A class of dimensionality-free metrics for the convergence of empirical measures

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Abstract

This paper concerns the convergence of empirical measures in high dimensions. We propose a new class of metrics and show that under such metrics, the convergence is free of the curse of dimensionality (CoD). Such a feature is critical for high-dimensional analysis and stands in contrast to classical metrics (e.g., the Wasserstein distance). The proposed metrics originate from the maximum mean discrepancy, which we generalize by proposing specific criteria for selecting test function spaces to guarantee the property of being free of CoD. Therefore, we call this class of metrics the generalized maximum mean discrepancy (GMMD). Examples of the selected test function spaces include the reproducing kernel Hilbert space, Barron space, and flow-induced function spaces. Three applications of the proposed metrics are presented: 1. The convergence of empirical measure in the case of random variables; 2. The convergence of $n$-particle system to the solution to McKean-Vlasov stochastic differential equation; 3. The construction of an $\epsilon$-Nash equilibrium for a homogeneous $n$-player game by its mean-field limit. As a byproduct, we prove that, given a distribution close to the target distribution measured by GMMD and a certain representation of the target distribution, we can generate a distribution close to the target one in terms of the Wasserstein distance and relative entropy. Overall, we show that the proposed class of metrics is a powerful tool to analyze the convergence of empirical measures in high dimensions without CoD.

Keywords: Generalized maximum mean discrepancy, curse of dimensionality, empirical measure, McKean-Vlasov stochastic differential equation, mean-field games.

1 Introduction

The convergence of empirical measures plays a crucial role in analyzing the efficiency of mean-field theory or mean-field games (MFGs), which are fundamental tools to approximate finite-particle or finite-agent systems of large numbers. In specific, mean-field theory studies the behavior of a high-dimensional stochastic particle system by considering the effect of all other particles approximated by an average single effect. MFGs are introduced independently by Lasry-Lions ([43, 44, 45]) and Huang-Malhamé-Caines ([36, 37]). MFGs study the decision-making problem of a continuum of agents, and is able to provide approximations to Nash equilibria of $n$-player game in which players interact through their empirical measure. We refer to the books [13, 14] and the references therein for further background on MFGs.

In stochastic analysis, there is a rich literature of studies on the convergence of the large-$n$ system to the corresponding limit (also known as the McKean-Vlasov system); See, for instance, recent developments in [42, 41, 38]. In general, the distance between an $n$-body empirical measure and its limit has the form of $n^{-c/d}$, where $d$ is the dimension of one body and $c$ is a constant. Such results have been established in various settings, from the simple case of $n$-independent samples drawn from a given distribution [20, 28, 60].
to complicated cases of the McKean-Vlasov system [19] and MGFs [12]. In many interesting applications (e.g., the construction of \(\varepsilon\)-Nash equilibria [12]), \(d\) can be so large that the resulting convergence rate is extremely slow. This phenomenon is referred to as the curse of dimensionality (CoD), the main challenge in high-dimensional analysis and algorithms. This challenge exists in general McKean-Vlasov systems and MFGs, although in some special cases where the \(n\)-particle interaction is through the empirical moments, one may have a dimension-free convergence rate (e.g., [48, 49]).

The analysis in [20, 28, 60, 19, 12] suggests that the CoD phenomenon is related to the usage of the Wasserstein metric. In fact, it is well-known that the convergence of empirical measures under the Wasserstein distance presents the CoD [20] for any distribution that is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^d\). In this paper, we propose a new class of dimensionality-free metrics for the convergence analysis of the mean-field problems. Specifically, we generalize the concept of maximum mean discrepancy [9] (MMD) by imposing a set of criteria for selecting the test function class to guarantee the convergence rate under various settings is dimensionality-free. The maximum mean discrepancy (MMD) was first proposed in [9] as a tool of statistical tests for checking if two sets of observations are generated by the same distribution. It is closely related to a proposal by [27], with the test function space chosen as the reproducing kernel Hilbert space for computational efficiency. Our criteria mainly build on the function class’s empirical Rademacher complexity, allowing the test functions to be the reproducing kernel Hilbert space (RKHS), Barron function space, and flow-induced function spaces, just to name a few. We call this class of metrics the generalized maximum mean discrepancy (GMMD).

Beating the CoD is also a central topic in the machine learning community. One of the core problems in the high-dimensional analysis of machine learning models is identifying an appropriate function space equipped with an appropriate norm that can control the approximation and estimation errors of a particular machine learning model. This perspective is closely related to the proposed GMMD, and we remark that all proposed test function classes originate from machine learning models. Reproducing kernel Hilbert space is particularly important in statistical learning theory due to the representer theorem [39]. Barron space [21] was introduced for analyzing two-layer neural network models where optimal direct and inverse approximation theorems hold, as well as a priori estimate [22]. The flow-induced function spaces [21] were introduced for analyzing residual neural networks [33], which have wide applications in computer vision and scientific machine learning.

Our main results are summarized as follows:

1. We propose a novel metric to measure the distance of probability measures by generalizing MMD with proper selection criteria (Assumption 1) for test functions, yielding a dimensionality-free metric for the convergence of empirical measures associated with independent samples drawn from a given distribution (Theorem 2.1);

2. We generalize the results in [62]: given a target distribution being a bias potential model and a distribution close to the target measured by GMMD, we can generate a distribution close to the target in terms of the Wasserstein distance and relative entropy (Theorem 2.2). In this sense, we can transform the empirical measure into a new distribution close to the target in the Wasserstein distance without CoD;

3. The convergence result is extended to independent identically distributed (i.i.d.) stochastic processes by imposing assumptions (Assumption 2) on their modulus of continuity (Theorem 2.3);

4. We give three classes of test functions (reproducing kernel Hilbert space, Barron space, and flow-induced function spaces) for which the criteria in Assumption 1 are satisfied (Theorems 3.1–3.3);

5. The convergence of the empirical measure associated with an \(n\)-particle system to the distribution of the McKean-Vlasov stochastic differential equation is shown free of CoD (Theorem 4.1);

6. We show that the construction of an \(\varepsilon\)-Nash equilibrium for a homogeneous \(n\)-player game by its mean-field limit has no CoD (Theorem 5.2), i.e. \(\varepsilon\) is independent of \(d\).
Notations. We use $\mathcal{P}(\mathbb{R}^d)$ to denote the space of probability measures on $\mathbb{R}^d$, $\mathcal{P}^p(\mathbb{R}^d)$ with $p \geq 1$ denotes a subspace of $\mathcal{P}(\mathbb{R}^d)$ of probability measures with finite $p^{th}$-moment, i.e., $\mu \in \mathcal{P}^p(\mathbb{R}^d)$ if

$$M_p(\mu) := \left( \int_{\mathbb{R}^d} \|x\|^p \, d\mu(x) \right)^{1/p} < +\infty.$$ 

We will primarily work with probability measures with finite first and second moments, that are, $\mathcal{P}^1(\mathbb{R}^d)$ and $\mathcal{P}^2(\mathbb{R}^d)$. We use $\| \cdot \|$ to denote the $l^2$ norm in Euclidean space and define the Lipschitz constant with respect to the $l^2$ norm:

$$\text{Lip}(f) = \sup_{x,y, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}.$$ 

We denote by $\delta_{x_0}$ the delta distribution at $x_0$.

2 Generalized Maximum Mean Discrepancy

To define a metric on the space of probability measures, one natural approach is to choose a suitable class $\Phi$ of test functions on $\mathbb{R}^d$ and consider

$$\sup_{f \in \Phi} \left| \int_{\mathbb{R}^d} f \, d(\mu - \mu') \right|,$$

for any $\mu, \mu' \in \mathcal{P}^1(\mathbb{R}^d)$. Common choices include:

- the class of 1-Lipschitz functions, which leads to 1-Wasserstein metric $W_1$;
- the unit ball of a reproducing kernel Hilbert space (RKHS), which yields the maximum mean discrepancy (MMD) defined in [9].

Following this idea, we introduce the following generalized maximum mean discrepancy (GMMD).

Definition 2.1. Let $\Phi$ be a class of measurable functions on $\mathbb{R}^d$ such that

$$\sup_{f \in \Phi} |f(x)| \leq C(1 + \|x\|),$$

for a constant $C > 0$ depending on $\Phi$. Then for any $\mu, \mu' \in \mathcal{P}^1(\mathbb{R}^d)$, we define the generalized maximum mean discrepancy (GMMD) $D_\Phi$ by:

$$D_\Phi(\mu, \mu') = \sup_{f \in \Phi} \left| \int_{\mathbb{R}^d} f \, d(\mu - \mu') \right|. \quad (1)$$

Remark 2.1. The above definition coincides with the maximum mean discrepancy defined in [9, Definition 2.1]. In the original paper, the authors soon focused on one specific class $\Phi$, the reproducing kernel Hilbert space (RKHS), for computational efficiency. Thereafter, the literature refers MMD to the definition (1) associated with RKHS. To distinguish the difference and emphasize the generality of the function class $\Phi$, we shall name (1) as the GMMD in this paper.

We are interested in the metrics with dimensionality-free properties, for instance, that the empirical measure obtained from $n$ independent samples from a given measure $\mu$ approaches $\mu$ with a convergence speed independent of $d$ (for a precise statement, see Theorem 2.1 (b)). The conditions on $\Phi$, presented in Assumption 1, will fulfill our goal. Later in Section 3, we shall discuss several classes of test functions, including RKHS, Barron space, and flow-induced function spaces, where Assumption 1 is satisfied.

Assumption 1 (function class). Assume $\Phi$ satisfies the following properties:

(a) If $\mu$ is a signed measure on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f \, d\mu = 0, \forall f \in \Phi \Rightarrow \mu \equiv 0;$$

(b) $\sup_{f \in \Phi} |f(x)| \leq C(1 + \|x\|)$ for some constant $C > 0$.

(c) $\text{Lip}(f) \leq C(1 + \|x\|)$ for some constant $C > 0$.

(d) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^2 \, d\mu < +\infty$.

(e) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^4 \, d\mu < +\infty$.

(f) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^6 \, d\mu < +\infty$.

(g) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^8 \, d\mu < +\infty$.

(h) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^10 \, d\mu < +\infty$.

(i) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^12 \, d\mu < +\infty$.

(j) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^14 \, d\mu < +\infty$. 

(k) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^16 \, d\mu < +\infty$. 

(l) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^18 \, d\mu < +\infty$. 

(m) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^20 \, d\mu < +\infty$. 

(n) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^22 \, d\mu < +\infty$. 

(o) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^24 \, d\mu < +\infty$. 

(p) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^26 \, d\mu < +\infty$. 

(q) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^28 \, d\mu < +\infty$. 

(r) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^30 \, d\mu < +\infty$. 

(s) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^32 \, d\mu < +\infty$. 

(t) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^34 \, d\mu < +\infty$. 

(u) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^36 \, d\mu < +\infty$. 

(v) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^38 \, d\mu < +\infty$. 

(w) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^40 \, d\mu < +\infty$. 

(x) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^42 \, d\mu < +\infty$. 

(y) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^44 \, d\mu < +\infty$. 

(z) $\sup_{f \in \Phi} \int_{\mathbb{R}^d} \|x\|^46 \, d\mu < +\infty$. 

{\text{(Continued on the next page)}}
(b) There exist constants $A_1, A_2 > 0$, such that for any $f \in \Phi$, $\text{Lip}(f) \leq A_1$ and $|f(0)| \leq A_2$;

(c) There exists a constant $A_3 > 0$, such that for any $\mathcal{X} = \{x^1, \ldots, x^n\} \subset \mathbb{R}^d$, the empirical Rademacher complexity satisfies

$$\text{Rad}_n(\Phi, \mathcal{X}) := \frac{1}{n} \mathbb{E} \sup_{f \in \Phi} \left| \sum_{i=1}^n \xi_i f(x^i) \right| \leq \frac{A_3}{n} \sqrt{\sum_{i=1}^n (\|x^i\|^2 + 1)},$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. random variables drawn from the Rademacher distribution, i.e., $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$.

**Remark 2.2.** Note that by taking $n = 1$ and $\mathcal{X} = \{0\}$ in the item (c) above, it is easy to see that $A_2 \leq A_3$. Nevertheless, it is often the case that $A_3$ is much larger than $A_2$. Therefore, we introduce $A_2$ for a more precise estimate. We point out that, by comparing claim (b) in Theorem 2.1 below and [23, Corollary 3.2, 3.4]), we know that in high dimensions, any class of functions satisfying the assumption above is relatively small compared to the class of all $A_1$-Lipschitz functions.

**Remark 2.3.** Rademacher complexity measures the richness of a function class with respect to a specific probability distribution. It is a powerful tool to bound the generalization error when learning the function in the class through empirical risk minimization. We refer to [5, 53] for more information about the Rademacher complexity.

The subsection below highlights properties of the metric $D_\Phi$ for random variables when the function class $\Phi$ satisfies Assumption 1. For comparison purpose, we recall the $p$-Wasserstein metric $W_p$ defined as follows

$$W_p(\mu, \mu') = \left( \inf_{\gamma \in \Gamma(\mu, \mu')} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p \, d\gamma(x, y) \right)^{1/p}, \quad (2)$$

where $\mu, \mu' \in \mathcal{P}^p(\mathbb{R}^d)$ and $\Gamma(\mu, \mu')$ denotes the collection of all probability distributions on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\mu'$ on the first and second arguments, respectively. Note that Definition 2.1 with $\Phi = \{\text{all continuous 1-Lipschitz functions from } \mathbb{R}^d \text{ to } \mathbb{R}\}$ admits the dual representation of (2) with $p = 1$ (cf. [6, Theorem 1.3]).

### 2.1 Convergence Analysis for Random Variables

**Theorem 2.1.** Under Assumption 1, we have:

(a) $D_\Phi$ is a metric on $\mathcal{P}^1(\mathbb{R}^d)$. In addition, if $\mu, \mu' \in \mathcal{P}^2(\mathbb{R}^d)$,

$$D_\Phi(\mu, \mu') \leq A_1 W_1(\mu, \mu') \leq A_1 W_2(\mu, \mu').$$

(b) Given $\mu \in \mathcal{P}^2(\mathbb{R}^d)$, let $X^1, \ldots, X^n$ be i.i.d. random variables drawn from the distribution $\mu$ and

$$\tilde{\mu}^n = \frac{1}{n} \sum_{i=1}^n \delta_{X^i},$$

be the empirical measure of $X^1, \ldots, X^n$. Then,

$$\text{ED}_\Phi(\mu, \tilde{\mu}^n) \leq \frac{2A_3}{\sqrt{n}} \sqrt{\int_{\mathbb{R}^d} \|x\|^2 + 1 \, d\mu(x)},$$

$$\mathbb{E} \left[ D_\Phi^2(\mu, \tilde{\mu}^n) \right] \leq \frac{A_3^2}{n} \int_{\mathbb{R}^d} \|x\|^2 \, d\mu(x) + \frac{4A_3^2}{n} \int_{\mathbb{R}^d} \|x\|^2 + 1 \, d\mu(x).$$

(c) If $\mu$ satisfies the $T_1$ inequality (cf. [30]),

$$W_1^2(\mu, \tilde{\mu}) \leq 2\kappa^2 \mathcal{H}(\tilde{\mu} | \mu) \quad \forall \tilde{\mu} \ll \mu,$$
where $H$ is the relative entropy $H(\mu|\nu) := \mathbb{E}^\nu\left(\frac{dp}{d\mu}\log\left(\frac{dp}{d\nu}\right)\right)$, (an equivalent condition of the $T_1$ inequality is that there exists a $\delta > 0$ such that $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(\delta \|x-y\|^2) d\mu(x) d\nu(y) < +\infty$; see [17, Theorem 2.3]), then,

$$
\mathbb{P}\left(D_\Phi(\mu, \bar{\mu}^n) - \frac{2A_3}{\sqrt{n}} \sqrt{\int_{\mathbb{R}^d} \|x\|^2 + 1} d\mu(x) \geq a\right) \leq \exp\left(-\frac{na^2}{2A_3^2 \kappa^2}\right).
$$

Proof. For claim (a), by definition $D_\Phi$ is symmetric and admits the triangle inequality. We only need to show that for any $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$, $D_\Phi(\mu, \mu') = 0$ leads to $\mu = \mu'$, which is ensured directly by Assumption 1 (a). In addition, from Assumption 1 (b), we deduce that $D_\Phi(\mu_1, \mu_2) \leq A_1 \mathcal{W}_1(\mu_1, \mu_2)$ by using the definition of $\mathcal{W}_1$. The relation between $\mathcal{W}_1$ and $\mathcal{W}_2$ is a classical result in the literature (e.g., see [58, Remark 6.6]).

