On the monomorphism category of $n$-cluster tilting subcategories

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Abstract Let $M$ be an $n$-cluster tilting subcategory of mod-$\Lambda$, where $\Lambda$ is an Artin algebra. Let $S(M)$ denote the full subcategory of $S(\Lambda)$, the submodule category of $\Lambda$, consisting of all the monomorphisms in $M$. We construct two functors from $S(M)$ to mod-$M$, the category of finitely presented additive contravariant functors on the stable category of $M$. We show that these functors are full, dense and objective and hence provide equivalences between the quotient categories of $S(M)$ and mod-$M$. We also compare these two functors and show that they differ by the $n$-th syzygy functor, provided $M$ is an $n\mathbb{Z}$-cluster tilting subcategory. These functors can be considered as higher versions of the two functors studied by Ringel and Zhang (2014) in the case $\Lambda = k[x]/(x^n)$ and generalized later by Eiríksson (2017) to self-injective Artin algebras. Several applications are provided.

Keywords submodule categories, $n$-abelian categories, $n$-cluster tilting subcategories

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1 Introduction

Let $R$ be a commutative artinian ring. Let $\Lambda$ be an Artin $R$-algebra of finite representation type. Let $\Gamma := \text{End}_A(E)^{op}$ be the Auslander algebra of $\Lambda$, where $E$ is an additive generator of mod-$\Lambda$, the category of finitely presented $\Lambda$ modules. Moreover, let $T_2(\Lambda)$ denote the upper triangular matrix algebra of $\Lambda$. It is known that mod-$T_2(\Lambda)$ is equivalent to $H(\Lambda)$, the morphism category of $\Lambda$. So the objects of mod-$T_2(\Lambda)$ can be considered as morphisms in mod-$\Lambda$. Using this equivalence, Auslander [2] introduced and studied a functor from mod-$T_2(\Lambda)$ to mod-$\Gamma$ by sending an object $f$ of mod-$T_2(\Lambda)$ to the cokernel of the induced map $\text{Hom}_\Lambda(E, f)$. This functor is usually denoted by $\alpha$. It then has been studied further by Auslander and Reiten [5, 6]. For more recent account, see [28, Section 3].

Let $n$ be a fixed positive integer. Let $S(n)$ be the submodule category of mod-$\Lambda_n$, where $\Lambda_n = k[x]/(x^n)$ and $k$ is a field. Having the functor $\alpha$ as an “essential tool”, Ringel and Zhang [28] introduced and studied...
two functors $F$ and $G$ from $S(n)$ to $\mod-\Pi_{n-1}$, where $\Pi_{n-1}$ denotes the preprojective algebra of type $A_{n-1}$. They showed that $\Pi_{n-1}$ is isomorphic to $\Gamma$, the stable Auslander algebra of $\Lambda$. Hence $F$ and $G$ are functors from $S(n)$ to $\mod-\Gamma$. They proved that these functors are full, dense and objective and hence they induce equivalences between the quotients of $S(n)$ by the ideals of the kernel objects of $F$ and $G$ and $\mod-\Gamma$. This, in particular, introduces quotients of $S(n)$ that are abelian categories with enough projective objects. They also provided a comparison of $F$ and $G$ and showed that they differ only by the syzygy functor on the stable module category of $\Gamma$. Later, Eiriksson [11] studied these functors in a more general setting of representation-finite self-injective Artin algebras, and thus he studied two functors from the submodule category of a representation-finite self-injective algebra $\Lambda$ to the module category of the stableAuslander algebra of $\Lambda$ (see [11, Section 4]).

Let us be a little bit more explicit. Consider the compositions

$$S(\Lambda) \xrightarrow{\eta} \mod-T_2(\Lambda) \xrightarrow{\alpha} \mod-\Gamma \xrightarrow{q} \mod-\Gamma,$$

where $\eta$ is the inclusion of $S(\Lambda)$ in $\mod-T_2(\Lambda)$, $\epsilon$ maps a morphism $f$ in $S(\Lambda)$ to $\text{Coker}(f)$ in $\mod-T_2(\Lambda)$, $\alpha$ is the functor introduced by Auslander [3], just recalled above, and $q$ is introduced in [11, Section 3]. Then the functors $F$ and $G$ are given by $F = q\eta\epsilon$ and $G = q\alpha\epsilon$. Note that the functor $F$ also was studied by Li and Zhang [22]. Eiríksson [11, Theorem 1] proved that $F$ induces an equivalence of categories $S(\Lambda)/\mathcal{U} \to \mod-\Gamma$, where $\mathcal{U}$ is the additive subcategory of $S(\Lambda)$ generated by all the objects of the form $M \to M$ and $M \to I$, where $M \in \mod-A$ and $I$ is an injective-projective module. Moreover, $G$ induces an equivalence of categories $S(\Lambda)/\mathcal{V} \to \mod-\Gamma$, where $\mathcal{V}$ is the smallest additive subcategory of $S(\Lambda)$ generated by all the objects of the form $M \to M$ and $0 \to M$, where $M \in \mod-A$.

Roughly speaking, our aim in this paper is to introduce the above two functors to the higher homological algebra. Higher homological algebra is born while Iyama [14,15] developed higher versions of Auslander’s correspondence and Auslander-Reiten theory for Artin algebras and related rings. The notions of $n$-cluster tilting modules and $n$-cluster tilting subcategories are fundamental in the Iyama’s theory (see, for example, [16]).

Although $n$-cluster tilting subcategories of abelian categories are not abelian, Jasso [18] proved that they have a very nice structure, known as $n$-abelian structure. In this new structure, special sequences of length $n + 2$ play the role of short exact sequences in abelian categories. Higher homological algebra is currently a very active area of research. Its importance stems from the many connections and applications cluster tilting theory has in many research areas: algebraic and quantum groups (total positivity and canonical bases), representation theory (in particular representations of quivers), geometry (Poisson geometry, Teichmüller spaces, integrable systems), combinatorics (Stasheff associahedra), algebraic geometry (Bridgeland’s stability conditions, Calabi-Yau algebras, Donaldson-Thomas invariants) and non-commutative geometry (non-commutative crepant resolutions). For basics of the theory, see Subsection 2.1 below.

Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\mod-\Lambda$ and $S(\mathcal{M})$ denote the subcategory of $S(\Lambda)$ consisting of all the monomorphisms in $\mathcal{M}$. In this paper, motivated by the works of [11,28], two functors $\Phi$ and $\Psi$ from $S(\mathcal{M})$ to $\mod-\mathcal{M}$ are introduced and studied, where $\mod-\mathcal{M}$ is the category of additive contravariant finitely presented functors from $\mathcal{M}$ to $\mathcal{A}$, the category of abelian groups.

The rest of this paper is organized as follows. In Section 2, we provide necessary backgrounds that are needed throughout the paper. In Section 3, we introduce and study the functor $\Phi$ and in Section 4, we investigate the functor $\Psi$. Section 5 contains a comparison of these two functors, when $\Lambda$ is a self-injective Artin algebra and $\mathcal{M}$ is an $n\mathbb{Z}$-cluster tilting subcategory of $\mod-\Lambda$. Section 6 is devoted to a list of the dual of the results of Sections 3 and 4. Since their proofs are similar, we just list the statements without proof. In the last section, we provide some applications. In particular, we present a duality from $\mod-\mathcal{M}$ to $\mathcal{M}$-mod (see Corollary 7.2), that could be considered as a higher version of the Auslander’s result [4] showing the existence of a duality between $\mod-\mathcal{A}$ and $\mathcal{A}$-mod, where $\mathcal{A}$ is an abelian category. We use this duality to prove a higher version of the Hilton-Rees theorem for $n$-cluster tilting subcategories (see Corollary 7.6), and a higher version of Auslander’s direct summand conjecture (see Corollary 7.8).
Moreover, we apply our results to reprove the existence of $n$-Auslander-Reiten translation $\tau_n = \tau^{\Omega_n-1}$ in $n$-cluster tilting subcategories (see Theorem 7.9).

2 Preliminaries

Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ be a full additive subcategory of $\mathcal{A}$. For an object $A \in \mathcal{A}$, let $\mathcal{A}(-,A)|_{\mathcal{M}}$ denote the functor $\mathcal{A}(-,A)$ restricted to $\mathcal{M}$. A right $\mathcal{M}$-approximation of $A$ is a morphism $\pi: M \to A$ with $M \in \mathcal{M}$ such that $\mathcal{A}(-,M)|_{\mathcal{M}} \to \mathcal{A}(-,A)|_{\mathcal{M}} \to 0$ is exact. $\mathcal{M}$ is called a contravariantly finite subcategory of $\mathcal{A}$ if every object of $\mathcal{A}$ admits a right $\mathcal{M}$-approximation. Dually, the notion of left $\mathcal{M}$-approximations and covariantly finite subcategories are defined. $\mathcal{M}$ is called a functorially finite subcategory of $\mathcal{A}$ if it is both a contravariantly finite and a covariantly finite subcategory.

A subcategory $\mathcal{M}$ of $\mathcal{A}$ is called a generating subcategory if for every object $A \in \mathcal{A}$, there exists an epimorphism $M \to A$ with $M \in \mathcal{M}$. Cogenerating subcategories are defined dually. $\mathcal{M}$ is called a generating-cogenerating subcategory of $\mathcal{A}$ if it is both a generating and a cogenerating subcategory.

2.1 Higher homological algebra

The concept of $n$-abelian categories is formalized and studied in [18] as a generalisation of the notion of abelian categories. Let us recall the basics. Let $\mathcal{M}$ be an additive category. Let $d^0 : M^0 \to M^1$ be a morphism in $\mathcal{M}$. An $n$-cokernel of $d^0$ is a sequence

$$M^1 \xrightarrow{d^1} M^2 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1}$$

of morphisms in $\mathcal{M}$ such that for every $M \in \mathcal{M}$, the induced sequence

$$0 \longrightarrow \mathcal{M}(M^{n+1},M) \xrightarrow{d^n} \cdots \xrightarrow{d^1} \mathcal{M}(M^1,M) \xrightarrow{d^0} \mathcal{M}(M^0,M)$$

of abelian groups is exact. Such an $n$-cokernel of $d^0$ will be denoted by $(d^1,d^2,\ldots,d^n)$. The notion of $n$-kernel of a morphism $d^n : M^n \to M^{n+1}$ is defined dually.

A complex

$$M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1}$$

of objects and morphisms in $\mathcal{M}$ is called $n$-exact [18, Definitions 2.2 and 2.4] if $(d^0,d^1,\ldots,d^{n-1})$ is an $n$-kernel of $d^n$ and $(d^1,d^2,\ldots,d^n)$ is an $n$-cokernel of $d^0$. An $n$-exact sequence like the above one will be denoted by

$$0 \longrightarrow M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow 0.$$

An additive category $\mathcal{M}$ is called $n$-abelian [18, Definition 3.1] if it is idempotent complete, each morphism in $\mathcal{M}$ admits an $n$-cokernel and an $n$-kernel, for every monomorphism $d^0 : M^0 \to M^1$ in $\mathcal{M}$ and for every $n$-cokernel $(d^1,d^2,\ldots,d^n)$ of $d^0$, the sequence

$$M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1}$$

is $n$-exact and finally for every epimorphism $d^n : M^n \to M^{n+1}$ in $\mathcal{M}$ and for every $n$-kernel $(d^0,d^1,\ldots,d^{n-1})$ of $d^n$, the sequence

$$M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1}$$

is $n$-exact.

