Abstract

We describe a $\mathcal{N}=2$ supersymmetric extension of the nonrelativistic (2+1)-dimensional model describing particles on the noncommutative plane with scalar (electric) and vector (magnetic) interactions. First, we employ the $N=2$ superfield technique and show that in the presence of a scalar $\mathcal{N}=2$ superpotential the magnetic interaction is implied by the presence of noncommutativity of position variables. Further, by expressing the supersymmetric Hamiltonian as a bilinear in $N=2$ supercharges we obtain two supersymmetric models with electromagnetic interactions and two different noncanonical symplectic structures describing noncommutativity. We show that both models are related to each other by a noncanonical transformation of phase space variables supplemented by a Seiberg-Witten map of the gauge potentials.
1 Introduction

A model of nonrelativistic classical mechanics in 2+1 dimensions with the following noncommutativity of position coordinates \((i, j = 1, 2)\)

\[
[x_i, x_j] = i \epsilon_{ij} \tilde{\theta},
\]

was proposed by the present authors in [1]. Note that the relation (1) does not violate the D=2+1 Galilean symmetry but the scalar parameter \(\tilde{\theta}\) introduces a second Galilean central charge [2]. In [1] the relation (1) was obtained from the quantization of the following extension of the classical D=2+1 free particle Lagrangian

\[
L^{(0)}(0) = \frac{m 2 x_i^2}{2} - k \epsilon_{ij} \dot{x}_i \dot{x}_j.
\]

By employing the Faddeev-Jackiw method [3] one can reexpress (2) as the following first order Lagrangian (we put \(m = 1\))

\[
L^{(0)} = P_i (\dot{x}_i - y_i) + \frac{y_i^2}{2} + \frac{\tilde{\theta}}{2} \epsilon_{ij} y_i y_j.
\]

Using the new variables [4]

\[
Q_i = \tilde{\theta}(y_i - P_i),
\]
\[
X_i = x_i + \epsilon_{ij} Q_j,
\]

one gets

\[
L^{(0)} = L^{(0)}_{\text{ext}} + L^{(0)}_{\text{int}},
\]

where

\[
L^{(0)}_{\text{ext}} = P_i \dot{X}_i + \frac{\tilde{\theta}}{2} \epsilon_{ij} P_i \dot{P}_j - H^{(0)}_{\text{ext}},
\]
\[
L^{(0)}_{\text{int}} = \frac{1}{2\tilde{\theta}} \epsilon_{ij} Q_i \dot{Q}_j - H^{(0)}_{\text{int}},
\]

\[
H^{(0)}_{\text{ext}} = \frac{1}{2} \dot{\vec{P}}^2, \quad H^{(0)}_{\text{int}} = -\frac{1}{2\tilde{\theta}} \dot{\vec{Q}}^2,
\]

\[\text{1}^{1}\text{In this paper, following other authors, we shall call it the free L.S.Z. model. We use this rather clumsy notation to distinguish us from LSZ which should be reserved for Lehmann Symanzik and Zimmermann.}\]
together with the following nonvanishing Poisson brackets (PBs):

\[
\{X_i, X_j\} = \tilde{\theta} \epsilon_{ij}, \quad \{X_i, P_j\} = \delta_{ij}, \quad \{Q_i, Q_j\} = -\tilde{\theta} \epsilon_{ij}.
\]

The variables \(\{X_i, P_j\}\) parametrize a noncommutative phase space, and the variables \(Q_i\) describe the internal structure of the noncommutative particle [4]. Therefore interactions which do not involve the internal structure should be given in terms of the noncommutative phase space variables.

