Non-Fermi liquid to Fermi liquid crossover inside a valley-polarized nematic state

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The interplay between strong correlations and non-trivial topology in twisted moiré systems can give rise to a rich landscape of ordered states that intertwine the spin, valley, and charge degrees of freedom. In this paper, we investigate the properties of a metal that undergoes a quantum phase transition to a valley-polarized nematic state. Besides breaking the threefold rotational symmetry of the triangular moiré superlattice, this type of order also breaks twofold rotational and time-reversal symmetries. At zero temperature, the ordered state displays a pseudo-Goldstone mode due to the existence of a dangerously irrelevant coupling $\lambda$ in the 6-state clock model that describes the valley-polarized nematic quantum critical point. Using a two-patch model, we compute the fermionic self-energy to show that down to very low energies, the Yukawa-like coupling between the pseudo-Goldstone mode and the electronic degrees of freedom promotes the emergence of non-Fermi liquid behavior. Below a crossover energy scale $\Omega^* \sim \lambda^{3/2}$, Fermi liquid behavior is recovered. We discuss possible experimental manifestations of this behavior, as well as the applicability of these results to other non-trivial nematic states, such as the spin-polarized nematic phase.

I. INTRODUCTION

The observation of electronic nematicity in the phase diagrams of twisted bilayer graphene [1–4] and twisted double-bilayer graphene [5, 6] provides a new setting to elucidate these electronic liquid crystalline states, which spontaneously break the rotational symmetry of the system. Shortly after nematicity was proposed to explain certain properties of high-temperature superconductors [7], it was recognized that the Goldstone mode of an ideal electronic nematic phase would have a profound impact on the electronic properties of a metal [8–10]. This is because, in contrast to other Goldstone modes such as phonons and magnons, which couple to the electronic density via a gradient term, the nematic Goldstone mode displays a direct Yukawa-like coupling to the electronic density [11]. As a result, it is expected to promote non-Fermi liquid (NFL) behavior, as manifested in the sub-linear frequency dependence of the imaginary part of the electronic self-energy [8, 12].

However, because the crystal lattice breaks the continuous rotational symmetry of the system, the electronic nematic order parameter realized in layered quantum materials has a discrete $Z_6$ symmetry, rather than the continuous XY (or O(2)) symmetry of its two-dimensional (2D) liquid crystal counterpart [13]. In the square lattice, the $Z_2$ (Ising-like) symmetry is associated with selecting one of the two orthogonal in-plane directions connecting either nearest-neighbor or next-nearest-neighbor sites [14]. In the triangular lattice, the $Z_3$ (3-state Potts/clock) symmetry refers to choosing one of the three bonds connecting nearest-neighbor sites [15, 16]. In both cases, the excitation spectrum in the ordered state is gapped, i.e. there is no nematic Goldstone mode. Consequently, NFL behavior is not expected to arise inside the nematic phase – although it can still emerge in the disordered state due to interactions mediated by possible quantum critical fluctuations [17–27].

In twisted moiré systems [28, 29], which usually display an emergent triangular moiré superlattice, another type of nematic order can arise due to the presence of the valley degrees of freedom: a valley-polarized nematic state [30]. Compared to the standard nematic state, valley-polarized nematic order breaks not only the threefold rotational symmetry of the lattice, but also “inversion” (more precisely, two-fold rotational) and time-reversal symmetries. Experimentally, while threefold rotational symmetry-breaking [1–4] and time-reversal symmetry-breaking [31–34] have been observed in different regions of the phase diagram of twisted bilayer graphene, it is not clear yet whether a valley-polarized nematic state is realized in this or any other twisted moiré system. Theoretically, the valley-polarized nematic order parameter has a $Z_6$ symmetry, which corresponds to the 6-state clock model [35]. Interestingly, it is known that the 6-state clock model transition belongs to the XY universality class in three spatial dimensions, with the sixfold anisotropy perturbation being irrelevant at the critical point [36–39].

Thus, at $T = 0$ and in a 2D triangular lattice, a valley-polarized nematic quantum critical point (QCP) should share the same universality class as a QCP associated with a hypothetical XY electronic nematic order parameter that is completely decoupled from the lattice degrees of freedom [40]. In other words, a 2D 6-state clock model exhibits a continuous phase transition at $T = 0$, that is described by a $(2+1)$D Ginzburg-Landau theory of an O(2) order parameter with a $Z_6$ anisotropic term – the latter is irrelevant in the renormalization group (RG) sense. In fact, the sixfold anisotropy term is dangerously irrelevant [36], as it becomes a relevant perturbation inside the ordered state [37, 41–45]. As
a result, the valley-polarized nematic phase displays a pseudo-Goldstone mode, i.e., a Goldstone mode with a small gap that satisfies certain scaling properties as the QCP is approached [46].

In this paper, we study the electronic properties of a metal inside the valley-polarized nematic phase at $T = 0$. First, we show that the phase fluctuations inside the valley-polarized phase couple directly to the electronic density. Then, using a two-patch model [19–22, 47, 48], we show that the electronic self-energy $\Sigma$ displays, along the hot regions of the Fermi surface and above a characteristic energy $\Omega^*$, the same NFL behavior as in the case of an “ideal” XY nematic order parameter $C_2$, i.e., $\Sigma(\nu_n) \sim i |\nu_n|^{2/3}$, where $\nu_n$ is the fermionic Matsubara frequency. Below $\Omega^*$, however, we find that $\Sigma(\nu_n) \sim i \nu_n$, and Fermi liquid (FL) behavior is restored. Moreover, the bosonic self-energy, describing the phase fluctuations, acquires an overdamped dynamics due to the coupling to the fermions.

Exploiting the scaling properties of the 6-state clock model, we argue that this NFL-to-FL crossover energy scale $\Omega^*$, which is directly related to the dangerously irrelevant variable $\lambda$ of the 6-state clock model via $\Omega^* \sim \lambda^{3/2}$, is expected to be much smaller than the other energy scales of the problem. As a result, we expect NFL behavior to be realized over an extended range of energies. We discuss possible experimental manifestations of this effect at finite temperatures, and the extension of this mechanism to the case of spin-polarized nematic order, which has been proposed to occur in moiré systems with higher-order van Hove singularities [49, 50].

The paper is organized as follows: Sec. II introduces the bosonic and fermionic actions that describe the system inside the valley-polarized nematic state. Sec. III introduces the patch method and presents the results for the electronic self-energy, focusing on the onset of NFL behavior. In Sec. IV, we discuss the implications of our results for the observation of NFL behavior in different types of systems.

