Amenability of bounded automata groups on infinite alphabets

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Abstract
We study the action of groups generated by bounded activity automata with infinite alphabets on their orbital Schreier graphs. We introduce an amenability criterion for such groups based on the recurrence of the first-level action. This criterion is a natural extension of the result that all groups generated by bounded activity automata with finite alphabets are amenable. Our motivation comes from the investigation of iterated monodromy groups of entire transcendental functions in holomorphic dynamics.

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1 | INTRODUCTION

One of the key geometric properties of a group is whether or not it is amenable. Nonamenability is the group-theoretic ingredient in the Banach–Tarski paradox. Groups such as abelian groups or groups with subexponential growth are amenable, while free groups are not. For some groups, amenability is easy to decide, while for others this is a difficult issue, and much work has been done on it (see, e.g., [9]).

Self-similar groups provide many examples of “exotic” amenable groups. The Grigorchuk group [5] was the first example of a group of intermediate growth. Groups of intermediate growth are always amenable, but not elementary amenable (see [4]). The basilica group is amenable [3], but not elementary subexponentially amenable [6].
Both the Grigorchuk group and the basilica group are examples of automata groups on a two-letter alphabet of bounded activity growth. They fit into the hierarchy of polynomial activity growth introduced in [16], where both finite and infinite alphabets are considered. Under certain assumptions (which are always satisfied for finite alphabets), these groups do not contain free subgroups (see [17]). This raises the question whether groups generated by polynomial activity growth automata are in fact amenable.

For finite alphabets, it is shown in [2] that every group generated by a bounded activity automaton is amenable.

A large family of such groups that arise naturally in holomorphic dynamics are iterated monodromy groups of postcritically finite polynomials [11]. Furthermore, in [1], it is shown that automata groups on finite alphabets of linear activity growths are amenable. The techniques of [2] and [1] have been conceptualized in [8].

In this paper and [14], we study the iterated monodromy groups (IMGs) of postsingularly finite entire transcendental functions. This is motivated by a surge of increased interest in transcendental dynamics in recent years. It is thus natural to investigate the groups arising from transcendental IMGs and to explore similarities or new phenomena. Indeed, while polynomial IMGs are always amenable, it turns out that this is not so in the transcendental world. One of our main results, shown in [14], is the following explicit criterion on whether the IMG of an entire transcendental function is amenable or not.

**Theorem 1** (Amenability of transcendental IMGs). Let \( f \) be a postsingularly finite entire transcendental function. Then, the iterated monodromy group of \( f \) is amenable if and only if the monodromy group of \( f \) is amenable.

This is a nontrivial distinction as there are entire transcendental functions with amenable IMGs such as the exponential family [15]. On the other hand, there are entire transcendental functions with monodromy group given by the free product \( C_2 \ast C_2 \ast C_2 \) (see appendix of [14]) and thus with nonamenable IMGs.

The main theorem of this paper provides the key group-theoretic part in the proof of Theorem 1; it is also of interest in its own right.

Main Theorem Amenability of bounded automata groups. Let \( X \) be a countably infinite set. Let \( P \) be an amenable subgroup of \( \text{Sym}(X) \). Suppose that the action of \( P \) on \( X \) is recurrent. Then, \( \text{Aut}_{f.s.}^\infty(X^*; P) \) is amenable.

See Section 2 for a precise definition of \( \text{Aut}_{f.s.}^\infty(X^*; P) \); it is roughly the group of bounded activity automata where every first-level action is in \( P \).

In [14], we show that iterated monodromy groups of postsingularly finite entire transcendental functions are given by bounded activity automata on countably infinite alphabets, so that we can apply the main theorem to deduce Theorem 1.

We note that Theorem 1 is our main motivation for the main theorem, but this paper does not logically depend on [14].

**Overview.** In Section 2, we start by introducing self-similar groups on infinite alphabets and related concepts, such as the space of ends. We continue in Section 3 with a discussion of recurrent random walks and how to pass from a recurrent action on the alphabet to a recurrent action of a bounded activity group on the space of ends. This will be a key ingredient to invoke the
amenability criterion of [8] in Section 4 to prove the main theorem. In Section 5, we briefly discuss
the forthcoming paper and further related open questions.

This paper is based on the third chapter of the author’s PhD thesis [13].

2 REGULAR TREES

In this section, we introduce self-similar groups and other relevant concepts and fix the notation.
We mostly follow the notation of [10]. See also [16, 17] for self-similar groups on infinite alphabets.

Definition 2.1. Let $X$ be a countably infinite set. The standard $X$-regular tree has as vertex set
$X^*$, the set of finite words in $X$. Its root is the empty word $\emptyset$. Its edges are all pairs $(v, vx)$ for $v \in X^*, x \in X$. By abuse of notation, we denote the standard $X$-regular tree also as $X^*$, and we denote
by Aut($X^*$) the group of rooted tree automorphisms of $X^*$. We denote the identity of Aut($X^*$) by 1.

