A First-Order Framework for Inquisitive Modal Logic

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Abstract

We present a natural standard translation of inquisitive modal logic InqML into first-order logic over the natural two-sorted relational representations of the intended models, which captures the built-in higher-order features of InqML. This translation is based on a graded notion of flatness that ties the inherent second-order, team-semantic features of InqML over information states to subsets or tuples of bounded size. A natural notion of pseudo-models, which relaxes the non-elementary constraints on the intended models, gives rise to an elementary, purely model-theoretic proof of the compactness property for InqML.

1 Introduction

Inquisitive logics have recently been expounded systematically by Ciardelli in [2], following up on previous work with Roelofsen [6] and earlier sources especially in the work of Groenendijk cited there. The fundamental motivation is to provide logics with expressive means to deal not just with assertions but also with questions. While the general programme can be carried out systematically for various logics, like propositional logic in [6] and first-order logic in [2, 3, 9], it certainly seems particularly natural also at the level of modal logics, as outlined in [2], where Ciardelli gives a first detailed account of inquisitive modal logic in 2016. In its epistemic interpretation, for instance, modal logic offers the natural classical framework for distinctions between different states of affairs (facts, about which basic assertions can be
made) and cognitive states (information states, about which more complex assertions, e.g. concerning knowledge, can be made). This is a very natural context in which one may want to give semantics also to questions. The study of questions is more generally well motivated – also at the more foundational philosophical or linguistic level – by considerations about language and logic in all kinds of scenarios that relate facts, knowledge and information. For instance one may want to account for the conceptual difference between ‘knowing that’ and ‘knowing whether’ something is the case. And indeed, inquisitive modal logic provides connectives and modal operators that neatly capture such distinctions; in particular it also offers, right at the propositional level, a non-trivial disjunction of alternatives $p$ and $\neg p$, whose semantics captures the idea of the question ‘whether $p$’, which is suggestively denoted as $?p$. This novel formula $?p$ is meant to specify, as a kind of $p/\neg p$ alternative, information states that support one of the admissible answers, but crucially without specifying which one. The semantics for inquisitive modal logic is given in terms of satisfaction of formulae in information states (support semantics in [6,2]), i.e. in sets of possible worlds rather than in individual possible worlds. This latter feature also accounts for the conceptual links between the semantics of inquisitive logics and team semantics for logics of dependence [11,15]. Not too surprisingly maybe, the semantic modelling for such phenomena in a modal framework involves not just possible worlds and relations between them (as is the case for basic modal logic) but includes information states as primary objects, together with relationships between (sets of) possible worlds and (sets of) information states.

In this sense, the setting of inquisitive modal logic puts an extra level of (set-theoretic) complexity on top of the familiar modal modelling (cf. [1,7]). For instance, in the epistemic setting where possible worlds are associated with states of affairs: where Kripke models assign to possible worlds sets of alternative possible worlds (information states), inquisitive models assign to possible worlds sets of such possible information states (inquisitive states), which may be thought of as possible answers, or possible information updates. Correspondingly, natural relational encodings involve both the set of worlds and a set of information states, as two relevant sorts on an equal footing. The way these sorts live in a base set and its power set already suggests a degree of logical complexity that might be more of a challenge for a direct first-order account of the semantics of InqML than the well-known standard translation for basic modal logic ML. Indeed, the relational encodings of the intended inquisitive models form a non-elementary class. So the straightforward compactness argument for ML, which just applies first-order compactness after standard translation, does not work: the basic idea needs to be adapted.
From a model-theoretic perspective, translating a non-classical logic like InqML into the classical framework of first-order logic FO provides a systematic advantage. With a standard translation we can investigate InqML in the well-known FO setting and explore model-theoretic features of InqML, such as compactness, in this context.

Compared to previous translations as used in [4, 5], the one we present here combines several advantages: it puts minimal requirements on the relational encodings of the inquisitive models; it is more directly defined by natural induction on the full syntax of InqML and is consequently more intuitive. At the technical level, the main novelty is an application of a concept we call graded flatness. The same concept was previously considered in (first-order) dependence logic in [12] and in the context of inquisitive first-order logic in [8] under the name of coherence, but its application in the modal context seems to represent an innovation. While compactness for InqML is obtained as a corollary to a completeness result in [2], our new route to compactness provides the first purely model-theoretic argument.

