Quantum Markovian Approximations for Fermionic Reservoirs

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Abstract

We establish a quantum functional central limit for the dynamics of a system coupled to a Fermionic bath with a general interaction linear in the creation, annihilation and scattering of the bath reservoir. Following a quantum Markovian limit, we realize the open dynamical evolution of the system as an adapted quantum stochastic process driven by Fermionic Noise.

1 Introduction & Statement Results

Models of quantum systems driven by Fermionic Wiener processes have been considered in [1], [2]. We extend the results of [3] concerning quantum Markov approximations to the case of a Fermi reservoir. The main technical difference comes about from the absence of coherent vectors in Fermi Fock space and, of course, now we have to control the sign which arises from the canonical anti-commutation relations (CAR). This in turn extends a result of [4] on the weak coupling limit.

We consider the evolution on a joint Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ governed by a time-dependent Hamiltonian $\mathcal{H}_t (\lambda) = \sum_{\alpha,\beta \in \{0,1\}} E_{\alpha\beta} \otimes [a^+_\alpha (\lambda)]^\alpha [a^-_\beta (\lambda)]^\beta$ where the fields $a^{\mp}_t (\lambda)$ are Fermionic fields on $\mathcal{H}_R$ and have an increasingly singular correlation as the parameter $\lambda \to 0$. That is $\lim_{\lambda \to 0} \{a^-_s (\lambda), a^+_t (\lambda)\} = \gamma \delta (s - t)$ for some $\gamma > 0$.

We show that the unitary family $U_t (\lambda) := \hat{T} \exp \left\{ -i \int_0^t ds \mathcal{H}_s (\lambda) \right\}$ converges in weak matrix elements to a unitary adapted quantum stochastic process $U_t$ satisfying the quantum stochastic differential equation driven by Fermionic Wiener processes $\lambda^\pm_t$ and by the gauge process $\Lambda_t$ [1], [2], [5]:

$$dU_t = \frac{1}{\gamma} (W - 1) U_t \otimes d\Lambda_t + LU_t \otimes d\lambda^+_t - L^1 W U_t \otimes d\lambda^-_t - KU_t \otimes dt \quad (1.1)$$
where \( W = \frac{1+i\kappa E_{11}}{1+i\kappa E_{11}} \) (unitary), \( L = -i(1+i\kappa E_{11})^{-1}E_{10} \) (bounded), and \( K = \frac{1}{\gamma}L^TL + iH \) with \( H = E_{00} + \text{Im} \kappa E_{01} + \frac{1}{L^TL}E_{10} \) (self-adjoint). Here \( \kappa \) is a complex damping constant having a microscopic origin with \( \gamma = 2\Re \kappa \).

We fix some notation. Let \( \Gamma (\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes n} \) be the “Full” Fock space over a fixed complex separable Hilbert space \( \mathfrak{h} \). The (anti)-symmetrization operators \( \mathfrak{P}_{\pm} \) are defined through linear extension of the relations \( \mathfrak{P}_{\pm}f_1 \otimes \cdots \otimes f_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (\pm 1)^{\sigma} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \), with \( f_j \in \mathfrak{h} \), \( \mathfrak{S}_n \) denotes the permutation group on \( \{1, \ldots, n\} \) and \( (-1)^{\sigma} \) is the parity of the permutation \( \sigma \). The Bose Fock space \( \Gamma_+ (\mathfrak{h}) \) and the Fermi Fock space \( \Gamma_- (\mathfrak{h}) \) having \( \mathfrak{h} \) as one-particle space are then defined as the subspaces \( \Gamma_{\pm} (\mathfrak{h}) := \mathfrak{P}_{\pm} \Gamma (\mathfrak{h}) \). As usual, we distinguish the Fock vacuum \( \Phi = (1,0,0,\ldots) \) which is common to both Bose and Fermi spaces.

Let \( h \in \mathfrak{h} \), \( U \) unitary and \( H \) self-adjoint on \( \mathfrak{h} \). We define the following operators on the Full Fock space

\[
C^+ (h) f_1 \otimes \cdots \otimes f_n := \sqrt{n+1} h \otimes f_1 \otimes \cdots \otimes f_n ; \\
C^- (h) f_1 \otimes \cdots \otimes f_n := \frac{1}{\sqrt{n}} \langle h | f_1 \rangle f_2 \otimes \cdots \otimes f_n ; \\
\Gamma (U) f_1 \otimes \cdots \otimes f_n := (U f_1) \otimes \cdots \otimes (U f_n) ; \\
\gamma (H) f_1 \otimes \cdots \otimes f_n := \sum_j f_1 \otimes \cdots \otimes (H f_j) \otimes \cdots \otimes f_n .
\]

Bose creation and annihilation fields are then defined on \( \Gamma_+ (\mathfrak{h}) \) as \( B^\pm (h) := \mathfrak{P}_{\pm} C^\pm (h) \mathfrak{P}_{\pm} \), while Fermi creation and annihilation fields are defined on \( \Gamma_- (\mathfrak{h}) \) as \( A^\pm (h) := \mathfrak{P}_{\mp} C^\mp (h) \mathfrak{P}_{\pm} \). Using the traditional conventions \([X, Y] = XY - YX \) and \( \{X, Y\} = XY + YX \), we have the canonical (anti)-commutation relations

\[
\text{CCR: } [B^- (f), B^+ (g)] = \langle f | g \rangle ; \quad \text{CAR: } \{A^- (f), A^+ (g)\} = \langle f | g \rangle . \tag{1.2}
\]

Second quantization operators are defined as \( \Gamma_{\pm} (U) := \mathfrak{P}_{\pm} \Gamma (U) \mathfrak{P}_{\pm} \) and differential second quantization operators as \( \gamma_{\pm} (H) := \mathfrak{P}_{\pm} \gamma (H) \mathfrak{P}_{\pm} \). We have the relation \( \exp \{ it \gamma_{\pm} (H) \} = \Gamma_{\pm} (e^{itH}) \). More generally, we may take the argument of the differential second quantizations to be bounded: for the rank-one operator \( H = |f \rangle \langle g| \), described in standard Dirac bra-ket notation, we have \( \gamma_+ (|f \rangle \langle g|) \equiv B^+ (f) B^- (g) \) and \( \gamma_- (|f \rangle \langle g|) \equiv A^+ (f) A^- (g) \).

In the Bose case, the exponential vector map \( \varepsilon : \mathfrak{h} \mapsto \Gamma_+ (\mathfrak{h}) \) is introduced as \( \varepsilon (f) = \bigoplus_{n=0}^{\infty} \frac{f^\otimes n}{\sqrt{n!}} \) with \( f^\otimes n \) the \( n \)-fold tensor product of \( f \) with itself. The Fock vacuum is, in particular, given by \( \Phi = \varepsilon (0) \). The exponential vectors are frequently used in the analysis of second quantized Boson fields and facilitate enormously dealings with creation, annihilation and conservation operators. \( \varepsilon (h) \) is a total subset in \( \Gamma_+ (\mathfrak{h}) \). There is no analogue in the Fermi case and here we have to treat \( n \)-particle vectors \( \mathfrak{P}_{\pm} f_1 \otimes \cdots \otimes f_n \), sometimes written in exterior algebra notation as \( f_1 \wedge \cdots \wedge f_n \), as a natural domain for investigations. The operations of Bose and Fermi second quantization have the natural functorial property \( \Gamma_{\pm} (\mathfrak{h}_1 \oplus \mathfrak{h}_2) \cong \Gamma_{\pm} (\mathfrak{h}_1) \otimes \Gamma_{\pm} (\mathfrak{h}_2) \), see [9].
2 The Model

We shall consider a quantum mechanical system $S$ with a state space $\mathcal{H}_S$ coupled to a quantum Fermi field reservoir $R$ with a state space $\mathcal{H}_R = \Gamma_- (\mathfrak{h}_R^1)$. The interaction between the system and reservoir will be given by the formal Hamiltonian

$$H^{(\lambda)} = H_S \otimes I_R + I_S \otimes H_R + \hbar H_{\text{int}}^{(\lambda)}.$$ 

The interaction Hamiltonian $H_{\text{int}}^{(\lambda)}$ is described by

$$H_{\text{int}}^{(\lambda)} = E_{11} \otimes A^+(g)A^-(g) + \lambda E_{10} \otimes A^+(g) + \lambda E_{01} \otimes A^-(g) + \lambda^2 E_{00} \otimes I_R,$$

where $E_{\alpha\beta}$ are bounded operators on $\mathcal{H}_S$ and $E_{\alpha\beta}^* = E_{\beta\alpha}$, $A^\pm$ are the Fermi creation/annihilation operators which lead to a representation of the CAR algebra over the Hilbert space $\mathfrak{h}_R^1$. We shall assume the following harmonic relations

$$e^{-\tau H_S/\hbar} E_{\alpha\beta} e^{\tau H_S/\hbar} = e^{i\omega_{\beta-\alpha}} E_{\alpha\beta},$$
$$e^{-\tau H_S/\hbar} A^\pm (g) e^{\tau H_S/\hbar} = A^\pm (S_\tau g).$$

