ASYMPTOTIC APPROXIMATION TO A SOLUTION OF A SINGULARLY PERTURBED LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM WITH SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION OF STATE VARIABLE

NGUYEN THI HOAI
Department of Mathematics, Mechanics and Informatics
University of Science, Vietnam National University, Hanoi
334 Nguyen Trai, Thanh Xuan, Ha Noi, Vietnam

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Abstract. The direct scheme method is applied to construct an asymptotic approximation of any order to a solution of a singularly perturbed optimal problem with scalar state, controlled via a second-order linear ODE and two fixed end points. The error estimates for state and control variables and for the functional are obtained. An illustrative example is given.

1. Introduction. For many practical problems, an exact solution may be unattainable. The numerical and asymptotic methods have been chosen as an alternative tool for approximating solutions to real-world problems. The models that describe processes in dynamical physics, chemical kinetics, and mathematical biology usually contain singularly perturbed equations and lots of them are differential with a small parameter multiplying the highest derivative. Since the standard numerical methods often fail to work when the small parameter is sufficiently small, special numerical methods for singularly perturbed equations have been developed (see, for example, [31, 5, 13] and the references therein) to provide accurate numerical solutions. In the case of a sufficiently small parameter, one can also use the asymptotic methods, which effectively find terms of the solution expansion. Different asymptotic techniques can be found in, for instance, [29, 38, 8, 9, 26, 3] and in many others.

The techniques of singular perturbations are widely applied in singularly perturbed optimal control problems. Since a boundary-value problem will arise from optimality conditions, the asymptotic methods play a specially important role instead of numerical solutions. There is a variety of works concerning singularly perturbed optimal control problems, namely, in reviews [18, 19, 14, 32] and also in [15, 25, 11].

In recent years, the direct scheme method (see, [3]) has been broadly used to find an asymptotic expansion of the solution of a singularly perturbed optimal control problem by virtue of its advantages toward the asymptotic analysis. This method is consisted of immediately substituting a postulated asymptotic series of

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the solution in the problem condition and determining a sequence of decomposed problems for finding the asymptotic terms. Using the method, on one hand, we can obtain an approximation series which is similar to the one given by the asymptotic analysis. On the other hand, it gives the opportunity to prove that the value of minimized functional does not increase when a higher-order approximation to the control variable is used. The application of the direct scheme method can be found in [18, 19, 20, 21, 12].

In our manuscript, we consider a linear-quadratic singularly perturbed optimal control problem over a trajectory given via a second-order linear ODE. As we know, second-order differential equations are one of the most widely studied classes of differential equations in engineering, mathematics, physics, and other sciences. In many domains, the discretization of a partial differential equation also leads to a system of ordinary differential equations of the second-order. Conventionally, the second-order can be converted to a system of first-order ODEs. However, it will increase the computational cost in terms of function evaluation and thus will affect the computational time. Hence many researchers have shown an interest in studying second-order differential equations directly. We mention here some of recent works concerning singularly perturbed boundary value problems in which different techniques of finding approximate solution are used.

The central interest is a singularly perturbed boundary value problem of the form

$$L(x(t)) := \varepsilon x''(t) + a(t)x'(t) + b(t)x(t) = f(t), \quad t \in (0, 1),$$

$$x(0) = \alpha, \quad x(1) = \beta,$$

where $\varepsilon$ is a small parameter, $\alpha, \beta$ are some finite constants, the coefficients $a(t), b(t)$ and $f(t)$ are sufficiently smooth real functions of $t$. The exhibition of boundary layers at one or both ends of the interval depending on the properties of $a(t)$. For solving this problem, one can use either of the most notable methods, such as the spline approximation methods [1, 23, 13], the asymptotic initial-value methods in [35, 10], the finite-difference methods [33, 24], the boundary value techniques [29, 30]. In the case of discontinuous coefficients or more complex boundary conditions, interested readers can refer to the works [36, 39, 28, 17]. A non-linear problem has been considered in [40].

Another interesting equations (reaction-diffusion equations) of the type

$$L(x(t)) := \varepsilon^2 x''(t) + f(t, x(t)) = 0, \quad t \in (0, 1),$$

$$x(0) = \alpha, \quad x(1) = \beta$$

have been investigated using the collocation method in [34], the dynamical technique in [16]. The linear case of this problem has been studied in [2] using higher-order spline scheme and in [4] using the variational difference scheme. A review of recent works on computational techniques for solving singularly perturbed boundary value problem is presented in [27].

Some singularly perturbed optimal control problems of the same type to the last equation with a constraint on control were considered in [6, 7]. In these works, the nonlinear term was replaced by a linear combination of the state variable, its first derivative and the control variable and the general theorems on approximation were obtained.

While most of the mentioned above papers deals with zero-order approximation, in the present paper, we use the direct scheme method to construct an asymptotic approximation up to any prescribed order to the solution of a control problem of the
same type to the last equation replacing the nonlinear term by a linear combination of state and control variables. The asymptotic expansion contains an outer series and two inner series in the vicinities of two fixed end-points. Moreover, we also obtain the error estimations of the asymptotic solution with respect to the control, trajectory, and functional. It is shown that the values of the considered functional do not increase when a higher-order approximation to the optimal control is applied. An example is given to illustrate the obtained result.

The paper is organized as follows. Section 2 contains a problem statement and the proposition of existence and uniqueness of the solution of the considered problem. The problem decomposition is presented in Section 3. The zero and the \( n \)th order terms of the postulated asymptotic expansion of optimal solution are determined in Sections 4 and 5, respectively. We obtain the error estimations for the state and control variables and for the functional in Section 6. Moreover, in this section, we also prove that the value of the performance index does not increase when the higher-order approximation of the control variable is used. An example of finding the zero and the first-order approximations are given in Section 7. The transformations of coefficients in the expansion of minimized functional are given in Appendices A and B. The last section is a conclusion of the manuscript.

2. Problem Statement. We consider the following problem

\[
P_\varepsilon: J_\varepsilon(u) = \frac{1}{2} \int_0^T \left( w(t,\varepsilon)x^2 + 2f(t,\varepsilon)x + r(t,\varepsilon)u^2 \right) dt \to \min_u
\]

\[
\varepsilon^2 \frac{d^2x}{dt^2} = a(t,\varepsilon)x + b(t,\varepsilon)u + g(t,\varepsilon), \quad t \in [0,T],
\]

\[
x(0,\varepsilon) = x^0, \quad x(T,\varepsilon) = x^T.
\]

Here \( T > 0 \) is fixed, \( x = x(t,\varepsilon) \in C^2[0,T] \) and \( u = u(t,\varepsilon) \in L^2(0,T) \) are scalar functions, the functions in (1), (2) are sufficiently smooth with respect to both parameters \( t \) and \( \varepsilon \), the functions \( a(t,0), w(t,0) \) and \( r(t,0) \) are positive.

For each sufficiently small fixed \( \varepsilon > 0 \), a unique optimal solution of problem (1) - (3) can be found due to the result

**Proposition 1.** For sufficiently small \( \varepsilon > 0 \), a control \( u_*(\cdot,\varepsilon) \) is optimal for the problem \( P_\varepsilon \) if and only if \( u_*(t,\varepsilon) \) is given by

\[
u_*(t,\varepsilon) = b(t,\varepsilon)\varphi(t,\varepsilon)/r(t,\varepsilon),
\]

where \( \varphi(\cdot,\varepsilon) \) is the solution of the problem

\[
\varepsilon^2 \frac{d^2\varphi}{dt^2} = a(t,\varepsilon)\varphi - w(t,\varepsilon)x_* - f(t,\varepsilon),
\]

\[
\varphi(0,\varepsilon) = 0, \quad \varphi(T,\varepsilon) = 0,
\]

and \( x_*(\cdot,\varepsilon) \) is the solution of system (2), (3) corresponding to \( u = u_* \). The problem \( P_\varepsilon \) is uniquely solvable.

