Analytical Solution of Covariance Evolution for Regular LDPC Codes

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Abstract—The covariance evolution is a system of differential equations with respect to the covariance of the number of edges connecting to the nodes of each residual degree. Solving the covariance evolution, we can derive distributions of the number of check nodes of residual degree 1, which helps us to estimate the block error probability for finite-length LDPC code. Amraoui et al. resorted to numerical computations to solve the covariance evolution. In this paper, we give the analytical solution of the covariance evolution.

I. INTRODUCTION

Gallager invented low-density parity-check (LDPC) codes [1] in 1963. LDPC codes are linear codes defined by sparse bipartite graphs. Luby et al. introduced the peeling algorithm (PA) [2], [4] for the binary erasure channel (BEC). PA is an iterative algorithm which is defined on Tanner graphs. PA and brief propagation (BP) decoder have the same decoding result. As PA proceeds, edges and nodes are progressively removed. The residual graphs consist of nodes and edges that are still unknown at each iteration. The decoding successfully halts if the graph vanishes.

Amraoui [3] showed that distributions of the number of check nodes of degree one in the residual graph convergences weakly to a Gaussian as blocklength tends to infinity. Amraoui also showed that block and bit error probability of finite-length LDPC codes are derived by the average and the variance of the number of check nodes of degree one in the residual graph.

The average number of check nodes of degree one in the residual graph is also determined from a system of equations, which was derived and solved by Luby et al. [2]. In the residual graph, the number of check nodes of degree one is determined from a system of differential equations called covariance evolution, which was derived by Amraoui et al. [3]. Since analytical solution of covariance evolution has not been known so far, we had to resort to numerical computations to solve the covariance evolution.

An alternative way to determine the variance of the number of check nodes of degree one was proposed in [3]. The variance of the number of check nodes of degree one in the residual graph can be computed by determining the variance of the number of erased messages of BP for BEC with parameter $\epsilon^*$ where $\epsilon^*$ is the threshold of the ensemble under BP decoding. This method is a valid approximation for the erasure probability close to $\epsilon^*$. Moreover, Ezri et al. extended to this method to more general channels [5]. However, if we solve the covariance evolution analytically, we can derive the variance of the number of check nodes of degree one in the residual graph for all $\epsilon$ where $\epsilon$ is the channel parameter for the BEC.

In this paper, we show an analytical solution of the covariance evolution for regular LDPC code ensembles.

II. COVARIANCE EVOLUTION [3]

In this section, we briefly review the covariance evolution and initial covariance in [3].

We consider the transmission over the BEC with channel erasure probability $\epsilon$ using LDPC codes in a $(b, d)$-regular LDPC code ensemble. Let $l$ denote the iteration round and $\xi$ be the total number of edges in the original graph. We define that

$$\tau := \frac{t}{\xi}. \tag{1}$$

Define a parameter $y$ such that $dy/d\tau = -1/(ey^{b-1})$ and $y = 1$ when $\tau = 0$. Let $b, t$ denote a random variable corresponding to the number of edges connecting to variable nodes of degree $b$ in the residual graph at the iteration round $t$. Let $r_{k, t}$ denote a random variable corresponding to the number of edges connecting to check nodes of degree $k$ in the residual graph at the iteration round $t$. Those random variables depends on the choice of the graph from $(b, d)$-regular LDPC code ensemble, the channel outputs and the random choices made by PA. We define

$$D_t := \{l_{b, t}, r_{1, t}, r_{2, t}, \ldots, r_{d-1, t}\}.$$

To simplify the notation, we drop the subscript $t$. For $i \in D_t$, we define $i(y)$ by

$$\tilde{i}(y) := \frac{E[i]}{\xi}.$$

We also define $\delta^{(i,j)}(y)$ by the covariance of $i$ and $j$ ($i, j \in D_t$) divided by the total number of edges in the original graph i.e.

$$\delta^{(i,j)}(y) := \frac{\text{Cov}[i, j]}{\xi}.$$
This system is referred to as covariance evolution.