The family $S_\tau$ will be a one-parameter group of unitaries on $\mathfrak{h}_R^1$ (operator $H_R$ is a differential second quantization of the Stone generator $H_R^\tau$ of this family). We remove the free dynamics by introducing the operator

$$U(\tau, \lambda) = e^{-\tau (H_S \otimes I_R + I_S \otimes H_R)/\hbar} e^{\tau H^{(\lambda)}/\iota}.$$ 

Operator $U$ transforms to the interaction picture and satisfies the equation

$$\frac{\partial}{\partial t} U(t/\lambda^2, \lambda) = -i \Upsilon_t (\lambda) U(t/\lambda^2, \lambda)$$

(2.3)

where $\Upsilon_t (\lambda) = E_{\alpha\beta} \otimes [a^+_t (\lambda)]^\alpha [a^-_t (\lambda)]^\beta$ and $a^+_t (\lambda) = 1/\lambda A^\pm (S^\omega_{t/\lambda^2} g)$, $S^\omega_t = e^{-i\tau \omega} S_\tau$. (We employ the summation convention that when the Greek indices $\alpha, \beta, \ldots$ are repeated then we sum each index over the values 0 and 1 - moreover we understand the index $\alpha$ in $[,]^\alpha$ to represent a power.) The operators $a^+_t (\lambda)$ have the following correlation in the Fock vacuum state for the reservoir:

$$\langle a^-_t (\lambda) a^+_t (\lambda) \rangle_R = G_\lambda (t-s) := \frac{1}{\lambda^2} \langle g \mid S^\omega_{(t-s)/\lambda^2} g \rangle.$$ 

(2.4)

3 The Wick ordering

Our main tool will be the Wick ordering of certain operator products. To simplify the calculation we need a special technique which we introduce in this section. Let $\mathfrak{h}$ be an arbitrary Hilbert space.

For any $\bar{f} = (f_1, \ldots, f_n), \bar{g} = (g_1, \ldots, g_n) \in \mathfrak{h}^n, \alpha\beta = \{0,1\}^n$, we consider the product

$$F_n \left( \bar{f}, \bar{g} ; \alpha\beta \right) = [A^+(f_n)]^{\alpha_n} [A^-(g_n)]^{\beta_n} \ldots [A^+(f_1)]^{\alpha_1} [A^-(g_1)]^{\beta_1}.$$ (3.1)
where \( A^\pm \) are the Fermi creation/annihilation operators satisfying the CAR over the Hilbert space \( \mathcal{H} \). We have the relations
\[
[A^-(g_i)]^{\beta_i}[A^+(f_k)]^{\alpha_k} = (-1)^{\beta_i\alpha_k}[A^+(f_k)]^{\alpha_k}[A^-(g_i)]^{\beta_i} + \beta_i\alpha_k (g_i|f_k) . \tag{3.2}
\]

**Definition 3.1:** Let \( S_n = \{1, \ldots, n\} \); we denote by \( \mathfrak{S}_n \) the class of all partitions (that is, mutually exclusive, collectively exhaustive collections of subsets) of \( S_n \) and let \( \mathfrak{S} = \bigcup_{n=1}^\infty \mathfrak{S}_n \). For \( \mathcal{G} = \{G_1, \ldots, G_m\} \in \mathfrak{S}_n \) we call each \( G_k \subset S_n \) a part of the partition \( \mathcal{G} \) and then set \( \mathcal{G}_{\text{out}} := \{\max G_k : k = 1, \ldots, m\} \) and \( \mathcal{G}_{\text{in}} := \{\min G_k : k = 1, \ldots, m\} \). For \( \alpha, \beta \in \{0, 1\}^n \), we define the sign function \( \xi \) on \( \mathfrak{S}_n \) by
\[
\xi(\mathcal{G}, \alpha, \beta) := \sum'_{j>i} \beta_j\alpha_i \tag{3.3}
\]
where the sum is over all pairs \((j, i)\) such that \( j > i \) and any of the following conditions hold:

1. there exists a \( i' \) in the same part as \( i \) and a \( j' \) in the same part as \( j \) with \( i' > j > i > j' \);
2. \( i \in \mathcal{G}_{\text{out}} \) and there exists a \( j' \) in the same part as \( j \) with \( j > i > j' \);
3. \( j \in \mathcal{G}_{\text{in}} \) and there exists a \( i' \) in the same part as \( i \) with \( i' > j > i \);
4. \( i \in \mathcal{G}_{\text{out}} \) and \( j \in \mathcal{G}_{\text{in}} \).

Now we are going to introduce the Feynman type diagrams related to product (3.1); they will facilitate the explanation of the Wick ordering of product (3.1).

We arrange the elements of \( S_n \) as vertices along the line in descending order (figure 1 gives a typical example). At each vertex \( i \in S_n \) we have two lines: one representing a annihilation of \( \beta_i = 0, 1 \) reservoir quanta coming in from the right and one representing a creation of \( \alpha_i = 0, 1 \) reservoir quanta going out to the left. We choose pairs \((i_1, j_1), \ldots, (i_k, j_k)\) which satisfy the conditions \( i_h < j_l \), \( i_h \neq i_l \), \( j_l \neq j_l \), where \( h, l = 1, \ldots, k \) and \( h \neq l \). Clearly, the number of pairs cannot exceed \( n-1 \), but may be zero as well. For each pair \((i_h, j_h)\) we join together (contract) the \( \beta_{i_h} \) and \( \alpha_{j_h} \) lines and get the so called internal line with label \( \alpha_{j_h}\beta_{i_h} \).

Each such diagram leads to an equivalence class on \( S_n \) as follows: we say that \( x \equiv y \) if \( x = y \) or if there exists a sequence of pairs connecting \( x \) to \( y \). The equivalence classes give is a partition of \( S_n \) and in this way every such diagram is uniquely associated with a partition in \( \mathfrak{S}_n \). Indeed, each partition \( \mathcal{G} \in \mathfrak{S}_n \) gives a set of pairs \( \{i_r, i_{r-1}, \ldots, i_1\} \); the inverse correspondence is obvious. Note that \( \mathcal{G}_{\text{in}} \) is the set \( \bigcup_{i_r, i_{r-1}, \ldots, i_1} \{i_1\} \) and \( \mathcal{G}_{\text{out}} \) is the set \( \bigcup_{i_r, i_{r-1}, \ldots, i_1} \{i_s\} \). The elements of set \( \mathcal{G}_{\text{in}}, \mathcal{G}_{\text{out}} \) we shall call the vertices with external incoming, outgoing lines respectively.

In the definition of the sign function \( \xi \) we see that the sum is taken over all line intersections, that is, the product \( \beta_j\alpha_i \) includes in summation if \( i < j \) and the \( \beta_j \) line and \( \alpha_i \) line intersect each other.
The diagram in figure 1 corresponds to the partition $\mathcal{G} = \{1, 4\}; \{2, 3, 5\}$; here function $\xi$ will be given by $\xi(G, \alpha, \beta) = \beta_2 \alpha_1 + \beta_4 \alpha_3 + \beta_5 \alpha_4$; $\mathcal{G}_{out} = \{4, 5\}$, $\mathcal{G}_{in} = \{1, 2\}$.

**Lemma 3.2:** For any $\mathbf{f} = (f_1, \ldots, f_n), \mathbf{g} = (g_1, \ldots, g_n) \in \mathfrak{h}^\times n$, and $\alpha, \beta \in \{0, 1\}^n$, we have that the normal ordered form of $F_n(\mathbf{f}, \mathbf{g}, \alpha, \beta)$ given in (3.1) is

$$\sum_{\mathcal{G} \in \mathfrak{S}_n} (-1)^{\xi(G, \alpha, \beta)} \prod_{\{i(r) > \ldots > i(1)\} \in \mathcal{G}} \left\{ \prod_{h=1}^{n-1} \beta_{i(h+1)} \alpha_{i(h)} \langle g_{i(h+1)} | f_{i(h)} \rangle \right\} \times \prod_{i \in \mathcal{G}_{out}} [A^+(f_i)]^{\alpha_i} \prod_{j \in \mathcal{G}_{in}} [A^-(g_j)]^{\beta_j}. \quad (3.4)$$

**Proof.** The proof is by induction on $n$. For $n = 1$ there is nothing to prove so we assume that $n > 1$. We have that

$$F_{n+1} = [A^+(f_{n+1})]^{\alpha_{n+1}} [A^-(g_{n+1})]^{\beta_{n+1}} F_n$$

Using the induction hypothesis, we write $F_n$ as shown in (3.4). Let us concentrate on the contribution from a fixed partition $\mathcal{G} \in \mathfrak{S}_n$. We see that the annihilator $[A^- (g_{n+1})]^{\beta_{n+1}}$ will be out of normal order and we use the relation (3.2) repeatedly to put it to normal order. We first of all anti-commute it with the nearest creator, which will be $[A^+(f_i)]^{\alpha_i}$ where $i = \max \mathcal{G}_{out}$ and study the additional term involving the factor $\beta_{n+1} \alpha_i (f_{n+1}) f_i$. This is exactly the contribution to $F_{n+1}$ that should come from the partition $\mathcal{G}' \in \mathfrak{S}_{n+1}$ obtained by inserting the new vertex element $n+1$ into the part of $\mathcal{G}$ containing $i$. Note that the sign function is unchanged and this is consistent with the fact that no new intersections arise.

