SOS APPROXIMATIONS OF NONNEGATIVE POLYNOMIALS VIA SIMPLE HIGH DEGREE PERTURBATIONS

JEAN B. LASSERRE AND TIM NETZER

Abstract. We show that every real polynomial \( f \) nonnegative on \([-1, 1]^n\) can be approximated in the \( l_1 \)-norm of coefficients, by a sequence of polynomials \( \{f_\varepsilon^r\} \) that are sums of squares. This complements the existence of s.o.s. approximations in the denseness result of Berg, Christensen and Ressel, as we provide a very simple and explicit approximation sequence.

Then we show that if the Moment Problem holds for a basic closed semi-algebraic set \( K_S \subset \mathbb{R}^n \) with nonempty interior, then every polynomial non-negative on \( K_S \) can be approximated in a similar fashion by elements from the corresponding preordering.

Finally, we show that the degree of the perturbation in the approximating sequence depends on \( \varepsilon \) as well as the degree and the size of coefficients of the nonnegative polynomial \( f \), but not on the specific values of its coefficients.

1. Introduction

Sums of squares (s.o.s.) polynomials are not only of self-interest, but are also of primary importance for practical computation, especially in view of their numerous potential applications, notably in polynomial optimization; see e.g. [5, 10, 12, 13]. Indeed, in the computational complexity terminology, checking whether a given polynomial is nonnegative is a NP-hard problem, whereas checking whether it is s.o.s. reduces to solving a (convex) semidefinite programming (SDP) problem which (up to arbitrary precision) can be done in time polynomial in the input size of the problem; for more detail on semidefinite programming, the interested reader is referred to Vandenberghe and Boyd [14].

It has been known for some time that the cone of s.o.s. polynomials is dense (for the \( l_1 \)-norm of coefficients) in the cone of polynomials nonnegative on the unit ball \([-1, 1]^n \subset \mathbb{R}^n\); see e.g. Berg, Christensen and Ressel [1] and Berg [2]. However, [1] is essentially an existence result.

Contribution. Our contribution is threefold:

(i) We first provide an explicit and very simple s.o.s. approximation of polynomials nonnegative on the unit ball \([-1, 1]^n\). Namely, let

\[
\Theta_r := 1 + \sum_{j=1}^n X_j^{2r} \in \mathbb{R}[X_1, \ldots, X_n].
\]

Then, given \( \varepsilon > 0 \) and a polynomial \( f \in \mathbb{R}[X_1, \ldots, X_n] \) nonnegative on \([-1, 1]^n\), the polynomial \( f_\varepsilon^r := f + \varepsilon \Theta_r \) is s.o.s. provided \( r \) is large enough, say \( r \geq r(f, \varepsilon) \). Of
course, \( \|f_{\varepsilon} - f\|_1 \to 0 \) as \( \varepsilon \to 0 \). Although our result is not completely constructive (as \( r(f, \varepsilon) \) is not known), it complements the pure existence result [1].

If \( f \) is nonnegative on the ball \([-l, l]^n\) for some \( l > 0 \), then for every \( \varepsilon > 0 \), the polynomial \( f + \varepsilon(1 + \sum_{j=1}^{n} (X_j/l)^{2r}) \) is s.o.s. provided \( r \) is sufficiently large (just use \( x \mapsto g(x) := f(lx) \geq 0 \) on \([-1,1]^n\)).

Note that the representation \( f + \varepsilon \Theta_r = q_{\varepsilon r} \) for some s.o.s. polynomial \( q_{\varepsilon r} \), is an obvious certificate of nonnegativity of \( f \) on \([-1,1]^n\). Indeed, for \( x \in [-1,1]^n \), one has

\[
f(x) + \varepsilon \Theta_r(x) = q_{\varepsilon r}(x) \geq 0,
\]

provided \( r \) is big enough. As all \( \Theta_r \) are bounded by \( n + 1 \) on \([-1,1]^n\), letting \( \varepsilon \downarrow 0 \) yields \( f(x) \geq 0 \).

Our s.o.s. approximation result states that to approximate (uniformly on \([-1,1]^n\)) a polynomial nonnegative on \([-1,1]^n\), it is enough to slightly perturb by a small \( \varepsilon > 0 \) its (maybe zero) coefficients of some even power of marginal monomials \( \{X_j^{2r}\} \).

The method of the proof is quite different and much simpler than that of [7] for s.o.s. approximation of nonnegative polynomials; in particular, it does not use Nussbaum’s deep result on moment sequences [9]. It also simplifies the approximating sequence obtained in [8] in the spirit of [7].

In addition, if one fixes à priori the degree \( r \) of the perturbation \( \Theta_r \), we also characterize the minimum value \( \varepsilon_r^* \) of the parameter \( \varepsilon \), to make \( f + \varepsilon \Theta_r \) a s.o.s. It is given by

\[
-\varepsilon_r^* := \min_L \{ L(f) \mid L : A_{2r} \to \mathbb{R} \text{ linear}, \; L(\Theta_r) \leq 1, \; L(h^2) \geq 0 \; \forall h \in A_r \},
\]

where \( A_r \) is the finite dimensional vector space of polynomials of degree at most \( r \).

(ii) We next obtain a similar approximation result for polynomials nonnegative on certain semi-algebraic sets. For a finite set \( S \subset \mathbb{R}[X_1, \ldots, X_n] \) of polynomials, denote by \( K_S \) the associated basic closed semi-algebraic set in \( \mathbb{R}^n \), and by \( T_S \) the preordering generated by \( S \). Assume that \( K_S \) has nonempty interior and the so-called Moment Problem holds for \( S \), that is, every linear form on \( \mathbb{R}[X_1, \ldots, X_n] \) which is nonnegative on \( T_S \) comes from a measure on \( K_S \). Then every polynomial \( f \) nonnegative on \( K_S \) is approximated in the \( l_1 \)-norm by the same sequence \( \{f_{\varepsilon r}\} \), which now lies in \( T_S \). In addition, if one uses the perturbation

\[
(2) \quad \theta_r := \sum_{i=1}^{n} \sum_{k=0}^{r} \frac{X_i^{2k}}{k!} \in \mathbb{R}[X_1, \ldots, X_n],
\]

instead of \( \Theta_r \) as in (1), one obtains a certificate of nonnegativity on \( K_S \). This is because when using \( \theta_r \), the fact that the (new) approximating sequence \( \{f_{\varepsilon r}\} \) lies in \( T_S \), also implies that \( f \) is nonnegative on \( K_S \). Therefore, one may use this property to detect whether some given \( f \) is nonnegative on \( K_S \).