For $v \in X^*$, let $v|X^*$ be the subtree of all descendants of $v$. If $g \in$ Aut($X^*$), $v \in X^*$, there is a
unique $g_{|v} \in$ Aut($X^*$) given by $g(vw) = g(v)g_{|v}(w)$. This is called the section of $g$ along $v$.

A set $S \subset$ Aut($X^*$) is called self-similar if it is closed under taking sections, that is, $g_{|v} \in S$ for
all $g \in S, v \in X^*$. We are mainly interested in self-similar groups, that is, subgroups $G \subset$ Aut($X^*$)
that are self-similar as sets.

For $g \in$ Aut($X^*$), we denote by $\alpha_n(g) \in \mathbb{N} \cup \{\infty\}$ the number of words $v$ of length $n$ for
which the section $g_{|v}$ is not trivial. We denote by Aut$_{\text{fin.}}(X^*)$ the set of automorphisms with finitely many
nontrivial sections on every level, that is, the set of automorphisms $g \in$ Aut($X^*$) with $\alpha_n(g) \in \mathbb{N}$
for all $n \in \mathbb{N}$. If $g \in$ Aut$_{\text{fin.}}(X^*)$ has a $n \in \mathbb{N}$ so that $g_{|v} = 1$ for all $v \in X^n$, we say that $g$ is finitary.
If $g \in$ Aut$_{\text{fin.}}(X^*)$ has a $c \in \mathbb{N}$ so that $\alpha_n(g) \leq c$ for all $n$, we say that $g$ has bounded activity.

We denote by Aut$_f(X^*)$ the set of automorphisms with bounded activity, and by Aut$_{f,r}(X^*)$ the
set of finitary automorphisms.

We also have maps $\rho_n \colon$ Aut($X^*$) $\to$ Sym($X^n$), which are induced by the action of
Aut($X^*$) on the $n$-th level of $X^*$. Let $P$ be a subgroup of Sym($X$). Let Aut($X^*$; $P$) denote the set of automorphisms such that $\rho_1(g_{|v}) \in P$ for all $v \in X^*$. We denote by Aut$_{\text{fin.}}(X^*$; $P$), Aut$_f(X^*$; $P$), Aut$_{f,r}(X^*$; $P$) the intersections of Aut$_{\text{fin.}}(X^*)$, Aut$_f(X^*)$, Aut$_{f,r}(X^*)$
with Aut($X^*$; $P$), respectively.

Since we consider infinite alphabets, let us fix notations for the two versions of wreath products.

Notation 2.2. Let $A$ and $B$ be groups, $L$ be a set with an $A$-left action. The unrestricted wreath product
$(\prod_{l \in L} B) \rtimes A$, which are induced by the action of
Aut($X^*$) on the $n$-th level of $X^*$. Let $P$ be a subgroup of Sym($X$). Let Aut($X^*$; $P$) denote the set of automorphisms such that $\rho_1(g_{|v}) \in P$ for all $v \in X^*$. We denote by Aut$_{\text{fin.}}(X^*$; $P$), Aut$_f(X^*$; $P$), Aut$_{f,r}(X^*$; $P$) the intersections of Aut$_{\text{fin.}}(X^*)$, Aut$_f(X^*)$, Aut$_{f,r}(X^*)$
with Aut($X^*$; $P$), respectively.

We will mainly work with the restricted wreath product. We denote the right factor embedding
$A \to B \rtimes_l A$ by $\iota$, and by $b@l$ the image of $b$ under the embedding of $B$ into the component indexed
by $l \in L$.

If additionally $M$ is a set with a $B$-left action, we will consider the action of $B \rtimes_l A$ on $L \times M$
given by $((b_I)_{I \in L}, a)(l, m) = (a(l), b_I(m))$.

For a subgroup $P$ of Sym($X$), we denote the $n$-th iterated restricted wreath product (along $X$)
by $P_n$. So $P_1 = P$ and $P_{n+1} = P_n \rtimes_X P$. Note that if $P$ is amenable, then all $P_n$ are amenable, as
amenability is stable under taking restricted wreath products. With this in mind, we have the following lemma.

**Lemma 2.3.**

\[ \text{Aut}(X^*; P) \to \text{Aut}(X^*; P) \wr_X P \]

\[ g \mapsto (x \mapsto g_{|x}, \rho_1(g)) \]

is an isomorphism of groups. It restricts to isomorphisms

\[ \text{Aut}_{\text{fin}}(X^*; P) \cong \text{Aut}_{\text{fin}}(X^*; P) \wr_X P, \]

\[ \text{Aut}_{\text{fin}}(X^*; P) \cong \text{Aut}_{\text{fin}}(X^*; P) \wr_X P, \]

\[ \text{Aut}_F(X^*; P) \cong \text{Aut}_F(X^*; P) \wr_X P. \]

