Electrical conductivity in the early universe

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Abstract

We solve numerically the Boltzmann equation in the early universe in the presence of a constant electric field and find the electrical conductivity $\sigma$ in the range $1 \text{ MeV} \lesssim T \lesssim M_W$. The main contribution to $\sigma$ is shown to be due to leptonic interactions. For $T \lesssim 100 \text{ MeV}$ we find $\sigma \simeq 0.76T$ while at $T \simeq M_W$ we obtain $\sigma \simeq 6.7T$.

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Recently, there has been much interest in primordial magnetic fields. Primordial fields might act as the seed field for the dynamo mechanism which may be responsible for the observed galactic magnetic fields [1]. These have been measured both in the Milky Way and in other spiral galaxies, including their halos (which perhaps can be viewed as an indication of the primordial origin of the seed field). Typically the strength of the observed field is of the order of $O(10^{-6})$ G.

It has been suggested that primordial magnetic fields might be generated as a consequence of the early cosmic phase transitions, such as the electroweak or the QCD phase transitions [2]. These would then be small scale random fields. They would be imprinted on the comoving plasma and dissipate very slowly [3] because the plasma of the early universe is known to have a very large electrical conductivity [4]. Indeed, on dimensional grounds alone one can argue that conductivity scales as $\sigma \sim T/\alpha$, where $\alpha$ is the coupling constant squared associated with the scattering processes of the charged particles in the plasma. As a consequence, the magnetic Reynolds number of the universe, given by $R \sim v\sigma H^{-1}$ where $H \equiv \dot{R}/R \sim T^2/M_{Pl}$ is the Hubble parameter and $v$ the bulk velocity, is huge. Despite its smallness, diffusion can play an important role in the subsequent evolution of the microscopic random magnetic field. The main issue here is just how the small scale field, the origin of which is microphysics, gets amplified to a large scale seed field. In a recent paper Baym, Bödeker and McLerran [5] have suggested that in a first order electroweak phase transition, turbulence generated by the colliding shock fronts of the phase transition bubbles could give rise to large fields. More generally, a recent numerical simulation [6] of the full magnetohydrodynamical equations in an expanding universe shows clearly a transfer of energy from small scales to large scales, albeit in 2 dimensions. Because of the very large Reynolds number, real numerical simulations are not presently realistic. However, in 1+3 dimensions one may apply the so-called shell (or cascade) model, which has proved very succesful in studies of hydrodynamical turbulence, to the early universe, with the result that the scale of the magnetic field fluctuations indeed increases very rapidly [3].

The previous estimates for the conductivity $\sigma$ have, however, been rough order of magnitude estimates only. The proper way to derive electrical conductivity of the early universe is to consider plasma in an electric field, and then use the Boltzmann equations to find the bulk motion, which serves to define conductivity through Ohm’s law. To be able to do this, one first has to compute the collision integral, which is the weighted sum of the matrix elements of all the relevant processes leading to microscopic diffusion of the bulk motion. In the present paper we compute the collision integrals for leptons and hadrons (quarks) separately and solve the Boltzmann equations in the approximation where bulk velocity is much smaller than the thermal velocity of the plasma particles. This means that we treat the effect of the electric field as a
perturbation in an otherwise homogeneous and isotropic background.

The Boltzmann equation for the distribution function \( f(x(\lambda), p(\lambda)) \) of a charged particle in an electromagnetic field is found by requiring that along the world line, parametrized by \( \lambda \), the total change in \( f \) equals the collision integral. Using the equation of motion of a point particle with electric charge \( q \), which in curved space reads

\[
\frac{dp^\mu}{d\lambda} = -\Gamma_{\alpha\beta}^\mu p^\alpha p^\beta + qF_{\beta}^\mu p^\beta ,
\]

one can cast the Boltzmann equation in the form

\[
\frac{\partial f}{\partial t} p^0 + \mathbf{p} \cdot \nabla f - \frac{\partial f}{\partial p^0} p^2 R \dot{R} - 2\frac{\dot{R}}{R} p^\beta \frac{\partial f}{\partial p^\beta} - q \frac{\partial f}{\partial p^\beta} \mathbf{E} \cdot \mathbf{p} - q \frac{\partial f}{\partial p^0} \mathbf{E}^i = C(p, t)p^0 ,
\]

where \( C(p, t) \) is the collision integral. Here we have assumed a Robertson-Walker metric with a scale factor \( R \) for the background, and we have defined \( F_i^0 = -E_i \) and \( F_i^j = \varepsilon^{ijk}B_k \). We also prefer to use co-moving coordinates and define

\[
\tilde{\dot{f}}(p, t) = \int \delta(p_0 - (p^2 R^2 + m^2)^{1/2}) f(p, p_0, t) dp_0 .
\]

Inserting this into Eq. (2) and integrating we find, making use of the local momentum defined as \( \tilde{\mathbf{p}} = R\mathbf{p} \),

\[
\frac{\partial \tilde{\dot{f}}}{\partial t} = \frac{\dot{R}}{R^3} \frac{\partial \tilde{\dot{f}}}{\partial p^0} - q \frac{\partial \tilde{\dot{f}}}{\partial p^0} \mathbf{E}^i = C(\tilde{\mathbf{p}}, t) .
\]

Because of the assumed isotropy of the background, the terms \( \mathbf{p} \cdot \nabla f \) and \( \mathbf{E} \cdot \mathbf{p} \frac{\partial f}{\partial p_0} \) have been dropped. This is justified if one assumes, as we do, that the effect of the electromagnetic field on the distributions can be considered as a perturbation. From now on, we also drop the tildes for brevity.

