Abstract. Let $G = \exp(g)$ be an exponential solvable Lie group and $\text{Ad}(G) \subset \mathbb{D}$ an exponential solvable Lie group of automorphisms of $G$. Assume that for every non-$\ast$-regular orbit $\mathbb{D} \cdot q$, $q \in g^\ast$, of $\mathbb{D} = \exp(d)$ in $g^\ast$, there exists a nilpotent ideal $n$ of $g$ containing $d \cdot g$ such that $\mathbb{D} \cdot q|_n$ is closed in $n^\ast$. We then show that for every $\mathbb{D}$-orbit in $g^\ast$ the kernel $\ker_{C^\ast}(\pi)$ of $\pi$ in the $C^\ast$-algebra of $G$ is $L^1$-determined, which means that $\ker_{C^\ast}(\pi)$ is the closure of the kernel $\ker_{L^1}(\pi)$ of $\pi$ in the group algebra $L^1(G)$. This establishes also a new proof of a result of Ungermann, who obtained the same result for the trivial group $\mathbb{D} = \text{Ad}(G)$. We finally give an example of a non-closed non-$\ast$-regular orbit of an exponential solvable group $G$ and of a coadjoint orbit $O \subset g^\ast$, for which the corresponding kernel $\ker_{C^\ast}(\pi_O)$ in $C^\ast(G)$ is not $L^1$-determined.

1. Introduction

1.1. $\ast$-regular Banach algebras

See [8]. Let $A$ be an involutive Banach algebra. We denote by $\text{Rep}(A)$ the set of unitary representations $(\pi, \mathcal{H}_\pi)$ of $A$ and by $\hat{A}$ the spectrum of $A$, i.e. the set of all equivalence classes of irreducible representations of $A$.

The $C^\ast$-algebra $C^\ast(A)$ of $A$ is the completion of $A$ for its $C^\ast$-norm,

$$\|a\|_{C^\ast} := \sup_{\pi \in \text{Rep}(A)} \|\pi(a)\|_{\text{op}}, \quad a \in A.$$ 

For every unitary representation $(\pi, \mathcal{H}_\pi)$ on a Hilbert space $\mathcal{H}_\pi$, there exists a unique extension to $C^\ast(A)$. We shall denote this extension also by $(\pi, \mathcal{H}_\pi)$. In this way the spectrum of $C^\ast(A)$ coincides with the spectrum of $A$.

Definition 1.1. Let for $\pi \in \text{Rep}(A)$, $\ker_A(\pi)$ (respectively $\ker_{C^\ast}(\pi)$) denote the kernel of $\pi$ in $A$ (respectively in $C^\ast(A)$).

We equip the spaces

$$\text{Prim}(A) := \{\ker_A(\pi), \ \pi \in \hat{A}\}, \quad \text{Prim}_{C^\ast}(A) := \{\ker_{C^\ast}(\pi), \ \pi \in \hat{A}\}$$

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with the hull kernel (or Fell) topology: a subset $L$ of $\text{Prim}(A)$ is closed in this topology if and only if there exists a two-sided ideal $I$ in $A$ such that $L$ is the hull $h(I)$ of $I$, which means that

$$L = h(I) := \{ J \in \text{Prim}(A); \ J \supset I \}.$$ 

For any subset $L$ of $\text{Prim}(A)$ let

$$\text{ker}(L) := \bigcap \{ J \in \text{Prim}(A); \ J \in L \}.$$ 

Then the closure $\overline{\text{Prim}}^L$ of a subset $L$ in $\text{Prim}(A)$ is the hull of $\text{ker}(L)$.

In all these definitions we can of course replace $A$ by $C^*(A)$.

**Definition 1.2.** We have a natural surjection $R : \text{Prim}^C(A) \to \text{Prim}(A)$, which is defined by

$$R(J) = J \cap A, \ J \in \text{Prim}^C(A).$$

This mapping $R$ is obviously continuous.

We say that $\text{Prim}^C(A)$ is $A$-determined if for every $\pi \in \hat{A}$ the closure $\overline{\ker_A(\pi)}^C$ in $C^*(A)$ of the ideal $\ker_A(\pi)$ is the ideal $\ker_C(\pi)$.

Obviously then $\text{Prim}^C(A)$ is $A$-determined if and only if for any two elements $\pi, \rho \in \hat{A}$ the relation $\ker_A(\pi) \subset \ker_A(\rho)$ implies that $\ker_C(\pi) \subset \ker_C(\rho)$.

Hence if $\text{Prim}^C(A)$ is $A$-determined, then the mapping $R$ is injective.

We say that the algebra $A$ has finite rank if for every irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of $A$ there exists $a \in A$ such that $\pi(a)$ has rank one. It is easy to see that for every Banach $*$-algebra $A$ with finite rank, the mapping $R$ is injective.

**Definition 1.3.** We say that the algebra $A$ is $*$-regular if the mapping $R$ is even a homeomorphism (see [2]).

**Remark 1.4.** Obviously, if $A$ is $*$-regular, then $\text{Prim}^C(A)$ is $A$-determined.

### 1.2. $*$-regular groups

Now let $G$ be a locally compact group. On $G$ there exists a unique (up to a positive constant) left translation-invariant Borel measure, the so-called Haar measure, which we denote by $dx$. Then we can equip the Banach space $L^1(G, dx) = L^1(G)$ of integrable complex-valued functions of $G$ with respect to Haar measure with the convolution product $*$,

$$F * F'(y) := \int_G F(x)F'(x^{-1}y) \, dx, \quad y \in G, \ F, F' \in L^1(G),$$

and we obtain a Banach algebra structure on $L^1(G)$. An isometric involution $*$ on $L^1(G)$ is given by

$$F^*(y) := \Delta_G(y)^{-1} \overline{F(y^{-1})}, \quad y \in G, \ F \in L^1(G).$$

Here $\Delta_G : G \to \mathbb{R}_{>0}$ is the unique continuous function on $G$ which satisfies

$$\int_G F(xy^{-1}) \, dx = \Delta_G(y) \int_G F(x) \, dx, \quad F \in L^1(G), \ y \in G.$$
For an irreducible unitary representation \((\pi, \mathcal{H})\) of \(G\), let \(\ker_{L^1}(\pi)\) be the kernel of the corresponding representation of the involutive Banach algebra \(L^1(G)\) (respectively \(\ker_{C^*}(\pi)\) the kernel of the representation \(\pi\) of the \(C^*\)-algebra \(C^*(G)\) of \(G\)).

Let as before \(\text{Prim}_{C^*}(G)\) be the space of all primitive ideals \(J = \ker_{C^*}(\pi)\) of \(C^*(G)\), and also equipped with Fell’s topology.

We have again the restriction mapping:

\[
R : \text{Prim}_{C^*}(G) \rightarrow \text{Prim}_{L^1}(G); \quad J \mapsto J \cap L^1(G).
\]

**Definition 1.5.** The group \(G\) is called \(*\)-regular if the mapping \(R\) is a homeomorphism.

The following has been proved in [2, Theorem 2].

**Theorem 1.6.** Suppose that a locally compact group \(G\) has polynomial growth, i.e. for every compact neighbourhood \(U\) of the identity \(e\), the Haar measure of the \(n\)th power \(|U^n|\) is polynomially bounded in \(n\). Then \(G\) is \(*\)-regular. In particular, nilpotent locally compact groups have polynomial growth and are therefore \(*\)-regular.

1.2.1 Induced from a normal subgroup. We need some formulas for induced representations of the form \(\pi = \text{ind}^G_H \pi_0\), where \(H\) is a closed normal subgroup of \(G\) and \((\pi_0, \mathcal{H}_0)\) is a unitary representation of \(H\).

Let \(d\hat{s}\) be the Haar measure of the group \(G/H\). Then the Hilbert space \(\mathcal{H}_\pi\) of \(\pi\) is the space

\[
L^2(G/H, \pi_0) := \left\{ \xi : G \rightarrow \mathcal{H}_0; \ \xi \text{ measurable}, \ \xi(gh) = \pi_0(h^{-1})(\xi(g)), \ g \in G, \ h \in H, \int_{G/H} \|\xi(g)\|^2 \, d\hat{g} < \infty \right\}.
\]

Hence for \(F \in L^1(G)\),

\[
\pi(F)\xi(t) = \int_G F(s)\xi(s^{-1}t) \, ds
= \int_G F(ts)\xi(s^{-1}) \, ds
= \int_{G/H} \Delta_G(s^{-1}) F(ts^{-1})\xi(s) \, ds
= \int_{G/H} \Delta_G(s^{-1}) \int_H \Delta_H(h^{-1}) F(th^{-1}s^{-1})\xi(sh) \, dh \, d\hat{s}
= \int_{G/H} \Delta_G(s^{-1}) \left( \int_H \Delta_H(h^{-1}) F(th^{-1}s^{-1})\pi_0(h^{-1})(\xi(s)) \, dh \right) \, d\hat{s}
= \int_{G/H} \Delta_G(s^{-1}) \left( \int_H F(th^{-1}s^{-1})\pi_0(h) \, dh \right)\xi(s) \, d\hat{s}
= \int_{G/H} F_{\pi_0}(t, s)\xi(s) \, d\hat{s},
\]

(1.2.1)
where \( F_{\pi_0}(t, s) := \Delta_G(s^{-1}) \int_H F(ths^{-1})\pi_0(h) \, dh \in B(H_{\pi_0}) \) is the kernel function of the operator \( \pi \).

