The initial singularity of ultrastiff perfect fluid spacetimes without symmetries

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Abstract

We consider the Einstein equations coupled to an ultrastiff perfect fluid and prove the existence of a family of solutions with an initial singularity whose structure is that of explicit isotropic models. This family of solutions is ‘generic’ in the sense that it depends on as many free functions as a general solution, i.e., without imposing any symmetry assumptions, of the Einstein-Euler equations. The method we use is that of a Fuchsian reduction.

1 Introduction

In general relativity, spacetime singularities can occur under very general conditions, see \cite{1}. However, mathematical theorems detailing the structure of these singularities usually assume a great deal of symmetry; either spatial homogeneity is assumed, or as in the Gowdy and $T^2$ cases, an Abelian two-parameter isometry group acting on spatial sections. A potentially powerful method to obtain theorems concerning non-homogeneous models is the method of Fuchsian reduction \cite{2}, which is a rigorous attempt to ‘expand’ solutions in the neighborhood of a singularity. A particular way (but by no means the only one \cite{2}) to obtain the leading order behavior of solutions toward a singularity is to formulate an ‘asymptotic system’ of equations that is derived from the Einstein equations by dropping terms, e.g., spatial derivatives, which are believed to be unimportant for the leading asymptotic dynamics. If this ‘asymptotic system’ truly captures the leading divergent behavior of solutions of the Einstein equations toward a singularity, then the remaining part of each solution obeys equations that are of so-called Fuchsian type. In this case, it is possible to prove the existence of a general class of solutions of the Einstein equations whose asymptotic behavior is governed by the behavior of solutions of the asymptotic system. This method of Fuchsian reduction was applied, e.g., in \cite{3} to the initial singularity of cosmological models with a stiff fluid or massless scalar field in $3+1$ dimensional spacetimes without symmetries, and to the Einstein-dilaton-$p$-form system in higher dimensions \cite{4}.

The structure of spacetime singularities is particularly interesting in the context of the Weyl curvature hypothesis, where the focus is on isotropic singularities \cite{5}. Spacetimes with such singularities are not generic among solutions of the Einstein-matter equations, i.e., an arbitrary matter/curvature distribution
at late times does not develop an isotropic initial singularity when traced back in time. However, this is expected to only hold for matter sources where the ratio of pressure to energy density is less than or equal to one; if this ratio is greater than one, isotropic singularities are expected to be generic. In the case of a perfect fluid matter source, such a matter model is called an ultrastiff perfect fluid.

The interest in ultrastiff fluids has come from attempts to obtain non-singular descriptions of spacetimes representing collapsing universes bouncing outward before a singularity occurs. This in the case in the ecpyrotic scenario or the cyclic brane world models [6], where the ultrastiff fluid is an effective description of a scalar field with negative potential energy during the contraction phase. The scalar field is in turn a representation of the size of a compactified dimension and the potential a representation of the force between two different brane worlds. These models have led to an interest in ultrastiff cosmologies in general [7] and especially in their structure in the final stages of collapse, or equivalently, initial stages of expansion in an expanding universe.

The structure of the initial singularity in cosmological models with an ultrastiff perfect fluid has been studied under the assumption that the singularity is ‘local’ [8, 9], which means that the time evolution at adjacent spatial points decouples. Under this a priori assumption of locality it has been demonstrated that the past asymptotic structure is described by an isotropic solution [8, 10] of the Friedmann-Robertson-Walker type, i.e., that anisotropies become small compared to the overall expansion of the universe close to the singularity. The situation is similar for cosmological models with several non-interacting perfect fluids, of which at least one possesses an ultrastiff equation of state [11].

In this paper we investigate the past asymptotic dynamics of cosmological spacetimes with ultrastiff perfect fluids. We do not impose any assumptions on the symmetry of the spacetimes and we do not make any a priori assumptions on the past asymptotic behavior (like asymptotic locality). We prove the existence of a family of solutions that converge to isotropic models toward the initial singularity; the past asymptotic behavior of these solution is characterized in detail, see Theorem 4.1. This family of solutions we construct is ‘generic’ in the sense that it depends on as many free functions as a general solution of the initial value problem connected with the Einstein-Euler equations.

The paper is organized as follows. In section 2 we give the Einstein-Euler equations in a Gaussian coordinate system; in section 3 we describe the asymptotic system and its role in the Fuchsian reduction. Section 3.1 is devoted to the special case of a perfect fluid with equation of state \( p = 3\mu \), for which the asymptotic system admits explicit solutions. We use these solutions to set up a Fuchsian system of equations; this is done in analogy with the analysis of the stiff fluid case [3]. In section 3.2 we give the treatment of the general ultrastiff fluid case, which is less explicit but follows the same principles. The main result is then stated in section 4, while section 5 gives a derivation of the reduced equations that are used to prove the main result in section 6.

### 2 The Einstein-Euler equations

Let \( {}^4M \) be a four-dimensional Lorentzian manifold with metric \( {}^4g_{\alpha\beta} \) (with \( \alpha, \beta = 0, 1, 2, 3 \)). We consider the Einstein-Euler equations for self-gravitating perfect fluids, i.e.,

\[
\begin{align*}
{}^4R_{\alpha\beta} - \frac{1}{2} {}^4R {}^4g_{\alpha\beta} &= T_{\alpha\beta} \\
\nabla_\alpha T^{\alpha\beta} &= 0,
\end{align*}
\]

where we assume a linear equation of state \( p = w\mu, \ w = \text{const} \), relating the energy density \( \mu \) (as measured in the fluid’s rest frame) and the pressure \( p \) of the fluid; \( u^\alpha \) is the four-velocity of the fluid. We use geometrized units, i.e., \( c = 1 \) and \( 8\pi G = 1 \), where \( c \) is the speed of light and \( G \) the gravitational constant.

We consider spacetimes that are diffeomorphic to \( \mathbb{R} \times M \), where \( M \) is three-dimensional, and metrics of
the form
\[ -dt^2 + g_{ab} \omega^a \omega^b , \]  
where \( g_{ab} = g_{ab}(t) \) denotes a one-parameter family of Riemannian metrics, which are naturally identified with the metrics induced on \( t = \text{const} \) hypersurfaces; the coframe \( \{ \omega^1, \omega^2, \omega^3 \} \) on \( M \) is arbitrary. The range of Latin indices \( a, b, \ldots \) is 1, 2, 3.

The Einstein-Euler equations become a first order system of equations with constraints when the extrinsic curvature \( k_{ab} = -\frac{1}{2} \partial_t g_{ab} \) of the \( t = \text{const} \) hypersurfaces is used \((3 + 1 \text{ split})\). For our purposes it is preferable to split \( k_{ab} \) into its trace (‘mean curvature’) and (the negative of) its traceless part, i.e.,
\[ k_{ab} = -\sigma_{ab} + \frac{1}{3} (\text{tr} \, k) \, g_{ab} . \]  
The tensor \( \sigma_{ab} \) coincides with the rate of shear tensor of the (geodesic) congruence orthogonal to the \( t = \text{const} \) hypersurfaces. Then the evolution equations are
\[ \partial_t g_{ab} = 2 g_{ac} \left( \sigma^c_b - \frac{1}{3} (\text{tr} \, k) \delta^c_b \right) , \]  
(4a)
\[ \partial_t \sigma^a_b = (\text{tr} \, k) \sigma^a_b - \left( R^a_b - \frac{1}{3} R \delta^a_b \right) + \left( S^a_b - \frac{1}{3} (\text{tr} \, S) \delta^a_b \right) , \]  
(4b)
\[ \partial_t (\text{tr} \, k) = R + (\text{tr} \, k)^2 + \frac{1}{2} (\text{tr} \, S) - \frac{3}{2} \rho , \]  
(4c)
and the constraint equations read
\[ R - \sigma^a_b \sigma^b_a + \frac{2}{3} (\text{tr} \, k)^2 = 2 \rho , \]  
(5a)
\[ - \nabla_a \sigma^a_b - \frac{2}{3} \nabla_a (\text{tr} \, k) = j_b , \]  
(5b)
where
\[ \rho = \mu (1 + (1 + w) u^a u_a) , \quad j_b = \mu (1 + w) (1 + u_a u^a)^{1/2} u_b , \quad S_{ab} = \mu ((1 + w) u_a u_b + w g_{ab}) . \]  
(6)
The quantity \( \rho \) is the energy density, \( j_b \) the current measured by the observer associated with the geodesic congruence. Equations (4) and (5), together with (6), correspond to (1a). The Euler equations (1b) read
\[ \partial_t \mu - (1 + w)(\text{tr} \, k) \mu = -(1 + w) \left[ u^2 \partial_t \mu + 2 \mu u^a \partial_t u_a - \mu \sigma_{ab} u^a u^b - \frac{2}{3} \mu (\text{tr} \, k) u^2 
+ \sqrt{1 + u^2} \, u^a \nabla_a \mu + \mu (1 + u^2)^{-1/2} u_a u^b \nabla_b u^a + \mu \sqrt{1 + u^2} \, \nabla_a u^a \right] , \]  
(7a)
\[ \partial_t u_a + w (\text{tr} \, k) u_a + \frac{w}{1 + w} \mu^{-1} \nabla_a \mu = -\mu^{-1} u_a \left[ \partial_t \mu - (1 + w) \mu (\text{tr} \, k) \right] - u_a (1 + u^2)^{-1} \left[ u^b \partial_t u_b - \sigma_{bc} u^b u^c \right] 
+ \frac{1}{3} u_a (1 + u^2)^{-1} (\text{tr} \, k) u^2 - \left[ (1 + u^2)^{-1/2} - 1 \right] \frac{w}{1 + w} \mu^{-1} \nabla_a \mu 
- (1 + u^2)^{-1/2} \left[ \mu^{-1} u_a u^b \nabla_b \mu + u_a \nabla_b u^b + u^b \nabla_b u_a \right] , \]  
(7b)
where we use the abbreviation \( u^2 = u_a u^a \).

