DEFINABLE K-HOMOLOGY OF SEPARABLE C*-ALGEBRAS

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ABSTRACT. In this paper we show that the K-homology groups of a separable C*-algebra can be enriched with additional descriptive set-theoretic information, and regarded as definable groups. Using a definable version of the Universal Coefficient Theorem, we prove that the corresponding definable K-homology is a finer invariant than the purely algebraic one, even when restricted to the class of UHF C*-algebras, or to the class of unital commutative C*-algebras whose spectrum is a 1-dimensional connected subspace of $\mathbb{R}^3$.

INTRODUCTION

Given a compact metrizable space $X$, the group Ext $(X)$ classifying extensions of the C*-algebra $C(X)$ by the C*-algebra $K(H)$ of compact operators was initially considered by Brown, Douglas, and Fillmore in their celebrated work [BDF77]. There, they showed that Ext $(−)$ is indeed a group, and that defining, for a compact metrizable space $X$,

$$\tilde{K}_p(X) := \begin{cases} \text{Ext} (X) & \text{if } p \text{ is odd,} \\ \text{Ext} (\Sigma X) & \text{if } p \text{ is even;} \end{cases}$$

where $\Sigma X$ is the suspension of $X$, yields a (reduced) homology theory that satisfies all the Eilenberg–Steenrod–Milnor axioms for Steenrod homology, apart from the Dimension Axiom; see also [KS77]. They furthermore observed, building on a previous insight of Atiyah [Ati70], that such a homology theory can be seen as the Spanier–Whitehead dual of topological K-theory [Ati89].

More generally, for an arbitrary separable unital C*-algebra $A$, one can consider a semigroup Ext $(A)$ classifying the essential, unital extensions of $A$ by $K(H)$. By Voiculescu’s non-commutative Weyl-von Neumann Theorem [Voi76, Arv77], the trivial element of Ext $(A)$ correspond to the class of trivial essential, unital extensions. The group Ext $(A)^{-1}$ of invertible elements of Ext $(A)$ corresponds to the essential, unital extensions that are semi-split. Thus, by the Choi–Effros lifting theorem [CE76], Ext $(A)$ is a group when $A$ is nuclear. One can extend K-homology to the category of all separable C*-algebras by setting

$$K^p(A) = \begin{cases} \text{Ext} (A^+)^{-1} & \text{if } p \text{ is odd,} \\ \text{Ext} (S(A)^+)^{-1} & \text{if } p \text{ is even;} \end{cases}$$

where $S A$ is the suspension of $A$ and $A^+$ is the unitization of $A$. This gives a cohomology theory on the category of separable C*-algebras, which can be recognized as the dual of K-theory via Paschke duality [Pas81, Hig95, KS17]. Kasparov’s bivariant functor KK $(−, −)$ simultaneously generalizes K-homology and K-theory, where $K^1(A)$ is recovered as KK $(A, \mathbb{C})$ and $K^0(A)$ as KK $(A, C_0(\mathbb{R}))$.

It was already noticed in the seminal work of Brown, Douglas, and Fillmore [BDF77, BDF73] that the invariant $K^p(X)$ for a compact metrizable space $X$ can be endowed with more structure than the purely algebraic group structure. Indeed, one can write $X$ as the inverse limit of a tower $(X_n)_{n \in \omega}$ of compact polyhedra, and endow $K^p(X)$ with the topology induced by the maps $K^p(X) \to K^p(X_n)$ for $n \in \omega$, where $K^p(X_n)$ is a countable group endowed with the discrete topology. This gives to $K^p(X)$ the structure of a topological group, which is however in general not Hausdorff.

The study of $K^p(A)$ as a topological group for a separable unital C*-algebra $A$ was later systematically undertaken by Dadarlat [Dad00, Dad05] and Schochet [Sch01, Sch02, Sch05] building on previous work of Salinas [Sal92]. (In fact, they consider more generally Kasparov’s KK-groups.) In [Sch01, Dad05] several natural topologies on $K^p(A)$,
corresponding to different ways to define K-homology for separable C*-algebras, are shown to coincide and to turn $K^p(A)$ into a pseudo-Polish group. This means that, if $K^p_w(A)$ denotes the closure of zero in $K^p(A)$, then the quotient of $K^p(A)$ by $K^p_w(A)$ is a Polish group. In [Sch02], for a C*-algebra $A$ satisfying the Universal Coefficient Theorem (UCT), the topology on $K^p(A)$ is related to the UCT exact sequence, and $K^p_w(A)$ is shown to be isomorphic to the group $P\text{Ext}(K_{1-p}(A),\mathbb{Z})$ classifying pure extensions of $K_{1-p}(A)$ by $\mathbb{Z}$. A characterization of $K^p_w(A)$ for an arbitrary separable nuclear C*-algebra $A$ is obtained in [Dad00]. For a separable quasidiagonal C*-algebra satisfying the UCT, $K^p_w(A)$ is shown to be the subgroup of $K^p(A) = \text{Ext}(A^+)$ corresponding to quasidiagonal extensions of $A^+$ by $K(H)$ [Sch02]; see also [Bro84] for the commutative case. The quotient $K^p_w(A)$ of $K^p(A)$ by $K^p_w(A)$ is the group $KL_p(A,\mathbb{C})$ introduced by Rordam [Ror95]. A universal multicoefficient theorem describing $K^p_w(A)$ in terms of the K-groups of $A$ with arbitrary cyclic groups as coefficients is obtained in [DE02] for all separable nuclear C*-algebras satisfying the UCT; see [Dad05, Theorem 5.4].

In many cases of interest, the topology on $K^p(A)$ turns out to be trivial, i.e. the closure of zero in $K^p(A)$ is the whole group. For example, the topology on $K^1(A)$ is trivial when $A$ is a UHF C*-algebra, despite the fact that $K^0(A)$ is not trivial, and in fact uncountable. Similarly, for every 1-dimensional solenoid $X$, the topology on $K_0(X)$ is trivial, although $K_0(X)$ is an uncountable group.

In this paper, we take a different approach and consider the group $K^p(A)$, rather than as a pseudo-Polish topological group, as a definable group. This should be thought of as a group $G$ explicitly defined as the quotient of a Polish space $X$ by a “well-behaved” equivalence relation $E$, in such a way that the multiplication and inversion operations in $G$ are induced by Borel functions on $X$. This is formally defined in Section 1.5, where the notion of well-behaved equivalence relation is made precise. The definition is devised to ensure that the category of definable groups has good properties, and behaves similarly to the category of standard Borel groups. A morphism in this category is a definable group homomorphism, namely a group homomorphism that lifts to a Borel function between the corresponding Polish spaces.

It has recently become apparent that several homological invariants in algebra and topology can be seen as functors to the category of definable groups. The homological invariants $\text{Ext}$ and $\text{lim}^1$ are considered in [BLP20], whereas Čech cohomology of locally compact second countable Hausdorff spaces is considered in [BLP]. The definable versions of these invariants are shown in [BLP20, BLP] to be finer than the purely algebraic versions.

In this paper, we show that, for an arbitrary separable C*-algebra $A$, $K^p(A)$ can be regarded as a definable group. Furthermore, different descriptions of $K^p(A)$—in terms of extensions, Paschke duality, Fredholm modules, and quasi-homomorphisms—yield naturally definably isomorphic definable groups. For C*-algebras that have a KK-filtration in the sense of Schochet [Sch96], we show that the definable subgroup $K^p_w(A)$ of $K^p(A)$ is definably isomorphic to $P\text{Ext}(K_{1-p}(A),\mathbb{Z})$. The latter is regarded as a definable group in [BLP20, Section 7]. In fact, $P\text{Ext}(K_{1-p}(A),\mathbb{Z})$ is the quotient of a Polish group by a Borel Polishable subgroup, and hence a group with a Polish cover in the parlance of [BLP20, Section 7].

Using this and the rigidity theorem for $P\text{Ext}(\Lambda,\mathbb{Z})$ from [BLP20, Section 7] where $\Lambda$ is a torsion-free abelian group without finitely-generated direct summands, we prove that definable K-homology provides a finer invariant than the purely algebraic (or topological) groups $K^p(A)$ for a separable C*-algebra $A$, even when one restricts to UHF C*-algebras or commutative unital C*-algebras whose spectrum is a 1-dimensional subspace of $\mathbb{R}^3$.

**Theorem A.** The definable $K^1$-group is a complete invariant for UHF C*-algebras up to stable isomorphism. In contrast, there exists an uncountable family of pairwise non stably isomorphic UHF C*-algebras with algebraically isomorphic $K^1$-groups (and trivial $K^0$-groups).

**Theorem B.** The definable $K_0$-group is a complete invariant for 1-dimensional solenoids up to homeomorphism. In contrast, there exists an uncountable family of pairwise non homeomorphic 1-dimensional solenoids with algebraically isomorphic $K_0$-groups (and trivial $K_1$-groups).

The historic evolution in the treatment of K-homology described above should be compared with the similar evolution in the study of unitary duals of second countable, locally compact groups or, more generally, separable C*-algebras. Given a separable C*-algebra $A$, its unitary dual $\hat{A}$ is the quotient of the Polish space $\text{Irr}(a)$ of the unitary irreducible representations of $A$ by the relation of unitary equivalence. This includes as a particular instance the case of second countable, locally compact groups, by considering the corresponding universal C*-algebra. While initially $\hat{A}$ was considered as a topological space endowed with the quotient topology, it was recognized in the seminal
work of Mackey, Glimm, and Effros [Mac57, Gli61, Eff65] that a more fruitful theory is obtained by considering \( \hat{A} \) endowed with the quotient Borel structure, called the Mackey Borel structure. This led to the notion of type I \( \text{C}^*\)-algebra, which precisely captures those separable \( \text{C}^*\)-algebras with the property that the Mackey Borel structure is standard. It was soon realized that, in the non type I case, the right notion of “isomorphism” of Mackey Borel structures on duals \( \hat{A}, \hat{B} \) corresponds to a bijection \( \hat{A} \to \hat{B} \) that is induced by a Borel function \( \text{Irr}(A) \to \text{Irr}(B) \).

In our terminology from Section 1.4, this corresponds to regarding a unitary dual \( \hat{A} \) as a definable set, where an isomorphism of Mackey Borel structures on \( \hat{A}, \hat{B} \) is a definable bijection \( \hat{A} \to \hat{B} \). For example, this approach is taken by Elliott in [Ell77], where he proved that the unitary duals of any two separable AF \( \text{C}^*\)-algebras that are not type I are isomorphic in the category of definable sets. It is a question of Dixmier from 1967 whether the unitary duals of any two non-type I separable \( \text{C}^*\)-algebras are isomorphic in the category of definable sets; see [Tho15, Far12, KLP10]. This problem was recently considered in the case of groups by Thomas, who showed that the unitary duals of any countable amenable non-type I groups are isomorphic in the category of definable sets [Tho15, Theorem 1.10]. Furthermore, the unitary dual of any countable groups admits a definable injection to the unitary dual of the free group on two generators [Tho15, Theorem 1.9].

The work of Mackey, Glimm, and Effros on unitary representations pioneered the application of methods from descriptive set theory to \( \text{C}^*\)-algebras. More recent applications have been obtained by Kechris [Kec98] and Farah–Toms–Törnquist [FTT14, FTT12], who studied the problem of classifying several classes of \( \text{C}^*\)-algebras from the perspective of Borel complexity theory; see also [Lup14, GL16, EFP+13].

The rest of this paper is organized as follows. In Section 1 we recall fundamental results from descriptive set theory about Polish spaces and standard Borel spaces, and make precise the notions of definable set, and the corresponding notion of definable group. In Section 2 we introduce the notion of strict \( \text{C}^*\)-algebra, which is a (not necessarily norm-separable) \( \text{C}^*\)-algebra whose unit ball is endowed with a Polish topology induced by bounded seminorms, called the strict topology, such that the \( \text{C}^*\)-algebra operators are strictly continuous on the unit ball. The main example we will consider are multiplier algebras of separable \( \text{C}^*\)-algebras, endowed with their usual strict topology, as well as Paschke dual algebras of separable \( \text{C}^*\)-algebras. In Section 3 we study the K-theory of a strict \( \text{C}^*\)-algebra or, more generally, the quotient of a strict \( \text{C}^*\)-algebra by a strict ideal, such as a corona algebra or the commutant of a separable \( \text{C}^*\)-algebra in a corona algebra. We observe that the K-theory groups of a strict \( \text{C}^*\)-algebra can be regarded as quotients of a Polish space by an equivalence relation. As such an equivalence relation is not necessarily well-behaved, they are in general only semidefinable groups, although they will be in fact definable groups in the case of Paschke dual algebras of separable \( \text{C}^*\)-algebras. In Section 4 definable \( \text{K}\)-homology for separable \( \text{C}^*\)-algebras is introduced, and shown to be given by definable groups by considering its description in terms of the K-theory of Paschke dual algebras. The equivalent descriptions of \( \text{K}\)-homology due to Cuntz and Kasparov are considered in Section 5, where they are shown to yield definably isomorphic groups. In Section 6 we discuss properties of definable \( \text{K}\)-homology, which can be seen as definable versions of the general properties that an abstract cohomology theory for separable, nuclear \( \text{C}^*\)-algebras in the sense of Schochet satisfies [Sch84]. A definable version of the Universal Coefficient Theorem of Brown, later generalized to KK-groups by Rosenberg and Schochet, is considered in Section 7. Theorem A is a consequence of the definable UCT, the classification of AF \( \text{C}^*\)-algebras by K-theory, and the rigidity result for definable PExt of torsion-free finite-rank abelian groups from [BLP20]. Finally, Section 8 considers definable \( \text{K}\)-homology for compact metrizable spaces, and Theorem B is obtained applying again the UCT and the rigidity theorem for definable PExt.

1. Polish spaces and definable groups

In this section we recall some fundamental notions concerning Polish spaces and Polish groups, as well as standard Borel spaces and standard Borel groups, as can be found in [BK96, Kec95, Gao09]. We also consider the notion of Polish category, which is a category whose hom-sets are Polish spaces and composition of morphisms is a continuous function, and establish some of its basic properties. Furthermore, we recall the notion of idealistic equivalence relation on a standard Borel space and some of its fundamental properties as established in [KM16, MR12]. We then define precisely the notion of (semi)definable set and (semi)definable group.

1.1. Polish spaces and standard Borel spaces. A Polish space is a second countable topological space whose topology is induced by a complete metric. A subset of a Polish space \( X \) is \( G_\delta \) if and only if it is a Polish space when endowed with the subspace topology. If \( X \) is a Polish space, then the Borel \( \sigma \)-algebra of \( X \) is the \( \sigma \)-algebra
generated by the collection of open sets. By definition, a subset of $X$ is Borel if it belongs to the Borel $\sigma$-algebra. If $X,Y$ are Polish spaces, then the product $X \times Y$ is a Polish space when endowed with the product topology. More generally, if $(X_n)_{n \in \omega}$ is a sequence of Polish spaces, then the product $\prod_{n \in \omega} X_n$ is a Polish space when endowed with the product topology. The class of Polish spaces includes all locally compact second countable Hausdorff spaces. We denote by $\omega$ the set of natural numbers including 0. We regard $\omega$ as a (Polish) topological space endowed with the discrete topology. The Baire space $\omega^\omega$ is the Polish space obtained as the infinite product of copies of $\omega$.

A standard Borel space is a set $X$ endowed with a $\sigma$-algebra (the Borel $\sigma$-algebra) that comprises the Borel sets with respect to some Polish topology on $X$. A function between standard Borel spaces is Borel if it is measurable with respect to the Borel $\sigma$-algebras. A subset of a standard Borel space $X$ is analytic if it is the image of a Borel function $f : Z \to X$ for some standard Borel space $Z$. This is equivalent to the assertion that there exists a Borel subset $B \subseteq X \times \omega^\omega$ such that $B = \text{proj}_X(A)$ is the projection of $A$ on the first coordinate. A subset of $X$ is co-analytic if its complement is analytic. One has that a subset of $X$ is Borel if and only if it is both analytic and co-analytic.

Given standard Borel spaces $X,Y$, we let $X \times Y$ be their product endowed with the product Borel structure, which is also a standard Borel space. If $(X_n)_{n \in \omega}$ is a sequence of standard Borel spaces, then their disjoint union $X$ is a standard Borel space, where a subset $A$ of $X$ is Borel if and only if $A \cap X_n$ is Borel for every $n \in \omega$. The product $\prod_{n \in \omega} X_n$ is also a standard Borel space when endowed with the product Borel structure. In the following proposition, we collect some well-known properties of the category of standard Borel spaces and Borel functions.

**Proposition 1.1.** Let $\text{SB}$ be the category that has standard Borel spaces as objects and Borel functions and morphisms.

1. If $X$ is a standard Borel space and $A \subseteq X$ is a Borel subset, then $A$ is a standard Borel space when endowed with the induced standard Borel structure;
2. If $X,Y$ are standard Borel spaces, $f : X \to Y$ is an injective Borel function, and $A \subseteq X$ is Borel, then $f(A)$ is a Borel subset of $Y$;
3. If $X,Y$ are standard Borel spaces, and $f : X \to Y$ is a bijective Borel function, then the inverse function $f^{-1} : Y \to X$ is Borel;
4. If $X,Y$ are standard Borel spaces, and there exist injective Borel functions $f : X \to Y$ and $g : Y \to X$, then there exists a Borel bijection $h : X \to Y$;
5. The category $\text{SB}$ has finite products, finite coproducts, equalizers, and pullbacks;
6. A Borel function is monic in $\text{SB}$ if and only if it is injective, and epic in $\text{SB}$ if and only if it is surjective;
7. An inductive sequence of standard Borel spaces and Borel functions has a colimit in $\text{SB}$.

A Polish group is a topological group whose topology is Polish. If $G$ is a Polish group, and $H$ is a closed subset of $G$, then $H$ and $G/H$ are also Polish groups when endowed with the subspace topology and the quotient topology, respectively. If $G_0,G_1$ are Polish groups, and $\varphi : G_0 \to G_1$ is a Borel function, then $\varphi$ is continuous. In particular, if $G$ is a Polish space, then it has a unique Polish group topology that induces its Borel structure. A Borel subgroup $H$ of a Polish group $G$ is Polishable if there is a (necessarily unique) Polish group topology on $H$ that induces the Borel structure on $H$ inherited from $G$. If $G$ is a Polish group, then a Polish $G$-space is a Polish space $X$ endowed with a continuous action of $G$. A Borel $G$-space is a standard Borel space $X$ endowed with a Borel action of $G$. Given a Borel $G$-space $X$, there exists a Polish topology $\tau$ on $X$ such that $(X,\tau)$ is a Polish $G$-space; see [BK96, Theorem 5.2.1].

A standard Borel group is a standard Borel space $G$ that is also a group, and such that the group operation on $G$ and the function $G \to G$, $x \mapsto x^{-1}$ are Borel; see [Kec95, Definition 12.23]. Clearly, every Polish group is, in particular, a standard Borel group.

The notion of Polish topometric space was introduced and studied in [BYM15, BYBM13, BY08, BYU10]. A topometric space is a Hausdorff space $X$ endowed with a topology $\tau$ and a $[0,\infty]$-valued metric $d$ such that:

1. the metric-topology is finer than $\tau$;
2. the metric is lower-semicontinuous with respect to $\tau$, i.e. for every $r \geq 0$ the set
   \[ \{(a,b) \in X \times X : d(a,b) \leq r\} \]
   is $\tau$-closed in $X \times X$. 
A Polish topometric space is a topometric space such that the topology $\tau$ is Polish and the metric is complete. A Polish topometric group is a Polish topometric space $(G, \tau, d)$ that is also a group, and such that $G$ endowed with the topology $\tau$ is a Polish group, and the metric on $G$ is bi-invariant.

1.2. Polish categories. By definition, we let a Polish category be a category $C$ enriched over the category of Polish spaces (regarded as a monoidal category with respect to binary products). Thus, for each pair of objects $a, b$ of $C$, $C(a, b)$ is a Polish space, such that for objects $a, b, c$, the composition operation $C(b, c) \times C(a, b) \to C(a, c)$ is continuous.

Suppose that $C$ is a Polish category. For objects $a, b$ of $C$, define $\text{Iso}_C(a, b) \subseteq C(a, b)$ be the set of $C$-isomorphisms $a \to b$. While $\text{Iso}_C(a, b)$ is not necessarily a $G_δ$ subset of $C(a, b)$, and hence not necessarily a Polish space when endowed with the subspace topology, $\text{Iso}_C(a, b)$ is endowed with a canonical Polish topology, defined as follows. For a net $(α_i)$ in $\text{Iso}_C(a, b)$ and $α \in \text{Iso}_C(a, b)$, set $α_i \to α$ if and only if $α_i \to α$ in $C(a, b)$ and $α_i^{-1} \to α^{-1}$ in $C(b, a)$. One can then easily show the following.

Lemma 1.2. Adopt the notations above. Then $\text{Iso}_C(a, b)$ is a Polish space.

It is clear from the definition that, for every object $a$ of $C$, $\text{Aut}_C(a) := \text{Iso}_C(a, a)$ is a Polish group. Furthermore, the canonical (right and left) actions of $\text{Aut}_C(a)$ and $\text{Aut}_C(b)$ on $C(a, b)$ are continuous.

Definition 1.3. Suppose that $C$ and $D$ are Polish categories, and $F : C \to D$ is a functor. We say that $F$ is continuous if, for every pair of objects $a, b$ of $C$, the map $C(a, b) \to D(F(a), F(b))$, $f \mapsto F(f)$ is continuous. We say that $F$ is a topological equivalence if it is continuous, and there exists a continuous functor $G : D \to C$ such that $GF$ is isomorphic to the identity functor $I_C$, and $FG$ is isomorphic to the identity functor $I_D$.

The notion of topological equivalence of categories is the natural analogue of the notion of equivalence of categories in the context of Polish categories; see [ML98, Section IV.4]. The same proof as [ML98, Section IV.4, Theorem 1] gives the following characterization of topological equivalences.

Lemma 1.4. Suppose that $C$ and $D$ are Polish categories, and $F : C \to D$ is a functor. The following assertions are equivalent:

(1) $F$ is a topological equivalence;

(2) each object of $D$ is isomorphic to one of the form $F(a)$ for some object $a$ of $C$, and for each pair of objects $c, d$ of $C$, the map $C(c, d) \to C(F(c), F(d))$ is a homeomorphism.

1.3. Idealistic equivalence relations. Suppose that $C$ is a set. A $σ$-filter on $C$ is a nonempty family $\mathcal{F}$ of subsets of $C$ that is closed under countable intersections, and such that $\emptyset \notin \mathcal{F}$ and if $A \subseteq B \subseteq C$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$. The dual notion is the one of $σ$-ideal. Thus, a nonempty family $\mathcal{I}$ of subsets of $C$ is a $σ$-ideal if it is closed under countable unions, $C \notin \mathcal{I}$, and $A \subseteq B \subseteq C$ and $B \in \mathcal{I}$ imply $A \in \mathcal{I}$. Clearly, if $\mathcal{F}$ is a $σ$-filter on $C$, then $\{C \setminus A : A \in \mathcal{F}\}$ is a $σ$-ideal on $C$, and vice-versa. Thus, one can equivalently formulate notions in terms of $σ$-filters or in terms of $σ$-ideals.

If $\mathcal{F}$ is a $σ$-filter on $C$, then $\mathcal{F}$ can be thought of as a notion of “largeness” for subsets of $C$. Based on this interpretation, we use the “$σ$-filter quantifier” notation “$\mathcal{F}x$, $x \in A$” for a subset $A \subseteq C$ to express the fact that $A \in C$. If $P(x)$ is a unary relation for elements of $C$, “$\mathcal{F}x$, $P(x)$” is the assertion that the set of $x \in C$ that satisfy $P(x)$ belongs to $\mathcal{F}$.

Example 1.5. Suppose that $C$ is a Polish space. A subset $A$ of $C$ is meager if it is contained in the union of a countable family of closed nowhere dense sets. By the Baire Category Theorem [Kec95, Theorem 8.4], meager subsets of $C$ form a $σ$-ideal $\mathcal{I}_C$. The corresponding dual $σ$-filter is the $σ$-filter $\mathcal{F}_C$ of comeager sets, which are the subsets whose complement is meager.

Suppose that $X$ is a standard Borel space. We consider an equivalence relation $E$ on $X$ as a subset of $X \times X$, endowed with the product Borel structure. Consistently, we say that $E$ is Borel or analytic, respectively, if it is a Borel or analytic subset of $X \times X$. In the following, we will exclusively consider analytic equivalence relations, most of which will in fact be Borel. For an element $x$ of $X$ we let $[x]_E$ be its corresponding $E$-class.

We now recall the notion of idealistic equivalence relation, initially considered in [Kec94]; see also [Gao09, Definition 5.4.9] and [KM16]. We will consider a slightly more generous definition than the one from [Kec94,
Gao09, KM16]. The more restrictive notion is recovered as a particular case by insisting that the function $s$ in Definition 1.6 be the identity function of $X$. In the following definition, for a subset $A$ of a product space $X \times Y$ and $x \in X$, we let $A_x = \{ y \in Y : (x, y) \in A \}$ be the corresponding vertical section.

**Definition 1.6.** A Borel equivalence relation $E$ on a standard Borel space $X$ is idealistic if there exist a Borel function $s : X \to X$ satisfying $s(x) Ex$ for every $x \in X$, and a function $C \mapsto \mathcal{F}_C$ that assigns to each $E$-class $C$ a $\sigma$-filter $\mathcal{F}_C$ of subsets of $C$ such that, for every Borel subset $A$ of $X \times X$, the set

$$A_{s, E} := \{ x \in X : \mathcal{F}_{[x]_E} s'(s(x), x') \in A \} = \{ x \in X : A_{s(x)} \in \mathcal{F}_{[x]_E} \}.$$

is Borel.

Idealistic equivalence relations arise naturally as orbit equivalence relations of Polish group actions. Suppose that $G \acts X$ is an action of a Polish group $G$ on a Polish space $X$. Let $E^X_G$ be the corresponding orbit equivalence relation, obtained by setting $xE^X_G y$ if and only if there exists $g \in G$ such that $g \cdot x = y$. Then $E^X_G$ is an idealistic equivalence relation, as witnessed by the identity function $s$ on $X$ and the function $C \mapsto \mathcal{F}_C$ where $A \in \mathcal{F}_C$ if and only if $\mathcal{F}_{g \cdot A}$, $g \cdot x \in A$ (where as in Example 1.5, $\mathcal{F}_A$ denotes the $\sigma$-filter of comeager subsets of $G$). In particular, if $G$ is a Polish group, and $H$ is a Polishable Borel subset of $G$, then the coset equivalence relation $E^H_G$ of $H$ in $G$ is Borel and idealistic.

Suppose that $E$ is an equivalence relation on a standard Borel space $X$. A Borel selector for $E$ is a Borel function $s : X \to X$ such that, for $x, y \in X$, $xEy$ if and only if $s(x) = s(y)$. If $E$ has a Borel selector, then $E$ is Borel and idealistic; see [Gao09, Theorem 5.4.11]. (Precisely, an equivalence relation has a Borel selector if and only if it is Borel, idealistic, and smooth [Gao09, Definition 5.4.1].)

### 1.4. Definable sets.

Definable sets are a generalization of standard Borel sets, and can be thought of as sets explicitly presented as the quotient of a standard Borel space by a “well-behaved” equivalence relation $E$.

**Definition 1.7.** A definable set $X$ is a pair $(\hat{X}, E)$ where $\hat{X}$ is a standard Borel space and $E$ is a Borel and idealistic equivalence relation on $\hat{X}$. We think of $(\hat{X}, E)$ as a presentation of the quotient set $X = \hat{X}/E$. Consistently, we also write the definable set $(\hat{X}, E)$ as $\hat{X}/E$. A subset $Z$ of $X$ is definable if there is an $E$-invariant Borel subset $\hat{Z}$ of $\hat{X}$ such that $Z = \hat{Z}/E$.

We now define the notion of morphism between definable sets. Let $X = \hat{X}/E$ and $Y = \hat{Y}/F$ be definable sets. A lift of a function $f : X \to Y$ is a function $\hat{f} : \hat{X} \to \hat{Y}$ such that $f([x]_E) = [\hat{f}(x)]_F$ for every $x \in \hat{X}$. We say that $f$ has a Borel lift if it admits a lift $\hat{f} : \hat{X} \to \hat{Y}$ that is a Borel function.

**Definition 1.8.** Let $X$ and $Y$ be definable sets. A function $f : X \to Y$ is Borel-definable if it has a Borel lift.

**Remark 1.9.** Since Borel-definability is the only notion of definability we will consider in this work, we will abbreviate “Borel-definable” to “definable”.

We consider definable sets as objects of a category $\text{DSet}$, whose morphisms are the definable functions. We regard a standard Borel space $X$ as a particular instance of definable set $X = \hat{X}/E$ where $X = \hat{X}$ and $E$ is the relation of equality on $X$. This renders the category of standard Borel spaces a full subcategory of the category of definable sets.

If $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are definable sets, then their product $X \times Y$ in $\text{DSet}$ is the definable set $X \times Y := (\hat{X} \times \hat{Y})/(E \times F)$, $E \times F$ being the equivalence relation on $\hat{X} \times \hat{Y}$ defined by setting $(x, y) (E \times F) (x', y')$ if and only if $xEx'$ and $yFy'$. (It is easy to see that $E \times F$ is Borel and idealistic if both $E$ and $F$ are Borel and idealistic.) Many of the good properties of standard Borel spaces, including all the ones listed in Proposition 1.1, generalize to definable sets.

**Proposition 1.10.** Let as above $\text{DSet}$ be the category that has definable as objects and definable functions as morphisms.

1. If $X$ is a definable and $A \subseteq X$ is a definable subset, then $A$ is itself a definable set;
2. If $X, Y$ are definable sets, $f : X \to Y$ is an injective definable function, and $A \subseteq X$ a definable subset, then $f(A)$ is a definable subset of $Y$;
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(3) If \(X, Y\) are definable sets, and \(f : X \to Y\) is a bijective definable function, then the inverse function \(f^{-1} : Y \to X\) is definable;

(4) If \(X, Y\) are definable sets, and there exist injective definable functions \(f : X \to Y\) and \(g : Y \to X\), then there exists a definable bijection \(h : X \to Y\);

(5) The category \(\mathbf{DSet}\) has finite products, finite coproducts, equalizers, and pullbacks;

(6) A definable function is monic in \(\mathbf{DSet}\) if and only if it is injective, and epic in \(\mathbf{DSet}\) if and only if it is surjective;

(7) An inductive sequence of definable sets and injective definable functions has a colimit in \(\mathbf{DSet}\).

Proof. (1) is immediate from the definition. (2) and (3) are consequences of [KM16, Lemma 3.7], after observing that the same proof there applies in the case of the more generous notion of idealistic equivalence relation considered here. (4) is a consequence of (2) and [MR12, Proposition 2.3]. Finally, (5), (6), and (7) are easily verified.

Occasionally we will need to consider quotients \(X = \hat{X}/E\) where \(\hat{X}\) is a standard Borel space \(E\) is an analytic equivalence relation on \(X\) that is not Borel and idealistic, or has not yet been shown to be Borel and idealistic. In this case, we say that \(X = \hat{X}/E\) is a semidefinable set. Clearly, every definable set is, in particular, a semidefinable set. The notion of definable subset and definable function are the same as in the case of definable sets. Thus, if \(X = \hat{X}/E\) and \(Y = \hat{Y}/F\) are semidefinable sets, \(Z \subseteq X\) is a subset and \(f : X \to Y\) is a function, then \(f\) is definable if it has a Borel lift \(\hat{f} : \hat{X} \to \hat{Y}\), and \(Z\) is definable if there is a Borel \(E\)-invariant subset \(\hat{Z}\) of \(\hat{X}\) such that \(Z = \hat{Z}/E\). The category \(\mathbf{SemiDSet}\) has semidefinable sets as objects and definable functions has morphisms. Notice that, in particular, an isomorphism from \(X\) to \(Y\) in \(\mathbf{SemiDSet}\) is a bijection \(f : X \to Y\) such that both \(f\) and the inverse function \(f^{-1} : Y \to X\) are definable.

Lemma 1.11. Suppose that \(X = \hat{X}/E\) is a definable set, \(Y = \hat{Y}/F\) is a semidefinable set. If \(X\) and \(Y\) are isomorphic in \(\mathbf{SemiDSet}\), then \(Y\) is a definable set.

Proof. Suppose that the Borel function \(s_X : \hat{X} \to \hat{X}\) and the assignment \(C \mapsto \mathcal{E}_C\) witness that \(E\) is idealistic. By assumption, there exists a bijection \(f : X \to Y\) such that \(f\) has a Borel lift \(\alpha : \hat{X} \to \hat{Y}\), and \(f^{-1}\) has a Borel lift \(\beta : \hat{Y} \to \hat{X}\). For \(y, y' \in \hat{Y}\) we have that \(yFy'\) if and only if \(\beta(y)E\beta(y')\), whence \(F\) is Borel. We now show that \(F\) is idealistic.

Define an assignment \(D \mapsto \mathcal{F}_D\) from \(D\)-classes to \(\sigma\)-filters, by setting \(S \in \mathcal{F}_D\) if and only if \(\alpha^{-1}(S) \in \mathcal{E}_C\) where \(f(C) = D\). Consider also the Borel map \(s_Y := \alpha \circ s_X \circ \beta : \hat{Y} \to \hat{Y}\). Then it is easy to verify that \(s_Y\) and the assignment \(D \mapsto \mathcal{F}_D\) witness that \(F\) is idealistic.

Lemma 1.12. Suppose that \(X = \hat{X}/E\) and \(Y = \hat{Y}/F\) are semidefinable sets. Assume that there exists a definable bijection \(f : X \to Y\) (which is not necessarily an isomorphism in \(\mathbf{SemiDSet}\)). If \(E\) is Borel, then \(F\) is Borel.

Proof. By assumption \(E \subseteq \hat{X} \times \hat{X}\) is Borel, and \(F \subseteq \hat{Y} \times \hat{Y}\) is analytic. Furthermore, \(f\) has a Borel lift \(\hat{f} : \hat{X} \to \hat{Y}\). Since \(f\) is a bijection, we have that, for \(y, y' \in \hat{Y}\),

\[yFy' \iff \forall x, x' \in \hat{X}, \text{ if } \hat{f}(x)Fy \text{ and } \hat{f}(x')FY', \text{ then } xEx'.\]

This shows that \(F\) is co-analytic. As \(F\) is also analytic, we have that \(F\) is Borel.

Lemma 3.7 in [KM16] can be stated as the following proposition, which generalizes items (2) and (3) in Proposition 1.10.

Proposition 1.13 (Kechris–Macdonald). Let \(X = \hat{X}/E\) be a definable set, \(Y = \hat{Y}/F\) be semidefinable set such that \(F\) is Borel, and \(f : X \to Y\) be a definable function. If \(f\) is injective, then the range of \(f\) a definable subset of \(Y\). If \(f\) is bijective, then the inverse function \(f^{-1} : Y \to X\) is definable.

The following result is a consequence of Lemma 1.12 and Proposition 1.13.

Corollary 1.14. Suppose that \(X = \hat{X}/E\) is a definable set, and \(Y = \hat{Y}/F\) is a semidefinable set. If \(f : X \to Y\) is a definable bijection, then \(Y\) is a definable set and \(f\) is an isomorphism in \(\mathbf{DSet}\).

Proof. By Lemma 1.12, \(F\) is Borel. Whence, by Proposition 1.13, \(f\) is an isomorphism in \(\mathbf{SemiDSet}\). Since \(X\) is a definable set, it follows from Lemma 1.11 that \(Y\) is also a definable set, and \(f\) is an isomorphism in \(\mathbf{DSet}\).
1.5. **Definable groups.** A definable group can be simply defined as a group in the category $\text{DSet}$ in the sense of [ML98, Section III.6]. Thus, a definable group is a definable set $G = \hat{G}/E$ that is also a group, and such that the group operation $G \times G \to G$ is definable, and the function $G \to G, x \mapsto x^{-1}$ is also definable. As in the case of sets, we regard a standard Borel group as a particular instance of definable group $G = \hat{G}/E$ where $G = \hat{G}$ is a standard Borel group and $E$ is the relation of equality on $\hat{G}$. Thus, standard Borel groups form a full subcategory of the category of definable groups.

Naturally, a semidefinable group will be a group in $\text{SemiDSets}$, i.e. a semidefinable set $G = \hat{G}/E$ that is also a group, and such that the group operation $G \times G \to G$ is definable, and the function $G \to G$ that maps every element to its inverse is definable.

**Lemma 1.15.** If $G = \hat{G}/E$ is a semidefinable group, then the equivalence relation $E$ is Borel if and only if the identity element of $G$, which is the $E$-class $[\ast]_E$ of some element $\ast$ of $\hat{G}$, is a Borel subset of $\hat{G}$.

**Proof.** Clearly, if $E$ is Borel, then $[\ast]_E$ is Borel. Conversely, suppose that $[\ast]_E$ is Borel. If $m : \hat{G} \times \hat{G} \to \hat{G}$ and $\zeta : \hat{G} \to \hat{G}$ are Borel lifts of the group operation in $X$ and of the function that maps each element to its inverse, respectively, then we have that $x Ey$ if and only if $m(\hat{X}, \zeta(y)) \in [\ast]_E$. This shows that $E$ is Borel. \hfill $\square$

**Corollary 1.16.** Suppose that $G = \hat{G}/E$ is a semidefinable group. If $E$ is the orbit equivalence relation of a Borel action of a Polish group $H$ on the standard Borel space $\hat{G}$, then $G$ is a definable group.

**Proof.** By [BK96, Theorem 5.2.1] one can assume that $\hat{G}$ is a Polish $H$-space, and $E$ is the orbit equivalence relation of a continuous $H$-action on $\hat{G}$. By [Ga09, Proposition 3.1.10], every $E$-class is Borel. Therefore $E$ is Borel by Lemma 1.15. Furthermore, $E$ is idealistic by [Ga09, Proposition 5.4.10]. \hfill $\square$

**Remark 1.17.** A particular instance of definable group is obtained as follows. Suppose that $G$ is a Polish group and $H$ is a Borel Polishable subgroup. Let $E_H^G$ be the coset equivalence relation of $H$ in $G$. The quotient group $G/H$ is the quotient of $G$ by the equivalence relation $E_H^G$. Since $H$ is Polishable, $E^G_H$ is the orbit equivalence relation of a Borel action of a Polish group on $G$. Thus, $G/H = G/E_H^G$ is a definable group by Corollary 1.16. The definable groups obtained in this way are called groups with a Polish cover in [BLP20].

## 2. Strict $\text{C}^*$-algebras

In this section we introduce the notion of strict Banach space and strict $\text{C}^*$-algebra and some of their properties. Briefly, a strict Banach space is a Banach space whose unit ball is endowed with a Polish topology (called the strict topology) that is coarser than the norm-topology and induced by a sequence of bounded seminorms. A suitable semicontinuity requirement relates the norm and the strict topology. A strict $\text{C}^*$-algebra is a strict Banach space that is also a $\text{C}^*$-algebra with some suitable continuity requirement relating the $\text{C}^*$-algebra operations and the strict topology. The name is inspired by the strict topology on the multiplier algebra of a separable $\text{C}^*$-algebra, which will be one of the main examples. Other examples are Paschke dual algebras of separable $\text{C}^*$-algebras.

### 2.1. Strict Banach spaces

Let $X$ be a Banach space. We denote by $\text{Ball}(X)$ its unit ball. A seminorm $p$ on $X$ is bounded if $\|p\| := \sup_{x \in \text{Ball}(X)} p(x) < \infty$. We say that $p$ is contractive if $\|p\| \leq 1$.

**Definition 2.1.** A strict Banach space is a Banach space $\mathcal{X}$ such that $\text{Ball}(\mathcal{X})$ is endowed with a topology (called the strict topology) such that, for some sequence $(p_n)$ of contractive seminorms on $\mathcal{X}$, letting $d$ be the pseudometric on $\text{Ball}(\mathcal{X})$ defined by

$$d(x, y) = \sum_{n \in \omega} 2^{-n} p_n(x - y),$$

one has that:

1. $d$ is a complete metric that induces the strict topology on $\text{Ball}(\mathcal{X})$;
2. $\text{Ball}(\mathcal{X})$ contains a countable strictly dense subset;
3. $\|x\| = \sup_{n \in \omega} p_n(x)$ for every $x \in \mathcal{X}$.

**Example 2.2.** Suppose that $X$ is a separable Banach space. Then $X$ is a strict Banach space where the strict topology on $\text{Ball}(X)$ is the norm topology.
Example 2.3. Suppose that $Y$ is a separable Banach space, and $Y^*$ is its Banach space dual. Then $Y^*$ is a strict Banach space where the strict topology on $Ball(Y^*)$ is the weak*-topology.

