Polynomial Time Reinforcement Learning in Correlated FMDPs with Linear Value Functions

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July 13, 2021

Abstract

Many reinforcement learning (RL) environments in practice feature enormous state spaces that may be described compactly by a “factored” structure, that may be modeled by Factored Markov Decision Processes (FMDPs). We present the first polynomial-time algorithm for RL with FMDPs that does not rely on an oracle planner, and instead of requiring a linear transition model, only requires a linear value function with a suitable local basis with respect to the factorization. With this assumption, we can solve FMDPs in polynomial time by constructing an efficient separation oracle for convex optimization. Importantly, and in contrast to prior work, we do not assume that the transitions on various factors are independent.

1 Introduction

Many important application domains of Reinforcement learning (RL) – such as resource allocation or complex games – feature large state spaces, for which existing theoretical guarantees are unsatisfactory. But, many of these domains are believed to be captured by a small dynamic Bayesian network (DBN) on factored state variables. Therefore, Factored MDPs (FMDPs) were introduced by Boutilier, Dearden, and Goldszmidt (2000) to take advantage of such a priori knowledge about independence and the structure of the transition function. Subsequently, efficient approximate FMDP planners were developed by Guestrin, Koller, Parr, and Venkataraman (2003), and RL in FMDPs was considered by Kearns and Koller (1999) assuming access to an efficient oracle FMDP planner.

More recently, Osband and Van Roy (2014) obtained near-optimal RL regret bounds in FMDPs assuming access to a stronger oracle planner which returns the optimistic solution to a family of FMDPs. No polynomial-time algorithm for such a planner is known. Moreover, planning for a single FMDP is intractable (Mundhenk, Goldsmith, Lusena, and Allender, 2000; Lusena, Goldsmith, and Mundhenk, 2001), and planning over a family is generally no easier.

FMDP planning is difficult in part because of a product constraint induced by the independence structure of the transition function. We argue the independent transition assumption is unnecessary for obtaining a

*Work performed while affiliated with UT Dallas and participating in an NSF REU at Washington University in St. Louis.
regret bound, and instead permit potentially correlated transitions on the state variables. We propose a polynomial time algorithm for RL for this family of Correlated FMDPs with bounded-norm and factored linear value functions, given an efficient variable elimination order for the induced cost network of the basis.\textsuperscript{1} Our algorithm does not assume access to an oracle planner or depend on the diameter of the FMDP, which indeed may be disconnected. Furthermore, in contrast to recent works we do not assume the transition model is linear (see Appx. A for an extended discussion).

Our RL algorithm is based on UCRL-Factored Osband and Van Roy (2014), which employs an oracle for an optimistic planner over a family of FMDPs as a subroutine. For general FMDPs, it is unclear whether such a planner with polynomial time and theoretical guarantees can exist. We propose a theoretically grounded planner for the correlated, factored linear value function case by modifying the imprecise FMDP planner of Delgado, de Barros, Cozman, and Sanner (2011). Our formulation has the reward functions $R$ and transition probabilities $P$ take unknown values from bounded convex sets centered on their empirical estimates.

Due to the independence assumption on transition probabilities in FMDP DBNs, and $P$ being variable, the original imprecise FMDP planner formulation of Delgado et al. (2011) inevitably leads to multi-linear programming, which is non-convex in general. We circumvent this difficulty by: 1) removing the independence assumption of the transition model – hence calling it correlated – and only computing estimates of the factored marginal transition probabilities which do not need to be consistent; 2) utilizing an optimistic formulation as required by UCRL-Factored, which is easier to formulate and solve than the pessimistic formulation of Delgado et al. (2011), which contained a difficult min max constraint; and 3) constructing an efficient separation oracle for the program by applying the variable elimination procedure proposed by Guestrin et al. (2003). Note that our planning problem is a convex program with an exponential number of constraints, which cannot simply be plugged into a standard LP solver to obtain a polynomial-time guarantee.

**Related Work** Xu and Tewari (2020) improve UCRL-Factored for the non-episodic setting by discretizing the confidence sets but still require an oracle planner. Tian, Qian, and Sra (2020) derive an optimal minimax regret bound for episodic RL in FMDPs, but utilize a subroutine VI\_OPTIMISM which performs value iteration to find an optimal policy, iterating over all exponentially many states. Importantly, our work builds on Jakšch, Ortner, and Auer (2010) and Osband and Van Roy (2014) by modifying the underlying structural assumptions, to show that exact polynomial-time planning is indeed possible while retaining RL regret bounds similar to their oracle-efficient ones.

Beyond Osband and Van Roy (2014) we also assume that the optimal value function is linear w.r.t. a particular basis of functions. Linear value functions and approximations have been well studied (Bradtke and Barto, 1996; Yu and Bertsekas, 2007; Parr, Taylor, Painter-Wakefield, and Littman, 2010; Osband, Roy, and Wen, 2016). The bounds obtained in these works are polynomial in the number of states, however, and the algorithms do not scale to large MDPs that may still have compact FMDPs. Weisz, Amortila, and Szepesvári (2021) prove an exponential lower bound for linearly-realizable MDPs, however their construction requires an exponential sized action space. We instead assume a polynomial sized action space for tractable planning.

Linear $Q$-functions have also seen a resurgence in study. Jin, Yang, Wang, and Jordan (2020) consider linear transitions and rewards in the non-discounted case. Yang and Wang (2019) consider RL in the discounted setting with a linear transition model. Both show this implies the optimal $Q$-function is linear, and is needed to avoid unbounded Bellman error. Our work is instead focused on the linear value function setting, where the two settings are fundamentally different (Appx. A). In particular, unlike a linear $Q$-function, a linear value function does not require that the transition probabilities are also linear functions. Wang, Salakhutdinov, and Yang (2020) instead focus on RL with general $Q$-function approximation, but unfortunately their results depend on the eluder dimension of function classes, which is at worst $\text{poly}(|S|)$.

Imprecise MDPs were first introduced by White and Eldeib (1994) to model transition functions that are imprecisely specified (i.e. could be any function within some convex transition set). Using techniques from Guestrin et al. (2003), Delgado et al. (2011) proposed a pessimistic planner for imprecise FMDPs but

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\textsuperscript{1}For example, we might design such a basis. Alternatively, although optimizing elimination orders is NP-hard, many practical heuristics have been proposed (some even with approximation guarantees). See Kjærulff (1990); Becker and Geiger (2001); Kask, Gelfand, Otten, and Dechter (2011), etc.
could not simultaneously guarantee correctness and efficiency. For the purpose of learning, we instead require (and thus construct) an optimistic planner for a family of FMDPs with imprecise transition and reward functions. Our setting is similar to the Bounded MDPs introduced by Givan, Leach, and Dean (2000), but with an exponential-sized state space, additional linear structure, and a less strict requirement on “well-formed transition functions”.

There is also a line of work on simultaneous FMDP structure and reinforcement learning (Strehl, Diuk, and Littman, 2007; Diuk, Li, and Leffler, 2009). We instead assume that such structure is given as input to the RL problem.

2 Preliminaries

Our work is based on the confidence set approach from Jaksch et al. (2010) and Osband and Van Roy (2014), both of whom consider RL in a non-discounted, cumulative episodic reward setting introduced by Burnetas and Katehakis (1997). Consequently, the value function may take different values at the same state at different points in the time horizon $\tau$. Therefore, any approach to RL in this setting must solve for a different value function at each time step.

Let $M = (S, A, R^M, P^M, \tau, \rho)$ be a finite horizon MDP. Each episode is a run of the MDP $M$ with the finite time horizon $\tau$. $R^M : S \times A \rightarrow \mathbb{R}$ is a reward distribution from $(s, a)$ pairs, $P^M(s'|s, a)$ is the transition probability over $S$ from $s \in S, a \in A$, and $\rho$ the initial distribution over $S$.

A deterministic policy $\mu$ is a mapping each state $s \in S$ to an action $a \in A$. For an MDP $M$ and policy $\mu$, we define the value function as: $V^\mu_M(s) := \mathbb{E}_{M, \mu}[\sum_{t=1}^{\tau} R^M(s_t, a_t) \mid s_1 = s]$, where $R^M(s, a)$ is the expected reward for taking action $a$ in state $s$. The subscripts of $\mathbb{E}$ denote that $a_t = \mu(s_t)$ and $s_{t+1} \sim P^M(\cdot | s_t, a_t)$ for each $i$. A policy $\mu$ is optimal if $V^\mu_M(s) = \max_{\mu'} V^{\mu'}_M(s)$ for all $s \in S$. We denote the optimal policy for an MDP $M$ by $\mu^M$.

The RL agent interacts with $M$ over episodes, where each episode begins at $t_k = (k-1)\tau + 1, k = 1, 2, \ldots$. At time step $t$, the agent selects an action $a_t$, observes a scalar reward $r_t$, then transitions to $s_{t+1}$. Let $H_t = (s_1, a_1, r_1, \ldots, s_{t-1}, a_{t-1}, r_{t-1})$ be the history of observed transitions prior to time $t$. An RL algorithm is a sequence of functions $\{\pi_k \mid k = 1, 2, \ldots\}$, each mapping $H_t$ to a probability distribution $\pi_k(H_t)$ over policies which the agent will employ in episode $k$. The regret incurred is defined as $\text{Regret}(T, \pi, M^*) := \sum_{k=1}^{\lceil T/\tau \rceil} \Delta_k$ where $\Delta_k$ is the regret over the $k$th episode.

\[ \Delta_k := V^*_{\mu^M}(s_{k+1}) - V^{\mu_k}_M(s_{k+1}) \]

with $\mu^* = \mu^M$, $\mu_k \sim \pi_k(H_{t_k})$, and $s_{k+1}$ being the first state of the $k$th episode.

**Definition 1.** Let $\mathcal{X} = X_1 \times \cdots \times X_n$. For any subset of indices $Z \subseteq [n]$, the scope operation of a set is defined as $\mathcal{X}[Z] := \bigotimes_{i \in Z} X_i$. For any $x \in \mathcal{X}$ we can define the scoped variable $x[Z] \in \mathcal{X}[Z]$ to be the values of the variables $x_i \in X_i$ with indices $i \in Z$. Similarly, we will use lower case letters to denote specific values.

We write $\mathcal{X} = S \times \mathcal{A}$ in RL, where we let $m$ be the number of state factors, $S = S_1 \times \cdots \times S_m$. Let $\mathcal{P}_{\mathcal{X}[Z], \mathbb{R}}$ be the set of functions mapping $\mathcal{X}[Z]$ to $\sigma$-subgaussian probability measures over the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with mean in $[0, C]$ and Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$.

**Definition 2.** The reward function class $\mathcal{R}$ is factored over $\mathcal{X} = X_1 \times \cdots \times X_n$ with scopes $Z_1^R, \ldots, Z_l^R$ iff for all $R \in \mathcal{R}, x \in \mathcal{X}$ there are functions $\{R_i \in \mathcal{P}_{\mathcal{X}[Z_i^R], \mathbb{R}}\}_{i=1}^l$ such that $\mathbb{E}[r] = \sum_{i=1}^l \mathbb{E}[r_i]$ where $r \sim R(x)$ is equal to $\sum_{i=1}^l r_i$ with each $r_i \sim R_i(x[Z_i^R])$ individually observed.

**Remark 1.** Traditionally, due to an assumption of locality of the reward, each of the individual reward function scopes is extremely small. The intuition is that without at least local structure, we cannot tractably learn the value of exponentially many states.

**Definition 3.** The Bellman operator $T^M_\mu$ for the MDP $M$, stationary policy $\mu$, and value function $V : S \rightarrow \mathbb{R}$, is defined by $T^M_\mu V(s) = R^M(s, \mu(s)) + \sum_{s' \in S} P^M(s'|s, \mu(s))V(s')$. 

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Definition 4. The linear value function class $V$ is factored over $S = S_1 \times \cdots \times S_m$ with scopes $Z^h_1, \ldots, Z^h_m$ iff for the optimal value function $V^M_{\mu^M}$ (with associated optimal policy $\mu^M$ for MDP $M$) we have, for basis functions $h_j : S[Z^h_j] \rightarrow \mathbb{R}$ and associated $w_j \in \mathbb{R}$ for $j = 1, \ldots, \phi$, $V^M_{\mu^M}(s) = \sum_{j=1}^{\phi} w_j h_j(s[Z^h_j])$, for all $s \in S$.

Typical FMDPs rely on a Dynamic Bayesian Network (DBN) to describe their transition function. The transition graph of DBN is a two-layered directed acyclic graph, where the nodes in one layer are given by state variables $S_i$ at time step $t - 1$, and the other layer’s nodes are state variables at time step $t$. The edges are all directed from step $t - 1$ to step $t$. They induce the FMDP transition independence assumption: there exist $P_t$ such that $P^M(s_t|s_{t-1}) = \prod_{i=1}^{m} P_i(s_t[i] \mid s_{t-1}[Z_i], a_{t-1})$ for any $s \in S$ with associated scopes $Z_i$ for $i = 1, \ldots, m$ (Osband and Van Roy, 2014). Here $Z_i$ denotes the parents of the state $i$ in the transition graph. This ensures transitioning from $s_{t-1}$ to $s_t$ with $a_{t-1}$ is independent on each state variable.

For planning purposes, we seek to drop the independence as it introduces a difficult product constraint during optimization. Our formulation retains most of the structural assumptions of the DBN, letting each state variable $S_k$ at each time step be fully determined by only its parents paired with the action $a$. However, we only request that the transition marginals on state variables be consistent, and do not necessarily enforce independence on each state variable.

Assumption 1. Each state variable $S_k$ depends only on a (small) subset of other state variables called the parents, denoted by $\text{Pa}(S_k)$, identically to the two-layer DBN assumption. More generally, we can denote the parents of state variables in the scope $Z^h_j$ by $\text{Pa}(Z^h_j)$.