For claim (b), we first notice that

$$
D_\Phi(\mu, \bar{\mu}^n) = \frac{1}{n} \sup_{f \in \Phi} \left| \sum_{i=1}^n \left[ f(X^i) - \mathbb{E} f(X^i) \right]\right| \leq \frac{2}{n} \mathbb{E} \sup_{f \in \Phi} \left| \sum_{i=1}^n \xi_i f(X^i) \right|,
$$

where in the last step we use Rademacher complexity to bound the largest gap between the expectation of a function and its empirical version (see, e.g., [53, Lemma 26.2]). Using Assumption 1 (c), one deduces

$$
\mathbb{E} D_\Phi(\mu, \bar{\mu}^n) \leq \frac{2A_3}{n} \mathbb{E} \left( \sum_{i=1}^n (\|X^i\|^2 + 1) \right) \leq \frac{2A_3}{\sqrt{n}} \sqrt{\int_{\mathbb{R}^d} \|x\|^2 + 1} d\mu(x). \quad (3)
$$

Now let $Y_1, \ldots, Y_n$ be i.i.d. random variables drawn from the distribution $\mu$ and be independent of $X_1, \ldots, X_n$. By Efron-Stein-Steele inequality (cf. [10, Theorem 5]), one has

$$
\begin{align*}
\text{Var}(D_\Phi(\mu, \bar{\mu}^n)) &= \text{Var} \left( \frac{1}{n} \sup_{f \in \Phi} \left| \sum_{i=1}^n \left[ f(X^i) - \mathbb{E} f(X^i) \right]\right| \right) \\
&\leq \frac{1}{2} \sum_{i=1}^n \mathbb{E} \left[ \sup_{f \in \Phi} \left| \sum_{j=1}^n \left[ f(X^j) - \mathbb{E} f(X^j) \right] \right| - \sup_{f \in \Phi} \left| \sum_{j=1,j \neq i}^n \left[ f(X^j) - \mathbb{E} f(X^j) \right] + f(Y^i) - \mathbb{E} f(X^i) \right| \right]^2 \\
&\leq \frac{A_1^2}{2n^2} \sum_{i=1}^n \mathbb{E} \|X^i - Y^i\|^2 = \frac{A_1^2}{n} \int_{\mathbb{R}^d} \|x\|^2 d\mu(x).
\end{align*}
$$

Therefore, we have

$$
\mathbb{E} \left[D_\Phi^2(\mu, \bar{\mu}^n)\right] = \text{Var}(D_\Phi(\mu, \bar{\mu}^n)) + [\mathbb{E} D_\Phi(\mu, \bar{\mu}^n)]^2 \leq \frac{A_1^2}{n} \int_{\mathbb{R}^d} \|x\|^2 d\mu(x) + \frac{4A_3^2}{n} \int_{\mathbb{R}^d} \|x\|^2 + 1 d\mu(x).
$$

For claim (c), we need some established concentration inequalities. Let $\mu^{\otimes n} \in \mathcal{P}(\mathbb{R}^{d \times n})$ be the $n^{th}$ times product of $\mu$ and the distance between $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n) \in \mathbb{R}^{d \times n}$ be

$$
\|x - y\| = \sum_{i=1}^n \|x^i - y^i\|.
$$

By Theorem 5.2 in [16], for any $\tilde{\mu} \ll \mu^{\otimes n}$, we have

$$
\mathcal{W}_1^2(\mu^{\otimes n}, \tilde{\mu}) \leq 2n \kappa^2 \mathcal{H} (\mu^{\otimes n}).
$$

Using the fact that

$$
|G(x) - G(y)| \leq \frac{A_1}{n} \|x - y\|
$$

5
for any $x$ and $y \in \mathbb{R}^{d \times n}$, and Theorem 5.1 in [16], we have

$$
\mathbb{P}(D_{\Phi}(\mu, \bar{\mu}^n) - ED_{\Phi}(\mu, \bar{\mu}^n) \geq a) = \mu^{\otimes n}(G(x) - \int_{\mathbb{R}^{d \times n}} G(x) \, d\mu^n(x) \geq a) \leq \exp\left(-\frac{na^2}{2A_1^2}\right).
$$

Finally, we conclude our proof by the observation

$$
ED_{\Phi}(\mu, \bar{\mu}^n) \leq \frac{2A_3}{\sqrt{n}} \sqrt[4]{\int_{\mathbb{R}^d} \left[||x||^2 + 1\right] \, d\mu(x)}
$$

from inequality (3).

The above theorem focuses on the convergence of $\bar{\mu}^n$ to $\mu$ in the sense of GMMD. It is well-known that the convergence of empirical measures under the Wasserstein distance faces the CoD $[\mathbb{C}]$, we define a mapping (called bias-potential model) $M_P$ (a subclass of $\Phi$) $\to \mathcal{P}(\mathbb{R}^d)$:

$$
\Phi^P = \left\{ V \in \Phi, \int_{\mathbb{R}^d} e^{-V(x)} \, dP(x) < +\infty \right\}, \quad \frac{dM_P(V)}{dP} = \frac{e^{-V}}{\int_{\mathbb{R}^d} e^{-V(x)} \, dP(x)}.
$$

Assume that there exists $V^* \in \Phi$ such that $\mu = M_P(V^*)$, and $\nu$ is a distribution such that $D_{\Phi}(\mu, \nu) \ll 1$. Let

$$
\text{Loss}(V) = \int_{\mathbb{R}^d} V(x) \, d\nu(x) + \log \left(\int_{\mathbb{R}^d} e^{-V(x)} \, dP(x)\right),
$$

for $V \in \Phi^P$ and

$$
V' = \arg \min_{V \in \Phi} \text{Loss}(V), \quad \nu' = M_P(V').
$$

Then,

(a) $\mathcal{H}(\mu|\nu') \leq 2D_{\Phi}(\mu, \nu)$.

(b) If $P$ satisfies the $T_1$ inequality,

$$
W_2^2(P, \check{P}) \leq 2K^2 \mathcal{H}(\check{P}|P), \forall \check{P} \ll P,
$$

then there exists a constant $C$ depending only on $A_1$, $A_2$ and $K$ such that

$$
W_1(\mu, \nu') \leq C \sqrt{D_{\Phi}(\mu, \nu)}, \quad W_2(\mu, \mu') \leq CD_{\Phi}(\mu, \nu)^{\frac{1}{2}}.
$$

(c) If $P$ satisfies the logarithmic-Sobolev inequality,

$$
\int_{\mathbb{R}^d} g^2 \log(g^2) \, dP - \int_{\mathbb{R}^d} g^2 \, dP \log \left(\int_{\mathbb{R}^d} g^2 \, dP\right) \leq 2K^2 \int_{\mathbb{R}^d} \|\nabla g\|^2 \, dP,
$$

then there exists a constant $C$ depending only on $A_1$, $A_2$ and $K$ such that

$$
W_2(\mu, \nu') \leq C \sqrt{D_{\Phi}(\mu, \nu)}.
$$
In particular, the logarithmic-Sobolev inequality, with a constant depending only on $A$, gives
\[ \text{Loss}(V') \leq \inf_{V \in \Phi} \text{Loss}(V) + \epsilon. \]

This results in replacing $V'$ with $V^*$ and an additional $\epsilon$ term in all the estimates in this theorem.

**Proof.** For claim (a), we compute
\[
\mathcal{H}(\mu|\nu') = \int_{\mathbb{R}^d} (V' - V^*) \, d\mu(x) - \log \left( \int_{\mathbb{R}^d} e^{-V^*(x)} \, dP(x) \right) + \log \left( \int_{\mathbb{R}^d} e^{-V'(x)} \, dP(x) \right) \\
\leq 2D_\Phi(\mu, \nu) + \text{Loss}(V') - \text{Loss}(V^*) \leq 2D_\Phi(\mu, \nu).
\]

For claim (b), we will use $C$ as a constant depending only on $A_1, A_2$ and $K$, which may vary from line to line. Theorem 2.3 in [17] gives
\[
\int_{\mathbb{R}^d} e^{\frac{x^2}{2}} \, dP(x) \leq C.
\]
Consequently,
\[
\int_{\mathbb{R}^d} e^{-A_2 x^2} \, dP(x) \geq e^{-A_2} P(\|x\| \leq 1) \geq e^{A_2} [1 - P(\|x\| \geq 1)] \geq e^{A_2} [1 - C e^{-\frac{x^2}{2}}] = \frac{1}{C}.
\]

Therefore, for any $V \in \Phi$, with $|V(x)| \leq A_1 \|x\| + A_2$ we have
\[
\int_{\mathbb{R}^d} e^{\frac{x^2}{2}} \, dP(V)(x) \leq \frac{\int_{\mathbb{R}^d} e^{\frac{x^2}{2} + A_1 \|x\| + A_2} \, dP(x)}{\int_{\mathbb{R}^d} e^{-A_2 \|x\|} \, dP(x)} \leq C \int_{\mathbb{R}^d} e^{\frac{x^2}{2} + C \|x\|^2 + C} \, dP(x) \leq C.
\]
In particular, $\int_{\mathbb{R}^d} \exp(\frac{x^2}{2C}) \, d\nu'(x) \leq C$. Now with Corollary 2.6 in [7] and claim (a), we deduce
\[
\mathcal{W}_1(\mu, \nu') \leq C \sqrt{\mathcal{H}(\mu|\nu')} \leq C \sqrt{D_\Phi(\mu, \nu)},
\]
\[
\mathcal{W}_2(\mu, \nu') \leq C[\sqrt{\mathcal{H}(\mu|\nu')} + \|\mu\|_{\mathcal{M}_P(V')}^\frac{1}{2}] \leq C[\sqrt{D_\Phi(\mu, \nu)} + D_\Phi(\mu, \nu)^\frac{1}{2}] \leq CD_\Phi(\mu, \nu)^\frac{1}{2}.
\]

For claim (c), using Lemma 4.1 and Theorem 4.3 in [1], we know that for any $V \in \Phi$, $\mathcal{M}_P(V)$ satisfies the logarithmic-Sobolev inequality, with a constant depending only on $A_1, A_2$ and $K$. Then by Theorem 1.5 in [15] or Theorem 1.1 in [59], we know that $\nu' = \mathcal{M}_P(V')$ satisfies the $T_2$ inequality
\[
\mathcal{W}_2(\nu, \nu') \leq C \sqrt{\mathcal{H}(\nu|\nu')} , \quad \forall \nu \ll \nu'.
\]
Choosing $\tilde{\nu} = \mu$ and together with claim (a), we conclude
\[
\mathcal{W}_2(\mu, \nu') \leq C \sqrt{\mathcal{H}(\mu|\nu')} \leq C \sqrt{D_\Phi(\mu, \nu)}.
\]

Bias potential model [55, 8, 62] is a large family of probability models. With Proposition 2.1 in [62] and Assumption 1, if $P$ is the uniform distribution on a compact subset $K \subset \mathbb{R}^d$, the set $\{\mu = \frac{e^{-\beta V}}{\mathbb{E}[e^{-\beta V}]} \, P : (\beta, V) \in \mathbb{R} \times \Phi\}$ is dense in $\mathcal{P}(K)$ under the Wasserstein distance. Besides the bias potential model we study here, another important form of distribution representations is the generative adversarial network (GAN) [29, 2], which assumes $\mu = \phi \circ P$ with a known base distribution $P \in \mathcal{P}(\mathbb{R}^d)$ and $\phi$ lies in certain function classes. In practice, GAN has shown astonishing power in learning distribution [11, 18]. However, how to establish similar theoretical results with respect to GAN is far from clear.
2.2 Convergence Analysis for Stochastic Processes

Theorems in the previous subsection focus on the measure $\mu$ on $\mathbb{R}^d$. When dealing with empirical measures of a stochastic process such as discussing the convergence of the $n$-particle dynamics to the McKean-Vlasov system (see Section 4 for details), i.e., $\mu \in \mathcal{P}(C([0,T];\mathbb{R}^d))$, we need to extend Theorem 2.1 to Theorem 2.3 below. The results are similar, except for an additional term $\phi(n)$ coming from the regularity of the process. To this end, we introduce the following assumption.

Assumption 2 (modulus of continuity). Given a probability measure $\mu$ on $C([0,T];\mathbb{R}^d)$, there exist constants $Q, \alpha, \beta > 0$, such that for any $h > 0$,

$$
\int_{C([0,T];\mathbb{R}^d)} \Delta(x, h) \, d\mu(x) \leq Qh^\alpha \log^\beta \left( \frac{2T}{h} \right),
$$

where $\Delta(f, h)$ denotes the modulus of continuity of $f \in C([0,T];\mathbb{R}^d)$:

$$
\Delta(f, h) = \sup_{t,s \in [0,T], |t-s| \leq h} \|f(t) - f(s)\|.
$$

Remark 2.5. The condition (5) is satisfied by many kinds of processes, e.g., Brownian motion and Itô diffusion processes with $\alpha = \beta = \frac{1}{2}$, under quite generic conditions on the drift and diffusion coefficients ([26, Theorem 1]).

**Theorem 2.3.** Under Assumptions 1–2,

(a) Let $\mu$ be a probability distribution on $C([0,T];\mathbb{R}^d)$ such that

$$
\int_{C([0,T];\mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x_t\|^2 \, d\mu(x) < +\infty.
$$

Denote by $X^1, \ldots, X^n$ i.i.d. random processes drawn from $\mu$, and

$$
\bar{\mu}_n^i = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^i}
$$

as the empirical measure of $X_1^1, \ldots, X_n^n$. Define $\mu_\ell = \mathcal{L}(X_1^\ell)$, then

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \text{D}_\Phi(\mu_\ell, \bar{\mu}_n^\ell) \right] \leq \phi(n),
$$

where

$$
\phi(n) = 2 \left[ \frac{A_3 + 8 \max\{A_1, A_2\}}{\sqrt{n}} + 8 \max\{A_1, A_2\} \sqrt{\log(2^{2\alpha}n)} \right] \left( \int_{C([0,T];\mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x_t\|^2 \, d\mu(x) + 1 \right)^{\frac{\beta}{2}}
$$

$$
+ \frac{2A_1QT^\alpha}{\sqrt{n}} \left( \frac{\log(2^{4\alpha}n)}{2\alpha} \right)^{\frac{\beta}{2}},
$$

$$
= O(n^{-\frac{\beta}{2}}(\log n)^{\max\{\beta, \frac{\beta}{2}\}}).
$$

In addition,

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \text{D}_\Phi^2(\mu_\ell, \bar{\mu}_n^\ell) \right] \leq \frac{A_3^2}{n} \int_{C([0,T];\mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x_t\|^2 \, d\mu(x) + \phi^2(n).
$$

(b) If $\mu$ satisfies the $T_1$ inequality, that is,

$$
\mathcal{W}_1^2(\mu, \bar{\mu}) \leq 2\kappa^2 \mathcal{H}(\bar{\mu} | \mu) \quad \forall \bar{\mu} \ll \mu,
$$

where $\mathcal{H}$ is the relative entropy defined in Theorem 2.1 (d), then

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} \text{D}_\Phi(\mu_\ell, \bar{\mu}_n^\ell) - \phi(n) \geq a \right) \leq \exp \left( -\frac{na^2}{2A_1^2\kappa^2} \right).
$$
Remark 2.6. The logarithmic term in $\phi(n)$ can be removed for many kinds of function classes $\Phi$, in particular for the function classes discussed in Section 3. We defer the proof of this claim to Appendix A.

Proof. The proof follows the same as in Theorem 2.1, except for the estimate of

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} D_{\Phi}(\mu_t, \hat{\mu}_t^n) \right],
$$

which is bounded by $\phi(n)$. To see this, following the same argument as in Theorem 2.1, it suffices to estimate

$$
\frac{1}{n} \mathbb{E} \sup_{t \in [0,T]} \sup_{f \in \Phi} \left| \sum^n_{i=1} \xi_i f(X_t^i) \right|,
$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. random variables drawn from the Rademacher distribution and are independent of the processes $X^1, \ldots, X^n$.