Let $X$ be a seminormed space, and consider the cone $\mathcal{S}(X)$ of bounded seminorms on $X$ as a complete metric space, with respect to the metric defined by $d(p, q) = \sup_{x \in Ball(X)} |p(x) - q(x)|$. For a subset $\mathcal{G} \subseteq \mathcal{S}(X)$, we let $\sigma(X, \mathcal{G})$ be the topology on $Ball(X)$ generated by $\mathcal{G}$. We denote by $Ball(\mathcal{G})$ the set of contractive seminorms in $\mathcal{G}$. If $p \in \mathcal{S}(X)$, $\mathcal{G} \subseteq \mathcal{S}(X)$, and $(x_n)$ is a sequence in $Ball(X)$, then we say that:

- $(x_n)_{n \in \omega}$ is $p$-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \omega$ such that, for $n, m \geq n_0$, $p(x_n - x_m) < \varepsilon$;
- $(x_n)_{n \in \omega}$ is $\mathcal{G}$-Cauchy if it is $p$-Cauchy for every $p \in \mathcal{G}$;
- $Ball(X)$ is $\mathcal{G}$-complete if, for every sequence $(x_n)_{n \in \omega}$ in $Ball(X)$, if $(x_n)_{n \in \omega}$ is $\mathcal{G}$-Cauchy, then $(x_n)_{n \in \omega}$ is $\sigma(X, \mathcal{G})$-convergent to some element of $Ball(X)$.

The following lemma is elementary.

Lemma 2.4. Suppose that $X$ is a seminormed space. Let $\tau$ be a topology on $Ball(X)$. Assume that $\mathcal{G}$ and $\mathcal{I}$ are two sets of bounded seminorms on $X$ such that the topologies $\sigma(X, \mathcal{G})$ and $\sigma(X, \mathcal{I})$ on $Ball(X)$ coincide with $\tau$. Then, for a sequence $(x_n)$ in $Ball(X)$, $(x_n)$ is $\mathcal{G}$-Cauchy if and only if it is $\mathcal{I}$-Cauchy. In this case, we say that $(x_n)$ is $\tau$-Cauchy. It follows that $Ball(X)$ is $\mathcal{G}$-complete if and only if it is $\mathcal{I}$-complete. In this case, we say that $Ball(X)$ is $\tau$-complete.

In view of Lemma 2.4 one can equivalently define a strict Banach space as follows.

Definition 2.5. A strict Banach space is a Banach space $\mathfrak{X}$ such that $Ball(\mathfrak{X})$ is endowed with a topology (called the strict topology) such that, for some separable cone $\mathcal{G}$ of bounded seminorms on $X$, one has that:

1. the strict topology on $Ball(\mathfrak{X})$ is the $\sigma(\mathfrak{X}, \mathcal{G})$-topology, and $Ball(\mathfrak{X})$ is strictly complete;
2. $Ball(\mathfrak{X})$ contains a countable strictly dense subset;
3. $\|x\| = \sup_{p \in Ball(\mathcal{G})} p(x)$ for every $x \in \mathfrak{X}$.

Let $\mathfrak{X}$ be a strict Banach space. Notice that, if $\mathfrak{Y}$ is a norm-closed subspace of $X$ such that $Ball(\mathfrak{Y})$ is strictly closed in $Ball(\mathfrak{X})$, then $\mathfrak{Y}$ is a strict Banach space with the induced norm and the induced strict topology.

Proposition 2.6. Suppose that $\mathfrak{X}$ is a strict Banach space. Then $Ball(\mathfrak{X})$ is a Polish topometric space when endowed with the strict topology and the norm-distance.

Proof. By definition, the strict topology on $Ball(\mathfrak{X})$ is Polish. Since the strict topology is induced by bounded seminorms on $X$, it is coarser than the norm topology. The function $(x, y) \mapsto \|x - y\|$ is strictly lower-semicontinuous, being the supremum of strictly continuous functions. Since the norm on $X$ is complete, the distance $(x, y) \mapsto \|x - y\|$ on $Ball(\mathfrak{X})$ is complete.

Suppose that $\mathfrak{X}$ is a strict Banach space. We extend the strict topology of $Ball(\mathfrak{X})$ to any bounded subset of $\mathfrak{X}$ by declaring the function

$$nBall(\mathfrak{X}) = \{z \in \mathfrak{X} : \|z\| \leq n\} \to Ball(\mathfrak{X}), z \mapsto \frac{1}{n}z$$

to be a homeomorphism with respect to the strict topology. Then we have that addition and scalar multiplication on $\mathfrak{X}$ are strictly continuous on bounded sets, and the norm is strictly lower-semicontinuous on bounded sets. In particular $nBall(X)$ is a strictly closed subspace of $mBall(X)$ for $n \leq m$.

Definition 2.7. Let $\mathfrak{X}$ be a strict Banach space. The (standard) Borel structure on $\mathfrak{X}$ is defined by declaring a subset $A$ of $\mathfrak{X}$ to be Borel if and only if $\frac{1}{n}A \cap Ball(\mathfrak{X})$ is Borel for every $n \geq 1$ or, equivalently $A \cap nBall(\mathfrak{X})$ is Borel for every $n \geq 1$.

Notice that the Borel structure on $\mathfrak{X}$ is standard, as by definition $\mathfrak{X}$ is Borel isomorphic to the disjoint union of the standard Borel spaces $(n + 1)Ball(\mathfrak{X}) \setminus nBall(\mathfrak{X})$ for $n \geq 1$.

Definition 2.8. If $\mathfrak{X}$ and $\mathfrak{Y}$ are strict Banach spaces. A bounded linear map $T : \mathfrak{X} \to \mathfrak{Y}$ is contractive if $\|T\| \leq 1$, and strict if it is strictly continuous on bounded sets. A bounded seminorm $p$ on $\mathfrak{X}$ is strict if it is strictly continuous on bounded sets.
Clearly, strict Banach spaces form a category where the morphisms are the strict contractive linear maps. Notice that a strict, bijective, and isometric linear map \( T : \mathfrak{X} \to \mathfrak{Y} \) between strict Banach spaces might not be a homeomorphism \( T : \text{Ball}(\mathfrak{X}) \to \text{Ball}(\mathfrak{Y}) \), and hence not an isomorphism in the category of strict Banach spaces. Nonetheless, \( T : \mathfrak{X} \to \mathfrak{Y} \) will be a Borel isomorphism, as both \( \mathfrak{X} \) and \( \mathfrak{Y} \) are standard Borel spaces.

**Definition 2.9.** Let \( \mathfrak{X} \) be a strict Banach space. Define \( \mathcal{S}_{\text{strict}}(\mathfrak{X}) \) to be the space of bounded, strict seminorms on \( \mathfrak{X} \).

Notice that \( \mathcal{S}_{\text{strict}}(\mathfrak{X}) \) is a closed subspace of the complete metric space \( \mathcal{S}(\mathfrak{X}) \). A sequence \( (x_n) \) in \( \text{Ball}(\mathfrak{X}) \) is strictly convergent if and only if it is \( \mathcal{S}_{\text{strict}}(\mathfrak{X}) \)-Cauchy. A bounded linear map \( T : \mathfrak{X} \to \mathfrak{Y} \) is strict if and only if \( p \circ T \in \mathcal{S}_{\text{strict}}(\mathfrak{Y}) \) for every \( p \in \mathcal{S}_{\text{strict}}(\mathfrak{Y}) \).

**Remark 2.10.** Suppose that \( \mathfrak{X} \) is a strict Banach space, and \( \mathcal{S} \) is a separable cone of bounded, strict seminorms on \( \mathfrak{X} \) that induces the strict topology on \( \text{Ball}(\mathfrak{X}) \). One can consider the globally defined topology \( \sigma(\mathfrak{X},\mathcal{S}) \) on \( \mathfrak{X} \), induced by all the seminorms in \( \mathcal{S} \). This topology coincides with the strict topology on \( \text{Ball}(\mathfrak{X}) \). However, it is not first countable on the whole of \( \mathfrak{X} \), unless \( \mathfrak{X} \) is a separable Banach space and the strict topology is equal to the norm topology. Indeed, if the \( \sigma(\mathfrak{X},\mathcal{S}) \)-topology on \( \mathfrak{X} \) is first-countable, then \( (\mathfrak{X},\sigma(\mathfrak{X},\mathcal{S})) \) is a Frechet space. By the Open Mapping Theorem for Frechet spaces [RR64, Theorem 8, page 120], any two comparable Frechet space topologies must be equal. Thus, \( \sigma(\mathfrak{X},\mathcal{S}) \) equals the norm topology. In particular, the norm-topology on \( \text{Ball}(\mathfrak{X}) \) is equal to the strict topology, and it has a countable dense subset. Hence the norm-topology on \( \mathfrak{X} \) is separable.

For future reference, we record the easily proved observation that a uniform limit of strictly continuous functions is strictly continuous.

**Lemma 2.11.** Suppose that \( \mathfrak{X} \) and \( \mathfrak{Y} \) are strict Banach spaces, and \( A \subseteq \text{Ball}(\mathfrak{X}) \). Suppose that \( f : A \to \mathfrak{Y} \) is a function. Assume that there exists an sequence \( (f_n) \) of strictly continuous function \( f_n : A \to \mathfrak{Y} \) such that

\[
\lim_{n \to \infty} \sup_{x \in A} \|f_n(x) - f(x)\| = 0.
\]

Then \( f \) is strictly continuous.

A standard Baire Category argument shows that one can characterize bounded subsets in terms of bounded, strict seminorms, as follows.

**Lemma 2.12.** Let \( \mathfrak{X} \) be a strict Banach space. If \( A \subseteq \mathfrak{X} \), then \( A \) is bounded if and only if, for every \( p \in \mathcal{S}_{\text{strict}}(\mathfrak{X}) \), \( p(A) \) is a bounded subset of \( \mathbb{R} \).

A natural way to obtain strict Banach spaces is via pairings.

**Definition 2.13.** A Banach pairing is a bounded bilinear map \( \langle \cdot, \cdot \rangle : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z} \), where \( \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \) are Banach spaces. Define the \( \sigma(\mathfrak{X},\mathfrak{Y}) \)-topology on \( \mathfrak{X} \) to be the locally convex vector space generated by the cone \( \mathcal{S}_\mathfrak{Y} \) of bounded seminorms \( x \mapsto \|\langle x, y \rangle\| \) for \( y \in \mathfrak{Y} \).

The following lemma is an immediate consequence of the definition of strict Banach space.

**Lemma 2.14.** Suppose that \( \langle \cdot, \cdot \rangle : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z} \) is a Banach pairing. Consider the cone \( \mathcal{S}_\mathfrak{Y} \) of bounded seminorms on \( \mathfrak{X} \) of the form \( x \mapsto \|\langle x, y \rangle\| \) for \( y \in \mathfrak{Y} \). Assume that:

- \( \mathfrak{Y}, \mathfrak{Z} \) are norm-separable Banach spaces;
- for every \( x_0 \in \mathfrak{X} \),
  \[ \|x_0\| = \sup_{y \in \text{Ball}(\mathfrak{Y})} \|\langle x_0, y \rangle\| ; \]
- \( \text{Ball}(\mathfrak{X}) \) is \( \mathcal{S}_\mathfrak{Y} \)-complete;
- \( \text{Ball}(\mathfrak{X}) \) when endowed with the \( \sigma(\mathfrak{X},\mathfrak{Y}) \)-topology has a dense countable subset.

Then \( \mathfrak{X} \) is a strict Banach space where the strict topology on \( \text{Ball}(\mathfrak{X}) \) is the \( \sigma(\mathfrak{X},\mathfrak{Y}) \)-topology.

Suppose that \( \mathfrak{X} \) is a norm-separable Banach space, and \( \mathfrak{Y} \) is a strict Banach spaces. A linear map \( T : \mathfrak{X} \to \mathfrak{Y} \) is bounded if it maps bounded sets to bounded sets or, equivalently,

\[
\|T\| = \sup_{x \in \text{Ball}(\mathfrak{X})} \|T(x)\| < \infty.
\]

This defines a norm on the space \( L(\mathfrak{X},\mathfrak{Y}) \) of bounded linear maps \( \mathfrak{X} \to \mathfrak{Y} \). We also define the strict topology on \( \text{Ball}(L(\mathfrak{X},\mathfrak{Y})) \) to be the topology of pointwise convergence in the strict topology of \( \text{Ball}(\mathfrak{Y}) \). Then one can easily show the following.
Proposition 2.15. Suppose that $X$ is a norm-separable Banach space, and $\mathcal{Y}$ is a strict Banach space. Then $L(X, \mathcal{Y})$ is a strict Banach space.

2.2. Strict $C^*$-algebras. We now introduce the notion of strict $C^*$-algebra. Given a $C^*$-algebra $A$, we let $A_{sa}$ be the set of its selfadjoint elements. We also denote by $M_n(A)$ the $C^*$-algebra of $n \times n$ matrices over $A$, which can be identified with the tensor product $M_n(C) \otimes A$. We refer to [Bla06, Dav96a, Mur90, Ped79] for fundamental notions and results in the theory of $C^*$-algebras.

Definition 2.16. A strict $C^*$-algebra is a $C^*$-algebra $\mathfrak{A}$ such that, for every $n \geq 1$, $M_n(\mathfrak{A})$ is also a strict Banach space satisfying the following properties:

(1) the $*$-operation and the multiplication operation on $M_n(\mathfrak{A})$ are strictly continuous on bounded sets;

(2) the strict topology on $\text{Ball}(M_n(\mathfrak{A}))$ is induced by the inclusion

$$\text{Ball}(M_n(\mathfrak{A})) \subseteq M_n(\text{Ball}(\mathfrak{A})),$$

where $\text{Ball}(\mathfrak{A})$ is endowed with the strict topology, and $M_n(\text{Ball}(\mathfrak{A}))$ is endowed with the product topology.

Example 2.17. Suppose that $A$ is a separable $C^*$-algebra. Then we have that $A$ is a strict $C^*$-algebra where, for every $n \geq 1$, $\text{Ball}(M_n(A))$ is endowed with the norm-topology.

Suppose that $\mathfrak{A}$ is a strict $C^*$-algebra. Then, for every $n \geq 1$, $M_n(\mathfrak{A})$ is also a strict $C^*$-algebra. If $\mathfrak{A}$ is a strict $C^*$-algebra, then we regard $\mathfrak{A}$ as a standard Borel space with respect to the standard Borel structure induced by the strict topology on $\text{Ball}(\mathfrak{A})$ as in Definition 2.7. We say that a subset of $\mathfrak{A}$ is Borel if it is Borel with respect to such a Borel structure. We have that the Borel structure on $M_n(\mathfrak{A})$ (as a strict $C^*$-algebra) coincides with the product Borel structure.

Definition 2.18. Suppose that $\mathfrak{A}$ is a strict unital $C^*$-algebra. A strict ideal of $\mathfrak{A}$ is a norm-closed proper two-sided $\mathfrak{A}$-borel ideal $\mathfrak{J}$ of $\mathfrak{A}$ that is also a strict $C^*$-algebra, and such that the inclusion map $\mathfrak{J} \to \mathfrak{A}$ is strict.

Remark 2.19. In order for $\mathfrak{J}$ to be a strict ideal of $\mathfrak{A}$, we do not require that $\text{Ball}(\mathfrak{J})$ be strictly closed in $\text{Ball}(\mathfrak{A})$ nor that the strict topology on $\text{Ball}(\mathfrak{J})$ be the subspace topology induced by the strict topology of $\text{Ball}(\mathfrak{A})$.

Example 2.20. Suppose that $\mathfrak{A}$ is a strict unital $C^*$-algebra and $J \subseteq \mathfrak{A}$ is a norm-closed and norm-separable proper two-sided ideal of $\mathfrak{A}$. Then $J$ is a strict ideal of $\mathfrak{A}$.

We regard strict (unital) $C^*$-algebras as objects of a category with strict (unital) $^*$-morphisms as morphisms. (Recall that a bounded linear map is strict if it is strictly continuous on bounded sets.) If $\mathfrak{A} \subseteq \mathfrak{B}$, then we say that $\mathfrak{A}$ is strictly dense in $\mathfrak{B}$ if $\text{Ball}(\mathfrak{A})$ is dense in $\text{Ball}(\mathfrak{B})$ with respect to the strict topology.

It follows from the axioms of a strict $C^*$-algebra that, if $\mathfrak{A}$ is a strict $C^*$-algebra, and $p(x_1, \ldots, x_n)$ is a $^*$-polynomial, then $p$ defines a function $\mathfrak{A}^n \to \mathfrak{A}$ that is strictly continuous on bounded sets. In particular, the sets of normal, self-adjoint, and positive elements of norm at most 1 are strictly closed in $\text{Ball}(\mathfrak{A})$. If $f : [-1,1]^n \to \text{nBall}(C)$ is a continuous function, then $f$ induces by continuous functional calculus and Lemma 2.11 a strictly continuous functions $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ from the strictly closed set of $n$-tuples of pairwise commuting self-adjoint elements in $\text{Ball}(\mathfrak{A})$ to $\text{nBall}(\mathfrak{A})$. Similarly, if $f : \mathbb{B} \to \text{nBall}(C)$ is a continuous function, where

$$\mathbb{B} = \{ z \in C : |z| \leq 1 \},$$

then $f$ induces by continuous functional calculus and Lemma 2.11 a strictly continuous function $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ from the strictly closed set of $n$-tuples of pairwise commuting normal elements in $\text{Ball}(\mathfrak{A})$ to $\text{nBall}(\mathfrak{A})$.

Suppose that $\mathfrak{A}$ is a strict $C^*$-algebra. Let $\text{Normal}(\mathfrak{A})$ be the Borel set of normal elements of $\mathfrak{A}$. For $a \in \text{Normal}(\mathfrak{A})$, the spectrum $\sigma(a)$ is a closed subset of $C$. We consider the space $\text{Closed}(C)$ of closed subsets of $C$ as a standard Borel space endowed with the Effros Borel structure [Kec95, Section 12.C]. If $X$ is a standard Borel space and $B$ is a basis of open subsets of $C$, then a function $\Phi : X \to \text{Closed}(C)$ is Borel if and only if, for every $U \in B$, $\{ x \in X : \Phi(x) \cap U \neq \emptyset \}$ is Borel. The proof of the following lemma is standard; see [Sim95, Lemma 1.6].

Lemma 2.21. Suppose that $\mathfrak{A}$ is a strict $C^*$-algebra. The function $\text{Normal}(\mathfrak{A}) \to \text{Closed}(C)$, $a \mapsto \sigma(a)$ is Borel.
Proof. It suffices to show that the map Normal (𝒜) ∩ Ball (𝒜) → Closed (C), \( a \mapsto \sigma(a) \) is Borel. Observe that C has a basis of open sets of the form \( U_f := \{ x \in C : f(x) > 0 \} \) where \( f : C \to [0,1] \) is a continuous function. For such a continuous function \( f : C \to [0,1] \), we have that
\[
\{ a \in \text{Normal (𝒜)} \cap \text{Ball (𝒜)} : \sigma(a) \cap U_f \neq \emptyset \}
= \{ a \in \text{Normal (𝒜)} \cap \text{Ball (𝒜)} : f(a) \neq 0 \},
\]
which is closed in Normal (𝒜) ∩ Ball (𝒜). This concludes the proof. \( \square \)

Suppose that \( \mathfrak{A} \) is a strict C*-algebra. Fix \( r \in (0,1) \) and consider the set
\[
X = \{ x \in \mathfrak{A} : \| 1 - x \| \leq r \} \subseteq 2\text{Ball (𝒜)}.
\]
Then, for \( x \in X \) we have that \( x \) is invertible, \( \| x^{-1} \| \leq \frac{1}{1 - r} \), and
\[
x^{-1} = \sum_{n \in \omega} x^n.
\]
It follows from Lemma 2.11 that the function \( X \rightarrow \frac{1}{1 - r}\text{Ball (𝒜)}, x \mapsto x^{-1} \) is strictly continuous.

More generally, suppose that \( \Omega \) is an open subset of \( C \), and \( f : \Omega \rightarrow C \) is a holomorphic function. Suppose that \( 0 \in \Omega \) and \( r > 0 \) is such that \( \{ z \in C : |z| \leq r \} \subseteq \Omega \). Then \( f \) admits a Taylor expansion
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
that converges uniformly for \( |z| \leq r \) [Ahl78, Chapter 5, Theorem 3 and Chapter 2, Theorem 2]. Fix \( b_0 \in \mathfrak{A} \) and set
\[
X := \{ x \in \mathfrak{A} : \| x - b_0 \| \leq r \} \subseteq (1 + \| b_0 \| )\text{Ball (𝒜)}
\]
Then for \( x \in X \),
\[
f(x - b_0) := \sum_{n=0}^{\infty} a_n (x - b_0)^n \in \mathfrak{A};
\]
see [Ped89, Lemma 4.1.11]. Furthermore, the function \( X \rightarrow c\text{Ball (𝒜)}, x \mapsto f(x - b_0) \) is strictly continuous on \( X \) by Lemma 2.11, where \( c = \sup \{ |f(z)| : |z| \leq r \} \).

2.3. Multiplier algebras. Suppose that \( A \) is a separable C*-algebra. A double centralizer for \( A \) is a pair \( (L, R) \) of bounded linear maps \( L, R : A \rightarrow A \) such that \( \| L \| = \| R \| \) satisfying \( L(x)y = xR(y) \) for \( x, y \in A \). Let \( M(A) \) be the set of double centralizers for \( A \). Then \( M(A) \) is a C*-algebra with respect to the operations
\[
(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2)
\]
\[
(L_1R_1)(L_2, R_2) = (L_1L_2, R_2R_1)
\]
\[
\lambda (L, R) = (\lambda L, \lambda R)
\]
\[
(L, R)^* = (R^*, L^*)
\]
and the norm
\[
\|(L, R)\| = \| L \| = \| R \|
\]
for \( (L, R), (L_1, R_1), (L_2, R_2) \in M(A) \) and \( \lambda \in \mathbb{C} \). The strict topology on \( \text{Ball (M(A))} \) is the topology of pointwise convergence, namely the topology induced by the seminorms
\[
p_a : (L, R) \mapsto \max \{ \| L(a) \|, \| R(a) \| \}
\]
for \( a \in A \).

An element \( a \in A \) can be identified with the multiplier \( (L_a, R_a) \in M(A) \) defined by setting \( L_a(x) = ax \) and \( R_a(x) = xa \) for \( x \in X \). This allows one to regard \( A \) as an essential ideal of \( M(A) \). (An ideal \( J \) of a C*-algebra \( B \) is essential if \( J^\perp := \{ b \in B : bJ = 0 \} \) is zero or, equivalently, \( J \) has nonzero intersection with every nonzero ideal of \( B \).) If \( (v_n)_{n \in \omega} \) is an approximate unit for \( A \) [HR00, Definition 1.7.1] then, by definition, \( (v_n) \) strictly converges to \( 1 \) in \( \text{Ball (M(A))} \). In particular, \( \text{Ball (A)} \) is strictly dense in \( \text{Ball (M(A))} \).

If \( (x_i)_{i \in \omega} \) is a strictly Cauchy sequence in \( \text{Ball (M(A))} \), in the sense that \( (x_i)_{i \in \omega} \) is \( p_a \)-Cauchy for every \( a \in A \), then setting
\[
L(a) := \lim_{i \to \omega} x_i a
\]
\[ R(a) := \lim_{i \to \infty} ax_i \]

for \( a \in A \) defines a double centralizer \( (L, R) \in \text{Ball}(M(A)) \) that is the strict limit of \( (x_i)_{i \in \omega} \) in \( \text{Ball}(M(A)) \). For \( n \geq 1 \), one can identify \( M_n(M(A)) \) with \( M(M_n(A)) \) and consider the corresponding strict topology. From the above remarks and Lemma 2.14, it easily follows the following; see [Far19, Chapter 13] or [WO93, Chapter 2].

**Proposition 2.22.** Let \( A \) be a separable C*-algebra. Then \( M(A) \) is a strict unital C*-algebra containing \( A \) as a strictly dense essential strict ideal where, for every \( n \geq 1 \), the strict topology on \( \text{Ball}(M(A)) \) is as described above, and \( M_n(M(A)) \) is identified with \( M(M_n(A)) \).

**Example 2.23.** When \( A \) is the algebra \( K(H) \) of compact operators on a separable Hilbert space, then \( M(A) = B(H) \) and the strict topology on \( \text{Ball}(B(H)) \) is the strong-* topology [Bla06, Proposition I.8.6.3].

**Example 2.24.** One can also regard \( B(H) \) as the dual space of the Banach space \( L^1(H) \) of trace-class operators. This turns \( B(H) \) into a strict Banach space, where the strict topology on \( B(H) \) is the weak*-topology [Bla06, Definition I.8.6.2]. As the identity map \( \text{Ball}(B(H)) \to \text{Ball}(B(H)) \) is strong-*–weak continuous, the strong-* and weak topologies on \( \text{Ball}(B(H)) \) define the same standard Borel structure on \( B(H) \).

One can define as above the strict topology on the whole multiplier algebra \( M(A) \) to be the topology of pointwise convergence of double multipliers. However, this topology on \( M(A) \) is not first countable whenever \( A \) is not unital; see Remark 2.10.

Suppose that \( A \) is a separable C*-algebra, and \( X \) is a compact metrizable space. One can then consider the separable C*-algebra \( C(X, A) \) of continuous functions \( X \to A \). Let also \( C_\beta(X, M(A)) \) be the C*-algebra of strictly continuous bounded functions \( X \to M(A) \). There is an obvious unital *-homomorphism \( C_\beta(X, M(A)) \to M(C(X, A)) \), where \( C_\beta(X, M(A)) \) acts on \( C(X, A) \) by pointwise multiplication. The unital *-homomorphism \( C_\beta(X, M(A)) \to M(C(X, A)) \) is in fact an *-isomorphism [APT73, Corollary 3.4]. We can thus identify \( C_\beta(X, M(A)) \) with \( M(C(X, A)) \) and regard it as a strict C*-algebra. Observe that, for \( t \in X \), the function \( \text{Ball}(C_\beta(X, M(A))) \to \text{Ball}(M(A)), f \mapsto f(t) \) is strictly continuous. We let \( C_t(X, M(A)) \) be the C*-algebra of norm-continuous functions \( X \to M(A) \), which is a C*-subalgebra of \( C_\beta(X, M(A)) \).

**Lemma 2.25.** Suppose that \( A \) is a separable C*-algebra, and \( X \) is a compact metrizable space. Then \( C(X, M(A)) \) is a Borel subset of \( C_\beta(X, M(A)) \).

**Proof.** Fix a compatible metric \( d \) on \( X \), and a countable dense subset \( X_0 \) of \( X \). Clearly, it suffices to show that \( \text{Ball}(C(X, M(A))) \) is a Borel subset of \( \text{Ball}(C_\beta(X, M(A))) \). Fix, for every \( k \in \omega \), a finite cover \( \{A_0^k, \ldots, A_{k-1}^k\} \) of \( X \) consisting of open sets of diameter less than \( 2^{-k} \), and fix elements \( t_i^k \in A_i^k \) for \( i < k \). We have that a strictly continuous function \( f : X \to \text{Ball}(M(A)) \) is norm-continuous if and only if, for every \( n \in \omega \) there exists \( k \in \omega \) such that, for every \( i < k \) and \( s \in A_i^k \), \( \| f(s) - f(t_i^k) \| \leq 2^{-k} \). Since \( 2^{-k} \text{Ball}(M(A)) \) is strictly closed and \( f \) is strictly continuous, we have that \( f \) is norm-continuous if and only if for every \( n \in \omega \) there exists \( k \in \omega \) such that, for every \( i < k \) and \( s \in A_i^k \cap X_0 \), \( \| f(s) - f(t_i^k) \| \leq 2^{-k} \). This shows that the set of norm-continuous functions is Borel.

**Corollary 2.26.** Suppose that \( A \) is a separable C*-algebra. Then the set \( C([0, 1], M(A)) \) of norm-continuous paths \( [0, 1] \to M(A) \) is a Borel subset of \( C_\beta([0, 1], M(A)) \).

Suppose that \( A \) and \( B \) are separable C*-algebras. A morphism from \( A \) to \( B \) in the sense of [Wor80, Wor91, WN92, Wor95] is a *-homomorphism \( \varphi : A \to M(B) \) such that \( \varphi(A)B \) is norm-dense in \( B \). This is called \( S \)-morphism in [Val85, Definition 0.2.7] and a nondegenerate *-homomorphism in [Lan95]. We recall the well-known fact that there is a correspondence between morphisms from \( A \) to \( B \) and strict unital *-homomorphisms \( M(A) \to M(B) \); see [Lan95, Proposition 2.1].

**Lemma 2.27.** Let \( A \) and \( B \) be separable C*-algebras.

- **Suppose that \( \psi : M(A) \to M(B) \) is a strict unital *-homomorphism. Then \( \psi|_A \) is a morphism from \( A \) to \( B \).**
- **Conversely, if \( \varphi \) is a morphism from \( A \) to \( B \), then \( \varphi \) extends to a unique strict unital *-homomorphism \( \bar{\varphi} : M(A) \to M(B) \). If \( \varphi \) is injective, then \( \bar{\varphi} \) is injective.**
If \((e_n)\) is an approximate unit for \(A\), then a *-homomorphism \(\varphi : A \to M(B)\) is a morphism from \(A\) to \(B\) if and only if \((\varphi(e_n))\) strictly converges to 1.

A further characterization of morphisms is provided in [Val85, Lemme 0.2.6] and [Iš0, Proposition 1.1]. It follows from Lemma 2.27 that the composition of morphisms \(A \to B\) and \(B \to C\) is meaningful, and it gives a morphism \(A \to C\).

Suppose that \(A, B\) are separable C*-algebras. A *-homomorphism \(\varphi : A \to M(B)\) is strict [Lan95, page 49] (also called quasi-unital [JT91, Definition 1.3.13]) if there exists a projection \(p_\varphi \in M(B)\), called the relative unit of \(\varphi\), such that \(\varphi(A)B = p_\varphi B\). One has the following generalization of Lemma 2.27; see [Lan95, Corollary 5.7].

Lemma 2.28. Let \(A\) and \(B\) be separable C*-algebra.

- Suppose that \(\psi : M(A) \to M(B)\) is a strict *-homomorphism. Then \(\psi|_A\) is a strict *-homomorphism from \(A\) to \(M(B)\) with relative unit \(\psi(1)\).
- Conversely, if \(\varphi\) is a strict *-homomorphism from \(A\) to \(M(B)\) with relative unit \(p_\varphi\), then \(\varphi\) extends to a unique strict *-homomorphism \(\bar{\varphi} : M(A) \to M(B)\) with \(\bar{\varphi} = p_\varphi\). If \(\varphi\) is injective, then \(\bar{\varphi}\) is injective.
- If \((e_n)\) is an approximate unit for \(A\), then a *-homomorphism \(\varphi : A \to M(B)\) is strict if and only if \((\varphi(e_n))\) is strictly Cauchy.

We now observe that the category of multiplier algebras of separable C*-algebras, regarded as a full subcategory of the category of strict unital C*-algebras, can be regarded as a Polish category; see Section 1.2. This means that, for every separable C*-algebras \(A\) and \(B\), the set Mor\((M(A), M(B))\) of strict unital *-homomorphisms \(M(A) \to M(B)\) is a Polish space, and composition of morphisms is a continuous function.

Following [Wor95] we consider Mor\((M(A), M(B))\) as endowed with the topology of pointwise strict convergence. This is the subspace topology induced by regarding, as in Lemma 2.27, Mor\((M(A), M(B))\) as a subspace of Ball\((L(A, M(B)))\), where \(L(A, M(B))\) is the space of bounded linear maps from \(A\) to \(M(B)\). (Recall that, if \(X\) is a Banach space and \(\mathfrak{A}\) is a strict Banach space, then the space \(L(X, \mathfrak{A})\) is a strict Banach space when Ball\((L(X, \mathfrak{A}))\) is endowed with the topology of pointwise strict convergence; see Proposition 2.15.) As Mor\((M(A), M(B))\) is a \(G_δ\) subset of Ball\((L(A, M(B)))\), it is a Polish space with the induced topology. It is easy to see that this turns the category of multiplier algebras of separable C*-algebras into a Polish category.

If \(A, B\) are separable C*-algebras, then the space Iso\((M(A), M(B))\) of isomorphisms \(M(A) \to M(B)\) in the category of strict unital C*-algebras endowed with the Polish topology as in Lemma 1.2 can be identified, via the correspondence given by Lemma 2.27, with the space Iso\((A, B)\) of *-isomorphisms \(A \to B\) endowed with the topology of pointwise norm-convergence.

Consider now the category of locally compact second countable Hausdorff spaces, where a morphism is simply a continuous map. Given locally compact second countable Hausdorff spaces \(X, Y\), let Mor\((X, Y)\) be the set of all continuous maps \(X \to Y\). This is endowed with a Polish topology called the compact-open topology, that has as subsbasis of open sets the sets of the form

\[
(K, U) := \{ f \in \text{Mor}(X, Y) : f(K) \subseteq U \}
\]

for a compact subset \(K\) of \(X\) and an open subset \(U\) of \(Y\). This turns the category of locally compact second countable Hausdorff spaces and continuous maps into a Polish category. We let Iso\((X, Y) \subseteq \text{Mor}(X, Y)\) be the set of homeomorphisms \(X \to Y\). The Polish topology induced on Iso\((X, Y)\) as in Lemma 1.2 was shown in [Are46, Theorem 5], where it is called the \(g\)-topology, to have as subsbasis of open sets the sets of the form \((K, Y \setminus L)\) where \(K, L\) are closed sets and at least one between \(K\) and \(L\) is compact. For a locally compact second countable Hausdorff space \(X\), let \(X^+\) be its one-point compactification, obtained by adjoining a point at infinity \(\infty_X\). Each \(f \in \text{Iso}(X, Y)\) admits a unique extension to \(f^+ \in \text{Iso}(X^+, Y^+)\) that fixes the point at infinity, in the sense that \(f^+(\infty_X) = \infty_Y\).

By [Are46, Theorem 5], the assignment \(f \mapsto f^+\) defines a homeomorphism from Iso\((X, Y)\) onto the closed subset of Iso\((X^+, Y^+)\) consisting of the homeomorphisms that fix the point at infinity.

Given a locally compact second countable Hausdorff space \(X\), we let \(C_0(X)\) be the separable C*-algebra of continuous complex-valued functions on \(X\) vanishing at infinity. Its multiplier algebra M\((C_0(X))\) is the algebra \(C_0(X)\) of bounded continuous complex-valued functions on \(X\). The unit ball Ball\((C_0(X))\) of \(C_0(X) = M(C_0(X))\) endowed with the strict topology can be identified with the space Mor\((X, \mathbb{D})\) of continuous functions \(X \to \mathbb{D}\) endowed with the compact-open topology, where \(\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}\) is the closed unit disk. Every separable
commutative C*-algebra $A$ is isomorphic to $C_0(\hat{A})$, where $\hat{A}$ is the locally compact second countable Hausdorff space of nonzero homomorphisms $A \to \mathbb{C}$ (the spectrum of $A$).

A continuous map $f : X \to Y$ induces a strict unital *-homomorphism $C_0(Y) \to C_0(X)$ given by $\varphi_f : C_0(Y) \to C_0(X)$, $a \mapsto a \circ f$. This defines a fully faithful contravariant functor from the category of locally compact second countable Hausdorff spaces to the category of strict unital C*-algebras. In fact, the assignment $\text{Mor}(X, Y) \to \text{Mor}(C_0(Y), C_0(X))$, $f \mapsto \varphi_f$ is a homeomorphism, where $\text{Mor}(X, Y)$ is endowed as above with the compact-open topology and $\text{Mor}(C_0(X), C_0(Y))$ is endowed with the topology of pointwise strict convergence. Thus, by Lemma 1.4, the assignment $X \to C_0(X)$ is a contravariant topological equivalence of categories from the Polish category of locally compact second countable Hausdorff spaces to the Polish category of multiplier algebras of commutative separable C*-algebras; see Definition 1.3.

2.4. Essential commutants and Paschke dual algebras. Suppose that $B$ is a separable C*-algebra, and $C \subseteq M(B)$ is a separable C*-subalgebra. Define then the essential commutant $\mathfrak{D}(C)$ of $C$ in $M(B)$ to be the C*-algebra

$$\{ x \in M(B) : \forall c \in B, [x, c] \in B \},$$

where $[x, c]$ is the commutator $xc - cx$. Define the strict topology on $\text{Ball}(\mathfrak{D}(C))$ to be the topology generated by the seminorms

$$x \mapsto \max \{\|xb\|, \|bx\|\}$$

and

$$x \mapsto \|[x, c]\|$$

for $c \in C$ and $b \in B$. If $(v_n)_{n \in \omega}$ is a approximate unit for $B$ that is approximately central for $C$ [HR00, Definition 3.2.4], then $(v_n)_{n \in \omega}$ converges strictly to 1 in $\text{Ball}(\mathfrak{D}(B))$.

We have that $\mathfrak{D}(C)$ is strictly complete. Indeed, consider a strictly Cauchy sequence $(x_i)_{i \in \omega}$ in $\text{Ball}(\mathfrak{D}(C))$. Then we have that $(x_i)_{i \in \omega}$ converges to some $x \in \text{Ball}(M(B))$ in the strict topology of $M(B)$. For every $c \in C$, the sequence $([x_i, c])_{i \in \omega}$ is norm-Cauchy in $B$, whence it norm-converges to some element of $B$, which must be equal to $[x, c]$. This shows that $x \in \text{Ball}(\mathfrak{D}(C))$ is the strict limit of $(x_i)_{i \in \omega}$ in $\text{Ball}(\mathfrak{D}(C))$. For $n \geq 1$, we can identify $M_n(\mathfrak{D}(C))$ with $\mathfrak{D}(\Delta_n(C)) \subseteq M(M_n(B))$, where $\Delta_n(C) \subseteq M_n(B)$ is the image of $C$ under the diagonal embedding $\Delta_n : B \to M_n(B)$. From the above remarks and Lemma 2.14 we thus obtain the following.

**Proposition 2.29.** Let $B$ be a separable C*-algebra, and let $C \subseteq M(B)$ be a separable C*-subalgebra. Let $\mathfrak{D}(C)$ be the corresponding essential commutant. Then $\mathfrak{D}(C)$ is a strict C*-algebra containing $B$ as a strictly dense essential strict ideal where, for every $n \geq 1$, $M_n(\mathfrak{D}(C))$ is identified with $\mathfrak{D}(\Delta_n(C))$, and Ball $\mathfrak{D}(\Delta_n(C))$ is endowed with the strict topology described above.

Suppose now as above that $B$ is a separable C*-algebra, and $C \subseteq M(B)$ is a separable C*-subalgebra. Let also $I \subseteq C$ be a closed two-sided ideal. Define the essential annihilator

$$\mathfrak{D}(C//I) = \{ x \in \mathfrak{D}(C) : \forall a \in I, xa \in B \},$$

which is a closed two-sided ideal of $\mathfrak{D}(C)$. The strict topology on Ball $\mathfrak{D}(C//I)$ is the topology generated by the seminorms

$$x \mapsto \max \{\|xb\|, \|bx\|\}$$

and

$$x \mapsto \|[x, c]\|$$

for $b \in B$, $c \in C$, and $a \in I$. A straightforward argument gives the following.

**Proposition 2.30.** Let $B$ be a separable C*-algebra, let $C \subseteq M(B)$ be a separable C*-subalgebra, and $I \subseteq C$ be a closed two-sided ideal. Let $\mathfrak{D}(C)$ be the corresponding essential commutant, and $\mathfrak{D}(C//I)$ be the essential annihilator. Then $\mathfrak{D}(C//I)$ is a strict ideal of $\mathfrak{D}(C)$, where for every $n \geq 1$, $M_n(\mathfrak{D}(C//I))$ is identified with $\mathfrak{D}(\Delta_n(C//\Delta_n(I)))$ and Ball $\mathfrak{D}(\Delta_n(C//\Delta_n(I)))$ is endowed with the strict topology described above.
Example 2.31. Suppose that \( A \) is a separable unital C*-algebra, \( J \) is a closed two-sided ideal of \( A \), and \( p : A \to B(H) \) is a nondegenerate representation of \( A \) that is \emph{ample}, in the sense that \( p(A) \cap K(H) = \{0\} \). We regard \( B(H) \) as the multiplier algebra of \( K(H) \). The Paschke dual \( \mathcal{D}_p(A) \) as defined in [HR00, Definition 5.1.1] is the essential commutant \( \mathcal{D}(\rho(A)) \) of \( \rho(A) \) inside \( B(H) \); see also [Pas81]. The relative dual algebra \( \mathcal{D}_p(A) / J \) as defined in [HR00, Definition 5.3.2] is the strict ideal \( \mathcal{D}(\rho(A) / \rho(J)) \) of \( \mathcal{D}_p(A) = \mathcal{D}(\rho(A)) \).

2.5. Homotopy of projections. Suppose that \( \mathfrak{A} \) is a strict unital C*-algebra. Recall that a strict ideal of \( \mathfrak{A} \) is a proper norm-closed Borel two-sided ideal \( \mathfrak{J} \) of \( \mathfrak{A} \) that is also a strict C*-algebra and such that the inclusion map \( \mathfrak{J} \to \mathfrak{A} \) is a strict *-homomorphism.

Definition 2.32. A strict (unital) C*-pair is a pair \((\mathfrak{A}, \mathfrak{J})\) where \( \mathfrak{A} \) is a strict (unital) C*-algebra and \( \mathfrak{J} \) is a strict ideal of \( \mathfrak{A} \).

We regard strict unital C*-pairs as objects of a category, where a morphism from \((\mathfrak{A}, \mathfrak{J})\) to \((\mathfrak{B}, \mathfrak{J})\) is a strict unital *-homomorphism \( \varphi : \mathfrak{A} \to \mathfrak{B} \) that maps \( \mathfrak{J} \) to \( \mathfrak{J} \).

