Generalized Symmetries of Partial Differential Equations and Quasiexact Solvability

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Using the adjoint action of the infinitesimal translations (with respect to some (in)dependant variables) on specific finite-dimensional subspaces of the space of generalized symmetries of some system of partial differential equations, we explicitly determine the dependance of coefficients of generalized symmetries from these subspaces on the above-mentioned variables. We establish the connection of our results with the theory of quasiexactly solvable models. Some generalizations of the approach proposed also are discussed.

Introduction

It is well known [1], [2] that the problem of finding of all the generalized symmetries (or non-Lie symmetries in terms of [2]) of a given system of partial differential equations (PDEs) is nontrivial and seldom admits the complete solution. More or less complete results in this field are obtained for some linear equations (e.g. free Dirac, Klein – Gordon and Schrödinger equations ([2] and references therein)), 1+1-dimensional evolution equations (vide, e.g., [3], [4]) and integrable systems ([5] and references therein).

On the other hand, the knowledge of the generalized symmetries of a given system of PDEs allows one to construct (partially) invariant solutions of this system (the famous examples of which are finite-gap solutions of the KdV equation and numerous spherically-symmetric, self-similar etc. solutions [3]), to find the conservation laws [1], [3], to check the integrability [4] and to separate variables in linear PDEs [1]. Thus, finding generalized symmetries of given system of PDEs is interesting problem of modern mathematical physics.

In this paper we consider the systems of PDEs, whose set of generalized symmetries possesses specific finite-dimensional subspaces \( V^{(q)} \) (refer to Section 2), which are invariant under the action of the set of infinitesimal symmetries.

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translations with respect to some (in)dependent variables. For the generalized symmetries from these subspaces we determine their dependance on these variables, i.e. partially solve the above-mentioned problem.

The plan of the paper is as follows. In Section 1 we briefly remind some basic notions concerning generalized symmetries. Section 2 presents the key idea of the proposed approach and some possible generalizations. Finally, in the Section 3 we discuss the connection of our results with the theory of quasiexact solvability.

1 Generalized symmetries: basic definitions

Let us consider the system of PDEs of the form:

$$F_\nu(x, u, \ldots, u^{(d)}) = 0, \quad \nu = 1, \ldots, f, \quad (1)$$

where \( u = u(x) \) is unknown vector-function \( u = (u_1, \ldots, u_n)^T \) of \( m \) independent variables \( x = (x_1, \ldots, x_m) \); \( u^{(s)} \) denotes the set of derivatives of \( u \) with respect to \( x \) of the order \( s \). (\( T \) denotes here and below matrix transposition).

**Definition 1** [1]. The operator \( Q \) of the form

$$Q = \sum_{i=1}^{m} \xi_i(x, u, \ldots, u^{(q)}) \partial/\partial x_i + \sum_{a=1}^{n} \eta_a(x, u, \ldots, u^{(q)}) \partial/\partial u_a \quad (2)$$

is called the generalized symmetry of order \( q \) of the system of PDEs (1) if its prolongation \( \text{pr}Q \) annihilates \( F_\nu \) on the set \( M \) of (sufficiently smooth) solutions of (1):

$$\text{pr}Q[F_\nu \mid M] = 0, \quad \nu = 1, \ldots, f. \quad (3)$$

Here \( \text{pr}Q = \sum_{l=1}^{m} \xi_l D_l + \sum_{a=1}^{n} \sum_{J}(D_J(\eta_a - \sum_{l=1}^{m} \xi_l u_{a,l})) \partial/\partial u_{a,J} \quad (4)$$

where the summation is extended to all the multiindices \( J = (j_1, \ldots, j_m) \) with non-negative integer \( j_s, \ s = 1, \ldots, m, \ u_{a,J} = \partial^{j_1 + \ldots + j_m} u_a / \partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_m^{j_m}, \ u_{a,l} \equiv \partial u_a / \partial x_l, \ D_l \) is the so-called total derivative \( \text{[1]} \) acting on the functions of \( x, u, u^{(1)}, \ldots \):

$$D_l = \partial/\partial x_l + \sum_J (\partial u_{a,J} / \partial x_l) \partial/\partial u_{a,J}$$
and $D_J = D_1^{j_1} D_2^{j_2} \ldots D_m^{j_m}$.

