ON THE IDEAL MAGNETOHYDRODYNAMICS IN THREE-DIMENSIONAL THIN DOMAINS: WELL-POSEDNESS AND ASYMPTOTICS

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ABSTRACT. We consider the ideal magnetohydrodynamics (MHD) subject to a strong magnetic field along $x_1$ direction in three-dimensional thin domains $\Omega_\delta = \mathbb{R}^2 \times (-\delta, \delta)$ with slip boundary conditions. It is well-known that in this situation the system will generate Alfvén waves. Our results are summarized as follows:

(i). We construct the global solutions (Alfvén waves) to MHD in the thin domain $\Omega_\delta$ with $\delta > 0$. In addition, the uniform energy estimates are obtained with respected to the parameter $\delta$.

(ii). We justify the asymptotics of the MHD equations from the thin domain $\Omega_\delta$ to the plane $\mathbb{R}^2$. More precisely, we prove that the 3D Alfvén waves in $\Omega_\delta$ will converge to the Alfvén waves in $\mathbb{R}^2$ in the limit that $\delta$ goes to zero. This shows that Alfvén waves propagating along the horizontal direction of the (3D) strip are stable and can be approximated by the (2D) Alfvén waves when $\delta$ is sufficiently small. Moreover, the control of the (2D) Alfvén waves can be obtained from the control of (3D) Alfvén waves in the thin domain $\Omega_\delta$ with aid of the uniform bounds.

The proofs of main results rely on the design of the proper energy functional and the null structures of the nonlinear terms. Here the null structures means two aspects: separation of the Alfvén waves ($z_+$ and $z_-$) and no bad quadratic terms $Q(\partial_3 z^h, \partial_3 z^h)$ where $z^\pm = (z^h_\pm, z_3^\pm)$ and $Q(\partial_3 z^h, \partial_3 z^h)$ is the linear combination of terms $\partial^\alpha \partial_3 z^h \partial^\beta \partial_3 z^h$ with $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^3$.

1. INTRODUCTION

The purpose of this article is to study the global well-posedness and asymptotics of incompressible ideal magnetohydrodynamics (MHD) in three-dimensional (3D) thin domains $\Omega_\delta \overset{\text{def}}{=} \mathbb{R}^2 \times (-\delta, \delta)$ with strong background magnetic fields. We remark that parameter $\delta > 0$ is sufficiently small. Thin domains are widely considered in the study of many problems in science, such as in solid mechanics (thin rod, plates and shells), in fluid dynamics (lubrication, meteorology problems, ocean dynamics), and in magnetohydrodynamics (wave heating in the solar and stellar atmosphere, solar tachocline, shallow-water MHD). Most of the above problems are described by partial differential equations (PDE) in thin domains. We refer to [5, 12, 13] for more details on physics background.

In the present article, we consider the incompressible ideal MHD equations in the thin plate $\Omega_\delta$. The ideal MHD equations in thin plate (or strip) read as

\begin{align}
\partial_t v + v \cdot \nabla v &= -\nabla p + (\nabla \times b) \times b, \quad \text{in} \quad \Omega_\delta \times \mathbb{R}^+ \\
\partial_t b + v \cdot \nabla b &= b \cdot \nabla v, \\
\div v &= 0, \\
\div b &= 0,
\end{align}

(1.1)

where $b = (b^1, b^2, b^3)^T$, $v = (v^1, v^2, v^3)^T$ and $p$ are the magnetic field, the velocity and scalar pressure of the fluid respectively. The slip boundary conditions are imposed on $v$ and $b$ as

\begin{align}
v^3_{x_3=\pm \delta} = 0, \quad b^3_{x_3=\pm \delta} = 0.
\end{align}

(1.2)
We can write the Lorentz force term \((\nabla \times \mathbf{b}) \times \mathbf{b}\) in the momentum equation in a more convenient form. Indeed, we have
\[
(\nabla \times \mathbf{b}) \times \mathbf{b} = \nabla (-\frac{1}{2}|\mathbf{b}|^2) + \mathbf{b} \cdot \nabla \mathbf{b}.
\]
The first term \(\nabla (-\frac{1}{2}|\mathbf{b}|^2)\) is called the magnetic pressure force since it is in the gradient form just as the fluid pressure does. The second term \(\mathbf{b} \cdot \nabla \mathbf{b} = \nabla \cdot (\mathbf{b} \otimes \mathbf{b})\) is the magnetic tension force, which is the only restoring source to generate Alfvén waves. Therefore, we can use \(p\) again in the place of \(p + \frac{1}{2}|\mathbf{b}|^2\). The momentum equation then reads
\[
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{b} \cdot \nabla \mathbf{b}.
\]
We study the most interesting situation when a strong background magnetic field \(\mathbf{B}_0\) presents (to generate Alfvén waves). Let \(\mathbf{B}_0 = |\mathbf{B}_0| \mathbf{e}_1\) be a uniform constant (non-vanishing) background magnetic field. The vector \(\mathbf{e}_1\) is the unit vector parallel to \(x_1\)-axis. In this case, the small initial perturbation will generate a stable Alfvén waves which propagate along the background magnetic field \(\mathbf{B}_0\). We shall study the global existence of the Alfvén waves in the thin plate and also the asymptotics of the system as the width of the strip goes to zero.

We introduce the Elsässer variables: \(Z_+ = v + b\) and \(Z_- = v - b\), to rewrite the MHD equations (1.1) as
\[
\begin{align*}
\partial_t Z_+ + Z_- \cdot \nabla Z_+ &= - \nabla p, & \text{in } \Omega_{\delta} \times \mathbb{R}^+ \\
\partial_t Z_- + Z_+ \cdot \nabla Z_- &= - \nabla p, \\
\text{div } Z_+ &= 0, \\
\text{div } Z_- &= 0.
\end{align*}
\]
(1.3)
Let \(\mathbf{B}_0 = |\mathbf{B}_0|(1,0,0)^T\) be a uniform background magnetic field and \(z_+ = Z_+ - \mathbf{B}_0\), \(z_- = Z_- + \mathbf{B}_0\). Then the MHD equations and boundary condition (1.5) can be reformulated as
\[
\begin{align*}
\partial_t Z_+ + Z_- \cdot \nabla z_+ &= - \nabla p, & \text{in } \Omega_{\delta} \times \mathbb{R}^+ \\
\partial_t Z_- + Z_+ \cdot \nabla z_- &= - \nabla p, \\
\text{div } z_+ &= 0, \\
\text{div } z_- &= 0.
\end{align*}
\]
(1.4)
with the boundary conditions
\[
z_+^3|_{x_3 = \pm \delta} = 0, \quad z_-^3|_{x_3 = \pm \delta} = 0.
\]
(1.5)
Suppose \(j_+ = \text{curl } z_+\) and \(j_- = \text{curl } z_-\). It is easy to see that \((j_+, j_-)\) satisfies
\[
\begin{align*}
\partial_t j_+ + Z_- \cdot \nabla j_+ &= - \nabla z_- \wedge \nabla z_+, & \text{in } \Omega_{\delta} \times \mathbb{R}^+ \\
\partial_t j_- + Z_+ \cdot \nabla j_- &= - \nabla z_+ \wedge \nabla z_-.
\end{align*}
\]
(1.6)
We remark that \(j_+\) and \(j_-\) are divergence free. The explicit expressions of the nonlinear terms on the righthand side (r.h.s) are
\[
\nabla z_- \wedge \nabla z_+ = \nabla z_-^k \times \partial_k z_+, \quad \nabla z_+ \wedge \nabla z_- = \nabla z_+^k \times \partial_k z_-.
\]
(1.7)
Here we use the Einstein’s convection: if an index appears once up and once down, it is understood to be summing over \(\{1, 2, 3\}\).
1.1. **Short review of the problem.** We first give a short review on the PDEs in thin domains as well as the incompressible MHD system with strong background magnetic fields.

For the incompressible Navier-Stokes(NS) system, we refer to [14] on the global strong solutions and attractors of NS system for large initial data and force term in $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$ with periodic boundary conditions in the horizontal direction, where $Q_2$ is the rectangle in $\mathbb{R}^2$. In [8], the authors considered the NS system in a non-flat thin domain $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : x_1, x_2 \in (0, 1), x_3 \in (0, \varepsilon g(x_1, x_2))\}$ with periodic boundary condition in the horizontal direction and Navier boundary conditions on the top and bottom. They proved the existence of global solutions to NS system for the initial data whose $H^1(\Omega_\varepsilon)$ norm is smaller than $C\varepsilon^{-3/2}$. They also verified that solutions to 3D NS converge to the 2D Navier-Stokes-like system (for $g = 1$, it is exactly 2D NS), provided that the initial data converge to the 2D vector fields as $\varepsilon$ goes to zero. For the Euler equations in $Q_\varepsilon = \Omega \times (0, \varepsilon)$ where $\Omega$ is a rectangle in $\mathbb{R}^2$, the authors in [11] considered the periodic condition in the horizontal direction and the slip boundary condition on the vertical direction. By assuming that the initial data is uniformly bounded in $\varepsilon$ in space $W^{2,q}(Q_\varepsilon)$ ($q > 3$), they obtained the classic solution in the interval $(0, T(\varepsilon))$, where $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We remark that the mean value operator in the thin direction plays an important role.

For incompressible MHD with strong background magnetic fields, Bardos, Sulem and Sulem [2] first obtained the global solutions for the ideal MHD in the Hölder space (not in the energy space). The work in [2] treated MHD system as 1D waves system and it relied on the convolution of the fundamental solutions. For MHD with only strong fluid viscosity, in [9] [15] the authors proved the global existence of the small (w.r.t viscosity) solution with some admissible condition for the initial data. The work in [15] (also [9]) regarded MHD system as an anisotropic damped wave equation in the Lagrangian coordinates, that is, $\partial_t^2 Y - \mu \Delta \partial_\gamma Y - \partial_\gamma^2 Y \sim 0$. When $\mu = 0$, it holds $\partial_t^2 Y - \partial_\gamma^2 Y \sim 0$ which shows the ideal MHD system is 1D waves system. Actually, in 1942, Alfvén [1] first discovered such 1D waves (the so-called Alfvén waves) by linear analysis, in the incompressible ideal MHD with strong background magnetic fields. Recently, the authors in [6] provided a rigorous mathematical proof for the existence, propagation and stability of the (ideal/viscous) Alfvén waves in the nonlinear setting. The approach is inspired by the stability of Minkowski spacetime [4]. We also refer to [3] and [16] for alternative proofs on the same subject.

Finally we give some comments on this short review:

(i). In [8] [11] [14], the estimate of the pressure $p$ can be neglected due to the Leray projection and the divergence free condition. While for the incompressible ideal MHD, to catch the propagation of the Alfvén waves, the spacetime weight is introduced and thus the estimates for the pressure is compulsory.

(ii). In [8] [11] [14], the authors addressed the problem in the bounded domains and regarded the asymptotic of the equations from 3D to 2D as a perturbation of the equations in 2D.

Different from the previous work, in this paper, we will consider the effect of the shape of the thin domain $\Omega_\delta$ with $\delta > 0$ on the Alfvén waves. Our main goals can be concluded as follows:

(i). We want to show that [16] with boundary conditions [16] is global well-posed in any thin domain $\Omega_\delta$ with $\delta > 0$. In addition, some kind of the uniform energy estimates with respect to $\delta$ can be obtained.

(ii). We want to investigate the asymptotics of the MHD equations from the thin domain $\Omega_\delta$ to the plane $\mathbb{R}^2$. More precisely, we want to prove that the 3D Alfvén waves in $\Omega_\delta$ will converge to the Alfvén waves in $\mathbb{R}^2$ in the limit that $\delta$ goes to zero. It shows that when the width of the strip is very small Alfvén waves propagating along the horizontal direction of the (3D) strip are stable and can be approximated by the (2D) Alfvén waves. Moreover, the control of the (2D) Alfvén waves can be obtained from the control of (3D) Alfvén waves in the thin domain $\Omega_\delta$ via the uniform bounds.

1.2. **Difficulties, key observations and the strategies.** The main difficulty of the problem results from the shape of the thin domain $\Omega_\delta$. For instance, in the thin domain $\Omega_\delta$, the constant in the Sobolev imbedding inequality is related to the parameter $\delta$ (see Lemma [2.6]). Thus to make clear the dependence
of the energy functional on the parameter $\delta$, it is natural to rewrite the MHD system in a fixed domain $\Omega_1$ by taking the proper scaling. To do that, we introduce the new functions defined by

$$z_{\pm}(t, x_h, x_3) \equiv z_{\pm}^h(t, x_h, \delta x_3), \quad z_{3\pm}^h(t, x_h, x_3) \equiv \delta^{-1} z_{3\pm}^h(t, x_h, \delta x_3), \quad p(\delta)(t, x_h, x_3) \equiv p(t, x_h, \delta x_3), \quad \text{for any } x \in \Omega_1.$$ 

(1.8)

It is easy to check that $z_{\pm}(x_h, x_3) \equiv z_{\pm}(x_h, 0)$ still verifies the divergence free condition. Similarly we have $z_{3\pm}^h(x, x_3) \equiv z_{3\pm}(x, 0)$.

Then due to (1.4) and (1.8), $(z_+, z_-, p(\delta))$ satisfies the following system

$$\begin{align*}
\partial_t z_+ + (e_1 + z_+) \cdot \nabla z_+ &= -\nabla \delta p(\delta), \\
\partial_t z_- + (e_1 + z_-) \cdot \nabla z_- &= -\nabla \delta p(\delta), \\
\nabla \cdot z_+ &= 0, \quad \nabla \cdot z_- = 0,
\end{align*}$$

(1.9)

where $\nabla \delta = (\partial_1, \partial_2, \partial_3)$. Now it is clear that the system (1.9) is anisotropic and moreover the pressure $p(\delta)$ satisfies the singular Laplace equation

$$-(\partial^2_1 + \partial^2_2 + \delta^{-2}\partial^2_3)p(\delta) = \nabla \cdot (z_- \cdot \nabla z_+).$$

(1.10)

We remark that these two properties induce the difficulties of the solvability of the well-posedness for the original system (1.4).

Now let us talk about the key observations which are crucial to overcome the difficulties. Thanks to the anisotropic property of the equations, by calculation, we see that

$$||\partial^\alpha_h \partial^\beta_{3\pm}(t)||_{L^2(\Omega_1)} = \delta^{-\frac{\alpha}{2}} ||\partial^\alpha_h \partial^\beta_{3\pm}(t)||_{L^2(\Omega_1)} = \delta^{-\frac{\alpha}{2}} ||\partial^\alpha_h \partial^\beta_{3\pm}(t)||_{L^2(\Omega_1)},$$

which give the hints on the construction of the energy functional. In fact, we will introduce

$$\sum_{\pm} \delta^2 ||(x_1 + t)1^{\alpha} \nabla^h \partial^\beta_{3\pm}(t)||^2_{L^2(\Omega_1)} + \delta^{-3} \delta^2 ||(x_1 + t)1^{\alpha} \nabla^h \partial^\beta_{3\pm}(t)||^2_{L^2(\Omega_1)}$$

(1.11)

in the total energy (see Theorem 1.1) which is compatible with the anisotropic property of the system.

To prove the propagation of Alfvén waves, we will use two kinds of the null structures inside the system. Similar to [6], in a fixed strip, we still have the separation property of Alfvén waves, since the background magnetic field $B_0 = |B_0|e_1$ parallels to the strip. To implement the idea, the estimate of the pressure is compulsory since the spacetime weight $x_1 + t$ is involved in the energy functional. Recalling that in (1.9), the pressure $p(\delta)$ verifies (1.10). Thus the first challenge is to give an explicit expression for the pressure in the thin domain $\Omega_1$. By construction of the Green function, we successfully obtain the explicit formula for the pressure and moreover get the upper bounds for the Green function (see Lemma 2.1 and Corollary 2.3). It is not surprise that the upper bounds for the Green function contains the singular factor $\delta^{-1}$ because of (1.10). But it will bring the trouble to close the energy estimates if the total energy functional only contains the norms in (1.11). To overcome the difficulty, we introduce another energy

$$\sum_{\pm} \delta^2 ||(x_1 + t)1^{\alpha} \nabla^h \partial^\beta_{3\pm}(t)||^2_{L^2(\Omega_1)}$$

to absorb the additional singular factor $\delta^{-1}$ coming from the pressure. Then all the difficulties are reduced to prove the propagation of this new energy. Our key observation lies in the second type of the null structure of the system that there is no linear combination of terms $\partial^\alpha \partial^\beta z_+^h \partial^\gamma \partial^\delta z_-^h$ with $\alpha, \beta \in (Z_{\geq 0})^3$ in the system. Thus the propagation of the energy can be proved and then we complete the energy estimates.
Based on the above observations, our strategy can be concluded as:

1) We first prove the global existence of the solutions (Alfvén waves) to the MHD system in 3D thin plates and derive some kind of the uniform energy estimates with respective to the width parameter $\delta$.

2) With these uniform energy estimates in hand, we consider the asymptotics of the equations from $\Omega_1$ to $\mathbb{R}^2$. We split the proof into two steps. In the first step, we show that the horizontal component of the Alfvén waves in $\Omega_1$ can be approximated by their mean average in height. Then in the second step, we prove that the mean average of the Alfvén waves in horizontal converges to the 2D Alfvén waves as $\delta$ goes to zero if the initial Alfvén waves converge to the 2D Alfvén waves.

1.3. Statement of the main results. In this subsection, we will state three results on the MHD system in the thin domain $\Omega_\delta$.

1.3.1. Global well-posedness. Before stating the existence result, we introduce the linear characteristic hypersurfaces

\[
C_{u_+}^\pm \overset{\text{def}}{=} \{(t, x) \in \mathbb{R}^+ \times \Omega_\delta \mid u_+ = x_1 - t = \text{constant}\},
\]

\[
C_{u_+}^+ \overset{\text{def}}{=} \{(t, x_h) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid u_+ = x_1 - t = \text{constant}\},
\]

where $x = (x_h, x_3)$ and $x_h = (x_1, x_2)$. $C_{u_+}^-$ and $C_{u_+}^{-, h}$ can be defined in a similar way. Here $u_\pm = u_\pm (t, x_1) = x_1 \mp t$. We remark that the hypersurfaces $C_{u_+}^\pm$ and $C_{u_+}^{-, h}$ are regarded as the level sets of functions $u_\pm (t, x_1)$ in $(0, t^*) \times \Omega_\delta$ and $(0, t^*) \times \mathbb{R}^2$ respectively.

For given multi-index $\alpha_h = (\alpha_1, \alpha_2) \in (\mathbb{Z}_{\geq 0})^2$ and $l \in \mathbb{Z}_{\geq 0}$, we set $\partial_{\alpha_h}^l \partial_t^j f = \partial_{\alpha_h}^{\alpha_1} \partial_t^{\alpha_2} \partial_t^j f$ and $|\nabla_{\alpha_h}^l \partial_t^j f|^2 = \sum_{|\alpha_h|=k} |\partial_{\alpha_h}^{\alpha_1} \partial_t^{\alpha_2} \partial_t^j f|^2$. Then the energy $E_{\pm}^{(\alpha_h, l)} (z_\pm (t))$ and flux $F_{\pm}^{(\alpha_h, l)} (z_\pm)$ are defined as follows:

\[
E_{\pm}^{(\alpha_h, l)} (z_\pm (t)) = \| \langle u_\pm \rangle^{1+\sigma} \partial_{\alpha_h}^{\alpha_1} \partial_t^{\alpha_2} z_\pm (t) \|_{L^2(\Omega_\delta)},
\]

\[
F_{\pm}^{(\alpha_h, l)} (z_\pm) = \int_0^t \int_{\Omega_\delta} \frac{\langle u_\pm \rangle^{2(1+\sigma)}}{\langle u_\pm \rangle} |\partial_{\alpha_h}^{\alpha_1} \partial_t^{\alpha_2} z_\pm |^2 dx dt,
\]

where $\langle u \rangle = (1 + |u|^2)^{\frac{1}{2}}$. We can also give the definitions to $E_{\pm}^{(\alpha_h, l)} (z_\pm^h (t))$, $E_{\pm}^{(\alpha_h, l)} (z_\pm^l (t))$, $E_{\pm}^{(\alpha_h, l)} (j_\pm (t))$, $F_{\pm}^{(\alpha_h, l)} (z_\pm^h)$, $F_{\pm}^{(\alpha_h, l)} (z_\pm^l)$ and $F_{\pm}^{(\alpha_h, l)} (j_\pm)$, etc., in a similar way.

For simplicity, we introduce the total energy and flux as follows:

\[
E_{\pm}^{(k, l)} (z_\pm) = \sup_{0 \leq t \leq \tau} E_{\pm}^{(k, l)} (z_\pm (t)) = \sup_{0 \leq t \leq \tau} \sum_{|\alpha_h|=k} E_{\pm}^{(\alpha_h, l)} (z_\pm (t)),
\]

\[
F_{\pm}^{(k, l)} (z_\pm) = \sum_{|\alpha_h|=k} F_{\pm}^{(\alpha_h, l)} (z_\pm).
\]

We remark that $E_{\pm}^{(k, l)} (z_\pm^h)$, $E_{\pm}^{(k, l)} (z_\pm^l)$, $E_{\pm}^{(k, l)} (j_\pm)$, $F_{\pm}^{(k, l)} (z_\pm^h)$, $F_{\pm}^{(k, l)} (z_\pm^l)$ and $F_{\pm}^{(k, l)} (j_\pm)$, etc., can be defined in a similar way.

Now we are in a position to state our main results. The first result is on the global well-posedness and uniform energy estimates (with respect to $\delta$) of (1.3) in the domain $\Omega_\delta$ for any $\delta \in (0, 1]$.

**Theorem 1.1.** Let $B_0 = (1, 0, 0)^T$ be a given background magnetic field and $N_\sigma = 2N$, $N \in \mathbb{Z}_{\geq 5}$, $\delta \in (0, 1]$ and $\sigma \in (0, \frac{1}{9})$. Suppose that the vector fields $(z_{+, 0} (x), z_{-, 0} (x))$ satisfies $\text{div} \ z_{\pm, 0} = 0$ and
Then by (1.15), (1.16) and (1.17), we have

\[
E(0) \overset{\text{def}}{=} \sum_{k,l} \delta^{2(l-\frac{1}{2})} \| \langle x \rangle^{1+\sigma} \nabla_k^h \partial_3 z_{\pm,0} \|_{L^2(\Omega_3)}^2 + \sum_{k \leq N_3} \delta^{-3} \| \langle x \rangle^{1+\sigma} \nabla_k^h \partial_3 z_{\pm,0} \|_{L^2(\Omega_3)}^2
\]

where

\[
\delta \overset{\text{def}}{=} (\Omega_3)^{\frac{1}{2}}, \quad \delta \to 0.
\]

The MHD system (1.3) with boundary conditions (1.5) admits a unique and global smooth solution. Moreover, there holds

\[
\sum_{k+l \leq N_3} \delta^{2(l-\frac{1}{2})} \| \langle x \rangle^{1+\sigma} \nabla_k^h \partial_3 z_{\pm,0} \|_{L^2(\Omega_3)}^2 + \sum_{k \leq N_3} \delta^{-3} \| \langle x \rangle^{1+\sigma} \nabla_k^h \partial_3 z_{\pm,0} \|_{L^2(\Omega_3)}^2
\]

\[
\leq \varepsilon_0^2 C \mathcal{E}(0).
\]

In particular, the constant \( \varepsilon_0 \) and \( C \) are independent of the parameter \( \delta \).

Remark 1.1. By (1.8), it is easy to check that \( \sum_{k \leq N_3} \| \langle x \rangle^{1+\sigma} \nabla_k^h z_{\pm,0} \|_{L^2(\Omega_3)}^2 \leq \delta^2 \varepsilon_0^2 \), which means that as \( \delta \to 0 \), the limit of \( z_{\pm,0} \) exists and is independent of the vertical variable \( x_3 \). By similar argument, Theorem 1.1 implies that for any time \( t \), the limit of solution \( z_{\pm,0} \) (if there exists) will be also independent of the vertical variable \( x_3 \). Thanks to \( z_{\pm,0} = 0 \), the limit of \( z_{\pm,0} \) will also be independent of \( x_3 \). It looks promising that the solutions to the 3D MHD in thin domains will converge to the solutions to the 2D MHD as \( \delta \) goes to 0.

1.3.2. Approximation theory. The second result is on the approximation of the solutions to MHD in thin domains. We first introduce the projection \( M_\delta \) from \( L^2(\Omega_3) \) to \( L^2(\mathbb{R}^2) \) as follows:

\[
M_\delta f(x_3) \overset{\text{def}}{=} \frac{1}{2\delta} \int_{-\delta}^{\delta} f(x_3, x_3) dx_3.
\]

Then there hold

\[
\nabla_h M_\delta f = M_\delta (\nabla_h f), \quad M_\delta (\partial_3 f) = \partial_3 M_\delta f = 0 \quad \text{for any} \quad f|_{x_3=\pm,0} = 0,
\]

\[
M_\delta^2 f = M_\delta f, \quad M_\delta (I - M_\delta) f = (I - M_\delta) M_\delta f = 0.
\]

Assume that \((z_+(t,x), z_-(t,x))\) is a smooth solution to (1.4) with boundary conditions (1.5) and initial data \( z_{\pm}(0,x) = z_{\pm,0}(x) \). We set

\[
\bar{z}_+^h = (M_\delta z_+^h)(t, x_3), \quad \bar{z}_-^h = (M_\delta z_-^h)(t, x_3), \quad \bar{z}_\pm = (\bar{z}^h_\pm, 0),
\]

where \( f^h \overset{\text{def}}{=} (f^1, f^2)^T \). We also set

\[
w_\pm^h = z_\pm^h - \bar{w}_\pm, \quad w_\pm^h = \bar{z}_\pm^h.
\]

Then by (1.16), (1.17) and (1.18), we have

\[
M_\delta w_\pm^h = 0.
\]

Now we want to derive the equations for \( \bar{z}_\pm^h \) and \( w_\pm^h \). To do that, we observe that

\[
z_\pm \cdot \nabla z_\pm^h = (z_\pm + w_\pm) \cdot \nabla (z_\pm^h + w_\pm^h)
\]

\[
= \bar{z}_\pm^h \cdot \nabla_h \bar{z}_\pm^h + \bar{z}_\pm^h \cdot \nabla_h w_\pm^h + w_\pm \cdot \nabla z_\pm^h.
\]
Using (1.18) and \( \text{div } z_{\pm} = 0 \), we have

\[
M_\delta(z_{\mp} \cdot \nabla z_{\pm}^h) = z_{\mp}^h \cdot \nabla h z_{\pm}^h + M_\delta(w_{\mp} \cdot \nabla z_{\pm}^h),
\]

\[
(I - M_\delta)(z_{\mp} \cdot \nabla z_{\pm}^h) = z_{\mp}^h \cdot \nabla h w_{\pm}^h + (I - M_\delta)(w_{\mp} \cdot \nabla z_{\pm}^h).
\]

(1.19)

Due to (1.15), \( z_{\pm} = 0 \) and \( z_{\pm}^3 |_{x_3 = \pm \delta} = 0 \), \( z_{\pm}^3 |_{x_3 = \pm \delta} = 0 \), we get

\[
\nabla_h \cdot z_{\pm}^h = 0, \quad \nabla \cdot w_{\pm} = 0, \quad w_{\pm}^3 |_{x_3 = \pm \delta} = 0.
\]

(1.20)

Thanks to (1.19) and (1.20), we deduce from (1.4) that

\[
\partial_t z_{\pm}^h \mp \partial_1 z_{\pm}^h \mp z_{\mp}^h \cdot \nabla_h z_{\pm}^h = -\nabla_h (M_\delta p) - M_\delta (w_{\mp} \cdot \nabla z_{\pm}^h), \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+
\]

\[
\nabla_h \cdot z_{\pm}^h = 0, \quad \frac{\partial z_{\pm}^h}{\partial t}|_{t=0} = M_\delta z_{\pm,0}^h \overset{\text{def}}{=} z_{\pm,0}^h.
\]

(1.21)

and

\[
\partial_t w_{\pm}^h \mp \partial_1 w_{\pm}^h \mp z_{\pm}^h \cdot \nabla_h w_{\pm}^h = -(I - M_\delta)(\nabla_h p) - (I - M_\delta)(w_{\mp} \cdot \nabla z_{\pm}^h), \quad \text{in } \Omega_\delta \times \mathbb{R}^+
\]

\[
\frac{\partial w_{\pm}^h}{\partial t}|_{t=0} = (I - M_\delta)z_{\pm,0}^h \overset{\text{def}}{=} w_{\pm,0}^h.
\]

(1.22)

To investigate the difference between the original system and the mean average system, we introduce the energy \( E^{(\alpha_h)}(z_{\pm}^h(t)) \) and the flux \( F^{(\alpha_h)}(z_{\pm}^h) \) involving the quantities on \( \mathbb{R}^2 \) as follows:

\[
E^{(\alpha_h)}_{\pm, h}(z_{\pm}^h(t)) = \| (u_{\mp})^{1+\sigma} \partial_\alpha_h z_{\pm}^h(t) \|^2_{L^2(\mathbb{R}^2)},
\]

\[
F^{(\alpha_h)}_{\pm, h}(z_{\pm}^h) = \int_0^t \int_{\mathbb{R}^2} \frac{(u_{\mp})^{1+\sigma}}{\langle u_{\pm} \rangle^{1+\sigma}} |\partial_\alpha_h z_{\pm}^h(t)|^2 dxdt.
\]

Then the total energy and flux are defined by

\[
E^{(k)}_{\pm, h}(z_{\pm}^h(t)) = \sup_{0 \leq t \leq T^*} E^{(k)}_{\pm, h}(z_{\pm}^h(t)) = \sup_{0 \leq t \leq T^*} \sum_{|\alpha_h|=k} E^{(\alpha_h)}_{\pm, h}(z_{\pm}^h(t)),
\]

\[
F^{(k)}_{\pm, h}(z_{\pm}^h) = \sum_{|\alpha_h|=k} F^{(\alpha_h)}_{\pm, h}(z_{\pm}^h).
\]

We remark that \( E^{(\alpha_h)}(w_{\pm}(\cdot, x_3)) \), \( F^{(\alpha_h)}(w_{\pm}(\cdot, x_3)) \), \( E^{(k)}(w_{\pm}(\cdot, x_3)) \), \( F^{(k)}(w_{\pm}(\cdot, x_3)) \), etc., can also be defined in a similar way.