At a conceptual level, our treatment may also suggest the relaxed class of models, which we call pseudo-models and whose relational counterparts form an elementary class, as an alternative to the intended models for InqML, whose relational counterparts form a non-elementary class. Somewhat surprisingly, pseudo-models faithfully reflect some of the most salient logical features. Analogous ideas appear to have been considered in neighbourhood semantics [14], where monotonic models do not translate into an elementary class, while general neighbourhood models do and in some connections can be used in the analysis of the former [10].

Our choice of terminology is meant to make contact with team semantic notions to reflect the close relationship between the team semantic treatment of dependence logics [15] and the setting of InqML. Sets of worlds (i.e. information states) can be seen as teams, and in our two-sorted view are treated as first-class objects (of the second sort) along with worlds (as objects of the first sort).

2 Inquisitive modal logic

By \((p_i)_{i \in I}\) we denote a set of propositional variables. Following the terminology in [6, 2] we associate the following two kinds of states with a given non-empty set \(W\) of possible worlds.

**Information states:** any subset \(s \subseteq W\) is called an information state; the set of information states over \(W\) is \(\mathcal{P}(W)\).

**Inquisitive states:** a non-empty set \(\Pi\) of information states, \(\Pi \subseteq \mathcal{P}(W)\), is an...
inquisitive state if it is closed w.r.t. set inclusion: \( s \in \Pi \) implies \( t \in \Pi \) for all \( t \subseteq s \); the set of inquisitive states over \( W \) is the set of those non-empty sets of information states in \( \mathcal{P}(\mathcal{P}(W)) \) that satisfy this characteristic downward closure condition.

**Definition 2.1** (inquisitive, modal) models.
Let \( W \) be a set of possible worlds and \( \Sigma : W \to \mathcal{P}(\mathcal{P}(W)) \setminus \{\emptyset\} \) be a function that assigns an inquisitive state \( \Sigma(w) \) to every world \( w \in W \) (an inquisitive assignment) and \( V : (p_i)_{i \in I} \to \mathcal{P}(W) \) a function that assigns a subset of \( W \) to every propositional variable (a propositional assignment). Then \( M = (W, \Sigma, V) \) is called an (inquisitive modal) model.

With a model \( M = (W, \Sigma, V) \) we associate an induced Kripke model \( K(M) = (W, \sigma, V) \), where \( \sigma : W \to \mathcal{P}(W) \) is defined as \( \sigma(w) := \bigcup \Sigma(w) \) (a modal assignment).

A state-pointed (inquisitive modal) model is a pair \( M, s \) which consists of a model \( M \) together with a distinguished information state \( s \subseteq W \). If \( s \) is a singleton information state, i.e. \( s = \{w\} \) for some \( w \in W \), we also speak of a world-pointed inquisitive model \( M, \{w\} \) for which we also write just \( M, w \).

Note that the associated Kripke structure reduces the inquisitive assignment (of inquisitive states \( \Sigma(w) \in \mathcal{P}(\mathcal{P}(W)) \)) to an assignment of single information states \( \sigma(w) = \bigcup \Sigma(w) \in \mathcal{P}(W) \) that can be cast as sets of successors w.r.t. a modal accessibility relation. The natural relational encoding of \( \sigma \) is in terms of the accessibility relation
\[
R = \{(w, w') : w' \in \sigma(w)\} \subseteq W \times W,
\]
so that \( \sigma(w) \) becomes the set of immediate successors of \( w \) w.r.t. \( R \), \( \sigma(w) = R[w] = \{w' : (w, w') \in R\} \). A corresponding, natural relational encoding of the inquisitive assignment will have to resort to a two-sorted encoding with a second sort of information states (from \( \mathcal{P}(W) \)) besides the first sort \( W \) of worlds (see Section 3.1 below). In this two-sorted scenario, however, the characteristic downward closure condition on the inquisitive states \( \Sigma(w) \) would remain non-elementary. This motivates the following relaxation of the notion of models to what we call pseudo-models, which may also be cast in the model-theoretic tradition of approximate or weak models that reduce the complexity of higher-order features, similar to, e.g. the use of weak models in topological model theory [16]. As we shall see below, this concept can serve here as a useful tool for the analysis of the intended, proper models.

