Entanglement entropies of the $J_1 - J_2$ Heisenberg antiferromagnet on the square lattice

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Using a modified spin-wave theory which artificially restores zero sublattice magnetization on finite lattices, we investigate the entanglement properties of the Néel ordered $J_1 - J_2$ Heisenberg antiferromagnet on the square lattice. Different kinds of subsystem geometries are studied, either corner-free (line, strip) or with sharp corners (square). Contributions from the $nG = 2$ Nambu-Goldstone modes give additive logarithmic corrections with a prefactor $nC/2$ independent of the Rényi index. On the other hand, corners lead to additional (negative) logarithmic corrections with a prefactor $l_q^C$ which does depend on both $nC$ and the Rényi index $q$, in good agreement with scalar field theory predictions. By varying the second neighbor coupling $J_2$ we also explore universality across the Néel ordered side of the phase diagram of the $J_1 - J_2$ antiferromagnet, from the frustrated side $0 < J_2/J_1 < 1/2$ where the area law term is maximal, to the strongly ferromagnetic regime $-J_2/J_1 ≫ 1$ with a purely logarithmic growth $S_q = nC/2 \ln N$, thus recovering the mean-field limit for a subsystem of $N$ sites. Finally, a universal subleading constant term $\gamma_q^{\text{ord}}$ is extracted in the case of strip subsystems, and a direct relation is found (in the large-$S$ limit) with the same constant extracted from free lattice systems. The singular limit of vanishing aspect ratios is also explored, where we identify for $\gamma_q^{\text{ord}}$ a regular part and a singular component, explaining the discrepancy of the linear scaling term for fixed width vs. fixed aspect ratio subsystems.

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I. INTRODUCTION

Entanglement properties of interacting quantum spin systems have recently attracted a lot of interest. In particular, great attention is paid to the universal information carried by bipartite entanglement measures such as the Rényi entanglement entropies (EEs) defined by

$$S_q = \frac{1}{1 - q} \ln \text{Tr} (\hat{\rho}_\Omega)^q,$$

(1.1)

where $\hat{\rho}_\Omega$ is the reduced density matrix of a given subsystem $\Omega$ (see Fig. 1) computed in the ground-state wave-function. Note that the special limit of $q \rightarrow 1$ corresponds to the von Neumann EE given by $S_1 = -\text{Tr} \rho_{\Omega} \ln \rho_{\Omega}$ and is always implicitly understood whenever we refer to $q = 1$. As a general result, at $T = 0$ the Rényi EEs follow an area law$^{1,2}$ in dimension $d$

$$S_q = a_q L^{d-1} + \cdots$$

(1.2)

where $L^{d-1}$ is the size of the boundary between subsystem $\Omega$ and the rest, and the ellipses are subleading corrections. Such corrections have been shown to carry universal information about topological order$^{3-6}$, or the presence of Nambu-Goldstone modes associated to the breaking of a continuous symmetry$^{7-10}$. In the latter case, Metlitski and Grover (MG)$^7$ have derived the following analytical expression in the case of smooth boundaries (no corner), as for instance depicted for $d = 2$ in Fig. 1 (a) for $L \times \ell$ strip subsystems:

$$S_q = a_q L^{d-1} + \frac{nG}{2} \ln \left( \frac{\rho_s}{\ell} L^{d-1} \right) + \gamma_q^{\text{ord}},$$

(1.3)

where $\rho_s$ is the stiffness, $v$ the velocity of the $nG$ Nambu-Goldstone modes, and $\gamma_q^{\text{ord}}$ a universal geometric constant. In the case of subsystems having sharp corners, as depicted in Fig. 1 (b), it is expected that$^7$:

$$S_q = a_q L^{d-1} + \frac{nG}{2} \ln \left( \frac{\rho_s}{\ell} L^{d-1} \right) + nG \sum_c l_q^C(\varphi_c) \ln \left( \frac{L}{a} \right) + \gamma_q^{\text{ord}},$$

(1.4)

where $a$ is a non-universal length scale, and the corner contribution depends on $nC$, the Rényi parameter $q$, and the number of corners $c$ of angle $\varphi_c$. The contributions $l_q^C(\varphi_c)$ from each corner come from the (free) Goldstone modes and can be computed, following the work of Casini and Huerta$^{11}$ on scalar field theory, by the numerical solution of a set of non-linear differential equations, valid for $\varphi_c \in [0, \pi]$ ($l_q(\varphi) = l_q(2\pi - \varphi)$) and $q \in \mathbb{N} \setminus \{1\}$.

Previous works have explored the scaling of the entanglement entropy in ground-states of systems that break
continuous symmetries in the thermodynamic limit. Subleading logarithmic corrections arising from the Goldstone modes have been observed in quantum Monte Carlo simulations of finite spin systems, even though the prefactor of this correction did not perfectly agree with the prediction $n_G/2$, until a very recent large-scale, low-temperature quantum Monte Carlo (QMC) investigation by Kulchytskyy et al. for the 2d XY model and $q = 2$. Logarithmic corrections have also been observed in finite-size SW calculations (similar to the ones presented in this manuscript), but not with a high-enough precision to again ascertain the prediction, except for the case a line-shaped subsystem for which the prefactor $n_G/2$ could be recovered assuming further subleading corrections (see also the recent work Ref. [15]). The existence of logarithmic corrections have also been discussed based on a phenomenological picture of the tower of low-lying states in the symmetry-broken phase of antiferromagnets. Logarithmic corrections due to corner contributions have on the other hand been identified and calculated precisely in free lattice systems, broken continuous symmetries as well as for various critical points using QMC, cluster expansions or tree tensor network techniques. In a recent work, predictions for the universality of corner contributions in various theories are also provided. Finally, Kulchytskyy et al. could also compute with QMC the subleading constant correction $\gamma_{\text{ord}}$ in the 2d XY model, finding a good agreement with the prediction of MG in Ref. 7.

In this paper, we provide a systematic high-precision study of the universal nature of three subleading terms of the Rényi EE appearing in Eq. (1.4) for a generic model of quantum antiferromagnetism in two dimensions ($d = 2$). This is achieved using a large-$S$ semi-classical approach, the modified linear spin-wave (SW) theory, where the rotational SU(2) symmetry, while practically broken, is artificially restored for finite size systems. We focus on the $J_1 - J_2$ spin-S antiferromagnet defined on a bipartite $L \times L$ square lattice by the following Hamiltonian

\[ H = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle \langle ij \rangle \rangle} S_i \cdot S_j + h \sum_i (-1)^i S^z_i, \quad (1.5) \]

where $S$ are spin-$S$ operators, interactions act between nearest neighbours $\langle ij \rangle$ and second nearest neighbours $\langle \langle ij \rangle \rangle$ along the diagonals of a square lattice (see Fig. 1), and $h$ is an external staggered field. We impose periodic boundary conditions in all directions. At $h = 0$ this model spontaneously breaks the SU(2) symmetry at zero temperature in the thermodynamic limit, and displays Néel order for $J_2 < J_1$, with $J_1 / J_2 \to J_1 / 2$ for $S \to \infty$. The restoration of zero sublattice magnetization in finite systems is made possible by tuning the small staggered field $h^\ast(L)$ such that on any site $\langle S^z_i \rangle = 0$. As first done in Refs. 8 and 10, this allows to correctly compute Rényi EEs on finite systems. Here we make a systematic and extensive study across the full Néel regime $-\infty < J_2 < J_1^\ast$ for various subsystem shapes and sizes in order to characterize contributions form (i) Nambu-Goldstone modes, (ii) corners, (iii) frustration effects $J_2 / J_1 > 0$, and (iv) geometric effects appearing through the universal constant $\gamma_{\text{ord}}$ in Eq. (1.3).

