Mathematics of Plott choice functions

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Abstract

This paper is devoted to a study of mathematical structures arising from choice functions satisfying the path independence property (Plott functions). We broaden the notion of a choice function by allowing of empty choice. This enables us to define a lattice structure on the set of Plott functions. Moreover, this lattice is functorially dependent on its base. We introduce a natural convex structure on the set of linear orders (or words) and show that Plott functions are in one-to-one correspondence with convex subsets in this set of linear orders. That correspondence is compatible with both lattice structures.

Keywords: Convex geometries, shuffle, linear orders, lattices, direct image, path independence, convex structure

Dedicated to the memory of Andrei Malishevski

1 Introduction

The paper is devoted to a study of choice functions satisfying the path independence property. This property was introduced by Plott [11] and we will call such functions Plott functions. He considered the concept of “path independence” as a mean of weakening the condition of rationality. The issue of ”rationality of the choice” was the main theme of investigations of Plott functions (see, for example, [11 2 9 10]). Because we do not discuss this

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issues. Our purpose is to show a very nice and rich mathematics related to Plott functions.

There are two starting points for this mathematics. The first one is the Aizerman-Malishevski theorem which states that the class of Plott functions is identical to the class of joint-extremal choice functions. This theorem was announced in [2]; apparently its first proof has appeared in [9]. The next important step was made by Koshevoy [7]. He observed that Plott functions can be identify with the so-called convex geometries (these geometries were introduced in [4]). This enables to give a transparent explanation of the Aizerman-Malishevski theorem. More important was that this correspondence hints on hidden mathematical structures of Plott functions.

One circumstance impedes progress in this direction. Historically, mainly due to psychological reasons, only choice functions with non-empty values have been considered. The elimination of this non-essential restriction allows us to define a lattice structure on the set of Plott functions. To our mind, this lattice is the third (or the fourth) most significant lattice associated with a finite set, after the Boolean lattice of subsets and the semimodular lattice of partitions.

The set of primitive (or join-irreducible) elements of this lattice consists of linear Plott functions. These functions correspond to objects which are very close to linear orders; namely, to linear orders on subsets. To give such an order on a set $X$ is the same as to give a simple (without repeated letters) word over the alphabet $X$. Let $\mathbf{SW}(X)$ denote the set of simple words. We define a convex structure on the set $\mathbf{SW}(X)$ and show that Plott functions can be identified with convex subsets of $\mathbf{SW}(X)$.

Another interest property of Plott functions is its functoriality with respect to a base change. If $\phi : X \rightarrow Y$ is a mapping of sets and $f$ is a Plott function on $X$ then we define a Plott function $\phi_*(f)$ on $Y$, the direct image of $f$. The direct image commutes with the join and pushes forward linear Plott functions into linear Plott functions. In terms of words this means that the corresponding map $\phi^* : \mathbf{SW}(X) \rightarrow \mathbf{SW}(Y)$ sends convex sets into convex sets. Similarly one can define the inverse image of Plott functions.

An important application of the direct image is related to the issue of rationalization. We show that every Plott function on a set $X$ possesses a rationalization by some partial order on some superset $X' \rightarrow X$. Moreover, we show that there exists a canonical (and minimal) rationalization.

The paper is organized as follows. In Section 2 we define Plott functions, give several important examples, and introduce the lattice of Plott functions. In Section 3 we recap the correspondence between Plott functions and convex geometries. In Section 4 we introduce the notion of a support of a Plott function. The central role in our analysis of Plott functions plays linear Plott
functions (or simple words). In Section 5 we associate to a Plott function its basement, a special subset of linear Plott functions. Sections 6 is devoted to the direct image of Plott functions. In Section 7 we discuss the notion of a superset rationalization and construct the canonical rationalization for each Plott function. In section 8 we introduce the shuffles and the melanges of simple words. Using the melange operation, we introduce in Section 8 a natural convex structure on the set $\text{SW}(X)$ of simple words. This structure enables us to identify Plott functions with convex subsets of $\text{SW}(X)$. In Section 9 we study the inverse image of Plott functions.

## 2 The Plott choice functions

Let $X$ be a finite set. A *choice function* on $X$ is a map $f : 2^X \to 2^X$ such that $f(A) \subseteq A$ for any $A \subseteq X$.

A choice function $f$ is a *Plott function* (or satisfies the *path independence* condition), if for any subsets $A$ and $B$ of $X$,

$$f(A \cup B) = f(f(A) \cup B).$$

Let us begin with examples of constructions of Plott functions.

**Example 1.** Let $R$ be a partial order on $X$ (i.e. a reflexive, transitive, and anti-symmetric binary relation). By $\text{Max}(A)$ (or $\text{Max}(R|A)$) we denote the set of best elements in $A$ under $R$. It is easy to check that the choice function $f_R$, $f_R(A) = \text{Max}(R|A)$, is a Plott function.

The following instance of this construction is important for us. It is the case, when $R = \leq$ is a linear order, that is a complete partial order. To define a linear order is the same as to define an enumeration of elements of $X$, i.e a bijection $\nu : \{1, \ldots, n\} \to X$. Here $\nu(1)$ denotes the best element of $X$, $\nu(2)$ the second best and so on all the way to the worst element $\nu(n)$. The choice under maximization of a linear order is a singleton-valued choice function, i.e. $|f(A)| = 1$ for any non-empty $A \subseteq X$. The reverse is also true, that is a singleton-valued Plott choice function is the maximization choice function of a linear order.

**Example 2.** Let $\leq$ be a linear order on $X$. The following choice function $f(A) = \{\max(A), \min(A)\}$ is a Plott function.

More general, let $(\leq_i, i \in I)$ be a family of linear orders on $X$. Then a joint-extremal choice function $f$ is given by the union of choices made under

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\[^1\] $A \subseteq B$ denotes the inclusion of sets.
maximization for each individual order \( \leq_i \). That is
\[
f(A) = \cup_{i \in I} \max(\leq_i |A).
\]

It is easy to check that so defined function is a Plott function. Moreover the following fact is well known \[3]\: 

**Lemma 1.** Let \( f \) and \( g \) be any two Plott functions on \( X \). Then the function \( f \cup g \) given by \((f \cup g)(A) = f(A) \cup g(A), A \subseteq X,\) is a Plott function.

A less trivial assertion is that any Plott function with non-empty values \((f(A) \neq \emptyset \text{ for any } A \neq \emptyset)\) is a joint-extremal choice function. This basic fact about Plott functions was announced in \[2\]. There are different ways to prove it and we will discuss one of them below.

**Example 3.** Let \( \leq \) be a linear order on \( X \), and our choice from a set \( A \) consists of the \( k \) best elements of \( A \) with respect to this order. One can check that this is a Plott function. It is a useful exercise to find an implementation of this choice function as a joint-extremal choice function (even for \( k = 2 \); see \[6\] for general \( k \)).

**Example 4.** Let \( S \) be a subset of \( X \). Let us associate to this subset the following choice function: \( \mathbf{1}_S(A) = A \cap S \). One can check that this choice function is a Plott function (which allows of empty choices).

The latter example has some interesting features for us. It hints us an idea that in some sense Plott functions resemble subsets. We will see later that a Plott function defines a certain subset of \( X \), the support of this function. Of course, a Plott function is more subtle object than its support (see Section 5 for more adequate object), but this analogy with subsets suggests to adopt a functorial view on Plott functions.