\[
\frac{d\hat{f}^{(i,j)}(y)}{dy} = -\frac{e(y)}{y} \sum_{k \in D} \left( \frac{\partial f^{(i)}}{\partial k} \delta^{(i,k)} + \frac{\partial \hat{f}^{(j)}}{\partial k} \delta^{(i,k)} \right) + \hat{f}^{(i,j)}(y),
\]

where

\[
\frac{\partial \hat{f}^{(l,b)}}{\partial l_b} = 0, \quad \frac{\partial \hat{f}^{(l,b)}}{\partial r_f} = 0,
\]

\[
\frac{\partial \hat{f}^{(r_j)}}{\partial r_j} = -j(b-1)\bar{\delta}^{r_j} + x_j y_j - \frac{x_j y_j}{l_b^2},
\]

\[
\frac{\partial \hat{f}^{(r_1, r_j)}}{\partial r_f} = \frac{b-1}{l_b} \left\{ \left( \tilde{r}_1 + \bar{\delta}^{r_1} - \bar{\delta}^{r_1} \right) \right\}.
\]

\[
\frac{\partial \hat{f}^{(r_1, r_j)}}{\partial r_f} = \frac{k_j(b-1)}{l_b} \left\{ (\tilde{r}_j + 1 - \bar{\delta}^{r_1}) (\tilde{r}_j + 1 - \bar{\delta}^{r_j}) \right\}.
\]

Theorem 1. Let \( \tau \) be the normalized iteration round of PA as defined in [1]. For a \((b,d)\)-regular LDPC code ensemble and \( j, k \in \{1,2,\ldots,d-1\} \), in the limit of the code length, we obtain the following.

\[
\delta^{(l,b)} = be\tilde{e},
\]

\[
\delta^{(l,b)} = -G_j \{ e\tilde{e} (b-1) y_j^{-1} + \tilde{e} x_j \} + \hat{I}_{(j=1)} be\tilde{e},
\]

\[
\delta^{(d,b)} = b - G_k G_j \{ e\tilde{e} (b-1) y_j^{-1} - (e - \tilde{e}) x_j y_j^{-1} + x_j^2 \}
\]

\[
- d \left( \frac{d-1}{b} \right) \left( \frac{d-1}{b-1} \right) x_j y_j^{-1} - \tilde{e} x_j - x_j^2,
\]

\[
\hat{I}_{(k=1)} G_j + \hat{I}_{(j=1)} G_k \{ e\tilde{e} (b-1) y_j^{-1} - \tilde{e} x_j + x_j^2 \},
\]

where \( G_j := \left( \frac{d-2}{b} \right) x_j^{-1} x_j^{-1} (x_j - 1) + \hat{I}_{(j=1)} \) and \( y \) is defined by \( dy/d\tau = -1/(e^{b-1}) \) with \( y = 1 \) when \( \tau = 0 \).

A. Scaling parameter \( \alpha \)

In [3], scaling parameter \( \alpha \) is given by

\[
\alpha = -\sqrt{\frac{\delta^{(r_1, r_1)}(e^* , y^*)}{\xi/n^2 E_{1,c^*}}},
\]

where \( e^* \) is the threshold of the ensemble under BP decoding , \( y^* \) is the non-zero solution of \( \hat{r}_1(y) \) at the threshold , \( n \) is the blocklength and \( \xi \) is the total number of edges in the original graph.

We define \( x^* := e^*(y^*)^{b-1} \) and \( \tilde{x}^* := 1 - x^* \). Since \( \hat{r}_1(e^*, y^*) = 0 \) and \( \hat{r}_1|_{e^*} y^* \), we see that \( y^* = 1-(\tilde{x}^*)^{b-1} \) and \( y^* = (b-1) x^*(\tilde{x}^*)^{b-2} \). Using those equations, we have from [5]

\[
\delta^{(r_1, r_1)}(e^* , y^*) = \frac{x^* y^b}{b-1}(y^* - x^*).
\]

(Note that \( G_1 = \frac{b-1}{y} \)). Recall that \( \hat{r}_1(e, y) = x y - 1 + \tilde{x}^{d-1} \). We see that

\[
\frac{\partial \hat{r}_1}{\partial e} |_{e^*} y^* = -\frac{x^* y^*}{e^*(b-1)}.
\]

From [6], we can obtain

\[
\alpha = \varepsilon \sqrt{\frac{b-1}{b} \frac{1}{\left( x^* - 1 / y^* \right)}}.
\]

This is the same result as in [3] for regular LDPC code ensembles.

B. Example of Solution of Covariance Evolution

Figure [1] shows the solution of the covariance evolution \( \delta^{(r_1, r_1)}(e, y) \), \( j \in \{1,2,\ldots,5\} \), as a function of \( y \) for (3,6)-regular LDPC code ensemble. Figure [2] shows the solution of the covariance evolution \( \delta^{(r_1, r_1)}(e, y) \), \( j \in \{1,2,3\} \), as a function of \( y \) for (2,4)-regular LDPC code ensemble.