Further, for brevity, all the arguments of the variables sometimes are omitted.

**Proof. Sufficiency** Let \( u(t,\varepsilon) \) be an arbitrary admissible control for the problem \( P_\varepsilon \) and \( x(t,\varepsilon) \) is the corresponding solution of (2), (3). We consider the difference

\[
J_\varepsilon(u) - J_\varepsilon(u_*) = \frac{1}{2} \int_0^T \left( w(t,\varepsilon)(x - x_*)^2 + r(t,\varepsilon)(u - u_*)^2 \right) dt + \Delta_1,
\]

(7)
where
\[ \Delta_1 = \int_0^T ((w(t, \varepsilon)x_\ast + f(t, \varepsilon))(x - x_\ast) + r(t, \varepsilon)u_\ast(u - u_\ast))dt. \] (8)

Using (5), (4), (2), the technique of integration by parts, and (3), (6), to construct \( \Delta_1 \), we get
\[ \Delta_1 = \int_0^T (a(t, \varepsilon)\varphi - \varepsilon^2 \frac{d^2 \varphi}{dt^2})(x - x_\ast) + b(t, \varepsilon)\varphi(u - u_\ast))dt 
= \int_0^T (-\varepsilon^2 \frac{d^2 \varphi}{dt^2}(x - x_\ast) + \varphi(a(t, \varepsilon)(x - x_\ast) + b(t, \varepsilon)(u - u_\ast))dt 
= \int_0^T (-\varepsilon^2 \frac{d^2 \varphi}{dt^2}(x - x_\ast) + \varphi(\frac{d^2 x}{dt^2} - \frac{d^2 x_\ast}{dt^2}))dt 
= \varepsilon^2 \left( \frac{d\varphi}{dt}(t, \varepsilon)(x(t, \varepsilon) - x_\ast(t, \varepsilon)) + \varphi(t, \varepsilon)(\frac{dx}{dt}(t, \varepsilon) - \frac{dx_\ast}{dt}(t, \varepsilon)) \right) \bigg|_0^T = 0. \]

Since \( \Delta_1 = 0 \) and \( w(t, 0) \) and \( r(t, 0) \) are positive, for sufficiently small \( \varepsilon > 0 \) it follows from (7) that \( J_\varepsilon(u) - J_\varepsilon(u_\ast) \geq 0 \) and therefore \( u_\ast(., \varepsilon) \) is optimal for the problem (1) - (3).

**Necessity** Let \( u_\ast(., \varepsilon) \) be optimal control and \( u(., \varepsilon) \) be any other admissible control for the problem \( P_\varepsilon \) and \( x_\ast(., \varepsilon) \) and \( x(., \varepsilon) \) be the solution of system (2), (3) corresponding to these control variables. We consider the difference (7) with \( \Delta_1 \) given by (8). Using (5), (2), the technique of integration by parts and the conditions (3), (6), we obtain
\[ \Delta_1 = \int_0^T (r(t, \varepsilon)u_\ast - b(t, \varepsilon)\varphi)(u - u_\ast)dt. \] (9)

We write out the formulas for \( x(t, \varepsilon) \) and \( x_\ast(t, \varepsilon) \) as follow
\[ x(t, \varepsilon) = C_1x_1(t, \varepsilon) + C_2x_2(t, \varepsilon) + \int_0^t \frac{1}{\varepsilon}G(t, \xi, \varepsilon)(b(\xi, \varepsilon)u + g(\xi, \varepsilon))d\xi, \]
\[ x_\ast(t, \varepsilon) = C_1x_1(t, \varepsilon) + C_2x_2(t, \varepsilon) + \int_0^t \frac{1}{\varepsilon}G(t, \xi, \varepsilon)(b(\xi, \varepsilon)u_\ast + g(\xi, \varepsilon))d\xi. \] (10)

Here \( C_1, C_2, C_1, \) and \( C_2, \) are some constants, \( x_1(t, \varepsilon) \) and \( x_2(t, \varepsilon) \) are fundamental set of solutions of the associated homogeneous equation to (2),
\[ G(t, \xi, \varepsilon) = \frac{x_1(\xi, \varepsilon)x_2(t, \varepsilon) - x_2(\xi, \varepsilon)x_1(t, \varepsilon)}{x_1(\xi, \varepsilon)x_2'(\xi, \varepsilon) - x_2(\xi, \varepsilon)x_1'(\xi, \varepsilon)} \]
where \( x_1'(\xi, \varepsilon) = \frac{dx_1}{dt}(t, \varepsilon)|_{t=\xi} \), \( x_2'(\xi, \varepsilon) = \frac{dx_2}{dt}(t, \varepsilon)|_{t=\xi} \).

Using the condition (3) and the assumption that system (2) and (3) has a unique solution with each admissible control \( u \) we find that
\[ C_1 - C_1 = \frac{x_2(0, \varepsilon)\int_0^T \frac{1}{\varepsilon}G(T, \xi, \varepsilon)b(\xi, \varepsilon)(u_\ast - u)d\xi}{x_2(0, \varepsilon)x_1(T, \varepsilon) - x_1(0, \varepsilon)x_2(T, \varepsilon)}, \]
\[ C_2 - C_2 = \frac{x_1(0, \varepsilon)\int_0^T \frac{1}{\varepsilon}G(T, \xi, \varepsilon)b(\xi, \varepsilon)(u_\ast - u)d\xi}{x_1(0, \varepsilon)x_2(T, \varepsilon) - x_2(0, \varepsilon)x_1(T, \varepsilon)}. \] (11)

It follows from (10) and (11) that
\[ ||x(., \varepsilon) - x_\ast(., \varepsilon)|| \leq c\max_{[0,T]} ||u(., \varepsilon) - u_\ast(., \varepsilon)|| \]
where $c$ is a constant. Thus we conclude that if $u(., \varepsilon)$ is sufficiently close to $u_*(., \varepsilon)$ the major term in the difference (7) is the term (9). Since $J_\varepsilon(u) - J_\varepsilon(u_*) \geq 0$ for $u \neq u_*$ then

$$r(t, \varepsilon)u_* - b(t, \varepsilon)\varphi = 0.$$  

From here, for sufficiently small $\varepsilon$, we obtain the formula (4) for optimal control variable $u_*(., \varepsilon)$.

**Uniqueness** The uniqueness of solution follows directly from the arguments used to prove necessity and sufficiency.

### 3. Problem Decomposition

We find the asymptotic series of a solution of the problem (1) - (3) in the form

$$z(t, \varepsilon) = \Xi(t, \varepsilon) + \Pi z(\tau, \varepsilon) + Qz(\sigma, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \left( \Xi_i(t) + \Pi_i z(\tau) + Q_i z(\sigma) \right),$$  

(12)

where $z = (x, u)'$, the prime means transposition; $\tau = \frac{t}{\varepsilon}$, $\sigma = \frac{t-T}{\varepsilon}$; the symbols $\Pi$ and $Q$ stand for boundary functions in the left and in the right end points of interval $[0, T]$, respectively. Let the following assumption be hold

**I.** All boundary functions attain the exponential character. It means that

$$\| \Pi_i z(\tau) \| \leq ce^{-\kappa \tau}, \quad \kappa \geq 0 \quad \text{and} \quad \| Q_i z(\sigma) \| \leq ce^{\kappa \sigma}, \quad \sigma \leq 0,$$

where $c$ and $\kappa$ are some positive constants.

Further, in the expansion of $w(t, \varepsilon)$ into positive power series with respect to $\varepsilon$: $w(t, \varepsilon) = \sum_{i \geq 0} w_i(t)\varepsilon^i$, we use the symbol $[w(t, \varepsilon)]_i$ to denote the coefficient of $\varepsilon^i$.

