A $C^1$ Anosov Diffeomorphism with a Horseshoe That Attracts Almost Any Point

C. Bonatti, S. S. Minkov, A. V. Okunev, and I. S. Shilin

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Abstract. We present an example of a $C^1$ Anosov diffeomorphism with a physical measure such that its basin has full Lebesgue measure and its support is a horseshoe of zero measure.

Key words: Anosov diffeomorphism, $C^1$-topology, physical measure, thick horseshoe.

1. Introduction. Consider a diffeomorphism $F$ that preserves a probability measure $\nu$. The basin of $\nu$ is the set of all points $x$ such that the sequence of measures

$$\delta^n_x := \frac{1}{n}(\delta_x + \cdots + \delta_{F^{n-1}(x)})$$

converges to $\nu$ in the weak-$*$ topology. A measure is called physical if its basin has positive Lebesgue measure. It is well known (see [1, Sec. 1.3]) that any transitive $C^2$ Anosov diffeomorphism (note that all known Anosov diffeomorphisms are transitive) has a unique physical measure and

- the basin of this measure has full Lebesgue measure,
- the support of this measure coincides with the whole phase space.

However, in $C^1$ dynamics there are many phenomena that are impossible in $C^2$. For example, Bowen [2] constructed an example of a $C^1$ diffeomorphism of the plane with a “thick” horseshoe (i.e., a horseshoe with positive Lebesgue measure). Using an analogue of this construction, we prove the following theorem.

Theorem 1. There exists a $C^1$ Anosov diffeomorphism $F_H$ of the 2-torus $T^2$ that admits a physical measure $\nu$ such that

- its basin has full Lebesgue measure,
- it is supported on a horseshoe $\mathcal{H}$,
- $\omega(x) = \mathcal{H}$ for a.e. $x$ with respect to the Lebesgue measure.

In our construction the horseshoe $\mathcal{H}$ will be semithick, i.e., $\mathcal{H}$ will be the product of a Cantor set with positive measure in the unstable direction and a Cantor set with zero measure in the stable direction. The dynamics on $\mathcal{H}$ in the unstable direction will be as in Bowen’s thick horseshoe, and in the stable direction, as in a linear horseshoe. Let us also mention that the last assertion of the theorem is not a corollary of the former two. Indeed, $\delta^n_x \to \nu$ does not imply that $\omega(x) \subset \text{supp} \nu$, since the orbit of $x$ may stay near $\text{supp} \nu$ almost all the time but still move far away from this set infinitely many times.

The Milnor attractor of a map $F$ (denoted by $A_M(F)$) is the minimal (by inclusion) closed set that contains the $\omega$-limit sets of almost all points with respect to the Lebesgue measure. This definition was introduced by Milnor in [4]. It was also proved in [4] that the Milnor attractor always exists. Since the definition of a Milnor attractor uses the Lebesgue measure, the Milnor attractor may change after conjugation by a homeomorphism. Let us say that the Milnor attractor of a $C^1$ diffeomorphism $F$ is topologically invariant in $C^1$ if, for any $C^1$-diffeomorphism $G$ such that $G$ and $F$ are conjugate by a homeomorphism ($G = H \circ F \circ H^{-1}$), we have $A_M(G) = H(A_M(F))$. The first example of a Milnor attractor which is not topologically invariant was constructed by
S. Minkov in his thesis and is similar to an example in [3, Sec. 3]. Our construction gives an open set of such examples.

**Corollary 2.** For any $C^1$ diffeomorphism that is sufficiently close to $F_\mathcal{H}$, the Milnor attractor is not topologically invariant in $C^1$.

**Proof.** Since $F_\mathcal{H}$ is structurally stable, it has a $C^1$ neighborhood $U$ such that all maps in $U$ are conjugate to each other. Since $C^2$ diffeomorphisms are dense in the space of $C^1$ diffeomorphisms, one may find a $C^2$ Anosov diffeomorphism $G \in U$. Any map $F \in U$ is conjugate to both $F_\mathcal{H}$ and $G$. Since $A_M(F_\mathcal{H}) = \mathcal{H}$, while $A_M(G) = \mathbb{T}^2$, the attractor of $F$ is not topologically invariant in $C^1$. \qed

2. The semithick horseshoe. The thick horseshoe constructed in [2] can be embedded in a $C^1$ Anosov diffeomorphism of $\mathbb{T}^2$ (see [5]). We need a similar example but with a semithick horseshoe, i.e., a horseshoe that is the product of a Cantor set with positive measure in the unstable direction and a Cantor set with zero measure in the stable direction. Consider any linear Anosov diffeomorphism $F_L$. In the sequel, we will refer to the unstable direction of $F_L$ as vertical and to the stable direction as horizontal.