The collision integral encompasses all the scattering processes relevant for the diffusion of the charged test particle. For electromagnetism the leading processes are \( 2 \to 2 \) reactions (we do not consider thermal effects such as the plasmon decay). Thus

\[
C(p, t) = \frac{1}{p_0} \int dP_b dP_c dP_d (2\pi)^4 \delta^4(p + p_b - p_c - p_d) |M(\text{Ab} \to \text{cd})|^2 \mathcal{F} ,
\]

where \( p \) is the four-momentum of the charged test particle \( A \), and we have defined \( dP_i \equiv d^3p_i/(2\pi)^32E(p_i) \). The factor \( \mathcal{F} \) contains the distribution functions of the initial and final states. For fermions one must include the Pauli blocking factors and for bosons the enhancement factors. Below are two examples of the \( \mathcal{F} \)-factor. The first is for the case when all the particles in the reaction are fermions, and in the second case particles \( b \) and \( c \) are bosons, such as is the case for Compton scattering:

\[
\mathcal{F} = \left\{ \begin{array}{ll}
[1 - f(p)][1 - f_b(p_b)]f_c(p_c)f_d(p_d) - [1 - f_c(p_c)][1 - f_d(p_d)]f(p)f_b(p_b) , \\
[1 - f(p)][1 + f_b(p_b)]f_c(p_c)f_d(p_d) - [1 - f_d(p_d)][1 + f_c(p_c)]f(p)f_b(p_b) .
\end{array} \right.
\]
Above the time dependence is not shown explicitly.

Let us now assume that there exists a constant electric field in the early universe (constant in the sense that its coherence length is larger than the mean free path of charged particles) and treat its effect as a perturbation on the distribution of the test particle. We write \( f = f_0 + \delta f \), where \( f_0 \) is the equilibrium distribution and \( \delta f \) the small perturbation. Inserting this into the collision integral Eq. (5) results in

\[
C(p, t) = -\frac{1}{p_0} \int dP_b dP_c dP_d (2\pi)^4 \delta^4(p + p_b - p_c - p_d) \times |M|^2 ([f_{0c}f_{0d}[1 - f_{0b}] + f_{0b}[1 - f_{0c}][1 - f_{0d}]) \delta f \equiv \hat{C}(p) \delta f(p, t). \tag{7}
\]

Eq. (7) assumes that all the particles in the reaction are fermions; generalization to other cases is straightforward.

We have assumed that all the particles \( b, c \) and \( d \) have an equilibrium distribution. Therefore, if also the test particle has an equilibrium distribution, the collision integral vanishes. Thus only the perturbation term in Eq. (7) survives. Note that the equilibrium distribution depends only on the momentum.

Treating the electric field as a small perturbation, the Boltzmann equation can be linearized and reads

\[
\frac{\partial \delta f}{\partial t} - \frac{\hat{R}}{R} \frac{\partial \delta f}{\partial p_i} - q \frac{\partial f_0}{\partial p^i} E^i = \hat{C}(p) \delta f(p, t) \tag{8}
\]

with \( \hat{C}(p) \) defined in Eq. (7). Finally, assuming that \( E^i \gg H |p| \equiv |\mathbf{p}| \), we may search for stationary perturbations, for which one easily obtains

\[
\delta f(p) = -\frac{q}{\hat{C}(p)} \frac{\partial f_0(p)}{\partial p^i} E^i. \tag{9}
\]

The induced current density is given by

\[
\mathbf{j} = q \int \delta f(p) \mathbf{p} d^3 \mathbf{p}, \tag{10}
\]

and conductivity \( \sigma_A \), associated with a given particle species \( A \), is defined by

\[
\mathbf{j}_A = \sigma_A \mathbf{E}. \tag{11}
\]

Thus the contribution of a single species \( A \) to conductivity is \( \sigma_A \sim 1/\sum |M(AX \rightarrow Y)|^2 \), where the sum is over all the processes which scatter the test particle \( A \). For the purpose of conductivity, we may view the mixture of different particle species, such as

\footnote{Of course, we do not claim such a field actually exists but rather use it as a probe of the plasma properties.}
is found in the early universe, a multicomponent fluid. The flow of each component contributes to the total current and adds up to the total conductivity, which reads

\[ \sigma_{\text{tot}} = \sum_A \sigma_A, \]  

(12)

where the sum is over all the relativistic charged species present in the thermal bath. Note that the total conductivity is dominated by the species that has the weakest interaction. This is because the weaker the interaction, the longer time the current flow is maintained.