Let us briefly summarize our main results. Using a large-$S$ approach, we have numerically extracted the three subleading corrections in the scaling of EEs Eq. (1.4) with $n_G = 2$ for SU(2) antiferromagnets. Universality has been tested in the entire Néel ordered regime of $J_1 - J_2$ Heisenberg model Eq. (1.5) for various $S$, even in the frustrated regime where QMC is inapplicable. In the case of subsystems having sharp corners, small negative corner terms $l_q^c$ are found, in perfect agreement with the predictions by Casini and Huerta for free scalar fields. The non-universal area-law term has also been studied as a function of the second neighbour coupling, showing remarkable behaviors both in the mean-field limit ($-J_2 / J_1 \to 1$) where it vanishes, and close to the frustrated critical point $J_2^\ast$ where the area law prefactor strongly increases, while log corrections due to Nambu-Goldstone modes are still present. Furthermore, the additional geometric constant $\gamma_{\text{ord}}$, which depends on the subsystem aspect ratio $c / L$, is extracted for various Rényi indices, and a simple relation with the free scalar field result is derived. We have also explored the limit of vanishing aspect ratios where a non-trivial slow singular behavior shows up as $\gamma_{\text{ord}}(c / L \ll 1) \to -\infty$.

The rest of the paper is organized as follows. In Section II we start by recalling the modified SW formalism for the $J_1 - J_2$ spin-$S$ antiferromagnet, and how it can be used to compute the Rényi EEs. We then turn to the results for EEs in Section III where we discuss several aspects: we first describe numerical diagonalization results, which can be conveniently performed up to subsystems of $\lesssim 10^5$ sites, for various shapes of subsystems including strips (Sec. IIIA and Fig. 1a) and squares (Fig. 1b), with a particular focus on the corner contributions (Sec. IIIB) and their dependence on the Rényi parameter $q$. In Section IIIC the dependence on the second neighbour coupling $J_2$ is studied, focussing on the non-universal area law prefactor $a_q$. In Section IV we discuss the constant term $\gamma_{\text{ord}}$ which is compared to the field-theory prediction of MG in Sec. IV A. An interesting connection to the free scalar field result is achieved in Sec IVB. We further explore the singular limit of vanishing aspect ratios in Sec. IVC using quasi-analytical results for single and double line subsystems where translation symmetry inside the subsystem allows to get an explicit expression for $S_q$. Finally we summarize and discuss our results in Section V. Details of spin-wave calculations are provided in Appendix A, analytical results for the mean-field limit $-J_2 / J_1 \gg 1$ are presented in Appendix B, and an analytical derivation for one-dimensional subsystems is given in Appendix C.
II. MODIFIED SPIN-WAVE APPROACH

A. Dyson-Maleev transformation and Bogoliubov diagonalization

We use the Dyson-Maleev formalism\textsuperscript{29,30} to map spin operators onto bosonic ones. For sites on sublattice A of the square lattice:

\[ S_i^z = S - b_i^\dagger b_i, \quad S_i^+ = (2S - b_i^\dagger b_i) b_i, \quad S_i^- = b_i, \]  
(2.1)

and for the sublattice B:

\[ S_i^z = b_i^\dagger b_i - S, \quad S_i^+ = -b_i^\dagger (2S - b_i^\dagger b_i), \quad S_i^- = -b_i. \]  
(2.2)

Truncating at 1/S order, the \( J_1 - J_2 \) Hamiltonian Eq. (1.5) becomes (up to a constant)

\[ \mathcal{H} = \sum_i (h + 4S[J_1 - J_2]) b_i^\dagger b_i + \sum_{\langle ij \rangle} S J_2 (b_i b_j^\dagger + b_j^\dagger b_i) - \sum_{\langle ij \rangle} S J_1 (b_i b_j + b_j^\dagger b_i). \]  
(2.3)

After a Fourier transformation, it reads

\[ \mathcal{H} = \sum_k A_k (b_k^\dagger b_{-k} + b_{-k}^\dagger b_k) + B_k (b_k^\dagger b_k + b_{-k}^\dagger b_{-k}), \]  
(2.4)

with

\[ A_k = 2S J_2 \cos k_x \cos k_y + 2S (J_1 - J_2) + \frac{h}{2}, \]  
(2.5)

\[ B_k = -S J_1 [\cos k_x + \cos k_y]. \]  
(2.6)

The quadratic part of the above Hamiltonian can be diagonalized via a standard Bogoliubov transformation:

\[ b_k = u_k \alpha_k - v_k \alpha_{-k}^\dagger, \quad b_k^\dagger = u_k \alpha_k^\dagger - v_k \alpha_{-k}. \]  
(2.7)

The quasiparticle operators \( \alpha_k \) and \( \alpha_{-k}^\dagger \) satisfy bosonic commutation relations provided \( u_k^2 - v_k^2 = 1 \), and diagonalize (2.4) if

\[ u_k^2 = \frac{1}{2} \left( \frac{A_k}{\sqrt{A_k^2 - B_k^2}} + 1 \right), \]  
(2.8)

\[ v_k^2 = \frac{1}{2} \left( \frac{A_k}{\sqrt{A_k^2 - B_k^2}} - 1 \right). \]  
(2.9)

In terms of Bogoliubov quasi-particles, the \( J_1 - J_2 \) Hamiltonian takes the simpler form

\[ \mathcal{H} = \sum_k \Omega_k \alpha_k^\dagger \alpha_k + \text{constant}, \]  
(2.10)

with the SW excitation spectrum \( \Omega_k = 2\sqrt{A_k^2 - B_k^2} \) (this spectrum is illustrated in Appendix A). In the vicinity of the two minima at \( k = (\pi, \pi) \) and \( (0,0) \), the dispersion is linear, with a velocity

\[ v_{sw} = 2\sqrt{2S} \sqrt{J_1 (J_1 - 2J_2)}, \]  
(2.11)

which is defined only on the AF side \( J_2 < J_1/2 \). The SW spectrum and velocity are illustrated in Fig. 13 and 14 of Appendix A.

In the thermodynamic limit, the continuous \( \text{SU}(2) \) symmetry of the original \( J_1 - J_2 \) Hamiltonian can be spontaneously broken, with the two associated Nambu-Goldstone modes at \( k = (\pi, \pi) \) and \( (0,0,0) \). The corresponding staggered magnetization order parameter is given at the 1/S order by

\[ m_{AF} = \lim_{h \to 0} \lim_{N \to \infty} \left| \langle S_i^z \rangle \right| = S + \frac{1}{2} - \frac{1}{8\pi^2} \int_{Bz} d^2k \frac{A_k}{\Omega_k}. \]  
(2.12)

In Appendix A, this expression is evaluated numerically to obtain the range of parameter space where Néel order is expected from this SW treatment.

B. Spin-Wave theory for finite size systems

The above SW approach assumes a classical ordered state as a starting point. This does not allow for a correct study of finite size effects since the spin rotational symmetry has to remain unbroken on finite-size lattices. In order to repair this, adding a staggered magnetic field to the quantum antiferromagnet allows to artificially restore zero sub-lattice (SW-corrected) magnetization, as originally proposed in Refs. 26 and 27. This will turn crucial to capture the subleading scaling terms in the entanglement entropy.

In this approach, one imposes that for any given finite size sample \( \langle S_i^z \rangle = 0 \quad \forall i \), which yields a staggered field \( h^* \) such that the number of Holstein-Primakoff bosons \( \langle n \rangle = S \). This leads to

\[ \sum_k A_k(h^*) \frac{1}{\Omega_k(h^*)} = N(2S + 1). \]  
(2.13)

This regularizing field is very small and scales rapidly to zero with the system size\textsuperscript{8}. Indeed, one can rewrite Eq. (2.13) as follows

\[ N(2S + 1) - \sum_{k \neq k_0} A_k(h^*) \frac{1}{\Omega_k(h^*)} = 2 \frac{A_{k_0}(h^*)}{\Omega_{k_0}(h^*)}, \]  
(2.14)

where \( k_0 = (0,0) \) and \( (\pi, \pi) \) are the singular modes where the dispersion vanishes in the absence of staggered field. The contributions from these two modes, divergent in the limit \( h^* \to 0 \), are similar:

\[ \frac{A_{k_0}}{\Omega_{k_0}} = \frac{4S J_1 + h^*}{\sqrt{h^*(h^* + 8S J_1)}}. \]  
(2.15)

Defining

\[ m^*(N, h^*) = S + \frac{1}{4} - \frac{1}{2N} \sum_{k \neq k_0} A_k(h^*) \frac{1}{\Omega_k(h^*)}, \]  
(2.16)
we obtain a self-consistent equation for \( h^* \)

\[
h^* = 4 S J_1 \left[ \frac{1}{\sqrt{1 - \left( \frac{1}{N m^*} \right)^2}} - 1 \right].
\] (2.17)

In the limit \( N \gg 1, m^* \rightarrow m_{AF} \), and we have

\[
h^* \approx \frac{2 S J_1}{m_{AF} N^2}.
\] (2.18)

As seen below, it is essential to determine the actual value of \( h^*(L) \) with a high precision in order to compute accurately various finite size correlations. Since the field \( h^* \) gets rapidly very small with increasing system sizes, we resort to a multiple precision evaluation of the self-consistent equation Eq. (2.17). In Fig. 2 we present the result showing the behavior of \( h^*(L) \) for some representative values of \( J_2 \) and \( S \). In all cases, the staggered field vanishes very fast and is well described by Eq. (2.18) at large enough \( L \).