As it is mentioned in [18, Remark 3.2], we may replace the last two conditions of the above definition by the weaker conditions that every monomorphism $d^0 : M^0 \to M^1$, respectively every epimorphism $d^n : M^n \to M^{n+1}$, can be completed to an $n$-exact sequence

$$0 \longrightarrow M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow 0.$$
Let $\mathcal{A}$ be an abelian category. An additive subcategory $\mathcal{M}$ of $\mathcal{A}$ is called an $n$-cluster tilting subcategory [18, Definition 3.14] if it is a functorially finite and generating-cogenerating subcategory of $\mathcal{A}$ such that $\mathcal{M}^{\perp n} = \mathcal{M} = \perp n \mathcal{M}$, where

\[ \mathcal{M}^{\perp n} := \{ A \in \mathcal{A} | \text{Ext}^i_{\mathcal{A}}(M, A) = 0 \text{ for all } 0 < i < n \} \]

and

\[ \perp n \mathcal{M} := \{ A \in \mathcal{A} | \text{Ext}^i_{\mathcal{A}}(A, M) = 0 \text{ for all } 0 < i < n \}. \]

It is known that every $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$ has a structure as an $n$-abelian category [18, Theorem 3.16]. On the other hand, every small $n$-abelian category $\mathcal{M}$ is equivalent to an $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$ (see [10, Theorem 4.3] and [21, Theorem 7.3]).

2.2 Morphism category

Let $\mathcal{A}$ be an abelian category. The morphism category of $\mathcal{A}$ denoted by $\mathcal{H}(\mathcal{A})$ is a category whose objects are morphisms in $\mathcal{A}$. Its morphisms are given by commutative diagrams and composition is defined naturally [27]. Let $f : A \to B$ be an object of $\mathcal{H}(\mathcal{A})$. Then $A$ (resp. $B$) is called the source (resp. target) of $f$ and is denoted by $s(f)$ (resp. $t(f)$). It is known that $\mathcal{H}(\mathcal{A})$ is an abelian category. A sequence

\[ 0 \longrightarrow f' \xrightarrow{\alpha} f \xrightarrow{\beta} f'' \longrightarrow 0 \]

of morphisms in $\mathcal{H}(\mathcal{A})$ is exact if and only if the induced sequences of sources and targets are exact in $\mathcal{A}$. It has two important full subcategories, i.e., the monomorphism category and the epimorphism category denoted, respectively by $\mathcal{S}(\mathcal{A})$ and $\mathcal{F}(\mathcal{A})$. As it is expected from their names, the objects of $\mathcal{S}(\mathcal{A})$ (resp. the objects of $\mathcal{F}(\mathcal{A})$) are monomorphisms (resp. epimorphisms) in $\mathcal{H}(\mathcal{A})$. They both are closed under extensions and summands, $\mathcal{S}(\mathcal{A})$ is closed under taking kernels and $\mathcal{F}(\mathcal{A})$ is closed under taking cokernels. By [20, Appendix A], a full extension closed subcategory of an abelian category is a Quillen exact category. Hence $\mathcal{S}(\mathcal{A})$ and $\mathcal{F}(\mathcal{A})$ are both exact categories. The conflations of $\mathcal{S}(\mathcal{A})$ (resp. $\mathcal{F}(\mathcal{A})$) are exact sequences

\[ 0 \longrightarrow f' \xrightarrow{\alpha} f \xrightarrow{\beta} f'' \longrightarrow 0 \]

in $\mathcal{H}(\mathcal{A})$ with terms in $\mathcal{S}(\mathcal{A})$ (resp. $\mathcal{F}(\mathcal{A})$).

2.3 Functor category

Let $\mathcal{X}$ be a skeletally small additive category. By definition, a (right) $\mathcal{X}$-module is a contravariant additive functor $F : \mathcal{X} \to \text{Ab}$, where Ab denotes the category of abelian groups. The $\mathcal{X}$-modules and natural transformations between them form an abelian category denoted by Mod-$\mathcal{X}$. A $\mathcal{X}$-module $F$ is called finitely presented if there exists an exact sequence

\[ \mathcal{X}(-, X) \longrightarrow \mathcal{X}(-, X') \longrightarrow F \longrightarrow 0 \]

with $X$ and $X'$ in $\mathcal{X}$. Finitely presented $\mathcal{X}$-modules form a full subcategory of Mod-$\mathcal{X}$ that is denoted by mod-$\mathcal{X}$. It is proved by Auslander [3, Chapter III, Section 2] that mod-$\mathcal{X}$ is an abelian category if and only if $\mathcal{X}$ admits weak kernels. This happens, for example, when $\mathcal{X}$ is a contravariantly finite subcategory of an abelian category $\mathcal{A}$.

Dually, the category of covariant additive functors $F : \mathcal{X} \to \text{Ab}$ is denoted by $\mathcal{X}$-Mod, called the category of left $\mathcal{X}$-modules. Moreover, $\mathcal{X}$-mod denotes its subcategory consisting of all the finitely presented left $\mathcal{X}$-modules.
2.4 Objective functors

Here, we recall some facts on objective functors. For a good reference, see [28, Appendix]. Let \( F : \mathcal{X} \to \mathcal{Y} \) be an additive functor between additive categories. \( F \) is called an objective functor if any morphism \( f \) in \( \mathcal{X} \) with \( F(f) = 0 \) factors through an object \( K \) with \( F(K) = 0 \). \( K \) is then called a kernel object of \( F \).

We say that the kernel of an objective functor \( F \) is generated by \( \mathcal{X} \) if \( \text{add-}\mathcal{X} \) is the class of all the kernel objects of \( F \), where \( \text{add-}\mathcal{X} \) denotes the full subcategory of \( \mathcal{X} \) consisting of all the direct summands of finite direct sums of objects of \( \mathcal{X} \).

Let \( F : \mathcal{X} \to \mathcal{Y} \) be a full, dense and objective functor and the kernel of \( F \) is generated by \( \mathcal{X} \). Then \( F \) induces an equivalence \( F : \mathcal{X}/\langle \mathcal{X} \rangle \to \mathcal{Y} \). Recall that for a class \( \mathcal{X} \) of objects of the category \( \mathcal{X} \), the ideal of \( \mathcal{X} \) generated by all the maps that factor through a direct sum of objects in \( \mathcal{X} \) is denoted by \( \langle \mathcal{X} \rangle \). Following [28], for the ease of notation, we just write \( \mathcal{X}/\langle \mathcal{X} \rangle \) instead of \( \mathcal{X}/\langle \mathcal{X} \rangle \).

The composition of objective functors is not necessarily objective but if, in addition, we know that they are both full and dense, then their composition is objective, full and dense.

**Notation 2.1.** Let \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of \( \text{mod-}\Lambda \). Let \( m \) be a natural number.

- We let \( \overline{\mathcal{M}}^{\leq m} \) be the subcategory of \( \text{mod-}\Lambda \) consisting of all the modules \( X \) admitting a \( \text{Hom}_{\Lambda}(\mathcal{M}, -) \)-exact sequence
  \[
  0 \to M^0 \to \cdots \to M^{m-1} \to M^m \to X \to 0
  \]
  with \( M^i \in \mathcal{M} \) for all \( i \in \{0, 1, \ldots, m\} \). In this case, we say that \( X \) has proper \( \mathcal{M} \)-dimension at most \( m \).

  Note that since \( \mathcal{M} \) contains projectives, the sequence itself is exact.

- Dually, we let \( \overline{\mathcal{M}}^{< m} \) be the subcategory of \( \text{mod-}\Lambda \) consisting of all the modules \( X \) admitting a \( \text{Hom}_{\Lambda}(-, \mathcal{M}) \)-exact sequence
  \[
  0 \to X \to M^0 \to M^1 \to \cdots \to M^m \to 0
  \]
  with \( M^i \in \mathcal{M} \) for all \( i \in \{0, 1, \ldots, m\} \). In this case, we say that \( X \) has coproper \( \mathcal{M} \)-dimension at most \( m \).

Since \( \mathcal{M} \) contains injectives, the sequence itself is exact.

- Let \( \mathcal{P}^{\leq 1}(\mathcal{M}) \) (resp. \( \mathcal{P}^{< 1}(\mathcal{M}^{\text{mod}}) \)) be the subcategory of \( \text{mod-}\mathcal{M} \) (resp. \( \mathcal{M} \)-mod) consisting of all the finitely presented functors of projective dimension at most one. Note that since \( \mathcal{M} \) is an \( n \)-cluster tilting subcategory of \( \text{mod-}\Lambda \), it admits weak kernels (resp. weak cokernels), so \( \text{mod-}\mathcal{M} \) (resp. \( \mathcal{M} \)-mod) is an abelian category.

3 The functor \( \Phi : \mathcal{S}(\mathcal{M}) \to \text{mod-}\mathcal{M} \)

Let \( \Lambda \) be an Artin algebra and \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of \( \text{mod-}\Lambda \), where \( n > 1 \) is a fixed positive integer. To define \( \Phi \), we need some preparations. In particular, we need to define auxiliary functors \( \Upsilon \) and \( i_\Lambda \). We do this in the following two subsections. Throughout the paper, the Hom functor \( \text{Hom}_{\Lambda}(-, M)|_\mathcal{M} \) will be denoted by \( \mathcal{M}(-, M) \), when \( M \in \mathcal{M} \).

3.1 The functor \( \Upsilon : \mathcal{S}(\mathcal{M}) \to \mathcal{P}^{\leq 1}(\mathcal{M}) \)

In this subsection, we study a restriction to \( \mathcal{S}(\mathcal{M}) \) of the functor \( \alpha \) introduced by Auslander [2].

Consider the subcategory
\[
\mathcal{S}(\mathcal{M}) := \{(M^0 \xrightarrow{f} M^1) \mid f \in \mathcal{S}(\Lambda) \text{ and } M^0, M^1 \in \mathcal{M}\}
\]
of \( \mathcal{S}(\Lambda) \). The assignment
\[
(M^0 \xrightarrow{f} M^1) \mapsto \text{Coker}(\mathcal{M}(-, M^0) \xrightarrow{\mathcal{M}(\cdot, f)} \mathcal{M}(-, M^1))
\]
defines a functor
\[
\Upsilon : \mathcal{S}(\mathcal{M}) \to \mathcal{P}^{\leq 1}(\mathcal{M}).
\]
We show that this functor is full, dense and objective. To this end, we consider the following functor. Throughout, let $Y_{\mathcal{M}} : \text{mod-} \Lambda \rightarrow \text{mod-} \mathcal{M}$ be the functor that maps $X \in \text{mod-} \Lambda$ to $\text{Hom}_\Lambda(-, X)|_{\mathcal{M}}$.

