Abstract

In this paper, we will analyze the energy dependent deformation of massive gravity using the formalism of massive gravity’s rainbow. So, we will use the Vainshtein mechanism and the dRGT mechanism for the energy dependent massive gravity, and thus analyze a ghost free theory of massive gravity’s rainbow. We study the energy dependence of a time-dependent geometry, by analyzing the radiating Vaidya solution in this theory of massive gravity’s rainbow. The energy dependent deformation of this Vaidya metric will be performed using suitable rainbow functions.

1 Introduction

It is expected that the usual energy-momentum dispersion relation will get deformed in the UV limit due to quantum gravitational effects. In fact, such a deformation of the usual energy-momentum dispersion relation has been observed to occur in loop quantum gravity [1]-[2], discrete spacetime [3], string field theory [4], spacetime foam [5], spin-networks [6], and non-commutative geometry [7]. As the usual energy-momentum dispersion relation is fixed by Lorentz symmetry, the deformation of the usual energy-momentum dispersion relation in the UV limit seems to indicate a breaking of Lorentz symmetry in the UV limit. In fact, such a violation of Lorentz symmetry can be used to explain anomalies in ultra-high energy cosmic rays and TeV photons [8]-[9]. It may be noted that the threshold anomalies are only predicted by deformations where the usual energy-momentum dispersion relation is deformed by a preferred reference frame, and they do not occur in deformation where no such preferred reference frame exists [10]. The deformation of the usual energy-momentum dispersion relation can be explained using the doubly special relativity (DSR) [11]-[12]. The DSR is an extension of the special theory of relativity, in which the Planck energy and the velocity of light are universal constants. So, just as in special relativity, no object can attain a velocity greater than the velocity of light, in DSR no object can have an energy greater than the Planck energy. The DSR can be generalized to curved spacetime, and the resulting theory is called gravity’s rainbow [13]. In this formalism, the spacetime geometry is described by a rainbow of energy dependent metrics, as the geometry of spacetime depends on the energy of the probe. The gravitational dynamics in gravity’s rainbow can be studied using rainbow functions [14]-[22]. The gravity’s rainbow has been used to study inflation [23]-[24], and a resolving of the Big Bang singularity [25]-[27]. It may be noted that gravity’s rainbow is related to Horava-Lifshitz gravity [28]-[29], and for a specific choice of rainbow functions, it produces the same results as produced by Horava-Lifshitz gravity [30].

The main motivation for gravity’s rainbow comes from the observation that the supergravity is a low energy approximation to the string theory [31]-[36]. This is because according to the renormalization group flow, constants depend on the scale at which a theory [37]-[38]. Furthermore, the scale at which a theory is measured will depend on the energy of the probe used to measure such a theory. Thus, as the constants in a
theory depend explicitly on the scale at which a theory is measured, they also depend implicitly on the energy of the probe used to measure such constants. Now string theory can be viewed as a two dimensional theory, and the target space metric can be regarded as a matrix of coupling constants of this two dimensional theory. As these coupling constants would flow and depend explicitly on the scale at which the theory is measured, they would implicitly depend on the energy of the probe used to perform such a measurement. This would make the metric of spacetime depend on the energy of the probe, and thus we would obtain gravity’s rainbow.

So, the gravity’s rainbow can be motivated from string theory, as the energy dependence of the spacetime metric can be motivated from the flowing of target space geometry in string theory. It may be also noted that various solutions obtained in string theory have been generalized to massive gravity, which is a theory with massive gravitons. In fact, massive Type IIA supergravity has been studied. The Fermionic T-duality has also been studied for massive type IIA supergravity. The relation between the massive IIA supergravity and M-theory has also been investigated. A relation between massive IIA/IIB supergravities has also been analyzed, and it has been demonstrated that a duality exists between such massive supergravities. Thus, it is possible to study massive supergravity in string theory, and so massive gravity is also important in string theory. It may be noted that other solutions motivated by string theory has been also studied in massive gravity. In fact, a brane in warped AdS spacetime has been constructed in massive gravity. This was done by analyzing the effect of the mass term for the graviton on an infrared brane. A nonextremal brane has also been analyzed in massive gravity. As there is a good motivation to both study the massive gravity and gravity’s rainbow from string theory, it is both interesting and important to study the rainbow deformation of massive gravity.