Given any $y^1, \ldots, y^n \in \mathbb{R}^d$, we define $F(\xi)$ for $\xi = (\xi_1, \ldots, \xi_n) \in \{-1, 1\}^n$ by

$$
F(\xi) := \frac{1}{n} \sup_{f \in \Phi} \left| \sum^n_{i=1} \xi_i f(y^i) \right|.
$$

By Assumption 1 (c), we immediately have

$$
\mathbb{E} F(\xi) \leq \frac{A_3}{n} \sqrt{n \left( \|y\|^2 + 1 \right)}. \tag{7}
$$

By definition, for any $\xi$ and $\epsilon > 0$, there exists a function $f^\xi \in \Phi$ such that

$$
F(\xi) \leq \frac{1}{n} \left| \sum^n_{i=1} \xi_i f^\xi(y^i) \right| + \epsilon.
$$

Note that, for any $1 \leq j \leq n$, one has

$$
D_j F(\xi) := F(\xi) - \min_{z \in \{0, 1\}} F(\xi_1, \ldots, \xi_{j-1}, z, \xi_{j+1}, \ldots, \xi_n)
\leq \epsilon + \frac{1}{n} \left| \sum^n_{i=1} \xi_i f^\xi(y^i) \right| - \min_{z \in \{0, 1\}} F(\xi_1, \ldots, \xi_{j-1}, z, \xi_{j+1}, \ldots, \xi_n)
\leq \epsilon + \frac{1}{n} \left| \sum^n_{i=1} \xi_i f^\xi(y^i) \right| - \frac{1}{n} \left| \sum^n_{i \neq j} \xi_i f^\xi(y^i) \right| + \epsilon + \frac{2}{n} |f^\xi(y^i)|.
$$

Summing the above inequality over $j$ and taking the supremum over all $\xi$ give,

$$
\sup_{\xi} \sum^n_{j=1} |D_j F(\xi)|^2 \leq \sup_{\xi} \sum^n_{j=1} \left( \epsilon + \frac{2}{n} |f^\xi(y^i)| \right)^2 \leq \sup_{f \in \Phi} \sum^n_{j=1} \left( \epsilon + \frac{2}{n} |f(y^i)| \right)^2,
$$

holding true for arbitrary $\epsilon$, which implies

$$
\sup_{\xi} \sum^n_{j=1} |D_j F(\xi)|^2 \leq \frac{4}{n^2} \sup_{f \in \Phi} \sum^n_{j=1} |f(y^i)|^2 \leq \frac{8}{n^2} \sum^n_{j=1} (A_1^2 \|y^i\|^2 + A_2^2).
$$

Therefore, by Theorem 3.18 in [56], we obtain

$$
\mathbb{P}(F(\xi) - \mathbb{E} F(\xi) \geq a) \leq \exp \left( -\frac{n^2 a^2}{64 \sum^n_{j=1} (A_1^2 \|y^i\|^2 + A_2^2)} \right). \tag{8}
$$
Given $n' \in \mathbb{N}^+, x^1, \ldots, x^n \in C([0, T]; \mathbb{R}^d)$ and $t_1, \ldots, t_{n'} \in [0, T]$, we define the constant

$$M := \sqrt{\max_{1 \leq p \leq n'} \sum_{i=1}^{n'} (\|x^i_{t_p}\|^2 + 1)},$$

based on the deterministic values $\{x^i_{t_p}\}_{p=1}^{n'}$, for $i = 1, \ldots, n$. By (7), we have

$$E \left[ \frac{1}{n} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(x^i_{t_p}) \right] \leq \frac{A_3}{n} M, \ \forall \ p = 1, \ldots, n'. \quad (9)$$

Using union bound and (8)-(9), we have the following estimation of the tail probability, for any $a > 0$,

$$\mathbb{P} \left( \frac{1}{n} \sup_{1 \leq p \leq n'} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(x^i_{t_p}) \right) \leq \frac{A_3}{n} M \geq \frac{8\sqrt{\log n'} \max\{A_1, A_2\}}{n} M + a$$

$$\leq n' \max_{1 \leq p \leq n'} \mathbb{P} \left( \frac{1}{n} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(x^i_{t_p}) \right) \leq \frac{A_3}{n} M \geq \frac{8\sqrt{\log n'} \max\{A_1, A_2\}}{n} M + a$$

$$\leq n' \exp \left( -\frac{n^2 a^2}{64 \max\{A_1^2, A_2^2\} M^2} \right)$$

$$\leq n' \exp(-\log(n')) \exp \left( -\frac{n^2 a^2}{64 \max\{A_1^2, A_2^2\} M^2} \right)$$

$$\leq \exp \left( -\frac{n^2 a^2}{64 \max\{A_1^2, A_2^2\} M^2} \right).$$

Therefore,

$$\frac{1}{n} E \max_{1 \leq p \leq n'} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(x^i_{t_p}) \leq \frac{A_3}{n} M + \frac{8\sqrt{\log n'} \max\{A_1, A_2\}}{n} M + \int_0^{+\infty} \exp \left( -\frac{n^2 a^2}{64 \max\{A_1^2, A_2^2\} M^2} \right) da$$

$$\leq \frac{A_3}{n} M + 8(\sqrt{\log n'} + 1) \max\{A_1, A_2\} M$$

$$\leq \frac{A_3 + 8(\sqrt{\log n'} + 1) \max\{A_1, A_2\}}{n} M$$

$$= \frac{A_3 + 8(\sqrt{\log n'} + 1) \max\{A_1, A_2\}}{n} \left( \max_{1 \leq p \leq n'} \sum_{i=1}^{n} (\|x^i_{t_p}\|^2 + 1) \right).$$

The above estimate is for deterministic functions in $C([0, T]; \mathbb{R}^d)$ evaluated at time points $t_1, \ldots, t_{n'}$. Applying it to i.i.d. stochastic processes $\{X^i\}_{i=1}^{n}$ with the law $\mu$, we obtain

$$\frac{1}{n} E \max_{1 \leq p \leq n'} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(X^i_{t_p}) = \frac{1}{n} E \left\{ E \left[ \max_{1 \leq p \leq n'} \sup_{f \in \Phi} \sum_{i=1}^{n} \xi_i f(X^i_{t_p}) \mid X^1, \ldots, X^n \right] \right\}$$

$$\leq \frac{A_3 + 8(\sqrt{\log n'} + 1) \max\{A_1, A_2\}}{n} E \left( \max_{1 \leq p \leq n'} \sum_{i=1}^{n} (\|X^i_{t_p}\|^2 + 1) \right)$$

$$\leq \frac{A_3 + 8(\sqrt{\log n'} + 1) \max\{A_1, A_2\}}{\sqrt{n}} \left( \int_{C([0, T]; \mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x^i\|^2 d\mu(x) + 1 \right)^{\frac{1}{2}}.$$
Using the above estimate and letting $t_p$ be equidistributed on $[0, T]$, that is, $t_p = \frac{(p-1)T}{n'}$ for $p = 1, \ldots, n'$, by Assumption 2, and with the notation $\Pi(t) = \frac{\eta(t)}{n'}$, one can deduce
\[
\left| \frac{1}{n} \mathbb{E} \sup_{t \in [0, T]} \sup_{f \in \Phi} \left| \sum_{i=1}^{n} \xi_i f(X_i^t) \right| - \frac{1}{n} \mathbb{E} \sup_{1 \leq p \leq n'} \sup_{f \in \Phi} \left| \sum_{i=1}^{n} \xi_i f(X_i^{t_p}) \right| \right| 
\leq \frac{1}{n} \mathbb{E} \sup_{t \in [0, T]} \sup_{f \in \Phi} \left| \sum_{i=1}^{n} \xi_i \left( f(X_i^t) - f(X_i^{\Pi(t)}) \right) \right| 
\leq A_1 \mathbb{E} \sup_{t \in [0, T]} \|X_i^1 - X_i^{\Pi(t)}\| \leq A_1 Q \left( \frac{T}{n'} \right)^{\alpha} \log^\beta(2n').
\]
Choosing $n' = \lfloor n^{\frac{1}{d}} \rfloor + 1$ gives
\[
\frac{1}{n} \mathbb{E} \sup_{t \in [0, T]} \sup_{f \in \Phi} \left| \sum_{i=1}^{n} \xi_i f(X_i^t) \right| 
\leq \left[ \frac{A_3 + 8 \max\{A_1, A_2\}}{\sqrt{n}} + 8 \max\{A_1, A_2\} \sqrt{\log(2^{2\alpha}n)} \right] \left( \mathbb{E} \sup_{C([0, T] \times \mathbb{R}^d)} \|x\|^2 d\mu(x) + 1 \right)^{\frac{1}{2}} + A_1 QT\frac{\log(2^{4\alpha}n)}{2\alpha} \right)^{\frac{1}{2}},
\]
Therefore we have proved (6). The rest of the proof follows Theorem 2.1, and we conclude the proof. \(\square\)

The next theorem shows that the law of the stochastic differential equation (SDE) satisfies the condition of Theorem 2.3, i.e., Assumption 2.

**Theorem 2.4.** Given a constant $T > 0$, a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ supporting an $m$-dimensional Brownian motion $W$ as well as a $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variable $\eta$. We consider the following SDE
\[
dX_t = B(t, X_t) dt + \Sigma(t, X_t) dW_t, \quad X_0 = \eta,
\]
where $B : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\Sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ satisfy:
\[
\|B(t, x) - B(t, x')\|^2 + \|\Sigma(t, x) - \Sigma(t, x')\|^2 \leq K^2 \|x - x'\|^2,
\]
\[
\|B(t, 0)\| + \|\Sigma(t, 0)\| \leq K,
\]
with $K$ being a positive constant and $\|\cdot\|_F$ denoting the Frobenius norm on $\mathbb{R}^{d \times m}$. It is well-known that the above SDE admits a unique strong solution (cf. [63, Theorem 3.3.1]). We denote by $\mu_0 := \mathcal{L}(\eta)$, $\mu := \mathcal{L}(X)$ the laws of $\eta$ and $X$, respectively.

(a) Assume that $\mathbb{E}\|\eta\|^2 \leq K^2$, then there exists a positive constant $C$ depending only on $K$ and $T$, such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|X_t\|^2 \leq C,
\]
and
\[
\int_{C([0, T] \times \mathbb{R}^d)} [\Delta(x, h)]^2 d\mu(x) \leq C h \log \left( \frac{2T}{h} \right).
\]

(b) Assume that
\[
\mathcal{W}_2^2(\mu_0, \bar{\mu}) \leq 2K^2 \mathcal{H}(\bar{\mu}|\mu_0), \quad \forall \bar{\mu} \ll \mu_0.
\]
Then, there exists a positive constant $C$ depending on $K$ and $T$, such that
\[
\mathcal{W}_2^2(\mu, \bar{\mu}) \leq C \mathcal{H}(\bar{\mu}|\mu), \quad \forall \bar{\mu} \ll \mu,
\]
where $\mathcal{H}$ is the relative entropy defined in Theorem 2.1 (d).
Consider a local martingale for $\lambda > 0$.

Consequently, it suffices to estimate the second term in (10). In the sequel, we will use short notations $\Sigma_t := \Sigma(t, X_t)$ and $Y_t := \int_0^t \Sigma_u \, dW_u$.

We first work on the case $\eta = \delta_{x_0}$ for a fixed $x_0 \in \mathbb{R}^d$. Fixing $s \in [0, T]$, one has

$$d\|Y_t - Y_s\|^2 = 2(Y_t - Y_s)^T \Sigma_t \, dW_t + \|\Sigma_t\|^2 \, dt.$$ 

Hence, for any $\lambda > 0$, $\exp \left( \lambda \|Y_t - Y_s\|^2 - \int_s^t \|\Sigma_u\|^2 \, du \right) - 2\lambda \int_s^t \|Y_u - Y_s\)^T \Sigma_u\|^2 \, du$ is a nonnegative local martingale for $t \in [s, T]$, thus a supermartingale. Fix $a > 0$ and let $\tau$ be a stopping time defined by

$$\tau = \inf\{ u \in [s, t] : \|Y_u - Y_s\| \geq a \} \wedge t, \quad \inf\{0\} = +\infty.$$ 

Then $E \left[ \exp \left( \lambda \|Y_T - Y_s\|^2 - \int_s^T \|\Sigma_u\|^2 \, du \right) - 2\lambda \int_s^T \|Y_u - Y_s\)^T \Sigma_u\|^2 \, du \right] \leq 1$. Noticing that for any $u \in [s, \tau]$, we have $\|Y_u - Y_s\| \leq a$, and

$$\int_s^\tau \|Y_u - Y_s\)^T \Sigma_u\|^2 \, du \leq a^2 \int_s^\tau \|\Sigma_u\|^2 \, du \leq Ca^2 \left( 1 + \sup_{0 \leq u \leq T} \|X_u\|^2 \right) \frac{(t - s)}{2}.$$ 

Consequently,

$$E \left[ \exp \left( \lambda \|Y_T - Y_s\|^2 - C(1 + \sup_{0 \leq u \leq T} \|X_u\|^2)(t - s) \right) - CA^2 \left( 1 + \sup_{0 \leq u \leq T} \|X_u\|^2 \right)(t - s) \right] \leq 1.$$ 

Now, let $S_X$ be the maximum of $X_t$ on $[0, T]$, i.e., $S_X := \sup_{0 \leq u \leq T} \|X_u\|$. For a fixed constant $M > 0$, one deduces

$$E \left[ \exp(\lambda \|Y_T - Y_s\|^2)1_{S_X \leq M} \right] \leq \exp(C\lambda(1 + M^2)(t - s) + CA^2 \left( 1 + \sup_{0 \leq u \leq T} \|X_u\|^2 \right)(t - s)).$$ 

Hence,

$$P(\|Y_T - Y_s\| \geq a, S_X \leq M) \leq P(\|Y_T - Y_s\| \geq a, S_X \leq M) \leq \exp(-\lambda a^2 + C\lambda(1 + M^2)(t - s) + CA^2 \left( 1 + \sup_{0 \leq u \leq T} \|X_u\|^2 \right)(t - s)).$$

Picking $\lambda = [2C(1 + M^2)(t - s)]^{-1}$, we know that

$$P(\|Y_T - Y_s\| \geq a, S_X \leq M) \leq C \exp \left( -\frac{a^2}{C(1 + M^2)(t - s)} \right), \quad \forall s, t \in [0, T].$$

Therefore, for any $(t_1, s_1), \ldots, (t_m, s_m) \in [0, T] \times [0, T]$, using Exercise 2.5.10 and Proposition 2.5.2 in [57], we obtain

$$E \left[ \max_{1 \leq i \leq m} \|Y_{t_i} - Y_{s_i}\|^4 1_{S_X \leq M} \right] \leq C \log^2(m)(1 + M^4) \max_{1 \leq i \leq m} |t_i - s_i|^2. \quad (11)$$
For any \( t \in [0, T] \) and any integer \( k \), define
\[
\Pi_k(t) = h2^{-k} \left[ \frac{2^k t}{h} \right].
\]

The continuity of \( Y_t \) together with inequality (11) gives
\[
\left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^4 \mathbf{1}_{S_X \leq M} \right] \right)^{\frac{1}{4}} \leq C \left( \mathbb{E} \left[ \sup_{t \in [0, T]} \| Y_t - Y_{n(t)} \|^4 \mathbf{1}_{S_X \leq M} \right] \right)^{\frac{1}{4}}
\leq C \sum_{k=0}^{\infty} \left( \mathbb{E} \left[ \sup_{t \in [0, T]} \| Y_{n_{k+1}(t)} - Y_{n_k(t)} \|^4 \mathbf{1}_{S_X \leq M} \right] \right)^{\frac{1}{4}}
\leq C(1 + M) \sum_{k=0}^{\infty} \left( \frac{h^2}{4^k} \log^2 \left( \frac{2^{k+2}T}{h} \right) \right)^{\frac{1}{4}}
\leq C(1 + M) \sqrt{h \log \left( \frac{2T}{h} \right)}.
\]

Then by the Cauchy-Schwartz inequality, one has
\[
\left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^2 \right] \right)^{\frac{1}{2}} \leq \sum_{k=1}^{+\infty} \left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^2 \mathbf{1}_{k-1 \leq S_X \leq k} \right] \right)^{\frac{1}{2}}
\leq \sum_{k=1}^{+\infty} \left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^4 \mathbf{1}_{S_X \leq k} \right] \mathbb{P}(S_X \geq k - 1) \right)^{\frac{1}{2}}
\leq C \sqrt{h \log \left( \frac{2T}{h} \right)} \sum_{k=1}^{+\infty} (k + 1) \mathbb{P}(S_X \geq k - 1).
\]

Using claim (b) and Theorem 5.1 in [16], we know that
\[
\mathbb{P}(\sup_{0 \leq u \leq T} \| X_u \| \geq n) \leq 2 \exp \left( -\frac{n^2}{C(1 + \| x_0 \|^2)} \right).
\]

Therefore,
\[
\sum_{k=1}^{+\infty} (k + 1) \mathbb{P}(S_X \geq k - 1) \leq C + C \sum_{k=1}^{+\infty} (k + 2) \exp \left( -\frac{k^2}{C(1 + \| x_0 \|^2)} \right)
\leq C \left[ 1 + \int_{0}^{+\infty} (a + 2) \exp \left( -\frac{a^2}{C(1 + \| x_0 \|^2)} \right) \, da \right] \leq C(1 + \| x_0 \|),
\]

which means
\[
\left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^2 \right] \right)^{\frac{1}{2}} \leq C(1 + \| x_0 \|) \sqrt{h \log \left( \frac{2T}{h} \right)}.
\]

For general \( \eta \), we use the above inequality to deduce
\[
\mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^2 \right] \leq \mathbb{E} \left( \mathbb{E} \left[ \sup_{s, t \in [0, T], |s-t| \leq h} \| Y_t - Y_s \|^2 \mid X_0 = x_0 \right] \right)
\leq C(1 + \mathbb{E} \| \eta \|^2) h \log \left( \frac{2T}{h} \right) \leq C h \log \left( \frac{2T}{h} \right).
\]

With inequality (10), we obtain the desired result. \( \square \)
3 Examples of Test Function Classes

In this section, we propose three classes of test functions that satisfy Assumption 1.

3.1 The Reproducing Kernel Hilbert Space (RKHS)

RKHS has developed into an essential tool in many areas, especially statistics and machine learning [34]. We first recall its definition: a Hilbert space of functions $f : \mathbb{R}^d \to \mathbb{R}$, is said to be an RKHS if all evaluation functionals are bounded and linear. A more intuitive definition is through the so-called reproducing kernel.

