Binary Sequent Calculi for Truth-invariance Entailment of Finite Many-valued Logics

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Abstract. In this paper we consider the class of truth-functional many-valued logics with a finite set of truth-values. The main result of this paper is the development of a new binary sequent calculi (each sequent is a pair of formulae) for many valued logic with a finite set of truth values, and of Kripke-like semantics for it that is both sound and complete. We did not use the logic entailment based on matrix with a strict subset of designated truth values, but a different new kind of semantics based on the generalization of the classic 2-valued truth-invariance entailment. In order to define this non-matrix based sequent calculi, we transform many-valued logic into positive 2-valued multi-modal logic with classic conjunction, disjunction and finite set of modal connectives. In this algebraic framework we define an uniquely determined axiom system, by extending the classic 2-valued distributive lattice logic (DLL) by a new set of sequent axioms for many-valued logic connectives. Dually, in an autoreferential Kripke-style framework we obtain a uniquely determined frame, where each possible world is an equivalence class of Lindenbaum algebra for a many-valued logic as well, represented by a truth value.

1 Introduction

A significant number of real-world applications in Artificial Intelligence has to deal with partial, imprecise and uncertain information, and that is the principal reason for introducing non-classic truth-functional many-valued logics, as for example, the fuzzy, bilattice-based and paraconsistent logics, etc.. The first formal semantics for modal logic was based on many-valuedness, proposed by Lukasiewicz in 1918-1920, and consolidated in 1953 in a 4-valued system of modal logic \[1\]. All cases of many-valued logics mentioned above are based on a lattice \((X, \leq)\) of truth values.

Sequent calculus, introduced by Gentzen \[2\] and Hertz \[3\] for classical logic, was generalized to the many-valued case by Rouseau \[4\] and others. The tableaux calculi were presented in \[5,6\]. The standard two-sides sequent calculi for lattice-based many-valued logics has been elaborated recently (with an autoreferential Kripke-style semantics for such logics) in two complementary ways in \[7,8,9,10\] as well.

In what follows we will consider, for a given logic language, with a set of ground formulae (without variables) \(\mathcal{L}\), with a set of many-valued logic connectives in \(\Sigma\) and predicate (or propositional) letters \(p, q, r,\ldots\), the truth-functional algebraic semantics.
many-valued valuation $v : \mathcal{L} \rightarrow X$ in this case is a homomorphism between the free syntax-algebra of this logic language and the algebra $(X, \Sigma)$ of truth-values. That is, for any n-ary logic connective $\odot$ in $\Sigma$, $\odot : X^n \rightarrow X$, it holds that $v(\odot(\phi_1, \ldots, \phi_n)) = \odot(v(\phi_1), \ldots, v(\phi_n))$, where $\phi_i \in \mathcal{L}$, $1 \leq i \leq n$ are logic formulae in $\mathcal{L}$. Notice that there is a non truth-functional many-valued semantic presented in [11] as well. It is based on "nondeterministic" connectives $\tilde{\odot} : X^n \rightarrow 2^X$ (where $A^B$ denotes the set of functions from $B$ to $A$, and $2 = \{0, 1\}$ is a complete lattice of classical 2-valued logic) for each ordinary connective $\odot \in \Sigma$, such that $v(\tilde{\odot}(\phi_1, \ldots, \phi_n)) \in \tilde{\odot}(v(\phi_1), \ldots, v(\phi_n))$, and so called, Nmatrices (non-deterministic matrices). But in this case the compositional property (the homomorphic property above) of a many-valued non-deterministic valuation $v$ is not valid. Thus, in what follows we will consider only the truth-functional many-valued logics where the valuations are homomorphic.

The well-known semantics of the many-valued logics is based on algebraic matrices $(X, D)$ where $D \subset X = \{x_1, \ldots, x_m\}$ is a strict subset of designated truth values, so that a valuation $v$ is a model for a formula $\phi \in \mathcal{L}$ iff $v(\phi) \in D$. This semantics is commonly used in practice, particularly when the number of truth-values is limited. As in the case of three-valued propositional logic, two different choices of the set of designated values for the same semantics give respectively Kleene logic, and a basic paraconsistent logic $J_3$ - both quite important for mathematical logics itself and for its applications.

The well known sequent system developed for such a many-valued logic with matrix-based semantics is an ad hoc system based on m-sequents, based on standard matrices. The more detailed information for interested readers can be found in [12,13]. But such an approach, through absolutely correct and useful, has some little drawbacks, that hopefully may be improved:

- It is much less known than those which are based on ordinary two-sided sequents.
  The framework for two-sided sequent calculi is well-understood, and a lot of programs have been made towards developing their efficient implementations.
- The use of two-sided sequents reflects the basic fact that logic is all about consequence relations, while in the m-sequent calculus only some characterization of the consequence relation can be done in a roundabout way.
- The use of two-sided sequents is universal and independent of any particular semantics, while the use of m-sequents relies in a particular way of a specific semantics for a given logic.

Consequently, it is interesting to consider a calculus for many-valued logics based on standard binary sequents. The previous work in this direction is recently proposed in [14] based on m-valued Nmatrices, signed formulae and a Rasiowa-Sikorski deduction system [15,16]. In the approach used in [14], an m-sequent calculus is transformed into the ordinary two-sided sequent system (where each sequent is of the form $\Gamma \vdash \Delta$, where $\Gamma, \Delta \subset \mathcal{L}$ are the finite subsets of logic formulae).

Notice that both sequent systems above are based on matrix semantics for logic entailment. This is not our case: we will use the truth-invariance semantics for many-valued logic [17] that is different from matrix-based semantics.

In this paper we propose this new approach to the semantics of many-valued logics by transforming the original many-valued logic into the 2-valued multi-modal logic [18].
and then, based on the classical 2-valued distributive lattice logic (DLL) [19] extended by a set of new axioms for this 2-valued transformation of many-valued logic, by applying Dunn’s binary-sequent approach. In the original Dunn’s approach each sequent is of the simple form $\Phi \vdash \Psi$ (here $\Phi$ and $\Psi$ are the 2-valued multi-modal formulae) and, consequently, his system is a particular sequent calculus where the left side of a sequent is not generally a set of 2-valued formulae but a single formula. Another differences between previous approaches and the approach used in this work are as follows:

- We will not use the many-valued Rosser-Turquette operators $J_k$ [20] that have been introduced a lot of time before the appearance of the Kripke semantics for algebraic modal operators, and we will replace them by modal operators as follows: here we will adopt the ontological encapsulation of many-valued logic into 2-valued multi-modal logic [18] with algebraic modal operators $[x] : X \rightarrow 2$ for any $x \in X$. In what follows we will use $[x]$ both as modal operator, i.e., a syntactic language entity for modality "having a truth-value $x$" (it express exactly the modality of truth of the formula $\phi$), and as a function on truth values, i.e., a semantic entity, which will be clear from the particular context where $[x]$ is used.

- We avoided to use the signed formulae used in [14]. In this way, by using modal approach and its standard Kripke semantics, we are not obligated to develop an unnecessary ad-hoc semantics for such a calculus as it will be shown in Section 4.

Notice that the main result of this work is the fact that we obtained a standard binary sequent calculi for finite many-valued logic with the truth-invariance semantics of logic entailment. Consequently, this work is not only another new reformulation of the same many-valued inference system based on matrices, but is substantially new inference system, different from the m-sequents and from the sequent calculi in [14] (that are mutually equivalent). Such an approach we will apply to many-valued predicate logic (without quantifiers), with the set of k-ary predicate letters in $P$, and to its particular propositional case (where all predicate letters have 0-arity).

This paper follows the following plan: After a brief introduction to the truth-invariance inference semantics, multi-modal predicate logics (without quantifiers), and a short introduction to binary sequents and bivaluations, in Section 2 we present the reduction of finite many-valued propositional (and predicate) logic language $\mathcal{L}$ into the 2-valued multi-modal algebraic logic language $\mathcal{L}_M$. Then we show the main properties for this positive logic (with standard 2-valued conjunction and disjunction and a modal operator $[x]$ for each truth value $x$ in a finite set $X$). We present a normal-forms reduction as well, where each obtained formula has the modal operators applied only to propositional letters in $\mathcal{L}$ (atoms in predicate logic with Herbrand base $H$). In Section 3 we develop the binary sequent system $G$, by extending the classic 2-valued distributive lattice logic (DLL) with the set of sequent axioms for each logic connective of a many-valued logic language $\mathcal{L}$. We show that this proof-theoretic sequent logic is sound and complete w.r.t. the model-theoretic semantics, based on many-valued valuations: each deduced sequent from $G$ and a given set of sequent assumptions $\Gamma$ is also a valid sequent (satisfied for every many-valued valuation), and viceversa.

Finally in Section 4 we develop an autoreferential Kripke-style semantics, based on Lindenbaum algebra of a many-valued logic language $\mathcal{L}$, and we define the Kripke frame.
for it with the set of possible worlds equal to the set of truth values $X$. Then we show that this semantics is correct (sound and complete) for the multi-modal logic language $\mathcal{L}_M$, that is, we demonstrate that for each many-valued algebraic model we obtain a correspondent Kripke model, so that a formula true in $\mathcal{L}_M$ is true in this Kripke model as well, and vice versa.

In what follows we denote by $B^A$ the set of all functions from $A$ to $B$, by $A^n$ a $n$-fold cartesian product $A \times \ldots \times A$ for $n \geq 1$, and by $\mathcal{P}(A)$ the set of all subsets of $A$.

### 1.1 Truth-invariance model-theoretic entailment

In the standard 2-valued model-theoretic semantics we say that a valuation $v : \mathcal{L} \rightarrow \{0, 1\}$ is a **model** of a sentence $\psi \in \mathcal{L}$ iff $v(\psi) = 1$ (here $\mathcal{L}$ denotes the set of all ground formulae of a given logic language). Consequently, a formula $\phi$ is deduced from the set of formulae $\Gamma \subseteq \mathcal{L}$, denoted by $\Gamma \models_1 \phi$, iff $\forall v \in \text{Mod}_\Gamma.(v(\phi) = 1)$, where $\text{Mod}_\Gamma$ is the set of all models of the formulae in $\Gamma$ (here we use the index $1 \in \{0, 1\}$ in the consequence relation $\models_1$ to indicate the deduction of the true formulae).

The set $\Gamma$ can be formally constructed by a subset of formulae $\Gamma_1$ that we want to be (always) true, and by a subset of formulae $\Gamma_0$ that we want to be (always) false, so that, $\Gamma = \Gamma_1 \cup \{\neg \phi \mid \phi \in \Gamma_0 \}$ where $\neg$ is the 2-valued negation operator ($\neg 1 = 0$, $\neg 0 = 1$).