Every strict unital C*-pair \((\mathfrak{A}, \mathfrak{J})\) determines a quotient unital C*-algebra \( \mathfrak{A} / \mathfrak{J} \). If \( \mathfrak{A} / \mathfrak{J} \) and \( \mathfrak{B} / \mathfrak{J} \) are two unital C*-algebras obtained in this way, then we say that a unital *-homomorphism \( \varphi : \mathfrak{A} / \mathfrak{J} \to \mathfrak{B} / \mathfrak{J} \) is \emph{definable} if it has a Borel lift (or a Borel representation in the terminology of [Far11, Gha15]). This is a Borel function \( f : \mathfrak{A} \to \mathfrak{B} \) (which is not necessarily a *-homomorphism) such that \( \varphi(a + \mathfrak{J}) = f(a) + \mathfrak{J} \) for every \( a \in \mathfrak{A} \). The notion of definable unital *-homomorphisms determines a category, whose objects are strict unital C*-pairs and whose morphisms are the definable unital *-homomorphisms. When the strict unital C*-pair \((\mathfrak{A}, \mathfrak{J})\) is considered as the object of this category, we call it a unital C*-algebra with a strict cover, and denote it by \( \mathfrak{A} / \mathfrak{J} \), as we think of it as a unital C*-algebra explicitly presented as the quotient of a strict unital C*-algebra by a strict ideal. The category of unital C*-algebras with a strict cover thus has unital C*-algebras with strict cover as objects and definable unital *-homomorphisms as morphisms. The notion of a unital C*-algebra with a strict cover is the analogue in the context of C*-algebras to the notion of group with a Polish cover considered in [BLP20]; see Remark 1.17.

Notice that every strict unital *-homomorphism \( (\mathfrak{A}, \mathfrak{J}) \to (\mathfrak{B}, \mathfrak{J}) \) between strict unital C*-pairs induces a definable unital *-homomorphism \( \mathfrak{A} / \mathfrak{J} \to \mathfrak{B} / \mathfrak{J} \) between the corresponding unital C*-algebras with a strict cover. This allows one to regard the category of strict unital C*-pairs as a subcategory of the category of unital C*-algebras with a strict cover. These categories have the same objects, but different morphisms.

If \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair and \( a, b \in \mathfrak{A} \), we write \( a \equiv b \mod \mathfrak{J} \) if \( a - b \in \mathfrak{J} \). If \( a \in M_n(\mathfrak{A}) \) and \( b \in M_k(\mathfrak{A}) \), then we set

\[
a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{n+k}(\mathfrak{A}).
\]

We let \( 1_n \) be the identity element of \( M_n(\mathfrak{A}) \) and \( 0_n \) be the zero element of \( M_n(\mathfrak{A}) \).

Suppose that \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair. A positive element of \( \text{Ball}(\mathfrak{A}) \) is a projection mod \( \mathfrak{J} \) if \( p^2 \equiv p \mod \mathfrak{J} \) or, equivalently, \( p + \mathfrak{J} \) is a projection in \( \mathfrak{A} / \mathfrak{J} \). Define the set \( \text{Proj}(\mathfrak{A}/\mathfrak{J}) \subseteq \text{Ball}(\mathfrak{A}) \) to be the Borel set of projections mod \( \mathfrak{J} \) in \( \mathfrak{A} \).

The Borel structure on \( \text{Proj}(\mathfrak{A}/\mathfrak{J}) \) is induced by the Polish topology defined by declaring a net \((p_i)_{i \in I}\) to converge to \( p \) if and only if \( p_i \to p \) strictly in \( \text{Ball}(\mathfrak{A}) \). (Recall that the strict topology on \( \text{Ball}(\mathfrak{A}) \) is the uniform topology induced by the strict topology on \( \text{Ball}(\mathfrak{A}) \).)

We also say that an element \( u \) of \( \text{Ball}(\mathfrak{A}) \) is a unitary mod \( \mathfrak{J} \) if \( uu^* \equiv 1 \mod \mathfrak{J} \) and \( u^*u \equiv 1 \mod \mathfrak{J} \) or, equivalently, \( u + \mathfrak{J} \) is a unitary in \( \mathfrak{A} / \mathfrak{J} \). We let \( U(\mathfrak{A}/\mathfrak{J}) \) be the Borel set of unitaries mod \( \mathfrak{J} \) in \( \mathfrak{A} \). The Borel structure on \( U(\mathfrak{A}/\mathfrak{J}) \) is induced by the Polish topology defined by declaring a net \((u_i)_{i \in I}\) to converge to \( u \) if and only if \( u_i \to u \) strictly in \( \text{Ball}(\mathfrak{A}) \), \( uu_i - 1 \to uu^* - 1 \) in \( \text{Ball}(\mathfrak{A}) \), and \( u_i^*u_i - 1 \to u^*u - 1 \) in \( \text{Ball}(\mathfrak{A}) \). Notice that, if \( u, w \in U(\mathfrak{A}/\mathfrak{J}) \), then \( uu_w \in U(\mathfrak{A}/\mathfrak{J}) \) and \( uu^*_w \in U(\mathfrak{A}/\mathfrak{J}) \). This turns \( U(\mathfrak{A}/\mathfrak{J}) \) into a *-monoid, where the *-monoid operations are continuous. Similarly, the function \( U(\mathfrak{A}/\mathfrak{J}) \times \text{Proj}(\mathfrak{A}/\mathfrak{J}) \to \text{Proj}(\mathfrak{A}/\mathfrak{J}) \), \((u, p) \mapsto u^*pu\) is continuous.

More generally, an element \( v \) of \( \text{Ball}(\mathfrak{A}) \) is called a \emph{partial unitary} mod \( \mathfrak{J} \) if \( uu^* \equiv uu^* \) and \( uu^* \) is a mod \( \mathfrak{J} \) projection or, equivalently, if \( v + \mathfrak{J} \) is a partial unitary in \( \mathfrak{A} / \mathfrak{J} \). As in [RLL00, 8.2.12], we let \( \text{PU}(\mathfrak{A}/\mathfrak{J}) \) be the Borel set of mod \( \mathfrak{J} \) partial unitaries in \( \mathfrak{A} \). In a similar fashion one can define the Borel set \( \text{PI}(\mathfrak{A}/\mathfrak{J}) \) of mod \( \mathfrak{J} \) partial isometries in \( \mathfrak{A} \), consisting of those \( v \in \text{Ball}(\mathfrak{A}) \) such that \( vv^* + v^*v \) are mod \( \mathfrak{J} \) projections.

In the rest of this section we record some lemmas about unitaries and projections modulo a strict ideal in a strict unital C*-algebra. The content of these lemmas can be summarized as the assertion that a homotopy between projections and unitaries in a unital C*-algebra with a strict cover is witnessed by certain unitary elements in the
path-component of the identity of the unitary group that can be chosen in a Borel fashion. The proofs follow standard arguments from the literature on K-theory for C*-algebras; see [RLL00, HR00, Bla98, WO93].

Given elements $y_1, \ldots, y_n$ of Ball ($\mathfrak{A}$), subject to a certain relation $P(y_1, \ldots, y_n)$, we say that an element $z \in \text{Ball} (\mathfrak{A})$ satisfying a relation $R(y_1, \ldots, y_n, z)$ can be chosen in a Borel fashion (from $y_1, \ldots, y_n$) if there is a Borel function $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n)$ that assigns to the $n$-tuple $(y_1, \ldots, y_n)$ in Ball ($\mathfrak{A}$) satisfying $P$ an element $z(y_1, \ldots, y_n)$ in Ball ($\mathfrak{A}$) such that $(y_1, \ldots, y_n, z(y_1, \ldots, y_n))$ satisfies $R$. In other words, the set of tuples $(y_1, \ldots, y_n, z) \in \text{Ball} (\mathfrak{A})^n \times \text{Ball} (\mathfrak{A})$ such that $(y_1, \ldots, y_n)$ satisfies $P$ and $(y_1, \ldots, y_n, z)$ satisfies $R$ has a Borel uniformization [Kec95, Section 18.A].

Suppose that $A$ is a unital C*-algebra. Let $\text{Proj}(A)$ be the set of projections in $A$. Two projections $p, q$ in $A$ are:

- Murray–von Neumann equivalent if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$, in which case we write $p \sim \text{MvN} q$;
- unitary equivalent if there exists $u \in U(A)$ such that $u^*qu = p$;
- homotopic if there is a norm-continuous path $(p_t)_{t \in [0,1]}$ in $\text{Proj}(M_n(A))$ with $p_0 = p$ and $p_1 = q$.

**Lemma 2.33.** Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital C*-pair, $d \in \mathbb{N}$, $u \in \text{dBall} (\mathfrak{A})$ satisfies $\|u - 1 + \mathfrak{J}\| \leq 1/2$. Then one can choose in a Borel way a selfadjoint element $y \in \text{Ball} (\mathfrak{A})$ such that $e^{iy} \equiv u \mod \mathfrak{J}$.

**Proof.** Consider $u - 1 + \mathfrak{J} \in \mathfrak{A}/\mathfrak{J}$, and observe that there exists $a \in \mathfrak{A}$ such that $\|a\| \leq 1/2$ and $a + \mathfrak{J} = u - 1 + \mathfrak{J}$, which can be chosen in a Borel way by strict continuity of the continuous functional calculus. Hence, setting $\tilde{u} := a + 1 \in (d+1) \text{Ball} (\mathfrak{A})$, we have that $\tilde{u} \equiv u \mod \mathfrak{J}$ and $\|\tilde{u} - 1\| \leq 1/2$. Thus, after replacing $d$ with $d + 1$ and $u$ with $\tilde{u}$, we can assume that $\|u - 1\| \leq 1/2$.

Let $\log : D \to \mathbb{C}$ be an holomorphic branch of the logarithm defined on $\{z \in \mathbb{C} : |z - 1| < 1\}$. Considering the holomorphic functional calculus, one can define the element $\log (u) \in \mathfrak{A}$. If

$$
\log (z) = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n}
$$

is the uniformly convergent power series expansion in $\{z \in \mathbb{C} : |z| \leq 1/2\}$, then we have that

$$
\log (u) = \sum_{n=0}^{\infty} \frac{(1-u)^n}{n}.
$$

In particular, we have that $\|\log (u)\| \leq 1$. Define

$$
y := \frac{\log (u) + \log (u)^*}{2} \in \text{Ball} (\mathfrak{A}_{sa}).
$$

Then we have that $y \equiv \log (u) \mod \mathfrak{J}$ satisfies $\exp (iy) \equiv u \mod \mathfrak{J}$. \hfill \Box

**Corollary 2.34.** Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital C*-pair, and $u, w \in \mathfrak{A}$ are mod $\mathfrak{J}$ unitaries. Then there following assertions are equivalent:

1. there is a norm-continuous path from $u + \mathfrak{J}$ to $w + \mathfrak{J}$ in $\mathfrak{A}/\mathfrak{J}$;
2. there exists $\ell \geq 1$ and $y_1, \ldots, y_\ell \in \text{Ball} (\mathfrak{A}_{sa})$ such that $e^{iy_1} \cdots e^{iy_\ell} u \equiv w \mod \mathfrak{J}$.

**Lemma 2.35.** Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital C*-pair, and $p, q, x \in \text{Ball} (\mathfrak{A})$ are such that $p, q$ are mod $\mathfrak{J}$ projections, $x^*x \equiv p \mod \mathfrak{J}$, and $xx^* \equiv q \mod \mathfrak{J}$. Then one can choose $Y_1, \ldots, Y_\ell \in \text{Ball} (M_2 (\mathfrak{A}_{sa}))$ in a Borel fashion from $p, q, x$ such that, setting $U := e^{iY_1} \cdots e^{iY_\ell}$, one has that

$$
U^* (q \oplus 0) U \equiv (p \oplus 0) \mod M_2 (\mathfrak{J})
$$

and

$$
(q \oplus 0) U (p \oplus 0) \equiv x \oplus 0 \mod M_2 (\mathfrak{J}),
$$

where $\ell \geq 1$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q, x$.

**Proof.** Consider the mod $M_2 (\mathfrak{J})$ unitary

$$
X := \begin{bmatrix}
    x & 1 - q \\
    1 - p & x^*
\end{bmatrix} \in M_2 (\mathfrak{A}).
$$
Notice that $X$ satisfies
\[ X^* (q \oplus 0_d) X \equiv (p \oplus 0_d) \text{ mod } M_2 (\mathfrak{J}) \]
and
\[ (q \oplus 0_d) X (p \oplus 0_d) \equiv x \oplus 0_d \text{ mod } M_2 (\mathfrak{J}). \]
Consider the norm-continuous path of mod $M_2 (\mathfrak{J})$ unitaries
\[ X_t := \begin{bmatrix} \cos \left( \frac{\pi t}{2} \right) x & 1 - \left( 1 - \sin \left( \frac{\pi t}{2} \right) \right) q \\ \left( 1 - \sin \left( \frac{\pi t}{2} \right) \right) p - 1 & \cos \left( \frac{\pi t}{2} \right) x^* \end{bmatrix} \]
for $t \in [0,1]$. Notice that the modulus of continuity of $(X_t)_{t \in [0,1]}$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $(p,q,x)$. Fix $\ell \geq 1$ such that if $t, s \in [0,1]$ satisfy $|s - t| \leq 1/\ell$, then
\[ \|X_t - X_s\| \leq 1/2. \]
Thus, for $i \in \{1, 2, \ldots, \ell\}$ we have that
\[ \|X_{i/\ell} - X_{(i+1)/\ell}\| \leq 1/2. \]
By Lemma 2.33 we can choose in a Borel fashion $Y_1 \in \text{Ball} (\mathfrak{A}_{sa})$ such that $e^{iY_1} \equiv X_{1/\ell} \text{ mod } M_2 (\mathfrak{J})$. Thus
\[ R_1 := \exp (iY_1) - X_{1/\ell} \in M_2 (\mathfrak{J}). \]
Consider now $X_{2/\ell}$ and the fact that
\[ \|X_{2/\ell} - X_{1/\ell}\| \leq 1/2. \]
Thus
\[ \|\exp (iY_1) - (X_{2/\ell} + R_1)\| \leq 1/2. \]
and
\[ \|1 - \exp (-iY_1) (X_{2/\ell} + R_1)\| \leq 1/2. \]
Thus by Lemma 2.33 one can choose in a Borel fashion $Y_2 \in \text{Ball} (\mathfrak{A}_{sa})$ such that
\[ \exp (iY_2) \equiv \exp (-iY_1) (X_{2/\ell} + R_1) \equiv \exp (-iY_1) X_{2/\ell} \text{ mod } M_2 (\mathfrak{J}) \]
and hence
\[ \exp (iY_1) \exp (iY_2) \equiv X_{2/\ell} \text{ mod } M_2 (\mathfrak{J}). \]
Proceeding recursively in this way, one can choose $Y_1, \ldots, Y_\ell \in \text{Ball} (\mathfrak{A}_{sa})$ in a Borel fashion such that
\[ \exp (iY_1) \cdots \exp (iY_\ell) \equiv X \text{ mod } M_2 (\mathfrak{J}). \]
Then we have that, setting $U := \exp (iY_1) \cdots \exp (iY_\ell)$,
\[ U^* (q \oplus 0_d) U \equiv X^* (q \oplus 0_d) X \equiv p \oplus 0_d \text{ mod } M_2 (\mathfrak{J}) \]
and
\[ (q \oplus 0_d) U (p \oplus 0_d) \equiv (q \oplus 0_d) X (p \oplus 0_d) \equiv x \oplus 0_d \text{ mod } M_2 (\mathfrak{J}). \]
This concludes the proof.

**Lemma 2.36.** Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital $C^*$-pair, and $p,q \in \mathfrak{A}_{sa}$ are mod $\mathfrak{J}$ projections such that $\|p - q\| \leq 1/2$. Then one can choose $y_1, \ldots, y_\ell \in \text{Ball} (\mathfrak{A}_{sa})$ in a Borel fashion from $p, q$ such that, setting $u := e^{iy_1} \cdots e^{iy_\ell}$, one has that $u^*qu \equiv p \text{ mod } \mathfrak{J}$, where $\ell \geq 1$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$.

**Proof.** As in the proof of [RLL00, Proposition 2.2.4], consider the norm-continuous path of mod $\mathfrak{J}$ projections $a_t := (1-t)p + tq$ for $t \in [0,1]$. Let also $K = [-1/4, 1/4] \cup [3/4, 5/4] \subseteq \mathbb{R}$, and $f : K \to \mathbb{C}$ be the continuous function that is 0 on $[-1/4, 1/4]$ and 1 on $[3/4, 5/4]$. Then $p_t := f (a_t)$ for $t \in [0,1]$ is a norm-continuous path of mod $\mathfrak{J}$ projections from $p$ to $q$. Notice that the uniform continuity modulus of $t \mapsto a_t$ and $t \mapsto p_t$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$.

Thus, there exists $k \in \omega$ (that does depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$) such that, for every $t, s \in [0,1]$ such that $|t - s| \leq 1/k$, one has that $\|p_t - p_s\| \leq 1/6$. Thus, $p_0 = p, p_1/k, p_2/k, \ldots, p_1 = q$ are mod $\mathfrak{J}$ projections (that depend in a Borel way from $p, q$ by strict continuity of the continuous functional calculus) such that $\|p_{(i+1)/k} - p_i/k\| \leq 1/6$.
for $i \in \{0, 1, \ldots, k - 1\}$ and $(p_{i(i+1)+1}/(i+1))_{s \in [0,1]}$ is a norm-continuous path from $p_{i/k}$ to $p_{i+1/k}$ (whose modulus of continuity does not depend on $\mathfrak{A}$ and $p, q \in \mathfrak{A}$) satisfying

$$\|p_{i(i+1)+1}/(i+1) - p_{i/k}\| \leq 1/6$$

for $s \in [0, 1]$.

Thus, we can assume without loss of generality that $\|p_t - p\| \leq 1/6$ for every $t \in [0, 1]$. We now proceed as in the proof of [NdK17, Proposition 2.17]. Define

$$x_t := (2p - 1)(p_t - p) + 1$$

By definition, we have that $x_0 = 1$. Notice that

$$x_t \equiv pp_t + (p - 1)(p_t - 1) \mod \mathfrak{J}$$

and

$$p_{x_t} \equiv pp_t \equiv x_t p_t \mod \mathfrak{J}$$

This implies that

$$p|x_t^*| \equiv |x_t^*| \mod \mathfrak{J}$$

and

$$p_t |x_t| \equiv |x_t| p_t \mod \mathfrak{J}$$

for $t \in [0, 1]$. We have that

$$\|x_t - 1\| = \|(2p - 1)(p_t - p)\| \leq \|2p - 1\| \|p_t - p\| \leq 1/2.$$}

Thus, $x_t$ is invertible. Let $x_t := u_t |x_t|$ be its polar decomposition, where $u_t$ is a unitary. Then we have that $pu_t \equiv u_t p_t \mod \mathfrak{J}$. Indeed,

$$pu_t \equiv px_t |x_t|^{-1} \equiv x_t p_t |x_t|^{-1} \equiv x_t |x_t|^{-1} p_t \equiv u_t p_t \mod \mathfrak{J}.$$}

Thus

$$u_t^* pu_t \equiv p_t \mod \mathfrak{J}$$

for $t \in [0, 1]$ and in particular $u_t^* pu_t \equiv q \mod \mathfrak{J}$.

Notice that $(u_t)_{t \in [0,1]}$ is a norm-continuous path, whose modulus of continuity does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$. Therefore, there exists $k \geq 1$ (that does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$) such that, whenever $s, t \in [0, 1]$ satisfy $|s - t| \leq 1/k$, we have $\|u_t - u_s\| \leq 1/2$. By Lemma 2.33 one can then choose in a Borel way $y_1, \ldots, y_k \in \text{Ball}(\mathfrak{A}_{sa})$ such that, setting $u := \exp(iy_1) \cdots \exp(iy_k)$, then $u \equiv u_1 \mod \mathfrak{J}$ and hence $u^* pu \equiv q \mod \mathfrak{J}$. This concludes the proof. □

Lemma 2.37. Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital $C^*$-pair, and $p, q \in \mathfrak{A}$ are mod $\mathfrak{J}$ projections that satisfy $pq \equiv qp \equiv 0 \mod \mathfrak{J}$. Then one can choose $Y_1, \ldots, Y_k \in \text{Ball}(M_2(\mathfrak{A}_{sa}))$ in a Borel fashion from $p, q$ such that, setting $U := e^{iy_1} \cdots e^{iy_k}$, one has that $U^* (p + q) U \equiv (p + q) \odot 0 \mod M_2(\mathfrak{J})$, where $\ell \geq 1$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $p, q$.

Proof. Consider the path

$$r_t := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} p + \begin{bmatrix} \cos^2 \left( \frac{\pi}{\ell} \right) & \cos \left( \frac{\pi}{\ell} \right) \sin \left( \frac{\pi}{\ell} \right) \\ \cos \left( \frac{\pi}{\ell} \right) \sin \left( \frac{\pi}{\ell} \right) & \sin^2 \left( \frac{\pi}{\ell} \right) \end{bmatrix} q$$

for $t \in [0, 1]$. This is a norm-continuous path of mod $M_2(\mathfrak{J})$ projections in $M_2(\mathfrak{A})$ from $(p + q) \odot 0$ to $p + q$, whose modulus of continuity does not depend on $\mathfrak{A}$ and $p, q$. Therefore, the conclusion follows from Lemma 2.36. □

Lemma 2.38. Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital $C^*$-pair, and $u, v \in \mathfrak{A}$ are mod $\mathfrak{J}$ unitaries. Then one can choose $y_1, \ldots, y_k \in \text{Ball}(M_2(\mathfrak{A}_{sa}))$ in a Borel fashion from $u$ and $v$ such that, $(u \odot v) \equiv e^{iy_1} \cdots e^{iy_k} (uv \odot 1) \mod M_2(\mathfrak{J})$, where $\ell \geq 1$ does not depend on $(\mathfrak{A}, \mathfrak{J})$ and $u, v$. 

Proof. Consider the path...
Proof. Fix a unitary path \((W_t)_{t \in [0,1]}\) in \(U(M_2(\mathbb{C}))\) from 1 to \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
Fix \(\ell \geq 1\) such that, for \(s, t \in [0,1]\), \(\|W_s - W_t\| \leq 1/2\). Then
\[
u_t := (u \oplus 1) W_t^* (v \oplus 1) W_t (uv \oplus 1)
\]
is a path of mod \(M_2(\mathbb{J})\) unitaries from 1 to \((u \oplus v) (uv \oplus 1)^*\) to 1. Then, as in the proof of 2.35, using Lemma 2.33 one can recursively choose in a Borel fashion \(y_1, \ldots, y_\ell \in \text{Ball}(M_2(\mathbb{J}))\) such that \(e^{iy_k} \equiv u_{k/\ell}^{(u_{k+1}/\ell) \mod M_2(\mathbb{J})}\) for \(k \in \{0,1,\ldots,\ell-1\}\) and hence \((u \oplus v) (uv \oplus 1)^* \equiv e^{iy_1} \cdots e^{iy_\ell} \mod M_2(\mathbb{J})\). \(\square\)

2.6. The Definable Arveson Extension Theorem. In the rest of this section, we present definable versions of some fundamental results in operator algebras, to be used in the development of definable K-homology. Suppose that \(H\) is a separable Hilbert space. We regard \(B(H)\) as the multiplier algebra of the \(C^*\)-algebra \(K(H)\) of compact operators on \(H\). The corresponding strict topology on \(\text{Ball}(B(H))\) is the strong-* topology. Consistently, we consider \(B(H)\) as a standard Borel space with respect to the induced standard Borel structure. If \(Z\) is a separable Banach space, we consider \(L(Z,B(H))\) as a strict Banach space, where \(\text{Ball}(L(Z,B(H)))\) is endowed with the topology of pointwise strong-* convergence. We denote by \(U(H)\) the unitary group of \(B(H)\), which is a Polish group when endowed with the strong-* topology.

Suppose that \(A\) is a separable unital \(C^*\)-algebra, and \(X \subseteq A\) is an operator system. Let \(H\) be a separable Hilbert space. Arveson’s Extension Theorem asserts that every contractive completely positive (ccp) map \(\phi : X \to B(H)\) [BO08, Section 1.5] admits a contractive completely positive extension \(\hat{\phi} : A \to B(H)\) [Pau02, Theorem 7.5]. We observe now that \(\hat{\phi}\) can be chosen in a Borel way from \(\phi\). Notice that the space \(\text{CCP}(X,B(H))\) of contractive completely positive maps is closed (hence, compact) in \(\text{Ball}(L(X,B(H)))\) endowed with the topology of pointwise weak convergence.

Lemma 2.39. Suppose that \(A\) is a separable unital \(C^*\)-algebra, and \(X \subseteq A\) is an operator system. Let \(H\) be a separable Hilbert space. Then there exists a Borel function \(\text{CCP}(X,B(H)) \to \text{CCP}(A,B(H))\), \(\phi \mapsto \hat{\phi}\) such that \(\hat{\phi}\) is an extension of \(\phi\).

Towards the proof of Lemma 2.39, we recall the following particular case of the selection theorem for relations with compact sections [Kec95, Theorem 28.8].

Lemma 2.40. Suppose that \(X,Y\) are compact metrizable spaces, and \(A \subseteq X \times Y\) is a Borel subset such that, for every \(x \in X\), the vertical section
\[
A_x = \{ y \in Y : (x,y) \in A \}
\]
is a closed nonempty set. Then there exists a Borel function \(f : X \to Y\) such that \((x,f(x)) \in A\) for every \(x \in X\).

Using this selection theorem, Lemma 2.39 follows immediately the Arveson Extension Theorem.

Proof of Lemma 2.39. We consider \(\text{CCP}(X,B(H))\) as a compact metrizable space, endowed with the topology of pointwise weak convergence. Consider the Borel set \(A \subseteq \text{CCP}(X,B(H)) \times \text{CCP}(A,B(H))\) of pairs \((\phi,\psi)\) such that \(\psi|_X = \phi\). Then by the Arveson Extension Theorem, the vertical sections of \(A\) are nonempty, and clearly closed. Thus, by Lemma 2.40 there exists a Borel function \(f : \text{CCP}(X,B(H)) \to \text{CCP}(A,B(H))\) such that \(f(\phi)|_X = \phi\) for every \(\phi \in \text{CCP}(X,B(H))\). \(\square\)

2.7. The Definable Stinespring Dilation Theorem. Suppose that \(A\) is a separable unital \(C^*\)-algebra, and \(H\) is a separable Hilbert space. Stinespring’s Dilation Theorem asserts that, for every contractive completely positive map \(\phi : A \to B(H)\), there exists a linear map \(V_\phi : H \to H\) with \(\|V_\phi\|^2 = \|\phi\|^2\) and a nondegenerate representation \(\pi_\phi\) of \(A\) on \(H\) such that \(\phi(a) = V_\phi^* \pi_\phi(a) V_\phi\) for every \(a \in A\). Notice that the set \(\text{Rep}(A,H)\) of nondegenerate representations of \(A\) on \(H\) is a \(G_2\) subset of \(\text{Ball}(L(A,B(H)))\), whence Polish with the subspace topology, where \(\text{Ball}(B(H))\) is endowed with the strong-* topology. It follows from the proof of the Stinespring Dilation Theorem, where \(V\) and \(\pi\) are explicitly defined in terms of \(\phi\), that they can be chosen in a Borel way from \(\phi\); see [Bla06, Theorem II.6.9.7]
Lemma 2.41. Suppose that $A$ is a separable unital C*-algebra, and $H$ is a separable Hilbert space. Then there exists a Borel function $CCP (A, B(H)) \to \text{Ball} (B(H) \times \text{Rep} (A, B(H)), \phi \mapsto (\phi_{\varphi}, \pi_{\varphi})$ such that $(\phi, a) = V_{\varphi}^{*} \pi_{\varphi} (a) V_{\varphi}$ for $a \in A$ and $\| V_{\varphi} \|^{2} = \| \phi \|$ for every contractively complete positively definite map $\phi : A \to B(H)$.

2.8. The Definable Voiculescu Theorem. Suppose that $A$ is a separable unital C*-algebra, and $\rho, \rho' : A \to B(H)$ are two maps. If $U \subseteq U(H)$ is a unitary operator, write $\rho' \equiv_{U} \rho$ if $\rho' (a) \equiv U^{*} \rho (a) U \text{ mod } K(H)$ for every $a \in A$. If $V : H \to H$ is an isometry, write $\rho' \preceq_{V} \rho$ if $\rho' (a) \equiv V^{*} \rho (a) V \text{ mod } K(H)$ for every $a \in A$. A nondegenerate representation $\rho$ of $A$ on $B(H)$ is ample if, for every $a \in B(H)$, $\rho (a) \in K(H)$ if $a = 0$. Notice that the set $\text{ARep} (A, H)$ of ample representations of $A$ on $B(H)$ is a $G_{\delta}$ subset of $\text{Ball}(L (A, B(H)))$. Similarly, the set $\text{Iso}(H)$ of isometries $H \to H$ is a $G_{\delta}$ subset of $\text{Ball}(B(H))$. A formulation of Voiculescu’s Theorem asserts that if $\rho : A \to B(H)$ is an ample representation, and $\sigma : A \to B(H)$ is a unital completely positive (ucp) map, then there exists an isometry $V : H \to H$ such that $\sigma \preceq_{V} \rho$; see [HR00, Theorem 3.4.3, Theorem 3.4.6, Theorem 3.4.7]. We will observe that one can select $V$ in a Borel fashion from $\rho$ and $\sigma$.

Lemma 2.42. Let $A$ be a separable unital C*-algebra, and $H$ a separable Hilbert space. There exists a Borel function $UCP (A, B(H)) \times \text{ARep} (A, H) \to \text{Iso} (H), (\sigma, \rho) \mapsto V_{\sigma, \rho}$ such that $\sigma \preceq_{V_{\sigma, \rho}} \rho$.

Towards obtaining a proof of Lemma 2.42, we argue as in the proof of Voiculescu’s theorem as expounded in [HR00, Chapter 3]. First, one considers the case of ucp maps $A \to B(H)$ where $H$ is finite-dimensional. The following can be seen a definable version of [HR00, Proposition 3.6.7]. Notice that the set $\text{Proj} (H)$ of orthogonal projections $H \to H$ is closed subset of $\text{Ball} (B(H))$. Let $\text{Proj}_{fd} (H)$ be the Borel subset of finite-dimensional projections. The following lemma is a consequence of [HR00, Proposition 3.6.7] itself and the Luzin–Novikov Uniformization Theorem for Borel relations with countable sections [Kec95, Theorem 18.10].

Lemma 2.43. Fix $d \geq 1$. For every finite subset $F$ of $A$ and $\varepsilon > 0$, there exists a Borel map $UCP (A, M_{d} (C^{d})) \times \text{ARep} (A, H) \times \text{Proj}_{fd} (H) \to \text{Ball} (H), (\sigma, \rho, P) \mapsto V$ such that $\text{Ran} (V)$ is orthogonal to $P(H)$ and $\| \sigma (a) - V^{*} \rho (a) V \| < \varepsilon$ for $a \in F$.

One then uses Lemma 2.43 to establish Lemma 2.42 in the case of block-diagonal maps. Recall that $\sigma$ is block-diagonal with respect to $(P_{n})_{n \in \omega}$ if $(P_{n})_{n \in \omega}$ is a sequence of pairwise orthogonal finite-rank projections $P_{n} \in B(H)$ such that $\sum_{n} P_{n} = I$ and $\sigma (a) = \sum_{n} P_{n} \sigma (a) P_{n}$ for every $a \in A$ (where the convergence is in the strong-* topology). Consider the set $\text{BlockUcp} (A, B(H))$ of pairs $(\sigma, (P_{n})_{n \in \omega}) \in UCP (A, B(H)) \times \text{Proj}_{fd} (H)$ such that $\sigma$ is block-diagonal with respect to $(P_{n})_{n \in \omega}$. The proof of [HR00, Lemma 3.5.2] shows the following.

Lemma 2.44. There exists a Borel function $\text{BlockUcp} (A, B(H)) \times \text{ARep} (A, H) \to \text{Iso} (H), (\sigma, (P_{n})_{n \in \omega}, \rho) \mapsto V$ such that $\sigma \preceq_{V} \rho$.

Finally, one shows that the general case of Voiculescu’s theorem can be reduced to the block-diagonal case, as in [HR00, Theorem 3.5.5].

Lemma 2.45. There exists a Borel function $UCP (A, B(H)) \to \text{BlockUcp} (A, B(H)) \times \text{Iso} (H), \sigma \mapsto (\sigma', (P_{n})_{n \in \omega}, V')$ such that $\sigma \preceq_{V', \sigma'}$.

Lemma 2.42 is then obtained by combining Lemma 2.44 and Lemma 2.45.

As a consequence of the definable Voiculescu theorem, one obtains the following; see [HR00, Theorem 3.4.6].

Lemma 2.46. Let $A$ be a separable unital C*-algebra, and $H$ a separable Hilbert space. There exist:

- a Borel map $\text{Rep} (A, H) \times \text{ARep} (A, H) \to U(H), (\rho', \rho) \mapsto U_{\rho', \rho}$ such that $\rho' \equiv_{U_{\rho', \rho}} \rho$;
- a Borel map $\text{ARep} (A, H) \times \text{ARep} (A, H) \to U(H), (\rho', \rho) \mapsto W_{\rho', \rho}$ such that $\rho \equiv_{W_{\rho', \rho}} \rho'$.

2.9. Spectrum. Suppose now that $\mathfrak{A}$ is a strict unital C*-algebra, and $J$ is a norm-separated closed two-sided ideal of $\mathfrak{A}$. One can consider the quotient $\mathfrak{A}^{*}$-algebra $\mathfrak{A}/J$ and, for $a \in \mathfrak{A}$, the spectrum $\sigma_{\mathfrak{A}/J} (a)$ of $a + J$ in $\mathfrak{A}/J$. We also set $\rho_{\mathfrak{A}/J} (a) = C \setminus \sigma_{\mathfrak{A}/J} (a)$. The following lemma is analogous to [AM15, Theorem 3.16].

Lemma 2.47. Suppose that $\mathfrak{A}$ is a strict C*-algebra, and $J$ a norm-separated closed two-sided ideal of $\mathfrak{A}$. Suppose that every invertible selfadjoint element of $\mathfrak{A}/J$ lifts to an invertible selfadjoint element of $\mathfrak{A}$. If $a \in \mathfrak{A}$, and $J_{0}$ is a countable dense subset of $J \cap \mathfrak{A}$, then

$$\sigma_{\mathfrak{A}/J} (a) = \bigcap_{d \in J_{0}} \sigma (a + d).$$
Proof. It suffices to prove that $\rho_{3/2}(a)$ is the union of $\rho(a + d)$ for $d \in J_0$. Clearly, $\rho(a + d) \subseteq \rho_{3/2}(a)$ for every $d \in J$, so it suffices to prove the other inclusion. Suppose that $\lambda \in \rho_{3/2}(a)$. We want to show that $\lambda \in \rho(a + d)$ for some $d \in J_0$. After replacing a with $a - \lambda$, it suffices to consider the case when $\lambda = 0$. In this case, $a + J$ is invertible in $\mathfrak{A}/J$. Therefore, by assumption there exists $d \in J$ such that $a + d$ is invertible in $\mathfrak{A}$. Since the set of invertible elements of $\mathfrak{A}$ is norm-open, there exists $d_0 \in J$ such that $a + d_0$ is invertible in $\mathfrak{A}$, and hence $0 \in \rho(a + d_0)$. □

Lemma 2.48. Suppose that $\mathfrak{A}$ is a strict unital $C^*$-algebra, and $J$ a norm-separable closed two-sided ideal of $\mathfrak{A}$. Suppose that every invertible selfadjoint element of $\mathfrak{A}/J$ lifts to an invertible selfadjoint element of $\mathfrak{A}$. Then the function $\mathfrak{A}_{sa} \to \text{Closed}(\mathbb{R}), a \mapsto \sigma_{3/2}(a)$ is Borel.

Proof. Fix a countable norm-dense subset $J_0$ of $J \cap \mathfrak{A}_{sa}$. Then by the previous lemma we have that, for $a \in \mathfrak{A}_{sa}$, 

$$\sigma_{3/2}(a) = \bigcap_{d \in J_0} \sigma(a + d).$$

As the function $\text{Closed}(\mathbb{R})^\omega \to \text{Closed}(\mathbb{R}), (F_n) \mapsto \bigcap_{n \in \omega} F_n$ is Borel, this concludes the proof. □

Lemma 2.49. The function $B(H)_{sa} \to K(\mathbb{R}), T \mapsto \sigma_{\text{ess}}(T) = \sigma_{\text{Q} (H)}(T)$ is Borel.

Proof. An operator $T \in B(H)$ induces an invertible element of $Q(H)$ if and only if it is Fredholm. If $T$ is Fredholm and self-adjoint, then it has index 0, and 0 is an isolated point of the spectrum of $T$ that is an eigenvalue with finite multiplicity. Thus, if $P$ is the finite-rank projection onto the eigenspace of 0 for $T$, then we have that $T + P$ is invertible and selfadjoint and induces the same element of $Q(H)$ as $T$. This shows that every invertible selfadjoint element of $Q(H)$ lifts to an invertible selfadjoint element of $B(H)$. Therefore, the conclusion follows from Proposition 2.48.

Suppose that $T \in \text{Ball}(B(H))$ is a positive operator satisfying that $T^2 \equiv T$ mod $K(H)$. Then it is well-known that there exists a projection $P \equiv T$ mod $K(H)$. We observe that one can choose such a $P$ in a Borel fashion from $T$; see [And20, Lemma 3.1].

Lemma 2.50. Consider the Borel set $\text{Proj}(B(H)/K(H))$ of mod $K(H)$ projections in $B(H)$. Then there is a Borel function $\text{Proj}(B(H)/K(H)) \to \text{Proj}(B(H)), T \mapsto P_T$ such that $T \equiv P_T$ mod $K(H)$ for every $T \in \text{Proj}(B(H)/K(H))$.

Proof. Suppose that $T \in \text{Proj}(B(H)/K(H))$. Observe $\sigma_{\text{ess}}(T) \subseteq \{0,1\}$. In particular, $\sigma_{\text{ess}}(T)$ is countable, with only accumulation points 0 and 1. From Lemma 2.49, the maps $T \mapsto \sigma_{\text{ess}}(T)$ and $T \mapsto \sigma(T)$ are Borel. If $\sigma_{\text{ess}}(T) = \{0\}$ then one can set $P_T = 0$. If $\sigma_{\text{ess}}(T) = \{1\}$, one can set $P_T = 1$.

Let us consider the case when $\{0,1\} = \sigma_{\text{ess}}(T')$. By [Kec95, Theorem 12.13] there exists a Borel map $\text{Proj}(B(H)/K(H)) \to [0,1]^\omega$, $T \mapsto (t_n)$ such that $(t_n)$ is an increasing enumeration of $\sigma(T)$. One can then choose in a Borel way $n_0 \in \omega$ such that $t_{n_0} < t_{n_0 + 1}$ and then a continuous function $f : [0,1] \to [0,1]$ such that 

$$f(t) = \begin{cases} 0 & \text{if } t \leq n_0, \\ 1 & \text{if } t \geq n_0 + 1. \end{cases}$$

One can then set $P_T = f(T)$. □

2.10. Polar decompositions. We now observe that the polar decomposition of an operator is given by a Borel function. We will use the following version of the selection theorem for relations with compact sections from [Kec95, Theorem 28.8].

Lemma 2.51. Suppose that $X$ is a standard Borel space, $Y$ is a compact metrizable space, and $A \subseteq X \times Y$ is a Borel subset such that, for every $x \in X$, the vertical section 

$$A_x = \{y \in Y : (x,y) \in A\}$$

is a closed nonempty set. Then the assignment $X \to \text{Closed}(Y)$, $x \mapsto A_x$ is Borel, where $\text{Closed}(Y)$ is endowed with the Effros Borel structure.
As an application, we obtain the following. Let $H$ be a separable Hilbert space. We consider the unit ball $\text{Ball}(H)$ of $H$ as a compact metrizable space endowed with the weak topology. We also consider $\text{Closed}(\text{Ball}(H))$ as a standard Borel space, endowed with the Effros Borel structure.

**Lemma 2.52.** The function $B(H) \to \text{Closed}(\text{Ball}(H))$, $T \mapsto \text{Ker}(T) \cap \text{Ball}(H)$, is Borel.

**Proof.** By Lemma 2.51, it suffices to show that the set

$$A = \{(T,x) \in B(H) \times \text{Ball}(H) : Tx = 0\}$$

is Borel. Fix a countable norm-dense subset $\{x_n : n \in \omega\}$ of $\text{Ball}(H)$. Then we have that, if $(T,x) \in B(H) \times \text{Ball}(H)$, then $(T,x) \in A$ if and only if $\forall k \in \omega \exists n \in \omega$ such that $\|x - x_n\| < 2^{-k}$ and $\|Tx_k\| < 2^{-k}$. Since the norm on $B(H)$ is strong-$*$ lower-semicontinuous, this shows that $A$ is Borel. □

Recall that, for an operator $T \in B(H)$, one sets $|T| := (T^*T)^{1/2}$. By strong-$*$ continuity on bounded sets of continuous functional calculus, the function $T \mapsto |T|$ is Borel. Furthermore, there exists a unique partial isometry $U$ with $\text{Ker}(U) = \text{Ker}(T)$ such that $T = U|T|$ [Ped89, Theorem 3.2.17]. The decomposition $T = U|T|$ is then called the polar decomposition of $T$.