A symmetric function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is called the reproducing kernel of $H$ if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k(x^i, x^j) \geq 0$$

holds for any $n \in \mathbb{N}, x^1, \ldots, x^n \in \mathbb{R}^d$, and $a_1, \ldots, a_n \in \mathbb{R}$. Moreover, it defines a unique Hilbert space $H_k$ such that: (a) $k(x, \cdot) \in H_k$, $\forall x \in \mathbb{R}^d$; (b) $f(x) = \langle f, k(x, \cdot) \rangle_{H_k}$, $\forall x \in \mathbb{R}^d$ and $f \in H_k$. The function $k$ is called the reproducing kernel of $H_k$ (cf. [3]). In particular, $k(x, y) = \langle k(x, \cdot), k(y, \cdot) \rangle_{H_k}$.

**Theorem 3.1.** Assume the reproducing kernel $k(\cdot, \cdot)$ satisfies:

(a) There exist constants $K_1, K_2 > 0$, such that $\forall x, y \in \mathbb{R}^d$, $k(x, x) + k(y, y) - 2k(x, y) \leq K_2^2\|x - y\|^2$ and $K_2 = \sqrt{k(0, 0)}$;

(b) If $\mu$ is a signed measure on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} k(x, y) \, d\mu(y) = 0, \forall x \in \mathbb{R}^d \Rightarrow \mu \equiv 0.$$

Then for any $\mu \in \mathcal{P}^1(\mathbb{R}^d)$, $\Phi = \{f \in H_k, \|f\|_{H_k} \leq 1\}$ satisfies Assumption 1 with $A_1 = K_1$, $A_2 = K_2$ and $A_3 = \sqrt{2}\max\{K_1, K_2\}$.

**Proof.** Assumption 1 (a) is implied by item (b) above. For Assumption 1 (b), $\forall f \in H_k$ such that $\|f\|_{H_k} \leq 1$, we compute

$$|f(x) - f(y)| = |\langle f, k(x, \cdot) - k(y, \cdot) \rangle_{H_k}| \leq \sqrt{|\langle k(x, \cdot) - k(y, \cdot), k(x, \cdot) - k(y, \cdot) \rangle_{H_k}|}$$

$$= \sqrt{k(x, x) + k(y, y) - 2k(x, y)} \leq K_1\|x - y\|,$$

for any $x, y \in \mathbb{R}^d$, which implies that the Lipschitz constant is $K_1$.

For Assumption 1 (c), we first derive an estimate for $k(x, x)$. To this end, let $n = 2$, $x^1 = x$, $x^2 = 0$ in inequality (12), we have

$$a_1^2k(x, x) + a_2^2k(0, 0) + 2a_1a_2k(x, 0) \geq 0,$$

for any $a_1, a_2$, implying $|k(x, 0)| \leq \sqrt{k(x, x)k(0, 0)}$. Therefore,

$$(\sqrt{k(x, x)} - \sqrt{k(0, 0)})^2 = k(x, x) + k(0, 0) - 2k(x, x)k(0, 0) \leq k(x, x) + k(0, 0) - 2k(x, 0) \leq K_1^2\|x\|^2,$$

and $k(x, x) \leq (\sqrt{k(0, 0)} + K_1 \|x\|)^2 = (K_2 + K_1 \|x\|)^2 \leq 2(K_2^2 + K_1^2\|x\|^2)$. Now we estimate the Rademacher
3.2 The Barron Space

Barron space was firstly introduced in [21, 22], which is designed to analyze the approximation and generalization properties of two-layer neural networks. It can be considered as the continuum analog of two-layer neural networks. See [21, Section 2.1] for a detailed discussion on Barron space.

**Definition 3.1.** We say $f : \mathbb{R}^d \to \mathbb{R}$ is a Barron function, if $f$ admits the following representation:

$$
f(x) = \int_{\mathbb{S}^d} \sigma(\omega \cdot x + b) \, d\rho(\omega, b),
$$

where $\sigma(x) = \max\{x, 0\}$ is the ReLU function, and $\rho$ is a finite signed measure on $\mathbb{S}^d = \{(\omega, b) \in \mathbb{R}^{d+1}, ||\omega||^2 + |b|^2 = 1\}$ with $|| \cdot ||$ being the $2$-norm. We will use the Barron space $\mathcal{B}$ to denote the collection of all Barron functions and define a norm $|| \cdot ||_{\mathcal{B}}$ on the Barron space as follows:

$$
\| f \|_{\mathcal{B}} = \inf_{\rho} \| \rho \|_{TV},
$$

where the infimum is taken over all $\rho$ for which (13) holds for all $x \in \mathbb{R}^d$, and $\| \cdot \|_{TV}$ is the total variation of $\rho$.

The following theorem reveals some useful properties of Barron space and shows that the unit ball of Barron space, denoted by $\mathcal{B}_1 := \{ f \in \mathcal{B}, \| f \|_{\mathcal{B}} \leq 1 \}$, can serve as a good choice of the test function class $\Phi$.

**Theorem 3.2.**

(a) Barron space is a Banach space.

(b) For any $f \in \mathcal{B}$, $f$ is Lipschitz continuous with the $l^2$ norm in $\mathbb{R}^d$ and $\text{Lip}(f) \leq \| f \|_{\mathcal{B}}$.

(c) Denote by $\mathcal{P}(\mathbb{S}^d)$ all probability measures on $\mathbb{S}^d$. For any $\pi \in \mathcal{P}(\mathbb{S}^d)$, let $k_\pi(x, x') : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be

$$
k_\pi(x, x') = \int_{\mathbb{S}^d} \sigma(\omega \cdot x + b)\sigma(\omega \cdot x' + b) \, d\pi(\omega, b),
$$

and $\mathcal{H}_\pi$ be the reproducing kernel Hilbert space (RKHS) associated with $k_\pi$. Then,

$$
\mathcal{B} = \bigcup_{\pi \in \mathcal{P}(\mathbb{S}^d)} \mathcal{H}_\pi,
$$

and

$$
\| f \|_{\mathcal{B}} = \inf_{\pi \in \mathcal{P}(\mathbb{S}^d)} \| f \|_{\mathcal{H}_\pi},
$$

with $\| \cdot \|_{\mathcal{H}_\pi}$ being the norm in $\mathcal{H}_\pi$. 

15
(d) Let \( f \) be a measurable function in \( L^2(\mathbb{R}^d) \) and
\[
\gamma(f) = \int_{\mathbb{R}^d} (1 + |\omega|^2) |\hat{f}(\omega)| \, d\omega < +\infty,
\]
where \( \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-i\omega \cdot x} \, dx \) is the Fourier transform of \( f \). Then \( f \in \mathcal{B} \) and
\[
\|f\|_\mathcal{B} \leq 4\gamma(f).
\]

(c) For any compact set \( K \subset \mathbb{R}^d \), the restriction of \( \mathcal{B} \) on \( K \) is dense in \( C(K) \).

(f) Let \( \mathcal{X} = \{x_1, \ldots, x_n\} \), where \( x^i \in \mathbb{R}^d \) and \( \Phi = \mathcal{B}_1 \). Then the empirical Rademacher complexity satisfies
\[
\text{Rad}_n(\Phi, \mathcal{X}) \leq \frac{2}{n} \sqrt{\sum_{i=1}^n (\|x^i\|^2 + 1)}.
\]

One can then conclude \( \Phi = \mathcal{B}_1 \) satisfies Assumption 1, by observing that Assumption 1 (a) is fulfilled by claim (e) above, and claims (b) and (f) together imply that \( \Phi \) satisfies Assumption 1 (b)–(c) with \( A_1 = A_2 = 1 \) and \( A_3 = 2 \).

Remark 3.1. In [4, Section 9], examples of functions with bounded \( \gamma(f) \) are provided (e.g., Gaussian, positive definite functions, linear functions, radial functions and functions in \( H^s(\mathbb{R}^d) \) with \( s > \frac{d}{2} + 1 \)). By claim (d), they all belong to the Barron space.

Proof. (a) Let \( \mathcal{M} \) be the set of all signed measures on \( \mathbb{S}^d \). With \( \|\rho\|_{\mathcal{M}} := \|\rho\|_{TV} \), \( \mathcal{M} \) is a Banach space.

Let \( \mathcal{N} = \{\rho \in \mathcal{M} : \int_{\mathbb{R}^d} \sigma(\omega \cdot x + b) \, d\rho(x, b) = 0, \forall x \in \mathbb{R}^d\} \), then \( \mathcal{N} \) is a closed subspace of \( \mathcal{M} \). It is evident that \( \mathcal{B} \equiv \mathcal{M}/\mathcal{N} \), thus we obtain the desired result [52, Theorem 1.41].

(b) For any \( (\omega, b) \in \mathbb{S}^d \), \( \text{Lip}(\sigma(\omega \cdot x + b)) \leq 1 \). Then our result follows from the subadditivitiy of Lip.

(c) See [21, Theorem 3].

(d) Our proof is similar to that in [40, Theorem 6]. By the property of Fourier transform, one has
\[
f(x) = x \cdot \nabla f(0) + f(0) + \int_{\mathbb{R}^d} (e^{i\omega \cdot x} - i\omega \cdot x - 1)\hat{f}(\omega) \, d\omega.
\]  

For \( |z| \leq c \), the following identity holds
\[
-\int_0^c [\sigma(z - u)e^{iu} + \sigma(-z - u)e^{-iu}] \, du = e^{iz} - iz - 1.
\]

Now let \( c = \|\omega\|, \, z = \omega \cdot x, \, \alpha(\omega) = \omega/\|\omega\| \) and \( u = \|\omega\|t \), we have:
\[
-\|\omega\|^2 \int_0^1 \sqrt{1 + t^2} \left[ \sigma \left( \frac{\alpha(\omega) \cdot x - t}{\sqrt{1 + t^2}} \right) e^{i|\omega|t} + \sigma \left( -\frac{\alpha(\omega) \cdot x - t}{\sqrt{1 + t^2}} \right) e^{-i|\omega|t} \right] \, dt = e^{i\omega \cdot x} - i\omega \cdot x - 1.
\]

In other words,
\[
e^{i\omega \cdot x} - i\omega \cdot x - 1 = \int_{\mathbb{S}^d} \sigma(\omega' \cdot x + b) \, d\rho(\omega', b)
\]
is in the Barron space \( \mathcal{B} \) with \( \|\rho\|_{TV} \leq 2\sqrt{2}\|\omega\|^2 \). Recall that equation (14) gives
\[
f(x) = \|\nabla f(0)\| \left[ \sigma \left( \frac{\nabla f(0) \cdot x}{\|\nabla f(0)\|} \right) + \sigma \left( -\frac{\nabla f(0) \cdot x}{\|\nabla f(0)\|} \right) \right] + f(0)\sigma(0 \cdot x + 1)
\]
\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{S}^d} \sigma(\omega' \cdot x + b) \, d\rho(\omega', b) \hat{f}(\omega) \, d\omega,
\]
and one concludes
\[
\|f\|_\mathcal{B} \leq 2\|\nabla f(0)\| + |f(0)| + 2\sqrt{2} \int_{\mathbb{R}^d} \|\omega\|^2 \hat{f}(\omega) \, d\omega \leq \int_{\mathbb{R}^d} [1 + 2\|\omega\| + 2\sqrt{2}\|\omega\|^2] |\hat{f}(\omega)| \, d\omega \leq 4\gamma(f).
\]
(e) For any \( f \in C_0^\infty(\mathbb{R}^d) \), we have \( \gamma(f) < +\infty \) and hence \( f \in B \). Because the restriction of \( C_0^\infty(\mathbb{R}^d) \) on \( K \) is dense in \( C(K) \), we easily conclude.

(f) Our proof here is similar to the one in [21, Theorem 6].

\[
\text{Rad}_n(\Phi, \mathcal{X}) = \frac{1}{n} \mathbb{E} \sup_{f \in \mathcal{Y}} \sum_{i=1}^{n} \xi_i f(x^i)
\]
\[
= \frac{1}{n} \mathbb{E} \sup_{\|\rho\|_{TV} \leq 1} \left| \int_{\mathbb{S}^d} \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b) \, d\rho(\omega, b) \right|
\]
\[
= \frac{1}{n} \mathbb{E} \sup_{(\omega, b) \in \mathbb{S}^d} \left| \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b) \right|
\]
\[
\leq \frac{1}{n} \left[ \mathbb{E} \max \left\{ \sup_{(\omega, b) \in \mathbb{S}^d} \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b), 0 \right\} + \mathbb{E} \max \left\{ \sup_{(\omega, b) \in \mathbb{S}^d} - \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b), 0 \right\} \right]
\]
\[
= \frac{2}{n} \mathbb{E} \sup_{(\omega, b) \in \mathbb{S}^d} \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b),
\]

where the last equality holds due to \( \sup_{(\omega, b) \in \mathbb{S}^d} \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot x^i + b) \geq 0 \) and the symmetry of \( \xi_1, \ldots, \xi_n \).

Then Lemma 26.9 in [53] gives

\[
\text{Rad}_n(\Phi, \mathcal{X}) \leq \frac{2}{n} \mathbb{E} \sup_{(\omega, b) \in \mathbb{S}^d} \sum_{i=1}^{n} \xi_i (\omega \cdot x^i + b) \leq \frac{2}{n} \mathbb{E} \left\| \sum_{i=1}^{n} \xi_i ((x^i)^T, 1)^T \right\| \leq \frac{2}{n} \mathbb{E} \left( \sum_{i=1}^{n} (\|x^i\|^2 + 1) \right).
\]

\[ \square \]

Remark 3.2. As the Barron space serves as the continuum analog of two-layer neural networks, the general-
ized Barron space or the Banach space associated with multi-layer networks introduced in [24] serves as
the continuum analog of multi-layer neural networks. By the fact that the unit ball of the Barron space \( B_1 \)
is a Polish space, one can define a signed Radon measure \( \rho \) on the Borel \( \sigma \)-algebra of \( B_1 \), and then define

\[
f_{\rho}(x) = \int_{B_1} \sigma(g(x)) \, d\rho(g),
\]

\[
\|f\|_{B^2} = \inf \{ \|\rho\|_{TV} : f = f_{\rho} \text{ on } \mathbb{R}^d \},
\]

\[
B^2 = \{ f \in C(\mathbb{R}^d), \|f\|_{B^2} < +\infty \},
\]

where the integral is in the sense of Bochner integrals. \( B^2 \) can be interpreted as the Banach space associated
with three-layer neural networks. We can then repeat this process and define \( B^L \) for any \( L \geq 2 \), which is
associated with \( (L + 1) \)-layer neural networks. Naturally we denote by \( B^1 \) the Barron space \( B \) defined via
(13). It can be proven that the unit ball of the generalized Barron space is also a suitable test function
class. For more technical issues and intuition about \( B^L \), we refer to [24, Section 2.2].

3.3 Flow-induced Function Spaces

Flow-induced function spaces introduced in [21] can serve as a continuum analog as the residual neural
networks (ResNet, [33]). Our definition here is slightly different from the original definition in [21] but
shares the same spirit. Given an integer \( D \geq d + 1 \), let \( \rho = \{ \rho_\tau \}_{0 \leq \tau \leq 1} \) be a class of vector signed measure
from \( \mathbb{S}^{D-1} = \{ \omega \in \mathbb{R}^D, \|\omega\| = 1 \} \) to \( \mathbb{R}^D \) such that the following ordinary differential equation (ODE)

\[
\frac{dZ_\tau}{d\tau} = \int_{\mathbb{S}^{D-1}} \sigma(\omega \cdot Z_\tau) \, d\rho_\tau(\omega), \quad \forall \ 0 \leq \tau \leq 1, \ Z_0 = z_0
\]

(15)
Taking $z$ to 1 gives
where

$$
\rho = (\rho_1, \rho_2, \ldots, \rho_D)
$$

is well-posed for any initial condition $z_0$, i.e., the solution exists, is unique, and is continuous with respect to the initial condition $z_0$. Let us denote by $\Psi$ the collection of all admissible $\rho$ satisfying the well-posedness condition. Then, for any $x \in \mathbb{R}^d$, let $Z_\rho(\tau, x)$ be the solution of (15) with the initial condition $z_0 = (x^T, 1, 0_{(d-d-1)})^T$, where $0_{(d-d-1)}$ denotes a vector of zeros of length $D - d - 1$. And for any $\rho = \{\rho_\tau\}_{0 \leq \tau \leq 1} \in \Psi$, we define

$$
\Lambda(\rho, \tau) = \sqrt{\sum_{i=1}^{D} \|\rho^i_\tau\|_{TV}^2},
$$

where $\rho^i_\tau$ is the $i$th component of $\rho_\tau$. To simplify the discussion, we in addition require that $\Lambda(\rho, \tau)$ is continuous with respect to $\tau$ for any $\rho \in \Psi$. Then, we can define the flow-induced function spaces as follows:

$$
f_{\rho, \alpha}(x) := \alpha^T Z_\rho(1, x), \quad \forall \rho \in \Psi, \quad \alpha \in \mathbb{R}^D,
$$

$$
\|f\|_D := \inf \left\{ \|\alpha\| \exp \left( \int_{0}^{1} \Lambda(\rho, \tau) \, d\tau \right) \mid f = f_{\rho, \alpha} \text{ on } \mathbb{R}^d \right\},
$$

$$
\mathcal{D} := \{ f \in C(\mathbb{R}^d), \|f\|_D < +\infty \}.
$$

The following theorem gives some useful properties of the flow-induced function spaces and indicates that the unit balls of the flow-induced function spaces are also appropriate test function classes $\Phi$.