Our setting captures interesting environments. Consider for example a gridworld, in which there is a penalty for colliding with other, randomly moving objects. If there is a safe policy, the optimal $V \equiv 0$ (and is thus linear), and the local movement ensures a compact DBN. Yet, the presence/absence of objects is not independent across positions.

Definition 5. $\text{Val}()$ is the assignment operator. $\text{Val}(Z^h_j)$ is the set of all assignments to state variables in $Z^h_j$, and $\text{Val}(Z^h_j)$ to variables $S_k \notin Z^h_j$, e.g. if $Z^h_j$ was three binary state variables, then we would have $\text{Val}(Z^h_j) = \{0, 1\}^3$.

By marginalizing similar to Koller and Parr (1999) and using Def. 4, we rewrite the second term of the Bellman operator:

$$
\sum_{s' \in S} P^M(s' \mid s, a) \sum_{j=1}^{\phi} w_j h_j(s' \mid Z^h_j) = \sum_{j=1}^{\phi} w_j \sum_{s' \in \text{Val}(Z^h_j)} h_j(s') \sum_{s' \in \text{Val}(Z^h_j)} P^M(s', s' \mid s, a) \\
= \sum_{j=1}^{\phi} w_j h_j(s') P^M(s' \mid s, a) \\
= \sum_{j=1}^{\phi} w_j h_j(s') P^M(s' \mid s \mid \text{Pa}(Z^h_j), a) \tag{4}
$$

Here we split $s'$ into $s'$ and $s'$. We first marginalized out $s' \in \text{Val}(Z^h_j)$, the parts of the state $s$ which are outside of the scope of the $j$th basis function (Def. 4), then condition $P^M$ on the variables of $s$ that occur in the parents w.r.t. $a = \mu(s) \in \mathcal{A}$ of the scope $Z^h_j$.

Therefore, instead of exponentially many transition probability values $P^M(s' \mid s, a)$ w.r.t. the full state space, we only need a collection $\mathcal{P}$ of marginal distributions $P^M(\cdot \mid s \mid \text{Pa}(Z^h_j), \mu(s))$ over associated domains $\text{Val}(Z^h_j)$ to fully describe the transition dynamics of the FMDP.

We can bound the cardinality of $\mathcal{P}$:

$$
|\mathcal{P}| \leq \sum_{j=1}^{\phi} |\mathcal{A}||\text{Val}(S_k)|^{|\text{Pa}(Z^h_j)|} \leq O(\text{poly}(m)|\text{Val}(S_k)|^\phi) \tag{5}
$$

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Algorithm 1 Correlated UCRL-Factored

for episode \( k = 1 \ldots K \) do
  \( d^R_i \leftarrow 4\sigma^2 \log(4|\mathcal{Y}|[Z^h_i]|k/\delta) \) for \( i = 1 \ldots l \)
  \( d^P_j \leftarrow 2|\text{Val}(Z^h_j)| \log(2) - 2 \log(\delta/(2N|\text{Pa}(Z^h_j)|k^2)) \) for \( j = 1 \ldots N \)

  \( \mathcal{M}_k = \{ M[R_i \in R^i_{\text{d}}(d^R_i), P_j \in P^j_{\text{d}}(d^P_j) \) \forall i, j \} \)

  \( \mu_k = \text{OptimisticPlanner}(\mathcal{M}_k, \epsilon = \sqrt{1/k}) \)

  sample initial state variables \( s^1_1, \ldots, s^m_1 \)
  for timestep \( t = 1 \ldots \tau \) do
    sample and apply \( a_t = \mu_k(s_t) \)
    observe \( r^1_t, \ldots, r^m_t \) and \( s^1_{t+1}, \ldots, s^m_{t+1} \)
  end for
end for

where \( \zeta \geq |\text{Pa}(Z^h_i)| \) is a scope size bound, and \( |\text{Val}(S_k)| \) represents the number of values a state variable can take. E.g., if \( S = \{0,1 \}^m \), \( |\text{Val}(S_k)| = 2 \). By decomposing the value function and applying the Bellman operator, we avoid the typical FMDP transition-independence assumption. Instead, \( \mathcal{P} \) is a set of correlated marginal transition functions.

Definition 6. A Correlated Factored MDP is an MDP with factored linear value and reward functions, as well as marginally consistent transition probabilities. It is given by:

\[
M = (\{S_i\}_{i=1}^m; \{X_i\}_{i=1}^l; \{Z_i^R\}_{i=1}^l; \{R_i\}_{i=1}^l; P^M; \{h_j\}_{j=1}^\phi; \{Z_j^h\}_{j=1}^\phi; \tau; \rho, \gamma)
\]

where \( Z_i^R \) is the scope for the \( i \)th reward function, and \( Z_j^h \) is the scope of the \( j \)th basis for the linear value function, \( P^M \) is a transition function with consistent marginals given by the decomposition in (4), \( \tau \) is the time horizon, \( \rho \) is the initial distribution over \( \mathcal{S} \), and \( \gamma \) is the discount factor.

Remark 2. Correlated FMDPs subsume FMDPs in the linear value function case, as our transition marginals can express independent and dependent transition functions.

3 Algorithm

For simplicity we first illustrate our algorithm in the discounted episodic RL setting with the discounted Bellman Operator: \( \mathcal{T}_\mu^M V(s) = \mathcal{R}_\mu^M(s, \mu(s)) + \gamma \sum_{s' \in S} P^M(s'|s, \mu(s))V(s') \). Each episode is finite but “long enough” for us to consider the Bellman operator time-invariant. Therefore, there is a single value function \( V^* \) we need to estimate. Afterwards, we discuss how to extend our result to the finite episode setting with a potentially different value function \( V_h \) for each step \( h \in \mathcal{T} \).

Our proposed algorithm modifies UCRL-Factored (Osband and Van Roy, 2014), keeping track of additional confidence sets around each \( R_i \) and marginal distribution \( P^M (\cdot|\text{Pa}(Z^h_i)), a \). We use the definition of Osband and Van Roy (2014): The confidence set at time \( t \) is centered at an empirical estimate \( \hat{f}_t \in \mathcal{M}_{\mathcal{X},\mathcal{Y}} \) at time \( t \) defined by \( \hat{f}_t(x) = \frac{1}{n_t(x)} \sum_{\tau < t, x_\tau = x} \delta_{y_\tau} \), where \( n_t(x) \) counts the number of occurrences of \( x \) in \( (x_1, \ldots, x_{t-1}) \) and \( \delta_{y_\tau} \) is the probability mass function over \( \mathcal{Y} \) which assigns all probability to outcome \( y_\tau \). Our sequence of confidence sets depends on a choice of norm \( \| \cdot \| \) and a non-decreasing sequence \( \{d_t : t \in \mathbb{N} \} \). For each \( t \), the confidence set \( \mathcal{F}_t = \mathcal{F}_t(\| \cdot \|, x^{t-1}_t, d_t) \) is defined as:

\[
\left\{ f \in \mathcal{F} \left| ||f - \hat{f}_t(x_t)|| \leq \sqrt{\frac{d_t}{n_t(x_t)}} \right. \forall i \in [t-1] \right\}
\]

We write \( \mathcal{R}^I_i(d^R_i) \) as shorthand for the reward confidence set \( \mathcal{R}^I_i(\mathbb{E}[\cdot], x^{t-1}_t|Z^R_i, d^R_i) \) and \( P^I_i(d^P_i) \) for \( \bigotimes_{a \in \mathcal{A}} P_i^a(\| \cdot \|, (s^{t-1}_t, \text{Pa}(Z^h_i), a^{t-1}_i), d^P_i) \), the set of \( |\mathcal{A}| \) confidence sets over \( (\text{jth marginal, action}) \) pairs.

UCRL-Factored iteratively refines confidence sets containing the true, factored reward and transition functions w.h.p., while using an oracle planner over these sets to obtain a policy for each episode of learning.
Algorithm 2 OptimisticPlanner

$w = w_0$ // set initial value function weights
$M_R \leftarrow$ optimistic rewards $R_i(z)$ with (8)
$M_P \leftarrow$ optimistic transition marginals (Alg. 3, Apx. B.2)
$\Omega \leftarrow$ Simplify constraints of (1) with variable elimination Alg. 4 and computed $M_R, M_P$

while $w$ does not satisfy constraints $\Omega$ do
  Use tightness to construct cutting-plane (Thm. 1)
  $w \leftarrow$ new ellipsoid centroid within the cutting-plane
  $\Omega =$ Simplify constraints of (1) with variable elimination Alg. 4 and computed $M_R, M_P$
end while

\[
\sum_{j=0}^{\phi} w_j h_j(s) \geq \sum_{i=1}^{l} R_i(s, a) + \gamma \sum_{j=0}^{\phi} \sum_{z' \in \text{Val}(Z_j^h)} w_j h_j(s') P_j(s'|s[Pa(Z_j^b)],a) \text{ for all } s \in S, a \in A
\]

$R_i, P_j(\cdot | s[Pa(Z_j^b)], a) = \arg \max_{\widehat{\mu}_i \in \mathcal{R}_i(d_i^{R_i})} \sum_{i=1}^{l} \widehat{R}_i(s, a) + \gamma \sum_{j=0}^{\phi} \sum_{z' \in \text{Val}(Z_j^b)} w_j h_j(s') \tilde{P}_j(s'|s[Pa(Z_j^b)],a)$

Figure 1: Constraints for the OptimisticPlanner optimization problem. The objective is $\min_{w} \sum_{s} \sum_{j=0}^{\phi} w_j h_j(s)$.

Let $N = |P|$ be the number of transition function marginals in (5). Alg. 1 gives our full RL algorithm which modifies UCRL-Factored by changing the number and choice of confidence set sequences, and using the OptimisticPlanner we construct next instead of an imagined oracle.

We formulate our MDP planning task as an LP solving for the optimal value function $V^*(s)$ over each state $s$ (Alg. 2). After computing $V^*$, we obtain the optimal policy by greedily taking actions. The seminal work by Guestrin et al. (2003) showed that the Approximate Linear Programming (ALP) formulation for planning in an FMDP gives the optimal value function $V^*$ iff $V^*$ exists within the subspace spanned by the chosen basis. Their optimization problem is defined as minimizing the objective $\sum_{s} \sum_{j=0}^{\phi} w_j h_j(s)$ subject to $|S \times A|$ constraints,

\[
\sum_{j=0}^{\phi} w_j h_j(s) \geq R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) \sum_{j=0}^{\phi} w_j h_j(s')
\]

for each $(s, a)$ pair. $R$ is a deterministic function.

In Alg. 1, the reward distribution and transition functions must be learned by successively updating the corresponding confidence sets for each reward component function $R_i(d_i^{R_i})$ and transition function marginal $P_j(d_i^{P_j})$. Combining the formulations of Guestrin et al. (2003) and Delgado et al. (2011), we obtain the imprecise FMDP formulation Fig. 1, where $R$ and $P$ are defined over bounded convex sets centered on an empirical estimate of the reward and transition functions (see Apx B.1).

Note that the arg max in Fig. 1 specifies an optimistic solution, which guarantees that the reward and transition function are set to the best possible value within their respective confidence sets. We also include a constant basis function $h_0$, which is required for convergence (Guestrin et al., 2003). The number of constraints in this program is a function of the exponential number of states. Importantly, since we no longer assume the transition factors are independent, we are able to drop the difficult product constraint in Delgado et al. (2011), ensuring we are left with a linear formulation. Correctness of this approach can be shown through a simple modification to Thm. 1 of Delgado et al. (2011).

Although Fig. 1 is presented as a non-trivial bilevel program, we argue that we can construct an efficient separation oracle to solve it with an algorithm such as the Ellipsoid method (Grötschel, Lovász, and Schrijver, 1988) (or a more efficient equivalent (Jiang, Lee, Song, and Wong, 2020)) in polynomial time. We accomplish
this by removing the bilevel constraints and adding a polynomial number of linear constraints describing all possible variations of $R$ and $P$ within their confidence sets. This reduces the problem to simply solving an LP over the exponential sized state space. Observe that the arg max for $R$, $P$ is the largest of the RHS for the family of constraints we generate, so the two programs are equivalent.

**Remark 3.** Each $P_j(\cdot|s[\text{Pa}(Z_j^h)], a)$ marginal has its own confidence set in $P^f_j(d^R_j)$, and only depends on the inner sum over $\text{Val}(Z_j^h)$ within each constraint in Fig. 1. This is essential.

### 3.1 Separation Oracle

To solve Fig. 1, we seek a separation oracle which either returns a cutting plane between some given point $w$ and the feasible set, or determines that the point $w$ lies inside the feasible set, satisfying all constraints. We repeat the following procedure for each action $a \in A$. Therefore, we need only consider the constraints over each state.

#### 3.1.1 Computing Optimistic Parameters

If all constraints in Fig. 1 are satisfied, the tightest constraint in particular is satisfied. If a constraint is not satisfied, then it determines a cutting plane in terms of $w$. Our separation oracle checks whether the following inequalities hold for each action $a$, obtained by rewriting the constraints in Fig. 1:

$$0 \geq \max_{s \in S, \mathcal{R}_j \in \mathcal{R}_i^f} \left[ \sum_{j=1}^l \mathcal{R}_j(s, a) + \gamma \sum_{j=0}^{\phi} w_j \left( -h_j(s) + \sum_{i' \in \text{Val}(Z_j^h)} h_j(i') P_j(i'|s[\text{Pa}(Z_j^h)], a) \right) \right]$$

(7)

Notice each $\mathcal{R}_i$ depends only on the subset of state variables given by its scope $Z_i^R$. We can thus precompute the optimal value of $\mathcal{R}_i(x[Z_i^R])$ for the polynomial number of assignments to $x[Z_i^R]$, represented by $z \in \text{Val}(Z_i^R)$, in $O(1)$ time by using the largest value within the confidence set:

$$\mathcal{R}_i(z) = \frac{1}{n_t(z)} \sum_{\tau \in \text{t}, x_{\tau} = x} \delta y_\tau + \sqrt{\frac{d_\tau}{n_t(z)}},$$

(8)

where $-n_t(z)$ denotes the number of visits to any $(s, a)$ which takes the values given by $z$ over state variables in $Z_i^R$, up until time $t - 1$. Notice that this allows us to remove the imprecise nature of the reward in $O(lm)$ time by creating a polynomial-sized lookup table for the value of $\mathcal{R}_i$ at any $(s, a)$ constraint.