**Lemma 2.53.** The function $B(H) \to B(H)$, $T \mapsto U$ that assigns to an operator the partial isometry $U$ in the polar decomposition of $T$ is Borel.

**Proof.** It suffices to notice that is graph, which is the set of pairs $(T,U)$ such that $U$ is a partial isometry with $\text{Ker}(U) = \text{Ker}(T)$ and $T = U|T|$, is a Borel set by Lemma 2.52. □

Consider the Borel set $U(B(H)/K(H))$ of mod $K(H)$ unitaries in $B(H)$. Thus, $T \in U(B(H)/K(H))$ if and only if $T^*T \equiv TT^* \equiv I \mod K(H)$. If $U$ is the partial isometry in the polar decomposition of $T$, then $U \equiv T \mod K(H)$ and $U$ is an essential unitary. In fact, one can easily define (in a Borel fashion from $T$) an isometry or co-isometry $V$ such that $T \equiv V \mod K(H)$. One has that $T$ is in particular a Fredholm operator. Its index is defined by

$$\text{index}(T) = \text{rank}(1 - V^*V) - \text{rank}(1 - VV^*) .$$

Thus, index$(T)$ is a Borel function of $T \in U(B(H)/K(H))$.

More generally, consider the Borel set of pairs $(P,T) \in \text{Ball}(B(H))^2$ such that $P$ is a projection, $PT = TP = T$ and $TT^* \equiv T^*T \equiv P$. If $V$ is the partial isometry in the polar decomposition of $T$, then $V \equiv T \mod K(H)$ and the index of $PTP$ regarded as an operator on $PH$ is given by the Borel function

$$\text{index}(PTP) = \text{rank}(P - V^*V) - \text{rank}(P - VV^*) .$$

3. K-theory of unital C*-algebras with a strict cover

In this section we explain how the $\mathbb{K}_0$ and $\mathbb{K}_1$ groups of a unital C*-algebra with a strict cover can be regarded as semidefinable groups. We also recall the definition of the index map and the exponential map between the $\mathbb{K}_0$ and $\mathbb{K}_1$ groups, and observe that they are definable homomorphisms. Finally, we consider the six-term exact sequence associated with a strict unital C*-pair, and observe that the connective maps are all definable group homomorphisms.

### 3.1. \(\mathbb{K}_0\)-group

Suppose that $\mathfrak{A}/\mathfrak{J}$ is a unital C*-algebra with a strict cover. Recall that $\text{Proj}(\mathfrak{A}/\mathfrak{J})$ denotes the Polish space of projections mod $\mathfrak{J}$ in $\mathfrak{A}$. Similarly, for $n \geq 1$ we have that $\text{Proj}(M_n(\mathfrak{A}/\mathfrak{J})) := \text{Proj}(M_n(\mathfrak{A})/M_n(\mathfrak{J}))$ is a Polish space. We say that an element of $\text{Proj}(M_n(\mathfrak{A}/\mathfrak{J}))$ for some $n \geq 1$ is a mod $\mathfrak{J}$ projection over $\mathfrak{A}$. We define $Z_0(\mathfrak{A}/\mathfrak{J})$ be the set of pairs of mod $\mathfrak{J}$ projections over $\mathfrak{A}$, which is the disjoint union of $Z_0^{(n)}(\mathfrak{A}/\mathfrak{J}) := \text{Proj}(M_n(\mathfrak{A}/\mathfrak{J})) \times \text{Proj}(M_n(\mathfrak{A}/\mathfrak{J}))$ for $n \geq 1$ endowed with the induced standard Borel structure. Two mod $\mathfrak{J}$ projections $p,q$ in $\mathfrak{A}$ are Murray–von Neumann equivalent (respectively, unitary equivalent, and homotopic) mod $\mathfrak{J}$ if and only if $p+\mathfrak{J}$ and $q+\mathfrak{J}$ are Murray–von Neumann equivalent (respectively, unitary equivalent, and homotopic) in $\mathfrak{A}/\mathfrak{J}$.

The $\mathbb{K}_0$-group $\mathbb{K}_0(\mathfrak{A}/\mathfrak{J})$ of $\mathfrak{A}/\mathfrak{J}$—see [HR00, Chapter 4]—is defined as a quotient of $Z_0(\mathfrak{A}/\mathfrak{J})$ by an equivalence relation $B_0(\mathfrak{A}/\mathfrak{J})$, defined as follows. For $(p,p'), (q,q') \in Z_0(\mathfrak{A}/\mathfrak{J})$, $(p,p') \sim (q,q')$ if and only if there exist $m,n \in \omega$ and $r \in \text{Proj}(M_{mn}(\mathfrak{A}/\mathfrak{J}))$ such that $p \oplus q \oplus r \oplus 0_n$ and $q \oplus p' \oplus r \oplus 0_m$ are Murray–von Neumann equivalent mod $\mathfrak{J}$. By Lemma 2.37, we have the following equivalent description of $B_0(\mathfrak{A}/\mathfrak{J})$. 


Lemma 3.1. Suppose that $\mathfrak{A}/\mathfrak{J}$ is a unital $C^*$-algebra with a strict cover, and $(p,p'), (q,q') \in Z_0(\mathfrak{A}/\mathfrak{J})$ where $p, p' \in M_d(\mathfrak{A}/\mathfrak{J})$ and $q, q' \in M_k(\mathfrak{A}/\mathfrak{J})$. Then $(p,p') B_0(\mathfrak{A}/\mathfrak{J}) (q,q')$ if and only if there exist $m, n \in \omega$ and $y_1, \ldots, y_\ell \in \text{Ball}(M_{d+k+m+n}(\mathfrak{A}))$ such that, setting $u := e^{iy_1} \cdots e^{iy_\ell}$, one has that
\[
u(p \oplus q' \oplus 1_m \oplus 0_n) \equiv q \oplus p' \oplus 1_m \oplus 0_n \text{ mod } \mathfrak{J},
\]
where $\ell \geq 1$ does not depend on $\mathfrak{A}/\mathfrak{J}$ and $(p,p'), (q,q') \in Z_0(\mathfrak{A}/\mathfrak{J})$.

The (commutative) group operation on $K_0(\mathfrak{A}/\mathfrak{J})$ is induced by the Borel function on $Z_0(\mathfrak{A}/\mathfrak{J})$, $((p,p'), (q,q')) \mapsto (p \oplus q, p' \oplus q')$. The neutral element of $K_0(\mathfrak{A}/\mathfrak{J})$ corresponds to $(0,0) \in Z_0(\mathfrak{A}/\mathfrak{J})$. The function that maps an element to its additive inverse is induced by the Borel function on $Z_0(\mathfrak{A}/\mathfrak{J})$ given by $(p,p') \mapsto (p',p)$. Thus, $K_0(\mathfrak{A}/\mathfrak{J})$ is in fact a semidefinable group.

If $\mathfrak{A}/\mathfrak{J}$ and $\mathfrak{B}/\mathfrak{J}$ are unital $C^*$-algebras with a strict cover, and $\varphi : \mathfrak{A}/\mathfrak{J} \to \mathfrak{B}/\mathfrak{J}$ is a definable unital $*$-homomorphism, then the induced group homomorphism $K_0(\mathfrak{A}/\mathfrak{J}) \to K_0(\mathfrak{B}/\mathfrak{J})$ is also definable. Thus, the assignment $\mathfrak{A}/\mathfrak{J} \to K_0(\mathfrak{A}/\mathfrak{J})$ gives a functor from the category of unital $C^*$-algebras with a strict cover to the category of semidefinable abelian groups.

Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital $C^*$-pair. We denote by $\mathfrak{J}^+$ the unitization of $\mathfrak{J}$, which can be identified with the $C^*$-subalgebra $\mathfrak{J}^+ = \text{span} \{\mathfrak{J}, 1\} \subseteq \mathfrak{A}$. Since $\mathfrak{J}$ is a proper ideal of $\mathfrak{A}$, we can write every element of $\mathfrak{J}^+$ uniquely as $a + \lambda 1$ where $a \in \mathfrak{J}$ and $\lambda \in \mathbb{C}$. More generally, every element of $M_n(\mathfrak{J}^+)$ can be written uniquely as $a + \alpha 1$ where $a \in M_n(\mathfrak{J})$ and $\alpha \in M_n(\mathbb{C})$. As the map $M_n(\mathfrak{J}^+) \to M_n(\mathbb{C})$, $a + \alpha 1 \mapsto \alpha$ is a unital $*$-homomorphism, we have that $\|a\| \leq \|a + \alpha 1\|$ and hence $\|a\| \leq \|a + \alpha 1\|$ for $a + \alpha 1 \in M_n(\mathfrak{J}^+)$. We define $\text{Proj}(M_n(\mathfrak{J}^+))$ to be the set of projections in $M_n(\mathfrak{J}^+)$, which we regard as a Borel subset of $2\text{Ball}(M_n(\mathfrak{J}^+)) \times \text{Ball}(M_n(\mathbb{C}))$. Similarly, the unitary group $U(M_n(\mathfrak{J}^+))$ is regarded as a Borel subset of $2\text{Ball}(M_n(\mathfrak{J}^+)) \times \text{Ball}(M_n(\mathbb{C}))$. Define also $Z_0(n)(\mathfrak{J})$ to be the Borel subset of $\text{Proj}(M_n(\mathfrak{J}^+)) \times \text{Proj}(M_n(\mathfrak{J}^+))$ consisting of pairs $(p,p')$ such that $p \equiv p' \text{ mod } M_n(\mathfrak{J})$. Finally, let $Z_0(\mathfrak{J})$ to be the disjoint union of $Z_0(n)(\mathfrak{J})$ for $n \geq 1$.

The $K_0$-group $K_0(\mathfrak{J})$ of $\mathfrak{J}$—see [HR00, Definition 4.2.1]—is defined as a quotient of $Z_0(\mathfrak{J})$ by an equivalence relation $B_0(\mathfrak{J})$, defined as follows. One has that, for $(p,p'), (q,q') \in Z_0(\mathfrak{J})$, $(p,p') B_0(\mathfrak{J}) (q,q')$ if and only if there exist $m, n \in \omega$ and $x \in \text{Proj}(M_m(\mathfrak{J}^+))$ such that $p \oplus q' \oplus x \oplus 0_n$ and $q \oplus p' \oplus x \oplus 0_m$ are Murray–von Neumann equivalent. For $p, p' \in Z_0(\mathfrak{J})$, we let $[p] - [p']$ be the corresponding element of $K_0(\mathfrak{J})$. The (commutative) group operation on $K_0(\mathfrak{J})$ is induced by the Borel function on $Z_0(\mathfrak{J})$, $((p,p'), (q,q')) \mapsto (p \oplus q, p' \oplus q')$. The neutral element of $K_0(\mathfrak{J})$ corresponds to $(0,0) \in Z_0(\mathfrak{J})$. The function that maps an element of $K_0(\mathfrak{J})$ to its additive inverse is induced by the Borel function on $Z_0(\mathfrak{J})$ given by $(p,p') \mapsto (p',p)$. Thus, $K_0(\mathfrak{J})$ is a semidefinable group.

If $(\mathfrak{A}, \mathfrak{J})$ are $(\mathfrak{B}, \mathfrak{K})$ are strict $C^*$-pairs, and $\varphi : (\mathfrak{A}, \mathfrak{J}) \to (\mathfrak{B}, \mathfrak{K})$ is a strict $*$-homomorphism, then it induces a strict $*$-homomorphism $\varphi|_\mathfrak{J} : \mathfrak{J} \to \mathfrak{K}$. In turn, this induces a definable group homomorphism $K_0(\mathfrak{J}) \to K_0(\mathfrak{K})$. This gives a functor $(\mathfrak{A}, \mathfrak{J}) \to K_0(\mathfrak{J})$ from strict $C^*$-pairs to semidefinable groups. The proof of the following lemma is taken from [RLL00, Proposition 4.2.2].

Lemma 3.2. Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital $C^*$-pair. Then there is a Borel map $Z_0(\mathfrak{J}) \to Z_0(\mathfrak{J})$, $(P,P') \mapsto (p,p')$ such that $[P] - [P'] = [p] - [p']$ and $p \in M_n(\mathbb{C})$.

Proof. Suppose that $(P,P') \in Z_0(\mathfrak{J})$. By definition, we have that for some $x, x' \in M_d(\mathfrak{J})$ and $\alpha \in M_d(\mathbb{C})$, $P = x + \alpha$ and $P' = x' + \alpha$. Thus, we can define
\[
p := \begin{bmatrix} P & 0 \\ 0 & 1_d - P' \end{bmatrix} \in M_{2d}(\mathfrak{J}^+)
\]
and
\[
p' := \begin{bmatrix} 0 & 0 \\ \alpha & 1_d - \alpha \end{bmatrix} \in M_{2d}(\mathbb{C}).
\]
Then we have that
\[
[p] - [p'] = [P] + [1_d - P'] + [\alpha] - [1_d - \alpha] = [P] - [P'].
\]
This concludes the proof. □
3.2. Relative $K_0$-group. Suppose now that $(\mathfrak{A}, \mathfrak{J})$ is a strict unital C*-pair. For $n \geq 1$, define $Z_0^{(n)}(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ to be the Borel set of triples $(p, q, x) \in \text{Ball}(\mathfrak{A})^3$ where $p, q$ are projections and $x \in \text{Ball}(M_n(\mathfrak{A}))$ satisfies $x^*x \equiv p \mod M_n(\mathfrak{J})$ and $xx^* \equiv q \mod M_n(\mathfrak{J})$. Define $Z_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ to be the disjoint union of $K_0^{(n)}(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ for $n \geq 1$ endowed with the induced standard Borel structure. The elements of $Z_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ are called relative $K$-cycles for $(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$; see [HR00, Definition 4.3.1]. If $(p, q, x) \in Z_0^{(n)}(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$, then we say that $(p, q, x)$ is a relative $K$-cycle of dimension $n$. A relative $K$-cycle $(p, q, x)$ for $(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ is degenerate if $x^*x = p$ and $xx^* = q$. Two relative $K$-cycles $(p, q, x)$ and $(p', q', x')$ of dimension $n$ are homotopic if there exists a norm-continuous path $\{(p_t, q_t, x_t)\}_{t \in [0,1]}$ of relative $K$-cycles for $(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ of dimension $n$ with $(p, q, x) = (p_0, q_0, x_0)$ and $(p', q', x') = (p_1, q_1, x_1)$.

Notice that if $(p, q, x)$ is a relative $K$-cycle of dimension $d$, and $(u_t)_{t \in [0,1]}$ is a path of unitaries in $M_d(\mathfrak{A})$ starting at $1$, then

$$(u_t^*pu_t, u_t^*qu_t, u_t^*x_t)$$

and

$$(p, u_t^*qu_t, u_t^*x_t)$$

are norm-continuous paths of relative $K$-cycles starting at $(p, q, x)$. If $p \equiv q \equiv x \mod M_d(\mathfrak{J})$, then

$$(p, q, tp + (1 - t)q)$$

is a norm-continuous path of relative cycles from $(p, q, x)$ to $(p, q, p)$. We have the following lemma; see [NdK17, Proposition 3.4].

**Lemma 3.3.** Suppose that $(p, q, x)$ is a relative cycle of dimension $n$ for $(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$. Then $r_0 := (p \oplus q, p \oplus q, p \oplus q)$ is homotopic to the degenerate cycle $(p \oplus q, p \oplus q, p \oplus q)$. The relative $K_0$-group $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ is defined to be the quotient of $Z_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ by the equivalence relation $B_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ defined as follows. For $(p, q, x), (p', q', x')$, set $(p, q, x) B_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J}) (p', q', x')$ if and only if there exists a degenerate relative $K$-cycle $(p_0, q_0, x_0), (p_0', q_0', x_0')$ such that $(p \oplus p_0, q \oplus q_0, x \oplus x_0)$ and $(p' \oplus p_0', q' \oplus q_0', x' \oplus x_0')$ are of the same dimension and homotopic. The group operations on $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ are induced by the Borel maps

$$((p, q, x), (p', q', x')) \mapsto (p \oplus p', q \oplus q', x \oplus x')$$

and

$$(p, q, x) \mapsto (q, p, x^*).$$

It follows from Lemma 3.3 that $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ is indeed a group. We let $[p, q, x]$ be the element of $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ represented by the relative $K$-cycle $(p, q, x)$. The trivial element of $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ is equal to $[p, q, x]$ where $(p, q, x)$ is any degenerate relative $K$-cycle. Let $\mathfrak{J}^+$ be the unitization of $\mathfrak{A}$, which we identify with $\text{span}(\mathfrak{J}, 1) \subseteq \mathfrak{A}$.

**Lemma 3.4.** There is a Borel function $Z_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J}) \to Z_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J}), (P, Q, X) \mapsto (p, q, p)$ such that $p \in M_n(\mathfrak{C})$, $q \in M_n(\mathfrak{J}^+)$, $p \equiv q \equiv x \equiv 0 \mod M_n(\mathfrak{J})$, and $[P, Q, X] = [p, q, p] \in K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$.

**Proof.** Notice that $(1 - P, 1 - P, 1 - P)$ is a degenerate relative $K$-cycle of dimension $d$. Consider then

$$(P \oplus (1 - P), Q \oplus (1 - P), X \oplus (1 - P)).$$

By Lemma 2.38 one can choose $Y_1, \ldots, Y_\ell \in \text{Ball}(M_{2d}(\mathfrak{A})_{sa})$ in a Borel way such that, setting $U := e^{Y_1} \cdots e^{Y_\ell}$, one has that $U(P \oplus (1 - P) U^*) = 1_d \oplus 0_d$, where $\ell \geq 1$ does not depend on $\mathfrak{J}$ and $(P, Q, X)$. Thus, after replacing $(P, Q, X)$ with $(U(P \oplus (1 - P) U^*), U(Q \oplus (1 - P) U^*), U(X \oplus (1 - P) U^*))$, we can assume without loss of generality that $P = 1_d \oplus 0_d$.

By Lemma 2.35 one can choose $Y_1, \ldots, Y_\ell \in \text{Ball}(M_{4d}(\mathfrak{A})_{sa})$ in a Borel fashion from $(P, Q, X)$ such that, setting $U := e^{Y_1} \cdots e^{Y_\ell}$, one has that

$$U^*(Q \oplus 0_{2d}) U \equiv (P \oplus 0_{2d}) \mod M_{2d}(\mathfrak{J}).$$

Thus, after replacing $(P, Q, X)$ with $(P \oplus 0_{2d}, U^* (Q \oplus 0_{2d}) U, U^*(X \oplus 0_{2d}))$ we can assume without loss of generality that $P = 1_d \oplus 0_{2d} \in M_{2d}(\mathfrak{C})$ and $Q \in M_{4d}(\mathfrak{A})$ satisfy $P \equiv Q \mod M_{4d}(\mathfrak{J})$ and hence $Q \in M_{4d}(\mathfrak{J}^+)$.

In this case, we have that $[P, Q, X] = [P, Q, P]$, since $(P_t Q_t, tp + (1 - t)X_t)_{t \in [0,1]}$ is a norm-continuous path of relative $K$-cycles from $(P, Q, X)$ to $(P, Q, P)$. This concludes the proof.

**Proposition 3.5.** Suppose that $(\mathfrak{A}, \mathfrak{J})$ is a strict C*-pair. Then $K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$ is a definable group. The assignment $K_0(\mathfrak{J}) \to K_0(\mathfrak{A}, \mathfrak{A}/\mathfrak{J})$, $[P] - [Q] \mapsto [P, Q, P]$ is a natural definable isomorphism, called the excision isomorphism.
Proof. By [NdK17, Theorem 3.9], the excision homomorphism \( K_0(\mathfrak{A}, \mathfrak{J}) \to K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \) is bijective; see also [HR00, Theorem 4.3.8]. Clearly, it is induced by a Borel function \( Z_0(\mathfrak{A}, \mathfrak{J}) \to Z_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \). By Lemma 3.4 the inverse homomorphism \( K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to K_0(\mathfrak{J}) \) is also induced by a Borel function \( Z_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to Z_0(\mathfrak{J}) \). Thus, \( K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \) is a definable group, and the excision isomorphism is a definable isomorphism. \( \square \)

There is a natural definable homomorphism \( K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to K_0(\mathfrak{A}) \) that is induced by the Borel map \( Z_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to Z_0(\mathfrak{A}) \) \( (p, q, x) \mapsto (p, q) \). We also have a natural definable homomorphism \( K_0(\mathfrak{A}) \to K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \) induced by the Borel map \( Z_0(\mathfrak{A}) \to Z_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \). We have the following result; see [HR00, Proposition 4.3.5].

**Proposition 3.6.** Suppose that \( (\mathfrak{A}, \mathfrak{J}) \) is a strict unital C*-pair. The (natural) sequence of definable groups and definable group homomorphisms

\[
K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/\mathfrak{J})
\]

is exact.

Combining the excision isomorphism \( K_0(\mathfrak{J}) \to K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \) with the natural definable homomorphism \( K_0(\mathfrak{A}, \mathfrak{J}/\mathfrak{J}) \to K_0(\mathfrak{A}) \), we obtain a natural definable group homomorphism \( K_0(\mathfrak{J}) \to K_0(\mathfrak{A}) \). This is defined by mapping \( (p, q) \in Z_0^{(n)}(\mathfrak{A}) \) to \( (p, q) \) regarded as an element of \( Z_0^{(n)}(\mathfrak{A}) \). Combining Proposition 3.5 with Proposition 3.6 we have the following.

**Corollary 3.7.** Suppose that \( \mathfrak{A} \) is a unital strict C*-algebra and \( \mathfrak{J} \) is a proper strict ideal of \( \mathfrak{A} \). Then the natural sequence

\[
K_0(\mathfrak{J}) \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/\mathfrak{J})
\]

is exact.

3.3. \( K_1 \) group. Suppose that \( (\mathfrak{A}, \mathfrak{J}) \) is a strict unital C*-pair. We can then consider the Borel set \( U(\mathfrak{A}/\mathfrak{J}) \) of elements of \( \text{Ball}(\mathfrak{A}) \) that are unitaries mod \( \mathfrak{J} \). We then let \( Z_1(\mathfrak{A}/\mathfrak{J}) \) to be the disjoint union of \( U(M_n(\mathfrak{A})/M_n(\mathfrak{J})) \) for \( n \geq 1 \). The equivalence relation \( B_1(\mathfrak{A}/\mathfrak{J}) \) on \( Z_1(\mathfrak{A}/\mathfrak{J}) \) is defined by setting \( aB_1(\mathfrak{A}/\mathfrak{J}) u' \) for \( u \in U(M_n(\mathfrak{A})/M_n(\mathfrak{J})) \)

\[
\text{and } u', \in U(M_n(\mathfrak{A})/M_n(\mathfrak{J})) \text{ if and only if there exist } k, k' \in \omega \text{ with } n + k = n' + k' \text{ and such that there is a norm-continuous path from } u + 1_k + M_{n+k}(\mathfrak{J}) \text{ to } u' + 1_{k'} + M_{n+k}(\mathfrak{J}) \text{ in the unitary group of the quotient unital C*-algebra } M_{n+k}(\mathfrak{A})/M_{n+k}(\mathfrak{J}) \text{ this equivalence relation is analytic by Corollary 2.34}. \]

The definable \( K_1 \)-group \( K_1(\mathfrak{A}/\mathfrak{J}) \) is then the semidefinable group obtained as quotient \( Z_1(\mathfrak{A}/\mathfrak{J})/B_1(\mathfrak{A}/\mathfrak{J}) \) with group operations defined as above. This defines a functor \( \mathfrak{A}/\mathfrak{J} \mapsto K_1(\mathfrak{A}/\mathfrak{J}) \) from unital C*-algebras with strict cover to semidefinable groups.

Given a strict unital C*-pair \( (\mathfrak{A}, \mathfrak{J}) \), we also consider the definable \( \text{K}_1 \)-group \( K_1(\mathfrak{J}) \) of \( \mathfrak{J} \). As above, we identify the unitization \( \mathfrak{J}^+ \) of \( \mathfrak{J} \) with span \( \{ 1, 0 \} \subset \mathfrak{A} \). For \( n \geq 1 \) we let \( U(M_n(\mathfrak{J}^+)) \) be the unitary group of \( M_n(\mathfrak{J}^+) \). Recall that every element of \( M_n(\mathfrak{J}^+) \) can be written uniquely as \( x + \alpha 1 \) where \( x \in M_{n}(\mathfrak{J}) \) and \( \alpha \in M_{n}(\mathbb{C}) \). We consider \( U(M_n(\mathfrak{J}^+)) \) as a Borel subset of \( \text{Ball}(M_n(\mathfrak{J})) \times \text{Ball}(\mathbb{C}(\mathfrak{J})) \). We then set \( Z_1(\mathfrak{J}) \) to be the disjoint union of \( U(M_n(\mathfrak{J}^+)) \) for \( d \geq 1 \), and let \( B_1(\mathfrak{J}) \) be the (analytic) equivalence relation on \( Z_1(\mathfrak{J}) \) obtained by setting \( aB_1(\mathfrak{J}) u' \) for \( u \in U(M_n(\mathfrak{J}^+)) \) and \( u' \in U(M_n(\mathfrak{J}^+)) \) if and only if there exist \( k, k' \in \omega \) with \( n + k = n' + k' \) and such that there is a norm-continuous path from \( u + 1_k \) to \( u' + 1_{k'} \) in \( U(M_{n+k}(\mathfrak{J}^+)) \). The definable \( K_1 \)-group \( K_1(\mathfrak{J}) \) is then the semidefinable group obtained as quotient \( Z_1(\mathfrak{J})/B_1(\mathfrak{J}) \) with group operations defined as above.

Suppose that \( (\mathfrak{A}, \mathfrak{J}) \) is a strict unital C*-pair. We have natural definable group homomorphisms

\[
K_1(\mathfrak{J}) \to K_1(\mathfrak{A}) \to K_1(\mathfrak{A}/\mathfrak{J})
\]

The definable group homomorphism \( K_1(\mathfrak{J}) \to K_1(\mathfrak{J}/\mathfrak{J}) \) is induced by the inclusion \( \mathfrak{J}^+ \subseteq \mathfrak{J} \), which induces Borel maps \( U(M_d(\mathfrak{J}^+)) \to U(M_d(\mathfrak{J})) \) for every \( d \geq 1 \). The definable group homomorphism \( K_1(\mathfrak{J}/\mathfrak{J}) \to K_1(\mathfrak{J}/\mathfrak{J}) \) is also induced by the Borel maps \( U(M_d(\mathfrak{J})) \to (M_d(\mathfrak{J})/M_d(\mathfrak{J})) \) for \( d \geq 1 \). We have the following result, which can be easily verified directly, and also follows from Corollary 3.7 via the Bott isomorphism theorem [HR00, Theorem 4.9.1].

**Proposition 3.8.** Suppose that \( (\mathfrak{A}, \mathfrak{J}) \) is a strict unital C*-pair. The sequence of natural definable homomorphisms

\[
K_1(\mathfrak{J}) \to K_1(\mathfrak{A}) \to K_1(\mathfrak{A}/\mathfrak{J})
\]

is exact.
3.4. The six-term exact sequence. Suppose that \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair. One can define a natural definable group homomorphism \(\partial_1 : K_1 (\mathfrak{A}/\mathfrak{J}) \to K_0 (\mathfrak{J})\) called the index map, as follows. An element of \(K_1 (\mathfrak{A}/\mathfrak{J})\) is of the form \([u]\) where \(u \in U (M_d (\mathfrak{A}) / M_d (\mathfrak{J}))\) for some \(d \geq 1\). Then define

\[
P := \begin{bmatrix}
    uu^* & u (1 - u^* u)^{1/2} \\
    u^* (1 - uu^*)^{1/2} & 1 - u^* u
\end{bmatrix} \in M_{2d} (\mathfrak{J}^+)\]

and

\[
Q := 1_d \oplus 0_d \in M_{2d} (\mathfrak{J}^+)\,.
\]

Then \(P, Q\) are projections such that \(P \equiv Q \mod M_{2d} (\mathfrak{J})\) and hence \((P, Q) \in Z_0 (\mathfrak{J})\). One then defines \(\partial_1 ([u]) = [P] - [Q]\); see [HR00, Proposition 4.8.10]. As \((P, Q)\) is obtained in a Borel fashion from \(u\), the boundary map \(\partial_1 : K_1 (\mathfrak{A}/\mathfrak{J}) \to K_0 (\mathfrak{J})\) is definable.

Equivalently, one can define \(\partial_1\) as follows. Given an element \([u]\) of \(K_1 (\mathfrak{A}/\mathfrak{J})\) for some \(u \in U (M_d (\mathfrak{A}) / M_d (\mathfrak{J}))\).

Consider the partial isometry \(v \in M_{2d} (\mathfrak{A})\) defined by

\[
v = \begin{bmatrix}
    u & 0 \\
    (1 - u^* u)^{1/2} & 0
\end{bmatrix}
\]

and observe that \(v \equiv u \oplus 0 \mod \mathfrak{J}\) [RLL00, Lemma 9.2.1]. Then \(1_{2d} - v^* v\) and \(1_{2d} - vv^*\) are projections in \(M_{2d} (\mathfrak{J}^+)\) such that \(1_{2d} - v^* v \equiv 1_{2d} - vv^* \equiv 0_d \oplus 1_d \in M_d (\mathfrak{C})\). Therefore, \((1_{2d} - v^* v, 1_{2d} - vv^*) \in Z_0 (\mathfrak{J})\). One has that \(\partial_1 [u] = [1_{2d} - u^* u] - [1_{2d} - vv^*] \in K_0 (\mathfrak{J})\); see [RLL00, Proposition 9.2.3]. Then we have the following; see [RLL00, Lemma 9.3.1 and Lemma 9.3.2].

**Proposition 3.9.** Suppose that \(\mathfrak{A}\) is a strict unital C*-algebra and \(\mathfrak{J}\) is a strict ideal of \(\mathfrak{A}\). Then the sequence

\[
K_1 (\mathfrak{A}) \to K_1 (\mathfrak{A}/\mathfrak{J}) \overset{\partial_1}{\to} K_0 (\mathfrak{J}) \to K_0 (\mathfrak{A})
\]

is exact.

Suppose that \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair. One can consider a natural definable homomorphism \(\partial_0 : K_0 (\mathfrak{A}/\mathfrak{J}) \to K_1 (\mathfrak{J})\) called the exponential map. This is defined as follows. Consider an element of \(K_0 (\mathfrak{A}/\mathfrak{J})\) of the form \([p] - [q]\) for some \(p, q \in \text{Proj} (M_n (\mathfrak{A}) / M_n (\mathfrak{J}))\). Then we have that \(\exp (2 \pi i p)\) and \(\exp (2 \pi i q)\) are unitary elements of \(M_n (\mathfrak{J}^+)\) such that \(\exp (2 \pi i p) \equiv \exp (2 \pi i q) \mod M_n (\mathfrak{J})\). Then one has that \(\partial_0 ([p] - [q]) = [\exp (2 \pi i p)] - [\exp (2 \pi i q)] \in K_1 (\mathfrak{J})\); see [RLL00, Proposition 12.2.2] and [HR00, Section 4.9]. From Proposition 3.9 one can obtain via the Bott isomorphism theorem [HR00, Theorem 4.9.1] the following.

**Proposition 3.10.** Suppose that \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair. Then the sequence

\[
K_0 (\mathfrak{A}) \to K_0 (\mathfrak{A}/\mathfrak{J}) \overset{\partial_0}{\to} K_1 (\mathfrak{J}) \to K_1 (\mathfrak{A})
\]

is exact.

Suppose that \((\mathfrak{A}, \mathfrak{J})\) is a strict unital C*-pair. Then as discussed above we have exact sequences

\[
K_0 (\mathfrak{J}) \to K_0 (\mathfrak{A}) \to K_0 (\mathfrak{A}/\mathfrak{J})
\]

and

\[
K_1 (\mathfrak{J}) \to K_1 (\mathfrak{A}) \to K_1 (\mathfrak{A}/\mathfrak{J})
\]

These are joined together by the index and exponential maps. From Proposition 3.10, Proposition 3.9, Corollary 3.7 and Proposition 3.8, one obtains the six-term exact sequence

\[
\begin{array}{cccccc}
K_1 (\mathfrak{J}) & \to & K_1 (\mathfrak{A}) & \to & K_1 (\mathfrak{A}/\mathfrak{J}) & \to \\
\partial_1 & & & & & \partial_0 \\
& K_0 (\mathfrak{A}/\mathfrak{J}) & \adjoint & K_0 (\mathfrak{A}) & \adjoint & K_0 (\mathfrak{J})
\end{array}
\]

for the strict unital C*-pair \((\mathfrak{A}, \mathfrak{J})\), where the vertical arrows are the index map and the exponential map; see [RLL00, Theorem 12.1.2].
4. DEFINABLE K-HOMOLOGY OF SEPARABLE C*-ALGEBRAS

In this section we recall the definition of the Ext invariant for separable unital C*-algebras, and its description
due to Paschke in terms of the K-theory of Paschke dual algebras as defined in [Hig95, HR00] or, equivalently,
of commutants in the Calkin algebra. Following [HR00, Chapter 3], we consider the group Ext(−)−1 defined in
terms of unital semi-split extensions. In the case of separable unital nuclear C*-algebras, every unital extension is
semi-split , and the group Ext(−)−1 coincides with the group Ext(−) defined in terms of unital extensions. Using
Paschke’s K-theoretical description of Ext from [Pas81], we show that Ext(−)−1 yields a contravariant functor from
separable unital C*-algebras to the category of definable groups.

We also recall the definition of the K-homology groups of separable C*-algebras as in [HR00, Chapter 5]. Using
their description in terms of Ext, we conclude that they can be endowed with the structure of definable groups,
in such a way that the assignments A → K0(A) and A → K1(A) are functors from the category of separable
C*-algebras to the category of definable groups. We call a pair (A, I) a separable C*-pair if A is a separable
C*-algebra and I is a closed two-sided ideal of A. A morphism (A, I) → (B, J) between separable C*-pairs is a
*,-homomorphisms A → B that maps I to J. Recall that a C*-algebra A is nuclear if the identity map of A is the
pointwise limit of contractive completely positive maps that factor through finite-dimensional C*-algebras; see
[HR00, Section 3.3]. We say that a separable C*-pair (A, I) is nuclear if A is nuclear. We also consider the relative
definable K-homology groups, and the six-term exact sequence in K-homology associated with a separable C*-pair.

4.1. C*-algebra extensions and the Ext group. Let H be a separable Hilbert space, and B(H) be the algebra
of bounded linear operators on H. We let K(H) ⊆ B(H) be the closed ideal of compact operators, and Q(H) be the
Calkin algebra, which is the quotient of B(H) by K(H). Let π : B(H) → Q(H) be the quotient map.

If U ∈ U(H) is a unitary operator, then U defines an automorphism Ad(U) : B(H) → B(H) given by
T → U*TU. As K(H) is Ad(U)-invariant, we have an induced automorphism of Q(H), still denoted by Ad(U).

Suppose that A is a unital, separable C*-algebra. A unital extension of A (by K(H)) is a unital *-homomorphism
ϕ : A → Q(H). A unital extension of A is injective or essential if it is an injective *,-homomorphism A → Q(H).
Two extensions ϕ, ϕ′ : A → Q(H) are equivalent if there exists U ∈ U(H) such that Ad(U) ◦ ϕ′ = ϕ. An injective,
unital extension ϕ : A → Q(H) is semi-split (or weakly nuclear in the terminology of [EK01]) if there exists a
unital completely positive (ucp) map σ : A → B(H) such that ϕ = π ◦ σ [HR00, Theorem 3.1.5]. An injective
unital extension ϕ : A → Q(H) is split or trivial if there exists a unital *,-homomorphism ˆϕ : A → B(H) such that
ϕ = π ◦ ˆϕ.

Every unital, essential extension ϕ : A → Q(H) determines an exact sequence

0 → K(H) → Eϕ → A → 0

where

Eϕ = \{(x, y) ∈ A ⊕ B(H) : ϕ(x) = π(y)\},

and K(H) is an essential ideal of Eϕ. The extension is split or trivial if the map Eϕ → A is a split epimorphism
in the category of unital C*-algebras and unital *,-homomorphisms.

Conversely, given an exact sequence

0 → K(H) → E → A → 0

where p : E → A is a unital *,-homomorphism and K(H) is an essential ideal of E, one can define an essential unital
extension ϕ : A → Q(H) as follows. Consider K(H) ⊆ E ⊆ B(H), then define ϕ(a) = π(ˆa) ∈ Q(H) for a ∈ A
where ˆa ∈ E is such that p(ˆa) = a. Again, we have that ϕ is trivial if and only if p : E → A is a split epimorphism.

Let A be a separable, unital C*-algebra. One defines Ext(A) to be the set of unitary equivalence classes of
unital, injective extensions of A by K(H); see [HR00, Definition 2.7.1], and Ext_nuc(A) = Ext(A)−1 to be the
subset of unitary equivalence classes of unital, injective semi-split (or weakly nuclear) extensions of A by K(H)
[Bla98, 15.7.2].

One can define a commutative monoid operation on Ext(A). The (additively denoted) operation on Ext(A) is
induced by the map (ϕ, ϕ′) → Ad(V) ◦ (ϕ ⊕ ϕ′) where V : H → H ⊕ H is a surjective linear isometry; see [HR00,
Proposition 2.7.2]. By Voiculescu’s Theorem [HR00, Theorem 3.4.3], one has that the neutral element of Ext(A)
is the set of split extensions, which form a single unitary equivalence class [HR00, Theorem 3.4.7]. Furthermore,
the set $\text{Ext}(A)^{-1}$ is equal to the set of elements of $\text{Ext}(A)$ that have an additive inverse, whence it forms a group [HR00, Definition 2.7.6].

When $A$ is a nuclear unital separable C*-algebra, by the Choi–Effros lifting theorem [HR00, Theorem 3.3.6], one has that $\text{Ext}(A) = \text{Ext}(A)^{-1}$. In particular, in this case $\text{Ext}(A)$ is itself a group.

Let $A$ be a separable unital C*-algebra. We regard $\text{Ext}(A)^{-1}$ as a definable group, as follows. Fix a separable Hilbert space $H$. Let us say that a ucp map $\phi : A \to B(H)$ is ample if $\|\pi \circ \phi(x)\| = \|x\|$ for every $x \in A$. Notice that the set $\text{AUCP}(A, B(H))$ of ample ucp maps $A \to B(H)$ is a $G_4$ subset of the space $\text{Ball}(L(A, B(H)))$ of bounded linear maps of norm at most 1 endowed with the topology of pointwise strong-* convergence. Thus, $\text{AUCP}(A, B(H))$ is a Polish space.

Let $\mathcal{E}(A) \subseteq \text{AUCP}(A, B(H))$ be the Borel set of ample ucp maps $\varphi : A \to B(H)$ such that

$$\varphi(xy) \equiv \varphi(x) \varphi(y) \mod K(H)$$

for $x, y \in A$. An injective, unital semi-split extension of $A$ by definition has a ucp lift, which is an element of $\mathcal{E}(A)$, and conversely every element of $\mathcal{E}(A)$ gives rise to an injective, unital semi-split extension of $A$. Thus, we can regard $\mathcal{E}(A)$ as the space of representatives of injective, unital semi-split extensions of $A$. We define a Polish topology on $\mathcal{E}(A)$ that induces the Borel structure on $\mathcal{E}(A)$ by declaring a net $(\varphi_i)_{i \in I}$ in $\mathcal{E}(A)$ to converge to $\varphi$ if and only if, for every $x, y \in X$, $(\varphi_i(x))_{i \in I}$ is a strong-* convergent to $\varphi(x)$, and $(\varphi_i(xy) - \varphi_i(x) \varphi_i(y))_{i \in I}$ norm-converges to $\varphi(xy) - \varphi(x) \varphi(y)$.

Two elements $\varphi, \varphi'$ of $\mathcal{E}(A)$ represent the same element $[\varphi]$ of $\text{Ext}(A)^{-1}$ if and only if there exists $U \in U(H)$ such that $U^* \varphi(a) U = \varphi'(a)$ mod $K(H)$ for every $a \in A$. This defines an analytic equivalence relation $\cong$ on $\mathcal{E}(A)$. We can thus regard $\text{Ext}(A)^{-1}$ as the semidefinable set $\mathcal{E}(A)/\cong$.

We now observe that the group operations on $\text{Ext}(A)^{-1}$ are definable, and thus this turns $\text{Ext}(A)^{-1}$ into a semidefinable group. We will later show in Proposition 4.12 that in fact $\text{Ext}(A)^{-1}$ is a definable group.

**Proposition 4.1.** Let $A$ be a separable unital C*-algebra. The addition operation $(x, y) \mapsto x + y$ and the additive inverse operation $x \mapsto -x$ on $\text{Ext}(A)^{-1}$ are definable functions. Thus, $\mathcal{E}(A)/\cong = \text{Ext}(A)^{-1}$ is a semidefinable group.

Proof. Fix a representation $A \subseteq B(H)$ such that $A \cap K(H) = \{0\}$. The assertion for addition is clear, as the Borel map $\mathcal{E}(A) \times \mathcal{E}(A) \to \mathcal{E}(A)$, $(\varphi, \varphi') \mapsto \text{Ad}(W) \circ (\varphi + \varphi')$ is a lift for the addition operation, where $W$ is a fixed surjective linear isometry $H \to H \oplus H$.