Proceeding to normal order in this manner we obtain all the new partitions arising in this way with the correct multiplicative factor $(-1)^{\beta_{n+1} \alpha_i}$, appearing at each stage. Eventually we are left with then $n+1$ creators and annihilators put to normal order and this is the contribution from the partition $\mathcal{G} \cup \{n+1\} \in \mathfrak{S}_{n+1}$: again the sign is consistent with the $n+1$ st. form of (3.4). If we sum over all partitions $\mathcal{G} \in \mathfrak{S}_n$ we see that we have the appropriate contributions from all partitions in $\mathfrak{S}_{n+1}$ as required. ■

**Definition 3.3:** Let $\mathcal{P}(S_m, S_n)$ denote the collection of all $P \subset S_m \times S_n$ such that if $(i_1, j_1), (i_2, j_2) \in P$ then $i_1 \neq i_2 \iff j_1 \neq j_2$. For any $P \in \mathcal{P}(S_m, S_n)$, we denote by $P|_{S_m}$ the set $\{i \in S_m : \exists j \in S_n, \text{ such that } (i, j) \in P\}$ and by $P|_{S_n}$ denote a set $\{j \in S_n : \exists i \in S_m, \text{ such that } (i, j) \in P\}$: hopefully no confusion should arise. We also set $P|_{S_m} := S_m \setminus P|_{S_m}$ and $P|_{S_n} := S_n \setminus P|_{S_n}$.

**Lemma 3.4:** We have the following normal ordering

$$\prod_{j \in S_n} [A^- (g_j)]^{\beta_j} \prod_{i \in S_m} [A^+(f_i)]^{\alpha_i} = \sum_{P \in \mathcal{P}(S_m, S_n)} (-1)^{\xi(P, \alpha, \beta)} \prod_{(i,j) \in P} \langle \beta_j g_j | \alpha_i f_i \rangle \prod_{i \in P|_{S_m}} [A^+(f_i)]^{\alpha_i} \prod_{j \in P|_{S_n}} [A^-(g_j)]^{\beta_j}. \quad (3.5)$$
This is an immediate corollary to lemma 3.2. We set \( \hat{f} = (f_1, \ldots, f_m, 0, \ldots, 0) \), \( \hat{\alpha} = (0, \ldots, 0, g_1, \ldots, g_n) \in \mathbb{R}^\times(n+m) \) and \( \hat{\beta} = (0, \ldots, 0, \beta_1, \ldots, \beta_n) \in \{0, 1\}^\times(n+m) \) then applying the general formula (3.1) to product (3.3) we can see that only those partitions of \( S_{n+m} \) consisting of singletons and pairs as parts give a non-zero contribution to the normal order formula, the creator element of pair has to belong to the first \( m \) indexes and the annihilator one has to belong to the last \( n \) indexes. Clearly, the set of such partitions has one-to-one relation with the set \( \mathcal{P}(S_m, S_n) \). We have used the notation \( \xi \) for the map whose exact definition is \( \xi(P, \alpha, \beta) := \xi(G_P, \hat{\alpha}, \hat{\beta}) \), where \( \hat{\alpha} \) and \( \hat{\beta} \) were constructed as above and \( G_P \) is a partition of \( S_{n+m} \) corresponded to the set of pairs \( P \).

4 The Dyson series expansion

By \( \mathfrak{f} \) denote the maximal subspace of \( h^1_P \) which satisfies the following condition:
\[
\int_\mathbb{R} |\langle f_1 | S_u^\omega f_2 \rangle| du < \infty \text{ whenever } f_1, f_2 \in \mathfrak{f}.
\]
The sesquilinear form on \( \mathfrak{f} \) is defined by \( \langle f_1 | f_2 \rangle := \int_\mathbb{R} \langle f_1 | S_u^\omega f_2 \rangle du \). We also denote by \( \mathfrak{f} \) the associated Hilbert space, i.e. the completion of the quotient of \( \mathfrak{f} \) by zero \((\cdot, \cdot)\)-norm elements. We shall suppose below that the test vector \( g \) appearing in the interaction Hamiltonian \( \mathcal{H} \) belongs to \( \mathfrak{f} \). Define
\[
\kappa = (g|g)_+ := \int_0^\infty G_1(u) du, \quad K := \int_0^\infty |G_1(u)| du
\]
then we have the Markovian limit for the two-point functions
\[
\lim_{\lambda \to 0} G_1(t - s) = \kappa \vartheta_+(t - s) + \kappa^* \vartheta_-(t - s), \tag{4.1}
\]
where \( \vartheta_\pm \) are generalized functions defined by (see [3])
\[
\langle \vartheta_\pm, f \rangle := \int \vartheta_\pm(t - s) f(s) ds = f(t^\pm). \tag{4.2}
\]

For any \( f \in \mathfrak{f} \) and \( S, T, \lambda \in \mathbb{R} \) define a collective vector \( \hat{f} \) by the rule
\[
\hat{f}(\lambda) := \frac{1}{\lambda} \int_S^T du S_u^\omega f. \tag{4.3}
\]

Let us fix vectors \( \varphi_1, \varphi_2 \in \mathcal{H}_S \), integers \( k_+, k_- \in \mathbb{N}_0 \), vectors \( f_1^+, \ldots, f_m^+, f_1^-, \ldots, f_m^- \in \mathfrak{f} \) and numbers \( T_1^+, S_1^+, \ldots, T_m^+, S_m^+, T_1^-, S_1^-, \ldots, T_m^-, S_m^- \in \mathbb{R} \) such that \( T_i^+ > S_i^+ \) and \( T_i^- > S_i^- \) for all \( i \). To each quadruple \((f_i^+, S_i^+, T_i^+, \lambda)\) we put into the correspondence the collective vector \( \hat{f}_i^+(\lambda) \) defined by (4.3). Denote
\[
h_i^+(t, \lambda) := \frac{1}{\lambda} \langle \hat{f}_i^+(\lambda) | S_t^\omega f_i^+ \rangle = \frac{1}{\lambda^2} \int_{S_t^\omega f_i^+}^T du \langle S_u^\omega f_i^+ | S_t^\omega f_i^+ \rangle. \tag{4.4}
\]
We shall be interested in the behavior of inner product

$$\langle \phi_1 \otimes \prod_{i \in S_{k_-}} A^+(f_i^-(\lambda)) \Phi | U(t/\lambda^2, \lambda) \phi_2 \otimes \prod_{i \in S_{k_+}} A^+(f_i^+(\lambda)) \Phi \rangle$$

(4.5)

in the limit $\lambda \to 0$ (the so called $\delta-$correlated noise limit). Here and below we use sign $\prod'$ to denote the product in inverse order. The operator $U(t/\lambda^2, \lambda)$ here, defined in (2.2), can be developed as a formal Dyson series:

$$U(t/\lambda^2, \lambda) = \sum_{n=0}^{\infty} (-i)^n D_n(t, \lambda)$$

where we have the multi-time integral

$$D_n(t, \lambda) = \int_{\Delta_n(t)} ds_n \ldots ds_1 \ U_{s_1}(\lambda) \ldots U_{s_n}(\lambda)$$

over the simplex $\Delta_n(t) := \{ (s_n, \ldots, s_1) : t > s_n > \ldots > s_1 > 0 \}$.