We would like to perform a similar procedure with the imprecise transition function. For each $j$ in (7), given $a$, there are multiple transition marginals to solve for, where each depend only on an assignment $z \in \text{Val}(\text{Pa}(Z_j^h))$ to the parents of the $j$th scope (Rmk. 3). Therefore, we have the following optimization problem over each $P_j(\cdot|s[\text{Pa}(Z_j^h)], a)$:

$$\max_{s' \in \text{Val}(Z_j^h)} \gamma w_j \left( -h_j(s) + \sum_{i' \in \text{Val}(Z_j^h)} h_j(i') P_j(i'|s[\text{Pa}(Z_j^h)], a) \right)$$

subject to the constraint that $P_j(\cdot|s[\text{Pa}(Z_j^h)], a) \in P^f_i$. As $P^f_i$ is a convex set (for a given marginal), we can essentially use Figure 2 of Jaksch et al. (2010) to solve this problem. To maximize a linear function over a convex polytope, we need only consider the polynomial number of polytope vertices. Our Alg. 3 given in Appx. B.2 simply greedily assigns resources to high valued $h_j(s_k')$ functions, while normalizing to ensure that $P$ remains a true probability distribution.

**Remark 4.** We only compute the optimistic parameters for both the reward and transition functions a single time before solving Fig. 1. Notice the optimistic reward did not depend on a particular setting of $w$, so we can use the resulting values for each call to the separation oracle. Similarly, optimistic transition probabilities depend only on $\text{sign}(w_j)$ in Alg. 3. For each of $N$ transition marginals, we compute and store both orderings based on $\text{sign}(w_j)$ in the lookup table. In the separation oracle (7), for a query $w$, use transition functions corresponding to the correct ordering in $O(1)$ by table lookup.
We have thus obtained a polynomial-size lookup table for each possible $R_i(s, a)[Z^R_i]$ and $P_j(s|\text{Pa}(Z^h_j), a)$. However, we are still left with a maximization over an exponential sized state space $S$. To ameliorate this, we utilize the procedure of variable elimination from probabilistic inference, which was applied to FMDPs by Guestrin et al. (2003).

Variable elimination constructs a new optimization problem, equivalent to (7), but over a tractable constraint space. Let some order over $S_1, \ldots, S_m$ be given, and assume that our state space is \{0, 1\}^m. Let $c_j$ be the term beside $w_j$ (inside the parenthesis) in (7). Motivated by the example in Delgado et al. (2011), suppose that the only scopes containing $S_1$ are $Z^R_i = \{S_1\}$, and $\text{Pa}(Z^h_j) = \{S_1, S_4\}$. Suppose that the first state variable to eliminate is $S_1$. Variable elimination rewrites (7) by moving the “relevant functions” inside (due to linearity):

$$\max_{s \in \mathbb{S}^m} \left[ \sum_{i=2}^l R_i(x[Z^R_i]) + \sum_{j=0,2 \ldots \phi} w_j c_j(s, a) + \max_{S_1} \left[ R_1(x[Z^R_1]) + w_1 c_1(s, a) \right] \right]$$

Next, we replace the inside max with a new LP variable $u_{S_1}^c$. However, to enforce $u_{S_1}^c$ to be the max, we need to add four additional constraints to the LP objective, one for each binary assignment to $S_1, S_4$. These constraints involve evaluating $R_1(x[Z^R_1])$ and $w_1 c_1(s, a)$ at each assignment, which simply uses our previously-constructed poly sized lookup table (taking into account $\text{sign}(w_1)$).

In the general case with $L$ relevant functions, the complexity of this problem has an exponential dependence on the width of the induced network of our scopes. Let the set of all scopes $Z = \{Z^R_i \mid i \in [l]\} \cup \{\text{Pa}(Z^h_j) \mid j \in [\phi]\}$ be given. We can construct a cost network over variables $S_1, \ldots, S_m$ s.t. there is an undirected edge between any two variables iff they appear together in any scope in $Z$. The width of this network is the longest path between any two variables.

**Remark 5.** In general, we may have network width $m$ and no complexity reduction from exponential time. Environment scopes must be small and also ensure the induced cost network has small diameter. Heuristics could be useful to find an order criterion, but an optimal choice is NP-hard in general.

**Theorem 1.** A polynomial-time (strong) separation oracle exists.

**Proof Sketch:** For a given $w$, obtain the simplified version $\Omega$ of the exponentially large LP formulation through variable elimination as above (Alg. 4). Given $w$, we can efficiently check the feasibility of the original LP by checking feasibility of $\Omega$. If $\Omega$ is infeasible for $w$, then we obtain a sequence of tight simplified linear constraints with the final exceeding the bound of (7). Since simplified constraints are obtained by iteratively maxing out state variables, from these tight constraints we can read off the corresponding state variable values $s^*$. The inequality in (7) with $s^*$ would be the one that $w$ violates, and we use this to define a separating hyperplane. Details and Alg. 4 are in Appx. B.3.

### 3.2 Completing the Cutting Plane Analysis

We have thus obtained a strong separation oracle, which returns $w$ if it lies in the solution set, or a strict separating hyperplane whose half-space contains the feasible solution set and does not contain the query point $w$. The strictness may be achieved by $\epsilon$-perturbation of a non-strict separator. We establish the remaining properties of the Ellipsoid Method for Fig. 1.

---

2Cost network and FMDP diameters are separate and unrelated.
First, recall that \( \min \sum_{s \in S} \sum_{j=0}^{\phi} w_j h_j(s) \) is the objective of our problem. A naïve summation over states may require exponential time, so we simplify: \( \sum_{j=0}^{\phi} w_j \sum_{s \in S} h_j(s) = \sum_{j=0}^{\phi} w_j g(Z_j^h) \sum_{s_h \in \text{Val}(Z_j^h)} h_j(s_k) \) where \( g : \{Z_j^h : j \in [\phi]\} \rightarrow Z^+ \) counts the number of states that are valued with \( h_j(s_k) \) by counting combinations of state variables which are not in the scope of \( h_j \). \( g(Z_j^h) = \prod_{i=1,...,m \in Z_j^h} |\text{Val}(S_i)| \). We can now evaluate our objective in polynomial time by iterating only over states within the scope of each \( h_j \). Next, we must assume that \( w \) lies in a bounded convex set to perform the optimization procedure. In particular, we will use \( \|w\|_1 \leq W \) for some \( W \in \mathbb{R} \) for reasons discussed further in Sec. 4. It is clear that if the MDP is well defined and has a bounded linear value function, then \( w \) must be bounded. Our main planning result follows.

**Theorem 2.** The Ellipsoid algorithm solves the optimization problem Fig. 1 in polynomial time.

This follows from the strong separation oracle of Thm. 1, but we defer the details to Appx. B.4.

### 3.3 Extension to Finite Episode Setting

We can solve the non-discounted, episodic RL setting by introducing a \( V_i \) for the \( i \)th step at each episode and adding new constraints. In particular, let \( V_{\tau+1} = 0 \) and add the following constraints: \( V_i(s) \geq T_i a V_{i+1} := (R(s,a) + \sum_{s' \in S} P(s'|s,a) V_{i+1}(s')) \), \( \forall s \in S, a \in A, i = 1,...,\tau \), which correspond to the time-dependent Bellman operator \( V_i(s) = \max_a \{ R(s,a) + \sum_{s' \in S} P(s'|s,a) V_{i+1}(s') \} \). Our constraint simplification algorithm still works because these new constraints and \( V(s) \geq R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V(s') \) in the discounted setting can all be reduced to the same structure as \( (7) \). We are now solving for \( \tau \) value functions instead of the single one in Fig. 1. Details available in Appx. B.5.

### 4 Regret Analysis

For simplicity of illustration, we first give an analysis for the discounted episodic setting, then obtain the regret for the non-discounted episodic setting by revising it. We sketch the regret analysis of Alg. 1, which is similar to the analysis of Osband and Van Roy (2014), and defer details to Appx. D. We simplify our notation by writing \( * \) in place of \( M^* \) or \( \mu^* \), and \( k \) in place of \( M_k \) and \( \mu_k \). Let \( s_{tk+1} \) be the first state in the \( k \)th episode. The regret of the \( k \)th episode is given by \( \Delta_k = V^*_k(s_{tk+1}) - V^*_k(s_{tk+1}) \) in \( (1) \). We split the regret into five bounded parts: (1) planning accuracy, (2) a martingale, (3) reward function estimate, (4) transition function estimate, and (5) PAC concentration bound. We combine these to derive a regret bound in terms of the number of episodes. Begin by adding and subtracting the imagined near optimal reward of policy \( \mu_k \), known by the agent:

\[
(V_k^*(s_{tk+1}) - V^*_k(s_{tk+1})) + (V^*_k(s_{tk+1}) - V^*_k(s_{tk+1})).
\]

The second term can be bounded by a choice of planning accuracy \( \epsilon = \sqrt{1/k} \) by optimism (Thm. 2). Deconstruct the first term through dynamic programming:

\[
\frac{1}{1-\gamma} \left( T_k - T_k^* \right) V^*_k(s_{tk+1}) + d_k \right),
\]

where \( d_k \) is a martingale difference bounded by \( \|w\|_1 \) and the max value of any \( h_j \) over all states \( s \in S \). Next, we apply Azuma–Hoeffding to \( d_k \) similarly to Jaksch et al. (2010) but without a dependence on the diameter due to our norm bounded linear basis approach. For the remaining Bellman term in (10), apply Cauchy-Schwarz to bound by:

\[
\leq |R_k(x_{k,1}) - R^*(x_{k,1})| + \sum_{j=1}^{\phi} |w_k^{(j)}(s) \sum_{s' \in S} (P^k(s'|x_{k,1}) - P^*(s'|x_{k,1})) h^j(s')| \]

The reward term in (11) is bounded akin to Osband and Van Roy (2014). For the transition term, we diverge by applying Hölder’s inequality with different choice of norms. We then apply Corollary 1 from Appx. C which bounds the widths of confidence sets over time. Lastly, using \( L_1 \) empirical deviation bounds, we show
that for a choice of $\delta$ at each episode, the true MDP lies within our confidence sets w.p. at least $1 - 2\delta$. We defer the general bound to Thm. 4 in Appx. D.3.

We then revise our analysis for the non-discounted episodic setting: after solving the LP with the new additional constraints, at each $i$th step $V_i = T_i a V_{i+1}$ will be tight for some action $a$. By repeatedly applying this time-dependent Bellman operator in our regret analysis, we verify that the only difference from the discounted setting is that we obtain a summation over $i = 1, \ldots, \tau$, which eventually introduces $\tau$. Due to the complex formulas within the main theorem, we present a simplified version for a symmetric case, which we present in Thm. 3 (details in Appx. D.4).

**Theorem 3.** Let $M^*$ be a Correlated FMDP with $V^*$ having a linear decomposition. Let $l + 1 \leq \phi$, $C = \sigma = 1$, $|S_i| = |X_i| = \kappa$, $|Z_i^b| = |Pa(Z_i^b)| = \zeta$ for all $i$, and let $J = \kappa \zeta$, $\|w\|_1 \leq W$, and $\max_j and $s \in Val(Z_j^b) |h_j(s)| \leq G$. Assuming $WG \geq 1$, and an efficient variable elimination ordering, then

$$\text{Regret}(T, \pi_\tau, M^*) \leq \tau \left(30\phi WG \sqrt{TJ(J \log(2) + \log(2N\zeta T^2/\delta))}\right) \text{w.p. at least } 1 - 3\delta.$$

**Discussion** Our bound is similar to Cor. 2 from Osband and Van Roy (2014), but not identical. Most importantly, we have provided an efficient planning algorithm which Osband and Van Roy (2014) assume as an oracle when computing their regret bound. We also have an extra $\sqrt{J}$ cost due to the support of each the transition marginal functions we are estimating being of size $J$ and not of size $\kappa$. This follows naturally from considering dependent transition functions.

Instead of a dependence on $m$, the number of state variables, we have a factor of $\phi$, the number of basis functions. Osband and Van Roy (2014) have a factor of the diameter in their simplified bound, which they replace with $\tau$ the time horizon. However, our bound does not rely on the diameter and instead depends only on the 1-norm of the basis vector $W$ and the max value that any basis function $G$. In fact, neither our planner nor regret analysis require the FMDP be completely connected as long as there is a reachable state in every connected component (see Rmk. 8 in Appx. D.3 for details). Our dependence on the horizon $\tau$ rather than diameter matches the recent minimax-regret of Tian et al. (2020). However, our regret bound can be obtained using a provably efficient algorithm.

## 5 Future Work

No lower bound for our problem setting is known. Our regret bound in Thm. 3 is polynomial in $\phi$, which represents the size of our basis. We also ask if it is possible to remove this dependency on $\phi$ in our transition function error analysis (11). This would allow for our approach to be utilized with kernels and a possibly infinitely sized basis. Correspondingly, we ask if there exists an efficient kernelized planning algorithm; if both could be resolved affirmatively, this would in turn enable the use of rich, kernelized value functions (as opposed to Q-functions) for RL in large FMDPs.

**Acknowledgements**

This research is partially supported by NSF awards IIS-1908287, IIS-1939677, and CCF-1718380, and associated REU funding.

