Some problems in mechanics of growing solids with applications to AM technologies

A V Manzhirov
Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia
Bauman Moscow State Technical University, Moscow, Russia
E-mail: manzh@inbox.ru

Abstract. Additive Manufacturing (AM) technologies are an exciting area of the modern industrial revolution and have applications in engineering, medicine, electronics, aerospace industry, etc. AM enables cost-effective production of customized geometry and parts by direct fabrication from 3D data and mathematical models. Despite much progress in AM technologies, problems of mechanical analysis for AM fabricated parts yet remain to be solved. This paper deals with three main mechanical problems: the onset of residual stresses, which occur in the AM process and can lead to failure of the parts, the distortion of the final shape of AM fabricated parts, and the development of technological solutions aimed at improving existing AM technologies and creating new ones. An approach proposed deals with the construction of adequate analytical model and effective methods for the simulation of AM processes for fabricated solid parts.

1. Specific features of growing solids
A continuously growing solid is a solid whose composition, mass or volume varies because of a continuous addition of material to its surface. The process of adding new material to the solid is called growth as well as in particular cases accretion, erection, building-up, healing, etc. For continuous growth, the following basic stages of its deformation are strictly followed: before, during, and after the growth process. Each of these stages is characterized by instants when it starts and ends. The first is characterized by the instant of application of a load to the solid and the instant when growth starts. The second by the instant when growth starts and the instant when it ends. Conversely, the third is characterized by the instant when growth ends and the instant when it starts. The solid on whose surface new material is deposited starting from the instant when accretion starts is called the basic or original solid. Note that growth can also occur without the basic solid, starting from an infinitesimal material element. The part of the surface where infinitesimal pieces of the material are deposited at the actual instant is called the growth surface. The growth surface may be disconnected, in general. In particular, it can be the whole surface of the solid. Finally, the part of the surface of the original or the growing solid that coincides with the growth surface at the instant when growth starts will be called the base surface. The base surface is clearly the part of the surface of the solid on which material is to be deposited during the next stage of continuous growth. At different stages, it coincides, as a rule, with the surface between the basic solid and the additional solid as well as with the surfaces between the sub-solids [1–3].
We assume that the basic solid, which is made from a viscoelastic ageing material (e.g., see [3]), occupies a domain $\Omega_0$ with surface $S_0$ and is free of stresses up to the time $\tau_0$ of application of the load. From $\tau_0$ up to time $\tau_1$ when growth starts the classical boundary conditions are given on $S_0$, the specific form of which is stated below. At $\tau_1$ the continuous growth of a solid begins due to the addition of material particles to the growth surface $S^*(t)$. As it grows, the solid occupies a domain $\Omega(t)$ with surface $S(t)$. It is obvious that $S^*(t) \subseteq S(t)$. The time when a particle characterized by a position vector $x$ is deposited on the solid will be denoted by $\tau^*(x)$ and called the time of deposition of the particle on the growing solid. The configuration of the accreted solid is completely determined by the function $\tau^*(x)$ depending on the spatial coordinates. Boundedness and piecewise continuity are the general conditions usually imposed on $\tau^*(x)$.

We denote by $\tau_1^*(x)$ the time when an element of the growing solid is formed and by $\tau_0(x)$ the time when a load is applied to it. Naturally, $\tau_1^*(x) \leq \tau_0(x) = \tau_0$ for the elements of the basic solid ($x \in \Omega_0$).

The vector equilibrium equation is obviously satisfied in the domain occupied by the growing solid at each instant of time. For quasistatic processes, it has the form

$$ x \in \Omega(t): \nabla \cdot T = 0, $$

where $T$ is the stress tensor and $\nabla$ is the Hamilton operator. (Here and below we use the conventional notation of tensor calculus.)

The Cauchy conditions and the compatibility equations for deformations are always satisfied in the domain occupied by the basic solid, i.e.,

$$ x \in \Omega_0: E = \frac{1}{2} [\nabla u + (\nabla u)^T], \quad \nabla \times (\nabla \times E)^T = 0, $$

where $E$ is the strain tensor and $u$ is the displacement vector. But in the domain $\Omega^*(t)$ occupied by the additional solid ($\Omega^*(t) = \Omega(t)/\Omega_0$) only their analogs involving the rates of change of the corresponding variables are satisfied,

$$ x \in \Omega^*(t): D = \frac{1}{2} [\nabla v + (\nabla v)^T], \quad \nabla \times (\nabla \times D)^T = 0, $$

$$ D = \frac{\partial E}{\partial t}, \quad v = \frac{\partial u}{\partial t}; $$

i.e. the strains are usually incompatible.

This reflects the fact that the deposited elements may be subject to deforming actions prior to the time of deposition on the basic solid independently of the processes taking place in the solid itself.

To study the stress-strain state of a growing solid one must know the laws of deformation of the basic solid from the instant $\tau_0$ when the load is applied up to the instant $\tau_1$ when growth starts and of the deposited material from the instant $\tau_0(x)$ when a load is applied to this material up to the instant $\tau^*(x)$ of their deposition on the growing solid. The state of the original solid is determined from the solution of the problem with fixed boundary. The initial state of new elements which represent deposited surfaces as well as the boundary condition on the moving surface of the growing body can be determined by solving an additional contact problem of interaction between the solid and the surface.

