Abstract

We extend vector formalism by including it in the algebra of split octonions, which we treat as the universal algebra to describe physical signals. The new geometrical interpretation of the products of octonionic basis units is presented. Eight real parameters of octonions are interpreted as the space-time coordinates, momentum and energy. In our approach the two fundamental constants, \( c \) and \( \hbar \), have the geometrical meaning and appear from the condition of positive definiteness of the octonion norm. We connect the property of non-associativity with the time irreversibility and fundamental probabilities in physics.

PACS numbers: 01.55.+b; 02.10.De; 04.50.+h

1 Introduction

Usually in physics the geometry is thought to be objective without any connection to the way it is observed. However, the properties of space-time (such as dimension, distances, etc.) are just reflections of the symmetries of physical signals we receive. Analyzing signals our brain can operate only via the classical associations and numbers. Algebras closely relates to the physical measurements, which consists not only with the actual receiving of signals, but with perceiving this signals to form the data as well. So we can introduce some kind of anthropic principle in mathematics, which means that investigating the algebras we study the way our brain abstracts physical observations. For example, algebra of rational numbers expresses our experience that there exist some classical objects with which we can exchange signals and these objects do not change much during 'algebraic' operations. Any observable quantity, which our brain could extract from a single measurement, is a real number, the norm (distance) formed by the multiplication of the numbers corresponding to the direct and reflected signals. In general one-way signal can be
expressed with any kind of number, important is to have the real norm. Introduction of some distance (norm) always means some comparison of two physical objects using one of them as an etalon (for example, simple counting). In the algebraic language all these features of our way of thinking mean that to perceive the real world our brain uses division algebras with the unit element over the field of real numbers.

Besides of the usual algebra of real numbers there are, according to the Hurwitz theorem, three unique division algebras, the algebra of complex numbers, quaternions and octonions [1]. Essential feature of all normed composition algebras is the existence of a real unit element and a different number of adjoined hyper-complex units. The square of the unit element is always positive while the squares of the hyper-complex units can be negative as well. In applications of division algebras mainly the elements with the negative square (similar to the complex unit $i$) are used. In this case norm of the algebra is positively defined. Using the vector-like elements (with the positive square) in division algebras leads to so-called split algebras having an equal number of terms with the positive and negative signs in the definition of their norms.

Because of importance of 'imaginary' and vector-like elements of algebras we want to recall the history of their introduction [2].

Word 'vector' was introduced by William Hamilton for the pure imaginary part of quaternions, discovered by him in 1843. He did not give geometrical meaning to the scalar part of a quaternion but kept it to have division property of algebra. Later Hamilton used also the word 'versors' (meaning 'rotators') for three quaternionic basis units with the negative square, since he understood that such elements could be interpreted as representing rotation of vectors, instead of something that corresponds to a straight line in space.

The algebra of Euclidean vectors was developed in 1880s by Gibbs and Heaviside from Hamilton’s quaternions when they tried to rewrite Maxwell’s quaternionic equations in a more convenient form. They removed scalar part of quaternions and kept Hamilton’s term 'vector' representing a pure quaternion. Since vectors have their origin in physical problems the definition of the products of vectors was obtained from the way in which such products occur in physical applications. Instead of the whole quaternion product (which has a scalar and a ‘vector’ part), Gibbs and Heaviside defined two different types of vector multiplications, the scalar and vector products. They also changed the sign from minus to plus in the scalar product, resulting to positive squares of basis units. This changing corresponds to a shift of interpretation of basis units from ‘versors’ to unit vectors.

By revising the process of addition of physical vectors it was found that using the three unit orthogonal basis elements $e_n$ ($n = 1, 2, 3$) any vector could be decomposed into components. One can introduce also conjugated (reflected) vectors $\bar{e}_n$, which differ from $e_n$ by the sign. The properties of scalar and vector products are encoded in the following algebra of the unit orthogonal basis elements

$$e_n^2 = 1, \quad \bar{e}_n = -e_n, \quad e_ne_m = -e_m e_n = \varepsilon_{nmk} e_k, \quad n, m = 1, 2, 3 \quad (1)$$

where $\varepsilon_{nmk}$ is fully anti-symmetric tensor. Or vice versa, we can say that postu-
lating the algebra \( \mathbb{1} \) one can recover the ordinary multiplication laws of physical vectors. There is ambiguity in choosing of the sign of \( \varepsilon_{nmk} \) in \( \mathbb{1} \) connected with the existence of the left-handed and right-handed coordinate systems. Note that anti-commutation of the basis units in \( \mathbb{1} \) is the result of the definition of vector product, and not an essential property of the vector basis elements \( e_n \) themselves.