(iii) Finally, we address the issue of identifying the factors that influence the degree \( r \) up to which one has to perturb \( f \) to obtain an s.o.s. We find that \( r \) depends only on \( \varepsilon \), the dimension \( n \), the degree and the size of the coefficients of \( f \), but not on the explicit choice of \( f \).

Link with related results. The s.o.s. approximation \( f + \varepsilon \Theta_r \) in (1) resembles the one in (2) recently introduced by the first author in [7], for polynomials nonnegative on the whole \( \mathbb{R}^n \); with \( \theta_r \) instead of \( \Theta_r \), it is proven in [7] that given a globally nonnegative polynomial \( f \) and \( \varepsilon > 0 \), the polynomial \( f + \varepsilon \theta_r \) is s.o.s. provided \( r \)
is large enough (and we also have \( \| f + \varepsilon \Theta_r - f \|_1 \to 0 \) as \( \varepsilon \to 0 \)). Notice that this latter result is also a certificate of nonnegativity on \( \mathbb{R}^n \) and is more than a denseness result for the \( l_1 \)-norm. Indeed, it also shows that every nonnegative polynomial can be approximated by s.o.s. polynomials uniformly on compact sets, a nice additional property.

So a polynomial \( f \) nonnegative on \( \mathbb{R}^n \) (hence also on \([-1, 1]^n\)) could be approximated either by \( f_{\varepsilon r} = f + \varepsilon \Theta_r \) or by \( f_{\varepsilon r} = f + \theta_r \) for sufficiently large \( r \in \mathbb{N} \); in both cases \( \| f - f_{\varepsilon r} \|_1 \to 0 \) as \( \varepsilon \to 0 \). However, the former approximation is not a certificate of nonnegativity of \( f \); in particular, it loses the nice property of uniform approximation on compact sets possessed by the latter.

In other words, the s.o.s. approximation \( f + \varepsilon \Theta_r \) is indeed specific for polynomials nonnegative on \([-1, 1]^n\). For polynomials nonnegative on \( \mathbb{R}^n \), the s.o.s. approximation \( f + \varepsilon \theta_r \) (although a little more complicated than \( f + \varepsilon \Theta_r \)) should be preferred.

The above mentioned Moment Problem for a finite set of polynomials \( S \subset \mathbb{R}[X_1, \ldots, X_n] \) is discussed in e.g. [3, 4], where the authors ask whether for each polynomial \( f \) nonnegative on the corresponding basic closed semi-algebraic set \( K_S \), there exists some polynomial \( q \in \mathbb{R}[X_1, \ldots, X_n] \) such that for every \( \varepsilon > 0 \), the polynomial \( f + \varepsilon q \) lies in the preordering \( T_S \) generated by \( S \). This is still an open problem. Our result is weaker, as the polynomial \( q \) (= \( \Theta_r \) or \( \theta_r \)) depends on \( \varepsilon \) via its degree \( r \).

Finally, the degree bounds that we discuss here have been already investigated in [8] in a similar context, but for the approximations obtained in [7].

The paper is organized as follows. After introducing some notation and definitions in §2, our results are presented in §3.1 for s.o.s. approximations of polynomials nonnegative on \([-1, 1]^n\), in §3.2 for related results on polynomials nonnegative on a basic closed semi-algebraic set \( K_S \subset \mathbb{R}^n \), and in §3.3 for results on the degree bounds. For ease of exposition, some technical proofs have been postponed in an Appendix in §4.

2. Notations and definitions

Let \( \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n] \) denote the ring of real polynomials, \( A_r \) the finite dimensional subspace of polynomials of degree at most \( r \) and \( s(r) = \binom{n+r}{n} \) its dimension. Let \( A_{\text{sos}} \subset A_r \) be the space of s.o.s. polynomials of degree at most \( r \).

We always fix the canonical monomial basis for \( A_r \) and \( \mathbb{R}[X_1, \ldots, X_n] \), if we consider them as real vector spaces. For \( \alpha \in \mathbb{N}^n \), we write \( X^\alpha \) for \( X_1^{\alpha_1} \cdots X_n^{\alpha_n} \), and \( |\alpha| \) for \( \sum_{i=1}^n \alpha_i \).

A linear form \( L \) on \( \mathbb{R}[X] \) is said to have a representing measure \( \mu \) if

\[
L(f) = \int_{\mathbb{R}^n} fd\mu \quad \forall f \in \mathbb{R}[X] .
\]

This is the same as saying that the sequence of values of \( L \) on the canonical monomial basis is the moment sequence of this measure \( \mu \).

Of course not every linear form has a representing measure. However, there is a sufficient condition to ensure that it is indeed the case.

**Definition 2.1.** A function \( \varphi : \mathbb{N}^n \to \mathbb{R}_+ \) is called an absolute value if

(i) \( \varphi(0) = 1 \);

(ii) \( \varphi(\alpha + \beta) \leq \varphi(\alpha) \varphi(\beta) \) for all \( \alpha, \beta \in \mathbb{N}^n \).
The following result is stated in Berg et al. [1].

**Theorem 2.2.** Let $L$ be a linear form on $\mathbb{R}[X]$ such that $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X]$. If there is an absolute value $\varphi$ and a constant $C > 0$ such that $|L(X^\alpha)| \leq C\varphi(\alpha)$ for all $\alpha \in \mathbb{N}^n$, then $L$ has exactly one representing measure $\mu$ on $\mathbb{R}^n$. The support of $\mu$ is contained in the set $\{x \in \mathbb{R}^n \mid |x^\alpha| \leq \varphi(\alpha) \forall \alpha \in \mathbb{N}^n\}$.