The traditional boundary conditions for the displacement vector and the vector of surface forces are given on the stationary sections of the surface of the growing solid.

Currently a great number of AM fabricated part are made from viscoelastic materials with complex properties. To describe the behavior of these materials one can use the constitutive equations for a viscoelastic-ageing solid (see, e.g., [1, 2]).
The description of the process of continuous growth of a viscoelastic ageing solid involves three characteristic instants: the instant $\tau_1^*(x)$ when the element with coordinate $x$ is formed, the instant $\tau_0(x)$ when a load is applied to this element, and the instant $\tau^*(x)$ when the element is deposited on the growing solid. These three instants are different, in general.

The deposition process is largely determined by specifying these three instants. If the processes of continuous concrete casting, ice formation, crystal growth, etc. are studied, then $\tau_1^*(x) = \tau_0(x) = \tau^*(x)$; i.e., the elements are deposited at the same instant as they are formed and a load is applied to them. If spray deposition or erection of a structure from a large number of blocks is modeled by a continuous growth process, then, as a rule, $\tau_0(x) = \tau^*(x)$, and the instant $\tau_1^*(x)$ when the elements are formed is arbitrary. If the deformation of elements begins as soon as they are formed and they are being added to the basic solid only over some time interval, then $\tau_1^*(x) = \tau_0(x) \neq \tau^*(x)$, and so on.

Suppose that a homogeneous viscoelastic ageing solid occupying a domain $\Omega_0$ with surface $S_0$ ($x \in \Omega_0$) is formed at instant $\tau_1^*(x) = 0$ and is free of stresses up to the instant $\tau_0 \geq 0$, when a load is applied. Starting from the latter instant, consider two kinds of boundary conditions on the surface of the solid (surface forces on $S_1(t)$ and displacements on $S_2(t)$).

The sections of the surface on which different boundary conditions are given do not intersect one another and cover the whole surface of the solid. The dependence of $S_i$ on $t$ enables us to take into account the possible evolution of the system of loads, punches, etc. on $S_0$, and is assumed to be piecewise constant. Unless the solid surface is closed, the behavior of stresses or strains at infinity is prescribed.

Continuous deposition of material formed simultaneously with the solid ($\tau_1^*(x) = 0$) starts at $\tau_1 \geq \tau_0$. The solid occupies a domain $\Omega$ with surface $S(t)$ during its growth. The growth surface $S^*(t)$ ($S^*(\tau_1) \subset S_0$) moves in space. In the general case the positions of $S(t)$ and $S^*(t)$ are unknown. The sections $S_i(t)$ ($i = 1, 2$) on which the common boundary conditions are given can vary because of the loading of the stationary surface of the additional solid. We assume that the growing surface is always free of outer loads and the newly deposited surfaces are loaded at the instant of their deposition.

At the instant $\tau_2 > \tau_1$, the growth of the solid ceases, and starting from this instant four kinds of boundary conditions are given on the sections $S_i(t)$ of the surface $S_1 = S(\tau_2)$ of the solid occupying the domain $\Omega_1 = \Omega(\tau_2)$. After some time, at the instant $\tau_3 > \tau_2$ the solid growth may start again.

2. Basic concepts in the theory of continuous surface growth

Recent research has shown that solids formed by growth processes differ in their properties essentially from solids in the traditional view. Moreover, the classical approaches of solid mechanics fail when modeling the growing solid behavior. They should be replaced by new ideas and methods of modern mechanics, mathematics, physics, and engineering sciences. The approach proposed here deals with the construction of an adequate model of surface growth processes of solids. This approach is based on the following statements:

- The surface growth of a solid is modeled by the motion of its boundary due to the influx of new material to the surface of the solid.
- The boundary conditions on the moving boundary (the growth surface) are found from an additional solid-surface contact problem depending on the specific features of the growth process.
- The strain rate tensor (or the stretch rate tensor) of a growing solid is compatible, while the strain tensor is, as a rule, incompatible. Accordingly, it is absolutely natural to take the stress rate tensor, the strain rate tensor, and the velocity vector to be the main variables in the problem.
In general, a boundary value problem for a growing solid contains three dependent controlled groups of values, namely, the surface and bulk loads, the stresses on the accretion surfaces, and the material influx rate on the accretion surfaces. We have consider the case in which the velocities of the boundary particles of the growing solid are much smaller than the influx rate of new particles and strains are small.

The process of accretion or deposition of new material to a solid is studied in the fundamental scientific area called Mechanics of Growing Solids. This area deals with all sorts of solid materials including elastic, viscoelastic, plastic, composite and graded materials.

