I. INTRODUCTION

There are few areas of physics in which one confronts the idea of time travel to the past. When it is discussed, time travel is often associated with particular ways of thinking about quantum mechanics and quantum field theory. One often hears the view that in quantum teleportation, for example, the teleported information travels back in time to the point at which entanglement was created, before proceeding forward in time to the recipient. Another example might be remarks about antiparticles in quantum field theory being akin to particles “travelling backwards in time”. The time travel considered in this work is rather different. Here it is assumed that time travel into the past is a possible physical process, and the question is asked: how must quantum mechanics be modified to account for time travel? The motivation for this comes not from within quantum theory itself, but rather from general relativity and the existence of spacetime solutions containing closed timelike curves (CTCs).

CTCs are paths through exotic spacetimes along which massive particles may travel, apparently forwards through time, only to return to their own past. As such they represent a mechanism for time travel to the past. The discovery that the Einstein field equations of general relativity permit spacetime solutions that contain CTCs came as something of a surprise. Many believe that such spacetimes containing CTCs should not occur in nature, however there is no consensus on the possibility of precluding such solutions. Spacetimes containing CTCs are globally non-hyperbolic and therefore are incompatible with standard relativistic quantum field theory. Such spacetimes typically contain no Cauchy surfaces and so the initial value problem is not well defined, making the analysis of matter in such spacetimes tricky.

Deutsch began a programme for investigating quantum mechanics along CTCs that uses the quantum circuit formalism of quantum computation. In this approach a circuit is used to send qubits back in time along classical paths between localised regions of spacetime where they evolve and interact according to unitary quantum theory; the relativistic effects of CTCs are present only in the ability to send qubits back in time. In this way, the resulting theories abstract away from the specific mechanism of time travel, and may therefore be applicable to any such mechanism, be it TARDIS, DeLorean, or CTC.

Deutsch’s theory of quantum mechanics with time travel has come to be known as the theory of D-CTCs. A second theory of quantum mechanics with time travel based on the same quantum circuit approach has come to be known as the theory of P-CTCs. P-CTCs are motivated by “teleporting” quantum states into the past and are equivalent to being able to postselect on quantum measurement outcomes.

Science fiction is rife with naïve classical examples of time travel to the past, almost inevitably leading to various paradoxes. To some extent, both D-CTCs and P-CTCs mitigate such paradoxes. However, both theories still contain undesirable features that may also be pathological or paradoxical.

The evolution of a state in a region involving time travel is expected to be both non-unitary and non-linear and non-linear. Presence of non-linearity fundamentally changes the structure of quantum theory. The standard proofs of most central theorems in quantum mechanics (including no-signalling, no-cloning, generalised uncertainty principles, and indistinguishably of non-orthogonal states) depend on the linearity of the theory.
It is therefore of little surprise that: D-CTCs and P-CTCs can both distinguish non-orthogonal states with a single measurement and (depending on one’s ontology) signal faster than light; D-CTCs can clone arbitrary pure states; and P-CTCs can delete arbitrary states. Linearity is also necessary for the equivalence of proper and improper mixed states in ordinary quantum theory \cite{10, 21} and the ability of quantum theory to be compatible with many different interpretations and ontologies \cite{52}. Non-linear extensions of quantum theory, such as D-CTCs and P-CTCs, therefore require careful examination of any ontological assumptions as the predictions of the theories generally depend on them.

Such issues surrounding theories of time travel in quantum circuits are subtle, and different theories give different perspectives. As well as containing undesirable features, both D-CTCs and P-CTCs are more difficult to interpret than standard quantum mechanics. Since they are constructed from the abstract quantum circuit formalism, any interpretational assumptions behind them are unclear. One may therefore ask, could there be a theory with fewer undesirable features? Could such a theory have a more robust physical motivation?

In this paper, the first of these questions is answered squarely in the affirmative. Several new theories of time travel in quantum mechanics, falling into two overlapping classes, are presented. One such theory, called the theory of T-CTCs, is thoroughly developed and seen to have far fewer problematic features than either D-CTCs or P-CTCs—in particular, cloning and deleting are impossible and non-orthogonal states cannot be distinguished with a single measurement. The second question is more subjective in its nature, but the theories presented here all have physical motivations that may be, depending one’s prejudices, at least as compelling as those of D-CTCs and P-CTCs.

This strongly suggests that the question as to how quantum mechanics behaves in the presence of time travel, and specifically CTCs, is far from decided. With so many potential answers, different insights and criteria are needed to convincingly select one theory to the exclusion of the others.

Sections \ref{sec:modelling} and \ref{sec:general} of this paper largely consist of a thorough review of time travel in the quantum circuit formalism, but they do also contain some new results and define the notation and concepts used in the later sections. In section \ref{sec:modelling} the framework of quantum circuits with time travel is developed and classical time travel discussed. In section \ref{sec:general} the theories of D-CTCs and P-CTCs are presented with a discussion of their features. As part of the development of D-CTCs, the equivalent circuit model—a proposed alternative model of D-CTCs \cite{50, 51}—is shown to contain errors. In section \ref{sec:designing} certain desirable features of any theory of quantum time travel are identified, and used to find two classes of new theories; from these new theories that of T-CTCs is selected and fully developed. Finally, in section \ref{sec:comparison} the theory of T-CTCs is compared to the previous theories, focussing on their motivation, shortcomings, and successes.

II. MODELLING TIME TRAVEL WITH QUANTUM CIRCUITS

A. The Standard Form of Quantum Circuits in with Time Travel

The quantum circuit approach to time travel is based on a particular form of qubit circuit introduced by Deutsch \cite{22}. This convenient building block, from which all other circuits involving time travel can be built, will be called the standard form circuit. It allows different circuits and theories to be concisely specified.

The advantage of the quantum circuit model in this context is that quantum evolution of qubit states is separated from spatial motion. Quantum interactions described by unitary gates are assumed to only occur in small, freely falling, non-rotating regions of spacetime so that they obey non-relativistic quantum mechanics. To include time travel to the past in such a quantum circuit model is therefore equivalent to saying that the classical paths qubits take between gates are allowed to go back in time. The restriction to circuits of qubits may seem unphysical but it is convenient and expected to be computationally universal \cite{22}; circuits with systems other than qubits may also be constructed by analogy. Within this approach, different theories are then defined by their behaviour sending qubits back in time. In this paper only finite-dimensional Hilbert spaces will be considered by considering only finite numbers of qubits.

The standard form circuit is a circuit of $n + m$ qubits (with both $n$ and $m$ finite), a single time travel event that occurs in a localised spacetime region \cite{64}, and a single unitary quantum interaction $U$. The $m$ qubits that travel back in time are called the chronology violating (CV) qubits, which arrive from their own future in the state $\tau_i$ and after the interaction are said to be in the state $\tau_f$. The remaining $n$ qubits are called the chronology respecting (CR) qubits, which arrive from the unambiguous past in the state $\rho_i$ and emerge into the unambiguous future in the state $\rho_f$. Therefore, a standard form circuit is completely specified by values for $n$, $m$, and $U$ while a theory for quantum mechanics with time travel is a specification of $\rho_f$ given $U$ and $\rho_i$. A generic standard form circuit is illustrated in Fig. \ref{fig:circuits}.

All theories of quantum time travel presented in this paper make one additional assumption: before the interaction $U$ the CR and CV systems are not entangled so that the state of all $n + m$ qubits is described by the product state $\rho_i \otimes \tau_i$ \cite{21}. In Ref. \cite{48} it is argued that this is unreasonable as the causal past of the CV qubits contains $\rho_i$ and entanglement should therefore be possible, but that possibility will not be considered here.
B. Time Travel Paradoxes and the Classical Model

In order define the types of possible paradox in time travel, it shall be useful to leave quantum mechanics to one side briefly, and concentrate on classical time travel. Consider a classical version of the standard form circuit, with classical states, $\rho_i,\rho_f$ and $\tau_i,\tau_f$, and some classical dynamical evolution, $\tilde{U}$, replacing their quantum counterparts. An ontic state of a system is a complete physical state according to some theory, while an epistemic state is a probability distribution over ontic states reflecting some observer’s uncertainty of the ontic state.

The standard way of introducing time travel into classical theories is to impose a consistency condition on ontic states that go back in time, that is $\tilde{\tau}_i = \tilde{\tau}_f \equiv \tilde{\tau}$, where $\tilde{\tau}$ is an ontic state. In other words, the ontic state that emerges in the past is required to be the same one that $\tau$ is an ontic state. In other words, the ontic state that $\rho_i$ and $\tilde{U}$, a consistent $\tilde{\tau}$ can be deduced, from which $\tilde{\rho}_f$ may be calculated.

There are two distinct types of paradox that may arise when considering time travel to the past classically: paradoxes of dynamical consistency and paradoxes of information. Loosely speaking, in this paper paradoxes are situations that render a theory unimaginable. As such, the view is taken that communication or information processing abilities of unprecedented power are not necessarily paradoxical. A physically consistent theory may predict “absurdly” powerful communication or computational abilities that are hard to accept, but do not preclude the theory. The rationale is that the “absurd” effect of time travel has been assumed, so one should expect some “absurd” conclusions.

A dynamical consistency paradox is a situation in which a consistent history of events is not possible \cite{22, 40, 41}. The “grandfather paradox”, where a grand-patricidal time traveller goes back to kill their infant grandfather, is the usual example of such a paradox. Generally, dynamical consistency paradoxes arise in a theory when there are initial states and evolutions that fail to specify a valid final state. Classically, this occurs because the consistency condition is rendered unsatisfiable: either the consistency condition is unsatisfiable for a particular set of possible $\rho_i$, or the consistency condition is unsatisfiable for all $\rho_i$. There are three options to avoid these paradoxes when they arise in a theory: disallow time travel in that theory; only allow time travel with certain interactions that avoid these paradoxes; or enforce retrospective constraints on the initial conditions. The first option is contrary to the object of study and is not considered. Retrospective constraints are insufficient to prevent paradox when interactions exist that preclude any consistent input, therefore such interactions must be disallowed from any theory of time travel. Ref. \cite{22} and references therein discuss whether such retrospective constraints are acceptable.