3 The asymptotic system

Alongside the Einstein-Euler equations we consider another system of equations for the same set of variables, which we denote by \( \{ 0 g_{ab}, \text{tr} \, 0 k, 0 \sigma^a_b, 0 \mu, 0 u_a \} \) in the present context. This asymptotic system consists of evolution equations for \( \{ 0 g_{ab}, \text{tr} \, 0 k, 0 \sigma^a_b, 0 \mu, 0 u_a \} \),
\[ \partial_t 0 g_{ab} = 2 g_{ac} \left( 0 \sigma^c_b - \frac{1}{3} (\text{tr} \, 0 k) \delta^c_b \right) , \]  
(8a)
\[ \partial_t 0 \sigma^a_b = (\text{tr} \, 0 k) 0 \sigma^a_b , \]  
(8b)
\[ \partial_t (\text{tr} \, 0 k) = (\text{tr} \, 0 k)^2 + \frac{2}{3} (w - 1) 0 \mu , \]  
(8c)
and evolution equations for the matter variables \((\varrho,\varrho_a)\),
\[
\partial_t \varrho - (1 + w)(\text{tr } \varrho) \varrho = 0, \quad (9a)
\]
\[
\partial_t \varrho_a + w (\text{tr } \varrho) \varrho_a + \frac{w}{1+w} \varrho^{-1} \varrho_a \mu = 0. \quad (9b)
\]

In addition, we consider the constraint equations
\[
- \varrho a^a b^b + \frac{2}{3} (\text{tr } \varrho) k = 0, \quad (10a)
\]
\[
- \varrho \nabla a^a b^b - \frac{2}{3} \varrho b^b (\text{tr } \varrho) k = 0, \quad (10b)
\]
where \(\varrho \equiv \varrho\) and \(\varrho a \equiv (1 + w) \varrho \varrho_a\). The symbol \(\nabla a\) denotes the covariant derivative associated with the metric \(g_{ab}\).

Note that the evolution equations (8) and (9) preserve the constraints (10). A computation shows that
\[
\partial_t \left( - \varrho a^a b^b + \frac{2}{3} (\text{tr } \varrho) k^2 - 2 \varrho \rho \right) = 2 (\text{tr } \varrho) \left( - \varrho a^a b^b + \frac{2}{3} (\text{tr } \varrho) k^2 - 2 \varrho \rho \right),
\]
\[
\partial_t \left( - \varrho \nabla a^a b^b - \frac{2}{3} \varrho b^b (\text{tr } \varrho) k - \varrho a \right) = (\text{tr } \varrho) \left( - \varrho a^a b^b - \frac{2}{3} \varrho b^b (\text{tr } \varrho) k - \varrho a \right)
\]
\[
- \frac{1}{2} \varrho b^b \left( - \varrho a^a b^b + \frac{2}{3} (\text{tr } \varrho) k^2 - 2 \varrho \rho \right),
\]
hence if the constraints are satisfied initially, they are satisfied for all times.

The asymptotic system (8)–(10) is a system of partial differential equations, where, however, the spatial derivatives merely enter in a passive manner, through the decoupled equations (9b) and (10b); hence in spirit, (8)–(10) are ODEs. It is clear that the asymptotic system is obtained from the Einstein-Euler system by dropping (a large number of) terms. The derivation of the asymptotic system from the Einstein-Euler system is intimately connected with considerations involving the concepts of asymptotic silence and locality; we refer to [9] and references therein. The aim of these considerations is to find a simple system of equations that reproduces the dynamics of the full Einstein-matter equations in the asymptotic limit toward a ‘generic spacelike singularity’. In other words, a simple (‘asymptotic’) system is sought that governs the asymptotic behavior toward a spacelike singularity of (typical) solutions of the Einstein-matter equation in the sense that each (typical) solution of the Einstein-matter equation is approximated, to a certain order, by a solution of the asymptotic system.

Whether the asymptotic system given in (8)–(10) indeed governs the asymptotic behavior of solutions of the full Einstein-Euler equations, is a non-trivial question. The considerations of [9] strongly indicate that a necessary condition is that the equation of state parameter satisfies \(w \geq 1\). (In the case \(w < 1\), the conjectured asymptotic system is considerably more complicated, because it must capture the conjectured oscillatory behavior, ‘Mixmaster behavior’, of solutions toward the singularity.) The case of a stiff fluid, i.e. \(w = 1\), has been treated in [3]: The asymptotic system given in [3] is shown to govern the asymptotic behavior of solutions of the Einstein-Euler equations with a stiff fluid. (Note, however, the comment in the symmetry subsection of section 3.2.)

In this paper we consider ultrastiff fluids, which are characterized by \(w > 1\). We prove that the asymptotic behavior of solutions (more specifically, of a full set of solutions in the sense of function counting) of the Einstein-Euler equations with an ultrastiff fluid is governed by the asymptotic system (8)–(10); in addition, we give detailed considerations about the order of the approximation.

### 3.1 A particular(ly simple) case of ultrastiff fluids: \(w = 3\)

We begin by solving the asymptotic system with \(w = 3\). The reason for this choice is one of convenience, since in the special case \(w = 3\), the solutions of the asymptotic system take a simple and explicit form (which enables us to stress the analogies of the present analysis with the analysis of [3]). The general (but less explicit) case \(w > 1\) is treated in subsection 3.2.
We are interested in solutions with $\partial_\mu > 0$ and $\text{tr} \, \partial_k < 0$, i.e., 'expanding' models with positive energy density. (Note that $\partial_\mu$ remains positive if it is positive initially, see (9a); likewise, if $\text{tr} \, \partial_k$ is negative at $t = t_0$, it remains negative for all $t < t_0$ because $\partial_\mu (\text{tr} \, \partial_k) > 0$, see (8c).) The basic observation is that

$$\partial_\mu (\sqrt{3} \, \partial_\mu \pm \text{tr} \, \partial_k) = \pm (\sqrt{3} \, \partial_\mu \pm \text{tr} \, \partial_k)^2.$$ (11)

The solutions of (11) with the minus sign are

$$\sqrt{3} \, \partial_\mu - \text{tr} \, \partial_k = \frac{1}{t + \psi(x)},$$ (12)

with $t > -\psi(x)$, where $\psi = \psi(x)$ is an arbitrary function of the spatial coordinates which we collectively denote by $x$. (The trivial solution is excluded by our assumptions.) The 'initial singularity' is signalled by the divergence of (12) at $t = -\psi$. We wish this initial singularity to occur at $t = 0$. To simplify further computations we thus choose our initial hypersurface to be a surface where $\psi = \text{const}$. This is not a restriction of the generality of the result since it is always possible to find such a surface between any given initial hypersurface and the singularity. The constant may be set to zero by a shift in the time coordinate.