**Theorem 1.2.** Let \( (z_+, z_-) \) be a solution obtained in Theorem 1.7 with \( N_\ast = 2N \), \( N \in \mathbb{Z}_{\geq 5} \). Suppose that \( w_{\pm} \) is defined by (1.17). Then there exists a constant \( \varepsilon_1 (\leq \varepsilon_0) \) such that if \( E(0) \leq \varepsilon_2^2 \) (see the definition of \( E(0) \) in (1.12)), it holds

\[
\sum_{\pm, x_3 \in (-\delta, \delta)} \left( \sum_{k \leq N} (E^{(k)}_{\pm, h}(w_{\pm}^h(x_3)) + F^{(k)}_{\pm, h}(w_{\pm}^h(x_3))) + \delta^{-2} \sum_{k \leq N-1} (E^{(k)}_{\pm, h}(w_{\pm}^3(x_3)) + F^{(k)}_{\pm, h}(w_{\pm}^3(x_3))) \right)
\]

\[
\leq C \sum_{\pm, k \leq N} \sum_{x_3 \in (-\delta, \delta)} E^{(k)}_{\pm, h}(w_{\pm}^h(0, x_3)) + C \delta^2 \varepsilon_1.
\]

(1.23)

**Remark 1.2.** Roughly speaking, the theorem implies that if \( z_{\pm}^h \) is close to the mean average \( M_\delta z_{\pm}^h \) in height at the initial time, then \( z_{\pm}^h \) will keep close to the mean average \( M_\delta z_{\pm}^h \) in height for all time. In other words, the horizontal part of the vertical solutions to MHD in 3D thin domains can be approximated by the mean average in height while the vertical part of the solution is close to zero.
1.3.3. **Asymptotics from 3D MHD to 2D MHD.** Based on the first two results, we are in a position to investigate the asymptotics from the 3D MHD system in thin domains \( \Omega_\delta \) to the 2D MHD in \( \mathbb{R}^2 \) as \( \delta \) goes to zero. We recall that the system in thin domain \( \Omega_\delta \) can be rewritten by the following system in \( \Omega_1 \):

\[
\begin{align*}
\partial_t z_+(\delta) + (-e_1 + z_-(\delta)) \cdot \nabla z_+(\delta) &= -\nabla \delta p(\delta), \quad \text{in } \Omega_1 \times \mathbb{R}^+ \\
\partial_t z_-(\delta) + (e_1 + z_+(\delta)) \cdot \nabla z_-(\delta) &= -\nabla \delta p(\delta), \\
\nabla \cdot z_+(\delta) &= 0, \quad \nabla \cdot z_-(\delta) = 0, \\
z_+(\delta)|_{t=0} &= z_{+(\delta),0}(x), \quad z_-(\delta)|_{t=0} = z_{-(\delta),0}(x),
\end{align*}
\]

where \( \nabla \delta = (\partial_1, \partial_2, \delta^{-2} \partial_3)^T \). By setting

\[
z^h_{\pm}(t, x_h) = M_h z^h_{\pm}(\delta), \quad w^h_{\pm}(\delta) = z^h_{\pm}(\delta) - \bar{z}^h_{\pm}(\delta), \quad w^3_{\pm}(\delta) = z^3_{\pm}(\delta),
\]

and with the help of \((1.21)\) and \((1.8)\), we see that \((z^h_{\pm}(\delta), \bar{z}^h_{\pm}(\delta), p(\delta))\) satisfies

\[
\begin{align*}
\partial_t z^h_+(\delta) - \partial_1 z^h_+(\delta) + \bar{z}^h_+(\delta) \cdot \nabla h \bar{z}^h_+(\delta) &= -\nabla h M_h p(\delta) - M_h (w^h_-(\delta) \cdot \nabla h z^h_+(\delta)), \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\
\partial_t z^h_-(\delta) + \partial_1 z^h_-(\delta) + \bar{z}^h_+(\delta) \cdot \nabla h \bar{z}^h_-(\delta) &= -\nabla h M_h p(\delta) - M_h (w^h_+(\delta) \cdot \nabla z^h_-(\delta)), \\
\nabla_h \cdot z^h_+(\delta) &= 0, \quad \nabla_h \cdot z^h_-(\delta) = 0, \\
z^h_+(\delta)|_{t=0} = M_1 z^h_{+(\delta),0} \quad \text{def} = z^h_{+(\delta),0}, \quad z^h_-(\delta)|_{t=0} = M_1 z^h_{-(\delta),0} \quad \text{def} = z^h_{-(\delta),0}.
\end{align*}
\]

Before stating the result, we introduce the energy \( E^{(k)}(\delta)(z_{\pm}(\delta)) \) and flux \( F^{(k)}(\delta)(z_{\pm}(\delta)) \) on domain \((0, t^*) \times \Omega_1\) which are defined in a similar way as those on domain \((0, t^*) \times \Omega_\delta\). For simplicity, we also use the notations:

\[
E^{(k)}_{\pm}(z_{\pm}(\delta)) = \sum_{k' + l' = k} E^{(k', l')}_{\pm}(z_{\pm}(\delta)), \quad F^{(k)}_{\pm}(z_{\pm}(\delta)) = \sum_{k' + l' = k} F^{(k', l')}_{\pm}(z_{\pm}(\delta)).
\]

We remark that \( F^{(k)}_{\pm}(w_{\pm}(\delta)), E^{(k)}_{\pm}(z_{\pm}(\delta)), F^{(k)}_{\pm}(z_{\pm}(\delta)) \), e.t.c. can be defined in a similar way.

Before stating the main result, let us give a remark on the energies defined for the original system \((1.3)\) and those for the rescaled system \((1.9)\).

**Remark 1.3.** By \((1.8)\), for any \( \alpha_h \in (\mathbb{Z}_{\geq 0})^2, \ l \geq 0, \) we have

\[
\begin{align*}
\langle u_\pm \rangle^{1+\sigma} \| \partial_h^{\alpha_h} \partial_{\delta_+}^{\alpha_h} z^h_{\pm}(\delta) \|_{L^2(\Omega_1)} &= \delta^{\frac{1}{2}} \| \langle u_\pm \rangle^{1+\sigma} \partial_h^{\alpha_h} \partial_{\delta_+}^{\alpha_h} z^h_{\pm} \|_{L^2(\Omega_1)}, \\
\langle u_\pm \rangle^{1+\sigma} \| \partial_h^{\alpha_h} z^h_{\pm}(\delta) \|_{L^2(\Omega_1)} &= \delta^{-\frac{1}{2}} \| \langle u_\pm \rangle^{1+\sigma} \partial_h^{\alpha_h} z^h_{\pm} \|_{L^2(\Omega_1)}, \\
\langle u_\pm \rangle^{1+\sigma} \| \partial_h^{\alpha_h} \partial_{\delta_+}^{\alpha_h} z^h_{\pm}(\delta) \|_{L^2(\Omega_1)} &= \delta^{\frac{1}{2}} \| \langle u_\pm \rangle^{1+\sigma} \partial_h^{\alpha_h} \partial_{\delta_+}^{\alpha_h} z^h_{\pm} \|_{L^2(\Omega_1)} \quad \text{for } \ l \geq 1,
\end{align*}
\]

i.e.,

\[
\begin{align*}
E^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) &= \delta^{2(l-\frac{1}{2})} E^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)), \quad E^{(\alpha_h, 0)}_{\pm}(z^h_{\pm}(\delta)) = \delta^{-3} E^{(\alpha_h, 0)}_{\pm}(z^h_{\pm}), \\
E^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) &= \delta^{2(l-\frac{1}{2})} E^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) = \delta^{2(l-\frac{3}{2})} E^{(\alpha_h, l-1)}_{\pm}(\nabla_h \cdot z^h_{\pm}) \quad \text{for } \ l \geq 1.
\end{align*}
\]

Similarly, we also have

\[
\begin{align*}
F^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) &= \delta^{2(l-\frac{1}{2})} F^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)), \quad F^{(\alpha_h, 0)}_{\pm}(z^h_{\pm}(\delta)) = \delta^{-3} F^{(\alpha_h, 0)}_{\pm}(z^h_{\pm}), \\
F^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) &= \delta^{2(l-\frac{1}{2})} F^{(\alpha_h, l)}_{\pm}(z^h_{\pm}(\delta)) = \delta^{2(l-\frac{3}{2})} F^{(\alpha_h, l-1)}_{\pm}(\nabla_h \cdot z^h_{\pm}) \quad \text{for } \ l \geq 1.
\end{align*}
\]
Then thanks to (1.26) and (1.27), using div $z_\pm = 0$ and (1.25), we obtain

\[
\mathcal{E}(\delta)(t) = \sum_{k \leq N_*} \left( E^{(k)}_\pm (z^h_\pm(\delta)) + F^{(k)}_\pm (z^h_\pm(\delta)) \right) + \sum_{k \leq N_* - 1} \left( E^{(k)}_\pm (z^3_\pm(\delta)) + F^{(k)}_\pm (z^3_\pm(\delta)) \right) + \delta^2 \sum_{k \leq N_* - 1} (E^{(k)}_\pm (\partial_3 z^3_\pm(\delta)) + F^{(k)}_\pm (\partial_3 z^3_\pm(\delta)))
\]

\[
\sim \sum_{k+l \leq N_*} \delta^{2(l+\frac{1}{2})} (E^{(k,l)}_\pm (z_\pm)) + F^{(k,l)}_\pm (z_\pm)) + \sum_{k \leq N_* - 1} \delta^{-3} (E^{(k,0)}_\pm (z^3_\pm) + F^{(k,0)}_\pm (z^3_\pm)) + \sum_{k+l \leq N_* + 2} \delta^{2(l-\frac{1}{2})} (E^{(k,l)}_\pm (\partial_3 z_\pm) + F^{(k,l)}_\pm (\partial_3 z_\pm)).
\]

(1.28)

**Definition 1.4.** We call that the sequence $\{z^h_\pm(\delta)(x)\}_{0 < \delta \leq 1}$ converges to $z^h_\pm(0)(x)$ in $H^N(\Omega_1)$ if it holds

\[
\lim_{\delta \to 0} \sum_{k \leq N} \|u_\pm\|^{1+\sigma} \nabla^k (z^h_\pm(\delta) - z^h_\pm(0)) \|_{L^2(\Omega_1)} = 0.
\]

(1.29)

Similarly, the sequence $\{z^h_\pm(\delta)(x)\}_{0 < \delta \leq 1}$ is said to converge to $z^h_\pm(0)(x)$ in $H^N(\mathbb{R}^2)$ if it holds

\[
\lim_{\delta \to 0} \sum_{k \leq N} \|u_\pm\|^{1+\sigma} \nabla^k (z^h_\pm(\delta) - z^h_\pm(0)) \|_{L^2(\mathbb{R}^2)} = 0.
\]

(1.30)

The convergence of the sequence $\{z^3_\pm(\delta)(\cdot)\}_{0 < \delta \leq 1}$ to $z^3_\pm(0)(\cdot)$ in $H^N(\Omega_1)$ and the sequence $\{z^3_\pm(\delta)(\cdot, x_3)\}_{0 < \delta \leq 1}$ to $z^3_\pm(\cdot, x_3)$ (for fixed $x_3 \in (-1, 1)$) in $H^N(\mathbb{R}^2)$ can be defined in a similar way.

**Theorem 1.3.** Let $N_* = 2N$, $N \in \mathbb{Z}_{\geq 5}$, $\delta \in (0, 1]$, $\sigma \in (0, \frac{1}{2})$ and $\varepsilon_1$ be the constant in Theorem 1.2. Suppose that the vector field sequence $\{(z^h_\pm(\delta), 0, z^3_\pm(\delta), 0)\}_{0 < \delta \leq 1}$ satisfies div $z^h_\pm(\delta), 0 = 0, z^3_\pm(\delta), 0|_{x_3 = \pm 1} = 0, z^3_\pm(\delta), 0|x_3 = 0 = 0$ and for all $\delta \in (0, 1]$,

\[
E_\delta(0) = \sum_{k \leq N_*} \left( E^{(k)}_\pm (z^h_\pm(\delta), 0) + E^{(k)}_\pm (z^3_\pm(\delta), 0) \right) + \delta^2 \sum_{k \leq N_* - 1} \left( E^{(k)}_\pm (\partial_3 z^3_\pm(\delta), 0) \right) \leq \varepsilon_1.
\]

(1.31)

Assume that

\[
(z^h_\pm(\delta), 0(x_3), z^3_\pm(\delta), 0(x_3)) \text{ converges to } (z^h_\pm(0), 0(x_3), 0) \text{ in } H^{N+1}(\Omega_1),
\]

(1.32)

with $\nabla \cdot z^h_\pm(0), 0 = 0$. Then if $(z^\pm(\delta)(t, x), z^\pm(\delta)(t, x))$ is a solution to MHD (1.14) with the initial data $(z^\pm(\delta), 0(x), z^\pm(\delta), 0(x))$, there exist functions $z^h_\pm(0)(t, x)$ such that for any $x_3 \in (-1, 1)$, $t > 0$

\[
\lim_{\delta \to 0} z^\pm_h(\delta)(t, x_3, x_3) = z^\pm_h(0)(t, x_3), \text{ in } H^N(\mathbb{R}^2),
\]

\[
\lim_{\delta \to 0} z^\pm_3(\delta)(t, x_3, x_3) = 0, \text{ in } H^{N-1}(\mathbb{R}^2).
\]

(1.33)

In particular, $(z^h_+(0)(t, x), z^h_-(0)(t, x))$ solves 2D version of the system (1.4) with the initial data $(z^h_+(0), 0(x), z^h_-(0), 0(x))$ and verifies

\[
\sum_{\pm, k \leq N} \left( E^{(k)}_\pm (z^h(\pm, 0) + F^{(k)}_\pm (z^h(\pm, 0)) \right) \leq C \sum_{\pm, k \leq N} E^{(k)}_\pm (z^h(\pm, 0)).
\]

(1.34)
Moreover the error estimate for the asymptotics can be concluded as follows:

\[
\sum_{k \leq N-1} \sup_{+,-} \sum_{x_3 \in (-1, 1)} \left( E_{\pm, h}^{(k)}(z_{\pm}(\delta), x_3) - z_{\pm}(0)(\cdot) \right) + F_{\pm, h}^{(k)}(z_{\pm}(\delta), x_3) - z_{\pm}(0)(\cdot)) \\
+ \sum_{k \leq N-1} \left( E_{\pm, h}^{(k)}(z_{\pm}(\delta), x_3) + F_{\pm, h}^{(k)}(z_{\pm}(\delta), x_3)) \right) \\
\leq C \sup_{k \leq N-1} \sup_{x_3 \in (-1, 1)} E_{\pm, h}^{(k)}(z_{\pm}(\delta), x_3) - z_{\pm}(0)(\cdot) + C\delta^2 z_{\cdot}^4.
\]

(1.35)

**Remark 1.5.** 1. Thanks to \((1.28)\), the initial condition \((1.31)\) implies \((1.12)\). Then by Theorem 1.1, we have the global existence for \(E_3(t)\) and get the uniform control of \(E_3(t)\) for all time.

2. The assumption \((1.31)\) shows that \(\sum_{k \leq N+2} \| (x_1)^{1+\sigma} \|_{L^2(\Omega)} \leq \delta^2 \varepsilon_1^2\), which implies \(\partial_3 z_{\pm}(0,0)(x_3) = 0\). Thus we have \(z_{\pm}(0,0)(x_3) = z_{\pm}(0,0)(x_3)\). We remark that this property will be kept for all time thanks to the uniform bound for \(E_3(t)\).

**Notations.** For any \(k \in \mathbb{Z}, H^k(\Omega)\) means the standard Sobolev spaces on domain \(\Omega\). For any \(k \in \mathbb{Z}\) and \(p \in [1, \infty)\), \(\Omega_\delta = \mathbb{R}^2 \times (-\delta, \delta) \in \mathbb{R}^3\) with \(\delta \in (0, 1]\), without confusion of the domain, the norms \(\| \cdot \|_{H^k} = \| \cdot \|_{H^k(\Omega)}, \| \cdot \|_{H^k_{\pm, h}} = \| \cdot \|_{H^k(\Omega)}\), \(\| \cdot \|_{H^k_{\pm, h}} = \| \cdot \|_{H^k(\Omega)}\), \(\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)}\) mean \(\| \cdot \|_{H^k(\Omega)}, \| \cdot \|_{H^k(\Omega)}, \| \cdot \|_{L^p(\Omega)}\) respectively. Similarly, we shall always use the notations \(\| \cdot \|_{L^p(\Omega)}\), \(\| \cdot \|_{L^p(\Omega)}\), \(\| \cdot \|_{L^p(\Omega)}\), \(\| \cdot \|_{L^p(\Omega)}\) etc. for any \(p, q, r \in [1, \infty]\), \(k \in \mathbb{Z}\). The notation \(A \lesssim B\) means \(A \leq c B\) while \(A \sim B\) means \(c^{-1}B \leq A \leq cB\) for some universal constant \(c\) independent of \(\delta\).

2. **The estimate for the pressure, technical lemmas and the characteristic geometry**

In this section, we will give the proof to some lemmas which will be used throughout the paper.

2.1. **Derivation of the pressure \(p\) in the domain \(\Omega_\delta\).** Taking divergence to the first or the second equation of \((1.4)\), and using conditions \(\mathrm{div} z_\pm = 0, z_\pm^3|_{x_3=\pm \delta} = 0\) and \(z_\pm^3|_{x_3=\pm \delta} = 0\), we have

\[
\Delta p = - \nabla \cdot (z_+ \cdot \nabla z_-) \quad \text{in} \quad \Omega_\delta, \\
\partial_3 p|_{x_3=\pm \delta} = 0.
\]

(2.1)

Due to \(\mathrm{div} z_\pm = 0\), the source term \(\nabla \cdot (z_+ \cdot \nabla z_-)\) can be also written in the forms

\[
\nabla \cdot (z_+ \cdot \nabla z_-) = \partial_3 z_+ \partial_3 z_- = \partial_3 \partial_3 (z_+ z_-).
\]

In the next context, we shall solve \(p\) from \((2.1)\).

**Lemma 2.1.** Given smooth vectors \((z_+, z_-)\), we have

\[
\nabla p(t, x) = \int_{\Omega_\delta} \nabla x G_\delta(x, y)(\partial_3 z_+ \partial_3 z_-)(t, y) dy,
\]

(2.2)

where

\[
\nabla x G_\delta(x, y) = \frac{1}{4\pi} \left[ \nabla x \frac{1}{|x-y|} + \sum_{k=1}^{\infty} \left( \nabla x \frac{1}{|x_+, k - y|} + \nabla x \frac{1}{|x_-, k - y|} \right) \right],
\]

(2.3)

with

\[
x_+, k = (x_k, (-1)^k(x_3 - 2k\delta)), \quad x_-, k = (x_h, (-1)^k(x_3 + 2k\delta)).
\]

(2.4)

**Remark 2.2.** It is easy to check that

\[
\partial_3^{\alpha_h} G_\delta(x, y) = (-1)^{|\alpha_h|} \partial_3^{\alpha_h} G_\delta(x, y),
\]

(2.5)

for any \(\alpha_h = (\alpha_1, \alpha_2) \in (\mathbb{Z}_{\geq 0})^2\) and \(|\alpha_h| \geq 1\).
Proof. Firstly, we set
\[ f(t, x) \overset{\text{def}}{=} (\partial_i z_i^+ \partial_j z_j^-)(t, x). \]

For continuous function \( f \in C(\overline{\Omega}_\delta) \), we extend \( f \) from \( \Omega_\delta \) to the whole space \( \mathbb{R}^3 \) in the following way
\[ \tilde{f}(t, x_h, x_3) = f(t, x_h, (1)^k(x_3 - 2k\delta)), \quad \text{if} \quad x_3 \in ((2k - 1)\delta, (2k + 1)\delta], \quad k \in \mathbb{Z}. \tag{2.6} \]

It is easy to check that \( \tilde{f} \in C(\mathbb{R}^3) \). Then we solve \( \tilde{p} \) (up to a constant) from the Laplacian equation \( \Delta \tilde{p} = -f \) as follows:
\[ \tilde{p}(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|}\tilde{f}(t, y)dy, \quad \text{for} \quad x \in \mathbb{R}^3. \tag{2.7} \]

Taking \( p = \tilde{p}|_{\Omega_\delta} \), we shall verify that \( p \) solves (2.1). By virtue of (2.6) and (2.7), we have
\[ \Delta p = -f, \quad \text{for} \quad x \in \Omega_\delta. \tag{2.8} \]

We only need to check that \( \partial_3 p|_{x_3=\pm \delta} = 0 \). To do so, we transform the integration over \( \mathbb{R}^3 \) in (2.7) to that over \( \Omega_\delta \). Due to (2.6) and (2.7), we have
\[
\nabla p(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla_x \frac{1}{|x - y|}\tilde{f}(t, y)dy
\]
\[ = \frac{1}{4\pi} \sum_{k = -\infty}^{\infty} \int_{(2k+1)\delta}^{(2k+1)\delta} \int_{\mathbb{R}^2} \nabla_x \left( \frac{1}{|x_h - y_h|^2 + |x_3 - y_3|^2} \right) f(t, y_h, (-1)^k(y_3 - 2k\delta))dy_hdy_3. \]

For \( I_k(t, x) \), setting \( \tilde{y}_3 = (-1)^k(y_3 - 2k\delta) \), we have
\[ y_3 = (-1)^k\tilde{y}_3 + 2k\delta, \quad x_3 - y_3 = x_3 - 2k\delta - (-1)^k\tilde{y}_3 = (-1)^k(-1)(x_3 - 2k\delta) - \tilde{y}_3, \]
and
\[ I_k = (-1)^k \int_{(-1)^{k+1} \delta}^{(-1)^k \delta} \int_{\mathbb{R}^2} \nabla_x \left( \frac{1}{|x_h - y_h|^2 + |(-1)^k(x_3 - 2k\delta) - \tilde{y}_3|^2} \right) f(t, y_h, \tilde{y}_3)dy_hd\tilde{y}_3
\]
\[ = \int_{-\delta}^{\delta} \int_{\mathbb{R}^2} \nabla_x \left( \frac{1}{|x_h - y_h|^2 + |(-1)^k(x_3 - 2k\delta) - y_3|^2} \right) f(t, y_h, y_3)dy_hdy_3. \]

Then we have
\[ \nabla p(t, x) = \int_{\Omega_\delta} \nabla_x G_\delta(x, y)f(t, y)dy, \tag{2.9} \]
where
\[ \nabla_x G_\delta(x, y) \overset{\text{def}}{=} \sum_{k = -\infty}^{\infty} \nabla_x \left( \frac{1}{|x_h - y_h|^2 + |(-1)^k(x_3 - 2k\delta) - y_3|^2} \right). \tag{2.10} \]

We remark that the r.h.s of (2.10) is summable (see the following Lemma 2.8).