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A Relation Between Linear Value and Linear Q-function

Many recent advances on provable polynomial RL algorithms assumes the state-action value function (Q-function) to be linear: a linear Q-function is defined as $Q^* = \sum_{i=1}^{\phi} w_i h_i(s, a)$, with basis elements $\{h_1, \ldots, h_\phi\}$. However, the linear Q-function assumption has its limitations. For example, Yang and Wang (2019) shows that a linear Q function requires the transition function to be linear in order to avoid unbounded Bellman error. (A similar argument appears in Proposition 2.3, 5.1 of Jin et al. (2020)). By contrast, we will show that a linear value function does not entail that the Q-function is linear, therefore, by using linear value function we can capture RL classes with more complicated Q functions that are not necessarily linear.

Intuitively, this is true because $V(s) = \max_a Q(s, a)$ and maximum function being linear does not necessarily infer that piece-wise functions are linear. Concretely, for a given state-action basis $\{h_i(s, a)\}_i$, we can provide an MDP for which there is no coefficient setting $w$ for which the optimal Q-function is linear, whereas this MDP will have an optimal linear value function $V^* = \sum_{i=1}^{\phi} w_i f_i(s)$ for any state value function basis $\{f_i(s)\}_i$.

**Proposition 1.** Let a state-action (Q-function) basis $\{h_1(s, a), \ldots, h_\phi(s, a)\}$ be given such that $\phi < N = 2^m$. Then there is an MDP family $M$ on $N$ states ($m$ binary factors) for which the optimal Q-function cannot be expressed as a linear combination of these basis functions with high probability $(1 - 2^{-N+\phi} \geq 1/2)$ for any MDP $M \in M$. On the other hand, every MDP $M \in M$ does admit a compact, optimal linear value function representation for any given basis set of state feature functions.

**Proof.** Consider a family of environments where there are $N$ states $S_1, \ldots, S_N$, and for simplicity the time horizon $\tau = 1$. Pick one of the states and call it $S_{opt}$. There are two actions everywhere within these MDPs: action $a_1$ takes any state $S_i \to S_{opt}$ for all $i \in [N]$ and gives reward 0; action $a_2$ takes $S_i \to S_{j(i)}$, $j(i) \neq opt$, and gives a reward from the set $\{-1, -1/2\}$. Call the family of all possible MDPs of this form $M$. We sample $M \in M$ uniformly at random—equivalently, by taking $j(i) \sim$ uniform$(N - 1)$ independently for each $i$, and the rewards independently and uniformly from $\{-1, -1/2\}$.

The optimal value function is a constant 0 for every state in every MDP $M \in M$. That is, $V(S_j) = 0$ for all $j \in [N]$ as the optimal policy simply takes the action $a_1$ everywhere—we can always obtain 0 by taking $a_1$ and the other action incurs negative reward in all states. Therefore, the value function can be represented with any basis by taking the zero linear combination.

On the other hand, consider the $\phi \times 2N$ matrix of Q-function basis feature representations for each $s, a$ pair:

$$B = \begin{bmatrix}
h_1(S_1, a_1) & h_1(S_2, a_1) & \ldots & h_1(S_N, a_1) & h_1(S_1, a_2) & \ldots & h_1(S_N, a_2) \\
h_2(S_1, a_1) & h_2(S_2, a_1) & \ldots & h_2(S_N, a_1) & h_2(S_1, a_2) & \ldots & h_2(S_N, a_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
h_\phi(S_1, a_1) & h_\phi(S_2, a_1) & \ldots & h_\phi(S_N, a_1) & h_\phi(S_1, a_2) & \ldots & h_\phi(S_N, a_2)
\end{bmatrix}$$

Choose a maximal ($d$) size set of states $S' = \{S'_1, \ldots, S'_d\}$ s.t. the col. vectors $[h_1(S'_1, a_2) \ h_2(S'_2, a_2) \ \ldots \ h_\phi(S'_d, a_2)]^T$ are linearly independent for all states $S'_j \in S'$ (naturally, $d \leq \phi$). Next, consider any assignment of rewards from $\{-1, -1/2\}$ for these $d$ states, and suppose for contradiction that there is a linear representation of every environment in $M$. By assumption, any state $\hat{S} \notin S'$ has $[h_1(\hat{S}, a_2) \ h_2(\hat{S}, a_2) \ \ldots \ h_\phi(\hat{S}, a_2)]^T$ determined by a linear combination of columns of states in $S'$, given by $\lambda_1, \ldots, \lambda_d$. In particular, supposing that for some choice of $\{w_1, \ldots, w_\phi\}$, $\sum_i w_i h_i(S'_j, a_2) = Q(S'_j, a_2)$ for all $j$, if these also represent $Q(\hat{S}, a_2)$, then

$$Q(\hat{S}, a_2) = \sum_{i=1}^{\phi} w_i h_i(\hat{S}, a_2) = \sum_{i=1}^{\phi} w_i \sum_{j=1}^{d} \lambda_j h_i(S'_j, a_2) = \sum_{j=1}^{d} \lambda_j \sum_{i=1}^{\phi} w_i h_i(S'_j, a_2) = \sum_{j=1}^{d} \lambda_j Q(S'_j, a_2).$$

i.e., $Q(\hat{S}, a_2)$ is therefore determined by rewards of states in $S'$, but we have two distinct, possible values for $Q(\hat{S}, a_2)$ in our family: $\{-1, -1/2\}$. Therefore, MDPs taking one of them cannot be captured by linear functions over the basis. Furthermore, for an MDP $M \in M$ chosen at random, since the reward of each $\hat{S}$ is
chosen independently, the $Q$-function is linear with probability only $2^{-(N-d)}$. Since $d \leq \phi < N$, this is at most $1/2$.

We emphasize that we are first given a basis, and are interested in understanding families of environments which may or may not be a linear combination of these bases elements. We do not state that a random MDP from the family we provide does not have its own linear $Q$-function representation. (Indeed, any basis that includes $Q(s,a)$ trivially represents the $Q$ function.) We only state that for a given basis, we can find an MDP $M$ whose optimal $Q$-function does not admit a linear decomposition with high probability.

**Prop. 1** demonstrates that there exist some RL environments where it is feasible to learn a compact linear value function but for which a compact linear $Q$-function is not expressive enough. We remark that conversely to **Prop. 1**, due to the relationship $V(s) = \max_a Q(s,a)$, there surely exist MDPs for which there is a compact linear $Q$-function but no compact linear value function. (It is in general only piecewise linear.) Therefore, we argue that the linear $Q$-function work is orthogonal to ours.

## B Planner Construction Derivation

### B.1 Relation to Previous Formulations

Imprecise MDPs are MDPs where the transition function may be defined imprecisely over a bounded convex set. Naturally, this leads to multiple notions of optimality. One such notion is pessimism, where we are interested in the optimal policy in the case where the transition function is always working “against us” (maximin). Delgado et al. (2011) formulate the maximin solution to imprecise FMDPs by extending (6) as follows.

$$\min_w \sum_x \sum_{i=0}^k w_i h_i(x)$$

subject to

$$\sum_{i=0}^k w_i h_i(x) \geq R(x, a) + \gamma \sum_{x' \in S} P(x'|x, a) \sum_{i=0}^k w_i h_i(x'), \forall x \in S, \forall a \in A$$

where

$$P(x'|x, a) = \arg \min_Q \sum_{x' \in S} Q(x'|x, a) \sum_{i=0}^k w_i h_i(x')$$

and

$$Q(x'_i | pa(X'_i), a) = \prod_i Q(x'_i | pa(X'_i))$$

(12)

where $K$ denotes a convex transition credal set.

Guestrin et al. (2003) give a simplification of ALP in the factored case, reducing the number of constraints to allow ALP to be tractable even with exponentially many states in the MDP. Delgado et al. (2011) applies a similar simplification to the imprecise case, allowing them to heuristically solve imprecise factored MDPs to allow ALP to be tractable even with exponentially many states in the MDP. Delgado et al. (2011) applies (13) to efficiently run a linear program to solve for the optimistic solution to the imprecise FMDP defined over our confidence sets.

Let $R_i(d_R^t)$ and $P_i(d_P^t)$, the reward function and transition function confidence sets at the $t$th time step, be given. Our goal is to generate an $\epsilon$-optimal planner which returns the optimistic solution to the set of MDPs given by these confidence sets. Formally, at the $k$th episode of our procedure we would like the optimistic solution to the set of MDPs $M_k$ given as follows.

$$M_k = \{ M | \bar{R}_i \in R_i(d_R^t), P_j \in P_j(d_P^t) \ \forall i, \forall j \}$$

(13)

Where $\bar{R}_i$ is the expected reward of the $i$th $\sigma$-subgaussian factored reward function.
Combining the formulations of Guestrin et al. (2003) and Delgado et al. (2011), we then write Fig. 1.

\subsection{B.2 Constructing a Separation Oracle}

Consider the stated separation oracle objective.

\begin{equation}
0 \geq \max_{s \in S, \tau_i \in R_i^t, P(|s|Pa(Z^h_s)), a \in P_i^t} \left[ \sum_{i=1}^{l} \mathcal{R}_i(s, a) + \gamma \sum_{j=0}^{\phi} w_j(-h_j(s) + \sum_{s' \in \text{Val}(Z^h_s)} h_j(s')P(s'|s|Pa(Z^h_s), a)) \right] \quad \forall a \in A
\end{equation}

Notice that maximizing over \(s \in S\) is the same as maximizing over \(S_1, \ldots, S_m\) individually as the state space is factored. We can then apply the methods from Delgado et al. (2011) and Guestrin et al. (2003) to simplify the maximization procedure. Checking whether (7) is satisfied can be done in two steps, first by solving the exponential sized LP given on the RHS for each \(a\), and second comparing the maximum over all \(a \in A\) to 0. We will focus on the first step, since the second is trivial. We can group and rewrite the program as follows.

\begin{equation}
\max_{A} \left[ \sum_{i=1}^{l} \mathcal{R}_i(x[Z^R_i]) + \sum_{j=0}^{\phi} w_j c_j(s, a) \right]
\end{equation}

Where \(A = S_1, \ldots, S_m\), \(\mathcal{R}_i \in R_i^t, P(|s|Pa(Z^h_s)), a \in P_i^t \quad \forall i = 1 \ldots l \forall j = 0 \ldots \phi\), the cartesian product of states, confidence sets for rewards, and confidence sets for marginal distributions. Furthermore, \(x = (s, a)\) is scoped on the \(i\)th reward scope, and \(c_j\) is the appropriate function from (7) inside the parenthesis next to \(w_j\), with \(\gamma\) brought inside.

We will use variable elimination to reduce the (15) to a tractable linear program. Let some order criterion \(O\) over \(1 \ldots m\) be given, where \(O(k)\) returns a variable to eliminate at time step \(k = 1 \ldots m\). Note that determining the optimal order \(O^*\) is in general NP-hard. At each iteration of variable elimination, we will bring the relevant state variable \(S_k\) inside the max. Algorithm 4 gives the full description of our proposed simplification, heavily based on Delgado et al. (2011) and Guestrin et al. (2003).

To illustrate the variable elimination procedure, we will work through the hypothetical example from Delgado et al. (2011) while noting differences along the way. Suppose that \(O(1) = S_1\) at the first iteration of simplification, and that the only scopes \(Z_i^R\) and \(Pa(Z^h_s)\) including \(S_1\) are \(Z_1^R = S_1\) and \(Pa(Z_1^h) = S_1 \times S_1\). Here, the function \(c_1\) is scoped on \(Pa(Z_1^h)\) due to the transition function being backprojected for simplification earlier (see (4)). Therefore, we can rewrite (15) as follows due to linearity of the objective.

\begin{equation}
\max_{A} \left[ \sum_{i=2}^{l} \mathcal{R}_i(x[Z^R_i]) + \sum_{j=0,2\ldots\phi} w_j c_j(s, a) + \max_{S_1, \tau_i \in R_i^t, P(|s|Pa(Z^h_s)), a \in P_i^t} \left[ \mathcal{R}_1(x[Z^R_1]) + w_1 c_1(s, a) \right] \right]
\end{equation}

Where \(A\) is as before, but with \(S_1, l = 1,\) and \(j = 1\) removed: \(A = S_2, \ldots, S_m, \tau_i \in R_i^t, P(|s|Pa(Z^h_s)), a \in P_i^t \quad \forall i = 2 \ldots l \forall j = 0,2\ldots\phi\). In general, we will have \(L\) relevant functions to pull into the second max each iteration, which we will rename as \(u_{Z_1}^{l_1}, \ldots, u_{Z_L}^{l_L}\). In our example, we have that \(u_{Z_1}^{l_1} = \mathcal{R}_1(x[Z^R_1])\) and \(u_{S_1}^{l_2} = w_1 c_1(s, a)\).

For each variable \(S_k\) we wish to eliminate, we select the \(L\) relevant functions and replace them with a maximization over \(S_k\) as follows. Here we diverge from Delgado et al. (2011) since they need only maximize over \(S_k\), but we still have a maximization over \(R, P\).

\begin{equation}
u^{e_k}_Z = \max_{S_k, \tau_i^t; P_i^t} \sum_{j=1}^{L} u_{Z_i}^{l_j}
\end{equation}

Where \(Z\) is the union of all variables appearing in any scope \(Z_i\) setminus the variable \(S_k\), since we maximize it out. Note that there may be none or any number of relevant reward and marginal distribution functions
(within c) in a single \( u^c_{Z_i} \), and we must include all relevant confidence sets within the maximization. Each confidence set will belong only to the relevant \( u^c_{Z_i} \) which is the first to pull it out of the larger max in (15) according to the elimination order criterion \( \hat{O} \). Note that \( u^c_{Z_i} \) is a new variable which we add to the optimization procedure.

