that we use is then:

\[
ds^2_d = -F(r)(dt + \sum_{i=1}^{p} 2N_i A_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p} (r^2 + N_i^2)g_{M_i}\]

where

\[A_i = \begin{cases} 
\cos \theta_i d\phi_i, & \text{for } \delta = 1 \text{ (sphere)} \\
\theta_i d\phi_i, & \text{for } \delta = 0 \text{ (torus)} \\
\cosh \theta_i d\phi_i, & \text{for } \delta = -1 \text{ (hyperboloid)} 
\end{cases}\]

By solving the vacuum Einstein equations with cosmological constant we obtain:

\[
F(r) = \frac{r}{\prod_{i=1}^{p} (r^2 + N_i^2)} \left[ \int^r \left( \delta_1 - \frac{d-1}{l^2} (s^2 + N_1^2) \right) \prod_{i=1}^{p} \frac{(s^2 + N_i^2)}{s^2} ds - 2m \right]
\]

while the constraints on the values of the nut parameters \(N_i\) and the cosmological constant \(\lambda\) can be expressed in the very simple form for every \(i, j = 1 \ldots p\):

\[
\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i
\]

If all the factors \(M_i\) coincide then we can satisfy this constraint in two ways: either we can take all the nut parameters to be equal \(N_i = N\) and keep the cosmological constant non-vanishing or else we can take \(\lambda = 0\) and keep the nut parameters independent. However, if at least two factors \(M_i\) are different, it is inconsistent to set \(\lambda = 0\). In this case all the nut parameters corresponding to identical \(M_i\) factors must remain equal, while those corresponding to different \(M_i\) factors must remain distinct such that the above constraints are still satisfied.

Note that while the factors in front of the \(M_i\) are never zero, this is not so for the Euclidean section of the metric. Therefore, when we shall address the possible singularities of the above metrics we shall focus mainly on their Euclidean sections, recognising that the Lorentzian versions are singularity-free – apart from quasi-regular singularities [31], which correspond to the end-points of incomplete and inextensible geodesics that spiral infinitely.
around a topologically closed spatial dimension. However, since the Riemann tensor and all its derivatives remain finite in all parallelly propagated orthonormal frames we take the point of view that these represent some of mildest of types of singularities and we shall ignore them when discussing the singularity structure of the Taub-NUT solutions. We also note that for asymptotically $dS$ spacetimes that have no bolts quasi-regular singularities are absent [32].

Scalar curvature singularities have the possibility of manifesting themselves only in the Euclidean sections. These are simply obtained by the analytic continuations $t \to i\tau$ and $N_j \to in_j$, and can be classified by the dimensionality of the fixed point sets of the Killing vector $\xi = \partial/\partial\tau$ that generates a $U(1)$ isometry group. In four dimensions, the Killing vector that corresponds to the coordinate that parameterizes the fibre $S^1$ can have a zero-dimensional fixed point set (we speak about a ‘NUT’ solution in this case) or a two-dimensional fixed point set (referred to as a ‘bolt’ solution). The classification in higher dimensions can be done in a similar manner. If this fixed point set dimension is $(d - 1)$ the solution is called a Bolt solution; if the dimensionality is less than this then the solution is called a NUT solution. If $d = 3$, Bolts have dimension 2 and NUTs have dimension 0. However if $d > 3$ then NUTs with larger dimensionality can exist [26, 27]. Note that fixed point sets need not exist; indeed there are parameter ranges of NUT-charged asymptotically $dS$ spacetimes that have no Bolts [32].

Since a singularity analysis for some of the lower-dimensional cases of these metrics has previously appeared [26] we shall limit ourselves to an outline its general features. The analysis is a direct application of the one given in [23]. In order to extend the local metrics presented above to global metrics on non-singular manifolds the idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters in the solution the singularities are not removable, corresponding to conical singularities in the manifold. We are mainly interested in the case of compact Einstein-Kähler manifolds $M_i$. Generically the Kähler forms $J_i$ on $M_i$ can be equal to $dA_i$ only locally. Hence we need to use a number of overlapping coordinate patches to cover the whole manifold. In order to render the 1-form $d\tau + \sum 2n_iA_i$ well-defined we need to identify $\tau$ periodically. In general this can be done if the ratios of all the parameters $n_i$ are rational numbers. If we choose them to be positive integers we can define $q = \gcd\{n_1, \ldots, n_p\}$ and require the period of $\tau$ to be given by:

$$\beta = \frac{8\pi q}{k}$$

(6)

where $k$ is a positive integer. It is also necessary to eliminate the singularities in the metric that appear as $r$ is varied over $M$. Attention must be paid to the so-called endpoint values of $r$: these are the values for which the metric components become zero or infinite. For a complete manifold $r$ must range between two adjacent endpoints – if any conical singularities occur at these points they must be eliminated. The finite endpoints occur at $r = \pm n_i$ or at the simple zeros of $F_E(r)$. In general $r = \pm n_i$ are curvature singularities unless $F_E = 0$ there as well. To eliminate a conical singularity at a zero $r_0$ of $F_E(r)$ we must restrict the periodicity of $\tau$ to be given by:

$$\beta = \frac{4\pi}{|F'_E(r_0)|}$$

(7)
and this will generally impose a restriction on the values of the parameters once we match it with (6). For compact manifolds the radial coordinate takes values between two finite endpoints and the regularity constraint must be imposed at both endpoints. If the manifold is noncompact then the cosmological constant is non-positive and the radial coordinate takes values between one finite endpoint \( r_0 \) and one infinite endpoint \( r_1 = \infty \). For our asymptotically locally flat or \((A)dS\) solutions the infinite endpoints are not within a finite distance from any points \( r \neq r_1 \) so there is no regularity condition to be imposed at \( r_1 \).

In this case the regularity conditions to be satisfied are that \( F_E(r) > 0 \) for \( r \geq r_0 \) and \( \beta = \frac{4\pi}{|F'_E(r_0)|} \).

Consider for example the six-dimensional case with a fibration over the base space \( S^2 \times S^2 \). If the cosmological constant is non-zero, \( \lambda = -5/l^2 \), then we must have \( n_1 = n_2 = n \). Regularity of the 1-form \( dr - 2nA \) forces the periodicity of \( \tau \) to be given by \( 8\pi n/k \), where \( k \) is an integer. We must match this periodicity with the one emerging by requiring absence of conical singularities at the root \( r_0 \) of \( F_E(r) \), which is

\[
F_E(r) = \frac{3r^6 + (l^2 - 15n^2)r^4 - 3n^2(2l^2 - 15n^2)r^2 - 6mrl^2 - 3n^4(l^2 - 5n^2)}{3l^2(r^2 - n^2)^2} \tag{8}
\]

from the Einstein equations using (4). The nut solution corresponds to \( r_0 = n \) in which case we obtain \( \frac{4\pi}{|F'_E(r)|} = 12\pi n \). As there is no integer \( k \) for which the periodicities can be matched, we conclude that this solution is singular. Indeed it is easy to check that \( r_0 = n \) is the location of a curvature singularity. To define a bolt solution it is sufficient to require \( r_0 > n \) and the regularity condition in this case is given by \( \frac{4\pi}{|F'_E(r)|} = \frac{8\pi n}{k} \), with \( k \) an integer. Solving this constraint we find

\[
r_0 = \frac{kl^2 \pm \sqrt{k^2l^4 - 80n^2l^2 + 400n^4}}{20n}.
\]

If the cosmological constant vanishes then we can have different values for the nut parameters. Without loss of generality, assume that \( n_1 > n_2 \) and that they are rationally related. Then it is easy to see that, in order to keep the metric positive definite, we have to restrict the range of the radial coordinate such that \( r > n_1 \). As above, the periodicity of the \( \tau \) coordinate is found to be \( 8\pi n_2/k \), where \( k \) is an integer. We have to match this with the periodicity imposed on \( \tau \) by eliminating the conical singularities at a root \( r_0 \) of \( F_E(r) \). We distinguish two types of solutions: a nut and a bolt. The nut solution corresponds to \( r_0 = n_1 \) and in this case the periodicity \( \frac{4\pi}{|F'_E(n_1)|} = 8\pi n_1 \) cannot be matched with \( 8\pi n_2/k \) for any integer value of \( k \). However note that \( r_0 = n_1 \) is not the location of a curvature singularity! On the other hand, the bolt solution corresponds to \( r \geq r_0 > n_1 \) and the periodicity is found to be \( \frac{4\pi}{|F'_E(n_1)|} = \frac{8\pi n_2}{p} \), where \( p \) is some integer. It is now possible to match it with \( 8\pi n_2/k \) with \( k \) an integer such that \( p/k = n_1/n_2 \). The bolt solution is then non-singular.

The situation changes considerably if we take \( B = CP^2 \) as the base space. In this case \( p = 1 \) in eq. (2) and the submanifold \( g_M, \) has the metric

\[
d\Sigma_{2}^{2} = \frac{du^{2}}{(1 + \frac{4u^2}{6})^{2}} + \frac{u^{2}}{4(1 + \frac{4u^2}{6})^{2}} (d\psi + \cos(\theta)d\phi)^{2} + \frac{u^{2}}{4(1 + \frac{4u^2}{6})} (d\theta^{2} + \sin^{2}\theta d\phi^{2}) \tag{9}
\]
with \( F_E(r) \) still given by (8). However the one-form \( 2nA \) is now given by

\[
A = \frac{u^2 n}{2 (1 + \frac{u^2}{6})} (d\psi + \cos \theta d\phi)
\]  

(10)