Information paradoxes are situations with consistent dynamics in which information appears that has no source. The prototypical example is the unproven theorem paradox: a mathematician reads the proof of a theorem, travels back in time and writes the proof down in a book; the younger version of the same mathematician learns this proof from said book and subsequently goes back in time in order to fulfill their obligation to write the proof down \cite{22, 41, 42, 43}. The theorem appears to have emerged from nowhere. A quantum circuit that models this paradox was given in Ref. \cite{41} and is illustrated in Fig. 2. The circuit consists of a CR “book” qubit $B$, a CR “mathematician” qubit $M$, and a CV “time traveller” qubit $T$. In the standard form this circuit has $n \equiv 2$, $m \equiv 1$, and $U = \text{SWAP}_{MT} \text{CNOT}_{BM} \text{CNOT}_{TB}$. The interpretation is that the time traveller writes the theorem in the book (CNOT$_{TB}$), the mathematician then reads the book (CNOT$_{BM}$), and finally the mathematician and time traveller swap places (SWAP$_{MT}$) so that the mathematician may go back in time to write the theorem. Using $N$ copies of the circuit ($n = 2N$ and $m = N$) enables the mathematician to encode a theorem in an $N$-bit binary string.

Information paradoxes arise when a theory has a
uniqueness ambiguity: there are choices of interactions and initial states that fail to specify a unique final state. In the classical unproven theorem paradox example this arises because there are many classical time travelling states $\tilde{\tau}$ consistent with the consistency condition, each of which would produce a different $\tilde{\rho}_f$. For example, there will be one consistent $\tilde{\tau}$ where the theorem written was an answer to $P = \text{NP}$, another where the theorem answered $\text{BPP} = \text{BQP}$, many where the theorem was nonsense, etc.

An alternative information paradox sometimes considered in the literature is one where a time travel circuit is constructed such that the only dynamically consistent solution reveals a fixed point of some given function [22, 42]. This is not a paradox in the sense used in this paper, but rather an instant solution to a problem that is hard if $P \neq \text{NP}$ [28] and as such this will not be considered further here. The key difference is uniqueness: information paradoxes appear to be capable of producing any arbitrary “theorem”, while the output of a fixed point solving circuit is uniquely determined by its structure and input [42]. Note, however, that both D-CTCs and P-CTCs are also capable of solving for fixed points in this way.

The uniqueness ambiguity is required to cause information paradoxes such as the unproven theorem paradox. If there were no uniqueness ambiguity then any information that appears as a result of time travel is uniquely specified by the structure of, and input to, the circuit. It is therefore reasonable to say that this circuit is acting as a type of computer that simply performs the calculations it is instructed to do, rather than in a paradoxical fashion.

III. D-CTCS AND P-CTCS

A. D-CTCs

One may construct the theory of D-CTCs rapidly by following Ref. [61] and assuming that reduced density operators are ontic states [62]. Just as in the classical model, a consistency condition $\tau_i = \tau_f \overset{ad}{=} \tau$ should be imposed on the ontic time travelling states. In the standard form this implies

$$\tau = \mathcal{D}(\tau) \overset{ad}{=} \text{Tr}_{CR}[U(\rho_i \otimes \tau)U^\dagger].$$

(1)

Since the partial trace is used to separate CR and CV systems a consistent approach then suggests that the final state should be

$$\rho_f = \text{Tr}_{CV}[U(\rho_i \otimes \tau)U^\dagger].$$

(2)

Essentially, the ingredient that makes D-CTCs different to ordinary unitary quantum mechanics is the map

$$U(\rho_i \otimes \tau)U^\dagger \rightarrow \rho_f \otimes \tau$$

(3)

which replaces the general quantum state after $U$ with the product of its reduced density operators. The point at which the novel map described in Eq. (3) is supposed to occur is not defined, and so there is a dynamical ambiguity in the theory. However, this ambiguity is entirely without observable consequence since all local observations on separated parts of a bipartite system are entirely dictated by the reduced density operators, and Eq. (3) simply maps a bipartite system onto the product of its reduced density operators.

Eqs. (1) define the theory of D-CTCs in its barest form: to find $\rho_f$ given $\rho_i$ and $U$ one solves Eq. (1) to obtain $\tau$ and then evaluates Eq. (2). Note that the equation of motion (2) is both non-linear and non-unitary for general $\rho_i$ and $U$.

The superoperator $\mathcal{D}$ is the Stinespring dilation form [11, 52] of an ordinary linear quantum channel. Schauder’s fixed point theorem [54, 58, 62] guarantees that any trace-preserving quantum channel between density operators has at least one fixed point for $\tau$. Therefore, any D-CTC with any unitary and input state can be solved for an output state $\rho_f$ and D-CTCs are not vulnerable to dynamical consistency paradoxes that are present classically, despite being based on a very similar consistency condition. Direct and instructive proofs of this can be found in Refs. [13, 22].

1. The Uniqueness Ambiguity in D-CTCs

Whilst there always exists a solution of Eq. (1) for $\tau$, this solution is not always unique and therefore D-CTCs contain the same uniqueness ambiguity present in classical time travel. As such, they are vulnerable to unproven theorem paradoxes. Most theorems involving D-CTCs in the literature use circuits that permit only a single solution for $\tau$ and thereby avoid this ambiguity, but it is present in general.

Consider the unproven theorem circuit of section II B.

Solving Eq. (1) one finds a family of CV states $\tau = \alpha|0\rangle\langle 0| + (1 - \alpha)|1\rangle\langle 1|$ and a corresponding family of output states

$$\rho_f = \alpha|0\rangle_B\langle 0| \otimes |0\rangle_M\langle 0| + (1 - \alpha)|1\rangle_B\langle 1| \otimes |1\rangle_M\langle 1|$$

(4)

for $0 \leq \alpha \leq 1$. As such, there is a one-parameter continuous family of possible solutions in the D-CTC case and it is seen that D-CTCs are vulnerable to the unproven theorem paradox.

In Ref. [22] the maximum entropy rule was suggested to resolve this ambiguity. The rule states that one should choose the consistent $\tau$ with the greatest von Neumann entropy $S(\tau)$. This always specifies a unique $\tau$ due to linearity of $\mathcal{D}$ and concavity of $S$ [20] and was motivated by a desire to limit the ability of D-CTCs to solve fixed point problems in the manner discussed in section II B.

However, since the maximum entropy rule has not been universally accepted and other possible principles exist
In this paper it is viewed as a non-essential extension to the theory of D-CTCs. In case of the unproven theorem circuit discussed above, the maximum entropy rule would require $\alpha = \frac{1}{2}$ so that $\tau = 1/2$ is uniquely specified. The interpretation of the unproven theorem circuit for this solution is that the “book” $B$ contains an equal mixture of all possible “theorems” (most of which will be nonsense) and is useless, so this also resolves the unproven theorem paradox for this example.

By allowing for some noise along the path of the CV system, it may be possible to avoid the uniqueness ambiguity. Note that since Eq. (1) is a consistency condition it must be exactly, rather than approximately, satisfied. In order to incorporate this noise into a D-CTC circuit one must modify the circuit, including a new quantum channel $N$ responsible for the noise. Eq. (1) is therefore modified to $\tau = N(D(\tau))$ where $D$ remains the quantum channel associated with the noiseless circuit. Suppose that the noise $N$ is modelled with a depolarising channel $\Delta$ such that depolarisation occurs with probability $0 < p < 1$. The consistency condition then becomes

$$\tau = \frac{p}{2m} 1 + (1-p) D(\tau). \quad (5)$$

Now suppose that two density operators satisfying Eq. (5) differ by $\Delta \tau$. By linearity of $D$, $\Delta \tau = (1-p) D(\Delta \tau)$ and therefore the solution to Eq. (5) can only be non-unique if $(1-p)^{-1}$ is an eigenvalue of $D$. In a finite-dimensional Hilbert space, such as those of finitely many qubits considered here, $D$ has a discrete spectrum of eigenvalues while $(1-p)^{-1}$ takes an arbitrary value from a continuum. Therefore $(1-p)^{-1}$ will not generally be an eigenvalue of $D$ so the addition of an arbitrary depolarising channel specifies $\tau$ uniquely. It therefore appears that the addition of depolarising noise to the CV system may solve the uniqueness ambiguity without need for the maximum entropy rule. The conjecture that by considering general noise on the CV system the uniqueness ambiguity may be avoided is supported by the observation in Ref. [18] that the subspace of unitaries that give rise to the uniqueness ambiguity is of measure zero. However, the uniqueness ambiguity is still present in the theory, even if it disappears under small noise.

2. Interpretation of D-CTCs

D-CTCs were introduced above by assuming that density operators are ontic, and reasoning that one should require consistency of ontic states in time travel. This interpretation is favoured in Ref. [60], in which D-CTCs are reviewed from an epistemic perspective and found to be inconsistent.

While issues with such an epistemic interpretation were recognised by Deutsch, he did not present any ontological assumptions from which the theory is derived. He did, however, favour an interpretation in terms of Everettian quantum theory [23, 53]. This provides an interpretation for the disentanglement process in D-CTCs expressed in Eq. (3): when a system traverses a D-CTC it passes between Everettian branches into a different “world” from the one from which it left.

The entanglement breaking map as presented has similarities with the “collapse” of the state on measurement in ordinary quantum mechanics. Whilst this is entirely unobservable, on an interpretational level this disentanglement process needs to be accounted for in a similar way to the post-measurement state collapse of standard quantum mechanics. The Everettian view of Ref. [22] is one way of approaching this.

3. The Equivalent Circuit Model

The equivalent circuit model of D-CTCs is a proposed alternative model for D-CTCs [50, 51] that attempts to remove two ambiguities from the theory: the uniqueness ambiguity and the treatment of mixed states by the theory (section III C). In this section it is demonstrated that, in its standard presentation, the equivalent circuit model fails to generally reproduce the theory of D-CTCs and is not able to resolve the uniqueness ambiguity.

The equivalent circuit model is defined for standard form circuits with $n = m$ and is outlined in Fig. 4. In practice, $n = m$ may be assumed for all standard form CTC circuits since one can simply introduce spectator qubits which do not interact ($U$ acts on them as the identity). The unitary in $V$ differs from the unitary $U$ of standard form by a swap gate so that CR and CV qubits are now the same. Unitarity is restored by “unwrapping” the circuit so that each time a system goes back in time, there is a new copy of the circuit, as shown in Fig. 4(b). These copies of the circuit form a finite “ladder”, the “rungs” of which are CV qubit paths. The unwrapped circuit therefore has infinitely many outputs from which one, infinitely far up the ladder, is selected.

