AUTOMATIC CONTINUITY OF SOME LINEAR MAPPINGS
FROM CERTAIN PRODUCTS OF BANACH ALGEBRAS

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Abstract. Let \( A \) and \( U \) be Banach algebras and \( \theta \) be a nonzero character on \( A \). Then the Lau product Banach algebra \( A \times_{\theta} U \) associated with the Banach algebras \( A \) and \( U \) is the \( l^1 \)-direct sum \( A \oplus U \) equipped with the algebra multiplication \( (a, u)(a', u') = (ab, \theta(a)u' + \theta(a')u + uu') \) \((a, a' \in A, u, u' \in U)\) and \( l^1 \)-norm. In this paper we shall investigate the derivations and multipliers from this Banach algebras and study the automatic continuity of these mappings. We also study continuity of the derivations for some special cases of Banach algebra \( U \) and Banach \( A \times_{\theta} U \)-bimodule \( X \) and establish various results on the continuity of derivations and give some examples.

1. Introduction

Let \( A \) be a Banach algebra (over the complex field \( \mathbb{C} \)), and \( X \) be a Banach \( A \)-bimodule. A linear mapping \( D : A \rightarrow X \) is called a derivation if \( D(ab) = aD(b) + D(a)b \) for all \( a, b \in A \). For any \( x \in X \), the mapping \( ad_x : A \rightarrow X \) given by \( ad_x(a) = ax - xa \) is a continuous derivation called inner. Let \( D : A \rightarrow A \) be a derivation. Then by a generalized \( \delta \)-derivation we mean a linear mapping \( \delta : X \rightarrow X \) satisfying \( \delta(xa) = \delta(x)a + xd(a) \) for all \( a \in A \) and \( x \in X \).

The problem of continuity of linear mapping between two Banach algebras (or Banach spaces, in general) lies in the theory of automatic continuity which is an important topic in mathematical analysis. This theory has been an active field of research during the last fifty years. The automatic continuity of the derivations on different structures in mathematics has attracted the attention of the researchers and, specifically, has been mainly developed in the context of Banach algebras and studied extensively (see for example, [8], [17], [18], [20], [22], [28]). The continuity of derivations from a Banach algebra into a Banach bimodule arises in a number of situations. In particular, it arises in cohomology theory of Banach algebras and also in the theory of extensions of Banach algebras. The reader is referred to [4] which is a detailed source in this context. Here we mention the most important established results concerning continuity of derivations. A celebrated theorem due to Johnson and Sinclair [12] states that every derivation on a semisimple Banach algebra is continuous. Ringrose [21] showed that every derivation from a \( C^* \)-algebra \( A \) into a Banach \( A \)-bimodule \( X \) is continuous. The automatic continuity of module derivations from some special classes of Banach algebras is studied by several authors; for instance, Bade and Curtis [2] studied the structure and continuity of the Banach algebra \( C^n(I) \) of \( n \) times continuously differentiable functions on an interval \( I \) into \( C^n(I) \)-Banach bimodules. By determining the value of the derivation

MSC(2010): 46H40; 13N15; 42A45; 46H25.

Keywords: automatic continuity, derivation, Lau product, multiplier.
on certain semigroups, in his paper [9], the author describes bounded derivations from commutative Banach algebras into commutative Banach bimodules. In [5], Christensen proved that every derivation from a nest algebra on the Hilbert space $\mathcal{H}$ into $\mathcal{B}(\mathcal{H})$ is continuous. Additionally, some results on automatic continuity of the derivations on prime Banach algebras have been established by Villena in [29] and [30].

Recall that for a Banach algebra $A$, a linear mapping $T: A \to A$ is called a multiplier on $A$ whenever $aT(b) = T(a)b$ for all $a, b \in A$. $A$ is said to be faithful, if for any $x \in A$, $Ax = \{0\} = xA$ implies that $x = 0$. It is well-known and easy to show that if $A$ is faithful, then every multiplier on $A$ is continuous. The notion of multiplier was originally introduced by Helgason [10], as a bounded continuous function $g$ defined on the regular maximal ideal space $\Delta(A)$ such that $g(\hat{A}) \subseteq \hat{A}$ where $\hat{A}$ denotes the Gelfand representation of $A$, and has been developed by Wang [31] and Birtal [3]. The theory of multipliers has an important applications in many areas of harmonic analysis and as well as in optimization theory, differential equations, probability, mathematical finance and economics. A good reference for this theory is the monograph of Larsen [14] (see also Laursen and Neumann [16]).

Let $A$ and $\mathcal{U}$ be Banach algebras and $\theta: A \to \mathbb{C}$ be a non-trivial character on $A$. We equip the space $A \times \mathcal{U}$ with the usual $\mathbb{C}$-module structure. Then the multiplication

$$(a, u)(a', u') = (aa', \theta(a)a + \theta(a')u + uu') \quad (a, a' \in A, u, u' \in \mathcal{U})$$

with the norm

$$||(a, u)|| = ||a|| + ||u||,$$

turn $A \times \mathcal{U}$ into a Banach algebra called Lau products of Banach algebras, denoted by $A \times_\theta \mathcal{U}$.

Note that we identify $A \times \{0\}$ with $A$, and $\{0\} \times \mathcal{U}$ with $\mathcal{U}$, thus $A$ is a closed subalgebra of $A \times_\theta \mathcal{U}$ while $\mathcal{U}$ is a closed ideal of it, and

$$\frac{A \times_\theta \mathcal{U}}{\mathcal{U}} \cong A \quad (\text{isometrically isomorphism}).$$

Indeed $A \times_\theta \mathcal{U}$ is equal to direct sum of $A$ and $\mathcal{U}$ as Banach spaces.

The Lau product was first introduced by T. Lau [15] for special classes of Banach algebras that are predual of a von Neumann algebra and for which the identity of the dual is a multiplicative linear functional. Monfared [19], has studied and verified some structural and topological properties of this special product. The reader can find more information in [19] and references therein.