Let us mention that the generalized symmetries of (1) of order 0 are usually called Lie symmetries \[1\].

**Definition 2** \[1\]. Let $Q_s$ be some differential operators of the form (2):

$$Q_s = \sum_{i=1}^{m} \xi_i^{(s)}(x, u, \ldots, u^{(q_s)}) \partial/\partial x_i + \sum_{\alpha=1}^{n} \eta_\alpha^{(s)}(x, u, \ldots, u^{(q_s)}) \partial/\partial u_\alpha$$

Then the operator $Q_3$ of the same form with the coefficients

$$\xi_i^{(3)} = \text{pr}Q_1[\xi_i^{(2)}] - \text{pr}Q_2[\xi_i^{(1)}]$$
$$\eta_\alpha^{(3)} = \text{pr}Q_1[\eta_\alpha^{(2)}] - \text{pr}Q_2[\eta_\alpha^{(1)}]$$

is called the Lie bracket of the operators $Q_1$, $Q_2$ and is denoted $Q_3 = [Q_1, Q_2]$.

**Proposition 1** \[1\]. If $Q_1$, $Q_2$ are generalized symmetries of (4), then so does $[Q_1, Q_2]$.

Let $Sym$ be the Lie algebra \[1\](with respect to Lie bracket $[,]$) of all the generalized symmetries of (1) of non-negative orders, $Sym^{(q)}$ the linear space of the generalized symmetries of (1) of order not higher than $q$ and $p^{(q)}$ its dimension, $Sym_q \equiv Sym^{(q)}/Sym^{(q-1)}$ ($q \neq 0$), $Sym_0 \equiv Sym^{(0)}$ and $p_0$ the dimension of $Sym_0$. It is straightforward to check \[1\] that $Sym_0$ is subalgebra of $Sym$ with respect to Lie bracket. The generalized symmetries of $Sym_0$ (i.e. Lie symmetries) may be considered as vector fields on the manifold of 0-jets $M^{(0)}$ \[1\] with local coordinates $z_A$: $z_i = x_i, i = 1, \ldots, m, z_{m+\alpha} = u_\alpha, \alpha = 1, \ldots, n$ (from now on the indices $A, B, C, D, \ldots$ will run from 1 to $m+n$ and we shall denote $\partial_A \equiv \partial/\partial z_A$). Moreover, the Lie bracket of two Lie symmetries of (1) $Q_1, Q_2$ coincides with the commutator of corresponding vector fields.

Let us mention that we choose as the basic field the field of complex numbers $\mathbb{C}$, i.e. $Sym$ is considered as Lie algebra over $\mathbb{C}$ and the coefficients $\xi_i, \eta_\alpha$ of generalized symmetries are complex \[1\].

\[2\]In fact, everywhere in our considerations (except the examples) $\mathbb{C}$ may be replaced by the arbitrary algebraically closed field $\mathbb{K}$, since the Corollary 1 of the Theorem 3' from §2 of Chapter VIII of \[1\], which we use in the proofs of our theorems, remains true for this case too.
2 The explicit formulas for the symmetries

Theorem 1 Let $W$ be some linear subspace of the linear space of the differential operators of the form $W_{A_1,\ldots,A_g}$ be the linear space of the operators, obtained from the operators from $W$ by setting $z_{A_1} = 0, \ldots, z_{A_g} = 0$ in their coefficients, $V = W \cap \text{Sym}$, for some $q_1$ the dimension of the subspace of $V \cap \text{Sym}^{(q_1)} \equiv V \cap \text{Sym}^{(q_1)}$ $v(q_1) < \infty$ and for any generalized symmetry of $[7]$ $Q \in V^{(q_1)} \partial Q/\partial z_{A_s} \in V, s = 1, \ldots, g$.