The solutions of (11) with the plus sign are

$$\sqrt{3} \, \partial_\mu + \text{tr} \, \partial_k = -\frac{1}{t + \phi(x)} \quad \text{and} \quad \sqrt{3} \, \partial_\mu + \text{tr} \, \partial_k \equiv 0,$$ (13)

where $\phi = \phi(x) > 0$ is an arbitrary function of the spatial coordinates. (The function $\phi$ is positive, since $2 \sqrt{3} \, \partial_\mu = t^{-1} - (t + \phi)^{-1} > 0$.) The trivial solution in (13) can be thought of as arising as a special case by taking the limit $\phi \to \infty$. We conclude that

$$\text{tr} \, \partial_k = \frac{1}{2} \left( \frac{1}{t} + \frac{1}{t + \phi} \right), \quad \partial_\mu = \frac{1}{12} \left( \frac{1}{t} - \frac{1}{t + \phi} \right)^2.$$ (14a)

Integration of the remaining evolution equations of (8) and (9) then yields

$$0_{\sigma^a_b} = \frac{0_{\sigma^a_b}}{\sqrt{t(1 + \phi^{-1}t)}},$$ (14b)

$$0_{u_a} = 0_{u_a} \left( t (1 + \phi^{-1}t) \right)^{3/2} - 3 \phi^{-2} (\nabla_a \phi) t^2 \left( 1 + \frac{2}{3} \phi^{-1} t \right),$$ (14c)

$$0_{g_{ab}} = 0_{g_{ac}} \left( t (1 + \phi^{-1}t) \right)^{1/3} \left( \frac{2}{t} + \sqrt{1 + \phi^{-1}} \right)^{1/2} \left( 1 + \frac{2}{3} \phi^{-1} t \right)^{4 \sqrt{3} \sigma^c_b},$$ (14d)

$$0_{g^{ab}} = (t (1 + \phi^{-1}t))^{-1/3} \left( \frac{2}{t} + \sqrt{1 + \phi^{-1}} \right)^{-1/2} \left( 1 + \frac{2}{3} \phi^{-1} t \right)^{-4 \sqrt{3} \sigma^a_c} 0_{g^{cb}}.$$ (14e)

The equations for the metric and its inverse use the matrix exponential $z^A := e^{(\log z)A}$, which is well defined for any positive scalar $z$ and square matrix $A$. Note that $(\sqrt{\phi^{-1}t} + \sqrt{1 + \phi^{-1}})^{\sqrt{\phi}} = e^{\sqrt{\phi}t}$ in the limit $\phi \to \infty$, hence $0_{g_{ab}} = 0_{g_{ac}} t^{1/3} \exp(4 \sqrt{T} 0_{\sigma^c_b})$ in that special case.

The constraints (10) give restrictions on the free functions of the spatial variables: $\phi$, $0_{\sigma^a_b}$, $0_{g_{ab}}$, $0_{u_a}$. We have $0_{\sigma^b_a} 0_{\sigma^a_b} = 2 \phi / 3$ from (10a). The constraint (10b) can be used to express, e.g., $0_{u_a}$ in terms of $0_{\sigma^a_b}$, $0_{g_{ab}}$, and $\phi$. For the Fuchsian analysis of Section 6 we will assume that the free functions are analytic in the spatial variables $x$.

### 3.2 Ultrastiff fluids with general $w > 1$

In the case of a general ultrastiff fluid with an arbitrary value of $w > 1$ the solutions of the asymptotic system, which are given by (14) in the case $w = 3$, cannot be given in explicit form. To derive the
solutions of the asymptotic system with $w \neq 3$ we resort to power series; the ideal framework is the Hubble-normalized dynamical systems approach.

Consider the asymptotic system (8)–(9). The equations (8a) for $a_{ab}$ and (9b) for $a_a$, together with (10b), decouple from this system, i.e., the asymptotic system reduces to equations for $\sigma_{ab}$, $\mathrm{tr} \, k$, and $\mu$, which are subject to the constraint (10a). We write these equations in the standard Hubble-normalized variables, i.e., we introduce

$$H = -\frac{1}{3} \mathrm{tr} \, k \quad \text{and} \quad \Sigma_a = \frac{\sigma_{ab}}{H}, \quad \Omega = \frac{\mu}{3H^2},$$

and an adapted time variable $\partial \tau = H^{-1} \partial t$. In these variables we obtain a decoupled evolution equation for $H$,

$$\partial \tau H = -\frac{3}{2} H (w + 1 - (w - 1)\Sigma^2),$$

and an evolution equation for $\Sigma_a$,

$$\partial \tau \Sigma_a = \frac{3}{4} (w - 1) \Sigma_b (1 - \Sigma^2),$$

where $\Sigma^2 = \frac{1}{3} \Sigma^b \Sigma_a$. The constraint simplifies to $\Omega = 1 - \Sigma^2$ (which makes the evolution equation for $\Omega$ redundant).

Equation (16b) can be solved straightforwardly; we obtain

$$\frac{\Sigma_a}{\sqrt{1 - \Sigma^2}} = \frac{\Sigma_b}{\sqrt{1 - \Sigma^2}} e^{\frac{3}{2}(w - 1)\tau},$$

where $\Sigma_a$ are functions of the spatial variables. Furthermore, (16) yields

$$H \sqrt{1 - \Sigma^2} = \dot{H} \sqrt{1 - \Sigma^2} e^{-\frac{3}{2}(w + 1)\tau},$$

where $\dot{H}$ depends on the spatial variables. Consequently,

$$\sigma_{ab} = \dot{H} \Sigma_a \Sigma_b e^{-3\tau},$$

$$\mathrm{tr} \, k = -3H = -3\dot{H} \sqrt{1 - \Sigma^2} e^{\frac{3}{2}(1 + w)\tau} \left(1 + \frac{\Sigma^2}{1 - \Sigma^2} e^{3(1 - w)\tau}\right)^{1/2},$$

and $\mu = 3H^2 (1 - \Sigma^2) e^{-3(w + 1)\tau}$. Accordingly, we obtain explicit expression for the solutions of the asymptotic system in the time variable $\tau$.

In order to express (18) in terms of cosmological time $t$ we must integrate the relation $dt = H^{-1} d\tau$ where $H$ is given by (18b). For large negative $\tau$ we can expand the square root in a power series in $e^{3(w + 1)\tau}$, and integrate it term by term. Requiring synchronous cosmological time, where $t = 0$ corresponds to the initial singularity, we obtain

$$t = \frac{2 e^{\frac{3}{2}(1 + w)\tau}}{3(1 + w)H \sqrt{1 - \Sigma^2}} \times$$

$$\times \left(1 - \frac{1 + w}{2(3w - 1)} \frac{\Sigma^2}{1 - \Sigma^2} e^{3(1 - w)\tau} + \frac{3(1 + w)}{8(5w - 3)} \left(\frac{\Sigma^2}{1 - \Sigma^2}\right)^2 e^{6(1 - w)\tau} + O(e^{9(1 - w)\tau})\right)$$

as $\tau \to -\infty$. Inverting this relation results in

$$e^{\frac{3}{2}(1 + w)\tau} = \frac{3}{2}(1 + w)H \sqrt{1 - \Sigma^2} \times$$

$$\times \left(1 + \frac{1 + w}{2} \frac{3(1 + w)}{2} (2w - 1) / (1 + w) \left[\Sigma^2 \dot{H}^2 (2w - 1) / (1 + w) / (3w - 1) (1 - \Sigma^2) / 2 / (1 + w) (2w - 1) / (1 + w) + O(t^{-1 + w})\right)\right)$$
in the limit \( t \to 0 \). A calculation then shows that

\[
0g^a_b = \hat{H}^2 \Sigma^a_b \left( \frac{3}{2} (1 + w) \hat{H} \right)^{-2/(1+w)} (1 - \Sigma^2)^{-1/(1+w)} t^{-2} (1 + O(t^{2/(1+w)}))
\]

\[
\equiv 0g^a_b \ t^{-2/(1+w)} \left( 1 + O(t^{2/(1+w)}) \right),
\]  

(19a)

\[
\text{tr } 0^k = -3H = -\frac{2}{1 + w} t^{-1} \left( 1 + \frac{w - 1}{3w - 1} \left( \frac{3(1 + w)}{2} \right)^{2(w-1)/(1+w)} \Sigma^2 \hat{H}^{2(w-1)/(1+w)} t^{-2} \right) + O(t^{4(w-1)/(1+w)})
\]

\[
\equiv -\frac{2}{1 + w} t^{-1} \left( 1 + \phi^{-1} \ t^{2(w-1)/(1+w)} + O(t^{4(w-1)/(1+w)}) \right),
\]  