For ease of notation, for the factored reward functions we will only refer to the state variables within their scope, since the action must be included in the scope anyways. Returning to the example, our \( Z \) will be \( \{S_1\} \cup \{S_1, S_4\} \setminus \{S_1\} \). So we have that

\[
u^c_{S_4} = \max_{S_1, \hat{R}_1 \in \hat{R}_1^i} \max_{P(1:1)} \left[u^{f_1}_{S_1} + u^{f_2}_{S_1}, S_4, S_4\right],
\]

and we can then rewrite (16) as

\[
\max_A \left[ \sum_{i=2}^{\tau=\tau_1} \hat{R}_1(x[z]) + \sum_{j=0,2,\ldots,\phi} w_j c_j(s, a) + u^c_{S_4} \right],
\]

with \( A = S_2, \ldots, S_m, \hat{R}_1 \in \hat{R}_1^i, P(1:1) \forall i = 2 \ldots \tau \forall j = 0, 2 \ldots \phi \). However, to enforce the definition of \( u^c_{S_4} \) in (18), we need four new inequality constraints, one for each combination of \( S_1 \) and \( S_4 \) (in the binary state variable case):

\[
u^c_{S_4} \geq u^{f_1}_{S_1} + u^{f_2}_{S_1}, S_4, S_4,
\]

\[
u^c_{S_4} \geq u^{f_1}_{S_1} + u^{f_2}_{S_1}, S_4, S_4,
\]

\[
u^c_{S_4} \geq u^{f_1}_{S_1} + u^{f_2}_{S_1}, S_4, S_4,
\]

\[
u^c_{S_4} \geq u^{f_1}_{S_1} + u^{f_2}_{S_1}, S_4, S_4.
\]

Furthermore, we need to also consider the relevant confidence sets \( \hat{R}_1^i \) and \( P^i_1 \). For example, consider \( u^{f_1}_{S_1} = \max_{\hat{R}_1^i} \hat{R}_1(\hat{S}_1, a) \). The appropriate confidence set \( \hat{R}_1^i \) has width based on how many times the pair \( \hat{S}_1, a \) has been observed up to time \( \tau \). Note that \( \hat{S}_1 \) here refers only to the value of the first state variable in the state vector (which is set to zero), the rest of the state values are arbitrary. However, since we are maximizing we can exactly set \( \hat{R}_1(\hat{S}_1, a) \) to the maximum value in the confidence set given by:

\[
\hat{R}_1(\hat{S}_1, a) = \hat{f}_1(\hat{S}_1, a) + \sqrt{\frac{d_i}{n_1(\hat{S}_1, a)}},
\]

\[
= \frac{1}{n_1(\hat{S}_1, a)} \sum_{\tau < t: x_r = x} \delta y_r + \sqrt{\frac{d_i}{n_1(\hat{S}_1, a)}},
\]

in \( O(1) \) time. In general, we can compute \( \hat{R}_i \) for any assignment \( z \in \text{Val}(Z^t_i) \) in \( O(1) \) time as follows:

\[
\hat{R}_i(z) = \frac{1}{n_i(z)} \sum_{\tau < t: x_r = x} \delta y_r + \sqrt{\frac{d_i}{n_i(z)}},
\]

Similarly, we must optimize for each assignment \( z \in \text{Val}(\text{Pa}(Z^t_j)) \), for example, \( u^{f_2}_{S_1, S_4} = \max_{\hat{R}_1^i} \hat{w}_1 c_1(s_1, S_4, a) \), where \( s_1 = 1 \) and \( s_4 = 1 \) is given. We can optimize for \( c_j \) w.r.t some assignment \( z \) by Algorithm 3, similar to Figure 2 of Jaksch et al. (2010) and originally given by Strehl and Littman (2008). A full proof is given in Jaksch et al. (2010).

**Lemma 1.** For all \( w \), we can precompute each function \( \hat{R}_i \) and \( c_j \) to remove the bounded nature of our MDP in polynomial time.
Algorithm 3 Transition Function Optimization

Optimal marginal transition function $P(\cdot | z, a)$ is returned for some assignment $z \in \text{Val}(\text{Pa}(Z_j^h))$.

Sort $S = \text{Val}(Z_i^h) = \{s'_1, \ldots, s'_K\}$ in descending order s.t. $h_j(s'_1) \geq \cdots \geq h_j(s'_K)$. Reverse order if $w_j < 0$.

Set $P(s'_1|z,a) := \min\{1, \hat{P}(s'_1|z,a) + \frac{1}{2} \sqrt{\frac{w_j}{\hat{\eta}(z,a)}}\}$

Set $P(s'_j|z,a) := \hat{P}(s'_j|z,a)$ for all states $s'_j$ s.t. $j > 1$.

Set $\ell := k$

while $\sum_{s'_j \in S} P(s'_j) > 1$ do

Reset $P(s'_k|z,a) := \max\{0, 1 - \sum_{s'_j \neq s'_k} P(s'_j|z,a)\}$

Set $\ell := \ell - 1$

end while

Proof. Let $w$ be fixed and given. Assume that some $R_i^*$ has restricted scope $Z_i^R$. For a given $s, a$ pair, we know that $R_i^* \in R_i^S$ since the width of the confidence set $R_i^S$ depends on the $s, a$ pair scoped on $Z_i^R$. However, the scope $Z_i^R$ can take only a polynomial number of different assignments. Therefore, we can iterate over all assignments $z \in \text{Val}(Z_i^R)$ and compute the maximum $R_i^*$ for each. Since $R_i^*$ is a single dimensional value, the maximum takes exactly the form (8).

We can do a similar procedure for each $c_j$, which is scoped on $\text{Pa}(Z_j^h)$, although optimization here is multidimensional. By iterating over all $\text{Val}(\text{Pa}(Z_j^h))$, we can solve the optimization problem given by (3.1.1) independently for both possible signs of $w$.

Since the number of confidence sets is polynomial, and solving over each is a polynomial time operation, we can remove the "imprecise" nature of our MDP in polynomial time by explicitly optimizing for the transition and reward functions.

B.3 Separation Oracle Proofs

We will prove that this reduction is tight, and that we can extract a state $s$ where the constraint is violated if $w$ lies outside the feasible set.

Lemma 2. Minimizing (27) will return a polynomial sized set of tight constraints $\omega \subseteq \Omega$ if $\kappa > 0$ where $\kappa$ is the objective value at the solution of the LP in (27).

Proof. Due to Lemma 1, the only difference between our algorithm and Guestrin et al. (2003) is that instead of adding (27) as a constraint relative to $\kappa$, we explicitly minimize over it. Once we retrieve its minimum objective value, we compare that to 0. If it is less than or equal to 0, then our current $w$ belongs in the feasible set, i.e. it satisfies the exponentially many constraints of our program by setting $\phi = 0$ in the induction proof of Theorem 4.4 of Guestrin et al. (2003). This follows from enforcing that each introduced variable must satisfy being at least as large as the sum of the relevant functions it represents.

Now assume that $\kappa > 0$. By minimization of a sum of LP variables, each $u_{j'}^{c_j}$ must be tight on at least one constraint by construction, given by an assignment to some subset of variables. Add this constraint to $\omega$ for each $j = 1 \ldots |\mathcal{F}|$. Since $|\Omega|$ is poly($m$) by Guestrin et al. (2003), so is $\omega \subset \Omega$.

A strong oracle is an oracle which returns either the point given to it if the point lies in the solution set, or a separating halfspace / hyperplane which completely contains the feasible solution set and does not contain the query point.

We restate Thm. 1 from the main text, and provide a proof:

Theorem 1. A polynomial-time (strong) separation oracle exists.

Proof. For each action $a$, run Algorithm 4. Take the maximum objective value $\kappa^*$ of (27) over all actions $a$. If $\kappa^* \leq 0$, then $w$ lies in the set described by the exponential number of state constraints. If $\kappa^* > 0$, then we have a set of tight constraints $\omega$ given by Lemma 2, since $\kappa^*$ is exactly the $\kappa$ for some action $a$. Any
Algorithm 4 Separation Oracle Objective Simplification

Optimal objective value (15) for a fixed action $a$ is returned.

// Data structure for constraints of LP
Let $\Omega = \{\}$

// Data structure for functions generated by variable elimination
Let $\mathcal{F} = \{\}$

// Generate equality constraints using lookup over pre-computed confidence set values
for $j = 1 \ldots \phi$ do
    for each assignment $z \in \text{Val}(\text{Pa}(Z^h_j))$ do
        Create a new LP variable $u^{f_j}_{z}$ and add the constraint to $\Omega$:
        $$u^{f_j}_{z} = \max_{P_f} w_j c_j (z, a)$$
        Plug in RHS from lookup table generated by Algorithm 3.
        Store new function $f_j$ to be used in variable elimination step: $\mathcal{F} = \mathcal{F} \cup \{f_j\}$.
    end for
end for

for $i = 1 \ldots l$ do
    for each assignment $z \in \text{Val}(Z^R_i)$ do
        Create a new LP variable $u^{f_i}_{z}$ and add the constraint to $\Omega$:
        $$u^{f_i}_{z} = \max_{R_i} R_i (z, a)$$
        Plug in RHS from lookup table generated by (8).
        Store new function $f_i$ to be used in variable elimination step: $\mathcal{F} = \mathcal{F} \cup \{f_i\}$.
    end for
end for

// Now, $\mathcal{F}$ and $\Omega$ contain all the functions and constraints we need to construct the simplified objective using variable elimination.

for $i = 1 \ldots m$ do
    // Next variable to be eliminated
    Let $l = O(i)$
    // Select the relevant functions from $\mathcal{F}$
    Let $e_1, \ldots, e_L$ be the functions in $\mathcal{F}$ whose scope contains $S_i$, and let $Z_j = \text{Scope}[e_j]$.
    // Introduce linear constraints for maximum over current variable $S_i$
    Define a new function $e$ with scope $Z = \cup_{j=1}^{L} Z_j - \{S_i\}$ to represent $\max_{S_i} \sum_{j=1}^{L} e_j$.
    // Add constraints $\Omega$ to enforce maximum.
    for each assignment $z \in \text{Val}(Z)$ do
        Add constraints to $\Omega$ to enforce max:
        $$u^{e}_{z} \geq \sum_{j=1}^{L} u^{e_j}_{(z,s_j)} \{Z_j\} \quad \forall S_i$$
    end for
    // Update set of functions.
    $\mathcal{F} = \mathcal{F} \cup \{e\} \setminus \{e_1, \ldots, e_L\}$
end for

// Now, all variables have been eliminated and all functions have empty scope.
Let $\kappa$ be the objective value at the solution of the following LP:

\[
\begin{align*}
\min_{j=1\ldots|\mathcal{F}|} & \sum_{z \in \mathcal{F}} u^{e_j}_{z} \\
\text{s.t.} & \quad \Omega
\end{align*}
\]  

(27)

Return $\kappa$.
state \( s = (s_1, \ldots, s_m) \) which is consistent with assignments within the tight constraints \( \omega \) will be a violating constraint in (1). This is due to the fact that the simplified tight constraint, when \( \kappa^* > 0 \), represents an \( s, a \) constraint violation in the original formulation (15).

We can then use the \( s, a \) and appropriately optimize for each \( R_i \) and \( P \) marginal described by this violating constraint as a separating hyperplane in terms of \( w \) as follows:

\[
hp(w) = \sum_{i=1}^{l} R_i(s, a) + \sum_{j=0}^{\phi} \gamma w_j \left( -h_j(s) + \sum_{\hat{s}' \in \text{Val}(Z^h_j)} h_j(\hat{s}') P(\hat{s}'|s[\text{Pa}(Z^h_j)], a) \right)
\]

(28)

B.4 Convergence of Ellipsoid Method

**Theorem 2.** The optimization problem (1) can be solved in polynomial time with the ellipsoid algorithm.

Proof. By Theorem 6.4.9 of Grötschel et al. (1988), the strong optimization problem of maximizing \( c^T w \) over some convex set \( P \) (which may require asserting that \( P \) is empty) can be solved given a strong separation oracle. However, the optimization problem must be over a “well-described polyhedron”, \( P \). By definition, \( P \) is well described if there exists a system of inequalities with rational coefficients that has a solution set \( P \) such that the encoding length of each inequality in the system is at most \( \gamma \) (Definition 6.2.2 Grötschel et al. (1988)).

Although our system is defined by an exponential number of state constraints (1), at the solution to the problem each reward and transition marginal function is fixed. Therefore, we can represent each inequality in binary with some bounded length \( \gamma \).

We also have a strong separation oracle by Theorem 1: an oracle which returns either the point \( w_t \) if given a point in \( P \) or a separating hyperplane completely containing \( P \). Lastly, to apply the ellipsoid algorithm to strong optimization in polynomial time, one binary searches for the minimum objective value \( d \) by solving a sequence of ellipsoid problems with \( c^T w \leq d \) added to the inequality set \( P \). This also has bounded encoding length. Therefore, our polyhedron \( P \) is well-described, and we can solve the strong optimization problem in polynomial time.