**Lemma 4.1:** For any $k_+, k_- \in \mathbb{N}$, $f_1^-, f_{k_-}, f_1^+, f_{k_+} \in \mathfrak{t}$, $\{ S_{k_-}^i, T_{k_+}^i \}_{i=1}^{k_+}$, $\{ S_{k_-}^i, T_{k_+}^i \}_{i=1}^{k_-} \subset \mathbb{R}$, $t \geq 0$, we have

$$\langle \phi_1 \otimes \prod_{i \in S_{k_-}} A^+(f_i^-(\lambda)) \Phi | D_n(t, \lambda) \phi_2 \otimes \prod_{i \in S_{k_+}} A^+(f_i^+(\lambda)) \Phi \rangle$$

$$= \sum_{\alpha, \beta \in \{0,1\}^n} \langle \phi_1 \mid E_{\alpha_0, \beta_n} \ldots E_{\alpha_1, \beta_1} \phi_2 \rangle \sum_{\mathcal{G} \in \mathcal{G}_n} \sum_{P_1 \in \mathcal{P}(S_{k_-}, \mathcal{G}_{out})} \sum_{P_2 \in \mathcal{P}(S_{k_+}, \mathcal{G}_{in})}$$

$$(-1)^{\xi(\mathcal{G}, \alpha, \beta) + \xi(P_1, \alpha_1) + \xi(P_2, 1, \beta)} \left( \sum_{i \in P_1 \setminus \mathcal{G}_{out}} \alpha_i + \sum_{i \in P_2 \setminus \mathcal{G}_{in}} \beta_i \right)$$

$$\times \langle \prod_{i \in P_1 \setminus \mathcal{G}_{out}} A^+(f_i^-(\lambda)) \Phi | \prod_{i \in P_2 \setminus \mathcal{G}_{in}} A^+(f_i^+(\lambda)) \Phi \rangle$$

$$\times \int_{\Delta_n(t)} ds_n \ldots ds_1 \ \prod_{\{i(r) > \cdots > i(1)\} \in \mathcal{G}} \left( \prod_{h=1}^{r-1} \beta_{i(h+1)} \alpha_{i(h)} G_\lambda(s_{i(h+1)} - s_{i(h)}) \right)$$

$$\times \prod_{(i_1, j_1) \in P_1} \alpha_{j_1} h_{i_1}^-(s_{j_1}, \lambda) \prod_{(i_2, j_2) \in P_2} (\beta_{j_2} h_{i_2}^+(s_{j_2}, \lambda))^*$$

(4.6)

**Proof.** Note that

$$\Upsilon_{s_n}(\lambda) \ldots \Upsilon_{s_1}(\lambda) = \sum_{\alpha, \beta \in \{0,1\}^n} E_{\alpha_0, \beta_n} \ldots E_{\alpha_1, \beta_1} [a_{\alpha_0}^+(\lambda)]^{\alpha_n} [a_{\alpha_1}^-(\lambda)]^{\beta_n} \ldots [a_{\alpha_1}^+(\lambda)]^{\alpha_1} [a_{\alpha_1}^-(\lambda)]^{\beta_1}.$$  

(4.7)

The proof consists of a direct application of lemma 3.2 to (4.4) and then we apply twice the formula (4.5). Note that we do not get a Wick ordering; each
additional term, arising from an application of the CAR, will include a product of operators of the form

$$\prod_{i \in P_2 | G_{in}} [a_{s_i}^+(\lambda)]^{\alpha_i} \prod_{i \in P_2 | G_{in}} A^- (\bar{f}_i^- (\lambda)) \prod_{i \in P_2 | G_{in}} A^+ (\bar{f}_i^+ (\lambda)) \prod_{i \in P_2 | G_{in}} [a_{s_i}^- (\lambda)]^{\beta_i}. $$

Taking the vacuum expectation we get zero whenever any $\alpha$ or $\beta$ from the correspondent sets equals to unity. So using the Kronecker delta $\delta_0 (.)$ we can write the vacuum expectation in the following form

$$\delta_0 \left( \sum_{i \in P_1 | G_{out}} \alpha_i + \sum_{i \in P_1 | G_{in}} \beta_i \right) \left( \prod_{i \in P_1 | G_{in}} A^+ (\bar{f}_i^- (\lambda)) \Phi \prod_{i \in P_2 | G_{in}} A^+ (\bar{f}_i^+ (\lambda)) \Phi \right).$$

\[\blacksquare\]

**Definition 4.2:** The partition $\mathcal{G} \in \mathfrak{G}_n$ is called the type $I$ partition if each subset $\{i(r) > \ldots > i(1)\} \in \mathcal{G}$ satisfies the condition $i(2) - i(1) = i(3) - i(2) = \ldots = i(r) - i(r - 1) = 1$. We shall denote by $\mathfrak{G}_{I,n}$ a set of all type $I$ partitions. The rest of partitions forms a set $\mathfrak{G}_{II,n}$, they are called the type $II$ partitions. Set also $\mathfrak{G}_I := \cup_n \mathfrak{G}_{I,n}$. Given $\mathcal{G} \in \mathfrak{G}$ we denote the number of vertices partitioned by $\mathcal{G}$ as $E(\mathcal{G})$ and the number of parts making up $\mathcal{G}$ as $N(\mathcal{G})$.

Note that type $I$ partitions correspond to diagrams where the contractions are between pairs of consecutive vertices only.

**Lemma 4.3:** For any $\mathcal{G} \in \mathfrak{G}_{II,n}$, $k, k_+ \in \mathbb{N}_0$, $f_1^-, \ldots, f_{k_+}^+$, $f_1^+, \ldots, f_{k}^-, \in \mathfrak{m}$, $\{S_i, T_i\}_{i=1}^k$, $\{S_i^+, T_i^+\}_{i=1}^{k_+} \in \mathbb{R}$, $t \geq 0$, we have

$$\lim_{\lambda \to 0} \int_{\Delta_n(t)} ds \ldots ds_1 \prod_{\{i(r) > \ldots > i(1)\} \in \mathcal{G}} \left\{ \prod_{h=1}^{r-1} \beta_{i(h+1)} \alpha_{i(h)} G_{\lambda}(s_{i(h+1)} - s_{i(h)}) \right\} \times |\Phi| \prod_{j \in S_{k_-}} A^- (\bar{f}_j^- (\lambda)) \prod_{j \in G_{out}} [a_{s_j}^-(\lambda)]^{\alpha_j} \prod_{j \in G_{in}} [a_{s_j}^-(\lambda)]^{\beta_j} \prod_{i \in S_{k_+}} A^+ (\bar{f}_i^+(\lambda)) \Phi| = 0. $$

(4.8)

**Proof.** The inner product in (4.8) is bounded for all $\lambda > 0$. The proof of the identity

$$\lim_{\lambda \to 0} \int_{\Delta_n(t)} ds \ldots ds_1 \prod_{\{i(r) > \ldots > i(1)\} \in \mathcal{G}} \left\{ \prod_{h=1}^{r-1} G_{\lambda}(s_{i(h+1)} - s_{i(h)}) \right\} = 0$$

for $\mathcal{G} \in \mathfrak{G}_{II,n}$ repeats the proof of \[\blacksquare\] Lemma 6.1.  

**Lemma 4.4:** For any $\tilde{f}_i^+(\lambda)$ and $\tilde{f}_j^-(\lambda)$ we have

$$\lim_{\lambda \to 0} \langle \tilde{f}_j^- (\lambda) | \tilde{f}_i^+ (\lambda) \rangle = \langle \chi_{[S_j^-, T_j^-]} , \chi_{[S_i^+, T_i^+]} | f_j^- | f_i^+ \rangle. $$

(4.9)
2. The functions $h_i^\pm(t, \lambda)$ defined in (4.1) will have the limits
\[
\lim_{\lambda \to 0} h_i^\pm(t, \lambda) = h_i^\pm(t) := 1_{[s_i^+, T_i^+]}(t)(f_i^\pm | g).
\] (4.10)

For the proof of first statement see [3] Lemma 3.2. The second statement can be proved similarly.

5 Uniform convergence of Dyson series

With each partition $G \in \mathfrak{G}_n$ we associate a sequence of occupation numbers $n = (n_j)_{j=1}^\infty$ where $n_j = 0, 1, 2, \ldots$ counts the number of $j$-tuples making up $G$ (see [3]). We put, by definition
\[
E(n) := \sum_j j n_j, \quad N(n) := \sum_j n_j.
\] (5.1)

Denote by $G_0(n)$ a partition where we have all 1-tuples in the beginning, then followed by all 2-tuples, etc. A permutation $\rho$ of set $S_{E(n)}$ is called admissible if it maps the partition $G_0(n)$ into another partition $\rho(G_0(n))$. We shall denote by $\mathfrak{S}_n^0$ the collection of all admissible permutations $\rho$.

We have the following inequalities:
\[
|h_i^+(t, \lambda)| \leq \int_{\mathbb{R}} |S_u^N f_i^\pm | g\rangle du \text{ for all } \lambda > 0, \ t \geq 0;
\]
\[
|(f_i^- | \tilde{f}_i^+)(\lambda)| \leq (T_i^- - S_i^-) \int_{\mathbb{R}} |(f_i^- | S_u^N f_i^+)| du \text{ for all } \lambda > 0.
\]

Denote $C_h := \max\{1, \max_{i} \int_{\mathbb{R}} |(S_u^N f_i^\pm | g\rangle du\}$, $C_f := \max\{1, \max_{i,j} (T_i^- - S_i^-) \int_{\mathbb{R}} |(f_i^- | S_u^N f_j^+)| du\}$, $C_{11} := ||E_{11}||$ and $C := \max\{|E_{11}|, |E_{10}|, |E_{01}|, |E_{00}|\}$.

Denote also
\[
I(n, G) := \int_{\Delta_n(t)} ds \ldots ds_1 \prod_{i(1)<\ldots<i(r)} \left\{ \prod_{h=1}^{r-1} G_{\lambda}(s_{i(h+1)} - s_{i(h)}) \right\}
\] (5.2)

**Theorem 5.1**: Suppose $K||E_{11}| < 1$; then the series
\[
\sum_n (-i)^n \left( \prod_{i \in S_{k-}} A^+(\tilde{f}_i^-)(\lambda) \Phi \right) D_n(t, \lambda) \prod_{i \in S_{k+}} A^+(f_i^+)(\lambda) \Phi
\] (5.3)

converges uniformly and absolutely in the pair $(\lambda, t) \in \mathbb{R}_+ \times [0, T]$ for any $T < \infty$.