In order to obtain a lift for the function $\text{Ext}(A)^{-1} \to \text{Ext}(A)^{-1}$, $x \mapsto -x$, one can use the definable Stinespring Dilation Theorem (Lemma 2.41). Thus, if $\varphi \in \mathcal{E}(A)$, and $\pi$ and $V$ are the nondegenerate representation $\pi$ of $A$ and the isometry $V : H \to H$ obtained from $\varphi$ in a Borel fashion as in Lemma 2.41, then defining the projection $P := I - V^* V \in B(H)$ and $\varphi' : A \to B(H)$, $a \mapsto W^* (P \pi(a) P + a) W$, one has that $\varphi' \in \mathcal{E}(A)$ represents $-[\varphi]$; see also [HR00, Theorem 3.4.7].

Suppose that $A, B \subseteq B(H)$ are separable unital C*-algebras. A unital *-homomorphism $\alpha : A \to B$ induces a definable group homomorphism $\text{Ext}(B)^{-1} \to \text{Ext}(A)^{-1}$, as follows. If $\varphi \in \mathcal{E}(B)$ is a representative for an injective, unital extension, then one can consider $\alpha^* (\varphi) \in \mathcal{E}(A)$ defined by $a \mapsto W^* (\alpha(a) \oplus a) W$ where $W : H \to H \oplus H$ is a fixed surjective isometry. This defines a Borel function $\alpha^* : \mathcal{E}(B) \to \mathcal{E}(A)$, which induces a definable group homomorphism $\alpha^* : \text{Ext}(B)^{-1} \to \text{Ext}(A)^{-1}$. Thus, Ext$(\cdot)^{-1}$ is a contravariant functor from the category of separable unital C*-algebras to the category of semidefinable groups.

### 4.2. $K_0$-group and the Voiculescu property

Let $\mathfrak{A}$ be a strict unital C*-algebra. Recall that two projections $p, q \in \mathfrak{A}$ are Murray–von Neumann (MvN) equivalent if there exists $v \in \mathfrak{A}$ such that $v^* v = p$ and $v v^* = q$, in which case we write $p \sim_{\text{MvN}} q$, and $v$ is called a partial isometry with support projection $p$ and range projection $q$. We say that a projection $p \in \mathfrak{A}$ is ample if $p \oplus 0$ is Murray–von Neumann equivalent to $p \oplus 1$, and co-ample if $1 - p$ is ample.

**Definition 4.2.** Let $\mathfrak{A}$ be a strict unital C*-algebra. We say that $\mathfrak{A}$ satisfies the Voiculescu property if the set of ample projections in $\mathfrak{A}$ is a Borel subset of $\text{Ball}(\mathfrak{A})$ containing 1, and there exist strict unital *-isomorphisms $\Phi_{k, n} : M_n(\mathfrak{A}) \to M_k(\mathfrak{A})$ for $n, k \geq 1$ such that, for $n, k, m, n_0, k_0, n_1, k_1 \geq 1$:

1. for $n > k$, $\Phi_{k, n}(p \oplus 0_{n-k}) \sim_{\text{MvN}} p$ for every projection $p \in M_k(\mathfrak{A})$;
Lemma 4.6. Let\( \Phi_{k,m} \) and \( \Phi_{m,n} \) are projections over \( \mathfrak{A} \) that are both ample and co-ample.

Remark 4.3. Let \( \mathfrak{A} \) be a strict C*-algebra that satisfies the Voiculescu property. Then for \( n > k \geq 1 \) and \( p \in M_n(\mathfrak{A}) \) we have \( \Phi_{k,n}(p) \oplus 0_{n-k} \sim_{\text{MvN}} p \). Indeed, by (1) we have that \( \Phi_{k,n}(\Phi_{k,n}(p) \oplus 0_{n-k}) \sim_{\text{MvN}} \Phi_{k,n}(p) \). Therefore, \( \Phi_{k,n}(p) \oplus 0_{n-k} \sim_{\text{MvN}} p \).

Lemma 4.4. Suppose that \( \mathfrak{A} \) is a strict unital C*-algebra that satisfies the Voiculescu property. Then \([1]\) is the neutral element of \( K_0(\mathfrak{A}) \).

Proof. Recall that, by Lemma 3.1, given projections \( p,q \) over \( \mathfrak{A} \), we have that \([p] = [q]\) if and only if there exist \( m,n,n' \in \omega \) such that \( p \oplus 1_m \oplus 0_n \) and \( q \oplus 1_m \oplus 0_{n'} \) are Murray–von Neumann equivalent. Suppose that \( p \) is a projection over \( \mathfrak{A} \). As

\[
1 \oplus 0 \sim_{\text{MvN}} 1 \oplus 1,
\]

we have that

\[
(p \oplus 0) \oplus 1 \sim_{\text{MvN}} (p \oplus 1) \oplus 1.
\]

Therefore, \( p \) and \( p \oplus 1 \) represent the same element of \( K_0(\mathfrak{A}) \). Therefore, we have that

\[
[p] + [1] = [p \oplus 1] = [p] + [1].
\]

This shows that \([1]\) is the neutral element of \( K_0(\mathfrak{A}) \). \( \square \)

Lemma 4.5. Suppose that \( \mathfrak{A} \) is a strict unital C*-algebra that satisfies the Voiculescu property. If \( p,q \in M_n(\mathfrak{A}) \), then \( \Phi_{1,2n+2}(p \oplus (1-q) \oplus 1 \oplus 0) \in Z_0^A(\mathfrak{A}) \) and \([p] - [q] = \Phi_{1,2n+2}(p \oplus (1-q) \oplus 1 \oplus 0)\) in \( K_0(\mathfrak{A}) \).

Proof. If \( p \in M_n(\mathfrak{A}) \) and \( q \in M_n(\mathfrak{A}) \) are projections over \( \mathfrak{A} \), then we have that

\[
[p] - [q] = [p] + [1] - [q] = [p] + [1] - q
\]

\[
= [p] + [1-q] + [1]
\]

\[
= [p \oplus (1-q) \oplus 1 \oplus 0].
\]

As, by Remark 4.3,

\[
\Phi_{1,2n+2}(p \oplus (1-q) \oplus 1 \oplus 0) \oplus 0_{2n+1} \sim_{\text{MvN}} p \oplus (1-q) \oplus 1 \oplus 0
\]

we have

\[
[p] - [q] = [p \oplus (1-q) \oplus 1 \oplus 0] = \Phi_{1,2n+2}(p \oplus (1-q) \oplus 1 \oplus 0)\]

We now show that \( \Phi_{1,2n+2}(p \oplus (1-q) \oplus 1 \oplus 0) \) is ample and co-ample. Set \( r := p \oplus (1-q) \). We have by (4) of Definition 4.2 and since \( 1 \) is ample,

\[
\Phi_{1,2n+2}(r \oplus 1 \oplus 0) \oplus 1 = \Phi_{1,2n+2}(r \oplus 1 \oplus 0) \oplus \Phi_{1,1}(1)
\]

\[
\sim_{\text{MvN}} \Phi_{2,2n+3}(r \oplus 1 \oplus 0 \oplus 1)
\]

\[
\sim_{\text{MvN}} \Phi_{2,2n+3}(r \oplus 1 \oplus 0 \oplus 0)
\]

\[
\sim_{\text{MvN}} \Phi_{1,2n+2}(r \oplus 1 \oplus 0) \oplus \Phi_{1,1}(0)
\]

\[
\sim_{\text{MvN}} \Phi_{1,2n+2}(r \oplus 1 \oplus 0) \oplus 0.
\]

This shows that \( \Phi_{1,2n+2}(r \oplus 1 \oplus 0) \) is ample. Considering that

\[
1 - \Phi_{2n+2}(r \oplus 1 \oplus 0) = \Phi_{2n+2}((1-r) \oplus 0 \oplus 1) \sim_{\text{MvN}} \Phi_{2n+2}((1-r) \oplus 1 \oplus 0)
\]

we have by the above that \( 1 - \Phi_{2n+2}(r \oplus 1 \oplus 0) \) is also ample, and hence \( \Phi_{2n+2}(r \oplus 1 \oplus 0) \) is co-ample. \( \square \)

Lemma 4.6. Let \( \mathfrak{A} \) be a strict unital C*-algebra that satisfies the Voiculescu property. If \( p,q \in Z_0^A(\mathfrak{A}) \) are ample and co-ample projections, then the following assertions are equivalent:

(1) \( p,q \) represent the same element of \( K_0(\mathfrak{A}) \);
(2) \( p,q \) are Murray–von Neumann equivalent;
(3) \( p,q \) are unitary equivalent.
Proof. The implications $(3) \Rightarrow (2) \Rightarrow (1)$ hold in general.

$(1) \Rightarrow (3)$ Suppose that $p, q \in \mathbb{Z}_0^A (\mathfrak{A})$ are such that $[p_1] = [p_2]$. Then there exist $n, k \in \omega$ such that

$$p \oplus 1_n \oplus 0_k \sim_{\text{MvN}} q \oplus 1_n \oplus 0_k.$$  

Since $p, q$ are ample, we have

$$p \oplus 0_{n+k} \sim_{\text{MvN}} p \oplus 1_n \oplus 0_k \sim_{\text{MvN}} q \oplus 1_n \oplus 0_k \sim_{\text{MvN}} q \oplus 0_{n+k}.$$  

Therefore, we have

$$p \sim_{\text{MvN}} \Phi_{n+k+1}(p \oplus 0_{n+k}) \sim_{\text{MvN}} \Phi_{n+k+1}(q \oplus 0_{n+k}) \sim_{\text{MvN}} q.$$  

Using the fact that $p, q$ are co-ample, the same argument applied to $1 - p$ and $1 - q$ shows that $1 - p \sim_{\text{MvN}} 1 - q$. Thus, $p, q$ are unitarily equivalent. \hfill \Box

Lemma 4.7. Let $\mathfrak{A}$ be a strict $C^*$-algebra that satisfies the Voiculescu property. If $p, q \in \mathbb{Z}_0^A (\mathfrak{A})$, then $\Phi_{1,2} (p \oplus q) \in \mathbb{Z}_0^A (\mathfrak{A}).$

Proof. We need to show that $\Phi_{1,2} (p \oplus q)$ is ample and co-ample. We have that

$$\Phi_{1,2} (p \oplus q) \oplus 1 = \Phi_{1,2} (p \oplus q) \oplus \Phi_{1,1} (1)$$

$$= \Phi_{2,3} (p \oplus q \oplus 1)$$

$$\sim_{\text{MvN}} \Phi_{2,3} (p \oplus q \oplus 0)$$

$$\sim_{\text{MvN}} \Phi_{1,2} (p \oplus q) \oplus \Phi_{1,1} (0)$$

$$\sim_{\text{MvN}} \Phi_{1,2} (p \oplus q) \oplus 0.$$  

This shows that $\Phi_{1,2} (p \oplus q)$ is ample. The same argument applied to $1 - (p \oplus q) = (1 - p) \oplus (1 - q)$ shows that $\Phi_{1,2} (p \oplus q)$ is co-ample. \hfill \Box

Suppose that $\mathfrak{A}$ is a strict unital $C^*$-algebra satisfying Voiculescu’s property. Consider the unitary group $U (\mathfrak{A})$, which is a strictly closed subset of Ball ($\mathfrak{A}$) and hence a Polish group when endowed with the strict topology, and the standard Borel space $\mathbb{Z}_0^A (\mathfrak{A})$ of projections in $\mathfrak{A}$ that are both ample and co-ample, which by assumption is a Borel subset of $\mathfrak{A}$ invariant under unitary conjugation. We can consider the Borel action $U (\mathfrak{A}) \curvearrowright \mathbb{Z}_0^A (\mathfrak{A})$ by conjugation. We let $B_0^A (\mathfrak{A})$ be the corresponding orbit equivalence relation, and $K_0^A (\mathfrak{A}) := \mathbb{Z}_0^A (\mathfrak{A}) / B_0^A (\mathfrak{A})$ be the corresponding semidefinite set. For $p \in \mathbb{Z}_0^A (\mathfrak{A})$, we let $[p]_B_0^A (\mathfrak{A})$ be the $B_0^A (\mathfrak{A})$-class of $p$. The Borel functions $(p, q) \mapsto \Phi_{1,2} (p \oplus q)$ and $p \mapsto 1 - p$ induce a semidefinite group structure on $K_0^A (\mathfrak{A})$ with trivial element $[\Phi_{1,2} (1 \oplus 0)]_{B_0^A (\mathfrak{A})}$. Since $B_0^A (\mathfrak{A})$ is the orbit equivalence relation associated with a Borel action of a Polish group on $B_0^A (\mathfrak{A})$, we have that $K_0^A (\mathfrak{A})$ is in fact a definable group by Corollary 1.16.

The following proposition is an immediate consequence of the lemmas above.

Proposition 4.8. Suppose that $\mathfrak{A}$ is a strict $C^*$-algebra satisfying Voiculescu’s property. Adopt the notation from Definition 4.2. Then $K_0^A (\mathfrak{A})$ and $K_0 (\mathfrak{A})$ are definably isomorphic definable groups.

Proof. By the above remarks, $K_0^A (\mathfrak{A})$ is a definable group. Furthermore, the Borel functions $\mathbb{Z}_0^A (\mathfrak{A}) \rightarrow \mathbb{Z}_0^A (\mathfrak{A})$, $p \mapsto (p, 0)$ and $\mathbb{Z}_0^A (\mathfrak{A}) \rightarrow \mathbb{Z}_0^A (\mathfrak{A})$, $(p, q) \mapsto \Phi_{1,2n+1} (p \oplus (1 - q) \oplus 1 \oplus 0)$ induce mutually inverse definable isomorphisms between $K_0^A (\mathfrak{A})$ and $K_0 (\mathfrak{A})$. By Lemma 1.11, this shows that $K_0 (\mathfrak{A})$ is also a definable group, definably isomorphic to $K_0^A (\mathfrak{A})$. \hfill \Box

Suppose that $A$ is a separable, unital $C^*$-algebra, and $\rho : A \rightarrow B (H)$ is a nondegenerate ample representation. Define the corresponding Paschke dual $\mathfrak{D}_\rho (A)$ as in Example 2.31 to be the algebra

$$\mathfrak{D}_\rho (A) = \{ T \in B (H) : \forall a \in B (H), T \rho (a) \equiv \rho (a) T \mod K (H) \}.$$  

Then, $\mathfrak{D}_\rho (A)$ is a strict unital $C^*$-algebra, with respect to the strict topology on Ball ($\mathfrak{D}_\rho (a)$) induced by the seminorms

$$T \mapsto \max\{ \| TS \|, \| ST \| \}$$  

for $S \in K (H)$, and

$$T \mapsto \| T \rho (a) - \rho (a) T \|.$$
for \( a \in A \). We now observe that, as a consequence of Voiculescu’s theorem, \( \mathcal{D}_\rho(A) \) satisfies the Voiculescu property; see Definition 4.2.

Let \( \rho^n : A \to B(H^n) \) be the \( n \)-fold direct sum of \( \rho \). Notice that, under the usual identification of \( B(H^n) \) with \( M_n(B(H)) \), \( \mathcal{D}_{\rho^n}(A) \) corresponds to \( M_n(\mathcal{D}_\rho(A)) \). For \( k, n \geq 1 \), as both \( \rho^k \) and \( \rho^n \) are ample representations of \( A \), by Voiculescu’s theorem there exists a surjective isometry \( V_{k,n} : H^k \to H^n \) such that \( \text{Ad}(V) : B(H^n) \to B(H^k) \) satisfies \( \text{Ad}(\rho^k(a)) \equiv \rho^n(a) \mod K(H) \) for every \( a \in A \). This implies that \( \text{Ad}(V) \) induces a strict *-isomorphism \( \Phi_{k,n} := \text{Ad}(V) : M_n(\mathcal{D}_\rho(A)) \to M_k(\mathcal{D}_\rho(A)) \). By Voiculescu’s theorem, \( \Phi_{k,n} \) does not depend, up to unitary equivalence, from the choice of the surjective isometry \( V_{k,n} : H^k \to H^n \). Thus, we have that \( \mathcal{D}_\rho(A) \) satisfies (2), (3), and (4) of Definition 4.2.

Every projection \( P \in \mathcal{D}_\rho(A) \) defines a unital extension \( \varphi_P : A \to B(PH) \), \( a \mapsto \pi(P\rho(a)|_{PH}) \). By [HR00, Lemma 5.1.2], we have the following.

**Lemma 4.9.** Suppose that \( P, P_1, P_2 \in \mathcal{D}_\rho(A) \) are projections. The following assertions are equivalent:

1. \( \varphi_{P_1}, \varphi_{P_2} \) are equivalent extensions;
2. \( P_1, P_2 \) are Murray–von Neumann equivalent.

Furthermore, the following assertions are equivalent:

1. \( P \) is ample;
2. \( \varphi_P \) is injective.

From Lemma 4.9 it is easy to deduce the following.

**Proposition 4.10.** Suppose that \( A \) is a separable, unital \( C^* \)-algebra, \( \rho : A \to B(H) \) is a nondegenerate ample representation of \( A \), and \( \mathcal{D}_\rho(A) \) is the corresponding Paschke dual. Then \( \mathcal{D}_\rho(A) \) satisfies Voiculescu’s property.

**Proof.** By Lemma 4.9, a projection \( P \in \mathcal{D}_\rho(A) \) is ample if and only if \( \varphi_P \) is injective. This is equivalent to the assertion that, for every self-adjoint \( a \in A \), every \( S \in K(H) \), and every \( \varepsilon > 0 \),

\[
\|P\rho(a) - S\| > \|a\| - \varepsilon.
\]

By strict lower semicontinuity of the norm in \( \mathcal{D}_\rho(A) \), this is an open condition. This shows that the set of ample projections is a \( G_\delta \) set.

Since \( \rho \) is an ample representation, and \( \varphi_I = \rho \), we have that \( I \in \mathcal{D}_\rho(\mathfrak{A}) \) is ample.

Finally, we need to verify (1) of Definition 4.2. For \( n > k \), and projection \( P \in M_k(\mathcal{D}_\rho(A)) = \mathcal{D}_{\rho^k}(A) \) we have

\[
Q := \Phi_{k,n}(P \oplus 0_{n-k}) = V_{k,n}^*(P \oplus 0_{n-k})V_{k,n}.
\]

Thus, \( \varphi_Q \) is equivalent to \( \varphi_{P\oplus 0_{n-k}} = \varphi_P \). Hence, by Lemma 4.9, \( Q \) and \( P \) are Murray–von Neumann equivalent. \( \square \)

As a consequence of Proposition 4.10 and Proposition 4.8 we have the following.

**Proposition 4.11.** Suppose that \( A \) is a separable \( C^* \)-algebra, and \( \rho : A \to B(H) \) is a nondegenerate ample representation of \( A \). Then \( K_0(\mathcal{D}_\rho(A)) \) and \( K_0^A(\mathcal{D}_\rho(A)) \) are definably isomorphic definable groups.

Suppose that \( A, B \) are separable unital \( C^* \)-algebras. Recall that, if \( \rho, \rho' \) are linear maps from \( A \) to \( B(H) \), and \( V : H \to H \) is an isometry, then we write \( \rho' \lesssim_V \rho \) if \( \rho'(a) \equiv V^*\rho(a)V \mod K(H) \) for every \( a \in A \). Suppose that \( A, B \) are separable unital \( C^* \)-algebras, \( \alpha : A \to B \) is a unital *-homomorphism. Let \( \rho_A, \rho_B \) be ample representations of \( A, B \) on a Hilbert space \( H \). An isometry \( V_\alpha : H \to H \) covers \( \alpha \) if \( \rho_A \lesssim_{V_\alpha} \rho_B \circ \alpha \).

We have that for every unital *-homomorphism \( \alpha : A \to B \) there exists an isometry \( V_\alpha : H \to H \) that covers \( \alpha \) [HR00, Lemma 5.2.3]. Furthermore, \( \text{Ad}(V_\alpha) \) induces a strict unital *-homomorphism \( \text{Ad}(V_\alpha) : \mathcal{D}_{\rho_B}(B) \to \mathcal{D}_{\rho_A}(A) \). In turn, \( \text{Ad}(V_\alpha) : \mathcal{D}_{\rho_B}(B) \to \mathcal{D}_{\rho_A}(A) \) induces a definable group homomorphism \( K_0(\mathcal{D}_{\rho_B}(B)) \to K_0(\mathcal{D}_{\rho_A}(A)) \).

This definable group isomorphism only depends on \( \alpha \), and not on the choice of the isometry \( V_\alpha \) that covers \( \alpha \) [HR00, Lemma 5.2.4]. This gives a contravariant functor \( A \mapsto K_0(\mathcal{D}_{\rho_A}(A)) \) from the category of separable unital \( C^* \)-algebras to the category of definable groups. Similarly, one can regard \( A \mapsto K_0^A(\mathcal{D}_{\rho_A}(A)) \) as a contravariant functor, naturally isomorphic to \( K_0(\mathcal{D}_{\rho_A}(A)) \).

Using Proposition 4.11 we can show the following.

**Proposition 4.12.** Suppose that \( A \) is a separable, unital \( C^* \)-algebra, and \( \rho : A \to B(H) \) is a nondegenerate ample representation of \( A \). Then \( \text{Ext}(A)^{-1} \) is a definable group, which is naturally definably isomorphic to \( K_0(\mathcal{D}_\rho(A)) \).
Lemma 4.14. Consider the corresponding Paschke dual $\mathfrak{D}_\rho (A)$. An ample and co-ample projection $P \in \mathfrak{D}_\rho (A)$ determines an extension $\varphi_P \in \mathcal{E} (A)$ of $A$, defined as follows. Choose a linear isometry $V : H \to H$ such that $VV^* = P$, and define $\varphi_P (a) = V^* \rho (a) V$. (Notice that $V$ can be chosen in a Borel fashion from $P$.) The Borel function $P \mapsto \varphi_P$ induces a definable group isomorphism

$$\gamma : K_0^A (\mathfrak{D}_\rho (A)) \to \mathcal{E} (A) / \approx = \text{Ext} (A)^{-1} ;$$

see [HR00, Proposition 5.1.6].

We claim that the inverse group homomorphism $\gamma^{-1} : \text{Ext} (A)^{-1} \to K_0^A (\mathfrak{D}_\rho (A))$ is definable as well. Indeed, if $\varphi \in \mathcal{E} (A)$ then, by the Definable Voiculescu Theorem (Lemma 2.42), one can choose in a Borel way an isometry $V_\varphi : H \to H$ such that $\varphi \leq_{V_\varphi} \rho$. Thus, we have that

$$\varphi (a) \equiv V_\varphi^* \rho (a) V_\varphi \mod K (H)$$

for every $a \in A$. If $P := V_\varphi V_\varphi^*$ then we have that $P$ is a projection in $\mathfrak{D}_\rho (A)$ such that $\varphi_P$ is equivalent to $\varphi$. As $P$ is not necessarily ample and co-ample, one can replace $P$ with $\Phi_{1,3} (P \oplus 1 \oplus 0)$ to obtain an ample and co-ample projection $P_\varphi \in Z_0 (\mathfrak{D}_\rho (A))$ such that $\varphi_{P_\varphi}$ is equivalent to $\varphi$. Thus the Borel function $\varphi \mapsto P_\varphi$ is a lift of the inverse map $\gamma^{-1} : \text{Ext} (A)^{-1} \to K_0^A (\mathfrak{D}_\rho (A))$. This shows that $\gamma^{-1}$ is also definable. Therefore, $\gamma$ is a natural isomorphism in the category of semidefinable groups.

As $K_0^A (\mathfrak{D}_\rho (A))$ is in fact a definable group, this implies that $\text{Ext} (A)^{-1}$ is a definable group. Since $K_0^A (\mathfrak{D}_\rho (A))$ is naturally definably isomorphic to $K_0 (\mathfrak{D}_\rho (A))$, we have that $\text{Ext} (A)^{-1}$ is naturally definably isomorphic to $K_0 (\mathfrak{D}_\rho (A))$ as well.

4.3. Definable K-theory of commutants in the Calkin algebra. Suppose that $A$ is a unital separable C*-algebra, and $\rho$ is an ample representation of $A$ on the infinite-dimensional separable Hilbert space $H$. Then $\rho$ induces an ample representation $\rho^+$ of the unitization $A^+$ on $H \oplus H$, defined by $\rho^+ (a) = \rho (a) \oplus 0$ for $a \in A$. We define the Paschke dual algebra to be the strict unital C*-algebra

$$\mathfrak{D}_\rho (A) := \{ T \in B (H) : \forall a \in A, T \rho (a) \equiv \rho (a) T \mod K (H) \} .$$

We also have the Paschke dual algebra

$$\mathfrak{D}_{\rho^+} (A^+) = \{ T \in B (H \oplus H) : \forall a \in A, T \rho^+ (a) \equiv \rho^+ (a) T \mod K (H \oplus H) \} .$$

Notice that

$$\mathfrak{D}_{\rho^+} (A^+) = \begin{bmatrix} \mathfrak{D}_\rho (A) & K (H) \\ K (H) & B (H) \end{bmatrix} ;$$

see [HR00, Section 5.2].

Define $\mathfrak{J}$ to be the strict ideal

$$\begin{bmatrix} K (H) & K (H) \\ K (H) & B (H) \end{bmatrix}$$

of $\mathfrak{D}_{\rho^+} (A^+)$. Let also $\mathfrak{D}_{\rho^+} (A//A)$ be the strict ideal

$$\{ T \in \mathfrak{D}_{\rho^+} (A) : \forall a \in A, T \rho (a) \equiv 0 \mod K (H) \}$$

of $\mathfrak{D}_{\rho^+} (A^+)$.  

Lemma 4.13. The C*-algebras $\mathfrak{J}$ and $\mathfrak{D}_{\rho^+} (A//A)$ defined above have trivial K-theory.

Proof. The assertion about $\mathfrak{J}$ follows by considering the six-term exact sequence in K-theory associated with the pair $(\mathfrak{J}, M_2 (K (H)))$; see also [HR00, Exercise 4.10.9]. The assertion about $\mathfrak{D}_{\rho^+} (A//A)$ is [HR00, Lemma 5.4.1].

Lemma 4.14. Suppose that $A$ is a separable unital C*-algebra. For $i \in \{0, 1\}$:

1. The definable group homomorphism $K_i (\mathfrak{D}_{\rho^+} (A^+)) \to K_i (\mathfrak{D}_{\rho^+} (A^+) / \mathfrak{J})$ is an isomorphism in the category of semidefinable groups;

2. The strict *-homomorphism $\varphi : \mathfrak{D}_\rho (A) \to \mathfrak{D}_{\rho^+} (A^+), x \mapsto x \oplus 0$ induces an isomorphism

$$K_i (\mathfrak{D}_\rho (A) / K (H)) \to K_i (\mathfrak{D}_{\rho^+} (A^+) / \mathfrak{J})$$

in the category of semidefinable groups.

3. The map $K_0 (\mathfrak{D}_\rho (A)) \to K_0 (\mathfrak{D}_\rho (A) / K (H))$ is an isomorphism in the category of semidefinable groups;
(4) The subgroup $G$ of $K_1(\mathcal{D}_p(A) / K(H))$, consisting of the kernel of the (surjective) index map
\[ \partial_0 : K_1(\mathcal{D}_p(A) / K(H)) \rightarrow K_0(K(H)) \cong \mathbb{Z} \]
is definable, and the definable group homomorphism $K_1(\mathcal{D}_p(A)) \rightarrow K_1(\mathcal{D}_p(A) / K(H))$ induces an isomorphism $K_1(\mathcal{D}_p(A)) \rightarrow G$ in the category of semidefinable groups.

Proof. (1) Since $\mathfrak{J}$ has trivial K-theory, the group homomorphism $K_i(\mathcal{D}_{p+}(A^+)) \rightarrow K_i(\mathcal{D}_{p+}(A^+) / \mathfrak{J})$ is an isomorphism. We need to prove that the inverse group homomorphism $K_i(\mathcal{D}_{p+}(A^+) / \mathfrak{J}) \rightarrow K_i(\mathcal{D}_{p+}(A^+))$ is definable.

Consider first the case $i = 0$. Consider $p \in \mathcal{D}_p(A^+) / \mathfrak{J}$. Thus, $p \in \text{Proj}(\mathcal{M}_d(\mathcal{D}_{p+}(A)) / \mathcal{M}_d(\mathfrak{J}))$ for some $d \geq 1$. After replacing $\rho$ with $\rho^d$ we can assume that $d = 1$. Thus, $p \in \mathcal{D}_{p+}(A)$ is a mod $\mathfrak{J}$ projection. This implies that
\[ p = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \]
where $p_{11} \in \mathcal{D}_p(A)$ is a mod $K(H)$ projection. Then by Lemma 2.50, one can choose in a Borel fashion from $p$ a projection $q \in \mathcal{D}_p(A)$ such that $q \equiv p_{11} \mod K(H)$ and hence $q \oplus 0 \equiv p \mod \mathfrak{J}$.

We now consider the case when $i = 1$. Consider $q \in \mathcal{D}_p(A^+) / \mathfrak{J}$. Thus, $q \in U\left(\mathcal{M}_d(\mathcal{D}_{p+}(A)) / \mathcal{M}_d(\mathfrak{J})\right)$ for some $d \geq 1$. After replacing $\rho$ with $\rho^d$ we can assume that $d = 1$. Thus,
\[ u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \]
where $u_{11} \in \mathcal{D}_{p+}(A)$ is a mod $K(H)$ unitary. Let $v \in \mathcal{D}_p(A)$ be the partial isometry in the polar decomposition of $u_{11}$, which depends in a Borel fashion from $u_{11}$ by Lemma 2.53. Then we have that
\[ v := \begin{bmatrix} v & I - vv^* \\ I - v^*v & v^* \end{bmatrix} \in \mathcal{D}_{p+}(A^+) \]
is a unitary such that $v \equiv u \mod \mathfrak{J}$.

(2) Since $\varphi$ induces a $*$-isomorphism $\mathcal{D}_p(A) / K(H) \rightarrow \mathcal{D}_{p+}(A^+) / \mathfrak{J}$, it induces a definable group isomorphism $K_1(\mathcal{D}_p(A) / K(H)) \rightarrow K_1(\mathcal{D}_{p+}(A^+) / \mathfrak{J})$. It is immediate to verify that the inverse group homomorphism is also definable, as it is induced by the Borel function
\[ \begin{bmatrix} x_{11} \\ x_{21} \\ x_{22} \end{bmatrix} \mapsto x_{11}. \]

(3) and (4): Under the isomorphism $K_0(K(H)) \cong \mathbb{Z}$, the definable group homomorphism $K_1(\mathcal{D}_p(A) / K(H)) \rightarrow K_0(K(H)) \cong \mathbb{Z}$ maps each $T \in U(\mathcal{D}_p(A) / K(H))$ to its Fredholm index, and in particular it is surjective. As the Fredholm index is given by a Borel map, and $K_1(K(H)) = \{0\}$, it follows from the six-term exact sequence in K-theory associated with $\mathcal{D}_p(A)$ and $K(H)$ that $K_0(\mathcal{D}_p(A) / K(H)) \cong \mathbb{Z}$ is a definable group isomorphism, and that $K_1(\mathcal{D}_p(A)) \rightarrow K_1(\mathcal{D}_p(A) / K(H))$ is an injective definable group homomorphism with range equal to $G$. The inverse $K_0(\mathcal{D}_p(A) / K(H)) \rightarrow K_0(\mathcal{D}_p(A))$ is definable by Lemma 2.50. The inverse $G \rightarrow K_1(\mathcal{D}_p(A))$ is definable by Lemma 2.53, considering that given $T \in U(B(H) / K(H))$ such that index$(T) = 0$, then the partial isometry $U$ in the polar decomposition of $T$ is a unitary such that $U \equiv T \mod K(H)$. \hfill \Box

Corollary 4.15. Suppose that $A$ is a separable unital $C^*$-algebra, and $\rho$ is an ample representation of $A$. Then $\text{Ext}(A)^{-1}$, $K_0(\mathcal{D}_p(A), F_0(\mathcal{D}_p(A) / K(H)), K_0(\mathcal{D}_{p+}(A^+) / \mathfrak{J})$, and $K_0(\mathcal{D}_{p+}(A^+))$ are definably isomorphic definable groups.

Proof. This is a consequence of Lemma 4.14, Proposition 4.12, and Corollary 1.14. \hfill \Box
Lemma 4.16. Suppose that $A$ is a separable unital $\mathcal{C}^*$-algebra. Then $K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$ is a definable group, definably isomorphic to the definable group $K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H))$.

Proof. A definable group isomorphism

$$K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H)) \to K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$$

is defined in [Pas81, Theorem 6], as follows. Let $p$ be a mod $K(H)$ projection in $\mathfrak{D}_{\rho_A}(\Sigma A)$. Then $f(p) := pU + (1-p) \in \mathfrak{D}_{\rho_A}(A)$ is a mod $K(H)$ unitary. A similar definition for mod $K(H)$ projections over $\mathfrak{D}_{\rho_A}(\Sigma A)$ defines a Borel function $Z_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H)) \to Z_1(\mathfrak{D}_{\rho_A}(A)/K(H))$. It is proved in [Pas81, Theorem 6] that this Borel function induces an isomorphism $K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H)) \to K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$.

By Corollary 4.15 we have that $K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H))$ is a definable group. Thus, by Proposition 4.14 we have that $K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$ is a definable group as well. □

Proposition 4.17. Suppose that $A$ is a separable unital $\mathcal{C}^*$-algebra, and $\rho$ is an ample representation of $A$. Then $\text{Ext}(\Sigma A)^{-1}$, $K_1(\mathfrak{D}_\rho(A)/K(H))$, $K_1(\mathfrak{D}_{\rho^+}(A^+)/\mathfrak{D})$, and $K_1(\mathfrak{D}_{\rho^+}(A^+))$ are definably isomorphic definable groups.

Proof. Let $\rho_A$ and $\rho_B$ be the ample representations of $\Sigma A$ and $A$, respectively, as in Lemma 4.16. Then by Corollary 4.15, $\text{Ext}(\Sigma A)^{-1}$ and $K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H))$ are definably isomorphic groups. By Lemma 4.16, $K_0(\mathfrak{D}_{\rho_A}(\Sigma A)/K(H))$ and $K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$ are definably isomorphic definable groups. By Voiculescu’s theorem, $\mathfrak{D}_{\rho_A}(A)/K(H)$ and $\mathfrak{D}_{\rho}(A)/K(H)$ are isomorphic in the category of unital $\mathcal{C}^*$-algebras with a strict cover; see Lemma 2.40. In particular, $K_1(\mathfrak{D}_{\rho_A}(A)/K(H))$ and $K_1(\mathfrak{D}_{\rho}(A)/K(H))$ are isomorphic in the category of semidefines groups. From this and Corollary 1.14, it follows that $K_1(\mathfrak{D}_{\rho}(A)/K(H))$ is a definable group. Finally, $K_1(\mathfrak{D}_{\rho^+}(A^+)/\mathfrak{D})$ and $K_1(\mathfrak{D}_{\rho^+}(A^+))$ are definable groups, definably isomorphic to $K_1(\mathfrak{D}_{\rho}(A)/K(H))$ by Lemma 4.14 and Corollary 1.14 again. □

4.4. Definable K-homology. Suppose that $A$ is a separable $\mathcal{C}^*$-algebra. Fix an ample representation $\rho^+$ of $A^+$, and define $\mathfrak{D}(A) := \mathfrak{D}_{\rho^+}(A^+)$. The K-homology groups of $A$ are the definable groups

$$K_1(A) := K_0(\mathfrak{D}(A)) \cong \text{Ext}(A^+)^{-1}$$

and

$$K^0(A) := K_1(\mathfrak{D}(A)) \cong \text{Ext}((SA)^+)^{-1};$$

see [HR00, Definition 5.2.7]. By Proposition 4.12, $K^p$ for $p \in \{0, 1\}$ is a contravariant functor from the category of separable $\mathcal{C}^*$-algebras to the category of definable abelian groups.

When $A$ is a separable unital $\mathcal{C}^*$-algebra, one can also define the reduced K-homology groups by considering an ample representation $\rho$ of $A$ (rather than $A^+$) and the corresponding Pachke dual algebra $\mathfrak{D}(A)$ and set

$$\tilde{K}_1(A) := K_0(\mathfrak{D}(A)) \cong \text{Ext}(A)^{-1}$$

and

$$\tilde{K}^0(A) := K_1(\mathfrak{D}(A));$$

see [HR00, Definition 5.2.1].

Suppose now that $A$ is a separable $\mathcal{C}^*$-algebra, and $J$ is a closed two-sided ideal of $A$. Fix as above an ample representation $\rho^+$ of $A^+$, define $\mathfrak{D}(A) := \mathfrak{D}_{\rho^+}(A^+)$ as above, and set $\mathfrak{D}(A//J)$ to be the strict ideal

$$\{T \in \mathfrak{D}(A) : \forall a \in J, T\rho(a) \equiv 0 \mod K(H)\}$$

of $\mathfrak{D}(A)$.

Lemma 4.18. Suppose that $A$ is a separable $\mathcal{C}^*$-algebra, and $i \in \{0, 1\}$. Then $K_i(\mathfrak{D}(A)//(A//A))$ is a definable group, definably isomorphic to $\tilde{K}^{1-i}(A)$.

Proof. By Lemma 4.13, $\mathfrak{D}(A//A)$ has trivial K-theory. Thus, by the six-term exact sequence in K-theory, the definable group homomorphism $K_i(\mathfrak{D}(A^+)) \to K_i(\mathfrak{D}(A^+)//\mathfrak{D}(A//A))$ is an isomorphism. Since $K_i(\mathfrak{D}(A^+))$ is a definable group by Proposition 4.17 and Corollary 4.15, the conclusion follows from Corollary 1.14.

Lemma 4.19. Suppose that $A$ is a separable $\mathcal{C}^*$-algebra, and $J$ is a closed two-sided ideal of $A$. Then the inclusion map $\mathfrak{D}(A) \subseteq \mathfrak{D}(J^+)$ induces an isomorphism $\mathfrak{D}(A)//\mathfrak{D}(J//J) \to \mathfrak{D}(J)//\mathfrak{D}(J//J)$ in the category of separable unital $\mathcal{C}^*$-algebras with a strict cover.
Proof. We identify \( A^+ \) with its image inside \( B(H) \) under \( \rho^+ \). It follows from the definition that \( \mathfrak{D}(A/\mathcal{J}) = \mathfrak{D}(J/\mathcal{J})/\mathfrak{D}(A) \). Thus, the inclusion map \( \mathfrak{D}(A) \subset \mathfrak{D}(J) \) induces a definable injective unital \(*\)-homomorphism \( \mathfrak{D}(A)/\mathfrak{D}(A/\mathcal{J}) \to \mathfrak{D}(J)/\mathfrak{D}(J/\mathcal{J}) \), which is in fact onto [HR00, Theorem 5.4.5]. It remains to prove that the inverse unital \(*\)-isomorphism \( \mathfrak{D}(J)/\mathfrak{D}(J/\mathcal{J}) \to \mathfrak{D}(A)/\mathfrak{D}(A/\mathcal{J}) \) is also definable. This amounts at noticing that the proof of [HR00, Theorem 5.4.5] via Kasparov’s Technical Theorem [HR00, Theorem 3.8.1] can be used to describe a Borel lift \( \mathfrak{D}(J) \to \mathfrak{D}(A) \) of the unital \(*\)-isomorphism \( \mathfrak{D}(J)/\mathfrak{D}(J/\mathcal{J}) \to \mathfrak{D}(A)/\mathfrak{D}(A/\mathcal{J}) \).

For \( T \in \mathfrak{D}(J) \) let \( E(T) \) be closed linear span of \( \{a, T \mid [a, T] : a \in A\} \). Fix a dense sequence \((j_m)\) in \( \mathcal{D}(J) \), a dense sequence \((a_m)\) in \( \mathcal{D}(A) \), and a dense sequence \((b_m)\) in \( \mathcal{D}(K(H)) \). Notice that, for \( j \in J \), and \( a \in A \), and \( T \in \mathfrak{D}(J) \), we have that

\[
j[a, T] = jaT - jTa = jaT - Tja = 0 \mod K(H).
\]

Fix an approximate unit \((u_n)\) for \( J \) such that, for \( m \leq n \),

\[
\|u_n a_m - a_m u_n\| \leq 2^{-n}.
\]

Fix an approximate unit \((w_n)_{n \in \omega}\) for \( K(H) \) such that, if we set

\[
d_n := (w_n - w_{n-1})^{1/2}.
\]

then we have, for \( m_1, m_2, m \leq n \),

\[
\|d_n b_m\| \leq 2^{-n},
\]

\[
\|d_n j_m - j_m d_n\| \leq 2^{-n},
\]

\[
\|d_n a_m - a_m d_n\| \leq 2^{-n}.
\]

One can see that such an approximate unit for \( K(H) \) exists by considering a approximate unit for \( K(H) \) that is quasicentral for \( J \) and \( A \) [HR00, Theorem 3.2.6] and then a suitable subsequence via a diagonal argument.