Compared to Definition 2.1, the following definition of pseudo-models just waives the downward closure requirement on inquisitive assignments.
Definition 2.2 (pseudo-models and inquisitive closure).
A pseudo-(inquisitive modal) model is a structure \( M = (W, \Sigma, V) \) over the set of possible worlds \( W \) with propositional assignment \( V : (p_i)_{i \in I} \rightarrow \mathcal{P}(W) \) and a function \( \Sigma : W \rightarrow \mathcal{P}(\mathcal{P}(W)) \setminus \{\emptyset\} \), which assigns a non-empty subset \( \Sigma(w) \subseteq \mathcal{P}(W) \) but not necessarily an inquisitive state to every world \( w \in W \).

With any pseudo-model \( M = (W, \Sigma, V) \) we associate its inquisitive closure \( M \downarrow := (W, \Sigma \downarrow, V) \), which is the proper model whose inquisitive assignment \( \Sigma \downarrow \) is induced by \( \Sigma \) according to

\[
\Sigma \downarrow : W \rightarrow \mathcal{P}(\mathcal{P}(W)),
\]

\[
w \mapsto \Sigma \downarrow(w) := \{ t \in \mathcal{P}(W) : t \subseteq s \text{ for some } s \in \Sigma(w) \}.
\]

We note that the distinction between a pseudo-model and its inquisitive closure is immaterial at the level of the associated Kripke models with their modal assignment \( \sigma(w) = \bigcup \Sigma(w) = \bigcup \Sigma \downarrow(w) \).

Definition 2.3 (InqML: syntax).
The basic syntax of InqML is given by the grammar

\[\varphi ::= p | \bot | (\varphi \land \varphi) | (\varphi \rightarrow \varphi) | (\varphi \lor \varphi) | \Box \varphi | \lozenge \varphi,\]

with negation, disjunction and diamond \( \lozenge \) treated as abbreviations according to

\[\neg \varphi := \varphi \rightarrow \bot, \varphi \lor \psi := \neg(\neg \varphi \land \neg \psi), \text{ and } \lozenge \varphi := \neg \Box \neg \varphi.\]

In [2] the symbol \( \lor \) is called intuitionistic disjunction, \( \Box \) the universal modality and \( \lozenge \) the inquisitive modality.

The following extends the standard definition of the semantics of InqML from [6, 2] to pseudo-models in a straightforward manner.

Definition 2.4 (InqML: semantics).
Let \( M = (W, \Sigma, V) \) be a model or a pseudo-model, \( s \subseteq W \) an information state. The semantics of InqML is defined as follows.

- \( M, s \models p : \iff s \subseteq V(p) \)
- \( M, s \models \bot : \iff s = \emptyset \)
- \( M, s \models \varphi \land \psi : \iff M, s \models \varphi \text{ and } M, s \models \psi \)
- \( M, s \models \varphi \rightarrow \psi : \iff \forall t \subseteq s : M, t \models \varphi \Rightarrow M, t \models \psi \)
- \( M, s \models \varphi \lor \psi : \iff M, s \models \varphi \text{ or } M, s \models \psi \)
- \( M, s \models \Box \varphi : \iff \forall w \in s : M, \sigma(w) \models \varphi \)
We note that the semantic clause for implication in Definition 2.4 refers to all subsets $t \subseteq s$, over models or pseudo-models alike. Similarly, the downward closure condition known as persistency, as discussed in the following observation, speaks about all subsets $t \subseteq s$ of the given information state $s$, also when interpreted in the non-standard setting of pseudo-models.

**Observation 2.5.** Over all models as well as pseudo-models $M$, InqML has the following properties, for all $\varphi \in \text{InqML}$:

(i) $M, s \models \varphi$ implies $M, t \models \varphi$ for all $t \subseteq s$;

(ii) $M, \emptyset \models \varphi$.

Property (i) is called persistency in [6, 2] (and usually referred to as downward closure in team semantic terminology), while (ii) is called semantic ex-falso (reflecting the empty team property).

The following gives a further indication that extension of the semantics of InqML beyond the intended inquisitive models is very natural. The proof is by straightforward syntactic induction following the clauses of Definition 2.4; for the $\Box$-case one uses persistency, and for the $\square$-case, which refers to the associated Kripke structure, one uses the fact that the associated modal assignment $\sigma$ is the same for the pseudo-model and its inquisitive closure.