In particular, suppose \( G = A \ltimes H \) is the semi-direct product of a locally compact group \( H \) with an abelian group \( A \). Then identifying \( G/H \) with \( A \), we have \( L^2(G/H, \pi_0) \cong L^2(A, H_{\pi_0}) \), and for \( F = F(t \cdot h) \in L^1(G) \) and \( \xi = \xi(t) \in L^2(A, H_{\pi_0}) \), \( t \in A, h \in H \), we have

\[
\pi(F)\xi(t) = \int_A \int_H F((t-s) \cdot h)\pi_0((-s) \cdot h \cdot s)(\xi(s)) \, ds \, dh
\]

where \( s \cdot \pi_0 \) is the unitary representation of \( H \) defined by \( s \cdot \pi_0(h) := \pi_0(s^{-1} \cdot h \cdot s) \) and \( F(s) \) is the function in \( L^1(H) \) defined by \( F(s)(h) := F(s \cdot h), \ h \in H, \ s \in A \).

**Lemma 1.7.** Let \( Z \subset G \) be a closed normal subgroup of the locally compact group \( G \) and let \( \chi : Z \to \mathbb{T} \) be a unitary character of \( Z \). Let \( \tau_\chi := \text{ind}^G_Z \chi \). Then for every unitary representation \( (\pi, H_{\pi}) \) of \( G \) such that its restriction to \( Z \) is a multiple of the character \( \chi \) we have that \( \ker L^1(\tau_\chi) \subset \ker L^1(\pi) \).

**Proof.** Indeed if \( F \in \ker L^1(\tau_\chi) \) then we have by (1.2.1) that

\[
\int_Z F(tz)\chi(z) \, dz = 0 \quad \text{for almost all } t \in G.
\]

Therefore

\[
\pi(F) = \int_G F(s)\pi(s) \, ds
\]

\[
= \int_{G/Z} \left( \int_Z F(sz)\chi(z) \, dz \right) \pi(s) \, ds
\]

\[
= \int_{G/Z} \left( \int_Z F(sz)\chi(z)I_H \, dz \right) \circ \pi(s) \, ds
\]

\[
= \int_{G/Z} 0 \, I_H \circ \pi(s) \, ds
\]

\[
= 0,
\]

where \( \pi|_Z = \chi I_H \).

\[ \square \]

2. **Exponential Lie group**

For this section, details, bibliography and proofs can be found for instance in [3].

Let \( G = \exp(g) \) be an exponential Lie group. This means that the exponential mapping \( \exp : g \to G \) is a diffeomorphism. Then \( G \) is connected and simply connected.

2.1. **Function spaces on an exponential group \( G \)**

Let \( G = \exp(g) \) be a solvable exponential Lie group. We denote by \( u(g) \) the complex enveloping algebra of the Lie algebra \( g \). For \( A \in g \) the left action (respectively the right action)
of $\mathfrak{g}$ on the space $C^\infty(G)$ of smooth functions on $G$ is defined by

$$A \ast f(g) := \left. \frac{d}{dt} f(\exp(-tA)g) \right|_{t=0}, \quad \text{respectively } f \ast A(g) := \left. \frac{d}{dt} f(g \exp(tA)) \right|_{t=0},$$

$$g \in G, \ f \in C^\infty(G).$$

**Definition 2.1.** Let $N = \exp(n)$ be a (closed connected) normal nilpotent subgroup of the exponential Lie group $G = \exp(\mathfrak{g})$ containing the subgroup $[G, G] = \exp([\mathfrak{g}, \mathfrak{g}])$. Let $\mathfrak{s}$ be any subspace of $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum of $n$ and $\mathfrak{s}$ and let $\| \|_p$ be a norm on $\mathfrak{g}$. Then $G$ is the topological product of $\mathfrak{s}$ and $N$ and

$$G = \exp(\mathfrak{s}) \cdot N.$$

For $c > 0$ in $\mathbb{R}$ define the function $e_c$ on $G$ by

$$e_c(\exp(S) \cdot n) := e^{c\|S\|}, \quad S \in \mathfrak{s}, \ n \in N.$$

We denote by $ES(G) = ES(G)^n$ the space of functions

$$ES(G) = \{ F : G \to \mathbb{C}; \ F \text{ smooth, } e_c \cdot p \cdot (A \ast F \ast B) \in L^\infty(G), \forall A, B \in u(g), \ c > 0, \ p \in \mathcal{P}(n) \}.$$

In particular, $ES(N)$ is the usual Schwartz space of the nilpotent Lie group $N$. The function space $ES(G)$ is a subalgebra [6] of the convolution algebra $L^1(G)$ and the injection $ES(G) \to L^1(G)$ is continuous.

### 2.2. The spectrum of $G$

The dual space $\widehat{G}$ of an exponential group $G = \exp(\mathfrak{g})$ can be described thanks to Kirillov’s orbit picture (which had been developed by Kirillov, Bernat, Pukanszky and Vergne), as the space $\mathfrak{g}^*/G$ of coadjoint orbits. For every $\ell \in \mathcal{O}$, there exist Pukanszky polarizations, i.e. subalgebras $\mathfrak{p}$ of $\mathfrak{g}$, such that $\langle \ell, [\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \rangle = \{0\}$, such that $\dim(\mathfrak{p}) = \dim(\mathfrak{g}) - \frac{1}{2} \dim(\mathcal{O})$ and such that $\Ad^*(\mathcal{O})\ell = \ell + \mathfrak{p}^\perp$, where $P = \exp(\mathfrak{p})$.

Define the unitary character $\chi_\ell$ of $P$ by

$$\chi_\ell(\exp(Y)) := e^{-i\langle \ell, Y \rangle}, \quad Y \in \mathfrak{p}.$$

We can then define the monomial representation $\pi_{\ell, p} := \text{ind}_P^G \chi_\ell$ of $G$, which acts by left translation on the space

$$L^2(G/P, \chi_\ell) := \left\{ \xi : G \to \mathbb{C}; \ \xi \text{ measurable, } \xi(gp) = \chi_\ell(p^{-1})\left( \frac{\Delta_H(p)}{\Delta_G(p)} \right)^{1/2} \xi(g), \forall g \in G, \ p \in P, \ \| \xi \|^2 := \int_{G/P} |\xi(g)|^2 \, dv(g) < \infty \right\}.$$

Here $\int_{G/P} \, dv$ denotes the unique (up to a positive constant) left-invariant positive linear form on the vector space

$$\mathcal{D}(G, P) := \left\{ \psi : G \to \mathbb{C}; \ \psi \text{ continuous, } \psi(gy) = \frac{\Delta_P(y)}{\Delta_G(y)} \psi(g), \forall g \in G, \ y \in P \right\}.$$
The representation \((\pi_{\ell, p}, \mathcal{H}_{\ell, p} := L^2(G/P, \chi(\ell)))\) is then irreducible and, up to equivalence, every irreducible unitary representation of \(G\) arises in this way. Furthermore, we have that \(\pi_{\ell, p}\) is equivalent to \(\pi_{\ell', p}\) if and only if \(\ell\) and \(\ell'\) are contained in the same coadjoint orbit. In this way the spectrum of \(G\) (and of \(L^1(G)\) and \(C^*(G)\)) is determined by the space of coadjoint orbits \(\mathfrak{g}^* / G\) of \(G\). The mapping

\[ K : \mathfrak{g}^* / G \to \hat{G}, \quad K(O) = [\pi_{\ell, p}] \quad (\ell \in \mathcal{O}) \]

is thus a bijection and even, according to [5], a homeomorphism.

It follows from [6] that for every exponential solvable Lie group, its \(L^1\)-algebra has finite rank. Hence the mapping \(R\) is injective for these groups.

**Remark 2.2.** Let \(G\) be a locally compact group, and let \(N\) be a normal subgroup of \(G\) contained in a closed subgroup \(H\) of \(G\). Let \(\chi : H \to \mathbb{T}\) be a unitary character of \(H\) such that \(\chi|_N\) is \(G\)-invariant, i.e. \(\chi(gng^{-1}) = \chi(n), \ n \in N, \ g \in G\). Let \(\pi_{\chi, H} := \text{ind}_{H}^{G} \chi\) be the representation of \(G\) obtained by inducing the character \(\chi\) of \(H\) to \(G\). This representation \(\pi_{\chi, H}\) is constructed in the same way as the representation \(\pi_{\ell, p}\) above. Then we have that

\[ \pi_{\chi, H}(n) = \chi(n)\mathbb{1}_{L^2(G/H, \chi)}, \quad n \in N. \]

Indeed, the group \(G\) acts by left translation on the functions \(\xi \in L^2(G/H, \chi)\) and so

\[
\pi_{\chi, H}(n)\xi(g) = \xi(g^{-1}n^{-1}g) = \chi(g^{-1}n^{-1}g) = \chi(n^{-1}g) = \chi(n)\xi(g), \quad g \in G, \ n \in N.
\]

**Definition 2.3.** For a Lie algebra \(\mathfrak{b}\) we consider the sequence of ideals \(\mathfrak{b}^1 := \mathfrak{b}, \ \mathfrak{b}^{k+1} := [\mathfrak{b}, \mathfrak{b}^k], \ k \in \mathbb{N}^+\) and we let \(\mathfrak{b}^\infty := \lim_{k \to \infty} \mathfrak{b}^k\).

Boidol has determined all \(*\)-regular exponential solvable Lie groups in [1]. The group \(G\) (which we call Boidol’s group) with Lie algebra \(\mathfrak{g} = \text{span}\{T, X, Y, Z\}\) and the Lie brackets

\[ [T, X] = -X, \quad [T, Y] = Y, \quad [X, Y] = Z, \]

is the lowest-dimensional non-\(*\)-regular exponential solvable Lie group.