**Lemma 3.** There exists a $C^1$ Anosov diffeomorphism $F_{\text{Init}} : \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties:

1. $F_{\text{Init}}$ preserves the unstable foliation of $F_L$ (we will call it the vertical foliation).
2. The restriction of $F_{\text{Init}}$ to some rectangle $UK$ is a horseshoe (see the picture). The boundary of $UK$ is formed by two vertical and two horizontal segments. The intersection of $UK$ with $F_{\text{Init}}^{-1}(UK)$ is the union of two stripes, $\Pi_+$ and $\Pi_-$. The map $F_{\text{Init}}$ has one fixed point inside both $\Pi_+$ and $\Pi_-$. These points lie strictly inside the two horizontal segments that form the upper and lower boundaries of $UK$.
3. Inside $\Pi_\pm$ the map $F_{\text{Init}}$ is the Cartesian product of a linear contraction in the horizontal direction and a nonlinear expansion in the vertical direction. This nonlinear expansion is as in Bowen’s thick horseshoe. The horseshoe $\mathcal{H} = \bigcap_{n=-\infty}^{\infty} F_{\text{Init}}^n(UK)$ is the product $C_{\text{thin}} \times C_{\text{thick}}$ of a horizontal thin Cantor set and a vertical thick Cantor set.

We do not present the proof of this lemma, because it is straightforward but has many technical details. Let us represent $UK$ as the product of a vertical segment $V$ and a horizontal segment $H$. Let $S = H \times C_{\text{thick}}$ denote the union of the stable leaves of the points of $\mathcal{H}$ for the restriction of $F_{\text{Init}}$ to $UK$. The set $S$ is foliated by the horizontal stable leaves of the points of the horseshoe, and so every point of $S$ is attracted to $\mathcal{H}$.

3. Plan of the proof of Theorem 1. We will construct a special class $\mathcal{C}^1$ of $C^1$ Anosov diffeomorphisms, and then we will prove that a generic map in this class has the properties claimed in Theorem 1.

The auxiliary class of homeomorphisms $\mathcal{C}$ consists of all homeomorphisms $F$ of the torus with the following properties:

1. $F$ coincides with $F_{\text{Init}}$ on $S$;
2. $F$ preserves the vertical foliation and expands the vertical leaves;
3. $F$ is bi-Lipschitz with constant $L$, where $L$ is a large number, the same for all maps in $\mathcal{C}$;
4. $F$ satisfies some technical requirements that we do not state here.

The special class of Anosov diffeomorphisms $\mathcal{C}^1$ is defined as the intersection of $\mathcal{C}$ with the closure of some small $C^1$-neighborhood of $F_{\text{Init}}$.

To prove Theorem 1, we need the following three facts. Let us endow $\mathcal{C}$ with the metric induced from $\text{Homeo}(\mathbb{T}^2)$ and $\mathcal{C}^1$ with the metric induced from $\text{Diff}^1(\mathbb{T}^2)$.

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*The subscript Init is a shorthand for the word “initial.” This map is not our example but some initial map that is a starting point of our construction.*
Proposition 4. \( \mathcal{C}_1 \) is a nonempty complete metric space.

Proof. \( \mathcal{C}_1 \) is a closed subspace of the complete metric space \( \text{Diff}^1(\mathbb{T}^2) \). It is nonempty, because it contains \( F_{\text{Init}} \).

For \( G \in \mathcal{C} \), let \( B(G) = \bigcup_{n \geq 0} G^{-n}(S) \) denote the union of the stable leaves of all points of \( \mathcal{H} \).

For \( \varepsilon > 0 \), we use \( A_\varepsilon \) to denote the set of all \( G \in \mathcal{C} \) with \( \text{Leb}(B(G)) > 1 - \varepsilon \), where \( \text{Leb} \) denotes the probability Lebesgue measure on \( \mathbb{T}^2 \).

Lemma 5. For any \( \varepsilon > 0 \), the set \( A_\varepsilon \) is open in \( \mathcal{C} \).

Sketch of the proof. Take \( F \in A_\varepsilon \); then \( \text{Leb}(B(F)) > 1 - \varepsilon \). Let us approximate \( B(F) = \bigcup_{n \geq 0} F^{-n}(S) \) by a set \( F^{-N}(\hat{S}) \), where \( N \) is a large number and \( \hat{S} \supset S \) is a finite union of segments that approximates \( S \) well (i.e., \( \text{Leb}(\hat{S} \setminus S) \) is small). If we slightly perturb \( F \), then the boundary of the set \( F^{-N}(\hat{S}) \) will shift slightly, so that the measure of this set will remain almost the same. Using that this set approximates \( B(F) \) well, one can show that \( \text{Leb}(B(F)) \) will remain almost the same after a small perturbation of \( F \). We use the bi-Lipschitz continuity of maps in \( \mathcal{C} \) to estimate from above the measure of \( F^{-N}(\hat{S} \setminus S) \) before and after the perturbation.