Then in each $V^{(q)}$ ($q = 0, \ldots, q_1$) there exists such a basis of linearly independent generalized symmetries $Q_{l}^{(q)}$, $l = 1, \ldots, q^{(q)}$, $\gamma = 1, \ldots, \rho^{(q)}$ ($\rho^{(q)} \leq v(q)$, $\sum_{\gamma=1}^{\rho^{(q)}} r^{(q)} = v^{(q)}$) that the generalized symmetries from this basis are given by the formulas

$$Q_{l}^{(q)} = \exp(\sum_{s=1}^{q} \lambda^{(q,A_s)}_{\gamma} z_{A_s}) \times \prod_{j_1=0}^{k_{\gamma}^{(q,A_1)}} \ldots \prod_{j_g=0}^{k_{\gamma}^{(q,A_g)}} (z_{A_1})^{j_1} (z_{A_2})^{j_2} \ldots (z_{A_g})^{j_g} C_{l,j_1,\ldots,j_g}^{(q,\gamma)}$$

where $C_{l,j_1,\ldots,j_g}^{(q,\gamma)}$ are some differential operators from $W_{A_1,\ldots,A_g}$ of order $q$ or lower; $\lambda^{(q,A_s)}_{\gamma} \in \mathbb{C}$ are some constants and $k_{\gamma}^{(q,A_s)}$, $s = 1, \ldots, g$ are some fixed numbers from the range $1, \ldots, r^{(q)}$.

Proof. Let $Q_{1}^{(s)}, \ldots, Q_{q}^{(s)}$ be some basis in $V_s$, where $V_s = V^{(s)}/V^{(s-1)}$, $s \neq 0, V_0 = V^{(0)}$.

According to the conditions of the Theorem $\partial Q_{l}^{(s)}/\partial z_{A_i}$ also are generalized symmetries of $[7]$, which obviously belong to $V^{(s)}$ (but not necessarily to $V_s$). Thus, the set of differential operators $\partial/\partial z_{A_i}$ possesses finite-dimensional invariant spaces $V^{(s)}$, $s = 0, \ldots, q_1$. Let us denote the finite-dimensional linear operator being the representation of $\partial/\partial z_{A_i}$ on $V^{(s)}$ by $G^{(s,A_i)}$. Since $\partial^2 Q_{l}^{(s)}/\partial z_{A_i} \partial z_{A_j} = \partial^2 Q_{l}^{(s)}/\partial z_{A_i} \partial z_{A_j}$, i.e. the operators $\partial/\partial z_{A_i}, \partial/\partial z_{A_j}$ commute, so do their representations:

$$G^{(s,A_i)} G^{(s,A_j)} = G^{(s,A_j)} G^{(s,A_i)}, \quad i, j = 1, \ldots, g, s = 0, \ldots, q_1.$$

Hence, according to the Corollary 1 of the Theorem 3’ from §2 of Chapter VIII of [7], the space $V^{(q)}$ ($q \leq q_1$) may be decomposed into the direct sum of

$^3$ $A_1, \ldots, A_g$ are fixed integers from the range $1, \ldots, m + n$, $g \leq m + n$.

$^4$ it is clear that this implies $\partial Q/\partial z_{A_s} \in V^{(q_1)}$, $s = 1, \ldots, g$.

$^5$ since $v^{(s)} \leq v^{(q_1)}$ for $s \leq q_1$. 

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such common invariant spaces \( I^{(q)}_\gamma \) of the linear operators \( G^{(q,A_1)}, \ldots, G^{(q,A_g)} \), that the minimal polynomials of \( G^{(q,A_s)} \) on \( I^{(q)}_\gamma \) are

\[
(G^{(q,A_s)} - \lambda^{(q,A_s)}_\gamma)k^{(q,A_s)}_\gamma = 0
\]

for some \( \lambda^{(q,A_s)}_\gamma \in \mathbb{C} \) and \( k^{(q,A_s)}_\gamma \) \( (k^{(q,A_s)}_\gamma) \) are fixed integers from the range \( 1, \ldots, r^{(q)}_\gamma \), where \( r^{(q)}_\gamma \) denotes the dimension of the space \( I^{(q)}_\gamma \), \( \gamma = 1, \ldots, p^{(q)} \), \( s = 1, \ldots, g \). In order to avoid possible ambiguity in the definition of the subspaces \( I^{(q)}_\gamma \) we set the following requirement: the representation \( G^{(q,A_1)}_\gamma \) of the operators \( G^{(q,A_s)} \) on each \( I^{(q)}_\gamma \), \( i = 1, \ldots, g \), must be indecomposable.

