Emergent topological phase transition in Cluster Ising model with dissipation

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We study a cluster Ising model with non-Hermitian external field which can be exactly solved in the language of free fermions. By investigating the second derivative of energy density and fidelity, the possible new critical points are tentatively located. String order parameter and staggered magnetization are then detected to reveal emergent phases of brand new characteristics. To categorize the exotic phases induced by non-Hermiticity, we calculate the variation mode of spin correlation function, which indicates the emergent critical points denote phase transitions without symmetry breaking. With the help of string order parameter and staggered magnetization, we find that there are four phases after introducing the non-Hermiticity—two symmetry-protected-topological (SPT) phases, one paramagnetic (PM) phase and one antiferromagnetic (AF) phase. A phase diagram is then presented to graphically illustrate the generation of three critical lines from a critical point as non-Hermitian strength increases, which correspond to SPT-SPT, SPT-PM and PM-AFM phase transition, respectively. Our theoretical work is expected to be realized in the experiment of ultra-cold atoms, pushing for progress in exploring novel topological phases and phase transitions.

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I. INTRODUCTION

According to conventional Landau-Ginzberg-Wilson paradigm [1], an ordered phase corresponds to certain symmetry breaking of the system while a disordered phase associates to the absence of symmetry breaking. The phase transition between ordered phases and disordered phases can be detected by certain local parameters. However, topological phase [2–4], which describes the global properties of the system, has to be detected by topological invariants [5, 6] that reflect certain hidden symmetry, including winding number and Chern number [7, 8] related to the Berry phase [9] of the model.

With the rapid development of quantum simulation and quantum computation, the marriage of traditional condensed matter physics and cutting-edge experimental techniques gives birth to various new topics. In highly pure and controllable ultra-cold atom platforms, a triangle optical lattice can be set up with atoms loaded in a unique way [10], so as to create an equivalent three-spin ring-exchange interaction in spin system that can be mapped to a “zig-zag chain”. Notably, the ground state of the system is cluster state [11], a disordered state whose spin spacial rotational symmetry is protected. However, research papers also have it that except for three-spin ring-interaction, two-spin interaction relating to symmetry breaking (SB) state is also observed in such a system, giving rise to the competition of two kinds of interaction. From the view of quantum information, researchers regard the two-spin interaction as a perturbation for it will damage the symmetry protected topological (SPT) cluster state and they are curious about SPT’s threshold of robustness [12–14]. From the perspective of condensed matter physics, the exotic continuous quantum phase transitions (QPT) between SPT phases and SB phases are fascinating for their own sake [15–17]. Therefore, cluster Ising model is put forward as a good toy model to investigate the phase transitions in such quantum many-body systems. Generally, SPT cluster phase can be characterized by global sting order [18, 19] while the SB antiferromagnetic (AFM) state can be identified by local staggered magnetization [20, 21].

The above discussion is based on traditional quantum mechanics, which requires the Hermiticity of observables to ensure that their eigenvalues are real numbers. However, non-Hermitian physics has currently attracted extensive research interest as the non-Hermitian experimental techniques grow mature in a wide range of table-top experimental platforms [22, 23], inclusive of ultra-cold atom system [24, 25], optical system [26–29], nitrogen-vacancy center [30], etc. Many novel non-Hermitian phenomena have been detected, such as parity-time (PT) symmetry [31–33], non-Hermitian skin effect [34, 35], new topological behaviors associated with exceptional points [23, 36–41], disorder induced by non-
Hermiticity [42–44], etc. Recently, intense attention has been paid on how non-Hermiticity influences quantum phase transitions in systems with rotation-time-reversal (\(R\bar{T}\)) symmetry [45–48], spin model with imaginary external field [49–51] and so on.

In this work, we try to uncover the connection between SPT-SB phase transition and non-Hermiticity without \(R\bar{T}\) or \(R\bar{T}\) symmetry. The main body of this paper is organized as follows: In Sec. II, we introduce cluster Ising model with on-site dissipations and analytically transform it into free fermion expression. In Sec. III, the observables and the methods we adopted to characterize quantum phase transition are presented. In Sec. IV, we give the theoretical results of observables and discuss the exotic phase diagram and phase transitions. We summarize our work in Sec. V.