Let us introduce some notations. Denote by \( \text{CF}(X) \) the set of choice functions on \( X \) and by \( \text{PF}(X) \) the subset of Plott functions. The set \( \text{CF}(X) \) (as well as \( \text{PF}(X) \)) has a natural partial order. Namely, for choice functions \( f \) and \( g \) on \( X \), we write \( f \leq g \) if \( f(A) \subseteq g(A) \) for any \( A \subseteq X \). The poset \( \text{CF}(X) \) is a lattice with \( f \cup g \) and \( f \cap g \) as the join and the meet of \( f \) and \( g \), respectively. According to Lemma 1, in the sub-poset \( \text{PF}(X) \) any pair of elements \( f \) and \( g \) has the least upper bound \( f \lor g = f \cup g \). Moreover, since \( \text{PF}(X) \) has the least element \( 0 \) (\( 0(A) = \emptyset \) for any \( A \subseteq X \)), the poset \( \text{PF}(X) \) is in fact a lattice. However, the greatest lower bound \( f \land g \) in \( \text{PF}(X) \) might differs of \( f \cap g \); it is equal to the greatest Plott function dominated by the \( f \cap g \). Later on we shall present more explicit description of this greatest lower bound.
More generally, for any choice function $f$ there exists the maximal Plott function $f^\#$ such that $f^\# \leq f$. One can say that $f^\#$ is the Plottization of $f$. So, $f \land g$ is the Plottization of $f \cap g$.

In the following figure we draw the lattice $\text{PF}(X)$ for the set $X$ of three elements $x, y, z$. In the centre of the picture is the largest Plott function $1$. The smallest Plott function $0$ is not drawn at all; it is situated “at the infinity.” White circles denote join-irreducible Plott functions. Words near the circles represent the corresponding Plott functions; see Section 5.

3 Connection with convex geometries

We have seen that the set $\text{PF}(X)$ of Plott functions on $X$ is a lattice. However, even on the individual level, each Plott function is canonically related to some special lattice, the so-called convex geometry. This construction was introduced in [7] (see also [8]). Let us recap some details of this construction, mainly to agree on notations.

Recall, that a collection $\mathcal{F}$ of subsets of $X$ (we will refer to the elements of $\mathcal{F}$ as to closed or convex subsets of $X$) forms a convex geometry, if $\mathcal{F}$ is stable with respect to the intersection, contains $X$, and possesses the Minkowski-
Krein-Mil'man (MKM) property. To formulate the latter property we have to introduce the notion of an extreme point. A point \( a \) of a set \( A \subset X \) is said to be \textit{extreme} (with respect to \( F \)), if \( a \) does not belong to the closure of the set \( A \setminus a \) (the closure of a set \( B \subset X \) is the minimal set of \( F \) which contains \( B \)). The set of extreme points of \( A \) is denoted by \( \text{ext}(A) \) (or \( \text{ext}_F(A) \), if we want to emphasize the closure system we are interested in). A collection \( F \) of subsets of \( X \) is said to satisfy the MKM property if the closure of \( A \) coincides with the closure of \( \text{ext}(A) \) for any \( A \subset X \). Roughly speaking, each subset has “enough” extreme points. Denote by \( \text{CG}(X) \) the set of all convex geometries on \( X \).

There are the following mutually inverse mappings between the introduced sets \([7]\) (the lattice isomorphisms in fact):

\[
\text{PF}(X) \to \text{CG}(X) \to \text{PF}(X).
\]

The mapping from Plott functions to geometries is defined by the following rule: the closure of a set \( A \) consists of all elements \( x \in X \), such that \( f(A) = f(A \cup x) \). The inverse mapping sends a convex geometry \( F \) to the choice function \( \text{ext}_F \).

This construction enables us to switch from one language to another. Having formulated something in one language, we will sometimes translate this into the other language, and sometimes we will leave such a translation to the reader. For example, above we considered the join of a pair of Plott functions. In the language of convex geometries, the join \( F \lor G \) consists of the sets \( F \cap G \) with \( F \in F \) and \( G \in G \). One more example: a Plott function \( f \) takes only non-empty values if and only if \( \emptyset \) is a closed set of the corresponding convex geometry \( F \).

\section{The support}

With each Plott function \( f \) we associate a set, the \textit{support} of \( f \). This set is given by the following formula:

\[
S = \text{supp}(f) = \{ x \in X, f(x) = x \}.
\]

**Proposition 1.** For any \( A \subset X \), there holds \( f(A) = f(A \cap S) \).

**Proof.** We proceed by induction on the number of elements of \( A \setminus S \). If there is no such elements, then \( A \cap S = A \) and the proposition is trivially true. Let \( x \notin S \) and \( A' = A \cup x \). Since \( A' \cap S = A \cap S \), we have to show \( f(A \cup x) = f(A) \). According to the path independence property, we have

\[
f(x \cup A) = f(f(x) \cup A) = f(\emptyset \cup A) = f(A).
\]

\[\square\]
Thus, we see that elements outside the support $S$ are irrelevant to the choice; in terms of convex geometries $X \setminus S$ is the minimal convex set (the closure of $\emptyset$). Actually such a function is defined on $S$. We will develop a formalism which provides this intuition.

It is clear that the mapping

$$\text{supp}: \text{PF}(X) \to 2^X$$

sending a choice function to its support is a $\lor$-morphism of lattices. That is a monotone mapping (with respect to the partial orders on these lattices) commuting with the join operation $\lor$. In general case, this map does not commute with $\land$. There is a natural "reverse" mapping (see Example 4), which also commutes with $\lor$.

The support of a Plott function is a simple, but a rather rough characteristic. It is exactly in this sense that we pointed out before that Plott functions remind us of subsets. For subsets we might speak about their image and inverse images under mappings. In Sections 6 and 9 we shall define direct and inverse images of Plott functions. That means that $\text{PF}(X)$ is functorial with respect to the base $X$.

5 Linear Plott functions and simple words

**Definition.** A Plott function $f$ is said to be linear if $f(A)$ contains at most one element for any $A \subset X$.

Example 1 show how one can construct linear Plott functions by the means of linear orders. Slightly more general construction is as follows. Fix a subset $S \subset X$ and a linear order $\leq_S$ on $S$ and define a choice function $l$ by the following formula: for $A \subset X$ $l(A)$ is the best element (with respect to $\leq_S$) of $A \cap S$ (if $A \cap S = \emptyset$ then $l(A) = \emptyset$). Obviously, $l$ is a linear Plott function with the support $S$.

The reverse is also true. For functions with the full support $X$ this was proven by Plott [11]. In the general case, we have to restrict a linear Plott function to its support.

To give a linear order on a subset of $X$ is the same as to give a simple word over the alphabet $X$. A word is a sequence $x(1)x(2)...x(k)$ of elements $x(i) \in X$, $k = 0, 1, ...$. A word is simple if no letter is repeated. A simple word $w = x(1)x(2)...x(k)$ define a linear order $\leq_w = (x(1) > x(2) > ... > x(k))$ on the subset $\text{supp}(w) = \{x(1), x(2), ..., x(k)\} \subset X$. The set of simple words is denoted by $\text{SW}(X)$. We have seen that the set $\text{SW}(X)$ can be identify with the set $\text{LPF}(X)$ of linear Plott functions.
It is easy to describe a "linear" convex geometry corresponding to a simple word \( w = x(1)x(2)...x(k) \). It consists of subsets
\[
X = \mathcal{L}(0) \supset \mathcal{L}(1) \supset ... \supset \mathcal{L}(k),
\]
where \( \mathcal{L}(j) = X \setminus \{x(1), ..., x(j)\}, \ j \leq k \). Vice versa, a chain \( \mathcal{L} \):
\[
X = \mathcal{L}(0) \supset \mathcal{L}(1) \supset ... \supset \mathcal{L}(k)
\]
of subsets is a convex geometry if and only if any two consecutive members of this chain differ exactly by a single element.