It may be noted that massive gravity can also be phenomenologically motivated from accelerated cosmic expansion. Even though there are problems with the massive gravity, these problems can be resolved using the the Vainshtein mechanism. However, the Vainshtein mechanism produces the Boulware-Deser ghosts. It is possible to resolve this with ghosts fields by using the dRGT mechanism. It is possible to have a well defined initial value formulation for massive gravity. In fact, initial value constraints for spherically symmetric deformations of flat space, in such a massive theory of gravity have been studied. It has been demonstrated that even though the energy can be negative and even unbounded from below in certain sector of the theory, there is a physical sector of the theory, in which the energy is positive and the ghosts are suppressed, and that the theory is stable. The negative energy sector remains disjointed, and does not have any effect on the physical sector of this theory. The initial values for cosmological solutions have also been studied in massive gravity. So, the theory has well defined posed initial value formulation, and can be used to analyze the effects of graviton mass on various physical phenomena. The cosmological solutions in massive gravity have also been used to obtain an upper bound on the graviton mass. The open FRW universes have been also studied in massive gravity, and it has been possible to obtain universes with standard curvature and an effective cosmological constant, such a theory of massive gravity. In fact, various different solutions in massive gravity have been studied, and the effect of such a mass on the physics of various systems has been discussed. So, massive gravity is a very important theory of modified gravity, and it is important to study different solutions in massive gravity.

In fact, as both rainbow gravity, and massive gravity are motivated from string theory and phenomenology, we will analyze a solution in the rainbow deformation of the massive gravity. We will study Vaidya solutions in this theory of massive gravity’s rainbow because Vaidya spacetime has used to study interesting physical system. The Vaidya spacetime in massive gravity has been constructed, and the AdS/CFT has been used to interstage field theory dual to a Vaidya-AdS solutions in massive gravity. The Vaidya spacetime is also important in string theory. As Vaidya soltion is important in string theory, and string theory can also be used to motivate a rainbow deformation of massive gravity, we will study the the Vaidya spacetime, massive gravity’s rainbow. It may be noted even though Vaidya solution has been studied in gravity’s rainbow it has not been studied in massive gravity’s rainbow, and so such it is interesting to analyze the Vaidya solution in massive gravity’s rainbow.

2 The Massive Gravity’s Rainbow

In this section, we study the time-dependent black hole solution using Vaidya metric. This metric will be made energy dependent using the framework of massive gravity’s rainbow. The four dimensional action for
such a massive theory of gravity, can be written as

$$I = \int d^4x \sqrt{-g} \left[ R + {\mathcal M}^2 \sum_{i=1}^{4} c_i \mathcal{U}_i(g, f) + \mathcal{L}_m \right],$$  \hspace{1cm} (1)$$

where \(\mathcal{M}\) is the mass parameter in the massive gravity. Here \(f\) is the reference metric, \(c_i\) are constants, and \(\mathcal{U}_i\) are symmetric polynomials of the eigenvalues of the \(d \times d\) matrix \(\mathcal{K}_{\mu \nu} = \sqrt{g^{\mu \nu}} \mathcal{F}_{\mu \nu}\). These symmetric polynomials can be written as

$$\mathcal{U}_1 = |\mathcal{K}|,$$
$$\mathcal{U}_2 = |\mathcal{K}|^2 - |\mathcal{K}^2|,$$
$$\mathcal{U}_3 = |\mathcal{K}|^3 - 3|\mathcal{K}| |\mathcal{K}^2| + 2|\mathcal{K}^3|,$$
$$\mathcal{U}_4 = |\mathcal{K}|^4 - 6|\mathcal{K}| |\mathcal{K}^2|^2 + 6|\mathcal{K}^3|^2 + 6|\mathcal{K}^4|.$$  \hspace{1cm} (2)