(19b)

where we have introduced the spatial functions \( 0g^a_b \) and \( \phi \). In addition, (8a) and (9) yield

\[
0g_{ab} = 0g_{ac} t^{4/(1+w)} \left( \delta^c_b + 2 \frac{w + 1}{w - 1} 0g^c_b \ t^{w-1} + O(t^{2/(1+w)}) \right),
\]  

(19c)

\[
0\mu = \frac{4}{3} \left( 1 + w \right)^2 t^{-2} \left( 1 - \frac{w + 1}{w - 1} \phi^{-1} \ 0g^c_b \ t^{2(w-1)/(1+w)} + O(t^{4(w-1)/(1+w)}) \right),
\]  

(19d)

\[
0u_a = 0u_a t^{2w - 4(1 + w)} \left( 1 + w \right)^2 \phi^{-1} \ 0g^c_b \ t^{w-1} + O(t^{2(w-1)/(1+w)}),
\]  

(19e)

and requiring that \( 0g^{ab} \) is the inverse of \( 0g_{ab} \) gives

\[
0g^{ab} = \frac{0g^{ac} t^{4/(1+w)} \left( \delta^c_b - 2 \frac{w + 1}{w - 1} 0g^c_b \ t^{w-1} + O(t^{2/(1+w)}) \right).}
\]  

(19f)

The expressions (19) replace (14) in the context of the case of general \( w > 1 \). It is not difficult to convince oneself that (19) reduces to (14) in the special case \( w = 3 \).

4 Main result

Stated in an informal way, the main result we will prove is that (‘typical’) solutions of the Einstein-Euler equations ‘look like’ the solutions (19) of the asymptotic system in the asymptotic limit toward the singularity. The rigorous statement is the following.

**Theorem 4.1.** Let \( \mathcal{W} = (0g_{ab}, \text{tr } 0^k, 0g^a_b, 0\mu, 0u_a) \) be a solution of the asymptotic system as given by (19). Then there exists a unique solution \( \mathcal{W} = (g_{ab}, \text{tr } k, \sigma^a_b, \mu, u_a) \) of the Einstein-Euler equations, whose asymptotic behavior, to leading order, is that of \( \mathcal{W} \). Specifically, there exists a set of quantities \( \mathcal{U} = (\gamma^a_b, \kappa, \zeta^a_b, \nu, \varsigma_a) \) that vanish in the limit \( t \to 0 \), together with their spatial derivatives, and an arbitrarily small \( \varepsilon > 0 \), such that

\[
g_{ab} = 0g_{ab} + t^{2 - 4/(1 + w) - 2\varepsilon} 0g_{ac} \gamma^c_b, \quad \text{and}
\]

\[
\sigma^a_b = 0\sigma^a_b + t^{1 - 4/(1 + w)} - \varepsilon / 2 \varsigma^a_b, \quad \mu = 0\mu + t^{1 - 10/(3 + 1) - \varepsilon} \nu, \quad u_a = 0u_a + t^{4 - 10/(1 + w) - 2\varepsilon} \varsigma_a.
\]

(20a)

The inverse metric is given by \( g^{ab} = 0g^{ab} + t^{2 - 4/(1 + w) - 2\varepsilon} \gamma^{ac} 0g_{cb} \), where \( \gamma^a_b \) shares the properties of \( \mathcal{U} \).
Remark. The number of free functions (of the spatial variables) in the set of solutions of the asymptotic system is the same as the number of free functions that determine initial data for the Einstein-Euler equations. Therefore, in the sense of function counting, Theorem 4.1 shows that ‘typical’ ('generic') solutions of the Einstein-Euler system behave according to (20).

5 The reduced system

The proof of Theorem 4.1 is based on techniques in connection with Fuchsian equations [2, 3, 12]. Our aim is to construct, from the Einstein-Euler evolution equations (4) and (7) and the asymptotic system (8)–(10), a system of (generalized) Fuchsian type, i.e. a system of the form

\[ t\partial_t \mathcal{U} + A \mathcal{U} = F(t, x, \mathcal{U}, \partial_t \mathcal{U}) + G(t, x, \mathcal{U}) t \partial_t \mathcal{U} \]  

(21)

where \( \mathcal{U} \) denotes the collection of the variables of Theorem 4.1 (and additional variables). In this context, \( A \) is required to be a time-independent matrix that depends analytically on the spatial coordinates, and \( F \) and \( G \) are functions that are continuous in \( t > 0 \) and analytic in the other variables. Sufficient conditions that the system (21) be a (generalized) Fuchsian system are that (i) the matrix \( A \) is the direct sum of a submatrix whose eigenvalues have positive real parts and the zero matrix, and that (ii) the functions \( F \) and \( G \) vanish like a positive power of \( t \) as \( t \to 0 \). If this is the case, then there exists a unique solution \( \mathcal{U} \) of (21) such that \( \mathcal{U} \) and its spatial derivatives vanish with \( t \); see [2, 3]. Our aim is to construct a system (21) and to show that the ‘Fuchsian conditions’ are met.

Let \( \mathcal{W} = (g_{ab}, g^a_b, \text{tr } k, \sigma^a_b, \mu, u_a) \) be a solution of the asymptotic system as given by (19). Let us (re)define \( \mathcal{U} \) as \( \mathcal{U} = (\gamma^a_b, \bar{\gamma}^a_b, \kappa, \varsigma^a_b, \nu, u_a, \lambda^a_{bc}, \bar{\lambda}^a_{bc}) \). We make the ansatz

\[
\begin{align*}
g_{ab} &= g_{ab} + t^{\alpha_\gamma} g_{ac} \gamma^c_b, \\
g^{ab} &= g^{ab} + t^{-\alpha_\gamma} \bar{g}^c_b \gamma^c_a, \\
\text{tr } k &= \text{tr } k + t^{\alpha_k + 1 - \frac{4}{3(w+1)}} \kappa, \\
\sigma^a_b &= \sigma^a_b + t^{-\alpha_\sigma} \varsigma^a_b, \\
\mu &= \mu + t^{\alpha_\mu - \frac{1}{3(w+1)}} \nu, \\
u_a &= \nu_a + t^{1/w + \alpha_\nu} v_a, \\
\gamma^a_{b,c} &= t^{-\alpha_\gamma} \lambda^a_{bc}, \\
\bar{\gamma}^a_{b,c} &= t^{-\alpha_\bar{\gamma}} \bar{\lambda}^a_{bc},
\end{align*}
\]

(22)

where \((\alpha_\gamma, \bar{\alpha}_\gamma, \alpha_\kappa, \alpha_\varsigma, \alpha_\nu, \alpha_\nu, \alpha_\lambda, \bar{\alpha}_\lambda)\) are constants.

The procedure is (almost) straightforward: Imposing the Einstein-Euler equations yields a system of equations for the variables \( \mathcal{U} \). The plan is then to find a range of the free constants such that this system is Fuchsian, which in turn implies the existence of a unique solution \( \mathcal{U} \) with the boundary condition \( \mathcal{U} \to 0 \) (and \( \partial_t \mathcal{U} \to 0 \)) as \( t \to 0 \).

However, a slight modification of the Einstein-Euler equations is needed for this purpose; cf. [3]. The problem concerns the symmetry of the metric tensor \( g_{ab} \). Obviously, for an initial value problem, the issue of symmetry does not arise: The Einstein evolution equations propagate the symmetry of the metric (and the second fundamental form). In the present context, however, the solution \( \mathcal{U} \) of the Fuchsian system we intend to obtain, is not determined by its initial value but by requiring the boundary condition \( \mathcal{U} \to 0 \) (\( t \to 0 \)). It is thus not clear, a priori, whether the tensor \( g_{ab} \) of (22a) will be a symmetric tensor, since we do not have control over the properties of \( \gamma^a_{b} \). This is turn suggests that special care is needed in connection with the curvature terms in the Einstein equations: In (4) and (5) we replace the curvature terms \( R^a_{b} \) and \( R \) by the corresponding terms that are definined from the symmetric part of the metric.
Having prepared the Einstein-Euler equations in this way we can insert the ansatz (22) into (4) and (7) without difficulty, and we obtain the reduced system, a system of equations for the variables \( \mathcal{U} \). (By construction, this system is well-defined irrespective of the symmetry properties of the variables \( \mathcal{U} \). We will return to the question of symmetry of the unique solution we subsequently construct in a separate subsection.)