Setting
\[ x_{+, k} = (x_h, (-1)^k(x_3 - 2k\delta)), \quad x_{-, k} = (x_h, (-1)^k(x_3 + 2k\delta)), \quad \text{for} \quad k \in \mathbb{N}, \]
we have
\[ \nabla_x G_\delta(x, y) = \frac{1}{4\pi} \left[ \nabla_x \frac{1}{|x - y|} + \sum_{k = 1}^{\infty} \left( \nabla_x \frac{1}{|x_{+, k} - y|} + \nabla_x \frac{1}{|x_{-, k} - y|} \right) \right]. \tag{2.11} \]
We remark that \( x_{+,k} \) is the reflection point of the point \( x_{+,k-1} \) in the plane \( \Gamma_{(-)}k^{-1} \) and \( x_{-,k} \) is the reflection point of the point \( x_{-,k-1} \) in the plane \( \Gamma_{(-)}k \) (here \((-)^m \) equals “+” if \( m \) is even while equals “−” if \( m \) is odd, and \( x_{+,0} = x_{-,0} \overset{\text{def}}{=} x \)). Denoting by 
\[
\phi_k(x, y) = \frac{1}{|x_{+,k} - y|} + \frac{1}{|x_{-,k} - y|}, \quad \text{for} \quad k \in \mathbb{N},
\]
we have
\[
\partial_{x_3} \phi_k(x, y) = (-1)^{k+1} \left( \frac{x_{+,k}^3 - y_3}{|x_{+,k} - y|^3} + \frac{x_{-,k}^3 - y_3}{|x_{-,k} - y|^3} \right),
\]
which implies
\[
\partial_{x_3} \phi_k(x, y)|_{x_3 = \delta} = (-1)^{k+1} \left( \frac{(-1)^{k+1}(2k - 1)\delta - y_3}{|(x_h, (-1)^{k+1}(2k - 1)\delta - y)|^3} + \frac{(-1)^k(2k + 1)\delta - y_3}{|(x_h, (-1)^k(2k + 1)\delta - y)|^3} \right),
\]
\[
\partial_{x_3} \phi_k(x, y)|_{x_3 = -\delta} = (-1)^{k+1} \left( \frac{(-1)^{k+1}(2k + 1)\delta - y_3}{|(x_h, (-1)^{k+1}(2k + 1)\delta - y)|^3} + \frac{(-1)^k(2k - 1)\delta - y_3}{|(x_h, (-1)^k(2k - 1)\delta - y)|^3} \right).
\]
Then we have
\[
\sum_{k=1}^{\infty} \partial_{x_3} \phi_k(x, y)|_{x_3 = \delta} = \frac{\delta - y_3}{|(x_h, \delta) - y|^3} + \lim_{k \to \infty} \frac{(-1)^{k+1}[(-1)^k(2k + 1)\delta - y_3]}{|(x_h, (-1)^k(2k + 1)\delta - y)|^3},
\]
\[
\sum_{k=1}^{\infty} \partial_{x_3} \phi_k(x, y)|_{x_3 = -\delta} = \frac{-\delta - y_3}{|(x_h, -\delta) - y|^3} + \lim_{k \to \infty} \frac{(-1)^{k+1}[(-1)^k(2k + 1)\delta - y_3]}{|(x_h, (-1)^k(2k + 1)\delta - y)|^3}.
\]
Since \( y_3 \in (-\delta, \delta) \), we have
\[
\lim_{k \to \infty} \frac{(-1)^{k+1}[(-1)^k(2k + 1)\delta - y_3]}{|(x_h, (-1)^k(2k + 1)\delta - y)|^3} = \lim_{k \to \infty} \frac{(-1)^{k+1}[(-1)^{k+1}(2k + 1)\delta - y_3]}{|(x_h, (-1)^{k+1}(2k + 1)\delta - y)|^3} = 0,
\]
which implies that
\[
\sum_{k=1}^{\infty} \partial_{x_3} \phi_k(x, y)|_{x_3 = \pm\delta} = \frac{\pm\delta - y_3}{|(x_h, \pm\delta) - y|^3} = -\partial_{x_3} \left( \frac{1}{|x - y|} \right)|_{x_3 = \pm\delta}.
\]
Due to (2.11), we obtain
\[
\partial_{x_3} G_\delta(x, y)|_{x_3 = \pm\delta} = 0,
\]
which along with (2.9) implies
\[
\partial_{\delta} p|_{x_3 = \pm\delta} = 0.
\]
Thus, \( p(t, x) \) obtained in the proof satisfies (2.1). Moreover, there holds (2.2). The lemma is proved. \( \square \)

**Corollary 2.3.** Assume that \( \nabla_\xi G_\delta(x, y) \) is the function on \( \Omega_3 \times \Omega_3 \) defined in (2.3). Then we have
\[
|\nabla_\xi^k G_\delta(x, y)| \leq \frac{1}{\delta} \frac{1}{|x_h - y_h|^k}, \quad k = 1, 2, 3, \ldots.
\]
Proof. By the definition (2.3), we have
\[
|\nabla_x G_\delta(x, y)| \leq \frac{1}{4\pi} \left|\frac{1}{|x - y|^2} + \sum_{k=1}^{\infty} \left(\frac{1}{|x_{+,k} - y|^2} + \frac{1}{|x_{-,k} - y|^2}\right)\right|
\]
\[
\leq \frac{1}{4\pi} \left[\frac{1}{|x_h - y_h|^2 + |x_3 - y_3|^2} + \sum_{k=1}^{\infty} \left(|x_h - y_h|^2 + |(-1)^k x_3 - 2(-1)^k \delta|^2\right)\right]
\]
\[
+ \frac{1}{|x_h - y_h|^2 + |(-1)^k x_3 - 2(-1)^k \delta|^2}\right)\right].
\]
Since \(x_3, y_3 \in (-\delta, \delta)\), we have
\[
|\nabla_x G_\delta(x, y)| \leq \frac{1}{4\pi} \left[\frac{1}{|x_h - y_h|^2 + |2(k - 1)\delta|^2}\right]
\]
\[
\leq \frac{3}{4\pi} \sum_{k=0}^{\infty} \frac{1}{|x_h - y_h|^2 + |2k\delta|^2} \leq \frac{3}{4\pi} \int_{0}^{\infty} \frac{1}{|x_h - y_h|^2 + 4\tau^2\delta^2} d\tau
\]
\[
\leq \frac{3}{4\pi} \frac{1}{2\delta|x_h - y_h|} \int_{0}^{\infty} \frac{1}{1 + \tau^2} d\tau = \frac{3}{16} \cdot \frac{1}{\delta} \cdot \frac{1}{|x_h - y_h|}.
\]
Then (2.12) holds for \(k = 1\). Similarly, we could prove that (2.12) holds for all \(k \in \mathbb{N}\). \(\square\)

2.2. Technical lemmas. We first state the following div-curl lemma.

Lemma 2.4 (div-curl lemma). Let \(\lambda(x)\) be a smooth positive function on \(\Omega_\delta\). Then for all smooth vector field \(v(x) = (v^1(x), v^2(x), v^3(x)) \in H^1(\Omega_\delta)\) with the following properties
\[
div v = 0, \quad \sqrt{\lambda} \nabla v \in L^2(\Omega_\delta), \quad \frac{\nabla|\lambda|}{\sqrt{\lambda}} v \in L^2(\Omega_\delta),
\]
it holds
\[
\|\sqrt{\lambda} \nabla v\|_{L^2}^2 \leq C\left(\|\sqrt{\lambda} \text{curl} v\|_{L^2}^2 + \||\nabla|\lambda|\sqrt{\lambda} v\|_{L^2}^2 + \int_{\partial \Omega_\delta} \lambda v \cdot \nabla v^3 dx_1 dx_2\right),
\]
where \(C\) is a universal constant independent of \(\delta\).

Proof. Since div \(v = 0\), we have
\[
-\Delta v = \text{curl} \text{curl} v.
\]
Multiplying the above identity by \(\lambda v\), and integrating over \(\Omega_\delta\), we have
\[
\int_{\Omega_\delta} -\Delta v \cdot \lambda v dx = \int_{\Omega_\delta} \text{curl} \text{curl} v \cdot \lambda v dx.
\]
By integration by parts, we have
\[
\int_{\Omega_\delta} -\Delta v \cdot \lambda v dx = \int_{\Omega_\delta} \lambda |\nabla v|^2 dx + \int_{\Omega_\delta} (\nabla \lambda \cdot \nabla)v \cdot v dx - \int_{\partial \Omega_\delta} \frac{\partial v}{\partial n} \cdot \lambda v ds,
\]
\[
\int_{\Omega_\delta} \text{curl} \text{curl} v \cdot \lambda v dx = \int_{\Omega_\delta} \lambda |\text{curl} v|^2 dx + \int_{\Omega_\delta} \text{curl} v \cdot (\nabla \lambda \times v) dx - \int_{\partial \Omega_\delta} \text{curl} v \times n \cdot \lambda v ds,
\]
where \(n\) is the unit outward normal to \(\partial \Omega_\delta\) and \(ds\) is the surface measure of \(\partial \Omega_\delta\). Then we obtain
\[
\int_{\Omega_\delta} \lambda |\nabla v|^2 dx = \int_{\Omega_\delta} \lambda |\text{curl} v|^2 dx - \int_{\Omega_\delta} (\nabla \lambda \cdot \nabla)v \cdot v dx
\]
\[
+ \int_{\Omega_\delta} \text{curl} v \cdot (\nabla \lambda \times v) dx - \int_{\partial \Omega_\delta} \left(\frac{\partial v}{\partial n} - \text{curl} v \times n\right) \cdot \lambda v ds.
\]
(2.14)
Lemma is proved.

Let $\; n \equiv (n_1, n_2, n_3)^T.\) Using Hölder inequality, we deduce from (2.14) and (2.15) that

$$
|\sqrt{\lambda} \nabla v|^2_{L^2} \leq C(\| \sqrt{\lambda} \text{curl } v \|^2_{L^2} + \| \nabla \lambda \|^2_{L^2} + \int_{\partial \Omega} |n_i \partial_j v^i v^j dS|).
$$

Noticing that $n \mid_{x_3 \pm \delta} = (0, 0, \pm 1)^T$ and $dS = dx_1 dx_2$, we obtain the desired inequality (2.13). The Lemma is proved.

As a consequence of Lemma 2.3, we have the following corollary.

**Corollary 2.5.** Let $\lambda(x)$ be a smooth positive function on $\Omega_\delta$ with $|\nabla \lambda| \leq C|\lambda|$. For all smooth vector field $v(x) \in H^k(\Omega_\delta)$ ($k \in \mathbb{N}$) with the following properties

$$
\text{div } v = 0, \quad v^3 \mid_{x_3 \pm \delta} = 0, \quad \sqrt{\lambda} \nabla^l v \in L^2(\Omega_\delta), \quad l = 0, 1, \ldots, k,
$$

we have

$$
\| \sqrt{\lambda} \nabla^k v \|^2_{L^2} \leq C(\sum_{l=0}^{k-1} \| \sqrt{\lambda} \nabla^l \text{curl } v \|^2_{L^2} + \| \sqrt{\lambda} v \|^2_{L^2}). \tag{2.17}
$$

**Proof.** We prove (2.17) by the induction method. For $k = 1$, using (2.13), we have

$$
\| \sqrt{\lambda} \nabla v \|^2_{L^2} \leq C(\| \sqrt{\lambda} \text{curl } v \|^2_{L^2} + \| \sqrt{\lambda} v \|^2_{L^2} + \int_{\partial \Omega} \lambda(v^h \cdot \nabla_h v^3 + v^3 \partial_3 v^3) d\Omega_\delta).
$$

Due to the boundary condition $v^3 \mid_{x_3 \pm \delta} = 0$ and $|\nabla \lambda| \leq C\lambda$, we get

$$
\| \sqrt{\lambda} \nabla v \|^2_{L^2} \leq C(\| \sqrt{\lambda} \text{curl } v \|^2_{L^2} + \| \sqrt{\lambda} v \|^2_{L^2}). \tag{2.18}
$$

Assume that for any $1 \leq m \leq k - 1$, it holds (2.17), i.e.,

$$
\| \sqrt{\lambda} \nabla^m v \|^2_{L^2} \leq C(\sum_{l=0}^{m-1} \| \sqrt{\lambda} \nabla^l \text{curl } v \|^2_{L^2} + \| \sqrt{\lambda} v \|^2_{L^2}). \tag{2.19}
$$

Then for $k$, we have

$$
\| \sqrt{\lambda} \nabla^k v \|^2_{L^2} \leq \| \sqrt{\lambda} \nabla^{k-1}(\partial_h v) \|^2_{L^2} + \| \sqrt{\lambda} \nabla^{k-1}(\partial_3 v) \|^2_{L^2},
$$

where $\partial_h$ is the horizontal derivative with respect to $x_h$. Since $\nabla_h v^3 \mid_{x_3 \pm \delta} = 0$, using (2.13), we have

$$
\| \sqrt{\lambda} \nabla^{k-1}(\partial_h v) \|^2_{L^2} \leq C(\sum_{l=0}^{k-2} \| \sqrt{\lambda} \nabla^l \text{curl } (\partial_h v) \|^2_{L^2} + \| \sqrt{\lambda} \nabla_h v \|^2_{L^2}).
$$

Using (2.18), we have

$$
\| \sqrt{\lambda} \nabla^{k-1}(\partial_h v) \|^2_{L^2} \leq C(\sum_{l=0}^{k-1} \| \sqrt{\lambda} \nabla^l \text{curl } v \|^2_{L^2} + \| \sqrt{\lambda} v \|^2_{L^2}). \tag{2.20}
$$

On the other hand, since

$$
\partial_3 v^1 = (\partial_3 v^1 - \partial_1 v^3) + \partial_1 v^3, \quad \partial_3 v^2 = (\partial_3 v^2 - \partial_2 v^3) + \partial_2 v^3, \quad \partial_3 v^3 = -\partial_1 v^1 - \partial_2 v^2.
$$

we have

$$
\| \sqrt{\lambda} \nabla^{k-1}(\partial_3 v) \|^2_{L^2} \leq \| \sqrt{\lambda} \nabla^{k-1}(\text{curl } v) \|^2_{L^2} + \| \sqrt{\lambda} \nabla^{k-1}(\partial_3 v) \|^2_{L^2}.\tag{2.21}
$$
Then using (2.20), we have
\[
\|\sqrt{\lambda} \nabla^{k-1}(\partial_3 v)\|_{L^2}^2 \leq C \left( \sum_{l=0}^{k-1} \|\sqrt{\lambda} \nabla^l \text{curl } v\|_{L^2}^2 + \|\sqrt{\lambda} v\|_{L^2}^2 \right). \tag{2.21}
\]

Thanks to (2.20) and (2.21), we obtain the desired result. \(\square\)

**Lemma 2.6 (Sobolev inequality).** (i) For any \( f(x) \in H^1_\delta (\mathbb{R}^2) \), we have
\[
\| f(x) \|_{L^2 (H^1_\delta (\mathbb{R}^2))} \leq C (\delta^{-\frac{1}{2}} \| f \|_{L^2 (\mathbb{R}^2)} + \delta^{\frac{1}{2}} \| \partial_3 f \|_{L^2 (\mathbb{R}^2)}).
\tag{2.22}
\]

(ii) For any \( f(x) \in H^2 (\Omega_\delta) \), we have
\[
\| f(x) \|_{L^\infty} \leq C \sum_{k+l \leq 2} \delta^{l-\frac{1}{2}} \| \nabla^k \partial_3^l f \|_{L^2}.
\tag{2.23}
\]

**Proof.** For any function \( f \) defined on \( \Omega_\delta \), we set
\[
\hat{f}(x_h, x_3) = f(x_h, \delta x_3), \quad x \in \Omega_1 = \mathbb{R} \times (-1, 1).
\]

Then the general Sobolev inequality shows that
\[
\| f(x) \|_{L^\infty (-\delta, \delta; H^1_\delta)} = \| \hat{f}(x) \|_{L^\infty (-\delta, \delta; H^1_\delta)} \leq C \left( \| \hat{f}(x) \|_{L^2 (-\delta, \delta; H^2_\delta)} + \| \partial_3 \hat{f}(x) \|_{L^2 (-\delta, \delta; H^2_\delta)} \right).
\]

Since
\[
\hat{f}(x_h, x_3) = f(x_h, \delta x_3), \quad \partial_3 \hat{f}(x_h, x_3) = \delta (\partial_3 f)(x_h, \delta x_3),
\]
we have
\[
\| f(x) \|_{L^\infty (-\delta, \delta; H^1_\delta)} \leq C (\delta^{-\frac{1}{2}} \| f \|_{L^2 (-\delta, \delta; H^2_\delta)} + \delta^{\frac{1}{2}} \| \partial_3 f \|_{L^2 (-\delta, \delta; H^2_\delta)}).
\]

Similarly, we have
\[
\| f(x) \|_{L^\infty (\Omega_\delta)} = \| \hat{f}(x) \|_{L^\infty (\Omega_\delta)} \leq C \sum_{k+l \leq 2} \| \nabla^k \partial_3^l \hat{f} \|_{L^2 (\Omega_\delta)} \leq C \sum_{k+l \leq 2} \delta^{l-\frac{1}{2}} \| \nabla^k \partial_3^l f \|_{L^2 (\Omega_\delta)}.
\]

The lemma is proved. \(\square\)

**2.3. The characteristic geometry.** We study the spacetime \([0, t^*] \times \Omega_\delta = [0, t^*] \times \{ \mathbb{R}^2_\delta \times (-\delta, \delta) \times 3 \} \) associated to solution \((v, b)\) or \((z_+, z_-)\) of MHD. We recall that there exists a natural foliation for \([0, t^*] \times \Omega_\delta\) as
\[
[0, t^*] \times \Omega_\delta = \bigcup_{0 \leq \ell \leq t^*} \Sigma_\ell,
\]
where \(\Sigma_\ell\) is the constant time slice (\(\Sigma_0\) is the initial time slice where the initial data are given).

Due to the linear characteristic hypersurface, there exist another foliations as
\[
[0, t^*] \times \Omega_\delta = \bigcup_{u_+ \in \mathbb{R}} C^+_{u_+} = \bigcup_{u_- \in \mathbb{R}} C^-_{u_-},
\]
where
\[
C^+_{u_+} \overset{\text{def}}{=} \{(t, x) \in [0, t^*] \times \Omega_\delta \mid u_+ = x_1 - t = \text{constant}\},
\]
\[
C^-_{u_-} \overset{\text{def}}{=} \{(t, x) \in [0, t^*] \times \Omega_\delta \mid u_- = x_1 + t = \text{constant}\},
\]
with \(C^+_{u_+}\) and \(C^-_{u_-}\) being the level sets \(\{u_+ = \text{constant}\}\) and \(\{u_- = \text{constant}\}\) respectively.

We denote by
\[
S^+_{t, u_+} = C^+_{u_+} \cap \Sigma_t, \quad S^-_{t, u_-} = C^-_{u_-} \cap \Sigma_t.
\]
Therefore, for time \(t\), there exist two foliations of \(\Sigma_t\) as follows
\[
\Sigma_t = \bigcup_{u_+ \in \mathbb{R}} S^+_{t, u_+} = \bigcup_{u_- \in \mathbb{R}} S^-_{t, u_-}.
\]
In particular, we have characteristic hypersurface as follows:
\[ \Sigma_t^{[u_1^+, a_1^+]} = \bigcup_{u_+ \in [u_1^+, a_1^+]} S_{t,u_+}^0, \quad \Sigma_t^{[u_1^-, a_1^-]} = \bigcup_{u_- \in [u_1^-, a_1^-]} S_{t,u_-}^0, \]
\[ W_t^{[u_1^+, a_1^+]} = \bigcup_{\tau \in [0, t]} \Sigma_t^{[u_1^+, a_1^+]}, \quad W_t^{[u_1^-, a_1^-]} = \bigcup_{\tau \in [0, t]} \Sigma_t^{[u_1^-, a_1^-]}, \quad W_t = \bigcup_{\tau \in [0, t]} \Sigma_t. \]

We shall also study the spacetime \([0, t^*) \times \mathbb{R}^2\) associated to the solution of MHD in the horizontal direction of the thin domain (or that of 2D MHD). There also exists a natural foliation of \([0, t^*) \times \mathbb{R}^2\) as \([0, t^*) \times \mathbb{R}^2 = \bigcup_{0 \leq \tau \leq t} \Sigma_{t,h}\), where \(\Sigma_{t,h}\) is the constant time slice of the horizontal direction. We also define the two-dimensional linear characteristic hypersurface as follows:
\[ C_{u_+,h}^+ \defeq \{(t, x_h) \in [0, t^*) \times \mathbb{R}^2 \mid u_+ = x_1 - t = \text{constant}\}, \]
\[ C_{u_-,h}^- \defeq \{(t, x_h) \in [0, t^*) \times \mathbb{R}^2 \mid u_- = x_1 + t = \text{constant}\}, \]
where \(C_{u_+,h}^+\) and \(C_{u_-,h}^-\) being the level sets \(\{u_+ = \text{constant}\}\) and \(\{u_- = \text{constant}\}\) in \([0, t^*) \times \mathbb{R}^2\) respectively. Then there exist another two foliations of \([0, t^*) \times \mathbb{R}^2\)
\[ [0, t^*) \times \mathbb{R}^2 = \bigcup_{u_+ \in \mathbb{R}} C_{u_+,h}^+ = \bigcup_{u_- \in \mathbb{R}} C_{u_-,h}^-. \]

We can also give the definitions to \(S_{t,u_+,h}^+, S_{t,u_-,h}^-, \Sigma_{t,h}, \Sigma_{t,h}^{[u_1^+, a_1^+]}, \Sigma_{t,h}^{[u_1^-, a_1^-]}, W_{t,h}^{[u_1^+, a_1^+]}, \) and \(W_{t,h}^{[u_1^-, a_1^-]}\) in a similar way.

3. Proof of Theorem 1.1

In this section, we will give a complete proof to Theorem 1.1.

3.1. The priori estimates of the linearized system. We consider the following linearized system
\[ \partial_t f_+ - \partial_t f_+ + z_+ \cdot \nabla f_+ = \rho_+, \quad \text{in } \Omega_\delta, \]
\[ \partial_t f_- + \partial_t f_- + z_- \cdot \nabla f_- = \rho_. \quad \text{(3.1)} \]
Here \(\text{div } z_\pm = 0\) and \(z_\pm^3 |_{x_3 = \pm \delta} = 0, z_\pm^3 |_{x_3 = \pm \delta} = 0\). We first have the following proposition.

Proposition 3.1. Suppose that \(\text{div } z_\pm = 0, z_\pm^3 |_{x_3 = \pm \delta} = 0, z_\pm^3 |_{x_3 = \pm \delta} = 0\) and
\[ \|z_\pm^1\|_{L^\infty_t L^2_x} \leq 1. \quad \text{(3.2)} \]

If \((f_+, f_-)\) is a smooth solution to (3.1), it holds
\[ \sup_{0 \leq \tau \leq t} \int_{\Sigma_t} \langle u_\pm \rangle^{2(1+\sigma)}|f_\pm|^2 \, dx + \int_0^t \int_{\Omega_\delta} \langle u_\pm \rangle^{1+\sigma}|f_\pm|^2 \, dx \, d\tau \]
\[ \lesssim \int_{\Sigma_0} \langle u_\pm \rangle^{2(1+\sigma)}|f_\pm|^2 \, dx + \int_0^t \int_{\Omega_\delta} \langle u_\pm \rangle^{2(1+2\sigma)} |z_\pm^1|^2 |f_\pm|^2 \, dx \, d\tau + \int_0^t \int_{\Omega_\delta} \rho_\pm |\langle u_\pm \rangle^{2(1+\sigma)}f_\pm| \, dx \, d\tau \quad \text{(3.3)} \]
\[ + \int_{\mathbb{R}} \langle u_\pm \rangle^{1+\sigma} \left( \int_{W_{t}} |\rho_\pm| \langle u_\pm \rangle^{2(1+\sigma)}f_\pm \, dx \right) \, du_\pm. \]

In particular, we have
\[ \sup_{0 \leq \tau \leq t} \int_{\Sigma_t} \langle u_\pm \rangle^{2(1+\sigma)}|f_\pm|^2 \, dx + \int_0^t \int_{\Omega_\delta} \langle u_\pm \rangle^{1+\sigma}|f_\pm|^2 \, dx \, d\tau \]
\[ \lesssim \int_{\Sigma_0} \langle u_\pm \rangle^{2(1+\sigma)}|f_\pm|^2 \, dx + \int_0^t \int_{\Omega_\delta} \langle u_\pm \rangle^{1+2\sigma}|z_\pm^1|^2 |f_\pm|^2 \, dx \, d\tau + \int_0^t \int_{\Omega_\delta} |\rho_\pm| |\langle u_\pm \rangle^{2(1+\sigma)}|f_\pm| \, dx \, d\tau. \quad \text{(3.4)} \]
Proof. We start with the estimates for $f_+$. Since we are noticing that $u_\pm = x_1 + t$, the left hand side(l.h.s) of the above equality equals

$$
\frac{1}{2} \int_0^t \int_{\Sigma_r} (\partial_t - \partial_1 + z_- \cdot \nabla) (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau = \frac{1}{2} \int_0^t \int_{\Sigma_r} \rho_+ \cdot (u_-)^{2(1+\sigma)} f_+ \, dx \,d\tau.
$$

Since $u_- = x_1 + t$, we have

$$
\frac{1}{2} \int_0^t \int_{\Sigma_r} (\partial_t - \partial_1 + z_- \cdot \nabla) (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau = \frac{1}{2} \int_0^t \int_{\Sigma_r} z_- \partial_1 (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau.
$$

We denote the vector field in the spacetime $[0, t^*] \times \Omega_\delta$ (also the operator) by

$$
L_\pm = \partial_t - \partial_1 + z_- \cdot \nabla, \quad T = \partial_t,
$$

and denote by $\text{div}$ the divergence of $\mathbb{R}^4$ with standard Euclidean metric. We have

$$
\frac{1}{2} \int_0^t \int_{\Sigma_r} L_-(u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau = \frac{1}{2} \int_0^t \int_{\Sigma_r} z_- \partial_1 (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau + \frac{1}{2} \int_0^t \int_{\Sigma_r} \rho_+ \cdot (u_-)^{2(1+\sigma)} f_+ \, dx \,d\tau.
$$

Since $\text{div} z_- = 0$, we have $\text{div} L_- = 0$. Then using the Stokes formula and $z_3^3 |_{x_3 = \pm \delta} = 0$, we have

$$
\frac{1}{2} \int_0^t \int_{\Sigma_r} L_-(u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau = \frac{1}{2} \int_0^t \int_{\Sigma_r} \text{div} (u_-)^{2(1+\sigma)} |f_+|^2 L_- \, dx \,d\tau
$$

$$
= \frac{1}{2} \int_{\Sigma_0} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \langle L_- \rangle \, dx - \frac{1}{2} \int_0^t \int_{\Sigma_0} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \langle L_- \rangle \, dx.
$$

Noticing that $\langle L_- \rangle = 1$, we have

$$
\frac{1}{2} \int_{\Sigma_t} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \, dx = \frac{1}{2} \int_{\Sigma_0} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \, dx + \frac{1}{2} \int_0^t \int_{\Sigma_r} z_- \partial_1 (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau
$$

$$
+ \frac{1}{2} \int_0^t \int_{\Sigma_r} \rho_+ \cdot (u_-)^{2(1+\sigma)} f_+ \, dx \,d\tau.
$$

To control the second term in the l.h.s of (3.3), we have to derive the local energy estimates. Multiplying $\langle u_- \rangle^{2(1+\sigma)} f_+$ to the first equation of (3.1), and then integrating over $W^t_{[u_+; \infty]}$, similarly to the derivation of (3.3), we obtain

$$
\frac{1}{2} \int \int_{W^t_{[u_+; \infty]}} L_-(u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau
$$

$$
= \frac{1}{2} \int \int_{W^t_{[u_+; \infty]}} z_- \partial_1 (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau + \int \int_{W^t_{[u_+; \infty]}} \rho_+ \cdot (u_-)^{2(1+\sigma)} f_+ \, dx \,d\tau.
$$

Using Stokes formula and the facts that $\text{div} L_- = 0$ and $z_3^3 |_{x_3 = \pm \delta} = 0$, we have

$$
\frac{1}{2} \int \int_{W^t_{[u_+; \infty]}} L_-(u_-)^{2(1+\sigma)} |f_+|^2 \, dx \,d\tau = \frac{1}{2} \int \int_{W^t_{[u_+; \infty]}} \text{div} (u_-)^{2(1+\sigma)} |f_+|^2 L_- \, dx \,d\tau
$$

$$
= \frac{1}{2} \int_{\Sigma_0} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \langle L_- \rangle \, dx - \frac{1}{2} \int_0^t \int_{\Sigma_0} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \langle L_- \rangle \, dx
$$

$$
+ \frac{1}{2} \int \int_{E_{u_+}^t} \langle u_- \rangle^{2(1+\sigma)} |f_+|^2 \langle L_- \rangle \, d\sigma_+.
$$
where $\nu^+$ is the unit outward normal to $C_{u_+}^+$ and

$$
\nu^+ = (1, -1, 0, 0)^T.
$$

Since $L_- = (1, -1 + z_1^+, z_2^+, z_3^-)^T$, we have

$$
\langle L_-, \nu^+ \rangle = 2 - z_1^-.
$$

Then using the fact $\langle L_-, T \rangle = 1$ and the assumption \(3.2\), we have

$$
\begin{align*}
\frac{1}{2} \int \int \left( \langle u_- \rangle \right)^{2(1+\sigma)} |f_+|^2 dx & + \frac{1}{2} \int \int_{C_{u_+}^+} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx \\
& \leq \frac{1}{2} \int \int_{\Sigma_{0}} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx + \frac{1}{2} \int \int_{W_i} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\tau \\
& \quad + \int \int_{W_i} \rho_+ \cdot \langle u_- \rangle^{2(1+\sigma)} f_+ d\tau d\sigma.
\end{align*}
$$

Multiplying $\frac{1}{\langle u_+ \rangle^{1+\sigma}}$ to both sides of \(3.6\), and integrating the resulting inequality over $R$, we have

$$
\begin{align*}
\frac{1}{2} \int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx & + \frac{1}{2} \int \int_{C_{u_+}^+} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx \\
& \leq \frac{1}{2} \int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx \\
& \quad + \frac{1}{2} \int \int_{W_i} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\tau \\
& \quad + \int \int_{W_i} \rho_+ \cdot \langle u_- \rangle^{2(1+\sigma)} f_+ d\tau d\sigma.
\end{align*}
$$

By the definition of $C_{u_+}^+$, we have $d\sigma = \sqrt{2} d\tau dx_2 dx_3$. Then we have

$$
\int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\sigma = \sqrt{2} \int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\sigma.
$$

We define the variables transformation $\Phi$ from $W_i$ to $W_1$ such that

$$
(\tau, x_1, x_2, x_3) \mapsto \Phi(\tau, x) = (\tau, u_+, x_2, x_3)
$$

with $u_+ = x_1 - \tau$. We have $\det(d\Phi) = 1$. Then we have

$$
\int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\sigma = \sqrt{2} \int \int \left( \langle u_+ \rangle \right)^{1+\sigma} \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\sigma.
$$

Since $\int \int \left( \langle u_+ \rangle \right)^{1+\sigma} d\sigma$ is finite, we deduce from \(3.7\) that

$$
\begin{align*}
\int_0^t \int \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\tau & \lesssim \int \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx \\
& + \int \int \langle u_- \rangle^{2(1+\sigma)} f_+^2 dx d\tau + \int \int \rho_+ \cdot \langle u_- \rangle^{2(1+\sigma)} f_+ d\tau d\sigma.
\end{align*}
$$
Combining (3.5) and (3.9), and using the fact that $|\partial_t ( (u_-)^{2(1+\sigma)}) | \lesssim (u_-)^{1+2\sigma}$, we obtain that

$$
\sup_{0 \leq t \leq T} \int_{\Sigma_t} (u_-)^{2(1+\sigma)} |f_+|^2 \, dx + \int_0^T \int_{\Sigma_t} (u_-)^{2(1+\sigma)} |f_+|^2 \, dx \, dt \\
\lesssim \int_{\Sigma_0} (u_-)^{2(1+\sigma)} |f_+|^2 \, dx + \int_0^T \int_{\Sigma_t} (u_-)^{1+2\sigma} |z_1|^2 |f_+|^2 \, dx \, dt + \int_0^T \int_{\Sigma_t} \rho_+ \cdot (u_-)^{2(1+\sigma)} f_+ \, dx \, dt \, dt |d\Sigma_+|.
$$

Similar estimate holds for $f_-$. Then we arrive at (3.3). Since $\int_{R} \frac{1}{(u_\pm)^{1+\sigma}} \, du_\pm < \infty$, we get (3.4). The proposition is proved.

