A new class of generalized Genocchi polynomials

N. I. Mahmudov
Eastern Mediterranean University
Gazimagusa, TRNC, Mersin 10, Turkey
Email: nazim.mahmudov@emu.edu.tr

February 2, 2012

Abstract

The main purpose of this paper is to introduce and investigate a new class of generalized Genocchi polynomials based on the $q$-integers. The $q$-analogues of well-known formulas are derived. The $q$-analogue of the Srivastava–Pintér addition theorem is obtained.

1 Introduction

Throughout this paper, we always make use of the following notation: $N$ denotes the set of natural numbers, $N_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The $q$-shifted factorial is defined by

$$ (a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}, \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, \quad a \in \mathbb{C}. $$

The $q$-numbers and $q$-numbers factorial is defined by

$$ [a]_q = \frac{1 - q^a}{1 - q}, \quad (q \neq 1); \quad [0]_q = 1; \quad [n]_q! = [1]_q [2]_q ... [n]_q \quad n \in \mathbb{N}, \quad a \in \mathbb{C} $$

respectively. The $q$-polynomial coefficient is defined by

$$ \left\lbrack \frac{n}{k} \right\rbrack_q = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k}. $$

The $q$-analogue of the function $(x + y)^n$ is defined by

$$ (x + y)^n_q := \sum_{k=0}^{n} \left\lbrack \frac{n}{k} \right\rbrack_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k, \quad n \in \mathbb{N}_0. $$

In the standard approach to the $q$-calculus two exponential function are used:

$$ e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q) q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1 - q|}, $$

$$ E_q(z) = \sum_{n=0}^{\infty} \frac{q^\frac{1}{2} n(n-1)\cdots(n-k) z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1 + (1 - q) q^k z), \quad 0 < |q| < 1, \quad z \in \mathbb{C}. $$
From this form we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(qz),$$

where $D_q$ is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}.$$

Carlitz has introduced the $q$-Bernoulli numbers and polynomials in [1]. Srivastava and Pinter proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [22]. They also gave some generalizations of these polynomials. In [13]-[17], Kim et al. investigated some properties of the $q$-Bernoulli polynomials and Genocchi polynomials. They gave some recurrence relations. In [2], Cenkci et al. gave $q$-Euler polynomials and Genocchi polynomials. They gave some generalizations.

**Definition 1** The $q$-Bernoulli numbers $B_{n,q}^{(\alpha)}$ and polynomials $B_{n,q}^{(\alpha)}(x,y)$ in $x, y$ of order $\alpha$ are defined by means of the generating function functions:

$$\left(\frac{t}{e_q(t) - 1}\right)^\alpha = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)} \frac{t^n}{q^n}, \quad |t| < 2\pi,$$

$$\left(\frac{t}{e_q(t) - 1}\right)^\alpha e_q(t) E_q(t) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y) \frac{t^n}{q^n}, \quad |t| < 2\pi.$$

**Definition 2** The $q$-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x,y)$ in $x, y$ are defined by means of the generating functions:

$$\left(\frac{2t}{e_q(t) + 1}\right)^\alpha = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{q^n}, \quad |t| < \pi,$$

$$\left(\frac{2t}{e_q(t) + 1}\right)^\alpha e_q(t) E_q(t) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y) \frac{t^n}{q^n}, \quad |t| < \pi.$$

It is obvious that

$$B_{n,q}^{(\alpha)} = B_{n,q}^{(\alpha)}(0,0), \quad \lim_{q \to 1^-} B_{n,q}^{(\alpha)}(x,y) = B_n^{(\alpha)}(x+y), \quad \lim_{q \to 1^-} B_{n,q}^{(\alpha)} = B_n^{(\alpha)},$$

$$G_{n,q}^{(\alpha)} = G_{n,q}^{(\alpha)}(0,0), \quad \lim_{q \to 1^-} G_{n,q}^{(\alpha)}(x,y) = G_n^{(\alpha)}(x+y), \quad \lim_{q \to 1^-} G_{n,q}^{(\alpha)} = G_n^{(\alpha)}.$$

Here $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ denote the classical Bernoulli and Genocchi polynomials of order $\alpha$ are defined by

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{tx} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^t + 1}\right)^\alpha e^{tx} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

The aim of the present paper is to obtain some results for the $q$-Genocchi polynomials. The $q$-analogues of well-known results, for example, Srivastava and Pinter [2], Cheon [3], etc., can be derived from these $q$-identities. The formulas involving the $q$-Stirling numbers of the second kind, $q$-Bernoulli polynomials and $q$-Bernstein polynomials are also given. Furthermore some special cases are also considered.