We need to find the smallest value of \( \int 2n dA \) over a closed 2-chain. Changing coordinates so that \( u = \sqrt{\frac{6}{\pi}} \tan \chi \) the \( CP^2 \) metric can be written as [33]

\[
d s^2 = \frac{6}{\delta} \left( d\chi^2 + \frac{\sin^2 \chi}{4} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \sin^2 \chi \cos^2 \chi (d\psi + \cos \theta d\phi)^2 \right)
\]

(11)

\[
A = \frac{3n}{2\delta} \sin^2 \chi (d\psi + \cos \theta d\phi)
\]

where \( 0 \leq \chi \leq \frac{\pi}{2} \), \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \), and \( 0 \leq \psi \leq 4\pi \). We see from (11) that \( \chi = 0 \) is a ‘nut’ in this subspace, and so there is no closed 2-chain on which to integrate \( 2n dA \). However at \( \chi = \frac{\pi}{2} \) the \((\theta, \phi)\) sector is a 2-dimensional bolt. Hence at \( \chi = \frac{\pi}{2} \) we obtain

\[
\int 2n dA = 2 \frac{3n}{2\delta} 4\pi = \frac{12\pi n}{\delta}
\]

implying that the periodicity of \( \tau \) can be \( 12\pi n/k \), where we use the normalisation \( \delta = 1 \). Equating this to \( |F_E(r=n)| = 12\pi n \) yields\(^1\) \( k = 1 \), and the geometry at \( r_0 = n \) is smooth. Thus we can obtain regular nut and bolt solutions if the base space is \( CP^2 \). More generally, for \( CP^q \) the periodicity is \( \frac{4\pi n(q+1)}{k\delta} \), with \( k \) an integer [12, 23].

### 3 A more general class of solutions

In this section we present a more general class of Taub-NUT metrics in even dimension. These spaces are constructed as complex line bundles over a product of Einstein-Kähler spaces \( M_i \), with dimensions \( 2q_i \) and metrics \( g_{M_i} \). Then the total dimension is \( d = 2(1 + \sum q_i) \). The metric ansatz that we use is the following:

\[
d s_d^2 = -F(r)(dt + \sum_{i=1}^p 2N_i A_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^p (r^2 + N_i^2)g_{M_i}
\]

Here \( J_i = dA_i \) is the Kähler form for the \( i \)-th Einstein-Kahler space \( M_i \) and we use the normalisation such that the Ricci tensor of the \( i \)-th manifold is \( R_{ab} = \delta_{ij}g_{ab} \). Then the general solution to Einstein’s field equations with cosmological constant \( \lambda = \pm(d - 1)/l^2 \) is given by:

\[
F(r) = \frac{r}{\prod_{i=1}^p (r^2 + N_i^2)^{q_i}} \left[ \int \left( \delta_1 + \frac{d - 1}{l^2} (s^2 + N_i^2) \right) \prod_{i=1}^p \frac{(s^2 + N_i^2)^{q_i}}{s^2} ds - 2m \right]
\]

\( ^1\)The parameter \( m = \frac{4\pi^2(6n^2-\ell^2)}{32\pi} \) is fixed by requiring that \( F_E(n) = 0 \).
while the constraints on the values of the nut parameters $N_i$ and the cosmological constant $\lambda$ can be expressed in the very simple form for every $i,j = 1,p$:

$$\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i$$  \hspace{1cm} (13)

It is easy to see that if the Einstein-Kähler spaces are two-dimensional, i.e. $q_i = 1$, we recover the solution from the previous section.

The singularity analysis of these metrics proceeds as described in the previous section. The Euclidian section of these metrics is obtained by analytical continuation of the time coordinate and of the nut parameters. From the general expression of the function $F_E(r)^2$ it is an easy matter to see that if the root $r_0 = n_j$ where $n_j$ is the nut parameter associated with an Einstein-Kähler manifold $M_j$ of dimension $2q_j$ then:

$$\frac{4\pi}{|F_E'(n_j)|} = \frac{4\pi n_j(q_j + 1)}{\delta_j}$$  \hspace{1cm} (14)

otherwise, for generic roots $r_0$ we deduce that:

$$\beta = \frac{4\pi}{|F_E'(r_0)|} = \frac{4\pi r_0}{\delta_1 - \lambda(r_0^2 - n_1^2)}$$  \hspace{1cm} (15)

These formulae are very useful in the singularity analysis of these metrics.