Let $\mathcal{X}$ be a Banach $(A \times_\theta \mathcal{U})$-bimodule. Then $\mathcal{X}$ is a Banach $A$-bimodule by defining module operations in a natural fashion;

$$a.x = (a, 0)x \quad , \quad x.a = x(a, 0) \quad (a \in A, x \in \mathcal{X}).$$

Similarly $\mathcal{X}$ turns into a Banach $\mathcal{U}$-bimodule with the module actions given by

$$u.x = (0, u)x \quad , \quad x.u = x(0, u) \quad (u \in \mathcal{U}, x \in \mathcal{X}).$$

A key notion to study the automatic continuity of a linear mapping $T: \mathcal{X} \to \mathcal{Y}$ between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is the separating space which is defined as

$$\mathcal{S}(T) := \{y \in \mathcal{Y} \mid \text{there is } \{x_n\} \subseteq \mathcal{X} \text{ with } x_n \to 0, T(x_n) \to y\}.$$
Note that by the closed graph theorem, \( T \) is continuous if and only if \( \mathcal{S}(T) = \{0\} \).
For a thorough discussion of the separating space one can refer to [6] and [24].

For a derivation \( D : \mathcal{A} \to \mathcal{X} \) the two-sided continuity ideal is defined to be
\[
\mathcal{I}(D) = \{ a \in \mathcal{A} : a\mathcal{S}(D) = \mathcal{S}(D)a = 0 \}.
\]
Note that a derivation need not be continuous on \( \mathcal{I}(D) \). But rather it is bounded as a bilinear form. However, if \( \mathcal{I}(D) \) has a bounded approximate identity, then the restriction of \( D \) to its continuity ideal \( \mathcal{I}(D) \) is continuous.

Let \( \mathcal{A} \) be a Banach algebra and \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach \( \mathcal{A} \)-bimodules. By \( Z(\mathcal{A}) \), we mean the center of \( \mathcal{A} \) while for \( \mathcal{S} \subseteq \mathcal{X} \) we have \( Z_\mathcal{S}(\mathcal{A}) = \{ s \in \mathcal{S} : s\mathcal{A} = \mathcal{A}s \} \). Also, the annihilator of \( \mathcal{A} \) over \( \mathcal{S} \), denoted by \( \text{ann}_\mathcal{S} \mathcal{A} \) is defined as
\[
\text{ann}_\mathcal{S} \mathcal{A} := \{ s \in \mathcal{S} : s \mathcal{A} = \mathcal{A}s = \{0\} \}.
\]
Similarly for a subset \( \mathcal{D} \subseteq \mathcal{A} \) we write,
\[
\text{ann}_\mathcal{D} \mathcal{X} := \{ x \in \mathcal{X} : x \mathcal{D} = \mathcal{D}x = \{0\} \}.
\]
A linear mapping \( \phi : \mathcal{X} \to \mathcal{Y} \) is said to be a left (resp. right) \( \mathcal{A} \)-module homomorphism if \( \phi(ax) = a\phi(x) \) (resp. \( \phi(xa) = \phi(x)a \) ) for all \( a \in \mathcal{A} \) and \( x \in \mathcal{X} \). \( \phi \) is called \( \mathcal{A} \)-module homomorphism, if it is both left and right \( \mathcal{A} \)-module homomorphism.

This paper is organized as follows. In section 2 we consider derivations of Lau products of Banach algebras and determine the general structure of them and by studying the properties of the appearing maps, we obtain conditions under which these mappings are automatically continuous and establish various results in this context. Since inner derivations form an important class of automatically continuous derivations, some of the results are devoted to investigating the inner-ness of the derivations. We also consider some special cases and study continuity of the derivations and give some examples. In section 3 we turn our attention to the multipliers of Lau products and obtain some results concerning automatic continuity of these mappings.

2. The Derivations

Our aim in this section is to study the derivations from \( \mathcal{A} \times_\theta \mathcal{U} \) and investigate automatic continuity of them. Also, we obtain several results on the continuity of the derivations for some special cases.
Throughout, \( \mathcal{A}, \mathcal{U} \) are Banach algebras, \( \theta \) is a nonzero character on \( \mathcal{A} \), \( \mathcal{A} \times_\theta \mathcal{U} \) denotes the corresponding Lau product and \( \mathcal{X} \) is a Banach \( (\mathcal{A} \times_\theta \mathcal{U}) \)-bimodule. In the following theorem we characterize the derivations from Lau products and determine the general structure of them.

Theorem 2.1. Let \( D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X} \) be a linear mapping. Then the following statements are equivalent.
(i) \( D \) is a derivation.
(ii) There are linear mappings \( \delta_1 : \mathcal{A} \to \mathcal{X} \) and \( \delta_2 : \mathcal{U} \to \mathcal{X} \) with
\[
D((a,u)) = \delta_1(a) + \delta_2(u) \quad (a \in \mathcal{A}, u \in \mathcal{U})
\]
such that \( \delta_1 \) and \( \delta_2 \) are derivations and satisfy the following conditions;
\[
a\delta_2(u) + \delta_1(a)u = \theta(a)\delta_2(u) = \delta_2(u)a + u\delta_1(a) \quad (a \in \mathcal{A}, u \in \mathcal{U})
\]
Proof. (i) \( \implies \) (ii) Suppose that \( D : \mathcal{A} \times _{\theta} \mathcal{U} \to \mathcal{X} \) is a derivation. Then put \( \delta _1 (a) = D((a,0)) \) and \( \delta _2 (u) = D((0,u)) \). Thus since \( D \) is linear, \( D((a,u)) = \delta _1 (a) + \delta _2 (u) \). If we apply \( D \) on the both sides of \( (a,u)(a',u') = (aa',\theta (a)u' + \theta (a')u + uu') \), then it is easy to see that \( \delta _1, \delta _2 \) satisfy the given equations.

(ii) \( \implies \) (i) If \( D \) is a linear mapping satisfying the conditions given in (ii), then it is routinely checked that \( D \) is a derivation. \( \square \)

In view of the above theorem, for every derivation \( D : \mathcal{A} \times _{\theta} \mathcal{U} \to \mathcal{X} \) we can write \( D = \delta _1 + \delta _2 \) where \( \delta _1, \delta _2 \) are as in the above theorem.

Since inner derivations are continuous ones, if we show that a given derivation \( D \) is inner this, in turn, implies that \( D \) is automatically continuous. In the following result we characterize inner derivations from \( \mathcal{A} \times _{\theta} \mathcal{U} \).

**Proposition 2.2.** Let \( D : \mathcal{A} \times _{\theta} \mathcal{U} \to \mathcal{X} \) be a derivation with \( D = \delta _1 + \delta _2 \). Then

(i) If \( D \) is inner, then \( \delta _1 \) and \( \delta _2 \) are inner.