3.2. The a priori estimates for the solutions to the MHD system (1.4). In this subsection, we shall use Proposition 3.3 to derive the a priori estimates for the solutions of (1.4). We shall derive the uniform estimates with respect to $\delta$. In order to do that, thanks to (1.8), we introduce the following energy functional

$$
sup_{t > 0} \left[ \sum_{k+l \leq N_\delta} \delta^{2-l-\frac{1}{2}} E^{(k,l)} (z_\pm (t)) + \sum_{k \leq N_\delta - 1} \delta^{-3} E^{(k,0)} (z_\pm^3 (t)) \right].
$$

However, to control the above norms, we need other energy in the l.h.s of (1.4).

3.2.1. The uniform estimates of $\nabla_h^k z_\pm$. The estimates for $\nabla_h^k z_\pm$ is stated in the following proposition.

Proposition 3.2. Assume that $(z_+, z_-)$ are the smooth solutions to (1.4). Let $N_\delta = 2N$, $N \in \mathbb{Z}_{\geq 5}$, and

$$
\| z_\pm^1 \|_{L^\infty_t L^\infty_x} \leq 1.
$$

Then we have

$$
\sum_{k \leq N_\delta} \delta^{-1} E^{(k,0)} (z_\pm) + E^{(k,0)} (z_\pm) \\
\lesssim \sum_{k \leq N_\delta} \delta^{-1} E^{(k,0)} (z_\pm, 0) + \left( \sum_{k+l \leq N_\delta} \delta^{l-\frac{1}{2}} E^{(k,l)} (z_\pm) \right)^{\frac{1}{2}} + \sum_{k \leq N_\delta - 1} \delta^{-\frac{3}{2}} E^{(k,0)} (z_\pm^3) \left( \sum_{k+l \leq N_\delta} \delta^{2-l-\frac{1}{2}} E^{(k,l)} (z_\pm) \right)^{\frac{1}{2}} + \sum_{k+l \leq N_\delta + 2} \delta^{2-l-\frac{1}{2}} E^{(k,l)} (z_\pm) \left( \sum_{k+l \leq N_\delta + 2} \delta^{2-l-\frac{1}{2}} E^{(k,l)} (z_\pm) \right) (\partial_3 z_\pm).
$$

Proof. We shall divide the proof into several steps.

Step 1. Linearized system. We shall denote $\alpha_h = (\alpha_1, \alpha_2) \in (\mathbb{Z}_{\geq 0})^2$ and

$$
z^{(\alpha_h)}_\pm \defeq \partial_{\alpha_1} \partial_{\alpha_2} z_\pm.
$$

Applying $\partial_{\alpha_h}^{\alpha_h}$ to both sides of (1.4), we have

$$
\partial_t z^{(\alpha_h)}_+ - \partial_1 z^{(\alpha_h)}_+ + z_- \cdot \nabla z^{(\alpha_h)}_+ = \rho^{(\alpha_h)}_+ - \partial_{\alpha_1}^{\alpha_h} \nabla p, \quad \text{in} \quad \Omega_3,
$$

$$
\partial_t z^{(\alpha_h)}_- + \partial_1 z^{(\alpha_h)}_- + z_+ \cdot \nabla z^{(\alpha_h)}_- = \rho^{(\alpha_h)}_- - \partial_{\alpha_1}^{\alpha_h} \nabla p,
$$

where

$$
\rho^{(\alpha_h)}_+ = -\partial_{\alpha_1}^{\alpha_h} (z_- \cdot \nabla z_+) + z_- \cdot \nabla \partial_{\alpha_1}^{\alpha_h} z_+,
$$

$$
\rho^{(\alpha_h)}_- = -\partial_{\alpha_1}^{\alpha_h} (z_+ \cdot \nabla z_-) + z_+ \cdot \nabla \partial_{\alpha_1}^{\alpha_h} z_-.
$$
We only give the estimates for $z_+^{(\alpha_h,0)}$. Applying Proposition 3.1 to the first equation of (3.12), we obtain that

$$
\sup_{0 \leq \tau \leq t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} |z_+^{(\alpha_h,0)}|^{2} dx + \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} \langle z_+^{(\alpha_h,0)} \rangle^{2} dx d\tau \\
\lesssim \int_{\Sigma_0} \langle u_- \rangle ^{2(1+\sigma)} |z_+^{(\alpha_h,0)}|^{2} dx + \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{1+2\sigma} |z_+^{(\alpha_h,0)}|^{2} dx d\tau \\
+ \left| \int_{0}^{t} \int_{\Sigma_{\tau}} \langle \rho_+^{(\alpha_h,0)} - \partial_h^{(\alpha_h,0)} \nabla p \cdot \langle u_- \rangle ^{2(1+\sigma)} z_+^{(\alpha_h,0)} \rangle dx d\tau \right| \\
+ \left| \int_{R} \frac{1}{\langle u_+ \rangle ^{1+\sigma}} \int_{W_{t}^{\{u_+ \to \infty \}}} \langle \rho_+^{(\alpha_h,0)} - \partial_h^{(\alpha_h,0)} \nabla p \cdot \langle u_- \rangle ^{2(1+\sigma)} z_+^{(\alpha_h,0)} \rangle dx d\tau \right| du_+.
$$

(3.14)

**Step 2. Estimates of the nonlinear terms.** In this step, we estimate the nonlinear terms in the r.h.s of (3.14) term by term.

**Step 2.1. Estimate of** $\int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} |z_+^{(\alpha_h,0)}|^{2} dx d\tau$. It is bounded by

$$
\| \langle u_+ \rangle ^{1+\sigma} z_+^{(\alpha_h,0)} \|_{L_{t}^{\infty} L_{x}^{\infty}} \lesssim \sum_{k=0}^{2} \delta^{-\frac{1}{2}} \left( E_{k}^{(E,k)} (z_+) \right)^{\frac{1}{2}} + \delta^{\frac{1}{2}} \left( E_{k}^{(E,k)} (z_+) \right)^{\frac{1}{2}} \cdot F_{+}^{(\alpha_h,0)} (z_+).
$$

Then we obtain that

$$
\delta^{-1} \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} |z_+^{(\alpha_h,0)}|^{2} dx d\tau \lesssim \sum_{k=1+j \leq 2N} \delta^{-\frac{1}{2}} \left( E_{k}^{(E,k)} (z_+) \right)^{\frac{1}{2}} \cdot \delta^{-1} F_{+}^{(\alpha_h,0)} (z_+).
$$

(3.15)

**Step 2.2. Estimate of term** $\int_{0}^{t} \int_{\Sigma_{\tau}} \partial_h^{(\alpha_h,0)} \nabla p \cdot \langle u_- \rangle ^{2(1+\sigma)} z_+^{(\alpha_h,0)} \rangle dx d\tau$. Firstly, we deal with the case $|\alpha_h| \geq 1$. Since $\text{div} z_+ = 0$, $z_+^{3}|_{x_3 = \pm \delta} = 0$, using the integration by parts, we have

$$
\left| \int_{0}^{t} \int_{\Sigma_{\tau}} \partial_h^{(\alpha_h,0)} \nabla p \cdot \langle u_- \rangle ^{2(1+\sigma)} z_+^{(\alpha_h,0)} \rangle dx d\tau \right| = - \int_{0}^{t} \int_{\Sigma_{\tau}} \partial_h^{(\alpha_h,0)} \nabla p \langle u_- \rangle ^{2(1+\sigma)} z_+^{(\alpha_h,0)} \rangle dx d\tau \\
\lesssim (\int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} \langle u_+ \rangle ^{1+\sigma} \| \partial_h^{(\alpha_h,0)} p \|_{L_{t}^{\infty} L_{x}^{\infty}} \rangle dx d\tau)^{\frac{1}{2}} \cdot (\int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} \langle u_+ \rangle ^{1+\sigma} |z_+^{(\alpha_h,0)}|^{2} dx d\tau)^{\frac{1}{2}}.
$$

(3.16)

Then we only need to control the term

$$
(\int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_- \rangle ^{2(1+\sigma)} \langle u_+ \rangle ^{1+\sigma} |\partial_h^{(\alpha_h,0)} p|^{2} dx d\tau)^{\frac{1}{2}} = \| \langle u_- \rangle ^{1+\sigma} \langle u_+ \rangle ^{1+\sigma} \partial_h^{(\alpha_h,0)} p \|_{L_{t}^{\infty} L_{x}^{\infty}}.
$$

Thanks to (2.2), using the facts that $\text{div} z_\pm = 0$ and $z_\pm^3|x_3 = \pm \delta = 0$, $z_\pm^3|x_3 = \pm \delta = 0$, and integrating by parts, we obtain by (2.5) that

$$
\partial_h^{(\alpha_h,0)} p(\tau, x) = (-1)^{|\alpha_h|} \int_{\Omega_{3}} \partial_{i} \partial_{j} \partial_h^{(\alpha_h)} G_{3}(x, y)(z_+^{i} z_-^{j})(\tau, y)dy = \int_{\Omega_{3}} \partial_{i} \partial_{j} G_{3}(x, y) \partial_h^{(\alpha_h)} (z_+^{i} z_-^{j})(\tau, y)dy.
$$

We choose a smooth cut-off function $\theta(r)$ so that

$$
\theta(r) = \begin{cases} 
1, & \text{for } |r| \leq 1, \\
0, & \text{for } |r| \geq 2.
\end{cases}
$$

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Then we split $\partial_{h}^{\alpha} p$ into two parts

\[
\partial_{h}^{\alpha} p(\tau, x) = \int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) \theta(|x_{h} - y_{h}|) \partial_{h}^{\alpha} (z_{+}^{i} z_{+}^{j})(\tau, y) dy \qquad \text{A}_{1}(\tau, x)
\]

\[
+ \int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) (1 - \theta(|x_{h} - y_{h}|)) \partial_{h}^{\alpha} (z_{+}^{i} z_{-}^{j})(\tau, y) dy . \qquad \text{A}_{2}(\tau, x)
\]

According to the decomposition, we have

\[
\| (u_{-})^{1+\sigma} (u_{+})^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha} p \|_{L^{2}_{t} L^{2}_{x}} \\
\lesssim \| (u_{-})^{1+\sigma} (u_{+})^{\frac{1}{2}(1+\sigma)} A_{1}(\tau, x) \|_{L^{2}_{t} L^{2}_{x}} + \| (u_{-})^{1+\sigma} (u_{+})^{\frac{1}{2}(1+\sigma)} A_{2}(\tau, x) \|_{L^{2}_{t} L^{2}_{x}} .
\]

(i) Estimate of $A_{1}$. Firstly, we have

\[
A_{1}(\tau, x) = \sum_{\beta_{h} \leq \alpha_{h}} C_{\alpha_{h}, \beta_{h}} \int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) \theta(|x_{h} - y_{h}|) (\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{i} \partial_{h}^{\beta_{h}} z_{+}^{j})(\tau, y) dy .
\]

If $|\beta_{h}| \leq N$, using the facts that $\text{div} \ z_{+} = 0$ and $z_{+}^{j}|_{x_{3}=\pm \delta} = 0$, and integrating by parts, we have

\[
A_{1, \alpha_{h}, \beta_{h}}(\tau, x) = -\int_{\Omega_{\delta}} \partial_{i} G_{\delta}(x, y) \partial_{j} \theta(|x_{h} - y_{h}|) (\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{+}^{i})(\tau, y) dy .
\]

For $A_{1, \alpha_{h}, \beta_{h}}$, using Corollary 2.3, we have

\[
|A_{1, \alpha_{h}, \beta_{h}}(\tau, x)| \lesssim \int_{-\delta}^{\delta} \int_{|x_{h} - y_{h}| \leq 2} \frac{1}{\delta} \frac{1}{|x_{h} - y_{h}|} \left( |\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{+}^{i}| \right)(\tau, y) dy_{h} dy_{3} . \quad (3.17)
\]

Before going further, we need the following auxiliary lemma concerning the weights.

**Lemma 3.3.** For $|x_{h} - y_{h}| \leq 2$, we have

\[
(u_{\pm}(t, x_{1})) \leq \sqrt{5} (u_{\pm}(t, y_{1})). \quad (3.18)
\]

**Proof.** Since $u_{\pm}(t, x_{1}) = x_{1} \mp t$, we have

\[
|u_{\pm}(t, x_{1})| \leq |u_{\pm}(t, y_{1})| + |x_{1} - y_{1}| \leq |u_{\pm}(t, y_{1})| + 2 .
\]

Then we obtain

\[
(u_{\pm}(t, x_{1})) \leq (1 + 2|u_{\pm}(t, y_{1})|^{2} + 4) \frac{1}{2} \leq \sqrt{5} (u_{\pm}(t, y_{1})).
\]

The lemma is proved.

Thanks to (3.17) and (3.18), we have

\[
(\langle u_{-} \rangle^{1+\sigma} (\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} A_{1, \alpha_{h}, \beta_{h}}(\tau, x)|
\lesssim \int_{-\delta}^{\delta} \int_{|x_{h} - y_{h}| \leq 2} \frac{1}{\delta} \frac{1}{|x_{h} - y_{h}|} \left( \langle u_{-} \rangle^{1+\sigma} (\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{+}^{i}|) \right)(\tau, y) dy_{h} dy_{3} .
\]
Using Young inequality for the horizontal variables $x_h$ and using also Hölder inequality, we have

$$
\|\langle u_+ \rangle^{1+\sigma} (\langle u_+ \rangle^{1+\sigma} A_1^{h_1, \alpha_h, \beta_h}) \|_{L^2_t L^2_x} \lesssim \delta^{-\frac{1}{2}} \| u_+ \|_{L^2_t (\| u_+ \|_{L^2_x} \leq 2)} \cdot \| \langle u_+ \rangle^{1+\sigma} (\partial_t^{\alpha_h \beta_h} \partial_j z^-) \|_{L^2_t L^2_x} \lesssim \delta^{-\frac{1}{2}} \left( \| \langle u_+ \rangle^{1+\sigma} (\partial_t^{\alpha_h \beta_h} \partial_j z^-) \|_{L^2_t L^2_x} \right).
$$

Then we have

$$
\left( \langle u_+ \rangle^{1+\sigma} (\partial_t^{\alpha_h \beta_h} \partial_j z^-) \right) \lesssim \delta^{-\frac{1}{2}} \left( \sum_{k \leq 2} E_{-}^{(\beta_h + k, 1)} (z_-) \right)^{\frac{1}{2}} \left( F_{+}^{(\alpha_h, 0)} (z_+) \right)^{\frac{1}{2}}.
$$

For $A_{2, \alpha_h, \beta_h}^2$, using Corollary 2.23 we have

$$
|A_{2, \alpha_h, \beta_h}^2 (\tau, x)| \lesssim \int_0^\tau \int_{|x_h - y_h| \leq 2} \frac{1}{|x_h - y_h|} \left| \partial_t^{\alpha_h \beta_h} \partial_j (z^-) \right| (	au, y)dy_h dy_3.
$$

Similar argument to $A_{1, \alpha_h, \beta_h}^2$ can be applied to $A_{1, \alpha_h, \beta_h}^2$ and then we have

$$
\left( \langle u_+ \rangle^{1+\sigma} \left( \sum_{k \leq 2} E_{-}^{(\beta_h + k, 1)} (z_-) \right) \right)^{\frac{1}{2}} \left( F_{+}^{(\alpha_h, 0)} (z_+) \right)^{\frac{1}{2}}.
$$

Thanks to (3.19) and (3.20) for $|\beta_h| \leq N$, we have

$$
\delta^{-\frac{1}{2}} \left( \langle u_+ \rangle^{1+\sigma} \left( \sum_{k \leq 2} E_{-}^{(\beta_h + k, 1)} (z_-) \right) \right)^{\frac{1}{2}} \lesssim \sum_{k \leq N+3} \delta^{-\frac{1}{2}} \left( E_{-}^{(k, 0)} (z_-) \right)^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}} \left( E_{-}^{(k, 0)} (z_-) \right)^{\frac{1}{2}}.
$$

Remark 3.4. Notice that term $\sum_{k \leq N+3} \delta^{-\frac{1}{2}} \left( E_{-}^{(k, 0)} (z_-) \right)^{\frac{1}{2}}$ of the second term on the r.h.s of (3.21) comes from the second term on the r.h.s of (3.19). Indeed, for $|\alpha_h| = N_0$, $|\beta_h| = 0$, we have no idea to give the upper bound for the term $\delta^{-\frac{1}{2}} \left( F_{+}^{(\alpha_h, 0)} (z_+) \right)^{\frac{1}{2}}$ in (3.19).

Similarly, for $|\beta_h| > N$, using the facts that $\text{div} z_- = 0$ and $z_3^- |_{x_3 = \pm \delta} = 0$, and integrating by parts, we have

$$
A_{1, \alpha_h, \beta_h} = - \int_{\Omega_3} \partial_j G_\delta (x, y) \theta (|x_h - y_h|) \partial_t^{\alpha_h \beta_h} \partial_j (z^-) \partial_j (z^-) (\tau, y)dy - \int_{\Omega_3} \partial_j G_\delta (x, y) \partial_j \theta (|x_h - y_h|) \partial_t^{\alpha_h \beta_h} \partial_j (z^-) \partial_j (z^-) (\tau, y)dy.
$$

Then for $|\beta_h| > N$, we have

$$
\delta^{-\frac{1}{2}} \left( \langle u_+ \rangle^{1+\sigma} \left( \sum_{k \leq N+3} \delta^{-\frac{1}{2}} \left( E_{-}^{(k, 0)} (z_-) \right)^{\frac{1}{2}} \right) \right) \lesssim \sum_{k \leq N+3} \delta^{-\frac{1}{2}} \left( E_{-}^{(k, 0)} (z_-) \right)^{\frac{1}{2}} + \sum_{k \leq N+1} \delta^{-\frac{1}{2}} \left( F_{+}^{(k, 0)} (\partial_3 z^-) \right)^{\frac{1}{2}}.
$$
Thanks to (3.21) and (3.22), notice that \( N \geq 5 \), we have
\[
\begin{align*}
\delta^{-\frac{1}{2}}\| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} A_1 \|_{L^2_{t}L^2_{x}} & \lesssim \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}}(E_{-k,0}^{(k,0)}(z_-))^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}}(E_{-k,0}^{(k,0)}(\partial_3 z_-))^{\frac{1}{2}} \right) \\
& \times \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}}(F_{+k,0}^{(k,0)}(z_+))^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}}(F_{+k,0}^{(k,0)}(\partial_3 z_+))^{\frac{1}{2}} \right). 
\end{align*}
\tag{3.23}
\]

(ii) Estimate of \( A_2 \). For \( A_2 \), integrating by parts, for \( \gamma_h \leq \alpha_h \) and \( |\gamma_h| = 1 \), we have
\[
A_2(\tau, x) = -\int_{\Omega_d} \partial_h \partial_i \partial_j G_\delta(x, y)(1 - \theta(|x_h - y_h|)) \partial_h^{\gamma_h - \gamma_h} (z_+^j z_+^i)(\tau, y) dy \\
+ \int_{\Omega_d} \partial_i \partial_j G_\delta(x, y) \partial_h^{\gamma_h} \theta(|x_h - y_h|) \partial_h^{\gamma_h - \gamma_h} (z_+^j z_+^i)(\tau, y) dy.
\]

For \( A_2^3 \), the integration happens for \( y \in \{|x_h - y_h| \leq 2\} \times (-\delta, \delta) \). Similar derivation as that for \( A_1 \), replacing \( \theta(|x_h - y_h|) \) and \( \alpha_h \) in \( A_1 \) by \( \partial_h^{\gamma_h} \theta(|x_h - y_h|) \) and \( \alpha_h - \gamma_h \), gives rise to
\[
\begin{align*}
\delta^{-\frac{1}{2}}\| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} A_2^3 \|_{L^2_{t}L^2_{x}} & \lesssim \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}}(E_{-k,0}^{(k,0)}(z_-))^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}}(E_{-k,0}^{(k,0)}(\partial_3 z_-))^{\frac{1}{2}} \right) \\
& \times \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}}(F_{+k,0}^{(k,0)}(z_+))^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}}(F_{+k,0}^{(k,0)}(\partial_3 z_+))^{\frac{1}{2}} \right). 
\end{align*}
\tag{3.24}
\]

For \( A_2^3 \), using Corollary 2.3, we have
\[
|A_2^3(\tau, x)| \lesssim \int_{-\delta}^{\delta} \int_{|x_h - y_h| \geq 1} \frac{1}{\delta |x_h - y_h|^3} \partial_h^{\gamma_h - \gamma_h} (z_+^j z_+^i)(\tau, y) dy dy_3.
\tag{3.25}
\]

To estimate \( A_2^1 \), we need the following lemma.

Lemma 3.5. For \( |x_h - y_h| \geq 1 \), we have
\[
\langle u_\pm(t, x_1) \rangle \leq \sqrt{3} |x_h - y_h| \langle u_\pm(t, y_1) \rangle.
\tag{3.26}
\]

Proof. Since \( u_\pm(t, x_1) = x_1 \mp t \), we have
\[
|u_\pm(t, x_1)| \leq |u_\pm(t, y_1)| + |x_1 - y_1|.
\]

Then we have for \( |x_h - y_h| \geq 1 \)
\[
\langle u_\pm(t, x_1) \rangle \leq (1 + 2|u_\pm(t, y_1)|^2 + 2|x_1 - y_1|^2)^{\frac{1}{2}} \leq \sqrt{3} |x_h - y_h| (1 + |u_\pm(t, y_1)|^2)^{\frac{1}{2}}.
\]

The lemma is proved. \qed

Thanks to (3.25) and (3.26), we have
\[
\langle u_-^{1+\sigma} \rangle \langle u_+^{\frac{1}{2}(1+\sigma)} A_2^1(\tau, x) \rangle \lesssim \int_{-\delta}^{\delta} \int_{|x_h - y_h| \geq 1} \frac{1}{\delta |x_h - y_h|^3 (1+\sigma)} \langle u_-^{1+\sigma} \rangle \langle u_+^{\frac{1}{2}(1+\sigma)} \rangle \partial_h^{\gamma_h} (z_+^j z_+^i)(\tau, y) dy dy_3.
\]

Using Young inequality and Hölder inequality, we have
\[
\begin{align*}
\| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} A_2^1 \|_{L^2_{t}L^2_{x}} \lesssim \delta^{-\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2_{t}L^2_{x}} \| A_2^1 \|_{L^2_{t}L^2_{x}} \\
\lesssim \delta^{-\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2_{t}L^2_{x}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \partial_h^{\gamma_h} (z_+^j z_+^i) \|_{L^2_{t}L^2_{x}}.
\end{align*}
\]
On one hand, since $\sigma \in (0, \frac{1}{3})$, $\|\frac{1}{|x_h|^{\frac{4}{3(1+\sigma)}}}\|_{L^2_t(x_h|_{x_h} \geq 1)} < \infty$. On the other hand, there holds

$$\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial^{\alpha_h-\gamma_h} \left( z_-^\gamma \right) \|_{L^2_t L^2_x} \lesssim \sum_{\beta_h \leq \alpha_h-\gamma_h} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial^{\alpha_h-\gamma_h-\beta_h} \partial^{\beta_h} \partial^{\gamma_h} \|_{L^2_t L^2_x}$$

Then we have

$$\delta^{-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} A_2 \|_{L^2_t L^2_x} \lesssim \sum_{k \leq 2N} \delta^{-\frac{1}{2}} \left( \langle E_-^{(k,0)} \rangle \right)^{\frac{1}{2}} + \sum_{k \leq 2N} \delta^{-\frac{1}{2}} \left( \langle F_+^{(k,0)} \rangle \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3.27)

Due to (3.24) and (3.27), we have

$$\delta^{-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} A_2 \|_{L^2_t L^2_x} \lesssim \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}} \left( \langle E_-^{(k,0)} \rangle \right)^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}} \left( \langle E_-^{(k,0)} \rangle \right)^{\frac{1}{2}} \right)$$

$$\times \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}} \left( \langle F_+^{(k,0)} \rangle \right)^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}} \left( \langle F_+^{(k,0)} \rangle \right)^{\frac{1}{2}} \right).$$  \hspace{1cm} (3.28)

Thanks to (3.24) and (3.27), we obtain that $\delta^{-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial^{\alpha_h} p \|_{L^2_t L^2_x}$ is bounded by the r.h.s of (3.28). Then due to (3.16), for $|\alpha_h| \geq 1$, we have

$$\delta^{-1} \int_0^t \int_{\Sigma_t} \partial^{\alpha_h} p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} dxd\tau$$

$$\lesssim \left( \sum_{k \leq 2N} \delta^{-\frac{1}{2}} \left( \langle E_-^{(k,0)} \rangle \right)^{\frac{1}{2}} + \sum_{k \leq N+2} \delta^{-\frac{1}{2}} \left( \langle E_-^{(k,0)} \rangle \right)^{\frac{1}{2}} \right)$$

$$\times \left( \sum_{k \leq 2N} \delta^{-1} \langle F_+^{(k,0)} \rangle \left( z_+ \right) + \sum_{k \leq N+2} \delta^{-1} \langle F_+^{(k,0)} \rangle \left( \partial_3 z_+ \right) \right).$$  \hspace{1cm} (3.29)

Actually, (3.29) holds also for $|\alpha_h| = 0$. Indeed, for case $|\alpha_h| = 0$, we have

$$\left| \int_0^t \int_{\Sigma_t} \nabla p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} dxd\tau \right|$$

$$\lesssim \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \nabla p \|_{L^2_t L^2_x} \cdot \left( \int_0^t \int_{\Sigma_t} \langle u_- \rangle^{2(1+\sigma)} \langle u_+ \rangle^{3(1+\sigma)} \langle z_+ \rangle^{2} dxd\tau \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3.30)

By similar derivation as that for $\delta^{-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial^{\alpha_h} p \|_{L^2_t L^2_x}$, we could bound $\delta^{-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \nabla p \|_{L^2_t L^2_x}$ by the r.h.s of (3.28). Then $\delta^{-1} \int_0^t \int_{\Sigma_t} \nabla p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} dxd\tau$ is bounded by the r.h.s of (3.29).