C. Outline of proof

1) Proof for \( \delta^{(l,b)} \): From [2], we get \( \frac{d\delta^{(l,b)}}{dy}(y) = 0 \). From initial covariance, we have \( \delta^{(l,b)} = be\tilde{e} \). This leads to [3].
As a function of the parameter \( \epsilon \), we have

\[
A^{(l_b, \Sigma)}(1) = \sum_{r=1}^{d-1} \delta^{(l_b, r)}(1) = be^\epsilon - bde^\epsilon. 
\]

We see that \( C_{l_b, \Sigma} = -de^\epsilon \). We have

\[
A^{(l_b, \Sigma)} = G_d\{ (b-1)e\bar{y}^{-1} + \bar{e}\bar{x} \} + be^\epsilon. 
\]

From (2) and (8), we get for \( j \in \{1, \ldots, d-1\} \)

\[
\frac{d\delta^{(l_b, r)}(y)}{dy} = \frac{b-1}{y} \{j\delta^{(l_b, r)} + D^{(l_b, r)}\},
\]

where

\[
D^{(l_b, r)} := j\{ \bar{r}_{j+1} - \bar{r}_j \} be^\epsilon - \delta^{(l_b, r+1)} I_{j \neq d-1}
\]

\[
+ (A^{(l_b, \Sigma)} - be^\epsilon)I_{j = d-1}. 
\]

Those equations are first order linear differential equations. The solutions are given by

\[
\delta^{(l_b, r)} = y^j(b-1) \left\{ \int \frac{(b-1)D^{(l_b, r)}}{y^{j(b-1)+1}} dy + C_{l_b, r} \right\},
\]

with constants \( C_{l_b, r} \) determined from the initial covariances. Those equations can be solved by mathematical induction for \( j \in \{2, 3, \ldots, d-1\} \).

We show that if \( \delta^{(l_b, r_j+1)} \) fulfill (4), then also \( \delta^{(l_b, r_j)} \) fulfill (4). Assume \( \delta^{(l_b, r_j+1)} = -G_{j+1}\{\epsilon(b-1) + \bar{e}\bar{x}\} \). Using the induction hypothesis, we can write

\[
D^{(l_b, r_j)} = j\bar{x}G_{j+1} + \epsilon\bar{y}^{-1}\{bG_{d-1} + (b-1)(d-1)G_d\}. 
\]

Using the same way in the induction step, we can obtain

\[
\delta^{(l_b, r_{d-1})} = -G_{d-1}\{b-1\epsilon\bar{y}^{-1} + \bar{e}\bar{x}\}. 
\]

Using the same way in the induction step, we can obtain

\[
\delta^{(l_b, r_{d-1})} = -G_{d-1}\{b-1\epsilon\bar{y}^{-1} + \bar{e}\bar{x}\}. 
\]

Similarly, we have

\[
y^j(b-1) \int \frac{(b-1)\epsilon\bar{y}^{-1}bG_{j+1}}{y^{j(b-1)+1}} dy
\]

\[
= (b-1)be^\epsilon \left\{ \frac{d-1}{j-1} x^j \sum_{s=0}^{d-j} (d-j-s)(-\epsilon)^{s-1}K_{s-1} \right\}. 
\]

Similarly, we have

\[
y^j(b-1) \int \frac{(b-1)^2\epsilon\bar{y}^{-1}bG_{j+1}}{y^{j(b-1)+1}} dy
\]

\[
= (b-1)^2 \epsilon^2 \left\{ \frac{d-1}{j-1} x^j \sum_{s=0}^{d-j} (d-j-s)(-\epsilon)^{s-1}K_{s-1} \right\}. 
\]
where 
\[ K_s := \frac{y^{s(b-1)}}{s(b-1)-1} I_{s(b-1)\neq 1} + \log y I_{s(b-1)=1}. \]

Note that
\[ \{s(b-1) - b\} K_{s-1} = y^{(s-1)(b-1)-1} - I_{(s-1)(b-1)=1}. \]