The square root in \(\mathcal{K}\) can be defined using \((\sqrt{\mathcal{A}})^{\mu}_{\nu} (\sqrt{\mathcal{A}})^{\nu}_{\lambda} = \delta^{\mu}_{\lambda}\) and \(\mathcal{K} = \mathcal{K}^{\mu}_{\mu}\). Now the equation of motion from this action, can be written as

$$G_{\mu \nu} + {\mathcal M}^2 \chi_{\mu \nu} = T_{\mu \nu},$$  \hspace{1cm} (3)$$

where \(G_{\mu \nu}\) is the Einstein tensor, and \(\chi_{\mu \nu}\) is given by

$$\chi_{\mu \nu} = \frac{c_1}{2}(U_1 g_{\mu \nu} - K_{\mu \nu}) - \frac{c_2}{2}(U_2 g_{\mu \nu} - 2U_1 K_{\mu \nu} + 2K_{\mu \nu}^2)$$
$$- \frac{C_3}{2}(U_3 g_{\mu \nu} - 3U_2 K_{\mu \nu} + 6U_1 K_{\mu \nu}^2 - 6K_{\mu \nu}^3)$$
$$- \frac{C_4}{2}(U_4 g_{\mu \nu} - 4U_3 K_{\mu \nu} + 12U_2 K_{\mu \nu}^2 - 24U_1 K_{\mu \nu}^3 + 24K_{\mu \nu}^4).$$  \hspace{1cm} (4)

We will analyze a spatial reference metric, in the basis \((t, r, \theta, \phi)\) \hspace{1cm} (5)

$$f_{\mu \nu} = \text{diag}(0, 0, c^2 h_{ij}).$$

where \(h_{ij}\) is two dimensional Euclidean metric and \(c\) is a positive constant. We will now write the Vaidya metric for this massive theory, deformed by gravity’s rainbow. So, we will analyze the rainbow deformation of the Vaidya metric, in the case of advanced time coordinate. These rainbow deformations of this metric can be expressed as \[7, 50\]

$$ds^2 = - \frac{1}{{\mathcal F}(E)} \left( 1 - \frac{m(t, r)}{r} \right) dt^2 + \frac{2}{{\mathcal F}(E) \mathcal{G}(E)} dt dr + \frac{1}{{\mathcal G}^2(E)} r^2 d\Omega_2^2,$$  \hspace{1cm} (6)$$

where \({\mathcal F}(E)\) and \({\mathcal G}(E)\) are known as the gravity’s rainbow functions. It may be noted that here \(E = E_s/E_p\), where \(E_s\) is the maximum energy that a probe in that system can take, and \(E_p\) is the Planck energy. So, as \(E_s/E_p \to 0\), \({\mathcal F}(E) = \mathcal{G}(E) = 1\), and the general relativity is recovered in the IR limit of the theory \[14-22\]. These rainbow functions are motivated from various theoretical and phenomenology considerations. The results from loop quantum gravity and \(\kappa\)-Minkowski noncommutative spacetime, have been used to motivate the following rainbow functions \[1\] \[2\]

$${\mathcal F}(E/E_p) = 1 \quad \text{and} \quad \mathcal{G}(E/E_p) = \sqrt{1 - a \left( \frac{E}{E_p} \right)^q}.$$  \hspace{1cm} (7)$$

The modified dispersion relation with constant velocity of light, has been used to motivate the following rainbow functions \[87\]

$${\mathcal F}(E/E_p) = \mathcal{G}(E/E_p) = \frac{1}{1 - a E/E_p},$$  \hspace{1cm} (8)$$

The hard spectra from gamma-ray burster’s, has been used to motivate the following rainbow functions \[5\]

$${\mathcal F}(E/E_p) = \frac{e^{a E/E_p} - 1}{a E/E_p} \quad \text{and} \quad \mathcal{G}(E/E_p) = 1.$$  \hspace{1cm} (9)$$