(ii) If \( \delta _1 = ad_{x_0} \) and \( \delta _2 = ad_{y_0} \), then \( D = ad_{z_0} \) for some \( z_0 \in \mathcal{X} \) if and only if \( z_0 - x_0 \in Z_\mathcal{X} (\mathcal{A}) \) and \( z_0 - y_0 \in Z_\mathcal{X} (\mathcal{U}) \).

**Proof.** (i) Since \( D \) is inner there exists some \( x_0 \in \mathcal{X} \) for which \( D = ad_{x_0} = (a,u)x_0 - x_0(a,u) \) for all \( a \in \mathcal{A} \) and \( u \in \mathcal{U} \). Substituting \( a = 0 \), we obtain \( \delta _2 (u) = uz_0 - x_0u \) for all \( u \in \mathcal{U} \). Similarly if \( u = 0 \), then we have \( \delta _1 (a) = ax_0 - x_0a \) for all \( a \in \mathcal{A} \). Thus \( \delta _1 \) and \( \delta _2 \) are inner.

(ii) Assume that \( \delta _1 = ad_{x_0} \) and \( \delta _2 = ad_{y_0} \) and there exists \( z_0 \in \mathcal{X} \) be such that \( D = ad_{z_0} \). We show that \( z_0 - y_0 \in Z_\mathcal{X} (\mathcal{U}) \). Note that \( (a,u)z_0 - z_0(a,u) = ax_0 - x_0a + uy_0 - y_0u \) for all \( a \in \mathcal{A} \) and \( u \in \mathcal{U} \). Substituting \( a = 0 \) we have \( u(z_0 - y_0) = (z_0 - y_0)u \) for all \( u \in \mathcal{U} \). Thus \( (z_0 - y_0) \in Z_\mathcal{X} (\mathcal{U}) \). By the same way one can show that \( z_0 - x_0 \in Z_\mathcal{X} (\mathcal{A}) \). The converse is clear and we omit it. \( \square \)

It is clear from the above proposition that if \( \delta _1, \delta _2 \) are inner derivations induced by the same element \( x_0 \) (i.e., \( \delta _1 = ad_{x_0} \) and \( \delta _2 = ad_{x_0} \)), then \( D \) is always inner since one can take \( z_0 = x_0 \). It may happen, however, that \( \delta _1 \) and \( \delta _2 \) are inner but \( D \) is not. The following example shows this.

**Example 2.3.** Consider that Banach algebra \( \mathcal{A} \) of upper triangular \( 3 \times 3 \) real matrices with \( 0 \) on the diagonal. So \( \mathcal{A}^3 = 0 \) but \( \mathcal{A}^2 \neq 0 \). Let \( a_0, b_0 \) be two non-central and central elements of \( \mathcal{A} \) respectively. Define \( \delta _1, \delta _2 : \mathcal{A} \to \mathcal{A} \) respectively by \( \delta _1 (a) = aa_0 - a_0a \) and \( \delta _2 (a) = ab_0 - b_0a \). Then \( \delta _1, \delta _2 \) are inner but \( D : \mathcal{A} \times _{\theta} \mathcal{A} \to \mathcal{A} \) with \( D((a,b)) = \delta _1 (a) + \delta _2 (b) \) is a derivation which is not inner. To show this, assume towards a contradiction that \( D = ad_{c_0} \) for some \( c_0 \in \mathcal{A} \). Then by Proposition 2.2 \( c_0 - a_0, c_0 - b_0 \in Z (\mathcal{A}) \) and so \( a_0 \in Z (\mathcal{A}) \), a contradiction.

It is worth noting that the above example can be extended to a more general case by considering \( \mathcal{A} \) to be any non-commutative Banach algebra with \( \mathcal{A}^3 = 0 \).

The next result is a consequence of Theorem 2.1

**Proposition 2.4.** Let \( \mathcal{A} \) and \( \mathcal{U} \) be Banach algebras and \( \mathcal{X} \) be a Banach \( (\mathcal{A} \times _{\theta} \mathcal{U}) \)-bimodule. Then
Proof. (i) It is clear that by the following module actions \( D \) is continuous if and only if \( \delta \) is continuous. Moreover, if \( \delta = ad_{x_0} \) for some \( x_0 \in Z_X(U) \), then so is \( \delta \).

(ii) Every derivation \( D : U \to X \) gives rise to a derivation \( \tilde{D} : A \times_U U \to X \). \( \tilde{D} \) is continuous if and only if \( D \) is continuous. Moreover, if \( D = ad_{x_0} \) for some \( x_0 \in Z_X(A) \), then so is \( \tilde{D} \).

Note that in the preceding proposition the inner-ness of \( \tilde{\delta} \) (resp. \( D \)) implies so is \( \delta \) (resp. \( \delta \)). This follows directly from Proposition 2.2 (i).

In this part of the paper, we investigate the relation between the separating spaces of \( \delta_1, \delta_2 \) and use the results to study the automatic continuity of the derivations.

**Theorem 2.5.** Let \( D : A \times_U U \to X \) be a derivation and \( \delta_1, \delta_2 \) be as in Theorem 2.4. Then

(i) \( \mathcal{S}(\delta_1) \) is an \( A \)-submodule of \( X \) and \( \mathcal{S}(\delta_1) \subseteq \text{ann}_X U \).

(ii) \( \mathcal{S}(\delta_2) \) is a symmetric \( A \)-submodule of \( X \) and \( \mathcal{S}(\delta_2) \subseteq Z_X(A) \). Moreover \( \mathcal{S}(\delta_2) \) is a \( U \)-submodule of \( X \), too.

Proof. (i) We only prove the given inclusion. Let \( x_0 \in \mathcal{S}(\delta_1) \). Then there exists some sequence \( a_n \) in \( A \) such that \( a_n \to 0 \) and \( \delta_1(a_n) \to x_0 \). So we have

\[
\delta_2(u)a_n + u\delta_1(a_n) = \theta(a_n)\delta_2(u)
\]

for all \( u \in U \). Letting \( n \) tend to infinity, we get \( u x_0 = 0 \) and similarly \( x_0 u = 0 \) for all \( u \in U \). Hence \( \mathcal{S}(\delta_1) \subseteq \text{ann}_X U \).
(ii) Let \( y_0 \in \mathcal{S}(\delta_2) \). By Theorem 2.5,
\[
\delta_2(u_n) a + u_n \delta_1(a) = a \delta_2(u_n) + \delta_1(a) u_n
\]
for some sequence \( u_n \) in \( \mathcal{U} \) with \( u_n \to 0 \) and all \( a \in \mathcal{A} \). By tending \( n \) to infinity, we obtain \( y_0 a = a y_0 \) for all \( a \in \mathcal{A} \). Thus \( \mathcal{S}(\delta_2) \subseteq Z_X(\mathcal{A}) \).
\( \square \)

**Theorem 2.6.** Let \( D: \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \to \mathcal{X} \) be a derivation and \( \delta_1, \delta_2 \) be as in Theorem 2.5.

