Disorder induced phase transition in an opinion dynamics model: results in 2 and 3 dimensions

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We study a model of continuous opinion dynamics with both positive and negative mutual interaction. The model shows a continuous phase transition between a phase with consensus (order) and a phase having no consensus (disorder). The mean field version of the model was already studied. Using extensive numerical simulations, we study the same model in 2 and 3 dimensions. The critical points of the phase transitions for various cases and the associated critical exponents have been estimated. The universality class of the phase transitions in the model is found to be same as Ising model in the respective dimensions.

I. INTRODUCTION

Social dynamics is being studied qualitatively and quantitatively, extensively at present [1–5]. This interdisciplinary area used to be the traditional ground for social scientists, but has seen increasing participation of statistical physicists recently. Social systems are quite interesting – they demonstrate rich emergent phenomena, resulting out of interaction of a large number of entities or agents [6]. These rich dynamical systems can be studied using various tools of statistical physics [3,5].

Our present study concerns the dynamics of opinions, and how consensus may or may not emerge out of interacting individual opinions, or, to be rather specific, out of interaction of individuals whose opinions evolve out of influence of others. There has been a series of studies on this topic [7–11], which have enriched our understanding in this regard. Opinions are usually modeled as discrete or continuous variables, which can undergo changes spontaneously or due to interaction with others, or even external factors. The interest lies in the study of dynamics of opinions, as well as the steady state properties – a phase with a spectrum of opinions and another phase where the majority have similar values. In continuous opinion models, opinions cluster around a single value (consensus), or two (polarization) or can even have several values (fragmentation).

The present model is similar to some simple models proposed recently [12–15], apparently inspired by the kinetic models of wealth exchange [16,17]. A symmetry breaking transition was observed in such models: the average opinion is nonzero in the symmetry broken phase, while the opinions of all individuals are identically zero indicating a ‘neutral state’ in the symmetric phase. The parameters representing conviction (self interaction) and influence (mutual interaction) in these models were considered either to be uniform (scalar) or in the generalized case different for each individual, i.e, as components of a vector. There is an added feature of the randomness in the influence term which effectively controlled the sharpness of the phase transitions in these models.

The model we study has conviction parameter set to unity. In absence of interaction, opinions here remain frozen, while any interaction, however small, leads to a state of all individuals having extreme opinions [13]. Allowing for negative values in interaction (influence) takes care of the situation where a pair has a disagreement. This model [18] uses the fraction of negative influences as the tuning parameter to study the phase transition behavior. The salient features of the phase transition in the mean field/infinite range version for the model is already known, including the universality class. There have been a few studies [19,20] extending this model to further realistic situations, by introduction of additional parameters. Although the mean field behavior of the model has been well investigated, the knowledge of the critical behavior of the model in finite dimensions can only ascertain its universality class, which remains to be investigated. In this paper, we study the model in 2 and 3 dimensions using extensive numerical simulations.

The rest of the paper is organized as follows: we introduce the model in Sec. III and report our results in Sec. III. We conclude with discussions in Sec. IV.

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II. THE MODEL

We study a model for emergence of consensus, as introduced in Ref. [18] (BCS model hereafter). Let \( o_i(t) \) be the opinion of an individual \( i \) at time \( t \). In a system of \( N \) individuals (referred to as the ‘society’ hereafter), opinions change out of pair-wise interactions via mutual influences/couplings \( \mu_{ij} \) as:

\[
o_i(t+1) = o_i(t) + \mu_{ij} o_j(t).
\]

