We propose a new experiment to measure the running of the fine-structure constant in the space-like region by scattering high-energy muons on atomic electrons of a low-Z target through the process $\mu e \rightarrow \mu e$. The differential cross section of this process, measured as a function of the squared momentum transfer $t = q^2 < 0$, provides direct sensitivity to the leading-order hadronic contribution to the muon anomaly $\alpha^{\text{HWO}}_\mu$. By using a muon beam of 150 GeV, with an average rate of $\sim 1.3 \times 10^7$ muon/s, currently available at the CERN North Area, a statistical uncertainty of $\sim 0.3\%$ can be achieved on $\alpha^{\text{HWO}}_\mu$ after two years of data taking. This direct measurement of $\alpha^{\text{HWO}}_\mu$ will provide an independent determination, competitive with the time-like dispersive approach, and consolidate the theoretical prediction for the muon $g-2$ in the Standard Model. It will allow therefore a firmer interpretation of the measurements of the future muon $g-2$ experiments at Fermilab and J-PARC.
I. INTRODUCTION

In searching for new physics, low-energy high-precision measurements are complementary to the LHC high-energy frontier. The long-standing \((3–4)\sigma\) discrepancy between the experimental value of the muon anomalous magnetic moment \(a_\mu = (g - 2)/2\) and the Standard Model (SM) prediction, \(\Delta a_\mu(\text{Exp} - \text{SM}) \sim (28 \pm 8) \times 10^{-10}\) \cite{1,2}, has been considered during these years as one of the most intriguing indications of physics beyond the SM. However, the accuracy of the SM prediction, \(5 \times 10^{-10}\), is limited by strong interaction effects, which cannot be computed perturbatively at low energies. Long time ago, by using analyticity and unitarity, it was shown \cite{3} that the leading-order (LO) hadronic contribution to the muon \(g\)-2, \(a_\mu^{\text{HLO}}\), could be computed via a dispersion integral of the hadron production cross section in \(e^+e^-\) annihilation at low-energy. The present error on \(a_\mu^{\text{HLO}}\), \(\sim 4 \times 10^{-10}\), with a fractional accuracy of 0.6\%, constitutes the main uncertainty of the SM prediction. An alternative evaluation of \(a_\mu^{\text{HLO}}\) can be obtained by lattice QCD calculations \cite{4}. Even if the current results are not yet competitive with those obtained with the dispersive approach via time-like data, their errors are expected to decrease significantly in the next few years \cite{5}. The \(O(\alpha^3)\) hadronic light-by-light contribution, \(a_\mu^{\text{HLbL}}\), which has the second largest error in the theoretical evaluation, contributing with an uncertainty of \((2.5–4) \times 10^{-10}\), cannot at present be determined from data and its calculation relies on the use of specific models \cite{6,7}.

From the experimental side, the error achieved by the BNL E821 experiment, \(\delta a_\mu^{\text{Exp}} = 6.3 \times 10^{-10}\) (corresponding to 0.54 ppm) \cite{8}, is dominated by the available statistics. New experiments at Fermilab and J-PARC, aiming at measuring the muon \(g\)-2 to a precision of \(1.6 \times 10^{-10}\) (0.14 ppm), are underway \cite{9,10}.

![FIG. 1. Comparison between \(a_\mu^{\text{SM}}\) and \(a_\mu^{\text{Exp}}\). DHMZ is Ref. 11, HLMNT is Ref. 12; SMXX is the average of the two previous values with a reduced error as expected by the improvement on the hadronic cross section measurement; BNL-E821 04 ave. is the current experimental value of \(a_\mu\); New \((g\)-2\) exp. is the same central value with a fourfold improved precision as planned by the future \(g\)-2 experiments at Fermilab and J-PARC 2.](image)

Fig. 1 from Ref. 11, shows the status of the \(g\)-2 discrepancy compared with what could be expected after the new \(g\)-2 measurements at Fermilab and J-PARC, assuming that the central value would remain the same. Together with a fourfold improved precision on the experimental side, an improvement on the LO hadronic contribution is highly desirable.