The maximum energy of the system depends on the physical systems being analyzed, and for black holes, this energy is equal to the energy of a quantum particle near the horizon. This is because such a particle can be viewed as a probe for the geometry of the black hole. In fact, we can use the uncertainty principle, \(\Delta p \geq 1/\Delta x\),
to obtain a bound on the energy of such a particle. So, we can write \(E_s \geq 1/\Delta x\), where \(\Delta x\) is the uncertainty in position of the particle near the horizon, and it is equal to the radius of the event horizon. Thus, the bound on the energy for a black hole can be written as

\[
E_s \geq 1/\Delta x \approx 1/r_+.
\]

It may be noted as the black hole evaporates due to the Hawking radiation, its radius reduces, and this changes the bound on this maximum energy. So, this energy is a dynamical function, and thus rainbow functions are also dynamical. Even though, we do not need the explicit dynamical behavior of rainbow functions, it is important to know that they are dynamical, and so they cannot be gauged away by rescaling of the metric.

Now, we assume the total energy-momentum tensor of the field equation (3), can be expressed in the following form

\[
T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)},
\]

where \(T_{\mu\nu}^{(n)}\) and \(T_{\mu\nu}^{(m)}\) are the energy-momentum tensor for the Vaidya null radiation and the energy-momentum tensor of the perfect fluid, respectively. They can be defined as

\[
T_{\mu\nu}^{(n)} = \sigma l_\mu l_\nu, \quad T_{\mu\nu}^{(m)} = (\rho + p)(l_\mu n_\nu + l_\nu n_\mu) + pg_{\mu\nu},
\]

where \(\sigma, \rho\) and \(p\) are null radiation density, energy density and pressure of the perfect fluid, respectively. In this regard, \(l_\mu\) and \(n_\mu\) are linearly independent future pointing null vectors,

\[
l_\mu = \left(\frac{1}{F(E)}, 0, 0, 0\right) \quad \& \quad n_\mu = \left(\frac{1}{2F(E)} \left(1 - \frac{m(t,r)}{r}\right), -\frac{1}{G(E)}, 0, 0\right),
\]

satisfying the following conditions

\[
l_\mu l^\mu = n_\mu n^\mu = 0 \quad \& \quad l_\mu n^\mu = -1.
\]

Therefore, the non-vanishing components of the total energy-momentum tensor can be written as

\[
T_{00} = \frac{\sigma}{F^2(E)} + \frac{\rho}{F^2(E)} \left(1 - \frac{m(t,r)}{r}\right), \quad T_{01} = -\frac{\rho}{F(E)G(E)},
\]

\[
T_{22} = \frac{pr^2}{G^2(E)}, \quad T_{33} = \frac{pr^2 \sin^2 \theta}{G^2(E)}.
\]

Using the metric ansatz (16), we obtain

\[
\mathcal{K}^\mu_{\nu} = \text{diag} \left(0, 0, \frac{cG(E)}{r}, \frac{cG(E)}{r}\right).
\]

Therefore, we find that

\[
(K^2)^\mu_{\nu} = K^\mu_{\alpha} K^\alpha_{\nu} = \text{diag} \left(0, 0, \frac{c^2 G^2(E)}{r^2}, \frac{c^2 G^2(E)}{r^2}\right),
\]

\[
(K^3)^\mu_{\nu} = K^\mu_{\alpha} K^\alpha_{\beta} K^\beta_{\nu} = \text{diag} \left(0, 0, \frac{c^3 G^3(E)}{r^3}, \frac{c^3 G^3(E)}{r^3}\right),
\]

\[
(K^4)^\mu_{\nu} = K^\mu_{\alpha} K^\alpha_{\beta} K^\beta_{\chi} K^\chi_{\nu} = \text{diag} \left(0, 0, \frac{c^4 G^4(E)}{r^4}, \frac{c^4 G^4(E)}{r^4}\right).
\]

We also obtain the following quantities

\[
[K] = K^\mu_{\mu} = \frac{2cG(E)}{r}, \quad [K^2] = (K^2)^\mu_{\mu} = \frac{2cG^2(E)}{r^2},
\]

\[
[K^3] = (K^3)^\mu_{\mu} = \frac{2c^3 G^3(E)}{r^3}, \quad [K^4] = (K^4)^\mu_{\mu} = \frac{2c^4 G^4(E)}{r^4}.
\]