In [5] we extended the free model described by the Lagrangian (6a) in the following two ways:

i) By adding to (6a) the term \((X_{\mu} = (X_i, t), \vec{X} = (X_1, X_2); c = 1)\):

\[
L^{\text{int}} = e A_{\mu}(\vec{X}, t) \dot{X}^\mu = e A_i(\vec{X}, t) \dot{X}_i + e A_0(\vec{X}, t),
\]

(9)
describing the Duval-Horvathy way of introducing the minimal electromagnetic interaction [6]. Then adding interaction (9) modifies the PBs (8) to [5, 6]:

\[
\{X_i, X_j\} = \tilde{\theta} \epsilon_{ij} 1 - e \tilde{\theta} B, \quad \{P_i, P_j\} = e B \epsilon_{ij} 1 - e \tilde{\theta} B, \\
\{X_i, P_j\} = \delta_{ij} 1 - e \tilde{\theta} B,
\]

(10)

which implies the consideration of values \(e \tilde{\theta} B \neq 1\) in order to avoid a singularity at tachyonic states after quantization.

ii) Another way of introducing the minimal electromagnetic interaction in (6a) is provided by the replacement

\[
H^{(0)}_{\text{ext}} \rightarrow H_{\text{ext}} = \frac{1}{2} (P_i - e A_i)^2 - e A_0.
\]

(11)

In such a case the PBs (8a) remain unchanged.

The two models with the additional gauge interaction are classically equivalent to each other [5]. To go between them one has to perform a classical Seiberg-Witten (SW) map of gauge potentials \(A_{\mu}\) together with a noncanonical transformation of the phase space variables \((X_i, P_i)\).

In this paper we employ the \(N = 2\) superfield technique to supersymmetrize these models. The \(N=1\) supersymmetrization of the free actions (2) and (6a) was discussed in [7, 8].

In Sect. 2 we introduce the \(N=2\) supersymmetrization of the model (6a) having added to it a scalar superpotential. Then we show that this procedure,
in the presence of a scalar (electric) interaction and for a nonvanishing non-commutativity parameter $\tilde{\theta}$, leads to the emergence of magnetic interactions. In Sect. 3 we use the Hamiltonian formulation and the standard superalgebra of the $N=2$ supersymmetric model to consider, in a unified way, the models with two ways of introducing the gauge coupling (see (9) and (11)). There we describe also the map relating both models (the noncanonical transformation of phase space variables plus a SW map for the gauge potentials derived in [5]). One should point out that the fermionic particle degrees of freedom are the same in both models.

In Sect. 4 we briefly present our conclusions and discuss some open questions.

## 2 $N=2$ Superfield Supersymmetrization

Here we supersymmetrize the Lagrangian (6a) with an additional interaction given by a scalar superpotential.

First, we introduce the covariant derivatives

$$D = \frac{\partial}{\partial \theta} - i \bar{\theta} \frac{\partial}{\partial t} \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + i \theta \frac{\partial}{\partial t},$$

with the property:

$$D^2 = \bar{D}^2 = 0.$$  \hspace{1cm} (13)

Next, we employ $N = 2$ superfields describing real supercoordinates

$$X_i(t) \rightarrow \Phi_i(t, \theta, \bar{\theta}) = X_i(t) + i \theta \psi_i(t) + \bar{\theta} \bar{\psi}_i(t) + \theta \theta F_i(t)$$ \hspace{1cm} (14)

and the following odd complex chiral $N=2$ superfields describing supermomenta

$$P_i(t) \rightarrow \Pi_i(t, \theta, \bar{\theta}) = i \chi_i(t) - i \theta (P_i(t) + i f_i(t)) - \theta \theta \dot{\chi}_i(t),$$ \hspace{1cm} (15)

which satisfy the chirality condition:

$$\bar{D} \Pi_i(t; \theta, \bar{\theta}) = 0.$$ \hspace{1cm} (16)

It is easily seen that

$$D \Phi_i = i \psi_i - \bar{\theta}(\dot{X}_i - i F_i) + \theta \theta \dot{\psi}_i,$$

$$\bar{D} \Phi_i = i \bar{\psi}_i - i \theta (\dot{X}_i + i F_i) - \theta \theta \dot{\bar{\psi}}_i.$$ \hspace{1cm} (17)