The identification of \( \text{SW}(X) \) and \( \text{LPF}(X) \) is compatible with orders on these sets if we set that a word is larger any of its prefixes. Picture 2 illustrates the poset \( \text{SW}(X) \) for a three-elements set \( X \).

**Definition.** The *basement* of a choice function \( f \) is the set \( \text{Bas}(f) \) of all linear Plott functions \( l \) such that \( l \leq f \).

Using the identification of linear Plott functions with simple words we can consider any basement as a subset of \( \text{SW}(X) \). As we shall see, any Plott function \( f \) is equal to the join of all linear Plott functions of the basement of \( f \). However, in the beginning we consider several examples.

**Example 4'.** Let \( f = 1_X \) be the identity choice function on \( X \) (\( f(A) \equiv A \)). Then the basement of this function is the whole set \( \text{LPF}(X) \) (or \( \text{SW}(X) \)).

**Example 5.** Suppose \( l_w \) is a linear Plott function corresponding to a simple word \( w = x(1)x(2)...x(k) \). The basement of \( l_w \) consists of all prefixes of \( w \).

**Example 2'.** Suppose that \( X = \{a, b, c\} \) and that \( f \) is the join of two linear Plott functions corresponding to the words \( abc \) and \( cba \). Then the basement of \( f \) consists of the following words:
where ∅ denotes the empty word.

Sometimes it is convenient to deal with special subset of $\text{Bas}(f)$.

**Definition.** The socle of a choice function $f$ is the set $\text{Soc}(f)$ of maximal elements of $\text{Bas}(f)$.

**Example 1’.** Let $R$ be a partial order on $X$ and let $f_R$ be the Plott function arising from maximization $R$. It is easy to check that the socle of $f_R$ consists of all linear Plott functions corresponding to all linear extensions of $R$.

For any choice function $f$ we have $f \geq \vee(\text{Bas}(f))$. The following result states that the equality holds for any Plott function $f^2$. It generalizes and refines the Aizerman-Malishevki theorem.

**Theorem 1.** Any Plott function $f$ is equal to the join of all linear Plott functions from $\text{Bas}(f)$. That is

$$f = \vee(\text{Bas}(f)).$$

In ”geometric” language this theorem asserts that any convex geometry is equal to the join of its ”linear” convex sub-geometries. We shall prove the theorem in this geometric form.

**Proof.** Consider a convex geometry $\mathcal{F}$ on $X$, and consider a maximal chain $\mathcal{L} = (L(0) \supset L(1) \supset ...)$, where all $L(k) \in \mathcal{F}$. We claim that $\mathcal{L}$ is a ”linear” convex sub-geometry. Since it is a chain, the linearity follows from the construction. We have to check that adjacent members of this chain differ exactly by a single element.

So, let $F = L(k)$ and $G = L(k+1)$ be any two consecutive entries of the chain $\mathcal{L}$. We claim that there exists an extreme point $x$ of the set $F$,

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$^2$Here is a straightforward analogy with the standard convexity. Linear orders correspond to linear functionals and a chain of orders corresponds to a chain of half-spaces. Similar to a polytope which is given by the intersection of half-spaces, our ”convex” sets will be given by the intersection of chains-”half-spaces”. Theorem 1 is similar to the assertion that a convex functions $f$ is supremum of linear (affine) functions which are less than or equal to $f$. 

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which does not belong to \( G \). Indeed, the set \( \text{ext}(F) \) can not be a subset of \( G \), otherwise by the MKM property, this would imply that \( F \) (as the closure of \( \text{ext}(F) \)) is a subset of \( G \). Consider the set \( F' = F \setminus x \). Since \( F \) is closed and \( x \) is an extreme point of \( F \), the set \( F' \) is closed, and \( F' \) is located between \( F \) and \( G \). Since the chain is maximal, we have \( F' = G \), that proves our assertion.

Now the theorem is obvious. We have proven an even more general result: a convex geometry \( \mathcal{F} \) is equal to the union of its ”linear” sub-geometries. This holds because any element of \( \mathcal{F} \) might be included in some maximal chain. In fact, any chain in \( \mathcal{F} \) might be condensed until it becomes maximal. ■

As a consequence of this theorem and Example 5 we obtain that linear Plott functions are indeed join-irreducible elements of the lattice \( \text{PF}(X) \).

Another consequence is that the Plottization of \( f \) is equal to the join of function from the basement of \( f \).

Theorem 1 demonstrates that any Plott function can be identified with the subset \( \text{Bas}(f) \subset \text{SW}(X) \). Later on we introduce a convex structure on \( \text{SW}(X) \) and show that the basements of Plott functions are precisely the convex subsets of \( \text{SW}(X) \). For this and other aims we study the behavior of Plott functions with respect to exchange of the base set \( X \).

6 The direct image

Until now we dealt with choice functions on a fixed set \( X \). Now we compare these functions when \( X \) varies.

Suppose we are given a mapping \( \phi: X \to Y \), and let \( f \) be a choice function \( f \) on \( X \). Then we may “push forward” \( f \) from \( X \) to \( Y \) (similar to subsets and linear orders). Specifically, define a choice function \( \phi_*(f) \) on \( Y \), sending \( B \subset Y \) to

\[
\phi_*(f)(B) = \phi(f(\phi^{-1}B)).
\]

The choice function \( \phi_*(f) \) is called the direct image of \( f \) under the map \( \phi \).

For each element \( y \in Y \), we will refer to the elements of the pre-image \( \phi^{-1}(y) \) as to clones of \( y \) and denote them \( y', y'', \ldots \) (however, some \( y \in Y \) might have no clones at all). Then an ”original element” \( y \in B \) belongs to the choice \( \phi_*(f)(B) \) if and only if at least one of the clones of \( y \) belongs to the choice under \( f \), i.e. \( \phi^{-1}(y) \cap f(\phi^{-1}(B)) \neq \emptyset \). It is clear that the map \( \phi_*: \text{CF}(X) \to \text{CF}(Y) \) is monotone and commutes with \( \lor \).

Example 4′′. Let \( f = 1_S \) be the identity choice function with the support \( S \subset X \). Then \( \phi_*(f) = 1_{\phi(S)} \).
Moreover, the support of the choice function $\phi_*(f)$ is equal to the image of the support of $f$.

We assert that the direct image $\phi_*(l)$ of a linear Plott function $l$ on $X$ is a linear Plott function on $Y$. Let a simple word $w$ generate $l$. Then we explicitly write a simple word generates $\phi_*(l)$.

For this we define a map

$$\mathbf{SW}(\phi) = \psi_* : \mathbf{SW}(X) \to \mathbf{SW}(Y).$$

For a word $w = x(1)\ldots x(k)$ let $\phi(w)$ denote a word $\phi(x(1))\ldots \phi(x(k))$. For a simple word $w$ we define $\phi_*(w)$ as a simplification of the word $\phi(w)$. Specifically, if $w = vx$ then

$$\phi_*(w) = \begin{cases} 
\phi_*(v)\phi(x) & \text{if } \phi(x) \text{ does not occur in } \phi(v), \\
\phi_*(v) & \text{otherwise.}
\end{cases}$$

Suppose, for example, $Y = \{a, b, c, d\}$, $X = \{a', a'', b', c', c'', c''\}$ and $\phi(a') = \phi(a'') = a$ and so on. Then for the word $w = c''b'c'a''a'$ its direct image is the word $\phi_*(w) = cba$.