From (11) and (12), we have
\[ y^{j(b-1)} \int \frac{(b-1)e^\varepsilon y^{-1}jG_j + (b-1)jjG_{j+1}}{y^{j(b-1)+1}} dy \]
\[ = -(b-1)e^\varepsilon \left( \frac{d-1}{j-1} \right) x^j \]
\[ \cdot \sum_{s=0}^{d-j} \left( \frac{d-j}{s} \right) (j+s)(s(b-1) - b) K_{s-1} \]
\[ = -(b-1)e^\varepsilon y^{-1} G_j + (b-1)e^\varepsilon x^j P_j, \tag{13} \]

where
\[ P_j := \left( \frac{d-1}{j-1} \right) \sum_{s=0}^{d-j} \left( \frac{d-j}{s} \right) (j+s)(-\varepsilon)^{-1} I_{(s-1)(b-1)=1}. \]

From (9), (10) and (13), we have
\[ \delta^{(l_0,r_j)} = -G_j \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \} + \frac{d}{d-1} \left( \frac{d}{j-1} \right) \varepsilon x^j \]
\[ + (b-1)e^\varepsilon x^j P_j + C_{l_0,r_j} y^{j(b-1)}. \]

From initial covariance, we have
\[ C_{l_0,r_j} = -\frac{d}{d-j} \left( \frac{d}{j-1} \right) e^\varepsilon - (b-1)e^\varepsilon x^j P_j. \]

Hence we obtain
\[ \delta^{(l_0,r_j)} = -G_j \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \}. \]

This leads to (4) for \( j \in \{2, 3, \ldots, d-1\}. \)

Note that \( \delta^{(l_0,r_j)} = A_{l_0,\Sigma} - \sum_{j=2}^{d-j} \delta^{(l_0,r_j)} \) and that \( G_1 = \sum_{j=2}^{d-j} G_j. \) We have
\[ \delta^{(l_0,r_j)} = -G_j \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \} + be^\varepsilon. \]

Hence we obtain (4).

3) Proof for \( B^{(\Sigma,\Sigma)} \). In order to solve \( \delta^{(r_j,r_k)} \), we define \( B^{(r_j,\Sigma)} := \sum_{k=1}^{d-j} \delta^{(r_j,r_k)} \) and \( B^{(\Sigma,\Sigma)} := \sum_{j=1}^{d-j} B^{(r_j,\Sigma)}. \) From (2), we get for \( j \in \{1, 2, \ldots, d-1\} \)
\[ \frac{dB^{(\Sigma,\Sigma)}}{dy} = \frac{b-1}{y} \left( D^{(\Sigma,\Sigma)} + 2dB^{(\Sigma,\Sigma)} \right), \tag{14} \]
\[ \frac{dB^{(r_j,\Sigma)}}{dy} = \frac{b-1}{y} \left\{ D^{(r_j,\Sigma)} + (d+j)B^{(r_j,\Sigma)} \right\}, \tag{15} \]

where
\[ D^{(\Sigma,\Sigma)} := d \frac{\tilde{\bar{r}}_d - \frac{\bar{I}_b}{b}}{b} (d\bar{r}_d + 2A_{l_0,\Sigma}), \]
\[ D^{(r_j,\Sigma)} := j \frac{\tilde{\bar{r}}_{j+1} - \frac{\bar{I}_b}{b}}{b} (d\bar{r}_d + A_{l_0,\Sigma}) + d \frac{\tilde{\bar{r}}_d - \frac{\bar{I}_b}{b}}{b} \delta^{(l_0,r_j)} \]
\[ - jB^{(r_{j+1},\Sigma)} I_{(j \neq d-1)} \]
\[ + (d-1)(B^{(\Sigma,\Sigma)} - dr_d - A_{l_0,\Sigma}) I_{(j=d-1)}. \]

The solution of (14) is given by
\[ B^{(\Sigma,\Sigma)} = \frac{y^{2d(b-1)}}{(b-1)^2} \left\{ \int (b-1) \frac{D^{(\Sigma,\Sigma)}}{y^{2d(b-1)+1}} dy + C_{\Sigma,\Sigma} \right\} \]
\[ = \frac{b-1}{b} G_d \{ \varepsilon(b(b-1)y^{-2} - (\varepsilon-\varepsilon)xy^{-1}) \}
\[ + 2G_d \{ (b-1)\varepsilon ey^{-1} + \varepsilon x \}
\[ + dx^d + be\varepsilon + C_{\Sigma,\Sigma} y^{2d(b-1)}, \]

with a constant \( C_{\Sigma,\Sigma} \) which determined from the initial covariance. From the initial covariance, we get
\[ B^{(\Sigma,\Sigma)}(1) = be\varepsilon - 2bde^2\varepsilon + d\varepsilon - de^2d + (b-1)d^2e^{2d-1}\varepsilon. \]