II. EFFECTIVE ACTION IN THE VALLEY-POLARIZED NEMATIC STATE

As discussed in Ref. [30], which dubbed the valley-polarized nematic phase as “compass” order, the corresponding order parameter can be parametrized in terms of a two-component bosonic field $\Phi = \Phi_1 + i \Phi_2$. We consider here a triangular moiré superlattice, described by the point group $D_6$. While $(\Phi_1, \Phi_2)$ transforms in the same way as the standard nematic order parameter under threefold rotations $C_{3z}$, it is odd under time-reversal and under twofold rotations $C_{2z}$. Note that, in the $D_6$ group, inversion is not a symmetry operation. Since the valley degrees of freedom are exchanged by $C_{2z}$, this symmetry operation plays a similar role as inversion. As a result, $(\Phi_1, \Phi_2)$ transforms as the $E_1^-$ irreducible representation of $D_6$, with the “−” superscript denoting that it is odd under time-reversal. It is convenient to define the complex bosonic field $\Phi = \Phi_1 + i \Phi_2$. Writing the complex field as $\Phi = |\Phi| e^{-i\alpha}$, the leading-order term in the low-energy effective action that depends on the phase $\alpha$ is a sixth-order term. This is in contrast with the case for the standard nematic order parameter, which transforms as $E_2^+$, and hence a cubic term is allowed [16]. The explicit form for the action of the field $\Phi$ is [30]:

$$S = \frac{1}{2} \int d^2 x \int d\tau \left[ e^{-\frac{i\alpha}{\rho}} |\partial_\tau \Phi|^2 + |\partial_\tau \Phi|^2 + r |\Phi|^2 - u |\Phi|^4 - \lambda (\Phi^6 + \Phi^*6) \right].$$

(2.1)

Here, $x$ denotes the position vector, $\tau$ the imaginary time, and $c$ the bosonic velocity. The quadratic coefficient $r$ tunes the system towards a QCP at $r = r_c$, and the quartic coefficient $u > 0$. In the absence of the sixfold anisotropy term with coefficient $\lambda$, the action has an $O(2)$ symmetry, and describes a continuous phase transition belonging to the XY universality class.

The $\lambda$-term is the lowest-order term in $\Phi$ that lowers the symmetry from $O(2)$ to $Z_6$. As a result, the action corresponds to a 6-state clock model. Indeed, minimization of the action with respect to the phase $\alpha$ leads to six different minima, corresponding to $\alpha = \frac{\pi}{6} n$ for $\lambda > 0$; and $\alpha = \frac{\pi}{6} (n + \frac{1}{2})$ for $\lambda < 0$ (with $n = 0, \ldots, 5$). At finite temperatures, the 2D 6-state clock model undergoes two Kosterlitz-Thouless transitions: the first one signals quasi-long-range order of the phase $\alpha$, whereas the second one marks the onset of discrete symmetry-breaking and long-range order [35].

Since the focus of this paper is on the valley-polarized nematic QCP, hereafter we set $T = 0$, and consider some non-thermal tuning parameter $r$. At $T = 0$, the model in Eq. (2.1) maps onto the three-dimensional (3D) 6-state clock model [38, 46]. One of the peculiarities of this well-studied model is that the $\lambda$-term is a dangerously irrelevant perturbation [37, 41–45]. Indeed, the scaling dimension $y$ associated with the $\lambda$ coefficient is negative; while an $\epsilon$-expansion around the upper critical dimension $d_c = 4$ gives $y = -2 - \epsilon$ [37], recent Monte Carlo simulations report $y \approx -2.55$ for the classical 3D 6-state clock model [42, 45].

To understand what happens inside the ordered state, it is convenient to write $\Phi = |\Phi_0| e^{-i\alpha}$ with fixed $|\Phi_0|$, and consider the action for the phase variable $\alpha$ only, as shown below:

$$S_\alpha = \frac{1}{2} \int d^2 x \int d\tau \left[ \rho_r |\partial_\tau \alpha|^2 + \rho_x |\partial_\tau \alpha|^2 - 2 \lambda |\Phi_0|^6 \cos(6\alpha) \right].$$

(2.2)

Here, $\rho_x$ and $\rho_r$ are generalized stiffness coefficients. Expanding around one of the minima, $\alpha_0$, of the last term
where a constant term is dropped, and \( \tilde{\alpha} \equiv \alpha - \alpha_0 \). It is clear that the \( \lambda \)-term, regardless of its sign, introduces a mass for the phase variable. Thus, while the \( \lambda \)-term is irrelevant at the critical point, which is described by the XY fixed point, it is relevant inside the ordered phase, which is described by a \( Z_6 \) fixed point, rather than the Nambu-Goldstone fixed point (that characterizes the ordered phase of the 3D XY model) [37, 45, 46].

Importantly, due to the existence of this dangerously irrelevant perturbation, there are two correlation lengths in the ordered state [41–44, 46]: \( \xi \), associated with the usual amplitude fluctuations of \( \Phi \); and \( \xi' \), associated with the crossover from continuous to discrete symmetry-breaking of \( \alpha \). Although both diverge at the critical point, they do so with different exponents \( \nu \) and \( \nu' \). Because \( \nu' > \nu \), there is a wide range of length scales (and energies, in the \( T = 0 \) case) for which the ordered phase behaves as if it were an XY ordered phase. In Monte Carlo simulations, this is signaled by the emergence of a nearly-isotropic order parameter distribution [41]. More broadly, this property is expected to be manifested as a small gap in the spectrum of phase fluctuations, characteristic of a pseudo-Goldstone mode [51].

For simplicity of notation, in the remainder of the paper, we rescale \( (\tau, \mathbf{x}) \) to absorb the stiffness coefficients. Moreover, we set \( \lambda > 0 \) and choose \( \alpha_0 = 0 \), such that \( \tilde{\alpha} = \alpha \). Defining \( m^2 \equiv 36 |\tilde{\alpha}|^2 \), and taking the Fourier transform, the phase action becomes:

\[
S_\alpha = \frac{1}{2} \int d^2 x d\tau \left[ \rho_\tau |\partial_\tau \tilde{\alpha}|^2 + \rho_x |\partial_x \tilde{\alpha}|^2 + 36 |\tilde{\alpha}|^4 \right], \tag{2.3}
\]

\[
\text{where } k = (\nu_n, \mathbf{k}), \alpha \text{ is the valley index, } \varepsilon_\alpha(k) \text{ is the electronic dispersion, and } \nu_n \text{ is the fermionic Matsubara frequency. It is now straightforward to construct the fermionic bilinear that has the same transformation properties as the valley-polarized nematic order parameter, and thus couples linearly with } \Phi, \text{ as captured in the action } [30]
\]

\[
S_{bf} = \gamma_0 \int \frac{d^2 k}{(2\pi)^2} \sum_{q=1}^6 (1)^{q+1} \psi_\alpha^q(k + q) \psi_\alpha(k) \times \left[ \frac{\Phi(k) + \Phi^*(k)}{2} \cos(2\theta_k) - \frac{\Phi(k) - \Phi^*(k)}{2i} \sin(2\theta_k) \right]. \tag{2.6}
\]