It is also possible to incorporate the above constraints directly in the metric. However, this would require the manifolds involved to be non-canonically normalised. Take for instance the 6-dimensional Taub-NUT fibration constructed over $S^2 \times S^2$. If we normalise the spheres such that their Einstein constant is $\delta_1 = 1$ then the constraint equation on the parameters takes the form $\lambda(n_1^2 - n_2^2) = 1 - 1 = 0$ and we can have a solution with non-vanishing cosmological constant only if the nut parameters are equal. Suppose now that we normalise the spheres such that their Einstein constants are $\delta_1$, respectively $\delta_2$. Then the constraint should read $\lambda(n_1^2 - n_2^2) = \delta_1 - \delta_2$. One way to change the Einstein constant in a general equation of the form $\text{Ricci}(M_i) = \lambda_i g(M_i)$ is to multiply the metric $g(M_i)$ by a constant factor $1/\delta_i$. This yields $\lambda_i/\delta_i$ as the normalised Einstein constant for the new rescaled metric$^3$. On the other hand, recall that for a $2q$-dimensional Einstein-Kähler with Kähler form $A_i$, the product $(dA_i)^q$ is proportional to its volume form. When rescaling the metric by $1/\delta_i$ the volume form gets rescaled by a factor $1/\delta_i^q$ – hence we must rescale $A_i$ by a factor of $1/\delta_i$ to obtain the Kähler form for the rescaled metric. For spheres we should then multiply $A_i$ by $1/\delta_i$ and the metric elements $d\Omega_i^2$ by $1/\delta_i$, for each $i = 1,2$. The expression for $F(r)$ remains unchanged in this process$^4$.

Applying this to the six-dimensional case, we obtain

$$ds^2 = -F(r) \left( dt - \frac{2n_1}{\delta_1} \cos \theta_1 d\phi_1 - \frac{2n_2}{\delta_2} \cos \theta_2 d\phi_2 \right)^2 + \frac{dr^2}{F(r)} + \frac{r^2 + n_1^2}{\delta_1}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{r^2 + n_2^2}{\delta_2}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)$$  \hspace{1cm} (16)

$^2$This is deduced from (12) by analytical continuation of all the nut charges $N_i \rightarrow in_i$.

$^3$The Ricci tensor is invariant under an overall rescaling of the metric by a constant.

$^4$It can be read from the general form (12) for general values of $\delta$’s.
while the constraint equation takes the form
\[ \lambda(n_1^2 - n_2^2) = \delta_1 - \delta_2 \]
Solving this equation for \( \delta_2 \) and replacing its value in the metric we obtain:
\[
 ds^2 = -F(r) \left( dt - 2n_1 \cos \theta_1 d\phi_1 - \frac{2n_2}{\delta_1 - \lambda(n_1^2 - n_2^2)} \cos \theta_2 d\phi_2 \right)^2 + \frac{dr^2}{F(r)} \\
+ \frac{r^2 + n_1^2}{\delta_1}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{r^2 + n_2^2}{\delta_1 - \lambda(n_1^2 - n_2^2)}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \tag{17} 
\]
which is a solution of Einstein field equations with cosmological constant \( \lambda \) for every value of the nut parameters \( n_1 \) and \( n_2 \). Notice that now the constraint equation is already encoded in the metric, and we can for convenience scale \( \delta_1 = 1 \). When \( n_2 = 0 \) it reduces to the cosmological 6-dimensional Taub-NUT solution obtained previously in \cite{26} \cite{27}.

It is interesting to note that the above form of the metric allows non-singular NUTs of intermediate dimensionality constructed over the base \( S^2 \times S^2 \). To see this let us notice that the absence of Misner string singularities can be accomplished if \( n_1/\delta_1 \) and \( n_2/\delta_2 \) are rationally related. Specifically, we can choose for example \( n_1 = 2n_2 \) and \( \delta_1 = 1 \) while \( \delta_2 = 1/2 \). To satisfy these relations it is enough to take \( \lambda n_2^2 = -1/6 \), where \( \lambda = -5/l^2 \) in 6-dimensions. Then regularity of the 1-form \( (d\tau - 2n_1/\delta_1 A_1 - 2n_2/\delta_2 A_2) \) requires the periodicity of \( \tau \) to be given by \( \frac{8\pi n_1}{k\delta_1} = \frac{8\pi n_2}{k\delta_2} \) where \( k \) is some integer. It is easy to see that we can match this periodicity with \( \frac{4\pi}{\lbrack F_E(w) \rbrack} = \frac{8\pi n_1}{n_1} \) if we take \( k = 1 \). Then there exists a nut at \( r = n_1 \) which is completely regular – it can be easily checked that there are no curvature singularities in this case! We conclude that the NUT solution of intermediate dimensionality constructed over the base space \( S^2 \times S^2 \) is regular.

4 Warped-type Fibrations and Generalized Eguchi-Hanson Solitons

We can find a very general class of solutions of Einstein’s field equations if we use the generalised ansatz \cite{26} \cite{27}
\[
 ds^2 = -F(r)(dt + \sum_{i=1}^{p} 2N_i A_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p}(r^2 + N_i^2)g_{M_i} + \alpha r^2 g_Y \tag{18} 
\]
As before \( J_i = dA_i \) is the Kähler form for the \( i \)-th Einstein-Kahler space \( M_i \), \( Y \) is a \( q \)-dimensional Einstein space with metric \( g_Y \) and we use the normalisation such that the Ricci tensor of the \( i \)-th manifold is \( R_{ab} = \delta_i g_{ab} \) and \( R_{Yab} = \delta_Y g_{Yab} \).