Now, using the Eqs. (17), (18), and Eq. (2), we obtain

\[
U_1 = \frac{2cG(E)}{r}, \quad U_2 = \frac{2cG^2(E)}{r^2},
\]

\[
U_3 = 0, \quad U_4 = 0.
\]
Using the Eqs. (16), (17), (18) and (19), we can obtain the non-vanishing components of the massive gravity term $\chi_{\mu\nu}$ in the field equation (3) as

$$
\chi_{00} = \frac{1}{r^2 F^2(E)} \left( \frac{c_2 c^2 G(E)}{r^2 F^2(E)} \right) \left( 1 - \frac{m}{r} \right), \\
\chi_{01} = \chi_{10} = -\frac{1}{r F(E)} \left( c_1 c + \frac{c_2 c^2 G(E)}{r} \right), \\
\chi_{22} = -\frac{c_1 c r}{2 G(E)}, \\
\chi_{33} = -\frac{c_1 c r \sin^2 \theta}{2 G(E)}.
$$

(20)

Then, for the 00 component of the field equation (3), we have

$$
\frac{G(E)}{r^2} \left[ r \dot{m} F(E) + r G(E) m' - G(E) m m' \right] = \sigma + \rho (1 - \frac{m}{r}) - M^2 \left[ \frac{c_1 G(E)}{r} + \frac{c_2 c^2 G(E)}{r^2} \right] \left( 1 - \frac{m}{r} \right),
$$

(21)

where dot and prime signs denote the derivative with respect to time and radial coordinates, respectively. For the 01 and 10 component, we have

$$
-\frac{G(E) m'}{r^2} = -\frac{\rho}{G(E)} + \frac{M^2}{r} \left( c_1 c + \frac{c_2 c^2 G(E)}{r} \right).
$$

(22)

Finally, for the 22 and 33 component, we obtain

$$
-\frac{1}{2} r m'' = \frac{p r^2}{G^2(E)} + \frac{M^2 c_1 c r}{2 G(E)}.
$$

(23)

Thus, we have been able to analyze the Einstein equation in gravity’s rainbow. In the next section, we will analyze Vaidya spacetime in this massive gravity’s rainbow.

### 3 Dynamics of the Collapsing System

In this section, we will first find a solution for the field equations describing this model. Then, we will analyze the dynamics of a collapsing system. The matter field will be assumed to follow a barotropic equation of state, which is given by

$$
p = k \rho,
$$

(24)

where $k$ is the barotropic parameter. Now we can use the Eqs. (18),(19) and (20), and obtain an equation describing the behavior of $m(t, r)$ for this system,

$$
r^2 m'' + 6k \frac{m}{G(E)} + \frac{(1 + 3k) M^2 c_1 c r}{G(E)} + 6k c^2 c^2 M^2 r - 6k f_1(t) = 0,
$$

(25)

where $f_1(t)$ is an arbitrary function of time. This differential equation can be solved to obtain a solution for $m(t, r)$,

$$
m(t, r) = f_2(t) r^\omega_1 + f_3(t) r^\omega_2 - \frac{M^2 c_1 c (1 + 3k) r}{(2 - \omega_1) (2 - \omega_2) G(E)} - c_2 c^2 M^2 r + f_1(t),
$$

(26)

where $\omega_1 = \frac{1}{2} (1 + \sqrt{1 - 24k}), \ \omega_2 = \frac{1}{2} (1 - \sqrt{1 - 24k})$. Here, $f_2(t)$ and $f_3(t)$ are arbitrary functions of time $t$. So, from these equations, we obtain the admissible range of $k$, which is $(-\infty, 1/24)$. Thus, the metric given in Eq. (6), can be expressed as

$$
d s^2 = \frac{1}{F^2(E)} \left( -1 + f_2(t) r^{\omega_1 - 1} + f_3(t) r^{\omega_2 - 1} - \frac{M^2 c_1 c (1 + 3k) r}{(2 - \omega_1) (2 - \omega_2) G(E)} - c_2 c^2 M^2 + \frac{f_1(t)}{r} \right) d t^2 + \frac{2 d t d r}{F(E) G(E)} + \frac{1}{G^2(E)} r^2 d \Omega_2^2.
$$