II. MODEL AND EXACT SOLUTION

In this section, we establish the non-Hermitian cluster Ising model (NHCIM) and conduct the diagonalization procedure to obtain exact solution of the ground state. Based on the conventional cluster Ising model, we build up our Hamiltonian by inserting dissipation, which is equivalent to an external imaginary field. The expression reads

\[
H = -\sum_{l=1}^{N} \sigma_{l-1}^{x} \sigma_{l+1}^{x} + \lambda \sum_{l=1}^{N} \sigma_{l}^{y} \sigma_{l}^{y} + i \frac{\Gamma}{2} \sum_{l=1}^{N} \sigma_{l}^{z},
\]

where \(\sigma_{l}^{x}, \sigma_{l}^{y}\) and \(\sigma_{l}^{z}\) are Pauli matrices of the \(j^{th}\) spin and \(\sigma^{x}\) denotes the matrix \(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) corresponding to the loss or gain, which is a simple way to involve non-Hermiticity in optical or atomic experiments [27, 43, 52, 53]. \(N\) is large enough for us to view \(N/2\) as a ‘decent half’ regardless of the parity of \(N\). There are two control parameters \(\lambda\) and \(\Gamma\) in our model, where the former indicates the competition between SPT phase and SB phase while the latter determines the strength of complex field (see Fig. 1).

The diagonalization process can be divided into three steps. First of all, we rewrite the Hamiltonian in free fermionic language using the standard Jordan-Wigner transformation, which is defined as

\[
\sigma_{l}^{+} = \prod_{j<l}(1 - 2c_{j}^{\dagger}c_{j})c_{l},
\]

where \(c_{l}^{\dagger}\) and \(c_{l}\) are the creation and annihilation operators, respectively. Then we can obtain the Hamiltonian in spinless fermion expression as

\[
H = \sum_{l=1}^{N} \left[ -(c_{l}^{\dagger} - c_{l})(c_{l+2}^{\dagger} + c_{l+2}) + \lambda (c_{l}^{\dagger} + c_{l})(c_{l+1}^{\dagger} - c_{l+1}) \right] + \frac{i \Gamma}{2} \sum_{l=1}^{N} \left( 1 - c_{l}^{\dagger}c_{l} \right).
\]

Secondly, a Fourier transformation \(c_{l} = \frac{1}{\sqrt{N}} \sum_{k=-\pi/2}^{\pi/2} e^{2\pi i k l / N} b_{k}\) is conducted and we have

\[
H = 2 \sum_{k} \left[ ik_{k} (c_{k}^{\dagger}c_{k} + c_{k}^{\dagger}c_{k} - 1) \right] + z_{k} (c_{k}^{\dagger}c_{k} + c_{k}^{\dagger}c_{k} - 1).
\]

Here, \(y_{k} = -\sin(\frac{2\pi}{N} k) - \lambda \sin(\frac{2\pi}{N} k)\) and \(z_{k} = -\cos(\frac{2\pi}{N} k) + \lambda \cos(\frac{2\pi}{N} k) + \frac{\Gamma}{2}\). Thirdly, a Bogoliubov transformation helps diagonalize the above equation, which reads

\[
b_{k} = \cos(\frac{\theta_{k}}{2}) \gamma_{k} + i \sin(\frac{\theta_{k}}{2}) \gamma_{k}^{\dagger},
\]

\[
b_{-k} = \cos(\frac{\theta_{k}}{2}) \gamma_{-k} - i \sin(\frac{\theta_{k}}{2}) \gamma_{-k}^{\dagger}.
\]

Eventually, a diagonalized solution is acquired as

\[
H = \sum_{k>0} A_{k} (\gamma_{k}^{\dagger} \gamma_{-k} - \frac{1}{2}),
\]

where

\[
A_{k} = \sqrt{y_{k}^{2} + z_{k}^{2}},
\]

\[
\tan(\theta_{k}) = -\frac{y_{k}}{z_{k}},
\]

and the ground state of the model is

\[
\langle G \rangle = \prod_{k>0} [\cos(\frac{\theta_{k}}{2}) + \sin(\frac{\theta_{k}}{2}) c_{k}^{\dagger} c_{-k}^{\dagger}] |\text{Vac}\rangle,
\]

where \(|\text{Vac}\rangle\) denotes the vacuum state of the free fermion.