Differently from the dispersive approach, which uses time-like data from annihilation cross sections, our proposal is to determine \(a_\mu^{\text{HLO}}\) from a measurement of the effective electromagnetic coupling in the space-like region, where the vacuum polarization is a smooth function of the squared momentum transfer. This approach to evaluate \(a_\mu^{\text{HLO}}\) has been recently proposed \cite{13} by using Bhabha scattering data. A method to determine the running of the electromagnetic coupling by using small-angle Bhabha scattering was proposed in \cite{14} and applied to LEP data in \cite{15}. The hadronic contribution to the running of \(\alpha\) could also be determined unambiguously by using the \(t\)-channel \(\mu e\) scattering process,
from which $a_\mu^{\text{HLO}}$ could be obtained, as detailed in the following.

The paper is organized as follows. After a short review of the theoretical framework in Sect. II, we present our experimental proposal in Sect. III. Preliminary considerations on the detector and systematic uncertainties are given in Sect. IV and Sect. V, respectively, while our conclusions are drawn in Sect. VI.

II. THEORETICAL FRAMEWORK

With the help of dispersion relations and the optical theorem, the LO hadronic contribution to the muon $g$-2 is given by the well-known formula [3, 16]

$$a_\mu^{\text{HLO}} = \left( \frac{\alpha m_\mu}{3\pi} \right)^2 \int_{4m_\mu^2}^{\infty} ds \frac{\tilde{K}(s)R_{\text{had}}(s)}{s^2},$$  \hspace{1cm} (1)

where $R_{\text{had}}(s)$ is the ratio of the total $e^+e^- \rightarrow \text{hadrons}$ and the Born $e^+e^- \rightarrow \mu^+\mu^-$ cross sections, $\tilde{K}(s)$ is a smooth function and $m_\mu$ ($m_\pi$) is the muon (pion) mass. We remark that $R_{\text{had}}(s)$ in the integrand function of Eq. (1) is highly fluctuating at low energy due to resonances and threshold effects. The dispersive integral in Eq. (1) is usually calculated by using the experimental value of $R_{\text{had}}(s)$ up to a certain value of $s$ [7, 17, 18] and by using perturbative QCD (pQCD) [19] in the high-energy tail. For the calculation of $a_\mu^{\text{HLO}}$, an alternative formula can also be exploited [13, 20], namely

$$a_\mu^{\text{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx \left( 1 - x \right) \Delta \alpha_{\text{had}}[t(x)],$$  \hspace{1cm} (2)

where

$$t(x) = \frac{x^2 m_\mu^2}{x - 1} < 0$$  \hspace{1cm} (3)

is a space-like (negative) squared four-momentum and $\Delta \alpha_{\text{had}}(t)$ is the hadronic contribution to the running of the fine-structure constant. In contrast with the integrand function of Eq. (1), the integrand in Eq. (2) is smooth and free of resonances.

By measuring the running of $\alpha(t)$,

$$\alpha(t) = \frac{\alpha(0)}{1 - \Delta \alpha(t)},$$  \hspace{1cm} (4)

where $t = q^2 < 0$ and $\alpha(0) = \alpha$ is the fine-structure constant in the Thomson limit, the hadronic contribution $\Delta \alpha_{\text{had}}(t)$ can be extracted by subtracting from $\Delta \alpha(t)$ the purely leptonic part $\Delta \alpha_{\text{lep}}(t)$, which can be calculated order-by-order in perturbation theory (it is known up to three loops in QED [21] and up to four loops in specific $q^2$ limits [22]).

Eq. (2) involves $\Delta \alpha_{\text{had}}(q^2)$ evaluated at negative space-like momenta $t < 0$.