Boidol’s criterion for \(*\)-regularity for exponential solvable Lie groups \(G = \text{exp}(\mathfrak{g})\) is the following. Let \(n\) be a nilpotent ideal of \(\mathfrak{g}\) containing \([\mathfrak{g}, \mathfrak{g}]\). Then \(G\) is \(*\)-regular if and only if every \(\ell \in \mathfrak{g}^*\) vanishes on the ideal \(m(\ell)^\infty\) of \(\mathfrak{g}\). Here

\[ m(\ell) := \mathfrak{g}(\ell) + n. \]

**Definition 2.4.** Let \(\mathbb{D} = \text{exp}(\mathfrak{d})\) be a solvable exponential Lie group of automorphisms of \(G\) acting in an exponential way on the Lie algebra \(\mathfrak{g}\) and containing the group \(\text{Ad}(G)\) of the inner automorphisms of \(G\). In particular the semi-direct product \(\mathbb{D} \rtimes G\) is an exponential group.

We denote by

\[ d \cdot g, \quad \text{respectively} \quad d \cdot X, \quad \text{respectively} \quad D \cdot X \]
the action of $\mathbb{D}$ on $G$ (respectively of $\mathbb{D}$ on $\mathfrak{g}$, respectively of $\mathfrak{d}$ on $\mathfrak{g}$). This also gives us a continuous action of $\mathbb{D}$ on the representations of the group $G$. For a representation $(\pi, \mathcal{H})$ of $G$ on a Hilbert space $\mathcal{H}$ we let

$$d \cdot \pi(g) := \pi(d^{-1} \cdot g), \quad g \in G, \ d \in \mathbb{D}.$$  

We then have automatically an action of $\mathbb{D}$ on $\text{Prim}(G)$ and on $\text{Prim}_{C^*}(G)$, which is continuous for the Fell topology. Suppose that the nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ contains $\mathfrak{d} \cdot \mathfrak{g}$. Let for $q \in \mathfrak{g}^*$:

$$\mathfrak{g}(q) := \{X \in \mathfrak{g}; \langle q, [X, n] \rangle = 0\}, \quad \mathfrak{g}^n(q_\mathfrak{n}) := \{X \in \mathfrak{g}; \langle q, [X, \mathfrak{n}] \rangle = 0\},$$

$$\mathfrak{d}(q) := \{D \in \mathfrak{d}; \langle q, D \cdot X \rangle = 0 \ \forall X \in \mathfrak{g}\}, \quad \mathfrak{d}(q_\mathfrak{n}) := \{D \in \mathfrak{d}; \langle q, D \cdot X \rangle = 0 \ \forall X \in \mathfrak{n}\},$$

$$\mathfrak{g}_\mathfrak{d} \mathfrak{g} := \{X \in \mathfrak{g}; \langle q, \mathfrak{d} \cdot X \rangle = 0\} \subset \mathfrak{g}(q),$$

and let

$$m(q) = n + \mathfrak{g}_\mathfrak{d} \mathfrak{g}. \quad \quad (2.2.1)$$

Then $\mathfrak{g}(q)$, $\mathfrak{g}(q_\mathfrak{n})$ and $\mathfrak{g}_\mathfrak{d} \mathfrak{g}$ are subalgebras of $\mathfrak{g}$, $\mathfrak{d}(q)$ is a subalgebra of $\mathfrak{d}$ and $m(q)$ is a $\mathfrak{d}$-invariant ideal of $\mathfrak{g}$. Also let

$$G(q) := \exp(\mathfrak{g}(q)),$$

$$G_\mathfrak{d} \mathfrak{g} := \exp(\mathfrak{g}_\mathfrak{d} \mathfrak{g}),$$

$$\mathbb{D}(q) := \{d \in \mathbb{D}; \ d \cdot q = q\},$$

$$\mathbb{D}(\mathfrak{d}(q)) := \{d \in \mathbb{D}; \ d \cdot q \in \mathfrak{d}(q)\} = \mathbb{D}(q) \cdot \text{Ad}(G) = \{d \in \mathbb{D}; \ d \cdot \ker_{C^*}(\pi_q) = \ker_{C^*}(\pi_q)\}.$$  

**Lemma 2.5.** We have for any $q \in \mathfrak{g}^*$ that

$$q + m(q) \perp \subset \Omega_q := \mathbb{D} \cdot q.$$  

**Proof.** Since $\mathbb{D}$ is an exponential group acting exponentially on $\mathfrak{g}$, by [3] the stabilizer $\mathbb{D}(q_\mathfrak{n})$ of $q_\mathfrak{n}$ in $\mathbb{D}$ is the subgroup

$$\mathbb{D}(q_\mathfrak{n}) = \exp(\mathfrak{d}(q_\mathfrak{n})).$$

The bilinear form

$$B_q : \mathfrak{g} \times \mathfrak{d} \rightarrow \mathbb{R}, \quad B_q(X, D) := \langle q, D \cdot X \rangle$$

establishes a duality between the spaces $\mathfrak{g}/\mathfrak{g}_\mathfrak{d} \mathfrak{g}$ and $\mathfrak{d}/\mathfrak{d}(q)$. Since the orthogonal of $\mathfrak{g}_\mathfrak{d} \mathfrak{g} + n$ in $\mathfrak{d}$ with respect to $B_q$ is the subspace $\mathfrak{d}(q_\mathfrak{n})$, it follows that the quotient spaces $\mathfrak{d}(q_\mathfrak{n})/\mathfrak{d}(q)$ and $\mathfrak{g}/(n + \mathfrak{g}_\mathfrak{d} \mathfrak{g})$ have the same dimension. Therefore $\text{ad}^*(\mathfrak{d}(q_\mathfrak{n}))q = (\mathfrak{g}_\mathfrak{d} \mathfrak{g} + n)^\perp$. On the other hand, for every $A \in \mathfrak{d}(q_\mathfrak{n})$ we have that

$$\exp(A) \cdot q = q + A \cdot q,$$

since $A^j \cdot q = 0$ for $j \geq 2$. Therefore

$$\mathbb{D}(q_\mathfrak{n}) \cdot q = q + (\mathfrak{g}_\mathfrak{d} \mathfrak{g} + n)^\perp = q + m(q)^\perp.$$  

□
LEMMA 2.6. Let \( q \in \mathfrak{g}^* \). If the \( \mathbb{D} \)-orbit \( \mathbb{D} \cdot q \nmid \subset \mathfrak{n}^* \) is closed, then also the \( \mathbb{D} \)-orbit \( \mathbb{D} \cdot q \subset \mathfrak{g}^* \) is closed.

Proof. Since the preceding Lemma 2.5 tells us that
\[
\mathbb{D} \cdot q \supset q + m(q)^\perp,
\]
it suffices to show that the \( \mathbb{D} \)-orbit \( \mathbb{D} \cdot q |_{\mathfrak{h}} \) is closed in \( \mathfrak{h}^* \), where \( \mathfrak{h} := m(q) \). So we can assume that \( \mathfrak{g} = m(q) \).

Let us show that \( \mathbb{D} \cdot q \) is closed in \( \mathfrak{g}^* \). Since \( \mathbb{D} \cdot q |_{\mathfrak{n}} \) is closed, it follows that for any \( p \in \mathbb{D} \cdot q \) we have that
\[
p |_{\mathfrak{n}} \in \mathbb{D} \cdot q |_{\mathfrak{n}} = \mathbb{D} \cdot q |_{\mathfrak{n}}
\]
and so \( p |_{\mathfrak{n}} = d \cdot q |_{\mathfrak{n}} \) for some \( d \in \mathbb{D} \). Hence
\[
p = d \cdot q + r
\]
for some \( r \in \mathfrak{n}^\perp \). This element \( r \) has the property that
\[
d \cdot r = r, \quad d \in \mathbb{D},
\]
since \( \langle r, \mathfrak{d} \cdot \mathfrak{g} \rangle = \{0\} \).

We now replace \( p \) by \( d^{-1} \cdot p \). Then for the new \( p \) we have that
\[
p = q + r \tag{2.2.2}
\]
Now \( p \in \mathbb{D} \cdot q \). Therefore
\[
p = \lim_{k \to \infty} d_k \cdot q = \lim_{k \to \infty} d_k \cdot (p - r)
\]
\[
= \lim_{k \to \infty} d_k \cdot p - r
\]
\[
\Rightarrow \quad p |_{\mathfrak{n}} = \lim_{k \to \infty} d_k \cdot p |_{\mathfrak{n}}
\]
for some sequence \( (d_k)_{k \in \mathbb{N}} \subset \mathbb{D} \). In this way we see that
\[
\lim_{k \to \infty} d_k \text{ mod } \mathbb{D}(p |_{\mathfrak{n}}) = e \text{ mod } \mathbb{D}(p |_{\mathfrak{n}}).
\]
But since \( \mathfrak{g} = \mathfrak{g}_{\mathfrak{d} \cdot \mathfrak{g}} + \mathfrak{n} \) it follows that \( \mathfrak{d}(q |_{\mathfrak{n}}) = \mathfrak{d}(q) \) and so
\[
\mathfrak{d}(p |_{\mathfrak{n}}) = \mathfrak{d}(q |_{\mathfrak{n}}) = \mathfrak{d}(q) = \mathfrak{d}(p)
\]
\[
\Rightarrow \quad \mathbb{D}(q) = \mathbb{D}(p) = \exp(\mathfrak{d}(p)) = \exp(\mathfrak{d}(p |_{\mathfrak{n}})) = \mathbb{D}(p |_{\mathfrak{n}}).
\]
In this way we see that
\[
\lim_{k \to \infty} d_k \text{ mod } \mathbb{D}(q) = e \text{ mod } \mathbb{D}(q)
\]
\[
\Rightarrow \quad p = \lim_{k \to \infty} d_k \cdot q = q.
\]
Hence by (2.2.2)
\[
q = p = q + r \quad \Rightarrow \quad r = 0.
\]
We see in this way that the \( \mathbb{D} \)-orbit \( \mathbb{D} \cdot q \) is closed in \( \mathfrak{g}^* \). \( \square \)
The first paper on the problem of the \( L^1 \)-determination for exponential solvable Lie groups appeared in 2010. (See [8].)