The Lagrangian (6a) is then supersymmetrized by

$$L_{\text{ext}}^{(0)} \rightarrow L_{(N=2)\text{ext}}^{(0)} = \frac{1}{2} \int d\theta d\bar{\theta} \left\{ \bar{D} \Phi_i \Pi_i + \Pi_i D \Phi_i \right\}$$

4
\[-\Pi_i \Pi_i + \frac{\tilde{\theta}}{2} \epsilon_{ij} (\Pi_i \tilde{\Pi}_j + \tilde{\Pi}_i \Pi_j) \right\},
\]

(18)
giving us

\[L^{(0)}_{(N=2)_{\text{ext}}} = L^{(0)}_{\text{ext}} - \frac{1}{2} f_i^2 + F_i f_i + \frac{\tilde{\theta}}{2} \epsilon_{ij} f_i \dot{f}_j \]
\[+ i(\psi \dot{X_i} - \dot{\psi} \chi_i) + i \dot{\chi_i} \chi_i - i \tilde{\theta} \epsilon_{ij} \dot{\chi}_i \dot{\chi}_j.\]  

(19)

For the interaction term we take

\[L^{\text{int}}_{(N=2)_{\text{ext}}} = - \int d\theta d\bar{\theta} W(\Phi_i(t; \theta, \bar{\theta}))\]
\[= -F_i \partial_i W(\dot{X}(t)) - \bar{\psi} \psi \partial_i \partial_j W(\dot{X}(t)).\]  

(20)

Introducing the complete Lagrangian

\[L_{(N=2)_{\text{ext}}} = L^{(0)}_{(N=2)_{\text{ext}}} + L^{\text{int}}_{(N=2)_{\text{ext}}},\]  

(21)

one gets the following Euler-Lagrange equations (EOM) for the auxiliary fields \(f_i\) and \(F_i\)

\[f_i(t) = \partial_i W(\dot{X}(t))\]  

(22a)
\[F_i(t) = f_i(t) - \tilde{\theta} \epsilon_{ij} \dot{f}_j(t) = \partial_i W(\dot{X}(t)) - \tilde{\theta} \epsilon_{ij} \partial_j W(\dot{X}(t)) \chi_k(t),\]  

(22b)

where, in (22b), we have used (22a) to eliminate the field \(f_i\) as well as its time derivative. By means of (22b), the auxiliary variables can be completely eliminated, “on-shell”, from the remaining EOM. It is easy to check that the resulting reduced system of EOM can also be obtained from an effective Lagrangian given by

\[L_{(N=2)_{\text{ext}}} = L^{(0)}_{\text{ext}} - \frac{1}{2} (\partial_i W)^2 + \dot{X}_i A_k\]
\[+ i(\psi \dot{X_i} - \dot{\psi} \chi_i) + i \dot{\chi_i} \chi_i - i \tilde{\theta} \epsilon_{ij} \dot{\chi}_i \dot{\chi}_j,\]  

(23)

where we have inserted (22a) into (22b). The vector potential \(A_k\) is given by

\[A_k = \frac{\tilde{\theta}}{2} \epsilon_{ij} \partial_i W \partial_j \partial_k W,\]  

(24)

and so the magnetic field \(B = \epsilon_{ij} \partial_i A_j\) takes the form

\[B = \frac{\tilde{\theta}}{2} \epsilon_{ik} \epsilon_{ij} (\partial_i \partial_k W)(\partial_j \partial_k W).\]  

(25)
We see that the scalar potential term in (23)

\[ A_0 = -\frac{1}{2}(\partial_i W)^2, \]  

is accompanied by a magnetic gauge field interaction with the vector potential \( A_k \), proportional to the noncommutativity parameter \( \tilde{\theta} \). Note that the static electric term (26) and the magnetic potentials (24) are not independent. In the rotation-invariant case (\( W = W(r); r = (X_i X_i)^{1/2} \)) we have

\[ A_0 = -\frac{1}{2}(W'(r))^2, \quad A_k = -\frac{\tilde{\theta}}{2} \epsilon_{kl} X_l \left( \frac{W'(r)}{r} \right)^2, \]  

i.e. one gets the relation

\[ A_k = \tilde{\theta} \epsilon_{kl} \frac{X_l}{r^2} A_0 \implies B(r) = -\frac{\tilde{\theta}}{r} A'_0(r). \]  