The reduced system reads

\[
\begin{align*}
    t \partial_t \gamma^a_b + \alpha_\gamma \gamma^a_b &= 2 t \left( \gamma^c_a 0_{\sigma c} - 0_{\sigma a} \gamma^c_b \right) + 2 t^2 (1+3w)/3(1+w)^{-\alpha_\gamma} \kappa \\
    t \partial_t \bar{\gamma}^a_b + \bar{\alpha}_\gamma \bar{\gamma}^a_b &= 2 t \left( \bar{\gamma}^c_a 0_{\sigma c} - 0_{\sigma a} \bar{\gamma}^c_b \right) - 2 t^2 (1+3w)/3(1+w)^{-\bar{\alpha}_\gamma} \kappa \\
    t \partial \kappa + (1 + \alpha_\kappa) \kappa &= \frac{1}{2} (1 + 3w) t^{\alpha_\kappa - \alpha_\nu} \nu = 2 t^{1-\alpha_\kappa - \alpha_\nu} 0_{\sigma \kappa} 0_{\sigma \nu} + \frac{4}{3(1+w)} \left[ 1 + \frac{4w}{3(1+w)} t \right] \kappa \\
    &\quad + (1 + w)^2 t^4 (1+w)^{-\alpha_\kappa} 0_{\sigma \kappa} 0_{\sigma \nu} + (1 + w)^2 t^4 (1+w)^{-\alpha_\kappa} 0_{\sigma \kappa} 0_{\sigma \nu} \kappa^2 \\
    &\quad + (1 + w)^2 t^4 (1+w)^{-\alpha_\kappa} 0_{\sigma \kappa} 0_{\sigma \nu} \kappa \nu ,
\end{align*}
\]

For brevity we do not write out explicitly the terms on the right hand sides in equations (23c) and (23f). The expressions in question are straightforwardly obtained by inserting the ansatz (22) into the r.h.s. of (7).
6 Fuchsian analysis

6.1 The reduced system as a Fuchsian system

In this section we perform the proof of Theorem 4.1. The first part of the proof is to show that there exist appropriate choices of the constants \((\alpha_\gamma, \bar{\alpha}_\gamma, \alpha_\kappa, \alpha_\nu, \alpha_v, \alpha_\lambda, \bar{\alpha}_\lambda)\) such that the reduced system (23) becomes a Fuchsian system.

The first step is to consider the l.h.s. of (23). As a prerequisite, the l.h.s. must be of the form \(t \partial_t U + \mathcal{A} U\), where \(U\) denotes the collection of variables, i.e., \(U = (\gamma^a_b, \bar{\gamma}^a_b, \kappa, \zeta^a_b, \nu, v, \lambda^a_{bc}, \bar{\lambda}^a_{bc})\). This requires setting \(\alpha_\kappa = \alpha_\nu =: \beta\).

To establish the conditions on the coefficient matrix \(\mathcal{A}\), we show that \(\mathcal{A}\) is the direct sum of a positive definite matrix and the zero matrix, provided that the constants \((\alpha_\gamma, \bar{\alpha}_\gamma, \alpha_\kappa =: \beta, \alpha_\nu =: \beta, \alpha_v, \alpha_\lambda, \bar{\alpha}_\lambda)\) are chosen appropriately.

Obviously, the only non-diagonal part of \(\mathcal{A}\), associated with the variables \((\kappa, \nu)\), corresponds to the submatrix

\[
\begin{pmatrix}
1 + \beta & -\frac{1}{3}(1 + 3w) \\
-\frac{1}{3}(1 + w)^{-1} & 2 - \frac{1}{3}(1 + w)^{-1} + \beta
\end{pmatrix}.
\]

The eigenvalues of this matrix are \(\beta\) and \(3 - \frac{1}{3}(1 + w)^{-1} + \beta\), which are positive if and only if \(\beta > 0\). Therefore, the conditions ensuring positive definiteness of the submatrix associated with \((\gamma^a_b, \bar{\gamma}^a_b, \kappa, \zeta^a_b, \nu, v, \lambda^a_{bc}, \bar{\lambda}^a_{bc})\) are

\[
a)\ \alpha_\gamma > 0 \quad b)\ \bar{\alpha}_\gamma > 0 \quad c)\ \alpha_\kappa = \alpha_\nu = \beta > 0 \quad d)\ \alpha_\zeta < 1 + \frac{2}{3(1 + w)} \quad e)\ \alpha_\nu > 0. \quad (24)
\]

In the second step we turn to the r.h.s. of (23), which is of the form \(\mathcal{F} + \mathcal{G} \partial_t U\), where \(\mathcal{F}\) depends on \(t\), the spatial variables, and \(U\) and its first spatial derivatives, and \(\mathcal{G} = \mathcal{G}(t, x, U)\). We need to show that \(\mathcal{F}\) and \(\mathcal{G} \in O(t^\delta)\) for some positive number \(\delta\), if the constants \((\alpha_\gamma, \bar{\alpha}_\gamma, \alpha_\kappa =: \beta, \alpha_\nu =: \beta, \alpha_v, \alpha_\lambda, \bar{\alpha}_\lambda)\) are chosen appropriately. We use the notation \(f(t, x) \in O(g(t))\) to compare the asymptotic behavior of a function \(f(t, x)\) and a function \(g(t)\) near \(t = 0\), if \(f(t, x) = O(g(t))\) as \(t \to 0\) uniformly on compacts subsets of the spatial variables \(x\).

Equations (23a) and (23b) imply the conditions

\[
a)\ \alpha_\gamma + \alpha_\zeta < \frac{2(1 + 3w)}{3(1 + w)} \quad b)\ \bar{\alpha}_\gamma + \alpha_\zeta < \frac{2(1 + 3w)}{3(1 + w)}. \quad (25)
\]

To treat (23c) we use that \(w^2\) differs from \(\theta u^2\) by a function of time that goes to zero as \(t \to 0\) faster than \(\theta u^2\) itself, cf. (22b), (22f), and (24b, e). The leading order is thus obtained by replacing \(w^2\) by \(\theta u^2\) in (23c). We get the conditions

\[
a)\ \alpha_\zeta + \beta < 1 - \frac{2}{1 + w} \quad b)\ 2\alpha_\zeta + \beta < 2 - \frac{4}{3(1 + w)} \quad c)\ \beta < 2 - \frac{4}{1 + w}. \quad (26)
\]

Analogously, (23d) leads to

\[
a)\ \alpha_\zeta + \beta > -(1 - \frac{2}{1 + w}) = \frac{1 - w}{1 + w} \quad b)\ \alpha_\zeta > -(2 - \frac{4}{1 + w}) = 2 \frac{1 - w}{1 + w}; \quad (27)
\]

the estimate of the curvature term in (23d) is not included in these conditions; we refer to Corollary 6.4.

The analysis of (23e) requires a thorough investigation of the terms on the r.h.s. of (7a). In this process, we may replace, in each term, the variables \((g_{ab}, g^{ab}, tr k, \sigma^a_b, \mu, u_v)\) by the solution of the asymptotic
system \( \left( 0_{ab}, 0_{ab, b}, \text{tr} \, 0_{b}, 0_{ab, a}, 0_{ab, b}, 0_{ab} \right) \). To see that this replacement is possible, consider, e.g., the term \( \mu u^a \partial_t u^a \) of (7a). Inserting (22) the term \( \mu u^a \partial_t u^a \) becomes

\[
\left( 0_\mu + t^{\beta - 4/3(1 + w)} \gamma \right) \left( 0_{ab} + t^{\bar{\gamma}_c \bar{\gamma}_c} 0_{bc} \right) 0_{ub} + t^{2w/(1 + w) + \alpha_v \nu} \nu_b \left( \partial_t 0_{ua} + t^{2w/(1 + w) + \alpha_v - 1} \text{const} \, v_a + t \partial_t v_a \right).
\]

Because of (24e), the power \( t^{2w/(1 + w) + \alpha_v} \) goes to zero as \( t \to 0 \) faster than \( 0_{ua} \); the same is true for \( t^{2w/(1 + w) + \alpha_v - 1} \) in comparison with \( \partial_t 0_{ua} \) and \( t^{\beta - 4/3(1 + w)} \gamma \) in comparison with \( 0_\mu \). We conclude that the term \( t^{1 + 4/3(1 + w) - \beta} \mu u^a \partial_t u^a \) of (23e) is of the required form \( \mathcal{F} + \mathcal{G} \, t \partial_t \mathcal{U} \), where \( \mathcal{F} \) and \( \mathcal{G} \) converge to zero as \( O(t^\delta) \) for some positive power \( \delta \). If \( t^{1 + 4/3(1 + w) - \beta} \mu u^a \partial_t u^a \in O(t^\delta) \),

We find that the term on the r.h.s. of (23e) with the slowest convergence is \( t^{1 + 4/3(1 + w) - \beta} \sqrt{1 + u^2} \nabla_u u^a \). This term thus yields the most restrictive condition on \( \beta \), which reads

\[
\beta < 1 - \frac{2}{1 + w} = \frac{w - 1}{w + 1}.
\]

Finally, in order for the r.h.s. of (23f) to be of the required form, we must impose the additional conditions

\[
a) \quad \alpha_v < 1 + \frac{2}{3(1 + w)} + \beta = \frac{3w + 5}{3(1 + w)} + \beta \quad \quad b) \quad \alpha_v < 3 - \frac{10}{3(1 + w)} = \frac{9w - 1}{3(1 + w)}.
\]

Equations (23g) and (23h) yield \( \alpha_\lambda > 0 \) and \( \tilde{\alpha_\lambda} > 0 \).