Here, \( \gamma_0 \) is a coupling constant, and \( \tan \theta_k = k_y/k_x \). Writing \( \Phi = |\Phi_0| e^{-i\alpha} \), we obtain the coupling between the phase variable and the electronic operators inside the valley-polarized nematic state. As before, we set \( \alpha_0 = 0 \), and expand around the minimum, to obtain:

\[
S_{af} = \gamma \int d^2 k \sum_{q=1}^6 (1)^{q+1} \psi_\alpha^q(k + q) \psi_\alpha(k) \times \cos(2\theta_k) (2\pi)^3 \delta^3(k - q) + \alpha(q) \sin(2\theta_k), \tag{2.7}
\]

where \( \alpha \equiv \gamma_0 |\Phi_0| \). The first term in the last line shows that long-range order induces opposite nematic distortions in the Fermi surfaces with opposite valley quantum numbers. The second term shows that the phase mode couples to the charge density directly via a Yukawa-like coupling. As discussed in Ref. [11], this is an allowed coupling when the generator of the broken symmetry does not commute with the momentum operator.

III. ELECTRONIC SELF-ENERGY IN THE PATCH MODEL

Our goal is to derive the properties of the electronic degrees of freedom in the valley-polarized nematic ordered phase. To compute the electronic self-energy in a controlled manner, we employ the patch method discussed in Ref. [19–22, 47, 48]. This relies on the fact that fermions from different patches of a Fermi surface interact with a massless order parameter with largely disjoint sets of momenta, and that the inter-patch coupling is small in the low-energy limit, unless the tangent vectors at the patches are locally parallel or anti-parallel. Thus, the advantage of this emergent locality in momentum space is that we can now decompose the full theory into a sum of two-patch theories, where each two-patch theory describes electronic excitations near two antipodal points, interacting with the order parameter boson with momentum along the local tangent. This formalism has been successfully used in computing the universal properties and scalings for various non-Fermi liquid systems, such as the Ising-nematic QCP [19–23, 52], the FFLO QCP [47], and a critical Fermi surface interacting with
transverse gauge field(s) [48]. The only scenario that breaks this locality in momentum space is the presence of short-ranged four-fermion interactions in the pairing channel [24, 25].

For our case of the valley polarized nematic order parameter, we consider two patches of Fermi surface, with opposite Fermi velocities, coupled with the bosonic field [20–22, 47, 48]. Here, we choose a patch centred at $\theta_k = \theta_0$, and construct our coordinate system with its origin at $\theta_0$. As explained above, we must also include the fermions at the antipodal patch with $\theta_k = \pi + \theta_0$. We denote the fermions living in the two antipodal patches as $\psi_+$ and $\psi_-$, as illustrated in Fig. 1. We note that the coupling constant remains the same for the fermions in the two antipodal points.

Expanding the spectrum around the Fermi surface patches up to an effective parabolic dispersion, and using Eqs. (2.4), (2.5), and (2.7), we thus obtain the effective field theory in the patch construction as:

$$ S_f = \int k \sum_{j=\pm} \psi^\dagger_{a,j}(k) [i \nu_n + s k_1 + \frac{k_2^2}{2k_F} + (-1)^{a+1} \gamma \cos(2\theta_0)] \psi_{a,j}(k), $$

$$ S_\alpha = \frac{1}{2} \int q \alpha(-q) \left( \omega_n^2 + q_1^2 + q_2^2 + m^2 \right) \alpha(q), $$

$$ S_{\alpha f} = \gamma \int k,q \sum_{s=\pm} (-1)^{a+1} \alpha(q) \psi^\dagger_{a,s}(k + q) \psi_{a,s}(k). $$

(3.1)

Here, for simplicity, we have assumed that the Fermi surface is the same for both valley quantum numbers. Note that the fermionic momenta are expanded about the Fermi momentum $k_F$ at the origin of the coordinate system of that patch. In our notation, shown in Fig. 1, $k_1$ is directed along the local Fermi momentum, whereas $k_2$ is perpendicular to it (or tangential to the Fermi surface). Note that the local curvature of the Fermi surface is given by $\frac{1}{k_F^2}$. Furthermore, $\psi_{a,+}$ ($\psi_{a,-}$) is the right (left)-moving fermion with valley index $a$, whose Fermi velocity along the $k_1$ direction is positive (negative).

Following the patch approach used in Refs. [20–22, 47, 48], we rewrite the fermionic fields in terms of the two-component spinor $\Psi$, where

$$ \Psi^T(k) = \begin{pmatrix} \psi_{1,+}(k) & \psi_{2,+}(k) & \psi_{1,-}^\dagger(-k) & \psi_{2,-}^\dagger(-k) \end{pmatrix}, $$

$$ \bar{\Psi}(k) = \Psi^\dagger(k) \sigma_2 \otimes \tau_0. $$

(3.2)

Here, $\sigma_i$ (with $i = 1, 2, 3$) denotes the $i^{th}$ Pauli matrix acting on the patch space (consisting of the two antipodal patches), whereas $\tau_i$ is the $i^{th}$ Pauli matrix acting on valley space (not to be confused with imaginary time $\tau$, which has no subscript). We use the symbols $\sigma_0$ and $\tau_0$ to denote the corresponding $2 \times 2$ identity matrices. In this notation, the full patch action becomes:

$$ S_f = \int k \bar{\Psi}^\dagger(k) \left[ i (\sigma_2 \nu_n + \sigma_1 \delta_k) \otimes \tau_0 \right] \Psi(k), \quad S_\alpha = \frac{1}{2} \int q \left( \omega_n^2 + q_1^2 + q_2^2 + m^2 \right) \alpha(q), $$

$$ S_{\alpha f} = \gamma \int k,q \bar{\Psi}(k + q) \left[ (2\pi)^3 \delta^3(k - q) \cos(2\theta_0) \sigma_2 + i \sin(\theta_0) \alpha(q) \sigma_1 \right] \otimes \tau_3 \Psi(k), \quad \delta_k = k_1 + \frac{k_2^2}{2k_F}. $$

(3.3)

For convenience, we have included the valley-dependent Fermi-surface distortion $\gamma \cos(2\theta_0)$ in the interaction action. The form of $S_f$ shows as if the fermionic energy disperses only in one effective direction near the Fermi surface, such that, as usual in the patch model, the $(2 + 1)$-dimensional fermions can be viewed as if they were a $(1 + 1)$-dimensional “Dirac” fermion, with the momentum along the Fermi surface interpreted as a continuous flavor.