Obviously there holds

**Lemma 2.** Let $l_w$ be a linear Plott function on $X$, corresponding to a simple word $w$. Then $\phi_*(l_w)$ is a linear Plott function on $Y$, corresponding to the word $\phi_*(w)$. ■

In other words, we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{LPF}(X) & \xrightarrow{\phi_*} & \mathbf{LPF}(X) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathbf{SW}(X) & \xrightarrow{\phi_*} & \mathbf{SW}(Y)
\end{array}$$

where the vertical maps identify linear Plott functions and simple words.

**Corollary.** The direct image of a Plott function is a Plott function.

**Proof.** Because of Theorem 1, a Plott function $f$ on $X$ is equal to the join of a set $B$ of linear Plott functions. Since $\phi_*$ commutes with $\lor$, $\phi_*(f)$ is equal to the join ($\lor$) of the linear Plott functions $\phi_*(l)$, $l \in B$. ■

It is clear that if $\phi : X \to Y$ and $\psi : Y \to Z$ are two maps, then there holds $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. This implies that $\mathbf{PF}$ is a covariant functor from the category of (finite) sets to the category of partially ordered sets (even more, $\lor$-lattices).
In particular, if $X$ is a subset of $Y$, we may speak about a “trivial”
extension of a choice function $f$ from $X$ to $Y$. We denote it by $f_Y$, and it is
set by the rule
$$f_Y(B) = f(B \cap X).$$
Obviously, the support of $f_Y$ is a subset of $X$. We can identify Plott functions
on $X$ and Plott functions on $Y$ with the support in $X$.

The following Theorem refines the previous corollary.

**Theorem 2.** $\text{Bas}(\phi_*(f)) = \phi_*(\text{Bas}(f))$.

Or, equivalently, if we understand the basements as subsets of $\text{SW}(X)$
and $\text{SW}(Y)$, then $\text{Bas}(\phi_*(f)) = \phi_2(\text{Bas}(f))$.

A proof of this theorem is proceeded in the language of convex geomet-
rices, and, therefore, in the beginning we give a ”geometric” description of
the direct image. Let $\phi : X \to Y$ be a mapping of sets, and $\mathcal{F}$ be a convex
geometry on $X$. Let us define
$$\mathcal{F}_Y = \{\phi_+(F), F \in \mathcal{F}\}.$$  
Here $\phi_+$ denotes the full image, that is for a subset $A \subset X$
$$\phi_+(A) = \{y \in Y, \phi^{-1}(y) \subset A\}.$$  
In particular, if $y \notin \phi(X)$ then $y \in \phi_+(A)$ for any $A \subset X$. The operation $\phi_+$
commutes with the intersection $\cap$.

**Proposition 2.** Let $\mathcal{F}$ be the convex geometry corresponding to a Plott
function $f$ on $X$. Then $\mathcal{F}_Y$ is the convex geometry on $Y$ which corresponds
to $\phi_*(f)$.

**Proof.** A) $\mathcal{F}_Y$ is a lattice. Indeed, $Y = \phi_+(X)$ belongs to $\mathcal{F}_Y$. Since $\phi_+$
commutes with the intersection, $\mathcal{F}_Y$ is stable with respect to the intersection.

B) Let us check that $\mathcal{F}_Y$ is a convex geometry.

For this we have to show that if $G \supset G'$ are adjacent closed sets of $\mathcal{F}_Y$
(i.e. there are no closed sets located between $G$ and $G'$), then they differ by
a single element. By the definition of $\mathcal{F}_Y$ there are convex sets $F$ and $F'$ (of
$\mathcal{F}$) such that $G = \phi_+(F)$ and $G' = \phi_+(F')$. Set $F'' = F \cap F'$. It is clear that
$F'' \subset F$. On the other hand
$$\phi_+(F'') = \phi_+(F) \cap \phi_+(F') = G \cap G' = G'.$$
Replacing, if needed, $F'$ by $F''$ we can suppose that $F \supset F'$. Let now
$F = F_0 \supset \ldots \supset F_k = F'$ be a condensend chain of convex sets between $F$
and $F'$. Since $G$ and $G'$ are adjacent, any set of the form $\phi_+(F_j)$ is either $G$ or $G'$. Since $\mathcal{F}$ is a convex geometry then neighbor members of the chain $F_j$ differ by a single element. Therefore $\phi_+(F_j)$ and $\phi_+(F_{j+1})$ differ by at most one element. It is now obvious that $G$ and $G'$ differ by a single element.

C) Now we describe extreme points of an arbitrary subset $B \subset Y$. By the definition, a point $b \in B$ is extreme if there exists a convex set $G \subset Y$ which contains $B \setminus b$ but does not contain $b$. That is there exists a convex set $F \subset X$ which contains $\phi^{-1}(B \setminus b)$ but not $\phi^{-1}(b)$. That means exactly that the fiber $\phi^{-1}(b)$ contains an extreme point of the set $\phi^{-1}(B)$. In other words, we have proven the following formula

$$\text{ext}_{F_Y}(B) = \phi(\text{ext}_{\Phi}(\phi^{-1}(B))).$$

**Proof of Theorem 2.** The inclusion $\supset$ is obvious. Inversely, let $l$ be a linear Plott function on $Y$ such that $l \leq \phi_+(f)$. Consider the corresponding chain $\mathcal{L} = (Y = G_0 \supset G_1 \supset ... \supset G_k)$ of convex sets in $Y$. Accordingly to the step B) of the previous proposition proof, this chain can be lifted to a chain $X = F_0 \supset F_1 \supset ... \supset F_k$ of convex sets in $X$. If we condense this chain we obtain a linear subgeometry of $\mathcal{F}$ which projects in $\mathcal{L}$. $\blacksquare$

7 Superset rationalizations of Plott functions

A Plott function $f$ on $X$ is called *rationalizable* by a partial order $R$ on $X$ if, for any $A \subset X$, $f(A)$ consists of maximal elements in $A$ with respect to this order $R$ (see Example 1). The partial order $R$ (if it exists) is uniquely defined by $f$ and is called the *rationalization* of $f$.

Not every Plott function is rationalizable. For example, the function from Example 2' is not rationalizable. In literature (especially in papers by Malishevski and by Nehring), it was considered another form of rationalization of Plott functions using transitive hyper-relations on $X$ (that are special relations between $2^X$ and $X$). Here we propose a rationalization of Plott functions by transitive relations on a ”superset” $X' \rightarrow X$.

**Definition.** A triple $(Y, \leq, \psi)$ (where $\leq$ is a partial order on $Y$ and $\psi: Y \rightarrow X$ is a mapping) is said to be a *SS-rationalization* of $f$ if $f = \psi_*(g)$, where $g$ is the Plott function on $Y$ rationalizable by $\leq$.