Interestingly this small staggered field opens a gap in the excitation spectrum

\[
\Delta^* \simeq \sqrt{2 S J_1 h^*} \simeq \frac{2 S J_1}{m_{AF} N}.
\] (2.19)

which scales in the same way as the Anderson tower of states. Therefore, the excitation spectrum has linearly dispersing Nambu-Goldstone (SW) modes with a level spacing \( \sim 1/L \) and a tower of states like finite size gap \( \sim 1/L^2 \) produced by the symmetry restoring staggered field.

We use this modified finite-size SW approach to compute the entanglement entropy as detailed below. In order to show that it reproduces fairly well the physics of finite-size systems, we also compare in Appendix A results for the finite-size structure factor for \( S = 1/2 \) and various \( J_2 < 0 \) to the ones obtained with the exact QMC method.

## C. Entanglement entropy

As the diagonalized Hamiltonian Eq. (2.10) is noninteracting, Wick’s theorem eases the computation of entanglement entropy, which can nicely be extracted from the correlation matrix\(^3\), an object which contains all two-body correlations within a block of sites. For completeness, we recapitulate here the essential formulae.

We first need to define single particle Green’s function \( \langle b_i^\dagger b_j \rangle = -\delta_{ij}^{\delta_k} + f_{ij} \) and \( \langle b_i b_j \rangle = g_{ij} \), with

\[
f_{ij} = \frac{1}{2N} \sum_k A_k \cos [k \cdot (r_i - r_j)]
\]

\[
g_{ij} = -\frac{1}{2N} \sum_k B_k \cos [k \cdot (r_i - r_j)].
\] (2.20)

We remark that \( g_{ij} = 0 \) (\( f_{ij} = 0 \)) if \( i \) and \( j \) belong to the same (different) sublattice(s).

The entanglement entropy of a region \( \Omega \) containing \( N_\Omega \) sites can then be extracted\(^33–35\) from the eigenvalues \( \nu_i^2 \) of the \( N_\Omega \times N_\Omega \) correlation matrix \( C \)

\[
C_{ij} = \sum_{i' \in \Omega} (f_{ii'} + g_{ii'})(f_{i'j} - g_{i'j})
\] (2.21)

where \( i, j \in \Omega \). Due to the sublattice properties of \( f \) and \( g \), we have that \( C_{ij} = C_{ji} \) if \( i \) and \( j \) belong to the same sublattice, \( C_{ij} = -C_{ji} \) otherwise.

The Rényi entanglement entropy is obtained as\(^32\)

\[
S_q = \frac{1}{q-1} \sum_{\ell=1}^{N_\Omega} \ln \left[ (\nu_\ell + \frac{1}{2})^q - (\nu_\ell - \frac{1}{2})^q \right],
\] (2.22)

which for \( q = 1 \) reads

\[
S_1 = \sum_{\ell=1}^{N_\Omega} \epsilon \ln \left( \nu_\ell + \frac{\epsilon}{2} \right),
\] (2.23)

and for \( q = \infty \)

\[
S_\infty = \sum_{\ell=1}^{N_\Omega} \ln \left( \nu_\ell + \frac{1}{2} \right).
\] (2.24)

As first shown by Srednicki\(^36\), Callan and Wilczek\(^37\), the entropy of a free massless bosonic field obeys a strict area law, which is what we observe (data not shown) in the absence of the regularizing staggered field \( h^* \). However, as we will see below, the finite staggered field which opens a finite size gap \( \sim 1/N \) leads to an additive logarithmic correction proportional to the number of Goldstone bosons.

## III. RESULTS FOR EE

### A. Strip geometry

Let us start with the case of an \( L \times \ell \) strip subsystem embedded in an \( L \times L \) torus, as depicted in Fig. 1
with associated to the breaking of SU(2) rotational symmetry.

The properties of the area behavior does not depend on the number of subsystem sites but is used, allowing the diagonalization procedure to reach large sizes. This plot clearly demonstrates the area law for systems up to $L = 10^4$ and the deviation decreases slowly as $L$ increases.

This leads us to the conclusion that, within our SW approach, we find $l_q = n_G/2 = 1$ to be independent on $q$ and the aspect ratio of the subsystem, in perfect agreement with the field theoretical result by MG.

### B. Square subsystems: corner contributions

In addition to the breaking of continuous symmetries, logarithmic corrections to the area law can also be caused by geometry: In particular, logarithmic corrections induced by sharp corners of the subsystem have been discussed in several works. The prefactor of the logarithmic corner correction term is expected to be universal for all systems with the same type of symmetry breaking/phase transition. However, such corrections are quite difficult to capture with QMC since the prefactor is very small. Together with the contributions coming from Nambu-Goldstone modes Eq. (1.4), we expect a total correction of the form

$$ n_G \left( \sum_c \frac{\ell^q_c(\varphi_c)}{q} + \frac{1}{2} \right) \ln L, \quad (3.2) $$

![Figure 3. Entanglement Rényi entropies for the strip subsystem with different aspect ratios $\ell/L$ (upper panel) and fit results for the prefactor $l_q$ of the logarithmic scaling term as a function of the minimal system size $L_{\text{min}}$ included in the fit (lower panel). The results displayed here have been obtained for $S = 1/2$ and $J_2 = 0$. Clearly, $l_q = 1$ independent of $q$ and the aspect ratio of the subsystem.](image)

(a). This geometry has no corner and we therefore expect the expression Eq. (1.3) to hold. Results obtained from the exact diagonalization of the correlation matrix $C$ for systems up to $\sim 10^6$ lattice sites are shown in the upper panel of Fig. 3 where the Rényi entropies for $q = 1, 2, 3$ are displayed for three representative aspect ratios $\ell/L = 1/2$, 1/4, 1/8. Note that for this strip geometry, translation symmetry of the subsystem is used, allowing the diagonalization procedure to reach large sizes. This plot clearly demonstrates the area law behavior $S_q \sim a_q L$ since the dominant scaling behavior does not depend on the number of subsystem sites but only on its perimeter $2L$, which is independent of the aspect ratio of the subsystem. The properties of the area law prefactor $a_q$ will be analyzed in detail in Sec. III C, and the universal additive constant $\gamma_q$ from Eq. (1.3) in Sec. IV.

Here, we want to focus on the logarithmic correction associated to the breaking of SU(2) rotational symmetry with $n_G = 2$ Nambu-Goldstone modes, expected to be $\frac{2n_G}{2} \ln L$. This correction is believed to be universal as it should not depend on the geometry and only reflect the nature of the continuous symmetry which is broken in the ground state. Therefore, we perform fits to the general scaling ansatz

$$ S_q(L) = a_q L + b_q \ln L + c_q/L + d_q/L^2, \quad (3.1) $$

over various fit ranges $[L_{\text{min}}, L_{\text{max}}]$. Results for $l_q$ are plotted in the lower panel of Fig. 3 for various values of the Rényi parameter and several aspect ratios. For $q = 1, 2$ we clearly observe that $l_q = 1$ over basically the whole range of $L_{\text{min}}$, whereas for larger values of $q$, the convergence is relatively slow as these results are to our experience hampered by more severe finite size effects.