We substitute (12) into (1) - (3) and present the integrand in (1) and the right-hand side in (2) as asymptotic sums of terms depending on $t, \tau, \sigma$ separately. By condition I, we can change the integrals in (1) of functions with respect to $\tau$ and $\sigma$ into infinite integrals of the same variables over the corresponding intervals.

From (2) and (3), equating the coefficients of like powers of $\varepsilon$ depending on $t, \tau$, and $\sigma$, separately, we obtain expressions for the terms of the series (12).

So we represent the minimized functional (1) as

$$J_\varepsilon(u) = \sum_{i=0}^{\infty} \varepsilon^i J_i.$$  

(13)

For the terms in (12), we have the equations

$$\frac{d^2 \Xi_i}{dt^2} = a_0(t)\Xi_i + b_0(t)\Xi_i + [(a(t, \varepsilon) - a_0(t))\Xi(t, \varepsilon)]_i$$  

$$+ [(b(t, \varepsilon) - b_0(t))\Xi(t, \varepsilon)]_i + [g(t, \varepsilon)]_i,$$  

(14)

$$\frac{d^2 \Pi_i x}{d\tau^2} = a_0(0)\Pi_i x + b_0(0)\Pi_i u + [(a(\tau, \varepsilon, \varepsilon) - a_0(0))\Pi x(\tau, \varepsilon)]_i$$  

$$+ [(b(\tau, \varepsilon, \varepsilon) - b_0(0))\Pi u(\tau, \varepsilon)]_i,$$  

(15)

$$\frac{d^2 Q_i x}{d\sigma^2} = a_0(T)Q_i x + b_0(T)Q_i u + [(a(T + \sigma, \varepsilon, \varepsilon) - a_0(T))Q x(\sigma, \varepsilon)]_i$$  

$$+ [(b(T + \sigma, \varepsilon, \varepsilon) - b_0(T))Q u(\sigma, \varepsilon)]_i,$$  

(16)

$$\Pi_i x(0) = x^0 - \Pi_0 x(0), \quad \Pi_i x(0) = -\Pi_i(0), \quad i \geq 1, \quad \Pi_i x(+\infty) = 0, \quad i \geq 0,$$  

(17)

$$Q_0 x(0) = x^T - \Pi_0 x(0), \quad Q_i x(0) = -\Pi_i x(T), \quad i \geq 1, \quad Q_i x(-\infty) = 0, \quad i \geq 0.$$  

(18)

Note that, the terms with negative sub-index are supposed to be zero and for simplicity the argument $t$ sometimes is omitted.
4. Construction of the zero-order terms. The regular terms of zero-order \( \bar{x}_0(t) \) and \( \bar{u}_0(t) \) in (12) are solution of the following problem \( \mathcal{P}_0 \), which consists of minimizing the functional

\[
\mathcal{J}_0 = J_0 = \frac{1}{2} \int_0^T \left( w_0(t)(\bar{x}_0)^2 + 2f_0(t)\bar{x}_0 + r_0(t)(\bar{u}_0)^2 \right) dt
\]

over the trajectory of the system (obtained from (14) with \( i = 0 \))

\[
0 = a_0(t)\bar{x}_0 + b_0(t)\bar{u}_0 + g_0(t).
\]

**Proposition 2.** A control \( \bar{u}_{0*}(\cdot) \) is optimal for the problem \( \mathcal{P}_0 \) if and only if it has the form

\[
\bar{u}_{0*} = b_0(t)\bar{\psi}_0/r_0(t),
\]

where \( \bar{\psi}_0(\cdot) \) is solution of equation

\[
0 = a_0(t)\bar{\psi}_0 - w_0(t)\bar{x}_{0*} - f_0(t)
\]

with \( \bar{x}_{0*}(\cdot) \) is solution of (20) when \( \bar{u}_0(\cdot) = \bar{u}_{0*}(\cdot) \). Furthermore, the problem (19), (20) has a unique solution.

The proof of Proposition 2 is quite straightforward and similar to the proof of Proposition 1. Thus, we have found \( \bar{x}_0(t) = \bar{x}_{0*}(t) \), \( \bar{u}_0(t) = \bar{u}_{0*}(t) \) and \( \bar{\psi}_0(t) \).

The coefficient \( J_1 \) in (13) is

\[
J_1 = \int_0^T \left( (w_0(t)\bar{x}_0 + f_0(t))(\bar{x}_1 + r_0(t)\bar{u}_1 + \zeta(t)) \right) dt
\]

\[
+ \int_0^{+\infty} \left( (w_0(0)\bar{x}_0(0) + f_0(0))\Pi_0 x + r_0(0)\bar{u}_0(0)\Pi_0 u \right) dt
\]

\[
+ \frac{1}{2} \int_0^{+\infty} \left( (w_0(0)\Pi_0 x)^2 + r_0(0)(\Pi_0 u)^2 \right) dt
\]

\[
+ \int_{-\infty}^0 \left( (w_0(T)\bar{x}_0(T) + f_0(T)Q_0 x + r_0(T)\bar{u}_0(T)Q_0 u) \right) d\sigma
\]

\[
+ \frac{1}{2} \int_{-\infty}^0 \left( (w_0(T)Q_0 x)^2 + r_0(T)(Q_0 u)^2 \right) d\sigma.
\]

Here and further the symbol \( \zeta(t) \) is used to collect all known summands, which are the functions of \( t \).

Using (22), (21) and (14) with \( i = 1 \), we obtain that the integral over the interval \([0, T] \) in \( J_1 \) is a known value.

Since the derivative of a function of exponential type is also a function of exponential type, then we have

\[
\left\| \frac{d\Pi_0 x}{d\tau}(\tau) \right\| \leq ce^{-\kappa \tau}, \quad \tau \geq 0 \quad \text{and} \quad \left\| \frac{dQ_0 x}{d\sigma}(\sigma) \right\| \leq ce^{\kappa \sigma}, \quad \sigma \leq 0.
\]

Using (22), (21) at \( t = 0 \), (15) with \( i = 0 \) and (23), we obtain

\[
\int_0^{+\infty} \left( (w_0(0)\bar{x}_0(0) + f_0(0))\Pi_0 x + r_0(0)\bar{u}_0(0)\Pi_0 u \right) dt = -\bar{\psi}_0(0) \frac{d\Pi_0 x}{d\tau}(0).
\]

Similarly, we have

\[
\int_{-\infty}^0 \left( (w_0(T)\bar{x}_0(T) + f_0(T)Q_0 x + r_0(T)\bar{u}_0(T)Q_0 u) \right) d\sigma = \bar{\psi}_0(T) \frac{dQ_0 x}{d\sigma}(0).
\]
So the construction of $J_1$, after neglecting known summands, is denoted as $\Pi_0 J + Q_0 J$, where
\[
\Pi_0 J = -\psi_0(0) d\Pi_0 x/d\tau(0) + \frac{1}{2} \int_0^{+\infty} (w_0(0)(\Pi_0 x)^2 + r_0(0)(\Pi_0 u)^2) \, d\tau,
\]
\[
Q_0 J = \overline{\psi}_0(T) dQ_0 x/d\sigma(0) + \frac{1}{2} \int_0^{-\infty} (w_0(T)(Q_0 x)^2 + r_0(T)(Q_0 u)^2) \, d\sigma.
\]