Then the general solution of Einstein’s field equations is given by
\[
 \alpha = \frac{\delta_Y}{\delta_1 - \lambda N_1^2} \\
 F(r) = \frac{r^{1-q}}{\prod_{i=1}^{p}(r^2 + N_i^2)^\eta} \left[ \int r \left( \delta_1 + \frac{d-1}{l^2} (s^2 + N_1^2) \right) s^{q-2} \prod_{i=1}^{p}(s^2 + N_i^2)^\eta ds - 2m \right] \tag{19} 
\]
where the constraints on the values of cosmological constant \( \lambda = \mp \frac{d-1}{l^2} \) and the nut parameters \( N_i \) can be expressed in the following simple form:

\[
\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i
\]

for every \( i, j \). For \( p = 1 \) we recover the general solution found by Lü, Page and Pope in ref. [27].

We can treat the case \( \delta_Y = 0 \) (or \( q = 1 \)) if we take the limit in which \( \lambda N_1^2 = \delta_1 \) in order to keep \( \alpha \) finite in the above expressions. In general it is not necessary to have all the nut parameters identical, though nut parameters \( N_j \) corresponding to Einstein-Kähler spaces that have the same Einstein constants \( \delta_j \) have to be equal.

The case \( q = 1 \) is particularly interesting to us, as it will provide a generalisation of Eguchi-Hanson metrics to arbitrary odd-dimensions [30]. For simplicity we shall work in the Euclidian sector. In this case the metric can be written as

\[
ds_\alpha^2 = F(r)(d\chi + \sum_{i=1}^p 2n_iA_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^p (r^2 - n_i^2)g_{M_i} + r^2dy^2
\]

and we use \( \delta_1 + \lambda n_1^2 = 0 \) such that:

\[
F(r) = \frac{1}{\prod_{i=1}^p (r^2 - n_i^2)^{q_i}} \left[ -\lambda \int_1^r \frac{(s^2 - n_i^2)^{q_i}}{sds} - 2m \right]
\]

while the constraints on the values of the nut parameters \( n_i \) and the cosmological constant \( \lambda \) take the form \( \lambda n_i^2 = -\delta_i \). A positive value for the cosmological constant can still be accommodated if we take \( \delta_i < 0 \) (for instance a product of hyperboloids, for which \( \delta_i = -1 \)). Let us take for simplicity a negative cosmological constant \( \lambda = -\frac{d-1}{l^2} \) and let us suppose that all the \( \delta_i \)'s are the same, \( i.e. \delta_i = \delta \) (for instance we can have a product of spheres or more generally products \( CP^{a_i} \) factors, for various values of \( a_i \), normalised such that their cosmological constant is \( \delta \)). Assume then that the base space contains a product of \( b_i CP^{a_i} \) factors. Then the dimension of the total space is \( d = \sum_i 2a_i b_i + 3 \), \( n_i^2 = \frac{\delta l^2}{d-1} \equiv n^2 \) and we have:

\[
F(r) = \frac{1}{(r^2 - n^2)\sum_i a_i b_i} \left[ -\lambda \int_1^r \frac{(s^2 - n^2)\sum_i a_i b_i}{sds} - 2m \right]
\]

\[
= \frac{1}{(r^2 - n^2)\sum_i a_i b_i} \left[ \frac{(r^2 - n^2)\sum_i a_i b_i + 1}{l^2} - 2m \right]
\]

It is convenient at this time to make the change of variables such that \( \rho^2 = r^2 - n^2 \):

\[
F(\rho) = \frac{\rho^2}{l^2} - \frac{2m}{\rho^{d-3}}
\]

\[
= \frac{\rho^2}{l^2} \left[ 1 - \frac{2ml^2}{\rho^{d-1}} \right] \equiv \frac{\rho^2}{l^2} g(\rho)
\]
It is now easy to see that the metric \(21\) with \(p = \sum b_i\) can be written in the following form:

\[
 ds^2 = \frac{4\delta \rho^2}{d-1}g(\rho)(d\chi + \sum_{i=1}^p A_i)^2 + \frac{(d-1)\rho^2}{\left(\delta + \frac{(d-1)\rho^2}{l^2}\right) g(\rho)} + \sum_{i=1}^p \rho^2 g_{Mi} + \frac{l^2}{d-1} \left(\delta + \frac{(d-1)\rho^2}{l^2}\right) dy^2
\]