What is not often underlined is that this standard 2-valued model-theoretic semantics **implicitly** defines the set of false sentences deduced from $\Gamma$ as well, denoted here explicitly by the new derived symbol $\models_0$ for the deduction of false sentences (with the index $0 \in \{0, 1\}$), by: $\Gamma \models_0 \phi$ iff $\Gamma \models_1 \neg \phi$, that is, iff $\forall v \in \text{Mod}_\Gamma.(v(\phi) = 0)$.

Consequently, the classic 2-valued truth-invariance semantics of logic entailment can be paraphrased by the following generalized entailment, denoted by $\Gamma \models \phi$:

(CL) "a formula $\phi$ is a logic consequence of the set $\Gamma$" iff $(\exists x \in \{0, 1\})(\forall v \in \text{Mod}_\Gamma).(v(\phi) = x)$.

Thus, classic 2-valued entailment deduces both true and false sentences if they have the same (i.e., invariant) truth-value in all models of $\Gamma$. The consequence relation $\models_1$ defines Tarskian closure operator $C : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$, such that $C(\Gamma) = \{\phi \mid \phi \in \mathcal{L}$ and $\Gamma \models_1 \phi\}$. In the 2-valued logics we do not need to use the consequence relation $\models_0$ because the set of false sentences deduced from $\Gamma$ is equal to the set $\{\neg \phi \mid \phi \in \mathcal{L}$ and $\Gamma \models_1 \phi\} = \{\neg \phi \mid \phi \in C(\Gamma)\}$. This particular property explains why in the classic 2-valued logic it is enough to consider only the consequence relation for deduction of true sentences, or alternatively the Tarskian closure operator $C$.

But in the case of many-valued logics it is not generally the case, and we need the consequence relations for the derivation of sentences that are not true as well. Consequently, this classic 2-valued model-theoretic truth-invariance semantics of logic entailment we will directly extend to many-valued logics by:

(MV) $\Gamma \models \phi$ iff $(\exists x \in X)(\forall v \in \text{Mod}_\Gamma).(v(\phi) = x)$, where $\text{Mod}_\Gamma$ is specified by prefixing a particular truth-value $y \in X$ to each formula $\psi \in \Gamma$, as we explained previously in the case of the 2-valued logic with two subsets $\Gamma_1$ and $\Gamma_0$ for prefixed true and false sentences.

The matrix-based inference is different and specified by:

(MX) $\Gamma \models \phi$ iff $(\forall v \in \text{Mod}_{\Gamma,D}).(v(\phi) \in D)$, with the set of models $\text{Mod}_{\Gamma,D}$ of $\Gamma$ defined as the set of valuations $v$ such that
It is easy to verify that both semantics, the truth-invariance (MV) and the matrix-based (MX), in the case of classic 2-valued logics, where \(D = \{1\}\), coincide.

This new truth-invariance semantics of logic entailment for many-valued logics has been presented first time in [17] and successively used for a new representation theorem for many-valued modal logics [10].

1.2 Introduction to multi-modal predicate logic

More exhaustive and formal introduction to modal logics and their Kripke models can be easily found in a literature, for example in [21]. Here will be given only a short informal version, in order to make more clear definitions used in next paragraphs.

A predicate multi-modal logic, for a language with a set of predicate symbols \(r \in P\) with arity \(ar(r) \geq 0\) and a set of functional symbols \(f \in F\) with arity \(ar(f) \geq 0\), is a standard predicate logic extended by a finite number of universal modal operators \(\Box_i, i \geq 1\). In this case we do not require that these universal modal operators are normal (that is, monotonic and multiplicative) modal operators as in the standard setting for modal logics, but we require that they have the same standard Kripke semantics.

We define the set of terms of this predicate modal logic as follows: all variables \(x \in Var\), and constants \(d \in S\) are terms; if \(f \in F\) is a functional symbol of arity \(k = ar(f)\) and \(t_1, \ldots, t_k\) are terms, then \(f(t_1, \ldots, t_k)\) is a term. We denote by \(\mathcal{T}_0\) the set of all ground (without variables) terms.

An atomic formula (atom) for a predicate symbol \(r \in P\) with arity \(k = ar(r)\) is an expression \(r(t_1, \ldots, t_k)\), where \(t_i, i = 1, \ldots, k\) are terms. Herbrand base \(H\) is a set of all ground atoms (atoms without variables). More complex formulæ, for a predicate multi-modal logic, are obtained as a free algebra obtained from set of all atoms and usual set of classic 2-valued binary logic connectives in \(\{\land, \lor, \Rightarrow\}\) for conjunction, disjunction and implication respectively (negation of a formula \(\phi\) is expressed by \(\phi \Rightarrow 0\), where 0 is used for an inconsistent formula (has constantly value 0 for every valuation), and a number of unary universal modal operators \(\Box_i\). We define \(\mathcal{N} = \{1, 2, \ldots, n\}\) where \(n\) is the maximal arity of symbols in the finite set \(P \cup F\).

Definition 1. We denote by \(\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i \mid 1 \leq i \leq k\}, S, V)\) a multi-modal Kripke model with finite \(k \geq 1\) modal operators with a set of possible worlds \(\mathcal{W}\), the accessibility relations \(\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}\), non empty set of individuals \(S\), and a function \(V: \mathcal{W} \times (P \cup F) \rightarrow \bigcup_{n \in \mathcal{N}}(2 \cup S)^{S^n}\), such that for any world \(w \in \mathcal{W}\),

1. For any functional letter \(f \in F\), \(V(w, f) : S^{ar(f)} \rightarrow S\) is a function (interpretation of \(f\) in \(w\)).
2. For any predicate letter \(r \in P\), the function \(V(w, r) : S^{ar(r)} \rightarrow 2\) defines the extension of \(r\) in a world \(w\), \(\|r\| = \{d = < d_1, \ldots, d_k \geq \in S^k \mid k = ar(r), V(w, r)(d) = 1\}\).

We denote by \(\mathcal{M} \models_{w, g} \varphi\) the fact that a formula \(\varphi\) is satisfied in a world \(w \in \mathcal{W}\) for a given assignment \(g : Var \rightarrow S\). For example, a given atom \(r(x_1, \ldots, x_k)\) is satisfied in \(w\) by assignment \(g\), i.e., \(\mathcal{M} \models_{w, g} r(x_1, \ldots, x_k)\), iff \(V(w, r)(g(x_1), \ldots, g(x_k)) = 1\).
The Kripke semantics is extended to all formulae as follows:

\[ M \models_{w,g} \varphi \land \psi \iff M \models_{w,g} \varphi \text{ and } M \models_{w,g} \psi, \]
\[ M \models_{w,g} \varphi \lor \psi \iff M \models_{w,g} \varphi \text{ or } M \models_{w,g} \psi, \]
\[ M \models_{w,g} \varphi \iff M \models_{w,g} \psi, \]
\[ M \models_{w,g} \Box_{w} \varphi \iff \forall w'((w, w') \in R, \text{ implies } M \models_{w', g} \varphi). \]

The existential modal operator \( \exists w \) can be defined as a derived operator by taking \( \neg \Box_i \neg \).

A formula \( \varphi \) is said to be true in a model \( M \) if for each assignment function \( g \) and possible world \( w, M \models_{w,g} \varphi \). A formula is said to be valid if it is true in every model.

We denote by \( \| \phi/g \| = \{ w \mid M \models w,g \varphi \} \) the set of all worlds where the ground formula \( \phi/g \) (obtained from \( \phi \) and an assignment \( g \)) is satisfied.

Remark: in this paper we will use the notation \([x]\) (for any truth value \( x \in X \)) for universal modal operators, instead of standard notation \( \Box_i \).

### 1.3 Introduction to binary sequents and bivaluations

Sequent calculus has been developed by Gentzen [2], inspired on ideas of Paul Hertz [3]. Given a propositional logic language \( L_A \) (a set of logic formulae) a sequent is a consequence pair of formulae, \( s = (\phi; \psi) \in L_A \times L_A \), denoted also by \( \phi \vdash \psi \).

A Gentzen system, denoted by a pair \( \mathcal{G} = (\mathbb{L}, \models) \) where \( \models \) is a consequence relation on a set of sequents in \( \mathbb{L} \subseteq L_A \times L_A \), is said normal if it satisfies the following conditions:

1. (reflexivity) if \( s \in \Gamma \) then \( \Gamma \models s \)
2. (transitivity) if \( \Gamma' \models s \) and for every \( s' \in \Gamma, \Theta \vdash s' \), then \( \Theta \vdash s \)
3. (finiteness) if \( \Gamma \vdash s \) then there is finite \( \Theta \subseteq \Gamma \) such that \( \Theta \vdash s \)
4. for any homomorphism \( \sigma \) from \( \mathbb{L} \) into itself (i.e., a substitution), if \( \Gamma \models s \) then \( \sigma[\Gamma] \models \sigma(s) \)
5. if \( \Gamma \models s \) and \( \Gamma \subseteq \Theta \), then \( \Theta \vdash s \)

We denote by \( C : \mathcal{P}(\mathbb{L}) \rightarrow \mathcal{P}(\mathbb{L}) \) the Tarskian closure operator such that \( C(\Gamma) = \text{def} \{ s \in \mathbb{L} \mid \Gamma \vdash s \} \), with the properties: \( \Gamma \subseteq C(\Gamma) \) (from reflexivity (1)); it is monotonic, i.e., \( \Gamma \subseteq \Gamma' \) implies \( C(\Gamma) \subseteq C(\Gamma') \) (from (5)), and an involution \( C(C(\Gamma)) = C(\Gamma) \) as well. Thus, we obtain that

6. \( \Gamma \models s \) iff \( s \in C(\Gamma) \).

Any sequent theory \( \Gamma \subseteq \mathbb{L} \) is said a closed theory iff \( \Gamma = C(\Gamma) \). This closure property corresponds to the fact that \( \Gamma \vdash s \) iff \( s \in \Gamma \).

Each sequent theory \( \Gamma \) can be considered as a bivaluation (a characteristic function) \( \beta : \mathbb{L} \rightarrow 2 \) such that for any sequent \( s \in \mathbb{L}, \beta(s) = 1 \) iff \( s \in \Gamma \).

### 2 Reduction of finite many-valued into 2-valued multi-modal logic

In what follows let \( L_P \) be a predicate logic language obtained as a free algebra, from connectives in \( \Sigma \) of an algebra with a set \( X \) of truth values (for example the many-valued conjunction, disjunction and implication \( \{ \land_m, \lor_m, \Rightarrow_m \} \subseteq \Sigma \) are binary operators, negation \( \neg_m \in \Sigma \) and other modal operators are unary operators, while each \( x \in X \subseteq \Sigma \) is a constant (nullary operator)), a set \( P \) of predicate symbols denoted by
We can use this 2-valued multi-modal logic language Ł

The constants considered as propositional letters.