Nevertheless, the resulting $l_q$ is already very close to 1 and the deviation decreases slowly as $L_{\text{min}}$ is increased.

![Figure 4. Difference of entanglement entropies for the $S = 1/2$ $J_2 = 0$ Heisenberg antiferromagnet of square and strip subsystems having the same boundary length. The remaining dominant scaling term is the logarithmic term which stems from the corners of the square subsystem. We show fits (full lines) to the form $S^q_{\text{sq}}(\pi/2) [\ln(L) + b_q + c_q/L + d_q/L^2]$. SW results (symbols) are shown for two different aspect ratios a for $q = 1, 2, 3$ and 4.](image)
where the sum is taken over all sharp corners inside the subsystem making an angle $\varphi_c$. Here we aim at numerically extracting $\ell^c_q(\pi/2)$ for a square subsystem (panel (b) of Fig. 1), expected to coincide with the result of a free scalar field (see Refs. 40 and 41 and Appendix B), allowing us to extract the leading term of this difference to be given only by the corner log contribution:

$$S^\text{square}_q - S^\text{strip}_q = 8\ell^c_q(\pi/2) \ln L + \cdots$$ (3.3)

Numerical results are plotted in Fig. 4 where we clearly see that the above difference Eq. (3.3) is indeed dominated by a logarithmic scaling which allows us to extract $\ell^c_q(\pi/2)$. Small variations of the results for different aspect ratios of the strips (see left and right panels of Fig. 4) can be used as a measure of the error due to finite size effects and fitting procedure. Our results are displayed in Table I where we compare to the free-field results by Casini and Huerta (CH)\textsuperscript{11}.

Interestingly, we can also study the dependence on the Rényi index for non-integer values of $q$. In Fig. 5 we show $\ell^c_q(\pi/2)$ versus the Rényi parameter $q$ for four different aspect ratios. For $q$ not too large, the estimates obtained after fitting our numerical data (see caption of Fig. 4), are clearly independent of the aspect ratio, as expected. This non-trivial $q$-dependence for a free scalar field can be compared to recent numerical results for O(1) and O(2) Wilson-Fisher critical points\textsuperscript{21}, featuring qualitatively similar behaviors.

### C. $J_2$-dependence and area law prefactor

Besides universal contributions arising from Nambu-Goldstone modes and corners, we now study the dominant part which governs the entanglement growth with the subsystem area. As already discussed in the beginning of the paper, the $J_1 - J_2$ spin-$S$ Heisenberg model on the square lattice is Néel ordered for $J_2/J_1 < 0.5$ in the large $S$ limit (see Appendix A and Fig. 15 for the critical value of $J_2$ as a function of $S$). Scanning across the entire Néel ordered regime, we have performed fits to the form Eq. (3.1) for various values of the second neighbor coupling $J_2$ and spin $S$ for the strip geometry (corner-free) with a 1/8 aspect ratio. Shown in Fig. 6, the area law coefficient $a_q$ displays a quite remarkable behavior. First, the results appear to be almost independent of the spin size $S$. Then, as expected from the mean-field limit $J_2/J_1 \to -\infty$ (see Refs. 40 and 41 and Appendix B), $a_q$ goes to zero in the limit $-J_2/J_1 \gg 1$. This is because the ground-state becomes more and more classical, with

| $q$  | $J_{11}$ | $J_{22}$ | $J_{33}$ | $J_{44}$ |
|------|---------|---------|---------|---------|
| $q = 1$ | 0.0118  | 0.0064  | 0.0051  | 0.0043  |
| $q = 2$ | 0.0064  | 0.0051  | 0.0043  | 0.0039  |
| $q = 3$ | 0.0051  | 0.0043  | 0.0039  | 0.0036  |
| $q = 4$ | 0.0043  | 0.0039  | 0.0036  | 0.0033  |

Table I. Prefactor $\ell^c_q(\pi/2)$ of the corner logarithmic correction obtained after fitting data in Fig. 4. A comparison with data of Casini and Huerta (CH)\textsuperscript{11} is also given.

Figure 5. Logarithmic contribution of a $\frac{\pi}{2}$ corner of the subsystem as a function of $q$. This result is obtained for $J_2 = 0$ and $S = \frac{1}{2}$ by subtracting the entanglement entropy of a strip subsystem with the same perimeter as the square subsystem with $4 \frac{\pi}{2}$ corners and fitting to the same form as shown in Fig. 4 for different aspect ratios of the strip. Up to slight deviations for larger Rényi indices $q$ due to finite size effects, the results do not depend on the aspect ratio.

Figure 6. Top: Area law coefficients $a_q$ for $q = 1, 2$ (extracted for an aspect ratio $\ell/L = 1/8$) as a function of the next neighbor coupling $J_2/J_1$. Approaching the critical point in the frustrated regime ($J_2 > 0$) the area law coefficient grows rapidly. Bottom: Prefactor $\ell_q$ of the log correction in Eq. (1.3) due to the two Goldstone modes of the antiferromagnet. The deviation of $\ell_q$ from 1 close to the critical point in the frustrated regime reveals the limitation of the SW approximation.
a very low entanglement. However, less is known when
frustration is turned on, and we observe a rapid growth of
the area law term when the critical point is approached, a
feature also observed for the unfrustrated Heisenberg bi-
layer\textsuperscript{14}. Note that the validity of the SW calculation
can be questioned when quantum fluctuations become large,
approaching the critical point $J_2^\ell$. Nevertheless, we be-
lieve the results to be under control if $1 - m_{AF}/S \ll 1$, a
condition which can be checked in Fig. 15 of Appendix A.
Moreover, as long as the logarithmic term in the entan-
glement entropies scaling is still present and equal to one
(due to the two Nambu-Goldstone modes, as shown in the
bottom panel of Fig. 6) thus confirming the universality
across the ordered phase), we believe the SW approxima-
tion correctly captures the behavior of EE. In practice, we
start to see a deviation of $l_q$ from unity only for the
very last points at $J_2/J_1 \geq 0.3$ in Fig. 6.

IV. UNIVERSAL ADDITIVE CONSTANT $\gamma_{q}^{\text{ord}}$

A. Direct extraction from large-$S$ data

Following MG\textsuperscript{7}, a universal additive constant
$\gamma_{q}^{\text{ord}}(\ell/L)$, depending on the aspect ratio $\ell/L$ of a strip,
appears in the Rényi EE scaling Eq. (1.3). Nevertheless,
there is also a non-universal term involving the spin stiff-
ness and the SW velocity in Eq. (1.3). It is therefore
much easier to work in the $S \to \infty$ limit of the $J_1 - J_2$
Heisenberg Hamiltonian where $\rho_s$ and $v$ are known
exactly. In such a limit and having shown above that the
logarithmic prefactor is exactly given by $l_q = n_G/2 = 1$
(the corner contribution vanishes for this geometry), we
expect the EE scaling for strips with an aspect ratio $\ell/L$
to be in the $S \gg 1$ limit

$$S_q = a_q L + \ln \left( S \sqrt{\frac{J_1 - 2J_2}{8J_1} L} \right) + \gamma_{q}^{\text{ord}}(\ell/L). \quad (4.1)$$

This large $S$ expression has been used to fit our nume-ical SW data obtained for $S = 100$ and $J_2/J_1 = 0, -1$.
Results for the additive constant $\gamma_{q}^{\text{ord}}(\ell/L)$ are plotted
in Fig. 7 as a function of the aspect ratio $\ell/L$ for various Rényi parameters $q$. The agreement with the result ex-
tracted from Ref. 7 is excellent for $q = 2$. The universal
character of $\gamma_{q}^{\text{ord}}(\ell/L)$ is also corroborated by the fact
that our estimates do not depend on the values of the
second neighbour coupling $J_2/J_1 = 0, -1$; the sole $J_2$
dependence being given by the two first terms in Eq. (4.1).