(i) If \( \text{ann}_X \mathcal{U} = \{0\} \) or \( \mathcal{U} \) has a left (or right) bounded approximate identity for \( \mathcal{X} \), then \( \delta_1 \) is continuous.

(ii) Suppose that \( Z_X(\mathcal{A}) = \{0\} \). Then \( \delta_2 : \mathcal{U} \to \mathcal{X} \) is a continuous derivation. In addition if \( \text{ann}_X \mathcal{U} = \{0\} \), then any derivation \( D: \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \to \mathcal{X} \) is continuous.

**Proof.**
(i) If \( \text{ann}_X \mathcal{U} = \{0\} \), by Theorem 2.5 it follows that \( \mathcal{S}(\delta_1) = \{0\} \). Thus \( \delta_1 \) is continuous.

(ii) If \( Z_X(\mathcal{A}) = \{0\} \), the second part of Theorem 2.5 implies \( \mathcal{S}(\delta_2) = \{0\} \). Hence \( \delta_2 \) is continuous.
\( \square \)

If \( \mathcal{X} \) and \( \mathcal{Y} \) are Banach \( \mathcal{A}, \mathcal{U} \)-bimodule respectively, it can be seen that the module actions
\[
(a, u) x = ax \quad , \quad x (a, u) = xa
\]
and
\[
(a, u) y = \theta(a) y + uy \quad , \quad y (a, u) = \theta(a) y + yu \quad (a \in \mathcal{A}, u \in \mathcal{U}, x \in \mathcal{X}, y \in \mathcal{Y}).
\]
transform \( \mathcal{X} \) into a Banach \( \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \)-bimodule. Now consider \( \mathcal{M} = \mathcal{X} \times \mathcal{Y} \). \( \mathcal{M} \) becomes a Banach \( \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \)-bimodule with the module actions given by
\[
(a, u) (x, y) = (ax, \theta(a) y + uy)
\]
\[
(x, y) (a, u) = (xa, \theta(a) y + yu) \quad (a \in \mathcal{A}, u \in \mathcal{U}, x \in \mathcal{X}, y \in \mathcal{Y}).
\]

**Theorem 2.7.** Let \( \mathcal{X}, \mathcal{Y} \) be Banach \( \mathcal{A}, \mathcal{U} \)-bimodules respectively and \( \mathcal{M} \) defined as above. Then \( D: \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \to \mathcal{M} \) is derivation if and only if
\[
D((a, u)) = d_1(a) + d_2(u) \quad (a \in \mathcal{A}, u \in \mathcal{U})
\]
where \( d_1 : \mathcal{A} \to \mathcal{X} \) and \( d_2 : \mathcal{U} \to \mathcal{Y} \) are derivations. Moreover, \( D \) is inner if and only if \( d_1, d_2 \) are inner in such a way if \( D = ad_z \) with \( z = (x, y) \in \mathcal{M} \), then \( d_1 = ad_x \) and \( d_2 = ad_y \) and vice versa.

**Proof.** It can be routinely checked that \( D \) is a derivation if and only if \( d_1, d_2 \) are derivations. For the second part, suppose that \( D = ad_z \) for some \( z = (x, y) \in \mathcal{M} \). Then we have
\[
D((a, u)) = (a, u)(x, y) - (x, y)(a, u) = (ax - xa, uy - yu)
\]
for all \( (a, u) \in \mathcal{A} \times_{\mathcal{\theta}} \mathcal{U} \) and \((x, y) \in \mathcal{M} \). So \( \delta_1(a) = ad_x \) and \( \delta_2(u) = ad_y \) are inner. The other direction can be done similarly so we omit its proof.

\( \square \)

**2.1 Special cases.** Let \( \mathcal{A}, \mathcal{U} \) and \( \mathcal{X} \) be as in the previous section. In this subsection we shall study automatic continuity of the derivations from Lau Banach algebras for some special cases of \( \mathcal{X} \) and \( \mathcal{U} \) and establish various results in this context.
**X is a simple Banach \((A \times_\theta U)\)-bimodule.** In this part we assume that \(X\) is a simple Banach \((A \times_\theta U)\)-bimodule and obtain some results as follows.

**Theorem 2.8.** Suppose that \(X\) is a simple Banach \((A \times_\theta U)\)-bimodule and \(D : A \times_\theta U \rightarrow X\) with \(D = \delta_1 + \delta_2\) is any derivation. Then either \(\delta_1\) is continuous or \(\text{ann}_X D = X\).

**Proof.** Since \(\delta_1\) is a derivation, by Theorem 2.7 \(\mathfrak{S}(\delta_1)\) is an \(A\)-subbimodule of \(X\). Since \(X\) is simple, we have either \(\mathfrak{S}(\delta_1) = \{0\}\), from which we conclude that \(\delta_1\) is continuous, or \(\mathfrak{S}(\delta_1) = X\). If \(\mathfrak{S}(\delta_1) = X\), the same Theorem implies that \(\text{ann}_X U = X\). \(\square\)

**Theorem 2.9.** Let \(X\) be a simple Banach \((A \times_\theta U)\)-bimodule and \(D : A \times_\theta U \rightarrow X\) be a derivation with \(D((a,u)) = \delta_1(a) + \delta_2(u)\). Then \(\delta_2\) is continuous or \(X\) is a symmetric Banach \(A\)-bimodule.