The functions $\Pi_0 x(\tau)$ and $\Pi_0 u(\tau)$ will be found by solving the problem $\Pi_0 P$ which consists of minimizing the functional $\Pi_0 J$ over the trajectory of the system (obtained from (15) and (17) with $i = 0$)
\[
\frac{d^2 \Pi_0 x}{d\tau^2} = a_0(0)\Pi_0 x + b_0(0)\Pi_0 u, \quad \tau \geq 0, \quad (24)
\]
\[
\Pi_0 x(0) = x^0 - \pi_0(0), \quad \Pi_0 x(+\infty) = 0. \quad (25)
\]
The functions $Q_0 x(\sigma)$ and $Q_0 u(\sigma)$ will be found by solving the problem $Q_0 P$ which consists of minimizing the functional $Q_0 J$ over the trajectory of the system (obtained from (16) and (18) with $i = 0$)
\[
\frac{d^2 Q_0 x}{d\sigma^2} = a_0(T)Q_0 x + b_0(T)Q_0 u, \quad \sigma \leq 0 \quad (26)
\]
\[
Q_0 x(0) = x^0 - \pi_0(T), \quad Q_0 x(-\infty) = 0. \quad (27)
\]

**Proposition 3.** A control $\Pi_0 u(\cdot)$ is optimal for the problem $\Pi_0 P$ if and only if it is has the form
\[
\Pi_0 u(\cdot) = b_0(0)\Pi_0 \psi/r_0(0), \quad (28)
\]
where $\Pi_0 \psi(\cdot)$ is the solution of the system
\[
\frac{d^2 \Pi_0 \psi}{d\tau^2} = a_0(0)\Pi_0 \psi - w_0(0)\Pi_0 x, \quad \tau \geq 0, \quad (29)
\]
\[
\Pi_0 \psi(+) = 0, \quad \Pi_0 \psi(-\infty) = -\overline{\psi}_0(0), \quad (30)
\]
where $\Pi_0 x(\cdot)$ is the solution of system (24), (25) corresponding to $\Pi_0 u(\cdot) = \Pi_0 u(\cdot)$. Furthermore, the problem $\Pi_0 P$ has a unique solution.

**Proposition 4.** A control $Q_0 u(\cdot)$ is optimal for the problem $Q_0 P$ if and only if it is has the form
\[
Q_0 u(\cdot) = b_0(T)Q_0 \psi/r_0(T), \quad (31)
\]
where $Q_0 \psi(\cdot)$ is the solution of the system
\[
\frac{d^2 Q_0 \psi}{d\sigma^2} = a_0(T)Q_0 \psi - w_0(T)Q_0 x, \quad \sigma \leq 0, \quad (32)
\]
\[
Q_0 \psi(-\infty) = 0, \quad Q_0 \psi(+) = -\overline{\psi}_0(T), \quad (33)
\]
where $Q_0 x(\cdot)$ is the solution of system (26), (27) corresponding to $Q_0 u(\cdot) = \Pi_0 u(\cdot)$. Furthermore, the problem $Q_0 P$ has a unique solution.

The proofs of Propositions 3 and 4 can be found in [22], in addition we use the fact that since $\Pi_0 \psi(\tau)$ and $Q_0 \psi(\sigma)$ satisfy condition I then theirs derivatives approach zero when their arguments tend to infinity. It means
\[
\frac{d\Pi_0 \psi}{d\tau}(\tau) \to 0 \quad \text{when} \quad \tau \to +\infty \quad \text{and} \quad \frac{dQ_0 \psi}{d\sigma}(\sigma) \to 0 \quad \text{when} \quad \sigma \to -\infty.
\]
Thus we have found $\Pi_0 x(\tau) = \Pi_0 x_0(\cdot)$, $\Pi_0 u(\tau) = \Pi_0 u_0(\cdot)$, $Q_0 x(\sigma) = Q_0 x_0(\cdot)$, $Q_0 u(\sigma) = Q_0 u_0(\cdot)$, $\Pi_0 \psi(\tau)$ and $Q_0 \psi(\sigma)$. 
5. **Higher-order approximations.** Suppose that the functions $\overline{u}_j(t)$, $\overline{v}_j(t)$ (at the same time the function $\overline{\nu}_j(t)$), the functions $\Pi_j x(\tau), \Pi_j u(\tau)$ (at the same time the function $\Pi_j \psi(\tau)$) and the functions $Q_j x(\sigma), Q_j u(\sigma)$ (at the same time the function $Q_j \psi(\sigma)$), $j < n$, have been determined.

We use the following notations

\[
\overline{v}_j(t, \varepsilon) = \sum_{i=0}^{j} \varepsilon^i \Pi_i \psi(\tau), \quad Q_j \psi(\sigma, \varepsilon) = \sum_{i=0}^{j} \varepsilon^i Q_i \psi(\sigma).
\]

Similarly, we can obtain problems $\overline{P}_n$, $\Pi_n P$ and $Q_n P$ to determine the $n$-th order terms of the series (12) ($n \geq 1$).

The construction of the coefficient $J_{2n}$ in (13) (See Appendix A), after neglecting all known summands, is denoted by $J_n$, where

\[
J_n = \frac{1}{2} \int_{0}^{T} (w_0(t)(\overline{x}_n)^2 + 2[(w(t, \varepsilon) - w_0(t))\overline{x}(t, \varepsilon) + f(t, \varepsilon)
- (a(t, \varepsilon) - a_0(t))\overline{v}_{n-1}(t, \varepsilon)]n \overline{x}_n + \frac{d^2 \overline{v}_{n-2}}{dt^2} \overline{x}_n + r_0(t)(\overline{v}_n)^2
+ 2[(r(t, \varepsilon) - r_0(t))\overline{v}(t, \varepsilon) - (b(t, \varepsilon) - b_0(t))\overline{v}_{n-1}(t, \varepsilon)]n \overline{v}_n dt.
\]  

(34)

The functions $\overline{x}_n(t)$ and $\overline{v}_n(t)$ are the solution of the problem $\overline{P}_n$ which consists of minimizing the functional (34) over the trajectory of system (14) with $i = n$.

**Theorem 5.1.** A control $\overline{x}_{n*}(\cdot)$ is optimal for problem $\overline{P}_n$ if and only if it has the form

\[
\overline{x}_{n*} = \frac{1}{r_0(t)}(b_0(t)\overline{v}_n + [(b(t, \varepsilon) - b_0(t))\overline{v}_{n-1}(t, \varepsilon) - (r(t, \varepsilon) - r_0(t))\overline{v}(t, \varepsilon)]n).
\]  

(35)

where the function $\overline{v}_{n*}(\cdot)$ is a solution of system

\[
\frac{d^2 \overline{v}_{n-2}}{dt^2} = a_0(t)\overline{v}_n - w_0(t)(\overline{x}_{n*} - [(w(t, \varepsilon) - w_0(t))\overline{x}(t, \varepsilon)
- f(t, \varepsilon)]n + [(a(t, \varepsilon) - a_0(t))\overline{v}_{n-1}(t, \varepsilon)]n.
\]  

(36)

Here $\overline{x}_{n*}(\cdot)$ is a solution of (14) when $\overline{x}_n(\cdot) = \overline{x}_{n*}(\cdot)$. The problem $\overline{P}_n$ is uniquely solvable.

Thus, we have found $\overline{x}_n(t) = \overline{x}_{n*}(t)$, $\overline{u}_n(t) = \overline{u}_{n*}(t)$ and $\overline{v}_n(t)$.