Following Ref. [51], the standard way of calculating with these unwrapped equivalent circuits is to find $\tau$ by starting the infinite ladder of circuits with a guess for $\tau$, which shall be called $\sigma_0$. The state of the CV system
on the \( N \)th “rung” of the ladder of circuits is given by
\[
\sigma_N = D^N(\sigma_0), \quad D(\sigma) = \text{Tr}_2[\rho(\rho_i \otimes \sigma)V]\]  \( (6) \)
where the partial trace is taken over the final \( n \) qubits in the tensor product (lower arm in each unit in Fig. 3(b)). Note that \( D \) is exactly as defined in Eq. 11. The claim of the equivalent circuit model is that since \( D \) always has fixed points, iterating \( D \) as in Eq. 6 will always cause \( \sigma_N \) to converge to a fixed point of \( D \), i.e. \( \tau = \lim_{N \rightarrow \infty} \sigma_N \). This is not generally correct and corner examples can be constructed. For example, suppose \( V \) is of the form \( V = (A \otimes B)\text{SWAP} \) for unitaries \( A \) and \( B \) so that \( D(\sigma) = A\sigma A^\dagger \) and the action of \( D \) reduces to just applying the unitary \( A \). Repeated action of a unitary does not result in convergence in general. So repeated action of \( D \) does not generally reproduce the consistency condition as claimed.

Another claim of the equivalent circuit model is that, given arbitrarily small decoherence along the CV system, the maximum entropy rule is reproduced and is equivalent to simply choosing \( \sigma_0 \propto \mathbb{1} \). Notwithstanding the comments of the previous section and the argument above, it shall now be shown that the argument justifying this claim presented in the appendix of Ref. [51] is mistaken.

Suppose that repeated action of \( D \) does cause convergence to a fixed point and write \( \tau(\rho_i, \sigma_0) \) for the fixed point converged to when the iteration starts from \( \sigma_0 \) and the input state is \( \rho_i \). In Eq. (A3) of Ref. [51] the entire process of convergence to a fixed point is written as a Kraus decomposition with Kraus operators \( \{E_j\} \) [33, 44]. The reasoning then follows
\[
\tau(\rho_i, \sigma_0) = \lim_{N \rightarrow \infty} D^N(\sigma_0) = \sum_j E_j \sigma_0 E_j^\dagger
\]
\[
\sum_j E_j \sigma_0 E_j^\dagger = \sum_{j,k} E_j E_k \sigma_0 E_k^\dagger E_j^\dagger \quad (7)
\]
\[
\sum_{j,k} E_j E_k \sigma_0 E_k^\dagger E_j^\dagger = \sum_{j,k} E_j E_k \sigma_0 E_k^\dagger E_j^\dagger
\]
where the second line follows since \( \tau \) is a fixed point. It is then claimed that this implies \( [E_j \sigma_0, E_j^\dagger] = 0 \). This is not true generally; for example, it would imply that \( \tau(\rho_i, \sigma_0) = \sigma_0 \) for all \( \sigma_0 \) and therefore that every density operator is a fixed point of \( D \). Thus, even if repeated action of \( D \) caused convergence in the zero noise case, this approach cannot be used to reproduce the maximum entropy rule when noise is included.

4. Computation with D-CTCs

In Ref. 2, it was proved that D-CTCs are able to compute any problem in the complexity class \( \text{PSPACE} \) in polynomial time. In the same paper, there is a proof that classical computers with time travel are also able to compute any problem in \( \text{PSPACE} \) in polynomial time. However, it should be noted that the model of classical time travel used in that proof is different from the model of section II B. In particular, in Ref. 2, classical time travel does not impose a consistency condition on ontic classical states \( \tilde{\tau} \), but rather on classical epistemic states of the time travelling system—viz. probability distributions over possible \( \tilde{\tau} \). Such a model for classical time travel does not seem physically motivated, as classical probability distributions are assumed to be entirely due to observer ignorance rather than anything intrinsically physical. It does, however, have the pleasing computational property of not suffering dynamical consistency paradoxes. Therefore, claims that classical computers with time travel have the power of \( \text{PSPACE} \), or that they are equivalent to quantum computers with time travel, should be taken with caution, as they are based on an unusual model of classical time travel.

It is worth noting that D-CTCs can also produce discontinuous maps from \( \rho_i \) to \( \rho_f \). For all practical purposes the theory loses predictive power when the input state is close to one of these discontinuities in a similar way that the onset of chaos causes loss of predictive power about a classical dynamical system.

B. P-CTCs

The theory of P-CTCs is due to Svetlichny [50], inspired by Coecke’s work on diagrammatic approaches to quantum mechanics [67], and Lloyd et al. [40, 41], based on the unpublished work of Bennett and Schumacher [11] and inspired by the Horowitz-Maldacena final state condition of black hole evaporation [32]. The core idea is to use the ordinary quantum mechanical teleportation protocol as a basis for teleporting qubits into the past. Ref. [19] contains an accessible introduction.

The theory of P-CTCs is defined by ignoring the mechanism behind time travel and postulating only that the result is mathematically equivalent to teleportation into the past, achieved by the following unphysical operational protocol schematically illustrated in Fig. 4.

Prepare \( 2m \) qubits, half labelled \( A \) and half \( B \), prepared in the maximally entangled state \( | \Phi \rangle = 2^{-\frac{m}{2}} \sum |i\rangle_B |i\rangle_A \), where \( \{|i\rangle\} \) is any orthonormal basis of states of \( m \) qubits. Treat the \( B \) qubits as the CV system and let them interact with the CR qubits as normal. After the interaction, measure the \( B \) and \( A \) qubits in a maximally entangled basis that includes \( |\Phi \rangle \), but postselect on the outcome \( |\Phi \rangle \). This is equivalent to simply projecting the tripartite system of CR, \( B \), and \( A \) qubits onto \( |\Phi \rangle \langle \Phi | \) and then renormalising the resulting state. Comparing this to the standard multi-qubit quantum teleportation protocol, the effect is to “teleport” the final state of the \( B \) qubits back onto the \( B \) qubits in the past. This protocol may be simulated in the laboratory by manual postselection of measurement outcomes [11, 50].

The effect of this protocol on CR qubits initially in the
pure state $|\psi_i\rangle$ is (up to normalisation)
\[ B_A \langle \Phi | U_{CR,B} | \psi_i \rangle | \Phi \rangle_{BA} = 2^{-m} \sum B \langle i | U_{CR,B} | i \rangle B | \psi_i \rangle \propto \text{Tr}_{CV}(U) | \psi_i \rangle \]

where subscripts denote the subsystems that states belong to or that operators act on. Generalising this result to mixed input $\rho_i$ and normalising the output to a valid unit-trace density operator shows that the evolution effected by a P-CTC is
\[ \rho_i \rightarrow \rho_f = \frac{P \rho_i P^\dagger}{\text{Tr}(P \rho_i P^\dagger)}, \quad P \doteq \text{Tr}_{CV}(U). \]

This evolution is non-linear due to the normalisation and non-unitary due to the partial trace. Considered as a map applied to the CR qubits, Eq. (9) completely specifies the action of P-CTCs, and as such one may prefer to forget about the protocol described above and instead define the action of a P-CTC to be given by Eq. (9) [68]. With this mindset, the protocol demonstrates that P-CTCs are equivalent to quantum teleportation to the past.

Since Eq. (9) maps each $\rho_i$ onto a specific $\rho_f$ without ambiguity P-CTCs do not suffer the uniqueness ambiguity present classically and in D-CTCs and so are not vulnerable to information paradoxes. Applying the P-CTC protocol to the unproven theorem circuit described in section 11B produces the output state $\rho_f = |f\rangle \langle f|$ where $|f\rangle = \sqrt{\frac{d}{2}} \left( |0\rangle_B |0\rangle_M + |1\rangle_B |1\rangle_M \right)$. The output of $N$ such circuits will always be an equal superposition of all possible theorems and the paradox is avoided.

The operator $P$ defined in Eq. (9) is not a unitary operator, but the partial trace of a unitary operator. It does not therefore preserve arbitrary vector norms, but vector norms are still bounded from above. Consider $P$ acting on a vector $|\psi\rangle$ of the CR system, and let $\{ |\alpha\rangle \}_{\alpha=0}^{d-1}$ be an orthonormal basis on the $d$-dimensional CV system. Using the triangle inequality and unitarity it is found that
\[ \| P |\psi\rangle \|^2 \leq \left( \sum_{\alpha} \| \langle \alpha | U | \psi \rangle | \alpha \rangle \|^2 \right)^2 \leq \left( \sum_{\alpha} \| U |\psi\rangle | \alpha \rangle \|^2 \right)^2 = d^2 \| \psi \|^2. \quad (10) \]

1. Dynamical Consistency and Noise

P-CTCs suffer from dynamical consistency paradoxes. The unphysical protocol described above necessarily fails when there is no $|\Phi\rangle$ component in the state on which to project, viz. when Alice has zero probability of obtaining $|\Phi\rangle$ as her measurement outcome. In this case $P \rho_i P^\dagger = 0$ and no output state is defined.

Most authors interpret this to mean that such evolutions “do not happen” [16, 40, 41]. In terms of the discussion in section 11B this is avoiding paradox by enforcing retrospective constraints to prevent inputs that would produce a paradox. However, there are also P-CTC circuits where all inputs lead to a dynamical consistency paradox. One simple, if contrived, example is to have $U = V \otimes W$ where $W$ is traceless, since then $P = V \text{Tr} W = 0$ regardless of the input state.

If one considers some arbitrary noise on the CV system then dynamical inconsistency is avoided. So whilst the paradoxes are in the theory, one would never expect a realistic system to encounter one. The reason for this is that noise will always introduce an arbitrarily small component to the state which is consistent. Consider the tripartite state in the operational protocol before postselection $\sigma = U (\rho_i \otimes |\Phi\rangle \langle \Phi| U^\dagger$ and consider some small noise channel $N$ defined so that $N (\sigma) = (\sigma + \epsilon \chi) / (1 + \epsilon)$ where $\chi$ is a valid density operator satisfying $\langle \Phi | \chi | \Phi \rangle \neq 0$ and $\epsilon$ is an arbitrarily small positive real number. It is easily seen that, even if $\langle \Phi | \sigma | \Phi \rangle = 0$, the resulting output state after postselection and renormalisation is
\[ \rho_f = \frac{\langle \Phi | \chi | \Phi \rangle}{\text{Tr} (\Phi | \chi | \Phi \rangle)} \]

independent of $\epsilon$. Noise of this type can never be totally eliminated and therefore dynamical inconsistency, whilst in the theory, would never be expected to be encountered.