For the $S = 1/2$ Heisenberg model ($J_2 = 0$), while we
have seen above that the logarithmic corrections are per-
fectly captured, a precise determination of the additional
constant $\gamma_{q}^{\text{ord}}(\ell/L)$ is less obvious. Indeed, as shown in
Fig. 8 for $q = 2$, using $\rho_s/v = 0.11675$ from previous $1/S$
estimates\textsuperscript{43}, the $\gamma_{q}^{\text{ord}}$ estimates are close but do not agree
with the $S = 100$ results. Taking instead the most recent
QMC estimate\textsuperscript{42} for this ratio $\rho_s/v = 0.10882(4)$, the
agreement is clearly better. We have also checked that re-
results on $\gamma_{q}^{\text{ord}}(\ell/L)$ obtained by taking into account higher
orders in $1/S$ from Ref. 43 give indeed an improvement
over the $1/S$ order, but are not as good as the QMC
result.

![Figure 7](image1)

Figure 7. Universal additive constant $\gamma_{q}^{\text{ord}}(\ell/L)$ for different Rényi indices $q$ as a function of the aspect ratio of the strip geometry for the $S = 100$ Heisenberg $J_1 - J_2$ antiferromagnet at $J_2 = 0$. The inset displays a zoom, showing that our result is in perfect agreement with the universal geometric constant obtained by MG\textsuperscript{7} for $q = 2$. Results for $J_2/J_1 = -1$
at $q = 2$, shown by red plus signs, agree perfectly with MG (black circles) and $J_2 = 0$ (black x), confirming universality.

![Figure 8](image2)

Figure 8. Results for $S = \frac{1}{2}$ and $J_2 = 0$ for $\gamma_{2}^{\text{ord}}$ obtained from fits using rectangular subsystems with fixed aspect ratios. For $S = \frac{1}{2}$, we show two sets of results using slightly different estimates for $\rho_s/v$. Data shown in blue use $\rho_s/v = 0.10882(4)$ from the most recent QMC estimate\textsuperscript{42}, while data shown in green use $\rho_s/v = 0.11675$ from a $1/S$ calculation\textsuperscript{43}. The use of the QMC result leads to a much better agreement with our $S = 100$ data.
B. Connection to the free scalar field result

In a (corner free) strip subsystem geometry with a finite aspect ratio $\ell/L$, one expects for gapped free bosons with a very large correlation length $\xi \gg L$ the following subleading corrections to the area law:\footnote{The reason for the disagreement between our large $S$ approach — expected to become unbiased in the limit $S \to \infty$ — and the result from Ref. 7 for $\gamma_q^{\text{ord}}$ is not clear at the moment.}

\[
\Delta S_q = \frac{1}{2} \ln \left( \frac{\xi}{L} \right) + \gamma_q^{\text{free}} (\ell/L),
\]

(4.2)

where $\gamma_q^{\text{free}} (\ell/L)$ is a universal geometric constant which depends non-trivially on both the Rényi parameter and the aspect ratio. By artificially gapping the linear SW Hamiltonian Eq. (2.4) with a very small staggered field $h$, the dispersion relation in the vicinity of its two minima reads

\[
\Omega(k) = \sqrt{8S J_1 h + 8S^2 J_1 (J_1 - 2J_2)|k|^2},
\]

(4.3)

thus leading to

\[
\xi = \sqrt{\frac{S(J_1 - 2J_2)}{h}}.
\]

(4.4)

The correction due to the two minima becomes

\[
\Delta S_q = \ln \left( \frac{\sqrt{S(J_1 - 2J_2)} h}{hL^2} \right) + 2\gamma_q^{\text{free}} (\ell/L),
\]

(4.5)

which is used to fit numerical SW results with a very small field $h = 10^{-18}$ to extract $\gamma_q^{\text{free}} (\ell/L)$, shown in Fig. 9.

Quite interestingly, from the above formulation we can infer a very simple and direct relation between $\gamma_q^{\text{ord}}$ and $\gamma_q^{\text{free}}$. Indeed, in the large $S$ limit the size-dependent staggered field (added to artificially restore zero sublattice magnetization) takes the exact form $h^* (L) = \frac{2J_1}{3L^2}$. Plugging this into Eq. (4.5) yields

\[
\gamma_q^{\text{ord}} \text{SU}(2) = 2\gamma_q^{\text{free}} + \ln 2,
\]

(4.6)

which agrees with MG\textsuperscript{7}, but only when $q = 2$. In Fig. 9, comparing $\gamma_q^{\text{free}}$ to $(\gamma_q^{\text{ord}} \text{SU}(2) - \ln 2)/2$ for $q = 1, 4$ gives a perfect agreement for a wide range of aspect ratios.

One can also repeat the same argument for the XY model with only one Nambu-Goldstone mode\textsuperscript{10} to get

\[
\gamma_q^{\text{ord}} \text{U(1)} = \gamma_q^{\text{free}} + 5 \text{\(\frac{4}{4}\ln 2\)}.
\]

(4.7)

The reason for the disagreement between our large $S$ approach — expected to become unbiased in the limit $S \to \infty$ — and the result from Ref. 7 for $\gamma_q^{\text{ord}}$ is not clear at the moment.

C. Limit of vanishing aspect ratio

In this section we shed light on the divergent behavior of $\gamma_q^{\text{ord}}$ for small aspect ratios by calculating $\gamma_q^{\text{ord}}$ in the extreme limit of $\ell/L \to 0$ using subsystems with a fixed number $\ell$ of lines and thus a varying aspect ratio as a function of $L$. In order to achieve this, we work with $S = 100$ at $J_2 = 0$, and we want to subtract all dominant terms, in particular the linear area law contribution $a_q L$.

Let us therefore start with a study of the dominant scaling contribution of $S_q$ by plotting $S_q/L$ vs. 1/$L$ as displayed in Fig. 10. We show the area law behavior by plotting $S_2/L$ vs. 1/$L$ and an extrapolation $L \to \infty$, which guarantees to eliminate all subleading terms. The figure shows two sets of curves. In the first one, each curve corresponds to subsystems with a constant aspect ratio, such that $\gamma_q^{\text{ord}}$ is a constant for each curve. These curves all yield identical area law prefactors $a_q^2 = 0.190216(1)$ as expected.

The second set of curves shows results corresponding to a fixed number $\ell$ of lines in the subsystem (i.e. a $\ell$-leg ladder), which implies that the aspect ratio of the subsystem is a function of $L$. The dominant linear prefactor $a_q^\ell$ found for the $\ell$-leg ladder subsystem is different from the fixed aspect ratio value, which is approached only for $\ell \gg 1$. The reason for this discrepancy lies in the divergent behavior of $\gamma_q^{\text{ord}}$ when $\ell/L$ tends to zero and the fact that the assumption that the only surviving term in the scaling of $S_q/L$ at large sizes is the area law is no longer true. In fact, as the aspect ratio of the subsystem changes constantly, $\gamma_q^{\text{ord}}$ seems to have a contribution that is linear in the inverse aspect ratio and hence leads to a shift or an effectively changed area law prefactor. As a next step, we will try to determine this contribution.

Fig 11 shows our data for $\gamma_2^{\text{ord}}$ as obtained in Sec. IV multiplied by the aspect ratio $L/\ell$ as a function of the aspect ratio in order to extract the singular contribution $\gamma_2^{\text{ord}}$ as the intercept at vanishing aspect ratio. The data
shows convincing evidence that this contribution indeed extrapolates to a nonvanishing value, which we determine by a cubic fit. With this information, we can now decompose $\gamma_q^{\text{ord}}$ in a singular and a regular component:

$$\gamma_q^{\text{ord}} \left( \frac{\ell}{L} \right) = \gamma_q^{\text{reg,ord}} \left( \frac{\ell}{L} \right) + \frac{L}{\ell} \gamma_q^*,$$

$$\lim_{\ell/L \to 0} \left( \frac{\ell}{L} \gamma_q^{\text{reg,ord}} \right) = 0. \quad (4.8)$$

For completeness, we provide a table of $\gamma_q^*$ for other Rényi indices $q$ in Tab. II.