**Lemma 3.1.** The functor $Y_{\mathcal{M}}$ is full and faithful. In addition, its restriction to $\mathcal{M}_{\leq 1}$ induces an equivalence of categories

$$Y_{\mathcal{M}}|_{\mathcal{M}_{\leq 1}} : \mathcal{M}_{\leq 1} \xrightarrow{\simeq} \widehat{\mathcal{P}}_{\leq 1}(\mathcal{M}).$$

**Proof.** Since $\mathcal{M}$ contains projective $\Lambda$-modules, it follows that $Y_{\mathcal{M}}$ is full and faithful. We show that its restriction to $\mathcal{M}_{\leq 1}$ maps to $\widehat{\mathcal{P}}_{\leq 1}(\mathcal{M})$ and induces an equivalence. To see this, pick $X \in \mathcal{M}_{\leq 1}$ and let

$$0 \rightarrow M^0 \rightarrow M^1 \rightarrow X \rightarrow 0$$

be an $\mathcal{M}$-proper resolution of $X$. It yields the exact sequence

$$0 \rightarrow \mathcal{M}(-, M^0) \rightarrow \mathcal{M}(-, M^1) \rightarrow \text{Hom}_\Lambda(-, X)|_{\mathcal{M}} \rightarrow 0.$$

This in turn implies that $\text{Hom}_\Lambda(-, X)|_{\mathcal{M}} \in \widehat{\mathcal{P}}_{\leq 1}(\mathcal{M})$. Hence, to complete the proof, we show that $Y_{\mathcal{M}}|$ is dense. Let $F \in \widehat{\mathcal{P}}_{\leq 1}(\mathcal{M})$. There exists a projective resolution

$$0 \rightarrow \mathcal{M}(-, M^0) \xrightarrow{(-, f)} \mathcal{M}(-, M^1) \rightarrow F \rightarrow 0 \quad (3.1)$$

in $\text{mod-} \mathcal{M}$, where $M^0, M^1 \in \mathcal{M}$. By Yoneda’s lemma, we obtain a monomorphism $f : M^0 \rightarrow M^1$. Set $C = \text{Coker} f$. We show that $Y_{\mathcal{M}}(C) = F$. Since $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\text{mod-} \Lambda$, it is $n$-abelian and hence $f$ extends to an $n$-exact sequence

$$0 \rightarrow M^0 \xrightarrow{f} M^1 \rightarrow M^2 \rightarrow \cdots \rightarrow M^{n+1} \rightarrow 0.$$

The $n$-exactness of the sequence induces the short exact sequence

$$0 \rightarrow \mathcal{M}(-, M^0) \xrightarrow{(-, f)} \mathcal{M}(-, M^1) \rightarrow \text{Hom}_\Lambda(-, C)|_{\mathcal{M}} \rightarrow 0. \quad (3.2)$$

By comparing the sequences (3.1) and (3.2), we obtain the result. \hfill \Box

**Proposition 3.2.** The functor $\Upsilon : S(\mathcal{M}) \rightarrow \widehat{\mathcal{P}}_{\leq 1}(\mathcal{M})$ is full, dense and objective.

**Proof.** The functor $\Upsilon$ can be considered as the composition

$$\Upsilon : S(\mathcal{M}) \xrightarrow{C} \mathcal{M}_{\leq 1} \xrightarrow{Y_{\mathcal{M}}|_{\mathcal{M}_{\leq 1}}} \widehat{\mathcal{P}}_{\leq 1}(\mathcal{M}),$$

where $C$ is the usual cokernel functor. In view of Lemma 3.1, $Y_{\mathcal{M}}|$ is full, faithful and dense. Since faithful functors are objective, to prove the result it is enough to show that the functor

$$C : S(\mathcal{M}) \rightarrow \mathcal{M}_{\leq 1}$$

is full, dense and objective. To see that it is full, let $(M^0 \xrightarrow{f} M^1)$ and $(M^0 \xrightarrow{f'} M^1)$ be two objects of $S(\mathcal{M})$ and $\alpha : C(f) \rightarrow C(f')$ be a morphism in $\mathcal{M}_{\leq 1}$. Consider the diagram

$$\begin{array}{cccc}
0 & \xrightarrow{f} & M^0 & \xrightarrow{C(f)} & 0 \\
\downarrow{\alpha^0} & & \downarrow{\alpha^1} & & \\
0 & \xrightarrow{f'} & M^0 & \xrightarrow{C(f')} & 0.
\end{array}$$

Since $\mathcal{M}$ is an $n$-cluster tilting subcategory with $n > 1$, rows are $\mathcal{M}$-proper and hence $\alpha$ can be lifted to a morphism $f \xrightarrow{\alpha} f'$.
in \( S(M) \). Hence \( C \) is full. By definition, it is dense. To see that it is objective, let \((\alpha^0, \alpha^1)\) be a morphism of \((M^0 \xrightarrow{f} M^1)\) to \((M^0' \xrightarrow{f'} M^1')\) such that \( C(\alpha^0, \alpha^1) = 0 \). Hence \((\alpha^0, \alpha^1)\) in the following diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & M^0 \\
\downarrow{\alpha^0} & & \downarrow{\alpha^1} \\
0 & \xrightarrow{f'} & M^0'
\end{array}
\quad \begin{array}{ccc}
0 & \xrightarrow{C(f)} & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{C(f')} & 0
\end{array}
\]

is null-homotopic. Therefore, there exists a morphism \( s : M^1 \rightarrow M^{0'} \) such that \( sf = \alpha^0 \) and \( f's = \alpha^1 \).

This in turn induces the following factorization of the morphism \((\alpha^0, \alpha^1)\):

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & M^0 \\
\downarrow{s} & & \downarrow{s} \\
0 & \xrightarrow{f'} & M^0'
\end{array}
\quad \begin{array}{ccc}
0 & \xrightarrow{\eta} & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\vartheta_F} & 0
\end{array}
\]

where the middle object is clearly a kernel object. This completes the proof of the proposition.

We have the following immediate corollary.

**Corollary 3.3.** With the above notations, there exists an equivalence of additive categories

\[
S(M)/K \cong \tilde{\mathcal{T}}^{<1}(M) \cong \tilde{\mathcal{M}}^{<1},
\]

where \( K \) is the full subcategory of \( S(M) \) generated by all the isomorphisms in \( S(M) \).

**Proof.** It follows from the definition of \( \Upsilon \) that its kernel objects are isomorphisms in \( S(M) \). Hence, the first equivalence follows from Subsection 2.4. The second one is just Lemma 3.1. \( \square \)

### 3.2 The functor \( i_\lambda : \text{mod-}M \rightarrow \text{mod-}M \)

Recollements for triangulated categories appeared first in [8] to study the construction of the category of perverse sheaves on a singular space. For a good account for studying recollements in abelian categories, see [25].

Let \( \Lambda \) be an Artin algebra and \( \mathcal{X} \) be a contravariantly finite subcategory of \( \text{mod-}\Lambda \) containing \( \text{prj-}\Lambda \), the full subcategory of \( \text{mod-}\Lambda \) consisting of all the projective \( \Lambda \)-modules. By [1, Theorem 3.5], we have a recollement

\[
\begin{array}{c}
\text{mod}_0 \mathcal{X} \\
\xrightarrow{i_\lambda} \xrightarrow{i} \text{mod-}\mathcal{X} \\
\xrightarrow{i'} \text{mod-}\mathcal{X} \\
\xrightarrow{\vartheta} \text{mod-}\Lambda
\end{array}
\]

in which \( \text{mod}_0 \mathcal{X} := \ker \vartheta \) is the full subcategory of \( \text{mod-}\mathcal{X} \) consisting of all the functors \( F \) such that \( \vartheta(F) = 0 \), equivalently, \( \text{mod}_0 \mathcal{X} \) consists of all the functors \( F \in \text{mod-}\mathcal{X} \) that vanish on \( \text{prj-}\Lambda \).

On the other hand, the canonical functor \( \mathcal{X} \rightarrow \mathcal{X}' \) induces the functor \( \tilde{\varphi} : \text{mod-}\mathcal{X} \rightarrow \text{mod-}\mathcal{X}' \). By [1, Proposition 4.1], the restriction of this functor to \( \text{mod}_0 \mathcal{X} \) induces an equivalence of categories \( \text{mod}_0 \mathcal{X} \cong \text{mod}_0 \mathcal{X}' \). Moreover, \( \text{mod}_0 \mathcal{X} \) is an abelian category. Throughout we treat \( \text{mod}_0 \mathcal{X} \) in this way. The above recollement will be denoted by \( \mathcal{R}(\mathcal{X}, \Lambda) \).

Our aim in this subsection is to study the functor \( i_\lambda \) in the above recollement for the case where \( \mathcal{X} = M \) is an \( n \)-cluster tilting subcategory of \( \text{mod-}\Lambda \). To this end, for every \( F \in \text{mod-}M \), we consider the exact sequence

\[
0 \rightarrow \ker(\eta_F) \rightarrow \vartheta_\lambda \vartheta(F) \xrightarrow{\eta_F} F \rightarrow ii_\lambda(F) \rightarrow 0,
\]
where \( \eta_F \) denotes the counit of adjunction (see [26, Proposition 2.8]). So, in order to know \( i_\lambda(F) \), it is enough to know \( \eta_F \), and for that we need to recall the definitions of the functors \( \vartheta \) and \( \vartheta_\lambda \).

Let \( F \in \text{mod-}M \) and \( M(-, M^0) \to M(-, M^1) \to F \to 0 \) be a minimal projective presentation of \( F \). Then by [1, p.333], \( \vartheta(F) \) is defined as the cokernel of the morphism \( M^0 \to M^1 \) in \( \text{mod-}\Lambda \). Let \( \varphi : F \to F' \) be a morphism in \( \text{mod-}M \). Then it can be lifted to the projective presentations of \( F \) and \( F' \) and hence, by applying Yoneda’s lemma, we obtain a morphism \( \vartheta(\varphi) : \vartheta(F) \to \vartheta(F') \).

Now let \( M \in \text{mod-}\Lambda \) be an arbitrary module. Let \( P \to Q \to M \to 0 \) be a minimal projective presentation of \( M \) in \( \text{mod-}\Lambda \). By [1, p.334], we set

\[
\vartheta_\lambda(M) := \text{Coker}(M(-, P) \to M(-, Q)).
\]

\( \vartheta_\lambda \) on morphisms is defined in an obvious way.

Finally, the counit \( \eta_F : \vartheta_\lambda \vartheta(F) \to F \) is defined as follows. Let \( P \to Q \to \vartheta(F) \to 0 \) be a minimal projective presentation of \( \vartheta(F) \) in \( \text{mod-}\Lambda \). So we have the commutative diagram

\[
\begin{array}{ccc}
P & \longrightarrow & Q & \longrightarrow \vartheta(F) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
M^0 & \longrightarrow & M^1 & \longrightarrow \vartheta(F) & \longrightarrow 0 \\
\end{array}
\]

in \( \text{mod-}\Lambda \). Applying the Yoneda functor to the left square of this diagram leads to the following commutative diagram in \( \text{mod-}M \):

\[
\begin{array}{ccc}
M(-, P) & \longrightarrow & M(-, Q) & \longrightarrow \vartheta_\lambda(\vartheta(F)) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \eta_F \\
M(-, M^0) & \longrightarrow & M(-, M^1) & \longrightarrow F & \longrightarrow 0.
\end{array}
\]

(3.3)

So we obtain the counit. Now \( i_\lambda(F) \) is the cokernel of \( \eta_F \).