Fix \( T \in \mathfrak{D}(J) \). Then using the Lusin-Novikov Selection Theorem [Kec95, Theorem 18.10] and [HR00, Theorem 3.2.6] one can see that one can recursively define, for \( n \in \omega, \ell^T \in \omega, k^T_{n,0}, \ldots, k^T_{n,\ell^T_n} \geq n, \) and \( t^T_{n,0}, \ldots, t^T_{n,\ell^T_n} \in [0, 1] \cap Q \) that depend in a Borel fashion from \( T \) such that, setting

\[
w_n^T := t^T_{n,0} w_{k^T_{n,0}} + \cdots + t^T_{n,\ell^T_n} w_{k^T_{n,\ell^T_n}},
\]

and

\[
d_n^T := (w_n^T - w_{n-1}^T)^{1/2},
\]

one has that \( w_n^T, d_n^T \in K(H) \) depend in a Borel fashion from \( T \) and, for \( m_1, m_2, m \leq n \),

\[
\|d_n^T [a_m, T] - [a_m, T] d_n^T\| \leq 2^{-n},
\]

\[
\|d_n^T j_m\| \leq 2^{-n},
\]

\[
\|d_n^T a_m\| \leq 2^{-n}.
\]

Furthermore, we also have from the choice of \((w_n)\) that, for \( m \leq n \),

\[
\|d_n^T b_m\| \leq 2^{-n},
\]

\[
\|d_n^T j_m - j_m d_n^T\| \leq 2^{-n},
\]

\[
\|d_n^T a_m - a_m d_n^T\| \leq 2^{-n}.
\]

As in the proof of Kasparov’s Technical Theorem [HR00, Theorem 3.8.1], one has that

\[
\sum_{n \in \omega} d_n^T u_n d_n^T
\]

converges in the strong-* topology to some positive element \( X_T \in \mathfrak{D}(B(H)) \). Furthermore, we have that

\[
(1 - X_T) j \equiv 0 \mod K(H)
\]

\[
X_T [T, a] \equiv 0 \mod K(H)
\]

\[
[X_T, a] \equiv 0 \mod K(H).
\]
for $j \in J$ and $a \in A$. Thus, $X_T T \in \mathcal{D}(A)$ and $(1 - X_T) T \in \mathcal{D}(J//J)$. Indeed, if $a \in A$ then we have that
\[
[X_T, a] = X_T T a - a X_T T = X_T T a - X_T a T + X_T a T - a X_T T = X_T [T, a] + [X_T, a] T \equiv 0 \mod K(H).
\]

If $j \in J$ then we have that
\[
(1 - X_T) T_j \equiv (1 - X_T) j T \equiv 0 \mod K(H).
\]

We have that the function $\mathcal{D}(J) \to K(H)$, $T \mapsto [X_T, a] T$ is Borel, being the pointwise limit of Borel functions
\[
T \mapsto \sum_{k=1}^{n} [d_k^T u_k d_k^T, a].
\]

Thus, the function $\mathcal{D}(J) \to K(H)$, $T \mapsto [X_T, a] T$ is Borel as well. For the same reasons, the function $\mathcal{D}(J) \to K(H)$, $T \mapsto X_T [T, a] T$ is Borel, and hence the function $\mathcal{D}(J) \to K(H)$, $T \mapsto [X_T, a] = X_T [T, a] + [X_T, a] T$ is Borel. A similar argument shows that the function $\mathcal{D}(J) \to K(H)$, $T \mapsto X_T T b$ is Borel for $b \in K(H)$. Therefore, the function $\mathcal{D}(J) \to \mathcal{D}(A), T \mapsto X_T T$ is Borel. Since $T - X_T T = (1 - X_T) T \in \mathcal{D}(J//J)$, we have that $T \mapsto X_T T$ is a lift of the unital *-isomorphism $\mathcal{D}(J)/\mathcal{D}(J//J) \to \mathcal{D}(A)/\mathcal{D}(A//J)$. This concludes the proof.

**Corollary 4.20.** Suppose that $A$ is a separable C*-algebra, and $J$ is a closed two-sided ideal of $A$. Fix $i \in \{0, 1\}$. Then $K_i(\mathcal{D}(A^+)/\mathcal{D}(A//J))$ is a definable group, definably isomorphic to $K_i(\mathcal{D}(J^+)/\mathcal{D}(J//J))$.

**Proof.** By Lemma 4.18, $K_i(\mathcal{D}(J^+)/\mathcal{D}(J//J))$ is a definable group. By Lemma 4.19, $K_i(\mathcal{D}(A^+)/\mathcal{D}(A//J))$ is isomorphic to $K_i(\mathcal{D}(J^+)/\mathcal{D}(J//J))$ in the category of semidefinable groups. Whence, the conclusion follows from Corollary 1.14.

Suppose as above that $A$ is a separable C*-algebra, and $J$ is a closed two-sided ideal of $A$. One defines for $i \in \{0, 1\}$ the relative K-homology groups
\[
K^i(A, A//J) := K_{1-i}(\mathcal{D}(A)/\mathcal{D}(A//J));
\]
see [HR00, Definition 5.3.4]. These are definable groups by Corollary 4.20. The assignment $(A, J) \mapsto K^i(A, A//J)$ gives a contravariant functor from the category of separable C*-pairs to the category of definable groups. Here, a separable C*-pair is a pair $(A, I)$ where $A$ is a separable C*-algebra and $I$ is a closed two-sided ideal of $A$. A morphism $(A, I) \to (B, J)$ of separable C*-pairs is a *-homomorphism $A \to B$ that maps $I$ to $J$. If $\alpha : (A, J) \to (B, I)$ is a morphism of C*-pairs, and $V : H \to H$ is an isometry that covers $\alpha^+ : A^+ \to B^+$, then we have that the corresponding strict unital *-homomorphism $\text{Ad}(V) : \mathcal{D}(B^+) \to \mathcal{D}(A^+)$ maps $\mathcal{D}(B//I)$ to $\mathcal{D}(A//I)$. Thus, it induces a definable unital *-homomorphism $\mathcal{D}(B)/\mathcal{D}(B//I) \to \mathcal{D}(A)/\mathcal{D}(A//I)$, and a definable group homomorphisms $K^i(B, B//I) \to K^i(A, A//J)$.

The natural definable isomorphisms
\[
K^i(A, A//J) = K_{1-i}(\mathcal{D}(A^+)/\mathcal{D}(A//J)) \cong K_{1-i}(\mathcal{D}(J)/\mathcal{D}(J//J)) \cong K_{1-i}(\mathcal{D}(J)) = K^i(J)
\]
from Lemma 4.19 and Lemma 4.14 give a natural definable isomorphism $K^i(A, A//J) \cong K^i(J)$ called the excision isomorphism; see [HR00, Theorem 5.4.5].

Suppose that $(A, J)$ is a separable C*-pair. We say that $(A, J)$ is semi-split if the short exact sequence
\[
0 \to J \to A \to A//J \to 0
\]
is semi-split in the sense of [HR00, Definition 5.3.6], namely the quotient map $A^+ \to A^+/J$ admits a ucp right inverse. In this case, if $V : H \to H$ is a linear isometry that covers the quotient map $A \to A//J$, then the unital *-homomorphism $\text{Ad}(V) : \mathcal{D}(A//J) \to \mathcal{D}(A//J)$ induces a natural definable isomorphism in K-theory [HR00, Proposition 5.3.7]. By the Choi–Effros lifting theorem [CE76], every nuclear separable C*-pair is semi-split.

Thus, under assumption that $(A, J)$ is semi-split, from the six-term exact sequence in K-theory
5.1. **Graded vector spaces and algebras.** Let $V$ be a vector space. A *grading* of $V$ is a decomposition $V = V^+ \oplus V^-$ as a direct sum of two subspaces, called the positive and negative part of $V$. The corresponding grading operator $\gamma_V$ is the involution of $V$ whose eigenvalues for 1 and $-1$ are $V^+$ and $V^-$, respectively. A vector space endowed with a grading is a *graded vector space*. The *opposite* of the graded vector space $V$ is the graded vector space $V^{op}$ obtained from $V$ by interchanging the positive and the negative part. An endomorphism $T$ of $V$ is *even* if $T(V^+) \subseteq V^+$ and $T(V^-) \subseteq V^-$ or, equivalently, $\gamma_V T = T \gamma_V$; it is *odd* if $T(V^+) \subseteq V^-$ and $T(V^-) \subseteq V^+$ or, equivalently, $\gamma_V T = -T \gamma_V$.

A *graded Hilbert space* is Hilbert space endowed with a grading whose positive and negative parts are closed orthogonal subspaces or, equivalently, the grading operator is a self-adjoint unitary.

A *graded algebra* is a complex algebra that is also a graded vector space, and such that:

$$A^+ \cdot A^+ \cup A^- \cdot A^- \subseteq A^+$$

and

$$A^+ \cdot A^- \cup A^- \cdot A^+ \subseteq A^-$$

or, equivalently, the grading operator $\gamma_A$ is an algebra automorphism of $A$. The elements of $A^+$ are *even* elements of the algebra, and the elements of $A^-$ are called *odd* elements of the algebra. An element is *homogeneous* if it is either even or odd. The degree $\partial a$ of an even element $a$ is 0, while the degree $\partial a$ of an odd element $a$ is 1. The graded commutator of elements of $A$ is defined for homogeneous elements by

$$[a, a'] = aa' - (-1)^{\partial a \partial a'} a'a$$

and extended by linearity.

A *graded C*-algebra* $A$ is a C*-algebra that is also a graded algebra and such that $A^+$ and $A^-$ are closed self-adjoint subspaces or, equivalently, the grading operator $\gamma_A$ is a C*-algebra automorphism of $A$.

**Example 5.1.** Suppose that $V$ is a graded vector space. Then the algebra $\text{End}(V)$ of endomorphisms of $V$ is a graded algebra, with $\text{End}(V)^+$ equal to the set of even endomorphisms of $V$, and $\text{End}(V)^-$ is the set of odd endomorphisms of $V$.

**Example 5.2.** If $H$ is a graded Hilbert space, then $B(H) \subseteq \text{End}(H)$ is a graded C*-algebra.
Example 5.3. Fix $n \geq 1$. Define $C_n$ to be the graded complex unital $*$-algebra generated by $n$ odd operators $e_1, \ldots, e_n$ such that, for distinct $i, j \in \{1, 2, \ldots, n\}$,
\[ e_i e_j + e_j e_i = 0, \]
\[ e_i^2 = -1, \]
\[ e_i^* = -e_i. \]
As a complex vector space, $C_n$ has dimension $2^n$, where monomials $e_{i_1} \cdots e_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq n$ and $0 \leq k \leq n$ comprise a basis. Declaring these monomials to be orthogonal defines an inner product on $C_n$. The left regular representation of $C_n$ on the Hilbert space $C_n$ turns $C_n$ into a graded $C^*$-algebra.

Suppose that $V_1$ and $V_2$ are graded vector spaces. The graded tensor product $V := V_1 \hat{\otimes} V_2$ is the tensor product of $V_1$ and $V_2$ equipped with the grading operator $\gamma_V := \gamma_{V_1} \otimes \gamma_{V_2}$. Thus, we have that
\[ V^+ = (V_1^+ \otimes V_2^+) \oplus (V_1^- \otimes V_2^-) \]
\[ V^- = (V_1^+ \otimes V_2^-) \oplus (V_1^- \otimes V_2^+). \]
If $A_1$ and $A_2$ are graded algebras, then the graded tensor product $A := A_1 \hat{\otimes} A_2$ (as graded vector spaces) is a graded algebra with respect to the multiplication operation defined on homogeneous elementary tensors by
\[ (a_1 \hat{\otimes} a_2) (a_1' \hat{\otimes} a_2') = (-1)^{\beta a_2 \partial a_1} (a_1 a_1' \hat{\otimes} a_2 a_2'). \]
When $V_1, V_2$ are graded vector spaces, then there is canonical inclusion
\[ \text{End} (V_1) \hat{\otimes} \text{End} (V_2) \subseteq \text{End} (V_1 \hat{\otimes} V_2) \]
obtained by setting, for homogeneous $T_i \in \text{End} (V_i)$ and $v_i \in V_i$,
\[ (T_1 \hat{\otimes} T_2) (v_1 \hat{\otimes} v_2) = (-1)^{\partial v_1 \partial T_2} (T_1 v_1 \otimes T_2 v_2). \]
We have that $\text{End} (V_1) \hat{\otimes} \text{End} (V_2) = \text{End} (V_1 \hat{\otimes} V_2)$ when $V_1, V_2$ are finite-dimensional.

Example 5.4. There is a canonical isomorphism $C_m \hat{\otimes} C_n \cong C_{m+n}$, obtained by mapping $e_i \hat{\otimes} 1$ to $e_i$ and $1 \hat{\otimes} e_j$ to $e_{m+j}$ for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

Fix $p \geq 0$. A $p$-graded Hilbert space is a graded Hilbert space endowed with $p$ odd operators $\varepsilon_1, \ldots, \varepsilon_p$ such that $\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$, $\varepsilon_i^2 = -1$, and $\varepsilon_i^* = -\varepsilon_j$ for distinct $i, j \in \{1, 2, \ldots, p\}$. Equivalently, a $p$-graded Hilbert space can be thought of as a graded right module over $C_p$, where one sets
\[ x e_i := \varepsilon_i (x) \]
for $i \in \{1, 2, \ldots, p\}$ and $x \in H$. A $0$-graded Hilbert space is simply a graded Hilbert space. By convention, a $(-1)$-graded Hilbert space is an ungraded Hilbert space. Suppose that $H_0$ and $H_1$ are $p$-graded Hilbert spaces. A $p$-graded bounded linear map $H_0 \to H_1$ is a bounded linear map that is also a right $C_p$-module map.

Example 5.5. Suppose that $H$ is a $p$-graded Hilbert space. Then $H^{op}$ is $p$-graded, where $\varepsilon_i^{H^{op}} = -\varepsilon_i^H$ for $1 \leq i \leq n$.

Example 5.6. Suppose that $H, H'$ are $p$-graded Hilbert spaces. Then $H \oplus H'$ is $p$-graded, where $\varepsilon_i^{H \oplus H'} = \varepsilon_i^H \oplus \varepsilon_i^{H'}$.

Example 5.7. If $H_1$ is $p_1$-graded and $H_2$ is $p_2$-graded, then considering the isomorphism $C_{p_1} \hat{\otimes} C_{p_2} \cong C_{p_1+p_2}$, and the inclusion $B (H_1 \hat{\otimes} H_2) \subseteq B (H_1) \otimes B (H_2)$, we have that $H_1 \hat{\otimes} H_2$ is $(p_1 + p_2)$-graded. Explicitly, for $i \in \{1, 2, \ldots, p_1\}$ and $j \in \{1, 2, \ldots, p_2\}$ we have that
\[ (x \otimes y) e_i = (-1)^{\partial y} (x e_i \otimes y) \]
\[ (x \otimes y) e_{p_1+j} = x \otimes ye_j. \]
A straightforward verification shows the following; see [HR00, Proposition A.3.4].

Proposition 5.8. For $p \geq 0$, the categories of $p$-multigraded and $(p+2)$-multigraded Hilbert spaces are equivalent. The categories of Hilbert spaces and linear maps and 1-graded Hilbert spaces and even 1-graded linear maps are equivalent.
5.2. Fredholm modules. Suppose that $A$ is a separable C*-algebra. We now recall the definition of Fredholm module over $A$; see [HR00, Definition 8.1.1]. For each dimension $d \in \omega \cup \{\aleph_0\}$ fix a Hilbert space $H_d$ of dimension $d$.

**Definition 5.9.** Suppose that $A$ is a separable C*-algebra. A Fredholm module over $A$ is a triple $(F, \rho, H)$ such that:

- $H$ is a separable Hilbert space $H_d$ for some $d \in \omega \cup \{\aleph_0\}$;
- $\rho : A \to B(H)$ is a *-homomorphism;
- $F \in B(H)$ satisfies $(F^2 - 1) \rho (a) \equiv (F - F^*) \rho (a) \equiv [F, \rho (a)] \equiv 0 \mod K(H)$ for every $a \in A$.

**Remark 5.10.** Recall that, if $H$ is a separable Hilbert space, then $B(H)$ is a standard Borel space with respect to the Borel structure induced by the strong-* topology on $\text{Ball}(B(H))$. Similarly, the Banach space $L(A, B(H))$ of bounded linear maps $A \to B(H)$ is a standard Borel space when endowed with the Borel structure induced by the topology of pointwise strong-* convergence on $\text{Ball}(L(A, B(H)))$. It easily follows that the set $F_{-1}(A)$ of Fredholm modules over $A$ is endowed with a standard Borel structure.

The definition of graded Fredholm module is similar, where one replaces Hilbert spaces with graded Hilbert spaces.

**Definition 5.11.** Suppose that $A$ is a separable C*-algebra. A graded Fredholm module over $A$ is a triple $(F, \rho, H)$ such that:

- $H$ is a separable graded Hilbert space of the form $(H_d, \gamma)$ for some $d \in \omega \cup \{\aleph_0\}$ and some grading operator $\gamma$ on $H_d$;
- $\rho : A \to B(H)$ is a *-homomorphism such that, for every $a \in A$, $\rho (a) \in B(H)^+$ is even, and hence $\rho = \rho^+ \oplus \rho^-$ for some representations $\rho^\pm$ of $A$ on $H^\pm$, where we regard $B(H)$ as a graded C*-algebra;
- $F \in B(H)$ is an odd operator that satisfies, for every $a \in A$, $(F^2 - 1) \rho (a) \equiv (F - F^*) \rho (a) \equiv [F, \rho (a)] \equiv 0 \mod K(H)$.

**Remark 5.12.** Again, we have that the set $F_0(A)$ of Fredholm modules over $A$ is endowed with a standard Borel structure.

The notions of graded and ungraded Fredholm modules can be recognized as particular instances (for $p = 0$ and $p = -1$, respectively) of the notion of $p$-graded Fredholm module; see [HR00, Definition 8.1.11].

**Definition 5.13.** Suppose that $A$ is a separable C*-algebra. A $p$-multigraded Fredholm module over $A$ is a triple $(F, \rho, H)$ such that:

- $H$ is a separable $p$-multigraded Hilbert space $H$ of the form $(H_d, \gamma, \varepsilon_1, \ldots, \varepsilon_p)$ for $d \in \omega \cup \{\aleph_0\}$, grading operator $\gamma$ on $H_d$, and odd operators $\varepsilon_1, \ldots, \varepsilon_p$ on $(H_d, \gamma)$;
- a *-homomorphism $\rho : A \to B(H)$ such that, for every $a \in A$, $\rho (a)$ is an even $p$-multigraded operator on $H$;
- $F \in B(H)$ is an odd $p$-multigraded operator on $H$ such that, for every $a \in A$, $(F^2 - 1) \rho (a) \equiv (F - F^*) \rho (a) \equiv [F, \rho (a)] \equiv 0 \mod K(H)$.

**Remark 5.14.** As in the cases $p = 0$ and $p = -1$, the set $F_p(A)$ of $p$-multigraded Fredholm modules over $A$ is a standard Borel space.

We recall the notion of degenerate $p$-multigraded Fredholm module; see [HR00, Definition 8.2.7].

**Definition 5.15.** Suppose that $A$ is a separable C*-algebra. A $p$-multigraded Fredholm module $(F, \rho, H)$ over $A$ is degenerate if $(F^2 - 1) \rho (a) = (F - F^*) \rho (a) = [F, \rho (a)]$ for every $a \in A$.

It is clear that the set $D_p(A)$ of degenerate $p$-multigraded Fredholm modules is a Borel subset of $F_p(A)$.

Given $p$-multigraded Fredholm modules $x = (\rho, H, F)$ and $x' = (\rho', H', F')$ over $A$, their sum is the $p$-multigraded Fredholm module $x \oplus x' = (\rho \oplus \rho', H \oplus H', F \oplus F')$. The opposite of $x$ is the $p$-multigraded Fredholm module $x^{op} = (\rho, H^{op}, -F)$. The sum and opposite define Borel functions $F_p(A) \times F_p(A) \to F_p(A)$, $(x, x') \mapsto x \oplus x'$ and $F_p(A) \to F_p(A)$, $x \mapsto x^{op}$.

Let $A$ be a separable C*-algebra, and fix $p \geq -1$. Suppose that $(\rho, H, F)$ and $(\rho', H', F')$ are $p$-multigraded Fredholm modules. Then $(\rho, H, F)$ and $(\rho, H', F')$ are:
• **unitarily equivalent** if there exists an even $p$-multigraded unitary linear map $U : H' \to H$ such that $F' = \text{Ad}(U)(F)$ and $\rho' = \text{Ad}(U) \circ \rho$ [HR00, Definition 8.2.1];

• **operator homotopic** if $\rho = \rho'$, $H = H'$, and there exists a norm-continuous path $(F_t)_{t \in [0,1]}$ in $B(H)$ such that, for every $t \in [0,1]$, $(\rho, H, F_t)$ is a $p$-multigraded Fredholm module over $A$ [HR00, Definition 8.2.2].

The notion of **stable homotopy** is defined in terms of unitary equivalence and operator homotopy; see [HR00, Proposition 8.2.12].

**Definition 5.16.** Suppose that $A$ is a separable $C^*$-algebra, and $p \geq -1$. The relation $B_p(A)$ of stable homotopy of $p$-multigraded Fredholm modules over $A$ is the relation defined by setting $x_{B_p(A)}(x')$ if and only if there exists a degenerate $p$-multigraded Fredholm module $x_0$ over $A$ such that $x \oplus x_0$ and $x' \oplus x_0$ are unitarily equivalent to a pair of operator homotopic $p$-multigraded Fredholm modules over $A$.

One has that $B_p(A)$ is an equivalence relation on $F_p(A)$; see [HR00, Proposition 8.2.12]. Furthermore, $B_p(A)$ is an analytic equivalence relation, as it follows easily from the definition and the fact that the set $D_p(A)$ of degenerate $p$-multigraded Fredholm modules is a Borel subset of $F_p(A)$, and the set of norm-continuous paths in $B(H)$ is a Borel subset of the $C^*$-algebra $C_b([0,1], B(H)) = M(C([0,1], K(H)))$ of strictly continuous bounded functions $[0,1] \to B(H)$ by Corollary 2.26.

5.3. **The Kasparov K-homology groups.** We now recall the definition of the Kasparov K-homology groups in terms of Fredholm modules; see [HR00, Definition 8.2.5 and Proposition 8.2.12].

**Definition 5.17.** Let $A$ be a separable $C^*$-algebra, and fix $p \geq -1$. The **Kasparov K-homology group** $K_{-p}(A; \mathbb{C})$ is the semidefinable abelian group obtained as the quotient of the standard Borel space $F_p(A)$ by the analytic equivalence relation $B_p(A)$ of stable homotopy of $p$-multigraded Fredholm modules, where the group operation on $K_{-p}(A; \mathbb{C})$ is induced by the Borel function $F_p(A) \times F_p(A) \to F_p(A)$, $(x, x') \mapsto x \oplus x'$, and the function $KK_{-p}(A; \mathbb{C}) \to KK_{-p}(A; \mathbb{C})$ that maps each element to its additive inverse is induced by the Borel function $F_p(A) \to F_p(A)$, $x \mapsto x^\text{op}$.

The fact that $KK_{-p}(A; \mathbb{C})$ is indeed a group is the content of [HR00, Proposition 8.2.10, Corollary 8.2.11, Proposition 8.2.12]. The trivial element of $KK_{-p}(A; \mathbb{C})$ is given by the $B_p(A)$-class of any degenerate Fredholm module. The assignment $A \mapsto KK_{-p}(A; \mathbb{C})$ is easily seen to be a contravariant functor from separable $C^*$-algebras to semidefinable groups. We will later show in Proposition 5.18 that $KK_{-p}(A; \mathbb{C})$ is in fact a definable group.

Suppose that $A$ is a separable $C^*$-algebra. Fix a representation $\rho_A : A \to B(H_A)$ of $A$ that is the restriction to $A$ of an amenable representation of the unitization $A^+$ of $A$. We then let $\rho_A \oplus \rho_A$ be the corresponding representation (by even operators) on the graded Hilbert space $H_A \oplus H_A$. We consider the Paschke dual algebra $\mathcal{D}(A) = \mathcal{D}_{\rho_A}(A)$ associated with $\rho_A$; see Section 2.10.

There is a natural definable group homomorphism $\Phi^1 : K_0(\mathcal{D}(A)) \to KK_1(A; \mathbb{C})$, $[P] \mapsto [x_P]$, defined as follows. Given a projection $P$ in $\mathcal{D}(A)$, define $x_P$ to be the ungraded Fredholm module $(\rho_A, H_A, 2P - I)$ over $A$; see [HR00, Example 8.1.7]. We also have a natural definable group homomorphism $\Phi^0 : K_1(\mathcal{D}(A)) \to KK_0(A; \mathbb{C})$, $[U] \mapsto [x_U]$, defined as follows. Given a unitary $U$ in $\mathcal{D}(A)$, define $x_U$ to be the graded Fredholm module $(\rho_A \oplus \rho_A, H_A \oplus H_A, F_U)$, where $H_A \oplus H_A$ is graded by $I_{H_A} \oplus (-I_{H_A})$, and

$$F_U = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix};$$

see [HR00, Example 8.1.7]. Then it is shown in [HR00, Theorem 8.4.3] that $\Phi^1$ and $\Phi^0$ are in fact group isomorphism. From this, we obtain the following.

**Proposition 5.18.** Suppose that $A$ is a separable $C^*$-algebra and $i \in \{0,1\}$. Then $KK_i(A; \mathbb{C})$ is a definable group, naturally definably isomorphic to $K^i(A)$.

**Proof.** Since $K_{1-i}(\mathcal{D}(A)) = K^i(A)$ is a definable group and $\Phi^i : K_{1-i}(\mathcal{D}(A)) \to KK_i(A; \mathbb{C})$ is a definable group isomorphism, it follows from Proposition 1.14 that $KK_i(A; \mathbb{C})$ is a definable group, naturally definably isomorphic to $K^i(A)$.

**Remark 5.19.** Suppose that $A$ is a separable $C^*$-algebra, and $p \geq -1$. A $p$-multigraded Fredholm module $(\rho, H, F)$ over $A$ is selfadjoint if $F$ is selfadjoint, and contractive if $F$ is contractive [HR00, Definition 8.3.1]. A selfadjoint,
contractive \( p \)-multigraded Fredholm module \((\rho, H, F)\) is involutive if \( F^2 = 1 \) [HR00, Definition 8.3.4]. Kasparov’s K-homology groups can be normalized by requiring that the Fredholm modules be involutive. This means that one obtain the same definable abelian group (up to a natural isomorphism) if one only considers in the definition of the Kasparov K-homology groups involutive Fredholm modules, where also stable homotopy is defined in terms of involutive Fredholm modules; see [HR00, Lemma 8.3.5].

A graded Fredholm module \((\rho, H, F)\) over \( A \) is balanced if there is a separable Hilbert space \( H' \) such that \( H = H' \oplus H'' \) is graded by \( I_{H'} \oplus (-I_{H''}) \), and \( \rho = \rho^+ \oplus \rho^- \), where \( \rho^+ \) and \( \rho^- \) are the same representation of \( A \) on \( H' \). Then one has that \( \text{KK}_0 (A; \mathbb{C}) \) can be normalized by requiring that the graded Fredholm modules be involutive and balanced [HR00, Proposition 8.3.12].

Suppose that \( A \) is a separable C*-algebra. Fix \( p \geq 0 \). If \( x = (\rho, H, F) \) is a \( p \)-multigraded Fredholm module over \( A \), then one can assign to it the \((p + 2)\)-multigraded Fredholm module \( x' = (\rho \oplus \rho^\op, H \oplus H^\op, F \oplus F^\op) \) where \( H \oplus H^\op \) is \((p + 2)\)-multigraded by the operators \( \varepsilon_i \oplus \varepsilon^\op_i \) for \( 1 \leq i \leq p \) together with

\[
\begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & iI \\
iI & 0
\end{bmatrix}.
\]

When \( p = -1 \) one can define \( x' \) to be the 1-graded Fredholm module \((\rho \oplus \rho, H \oplus H, F \oplus F)\) where \( H \oplus H \) is graded by \( I_{H} \oplus (-I_H) \) and 1-multigraded by the odd operator

\[
\begin{bmatrix}
0 & iI \\
iI & 0
\end{bmatrix}.
\]

This gives for \( p \geq -1 \) a natural definable group homomorphism \( \text{KK}_{-p} (A; \mathbb{C}) \to \text{KK}_{-p-2} (A; \mathbb{C}) \), which is an isomorphism by [HR00, Proposition 8.2.13]. From this, Proposition 1.14, and Proposition 5.18, we obtain the following.

**Proposition 5.20.** Suppose that \( A \) is a separable C*-algebra. For \( p \geq -1 \), \( \text{KK}_{-p} (A; \mathbb{C}) \) is a definable group, naturally definably isomorphic to \( K^{-p-2} (A) \).

### 5.4. Relative Kasparov K-homology

The relative Kasparov K-homology groups are defined as above, by replacing Fredholm modules with relative Fredholm modules; see [HR00, Definition 8.5.1].

**Definition 5.21.** Suppose that \((A, J)\) is a separable C*-pair. An ungraded Fredholm module over \((A, J)\) is a triple \((\rho, H, F)\) where:

- \( H \) is a separable Hilbert space \( H_d \) for some \( d \in \omega \cup \{\aleph_0\} \);
- \( \rho : A \to B (H) \) is a *-homomorphism;
- \( F \in B (H) \) satisfies \((F^2 - 1) \rho (j) \equiv (F - F^*) \rho (j) \equiv [F, \rho (a)] \equiv 0 \mod K (H)\) for every \( a \in A \) and \( j \in J \).

A graded Fredholm module over \((A, J)\) is a triple \((\rho, H, F)\) where:

- \( H \) is a separable graded Hilbert space of the form \((H_d, \gamma)\) for some \( d \in \omega \cup \{\aleph_0\} \) and some grading operator \( \gamma \) on \( H_d \);
- \( \rho : A \to B (H) \) is a *-homomorphism such that, for every \( a \in A \), \( \rho (a) \in B (H)^+ \) is even;
- \( F \in B (H) \) is an odd operator that satisfies \((F^2 - 1) \rho (j) \equiv (F - F^*) \rho (j) \equiv [F, \rho (a)] \equiv 0 \mod K (H)\) for every \( a \in A \) and \( j \in J \).

As above, one can consider the definable group \( \text{KK}_{-1} (A, J; \mathbb{C}) \) whose elements are stable homotopy equivalence classes of Fredholm modules over \((A, J)\). Considering graded Fredholm modules over \((A, J)\) one obtains the definable group \( \text{KK}_0 (A, J; \mathbb{C}) \). These groups are called the relative Kasparov K-homology groups of the pair \((A, J)\), and turn out to be naturally isomorphic to the relative K-homology groups \( K^1 (A, J) \) and \( K^0 (A, J) \); see [HR00, Section 8.5]. More generally, one can define \( \text{KK}_{-p} (A, J; \mathbb{C}) \) in terms of \( p \)-multigraded Fredholm modules over \((A, J)\).

In the Kasparov picture, the excision isomorphism \( \text{KK}_{-p} (A, J; \mathbb{C}) \to \text{KK}_{-p} (J; \mathbb{C}) \) is induced by the (Borel) inclusion map from the set of \( p \)-multigraded Fredholm modules over \((A, J)\) into the set of \( p \)-multigraded Fredholm modules over \( J \). (Notice that, indeed, a Fredholm module over \((A, J)\) is, in particular, a Fredholm module over \( J \).)
5.5. **KKₜ-cycles and K-homology.** Suppose that \( A \) is a separable C*-algebra. Let \( H \) be the separable infinite-dimensional Hilbert space. A KKₜ-cycle for \( A \) is a pair \( (φ_+, φ_-) \) of *-homomorphisms \( A \to B(H) \) such that \( φ_+(a) ≡ φ_-(a) \mod K(H) \) for every \( a \in A \); see [JT91, Definition 4.1.1]. Define \( F(A; C) \) to be the standard Borel space of KKₜ-cycles for \( A \). The standard Borel structure on \( F(A; C) \) is induced by the Polish topology obtained by setting \( (φ_+^{(1)}, φ_-^{(1)}) \to (φ_+, φ_-) \) if and only if, for every \( a \in A \), \( φ_+^{(1)}(a) \to φ_+(a) \) and \( φ_-^{(1)}(a) \to φ_-(a) \) in the strong-* topology, and \( (φ_+^{(i)} - φ_-^{(i)}) (a) \to (φ_+ - φ_-) (a) \) in norm. We regard \( F(A; C) \) as a Polish space with respect to such a topology.

Define \( ∼ \) to be the relation of homotopy for elements of the Polish space \( F(A; C) \). Thus, for \( x, x' \in F(A; C) \), \( x \sim x' \) if and only if there exists a continuous path \( (x_t)_{t \in [0,1]} \) in \( F(A; C) \) such that \( x_0 = x \) and \( x_1 = x' \); see [JT91, Definition 4.1.2]. As discussed in Section 2.3 one can regard a strong-* continuous path \( (φ_t)_{t \in [0,1]} \) of *-homomorphisms \( A \to B(H) \) as an element of the unit ball of \( C_β([0,1], B(H)) = M(C([0,1], K(H))) \). This allows one to regard the set of such paths as a Polish space endowed with the strict topology on Ball \( (C_β([0,1], B(H))) \), such that norm-continuous paths form a Borel subset by Corollary 2.26. It can be deduced from these observations that the relation \( ∼ \) of homotopy in \( F(A; C) \) is an analytic equivalence relation.

One lets \( KKₜ(A; C) \) be the semidefinable set obtained as a quotient of the Polish space \( F(A; C) \) by the analytic equivalence relation \( ∼ \) [JT91, Definition 4.1.3]. One has that \( KKₜ(A; C) \) is a semidefinable group, where the group operation is induced by the Borel function \( F(A; C) \times F(A; C) \to F(A; C) \), \( ((φ_+, φ_-), (ψ_+, ψ_-)) \mapsto (Ad(V) ◦ (φ_+ ⊕ ψ_+), Ad(V) ◦ (φ_+ ⊕ ψ_-)) \), where \( V : H ⊕ H \to H \) is a fixed surjective linear isometry, and the function mapping each element to its additive inverse is induced by the Borel function \( KKₜ(A; C) \to KKₜ(A; C), (φ_+, φ_-) \mapsto (φ_-, φ_+) \); see [JT91, Proposition 4.1.4]. The trivial element of \( KKₜ(A; C) \) is the homotopy class of \((0,0)\). The assignment \( A \mapsto KKₜ(A; C) \) gives a contravariant functor from separable C*-algebras to semidefinable groups.

Let \( A \) be a separable C*-algebra. We now observe that \( KKₜ(A; C) \) is in fact a definable group, definably isomorphic to \( KK₀(A; C) \) and hence to \( K₀(A) \). There is a natural definable isomorphism \( Ψ : KKₜ(A; C) \to KK₀(A; C) \) defined as follows; see [JT91, Theorem 4.1.8]. Suppose that \( (φ_+, φ_-) \in F(A; C) \). Then one can consider the graded Kasparov module over \( A \) defined as \( (φ_+ ⊕ φ_-, H ⊕ H, F) \) where \( H ⊕ H \) is graded by \( I_H ⊕ (-I_H) \) and

\[
F = \begin{bmatrix}
0 & I_H \\
I_H & 0
\end{bmatrix}.
\]

Then one sets \( Ψ([φ_+, φ_-]) = [φ_+ ⊕ φ_-, H ⊕ H, F] \).

We now observe that the inverse function \( Ψ⁻¹ : KK₀(A; C) \to KKₜ(A; C) \) is also definable, as it follows from the proof of [JT91, Theorem 4.1.8]. Let \((ρ₀, H₀, F₀)\) be a graded Kasparov module, which can be assumed to be involutive and balanced by normalization and where we can assume \( H₀ \) to be infinite-dimensional; see [HR00, Proposition 8.3.12]. Then we have that \( H₀ = H ⊕ H \) is graded by \( I_H ⊕ (-I_H) \) and \( ρ₀ = ρ₀^* \) are the same representation of \( A \) on \( H₀ \), and

\[
F₀ = \begin{bmatrix}
0 & u^* \\
u & 0
\end{bmatrix}
\]

for some unitary \( u \in B(H') \). Then by [JT91, E 2.1.3], the Kasparov modules \((ρ₀, H₀, F₀)\) and \(((Ad(u) ◦ ρ^+) ⊕ ρ^-, H ⊕ H, F)\) represent the same element of \( KK₀(A; C) \), where as above

\[
F = \begin{bmatrix}
0 & I_H \\
I_H & 0
\end{bmatrix}.
\]

One has that \( Ψ⁻¹[ρ₀, H₀, F₀] = [(Ad(u) ◦ ρ^+), ρ^-] \). As the assignment \((ρ₀, H₀, F₀) \mapsto ((Ad(u) ◦ ρ^+), ρ^-) \) is given by a Borel function, this shows that the inverse \( Ψ⁻¹ : KK₀(A; C) \to KKₜ(A; C) \) is definable. We can thus obtain the following.

**Proposition 5.22.** Let \( A \) be a separable C*-algebra. Then \( KKₜ(A; C) \) is a definable group, naturally definably isomorphic to \( K₀(A) \).

**Proof.** By the above discussion, the natural definable homomorphism \( KKₜ(A; C) \to KK₀(A; C) \) is an isomorphism in the category of semidefinable groups. Therefore, \( KKₜ(A; C) \) is also a definable group, naturally isomorphic to \( KK₀(A; C) \). As in turn \( KK₀(A; C) \) is naturally definably isomorphic to \( K₀(A) \), the conclusion follows. \( □ \)
5.6. Cuntz’s K-homology. Suppose that $A, B$ are separable C*-algebras. Let $\text{Hom}(A, B)$ be the set of *-homomorphisms $A \to B$. Then $\text{Hom}(A, B)$ is a Polish space when endowed with the topology of pointwise norm-convergence. Two *-homomorphisms $\phi, \phi' : A \to B$ are homotopic, in which case we write $\phi \sim \phi'$, if they belong to the same path-connected component of $\text{Hom}(A, B)$. Thus, two *-homomorphism $\phi, \phi' : A \to B$ satisfy $\phi \sim \phi'$ if and only if there exists a continuous path $(\lambda_t)_{t \in [0,1]}$ in $\text{Hom}(A,B)$ such that $\lambda_0 = \phi$ and $\lambda_1 = \phi'$; see [JT91, Definition 1.3.10]. Such a path $(\lambda_t)_{t \in [0,1]}$ can be thought of as a *-homomorphism $\lambda : A \rightarrow C([0,1], B)$, where $C([0,1], B) \cong \mathcal{K}([0,1]) \otimes B$ is the C*-algebra of continuous functions $[0,1] \to B$. This shows that the relation $\sim$ of homotopy in $\text{Hom}(A, B)$ is an analytic equivalence relation. We let $[A, B]$ be the semidefinable set of homotopy classes of *-homomorphisms $A \to B$.

Recall that a separable C*-algebra $B$ is stable if $B \otimes K$ is *-isomorphic to $B$. Suppose in the following that $B$ is stable. Thus we have that $M(B) \otimes M(K(H)) \subseteq M(B \otimes K(H)) \cong M(B)$. This implies that one can choose isometries $w_0, w_1 \in M(B)$ satisfying $w_0 w_0^* + w_1 w_1^* = 1$ and $w_i^* w_j = 0$ for $i, j \in \{0, 1\}$ distinct. (This is equivalent to the assertion that $w_0, w_1$ generate inside $M(B)$ a copy of the Cuntz algebra $O_2$.) One can then define a *-isomorphism $\theta : M_2(B) \rightarrow B$, $x \mapsto w_0 x w_0^* + w_1 x w_1^*$. A *-isomorphism of this form is called inner; see [JT91, Definition 1.3.8]. Any two inner *-isomorphisms $\theta, \theta' : M_2(B) \rightarrow B$ are unitary equivalent, namely there exists a unitary $u \in M(B)$ such that $\text{Ad}(u) \circ \theta = \theta'$. [JT91, Lemma 1.3.9].

Under the assumption that $B$ is stable, one can endow the semidefinable set $[A, B]$ with the structure of semidefinable semigroup. The operation on $[A, B]$ is induced by the Borel function $\text{Hom}(A, B)^2 \rightarrow \text{Hom}(A, B)$, $(\phi, \psi) \mapsto \theta \circ (\phi \oplus \psi)$, where $\theta$ is a fixed inner *-isomorphism $M_2(B) \rightarrow B$; see [JT91, Lemma 1.3.12]. The trivial element in $[A, B]$ is the homotopy class of the zero *-homomorphism. Furthermore, the argument of [JT91, E 4.1.4] shows that $[A, B]$ is isomorphic to $[K(H) \otimes A, B]$ in the category of semidefinable semigroups.

Suppose that $A$ is a separable C*-algebra. Define $QA$ to be the separable C*-algebra $A + A$, where $A + A$ denotes the free product of $A$ with itself. We let $i, \overline{i}$ be the two canonical inclusions of $A$ inside $QA$. Let $q_A$ be the closed two-sided ideal of $QA$ generated by the elements of the form $i(a) - \overline{i}(a)$ for $a \in A$; see [JT91, Definition 5.1.1].

If $B$ is a separable C*-algebra, and $\phi, \psi : A \rightarrow B$ are *-homomorphism, then there is a unique *-homomorphism $Q(\phi, \psi) : QA \rightarrow B$ such that $Q(\phi, \psi) \circ i = \phi$ and $Q(\phi, \psi) \circ i = \psi$. The restriction of $Q(\phi, \psi)$ to $q_A$ is denoted by $q(\phi, \psi)$. One has that the range of $q(\phi, \psi)$ is contained in an ideal $J$ of $B$ if and only if $q(\phi) \equiv q(\psi)$ mod $J$ for every $a \in A$, in which case $q(\phi, \psi) \in \text{Hom}(q_A, J)$. One has that $q_A$ is the kernel of the map $Q(id_A, id_A) : QA \rightarrow A$; see [JT91, Lemma 5.1.2].