In general, we can assume that other subdominant terms show pathologic behavior in the limit of vanishing (and non constant) aspect ratios, i.e. for fixed width $\ell$ subsystems, we will for the moment assume that they could produce a total correction to the area law of the form $\eta^{\text{lin}} \ell^2$ in total. The scaling of the EE then reads:

$$S_q = \left( a_q^* + \frac{\gamma_q^*}{\ell} + \frac{n_q^*}{\ell^2} \right) L + \frac{n_G}{2} \ln \left( \frac{L^2}{v} \rho_s \right) + \gamma_q^{\text{reg,ord}} + \ldots \quad (4.9)$$

| $q$   | 1   | 2   | 3   | 4   |
|-------|-----|-----|-----|-----|
| $\gamma_q^*$ | -0.07677 | -0.04579 | -0.03530 | -0.03123 |

Table II. Values of $\gamma_q^*$ for different Rényi indices.

Clearly, for fixed aspect ratio subsystems the terms $\gamma_q^{\text{ord}}$ and $\eta_q^*$ become irrelevant for the area law for large system sizes. However, for fixed width subsystems, the effective linear (in $L$) scaling coefficient $a_q^{\ell}$ is in fact given by

$$a_q^{\ell} = a_q^* + \frac{\gamma_q^*}{\ell} + \frac{n_q^*}{\ell^2}. \quad (4.10)$$

We can therefore obtain (in the limit of large $L$) $\gamma_q^{\text{ord}}$ from fixed width subsystems by subtracting several terms from $S_q$: Obviously we need to subtract $(a_q^* - \gamma_q^*/\ell) L$ to eliminate the linear contribution (note how this automatically takes care of the unknown terms $\eta_q^*$).

The second term that we have to subtract from the EE is the logarithmic term which is due to the spontaneous breaking of SU(2) symmetry. We have argued above alongside with several works that its value is $n_G/2 = 1$ for the case of fixed aspect ratio subsystems and shown in Ref. 10 that this is also true for fixed width $\ell$ subsystems, such as the single line with $\ell = 1$, we therefore subtract the term $n_G/2 \ln(\ell^2 L)$, taking also care of the constant stemming from $\rho_s/v$, that we know with great accuracy for the case of $S = 100$ at $J_2 = 0$.

Remaining subleading terms are expected to die off in the limit of $\ell/L \to 0$ and are therefore unimportant in the region of interest.

In total, for the limit of $L \to \infty$ and a fixed width $\ell$ of the subsystem, we obtain $\gamma_q^{\text{ord}}$ through:

$$\gamma_q^{\text{ord}} \left( \frac{\ell}{L} \right) = S_q - \left( a_q^{\ell} - \gamma_q^*/\ell \right) L - \frac{n_G}{2} \ln \left( \frac{\rho_s}{v} L \right). \quad (4.11)$$
We can now apply Eq. (4.11) to calculate $\gamma_{\text{ord}}^q$ in the small aspect ratio regime from fixed width subsystems of width $\ell$. Fig. 12 shows our result in comparison the previously obtained values of $\gamma_{\text{ord}}^q$ from fixed aspect ratio subsystems (strips). SW results for $\ell = 1$ and $\ell = 2$ are built on an analytical derivation (presented in Appendix C) obtained exploiting the fully symmetric nature of such subsystems. The perfect agreement of the results obtained with different methods and in particular the agreement of the results for different $\ell$ is a strong evidence for the reliability of this result and therefore demonstrates also the singular nature of $\gamma_{\text{ord}}^q$ given by the singular component $\gamma_q^*$.

| $\ell$ | $a_q^*$ | $a_q^* + \gamma_{\text{ord}}^q/\ell + \eta_2^*/\ell^2$ | $a_q^* + \gamma_{\text{ord}}^q/\ell$ |
|-------|---------|---------------------------------|------------------|
| 1     | 0.156142| 0.155936                        | 0.144426         |
| 2     | 0.170287| 0.170199                        | 0.167321         |
| 3     | 0.176265| 0.176232                        | 0.174953         |
| 4     | 0.179543| 0.179488                        | 0.178768         |
| 5     | 0.181492| 0.181518                        | 0.181058         |

Table III. Comparison of the directly obtained linear scaling factor $a_q^*$ of fixed width subsystems to the result obtained using the singular contributions $\gamma_2^*$ and $\eta_2^*$ from subdominant terms. The inclusion of $\eta_2^*$ significantly improves the result and provides numerical evidence for the correctness of Eq. (4.10).

Can higher subleading terms generate corrections to the area law coefficient? It is certainly legal to assume that pathological behavior in the limit of $\ell/L \rightarrow 0$ is not only present in the scaling constant $\gamma_{\text{ord}}^q$ but also in higher terms, such as $c_q/L$ and $d_q/L^2$. However, for them to modify the area law coefficient, they have to diverge much faster, i.e., as $\ell^2/L^2$ for the case of $c_q$. In order to investigate this possibility, we plot $c_2\frac{\ell^2}{L^2}$ in black in Fig. 11 and observe that a (small) nonzero contribution to the linear scaling in $L$ is indeed present which we call $\eta_2^*$ (here we will neglect the contributions to $\eta$ from even higher terms, which are difficult to access through fits to numerical data). Let us finally plug all the information together and see if the singular contributions of subdominant terms can explain the discrepancy between $a_q^*$ and $a_q^*$ observed in Fig. 10 by comparing in Tab. III results for $a_q^*$ as obtained from a direct fit to fixed width EEs and for $a_q^* + \gamma_{\text{ord}}^q/\ell + \eta_2^*/\ell^2$. The left column shows the total linear scaling prefactor $a_q^*$ for fixed width subsystems as displayed in Fig. 10, while the rightmost column shows the fixed aspect ratio linear scaling prefactor $a_q^*$ corrected by the singular contribution of $\gamma_{\text{ord}}^q$, giving reasonable agreement. The middle column takes into account the next subdominant singular contribution $\eta_2^*$ from the term $c_q/L$ as discussed above and reproduces the direct fit result to very high accuracy, thus providing strong evidence for the correctness of Eq. (4.10). We expect that even less dominant terms, such as $d_q^*/L^2$ will provide further corrections, which are relevant for small widths $\ell$ and should account for the remaining discrepancy, these terms are however very small and extremely difficult to extract numerically.

V. DISCUSSIONS AND CONCLUSIONS

In this work, we have performed a high-precision SW study of the $J_1 - J_2$ Heisenberg SU(2) antiferromagnet on the square lattice in order to investigate its quantum entanglement properties. Numerical calculations on finite size systems have been performed with an artificial restoration of zero sub-lattice magnetization using a small size-dependent staggered field $h^*(L)$. Several situations have been explored, and we have obtained finite size scaling results at large enough size such that the various terms appearing in the entanglement entropies have been precisely computed.

The universal logarithmic correction due to Nambu-Goldstone modes associated with the breaking of a continuous symmetry (SU(2) in the present case) is well captured, giving a correction perfectly fitted by $\frac{\pi^2}{24} \ln L$, independent of the Rényi index $q$. In the case of subsystems having sharp corners, additional (negative) logarithmic corrections have been precisely evaluated, in perfect agreement with scalar field theory predictions. The $J_1 - J_2$ model also offers a nice playground where we could check universality of the logarithmic term across the entire ordered regime but where we could further study the non-universal area law part which exhibits a non-trivial behavior, with a noticeable growth approaching the critical point in the frustrated regime. In the opposite limit of strong ferromagnetic second neighbor coupling, the mean-field limit is recovered with a vanish-
ing area law term and a smooth crossover to a purely logarithmic scaling of the entropies.