In the remainder of this subsection we consider the most complicated expressions of the r.h.s. of (23), which are the curvature terms in (23d). The estimates of the curvature terms are presented in a succession of lemmas. Throughout we use a coordinate frame to perform the computations.

**Lemma 6.1.**

\[
0_{g_{ab}}, 0_{g_{ab, c}}, 0_{g_{ab, cd}} \in O(t^{\frac{4}{3(1 + w)}}), \quad 0_{g_{ab}}, 0_{g_{ab, c}} \in O(t^{\frac{4}{3(1 + w)}}).
\]

**Proof.** The statements follow directly from the series expansions equations (19c), (19f) and the smoothness of the initial data.

\[
0^a R_{ab} \in O(1).
\]

\[
0^b R_{ab} \in O(1).
\]

**Proof.** In a coordinate frame we have

\[
0^2 \Gamma^a_{bc} = \frac{1}{t^{\frac{4}{3(1 + w)}}} \left( 0_{g_{bc, e}} + 0_{g_{dc, b}} - 0_{g_{bc, d}} \right) \in O(1),
\]

\[
0^2 \Gamma^a_{bc, d} = \frac{1}{t^{\frac{4}{3(1 + w)}}} \left( 0_{g_{ac, d}} + 0_{g_{bc, d}} - 0_{g_{ac, d}} \right) \in O(1),
\]

which gives \( 0^a R_{ab} = 0^a \Gamma^c_{ab, c} - 0^a \Gamma^c_{ac, b} + 0^a \Gamma^c_{dc, b} \Gamma^d_{ab} - 0^a \Gamma^c_{db} 0^d \Gamma^c_{ac} \in O(1) \).

**Lemma 6.3.** If \( \alpha_\gamma > \alpha_\lambda > 0 \), and \( \bar{\alpha}_\gamma > \bar{\alpha}_\lambda > 0 \) then

\[
S^a R_{ab} \in O(1).
\]

**Proof.** Assume \( \alpha_\gamma > \alpha_\lambda > 0 \), and \( \bar{\alpha}_\gamma > \bar{\alpha}_\lambda > 0 \). We have

\[
S^a g_{ab} = 0^a g_{ab} + \frac{1}{2} \alpha_\gamma \left( 0_{g_{ac, c}} g_{ab} + 0_{g_{bc, a}} g_{ab} \right) \in O(0 g_{ab}) \subseteq O(t^{\frac{4}{3(1 + w)}}),
\]

\[
S^a g_{ab} = 0^a g_{ab} + t^{\bar{\alpha}_\gamma} \left( 0^a g_{ab} + 0^a g_{ab} \right) \in O(0 g_{ab}) \subseteq O(t^{\frac{4}{3(1 + w)}}),
\]
\[ S_{gab,c} = \frac{\partial}{\partial t} g_{ab,c} = \frac{1}{2} t^{\alpha}(0 g_{ab,cd} \gamma^d_c + 0 g_{ab,c} \gamma^a_d) + \frac{1}{2} t^{\alpha}(0 g_{ab} \lambda^d c + 0 g_{ab} \lambda^c d) \in O(t^{\frac{4}{3(1+w)}}), \]

\[ S_{gab} = \frac{\partial}{\partial t} g_{ab} = \frac{1}{2} t^{\alpha}(0 g_{abcd} \gamma^c_b + 0 g_{bc,cd} \gamma^c_a) + \frac{1}{2} t^{\alpha}(0 g_{ac} \lambda^d c + 0 g_{ac} \lambda^c d) \in O(t^{\frac{4}{3(1+w)}}), \]

\[ S_{gab,cd} = \frac{\partial}{\partial t} g_{ab,cd} = \frac{1}{2} t^{\alpha}(0 g_{ae,cd} \gamma^c_b + 0 g_{ae,cd} \gamma^c_a) + \frac{1}{2} t^{\alpha}(0 g_{ae} \lambda^d c + 0 g_{ae} \lambda^c d) \in O(t^{\frac{4}{3(1+w)}}). \]

The result then follows as in the proof of Lemma 6.2. \( \square \)

**Corollary 6.4.** The curvature term \( t^{4/3(1+w)}+\alpha S^{R^a_b} - \frac{1}{3} R^a_b \) on the r.h.s. of equation (23d) vanishes with \( t \) if \( \alpha \) is any positive constant and the conditions of Lemma 6.3 are met.

It remains to collect conditions (24)-(29) and the result of Corollary 6.4: The reduced system (23) is a Fuchsian system of the form (21) that satisfies the requirements on the matrix and the functions if the constants \( (\alpha, \beta, \alpha_\mid = \beta), \alpha, \alpha_\mid, \alpha_\mid, \alpha, \alpha, \alpha \) are chosen according to the conditions

\[
0 < \alpha_\mid < \alpha, \quad 0 < \bar{\alpha}_\mid < \bar{\alpha}, \quad \alpha + \alpha_\mid < 2 - \frac{4}{3(1+w)}, \quad \bar{\alpha} + \alpha_\mid < 2 - \frac{4}{3(1+w)}, \quad \beta < 1 - \frac{2}{1+w}, \quad \alpha_\mid + \beta < 1 - \frac{2}{1+w}, \quad \alpha_\mid < 1 + \frac{2}{3(1+w)} + \beta. \tag{33}
\]

Under these conditions, there exists, for each choice of solution \( \mathcal{W} = (g_{ab}, \kappa, \sigma_{ab}, \mu, u_\alpha) \) of the asymptotic system, a unique solution \( \mathcal{U} \) of the reduced system that vanishes as \( t \to 0 \).

To obtain the statement (20) of Theorem 4.1 we set

\[
\alpha = \bar{\alpha} = 2 - \frac{4}{3(1+w)} - 2\varepsilon, \quad \alpha_\mid = \alpha_\mid = \beta = 1 - \frac{2}{1+w} - \varepsilon, \quad \alpha_\mid = \varepsilon/2, \quad \alpha_\mid = 2 - \frac{4}{3(1+w)} - 2\varepsilon \tag{34}
\]

which is compatible with (33) for arbitrarily small \( \varepsilon > 0 \), and insert these constants into (22).

### 6.2 Symmetry of the solution of the Fuchsian system

In the second part of the proof of Theorem 4.1 we need to come back to the problem of symmetry: Using the reduced system and its unique solutions that vanish as \( t \to 0 \), we have constructed solutions \( \mathcal{W} = (g_{ab}, \kappa, \sigma_{ab}, \mu, u_\alpha) \) of the Einstein-Euler evolution equations (4) and (7) with modified curvature terms, \( S^{R^a_b} \) and \( S^R \), cf. the remarks following (22). A priori, it is not obvious that the constructed tensors \( g_{ab} \) and \( \sigma_{ab} \) are symmetric tensors; however, if they are, then the solutions \( \mathcal{W} \) we have obtained are solutions of the proper Einstein-Euler evolution equations. (The Einstein-Euler constraint equations are analyzed in the subsequent subsection.)