From Eq. (3.3), the bare fermionic propagator can be
readily obtained as:

\[ G_0(k) = -i \frac{\sigma_2 \nu_n + \sigma_1 \delta_k}{\nu_n^2 + \delta_k^2} \otimes \tau_0, \tag{3.4} \]

We note that the strength of the coupling constant between the bosons and the fermions, given by \( \gamma \sin(2\theta_0) \), depends on the value of \( \theta_0 \). For the patch centered at \( \theta_0 = \theta_0 \), the leading order term from the loop integrals can be well-estimated by assuming \( \theta = \theta_0 \) for the entire patch, as long as \( \sin(2\theta_0) \neq 0 \). However, for \( \sin(2\theta_0) = 0 \), we need to go beyond the leading order terms (which are zero), while performing the loop integrals. The patches centered around \( \theta_0 = \theta_0 \), with \( \sin(2\theta_0) \sim 0 \), are the so-called “cold spots”; we will refer to the other patches as belonging to the “hot regions” of the Fermi surface.

We first compute the one-loop bosonic self-energy \( \Pi_1 \), which takes the form:

\[ \Pi_1(q) = - (i \gamma)^2 \int \frac{d^3k}{(2\pi)^3} \left[ \sin^2(2\theta_0) + \frac{4k^2 \cos(4\theta_0)}{k_F^2} + \frac{2k_2 \sin(4\theta_0)}{k_F} \right] \text{Tr} \left[ \sigma_1 G_0(k + q) \sigma_1 G_0(k) \right] \]

This result is obtained by considering a patch centered around \( \theta_0 = \theta_0 \), and then expanding \( \sin^2(2\theta_0 + 2\tilde{k}_2/k_F) \) in inverse powers of \( k_F \). In the limits \( \omega_{q_2} \ll 1, k_F \gg |q|, \) and \( |q| \rightarrow 0 \), we have, to leading order:

\[ \Pi_1(q) \bigg|_{hs} = - \frac{|\omega_n| \gamma^2 \sin^2(2\theta_0) k_F}{|q_2|}, \tag{3.6} \]

as long as \( \sin(2\theta_0) \neq 0 \) (i.e., at the hot regions). For the cold spots, the leading-order term is given by

\[ \Pi_1(q) \bigg|_{cs} = - \frac{8 \gamma^2 \cos(4\theta_0) k_F |\omega_n^2|}{q_2^2}, \tag{3.7} \]

Here, the subscript hr (cs) denotes hot regions (cold spots). A similar result was previously obtained in Refs. [8, 10] using a different approach, and for the case of an XY nematic order parameter (see also [53]). We, therefore, conclude that the pseudo-Goldstone mode in the valley-polarized nematic phase is overdamped at the hot regions.

We can now define the dressed bosonic propagator, that includes the one-loop bosonic self-energy, as:

\[ D_1(q) = \frac{1}{q^2 + m^2 - \Pi_1(q)}. \tag{3.8} \]

The one-loop fermionic self-energy \( \Sigma_1(k) \) can then be expressed in terms of \( \tilde{\Sigma} \), defined as:

\[ \tilde{\Sigma}(k) \equiv \Sigma_1(k) - \gamma \cos(2\theta_0) \sigma_2 \otimes \tau_3 \]

\[ = - \gamma^2 \sin^2(2\theta_0) \int_q (\sigma_1 \otimes \tau_3) G_0(k + q) (\sigma_1 \otimes \tau_3) D_1(-q). \tag{3.9} \]

Here, \( q = (\omega_n, q) \). In order to be able to perform the integrals, we have neglected the \( q^2_2 \) and \( \omega_n^2 \) contributions in the bosonic propagator, which are anyway irrelevant in the RG sense [20, 21]. This is justified because the contributions to the integral are dominated by \( q_1 \sim \nu_n, \) \( \omega_n \sim \nu_n, \) and \( q_2 \sim |\nu_n|^{1/3} \), and we are interested in the small \( |\nu_n| \) limit (where \( \nu_n \) is the external fermionic Matsubara frequency). In the limit \( m \rightarrow 0 \), we can obtain analytical expressions for \( \tilde{\Sigma}(k) \) as follows:

\[ \tilde{\Sigma}(k) \bigg|_{hr,m \rightarrow 0} = - \gamma^2 \sin^2(2\theta_0) \int_q (\sigma_1 \otimes \tau_3) G_0(k + q) (\sigma_1 \otimes \tau_3) D_1(-q) \]

\[ = - \frac{i [\gamma \sin(2\theta_0)]^{4/3} \text{sgn}(\nu_n) |\nu_n|^{2/3}}{2 \sqrt{3} \pi^{2/3} k_F^{1/3}} \sigma_2 \otimes \tau_0, \tag{3.10} \]

\[ \tilde{\Sigma}(k) \bigg|_{cs,m \rightarrow 0} = - \frac{i \gamma^{3/2} \cos^4(4\theta_0) \text{sgn}(\nu_n) |\nu_n|^{1/2} k_F^2}{2^{1/4} \sqrt{\pi} k_F^{9/4}} \sigma_2 \otimes \tau_0. \tag{3.11} \]

The one-loop corrected self-energy is then given by:

\[ G^{-1}(k) = G_0^{-1}(k) - \Sigma_1(k). \]  

The frequency dependence of \( \tilde{\Sigma} \) at the hot regions, in the limit \( m \rightarrow 0 \), corresponds to an NFL behavior, since the fermionic lifetime has a sublinear dependence on frequency, implying the absence of well-defined quasiparticles. The same \( |\nu_n|^{2/3} \) dependence on the frequency was found in the case of an ideal XY nematic in Refs. [8, 12]. However, for the valley-polarized nematic state, \( m \) is not zero in the ordered state, as it is proportional to the dangerously irrelevant variable \( \lambda \) in the bosonic action. The limit of large \( m \) is straightforward to obtain, and gives an FL-correction to the electronic Green’s function, because

\[ \tilde{\Sigma}(k) \bigg|_{hr,m \gg} = \frac{3 \sqrt{\pi^2 k_F^3 m^2} (2\theta_0) |\nu_n|}{1/3} \]

\[ = \frac{(2 + 2^{2/3}) \gamma^2 \sin^2(2\theta_0)}{8 \pi m} i \nu_n \sigma_2 \otimes \tau_0. \tag{3.12} \]
From Eqs. (3.4), (3.8), and (3.9), we find that the crossover from NFL to FL behaviour occurs when \( m^2 > -\Pi_1(q) \), i.e., \( m^2 > |\omega_n| \gamma_k \pi \sin^2(2\theta_0) \), in the one-loop corrected bosonic propagator \( D_1(q) \) inside the integral. In that situation, the dominant contribution to the integral over \( q_2 \) comes from \( q_2 \sim m \). On the other hand, considering the fermionic propagator contribution to the integrand, the dominant contribution comes from \( \omega_n' \sim \nu_n \) for the \( \omega_n' \)-integral. Hence, the relevant crossover scale for the fermionic frequency \( \nu_n \) is approximately \( \Omega^* = \frac{m^3}{\pi k_F \sin^2(2\theta_0)} \). Because \( m^2 \sim \lambda \), it follows that \( \Omega^* \sim \lambda^{3/2} \).