**Remark 3.4.** Let \( F \in \text{mod-}M \) be such that the module \( M^1 \) in its projective presentation \( M(-, M^0) \to M(-, M^1) \to F \to 0 \) is projective. Then it follows from the construction of the functor \( i_\lambda \) presented above that \( i_\lambda(F) = 0 \). To see this, one just should note that in this case, the diagram (3.3) will be as follows:

\[
\begin{array}{ccc}
M(-, P) & \longrightarrow & M(-, Q) & \longrightarrow \vartheta_\lambda(\vartheta(F)) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \eta_F \\
M(-, M^0) & \longrightarrow & M(-, Q) & \longrightarrow F & \longrightarrow 0,
\end{array}
\]

and hence the cokernel of \( \eta_F \) vanishes.

### 3.3 The functor \( \Phi : S(M) \to \text{mod-}M \)

Now we have the necessary ingredients to introduce the functor \( \Phi \). In this subsection, we assume that \( \Lambda \) is a self-injective Artin algebra.

Using the notations of the previous subsections, consider the composition

\[
S(M) \xrightarrow{\Upsilon} \widehat{\mathcal{P}}^{<1}(M) \xrightarrow{i_\lambda} \text{mod-}M,
\]

and set \( \Phi := i_\lambda \circ \Upsilon \).

**Theorem 3.5.** The functor \( \Phi : S(M) \to \text{mod-}M \) is full, dense and objective.

**Proof.** We start the proof by showing that \( \Phi \) is full. Since we have already seen that \( \Upsilon \) is full, we just need to show that \( i_\lambda \) is full. Let \( F \) and \( G \) be functors in \( \widehat{\mathcal{P}}^{<1}(M) \) and \( \gamma : i_\lambda(F) \to i_\lambda(G) \) be a morphism
in mod-$\mathcal{M}$. Consider the diagram

$$
\begin{array}{cccc}
\vartheta_\lambda \vartheta(F) & \xrightarrow{\eta_F} & F & \xrightarrow{i_\lambda(F)} 0 \\
\downarrow \delta & & \downarrow \gamma & \\
\vartheta_\lambda \vartheta(G) & \xrightarrow{\eta_G} & G & \xrightarrow{i_\lambda(G)} 0
\end{array}
$$

with exact rows. Let $K$ be the kernel of $\xi$. We show that $\text{Ext}^1_{\mathcal{M}}(F, K) = 0$. This implies the fullness of $\iota_{\lambda}[\cdot]$, as in this case we deduce that there exists a morphism $\delta : F \to G$ such that the right square of the above diagram is commutative, i.e., $\iota_{\lambda}[\delta] = \gamma$. To show the vanishing of $\text{Ext}^1_{\mathcal{M}}(F, K)$, we apply the known isomorphism

$$
\text{Ext}^1_{\mathcal{M}}(F, K) \cong \text{Hom}_K(P_F, P_K[1]),
$$

where $K$ denotes the homotopy category of complexes of functors of mod-$\mathcal{M}$ and $P_F$ and $P_K$ denote the deleted projective resolutions of $F$ and $K$, respectively. Since $F \in \mathcal{P} \subseteq \mathcal{M}$, a projective resolution of it is of the form

$$
0 \longrightarrow \mathcal{M}(-, M^0) \xrightarrow{M(-, f)} \mathcal{M}(-, M^1) \longrightarrow F \longrightarrow 0,
$$

where $f : M^0 \to M^1$ is a monomorphism. Moreover, by the construction of $\vartheta_\lambda$, there is a projective presentation $\mathcal{M}(-, P) \to \mathcal{M}(-, Q) \to \vartheta_\lambda \vartheta(G) \to 0$ of $\vartheta_\lambda \vartheta(G)$ such that $P, Q \in \text{prj}-\Lambda$. Because of the epimorphism $\vartheta_\lambda \vartheta(G) \to K \to 0$, we can choose a deleted projective resolution of $K$:

$$
P_K : \cdots \to \mathcal{M}(-, N) \to \mathcal{M}(-, M) \to \mathcal{M}(-, Q) \to 0,
$$

where $\mathcal{M}(-, Q)$ is its zero's term. Now consider a chain map

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{M}(-, M^0) & \longrightarrow & \mathcal{M}(-, M^1) & \longrightarrow & 0 \\
\downarrow \cdot \cdot \cdot & & \downarrow \cdot \cdot \cdot & & \downarrow \mathcal{M}(-, g) & & \downarrow \cdot \cdot \cdot & & \downarrow \cdot \cdot \cdot & & \downarrow \cdot \cdot \cdot \\
\cdots & \longrightarrow & \mathcal{M}(-, N) & \longrightarrow & \mathcal{M}(-, M) & \longrightarrow & \mathcal{M}(-, Q) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

from $P_F$ to $P_K[1]$. Since $Q$ is a projective module and $\Lambda$ is a self-injective algebra, $Q$ is injective and hence, in view of Yoneda’s lemma, the morphism $g : M^0 \to Q$ can be extended to a morphism $h : M^1 \to Q$. Another use of Yoneda’s lemma implies the extension of the morphism $\mathcal{M}(-, g)$ to a morphism from $\mathcal{M}(-, M^1)$ to $\mathcal{M}(-, Q)$. This in turn implies that the chain map is null-homotopic. Hence $\text{Hom}_K(P_F, P_K[1]) = 0$, so we obtain the result.

Now we show that $\Phi$ is dense. Pick $F \in \text{mod}-\mathcal{M}$. Hence $F \in \text{mod}-\mathcal{M}$ and $F|_{\text{prj}-\Lambda} = 0$. Consider a projective presentation

$$
\mathcal{M}(-, M^0) \xrightarrow{M(-, f)} \mathcal{M}(-, M^1) \longrightarrow F \longrightarrow 0
$$

of $F$. Let $i : M^0 \to I$ be the injective envelop of $M^0$. Consider the object $[f i] : M^0 \to M^1 \oplus I$ in $\mathcal{S}(M)$. We claim that $\Phi([f i]) = F$. To see this, set

$$
G := \Upsilon([f i]) = \text{Coker}(\mathcal{M}(-, M^0) \longrightarrow \mathcal{M}(-, M^1 \oplus I)).
$$

Hence $\vartheta(\Upsilon([f i])) = \text{Coker}(M^0 \to M^1 \oplus I)$. Let $Q \to M^1$ be a projective cover of $M^1$. Hence we obtain an exact sequence

$$
P \longrightarrow Q \oplus I \longrightarrow \vartheta(G) \longrightarrow 0
$$

for some projective module $P$. Therefore, we obtain the following commutative diagram:

$$
\begin{array}{cccccccc}
\mathcal{M}(-, P) & \longrightarrow & \mathcal{M}(-, Q \oplus I) & \longrightarrow & \vartheta_\lambda \vartheta(G) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}(-, M^0) & \longrightarrow & \mathcal{M}(-, M^1 \oplus I) & \longrightarrow & G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{M}(-, M^0) & \longrightarrow & \mathcal{M}(-, M^1) & \longrightarrow & F & \longrightarrow & 0.
\end{array}
$$
Hence $\Phi([f, i]) = \text{Coker}(\vartheta_\lambda \vartheta(G) \to G) = F$.

Finally, we show that $\Phi$ is objective. Since by Proposition 3.2, $\Upsilon$ is full, dense and objective, by the last paragraph of Subsection 2.4, we just need to show that the restricted functor $i_\lambda|_{\widehat{P} \leq 1(M)}$ is objective. To this end, let $\delta : F \to G$ be a morphism in $\widehat{P} \leq 1(M)$ such that $i_\lambda|_{\widehat{P} \leq 1(M)}(\delta) = 0$. Hence we have the commutative diagram

$$
\begin{array}{c}
\vartheta_\lambda \vartheta(F) \to F \xrightarrow{i} i_\lambda(F) \to 0 \\
\vartheta_\lambda \vartheta(G) \to G \xrightarrow{i} i_\lambda(G) \to 0.
\end{array}
$$

Since $\xi \delta = 0$, $\delta$ factors through $K = \text{Ker} \xi$, say $\delta = \beta \alpha$. As we saw in the proof of the fullness of $i_\lambda$, at the first part of this proof, we may choose a projective presentation

$$
\mathcal{M}(-, M) \xrightarrow{(-, h)} \mathcal{M}(-, Q) \to K \to 0
$$

of $K$ such that $Q$ is a projective $\Lambda$-module. So we can complete the following diagram by lifting $\beta$ and $\alpha$ to the corresponding projective resolutions:

$$
\begin{array}{c}
0 \to \mathcal{M}(-, M^0) \xrightarrow{(-, f)} \mathcal{M}(-, M^1) \to F \to 0 \\
\downarrow (-, \alpha^0) \quad \downarrow (-, \alpha^1) \\
\mathcal{M}(-, M) \xrightarrow{(-, h)} \mathcal{M}(-, Q) \to K \to 0 \\
\downarrow (-, \beta^0) \quad \downarrow (-, \beta^1) \\
0 \to \mathcal{M}(-, N^0) \xrightarrow{(-, g)} \mathcal{M}(-, N^1) \to G \to 0.
\end{array}
$$

By applying Yoneda’s lemma, we obtain the following commutative diagram:

$$
\begin{array}{c}
0 \to M^0 \xrightarrow{f} M^1 \\
\downarrow \alpha^0 \quad \downarrow \alpha^1 \\
M \xrightarrow{h} Q \\
\downarrow \beta^0 \quad \downarrow \beta^1 \\
0 \to N^0 \xrightarrow{g} N^1.
\end{array} \quad (3.4)
$$

Let $\ell : M \to I$ be the injective envelope of $M$. The injectivity of $I$ implies the existence of a morphism $\zeta : M^1 \to I$ such that $\zeta f = t \alpha^1$. Using this map, in view of the diagram (3.4), we obtain the diagram

$$
\begin{array}{c}
0 \to M^0 \xrightarrow{f} M^1 \\
\downarrow \alpha^0 \quad \downarrow [\alpha^1, \zeta]^t \\
M \xrightarrow{h, \ell^t} Q \oplus I \\
\downarrow \beta^0 \quad \downarrow [\beta^1, 0] \\
0 \to N^0 \xrightarrow{g} N^1.
\end{array} \quad (3.5)
$$

Note that $[\beta^1, 0] \circ [\alpha^1, \zeta]^t = [\beta^1 \circ \alpha^1]$. By applying Yoneda's lemma to the diagram (3.5) and taking cokernels, we obtain a factorization of $\delta$ as $F \to K' \to G$, where $K'$ fits to the exact sequence

$$
0 \to \mathcal{M}(-, M) \xrightarrow{(-, h, \ell^t)} \mathcal{M}(-, Q \oplus I) \to K' \to 0.
$$

Hence $K' \in \widehat{P} \leq 1(M)$ and Remark 3.4 is a kernel object of $i_\lambda$, i.e., $i_\lambda(K') = 0$. The proof is therefore completed. □
Corollary 3.6. With the above notations, there exists an equivalence of abelian categories
\[ S(\mathcal{M})/\mathcal{U} \simeq \text{mod-}\mathcal{M}, \]
where \( \mathcal{U} \) is the subcategory of \( S(\mathcal{M}) \) generated by the objects of the form \( (M \xrightarrow{\eta} M) \) and \( (M \xrightarrow{f} P) \), where \( M \in \mathcal{M} \) and \( P \in \text{prj-}\Lambda \).