For the first line, see, for example, [16]. By iteration, we also get isomorphisms

\[ \text{Aut}_{\text{fin}}(X^*; P) \to \text{Aut}_{\text{fin}}(X^*; P) \wr_{X_n} P_n, \]

\[ g \mapsto (v \mapsto g_{|v}, \rho_n(g)), \]

and \( \text{Aut}_{\text{fin}}(X^*; P) \cong \text{Aut}_{\text{fin}}(X^{n*}; P_n). \)

### 2.1 Action on the space of ends \( X^\omega \)

We will also use the action of \( \text{Aut}(X^*) \) on the space of ends of \( X^\omega \). The set of ends of \( X^\omega \) can be identified with \( X^\omega \), the set of right infinite words in \( X \). For a word \( v \in X^n \), the *open cylinder set* \( C(v) = \{vw : w \in X^\omega\} \) is the set of right infinite words that have \( v \) as a prefix. The open cylinder sets form a basis of the end topology on \( X^\omega \). Since \( X \) is countably infinite, \( X^\omega \) is homeomorphic to the Baire space \( \mathbb{N}^\mathbb{N} \), in particular \( X^\omega \) is Hausdorff, but not locally compact. The action of \( \text{Aut}(X^*) \) on \( X^\omega \) is faithful, so we can also think of elements of \( \text{Aut}(X^*) \) as homeomorphisms on \( X^\omega \).

We will use the language of *germs*: These are equivalence classes of pairs \((g, w) \in \text{Aut}(X^*) \times X^\omega\), where \((g, w) \sim (h, w')\) if \( w = w' \) and \( g \) and \( h \) agree on a neighborhood of \( w \). Since we only consider germs of \( \text{Aut}(X^*) \), and \( \{C(v) : v \text{ is a prefix of } w\} \) forms a neighborhood basis of \( w \), \((g, w) \sim (h, w')\) if and only if \( w = w' \), \( g(w) = h(w) \) and \( g_{|v} = h_{|v} \) for some prefix \( v \) of \( w \). Given a subgroup of \( G \) of \( \text{Aut}(X^*) \), its associated *groupoid of germs* \( \mathcal{G} \) has as object set \( X^\omega \) and as morphisms germs represented by pairs \((g, w) \in G \times X^\omega\), going from \( w \) to \( g(w) \), with composition \([(g, w)] \circ [(h, w')] = [(gh, w')]\) under the condition \( h(w') = w \). For an end \( w \in X^\omega \), we will be particularly interested in the *isotropy group* \( \mathcal{G}_w \), the group of germs going from \( w \) to itself. We denote by \( \mathcal{T} \) the groupoid of germs of tail equivalences, that is germs of the form \((g, w)\) with \( g_{|v} \) trivial for some prefix \( v \) of \( w \). Given a groupoid of germs \( \mathcal{H} \), we denote by \([[[H]]]\) the set of homeomorphisms of \( X^\omega \) whose germs belong all to \( \mathcal{H} \).
If \( w \in X^\omega \) can be factored as \( w = vu \) with \( v \in X^n, u \in X^\omega \), we say that \( u \) is the \( n \)-tail of \( w \). If \( w, w' \in X^\omega \) have the same \( n \)-tail, we say that \( w \) and \( w' \) are \( n \)-tail equivalent. We say that \( w \) and \( w' \) are tail equivalent (or cofinal) if they are \( n \)-tail equivalent for some \( n \). The \( n \)-tail equivalence class of \( w \) is denoted by \( T_n(w) \) and \( T(w) = \bigcup_{n \in \mathbb{N}} T_n(w) \) is the cofinality class of \( w \).

**Lemma 2.4.** Let \( g \in \text{Aut}_B(X^*) \). There are only finitely many \( w \) such that the germ \((g, w)\) is not in \( \mathcal{T} \). Moreover, \( g \in \text{Aut}_B(X^*) \) if and only if \( g \in [[\mathcal{T}]] \). If \((g, w)\) is in \( \mathcal{T} \), then \( w \) and \( g(w) \) are cofinal.

*Proof.* The ends where the germ of \( g \) is not in \( \mathcal{T} \) are those where the sections along all prefixes are nontrivial. So they can be identified with the projective limit \( \lim\limits_{\leftarrow} \left\{ v \in X^n : g|_v \neq 1 \right\} \). Since \( g \in \text{Aut}_B(X^*) \), the sets in the limit construction have uniformly bounded cardinality. Hence the projective limit is also finite. This proves the first claim.