B.5 Extension of Planning to Non-Discounted Setting

It’s easy to switch to undiscounted episodic setting because we can introduce a different value function \( V_i \) for the \( i \)th step in each episode. Concretely, for the undiscounted episodic setting, we need to solve the following multi-level linear problem with the following constraints (for simplicity we didn’t write out the linear constraints that \( R, P \) must be within their confidence sets):

\[
\min_{V_1} \sum_s V_1(s) \quad \text{s.t.} \quad V_1(s) \geq R(s, a) + \sum_{s'} P(s'|s, a) V_2(s'), \quad \forall s \in \mathcal{S}, a \in \mathcal{A},
\]

where \( V_2 \) is the solution of

\[
\min_{V_2} \sum_s V_2(s) \quad \text{s.t.} \quad V_2(s) \geq R(s, a) + \sum_{s'} P(s'|s, a) V_3(s'), \quad \forall s \in \mathcal{S}, a \in \mathcal{A},
\]

where \( V_3 \) is the solution of subsequent subproblem involving \( V_1 \) with the same structure, and so on. This multi-level linear problem ends with

\[
\min_{V_\tau} \sum_s V_\tau(s) \quad \text{s.t.} \quad V_\tau(s) \geq R(s, a) + \sum_{s'} P(s'|s, a) V_{\tau+1}(s'), \quad \forall s \in \mathcal{S}, a \in \mathcal{A},
\]
where \( V_{\tau+1}(s) = 0, \forall s \in \mathcal{S} \) because each episode only has \( \tau \) steps. These linear programming formulations are equivalent to the step-wise sequential relationship:

\[
V_i(s) = \max_a \left\{ R(s, a) + \sum_{s'} P(s'|s, a) V_{i+1}(s') \right\}, \quad i = 1, \ldots, \tau.
\]

By inductively following the same argument as Lemma 1. of Delgado et al. (2011), we can see that this multi-level linear programming problem is equivalent to the following linear programing problem:

\[
\min_{V_i} \sum_s V_i(s)
\]

\[
\text{s.t. } V_i(s) \geq R(s, a) + \sum_{s'} P(s'|s, a) V_{i+1}(s'), \quad \forall s \in \mathcal{S}, a \in \mathcal{A}, \quad i = 1, \ldots, \tau,
\]

\[
V_{\tau+1}(s) = 0, \quad \forall s \in \mathcal{S}.
\]

Here each \( V_i(s) \) has the factored linear form \( \sum_j w_j h_j(s) \). So essentially we are just introducing extra \( w_j \) variables to linear programming problem. Since (29) has the same linear programming structure as the discounted episodic setting (with single value function \( V \)), we can still solve it by using our Ellipsoid-based planning algorithm.

### C Statement and Proof of Corollary 1

**Corollary 1.** For all finite sets \( \mathcal{X} \), measurable spaces \((\mathcal{Y}, \Sigma_\mathcal{Y})\), function classes \( \mathcal{F} \subseteq \mathcal{M}_{\mathcal{X}, \mathcal{Y}} \) with uniformly bounded widths \( w_\mathcal{F} \leq C_\mathcal{F} \forall x \in \mathcal{X} \) and non-decreasing sequences \( \{d_t : t \in \mathcal{N}\} \):

\[
\sum_{k=1}^{T} w_{\mathcal{F}_t}(x_{t_k+1}) \leq 4(C_\mathcal{F} |\mathcal{X}| + 1) + 4\sqrt{2d_T |\mathcal{X}| T},
\]

where \( x_{t_k+1} \) is the first \( x \in \mathcal{X} \) for episode \( k \).

We need auxiliary results before tackling the proof. First, we restate a result from the appendix of Osband and Van Roy (2014).

**Lemma 3.** For all finite sets \( \mathcal{X} \) and any \( d_T, \epsilon > 0 \):

\[
\sum_{t=1}^{T} 1\left\{ \sqrt{d_T/n_t(x_t)} > h(d_t, \epsilon) \right\} \leq \sum_{t=1}^{T} 1\left\{ \sqrt{d_T/n_t(x_t)} > \epsilon \right\} + |\mathcal{X}|
\]

where \( h(d_T, \epsilon) := \sqrt{d_T \epsilon^2/(d_T + \epsilon^2)} \).

Define the radius of each confidence set at each \( x \in \mathcal{X} \) to be \( r_{\mathcal{F}_t}(x) := \sup_{f \in \mathcal{F}_t} \| (f - \bar{f}_t)(x) \| \). Notice that by the triangle inequality, we have that \( w_{\mathcal{F}_t}(x) \leq 2r_{\mathcal{F}_t}(x) \) for all \( x \in \mathcal{X} \). Next, we present a similar result to Lemma 5 of Osband and Van Roy (2014).

**Lemma 4.** For all finite sets \( \mathcal{X} \), function classes \( \mathcal{F} \subseteq \mathcal{M}_{\mathcal{X}, \mathcal{Y}} \), non-decreasing sequences \( \{d_t : t \in \mathcal{N}\} \), any \( T \in \mathcal{N} \) and \( \epsilon > 0 \):

\[
\sum_{t=1}^{T} 1\{ r_{\mathcal{F}_t}(x_t) > \epsilon \} \leq \left( \frac{d_T}{\epsilon^2} + 1 \right) |\mathcal{X}|
\]

**Proof.** By construction of each confidence set, we have that

\[
\sum_{t=1}^{T} 1\{ r_{\mathcal{F}_t}(x_t) > \epsilon \} \leq \sum_{t=1}^{T} 1\{ \sqrt{d_T/n_t(x_t)} > \epsilon \}.
\]
Let \( g(\epsilon) = \sqrt{\frac{d_T \epsilon}{T}} \) be the \( \epsilon \)-inverse of \( h(d_T, \epsilon) \) such that \( g(h(d_T, \epsilon)) = \epsilon \) for all \( d_T, \epsilon \). We can apply the function \( g \) to \( h(d_T, \epsilon) \) on the LHS and \( \epsilon \) on the RHS in Lemma 3 \( n \)-times to be left with:

\[
\sum_{i=1}^{T} \mathbb{1}\{ \sqrt{d_T/n_i(x_i)} > \epsilon \} \leq \sum_{i=1}^{T} \mathbb{1}\{ \sqrt{d_T/n_i(x_i)} > g^{(n)}(\epsilon) \} + n|X|,
\]

where \( g^{(n)}(\epsilon) \) is the \( n \)-fold composition of \( g \) on \( \epsilon \). Note that \( g^{(1)}(\epsilon) < \cdots < g^{(n)}(\epsilon) \). Take \( n \) to be the lowest integer such that \( g^{(n)}(\epsilon) > \sqrt{d_T} \), then \( \sum_{i=1}^{T} \mathbb{1}\{ \sqrt{d_T/n_i(x_i)} > g^{(n)}(\epsilon) \} \leq |X| \), since for each \( x \in X \), \( n_i(x) < 1 \) at most one time over the entire history \( T \) of the indicator event. Therefore, we can bound (34) by \((n+1)|X|\).

In general, one can show that \( g^{(n)}(\sqrt{d_T/N}) = \sqrt{d_T/(N-n)} \) by induction. For a given \( \epsilon > 0 \), let \( \epsilon = \sqrt{d_T/N} \) for some \( N \in \mathbb{R}_+ \). We can choose \( n \) such that \( N > n + 1 \), then \( g^{(n)}(\sqrt{d_T/N}) = \sqrt{d_T} \) which is the desired result. Furthermore, we have that by solving for \( N \) in \( \epsilon = \sqrt{d_T/N} \), we are left with \( n < N = \frac{d_T}{\epsilon^2} \) to complete the proof. \( \square \)

We can now prove Corr. 1, which is a modification of Thm. 3 from Osband and Van Roy (2014).

**Proof.** Let \( x_k := x_{t_k+1} \). First, we remark that \( \sum_{k=1}^{T} |T(x_k)| \leq \sum_{k=1}^{T} 2 |T(x_k)| \). Let \( r_k = \epsilon \mathbb{1}_{\epsilon \leq |T(x)|} \), and reorder the sequence \( (r_1, \ldots, r_T) \) such that \( r_1 \geq \cdots \geq r_T \). Then we have that:

\[
\sum_{k=1}^{T} r_k \mathbb{1}_{\epsilon \leq |T(x)|} = \sum_{k=1}^{T} r_k \leq 1 + \sum_{k=1}^{T} r_k \mathbb{1}_{r_k \geq T^{-1}},
\]

since \( r_k \leq T^{-1} \) can happen at most \( T \) times and is upper bounded by \( T + T^{-1} = 1 \). Now observe the following: since we have ordered the sequence \( r_k \) to be decreasing, then for \( r_t \), \( t \in [T] \), we know that \( r_t \geq T^{-1} \) iff \( \sum_{k=1}^{T} \mathbb{1}_{r_k \geq T^{-1}} \geq t \). That is, it must be the case that if we have \( r_k \geq T^{-1} \) happen at least \( t \) times, then \( r_t \geq T^{-1} \) since \( r \) is a decreasing sequence.

By letting \( r_t > \epsilon > T^{-1} \) for some \( \epsilon > 0 \), we are left with \( \sum_{k=1}^{T} \mathbb{1}_{r_k > \epsilon} \geq t \). Then we can apply Lemma 4 to say that \( t \leq (\frac{d_T}{\epsilon^2} + 1)|X| \), so \( \epsilon \leq \sqrt{\frac{d_T |X|}{t - |X|}} \). This means that \( r_k \leq \min\{ C, \sqrt{\frac{d_T |X|}{t - |X|}} \} \) for all \( k \in [T] \). Therefore,

\[
\sum_{k=1}^{T} r_k \mathbb{1}_{r_k \geq T^{-1}} \leq C_{\mathbb{F}} |X| + \sum_{k=\lceil |X|+1 \rceil}^{T} \sqrt{\frac{d_T |X|}{t - |X|}} \\
\leq C_{\mathbb{F}} |X| + \sum_{k=0}^{T} \sqrt{\frac{d_T |X|}{t}} dt \\
\leq C_{\mathbb{F}} |X| + \int_{k=0}^{T} \sqrt{\frac{d_T |X|}{t}} dt \\
\leq C_{\mathbb{F}} |X| + 2\sqrt{d_T |X| T}
\]

Adjusting for the original triangle inequality relation between the confidence set width and radius completes the proof. \( \square \)

## D Full Regret Analysis

We begin the full regret analysis of our algorithm. We simplify our notation by writing \( * \) in place of \( M^* \) or \( \mu^* \), and \( k \) in place of \( M_k \) and \( \mu_k \). We begin by adding and subtracting the imagined near optimal reward of policy \( \hat{\mu}_k \), which is known by the agent. Let \( s_{t_k+1} \) be the first state in the \( k \)th episode.

\[
\Delta_k = V^*_a(s_{t_k+1}) - V^*_k(s_{t_k+1}) = \left( V_a^*(s_{t_k+1}) - V_k^*(s_{t_k+1}) \right) + \left( V_a^*(s_{t_k+1}) - V_k^*(s_{t_k+1}) \right)
\]

(36)
The term $V^*(s_{t_k+1}) - V^*_k(s_{t_k+1})$ relates the optimal rewards of the MDP $M^*$ to those near optimal for $\hat{M}$. We can bound this difference by planning accuracy $\epsilon = \sqrt{1/k}$ by optimism. Importantly, this term can only become more negative when we don’t insist that the transition marginals are consistent (in that they represent the marginals of a real distribution). This is what allowed us to relax enforcing that the marginals are consistent within our proposed oracle.

We then decompose the first term by repeated application of dynamic programming Osband, Russo, and Roy (2013):

\[
(V^k - V^*_k)(s_{t_k+1}) = (T^k V^k - T^*_k V^*_k)(s_{t_k+1})
\]

\[
= (T^k - T^*_k)V^k(s_{t_k+1}) + T^*_k V^*_k(s_{t_k+1}) - T^*_k V^*_k(s_{t_k+1})
\]

\[
= (T^k - T^*_k)V^k(s_{t_k+1}) + \sum_{s' \in S} P^*(s'|s_{t_k+1}, \mu_k(s_{t_k+1}))(V^k(s') - V^*_k(s'))
\]

\[
= (T^k - T^*_k)V^k(s_{t_k+1}) + \sum_{s' \in S} (P^*(s'|s_{t_k+1}, \mu_k(s_{t_k+1})))^\gamma (V^k(s') - V^*_k(s'))
\]

\[
= (T^k - T^*_k)V^k(s_{t_k+1}) + \sum_{s' \in S} (P^*(s'|s_{t_k+1}, \mu_k(s_{t_k+1})))^\gamma (V^k(s') - V^*_k(s')) + d_{t_k}
\]

Where $d_{t_k} := \gamma \sum_{s' \in S} (P^*(s'|s_{t_k+1}, \mu_k(s_{t_k+1})))^\gamma (V^k(s') - V^*_k(s')) - (T^k - T^*_k)V^k(s_{t_k+1})$. Continuing, we have that:

\[
(V^k - V^*_k)(s_{t_k+1}) = \frac{1}{1 - \gamma} (T^k - T^*_k)V^k(s_{t_k+1}) + d_{t_k}
\]

**Lemma 5.** The quantity $d_{t_k}$ is a bounded martingale difference.

**Proof.** Let $(s_{t_k+1}, \mu_k(s_{t_k+1})) = x_{k,1}$, then we have that:

\[
\mathbb{E}[d_{t_k}] = \mathbb{E}\left[\gamma \sum_{s \in S} \left\{ P^*(s|x_{k,1})(V^k - V^*_k)(s) \right\} \right] - \mathbb{E}\left[\gamma (V^k - V^*_k)(s_{t_k+1}) \right]
\]

\[
= \gamma \left\{ \sum_{s \in S} \left\{ P^*(s|x_{k,1})(V^k - V^*_k)(s) \right\} \right\} - \gamma \left\{ \sum_{s \in S} \left\{ P^*(s|x_{k,1})(V^k - V^*_k)(s) \right\} \right\} = 0,
\]

since the first term already takes the expectation, so $d_{t_k}$ is a martingale difference. Furthermore, we can show that is bounded as follows.

\[
d_{t_k} = \gamma \sum_{s \in S} \left\{ P^*(s|x_{k,1})(V^k - V^*_k)(s) \right\} - \gamma (V^k - V^*_k)(s_{t_k+1})
\]

\[
\leq \sum_{s \in S} \left\{ P^*(s|x_{k,1})(V^k - V^*_k)(s) \right\}
\]

\[
\leq \max_{s \in S} (V^k - V^*_k)(s)
\]

\[
\leq \max_{s \in S} |V^k(s)| = \max_{s \in S} \left| \sum_{j} w_j h_j(s) \right|
\]

\[
\leq \|w\|_1 \max_{s \in S} \max_{j} |h_j(s)| = \|w\|_1 \max_{j} \max_{s \in \text{Val}(Z_j)} |h_j(s)|
\]

The last fact is proven by Hölder’s inequality. Note that we do not use or assume a linear expansion of $V^*_k$. 

\[
\]