The following elementary properties of the $q$-Genocchi polynomials $G_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$ are readily derived from Definition. We choose to omit the details involved.

**Property 1.** Special values of the $q$-Genocchi polynomials of order $\alpha$:

$$G_{n,q}^{(0)}(x,0) = x^n, \quad G_{n,q}^{(0)}(0,y) = q^\frac{1}{2} n^{(n-1)} y^n.$$
Property 2. Summation formulas for the $q$-Genocchi polynomials of order $\alpha$:

\[
\mathfrak{G}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\alpha-k} \mathfrak{G}_k(x,y), \quad \mathfrak{G}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\alpha-k} \mathfrak{G}_k(x,y),
\]

\[
\mathfrak{G}_n^{(\alpha)}(0,y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\alpha-k} \mathfrak{G}_k(0,y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\alpha-k} \mathfrak{G}_k(0,y).
\]

Property 3. Difference equations:

\[
\mathfrak{G}_n^{(\alpha)}(1,y) + \mathfrak{G}_n^{(\alpha)}(0,y) = 2 [n]_q \mathfrak{G}_n^{(\alpha-1)}(0,y), \quad \mathfrak{G}_n^{(\alpha)}(x,0) + \mathfrak{G}_n^{(\alpha)}(x,-1) = 2 [n]_q \mathfrak{G}_n^{(\alpha-1)}(x,-1).
\]

Property 4. Differential relations:

\[
D_{q,x} \mathfrak{G}_n^{(\alpha)}(x,y) = [n]_q \mathfrak{G}_n^{(\alpha-1)}(x,y), \quad D_{q,y} \mathfrak{G}_n^{(\alpha)}(x,y) = [n]_q \mathfrak{G}_n^{(\alpha)}(x,qy).
\]

Property 5. Addition theorem of the argument:

\[
\mathfrak{G}_n^{(\alpha+\beta)}(x,y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\alpha-k} \mathfrak{G}_k(x,0) \mathfrak{G}_k^{(\beta)}(0,y).
\]

Property 6. Recurrence Relationships:

\[
\mathfrak{G}_n^{(\alpha)}(\frac{1}{m},y) + \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q \left( \frac{1}{m} - 1 \right)^{n-k} \mathfrak{G}_k^{(\alpha)}(0,y) = 2 [n]_q \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] q \left( \frac{1}{m} - 1 \right)^{n-1-k} \mathfrak{G}_k^{(\alpha-1)}(0,y).
\]

2 Explicit relationship between the $q$-Genocchi and the $q$-Bernoulli polynomials

In this section we prove an interesting relationship between the $q$-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x,y)$ of order $\alpha$ and the $q$-Bernoulli polynomials. Here some $q$-analogues of known results will be given. We also obtain new formulas and their some special cases below.

Theorem 3 For $n \in \mathbb{N}_0$, the following relationship

\[
\mathfrak{G}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{1}{m^{k+1} - 1} \left[ k + 1 \right]_q \sum_{j=0}^{k} \frac{1}{m^{k-j} - 1} \mathfrak{G}_j^{(\alpha-1)}(x,-1) - \mathfrak{G}_n^{(\alpha)}(x,-1) - \mathfrak{G}_n^{(\alpha)}(x,0)\right] \mathfrak{B}_{n-k,q}(0,my).
\]

holds true between the $q$-Genocchi and the $q$-Bernoulli polynomials.

Proof. Using the following identity

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx) E_q(ty) = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx) \cdot \frac{e_q \left( \frac{t}{m} \right) - 1}{t} = \frac{t}{e_q \left( \frac{t}{m} \right) - 1} \cdot E_q \left( \frac{t}{m} \right)
\]

3
we have
\[\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = -\sum_{n=1}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[n-k]_q!} \left( \frac{1}{[n-k]_q} \mathcal{G}_{k,q}^{(\alpha)}(x,0) - \mathcal{G}_{n,q}^{(\alpha)}(x,0) \right) \right) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(0,my) \frac{t^n}{m^n [n]_q!} \]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{[k+1]_q} \left( \sum_{j=0}^{k+1} \binom{k+1}{j} m^j \mathcal{G}_{j,q}^{(\alpha)}(x,0) - m^{k+1} \mathcal{G}_{k+1,q}^{(\alpha)}(x,0) \right) \mathcal{B}_{n-k,q}(0,my) \frac{t^n}{[n]_q!}.\]

It remains to use Property 6. 

