On possible sums from multiset of powers of $d$

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Abstract

Let $A$ be a finite multiset of powers of a positive integer $d > 1$. We describe the structure of the set $\text{span}(A)$ of all sums of submultisets of $A$, and in particular give a criterion of $\text{span}(A) = \text{span}(B)$ for two multisets $A, B$.

1 Introduction

Let $A$ be a finite multiset wherein every element of $A$ is a positive integer. Denote by $\text{sum}(A)$ the sum of all elements of $A$ (respecting multiplicity), and by $\text{span}(A)$ the set (not a multiset) of all sums $\text{sum}(C)$, where $C$ is a submultiset of $A$.

This setting describes a finite collection of tokens of positive integer costs, and $\text{span}(A)$ means the set of values which may be obtained using this collection of tokens.

In the general case describing the structure of $\text{span}(A)$ seems difficult. Hereafter we fix an integer $d > 1$ and consider the case when each element of $A$ is in the form $d^k$, for some integer $k \geq 0$.

A more convenient notation would be $A := (a_0, a_1, \ldots, a_k)$, where $a_i \in \mathbb{Z}_{\geq 0}$ and $a_k \neq 0$, where $a_i$ is the multiplicity of $d^i$ in $A$. We will from here on call this a $d$-collection and let $A_d$ denote the family of $d$-collections. Let $\text{comb}(A)$ denote \{$(c_0, c_1, \ldots, c_k) \in \mathbb{Z}^{k+1} : 0 \leq c_i \leq a_i$\}.

Then $\text{span}(A) = \{\sum_{i=0}^{k} c_id^i : (c_0, c_1, \ldots, c_k) \in \text{comb}(A)\}$.

Here, we will describe and prove the correctness of an algorithm to decide for arbitrary $A, B \in A_d$ whether or not $\text{span}(A) = \text{span}(B)$, as well as algorithm to decide for arbitrary $n \in \mathbb{Z}_{\geq 0}$ if $n \in \text{span}(A)$.

An elementary exchange of $(a_0, a_1, \ldots, a_k)$ picks some $i$ such that $a_i \geq d$, sets $a_i \leftarrow a_i - d$ and $a_{i+1} \leftarrow a_{i+1} + 1$. In other words, $d$ tokens of cost $d^i$ are...
replaced with one token of cost $d^{i+1}$. We let $e_i$ denote the elementary exchange wherein the $i$-th place is exchanged.

For $A = (a_0, a_1, \ldots, a_k), B = (b_0, b_1, \ldots, b_k) \in A_d$, we say that $A \preceq B$ if and only if $a_i \leq b_i$ for all $i$.

**Proposition 1.1.** If $A \preceq B$, then $\text{span}(A) \subseteq \text{span}(B)$.

**Proof.** That $A \preceq B$ implies $\text{comb}(A) \subseteq \text{comb}(B)$, from which the result follows. \qed

**Proposition 1.2.**

\[ \text{span}(e_i(A)) \subseteq \text{span}(A) \]

**Proof.** Instead of using the new token of cost $d^{i+1}$ you could use $d$ tokens of cost $d^i$ which it was changed for. \qed

### 2 Condition for span invariance with respect to elementary exchanges

**Proposition 2.1.** For $A = (a_0, a_1, \ldots, a_k)$, if $a_i > 2(d-1)$, then $\text{span}(A) = \text{span}(e_i(A))$.

**Proof.** It suffices to check that $\text{span}(A) \subset \text{span}(e_i(A))$, since the opposite inclusion is proved in Proposition 1.2. Consider an arbitrary submultiset $B \subset A$ and prove that $\text{sum}(B) \in \text{span}(e_i(A))$. If $B$ contains at most $d-1$ tokens $d^i$, it is also a submultiset of $e_i(A)$, since $e_i(A)$ contains more than $2(d-1) - d = d-2$ tokens $d^i$. If $B$ contains at least $d$ tokens $d^i$, replace $d$ such tokens by a token $d^{i+1}$ and get a submultiset $e_i(B) \subset e_i(A)$ with $\text{sum}(e_i(B)) = \text{sum}(B)$. Thus in both cases $\sum(B) \in \text{span}(e_i(A))$, as needed. \qed

From now on, we will call an elementary exchange at an $i$ where the hypothesis of the above proposition is satisfied to be a **proper elementary exchange**.