**Proof.** \(\mathfrak{S}(\delta_2)\) is an \(U\)-subbimodule of \(X\). Since \(X\) is simple, so either \(\mathfrak{S}(\delta_2) = \{0\}\) or \(\mathfrak{S}(\delta_2) = X\). The former implies that \(\delta_2\) is continuous. If \(\mathfrak{S}(\delta_2) = X\), then since 
\(\mathfrak{S}(\delta_2) \subseteq Z_X(A), \) then \(Z_X(A) = X\) or \(A \times X = XA\). \(\square\)

By the above theorem we conclude that if there are two non-commuting elements \(a_0 \in A, x_0 \in X\), then \(\delta_2\) is continuous.

As a consequence of Theorems 2.8 and 2.9 we state the following result.

**Corollary 2.10.** Suppose \(X\) is a simple Banach \((A \times_\theta U)\)-bimodule and \(D : A \times_\theta U \rightarrow X\) is a derivation with \(D = \delta_1 + \delta_2\). Then \(D\) is continuous if either of the following conditions holds.

(i) \(\text{ann}_X U \neq X\) and \(Z_X(A) \neq X\).

(ii) \(\text{ann}_X U = \{0\}\) and \(Z_X(A) \neq X\).

**Proof.**

(i) It is clear by Theorems 2.8 and 2.9 that \(D\) is continuous.

(ii) Follows directly from Theorem 2.9 and Corollary 2.6 \(\square\)

**The case \(U = A\).** In this part we study derivations \(D : A \times_\theta A \rightarrow X\) where \(A\) is a Banach algebra and \(X\) a Banach \((A \times_\theta A)\)-bimodule.

According to Theorem 2.9 if \(Z_X(A) = \{0\}\), then since \(\text{ann}_X A \subseteq Z_X(A)\), every derivation \(D : A \times_\theta A \rightarrow X\) is continuous. We give the following example satisfying these conditions.

**Example 2.11.**

(i) Let \(A = K(X)\), the compact operators on an infinite dimensional Banach space \(X\). Then since \(Z(A) = \{0\}\), any derivation \(D : K(X) \times_\theta K(X) \rightarrow K(X)\) is automatically continuous.

(ii) Suppose that \(B\) is a rectangular band and \(A = \ell^1(B)\). Then every derivation \(D : \ell^1(B) \times_\theta \ell^1(B) \rightarrow \ell^1(B)\) is continuous.

**Lemma 2.12.** Let \(D : A \times_\theta A \rightarrow X\) be a derivation with \(D = \delta_1 + \delta_2\). Then \(\mathcal{I}(\delta_1) = A\).

**Proof.** The result immediately follows from the definition of a continuity ideal with Theorem 2.5(i). \(\square\)

**Corollary 2.13.** Let \(D : A \times_\theta A \rightarrow X\) be a derivation with \(D = \delta_1 + \delta_2\). Then for any \(a \in A\), the linear mapping \(d_a : A \rightarrow X\) given by \(D_a(b) = a \delta_1(b)\) is a continuous derivation.
Let $\mathcal{A}$ and $\mathcal{U}$ be two Banach algebras. Then it is easy to see that by module actions
\[ a.u = \theta(a)u, \quad u.a = \theta(a)u, \]
for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. Thus, $\mathcal{U}$ becomes a Banach $\mathcal{A}$-bimodule. Therefore using Theorem 2.12, $\delta_2$ is a generalized $\delta_1$-derivation that satisfies
\[ \delta_2(ua) = \delta_2(u)a + u\delta_1(a) \]
for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. The generalized derivation $\delta_2$ appeared naturally in the decomposition of derivations $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{X}$. The next theorem connects the continuity of $\delta_1, \delta_2$ for the case $\mathcal{U} = \mathcal{A}$ to that of $\mathcal{A}$-bimodule homomorphisms.

**Theorem 2.14.** Suppose that $D : \mathcal{A} \times_{\theta} \mathcal{A} \to \mathcal{X}$ is a derivation with $D = \delta_1 + \delta_2$. Then $\delta_2$ is a generalized $\delta_1$-derivation if and only if $\delta_2 - \delta_1$ is a right $\mathcal{A}$-bimodule homomorphism.

**Proof.** First suppose that $\delta_2$ is a generalized $\delta_1$-derivation. Then we have
\[
(\delta_2 - \delta_1)(ab) = \delta_2(ab) = \delta_2(a)b + a\delta_1(b) - \delta_1(ab) = \delta_2(a)b - \delta_1(ab)
\]
for all $a, b \in \mathcal{A}$. Conversely, if $\delta_2 - \delta_1$ is a right $\mathcal{A}$-bimodule homomorphism, then by an easy calculation it can be seen that $\delta_2$ is a generalized $\delta_1$-derivation.

**Remark 2.15.** Note that a direct application of the Cohen factorization theorem shows that if $\mathcal{A}$ has a bounded approximate identity for $\mathcal{X}$, then every $\mathcal{A}$-bimodule homomorphism $\phi : \mathcal{A} \to \mathcal{X}$ is continuous. Indeed, let $(a_n) \subseteq \mathcal{A}$ be a sequence with $a_n \to 0$. Then by Cohen factorization theorem there exist a sequence $(b_n)$ in $\mathcal{A}$ converging to zero and some $c \in \mathcal{A}$ such that $a_n = cb_n$, so $\phi(a_n) = \phi(c)b_n \to 0$. Thus $\phi$ is continuous.

**Corollary 2.16.** Let $\mathcal{A}$ be a Banach algebra which has a bounded approximate identity and $D : \mathcal{A} \times_{\theta} \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then $\delta_1$ is continuous if and only if $\delta_2$ is continuous.

**Proof.** We know $\delta_2$ satisfies
\[
\delta_2(ab) = \delta_2(a)b + a\delta_2(b) \quad \text{and} \quad \delta_2(ab) = \delta_2(a)b + a\delta_1(b)
\]
for all $a, b \in \mathcal{A}$. So $\delta_2(a)b = \delta_1(a)b$ ($a, b \in \mathcal{A}$). Thus $(\delta_2 - \delta_1)(\mathcal{A}) \subseteq \text{ann}_X \mathcal{A}$. 

In the case where $\text{ann}_X \mathcal{A} = \{0\}$, $\delta_1, \delta_2$ agree on $\mathcal{A}$. For instance if $\mathcal{A}$ has a bounded approximate identity for $\mathcal{X}$, then $\text{ann}_X \mathcal{A} = \{0\}$.