**Step 2.3. Estimate of term** $\int_{C_{\alpha_h}^+} \frac{1}{\langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} d\tau}$. Using the facts that $\text{div} \ z_+ = 0$ and $z_+^{\pm \frac{1}{2}} |_{x_3=\pm 3} = 0$, and integrating by parts, for $|\alpha_h| \geq 1$, we have

$$\int \int_{W^1_{\alpha_h} \to \infty} \partial^{\alpha_h} p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} dxd\tau$$

$$= - \int \int_{W^1_{\alpha_h} \to \infty} \partial^{\alpha_h} p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} dxd\tau - \int_{C_{\alpha_h}^+} \partial^{\alpha_h} p \cdot \langle u_- \rangle^{2(1+\sigma)} \langle z_+ \rangle^{(\alpha_h,0)} d\sigma_+.$$
Then we have
\[
\int_{R} \frac{1}{(u_{+})^{1+\sigma}} \left| \int_{W_{t}[u_{+}, \infty]} \partial_{h}^{\alpha_{n}} \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \right| du_{+} + \int_{W_{t}[u_{+}, \infty]} \left| \langle \partial_{h}^{\alpha_{n}} p \rangle \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} \right| dx \tau \\
+ \int_{C_{n+}^{+}} \partial_{h}^{\alpha_{n}} p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} d\sigma_{+} + \int_{C_{n+}^{+}} \partial_{h}^{\alpha_{n}} p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} d\sigma_{+} du_{+}.
\]

Thanks to the facts $d\sigma_{+} = \sqrt{2} dx_{2} dx_{3} d\tau$ and (3.8), we have
\[
\int_{R} \frac{1}{(u_{+})^{1+\sigma}} \int_{C_{n+}^{+}} \partial_{h}^{\alpha_{n}} p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} d\sigma_{+} + \int_{t}^{t} \int_{[u_{+}, \infty]} \partial_{h}^{\alpha_{n}} p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \tau.
\]

Thus for $|\alpha_{h}| \geq 1$, we have
\[
\int_{R} \frac{1}{(u_{+})^{1+\sigma}} \left| \int_{W_{t}[u_{+}, \infty]} \partial_{h}^{\alpha_{n}} \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \right| du_{+} + \int_{t}^{t} \int_{[u_{+}, \infty]} \left| \partial_{h}^{\alpha_{n}} p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} \right| dx \tau.
\]

And for $|\alpha_{h}| = 0$, we have
\[
\int_{R} \frac{1}{(u_{+})^{1+\sigma}} \left| \int_{W_{t}[u_{+}, \infty]} \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+} dx \right| du_{+} + \int_{t}^{t} \int_{[u_{+}, \infty]} \left| \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+} \right| dx \tau.
\]

Therefore, the nonlinear term $\delta^{-1} \int_{R} \frac{1}{(u_{+})^{1+\sigma}} \left| \int_{W_{t}[u_{+}, \infty]} \partial_{h}^{\alpha_{n}} \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \right| du_{+}$ has the same estimates as $\delta^{-1} \left| \int_{t}^{t} \int_{[u_{+}, \infty]} \partial_{h}^{\alpha_{n}} \nabla p \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \right|$ which is bounded by the r.h.s of (3.29).

**Step 2.4. Estimate of term** $\int_{t}^{t} \int_{[u_{+}, \infty]} \rho_{+}^{(\alpha_{n}, 0)} \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \tau$. Firstly, using Hölder inequality, we have
\[
\left| \int_{t}^{t} \int_{[u_{+}, \infty]} \rho_{+}^{(\alpha_{n}, 0)} \cdot \langle u_{-}\rangle^{2(1+\sigma)} z_{+}^{(\alpha_{n}, 0)} dx \right| \leq \left\| \langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{2(1+\sigma)} \rho_{+}^{(\alpha_{n}, 0)} \right\| L_{t}^{2} L_{x}^{2} \cdot \left\| \frac{\langle u_{-}\rangle^{1+\sigma}}{(u_{+})^{2(1+\sigma)}} \right\| L_{t}^{2} L_{x}^{2}.
\]

We only need to bound the term $\left\| \langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{2(1+\sigma)} \rho_{+}^{(\alpha_{n}, 0)} \right\| L_{t}^{2} L_{x}^{2}$.

The expression of $\rho_{+}^{(\alpha_{n}, 0)}$ shows that
\[
|\rho_{+}^{(\alpha_{n}, 0)}| \leq \sum_{\beta_{h} < \alpha_{n}} \left| \partial_{h}^{\alpha_{n}-\beta_{h} z_{-} \cdot \nabla \partial_{h}^{\beta_{h} z_{+}} \right| \leq \sum_{\beta_{h} < \alpha_{n}} \left( \left| \partial_{h}^{\alpha_{n}-\beta_{h} z_{-} \cdot \nabla \partial_{h}^{\beta_{h} z_{+}} \right| + \left| \partial_{h}^{\alpha_{n}-\beta_{h} z_{-} \cdot \nabla \partial_{h}^{\beta_{h} z_{+}} \right| \right).
\]

For $I_{1}$, if $|\beta_{h}| < N$, by Hölder inequality, we have
\[
\left\| \langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{2(1+\sigma)} I_{1} \right\| L_{t}^{2} L_{x}^{2} \leq \left\| \langle u_{+}\rangle^{1+\sigma} \partial_{h}^{\alpha_{n}-\beta_{h} z_{+}} \right\| L_{t}^{\infty} L_{x}^{2} \left\| \frac{\langle u_{-}\rangle^{1+\sigma}}{(u_{+})^{2(1+\sigma)}} \right\| L_{t}^{2} L_{x}^{\infty}.
\]

Thanks to the Sobolev inequality (see (2.28)), we have
\[
\left\| \langle u_{-}\rangle^{1+\sigma} \partial_{h}^{\beta_{h} z_{+}} \right\| L_{t}^{2} L_{x}^{\infty} \leq \sum_{k+1 \leq 2} |\beta_{h}| + 1 + k \partial_{h}^{\beta_{h} z_{+}} \right\| L_{t}^{2} L_{x}^{\infty}.
\]
Then for $|\beta_h| < N$, we obtain that
\[
\delta^{-\frac{1}{2}}\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_1 \|_{L^2_t L^2_x} \lesssim \delta^{-\frac{1}{2}} (E_-^{(\alpha_h-\beta_h,0)}(z_-))^{\frac{1}{2}} \cdot \sum_{k+l \leq N+2} \delta^{l-\frac{1}{2}} (F_+^{(k,l)}(z_+))^{\frac{1}{2}}.
\] (3.32)

Similarly, for $|\beta_h| < N$, we have
\[
\delta^{-\frac{1}{2}}\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_2 \|_{L^2_t L^2_x} \lesssim \delta^{-\frac{1}{2}} (E_-^{(\alpha_h-\beta_h,0)}(z_-))^{\frac{1}{2}} \cdot \sum_{k+l \leq N+1} \delta^{l-\frac{1}{2}} (F_+^{(k,l)}(\partial_3 z_+))^{\frac{1}{2}}.
\] (3.33)

If $N \leq |\beta_h| \leq |\alpha_h| - 1 \leq 2N - 1$, by the similar argument as that for (3.32), it holds that
\[
\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_1 \|_{L^2_t L^2_x} \lesssim \|\langle u_+ \rangle^{1+\sigma} \partial_h \alpha_h \|_{L_t^\infty L_x^\infty} \|\langle u_- \rangle^{1+\sigma} \|_{L^2_t L^2_x} \|\partial_h \|_{L^2_t L^2_x}
\]
and then
\[
\delta^{-\frac{1}{2}}\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_1 \|_{L^2_t L^2_x} \lesssim \sum_{k+l \leq N+2} \delta^{l-\frac{1}{2}} (E_-^{(k,l)}(z_-))^{\frac{1}{2}} \cdot \delta^{-\frac{1}{2}} (F_+^{(\beta_h+1,0)}(z_+))^{\frac{1}{2}}.
\] (3.34)

For $N \leq |\beta_h| \leq |\alpha_h| - 1 \leq 2N - 1$, we also have
\[
\delta^{-\frac{1}{2}}\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_2 \|_{L^2_t L^2_x} \lesssim \sum_{k+l \leq N+2} \delta^{l-\frac{1}{2}} (E_-^{(k,l)}(z_-))^{\frac{1}{2}} \cdot \delta^{-\frac{1}{2}} (F_+^{(\beta_h+1,0)}(z_+))^{\frac{1}{2}}
\]
\[
div \cdot = 0 \left( \sum_{k \leq N+2} \delta^{l-\frac{1}{2}} (E_-^{(k,0)}(z_-))^{\frac{1}{2}} + \sum_{k+l \leq N+2} \delta^{l-\frac{1}{2}} (E_-^{(k,l)}(z_-))^{\frac{1}{2}} \right) \cdot \delta^{-\frac{1}{2}} (F_+^{(\beta_h+1,0)}(z_+))^{\frac{1}{2}}.
\] (3.35)

Thanks to (3.32), (3.33), (3.34) and (3.35), for $|\alpha_h| \leq 2N$, we obtain that
\[
\delta^{-\frac{1}{2}}\|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \rho_+^{(\alpha_h,0)} \|_{L^2_t L^2_x} \lesssim \left( \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E_-^{(k,l)}(z_-))^{\frac{1}{2}} \right) + \left( \sum_{k \leq 2N-1} \delta^{l-\frac{1}{2}} (E_-^{(k,0)}(z_-))^{\frac{1}{2}} \right) \cdot \delta^{-\frac{1}{2}} (F_+^{(\beta_h+1,0)}(z_+))^{\frac{1}{2}}.
\] (3.36)

Due to (3.31) and (3.36), we obtain that
\[
delta^{-1} \left| \int_0^t \int_{\Sigma_r} \rho_+^{(\alpha_h,0)} \cdot \langle u_- \rangle^{2(1+\sigma)} z_+^{(\alpha_h,0)} \right| d\tau dx d\tau \lesssim \left( \sum_{k \leq 2N} \delta^{2(l-\frac{1}{2})} F_+^{(k,l)}(z_+) \right) + \left( \sum_{k+l \leq 2N} \delta^{2(l-\frac{1}{2})} F_+^{(k,l)}(\partial_3 z_+) \right).
\] (3.37)

**Step 2.5. Estimate of $\int_{\mathbb{R}} \frac{1}{(u_+)^{\sigma+}} \langle u_- \rangle^{2(1+\sigma)} z_+^{(\alpha_h,0)} d\tau dx$. Since $\int_{\mathbb{R}} \frac{1}{(u_+)^{\sigma+}} dx < \infty$, the quantity is bounded by**
\[
\int_0^t \int_{\Sigma_r} |\rho_+^{(\alpha_h,0)}| \cdot \langle u_- \rangle^{2(1+\sigma)} z_+^{(\alpha_h,0)} d\tau dx.
\]

Then it is has the same estimate as that for $\int_0^t \int_{\Sigma_r} \rho_+^{(\alpha_h,0)} \cdot \langle u_- \rangle^{2(1+\sigma)} z_+^{(\alpha_h,0)} d\tau dx$. 

Step 3. The a priori estimate for $\nabla_h^k z_+$ for $k \leq 2N$. Combining (3.14) and the results from Step 2.1 to Step 2.5, we finally arrive at
\[
\sum_{k \leq N_\ast} \delta^{-1} [E_+^{(k,0)}(z_+) + F_+^{(k,0)}(z_+)] \\
\lesssim \sum_{k \leq N_\ast} \delta^{-1} E_+^{(k,0)}(z_{+,0}) + \left( \sum_{k+1 \leq N_\ast} \delta^{-1/2} (E_+^{(k,l)}(z_{-,0}))^{1/2} + \sum_{k \leq N_\ast-1} \delta^{-1/2} (E_-^{(k,0)}(z_-^{3}))^{1/2} \right) \\
+ \sum_{k \leq N_\ast+2} \delta^{-1/2} (E_-^{(k,0)}(\partial_3 z_-))^{1/2} \cdot \left( \sum_{k+1 \leq N_\ast} \delta^{2(l-1/2)} F_+^{(k,l)}(z_+) + \sum_{k \leq N_\ast+2} \delta^{2(l-1/2)} F_+^{(k,l)}(\partial_3 z_+) \right).
\]
This is exactly the inequality (3.11) for $z_+$. Similar estimate holds for $z_-$. Then the proposition is proved. \(\square\)

3.2.2. The uniform estimates of $\nabla_h^k \partial_3^l z_\pm$ for $k + l \leq N_\ast = 2N$. In this subsection, we derive the uniform estimates concerning $\nabla_h^k \partial_3^l z_\pm$ with the coefficient $\delta^{1/2}$. 

**Proposition 3.6.** Assume that $(z_+, z_-)$ are the smooth solutions to (1.4). Let $N_\ast = 2N, N \in \mathbb{Z}_{\geq 5}$ and \(\mathbb{R}^2\) hold. We have
\[
\sum_{k+l \leq N_\ast} \delta^{2(l-1/2)} [E_\pm^{(k,l)}(z_\pm) + F_\pm^{(k,l)}(z_\pm)] \\
\lesssim \sum_{k+l \leq N_\ast} \delta^{2(l-1/2)} E_\pm^{(k,l)}(z_{+,0}) + \left( \sum_{k+l \leq N_\ast} \delta^{-1/2} (E_\pm^{(k,l)}(z_\pm))^{1/2} + \sum_{k \leq N_\ast-1} \delta^{-1/2} (E_-^{(k,0)}(z_-^{3}))^{1/2} \right) \\
+ \sum_{k+l \leq N_\ast+2} \delta^{-1/2} (E_-^{(k,0)}(\partial_3 z_-))^{1/2} \cdot \left( \sum_{k+l \leq N_\ast} \delta^{2(l-1/2)} F_\pm^{(k,l)}(z_\pm) + \sum_{k \leq N_\ast+2} \delta^{2(l-1/2)} F_\pm^{(k,l)}(\partial_3 z_\pm) \right). \tag{3.38}
\]

**Proof.** We only prove (3.38) for $z_+$. We shall divide the proof into several steps.

**Step 1. Reduction of the uniform estimates.** For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$, since $\text{div } \partial_h^{\alpha_h} z_+ = 0$ and $\partial_h^{\alpha_h}(z_+^3)|_{z_+ = \pm \delta} = 0$, using div-curl formula (2.17), for $k + l \leq N_\ast = 2N$ and $l \geq 1$, we have
\[
\| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} \partial_3^l z_+ \|_{L_2^2} \lesssim \| \langle u_- \rangle^{1+\sigma} \nabla^l (\partial_h^{\alpha_h} z_+) \|_{L_2^2} \\
\lesssim \sum_{l_1 = 0}^{l-1} \| \langle u_- \rangle^{1+\sigma} \nabla_1 \partial_h^{\alpha_h} (\text{curl } z_+) \|_{L_2^2} + \| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} z_+ \|_{L_2^2}.
\]
Then we obtain that
\[
\delta^{-1/2} \| \langle u_- \rangle^{1+\sigma} \nabla_h^k \partial_3^l z_+ \|_{L_2^2} \\
\lesssim \sum_{k+l \leq l-1} \delta^{1+{1/2}} \| \langle u_- \rangle^{1+\sigma} \nabla_h^{k+l} (\text{curl } z_+) \|_{L_2^2} + \delta^{-1/2} \| \langle u_- \rangle^{1+\sigma} \nabla_h^{k+1} \|_{L_2^2}. \tag{3.39}
\]
Therefore we only need to give the uniform estimate of the term
\[
\delta^{-1/2} \| \langle u_- \rangle^{1+\sigma} \nabla_h^k \partial_3^l j_+ \|_{L_2^2}, \quad \text{for } k + l \leq 2N - 1,
\]
where $j_+ = \text{curl } z_+$ satisfies the first equation of (1.4), that is
\[
\partial_t j_+ - \partial_1 j_+ + z_- \cdot \nabla j_+ = -\nabla z_- \cdot \nabla z_+, \tag{3.40}
\]
with $\nabla z_- \cdot \nabla z_+ = -\nabla z_+^k \times \partial_k z_+$.

Let $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$, $l \in \mathbb{Z}_{\geq 0}$ and $|\alpha_h| + l \leq 2N - 1$. Applying $\partial_h^{\alpha_h} \partial_3^l$ to both sides of (3.40), we have
\[
\partial_t j_+^{(\alpha_h,l)} - \partial_1 j_+^{(\alpha_h,l)} + z_- \cdot \nabla j_+^{(\alpha_h,l)} = \rho_{+,1}^{(\alpha_h,l)} + \rho_{+,2}^{(\alpha_h,l)}, \tag{3.41}
\]
where \( j_{+}^{(\alpha,l)} \) denotes \( \partial_{h}^{\alpha} \partial_{j_{+}^{l}} \) and
\[
\rho_{+}^{(\alpha,l)} = -\partial_{h}^{\alpha} \partial_{j_{+}^{l}} (\nabla z_{-} \wedge \nabla z_{+}), \\
\rho_{+}^{(\alpha,l)} = -\partial_{h}^{\alpha} \partial_{j_{+}^{l}} (z_{-} \cdot \nabla j_{+}) + z_{-} \cdot \nabla \partial_{h}^{\alpha} \partial_{j_{+}^{l}}.
\]

Thanks to Proposition 3.1, we have
\[
\sup_{0 \leq \tau \leq t} \int_{\Sigma_{\tau}} \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}|^{2} \, dx + \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}|^{2} \, dx d\tau \\
\lesssim \int_{\Sigma_{0}} \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}|^{2} \, dx + \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_{-} \rangle^{2(1+\sigma)} |z_{-}^{1} \cdot |j_{+}^{(\alpha,l)}|^{2} \, dx d\tau \\
+ \int_{0}^{t} \int_{\Sigma_{\tau}} (|\rho_{+}^{(\alpha,l)}| + |\rho_{+}^{(\alpha,l)}|) \cdot \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}| \, dx d\tau.
\]

**Step 2. Estimates of the nonlinear terms in the r.h.s. of (3.42).**

**Step 2.1. Estimate of term \( \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_{-} \rangle^{2(1+\sigma)} |z_{-}^{1} \cdot |j_{+}^{(\alpha,l)}|^{2} \, dx d\tau \).** It is bounded by
\[
\| \langle u_{+} \rangle^{1+\sigma} z_{-}^{1} \|_{L_{x}^{2} L_{t}^{2}} \| \langle u_{+} \rangle^{1+\sigma} |j_{+}^{(\alpha,l)}| \|_{L_{x}^{2} L_{t}^{2}}^{2} \lesssim \sum_{k_{1}+l_{1} \leq 2} \delta_{1}^{-1} \langle E_{-}^{(k_{1},l_{1})}(z_{-}) \rangle^{\frac{1}{2}} \cdot F_{+}^{(\alpha,l)}(j_{+}).
\]

Then we have
\[
\delta_{2(l+1)} \int_{0}^{t} \int_{\Sigma_{\tau}} \langle u_{-} \rangle^{2(1+\sigma)} |z_{-}^{1} \cdot |j_{+}^{(\alpha,l)}|^{2} \, dx d\tau \lesssim \sum_{k_{1}+l_{1} \leq 2} \delta_{1}^{-1} \langle E_{-}^{(k_{1},l_{1})}(z_{-}) \rangle^{\frac{1}{2}} \cdot \delta_{2(l+1)} F_{+}^{(\alpha,l)}(j_{+}).
\]

**Step 2.2. Estimate of term \( \int_{0}^{t} \int_{\Sigma_{\tau}} |\rho_{+}^{(\alpha,l)}| \cdot \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}| \, dx d\tau \).** By Hölder inequality, we have
\[
\int_{0}^{t} \int_{\Sigma_{\tau}} |\rho_{+}^{(\alpha,l)}| \cdot \langle u_{-} \rangle^{2(1+\sigma)} |j_{+}^{(\alpha,l)}| \, dx d\tau \lesssim \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \rho_{+}^{(\alpha,l)} \|_{L_{x}^{2} L_{t}^{2}} \cdot (F_{+}^{(\alpha,l)}(j_{+}))^{\frac{1}{2}}.
\]

Now, we want to give the upper bound of \( \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \rho_{+}^{(\alpha,l)} \|_{L_{x}^{2} L_{t}^{2}} \). By the definition of \( \nabla z_{-} \wedge \nabla z_{+} \), and using the fact that \( \nabla z_{-} = 0 \), we have
\[
|\rho_{+}^{(\alpha,l)}| \lesssim \sum_{l_{1} \leq l_{0} \leq \alpha} \sum_{l_{0} \leq \alpha} \left( |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{1}}} \nabla h_{z_{-}}| \cdot |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{2}}} \nabla h_{z_{+}}| + |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{1}}} \nabla h_{z_{-}}| \cdot |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{2}}} \nabla h_{z_{+}}| \right)
\]
\[
+ \sum_{l_{1} \leq l_{0} \leq \alpha} \sum_{l_{0} \leq \alpha} \left( |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{1}}} \nabla h_{z_{-}}| \cdot |\partial_{h}^{\alpha} \partial_{j_{+}^{l_{2}}} \nabla h_{z_{+}}| \right).
\]

For \( I_{\beta_{+},l_{1}}^{1} \), if \( |\beta_{+}| + l_{1} + 1 \leq N \), we have
\[
\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_{+},l_{1}}^{1} \|_{L_{x}^{2} L_{t}^{2}} \lesssim \| \langle u_{+} \rangle^{1+\sigma} \nabla h_{\beta_{+} + \partial_{j_{+}^{l_{1}}} z_{-}} \|_{L_{x}^{\infty} L_{t}^{2}} \cdot \| \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \nabla h_{\beta_{+} + \partial_{j_{+}^{l_{1}}} z_{+}} \|_{L_{x}^{\infty} L_{t}^{2}}.
\]

By virtue of Lemma 2.4.1 (ii), we have
\[
\| \langle u_{-} \rangle^{1+\sigma} \nabla h_{\beta_{+} + \partial_{j_{+}^{l_{1}}} z_{+}} \|_{L_{x}^{2} L_{t}^{2}} \lesssim \sum_{k_{2}+l_{2} \leq 2} \delta_{1}^{-1} \langle F_{+}^{(k_{2},l_{2})(z_{+})} \rangle^{\frac{1}{2}}
\]
\[
\lesssim \delta_{2}^{-1} \sum_{k_{2}+l_{2} \leq N+2} \delta_{1}^{-1} \langle F_{+}^{(k_{2},l_{2})(z_{+})} \rangle^{\frac{1}{2}}.
\]
Since
\[
\left\| \langle u_+ \rangle^{1+\sigma} \nabla_h^{\alpha_h-\beta_h+1} \partial_3^{l-1} z_h \right\|_{L_t^\infty L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \left( E_{-}^{(k_1,l-1)}(z_-) \right)^{\frac{1}{2}},
\]
if \(|\beta_h| + l_1 + 1 \leq N\), then we have
\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^1 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \delta^{l-\frac{1}{2}} \left( E_{-}^{(k_1,l-1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq N+2} \delta^{l_2-\frac{1}{2}} \left( E_{+}^{(k_2,l_2)}(z_+) \right)^{\frac{1}{2}}.
\]
(3.46)

If \(N + 1 \leq |\beta_h| + l_1 + 1 \leq |\alpha_h| + l + 1 \leq 2N\), we have
\[
\left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^1 \right\|_{L_t^2 L_x^2} \lesssim \left\| \langle u_- \rangle^{1+\sigma} \nabla_h^{\alpha_h-\beta_h+1} \partial_3^{l-1} z_h \right\|_{L_t^\infty L_x^2} \cdot \left\| \frac{\langle u_- \rangle^{1+\sigma} \nabla_h^{\beta_h-1} \partial_3^l z_h}{\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)}} \right\|_{L_t^2 L_x^2}.
\]

Following the similar derivation as that for (3.46), we have
\[
\left\| \langle u_- \rangle^{1+\sigma} \nabla_h^{\alpha_h-\beta_h+1} \partial_3^{l-1} z_h \right\|_{L_t^\infty L_x^2} \lesssim \sum_{k_2+l_2 \leq N+2} \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}},
\]
and
\[
\left\| \frac{\langle u_- \rangle^{1+\sigma} \nabla_h^{\beta_h-1} \partial_3^l z_h}{\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)}} \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \left( F_{+}^{(k_1,l_1)}(z_+) \right)^{\frac{1}{2}}.
\]

Then we obtain that for \(N + 1 \leq |\beta_h| + l_1 + 1 \leq |\alpha_h| + l + 1 \leq 2N\), it holds
\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^1 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \delta^{l-\frac{1}{2}} \left( E_{-}^{(k_1,l-1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq N+2} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2+1)}(z_+) \right)^{\frac{1}{2}}.
\]
(3.47)

Similarly, for \(I_{\beta_h,l_1}^2\) and \(I_{\beta_h,l_1}^3\), if \(|\beta_h| + l_1 + 1 \leq N\), then we have
\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^2 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \delta^{l_1-\frac{1}{2}} \left( E_{-}^{(k_1,l-1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq N+2} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2+1)}(z_+) \right)^{\frac{1}{2}},
\]
(3.48)

\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^3 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_1 \leq |\alpha_h|+1} \delta^{l_1-\frac{1}{2}} \left( E_{-}^{(k_1,l-1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq N+2} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2+1)}(z_+) \right)^{\frac{1}{2}}.
\]
(3.49)

While for \(N + 1 \leq |\beta_h| + l_1 + 1 \leq |\alpha_h| + l + 1 \leq 2N\),
\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^4 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_2+l_2 \leq N+2} \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_1 \leq |\alpha_h|} \delta^{l_1+\frac{1}{2}} \left( F_{+}^{(k_1,l_1+1)}(z_+) \right)^{\frac{1}{2}}.
\]
(3.50)

\[
\delta^{l+\frac{1}{2}} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} I_{\beta_h,l_1}^5 \right\|_{L_t^2 L_x^2} \lesssim \sum_{k_2+l_2 \leq N+1} \delta^{l_2+\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_1 \leq |\alpha_h|+1} \delta^{l_1-\frac{1}{2}} \left( F_{+}^{(k_1,l_1)}(z_+) \right)^{\frac{1}{2}}.
\]
(3.51)
Thanks to (3.45) and inequalities (3.46) to (3.51), noticing that $|\alpha_h| + l \leq 2N - 1$ and $N \in \mathbb{Z}_{\geq 5}$, we have
\[
\delta^{l+\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+) \|_{L^2_t L^2_x} \lesssim \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} \left( E_{-}^{(k_1,l_1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} \left( E_{+}^{(k_1,l_1)}(z_+) \right)^{\frac{1}{2}}.
\]
Then due to (3.44) and the inequality
\[
F^{(\alpha_h,l)}_{+}(j_+) \lesssim F^{(|\alpha_h|+1,l)}_{+}(z_+) + F^{(|\alpha_h|,l+1)}_{+}(z_+),
\]
we have
\[
\delta^{2(l+\frac{1}{2})} \int_0^t \int_{\Sigma^r} |\rho^{(\alpha_h,l)}_{+,1} \cdot (u_-)^{2(1+\sigma)} j^{(\alpha_h,l)}_{+,1}| dx \, d\tau \lesssim \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} \left( E_{-}^{(k_1,l_1)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_1+l_1 \leq 2N} \delta^{2(l_1-\frac{1}{2})} F_{+}^{(k_1,l_1)}(z_+).
\]

**Step 2.3. Estimate of term** \[
\int_0^t \int_{\Sigma^r} |\rho^{(\alpha_h,l)}_{+,2} \cdot (u_-)^{2(1+\sigma)} j^{(\alpha_h,l)}_{+,2}| dx \, d\tau \leq \| (u_-)^{1+\sigma} (u_+) \|_{L^2_t L^2_x} \cdot \left( F^{(\alpha_h,l)}_{+}(j_+) \right)^{\frac{1}{2}}.
\]

For $\rho^{(\alpha_h,l)}_{+,2}$, we have
\[
|\rho^{(\alpha_h,l)}_{+,2}| \lesssim \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right) \left( \sum_{l_1 \leq l_2} \sum_{\beta_h < \alpha_h} \sum_{l_1 \leq l_2} \sum_{\beta_h \leq \alpha_h} \right)
\]

For $II_{\beta_h,l_1}$, if $|\alpha_h - \beta_h| + l - l_1 \leq 2N - 3$ (that is $|\beta_h| + l_1 \geq |\alpha_h| + l - 2N + 3$), by similar derivation, we have
\[
\delta^{l+\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+) \|_{L^2_t L^2_x} \lesssim \sum_{k_2+l_2 \leq 2N} \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2)}(j_+) \right)^{\frac{1}{2}}.
\]

Since $|\alpha_h - \beta_h| + l - l_1 \leq 2N - 3$, $|\beta_h| + l_1 \leq |\alpha_h| + l - 1$ and $k_2 + l_2 \leq 2$, we have $|\alpha_h - \beta_h| + k_2 + l_2 \leq 2N - 1$, $|\beta_h| + l_1 + 1 \leq |\alpha_h| + l \leq 2N - 1$.