Part of this work was also devoted to the study of the additional constant term $\gamma_q^{\text{ord}}$, expected to be universal for strip subsystems\textsuperscript{7}, only depending on their aspect ratio. It then appeared crucial to impose zero sublattice magnetization in the finite size SW theory, yielding a unique size-dependent staggered field $h^s(L)$ which (i) mimics a tower of state gap $\sim 1/L^2$ in the excitation spectrum (responsible for the logarithmic correction), and (ii) leads to the correct additional geometric constant $\gamma_q^{\text{ord}}$, in perfect agreement with MG\textsuperscript{7}, at least for $q = 2$. A simple and direct relation with non-interacting bosons was also derived. Finally we have precisely investigated the limit of vanishing aspect ratios using very large ladder subsystems in the limit of finite number of legs, discovering that the geometric constant contains both a regular and a singular component in this limit. Our study is intended that the geometric constant contains both a regular and a singular component in this limit. Our study is

\begin{itemize}
  \item Part of this work was also devoted to the study of the additional constant term $\gamma_q^{\text{ord}}$, expected to be universal for strip subsystems\textsuperscript{7}, only depending on their aspect ratio.
  \item It then appeared crucial to impose zero sublattice magnetization in the finite size SW theory, yielding a unique size-dependent staggered field $h^s(L)$ which (i) mimics a tower of state gap $\sim 1/L^2$ in the excitation spectrum (responsible for the logarithmic correction), and (ii) leads to the correct additional geometric constant $\gamma_q^{\text{ord}}$, in perfect agreement with MG\textsuperscript{7}, at least for $q = 2$. A simple and direct relation with non-interacting bosons was also derived.
  \item Finally we have precisely investigated the limit of vanishing aspect ratios using very large ladder subsystems in the limit of finite number of legs, discovering that the geometric constant contains both a regular and a singular component in this limit. Our study is concluded by showing that singular components of even less dominant terms explain perfectly the discrepancy of the area law terms obtained from fixed width vs. fixed aspect ratio subsystems.
\end{itemize}

Among the potentially interesting future directions, a quantitative study of the geometric constant using exact Monte Carlo, while very challenging, appears to be a very important point in order to test the validity of our prediction for $q > 2$. It may also be interesting to extend the present SW approach to other continuous symmetries like SU(N) models using modified flavor-wave theory for instance. Other geometries or $d = 3$ are certainly of great interest also, with a larger choice of subsystem shapes.

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Appendix A: Details of spin-wave calculations

This appendix provides details of spin-wave calculations which are not crucial for the computation of entanglement entropy, but which are nonetheless useful for an understanding of the method and its range of validity. We also provide a comparison between the finite-size SW approach and direct QMC computations for $S = 1/2$ for the antiferromagnetic structure factor in the ferromagnetic range of next neighbor coupling $J_2 < 0$.

1. Spin-wave spectrum and velocity

We present in Fig. 13 the spectrum $\Omega_k = 2\sqrt{A_k^2 - B_k^2}$ in the direction $k_x = k_y$ (obtained from expressions Eq. (2.6)) as a function of $k_x$, for different coupling strengths $J_2$ and for a spin value $S = 1/2$. The inset represents the spin-wave velocity, Eq. (2.11), as a function of $J_2 / J_1$. We see that the velocity vanishes at $J_2 / J_1 = 0.5$ where the SW spectrum features a continuous line of minima at $k_x = 0$ and $k_y = 0$, as depicted in Fig. 14.

Figure 13. SW spectrum at $h = 0$ for various $J_2/J_1$ plotted along the $k_x = k_y$ direction. Inset: SW velocity $v_{sw}$.

Figure 14. 2D color map of the SW spectrum at $h = 0$ for $J_2/J_1 = 1/2$ and $S = 1$. A line of minima is visible along the $k_x = 0$ and $k_y = 0$ directions.

2. Range of non-vanishing staggered magnetization

a. Antiferromagnetic order parameter

Eq. (2.12) can be evaluated numerically for different values of the spin size $S$ and second neighbor coupling strength $J_2/J_1$, in order to probe the range of validity of the spin-wave approach, which assumes an ordered ground-state. This is illustrated in Fig. 15 where the AF order parameter is represented, and as expected, is clearly enhanced by ferromagnetic diagonal coupling $J_2/J_1 < 0$ while it decreases towards zero when $J_2/J_1$ approaches 1/2. The critical frustration $J_2^c$ (in units of $J_1$), above which the SW-corrected order parameter vanishes, is also represented in the inset of Fig. 15 as a function of $S$ where we observe that $J_2^c \rightarrow 1/2$ when $S$ gets larger.

Figure 15. SW results for the AF order parameter Eq. (2.12) of the $J_1 - J_2$ model on the square lattice for various spin sizes $S$. Inset: Critical frustration $J_2^c$ (in units of $J_1$) plotted against the spin length $S$.

b. Finite size SW: AF structure factor

To illustrate the interest of using a formulation of SW which treats more correctly finite-size systems, we present results for the computation of the staggered structure factor per site on finite square lattices $L \times L = N$:

$$s(\pi, \pi) = \frac{1}{N^2} \sum_{ij} (-1)^{i-j} \langle S_i \cdot S_j \rangle. \quad (A1)$$

Using Wick’s theorem, all two-spin correlators can be computed in terms of the $f_{ij}$ and $g_{ij}$ functions defined in...
Figure 16. Staggered structure factor per site \(s(\pi, \pi)\) of the \(J_1 - J_2\) antiferromagnet Eq. (1.5) plotted against the inverse system length \(1/L\) for 3 values of \(J_2\), with \(J_1 = 1\). Symbols show QMC results, dashed lines are quadratic fits of the form Eq. (A3), and the full lines are modified SW results using Eq (A2).

Eq. (2.20) of the main text. Imposing that \(\langle S_i \cdot S_j \rangle = \langle S_i^x S_j^y \rangle\) (because the theory is strictly speaking not rotationally invariant), we obtain:

\[
s(\pi, \pi) = \frac{1}{N^2} \sum_{ij} (f_{ij}^2 + g_{ij}^2) - \frac{1}{4N}.
\]  

(A2)

A quantitative comparison between the above SW expectation and exact quantum Monte Carlo simulations is shown in Fig. 16. Ground-state expectation values for \(s(\pi, \pi)\) of the \(J_1 - J_2\) Hamiltonian Eq. (1.5) with \(S = 1/2\) and \(J_2 = 0, -1, -5\) have been obtained for various square lattices \(L \times L\) using the stochastic series expansion algorithm 41. One sees in Fig. 16 that the agreement is fairly good, in particular for strong second neighbor ferromagnetic coupling \(J_2/J_1 = -5\). Interestingly, the finite size scaling behavior, expected from previous works 45,46

\[
s(\pi, \pi) = m_{2F}^2 + m_1/L + m_2/L^2 + \cdots
\]  

(A3)

is very well captured by SW calculations, as visible in Table IV where QMC and SW estimates for \(m_{2F}\), \(m_1\), and \(m_2\) are compared and show a good agreement.

| \(J_2/J_1\) | \(m_{2F}^2\) QMC | \(m_1\) QMC | \(m_2\) QMC |
|-----------|----------------|-------------|-------------|
| 0         | 0.092 / 0.093(1) | 0.55 / 0.60(1) | 0.8 / 0.61(1) |
| -1        | 0.175 / 0.167(1) | 0.42 / 0.47(2) | 0.4 / 0.15(9) |
| -5        | 0.225 / 0.223(1) | 0.25 / 0.26(2) | 0.2 / 0.2(1) |

Table IV. Fit parameters from Eq. (A3).

The fact that finite size corrections are well captured by this modified SW formalism is a confirmation that it is a good starting point to study ground-state properties on finite systems and in particular the finite size scaling of the entanglement entropy, as discussed in the main text.

**Appendix B: Mean-field limit**

In the limit \(-J_2/J_1 \gg 1\) one should recover the mean-field result obtained for example for the Lieb-Mattis model 40,41. In such a limit, perfect ferromagnetic correlations between spins belonging to the same sublattice imply \(f_{ij} = S\) for \(i \neq j\) both on the same sublattice \((g_{ij} = 0)\) and \(f_{ii} = S + 1/2\). Antiferromagnetic correlations between opposite sublattices yield

\[
\sum_{ij} f_{ij} - g_{ij} = 0,
\]  

(B1)

thus leading to \(g_{ij} = S + 1/N\) for \(i \neq j\) on opposite sublattices \((f_{ij} = 0)\). Therefore non-zero matrix elements of the correlation \(C\) (for \(i\) and \(j\) on the same sublattice) are given by

\[
C_{ij} = S(1 - r_\Omega) + \frac{1}{4},
\]  

(B2)

where \(r_\Omega = N_{\Omega}/N\) is the ratio between the number of sites inside the sub-system \(N_\Omega\) and the total number of sites \(N\). The spectrum of the correlation matrix \(C\) is