**Theorem 3.3.** Fix an integer $D \geq d + 1$.

(a) For any $f \in \mathcal{D}$, $\text{Lip}(f) \leq \|f\|_D$ and $|f(0)| \leq \|f\|_D$.

(b) If $D \geq d + 2$, then for any $f \in \mathcal{B}$, $f \in \mathcal{D}$ and $\|f\|_D \leq c \|f\|_\mathcal{B}$. Hence, by Theorem 3.2 (e), the restriction of $\mathcal{D}$ on any compact set $K \subset \mathbb{R}^d$ is dense in $C(K)$.

(c) Let $\mathcal{X} = \{x^1, \ldots, x^n\}$, where $x^i \in \mathbb{R}^d$ and $\Phi = \{ f \in \mathcal{D}, \|f\|_D \leq 1 \}$. Then, the empirical Rademacher complexity satisfies

$$
\text{Rad}_n(\Phi, \mathcal{X}) \leq \frac{c^2}{n} \left\{ \sum_{i=1}^{n} (\|x^i\|^2 + 1) \right\}.
$$

Then, claims (a) and (c) together imply that $\Phi$ satisfies the Assumption 1 (b)–(c) with $A_1 = A_2 = 1$ and $A_3 = c^2$ and claim (b) implies $\Phi$ satisfies Assumption 1 (a) when $D \geq d + 2$.

**Proof.** For claim (a), given $\rho \in \Psi$, let $Z_\tau, Z'_\tau$ be the solutions to the ODE (15) with the initial conditions $z_0, z'_0 \in \mathbb{R}^D$, respectively. Then,

$$
\frac{d}{d\tau} \|Z_\tau - Z'_\tau\|^2 = 2 \int_{\mathbb{S}^{D-1}} [\sigma(\omega \cdot Z_\tau) - \sigma(\omega \cdot Z'_\tau)] \, d[Z_\tau - Z'_\tau]^T \rho_\tau(\omega)
$$

$$
\leq 2 \int_{\mathbb{S}^{D-1}} |\omega \cdot (Z_\tau - Z'_\tau)| \, d|[Z_\tau - Z'_\tau]^T \rho_\tau(\omega)|
$$

$$
\leq 2 \Lambda(\rho, \tau) \|Z_\tau - Z'_\tau\|^2,
$$

where $|\cdot|$ is obtained by taking element-wise absolute values of a vector. Integrating both sides from $\tau = 0$ to 1 gives

$$
\|Z_1 - Z'_1\| \leq \|z_0 - z'_0\| \exp \left( \int_{0}^{1} \Lambda(\rho, \tau) \, d\tau \right).
$$

Taking $z_0 = (x^T, 1, 0_{(D-d-1)})$ and $z'_0 = ((x')^T, 1, 0_{(D-d-1)})$, we deduce

$$
|\alpha^T Z_\rho(1, x) - \alpha^T Z_\rho(1, x')| \leq \|\alpha\| \exp \left( \int_{0}^{1} \Lambda(\rho, \tau) \, d\tau \right) \|x - x'\|,
$$

18
and by taking $z_0 = (0_d, 1, 0_{(D-d-1)})^T$ and $z'_0 = 0^T_{1_d}$, we have

$$|\alpha^T Z_{\rho}(1,0)| \leq \|\alpha\| \exp \left( \int_0^1 \Lambda(\rho, \tau) \, d\tau \right).$$

Now by the definition of $f$ in $D$ and $\|f\|_D$, we easily conclude.

For claim (b), without loss of generality, assume $D = d + 2$. For any $f \in B$ and $\epsilon > 0$, let

$$f(x) = \int_{S^d} \sigma(\omega \cdot x + b) \, d\rho(\omega, b)$$

with $0 < \|\rho\|_{TV} \leq \|f\|_{B} + \epsilon$. It is easy to show that $f(x) = \|\rho\|_{TV} 1^T_{d+2} Z_{\rho}(1, x)$ where $1_{d+2} = (0_{d+1}, 1)^T$ and

$$\hat{\rho}^i = 0, \quad 1 \leq i \leq d + 1, \quad \text{and} \quad \hat{\rho}^{d+2} = \frac{\rho}{\|\rho\|_{TV}}, \quad \text{for} \quad 0 \leq \tau \leq 1.$$

Therefore

$$\Lambda(\rho, \tau) \equiv 1, \quad 0 \leq \tau \leq 1,$$

and $\|f\|_D \leq \epsilon \|\rho\|_{TV} \leq \epsilon \|f\|_{B} + \epsilon$ for any $\epsilon > 0$, which concludes our proof.

For claim (c), we first prove that for any $f \in \Phi = \{f \in D, \|f\|_D \leq 1\}$ and $\epsilon > 0$, there exist $\overline{\rho} \in \Psi$ and $\alpha \in \mathbb{R}^D$ such that $f(x) = \alpha^T Z_{\overline{\rho}}(1, x)$ with

$$\|\alpha\| \exp \left( \int_0^1 \Lambda(\overline{\rho}, \tau) \, d\tau \right) \leq 1 + \epsilon, \quad \|\alpha\| \sup_{0 \leq \tau \leq 1} \Lambda(\overline{\rho}, \tau) \exp \left( \int_0^\tau \Lambda(\overline{\rho}, \tau') \, d\tau' \right) \leq 1 + \epsilon.$$ (16)

First, we choose $\alpha \in \mathbb{R}^D$ and $\rho \in \Psi$ satisfying that

$$f(x) = f_{\rho, \alpha}(x) = \alpha^T Z_{\rho}(1, x), \quad \text{and} \quad \|\alpha\| \exp \left( \int_0^1 \Lambda(\rho, \tau) \, d\tau \right) \leq 1 + \frac{\epsilon}{2}.$$

Given a strictly increasing and continuously differentiable function $F(\tau) : [0,1] \rightarrow [0,1]$ satisfying that $F(0) = 0$, $F(1) = 1$, we define

$$\overline{\rho}_\tau := F'(\tau) \rho_{F(\tau)}.$$

Then, from equation (15) one has

$$\frac{dZ_{\rho}(F(\tau), x)}{d\tau} = \int_{\mathcal{S}_{d-1}} \sigma(\omega \cdot Z_{\rho}(F(\tau), x)) \, d\overline{\rho}_\tau(\omega).$$

In addition, $Z_{\overline{\rho}}(\tau, x) = Z_{\rho}(F(\tau), x)$ and $f_{\overline{\rho}, \alpha} = f_{\rho, \alpha} = f$ on $\mathbb{R}^d$. Note that the wellposedness of the above ODE can be deduced from the fact that $F$ is an isomorphism on $[0,1]$. Noticing that

$$\int_0^1 \Lambda(\overline{\rho}, \tau) \, d\tau = \int_0^1 F'(\tau) \Lambda(\rho, F(\tau)) \, d\tau = \int_0^1 \Lambda(\rho, \tau) \, d\tau,$$

we have

$$\|\alpha\| \exp \left( \int_0^1 \Lambda(\overline{\rho}, \tau) \, d\tau \right) = \|\alpha\| \exp \left( \int_0^1 \Lambda(\rho, \tau) \, d\tau \right) \leq 1 + \frac{1}{2} \epsilon,$$

$$\|\alpha\| \sup_{0 \leq \tau \leq 1} \Lambda(\overline{\rho}, \tau) \exp \left( \int_0^\tau \Lambda(\overline{\rho}, \tau') \, d\tau' \right) = \|\alpha\| \sup_{0 \leq \tau \leq 1} F'(\tau) \Lambda(\rho, F(\tau)) \exp \left( \int_0^{F(\tau)} \Lambda(\rho, \tau') \, d\tau' \right).$$

Next, assume without loss of generality that $\exp(\int_0^\tau \Lambda(\rho, \tau') \, d\tau')$ strictly increases in $\tau$. Then, there exists a continuous and strictly increasing function $F^* : [0,1] \rightarrow [0,1]$ with $F^*(0) = 0$, $F^*(1) = 1$ and

$$\exp \left( \int_0^{F^*(\tau)} \Lambda(\rho, \tau') \, d\tau' \right) = 1 - \tau + \tau \exp \left( \int_0^1 \Lambda(\rho, \tau') \, d\tau' \right), \quad \forall \tau \in [0,1].$$
which gives (by implicit function theorem), when \( \Lambda(\rho, F^*(\tau)) \neq 0 \),

\[
(F^*)'(\tau) \Lambda(\rho, F^*(\tau)) \exp \left( \int_0^{F^*(\tau)} \Lambda(\rho, \tau') \, d\tau' \right) = \exp \left( \int_0^{1} \Lambda(\rho, \tau') \, d\tau' \right) - 1.
\]

Then equation (16) is obtained by approximating \( F^* \) through a continuously differentiable isomorphism on \([0,1]\).

Return to the proof of claim (c), for any \( \epsilon > 0 \), we define

\[\mathcal{Q}^\epsilon = \left\{ (\alpha, \rho) \in \mathbb{R}^D \times \Psi : \|\alpha\| \exp \left( \int_0^{1} \Lambda(\rho, \tau) \, d\tau \right) \leq 1 + \epsilon, \|\alpha\| \sup_{0 \leq \tau \leq 1} \Lambda(\rho, \tau) \exp \left( \int_0^{1} \Lambda(\rho, \tau') \, d\tau' \right) \leq 1 + \epsilon \right\},\]

and

\[R^\epsilon_\tau = \frac{1}{n} \mathbb{E} \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \sum_{i=1}^{n} \xi_i \alpha^T Z(\tau, x^i).
\]

With (16), we have \( \text{Rad}_n(\Phi, \mathcal{X}) \leq R^\epsilon_1 \) for any \( \epsilon \). So it suffices to deduce an upper bound for \( R^\epsilon_1 \).

A straightforward calculation gives

\[|R^\epsilon_\tau - R^\epsilon_0| \leq \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \max_{1 \leq i \leq n} |\alpha^T[Z(\tau, x^i) - Z(\tau', x^i)]|,
\]

and hence \( R^\epsilon_\tau \) is continuous. Next we fix \( \tau \in (0,1) \) and compute

\[
\lim_{h \to 0} \frac{R^\epsilon_{\tau+h} - R^\epsilon_{\tau}}{h} \leq \frac{1}{n} \mathbb{E} \lim_{h \to 0} \left[ \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \sum_{i=1}^{n} \xi_i \alpha^T Z(\tau + h, x^i) - \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \sum_{i=1}^{n} \xi_i \alpha^T Z(\tau, x^i) \right]
\]

\[
\leq \frac{1}{n} \mathbb{E} \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \left[ \sum_{i=1}^{n} \xi_i \alpha^T d\sigma(\omega \cdot Z(\tau, x^i)) \right]
\]

\[
= \frac{1}{n} \mathbb{E} \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \sum_{i=1}^{n} \xi_i \int_{\mathbb{R}^D} \sigma(\omega \cdot Z(\tau, x^i)) \, d\rho(\omega)
\]

\[
\leq \frac{1}{n} \sup_{(\alpha, \rho) \in \mathcal{Q}^\epsilon} \|\alpha\| \Lambda(\rho, \tau) \exp \left( \int_0^{\tau} \Lambda(\rho, \tau') \, d\tau' \right) \mathbb{E} \left[ \sup_{\|\omega\| \leq 1, \rho \in \Psi} \left| \sum_{i=1}^{n} \xi_i \sigma(\omega \cdot Z(\tau, x^i)) \right| \right]
\]

\[
\leq \frac{2(1 + \epsilon)}{n} \mathbb{E} \sup_{\|\omega\| \leq 1, \rho \in \Psi} \sum_{i=1}^{n} \xi_i \left( \frac{\omega^T Z(\tau, x^i)}{\exp(\int_0^{\tau} \Lambda(\rho, \tau') \, d\tau')} \right).
\]

For any \( \rho \in \Psi \), using a similar argument to the proof of equation (16), we can find another \( \overline{\rho} \in \Psi \) such that \( Z(\tau, x) = Z(\overline{\rho}(\tau, x)) \) and

\[
\exp \left( \int_0^{\tau} \Lambda(\overline{\rho}, \tau') \, d\tau' \right) \leq (1 + \epsilon) \exp \left( \int_0^{\tau} \Lambda(\rho, \tau') \, d\tau' \right),
\]

\[
\sup_{0 \leq u \leq \tau} \Lambda(\overline{\rho}, u) \exp \left( \int_0^{u} \Lambda(\overline{\rho}, \tau') \, d\tau' \right) \leq (1 + \epsilon) \exp \left( \int_0^{\tau} \Lambda(\rho, \tau') \, d\tau' \right).
\]

We can then find another \( \hat{\rho} \in \Psi \) such that \( \hat{\rho}(\tau, x) = \overline{\rho}(\tau, x) \) for all \( 0 \leq \tau' \leq \tau \) and

\[
\exp \left( \int_0^{1} \Lambda(\hat{\rho}, \tau') \, d\tau' \right) \leq (1 + \epsilon)^2 \exp \left( \int_0^{\tau} \Lambda(\rho, \tau') \, d\tau' \right),
\]

\[
\sup_{0 \leq u \leq 1} \Lambda(\hat{\rho}, u) \exp \left( \int_0^{u} \Lambda(\hat{\rho}, \tau') \, d\tau' \right) \leq (1 + \epsilon)^2 \exp \left( \int_0^{\tau} \Lambda(\rho, \tau') \, d\tau' \right).
\]
Hence, $Z_\rho(\tau, x) = Z_\rho(\tau, x)$, and for any $\omega$ with $\|\omega\| \leq 1$,

$$
\left( \frac{\omega}{(1 + \varepsilon)^2 \exp(\int_0^\tau \Lambda(\rho, \tau') \, d\tau')} \right) \in \mathcal{Q}^c.
$$

Therefore,

$$
\lim_{h \to 0} \frac{R_{\tau+h}^\varepsilon - R_\tau^\varepsilon}{h} \leq \frac{2(1 + \varepsilon)^3}{n} \mathbb{E} \left[ \sup_{(\rho, \rho') \in \mathcal{Q}^c} \sum_{i=1}^n \xi_i \alpha^T Z_\rho(\tau, x^i) \right] = 2(1 + \varepsilon)^3 R_\tau^\varepsilon,
$$

which means

$$
\lim_{h \to 0} \frac{\exp[-2(1 + \varepsilon)^3(\tau+h)] R_{\tau+h}^\varepsilon - \exp[-2(1 + \varepsilon)^3] R_\tau^\varepsilon}{h} \leq 0.
$$

For any $\varepsilon' > 0$, let $P_\tau = \exp[-2(1 + \varepsilon)^3 R_\tau^\varepsilon - \varepsilon' \tau]$, then $P_\tau$ is continuous in $\tau$ and satisfies

$$
\lim_{h \to 0} \frac{P_{\tau+h} - P_\tau}{h} \leq -\varepsilon' < 0,
$$

which means that $P_\tau$ is decreasing. Therefore,

$$
\exp[-2(1 + \varepsilon)^3 R_1^\varepsilon - \varepsilon' \leq R_0^\varepsilon,
$$

or $R_1^\varepsilon \leq e^{2(1+\varepsilon)} R_0^\varepsilon$ by letting $\varepsilon' \to 0$. We can conclude our proof by computing

$$
R_0^\varepsilon = \frac{1}{n} \mathbb{E} \sup_{\|\alpha\| \leq 1 + \varepsilon} \sum_{i=1}^n \xi_i ((x^i)^T, 1, 0, D_{-d-1}) \| \frac{1 + \varepsilon}{n} \mathbb{E} \sum_{i=1}^n \xi_i ((x^i)^T, 1, 0, D_{-d-1}) \| \leq \frac{1 + \varepsilon}{n} \left( \sum_{i=1}^n (x^i)^2 + 1 \right),
$$

and letting $\varepsilon \to 0$ in $\text{Rad}_n(\Phi, \mathcal{X}) \leq R_1^\varepsilon \leq e^{2(1+\varepsilon)} R_0^\varepsilon$. \hfill \Box

## 4 Application to McKean-Vlasov SDE

This section presents an application of GMMD proposed in Definition 2.1 to McKean-Vlasov Stochastic Differential Equation (SDE). Throughout this section, we fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, supporting $n+1$ independent $m$-dimensional Brownian motions $\{W^i\}_{i=1}^m$ and $W$, as well as i.i.d. $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variables $\{\eta_i\}_{i=1}^m$ with law $\eta$ and $\mathbb{E}\|\eta\|^2 < +\infty$. We are interested in the rate of convergence as $n \to \infty$ of an $n$-interacting particle system satisfying:

$$
\begin{align*}
\frac{dX_t^{n,i}}{dt} &= B(t, X_t^{n,i}, \mu_t^n) \, dt + \Sigma(t, X_t^{n,i}, \mu_t^n) \, dW_t^i, \quad X_0^{n,i} = \eta_i, \quad i \in \mathcal{I} := \{1, \ldots, n\}, \\
\mu_t^n := \frac{1}{n} \sum_{i=1}^n \delta_{X_t^{n,i}}.
\end{align*}
$$

More precisely, let $X_t$ solve the McKean-Vlasov stochastic differential equation:

$$
\begin{align*}
\frac{dX_t}{dt} &= B(t, X_t, \mu_t) \, dt + \Sigma(t, X_t, \mu_t) \, dW_t, \quad X_0 = \eta, \\
\mu_t := \mathcal{L}(X_t),
\end{align*}
$$

where $\mathcal{L}(X_t)$ denotes the law of $X_t$, we are interested in quantifying $\mathbb{E} \left[ \sup_{0 \leq t \leq T} D_2^2(\mu_t, \mu_t^n) \right]$. To this end, we consider the following assumption.