2. Physicality

The P-CTC formalism as described does not address the mechanism of time travel. Moreover the protocol described above to calculate the effect of P-CTC circuits is manifestly unphysical. Svetlichny described the formalism as only “effective quantum time travel” [50] that one might achieve in the lab by manual postselection, while Lloyd et al. motivated the approach stating that a
CTC is a communication channel into the past and that quantum teleportation represents a quantum communication channel which can be made to communicate to the past as described above [40].

Despite the unphysicality of approach, the introduction of noise in section III B is consistent with the assumption that P-CTC theory produces real effects from real states and unitaries. In the same way as for D-CTC circuits, noise may be introduced to P-CTC circuits by modifying the states and unitaries used accordingly. Inserting such modified elements is equivalent to modifying them in the P-CTC protocol, even if the protocol is merely a calculational tool.

The path integral approach of Politzer does give some physical motivation to the theory of P-CTCs [18]. Path integrals form an alternative to quantum circuits as a general approach to theories of quantum mechanics along CTCs [12, 24, 31, 48]. In Ref. [40] it is shown that P-CTC dynamics is compatible with the Politzer path integral approach. Neither standard P-CTCs nor Politzer path integrals assign states to particles along the CTC. For certain mechanisms of time travel this may not be problematic, but it would be puzzling if such a state was not defined for a CTC.

One could argue that the theory of P-CTCs has a similar dynamical ambiguity to D-CTCs since Eq. (9) defines a physical change to the state and the point at which this change is made is not defined in the theory. Just as with D-CTCs, this dynamical ambiguity has no observable consequences. Also, similarly to the case with D-CTCs, there is an ambiguity about how to treat mixed states in this non-linear theory, which is addressed in section III C.

3. Computation

P-CTCs are able to calculate any problem in PP in polynomial time [10]. This is because quantum mechanics with P-CTCs is computationally equivalent to quantum mechanics with postselection [16] and quantum mechanics with postselection has the computational power of PP [1]. PP is contained within PSPACE and is thought to be strictly smaller [8]. Therefore, D-CTCs are at least as computationally powerful as P-CTCs, and thought to be strictly more powerful.

Unlike D-CTCs, P-CTCs are unable to produce discontinuous evolutions because of the more mild form of non-linearity exhibited. It is shown in section III C that any theory with the same type of non-linearity as P-CTCs is unable to produce discontinuous evolutions.

C. Mixed states and Non-linearity

1. Non-linearity

The most general form of non-linearity in an equation of motion is due to the input state appearing in the equation at quadratic or higher orders. D-CTCs are an example of this: Eq. (2) depends on \( \rho_i \) directly, and also because \( \tau \) depends on \( \rho_i \), so that the equation is generally at least quadratic in \( \rho_i \).

A special case of non-linearity that is more mild is renormalisation non-linearity, where the equation of motion is linear, except for an overall scalar constant that simply normalises the final state. P-CTCs are an example of this, since Eq. (9) would be entirely linear were it not for the denominator, which is simply a normalisation constant. An equation of motion is said to be polynomial non-linear if it is non-linear but not renormalisation non-linear.

Renormalisation non-linear equations of motion are not able to lead to discontinuous evolutions. To see this consider some general input \( \rho_i \) belonging to some vector space \( V \) of possibilities and some linear equation of motion \( M(\rho_i) \) acting on that input. The output \( \rho_f \) is required to be normalised such that \( N(\rho_f) = 1 \) for some linear function \( N: V \rightarrow \mathbb{R} \). Thus, after \( M \) the final state is obtained by renormalisation

\[ \rho_f(\rho_i) = \frac{M(\rho_i)}{N(M(\rho_i))} \]

for which \( N(\rho_f(\rho_i)) = 1 \) as required. Such an evolution is always continuous since for any \( \sigma \)

\[ \lim_{\epsilon \to 0} \rho_f(\rho_i + \epsilon \sigma) = \rho_f(\rho_i). \]

2. Types of Mixed State

In quantum theory one normally identifies two possible types of mixed state: proper mixtures as a result of ignorance of an observer regarding the ontic state of a system and improper mixtures as a result of only having access to part of an entangled system. Both are described using density operators. Mixtures giving rise to the same evolution is always continuous since for any \( \sigma \)

\[ \lim_{\epsilon \to 0} \rho_f(\rho_i + \epsilon \sigma) = \rho_f(\rho_i). \]

whereas an objective collapse interpretation would allow both types of mixture. This equivalence also precludes superluminal signalling in objective collapse models. If there were an entanglement detector able to reliably distinguish proper from improper mixtures then if Alice
and Bob shared an entangled state they would be able to instantaneously signal to one another, since a local measurement by Alice would instantaneously disentangle Bob’s system, which Bob would be able to detect using the entanglement detector.

In a theory with non-linearity, however, it is not valid to describe proper mixtures using density operators. This follows immediately because, for a non-linear equation of motion \( M(\cdot) \) acting on some ensemble of states and corresponding probabilities \( \{ (\rho_j, p_j) \}_j \), applying \( M \) to the initial density operator is not generally the same as the density operator obtained by applying \( M \) to each individual state in the ensemble:

\[
M(\sum_j p_j \rho_j) \neq \sum_j p_j M(\rho_j). \tag{14}
\]

On the other hand, density operators are the correct way to describe improper mixtures under non-linear evolution. By examining the derivation of reduced density operators, as given in [44] for example, it is easily seen that linearity of operations is not assumed at any point.

There is a third type of mixed state, identified in Refs. [10, 22], that is rarely discussed but must be taken into account when discussing time travel. Normally, one assumes that the primitive states of quantum theory are pure states, implying that the state of the entire universe is pure, a claim that is mathematically justified due to the purification theorem [44]. One can relax this assumption and allow mixed states to be primitive states, such mixtures are called true mixtures. Ontologically these are distinct from both proper and improper mixtures since there is no reference system with which a true mixture is entangled yet observers with full knowledge of the state would still describe it as mixed. Note that D-CTCs are capable of sending pure states to mixed states (see, for example, the resolution to the unproven theorem paradox discussed in section III A 1) and therefore one necessarily needs to consider true mixtures in a theory containing D-CTCs. In fact, the state \( \tau \) defined by a D-CTC circuit cannot generally be purified and must be thought of as a true density operator [46].

Because reduced density operators are valid ways of describing improper mixtures even when non-linear evolutions are possible, it follows that non-linearity does not treat true and improper mixtures differently. None of the theories of time travel presented here distinguish between a true and improper mixtures as inputs to a circuit. So whilst theories of time travel may introduce true mixtures conceptually, they do not affect the way in which calculations are performed.

Because of this, the purification theorem still holds for \( \rho_i \), and one may always assume that \( \rho_i = |\psi_i\rangle\langle\psi_i| \) by simply extending \( U \) to act on the purification ancilla as the identity.

### 3. Non-linearity and Information Processing

The most contentious ambiguity related to theories of quantum mechanical time travel is how non-linearity affects information processing.

In Ref. [10] it is argued that D-CTC circuits that implement information processing tasks on pure state inputs are impotent in reality. The reason for this conclusion is that an information processing machine is only useful when the input state is uncertain. That is to say, it should be possible for it to act on one of a range of possible inputs, each with an associated probability. It is argued that this means that the input to such a machine is generally a mixed state input. However, if a non-linear machine has been proved to produce desired outputs when acting on pure state inputs it does not follow that the output produced when acting on a mixture of those inputs is a mixture of the corresponding desired outputs. This is called the “linearity trap” [10].

To resolve this tension, careful attention needs to be paid to assumptions regarding the ontology of states [16–18, 51]. In ordinary quantum mechanics one does not generally need to make such ontological assumptions explicit in order to calculate correctly, but since different types of mixture are no longer mathematically equivalent in non-linear theories ontology becomes important.

A guiding principle may be stated as follows:

In any extension of quantum theory including non-linearity, the dynamics of a system \( S \) may generally depend on more than its (possibly mixed) state according to standard linear quantum mechanics. One may need to take into account the entire ontic system of which \( S \) is a part and also the manner in which it was prepared in order to consistently deduce the dynamics of \( S \).

In particular, one should be explicit as to when a mixed state is a proper mixture.

So what of the information processing abilities of D-CTCs (and other similar non-linear theories)? By considering the discussion in section III C 2, if the “realistic” mixed input to such a machine is ontologically a proper mixture then the action on pure state inputs should generalise to the mixed state input, and the machine behaves as advertised. On the other hand, if the mixed input is improper then one cannot generalise in this way.

To illustrate, suppose that the input to some non-linear information processing machine is selected by performing some quantum measurement on a separate system. In Everettian quantum theory the resulting mixed state is a macroscopic superposition of which only one branch is experienced, and so the input to the machine is ontologically an improper mixture. On the other hand, in quantum theory with objective state collapse the resulting mixture is a proper mixture. The behaviour of such a non-linear machine would therefore be different depending on the ontology.
Since all non-linear evolutions treat proper and improper mixtures differently, then so long as this difference is observable it will lead to an entanglement detector. With an entanglement detector then, if one uses an interpretation that involves instantaneous disentanglement on measurement, one can signal instantly as discussed above. This occurs in both D-CTCs and P-CTCs.

As well as this, P-CTCs seem to allow another form of non-locality due to dynamical consistency paradoxes. If one accepts that evolutions leading to dynamical consistency paradoxes are disallowed, then this is to say that the future existence of a P-CTC will affect measurements before that P-CTC comes into existence in order to avoid the paradox \[10\]. This type of non-locality is not observable, however, as dynamical consistency paradoxes with P-CTCs disappear as soon as noise is added. D-CTCs are not vulnerable to dynamical consistency paradoxes and therefore do not lead to this type of non-locality.

### 5. Distinguishing Non-orthogonal States

Both D-CTCs and P-CTCs are capable of distinguishing non-orthogonal pure states using a single measurement \[15, 16\], an impossibility in standard quantum theory \[44\]. This is illustrated neatly by considering the single-qubit distinguishing circuit: a standard form circuit with \(m = n = 1\) and \(U = \text{CH}_{\text{CR,CV}}\text{SWAP}\), where \(\text{CH}_{\text{CR,CV}}\) is the controlled-Hadamard gate controlled on the CR qubit \[12, 10\].

Implementing the distinguishing circuit with a D-CTC, it is simple to check that \(\rho_1 = |0\rangle\langle 0| \rightarrow \tau = |0\rangle\langle 0| \Rightarrow \rho_f = |0\rangle\langle 0|\), and \(\rho_i = |−⟩⟨−| \Rightarrow \tau = |1\rangle\langle 1| \Rightarrow \rho_f = |1\rangle\langle 1|\). This D-CTC therefore renders the non-orthogonal states \(|0\rangle\) and \(|−⟩\) distinguishable by a computational basis measurement.