The construction of the coefficient $J_{2n+1}$ in (13) (see Appendix B), after neglecting all known summands, is denoted by $\Pi_n J + Q_n J$, where

\[
\Pi_n J = -\overline{v}_n(0)\overline{w}(0)(\Pi_{n+1})^2 + \frac{1}{2} \int_{0}^{t+\infty} w_0(0)(w_0(0)(\Pi_{n+1})^2 + 2[(w(T, \varepsilon), \varepsilon)
- w_0(0))\Pi_x(\tau, \varepsilon) - (a(\tau, \varepsilon) - a_0(0))\Pi_{n-1}(\tau, \varepsilon)]n \Pi_{n+1} x + r_0(0)(\Pi_{n+1} u)^2
+ 2[(r(T, \varepsilon) - r_0(T))\Pi_u(\tau, \varepsilon) - (b(\tau, \varepsilon) - b_0(0))\Pi_{n-1}(\tau, \varepsilon)]n \Pi_{n+1} u) d\tau.
\]  

(37)

\[
Q_n J = -\overline{v}_n(T)\overline{w}(0)(\Pi_{n+1})^2 + \frac{1}{2} \int_{0}^{t+\infty} w_0(0)(Q_{n+1} x)^2 + 2[(w(T + \sigma, \varepsilon) + w_0(T))Q_{n+1} x(\sigma, \varepsilon)
- a(T + \sigma, \varepsilon) - a_0(T))Q_{n-1}(\sigma, \varepsilon)]n Q_{n+1} x + r_0(T)(Q_{n+1} u)^2 + 2[(r(T + \sigma, \varepsilon) - r_0(T))Q_u(\sigma, \varepsilon)
- (b(T + \sigma, \varepsilon) - b_0(T))Q_{n-1}(\sigma, \varepsilon)]n Q_{n+1} u) d\sigma.
\]  

(38)
The functions \( \Pi_n x(\tau) \) and \( \Pi_n u(\tau) \) are the solution of the problem \( \Pi_n P \) that consists of minimizing the functional (37) over trajectory of the system (15) with the conditions (17) with \( i = n \).

The functions \( Q_n x(\sigma) \) and \( Q_n u(\sigma) \) are solution of the problem \( Q_n P \) that consists of minimizing the functional (38) over trajectory of the system (16) with the conditions (18) with \( i = n \).

**Theorem 5.2.** A control \( \Pi_n u_\ast \) is optimal for the problem \( \Pi_n P \) is an only if it has the form

\[
\Pi_n u_\ast = \frac{1}{t_0(0)} (b_0(0)\Pi_n \psi + [(b(\varepsilon \tau, \varepsilon) - b_0(0))\Pi_{n-1} \psi(\tau, \varepsilon)]_n
- [(r(\varepsilon \tau, \varepsilon) - r_0(0))\Pi u(\tau, \varepsilon)]_n)
\]

where the function \( \Pi_n \psi(\cdot) \) is a solution of the following system

\[
\begin{align*}
\frac{d^2 \Pi_n \psi}{d\tau^2} &= a_0(0)\Pi_n \psi - w_0(0)\Pi_n x_\ast - [(w(\varepsilon \tau, \varepsilon)
- w_0(0))\Pi x(\tau, \varepsilon)]_n + [(a(\varepsilon \tau, \varepsilon) - a_0(0))\Pi_{n-1} \psi(\tau, \varepsilon)]_n, \\
\Pi_n \psi(0) &= -\psi_n(0), \quad \Pi_n \psi(+\infty) = 0,
\end{align*}
\]

here \( \Pi_n x_\ast(\cdot) \) is the solution of (15), (17) with \( i = n \) and \( \Pi_n u(\cdot) = \Pi_n u_\ast(\cdot) \). Moreover, the problem \( \Pi_n P \) is uniquely solvable.

**Theorem 5.3.** A control \( Q_n u_\ast \) is optimal for the problem \( Q_n P \) is an only if it has the form

\[
Q_n u_\ast = \frac{1}{t_0(T)} (b_0(T)Q_n \psi + [(b(T + \varepsilon \sigma, \varepsilon) - b_0(T))Q_{n-1} \psi(\sigma, \varepsilon)]_n
- [(r(T + \varepsilon \sigma, \varepsilon) - r_0(T))Q u(\sigma, \varepsilon)]_n)
\]

where the function \( Q_n \psi(\cdot) \) is a solution of the following system

\[
\begin{align*}
\frac{d^2 Q_n \psi}{d\sigma^2} &= a_0(T)Q_n \psi - w_0(T)Q_n x_\ast - [(w(T + \varepsilon \sigma, \varepsilon)
- w_0(T))Q x(\sigma, \varepsilon)]_n + [(a(T + \varepsilon \sigma, \varepsilon) - a_0(T))Q_{n-1} \psi(\sigma, \varepsilon)]_n, \\
Q_n \psi(0) &= -\psi_n(T), \quad Q_n \psi(-\infty) = 0,
\end{align*}
\]

here \( Q_n x_\ast(\cdot) \) is the solution of (16), (18) with \( i = n \) and \( Q_n u(\cdot) = Q_n u_\ast(\cdot) \). The problem \( Q_n P \) is uniquely solvable.

Thus, we have found the \( n \)-th order boundary terms \( \Pi_n z(\tau) \) and \( Q_n z(\sigma) \) of expansion (12).

6. Error Estimates. We use the following notation

\[
\tilde{z}_n(t, \varepsilon) = \sum_{i=0}^{n} \varepsilon^i (z_i(t) + \Pi_i z(\tau) + Q_i z(\sigma)).
\]

**Theorem 6.1.** For sufficiently small \( \varepsilon > 0 \) the following estimations are hold

\[
\|z_\ast(t, \varepsilon) - \tilde{z}_n(t, \varepsilon)\| \leq \varepsilon^{n+1},
\]

\[
J_\varepsilon (\tilde{u}_n(t, \varepsilon)) - J_\varepsilon (u_\ast(t, \varepsilon)) \leq \varepsilon^{2(n+1)}.
\]

Here \( z_\ast(t, \varepsilon) \) is the optimal solution of the problem \( P_\varepsilon \), \( \tilde{z}_n(t) \), \( \Pi_n z(\tau) \) and \( Q_n z(\sigma) \) are the optimal solutions of the problems \( \overline{P}_n \), \( \Pi_n P \), and \( Q_n P \), respectively.
Proof. Let \( \varphi(t, \varepsilon) \) found in Proposition 1 be expressed in the form

\[
\varphi(t, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i (\varphi_i(t) + \Pi_i \varphi(\tau) + Q_i \varphi(\sigma)).
\]

It is easy to see that \( \varphi_i(t) \equiv \widetilde{\psi}_i(t), \Pi_i \varphi(\tau) \equiv \Pi_i \varphi(\tau) \) and \( Q_i \varphi(\sigma) \equiv Q_i \varphi(\sigma) \) for \( i = 0, 1, \ldots, n \).

Given the following nomenclatures

\[
\Delta z(t, \varepsilon) = z_n(t, \varepsilon) - \tilde{z}_n(t, \varepsilon), \quad \Delta \varphi(t, \varepsilon) = \varphi(t, \varepsilon) - \tilde{\varphi}_n(t, \varepsilon),
\]

where \( \tilde{\varphi}_n(t, \varepsilon) \equiv \psi_n(t, \varepsilon) = \sum_{i=0}^{n} \varepsilon^i (\psi_i(t) + \Pi_i \varphi(\tau) + Q_i \varphi(\sigma)). \)

Substitute \( z_i(t, \varepsilon) = \Delta z(t, \varepsilon) + \tilde{z}_n(t, \varepsilon), \) \( \varphi(t, \varepsilon) = \Delta \varphi(t, \varepsilon) + \tilde{\varphi}_n(t, \varepsilon) \) into (2) - (6), taking into account (14)-(18) with \( i = 0, 1, \ldots, n \), (35), (36), (39) - (44), we obtain the following system of \( \Delta x(t, \varepsilon), \Delta u(t, \varepsilon) \) and \( \Delta \varphi(t, \varepsilon) \)

\[
\varepsilon^2 \frac{d^2 \Delta x}{dt^2} = a(t, \varepsilon) \Delta x + s(t, \varepsilon) \Delta \varphi + O(\varepsilon^{n+1}), \quad (47)
\]

\[
\varepsilon^2 \frac{d^2 \Delta \varphi}{dt^2} = a(t, \varepsilon) \Delta \varphi - w(t, \varepsilon) \Delta x + O(\varepsilon^{n+1}),
\]

\[
\Delta u = b(t, \varepsilon) \Delta \varphi/r(t, \varepsilon) + O(\varepsilon^{n+1}),
\]

\[
\Delta x(0, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i Q_i x(-T/\varepsilon), \quad \Delta x(T, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i \Pi_i x(T/\varepsilon),
\]

\[
\Delta \varphi(0, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i Q_i \varphi(-T/\varepsilon), \quad \Delta \varphi(T, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i \Pi_i \varphi(T/\varepsilon).
\]

Here \( s(t, \varepsilon) = b(t, \varepsilon)^2/r(t, \varepsilon) \).