Proof. By the above theorem, \( \Phi \) is full, dense and objective. Hence we just should note that the kernel objects of \( \Phi \) are exactly those in the additive closure of a subcategory generated by all the monomorphisms as in the statement. This follows easily from the definition of \( \Phi \). So we are done. \( \Box \)

4 The functor \( \Psi : S(\mathcal{M}) \to \text{mod-}\mathcal{M} \)

In this section, we introduce the second functor on \( S(\mathcal{M}) \). Throughout assume that \( \Lambda \) is an arbitrary Artin algebra and \( \mathcal{M} \) is an \( n \)-cluster tilting subcategory of \( \text{mod-}\Lambda \).

Let \( (M^0 \xrightarrow{f} M^1) \) be an object of \( S(\mathcal{M}) \). Since \( \mathcal{M} \) is an \( n \)-cluster tilting subcategory, we may take an \( n \)-cokernel of \( f \) which results to an \( n \)-exact sequence
\[ 0 \to M^0 \xrightarrow{f} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \to \cdots \xrightarrow{d^n} M^{n+1} \to 0. \]

Hence the following induced sequence:
\[ 0 \to \mathcal{M}(-, M^0) \to \mathcal{M}(-, M^1) \to \mathcal{M}(-, M^2) \to \cdots \to \mathcal{M}(-, M^{n+1}) \to F \to 0 \]
is exact, where \( F \) is the cokernel of the morphism \( \mathcal{M}(-, M^n) \to \mathcal{M}(-, M^{n+1}) \). Clearly, \( F \) vanishes on projective modules and so \( F \in \text{mod-}\mathcal{M} \). We define a functor
\[ \Psi : S(\mathcal{M}) \to \text{mod-}\mathcal{M} \]
by setting \( \Psi(M^0 \xrightarrow{f} M^1) = F \). First of all, since every two \( n \)-cokernels of \( f \) are homotopy equivalent, we deduce that the definition of \( \Psi \) is independent of the choice of the \( n \)-cokernel of \( F \). Now let \( (M^0 \xrightarrow{f'} M^1) \) and \( (M^0 \xrightarrow{f} M^1) \) be two objects of \( S(\mathcal{M}) \) and consider a morphism
\[ f \xrightarrow{\alpha^0} f'. \]

By the property of \( n \)-exact sequences, we deduce that \((\alpha^0, \alpha^1)\) lifts to the following morphism of \( n \)-exact sequences:
\[ \begin{array}{cccccccccc}
0 & \to & M^0 & \xrightarrow{f} & M^1 & \xrightarrow{d^1} & M^2 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^n} & M^{n+1} & \to & 0 \\
\downarrow{\alpha^0} & & \downarrow{\alpha^1} & & \downarrow{\alpha^2} & & \downarrow{\alpha^n} & & \downarrow{\alpha^{n+1}} & & \downarrow{\alpha^{n+1}} & \\
0 & \to & M'^0 & \xrightarrow{f'} & M'^1 & \xrightarrow{d'^1} & M'^2 & \xrightarrow{d'^2} & \cdots & \xrightarrow{d'^n} & M'^{n+1} & \to & 0.
\end{array} \tag{4.1} \]

Yoneda’s lemma now comes to play to induce the commutative diagram
\[ \begin{array}{cccccccccc}
0 & \to & \mathcal{M}(-, M^0) & \to & \mathcal{M}(-, M^1) & \to & \cdots & \to & \mathcal{M}(-, M^{n+1}) & \to & F & \to & 0 \\
\downarrow{\mathcal{M}(-, \alpha^0)} & & \downarrow{\mathcal{M}(-, \alpha^1)} & & \downarrow{\mathcal{M}(-, \alpha^n)} & & \downarrow{\mathcal{M}(-, \alpha^{n+1})} & & \downarrow{\eta} & & \downarrow{\eta'} & \\
0 & \to & \mathcal{M}(-, M'^0) & \to & \mathcal{M}(-, M'^1) & \to & \cdots & \to & \mathcal{M}(-, M'^{n+1}) & \to & F' & \to & 0.
\end{array} \]

We set \( \Psi(\alpha^0, \alpha^1) = \eta \). The comparison lemma [18, Lemma 2.1] implies that \( \eta \) is independent of the lifting morphism \( \{\alpha^i\}_{2 \leq i \leq n+1} \).

Theorem 4.1. The functor \( \Psi : S(\mathcal{M}) \to \text{mod-}\mathcal{M} \) is full, dense and objective.
Proof. Let \((M^0 \xrightarrow{f} M^1)\) and \((M'^0 \xrightarrow{f'} M'^1)\) be two objects of \(\mathcal{S}(\mathcal{M})\) with \(\Psi(f) = F\) and \(\Psi(f') = F'\). Let \(\eta : F \to F'\) be a morphism in \(\text{mod-\mathcal{M}}\). So we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}(-, M^0) & \longrightarrow & \mathcal{M}(-, M^1) & \longrightarrow & \mathcal{M}(-, M^2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{M}(-, M^{n+1}) & \longrightarrow & F & \longrightarrow & 0 \\
\alpha^0 & & \downarrow \mathcal{M}(-, \alpha^0) & & \downarrow \mathcal{M}(-, \alpha^1) & & \downarrow \mathcal{M}(-, \alpha^2) & & \cdots & & \downarrow \mathcal{M}(-, \alpha^{n+1}) & & \eta & \\
0 & \longrightarrow & \mathcal{M}(-, M'^0) & \longrightarrow & \mathcal{M}(-, M'^1) & \longrightarrow & \mathcal{M}(-, M'^2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{M}(-, M'^{n+1}) & \longrightarrow & F' & \longrightarrow & 0.
\end{array}
\] (4.2)

Since rows are projective resolutions, \(\eta\) lifts to a morphism of the resolutions and hence, by Yoneda’s lemma, we conclude that \(\Psi\) is full.

To see that \(\Psi\) is dense, pick \(F \in \text{mod-\mathcal{M}}\). So there exists an exact sequence

\[\mathcal{M}(-, M^n) \to \mathcal{M}(-, M^{n+1}) \to F \to 0\]

such that \(M^n \to M^{n+1}\) is an epimorphism. By taking an \(n\)-kernel of this morphism, we obtain a monomorphism \(M^0 \xrightarrow{f} M^1\) with \(\Psi(f) = F\).

So it remains to prove that \(\Psi\) is objective. Let

\[f \xrightarrow{\alpha^0, \alpha^1} f'\]

be a morphism of \((M^0 \xrightarrow{f} M^1)\) to \((M'^0 \xrightarrow{f'} M'^1)\) in \(\mathcal{S}(\mathcal{M})\) such that \(\eta = \Psi(\alpha^0, \alpha^1) = 0\). Then the lifting of \(\eta\) as in the diagram (4.2) above is null-homotopic. In particular, by applying Yoneda’s lemma, we obtain the morphisms \(s^0 : M^1 \to M^0\) and \(s^1 : M^2 \to M^1\), i.e.,

\[
\begin{array}{ccc}
0 & \longrightarrow & M^0 \\
& \downarrow \alpha^0 & \downarrow \sigma^0 \\
0 & \longrightarrow & M^1 \\
& \downarrow \alpha^1 & \downarrow \sigma^1 \\
0 & \longrightarrow & M^2
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & M^0 \\
& \downarrow \alpha^0 & \downarrow [s^0, s^1, d^1] \\
0 & \longrightarrow & M^0 \oplus M^1 \\
& \downarrow 1 & \downarrow [f', 1] \\
0 & \longrightarrow & M^1
\end{array}
\]

such that \(\alpha^0 = s^0f\) and \(\alpha^1 = f's^0 + s^1d^1\). Therefore, \((\alpha^0, \alpha^1)\) factors through

\[\begin{array}{ccc}
(M^0 \xrightarrow{(1, 0)} M^0 \oplus M^1)
\end{array}\]

via the following maps:

\[
\begin{array}{ccc}
0 & \longrightarrow & M^0 \\
& \downarrow \alpha^0 & \downarrow [s^0, s^1, d^1] \\
0 & \longrightarrow & M^0 \oplus M^1 \\
& \downarrow 1 & \downarrow [f', 1] \\
0 & \longrightarrow & M^1
\end{array}
\]

Since the morphism \((1, 0)\) in the middle row is a split monomorphism, by [18, Proposition 2.6], the middle object is a kernel object. The proof is hence completed.

Corollary 4.2. With the above notations, there exists an equivalence of additive categories

\[\mathcal{S}(\mathcal{M})/\mathcal{V} \simeq \text{mod-\mathcal{M}},\]

where \(\mathcal{V}\) is the full subcategory of \(\mathcal{S}(\mathcal{M})\) generated by all the finite direct sums of objects of the form \((M \xrightarrow{i} M)\) and \((0 \to M)\), where \(M\) runs over objects of \(\mathcal{M}\).
Proof. In view of Subsection 2.4 and the above theorem, we just should show that $\mathcal{V}$ is generated by the kernel objects of $\Psi$. Let $M^0 \xrightarrow{f} M^1$ be a kernel object of $\Psi$, i.e., $\Psi(f) = 0$. We show that $f$ is a split monomorphism. By definition, $f$ extends to an $n$-exact sequence
\[ 0 \rightarrow M^0 \xrightarrow{f} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \rightarrow \cdots \xrightarrow{d^n} M^n \xrightarrow{d^{n+1}} M^{n+1} \rightarrow 0, \tag{4.3} \]
which in turn induces the exact sequence
\[ 0 \rightarrow \mathcal{M}(-, M^0) \rightarrow \mathcal{M}(-, M^1) \rightarrow \mathcal{M}(-, M^2) \rightarrow \cdots \rightarrow \mathcal{M}(-, M^{n+1}) \rightarrow 0. \tag{4.4} \]
For $i \in \{2, \ldots, n\}$, set $K^i = \text{Ker}(M^i \rightarrow M^{i+1})$. So we have the short exact sequences
\[ \varepsilon^i : 0 \rightarrow K^i \rightarrow M^i \rightarrow K^{i+1} \rightarrow 0, \quad i \in \{2, \ldots, n\} \]
and $0 \rightarrow M^0 \rightarrow M^1 \rightarrow K^2 \rightarrow 0$, where $K^{n+1} := M^{n+1}$. Apply the exact sequence (4.4) on $K^{n+1} = M^{n+1}$ implies that $\varepsilon^n$ splits that in turn implies that $K^n$ as a summand of $M^n$ belongs to $\mathcal{M}$. Therefore, we obtain the exact sequence
\[ 0 \rightarrow \mathcal{M}(-, M^0) \rightarrow \mathcal{M}(-, M^1) \rightarrow \cdots \rightarrow \mathcal{M}(-, M^{n-1}) \rightarrow \mathcal{M}(-, K^n) \rightarrow 0, \]
which is exact on $\mathcal{M}$. Apply this later sequence to $K^n$ and follow the argument step by step to obtain the exact sequence
\[ 0 \rightarrow \mathcal{M}(-, M^0) \rightarrow \mathcal{M}(-, M^1) \rightarrow \mathcal{M}(-, K^2) \rightarrow 0 \]
with $K^2 \in \mathcal{M}$ and which is exact on $\mathcal{M}$. Now apply this last sequence on $K^2$ to deduce that the short exact sequence $0 \rightarrow M^0 \xrightarrow{f} M^1 \rightarrow K^2 \rightarrow 0$ splits. Hence $f$ is a split monomorphism, as it was claimed. \hfill $\square$

Here, we provide two interesting applications of the equivalence of Corollary 4.2. Recall that an additive category $\mathcal{X}$ is called of finite type if the set of all iso-classes of indecomposable objects of $\mathcal{X}$ is finite. If mod-$\Lambda$ is of finite type, where $\Lambda$ is an Artin algebra, then $\Lambda$ is called of finite representation type.