We denote by ŁΦ,Ψ4.

Definition 2.

The set of truth values of the formulae of ŁΦ,Ψ.

The set of truth values of the modal 2-valued logic language ŁΦ,Ψ.

The set of all ground atoms (without variables).

We define a (many-valued) modality ŁΦ,Ψ.

The set of the ground atoms in ŁΦ,Ψ.

Based on this propositional many-valued logic language ŁΦ,Ψ, we are able to define the modal non-standard non-monotonic algebraic truth-functional operators. The set of all ground atoms obtained from a term c by an assignment g.

Any atom is a logic formula. The combination of logic formulae by logical connectives in Ł is another logic formula.

We will denote by Ł the subset of the logic language ŁΦ,Ψ composed of only ground formulae. In this case we can nominate each ground atom by a particular propositional letter, so that this logic language Ł is equivalent to the propositional logic language ŁΦ,Ψ.

We will use the letters φ, ψ for formulae of ŁΦ,Ψ.

We define a (many-valued) valuation v as a mapping v : H → X, which is uniquely extended in standard way to the homomorphism v : Ł → X (for example, for any A, B ∈ H, v(A ⊕ B) = v(A) ⊕ v(B), ⊕ ∈ {∧, ∨, ⇒, ¬}) and v(¬c) = ¬v(c), where ∧, ∨, ⇒, ¬ are the many-valued conjunction, disjunction, implication and negation respectively. The set of all many-valued valuations is a strict subset \∀m of the functional space \XL that satisfies the homomorphic conditions above.

Based on this propositional many-valued logic language ŁΦ,Ψ, we are able to define the following multi-modal 2-valued algebraic logic language ŁM, by introduction of modal non-standard (non monotonic) algebraic truth-functional operators [x] : Y → 2, where Y = X \cup 2, such that for any x, y ∈ Y, [x](y) = 1 iff x = y. The intersection of X and 2 can be non empty as well.

The set of truth values X is a finite set, so that the number of these algebraic modal operators n = |Y| ≥ 2 is finite as well (|Y| is the cardinality of the set Y).

Definition 2. Syntax: Let ŁΦ,Ψ be a predicate many-valued logic language with a set of truth values X and (2, ∧, ∨) the complete distributive two-valued lattice. The multi-modal 2-valued logic language ŁM, is the set of all modal formulae (we will use letters Φ, Ψ, for the formulae of ŁM) defined as follows:

1. 2 ⊆ ŁM.
2. [x]φ ∈ ŁM, for any x ∈ X, φ ∈ ŁΦ,Ψ.
3. [x]φ ∈ ŁM, for any x ∈ 2, φ ∈ ŁM.
4. Φ, Ψ ∈ ŁM implies Φ ∧ Ψ, Φ ∨ Ψ ∈ ŁM.

We denote by ŁM the sublanguage of ŁM without variables (ground atoms in ŁM are considered as propositional letters).

The constants 0, 1 correspond to the tautology and contradiction proposition respectively, and can be considered as nullary operators in ŁM.

We can use this 2-valued multi-modal logic language ŁM in order to define the sequents
as elements of the cartesian product $L_M \times L_M$, i.e., each sequent $s$, denoted by $\Phi \vdash \Psi$, where $\Phi, \Psi \in L_M$.

**Definition 3.** Semantics: For any many-valued valuation $v \in \mathbb{V}_m$, $v : L \rightarrow X$, we define the 'modal valuation' $\alpha : L_M \rightarrow 2$ as follows:

1. $\alpha(0) = 0$, $\alpha(1) = 1$.
2. $\alpha([x] \phi) = 1$ iff $x = v(\phi)$, for any $x \in X$, $\phi \in L$.
3. $\alpha([x] \phi) = 1$ iff $x = v(\phi)$, for any $x \in 2$, $\phi \in L_M$.
4. $\alpha(\Phi \land \Psi) = \alpha(\Phi) \land \alpha(\Psi)$, $\alpha(\Phi \lor \Psi) = \alpha(\Phi) \lor \alpha(\Psi)$, for any $\Phi, \Psi \in L_M$.

This transformation from many-valued valuations into modal valuations can be expressed by the mapping $\tilde{\mathfrak{F}} : \mathbb{V}_m \rightarrow \mathcal{V}$, where $\mathcal{V} \subset 2^{L_M}$ denotes the set of all modal valuations.

It is easy to verify that the mapping $\tilde{\mathfrak{F}}$ is a bijection, with its inverse $\tilde{\mathfrak{F}}^{-1}$ defined as follows: for any modal valuation $\alpha \in \mathcal{V}$, given in Definition 3, we have that the many-valued valuation $v = \tilde{\mathfrak{F}}^{-1}(\alpha) : L \rightarrow X$ is defined for any $\phi \in L$, by $v(\phi) = x \in X$ iff $\alpha([x] \phi) = 1$.

A many-valued valuation $v : L \rightarrow X$, $v \in \mathbb{V}_m$, satisfies a 2-valued multi-modal formula $\Phi \in L_M$ iff $\tilde{\mathfrak{F}}(v)(\Phi) = 1$.

Given two formulae $\Phi, \Psi \in L_M$, the sequent $\Phi \vdash \Psi$ is satisfied by $v$ if $\tilde{\mathfrak{F}}(v)(\Phi) \leq \tilde{\mathfrak{F}}(v)(\Psi)$.

A sequent $\Phi \vdash \Psi$ is an axiom if it is satisfied by every valuation $v \in \mathbb{V}_m \subset X_L$.

From this definition of satisfaction for sequents we obtain the reflexivity (axiom) $\Phi \vdash \Phi$ and transitivity (cut) inference rule, i.e., from $\Phi \vdash \Psi$ and $\Psi \vdash \ Tau$ we deduce $\Phi \vdash \ Tau$.

Let us define the set of 2-valued multi-modal literals (or modal atoms), $P_{mm} = \{x_1, \ldots, x_k | A \in L_M | k \geq 1 \text{ and } A \in H\}$.

For example, if $v(\phi) = x$ then $\tilde{\mathfrak{F}}(v)([1] [x] \phi) = 1$, while if $v(\phi) \neq x$ then $\tilde{\mathfrak{F}}(v)([0] [x] \phi) = 1$. Notice that the number of nested modal operators can be reduced from the fact that $[0][0]$ and $[1][1]$ are identities for the formulae in $L_M$. For example, for $[x] \phi \in L_M$, we have $[0][1][1][0] [x] \phi \equiv [0][0][0] [x] \phi \equiv [x] \phi$, where $\equiv$ is a standard logical equivalence.

Then, given a formula $\phi \in L$, we have that the modal formula $[x] \phi \in L_M$ can be naturally reduced into an equivalent formula, denoted by $\widehat{[x] \phi}$, where the modal operators $[x]$ are applied only to ground atoms (considered as propositional letters) in $H$.

Moreover, for any formula $\Phi \in L_M$ there is an equivalent formula $\tilde{\Phi}$ composed by logical connectives $\land, \lor$, and by multi-modal literals in $P_{mm}$. A canonical formula can be obtained by the following reduction:

**Definition 4.** Canonical Reduction: Let us define the following reduction rules:

1. For any unary operator $\sim \in \Sigma$, $\phi \in L$, and a value $x \in X$,

$$[x](\sim \phi) \mapsto \bigvee_{y \in X, x = \sim y} [y] \phi.$$

2. For any binary operator $\odot \in \Sigma$, $\phi, \psi \in L$, and a value $x \in X$,

$$[x](\phi \odot \psi) \mapsto \bigvee_{y, z \in X, x = y \odot z} ([y] \phi \land [z] \psi).$$

3. For any binary operator $\odot \in \{\land, \lor\}$, $\Phi, \Psi \in L_M$, and a value $x \in 2$,

$$[x](\Phi \odot \Psi) \mapsto \bigvee_{y, z \in 2, x = y \odot z} ([y] \Phi \land [z] \Psi).$$

We denote by $\widehat{[x] \phi}$ the canonical formula obtained by applying recursively these reduction rules to the formula $[x] \phi$.

**Proposition 1** The normal reductions in Definition 4 are truth-preserving, that is, for any $x \in X$ and $\phi \in L$ we have that $[x] \phi$ is logically equivalent to $\widehat{[x] \phi}$.
Analogously, for any \( x \in 2 \) and \( \Phi \in L_M \) we have that \( [x]\Phi \) is logically equivalent to \( [x]\Phi \).

**Proof:** Let us show that the steps of canonical reduction in Definition 4 are truth-preserving:

1. The first case of reduction: let us suppose that \( [x](\sim \phi) \) is true but \( \bigvee_{y \in X,x=\sim y} [y]\phi \) is false. From the truth of \( [x](\sim \phi) \) we obtain that \( x = v(\sim \phi) \) and, consequently, \( [z]\phi \) is true, then, from the truth-functional connective \( \sim \) we conclude that \( x = \sim z \) and from the fact that \( [z]\phi \) is true, we conclude that \( \bigvee_{y \in X,x=\sim y} [y]\phi \) must be true, which is a contradiction. Consequently \( [x](\sim \phi) \) implies \( \bigvee_{y \in X,x=\sim y} [y]\phi \).

2. Analogously, for the second case (and 3rd as well) we obtain that \( [x](\phi \land \psi) \) iff \( \bigvee_{y \in X,x=\sim y} [y]\phi \land [z]\psi \).

From the fact that both steps are also truth-preserving, we deduce that any consecutive execution of them is truth-preserving, and, consequently, \( [x]\phi \) iff \( \bigvee_{y \in X,x=\sim y} [y]\phi \).

The following proposition shows that the result of the canonical reduction of a formula \( \Phi \in L_M \) is a disjunction of modal conjunctions, which in the case of the formulae without nested modal operators is a simple disjunctive modal formula.

**Proposition 2** Any 2-valued logic formulae \( \Phi \in L_M \) is logically equivalent to disjunctive modal formula \( \bigvee_{1 \leq i \leq m} (\land_{1 \leq j \leq m, \}[y_{ij}] \ldots [y_{ijk_i}] A_{ij}) \), where for all \( 1 \leq i \leq m \), and \( 1 \leq j \leq m_i \), we have that \( 1 \leq k_{ij}, \ [y_{ijk_i}] \in X \), and \( A_{ij} \in H \).