Introduction of vector algebra in physics was successful, however because of removing of the scalar part of quaternions the division operation is not defined for vectors. There was lost also the property of ‘versors’, that they are rotation generators and expresses not only the final state achieved after a rotation, but the direction in which this rotation has been performed. It is this direction of rotation that the standard matrix representation of the rotation group fails to give.

It is known that Hamilton’s pure quaternions are not equivalent to vectors \( \mathbb{3} \). For ordinary Euclidean vectors, basis elements are three orthogonal unit polar vectors, while for Hamilton’s quaternions they are unit ‘versor’ (imaginary units) and with respect to coordinate transformations behave similar to axial vectors. There are differences also in the proprieties of their products. The product of quaternions is associative, while both types of vector products are not. So the division algebra, which includes the formalism of Euclidean 3-vectors, should be wider then the algebra of quaternions.

In this paper we present extension of vector algebra \( \mathbb{1} \) by embedding it in the algebra of split-octonions. In this sense our approach is a generalization of the well-developed formalism of geometric algebras \( [4, 5] \). Split octonions contain exactly three vector-like orthogonal elements needed to describe special dimensions \( [6] \).

2 Octonionic Intervals

Since their discovery in 1844-1845 by Graves and Cayley there were various attempts to find appropriate uses for octonions in physics \( [7] \). Recently the assumption that our Universe is made of pairs of octonions become an important idea in string theory also \( [8] \). As distinct from string models in present paper we want to apply octonions to describe space-time geometry and not only internal spaces.

We want to describe the physical signal \( s \) by a 8-dimensional number, the element of octonionic algebra,

\[
s = ct + x_n J^n + h \lambda_n j^n + c h \omega I , \quad n = 1, 2, 3
\]

(2)

where by the repeated indexes summing is considered as in standard tensor calculus. The eight scalar parameters in (2) we treat as the time \( t \), special coordinates \( x^n \), quantities \( \lambda^n \) with the dimension of the momentum\(^{-1}\) and \( \omega \) with the dimension of energy\(^{-1}\). In (2) two fundamental constants of physics, the velocity of light \( c \) and Plank’s constant \( h \), are presented also \( [6] \).

The orthogonal basis of split-octonions \( [2] \) is formed by the unit scalar element (which we denote by \( 1 \)) and by three different types (totally seven) of orthogonal hyper-complex units: the three vector-like elements \( J_n \), three ‘versor’-like units \( j_n \).
and one pseudo-scalar $I$. Similar quantity, represented as a sum of the elements with the different properties, is called para-vector in Clifford algebras \[9\].

The squares of octonionic basis units

$$J_n^2 = 1, \quad j_n^2 = -1, \quad I^2 = 1, \quad n = 1, 2, 3 \quad (3)$$

are always inner products resulting unit element. However, multiplication of different basis units should be defined as the skew products

$$J_nJ_m = -J_mJ_n, \quad j_nj_m = -j_mj_n, \quad J_nI = -IJ_n, \quad j_nI = -IJ_n. \quad (n \neq m) \quad (4)$$

As distinct from the geometric algebra approaches \[4, 5\], here we do not need to introduce different types of brackets for inner and outer products, similar to the algebra of Euclidean vectors \(1\).

To generate complete 8-dimensional basis of split-octonions \((1, J_n, j_n, I)\) the multiplication and distribution laws of only three vector-like elements \(J_n\) are needed \(\left(2^n = 8\right)\). Then our imagination about 3-dimensional character of the space can be the result of existing of only three vector-like basis units. The fundamental basis elements \(J_n\) geometrically can be presented as the unit orthogonal Euclidean vectors

$$J_1 = \longleftrightarrow, \quad J_2 = \uparrow, \quad J_3 = \nwarrow, \quad (5)$$
directed towards the positive directions of \(x, y\) and \(z\) axis respectively. As for the vector algebra \(1\), conjugated elements \(\tilde{J}_n\) can be understood as the reflected vectors

$$\tilde{J}_1 = -J_1 = \longleftrightarrow, \quad \tilde{J}_2 = -J_2 = \uparrow, \quad \tilde{J}_3 = -J_3 = \nwarrow. \quad (6)$$

