Appendix: On linear subspaces contained in the secant varieties of a projective curve

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1. Introduction.

If $C \subset \mathbb{P}^N$ is a curve imbedded in projective space, one can consider the secant variety
$\Sigma_d = \bigcup_{Z \in C^{(d)}} \langle Z \rangle$ swept out by the linear spans of $d$-uples of points of $C$. This $\Sigma_d$ contains
the $\mathbb{P}^{d-1}$’s parametrized by $Z \in C^{(d)}$ (here we are assuming that $d$ is not large with respect
to $N$). More precisely, $\Sigma_d$ is birational to a projective bundle of rank $d - 1$ over $C^{(d)}$. On
the other hand, if $d$ is large enough, $C^{(d)}$ also contains positive dimensional projective
spaces, corresponding to linear systems on $C$. Deciding whether or not $\Sigma_d$ contains linear
subspaces other than those contained in some of the $\mathbb{P}^{d-1}$’s is thus a non trivial problem.

Some time ago, C. Soulé obtained estimates for the maximal dimension of a linear
subspace contained in $\Sigma_d$, and asked me whether an ad hoc geometric argument would
lead to other results.

One answer in this direction is as follows:

We assume that $C$ is smooth of genus $g > 0$ and that the embedding $C \subset \mathbb{P}^N$ is given
by the sections of a line bundle $L \otimes \omega_C$, with $\deg(L) = m$. We then show:

Theorem. If $m \geq 2d + 3$, and $\delta \geq d - 1$, any $\mathbb{P}^\delta$ contained in $\Sigma_d$ is one of the $\mathbb{P}^{d-1} = \langle Z \rangle$,
$Z \in C^{(d)}$. In particular, $\Sigma_d$ contains no projective space $\mathbb{P}^\delta$, for $\delta \geq d$.

Thanks. I wish to thank Christophe Soulé for interesting discussions and for providing
the motivation to write this Note.

2. Proof of the theorem.

We first recall a few basic facts about secant varieties of curves (see [1]). First of all,
since $m \geq 2d + 1$, for any effective divisor $Z$ of degree $k \leq 2d$ on $C$, we have $H^1(L \otimes \omega_C(-Z)) = 0$, hence the linear span of $Z$ is of dimension $k - 1$. Let now $E \rightarrow C^{(d)}$
be the vector bundle with fiber $H^0(L \otimes \omega_C|Z)$ at $Z \in C^{(d)}$. Since the restriction map
$H^0(L \otimes \omega_C) \rightarrow H^0(L \otimes \omega_C|Z)$ is surjective for any $Z \in C^{(d)}$, there is a well defined
morphism $\alpha : \mathbb{P}(E^*) \rightarrow \mathbb{P}^N$, whose image is exactly the secant variety $\Sigma_d$. Since sections
of $L \otimes \omega_C$ separates any $2d$ points on $C$, it follows that $\alpha$ is one to one over $\Sigma_d - \Sigma_{d-1}$.
An easy computation shows that for any $Z \in C^{(d)}$, and for any $x$ in the linear span of $Z$, but not in the linear span of any $Z' \not\subseteq Z$, the differential of $\alpha$ is of maximal rank, so that $\Sigma_d \setminus \Sigma_{d-1}$ is smooth of dimension $2d - 1$. The projectivized tangent space to $\Sigma_d$ at $\alpha(x)$ is easy to describe, at least when $Z$ is a reduced divisor $\sum z_i$; indeed this is a $\mathbb{P}^{2d-1}$ which contains $\langle Z \rangle$ and also each projective line tangent to $C$ at some point $z_i \in Z$, as one sees by deforming $Z$ fixing $z_j$, $j \neq i$. It follows that it must be equal to the linear span of the divisor $2Z$. By continuity, this description of the projectivized tangent space to $\Sigma_d$ remains true at any point of $\Sigma_d \setminus \Sigma_{d-1}$.

We now start the proof of the theorem. We suppose that $\delta \geq d - 1$, and assume that some projective space $\mathbb{P}^\delta$ is contained in $\Sigma_d$. Assuming $\mathbb{P}^\delta$ is not contained in one of the $\mathbb{P}^{d-1}$'s we shall derive a contradiction.