Then for $|\beta_h| + l_1 \geq |\alpha_h| + l - 2N + 3$, we get
\[
\delta^{l+\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+) \|_{L^2_t L^2_x} \lesssim \sum_{k_2+l_2 \leq 2N-1} \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq 2N-1} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2)}(j_+) \right)^{\frac{1}{2}}.
\]

While if $|\beta_h| + l_1 \leq |\alpha_h| + l - 2N + 2 \leq 1$, we have
\[
\delta^{l+\frac{1}{2}} \| (u_-)^{1+\sigma} (u_+) \|_{L^2_t L^2_x} \lesssim \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq 2N-1} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2)}(j_+) \right)^{\frac{1}{2}}.
\]

\[
\sum_{k_2+l_2 \leq 2N-1} \delta^{l_2-\frac{1}{2}} \left( E_{-}^{(k_2,l_2)}(z_-) \right)^{\frac{1}{2}} \cdot \sum_{k_2+l_2 \leq 4} \delta^{l_2+\frac{1}{2}} \left( F_{+}^{(k_2,l_2)}(j_+) \right)^{\frac{1}{2}}.
\]
Thanks to (3.56), (3.57) and the fact $N \in \mathbb{Z}_{\geq 5}$, we get
\[
\delta^{l + \frac{1}{2}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2}^2 \lesssim \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2, l_2} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2, l_2} \right) \right)^{\frac{1}{2}}.
\]

Similarly, for $II_{k_0, l_0}^2$ if $|\alpha_h - \beta_h| + l - l_1 \leq 2N - 3$ (that is $|\beta_h| + l_1 \geq |\alpha_h| + l - 2N + 3$), it holds
\[
\delta^{l + \frac{1}{2}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2}^2 \lesssim \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2, l_2} \right)^{\frac{1}{2}}.
\]

While for $|\beta_h| + l_1 \leq |\alpha_h| + l - 2N + 2 \leq 1$, one has
\[
\| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2}^2 \lesssim \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 3} \delta^{l_2, l_2} \right)^{\frac{1}{2}}.
\]

Since $\text{div } z_- = 0$, we have
\[
\sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2, l_2} \right)^{\frac{1}{2}} \lesssim \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2, l_2} \right)^{\frac{1}{2}}.
\]

Then for $N \in \mathbb{Z}_{\geq 5}$, we obtain that
\[
\delta^{l + \frac{1}{2}} \| (u_-)^{1+\sigma} (u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2}^2 \lesssim \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2 - \frac{1}{2}} \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-2} \delta^{l_2, l_2} \right)^{\frac{1}{2}} \right) \cdot \left( \sum_{k_2 \leq 2N-1} \sum_{k_2 + l_2 \leq 2N-1} \delta^{l_2, l_2} \right)^{\frac{1}{2}}.
\]

Thanks to (3.56), (3.58) and (3.59), using (3.52), we have
\[
\delta^{2l + \frac{1}{2}} \left( 0 \right) \int_{\Sigma_t} |\rho^{(\alpha_h, l)}| \cdot \langle u_- \rangle^{1+\sigma} \cdot |\langle u_+ \rangle^{\alpha_h, l}| \cdot \text{d}x \cdot \text{d}t \lesssim \left( \sum_{k_1 + l_1 \leq 2N} \delta^{l_1 - \frac{1}{2}} \left( \sum_{k_1 \leq 2N-1} \sum_{k_1 \leq 2N-2} \delta^{l_1, l_1} \right)^{\frac{1}{2}} \right) \cdot \left( \sum_{k_1 + l_1 \leq 2N} \delta^{2l_1, l_1} \right)^{\frac{1}{2}}.
\]

**Step 3.** The *a priori estimate of $\nabla_t \partial_t^2 z_+$ for $k + l \leq 2N$ and $l \geq 1$.** Combining (3.42), (3.43), (3.53), (3.61) together with (3.52), for $k + l \leq 2N - 1$, we obtain that
\[
\delta^{2l + \frac{1}{2}} \left( \sum_{k_1 \leq 2N-1} \sum_{k_1 \leq 2N-1} \delta^{l_1, l_1} \right)^{\frac{1}{2}} \lesssim \delta^{2l_1, l_1} \left( \sum_{k_1 + l_1 \leq 2N} \delta^{l_1, l_1} \right)^{\frac{1}{2}} \cdot \left( \sum_{k_1 + l_1 \leq 2N} \delta^{2l_1, l_1} \right)^{\frac{1}{2}}.
\]

Thanks to (3.61), (3.33), (3.11) and (3.52), we obtain the desired inequality (3.33) for $z_+$. Similar estimate holds for $z_-$. Then it ends the proof of the proposition.
3.2.3. The uniform estimates of $\nabla_h^k \partial_t^l (\partial_3 z_{\pm})$ for $k + l \leq N + 2$. In this subsection, we want to derive the uniform estimates concerning $\nabla_h^k \partial_t^l (\partial_3 z_{\pm})$ with the lower order coefficient $\delta^{l - \frac{1}{2}}$. The main estimate is presented in the following proposition.

**Proposition 3.7.** Assume that $(z_+, z_-)$ are the smooth solutions to (3.6). Let $N_*=2N$, $N \in \mathbb{Z}_{\geq 5}$, and (3.2) hold. We have

$$
\sum_{k+l \leq N+2} \delta^{2(l-\frac{1}{2})} |E_\pm^{(k,l)}(\partial_3 z_{\pm}) + F_\pm^{(k,l)}(\partial_3 z_{\pm})| 
\lesssim \sum_{k+l \leq N+2} \delta^{2(l-\frac{1}{2})} E_\pm^{(k,l)}(\partial_3 z_{\pm}, \partial_3 z_{\pm}) + \sum_{k+l \leq N_*} \delta^{2(l-\frac{1}{2})} \sum_{k+l \leq N+2} \delta^{2(l-\frac{1}{2})} E_\pm^{(k,l)}(z_{\pm}, 0) 
+ \left( \sum_{k+l \leq N_*, l \leq N_1} \int \frac{1}{\nabla_h^l \partial_t^l} \nabla_h^l \partial_t^l \nabla_h^l \partial_3 z_{\pm} \right) \frac{1}{2} + \sum_{k+l \leq N+2} \delta^{2(l-\frac{1}{2})} F_\pm^{(k,l)}(\partial_3 z_{\pm}) \right). \tag{3.62}
$$

**Proof.** To prove the proposition, we only need to modify the proof of Proposition 3.6. 

**Step 1.** Estimates of nonlinear terms in the r.h.s of (3.32) for $|\alpha_h|+l \leq N+2$.

**Step 1.1.** Estimate of nonlinear term $\int_{[r_0]} \langle u_- \rangle^{-2(1+\sigma)} |z_{\pm}| \cdot |j_+^{(\alpha_h,l)}|^2 dx dt$. By virtue of (3.43), we have

$$
\sum_{k=0}^2 \left( \delta^{\frac{1}{2}} (E_0^{(k,0)}(z_{\pm})) \frac{1}{2} + \delta^{\frac{3}{2}} (E_0^{(k,1)}(z_{\pm})) \frac{1}{2} + \sum_{k=0}^2 \delta^{2(l-\frac{1}{2})} F_+^{(\alpha_h,l)}(j_+) \right). 
$$

**Step 1.2.** Estimate of term $\int_{[r_0]} \langle u_- \rangle^{-2(1+\sigma)} |\alpha_h^{(\alpha_h,l)}| \cdot |j_+^{(\alpha_h,l)}|^2 dx dt$. Thanks to (3.44), we only need to derive the bound of $\| \langle u_- \rangle^{-1+\sigma} (u_+) \frac{1}{2} \|_{L^2_t \mathcal{L}^2_x}$. 

For $I_{\beta_h,l_1} = \partial_h^{\alpha_h - \beta_h} \partial_3^{l_1} \nabla_h \partial_3 z_{\pm} \cdot |\partial_h^{\alpha_h} \partial_3^{l_1} \nabla_h \partial_3 z_{\pm}|$ and $I_{\beta_h,l_1} = \partial_h^{\alpha_h - \beta_h} \partial_3^{l_1} \nabla_h \partial_3 z_{\pm}$ in (3.45), by similar derivation of (3.47), we have

$$
\delta^{l-\frac{1}{2}} \| (u_-)^{-1+\sigma} (u_+) \frac{1}{2} \|_{L^2_t \mathcal{L}^2_x} \lesssim \sum_{k_2+l_2 \geq |\alpha_h|+l+3} \delta^{l_2-\frac{1}{2}} \left( E_0^{(k_2,l_2)}(z_{\pm}) \right) \frac{1}{2} \sum_{k_1 \leq |\alpha_h|+1} \delta^{l_1-\frac{1}{2}} \left( F_+^{(k_1,l_1)}(z_{\pm}) \right) \frac{1}{2}, \tag{3.64}
$$

and

$$
\delta^{l-\frac{1}{2}} \| (u_-)^{-1+\sigma} (u_+) \frac{1}{2} \|_{L^2_t \mathcal{L}^2_x} \lesssim \sum_{k_2+l_2 \geq |\alpha_h|+l+3} \delta^{l_2-\frac{1}{2}} \left( E_0^{(k_2,l_2)}(z_{\pm}) \right) \frac{1}{2} \sum_{k_1 \leq |\alpha_h|+1} \delta^{l_1-\frac{1}{2}} \left( F_+^{(k_1,l_1)}(z_{\pm}) \right) \frac{1}{2}. \tag{3.65}
$$

For $I_{\beta_h,l_1} = |\partial_h^{\alpha_h - \beta_h} \partial_3^{l_1} \nabla_h \partial_3 z_{\pm} | \cdot |\partial_h^{\alpha_h} \partial_3^{l_1} \nabla_h \partial_3 z_{\pm}|$ in (3.45), by similar derivation of (3.46), we have

$$
\delta^{l-\frac{1}{2}} \| (u_-)^{-1+\sigma} (u_+) \frac{1}{2} \|_{L^2_t \mathcal{L}^2_x} \lesssim \sum_{k_2+l_2 \geq |\alpha_h|+l+3} \delta^{l_2-\frac{1}{2}} \left( E_0^{(k_2,l_2)}(z_{\pm}) \right) \frac{1}{2} \sum_{k_2+l_2 \geq |\alpha_h|+l+3} \delta^{l_2-\frac{1}{2}} \left( F_+^{(k_2,l_2)}(z_{\pm}) \right) \frac{1}{2}. \tag{3.66}
$$
Thanks to (3.64), (3.69) and (3.69), we obtain the estimate of $\delta^{l-\frac{1}{2}} \| (u_-)^{1+\sigma}(u_+)^{\frac{1}{2}(1+\sigma)} \rho_{+,l}^{(\alpha_h,l)} \|_{L^2_t L^2_x}$. Then using (3.44) and inequality (3.52), for $|\alpha_h| + l \leq N + 2$ and $N \in \mathbb{Z}_{\geq 5}$, we get

$$\delta^{2(l-\frac{1}{4})} \int_0^t \int_{\Sigma_\tau} |\rho_{+,l}^{(\alpha_h,l)}| \cdot (u_-)^{2(1+\sigma)} |J_+^{(\alpha_h,l)}| dx dt$$

$$\lesssim \left( \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} (E_{(k_1,l_1)}^+(z_-))^\frac{1}{2} + \sum_{k_1+l_1 \leq N+2} \delta^{l_1-\frac{1}{2}} (E_{(k_1,l_1)}^-(\partial_3 z_-))^\frac{1}{2} \right)$$

$$\cdot \left( \sum_{k_1+l_1 \leq 2N} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(z_+) + \sum_{k_1+l_1 \leq N+2} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(\partial_3 z_+) \right).$$

(3.67)

**Step 1.3. Estimate of term $\int_0^t \int_{\Sigma_\tau} |J_+^{(\alpha_h,l)}| |\rho_{+,l}^{(\alpha_h,l)}|^\frac{1}{2}(1+\sigma) |dx dt$.** By virtue of (3.54), we only need to control $\delta^{l-\frac{1}{2}} \| (u_-)^{1+\sigma}(u_+)^{\frac{1}{2}(1+\sigma)} \rho_{+,l}^{(\alpha_h,l)} \|_{L^2_t L^2_x}$.

For $I_{\beta_h,l_1}^1 = |\partial^\alpha_h \beta_h | \cdot \nabla_k \partial^\alpha_h \beta_h | \cdot |\partial^\beta_h \beta_h |$ and $I_{\beta_h,l_1}^2 = |\partial^\alpha_h \beta_h | \cdot |\partial^\beta_h \beta_h |$ in (3.55), by similar derivation as that for (3.56), we have

$$\delta^{l-\frac{1}{2}} \| (u_-)^{1+\sigma}(u_+)^{\frac{1}{2}(1+\sigma)} \|_{L^2_t L^2_x}$$

$$\lesssim \sum_{k_2+l_2 \leq |\alpha_h| + l + 2} \delta^{l_2-\frac{1}{2}} (E_{(k_2,l_2)}^+(z_3))^\frac{1}{2} \cdot \delta^{1+\frac{1}{2}} (F_+^{(\beta_h,l_1+1)}(j_+))^\frac{1}{2}.$$

(3.68)

$$\text{div } z_-=0 \left( \sum_{k_2+l_2 \leq |\alpha_h| + l + 2} \delta^{l_2-\frac{1}{2}} (E_{(k_2,l_2)}^+(z_3))^\frac{1}{2} + \sum_{k_2 \leq |\alpha_h| + l + 2} \delta^{l_2-\frac{1}{2}} (E_{(k_2,0)}^+(z_3))^\frac{1}{2} \right) \cdot \delta^{1+\frac{1}{2}} (F_+^{(\beta_h,l_1+1)}(j_+))^\frac{1}{2}.$$

(3.69)

Thanks to (3.64) and (3.69), we obtain the estimates of $\delta^{l-\frac{1}{2}} \| (u_-)^{1+\sigma}(u_+)^{\frac{1}{2}(1+\sigma)} \rho_{+,l}^{(\alpha_h,l)} \|_{L^2_t L^2_x}$. Since $|\alpha_h| + l \leq N + 2$, $|\beta_h| + l_1 \leq |\alpha_h| + l - 1$ and $N \in \mathbb{Z}_{\geq 5}$, by using (3.54) and (3.54), we obtain that

$$\delta^{2(l-\frac{1}{2})} \int_0^t \int_{\Sigma_\tau} |J_+^{(\alpha_h,l)}| dx dt$$

$$\lesssim \left( \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} (E_{(k_1,l_1)}^+(z_-))^\frac{1}{2} + \sum_{k_1+l_1 \leq N+2} \delta^{l_1-\frac{1}{2}} (E_{(k_1,0)}^+(z_-))^\frac{1}{2} \right)$$

$$\times \left( \sum_{k_1+l_1 \leq 2N} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(z_+) + \sum_{k_1+l_1 \leq N+2} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(\partial_3 z_+) \right).$$

(3.70)

**Step 2. The a priori estimate of $\nabla^k \partial^\alpha_j \partial_3 z_+$ for $k + l \leq N + 2$.** Combining (3.22), (3.68), (3.67), (3.70) together with (3.54), we obtain that for $k + l \leq N + 2$, it holds

$$\delta^{2(l-\frac{1}{2})} (E_+^{(k,l)}(j_+) + F_+^{(k,l)}(j_+)) \lesssim \delta^{2(l-\frac{1}{2})} (E_+^{(k,l)}(\text{curl } z_{+0})) + \left( \sum_{k_1+l_1 \leq 2N} \delta^{l_1-\frac{1}{2}} (E_{(k_1,l_1)}^+(z_-))^\frac{1}{2} \right)$$

$$+ \sum_{k_1 \leq 2N-1} \delta^{l_1-\frac{1}{2}} (E_{(k_1,0)}^+(z_-))^\frac{1}{2} + \sum_{k_1+l_1 \leq N+2} \delta^{l_1-\frac{1}{2}} (E_{(k_1,l_1)}^-(\partial_3 z_-))^\frac{1}{2}$$

$$\times \left( \sum_{k_1+l_1 \leq 2N} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(z_+) + \sum_{k_1+l_1 \leq N+2} \delta^{2(l_1-\frac{1}{4})} F_+^{(k_1,l_1)}(\partial_3 z_+) \right).$$

(3.71)
Thanks to (3.71) and (3.39), by using (3.11) and (3.52), we obtain the desired inequality (3.62) for $z_+$. Similar estimate holds for $z_-$. Then it ends the proof of the proposition.

### 3.3. Proof of main uniform a priori estimates and Theorem 1.1

The existence part of Theorem 1.1 follows the standard method of continuity. We shall only prove the uniform energy estimate (1.13).

**Step 1. Ansatz and closure of the continuity argument.** To use the method of continuity, we first need the following ansatz. We assume that

\[ \| z_\pm \|_{L^\infty_t L^N_x} \leq 1, \quad (3.72) \]

and

\[ \delta^{2(l-\frac{1}{2})}E_{\pm}^{(k,l)}(z_\pm) \leq 2C_1\varepsilon^2, \quad \delta^{2(l-\frac{1}{2})}F_{\pm}^{(k,l)}(z_\pm) \leq 2C_1\varepsilon^2, \quad \text{for } k+l \leq N, \]

\[ \delta^{-3}E_{\pm}^{(k,0)}(z_\pm^3) \leq 2C_1\varepsilon^2, \quad \delta^{-3}F_{\pm}^{(k,0)}(z_\pm^3) \leq 2C_1\varepsilon^2 \quad \text{for } k \leq N-1, \]

\[ \delta^{2(l-\frac{1}{2})}E_{\pm}^{(k,l)}(\partial_3 z_\pm) \leq 2C_1\varepsilon^2, \quad \delta^{2(l-\frac{1}{2})}F_{\pm}^{(k,l)}(\partial_3 z_\pm) \leq 2C_1\varepsilon^2, \quad \text{for } k+l \leq N+2, \]

where $C_1$ would be determined by the energy estimates.

To close the continuity argument, we need to prove that there exists a small enough $\varepsilon_0$ such that for all $\varepsilon \leq \varepsilon_0$ the constant $2$ in (3.73) can be improved to be $1$, i.e.,

\[ \delta^{2(l-\frac{1}{2})}E_{\pm}^{(k,l)}(z_\pm) \leq C_1\varepsilon^2, \quad \delta^{2(l-\frac{1}{2})}F_{\pm}^{(k,l)}(z_\pm) \leq C_1\varepsilon^2, \quad \text{for } k+l \leq N, \]

\[ \delta^{-3}E_{\pm}^{(k,0)}(z_\pm^3) \leq C_1\varepsilon^2, \quad \delta^{-3}F_{\pm}^{(k,0)}(z_\pm^3) \leq C_1\varepsilon^2 \quad \text{for } k \leq N-1, \]

\[ \delta^{2(l-\frac{1}{2})}E_{\pm}^{(k,l)}(\partial_3 z_\pm) \leq C_1\varepsilon^2, \quad \delta^{2(l-\frac{1}{2})}F_{\pm}^{(k,l)}(\partial_3 z_\pm) \leq C_1\varepsilon^2, \quad \text{for } k+l \leq N+2. \]

For $\varepsilon \leq \varepsilon_0$, we also improve the ansatz (3.72) to $\| z_\pm \|_{L^\infty_t L^N_x} \leq \frac{1}{2}$.

Actually, by Sobolev inequality, we have

\[ \| z_\pm \|_{L^\infty_t L^N_x} \leq C_2 \sum_{k+l \leq 2} \left( \delta^{l-\frac{1}{2}}(E_{\pm}^{(k,l)}(z_\pm)) \right)^{\frac{1}{2}} \leq 6\sqrt{C_1}C_2\varepsilon_0. \]

Then by taking $\varepsilon_0$ small enough, we could prove that $\| z_\pm \|_{L^\infty_t L^N_x} \leq \frac{1}{2}$.

**Step 2. The uniform a priori estimate under the ansatz (3.72) and (3.73).** Since div $z_\pm = 0$ and $z_\pm^3|_{x_3=\pm\delta} = 0$, $z_\pm^3|_{x_3=\pm\delta} = 0$, we have

\[ z_\pm^3(t,x_h,x_3) = \int_{-\delta}^{x_3} (\partial_3 z_\pm^3)(t,x_h,s)ds = -\int_{-\delta}^{x_3} (\nabla_h \cdot z_\pm^h)(t,x_h,s)ds. \]

which implies

\[ |\nabla_h^{k+1} z_\pm^h(t,x_h,x_3)| \leq \int_{-\delta}^{\delta} |\nabla_h^{k+1} z_\pm^h(t,x_h,x_3)|dx_3, \quad \text{for any } k \in \mathbb{Z}_{\geq 0}. \]

Then by Hölder inequality, for any $k \in \mathbb{Z}_{\geq 0}$, we obtain that

\[ F_{\pm}^{(k,0)}(z_\pm^3) \lesssim \delta F_{\pm}^{(k+1,0)}(z_\pm^h), \quad F_{\pm}^{(k,0)}(z_\pm^3) \lesssim \delta F_{\pm}^{(k+1,0)}(z_\pm^h). \quad (3.74) \]
Thanks to (3.74), (3.11) of Proposition 3.2, (3.38) of Proposition 3.6 and (3.02) of Proposition 3.7 we obtain that
\[
E(t^*) \equiv \sum_{\pm} \left( \sum_{k+l \leq N_0} \delta^{2(l+1)} \left[ E^{(k,l)}_\pm (z_\pm) + F^{(k,l)}_\pm (z_\pm) \right] + \sum_{k \leq N_0-1} \delta^{-3} \left[ E^{(k,0)}_\pm (z_\pm) + F^{(k,0)}_\pm (z_\pm) \right] \right) + \sum_{k+l \leq N+2} \delta^{2(l+1)} \left[ E^{(k,l)}_\pm (\partial_3 z_\pm) + F^{(k,l)}_\pm (\partial_3 z_\pm) \right] 
\]
\[
\leq C_3 E(0) + C_3 \sum_{\pm} \left[ \left( \sum_{k+l \leq N_0} \delta^{l+1} (E^{(k,l)}_\pm) (z_\pm) \right)^{\frac{1}{2}} + \sum_{k \leq N_0-1 \leq N+2} \delta^{-\frac{3}{2}} (E^{(k,0)}_\pm (z_\pm)) \right] + \sum_{k+l \leq N+2} \delta^{l+1} (E^{(k,l)}_\pm) (\partial_3 z_\pm) \right].
\]
Using (3.73), we could obtain that
\[
E(t^*) \leq C_3 E(0) + C_3 C_1 \varepsilon E(t^*).
\]
Taking \( \varepsilon_0 \leq \frac{1}{C_3 C_4^{-1}} \), we obtain for any \( \varepsilon \leq \varepsilon_0 \), \( t^* < \infty \)
\[
E(t^*) \leq 2C_3 E(0) \leq C_1 \varepsilon^2,
\]
where we choose \( C_1 \geq 2C_3 \). Then Theorem 1 is proved.

4. PROOF OF THEOREM 1.2

In this section, we give the proof to Theorem 1.2. To do that, we first consider the following linearized system
\[
\partial_t f_+ - \partial_1 f_+ + z_h^\pm \cdot \nabla_h f_+ = \rho_+, \quad \text{in } \Omega_\delta,
\]
\[
\partial_t f_- - \partial_1 f_- + z_h^\pm \cdot \nabla_h f_- = \rho_-.
\]
(Note that \( \text{div}_h z_h^\pm = \nabla_h \cdot z_h^\pm = 0 \). Thanks to the proof of Proposition 3.1, we obtain the following proposition.

Proposition 4.1. Assume that \( \text{div}_h z_h^\pm = 0 \) and it holds
\[
\|z_h^\pm\|_{L_t^\infty L_x^\infty} \leq 1.
\]
Then for any smooth solutions \( (f_+, f_-) \) to (3.1), we have
\[
\sup_{0 \leq t \leq t_0} \int_{\Sigma_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h + \int_0^t \int_{\Sigma_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h dt 
\]
\[
\lesssim \int_{\Omega_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h + \int_0^t \int_{\Sigma_{r,h}} \langle u_\pm \rangle (1+\sigma)|z_\pm|^2 |f_\pm|^2 dx_h dt + \int_0^t \int_{\Sigma_{r,h}} \rho_\pm \cdot \langle u_\pm \rangle^2 (1+\sigma) f_\pm dx_h dt
\]
\[
+ \int_{\mathbb{R}} \langle u_\pm \rangle^2 (1+\sigma) \left| \int_{W_{\varepsilon,0}^-} \mu_{\varepsilon,0}^- \cdot \langle u_\pm \rangle \langle u_\pm \rangle^2 (1+\sigma) f_\pm dx_h dt \right| du_\pm.
\]
In particular, it holds
\[
\sup_{0 \leq t \leq t_0} \int_{\Omega_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h + \int_0^t \int_{\Sigma_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h dt 
\]
\[
\lesssim \int_{\Omega_{r,h}} \langle u_\pm \rangle^2 (1+\sigma) |f_\pm|^2 dx_h + \int_0^t \int_{\Sigma_{r,h}} \langle u_\pm \rangle (1+\sigma)|z_\pm|^2 |f_\pm|^2 dx_h dt + \int_0^t \int_{\Sigma_{r,h}} |\rho_\pm| \cdot \langle u_\pm \rangle^2 (1+\sigma) |f_\pm| dx_h dt.
\]
With Proposition 1.1 we now present the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall divide the proof into several steps.