In the case when we have no nested modal operators then \( k_{ij} = 1 \) for all \( i, j \). Consequently, \( \Phi \in L_M \) is logically equivalent to a disjunctive modal formula \( \bigvee_{1 \leq i \leq m} [x_i]\Phi_i \), where for all \( 1 \leq i \leq m \), \( x_i \in X \), \( \Phi_i \in L \).

**Proof:** From Proposition 4 and from the canonical reduction in Definition 4 it is easy to see that each logically equivalent reduction moves a modal operator toward propositional letters in \( H \), by introduction of conjunction and disjunction logic operators only. Thus, when this reduction is completely realized we obtain a positive propositional logic formula with modal propositions in \( P_{mm} \), and logic operators \( \land \) and \( \lor \). It is well known that such a positive propositional formula can be equivalently (but not uniquely) represented as a disjunction of conjunctions of modal propositions \( ([y_{ij1}] \ldots [y_{ijk_i}]) A_{ij} \in P_{mm} \).

Let us consider now a formula \( \Phi \in L_M \) without nested modal operators. Then \( k_{ij} = 1 \) for all \( i, j \) and, from the result above, a formula \( \Phi \in L_M \) can be equivalently (but not uniquely) represented as a disjunction of conjunctive forms \( \bigvee_{1 \leq i \leq m} (\land_{1 \leq j \leq m_i} [y_{ij1}] A_{ij}). \)

But, we have that \( \land_{1 \leq j \leq m_i} [y_{ij1}] A_{ij} = [x_i] \Phi_i \), where for any binary operator \( \circ \in \Sigma \), \( x_i = y_{i11} \circ \ldots \circ y_{im_i} X \), and \( \Phi_i \in A_{i1} \circ \ldots \circ A_{im_i} \in L \). It holds because \( \alpha([x_i] \Phi_i) = 1 \) iff \( x_i = v(\Phi_i) = v(A_{i1} \circ \ldots \circ A_{im_i}) = v(A_{i1}) \circ \ldots \circ v(A_{im_i}) = y_{i11} \circ \ldots \circ y_{im_i} \).

That is, we obtain that holds: \( \Phi \) iff \( \bigvee_{1 \leq i \leq m} [x_i] \Phi_i \).
With this normal reduction, by using truth-value tables of many-valued logical connectives, we introduced the structural compositionality and truth preserving for the 2-valued modal encapsulation of a many-valued logic \( \mathcal{L} \) as well. In fact, the following property is valid:

**Proposition 3** Given a many-valued valuation \( v : \mathcal{L} \to X \) and a formula \( \phi \in \mathcal{L} \), the normal reduct formula \( \bar{x} \bar{\phi} \in \mathcal{L}_m \) is satisfied by \( v \) iff \( x = v(\phi) \).

**Proof:** By structural induction on \( \phi \): let \( \alpha = \#(v) \), then

1. Case when \( \phi = A \in H \):
   If \( [x]A \) is satisfied than \( \alpha([x]A) = \alpha([x]A) = 1 \) (from Definition 3 and Proposition 1), thus \( x = v(A) = v(\phi) \).

2. Case when \( \phi = \sim \psi \), where \( \sim \) is an unary operator in \( \Sigma \), with \( y = v(\psi) \) and \( x = \sim y \):
   Suppose, by structural induction, that \( \bar{y}\bar{\psi} \) is satisfied, i.e., \( y = v(\psi) \), so that \( [y]\psi \) is true (from Definition 3). Thus, \( \bigvee_{y \in X, x = \sim y} [y]\psi \) is true and, by the truth-preservation (from the reduction 1 in Definition 3) we obtain that \( [x](\sim \psi) \) is true, so that \( [x]\phi \) is true, i.e., (by Definition 3) \( x = v(\phi) \), and (from Proposition 1) \( [x]\phi \) is true. With \( x = \sim y = \sim (v(\psi)) = (from the homomorphism of \( v) = v(\sim \psi) = v(\phi) \).

3. Case when \( \phi = \psi \circ \varphi \), where \( \circ \) is a binary operator in \( \Sigma \), with \( y = v(\psi) \), \( z = v(\varphi) \) and \( x = y \circ z \):
   Then, by structural induction hypothesis, \( \bar{y}\bar{\psi} \) and \( \bar{z}\varphi \) are satisfied, so that from Proposition 1 and Definition 3 we have that \( 1 = \alpha([y]\psi) = \alpha([y]\psi) \) and \( 1 = \alpha([z]\varphi) = \alpha([z]\varphi) \). Consequently, \( 1 = \alpha([y]\psi) \land \alpha([z]\varphi) = (from the homomorphism 2 in Definition 3) \alpha([y]\psi \land [z]\varphi) = \alpha(\bigvee_{y \in X, x = y \circ z} ([y]\psi \land [z]\varphi)) = (from the reduction 2 in Definition 3) \alpha([x](\psi \circ \varphi)) = \alpha([x]\phi) = (by Proposition 1) \alpha([x]\phi). \) Thus, \( [x]\phi \) is satisfied by \( v \) and \( x = y \circ z = v(\psi) \circ v(\varphi) = (from the homomorphism of \( v) = v(\psi \circ \varphi) = v(\phi) \).

Thus, as a consequence, for any \( \phi \in \mathcal{L} \) and a many-valued valuation \( v \in \mathbb{V}_m \), we have that \( \#(v)([x]\phi) = 1 \) iff \( x = v(\phi) \).

### 3 Many-valued truth and model theoretic semantics

The Gentzen-like system \( \mathcal{G} \) of the 2-valued propositional logic \( \mathcal{L}_M \) (where the set of propositional letters corresponds to the set \( P = \{ [x_1] \ldots [x_k] | A \in \mathcal{L}_M | k \geq 1 \) and \( A \in H \) }, is a 2-valued distributive logic (DLL in [19]), that is, \( 2 \subseteq \mathcal{L}_M \), extended by the set of sequent axioms, defined for each many-valued logic connective in \( \Sigma \) of the original many-valued logic \( \mathcal{L} \). It contains the following axioms (sequents) and rules:

(AXIOMS) The Gentzen-like system \( \mathcal{G} = \langle \mathcal{L}, \vdash \rangle \) contains the following sequents in \( \mathcal{L} \), for any \( \Phi, \Psi, T \in \mathcal{L}_M \):

1. \( \Phi \vdash \Phi \) (reflexive)
2. \( \Phi \vdash 1 \), \( 0 \vdash \Phi \) (top/bottom axioms)
3. \( \Phi \land \Psi \vdash \Phi \), \( \Phi \land \Psi \vdash \Psi \) (product projections: axioms for meet)
4. $\Phi \vdash \Phi \lor \Psi$, $\Phi \vdash \Psi \lor \Phi$ (coproduct injections: axioms for join)
5. $\Phi \land (\Psi \lor \tau) \vdash (\Phi \land \Psi) \lor (\Phi \land \tau)$ (distributivity axiom)
6. The set of Introduction axioms for many-valued connectives:
   6.1 $\forall y,z \in [y] \phi \land [z] \psi \vdash [x] (\phi \circ \psi)$, for any binary operator $\circ \in \{\land, \lor\}$.
   6.2 $\forall x \in X, x = \sim y [y] \phi \vdash [x] (\sim \phi)$, for any unary operator $\sim \in \Sigma$ and $\phi \in L$.
   6.3 $\forall x \in X, x = \sim y [y] \phi \land [z] \psi \vdash [x] (\phi \circ \psi)$, for any binary operator $\circ \in \Sigma$ and $\phi, \psi \in L$.

7. The set of Elimination axioms for many-valued connectives:
   7.1 $[x_1] (\phi \circ \psi) \vdash \forall y, z \in [y] \phi \land [z] \psi$, for any binary operator $\circ \in \{\land, \lor\}$.
   7.2 $[x] (\sim \phi) \vdash \forall y \in X, x = \sim y [y] \phi$, for any unary operator $\sim \in \Sigma$ and $\phi \in L$.
   7.3 $[x] (\phi \circ \psi) \vdash \forall y, z \in [y] \phi \land [z] \psi$, for any binary operator $\circ \in \Sigma$ and $\phi, \psi \in L$.

(INFEERENCE RULES) $G$ is closed under the following inference rules:
1. $\frac{\phi \lor \psi, \phi \lor \tau}{\phi \lor \psi \lor \tau}$ (cut/transitivity rule)
2. $\frac{\phi \lor \psi, \phi \lor \tau}{\phi \lor \psi \lor \tau}$ (lower/upper bound rules)
3. $\frac{\sigma(\phi \circ \psi)}{\sigma(\phi \circ \psi)}$ (substitution rule: $\sigma$ is substitution ($\gamma/p$)).

The axioms from 1 to 5 and the rules 1 and 2 are taken from [19] for the $DLL$ and it was shown that this sequent based Gentzen-like system is sound and complete. The new axioms in points 6 and 7 corresponds to the canonical (equivalent) reductions in Definition 4.

The set of sequents that define the poset of the classic 2-valued lattice of truth values $\{2, \leq\}$ is a consequence of the top/bottom axioms: for any two $x, y \in 2$, if $x \leq y$ then $x \vdash y \in G$.

Thus, for many-valued logics we obtain a normal modal Gentzen-like deductive system, where each sequent is a valid truth-preserving consequence-pair defined by the poset of the complete lattice $\{2, \leq\}$ of classic truth values (which are also the constants of this positive propositional language $L_M$), so that each occurrence of the symbol $\vdash$ can be substituted by the partial order $\leq$ of this complete lattice $\{2, \leq\}$.