**Definition 2.7.** An element \( q \in \mathfrak{g}^\ast \) is called \( \mathbb{D} \)-non-\( \ast \)-regular if for the \( \mathbb{D} \)-invariant ideal \( m(q) = g_{0,q} + n \) of \( \mathfrak{g} \) (where \( n \) is as in (2.2.1)) we have that
\[
\langle q, m(q)^\infty \rangle \neq \{0\}.
\]

Denoting by \( \text{Prim}_{\mathbb{D}^*}(G) \) the space of primitive ideals \( \text{Prim}_{C^*}(G) \) equipped with the action of \( \mathbb{D} \), we say that \( \text{Prim}_{\mathbb{D}^*}(G) \) is \( L^1 \)-determined if for every \( \mathbb{D} \)-orbit \( \Omega \) in \( \mathfrak{g}^* \) the ideal \( \ker_{C^*}(\Omega) := \bigcap_{q \in \Omega} \ker_{C^*}(\pi_q) \) is the closure of \( \ker_{L^1}(\Omega) := \bigcap_{q \in \Omega} \ker_{L^1}(\pi_q) \).

### 3. \( L^1 \)-determination of \( \text{Prim}_{\mathbb{D}^*}(G) \)

We want to show that \( \text{Prim}_{\mathbb{D}^*}(G) \) is \( L^1 \)-determined provided that the restriction to some ideal \( n \supset D \cdot q \) of every non-\( \ast \)-regular \( \mathbb{D} \)-orbit \( \Omega \) is closed in \( n^\ast \). This also gives a new proof of the result of Ungermann which states that for every exponential Lie group \( G \) with closed non-\( \ast \)-regular \( G \)-orbits in \( n^\ast \), the space \( \text{Prim}_{C^*}(G) \) is \( L^1 \)-determined.

In order to prove that \( \text{Prim}_{\mathbb{D}^*}(G) \) is \( L^1 \)-determined, one must show that for an irreducible representation class \( \pi_O \) of such a group, we have that \( \ker(D \cdot \pi_O) \cap L^1(G) \) is dense in \( \ker_{C^*}(D \cdot \pi_O) \), which is equivalent to the property that
\[
\ker_{L^1}(D \cdot \pi_q) \subset \ker_{L^1}(\pi_\ell) \quad \Rightarrow \quad \ker_{C^*}(D \cdot \pi_q) \subset \ker_{C^*}(\pi_\ell) \quad \forall \ell, q \in \mathfrak{g}^\ast.
\]

In other words we have to prove that
\[
\rho \in \hat{G}, \quad \rho(\ker_{C^*}(D \cdot \pi_O)) \neq \{0\} \quad \Rightarrow \quad \rho(\ker_{L^1}(D \cdot \pi_O)) \neq \{0\}.
\]

Since the Fell topology in \( \text{Prim}_{C^*}(G) \) corresponds to the natural topology of \( \mathfrak{g}^\ast / G \), we have for two coadjoint orbits \( O_q \) and \( O_\ell \):
\[
\ker_{C^*}(D \cdot \pi_q) \subset \ker_{C^*}(\pi_\ell) \quad \Leftrightarrow \quad \pi_\ell \in D \cdot \pi_q,
\]
\[
\Leftrightarrow \quad O_\ell \subset D \cdot O_q.
\]

Hence, we see that \( \text{Prim}_{\mathbb{D}^*}(G) \) is \( L^1 \)-determined if and only if for all \( \ell, q \in \mathfrak{g}^\ast \)
\[
\ell \notin D \cdot O_q \quad \Rightarrow \quad \pi_\ell(\ker_{L^1}(D \cdot \pi_q)) \neq \{0\}.
\]

**Lemma 3.1.** Let \( \mathbb{D} = \exp(\mathfrak{d}) \supset \text{Ad}(G) \) be an exponential group of exponential automorphisms of the exponential solvable Lie group \( G = \exp(\mathfrak{g}) \). Suppose that \( q \in \mathfrak{g}^* \) is \( \mathbb{D} \)-\( \ast \)-regular, that is \( \langle q, m(q)^\infty \rangle = \{0\} \) (where \( m(q) \) is as in (2.2.1)), and that \( \ell \notin \overline{\Omega}_q \). Then \( \pi_\ell(\ker_{L^1}(D \cdot \pi_q)) \neq \{0\} \).

**Proof.** Let \( a := m(q)^\infty \). We proceed by induction on the dimension \( \dim(\mathfrak{g}) \) of \( \mathfrak{g} \). If \( \mathfrak{g} \) is nilpotent, then the group \( G \) is \( \mathbb{D} \)-\( \ast \)-regular and we are done. This is the beginning of the induction procedure.

If \( a \neq \{0\} \), then \( a \) is a non-trivial \( \mathbb{D} \)-invariant nilpotent ideal of \( \mathfrak{g} \) and \( m(q)/a \) is nilpotent. Let \( A = \exp(a) \). Since \( \langle q, a \rangle = \{0\} \), \( q \) being \( \ast \)-regular, the representations \( \pi_{d,q}, d \in \mathbb{D} \), are trivial on \( A \). Hence for any \( F \in L^1(G) \) and any \( a \in A \), we have that the function \( F_a := \)}
$l_a F - F$ is contained in $\ker_D(\mathbb{D} \cdot \pi_q)$. Now if $\langle \ell, a \rangle \neq \{0\}$, then we find $a \in A$ such that $\pi_\ell(a)$ is not the identity operator and we then have many $F \in L^1(G)$, such that $\pi_\ell(F_a) = \pi_\ell(a) \cdot \pi_\ell(F) - \pi_\ell(F) \neq 0$.

If $\ell \neq 0$ vanishes also on $a$, then we pass to the algebra $\tilde{\mathfrak{g}} := \mathfrak{g}/a$ and the group $\tilde{G} := G/A = \exp(\tilde{\mathfrak{g}})$. We consider the group of automorphisms $\mathbb{D}$ on $\tilde{G}$ defined by $\mathbb{D}$. Since $\tilde{\mathfrak{g}} = (\mathfrak{g} + a)/a$ we have that $\tilde{\mathfrak{g}}$ is also $*$-regular. We can use the induction hypothesis for $\tilde{G}$, $\tilde{\mathbb{D}}$ and $\tilde{n} := n/a$. There exists then an $\tilde{F} \in L^1(\tilde{G})$ such that $\pi_{\tilde{\mathbb{D}}}(\tilde{F}) = \{0\}$ and such that $\pi_\ell(\tilde{F}) \neq 0$. We now choose any $F \in L^1(G)$, such that

$$\int_A F(ga) \, da = \tilde{F}(g), \quad g \in G.$$ 

Then we have that

$$\pi_{d,q}(F) = 0, \quad d \in \mathbb{D},$$

$$\pi_\ell(F) \neq 0.$$

If $a = \{0\}$, then $m(q)$ is a nilpotent $\mathbb{D}$-invariant ideal of $\mathfrak{g}$, such that $d \cdot q + m(q) \subseteq \mathbb{D} \cdot q$ for every $d \in \mathbb{D}$ by Lemma 2.5. Hence we have that

$$\ell|_{m(q)} \notin \mathbb{D} \cdot q|_{m(q)}.$$

Let $\ell_0 := \ell|_b$, $q_0 = q|_b$, $\mathbb{D}_0 := \mathbb{D}|_H$, $b := m(q)$, $H = \exp(b)$. There exists $f \in L^1(H)$ such that $\pi_{d,q_0}(f) = 0$, $d \in \mathbb{D}_0$, and $\pi_{\ell_0}(f) \neq 0$, since $H$ is $* -$regular. For any $a \in C_c(G)$, the space of continuous functions on $G$ with compact support, let

$$F = F_a := a \ast f \in L^1(G),$$

where

$$a \ast f(g) := \int_H a(gh^{-1}) f(h) \Delta_H(h^{-1}) \, dh, \quad g \in G.$$ 

For $p \in \mathfrak{g}^*$ the representation $\pi_{p|H}$ is supported by the $G$-orbit of $p|_b$ (see [5]). Hence $\pi_{d,q}(F_a) = \pi_{d,q}(a) \circ \pi_{d,q}|_H(f) = \pi_{d,q}(a) \circ 0 = 0$ for any $d \in \mathbb{D}$. Furthermore, since $\pi_{\ell_0}(f) \neq 0$, we have that $\pi_{d,\ell_0}(f) \neq 0$ in a neighbourhood of the identity in $\mathbb{D}$ and so $\pi_{\ell|H}(f) \neq 0$. Hence $\pi_\ell(F_a) = \pi_\ell(a) \circ \pi_{\ell|H}(f)$ is different from zero for many of the $a$. \hfill $\Box$

Next we study the non-*$-$regular elements of $\mathfrak{g}^*$, for which $\mathbb{D} \cdot q|_n$ is a closed subset of $n^*$. This also gives a new proof of Proposition 4.14 of [8].

**Proposition 3.2.** Let $\mathbb{D} = \exp(\mathfrak{d}) \supset \text{Ad}(G)$ be an exponential group of exponential automorphisms of the exponential solvable Lie group $G = \exp(\mathfrak{g})$. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $\mathfrak{d} \cdot \mathfrak{g}$. Let $q, \ell \in \mathfrak{g}^*$. Suppose that the $\mathbb{D}$-orbit $\mathbb{D} \cdot q|_n$ is closed in $n^*$ and that $\ell$ is not contained in the $\mathbb{D}$-orbit $Q_q = \mathbb{D} \cdot q \subseteq \mathfrak{g}^*$, which is closed by Lemma 2.6. Then there exists $F \in L^1(G)$ such that

$$\pi_{d,q}(F) = 0, \quad \forall d \in \mathbb{D}, \pi_\ell(F) \neq 0.$$
Proof. Let \( N := \exp(n) \).

For our \( q \in g^* \) let

\[ O_q := G \cdot q \subset g^* \]

be the \( G \)-orbit of \( q \).