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**Algorithm 1** Proper elementary exchange sequence

```plaintext
procedure RUNPROPERELEMENTARYEXCHANGES(A)

Require: for all $i$, $A[i] = a_i \cdot d^i, 0 \leq a_i < d$

Ensure: for all $i$, $A[i] \leq 2(d-1)$

\n
1. $i \leftarrow 0$
2. while exists untraversed non-zero element of $A$ do
3. \hspace{1em} while $A[i] > 2(d-1)$ do
4. \hspace{2em} $A[i] \leftarrow A[i] - d$
5. \hspace{2em} $A[i+1] \leftarrow A[i+1] + 1$
6. \hspace{1em} $i \leftarrow i + 1$
7. return $A$
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Corollary 2.1. Let $B$ be the $d$-collection returned by running Algorithm 1 on arbitrary $d$-collection $A$. Then, $\text{span}(A) = \text{span}(B)$.

Proof. Let $A, A_1, A_2, \ldots, A_m, B$ be the sequence of $d$-collections obtained from running the algorithm on $A$. This then follows from Proposition 2.1 applied to adjacent elements of this sequence along with transitivity of set equality. □

The converse of Proposition 2.1 does not hold. A simple counterexample would be for $d = 2$, $A = (3,2,0)$ and an elementary exchange that results in $A' = (3,0,1)$ in which case, $2(d-1) = 2 \geq d-1$, but all 7 values can be assumed by both $A$ and $A'$.

We say that $d$-collection $A$ is normal if and only if $a_i \leq 2(d-1)$ for all $i$ and that $d$-collection $A$ is $j$-normal if and only if $d-1 \leq a_i \leq 2(d-1)$ for all $i < j$ and $a_j < d-1$. We also define the normalization and $j$-normalization of a $d$-collection $A$ in the same manner per Algorithm 1 (with the latter running a modified version that does not exchange at any place $\geq j$), which we denote with $\text{norm}(A)$ and $\text{norm}_j(A)$ respectively. $[\text{norm}(A)]_i$ will denote the value at its $i$th place (such subscripting can be applied to any expression evaluating to a $d$-collection type).

Proposition 2.2. If $\text{norm}(A) = \text{norm}(B)$, then $\text{span}(A) = \text{span}(B)$.

Proof. Follows from Proposition 2.1 along with transitivity of set equivalence. □

Lemma 2.1. If a $d$-collection $A = (a_0, a_1, \ldots, a_k)$ is such that $a_i \geq d - 1$ for all $i < k$, then $\text{span}(A) = \{0,1,\ldots,\text{sum}(A)\}$.

Proof. We prove this by induction by $\text{sum}(A)$. The base case follows trivially. The inductive hypothesis applied to $A \setminus \{d^k\}$ implies that all elements up to $M := \text{sum}(A) - d^k$ belong to $\text{span}(A)$. Thus so are all elements from $\text{sum}(A) - M$ to $\text{sum}(A)$, since the set span$(A)$ enjoys a symmetry $x \rightarrow \text{sum}(A) - x$. It remains to note that $\text{sum}(A) \geq d^k + (d-1)(1 + d + \ldots + d^{k-1}) = 2d^k - 1$, thus $\text{sum}(A) - M = d^k \leq \text{sum}(A) - d^k + 1 = M + 1$. □

Proposition 2.3. If a $d$-collection $A = (a_0, a_1, \ldots, a_k)$ is $j$-normal, then the smallest value not in $\text{span}(A)$ is $X := 1 + \sum_{i=0}^{j} a_i d^i$, which is necessarily $< d^{i+1}$.

Proof. That $A$ is $j$-normal implies that

$$X - 1 = a_j d^j + \sum_{i=0}^{j-1} a_i d^i \leq 2(d^j - 1) + a_j d^j \leq 2(d^j - 1) + (d - 2)d^j = d^{j+1} - 2.$$ 