If $B$ is a separable C*-algebra, then the semidefinable semigroup $[q_A, K(H) \otimes B]$ is in fact a semidefinable group, where the function that maps each element to its additive inverse is induced by the Borel function $\text{Hom}(q_A, K(H) \otimes B), \phi \mapsto -\phi$; [JT91, Theorem 5.1.6]. The proof of [JT91, Theorem 5.1.12] shows that $[q_A, K(H) \otimes B]$ is isomorphic in the category of semidefinable groups to $[q_A, K(H) \otimes q_B]$. In turn, $[q_A, K(H) \otimes q_B]$ is isomorphic to $[K(H) \otimes q_A, K(H) \otimes q_B]$ in the category of semidefinable groups by [JT91, E 4.1.4].

Observe that, for a fixed C*-algebra $B$, the assignment $A \mapsto [q_A, K(H) \otimes B]$ is a contravariant functor from C*-algebras to semidefinable groups. Suppose that $A$ is a separable C*-algebra. Then there is a natural definable homomorphism $S : KK_b(A; \mathbb{C}) \rightarrow [q_A, K(H)]$ defined by setting $S([\phi_+, \phi_-]) = [\psi]$ where $\psi = q(\phi_+, \phi_-) \in \text{Hom}(q_A, K(H))$. One has that in fact $S$ is a group isomorphism [JT91, Theorem 5.2.4]. Therefore, we obtain from Proposition 5.22 and Corollary 1.14 the following.

**Proposition 5.23.** Suppose that $A$ is a separable C*-algebra. Then $[q_A, K(H)]$ is a definable group, naturally definably isomorphic to $K^0(A)$.

This description of K-homology is called Cuntz’s picture, as it was introduced by Cuntz in [Cun87]; see also [Bla86, Section 17.6] and [Cun83, Cun84, Zek89]. Using the Cuntz picture, one can easily define the more general Kasparov KK-groups $KK_0(A, B)$ for separable C*-algebras $A, B$, by setting $KK_0(A; B) = [q_A, K(H) \otimes B] \cong [K(H) \otimes q_A, K(H) \otimes q_B]$.

These are semidefinable groups, although we do not know whether they are definable groups when $B$ is an arbitrary separable C*-algebra. In particular, one has that $KK_0(A; \mathbb{C}) \cong K^0(A)$ and $KK_0(SA; \mathbb{C}) \cong K^1(A)$. The K-theory
groups are also recovered as particular instances of the KK-groups, via KK\(_0 (\mathbb{C}; A) \cong K^0 (A)\) and KK\(_0 (\mathbb{C} (\mathbb{R}); A) \cong K^1 (A)\).

Given separable C*-algebras \(A, B, C\), composition of *-homomorphisms \(K (H) \otimes qA \to K (H) \otimes qB\) and \(K (H) \otimes qB \to K (H) \otimes qC\) induces a definable bilinear pairing (Kasparov product)

\[
KK_0 (A; B) \times KK_0 (B; C) \to KK_0 (A; C) .
\]

In particular, \(KK_0 (A; A)\) is a (semidefinable) ring, with identity element \(1_A\) corresponding to the identity map of \(K (H) \otimes qA\). The KK-category of C*-algebras is the category enriched over the category of (semidefinable) abelian groups that has separable C*-algebras as objects and KK-groups as hom-sets. Two separable C*-algebras are KK-equivalent if they are isomorphic in the KK-category of C*-algebras.

By way of the Kasparov product and the natural isomorphisms \(K^0 (A) \cong KK_0 (A; \mathbb{C})\) and \(K^1 (A) \cong KK_0 (\mathbb{C}; A)\), one can regard K-homology as a contravariant functor from the KK-category of separable C*-algebras to the category of definable groups, and K-theory as a covariant functor from the KK-category of separable C*-algebras to the category of countable groups. In particular, KK-equivalent C*-algebras have definably isomorphic K-homology groups, and isomorphic K-theory groups.

6. Properties of definable K-homology

In this section we consider several properties of definable K-homology, which can be seen as definable versions of the properties of an abstract cohomology theory for separable, nuclear C*-algebra in the sense of [Sch84] that is C*-stable in the sense of [Cun87].

6.1. Products. Suppose that \((X_i)_{i \in \omega}\) is a sequence of semidefinable sets \(X_i = \hat{X}_i / E_i\). Then the product \(\prod_{i \in \omega} X_i\) is the semidefinable set \(\hat{X} / E\) where \(\hat{X} = \prod_{i \in \omega} \hat{X}_i\) and \(E\) is the (analytic) equivalence relation on \(\hat{X}\) defined by setting \((x_i) E (y_i)\) if and only if \(i \in \omega\), \(x_i E_i y_i\). If, for every \(i \in \omega\), \(G_i\) is a semidefinable group, then \(\prod_{i \in \omega} G_i\) is a semidefinable group when endowed with the product group operation.

Suppose that \((A_i)_{i \in \omega}\) is a sequence of separable C*-algebra. Define the direct sum \(\bigoplus_{i \in \omega} A_i\) to be the C*-algebra \(A\) consisting of the sequences \((a_i)_{i \in \omega} \in \prod_{i \in \omega} A_i\) such that \(\|a_i\| \to 0\); see [HR00, Definition 7.4.1]. If \(B\) is a separable C*-algebra, then the canonical maps \(A_i \to A\) induce an isomorphism of Polish spaces

\[
\text{Hom} (A, B) \to \prod_{i \in \omega} \text{Hom} (A_i, B) .
\]

When \(A_i\) is commutative with spectrum \(X_i\), then \(A\) is commutative with spectrum the disjoint union of \(X_i\) for \(i \in \omega\). The following result can be seen as a noncommutative version of the Cluster Axiom for a homology theory for pointed compact spaces from [Mil95].

**Proposition 6.1.** Suppose that \((A_i)_{i \in \omega}\) is a sequence of separable C*-algebras, and set \(A = \bigoplus_{i \in \omega} A_i\). Fix \(p \in \{0, 1\}\). Then \(\prod_{i \in \omega} K^p (A_i)\) is a definable group. Furthermore the canonical maps \(A_i \to A\) for \(i \in \omega\) induce a natural definable isomorphism

\[
K^p (A) \to \prod_{i \in \omega} K^p (A_i) .
\]

**Proof.** Since \(K^p (A)\) is a definable group, it suffices to prove the second assertion. After replacing \(A\) with its suspension, it suffices to consider the case when \(p = 0\). In this case, we can replace \(K^0\) with \(KK_h\) by Proposition 5.22. Recall that we let \(\mathbb{F} (A; \mathbb{C})\) be the space of \(KK_h\)-cycles for \(A\). The canonical maps \(A_i \to A\) induce an isomorphism of Polish spaces

\[
\mathbb{F} (A; \mathbb{C}) \to \prod_{i \in \omega} \mathbb{F} (A_i; \mathbb{C}) .
\]

In turn, this induces a definable isomorphism of the spaces of homotopy classes.

\[
KK_h (A) \to \prod_{i \in \omega} KK_h (A_i) .
\]

This concludes the proof. □
6.2. Homotopy-invariance. Suppose that \( A, B \) are separable C*-algebra. Recall that \( \text{Hom}(A, B) \) is a Polish space when endowed with the topology of pairwise convergence. Thus, \( \alpha, \beta \in \text{Hom}(A, B) \) if there exists a path in \( \text{Hom}(A, B) \) from \( \alpha \) to \( \beta \); see [HR00, Definition 4.4.1]. This can be thought of as an element \( \gamma \) of \( \text{Hom}(A, IB) \) such that \( \text{ev}_0 \circ \gamma = \alpha \) and \( \text{ev}_1 \circ \gamma = \beta \) where \( IB = C([0, 1], B) \) and \( \text{ev}_t : IB \to B, f \mapsto f(t) \) for \( t \in [0, 1] \). We let \([A, B]\) be the semidefinable set of homotopy classes of *-homomorphisms \( A \to B \). The homotopy category of C*-algebras has separable C*-algebras as objects and homotopy classes of *-homomorphisms as morphisms. Two C*-algebras are homotopy equivalent if they are isomorphic in the homotopy category of C*-algebras [HR00, Definition 4.4.7].

**Proposition 6.2.** For \( p \in \{0, 1\} \), the K-homology functor \( K^p(–) \) from separable C*-algebras is homotopy-invariant.

**Proof.** As in the case of the proof of Proposition 6.1, it suffices to show that the functor \( \text{KK}_h(–) \) is homotopy invariant, which is an immediate consequence of the definition. \( \square \)

Suppose that \( B \) is a separable C*-algebra. Recall that the suspension \( SB \) of \( B \) is the C*-subalgebra of \( IB \) consisting of \( f \in IB \) such that \( f(0) = f(1) = 0 \). Then \([A, SB]\) is a semidefinable abelian group, where the group operation is induced by the Borel function \( (f, g) \mapsto m(f, g) \) where
\[
m(f, g)(t) = \begin{cases} f(2t) & t \in [0, 1/2], \\ g(2(t - 1/2)) & t \in [1/2, 1]. \end{cases}
\]

The function that assigns each element to its additive inverse is induced by the Borel function \( f \mapsto \hat{f} \) where
\[
\hat{f}(t) = f(1 - t).
\]

The trivial element of \([A, SB]\) is the homotopy class of 0. For \( p \in \{0, 1\} \), there map \([A, SB] \to K^p(SB, A)\) is a group homomorphism [Sch84, Proposition 6.3].

A separable C*-algebra \( A \) is contractible if it is homotopy equivalent to the zero C*-algebra; see [HR00, Definition 4.4.4]. By homotopy invariance, \( K^p(A) = \{0\} \) whenever \( A \) is contractible and \( p \in \{0, 1\} \). In particular, if \((A, J)\) is a separable semi-split C*-pair such that \( A \) is contractible, the boundary homomorphism \( K^p(J) \to K^p(A/J) \) is a definable isomorphism.

If \( A \) is a separable, nuclear C*-algebra, then its cone \( CA \) is the C*-subalgebra of \( IA \) consisting of \( f \in IA \) such that \( f(1) = 0 \). This is a contractible C*-algebra [HR00, Example 4.4.6], and
\[
0 \to SA \to CA \to A \to 0
\]
is an exact sequence, where \( CA \to A \) is the map \( \text{ev}_0 \). The boundary homomorphism \( \sigma^A : K^p(SA) \to K^{p+1}(A) \) is thus an isomorphism; see [Sch84, Theorem 6.5].

6.3. Mapping cones. Suppose that \( A, B \) are separable, nuclear C*-algebras, and \( f : A \to B \) is a *-homomorphism. The mapping cone
\[
Cf = \{(x, y) \in CB \oplus A : f(y) = \text{ev}_0(x)\}
\]
of \( f \) is obtained as the pullback of \( \text{ev}_0 : CB \to B \) and \( f : A \to B \). As such, it is endowed with canonical *-homomorphisms \( Cf \to CB \) and \( Cf \to A \); see [Sch84, Definition 2.1]. We have a natural exact sequence
\[
0 \to SB \to Cf \to A \to 0
\]
where \( SB \to Cf, x \mapsto (x, 0) \). This induces a boundary homomorphism \( K^p(SB) \to K^{p+1}(A) \).

Considering the exact sequences
\[
0 \to SB \to Cf \to A \to 0 \\
0 \to SB \to CB \to B \to 0
\]
and the morphism between exact sequences
\[
\begin{array}{cccccc}
0 & \to & SB & \to & Cf & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow f & & \\
0 & \to & SB & \to & CB & \to & B & \to & 0
\end{array}
\]
where \( SB \to SB \) is the identity map, and \( Cf \to CB, (x, y) \mapsto x \), we obtain by naturality of the six-term exact sequence that the boundary morphism \( K^p(SB) \to K^{p+1}(A) \) is equal to the composition
\[
K^{p+1}(f) \circ \sigma^B : K^p(SB) \to K^{p+1}(B) \to K^{p+1}(A).
\]
The same argument together with the Five Lemma [Rot09, Proposition 2.72] shows that $K^p(f)$ is an isomorphism for every $p \in \{0, 1\}$ if and only if $K^p(Cf) = \{0\}$ for every $p \in \{0, 1\}$; see [Sch84, Theorem 6.5].

If $f : A \to B$ is a surjective $*$-homomorphism with kernel $J$, then considering the exact sequence

$$0 \to J \to Cf \to CB \to 0$$

one sees that the map $J \to Cf$ induces an isomorphism $K^p(J) \to K^p(Cf)$. Similarly, if $J$ is an ideal of $A$ and $f : J \to A$ is the inclusion, then considering the exact sequence

$$0 \to CJ \to Cf \to S(A/J) \to 0$$

shows that the map $Cf \to S(A/J)$ induces an isomorphism $K^p(Cf) \to K^p(S(A/J))$; see [Sch84, Proposition 6.6].

6.4. **Long exact sequence of a triple.** Consider a triple $J \subseteq H \subseteq A$ where $A$ is a separable, nuclear C*-algebra and $J$ and $H$ are closed two-sided ideals of $A$. Then we have exact sequences

$$0 \to H \to A \to A/H \to 0$$

and

$$0 \to H/J \to A/J \to A/H \to 0$$

and a natural morphism from the first one to the second one. By naturality of the six-term exact sequence in K-homology, we have that the boundary map

$$K^p(H/J) \to K^{1-p}(A/H)$$

is equal to the composition of the map $K^p(H/J) \to K^p(H)$ induced by the quotient map with the boundary map $K^p(H) \to K^{1-p}(A/H)$; see [Sch84, Theorem 6.10].

6.5. **Mayer–Vietoris sequence.** Consider separable, nuclear C*-algebras $P, A_1, A_2, B$, and $*$-homomorphisms $f_i : A_i \to B$ and $g_i : P \to A_i$ for $i \in \{1, 2\}$. Suppose that $f_1, f_2$ are surjective, and

$$
\begin{array}{ccc}
P & \xrightarrow{g_1} & A_1 \\
\downarrow{g_2} & & \downarrow{f_1} \\
A_2 & \xrightarrow{f_2} & B
\end{array}
$$

is a pushout diagram. Then there is a six-term exact sequence of definable group homomorphisms

$$
\begin{array}{cccc}
K^0(B) & \xrightarrow{\partial^p} & K^0(A_1) \oplus K^0(A_2) & \xrightarrow{\partial^p} & K^0(P) \\
\uparrow{\partial^1} & & \uparrow{\partial^1} & & \uparrow{\partial^1} \\
K^1(P) & \xleftarrow{\partial^p} & K^1(A_1) \oplus K^1(A_2) & \xleftarrow{\partial^p} & K^1(B)
\end{array}
$$

see [Sch84, Theorem 6.11]. The definable group homomorphism $K^p(B) \to K^p(A_1) \oplus K^p(A_2)$ is $(-K^p(f_1), K^p(f_2))$, the definable group homomorphism $K^p(A_1) \oplus K^p(A_2) \to K^p(P)$ is $K^p(g_1) + K^p(g_2)$. Furthermore, the definable group homomorphism $\partial^p : K^p(P) \to K^{1-p}(B)$ is defined as follows. Let $g : P \to A_1 \oplus A_2$ be defined by $x \mapsto (g_1(x), g_2(x))$. Consider the corresponding mapping cone $Cg$. We can regard $Cg$ as the set of triples $(\xi_1, \xi_2, x) \in CA_1 \oplus CA_2 \oplus P$ such that $(\xi_1(0), \xi_2(0)) = g(x)$. We have a $*$-homomorphism $\psi : Cg \to SB$ defined by setting

$$\psi(\xi_1, \xi_2, x)(t) = \begin{cases} f_1(\xi_1(1 - 2t)), & t \in [0, 1/2]; \\ f_2(\xi_2(2t - 1)), & t \in [1/2, 1]. \end{cases}$$

Then we have a natural short exact sequence

$$0 \to CJ_1 \oplus CJ_2 \to Cg \xrightarrow{\psi} SB \to 0$$

where $J_i = \text{Ker}(f_i)$ for $i \in \{1, 2\}$; see [Sch84, Proposition 4.5]. Thus, $\psi$ induces a definable isomorphism $K^p(SB) \to K^p(Cg)$. The definable group homomorphism $\partial^p : K^p(P) \to K^{1-p}(B)$ is defined as the composition of definable homomorphisms

$$K^p(P) \to K^p(Cg) \to K^p(SB) \to K^{1-p}(B)$$

where the map $K^p(P) \to K^p(Cg)$ is associated with the canonical $*$-homomorphism $Cg \to P$ as in the definition of mapping cone, the map $K^p(Cg) \to K^p(SB)$ is the inverse of the definable isomorphism $K^p(SB) \to K^p(Cg)$.
induced by \( \psi \), and the map \( \sigma^B : K^p(SB) \to K^{1-p}(B) \) is the suspension isomorphism; see the proof of [Sch84, Theorem 6.11].

6.6. The Milnor sequence of an inductive sequence. A tower of countable abelian groups is a sequence \( A = (A(n), p^{(n,n+1)}) \) of countable abelian groups and group homomorphism \( p^{(n,n+1)} : A(n+1) \to A(n) \). Given such a tower we let \( p^{(n,n)} \) be the identity map of \( A(n) \) and, for \( n < m \), \( p^{(n,m)} \) be the composition \( p^{(n,n+1)} \circ \cdots \circ p^{(m-1,m)} \).

Towers of countable groups form a category. A morphism from \( A = (A(n), p^{(n,n+1)}) \) to \( B = (B(k), p^{(k,k+1)}) \) is represented by a sequence \( (n_k, f^{(k)})_{k \in \omega} \) where \( (n_k) \) is an increasing sequence in \( \omega \) and \( f^{(k)} : A(n_k) \to B(k) \) is a group homomorphism. Two such sequences \( (n_k, f^{(k)})_{k \in \omega} \) and \( (n_k', f^{(k')})_{k \in \omega} \) represent the same morphism if there exists an increasing sequence \( (n_k'')_{k \in \omega} \) in \( \omega \) such that \( n_k'' \geq \max \{ n_k, n_k' \} \) and \( f^{(k)} p^{(n_k,n_k'')} = f^{(k')}(n_k,n_k'') \) for every \( k \in \omega \). The identity morphism and composition of morphisms are defined in the obvious way.

Given a tower \( A \) of countable abelian groups, one lets \( \lim^1 A \) be the definable group, which is in fact a group with Polish cover (see Remark 1.17), defined as follows. Consider \( Z^1(A) \) to be the product group

\[
\prod_{n \in \omega} A(n)
\]

endowed with the product topology, where each \( A(n) \) is endowed with the discrete topology. Define \( B^1(A) \) to be the Polishable Borel subgroup of \( Z^1(A) \) obtained as an image of the continuous group homomorphism

\[
\Phi_A : \prod_{n \in \omega} A(n) \to Z^1(A), \ (x_n) \mapsto (x_n - p^{(n,n+1)}(x_{n+1}))_{n \in \omega}.
\]

Then \( \lim^1 A \) is the corresponding definable group \( Z^1(A)/B^1(A) \). The assignment \( A \mapsto \lim^1 A \) is easily seen to be a functor from the category of towers of countable abelian groups to the category of definable groups; see also [BLP20, Section 5].

Given a tower \( A \) of countable abelian groups, we can also consider the (inverse) limit \( \lim A \). This is the Polish abelian group obtained as the kernel of the continuous group homomorphism \( \Phi_A \) described above. The assignment \( A \mapsto \lim A \) is a functor from the category of towers of countable abelian groups to the category of Polish abelian groups.

Suppose that \( (A_n, \varphi_n)_{n \in \omega} \) is an inductive sequence of separable, nuclear C*-algebras, and let \( A = \text{colim}_n (A_n, \varphi_n) \) be its inductive limit. If \( K^p(A_n) \) is countable for every \( n \in \omega \), then \( (K^p(A_n))_{n \in \omega} \) is a tower of countable abelian groups, where \( p^{(n,n+1)} : K^p(A_{n+1}) \to K^p(A_n) \) is induced by \( \varphi_n : A_n \to A_{n+1} \). The assignment \( (A_n, \varphi_n)_{n \in \omega} \mapsto (K^p(A_n))_{n \in \omega} \) defines a functor from the category of inductive sequences of separable C*-algebras with countable K-homology groups to the category of towers of countable abelian groups. The Milnor sequence for \( (A_n, \varphi_n)_{n \in \omega} \) describes \( K^p(A) \) as an extension of groups defined in terms of \( (K^p(A_n))_{n \in \omega} \); see [Sch84, Theorem 7.1]. The proof is inspired by Milnor’s argument for the corresponding result about Steenrod homology [Mil95]; see also [Mil62].

We let \( \mathbb{N} \) denote the set of natural numbers not including zero, and \( \omega = \mathbb{N} \cup \{ 0 \} \).

**Proposition 6.3.** Suppose that \( (A_n, \varphi_n)_{n \in \mathbb{N}} \) is an inductive sequence of separable, nuclear C*-algebras with countable K-homology groups, and \( A \) is the inductive limit of \( (A_n, \varphi_n)_{n \in \omega} \). Then for \( p \in \{ 0, 1 \} \) there is a natural short exact sequence of definable group homomorphisms

\[
0 \to \lim^1_n K^{1-p}(A_n) \to K^p(A) \to \lim_n K^p(A_n) \to 0
\]

where the homomorphism \( K^p(A) \to \lim_n K^p(A_n) \) is induced by the canonical maps \( A_n \to A \).

The assertion that the group homomorphisms in Proposition 6.3 are definable is a consequence of the proof of [Sch84, Theorem 7.1]. This involves the notion of mapping telescope \( T(A) \) of an inductive sequence of \( A = (A_n, \varphi_n)_{n \in \mathbb{N}} \) of separable C*-algebras; see [Sch84, Definition 5.2]. Without loss of generality, we can assume that \( A_0 = \{ 0 \} \). Let \( A \) be the corresponding direct limit and \( \varphi_{\infty, n} : A_n \to A_n \) be the canonical maps. For \( n < m \) set \( \varphi_{m,n} : A_n \to A_m, \varphi_{m,n} = \varphi_m \circ \varphi_{m+1} \circ \cdots \circ \varphi_{n-1} \). We also let \( \varphi_{n,n} \) be the identity of \( A_n \). One fixes an increasing sequence \( (t_n)_{n \in \omega} \) in \( [0, 1) \) with \( t_0 = 0 \) converging to 1. Let \( \prod_{n \in \omega} C([t_n, t_{n+1}], A_{n+1}) \) be the product of \( C([t_n, t_{n+1}], A_{n+1})_{n \in \omega} \) in the category of C*-algebras. Define then \( \hat{T}(A) \) to be the C*-subalgebra of \( \prod_{n \in \omega} C([t_n, t_{n+1}], A_{n+1}) \) consisting of those elements \( (\xi_n)_{n \in \omega} \) such that, for every \( n \in \omega, \varphi_{n+1}(\xi_n(t_{n+1})) = \xi_{n+1}(t_{n+1}) \).

An element \( (\xi_n)_{n \in \omega} \) of \( \hat{T}(A) \) can be seen as a function \( \xi : [0, 1) \to \bigcup_{n \in \omega} I_{A_{n+1}} \) where, for \( n \in \omega \)
and \( t \in [t_n, t_{n+1}) \) one sets \( \xi(t) := \xi_n(t) \). The function \( \xi^\infty : [0, 1) \to A \) defined by \( \xi^\infty(t) = \varphi_{(n, n+1)}(\xi(t)) \) for \( t \in [t_n, t_{n+1}) \) is then continuous.

The mapping telescope \( T(A) \) consists of the set of pairs \((\xi, a) \in \tilde{T}(A) \oplus A\) such that:

1. For every \( \varepsilon > 0 \) there exists \( n_0 \in \omega \) such that, for \( n \geq m \geq n_0 \) and for \( t \in [t_n, t_{n+1}] \) and \( s \in [t_m, t_{m+1}] \),

\[
\| \varphi_{(n+1, m+1)}(\xi_m(s)) - \xi_n(t) \| < \varepsilon,
\]

and

2. \( \lim_{t \to 1} \xi^\infty(t) = a \).

Then one has that \( T(A) \) is a contractible separable C*-algebra; see [Sch84, Lemma 5.4]. Define the surjective *-homomorphism \( e : T(A) \to A \), \((\xi, a) \mapsto a\), and set \( J = \ker(e) \subseteq T(A) \). We also have a map \( p : J \to \bigoplus_{n \in \omega} A_{n+1}, \xi \mapsto (\xi_n(t_{n+1}))_{n \in \omega} \); see [Sch84, Lemma 5.5]. As \( T(A) \) is contractible, the short exact sequence

\[ 0 \to J \to T(A) \to A \to 0 \]

gives rise to a definable boundary isomorphism \( \partial : K^{1-p}(J) \to K^p(A) \).

For \( n \in \omega \) define \( M_n \subseteq C([t_n, t_{n+1}], A_{n+1}) \oplus A_n \) to be the C*-subalgebra consisting of \((\xi, a)\) such that \( \xi(t_n) = \varphi_n(a) \). The *-homomorphism \( M_n \to A_n \), \((\xi, a) \mapsto a\) is a homotopy equivalence with homotopy inverse \( A_n \to M_n \), \( a \mapsto (\xi, a) \) where \( \xi(t) = \varphi_n(a) \) for \( t \in [0, 1] \). Then we have that the composition \( A_n \to M_n \to A_n \) is the identity, while the composition \( M_n \to A_n \to M_n \) maps \((\xi, a)\) to \((\xi', a)\) where \( \xi'(t) = \xi(t_n) = \varphi_n(a) \) for \( t \in [0, 1] \). This map is homotopic to the identity via the homotopy \((\phi_t)_{t \in [0, 1]}\) defined by \( \phi_t(\xi, a) = (\xi_t, a) \) where

\[
\xi_t(t_n + t(t_{n+1} - t_n)) = \xi(t_n + st(t_{n+1} - t_n))
\]

for \( s, t \in [0, 1] \).

Define

\[
D_1 := \bigoplus_{n \in \omega} M_{2n+1}
\]

\[
D_2 := \bigoplus_{n \in \omega} M_{2n}
\]

\[
B := \bigoplus_{n \in \omega} A_n
\]

As in [Sch84, Lemma 5.7], we have a pullback diagram

\[
\begin{array}{ccc}
cccJ & \xrightarrow{g_1} & D_1 \\
\downarrow g_2 & & \downarrow f_1 \\
D_2 & \xrightarrow{f_2} & B
\end{array}
\]

where:

- \( g_1 : J \to D_1 \) is defined by

\[
(\xi_k)_{k \in \omega} \mapsto (\eta_n, b_n)_{n \in \omega}
\]

where \( \xi_k \in C([t_k, t_{k+1}], A_{k+1}) \) for \( k \in \omega \) and \((\eta_n, b_n) = (\xi_{2n+1}, \xi_{2n}(t_{2n+1})) \in M_{2n+1} = C([t_{2n+1}, t_{2n+2}], A_{2n+2}) \oplus A_{2n+1} \) for \( n \in \omega \);

- \( g_2 : J \to D_2 \) is defined by

\[
(\xi_k)_{k \in \omega} \mapsto (\eta_n, b_n)_{n \in \omega}
\]

where \( \xi_k \in C([t_k, t_{k+1}], A_{k+1}) \) for \( k \in \omega \), and \((\eta_n, b_n) = (\xi_{2n}, \xi_{2n-1}(t_{2n})) \in M_{2n} = C([t_{2n}, t_{2n+1}], A_{2n+1}) \oplus A_{2n} \);

- \( f_1 : D_1 \to B \) is defined by

\[
(\eta_n, b_n)_{n \in \omega} \mapsto (c_n)_{n \in \omega}
\]

where \((\eta_n, b_n) \in M_{2n+1} = C([t_{2n+1}, t_{2n+2}], A_{2n+2}) \oplus A_{2n+1}, c_0 = 0, c_{2n+1} = b_n\), and \( c_{2n+2} = \eta_n(t_{2n+2}) \) for \( n \in \omega \);

- \( f_2 : D_2 \to B \) is defined by

\[
(\eta_n, b_n)_{n \in \omega} \mapsto (c_n)_{n \in \omega}
\]

where \((\eta_n, b_n) \in M_{2n} = C([t_{2n}, t_{2n+1}], A_{2n+1}) \oplus A_{2n}, c_{2n} = b_n\), and \( c_{2n+1} = \eta_n(t_{2n+1}) \) for \( n \in \omega \).
We thus have a corresponding Mayer–Vietoris definable six-term exact sequence

\[
\begin{array}{cccc}
K^0(B) & \longrightarrow & K^0(D_1) \oplus K^0(D_2) & \longrightarrow & K^0(J) \\
\phi^0 \downarrow & & \downarrow \phi^0 & & \\
K^1(J) & \longleftarrow & K^1(D_1) \oplus K^1(D_2) & \longleftarrow & K^1(B)
\end{array}
\]

associated with it. Combining this with the definable isomorphism \(K^{1-p}(J) \to K^p(A)\) as above, and with the definable isomorphisms

\[
K^p(B) \cong \prod_{n \in \omega} K^p(A_n)
\]

\[
K^p(D_1) \oplus K^p(D_2) \cong \prod_{n \in \omega} K^p(A_{2n}) \oplus \prod_{n \in \omega} K^p(A_{2n+1}) \cong \prod_{n \in \omega} K^p(A_n)
\]

obtained from Proposition 6.1 and from the homotopy equivalences \(M_n \to A_n\) for \(n \in \omega\), one obtains a definable six-term exact sequence

\[
\prod_{n \in \omega} K^0(A_n) \xrightarrow{\phi^0} \prod_{n \in \omega} K^0(A_n) \to K^1(A) \xrightarrow{\partial} \prod_{n \in \omega} K^1(A_n)
\]

As in the proof of [Sch84, Theorem 7.1], the group homomorphism

\[
\Phi^p : \prod_{n \in \omega} K^p(A_n) \to \prod_{n \in \omega} K^p(A_n)
\]

for \(p \in \{0, 1\}\) is given by

\[
(x_n) \mapsto (x_n - K^p(\varphi_n(x_{n+1})))_{n \in \omega}
\]

whereas the boundary homomorphism \(K^1(A) \to \prod_{n \in \omega} K^1(A_n)\) is induced by the canonical maps \(A_n \to A\). Thus, by definition of \(\lim^1\) and \(\lim^1\) of the tower \((K^p(A_n))_{n \in \omega}\), we have that \(\Phi^0\) and \(\Phi^1\) yield a definable exact sequence

\[
0 \to \lim^1_n K^{1-p}(A_n) \to K^p(A) \to \lim_n K^p(A_n) \to 0.
\]

This concludes the proof of Proposition 6.3.

6.7. C*-stability. Suppose that \(A\) is a separable C*-algebra, and \(H\) is a (not necessarily infinite-dimensional) separable Hilbert space. If \(e \in K(H)\) is a rank one projection, then we can define a *-homomorphism \(e_A : A \to K(H) \otimes A\), \(a \mapsto e \otimes a\). In turn, this induces a definable homomorphism \(K^p(K(H) \otimes A) \to K^p(A)\). The stability—or C*-stability [Cun87]—property of K-homology asserts that such a definable homomorphism \(K^p(K(H) \otimes A) \to K^p(A)\) is in fact an definable isomorphism; see [HR00, Theorem 9.4.1].

**Proposition 6.4.** Suppose that \(A\) is a separable C*-algebra, \(H\) is a separable Hilbert space, \(e \in K(H)\) is a rank one projection, and \(e_A : A \to K(H) \otimes A\) is the *-homomorphism defined by \(a \mapsto e \otimes a\). Then the induced map \(K^p(e_A) : K^p(K(H) \otimes A) \to K^p(A)\) is a definable isomorphism.

**Proof.** It is easy to see that one can reduce to the case when \(H\) is infinite-dimensional. After replacing \(A\) with its stabilization, we can assume that \(p = 0\). As \(K^0(\cdot)\) is naturally isomorphic to \(KK_h(\cdot; \mathbb{C})\), it suffices to prove the corresponding statement for \(KK_h(\cdot; \mathbb{C})\). One can then proceed as in [JT91, E 4.1.3]. Fix an infinite-dimensional separable Hilbert space \(H\), and let \(KK_h(\cdot; \mathbb{C})\) be defined with respect to \(H\). Consider the canonical inclusions \(K(H) \otimes B(H) \subseteq B(H) \otimes B(H) \subseteq B(H \otimes H)\) and the injective *-homomorphism \(e_{K(H)} : K(H) \to K(H) \otimes K(H)\), \(x \mapsto e \otimes x\). Consider also the *-isomorphism \(\lambda : K(H) \to K(H) \otimes K(H) \cong K(H \otimes H)\) defined by setting \(\lambda = \text{Ad}(V) \circ e_{K(H)}\) where \(V \in B(H \otimes H)\) is an isometry with \(VV^* = e \otimes I\). Then \(\lambda\) extends to strict *-isomorphism \(\bar{\lambda} = \text{Ad}(V) \circ e_{K(H)} : B(H) \to B(H \otimes H)\), where \(e_{K(H)} : B(H) \to B(H \otimes H)\) is the strict extension of \(e_{K(H)} : K(H) \to K(H) \otimes K(H) \cong K(H \otimes H)\).

One can then consider the (definable) homomorphism \(G : KK_h(A; \mathbb{C}) \to KK_h(K(H) \otimes A; \mathbb{C})\) induced by the (Borel) function \(\mathcal{F}(A; \mathbb{C}) \to \mathcal{F}(K(H) \otimes A; \mathbb{C})\),

\[
(\phi_+, \phi_-) \mapsto (\bar{\lambda}^{-1} \circ (\text{id}_{K(H)} \otimes \phi_+) \circ \bar{\lambda}^{-1} \circ (\text{id}_{K(H)} \otimes \phi_-)).
\]
Then we have that $\text{KK}_h (e_A; \mathbb{C}) \circ G : \text{KK}_h (A; \mathbb{C}) \to \text{KK}_h (A; \mathbb{C})$ is equal to the identity map. Indeed, $\text{KK}_h (e_A; \mathbb{C}) \circ G$ is induced by the function $F (A; \mathbb{C}) \to F (A; \mathbb{C}),$

$$(\phi_+, \phi_-) \mapsto (\lambda^{-1} \circ (\text{id}_{K(H)} \otimes \phi_+) \circ e_A, \tilde{\lambda}^{-1} \circ (\text{id}_{K(H)} \otimes \phi_-) \circ e_A)$$

$$= (\lambda^{-1} \circ \tilde{e}_{K(H)} \circ \phi_+, \tilde{\lambda}^{-1} \circ \tilde{e}_{K(H)} \otimes \phi_-).$$

We have that

$$(\tilde{\lambda}^{-1} \circ \tilde{e}_{K(H)} \circ \phi_+, \tilde{\lambda}^{-1} \circ \tilde{e}_{K(H)} \otimes \phi_-) \sim (\phi_+, \phi_-)$$

in $F (A; \mathbb{C})$ or, equivalently,

$$(\tilde{e}_{K(H)} \circ \phi_+, \tilde{\lambda}^{-1} \circ \tilde{e}_{K(H)} \otimes \phi_-) \sim (\lambda \circ \phi_+, \tilde{\lambda} \circ \phi_-)$$

in $F_{H \otimes H} (A; \mathbb{C})$, where $F_{H \otimes H} (A; \mathbb{C})$ is defined as $F (A; \mathbb{C})$ by replacing $H$ with $H \otimes H$. We have, by definition of $\lambda$,

$$(\lambda \circ \phi_+, \tilde{\lambda} \circ \phi_-) = (\text{Ad} (V) \circ \tilde{e}_{K(H)} \circ \phi_+, \text{Ad} (V) \circ \tilde{e}_{K(H)} \circ \phi_-).$$

By [JT91, Lemma 1.3.7] there exists a strictly continuous path $(V_t)_{t \in [0,1]}$ of isometries in $B (H \otimes H)$ connecting $I$ to $V$. Thus,

$$(\text{Ad} (V_t) \circ \tilde{e}_{K(H)} \circ \phi_+, \text{Ad} (V_t) \circ \tilde{e}_{K(H)} \circ \phi_-)$$

is a continuous path in $F_{H \otimes H} (A; \mathbb{C})$ connecting $(\tilde{e}_{K(H)} \circ \phi_+, \tilde{\lambda}^{-1} \circ \tilde{e}_{K(H)} \otimes \phi_-)$ to $(\lambda \circ \phi_+, \tilde{\lambda} \circ \phi_-)$. This concludes the proof that $\text{KK}_h (e_A; \mathbb{C}) \circ G$ is the identity of $\text{KK}_h (A; \mathbb{C})$.

We now show that $G \circ \text{KK}_h (e_A; \mathbb{C})$ is the identity of $\text{KK}_h (K(H) \otimes A; \mathbb{C})$. We have that $G \circ \text{KK}_h (A; \mathbb{C})$ is the (definable) group homomorphism induced by the Borel function $F (K(H) \otimes A; \mathbb{C}) \to F (K(H) \otimes A; \mathbb{C}),$

$$(\psi_+, \psi_-) \mapsto (\tilde{\lambda}^{-1} \circ \text{id}_{K(H)} \otimes (\psi_+ \circ e_A), \tilde{\lambda}^{-1} \circ \text{id}_{K(H)} \otimes (\psi_- \circ e_A)).$$

We claim that

$$(\tilde{\lambda}^{-1} \circ \text{id}_{K(H)} \otimes (\psi_+ \circ e_A), \tilde{\lambda}^{-1} \circ \text{id}_{K(H)} \otimes (\psi_- \circ e_A)) \sim (\psi_+, \psi_-)$$

in $F (K(H) \otimes A; \mathbb{C})$ or, equivalently

$$(\text{id}_{K(H)} \otimes (\psi_+ \circ e_A), \text{id}_{K(H)} \otimes (\psi_- \circ e_A))$$

$$\sim (\tilde{\lambda} \circ \psi_+, \tilde{\lambda}^{-1} \circ \psi_-)$$

$$= (\text{Ad} (V) \circ \tilde{e}_{K(H)} \circ \psi_+, \text{Ad} (V) \circ \tilde{e}_{K(H)} \circ \psi_-)$$

in $F_{H \otimes H} (K(H) \otimes A; \mathbb{C})$. Indeed, define $\sigma_1, \sigma_2 : K(H) \otimes A \to K(H) \otimes K(H) \otimes A$ be the (strict) *-homomorphisms given by

$$T \otimes a \mapsto T \otimes e \otimes a$$

and

$$T \otimes a \mapsto e \otimes T \otimes a.$$ We can consider their strict extensions $\sigma_1, \sigma_2 : M (K(H) \otimes A) \to M (K(H) \otimes K(H) \otimes A)$. Then we have that

$$\text{id}_{K(H)} \otimes (\psi_+ \circ e_A) = \psi_+ \circ \sigma_1 : K(H) \otimes A \to B (H \otimes H)$$

and

$$\tilde{e}_{K(H)} \circ \psi \pm = (\text{id}_{K(H)} \otimes \psi_+ \circ e_A) \circ \sigma_2 : K(H) \otimes A \to B (H \otimes H).$$

We have that $\tilde{\sigma}_1 = \text{Ad} (U \otimes 1) \circ \tilde{\sigma}_2$ for some unitary $U \in M (K(H) \otimes K(H) \otimes A)$. Since $M (K(H) \otimes K(H) \otimes A)$ is connected in the strict topology [JT91, Lemma 1.3.7], we have that

$$\sigma_1 = \sigma_2 = (\text{id}_{K(H)} \otimes \psi_+ \circ e_A) \circ (\text{id}_{K(H)} \otimes (\psi_- \circ e_A)).$$

This concludes the proof.
6.8. Split exactness. Suppose that
\[ 0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \]
is an exact sequence of definable groups and definable group homomorphisms. We say that it is definably split if \( p \) is a split epimorphism in the category of definable groups, namely there exists a definable group homomorphism \( f : C \to B \) such that \( f \circ p \) is equal to the identity of \( C \). This is equivalent to the assertion that \( i : A \to B \) is a split monomorphism in the category of definable groups, namely there exists a definable group homomorphism \( g : C \to B \) such that \( g \circ f \) is equal to the identity of \( A \). In turn, this is equivalent to the assertion that there exists a definable isomorphism \( \gamma : B \to A \times C \) that makes the diagram
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow{\text{id}_A} & & \downarrow{\gamma} \\
A & \longrightarrow & B \\
\end{array}
\]
commute.

If \((A, J)\) is a separable \( C^*\)-pair such that the exact sequence
\[ 0 \to J \to A \to A/J \to 0 \]
splits, then \((A, J)\) is, in particular, semi-split. Thus, there is a corresponding six-term exact sequence in K-homology. This reduces to two definably split exact sequences of definable groups and definable group homomorphisms
\[ 0 \to K^p (J) \to K^p (A) \to K^p (A/J) \to 0 \]
for \( p \in \{0, 1\} \). This is the split-exactness property of definable K-homology in the sense of [Cun87].

7. A definable Universal Coefficient Theorem

In this section we consider a definable version of the Universal Coefficient Theorem for K-homology due to Brown [Bro84], later generalized by Rosenberg and Schochet to KK-theory [RS87]. We also consider the fine structure of the definable K-homology groups as in [Sch96] in terms of the notion of filtration for a separable nuclear \( C^*\)-algebra introduced therein. As an application, we show that definable K-homology is a complete invariant for UHF \( C^*\)-algebras up to stable isomorphism, while the same conclusion does not hold for the purely algebraic K-homology.