**Proof.** We have to estimate the absolute value of right side of equality (4.6).

First, for given $\alpha$, $\beta$ and $G$ we need to estimate the maximal number of sets $P_i \in \mathcal{P}(S_{k-}, G_{\text{out}})$, which give non-zero contribution (we call them non-trivial sets). Denote $|G_{\text{out}}| := \sum_{i \in G_{\text{out}}} \alpha_i$. Clearly, if we have $|G_{\text{out}}| > k_-$ then all $P_i \in \mathcal{P}(S_{k-}, G_{\text{out}})$ gives a zero contribution (it is provided by function $\delta_0$). So
we can consider only the case \(|G_{out}| \leq k_-.\) We can also deduce that a non-trivial set \(P_1\) cannot include a pair \((i, j)\) such that \(\alpha_j = 0\). It means that the number of non-trivial sets \(P_1 \in \mathcal{P}(S_{k_-}, G_{out})\) cannot exceed the cardinality of \(\mathcal{P}(S_{k_-}, S_{k_-})\), which is equal to \(\sum_{i=0}^{k_-} \frac{(k_-)!}{i!(k_- - 2i)!} 2^i \leq (k_-)!2^{k_-}\). Similarly, we have the upper estimate \((k_+)!2^{k_+}\) for the maximal number of non-trivial sets \(P_2 \in \mathcal{P}(S_{k_+}, G_{in})\).

We have the following estimates:

\[
|\langle \prod_{i \in P_1 \setminus S_{k_-}} A^+(\tilde{f}_i^- (\lambda)) \Phi | \prod_{i \in P_2 \setminus S_{k_+}} A^+(\tilde{f}_i^+ (\lambda)) \Phi \rangle| \leq (k_+ \land k_-)! \langle C_f \rangle^{k_+ \land k_-},
\]

\[
|\prod_{(i_1, j_1) \in P_1} \alpha_{j_1} R_{i_1} (s_{j_1}, \lambda) \prod_{(i_2, j_2) \in P_2} (\beta_{j_2} R_{i_2} (s_{j_2}, \lambda))^\dagger| \leq (C_h)^{k_-} - (C_h)^{k_+}.
\]

Hence, we obtain

\[
|\langle \prod_{i \in S_{k_-}} A^+(\tilde{f}_i^- (\lambda)) \Phi | D_n (t, \lambda) \prod_{i \in S_{k_+}} A^+(\tilde{f}_i^+ (\lambda)) \Phi \rangle| \leq (k_+ \land k_-)! \langle C_f \rangle^{k_+ \land k_-}
\]

\[
\times (k_-)! \langle k_+ \rangle! (2C_h)^{k_- + k_+} \sum_{\mathcal{G} \in \mathfrak{S}_n} \sum_{\alpha, \beta \in \{0, 1\}^n} \langle \varphi_1 | E_{n_{\alpha\beta}} \ldots E_{n_{\alpha\beta}} \varphi_2 \rangle I(n, \mathcal{G})
\]

\[
= (k_+ \land k_-)! \langle C_f \rangle^{k_+ \land k_-} (k_-)! \langle k_+ \rangle! (2C_h)^{k_- + k_+}
\]

\[
\times \sum_{E(n) = n} \sum_{\rho \in \mathfrak{S}_n \rho(n)} I(n, \rho(\mathcal{G}(n))) \sum_{\alpha, \beta \in \{0, 1\}^n} \langle \varphi_1 | E_{n_{\alpha\beta}} \ldots E_{n_{\alpha\beta}} \varphi_2 \rangle.
\]

Here \(\sum'\) means that we sum up only over those \(\alpha\) and \(\beta\) which give a non-zero contribution for given \(\mathcal{G} \in \mathfrak{S}_n\). Indeed, for \(\mathcal{G} = \mathcal{G}(n)\) the number of fixed variables \(\alpha\) and \(\beta\) is equal to \(2(E(n) - N(n))\) (they equal to unity). Note that the estimate of \(|\varphi_1 | E_{n_{\alpha\beta}} \ldots E_{n_{\alpha\beta}} \varphi_2 |\) depends only on \(n\). We can vary only \(N(n)\) values of \(\alpha\) and \(N(n)\) values of \(\beta\). When we have all these unfixed variables equal to zero we obtain the upper bound \(c_{11}^{-2(n) + n_1} C_{2N(n) - n_1}\). When we set any variable equal to unity it means that we replace 1) an \(E_{00}\) operator by \(E_{01}\) or \(E_{10}\), or 2) an \(E_{10}\) or \(E_{01}\) operator by \(E_{11}\). In the first case the estimate stays the same; in the second case we get the estimate which is a product of the previous one and a factor \(C_{11} C^{-1}\). Note that this factor does not exceed the unity so the previous estimate is valid also. Thus, for given \(n\), we have the estimate

\[
|\varphi_1 | E_{n_{\alpha\beta}} \ldots E_{n_{\alpha\beta}} \varphi_2 | \leq c_{11}^{E(n) - 2N(n) + n_1} C_{2N(n) - n_1} \leq c_{11}^{E(n) - 2N(n)} C_{2N(n)},
\]

independent of the variables \(\alpha\) and \(\beta\). Clearly, we have \(2^{2N(n)}\) possible values of \(\alpha\) and \(\beta\), so for given \(n\) the total estimate of \(\sum'\) is

\[
\sum_{\alpha, \beta \in \{0, 1\}^n} |\varphi_1 | E_{n_{\alpha\beta}} \ldots E_{n_{\alpha\beta}} \varphi_2 | \leq c_{11}^{E(n) - 2N(n)} (2C)^{2N(n)}.
\]
We have obtained the following estimate for $n$-th term of series (5.3) and (6.1):

$$\lim_{\lambda \to 0} \left| \frac{\prod_{i \in S_{k_-}} A^+(f_i^-)}{D_n(t, \lambda)} \prod_{i \in S_{k_+}} A^+(f_i^+)(\lambda)\Phi \right| \leq (k_+ \wedge k_-)! (C_f)^{k_+ + k_-} \times (k_-)! (2C_f)^{k_+ - k_-} \sum_{n \in E(n) = n} C_n^{E(n) - 2N(n)} (2C)^{2N(n)} \sum_{\rho \in \vartheta(n)} I(n, \rho(\mathcal{G}(n))).$$

(5.4)

The uniform convergence in the pair $(\lambda, t) \in \mathbb{R}^+ \times [0, T]$ for any $T < \infty$ of series with $n$-th term given by right side of inequality (5.3) under the condition $KC_{11} < 1$ was proved in [3, Section 7]. It proves the absolute and uniform convergence of series (6.1) under the same condition.

6 Limit QSDE

Let $A^+ (\cdot)$ and $\Lambda (\cdot)$ be the Fermionic creation/annihilation and differential second quantization fields with test functions in $\mathfrak{t} \otimes L^2 (\mathbb{R}^+)$. Let $\Psi$ be a vacuum vector in $\Gamma_- (\mathfrak{t} \otimes L^2 (\mathbb{R}^+))$.

**Theorem 6.1:** Suppose $K \|[E_{11}]| < 1$, then for any $\varphi_1, \varphi_2, \in \mathcal{F}_S, k_+, k_- \in \mathbb{N}_0, f_{i_1}, \ldots, f_{i_k}, f_{i_k}^+, f_{i_k}^- \in \mathfrak{t}, \{S_i, T_i\}_{i=1}^{k_+}, \{S_i^+, T_i^+\}_{i=1}^{k_-} \subset \mathbb{R}, t \geq 0$, we have

$$\lim_{\lambda \to 0} \left( \phi_1 \otimes \prod_{i \in S_{k_-}} A^+ (f_i^-)(\lambda) \Phi \mid U(t/\lambda^2, \lambda) \phi_2 \otimes \prod_{i \in S_{k_+}} A^+ (f_i^+)(\lambda) \Phi \right)$$

$$= \sum_{\varrho \in \vartheta} \sum_{\alpha, \beta \in \{0, 1\}^E(\varrho)} \langle \varphi_1 | E_{\alpha}(\varrho) E_{\beta}(\varrho) \cdots E_{\alpha_1}(\varrho) \varphi_2 \rangle \sum_{P_1 \in \mathcal{P}(S_{k_-}, \varrho_{out})} \sum_{P_2 \in \mathcal{P}(S_{k_+}, \varrho_{in})}$$

$$\times (-i)^{|E(\varrho)|} (-1)^{|\xi(\Lambda, \alpha, \beta) + \xi(P_1, \alpha_1) + \xi(P_2, \beta)|} \sum_{\alpha_1 \in P_1, \beta_i \in P_2} \alpha_1 + \sum_{\beta_i} \beta_i$$