For a finite set $S = \{g_1, \ldots, g_s\}$ of polynomials, denote by $K_S$ the basic closed semi-algebraic set $K_S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \forall i = 1, \ldots, s\}$, and by $T_S$ the preordering generated by $S$, i.e the set of all finite sums of polynomials of the form

$$\sigma_e g_1^{e_1} \cdots g_s^{e_s},$$

where $e \in \{0, 1\}^s$ and $\sigma_e$ is s.o.s. Further, let $T_r$ be the set of all finite sums of such elements $\sigma_e g_1^{e_1} \cdots g_s^{e_s}$ of degree at most $r$. Note that this is different from $T_S \cap A_r$ in general, as cancellation of leading forms could result in a polynomial of degree at most $r$, without the single polynomials having this property.

For the degree bound issue addressed in §3.3, one needs some elementary notions from the theory of real closed fields and valuation theory. Given a real closed extension field $R$ of $\mathbb{R}$, denote by $\mathcal{O}$ the convex hull of $\mathbb{Z}$ in $R$, i.e

$$\mathcal{O} = \{x \in R \mid \exists m \in \mathbb{N} : |x| \leq m\}.$$

$\mathcal{O}$ is a valuation ring of $R$ with maximal ideal

$$m = \{x \in \mathcal{O} \mid \forall n \in \mathbb{N} \setminus \{0\} : |x| \leq \frac{1}{n}\}.$$

Let $\overline{R} := \mathcal{O}/m$ denote the residue field and $\sigma : \mathcal{O} \to \overline{R}$ the order preserving residue map. We have $\overline{R} = \mathbb{R}$ and $\sigma$ is the identity on $\mathbb{R}$. In fact, for every $\beta \in \mathcal{O}$ there is exactly one $b \in \mathbb{R}$ such that $\beta \equiv b \mod m$.

### 3. Main results.

In this section we prove our main results, whereas for ease of exposition, some technical proofs are postponed in §4. We first consider polynomials nonnegative on the unit ball $[-1, 1]^n$.

#### 3.1. Nonnegativity on the ball $[-1, 1]^n$.

We begin with the following result of its own interest.

**Theorem 3.1.** Let $f \in \mathbb{R}[X]$ be a polynomial of degree $r_f$, and let $\Theta_r \in \mathbb{R}[X]$ be as in (1). Let $r_f \leq 2r \in \mathbb{N}$ be fixed and consider the semidefinite program

$$\min \{L(f) \mid L : A_{2r} \to \mathbb{R} \text{ linear}, L(\Theta_r) \leq 1, L(h^2) \geq 0 \forall h \in A_r\} =: \varepsilon_r^*.$$

Then

(i) $\varepsilon_r^* < \infty$ and (3) is solvable, i.e. $\varepsilon_r^* = L(f)$ for some feasible $L$.

(ii) The polynomial $f_{\varepsilon r} := f + \varepsilon \Theta_r$ is s.o.s. if and only if $\varepsilon \geq -\varepsilon_r^*$.

(Note that the condition $L(h^2) \geq 0 \forall h \in A_r$ translates to the positive semidefiniteness of the matrix which represents the bilinear form $(p, q) \mapsto L(pq)$. Therefore (3) is an SDP.)
Proof. (i) The set of feasible solutions for (3) is nonempty, take for example the zero form. So \( \varepsilon^*_r < \infty \). Furthermore, the set of feasible solutions is compact (if we consider each linear form on \( \mathcal{A}_{2r} \) as the \( (2r) \)-vector of its values on the monomial basis). Indeed, the constraint \( L(\Theta_r) \leq 1 \) implies that
\[
L(1) \leq 1; \quad L(X_r^{2r}) \leq 1, \quad i = 1, \ldots, n.
\]
As \( L(p^2) \geq 0 \) for all \( p \in \mathcal{A}_r \), by Lemma 4.1 and Lemma 4.3 from the appendix, one has \( |L(X^n)| \leq 1 \) for all \( |\alpha| \leq 2r \). So the set of feasible solutions in \( \mathbb{R}^{s(2r)} \) is bounded. As it is obviously closed as well, it is compact. Since the objective function is linear and therefore continuous, there always exists an optimal solution.

(ii) By definition, the minimum value \( \varepsilon'_r \) for which \( f_{\varepsilon} \) is s.o.s. is given by
\[
\varepsilon'_r = \min \{ \varepsilon \mid f + \varepsilon \Theta_r \in \mathcal{A}_{2r}^{\text{sos}} \}.
\]
But (4) is an SDP whose dual reads
\[
\max_L \{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R}, \quad L(\Theta_r) \leq 1, \quad L(h^2) \leq 0 \forall h \in \mathcal{A}_r \}\.
\]
Equivalently, with the change of variable \( L \to -L \),
\[
\tag{5} - \min_L \{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R}, \quad L(\Theta_r) \leq 1, \quad L(h^2) \geq 0 \forall h \in \mathcal{A}_r \}\.
\]
One next proves that there is no duality gap between the respective primal and dual problems (4) and (5), that is, their respective optimal values are equal.

Let \( \mu \) be a measure on \( \mathbb{R}^n \) with all moments up to order \( 2r \) finite and with a strictly positive density. One may scale \( \mu \) to satisfy \( \int_{\mathbb{R}^n} \Theta_r d\mu < 1 \). Let \( L \) be integration with respect to \( \mu \). As \( \mu \) has strictly positive density, we must have \( L(p^2) > 0 \) for all \( p \in \mathcal{A}_r \setminus \{0\} \), and so \( L \) is a strictly feasible solution for the SDP in (5), that is, Slater’s condition holds, which in turn implies that both SDP problems in (4) and (5) have the same optimal value \( \varepsilon'_r = -\varepsilon^*_r \); see e.g. [14].

So the only if part in (ii) follows from the definition of \( \varepsilon'_r \). Now let \( \varepsilon \geq -\varepsilon^*_r \) and write
\[
f + \varepsilon \Theta_r = f - \varepsilon^*_r \Theta_r + (\varepsilon + \varepsilon^*_r) \Theta_r,
\]
and use that \( f - \varepsilon^*_r \Theta_r \) as well as \( (\varepsilon + \varepsilon^*_r) \Theta_r \) are s.o.s. to obtain the result. \( \square \)

Observe that \( \varepsilon^*_r = 0 \) whenever \( f \) is a s.o.s., because then \( L(f) \geq 0 \) for every feasible \( L \) and the zero linear form is feasible. If \( f \) is not s.o.s. (so \( \varepsilon^*_r < \infty \)), then the inequality constraint \( L(\Theta_r) \leq 1 \) in (3) can be replaced with the equality constraint \( L(\Theta_r) = 1 \), since by linearity, given a feasible solution \( L \) with \( L(\Theta_r) < 1 \) and with value \( L(f) < 0 \), one always obtains a better feasible solution \( L' = qL \) with \( L'(\Theta_r) = 1 \) (note that \( L(\Theta_r) = 0 \) implies \( L = 0 \)).