If \( G \) is nilpotent, then \( G \) is \(*\)-regular. Even more, by [7], since \( \ell \not\in \Sigma_q \), we can find a Schwartz function \( F \) on \( G \), such that \( \pi_d \cdot q(F) = 0 \), \( d \in D \), and \( \pi_\ell(F) \neq 0 \).

We proceed by induction on the dimension of \( g \). If \( n = g \), then \( g \) is nilpotent and we are done. This is the beginning of the induction process. Hence we can assume that \( n \) is a proper (\( \mathfrak{o} \)-invariant, \( D \)-invariant and nilpotent) ideal of \( g \).

We now consider the two cases \( g \neq m(q) \) and \( g = m(q) \) separately. We shall make use of the subalgebras and subgroups defined in Definition 2.4.

3.1. The case \( g \neq m(q) \)

Consider first the case where

\[ g \neq m(q) = g_{o \cdot q} + n. \]

Then, according to Lemma 2.5 we have that

\[ \Omega_q = \Omega_q + m(q)^\perp. \]

Hence for \( q_0 := q|m(q) \), since \( \ell \not\in \mathfrak{D} \cdot q \) we must also have that

\[ \ell_0 := \ell|m(q) \not\in \mathfrak{D} \cdot q_0. \]

We can apply the induction hypothesis to \( H = \exp(h) \), \( h = m(q) \), \( n \) and \( \mathfrak{D}|_H \), since \( h_{o \cdot q} = g_{o \cdot q} \) and since the \( \mathfrak{D}|_H \)-orbit of \( q|_H \) is also the \( \mathfrak{D} \)-orbit of \( q|_n \).

Let \( \ell_0 := \ell|_h \), \( q_0 = q|_h \), \( D_0 := \mathfrak{D}|_H \). There exists therefore \( f \in L^1(H) \) such that \( \pi_d \cdot q_0(f) = 0 \), \( d \in D_0 \), and \( \pi_\ell_0(f) \neq 0 \). For any \( a \in C_c(G) \), the space of continuous functions on \( G \) with compact support, let

\[ F = F_a := a \ast f \in L^1(G), \]

where

\[ a \ast f(g) := \int_H a(gh^{-1}) f(h) \Delta_H(h^{-1}) dh, \quad g \in G. \]

We finish as in the proof of Lemma 3.1.

3.2. The case \( g = m(q) \)

Suppose now that \( g = m(q) \). Then, since \( m(q) = m(d \cdot q) \), \( d \in D \), we also have for every \( d \in D \)

\[ m(d \cdot q) = g. \]

Because for \( d \in D \), we have that \( g(\mathfrak{o} \cdot (d \cdot q)) \subset g(d \cdot q) \) and since now by assumption \( G = \exp(g_{o \cdot q}) \cdot N \), it follows that \( G = G(d \cdot q) \cdot N \) and so in particular the \( G \)-orbits \( O_{d \cdot q} = G \cdot (d \cdot q) = N \cdot (d \cdot q) \) are closed in \( g^* \).
Choose any $\mathfrak{d}$-invariant ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one containing $\mathfrak{n}$ and let

$$H := \exp(\mathfrak{h}).$$

Then $H$ is a closed connected $\mathbb{D}$-invariant normal subgroup of $G$. Let

$$q_0 := q|_{\mathfrak{h}}, \quad \ell_0 := \ell|_{\mathfrak{h}}.$$

Since $G = G(d \cdot q) \cdot H$, for any $d \in \mathbb{D}$, we have that the representations $\pi_{d \cdot q}$ of $G$ are extensions of the representation $\pi_{d \cdot q_0}$ of $H$.

We can take a Vergne polarization $p$ at $q|_{\mathfrak{n}}$ in $\mathfrak{n}$, which is $G(q)$-invariant. The quotient space $N/P$ admits a (unique) left-invariant measure, since $N$ is nilpotent. We can realize the representation $\pi_q$ of $G$ on the space

$$H_q = L^2(G/P G(q), \chi_q) \simeq L^2(N/P, \chi_{q|n})$$

$$:= \left\{ \xi : N \mapsto \mathbb{C}, \text{ measurable, } \xi(np) = \chi_q(p^{-1})\xi(n), \quad p \in P, \quad n \in N, \quad \int_{N/P} |\xi(n)|^2 d\hat{n} < \infty \right\}$$

by left translation on $N$. The subgroup $G(q)$ then acts on $H_q$ by

$$\pi_q(s)\xi(n) = \left( \frac{\Delta_G(s)}{\Delta_P(s)} \right)^{1/2} \chi_q(s)\xi(s^{-1}ns), \quad n \in N, \quad s \in G(q).$$

3.2.1. The subcase $\mathfrak{g} = \mathfrak{m}(q), \ell_0 \not\subseteq \mathbb{D} \cdot q_0$. Suppose first that $\ell_0 \not\subseteq \mathbb{D} \cdot q_0$.

We can apply the induction hypothesis to $H := \exp(\mathfrak{h}), \mathbb{D}_0 := \mathbb{D}|_{\mathfrak{h}}, \mathfrak{n}$ and the linear forms $\ell_0, q_0$. There exists $f \in L^1(H)$ such that $\pi_{d \cdot q_0}(f) = 0$ for all $d \in \mathbb{D}_0$ and $\pi_{\ell_0}(f) \neq 0$.

Let $S \in \mathfrak{g} \setminus \mathfrak{h}$. Then

$$G = \exp(\mathbb{R}S) \cdot H$$

as topological product. We consider for every $\alpha \in C_c^\infty(\mathbb{R})$ the function $F_\alpha$ defined on $G$ by

$$F_\alpha(\exp(sS) \cdot h) := \alpha(s) f(h), \quad h \in H, \quad s \in \mathbb{R}.$$ 

Then obviously $F_\alpha \in L^1(G)$ and for every $d \in \mathbb{D}$ we have that

$$\pi_{d \cdot q}(F_\alpha) = \int_{\mathbb{R}} \int_{\mathbb{H}} \alpha(s) f(h)\pi_{d \cdot q}(\exp(sS) h) \, dh \, ds$$

$$= \int_{\mathbb{R}} \alpha(s)\pi_{d \cdot q}(\exp(sS)) \circ \int_{\mathbb{H}} f(h)\pi_{d \cdot q_0}(h) \, dh \, ds$$

$$= \int_{\mathbb{R}} \alpha(s)\pi_{d \cdot q}(\exp(sS)) \circ 0 \, ds$$

$$= 0.$$
Furthermore, if $\pi_\ell$ is an extension of $\pi_{\ell_0}$, i.e. if $O_\ell$ is not saturated with respect to $\mathfrak{h}$, then we have that

\[ \pi_\ell(F_\alpha) = \int_{\mathbb{R}} \alpha(s)\pi_\ell(\exp(sS)) \int_{H} f(h)\pi_{\ell_0}(h) \, dh \, ds \]

\[ = \int_{\mathbb{R}} \alpha(s)\pi_\ell(\exp(sS)) \, ds \circ \pi_{\ell_0}(f) \]

\[ \neq 0 \]

for many $\alpha$.

Now if the $G$-orbit of $\ell$ is saturated with respect to $\mathfrak{h}$, i.e. if $\pi_\ell = \text{ind}^{G}_{H}\pi_{\ell_0}$, then we use Section 1.2.1. The abelian group $A$ is now given by $\exp(\mathbb{R}S)$ and $\pi_0 = \pi_{\ell_0}$. Hence for $F \in L^1(G)$, $\xi \in L^2(\mathbb{R}, H_{\ell_0})$ and $t \in \mathbb{R}$:

\[ \pi_\ell(F)\xi(t) = \int_{\mathbb{R}} \pi_{\exp(sS)\cdot\ell_0}(F(t-s))\xi(s) \, ds. \]

For our function

\[ F_\alpha(\exp(sS)h) := \alpha(s)f(h), \quad h \in H, \ s \in \mathbb{R}, \ \alpha \in C_c(\mathbb{R}), \]

we then have

\[ \pi_\ell(F_\alpha)\xi(t) = \int_{\mathbb{R}} \alpha(t-s)\pi_{\exp(sS)\cdot\ell_0}(f)\xi(s) \, ds, \quad t \in \mathbb{R}, \ \xi \in H_\ell. \]

In particular for $t = 0$ this gives us

\[ \pi_\ell(F_\alpha)\xi(0) = \int_{\mathbb{R}} \alpha(-s)\pi_{\exp(sS)\cdot\ell_0}(f)\xi(s) \, ds, \quad t \in \mathbb{R}, \ \xi \in H_\ell. \]

If we choose an $\alpha$ supported in small neighbourhoods of zero with $\alpha(0) = 1$, we see that $\pi_\ell(F_\alpha) \neq 0$ for many $\alpha$ in $C_c(G)$, since $\pi_{\ell_0}(f) \neq 0$.

3.2.2. The subcase $g = m(q)$, $\ell_0 \in \overline{D \cdot q_0}$. Finally we have to consider the case

\[ m(q) = g \quad \text{and} \quad \ell_0 \in \overline{D \cdot q_0}. \]

It follows from Lemma 2.6 that $D \cdot q_0$ is closed in $\mathfrak{h}^\ast$. In particular

\[ \ell_0 = d_0 \cdot q_0 \]

for some $d_0 \in D$.

By Kirillov’s theorem [4], the smooth vectors $\xi \in H_{q_0} := H_q$ for the representation $\pi_{q_0}$ of $N$ are the Schwartz class functions in $H_{q_0}$ and any smooth operator $a$ for $N$ on $H_{q_0}$ is a compact operator whose kernel function is a Schwartz class on $N/P \times N/P$.