(27)

This metric is the generalized Vaidya metric in Massive gravity’s rainbow.
In this generalized Vaidya spacetime, the singularity can be either a naked singularity or a black hole. The nature of this singularity is determined by the existence of outgoing radial null geodesics, which end in the past central singularity at $r = 0$. Such geodesics exist for a locally naked singularity, and do not exist for a black hole. So, in massive gravity’s rainbow, the singularity formed from gravitational collapse can be either a naked singularity or a black hole. In general relativity, the cosmic censorship hypothesis states that the gravitational singularity must necessarily be covered by an event horizon. So, according to cosmic censorship hypothesis only black hole can form from a collapsing system. However, it has been demonstrated that inhomogeneous dust cloud may form a naked singularity [90]. Interesting results have also been obtained by studying fluid whose equation of state is different from the equation of state of dust [90]. So, it is possible to generalize the cosmic censorship hypothesis [91]. As we have to investigate the nature of singularities in massive gravity’s rainbow, we can use such a generalization of the cosmic censorship hypothesis.

As this system is described by a time-dependent geometry, the radius of shell at $r$, will also be a function of time $t$. We will describe such a radius by $R(t, r)$. This system starts from an initial time $t = 0$, and at that time, we have $R(0, r) = r$. It may be noted that for a inhomogeneous system, different shells may become singular at different times. Now, for this system, we can have future directed radial null geodesics coming out of the singularity. These will have a well defined tangent at the singularity. So, for this system, $\frac{dt}{dr}$ must tend to a finite limit, as the system approach the past singularity. It is possible for the system to reach the points $(t_0, r) = 0$. At this point, the singularity $R(t_0, 0) = 0$ occurs and the matter shells are crushed to a zero radius. This singularity at $r = 0$, is called a central singularity.

Now a naked singularity will form in this system, if future directed curves end in the past singularity. So, for such a system, the outgoing null geodesics will end in the past central singularity, which is at $r = 0$ and $t = t_0$. At such a point, $R(t_0, 0) = 0$, and so for these geodesics, we have $R \to 0$ as $r \to 0$ [92]. The equation for these outgoing radial null geodesics can be obtained from the Eq. (6). Thus, by putting $ds^2 = 0$ and $d\Omega_2^2 = 0$, we obtain

$$\frac{dt}{dr} = \frac{2F(E)}{G(E) \left(1 - \frac{m(t,r)}{r}\right)}.$$  \hspace{1cm} (28)

Here $r = 0, t = 0$ corresponds to a singularity in this equation. Now if $X = \frac{r}{t}$, then we can analyze the limiting behavior of $X$, as the system approaches $r = 0, t = 0$. So, if this limiting value of $X$ is denoted by $X_0$, then we can write

$$X_0 = \lim_{t \to 0, r \to 0} X = \lim_{t \to 0, r \to 0} \frac{t}{r} = \lim_{t \to 0, r \to 0} \frac{dt}{dr} = \lim_{t \to 0, r \to 0} \frac{2F(E)}{G(E) \left(1 - \frac{m(t,r)}{r}\right)}.$$  \hspace{1cm} (29)

We also use Eqs. (25) and (28), and obtain

$$\frac{2}{X_0} = \lim_{t \to 0, r \to 0} \frac{G(E)}{F(E)} \left[1 - f_2(t)r^{\omega_1-1} - f_3(t)r^{\omega_2-1} \right] + \frac{M^2c_1c(1 + 3k)}{(1 + \omega_1)(1 + \omega_2)G(E)} + c_2e^2M^2 - \frac{f_1(t)}{r}.$$  \hspace{1cm} (30)