Implementing the same distinguishing circuit with a P-CTC one can easily verify that the P-CTC operator \(P\) of Eq. \(9\) is \(P = \text{Tr}_{\text{CV}} U = |0\rangle\langle 0| + |1\rangle\langle −|\). Therefore this P-CTC maps \(|+⟩ \rightarrow |0⟩\) and \(|1⟩ \rightarrow |1⟩\), rendering the non-orthogonal states \(|+⟩\) and \(|1⟩\) distinguishable.

Refs. \[15, 16\] give general recipes for constructing maps for D-CTC and P-CTC circuits respectively that will map sets of non-orthogonal states onto sets of orthogonal states, so that the original states can be distinguished by a single measurement. In this way, D-CTCs can distinguish non-orthogonal states from any specified finite set of pure states, allowing D-CTCs to violate the Holevo bound of classical communication over a quantum channel \[12, 32\]. Since P-CTCs are only renormalisation non-linear, they are limited to only distinguishing pure states from a linearly independent set and therefore cannot break the Holevo bound in this way.

Distinguishability of non-orthogonal states from an arbitrarily large set enables arbitrarily precise identification of the state. Therefore cloning with D-CTCs is possible with arbitrarily high fidelity by simply preparing a new copy of the identified state. A more practical approach to cloning using D-CTCs was presented in Ref. \[3\] by extending the distinguishing circuits of Ref. \[15\]. However, note that neither cloning method produces clones with the same entanglement correlations as the original, which is forbidden by monogamy of entanglement \[19\].

It is possible to use a P-CTC circuit with a single CV qubit to perform any postselected quantum measurement \[16\]. It is for this reason that P-CTCs have the computational power of PP, as discussed in section \[III.B.3\]. This also means that P-CTCs trivially violate the no-cloning theorem \[15\] which forbids any process that takes two copies of an unknown pure quantum state and results in a single copy of the same state with the second system in some standard “blank” state. By simply performing a postselected measurement, a P-CTC could project the second system onto the “blank” state, a process that would work even if the original state was orthogonal to the blank state by virtue of any small noise, as discussed in section \[III.B.1\].

The processes of copying and deleting are clearly dual to one another, and the impossibility of each in ordinary quantum theory follows due to linearity.

### IV. NEW THEORIES

Having thoroughly developed the theories of D-CTCs and P-CTCs, the question now arises as to what other theories of quantum mechanics with time travel there might be and how they might compare to these existing examples. Before developing some new theories, it shall be useful to first review some background on integrating over quantum states, since this shall be used extensively in what follows.

#### A. Integrals over Quantum States

##### 1. Pure States

Given a quantum system, pure states of that system are described as unit vectors \(|φ⟩\) in a Hilbert space \(H\), such that unit vectors equal up to a phase factor are considered equivalent: \(|φ⟩ \sim e^{iθ}|φ⟩\). For finite \(\dim H = d\), the space \(P(H) \equiv CP^{d−1}\) contains all of the distinct pure states of that system \[2\].

For some scalar function \(I : H \rightarrow \mathbb{C}\) the integral over the physical states can be considered
where the integration measure \( d[\phi] \) is yet to be defined. Conveniently, there exists a unique natural measure over \( \mathcal{P}(\mathcal{H}) \), invariant under unitary transformations, given by taking a random unitary matrix distributed according to the Haar measure on the group \( U(d) \) \[65\]. Such a measure is usefully written in the Hurwitz parametrisation \[9, 38, 63\] defined with respect to an orthonormal basis \( \{ |\alpha\rangle \}_{\alpha=0}^{d-1} \) on \( \mathcal{H} \) such that any pure state \( |\phi\rangle \) may be uniquely written in the form

\[
|\phi\rangle = \prod_{\beta=d-1}^{1} \sin \theta_\beta |0\rangle + \sum_{\alpha=1}^{d-2} e^{i\varphi_\alpha} \cos \theta_\alpha \prod_{\beta=d-1}^{\alpha+1} \sin \theta_\beta |\alpha\rangle + e^{i\varphi_{d-1}} \cos \theta_{d-1} |d-1\rangle \tag{16}
\]

with parameters \( \theta_\alpha \in (0, \pi/2) \) and \( \varphi_\alpha \in [0, 2\pi) \). In this parametrisation, the integration measure takes the form

\[
d[\phi(\theta_\alpha, \varphi_\alpha)] = \prod_{\alpha=1}^{d-1} \cos \theta_\alpha (\sin \theta_\alpha)^{2\alpha-1} d\theta_\alpha d\varphi_\alpha. \tag{17}
\]

Eq. (17) is unique up to a multiplicative constant, which is left as unity here.

As noted above, this natural measure is invariant under unitary operations, so that under \( |\phi\rangle \to U|\phi\rangle \) the measure transforms as \( d[\phi] \to d[U|\phi] = d[\phi] \). It may be useful to observe that the Hurwitz parametrisation is a generalisation of the Bloch sphere parameterisation often used for qubits \( \mathcal{H} = \mathbb{C}^2 \). For qubits the measure is, up to a scalar, the rotationally-invariant area measure on a sphere

\[
|\phi\rangle = \sin \theta |0\rangle + e^{i\varphi} \cos \theta |1\rangle \tag{18}
\]

\[
d[\phi] \propto \sin(2\theta) d(2\theta) d\varphi. \tag{19}
\]

2. Mixed States

The mixed states of a quantum system with Hilbert space \( \mathcal{H} \) and \( d = \dim \mathcal{H} \) are described by density operators on \( \mathcal{H} \) that are Hermitian, unit trace, and positive semi-definite. The space of valid density operators on \( \mathcal{H} \) is called \( \mathcal{D}(\mathcal{H}) \) and each operator \( \rho \in \mathcal{D}(\mathcal{H}) \) describes a distinct state.

For some scalar valued function \( \mathcal{I} : \mathcal{D}(\mathcal{H}) \to \mathbb{C} \) consider the integral over the density operators on \( \mathcal{H} \)

\[
I = \int_{\mathcal{D}(\mathcal{H})} d[\tau] \mathcal{I}(\tau) \tag{20}
\]

for some integration measure \( d[\tau] \). Unlike \( \mathbb{CP}^{d-1} \) there is no unique natural measure on \( \mathcal{D}(\mathcal{H}) \) \[9\] and so one has to be chosen, along with a useful way to parametrise \( \tau \). As a result there is no unique natural way to define \( I \), it will depend on the choice of measure used.

B. Desirable Features

When considering how a new theory of quantum mechanics with time travel might be developed, it is useful to consider how it might be desirable for such a theory to behave. Having already developed D-CTCs and P-CTCs fully it is easier to anticipate potential shortcomings of any new theory. A list of desirable features is given below. Of course, all desiderata are linked to various philosophical prejudices, but there is still utility in considering them.

1. The theory should have physical motivation and have a physical interpretation.

2. The theory should reproduce standard quantum mechanics well enough to be consistent with current observations. In the case of CTCs it should reproduce quantum mechanics locally along the CTC, as well as in spacetime regions far from the CTC. It is also expected to be locally approximately consistent with special relativity and, specifically, to not allow superluminal signalling.

3. The theory should be dynamically consistent for all choices of \( U \) and \( \rho_i \). In other words, it should not have disallowed evolutions that lead to dynamical consistency paradoxes.

4. The theory should specify \( \rho_f \) uniquely given \( U \) and \( \rho_i \). If a range of possible output states are considered, then probabilities for each of these should be specified. In other words, it should not have uniqueness ambiguities that lead to information paradoxes.

5. The theory should specify a state \( \tau \) that travels back in time, this should either be uniquely specified or an ensemble of possibilities with corresponding probabilities should be uniquely specified.

6. Given a pure \( \rho_i \), prejudice might require that either of \( \rho_f \) or \( \tau \) are also pure.

7. The theory should not be able to distinguish non-orthogonal states in a single measurement, neither should it be able to clone arbitrary quantum states.

Feature 1 is the most subtle of these, and is discussed for D-CTCs and P-CTCs in sections \[\text{III A}2\] and \[\text{III B}2\] respectively.

D-CTCs have feature 2 so long as ontological assumptions regarding collapse are made that rule out superluminal signalling. P-CTCs only have feature 2 in the presence of finite noise, and even then similar assumptions
about collapse are required to rule out signalling. However, as noted in section III.C adding any non-linear evolution to quantum mechanics opens up the possibility of signalling in this way.

Feature 3 is clearly owned by D-CTCs but not P-CTCs, while feature 4 is definitely owned by P-CTCs but is only owned by D-CTCs by adding an extra postulate (uniqueness with D-CTCs may also be gained by finite noise, as conjectured in section III.A).

Neither D-CTCs nor P-CTCs fully have feature 5. P-CTCs do not specify any $\tau$, while D-CTCs specify $\tau$, but it is not necessarily unique. Feature 6 is perhaps the least compelling feature listed, and is one that neither D-CTCs nor P-CTCs have. Feature 7 is also clearly not one respected by either P-CTCs or D-CTCs.

It is a lot to ask for a theory to have these features. Notably, the standard way of introducing time travel into classical mechanics does not have features 2, 3, 4, or 5. However, since it is not clear how to proceed with constructing its quantum analogue, such a list may be a helpful guide.

C. A Selection of New Theories

With the above desirable features in mind, new theories of quantum mechanics with time travel may be constructed. In this section two overlapping classes of new theories are considered: weighted D-CTCs and transition probability theories, and an example of the latter, dubbed T-CTCs, is selected for further study.

Weighted D-CTCs represent an extension of the theory of D-CTCs. These are described by parametrising the convex subset of density operators $\tau_\alpha$ allowed by the consistency condition $[2]$ with $\alpha$, and then assigning a weight $w_\alpha \geq 0$ to each. The weighted mixture of these is then used for $\tau = \int \text{d} \alpha w_\alpha \tau_\alpha / \int \text{d} \alpha w_\alpha$. The D-CTC protocol can then be used with this uniquely determined choice of $\tau$. One basic example would be to weight all possibilities equally $w_\alpha = 1$, giving a uniform weighted D-CTC theory. There is a whole class of theories on this theme, from different choices of the weights to only taking the mixture over a subset of the $\tau_\alpha$ (perhaps taking a mixture of the $\tau_\alpha$ with the minimum entropy). In terms of the desirable features listed, this theory would gain features 4 and 5 at least, possibly at the expense of feature 1 depending on the details and motivation of the theory. Such a theory is essentially that of D-CTCs, with an alternative to the maximum entropy rule.