**Example 2'**. Let $X = \{a, b, c\}$ and let a choice function $f$ be the join of two linear orders, which correspond to the words $abc$ and $cba$ (see Example 2'). Then the poset

$$a' \rightarrow b' \rightarrow c'$$

$$c'' \rightarrow b'' \rightarrow a''$$
is a rationalization of \( f (\psi(x') = \psi(x'') = a, \ x \in \{a, b, c\}) \). However there exists a rationalization of \( f \) by the smaller poset

\[
a' \to b', c' \to b''.
\]

Let us explain a meaning of the SS-rationalization. A main idea of the binary rationalization consists in that that a rationalizable choice can be made on a pairwise base and the binary relation defined by this pairwise choice is an order. Then an element \( x \) is rejected in \( A \), if the considered opportunity set \( A \) contains an element \( x' \) which dominates \( x \) (with respect to the order). We follow this idea when we deal with a superset. Namely, each original \( x \) is split into several splinters, and all these splinters form a superset \( X' \). An order on \( X' \) is defined such that its direct image (under the natural mappings sending splinters to their original) is exactly the choice function under consideration. Now, the original \( x \) is rejected if and only if EVERY of its splinters is dominated.

In other words, each alternative has many aspects or qualities. And if an alternative is not dominated with respect to some of its quality, then we have a rationale to include this alternative in our choice. And conversely, if our alternative \( x \) is dominated with respect to any of its qualities, we have a rational to exclude it from our choice. Of course, to exclude an alternative we have to use several other alternatives. So, in Example 2’’ we exclude \( b \) from the set \( \{a, b, c\} \) because all of its splitters \( b' \) and \( b'' \) are dominated.

It is easy to prove (using Theorem 1) that any Plott function possesses (a lot of) SS-rationalizations. Conversely, an SS-rationalizable choice function is a Plott function indeed. Thus, Plott functions are precisely SS-rationalizable choice functions; this assertion was proven by [9]. We shall not digress to prove this assertion because we provide any Plott function with a canonical (and minimal, in a sense) SS-rationalization. For this we define a canonical splitting of elements of \( X \) into aspects or pieces.

The following notion will be of use in this section. Let \( f \) be a Plott function on \( X \) and let \( \mathcal{F} \) be the corresponding convex geometry on \( X \). Recall, that elements of \( \mathcal{F} \) we call convex sets.

**Definition.** A closed set \( P \in \mathcal{F} \) is said to be a *piece* of an element \( x \in X \) if \( x \notin P \) and \( x \in Q \) for any proper closed superset \( Q \in \mathcal{F} \) of \( P \) (\( P \subset Q \) and \( P \neq Q \)).

It immediately follows from the definition that a piece of \( x \) is a maximal convex set (under inclusion) not containing \( x \). In particular, if \( x \) does not belong to the closure (in \( \mathcal{F} \)) of a set \( A \subset X \), then there exists a piece of \( x \) which contains \( A \).
A convex set $P$ is said to be a *piece* if it is a piece of an element. It is important that any piece is exactly a piece of a unique element of $X$.

**Lemma 3.** Let $P$ be a piece of $x$. Then the set $P \cup x$ is convex.

**Proof.** Let $Q$ covers $P$ in the lattice $F$ (i.e. for any $Z \in F$ such that $Q \supset Z \supset P$ either $Z = Q$ or $Z = P$). Then $x \in Q$. Because $Q$ and $P$ differ by a single element (the MKM property), we have $Q = P \cup x$. ■

**Corollary.** Let $P$ be a piece of $x$ and a piece of $x'$. Then $x = x'$.

In fact, the set $P \cup x'$ is convex and contains $P$. Hence $P \cup x'$ contains $x$. That implies $x = x'$. ■

Let $f$ be a Plott function. Denote by $\mathcal{P}(f)$ the set of all pieces of the corresponding convex geometry $F$. It is easy to see that any convex set can be given as the intersection of some pieces. On the other hand, any piece $P$ cannot be presented as the intersection of convex sets, each of which properly contains $P$. In fact, any large convex set will contain $P \cup x$, if $P$ is a piece of $x$. Thus $\mathcal{P}(f)$ is a minimal basis (with respect to the meet operation) of the lattice $F$.

Since each piece is a piece of a unique element, the natural map $\phi \colon \mathcal{P}(f) \to X$, which sends $P \in \mathcal{P}(f)$ to the unique element $\phi(P) \in X$ for which $P$ is a piece, is correctly defined. The inverse image $\phi^{-1}(x)$ is constituted of all pieces of $x$. Because each element $x \in \text{supp}(f)$ has a piece, we have $\phi(\mathcal{P}(f)) = \text{supp}(f)$.

**Example 2$''$.** Consider the choice function from Example 2$''$. The element $a$ has exactly one piece $a' = \{b, c\}$, the element $c$ also has a unique piece $c' = \{a, b\}$. While the element $b$ has two pieces $b' = \{a\}$ and $b'' = \{c\}$.

The set $\mathcal{P}(f)$ endowed with the inclusion order $\subset$ is a poset. We state that the poset $\mathcal{P}(f)$ with the natural map $\phi \colon \mathcal{P} \to X$ is an SS-rationalization of $f$.

**Theorem 3.** Let $f$ be a Plott function. Then the triple $(\mathcal{P}(f), \subset, \phi)$ is an SS-rationalization of $f$.

**Proof.** Denote by $\tilde{f}$ the choice function on $\mathcal{P}(f)$, which is rationalizable by the order $\subset$. We have to check that $f = \phi \circ (\tilde{f})$, i.e. for any subset $A \subset X$ the following equality holds

$$f(A) = \phi(\tilde{f}(\phi^{-1}(A))).$$

Let $a \in f(A)$, that is $a$ is an extreme point of $A$. By the definition of an extreme point $a$ does not belong to the closure of the set $A \setminus a$. Denote by $P$ (any) piece of $a$, which contains $A \setminus a$. We claim that $P$ is not contained
in any element (of \(\mathcal{P}(f)\)) of \(\phi^{-1}(A)\). In fact, assume there exists a piece \(P'\) of some element \(a'\) of \(A \setminus a\), such that \(P' \supseteq P\). Then \(a' \in A \setminus a \subset P \subset P'\), that contradicts to \(a' \notin P'\). Thus \(a \in (\phi_*(\tilde{f}))(A)\).

Now, let \(a \notin f(A)\). Let us show that any piece \(P\) of \(a\) is a subset of some element of \(\phi^{-1}(A \setminus a)\). Because \(f\) is a Plott function, \(a\) does not belong to the set of extreme points of \(A\). Therefore the closure of \(A \setminus a\) contains \(a\). Thus \(A \setminus a\) is not a subset of \(P\) (otherwise, the closure of \(A \setminus a\) is a subset of \(P\), and hence \(a \in P\), that is not the case). Denote by \(a'\) any point of \(A \setminus a\), which does not belong to \(P\). Having expanded \(P \cup a\) to a maximal convex set not containing \(a'\), we obtain a piece \(P'\) of \(a'\), which contains \(P \cup a\). ■

**Definition.** The SS-rationalization \((\mathcal{P}(f), \subset, \phi)\) is said to be the canonical rationalization of a Plott function \(f\).

**Example 1**. Let \(f\) be rationalizable by a partial order \(R\). Then the corresponding convex geometry \(\mathcal{F}\) consists of ideals of the partial order. Since a lattice of ideals \(\mathcal{F}\) is closed under the union, for any element \(x\) there exists a maximal ideal \(AF(x)\), which does not contain \(x\). That is any element \(x\) has a unique piece \(AF(x)\). We obtain that \(\mathcal{P}(f)\) coincides (as a poset) with \(X\).