3. Mechanical analysis of growing solids
Consider the stress-strain state of a viscoelastic ageing solid \( \Omega_0 \) in the time interval \([\tau_0, \tau_1]\). We write out the equilibrium equation in the form,

\[
\nabla \cdot \bm{T} = 0.
\]

(1)

We represent the boundary conditions described above as follows

\[
\mathbf{x} \in S_1(t) : \mathbf{n} \cdot \bm{T} = p_0, \quad \mathbf{x} \in S_2(t) : \mathbf{u} = u_0,
\]

(2)

where \( p_0 \) and \( u_0 \) are given vectors of surface forces and strains and \( \mathbf{n} \) is the unit vector normal to the solid surface. The Cauchy conditions are written as follows:

\[
\mathbf{E} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T].
\]

(3)

We take the constitutive equations in the form in which the transverse contraction (Poisson’s) ratios of the instantaneous elastic strain and the creep strain of the ageing material are identical and are equal to \( \nu \). Then we have (see [1–3])

\[
\bm{T} = G(\mathbf{I} + \mathbf{N}(\tau_0, t))[2\mathbf{E} + (K-1)I_1(\mathbf{E})\mathbf{1}],
\]

(4)

where

\[
(\mathbf{I} + \mathbf{N}(\tau_0, t))^{-1} = (\mathbf{I} - \mathbf{L}(\tau_0, t)),
\]

\[
2G = E(1+\nu)^{-1}, \quad K = (1-2\nu)^{-1},
\]

\[
\mathbf{L}(\tau_0, t)f(t) = \int_{\tau_0}^{t} f(\tau)K(t, \tau)\,d\tau,
\]

\[
\omega(t, \tau) = 2C(t, \tau)(1+\nu),
\]

\[
K(t, \tau) = E(\tau)\frac{\partial}{\partial \tau}[E^{-1}(\tau) + C(t, \tau)] = K_1(t, \tau) = G(\tau)\frac{\partial}{\partial \tau}[G^{-1}(\tau) + \omega(t, \tau)],
\]

with \( E = E(t) \) and \( G = G(t) \) are the elastic moduli under tension and shear, \( C(t, \tau) \) and \( \omega(t, \tau) \) are the creep measures under tension and shear, \( K(t, \tau) \) is the creep function under tension, and \( \mathbf{1} \) is the unit tensor. The arguments are omitted in a number of obvious cases above. They will also be omitted in what follows and will be used only in those cases when their absence may be misleading.

Thus, Eqs. (1)–(4) constitute the boundary value problem of the linear theory of elasticity for a homogeneous ageing basic solid, the stress-strain state of which can be described by the solution of the system for \( t \in [\tau_0, \tau_1] \).
We transform the boundary value problem for the basic solid. Let us introduce the notation

\[ N^0 = H(\tau_0, t)N^{-1}, \quad a^0 = H(\tau_0, t)a^{-1}, \]

\[ H(\psi, t) = (I - L(\psi, t)), \] (5)

where \( N \) and \( a \) are an arbitrary tensor and arbitrary vector, respectively. We apply the operator \( H(\tau_0, t) \) to the relations in (1)–(4) containing \( T \) after dividing them by \( G \). Then, since \( H(\tau_0, t) \) commutes with the Hamilton operator, we obtain the following boundary value problem using (5) \((\tau_0 \leq t \leq \tau_1)\):

\[ \nabla \cdot T^0 = 0, \]

\[ x \in S_1(t): \quad n \cdot T^0 = p_0^0, \]

\[ x \in S_2(t): \quad u = u_0. \] (6)

Unlike (1)–(4), time occurs in the boundary value problem (6) as a parameter. The latter is mathematically equivalent to the boundary value problem of the theory of elasticity with a parameter \( t \). All analytic and numerical methods of the theory of elasticity can be used when constructing the solution of such a problem, which undoubtedly lends itself better to investigation than the problem of the theory of viscoelasticity.

To obtain the solution \( T, E, u \) of the boundary value problem (1)–(4), one must find the solution \( T^0, E, u \) of the boundary value problem (6) and use the relation as follows:

\[ T(x, t) = G(t)[T^0(x, t) + \int_{\tau_0}^{t} T^0(x, \tau)R(t, \tau) d\tau], \] (7)

Here \( R(t, \tau) \) is the resolvent of the kernel \( K(t, \tau) \).

Now consider the process of continuous growth of a solid \((\tau_1 \leq t \leq \tau_2)\). For a growing solid, we have the equilibrium equation in the form

\[ \nabla \cdot T = 0. \] (8)

The boundary conditions on the stationary part of the surface can be written as follows:

\[ x \in S_1(t): \quad n \cdot T = p_0, \]

\[ x \in S_2(t): \quad u = u_0. \] (9)

What regards the growth surface, we formulate the quasistatic and kinematic boundary conditions on it. The quasistatic boundary condition on the growth surface can be obtained from the solution of a contact problem for a solid and a surface provided that they interact without friction (see figure 1). Taking into account the following relations for small angles \( \alpha_1 \) and \( \alpha_2 \):

\[ 2\sigma_1 dh \sin \frac{d\alpha_1}{2} + 2\sigma_2 dh \sin \frac{d\alpha_2}{2} + d\sigma ds_1 dv_1 s_1 dT_s : L = 0, \]

\[ \frac{d\sigma}{dt} = -s_n \left( \sigma_1 \frac{d\alpha_1}{ds_1} + \sigma_2 \frac{d\alpha_2}{ds_2} \right), \quad \frac{d\alpha_1}{ds_1} = \frac{1}{\rho_1}, \quad \frac{d\alpha_2}{ds_2} = \frac{1}{\rho_2}, \] (10)