7.1. Index pairing for K-homology. Suppose that \( A \) is a separable \( C^*\)-algebra. Fix \( i \in \{0, 1\} \). Then one can define a natural definable index pairing \( K_i (A) \times K_i (A) \to \mathbb{Z} \), where \( \mathbb{Z} \) and the countable group \( K_i (A) \) are regarded as standard Borel spaces with respect to the trivial Borel structure. Suppose that \( A \subseteq B (H) \) such that the inclusion map is an ample representation and let \( \mathcal{D} (A) \) be the corresponding Paschke dual algebra. For \( i = 1 \) the pairing is defined by
\[ \langle [u_\ast], [P] \rangle = \text{Index}_{pH^k} (P^{\oplus k} u P^{\oplus k}) \]
where \( k \geq 1 \), \( u \in U (M_k (A^+)) \) is a unitary, \( P \in \mathcal{D} (A) \) is a projection, \( P^{\oplus k} \) is the \( k \)-fold direct sum of \( P \), \( P^{\oplus k} u P^{\oplus k} \in B (H^k) \) satisfies
\[ (P^{\oplus k} u P^{\oplus k})^* (P^{\oplus k} u P^{\oplus k}) \equiv (P^{\oplus k} u P^{\oplus k})^* (P^{\oplus k} u P^{\oplus k}) \equiv P^{\oplus k} \mod K (H^k) \]
and \( \text{Index}_{pH^k} (P^{\oplus k} u P^{\oplus k}) \) is its Fredholm index when regarded as a Fredholm operator on \( P^{\oplus k} H^k \); see [HR00, Definition 7.2.1]. Such a pairing is definable, in the sense that it is induced by a Borel function \( K_1 (A) \times \mathbb{Z} \to \mathcal{D} (A) \to \mathbb{Z} \), considering that the Fredholm index is given by a Borel map; see Section 2.10.

The index pairing \( K_0 (A) \times K^0 (A) \to \mathbb{Z} \) is defined by
\[ \langle [p] - [q], [U] \rangle = \text{Index}_{pH^k} (pU^{\oplus k} p) - \text{Index}_{qH^k} (qU^{\oplus k} q) \]
where \( k \geq 1 \), \( p, q \in M_k (A^+) \) are projections that satisfy \( p \equiv q \mod M_k (A) \) and \( pU^{\oplus k} p \in B (H^k) \) satisfies
\[ (pU^{\oplus k} p)^* (pU^{\oplus k} p) \equiv (pU^{\oplus k} p)^* (pU^{\oplus k} p) \equiv p \mod K (H^k) \]
and \( \text{Index}_{pH^k} (pU^{\oplus k} p) \) is the Fredholm index of \( pU^{\oplus k} p \) regarded as a Fredholm operator on \( pH^k \) and similarly for \( qU^{\oplus k} q \); see [HR00, Definition 7.2.3]. Again, this pairing is definable as the Fredholm index is given by a Borel map.
7.2. Extensions of groups. Suppose that $C, D$ are countable abelian groups. A (2-)cocycle on $C$ with coefficients in $D$ is a function $c : C \times C \to D$ such that, for every $x, y, z \in C$:

- $c(x, y) + c(x + y, z) = c(x, y + z) + c(y, z)$;
- $c(x, y) = c(y, x)$.

A cocycle is a coboundary if it is of the form $(x, y) \mapsto h(x) + h(y) - h(x, y)$ for some function $h : C \to D$. The set $Z(C, D)$ of cocycles on $C$ with coefficients in $D$ is a closed subgroup of the Polish group $D^{C \times C}$ endowed with the product topology (where $D$ is endowed with the discrete topology). The set $B(C, D)$ of coboundaries is a Polishable Borel subgroup of $Z(C, D)$. A weak coboundary is a cocycle $c$ such that, for every finite (or, equivalently, for every finitely-generated) subgroup $S$ of $C$, the restriction of $c$ to $S \times S$ is a coboundary for $S$. Weak coboundaries form a closed subgroup $B_w(C, D)$ of $Z(C, D)$.

The group $\text{Ext}(C, D)$ is the definable group, which is in fact a group with Polish cover (see Remark 1.17), obtained as the quotient $Z(C, D)/B(C, D)$; see [BLP20, Section 7]. The group $\text{PExt}(C, D)$ is the definable subgroup of $\text{Ext}(C, D)$ obtained as $B_w(C, D)/B(C, D)$; see [Sch03]. We also define $\text{Ext}_w(C, D)$ to be the Polish group obtained as the quotient of the Polish group $Z(C, D)$ by the closed subgroup $B_w(C, D)$. By definition, we have a short exact sequence of definable groups

$$0 \to \text{PExt}(C, D) \to \text{Ext}(C, D) \to \text{Ext}_w(C, D) \to 0.$$  

The terminology is due to the fact that every cocycle on $C$ with coefficients in $D$ gives rise to an extension of $C$ by $D$, in such a way that two cocycles differ by a coboundary if and only if the corresponding extensions are isomorphic. Furthermore, every extension of $C$ by $D$ arises from a cocycle in this fashion. Explicitly, if

$$0 \to D \xrightarrow{i} E \xrightarrow{p} C \to 0$$

is an extension of $C$ by $D$, the corresponding cocycle $c$ is defined as follows. For $x, y \in C$, pick lifts $\hat{x}, \hat{y}, \hat{z} \in E$ such that $p(\hat{x}) = x$, $p(\hat{y}) = y$, and $p(\hat{z}) = x + y$. Then we have that $\hat{x} + \hat{y} - \hat{z} \in \ker(p) = \operatorname{ran}(i)$ and hence $c(x, y) := i^{-1}(\hat{x} + \hat{y} - \hat{z}) \in D$. Conversely, given a cocycle $c$ on $C$ with coefficients in $D$ one can define an extension as above, where $E = D \times C$ is endowed with the operation defined by

$$(x, y) + (x', y') = (x + x', c(y, y'), y + y').$$

The weak coboundaries correspond in this way to extension of $C$ by $D$ that are pure, i.e. such that $i(D)$ is a pure subgroup of $E$; see [Fuc70, Section V.29].

If $(C_i)_{i \in \omega}$ is an inductive sequence of finitely-generated abelian groups and $C = \operatorname{colim}_{i \in \omega} C_i$ is the corresponding inductive limit (colimit), then the definable Jensen theorem asserts that $\text{PExt}(C, D)$ is naturally definably isomorphic to $\operatorname{lim}^\omega \text{Hom}(C_i, D)$, and $\text{Ext}_w(C, D)$ is naturally isomorphic as a Polish group to $\operatorname{lim}^\omega \text{Ext}(C_i, D)$; see [BLP20, Theorem 7.4] and [Sch03, Theorem 6.1].

7.3. The Universal Coefficient Theorem. The definable index pairing $K_1(A) \times K^1(A) \to \mathbb{Z}$ induces a definable homomorphism

$$\text{Index}_A : K^1(A) \to \text{Hom}(K_1(A), \mathbb{Z}),$$

where we adopt the notation from [HR00, Definition 7.2.3]. Recall that $K^1(A)$ is defined as $\text{Ext}(A^+)^{-1}$ where $A^+$ is the unitization of $A$. The definable homomorphism $\text{Index}_A : K^1(A^+)^{-1} \to \text{Hom}(K_1(A), \mathbb{Z})$ can be equivalently described as follows; see [RS81]. If $\tau$ is an injective unital extension of $A^+$ by $K(H)$

$$0 \to K(H) \to E \to A^+ \to 0,$$

then it gives rise to a six-term exact sequence in K-theory

$$\xymatrix{ K_0(K(H)) = \mathbb{Z} \ar[r] & K_0(E) \ar[r] & K_0(A^+) \ar[d]^{\partial^p} \ar[l]^-{\partial^f} \cr K_1(A^+) = K_1(A) \ar@{<->}[r] & K_1(E) \ar@{<->}[r] & K_1(K(H)) = \{0\}.}$$

The group homomorphism $K_1(A) \to \mathbb{Z}$ induced by $\tau$ in the diagram above depends only on the corresponding element $[\tau] \in \text{Ext}(A^+)^{-1}$, and it is equal to $\text{Index}_A([\tau])$. As in [HR00, Definition 7.6.7], we let $\partial K^1(A)$ be the definable subgroup of $K_1(A)$ obtained as the kernel of the index homomorphism $\text{Index}_A : K^1(A) \to \text{Hom}(K_1(A), \mathbb{Z})$. 


There is also a definable group homomorphism $\kappa_A : \ast K^1 (A) \to \text{Ext} (K_0 (A), \mathbb{Z})$, defined as follows. Suppose that $\tau$ is an injective unital extension of $A^+$ by $K (H)$ as above, such that moreover $[\tau] \in \text{Ker} (\gamma_A)$. Then the six-term exact sequence above reduces to a short exact sequence

$$0 \to K_0 (K (H)) = \mathbb{Z} \to K_0 (E) \to K_0 (A^+) \to 0$$

This defines an element of $\text{Ext} (K_0 (A^+), \mathbb{Z})$, which in turn defines an element of $\text{Ext} (K_0 (A), \mathbb{Z})$ via the inclusion $K_0 (A) \to K_0 (A^+)$. This element of $\text{Ext} (K_0 (A), \mathbb{Z})$ depends only on the class $[\tau]$ in $\text{Ext} (A^+, A^-)$ of the extension $\tau$, and defines a group homomorphism $\kappa_A : \text{Ker} (\gamma_A) \to \text{Ext} (K_0 (A), \mathbb{Z})$. We notice that $\kappa_A$ is a definable group homomorphism. Indeed, adopting the notation from Section 4.1, suppose that $\tau$ corresponds to the element $\kappa$ and $\gamma$ are the same element of $K_0 (A)$. One then has that, for $p, q, p', q' \in M_n (A^+)$ for some $n \geq 1$, we have that

$$(p - q + [p'] - [q']) = [p \oplus p'] - [q \oplus q'].$$

Then by Lemma 2.50 one can choose, in a Borel way from $p, q, p', q'$, and $\tau$, a unital injective *-homomorphism. A cocycle $c_{\tau} \in Z (K_0 (A), \mathbb{Z})$ that corresponds to the element $\kappa_A ([\tau])$ of $\text{Ext} (K_0 (A), \mathbb{Z})$ can be defined in a Borel fashion from $\tau$ as follows. The extension corresponding to $\tau$ is

$$0 \to K (H) \to E \to A^+ \to 0$$

where

$$E = \{(x, y) \in B (H) \oplus A^+ : q (x) \equiv (y) \mod K (H) \}$$

and $K (H) \to E$, $x \mapsto (x, 0)$.

Given a pair $([p] - [q], [p'] - [q'])$ of elements of $K_0 (A^+)$, where $p, q, p', q' \in M_n (A^+)$ for some $n \geq 1$, we have that

$$c_{\tau} ([p] - [q], [p'] - [q']) = n$$

if and only if for some $k \geq 1$ and pair of projections $x, y \in M_k (K (H))$, one has that

$$[\tau] - [(x, 0)] - [(y, 0)]$$

and

$$[\tilde{p}, p] - [(\tilde{q}, q)] + [(\tilde{p}', p')] - [(\tilde{q}', q')] - [(\tilde{r}, p \oplus p')] + [(\tilde{s}, q \oplus q')]$$

are the same element of $K_0 (E)$, and rank $(x) - \text{rank} (y) = n$.

In a similar fashion as above, by replacing $A$ with its suspension, one can consider a definable group homomorphism

$$\text{Index}_A : K^0 (A) \to \text{Hom} (K_0 (A), \mathbb{Z})$$

with kernel $\ast K^0 (A)$, and a definable group homomorphism

$$\kappa_A : \ast K^0 (A) \to \text{Ext} (K_1 (A), \mathbb{Z}).$$

We recall the following definition of a C*-algebra satisfying the Universal Coefficient Theorem (UCT); see [RS81, Definition 4.4].

**Definition 7.1.** A separable C*-algebra $A$ is said to satisfy the *Universal Coefficient Theorem* (UCT) for $C$, or the pair $(A, C)$ satisfies the UCT, if for $p \in [0, 1]$ the group homomorphisms $\text{Index}_A : K^p (A) \to \text{Hom} (K_p (A), \mathbb{Z})$ is surjective, and the group homomorphism $\kappa_A : \text{Ker} (\gamma_A) = \ast K^0 (A) \to \text{Ext} (K_1 (A), \mathbb{Z})$ is an isomorphism.
It is proved in [Bro84] that all the separable nuclear C*-algebras in the so-called bootstrap class satisfy the UCT for C; see also [Bro75]. In fact one can more generally consider the UCT for B, where B is any separable C*-algebra, defined in terms of Kasparov’s KK-groups; see [RS87]. It is unknown whether there exists a separable nuclear C*-algebra that does not satisfy the UCT.

7.4. Weak and asymptotic K-homology groups. We now recall the notion of a filtration (or KK-filtration) for a separable nuclear C*-algebra as in [Sch96, Definition 1.4], and we define the weak and asymptotic K-homology groups for C*-algebras with a filtration.

**Definition 7.2.** Suppose that A is a separable, nuclear C*-algebra. An inductive sequence \((A_n, \eta_n)_{n \in \omega}\) of separable, nuclear C*-algebras is a filtration of A if:

- for every \(n \in \omega\), \(A_n\) satisfies the Universal Coefficient Theorem for C (as in Definition 7.1);
- for every \(n \in \omega\) and \(p \in \{0, 1\}\), \(K_p(A_n)\) is a finitely generated group;
- \(A\) is KK-equivalent to the inductive limit of the sequence \((A_n, \eta_n)_{n \in \omega}\).

**Remark 7.3.** A slightly more restrictive definition is considered in [Sch96, Definition 1.4], where \(A_n\) is supposed to be commutative.

We let \(C\) be the category that has separable, nuclear C*-algebras with a filtration as objects, and *-homomorphisms as morphisms.

Suppose that \(A\) is a separable, nuclear C*-algebra with a filtration \((A_n)_{n \in \omega}\). Then the inductive limit \(\text{colim}_n A_n\) of the sequence \((A_n)_{n \in \omega}\) satisfies the UCT for \(C\) by [Sch96, Theorem 4.1]. Since the UCT is invariant under KK-equivalences, \(A\) satisfies the UCT as well. Thus, the definable group homomorphism \(\kappa_A : \circ K^p(A) = \text{Ker}(\text{Index}_A) \to \text{Ext}(K_p(A), C)^\omega\) is an isomorphism. After replacing \(A\) with \(\text{colim}_n A_n\) we can assume that \(A = \text{colim}_n A_n\). Notice that, as \(K_0(A_n)\) and \(K_1(A_n)\) are finitely-generated and \(A_n\) satisfies the UCT for \(C\), it follows that \(K^0(A_n)\) and \(K^1(A_n)\) are countable groups.

We thus have a surjective definable homomorphism \(K^p(A) \to \text{lim}_n K^p(A_n)\) as in Milnor’s exact sequence. We define the asymptotic K-homology group \(K^p_\infty(A)\) to be the kernel of the definable homomorphism \(K^p(A) \to \text{lim}_n K^p(A_n)\). As \(A\) satisfies the UCT for \(C\), the definable isomorphism \(\kappa_A : K^p(A) \to \text{Ext}(K_1-p(A), C)^\omega\) is an isomorphism. Since, for every \(n \in \omega\), \(A_n\) satisfies the UCT for \(C\), \(K^p_\infty(A) \subseteq \text{Ker}(\text{Index}_A)\) is equal to the inverse image of \(\text{PExt}(K_{1-p}(A), C)^\omega\) under \(\kappa_A\). In particular, this shows that \(K^p_\infty(A)\) does not depend on the choice of the filtration for \(A\). The assignment \(A \mapsto K^p_\infty(A)\) defines a homotopy-invariant functor from \(C\) to the category of definable groups. As noticed above, \(K^p_\infty(A)\) is naturally definably isomorphic to \(\text{PExt}(K_{1-p}(A), C)^\omega\). We also have that \(K_{1-p}(A) = \text{colim}_n K_{1-p}(A_n)\), and hence by the definable Jensen theorem \(K^p_{\infty}(A)\) is definably isomorphic to \(\text{lim}_n \text{Hom}(K_{1-p}(A_n), C)^\omega\).

We define the weak K-homology group \(K^p_\infty(A)\) to be the quotient of \(K^p(A)\) by \(K^p_{\infty}(A)\). As \(K^p(A)\) is a definable group and \(K^p_{\infty}(A)\) is a definable subgroup, \(K^p_\infty(A)\) has an induced structure of semidefinable group. We now observe that \(K^p_\infty(A)\) is a definable group, definably isomorphic to the Polish group \(\text{lim}_n K^p(A_n)\).

**Lemma 7.4.** Suppose that \(A\) is a separable, nuclear C*-algebra, and \((A_n, \eta_n)_{n \in \omega}\) is a filtration of \(A\). Then \(K^n_\infty(A)\) is a definable group, definably isomorphic to the Polish group \(\text{lim}_n K^p(A_n)\).

**Proof.** After replacing \(A\) with its suspension, we can assume that \(p = 1\). Furthermore, after replacing \(A\) with \(\text{colim}_n A_n\), we can assume that \(A = \text{colim}_n A_n\). Finally, after replacing \(A\) with \(A^+\) and \(A_n\) with \(A_n^+\), we can assume that \(A\) and \(A_n\) for \(n \in \omega\) are unital, and \(\eta_n : A_n \to A_{n+1}\) is a unital *-homomorphism. In this case, we have that \(K^1(A) = \text{Ext}(A)^{-1}\) and \(K^1(A_n) = \text{Ext}(A_n)^{-1}\) for \(n \in \omega\). The surjective definable homomorphism \(\text{Ext}(A)^{-1} \to \text{lim}_n \text{Ext}(A_n)^{-1}\) induced by the canonical maps \(A_n \to A\) induces a definable group isomorphism \(K^p_\infty(A) \to \text{lim}_n \text{Ext}(A_n)^{-1}\). It suffices to show that such a definable group isomorphism is an isomorphism in the category of semidefinable groups. In other words, we need to show that the inverse group homomorphism \(\text{lim}_n \text{Ext}(A_n)^{-1} \to K^p_\infty(A)\) is also definable. Recall that \(\text{Ext}(A)^{-1}\) is the quotient of the Polish space \(E(A)\) by the equivalence relation \(\approx\) as in Section 4.1.

Fix, for every \(\ell \in \omega\) an enumeration \((x^{(l)}_n)_{n \in \omega}\) of \(\text{Ext}(A)^{-1}\). For \(\ell_0 < \ell_1\) define the bonding map \(\eta_{(\ell_1, \ell_0)} : A_{\ell_0} \to A_{\ell_1}\).
and set $\eta_{(\ell, \ell)} = \text{id}_{A_\ell}$ for $\ell \in \omega$. Define

$$\eta_{(\infty, \ell)} : A_\ell \to A$$

to be the canonical map. Let also $p^{(\ell_0, \ell_1)} : \text{Ext}(A_{\ell_1})^{-1} \to \text{Ext}(A_{\ell_0})^{-1}$ be the group homomorphism induced by the bonding map $\eta_{(\ell_1, \ell_0)} : A_{\ell_0} \to A_{\ell_1}$. Then an element of $\lim_{\ell} \text{Ext}(A_\ell)^{-1}$ is a sequence $(x_\ell^{(\ell)})_{\ell \in \omega}$ such that, for $\ell_0 < \ell_1$, $p^{(\ell_0, \ell_1)}(x_{\ell_1}^{(\ell_1)}) = x_{\ell_0}^{(\ell_0)}$. For every $\ell, n \in \omega$ fix $\varphi_n^{(\ell)} \in \mathcal{E}(A_\ell)$ such that $[\varphi_n^{(\ell)}] = x_n^{(\ell)}$. For every $\ell \in \omega$ and $n, m \in \omega$ such that $p^{(\ell-1, \ell)}(x_n^{(\ell)}(m)) = p^{(\ell-1, \ell)}(x_m^{(\ell)}(n))$ fix $\varphi_n^{(m)} \in \mathcal{E}(A_{\ell-1})$ such that $\text{Ad}(U_{n,m}^{(\ell)}(\ell)) \circ \varphi_n^{(m)} \circ \eta_{\ell-1} = \varphi_n^{(m)} \circ \eta_{\ell-1}$. If $(x_n^{(\ell)})_{n \in \omega}$ is an element of $\lim_{n} \text{Ext}(A_n)^{-1}$, then setting $\psi^{(\ell)} := \text{Ad}(U_{n,m}^{(\ell)}(\ell)) \circ \varphi_n^{(m)} \circ \eta_{n}^{(\ell)} \in \mathcal{E}(A_\ell)$, one obtains a sequence $(\psi^{(\ell)})_{\ell \in \omega}$ such that $\psi^{(\ell)} \circ \eta_{\ell-1} = \psi^{(\ell-1)}$ for every $\ell > 0$. Therefore, setting $\psi = \text{colim}_{\ell} \psi^{(\ell)} : A \to B(H)$ defines an element of $\mathcal{E}(A)$ such that $[\psi \circ \eta_{(\infty, \ell)}] = x_\ell^{(\ell)}$ for every $\ell \in \omega$, and hence the image of $[\psi] \in K^p_\omega(A)$ under the definable homomorphism $K^p_\omega(A) \to \lim_{n} \text{Ext}(A_n)^{-1}$ is equal to $(x_\ell^{(\ell)})_{\ell \in \omega}$. This construction describes a Borel lift for the isomorphism $\lim_n \text{Ext}(A_n)^{-1} \to K^p_\omega(A)$, showing that it is definable.

The assignment $A \mapsto K^p_\omega(A)$ defines a functor from $\mathcal{C}$ to the category of definable groups. The weak K-homology group $K^1_B(A)$ is isomorphic to the group $\text{KL}(A, \mathbb{C})$ from [Rør95, Section 4]; see also [RS02, 2.4.8] and [Sch96, Corollary 3.8]. A description of $K^1_B(A)$ in terms of the sum $K(A)$ of all the K-theory groups of $A$ in all degrees and all cyclic coefficient groups is obtained in [DL96]; see also [Sch96, Theorem 3.10].

Suppose that $A$ is a separable, nuclear $\mathcal{C}^*$-algebra with a filtration $(A_n, \eta_n)_{n \in \omega}$. The index homomorphisms

$$\text{Index}_{A_n} : K^p(A_n) \to \text{Hom}(K_p(A_n), \mathbb{Z})$$

for $n \in \omega$ induce a definable group homomorphism

$$\text{Index}_A : K^p_\omega(A) \to \text{Hom}(K_p(A), \mathbb{Z}) = \lim_n \text{Hom}(K_p(A_n), \mathbb{Z}) .$$

Similarly the definable group homomorphisms

$$\nu_{A_n}^{-1} : \text{Ext}(K_p(A_n), \mathbb{Z}) \to K^p(A_n)$$

for $n \in \omega$ induce a definable group homomorphism

$$\lim_n \text{Ext}(K_p(A_n), \mathbb{Z}) = \text{Ext}_w(K_p(A), \mathbb{Z}) \to K^p_\omega(A) .$$

This gives a short exact sequence of definable groups

$$0 \to \text{Ext}_w(K_p(A), \mathbb{Z}) \to K^p_\omega(A) \to \text{Hom}(K_p(A), \mathbb{Z}) \to 0 .$$

By definition of PExt and Ext, we also have a short exact sequence of definable groups

$$0 \to \text{PExt}(K_p(A), \mathbb{Z}) \to \text{Ext}(K_p(A), \mathbb{Z}) \to \text{Ext}_w(K_p(A), \mathbb{Z}) \to 0$$

where PExt($K_p(A), \mathbb{Z}) \to \text{Ext}(K_p(A), \mathbb{Z})$ is the inclusion map and Ext($K_p(A), \mathbb{Z}) \to \text{Ext}_w(K_p(A), \mathbb{Z})$ is the quotient map.

**Proposition 7.5.** Suppose that $A$ is a separable, nuclear $\mathcal{C}^*$-algebra with a filtration and $p \in \{0, 1\}$. If $K_p(A)$ is torsion-free, then $K^p_\omega(A)$ is naturally isomorphic to $\text{Ext}(K_{1-p}(A), \mathbb{Z})$, and $K^p_\omega(A)$ is naturally isomorphic to $\text{Hom}(K_p(A), \mathbb{Z})$.

**Proof.** Since $K_p(A)$ is torsion-free, we have that PExt($K_p(A), \mathbb{Z}) = \text{Ext}(K_p(A), \mathbb{Z})$. Therefore,

$$K^p_\omega(A) \cong \text{PExt}(K_p(A), \mathbb{Z}) = \text{Ext}(K_p(A), \mathbb{Z}) .$$

From the exact sequence

$$0 \to \text{PExt}(K_p(A), \mathbb{Z}) \to \text{Ext}(K_p(A), \mathbb{Z}) \to \text{Ext}_w(K_p(A), \mathbb{Z}) \to 0$$

we conclude that

$$\text{Ext}_w(K_p(A), \mathbb{Z}) = \{0\} .$$

From this and the exact sequence

$$0 \to \text{Ext}_w(K_p(A), \mathbb{Z}) \to K^p_\omega(A) \to \text{Hom}(K_p(A), \mathbb{Z}) \to 0$$

we conclude that

$$K^p_\omega(A) \cong \text{Hom}(K_p(A), \mathbb{Z}) .$$
This concludes the proof. □

**Corollary 7.6.** Suppose that $A$ is a separable, nuclear $C^*$-algebra with a filtration and $p \in \{0,1\}$ is such that $K_p(A)$ is a finite-rank torsion-free abelian group and $K_{1-p}(A)$ is trivial. We can write

$$K_p(A) = \Lambda \oplus \Lambda'$$

where $\Lambda'$ is finitely-generated and $\Lambda$ has no finitely-generated direct summand. Then

$$K^p(A) \cong \text{Hom}(K_p(A),\mathbb{Z}) \cong \text{Hom}(\Lambda',\mathbb{Z})$$

and

$$K^{1-p}(A) \cong \text{Ext}(K_p(A),\mathbb{Z}) \cong \text{Ext}(\Lambda,\mathbb{Z})$$

as definable groups.

**Proof.** After replacing $A$ with $SA$, we can assume that $p = 0$. We have that

$$K^0_w(A) \cong \text{PExt}(K_1(A),\mathbb{Z}) \cong \{0\}.$$ 

Therefore,

$$K^0(A) \cong K^0_w(A) \cong \text{Hom}(K_0(A),\mathbb{Z}) \cong \text{Hom}(\Lambda',\mathbb{Z}).$$

Similarly, we have that

$$K^1(A) \cong K^1_w(A) \cong \text{PExt}(K_0(A),\mathbb{Z}) \cong \text{Ext}(K_0(A),\mathbb{Z}) \cong \text{Ext}(\Lambda,\mathbb{Z}).$$

This concludes the proof. □

**Corollary 7.7.** Suppose that $p \in \{0,1\}$ and $A,B$ are separable, nuclear $C^*$-algebras with a filtration, such that $K_p(A)$ and $K_p(B)$ are finite-rank torsion-free abelian groups, and $K_{1-p}(A)$ and $K_{1-p}(B)$ are trivial. Then the following assertions are equivalent:

1. $K^i(A)$ and $K^i(B)$ are definably isomorphic for $i \in \{0,1\}$;
2. $K_p(A)$ and $K_p(B)$ are isomorphic.

If furthermore $K_p(A)$ and $K_p(B)$ have no nonzero finitely-generated direct summand, then the following assertions are equivalent:

1. $K^{1-p}(A)$ and $K^{1-p}(B)$ are definably isomorphic;
2. $K_p(A)$ and $K_p(B)$ are isomorphic.

**Proof.** After passing to the suspension, we can assume that $p = 0$. Since $K_0(A)$ and $K_0(B)$ are finite-rank torsion-free abelian groups, we can write

$$K_0(A) = \Lambda_A \oplus \Lambda'_A$$

$$K_0(B) = \Lambda_B \oplus \Lambda'_B$$

where $\Lambda_A, \Lambda_B$ have no nonzero finitely-generated direct summand, and $\Lambda'_A, \Lambda'_B$ are finitely-generated. Then we have that $K_0(A) \cong K_0(B)$ if and only if $\Lambda_A \cong \Lambda_B$ and $\Lambda'_A \cong \Lambda'_B$. We have that $\Lambda'_A \cong \Lambda'_B$ if and only if

$$\text{Hom}(\Lambda'_A,\mathbb{Z}) \cong \text{Hom}(\Lambda'_B,\mathbb{Z})$$

Furthermore, by [BLP20, Corollary 7.6], we have that $\Lambda_A \cong \Lambda_B$ if and only if $\text{Ext}(\Lambda_A,\mathbb{Z})$ and $\text{Ext}(\Lambda_B,\mathbb{Z})$ are definably isomorphic. The conclusion thus follows from Corollary 7.6. □

We now show that Corollary 7.7 does not hold if $K^p(A)$ and $K^p(B)$ are merely asked to be isomorphic, rather than *definably* isomorphic; see Theorem 7.8.
7.5. Stable isomorphism of UHF algebras. Recall that a uniformly hyperfinite (UHF) C*-algebra is an infinite-dimensional separable unital C*-algebra that is the limit of an inductive sequence of full matrix algebras [Dav96b, Example III.5.1]. Since finite-dimensional C*-algebras are nuclear, satisfy the UCT for $\mathbb{C}$, and have finitely-generated $K_0$ and $K_1$ groups, UHF C*-algebras are nuclear and have a filtration. If $A$ is a UHF C*-algebra, then $K_0 (A)$ is a rank 1 torsion-free abelian group that is not isomorphic to $\mathbb{Z}$, while $K_1 (A)$ is trivial. Given a rank 1 torsion-free abelian group $\Lambda$ that is not isomorphic to $\mathbb{Z}$, there exists a UHF C*-algebra $A_\Lambda$ such that $K_0 (A_\Lambda) \cong \Lambda$. By Proposition 7.5, we have that $K_1 (A_\Lambda)$ is definably isomorphic to $\text{Ext} (\Lambda, \mathbb{Z})$, while $K_0 (A_\Lambda)$ is trivial.

Recall that a uniformly hyperfinite (UHF) C*-algebra is an infinite-dimensional separable unital C*-algebra [Gli60]; see also [RLL00, Chapter 7]. Recall that two separable C*-algebras $A, B$ are stably isomorphic (or, equivalently, Morita-equivalent; see [RW98, Definition Theorem 5.55]) if $A \otimes K (H) \cong B \otimes K (H)$, where $K (H)$ is the C*-algebra of compact operators on the separable infinite-dimensional Hilbert space.

**Theorem 7.8.** Definable $K^1$ is a complete invariant for UHF C*-algebras up to stable isomorphism. In contrast, there exists an uncountable family of pairwise non stably isomorphic UHF C*-algebras whose $K^1$-groups are algebraically isomorphic (but not definably isomorphic).

**Proof.** It follows from the classification of AF C*-algebras by K-theory that the (unordered) $K_0$-group is a complete invariant for UHF C*-algebras up to stable isomorphism; see [Dav96b, Chapter IV]. From this and Corollary 7.7, it follows that the definable $K^1$-group is also a complete invariant for UHF C*-algebras up to stable isomorphism.

If, adopting the notations above, $(\Lambda_i)_{i \in \mathbb{R}}$ is an uncountable family of pairwise nonisomorphic rank 1 torsion-free abelian groups not isomorphic to $\mathbb{Z}$ such that $\text{Ext} (\Lambda_i, \mathbb{Z}) \cong \text{Ext} (\Lambda_j, \mathbb{Z})$ for $i, j \in \mathbb{R}$, then $(A_{\Lambda_i})_{i \in \mathbb{R}}$ is an uncountable family of pairwise non stably isomorphic UHF C*-algebras whose $K^1$-groups are algebraically isomorphic but not definably isomorphic. \[\square\]

8. Definable K-homology of compact metrizable spaces

In this section, we consider definable K-homology of compact metrizable spaces, which can be seen as a particular instance of definable K-homology when restricted to unital, commutative, separable C*-algebras. As another application of the definable Universal Coefficient Theorem, we show that definable K-homology of compact metrizable spaces is a finer invariant than the its purely algebraic version, even when restricted to connected 1-dimensional subspaces of $\mathbb{R}^3$.

8.1. K-homology and topological K-theory of spaces. The notion (definable) of K-homology for compact metrizable spaces is obtained as a particular instance of the corresponding notion for separable C*-algebras, by considering the contravariant functor $X \mapsto C (X)$ assigning to a compact metrizable space the separable unital C*-algebra $C (X)$ of continuous complex-valued functions on $X$. Thus, if $X$ is a compact metrizable space, its definable K-homology groups are given by

$$K_p (X) := K^p (C (X))$$

for $p \in \{0, 1\}$; see [HR00, Chapter 7]. The reduced definable K-homology groups are similarly defined by

$$\tilde{K}_p (X) := \tilde{K}^p (C (X)).$$

In particular, one sets

$$\text{Ext} (X) := \tilde{K}_1 (X) = \tilde{K}^1 (C (X)) \cong \text{Ext} (C (X)).$$

Similarly, the topological K-theory groups of $X$ can be defined in terms of the K-theory of $C (X)$ by setting

$$K^p (X) := K_p (C (X));$$
Proposition 8.1. Equivalently, these groups can be defined in terms of vector bundles over $X$; see [Kar08, Chapter II] and [WO93, Chapter 13]. One can also define the reduced K-group $\tilde{K}^p(X)$ to be the quotient of $K^p(X)$ by the subgroup obtained as the image of $K^p(\{\ast\})$ under the homomorphism induced by the map $X \to \{\ast\}$. (Notice that $K^1(\{\ast\})$ is trivial and $K^0(\{\ast\}) \cong \mathbb{Z}$.)

8.2. The Universal Coefficient Theorem. Recall that a compact polyhedron is a compact metrizable space $P$ that is obtained as the topological realization of a finite simplicial complex; see [MS82, Appendix 1]. (In the following, we assume that all the polyhedra are compact.) The topological K-groups of a polyhedron are finitely-generated [HR00, Proposition 7.14]. Furthermore, if $P$ is a polyhedron, then it can be proved by induction on the number of simplices of the corresponding simplicial complex that the unital C*-algebra $C(P)$ satisfies the UCT for $C$ [Bro75, Bro84].

If $X$ is a compact metrizable space, then one can write $X$ as the (inverse) limit of a tower $(X_n)_{n \in \omega}$ of compact polyhedra [MS82, Section 1.6]. Such a tower, called a polyhedral resolution of $X$ in [MS82], can be obtained by considering the topological realizations of the nerves of a sequence of finite open covers of $X$ that is dense in the ordered set of finite open covers of $X$. If $(X_n)$ is a polyhedral resolution for $X$, then $(C(X_n))_{n \in \omega}$ is a filtration for $C(X)$ in the sense of Definition 7.2. Thus, one can consider the weak K-homology group

$$K^w_p(X) := K^p_\ast(C(X)) \cong \lim_{n} K_p(X_n)$$

and the asymptotic K-homology groups

$$K^\infty_p(X) := K^p_\ast(C(X)) \cong P\text{Ext}(K^{1-p}(X), \mathbb{Z}).$$

We can also consider their reduced versions, by letting $\tilde{K}^w_p(X)$ be the kernel of the definable group homomorphism $K^w_p(X) \to K^w_p(\{\ast\})$ induced by the map $X \to \{\ast\}$, and similarly for $\tilde{K}^\infty_p(X)$. It is then easy to see that

$$\tilde{K}^w_p(X) = \lim_{n} \tilde{K}_p(X_n)$$

and

$$\tilde{K}^\infty_p(X) \cong K^\infty_p(X).$$

By definition, we have short exact sequences of definable groups

$$0 \to K^\infty_p(X) \to K_p(X) \to K^w_p(X) \to 0$$

and

$$0 \to \tilde{K}^\infty_p(X) \to \tilde{K}_p(X) \to \tilde{K}^w_p(X) \to 0.$$

As particular instances of Proposition 7.5, Corollary 7.6, and Corollary 7.7 (or, precisely, their analogues for reduced K-homology), one obtains the following.

**Proposition 8.1.** Suppose that $X$ is a compact metrizable space and $p \in \{0, 1\}$. If $\tilde{K}^p(X)$ is torsion-free, then $K^\infty_p(X)$ is naturally definably isomorphic to $\text{Ext}(K^{1-p}(X), \mathbb{Z})$, and $K^w_p(X)$ is naturally isomorphic to $\text{Hom}(\tilde{K}^p(X), \mathbb{Z})$.

**Corollary 8.2.** Suppose that $X$ is a compact metrizable space and $p \in \{0, 1\}$ is such that $\tilde{K}^p(X)$ is a finite-rank torsion-free abelian group and $K^{1-p}(X)$ is trivial. We can write

$$\tilde{K}^p(X) = \Lambda \oplus \Lambda'$$

where $\Lambda'$ is finitely-generated and $\Lambda$ has no finitely-generated direct summand. Then

$$\tilde{K}^w_p(X) \cong \text{Hom}(\tilde{K}^p(X), \mathbb{Z}) \cong \text{Hom}(\Lambda', \mathbb{Z})$$

and

$$\text{Ext}(\tilde{K}^p(X), \mathbb{Z}) \cong \text{Ext}(\Lambda, \mathbb{Z})$$

as definable groups.

**Corollary 8.3.** Suppose that $p \in \{0, 1\}$, and $X, Y$ are compact metrizable spaces, such that $\tilde{K}^p(X)$ and $\tilde{K}^p(Y)$ are finite-rank torsion-free abelian groups, and $K^{1-p}(X)$ and $K^{1-p}(Y)$ are trivial. Then the following assertions are equivalent:

1. $\tilde{K}_i(X)$ and $\tilde{K}_i(Y)$ are definably isomorphic for $i \in \{0, 1\}$;
2. $\tilde{K}^p(A)$ and $\tilde{K}^p(B)$ are isomorphic.
If furthermore $K^p(X)$ and $K^p(Y)$ have no nonzero finitely-generate direct summand, then the following assertions are equivalent:

1. $\tilde{K}_{1-p}(X)$ and $\tilde{K}_{1-p}(Y)$ are definably isomorphic;
2. $K^p(A)$ and $K^p(B)$ are isomorphic.

8.3. Solenoids. A (1-dimensional) solenoid is a compact metrizable space $X$ that is homeomorphic to a 1-dimensional compact connected abelian group other than $T$. Thus, if $\Lambda$ is a rank 1 torsion-free abelian group (or, equivalently, a subgroup of $\mathbb{Q}$) other than $\mathbb{Z}$, then its Pontryagin dual group $X_\Lambda := \Lambda^*$ is a solenoid, and every solenoid arises in this fashion (up to homeomorphism). A solenoid $X$ can be realized as a compact subset of $\mathbb{R}^3$ (but not of $\mathbb{R}^2$) [ES52, Exercise VIII.E]; see also [JWZZ11, JWZ08, Bog88b, Bog88a]. Solenoids were originally considered by Vietoris [Vie27] and van Dantzig [vD32]. They arise in the context of dynamical systems, and they provided in the work of Smale the first examples of attractors of dynamical systems that are strange [Rue06, Sma67, Wil74].

If $T$ is the circle, then one has that $K^1(T) = \mathbb{Z}$ and $K^1(T) = \{0\}$. Furthermore, if $\varphi : T \to T$ is a continuous map of degree $n \in \mathbb{Z}$, then the induced map $\varphi^* : K^0(T) \to K^0(T)$ is given by $x \mapsto nx$. It follows easily from this that, if $A$ is a subgroup of $\mathbb{Q}$, then $K^1(X_A) \cong \mathbb{A}$ and $K^0(X_A) \cong \{0\}$. Thus, by Proposition 8.1, we have that $K_0(X) \cong \text{Ext} (\Lambda \times \mathbb{Z})$ and $K_1(X) \cong \{0\}$ as definable groups. When $\Lambda = \mathbb{Z}[1/p]$ for some prime number $p$, then the corresponding solenoid $X_\Lambda$ is called the $p$-adic solenoid. Its $K$-homology groups are also computed in [KS77, Theorem 6.8]. As in the proof of Theorem 7.8, we have the following.

**Theorem 8.4.** Definable $K_0$ is a complete invariant for 1-dimensional solenoids up to homeomorphism. In contrast, there exist uncountably many pairwise non homeomorphic 1-dimensional solenoids whose $K_0$-groups are algebraically isomorphic (but not definably isomorphic).

**Proof.** If $\Lambda$ is a 1-dimensional solenoid, then $K^1(X_\Lambda) \cong \Lambda$ and $K^0(X_\Lambda) \cong \{0\}$. It follows from this and Corollary 8.3 that definable $K_0$ is a complete invariant for 1-dimensional solenoids up to homeomorphism.

If $(\Lambda_i)_{i \in \mathbb{R}}$ is an uncountable family of pairwise nonisomorphic rank 1 torsion-free abelian groups such that $\text{Ext} (\Lambda_i \times \mathbb{Z}) \cong \text{Ext} (\Lambda_j \times \mathbb{Z})$ for $i, j \in \mathbb{R}$, then $(X_\Lambda_i)_{i \in \mathbb{R}}$ is an uncountable family of pairwise non homeomorphic solenoids whose $K_0$-groups are algebraically isomorphic but not definably isomorphic. \qed

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