For the second claim, we already observed that \( \text{Aut}_B(X^*) \subset [[\mathcal{T}]] \). In the other direction, if \( g \) is also in \([[[\mathcal{T}]]]\), then the projective limit must be empty. As all sets in the limit construction are finite, by König's lemma one of the sets in the limit construction must be empty. So \( g \) is also in \( \text{Aut}_B(X^*) \). This proves the second claim. For the last claim, if \((g, w)\) is in \( \mathcal{T} \) then \( w \) factors as \( vu \) with \( g|_v \) trivial, so \( g(w) = g(vu) = g(v)g|_u(u) = g(v)u \), so \( w \) and \( g(w) \) are cofinal. □

In particular, we have shown that \( \text{Aut}_B(X^*) \cap [[\mathcal{T}]] = \text{Aut}_B(X^*) \). In fact, the proof shows \( \text{Aut}_{\text{fin.}}(X^*) \cap [[\mathcal{T}]] = \text{Aut}_B(X^*) \), as we only need finiteness of every set in the limit construction.

### 2.2 Bounded finite state automorphisms

**Definition 2.5.** An automorphism \( g \in \text{Aut}(X^*) \) is called a finite state automorphism if the set of sections \( \{ g|_v : v \in X^* \} \) is finite.

We denote by \( \text{Aut}_{f.s.}(X^*; P) \) the subgroup of finite state automorphisms in \( \text{Aut}_B(X^*; P) \). Note that every \( g \in \text{Aut}_B(X^*) \) is a finite state automorphism.

A nontrivial automorphism \( g \in \text{Aut}_{\text{fin.}}(X^*) \) is called directed if there is a word \( v \in X^n \) with \( g|_v = g \) and \( g|_u \in \text{Aut}_B(X^*) \) for all \( u \in X^n, u \neq v \).

**Remark 2.6.** Finite state automorphisms are exactly the automorphisms that can be defined via finite state automata, see the discussion in [17, section 2.2] for infinite alphabets. The notion of activity translates to automata such that bounded activity finite state automorphisms are those defined by a finite state automaton of bounded activity. Multiple finite state automata can define the same finite state automorphism, so we will use the language of finite state automorphisms in the rest of this paper for simplicity.

A directed automorphism has bounded activity growth. We will use the following structural result about finite state automorphisms of bounded activity growth, see [16, Lemma 17] or [10, Proposition 3.9.11] for finite alphabets, and [17, section 2.2] for the extension to infinite alphabets.

**Lemma 2.7.** Let \( g \in \text{Aut}_{f.s.}(X^*) \) be a finite state automorphism. Then, there exists a level \( n \) such that for all \( v \in X^n \), \( g|_v \) is either directed or finitary.
3 | RANDOM WALKS

3.1 | Potential-theoretic background

We will use the potential-theoretic setting as in [18, section I.2]: There are many ways to define networks, it will be convenient for us to start from a conductance function, as we will work mostly in that language:

**Definition 3.1.** Let \( X \) be a countably infinite set, let \( a : X \times X \to [0,\infty) \) be a symmetric function such that the sum \( m(x) := \sum_{y \in X} a(x, y) \) is positive and finite for all \( x \in X \). Let \( E \subset X \times X \) be the support of \( a \), that is, \((x,y) \in E \) if and only if \( a(x,y) > 0 \). We think of \( (X,E) \) as a simply undirected graph with possible loops. Consider the function \( r : E \to (0,\infty) \) given by \( r(e) = 1/\sum_{(x,y) \in E} a(x,y) \). If the graph \( (X,E) \) is connected, then we call the triple \( \mathcal{N} := (X,E,r) \) the associated network to \( X \) and \( a \). We call \( r(e) \) the resistance of \( e \), \( a(e) \) the conductance of \( e \), and \( m(x) \) the total conductance at \( x \). The associated Markov chain on \( X \) has transition probabilities \( p(x,y) = a(x,y)/m(x) \). We say that \( \mathcal{N} \) is recurrent if the associated Markov chain is recurrent for all starting points \( x \in X \).

**Notation 3.2.** Let \( (X,E,r) \) be a network and \( Y \) be a subset of \( X \). We denote by \( \chi_Y : X \to \{0,1\} \) the characteristic function of \( Y \). Moreover, the edge boundary \( \partial^e Y \) is given by the set of edges between \( Y \) and \( X \setminus Y \), and the vertex boundary \( \partial^v Y \) is the set of vertices in \( X \setminus Y \) that share an edge with an element of \( Y \).

**Example 3.3.** If \( G \) is a group acting transitively on \( X \), and \( \lambda \) is a finitely supported symmetric finite measure on \( G \), such that the support of \( \lambda \) generates \( G \), then we can define a network on \( X \) with conductances

\[
a(x,y) = \sum_{g(x)=y} \lambda(g).
\]

In this case, the total conductance at every point is equal to the total mass of \( \lambda \).

In particular, if \( S \) is a finite symmetric generating set of \( G \), we can take \( \lambda \) to be the counting measure on \( S \) and obtain as the network the Schreier graph \( \Gamma(G,S,X) \). In the Schreier graph, there is an edge from \( x \) to \( y \) if and only if there is an \( s \in S \) with \( s(x) = y \). Note that by following the convention of [18], \( \Gamma(G,S,X) \) can have loops, but no parallel edges, but if there are multiple generators sending \( x \) to \( y \), then the edge \((x,y)\) will have the appropriate higher conductance. By the uniform random walk on Schreier graphs, we mean the random walk arising from this construction.