\[ \frac{d\sigma}{dt} = -s_n \left( \sigma_1 \frac{\rho_1}{\rho_2} + \sigma_2 \right) = -s_n (T_s : L), \quad dh = s_n dt, \quad s_n = v \cdot n, \]
one can obtain this condition in the form
\[ x \in S^*(t): \mathbf{n} \cdot \mathbf{S} = p \mathbf{n}, \quad \mathbf{S} = \frac{\partial (\mathbf{HT})}{\partial t}, \]
\[ p = -\frac{s_n}{G}(T_s : \mathbf{L}), \quad \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} (t = \tau^*(x)), \tag{11} \]
where \( \sigma_{11} \) and \( \sigma_{12} \) are the 2D principal stresses, \( T_s \) is the 2D tensor of the deposited elastic surface tension, \( \rho_1 \) and \( \rho_2 \) are the principal radii of curvature of the contacting surface, and \( \mathbf{L} \) is the 2D tensor of curvature of this surface.

The kinematic boundary condition is a form for the composition conservation law for a solid. It can be written in the form
\[ x \in S^*(t): \quad \mathbf{v} = \mathbf{v}_{\text{def}} + \mathbf{v}_{\text{dep}}, \tag{12} \]
where \( \mathbf{v} \) is the velocity of growth surface, \( \mathbf{v}_{\text{def}} \) is the velocity of growth surface due to the deformation of a solid under the action of surface and volumetric forces as well as under the tension of new elements, and \( \mathbf{v}_{\text{dep}} \) is the velocity of deposition of new elements to the growth surface. Here we suppose that \( \mathbf{v}_{\text{def}} \) and \( \mathbf{v}_{\text{dep}} \) are independent of each other. Hereinafter we study the case in which \( \mathbf{v}_{\text{def}} \ll \mathbf{v}_{\text{dep}} \). In this case we can set \( \mathbf{v}_{\text{def}} = 0 \) in (11) with high accuracy and use the equation of the growth surface in the form \( t = \tau^*(x) \). The latter equation gives us the value of \( s_n \) from (10). One can note that for the chosen case of problems statement we obtain the single boundary condition on the growth surface in the form (11).

The relation between the strain and displacement rate tensors is given by the equality
\[ \mathbf{D} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T], \tag{13} \]

The constitutive equation can be written as follows:
\[ \mathbf{T} = G(\mathbf{I} + \mathbf{N}(\tau_0(x), t))[2\mathbf{E} + (K - 1)I_1(\mathbf{E})\mathbf{1}], \tag{14} \]
\[ \tau_0(x) = \begin{cases} \tau_0, & x \in \Omega_0, \\ \tau^*(x), & x \in \Omega^*(t). \end{cases} \]
Relations (8)–(14) form a general noninertial initial–boundary value problem for a continuously growing solid, where the operator \((\mathbf{I} - \mathbf{L}(\tau_0(\mathbf{x}), t)) = \mathbf{H}(\tau_0(\mathbf{x}), t)\) and its inverse operator \((\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))\) can be determined from (4) with \(\tau_0\) replaced by \(\tau_0(\mathbf{x})\). Let us transform the initial–boundary value problem for a continuously accreted viscoelastic ageing solid into a problem with time parameter that has the same form as the boundary value problem of the theory of elasticity. Omitting technical details we present the final result in the form of a boundary value problem as follows

\[
\nabla \cdot \mathbf{S} = \mathbf{O},
\]

\[
\mathbf{x} \in S_1(t): \quad \mathbf{n} \cdot \mathbf{S} = \mathbf{Rp}_0,
\]

\[
\mathbf{x} \in S_2(t): \quad \mathbf{v} = \mathbf{v}_0,
\]

\[
\mathbf{x} \in S^*(t): \quad \mathbf{n} \cdot \mathbf{S} = p \mathbf{n} \quad (t = \tau^*(\mathbf{x})),
\]

\[
\mathbf{D} = \frac{1}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right],
\]

\[
\mathbf{S} = 2\mathbf{D} + (K - 1)\mathbf{I}_1(\mathbf{D})\mathbf{1},
\]

where \(\mathbf{R}\) acts on an arbitrary vector \(\mathbf{a}(\mathbf{x}, t)\) by the rule

\[
\mathbf{R} \mathbf{a}(\mathbf{x}, t) = \frac{1}{G(t)} \frac{\partial \mathbf{a}(\mathbf{x}, t)}{\partial t} + \int_{\tau_0(\mathbf{x})}^{t} \frac{\partial \mathbf{a}(\mathbf{x}, \tau)}{\partial \tau} \frac{\partial \mathbf{w}(\tau, t)}{\partial \tau} d\tau + \mathbf{a}(\mathbf{x}, \tau_0(\mathbf{x})) \frac{\partial \mathbf{w}(t, \tau_0(\mathbf{x}))}{\partial t}.
\]

Note that the conditions on \(S_1(t)\) and \(S^*(t)\) are identical.