Using this result allows us to restrict ourselves to considering the only \( r^{(q)}_\gamma \)-dimensional subspace \( I^{(q)}_\gamma \) of \( V^{(q)} \) and some basis \( Q^{(q)} \gamma, l = 1, \ldots, r^{(q)}_\gamma \) in it. Let \( R^{(q,\gamma)} = (Q^{(q)}_1, \ldots, Q^{(q)}_l, \ldots, Q^{(q)}_{r^{(q)}_\gamma})^T \) and let \( G^{(q,A_s)}_\gamma \) denote the restriction of \( G^{(q,A_s)} \) on \( I^{(q)}_\gamma \).

Then we obtain

\[
\partial R^{(q,\gamma)}/\partial z_{A_i} = G^{(q,A_s)}_\gamma R^{(q,\gamma)},
\]

where we have identified the operator \( G^{(q,A_s)}_\gamma \) with its matrix in the basis \( Q^{(q)}_l, l = 1, \ldots, r^{(q)}_\gamma \); each such system is compatible, since the matrices \( G^{(q,A_s)}_\gamma \) commute in virtue of (6), and its general solution is:

\[
R^{(q,\gamma)} = \prod_{i=1}^g \exp(G^{(q,A_s)}_\gamma z_{A_i})C^{(q,\gamma)},
\]

where \( C^{(q,\gamma)} \) is the vector of the operators from \( W \) of order \( q \) or lower, whose coefficients are independant from \( z_{A_1}, \ldots, z_{A_g} \) (i.e. \( C^{(q,\gamma)} \in W_{A_1,\ldots,A_g} \)). In virtue of (7) we obtain

\[
\exp(G^{(q,A_s)}_\gamma z_{A_i}) = \exp(\lambda^{(q,A_s)}_\gamma z_{A_i})\exp((G^{(q,A_s)}_\gamma - \lambda^{(q,A_s)}_\gamma)z_{A_i}) = \exp(\lambda^{(q,A_s)}_\gamma z_{A_i})\sum_{s=0}^{k^{(q,A_s)}_\gamma-1} \frac{(z_{A_s})^s}{s!} (G^{(q,A_s)}_\gamma - \lambda^{(q,A_s)}_\gamma)^s.
\]

The substitution of (8) into (3) evidently yields (3). \( \triangleright \)

**Remark 1.** Often (e.g. if \( V \) is an ideal in Sym or if \( V \) is Lie subalgebra of Sym, containing \( \partial/\partial z_{A_i} \) or generalized symmetries, which are equivalent to them) the sufficient condition for \( \partial Q/\partial z_{A_s} \in V, s = 1, \ldots, g \) (if \( Q \in V^{(q)}_1 \)) is (in virtue of the Proposition 1) the existence of Lie symmetries \( \partial/\partial z_{A_1}, \partial/\partial z_{A_2}, \ldots, \partial/\partial z_{A_g} \) for the system (1).
Remark 2. In the above Theorem it is possible to set \( q_1 = \infty \), taking into account that by construction \( V^{(\infty)} = V \) and modifying its conditions in the following way: \( v^{(q_1)} < \infty \) is replaced by \( v_q < \infty \) for all \( q = 0, 1, \ldots \), where \( v_q \) is the dimension of \( V_q \).

Remark 3. If \( W \) is the space of all the Lie vector fields on \( M^{(0)} \), \( V \equiv W \cap Sym = Sym^{(0)} \), \( v^{(0)} < \infty \) and the system (3) admits Lie symmetries \( \partial/\partial z_A, A = 1, \ldots, m + n \) the Theorem 1 allows us to find the dependance of all the Lie symmetries on all the variables \( x, u \) they depend from, i.e. to reduce the problem of description of Lie symmetries of (3) to solving algebraic equations.

From the merely technical point of view our results mean that if the conditions of the Theorem 1 are valid, one may seek for all the generalized symmetries of (1) from \( V^{(q)} (q \leq q_1) \) without loss of generality in the form

\[
Q = \exp(\sum_{s=1}^{q} \lambda^{(q,A_s)} z_{A_s}) \times \sum_{j_1=0}^{v^{(q)}-1} \cdots \sum_{j_g=0}^{v^{(q)}-1} (z_{A_1})^{j_1} (z_{A_2})^{j_2} \cdots (z_{A_g})^{j_g} C_{j_1,\ldots,j_g},
\]

(10)

where \( \lambda^{(q,A_s)} \in \mathbb{C} \) and \( C_{j_1,\ldots,j_g} \) are differential operators from \( W_{A_1,\ldots,A_g} \) of order \( q \) or lower. Thus, the substitution of (10) into (3) yields the equations for the coefficients of \( C_{j_1,\ldots,j_g} \), and if one is able to find all the independant solutions of these equations, the substitution of them into (10) gives all the linearly independant symmetries of (1) from \( V^{(q)} \).