Making now the change of variables \((d-1)\rho^2 \rightarrow \rho^2\), defining \(a^{d-1} = 2ml^2(d-1)^{\frac{d-1}{2}}\) and rescaling \(y\) to absorb the constant factor \(\frac{l^2}{d-1}\) we eventually obtain

\[
 ds^2 = \frac{4\delta \rho^2}{(d-1)^2}\left(1 - \frac{a^{d-1}}{\rho^{d-1}}\right)(d\chi + \sum_{i=1}^p A_i)^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{l^2} + \delta\right)\left(1 - \frac{a^{d-1}}{\rho^{d-1}}\right)} + \sum_{i=1}^p \frac{\rho^2}{d-1^2}g_{Mi}
\]

\[+ \left(\frac{\rho^2}{l^2} + \delta\right) dy^2 \tag{25}\]

which is the most general (Euclidian) form of the odd-dimensional Eguchi-Hanson solitons \([30]\), whose base space contains \(b_i\) factors \(CP^{a_i}\). The general solution whose base space contains a number of unit curvature spheres \(CP^1 = S^2\) has been analysed in \([30, 34]\). More generally we can replace the \(CP^{a_i}\) factors by arbitrary Einstein-Kähler manifolds \(M_i\) normalised such that their Einstein constants are equal \(\delta_i = \delta\) for all \(i = 1..p\). The parameter \(\delta\) is not essential and it can be absorbed by an appropriate rescaling of the radial coordinate and redefinition of the parameter \(a\). Without losing generality we can then set \(\delta = 1\).

It is interesting to note that while the Eguchi-Hanson solitons constructed over Einstein-Kähler spaces are in general nonsingular there are also Lorentzian section in odd-dimensions for which the curvature singularities at the origin can be easily avoided. Take for instance the five-dimensional metric:\(^5\)

\[
 ds^5 = -\frac{\rho^2}{4}\left(1 - \frac{a^4}{\rho^4}\right)(dt - \cosh \theta d\phi)^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{l^2} - 1\right)\left(1 - \frac{a^4}{\rho^4}\right)} + \frac{\rho^2}{4}(d\theta^2 + \sinh^2 \theta d\phi^2) + \left(\frac{\rho^2}{l^2} - 1\right) dy^2
\]

which is a solution of vacuum Einstein field equations with negative cosmological constant \(\lambda = -\frac{4}{l^2}\). In order to keep the signature of the metric Lorentzian we must restrict the values of the radial coordinate such that \(\rho > l\). Depending on the sign of the parameter \(a^4\) we can have a horizon located at \(\rho = a\) and in both situations the curvature singularity located at origin is avoided. In the limit in which the cosmological constant vanishes, \(i.e.\ l \rightarrow \infty\), the metric describes the product of a four-dimensional Eguchi-Hanson-like metric with a flat direction and there is no way to avoid the curvature singularity at \(r = 0\) while keeping the signature of the metric Lorentzian.\(^6\)

5 Conclusions

We have considered here higher dimensional solutions of the vacuum Einstein field equations with and without cosmological constant. These solutions are constructed as radial extensions\(^5\)

\(^5\)This metric can be formally obtained from \([20]\) by setting \(p = 1\) and \(\delta = -1\) and replacing \(CP^1\) by \(H^2\) (see also \([35]\)).

\(^6\)Or at least allow it to be Riemannian.
of circle fibrations over even dimensional spaces that can be factored in general as products of Einstein-Kähler spaces. The novelty of our solutions is that by associating a NUT charge $N$ with every such factor of the base space we have obtained higher dimensional generalizations of Taub-NUT spaces that can have quite generally multiple NUT parameters. In our work we have given the Lorentzian form of the solutions however, in order to understand the singularity structure of these spaces we have concentrated mainly on the their Euclidian sections. In most of the cases the Euclidean section is simply obtained using the analytic continuations $t \rightarrow it$ and $N_j \rightarrow in_j$. When continuing back the solutions to Lorentzian signature the roots of the function $F(r)$ will give the location of the chronology horizons since across these horizons $F(r)$ will change the sign and the coordinate $r$ changes from spacelike to timelike and vice-versa.

To render such metrics regular one follows a procedure [23] in which the basic idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters the metrics are singular – it is only for careful choices of the parameters that they become regular. In order to globally define the 1-form $d\tau + \sum_{i=1}^{p} 2n_i A_i$ we use various coordinate patches to cover the manifold, defining the 1-form on each patch. This can be done consistently only if we identify $\tau$ periodically, while the nut parameters $n_i$ must be rationally related. On the other hand, $r = \pm n_i$ correspond to curvature singularities, unless we also require that $F_E = 0$ there as well. Now by removing the possible conical singularities at the roots of $F_E(r)$ (be they at $r = \pm n_i$ or elsewhere) we get another periodicity for the Euclidian time $\tau$. By matching the two periodicities we obtained for $\tau$ we get another restriction on the value of the parameters appearing in the metric. As an example of this analysis we have considered the 6-dimensional Taub-NUT spaces constructed over both $S^2 \times S^2$ and $CP^2$. While the fibration over $CP^2$ is in general non-singular, we found that only the bolt solution was non-singular for the fibration over $S^2 \times S^2$, with distinct nut parameters.