We rewrite the system (47)-(49) in the matrix form

\[
\varepsilon^2 \frac{d^2 \Delta Y}{dt^2} = M(t, \varepsilon) \Delta Y + O(\varepsilon^{n+1}), \quad (50)
\]

\[
\Delta Y(0, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i Q_i Y(-T/\varepsilon), \quad \Delta Y(T, \varepsilon) = - \sum_{i=0}^{n} \varepsilon^i \Pi_i Y(T/\varepsilon), \quad (51)
\]

where \( \Delta Y = (\Delta x, \Delta \varphi)', \ M(t, \varepsilon) = \begin{bmatrix} a(t, \varepsilon) & s(t, \varepsilon) \\ -w(t, \varepsilon) & a(t, \varepsilon) \end{bmatrix}, Q_i Y = (Q_i x, Q_i \varphi)', \Pi_i Y = (\Pi_i x, \Pi_i \varphi)', \ i = 0, 1, \ldots, n. \)

Since \( a(t, 0) \) and \( w(t, 0) \) are positive and \( s(t, 0) \) is non-negative for all \( t \in [0, T] \), then for sufficiently small \( \varepsilon > 0 \) the matrix \( M(t, \varepsilon) \) is positive definite for all \( t \in [0, T] \).

As \( - \sum_{i=0}^{n} \varepsilon^i Q_i Y(-T/\varepsilon) \) and \( - \sum_{i=0}^{n} \varepsilon^i \Pi_i Y(T/\varepsilon) \) are exponential small, so from (51), it could be considered that

\[
\|\Delta Y(0, \varepsilon)\| \leq c \varepsilon^{n+1}, \quad \|\Delta Y(T, \varepsilon)\| \leq c \varepsilon^{n+1}.
\]

Using the result obtained in [37], we obtain

\[
\|\Delta Y(t, \varepsilon)\| \leq c \varepsilon^{n+1} \text{ for all } t \in [0, T].
\]

Hence, we obtain the estimation (45).

Denote by \( \bar{x} \) the solution of system (2), (3) at \( u = \bar{u}_n \). The optimal solution is represented as \( x = \bar{x} + \delta x, \) \( u = \bar{u}_1 + \Delta u \). Then \( \delta x \) is the solution of the problem

\[
\varepsilon^2 \frac{d^2 \delta x}{dt^2} = a(t, \varepsilon) \delta x + b(t, \varepsilon) \Delta u,
\]
\[
\delta x(0, \varepsilon) = 0, \quad \delta x(T, \varepsilon) = 0.
\]
In view of (45), \(\|\Delta u(t, \varepsilon)\| \leq c \varepsilon^{n+1}\), the solution of last problem satisfies
\[
\|\delta x(t, \varepsilon)\| \leq c \varepsilon^{n+1}.
\] (52)

The proof of Proposition 1 implies that
\[
J_\varepsilon(\overline{u}_1) - J_\varepsilon(u_*) = \frac{1}{2} \int_0^T (w(t)(\delta x) + r(t)(\Delta u)^2) \, dt.
\]

Taking into account the estimates (45) and (52), we obtain (46). \(\square\)

**Theorem 6.2.** For sufficiently small \(\varepsilon > 0\), we have
\[
J_\varepsilon(\overline{u}_{n-1}) \geq J_\varepsilon(\overline{u}_n) \geq J_\varepsilon(u_*).
\]

The proof of Theorem 6.2 can be found in [22]. Note that, only the direct scheme reveals this nonincreasing nature of the functional.

7. **Illustrative Example.** We consider the problem
\[
P_\varepsilon: J_\varepsilon(u) = \frac{1}{2} \int_0^1 (x^2 + u^2) \, dt \rightarrow \min_u,
\]
on over the trajectory of the system
\[
\varepsilon^2 \frac{d^2x}{dt^2} = x + u + t^2 + \varepsilon, \quad (53)
\]
\[
x(0, \varepsilon) = 1, \quad x(1, \varepsilon) = -1. \quad (54)
\]

In view of Proposition 1, the solution of the problem \(P_\varepsilon\) is obtained from solving the system consisting of (53), (54) and the following
\[
\varepsilon^2 \frac{d^2\varphi}{dt^2} = \varphi - x, \quad \varphi = u, \quad (55)
\]
\[
\varphi(0, \varepsilon) = 0, \quad \varphi(1, \varepsilon) = 0. \quad (56)
\]

The solution of (53), (55) is
\[
x(t, \varepsilon) = -\frac{1}{2} \varepsilon - \frac{1}{2} t^2 + ((C_1 A + C_2 B) \cos(\beta t) + (C_1 B - C_2 A) \sin(\beta t)) e^{\alpha t}
\]
\[
+ ((-C_3 A + C_4 B) \cos(\beta t) + (C_3 B + C_4 A) \sin(\beta t)) e^{-\alpha t},
\]
\[
u(t, \varepsilon) = \varphi(t, \varepsilon) = -\frac{1}{2} \varepsilon - \varepsilon^2 - \frac{1}{2} t^2 + ((C_1 B - C_2 A) \cos(\beta t) - (C_1 A
\]
\[
+ C_2 B) \sin(\beta t)) e^{\alpha t} - ((C_3 B + C_4 A) \cos(\beta t) + (C_3 A - C_4 B) \sin(\beta t)) e^{-\alpha t},
\]
where \(A = \frac{\xi}{4} \left( \sqrt{2 + 2\sqrt{2}} + \sqrt{-2 + 2\sqrt{2}} \right), B = \frac{\xi}{4} \left( \sqrt{2 + 2\sqrt{2}} - \sqrt{-2 + 2\sqrt{2}} \right), \beta = \frac{\sqrt{-2 + 2\sqrt{2}}}{2\xi}, \alpha = \frac{\sqrt{2 + 2\sqrt{2}}}{2\xi}\) and \(C_1 - C_4\) are some constants which are found by (54) and (56).

Let \(\varepsilon = 0\), we obtain the degenerate problem, that consists of minimizing the functional \(\frac{1}{2} \int_0^1 (x^2 + u^2) \, dt\) over the trajectory of the system
\[
0 = x + u + t^2.
\]
We solve this degenerate problem and obtain \(x(t) = u(t) = -\frac{t^2}{2}\).

Let \(a_1 = \frac{1}{2} \sqrt{-2 + 2\sqrt{2}}\) and \(b_1 = -\frac{1}{2} \sqrt{2 + 2\sqrt{2}}\). Applying the results obtained in Sections 4 and 5 we obtain the following terms:
\[
\pi_0(t) = -\frac{t^2}{2}, \quad \sigma_0(t) = -\frac{t^2}{2}, \quad \Pi_0 x(\tau) = \cos(a_1 \tau) e^{b_1 \tau}, \quad \Pi_0 u(\tau) = \sin(a_1 \tau) e^{b_1 \tau}, \quad Q_0 x(\sigma) =
\]
The Figs. 1 and 2 illustrate the solution of $P_\varepsilon$ and some of its approximations. Here the solid line represents the exact solution, the dashed line - the degenerate solution, the lines consisted of crosses and boxes - the zero and the first approximations, respectively, to the exact one. The numerical illustration of Theorem 6.2 is given in Table 1.
8. Conclusion. We can see from the paper that the asymptotic expansions of the optimal solution to the considered problem can be constructed more easily than reducing the equation into the first-order ODEs system. Furthermore, in comparison with other singularly perturbed problems with first-order state constraint, in this work, the Lyapunov’s stability is not required (the eigenvalues of the coefficient in front of fast variable have negative real part)

Because of the variational nature of each order approximation, specific optimal control computing methods can be applied. The results obtained in the example show that the constructed asymptotic solution becomes better if the smaller value of $\varepsilon$ and a higher-order asymptotic expansion are used.