Now we are going to show that the SS-rationalization \((\mathcal{P}(f), \subset, \phi)\) is not only canonical (that is uniquely constructed by \(f\)), but in some sense it occupies a “central” location among all SS-rationalizations of \(f\). A perfect situation would occur if for any SS-rationalization \((Y, \leq, \psi)\) of \(f\) would exist a mapping \(\alpha : Y \to \mathcal{P}(f)\), such that \(\phi \circ \alpha = \psi\) and \(\alpha_*(g) = \tilde{f}\). Unfortunately, this is not the case, as the following example from ([6]) demonstrates.

**Example 6**. Let \(X = \{a, b, c, d\}\) and let us define the Plott function \(f\) by the lattice of convex sets

\[
\begin{array}{c}
abcd \\
acd \\
abd \\
bc \\
bd \\
ac \\
ab \\
\end{array}
\]

In other words, the element \(d\) is chosen only from \(\{d\}\). For sets \(A \neq X\) and \(A \neq X \setminus d\), the choice set is given by \(f(A) = A \setminus d\). Finally, \(f(X) = f(X \setminus d) = \{a, b\}\). The poset \(\mathcal{P}(f)\) is the poset
However, if we consider a weaker poset \( Y \) (\( c' \) and \( c'' \) are sent into \( c \), \( a' \) into \( a \) and so on), then we obtain another SS-rationalization of \( f \). The mapping \( \alpha \) is obviously the identity mapping, but the corresponding Plott function \( g \) (on \( Y \)) is different from \( \tilde{f} \) (\( d' \in g(\{d', c''\}) \) but \( d' \not\in \tilde{f}(\{d', c''\}) \)).

However, the following weaker property holds.

**Proposition 3.** Let \( (Y, \leq, \psi) \) be an SS-rationalization of a Plott function \( f \). Then there exists a mapping \( \alpha: Y \to \mathcal{P}(f) \), such that \( \phi \circ \alpha = \psi \) and \( \alpha_*(g) \geq \tilde{f} \).

Before proving this proposition we state its important corollary, that the canonical rationalization is of minimal cardinality.

**Corollary.** \( |Y| \geq |\mathcal{P}(f)| \).

In fact, because the support of \( \tilde{f} \) is the whole \( \mathcal{P} \), the support of \( \alpha_*(g) \) is also the whole \( \mathcal{P} \), and hence the mapping \( \alpha \) is a surjection. ■

**Proof of Proposition 3.** For a point \( y \in Y \) denote by \( AF(y) \) the set of points \( y' \in Y \) which are not dominated (with respect to \( \leq \)) by \( y \). In other words, \( AF(y) \) is the complement in \( Y \) to the principal filter \( F(y) = \{y', y' \geq y\} \). The set \( AF(y) \) is an ideal with respect to the partial order \( \leq \) on \( Y \). Therefore, the set \( \psi_+(AF(y)) \) is a convex set in \( X \). This set does not contain \( \psi(y) \), therefore, there exists a piece \( P \) of the element \( \psi(y) \), which contains \( \psi_+(AF(y)) \). We define \( \alpha: Y \to \mathcal{P}(f) \) by setting \( \alpha(y) = P \) (if there are several pieces containing \( \psi_+(AF(y)) \), we pick up any). Because \( \phi(P) = \psi(y) \), we obtain \( \phi \circ \alpha = \psi \).

In order to show \( \alpha_*(g) \geq f \) it suffices to check that the direct image (under \( \alpha \)) of the partial order \( \leq \) (on \( Y \)) is weaker than the partial order \( \subset \) on \( \mathcal{P}(f) \). That is we have to check that if \( P \) and \( Q \) are two different pieces
and any element of $\alpha^{-1}(P)$ is dominated (with respect to $\leq$) by some element of $\alpha^{-1}(Q)$, then $P \subset Q$.

Let us prove this claim. Let $P$ be a piece of $x$. Then $x$ is an extreme point of $P \cup x$, and hence $x \in f(P \cup x)$. Since $f = \psi_+(g)$, we have, by the definition of the direct image, a point $y$ of $\psi^{-1}(x)$ which is not dominated by any point of $\psi^{-1}(P)$. That is $\psi^{-1}(P) \subset AF(y)$, or, equivalently, $P$ is a subset of the full image of $AF(y)$, $P \subset \psi_+(AF(y))$. Because the convex set $\psi_+(AF(y))$ does not contain the point $\psi(y) = x$ and $P$ is a piece of $x$, we have the equality $P = \psi_+(AF(y))$ indeed. Thus, we have $\alpha(y) = P$.

Now, any element of $\alpha^{-1}(P)$ (and, hence, $y$) is dominated by at least one element of $\alpha^{-1}(Q)$. Denote by $y'$ the element of $\alpha^{-1}(Q)$, which dominates $y$, that is $y < y'$ and $\alpha(y') = Q$. Due to the domination $y < y'$, we have the inclusion $AF(y) \subset AF(y')$, and, hence, $\psi_+(AF(y)) \subset \psi_+(AF(y'))$ holds. By the definition of the mapping $\alpha$, $\alpha(y') = Q$ implies the inclusion $\psi_+(AF(y')) \subset Q$. Since $P = \psi_+(AF(y))$, we obtain the desired inclusion $P \subset Q$. ■

8 Convex structure on $\text{SW}(X)$

In this section we define a convex structure on the set $\text{SW}(X)$ of simple words over $X$ (or on the set $\text{LPF}(X)$ of linear Plott functions).

Let $X$ and $Y$ be two disjoint sets, and let $f$ and $g$ be choice functions on $X$ and $Y$ correspondingly. The direct sum of $f$ and $g$ is the following choice function $f \coprod g$ on the disjoint union $X \coprod Y$: for a subset $A \coprod B$ of $X \coprod Y$

$$(f \coprod g)(A \coprod B) = f(A) \coprod g(B).$$

Obviously, $f \coprod g$ is a Plott function if $f$ and $g$ are Plott functions.

Let now $f$ and $g$ be linear Plott functions corresponding to simple words $w$ and $v$ over $X$ and $Y$. We want to describe the socle of $f \coprod g$. Without loss of generality we can assume that $w$ and $v$ are words of maximal length (that is the supports of $f$ and $g$ are equal to $X$ and $Y$, respectively). Let $\leq_X$ and $\leq_Y$ denote the corresponding linear orders on $X$ and $Y$. Then we can consider the partial order $R = \leq_X \coprod \leq_Y$ on $X \coprod Y$. As we have yet seen in Example 1, the socle of $f \coprod g$ consists of all linear extensions of the partial order $R$.

**Definition.** A word (over the alphabet $X \coprod Y$) corresponding to a linear extension of the partial order $R$ is called a *shuffle* of $w$ and $v$. The set of shuffles of $w$ and $v$ will be denoted by $\text{Sh}(w,v)$.
A shuffle has a form
\[ w_1v_1w_2v_2\ldots w_kv_k, \]
where \( w = w_1w_2\ldots w_k \) and \( v = v_1v_2\ldots v_k \); some of sub-words \( w_i, v_j \) are allowed to be empty. In words, a shuffled word of two words is the following composition: To compose it we have to take an initial piece of one of the words, then to append it by an initial piece of another word, then to return to the first word and to take the next piece starting from the interruption place, then again to switch to the another word and so on. We illustrate this on the following example, let \( w = xyz \) and \( v = abcd \). Then the word \( xabycdz \) is one of the shuffles.

The shuffle operation enables us to define a (multi-valued) operation
\[
Sh : \text{SW}(X) \times \text{SW}(Y) \rightarrow \text{SW}(\bigsqcup Y).
\]
Obviously, it is commutative. But it is also associative. To see this we note that the preceding construction can be done for arbitrary number of words \( w_i \) over disjoint sets \( X_i, i \in I \).