It will be also convenient to consider the reduced Schreier graph \( \bar{\Gamma}(G,S,X) \) that is obtained from \( \Gamma(G,S,X) \) by removing all loops, and assigning constant conductance to each edge. The resulting random walk is the simple random walk on the reduced Schreier graph. As \( S \) is finite, it follows from [18, Corollary I.3.5] that the uniform random walk is recurrent if and only if the simple random walk is. While the uniform random walk is closer connected to random walks induced by group actions, the simple random walk will have its use in shorting.

If the action of \( G \) on \( X \) is not transitive, for \( x \in X \) the orbital Schreier graph of \( x \) is the Schreier graph on the orbit of \( x \).

We are mostly interested in the space \( D(\mathcal{N}) \) of functions \( f : X \to \mathbb{R} \) with finite Dirichlet energy

\[
D(f) = \sum_{e \in E} a(e) \left( f(e^+) - f(e^-) \right)^2.
\]

For any choice of base point \( o \), \( D(\mathcal{N}) \) is a Hilbert space with
norm \( \|f\|_{D,\rho}^2 = D(f) + \|f(o)\|^2 \). All choices of \( o \) give equivalent norms, so there is a well-defined topology on \( D(\mathcal{N}) \), so that \( f_n \) converges to \( f \) if and only if \( \lim_n D(f_n - f) = 0 \) and \( f_n \) converges to \( f \) point-wise.

Let \( D_0(\mathcal{N}) \) be the closure of functions with finite support in \( D(\mathcal{N}) \). By [18, Theorem I.2.12], the random walk on \( \mathcal{N} \) is recurrent if and only if \( \chi_X \in D_0(\mathcal{N}) \). We also use \( D_0(\mathcal{N}) \) to get the following shorting criterion.

**Lemma 3.4** [18, Theorem I.2.19]. Let \( X = \bigcup_{i \in I} X_i \) be a partition of \( X \) such that \( \chi_{X_i} \in D_0(\mathcal{N}) \) for all \( i \in I \). Consider the **shorted network** \( \mathcal{N}' \) with vertex set \( I \) and conductivity \( a'(i,j) = \sum_{x \in X_i, y \in X_j} a(x,y) \) for \( i \neq j \), \( a'(i,i) = 0 \). If \( \mathcal{N}' \) is recurrent then so is \( \mathcal{N} \).

As a special case we want to mention the Nash-Williams criterion [12]:

**Lemma 3.5.** Let \( Y_0 \subset Y_1 \subset ... \) be an increasing chain of subsets of \( X \) with \( \chi_{Y_i} \in D_0(\mathcal{N}) \), such that \( \partial^s Y_i \subset Y_{i+1} \), and \( \bigcup Y_i = X \). Let \( a'_i := \sum_{e \in \partial^s Y_i} a(e) \). If \( \sum \frac{1}{a_i} = \infty \), then \( \mathcal{N} \) is recurrent.

**Proof.** Let \( X_0 = Y_0, X_{n+1} = Y_{n+1} \setminus Y_n \). Then also the characteristic functions of the \( X_i \) are in \( D_0(\mathcal{N}) \), so we can apply the previous lemma to short. The resulting shorted network is the nearest neighbor walk on \( \mathbb{N} \) with conductances \( a'_i \), so by [18, Paragraph I.2.16], the shorted network is recurrent, hence \( \mathcal{N} \) is also recurrent. \( \square \)

We will also use the following lemma.

**Lemma 3.6.** Let \( \mathcal{N}' = (X,E,r) \) be a network, \( Y \subset X \) with \( \partial^s Y \) finite. Suppose \( \mathcal{N}' = (Y,E',r') \) is a network on \( Y \) obtained from \( \mathcal{N} \) by restricting to \( Y \) and adding and removing finitely many edges and changing finitely many resistances.

Suppose \( \mathcal{N}' \) is a recurrent network. Then \( \chi_Y \) is in \( D_0(\mathcal{N}) \).

**Proof.** Since \( \mathcal{N}' \) is recurrent, \( \chi_Y \) is in \( D_0(\mathcal{N}') \). So there is a sequence \( f_n : Y \to \mathbb{R} \) such that \( \lim_n D_{\mathcal{N}'}(f_n - \chi_Y) = 0 \) and \( f_n \to 1 \) point-wise on \( Y \).