In the future the author hopes to develop the algorithm for constructing an asymptotic solution to the problem (1) - (3) in the case of a small parameter in front of the second derivative is of the power 1. In this case, the asymptotic expansion of the solution will have a more complicated form.

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APPENDICES

Appendix A. The construction of $J_{2n}$. The coefficient $J_{2n}$ in (13) is

$$J_{2n} = \frac{1}{2} \int_0^T (w_0(t)\pi_n)^2 + 2[w(t,\varepsilon) - w_0(t)]\pi(t,\varepsilon) + f(t,\varepsilon)]_n \pi_n + r_0(t)\pi_n^2$$

+ $2[(r(t,\varepsilon) - r_0(t))\pi(t,\varepsilon)]_n \pi_n + \zeta(t)]dt + J_{2n} + \Pi_1 J_{2n-1} + \Pi_2 J_{2n-1} + \Pi J_{2n-1}$

+ $\int_0^{+\infty} \Pi \zeta(\tau) d\tau + Q_1 J_{2n-1} + Q_2 J_{2n-1} + Q J_{2n-1} + \int_{-\infty}^0 Q(\sigma) d\sigma$,

where

$$\mathcal{J}_{2n} = \int_0^T \sum_{k=0}^{n-1} [w(t,\varepsilon)\pi(t,\varepsilon) + f(t,\varepsilon)]_k \pi_{2n-k} + [r(t,\varepsilon)\pi(t,\varepsilon)]_k \pi_{2n-k}] dt,$$

$$\Pi_1 J_{2n-1} = \int_0^{+\infty} \sum_{k=0}^{n-1} [w(\tau,\varepsilon)\pi(\tau,\varepsilon) + f(\tau,\varepsilon)]_k \pi_{2n-1-k} x + [r(\tau,\varepsilon)\pi(\tau,\varepsilon)]_k \pi_{2n-1-k} u] d\tau,$$

$$\Pi_2 J_{2n-1} = \int_0^{+\infty} \sum_{k=0}^{n-1} [w(\tau,\varepsilon)\Pi x(\tau,\varepsilon)]_k \sum_{i=0}^{n-1-k} \frac{\tau^i d^i \pi_{2n-1-k-i}}{dt^i}(0)$$

+ $[r(\tau,\varepsilon)\Pi u(\tau,\varepsilon)]_k \sum_{i=0}^{n-1-k} \frac{\tau^i d^i \pi_{2n-1-k-i}}{dt^i}(0)] d\tau,$
$$\Pi J_{2n-1} = \int_0^{+\infty} \sum_{k=0}^{n-1} \left( [w(\tau, \varepsilon) \Pi x(\tau, \varepsilon)]_k \Pi_{2n-1-k} x \right) d\tau + \left[ r(\tau, \varepsilon) \Pi u(\tau, \varepsilon)]_k \Pi_{2n-1-k} u \right] d\tau,$$

the expressions $Q_1 J_{2n-1}$, $Q_2 J_{2n-1}$ and $Q J_{2n-1}$ have the form like $\Pi_1 J_{2n-1}$, $\Pi_2 J_{2n-1}$ and $\Pi J_{2n-1}$, respectively, replacing the interval of integration over $[0, +\infty]$, the symbol $\tau \varepsilon$, the symbols $\Pi$ and $\tau$ by the interval of integration over $(-\infty, 0]$, the symbol $T+\sigma \varepsilon$, the symbols $Q$ and $\sigma$, respectively. The symbols $\Pi(\tau)$ and $Q(\sigma)$ here and further are used to denote the collections of known summands that are functions of $\tau$ and $\sigma$, respectively.

We also mean that after the symbol $\sim$ is an construction of the considered expression after omitting known summands.

Using (36), (35), and equation (14) with $i = n$ to construct the value $J_{2n}$. It follows that

$$J_{2n} \sim \sum_{k=0}^{n-2} \left( \frac{d\tau_{2n-k}}{dt} \bar{\psi}_k(t) - \tau_{2n-k}(t) \frac{d\bar{\psi}_k(t)}{dt} \right)_{\tau}^{T} + \int_0^T \left( [(a(t, \varepsilon) - a_0(t)) \bar{\omega}_{n-1}(t, \varepsilon)]_{\tau} - \frac{d^2 \psi_{n-2}}{dt^2} \right)_{\tau} + \int_0^T \left( [(b(t, \varepsilon) - b_0(t)) \bar{\psi}_{n-1}(t, \varepsilon)]_{\tau} \right) d\tau.$$

We construct the term $\Pi_1 J_{2n-1}$. Using (36), (35) and these equations with appropriate time of differentiation at $t = 0$, the equation (15), the technique of integration by parts, and for the boundary functions of exponential type (23) the following properties are hold

$$\lim_{\tau \to +\infty} \tau^i \frac{d^i \Pi_n x}{d\tau^i} (\tau) = 0, \quad \text{and} \quad \lim_{\sigma \to -\infty} \sigma^j \frac{d^j Q_n x}{d\sigma^j} (\sigma) = 0, \quad (A1)$$

for any $i \geq 0$ and $j \geq 0$, we obtain the following construction for $\Pi_1 J_{2n-1}$

$$\Pi_1 J_{2n-1} \sim \sum_{k=0}^{n-1} \bar{\psi}_k(0) \frac{d\Pi_{2n-1-k} x}{d\tau} (\tau) - \frac{d\bar{\psi}_k(0)}{dt} \Pi_{2n-1-k} x (\tau) \biggr|_0^{+\infty}. \quad (A2)$$

We continue to construct the term $\Pi_2 J_{2n-1}$. Using (40), (39), equation (14) with appropriate times of differentiation at $t = 0$, the property (A1), we obtain the following construction for this term

$$\Pi_2 J_{2n-1} \sim \sum_{k=0}^{n-1} \left( \frac{d\tau_{2n-k}}{dt} (0) \Pi_k \psi (\tau) - \tau_{2n-k}(0) \frac{d\Pi_k \psi}{d\tau} (\tau) \right)_{\tau}^{+\infty}. \quad (A3)$$

Lastly we construct the term $\Pi J_{2n-1}$. Using (40), (39), equation (15), the property (A1), we obtain the following construction for this term

$$\Pi J_{2n-1} \sim \sum_{k=0}^{n-1} \left( \Pi_k \psi (\tau) \frac{d\Pi_{2n-1-k} x}{d\tau} (\tau) - \frac{d\Pi_k \psi}{d\tau} (\tau) \Pi_{2n-1-k} x (\tau) \right)_{\tau}^{+\infty}. \quad (A4)$$
Similarly to (A2), (A3), and (A4), we obtain the following constructions for \( Q_1J_{2n-1}, Q_2J_{2n-1} \) and \( QJ_{2n-1} \)

\[
Q_1J_{2n-1} \sim \sum_{k=0}^{n-1} \left( \frac{dQ_{2n-1-k}x}{d\sigma}(\sigma) - \frac{d\psi_{2n-1-k}}{dt}(T)Q_{2n-1-k}x(\sigma) \right) \bigg|_{-\infty}^{0},
\]

\[
Q_2J_{2n-1} \sim \sum_{k=0}^{n-1} \left( \frac{dQ_{2n-1-k}x}{d\sigma}(\sigma) - \frac{d\psi_{2n-1-k}}{dt}(T)Q_{2n-1-k}x(\sigma) \right) \bigg|_{-\infty}^{0}, \quad (A5)
\]

\[
QJ_{2n-1} \sim \sum_{k=0}^{n-1} \left( Q_k\psi(\sigma) \frac{dQ_{2n-1-k}x}{d\sigma}(\sigma) - \frac{dQ_k\psi(\sigma)}{d\sigma}(T)Q_{2n-1-k}x(\sigma) \right) \bigg|_{-\infty}^{0}.
\]

Taking into account the construction for \( J_{2n} \), the obtained formulas (A2)-(A5) and the equalities (17), (18), (41) and (44), we finally get the construction denoted as \( J_n \) for \( J_{2n} \) in (13), which has the form (34).