Transition probability theories make use of some useful intuition from standard quantum mechanics. It is common to say that the probability of an initial state $|I\rangle$ to transition into a final state $|F\rangle$ under the unitary transformation $V$ is given by the transition probability $|\langle F|V|I\rangle|^2$. More precisely, what is meant is that $|\langle F|V|I\rangle|^2$ is the Born rule probability of finding the system in state $|F\rangle$ if one were to measure the system to see if it were in state $|F\rangle$ after the transformation. As an example of this useful way of thinking consider starting with a bipartite system, initially in state $|\psi_i\rangle|\phi\rangle$, and act upon it with the unitary $U$: the “probability of finding the second system in $|\phi\rangle$” after the transformation is $p(\phi) = ||\langle \phi|U|\psi_i\rangle|\phi||^2$ [20]. Generalising this to mixed states, one could say that the transition probability for a bipartite system initially in the state $\rho_i \otimes \tau$ to have the second system found in state $\tau$ after some unitary transformation $U$ is given by $p(\tau) = \text{Tr} [\tau(U(\rho_i \otimes \tau)U^\dagger)]$ [7].

The transition probability theories are the specific theories obtained by applying these ideas to time travel. The choices that need to be made to define a specific theory include: whether pure or mixed $\tau$ are used, which $|\phi\rangle$ or $\tau$ are to be considered, and how is $\rho_f$ to be separated from the CV system?

The following is an example of a transition probability theory that is also a weighted D-CTC theory. That is, choose the weights of a weighted D-CTC theory to be the transition probabilities: $w_\alpha = p(\tau_\alpha) = \text{Tr}[\tau_\alpha^2]$ (this holds when $\tau_\alpha$ solve the D-CTC consistency condition Eq. (1)). Therefore in this theory the equation of motion becomes

$$\rho_f = \frac{\int \text{d} \alpha \text{Tr} [\tau_\alpha^2] \text{Tr}_{CV} [U(\rho_i \otimes \tau_\alpha)U^\dagger]}{\int \text{d} \alpha \text{Tr} [\tau_\alpha^2]}. \quad (21)$$

This is an example of a specific theory that belongs to both classes.

There is a subset of the transition probability theories that have a clear physical interpretation. Consider a standard form time travel setup with $\rho_i = |\psi_i\rangle\langle \psi_i|$. Suppose it is known that the primitive states of quantum theory are pure states, and so the unknown state emerging from the future is a pure state $|\phi\rangle$. However, since $|\phi\rangle$ is unknown, the state $\tau_\alpha$ is a proper mixture reflecting this ignorance. It was argued above that the transition probability $p(\phi) = ||\langle \phi|U|\psi_i\rangle|\phi||^2$ gives the probability for consistency after the interaction. Therefore, an observer seeing the time travelling system emerging into the past can describe its state using the proper mixture

$$\tau_\alpha = Z^{-1} \int \text{d}|\phi\rangle p(\phi)|\phi\rangle \langle \phi|$$

where the constant $Z > 0$ normalises this state and the integral is over $P(H_{CV})$. After applying $U$ to $\rho_i \otimes \tau_\alpha$ the question then remains as to how to mathematically separate the CV system from the CR system. One possibility is to say that CR and CV systems become spatially separated, so one should use the partial trace to describe the CR subsystem as in ordinary quantum theory. A second possibility is to argue that since the probabilities $p(\phi)$ are found by projecting $U|\psi_i\rangle|\phi\rangle$ onto $|\phi\rangle$ for each $|\phi\rangle \in P(H_{CV})$, the same process should also give the final state corresponding to each $|\phi\rangle$.

Each of these options gives rise to a different transition probability theory. The equation of motion for the theory
corresponding to the first option is

$$\rho_f = Z^{-1} \int d[\phi] \, p(\phi) \, \text{Tr}_{CV} \left[ U(|\psi_i\rangle\langle\psi_i| \otimes |\phi\rangle\langle\phi|) U^\dagger \right].$$

(22)

However, in this paper the second option will be taken and fully developed in the following section into the theory of T-CTCs.

Clearly, variations exist on all of the theories presented above. For example, one could construct theories analogous to T-CTCs and Eq. (22), but integrating over mixed states \( \tau \in \mathcal{D}(\mathcal{H}_{CV}) \) instead; there are many such variations, especially since there is no unique choice of integration measure over \( \mathcal{D}(\mathcal{H}_{CV}) \).

### D. The Uniqueness Ambiguity and Epistemic Reasoning

Before proceeding to detail the theory of T-CTCs, some remarks are in order about the uniqueness ambiguity. In section IIIA this ambiguity was introduced as a necessary and sufficient condition for a theory to suffer information paradoxes. The argument is that, if the final state is uniquely determined by the initial state and dynamics, what has occurred can be regarded as a (possibly very powerful) computation and is therefore not paradoxical according to the meaning used in this paper.

This argument still holds if the unique final state is an epistemic state, so long as the probabilities in the epistemic state are determined by the physics, rather than by appealing to ignorance. If the probabilities are physically determined then any new information obtained can be viewed as being due to a probabilistic computation. For any particular final state to be likely, the physics must not only establish that final state as a possibility, but also that the corresponding probability is sufficiently high. Models for probabilistic computation are well-established and certainly not paradoxical.

Compare this to D-CTCs without noise or the maximum entropy rule. In section IIIA it was claimed that the theory of D-CTCs contains the uniqueness ambiguity and therefore suffers from information paradoxes. The difference to a probabilistic computation is that D-CTCs assign no probabilities to the possible final states, they are merely left as possibilities.

This discussion illustrates the fact that uniqueness is assured even when it is a unique epistemic state that is specified, rather than a unique ontic state, so long as the probabilities are physically determined. It is for this reason that requirements 4 and 5 of section IVB allow for uniquely specified epistemic states.

### E. T-CTCs

The physical story that T-CTCs tell is a compelling one. Consider this story from the point of view of a CR observer watching a standard form time travel circuit evolve.

This observer sees a CV system emerge from the future and, believing the primitive states of a quantum system to be pure states, knows that it is in some pure state \( |\phi\rangle \in \mathcal{P}(\mathcal{H}_{CV}) \) but does not know which one beyond the fact that it must be consistent with what is about to happen. The observer then sees an interaction between this CV system and a CR system that was initially prepared in state \( |\psi_i\rangle \), so that together the bipartite system is in some state of the form \( U(|\psi_i\rangle|\phi) \). The CV system is then observed to head back in time by some mechanism. At this point, the CR observer can calculate the probability for consistency for each \( |\phi\rangle \) to be the transition probability \( p(\phi) = ||(\phi|U|\psi_i\rangle|\phi)||^2 \), but also knows that if any given \( |\phi\rangle \) actually did emerge in the past then it has to have been found to be consistent. Thus, for each \( |\phi\rangle \) when the CV system heads back in time the CR observer knows that to check if the state of it is consistent has only possible answer: it is consistent. So the CR observer can describe the final state of the CR system to be the projection \( \langle\phi|U|\psi_i\rangle|\phi\rangle/||\langle\phi|U|\psi_i\rangle|\phi||^2 \). Finally, since the CR observer does not know which state \( |\phi\rangle \) actually emerged, a proper mixture of the possibilities is taken, each weighted by the corresponding probability \( ||(\phi|U|\psi_i\rangle|\phi)||^2 \). The resulting final state for the CR system is therefore

$$\rho_f = Z^{-1} \int d[\phi] \, U_\phi |\psi_i\rangle\langle\psi_i| U^\dagger_\phi$$

(23)

$$U_\phi = \langle\phi|U|\phi\rangle$$

(24)

$$Z = \int d[\phi] \, |\langle\psi_i|U^\dagger_\phi|\psi_i\rangle|^2$$

(25)

where the operator \( U_\phi \) acts only on \( \mathcal{H}_{CR} \) and the constant \( Z > 0 \) is defined to normalise \( \rho_f \).

Several features of the theory immediately follow from the definition in Eqs. (23-25). First, it is a non-unitary and non-linear theory. Second, it is only renormalisation non-linear and as such it only gives rise to continuous evolutions. Third, there is no ambiguity in the equation of motion (23) so there is no uniqueness ambiguity and no information paradoxes. Before proceeding to consider what other features T-CTCs may have, it is first necessary to show that T-CTCs satisfy some basic consistency requirements. It shall also be convenient to re-write Eq. (23) in a simpler form.

#### 1. Basic Requirements

Any reasonable theory of quantum mechanics with time travel should satisfy some basic consistency criteria, and it is necessary to show that T-CTCs satisfy them too before proceeding. Needless to say, both D-CTCs and P-CTCs satisfy these criteria.

The first is that if the identity is applied to the CV system, then the equation of motion should just reduce to...
ordinary quantum unitary evolution. To see this suppose \( U = V \otimes W \) separates on the CV and CR systems. Then \( \tilde{U}_\alpha = \langle \phi | W | \phi \rangle V \) and the CR system factorises out of the integral so that \( \rho_f = V | \psi_i \rangle \langle \psi_i | V^\dagger \) as required.

The second is to show that the point at which the projection occurs makes no difference. This is equivalent to saying that there is no observable dynamical ambiguity in this theory; in the sense discussed in sections III A and III B 2 for D-CTCs and P-CTCs respectively. This can be seen by considering the transformations \( U \rightarrow U (1 \otimes W) \) and \( U \rightarrow (1 \otimes W) U \) for some \( W \) that only acts on the CV system. By unitary invariance of the measure, one sees that both transformations have the same result, so it does not matter whether one applies \( W \) before or after \( U \) (equivalently, before or after the transition probabilities were calculated).

2. Simplification

The equation of motion (23) for T-CTCs in its current form is rather opaque. In order to more easily calculate with the theory it is useful to perform the integration in generality and therefore simplify the equation.

Let \( \{ |\alpha\rangle \}_{\alpha=0}^d \) be an orthonormal basis for the \( d \)-dimensional CV system and expand the unitary \( U \) in the Kronecker product form in this basis \( U = \sum_{\alpha,\beta} A_{\alpha\beta} \otimes | \alpha \rangle \langle \beta | \), where \( A_{\alpha\beta} \) are operators on the CR system.