Now let $\Omega_{q_0} := D \cdot \pi_{q_0}$ be the $D$-orbit of the representation $\pi_{q_0}$ in $\hat{H}$. Then by the orbit picture, the topological space $\Omega_{q_0}$ is homeomorphic to the quotient space $D/\overline{D \cdot q_0}$

Since $D\gamma_{q_0} = D\pi_{q_0} = Dq_0 \cdot \text{Ad}(G)$, this subgroup $D\gamma_{q_0}$ of $D$ is closed and connected. Let $\mathfrak{d}\gamma_{q_0} = \text{Lie}(D\gamma_{q_0}) \subset \mathfrak{d}$ be its Lie algebra. Let $\mathfrak{d} := \dim(\mathfrak{d}/\mathfrak{d}\gamma_{q_0})$. We choose a coexponential basis $\mathcal{X} = \{X_1, \ldots, X_d\}$ of $\mathfrak{d}$ modulo $\mathfrak{d}\gamma_{q_0}$. Then the mapping

\[ E_{\mathcal{X}} : \mathbb{R}^d \to \Omega_{q_0}, \quad E_{\mathcal{X}}(t_1, \ldots, t_d) = (\exp(t_1X_1) \cdots \exp(t_dX_d)) \cdot \pi_{q_0} \]
is a homeomorphism. Let us write for \( t \in \mathbb{R}^d \)
\[
\pi_{q_t} = E(t) \cdot \pi_{q_0} = \pi_{E(t) \cdot q_0} \in \mathbb{H}_d.
\]
There exists \( t_0 \in \mathbb{R}^d \) such that
\[
\pi_{q_0} = E(t_0) \cdot \pi_{q_0} = \pi_{\varepsilon_0}.
\]
The representations \( \pi_{q_t} \) are all acting on the same space \( \mathcal{H}_{q_0} \simeq L^2(\mathbb{R}^{\dim(a/p)}) \) and since \( \pi_{q_t} = \pi_{q_0} \circ E(t)^{-1} \), the smooth vectors and smooth operators do not depend on \( t \), i.e.
\[
\mathcal{H}_{q_t} = \mathcal{H}_{q_0}, \quad \mathcal{B}(\mathcal{H}_{q_t})^\infty = \mathcal{B}(\mathcal{H}_{q_0})^\infty, \quad t \in \mathbb{R}^d.
\]

Denote by
\[
\mathcal{B}(\mathbb{D} \cdot q)_0^\infty
\]
the space of smooth mappings \( \mathbb{R}^d \ni t \mapsto a(t) \in \mathcal{B}(\mathcal{H}_{q_t})^\infty \) with compact support.

By [6, Theorem 3] there exists for any operator field \( a \in \mathcal{B}(\mathbb{D} \cdot q)_0^\infty \) a function \( f_a \in ES(H) \) for which \( \pi_{q_t}(f_a) = a(t), \ t \in \mathbb{R}^d \). Furthermore the mapping \( \mathcal{B}(\mathbb{D} \cdot q)_0^\infty \ni a \mapsto f_a \in ES(H) \) is linear and continuous.

Now choose such an \( a \in \mathcal{B}(\mathbb{D} \cdot q)_0^\infty \). Let for \( s \in \mathbb{R} \) and \( t \in \mathbb{R}^d \)
\[
a^s(t) := \pi_{E(t) \cdot q}(\exp(-sS)) \circ a(t). \tag{3.2.1}
\]
It follows that the mapping \( \mathbb{R} \ni s \mapsto a^s \) is smooth. Therefore the corresponding mapping \( \mathbb{R} \ni s \mapsto f_{a^s} \in ES(H) \) is continuous. In particular we have
\[
\pi_{q_t}(f_{a^s}) = \pi_{E(t) \cdot q}(\exp(-sS)) \circ a(t), \quad s \in \mathbb{R}, \ t \in \mathbb{R}^d. \tag{3.2.2}
\]

We now define for \( \alpha \in C_c(\mathbb{R}) \) the \( L^1(G) \)-function \( F_{a}^{\alpha} \) by
\[
F_{a}^{\alpha}(\exp(sS)h) := \alpha(s)f_{a^s}(h), \quad h \in H, \ s \in \mathbb{R}.
\]
Therefore for \( d = E(t), \ t \in \mathbb{R}^d \), we have that
\[
\pi_{d,q}(F_{a}^{\alpha}) = \int_\mathbb{R} \int_H \pi_{E(t) \cdot q}(\exp(sS)) \circ \pi_{q_t}(h) \alpha(s)f_{a^s}(h) \, dh \, ds
\]
\[
= \int_\mathbb{R} \alpha(s) \pi_{E(t) \cdot q}(\exp(sS)) \circ \pi_{q_t}(f_{a^s}) \, ds
\]
\[
= \int_\mathbb{R} \alpha(s) \pi_{E(t) \cdot q}(\exp(sS)) \circ \pi_{E(t) \cdot q}(\exp(-sS)) \circ a(t) \, ds
\]
\[
= \left( \int_\mathbb{R} \alpha(s) \, ds \right) a(t).
\]
Hence, whenever \( \int_{\mathbb{R}} \alpha(s) \, ds = 0 \), the function \( F_{a}^{\alpha} \) is contained in \( \ker(\pi_{d,q}) \) for all \( d \in \mathbb{D} \).

Let
\[
C_c(\mathbb{R})^0 := \left\{ \alpha \in C_c(\mathbb{R}), \int_{\mathbb{R}} \alpha(s) \, ds = 0 \right\}.
\]
We now have to distinguish the two cases.
$L^1$-determined primitive ideals

**Case 1.** $\mathcal{O}_\ell$ is not saturated with respect to $\mathfrak{h}$, i.e.

$$\pi_\ell|_{\mathfrak{h}} \cong \pi_{\ell_0}.$$ 

In case 1, we have that $\pi_\ell$ is an extension of the irreducible representation $\pi_{\ell_0} = \pi_{q_0}$. Hence relation (3.2.2) has the following form:

$$\pi_{\ell_0}(f_{a \cdot t}) = \pi_\ell(\exp(sS)) \circ \pi_{\ell_0}(f_a), \quad s \in \mathbb{R}, \ a \in \mathcal{B}(\mathbb{D} \cdot q)_0^\infty. \quad (3.2.3)$$

We now use the fact that $\ell_0 = E(t_0) \cdot q_0 \in \mathbb{D} \cdot q_0$. Since $\ell \notin \mathbb{D} \cdot q$ we have that

$$\ell = E(t_0) \cdot q + \rho S^*$$

for some $\rho \neq 0$. Here $S^*$ is the element of $\mathfrak{g}^*$ defined by

$$(S^*, S) = 1, \quad S^*_\mathfrak{h} = 0.$$ 

Therefore

$$\pi_\ell(\exp(sS)) = e^{-i\rho s} \pi_{E(t_0)q}(\exp(sS)), \quad s \in \mathbb{R}.$$ 

Whence for any $a \in \mathcal{B}(\mathbb{D} \cdot q)_0^\infty$ satisfying (3.2.3) we have that

$$\pi_\ell(F^a_{\ell_0}) = \int_{\mathbb{R}} \int_{\mathcal{H}} e^{-i\rho s} \pi_{E(t_0)q}(\exp(sS)) \circ \pi_{q_0}(h) \alpha(s)f_{a^t}(h) \, dh \, ds$$

$$= \int_{\mathbb{R}} e^{-i\rho s} \alpha(s) \pi_{E(t_0)q}(\exp(sS)) \circ (\pi_{q_0}(f_{a^t})) \, ds$$

$$= \int_{\mathbb{R}} e^{-i\rho s} \alpha(s) \pi_{E(t_0)q}(\exp(sS)) \circ \pi_{E(t_0)q}(\exp(-sS)) \circ a(t_0) \, ds$$

$$= \left( \int_{\mathbb{R}} e^{-i\rho s} \alpha(s) \, ds \right) a(t_0).$$

Therefore, if we choose $\alpha \in C_c(\mathbb{R})^0$ such that

$$\hat{\alpha}(\rho) \neq 0$$

and $a$ such that $a(t_0) \neq 0$, then $F^a_{\ell_0} \in \ker(\mathbb{D} \cdot \pi_q)$ but $\pi_\ell(F^a_{\ell_0}) \neq 0$.

**Case 2.** $\mathcal{O}_\ell$ is saturated with respect to $\mathfrak{h}$.

In case 2 the representation $\pi_\ell$ is induced from $\pi_{\ell_0}$, $\pi_\ell = \text{ind}^G_H \pi_{\ell_0}$.

We choose as before $f \in L^1(H)$ such that the operator field $a(t) := \pi_{q_0}(f), \ t \in \mathbb{R}^d$ is contained in $\mathcal{B}(\mathbb{D} \cdot q)_0^\infty$ and such that $\pi_{\ell_0}(f)$ is not zero.

For $F \in L^1(G)$, we have again by Section 1.2.1 that

$$\pi_\ell(F)\xi(t) = \int_{\mathbb{R}} \pi_{\exp(sS) \cdot \ell_0}(F(t - s))\xi(s) \, ds.$$ 

For our function

$$F^a_{\ell_0}(\exp(sS)h) := \alpha(s)f_{a^t}(h), \quad h \in H, \ s \in \mathbb{R}, \ \alpha \in C_c(\mathbb{R})^0,$$

we then get

$$\pi_\ell(F^a_{\ell_0})\xi(t) = \int_{\mathbb{R}} \alpha(t - s)\pi_{\exp(sS) \cdot \ell_0}(f_{a^t})\xi(s) \, ds, \quad t \in \mathbb{R}, \ \xi \in \mathcal{H}_\ell. \quad (3.2.4)$$
Let us take a $\xi$ of the form $\xi(t) = \beta(t)\xi_0$, $t \in \mathbb{R}$, for some $\xi_0 \in \mathcal{H}_{\ell_0} = \mathcal{H}_{q_0}$ for which $\pi_{\ell_0}(f)\xi_0 \neq 0$ and $\beta \in C_c(\mathbb{R})$. Evaluating (3.2.4) for $t = 0$, this gives us

$$\pi_t(F^\alpha_g)\xi(0) = \int_{\mathbb{R}} \alpha(-s)\beta(s)\pi_{\exp(s\mathfrak{g})\ell_0}(f_{a^{-s}})\xi_0 \, ds.$$ 

If we choose $\beta$ supported in small neighbourhoods of zero such that $\beta(0) = 1$ and also $\alpha \in C_c(\mathbb{R})^0$ with $\alpha(0) = 1$, we have that $\pi_t(F^\alpha_g) \neq 0$.