Next, we obtain the following crucial result.

Theorem 3.2. Let \( f \in \mathbb{R}[X] \) be a polynomial of degree \( rf \), nonnegative on \([-1,1]^n \), and let \( \Theta_r \in \mathbb{R}[X] \) be as in (1). Let \( \varepsilon^*_r \) be the optimal value of the semidefinite program defined in (3), for all \( 2r \geq rf \). Then \( \varepsilon^*_r \to 0 \) as \( r \to \infty \).

Proof. From Theorem 3.1, \( \varepsilon^*_r = L^{(r)}(f) \leq 0 \) for some optimal solution \( L^{(r)} \) of the semidefinite program (3), whenever \( 2r \geq rf \). From the proof of Theorem 3.1, it follows that \( |L^{(r)}(X^n)| \leq 1 \) for all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq 2r \). Next, complete the vector of the values of \( L^{(r)} \) on the monomial basis of \( \mathcal{A}_r \) with zeros to make it an element in \( \mathbb{R}^{N_{2r}} \), and in fact even an element of \([-1,1]^{N_{2r}} \). By Tychonoff’s Theorem, we find
a subsequence \( r_k \) such that the sequence \( L^{(r_k)} \) converges to some \( y^* \in [-1, 1]^n \) in the product topology, and in particular pointwise convergence holds, i.e.
\[
L^{(r_k)}(x^\alpha) \to y^*_\alpha \quad \forall \alpha \in \mathbb{N}^n.
\]

Let \( L^* \) be the linear form on \( \mathbb{R}[X] \) defined by \( L^*(X^n) := y^*_n \). From the pointwise convergence in (6) we obtain \( L^*(p^2) \geq 0 \) for all \( p \in \mathbb{R}[X] \). This, together with \( y^* \in [-1, 1]^n \), implies that \( L^* \) has a representing measure \( \mu^* \) with support contained in \([-1, 1]^n \) (see Theorem 2.2). Now again from the pointwise convergence (6),
\[
\varepsilon_{rk} = L^{(r_k)}(f) \to L^*(f) = \int_{[-1,1]^n} f d\mu^* \geq 0,
\]
where the inequality uses nonnegativity of \( f \) on \([-1, 1]^n \). Since all \( \varepsilon_{rk} \leq 0 \), we get \( \varepsilon_{rk} \to 0 \). And as the converging subsequence \( r_k \) was arbitrary, this shows the desired result.

Therefore, we finally obtain:

**Corollary 3.3.** Let \( f \in \mathbb{R}[X] \) be a polynomial nonnegative on \([-1, 1]^n \) and let \( \Theta_r \in \mathbb{R}[X] \) be as in (1). Let \( \varepsilon > 0 \) be fixed. Then there exists some \( r(f, \varepsilon) \in \mathbb{N} \) such that for every \( r \geq r(f, \varepsilon) \), the polynomial \( f_r := f + \varepsilon \Theta_r \) is a s.o.s.

**Proof.** From Theorem 3.2 we know that the sequence \( \{\varepsilon_r^*\} \) with \( \varepsilon_r^* \) defined in (3) converges to 0 as \( r \to \infty \). So there is an \( r(f, \varepsilon) \) such that for all \( r \geq r(f, \varepsilon) \) we have \( \varepsilon_r^* \geq -\varepsilon \). By Theorem 3.1 the polynomial \( f - \varepsilon_r^* \Theta_r \) is a s.o.s., and so
\[
f + \varepsilon \Theta_r = f - \varepsilon_r^* \Theta_r + (\varepsilon + \varepsilon_r^*) \Theta_r
\]
is a s.o.s. as well, since \( (\varepsilon + \varepsilon_r^*) \Theta_r \) is also a s.o.s. \( (\varepsilon_r^* \geq -\varepsilon) \).

Corollary 3.3 refines the *denseness* result of Berg [2], because it provides an explicit approximation sequence. In addition, this approximation sequence is extremely simple, as the perturbation polynomial \( \Theta_r \) contains only the constant and the marginal monomials \( X_i^{2r}, i = 1, \ldots, n \). In addition, it provides a *certificate* of nonnegativity of \( f \) on \([-1, 1]^n \); indeed, if \( x \in [-1, 1]^n \), then for every \( r \geq r(f, \varepsilon) \) one has \( f(x) + \varepsilon \Theta_r(x) \geq 0 \). Letting \( \varepsilon \to 0 \) yields \( f(x) \geq 0 \).

It is straightforward to extend Corollary 3.3 to the case of a polynomial \( f \) nonnegative on the ball \([-l, l]^n \subset \mathbb{R}^n \) for some \( l > 0 \). Indeed, it suffices to apply Corollary 3.3 to the polynomial \( x \mapsto g(x) := f(lx) \) which is nonnegative on \([-1, 1]^n \). In this case the polynomial \( f + \varepsilon(1 + \sum_{j=1}^n (X_j/l)^{2r}) \) provides an s.o.s. approximation.

In some specific examples, one may even obtain a more precise result. Namely, given \( r \) fixed, one may provide an explicit bound \( \varepsilon_r > 0 \), such that the polynomial \( f_{r'} := f + \varepsilon \Theta_r \) is s.o.s. This is illustrated in the following nice two examples, kindly provided by Bruce Reznick.