22
Importantly, the above bound is not dependent on the diameter of the MDP, which may be exponential in general. With a bounded martingale difference, we may then use the Azuma-Hoeffding inequality to obtain the following concentration guarantee Osband and Van Roy (2014), Jaksch et al. (2010):

\[ P\left( \sum_{k=1}^{[T/\tau]} dt_k > \|w\|_1 \max_j \max_{s \in \text{Val}(Z_j^R)} |h_j(s)| \sqrt{2[T/\tau \log(2/\delta)} \right) \leq \delta. \]  

(52)

The remaining first term of (44) is the one step Bellman error of the imagined MDP \( \tilde{M}_k \), which depends only on observed states and actions \( x_{k,i} \). Using Cauchy-Schwartz repeatedly we have the following.

\[
\frac{1}{1-\gamma} (T_k^k - T_k) V^k_k(s_{t+1}) 
= \frac{1}{1-\gamma} (T_k^k - T_k) \sum_{j=1}^{\phi} w_{k,j} h_j(s_{t+1}) 
= \frac{1}{1-\gamma} \left[ (\tilde{R}_k(x_{k,1}) - \tilde{R}_k^*(x_{k,1})) + \sum_{s' \in S} P^k(s' | x_{k,1}) \sum_{j=1}^{\phi} w_{k,j} h_j(s') - \sum_{s' \in S} P^*(s' | x_{k,1}) \sum_{j=1}^{\phi} w_{k,j} h_j(s') \right] 
\leq \frac{1}{1-\gamma} \left[ |\tilde{R}_k(x_{k,1}) - \tilde{R}_k^*(x_{k,1})| + \sum_{j=1}^{\phi} w_{k,j} \sum_{s' \in S} (P^k(s' | x_{k,1}) - P^*(s' | x_{k,1})) h_j(s') \right] 
\]

(56)

Since \( x_{k,1} = (s_{t+1}, \mu_k(s_{t+1})) \) we can simplify further. Denote \( \mu_k(s_{t+1}) \) as \( a_{k,1} \) and we have the following for the rightmost transition function term by Hölder’s inequality.

\[
\sum_{j=1}^{\phi} w_{k,j} \sum_{s' \in S} (P^k(s' | x_{k,1}) - P^*(s' | x_{k,1})) h_j(s') 
= \sum_{j=1}^{\phi} w_{k,j} \sum_{s' \in \text{Val}(Z_j^R)} (P^k(s' | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1}) - P^*(s' | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1}) h_j(s') 
\leq \|w_k^R\|_1 \max_j \left[ \sum_{s' \in \text{Val}(Z_j^R)} (P^k(s' | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1}) - P^*(s' | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1}) h_j(s') \right] 
\leq \|w_k^R\|_1 \max_j \left[ \max_{s' \in \text{Val}(Z_j^R)} \left( h_j(s') \right) \|P^k(\cdot | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1}) - P^*(\cdot | s_{t+1} | \text{Pa}(Z_j^R)), a_{k,1})) \|_1 \right] 
\]

(60)

This shows that the one step Bellman error is bounded by the diameter of our convex set for \( w \) and a maximum over all basis function transition confidence set accuracy products. Finally, we can also bound the reward function term factor by factor by the triangle inequality:

\[ |\tilde{R}_k(x_{k,1}) - \tilde{R}_k^*(x_{k,1})| \]

(61)

\[ = \sum_{i=1}^{l} |R_i^k(x_{k,1}) - R_i^*(x_{k,1})| \]

(62)

\[ \leq \sum_{i=1}^{l} |R_i^k(x_{k,1} | Z_i^R) - R_i^*(x_{k,1} | Z_i^R)|. \]

(63)

D.1 Concentration Guarantees

We will use the guarantees provided by Osband and Van Roy (2014).
Lemma 6. For all finite sets \( \mathcal{X} \), finite sets \( \mathcal{Y} \), function classes \( \mathcal{P} \subseteq \mathcal{P}_{\mathcal{X},\mathcal{Y}} \), then for any \( x \in \mathcal{X} \), \( \epsilon > 0 \) the deviation of the true distribution \( P^* \) to the empirical estimate after \( t \) samples \( \hat{P}_t \) is bounded:

\[
\mathbb{P}(\|P^*(x) - \hat{P}_t(x)\|_1 \geq \epsilon) \leq \exp \left( \frac{\|\mathcal{Y}\| \log(2) - \frac{n_t(x)^2}{2}}{2} \right)
\]  

(64)

Proof. Osband and Van Roy (2014) claims that this is a relaxation of a proof by Weissman, Ordentlich, Seroussi, Verdú, and Weinberger (2003).

One can show Lemma 6 ensures that for any \( x \in \mathcal{X} \) \( \mathbb{P}(\|P_j^*(x) - \hat{P}_j(x)\|_1 \geq \sqrt{\frac{2\|\text{Val}(Z_j^h)\| \log(2) - 2 \log(\delta')}{n_t(x)}}) \leq \delta' \). Note that previous analysis in Osband and Van Roy (2014) has a minor technical error which changes the choice of \( \epsilon \) (Appendix D.2).

The number of marginal transition function confidence sets that we have is given by \( \delta = |\mathcal{A}| \sum_{j=1}^\phi |\text{Val}(\text{Pa}[Z_j^h])| \). Let us give them some ordering \( i \in [N] \). Then we define a sequence for each confidence set at each episode \( d_{tk}^j = 2|\text{Val}(Z_j^h)| \log(2) - 2 \log(\delta_{k,i}) \), where \( \delta_{k,i} = \delta/(2N|\text{Pa}[Z_j^h]|k^2) \). Now with a union bound over all confidence set events over all time steps \( k \) we have that:

\[
\bigcup_{i=1}^N \bigcup_{k=1}^\infty \mathbb{P}(P_i^* \notin \mathcal{P}_i(d_{tk}^j)) \leq \bigcup_{i=1}^N \bigcup_{k=1}^\infty \sum_{\delta_{k,i} = \frac{\delta}{2N|\text{Pa}[Z_j^h]|k^2}} \sum_{i=1}^\infty \frac{\delta}{\sum_{i=1}^\infty \frac{\delta}{2N|\text{Pa}[Z_j^h]|k^2}} \leq \delta
\]  

(65)

Then we define a sequence for each confidence set at each episode \( d_{tk}^j = 2|\text{Val}(Z_j^h)| \log(2) - 2 \log(\delta_{k,i}) \), where \( \delta_{k,i} = \delta/(2N|\text{Pa}[Z_j^h]|k^2) \). Now with a union bound over all confidence set events over all time steps \( k \) we have that:

\[
\bigcup_{i=1}^N \bigcup_{k=1}^\infty \mathbb{P}(P_i^* \notin \mathcal{P}_i(d_{tk}^j)) \leq \sum_{i=1}^\infty \sum_{k=1}^\infty \delta_{k,i} \leq \delta
\]  

(66)

So we have that \( \mathbb{P}(P_i^* \in \mathcal{P}_i(d_{tk}^j) \forall k \in \mathbb{N}, \forall j \in [N]) \geq 1 - \delta. \)

Lemma 7. If \( \{\epsilon_i\} \) are all independent and sub \( \sigma \)-gaussian, then \( \forall \beta > 0 \):

\[
\mathbb{P}\left(\frac{1}{n} \sum_{z=1}^n \epsilon_z > \beta \right) \leq \exp \left( \frac{\log(2) - \frac{n \beta^2}{2\sigma^2}}{2} \right).
\]  

(67)

In particular, we may use Lemma 7 to say that for any \( x \in \mathcal{X} \):

\[
\mathbb{P}\left(\frac{1}{n_t(x)} \sum_{z=1}^{n_t(x)} \hat{R}_{i,z}(x) - \mathcal{R}_i^*(x) > \sqrt{\frac{\sigma^2 \log(2\pi)}{n_t(x)}} \right) \leq \delta'.
\]  

(68)

Where the sub \( \sigma \)-gaussian random variable \( \hat{R}_{i,z} \) represents the empirical value of the \( i \)th component of the reward function at the \( z \)th time the pair \( x = (s, a) \) was observed before time \( t \). Recall that the true mean of the \( i \)th reward component \( \mathcal{R}_i^*(x) \) is a fixed scalar value. Now for each component of the factored reward function \( i = 1, \ldots, l \), define the sequence \( \delta_{k,i} = \delta/(2l|\mathcal{X}|Z_j^h)|k^2) \). With the same union bound as (65) over all confidence set events over all time steps \( k \), we have that:

\[
\bigcup_{i=1}^l \bigcup_{k=1}^\infty \mathbb{P}(\mathcal{R}_i^* \notin \mathcal{R}_i^*(d_{tk}^j)) \leq \sum_{i=1}^l \sum_{k=1}^\infty \delta_{k,i} \leq \delta.
\]  

(69)

Combining (65) and (69), we have that:

\[
\mathbb{P}\left(M^* \in \mathcal{M}_k \forall k \in \mathbb{N} \right) \geq 1 - 2\delta.
\]  

(70)
D.2 Aside: Technical Error in Osband and Van Roy

We point out a minor technical error in Osband and Van Roy (2014) which changes the analysis and simplification of the regret. In their Section 7.2, they claim that they may use $\epsilon = \sqrt{\frac{2|S_j|}{n_t(x)}} \log(\frac{2}{\delta'})$ with their Lemma 2 (our Lemma 6) to obtain the following: for any $x \in \mathcal{X}$ $\mathbb{P}\left(\|P_j^*(x) - \hat{P}_j_t(x)\|_1 \geq \epsilon \right) \leq \delta'$.

Plugging their choice of $\epsilon$ into Lemma 6, we get the following.

\[
P\left(\|P_j^*(x) - \hat{P}_j_t(x)\|_1 \geq \epsilon \right) \leq \exp\left(|S_j| \log(2) - \frac{n_t(x)\epsilon^2}{2}\right)
\]

(71)

\[
= \exp\left(|S_j| \log(2) - \frac{n_t(x)\frac{2|S_j|}{n_t(x)} \log(\frac{2}{\delta'})}{2}\right)
\]

(72)

\[
= \exp\left(|S_j| \log(2) - |S_j| \log(\frac{2}{\delta'})\right)
\]

(73)

\[
= \exp\left(|S_j| \log(\delta')\right)
\]

(74)

In Osband and Van Roy (2014), $|S_j| \in \mathbb{N}$ is the size of the scope for the $j$th transition function. In general, $|S_j| > 1$ implies $\exp\left(|S_j| \log(\delta')\right) > \delta'$. Therefore, they are assuming more tightness than they should with their empirical estimates of the transition functions. They subsequently use $d_{i_k} = 2|S_j| \log(\frac{2}{\delta'_{i_k,j}})$ as their increasing sequence, which incorrectly assumes the result above.

If we wish to end up with $\delta'$, we can solve for the correct $\epsilon$ as follows.

\[
\delta' = \exp\left(|S_j| \log(2) - \frac{n_t(x)\epsilon^2}{2}\right)
\]

(75)

\[
\log(\delta') = |S_j| \log(2) - \frac{n_t(x)\epsilon^2}{2}
\]

(76)

\[
\epsilon = \sqrt{\frac{2|S_j| \log(2) - 2\log(\delta')}{n_t(x)}}
\]

(77)

Now we let $d_{i_k} = 2|S_j| \log(2) - 2\log(\delta'_{i_k,j})$, where $\delta'_{k,j} = \delta/(2m|\mathcal{X}|Z_{k,j}^P)$ which is the same $\delta'_{k,j}$ value as from Osband and Van Roy (2014). Therefore as $k$ increases, so does $d_{i_k}$, and we still have the increasing sequence required for applications of Corollary 1.
D.3 Regret Bound

We can now analyze the regret bounds for our algorithm.

\[
\text{Regret}(T, \pi, M^*) = \sum_{k=1}^{\lceil T/\tau \rceil} \Delta_k = \sum_{k=1}^{\lceil T/\tau \rceil} \left[ \left( V_k^k(s_{t_k+1}) - V_k^* \right) + \left( V_\pi^k(s_{t_k+1}) - V_k^k(s_{t_k+1}) \right) \right]
\]

\[
\leq \sum_{k=1}^{\lceil T/\tau \rceil} \left[ \sqrt{\frac{1}{k}} \right] + \frac{1}{1-\gamma} \left[ \|w\|_1 \max_j \max_{s \in \text{Val}(Z^k)} |h_j(s)| \sqrt{2|T/\tau| \log(2/\delta)} \right]
\]

\[
+ \sum_{k=1}^{\lceil T/\tau \rceil} \sum_{i=1}^l |\mathcal{R}_i^k(x_{k,1}[Z_i^R]) - \mathcal{R}_i^k(x_{k,1}[Z_i^R])|
\]

\[
+ \sum_{k=1}^{\lceil T/\tau \rceil} \|w_k^k\|_1 \max_j \left[ \max_{s' \in \text{Val}(Z^k)} (|h_j(s')| \|P^k(s_{t_k+1}[Pa(Z_j^h)], a_{k,1}) - P^*(s_{t_k+1}[Pa(Z_j^h)], a_{k,1})\|_1) \right]
\]

With probability at least \(1 - \delta\) (PAC regret bound), and where \((1)\) is the planning oracle error contribution, \((2)\) is the contribution of the bounded martingale (Lemma 5) over all episodes with the Azuma-Hoeffding inequality from (52), \((3)\) is the contribution of the reward functions in the one step Bellman error, and \((4)\) is contribution from the marginal transition functions from (57). We begin by bounding \((1)\) ≤ \(2\sqrt{|T/\tau|}\) by integral sum bound. Next, let \(\max_j \max_{s \in \text{Val}(Z^k)} |h_j(s)| \leq G\) be some global bound on all the basis functions which must exist as the value function is bounded over a finite set. Then we can say:

\[(2) \leq \|w\|_1 G \sqrt{2|T/\tau| \log(2/\delta)}. \quad (78)\]