It is therefore expected that, for finite \( m \), above the characteristic energy scale \( \Omega^* \), the self-energy displays NFL behavior, captured by \( \Sigma \sim i \, \text{sgn}(\nu_n) |\nu_n|^{2/3} \). For low enough energies, such that \( |\nu_n| \ll \Omega^* \), the regular FL behavior with \( \Sigma \sim i \nu_n \) should be recovered. The crucial point is that, because \( \Omega^* \) depends on the dangerously irrelevant coupling constant \( \lambda \), it is expected to be a small energy scale. This point will be discussed in more depth in the next section. To proceed, it is convenient to write the complete expression for \( \tilde{\Sigma} = \Sigma \times (\sigma_2 \otimes \tau_0) \) for the case of an arbitrary \( m \):

\[
\left. i \tilde{\Sigma}(k) \right|_{\text{hr}} = -\int d\omega_{n'} \frac{m^3}{4 \pi^2} \frac{\text{sgn}(\tilde{\nu}_n + \tilde{\omega}_{n'}) \zeta_j(\tilde{\omega}_{n'}) \ln(-\zeta_j(\tilde{\omega}_{n'}))}{m^2 + 3 \zeta_j^2(\tilde{\omega}_{n'})},
\]

where \( \tilde{\nu}_n \equiv \nu_n/\Omega^* \), \( \tilde{\omega}_{n'} \equiv \omega_{n'}/\Omega^* \), and \( \zeta_j \) is the \( j^{th} \) root of the cubic-in-\( q_2 \) polynomial \( \pi q_2 (q_2^2 + m^2) + m^3 k_F |\tilde{\omega}_{n'}| \).

To confirm that indeed \( \Omega^* \) is the energy scale associated with the crossover from NFL to FL behavior, we have solved the integral in Eq. (3.13) numerically to obtain the self-energy for arbitrary \( m \). As shown in Fig. 2, \( \Omega^* \) separates the two asymptotic behaviors for the self-energy \( \Sigma \): (1) NFL, given by Eq. (3.10), and present for \( \nu_n \gg \Omega^* \); (2) FL, given by Eq. (3.12), and present for \( \nu_n \ll \Omega^* \).

### IV. DISCUSSION AND CONCLUSIONS

Our calculations with the patch model show that the energy scale \( \Omega^* \), associated with the NFL-to-FL crossover, is directly related to the dangerously irrelevant coupling constant \( \lambda \) of the 6-state clock model, according to \( \Omega^* \sim \lambda^{3/2} \). This has important consequences for the energy range in which the NFL is expected to be observed in realistic settings. In the classical 3D \( Z_6 \) clock model, it is known that the dangerously irrelevant variable \( \lambda \) introduces a new length scale \( \xi' \) in the ordered phase [41–44]. It is only beyond this length scale that the discrete nature of the broken symmetry is manifested; below it, the system essentially behaves as if it were in the ordered state of the XY model. Like the standard correlation length \( \xi \), which is associated with fluctuations of the amplitude mode, \( \xi' \) also diverges upon approaching the QCP from the ordered state. However, its critical exponent \( \nu' \) is larger than the XY critical exponent \( \nu \), implying that \( \xi' \gg \xi \) as the QCP is approached. As a result, there is a wide range of length scales for which the ordered state is similar to that of the XY model.

Applying these results to our quantum model, we therefore expect a wide energy range for which the fermionic self-energy displays the same behavior as fermions coupled to a hypothetical XY nematic order parameter, i.e., the NFL behavior \( \Sigma \sim i \, \text{sgn}(\nu_n) |\nu_n|^{2/3} \).
Thus, the actual crossover energy scale $\Omega^*$ should be very small compared to other energy scales of the problem. This analysis suggests that the valley-polarized nematic state in a triangular lattice is a promising candidate to display the strange metallic behavior predicted originally for the “ideal” (i.e., hypothetically uncoupled from the lattice) XY nematic phase in the square lattice [8].

It is important to point out a caveat with this analysis. Although the aforementioned critical behavior of the $Z_6$ clock model has been verified by Monte Carlo simulations for both the 3D classical case and the 2D quantum case [46], the impact of the coupling to the fermions remains to be determined. The results of our patch model calculations for the bosonic self-energy show the emergence of Landau damping in the dynamics of the phase fluctuations, which is expected to change the universality class of the QCP – and the value of the crossover exponent $\nu$ – from 3D XY to Gaussian, due to the reduction of the upper critical dimension. The impact of Landau damping on the crossover exponent $\nu^*$ is a topic that deserves further investigation, particularly since even in the purely bosonic case, there are different proposals for the scaling expression for $\nu^*$ (see Ref. [46] and references therein).

We also emphasize the fact that our results have been derived for $T = 0$. Experimentally, NFL behavior is often probed by the temperature dependence of the resistivity. Leaving aside the important differences between quasiparticle inverse-lifetime and transport scattering rate [53–55], it is therefore important to determine whether the NFL behavior of the self-energy persists at a small nonzero temperature. At first sight, this may seem difficult, since in the classical 2D $Z_6$ clock model, the $\lambda$-term is a relevant perturbation. In fact, as discussed in Sec. II, the system in 2D displays two Kosterlitz-Thouless transitions, with crossover temperatures of $T_{KT,1}$ and $T_{KT,2}$, with $Z_6$ symmetry-breaking setting on below $T_{KT,2}$ [35]. However, a more in-depth analysis, as outlined in Ref. [44], indicates that as the QCP is approached, a new crossover temperature $T^* < T_{KT,2}$ emerges, below which the ordered state is governed by the QCP (rather than the thermal transition). Not surprisingly, the emergence of $T^*$ is rooted on the existence of the dangerously irrelevant perturbation along the $T = 0$ axis. Therefore, as long as $\Omega^* < T^*$, the NFL behavior is expected to be manifested in the temperature dependence of the transport and thermodynamic quantities.