Combining Lemma 3.1 and Proposition 3.2 we finally obtain the following result.

**Theorem 3.3.** Let $G = \exp(\mathfrak{g})$ be an exponential solvable Lie group and $\mathbb{D} = \exp(\mathfrak{d}) \supset \text{Ad}(G)$ be an exponential group of exponential automorphisms of $G$. Suppose that for every $\mathbb{D}$-non-$*$-regular $q \in \mathfrak{g}^*$, there exists a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that the $\mathbb{D}$-orbit $\mathbb{D} \cdot q|_{\mathfrak{n}}$ is closed in $\mathfrak{n}^*$. Then $\text{Prim}^\mathbb{D}_{C^*}(G)$ is $L^1$-determined.

As a corollary, we obtain a result of Ungermann as follows.

**Theorem 3.4.** (Ungermann [8]) Let $G = \exp(\mathfrak{g})$ be an exponential solvable Lie group. Suppose that for every non-$*$-regular $q \in \mathfrak{g}^*$, there exists a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ such that the $G$-orbit $G \cdot q|_{\mathfrak{n}}$ is closed in $\mathfrak{n}^*$. Then $\text{Prim}^\mathbb{D}_{C^*}(G)$ is $L^1$-determined.

**4. Examples**

Let us give two examples.

**4.1. Boidol’s group**

The first one is Boidol’s group, $B = \exp(\mathfrak{b})$, which is well known. This group is not $*$-regular; as shown in [1], it has closed non-$*$-regular orbits. By Ungermann’s result [8], $\text{Prim}^\mathbb{D}_{C^*}(B)$ is $L^1$-determined. We recall the structure of its orbit space $\mathfrak{b}^*/B$ before the introduction of the second example, which shows that the condition on the closure of $\mathbb{D} \cdot q|_{\mathfrak{n}}$ in Theorem 3.3 is not unnecessary.

Let

$$\mathfrak{b} := \{T, X, Y, Z\},$$

$$[T, X] = -X, \quad [T, Y] = Y, \quad [X, Y] = Z.$$  

The simply connected connected group $B$ with Lie algebra $\mathfrak{b}$ can be realized on $\mathbb{R}^4$ with the multiplication

$$(t, x, y, z) \cdot (t', x', y', z) = (t + t', e^{t'} x + x', e^{-t'} y + y', z + z' + \frac{1}{2}(e^{t'} xy' - e^{-t'} x'y)).$$

The inverse of $(t, x, y, z)$ is given by

$$(t, x, y, z)^{-1} = (-t, -e^{-t}x, -e^{-t}y, -z).$$

For $(\rho, 0, 0, \lambda) \equiv \rho T^* + \lambda Z^* \in \mathfrak{b}^*$ and $(t, x, y, z) \in B$ we have that

$$\left\langle \text{Ad}^*(\exp(tX + yY)) (\rho, 0, 0, \lambda), T \right\rangle = \rho - xy\lambda,$$

$$\left\langle \text{Ad}^*(\exp(tX + yY)) (\rho, 0, 0, \lambda), X \right\rangle = y\lambda,$$

$$\left\langle \text{Ad}^*(\exp(tX + yY)) (\rho, 0, 0, \lambda), Y \right\rangle = -x\lambda,$$

$$\left\langle \text{Ad}^*(\exp(tX + yY)) (\rho, 0, 0, \lambda), Z \right\rangle = \lambda.$$
For \((0, \omega, \pm, 0) \equiv \omega X^* \pm Y^*, \, \omega \neq 0\), we have for \(t, x, y \in \mathbb{R}\) that
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\omega X^* \pm Y^*), \, T \rangle = -\omega x \pm y,
\]
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\omega X^* + Y^*), \, X \rangle = e^t \omega,
\]
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\omega X^* \pm Y^*), \, Y \rangle = e^{-t}.
\]

For \((0, 0, \beta, 0) \equiv \beta Y^*, \beta \neq 0\), we have for \(t, x, y \in \mathbb{R}\) that
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\beta Y^*), \, T \rangle = \beta y,
\]
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\beta Y^*), \, X \rangle = 0,
\]
\[
\langle \text{Ad}^*(e^{tT}e^{\omega X}e^{\gamma Y}) (\beta Y^*), \, Y \rangle = \beta e^{-t}.
\]

This gives us the following partition of \(\mathfrak{b}^*\):
\[
\mathfrak{b}^* = \Gamma_3 \cup \Gamma_2 \cup \Gamma_1 \cup \Gamma_0,
\]
where
\[
\Gamma_3 = \{ O_{\rho, \lambda} \mid \rho \in \mathbb{R}, \, \lambda \in \mathbb{R}^* \}
\]
and where
\[
O_{\rho, \lambda} = \left\{ \left( \rho + \frac{uv}{\lambda}, u, v, \lambda \right) ; u, v \in \mathbb{R} \right\}, \quad \ell_{\rho, \lambda} := (\rho, 0, 0, \lambda).
\]
Let
\[
\Gamma_2 := \bigcup_{\pm, \pm} \Gamma_{2, \pm, \pm} := \{ O_{\pm \omega, \pm} \mid \omega \in \mathbb{R}_+^* \},
\]
where
\[
O_{\pm \omega, \pm} := \{ (u, \pm \omega e^t, \pm e^{-t}, 0) ; u, t \in \mathbb{R} \}, \quad \ell_{\pm \omega, \pm} := (0, \pm \omega, \pm 1, 0).
\]
Let
\[
\Gamma_1 = O_{2, \pm} \cup O_{3, \pm},
\]
where
\[
O_{2, \pm} := \{ (u, \pm e^t, 0, 0) ; t, u \in \mathbb{R} \}, \quad O_{3, \pm} := \{ (u, 0, \pm e^t, 0) ; t, u \in \mathbb{R} \}.
\]
Finally let
\[
\Gamma_0 := \{ \ell_\tau \mid \tau \in \mathbb{R} \},
\]
where
\[
\ell_\tau := (\tau, 0, 0, 0).
\]

Letting the nilpotent ideal \(\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]\), we see that the \(\mathcal{B}\)-orbits of \(\ell_{\rho, \lambda}\), \(\lambda \in \mathbb{R}^*, \rho \in \mathbb{R}\), are all closed in \(\mathfrak{n}^*\). Furthermore the stabilizer of the element \(\ell_{\rho, \lambda}\) in \(\mathfrak{b}^*\) is the subalgebra
\[
\mathfrak{b}(\ell_{\rho, \lambda}) = \mathbb{R} \mathcal{T} + \mathbb{R} \mathcal{Z}
\]
and the ideal \( m(\ell,\rho,\lambda) \) is then given by
\[
m(\ell,\rho,\lambda) = b(\ell,\rho,\lambda) + [b, b]
= \mathbb{R}T + \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z = b.
\]
Hence
\[
m(\ell,\rho,\lambda)^\infty = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z = [b, b]
\]
and so \( \ell,\rho,\lambda \) is always non-\(*\)-regular. On the other hand, whenever \( \ell \in b^* \) vanishes on \( Z \), then \( b(\ell) \) is contained in \([b, b]\), if \( \ell \) is not a character, or \( b(\ell) = b \) and \( \langle \ell, [b, b] \rangle = 0 \). Hence the elements in \( \Gamma_2 \cup \Gamma_1 \cup \Gamma_0 \) are \(*\)-regular, and the elements in \( \Gamma_3 \) are non-\(*\)-regular and their \( B \)-orbits are closed in \( n^* \). Therefore Lemma 3.1 and Proposition 3.2 imply that the property (3.0.1) holds for all elements in \( b^* \), that is, \( \text{Prim}^{\ast}(B) \) is \( L^1 \)-determined (see [8]).

4.2. **Exponential group with non-closed non-\(*\)-regular orbits**

The second example is an exponential group, which has some non-\(*\)-regular orbits, which are not closed.

Let
\[
g := \{ Q, R, S, T, X, Y, U, Z, V \},
\]
\[
[S, X] = X, \quad [S, Z] = Z, \quad [T, X] = -X, \quad [T, Y] = Y, \quad [X, Y] = Z,
\]
\[
[R, Q] = Q, \quad [R, U] = U, \quad [R, V] = 2V, \quad [Q, T] = U, \quad [Q, U] = V.
\]
Let us test the Jacobi relation:
\[
[U, [Q, T]] = [U, U] = 0 = 0 + 0 = [-V, T] + [Q, 0] = [[U, Q], T] + [Q, [U, T]]
\]
and
\[
[R, [Q, T]] = [R, U] = U = [Q, T] + 0 = [[R, Q], T] + [Q, 0]
= [[R, Q], T] + [Q, [R, T]].
\]

Then \( g \) is the sum of the abelian ideal \( n = \text{span}\{Y, U, Z, V\} \), the subalgebra \( d = \text{span}\{R, S, X, Q\} \) (which is itself the sum of two abelian subalgebras of dimension two) and the one-dimensional space \( \mathbb{R}T \). Let \( G := \exp(g) \). We take the linear form \( \ell := V^* + Z^* \).