Henceforth, let \(|T/\tau| = K\) be the number of true episodes. For \((3)\), we apply Corollary 1 with \(T = K\), and plug in \(C_F = C\) as a width bound of each reward confidence set and \(d_{K}^R\) as our sequence:

\[(3) = \sum_{i=1}^{l} \sum_{k=1}^{K} |\mathcal{R}_i^k(x_{k,1}[Z_i^R]) - \mathcal{R}_i^k(x_{k,1}[Z_i^R])|
\]

\[= \sum_{i=1}^{l} \left[ 4(C|X[Z_i^R]| + 1) + 4\sqrt{2\sigma^2 \log(2/(\delta/2l|X[Z_i^R]|K^2))|X[Z_i^R]|K} \right]
\]

\[\leq \sum_{i=1}^{l} \left[ 5C|X[Z_i^R]| + 8\sigma \sqrt{|X[Z_i^R]|K \log(4l|X[Z_i^R]|K^2/\delta)} \right]
\]
We can bound the confidence sets of (4) by again applying Corollary 1.

\[
4 \leq \|w\|_1 G \sum_{k=1}^{K} \max_j \left[ \|P^k(\cdot|s_{t_k+1}|\text{Pa}(Z_j^h)) - P^*(\cdot|s_{t_k+1}|\text{Pa}(Z_j^h))\|_1 \right]
\]

\[
\leq \|w\|_1 G \sum_{k=1}^{K} \sum_{j=1}^{\phi} \left[ \|P^k(\cdot|s_{t_k+1}|\text{Pa}(Z_j^h)) - P^*(\cdot|s_{t_k+1}|\text{Pa}(Z_j^h))\|_1 \right]
\]

\[
\leq \|w\|_1 G \sum_{j=1}^{\phi} \left[ 5\|\text{Val}(Z_j^h)\| + 4\sqrt{4\|X[\text{Pa}(Z_j^h)]\|K[\|\text{Val}(Z_j^h)\| \log(2) - \log(\delta/2N\|\text{Pa}(Z_j^h)\|K^2))]} \right]
\]

(82)

(83)

(84)

(85)

Where \( \phi \) is the number of basis functions, and \( d_K^j = 2\|\text{Val}(Z_j^h)\| \log(2) - 2 \log(\delta/2N\|\text{Pa}(Z_j^h)\|K^2)) \) from our union bound.

Remark 6. Note that from (82) to (83) we do not have a dependence on the number of confidence sets \( N \) because we are conditioning on historical state action observations, with respect to individual basis function scopes \( Z_j^h \).

Theorem 4. Let \( M^* \) be an MDP with our special factored structure as well as an exactly linear factored optimal value function, and an efficient variable elimination ordering \( O \) be given. Using our procedure, we can bound the regret over \( T \) iterations (\( K \) episodes) for any \( M^* \), \( \text{Regret}(T, \pi_\tau, M^*) \)

\[
\leq 2 \sqrt{K} + \frac{1}{1-\gamma} \left[ \|w\|_1 G \sqrt{2K \log(2/\delta)} + \sum_{i=1}^{l} \left[ 5C|X[Z_i^R]| + 8\sigma \sqrt{|X[Z_i^R]|K \log(4l|X[Z_i^R]|K^2/\delta)} \right] \right]
\]

\[
+ \|w\|_1 G \sum_{j=1}^{\phi} \left[ 5\|\text{Val}(Z_j^h)\| + 4\sqrt{4\|X[\text{Pa}(Z_j^h)]\|K[\|\text{Val}(Z_j^h)\| \log(2) - \log(\delta/2N\|\text{Pa}(Z_j^h)\|K^2))]} \right]
\]

(86)

(87)

with probability at least \( 1 - \delta \).

We will simplify the bound in the symmetric case similar to Osband and Van Roy (2014).

Theorem 5. Let \( l + 1 \leq \phi, \sigma = 1, |S_i| = |X_i| = \kappa, |Z_i^R| = |\text{Pa}(Z_i^h)| = \zeta \) for all \( i \), and let \( J = \kappa^c \), and \( \|w\|_1 \leq W \). Then we have that:

\[
\text{Regret}(T, \pi_\tau, M^*) \leq \frac{1}{1-\gamma} \left( 30\phi W G \sqrt{KJ(J \log(2) + \log(2N\zeta K^2/\delta))} \right)
\]

(88)

with probability at least \( 1 - 3\delta \), where \( K = \lceil T/\tau \rceil \) is the number of observed episodes.

Proof. Assume \( WG \geq 1 \), then

\[
\text{Regret}(T, \pi_\tau, M^*) \leq 2 \sqrt{K} + \frac{1}{1-\gamma} \left( W G \sqrt{2K \log(2/\delta)} + \phi \left[ 5J + 8\sqrt{JK \log(4\phi JK^2/\delta)} \right] \right)
\]

\[
+ W G \phi \left[ 5J + 4\sqrt{JK(J \log(2) - \log(\delta/2N\zeta K^2))} \right]
\]

\[
\leq \frac{1}{1-\gamma} \left( 5J(1 + WG) + \sqrt{K} \left[ 2 + W G \sqrt{2\log(2/\delta)} \right. \right. \right.

\[
+ W G \phi \left( J \log(4\phi JK^2/\delta) + W G \phi \sqrt{J^2 \log(2)} + J \log(2N\zeta K^2/\delta) \right) \right)
\]

(89)

(90)

(91)

(92)
To combine the two rightmost square root terms, we compare the terms inside the logarithms:

\[ 2\zeta N \geq 4\phi J \] (93)

\[ 2\zeta|\mathcal{A}| \sum_{j=1}^{\phi} |\text{Val}(\mathcal{P}_j[Z])| = 2\zeta|\mathcal{A}|\phi J \geq 4\phi J \] (94)

\[ |\mathcal{A}| \geq \frac{2}{\zeta} \] (95)

Which is true for any non-trivial MDP with more than a single action. Therefore:

\[ \leq \frac{1}{1-\gamma} \left( 10\phi JW G + \sqrt{K} \left[ 2 + W G \sqrt{2 \log(2/\delta)} + 16\phi W G \sqrt{J^2 \log(2) + J \log(2N\zeta K^2/\delta)} \right] \right) \] (96)

\[ \leq \frac{1}{1-\gamma} \left( 10\phi JW G + 18\phi W G \sqrt{K(J^2 \log(2) + J \log(2N\zeta K^2/\delta))} \right) \] (97)

\[ \leq \frac{1}{1-\gamma} \left( 10\phi W G \sqrt{KJ^2} + 18\phi W G \sqrt{KJ(J \log(2) + \log(2N\zeta K^2/\delta))} \right) \] (98)

\[ \leq \frac{1}{1-\gamma} \left( 30\phi W G \sqrt{KJ(J \log(2) + \log(2N\zeta K^2/\delta))} \right) \] (99)

**Remark 7.** Our bound is similar to Corollary 2 from Osband and Van Roy (2014), but not identical. First, since we are considering a modified discounted scenario, we have a natural \( \frac{1}{1-\gamma} \) term in our bound. Next, we are dependent on \( \sqrt{K} \), the number of true episodes, rather than the number of total time steps \( T \). We have an extra \( \sqrt{J} \) cost due to the support of each transition marginal function we are estimating being of size \( J \) and not of size \( \kappa \). This follows naturally from considering non-independent transition functions. Instead of counting the number of state variables \( m \) in our regret, we instead have a factor of \( \phi \), the number of basis functions. Osband and Van Roy (2014) has a factor of the diameter in their simplified bound, which they replace with \( \tau \), the time horizon. On the other hand, our bound does not rely on the diameter and instead depends only on the 1-norm of the domain of the basis vector \( W \), and the max value that any basis function can take over its domain \( G \). Our only dependence on \( \tau \) is to determine the number of episodes, \( K = \lceil T/\tau \rceil \). Finally, our proposed fix for the aforementioned technical error (Appendix D.2) leads to the extra RHS term within the square root.

**Remark 8.** According to our regret bounds (87), (D.4), we don’t have dependency on the diameter \( D \) of MDP, which is the maximum number of expected time steps to get between any two states. Such dependency is present in Osband and Van Roy (2014). The only place in our analysis that could have potentially introduced this dependency is the argument following (60), (63). Due to triangle inequality, these two bounds rely on the empirical observations of the reward and transition probability at \( x_{k,1} \), since the confidence bounds are centered around the empirical observations. If one such \( x_{k,1} \) has not been observed then it will introduce trivial confidence bounds because \( n_{t_k+1}(x_{k,1}) = 0 \) in \( \sqrt{d_{t_k+1}/n_{t_k+1}(x_{k,1})} \). Because of this, one could have argued that \( T \) should be at least as large as \( D \) so we can visit every \( x \) to avoid such trivial bound. However, since each \( x_{k,1} \) has already appeared in our history up to \( t_k + 1 \), we have \( n_{t_k+1}(x_{k,1}) \geq 1 \). So such concern is dismissed. A consequence of our diameter independency is that we do not need to assume connectedness within the MDP \( M^* \). Indeed, if some state \( s' \) is not reachable, then the empirical observation of the transition from \( s_{t_k+1} \) to \( s' \) would simply be 0, and this does not change the analysis.

### D.4 Extension to Episodic Setting

We restate our main result.
Theorem 3. Let $M^*$ be a Correlated FMDP with $V^*_t$ having a linear decomposition. Let $l + 1 \leq \phi$, $C = \sigma = 1$, $|S_t| = |X_t| = \kappa$, $|Z_t^R| = \zeta$ for all $t$, and let $J = \kappa^3$, $\|\mathbf{w}\| \leq W$, and $\max_j \max_{s \in \text{Val}(Z_t^R)} |h_j(s)| \leq G$. Assuming $WG \geq 1$, and an efficient variable elimination ordering, then

\[
\text{Regret}(T, \pi, M^*) \leq \tau \left( 30\phi W G \sqrt{T J (J \log(2) + \log(2N\zeta T^2/\delta))} \right) \quad \text{w.p. at least } 1 - 3\delta.
\]

Proof. We follow exactly the same regret analysis presented within the earlier sections of Appx. D. The difference here is mainly one of notation. For the full algebra please see the proofs of Thm. 4 and Thm. 5. In particular, substitute the time-independent Bellman operator with the time dependant one:

\[
T^M_\mu V(s) := R^M(s, \mu(s)) + \sum_{s' \in S} P^M(s'|s, \mu(s))V(s')
\]

which is the same as the bellman operator defined in Sec. 3 but without the $\gamma$. We have already shown that our planner works in this setting (see Sec. 3.3), so we begin our regret analysis. This heavily follows the notation and proofs in Osband and Van Roy (2014), which we refer the interested reader to.

As in Osband and Van Roy (2014), let

\[
V^M_{\mu,i} := \mathbb{E}_{M,\mu} \left[ \sum_{j=i}^{\tau} R^M_j(s_j, a_j) \big| s_i = s \right]
\]

be the value function following policy $\mu$ starting at time step $i \in [\tau]$ in MDP $M$. Osband and Van Roy (2014) show that regret at step $k$ in this setting can be decomposed as follows.

\[
\Delta_k = V^*_{k+1}(s) - V^*_k(s) = \left( V^k_{k+1}(s) - V^*_k(s) \right) + \left( V^*_{k+1}(s) - V^k_{k+1}(s) \right)
\]

Again, the second term can be bounded by our choice of planning accuracy $\sqrt{1/k}$ through optimism. Decompose the first term as follows.

\[
(V^k_{k+1} - V^*_k)(s_{t+1}) = \sum_{i=1}^{\tau} (T^k_{k,i} - T^*_{k,i}) V^k_{k,i+1}(s_{t_1+i}) + \sum_{i=1}^{\tau} d_{t+i}
\]

\[
d_{t+i} := \sum_{s \in S} \left( P^*(s|x_{k,i})(V^*_{k,i+1} - V^k_{k,i+1})(s) \right) - (V^*_{k,i+1} - V^k_{k,i+1})(s_{t_1+i}) \text{ is the martingale difference which we bound with Lemma. 5 by dropping the } \gamma \text{ from the proof and replacing } [T/\tau] \text{ everywhere with just } T \text{ (see for example (52)).}
\]

We can again utilize the result in equation (8) of Osband and Van Roy (2014), and also all their confidence set results. The only place we diverge from them is in our application of Hölder’s. Starting from the analogous place to (56), where instead of a $\frac{1}{\sqrt{\gamma}}$, we have a sum over $\tau$ steps within the episode. For the first term above, we can bound by the following.

\[
\sum_{i=1}^{\tau} (T^k_{k,i} - T^*_{k,i}) V^k_{k,i+1}(s_{t_1+i})
\]

\[
\leq \sum_{i=1}^{\tau} \left[ |R^k(x_{k,i}) - R^* (x_{k,i})| + \phi \left| \sum_{j=1}^{\phi} u^{k,j}_{k}(s') \right| \sum_{j=1}^{\phi} (P^k(s'|x_{k,i}) - P^*(s'|x_{k,i})) \|h^{j}(s')\| \right]
\]

\[
\leq \sum_{i=1}^{\tau} \left[ \|\mathbf{w}\| \max_j \max_{s' \in \text{Val}(Z_{t+1}^R)} \left( \|h^{j}(s')\| \left( \|P^k(\cdot|s_{t+1}^{k}[\text{Pa}(Z_{t+1}^R)], a_{k,1}) - P^*(\cdot|s_{t+1}^{k}[\text{Pa}(Z_{t+1}^R)], a_{k,1})\| \right) \right] + \sum_{i=1}^{\tau} \left| R^k_{k,i}(x_{k,i}|Z_{t+1}^R) - R^*_{k,i}(x_{k,i}|Z_{t+1}^R) \right|
\]

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See (57) and (61) for the details. Notice that we have arrived at the exact start of Appx. D.3, except we have replaced $K$ with $T$ and $\frac{1}{1-\gamma}$ with $\tau$ in the final bound because of the outer sum $\sum_{i=1}^{\tau}$. There are no further deviations.