Fig. 2 (left) shows $\Delta \alpha_{\text{lep}}$ and $\Delta \alpha_{\text{had}}$ as functions of the scale $t(x)$ defined in Eq. (3). The integrand function of Eq. (2), $(1 - x)\Delta \alpha_{\text{had}}[t(x)]$, is plotted in Fig. 2 (right), using the output of the routine hadr5n12 [23] (which uses time-like hadroproduction data and perturbative QCD). The range $x \in (0, 1)$ corresponds to $t \in (-\infty, 0)$, with $x = 0$ for $t = 0$. The peak of the integrand occurs at $x_{\text{peak}} \simeq 0.914$ ($t_{\text{peak}} \simeq -0.108$ GeV$^2$) and $\Delta \alpha_{\text{had}}(t_{\text{peak}}) \simeq 7.86 \times 10^{-4}$ (see Fig. 2 (right)).

III. EXPERIMENTAL PROPOSAL

We propose to use Eq. (2) to determine $a_\mu^{\text{HLO}}$ by exploiting the measurement of $\Delta \alpha_{\text{had}}(t)$ in the space-like region.

Measuring the running of $\alpha(t)$ in the space-like region, using a muon beam with $E_\mu \simeq 150$ GeV on a fixed electron target, with a technique similar to the one described in [25] for a measurement of the pion form factor, is very appealing for the following reasons:

- it is a $t$-channel process, making the dependence on $t$ of the differential cross section proportional to $|\alpha(t)/\alpha(0)|^2$:

$$\frac{d\sigma}{dt} = \frac{d\sigma_0}{dt} \left| \frac{\alpha(t)}{\alpha(0)} \right|^2,$$ \hspace{1cm} (5)
where the first factor on the right-hand side $d\sigma_0/dt$ refers to the effective Born cross section including virtual and soft photons, analogously to Ref. [26], where small-angle Bhabha scattering at high energy was considered. The vacuum polarization effect, in the leading photon and soft photons, analogously to Ref. [26], where small-angle Bhabha scattering at high energy was considered. The vacuum polarization effect, in the leading photon $t$-channel exchange, is incorporated in the running of $\alpha$ and gives rise to the second factor $|\alpha(t)/\alpha(0)|^2$. It is understood that for a high precision measurement also higher-order radiative corrections must be included. For a detailed discussion see Refs. [14, 26];

- given the incoming muon energy $E_\mu^i$, in a fixed-target experiment the $t$ variable is related to the energy of the scattered electron $E_e^f$ or its angle $\theta_e^f$:

\begin{equation}
 t = (p_\mu^i - p_\mu^f)^2 = (p_e^i - p_e^f)^2 = 2m_e^2 - 2m_eE_e^f, \tag{6}
\end{equation}

\begin{equation}
 s = (p_\mu^i + p_\mu^f)^2 = (p_e^i + p_e^f)^2 = m_\mu^2 + m_e^2 + 2m_eE_\mu^i, \tag{7}
\end{equation}

\begin{equation}
 E_e^f = m_e \frac{1 + r^2c_e^2}{1 - r^2c_e^2}, \quad \theta_e^f = \arccos \left( \frac{1}{r} \sqrt{\frac{E_e^f - m_e}{E_e^f + m_e}} \right), \tag{8}
\end{equation}

where

\[ r \equiv \frac{\sqrt{(E_\mu^i)^2 - m_\mu^2}}{E_\mu^i + m_e}, \quad c_e \equiv \cos \theta_e^f; \]

- the boosted kinematics of the collision allows all the scattering angles to be accessed by a single detector element in the laboratory system. The angle $\theta_e^f$ spans the range $0$–$31.85$ mrad for the electron energy $E_e^f$ in the range $(1–139.8)$ GeV (the low-energy cut at 1 GeV is arbitrary). By using the same detector to cover the whole acceptance, many systematic errors, e.g. on the efficiency, will cancel out (at least at first order) in the relative ratios of event counts in the high and low $q^2$ regions (signal and normalization regions);