Relations (15)–(16) supplemented with the initial conditions for the basic solid at \(t = \tau_1\) form a boundary value problem with \(t\) as a parameter.

To obtain the solution \(\mathbf{T}, \mathbf{E}, \) and \(\mathbf{u}\) of the initial–boundary value problem (8)–(14) one must find the solution \(\mathbf{S}, \mathbf{D}, \) and \(\mathbf{v}\) of (15)–(16) and use the relations as follows

\[
\mathbf{T}(\mathbf{x}, t) = G(t) \left\{ \frac{\mathbf{T}(\mathbf{x}, \tau_0(\mathbf{x}))}{G(\tau_0(\mathbf{x}))} \left[ 1 + \int_{\tau_0(\mathbf{x})}^{t} R(t, \tau) d\tau \right] + \int_{\tau_0(\mathbf{x})}^{t} \mathbf{S}(\mathbf{x}, \tau) d\tau \right\},
\]

\[
\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \tau_0(\mathbf{x})) + \int_{\tau_0(\mathbf{x})}^{t} \mathbf{v}(\mathbf{x}, \tau) d\tau.
\]

Hence the solution of the growth problem for a viscoelastic ageing solid can be obtained by the solution of the mathematically identical problems with a parameter \(t\), which have the same form as the boundary value problem of the classical theory of elasticity. Then the true stresses and displacements in the growing solid can be reconstructed using (17).

Suppose that the solid ceases to grow at instant \(\tau_2\). At this instant, it occupies a domain \(\Omega_1\) with surface \(S_1\) on which two kinds of boundary conditions are specified, as in the case of the problem for the basic solid. Moreover, \(S^*(\tau_2) = S_i^* \subseteq \cup_i S_i(t)\) for \(i = 1, 2\). In this case the problem for the invariable solid occupying \(\Omega_1\) is similar to (8)–(14) without the initial–boundary condition on \(S^*(t)\),

\[
\nabla \cdot \mathbf{T} + \mathbf{f} = \mathbf{O},
\]

\[
\mathbf{x} \in S_1(t): \quad \mathbf{n} \cdot \mathbf{T} = \mathbf{p}_0, \quad \mathbf{x} \in S_2(t): \quad \mathbf{u} = \mathbf{u}_0,
\]

\[
\mathbf{D} = \frac{1}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right],
\]

\[
\mathbf{T} = G(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))[2\mathbf{E} + (K - 1)\mathbf{I}_1(\mathbf{E})\mathbf{1}],
\]
with \( \tau^*(x) = \tau_2 \) for \( x \in S_1^\ast \). The stresses, strains and displacements at \( t = \tau_2 \) found by solving the growth problem at the previous step serve as the initial conditions.

One can obtain the following boundary value problem:

\[
\begin{align*}
\nabla \cdot S &= 0, \\
x \in S_1(t): \quad n \cdot S &= Rp_0, \\
x \in S_2(t): \quad v &= v_0, \\
D &= \frac{1}{2}[\nabla v + (\nabla v)^T], \\
S &= 2D + (K - 1)I_1(D)1,
\end{align*}
\]

where the initial conditions remain as before.

For \( T, E, \) and \( u \) to be solutions of the initial–boundary value problem (18) it is necessary and sufficient that \( S, D, \) and \( v \) form a solution of the boundary value problem (19) and that relations (17) be satisfied.

Thus, to construct the solution of the problem over the time interval \([\tau_0, \tau_3]\) one has to construct the solutions of the following three identical problems (having the same form as the boundary value problem of the theory of elasticity with a parameter \( t \)): problem (6) for \( t = \tau_0 \) as well as problems (13)–(14) and (17). The stress-strain state of the growing solid can then be reconstructed for any \( t \in [\tau_0, \tau_3] \) from (7) and (15).

Among a great number of papers on AM technologies written by technologists, chemists, and physicists (see, e.g., [4–26]). there are few devoted to mechanical aspects and analysis of AM fabricated parts and similar problematics (see, e.g., [27–42]). In what follows we consider some mechanical problems of AM technologies in civil and structural engineering.

4. Additive manufacturing of a concrete dam

We consider a problem for a growing dam in the form of wedge which is manufactured from concrete. We use standard material moduli and functions for concrete and some plastics (e.g., see [1–3]). These problems differ one from each other by the rate as well as by the type of additive process. We study three types of growth for a wedge: angle growth (figure 2 a), translational growth (figure 2 b), and complex growth (figure 2 c). We have found that stress states for growing dams sufficiently differ from each other and from the classical case of fixed dam.

4.1. Translational type of building up

To be definite let us consider a problem for a dam, which in the process of its manufacturing, change the shape from a quarter-plane up to a plane (figure 2 d). Let us consider the problem of translational growth on a quarter-plane of non-stressed elements at a constant rate. Let a constant indenting force \( P(t) \) and zero tilt angle of a punch which can be treated as an engineering block of a dam be given, and let us assume that the base of the punch is flat. The configuration of the body at any instant of time is specified by the coordinate of the tip of the quarter \( x = -f(t) \) \((f(t) = 0, \text{when } \tau_0 \leq t \leq \tau_0)\). We assume that the process of growth does not stop \((\tau_2 = \infty)\), i.e. in the limit the body becomes a half-plane.