**Proposition 2.6.** Let $M$ be a pseudo-model, $M \downarrow$ its inquisitive closure, and $s \subseteq W$ any information state over their common universe $W$ of possible worlds. Then for $\varphi \in \text{InqML}$ we have

$$M \downarrow, s \models \varphi \iff M, s \models \varphi.$$ 

This indicates that, as far as e.g. deductive reasoning is concerned, InqML might as well be cast in the extended setting of pseudo-models. The difference is important, though, e.g. in issues concerning the natural habitat for the key notion of model equivalence, viz. inquisitive bisimulation equivalence [4], or how InqML embeds into classical logics of reference like FO.

### 3 Standard translation

A standard translation serves as a semantically adequate (one could say, truthful) translation between logical frameworks.

Recall the well-known standard translation of modal logic ML into first-order logic FO. It is based on the straightforward transcription of the clauses for the Kripke semantics of ML into their natural first-order rendering over
Kripke models viewed as ordinary relational first-order structures. The situation for InqML is different, because inquisitive (pseudo-)models are naturally rendered as two-sorted rather than ordinary single-sorted relational structures. Since an inquisitive assignment is a function from the set $W$ of possible worlds to sets of sets of possible worlds, it is of inherently higher type than a modal assignment. Its natural relational encoding consists of a binary relation not over $W$ itself, but between $W$ (as a first sort) and a set $S$ of information states (as a second sort) where $S \subseteq \mathcal{P}(W)$. In the following we discuss a setting and format for a standard translation of InqML into FO in the natural two-sorted relational framework that is similar in spirit to that in [4] but more liberal and more uniform. The technical novelty underpinning this new approach is the application of a graded notion of flatness, graded flatness, to InqML, as independently developed by the first author in [13].

We want to associate the semantics of inquisitive modal logic InqML over models (or even pseudo-models) with the semantics of first-order logic over associated two-sorted relational (pseudo-)models. As usual, this task involves two translation levels that need to go hand in hand: transformations linking the underlying (pseudo-)models $M$ to relational representations $\mathcal{M}$, from which the underlying (pseudo-)models $M$ can be recovered as $M = \text{M}(\mathcal{M})$; and a translation of formulae $\varphi \in \text{InqML}$ into formulae $\varphi^* \in \text{FO}$ such that

$$\text{M}(\mathcal{M}), s \models \varphi \iff \mathcal{M}, s \models \varphi^*.$$ 

### 3.1 Relational representations of models

As relational counterparts of inquisitive (pseudo-)models we consider two-sorted relational structures of the form

$$\mathcal{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$$

with some non-empty sets $W$ and $S$ as first and second sorts, linked by two mixed-sorted binary relations $\epsilon, E \subseteq W \times S$, and with unary predicates $P_i \subseteq W$ over the first sort for all $p_i, i \in I$, that encode the propositional assignment as usual. The intended rôles of $\epsilon$ and $E$ are as follows: $\epsilon \subseteq W \times S$ encodes membership of possible worlds in information states, so that $s \in S$ can be associated with $s := \{w \in W: (w, s) \in \epsilon\} \in \mathcal{P}(W)$; and $E \subseteq W \times S$ encodes the inquisitive assignment as a relation that associates the set $E[w] := \{s \in S: (w, s) \in E\} \in \mathcal{P}(S)$ with each world $w \in W$.\footnote{This aspect of two-sortedness is similar to the treatment of neighbourhood models [14], but InqML modelling imposes a different and in some sense tighter link between sorts.}
Definition 3.1 (relational (pseudo-)models).
A structure $\mathcal{M}$ of the type above is a relational (inquisitive modal) model if the following conditions are satisfied for all $w \in W$, $s, t \in S$ and $a \subseteq S$:

(i) $s = t \iff \underline{s} = \underline{t}$ (extensionality);

(ii) $E[w] \neq \emptyset$ (non-emptiness);

(iii) if $a \subseteq \underline{s}$ for $s \in E[w]$, then $a = \underline{t}$ for some $t \in E[w]$ (downward closure).

Correspondingly, a structure $\mathcal{M} = (W, S, \epsilon, E, (P_i)_{i \in I})$ of the same format is a relational (inquisitive modal) pseudo-model if it satisfies (i) and (ii).