**Corollary 2.17.** Let $D : \mathcal{A} \times_{\theta} \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. If $\text{ann}_X \mathcal{A} = \{0\}$, then $\delta_2 = \delta_1$. In this case any derivation $D : \mathcal{A} \times_{\theta} \mathcal{A} \to \mathcal{X}$ can be written as $D((a, b)) = \delta_1(a) + \delta_1(b) = \delta_1(a + b)$.
The case $X = U$.

As we noted before, $U$ is an ideal in $A \times_\theta U$. So $U$ is a Banach $(A \times_\theta U)$–bimodule as well. The following proposition is a special case of Theorem 2.1.

**Proposition 2.19.** Let $A$ and $U$ be two Banach algebras. Then $D : A \times_\theta U \to U$ is a derivation if and only if $D = \delta_1 + \delta_2$ such that $\delta_1 : A \to U$ and $\delta_2 : U \to U$ are derivations and

$$\theta(a)\delta_2(u) = \delta_1(a)u + a\delta_2(u) = u\delta_1(a) + \delta_2(u)a \quad (a \in A, u \in U).$$

In the next theorem we state some results similar to those of Theorem 2.5 and Corollary 2.21. Using the results we study the continuity of the derivations $D : A \times_\theta U \to U$.

**Theorem 2.20.** Let $D : A \times_\theta U \to U$ be a derivation with $D = \delta_1 + \delta_2$. Then

(i) $S(\delta_1)$ is an $A$-submodule of $U$ and $S(\delta_2)$ is an ideal in $U$. In particular, $A S(\delta_2) = S(\delta_2)A = S(\delta_2)$. 

(ii) $U$ annihilates $S(\delta_1)$; that is, $US(\delta_1) = S(\delta_1)U = \{0\}$.

**Corollary 2.21.** Suppose that $D : A \times_\theta U \to U$ is a derivation with $D = \delta_1 + \delta_2$ and $\text{ann}_U U = \{0\}$, then $\delta_1$ is continuous. In this case $D$ is continuous if and only if $\delta_2$ is continuous.

**Proof.** By the preceding theorem, $S(\delta_1)U = US(\delta_1) = \{0\}$. So $S(\delta_1) = \{0\}$ and thus $\delta_1$ is continuous. □

Note that any Banach algebra $U$ possessing a bounded approximate identity satisfies the hypothesis of the above corollary; since in this case $\text{ann}_U U = \{0\}$. For the case $U$ is semisimple, a well-known result of Johnson implies the continuity of $\delta_2$.

**Proposition 2.22.** Let $A$ and $U$ be Banach algebras such that $U$ is semisimple. Then every derivation $D : A \times_\theta U \to U$ is continuous.

**Proof.** Since $U$ is semisimple, by the Johnson theorem $\delta_2 : U \to U$ is continuous. On the other hand, $\text{ann}_U U = \{0\}$, thus by Proposition 2.1(i), $\delta_1$ is continuous as well. Therefore every derivation $D : A \times_\theta U \to U$ is continuous. □

All $C^*$-algebras, semigroup algebras, measure algebras and unital simple algebras are semisimple Banach algebras with a bounded approximate identity. Thus these classes of Banach algebras satisfy the hypothesis of the above proposition. Consequently, we have the following result.

**Corollary 2.23.** Suppose that $A$ is a Banach algebra and $U$ is a $C^*$-algebra. Then every derivation $D : A \times_\theta U \to U$ is continuous.

The last case which will be discussed is the case where Banach $(A \times_\theta U)$-bimodule $X$ is $A \times_\theta U$ itself.

**Theorem 2.24.** Let $D : A \times_\theta U \to A \times_\theta U$ be a mapping. Then the following statements are equivalent.
Theorem [26], for every derivation $U$ and $u \in \mathcal{U}$ where
\( \delta_1 : \mathcal{A} \to \mathcal{A}, \delta_2 : \mathcal{A} \to \mathcal{U} \) are derivations such that
\[
\theta(\delta_1(a))u + \delta_2(a)u = 0 \quad \text{and} \quad \theta(\delta_1(a))u + u\delta_2(a) = 0 \quad (a \in \mathcal{A}, u \in \mathcal{U}).
\]
(i) $D$ is a derivation.
(ii) $D((a, u)) = (\delta_1(a) + \tau_1(u), \delta_2(a) + \tau_2(u))$
for all $a \in \mathcal{A}, u \in \mathcal{U}$ where
\[
\delta_1 : \mathcal{A} \to \mathcal{A}, \delta_2 : \mathcal{A} \to \mathcal{U}
\]
are derivations.

Corollary 2.25. Suppose that $\delta_1 : \mathcal{A} \to \mathcal{A}, \delta_2 : \mathcal{A} \to \mathcal{U}, \tau_1 : \mathcal{U} \to \mathcal{A}$ and $\tau_2 : \mathcal{U} \to \mathcal{U}$ are linear mappings. Then

(i) $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{A} \times_{\theta} \mathcal{U}$ defined by $D((a, u)) = (\delta_1(a), 0)$ is a derivation if and only if $\delta_1$ is a derivation. Also, $D$ is inner if and only if $\delta_1$ is inner.

(ii) $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{A} \times_{\theta} \mathcal{U}$ with $D((a, u)) = (0, \delta_2(a))$ is a derivation if and only if $\delta_2$ is a derivation and $D(\mathcal{A}) \subseteq \text{ann}_\mathcal{U} \mathcal{U}$. Moreover, if $\delta_2 = \text{ad}_{u_0}$ is inner and $u_0 \in Z(\mathcal{U})$, then $D$ is inner.

(iii) $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{A} \times_{\theta} \mathcal{U}$ with $D((a, u)) = (\tau_1(u), 0)$ is a derivation if and only if $\tau_1(u)u' = 0, u\tau_1(u') + \tau_1(u)u' = 0 (u, u' \in \mathcal{U})$. In this case $D$ is inner if and only if $\tau_1 = 0$ and $Z(\mathcal{A}) \neq \emptyset$.

(iv) $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{A} \times_{\theta} \mathcal{U}$ by $D((a, u)) = (0, \tau_2(u))$ is (inner)derivation if and only if $\tau_2$ is (inner) derivation.