Note that by induction on $d$, we may assume that $\mathbb{P}^\delta$ is not contained in $\Sigma_{d-1}$. Let $\overline{\mathbb{P}}^\delta$ be the closure of $\alpha^{-1}(\mathbb{P}^\delta \setminus \Sigma_{d-1})$ in $\mathbb{P}(E^*)$. Denote by $\pi : \overline{\mathbb{P}}^\delta \to C^{(d)}$ the restriction to $\overline{\mathbb{P}}^\delta$ of the structural projection $\mathbb{P}(E^*) \to C^{(d)}$. Let $W := \pi(\overline{\mathbb{P}}^\delta)$ and $w := \dim W$. Our assumption is that $w > 0$. We shall denote by $P_v$ the fiber $\pi^{-1}(v)$. It is a projective space $\mathbb{P}^\delta \cap \langle Z_v \rangle$, which is generically of dimension $s = \delta - w$.

We start with the following observation:

**Lemma 1.** Under our assumption $\dim W > 0$ we have the inequality

\[(1) \quad w > \delta - w.\]

**Proof.** Indeed, we may assume that for $v, v'$ two generic distinct points of $W$, the supports of the associated divisors $Z_v, Z_{v'}$ of $C$ are disjoint. Otherwise, $Z_v$ would contain a fixed point $x \in C$, for any $v \in W$. But projecting $C$ from $x$, we then get a curve $C' \subset \mathbb{P}^{N-1}$, such that $\Sigma_{d-1}'$ contains a $\mathbb{P}^{d-1}$ which is not a $\mathbb{P}^{d-2}_Z$; since we may assume the theorem proven for $(m - 1, d - 1)$, this is impossible.

Now choose $v, v'$ as above. The projective spaces $\langle Z_v \rangle$ and $\langle Z_{v'} \rangle$ do not meet, hence the projective spaces $P_v = \langle Z_v \rangle \cap \mathbb{P}^\delta, P_{v'} = \langle Z_{v'} \rangle \cap \mathbb{P}^\delta$ do not meet. Since they are of dimension $s$ in a $\mathbb{P}^\delta$, it follows that $2s < \delta$, or $w > \delta - w$. \[\square\]

Next we observe that, at each point $\alpha(x, Z)$ of $\mathbb{P}^\delta \setminus (\mathbb{P}^\delta \cap \Sigma_{d-1})$, $\mathbb{P}^\delta$ is contained in the projectivized tangent space of $\Sigma_d$ at $\alpha(x, Z)$, that is in $\langle 2Z \rangle$. Hence for any $v \in W$, the corresponding divisor $Z_v \in C^{(d)}$ satisfies

\[\mathbb{P}^\delta \subset \langle 2Z_v \rangle.\]
We next study the infinitesimal variation of \( \langle 2 Z_v \rangle \subset \mathbb{P}^N \). Let \( H := O_{\mathbb{P}^N}(1) \). Then we have the identification

\[
H^0(\mathbb{P}^N, H) \simeq H^0(C, L \otimes \omega_C),
\]

which by definition of the linear span, induces an identification

\[
H^0(\mathbb{P}^N, H \otimes I_{\langle 2Z_v \rangle}) \simeq H^0(C, L \otimes \omega_C(-2Z_v)).
\]

If \( h \in T_{W,v} \), the infinitesimal deformation of \( \langle 2 Z_v \rangle \) in the direction \( h \) is described by an homomorphism:

\[
\varphi_h : H^0(\mathbb{P}^N, H \otimes I_{\langle 2Z_v \rangle}) \to H^0(\langle 2 Z_v \rangle, H_{\langle 2Z_v \rangle}).
\]

We have now an isomorphism induced by (2) and (3):

\[
H^0(\langle 2 Z_v \rangle, H_{\langle 2Z_v \rangle}) \simeq H^0(L \otimes \omega_C|2Z_v).
\]

We have the following

**Lemma 2.** Under the isomorphisms (3) and (4), if we identify \( h \) to an element \( u_h \in H^0(O_C(Z_v)|Z_v) \), \( \varphi_h \) identifies to the multiplication

\[
u_h : H^0(C, L \otimes \omega_C(-2Z_v)) \to H^0(Z_v, L \otimes \omega_C(-Z_v)|Z_v)
\]

followed by the inclusion

\[
H^0(Z_v, L \otimes \omega_C(-Z_v)|Z_v) \hookrightarrow H^0(2Z_v, L \otimes \omega_C|2Z_v).
\]

The proof is straightforward once we recall the construction of \( \varphi_h \) by differentiating under the parameters the equations vanishing on \( \langle 2 Z_v \rangle \).