Step 1. Linearized system of $w^h_\pm$. For any $\alpha_h = (\alpha_1, \alpha_2) \in (\mathbb{Z}_{\geq 0})^2$, we set

$$w^h_\pm \overset{\text{def}}{=} \partial^{\alpha_h}_h w^h_\pm = \partial_1^{\alpha_1} \partial_2^{\alpha_2} w^h_\pm.$$

Applying $\partial^{\alpha_h}_h$ to both sides of equations for $w^h_\pm$ in (4.22), we have

$$\begin{align*}
\partial_t w^h_+(\alpha_h) - \partial_t w^h(-\alpha_h) + z^h_+ \cdot \nabla_h w^h_+ &= \rho^h_+(\alpha_h), \\
\partial_t w^h_-(\alpha_h) + \partial_t w^h(-\alpha_h) + z^h_- \cdot \nabla_h w^h_- &= \rho^h_-(\alpha_h),
\end{align*}$$

where

$$\rho^h_\pm(\alpha_h) = -(I - M_\delta)(\partial^{\alpha_h}_h \nabla_h p) - (I - M_\delta)[\partial^{\alpha_h}_h (w^h_\pm \cdot \nabla_h z^h_\pm)] + \partial^{\alpha_h}_h (z^h_\mp \cdot \nabla_h w^h_\pm) + z^h_\mp \cdot \nabla_h \partial^{\alpha_h}_h w^h_\pm.$$

We shall only give the estimates for $w^h_+(\alpha_h)$. Applying Proposition 1.1 to the first equation of (4.5), for any $x_3 \in (-\delta, \delta)$, we have

$$\begin{align*}
\sup_{0 \leq t \leq T} \int_{\Sigma_{t,h}} &\langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)|^2 \, dx_h + \int_0^t \int_{\Sigma_{t,h}} \langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)|^2 \, dx_h \, d\tau \\
\lesssim &\int_{\Sigma_{t,h}} \langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)|^2 \, dx_h + \int_0^t \int_{\Sigma_{t,h}} \langle u_\theta \rangle^{1+2\sigma} |z^h_\pm|^1 |w^h_+(\alpha_h)(\cdot, x_3)|^2 \, dx_h \, d\tau \\
&+ \int_0^t \int_{\Sigma_{t,h}} \rho^h_+(\alpha_h)(\cdot, x_3) \cdot \langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)| \, dx_h \, d\tau.
\end{align*}$$

Step 2. Estimates of the nonlinear terms in the r.h.s of (4.6).

Step 2.1. Estimate of the second terms on the r.h.s of (4.6). Using Hölder and Sobolev inequalities, we have

$$\begin{align*}
\int_0^t \int_{\Sigma_{t,h}} |\langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)|^2 |dx_h \, d\tau &\lesssim \sum_{k \leq 2} \langle E_{k,h}(\langle z^1 \rangle) \rangle_1^{\frac{1}{2}} \cdot F_{k,h}(\langle w^h_+(\cdot, x_3) \rangle).
\end{align*}$$

Step 2.2. Estimate of the last terms on the r.h.s of (4.6). By Hölder inequality, we first have

$$\begin{align*}
\int_0^t \int_{\Sigma_{t,h}} |\rho^h_+(\alpha_h)(\cdot, x_3) \cdot \langle u_\theta \rangle^{2(1+\sigma)} |w^h_+(\alpha_h)(\cdot, x_3)| \, dx_h \, d\tau \\
\lesssim &\|\langle u_\theta \rangle^{2(1+\sigma)} \|_{L^1_t L^1_x}^{(1+\sigma)} \cdot \rho^h_+(\alpha_h)(\cdot, x_3) \|_{L^2_t L^2_x} \cdot \langle F_{k,h}(\langle w^h_+(\cdot, x_3) \rangle) \rangle^{\frac{1}{2}}.
\end{align*}$$

We only need to control $\|\langle u_\theta \rangle^{2(1+\sigma)} \|_{L^1_t L^1_x}^{(1+\sigma)} \cdot \rho^h_+(\alpha_h)(\cdot, x_3) \|_{L^2_t L^2_x}$. For $\rho^h_+(\alpha_h)(\cdot, x_3)$, since

$$\| (I - M_\delta) f(x) \| = \frac{1}{2\delta} \int_{-\delta}^{\delta} \| f(x, h_3) - f(x, y_3) \| dy_3 \leq \delta \| \partial_3 f(x, h_3) \|_{L^\infty},$$

(4.9)
we have
\[
\| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \rho_h^{\alpha_h} \|_{L^2 \chi^2} = \| (I - M_h) \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} \nabla_p (\cdot, x_3) \|_{L^2 \chi^2} \\
\leq \delta \| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} \|_{L^2 \chi^2}.
\]

(4.10)

Thanks to (2.22), for any \( \alpha_h \in (\mathbb{Z}_\geq 0)^2 \), we have
\[
\partial_h^{\alpha_h} \partial_3 p(\tau, x) = \int_{\Omega_h} \partial_3 G_3(x, y) \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i)(\tau, y) dy.
\]

Let \( \theta(r) \) be the smooth cut-off function defining in Step 2.1 of the proof to Proposition 3.2. Using (2.10) and integrating by parts, we have
\[
\partial_h^{\alpha_h} \partial_3 p(\tau, x) = \int_{\Omega_h} \partial_3 G_3(x, y) \theta(|x_h - y_h|) \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i)(\tau, y) dy + \int_{\Omega_h} \partial_3 G_3(x, y) (1 - \theta(|x_h - y_h|)) \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i)(\tau, y) dy.
\]

(i) Estimate of \( \| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_1 A_1(\tau, x) \|_{L^2 \chi^2 L^\infty} \). Firstly, by virtue of (2.12) and (3.18), we have
\[
\langle u_+ (\tau, x_1) \rangle^{\frac{1}{2}} \langle u_- (\tau, x_1) \rangle^{1+\sigma} |A_1(\tau, x)| \\
\leq \frac{1}{\delta} \int_0^3 \int_{|x_h - y_h| \leq 2} \frac{1}{x_h - y_h} \langle \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i) \rangle(\tau, y) dy_h dy_3.
\]

Using Young inequality for the horizontal variables \( x_h \) and then Sobolev inequality for the vertical variable \( x_3 \), we have
\[
\| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_1 A_1(\tau, x) \|_{L^2 \chi^2 L^\infty} \\
\lesssim \| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i) \|_{L^2 \chi^2 L^\infty} \\
\leq \sum_{l=1}^{1} \delta^{-\frac{1}{2}} \| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} \partial_3 (\partial_i z^i_+ \partial_j z^-_i) \|_{L^2 \chi^2}.
\]

(4.11)

Thanks to H"older inequality and the condition \( \div z \pm = 0 \), we have
\[
\| \langle u_+ \rangle^{\frac{1}{2}} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} (\partial_i z^i_+ \partial_j z^-_i) \|_{L^2 \chi^2} \\
\lesssim \sum_{\beta_h \leq \alpha_h} \| \langle u_+ \rangle^{1+\sigma} \partial_h^{\beta_h} \partial_j z^-_i \|_{L^2 \chi^2} \| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h - \beta_h} \partial_i z^i_+ \|_{L^2 \chi^2} \\
\leq \sum_{\beta_h \leq \alpha_h} \left( \| \langle u_+ \rangle^{1+\sigma} \partial_h^{\beta_h} \nabla_h z^h_+ \|_{L^\infty \chi^2} \| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h - \beta_h} \nabla_h z^h_+ \|_{L^2 \chi^2} \\
+ \| \langle u_+ \rangle^{1+\sigma} \partial_h^{\beta_h} \partial_3 z^h_+ \|_{L^\infty \chi^2} \| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h - \beta_h} \partial_3 z^h_+ \|_{L^2 \chi^2} \\
+ \| \langle u_+ \rangle^{1+\sigma} \partial_h^{\beta_h} \nabla_h z^h_+ \|_{L^\infty \chi^2} \| \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h - \beta_h} \nabla_h z^h_+ \|_{L^2 \chi^2} \right).
\]
Similarly, for Sobolev inequality (2.23), we have
\[
\delta^{-\frac{1}{2}} \| (u_+)^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} \partial_h^{\alpha_h} (\partial_i z_+^j \partial_j z_-^l) \|_{L^2_h L^2_x}
\]
\[\lesssim \sum_{k+l \leq |\alpha_h|+3} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_-))^{\frac{1}{3}} \cdot \sum_{k \leq |\alpha_h|+1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+))^{\frac{1}{2}}
\]
\[+ \sum_{k+l \leq |\alpha_h|+2} \delta^{-\frac{1}{3}} (E_-^{(k,l+1)}(z_-))^{\frac{1}{3}} \cdot \sum_{k \leq |\alpha_h|+1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+^3))^{\frac{1}{2}}
\]
\[+ \sum_{k+l \leq |\alpha_h|+3} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_+^3))^{\frac{1}{3}} \cdot \sum_{k \leq |\alpha_h|} \delta^{\frac{1}{2}} (F_+^{(k,1)}(z_+))^{\frac{1}{2}}.
\]
Then for $|\alpha_h| \leq 2N - 4$, using $\text{div} \, z_- = 0$, we have
\[
\delta^{-\frac{1}{3}} \| (u_+)^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} \partial_h^{\alpha_h} (\partial_1 z_+^2 \partial_2 z_-^1) \|_{L^2_h L^2_x}
\]
\[\lesssim \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_-))^{\frac{1}{3}} \right) + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+^3))^{\frac{1}{2}}
\]}
\[\times \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_+))^{\frac{1}{3}} \right) + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+))^{\frac{1}{2}}.
\]
Similarly, for $|\alpha_h| \leq 2N - 4$, we have the same estimate for the term
\[
\delta^{\frac{1}{2}} \| (u_+)^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} \partial_h^{\alpha_h} \partial_3 (\partial_1 z_+^1 \partial_2 z_-^1) \|_{L^2_h L^2_x}.
\]
With the help of (4.11) and (4.12), for $|\alpha_h| \leq 2N - 4$, we have
\[
\| (u_+)^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} A_2 \|_{L^\infty_h L^2_x L^2_h}
\]
\[\lesssim \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_-))^{\frac{1}{3}} \right) + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+^3))^{\frac{1}{2}}
\]}
\[\times \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{3}} (E_-^{(k,l)}(z_+))^{\frac{1}{3}} \right) + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (F_+^{(k,0)}(z_+))^{\frac{1}{2}}.
\]
(ii) Estimate of $\| (u_+)^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} A_2(\tau, x) \|_{L^2_h L^2_x L^\infty_t}$. For $A_2$, using $\text{div} \, z = 0$, $z_\pm^3 |_{x_3=\pm \delta} = 0$, $z_\pm^3 |_{x_3=x} = 0$ and (2.25), and integrating by parts, we have
\[
|A_2(\tau, x)| \lesssim \int_{\Omega_3} |\partial_1 \partial_2 \partial_3 \partial_h G_0(x, y)(1 - \theta(|x_h - y_h|))| \partial_h^{\alpha_h} (z_+^3 z_-^1)(\tau, y) dy
\]
\[+ \int_{\Omega_3} (\partial_3 G_0(x, y)||\theta'|||x_h - y_h|) \partial_h^{\alpha_h} (z_+^3 z_-^1)(\tau, y) dy.
\]
Due to (2.12), we have
\[
|A_2(\tau, x)| \lesssim \frac{1}{\delta} \int_{-\delta}^{\delta} \int_{|x_h - y_h| \geq 1} \frac{1}{|x_h - y_h|^3} \partial_h^{\alpha_h} (z_+^3 z_-^1)(\tau, y) dy_h dy_3
\]
\[+ \frac{1}{\delta} \int_{-\delta}^{\delta} \int_{|x_h - y_h| \leq 2} \frac{1}{|x_h - y_h|^3} \partial_h^{\alpha_h} (z_+^3 z_-^1)(\tau, y) dy_h dy_3.
\]
Similarly to (4.13), for $|\alpha_h| \leq 2N - 4$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} A_{22} \|_{L^2_t L^2_x L^\infty} \lesssim \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_-))^{\frac{1}{2}} \cdot \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}}. \tag{4.14}
\]

Using (3.20), we have
\[
\langle u_+ (\tau, x_1) \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- (\tau, x_1) \rangle^{1+\sigma} A_{21} (\tau, x) \\
\approx \frac{1}{\delta} \int_{-\delta}^\delta \int_{|x_h - y_h| \geq 1} \frac{1}{|x_h - y_h|^{\frac{3}{2}(1-\sigma)}} \left( \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} |\partial_h^{\alpha_h} (z_- z_+) \rangle \right) (\tau, y) dy_h dy_3.
\]

Thanks to Young inequality and the fact $\sigma \in (0, \frac{1}{2})$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} A_{21} \|_{L^2_t L^2_x L^\infty} \lesssim \delta^{-1} \| \frac{1}{|x_h|^{1(1-\sigma)}} \|_{L^2_t (|x_h| \geq 1)} \|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} (z_- z_+) \|_{L^2_t L^2_x} \\
\lesssim \sum_{\beta_h \leq h} \delta^{\frac{1}{2}} \|\langle u_+ \rangle^{1+\sigma} \partial_h^{\alpha_h} - \beta_h z_- \|_{L^\infty_t L^2_x} \delta^{-\frac{1}{2}} \| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\beta_h} z_+ \|_{L^\infty_t L^2_x}.
\]

Then for $|\alpha_h| \leq 2N - 4$, we obtain that
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} A_{21} \|_{L^2_t L^2_x L^\infty} \lesssim \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_-))^{\frac{1}{2}} \cdot \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}}. \tag{4.15}
\]

Thanks to (4.14) and (4.15), for $|\alpha_h| \leq 2N - 4$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} A_{22} \|_{L^2_t L^2_x L^\infty} \lesssim \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_-))^{\frac{1}{2}} \cdot \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}}. \tag{4.16}
\]

Due to (4.13) and (4.16), for $|\alpha_h| \leq 2N - 4$, we obtain that
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} \|_{L^2_t L^2_x L^\infty} \\
\lesssim \left( \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_-))^{\frac{1}{2}} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E^{(k,0)}_+(z_-))^{\frac{1}{2}} \right) \\
\times \left( \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E^{(k,0)}_+(z_+))^{\frac{1}{2}} \right). \tag{4.17}
\]

Then by virtue of (4.17) and (4.10), for $|\alpha_h| \leq 2N - 5$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \rho^h_{+,1} (\cdot, x_3) \|_{L^2_t L^2_x} \\
\lesssim \delta \left( \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_-))^{\frac{1}{2}} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E^{(k,0)}_+(z_-))^{\frac{1}{2}} \right) \\
\times \left( \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E^{(k,0)}_+(z_+))^{\frac{1}{2}} \right). \tag{4.18}
\]

**Step 2.2.2. Estimate of** $\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \rho^h_{+,2} (\cdot, x_3) \|_{L^2_t L^2_x}$. By the definition of $\rho^h_{+,2}$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \rho^h_{+,2} (\cdot, x_3) \|_{L^2_t L^2_x} \lesssim \|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_h} (w^-_h \cdot \nabla h z^+_h) \|_{L^\infty_t L^2_x}.
\]

Similarly as (4.12), for $|\alpha_h| \leq N (\leq 2N - 4)$ and $N \in \mathbb{Z}_{\geq 5}$, we have
\[
\|\langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \rho^h_{+,2} (\cdot, x_3) \|_{L^2_t L^2_x} \lesssim \sup_{k \leq N} \|\langle E^{(k,0)}_-(w^-_h (\cdot, x_3)) \rangle^{\frac{1}{2}} \cdot \sum_{k+l \leq 2N} \delta^{l-\frac{1}{2}} (E^{(k,l)}_+(z_+))^{\frac{1}{2}}. \tag{4.19}
\]
Step 2.2.3. Estimate of $\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\$. By the definition of $\rho_{h(\alpha)}$ and $w^\alpha$, as well as the facts (4.9) and div $z_- = 0$, we have

$$
\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\|^2 \lesssim \delta \|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|\partial_3 (z^3 \partial_3 z^h)\|L^2_{L^2}\|^2.
$$

Following similar arguments as that for (4.12), we have

$$
\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|\nabla_h \cdot z^h \partial_3 z^h\|L^2_{L^2}\|^2 \lesssim \delta \|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|\nabla_h \cdot z^h \partial_3 z^h\|L^2_{L^2}\|^2 \lesssim \delta \|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|\nabla_h \cdot z^h \partial_3 z^h\|L^2_{L^2}\|^2.
$$

Therefore for $|\alpha| \leq N, N \in \mathbb{Z}_{\geq 5}$, we obtain that

$$
\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\|^2 \lesssim \delta \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{2}} (E_{-}^{(k,l)}(z_-))^{\frac{1}{2}} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E_{-}^{(k,0)}(z_3))^{\frac{1}{2}} \right).
$$

Step 2.2.4. Estimate of $\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\$. By the definition of $\rho_{h(\alpha)}$, we have

$$
\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\|^2 \lesssim \sum_{\beta h < \alpha} \|u_+\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\|^2 \cdot \|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|\nabla_h \partial_3 w^h\|L^2_{L^2}\|^2 \lesssim \sum_{k \leq |\alpha| + 1} \|E_{-}^{(k)}(z_-)\|^{\frac{1}{2}} \sum_{k \leq |\alpha|} \sup_{x \in (\delta, \delta)} (F_{+}^{(k)}(w^h(\cdot, x_3)))^{\frac{1}{2}}.
$$

Since $z_-^h(\cdot) = \frac{1}{2\delta} \int_{-\delta}^{\delta} z_-^h(\cdot, x_3) dx_3$, for any $k \in \mathbb{Z}_{\geq 0}$, we have

$$
(E_{-}^{(k)}(z_-^h))^{\frac{1}{2}} \lesssim \delta^{-\frac{1}{2}} (E_{-}^{(k,0)}(z_3))^{\frac{1}{2}}.
$$

Then for $|\alpha| \leq N, N \in \mathbb{Z}_{\geq 5}$, we obtain that

$$
\|u_+\|^{1+(\sigma)}\|u_-\|^{1+\sigma}\rho_{h(\alpha)}\|L^2_{L^2}\|^2 \lesssim \sum_{k, l \leq 2N} \delta^{-\frac{1}{2}} (E_{-}^{(k,l)}(z_-^h))^{\frac{1}{2}} \cdot \sum_{k \leq N} \sup_{x \in (-\delta, \delta)} (F_{+}^{(k)}(w_+^h(\cdot, x_3)))^{\frac{1}{2}}.
$$
Thanks to (4.19), (4.20) and (4.22), we obtain the estimate for \( \| u_+^{\frac{1}{2}(1+\sigma)} (u_-)^{1+\sigma} h^{(\alpha_h)} (\cdot, x_3) \|_{L^2_t L^2_h} \).

Then using (4.8), for \(|x_3| \leq N, N \in \mathbb{Z}_{\geq 5}\), we have

\[
\int_0^t \int_{\Sigma_{r, h}} \rho^{h^{(\alpha_h)}} (\cdot, x_3) \cdot (u_-) \cdot u_+^{h^{(\alpha_h)}} (\cdot, x_3) dx_d dt \\
\leq \delta \left( \sum_{k+l \leq 2N} \delta^{-\frac{1}{2}} (E_{+}^{(k,l)} (z_-))^\frac{1}{2} + \sum_{k \leq 2N-1} \delta^{-\frac{1}{2}} (E_{+}^{(k,0)} (z_-))^\frac{1}{2} + \sum_{k+l \leq 2N} \delta^{-\frac{1}{2}} (F_{+}^{(k,l)} (z_+))^\frac{1}{2} \right) \\
+ \left( \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} (E_{+}^{(k)} (w_h^h (\cdot, x_3))^\frac{1}{2} + \sum_{k+l \leq 2N} \delta^{-\frac{1}{2}} (F_{+}^{(k,l)} (z_+))^\frac{1}{2} \right) \\
+ \sum_{k+l \leq 2N} \delta^{-\frac{1}{2}} (F_{+}^{(k,l)} (z_+))^\frac{1}{2} \sup_{k \leq N} \left( F_{+}^{(k)} (w_h^h (\cdot, x_3))^\frac{1}{2} + F_{+}^{(k)} (w_h^h (\cdot, x_3))^\frac{1}{2} \right).
\]

**Step 3.** The _a priori_ estimate for \( w_{\pm}^h \). Thanks to (1.13), (4.10), (4.17), (4.21) and (4.22), for \(|x_3| \leq N, N \in \mathbb{Z}_{\geq 5}\), we obtain that

\[
\sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right) \\
\leq \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} E_{+}^{(k)} (w_h^h (\cdot, x_3)) + \| E(0) \|_{H^0} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( F_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right) \\
+ \| E(0) \|_{H^0} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right).
\]

Similar estimate holds for \( w_{\pm}^h \). Then using Hölder inequality, we have

\[
\sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right) \\
\leq C_1 \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} E_{+}^{(k)} (w_h^h (\cdot, x_3)) + C_1 \| E(0) \|_{H^0}^2 \\
+ C_1 \| E(0) \|_{H^0}^2 \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right).
\]

Taking \( \varepsilon_1 \leq \varepsilon_0 \) small enough, for \( E(0) \leq \varepsilon_1^2 \), we have

\[
\sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{+}^{(k)} (w_h^h (\cdot, x_3)) + F_{+}^{(k)} (w_h^h (\cdot, x_3)) \right) \\
\leq 2C_1 \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} E_{+}^{(k)} (w_h^h (\cdot, x_3)) + 2C_1 \| E(0) \|_{H^0}^2 \varepsilon_1.
\]

Since \( \text{div} \ w_{\pm} = 0, w_{\pm}^3 |_{x_3 = \pm \delta} = 0 \) and \( w_{\pm}^3 |_{x_3 = \pm \delta} = 0 \), it holds

\[
w_{\pm}^3 (t, x) = \int_{-\delta}^{x_3} \partial_3 w_{\pm}^3 (t, x_h, y_3) dy_3 = - \int_{-\delta}^{x_3} \nabla_h \cdot w_{\pm}^3 (t, x_h, y_3) dy_3,
\]
which together with \(4.24\) implies that
\[
\delta^{-2} \sum_{+,-} \sum_{k \leq N-1} \sup_{x_3 \in (-\delta, \delta)} \left( E_{\pm,h}^{(k)}(w_{\pm}^3(\cdot, x_3)) + F_{\pm,h}^{(k)}(w_{\pm}^3(\cdot, x_3)) \right) 
\leq C \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-\delta, \delta)} \left( E_{\pm,h}^{(k)}(w_{\pm,0}^3(\cdot, x_3)) + C\delta^2 \epsilon_1^4 \right) .
\]

Thanks to \(4.24\) and \(4.25\), we arrive at \(4.23\). Then it ends the proof of Theorem 1.2.

5. PROOF OF THEOREM 1.3

In this section, we shall use Theorem 1.1 and Theorem 1.2 to prove Theorem 1.3.

**Proof of Theorem 1.3.** Since \(z_{\pm(\delta)}^h = z_{\pm(\delta)}^h + u_{\pm(\delta)}^h\) and \(z_{\pm(\delta)}^3 = w_{\pm(\delta)}^3\), the investigation of the asymptotics from 3D MHD to 2D MHD can be reduced to prove
\[
\lim_{\delta \to 0} w_{\pm(\delta)}^h(t, x_h, x_3) = 0, \quad \text{in } H^N(\mathbb{R}^2), \quad \text{uniform for } t > 0, \quad x_3 \in (-1, 1),
\]
\[
\lim_{\delta \to 0} v_{\pm(\delta)}^3(t, x_h, x_3) = 0, \quad \text{in } H^{N-1}(\mathbb{R}^2), \quad \text{uniform for } t > 0, \quad x_3 \in (-1, 1),
\]
and
\[
\lim_{\delta \to 0} z_{\pm(\delta)}^h(t, x_h) = z_{\pm(0)}^h(t, x_h), \quad \text{in } H^N(\mathbb{R}^2) \quad \text{uniform for } t > 0.
\]

We split the proof into several steps.

**Step 1. Uniform estimate of \(w_{\pm(\delta)}^3\) and the proof of \(4.3\).** Thanks to \(1.13\) and \(1.28\), for any \(\delta \in (0, 1]\), we obtain that
\[
\mathcal{E}_\delta(t) \leq C\mathcal{E}_\delta(0)
\]
Since
\[
z_{\pm(\delta)}^h,0(x_h) = \frac{1}{2} \int_{-1}^1 z_{\pm(\delta),0}(x_h, x_3) dx_3, \quad w_{\pm(\delta),0}^3 = z_{\pm(\delta),0}^h - z_{\pm(\delta),0}^3,
\]
we obtain that
\[
E_{\pm,h}^{(k)}(z_{\pm(\delta),0}^h - z_{\pm(0),0}^h,0) \lesssim \sup_{x_3 \in (-1, 1)} E_{\pm,h}^{(k)}(z_{\pm(\delta),0}(\cdot, x_3) - z_{\pm(0),0}(\cdot)),
\]
\[
\sup_{x_3 \in (-1, 1)} E_{\pm,h}^{(k)}(w_{\pm(\delta),0}^3(\cdot, x_3)) \lesssim \sup_{x_3 \in (-1, 1)} E_{\pm,h}^{(k)}(z_{\pm(\delta),0}(\cdot, x_3) - z_{\pm(0),0}(\cdot)).
\]

Thanks to \(1.23\), \(1.8\) and \(5.4\), for \(\delta \in (0, 1]\), we obtain that
\[
\sum_{+,-} \sup_{x_3 \in (-1, 1)} \left( \sum_{k \leq N} (E_{\pm,h}^{(k)}(w_{\pm(\delta)}^3(\cdot, x_3)) + F_{\pm,h}^{(k)}(w_{\pm(\delta)}^3(\cdot, x_3)))
\right)
\leq \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-1, 1)} (E_{\pm,h}^{(k)}(w_{\pm(\delta)}^3(\cdot, x_3)) + F_{\pm,h}^{(k)}(w_{\pm(\delta)}^3(\cdot, x_3)))
\lesssim \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-1, 1)} (E_{\pm,h}^{(k)}(w_{\pm(\delta)}^3(\cdot, x_3)) + \delta^2 \epsilon_1^4)
\]
\[
\lesssim \sum_{+,-} \sum_{k \leq N} \sup_{x_3 \in (-1, 1)} (E_{\pm,h}^{(k)}(z_{\pm(\delta),0}(\cdot, x_3) - z_{\pm(0),0}(\cdot))) + \delta^2 \epsilon_1^4.
\]

Since
\[
\sum_{k \leq N} \sup_{x_3 \in (-1, 1)} (E_{\pm,h}^{(k)}(z_{\pm(\delta),0}(\cdot, x_3) - z_{\pm(0),0}(\cdot))) \lesssim \sum_{k \leq N+1} E_{\pm}^{(k)}(z_{\pm(\delta),0} - z_{\pm(0),0}),
\]
using \(1.32\) and \(5.5\), we arrive at \(5.1\).
Step 2. Uniform estimate of $z^h_{\pm(\delta)}$. Since
\[ z^h_{\pm(\delta)}(t, x_h) = \frac{1}{2} \int_{-1}^{1} z^h_{\pm(\delta)}(t, x_h, x_3) dx_3, \]
we deduce from (1.31) and (5.3) that
\[ \sum_{k \leq N} (E^{(k)}_{\pm, h}(z^h_{\pm(\delta)}) + F^{(k)}_{\pm, h}(z^h_{\pm(\delta)})) \leq \sum_{k \leq N} (E^{(k)}_{\pm}(z^h_{\pm(\delta)}) + F^{(k)}_{\pm}(z^h_{\pm(\delta)})) \leq z^2_1. \quad (5.6) \]

Step 3. Convergence of the sequence \( \{z^h_{\pm(\delta)}\}_{0 < \delta \leq 1} \).

Step 3.1. Estimate of the difference $z^h_{\pm(\delta)} - z^h_{\pm(\delta')}$.

By setting \( z^h_{\pm(\delta)} \) \( \defeq \) \( z^h_{\pm(\delta)} \), \( p^{(\delta, \delta')} \) \( \defeq M_1p_1(\delta) - M_1p_1(\delta') \), we deduce from (1.24) and (5.4) that
\[ \sum_{k \leq N}(E^{(k)}_{\pm, h}(z^h_{\delta}) + F^{(k)}_{\pm, h}(z^h_{\delta})) \leq \sum_{k \leq N}(E^{(k)}_{\pm}(z^h_{\delta}) + F^{(k)}_{\pm}(z^h_{\delta})) \leq z^2_1. \quad (5.6) \]

Step 3.2. Uniform estimate of $\bar{\alpha}$.

Step 3.3. Estimate of $\bar{\alpha}$.