An obvious candidate to display a valley-polarized nematic state is twisted bilayer graphene [32, 56–58] – and twisted moiré systems more broadly [28, 29]. In these devices, the existence of correlations, larger than the narrow bandwidth of the moiré bands [59, 60], enables the emergence of a wide range of possible ordered states involving the spin, valley, and sublattice degrees of freedom [30, 61–82]. Experimentally, a valley-polarized nematic state would be manifested not only by the breaking of the threefold rotational symmetry of the moiré superlattice, but also by the appearance of valley polarization. While $C_{3z}$ [1–4] and time-reversal symmetry-breakings have separately been experimentally detected in the phase diagram of twisted bilayer graphene, it remains to be seen whether a valley-polarized nematic phase emerges. The fact that both nematic order (described by the complex order parameter $\psi = \psi_1 + i \psi_2$) and valley polarization (described by the real order parameter $\eta$) are condensed in the valley-polarized nematic state, can be seen by noting that, in terms of the irreducible representations of the $D_6$ group, $(\Phi_1, \Phi_2)$ transforms as $E^+_3$, $(\psi_1, \psi_2)$ transforms as $E^+_3$, and $\eta$ as $B^-_2$. Consequently, the following third-order terms are allowed in the action, implying that a non-zero $\Phi$ triggers also a non-zero $\psi$ and a non-zero $\eta$ (see also Ref. [30]):

$$S_{\Phi \psi} \sim \Phi^2 \psi + (\Phi^*)^2 \psi^* ,$$

$$S_{\Phi \psi \eta} \sim i \eta (\Phi \psi^* - \Phi^* \psi) .$$

(4.1)

A somewhat related type of order, which has also been proposed to be realized in twisted bilayer graphene and other systems with higher-order van Hove singularities [49, 50], is the spin-polarized nematic order. It is described by an order parameter of the form $\vec{\varphi} = (\varphi_1, \varphi_2)$, where the indices denote the two d-wave components associated with the irreducible representation $E_2$ of the point group $D_6$. The arrows denote that these quantities transform as vectors in spin space. The main difference of $\vec{\varphi}$ with respect to the valley-polarized nematic state is that the spin-polarized nematic state does not break the $C_{2z}$ symmetry. It is therefore interesting to ask whether our results would also apply for this phase. The main issue is that $\vec{\varphi}$ is not described by a 6-state clock model, since an additional quartic term is present in the action (see Ref. [49]), which goes as:

$$S_{\vec{\varphi}} \sim (\varphi_1 \cdot \varphi_2^2) - |\varphi_1|^2 |\varphi_2|^2 .$$

(4.2)

However, if spin-orbit coupling is present in such a way that $\vec{\varphi}$ becomes polarized along the z-axis, this additional term vanishes. The resulting order parameter $\varphi^z = (\varphi^z_1, \varphi^z_2)$ transforms as the $E_2$ irreducible representation, and its corresponding action is the same as Eq. (2.1), i.e., a 6-state clock model. Moreover, the coupling to the fermions has the same form as in Eq. (2.6), with $a$ now denoting the spin projection, rather than the valley quantum number. Consequently, we also expect an NFL-to-FL crossover inside the Ising spin-polarized nematic state. Of course, in graphitic systems, the spin-orbit coupling is negligibly small. Nevertheless, in twisted bilayer graphene, there is evidence for Ising ferromagnetic order, which presumably arises not from spin, but from “orbital” magnetic degrees of freedom [31, 32, 34]. Analogous to the Ising spin-polarized nematic order, an orbital-magnetic polarized nematic state would therefor also display the NFL-to-FL crossover.

In summary, we showed that when a metallic system undergoes a quantum phase transition to a valley-polarized nematic state, the electronic self-energy at
provide an interesting route to realize NFL behavior in twisted moiré systems.

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[1] Y. Jiang, X. Lai, K. Watanabe, T. Taniguchi, K. Haule, J. Mao, and E. Y. Andrei, Charge order and broken rotational symmetry in magic-angle twisted bilayer graphene, Nature 573, 91–95 (2019).

[2] A. Kerelsky, L. J. McGilly, D. M. Kennes, L. Xian, M. Yankowitz, S. Chen, K. Watanabe, T. Taniguchi, J. Hone, C. Dean, A. Rubio, and A. N. Pasupathy, Maximized electron interactions at the magic angle in twisted bilayer graphene, Nature 572, 95 (2019).

[3] Y. Choi, J. Kemmer, Y. Peng, A. Thomson, H. Arora, R. Polski, Y. Zhang, H. Ren, J. Aicea, G. Refael, F. von Oppen, K. Watanabe, T. Taniguchi, and S. Nadj-Perge, Electronic correlations in twisted bilayer graphene near the magic angle, Nature Physics 15, 1174 (2019).

[4] Y. Cao, D. Rodan-Legrain, J. M. Park, N. F. Q. Yuan, K. Watanabe, T. Taniguchi, R. M. Fernandes, L. Fu, and P. Jarillo-Herrero, Nematicity and competing orders in superconducting magic-angle graphene, Science 372, 264 (2021).

[5] C. Rubio-Verdú, S. Turkel, Y. Song, L. Klebl, R. Samajdar, M. S. Scheurer, J. W. F. Venderbos, K. Watanabe, T. Taniguchi, H. Ochoa, L. Xian, D. M. Kennes, R. M. Fernandes, A. Rubio, and A. N. Pasupathy, Moiré nematic phase in twisted double bilayer graphene, Nature Physics 10.1038/s41567-021-01438-2 (2021).

[6] R. Samajdar, M. S. Scheurer, S. Turkel, C. Rubio-Verdú, A. N. Pasupathy, J. W. F. Venderbos, and R. M. Fernandes, Electric-field-tunable nematic electronic order in twisted double-bilayer graphene, 2D Materials 8, 034005 (2021).

[7] S. A. Kivelson, E. Fradkin, and V. J. Emery, Electronic liquid-crystal phases of a doped Mott insulator, Nature 393, 550 (1998).

[8] V. Oganesyan, S. A. Kivelson, and E. Fradkin, Quantum theory of a nematic Fermi fluid, Phys. Rev. B 64, 195109 (2001).

[9] Y. B. Kim and H.-Y. Kee, Pairing instability in a nematic Fermi liquid, Journal of Physics: Condensed Matter 16, 3139 (2004).

[10] M. Zacharias, P. Wölfle, and M. Garst, Multiscale quantum criticality: Pomeranchuk instability in isotropic metals, Phys. Rev. B 80, 165116 (2009).

[11] H. Watanabe and A. Vishwanath, Criterion for stability of goldstone modes and Fermi liquid behavior in a metal with broken symmetry, Proceedings of the National Academy of Sciences 111, 16314 (2014).

[12] M. Garst and A. V. Chubukov, Electron self-energy near a nematic quantum critical point, Phys. Rev. B 81, 235105 (2010).