We know that the spaces \( \langle 2 Z_v \rangle \), for \( v \in W \), contain \( \mathbb{P}^\delta \). Infinitesimally, this translates into the fact that for any \( h \in T_{W,v} \), the image of \( \varphi_h \) vanishes on \( \mathbb{P}^\delta \), that is, is contained in

\[
\text{Ker}(H^0(\langle 2 Z_v \rangle, H_{\langle 2Z_v \rangle}) \to H^0(\mathbb{P}^\delta, H_{\mathbb{P}^\delta})).
\]

From the description of \( \varphi_h \) given in Lemma 2, we see that \( \text{Im} \varphi_h \) is contained in

\[
K := \text{Ker}(H^0(\langle 2 Z_v \rangle, H_{\langle 2Z_v \rangle}) \to H^0(\langle Z_v \rangle, H_{\langle Z_v \rangle})).
\]

Indeed, via the isomorphism (4), \( K \) identifies to

\[
\text{Ker}(H^0(L \otimes \omega_C|2Z_v) \to H^0(L \otimes \omega_C|Z_v)) = \text{Im} H^0(L \otimes \omega_C(-Z_v)|Z_v) \to H^0(L \otimes \omega_C|2Z_v).
\]
Finally, note that the restriction map $K \to H^0(\mathbb{P}^\delta, H_{|\mathbb{P}^\delta})$ has rank equal to the dimension of

$$\text{Ker}(H^0(\mathbb{P}^\delta, H_{|\mathbb{P}^\delta}) \to H^0(\mathbb{P}^\delta \cap \langle Z_v \rangle, H_{|\mathbb{P}^\delta \cap \langle Z_v \rangle}),$$

which is equal to $\delta - s$, since $\mathbb{P}^\delta \cap \langle Z_v \rangle = P_v$ is of dimension $s$.

Denote now by $V \subset H^0(\mathcal{O}_C(Z_v)_{|Z_v})$ the tangent space to $W$ at $v$. Lemma 2 and the estimate above give us the following conclusion:

**Lemma 3.** Under our assumptions, the multiplication map

$$\mu : V \otimes H^0(C, L \otimes \omega_C(-2Z_v)) \to H^0(L \otimes \omega_C(-Z_v)_{|Z_v})$$

has its image contained in a subspace of codimension at least $w$. \( \square \)

We now derive a contradiction. We observe first that since $\mathbb{P}^\delta$ is a rational variety dominating $W$, $W$ is contained in a linear system $|D| \subset C^{(d)}$. Hence $\mathcal{O}_C(D) = \mathcal{O}_C(Z_v)$ for all $v \in W$, and the fact that $W \subset |D|$ translates infinitesimally into the fact that $V = T_{W,v}$ is contained in the image of the restriction map:

$$H^0(\mathcal{O}_C(Z_v)) \to H^0(\mathcal{O}_C(Z_v)_{|Z_v}).$$

Let now $\tilde{V}$ be the inverse image of $V$ under this restriction map. Then $\text{rk } \tilde{V} = w + 1$, and Lemma 3 shows that the multiplication map

$$\tilde{\mu} : \tilde{V} \otimes H^0(C, L \otimes \omega_C(-2Z_v)) \to H^0(C, L \otimes \omega_C(-Z_v))$$

has its image contained in a space of codimension at least $w$.

Now we have the equality:

$$\text{rk } H^0(C, L \otimes \omega_C(-Z_v)) = d + \text{rk } H^0(C, L \otimes \omega_C(-2Z_v)),$$

since $H^1(C, L \otimes \omega_C(-2Z_v)) = 0$. So we conclude that

$$\text{rk } \tilde{\mu} \leq h^0(C, L \otimes \omega_C(-2Z_v)) + d - w. \quad (5)$$

On the other hand, we can apply Hopf lemma to $\tilde{\mu}$, and the inequality in Hopf lemma must be strict here, since the line bundle $L \otimes \omega_C(-2Z_v)$ is very ample, being of degree at least $2g + 1$, and $C$ is not rational. This gives us:

$$\text{rk } \tilde{\mu} > w + 1 + h^0(C, L \otimes \omega_C(-2Z_v)) - 1. \quad (6)$$

Combining (5) and (6), we get:

$$d - w > w. \quad (7)$$

But this contradicts inequality (1), since $\delta \geq d - 1$. \( \square \)

**References.**

[1] A. Bertram : Moduli of rank 2 vector bundles, theta divisors, and the geometry of curves in projective space, *J. Diff. Geom.* 35, 1992, 429-469.