- for $E_\mu^i = 150$ GeV, $s \simeq 0.164$ GeV$^2$ and $-0.143$ GeV$^2 \lesssim t \lesssim 0$ GeV$^2$ $(-\lambda(s, m_\mu^2, m_e^2)/s < t < 0$, where $\lambda(x, y, z)$ is the Källén function), which implies that $t$ spans the peak region of the integrand function of Eq. (2), as visible in Fig. 2 (right), corresponding to an electron scattering angle $\theta_e^f \simeq 1.5$ mrad;
• the angle of the scattered electron and muon are correlated as shown in Fig. 3 (for muons of 150 GeV). This constraint is extremely important to select elastic scattering events, rejecting background events from radiative or inelastic processes and to minimize systematic effects in the determination of $t$. Note that for scattering angles of (2–3) mrad there can be an ambiguity between the outgoing electron and muon, as their angles and momenta are similar. To associate them correctly it is necessary to identify the two particles by means of downstream dedicated detectors (calorimeter and muon detectors).

![Figure 3](image.png)

**FIG. 3.** The relation between the muon and electron scattering angles for 150 GeV incident muon momentum.

We estimate the statistical sensitivity of this experiment (supplemented with large $|t|$ contributions derived from pQCD) to be $\sim 0.3\%$ on the value of $a_{HEL}^\mu$ after two years of data taking, using 30 experimental points in $x$, assuming a running time of $2 \times 10^7$ s/yr, and a muon beam of 150 GeV with an average intensity of $\sim 1.3 \times 10^7$ muon/s. Such a beam is currently available at the CERN North Area [27], and a preliminary detector layout is described in the following section.

**IV. PRELIMINARY CONSIDERATIONS ON THE DETECTOR**

In order to perform the planned measurement to the required precision, a dedicated detector is necessary. We describe here a possible setup to measure the following observables:

- direction and momentum of the incident muon;
- directions of the outgoing electron and muon.

The CERN muon beam M2, used at 150 GeV, has the characteristics needed for such a measurement. The beam intensity appears to be adequate to provide the required event yield. The beam time structure allows to tag the incident muon while keeping low the background related to incoming particles (e.g. electrons). The electrons contamination is very small. The beam provides both positive and negative muons, which we plan to use.

The target consists of atomic electrons. To reach the required statistics, the target must consist of an adequate amount of material to give a sufficient number of electron scattering centres. The target has to be made of a low-Z material to minimize the impact of multiple scattering and the background due to bremsstrahlung and pair production processes. A promising idea for the detector is to use 20 layers of Be (or C) coupled to Si planes, spaced by intermediate air gaps, located at a relative distance of one meter from each other. Fig. 4 shows the basic layout, presently under study.

The arrangement provides both a distributed target with low-Z and the tracking system. As downstream particle identifiers we plan to use a calorimeter for the electrons and a muon system for the muons (a filter plus active planes).
FIG. 4. Scheme of a possible detector layout. (a) The detector is a modular system. Each module consists of a low-$Z$ target (Be or C) and two silicon tracking stations located at a distance of one meter. (b) To perform the $\mu/e$ discrimination in the case of small scattering angles (both $\theta_\mu$ and $\theta_e$ below 5 mrad) the detector is equipped with an electromagnetic calorimeter and a muon detector.

This particle identifier system is required to solve the muon-electron ambiguity for electron scattering angles around (2–3) mrad (cf. Fig. 3). The preliminary studies of such an apparatus, performed by using GEANT4, indicate an angular resolution for the outgoing particles of $\sim$0.02 mrad.

The detector acceptance must cover the region of the signal, with the electron emitted at extremely forward angles and high energies, as well as the normalization region, where the electron has much lower energy (around 1 GeV) and an emission angle of some tens of mrad. The boosted kinematics of the process allows the detector to cover almost 100% of the acceptance.

The incoming muons have to be tagged and their direction and momentum precisely measured. To this purpose, a detector similar to those used by COMPASS [28] or NA62 [29] can be employed.