### Assumption 3.

Assume $B : [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\Sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ are Lipschitz in $(x, \mu)$:

$$
\|B(t, x, \mu) - B(t, x', \mu')\|^2 + \|\Sigma(t, x, \mu) - \Sigma(t, x', \mu')\|^2 \leq K^2(\|x - x'\|^2 + D_2^2(\mu, \mu')).
$$

Here the distance between $\mu$ and $\mu'$ are measured by GMMD associated with a test function class $\Phi$, and

$$
\| \cdot \|_F \text{ denotes the Frobenius norm on } \mathbb{R}^{d \times m}.
$$

Also, assume that

$$
\sup_{t \in [0, T]} \|B(t, 0, \delta_0)\| + \|\Sigma(t, 0, \delta_0)\|_F \leq K.
$$
Remark 4.1. Inequality (19) is satisfied, for instance, when $B$ and $\Sigma$ are of the form

$$(B, \Sigma) = (B, \Sigma) \left( t, x, \int_{\mathbb{R}^d} f^1(x) \, d\mu(x), \int_{\mathbb{R}^d} f^2(x) \, d\mu(x), \ldots , \int_{\mathbb{R}^d} f^k(x) \, d\mu(x) \right),$$

and are Lipschitz in $\int_{\mathbb{R}^d} f^i(x) \, d\mu(x)$ with $f^i(x)$ in the class of test functions $\Phi$.

**Theorem 4.1.** Under Assumptions 1 and 3, and the assumption that $\mathbb{E}[\eta^2] \leq K^2$, we have:

(a) There exist unique adapted $L^2$-solutions for the $n$-body SDE (17) and the McKean-Vlasov SDE (18).

(b) There exists a constant $C > 0$, depending only on $K$ and $T$, such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|X_t\|_2^2 \leq C, \quad \mathbb{E} \sup_{0 \leq t \leq T} \|X_t^n\|_2^2 \leq C.$$

(c) There exists a constant $C > 0$, depending only on $K, T, A_1, A_2$ and $A_3$, such that

$$\mathbb{E} \sup_{0 \leq t \leq T} D_\Phi^2(\mu_t, \bar{\mu}_t^n) \leq C \frac{\log n}{n}.$$

Remark 4.2. As will seen in the proof, Theorem 4.1 (c) relies on the estimates in Theorems 2.3, about which we have mentioned in Remark 2.6 that the logarithm term can be removed from $\phi(n)$ for all the examples of $\Phi$ mentioned in Section 3. Consequently, claim (c) in Theorem 4.1 can be further improved to be bounded by $C/n$ when using test functions in Section 3.

**Proof.** Throughout the proof, we will use $C$ as a generic positive constant depending only on $K, T, A_1, A_2$ and $A_3$, which may vary from line to line.

By the relation between $D_\Phi$ and $W_2$ stated in Theorem 2.1 (a), Claim (a) follows from Lemma 3.2 and Theorem 3.3 in [41].

For claim (b), define $x = [x^1, \ldots , x^n]^T, L_n(x) = \frac{1}{n} \sum_{i=1}^n \delta_{x^i}$, and

$$B(t, x) = [B(t, x^1, L_n(x)), \ldots , B(t, x^n, L_n(x))]^T,$$

$$\Sigma(x) = \begin{bmatrix} \Sigma(t, x^1, L_n(x)) \\ \Sigma(t, x^2, L_n(x)) \\ \vdots \\ \Sigma(t, x^n, L_n(x)) \end{bmatrix},$$

where $\Sigma$ has zero entries except for the $n$ blocks of size $d \times m$ on the main diagonal. Then we can rewrite the $n$-body SDE (17) as

$$dX_t^n = B(t, X_t^n) \, dt + \Sigma(t, X_t^n) \, dW_t,$$  \hspace{1cm} (20)

where $X_t^n = [X_t^{1, n}, \ldots , X_t^{n, n}]^T$ and $W_t = [W_1^n, \ldots , W_n^n]^T$. Following Lemma 3.2 in [41], we obtain that $B$ and $\Sigma$ are $2L$-Lipschitz. Standard SDE estimates (cf. [63, Theorem 3.2.2]) give

$$\sup_{0 \leq t \leq T} \mathbb{E} \sum_{i=1}^n \|X_t^{i,n}\|^2 \leq Cn.$$

Notice that $X_t^{1,n}, \ldots , X_t^{n,n}$ are symmetric, one has

$$\sup_{0 \leq t \leq T} \mathbb{E}\|X_t^{1,n}\|^2 \leq C.$$

Then, using the Burkholder-Davis-Gundy inequality (cf. [63, Theorem 2.4.1]), we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \|X_t^{1,n}\|^2 \leq C \left[ \mathbb{E}\|\eta\|^2 + \mathbb{E} \int_0^T \|B(t, X_t^{1,n}, \bar{\mu}_t^n)\|^2 \, dt + \mathbb{E} \sup_{0 \leq u \leq T} \mathbb{E} \int_0^T \Sigma(t, X_t^{1,n}, \bar{\mu}_t^n) \, dW_t \right]^2 \leq C \left[ \mathbb{E}\|\eta\|^2 + \mathbb{E} \int_0^T \|B(t, X_t^{1,n}, \bar{\mu}_t^n)\|^2 \, dt + \mathbb{E} \int_0^T \|\Sigma(t, X_t^{1,n}, \bar{\mu}_t^n)\|^2 \, dt \right] \leq C \left[ 1 + \sup_{0 \leq t \leq T} \mathbb{E}\|X_t^{1,n}\|^2 + \sup_{0 \leq t \leq T} \mathbb{E} D_\Phi^2(\bar{\mu}_t^n, \delta_0) \right] \leq C \left[ 1 + \sup_{0 \leq t \leq T} \mathbb{E}\|X_t^{1,n}\|^2 \right] \leq C.$$
By [41, Theorem 3.3], we know that, as $n \to \infty$,
\[ X^{n,1} \Rightarrow X, \text{ in distribution in } C([0,T];\mathbb{R}^d). \]

We then obtain the second inequality in claim (b) through the Fatou’s Lemma.

For claim (c), let
\[ dY_t^{n,i} = B(t, Y_t^{n,i}, \mu_t) dt + \Sigma(t, Y_t^{n,i}, \mu_t) dW_t^{i}, \quad Y_0^{n,i} = \eta^i. \]

Then, \( \{Y_t^{n,i}\}_{i=1}^n \) are i.i.d. copies of \( X_t \). Following [63, Theorem 3.2.4], we obtain, \( \forall t \in [0,T], \)
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|X_s^{n,i} - Y_s^{n,i}\|^2 \right] \leq C \mathbb{E} \int_0^t \left[ \|B(s, X_s^{n,i}, \tilde{\mu}_s^{n}) - B(s, Y_s^{n,i}, \mu_s)\|^2 + \|\Sigma(s, X_s^{n,i}, \tilde{\mu}_s^{n}) - \Sigma(s, Y_s^{n,i}, \mu_s)\|^2 \right] ds. \]

Using the Lipschitz condition (19) in Assumption 3, we deduce
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|X_s^{n,i} - Y_s^{n,i}\|^2 \right] \leq C \mathbb{E} \int_0^t \left[ \sup_{0 \leq u \leq s} \|X_u^{n,i} - Y_u^{n,i}\|^2 + D_\Phi^2(\tilde{\mu}_s^{n}, \mu_s) \right] ds. \]

Then Gronwall’s inequality gives
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|X_s^{n,i} - Y_s^{n,i}\|^2 \right] \leq C \mathbb{E} \left[ \int_0^t D_\Phi^2(\tilde{\mu}_s^{n}, \mu_s) \, ds \right]. \]

Let \( \tilde{\mu}_t^n \) be the empirical measure of \( \{Y_t^{n,i}\}_{i=1}^n \), i.e.,
\[ \tilde{\mu}_t^n := \frac{1}{n} \sum_{i=1}^n \delta_{Y_t^{n,i}}. \]

With \( \sup_{0 \leq t \leq T} D_\Phi(\mu_t, \delta_0) \leq A_1 \sup_{0 \leq t \leq T} \mathbb{E} \|X_t\| \leq C \), we obtain
\[ \sup_{0 \leq t \leq T} \left[ \|B(t, 0, \mu_t)\| + \|\Sigma(t, 0, \mu_t)\|_F \right] \leq C. \]

Viewing \( \mu_t \) as a given function of \( t \), the Mckean-Vlasov SDE (18) satisfies the conditions in Theorem 2.4. Thus, combining results in Theorems 2.3 and 2.4, we have
\[ \mathbb{E} \sup_{0 \leq t \leq T} D_\Phi^2(\mu_t, \tilde{\mu}_t^n) \leq C \frac{\log n}{n}. \]

By the definition of \( D_\Phi \), one has
\[ D_\Phi^2(\tilde{\mu}_t^n, \tilde{\mu}_t^n) \leq \left[ \frac{A_1}{n} \sum_{i=1}^n \|X_t^{n,i} - Y_t^{n,i}\|^2 \right] \leq \frac{A_2^2}{n} \sum_{i=1}^n \|X_t^{n,i} - Y_t^{n,i}\|^2. \]

Therefore
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} D_\Phi^2(\mu_s, \tilde{\mu}_s^n) \right] \leq 2\mathbb{E} \left[ \sup_{0 \leq s \leq t} D_\Phi^2(\tilde{\mu}_s^n, \tilde{\mu}_s^n) + \sup_{0 \leq s \leq t} D_\Phi^2(\mu_s, \tilde{\mu}_s^n) \right] \leq C \mathbb{E} \left[ \int_0^t D_\Phi^2(\mu_s, \tilde{\mu}_s^n) \, ds \right] + C \frac{\log n}{n}. \]

With Gronwall’s inequality, we can obtain the desired result. \[ \square \]

We can furthermore establish a concentration inequality for \( \sup_{0 \leq t \leq T} D_\Phi(\mu_t, \tilde{\mu}_t^n) \).

**Theorem 4.2.** Under Assumptions 1 and 3, and assume that
\[ W_1^2(\mu_0, \tilde{\mu}) \leq 2K^2 \mathcal{H}(\tilde{\mu} | \mu_0) \quad \forall \tilde{\mu} \ll \mu_0, \text{ and } \mathbb{E} \|\eta\|^2 \leq K^2, \]
then
where $H$ denotes the relative entropy. Then, there exists a constant $C > 0$, depending only on $T$, $K$ and $A_1$, such that

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} D\Phi(\mu_t, \bar{\mu}_T^n) - \mathbb{E} \sup_{0 \leq t \leq T} D\Phi(\mu_t, \bar{\mu}_T^n) \geq a \right) \leq \exp\left(-\frac{na^2}{C}\right).
$$

Combining with Theorem 4.1, we obtain that for any $\delta \in (0, 1)$, with probability at least $1 - \delta$

$$
\sup_{0 \leq t \leq T} D\Phi(\mu_t, \bar{\mu}_T^n) \leq C(\sqrt{\log n} + \sqrt{-\log \delta})n^{-\frac{1}{2}},
$$

where $C$ may depend on $K$, $T$, $A_1$, $A_2$ and $A_3$.

Proof. Throughout this proof, we will still use $C$ as a positive constant depending only on some constants clearly mentioned in the above theorem, which may vary from line to line.

Recall that the $n$-particle system can be rewritten as in (20), and the results from [16, Theorem 5.5]: there exists a constant $C > 0$, depending only on $K$ and $T$, such that

$$
\mathbb{P}(F(X^n) - \mathbb{E}F(X^n) \geq a) \leq \exp\left(-\frac{a^2}{nMC^2}\right), \quad (21)
$$

for any function $F : C([0, T]; \mathbb{R}^{d+n}) \to \mathbb{R}$ being $M$-Lipschitz in the sense that

$$
|F(x) - F(y)| \leq M \sum_{i=1}^n \sup_{0 \leq t \leq T} \|x_i^t - y_i^t\|,
$$

for any $x := (x^1, \ldots, x^n), y := (y^1, \ldots, y^n)$ with $x^i, y^i \in C([0, T]; \mathbb{R}^d)$.

Now, for any $x = (x^1, \ldots, x^n)$ with $x^i \in C([0, T]; \mathbb{R}^d)$, we define

$$
G(x) = \sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{i=1}^n f(x_i^t) - \mathbb{E}f(X_t) \right|,
$$

then $G(X^n) = \sup_{0 \leq t \leq T} D\Phi(\mu_t, \bar{\mu}_T^n)$, and

$$
|G(x) - G(y)| \leq \frac{1}{n} \sup_{0 \leq t \leq T} \sup_{f \in \Phi} \left| \sum_{i=1}^n f(x_i^t) - \sum_{i=1}^n f(y_i^t) \right| \\
\leq \frac{A_1}{n} \sum_{i=1}^n \sup_{0 \leq t \leq T} \|x_i^t - y_i^t\|.
$$

Then, our conclusion follows from the last equation and equation (21).

Remark 4.3. If one applies Theorem 2.2 to $\mu_t$ to improve the convergence rate under the Wasserstein metric or relative entropy, a question is whether $\mu_t$ satisfies the bias potential model. For instance, when $B$ and $\Sigma$ in (18) have the forms $B(t, x, \mu) = a \int \varphi(y) d\mu(y) + bx$ and $\Sigma(t, x, \mu) = \sigma$, then $\mu_t$ is a normal distribution with mean $x_0 e^{(a+b)t}$ and variance $\frac{b}{2\alpha}\sigma^2(e^{2bt} - 1)$. If we are interested in approximating this distribution on a compact subset $K$, we can define the base distribution $P$ to be uniform on $K$, then the distribution can be approximated well with a dimensionality-free rate. For more complicated processes, whether the same convergence rate holds remains an open question.

5 Application to Mean-Field Games

In this section, we shall show that, for a homogeneous $n$-player game, the strategy derived by its mean-field counterpart produces an $\varepsilon$-Nash equilibrium, where $\varepsilon$ is free of the dimension of the state processes.

Following the setup in [12], we consider a homogeneous $n$-player stochastic differential game

$$
dX_i^t = b(t, X_i^t, \nu_i^n, \alpha_i^t) \, dt + \sigma(t, X_i^t, \nu_i^n, \alpha_i^t) \, dW_i^t, \quad 0 \leq t \leq T, \quad i \in \mathcal{I} \equiv \{1, \ldots, n\},
$$

24
where each player $i$ controls her private state $X^i_t \in \mathbb{R}^d$ through an $\mathbb{R}^k \supseteq A$-valued action $\alpha^i_t$, $W^i_t = (W^i_t)_{0 \leq t \leq T}$ are $m$-dimensional independent Brownian motions, $b$ and $\sigma$ are deterministic measurable functions, $(b, \sigma): [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \to (\mathbb{R}^d, \mathbb{R}^{d \times m})$, and $\nu^n_i$ is the empirical measure of $(X^1_t, \ldots, X^n_t)$ defined by

$$\nu^n_i(dx) := \frac{1}{n} \sum_{i=1}^n \delta_{X^i_t}(dx).$$

Each player aims to minimize the expected cost over the period $[0, T]$ by taking her action $\alpha^i \in \mathbb{A}$:

$$J^i(\alpha) := \mathbb{E} \left[ \int_0^T f(t, X^i_t, \nu^n_i, \alpha^i_t) \, dt + g(X^i_T, \nu^n_T) \right],$$

where $\mathbb{A}$ denotes the set of all admissible strategies:

$$\mathbb{A} = \left\{ \text{A-valued progressively measurable processes } (\alpha_t)_{0 \leq t \leq T} : \mathbb{E} \left[ \int_0^T |\alpha_t|^2 \, dt \right] < \infty \right\},$$

and $f$ and $g$ are deterministic measurable functions, $f : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \to \mathbb{R}$, $g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$.

Since the players interact through their empirical measure $\nu^n_i$, which depends on all players’ strategy $\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathbb{A}^n$, so does the cost functional for player $i$, $J^i(\alpha)$. Here $\mathbb{A}^n$ is the product space of $n$ copies of $\mathbb{A}$.