Figure 17. Mean-field limit for the Rényi entropies \(S_q\). The symbols show numerical results for \(J_2/J_1 = -10^5\) with different geometries and spin lengths \((S = 1, 10)\), plotted against the number \(N_\Omega\) of sites in the subsystem \(\Omega\). The numerical results (symbols) are compared to the analytical expression Eq. (B5) with \(r_\Omega \approx 3/8\) (a) or \(r_\Omega \approx 1/2\) (b), shown by the full lines.
then straightforwardly given by
\[
\nu_{1,2}^2 = \frac{N_\Omega}{2} S(1 - r_\Omega) + \frac{1}{4} \quad (B3)
\]
\[
\nu_l^2 = \frac{1}{4}, \quad (l = 3, \ldots, N_\Omega). \quad (B4)
\]

one sees that only two eigenvalues contribute, in a macroscopic way. We then compute directly the Rényi entropies for any partition \(N_\Omega\) and any \(q \geq 1\)
\[
S_q = \ln(N_\Omega) + \ln \left( \frac{S(1 - r_\Omega)}{2} \right) + 2 \frac{\ln q}{q - 1}. \quad (B5)
\]

The area law term vanishes, and the dominant scaling is now a pure logarithm of the number of sites \(N_\Omega\). This exact expression can be compared to the numerical solution of the SW Hamiltonian for very large negative values of \(J_2\). In Fig. 17 we show numerical results for \(J_2/J_1 = -100000\) for two values of \(S\) and two different geometries, which compares extremely well with the MF limit expression Eq. (B5). Note that the lines are not fitting functions. If we try instead to fit to the general form \(a_q L + b_q \ln L + c_q/L + d_q/L^2\), we end up with \(a_q \sim 10^{-11}\) and \(l_q = 2\).

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**Appendix C: Analytical derivation for one-dimensional subsystems**

A great simplification for the computation of entanglement entropy is possible if all sites \(i\) and \(j\) inside a subsystem \(\Omega\) are equivalent, or in other words if the matrix elements \(C_{ij}\) only depend on the relative distance \(|r_i - r_j|\). In such a case, for sites on different sublattices \(C_{ij} = 0\). This situation is achieved for one-dimensional subsystems with one or two lines (Fig. 1 (a) with \(\ell = 1, 2\)). In these specific situations, we can derive analytic expressions for the eigenvalues of \(C\), avoiding a numerical diagonalization. This has first been discussed in Ref. 10, and we provide here details of this calculation, starting with the case of a line-shaped subsystem.

This subsystem being invariant under translations along the \(x\) direction, the functions \(f_x\) and \(g_x\) defined in Eqs. (2.20) only depend on the distance \(x = x_i - x_j\) along the subsystem. They reduce consequently to
\[
f_x = \frac{1}{2N} \sum_{k_x} \cos(k_x x) \alpha_{k_x}, \quad g_x = \frac{1}{2N} \sum_{k_x} \cos(k_x x) \beta_{k_x} \quad (C1)
\]

with
\[
\alpha_{k_x} := \sum_{k_y} \frac{A_{k}}{\Omega_{k}} \quad \text{and} \quad \beta_{k_x} := \sum_{k_y} \frac{B_{k}}{\Omega_{k}} \quad (C2)
\]

which satisfy the property \(\alpha_{k_x + \pi} = \alpha_{k_x}\), \(\beta_{k_x} = -\beta_{k_x + \pi}\). Since the functions \(f_x\) and \(g_x\) possess translation and reflection symmetries
\[
f_x = f_{L-x} = f_{L+x}, \quad \text{and} \quad g_x = g_{L-x} = g_{L+x}, \quad (C3)
\]
so does the correlation matrix: \(C_{ij} = C(|x_i - x_j|) = C(L - i) = C(l - L)\). Since furthermore \(C(x)\) vanishes for odd distances, it is convenient to re-index all sites on one sublattice from 1 to \(L/2\) (say, blue sites in Fig. 1a for \(\ell = 1\)), and sites on the other sublattices from \(L/2 + 1\) to \(L\) (say, orange sites in Fig. 1a) to block-diagonalize \(C\) onto two identical blocks of size \(L/2 \times L/2\). The translation invariance ensures that each block is circulant, with matrix elements \(C(l) = \sum_{x_{even}} f_x f_{x-l} - \sum_{x_{odd}} g_x g_{x-l}\). The eigenvalues \(\nu_l^2\) of \(C\) are given by the properties of circulant matrices:
\[
\nu_l^2 = c(0) + (-1)^l c \left( \frac{L}{2} \right) \sum_{j=1}^{\left\lfloor L/4 - 1 \right\rfloor} c(2j) \cos \left( \frac{4\pi j l}{L} \right),
\]
\[
l \in \{0, 1, \ldots, L/2 - 1\}, \quad c \left( \frac{L}{2} \right) = 0 \text{ if } \frac{L}{2} \text{ mod } 2 = 1. \quad (C4)
\]

We can even simplify calculations by noticing that \(f_x\) and \(g_x\) are discrete Fourier transforms of \(\alpha_{k_x}\) and \(\beta_{k_x}\) respectively. Using the convolution theorem on \(C(l)\), we

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**Figure 18.** Area law coefficient \(a_q\) as a function of the normalized order parameter \(S - m_{AF}\), which is small in the ordered phase. Here, we show results for \(S = \frac{1}{2}\). The inset depicts the prefactor of the logarithmic entanglement entropy scaling term, which has a plateau at \(l_q = 1\) for intermediate \(J_2\) and evolves to the mean field limit of \(J_2 \rightarrow -\infty\). It is unclear if the behavior in the crossover region is a finite size effect or an artefact of the spin wave method.

Before concluding, we want to briefly comment on the crossover to the MF limit when the ferromagnetic second neighbour is turned on towards very large values. This is illustrated in Fig. 18 where the rapid decrease of the area law coefficient \(a_q\) (for \(q = 1, 2\)) is shown versus the quantum depletion of the AF order parameter \(S - m_{AF}\). In the same time, the log coefficients \(l_1\) and \(l_2\), plotted in the inset of Fig. 18, crossover from \(l_q = 1\) up to \(l_q = 2\) in the limit of vanishing quantum fluctuations \(m_{AF} \rightarrow S\).
arrive at the final expression for the $L$ eigenvalues of $C$ of the single-line subsystem:

\[
\nu_q^2 = \frac{1}{4N}(\alpha_q^2 - \beta_q^2),
\]  
((C5))

with $q \in \{-\pi + \frac{2\pi}{L}, \ldots, \pi\}$.

A very similar reasoning can be applied to the case of a 2-line (ladder) subsystem with $2L$ sites ($\ell = 2$ in Fig. 1a). It is convenient to re-index sites by labelling (in a zig-zag fashion) all (say, blue in in Fig. 1a) sites of one sublattice from 1 to $L$, and (orange in Fig. 1a) sites from the other sublattice from $L + 1$ to $2L$. Again, $C$ is block-diagonal with identical circulants blocks with matrix elements $C(l) = \sum_{x=0}^{L-1} f_x^+ f_{x-l}^+ - \sum_{x} \tilde{g}_x^+ \tilde{g}_{x-l}^+$ with $f_x^+ = f(x, 0) + f(x, 1)$ and $g_x^+ = g(x, 0) + g(x, 1)$. We can now introduce the discrete Fourier transform of the newcomers $f(x, 1)$ and $g(x, 1)$

\[
\tilde{\alpha}_k := \sum_{k_y} \frac{A_k \cos(\Omega k_y)}{\Omega_k} \quad \text{and} \quad \tilde{\beta}_k := \sum_{k_y} \frac{B_k \cos(\Omega k_y)}{\Omega_k},
\]  
((C6))

to again be able to apply the convolution theorem. We finally obtain that the $L$ eigenvalues of one block of $C$ for the ladder subsystem are given by:

\[
\nu_q^2 = \frac{1}{4N} \left[ (\alpha_q - \tilde{\alpha}_q)^2 - (\beta_q - \tilde{\beta}_q)^2 \right],
\]  
((C7))

with $q \in \{-\pi + \frac{2\pi}{L}, \ldots, \pi\}$. Since $C$ has two identical blocks, the $2L$ eigenvalues for the ladder subsystem are obtained by doubling the above spectrum.