In particular, if \( W = \frac{\omega^2}{2} r^2 \), we have the case of a harmonic potential and then

\[ A_k = -\frac{\tilde{\theta} \omega^2}{2} \epsilon_{kl} X_l \implies B = \omega^2 \tilde{\theta}. \]  

We see that in such a case the noncommutativity generates a constant magnetic field.\(^2\)

3 \( \text{N=2 Supersymmetrization Using Hamiltonian Approach} \)

In the previous section we used the N=2 superfield method to derive a classical Lagrangian describing the supersymmetrization of the Duval-Horvathy gauge coupling scheme (DH approach; see [6]), which then contains a scalar potential and a magnetic interaction term with a definite relation between them. Unfortunately, we did not find a superfield ansatz leading directly to the supersymmetrization of the gauge model with generalized gauge transformation provided by the substitution (11) introduced in [5] (we shall refer to this as the L.S.Z.-approach). So, below, we present a supersymmetrization of the L.S.Z.-approach using the supersymmetric version of the Hamiltonian framework. In such a case the supersymmetrization of the two minimal gauge coupling schemes can be treated on the same footing.

\(^2\)In the special case of a noncommutative harmonic oscillator this effect has recently been mentioned in [9].
We start with the common structure of the bosonic Hamiltonian for both models (we put $e = 1$)

$$H_b = \frac{1}{2}(P_i^2 + W_i^2(\vec{X})),$$  \hspace{1cm} (30)

where $P_i = P_i$ in the DH approach, and

$$P_i = P_i - A_i(\vec{X}),$$  \hspace{1cm} (31)

for the L.S.Z. approach with the Hamiltonian (11).

Note that the potential term in (30) is chosen to be positive for the supersymmetrization to be possible, i.e. in (9) and (11) we put (cp (26))

$$A_0 = -\frac{1}{2} W_i^2.$$  \hspace{1cm} (32)

In order to supersymmetrize (30) we supplement the bosonic phase space variables $X_i, P_i$ with fermionic coordinates $\psi_i$ and their hermitian conjugates $\bar{\psi}_i$ satisfying the canonical PBs

$$\{\psi_i, \bar{\psi}_j\} = -i \delta_{ij}.$$  \hspace{1cm} (33)

All other PBs involving $\psi_i(\bar{\psi}_i)$ do vanish.

Next we introduce the basic object in the N=2 supersymmetric Hamiltonian approach, a complex-valued fermionic supercharge $Q$ linearly dependent\(^3\) on the $\psi_i$. The Hamiltonian $H$ is provided by the formula

$$H = i \frac{\{Q, \bar{Q}\}},$$  \hspace{1cm} (34)

where

$$H = H_b + H_f,$$  \hspace{1cm} (35)

and $H_b$ is given by (30). The fermionic part $H_f$ will be determined below.

The standard N=2 superalgebra implies that

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0,$$  \hspace{1cm} (36)

i.e. $Q(\bar{Q})$ after quantization become nilpotent operators. Of course, the conservation of $Q(\bar{Q})$ i.e. $\{Q, H\} = 0$ is a consequence of (34) and (30).

In order to obtain formula (30) we assume

$$Q = i(P_i + iW_i(\vec{X}))\psi_i.$$  \hspace{1cm} (37)

\(^3\)A non-linear dependence of $Q$ on the $\psi_i(\bar{\psi}_i)$ has been considered by Gosh [10]. In this paper we restrict ourselves to linear realizations of the superalgebra.
¿From (37) it follows by a straightforward calculation that (36) is valid only if the following two conditions are satisfied

i) \( \{ P_i, P_j \} = \{ W_i, W_j \} \), \hspace{1cm} (38a)

ii) \( \{ P_i, W_j \} = \{ P_j, W_i \} \). \hspace{1cm} (38b)

Note that (38a–38b) should be valid for both approaches of introducing the minimal gauge couplings.