[13] E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein, and A. P. Mackenzie, Nematic Fermi fluids in condensed matter physics, Annual Review of Condensed Matter Physics 1, 153 (2010).

[14] R. M. Fernandes, A. V. Chubukov, and J. Schmalian, What drives nematic order in iron-based superconductors?, Nature Physics 10, 97–104 (2014).

[15] M. Hecker and J. Schmalian, Vestigial nematic order and superconductivity in the doped topological insulator Cu2Bi2Se3, npj Quantum Materials 3, 26 (2018).

[16] R. M. Fernandes and J. W. F. Venderbos, Nematicity with a twist: Rotational symmetry breaking in a moiré superlattice, Science Advances 6, eaba8834 (2020).

[17] W. Metzner, D. Rohe, and S. Andergassen, Soft Fermi surfaces and breakdown of Fermi-liquid behavior, Phys. Rev. Lett. 91, 066402 (2003).

[18] J. Rech, C. Pépin, and A. V. Chubukov, Quantum critical behavior in itinerant electron systems: Eliashberg theory and instability of a ferromagnetic quantum critical point, Phys. Rev. B 74, 195126 (2006).

[19] M. A. Metlitski and S. Sachdev, Quantum phase transitions of metals in two spatial dimensions. i. Ising-nematic order, Phys. Rev. B 82, 075127 (2010).

[20] D. Dalidovich and S.-S. Lee, Perturbative non-Fermi liquids from dimensional regularization, Phys. Rev. B 88, 245106 (2013).

[21] I. Mandal and S.-S. Lee, Ultraviolet/infrared mixing in non-Fermi liquids, Phys. Rev. B 92, 035141 (2015).

[22] I. Mandal, UV/IR mixing in non-Fermi liquids: higher-loop corrections in different energy ranges, European Physical Journal B 89, 278 (2016).

[23] A. Eberlein, I. Mandal, and S. Sachdev, Hyperscaling violation at the Ising-nematic quantum critical point in two-dimensional metals, Phys. Rev. B 94, 045133 (2016).

[24] M. A. Metlitski, D. F. Mross, S. Sachdev, and T. Senthil, Cooper pairing in non-Fermi liquids, Phys. Rev. B 91, 115111 (2015).

[25] I. Mandal, Superconducting instability in non-Fermi liquids, Phys. Rev. B 94, 115138 (2016).

[26] S. Lederer, Y. Schattner, E. Berg, and S. A. Kivelson, Enhancement of superconductivity near a nematic quantum critical point, Phys. Rev. Lett. 114, 097001
J. Lou, A. W. Sandvik, and L. Balents, Superconductivity near a nematic quantum critical point: Interplay between hot and lukewarm regions, Phys. Rev. B 98, 220501 (2018).

E. Y. Andrei and A. H. MacDonald, Graphene bilayers with a twist, Nature Materials 19, 1265 (2020).

L. Balents, C. R. Dean, D. K. Efetov, and A. F. Young, Superconductivity and strong correlations in moiré flat bands, Nature Physics 16, 725 (2020).

Y. Xu, X.-C. Wu, C.-M. Jian, and C. Xu, Orbital order and possible non-Fermi liquid in moiré systems, Phys. Rev. B 101, 205426 (2020).

A. L. Sharpe, E. J. Fox, A. W. Barnard, J. Finney, K. Watanabe, T. Taniguchi, M. A. Kastner, and D. Goldhaber-Gordon, Emergent ferromagnetism near three-quarters filling in twisted bilayer graphene, Science 365, 605 (2019).

X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Superconductors, orbital magnets and correlated states in magic-angle bilayer graphene, Nature 574, 653–657 (2019).

M. Serlin, C. L. Tschirhart, H. Polshyn, Y. Zhang, J. Zhu, K. Watanabe, T. Taniguchi, L. Balents, and A. F. Young, Intrinsic quantized anomalous hall effect in a moiré heterostructure, Science 367, 900–903 (2020).

C. L. Tschirhart, M. Serlin, H. Polshyn, A. Shragai, Z. Xia, J. Zhu, Y. Zhang, K. Watanabe, T. Taniguchi, M. E. Huber, and A. F. Young, Imaging orbital ferromagnetism in a moiré chern insulator, Science 372, 1323 (2021).

J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model, Phys. Rev. B 16, 1217 (1977).

D. J. Amit and L. Peliti, On dangerous irrelevant operators, Annals of Physics 140, 207 (1982).

M. Oshikawa, Ordered phase and scaling in Z_n models and the three-state antiferromagnetic Potts model in three dimensions, Phys. Rev. B 61, 3430 (2000).

J. Hove and A. Sudbø, Criticality versus q in the (2+1)-dimensional Z_q clock model, Phys. Rev. E 68, 046107 (2003).

F. Fucito and G. Parisi, On the range of validity of the G − ε expansion for percolation, Journal of Physics A: Mathematical and General 14, L507 (1981).

K. Ishida, M. Tsuji, S. Hosoi, Y. Mizukami, S. Ishida, A. Iyo, H. Eisaki, T. Wolf, K. Grube, H. v. Löhneysen, R. M. Fernandes, and T. Shibauchi, Novel electronic nematicity in heavily doped iron pnictide superconductors, Proceedings of the National Academy of Sciences 117, 6424 (2020).

J. Lou, A. W. Sandvik, and L. Balents, Emergence of U(1) symmetry in the 3D XY model with Z_q anisotropy, Phys. Rev. Lett. 99, 207203 (2007).

T. Okubo, K. Oshikawa, H. Watanabe, and N. Kawashima, Scaling relation for dangerously irrelevant symmetry-breaking fields, Phys. Rev. B 91, 174417 (2015).

F. Léonard and B. Delamotte, Critical exponents can be different on the two sides of a transition: A generic mechanism, Phys. Rev. Lett. 115, 200601 (2015).

D. Podolsky, E. Shimshoni, G. Morigi, and S. Fishman, Buckling transitions and clock order of two-dimensional Coulomb crystals, Phys. Rev. X 6, 031025 (2016).

H. Shao, W. Guo, and A. W. Sandvik, Monte carlo renormalization flows in the space of relevant and irrelevant operators: Application to three-dimensional clock models, Phys. Rev. Lett. 124, 080602 (2020).

P. Patil, H. Shao, and A. W. Sandvik, Unconventional U(1) to Z_q crossover in quantum and classical q-state clock models, Phys. Rev. B 103, 054418 (2021).

D. Pimenov, I. Mandal, F. Piazza, and M. Funk, Non-Fermi liquid at the FFLO quantum critical point, Phys. Rev. B 98, 024510 (2018).

I. Mandal, Critical Fermi surfaces in generic dimensions arising from transverse gauge field interactions, Phys. Rev. Research 2, 043277 (2020).