\[\text{FIG. 1. (Color online). Graphic demonstration of cluster Ising model with dissipation. Where } \lambda \text{ is the ratio of Ising exchange strength to cluster exchange strength and } \Gamma \text{ is the strength of loss or gain.}\]
III. OBSERVABLES AND METHODS

A. Ground state energy density and its second-order derivative

The non-analyticity of ground state energy density and its derivative imply the QPT in zero temperature. According to the above solution, the ground state energy can be calculated analytically and numerically via the equation

\[ U_g = -\frac{2}{N} \sum_{k>0} \sqrt{y_k^2 + z_k^2} = -\frac{1}{\pi} \int_0^\pi \sqrt{y_k^2 + z_k^2} dk, \] (12)

and we can easily obtain the second derivative of \( U_g \) with respect to \( \lambda \), i.e., \( \frac{\partial^2 U_g}{\partial \lambda^2} \).

B. Fidelity and fidelity susceptibility

As the inner product of two wave functions with a tiny difference in parameters, fidelity is also efficient in indicating the critical points of QPT, whose expression reads

\[ F(\lambda, \lambda + \epsilon) = \langle G(\lambda) \mid G(\lambda + \epsilon) \rangle = \prod_{k>0} F_k, \] (13)

with

\[ F_k = \cos \left( \frac{\theta_k(\lambda) - \theta_k(\lambda + \epsilon)}{2} \right). \] (14)

It is noticeable that \( \epsilon \) should take a small value. However, some arbitrariness in choosing the value of \( \epsilon \) can be accepted within a reasonable range, and the selection rule is just as what we have discussed before [17]. Here, we set \( \epsilon = 10^{-5} \), which can ensure the stability of results under different parameters.

C. Order parameters

Two kinds of order parameters will be investigated in this section, i.e., non-local string order parameter characterizing SPT phase and local staggered magnetization characterizing AFM phase, respectively. These order parameters will be non-zero as long as the system is in the corresponding phase.

Let us begin with the string order parameter, which is defined as

\[ \mathcal{O}^x = \lim_{N \to \infty} (-1)^N \left\langle \sigma_1^x \sigma_2^x \prod_{k=3}^{N-2} \sigma_k^x \sigma_{N-1}^x \sigma_N^x \right\rangle_0. \] (15)

Using the technique in Ref. [54], we can express it by the product of \( A_j = c_j^+ + c_j \) and \( B_j = c_j - c_j^+ \) like

\[ \mathcal{O}^x = \lim_{r \to \infty} \left\langle B_2 A_3 B_3 \ldots A_r B_r A_{r+1} B_{r+1} A_{r+2} \right\rangle. \] (16)
Then, with the help of Wick’s theorem [54], we can go on expanding it by the contractions $\langle A_j A_l \rangle$, $\langle B_j B_l \rangle$ and $\langle B_j A_l \rangle$, whose expression can be acquired using the ground state function. We have

$$\langle A_j A_l \rangle = \delta_{jl}, \quad (17)$$

$$\langle B_j B_l \rangle = -\delta_{jl}, \quad (18)$$

$$\langle B_j A_l \rangle = G_{j,l} = G_r$$

$$= \frac{1}{\pi} \int_0^\pi dk \{ \cos(kr) \cos \theta_k + \sin(kr) \sin \theta_k \}, \quad (19)$$

where $r = j - l$. Since $\langle A_j A_l \rangle$ and $\langle B_j B_l \rangle$ are always equal to zero and by considering the characteristics of their expression, $O^x$ can be transformed to a Toeplitz determinant,

$$O^x = \lim_{r \to \infty} \frac{G_{-r-1} G_{-r} \cdots G_{-r+1} \cdots G_{-r+1}}{G_{-r-1} G_{-r} \cdots G_{-r+1} \cdots G_{-r+1}}. \quad (20)$$

From the perspective of numerical calculation, we can take finite $r$ and calculate finite number of integrals to obtain $O^x$. But it is worth mentioning that $r$ should be as large as possible in order to approach the thermodynamic limit.