We extend \( f_n \) to \( X \) by 0. Then \( D_{\mathcal{N}'}(f_n - \chi_Y) \) and \( D_{\mathcal{N}'}(f_n - \chi_Y) \) differ in only finitely many summands, and these go to 0 by point-wise convergence of the sequence \( f_n \). So we have \( \lim_n D_{\mathcal{N}'}(f_n - \chi_Y) = 0 \) and thus \( \chi_Y \in D_0(\mathcal{N}) \). \( \square \)

### 3.2 Recurrence on orbital Schreier graphs

**Definition 3.7.** Let \( A \) be a group, \( L \) a left \( A \)-set. We say that the action of \( A \) on \( L \) is recurrent if for all finitely supported symmetric measures \( \lambda \) on \( A \), the random walk on \( L \) induced by \( \lambda \) is recurrent for all starting points \( l_0 \in L \).

**Remark 3.8.** If \( A \) is finitely generated, it is enough to show this for one finitely supported symmetric measure whose support generates \( A \). If \( S \) is a finite symmetric generating set of \( A \), it is enough to consider the uniform random walk on the Schreier graph \( \Gamma(A,S,L) \) or the simple random walk on \( \bar{\Gamma}(A,S,L) \). See, for example, [8, Lemma 6]. With this definition it is also clear that recurrent actions are closed under taking subgroups.
Lemma 3.9. Let $A, B$ be groups. Suppose that $L$ is a left $A$-set and $M$ is a left $B$-set such that the actions are both recurrent. Then the action of $B \wr L A$ on $L \times M$ is also recurrent.

Proof. Let us first reduce to the case where $A$ and $B$ are both finitely generated and both actions are transitive:

Let $(l, m) \in L \times M, \lambda$ be a symmetric finitely supported measure on $B \wr L A$. Then there are finitely generated subgroups $A' \subset A, B' \subset B$ such that $\text{supp}(\lambda) \subset B' \wr L A'$. So without loss of generality let $A$ and $B$ be finitely generated. Let $L'$ be the orbit of $l$ under $A$. Then we have a quotient map $\pi : B \wr L A \to B \wr L' A$ and we can replace $\lambda$ by $\pi^*(\lambda)$ to assume without loss of generality that the action of $A$ on $L$ is transitive. We can easily replace $M$ with the orbit of $m$ under $B$.

We can now assume that $S$ and $T$ are finite symmetric generating sets of $A$ and $B$, respectively, and both actions are transitive. Instead of showing recurrence for arbitrary $\lambda$, we can now fix a preferred generating set of $B \wr L A$ and show recurrence of the simple random walk on the associated reduced Schreier graph.

Fix any base point $l_0 \in L$. We take as our generating set of $B \wr L A$ the union of $\epsilon(S)$ and $T@l_0 = \{t@l_0 : \ t \in T\}$ (recall our notation 2.2 for $t : A \to B \wr L A$ and the embedding of $B$ at $l_0$), let $\mathcal{N}$ be the resulting network on the reduced Schreier graph with constant resistance. The generating set acts now as follows:

$$t(s)(l, m) = (s(l), m),$$
$$t@l_0(l_0, m) = (l_0, t(m)),$$
$$t@l_0(l', m) = (l', m) \text{ for } l' \neq l_0.$$

We use the shorting criterion by partitioning $L \times M = \bigcup_{m \in M} L \times m$. For every $m$, we see that the induced subgraph on $L \times m$ is isomorphic to the reduced Schreier graph $\hat{\Gamma}(A, S, L)$. Since the action of $A$ on $L$ is recurrent, the simple random walk on $\hat{\Gamma}(A, S, L)$ is recurrent. Moreover $\partial_e(L \times m)$ is a finite collection of edges at $(l_0, m)$, so by Lemma 3.6, we obtain $\chi_{L \times m} \in D_0(\mathcal{N})$. The shorted network with respect to the partition is isomorphic to the reduced Schreier graph $\hat{\Gamma}(B, T, L)$, so it is also recurrent. By Lemma 3.4, the network $\mathcal{N}$ is then also recurrent. □

Lemma 3.10. Let $G$ be a finitely generated subgroup of $\text{Aut}_{\text{fin}}(X^\omega)$. Assume that the action of $G$ on every finite level is recurrent. Then the action of $G$ on every component of the Schreier graph of the action of $G$ on $X^\omega$ is recurrent.

Proof. Let $S$ be a finite symmetric generating set of $G$. Let $K > 0$ be a uniform bound on $\alpha_n(s)$ for all $n \in \mathbb{N}, s \in S$. Let $\Omega$ be a component of the Schreier graph on $X^\omega$. Let $\mathcal{N}$ be the network associated with the uniform random walk on $\Omega$.

Let $E$ be the set of edges in $\mathcal{N}$ that go between different cofinality classes. By Lemma 2.4, $E$ is finite. Since $\Omega$ is connected, its vertex set must be contained in finitely many cofinality classes $C_1, \ldots, C_k$. Choose representatives $w_i \in C_i \cap \Omega$.