The left multiplication of any hyper-complex element on the vector-like element \(J_n\) we visualize geometrically as sweeping of this element along \(J_n\). In this picture square of \(J_n\) means sweeping of \(J_n\) along the other \(J_n\) taking its origin to the end value (which is equal to 1)

$$J_nJ_n = J_n^2 = \longleftrightarrow \times \longleftrightarrow = 0 \rightarrow 1 = 1, \quad J_n\tilde{J}_n = -J_n^2 = \longleftrightarrow \times \longleftrightarrow = -1 \leftarrow 0 = -1. \quad (7)$$

Another class of basis elements \(j^n\) in \(2\) (which are dual to \(J^n\)) is defined as the skew product of two fundamental basis units \(J^n\)

$$j_n = \frac{1}{2} \epsilon_{nmk} J^m J^k, \quad m, n, k = 1, 2, 3 \quad (8)$$

The elements \(j_n\) are neither a scalar nor a vector, we interpret them as bi-vectors, similar to \[4, 5\]. Since \(j_n^2 = -1\), bi-vectors behave like pure imaginary objects. There are differences, however. Here we have three anti-commuted bi-vectors and also they anti-commute with the vector-like objects \((j_nJ_m = -J_mj_n)\). We do not get behavior like this with complex numbers alone. The feature which imitates complexity and
results in "imaginary" properties of \( j_n \) is the definition of the products of \( J_n \). So the objects \( j_n \), similar to Hamilton’s quaternionic units, are 'versors' and can be used to represent rotations. As for the vector products (11), we have ambiguity in choosing the sign of \( \epsilon_{nmk} \) in the relation (8). In quaternionic and vector algebras this ambiguity is often understood as connected with the left-handed and right-handed coordinate systems. In our case elements \( j_n \) encodes the notion of an oriented plane without relying on the notion of a vector perpendicular to it. In the relation (8) we also choose positive sign, however, when considering products of octonions corresponding to the different signals, the ambiguity still remains and in physical applications could give two-value wave-functions, corresponding to the introduction of spin. Also the elements \( j_n \) are useful to define momentum operators similar as done in quaternionic quantum mechanics [10]. This is one of the justifications of appearance of Plank’s constant in the definition (2).

We want to visualize \( j_n \) geometrically as the orientated planes obtained by the sweeping of the second fundamental vector in the definition (8) along the first. For example,

\[
j_3 = J_1 J_2 = \overrightarrow{\pmb{1}} \times \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}}.
\]

(9)

Changing the order of vectors in (9) reverses the orientation of the plane

\[
J_2 J_1 = - j_3 = \overrightarrow{\pmb{1}} \times \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}}.
\]

(10)

Analogously for other two bi-vectors \( j_1 \) and \( j_2 \) we find

\[
J_1 J_3 = - j_2 = \overrightarrow{\pmb{1}} \times \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}},
\]

\[
J_2 J_3 = j_1 = \overrightarrow{\pmb{1}} \times \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}}.
\]

(11)

In our geometric approach sub-algebra of products of the only two vector-like units \( J_n \) can be easily found from the rotation of the figures (9), (10) and (11). For example, the commutation laws

\[
J_1 J_2 = J_2 \tilde{J}_1 = \tilde{J}_1 J_2 = \tilde{J}_2 J_1,
\]

(12)

have the geometrical interpretation as the rotations of the figure (9) in the \((x - y)\)-plane

\[
\overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}}.
\]

(13)

Analogously the laws

\[
J_2 J_1 = J_1 \tilde{J}_2 = \tilde{J}_2 J_1 = \tilde{J}_1 J_2,
\]

(14)

correspond to the rotations of the oppositely orientated figure (10),

\[
\overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}} = \overrightarrow{\pmb{1}}.
\]

(15)

With the similar rotations of (11) in the \((x - z)\)- and \((y - z)\)-planes we recover binary products of all three vector-like basis units \( J_n \) and their conjugates \( \tilde{J}_n \).
Non-associativity of octonions, which results in non-equivalence of left and right products for the expression containing more than two basis units \( J_n \), physically can be interpreted as the causality. To the direction from the past to the future we correspond one orientation of multiplication, for example the left product.