Let $n \geq 2$. By [9, Lemma 2.3], if $\mathcal{M}$ is an $n$-cluster tilting subcategory of an exact Krull-Schmidt, Frobenius $k$-category $\mathcal{F}$, then mod-$\mathcal{M}$ is of finite type if so is $\mathcal{F}$. Our first application deals with the finiteness type of mod-$\mathcal{M}$. Note that here we do not need $\Lambda$ to be self-injective.

**Proposition 4.3.** Let $n \geq 2$. Let $M \in \text{mod-}\Lambda$ be an $n$-cluster tilting module. Then $\Gamma = \text{End}_{\Lambda}(M)$ is of finite representation type provided so is $\Lambda$.

**Proof.** Set $\mathcal{M} = \text{add-}M$. Then mod-$\mathcal{M} \simeq \text{mod-End}_{\Lambda}(M)$. We show that mod-$\mathcal{M}$ is of finite type. By Corollary 4.2, we observe that mod-$\mathcal{M}$ is of finite representation type if and only if so is $S(\mathcal{M})$.

Now since $\Lambda$ is of finite representation type, obviously every subcategory of mod-$\Lambda$ is of finite type. In particular, $\mathcal{M}^{\leq 1}$ is of finite type. Therefore we deduce from Corollary 3.3 that $S(\mathcal{M})$ is of finite representation type. Hence we obtain the result. \hfill $\square$

It is known that mod-$\Lambda$ can be considered as an exact category whose conflations are all the $\mathcal{M}$-proper short exact sequences, i.e., short exact sequences of $\Lambda$-modules that are exact under the functor $\text{Hom}_{\Lambda}(\cdot, -)$ for all $M \in \mathcal{M}$. Note that the class of all such proper extensions corresponds to a sub-bifunctor $F_{\mathcal{M}}$ of the bifunctor $\text{Ext}^1_{\Lambda}(\cdot, -)$ (see [7]). Let $\mathcal{P}(F_{\mathcal{M}})$ be the subcategory of all the relative projective modules in mod-$\mathcal{M}$ with respect to this exact structure. Moreover, we denote by $\mathcal{P}(\mathcal{M})$ the full subcategory of mod-$\mathcal{M}$ consisting of all the projective objects.

**Proposition 4.4.** Let $n \geq 2$. Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of mod-$\Lambda$. Then there exist equivalences of categories
\[ \text{mod-}\mathcal{M} \simeq \overline{\mathcal{P}}^{\leq 1}(\mathcal{M}) \simeq \overline{\mathcal{M}}^{\leq 1}_{F_{\mathcal{M}}}, \]
where $\overline{\mathcal{P}}^{\leq 1}(\mathcal{M})$ is the stable category of the subcategory $\overline{\mathcal{P}}^{\leq 1}(\mathcal{M})$ of mod-$\mathcal{M}$ in the usual sense, and $\overline{\mathcal{M}}^{\leq 1}_{F_{\mathcal{M}}}$ is the stable category of the subcategory $\overline{\mathcal{M}}^{\leq 1}$ of mod-$\Lambda$ with respect to the exact structure induced by the sub-bifunctor $F_{\mathcal{M}}$ of $\text{Ext}^1_{\Lambda}(\cdot, -)$. 

Proof. By Proposition 3.2, the functor $\Upsilon : S(\mathcal{M}) \to \tilde{\mathcal{P}}^{\leq 1}(\mathcal{M})$ is full, dense and objective. Moreover, it sends objects of $\mathcal{V}$ to projective objects of mod-$\mathcal{M}$. Hence we induce an equivalence

$$S(\mathcal{M})/\mathcal{V} \simeq \tilde{\mathcal{P}}^{\leq 1}(\mathcal{M}).$$

So the first equivalence follows from Corollary 4.2. The second equivalence follows from Lemma 3.1 in view of the fact that the functor $\Upsilon_{\mathcal{M}}$ also provides an equivalence between $\mathcal{P}(\mathcal{M})$ and $\mathcal{P}(\mathcal{F}_{\mathcal{M}})$.

We have the following immediate corollary for $n = 2$.

**Corollary 4.5.** Let $\mathcal{M}$ be a 2-cluster tilting subcategory of mod-$\Lambda$. Then there exist the following equivalences of categories:

$$\text{mod-}\mathcal{M} \simeq \tilde{\mathcal{P}}^{\leq 1}(\mathcal{M}) \simeq \text{mod-}\Lambda_{F_{\mathcal{M}}}.$$

**Proof.** Let $X \in \text{mod-}\Lambda$ and consider a right $\mathcal{M}$-approximation $M \to X$ of $X$. Since $\mathcal{M}$ is a 2-cluster tilting subcategory of mod-$\Lambda$, $\text{Ext}^1_\mathcal{M}(M, \mathcal{M}) = 0$. This implies that $\text{Ext}^1_\mathcal{M}(M, \text{Ker} \alpha) = 0$. Hence $\text{Ker} \alpha \in \mathcal{M}$ and so $X \in \tilde{\mathcal{M}}^{\leq 1}$. Therefore mod-$\Lambda = \tilde{\mathcal{M}}^{\leq 1}$. Now the result follows from the above proposition.

## 5 Comparison

In this section, we compare the functors $\Phi$ and $\Psi$. Such comparison is inspired by [28, Theorem 2] and [11, Theorem 4.2]. Throughout assume that $\Lambda$ is a self-injective Artin algebra.

Let $\Omega_{\mathcal{M}} : \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}$ be the syzygy functor. Note that since mod-$\mathcal{M}$ is semi-perfect, we can assume that $\Omega_{\mathcal{M}}(F)$ is the kernel of a projective cover of $F$ in mod-$\mathcal{M}$. Let $\mathcal{W}$ be the smallest additive subcategory of $S(\mathcal{M})$ generated by all the objects of the form $(M \to M)$, $(0 \to M)$ and $(M \to P)$ with $M \in \mathcal{M}$ and $P \in \text{prj-}\Lambda$.

### 5.1 The functor $\Phi : S(\mathcal{M})/\mathcal{W} \to \text{mod-}\mathcal{M}$

There exists an induced functor $S(\mathcal{M})/\mathcal{W} \to \text{mod-}\mathcal{M}$ that will be denoted by $\Phi$. To see this, we show that $\Phi$ maps all the generators of $\mathcal{W}$ to zero. First, note that by definition $\Phi(M \to P) = 0$, where $P$ is a projective module. Moreover, Remark 3.4 implies that $\Phi(M \to P) = 0$, where $P$ is a projective module. Finally, we show that $\Phi(0 \to M) = F_{\mathcal{M}}(-, M)$, $\Phi(0 \to M) = \mathcal{M}(-, M)$ and $\Phi(-, M) = M$. So if we let $Q \to P \to M \to 0$ be a projective presentation of $M$, the claim follows from the following commutative diagram:

\[ \begin{array}{ccccccccc}
\mathcal{M}(-, Q) & \longrightarrow & \mathcal{M}(-, P) & \longrightarrow & \partial \lambda \vartheta(-, M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}(-, M) & \longrightarrow & \mathcal{M}(-, M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}(-, M) & \stackrel{\simeq}{\longrightarrow} & i_{\lambda}(-, M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 
\end{array} \]

### 5.2 The functor $\Psi : S(\mathcal{M})/\mathcal{W} \to \text{mod-}\mathcal{M}$

The notion of $n\mathbb{Z}$-cluster tilting subcategories was introduced by Iyama and Jasso [17], as subcategories that are closed under $n$-syzygies and $n$-cosyzygies, so are “better behaved from the viewpoint of higher homological algebra”. By [17, Definition 2.22], an $n$-abelian category $\mathcal{M}$ has $n$-cosyzygies if for every $M \in \mathcal{M}$, there exists an $n$-exact sequence

$$0 \longrightarrow M \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow L \longrightarrow 0$$
such that $I^i$ is injective for $i \in \{1, \ldots, n\}$, and $L \in \mathcal{M}$. By abuse of notation, we say that $L$ is the $n$-cosyzygy of $M$ and denote it by $\Omega^n_M$. The notion of $n$-syzygies is defined dually. We denote the $n$-syzygy of $M$ by $\Omega^n_M$.

We say that an $n$-cluster tilting subcategory $\mathcal{M}$ of mod-$\Lambda$ is an $n\mathbb{Z}$-cluster tilting if it admits $n$-syzygies, i.e., $\Omega^n(\mathcal{M}) \subseteq \mathcal{M}$ or equivalently, if it admits $n$-cosyzygies, i.e., $\Omega^{-n}(\mathcal{M}) \subseteq \mathcal{M}$ (see [17, Definition-Proposition 2.15] for more equivalent statements).

Now assume that $\mathcal{M}$ is an $n\mathbb{Z}$-cluster tilting subcategory. Then by definition, $\Psi(0 \to M) = 0$, $\Psi(M \to M) = 0$, and by using the fact that $\Lambda$ is self-injective, $\Psi(M \to P) = \mathcal{M}(-, \Omega^n_M(M))$. Hence in this case we also have the induced functor $\Phi : \mathcal{S}(\mathcal{M})/\mathcal{W} \to \text{mod-}\mathcal{M}$.

Now we are in a position to prove the main result of this section.

**Theorem 5.1.** Let $\Lambda$ be a self-injective Artin algebra and $\mathcal{M}$ be an $n\mathbb{Z}$-cluster tilting subcategory of mod-$\Lambda$. Then with the above notations, we have

$$\Phi = \Omega^n_{\mathcal{M}} \circ \Psi,$$

i.e., the functors $\Phi$ and $\Psi$ differ by the $n$-syzygy functor on mod-$\mathcal{M}$.