Let us illustrate these ideas by the following examples:

Example 1. Let \( m = 2, n = 1, u_1 \equiv u, x = (x_1 \equiv t, x_2 \equiv y), u(l) = \partial^l u/\partial y^l \) and \( u \) satisfies the evolution equation

\[
\partial u/\partial t = G(u, u_{(1)}, \ldots, u_{(d)}), \quad d \geq 2,
\]

(11)

\( W \) be the linear space of the differential operators of the form (2) with \( \xi_i \equiv 0 \), whose coefficient \( \eta \equiv \eta_1 \), which is called the characteristics of the symmetry (3), depends only on \( y, u, u_{(1)}, u_{(2)}, \ldots \). In [4] it is proved that for such a \( V \) \( v^{(q)} \leq v^{(1)} + q - 1 \) for \( q = 1, 2, \ldots \) and \( v^{(1)} \leq d + 3 \).

It is clear that for any \( Q \in V \) \( \partial Q/\partial y \in V \). In virtue of the Theorem 1 and the above formula (10) without loss of generality we may suppose that the characteristics of the generalized symmetries of (3) from \( W \) of order

\[\text{[6]Let us mention that in this point our results are completely analogous to those of the theory of quasiexact solvability (see also Section 3).}\]
not higher than \( q \) has the form
\[
\eta = \exp(\lambda y) \sum_{j=0}^{v(q)-1} \eta_j(u, u(1), \ldots, u(q)) y^j,
\]
(12)
where \( \lambda \in \mathbb{C} \).

Since \([4]\) for any generalized symmetry \( Q \) of order \( q \geq 2 \) from \( W \)
\[
\frac{\partial \eta}{\partial u(q)} = \text{const} \left( \frac{\partial G}{\partial u(d)} \right)^{q/d}
\]
and \( G \) is independant from \( y \), the comparison of (12) and (13) shows that in such a case \( \lambda = 0 \) in (12), i.e. the generalized symmetries of order \( q \geq 2 \) from \( W \) depend on \( y \) as polynomials of order not higher than \( v(q) - 1 \leq v(1) + q - 2 \).

**Example 2.** Let us consider one-dimensional free Schrödinger equation \( (m = 2, n = 1, u_1 \equiv \psi, x_1 \equiv t, x_2 \equiv x) \):
\[
L\psi \equiv \left( i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) \psi = 0.
\]
(14)
and its symmetry operators \([2]\) of the form \( \{A, B\} \equiv AB + BA, p \equiv -i \frac{\partial}{\partial x} \)
\[
R = \sum_{j=0}^{q} \{ \ldots \{ h_j(x, t), p \}, \ldots p \} \equiv \sum_{j=0}^{q} a_j(x, t) \frac{\partial^j}{\partial x^j},
\]
(15)
which should commute with \( L \):
\[
[R, L] \equiv RL - LR = 0,
\]
what yields the following equations for \( h_j \) \([8]\):
\[
\dot{h}_j = h'_{j-1}, \quad j = 1, \ldots, q,
\]
\[
\dot{h}_0 = 0,
\]
\[
h'_q = 0,
\]
(16)
where dot and prime denote partial derivatives with respect to \( t \) and \( x \).

The symmetry operators of the form \( \{A, B\} \) of (14) may be considered as the elements of \( \text{Sym} \) \([1]\).