$$\times \left\{ \prod_{i \in P_1 \setminus S_{k_-}} A^+ (f_i^- \otimes 1_{[S_i^- \cdot T_i^-]}) \Psi \prod_{i \in P_2 \setminus S_{k_+}} A^+ (f_i^+ \otimes 1_{[S_i^+ \cdot T_i^+]}) \Psi \right\}$$

$$\times \int_{\Delta_n(t)} ds_{E(\varrho)} \cdots ds_{S_{i_1}} \prod_{(i_1, j_1) \in P_1} \alpha_{j_1} h^{(j_1)}_{i_1} (s_{j_1}) \prod_{(i_2, j_2) \in P_2} (\beta_{j_2} h^{(j_2)}_{i_2} (s_{j_2}))^*$$

$$\times \prod_{\{i(r) > \cdots > i(3)\} \in \varrho} \kappa^{-1} \left\{ \prod_{h=1}^{r-1} \beta_{i(h+1)} \alpha_{i(h)} \Phi + (s_{i(h+1)} - s_{i(h)}) \right\}.$$
Proof. By theorem 5.1 under the condition $K||E_{11}|| < 1$ the series

$$
\sum_{n=1}^{\infty} (-i)^n \langle \phi_1 \otimes \prod_{i \in I_{k-}} A^+(\tilde{f}^-_i (\lambda)) \Phi | D_n(t, \lambda) \phi_2 \otimes \prod_{i \in I_{k+}} A^+(\tilde{f}^+_i (\lambda)) \Phi \rangle
$$

converges uniformly and absolutely in the pair $(\lambda, t) \in \mathbb{R}_+ \times [0, T]$ for any $T < \infty$. This means that we can pass to limit under the summation and change the order of summation.

Let us calculate the limit of right part of equation (4.6) as $\lambda \to 0$. It follows from lemma 4.3 that only type $I$ terms will survive under this limit. The equality

$$
\lim_{\lambda \to 0} \langle \phi_1 \otimes \prod_{i \in I_{k-}} A^+(\tilde{f}^-_i (\lambda)) \Phi | U(t/\lambda^2, \lambda) \phi_2 \otimes \prod_{i \in I_{k+}} A^+(\tilde{f}^+_i (\lambda)) \Phi \rangle
$$

is the immediate consequence of (4.3). The equality $\lim_{\lambda \to 0} h^\pm_i(t, \lambda) = h^\pm_i(t)$ is due to definition (1.10). The limit of $G_\lambda$ is given by (1.11), but, since we integrate over the simplex, only the future part $\kappa \delta_+ (u)$ survives. At last, we change the order of summation to $\sum_{\bar{\gamma} \in \mathcal{S}_I, n} \sum_{\alpha, \beta \in \{0, 1\}^n}$ and replace $\sum_{n=1}^{\infty} \sum_{\bar{\gamma} \in \mathcal{S}_I, n}$ by $\sum_{\bar{\gamma} \in \mathcal{S}_I}$. 

Our next goal is to prove the convergence in the sense of matrix elements of $U(t/\lambda^2, \lambda)$ to the solution $U_t$ of some QSDE. It is hard work to obtain the explicit form of this equation from (6.1). Fortunately, we already have this equation for Bosonic case [3]. The Bosonic version of (6.1) which differs only in the absence of the Fermionic signs. We can readily predict that QSDE will take the same form again in the Fermionic case; we shall show below that this indeed is the case.

Suppose $||\kappa E_{11}|| < 1$, for $\alpha, \beta \in \{0, 1\}$ we define the operators $L_{\alpha \beta} : \mathcal{H}_S \to \mathcal{H}_S$ by the rule

$$
L_{\alpha \beta} := -i \alpha E_{\alpha \beta} - \kappa E_{\alpha 1}(1 + i \kappa E_{11})^{-1} E_{1 \beta}
$$

By $\mathcal{G}_I(N)$ denote a set of type $I$ partitions $\mathcal{G}$ such that $N(\mathcal{G}) = N$. Clearly, $\mathcal{G}_I = \cup_{N=1}^{\infty} \mathcal{G}_I(N)$.

Lemma 6.2: For any $\alpha, \beta \in \{0, 1\}^N$ we have

$$
L_{\alpha_n \beta_n} \ldots L_{\alpha_1 \beta_1} = \sum_{\mathcal{G} = \{G_n \gg \ldots \gg G_1\} \in \mathcal{G}_I} L_{\alpha_n \beta_n} (r_{G_n}) \ldots L_{\alpha_1 \beta_1} (r_{G_1})
$$

(6.3)
where parts $G_1, \ldots, G_n \in G$ are labelled in the obvious way ($G_j > G_i$ if all the elements of $G_j$ are greater than those of $G_i$ - as the partition is type $I$ we must either have $G_j > G_i$ or $G_j < G_i$ for different parts), $r_G$ gives the size of a part $G$, and we introduce the operators $L_{\alpha \beta}(r)$

$$L_{\alpha \beta}(r) := \begin{cases} -iE_{\alpha \beta}, & r = 1; \\ -\kappa E_{\alpha 1}(-i\kappa E_{11})^{r-2}E_{1\beta}, & r > 1. \end{cases}$$

(Note that the summation in (6.3) is over all type $I$ partitions having $n$ parts.)

**Proof.** Using definition (6.2) of $L_{\alpha \beta}$ we can write

$$L_{\alpha_n \beta_n} \cdots L_{\alpha_1 \beta_1} = \left[ -iE_{\alpha_n \beta_n} - \kappa \sum_{r_n=2}^{\infty} E_{\alpha_n 1}(-i\kappa E_{11})^{r_n-2}E_{1\beta_n} \right]$$

$$\times \ldots \times \left[ -iE_{\alpha_1 \beta_1} - \kappa \sum_{r_1=2}^{\infty} E_{\alpha_1 1}(-i\kappa E_{11})^{r_1-2}E_{1\beta_1} \right]$$

$$= \sum_{r_1, \ldots, r_n=1}^{\infty} L_{\alpha_n \beta_n}(r_n) \cdots L_{\alpha_1 \beta_1}(r_1)$$

(6.4)

All the series converge absolutely so we can multiply them term by term. Taking into account that there exists an obvious one-to-one correspondence between all sets $\{r_1, \ldots, r_n\}$ and the partitions of $\Theta_I$ having $n$ parts, we see that (6.3) and (6.4) coincide. □

Define the following four operator processes $A_i^{10} = A^+(g \otimes 1_{[0,t]}), A_i^{01} = A^-(g \otimes 1_{[0,t]}), A_i^1 = A(P_g \otimes \chi_{[0,t]}), A_i^0 = t$.

**Lemma 6.3:** For any $k_+, k_- \in \mathbb{N}_0, f_1^-, f_1^+, f_2^-, f_2^+, \ldots, f_k^-, f_k^+ \in \mathfrak{t}, \{S_i^+, T_i^\}^{-k_-}_{i=1}, \{S_i^-, T_i^\}^{k_+}_{i=1} \subset \mathbb{R}$, $t \geq 0$, we have

\[
\langle \prod_{i \in S_{k_-}} A_i^+(f_i^- \otimes 1_{[S_i^-, T_i^\]})| dA_{\alpha_n \beta_n} \cdots dA_{\alpha_1 \beta_1} \prod_{i \in S_{k_+}} A_i^+(f_i^+ \otimes 1_{[S_i^+, T_i^\]})\Psi \rangle = \sum_{P_k \in \mathcal{P}(S_{k_-}, S_{n})} \sum_{P_2 \in \mathcal{P}(S_{k_+}, S_{n})} (-1)^{\xi(G^0_n, \alpha_\beta) + \xi(P_1, \alpha_1) + \xi(P_2, \beta)}
\]

\[
\times \delta_0 \left( \sum_{i \in P_1 | S_{n}} \alpha_i + \sum_{i \in P_2 | S_{n}} \beta_i \right) \prod_{(i_1, j_1) \in P_1} \alpha_{j_1} h_{i_1}(s_{j_1}) \prod_{(i_2, j_2) \in P_2} (\beta_{j_2} h_{i_2}(s_{j_2}))^* \times \langle \prod_{i \in P_1 | S_{k_-}} A_i^+(f_i^- \otimes 1_{[S_i^-, T_i^\]})\Psi | \prod_{i \in P_2 | S_{k_+}} A_i^+(f_i^+ \otimes 1_{[S_i^+, T_i^\]})\Psi \rangle, \tag{6.5}
\]

where $G^0_n$ is the partition of $S_n$ which consists of singletons only.
Proof. The operator $dA_i^\alpha\beta$ satisfy the following commutational relations

$$dA_i^\alpha\beta A^+(f \otimes 1_{[S,T]}) = (-1)^{\alpha+\beta} A^+(f \otimes 1_{[S,T]}) dA_i^\alpha\beta + \beta \langle g \mid f \rangle 1_{[S,T]}(t) dA_i^{\alpha\alpha_0}. \tag{6.6}$$

It is easy to see that the operator $dA_i^\alpha\beta$ acts here as a product $[dA_i^+]^\alpha [dA_i^-]^\beta$. Using this fact we can informally justify the equality (6.5) as follows.