By Lemma 2.1, any $n \leq \sum_{i=0}^{j} a_i d^i = X - 1$ is in $\text{span}(A)$. It remains to prove that $X \notin \text{span}(A)$. Since $X < d^{j+1}$, we can use only tokens not exceeding $d^j$ for collecting $X$, but the sum of such tokens in $A$ equals $X - 1$, thus indeed $X \notin \text{span}(A)$. □
3 Decomposition of normal $d$-collections

We now describe an algorithm for decomposing an arbitrary normal $d$-collection $A$ to submultisets which partition $A$. Call an index $j \leq k$ critical if either $j = k$ or $a_j < d - 1$. Let $j_1 < j_2 < \ldots < j_s = k$ be all critical indices. Define the $d$-collections $A_1, \ldots, A_s$ as follows: $A_i$ consists of those tokens $d^i$ from $A$, for which $j_{i-1} < t \leq j_i$ (for $i = 1$, we put $j_0 = -1$ by agreement). If $s = 1$, we say that $A$ is an irreducible $d$-collection.

**Theorem 3.1.** In above notations,

$$\text{span}(A) = \bigoplus_{i=1}^d d^{i-1+1} \cdot \{0, 1, \ldots, d^{-1-j_i-1}\text{sum}(A_j)\},$$

where the $\oplus$ sign means the direct sum: any element in $\text{span}(A)$ is uniquely represented as $x_1 + \ldots + x_s$ for $x_i \in d^{i-1+1} \cdot \{0, 1, \ldots, d^{-1-j_i-1}\text{sum}(A_j)\}$.

**Proof.** It follows from Lemma 2.1 and scaling that

$$\text{span}(A_i) = d^{i-1+1} \cdot \{0, 1, \ldots, d^{-1-j_i-1}\text{sum}(A_j)\}.$$ 

Also, by 2.3 we get $\text{sum}(A_i) \leq d^{i+1} - d^{i-1+1}$ for all $i < s$. Since any element $x \in \text{span}(A)$ may be represented as $x = \sum x_i$ for $x_i \in \text{span}(A_i)$, it remains to prove the direct sum condition. Assume that on the contrary that $x = \sum x_i = \sum y_i$ for $x_i, y_i \in \text{span}(A_i)$ but there exists $i$ for which $x_i \neq y_i$. Choose minimal such $i$, obviously we have $i < s$. Then $x_i - y_i = (y_{i+1} + \ldots + y_s) - (x_{i+1} + \ldots + x_s)$ is divisible by $d^{i+1}$, but both $x_i, y_i$ belong to $\text{sum}(A_i) \subset \{0, 1, \ldots, d^{i-1+1} - 1\}$, a contradiction. \qed

**Proposition 3.1.** If two irreducible $d$-collections $A, B$ are distinct, then $\text{sum}(A) \neq \text{sum}(B)$.

**Proof.** Let $A = (a_0, \ldots, a_k)$, $B = (b_0, \ldots, b_l)$. Assume that $k \neq l$, say, $l > k$. Then $\text{sum}(B) \geq (d - 1)(1 + d + \ldots + d^{k-1}) + d^k = 2d^k - 1$ while $\text{sum}(A) \leq 2(d - 1)(1 + d + \ldots + d^k) = 2(d^{k+1} - 1) \leq 2d^k - 2 < \text{sum}(B)$. If $l = k$, then choose the minimal index $i$ for which $a_i \neq b_i$. If $i = k$, then $\text{sum}(A) \neq \text{sum}(B)$ is clear. If $i < k$, then both $a_i, b_i$ belong to $\{d - 1, \ldots, 2d - 2\}$, so $a_i$ and $b_i$ differ modulo $d$, thus $\text{sum}(A)$ and $\text{sum}(B)$ differ modulo $d^{i+1}$. \qed

**Proposition 3.2.** If $A$ and $B$ are two normal $d$-collections, and $\text{span}(A) = \text{span}(B)$, then $A = B$.

**Proof.