To solve such games, we are interested in the concept of Nash equilibrium. That is a tuple $\alpha^* = (\alpha^1, \ldots, \alpha^n) \in \mathbb{A}^n$ such that

$$\forall i \in I, \ \alpha^i \in \mathbb{A}, \ \ J^i(\alpha^*) \leq J^i(\alpha^1, \ldots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \ldots, \alpha^n).$$

For homogeneous games with large $n$, if the system lacks tractability and needs to rely on numerical methods for Nash equilibrium, the conventional algorithms soon lose its efficiency and one may resort to recently developed machine learning tools [35, 31, 32]. On the other hand, one could utilize its limiting mean-field strategy to approximate the Nash equilibrium. More precisely, one can first obtain the optimal control $\alpha$ from the mean field games using the following steps:

(i) Fixed a deterministic measure $\mu_t \in \mathcal{P}(\mathbb{R}^d), \forall t \in [0, T]$;

(ii) Solve the standard stochastic control problem:

$$\inf_{\alpha \in \mathbb{A}} \mathbb{E} \left[ \int_0^T f(t, X_t, \mu_t, \alpha_t) \, dt + g(X_T, \mu_T) \right]$$

subject to: $dX_t = b(t, X_t, \mu_t, \alpha_t) \, dt + \sigma(t, X_t, \mu_t, \alpha_t) \, dW_t, \ X_0 = x_0$;

(iii) Determine the flow of measures $\mu_t$ such that $\forall t \in [0, T], \mathcal{L}(X_t^{*; \mu}) = \mu_t$, where $X_t^{*; \mu}$ denotes the state process associated to the optimal control given $\mu_t$ in step (ii).

Then one can construct an $\varepsilon$-Nash equilibrium from it if the optimal control $\alpha$ given by the fixed-point argument (step (iii)) is in a feedback form. We will make this statement rigorous in Theorem 5.2.

Throughout this section, the following assumptions are in force.

**Assumption 4.** (a) The drift $b$ is an affine function of $\alpha$ and $x$:

$$b(t, x, \mu, \alpha) = b_0(t, \mu) + b_1(t) x + b_2(t) \alpha,$$

where $b_0 \in \mathbb{R}^d$, $b_1 \in \mathbb{R}^{d \times d}$, $b_2 \in \mathbb{R}^{d \times k}$ are measurable functions and bounded by $K$. Moreover, for any $\mu, \mu' \in \mathcal{P}^2(\mathbb{R}^d)$: $|b_0(t, \mu) - b_0(t, \mu')| \leq K D\Phi(\mu, \mu')$. The volatility $\sigma(t, x, \mu, \alpha) \in \mathbb{R}^{d \times m}$ is a constant matrix.
(b) There exist two constants \( \lambda \) and \( K \), such that for any \((t, \mu) \in [0, T] \times \mathcal{P}^2(\mathbb{R}^d)\), the function \( f(t, \cdot, \mu, \cdot) \in \mathbb{R} \) is once continuously differentiable with Lipschitz-continuous derivatives, with the Lipschitz constants being bounded by \( K \). Moreover, it satisfies the convexity assumption:

\[
\|f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) - \langle (x' - x, \alpha' - \alpha), \partial_{(x, \alpha)} f(t, x, \mu, \alpha) \rangle \| \leq \lambda |\alpha' - \alpha|^2.
\]

The functions \( f, \partial_x f, \) and \( \partial_\alpha f \) are locally bounded. The functions \( f(\cdot, 0, \delta_0, 0), \partial_x f(\cdot, 0, \delta_0, 0) \) and \( \partial_\alpha f(\cdot, 0, \delta_0, 0) \) are bounded by \( K \), and for all \( t \in [0, T] \), \( x, x' \in \mathbb{R}^d \), \( \alpha, \alpha' \in \mathbb{R}^k \) and \( \mu, \mu' \in \mathcal{P}^2(\mathbb{R}^d) \), it holds:

\[
|\langle f, g \rangle(t, x', \mu', \alpha') - \langle f, g \rangle(t, x, \mu, \alpha)\| \leq K \left( 1 + |x'| + |x| + M_2(\mu) + M_2(\mu') \right) + D(|x'| - |x|) + D_\Phi(\mu', \mu).
\]

(c) The function \( g(\cdot, \cdot) \) is locally bounded, and for any \( \mu \in \mathcal{P}^2(\mathbb{R}^d) \), the function \( g(\cdot, \mu) \) is once continuously differentiable, convex, and has a \( K \)-Lipschitz-continuous first order derivative.

(d) For all \((t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d)\), \( |\partial_x f(t, x, \mu, 0)| \leq K \).

(e) For all \((t, x) \in [0, T] \times \mathbb{R}^d\), \( |x, \partial_x f(t, 0, \delta_x, 0)| \geq -K(1 + |x|), \langle x, \partial_x g(t, 0, \delta_x) \rangle \geq -K(1 + |x|) \).

In the sequel, a constant \( C \) will frequently appear in the theorems and proofs. It may depend on the bounds that appear in the above assumption \((\lambda, K, b_0, b_1, A_1, A_2, A_3, \ldots)\) and possibly vary from line to line. But it will be independent from the dimensionality \( d \) of the state process \( X_t \) and the number of players \( n \) in the game.

Remark 5.1. Let \( H \) be the Hamiltonian associated to the problem, with uncontrolled volatility, it reads

\[
H(t, x, \mu, p, \alpha) = \langle b(t, x, \mu, \alpha), p \rangle + f(t, x, \mu, \alpha).
\]

Items (a)–(b) in Assumption 4 ensure the uniqueness of minimizer \( \hat{\alpha} \) of \( H \), the measurability, local boundedness, Lipschitz-continuity of \( \hat{\alpha}(t, x, \mu, y) \) in \( (x, y) \) uniformly in \((t, \mu) \in [0, T] \times \mathcal{P}^2(\mathbb{R}^d)\). Moreover, the Lipschitz constant is free of \( d \). A repeatedly used estimate is \( |\hat{\alpha}(t, x, \mu, y)| \leq \lambda^{-1}(|\partial_x f(t, x, \mu, 0)| + |b_2(t)| |y|) \).

For detailed proof, see [12, Lemma 1].

The probabilistic approach of (i)–(iii) results in solving the following McKean-Vlasov forward backward stochastic differential equations (FBSDEs):

\[
\begin{align*}
dX_t &= b(t, X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t)) \, dt + \sigma \, dW_t, \\
dY_t &= -\partial_x H(t, X_t, \mathcal{L}(X_t), Y_t, \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t)) \, dt + Z_t \, dW_t,
\end{align*}
\]

with the initial condition \( X_0 = x_0 \in \mathbb{R}^d \) and the terminal condition \( Y_T = \partial_x g(X_T, \mathcal{L}(X_T)) \). More precisely, the following result holds.

Theorem 5.1. Under Assumption 4, the FBSDE system (25) has a solution \((X_t, Y_t, Z_t)\), and there exists the FBSDE value function \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) such that it has linear growth and Lipschitz in \( x \):

\[
|u(t, x)| \leq C(1 + |x|), \quad |u(t, x) - u(t, x')| \leq C|x - x'|,
\]

for some constant \( C \geq 0 \), and such that \( Y_t = u(X_t) \) \( \mathbb{P} \)-a.s., \( \forall t \in [0, T] \) and \( x, x' \in \mathbb{R}^d \). Moreover, for any \( \ell \geq 1 \), \( \mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^\ell] < \infty \), and the optimal cost \( J \) of the limiting mean-field problem (i)–(iii) is given by

\[
J = \mathbb{E} \left[ g(X_T, \mathcal{L}(X_T)) + \int_0^T f(t, X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t)) \, dt \right],
\]

where \( \hat{\alpha} \) is the minimizer of \( H \) defined in (24).

Proof. The existence of solution to (25) and related estimates follow from [12, Theorem 2], because the assumptions therein are satisfied using Theorem 2.1 (a) based on the Wasserstein distance and GMMD. The statement on \( J \) is a consequence of the stochastic maximum principle when the frozen flow of measures is \( \mathcal{L}(X_t) \), for instance see [12, Theorem 1]. \( \square \)
Theorem 5.2. Let \((X_t, Y_t, Z_t)\) be a solution of (25), \(u\) be the corresponding FBSDE value function, and \(\mu_t = \mathcal{L}(X_t)\) be the marginal probability measure, then

\[
\hat{\alpha}_t^{n,i} = \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i)), \quad i \in \mathcal{I},
\]

where \(X_t^i\) follows (22) with strategy \(\hat{\alpha}_t^{n,i}:

\[
dX_t^i = b(t, X_t^i, \nu_t^n, \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i))) dt + \sigma dW_t^i, \quad \nu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i},
\]

is an \(\varepsilon_n\)-Nash equilibrium of the \(n\)-player problem (22)–(23) with \(\varepsilon_n = C/\sqrt{n}\), i.e., for any progressively measurable strategy \(\beta^i\) such that \(\mathbb{E}[\int_0^T |\beta^i|^2 dt] < \infty\), we have

\[
J^{n,i}(\hat{\alpha}^{n,i}, \ldots, \hat{\alpha}^{n,i-1}, \beta^i, \hat{\alpha}^{n,i+1}, \ldots, \hat{\alpha}^{n,n}) \geq J^{n,i}(\hat{\alpha}^{n,1}, \ldots, \hat{\alpha}^{n,n}) - \varepsilon_n.
\]

Proof. We first claim that the SDE (29) is well defined, by the Lipschitz property and linear growth of \(\hat{\alpha}\) in \((x, y)\) and \(u\) in \(x\) (cf. Remark 5.1 and (26)). With Assumption 4 (a), we also have

\[
\sup_{n \geq 1} \max_{1 \leq i \leq n} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| X_t^i \right|^2 \right] + \mathbb{E} \int_0^T \left| \hat{\alpha}_t^{n,i} \right|^2 dt \leq C.
\]

The proof of (30) consists of two steps, and by symmetry we only need to prove it for \(i = 1\).

Step 1: MFG vs. \(N\)-player game using \((\hat{\alpha}^{n,1}, \ldots, \hat{\alpha}^{n,n})\). To this end, we introduce \(n\) independent copies of the mean-field states \(X_t^i\) in (25):

\[
d\tilde{X}_t^i = b(t, \tilde{X}_t^i, \mu_t, \hat{\alpha}(t, \tilde{X}_t^i, \mu_t, u(t, \tilde{X}_t^i))) dt + \sigma dW_t^i, \quad t \in [0, T], \quad i \in \mathcal{I}.
\]

Note that \(\mathcal{L}((\tilde{X}_t^i)_{i \in \mathcal{I}}) = \mu_t\), and we have similar estimates for \((\tilde{X}_t^i, \hat{\alpha}_t^i)\) as in (31). Let \(\tilde{\mu}_t^n\) be the empirical measure of \(\tilde{X}_t^i\) and define,

\[
\hat{\alpha}_t^i = \hat{\alpha}(t, \tilde{X}_t^i, \mu_t, u(t, \tilde{X}_t^i)),
\]

we then compute, by the regularity of \(b, u\) and \(\hat{\alpha}\), that for \(t \in [0, T]\):

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| X_t^i - \tilde{X}_t^i \right|^2 \right] \leq C \mathbb{E} \left[ \int_0^t \left| b(s, X_s^i, \nu_s^n, \hat{\alpha}_s^i) - b(s, \tilde{X}_s^i, \nu_s^n, \hat{\alpha}_s^i) \right|^2 ds \right]
\]

\[
\leq C \mathbb{E} \left[ \int_0^t \left| X_s^i - \tilde{X}_s^i \right|^2 + D_2^2(\nu_s^n, \mu_s) + \left| \hat{\alpha}_s^i - \hat{\alpha}_s^i \right|^2 ds \right]
\]

\[
\leq C \mathbb{E} \left[ \int_0^t \left| X_s^i - \tilde{X}_s^i \right|^2 + D_2^2(\nu_s^n, \mu_s) ds \right].
\]

Then Gronwall’s inequality gives

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| X_t^i - \tilde{X}_t^i \right|^2 \right] \leq C \mathbb{E} \left[ \int_0^t D_2^2(\nu_s^n, \mu_s) ds \right], \forall i \in \mathcal{I} \text{ and } t \in [0, T].
\]

A similar proof as in Theorem 2.1 gives

\[
\mathbb{E} D_2^2(\mu_t, \tilde{\mu}_t^n) \leq C, \quad \forall t \in [0, T],
\]

and by the definition of \(D_\Phi\) we have

\[
D_2^2(\nu_t^n, \tilde{\mu}_t^n) \leq \frac{A^2}{n} \sum_{i=1}^n \left| X_t^i - \tilde{X}_t^i \right|^2, \quad \forall t \in [0, T].
\]
Thus one deduces
\[
\mathbb{E}[D^2_\Phi(\mu, \tilde{\nu}^n_t)] \leq 2\mathbb{E}[D^2_\Phi(\mu_t, \tilde{\mu}^n_t)] + 2\mathbb{E}[D^2_\Phi(\tilde{\mu}^n_t, \tilde{\nu}^n_t)] \\
\leq \frac{C}{n} + C\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{\infty} |X^i - \tilde{X}_i|^2 \right] \leq \frac{C}{n} + C\mathbb{E} \left[ \int_0^T D^2_\Phi(\nu^s_t, \mu_s) \, ds \right].
\]

Applying Gronwall’s inequality again yields
\[
\mathbb{E}[D^2_\Phi(\mu_t, \tilde{\nu}^n_t)] \leq \frac{C}{n}, \quad \forall t \in [0, T],
\]
and consequently
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X^i_s - \tilde{X}_i|^2 \right] \leq \frac{C}{n}, \quad \forall i \in I, \quad \forall t \in [0, T].
\]

We are now ready to compare \( J^{n,i}(\tilde{\alpha}^{n,1}, \ldots, \tilde{\alpha}^{n,n}) \) with the mean-field problem value \( J \) defined in (27), which coincide with \( \mathbb{E} \left[ \int_0^T f(t, \tilde{X}_i^t, \mu_t, \bar{\alpha}_t) \, dt \right] \), as \( \tilde{X}_i^t \overset{D}{=} X_i^t \). By Assumption 4 (b), the Cauchy-Schwarz inequality, the boundedness of \( (X^i_t, \tilde{X}_i^t, \bar{\alpha}_t^{n,1}, \bar{\alpha}_t^{n}) \) in expectation (cf. Theorem 5.1 and estimate (31)), the Lipschitz property of \( \tilde{\alpha} \) and \( u \), we have
\[
|J - J^{n,i}(\tilde{\alpha}^{n,1}, \ldots, \tilde{\alpha}^{n,n})| \leq C \mathbb{E} \left[ |X^i_T - \tilde{X}_i^T|^2 + D^2_\Phi(\mu_T, \tilde{\nu}^n_T) \right]^{1/2} \\
+ C \left( \int_0^T \mathbb{E} \left[ |\tilde{X}_i^t - X_i^t|^2 + D^2_\Phi(\mu_t, \tilde{\nu}^n_t) \right] \, dt \right)^{1/2},
\]
and then conclude
\[
J^{n,i}(\tilde{\alpha}^{n,1}, \ldots, \tilde{\alpha}^{n,n}) = J + C/\sqrt{n},
\]
by the estimates (33) and (34). This suggest that, in order to prove (30), we only need to compare \( J^{n,i}(\tilde{\alpha}^{1}, \tilde{\alpha}^{n,2}, \ldots, \tilde{\alpha}^{n,n}) \) with \( J \).