V. CONSIDERATIONS ON SYSTEMATIC UNCERTAINTIES

Significant contributions of the hadronic vacuum polarization to the $\mu e \rightarrow \mu e$ differential cross section are essentially restricted to electron scattering angles below 10 mrad, corresponding to electron energies above 10 GeV. The net effect of these contributions is to increase the cross section by a few per mille: a precise determination of $q_\mu^{HLO}$ requires not only high statistics, but also a high systematic accuracy, as the final goal of the experiment is equivalent to a determination of the differential cross section with $\sim$10 ppm systematic uncertainty at the peak of the integrand function (cf. Fig. 2).

Such an accuracy can be achieved if the efficiency is kept highly uniform over the entire $q^2$ range, including the normalization region, and over all the detector components. This motivates the choice of a purely angular measurement: an acceptance of tens of mrad can be covered with a single sensor of modern silicon detectors, positioned at a distance of about one meter from the target. It has to be stressed that particle identification (electromagnetic calorimeter and muon filter) is necessary to solve the electron-muon ambiguity in the region below 5 mrad. The wrong
assignment probability can be measured with the data by using the rate of muon-muon and electron-electron events. Another requirement for reaching very high accuracy is to measure all the relevant contributions to systematic uncertainties from the data themselves. An important effect, which distinguishes the normalization from the signal region, is multiple scattering, as the electron energy in this region is as low as 1 GeV. Multiple scattering breaks the muon-electron two-body angular correlation, moving events out of the kinematic line in the 2D plot of Fig. 3. In addition, multiple scattering in general causes acoplanarity, while two-body events are planar, within resolution. These facts allow multiple scattering effects to be modelled and measured by using data. An additional handle on multiple scattering could be the inclusion of a thin layer in the apparatus, made of the same material as the main target modules. This possibility will be studied in detail with simulation.

In experiments dedicated to high-precision measurements, several systematic effects can be explored within the experiment itself. In this respect the proposed modularity of the apparatus will help. A test with a single module could provide a proof-of-concept of the proposed methods.

From the theoretical point of view, the control of the systematic uncertainties requires the development of high-precision Monte Carlo tools, including the relevant radiative corrections to reach the needed theoretical precision. To this aim, QED radiative corrections at leading-logarithmic level resummed at all orders of perturbation theory and matched to the exact $\mathcal{O}(\alpha)$ correction (as for instance implemented in the BabaYaga event generator [30] for Bhabha scattering) are mandatory, ensuring a theoretical precision at the level of $\mathcal{O}(10^{-4})$ on the differential cross section. Moreover, by using the ratio of the cross sections in the signal and normalization regions, we expect that the theoretical uncertainties will be further reduced to the level of $\mathcal{O}(10^{-5})$, due to partial cancellation of common radiative corrections. Work is in progress to extend BabaYaga to $\mu e \rightarrow \mu e$ scattering and to quantify the actual accuracy on the computation of the ratio of cross sections by means of dedicated Monte Carlo simulations. Any further improvement in the theoretical accuracy would require the matching of QED resummation with exact two-loop corrections, which are not available at present for the $\mu e \rightarrow \mu e$ process but are within reach.

VI. CONCLUSIONS

We presented a novel approach to determine the running of $\alpha(t)$ in the space-like region and $a_\mu^{\text{HLO}}$, the leading hadronic contribution to the muon $g-2$, by scattering high-energy muons on atomic electrons of a low-$Z$ target through the process $\mu e \rightarrow \mu e$. The experiment is primarily based on a precise measurement of the scattering angles of the two outgoing particles as the $q^2$ of the muon-electron interaction can be directly determined by the electron (or muon) scattering angle.

An advantage of the muon beam is the possibility of employing a modular apparatus, with the target subdivided in subsequent layers. A low-$Z$ solid target is preferred in order to provide the required event rate, limiting at the same time the effect of multiple scattering as well as of other types of muon interactions (pair production, bremsstrahlung and nuclear interactions).