**Example 1:** Let us consider the Godel’s 3-valued logic $X = \{0, \frac{1}{2}, 1\}$, and its 3-valued implication logic connective $\Rightarrow$ given by the following truth-table:

| x  | $\frac{1}{2}$ | 1 |
|----|--------------|---|
| 0  | 1            | 1 |
| $\frac{1}{2}$ | 0            | 1 |
| 1  | 0            | 1 |

One of the possible m-sequents for introduction rules for this connective, taken from [12], (each rule corresponds to the conjunction of disjunctive forms, where each disjunctive form is one m-sequent in the premise), is

\[
\left( \vDash \Delta \phi \rightarrow \phi \right) \quad \left( \vDash \Delta \psi \rightarrow \psi \right) \quad \Rightarrow: 0,
\left( \vDash \Delta \phi \rightarrow \phi \right) \quad \left( \vDash \Delta \psi \rightarrow \psi \right) \quad \Rightarrow: 1
\]
while in our approach we obtain the unique set of binary sequent introduction axioms:

\[
(\frac{1}{2}\phi \land [0] \psi) \lor ((1] \phi \land [0] \psi) \vdash [0](\phi \Rightarrow \psi) \\
1] \phi \land \frac{1}{2}\psi \vdash \frac{1}{2}(\phi \Rightarrow \psi) \\
\bigvee_{x,y \in X \land (x,y) \notin S} ([x] \phi \land [y] \psi) \vdash [1](\phi \Rightarrow \psi)
\]

where \( S = \{(\frac{1}{2}, 0), (1, 0), (1, \frac{1}{2})\} \), and elimination axioms:

\[
(0](\phi \Rightarrow \psi) \vdash (\frac{1}{2}\phi \land [0] \psi) \lor ([1] \phi \land [0] \psi) \\
\frac{1}{2}(\phi \Rightarrow \psi) \vdash [1]\phi \land \frac{1}{2}\psi \\
[1](\phi \Rightarrow \psi) \vdash \bigvee_{x,y \in X \land (x,y) \notin S} ([x] \phi \land [y] \psi).
\]

\[\square\]

**Definition 5.** For any two formulae \( \Phi, \Psi \in L_M \) when the sequent \( \Phi \vdash \Psi \) is satisfied by a 2-valued modal valuation \( \alpha : L_M \rightarrow 2 \) from Definition 3 (that is, when \( \alpha(\Phi) \leq \alpha(\Psi) \) as in standard 2-valued logics), we say that it is satisfied by the many-valued valuation \( v = \widehat{\alpha}^{-1}(\alpha) : L \rightarrow X \).

This sequent is a tautology if it is satisfied by all modal valuations \( \alpha \in \mathcal{V}, \) i.e., when \( \forall \alpha, [\mathcal{V}]\mathcal{F}(\Phi) \leq [\mathcal{V}]\mathcal{F}(\Psi) \).

For a normal Gentzen-like sequent system \( \mathcal{G} = \langle L, \vdash \rangle \) of a many-valued logic language \( L \), with the set of sequents \( \mathcal{L} = L_M \times L_M \), we tell that a many-valued valuation \( v \) is its model if it satisfies all sequents in \( \mathcal{G} \).

The set of all models of a given set of sequents (theory) \( \Gamma \) is:

\[
Mod_{\Gamma} = \{v \in \mathcal{V}_m \mid \forall(\Phi \vdash \Psi) \in \Gamma([\mathcal{V}]\mathcal{F}(\Phi) \leq [\mathcal{V}]\mathcal{F}(\Psi)) \subseteq \mathcal{V}_m \subseteq X^L\}
\]

**Proposition 4** SEQUENT’S BIVALUATIONS AND SOUNDNESS: Let us define the mapping \( \mathfrak{B} : \mathcal{V}_m \rightarrow 2^{L_M \times L_M} \) from valuations into sequent bivaluations, such that for any valuation \( v \in \mathcal{V}_m \), we obtain the sequent bivaluation \( \beta = \mathfrak{B}(v) = ev \circ \pi_1, \land > o([\mathcal{V}]\mathcal{F}(v) \times [\mathcal{V}]\mathcal{F}(v)) : L_M \times L_M \rightarrow 2 \), where \( \pi_1 \) is the first projection, \( o \) is the functional composition and \( ev : 2 \times 2 \rightarrow 2 \) is the equality mapping such that \( ev(x, y) = 1 \) iff \( x = y \).

Then, a sequent \( s = (\Phi \vdash \Psi) \) is satisfied by \( v \) iff \( \beta(s) = \mathfrak{B}(v)(s) = 1 \).

All axioms of the Gentzen like sequent system \( \mathcal{G} \), of a many-valued logic language \( L \) based on a set \( X \) of truth values, are the tautologies, and all its rules are sound for model satisfiability and preserve the tautologies.

**Proof:** From the definition of a bivaluation \( \beta \) we have that

\[
\begin{align*}
\beta(\Phi \vdash \Psi) &= \beta(\Phi \vdash \Psi) = ev \circ \pi_1, \land > o([\mathcal{V}]\mathcal{F}(\Phi) \times [\mathcal{V}]\mathcal{F}(\Psi)) \\
&= ev \circ \pi_1, \land > (\mathcal{F}(v)(\Phi) \times \mathcal{F}(v)(\Psi)) \\
&= ev \circ \pi_1 (\mathcal{F}(v)(\Phi), \mathcal{F}(v)(\Psi)), \land (\mathcal{F}(v)(\Phi), \mathcal{F}(v)(\Psi)) > \\
&= ev(\mathcal{F}(v)(\Phi), \mathcal{F}(v)(\Psi) \land \mathcal{F}(v)(\Psi)).
\end{align*}
\]

Thus \( \beta(\Phi \vdash \Psi) = 1 \) iff \( \mathcal{F}(v)(\Phi) \leq \mathcal{F}(v)(\Psi) \), i.e., when this sequent is satisfied by \( v \).

It is straightforward to check that all axioms in \( \mathcal{G} \) are tautologies (all constant sequents specify the poset of the complete lattice \( (2, \leq) \) of classic 2-valued logic, thus are tautologies). It is straightforward to check that all rules preserve the tautologies. Moreover, if all premises of a given rule in \( \mathcal{G} \) are satisfied by a given many-valued valuation \( v : L \rightarrow X \), then also the deduced sequent of this rule is satisfied by the same valuation,
i.e., the rules are sound for the model satisfiability.

\[\Box\]

It is easy to verify, that this entailment is equal to the classic propositional entailment.

Remark: It is easy to observe that each sequent is, from the logic point of view, a 2-valued object so that all inference rules are embedded into the classic 2-valued framework, i.e., given a bivaluation \(\beta = \mathcal{B}(v) : \mathcal{L}_M \times \mathcal{L}_M \to 2\), we have that a sequent \(s = \Phi \vdash \Psi\) is satisfied, \(\beta(s) = 1\) iff \(\mathcal{G}(v)(\Phi) \leq \mathcal{G}(v)(\Psi)\), so that we have the direct relationship between sequent bivaluations \(\beta\) and many-valued valuations \(v\).

This sequent feature, which is only an alternative formulation for the 2-valued classic logic, is fundamental in the framework of many-valued logics where the semantics for the entailment based on the algebraic matrices \((X, D)\) is often subjective and arbitrary.

Let us consider, for example, the fuzzy logic with the uniquely fixed semantics for all logical connectives, where the subset of designated elements \(D \subseteq X\) is an arbitrary-subjective choice between the infinite number of closed intervals \([a, 1]\) also for very restricted interval, for example, \(0.83 \leq a \leq 0.831\). It does not happen in the classic 2-valued logics where a different logic semantics (entailment) is obtained by adopting only different semantics for some of its logical connectives, usually for negation operator. This property of the classic 2-valued logic can be propagated to many-valued logics by adopting the principle of classic truth-invariance for the entailment: in that case for fixed semantics of all logical connectives of a given language we obtain a unique logic.

The definition of the 2-valued entailment in the sequent system \(\mathcal{G}\), given in Definition 5, can replace the current entailment based on algebraic matrices \((X, D)\) where \(D \subseteq X\) is a subset of designated elements. Thus, we are able now to introduce the many-valued valuation-based (i.e., model-theoretic) semantics for many-valued logics:

Definition 6. A many-valued model-theoretic semantics of a given many-valued logic \(L\) with a Gentzen system \(\mathcal{G} = \langle \mathcal{L}, \vdash \rangle\), is the semantic deducibility relation \(\models_m\), defined for any \(\Gamma = \{\Phi_i \vdash \Psi_i \mid i \in I\}\) and a sequent \(s = (\Phi \vdash \Psi) \in \mathcal{L} \subseteq \mathcal{L}_M \times \mathcal{L}_M\) by:

\(\Gamma \models_m s\) iff “all many-valued models of \(\Gamma\) are the models of \(s\)”, that is, iff \(\forall v \in \mathcal{V}_m(\forall (\Phi_i \vdash \Psi_i) \in \Gamma(\mathcal{G}(v)(\Phi_i) \leq \mathcal{G}(v)(\Psi_i)) \text{ implies } \mathcal{G}(v)(\Phi) \leq \mathcal{G}(v)(\Psi))\).

Lemma 1. For any \(\Gamma = \{\Phi_i \vdash \Psi_i \mid i \in I\}\) and a sequent \(s = (\Phi \vdash \Psi)\) we have that \(\Gamma \models_m s\) iff \(\forall v \in \mathcal{M}_r(\mathcal{G}(v)(s) = 1)\).

Proof: We have that \(\Gamma \models_m s\) iff 
\(\forall v \in \mathcal{M}_r(\forall (\Phi_i \vdash \Psi_i) \in \Gamma(\mathcal{G}(v)(\Phi_i) \leq \mathcal{G}(v)(\Psi_i) \text{ implies } \mathcal{G}(v)(\Phi) \leq \mathcal{G}(v)(\Psi))\)
iff \(\forall v \in \mathcal{M}_r(\mathcal{G}(v)(\Phi) \leq \mathcal{G}(v)(\Psi))\)
iff \(\forall v \in \mathcal{M}_r(\mathcal{G}(v)(s) = 1)\).
\[\Box\]

It is easy to verify that any many-valued logic has a Gentzen-like system \(\mathcal{G} = \langle \mathcal{L}, \vdash \rangle\) (see the definition at the beginning of this Section) that is a normal logic.

Theorem 1 Many-valued model-theoretic semantics is an adequate semantics for a many-valued logic \(L\) specified by a Gentzen-like logic system \(\mathcal{G} = \langle \mathcal{L}, \vdash \rangle\), that is, it is sound and complete. Consequently, \(\Gamma \models_m s\) iff \(\Gamma \vdash s\).
Proof: Let us prove that for any many valued model \( v \in Mod_{\Gamma} \), the obtained sequent bivaluation \( \beta = \mathfrak{B}(v) : L_M \times L_M \rightarrow 2 \) is the characteristic function of the closed theory \( \Gamma_v = C(T) \) with \( T = \{ \Phi \vdash 1, 1 \vdash \Phi \mid \Phi \in P_{mm}, \mathfrak{s}(v)(\Phi) = 1 \} \cup \{ \Phi \vdash 0, 0 \vdash \Phi \mid \Phi \in P_{mm}, \mathfrak{s}(v)(\Phi) = 0 \} \).