**Appendix B. The construction of** \( J_{2n+1} \). The coefficient \( J_{2n+1} \) is

\[
J_{2n+1} = J_{2n+1} + \int_0^T \zeta(t)dt + \Pi_1J_{2n} + \Pi_2J_{2n} + \Pi J_{2n}
\]

\[
+ \int_0^{+\infty} \left( \frac{1}{2}w_0(0)(\Pi_nx)^2 + [(w(\tau,\epsilon) - w_0(0))\Pi x(\tau,\epsilon)]_n\Pi_nx + \frac{1}{2}r_0(0)\Pi_nx \right)^2
\]

\[
+ [(r(\tau,\epsilon) - r_0(0))\Pi u(\tau,\epsilon)]_n\Pi_nu)\Pi^2 \zeta dt + \int_0^{+\infty} \Pi \zeta(\tau) d\tau + Q_1J_{2n} + Q_2J_{2n}
\]

\[
+ QJ_{2n} + \int_{-\infty}^{0} \left( \frac{1}{2}w_0(T)(Q_nx)^2 + [(w(T + \sigma,\epsilon) - w_0(T))Q x(\sigma,\epsilon)]_nQ_nx
\]

\[
+ \frac{1}{2}r_0(T)(Q_nx)^2 + [(r(T + \sigma,\epsilon) - r_0(T))Q u(\sigma,\epsilon)]_nQ_nu)\Pi^2 \zeta(\sigma) d\sigma
\]

where

\[
J_{2n+1} = \int_0^T \sum_{k=0}^n \left( [w(t,\epsilon)\overline{\tau}(t,\epsilon) + f(t,\epsilon)]_k \overline{\tau}_{2n+1-k} + [r(t,\epsilon)\overline{u}(t,\epsilon)]_k \overline{u}_{2n+1-k} \right) dt,
\]

\[
\Pi_1J_{2n} = \int_0^{+\infty} \sum_{k=0}^n \left( [w(\tau,\epsilon)\overline{\tau}(\tau,\epsilon) + f(\tau,\epsilon)]_k \Pi_{2n-k}x \right.
\]

\[
+ [r(\tau,\epsilon)\overline{u}(\tau,\epsilon)]_k \Pi_{2n-k}u)\Pi^2 \zeta d\tau,
\]

\[
\Pi_2J_{2n} = \int_0^{+\infty} \sum_{k=0}^n \left( [w(\tau,\epsilon)\Pi x(\tau,\epsilon)]_k \sum_{i=0}^{n-k} \frac{\tau^i}{i!} \frac{d^i \overline{\tau}_{2n-k-i}(0)}{dt^i} \right)
\]

\[
+ [r(\tau,\epsilon)\Pi u(\tau,\epsilon)]_k \sum_{i=0}^{n-k} \frac{\tau^i}{i!} \frac{d^i \overline{u}_{2n-k-i}(0)}{dt^i} d\tau,
\]

\[
\Pi J_{2n} = \int_0^{+\infty} \sum_{k=0}^n \left( [w(\tau,\epsilon)\Pi x(\tau,\epsilon)]_k \Pi_{2n-k}x + [r(\tau,\epsilon)\Pi u(\tau,\epsilon)]_k \Pi_{2n-k}u \right) d\tau,
\]

the expressions \( Q_1J_{2n}, Q_2J_{2n}, \) and \( QJ_{2n} \) have the form like \( \Pi_1J_{2n}, \Pi_2J_{2n} \) and \( \Pi J_{2n} \), respectively, replacing the interval of integration over \([0, +\infty]\), the symbol
τε, the symbols Π and τ by the interval of integration over (−∞, 0], the symbol T + σε, the symbols Q and σ, respectively.

Using (36), (35) and (14) to construct the term J2n+1 in J2n+1. In the result, it follows

\[ J_{2n+1} \sim \sum_{k=0}^{n} \left( \frac{d\tau_{2n-k}}{dt}(0) \frac{d\psi_{k}(t)}{dt} - \tau_{2n-k}(0) \frac{d\psi_{k}}{dt}(t) \right) \bigg|_{0}^{T} \]  

(B1)

Absolutely similar to the process of construction of Π1J2n−1, we obtain the following result of constructing Π1J2n

\[ \Pi_{1}J_{2n} \sim \sum_{k=0}^{n} \left( \psi_{k}(0) \frac{d\Pi_{2n-k}x}{d\tau}(\tau) - \frac{d\psi_{k}}{d\tau}(0)\Pi_{2n-k}x(\tau) \right) \bigg|_{0}^{+\infty} \]  

(B2)

Similar to the result of construction of Π2J2n−1, it follows the following construction of Π2J2n

\[ \Pi_{2}J_{2n} \sim \sum_{k=0}^{n} \left( \Pi_{k}\psi(\tau) \frac{d\Pi_{2n-k}x}{d\tau}(\tau) - \frac{d\Pi_{k}}{d\tau}(\tau)\Pi_{2n-k}x(\tau) \right) \bigg|_{0}^{+\infty} \]  

(B3)

And lastly, using (40), (39), equation (15), the technique of integration by parts, the property (A1), w obtain the following construction for ΠJ2n

\[ \Pi_{J_{2n}} \sim \sum_{k=0}^{n-1} \left( \Pi_{k}\psi(\tau) \frac{d\Pi_{2n-k}x}{d\tau}(\tau) - \frac{d\Pi_{k}}{d\tau}(\tau)\Pi_{2n-k}x(\tau) \right) \bigg|_{0}^{+\infty} \]  

(B4)

Similarly to (B2), (B3), and (B4), we obtain the following constructions for Q1J2n, Q2J2n and QJ2n

\[ Q_{1}J_{2n} \sim \sum_{k=0}^{n} \left( \psi_{k}(T) \frac{dQ_{2n-k}x}{d\sigma}(\sigma) - \frac{d\psi_{k}}{d\sigma}(T)Q_{2n-k}(\sigma) \right) \bigg|_{0}^{+\infty} \]  

(B5)

\[ Q_{2}J_{2n} \sim \sum_{k=0}^{n-1} \left( \frac{d\tau_{2n-k}}{dt}(T)Q_{k}\psi(\sigma) - \tau_{2n-k}(T)\frac{dQ_{k}}{d\sigma}(\sigma) \right) \bigg|_{-\infty}^{0} \]

\[ Q_{J_{2n}} \sim \sum_{k=0}^{n-1} \left( Q_{k}\psi(\sigma) \frac{dQ_{2n-k}x}{d\sigma}(\sigma) - \frac{dQ_{k}}{d\sigma}(\sigma)Q_{2n-k}(\sigma) \right) \bigg|_{-\infty}^{0} \]

From (B1)-(B5), in view of equalities (17), (18), (41) and (44), we finally get the construction denoted as ΠnJ + QnJ for J2n+1 in (13), where ΠnJ has the form (37) and QnJ has the form (38).

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E-mail address: nguyenthoai@hus.edu.vn