Let \( \mathcal{W} = (g_{ab}, \kappa, \sigma_{ab}, \mu, u_\alpha) \) be a solution of the asymptotic system as given by (14). Consider the unique solution \( \mathcal{U} \) of the reduced system (23) that vanishes with \( t \) and define, through (22), the quantities \( W = (g_{ab}, \kappa, \sigma_{ab}, \mu, u_\alpha) \). By construction, \( \mathcal{W} \) satisfies the Einstein-Euler evolution equations with modified curvature terms. These equations imply equations for the antisymmetric parts of \( g_{ab}, \sigma_{ab}, \) and \( \sigma_{ab} \). We have

\[
\partial_t g_{[ab]} = 2\sigma_{[ab]} - \frac{2}{3}(tr k)g_{[ab]}, \tag{35a}
\]

\[
\partial_t g^{[ab]} = -2\sigma^{[a}c g^{c]b} + \frac{2}{3}(tr k)g^{[ab]}, \tag{35b}
\]

\[
\partial_t \sigma_{[ab]} = \frac{1}{3}(tr k)\sigma_{[ab]} + 2\sigma_{[ac}\sigma^c_{b]} + \frac{1}{3}(w + 1)\mu u_2 g_{[ab]} + \frac{1}{3} R g_{[ab]}. \tag{35c}
\]
The terms $\sigma^{[a} g^{bc]}$ and $\sigma^{[a} \sigma^{bc]}$ in (35b) and (35c) can be expanded by splitting both $\sigma_{ab}$ and $g^{ab}$ into its symmetric and antisymmetric part. Each term in this expansion then contains at least one of the anti-symmetric tensors $\sigma_{[ab]}$ or $g_{[ab]}$. We have

\begin{align}
\sigma^{[a} g^{bc]} &= g^{[ac]} \sigma_{cd} g^{[db]} + g^{[ac]} \sigma_{cd} g^{[db]} + g^{[ac]} \sigma_{cd} g^{[db]} + g^{[ac]} g^{[bd]} \sigma_{cd}, \quad (35d) \\
\sigma^{[a} \sigma^{bc]} &= \sigma_{[ac]} g^{[cd]} \sigma_{[db]} + \sigma_{[ac]} g^{[cd]} \sigma_{[db]} + \sigma_{[ac]} g^{[cd]} \sigma_{[db]} + \sigma_{[ac]} \sigma_{[bd]} g^{[cd].} \quad (35e)
\end{align}

Let $g$, $\bar{g}$, and $\sigma$ be constants and define the antisymmetric tensors $\Omega_{ab}^g := t \bar{g} g_{[ab]}$, $\Omega_{ab}^\sigma := t \sigma g_{[ab]}$, and $\Omega_{ab}^\sigma := t \sigma^2 \sigma_{[ab]}$. From (35) we obtain a system of equations for these tensors,

\begin{align}
t \partial_k \Omega_{ab}^g + (\bar{g} - \frac{4}{3} \Omega_{ab}^\sigma) \Omega_{ab}^g &= 2t^{1+g-\sigma} \Omega_{ab}^g - \frac{4}{3(1+w)} [1 + \frac{1+w}{2} t \text{tr} k] \Omega_{ab}^g, \quad (36a) \\
t \partial_k \Omega_{ab}^\sigma + (\bar{g} - \frac{4}{3} \Omega_{ab}^\sigma) \Omega_{ab}^\sigma &= -2t^{1+g-\sigma} \Omega_{ab}^\sigma + \frac{4}{3(1+w)} [1 + \frac{1+w}{2} t \text{tr} k] \Omega_{ab}^\sigma, \quad (36b) \\
t \partial_k \Omega_{ab}^\sigma + (-\sigma + \frac{4}{3} \Omega_{ab}^\sigma) \Omega_{ab}^\sigma &= 2t^{1+\sigma} \Omega_{ab}^\sigma + \frac{4}{3(1+w)} [1 + \frac{1+w}{2} t \text{tr} k] \Omega_{ab}^\sigma + \frac{4}{3(1+w)} t^{1+\sigma} \Omega_{ab}^\sigma, \quad (36c)
\end{align}

where we use (35d) and (35e) to express $t^{1+g} \sigma_{[a} g^{[bc]}$ and $t^{1+\sigma} \sigma_{[a} \sigma^{bc]}$ in terms of $\Omega_{ab}^g$, $\Omega_{ab}^\sigma$, and $\Omega_{ab}^\sigma$, according to

\begin{align}
t^{1+\bar{g}} \sigma_{[a} g^{[bc]} &= t g^{[ac]} \sigma_{cd} \Omega_{[db]}^{cd} + \sigma_{[ac]} g^{[cd]} \Omega_{[db]}^{cd} + t^{1-\sigma-\bar{g}} \Omega_{[ac]}^{cd} \Omega_{[db]}^{cd} + t^{1+\bar{g}} \sigma_{[a} (g^{[bc]} \Omega_{cd}^{cd}), \quad (36d) \\
t^{1+\sigma} \sigma_{[a} \sigma^{bc]} &= t g^{[ac]} \sigma_{cd} \Omega_{[db]}^{cd} + t^{1-\sigma-\bar{g}} \Omega_{[ac]}^{cd} \Omega_{[db]}^{cd} + t^{1+\bar{g}} \sigma_{[a} \sigma^{bc]} \Omega_{cd}^{cd} + t^{1+\bar{g}+\sigma} \sigma_{[a} \sigma^{bc]} \Omega_{cd}^{cd}. \quad (36c)
\end{align}

By an appropriate choice of the constants $g$, $\bar{g}$, and $\sigma$ we are able to achieve that the system (36) is of the Fuchsian form (21), i.e., (i) the factors in front of $\Omega_{ab}^g$, $\Omega_{ab}^\sigma$, and $\Omega_{ab}^\sigma$ on the l.h.s. sides of (36) are positive, and (ii) the factors of $\Omega_{ab}^g$, $\Omega_{ab}^\sigma$, and $\Omega_{ab}^\sigma$ on the r.h.s. side vanish like a power of $t$ with $t \to 0$. (As we have established, the leading asymptotic behavior as $t \to 0$ of the terms on the r.h.s. of (36) is given by replacing $g_{[ab]}$ by $\bar{g} g_{[ab]}$ and using (19), and analogously for the remaining expressions.) From the l.h.s. of (36) we have

\begin{align}
g < -\frac{4}{3(1+w)}, \quad \bar{g} < \frac{4}{3(1+w)}, \quad \sigma < \frac{2}{3(1+w)}. \quad (37a)
\end{align}

From the r.h.s. of equations (36a)–(36c) we obtain the conditions

\begin{align}
1 + g - \sigma > 0, \quad 1 - \sigma - \bar{g} > 0, \quad 1 + \bar{g} - \sigma - \frac{8}{3(1+w)} > 0, \quad (37b) \\
1 + \sigma - \bar{g} - \frac{4}{3(1+w)} > 0, \quad 1 + \sigma - g - \frac{4}{3(1+w)} > 0. \quad (37c)
\end{align}

The third set of conditions we impose on the constants $g$, $\bar{g}$, and $\sigma$ is that $\Omega_{ab}^g$, $\Omega_{ab}^\sigma$, and $\Omega_{ab}^\sigma$ vanish in the limit $t \to 0$. E.g., from (19) and (20) we have

\begin{align}
\Omega_{ab}^g = t^{g+\alpha}, \quad \Omega_{[a\sigma]}^{\sigma b} \in O(t^{g+2-2\varepsilon}),
\end{align}

while it turns out that $\Omega_{ab}^\sigma \in O(t^{g+\frac{8}{3(1+w)}-2\varepsilon})$ and $\Omega_{ab}^\sigma \in O(t^{\sigma+1-\varepsilon/2})$. This gives

\begin{align}
g + 2 - 2\varepsilon > 0, \quad \bar{g} + 2 - \frac{8}{3(1+w)} - 2\varepsilon > 0, \quad \sigma + 1 - \varepsilon > 0. \quad (37d)
\end{align}
The positive constant $\varepsilon$ can always be chosen small enough so that it is possible to satisfy the collection of conditions (37) for an arbitrary $w > 1$. An explicit choice is

\[ g = -\frac{5 + 3w}{3(1 + w)}, \quad \bar{g} = \frac{1 - w}{1 + w}, \quad \sigma = -\frac{1 + 3w}{6(1 + w)} \]

Since (36) is a Fuchsian system of equations for the variables $\Omega^a_{\bar{g}}, \tilde{\Omega}_g^{ab}, \tilde{\Omega}^a_{ab}$, the solution $\Omega^a_{\bar{g}} = t^{\tilde{g}} g^{[ab]}, \tilde{\Omega}^a_{ab} = t^{\tilde{g}} g^{[ab]}$, where $\tilde{g} = t^2 g^{[ab]}$, $\tilde{\Omega}^a_{ab} = t^2 \sigma_{[ab]}$ we have in our hands, is the unique solution of (36) that vanishes as $t \to 0$. However, the trivial solution is an explicit solution of (36) with the obvious property that it vanishes in the limit $t \to 0$. The uniqueness result thus ensures that $\Omega^a_{\bar{g}} = t^{\tilde{g}} g^{[ab]} \equiv 0, \tilde{\Omega}^a_{ab} = t^2 \sigma_{[ab]} \equiv 0$, and $\Omega^a_{ab} = t^2 \sigma_{[ab]} \equiv 0$. We thus obtain symmetry of the metric and the extrinsic curvature, which are thus solution of the proper Einstein-Euler evolution equations.