If $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is commutative, then by Thomas’ theorem [26], for every derivation $D$ on $\mathcal{A} \times_{\theta} \mathcal{U}, \delta_1(\mathcal{A}) \subseteq \text{rad}(\mathcal{A}) \subseteq \ker \theta = \text{ann}_\mathcal{U} \mathcal{U}$ and $\delta_2(\mathcal{A}) \subseteq \text{ann}_\mathcal{U} \mathcal{U}$. Also, in this case $D = D_1 + D_2 + D_3$ where $D_1((a, u)) = (\delta_1(a), 0), D_2((a, u)) = (0, \delta_2(a))$ and $D_3((a, u)) = (\tau_1(u), \tau_2(u))$ are derivations on $\mathcal{A} \times_{\theta} \mathcal{U}$.

It is clear that if $\tau_1 = 0$, then by Theorem 2.24 $\tau_2$ becomes a derivation. Some conditions on $\mathcal{U}$ that force $\tau_1$ to be zero map are: $\mathcal{U}$ has a bounded approximate identity, $\mathcal{U}$ is unital and $\text{ann}_\mathcal{U} \mathcal{U} = \{0\}$. For instance, if $\mathcal{U}$ is a faithful, a semisimple or any Banach algebra having an approximate identity, then $\text{ann}_\mathcal{U} \mathcal{U} = \{0\}$. By Corollary 2.25(iv), the continuity of derivations on $\mathcal{A} \times_{\theta} \mathcal{U}$ implies the continuity of the derivations on $\mathcal{U}$. Particularly, if every derivation on $\mathcal{A} \times_{\theta} \mathcal{A}$ is continuous, then so is every derivation on $\mathcal{A}$.

Proposition 2.26. Let $\mathcal{A}$ and $\mathcal{U}$ be semisimple Banach algebras. Then every derivation $D : \mathcal{A} \times_{\theta} \mathcal{U} \to \mathcal{A} \times_{\theta} \mathcal{U}$ is continuous.

Proof. By [19], Theorem 3.1, $\mathcal{A} \times_{\theta} \mathcal{U}$ is semisimple if and only if both $\mathcal{A}$ and $\mathcal{U}$ are semisimple. Now the result follows from Johnson’s theorem.
Corollary 2.27. Suppose that $A$ is a $C^*$-algebra and $U$ is a Banach algebra with a bounded approximate identity. Then every derivation on $A \times \theta U$ is continuous if and only if every derivation on $U$ is continuous.

Proof. First suppose that every derivation on $U$ is continuous. Thus for every derivation $D : A \times \theta U \to A \times \theta U$ we have

$$D((a, u)) = (\delta_1(a) + \tau_1(u), \delta_2(a) + \tau_2(u)),$$

such that $\delta_1, \delta_2$ are derivations and by Ringrose’s result [21] are continuous. Since $U$ has a bounded approximate identity, $\tau_1 = 0$ and $\tau_2$ becomes a derivation. Thus $D$ is continuous if and only if $\tau_2$ is continuous. The converse is clear so omitted. □

Remark 2.28. Let $A$ be a commutative and $U$ a semisimple Banach algebra with $U$ having a bounded approximate identity. Then every derivation $D : A \times \theta U \to A \times \theta U$ is of the form $D = D_1 + D_2$ where $D_1((a, u)) = (\delta_1(a), 0)$ and $D_2((a, u)) = (0, \delta_2(u))$ are derivations on $A \times \theta U$ such that $D_2$ is continuous. Moreover, $D_1$ is continuous if $A \times \theta U$ is continuous.

Theorem 2.29. Let $A$ and $U$ be Banach algebras with bounded approximate identity. If $A$ is commutative. Then every derivation $D : A \times \theta U \to A \times \theta U$ can be written as $D = D_1 + D_2$ where $D_1((a, u)) = (\delta_1(a), \tau_2(u))$ and $D_2((a, u)) = (0, \delta_2(a))$ are derivations on $A \times \theta U$. Moreover, $D_1$ is continuous if either of the following conditions holds.

(i) There is a surjective $A$-module homomorphism $\phi : A \to U$ and $\delta_1$ is continuous.

(ii) There is an injective $A$-module homomorphism $\phi : A \to U$ and $\tau_2$ is continuous.

Proof. (i) Define $\psi : A \to U$ by $\psi = \tau_2 \circ \phi - \phi \circ \delta_1$. It is easy to see that $\psi$ is a continuous left $A$-module homomorphism. Similarly, $\phi \circ \delta_1$ is continuous and so is $\tau_2 \circ \phi$. On the other hand, $\phi$ is surjective. So $\mathcal{G}(\tau_2 \circ \phi) = \mathcal{G}(\tau_2) = \{0\}$ by [6], Proposition 5.2.2. Thus $\tau_2$ is continuous.

(ii) The proof is similar to part (i).

Remark 2.30. For every continuous derivation $\delta : A \to A$, we have $\delta(A) \subseteq \ker \theta = \text{ann}_A U$. Since Sinclair’s theorem implies that $\delta(P) \subseteq P$ for every primitive ideal $P$ of $A$. So in this case for continuous derivation $D : A \times \theta U \to A \times \theta U$ we have $\delta_1(A) \subseteq \ker \theta = \text{ann}_A U$ and $\delta_2(A) \subseteq \text{ann}_U U$. Also, $D = D_1 + D_2 + D_3$ where $D_1((a, x)) = (\delta_1(a), 0)$, $D_2((a, x)) = (0, \delta_2(a))$ and $D_3((a, x)) = (\tau_1(x), \tau_2(x))$ are all continuous.

Note that if $\delta : A \to U$ is a continuous derivation, this does not necessarily imply $\delta(A) \subseteq \text{ann}_U U$, as the following example shows.

Example 2.31. Assume that $G$ is a non-discrete abelian group. It has been shown in [4] that there is a nonzero continuous point derivation $d$ at a nonzero character $\theta$ on $M(G)$. Now consider $M(G) \times \theta \mathbb{C}$. Every derivation from $M(G)$ into $\mathbb{C}_\theta$ is a point derivation at $\theta$. It is clear that $\text{ann}_{\mathbb{C}} \mathbb{C} = \{0\}$. But $d \in Z^1(M(G), \mathbb{C}_\theta)$ is a nonzero derivation such that $d(M(G)) \not\subseteq \text{ann}_{\mathbb{C}} \mathbb{C} = \{0\}$.
3. The Multipliers

In this section we turn our attention to the multipliers of Lau products. As before, $A, U$ are Banach algebras, $\theta \in \Delta(A)$ is a nonzero character and $A \times_{\theta} U$ is the associated Lau Banach algebra.