We consider the temperature interval \(1 \text{ MeV} \lesssim T \lesssim M_W\), and make the simple, crude assumption that particles appear in the thermal bath only when temperature is greater than their mass. Thus below 100 MeV, for example, the only charged particles present are the electrons and positrons. When \(T \gtrsim T_{\text{QCD}}\), also the quarks should be counted in. Their main interactions are strong, so that their electromagnetic interactions may be neglected. The list of the relevant reactions for the leptons is presented in Table 1, and in Table 2 the same for the quarks, together with the matrix element squared (modulo factors of \(\pi\) and \(\alpha\)). A technical point is that for the \(t\)-channel reactions there arises an infrared singularity, which is known to be regulated by thermal effects. These we approximate by giving fermions, photons and gluons a Debye mass in the \(t\)-channel and \(u\)-channel propagator; otherwise all fermions are assumed to be massless (consistency requires that the external particle masses are kept non-thermal). The thermal masses are given by

\[
\begin{align*}
    m_{l}^2 &= \frac{e^2}{8} T^2 \simeq 0.0115 T^2, \\
    m_{q}^2 &= \frac{g_s^2}{6} T^2 \simeq 0.251 T^2, \\
    m_{\gamma}^2 &= \frac{e^2}{3} (\Sigma_l Q_l^2 + 3 \Sigma_q Q_q^2) T^2 \simeq 0.0306 (\Sigma_l Q_l^2 + 3 \Sigma_q Q_q^2) T^2, \\
    m_{g}^2 &= g_s^2 (3 + \frac{N_f}{3}) T^2 \simeq 1.508 (3 + \frac{N_f}{3}) T^2.
\end{align*}
\]  

(13)

where \(N_f\) is the number of quark families present, the sums are over all particles with \(m \leq T\), and we adopt the values \(g_s^2 = 4\pi\alpha_s \simeq 1.508\) and \(e^2 = 4\pi\alpha_e \simeq 0.0917\), and \(T_{\text{QCD}} = 200\) MeV.

We have computed the perturbation \(\delta f(p)\) numerically by evaluating the collision integral by a simple Monte Carlo simulation. The result agrees with the expected naive scaling law \(\sigma \sim T\). The form of \(\delta f(p)\) is very similar for both leptons and quarks. As an example, in Fig. 1 we show \(dj/dp = qp^3 \delta f(p)\) for the electron at \(T \lesssim m_\mu\), where
the peak at \( p \approx 3T \) is evident. In Fig. 1 we also compare \( \delta f(p) \) with the equilibrium distribution \( f_0(p) \) to demonstrate the region of validity of our calculation, where we have chosen \( E = 10^{-3}T^2 \) for definiteness. For most part, \( \delta f(p)/f_0 \) is a constant, except when \( p \to 0 \) where \( \delta f(p) \) vanishes. This behaviour is due to Compton scattering (and at higher temperatures also \( l^+l^- \) annihilation), for which the integrated matrix element squared gives rise to \( p \)-dependence slower than the one inherent in \( \hat{C}(p) \). We should also point out that our result is sensitive to the infrared cut-off provided by the thermal masses of the internal propagators, given by Eq. (13), and hence not strictly valid at very small \( p \). This is because it is the t-channel reactions which give the dominant contribution to the collision integral Eq. (7), and a straightforward numerical calculation gives an inversely linear dependence on the regulator in \( \hat{C}(p) \).

With this caveat, the total leptonic conductivity, together with the total hadronic conductivity, is shown in Fig. 2 for the range \( 1 \text{ MeV} \lesssim T \lesssim M_W \), above which one would have to account for the charged W’s. However, purely electromagnetic processes continue to dominate conductivity also well beyond the Z-pole. The reader may compare the result presented in Fig. 2 with the text-book estimate \[ \sigma \approx 14T \] for relativistic electron gas scattering off heavy ions, which yields \( \sigma \approx 14T \). The steps in the leptonic \( \sigma \) reflect both the appearance of new leptons in the thermal bath, as well as change in the regulator (thermal photon mass) due to the appearance of new quarks and leptons.

Introducing a thermal mass into the propagator is only an approximation of the full thermal screening, which includes also initial state and vertex corrections. Such corrections should remove the spurious infrared singularities in \( \hat{C}(p) \). Our approxima-
Figure 1: $dj/dp$ and $\delta f/f_0$ for the electron at $T \lesssim m_\mu$. Here $E = 10^{-3}T^2$.

Figure 2: $\sigma/T$ as a function of temperature.

A reliable order-of-magnitude estimate should, however, be sufficient to produce a reliable order-of-magnitude estimate of the electrical conductivity in the very early universe.

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