One considers a similar equation for \( o_j(t+1) \). The choice of pairs \( \{i, j\} \) is unrestricted, and hence the model is originally defined on a fully connected graph, or in other words, of infinite range (mean field). There is simply a pair-wise interaction and we imply no sum over the index \( j \). Here \( \mu_{ij} \) are real. Following the above dynamics (Eq. (1)), agent \( i \) updates his/her opinion by interacting with agent \( j \) and is influenced by the mutual influence term \( \mu_{ij} \). The opinions are bounded, i.e., \(-1 \leq o_i(t) \leq 1\). If the opinion value of an agent becomes higher (lower) than \(+1\) (\(-1\)), then it is made equal to \(+1\) (\(-1\)) to preserve this bound. This bound, along with Eq. (1) defines the dynamics of the model. Two specific cases were studied, namely discrete and continuous \( \mu \). In the first case, the discrete \( \mu \) case, \( \mu \in \{-1, +1\} \) i.e., takes values \pm 1 only, while in the second case, the continuous \( \mu \) case, \( \mu \in [-1, +1] \), i.e., takes any real value between \(-1\) and \(+1\). The ordering in the system is measured by the quantity \( O = \sum_i o_i/N \), the average opinion, which is the order parameter for the system. Changing the fraction of negative opinions \( p \) one can observe a symmetry breaking transition between an ordered and a disordered phases – below a certain value \( p_c \) of the parameter \( p \), the system orders (giving a non zero, finite value of the order parameter \( O \)), while a disordered phase exists above \( p_c \) (\( O = 0 \)). For the discrete \( \mu \) case, it was shown following analytical calculations that the critical point is \( p_c = \frac{1}{4} \) and \( \beta = \frac{1}{2} \), while Monte Carlo simulations confirmed the results, additionally finding \( \gamma = 1 \) and \( \nu = d \beta \). \( \beta \) and \( \gamma \) are scaling exponents for order parameter and susceptibility respectively. Thus \( \nu = \frac{1}{2} \), if the upper critical dimension is taken as \( d = 4 \). However, for the continuous \( \mu \) case, Monte Carlo simulations gave \( p_c \approx 0.34 \), while the critical exponents were the same as in the discrete case.

III. RESULTS

In one dimension, the model shows no phase transition at non-zero value of \( p \). In our study, we investigate the same model in 2 and 3 dimensions. We imagine agents to be fixed at the vertices of a hypercubic lattice of dimension \( d \). During the dynamics, a vertex \( i \) is chosen at random and one of its \( 2d \) neighbors \( (j) \) is randomly chosen to interact according to Eq. (1). For Monte Carlo simulations in 2 and 3-dimensions, we realize the model on a square lattice and a cubic lattice respectively, and use helical boundary conditions. We simulate both the discrete and the continuous \( \mu \) models.

We perform Monte Carlo simulations of the BCS model [18] in 2 and 3 dimensions. The observed phase transition is quite similar to a thermally driven ferromagnetic-paramagnetic transition in magnetic systems. We compute the following quantities:

(a) the average order parameter \( \langle O \rangle \), \( \langle \ldots \rangle \) means average over configurations.

(b) \( V = N \left[ \langle O^2 \rangle - \langle O \rangle^2 \right] \), analogous to susceptibility per agent,

(c) \( U = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2} \), the fourth order Binder cumulant.

The critical point is calculated from the crossing of the Binder cumulant curves \( U(L, p) \) for different system sizes \( L \). It is known that the value of the Binder cumulant at the critical point for a continuous phase transition is independent of the system size \( L \), and we call this the critical Binder cumulant \( U^* \).

The order parameter \( O \) behaves as \( O \sim |p - p_c|^\beta \) and the susceptibility \( V \) as \( V \sim |p - p_c|^{-\gamma} \) near the critical point, i.e., for small values of \( |p - p_c| \). In order to calculate the critical exponents, we apply finite-size scaling (FSS) theory. We expect, for large system sizes, an asymptotic FSS behavior of the form

\[
O = L^{-\beta/\nu} F_O(x) [1 + \ldots]
\]

\[
V = L^{\gamma/\nu} F_V(x) [1 + \ldots],
\]

where \( F \) are scaling functions with \( x = (p - p_c)L^{1/\nu} \) as the scaling variable. The dots \([1 + \ldots]\) indicate the corrections to scaling terms.
The respective critical exponents were estimated to be $\nu = 1$ and $\gamma = 1$. The respective critical exponents were estimated to be $\nu = 0.99 \pm 0.01$ and $\gamma = 0.122 \pm 0.002$ (Fig. 2). Inset shows unscaled data for $O$ with $p$. (c) Scaling collapse of $V$ with $\gamma = 1.75 \pm 0.01$. Inset shows unscaled data for $V$ with $p$.