Remark. The proof of the symmetry of the constructed solution that is given here is inspired by the considerations of [3] for the stiff fluid case, i.e., $w = 1$. However, the relevant arguments in [3] are incorrect, because the equations in [3], which are a variant of (35), lack a term that is quadratic in the extrinsic curvature. This problem can apparently be fixed, see [4]; we note, however, that the arguments we follow here for $w > 1$ fail for $w = 1$. The conditions (37) lead to a contradiction for $w = 1$, because in that case we find

\[ \bar{g} - \sigma > -(1 - \frac{8}{3(1 + w)}) = \frac{1}{3}, \quad \bar{g} - \sigma < 1 - \frac{4}{3(1 + w)} = \frac{1}{3}. \]

In addition, the first term on the r.h.s. of (36d) and (36e) behaves like $t^{(w-1)/(1+w)}$; this function vanishes for $t \to 0$ in the case of an ultrastiff fluid, i.e., $w > 1$, but does not vanish when $w = 1$.

6.3 The constraints

To conclude the proof of Theorem 4.1 we consider the Einstein-Euler constraints (5). We need to show that the constructed solutions $\mathcal{W} = \{(g_{ab}, \mathfrak{tr} k, \sigma^a_{ab}, \mu, u_a)\}$ of the Einstein-Euler evolution equations satisfy the Einstein-Euler constraints (5). If this is the case, then the solutions $\mathcal{W}$ we have obtained are solutions of the Einstein-Euler equations (4)–(7) and Theorem 4.1 is proved.

Consider $\mathcal{W} = \{(g_{ab}, \mathfrak{tr} k, \sigma^a_{ab}, \mu, u_a)\}$ and let

\[ \Phi = -\sigma^a_{ab} \sigma^b_{a} + \frac{3}{8} (\mathfrak{tr} k)^2 + R - 2 \rho, \quad \Psi_a = -\nabla^b \sigma^b_{a} - \frac{3}{8} \nabla_a \mathfrak{tr} k - j_a, \]

where $\rho$ and $j_a$ are given by (6). The Einstein-Euler evolution equations imply equations for these constraint quantities. We have

\[ \partial_t \Phi = 2 \mathfrak{tr} k \Phi + \nabla^a \Psi_a, \quad \partial_t \Psi_a = \mathfrak{tr} k \Psi_a - \frac{1}{2} \nabla_a \Phi. \]

Let $\varphi$ and $\psi$ be constants and define $\Omega^\varphi = t^{\varphi} \Phi$ and $\Omega^\psi = t^{\psi} \Psi_a$. From (39) we obtain the system

\begin{align}
\partial_t \Omega^\varphi + \frac{4}{1+w} (\varphi - \varphi) \Omega^\psi &= \frac{4}{1+w} \left[ 1 + \frac{1+w}{2} \mathfrak{tr} k \right] \Omega^\varphi + t^{1+\varphi-\psi} \nabla^a \Omega^\varphi, \\
\partial_t \Omega^\psi + \frac{2}{1+w} (\varphi - \psi) \Omega^\psi &= \frac{2}{1+w} \left[ 1 + \frac{1+w}{2} \mathfrak{tr} k \right] \Omega^\psi - \frac{1}{2} t^{1+\varphi-\psi} \nabla_a \Omega^\varphi,
\end{align}

where $\nabla_a \Omega^\varphi$ can be expressed as

\[ \nabla^a \Omega^\varphi = (\partial_a g_{ab}) \Omega^\psi_b + g^{ab} \partial_a \Omega^\psi_b + \frac{1}{2} g^{ad} b^e (\partial_e g_{bc}) \Omega^\psi_d. \]

It is straightforward to show that the system (40) is Fuchsian when the constants $\varphi$ and $\psi$ satisfy a number of conditions. (These conditions follow from (40) in the usual manner; in (41) we use the constructed
behavior of the metric and its spatial derivatives, see (20).) In particular, there exist appropriate choices of \( \varphi \) and \( \psi \) that are compatible with the additional conditions that \( \Omega^\Phi = t^\varphi \Phi \) and \( \Omega^\Psi = t^\psi \Psi \) vanish in the limit \( t \to 0 \). To obtain these conditions we note that \(-\sigma^a_b \sigma^b_a + \frac{2}{3} (\text{tr } k^2)^2 - 2\mu = 0 \), cf. (10), which results in

\[
\Omega^\Phi = t^\varphi \left( -\sigma^a_b \sigma^b_a + \frac{2}{3} (\text{tr } k^2)^2 + R - 2\rho \right)
= t^\varphi \left(-2 t^{1-4/(1+w)-\varepsilon/2} 0_{a}^{b} \sigma^b_a + \frac{4}{3} t^{2-10/(1+w)-\varepsilon} (\text{tr } 0_k) \kappa + 0 R - 2 t^{1-10/(1+w)-\varepsilon} - 2(1+w)^0 \mu 0_{a}^{2} + \text{higher order in } t \right),
\]

where “higher order in \( t \)” denotes terms that converge to zero faster than at least one of the terms that are given explicitly. From (19) we see that it is the curvature term that determines the leading behavior of \( \Omega^\Phi \) in \( t \). We obtain \( \Omega^\Phi \in O(t^{\varphi-4/(1+w)}) \). Analogously, we have

\[
\Omega^\Psi_a = t^\psi \left(-\nabla_b \sigma^b_a - \frac{2}{3} \nabla_a \text{tr } k - j_a \right)
= t^\psi \left( - (\nabla_b - 0_{a}^{b} \nabla^b_a - t^{1-4/(1+w)-\varepsilon/2} 0_{a}^{b} \nabla^b_a - \frac{2}{3} t^{2-10/(1+w)-\varepsilon} \nabla_a \kappa
- (1+w)^0 \mu (\sqrt{1+w^2} - 1) 0_{a}^{b} - (1+w) t^{4-10/(1+w)-2\varepsilon} 0_{a}^{b} \sqrt{1+w^2} \nu_a
- (1+w) t^{1-10/(1+w)-\varepsilon} \mu 0_{a}^{2} + \text{higher order in } t \right). \]

The leading order comes from the term \( 0_{a}^{b} \nabla^b_a \). We thus obtain \( \Omega^\Psi_a \in O(t^{\psi+(3w-1)/(1+w)-\varepsilon/2}) \). The required conditions are satisfied if the constants \( \varphi \) and \( \psi \) obey the inequalities

\[
\frac{4}{3(1+w)} < \varphi < \frac{4}{1+w} = \frac{12}{3(1+w)}, \quad -\left(1 - \frac{4}{3(1+w)} - \frac{\varepsilon}{2}\right) < \psi < \frac{2}{1+w} = \frac{6}{3(1+w)}, \quad (42a)
1 + \varphi - \psi - \frac{4}{3(1+w)} > 0, \quad 1 + \psi - \varphi > 0. \quad (42b)
\]

Assuming that \( \varepsilon \) is small enough (which we can always do) we are able to ensure that the required conditions are satisfied; an explicit choice of constants is \( \varphi = \psi = 5/3(1+w) \). Then (40) is a Fuchsian system of equations for the variables \( \Omega^\Phi, \Omega^\Psi_a \), and \( \Omega^\Psi = t^\varphi \Phi \) and \( \Omega^\Psi_a = t^\psi \Psi_a \) represent a solution of that system that vanishes as \( t \to 0 \). However, since the trivial solution is an explicit solution of (40), the uniqueness result guarantees that \( \Omega^\Phi = t^\varphi \Phi \equiv 0 \) and \( \Omega^\Psi = t^\psi \Psi_a \equiv 0 \), i.e., the constraints \( \Phi \equiv 0 \) and \( \Psi_a \equiv 0 \) of the Einstein-Euler system are identically satisfied.

The solution \( W = (g_{ab}, \text{tr } k, \sigma^a_b, \mu, u_a) \) of (20) is thus a solution of the Einstein-Euler equations (4)–(7) and Theorem 4.1 is proved.

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