Next, we can acquire staggered magnetization at temperature $T$ via the calculation of spin correlation function

$$R^y_{jl}(T) = \langle \sigma^o_j \sigma^o_l \rangle_T, \quad (21)$$

where $\alpha$ can be $x$, $y$ or $z$. Let us take $R^y_{jl}(T)$ as an example. Similar to the procedure of calculating string correlation function, $R^x_{jl}(T)$ can also be expanded by $A_j = c^+_j + c_j, B_j = c_j - c^+_j$, that is,

$$R^x_{jl}(T) = \langle c_j - c^+_j \prod_{j=1}^{j<m<l} (1 - 2c_m c^*_m) \rangle_T$$

$$= \langle B_j A_{j+1} B_{j+1} \cdots A_{l-1} B_{l-1} A_l \rangle_T. \quad (22)$$

By using Wick theorem again we can also convert $R^x_{jl}(T)$ into a Toeplitz determinant

$$R^x_{jl}(T) = \left| \begin{array}{cccc} D(-1, T) & D(-2, T) & \cdots & D(-r, T) \\ D(0, T) & D(-1, T) & \cdots & D(-r + 1, T) \\ \vdots & \vdots & \ddots & \vdots \\ D(r - 2, T) & D(r - 3, T) & \cdots & D(-1, T) \end{array} \right|, \quad (23)$$

where the elements of the determinant reads

$$\langle B_j A_l \rangle_T = D_{jl}(T) = D(j - l, T) = D(r, T). \quad (24)$$

Similarly, $R^y_{jl}(T)$ can be expressed by

$$R^y_{jl}(T) = \left| \begin{array}{cccc} D(1, T) & D(0, T) & \cdots & D(-r + 2, T) \\ D(2, T) & D(1, T) & \cdots & D(-r + 3, T) \\ \vdots & \vdots & \ddots & \vdots \\ D(r, T) & D(r - 1, T) & \cdots & D(1, T) \end{array} \right| \quad (25)$$

With $R^y_{jl}$, we can calculate staggered magnetization $m_y$ with the definition

$$\lim_{r \to \infty} (-1)^r R^y_{jl}(0) = m^2. \quad (26)$$

Overall, with the above expression, we can investigate the influence of non-Hermiticity on the CIM’s phase distribution at certain temperature.

**IV. RESULTS AND DISCUSSIONS**

In this section, we are going to illustrate the influence of non-Hermiticity on different observables. Starting with the second derivative of ground state energy density, as it is shown in Fig. 2 (a), the singularity emerges at $\lambda = 1$ when non-Hermitian strength $\Gamma = 0$, which corresponds to the critical point of standard hermitian Cluster-Ising model. However, with the involvement of non-Hermiticity (see Fig. 2 (b) and (c)), the singular point turns into three points and the distance between them increases as $\Gamma$ rises from 0.6 to 1.6. The emergence of new non-analytical points indicates possible new critical behaviors, though the category of which still remains unknown.

We also demonstrate the behaviors of fidelity whose singularity characterizes phase transition as well. As it shown in Fig. 3 (a), in Hermitian case, the singular point also works well in characterizing the SPT-AFM phase transition at $\lambda = 1$. Then, we increase $\Gamma$ to 0.6 and 1.6 again (see Fig. 3 (b) and (c)). The behavior of singularity is in good agreement with that of $U_y$’s second derivative. We can observe two more emerging singular points and witness them moving away from each other as non-Hermiticity increases, which denotes the emergence of unknown phase transitions.

The calculation of string order parameter $O_z$ and staggered magnetization $m_y$ is a standard procedure to investigate SPT-AFM phase transition in Hermitian Cluster-Ising model. One can directly recognize the domination of SPT (AFM) phase with the help of global (local) order parameter string correlation function (staggered magnetization) provided it is non-zero. We also investigate the distribution of these two order parameters under different non-Hermitian strengths. In Hermitian case (see Fig. 4 (a)), the behaviors of $O_z$ and $m_y$ are the same as the previous researches, where $O_z$ ($m_y$) is non-zero when $\lambda < 1$.
From the above discussions, it is obvious that the introduction of non-Hermitian term leads to new phases and phase transitions that beyond the framework of Hermitian case. After that, we want to classify the emergent phases and phase transitions from the perspective of symmetry. In phase transition theory, the spin correlation function $R_q(r)$ decreases to a fixed non-zero value as $r$ increases in symmetry breaking phase, comparing to the exponential decay to zero in symmetry protected disorder phase. At the critical point between these two phases, $R_q(r)$ exhibits a unique algebraic decay behavior. However, if there is not such a transition in the variation mode of $R_q(r)$ from exponential decay to dropping to a non-zero value while crossing the critical point, it should be a topological phase transition without symmetry breaking.