**Lemma 4.** Let \( w \) be a shuffle of simple words \( w_1, w_2, \ldots, w_n \) over \( X_1, \ldots, X_n \). Then there exists a shuffle \( w_2\ldots n \) of \( w_2, ..., w_n \) such that \( w \) is a shuffle of \( w_1 \) and \( w_2\ldots n \).

**Proof.** The linear order \( \leq \) corresponding to \( w \) is an extension of the order \( \leq_1 \bigsqcup \ldots \bigsqcup \leq_n \) on \( X_1 \bigsqcup \ldots \bigsqcup X_n \). Define \( \leq_{2\ldots n} \) as the restriction of \( \leq \) on the set \( X_2 \bigsqcup \ldots \bigsqcup X_n \). Obviously, \( \leq \) is an extension of the disjoint union of \( \leq_{2\ldots n} \) and \( \leq_1 \). Now define \( w_2\ldots n \) as the simple word corresponding to \( \leq_{2\ldots n} \).

Suppose now that \( Y \) is a copy of \( X \), and \( \delta : \bigsqcup X \rightarrow X \) is the co-diagonal mapping. A *melange* of two simple words \( w_1 \) and \( w_2 \) over \( X \) is a simple word \( \delta \circ (w) \) where \( w \) is a shuffle of \( w_1 \) and \( w_2 \).

Let us give an example. Suppose we have two words \( xyzab \) and \( zacyd \). Then the word \( zaxybcdb \) is a melange and it is obtained by the simplification of the following shuffle \( (za)xyz(cy)(a)(d)(b) \).

Similarly we can define a melange of any family \( w_i, i \in I \), of simple words.

**Proposition 4.** Let \( l_i \) be the linear Plott functions corresponding to simple words \( w_i, i \in I \). Then the set of melanges of \( w_i \) coincides with the socle of \( \vee_{i \in I} l_i \).

**Proof.** Let \( \delta \) be the co-diagonal mapping of \( X \times I \) onto \( X \), \( \delta(x, i) = x \). It is obvious that \( \delta \circ (\bigsqcup l_i) = \vee_{i \in I} l_i \). Now the assertion follows from Theorem 1.

19
As well as the shuffle, the melange is (multi-valued) commutative and associative operation on the set $\text{SW}(X)$. Using this operation we can define a convex structure on $\text{SW}(X)$.

Suppose that $w$ and $v$ are two simple words over $X$. Let $co(w, v)$ be the set of all prefixes of all melanges of $w$ and $v$. In other terms, $co(f, g)$ is the basement of $l_w \vee l_v$. We may understand $co(w, v)$ as a "segment" joining the points $w$ and $v$ in $\text{SW}(X)$.

**Definition.** A subset $C$ of $\text{SW}(X)$ is said to be convex if it contains $co(w, v)$ for every $w, v \in C$.

Lemma 4 has the following immediate consequence:

**Corollary.** Let $C$ be a convex subset of $\text{SW}(X)$ and $c_1, \ldots, c_n \in C$. If a simple word $w$ is a melange of $c_1, \ldots, c_n$ then $w \in C$.

Indeed, by Lemma 3 $w$ is a melange of $c_1$ and $c_2 \ldots n$ where $c_2 \ldots n$ is a melange of $c_2, \ldots, c_n$. By induction $c_2 \ldots n$ is in $C$. Then $w$ is in $C$. $\blacksquare$

**Theorem 4.** Let $C \subset \text{SW}(X)$. The following two assertion are equivalent:

1) $C$ is a convex subset of $\text{SW}(X)$;
2) $C = \text{Bas}(f)$ for some Plott function $f$ on $X$.

**Proof.** It is almost obvious that the basement $\text{Bas}(f)$ of any Plott function is a convex subset of $\text{SW}(X)$. Indeed, let $w$ and $v$ be two simple words in $\text{Bas}(f)$, and let $l_w$ and $l_v$ be the corresponding linear Plott functions. By the definition this means that $l_w \leq f$ and $l_v \leq f$. Therefore $l_w \vee l_v \leq f$ and $co(w, v) = \text{Bas}(l_w \vee l_v) \subset \text{Bas}(f)$.

Conversely, let $C$ be a convex subset of $\text{SW}(X)$. Let us define $f$ to be the join of the linear Plott functions $l_c$ where $c \in C$. Obviously $C \subset \text{Bas}(f)$. Let us check the inverse inclusion. Let $w$ be in $\text{Bas}(f)$ and even let $w$ be in the socle of $f$. By Proposition 4, $w$ is a melange of several words $c_1, \ldots, c_n \in C$. By the previous Corollary $w \in C$. $\blacksquare$

Thus, we obtain a bijection between the Plott functions on $X$ and the convex subsets of the convex space $\text{SW}(X)$. This bijection is compatible with the order structure. In particular the lattice $\text{PF}(X)$ of Plott functions on $X$ is isomorphic to the lattice of convex sets of the set $\text{SW}(X)$.

**Remark.** Because of this theorem and the bijection between Plott functions and convex geometries, we obtain the bijection between the lattice $\text{CG}(X)$ and the lattice of convex subsets of $\text{SW}(X)$. Let us explain how to construct the convex geometry $\mathcal{F}$ corresponding to a given convex subset $C$ of $\text{SW}(X)$. $\mathcal{F}$ consists of the complements of the sets $\text{supp}(w)$ where $w$ runs over $C$ and $\text{supp}(w)$ denotes the set of letters of a word $w$. 

20
As a consequence of Theorem 2, we obtain a more explicit description of the meet of a pair of Plott functions \( f_1 \) and \( f_2 \). Let \( C_1 \) and \( C_2 \) be the basements of \( f_1 \) and \( f_2 \). Then the function \( f_1 \wedge f_2 \) corresponds to the convex set \( C_1 \cap C_2 \). Similarly, \( f_1 \vee f_2 \) corresponds to the convex hull of \( C_1 \cup C_2 \).

**Example 7.** Let \( X = \{a, b, c\} \). Consider two linear Plott functions \( f_1 \) and \( f_2 \), which correspond to words \( abc \) and \( bac \). The basement of \( f_1 \) consists of words: \( abc \), \( ab \), \( a \) and empty word \( \emptyset \). The basement of \( f_2 \) consists of \( bac \), \( ba \), \( b \) and \( \emptyset \). The intersection of these basements consists of the single element \( \emptyset \). That is \( f_1 \wedge f_2 = 0 \).

**Example 8.** Let \( X \) be as in the above example. Consider other two orders \( abc \) and \( acb \), and let \( f_1 \) and \( f_2 \) be the corresponding linear Plott functions. The intersection of their basements consists of two elements: the empty word and the word \( a \). Hence, \( f_1 \wedge f_2 \) is Plott function with the support \( \{a\} \) (the choice of a subset \( A \subset X \) is equal to \( a \), if \( a \in A \) and empty set else).

**Remarks.** 1. Similarly one can define a convex structure on the set \( L(X) \) of linear orders on \( X \): a set \( C \subset L(X) \) is convex if it contains all melanges of its elements. We have used \( SW(X) \) in order to tame arbitrary Plott functions.

2. The lattice of convex sets in \( SW(X) \) is not a convex geometry. (In other words, the lattice of all convex geometries is not a convex geometry.) That might be shown on the three-elements set \( X \). Consider the whole set \( SW(X) \). It is a convex set. But this set have no extreme points. In fact, the word \( bca \) is a melange of the words \( bac \) and \( cba \). Similarly, any (complete) word is a melange of two other words. Therefore \( SW(X) \setminus w \) is not convex for any word \( w \).