To find the expression of products of \( J_n \) on \( J_m \) we can use the property of alternativity of the octonions. In the language of basis elements alternativity means that in the expressions of the multiplication of several basis units, where only two different fundamental basis elements are involved, the order of products is arbitrary. Physically this means that for the physical processes taking place in a single plane time is reversible. So products of \( J^n \) and \( j^m \) (when \( n \neq m \)) can be defined as

\[
J_n j_m = -\epsilon_{nmk} J^k .
\] (16)

The products of \( J_n \) by \( j_m \) we visualize geometrically as rotations of the vectors \( J_n \). For example, left product of \( j^3 \) on \( J^2 \) (or \( J_1 \)) is the operation meaning that \( j^3 \) applied to \( J^2 \) (or \( J_1 \)) rotates it clockwise by \( \pi/2 \) in the \((x−y)\)-plane, resulting to \( J^1 \) (or \(-J_2\))

\[
j_3 J_2 = (J_1 J_2) J_2 = J_1 = \begin{pmatrix} \uparrow \end{pmatrix} \times \begin{pmatrix} \uparrow \end{pmatrix} = \frac{\pi}{2} \wedge \begin{pmatrix} \uparrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix} ,
\]

\[
j_3 J_1 = (J_1 J_2) J_1 = -J_2 = \begin{pmatrix} \rightarrow \end{pmatrix} \times \begin{pmatrix} \rightarrow \end{pmatrix} = \frac{\pi}{2} \wedge \begin{pmatrix} \rightarrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix}.
\] (17)

Right products of \( J_n \) by \( j_m \) (\( n \neq m \)) can be considered as anti-clockwise rotations, or alternatively as the left sweepings of the oriented planes \( j_n \) along the vectors \( J_m \). For example, to the products reverse to (17) correspond

\[
J_1 j_3 = J_1 (J_1 J_2) = \begin{pmatrix} \rightarrow \end{pmatrix} \times \begin{pmatrix} \uparrow \end{pmatrix} = \frac{\pi}{2} \wedge \begin{pmatrix} \uparrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix} ,
\]

\[
J_2 j_3 = J_2 (J_1 J_2) = \begin{pmatrix} \uparrow \end{pmatrix} \times \begin{pmatrix} \uparrow \end{pmatrix} = \frac{\pi}{2} \wedge \begin{pmatrix} \uparrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix} .
\] (18)

Pictures similar to (17) and (18) we can paint also in the \((x−z)\)- and \((y−z)\)-planes.

The seventh octonionic basic element \( I \) (dual to the scalar unit 1) we treat as oriented 3-vector, maximal multi-vector generated by three elements \( J^n \). We define \( I \) as the left product of all fundamental vectors \( J_n \),

\[
I = J_n j_n = -j_n J_n .
\] (19)

There is no summing in this formula. From (19) we see that \( I \) has the three equivalent representations

\[
I = J_1 j_1 = J_2 j_2 = J_3 j_3 ,
\] (20)

which can be understood as the expression of the volume invariance. The element \( I \) can be visualized as 3-dimensional oriented cube obtained by the sweeping of oriented plane \( j_n \) along the vector \( J_n \). For example,

\[
I = J_1 j_1 = J_1 (J_2 J_3) = \begin{pmatrix} \rightarrow \end{pmatrix} \times \begin{pmatrix} \rightarrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix} = \begin{pmatrix} \rightarrow \end{pmatrix} .
\] (21)
The other two equivalent representations of \( I \)

\[
\begin{align*}
J_2(J_3J_1) &= J_2j_2 = \uparrow \times \leftarrow = \square , \\
J_3(J_1J_2) &= J_3j_3 = \leftarrow \times \uparrow = \square ,
\end{align*}
\]

(22)
correspond to the different rotations of the figure (21).