In the following theorem we characterize the multipliers on $A \times_{\theta} U$.

**Theorem 3.1.** A linear mapping $T : A \times_{\theta} U \to A \times_{\theta} U$ is a multiplier if and only if there are some linear mappings $R_1 : A \to A, R_2 : A \to U, S_1 : U \to A$ and $S_2 : U \to U$ with

$$T((a, u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u)) \quad (a \in A, u \in U)$$

satisfying the following conditions:

(i) $R_1 : A \to A$ is a multiplier,
(ii) $aS_1(u) = S_1(u)a = 0$,
(iii) $\theta(a)R_2(a') = \theta(a')R_2(a)$,
(iv) $\theta(a)S_2(u) = \theta(R_1(a)u + R_2(a)u = \theta(R_1(a))u + uR_2(a)$,
(v) $\theta(S_1(u))u' + S_2(u)u' = \theta(S_1(u))u + uS_2(u')$,

for all $a, a' \in A$ and $u, u' \in U$.

**Proof.** First suppose that $T$ is a multiplier. Since $T$ is linear, there exist some linear mappings $R_1 : A \to A, R_2 : A \to U, S_1 : U \to A$ and $S_2 : U \to U$ with

$$T((a, u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u))$$

for all $a \in A$ and $u \in U$. By the definition, $T((a, u))(a', u') = (a, u)T((a', u'))$ For all $a, a' \in A$ and $u, u' \in U$. If we substitute $u = u' = 0$, then we deduce that $R_1$ is a multiplier and $\theta(a)R_2(a') = \theta(a')R_2(a)$ for all $a, a' \in A$. Similarly, substituting $a = a' = 0$ yields (v). If we put $a' = 0, u = 0$ and $a = 0, u' = 0$ respectively, we obtain equalities given in (iv). Also putting $a' = u' = 0$, we conclude that $aS_1(u) = S_1(u)a = 0$ for all $a \in A$ and $u \in U$.

The converse is straightforward and is left for the reader. \qed

In view of the above theorem, in the sequel we consider any multiplier $T : A \times_{\theta} U \to A \times_{\theta} U$ as

$$T((a, u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u)) \quad (a \in A, u \in U)$$

in which the mentioned maps satisfy the conditions (i) -- (v). Part (iv) of the above theorem implies that $R_2$ always maps $A$ into the center of $U$ (that is, $R_2(A) \subseteq Z(U)$) and also if we put $u = u'$ in (v), we conclude that $S_2(u)u = uS_2(u)$ for all $u \in U$. Moreover by (ii), $S_1(A) \subseteq \text{ann}_A A$.

We now state the following theorem.

**Theorem 3.2.** Suppose that $T : A \times_{\theta} U \to A \times_{\theta} U$ is a multiplier with

$$T((a, u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u)) \quad (a \in A, u \in U).$$

Then

(i) The maps $R_2 : A \to U$ and $S_2 : U \to U$ are automatically continuous.
(ii) $\mathcal{S}(R_1), \mathcal{S}(S_1) \subseteq \ker \theta$.

**Proof.** (i) Let $s \in \mathcal{S}(S_2)$. Thus there exists some $u_n$ in $U$ for which $u_n \to 0$ and $S_2(u_n) \to s$. By Theorem 3.1(iv) we have

$$\theta(a)S_2(u_n) = \theta(R_1(a))u_n + R_2(a)u_n$$

for all $a \in A$. Since $\mathcal{S}(S_2) \subseteq \ker \theta$, it follows that $u_n \to 0$. Hence $R_2(a) \to 0$ for all $a \in A$. Thus $R_2$ is continuous.

(ii) Since $\mathcal{S}(S_1) \subseteq \ker \theta$, it follows that $S_1(u) = 0$ for all $u \in U$. Therefore $S_2(u) = 0$ for all $u \in U$. Hence $\mathcal{S}(S_1) \subseteq \ker \theta$. \qed
for all $a \in A$. Letting $n$ tend to infinity, we obtain $\theta(a)s = 0$ for all $a \in A$. So $s = 0$ and hence $S_2$ is continuous. For the continuity of $R_2$, suppose that $s' \in \mathfrak{S}(R_2)$ and $a_n$ is a sequence in $A$ with $a_n \to 0$ and $R_2(a_n) \to s'$. By part $(ii)$ of the preceding theorem,

$$\theta(a')R_2(a_n) = \theta(a_n)R_2(a')$$

for all $a' \in A$. By taking limits one obtains $s' = 0$. Thus $R_2$ is continuous.

(ii) Let $a_0 \in \mathfrak{S}(R_1)$. Then there exists $a_n \subseteq A$ such that $a_n \to 0$ and $R_1(a_n) \to a_0$. Therefore $\theta(a_n)S(u) = \theta(R_1(a_n))u + R_2(a_n)u$ for all $y \in U$. So $\theta(a_0)u = 0$ for all $u \in U$. Thus $a_0 \in \ker \theta$. The other inclusion is similar.

□

If $A$ is faithful Banach algebra, then every multiplier on $A$ is continuous. On the other hand, in this case $S_1 = 0$, since by Theorem 3.1-$(ii)$, $S_1(A) \subseteq \text{ann}_A A = \{0\}$, so $S_2 : U \to U$ is then a multiplier on $U$. Therefore we deduce the following result.

**Theorem 3.3.** Suppose that $A$ and $U$ are Banach algebras such that $A$ is faithful. Then every multiplier on $A \times_\theta U$ is continuous.

It is well-known in each of the following cases the Banach algebra $A$ is faithful:

1. $A$ is unital.
2. $A$ has an approximate identity (for example, $A$ is a $C^*$-algebra).
3. $A$ is semiprime.
4. $A$ is a semisimple.

Suppose that $A, U$ are Banach algebras such that $A$ is a faithful. Then an easy calculation shows that

$$\text{ann}_1(A \times_\theta U) = \{0, u : u \in \text{ann}_U U\} \cong \text{ann}_U U.$$

Therefore if $A$ is faithful, $A \times_\theta U$ is faithful if and only if $U$ is so. Therefore if we assume that $U$ is not faithful, then $A \times_\theta U$ is not faithful as well and by Theorem 3.3 all multipliers $T : A \times_\theta U \to A \times_\theta U$ are continuous. This result can be interesting by itself since it can provide non-faithful Banach algebras on which every multiplier is continuous.

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