Let us determine the contact pressure \( q \) and the eccentricity \( e \) of the point of application of the force, ensuring that there is no a tilt of the punch at different rates of translational growth. In what follows, we use dimensionless variables and formulate general qualitative conclusions. We set \( P(t) = 1, \quad \alpha = \pi/2 \). We consider 1) slow translational growth: \( f(t) = t - 2, f'(t) = 1 \) \((t \leq \tau_0)\); 2) rapid translational growth \( f(t) = 6(t - 2), f'(t) = 6 \) \((t \leq \tau_1)\); (the rate of growth \( f'(t) \) is 6 times as high as in the case 1)).

Figures 3 a shows the limit distribution of the contact stresses at various rates \( f'(t) \). The solid asymmetric line shows the initial distribution of the contact stresses when the dam has the shape of quarter-plane. The solid symmetric line demonstrates the contact stresses distribution
Figure 2. Types of additive manufacturing for a dam.

Figure 3. Contact stresses for the translational type of building up.
for the classical problem of the interaction between a flat punch and a half-plane. The dashed-and-dot and dashed lines show the limiting distribution of contact stresses for slow and rapid translational growth respectively.

Figure 3b shows the limiting values of the force eccentricity at various rates \( f'(t) \). The dash-dot and dashed lines describe the behavior of the force eccentricity on time for slow and rapid translational growth respectively.

Figures 3c and 3d show the variation with time of the contact stresses \( q_1 \) at the left edge of the punch \( (x = 0.1) \) and \( q_2 \) at the right edge \( (x = 8.9) \) for slow (dash-dot lines) and rapid (dashed lines) growth respectively.

We note that in case 1) the stresses, e.g. at the left edge, increase by a factor of 1.4 and the eccentricity is reduced by a factor of 1.2. In case 2) these relations are equal to 2.7 and 1.5, respectively. When \( t = t^o \approx 8 \) and the rate of growth is arbitrary, the process becomes a steady-state process and further growth of the body does not affect the contact stresses and the eccentricity.

4.2. Angle type of building up

Let us now consider the problem of angle growth on a quarter-plane of non-stressed elements at a constant rate. The initial aperture angle of the wedge is \( \alpha_0 = \pi/2 \), becoming \( \alpha_1 = \pi \) at the instant of termination. It is necessary to find contact stresses and the eccentricity of the indenting force which ensure that the punch is not misaligned. In the case of constant growth rate the instants of the onset \( \tau_1 \) and termination \( \tau_2 \) of growth completely determine the function

\[
\alpha(t) = \frac{1}{2} \pi \frac{t + \tau_2 - 2\tau_1}{\tau_2 - \tau_1}
\]

and thus configuration of the body at any instant of time. We chose concrete as the material of the wedge and use dimensionless quantities (see figure 4). Let us consider the case of a slow \( (\tau_1 = 2, \tau_2 = 10) \) and rapid \( (\tau_1 = 2, \tau_2 = 2) \) side growth of the wedge. From now on the dot-dash lines in the figures will present the basic characteristics during the slow growth, and dashed lines during the rapid growth (at a rate four times as high as that of the slow growth).

Figure 5a shows the limit distributions of contact pressures under the punch \( q \) as \( t \to \infty \) at different growth rates (dashed and dash-dot lines), and the distribution from the instant of the loading \( \tau_0 = 1 \) to the instant of onset of growth \( \tau_1 \), constant over this time interval (solid line). The distribution functions have singularities at the punch edges, but in order to make the graphs easier to construct, we have eliminated the region \( r < 0.1 \) and indicated the values of the stresses at \( r = 0.1 \), by dots.
Figure 5. Contact stresses for the angle type of building up.

Figure 5 b shows the dependence of the eccentricity $e$ of the point of application of the force on time $t$.

Figures 5 c and 5 d shows the variation with time of the contact stresses $q_1$ at the left edge of the punch ($r = 0.1$) and $q_2$ at the right edge ($r = 8.9$) for the two processes in question. We see that the processes characterized by the rate of growth show substantial qualitative and quantitative differences between each other. During the rapid growth the most intense change in the stresses and the eccentricity occurs in the interval $t \in [\tau_1, \tau_2]$ and continues after the growth has ceased. The stresses, e.g. at the left edge, increase by a factor of 3.5 and the eccentricity decreases by almost a half.

When the growth is slow, the characteristics change more smoothly and, beginning at a certain instant of time, the stress-strain state of the body becomes practically indifferent to the process of growth, i.e. under the conditions of limited creep the constant force acting on the punch becomes exhausted and further growth or its termination does not alter the already established values of the stresses and the eccentricity. We note for comparison that in case of a slow growth the stresses at the left edge increase by a factor of 2.4, and the eccentricity decreases by a factor of 1.5. By studying the process of slow growth we also broaden our understanding of the law governing the formation of the displacement and stress fields in a growing viscoelastic body acted upon by stationary forces.