The stabilizer \( g(\ell) \) of \( \ell \) is one-dimensional and is given by
\[
g(\ell) = \mathbb{R}T.
\]
A Pukanszky polarization \( p \) at \( \ell \) is given by
\[
p = \text{span}\{T, Y, U, Z, V\} = \mathbb{R}T \oplus n.
\]
Let
\[
D = \exp(d), \quad N := \exp(n), \quad P = \exp(p).
\]

The coadjoint orbit of \( \ell \) is given by
\[
\langle \text{Ad}^*(e^{R}e^{S}e^{Q}e^{X}e^{U}e^{Y}e^{Z}e^{V})\ell, T \rangle = \langle \ell, T - qU + \frac{1}{2}q^2V - xX + yY - xyZ \rangle
= \frac{1}{2}q^2 - xy.
\]
We can also restrict $\pi_\ell$. The irreducible representation $\pi_\ell$ of the orbit $\langle \ell, R + q Q + u U + uq V + 2v V \rangle$ is given by

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, R \rangle = \langle \ell, S + x X + y Z + z Z \rangle = xy + z,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, S \rangle = \langle \ell, e^{-r} Q + e^{-r} u V \rangle = e^{-r} u,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, Q \rangle = \langle \ell, e^{-r} U - q e^{-r} V \rangle = -e^{-r} q,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, U \rangle = \langle \ell, e^{-r} U - q e^{-r} V \rangle = -e^{-r} q,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, X \rangle = \langle \ell, e^{-r} U - q e^{-r} V \rangle = -e^{-r} q,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, Y \rangle = \langle \ell, e^{-r} U - q e^{-r} V \rangle = -e^{-r} q,$$

$$\langle Ad^* (e^R e^S e^Q e^X e^U e^Y e^Z e^V ) \ell, Z \rangle = \langle \ell, e^{-r} U - q e^{-r} V \rangle = -e^{-r} q.$$
(where $h := \mathfrak{d} + n$), which gives us an irreducible representation $\pi_{\ell'}$ of this group. Then

$$\pi_{\ell'} = \text{ind}_N^H \chi_{\ell'}, \quad \ell' := \ell h,$$

and $\pi_{\ell}$ is then an extension of $\pi_{\ell'}$ to $G$.

The representation $\pi_{\ell'}$ acts on $L^2(\mathbb{R}^4)$ by identifying $\mathbb{R}^4$ with the group $D$ and then

$$\pi_{\ell'}(\exp(tT))\xi(e^{tR}e^{sS}e^{xQ}e^{tT}) = e^{t/2}\xi(e^{-tT}e^{sS}e^{xQ}e^{tT})$$

$$= e^{t/2}\xi(e^{sS}e^{xQ}e^{tU})$$

$$= e^{t/2}\xi(e^{sS}e^{xQ}e^{tU}e^{-1/2t^2V})$$

$$= e^{t/2}e^{-1/2t^2}e^{xQ}e^{xQ}e^{tQ}.$$  

Furthermore we have for $\xi \in H_{\ell} \simeq L^2(\mathbb{R}^4)$ and $g = d \cdot n \in H$, $d \in D$, $n \in N$ and $c \in D$:

$$\pi_{\ell}(dn)\xi(c) = \xi(n^{-1}d^{-1}c)$$

$$= \xi((d^{-1}c) \cdot (c^{-1}d \cdot n^{-1}d^{-1}c))$$

$$= \chi_{\ell}(c^{-1}d \cdot nd^{-1}c)\xi(d^{-1}c).$$

Hence for $F \in L^1(G)$ we have that

$$\pi_{\ell}(F)\xi(c) = \int_\mathbb{R} \int_H \int_D \int_N F(e^{tH})\pi_{\ell}(e^{tT})\pi_{\ell}(h)\chi_{\ell}(c) \, dh \, dt$$

$$= \int_\mathbb{R} \pi_{\ell}(e^{tT}) \int_D \int_N \int_D \int_N F(e^{tH})\chi_{\ell}(c^{-1}d \cdot nd^{-1}c)\xi(d^{-1}c) \, dn \, dd \, dt$$

$$= \int_\mathbb{R} \pi_{\ell}(e^{tT}) \int_D \int_N \int_D \int_N F(e^{tH}c)\chi_{\ell}(c)\xi(d^{-1}c) \, dn \, dd \, dt$$

$$= \int_\mathbb{R} \pi_{\ell}(e^{tT}) \int_D \Delta_D(d^{-1})\tilde{F}^2(e^{tT}cd^{-1}) \, d \cdot \ell \cdot n \xi(d) \, dd \, dt,$$

where

$$\tilde{F}^2(g, \lambda) := \int_N \chi_{\lambda}(n) F(g \cdot n) \, dn, \quad \lambda \in n^*,$$

where $d \cdot \ell \cdot n := \text{Ad}^*(d)\ell \cdot n$ and where $\Delta_D$ is the modular function of the group $D$.

Now write $d = e^{rR}e^{sS}e^{uQ}e^{xX}$, $c = e^{r'K}e^{s'S}e^{u'Q}e^{x'X}$ and so

$$\pi_{\ell}(F)\xi(e^{rR}e^{sS}e^{uQ}e^{xX})$$

$$= \int_\mathbb{R} e^{t/2}e^{-i/2t(q'q)^2} \int_D e^{r+s}\tilde{F}^2(e^{tH}e^{(s'q'-r)}e^{sS}e^{q'-u'q}e^{x'X}) \, dt \, dr \, ds \, du \, dx.$$
We can replace $F$ by $K \ast F \ast K$ for some $K \in C_c^\infty(G)$ and then $F$ is contained in the kernel of $\pi_\ell$ if and only if

$$0 = \int_{\mathbb{R}} e^{\frac{1}{2} t} e^{-\frac{1}{2} t} (q')^2 \hat{F}^2(e^{i t} e^{i x} e^{(r' - r) S} e^{q' e^x} e^{(x' e^{x + s} - e^x) X},$$

$$- e^{-r} u U^* - x Y^* + e^{-2r} V^* + e^{-s} Z^*) \, dt$$

$$\equiv \int_{\mathbb{R}} e^{\frac{1}{2} t} e^{-\frac{1}{2} t} (q')^2 \hat{F}^2(e^{i t} e^{i x} e^{s' S} e^{q' e^x} e^{(x' e^{x + s} - e^x) X},$$

$$- x Y^* + e^{-2r} V^* + e^{-s} Z^*) \, dt$$

for every $r, r', s', u, q', x, x' \in \mathbb{R}$. If we let $r$ tend to $+\infty$ then we have even the relation

$$0 = \int_{\mathbb{R}} e^{\frac{1}{2} t} e^{-\frac{1}{2} t} u^2 \hat{F}^2(e^{i t} e^{i x} e^{s' S} e^{q' e^x} e^{(x' e^{x + s} - e^x) X},$$

$$- x Y^* + e^{-s} Z^*) \, dt$$

for every $r', s', u, q', x, x' \in \mathbb{R}$. Since for almost any $r', q', x, x', x, s' \in \mathbb{R}$ the function

$$\mathbb{R} \to \mathbb{C}; \quad \varphi_{r',q',x,x',s',s'}(\alpha) := \int_{\mathbb{R}} e^{\frac{1}{2} t} e^{-\frac{1}{2} t} \alpha \hat{F}^2(e^{i t} e^{i x} e^{s' S} e^{q' e^x} e^{(x' e^{x + s} - e^x) X},$$

$$- x Y^* + e^{-s} Z^*) \, dt$$

can be extended into the complex domain $|\text{Im} \alpha| < \frac{1}{2}$ as an analytic function and this extension is identically zero on $\mathbb{R}_+$, it follows that it is identically zero on $\mathbb{R}$ and so we have that

$$0 = \hat{F}^2(e^{i t} e^{i x} e^{s' S} e^{q' e^x} e^{(x' e^{x + s} - e^x) X},$$

$$- x Y^* + e^{-s} Z^*)$$

(4.2.2)

is identically zero for every $r', t, q', x, x', s, s' \in \mathbb{R}$.

Now let

$$\mathfrak{z} := \mathbb{R} U + \mathbb{R} V + \mathbb{R} Z, \quad Z = \exp(\mathfrak{z}),$$

and for any unitary character $\chi_\beta$ of $Z$ let

$$\hat{F}^3(g, \beta) := \int_{\mathbb{Z}} F(gu) \chi_\beta(u) \, du, \quad g \in G.$$

Since relation (4.2.2) is true for all $x \in \mathbb{R}$, it follows that

$$0 = \hat{F}^3(e^{i t} e^{i x} e^{s' S} e^{q' e^x} e^{x X} e^{x Y}, e^{-s} Z^*)$$

for every $r', t, q', x, x', s, s' \in \mathbb{R}$. Hence, using Section 1.2.1, we have that $F$ is contained in the kernel of all the representations $\tau_\beta := \text{ind}_Z^G \chi_\beta$, where $\beta = b Z^* \in \mathfrak{z}^*$ satisfying $b \geq 0$.

Now, since $[g, \mathfrak{z}] = \mathfrak{z}$, it follows that $0 \in \mathfrak{z}^*$ is $G$-invariant. We then see by Lemma 1.7 and Remark 2.2 that

$$\ker L^1(\pi_\ell) \subset \ker L^1(\tau_{\mathfrak{z}^*}) \subset \ker L^1(\pi_\ell)$$

for $\ell = V^* + Z^*$ and any $\lambda \in G^*$ satisfying $\lambda_{\mathfrak{z}^*} = 0$, in particular for

$$\lambda_{x,y} := x X^* + y Y^*.$$

If we now take this $\lambda_{x,y}$ with $x \cdot y > 0$ then by (4.2.1) we have that $\lambda_{x,y}$ is not contained in $\mathbb{Q}_\ell$. But the fact that $\ker L^1(\pi_\ell) \subset \ker L^1(\pi_{\lambda_{x,y}})$ tells us that $\ker C^*(\pi_\ell)$ is not $L^1$-determined (see (3.0.1)).
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