Let \( W \) be the set of all the linear differential operators of the form \( \{A, B\} \). It is known \([8]\) that for such a \( W \) the dimensions \( v(q) \) of the subspaces \( V(q) = W \cap \text{Sym}(q) \) (as usual, \( V = W \cap \text{Sym} \)) are \( v(q) = (q + 1)(q + 2)/2 \). Since \( \{A, B\} \) admits Lie symmetries \( \partial/\partial t \) and \( \partial/\partial x \) and if \( R \) is symmetry operator
of (14), then so do $\partial R/\partial t$ and $\partial R/\partial x$, all the conditions of the Theorem 1 are fulfilled. Using the formulas (10), we obtain the following expression for the generic symmetry operator of (14) of order $q$:

$$R = \exp(\lambda t + \mu x) \sum_{k=0}^{v(q)-1} \sum_{l=0}^{v(q)-1} \sum_{s=0}^{q} C_{kls} t^k x^l \partial_s / \partial x^s.$$  \(17\)

where $\lambda, \mu, C_{kls} \in \mathbb{C}$. Hence, $h_j$ for $R (17)$ have the form:

$$h_j = \exp(\lambda t + \mu x) \sum_{k=0}^{v(q)-1} \sum_{l=0}^{v(q)-1} h_{jkl} t^k x^l, \quad h_{jkl} \in \mathbb{C}.$$  \(18\)

The substitution of (18) into (16) yields $\lambda = \mu = 0$ for $R \not\equiv 0$ (i.e. $h_j$ are polynomials with respect to $t$ and $x$) and recurrent relations for $h_{jkl}$. Thus, we partially recovered (by different means) the results of [8]. The results concerning the general solution of (17) may be found there. Let us only mention that $h_j$ is polynomial of order $j$ with respect to $t$ and of order $q - j$ with respect to $x$.

The above examples show that our method is rather efficient tool for the reduction of the equation (3) for the coefficients of generalized symmetries with respect to some "selected" variables, which do not enter explicitly in the system of PDEs under consideration.

The requirement that $\partial Q/\partial z_{A_i} \in V, i = 1, \ldots, g$ for $Q \in V$ in fact may be replaced by the weaker one: there exist $g$ Lie vector fields $K_s = \sum_{A=1}^{m+n} \omega_A^{(s)} (z) \partial/\partial z_A, s = 1, \ldots, g$, such that $[K_i, K_j] = 0$ for all $i, j = 1, \ldots, g$, $[K_i, Q] \in V$ for any $Q \in V, i = 1, \ldots, g$, and in the generic point of $M(0)$ rank $\| \omega_A^{(s)} \|_{A=1,m+n,s=1,g} = g$. Really, in this case there exists [3] such replacement of coordinates on $M(0)$ (which, however, may be defined only locally in each chart of $M(0)$ but not globally) $z \to z'$ that in new coordinates $K_s = \partial/\partial z'_{A_s}, A_s \in \{1, \ldots, m+n\}, s = 1, \ldots, g$ and $[K_i, Q] = \partial Q/\partial z'_{A_i} \in V, i = 1, \ldots, g$ for any $Q \in V$, i.e. we come back to the situation, described in the Theorem 1.

3 Conclusions and Discussion

The general idea of this work consists in the study of adjoint action of Lie symmetries on $\text{Sym}$ (since $\partial Q/\partial z_A = [\partial_A, Q]$!). The peculiarity of Lie symmetries is that the Lie bracket of the Lie symmetry $L$ and some generalized
symmetry $Q$ of (1) of order $q$ is generalized symmetry of order not higher than $q$, i.e. $\text{ad}_{L} \equiv [L, \cdot] : \text{Sym}^{(q)} \rightarrow \text{Sym}^{(q)}$ for any $q$. If the space $\text{Sym}^{(q)}$ (or, in more general situation, $V^{(q)}$) is finite-dimensional for some $q = q_1$, the operator $\text{ad}_{L}$ possesses matrix representation on it. This observation (for $L = \partial_{A_s}$, $s = 1, \ldots, g$) explains the analogy between the formulas (5) and the form of the ansatz for eigenfunctions of Hamiltonian in quasiexactly solvable models since these eigenfunctions belong to the common invariant space of the representation of some Lie algebra by the first order linear differential operators, some of which are just of the form $\partial/\partial z_A$.

The above analogy with the theory of quasiexact solvability shows that it would be very interesting to generalize the results of this paper to the case of non-commutative algebra of Lie symmetries of (1). We intend to do it in further publications.

I am sincerely grateful to Profs. A.G. Nikitin and R.Z. Zhdanov for the fruitful discussion of the results of this work.

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