**Proof.** For every $X, Y \in \text{mod-}\Lambda$ and each $i > 0$, there exists a functorial isomorphism

$$\text{Hom}_{\Lambda}(\Omega^i(X), Y) \simeq \text{Ext}_\Lambda^i(X, Y).$$

Using these isomorphisms, one can show that $\mathcal{M}$ is a $n$-cluster tilting subcategory of the triangulated category mod-$\Lambda$. On the other hand, since $\mathcal{M}$ is an $n\mathbb{Z}$-cluster tilting subcategory, it is closed under $n$-cosyzygies and hence by [12, Theorem 1], its stable category $\mathcal{M}$ is an $(n + 2)$-angulated category. Moreover, any $n$-exact sequence in $\mathcal{M}$ induces an $(n + 2)$-angle in $\mathcal{M}$. Let $(M^0 \xrightarrow{f} M^1)$ be an object of $\mathcal{S}(\mathcal{M})$. It completes to an $n$-exact sequence

$$0 \to M^0 \xrightarrow{f} M^1 \xrightarrow{d^1} M^2 \to \cdots \to M^n \xrightarrow{d^n} M^{n+1} \to 0.$$

By abuse of notation, we say that $\mathcal{M}$ is the $(n+2)$-angle. In the above $(n+2)$-angle, $M^0$ is the $(n+2)$-angle of $M^0$. For every $M^0 \xrightarrow{f} M^1$, the above $(n+2)$-angle induces a long exact sequence of functors in mod-$\mathcal{M}$:

$$\cdots \to (-, \Omega^n_M(M^{n+1})) \to (-, \Omega^n_M(M^0)) \to \cdots \to (-, \Omega^n_M(M^{n+1})) \to (-, \Omega^n_M(M^0)) \to \cdots.$$

Note that for brevity, in the above sequence we have used $(-, \Omega^n_M(M^{n+1}))$ instead of $\mathcal{M}(-, \Omega^n_M(M^{n+1}))$. Set $F = \Psi(M^0 \xrightarrow{f} M^1)$ and $G = \Phi(M^0 \xrightarrow{f} M^1)$. By definitions of $\Psi$ and $\Phi$, one can see that $F$ is just the cokernel of the morphism $\mathcal{M}(-, \Omega^n_M(M^0)) \to \mathcal{M}(-, \Omega^n_M(M^{n+1}))$ appeared in the above long exact sequence while $G$ is the cokernel of the morphism $\mathcal{M}(-, \Omega^n_M(M^0)) \xrightarrow{\mathcal{M}(\cdot, -f)} \mathcal{M}(-, M^1)$ appeared in the same exact sequence. Hence, in view of this long exact sequence, we obtain the exact sequence

$$0 \to G \to \mathcal{M}(-, M^2) \to \cdots \to \mathcal{M}(-, M^n) \to \mathcal{M}(-, M^{n+1}) \to F \to 0$$

in mod-$\mathcal{M}$ that completes the proof. \qed

### 6 Dual statements on $\mathcal{F}(\mathcal{M})$

In order to provide some applications, we need to have the dual of the results we have had so far. Since almost all of the proofs are dual, we just summarize the dual statements without the proof.
6.1 The functor $\Phi': \mathcal{F}(\mathcal{M}) \to \overline{\mathcal{M}}\text{-mod}$

To define $\Phi'$, similar to what we did in Section 3, we need to define the functors $\Upsilon'$ and $i'_\lambda$.

The functor $\Upsilon': \mathcal{F}(\mathcal{M}) \to \widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}})$.

The functor $Y'_M: \text{mod-}\Lambda \to \mathcal{M}\text{-mod}$ defined by $X \mapsto \text{Hom}_\Lambda(X, -)|_\mathcal{M}$ is full and faithful. In addition, the restricted functor $Y'_M: \overline{\mathcal{M}}^{\leq 1} \to \widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}})$ is a duality. Compare with Proposition 3.2 and Corollary 3.3.

**Proposition 6.1** (Compare with Proposition 3.2 and Corollary 3.3). The contravariant functor $\Upsilon': \mathcal{F}(\mathcal{M}) \to \widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}})$ defined by the composition

$$
\Upsilon': \mathcal{F}(\mathcal{M}) \xrightarrow{K} \overline{\mathcal{M}}^{\leq 1} \xrightarrow{Y'_M} \widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}}),
$$

which maps $(M^n \xrightarrow{f} M^{n+1})$ to $\text{Coker}(\mathcal{M}(M^{n+1}, -) \xrightarrow{(f, -)} \mathcal{M}(M^n, -))$ is full, dense and objective. In particular, there is an equivalence of additive categories

$$
\mathcal{F}(\mathcal{M})/\mathcal{K}' \simeq (\widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}}))^{\text{op}} \simeq \overline{\mathcal{M}}^{\leq 1},
$$

where $\mathcal{K}'$ is the subcategory of $\mathcal{F}(\mathcal{M})$ generated by all the isomorphisms in $\mathcal{F}(\mathcal{M})$.

The functor $i'_\lambda: \text{mod-}\Lambda \to \overline{\mathcal{M}}\text{-mod}$.

Let $\Lambda$ be an Artin algebra and $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\text{mod-}\Lambda$. By [1, Theorem 3.8], there exists a recollement

$$
\begin{array}{ccc}
\overline{\mathcal{M}}\text{-mod} & \xrightarrow{i'_\lambda} & \mathcal{M}\text{-mod} \\
\rotatebox[origin=c]{90}{$\hookleftarrow$} & & \rotatebox[origin=c]{90}{$\hookrightarrow$} \\
i'_\nu & \xrightarrow{\phi'_\lambda} & (\text{mod-}\Lambda)^{\text{op}}
\end{array}
$$

of abelian categories, where $\overline{\mathcal{M}}\text{-mod} = \text{Ker}\phi'$ is the full subcategory of $\mathcal{X}\text{-mod}$ consisting of all the functors that vanish on injective modules.

Therefore, similar to Subsection 3.2, one can define $i'_\lambda$ explicitly and then define the contravariant functor $\Phi'$ as the following composition:

$$
\Phi': \mathcal{F}(\mathcal{M}) \xrightarrow{\Upsilon'} \widehat{\mathcal{M}}^{\leq 1}(\mathcal{M}^{\text{op}}) \xrightarrow{i'_\lambda} \overline{\mathcal{M}}\text{-mod},
$$

i.e., $\Phi' := i'_\lambda| \circ \Upsilon'$.

**Theorem 6.2** (Compare with Theorem 3.5 and Corollary 3.6). Let $\Lambda$ be a self-injective Artin algebra. The contravariant functor $\Phi': \mathcal{F}(\mathcal{M}) \to \overline{\mathcal{M}}\text{-mod}$ is full, dense and objective. In particular, there exists an equivalence of abelian categories

$$
\mathcal{F}(\mathcal{M})/\mathcal{U}' \simeq (\overline{\mathcal{M}}\text{-mod})^{\text{op}},
$$

where $\mathcal{U}'$ is the subcategory of $\mathcal{F}(\mathcal{M})$ generated by all the objects of the form $(M \xrightarrow{f} M)$ and $(I \xrightarrow{f} M)$, where $M \in \mathcal{M}$ and $I \in \text{inj-}\Lambda$.

6.2 The functor $\Psi': \mathcal{F}(\mathcal{M}) \to \overline{\mathcal{M}}\text{-mod}$

In analog with the definition of the functor $\Psi$ in Section 4, we can define the contravariant functor $\Psi'$.

Let us review the definition briefly:

Pick an epimorphism $(M^n \xrightarrow{f} M^{n+1})$ of $\mathcal{F}(\mathcal{M})$. By taking $n$-kernel in $\mathcal{M}$, we obtain the $n$-exact sequence

$$
0 \longrightarrow M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} \cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{f} M^{n+1} \longrightarrow 0.
$$

The $n$-exactness induces the exact sequence

$$
0 \longrightarrow \mathcal{M}(M^{n+1}, -) \longrightarrow \mathcal{M}(M^n, -) \longrightarrow \cdots \longrightarrow \mathcal{M}(M^1, -) \longrightarrow \mathcal{M}(M^0, -) \longrightarrow F \longrightarrow 0.
$$

Define $\Psi'(M^n \xrightarrow{f} M^{n+1}) := F$. Since the restriction of $F$ on injective modules is zero, $F$ is indeed an object of $\overline{\mathcal{M}}\text{-mod}$. The action of $\Psi'$ on morphisms can be defined naturally.
Theorem 6.3 (Compare with Theorem 4.1 and Corollary 4.2). The contravariant functor $Ψ : F(M) → \mathcal{M}\text{-mod}$ is full, dense and objective. In particular, there exists an equivalence of abelian categories

$$F(M)/\mathcal{V} ≃ (\mathcal{M}\text{-mod})^\text{op},$$

where $\mathcal{V}$ is the full subcategory of $F(M)$ generated by all the finite direct sums of objects of the form $(M \rightarrow M)$ and $(M \rightarrow 0)$, where $M$ runs over objects of $\mathcal{M}$.

As a dual of Proposition 4.4, we have the following result.

Proposition 6.4. Let $n ≥ 2$. Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\text{mod-Λ}$. Then there exist equivalences of categories

$$\mathcal{M}\text{-mod} ≃ (\mathcal{P}^{≤1}(\mathcal{M}^{\text{op}}))^\text{op} ≃ \mathcal{M}^{<1}_{PM},$$

where $\mathcal{M}^{<1}_{PM}$ is the stable category of the subcategory $\mathcal{M}^{≤1}$ of $\text{mod-Λ}$ with respect to the exact structure induced by the sub-bifunctor $F^{\mathcal{M}}$ of $\text{Ext}^{1}_{\Lambda}(−, −)$.

6.3 Comparison

Finally, we compare the functors $Φ'$ and $Ψ'$.

Theorem 6.5 (Compare with Theorem 5.1). Let $Λ$ be a self-injective Artin algebra and $\mathcal{M}$ be an $n\mathbb{Z}$-cluster tilting subcategory of $\text{mod-Λ}$. Then with the above notations, we have

$$Φ' = Ω_{-n}^M ◦ Ψ',$$

i.e., the functors $Φ'$ and $Ψ'$ differ by the $n$-cosyzygy functor on $\text{mod-}\mathcal{M}$.

7 Applications

In this section, we present some applications of our results. Let us begin by recalling the definition of a functor has already been observed by Auslander [4]. Define a functor $Θ : F(M) → \text{mod-}\mathcal{M}$ by

$$Θ(M^{1} \rightarrow M^{2}) = \text{Coker}(\mathcal{M}(−, M^{1}) \rightarrow \mathcal{M}(−, f) \rightarrow \mathcal{M}(−, M^{2})).$$

Note that since $f$ is an epimorphism, the cokernel is an object of $\text{mod-}\mathcal{M}$.

The proof of the following theorem is similar to the proof of Theorem 4.1.

Theorem 7.1. The functor $Θ$ is full, dense and objective. In particular, it induces an equivalence of categories

$$F(M)/\mathcal{V} ≃ \text{mod-}\mathcal{M},$$

where $\mathcal{V}$ is the full subcategory of $F(M)$ generated by all the objects of the form $(M^{1} \rightarrow M)$ and $(M \rightarrow 0)$ and $M$ runs over all the objects of $\mathcal{M}$.

As a consequence, we have the following interesting result.

Corollary 7.2. There are equivalences

$$S(M)/\mathcal{V} ≃ \text{mod-}\mathcal{M} ≃ (\mathcal{M}\text{-mod})^\text{op} ≃ F(M)/\mathcal{V}$$

of abelian categories.

Proof. The proof follows immediately from Corollary 4.2 and Theorem 6.3 in conjunction with the above theorem.