Relational (pseudo-)models with distinguished states $s \in S$ are described as state-pointed or, in the case of singleton states $\underline{s} = \{w\}$ as world-pointed, in analogy with the terminology for models. Due to extensionality (i), we shall identify states $s \in S$ with sets of worlds $\underline{s} \subseteq W$ and regard the second sort $S$ as a subset of the power set $\mathcal{P}(W)$, with $\epsilon$ as the actual membership relation between $W$ and $S \subseteq \mathcal{P}(W)$.

Observation 3.2. The class $\mathcal{C}$ of all state-pointed (respectively world-pointed) relational pseudo-models is $\Delta$-elementary, i.e. there exists a set of formulae $\Phi \subseteq \text{FO}$ such that $\mathcal{C} = \text{Mod}(\Phi)^2$.

It is fairly easy to see that the class of all relational models cannot be $\Delta$-elementary, as downward closure (condition (iii) in Definition 3.1) cannot be expressed without reference to arbitrary subsets if the first sort. Indeed, if it was $\Delta$-elementary then so would be the class of all full relational models, defined by the additional condition that $S = \mathcal{P}(W)$ be the full power set. That, however, is ruled out by the observation that FO does not satisfy compactness over this class of all full relational models: over that class, FO captures the full power of monadic second-order logic MSO over the first sort; so it can, e.g., define the class corresponding to Kripke models that satisfy the well-foundedness condition of L"ob frames (cf. [4]).

From relational (pseudo-)models to (pseudo-)models.
With any relational (pseudo-)model $\mathcal{M} = (W, \epsilon, E, (P_i)_{i \in I})$ we associate the (pseudo-)model $\mathbb{M}(\mathcal{M}) = (W, \Sigma, V)$ that decodes the relational information in $\mathcal{M}$ into functional assignments according to

\[ \Sigma: w \mapsto \{ s \in S: (w, s) \in E \}, \]
\[ V: p_i \mapsto \{ w \in W: w \in P_i \}. \]

\textsuperscript{2}If the set of propositions is finite, then $\mathcal{C}$ is even elementary, i.e. definable by a single FO-formula.
We observe that the actual extension of the second sort \( S \subseteq \mathcal{P}(W) \) in \( \mathcal{M} \) is immaterial in as far as it may go beyond the range of \( \Sigma \).

**From (pseudo-)models to relational (pseudo-)models.**

With a (pseudo-)model \( \mathcal{M} = (W, \Sigma, V) \) and a distinguished state \( s \subseteq W \), we associate as a relational representation any relational (pseudo-)model \( \mathcal{N} = (W, S, \epsilon, E, (P_i)_{i \in I}) \) that encodes \( \Sigma \) and \( V \) over sorts \( W \) and \( S \), where \( S \subseteq \mathcal{P}(W) \) is rich enough to represent the image \( \Sigma(w) \subseteq \mathcal{P}(W) \) for all \( w \in W \) as well as the distinguished state \( s \):

\[
S \subseteq \mathcal{P}(W) \text{ with } S \supseteq \{s\} \cup \bigcup_{w \in W} \Sigma(w),
\]

\[
\epsilon := \{(w, s) \in W \times S : w \in s\} = \in\restriction(W \times S),
\]

\[
E := \{(w, s) \in W \times S : s \in \Sigma(w)\},
\]

\[
P_i := \{w \in W : w \in V(p_i)\} \text{ for } i \in I.
\]

Note that \( \mathcal{M} \) is fully determined by \( \mathcal{M} \) once the actual extension of the second sort \( S \) is fixed; that however is naturally only subject to a richness condition. (We argued above that insistence on fullness, i.e. on the maximal extension \( S = \mathcal{P}(W) \), may not be advisable.)

**Definition 3.3** (relational representations).

A relational (pseudo-)model \( \mathcal{N} \) is a relational representation of a given (pseudo-)model \( \mathcal{M} \) precisely for \( \mathcal{M} = \mathcal{M}(\mathcal{N}) \).

A state-pointed relational (pseudo-)model \( \mathcal{N}, s \) is a relational representation of the state-pointed (pseudo-)model \( \mathcal{M}, s \) if in addition the distinguished state \( s \) is represented as an element of its second sort \( S \).

It is clear from the above that every state-pointed (pseudo-)model admits relational representations, and that every relational (pseudo-)model \( \mathcal{N} \) represents a unique (pseudo-)model, viz. \( \mathcal{M}(\mathcal{N}) \).