Therefore, we investigate the variation of $R_q(r)$ at different $\lambda$ under different non-Hermitian strength $\Gamma$. When $\Gamma = 0$, the critical point is at $\lambda = 1$, which denotes the SPT-AFM phase transition with symmetry breaking. From Fig. 5 (a), we can see that the curves before $\lambda = 1$ exhibit the exponential decay and those after $\lambda = 1$ decrease to fixed non-zero values. Notably, as it shown in the subset, the curve at critical point shows an algebraic decay, corresponding to the phase transition with symmetry breaking.

Interestingly, with the increasing of $\Gamma$ from 0 to 0.6 and 1.6 (see Fig. 5 (b) and (c)), despite
one can still observe such a change in $R_y(r)$’s variation mode at one shifting critical point, one can not observe such a change of variation mode around the two emergent critical points, which indicates that the one of the three critical points still denotes a phase transition with symmetry breaking while the other two do not. Analysing with the distributions of $O_x$ and $m_y$ shown in Fig. 4, we find that, as $\Gamma$ increases, the original critical point of Ising universal class diverges into three critical points, separating the graph into four zones. Critical point between zone III and zone IV is still a phase transition with symmetry breaking, while the phase transitions between zone I, II and III do not accompany with symmetry breaking. Considering $O_x$ is zero in zone III and non-zero in zone I and II, zone III should be a paramagnetic (PM) phase while zone I and II are SPT phase. Therefore, the phase transition between zone I and II, zone II and III as well as zone III and IV are emergent topological phase transition, SPT-PM phase transition and PM-AFM phase transition, respectively. To conclude, the introducing of non-Hermitian term gives rise to new novel phases and phase transitions as well as the shift of critical points.

To be more comprehensive and intuitive, we investigate the $\Gamma - \lambda$ phase diagram according to the non-anlytical points of $U_g$’s second derivative (see Fig. 6). When $\Gamma = 0$, there is only one SPT-AFM phase transition of Ising universal class at $\lambda = 1$. As $\Gamma$ grows, the critical point extends to three consecutive critical lines. Blue, red and yellow line correspond to topological phase transition, SPT-PM phase transition and PM-AFM phase, respectively. Notably, only phase transition on yellow critical line still accompanies with symmetry breaking.

V. SUMMARY

The phase transition in standard Cluster-Ising model is a typical Ising universality class. The influence of non-Hermiticity on Cluster-Ising model is investigated in this work. We first detect the singular behaviors of second derivative of energy density and fidelity, finding that new critical points may emerge and move away from each other. Next, we numerically investigate string order parameter and staggered magnetization, and the results show that the introduction of non-Hermiticity will give rise to a very rich phase diagram featuring the transformation from two phases to four. Therefore, in order to characterize them, we then investigate the variation mode of spin correlation function and observe the symmetry protection of the emergent phase transitions. Combining the result of string order parameter and staggered magnetization, we know that the phase transitions are SPT-SPT topological phase transition, SPT-PM phase transition and PM-AFM phase transition, respectively. Eventually, we give the $\Gamma - \lambda$ phase diagram to visualize the emergence of two extra critical lines without symmetry breaking and the maintenance of one conventional critical line with symmetry breaking. Our work can be realized in ultra-cold atom experiment and will shed light on experimental construction of novel topological phases and phase transition in open quantum many body systems.

![FIG. 6. (Color online). The $\Gamma - \lambda$ phase diagram according to $U_g$’s second derivative. In Hermitian case ($\Gamma = 0$), there is a conventional critical point of Ising universal class at $\lambda = 1$. With the increase of $\Gamma$, the critical point separates into 3 critical points, denoting the phase transitions between two SPT phases, PM phase and AFM phase.](image)

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