While one could think that more generally there are no regular fibrations with distinct nut parameters over base spaces that are products of identical factors, it turns out that this is not the case for fibrations over products of distinct manifolds. Take for instance the 8-dimensional metric constructed as a fibration over $CP^2 \times S^2$. If the cosmological constant is zero we can have in general two distinct nut parameters. There then exists a NUT of intermediate dimensionality: assuming that the nut parameter corresponding to the $CP^2$ factor is $n_1$, while the one corresponding to $S^2$ is $n_2$, then the periodicity of the Euclidian time can be set to $8\pi n_2 = 12\pi n_1$. There exists a regular 4-dimensional nut located at $r = n_2 = \frac{3}{2} n_1$.

As discussed in Section 3, there exists another way to obtain NUTs of intermediate dimensionality constructed over spaces of the same nature. However the price to pay is the use of non-canonically normalised Einstein-Kähler manifolds.

We would also like to take the opportunity and comment at this point on the existence of Misner string singularities in cases where the base space contains 2-dimensional hyperboloids $H^2$ (respectively planar geometries $T^2$). In the literature it is often stated that in these cases there are no hyperbolic (respectively planar) Misner strings [4, 13, 14]. From the general discussion in Section 2 we can see that this statement is true only if the Einstein-Kähler
geometries $H^2$ (respectively $T^2$) are not compact. Otherwise, we find that the integral of the 2-form $2ndA$ over closed 2-cycles in $H^2$ (respectively $T^2$) can have a finite value. This implies that the Euclidian time $\tau$ must have (under appropriate normalization of the compact space) periodicity $8\pi n/k$, for an integer $k$; we can therefore speak about hyperbolic (planar) Misner strings.

Our construction applies more generally, yielding multiple nut-charged generalizations of inhomogeneous Einstein metrics on complex line bundles [26, 27]. In this case we replace the Einstein-Kähler manifold $M$ by a product of Einstein-Kähler manifolds $M_i$ with arbitrary even-dimensions and to each such factor we associate a nut parameter $N_i$. As is was conjectured in [26], we find that, quite generally, in higher dimensions there are various constraints to be imposed on the possible values of the cosmological constant $\lambda$, the nut parameters $N_i$ and the values of the various $\delta$’s. These solutions represent the multiple nut parameter extension of the inhomogeneous Einstein metrics on complex line-bundles described in [23]. It is also possible to cast these solutions into a different form, by explicitly encoding the constraint conditions into the metric. However this requires us to resort to non-canonically normalised Einstein-Kähler manifolds.

In Section 4 we presented the multiple nut parameter extension of the metrics constructed by Lü, Page and Pope in [27]. In this case we replaced the Einstein-Kähler manifold $M$ by a product of Einstein-Kähler manifolds $M_i$ with arbitrary even-dimensions and to each such factor we associated a nut parameter $N_i$. The case in which $Y$ is one-dimensional is particularly interesting to us since it provided us with the most general form of the odd-dimensional Eguchi-Hanson-type instantons found recently by Clarkson and Mann [30].

Leaving a more detailed study of these solutions for future work, it is worth mentioning that our solutions can be used as test-grounds for the AdS/CFT correspondence and more generally in context of gauge/gravity dualities. For the present solutions the boundary is generically a circle fibration over base spaces that, being products of general Einstein-Kähler manifolds, can have exotic topologies. In particular, one should be able to understand the thermodynamic phase structure of such dual field theories by working out the corresponding phase structure for our gravity solutions in the bulk.

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References

[1] C. W. Misner, J. Math. Phys. 4 (1963) 924; C. W. Misner, in Relativity Theory and Astrophysics I: Relativity and Cosmology, edited by J. Ehlers, Lectures in Applied Mathematics, vol. 8 (American Mathematical Society, Providence, RI, 1967), p. 160.

[2] S. W. Hawking, C. J. Hunter and D. N. Page “Nut Charge, Anti-de Sitter Space and Entropy” Phys. Rev. D 59 (1999) 044033 [hep-th/9809035].

[3] R. B. Mann, “Misner string entropy,” Phys. Rev. D 60, 104047 (1999) arXiv:hep-th/9903229.
[4] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Large N phases, gravitational instantons and the NUTs and bolts of AdS holography,” Phys. Rev. D 59, 064010 (1999) [arXiv:hep-th/9808177].

[5] R. Clarkson, L. Fatibene and R. B. Mann, “Thermodynamics of \((d + 1)\)-dimensional NUT-charged AdS Spacetimes”, Nucl. Phys. B652 348 (2003) [hep-th/0210280].

[6] C. V. Johnson and H. G. Svendsen, “An exact string theory model of closed time-like curves and cosmological singularities,” Phys. Rev. D 70, 126011 (2004) [arXiv:hep-th/0405141].

[7] A. Strominger, “The dS/CFT correspondence,” JHEP 0110, 034 (2001) [arXiv:hep-th/0106113].

[8] R. Clarkson, A. M. Ghezelbash and R. B. Mann, “Entropic N-bound and maximal mass conjectures violation in four dimensional Taub-Bolt(NUT)-dS spacetimes,” Nucl. Phys. B 674, 329 (2003) [arXiv:hep-th/0307059].