We claim that the conductance of $\partial_e T_n(w_i)$ is uniformly bounded by $K |S|$ for every $n \in \mathbb{N}$: in fact, if $u$ is the $n$-tail of $w_i$, then $\partial_e T_n(w_i)$ (with multiplicities) can be identified with the set $\{(s, v) \in S \times X^n : s|_u(u) \neq u\}$. This set is contained in $\{(s, v) \in S \times X^n : s|_u \neq 1\}$, so the bound is clear.
Since \( \Omega \) is a connected component of the Schreier graph, the conductance of \( \partial^e(T_n(w_i) \cap \Omega) \) is also uniformly bounded by \( K |S| \) and \( T_n(w_i) \cap \Omega \) has only finitely many components. Each such component may be viewed as a subnetwork of the (recurrent) random walk of \( G \) on level \( n \), so the random walk on each component is recurrent by [18, Corollary 1.2.15], and finally by Lemma 3.6, their characteristic functions are in \( \mathcal{R}_0(\mathcal{N}) \).

Let \( Y_n := \bigcup_{1 \leq i \leq k} T_n(w_i) \cap \Omega \). Then, \( \chi_{Y_n} \) is the finite sum of the characteristic functions of components of \( T_n(w_i) \cap \Omega \), so we obtain \( \chi_{Y_n} \in \mathcal{D}_0(\mathcal{N}) \). Also, \( \partial^e Y_n \subset \bigcup_{1 \leq i \leq k} \partial^e(T_n(w_i) \cap \Omega) \), so the conductance of \( \partial^e Y_n \) is uniformly bounded by \( kK |S| \).

We can now take a subsequence \( Y_{n_i} \) such that \( \partial Y_{n_i} \) is properly contained in \( Y_{n_{i+1}} \). By applying Lemma 3.5 to the sequence \( Y_{n_i} \), the random walk on \( \mathcal{N} \) is recurrent. \( \square \)

4 AMENABILITY OF GROUPS GENERATED BY BOUNDED ACTIVITY AUTOMATA

In this section, we will prove the main theorem. We will use the following criterion.

**Theorem 2** (Theorem 11 in [8]). Let \( G \) be a finitely generated group of homeomorphisms of a topological space \( Y \), and \( \mathcal{H} \) be its groupoid of germs. Let \( \mathcal{H}_T \) be a groupoid of germs of homeomorphisms of \( Y \). Suppose that the following conditions hold:

1. The group \( [[\mathcal{H}]] \cap G \) is amenable.
2. For every generator \( g \in G \), the germ of \( g \) at \( y \) belongs to \( \mathcal{H} \) for all but finitely many \( y \in Y \). We say that \( y \in Y \) is singular if there exists \( g \in G \) such that \( (g, y) \notin \mathcal{H} \). If \( Y \) is singular.
3. For every singular point \( y \in Y \), the action of \( G \) on the orbit of \( y \) is recurrent.
4. The groups of germs \( \mathcal{G}_y \) are amenable for all \( y \in Y \).

Then the group \( G \) is amenable.

**Remark 4.1.** This is almost Theorem 11 in [8], but we weakened the condition (1) from \( [[\mathcal{H}]] \cap G \) being amenable to \( [[\mathcal{H}]] \cap G \) being amenable. In the last step of the original proof, a certain subgroup \( K \) of \( G \) is expressed as an extension of a subgroup of \( [[\mathcal{H}]] \) by a direct product of (finitely many) isotropy groups \( \mathcal{G}_y \), and it remains to show that \( K \) is amenable. But \( K \subset G \) is in fact an extension of a subgroup of \( [[\mathcal{H}]] \cap G \) by a direct product of isotropy groups \( \mathcal{G}_y \), so it is amenable also under the weakened condition (1) together with condition (4). The original proof only used the weaker condition.

**Proof of the main theorem.** In order to show amenability of \( \operatorname{Aut}^{f.s.}_B(X^*;P) \), it is enough to show amenability of every finitely generated subgroup of \( \operatorname{Aut}^{f.s.}_B(X^*;P) \). So let \( G = \langle S \rangle \) be a finitely generated subgroup of \( \operatorname{Aut}^{f.s.}_B(X^*;P) \). We will use Theorem 2 with \( G \) acting on \( X^\omega \), and \( \mathcal{H} \) being the groupoid of tail equivalences \( \mathcal{T} \). We will show that each condition of Theorem 2 is satisfied.

1. The group \( [[\mathcal{T}]] \cap G \) is amenable: In fact \( [[\mathcal{T}]] \cap \operatorname{Aut}^{f.s.}_B(X^*;P) = [[\mathcal{T}]] \cap \operatorname{Aut}_\rho(X^*;P) = \operatorname{Aut}_\rho(X^*;P) \). This follows easily from Lemma 2.4. Now \( \operatorname{Aut}_\rho(X^*;P) \) is the direct limit of iterated wreath products of \( P \), so it is amenable, hence \( [[\mathcal{T}]] \cap G \) is amenable.
2. This follows directly from Lemma 2.4.
3. By inductive application of Lemma 3.9, we see that the action of the \( n \)-th iterated wreath product of \( P \) on \( X^n \) is recurrent. The image of \( G \) under \( \rho_n \) lies in \( P_n \), hence by Remark 3.8,
the action of $G$ on every level is recurrent. Since $G \subset \text{Aut}_{f,s}^{f.s.}(X^*;P) \subset \text{Aut}_{f,s}(X^*)$, we get by Lemma 3.10 that $G$ acts recurrently on all orbital Schreier graphs.