Irrespective of the fact that the body has the form of a halfplane at the instant of termination, the distribution of contact pressures under the punch is not symmetrical at any $t \geq \tau_2$, and the eccentricity of the point of application of the force is very far from zero, i.e. the idea of a body which has grown to a half-plane may lead directly to incommensurable values of the fundamental characteristics.
5. Additive manufacturing of an arch structures

Some numerical results for AM fabricated concrete structures are demonstrated in this section. We suppose that concrete is a viscoelastic aging material, i.e., the material with time dependent mechanical properties (e.g., see [1–3]).

5.1. A Buried Arch
Consider now an arch structure manufactured from concrete and buried in a loose soil backfill (see figure 6a). In order to strengthen this structure the process of growth inside it is initiated. During the growth the thickness of the arch wall doubles. Figure 6b shows the limiting contact stress distribution ($t \to \infty$) on the end faces of the arch in the processes of slow (dashed line) and rapid (dash-dotted line) growth. Solid line shows the contact stress distribution on the end faces for a fixed arch with doubled wall thickness. One can easily see the fundamental difference in the behavior pattern of mechanical characteristics of growing solid as compared with a fixed one.

5.2. An Arch under Gravity
Consider one more AM process for an arch structure under gravity (see figure 7a). This process is close to the previous one so we save all designations for variables and plots lines, i.e., in figure 7b we use dashed and dash-dotted lines for the limiting contact stresses on the end faces of the arch for the modes of slow and rapid growth respectively and solid line for the analogous
contact stresses in the mode of fixed arch.

One can see a very strong difference between the patterns of arch mechanical behavior for both problems and chosen modes.

It is quite clear that one can also obtain the solution for a buried arch under gravitation by using the superposition principle.

6. Conclusions

- Fundamentals of mechanical analysis for AM fabricated parts have been developed.
- New general nonclassical boundary value problems have been formulated including a pair of specific quasistatic and kinematic conditions on the growth surface.
- The results obtained can also serve as a basis for the numerical research of AM processes in the case in which the velocity of growth surface due to the deformation of a solid under the action of surface and volumetric forces as well as under the tension of new elements is much smaller than the velocity of deposition of new elements to the solid’s surface.
- New mechanical effects for AM fabricated parts and structures have been found out. Such effects can never arise in the case of another manufacturing processes.

Acknowledgments

The work was supported by the Russian Science Foundation under grant No. 17-19-01257.