**3.2 Graded flatness and the standard translation**

Compared to the well-known standard translation for plain modal logic over Kripke models (for which \( \mathcal{M} \) and \( \mathcal{N} \) are practically identical), InqML involves challenges associated with the semantics of implication and \( \Box \). The corresponding clauses in Definition 2.4 involve reference to information states that might not necessarily be directly available in the second sort of \( \mathcal{N} \).

There are different suggestions to overcome this problem. A costly elimination of \( \Box \) is possible via so-called resolutions \[2\] or via the \( \Box \)-free characteristic formulae for finitary bisimulation classes; then a straightforward
translation can be given, which may be based on stronger closure conditions on the universe of information states in the relational encodings of the inquisitive models which need to give FO free access to the relevant information states, cf. [4]. Such stronger closure conditions may further interfere with compactness over the required classes of models, as mentioned in § 3.1. A straightforward translation without elimination of □ is also possible [5], but again requires even stronger closure conditions on the class of relational models.

Compared to [4], the present proposal is more general, more uniform and more direct. It relies on the following concept of graded flatness, which had also been investigated in the team-semantic context of dependence logic (cf. [12]) and in the context of inquisitive first-order logic (cf. [8]) under the name of coherence. For our context it was independently (re-)discovered and put to this new use in [13]. Our preferred terminology of graded flatness derives from the notion of flatness in team semantics. If we think of information states (sets of worlds) as teams, then a formula ϕ ∈ InqML would be flat (in the team semantic sense) if its truth in s is equivalent to truth in {w} for all w ∈ s. Graded flatness generalises this idea to quantitative bounds on the size of subsets s′ ⊆ s that need to be investigated, rather than singleton subsets. Such a size bound can be obtained as a syntactic parameter as follows.

**Definition 3.4 (flatness grade).**
The flatness grade ♭(ϕ) ∈ ℤ of ϕ ∈ InqML is defined by syntactic induction, for all ϕ, χ ∈ InqML, according to

- ♭(ϕ) := 0 for atomic ϕ and all ϕ of the form □ψ or ⊕ψ;
- ♭(ϕ ∧ χ) := max{♭(ψ), ♭(χ)};
- ♭(ϕ → χ) := ♭(χ);
- ♭(ϕ ∀ χ) := ♭(ψ) + ♭(χ) + 1.

**Proposition 3.5 (graded flatness).**
Inquisitive modal logic InqML satisfies the following graded flatness property. For all ϕ ∈ InqML and state-pointed (pseudo-)models M, s:

\[ M, s \models ϕ \iff M, t \models ϕ \text{ for all } t \subseteq s \text{ of size } |t| \leq ♭(ϕ) + 1. \]

**Proof.** The direction from left to right follows immediately from persistency for InqML. The implication from right to left is shown by syntactic induction on ϕ. We illustrate the ∀-case, which is the most interesting.
For \( \phi = \psi \lor \chi \) let \( m := b(\psi) \) and \( n := b(\chi) \) so that \( b(\phi) = m + n + 1 \), and assume that

\[ \mathcal{M}, t \models \psi \lor \chi \]

for all \( t \subseteq s \) of size \( |t| \leq m + n + 2 \). By Definition 2.4 we get that, for all such \( t \), \( \mathcal{M}, t \models \psi \) or \( \mathcal{M}, t \models \chi \), which, together with the induction hypothesis, further implies that, for all \( t \subseteq s \) with \( |t| \leq m + n + 2 \):

\[ \forall a \subseteq t \text{ with } |a| \leq m + 1 : \mathcal{M}, a \models \psi \]

or

\[ \forall a \subseteq t \text{ with } |a| \leq n + 1 : \mathcal{M}, a \models \chi. \]

It follows that

\[ \forall t \subseteq s \text{ with } |t| \leq m + 1 : \mathcal{M}, t \models \psi \]

or

\[ \forall t \subseteq s \text{ with } |t| \leq n + 1 : \mathcal{M}, t \models \chi. \]

Indeed, if this were false, there would exist information states \( t_1 \subseteq s \) with \( |t_1| \leq m + 1 \) and \( t_2 \subseteq s \) with \( |t_2| \leq n + 1 \) such that \( \mathcal{M}, t_1 \not\models \psi \) and \( \mathcal{M}, t_2 \not\models \chi \). But then, for \( t_0 := t_1 \cup t_2 \), the previous statement would be false: we have \( |t_0| \leq m + n + 2 \) but \( t_1 \subseteq t_0 \) violates the first disjunct and \( t_2 \subseteq t_0 \) violates the second disjunct.