**Example 3.4.** Consider the univariate polynomial \( f = 1 - X^2 \), obviously nonnegative on \([-1, 1] \). If \( \varepsilon \geq \varepsilon_r^* := (r-1)^{-1}/r^r \), the polynomial
\[
f_{r'} := 1 - X^2 + \varepsilon X^{2r}
\]
is globally nonnegative and therefore a s.o.s. Indeed, its minimum occurs when \(-2\varepsilon + 2\varepsilon x^{2r-1} = 0 \), i.e. at \( x_r := (1/r\varepsilon)^{1/(2r-2)} \). Hence, the value at \( x_r \) is
\[
1 - x_r^2 + \varepsilon x_r^2 x_r^{2r-2} = 1 - x_r^2 (r-1)/r,
\]

which is nonnegative if and only if
\[ x^2 \leq r/(r-1) \Leftrightarrow x^{2r}-2 \leq (r/(r-1))^{r-1} \Leftrightarrow 1/(r\varepsilon) \leq (r/(r-1))^{r-1}, \]
i.e. if and only if \( \varepsilon \geq (r-1)^{-1}/r^r = \varepsilon^*_r. \)

**Example 3.5.** On the other hand, consider the Motzkin polynomial \( f = 1 + X^2Y^2(X^2 + Y^2 - 3) \in \mathbb{R}[X,Y] \) which is nonnegative but not a s.o.s. Then, for all \( r \geq 3 \) and \( \varepsilon := 2^{4-2r} \), the polynomial \( f_{\varepsilon r} := f + \varepsilon X^{2r} \) is a s.o.s., and \( \|f - f_{\varepsilon r}\|_1 \to 0 \) as \( r \to \infty \). To prove this, write
\[ f = (XY^2 + X^3/2 - 3X/2)^2 + p, \]
where \( p = 1 - (X^3/2 - 3X/2)^2 = (1 - X^2)^2(1 - X^2/4) \). Next, the univariate polynomial \( q = p + 2^{4-2r}X^{2r} \) is nonnegative on \( \mathbb{R} \), hence a sum of squares. Indeed, if \( x^2 \leq 4 \), then \( p \geq 0 \) and so \( q \geq 0 \). If \( x^2 > 4 \), then \( |p(x)| \leq (x^2)^2x^2/4 = x^6/4. \) From
\[ q(x) \geq 2^{4-2r}x^{2r} - |p(x)| \geq \frac{x^6}{4}((x^2/4)^{r-3} - 1), \]
and the fact that \( n \geq 3, x^2 > 4 \), we deduce that \( q(x) \geq 0 \).

In Example 3.4, one approximates \( 1 - X^2 \) (uniformly on \([-1,1]\)) by the s.o.s. \( 1 - X^2 + \varepsilon X^{2r} \). In Example 3.5, the Motzkin polynomial can also be approximated in the \( l_1 \)-norm by \( f + \varepsilon (X^{2r} + Y^{2r}) \), but not uniformly on compact sets. For the latter property to hold, one needs the perturbation \( f + \varepsilon \sum_{j=1}^n \sum_{k=0}^r X_j^{2k}/k! \) introduced in [7].

### 3.2. Nonnegativity on basic closed semi-algebraic sets

We next prove the second announced result, namely the approximation of polynomials nonnegative on basic closed semi-algebraic sets. Let \( S \subseteq \mathbb{R}[X] \) be a finite set of polynomials and suppose the Moment Problem is solvable for \( S \), which means that every linear form on \( \mathbb{R}[X] \) which is nonnegative on the preordering \( T_S \), is integration with respect to some measure on \( K_S \). Further suppose \( K_S \) has nonempty interior, and let \( f \in \mathbb{R}[X] \) be nonnegative on \( K_S \).

With same notation as in §3.1, consider the semidefinite program
\[
(7) \quad \varepsilon^*_r := \min_L \{ L(f) \mid L : \mathcal{A}_{2r} \to \mathbb{R} \text{ linear, } L(\Theta_r) \leq 1, \ L(t) \geq 0 \ \forall t \in T_{2r} \}.
\]

Its dual reads
\[
(8) \quad \max_\varepsilon \{ \varepsilon \mid f - \varepsilon \Theta_r \in T_{2r} \}.
\]

Proceeding exactly as in the proof of Theorem 3.1, one constructs a strictly feasible solution for (7) as integration with respect to some (suitably scaled) measure on a ball in \( K_S \). Hence, with same arguments, the SDP (7) is also always solvable (note that \( \mathcal{A}_{2r}^{\text{gen}} \subseteq T_{2r} \)), and there is no duality gap between the SDPs (7) and (8), i.e., their optimal values are equal.

Again, every sequence of optimal solutions for (7) (with \( r \) growing) has a subsequence that converges pointwise to some \( \mathbf{y}^* \in [-1,1]^{2r} \) which is the moment sequence of some measure on \( K_S \), this time using the fact that the moment problem holds for \( S \). So, as in the proof of Theorem 3.2, the sequence \( \{\varepsilon^*_r\} \) converges to 0, since \( f \) is nonnegative on \( K_S \). Hence, as in Corollary 3.3, we get the following result:
Corollary 3.6. Let $S \subseteq \mathbb{R}[X]$ be a finite set of polynomials and suppose that the Moment Problem is solvable for $S$. Further, suppose that $K_S$ has a nonempty interior. Let $f \in \mathbb{R}[X]$ be nonnegative on $K_S$ and let $\Theta_r \in \mathbb{R}[X]$ be as in (1). Let $\varepsilon > 0$ be fixed. Then there is some $r(f, \varepsilon, S)$ such that for every $r \geq r(f, \varepsilon, S)$, the polynomial $f_r := f + \varepsilon \Theta_r$ lies in $T_S$.

Note that the pointwise limit $y^*$ from above is the moment sequence of a measure on $K_S$ as the Moment Problem holds for $S$, but on the other hand it is also the moment sequence of a measure on $[-1,1]^n$, as $y^* \in [-1,1]^n$ (Theorem 2.2). But by Theorem 2.2, $y^*$ is the moment sequence of exactly one measure. So the measure must be supported by $K_S \cap [-1,1]^n$. This leads to the fact that in Corollary 3.6, the polynomial $f$ must only be nonnegative on $K_S \cap [-1,1]^n$ for the statement to hold. So for example if $[-1,1]^n \cap K_S = \emptyset$, it holds for every polynomial $f$.