1. Let us show that for any sequent \( s \in \Gamma_v \) implies \( \beta(s) = 1 \):
First of all, for any sequent \( s \in T \) we have that: if it is of the form \( \Phi \vdash 1 \) or \( 1 \vdash \Phi \) (where \( \Phi \in P_{mm} \)) we have that \( \mathfrak{s}(v)(\Phi) = 1 \), thus \( s \) is satisfied by \( v \) (it holds that \( 1 \leq 1 \) in both cases); if it is of the form \( \Phi \vdash 0 \) or \( 0 \vdash \Phi \) we have that \( \mathfrak{s}(v)(\Phi) = 0 \), thus \( s \) is satisfied by \( v \) (it holds that \( 0 \leq 0 \) in both cases). Consequently, all sequents in \( T \) are satisfied by \( v \).

From Proposition 3 we have that all inference rules in \( \mathcal{G} \) are sound w.r.t. the model satisfiability. Thus for any deduction \( T \vdash s \) (i.e., \( s \in \Gamma_v \)) where all sequents in premises are satisfied by the many-valued valuation (model) \( v \), the deduced sequent \( s = (\Phi \vdash \Psi) \) must be satisfied as well, that is, it must hold that \( \mathfrak{s}(v)(\Phi) \leq \mathfrak{s}(v)(\Psi) \), i.e., \( \beta(s) = 1 \).

2. Let us show that for any sequent \( s = (\Phi \vdash \Psi) \in L_M \times L_M \) if \( \beta(s) = 1 \) implies \( s \in \Gamma_v \). For any sequent \( s = (\Phi \vdash \Psi) \in L_M \times L_M \) if \( \beta(s) = 1 \) we have one of the two possible cases:
2.1 Case when \( \mathfrak{s}(v)(\Phi) = 0 \). Then (from Proposition 2) \( \Phi \) can be substituted by \( \forall_{1 \leq i \leq m} (\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij}) \), i.e., \( \mathfrak{s}(v)(\forall_{1 \leq i \leq m} (\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij})) = 0 \), that is, for every \( 1 \leq i \leq m \), \( \mathfrak{s}(v)(\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij}) = 0 \), i.e., \( (\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij}) \vdash 0 \) \( \in T \). Consequently, by applying the inference rule 2b, we deduce \( T \vdash (\bigwedge_{1 \leq i \leq m} (\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij}) \vdash 0) \), that is, by the substitution inference rule (for \( \sigma : \forall_{1 \leq i \leq m} (\bigwedge_{1 \leq j \leq m} ([y_{ij_1}] \ldots [y_{ij_k}]) A_{ij}) \rightarrow \Phi \) we obtain that \( T \vdash (\Phi \vdash 0) \). From the fact that \( 0 \vdash \Psi \) is an axiom in \( \mathcal{G} \), and by applying the transitive inference rule, we obtain that \( T \vdash (\Phi \vdash \Psi) \), i.e., \( s \in C(T) = \Gamma_v \).

2.2 Case when \( \mathfrak{s}(v)(\Phi) = 1 \). In this case, from the fact that this sequent is satisfied, \( \mathfrak{s}(v)(\Psi) = 1 \) must be true as well. Thus, we can substitute \( \Psi \) by \( 1 \), so that we obtain the axiom \( 1 \vdash \Phi \) in \( \mathcal{G} \), and, consequently, by applying the substitution inference rule (for \( \sigma : 1 \rightarrow \Psi \) we obtain \( T \vdash (1 \vdash \Psi) \). From the fact that \( \Phi \vdash 1 \) is an axiom in \( \mathcal{G} \) and by applying the transitive inference rule we obtain that \( T \vdash (\Phi \vdash \Psi) \), i.e., \( s \in C(T) = \Gamma_v \).

So, from (1) and (2) we obtain that \( \beta(s) = 1 \) iff \( s \in \Gamma_v \), i.e., the sequent bivaluation \( \beta \) is the characteristic function of a closed set. Consequently, any many-valued model \( v \) of this many-valued logic \( L \) corresponds to the closed bivaluation \( \beta \) which is a characteristic function of a closed theory of sequents: we define the set of all closed bivaluations obtained from the set of many-valued models \( v \in Mod_{\Gamma} \):
\[ Biv_{\Gamma} = \{ \Gamma_v \mid v \in Mod_{\Gamma} \} \]. From the fact that \( \Gamma \) is satisfied by every \( v \in Mod_{\Gamma} \) we have that for every \( \Gamma_v \in Biv_{\Gamma} \), \( \Gamma \subseteq \Gamma_v \), so that \( C(\Gamma) = \bigcap Biv_{\Gamma} \) (an intersection of closed sets is a closed set also). Thus, for \( s = (\Phi \vdash \Psi) \), \( \Gamma \vdash_m s \) iff \( \forall v \in Mod_{\Gamma} (\forall \Phi_i \vdash \Psi_i) \in C(\mathfrak{s}(v)(\Phi_i) \leq \mathfrak{s}(v)(\Psi_i)) \) implies \( \mathfrak{s}(v)(\Phi_i) \leq \mathfrak{s}(v)(\Psi_i) \),
iff \( \forall v \in Mod_{\Gamma} (\forall \Phi_i \vdash \Psi_i) \in C((\beta(\Phi_i \vdash \Psi_i) = 1) \) implies \( \beta(\Phi_i \vdash \Psi_i) = 1) \),
iff \( \forall \Gamma_v \in Biv_{\Gamma} (\Gamma \subseteq \Gamma_v \) implies \( s \in \Gamma_v \),
iff \( \forall \Gamma_v \in Biv_{\Gamma} (s \in \Gamma_v \), because \( \Gamma \subseteq \Gamma_v \) for each \( \Gamma_v \in Biv_{\Gamma} \),
iff \( s \in \bigcap Biv_{\Gamma} (C(\Gamma), \) that is, \( \Gamma \vdash \Phi \) s.
Thus, in order to define the model-theoretic semantics for many-valued logics we do not need to use the matrices: we are able to use only the many-valued valuations, and many-valued models (i.e., the valuations which satisfy all sequents in \( \Gamma \) of a given many-valued logic \( \mathbb{L} \)). This point of view is used also for the definition of a new representation theorem for many-valued complete lattice based logics in [10,7].

Here, in a many-valued logic \( \mathbb{L} \), specified by a set of sequents in \( \Gamma \), for a formula \( \phi \in \mathbb{L} \) that has the same value \( x \in X \) (for any truth value \( x \)) for all many-valued models \( v \in \text{Mod}_\mathbb{L} \), we have that its modal version \( [x] \phi \) is a theorem; that is, \( \forall v \in \text{Mod}_\mathbb{L} (v(\phi) = x) \iff \Gamma \vdash (1 \vdash [x] \phi) \),

that corresponds to the truth-invariance many-valued entailment (MV) in subsection 1.1. Such a value \( x \in X \) does not need to be a designated element \( x \in D \), as in the matrix semantics for a many-valued logic. This fact explains why we do not need a semantic specification by matrix designated elements.

**Example 2:** Let us prove that, given an assumption \( \Gamma = \{ 1 \vdash [x] \phi, 1 \vdash [y] \psi \} \), then \( [z] (\phi \circ \psi) \) for \( z = x \odot y \), is deduced from \( \Gamma \), that is, \( \Gamma \vdash ([z] (\phi \circ \psi)) \); or equivalently if \( [x] \phi \) and \( [y] \psi \) are valid (i.e., for every valuation \( v \in \text{Val}_\mathbb{L}, v : X \rightarrow X \), the values of \( \phi \) and \( \psi \) are equal to \( x \) and \( y \) respectively, i.e. \( \forall v \in \text{Val}_\mathbb{L} (v(\phi) = x \text{ and } v(\psi) = y) \)), then \( [z] (\phi \circ \psi) \) is valid as well.

As first step we introduce the equivalence relation \( \equiv \), such that \( \phi \equiv \psi \) if and only if \( \phi \vdash \psi \) and \( \psi \vdash \phi \) (i.e., when \( \phi \iff \psi \)). Consequently, from the reflexivity axiom in \( \mathbb{G} \) we have that \( \phi \equiv \phi \), as for example \( 1 \equiv 1 \). The equivalent formulae can be used in the substitution inference rule: if \( \phi \equiv \Psi \) then we can use the substitution of \( \phi \) by \( \Psi \), that is, the substitution \( \sigma : \phi \mapsto \Psi \).

Let us show the simple equivalence \( \phi \land \phi \equiv \phi \):
from the reflexivity axiom \( \phi \vdash \phi \), by using the upper bound inference rule (when \( \Psi = \Phi \)) we deduce \( \phi \land \phi \vdash \phi \). Then, from the axioms for join \( \phi \vdash \phi \lor \phi \), we obtain \( \phi \lor \phi \equiv \phi \).

Now, from the assumptions \( 1 \vdash [x] \phi, 1 \vdash [y] \psi \in \Gamma \), by the lower bound inference rule in \( \mathbb{G} \), we obtain \( (a) \vdash [x] \phi \land [y] \psi \), i.e., \( \Gamma \vdash (1 \vdash [x] \phi \land [y] \psi) \). Let \( z = x \odot y \) and let us denote \( [z] (\phi \land [y] \psi) \) by \( \Phi \), so that \( (a) \) becomes \( (a') \vdash \Phi \). Now we can take the axiom for join, (b) \( \phi \vdash \phi \lor v \in X, v \odot w = z ([v] \phi \land [w] \psi) \), so from \( (a') \) and (b) and the transitivity rule we obtain \( 1 \vdash \phi \lor v \in X, v \odot w = z ([v] \phi \land [w] \psi) \), i.e., (c) \( 1 \vdash \phi \lor \phi \lor \psi, \text{where } \psi = v \in X, v \neq x, v \neq y, v \odot w = z ([v] \phi \land [w] \psi) \). Thus, by substitution \( \sigma : \{ 1 \mapsto 1, \phi \lor \psi \mapsto \Phi \} \) and by applying the substitution rule to (c) we deduce the sequent \( 1 \vdash \phi \lor \psi \), that is \( (d) \vdash v \in X, v \odot w = z ([v] \phi \land [w] \psi) \) (from \( \phi \lor \psi \equiv v \in X, v \odot w = z ([v] \phi \land [w] \psi) \)). Consequently, by applying the transitivity rule to the sequent (d) and the introduction axiom \( v \in X, v \odot w = z ([v] \phi \land [w] \psi) \vdash (1 \vdash [z] (\phi \circ \psi)) \), we deduce \( 1 \vdash [z] (\phi \circ \psi) \), that is \( \Gamma \vdash (1 \vdash [z] (\phi \circ \psi)) \).