Finally, by the induction hypothesis, \( \mathcal{M}, s \models \psi \) or \( \mathcal{M}, s \models \chi \), whence \( \mathcal{M}, s \models \psi \lor \chi \).

We use the following notation. Generally, we take variable symbols \( x, y, \ldots \) to be interpreted over the first sort (worlds), and variable symbols \( \lambda, \mu, \ldots \) over the second sort (information states). A tuple of length \( n \) is denoted as \( x = (x_1, \ldots, x_n) \), the set of its components as \( \{x\} = \{x_1, \ldots, x_n\} \). If the length of a tuple is determined by some flatness grade \( b(\phi) \), we write \( x_\phi \) for the tuple \( (x_1, \ldots, x_{b(\phi)+1}) \) and \( \{x_\phi\} \) for the associated set.

Our standard translation \( \varphi \mapsto \varphi^*(\lambda) \) is defined below, by syntactic induction on \( \varphi \in \text{InqML} \). The interesting, somewhat non-standard feature involves the necessary passage between sorts: while \( \varphi \) is translated into the first-order formula \( \varphi^*(\lambda) \) in a free variable \( \lambda \) of the second sort, the core induction deals with auxiliary formulae \( ST[\varphi](x) \) in tuples \( x \) of free variables of the first sort that capture the semantics of \( \psi^*(\mu) \) for \( \mu = \{x\} \).

**Definition 3.6** (standard translation).

For \( \varphi \in \text{InqML} \) define its standard translation \( \varphi^*(\lambda) \in \text{FO} \) in one free state variable \( \lambda \) as

\[
\varphi^*(\lambda) := \forall x_{\varphi} \left( \bigwedge_{k=1}^{b(\varphi)+1} x_k \in \lambda \rightarrow ST[\varphi](x_{\varphi}) \right),
\]

where the auxiliary first-order formulae \( ST[\varphi](x) \), with free world variables among \( x \), are defined by syntactic induction according to:
Proposition 3.7 (InqML as a fragment of FO).

Let $\phi \in \text{InqML}$ and let $\phi^*(\lambda) \in \text{FO}$ be its standard translation. Let $M, s$ be a (pseudo-)model and $\mathcal{M}, s$ be a relational representation of $M, s$. Then

$$M, s \models \phi \iff \mathcal{M}, s \models \phi^*(\lambda).$$

Proof. We show below that, for $\phi \in \text{InqML},$

$$M, \{w\} \models \phi \iff \mathcal{M}, w \models ST[\phi](x)$$

for all finite tuples of worlds $w$ from $W$ and matching tuples $x$ of variables.

From this we obtain the claim $M, s \models \phi \iff \mathcal{M}, s \models \phi^*(\lambda)$ of the proposition as follows. By persistency and graded flatness, $M, s \models \phi$ is equivalent to

$$\forall t \subseteq s \text{ with } |t| \leq b(\phi) + 1 : M, t \models \phi,$$

which is further equivalent to

$$\forall w_\phi \in s : M, \{w_\phi\} \models \phi.$$
(Note that the set of all tuples \( w_\varphi \in s \) does not contain the empty tuple; but we get the empty information state as a subset of \( s \) in the first expression; this is no problem for the equivalence, since semantic ex-falso gives the empty state for free.) With \((*)\) we find that \( M, s \models \varphi \) is equivalent to

\[
\forall w_\varphi \in s : M, w_\varphi \models ST[\varphi](x_\varphi),
\]

which translates equivalently into

\[
M, s \models \forall x_\varphi (\bigwedge_{k=1}^{b(\varphi)+1} x_k \in \lambda \rightarrow ST[\varphi](x_\varphi)),
\]

which is the same as \( M, s \models \varphi^*(\lambda) \), by Definition 3.6.