Now, choosing $f_1(t) = \gamma t$, $f_2(t) = \alpha t^{1-\omega_1}$ and $f_3(t) = \beta t^{1-\omega_2}$, we obtain the algebraic equation for $X_0$, which can be written as

$$\alpha X_0^{1+\omega_2} + \beta X_0^{1+\omega_1} + \gamma X_0^2 - \left(1 + c_2e^2M^2\right)X_0 - \frac{(1 + 3k)M^2c_1c}{(2 - \omega_1)(2 - \omega_2)G(E)} + \frac{2F(E)}{G(E)} = 0,$$  \hspace{1cm} (31)

where $\alpha$, $\beta$ and $\gamma$ are constants. If we only obtain the non-positive solution of the equation, then a black hole will form in this system. However, a naked singularity can form for positive roots of this equation. Since this equation is highly complicated, it is extremely difficult to find out an analytic solution of $X_0$. So, we will use numerical methods to find a numerical solutions of $X_0$. This will be done by assigning particular numerical values to the associated variables. In fact, as a specific rainbow function has been well motivated [3 9], we will use this rainbow functions for analyzing this system,

$$F(E) = 1, \quad G(E) = \sqrt{1 - \eta \left(\frac{E_1}{E_p}\right)}.$$  \hspace{1cm} (32)

In the above expressions, $E_p$ is the planck energy, given by $E_p = 1/\sqrt{G} = 1.221 \times 10^{19}$ GeV, where $G$ is the gravitational constant and $E_1 = 1.42 \times 10^{-13}$ [3 9]. The value of $\eta$ has been estimated to be $\eta \approx 1$ [3], and so in our study, we will use $\eta = 1$.  

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4 Conclusion and Discussion

Now, we will comment on the numerical results obtained in this paper. In Fig. 1, the contours $k - X_0$ were obtained for different numerical values of $\alpha$, where other parameters were fixed, in the massive gravity’s rainbow. The admissible plot range for the equation of state parameter, $k$ is $(-\infty, 1/24]$. In the figure, the plot range for $k$ has been taken as $-2 < k < 1/24$ (from late to early universe). We observe that the trajectories for different values of $\alpha$ almost coincide with each other from $k = -2$ till around $k = -1/3$, i.e., the quintessence and phantom regime (dark energy). But for $k > -1/3$, we observe that this coincidence gradually disappears, and the red line ($\alpha = 0.5$) diverges. However, this does not change the physics of the system much, as all the trajectories remain in the positive level of $X_0$. So, the singularity formed is a naked singularity. At around $k = 0$, the separation of the trajectories becomes more pronounced. For greater values of $\alpha$ (blue line), we see that there is a decreased tendency of formation of naked singularity compared to lower values of $\alpha$. Even we see that the blue line starts to take a dip around $k = 0$. The increased significance of $\alpha$ directly reflects on the function $f_2(t)$. Physically $k = 0$ represents the dust regime and $k > 0$ corresponds to early universe. So, the $\alpha$ dependence of the system will be more significant in the earlier than in the later stages of the evolution of the universe. This is because in massive gravity’s rainbow, the spacetime is energy dependent, and the energy in the earlier stages of the evolution of the universe is more than the energy at the later stages of the evolution of the universe. So, the rainbow functions are more important in the physics of the early stages of the universe. This is the reasons that the significance of $\alpha$ decreases at the later stages of the evolution of the universe.

In Fig. 2, similar figures are obtained for different values of $\beta$, where the other parameters are fixed. We observe that as the values of $\beta$ increase, the trajectories push downwards towards the $k$-axis. This indicate an increase in the tendency to form black holes. However, for both figs. 1 and 2, it is clear that the trajectories remain in the positive $X_0$ region, and a naked singularity forms from the collapse of this system. In Figs. 3 and 4, the $k - X_0$ plots are obtained for different values of $\gamma$ and $\mathcal{M}$, respectively. These plots also indicate that a naked singularity is formed from the collapse of this system. We can observe from Fig. 3, the tendency to form a black hole increases with increase in the value of $\gamma$. We can also observe from Fig. 4, the tendency to form a black hole decreases with the increase in the value of $\mathcal{M}$. So, the system can form a naked singularity by increasing the value of $\mathcal{M}$, and decreasing the value of $\gamma$.