Notice that in such deductions no one of the values \( x, y, z \) has to be designated value. We do not make any distinction for the truth values in \( X \).

\[ \square \]

**Remark:** There is also another way, alternative to 2-valued sequent systems, to reduce the many-valued logics into "meta" 2-valued logics: it is based on Ontological...
encapsulation [22,18], where each many-valued proposition (or many-valued ground atom \( p(a_1, \ldots, a_n) \)) is ontologically encapsulated into the "flattened" 2-valued atom \( \approx \langle \neg p(a_1, \ldots, a_n, x) \rangle \) (by enlarging original atoms with a new logic variable whose domain of values is the set of truth values of the complete lattice \( x \in X \)): roughly, "\( p(a_1, \ldots, a_n) \) has a value \( x \)" iff \( p_F(a_1, \ldots, a_n, x) \) is true. In fact, such a flattened atom is logically equivalent to the sequent \( (1 \vdash [x]p(a_1, \ldots, a_n)) \).

4 Autoreferential Kripke-style semantics

We are able to define an equivalence relation \( \approx_L \) between the formulae of any many-valued logic based on the set of truth values \( X \), in order to define the Lindenbaum algebra for this logic, \( (L/ \approx_L) \), where for any two formulae \( \phi, \psi \in L \), \( \phi \approx_L \psi \) iff \( \forall v \in V_M (v(\phi) = v(\psi)) \).

Thus, the elements of this quotient algebra \( L/ \approx_L \) are the equivalence classes, denoted by \( \langle \phi \rangle \).

In particular we will consider an equivalence class \( \langle \phi \rangle \) (the set of all equivalent formulae to \( \phi \) w.r.t. \( \approx_L \)) that has exactly one constant \( x \in X \), which is an element of this equivalence class (we abuse a denotation here by denoting by \( x \) a formula, such that has a constant logic value \( x \in X \) for every interpretation \( v \), as well), and we can use it as the representation element for this equivalence class \( \langle x \rangle \). Thus, every formula in this equivalence class has the same truth-value as this constant. Consequently, we have the injection \( i_X : X \rightarrow L/\approx_L \) between elements in the complete lattice \( (X, \leq) \) and elements in the Lindenbaum algebra, such that for any logic value \( x \in X \), we obtain the equivalence class \( \langle x \rangle = i_X(x) \in L/\approx_L \). It is easy to extend this injection into a monomorphism between the original algebra and this Lindenbaum algebra, by definition of correspondent connectives in this Lindenbaum algebra. For example: \( \langle x \land y \rangle = i_X(\langle x \land y \rangle) = i_X(\langle x \rangle, i_X(\langle y \rangle)) = \langle \langle x \rangle \land \langle y \rangle \rangle, \langle \neg x \rangle = i_X(\langle \neg x \rangle) = \langle \neg_L i_X(\langle x \rangle) \rangle = \langle i_X(\langle x \rangle) \rangle \), etc.

In an autoreferential semantics [7,8,23] we will assume that each equivalence class of formulae \( \langle \phi \rangle \) in this Lindenbaum algebra corresponds to one "state - description". In particular, we are interested to the subset of "state - descriptions" that are invariant w.r.t. many-valued interpretations \( v \), so that can be used as the possible worlds in the Kripke-style semantics for the original many-valued modal logic. But from the injection \( i_X \) we can take for such an invariant "state -description" \( \langle x \rangle \in L/\approx_L \) only its inverse image \( x = i_X^{-1}(\langle x \rangle) \subseteq X \).

Consequently, the set of possible worlds in this autoreferential semantics corresponds to the set of truth values \( X \).

Now we will consider the Kripke model (introduced in subsection 1.2) for the 2-valued multi-modal logic language \( L_M \) given in Definition 2 obtained from the many-valued predicate logic language \( L_P \) (defined at the beginning of Section 2):

**Definition 7. Kripke Semantics:** Let \( L_P \) be a many-valued predicate logic language, based on a set \( X \) of truth values, with a set of predicate letters \( P \) and Herbrand base \( H \). Let \( M_w = (F, S, \nu) \) be a Kripke model of its correspondent 2-valued multi-modal logic language \( L^*_M \), where the frame \( F = (W, \{R_w = W \times \{w\} \mid w \in W\}) \) where \( W = X \cup 2 \) and with mapping \( V : W \times P \rightarrow \bigcup_{n \leq \omega} 2^{2^n} \), such that for any
n-ary predicate \( p \in P \), and tuple \( (c_1, \ldots, c_n) \in S^n \), there exists a unique \( w \in W \), such that \( V(w, p)(c_1, \ldots, c_n) = 1 \).

It defines the Herbrand interpretation \( v : H \to X \), such that \( v(p(c_1, \ldots, c_n)) = w \) iff \( V(w, p)(c_1, \ldots, c_n) = 1 \), and its unique homomorphic extension to all ground formulae \( v : L \to X \).

Let \( g : \text{Var} \to S \) be an assignment for object variables \( v_i \in \text{Var} \), \( i = 1, 2, \ldots \), and \( w \in W \), then the satisfaction relation \( \models \) is defined by \( \mathcal{M}_v \models_{g,w} \varphi \) iff \( v(\varphi/g) = w \), for any many-valued formula \( \varphi \in L_p \). It is extended to all modal formulae in \( L_M \) as follows:

1. \( \mathcal{M}_v \models_{g,w} 1 \) and \( \mathcal{M}_v \not\models_{g,w} 0 \), for tautology and contradiction respectively.
2. \( \mathcal{M}_v \models_{g,w} [x] \Phi \) iff \( \forall y((w, y) \in R_x \implies \mathcal{M}_v \models_{g,y} \Phi) \), for any \( \Phi \in L_M^* \) or \( \Phi \in L_p \).
3. \( \mathcal{M}_v \models_{g,w} \Phi \land \Psi \) iff \( \mathcal{M}_v \models_{g,w} \Phi \) and \( \mathcal{M}_v \models_{g,w} \Psi \), for \( \Phi, \Psi \in L_M^* \).
4. \( \mathcal{M}_v \models_{g,w} \Phi \lor \Psi \) iff \( \mathcal{M}_v \models_{g,w} \Phi \) or \( \mathcal{M}_v \models_{g,w} \Psi \), for \( \Phi, \Psi \in L_M^* \).

Notice that, based on this Kripke model \( \mathcal{M}_v \), it is defined a many-valued valuation \( v : H \to X \), with a unique standard homomorphic extension \( v : L \to X \), as follows from definition above: for any ground atom \( p(c_1, \ldots, c_n) \in H \), we define \( v(p(c_1, \ldots, c_n)) = w \) where \( w \in X \) is a unique value which satisfies \( V(w, p)(c_1, \ldots, c_n) = 1 \).

Vice versa, given a many-valued model \( v : H \to X \) for a many-valued predicate logic language \( L_p \), we define a Kripke model with mapping \( V \) such that for any \( w \in W \), \( n \)-ary \( p \in P \), and a tuple \( (c_1, \ldots, c_n) \in S^n \), \( V(w, p)(c_1, \ldots, c_n) = 1 \) iff \( v(p(c_1, \ldots, c_n)) = w \).

\( \square \)

Let \( \Psi/g \in L_M \) be a ground formula obtained from \( \Psi \in L_M^* \) by assignment \( g \), then we denote by \( \| \Psi/g \| \) the set of worlds where the ground formula \( \Psi/g \in L_M \), is satisfied, with \( \| p(v_1, \ldots, v_n)/g \| = \{ v(p(g(v_1), \ldots, g(v_n))) \} \) and \( \| \phi/g \| = \{ v(\phi/g) \} \), \( \phi \in L \).

Thus, differently from the original many-valued ground atoms in \( L \) which can be satisfied only in one single world, the modal atoms in \( L_M \) have the standard 2-valued property, that is, they are true or false in these Kripke models, and, consequently, are satisfiable in all possible worlds, or absolutely not satisfiable in any world. Thus, our positive multi-modal logic with modal atoms \( L_M \) satisfies the classic 2-valued properties:

**Proposition 5** For any ground formula \( \Phi/g \) of the positive multi-modal logic \( L_M \) defined in Definition 2, we have that \( \| \Phi/g \| \in \{ \emptyset, W \} \), where \( \emptyset \) is the empty set.

**Proof:** By structural induction:

1. \( \| 1 \| = W \) and \( \| 0 \| = \emptyset \).
2. \( \| x \phi/g \| = W \) if \( x = v(\phi/g) \); \( \emptyset \), otherwise.

Let \( \Phi, \Psi \) be the two atomic modal formulae such that, by inductive hypothesis \( \| \Phi/g \|, \| \Psi/g \| \in \{ \emptyset, W \} \). Then,

3. \( \| [x] \Phi \| = \{ w \in W \mid x \in \| \Phi/g \| \} = W \) if \( \| \Phi/g \| = W \); \( \emptyset \), otherwise.
4. \( \| (\Phi \land \Psi)/g \| = \{ \Phi/g \| \cap \{ \Psi/g \| \in \{ \emptyset, W \} \.
5. \( \| (\Phi \lor \Psi)/g \| = \{ \Phi/g \| \cup \{ \Psi/g \| \in \{ \emptyset, W \} \.

Thus, from the fact that any formula \( \Phi \in L_M^* \) is logically equivalent to disjunctive modal formula \( V_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([\gamma_{ij1}] \ldots [\gamma_{ijk_i}])A_{ij}) \) (from Proposition 2), where
each $A_{ij} \in H$ is a ground atom (such that by inductive hypothesis for any modal atom it holds that $\|([y_{ij1} \ldots y_{ijk_i}])A_{ij}/g\| \in \{0, W\}$), and from points 3 and 4 above, we obtain that 
\[
\|\Phi/g\| = \bigvee_{1 \leq i \leq m}(\wedge_{1 \leq j \leq m_i}(\|[y_{ij1} \ldots y_{ijk_i}]A_{ij}/g\|)) \in \{0, W\}.
\]
\[\square\]

The following proposition demonstrates the existence of a one-to-one correspondence between the unique many-valued model of a many-valued logic $L$ and the Kripke model of a multi-modal positive logic $L_M$.

**Proposition 6**
For any many-valued formula $\phi/g \in L_P$, we have that \( v(\phi/g) = x \) iff \( \mathfrak{F}(v)([x]\phi/g) = 1 \) iff \( \| [x]\phi/g \| = W \).

**Proof:** For any ground formula, we have that \( \mathfrak{F}(v)([x]\phi/g) = 1 \) iff (by Proposition 1) \( \mathfrak{F}(v)([x]\phi/g) = 1 \) iff (by Proposition 3) \( v(\phi/g) = x \).

Thus, it is enough to prove that \( \alpha([x]\phi/g) = 1 \) iff \( \|[x]\phi/g\| = W \), where \( \alpha = \mathfrak{F}(v) \).