Oppositely orientated 3-cubes are received by the left products of \( j_n \) on \( J_m \). When \( n \neq m \) similar left products of \( j_n \) on \( J_m \) we had interpreted as the rotation of \( J_m \). However, when \( n = m \) these elements 'align', \( j_n \) can’t rotate \( J_n \) and we visualize this product as attaching of \( J_n \) by \( j_n \)

\[
\begin{align*}
j_1J_1 &= (J_2J_3)J_1 = \uparrow \times \rightarrow = \square , \\
j_2J_2 &= (J_3J_1)J_2 = \leftarrow \times \uparrow = \square , \\
j_3J_3 &= (J_1J_2)J_3 = \downarrow \times \leftarrow = \square .
\end{align*}
\]

(23)

The essential property of octonions, non-associativity, is the direct result of volume invariance requirement (20). Indeed,

\[
J_1(J_2J_3) - (J_1J_2)J_3 = J_1j_1 - j_3J_3 = 2I = \square - \uparrow = \square \neq 0 .
\]

(24)

Non-associativity is another property which shows that the octonionic fundamental basis units \( J^n \) are similar to the ordinary 3-vectors, having non-associative products also.

Adopting non-associativity (24), at the same time we need the rules to receive definite results for products of all seven octonionic basis units and their conjugates to form octonionic algebra. This requirement forces us to introduce special rule for opening of brackets in product of the vector-like elements \( J^n \) and the 'versor'-like elements \( j^m \) with the seventh basis unit \( I \). The products of \( J^n \) and \( j^n \) with \( I = (J_mj_m) \) when \( n = m \), because of alternativity, gives

\[
\begin{align*}J_nI &= j_n , \quad j_nI &= J_n .
\end{align*}
\]

(25)

However, when \( n \neq m \) these products contains all three fundamental basis units. If we ignore orientated feature of the products and will try to remove the brackets, then, because of anti-associativity (24), this products become two-valued. For example,

\[
IJ_1 = (J_2j_2)J_1 = \begin{cases} J_2(j_2J_1) = J_2j_3 = +j_1 , \\
- j_2(J_2J_1) = j_2j_3 = -j_1 .
\end{cases}
\]

(26)

From this relation we conclude that we should introduce negative sign when removing the brackets and multiply different kinds of basis units \( J^n \) and \( j^m \). Then in the first case of the relation (26), because of the product \( j_2J_1 \), extra minus arises and both cases give the same result \( -j_1 \). Using this rule of opening of brackets, the explicit calculation of the square of seventh basis unit becomes single valued for any
representation from (20) and we receive $I^2 = 1$. However, for the octonionic products for different physical signals this ambiguity remains and possibly corresponds to the quantum probabilities and some kind of spin.

Note that, as for ordinary vectors (1), anti-commuting features of octonionic units is not the property of the basis elements but their binary products. So in the expressions of the products of $j_n$ with $j_m$ we should not remove all brackets immediately leading to the mixing of four vector-like basis units. We should consider binary product of $j_n$ with one of the vector-like element $J_k$ forming $j_m$ and then with the second one, that gives the definite result

$$j_k = -\frac{1}{2} \epsilon_{knm} J^n J^m .$$  \hspace{1cm} (27)

As above we interpret the left product by $j_n$ as the clockwise and right product as the counter-clockwise rotations. Then the example of geometrical interpretation of (27) is

$$j_3 j_1 = -j_2 = \overrightarrow{\text{S}} \times \overrightarrow{\text{A}} = \frac{\pi}{2} \overset{\frown}{\text{A}} = \frac{\pi}{2} \underset{\frown}{\text{A}} = \overrightarrow{\text{F}} .$$  \hspace{1cm} (28)

Similar pictures we have for two other products $j_2 j_3 = -j_1$ and $j_1 j_2 = -j_3$.

Conjugations, when the vectors $J_n$ change their directions on the opposite, reverse also the order of elements in any given expression (physically this means the replacing of causes by the results)

$$\widetilde{J}_n = -J_n ,$$

$$\widetilde{j}_n = \frac{1}{2} \epsilon_{nmk} \widetilde{J}^m J^k = \frac{1}{2} \epsilon_{nmk} \widetilde{J}^k J^m = -j_n ,$$  \hspace{1cm} (29)

$$\widetilde{I} = \widetilde{J}_n j_n = \widetilde{j}_n \widetilde{J}_n = -I .$$

Finally the formulae (3), (4), (8), (16), (25), (27) and (29) give multiplication table of all basis elements and thus form the algebra of split-octonions.