References

[1] Aruyunyan N Kh, Manzhirov A V, and Naumov V E 1991 Contact Problems in Mechanics of Growing Bodies (Moscow: Nauka) p 172 [in Russian]
[2] Aruyunyan N Kh and Manzhirov A V 1999 Contact Problems in the Theory of Creep (Moscow: Nauka) p 320 [in Russian]
[3] Manzhirov A V 1995 The general non-inertial initial-boundary value problem for a viscoelastic ageing solid with piecewise-continuous accretion J. Appl. Math. Mech. 59 (5) 805–16
[4] Gibson I, Rosen D W, and Stucker B 2010 Additive Manufacturing Technologies: Rapid Prototyping to Direct Digital Manufacturing (New York: Springer Verlag, LLC) p 2X1+498
[5] Gebhardt A and Hoetter J S 2016 Additive Manufacturing: 3D Printing for Prototyping and Manufacturing (Munich: Carl Hanser Verlag) p 611
[6] Guo N and Leu M C 2013 Additive manufacturing: technology, applications and research needs Frontiers of Mech. Engng 8 (3) 215–43
[7] Bikas H, Stavropoulos P, and Chryssolouris G 2016 Additive manufacturing methods and modelling approaches: a critical review Int. J. Adv. Manufact. Techn. 83 (1-4) 389–405
[8] Denlinger E R, Irwin J, and Michaleris P 2014 Thermomechanical modeling of additive manufacturing large parts J. Manufact. Sci. Engng 136 (6), 061007
[9] Frazier W E 2014 Metal additive manufacturing: A review J. Mater. Engng Perform. 23 (6), 1917–28
[10] Huang Y, Leu M C, Mazumder J, and Dommez A 2015 Additive manufacturing: current state, future potential, gaps and needs, and recommendations J. Manufact. Sci. Engng 137 (1), 014001
[11] Gibson I, Rosen D, and Stucker B (2010) Additive Manufacturing Technologies (New York: Springer) p 238
[12] Heigel J C, Michaleris P, and Reutzel E W 2015 Thermo-mechanical model development and validation of directed energy deposition additive manufacturing of Ti–6Al–4V Addit. Manufact. 5, 9–19
[13] Chua C K and Leong K F 2014 3D Printing and Additive Manufacturing: Principles and Applications (with Companion Media Pack), Fourth Edition of Rapid Prototyping (World Scientific Publishing Co Inc)
[14] Biswas A, Shapiro V, and Tsukanov I 2004 Heterogeneous material modeling with distance fields Comp. Aided Geom. Des. 21 (3) 215–42
[15] Roberts I A, Wang, C J, Esterlein R, et al. 2009 A three-dimensional finite element analysis of the temperature field during laser melting of metal powders in additive layer manufacturing Int. J. Machine Tools Manufact. 49 (12) 916–23
[16] Melchels F P W, Domingos M A N, Klein T J, et al. 2012 Additive manufacturing of tissues and organs Progress in Polymer Sci. 37 (8) 1079–104
[17] Zhou J, Zhang Y, and Chen J K 2009 Numerical simulation of random packing of spherical particles for powder-based additive manufacturing J. Manufact. Sci. Engng 131 (3) 031004
[18] Parteli E J R 2013 DEM simulation of particles of complex shapes using the multisphere method: application for additive manufacturing AIP Conf. Proc. 1542 (1) 185–8
[19] Parteli E J R and Pöschel T 2016 Particle-based simulation of powder application in additive manufacturing Powder Technol. 288 96–102
[20] Khajavi S H, Partanen J, and Holmstrom J 2014 Additive manufacturing in the spare parts supply chain Computers in Industry 65 (1) 50–63
[21] Patil N, Pal D, and Stucker B 2013 A new finite element solver using numerical Eigen modes for fast simulation of additive manufacturing processes In Solid Freeform Fabrication Proceedings, August 2013 (The University of Texas at Austin) pp 535–43
[22] Kalpakjian S and Schmid S R 2014 Manufacturing Engineering and Technology (Upper Saddle River, NJ: Pearson)
[23] Hu D and Kovacevic R 2003 Modelling and measuring the thermal behaviour of the molten pool in closed-loop controlled laser-based additive manufacturing Proc. Inst. Mech. Engineers. Part B: J. Engng Manufact. 217 (4) 441–52
[24] Salmi M, Palohéimo K S, Tuomi J, et al. 2013 Accuracy of medical models made by additive manufacturing (rapid manufacturing) J. Cranio-Maxillofacial Surgery 41 (7) 603–9
[25] Li Y and Gu D 2014 Parametric analysis of thermal behavior during selective laser melting additive manufacturing of aluminum alloy powder Mater. Des. 63 856–67
[26] Turner B N, Strong R, and Gold S A 2014 A review of melt extrusion additive manufacturing processes: I. Process design and modeling Rapid Prototyping J. 20 (3) 192–204
[27] Manzhirov A V 2013 Mechanics of growing solids and phase transitions Key Engng Mater. 535-536 89–93
[28] Manzhirov A V 2017 Advances in the theory of surface growth with applications to additive manufacturing processes In Lecture Notes in Engineering and Computer Science: World Congr. on Engineering 2014, WCE 2014, 2–4 July, 2014, London, U.K. (London: IAENG) pp 1404–9
[29] Manzhirov A V 2015 Mechanical design of viscoelastic parts fabricated using additive manufacturing technologies In Lecture Notes in Engineering and Computer Science: Proc. of The World Congr. Engineering 2015, WCE 2015, 1–3 July, 2015, London, U.K. (London: IAENG) pp 710–4
[30] Manzhirov A V and Parshin D A 2015 Influence of the egression regime on the stress state of a viscoelastic arched structure erected by an additive technology under the force of gravity Mech. Solids 50 (6) 657–75
[31] Manzhirov A V and Parshin D A 2016 Application of prestressed structural elements in the erection of heavy viscoelastic arched structures with the use of an additive technology Mech. Solids 51 (6) 692–700
[32] Manzhirov A V and Parshin D A 2016 Accretion of spherical viscoelastic objects under self-gravity Lect. Notes Engng Comp. Sci. 2224 (1) 1131–5
[33] Manzhirov A V 2016 A method for mechanical design of AM fabricated viscoelastic parts In Transactions on Engineering Technologies (Singapore: Springer) pp 223–35
[34] Manzhirov A V 2016 Fundamentals of mechanical design and analysis for AM fabricated parts Proc. Manufact. 7 59–65
[35] Manzhirov A V 2017 Advances in the theory of surface growth with applications to additive manufacturing processes Proc. Engng 173 11-6
[36] Manzhirov A V 2017 Mechanical analysis of an AM fabricated viscoelastic shaft under torsion by rigid disks In Lecture Notes in Engineering and Computer Science: Proc. of The World Congr. Engineering 2017, WCE 2017, 5–7 July, 2017, London, U.K., Vol. II (London: IAENG) pp 856–60
[37] Manzhirov A V, Parshin D A, and Romanov A A 2017 Additive manufacturing of conical viscoelastic parts under axial tension-compression disks In Lecture Notes in Engineering and Computer Science: Proc. of The World Congr. Engineering 2017, WCE 2017, 5–7 July, 2017, London, U.K., Vol. II (London: IAENG) pp 934–939
[38] Manzhirov A V 2017 Mechanical design of AM fabricated prismatic rods under torsion MATEC Web of Conf. 95 12002
[39] Manzhirov A V and Gupta N K 2017 Fundamentals of continuous growth processes in technology and nature Proc. IUTAM 23 1–12
[40] Parshin D A and Manzhirov A V 2017 Quasistatic problems for piecewise-continuously growing solids with integral force conditions on surfaces expanding due to additional material influx J. Phys.: Conf. Ser. 937 012038
[41] Bychkov P S, Kozintsev V M, Manzhirov A V, and Popov A L 2017 Determination of residual stresses in products in additive production by the layer-by-layer photopolymerization method Mech. Solids 52 (5) 524–9