Lemma 6. For any \( \varepsilon > 0 \), the set \( A_\varepsilon \cap \mathcal{C}_1 \) is dense in \( \mathcal{C}_1 \).

Sketch of the proof. For a fixed map in \( \mathcal{C} \), we define the segments of level 0 as the connected components of the intersections of \( \mathbb{T}^2 \setminus \mathcal{UK} \) with the vertical leaves and the segments of level \( n \) as their \( n \)th preimages. If, for some \( N \), a map \( G \) is linear on all segments of level greater than \( N \), then \( \text{Leb}(B(G)) = 1 \). Let us sketch the proof of this fact.

- Each segment of level \( N \) after a bounded number of iterations is expanded onto some segment of a vertical leaf that cuts all the way through \( S \) (i.e., intersects all horizontal leaves that constitute \( S \)).
- Thus, there is a constant \( c > 0 \) such that the proportion of points of \( B(G) \) in each segment of level \( N \) is at least \( c \). This also holds for the segments of level greater than \( N \), because they are linearly expanded to segments of level \( N \).
- This leads us to a contradiction with the fact that the restriction of \( \mathbb{T}^2 \setminus B(G) \) to some vertical leaf has a density point. Hence \( \text{Leb}(B(G)) = 1 \).

Let us prove that \( A_\varepsilon \) is dense. For each \( F_0 \in \mathcal{C}_1 \), we need to find a map \( F \in (A_\varepsilon \cap \mathcal{C}_1) \) that is \( C^1 \)-closed to \( F_0 \). We will do this in two steps. First, we pick large \( N \) and build a nonsmooth map \( F_{PL} \in \mathcal{C} \) that is linear on all segments of level greater than \( N \). Second, we pick small \( \delta > 0 \) and, smoothing \( F_{PL} \), obtain a diffeomorphism \( F \in \mathcal{C}_1 \) that is \( \delta \)-close to \( F_{PL} \) in the topology of \( \text{Homeo}(\mathbb{T}^2) \). Since \( F_{PL} \in A_\varepsilon \) and this set is open, for small \( \delta \), we have \( F \in (A_\varepsilon \cap \mathcal{C}_1) \). It is possible to show that, for large \( N \) and small \( \delta \), the map \( F \) is \( C^1 \)-close to \( F_0 \).

Proof of Theorem 1. Consider the set \( A_0 := \bigcap_{\varepsilon=1/n} (A_\varepsilon \cap \mathcal{C}_1) \). By Lemmas 5 and 6 \( A_0 \) is a residual subset of \( \mathcal{C}_1 \). By the Baire theorem it is nonempty. Let us take any diffeomorphism in \( A_0 \) for \( F_\mathcal{H} \). It follows from the definition of \( A_0 \) that \( \text{Leb}(B(F_\mathcal{H})) = 1 \).

Let \( \chi : \{0,1\}^\mathbb{Z} \to \mathcal{H} \) denote the symbolic encoding map of the horseshoe. The physical measure \( \nu \) will be the \( \chi \)-image of the \((1/2,1/2)\)-Bernoulli measure. Clearly, \( \text{supp} \nu = \mathcal{H} \). It follows from Bowen’s construction of a thick horseshoe that the projection of \( \nu \) to the vertical axis coincides (up to a multiplicative constant) with the projection of the Lebesgue measure on \( S \). Using this property and Birkhoff’s ergodic theorem, we can easily prove that \( \text{Leb} \)-almost any point of \( S \) lies in the basin of \( \nu \). Thus, this basin also contains almost any point of \( F_\mathcal{H}^{-n}(S) \) for any \( n \) and, therefore, almost any point in \( B(F_\mathcal{H}) \). Since \( \text{Leb}(B(F_\mathcal{H})) = 1 \), this means that the basin of \( \nu \) has full Lebesgue measure.

Let us prove that \( \omega(x) = \mathcal{H} \) for almost any \( x \). Indeed, for any \( x \) in the basin of \( \nu \), we have \( \omega(x) \supset \mathcal{H} \), while for any \( x \in B(F_\mathcal{H}) \), we have \( \omega(x) \subset \mathcal{H} \).
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Université de Bourgogne, Dijon, France
e-mail: bonatti@u-bourgogne.fr

Moscow Center of Continuous Mathematical Education, Moscow, Russia
e-mail: stanislav.minkov@yandex.ru

National Research University Higher School of Economics, Moscow, Russia
e-mail: aokunev@list.ru

Moscow Center of Continuous Mathematical Education, Moscow, Russia
e-mail: shilin@yandex.ru

Translated by C. Bonatti, S. S. Minkov, A. V. Okunev, and I. S. Shilin