We see that \( C_{\Sigma,\Sigma} = \frac{b-1}{b} d^2e^{2d}d - d^2e^d. \) Hence we have
\[ B^{(\Sigma,\Sigma)} = \frac{b-1}{b} G_d \{ (b-1)e^\varepsilon y^{-2} - (\varepsilon-\varepsilon)xy^{-1} + x^2 \}
\[ + 2G_d \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \}
\[ + dx^d - dx^{2d} + be\varepsilon. \tag{16} \]

From (15), we get
\[ B^{(r_j,\Sigma)} = \frac{y^{d+j}(b-1)}{y^{(d+j)(b-1)+1}} \left\{ \int (b-1) \frac{D^{(r_j,\Sigma)}}{y^{(d+j)(b-1)+1}} dy + C_{r_j,\Sigma} \right\}, \]

with constants \( C_{r_j,\Sigma}. \) For \( j \in \{2, 3, \ldots, d-1\} \), those equation are solved by mathematical induction as the proof for \( \delta^{(l_0,r_j)} \).

From the initial covariances, note that
\[ B^{(r_j,\Sigma)}(1) = d(b-1) \left( \frac{d}{j-1} \right) e^{d+j-1}\varepsilon x^j (de-j)
\[ + d \left( \frac{d}{j-1} \right) e^{d+j}\varepsilon x^{j-1} - b \left( \frac{d}{j-1} \right) e^{d/j} x^{d-j} (de-j). \]

We have
\[ B^{(r_j,\Sigma)} = -\frac{b-1}{b} G_dG_j \{ (b-1)e^\varepsilon y^{-2} - (\varepsilon-\varepsilon)xy^{-1} + x^2 \}
\[ - G_j \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \} + \left( \frac{d}{j-1} \right) dx^d + jdx^{2d} + be\varepsilon. \tag{17} \]

Using \( B^{(r_j,\Sigma)} = B^{(\Sigma,\Sigma)} - \sum_{j=2}^{d-1} B^{(r_j,\Sigma)} \), we have
\[ B^{(r_j,\Sigma)} = -\frac{b-1}{b} G_dG_j \{ (b-1)e^\varepsilon y^{-2} - (\varepsilon-\varepsilon)xy^{-1} + x^2 \}
\[ - G_j \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \} + dx^{d+1} x^{d-1} + 2G_d \{ (b-1)e^\varepsilon y^{-1} + \varepsilon x \} + dx^d + \varepsilon x^{2d} + be\varepsilon. \tag{18} \]

4) Proof for \( \delta^{(r_k,r_j)} \). From (2), we get for \( k, j \in \{1, 2, \ldots, d-1\} \)
\[ \frac{d\delta^{(r_k,r_j)}}{dy} = \frac{b-1}{y} \left\{ (k+j)\delta^{(r_k,r_j)} + D^{(r_k,r_j)} \right\}, \]

where
\[ D^{(r_k,r_j)} := H_{k,j} + H_{j,k} - \frac{\bar{I}_b}{b} \int f^{(r_k,r_j)}, \]
\[ H_{k,j} := \frac{\tilde{r}_{k+1} - \bar{r}_k}{b} \delta^{(l_0,r_j)} - k\delta^{(r_k+1,r_j)} I_{(k \neq d-1)} \]
\[ - (d-1)(\delta^{(l_0,r_j)} - B^{(r_j,\Sigma)} I_{(k=d-1)}). \]
The solutions of those differential equations are given by

$$
\delta^{(r_k, r_j)} = y^{(k+j)(b-1)} \left\{ \frac{(b-1)D^{(r_k, r_j)}}{y^{(k+j)(b-1)+1}} dy + C_{r_k, r_j} \right\},
$$

with constants $C_{r_k, r_j}$. Using (17), we can solve those equations by mathematical induction for $j, k \in \{2, 3, \ldots, d-1\}$. We have

$$
\delta^{(r_k, r_j)} = \frac{b-1}{b} G_k G_j \{ \tilde{c}(b-1)y^{-2} - (\epsilon - \tilde{c})xy^{-1} + x^2 \}
$$

$$
- d \left( \frac{d-1}{k-1} \right) \left( \frac{d-1}{j-1} \right) x^{k+j-2+d-k-j}
$$

$$
+ I_{(k=j)} \left( \frac{d-1}{j-1} \right) jx^2 x^{-d-j}.
$$

Note that $\delta^{(r_k, r_k)} = B^{(r_k, \Sigma)} - \sum_{j=2}^{d-1} \delta^{(r_k, r_j)}$. We have for $k \in \{2, \ldots, d-1\}$