In this form the equation of motion is

\[
\rho_f = Z^{-1} \sum_{\alpha,\beta,\gamma,\delta} I_{(\alpha\beta)(\gamma\delta)} A_{\alpha\beta} | \psi_\alpha \rangle | \psi_\beta \rangle A_{\gamma\delta}^\dagger \langle \psi_\gamma | \langle \psi_\delta | \tag{26}
\]

having defined the integrals

\[
I_{(\alpha\beta)(\gamma\delta)} = \int d|\phi\rangle \langle \phi | \langle \phi | \langle \phi | | \psi_\alpha \rangle | \psi_\beta \rangle \langle \psi_\gamma | \langle \psi_\delta | \tag{27}
\]

Now consider expanding both \( d|\phi\rangle \) and \(|\phi\rangle \) in the Hurwitz parametrisation, Eqs. 16-18, with respect to the same basis. Since, for each \( \alpha \), \( d|\phi\rangle \) factorises out of the measure, any integrand in which the only \( \varphi_\alpha \)-dependence is an integer power of \( e^{i \varphi_\alpha} \) will integrate to zero. Considering the integrals in Eq. (27), every integrand will have such a phase factor unless one or both of the following conditions is met: \( \alpha = \beta \) and \( \gamma = \delta \), or \( \alpha = \gamma \) and \( \beta = \delta \). In these cases, all phase factors will cancel out and the phase integrals will not come to zero. Discarding the zero integrals in Eq. (26) it is therefore found that

\[
\rho_f = Z^{-1} \left( \sum_{\alpha \neq \beta} I_{(\alpha\beta)(\alpha\beta)} A_{\alpha\beta} | \psi_\alpha \rangle | \psi_\alpha \rangle A_{\alpha\beta}^\dagger \right. \\
+ \sum_{\alpha \neq \beta} I_{(\alpha\alpha)(\beta\beta)} A_{\alpha\alpha} | \psi_\alpha \rangle | \psi_\alpha \rangle A_{\beta\beta}^\dagger \\
+ \left. \sum_\alpha I_{(\alpha\alpha)(\alpha\alpha)} A_{\alpha\alpha} | \psi_\alpha \rangle | \psi_\alpha \rangle A_{\alpha\alpha}^\dagger \right) . \tag{28}
\]

By unitary invariance of the integration measure the components of \( |\phi\rangle \) in the integrands can be selected so that, for \( \alpha \neq \beta \),

\[
I_{(\alpha\beta)(\alpha\beta)} = I_{(\alpha\alpha)(\beta\beta)} = \int d|\phi\rangle |\langle \phi | \langle \phi | (| \phi \rangle - 1)|^2 |\langle \phi | (\phi - 2)|^2 \\
= (2\pi)^{d-1} \left( \int \prod_{\gamma=1}^{d-3} \cos \theta_\gamma (\sin \theta_\gamma)^{2\gamma-1} d\theta_\gamma \right) \\
\times \int d\theta_d-1 d\theta_d-2 \cos^3 \theta_d-1 \cos^3 \theta_d-2 \sin^{2d-1} \theta_d-1 \sin^{2d-5} \theta_d-2 . \tag{29}
\]

where in the final line the integrand has been expanded out in the Hurwitz parametrisation. Similarly

\[
I_{(\alpha\alpha)(\alpha\alpha)} = \int d|\phi\rangle |\langle \phi | \langle \phi | \tag{30}
\]

By evaluating these integrals it is seen that, for \( \alpha \neq \beta \), \( I_{(\alpha\beta)(\alpha\beta)} / I_{(\alpha\alpha)(\beta\beta)} = 2 \). So by factorising out \( I_{(\alpha\beta)(\alpha\beta)} \) from Eq. (29) one finds

\[
\rho_f \propto \sum_{\alpha,\beta} \left( A_{\alpha\beta} | \psi_\alpha \rangle | \psi_\beta \rangle A_{\alpha\beta}^\dagger + A_{\alpha\alpha} | \psi_\alpha \rangle | \psi_\alpha \rangle A_{\beta\beta}^\dagger \right) . \tag{31}
\]

Finally, note the following identities, which may readily be verified by expanding the traces: \( P \equiv \text{Tr}_{CV} U = \sum_\alpha A_{\alpha\alpha} \) and \( \sum_{\alpha,\beta} A_{\alpha\beta} | \psi_\alpha \rangle | \psi_\beta \rangle A_{\alpha\beta}^\dagger = \text{Tr}_{CV} [U (| \psi_\alpha \rangle \langle \psi_\alpha | \otimes \mathbb{1} U^\dagger) \right] \). Using these, and introducing a normalising constant \( z > 0 \) (which is generally different from \( Z \) used before) the final form of the equation of motion becomes

\[
\rho_f = z^{-1} \left( P | \psi_i \rangle | \psi_i \rangle P^\dagger + d \text{Tr} \left[ U (| \psi_i \rangle \langle \psi_i | \otimes \mathbb{1} \right. \left. U^\dagger) \right] \right) . \tag{32}
\]

Eq. (32) is in a much more revealing form than Eq. (23). It shows that the T-CTC equation of motion is a weighted mixture of the corresponding P-CTC equation of motion (9) with an ordinary quantum channel.

3. Dynamical Consistency

The theory of T-CTCs is always dynamically consistent. This may be seen directly from the equation of motion in the form (32). Even though it is possible for \( P | \psi_i \rangle = 0 \), the second term will always give rise to a non-zero density operator. Since it has already been seen that T-CTCs do not suffer uniqueness ambiguities, it follows that T-CTCs contain neither type of paradox identified
in section IIIB. Unlike P-CTCs and D-CTCs, no noise or extra rule is required to avoid these paradoxes.

For example, consider applying Eq. (32) to the unproven theorem circuit of section IIIB. For this circuit

\[
P = |0\rangle_B (0 \otimes |0\rangle_M \langle 0| + |0\rangle_B (1 \otimes |1\rangle_M \langle 1| + |1\rangle_B (1 \otimes |0\rangle_M \langle 1| + |1\rangle_B (0 \otimes |1\rangle_M \langle 0|)
\]

and therefore \( P(00)_{BM} = |00\rangle_{BM} + |11\rangle_{BM} \). It is also easily found that \( T_{CTC} \left[ U(|00\rangle_{BM} \langle 0|) \otimes \frac{1}{\sqrt{d}}U \right] = |00\rangle_{BM} \langle 01| + |11\rangle_{BM} \langle 11| \). Substituting these into Eq. (32) the output the unproven theorem T-CTC circuit is therefore

\[
\rho_f = \frac{1}{2}|00\rangle_{BM} \langle 00| + \frac{1}{4}|00\rangle_{BM} \langle 11| + \frac{1}{4}|11\rangle_{BM} \langle 00| + \frac{1}{2}|11\rangle_{BM} \langle 11|.
\]

This same result may also be found, with rather more effort, directly from the integral expression Eq. (23).

4. Computation

Basic conclusions on the computational power of T-CTCs follow directly from Eq. (32). The second term is realizable in ordinary quantum mechanics, and so is limited to the power of BQP, while the first term is the P-CTC equation of motion. Since BQP \( \subseteq \text{PP} \) it follows that T-CTCs cannot quickly solve any problems that are not contained within PP. There may be problems in PP that they are not able to solve quickly, so P-CTCs are at least as powerful as T-CTCs and may be strictly more powerful.

The form of Eq. (32) also suggests that T-CTCs may be less powerful than P-CTCs. This is because, for a T-CTC, \( \rho_f \) is only a pure state if either \( P(\psi) = 0 \) or if the two terms in Eq. (32) are equal. So every T-CTC algorithm that outputs a pure state is achievable on an ordinary quantum computer in exactly the same way. It would require great cunning to design an algorithm for a T-CTC-equipped computer that made computational use of the first term in Eq. (32). This observation also prevents T-CTCs from being able to perform an arbitrary postselected quantum measurement, since many postselected measurement outcomes are pure states. Therefore, one cannot prove that T-CTCs have the power of PP in the same way as seen for P-CTCs in section IIIB.3

5. Mixed States and Non-linearity

In section IIIC the point was made that non-linearity does not prevent the validity of describing both improper and true mixtures with density operators, and that the purification theorem still holds for them on \( \rho \); (so that \( \rho_i = |\psi_i\rangle \langle \psi_i| \) may always be assumed). This remains true in the theory of T-CTCs. It also remains true that proper mixtures are not validly described by density operators, since T-CTCs are non-linear. Non-linearity does, however, open the possibility of creating an entanglement detector with a T-CTC and therefore the possibility of signalling exactly as with D-CTCs and P-CTCs.

Another consequence of non-linearity is that D-CTCs and P-CTCs are both capable of distinguishing non-orthogonal states in single measurement. However, it shall now be seen that this is not the case with T-CTCs.

Consider the problem of distinguishing between two states \( \rho \) and \( \sigma \). The probability of success when using a single optimal measurement is given by \( \frac{1}{2}(1 + D(\rho, \sigma)) \) where \( D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma| \) is the trace distance between the states. Therefore, \( \rho \) and \( \sigma \) are perfectly distinguishable in a single measurement if and only if \( D(\rho, \sigma) = 1 \).

Another measure of distinguishability of states is the fidelity between \( \rho \) and \( \sigma \), defined as

\[
F(\rho, \sigma) = \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}.
\]

In the case of pure states \(|a\rangle \) and \(|b\rangle \), the Fidelity takes on the particularly simple form of the overlap \( F(|a\rangle, |b\rangle) = |\langle a|b\rangle| \).

So suppose one wishes to distinguish quantum states using a T-CTC. Only pure state inputs need be considered, so what is required is a bound on the distinguishability of the output states of some T-CTC circuit, \( \rho^a_f \) and \( \rho^b_f \), for which the input states were \(|a\rangle \) and \(|b\rangle \) respectively. Note first that trace distance is bounded by fidelity [14]

\[
D(\rho^a_f, \rho^b_f) \leq \sqrt{1 - F(\rho^a_f, \rho^b_f)^2}.
\]

It is then useful to separate the P-CTC term, seen in Eq. (32) in \( \rho^a_{f,b} \) from the rest. Therefore write \( \rho^a_f = (1 - \lambda^a)\sigma^a + \lambda^a \tau^a \) where \( \tau^a = \text{Tr}[U(|\psi\rangle \langle \psi| \otimes \frac{1}{\sqrt{d}}U^\dagger)] \) and

\[
\lambda^a = \frac{d}{d + \langle \psi | P^a | \psi \rangle} \geq \frac{1}{d + 1}.
\]

Using strong concavity and monotonicity of fidelity under quantum operations [14] it is seen that

\[
F(\rho^a_f, \rho^b_f) \geq \sqrt{\lambda^a \lambda^b F(\tau^a, \tau^b)} \geq \sqrt{\lambda^a \lambda^b F(|a\rangle, |b\rangle)} \geq \frac{1}{d + 1} |\langle a|b\rangle|.
\]

Finally, observe that by Eqs. (36) to (38) \n
\[
D(\rho^a_f, \rho^b_f) \leq \sqrt{1 - \frac{|\langle a|b\rangle|^2}{(d + 1)^2}} \leq 1
\]

with equality to unity only possible if \( |\langle a|b\rangle| = 0 \).