The normalization of the cross section is provided by the very same $\mu e \rightarrow \mu e$ process in the low-$q^2$ region, where the effect of the hadronic corrections on the fine-structure constant is negligible. Such a simple and robust technique has the potential to keep systematic effects under control, aiming to reach a systematic uncertainty of the same order as the statistical one. For this purpose a preliminary detector layout has been described. By considering a beam of 150 GeV muons with an average intensity of $\sim 1.3 \times 10^7$ muons/s, currently available at the CERN North Area, a statistical uncertainty of $\sim 0.3\%$ can be achieved on $a_\mu^{\text{HLO}}$ in two years of data taking.

A test of a single module could provide a validation of the proposed method.

ACKNOWLEDGMENTS

We would like to thank Carlo Brogolini, Lau Gatignon and Fred Jegerlehner for fruitful discussions, and Fedor Ignatov for fruitful discussions and help in the simulation. This work was supported in part by the Italian Ministry of University and Research under the PRIN project 2010YJ2NYW. M.P. acknowledges partial support by FP10 ITN Elusives (H2020-MS-CA-ITN-2015-674896) and Invisibles-Plus (H2020-MS-CA-RISE-2015-690575).

[1] T. Blum, A. Denig, I. Logashenko, E. de Rafael, B. Lee Roberts, T. Teubner and G. Venanzoni, arXiv:1311.2198 [hep-ph].
[2] F. Jegerlehner, EPJ Web Conf. 118 (2016) 01016.
C. Bouchiat, L. Michel, J. Phys. Radium 22 (1961) 121;  
L. Durand, Phys. Rev. 128 (1962) 441 [Erratum-ibid. 129 (1963) 2835];  
M. Gourdin, E. De Rafael, Nucl. Phys. B 10 (1969) 667.

C. Aubin, T. Blum, Phys. Rev. D 75 (2007) 114502;  
P. Boyle, L. Del Debbio, E. Kerrane, J. Janot, M. Petschlies, D.B. Renner, Phys. Rev. Lett. 107 (2011) 081802;  
M. Della Morte, B. Jager, A. Juttner, H. Wittig, JHEP 1203 (2012) 055;  
T. Blum et al., Phys. Rev. Lett. 116 (2016) no.23, 232002;  
B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen and G. P. Lepage. arXiv:1601.03071 [hep-lat].

C. Aubin, T. Blum, Phys. Rev. D 75 (2007) 114502;  
P. Boyle, L. Del Debbio, E. Kerrane, J. Janot, M. Petschlies, D.B. Renner, Phys. Rev. Lett. 107 (2011) 081802;  
M. Della Morte, B. Jager, A. Juttner, H. Wittig, JHEP 1203 (2012) 055;  
T. Blum et al., Phys. Rev. Lett. 116 (2016) no.23, 232002;  
B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen and G. P. Lepage. arXiv:1601.03071 [hep-lat].

C. Aubin, T. Blum, Phys. Rev. D 75 (2007) 114502;  
P. Boyle, L. Del Debbio, E. Kerrane, J. Janot, M. Petschlies, D.B. Renner, Phys. Rev. Lett. 107 (2011) 081802;  
M. Della Morte, B. Jager, A. Juttner, H. Wittig, JHEP 1203 (2012) 055;  
T. Blum et al., Phys. Rev. Lett. 116 (2016) no.23, 232002;  
B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen and G. P. Lepage. arXiv:1601.03071 [hep-lat].

C. Aubin, T. Blum, Phys. Rev. D 75 (2007) 114502;  
P. Boyle, L. Del Debbio, E. Kerrane, J. Janot, M. Petschlies, D.B. Renner, Phys. Rev. Lett. 107 (2011) 081802;  
M. Della Morte, B. Jager, A. Juttner, H. Wittig, JHEP 1203 (2012) 055;  
T. Blum et al., Phys. Rev. Lett. 116 (2016) no.23, 232002;  
B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen and G. P. Lepage. arXiv:1601.03071 [hep-lat].