In Figs. 5 and 6, we compare the $k - X_0$ contours of both massive gravity and massive gravity’s rainbow. In Fig. 5, the trajectories are for different values of $\alpha$. From the plot, we can observe that for massive gravity’s rainbow, $\alpha$ does not play an important role in the collapsing system, when $k < -1/3$. However, for pure massive gravity $\alpha$ does not play an important role throughout the domain. Besides, in gravity’s rainbow, the tendency to form black holes is greater than that in pure massive gravity. In Fig. 6, similar plots are obtained for different values of $\beta$. Here, it is also confirmed that in gravity’s rainbow, there is a greater tendency to form a black hole. The above observation is again established in Figs. 7 and 8, where similar plots are generated by varying $\gamma$ and $\mathcal{M}$, respectively. Finally we observe that in all the figures, there are small portions of lines which nearly vanish around the $k$-axis, for small values of $k$. As this system was very complicated, we could not find an analytical solution for Eq. (30). So, we obtained numerical solution for this equation, using particular values for the parameters. The vanishing lines in the $k - X_0$ plane, are produced from the noise in the numerical solution, and do not have physical significance.

In this paper, we have constructed a theory of massive gravity’s rainbow. This was done by analysing the energy dependent deformation of massive gravity. In the construction of massive gravity, we have used the Vainshtein mechanism and the dRGT mechanism. Then, this theory has been deformed by rainbow functions. We have analyzed radiating Vaidya black hole solution in this theory of massive gravity’s rainbow. The effects of both the graviton mass and rainbow deformation have been studied for a time-dependent system. It may be noted that the AdS solution in massive gravity, and the AdS/CFT correspondence corresponding to this AdS solution have been studied [93-94]. In fact, the holographic entanglement entropy for massive gravity has also been studied [95], and it has been demonstrated that for such systems both first order and second order phase transitions can occur. The holographic complexity for massive gravity has also been studied [96]. This holographic complexity of a boundary theory is dual to a volume in the bulk, just as the holographic entanglement entropy is dual to an area in the bulk. It would be interesting to study the rainbow deformation of such solutions. This can be done by making the bulk metric to depend on the energy of the probe. Then, deformation of the bulk metric can be done using suitable rainbow functions. It would be interesting to investigate the holographic entanglement entropy and holographic complexity of massive gravity deformed by suitable rainbow functions.
**Figs 1 and 2** show the variation of $X_0$ with $k$ for different values of $\alpha$ and $\beta$ respectively in massive gravity’s rainbow.

In Fig.1 the other parameters are fixed at $\beta = 2$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $\mathcal{M} = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

In Fig.2 the other parameters are taken as $\alpha = 0.5$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $\mathcal{M} = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

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**Figs 3 and 4** show the variation of $X_0$ with $k$ for different values of $\gamma$ and $M$ respectively in massive gravity’s rainbow.

*In Fig.3 the other parameters are fixed at $\alpha = 0.5$, $\beta = 2$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $M = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.  

In Fig.4 the other parameters are taken as $\alpha = 0.5$, $\beta = 2$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.  

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In Fig.5 the other parameters are fixed at $\beta = 2$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $M = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

In Fig.6 the other parameters are taken as $\alpha = 0.5$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $M = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

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Figs 7 and 8 show the variation of $X_0$ with $k$ for different values of $\gamma$ and $\mathcal{M}$ respectively in a comparative scenario between Massive gravity and Massive gravity’s rainbow.

In Fig.7 the other parameters are fixed at $\alpha = 0.5$, $\beta = 2$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $\mathcal{M} = 5$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

In Fig.8 the other parameters are taken as $\alpha = 0.5$, $\beta = 2$, $\gamma = 3$, $c = 0.8$, $c_1 = 4$, $c_2 = 2$, $\eta = 1$, $E_1 = 1.42 \times 10^{-13}$, $E_p = 1.221 \times 10^{19}$.

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