For 2-dimensions, we simulated the model for system sizes $N = L^2$, with $L = 12, 16, 24, 32, 48, 64, 96$, with averages over a number of configurations ranging from 2000 for $L = 12$ to 1000 for $L = 96$.

For the discrete $\mu$ case, we estimated $p_c = 0.1340 \pm 0.0001$ with critical Binder cumulant $U^* = 0.561 \pm 0.001$ (Fig. 1). The respective critical exponents were estimated to be $\nu = 0.99 \pm 0.01$, $\beta = 0.122 \pm 0.002$ (Fig. 1) and $\gamma = 1.75 \pm 0.01$ (Fig. 1).

For the continuous $\mu$ case, we estimated $p_c = 0.2266 \pm 0.0001$ with critical Binder cumulant $U^* = 0.559 \pm 0.001$ (Fig. 2). The respective critical exponents were estimated to be $\nu = 0.99 \pm 0.01$, $\beta = 0.125 \pm 0.001$ (Fig. 2) and $\gamma = 1.75 \pm 0.01$ (Fig. 2).

For 3-dimensions, we simulated the model for system sizes $N = L^3$, with $L = 8, 12, 16, 24, 32$, with averages over a number of configurations ranging from 2000 for $L = 8$ to 1700 for $L = 32$.

For the discrete $\mu$ case, we estimated $p_c = 0.1992 \pm 0.0002$ with critical Binder cumulant $U^* = 0.476 \pm 0.004$ (Fig. 3). The respective critical exponents were estimated to be $\nu = 0.63 \pm 0.01$, $\beta = 0.310 \pm 0.002$ (Fig. 3), and $\gamma = 1.255 \pm 0.005$ (Fig. 3).

For the continuous $\mu$ case, we estimated $p_c = 0.2854 \pm 0.0001$ with critical Binder cumulant $U^* = 0.485 \pm 0.002$ (Fig. 4). The respective critical exponents were estimated to be $\nu = 0.63 \pm 0.01$, $\beta = 0.310 \pm 0.002$ (Fig. 4), and $\gamma = 1.26 \pm 0.01$ (Fig. 4).
FIG. 3: Finite size scaling behavior for discrete $\mu$ case in $d = 3$: (a) Scaling collapse of Binder cumulant with $p_c = 0.1992 \pm 0.0002$ estimated from the crossing for different sizes $L$ (inset). $\nu = 0.63 \pm 0.01$ is estimated from the scaling collapse. Critical Binder cumulant value is $U^* = 0.476 \pm 0.004$. (b) Scaling collapse of order parameter $O$ for $\beta = 0.310 \pm 0.002$. Inset shows unscaled data for $O$ with $p$. (c) Scaling collapse of $V$ with $\gamma = 1.255 \pm 0.005$. Inset shows unscaled data for $V$ with $p$.

FIG. 4: Finite size scaling behavior for continuous $\mu$ case in $d = 3$: (a) Scaling collapse of Binder cumulant with $p_c = 0.2854 \pm 0.0001$ estimated from the crossing for different sizes $L$ (inset). $\nu = 0.63 \pm 0.01$ is estimated from the scaling collapse. Critical Binder cumulant value is $U^* = 0.485 \pm 0.002$. (b) Scaling collapse of order parameter $O$ for $\beta = 0.310 \pm 0.002$. Inset shows unscaled data for $O$ with $p$. (c) Scaling collapse of $V$ with $\gamma = 1.26 \pm 0.01$. Inset shows unscaled data for $V$ with $p$.