This proves that the output states of a T-CTC circuit are only perfectly distinguishable from one another in a single measurement if the input states were. It leaves
open the question as to whether one may use T-CTCs to probabilistically distinguish non-orthogonal states with more success than standard quantum mechanics.

It is comparatively very simple to observe that T-CTCs are incapable of cloning pure states. A pure state cloning machine always outputs a pure state. Since any T-CTC outputting a pure state can be simulated exactly by an ordinary quantum operation then the no-cloning theorem for T-CTCs is simply a result of the no-cloning theorem in ordinary quantum theory. In exactly the same way, it also immediately follows that T-CTCs are incapable of deleting arbitrary pure states. However, the question as to whether mixed states can be broadcast is left open.

6. Relation to Alternative Theories

Several of the above results for T-CTCs are easily modified to apply to some of the closely related theories introduced in section IV C.

Consider the modification to T-CTCs where, instead of integrating over pure CV states, mixed CV states are integrated over. This represents a class of theories since there is no unique natural choice for the integration measure, but it is still possible to deduce some general properties. Assuming that the chosen integration measure is unitarily invariant, this theory satisfies the basic requirements considered in section IV E. This theory is also only renormalisation non-linear and so continuity follows immediately. It is also possible to show that this theory always defines a unique non-zero $\rho_f$ for every $\rho_i$ and $U$, so that the theory suffers neither dynamical consistency nor information paradoxes.

Now consider the modification to T-CTCs expressed in Eq. (22) where, instead of separating CV and CR systems by a projection, they are separated by a partial trace. This theory satisfies the basic requirements of section IV E and also always defines a unique non-zero $\rho_f$, thus avoiding both dynamical consistency and information paradoxes. It is not, however, renormalisation non-linear but polynomial non-linear.

Finally consider the modification to Eq. (22) where mixed CV states are integrated over, rather than pure CV states. This similarly satisfies the basic requirements of section IV E, and also always defines a unique non-zero $\rho_f$. It is also polynomial non-linear.

The purpose of this discussion is to show that whilst T-CTCs were concentrated on above, the other theories mentioned in section IV C also have reasonable properties and may be worthy of further development.

V. DISCUSSION

Non-linear extensions of quantum theory are subtle since the long standing plurality of co-existing interpretations is broken. When considering time travel this manifests itself in two ways. The first is in the development and motivation of various possible theories: ontological bias will affect decisions made. The second is in using those theories: since mixed states with ontological differences but the same density operator may behave differently, as discussed in section III C. Neither of these issues arise when considering time travel classically, since ontology is generally clear and non-linear evolutions are common.

This uniquely quantum issue has both positive and negative effects on the theories. The way in which quantum theory works allows theories of time travel that do not suffer from the crippling paradoxes that are present classically, but which generally break some of the central structure of quantum theory. Distinguishability of non-orthogonal states, state cloning/deleting, and the spectre of superluminal signalling all present themselves. It also appears that computational power is greatly increased beyond that even of quantum computers.

Having set out the current state of research into the quantum circuit approach to time travel, the shortcomings of D-CTCs and P-CTCs are summarised in section IV B. Most troubling is that both D-CTCs and P-CTCs suffer from paradoxes (information and dynamical consistency respectively), and whilst both may be eliminated by arbitrarily small noise the theories themselves remain paradoxical. The two classes of new theories presented in section IV C were designed to avoid these paradoxes, and hopefully also satisfy many of the other desiderata of section IV B.

Of the new theories, that of T-CTCs has been selected due to its physical motivation. To illustrate the validity of the physical story told in section IV E, consider applying the same reasoning in a classical context.

A CR observer watching a classical time travel circuit sees a CV system in an unknown ontic state $\tau_i$ emerge from the future, interact with a CR system in a known state and then disappear back to the past in the ontic state $\tau_f$. For each $\tau_i$, the observer knows that when it heads back in time it must be found to be in the same state. Whilst in quantum mechanics the probability of finding a system in a given state is given by the transition probability, the corresponding probability classically is either unity or zero: either $\tau_i = \tau_f$ or $\tau_i \neq \tau_f$. So when the CR observer takes a probability distribution over all possible CV states $\tau_i$, the only states to which non-zero probabilities are assigned are precisely those states for which $\tau_i = \tau_f$ after the interaction. What is missing from this account is a way of specifying the exact probabilities to CV states. One might choose to use the principle of indifference, and weight each possibility equally, but this in not necessary.

Of course, this not a cast-iron argument for T-CTCs. A very similar argument could be used to argue in favour of D-CTCs, for example, by demanding exact equality of reduced density operators rather than consistency via the transition probability. In this case, using the principle of indifference would lead to a result that is equivalent to the uniform weighted D-CTCs mentioned in section
But by accepting the interpretation of transition probabilities, and supposing that only pure states are primitive, T-CTCs do have a clear physical motivation. These arguments for the classical model and uniform weighted D-CTCs differ from that for T-CTCs in an important respect. The probabilities assigned to the different possible histories with T-CTCs are physically determined: they are proportional to the transition probabilities. On the other hand, in the above discussion of classical model and D-CTCs using the same narrative there is no physical assignment of probabilities. The use of principle of indifference is an epistemic move, not a physical one.

Consider the desirable features listed in section IV.B in the light of the theory of T-CTCs. By the above argument, feature 1 is satisfied and, so long as instantaneous collapse is not assumed, feature 2 is also satisfied. Features 3, 4, and 5 are satisfied without condition, as discussed in section IV.E. Feature 6 is partially satisfied, in that both $\tau$ and $\rho_f$ are considered ontologically pure, but since a proper mixture is taken over so many possibilities, the mathematical form of either is very rarely pure. In section IV.E.5 it is shown that feature 7 is satisfied, although the related questions of distinguishing non-orthogonal states with greater-than-quantum fidelity and broadcasting of mixed states are left open.

Does this mean that T-CTCs are a better model for quantum mechanics with time travel or, more specifically, for quantum mechanics in the presence of CTCs? Not necessarily. The physical motivation for T-CTCs is far from a “first principles” argument and there is still the question as to how the state projection occurs. However, both D-CTCs and P-CTCs have incomplete physical motivations and both leave questions as to how, and when, exactly a proposed physical change occurs—all three theories share the unobservable dynamical ambiguity.

The theory of D-CTCs is weakened by the failure of the equivalent circuit model to hold up to scrutiny, since that model claimed to provide motivation for the D-CTC consistency condition and the maximum entropy rule. However, the discussion of section III.A.1 shows that there is evidence that the maximum entropy rule may nonetheless arise as a result of noise. It is an interesting open problem to see if this can be shown to be true generally, but it would not rid the theory of the fact that the uniqueness ambiguity is essentially present.

What has been comprehensively shown is that there is a whole landscape of other theories out there. The quantum circuit approach to quantum mechanics with time travel may be very attractive, in that it abstracts away from knotty problems with spacetime geometry or any other exact mechanism for time travel, but it is perhaps too general for the problem at hand. In order to identify a more robustly physical solution to quantum mechanics with time travel it may be necessary to use a different approach, such as path integral or field theoretic ideas. Alternatively, by very carefully committing to a specific ontology for quantum mechanics it may be possible to identify the corresponding theory of time travel. When non-linearity is present vagueness on this point is problematic.

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Of course, any Everettian observer could still choose to be ignorant, but this does not give rise to mixed states in the same way. To clarify, consider how proper mixtures arise in objective collapse models. An observer \( \mathcal{O} \) sets up a measurement \( \mathcal{M} \) on a system \( \mathcal{S} \), but chooses (by not looking, or what have you) to remain ignorant of the outcome of \( \mathcal{M} \). In an objective collapse model, \( \mathcal{O} \) now knows (assuming sufficient understanding of \( \mathcal{S} \), \( \mathcal{M} \), and quantum theory) that the state of the universe has now collapsed into one of multiple possible states with corresponding probabilities; this ensemble forms a proper mixture which \( \mathcal{O} \) uses to describe \( \mathcal{S} \) after \( \mathcal{M} \). In an Everettian model, however, \( \mathcal{O} \) now knows that the universal state has evolved into some macroscopic superposition, the only uncertainty is about which branch of this \( \mathcal{O} \) will experience upon discovering the outcome of \( \mathcal{M} \). For more on this see, for example, Ref. [4, especially §3.2].

One needs to be careful with expressions such as these since different states belong on different Hilbert subspaces, while \( U \) acts on the entire Hilbert space. Throughout this paper, \( |\phi\rangle \) is used for a pure state on the CV Hilbert space and \( \tau \) used for a mixed state on the same space, while \( |\psi_{i,f}\rangle \) and \( \rho_{i,f} \) are both used for pure and mixed states respectively on the CR Hilbert space. It is hoped that this will avoid confusion as to how such expressions fit together.

That is to say, interpreting the density operator \( \tau \) as a proper mixture, the probability of some state \( \rho \) being found in an eigenstate of \( \tau \) and \( \tau \) being a realisation of that eigenstate is given by \( \text{Tr}[\rho \tau] \). Equivalently, this is to say that if a state \( \rho \) is measured with some POVM which has an element \( \tau \) then the probability of getting the corresponding measurement result is \( \text{Tr}[\rho \tau] \). It is in these senses that this generalises the transition probability.

The proof that \( \rho_f \) is non-zero follows by showing that the integrand is positive semi-definite and that there exist some \( \tau \) for which the integrand is non-zero. Importantly, for this to go through an assumption does need to be made about the positivity of the chosen measure.

The proof that \( \rho_f \) is always non-zero follows similarly to the previous case, by proving that the integrand is always positive semi-definite and there always exist CV states \( |\phi\rangle \) for which both the integrand and \( d[|\phi\rangle] \) are non-zero.

Subject to reasonable assumptions about the measure used.