[9] R. Clarkson, A. M. Ghezelbash and R. B. Mann “A Review of the N-bound and the Maximal Mass Conjectures Using NUT-Charged dS Spacetimes” Int. J. Mod. Phys. A19 (2004) 3987-4036 [arXiv:hep-th/0408058].

[10] R. Clarkson, A. M. Ghezelbash and R. B. Mann “A Review of the N-bound and the Maximal Mass Conjectures Using NUT-Charged dS Spacetimes” Int. J. Mod. Phys. A19 (2004) 3987-4036 [arXiv:hep-th/0408058].

[11] S. W. Hawking, C. J. Hunter “Gravitational Entropy and Global Structure” Phys. Rev. D 59 (1999) 044025 [hep-th/9808085].

[12] M. M. Taylor-Robinson, “Higher dimensional Taub-Bolt solutions and the entropy of non compact manifolds,” [hep-th/9809041].

[13] D. Astefanesei, R. B. Mann and E. Radu, “Breakdown of the entropy / area relationship for NUT-charged spacetimes,” [arXiv:hep-th/0406050].

[14] D. Astefanesei, R. B. Mann and E. Radu, “Nut charged space-times and closed timelike curves on the boundary,” [arXiv:hep-th/0407110].

[15] E. Radu and D. Astefanesei, “Quantum effects in a rotating spacetime,” Int. J. Mod. Phys. D 11, 715 (2002) [arXiv:gr-qc/0112029].

[16] A. H. Taub “Empty Space-Times Admitting a Three Parameter Group of Motions” Annal. Math. 53 (1951) 472.

[17] E. Newman, L. Tamburino, and T. Unti “Empty-space generalization of the Schwarzschild metric” J. Math. Phys. 4 (1963) 915.

[18] M. Demiansky and E. T. Newman, “A combined Kerr-NUT solution of the Einstein field equations” Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys. 14 653(1966).
[19] G. W. Gibbons and M. J. Perry, “New Gravitational Instantons And Their Interactions,” Phys. Rev. D 22, 313 (1980).

[20] N. Alonso-Alberca, P. Meessen and T. Ortin, “Supersymmetry of topological Kerr-Newman-Taub-NUT-adS spacetimes,” Class. Quant. Grav. 17, 2783 (2000) [arXiv:hep-th/0003071].

[21] D. Klemm, “Rotating black branes wrapped on Einstein spaces,” JHEP 9811, 019 (1998) [arXiv:hep-th/9811126].

[22] F. A. Bais and P. Batenburg, “A New Class Of Higher Dimensional Kaluza-Klein Monopole And Instanton Solutions,” Nucl. Phys. B 253, 162 (1985).

[23] D. N. Page and C. N. Pope, “Inhomogeneous Einstein Metrics On Complex Line Bundles,” Class. Quant. Grav. 4, 213 (1987).

[24] M. M. Akbar and G. W. Gibbons, “Ricci-flat metrics with U(1) action and the Dirichlet boundary-value problem in Riemannian quantum gravity and isoperimetric inequalities,” Class. Quant. Grav. 20, 1787 (2003) [arXiv:hep-th/0301026].

[25] A. Awad and A. Chamblin, “A bestiary of higher dimensional Taub-NUT-AdS spacetimes,” Class. Quant. Grav. 19, 2051 (2002), [hep-th/0012240].

[26] R. Mann and C. Stelea, “Nuttier (A)dS black holes in higher dimensions,” Class. Quant. Grav. 21, 2937 (2004), [arXiv:hep-th/0312285].

[27] H. Lu, D. N. Page and C. N. Pope, “New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kaehler manifolds,” [arXiv:hep-th/0403079].

[28] W. Chen, H. Lu, C. N. Pope and J. F. Vazquez-Poritz, “A note on Einstein-Sasaki metrics in D \( \geq 7 \),” [arXiv:hep-th/0411218].

[29] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Supersymmetric AdS backgrounds in string and M-theory,” [arXiv:hep-th/0411194].

[30] R. Clarkson and R. B. Mann, “Eguchi-Hanson Solitons” [arXiv:hep-th/0508109].

[31] D.A. Konkowski, T.M. Helliwell and L.C. Shepley, Phys. Rev. D31, 1178 (1985); D.A. Konkowski and T.M. Helliwell, Phys. Rev. D31, 1195 (1985).

[32] M. T. Anderson, “Existence and stability of even dimensional asymptotically de Sitter spaces,” [arXiv:gr-qc/0408072].

[33] M. M. Akbar and P. D. D’Eath, “CP(2) and CP(1) sigma models in supergravity: Bianchi type IX instantons and cosmologies,” Class. Quant. Grav. 21, 2407 (2004) [arXiv:hep-th/0401194].

[34] R. Clarkson and R. B. Mann,”Eguchi-Hanson Solitons in Odd Dimensions” [arXiv:hep-th/0508200].
[35] D. Astefanesei, R. B. Mann and C. Stelea, “Nuttier bubbles,” arXiv:hep-th/0508162.