However, notice that $"f + \varepsilon \Theta_r$ lies in $T_S$" provides a certificate of nonnegativity of $f$ on $K_S \cap [-1,1]^n$ only, and not on $K_S$. So Corollary 3.6 is useful when one already knows that $f$ is nonnegative on $K_S$ and one wishes to obtain an $l_1$-norm approximation in $T_S$. If one wishes to test whether $f$ is indeed nonnegative on $K_S$, then the following result provides a certificate of nonnegativity on $K_S$.

Corollary 3.7. Let $S \subseteq \mathbb{R}[X]$ be a finite set of polynomials and suppose that the Moment Problem is solvable for $S$. Further, suppose that $K_S$ has a nonempty interior. Let $f \in \mathbb{R}[X]$ be nonnegative on $K_S$ and let $\theta_r \in \mathbb{R}[X]$ be as in (2). Let $\varepsilon > 0$ be fixed. Then there is some $r(f, \varepsilon, S)$ such that for every $r \geq r(f, \varepsilon, S)$, the polynomial $f_r := f + \varepsilon \Theta_r$ lies in $T_S$.

The proof is similar to that of Corollary 3.6, except that in the semidefinite program (7) we now have the constraint $L(\theta_r) \leq 1$ (instead of $L(\Theta_r) \leq 1$). In this case, every sequence of optimal solutions for (7) (with $r$ growing) has a subsequence that converges pointwise to some $y^* \in \mathbb{R}^n$ (rather than $y^* \in [-1,1]^n$). To prove this result, and as one cannot use Theorem 2.2 any more, one now invokes Nussbaum’s result [9] on moment sequences, which, in the present context, states that if

$$\sum_{i=1}^{n} \sum_{k=1}^{\infty} L(X_i^{2k})^{-1/2k} = +\infty, \quad i = 1, \ldots, n,$$

then $L$ is integration with respect to some measure on $\mathbb{R}^n$; see also Berg [2, Theorem 8]. The rest of the proof is identical.

That Corollary 3.7 provides a certificate of nonnegativity of $f$ on $K_S$, follows from the fact that $\theta_r(x)$ is bounded by $\sum_{i=1}^{n} \exp(x_i^2)$, for all $x \in \mathbb{R}^n$. Therefore, fix $x \in K_S$; as $f + \varepsilon \Theta_r$ lies in $T_S$, one has $f(x) + \varepsilon \theta_r(x) \geq 0$. Letting $\varepsilon \to 0$ yields $f(x) \geq 0$, the desired result.

The result in Corollary 3.6 (resp. in Corollary 3.7) is weaker than the condition $f + \varepsilon q \in T_S$ for some fixed $q$ and all $\varepsilon > 0$, as our $\Theta_r$ (resp. $\theta_r$) depends on $\varepsilon$ (via $r$). Whether the Moment Problem implies even this stronger version is an open problem, see for example [3, 4].

3.3. The degree of the perturbation. We are now concerned with the last announced result. We prove that the degree $r(f, \varepsilon)$ in Corollary 3.3 does not depend on the explicit choice of the polynomial $f$ but only on
• $\varepsilon$ and the dimension $n$,
• the degree and the size of the coefficients of $f$.

Therefore, if we fix these four parameters, we find an $r$ such that the statement of Corollary 3.3 holds for any $f$ nonnegative on $[-1,1]^n$, whose degree and size of the coefficients do not exceed the fixed parameters.

We first generalize Corollary 3.3 to real closed extension fields of $\mathbb{R}$ and then use the result in an ultrapower of $\mathbb{R}$. This approach towards degree bounds is similar to the one in [11].

Let $\Theta_r$ be as in (1). We first write the strict duality of the SDP problems (4) and (5) as a first order logic formula in the language of ordered rings with coefficients from $\mathbb{R}$. We just say that for every polynomial $f$ of some fixed maximum degree $2r$, there is a linear form $L$ on $A_{2r}$ (indeed a $s(2r)$-tuple of values) which is nonnegative on $A_{2r}^{\text{g.s.}}$ and which is less than or equal to 1 on $\Theta_r$. We also demand that all the values of $L$ on the monomial basis are bounded by 1 (as we have seen, this follows from the other conditions anyway). Further, we say that there exists some $\varepsilon$ such that $f+\varepsilon \Theta_r$ is a s.o.s and $\varepsilon = -L(f)$ with $L$ from above. All this can be done, using the known fact that every polynomial in $A_{2r}^{\text{g.s.}}$ is already a sum of $s(2r)$ squares of polynomials from $A_r$.

So, by Tarski’s Transfer Principle, for every $r \in \mathbb{N}$, this formula holds in every real closed extension field of $\mathbb{R}$. We use this in the following theorem:

**Theorem 3.8.** Let $R$ be a real closed extension field of $\mathbb{R}$, and denote by $\mathcal{O}$ the convex hull of $\mathbb{Z}$ with respect to the unique ordering in $R$. Let $m$ denote the unique maximal ideal in the valuation ring $\mathcal{O}$, and fix some $\varepsilon \in R, \varepsilon > 0$ and $\varepsilon \notin m$. Suppose $f \in \mathcal{O}[X]$ is nonnegative on $[-1,1]^n \subset R^n$. Then there exists $r \in \mathbb{N}$ such that the polynomial $f_{cr} = f + \varepsilon \Theta_r$ is a s.o.s. in $R[X]$.

**Proof.** Let $\overline{f}$ be the real polynomial obtained from $f$ by applying the residue map $\sigma: \mathcal{O} \to \mathcal{O}/m = \mathbb{R}$ to the coefficients of $f$. As $f \geq 0$ on $[-1,1]^n \subset R^n$, we have $\overline{f} \geq 0$ on $[-1,1]^n \subset \mathbb{R}^n$.

Next, consider the SDP problems from (3) associated with $\overline{f}$. From Theorem 3.2, there exists some $r$ such that $\varepsilon_r^* > -\sigma(\varepsilon)$ ($\varepsilon > 0, \varepsilon \notin m$ implies $\sigma(\varepsilon) > 0$). With that $r$ fixed, we now use that the formula described above holds in $R$. That is, we first get a linear form $L$ on the subspace of polynomials of $R[X]$ with degree at most $2r$, whose values on the monomial basis are bounded by 1 (and therefore, are in $\mathcal{O}$), which is nonnegative on the s.o.s. polynomials. Further, we also have $L(\Theta_r) \leq 1$. In addition, we get an $\varepsilon'$ such that $f + \varepsilon' \Theta_r$ is a s.o.s. in $R[X]$ and $\varepsilon' = -L(f)$.