Now, for $\alpha_\delta \in (Z_{\geq 0})^2$, applying $\partial_\beta^{\alpha_\delta}$ to both sides of the first equation in (5.6), we have
\[ \partial_t(\partial_\beta^{\alpha_\delta} \bar{z}^h_{\pm(\delta)}) = -\Delta_h \bar{z}^h_{\pm(\delta)} - \bar{z}^h_{\pm(\delta')} + \bar{R}(t, x_h). \quad (5.9) \]

Moreover, it is easy to check that
\[ \bar{R}_{(\delta)}(t, x_h) = \partial_\alpha \partial_\beta M_1(z^h_{\pm(\delta)} z^h_{\pm(\delta')}) = \partial_\alpha \partial_\beta (z^h_{\pm(\delta)} z^h_{\pm(\delta')}). \]

Using the Green function on $\mathbb{R}^2$, we have (up to a constant)
\[ M_1p_1(\delta, t, x_h) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x_h - y_h| \cdot (\partial_\alpha \partial_\beta z^h_{\pm(\delta)} + \bar{R}(\delta))(t, y_h) dy_h. \quad (5.8) \]

Then we obtain that
\[ p^{(\delta, \delta')} (t, x_h) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x_h - y_h| \cdot (\partial_\alpha \partial_\beta z^h_{\pm(\delta)}) (t, y_h) dy_h \]
\[ -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x_h - y_h| \cdot (\partial_\alpha \partial_\beta z^h_{\pm(\delta')}) (t, y_h) dy_h \]
\[ = p_{11}(t, x_h) + p_{12}(t, x_h) - p_{21}(t, x_h) + p_{22}(t, x_h). \quad (5.9) \]

Now, for $\alpha_\delta \in (Z_{\geq 0})^2$, applying $\partial_\beta^{\alpha_\delta}$ to both sides of the first equation in (5.6), we have
\[ \partial_t(\partial_\beta^{\alpha_\delta} \bar{z}^h_{\pm(\delta)}) = -\Delta_h \bar{z}^h_{\pm(\delta)} - \bar{z}^h_{\pm(\delta')} + \bar{R}(t, x_h). \quad (5.10) \]
where
\[ \rho_+^{(\alpha_h)} = -\partial_h^{\alpha_h} (\bar{z}_h^{(\delta,0)} \cdot \nabla_h \bar{z}_+^{(\delta,0)}) + \bar{z}_h^{(\delta,0)} \cdot \nabla_h \partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)} - \partial_h^{\alpha_h} (\bar{z}_h^{(\delta,0)} \cdot \nabla_h \bar{z}_+^{(\delta,0)}) - \partial_h^{\alpha_h} (R_0 - R_0^{(\delta,0)}). \]

Since \( \text{div}_h \bar{z}_-^{(\delta)} = \nabla_h \cdot \bar{z}_-^{(\delta)} = 0 \), applying Proposition 4.1 to (5.10), we have
\[
\sup_{0 \leq \tau \leq t} \int_{\Sigma_{\tau, h}} \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau + \int_0^t \int_{\Sigma_{\tau, h}} \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau \\
+ \int_0^t \int_{\Sigma_{\tau, h}} \nabla_h \partial_h^{\alpha_h} \rho^{(\delta,0)} \cdot \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau \\
+ \int_0^t \int_{\Sigma_{\tau, h}} \rho_+^{(\alpha_h)} \cdot \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau. 
\]

**Step 3.2. Estimates of the nonlinear terms on the r.h.s of (5.11)**

(i) For the second term on the r.h.s of (5.11), we have
\[
\int_0^t \int_{\Sigma_{\tau, h}} \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau \leq \sum_{k \leq \delta} (E_{k, \tau, h}^{(\alpha_h)} (\bar{z}_+^{(\delta,0)}))^k : F_{k, \tau, h}^{(\alpha_h)} (\bar{z}_+^{(\delta,0)}). 
\]

(ii) For the third term involving \( p^{(\delta,0)} \) on the r.h.s of (5.11), we only consider the case \( |\alpha_h| \geq 1 \). In fact, notice that for \( |\alpha_h| = 0 \)
\[
\left| \int_0^t \int_{\Sigma_{\tau, h}} \nabla_h p^{(\delta,0)} \cdot \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau \right| \leq \int_0^t \int_{\Sigma_{\tau, h}} |\nabla_h p^{(\delta,0)}| \cdot \langle u_- \rangle^{(\alpha_h)} |\bar{z}_+^{(\delta,0)}| dx \, d\tau.
\]

Then the estimate in the case of \( |\alpha_h| = 0 \) can be reduced to that in the case of \( |\alpha_h| \geq 1 \).

Integrating by parts and using the condition \( \text{div}_h \bar{z}_+^{(\delta,0)} = 0 \), we obtain that
\[
\int_0^t \int_{\Sigma_{\tau, h}} \nabla_h p^{(\delta,0)} \cdot \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau = \int_0^t \int_{\Sigma_{\tau, h}} \partial_h^{\alpha_h} p^{(\delta,0)} \cdot \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} \bar{z}_+^{(\delta,0)}| dx \, d\tau \\
\leq \| \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} p^{(\delta,0)}| \|_{L^2_h} \cdot \| F_{k, \tau, h}^{(\alpha_h)} (\bar{z}_+^{(\delta,0)}) \|_{L^2_h}^{\frac{1}{2}}. 
\]

We only need to bound \( \| \langle u_- \rangle^{(\alpha_h)} |\partial_h^{\alpha_h} p^{(\delta,0)}| \|_{L^2_h} \), and we split \( p^{(\delta,0)} \) into four parts. We first deal with the term involving \( p_{11}^{(\delta,0)} \). Let \( \theta(r) \) be the cut-off function defined in Step 2 of the proof to Proposition 3.2. Using
\( \text{div}_h z_+^{h(\delta, \sigma)} = 0 \), integrating by parts, we have

\[
\partial^\alpha_h p_{11}^{(\delta, \sigma)}(\tau, x_h) = \frac{1}{2\pi} \int_{R^2} \partial_h \log |x_h - y_h| \cdot \theta(|x_h - y_h|) \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)}) (\tau, y_h) dy_h
\]

\[
+ \frac{1}{2\pi} \int_{R^2} \partial_h \log |x_h - y_h| \cdot (1 - \theta(|x_h - y_h|)) \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)}) (\tau, y_h) dy_h
\]

\[
= I_1(\tau, x_h) + I_2(\tau, x_h).
\]

For \( I_1(\tau, x_h) \), using (3.18), we have

\[
\|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} I_1\|_{L^1_h(L^2_h)} \lesssim \int_{|x_h - y_h| \leq 2} \frac{1}{|x_h - y_h|} \|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)})\| (\tau, y_h) dy_h \|_{L^1_h(L^2_h)}
\]

Young \( \lesssim \frac{1}{|x_h|} \sum_{|\beta_\alpha| \leq \alpha_h} \|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} \partial_{\beta_\alpha} \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)})\|_{L^1_h(L^2_h)}
\]

\[
\lesssim \sum_{\beta_\alpha \leq \alpha_h} \|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} \partial_{\beta_\alpha} (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)})\|_{L^1_h(L^2_h)} \sum_{k \leq |\alpha_h| + 3} \left( \frac{E(k)^{\alpha_h}(z_+^{\beta(\delta, \sigma)})}{|x_h|} \right)^{\frac{1}{2}}.
\]

(5.14)

For \( I_2(\tau, x_h) \), using the condition \( \text{div}_h z_+^{h(\delta, \sigma)} = 0 \) and integrating by parts, we have

\[
I_2(\tau, x_h) = \frac{1}{2\pi} \int_{R^2} \partial^\alpha_h \partial_\alpha \log |x_h - y_h| \cdot (1 - \theta(|x_h - y_h|)) \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)}) (\tau, y_h) dy_h
\]

\[
- \frac{1}{2\pi} \int_{R^2} (\partial^\alpha_h \partial_\alpha \log |x_h - y_h| \cdot \partial_\beta \theta(|x_h - y_h|) + \partial_\beta \partial_\alpha \log |x_h - y_h| \cdot \partial^\alpha_h \theta(|x_h - y_h|)
\]

\[
+ \partial_\alpha \log |x_h - y_h| \cdot \partial^\alpha_h \partial_\beta \theta(|x_h - y_h|) \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)}) (\tau, y_h) dy_h
\]

\[
def I_{21}(\tau, x_h) + I_{22}(\tau, x_h),
\]

where \( \gamma_h \leq \alpha_h \) and \( |\gamma_h| = 1 \).

For \( I_{22}(\tau, x_h) \), we have

\[
|I_{22}| \lesssim \int_{|x_h - y_h| \leq 2} \frac{1}{|x_h - y_h|} \cdot \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)}) (\tau, y_h) dy_h.
\]

By the similarly derivation as that for \( I_1 \), we have

\[
\|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} I_{22}\|_{L^1_h(L^2_h)} \lesssim \sum_{k \leq |\alpha_h| - 1} \left( \frac{E(k)^{\alpha_h}(z_+^{\beta(\delta, \sigma)})}{|x_h|} \right)^{\frac{1}{2}} \sum_{k \leq |\alpha_h| + 1} \left( \frac{E(k)^{\alpha_h}(z_+^{\beta(\delta, \sigma)})}{|x_h|} \right)^{\frac{1}{2}}.
\]

(5.15)

For \( I_{21}(\tau, x_h) \), using (3.20), we have

\[
\|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} I_{21}\|_{L^1_h(L^2_h)} \lesssim \int_{|x_h - y_h| \geq 1} \frac{1}{|x_h - y_h|^{\frac{1}{2}(1-\sigma)}} \|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)})\| (\tau, y_h) dy_h \|_{L^1_h(L^2_h)}
\]

Young \( \lesssim \frac{1}{|x_h|^{\frac{1}{2}(1-\sigma)}} \|(u_+)^{\frac{1}{2}(1+\sigma)}(u_-)^{1+\sigma} \partial^\alpha_h (z_+^{\beta(\delta, \sigma)} \partial_\beta z_+^{\alpha(\delta)})\|_{L^2_h(L^2_h)}.
\]
Since $\sigma \in (0, \frac{1}{2})$, we have $\| \frac{1}{|x|^{2(1-\sigma)}} \|_{L^2(|x| \geq 1)} < \infty$. Then we obtain

$$
\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} I_{21} \|_{L^2_L^2} \lesssim \sum_{k \leq |\alpha_h| - 1} \left( E_{-h}^k (\bar{z}^{h(\delta, \delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| - 1} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}}.
$$

(5.16)

Thanks to (5.14), (5.15) and (5.19), we have

$$
\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_1} p_{11}^{(\delta, \delta')} \|_{L^2_L^2} \lesssim \sum_{k \leq |\alpha_h|} \left( E_{-h}^k (\bar{z}^{h(\delta, \delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| + 3} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}}.
$$

(5.17)

Since $z^h_{+}(\delta) = \frac{1}{2} \int_{-1}^1 z^h_{+}(\delta)(x, x_3) dx_3$, we have

$$
\left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}} \leq \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}}.
$$

Then we have

$$
\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_1} p_{11}^{(\delta, \delta')} \|_{L^2_L^2} \lesssim \sum_{k \leq |\alpha_h|} \left( E_{-h}^k (\bar{z}^{h(\delta, \delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| + 3} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}}.
$$

(5.18)

For term involving $p_{2(\delta)}$ and $p_{2(\delta')}$, since div $w_{-}(\delta) = 0$ and $w_{\delta}(x_3 = \pm 1) = 0$, we have

$$
\bar{R}(\delta) = \nabla_h \cdot \frac{1}{2} \int_{-1}^1 \sum_{i=1}^3 \partial_i (w^h_{-}(\delta)(x_3)) dx_3 = \nabla_h \cdot \sum_{\alpha=1}^2 \partial_\alpha (h_{\delta}^{(\alpha)} (w^h_{-}(\delta)(x_3))).
$$

(5.19)

Following the similar argument applied to (5.17) and using Sobolev inequality, we have

$$
\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_1} p_{2(\delta)} \|_{L^2_L^2} \lesssim \sum_{k \leq |\alpha_h| + 3} \sup_{x_3 \in (-1, 1)} \left( E_{-h}^k (w^h_{-}(\delta)(x_3)) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| + 4} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}}.
$$

(5.20)

Similar estimate holds for $\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_1} p_{2(\delta')} \|_{L^2_L^2}$.

Due to (5.17), (5.18) and (5.20), we obtain the estimate of $\| \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \langle u_- \rangle^{1+\sigma} \partial_h^{\alpha_1} p^{(\delta, \delta')} \|_{L^2_L^2}$ for $|\alpha_h| \geq 1$. Similar estimate holds for $|\alpha_h| = 0$. Then using (5.13), for $|\alpha_h| \geq 0$, we have

$$
\left| \int_0^t \int_{\Sigma_r} \nabla_h \partial_h^{\alpha_1} p^{(\delta, \delta')} \cdot \langle u_- \rangle \partial_h^{\alpha_1} \bar{z}^{h(\delta, \delta')} dx_h dt \right|
\leq \left( \sum_{k \leq \max \{|\alpha_h|, 1\}} \left( E_{-h}^k (\bar{z}^{h(\delta, \delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq \max \{|\alpha_h|, 1\}} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}} + \sum_{k \leq \max \{|\alpha_h|, 1\}} \left( E_{-h}^k (\bar{z}^{h(\delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq \max \{|\alpha_h|, 1\}} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}} + \sum_{k \leq \max \{|\alpha_h|, 1\}} \sum_{x_3 \in (-1, 1)} \sup_{x_3 \in (-1, 1)} \left( E_{-h}^k (w^h_{-}(\delta)(x_3)) \right)^{\frac{1}{2}} \cdot \sum_{k \leq \max \{|\alpha_h|, 1\}} \left( F_{+h}^k (\bar{z}^{h(\delta)}) \right)^{\frac{1}{2}} \right).
$$

(5.21)
(iii) For the fourth term on the r.h.s of (6.11), integrating by parts, for $|\alpha_h| \geq 1$, we have
\[
\int W_{[\alpha_h]} \nabla_h \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial^h \zeta^{1(\delta, \delta')} \, dx \, dt
\]
\[
= -\int W_{[\alpha_h]} \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot \partial_th \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt - \int \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt.
\]
Following the similar argument applied in Step 2.2 of the proof of Proposition 3.2, for $|\alpha_h| \geq 1$, we have
\[
\int W_{[\alpha_h]} \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt \lesssim \int W_{\alpha_h} \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt.
\]
Similar estimate holds for $|\alpha_h| = 0$. Then for $|\alpha_h| \geq 0$, we have
\[
\int W_{[\alpha_h]} \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt \lesssim \text{r.h.s of (5.21)}.
\]
(iv) For the last term on the r.h.s of (5.11), we have
\[
\int_{\Sigma, +} \frac{1}{\langle u_+ \rangle^{(1+\alpha)} t} \int W_{[\alpha_h]} \nabla_h \partial_h \rho_h^\alpha p^h(\delta, \delta') \cdot (u_-)^{(1+\alpha)} \partial_h \zeta^{1(\delta, \delta')} \, dx \, dt \lesssim \langle u_+ \rangle^{(1+\alpha)}(u_-)^{(1+\alpha)} \rho_h^{(\alpha_h)} \|F_{\rho_h^{(\alpha_h)}}(z^h(\delta, \delta'))\|^\delta.
\]
We only need to estimate $\|\langle u_+ \rangle^{(1+\alpha)}(u_-)^{(1+\alpha)} \rho_h^{(\alpha_h)} \|L^2_t L^2_h$.
For $\rho_h^{(\alpha_h)}$, using conditions that $\text{div } w^h(\delta) = 0$, $w^h_{3}(\delta) x_3 = 0 = 0$ and $z^h(\delta) = z^h_+ + w^h(\delta)$, and integrating by parts, we have
\[
R_+^{(\delta)} = M_1 \left( w^h_-(\delta) \cdot \nabla_h z^h_+ + \frac{1}{2} \int_{l=1} \nabla_h w^h_+ \partial_3 w^h_+ \, dx \right)
= M_1 \left( w^h_-(\delta) \cdot \nabla_h z^h_+ + \nabla_h w^h_+ \right).
\]
Using Sobolev inequality, we also have
\[
\|\langle u_+ \rangle^{(1+\alpha)}(u_-)^{(1+\alpha)} \rho_h^{(\alpha_h)} \|L^2_t L^2_h \lesssim \sum_{\delta=\delta, \delta'} \left( \sup_{k \leq |\alpha_h| + 2} \left( E_{-h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2} \cdot \sum_{k \leq |\alpha_h| + 2} \left( E_{+h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2} \right)
\]
\[
+ \sum_{k \leq |\alpha_h| + 3} \sup_{(\delta, \delta')} \left( E_{-h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2} \cdot \sum_{k \leq |\alpha_h| + 3} \sup_{(\delta, \delta')} \left( E_{+h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2}.
\]
Noticing that $z^h_+ (\delta) = \frac{1}{2} \int_{l=1} \nabla_h w^h_+ \partial_3 w^h_+ \, dx$ and $w^h_{\pm} (\delta) = z^h_{\pm} (\delta) - z^h_{\pm} (\delta)$. Then it holds
\[
\left( E_{+h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2} \lesssim \left( E_{+h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2}, \quad \left( E_{-h}^{(k)}(w^h_+(\delta), x_3) \right)^\frac{1}{2} \lesssim \left( E_{+h}^{(k)}(z^h_+ \delta, \delta') \right)^\frac{1}{2}.
\]
Similar estimates hold for $\left( E_{+h}^{(k)}(z^h_{\pm} \delta, \delta') \right)^\frac{1}{2}$ and $\left( E_{+h}^{(k)}(w^h_{\pm} \delta, x_3) \right)^\frac{1}{2}$.
Thanks to (5.23), (5.24), (5.25), (5.26) and (5.27), we have

$$\int_0^t \int_{\Sigma_{t,h}} |\rho_{+}^{(\alpha)}| \cdot |u_{-}|^{2(1+\sigma)} |\partial_h^{\alpha} z_{+}^{h(\delta,\delta')}| dx_h dt$$

$$\lesssim \left( \sum_{k \leq |\alpha_h| + 3} \left( E_{-}^{(k)} (z_{+}^{h(\delta,\delta')} \right) \right)^{\frac{1}{2}} + \sum_{k \leq |\alpha_h| + 3} \left( F_{-}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)^{\frac{1}{2}} \cdot \sum_{\delta \leq |\alpha_h| + 3} \left( F_{+}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)^{\frac{1}{2}}$$

$$+ \sum_{\delta = \delta' \leq |\alpha_h|, k \leq |\alpha_h|, x \in (-1,1)} \sup_{h,\bar{\alpha}} \left( E_{-}^{(k)} (w_{+}^{h(\delta,\delta')} \cdot \bar{\alpha}, x_3) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| + 3} \left( F_{+}^{(k)} (z_{-}^{h(\delta,\delta')}) \right)^{\frac{1}{2}}$$

$$+ \sum_{k \leq |\alpha_h| + 3} \left( E_{-}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)^{\frac{1}{2}} \cdot \sum_{k \leq |\alpha_h| + 3} \left( F_{+}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)^{\frac{1}{2}}.$$

### Step 3.3. Convergence of the sequence \( \{z_{\pm}^{h(\delta,\delta')}\}_{0 < \delta \leq 1} \)

Thanks to (5.11), (5.12), (5.21), (5.22) and (5.28), using (5.27) and (5.6), we obtain that

$$\sum_{k \leq N} \left( E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) + F_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)$$

$$\leq C_1 \sum_{k \leq N} \left( E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) + F_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right) + C_1 \epsilon \sum_{k \leq N} \left( E_{-,h}^{(k)} (w_{-}^{h(\delta,\delta')} \cdot \bar{\alpha}, x_3) + F_{-,h}^{(k)} (w_{+}^{h(\delta,\delta')} \cdot \bar{\alpha}, x_3) \right) \cdot \sum_{k \leq N} \left( F_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right)^{\frac{1}{2}}.$$

Similar estimate holds for \( z_{-,h}^{h(\delta,\delta')} \).

Taking \( \epsilon_1 \) sufficiently small, using Hölder inequality, we deduce from (5.24) that

$$\sum_{k \leq N} \left( E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) + F_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right) \leq 2C_1 \sum_{k \leq N} E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')})$$

$$+ C_2 \epsilon \sum_{\delta = \delta', k \leq N} \sup_{x \in (-1,1)} \left( E_{+,h}^{(k)} (w_{-}^{h(\delta,\delta')} \cdot \bar{\alpha}) + F_{+,h}^{(k)} (w_{+}^{h(\delta,\delta')} \cdot \bar{\alpha}, x_3) \right).$$

Thanks to (5.5) and (5.30), we have

$$\sum_{k \leq N} \left( E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) + F_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right) \leq 2C_1 \sum_{k \leq N} E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')})$$

$$+ C_3 \epsilon \sum_{k \leq N} \left( \sup_{x \in (-1,1)} E_{+,h}^{(k)} (z_{+}^{h(\delta,\delta')}) \right) + \delta^2 \epsilon_1^4.$$

By virtue of (1.32), (5.4) and (5.31), we obtain that \( \{z_{\pm}^{h(\delta,\delta')} (t, x_h)\}_{0 < \delta \leq 1} \) is a Cauchy sequence in \( H^N (\mathbb{R}^2) \) for \( t > 0 \).

### Step 4. Limit of the sequence \( \{z_{\pm}^{h(\delta,\delta')}\}_{0 < \delta \leq 1} \).

Since \( \{z_{\pm}^{h(\delta,\delta')} (t, x_h)\}_{0 < \delta \leq 1} \) is a Cauchy sequence in \( H^N (\mathbb{R}^2) \), there exists a unique \( z_{\pm}^{h(\delta,\delta')} (t, x_h) \) such that

$$\lim_{\delta \to 0} z_{\pm}^{h(\delta,\delta')} (t, x_h) = z_{\pm}^{h(\delta,\delta')} (t, x_h), \quad \text{in} \quad H^N (\mathbb{R}^2).$$

(5.32)
Taking $\delta' \to 0$ in (5.31), using (5.1) and (5.3), we obtain that
\[
\sum_{+, -, k \leq N} \left( E_{\pm, k}^{(k)}(z_{\pm}(\delta) - z_{\pm}(0)) + F_{\pm, k}^{(k)}(z_{\pm}(\delta) - z_{\pm}(0)) \right) 
\leq C_4 \sum_{+, -, k \leq N} \sup_{x_3 \in (-1, 1)} \left( E_{\pm, k}^{(k)}(z_{\pm}(\delta), 0, x_3) - z_{\pm}(0, 0, \cdot) \right) + C_4 \delta^2 \varepsilon_1^2.
\]  
(5.33)

By virtue of (5.32) and (5.33) and by using Sobolev inequality, we arrive at (5.2).

**Step 5. Derivation of the limit system.** For system (1.24) involving $z^h_\pm(\delta)$, we shall take $\delta \to 0$ and derive the limit system. From (1.24), we first have
\[
\partial_t z^h_{\pm}(\delta) = \mp \partial_z z^h_{\pm}(\delta) \cdot \nabla_h z^h_{\pm}(\delta) - \nabla_h (M_1 p(\delta)) - M_1 (u_{(\cdot)} \cdot \nabla z^h_{\pm}(\delta))
\]

Thanks to (5.8) and the derivation of $\|u_{(\cdot)}\|^2 \mp \|u_{(\cdot)}\|^2 \delta_{\pm}^{h,\delta'} \|_{L^2_e L^2_t}$ (see (5.17), (5.18) and (5.20)), by (5.1), (5.2) and (5.6), we obtain that
\[
\{ \nabla_h (M_1 p(\delta)) \}_{0 < \delta \leq 1} \text{ is a Cauchy sequence in } H^{N-1}(\mathbb{R}^2).
\]
Then we have
\[
\{ \partial_t z^h_{\pm}(\delta) \}_{0 < \delta \leq 1} \text{ is a Cauchy sequence in } H^{N-1}(\mathbb{R}^2).
\]
Moreover, by virtue of (5.3), we have
\[
\sum_{k \leq N} \left( E_{\pm, k}^{(k)}(\partial_t z^h_{\pm}(\delta)) + F_{\pm, k}^{(k)}(\partial_t z^h_{\pm}(\delta)) \right) \preceq \varepsilon_1^2.
\]  
(5.34)

Then by the uniqueness of limit, we have
\[
\lim_{\delta \to 0} \partial_t z^h_{\pm}(\delta) = \partial_t z^h_{\pm}(0), \text{ in } H^{N-1}(\mathbb{R}^2).
\]  
(5.35)

Due to (5.1), (5.2) and (5.3), taking $\delta \to 0$, we deduce from (1.24) in the sense of $H^{N-1}(\mathbb{R}^2)$, $(z^h_{\pm}(0, t), x_h, z^h_{\pm}(0, t), x_h)$ satisfy
\[
\partial_t z^h_{\pm}(0) - \partial_z z^h_{\pm}(0) \cdot \nabla_h z^h_{\pm}(0) = -\nabla_h p(0), \text{ in } \mathbb{R}^2 \times \mathbb{R}^+
\]
\[
\partial_t z^h_{\pm}(0) + \partial_z z^h_{\pm}(0) \cdot \nabla_h z^h_{\pm}(0) = -\nabla_h p(0),
\]
(5.36)
\[
\nabla_h \cdot z^h_{\pm}(0) = 0, \quad \nabla_h \cdot z^h_{\pm}(0) = 0,
\]
where
\[
p(0)(t, x_h) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x_h - y_h| \cdot (\partial_\alpha z^h_{\pm}(0) \partial_\beta z^h_{\pm}(0))(t, y_h) dy_h.
\]
We remark that (5.36) is exactly the two dimensional ideal MHD system.

Now we only need to prove that $z^h_{\pm}(0)$ is continuous with respect to $t$ and verifies the initial data
\[
z^h_{\pm}(0)|_{t=0} = z^h_{\pm}(0, 0).
\]  
(5.37)

Thanks to (5.2), (5.34) and (5.37), we have
\[
\sum_{k \leq N} \left( E_{\pm, k}^{(k)}(z^h_{\pm}(0)) + F_{\pm, k}^{(k)}(z^h_{\pm}(0)) \right) + \sum_{k \leq N-1} \left( E_{\pm, k}^{(k)}(\partial_t z^h_{\pm}(0)) + F_{\pm, k}^{(k)}(\partial_t z^h_{\pm}(0)) \right) \preceq \varepsilon_1^2.
\]

We obtain that
\[
z^h_{\pm}(0) \in C([0, \infty); H^{N-1}(\mathbb{R}^2)),
\]
and then (5.37) holds.
It remains to derive the a priori estimate (1.34) for the limit system (5.36)-(5.37). Setting \( z^h = 0 \) in Step 2, by virtue of (5.30), (5.1) and the assumption of the initial data, we obtain that

\[
\sum_{+, -} \sum_{k \leq N} \left( E^h_{\pm, h}(z^h_{\pm}(0)) + F^h_{\pm, h}(z^h_{\pm}(0)) \right) \leq C \sum_{+, -} \sum_{k \leq N} E^h_{\pm, h}(z^h_{\pm}(0, \eta)).
\]

This is exactly (1.34).

Step 6. Asymptotics from 3D MHD to 2D MHD. Thanks to (5.5) and (5.33), we have

\[
\sum_{+, -} \sum_{k \leq N} \sup_{x_3 \in (-1, 1)} \left( E^h_{\pm, h}(z^h_{\pm}(\cdot, x_3) - z^h_{\pm}(0)) \right) + F^h_{\pm, h}(z^h_{\pm}(\cdot, x_3) - z^h_{\pm}(0)) \right) \leq C \delta \sum_{+, -} \sum_{k \leq N} E^h_{\pm, h}(z^h_{\pm}(\cdot, x_3) - z^h_{\pm}(0)) + C \delta^2 \varepsilon^4.
\]

By virtue of (1.32), (5.38) and Sobolev inequality, we get (1.33). Then Theorem 1.3 is eventually proved.

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