It remains to show \((*)\) by syntactic induction. We treat the \( \Box \)-step, and show the implication from left to right, as an example. Let \( \varphi = \Box \psi \) and \( M, \{w\} \models \Box \psi \). Then by Definition 2.4

\[
\forall w \in \{w\} \forall \sigma(w) : M, \sigma(w) \models \psi
\]

and by Proposition 3.5 we get

\[
\forall w \in \{w\} \forall t \subseteq \sigma(w) \text{ with } |t| \leq b(\psi) + 1 : M, t \models \psi.
\]

This is equivalent to

\[
\forall w \in \{w\} \forall u_\psi \in \sigma(w) : M, {u_\psi} \models \psi.
\]

By induction hypothesis we get

\[
\forall w \in \{w\} \forall u_\psi \in \sigma(w) : M, u_\psi \models ST[\psi](y_\psi),
\]

which is expressible in FO according to

\[
M, w \models \bigwedge_{k \leq n} (\forall y_\psi \forall \mu_\psi ((\bigwedge_{l=1}^{Ew_k \mu_l \land y_l \in \mu_l}) \rightarrow ST[\psi](y_\psi))
\]

which according to Definition 3.6 is just \( M, w \models ST[\Box \psi](x) \).

\[ \square \]

4 Compactness for InqML

It is known that InqML has a sound and strongly complete proof calculus and therefore satisfies compactness (cf. [2]). We use our standard translation to give a new, purely model-theoretic proof, essentially by reduction
to first-order compactness. But while the corresponding reduction is totally straightforward for basic modal logic ML over Kripke structures, we here need to deal with the additional complication that the class of relational models is not ∆-elementary. Correspondingly, a detour through pseudo-models plays an essential role in our proof. Moreover, the proof also shows that compactness over the class of all pseudo-models works in a straightforward manner. We interpret this as an additional, natural indication that InqML could also be explored over the extended class of all pseudo-models.

We consider the satisfiability version of compactness. Of course, by semantic ex-falso (cf. Observation 2.5), any set Φ ⊆ InqML is trivially satisfied by any model M, ∅. For a non-trivial statement we need to exclude the empty state.

**Proposition 4.1 (compactness).**
InqML satisfies compactness, i.e. a set of formulae Φ ⊆ InqML is satisfiable by some non-empty state of some (pseudo-)model if, and only if, every finite subset of Φ is satisfiable by some non-empty state of some (pseudo-)model.

**Proof.** Let Φ ⊆ InqML be a set of formulae such that any finite subset Φ₀ ⊆ Φ is satisfiable by a state-pointed model M₀, s₀ such that s₀ ≠ ∅. We want to show that Φ is satisfiable as well.

Let Φ̂(λ) ⊆ FO (respectively Φ̂₀(λ) ⊆ FO) be the set of all standard translated formulae of Φ (respectively Φ₀) and let for each Φ₀ ⊆ Φ the relational model M₀, s₀ be a relational representation of M₀, s₀. Then Proposition 3.7 yields M₀, s₀ | |= Φ₀ for all finite Φ₀ ⊆ Φ̂₀.

We let ∆ = ∆(λ) ⊆ FO be a set of formulae defining the class of all state-pointed relational pseudo-models (see Observation 3.2) with non-empty state (λ ≠ ∅) and let ̂Φ := Φ̂ ∪ ∆. Since M₀, s₀ | |= ∆ for all finite Φ₀ ⊆ Φ, any finite subset of ̂Φ is satisfiable. Hence by compactness of FO, ̂Φ is satisfiable by some relational pseudo-model M, s.

Then Proposition 3.7 yields M, s | |= ̂Φ for M, s := M(M), s and Proposition 2.6 entails M ↓, s | |= ̂Φ for the inquisitive closure M ↓, s of M, s. □

With persistency it is easy to see that compactness over the class of all state-pointed (pseudo-)models implies compactness over the class of all world-pointed (pseudo-)models.

**5 Conclusion**

Our new standard translation combines several advantages: it is defined directly and inductively, and it works with minimal requirements on the
corresponding class of relational models. The basic idea of the standard translation and its use in a model-theoretic compactness proof requires a non-trivial adaptation of the well-known treatment of basic modal logic ML, due to the inherent two-sortedness of the relational representations of the intended models for inquisitive modal logic InqML; the notion of graded flatness plays a key rôle in taming the salient second-order features. Other than for ML, the relational counterparts of the intended models for InqML do not form an elementary class. Nevertheless, with our standard translation we could give a purely model-theoretic compactness proof for InqML over the class of all relational inquisitive models, as well as over the class of all relational pseudo-models. Our findings may suggest the class of pseudo-models as a suitable alternative for the usual basic class of models of InqML in other contexts too.

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