In order to obtain the consequences of (38a–38b) for the concrete models we need the respective PBs for the variables \( X_i, P_i \). They are given for the DH-approach by (10) and for the L.S.Z.-approach we get from (8a) and (31) the following PBs

\[
\{ P_i, P_j \} = \epsilon_{ij} B(\vec{X}), \\
\{ X_i, P_j \} = e_{ji}(\vec{X}), \\
\{ X_i, X_j \} = \tilde{\theta} \epsilon_{ij},
\]

with the inverse dreibeins (cp. [5]) given by

\[
e_{ji}(\vec{X}) = \delta_{ij} + \tilde{\theta} \epsilon_{li} \partial_l A_j(\vec{X}).
\]

(40)

Using the two choices of PBs it is easy to see that the condition (38a) fixes the magnetic field \( B \) for both approaches to be given by the same expression in terms of \( W_i \)

\[
B(\vec{X}) = \frac{\tilde{\theta}}{2} \epsilon_{ij} \epsilon_{kl} W_i(\vec{X}) \partial_k W_j(\vec{X}).
\]

(41)

The relation (41) is one of the main results of the present paper.

In order to compare (41) with the result (25) obtained in Section 3 for the DH-approach we have to identify

\[
W_i(\vec{X}) = \partial_i W(\vec{X}).
\]

(42)

Now let us examine the condition (38b). For the DH-approach using (42) and the PBs (10) we get

\[
\{ P_i, W_j \} = -\frac{\partial_i \partial_j W}{1 - \tilde{\theta} B},
\]

(43)

We see that (38b) is satisfied as the l.h.s. of (43) is symmetric w.r.t. \( i \leftrightarrow j \).

The validity of (38b) for the L.S.Z.-approach is more involved. From (39b) we obtain

\[
\{ W_j, P_i \} = \epsilon_{ik} \partial_k W_j =: f_{ij}(\vec{X}).
\]

(44)
A large class of models for which $f_{ij}$ is symmetric can be obtained in the case of rotational invariance. Then we have

$$W_i(\vec{X}) = \partial_i W(r),$$

and the vector potential in the Coulomb gauge is given by

$$A_i(\vec{X}) = \epsilon_{ij} \partial_j f(r).$$

In the case of the L.S.Z.-approach the relations (34), (37) and (39a-c) give us the following expression for the fermionic part $H_f$ of our Hamiltonian:

$$H_f^{L.S.Z.} = i B(\vec{X}) \epsilon_{ij} \bar{\psi}_i \psi_j + f_{ij}(\vec{X}) \bar{\psi}_i \psi_j.$$  

The first term in (47) describes the coupling of a non-anomalous magnetic moment ($g = 2$) to a magnetic field $B$.

On the other hand, in the case of the DH-approach we obtain

$$H_f^{DH} = \frac{1}{1 - \theta B(\vec{X})} \left( i B(\vec{X}) \epsilon_{ij} \bar{\psi}_i \psi_j + \partial_i \partial_j W(\vec{X}) \bar{\psi}_i \psi_j \right).$$

Let us now see how the supersymmetrized versions of the two approaches are related to each other.

In [5] we showed that the bosonic parts in the DH-approach and in the L.S.Z.-approach are related to each other by a noncanonical transformation of the phase-space variables\footnote{Fields in the L.S.Z. approach, from now onwards, are denoted by a hat ($\hat{B}, \hat{A}_\mu, \hat{P}_i$ etc).}

$$\mathcal{P}_i \rightarrow \hat{\mathcal{P}}_i,$$  

$$X_i \rightarrow X_i + \tilde{\theta} \epsilon_{ij} \hat{A}_j(\vec{X}) =: \eta_i(\vec{X}),$$

supplemented by a classical Seiberg-Witten (SW) map between the corresponding gauge potentials [5]. This SW-map provides the following relation between magnetic fields in the DH and L.S.Z. approaches:

$$\hat{B}(\vec{X}) = \frac{B(\vec{\eta})}{1 - \theta B(\vec{\eta})},$$

where

$$\hat{B}(\vec{X}) = \epsilon_{kl} \left( \partial_k \hat{A}_l(\vec{X}) + \frac{\tilde{\theta}}{2} \epsilon_{ij} \partial_i \hat{A}_k(\vec{X}) \partial_j \hat{A}_l(\vec{X}) \right),$$

and

$$B(\vec{\eta}) = \epsilon_{ik} \partial_i \eta_k A_k(\vec{\eta}).$$
For the case of static electric potentials the SW map is trivial \[5\]

\[
\hat{A}_0(\vec{X}) = A_0(\vec{\eta}(\vec{X})).
\] (54)

In \[5\] it was shown that the change of variables \(49, 50\) together with the SW map of gauge potentials leads to the equality between \(L_b\) (see (6a) and (9)) and \(\hat{L}_b\) (see (6a) and (11)) as functions of the corresponding variables.

Let us rewrite (54) in terms of \(\hat{W}_i\) and \(\partial_i W\) as

\[(\hat{W}_i(\vec{X}))^2 = (\partial_i W(\vec{\eta}))^2.\]

(55)

We shall take the simplest solution of (55)

\[
\hat{W}_i(\vec{X}) = \partial_i W(\vec{\eta}),
\]

(56)

and impose the triviality of the phase space transformation in the fermionic sector

\[
\hat{\psi}_i(t) = \psi_i(t).
\]

(57)

This ensures the equality of the two fermionic Hamiltonians \(47\) and \(48\) as functions of the corresponding variables i.e. the following relation holds:

\[
H^{L.S.Z.}_f\left(B(\vec{X}), f_{ij}(\vec{X}), \hat{\psi}_i\right) = H^{DH}_f\left(B(\vec{\eta}), W(\vec{\eta}), \psi_i\right).\]

(58)

Note that (56) leads always to a symmetric \(f_{ij}\)

\[
f_{ij}(\vec{X}) = \frac{\partial_i \partial_j W(\vec{\eta})}{1 - \tilde{\theta}B(\vec{\eta})}
\]

(59)

which solves the condition \(38b\) for an arbitrary potential \(W(\vec{\eta})\).

Furthermore, the maps \(49\), (56) and (57) respectively for the momenta, \(\hat{W}_i\) and \(\psi_i\) lead to the equality of the supercharges \(37\) for both approaches as functions of the corresponding variables i.e. we have

\[
Q = i(\hat{P}_i + i \partial_i W(\vec{\eta}))\psi_i = i(\hat{P}_i + i \hat{W}_i(\vec{X}))\hat{\psi}_i = \hat{Q}.
\]

(60)

The result (60) shows that not only Hamiltonians but also the supercharges in respective variables can be identified in both models.

### 4 Conclusions

The main results of this paper include the demonstration of the appearance of effective magnetic interaction, generated by a nonvanishing noncommutativity parameter \(\tilde{\theta}\) in the presence of supersymmetry (see formulae (25) and (41)).
The original higher order Lagrangian contains an external variable sector, describing position and momenta of planar particles as well as an internal sector, together describing (2+1)-dimensional anyon dynamics. In this paper we have studied only the gauge interactions in the external sector. Recently, the effects of gauge coupling in the internal sector have also been investigated [11].

In this paper the origin of noncommutativity of position coordinates stems from higher order time derivative terms, present in the free L.S.Z. Lagrangian [1]. However even in the free particle model by choosing a nonstandard reparametrization gauge one gets the noncommutative particle coordinates [12, 13]. Recently it has also been shown that the noncommutativity of planar particle positions can also be achieved by the coupling of relativistic planar particles to the \( D = 2 + 1 \) quantum gravity [14, 15]. A study of the interplay between these origins of noncommutativity would be an interesting subject of further investigations.

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