Since \(\mathcal{G}_{n,q}^{(\alpha)}(x,y)\) is not symmetric with respect to \(x\) and \(y\) we can prove a different from of the above theorem. It should be stressed out that Theorems 3 and 4 coincide in the limiting case when \(q \to 1^+\).

**Theorem 4** For \(n \in \mathbb{N}_0\), the following relationship
\[\mathcal{G}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[n-k]_q!} \left[ 2 \frac{k + 1}{q} \mathcal{G}_{k,q}^{(\alpha)}(x,0) - \mathcal{G}_{n,q}^{(\alpha)}(x,0) \right] \mathcal{B}_{n-k,q}(mx,0) \]
holds true between the q-Genocchi and the q-Bernoulli polynomials.

**Proof.** The proof is based on the following identity
\[\left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx) E_q(ty) = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha E_q(ty) \cdot \frac{e_q(tm/x - 1)}{t} \cdot \frac{t}{e_q(tm/x - 1)} \cdot e_q(tm/x).\]

Next we discuss some special cases of Theorems 3 and 4 By noting that
\[\mathcal{G}_{j,q}^{(0)}(0,y) = q^{\frac{j}{m} (y-1)} y^j, \quad \mathcal{G}_{j,q}^{(0)}(x,-1) = (x-1)^j q^j\]
we deduce from Theorems 3 and 4 Corollary 5 below.

**Corollary 5** For \(n \in \mathbb{N}_0, m \in \mathbb{N}\) the following relationship
\[\mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[n-k]_q!} \left[ 2 \frac{k + 1}{q} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{m} - 1 \right)^q q^{\frac{j}{m} (y-1)} y^j \right] \mathcal{B}_{n-k,q}(mx,0),\]
\[\mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[n-k]_q!} \left[ 2 \frac{k + 1}{q} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{m} - 1 \right)^q m^{k-j}(x-1)^j \right] \mathcal{B}_{n-k,q}(x-1),\]
\[\mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[n-k]_q!} \left[ 2 \frac{k + 1}{q} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{m} - 1 \right)^q m^{k-j}(x-1)^j \right] \mathcal{B}_{n-k,q}(x,0),\]
holds true between the q-Bernoulli polynomials and q-Euler polynomials.
Corollary 6 For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.

\begin{align*}
G_n(x + y) &= \sum_{k=0}^{n} \binom{n}{k} \frac{2}{k+1} ((k+1) y^k - G_{k+1,q}(y)) B_{n-k}(x), \\
G_n(x + y) &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}(k+1)} \left[ 2(k+1) G_k \left( y + \frac{1}{m} - 1 \right) - G_{k+1} \left( y + \frac{1}{m} - 1 \right) - G_{k+1}(y) \right] B_{n-k,q}(mx) \tag{1}.
\end{align*}

between the classical Genocchi polynomials and the classical Bernoulli polynomials.

Note that the formula (2) is new for the classical polynomials.

In terms of the $q$-Genocchi numbers $G_{n,q}^{(0)}$, by setting $y = 0$ in Theorem 3, we obtain the following explicit relationship between the $q$-Genocchi polynomials $G_{n,q}^{(0)}$ of order $\alpha$ and the $q$-Bernoulli polynomials.