$$
\delta^{(r_k, r_k)} = \frac{b-1}{b} G_k G_1 \{ \tilde{c}(b-1)y^{-2} - (\epsilon - \tilde{c})xy^{-1} + x^2 \}
$$

$$
- d \left( \frac{d-1}{k-1} \right) x^{k-1} x^{-2d-k-1}
$$

$$
- G_k \{ (b-1)\tilde{c}y^{-1} - \epsilon x + x^2 \}.
$$

Since $\delta^{(r_1, r_1)} = B^{(r_1, \Sigma)} - \sum_{j=2}^{d-1} \delta^{(r_1, r_j)}$, we have

$$
\delta^{(r_1, r_1)} = \frac{b-1}{b} G_k G_1 \{ \tilde{c}(b-1)y^{-2} - (\epsilon - \tilde{c})xy^{-1} + x^2 \}
$$

$$
- dx^2 x^{-2d-2} - x^{-d-1}
$$

$$
- 2G_1 \{ (b-1)\tilde{c}y^{-1} - \epsilon x + x^2 \} + (b\tilde{c} - x\tilde{c}).
$$

Thus, we can obtain (5).

IV. RELATIONSHIP TO STABILITY CONDITION

In this section, we consider the relationship between the stability condition (6), (4) and $\lim_{y \to 0} \delta^{(b, r_j)} (\epsilon, y)$.

For a $(b, d)$-regular LDPC code ensemble $(b \geq 3)$, we see from (4) and (5) that

$$
\lim_{y \to 0} \delta^{(b, r_j)} (\epsilon, y) = \tilde{c} \tilde{c} I_{(j=1)},
$$

$$
\lim_{y \to 0} \delta^{(r_1, r_j)} (\epsilon, y) = \tilde{c} \tilde{c} I_{(j=1)}.
$$

For a $(2, d)$-regular LDPC code ensemble, we see from (4) and (5) that

$$
\lim_{y \to 0} \delta^{(b, r_j)} (\epsilon, y) = \begin{cases} 
2\tilde{c}(d-1) \epsilon, & \text{if } j = 1 \\
2\tilde{c} \epsilon, & \text{if } j = 2 \\
0, & \text{otherwise},
\end{cases}
$$

$$
\lim_{y \to 0} \delta^{(r_1, r_j)} (\epsilon, y) = \begin{cases} 
2\tilde{c}(1 - (d-1) \epsilon)^2, & \text{if } j = k = 1 \\
2\epsilon (d-1)^2 \epsilon, & \text{if } (j, k) = (1, 2), (2, 1) \\
2\tilde{c} (d-1)^2 \epsilon^2, & \text{if } j = k = 2 \\
0, & \text{otherwise},
\end{cases}
$$

If we define the correlation coefficient for $i, j \in D$ by

$$
\rho_{i, j}(\epsilon) := \lim_{y \to 0} \frac{\delta^{(i, j)} (\epsilon, y)}{\sqrt{\delta^{(i, i)} (\epsilon, y) \delta^{(j, j)} (\epsilon, y)}},
$$

we obtain

$$
\rho_{b, r_1}(\epsilon) = \begin{cases} 
1, & \text{if } I_{(b=2)} (d-1) \epsilon \leq 1 \\
-1, & \text{if } I_{(b=2)} (d-1) \epsilon > 1.
\end{cases}
$$

Note that $I_{(b=2)} (d-1) \epsilon \leq 1$ agree with the stability condition for regular LDPC code ensembles.

Figure 3 shows the solution of covariance evolution $\lim_{y \to 0} \delta^{(j, r_1)} (\epsilon, y)$, $j \in \{l_2, r_1, r_2\}$, as a function of the channel parameter $\epsilon$ for the $(2,4)$-regular LDPC code ensemble. From Figure 3 we see that $\delta^{(r_2, r_1)} > 0$ and $\delta^{(l_2, r_1)} > 0$ when $(d-1) \epsilon < 1$. Also we see that $\delta^{(r_2, r_1)} < 0$ and $\delta^{(l_2, r_1)} < 0$ when $(d-1) \epsilon > 1$.

V. CONCLUSION AND FUTURE WORK

In this paper, we have solved analytically the covariance evolution for regular LDPC code ensembles. Moreover we have derived the relationship between stability condition.

As a future work, we will derive an analytical solution of the covariance evolution for irregular LDPC code ensembles.

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