IV. DISCUSSIONS

We have studied the BCS model of opinion dynamics in 2 and 3 dimensions. The model, originally proposed in infinite dimensions, has been extensively studied and shown to exhibit a continuous phase transition between a phase with order (full or partial order) and disorder (no consensus) \cite{18}. We used extensive Monte Carlo simulations to study the nature of the phase transition and critical behavior, estimate the associated critical exponents. Our findings indicate that the critical behavior of the model is same as that of the Ising model in the corresponding dimensions. Although the nature of randomness in the disorder parameter $p$ affects the critical point in both the discrete and continuous (mutual influence parameter) $\mu$ versions of the model, the values of the critical exponents $\nu$, $\beta$ and $\gamma$ are very close to the values known for the Ising model in the respective dimensions.

TABLE I: Comparing the critical exponents of the model studied, with Ising model in different dimensions. Mean field exponents for the model are taken from Ref. \cite{18}, while exponents of Ising model are taken from Ref. \cite{27} ($d = 2$, exact results) and Ref. \cite{28} ($d = 3$).

| dimension       | $p_c$       | $\nu$       | $\beta$       | $\gamma$ |
|-----------------|-------------|-------------|---------------|----------|
| mean field, discrete $\mu$ | $0.250 \pm 0.001$ \cite{18} | $2.00 \pm 0.01$ | $0.50 \pm 0.01$ \cite{18} | $1.00 \pm 0.05$ \cite{18} |
| mean field, continuous $\mu$ | $0.3404 \pm 0.0002$ \cite{18} | $2.00 \pm 0.01$ \cite{18} | $0.50 \pm 0.01$ \cite{18} | $1.00 \pm 0.05$ \cite{18} |
| $d = 2$, discrete $\mu$ | $0.1340 \pm 0.0001$ | $0.99 \pm 0.01$ | $0.122 \pm 0.002$ | $1.75 \pm 0.01$ |
| $d = 2$, continuous $\mu$ | $0.2266 \pm 0.0001$ | $0.99 \pm 0.01$ | $0.125 \pm 0.001$ | $1.75 \pm 0.01$ |
| $d = 3$, discrete $\mu$ | $0.1992 \pm 0.0002$ | $0.63 \pm 0.01$ | $0.310 \pm 0.002$ | $1.255 \pm 0.005$ |
| $d = 3$, continuous $\mu$ | $0.2854 \pm 0.0001$ | $0.63 \pm 0.01$ | $0.310 \pm 0.002$ | $1.26 \pm 0.01$ |
| mean field Ising | $\nu = \frac{\gamma}{2}$ ($d = 4$ \cite{27}) | $\frac{\gamma}{2}$ \cite{27} | $\frac{\gamma}{4}$ \cite{27} | $\frac{\gamma}{4}$ \cite{27} |
| $d = 2$ Ising | $1$ \cite{27} | $1$ \cite{27} | $1$ \cite{27} | $1$ \cite{27} |
| $d = 3$ Ising | $0.63012$ \cite{28} | $0.32865$ \cite{28} | $1.2373$ \cite{28} | $1.2373$ \cite{28} |
In Table I, we list the critical points and compare the critical exponents of the BCS model with those in the Ising model in $d = 2, 3$ as well as the mean field case. However, one cannot comment on the upper critical dimension in such a model unless the set of critical exponents for $d \geq 4$ are computed. For example, the upper critical dimension of the majority voter model was found to be $6$ \[29\], although critical exponents for that model in 2 and 3 dimensions reasonably matched Ising model exponents. It is also important to note that the model studied is not defined by a Hamiltonian, but just by the microscopic dynamical rules. The opinions are not necessarily discrete, yet the dynamics is sufficient to demonstrate a critical behavior similar to one of the most studied models of statistical physics. It may be noted that this is perhaps the first indication that a kinetic exchange type model can lead directly (from kinetic theory) to the celebrated cooperative (Ising) model universality class. This issue will be addressed more generally in a forthcoming paper \[30\].

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