Corollary 7 The following relationship holds true:

\[
G_{n,q}^{(\alpha)}(x, 0) = \sum_{k=0}^{n} \binom{n}{k} q^{2[k+1]} \left[ k+1 \right] q \left[ \sum_{j=0}^{k} \binom{k}{j} q^{j} \left[ \frac{1}{m} - 1 \right] q G_{j,q}^{(\alpha-1)} - G_{j+1,q}^{(\alpha)} \right] B_{n-k,q}(mx, 0).
\]

Corollary 8 For $n \in \mathbb{N}_0$ the following relationship holds true.

\[
G_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q^{2[k+1]} \left[ k+1 \right] q \left[ \sum_{j=0}^{k} \binom{k}{j} q^{j} G_{k+1,q}^{(0)} - G_{k+1,q}^{(0)} \right] B_{n-k,q}(mx, 0).
\]

Corollary 9 For $n \in \mathbb{N}_0$ the following relationship holds true.

\[
G_{n,q}(x, 0) = -\sum_{k=0}^{n} \binom{n}{k} q^{2[k+1]} G_{k+1,q} B_{n-k,q}(x, 0),
\]

\[
G_{n,q} = -\sum_{k=0}^{n} \binom{n}{k} q^{2[k+1]} G_{k+1,q} B_{n-k,q}.
\]

References

[1] L. Carlitz, $q$-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948), 987-1000.
[2] M. Cenkci, M. Can, and V. Kurt, $q$-extensions of Genocchi numbers, J. Korean Math. Soc. 43 (2006), no. 1, 183-198.
[3] G. Gasper, Lecture notes for an introductory minicourse on $q$-series, arXiv.math.CA/9509223.
[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.
[5] A. S. Hegazi and M. Mansour, A note on $q$-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys. 13 (2006), no. 1, 9-18.
[6] L. C. Jang and T. Kim, $q$-Genocchi numbers and polynomials associated with fermionic $p$-adic invariant integrals on $\mathbb{Z}_p$, Abstr. Appl. Anal. 2008 (2008), Art. ID 232187, 8 pp. doi:10.1155/2008/232187.

[7] L. C. Jang, T. Kim, D. H. Lee, and D. W. Park, An application of polylogarithms in the analogue of Genocchi numbers, NNTDM 7 (2000), 66(70).

[8] V. Kac and P. Cheung, Quantum Calculus, Springer Verlag, New York, 2002.

[9] T. Kim, $q$-generalized Euler numbers and polynomials, Russ. J. Math. Phys. 13 (2006), no. 3, 293(298).

[10] T. Kim, On the $q$-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007), no. 2, 1458(1465).

[11] T. Kim, A note on $p$-adic $q$-integral on $\mathbb{Z}_p$ associated with $q$-Euler numbers, Adv. Stud. Contemp. Math. (Kyungshang) 15 (2007), no. 2, 133(137).

[12] T. Kim, $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals, J. Nonlinear Math. Phys. 14 (2007), no. 1, 15(27).

[13] T. Kim, A note on the $q$-Genocchi numbers and polynomials, J. Inequal. Appl. 2007 (2007), Art. ID 71452, 8 pp. doi:10.1155/2007/71452.

[14] T. Kim, On the multiple $q$-Genocchi and Euler numbers, Russ. J. Math. Phys. 15 (2008), no. 4, 481(486).

[15] T. Kim, Note on $q$-Genocchi numbers and polynomials, Adv. Stud. Contemp. Math. (Kyungshang) 17 (2008), no. 1, 9(15).

[16] T. Kim, L.-C. Jang, and H.-K. Pak, A note on $q$-Euler and Genocchi numbers, Proc. Japan Acad. Ser. A Math. Sci. 77 (2001), no. 8, 139(141).

[17] T. Kim, M.-S. Kim, L.C. Jang, and S.-H. Rim, New $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals, Adv. Stud. Contemp. Math. (Kyungshang) 15 (2007), no. 2, 243(252).

[18] T. H. Koornwinder, Special functions and $q$-commuting variables, Special functions, q-series and related topics (Toronto, ON, 1995), 131(166, Fields Inst. Commun., 14, Amer. Math. Soc., Providence, RI, 1997.

[19] B. A. Kupershmidt, Reflection symmetries of $q$-Bernoulli polynomials, J. Nonlinear Math. Phys. 12 (2005), suppl. 1, 412(422).

[20] Y. Simsek, I. N. Cangul, V. Kurt, and D. Kim, $q$-Genocchi numbers and polynomials associated with $q$-Genocchi-type l-functions, Adv. Difference Equ. 2008 (2008), Art. ID 815750, 12 pp. doi:10.1155/2008/85750.

[21] A. De Sole and V. Kac, On integral representations of $q$-gamma and $q$-beta functions, arXiv:math QA/0302032.

[22] H. M. Srivastava and A. Pinter, Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett. 17 (2004), no. 4, 375(380).