But now, we can apply the residue map $\sigma$ to the values of $L$ on the monomial basis and get a linear form $\overline{L}$ which is feasible for the optimization problem from (3) associated with $\overline{f}$ and $r$. So

$$-\sigma(\varepsilon) < \varepsilon_r^* \leq \overline{T}(\overline{f}) = \sigma(L(f)) = -\sigma(\varepsilon').$$

This shows $\varepsilon' < \varepsilon$, and as $f + \varepsilon' \Theta_r$ is a s.o.s. in $R[X]$, so is $f + \varepsilon \Theta_r$. \hfill $\Box$

Once we have this result, the rest follows from a standard ultrapower argument. We use the result in

$$\mathbb{R}^* = (\prod_{N} \mathbb{R})/\mathcal{U},$$
where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

Fix some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and define by a first order logic formula $\Phi$ in the language of ordered rings, the set of all polynomials $f$ of degree at most $d$, with coefficients bounded by some $N \in \mathbb{N}$, and which are nonnegative on $[-1,1]^n$.

Next, for every $r \in \mathbb{N}$, define by a formula $\varphi_r$, the set of all polynomials $f$ of degree at most $d$, such that $f + \varepsilon \Theta_r$ is a s.o.s.

Notice that boundedness of the coefficients of a polynomial $f$ by some $N \in \mathbb{N}$, implies $f \in \mathcal{O}[X]$, and so, by Theorem 3.8, one has

$$\Phi \rightarrow \bigvee_{r \in \mathbb{N}} \varphi_r.$$ 

Now the $\mathcal{N}_1$-saturation of $\mathbb{R}^*$ yields

$$\Phi \rightarrow \varphi_{r'},$$

for some $r'$ depending on the formulas used, i.e. on $d, N, n, \varepsilon$. Therefore, in $\mathbb{R}^*$ one may choose the degree $r$ in Theorem 3.8 to depend only on $d, N, n, \varepsilon$. As this can be again formulated as a first order logic formula, it holds in $\mathbb{R}$ as well:

**Theorem 3.9.** Let $n, N, d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$ be given. Then there exists $r = r(n, N, d, \varepsilon) \in \mathbb{N}$ such that for every $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $d$, with coefficients bounded by $N$, and nonnegative on $[-1,1]^n$, the polynomial $f + \varepsilon \Theta_r$ is a s.o.s. (and so are $f + c\Theta_r$ for all $r' \geq r$).

4. APPENDIX

In this section we derive auxiliary results that are helpful in the proofs of the main section.

**Lemma 4.1.** Let $n = 1$ and let $L : \mathcal{A}_2 \rightarrow \mathbb{R}$ be a linear form such that $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$. Then $L(X^{2k}) \leq \max[L(1), L(X^{2r})]$ for all $k = 0, \ldots, r$.

**Proof.** The proof is by induction on $r$. Indeed for $r = 0$ and $r = 1$ the statement is trivial. So we assume the statement of Lemma 4.1 is true for some $r$ and we prove it for $r + 1$.

Let $L$ be a linear form on $\mathcal{A}_{2r+2}$ as stipulated. From $L(p^2) \geq 0$ for all $p \in \mathcal{A}_{r+1}$ we have

$$L(X^{2r+2}) \leq L(X^{2r+1})L(X^{2r-2}).$$

By the induction hypothesis, we have

$$L(X^{2k}) \leq \max[L(1), L(X^{2r})], \quad k = 0, \ldots, r.$$ 

Suppose first that $L(1) = \max[L(1), L(X^{2r})]$. Then obviously $L(X^{2k}) \leq \max[L(1), L(X^{2r+2})]$ for all $k \leq r + 1$ and we are done. Next, suppose $L(X^{2r}) = \max[L(1), L(X^{2r})]$. Then from (9) we obtain

$$L(X^{2r})^2 \leq L(X^{2r+2})L(X^{2r-2}) \leq L(X^{2r+2})L(X^{2r}),$$

so that $L(X^{2r}) \leq L(X^{2r+2})$. Therefore again $L(X^{2k}) \leq \max[L(1), L(X^{2r+2})]$ for all $k = 0, \ldots, r + 1$, the desired result.

**Lemma 4.2.** Let $n = 2$ and $L : \mathcal{A}_{2r} \rightarrow \mathbb{R}$ be a linear form and suppose $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$. Then all values $L(X^{2\alpha})$ where $0 \leq |\alpha| \leq r$ are bounded by $\max_{k=0,\ldots,r} \{L(X_1^{2k}), L(X_2^{2k})\}$.
Proof. What we will actually show is that all $L(X^{2\alpha})$, where $|\alpha| = k$, are bounded by $\max\{L(X_{2}^{2k}), L(X_{2}^{2k})\}$.

Let $p \in \mathbb{N}$ be such that either $k = 2p$ (if $k$ is even) or $k = 2p + 1$ (if $k$ is odd) and define $\Gamma := \{(2a, 2b) \mid a + b = k; \ a, b \neq 0\}$. One has $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 := \{(k, 0) + (k - 2i, 2i) \mid i = 1, ..., p\}$$

$$\Gamma_2 := \{(0, k) + (2j, k - 2j) \mid j = 1, ..., p\}.$$ 

If $k$ is odd, then this union is disjoint, else $\Gamma_1 \cap \Gamma_2 = \{(2p, 2p)\}$. For $s := \max\{L(X^{\gamma}) \mid \gamma \in \Gamma\}$, we get $s = L(X^{\gamma})$ for some $\gamma \in \Gamma_1$ or $\gamma \in \Gamma_2$.