We replace all $dA_{s_{k+1}}$ by products $[dA_i^+]^\alpha_i [dA_i^-]^{\beta_i}$. Using the anticommutational relations we get

$$[dA_{s_n}^+]^\alpha_n [dA_{s_n}^-]^\beta_n \ldots [dA_{s_1}^+]^\alpha_1 [dA_{s_1}^-]^{\beta_1} = (-1)^{\xi(G,\omega_2)} [dA_{s_n}^+]^\alpha_n \ldots [dA_{s_1}^+]^\alpha_1 [dA_{s_1}^-]^{\beta_1} \ldots [dA_{s_1}^-]^{\beta_1}. \tag{6.4}$$

To bring the expression $[dA_{s_n}^-]^{\beta_n} \ldots [dA_{s_1}^-]^{\beta_1} \prod_{i \in S_{k+1}} A^+(f_i^+ \otimes 1_{[S_i^+, T_i^+]})$ to normal order we can use the equality (6.3). Taking into account that in this case the inner product $\langle \beta | g_j | \alpha_i f_i \rangle$ becomes $\langle \beta | g_j | f_i^+ \rangle 1_{[S_i^+, T_i^+]}(s_i)$, which is exactly $(\beta, h_i^+(s_i))$, we obtain (6.5).

The formal proof are very similar. Note, first of all, that the representation of $dA_i^\alpha\beta$ as a product is justified when $\alpha$, $\beta$, or both equals to zero. We only need to show that we can also factorize $dA_{s_1}^{11}$.

To obtain the equality (6.3) from

$$\langle \Psi \mid \prod_{i \in S_{k+1}} A^-(f_i^- \otimes 1_{[S_i^-, T_i^-]}) dA_{s_1}^{\alpha_1\beta_1} \ldots dA_{s_1}^{\alpha_1\beta_1} \prod_{i \in S_{k+1}} A^+(f_i^+ \otimes 1_{[S_i^+, T_i^+]}) \Psi \rangle$$

we need to move $dA_{s_1}^{\alpha_1\beta_1}$ to the right if $\alpha_1 = 0$ and to the left if $\beta_1 = 0$ (and keep in place if both are zero). The only problem is $dA_{s_1}^{11}$ which we have to move to the right and to the left. Consider the term

$$C \times \ldots dA_{s_{i-1}}^{\alpha_i \beta_i} [dA_{s_{i-1}}^+]^{\alpha_{i-1}} \ldots [dA_{s_1}^+]^{\alpha_1} \prod_{i \in I \subset S_{k+1}} A^+(f_i^+ \otimes 1_{[S_i^+, T_i^+]}) [dA_{s_i}^+]^{\beta_i} \ldots [dA_{s_1}^-]^{\beta_1},$$

which appears after we have moved all $[dA_{s_i}^-]^{\beta_j}$, $j = 1, \ldots, i - 1$, to the right. Here $C$ is some factor, $I$ is a subset of $S_{k+1}$ with $i - 1 - j_r + |I| = k_+$. We suppose that $i$ is the least index with $\alpha_i = \beta_i = 1$. Moving $dA_{s_i}^{\alpha_i \beta_i}$ to the right and using the commutational relations (6.6), we represent this term as a sum of $|I| + 1$ terms; $|I|$ terms contain contraction and one term is without contraction. The terms with contraction have the form

$$C \times \ldots [dA_{s_{i-1}}^+]^{\alpha_{i-1}} \ldots [dA_{s_1}^+]^{\alpha_1} A^+(f_{I(1)}^+ \otimes 1_{[S_{I(1)}^+, T_{I(1)}^+]}) \ldots$$

$$\langle \beta_i h_i^+(s_i) \rangle^* dA_{s_i}^{\alpha_0 \beta_i} \ldots A^+(f_{I(U)}^+ \otimes 1_{[S_{I(U)}^+, T_{I(U)}^+]}) [dA_{s_i}^+]^{\beta_i} \ldots [dA_{s_1}^-]^{\beta_1},$$

where $I(k)$ is a $k$-th element of set $I$. Now we have to move the operator $dA_{s_i}^{\alpha_0 \beta_i}$ back to the position which were occupied by $dA_{s_i}^{\alpha_1 \beta_1}$ before. Operator $dA_{s_i}^{\alpha_0 \beta_i}$
anticommute with all \( d\mathbb{A}_i^+ \) and \( A^+ \) so we get

\[
( -1 )^{k_1 + 1} A^{+} C \ldots d\mathbb{A}_{s_1}^+ [ d\mathbb{A}_{s_i-1}^+ ]^{\alpha_i-1} \ldots [ d\mathbb{A}_{s_i}^+ ]^{\alpha_1} A^+ ( f_{i}(1) \otimes 1_{[S_{i(1)}, T_{i(1)}]} ) \ldots ( [ h_{i(k_i)} ]_t ( s_i ) )^* \ldots A^+ ( f_{i(1)} \otimes 1_{[S_{i(1)}, T_{i(1)}]} ) [ d\mathbb{A}_{s_i-1} ]^{\beta_j} \ldots [ d\mathbb{A}_{s_1} ]^{\beta_1}.
\]

This is exactly the same result which we would obtain if we initially had \( [ d\mathbb{A}_{s_i}^+ ]^{\alpha_i} [ d\mathbb{A}_{s_1}^- ]^{\beta_i} \) instead of \( d\mathbb{A}_{s_i}^{\alpha_i}\beta_i \) and moved only the \( [ d\mathbb{A}_{s_1} ]^{\beta_1} \) to the right. We can not bring the term without contraction to the same form but actually we do not need to do this because the vacuum expectation of this term equals to zero and so its exact form is of no importance. Thus, we can suppose again for generality that we had \( [ d\mathbb{A}_{s_i}^+ ]^{\alpha_i} [ d\mathbb{A}_{s_1}^- ]^{\beta_i} \), instead of \( d\mathbb{A}_{s_i}^{\alpha_i}\beta_i \).

Now each of new terms has the same form as the initial one and we can repeat the above procedure for the next \( d\mathbb{A}_{s_1}^{11} \). This proves lemma.

**Theorem 6.4:** Suppose \( \| E_{11} \| < 1 \), then for any \( \varphi_1, \varphi_2 \in \mathcal{H}_S, k_+, k_- \in \mathbb{N}_0, f^-_1, \ldots, f^-_{k_-}, f^+_1, \ldots, f^+_{k_+} \in \mathbb{t}, \{ S^-_i, T^-_i \}_{i=1}^{k_-}, \{ S^+_i, T^+_{i} \}_{i=1}^{k_+} \subset \mathbb{R}, t \geq 0, \) we have

\[
\lim_{\lambda \to 0} \langle \phi_1 \otimes \prod_{i \in S_{k_-}} A^+ ( f^-_i ( \lambda ) ) | U(t/\lambda^2, \lambda) \phi_2 \otimes \prod_{i \in S_{k_+}} A^+ ( f^+_i ( \lambda ) ) \Phi \rangle = \\
\langle \phi_1 \otimes \prod_{i \in S_{k_-}} A^+ ( f^-_i \otimes 1_{[S^-_i, T^-_i]} ) | U t \phi_2 \otimes \prod_{i \in S_{k_+}} A^+ ( f^+_i \otimes 1_{[S^+_i, T^+_i]} ) \Psi \rangle , \quad (6.7)
\]

where \( (U_t)_t \) is a unitary adapted quantum stochastic process on \( \mathcal{H}_S \otimes \Gamma_-(\mathbb{t} \otimes L^2(\mathbb{R}^+)) \) satisfying the quantum stochastic differential equation

\[
dU_t = L_{\alpha \beta} U_t \otimes d\mathbb{A}_t^{\alpha \beta} \quad (6.8)
\]

with \( U_0 = 1 \) and \( L_{\alpha \beta} \) given by (6.2).