Remark 7.3. We let

$$Σ = Ψ' ◦ Θ^{-1} : \text{mod-}\mathcal{M} → \mathcal{M}\text{-mod}$$

be the duality between $\text{mod-}\mathcal{M}$ and $\mathcal{M}\text{-mod}$, introduced in Corollary 7.2. Note that Auslander [4] has proved the existence of a duality between $\text{mod-}\mathcal{A}$ and $\mathcal{A}\text{-mod}$, where $\mathcal{A}$ is an abelian category. Hence, the duality $Σ$ of the above corollary can be thought of as a higher version of the Auslander’s duality.
We study the duality $\Sigma$ a little bit more. To this end, let us recall the notion of the defect of an $n$-exact sequence [19, Definition 3.1]. Let $\mathcal{M}$ be a subcategory of mod-$\Lambda$ and
\[
\delta : 0 \rightarrow M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \rightarrow 0
\]
be an $n$-exact sequence in $\mathcal{M}$. The contravariant defect of $\delta$ denoted by $\delta^*$ is defined by the exact sequence
\[
\text{Hom}_{\Lambda}(-, M^n) \rightarrow \text{Hom}_{\Lambda}(-, M^{n+1}) \rightarrow \delta^* \rightarrow 0
\]
of functors. Dually, the covariant defect of $\delta$ denoted by $\delta_*$ is defined by the following exact sequence of functors:
\[
\text{Hom}_{\Lambda}(M^1, -) \rightarrow \text{Hom}_{\Lambda}(M^0, -) \rightarrow \delta_* \rightarrow 0.
\]

We also need the following easy lemma.

**Lemma 7.4.** Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of mod-$\Lambda$. Let
\[
\delta : 0 \rightarrow M^0 \xrightarrow{d^0} M^1 \rightarrow \cdots \rightarrow M^n \xrightarrow{d^n} M^{n+1} \rightarrow 0
\]
be an $n$-exact sequence in $\mathcal{M}$. The following assertions hold:

(i) If $M^n$ belongs to $\text{prj-}\Lambda$, then $\delta^* \cong \text{M}(-, M^{n+1})$ and $\delta_* \cong \text{Ext}^n_{\Lambda}(M^{n+1}, -)|_{\mathcal{M}}$.

(ii) If $M^1$ belongs to $\text{inj-}\Lambda$, then $\delta^* \cong \text{Ext}^1_{\Lambda}(-, M^0)|_{\mathcal{M}}$ and $\delta_* \cong \text{M}(M^0, -)$.

**Proof.** We just prove the statement (i). The statement (ii) follows similarly. Assume that $M^n$ is a projective $\Lambda$-module. In this case, the contravariant defect follows by definition. For the covariant defect, set $K^i := \text{Ker}(d^i)$ for $i \in \{1, \ldots, n\}$. Since $M$ is an $n$-cluster tilting subcategory of mod-$\Lambda$, we have $\text{Ext}^i_{\Lambda}(M, \mathcal{M}) = 0$ for $i \in \{1, \ldots, n - 1\}$. This implies that $\delta_* \cong \text{Ext}^1_{\Lambda}(K^2, -)|_{\mathcal{M}}$ and then the dimension shifting argument in view of the short exact sequences
\[
0 \rightarrow K^i \rightarrow M^i \rightarrow K^{i+1} \rightarrow 0, \quad i \in \{2, \ldots, n\}
\]
applies to show that $\delta_* \cong \text{Ext}^n_{\Lambda}(M^{n+1}, -)|_{\mathcal{M}}$, where by convention $K^{n+1} = M^{n+1}$.

**Proposition 7.5.** Let $\Sigma : \text{mod-}\Lambda \rightarrow \text{mod}_{\Lambda}$ be the duality of Remark 7.3, where $\mathcal{M}$ is an $n$-cluster tilting subcategory of mod-$\Lambda$. Then for every $n$-exact sequence $\delta$ of $\mathcal{M}$, $\Sigma(\delta^*) = \delta_*$ and $\Sigma(\delta_*) = \delta^*$. In particular, we have the following statements:

(i) If $\Sigma|_\mathcal{M} : \text{prj-}\mathcal{M} \rightarrow \text{inj-}\mathcal{M}$ is defined by $\Sigma|_\mathcal{M}(\mathcal{M}(-, X)) = \text{Ext}^n_{\Lambda}(X, -)|_{\mathcal{M}}$ and is a duality.

(ii) $\Sigma^{-1}|_* : \text{prj-}\mathcal{M} \rightarrow \text{inj-}\mathcal{M}$ is defined by $\text{Ext}^n_{\Lambda}(-, X)|_{\mathcal{M}}$ and is a duality.

**Proof.** The facts that $\Sigma(\delta^*) = \delta_*$ and $\Sigma(\delta_*) = \delta^*$ follow directly by the definition of $\Sigma$. For the second part, we just prove the statement (i). The statement (ii) follows similarly. It is known that the projective functors of mod-$\mathcal{M}$ are just representable functors. Let $\mathcal{M}(-, X)$ be a projective object. Consider the projective cover $P \rightarrow X$ of $X$. By taking its $n$-kernel, we obtain an $n$-exact sequence. Now Lemma 7.4(i) in view of the first part of the proposition implies the result.

As an immediate consequence of the above proposition, we prove a higher version of the Hilton-Rees theorem for $n$-cluster tilting subcategories. For a recent account on the Hilton-Rees theorem and a “short and straightforward proof” of it, see [24, Section 4].

**Corollary 7.6** (Higher Hilton-Rees). Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of mod-$\Lambda$, and $X$ and $Y$ be in $\mathcal{M}$.

(i) There is an isomorphism between $\mathcal{M}(X, Y)$ and the group of the natural transformations from $\text{Ext}^n_{\Lambda}(X, -)|_{\mathcal{M}}$ to $\text{Ext}^n_{\Lambda}(Y, -)|_{\mathcal{M}}$.

(ii) There is an isomorphism between $\mathcal{M}(X, Y)$ and the group of the natural transformations from $\text{Ext}^n_{\Lambda}(-, X)|_{\mathcal{M}}$ to $\text{Ext}^n_{\Lambda}(-, Y)|_{\mathcal{M}}$.

**Proof.** The statement (i) follows from the duality $\Sigma|_\mathcal{M}$ of the statement (i) of the above proposition in view of Yoneda’s lemma. The statement (ii) follows similarly.
Remark 7.7. The above two results are known over an $n$-abelian category with enough projective and enough injective objects (see [23, Proposition 4.35 and Theorem 4.36]). So one can conclude that they also hold true for $n\mathbb{Z}$-cluster tilting subcategories. In fact, here we extend them to any $n$-cluster tilting subcategory of $\text{mod-}\Lambda$.

Our next aim is to state and prove a higher version of Auslander’s direct summand conjecture [2]. The conjecture says that for an object $A$ of an abelian category $\mathcal{A}$ with enough projective objects, any direct summand $F$ of $\text{Ext}^1_\mathcal{A}(A, -)$ is of the form $\text{Ext}^1_\mathcal{A}(B, -)$ for some $B$ in $\mathcal{A}$. For a review of the conjecture and the related results, see the introduction of [24]. We just mention that as Auslander proved [2, Proposition 4.3], if the above conjecture holds true, then functors of the form $\text{Ext}^1_\mathcal{A}(A, -)$ are the only injectives in $\text{mod-}\mathcal{A}$. A relative version of this conjecture is proved in [13, Theorem 3.10].

Corollary 7.8. Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\text{mod-}\Lambda$.

(i) If $F$ is a direct summand of $\text{Ext}^n_\mathcal{M}(A, -)|_\mathcal{M}$, then there exists $B \in \mathcal{M}$ such that $F \cong \text{Ext}^n_\mathcal{M}(B, -)|_\mathcal{M}$.

(ii) If $F$ is a direct summand of $\text{Ext}^n_\mathcal{M}(-, A)|_\mathcal{M}$, then there exists $B \in \mathcal{M}$ such that $F \cong \text{Ext}^n_\mathcal{M}(-, B)|_\mathcal{M}$.

Proof. (i) Let $F$ be a direct summand of $\text{Ext}^n_\mathcal{M}(A, -)|_\mathcal{M}$. By Proposition 7.5, $\text{Ext}^n_\mathcal{M}(A, -)|_\mathcal{M}$ is isomorphic to $\Sigma(\mathcal{M}(-, A))$. Since $\Sigma$ is an additive functor, there exists a summand $G$ of $\mathcal{M}(-, A)$ such that $\Sigma(G) = F$. But $G$, as a summand of $\mathcal{M}(-, A)$ should be of the form $\mathcal{M}(-, B)$ for some summand $B$ of $A$, because $\mathcal{M}$ is closed under direct summands. Hence $F \cong \text{Ext}^n_\mathcal{M}(B, -)|_\mathcal{M}$, as desired.

The last result of the paper reproves the existence of the $n$-Auslander-Reiten translation $\tau_n = \tau^{\Omega^-_\Lambda}$ that is already known by [14]. Our proof provides a functorial approach for the existence of the $n$-Auslander-Reiten translation.

Theorem 7.9. Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\text{mod-}\Lambda$. Then there is an equivalence $\tau_n: \mathcal{M} \rightarrow \overline{\mathcal{M}}$ such that for every $X, Y \in \mathcal{M}$,

$$\text{Ext}^n_\mathcal{M}(X, Y) \cong D\overline{\mathcal{M}}(Y, \tau_n(X)).$$

Proof. Since $\mathcal{M}$ is a functorially finite subcategory of $\text{mod-}\Lambda$, $\overline{\mathcal{M}}$ is a $k$-dualizing variety. Hence there exists a duality $D: \overline{\mathcal{M}}-\text{mod} \rightarrow \text{mod-}\overline{\mathcal{M}}$. Consider the composition

$$D \circ \Sigma: \text{mod-}\mathcal{M} \xrightarrow{\Sigma} \overline{\mathcal{M}}-\text{mod} \xrightarrow{D} \text{mod-}\overline{\mathcal{M}}.$$ 

It is obvious that the restriction of this composition to projective objects

$$D \circ \Sigma|: \text{prj-}\text{mod-}\mathcal{M} \rightarrow \text{prj-}\overline{\mathcal{M}}-\text{mod}$$

is also an equivalence. On the other hand, by Yoneda’s lemma, we have the equivalences $\mathcal{Y}: \mathcal{M} \cong \text{prj-}\text{mod-}\mathcal{M}$ and $\mathcal{Y}': \overline{\mathcal{M}} \cong (\text{prj-}\overline{\mathcal{M}}-\text{mod})^{op}$. So, altogether we obtain the equivalence $\tau_n: \mathcal{M} \rightarrow \overline{\mathcal{M}}$ as $\tau_n := \mathcal{Y}'^{-1} \circ D \circ \Sigma| \circ \mathcal{Y}$. The following commutative diagram explains what have been done:

$$\begin{array}{ccc}
\text{mod-}\mathcal{M} & \xrightarrow{\Sigma} & \overline{\mathcal{M}}-\text{mod} \\
\downarrow{\ell} & & \downarrow{\ell'} \\
\text{prj-}\text{mod-}\mathcal{M} & \xrightarrow{D \circ \Sigma|} & \text{prj-}\overline{\mathcal{M}}-\text{mod} \\
\mathcal{M} & \xrightarrow{\tau_n} & \overline{\mathcal{M}} \\
\mathcal{Y} & \xrightarrow{\mathcal{Y}'} & \mathcal{Y}' & \\
\end{array}$$

Now the commutativity of the diagram implies that

$$\Sigma(\mathcal{M}(-, X)) = D(\overline{\mathcal{M}}(-, \tau_n(X)))$$

for each $X \in \mathcal{M}$. This in view of Proposition 7.5 implies that

$$\text{Ext}^n_\mathcal{M}(X, -)|_\mathcal{M} \cong D(\overline{\mathcal{M}}(-, \tau_n(X))),$$

which is the desired isomorphism. The proof is hence completed. \qed
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