L. Classen, A. V. Chubukov, C. Honerkamp, and M. M. Scherer, Competing orders at higher-order van hove points, Phys. Rev. B 102, 125141 (2020).

D. V. Chichinadze, L. Classen, and A. V. Chubukov, Valley magnetism, nematicity, and density wave orders in twisted bilayer graphene, Phys. Rev. B 102, 125120 (2020).

C. Burgess, Goldstone and pseudo-goldstone bosons in nuclear, particle and condensed-matter physics, Physics Reports 330, 193 (2000).

I. Mandal, Zero sound and plasmon modes for non-Fermi liquids, (2021), arXiv:2108.09480 [cond-mat.str-el].

V. S. de Carvalho and R. M. Fernandes, Resistivity near a nematic quantum critical point: Impact of acoustic phonons, Phys. Rev. B 100, 115103 (2019).

D. L. Maslov, V. I. Yudson, and A. V. Chubukov, Resistivity of a non-galilean–invariant Fermi liquid near pomeranchuk quantum criticality, Phys. Rev. Lett. 106, 106403 (2011).

S. A. Hartnoll, R. Mahajan, M. Funk, and S. Sachdev, Transport near the Ising-nematic quantum critical point of metals in two dimensions, Phys. Rev. B 89, 155130 (2014).

Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, J. Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, R. C. Ashoori, and P. Jarillo-Herrero, Correlated insulator behaviour at half-filling in magic-angle graphene superlattices, Nature 556, 80–84 (2018).

Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, Nature 556, 43 (2018).

M. Yankowitz, S. Chen, H. Polshyn, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, Tuning superconductivity in twisted bilayer graphene, Science 363, 1059 (2019).

R. Bistritzer and A. H. MacDonald, Moiré bands in twisted double-layer graphene, Proceedings of the National Academy of Sciences 108, 12233 (2011).

G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, Origin of Magic Angles in Twisted Bilayer Graphene, Phys. Rev. Lett. 122, 106405 (2019).

J. F. Dodaro, S. A. Kivelson, Y. Schattner, X. Q. Sun, and C. Wang, Phases of a phenomenological model of twisted bilayer graphene, Phys. Rev. B 98, 075154 (2018).

L. Rademaker and P. Mellado, Charge-transfer insu-
lation in twisted bilayer graphene, Phys. Rev. B 98, 235158 (2018).

[63] H. Isobe, N. F. Q. Yuan, and L. Fu, Unconventional Superconductivity and Density Waves in Twisted Bilayer Graphene, Phys. Rev. X 8, 041041 (2018).

[64] D. M. Kennes, J. Lischner, and C. Karrasch, Strong correlations and $d + id$ superconductivity in twisted bilayer graphene, Phys. Rev. B 98, 241407 (2018).

[65] J. W. F. Venderbos and R. M. Fernandes, Correlations and electronic order in a two-orbital honeycomb lattice model for twisted bilayer graphene, Phys. Rev. B 98, 245103 (2018).

[66] Y. Sherkunov and J. J. Betouras, Electronic phases in twisted bilayer graphene at magic angles as a result of Van Hove singularities and interactions, Phys. Rev. B 98, 205151 (2018).

[67] A. Thomson, S. Chatterjee, S. Sachdev, and M. S. Scheurer, Triangular antiferromagnetism on the honeycomb lattice of twisted bilayer graphene, Phys. Rev. B 98, 075109 (2018).

[68] J. Kang and O. Vafek, Strong Coupling Phases of Partially Filled Twisted Bilayer Graphene Narrow Bands, Phys. Rev. Lett. 122, 246401 (2019).

[69] K. Seo, V. N. Kotov, and B. Uchoa, Ferromagnetic Mott state in Twisted Graphene Bilayers at the Magic Angle, Phys. Rev. Lett. 122, 246402 (2019).

[70] N. F. Q. Yuan, H. Isobe, and L. Fu, Magic of high-order van Hove singularity, Nature Communications 10, 5769 (2019).

[71] J. M. Pizarro, M. J. Calderón, and E. Bascones, The nature of correlations in the insulating states of twisted bilayer graphene, Journal of Physics Communications 3, 035024 (2019).

[72] W. M. H. Natori, R. Nutakki, R. G. Pereira, and E. C. Andrade, SU(4) Heisenberg model on the honeycomb lattice with exchange-frustrated perturbations: Implications for twistronics and Mott insulators, Phys. Rev. B 100, 205131 (2019).

[73] J. Kang and O. Vafek, Non-abelian Dirac node braiding and near-degeneracy of correlated phases at odd integer filling in magic-angle twisted bilayer graphene, Phys. Rev. B 102, 035161 (2020).

[74] N. Bullinck, E. Khalaf, S. Liu, S. Chatterjee, A. Vishwanath, and M. P. Zaletel, Ground state and hidden symmetry of magic-angle graphene at even integer filling, Phys. Rev. X 10, 031034 (2020).

[75] M. Xie and A. H. MacDonald, Nature of the correlated insulator states in twisted bilayer graphene, Phys. Rev. Lett. 124, 097601 (2020).

[76] T. Cea and F. Guinea, Band structure and insulating states driven by Coulomb interaction in twisted bilayer graphene, Phys. Rev. B 102, 045107 (2020).

[77] M. Christos, S. Sachdev, and M. S. Scheurer, Superconductivity, correlated insulators, and wess–zumino–witten terms in twisted bilayer graphene, Proceedings of the National Academy of Sciences 117, 29543 (2020).

[78] F. Xie, A. Cowsik, Z.-D. Song, B. Lian, B. A. Bernevig, and N. Regnault, Twisted bilayer graphene. vi. an exact diagonalization study at nonzero integer filling, Phys. Rev. B 103, 097601 (2021).

[79] E. Brillaux, D. Carpentier, A. A. Fedorenko, and L. Savary, Nematic insulator at charge neutrality in twisted bilayer graphene, arXiv e-prints , arXiv:2008.05041 (2020), arXiv:2008.05041 [cond-mat.str-el].

[80] I. Mandal, J. Yao, and E. J. Mueller, Correlated insulators in twisted bilayer graphene, Phys. Rev. B 103, 125127 (2021).

[81] D. V. Chichinadze, L. Classen, Y. Wang, and A. V. Chubukov, SU(4) symmetry in twisted bilayer graphene - an itinerant perspective, arXiv e-prints , arXiv:2108.05334 (2021), arXiv:2108.05334 [cond-mat.str-el].

[82] Z.-D. Song and B. A. Bernevig, MATBG as Topological Heavy Fermion: I. Exact Mapping and Correlated Insulators, arXiv e-prints , arXiv:2111.05865 (2021), arXiv:2111.05865 [cond-mat.str-el].