**Proof.** We have the following expansion for \( U_t \):

\[
U_t = \sum_{n=0}^{\infty} \sum_{\alpha_1, \beta_1 \in \{0, 1\}^n} \int_{\triangle_\alpha(t)} L_{\alpha_n \beta_n} \ldots L_{\alpha_1 \beta_1} \otimes d\mathbb{A}_n^{\alpha_n \beta_n} \ldots d\mathbb{A}_1^{\alpha_1 \beta_1} . \quad (6.9)
\]
From (6.3) and (6.5) we obtain

$$\langle \phi_1 \otimes \prod_{i \in S_k} A^+(f_i^- \otimes 1_{[S_i^-, T_i^-]}) | U_\alpha \phi_2 \otimes \prod_{i \in S_k} A^+(f_i^+ \otimes 1_{[S_i^+, T_i^+]}) \rangle$$

$$= \sum_{n=0}^{\infty} \sum_{\alpha, \beta \in \{0,1\}^n} N(G) = n \sum_{E(G) \in \Phi_I} \langle \varphi_1 | L_{\alpha_n \beta_n} (r_G) \cdots L_{\alpha_1 \beta_1} (r_G) \varphi_2 \rangle$$

$$\times \sum_{P_1 \in \mathcal{P}(S_{k-1}, S_n)} \sum_{P_2 \in \mathcal{P}(S_{k+1}, S_n)} (-1)^{\xi(\alpha_1, 0)} \xi(P_2, 1) + \xi(P_2, 1) + \xi(P_2, 1) \delta_0 \left( \sum_{i \in P_1 \mid s_n} \alpha_i \right)$$

$$+ \sum_{i \in P_2 \mid s_n} \beta_i \left( \prod_{i \in P_1 \mid s_{k-1}} A^+(f_i^- \otimes 1_{[S_i^-, T_i^-]}) \right) \prod_{i \in P_1 \mid s_{k+1}} A^+(f_i^+ \otimes 1_{[S_i^+, T_i^+]})$$

$$\times \int_{\Delta_n(t)} ds_n \cdots ds_1 \prod_{(i_1 \cdots j)} \alpha_{i_1 h_{i_1}^+(s_{j_1})} \prod_{(i_2 \cdots j_2)} (\beta_j h_{i_2}^+(s_{j_2}))^*.$$ (6.10)

Now we make a slight change of notation. Suppose \(G \in \Phi_I\) consists of parts with sizes \(r_1, \ldots, r_n\). The \(j\)th part will have \(q_j = 1 + \sum_{h < j} r_h\) as minimum element and \(p_j = \sum_{h \leq j} r_h\) as maximum element. Given \(\alpha, \beta \in \{0,1\}^n\) we define \(\hat{\alpha}(G), \hat{\beta}(G) \in \{0,1\}^{E(G)}\) by \(\hat{\alpha}_j(G) = \hat{\beta}_j(G) = 1\) with the exceptions

$$\hat{\alpha}_{p_j}(G, \alpha) = \alpha_j; \quad \hat{\beta}_q(G, \alpha) = \beta_j,$$ (6.11)

Note that \(\{q_j : j = 1, \ldots, n\} = G_{in}\) and \(\{p_j : j = 1, \ldots, n\} = G_{out}\). We then can write

$$\sum_{\alpha, \beta \in \{0,1\}^n} N(G) = n \sum_{E(G) \in \Phi_I} \langle \varphi_1 | L_{\hat{\alpha}_n \hat{\beta}_n} (r_G) \cdots L_{\hat{\alpha}_1 \hat{\beta}_1} (r_G) \varphi_2 \rangle$$

$$\sum_{\alpha, \beta \in \{0,1\}^n} N(G) = n \sum_{E(G) \in \Phi_I} \langle \varphi_1 | E_{\hat{\alpha}_n \hat{\beta}_n} (r_G) \cdots E_{\hat{\alpha}_1 \hat{\beta}_1} (r_G) E_G \varphi_2 \rangle$$

$$\times \prod_{\{i(r) > \cdots > i(1)\}} \kappa_r^{-1} \hat{\beta}_{i(r)}(G) \hat{\alpha}_{i(r-1)}(G) \hat{\beta}_{i(r-1)}(G) \cdots \hat{\alpha}_{i(2)}(G) \hat{\beta}_{i(2)}(G) \hat{\alpha}_{i(1)}(G)$$

$$\sum_{\alpha, \beta \in \{0,1\}^n} N(G) = n \sum_{E(G) \in \Phi_I} \langle \varphi_1 | E_{\hat{\alpha}_n \hat{\beta}_n} (r_G) \cdots E_{\hat{\alpha}_1 \hat{\beta}_1} (r_G) \varphi_2 \rangle$$

$$\times \prod_{\{i(r) > \cdots > i(1)\}} \kappa_r^{-1} \hat{\beta}_{i(r)}(G) \hat{\alpha}_{i(r-1)}(G) \hat{\beta}_{i(r-1)}(G) \cdots \hat{\alpha}_{i(2)}(G) \hat{\beta}_{i(2)}(G) \hat{\alpha}_{i(1)}(G)$$

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The multiplication by $\prod_{\{i(r)\}>\ldots>i(1)\} \in \mathcal{G}} \hat{\beta}_i(\mathcal{G}) \ldots \hat{\alpha}_i(\mathcal{G})$ does not affect the equality because it equals to unity; we are then able to extend the summation over all $\hat{\alpha}_i, \hat{\beta}_i \in \{0, 1\}^{E(\mathcal{G})}$ because this multiplier makes all additional terms equal to zero.

Now we are going to rewrite (6.9) in terms of extended parameters $\hat{\alpha}_i, \hat{\beta}_i$, which appeared from the original ones $\alpha_i, \beta_i$ due to a given $\mathcal{G} \in \mathfrak{S}_f$. To do this we only need to “restore the indices” to $\alpha_i, \beta_i$ in $\hat{\alpha}_i, \hat{\beta}_i$, that is, we have to replace $P_1 \in \mathcal{P}(S_{k_\ldots}, S_n)$ by $\hat{P}_1 \in \mathcal{P}(S_{k_\ldots}, \mathcal{G}_{out})$ and replace $P_2 \in \mathcal{P}(S_{k_\ldots}, S_n)$ by $\hat{P}_2 \in \mathcal{P}(S_{k_\ldots}, \mathcal{G}_{in})$. Clearly,

$$\delta_0 \left( \sum_{i \in \hat{P}_1|\hat{G}_{out}} \hat{\alpha}_i + \sum_{i \in \hat{P}_1|\hat{G}_{in}} \hat{\beta}_i \right) = \delta_0 \left( \sum_{i \in P_1|\hat{s}_n} \alpha_i + \sum_{i \in P_2|\hat{s}_n} \beta_i \right).$$

It is easy to check that $\xi(\mathcal{G}_n, \alpha, \beta) = \xi(\mathcal{G}, \hat{\alpha}(\mathcal{G}), \hat{\beta}(\mathcal{G}))$ for all $\mathcal{G} \in \mathfrak{S}_f$ such that $N(\mathcal{G}) = n$, $\xi(\hat{P}_1, \hat{\alpha}(\mathcal{G}), 1) = \xi(P_1, \alpha, 1)$ and $\xi(\hat{P}_2, 1, \hat{\beta}(\mathcal{G})) = \xi(P_2, 1, \beta)$.

The subintegral expression we shall write as follows: First, we rename the integral variables to $s_1, s_2, \ldots, s_{g_0}$. Then we extend the integration to $\triangle_{E(\mathcal{G})}$ by adding the $\delta_+\text{-functions} \prod_{\{i(r)\}>\ldots>i(1)\} \in \mathcal{G}} \prod_{h=1}^{r-1} \delta_+(s_{i(h+1)} - s_{i(h)})$ where we multiply out over all parts of $\mathcal{G}$. Combining all together finally we get (we now omit the “hats”)

$$\langle \phi_1 \otimes \prod_{i \in S_{k_-}} A^+\big(f_i^- \otimes 1_{[\hat{s}_{i-}, \hat{r}_{i-}^-]_1}\big) \Psi | U_t \phi_2 \otimes \prod_{i \in S_{k_+}} A^+\big(f_i^+ \otimes 1_{[\hat{s}_{i+}, \hat{r}_{i+}^+]_1}\big) \Psi \rangle \ni \sum_{\mathcal{G} \in \mathfrak{S}_f} \sum_{\alpha, \beta \in \{0, 1\}^{E(\mathcal{G})}} \langle \varphi_1 | E_{\alpha(\mathcal{G}), \beta(\mathcal{G})} \ldots E_{\alpha_1(\mathcal{G}), \beta_1(\mathcal{G})} \rangle \sum_{P_1 \in \mathcal{P}(S_{k_-}, \mathcal{G}_{out})} \sum_{P_2 \in \mathcal{P}(S_{k_+}, \mathcal{G}_{in})} \left( \sum_{i \in \hat{P}_1|\hat{G}_{out}} \hat{\alpha}_i + \sum_{i \in \hat{P}_2|\hat{G}_{in}} \hat{\beta}_i \right)

\times (-i)^{E(\mathcal{G})} \langle \varphi_2 | U_t \phi_2 \rangle \prod_{i \in \hat{P}_1|\hat{s}_n} \hat{\alpha}_i + \sum_{i \in \hat{P}_2|\hat{s}_n} \hat{\beta}_i \rangle \prod_{\{i(r)\}>\ldots>i(1)\} \in \mathcal{G}} \int_{\triangle_{E(\mathcal{G})}(t)} ds_{n}(\mathcal{G}) \ldots ds_1 \prod_{(i_1, j_1) \in P_1} \alpha_{j_1} h_{i_1}^{-1}(s_{j_1}) \prod_{(i_2, j_2) \in P_1} \beta_{j_2} h_{i_2}^{+1}(s_{j_2})^* \times \prod_{h=1}^{r-1} \prod_{i(h)}^{r-1} \left\{ \beta_{i(h+1)} \alpha_{i(h)} \delta_+\big(s_{i(h+1)} - s_{i(h)}\big) \right\}. \tag{6.12}$$

The right sides of equalities (6.12) and (6.11) are the same and this completes the proof.
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