Another example. Let us consider the segment \( co(xzy, zxy) \). It consists of seven words \( xzy, xz, x, \emptyset, z, xx, zxy \). But the word \( xzy \) is a melange of \( x \) and \( zxy \). Therefore it is not an extreme point of this segment. Similarly the order \( zxy \) is not an extreme point of this segment.

9 The inverse image

Here we define the inverse image functor, the right conjugate to the direct image; see a definition in \([6]\).

Let \( \phi : X \to Y \) be a mapping of sets and let \( g \) be a choice function on \( Y \).

**Definition.** The *inverse image* \( \phi^*(g) \) of \( g \) is the following Plott function on \( X \):

\[
\phi^*(g) = \wedge(f \in PF(X), \phi_*(f) \leq g).
\]
As the join of Plott functions, \( \phi^*(g) \) is a Plott function. Actually, \( \phi^*(g) \) is the maximal Plott function on \( X \) whose direct image is \( \leq g \). In particular, if \( \phi = id_X \) is identical mapping of \( X \), \( \phi^*(g) = g^\sharp \) is the Plottization of \( g \).

It is clear that
\[
\phi_*(\phi^*(g)) \leq g.
\]

Example 4″. Let \( g = 1_T \) be the function of the "identical" choice from a set \( T \subset Y \). Then \( \phi^*(g) = 1_S \) where \( S = \phi^{-1}(T) \).

Indeed, on one hand, \( \phi_*(1_S) = 1_T \leq g \) so that \( \phi^*(g) \geq 1_S \). On other hand, the support of \( \phi^*(g) \) should be contained in \( S \). Hence, \( \phi^*(g) \leq 1_S \).

Proposition 5. (Conjugation). Suppose \( \phi : X \to Y \) is a mapping, \( f \) is a Plott function on \( X \), and \( g \) is a choice function on \( Y \). The following assertions are equivalent:

1) \( \phi_*(f) \leq g \),
2) \( f \leq \phi^*(g) \).

Proof. 1) implies 2) by the definition. Inversely, let \( f \leq \phi^*(g) \). Then applying \( \phi_* \) to the both sides, we obtain \( \phi_*(f) \leq \phi_*(\phi^*(g)) \leq g \).

Proposition 6. Let \( g \) be a choice function on \( Y \). Then \( \text{Bas}(\phi^*(g)) = \phi^{-1}_* \text{Bas}(g) \).

Proof. Suppose that \( l \) is a linear Plott function on \( X \). If \( l \leq \phi^*(g) \) then by Proposition 5, \( \phi_*(l) \leq g \). Moreover \( \phi_*(l) \) is a linear Plott function (Lemma 2). Therefore it belongs to the basement of \( g \). Inversely, if \( \phi_*(l) \leq g \) then by the definition \( l \leq \phi^*(g) \).

Since the meet of Plott functions corresponds to the intersection of the basements, and \( \phi^{-1}_* \) commutes with the intersection, we obtain the following

Corollary. The functor \( \phi^* \) commutes with \( \wedge \).

Proposition 7. (The projection formula). Let \( \phi : X \to Y \) be a map of sets, let \( f \) be a Plott function on \( X \), and let \( g \) be a Plott function on \( Y \). Then
\[
\phi_*(f \wedge \phi^*(g)) = \phi_*(f) \wedge g.
\]
Proof. The inequality ≤ follows from monotonicity of the operator φ∗.
Let us prove the inverse inequality ≥. Suppose that l ∈ Bas(φ∗(f) ∧ g).
Then l ∈ Bas(φ∗(f)) and l ∈ Bas(g). By Theorem 2, there exists a linear
Plott function l′ ≤ f such that φ∗(l′) = l. Since l ≤ g, l′ ≤ φ∗(g). Therefore
l′ ≤ f ∧ φ∗(g) and l = φ∗(l′) ≤ φ∗(f ∧ φ∗(g)). □

In particular, if f = 1X then we obtain the following formula (where g is
an arbitrary Plott function on Y):
\[ \phi∗(φ∗(g)) = 1_{φ(X)} ∧ g. \]

For example, if φ is a surjective map then φ(X) = Y and φ∗(φ∗(g)) = g
for every Plott function g on Y. Note, that the inequality φ∗(φ∗(g)) ≤ g can
be strict when φ is not surjective.

Example 9. Let Y consists of three elements a, b, c and let g be the
linear Plott function corresponding to the word abc. Let X = \{b, c\} and let
φ be the natural inclusion of X into Y. Because we cannot make a prefix of
the word abc from letters b and c, we obtain that the inverse image of g is
equal to 0X. Therefore φ∗(φ∗(g)) = 0Y.

We give below two applications of the inverse image: to a construction of
direct products of Plott functions, and to a construction of natural transfor-
mations of Plott functions.

The direct product
Let X and Y be sets, and let α and β be the natural projections of X × Y
onto X and Y. Let f and g be Plott functions on X and Y, correspondingly.
Then we define the direct product \( f \prod g \) as the following Plott function on
X × Y:
\[ f \prod g = α∗(f) ∧ β∗(g). \]

Example 10. Let X = \{x, x′\} and Y = \{y, y′\}. Suppose that f is the
linear Plott function corresponding to the word xx'; similarly g corresponds
to the word yy'. Let us compute the direct product \( f \prod g \). For this we
describe its socle. The socle consists A simple word w (over the alphabet
X × Y) belongs to the socle iff its projection onto X is equal to xx' and
its projection onto Y is equal to yy'. That means that any such w has
(x, y) ∈ X × Y as its beginning letter. Therefore (for a set Z ⊂ X × Y)
\[ (f \prod g)(Z) = \begin{cases} \{(x, y)\} & \text{if } (x, y) ∈ Z, \\ Z & \text{if } (x, y) \notin Z. \end{cases} \]

Plott correspondences

23
Using the direct and inverse images and the meet $\land$, one can construct very general transformations of Plott functions. Namely, we call a Plott correspondence from $X$ to $Y$ a Plott function $h$ on a set $Z$ and two maps $\phi : Z \to X$ and $\psi : Z \to Y$. If now $f$ is a Plott function on $X$ then define

$$h(f) = \psi_\ast(h \land \phi^\ast(f)).$$

As a result we obtain a map (of sets or of posets)

$$h : \text{PF}(X) \to \text{PF}(Y).$$

In the "word" interpretation this map looks as follows. We take a convex subset $C$ in $\text{SW}(X)$, lift it on $\text{SW}(Z)$, intersect with the basement of $h$, and then descent the intersection onto $\text{SW}(Y)$.

Note that we always can assume that $Z = X \times Y$. Indeed, let $\alpha$ and $\beta$ be the natural projections of $X \times Y$ onto $X$ and $Y$, and let $\pi : Z \to X \times Y$ be a mapping such that $\phi = \alpha \circ \pi$ and $\psi = \beta \circ \pi$. Then

$$h(f) = \psi_\ast(h \land \phi^\ast(f)) = \beta_\ast \pi_\ast(h \land \pi^\ast \alpha^\ast(f)) = \beta_\ast(\pi_\ast(h) \land \alpha^\ast(f)) = (\pi_\ast(h))(f).$$

That is, as the operators from $\text{PF}(X)$ to $\text{PF}(Y)$, $h = \pi_\ast(h)$.

For example, if $Z$ is a subset of $X \times Y$, we can take $h = 1_Z$.

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