The ‘square length’ (norm) of a para-vector (2), or (31),

$$s^2 = c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2 ,$$  \hspace{1cm} (30)

has (4+4)-signature and in general is not positively defined. A split-octonion does not have an inverse when its norm (30) is zero. To the two critical rotations, in the planes $(t - x)$, or $(\lambda - \omega)$ and $(t - \omega)$, or $(x - \lambda)$, we had corresponded the two different universal constants $c$ and $\hbar$ in (2). It marks an important departure from the Euclidean space and shows that split-octonions can be used for study of properties of the real space-time. In the classical limit $\hbar \rightarrow 0$ the expression (30) reduces to the ordinary formula for space-time intervals.

Using the multiplication laws (25), a unique mapping of coordinates of any event into octonion algebra (2) can be equivalently determined as

$$s = c(t + \hbar I \omega) + J_n (x^n + \hbar I \lambda^n) .$$  \hspace{1cm} (31)
From this formula we see that pseudo-scalar $I$ introduces the 'quantum' term corresponding to some kind of uncertainty of space-time coordinates. This terms disappear in the classical limit $\hbar \to 0$. Note that physical limits on masses of reference systems cause uncertainties even for large distances [11].

Differentiating (31) by the proper time $\tau$ the proper velocity of a particle can be obtained

$$\frac{ds}{d\tau} = \frac{dt}{d\tau} \left[ c \left( 1 + \hbar I \frac{d\omega}{dt} \right) + J_n \left( \frac{dx_n}{dt} + \hbar I \frac{d\lambda^n}{dt} \right) \right]. \quad (32)$$

From this formula we see that for the critical signals corresponding to zero norm (30) we have the following relations

$$\frac{\partial}{\partial \lambda^n} = I \hbar \frac{\partial}{\partial x^n}, \quad \frac{\partial}{\partial \omega} = I \hbar \frac{\partial}{\partial t}, \quad (33)$$

that is similar to introduction of energy and momentum operators in quantum mechanics.

The invariance of the norm (30) gives the relation

$$\frac{d\tau}{dt} = \sqrt{1 - \hbar^2 \left( \frac{d\omega}{dt} \right)^2} - \frac{v^2}{c^2} \left[ 1 - \hbar^2 \left( \frac{d\lambda^n}{dx^n} \right)^2 \right], \quad (34)$$

where

$$v^2 = \frac{dx_n}{dt} \frac{dx^n}{dt} \quad (35)$$

is the velocity measured by the observer (31). So the Lorentz factor (31) contains extra terms and the dispersion relation in the (4+4)-space (30) has a form similar to that of double-special relativity models [12].

From the requirement to have the positive norm (30) from (34) we receive several relations

$$v^2 \leq c^2, \quad \frac{dx_n}{dt} \geq \hbar, \quad \frac{dt}{d\omega} \geq \hbar. \quad (36)$$

Recalling that $\lambda$ and $\omega$ have dimensions of momentum$^{-1}$ and energy$^{-1}$ respectively, we conclude that uncertainty principle probably has the same geometrical meaning as the existence of the maximal velocity [6].

So some characteristics of physical world (such as dimension, causality, maximal velocities, quantum behavior, etc.) possibly connected with the using of normed split algebras, or with our way of apprehend of reality.

### 3 Conclusion

In this paper we wanted to introduce some kind of anthropic principle in mathematics: some characteristics we usually attributed to physical world connected with our way to apprehend reality. To have advantages of division algebras we suggested to extend vector formalism by embedding it in the algebra of split-octonions.
Our approach is related with the geometric algebras [4, 5] in the sense that we also emphasized the geometric significance of vector products and avoided matrices and tensors. In distinct from these models we used non-associative oriented products and tried to give the physical interpretation to this property. Non-associativity, which results in difference of left and right products (what we correspond to causes and effects) could mean time irreversibility. Also, since the result of product of several octonions is not single valued, there should appear fundamental probabilities in calculations of physical processes if they are done by octonions.

We connected eight real parameters of octonions with the space-time coordinates, momentum and energy. To generate complete basis of split-octonionic the multiplication and distribution laws of only three vector-like elements are needed, that could result in our imagination about 3-dimensional character of the space. We found the Minkowski metric as the natural metric of octonionic space in the classical limit \( \hbar \to 0 \).

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