**Step 2:** MFG vs. N-player game using \((\beta^1, \tilde{\alpha}^{n,2}, \ldots, \tilde{\alpha}^{n,n})\). Denote by \((U^1_t, \ldots, U^n_t)\) the solution to (22) using strategy \((\beta^1, \tilde{\alpha}^{n,2}, \ldots, \tilde{\alpha}^{n,n})\), and \( \tilde{\nu}^n_t \) the empirical measure of \((U^1_t, U^2_t, \ldots, U^n_t)\), and \( \tilde{\nu}^{n-1} \) the empirical measure of \((U^2_t, \ldots, U^n_t)\). By the boundedness of \( b_0, b_1 \) and \( b_2 \), the admissibility of \((\beta^1, \tilde{\alpha}^{n,2}, \ldots, \tilde{\alpha}^{n,n})\), and Gronwall’s inequality, we have the following estimates:
\[
\mathbb{E}[\sup_{0 \leq s \leq T} |U^j_s|^2] \leq C(1 + \mathbb{E} \int_0^T |\beta^j_s|^2 \, dt), \quad \mathbb{E}[\sup_{0 \leq s \leq T} |U^j_s|^2] \leq C, \quad j \in I \setminus \{1\},
\]
\[
\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[\sup_{0 \leq s \leq T} |U^j_s|^2] \leq C(1 + \frac{1}{n} \mathbb{E} \int_0^T |\beta^j_s|^2 \, dt).
\]

**Step 2.1:** Controlling \( D^2_\Phi(\tilde{\nu}^n_t, \mu_t) \). By triangle inequality of GMMD, one has
\[
\mathbb{E}[D^2_\Phi(\tilde{\nu}^n_t, \mu_t)] \leq C \left\{ \mathbb{E}[D^2_\Phi(\tilde{\nu}^n_t)] + \mathbb{E}[D^2_\Phi(\tilde{\nu}^{n-1}_t)] + \mathbb{E}[D^2_\Phi(\tilde{\mu}^{n-1}_t, \mu_t)] \right\},
\]
and the last term is \( O(1/n) \) by (32). For the first term, we have
\[
\mathbb{E}[D^2_\Phi(\tilde{\nu}^n_t)] \leq \frac{C}{n(n-1)} \sum_{j=2}^{n} \mathbb{E}[|U^j_t - U^j_t|^2],
\]
and is \( O(1/n) \) using the estimate (36). By definition, the second term in (37) is bounded by
\[
\mathbb{E}[D^2_\Phi(\tilde{\nu}^{n-1}_t, \mu_t)] \leq \frac{C}{n-1} \sum_{j=2}^{n} \mathbb{E}[|U^j_t - \tilde{X}^j_t|^2].
\]
For $2 \leq j \leq n$, we deduce that
\[
E[|U_i^t - \bar{X}_i^t|^2] \leq 2E[|U_i^t - X_i^t|^2] + 2E[|X_i^t - \bar{X}_i^t|^2] \leq \frac{C}{n},
\]
by (34), the estimates (following the derivation of (58) in [12]):
\[
\sup_{0 \leq t \leq T} E[|U_i^t - X_i^t|^2] \leq \frac{C}{n} E \int_0^T |\beta_i^t - \bar{\alpha}_i^{n,i}|^2 dt, \quad 2 \leq i \leq n,
\]
and boundedness of moments of $\beta^1$ and $\bar{\alpha}^{n,i}$.

**Step 2.2: MFG using $\beta^1$ vs. N-player game using $(\beta^1, \bar{\alpha}^{n,2}, \ldots, \bar{\alpha}^{n,n})$.** To compare $J^{n,1}(\beta^1, \bar{\alpha}^{n,2}, \ldots, \bar{\alpha}^{n,n})$ with the mean field cost $J$ given in (27), we define the process $\bar{U}_i^t$ associated with the mean-field flow $\mu_t = \mathcal{L}(X_t)$ and strategy $\beta^1$:
\[
d\bar{U}_i^t = b(t, \bar{U}_i^t, \mu_t, \beta_i^t) dt + \sigma dw_t, \quad 0 \leq t \leq T.
\]
Comparing it with $U_i^t$, and using the boundedness of $b_1$, Assumption 4 (a), the estimate of $ED_n(\bar{\nu}_t^n, \mu_t)$ and Gronwall’s inequality, we deduce
\[
\sup_{0 \leq t \leq T} E[|U_i^t - \bar{U}_i^t|^2] \leq \frac{C}{n}.
\]
Therefore, a similar derivation as in Step 1 gives (replacing $X$ by $U$ and $\bar{\nu}$ by $\bar{\nu}$)
\[
|J(\beta^1) - J^{n,1}(\beta^1, \bar{\alpha}^{n,2}, \ldots, \bar{\alpha}^{n,n})| \leq \frac{C}{\sqrt{n}}
\]
where $J(\beta^1)$ is the mean-field cost using $\beta^1$:
\[
J(\beta^1) = E \left[ g(\bar{U}_T^n, \mu_T) + \int_0^T f(t, \bar{U}_t^n, \mu_t, \beta_t^1) dt \right].
\]
As $J$ is the optimal cost of the mean-field game, any strategy $\beta^1$ other than $\hat{\alpha}(t, X_t, \mu_t, u(t, X_t))$ will produce a higher cost, i.e., $J(\beta^1) \geq J$. Therefore, one has
\[
J^{n,i}(\beta^1, \bar{\alpha}^{n,2}, \ldots, \bar{\alpha}^{n,n}) \geq J(\beta^1) - \frac{C}{\sqrt{n}} \geq J - \frac{C}{\sqrt{n}} \tag{38}
\]
Combining (35) and (38) gives the desired result (28).

### 6 Conclusion

A new class of metrics, named generalized maximum mean discrepancy (GMMD), is proposed in this paper to study the convergence of empirical measures in high-dimensional spaces. We generalize the standard definition of maximum mean discrepancy by imposing specific criteria for selecting the test function space to guarantee the property of being free of the CoD. Examples of test function spaces include reproducing kernel Hilbert space, Barron function space, and flow-induced function spaces. Under the proposed metrics, we can show the following three cases of convergence are dimensionality-free: 1. The convergence of empirical measure drawn from a given distribution; 2. The convergence of $n$-particle system to the solution to McKean-Vlasov stochastic equation; 3. The construction of an $\varepsilon$-Nash equilibrium for a homogeneous $n$-player game by its mean-field limit. We also generalize the results in [62] and show that, given a distribution close to the target distribution measured by GMMD and the certain representation of the target distribution, we can generate a distribution close to the target one in terms of the Wasserstein distance and relative entropy.

As future work, we shall deepen the study of GMMD by investigating the mean-field limit of the $n$-player stochastic differential games in high dimensions, whose Nash equilibria can be given by the deep
fictitious theory and algorithms [35, 31, 32, 61]. Besides, we are interested in developing a similar theory (cf. Theorem 2.2) for models other than the bias potential type, for instance, the generative adversarial network models, which are useful and important in the machine learning community. We also plan to improve the convergence rate of the n-particle system (17) to the McKean-Vlasov SDE (18) by applying Theorem 2.2. However, to what extent forms of $B$ and $\Sigma$ the theorem can be applied remains an open question.

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Appendix

We show in this appendix that, under a slightly stronger condition (39) compared to (5), the logarithm term in \( \phi(n) \) defined in Theorem 2.3 can be removed, for all the examples of the test function classes discussed in Section 3.

Throughout the appendix, we assume \( \mu \) to be a distribution on \( C([0, T]; \mathbb{R}^d) \) such that

\[
\int_{C([0, T]; \mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x_t\|^2 \, d\mu(x) < +\infty,
\]

and

\[
\int_{C([0, T]; \mathbb{R}^d)} [\Delta(x, h)]^2 \, d\mu(x) \leq Q h^{\alpha} \log^\beta \left( \frac{2T}{h} \right),
\]

(39)

for any \( h > 0 \), where \( Q, \alpha, \beta > 0 \) are positive constants. Still, we will use \( C \) to denote a positive constant depending only on \( \alpha \) and \( \beta \), which may vary from line to line.

The first result is established for the reproducing kernel Hilbert space (RKHS).

**Proposition A.1.** Assume the kernel \( k \) satisfies the condition (a) in Theorem 3.1 and \( \Phi \) is the unit ball of \( \mathcal{H}_k \), then

\[
E \sup_{t \in [0, T]} D_\Phi(\mu_t, \bar{\mu}_n^t) \leq 2 \sqrt{\frac{2}{n} K_1^2 \int_{C([0, T]; \mathbb{R}^d)} \sup_{0 \leq t \leq T} \|x_t\|^2 \, d\mu(x) + K_2^2} + CK_1 \sqrt{\frac{QT^\alpha n}{n}}.
\]

**Proof.** Let \( X^1, \ldots, X^n \) be i.i.d. processes drawn from the distribution \( \mu \). Following [53, Lemma 26.2], we immediately have

\[
E \sup_{t \in [0, T]} D_\Phi(\mu_t, \bar{\mu}_n^t) \leq \frac{2}{n} E \sup_{t \in [0, T]} \sup_{\|f\|_{\mathcal{H}_k} \leq 1} \left| \sum_{i=1}^n \xi_i f(X_i^t) \right|.
\]
For a fixed integer $n' \geq 2$ and any $t_p, s_p \in [0, T], \ p = 1, \ldots, n'$, we first compute

$$\frac{1}{n} \mathbb{E} \max_{1 \leq p \leq n'} \sup_{\|f\|_{\mathcal{H}_k} \leq 1} \left| \sum_{i=1}^{n} \xi_i [f(X_{t_p}^i) - f(X_{s_p}^i)] \right|$$

$$= \frac{1}{n} \mathbb{E} \max_{1 \leq p \leq n'} \sup_{\|f\|_{\mathcal{H}_k} \leq 1} \left| \langle f, \sum_{i=1}^{n} \xi_i (k(X_{t_p}^i, \cdot) - k(X_{s_p}^i, \cdot)) \rangle_{\mathcal{H}_k} \right|$$

$$\leq \frac{1}{n} \mathbb{E} \max_{1 \leq p \leq n'} \sup_{\|f\|_{\mathcal{H}_k} \leq 1} \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_j k(X_{t_p}^i, X_{s_p}^j) + k(X_{s_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{t_p}^j) \right|. \quad (40)$$

Let $S$ be a nonnegative-definite matrix, then there exists a nonnegative-definite $n \times n$ matrix $F$ such that $S = F^T F$. Let $\xi = (\xi_1, \ldots, \xi_n)^T$, then there exists a universal constant $C > 0$ (cf. [51, Theorem 2.1] and [57, Exercise 2.5.10]), such that for any $a \geq 0$,

$$\mathbb{P}(\sqrt{\xi^T S \xi} - \sqrt{\text{Trace}(S)} \geq a) = \mathbb{P}(\|F\xi\| - \|F\|_{\text{HS}} \geq a) \leq 2 \exp\left(-\frac{a^2}{C \text{Trace}(S)}\right), \quad (41)$$

where $\|F\|_{\text{HS}} = \sqrt{\text{Trace}(F^T F)}$. Therefore, for any positive definite matrices $S_1, \ldots, S_{n'}$, [57, Exercise 2.5.8]) gives

$$\mathbb{E} \sup_{1 \leq p \leq n'} \sqrt{\xi^T S_p \xi} \leq C \max_{1 \leq p \leq n'} \sqrt{\text{Trace}(S_p) \log n'}.$$

Since for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j [k(X_{t_p}^i, X_{s_p}^j) + k(X_{s_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{t_p}^j)]$$

$$= (\sum_{i=1}^{n} a_i [k(X_{t_p}^i, \cdot) - k(X_{s_p}^i, \cdot)]) \left[ \sum_{i=1}^{n} a_i [k(X_{t_p}^i, \cdot) - k(X_{s_p}^i, \cdot)] \right]_{\mathcal{H}_k} \geq 0,$$

using (40) and (41) with $(K_p)_{i,j} = k(X_{t_p}^i, X_{t_p}^j) + k(X_{s_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{s_p}^j) - k(X_{t_p}^i, X_{t_p}^j)$, we deduce

$$\frac{1}{n} \mathbb{E} \max_{1 \leq p \leq n'} \sup_{\|f\|_{\mathcal{H}_k} \leq 1} \left| \sum_{i=1}^{n} \xi_i [f(X_{t_p}^i) - f(X_{s_p}^i)] \right|$$

$$\leq C \sqrt{n} \mathbb{E} \max_{1 \leq p \leq n'} \left| \sum_{i=1}^{n} k(X_{t_p}^i, X_{t_p}^i) + k(X_{s_p}^i, X_{s_p}^i) - 2k(X_{t_p}^i, X_{s_p}^i) \right|$$

$$\leq C \sqrt{\log n'} \mathbb{E} \max_{1 \leq p \leq n'} [k(X_{t_p}^1, X_{t_p}^1) + k(X_{s_p}^1, X_{s_p}^1) - 2k(X_{t_p}^1, X_{s_p}^1)]$$

$$\leq CK_1 \sqrt{n} \mathbb{E} \max_{1 \leq p \leq n'} \|X_{t_p}^1 - X_{s_p}^1\|^2, \quad (42)$$

where $K_1$ is the constant defined in Theorem 3.1.

We now use the chaining method to estimate the Rademacher complexity. For any $t \in [0, T]$ and integer $k$, we define

$$\Pi_k(t) = T^{-k} \left\lfloor \frac{2kT}{T} \right\rfloor.$$
We first compute
\[
\left| \frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_t) - \frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_{\Pi(t)}) \right| \right| 
\leq \frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_t) - f(X^i_{\Pi(t)}) \right| 
\leq K_1 \mathbb{E} \sup_{i \in [0,T]} \|X^i_t - X^i_{\Pi(t)}\| \leq C K_1 Q \left( \frac{T}{2^k} \right)^{\alpha/2} (k + 1)^{\beta/2} \to 0,
\]
as \( k \to +\infty \). We have also derived in Theorem 3.1 that
\[
\frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_0) \right| \leq \sqrt{\frac{2}{n} K_2^2 \int_{C([0,T];\mathbb{R}^d)} \|x_0\|^2 d\mu(x) + K_2^2}.
\]
Together with (42), we finally achieve
\[
\frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_t) \right| \leq \frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_0) \right| + \frac{1}{n} \mathbb{E} \sup_{i \in [0,T]} \left| \sum_{i=1}^{n} \xi_i f(X^i_{\Pi(t)}) \right| 
\leq \sqrt{\frac{2}{n} K_2^2 \int_{C([0,T];\mathbb{R}^d)} \|x_0\|^2 d\mu(x) + K_2^2} + C K_1 \sqrt{\frac{\log(2k+1)}{n} \mathbb{E} \sup_{i \in [0,T]} \|X^i_{\Pi(t)} - X^i_{\Pi(t)}\|^2} 
\leq \sqrt{\frac{2}{n} K_2^2 \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 d\mu(x) + K_2^2} + C K_1 \sqrt{\frac{\log(2k+1)}{n} \mathbb{E} \sup_{i \in [0,T]} \|X^i_{\Pi(t)} - X^i_{\Pi(t)}\|^2}
\leq \sqrt{\frac{2}{n} K_2^2 \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 d\mu(x) + K_2^2} + C K_1 \sqrt{\frac{\log(2k+1)}{n} \mathbb{E} \sup_{i \in [0,T]} \|X^i_{\Pi(t)} - X^i_{\Pi(t)}\|^2}
\leq \sqrt{\frac{2}{n} K_2^2 \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 d\mu(x) + K_2^2} + C K_1 \sqrt{\frac{\log(2k+1)}{n} \mathbb{E} \sup_{i \in [0,T]} \|X^i_{\Pi(t)} - X^i_{\Pi(t)}\|^2},
\]
where we have used \( \sum_{k=0}^{+\infty} \sqrt{\frac{(k+1)^{\alpha+1}}{2^{k\alpha}}} < +\infty \).

To establish results for the Barron space and flow-induced function spaces, we first present the following lemma.

\textbf{Lemma A.1.} Let \( X^1, \ldots, X^n \) be i.i.d. processes drawn from \( \mu \in \mathcal{P}^2(C([0,T];\mathbb{R}^d)) \) and \( \xi_1, \ldots, \xi_n \) be i.i.d. Rademacher variables which are independent of \( X^1, \ldots, X^n \). Then,
\[
\frac{1}{n} \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j \langle [X^i_t] X^j_t \rangle + 1 \right| \leq \sqrt{\frac{2}{n} \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 + 1} d\mu(x) + C \sqrt{\frac{QT^\alpha}{n}}.
\]

\textbf{Proof.} Taking \( k(x,x') = x^T x' + 1 \) for any \( x, x' \in \mathbb{R}^d \), this lemma can be derived using the proof of Proposition A.1. \( \square \)

\textbf{Proposition A.2.} 1. Let \( \Phi = B_1 \) be the unit ball of Barron space \( B \), then
\[
\mathbb{E} \sup_{0 \leq t \leq T} D_\Phi(\mu_t, \mu^\alpha_t) \leq 4 \sqrt{\frac{2}{n} \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 + 1} d\mu(x) + C \sqrt{\frac{QT^\alpha}{n}}.
\]

2. Let \( \Phi = \{ f \in \mathcal{D}, \|f\|_{\mathcal{D}} \leq 1 \} \) be the unit ball of flow-induced function spaces \( \mathcal{D} \), then
\[
\mathbb{E} \sup_{0 \leq t \leq T} D_\Phi(\mu_t, \mu^\alpha_t) \leq 2e^2 \sqrt{\frac{2}{n} \int_{C([0,T];\mathbb{R}^d)} \|x_t\|^2 + 1} d\mu(x) + C \sqrt{\frac{QT^\alpha}{n}}.
\]

35
Proof. The proof of these arguments is quite similar with the proof of claim (f) in Theorem 3.2 and claim (c) in Theorem 3.3 with the above Lemma A.1.

With [53, Lemma 26.2], and following the proofs of claim (f) in Theorem 3.2 and claim (c) in Theorem 3.3, we obtain

\[ \mathbb{E} \sup_{0 \leq t \leq T} D_{\Phi}(\mu_t, \bar{\mu}^n_t) \leq K_{\Phi} \frac{1}{n} \mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j [(X^i_t)^T X^j_t + 1] \right), \]

where \( K_{\Phi} = 4 \) in case 1, and \( K_{\Phi} = 2e^2 \) in case 2. Then we conclude our results by applying Lemma A.1. \( \square \)