(4) We encapsulate the proof in the following lemma. 

**Lemma 4.2.** Let $P$ be an amenable subgroup of $\text{Sym}(X)$. Let $G$ be a finitely generated subgroup of $\text{Aut}_{f,s}^{f.s.}(X^*;P)$. Then the group of germs $G_w$ is amenable for every $w \in X^\omega$.

**Proof.** By replacing $X$ with $X^N$ and $P$ with $P_N$ for sufficiently large $N \in \mathbb{N}$ and possibly enlarging the group $G$ itself, we can use Lemma 2.7 to assume without loss of generality the following:

- $G$ has a symmetric generating set $S$ that is self-similar as a set.
- For all $s \in S$ and $x \in X$, the section $s_{|x}$ is either finitary or directed.
- For every directed $s \in S$, there is an $x \in X$ with $s_{|x} = s$. For every $y \neq x$, $s_{|y}$ is finitary. So every directed generator is directed along a constant path.

See, for example, [10, Proposition 3.9.11] for more details.

Let $\Omega = \{w \in X^\omega : w \text{ is eventually constant}\}$. Then $\Omega$ is invariant under the action of every generator in $S$, so $X^\omega \setminus \Omega$ is also invariant under the action of $G$. For every generator, the germs in $X^\omega \setminus \Omega$ are contained in $T$, so for $w \in X^\omega \setminus \Omega$, the group of germs $G_w$ is contained in $T_w = 1$, so it is trivial.

For an end $w \in \Omega$, and a group element $g \in G$, let $v_n$ be the prefix of $w$ of length $n$ and consider the sequence $g_{|v_n}$. We claim that this sequence is eventually constant, and if $w$ is eventually constant equal to the letter $x$, then for all $y \in X \setminus \{x\}$, the section $g_{|v_n,y}$ is contained in $\text{Aut}_P(X^*;P)$ for $n$ large enough.

We show this by induction for every group element $g \in G$ over the word length of $g$. The statement is clear for the identity element. For a generator $s \in S$, either $s_{|w_n}$ is trivial for some $n$, and then the statement is clear, or $w$ is of the form $z(x^\omega)$, with $s_{|x}$ directed along $x$, and the statement also follows. Now if $g, h \in G$ both satisfy the statement for all ends and $w$ is an end eventually constantly equal to some letter $x$, then $w' := h(w)$ is in $\Omega$, with $v_n'$ as the prefix of $w'$ of length $n$, so that it is eventually constant to some letter $x'$, and

$$(gh)_{|v_n} = g_{|v'_n} h_{|v_n}$$

is eventually constant, and for every $y \neq x$, we have that $y' := h_{|v_n}(y)$ is well defined and different from $x'$, and so

$$(gh)_{|v_n,y} = g_{|v'_n,y} h_{|v_n,y}$$

is a product of two element of $\text{Aut}_P(X^*;P)$, so it is also in $\text{Aut}_P(X^*;P)$. This finishes the inductive step.

In particular, for $w \in \Omega$ eventually constantly equal to the letter $x \in X$, we get a group homomorphism

$$G_w \to \text{Aut}_P(X^*;P) \wr X \setminus \{x\} P_x$$

$$[g] \mapsto \left(y \mapsto g_{|v_n,y}, \rho_1(g_{|v_n})\right) \text{ for } n \text{ large enough.}$$
Here $P_x$ is the stabilizer of $x \in X$ for the action of $P$ on $X$. The group homomorphism is injective, and the codomain is amenable, so it follows that the group of germs $G_w$ is amenable as well. □

5 | OUTLOOK

We apply our main theorem in [14] to give a sufficient condition for amenability of iterated monodromy groups of of postsingularly finite entire functions (Theorem 1).

In the proof of our main theorem, we use the version of Theorem 2 from [8], which imposes a recurrence condition on the random walk on the orbital Schreier graphs. This recurrence condition was generalized to an extensive amenability condition in [7]. It is shown in [7] that every recurrent action is also extensive amenable. In our main theorem, it would be interesting to see whether we could weaken the recurrence condition to a condition about extensive amenability. Another direction to generalize is to step up in the hierarchy of automata with polynomial activity growth. In [1,8], it is shown that the group of automata of linear activity growth acting on a finite alphabet is amenable. Again, a crucial ingredient here is the recurrence of the random walk on the orbital Schreier graphs. It is not clear how this generalizes to infinite alphabets, as it seems that the estimates given in [1,8] to show recurrence used finiteness of the alphabet at an important point.

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