In the first step, proceeding from left to right, we will demonstrate it by structural induction on the length (number of logic connectives) of the formula $\phi$:

1. The simplest case when $\phi/g = p(c_1, \ldots, c_k), c_i = g(\nu_i) \in S, 1 \leq i \leq k$, is a ground atom for the $k$-ary predicate letter $p \in P$. Then, if $\alpha([x]\phi/g) = \alpha([x]p(c_1, \ldots, c_k)) = \alpha([x]p(c_1, \ldots, c_k)) = 1$, then (by Proposition 3) \( x = v(p(c_1, \ldots, c_k)) \), and from Definition 7 we have that \( \|[x]\phi/g\| = W \).

2. Let us now suppose, by inductive hypothesis, that it holds for all formulae with $N$ logical connectives in $\Sigma$. Then, for any formula $\phi \in L_P$ with $N + 1$ logical connectives we have the following two cases:

2.1. Case when $\phi = \neg \phi_1$ where $\neg \in \Sigma$ is a unary connective. Then, if $\alpha([x]\phi/g) = \alpha([y]\phi_1/g) = \alpha([y]\phi_1/g) = 1$, and from the inductive hypothesis for this $y$ we obtain (a) \( \|[y]\phi_1/g\| = W \). So, we obtain \( \| [x]\phi/g \| = \| \bigvee_{y \in X, x \neq y} [y]\phi_1/g \| = (\| [y]\phi_1/g \|) = (\| [y]\phi_1/g \|) = \| y \| = W \).

2.2. Case when $\phi = \phi_1 \circ \phi_2$, where $\circ \in \Sigma$ is a binary connective.

Then, if $\alpha([x](\phi_1 \circ \phi_2)/g) = \alpha([y] \phi_1/g \circ [z] \phi_2/g) = \alpha([y] \phi_1/g \circ [z] \phi_2/g) = 1$, then, there exist $y, z \in X$ such that $x = y \circ z$ with $\alpha([y]\phi_1/g) = 1$ and $\alpha([z]\phi_2/g) = 1$. That is, from Proposition 1 $\alpha([y] \phi_1/g) = 1$, and from inductive hypothesis for this $y$ we obtain (b) $\|[y]\phi_1/g \| = W$ and $\|[z]\phi_2/g \| = W$, that is (b) $\|[y]\phi_1/g \| = W$ and $\|[z]\phi_2/g \| = W$. So, we obtain \( \| [x]\phi/g \| = \| \bigvee_{y, z \in X, x = y \circ z} ([y]\phi_1/g \land [z]\phi_2/g) \| = \| [y]\phi_1/g \| \land \|[z]\phi_2/g\| = W \). Consequently, we have shown that $\alpha([x]\phi/g) = 1$ implies $\|[x]\phi/g\| = W$. Vice versa, the proof from right to left is analogous.
Let us show now that for any ground formula \( \Phi/g \in L_M \) it holds that \( \alpha(\Phi/g) = 1 \) iff \( \|\Phi/g\| = \mathcal{W} \):

From the fact that any \( \Phi \in L_M \) is logically equivalent to disjunctive modal formula
\[
\bigvee_{1 \leq i \leq m} \left( \bigwedge_{1 \leq j \leq m_i} \left( (y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} \right) \right),
\]
we obtain that,
\[
\alpha(\Phi/g) = \alpha(\bigvee_{1 \leq i \leq m} \left( \bigwedge_{1 \leq j \leq m_i} \left( (y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g \right) \right)) = \bigvee_{1 \leq i \leq m} \left( \bigwedge_{1 \leq j \leq m_i} \alpha((y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g) \right) = 1,
\]
then there exists \( i, (1 \leq i \leq m) \), such that for all \( 1 \leq j \leq m_i \), \( \alpha((y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g) = 1 \), i.e., \( \|((y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g)\| = \mathcal{W} \). Thus, \( \|\bigwedge_{1 \leq j \leq m_i} ((y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g)\| = \|\bigwedge_{1 \leq j \leq m_i} ((y_{ij1})\ldots(y_{ijk_{ij}})A_{ij} / g)\| = \mathcal{W} \), and consequently, \( \|\Phi/g\| = \mathcal{W} \).

And vice versa.

\[\Box\]

**Corollary 1**

*Given a many-valued model \( v \) for a many-valued logic language \( L_P \), for any ground formula \( \Phi/g \in L_M \) holds that \( \overline{S}(v)(\Phi/g) = 1 \) iff \( \|\Phi/g\| = \mathcal{W} \) (i.e., \( \Phi/g \) is true in the Kripke model in Definition [7]).*

**Proof:** By structural recursion and by Propositions [5] and [6]

From this corollary we obtain that any true formula \( \Phi \in L_M \) is true also in the Kripke model, and vice versa. That is, the autoreferential Kripke-style semantics for the multimodal logic \( L_M \), defined in Definition [7], is sound and complete.

### 5 Conclusion

The main goal of this paper has been the development of a new binary sequent calculi, with truth-invariance entailment, for a many-valued predicate logic language \( L_P \) with a finite set of truth values \( X \), and the definition of Kripke-like semantics for it, both sound and complete. We did not use any ordering of truth values in \( X \) nor any algebraic matrix with a strict subset of designated truth values. So, from this point of view, it is the most general semantic approach for the many-valued logics. In more specific cases, when the set \( X \) is a complete lattice of truth values, there is also another non-matrix based approach (with truth-preserving entailment) presented in [7,10] where the sequent system is based on the lattice poset of truth values.

In comparison with the standard historical approach based on m-sequents, this approach is deterministic, in the way that the axiomatic sequent system is uniquely determined by the set of many-valued logical connectives. It is more compact and is a particular implementation of standard two-sided sequent systems, where the left side of each sequent is just a single formula, as well. Moreover, it is not matrix-based and does need any definition of a subset of designated elements in \( X \), as in other approaches where for each subjectively defined subset of designated truth values (consider for example the logic with \( n = 10^3 \) truth-values, \( X = \{ \frac{i}{n} \mid 1 \leq i \leq n \} \)), for the same logic language \( L \) and the same semantics for its logical connectives, we obtain the different deductive system. Here this subjectiveness is avoided, based on the generalization of the 2-valued truth-invariance principle for the logic entailment, and the resulting deductive system for a many-valued logic with fixed semantics of its logical connectives is general and...
uniquely defined as in all cases of the 2-valued logics.
Differently from other approaches we defined also a Kripke-like semantics for this
many-valued deductive system, because our encapsulation of many-valued logic into
the 2-valued sequent system is based on the introduction of the finite set of modal oper-
ators, for each truth value in $X$, that is, this 2-valued encapsulation is modal as in [18].
The frame in this autoreferential Kripke semantics, based on the Lindenbaum algebra
considerations, is finite and uniquely determined by the set of truth values in $X$.

References

1. J.Lukasiewicz, “A system of modal logic,” Journal of Computing Systems, vol. 1, pp. 111–
149, 1953.
2. G.Gentzen, “Über die Existenz unabhängiger Axiomensysteme zu unendlichen
Satzsystemen,” Mathematische Annalen, 107, pp. 329–350, 1932.
3. P.Hertz, “Über Axiomensysteme für beliebige Satzsysteme,” Mathematische Annalen, 101,
pp. 457–514, 1929.
4. G.Rousseau, “Sequents in many valued logic I,” Fund.Math., 60, pp. 23–33, 1967.
5. W.A.Carnielli, “Systematization of finite many-valued logics through the method of tableaux,”
J.Symbolic Logic, 52(2), pp. 473–493, 1987.
6. R.Hähnle, “Uniform notation of tableaux rules for multiple-valued logics,” In Proc.Int.
Symposium on Multiple-valued Logic, pp. 238–245, 1991.
7. Z.Majkić, “Autoreferential semantics for many-valued modal logics,” Journal of Applied
Non-Classical Logics (JANCL), Volume 18- No.1, pp. 79–125, 2008.
8. Z.Majkić, “Weakening of intuitionistic negation for many-valued paraconsistent da Costa
system,” Notre Dame Journal of Formal Logics, Volume 49, Issue 4, pp. 401–424, 2008.
9. Z.Majkić, “On paraconsistent weakening of intuitionistic negation,” arXiv: 1102.1935v1, 9
February, pp. 1–10, 2011.
10. Z.Majkić, “A new representation theorem for many-valued modal logics,” arXiv:
1103.0248v1, 01 March, pp. 1–19, 2011.
11. A.Avron, “Classical Gentzen-type methods in propositional many-valued logics.” Studies in
Fuzziness and Soft Computing, Vol.114, pp. 117–155, 2003.
12. M.Baaz, C.G.Fermüller, and R.Zach, “Systematic construction of natural deduction systems
for many-valued logics,” 23rd Int.Symp. on Multiple Valued Logic, pp. 208–213, 1993.
13. M.Baaz, C.G.Fermüller, and G.Salzer, “Automated deduction for many-valued logics,” In
Handbook of Automated Reasoning, Elsevier Science Publishers, 2000.
14. A.Avron, J.Ben-Naim, and B.Konikowska, “Cut-free ordinary sequent calculi for logic having
generalized finite-valued semantics,” Logica Universalis 1, pp. 41–69, 2006.
15. H.Rasiowa and R.Sikorski, “The mathematics of metamathematics,” PWN- Polisch Scientific
Publishers, Warsaw, 3rd edition, 1970.
16. B.Konikowska, “Rasiowa-Sikorski deduction systems in computer science applications,”
Theoretical Computer Science 286, pp. 323–266, 2002.
17. Z.Majkić, “Many-valued intuitionistic implication and inference closure in a bilattice based
logic,” 35th International Symposium on Multiple-Valued Logic (ISMVL 2005), May 18-21,
Calgary, Canada, 2005.
18. Z.Majkić, “Reduction of many-valued logic programs into 2-valued modal logics,” arXiv:
1103.0920, 04 March, pp. 1–27, 2011.
19. J.M.Dunn, “Positive Modal Logic,” Studia Logica, vol. 55, pp. 301–317, 1995.
20. J.B.Rosser and A.Turquette, “Many-valued logics,” North Holland Publ.Company, Amster-
dam, 1952.
21. P. Blackburn, J. F. Benthem, and F. Wolter. “Handbook of modal logic,” Volume 3 (Studies in Logic and Practical Reasoning. Elsevier Science Inc., 2006.

22. Z. Majkić, “Ontological encapsulation of many-valued logic,” 19th Italian Symposium of Computational Logic (CILC04), June 16-17, Parma, Italy, 2004.

23. Z. Majkić and B. Prasad, “Lukasiewicz’s 4-valued logic and normal modal logics,” 4th Indian International Conference on Artificial Intelligence (IICAI-09), December 16-18, Tumkur, India.