From $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$, we have
$$L(X^{\alpha + \beta}) \leq L(X^{2\alpha})L(X^{2\beta}).$$

So in our case, we obtain

$$L(X_1^{2k}) \cdot L(X_1^{2k-4i}X_2^{4i}) \geq L(X_1^{2k-2i}X_2^{2i}), \ i = 1, ..., p \tag{10}$$

$$L(X_2^{2k}) \cdot L(X_1^{2j}X_2^{2k-4j}) \geq L(X_1^{2j}X_2^{2k-2j}), \ j = 1, ..., p. \tag{11}$$

With $s_k := \max\{L(X_1^{2k}), L(X_2^{2k})\}$, by (10) and (11), one gets either
$$s_k \cdot s \geq L(X_1^{2k}) \cdot L(X^{\gamma^*}) \geq L(X^{\gamma^*})^2 = s^2$$

or
$$s_k \cdot s \geq L(X_2^{2k}) \cdot L(X^{\gamma^*}) \geq L(X^{\gamma^*})^2 = s^2.$$

In any case $s_k \geq s$. \hfill \Box

Lemma 4.3. Let $n$ be arbitrary and $L : \mathcal{A}_r \to \mathbb{R}$ be a linear form and suppose $L(p^2) \geq 0$ for all $p \in \mathcal{A}$. Assume that for all $i=1, ..., n$ and $k=0, ..., r$, the values $L(X_1^{2k})$ are bounded by some $\tau$. Then all values $L(X^{\alpha})$, where $|\alpha| \leq 2r$, satisfy $|L(X^{\alpha})| \leq \tau$.

Proof. We only need to show that all values $L(X^{2\alpha})$, where $|\alpha| \leq r$, are bounded by $\tau$. Indeed, from $L(p^2) \geq 0$ for all $p \in \mathcal{A}_r$, we have $L(X^{\alpha + \beta}) \leq L(X^{2\alpha})L(X^{2\beta})$, and therefore, if all the values $L(X^{2\gamma})$ are bounded by $\tau$, one gets $|L(X^{\alpha})| \leq \tau$ for all $0 \leq |\alpha| \leq 2r$.

The proof is by induction on the number $n$ of variables.

$n = 1$ : Nothing is to be shown in this case, as all the values $L(X^{2\alpha})$ are bounded by $\tau$ by the assumption.

$n = 2$ : This is an immediate result of Lemma 4.2.

$n - 1 \rightarrow n, n > 2$ : By the induction hypothesis, the claim is true for all $L(X^{2\alpha})$, where $|\alpha| \leq r$ and some $\alpha_1 = 0$. Indeed, $L$ restricts to a linear form on the ring of polynomials with $n - 1$ indeterminates and satisfies all the assumptions needed. So the induction hypothesis gives the boundedness of all those values $L(X^{2\alpha})$.

Now take $L(X^{2\alpha})$, where $|\alpha| \leq r$ and all $\alpha_i \geq 1$. With no loss of generality, assume $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots \leq \alpha_n$. Consider the two elements
$$\gamma := (2\alpha_1, 0, \alpha_3 + \alpha_2 - \alpha_1, \alpha_4, ..., \alpha_n) \in \mathbb{N}^n$$

$$\gamma' := (0, 2\alpha_2, \alpha_3 + \alpha_1 - \alpha_2, \alpha_4, ..., \alpha_n) \in \mathbb{N}^n.$$ 

We have $|\gamma|, |\gamma'| \leq r$ and $\gamma_2 = \gamma_1' = 0$. Therefore, by the above result, we get
$$L(X^{2\gamma}) \leq \tau \text{ and } L(X^{2\gamma'}) \leq \tau.$$
As $L(p^2) \geq 0$ for all $p \in A_r$, one has
\[ L\left(X^{\alpha+\gamma'}\right)^2 \leq L\left(X^{2\gamma}\right) \cdot L\left(X^{2\gamma'}\right) \leq \tau^2, \]
which yields
\[ |L\left(X^{2\alpha}\right)| \leq \tau. \]
\[ \square \]

Acknowledgements

Both authors wish to thank M. Schweighofer for many interesting and helpful discussions on the topic. The work of the first author is partly supported by ANR Grant NT05-3-41612, while that of the second author is supported by the Land Baden-Württemberg through a Landesgraduiertenstipendium.

References

[1] C. Berg, J. P. R. Christensen, P. Ressel: Positive definite functions on abelian semigroups, Math. Ann. 223 (1976), pp. 253-274.
[2] C. Berg: The multidimensional moment problem and semigroups, Proc. Symp. Appl. Math. 37 (1987), 110-124.
[3] S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multidimensional moment problem, Trans. Amer. Math. Soc. 354 (2002), pp. 4285-4301.
[4] S. Kuhlmann, M. Marshall, N. Schwartz, Positivity, sums of squares and the multi-dimensional moment problem II, Advances in Geometry, to appear.
[5] J.B. Lasserre, Global optimization with polynomials and the problem of moments, SIAM J. Optim. 11 (2001), pp. 796-817.
[6] J. B. Lasserre, S.O.S. approximation of polynomials nonnegative on a real algebraic set, SIAM J. Optim., to appear.
[7] J. B. Lasserre, A sum of squares approximation of nonnegative polynomials, SIAM J. Optim., to appear.
[8] T. Netzer, High Degree Perturbations of Nonnegative Polynomials, Diploma Thesis, Department of Mathematics and Statistics, University of Konstanz, Germany, June 2005.
[9] A. E. Nussbaum: Quasi-analytic vectors, Ark. Mat. 6 (1966), pp. 179-191.
[10] P. A. Parrilo, Semidefinite programming relaxations for semialgebraic problems, Math. Progr. Ser. B 96 (2003), pp. 293-320.
[11] A. Prestel, C. N. Delzell: Positive polynomials, Springer, Berlin (2001).
[12] C. Scheiderer, Positivity and sums of squares: A guide to some recent results, Department of Mathematics, University of Duisburg, Germany.
[13] M. Schweighofer: Optimization of polynomials on compact semialgebraic sets, SIAM J. Optim. 15 (2005), pp. 805-825.
[14] L. Vandenberghe, S. Boyd: Semidefinite programming, SIAM review 38 (1996), pp. 49-95.

LAAS-CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse Cédex 4, France. Also associate member of IMT, the Institute of Mathematics of Toulouse.

E-mail address: lasserre@laas.fr

Universität Konstanz, Fachbereich Mathematik und Statistik, 78457 Konstanz, Germany

E-mail address: tim.netzer@uni-konstanz.de