** Let $A = A_1 \sqcup A_2 \sqcup \ldots \sqcup A_s$ be a decomposition of $A$, $B = B_1 \sqcup B_2 \ldots \sqcup B_t$ that of $B$. The minimal non-element of $\text{span}(A)$ equals $\sum(A_1) + 1$ by Proposition 2.3 (if $s = 1$, apply Lemma 2.1 instead). Analogously for $B$, so we get $\text{sum}(A_1) = \text{sum}(B_1)$ and thus $A_1 = B_1$ by Proposition 3.1. Now if $t$ or $s$ equals 1 we get from $\text{sum}(A) = \text{sum}(B)$ that $s = t = 1$ and $A = B = A_1$. If both $t, s$ are greater than 1, denote by $d'$ the maximal element of $A_1$, note that $\text{sum}(A_1) < d^{j+1}$ again by Proposition 2.3. Therefore, if we consider the map $\psi : x \to [x/d^{j+1}]$ on the set $\text{span}(A) = \text{span}(A_1) + \text{span}(A_2) + \ldots + \text{span}(A_s)$, its value does not depend on
the component in $\text{span}(A_1)$, and we get $\psi(\text{span}(A)) = d^{-j-1}(\text{span}(A_2) + \ldots + \text{span}(A_s))$. Use the same for $B$ to get $\text{span}(A_2) + \ldots + \text{span}(A_s) = \text{span}(B_2) + \ldots + \text{span}(B_t)$ and proceed by induction to conclude that $A_2 = B_2$ and so on.

**Theorem 3.2.** $\text{span}(A) = \text{span}(B)$ if and only if $\text{norm}(A) = \text{norm}(B)$

**Proof.** Proposition 2.2 and Proposition 3.2 are the two directions.

**Corollary 3.1.** Start with arbitrary $d$-collection $A$ and apply elementary exchanges $e_i$ if $a_i \geq 2d - 1$, in arbitrary order. Then this process terminates after finitely many steps, and the resulted normalized $d$-collection does not depend on the order.

**Proof.** Note that we may get only finitely many different $d$-collections, since the sum does not change. And there is a semiinvariant, for example, the sum of squares of elements, which increases after every operation. Thus the process terminates. By Proposition 2.1, these operations do not change the span. Since normalized $d$-collection is uniquely determined by its span, the independence of the order follows.

We conclude with the criterion used to determine if an elementary exchange breaks the desired span invariant.

**Proposition 3.3.** $\text{span}(e_i(A)) = \text{span}(A)$ if and only if $[\text{norm}_i(A)]_i > 2(d - 1)$

**Proof.** $\Leftarrow$: Suppose $[\text{norm}_i(A)]_i \leq 2(d - 1)$, in which case a run of Algorithm 1 would not perform any exchanges at $i$, which means that $[\text{norm}(A)]_i = [\text{norm}_i(A)]_i > [\text{norm}(e_i(A))]_i$, from which follows by one direction of Proposition 3.2 that the spans differ.

$\Rightarrow$: If $[\text{norm}_i(A)]_i > 2(d - 1)$, then Algorithm 1 which returns $\text{norm}(A)$ necessarily performs at least one elementary exchange at $i$. Upon $e_i(A)$, we’ve performed the first elementary exchange there doing so on the places less significant than $i$. Thus, once that algorithm run on $e_i(A)$ reaches $i$, the state of the $d$-collection will be same as the state of $d$-collection right after the first elementary exchange on $i$ when the algorithm is run on $A$. Thus, $\text{norm}(e_i(A)) = \text{norm}(A)$ in this case, from which follows by the other direction of Proposition 3.2 that the spans in this case are the same.

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References

[AE04] George E. Andrews and Kimmo Eriksson. *Integer Partitions*. Cambridge University Press, 2004.