Dynamical torsion and torsion potential

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Abstract

We introduce a generalized tetrad which plays the role of a potential for torsion and makes torsion dynamic. Starting from the Einstein-Cartan action with torsion, we get two field equations, the Einstein equation and the torsion field equation by using the metric tensor and the torsion potential as independent variables; in the former equation the torsion potential plays the role of a matter field. We also discuss properties of local linear transformations of the torsion potential and give a simple example in which the torsion potential is described by a scalar field.
1. Introduction

The gauge theory of gravitational field was first proposed by Utiyama [1] and later developed by Kibble [2] and Sciama [3]. The Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity uses the Einstein-Cartan Lagrangian $\sqrt{-g}R$, and it does not require torsion to vanish. Rather, the torsion is treated as an independent variable along with the metric. The torsion is not dynamic in the ECSK theory, however, being algebraically determined by the local spin density: Namely, the torsion is always a pointwise function of the source field, and it is usually called frozen torsion.

If we are to consider “dynamical torsion” — torsion that can propagate in vacuum — then we must depart from the ECSK theory. So far two attempts have been proposed to construct gravitational theory with dynamical torsion. One is the quadratic Lagrangian approach to Poincaré gauge theory initiated by Hayashi [4] in 1968. Further Hayashi and Shirafuji [5-7], and Hehl and Von der Heyde [8] pursued this approach. In this theory the torsion satisfies second-order differential equations and can propagate in vacuum. The other is the theory of new general relativity investigated from geometrical and observational point of view by Hayashi and Shirafuji [9]. In this theory the concept of absolute parallelism plays the fundamental role: It requires a vanishing local connection and hence a vanishing curvature, and attributes gravity to the torsion alone.

In this paper we propose a new approach to dynamical torsion. We suppose that spacetime has a locally Lorentzian metric and a nonsymmetric connection, and introduce sixteen fields forming a quadruplet of basis vectors, which are not necessarily orthogonal with each other. Next we require a symmetric condition for the local connection with respect to this basis. Then we find that torsion can be expressed in terms of first-order derivatives of the sixteen fields: Accordingly, we call the sixteen fields “torsion potential”. The curvature is not vanishing in the present theory in contrast with new general relativity based on absolute parallelism. Starting from the Einstein-Cartan action with torsion, but regarding the metric tensor and the torsion potential as independent variables, we obtain the Einstein equation and the torsion field equation.

In Sec.2 we show how to introduce the torsion potential and how to represent torsion and connection by it. In Sec.3 we derive field equations for the metric tensor and the torsion potential. In Sec.4 we discuss local linear transformations of the
torsion potential, which differ from local Lorentz transformations. In Sec. 5 we give a simple example in which the torsion potential is constructed from a scalar field. In the final section we summarize our results.

2. Torsion potential

Let us suppose that spacetime has a locally Lorentzian metric $g$ and a nonsymmetric connection $\Gamma$. We write the connection coefficients $\Gamma^\mu_{\rho\sigma}$ as

$$\Gamma^\mu_{\rho\sigma} = \overset{\circ}{\Gamma}^\mu_{\rho\sigma} + S^\mu_{\rho\sigma}, \quad (2.1)$$

where $\overset{\circ}{\Gamma}^\mu_{\rho\sigma}$ is the torsionless connection coefficients symmetric in $\rho$ and $\sigma$, and $S^\mu_{\rho\sigma}$ is the contorsion tensor defined by the torsion tensor $T^\mu_{\rho\sigma} (:= \Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho})$ in the form

$$S_{\mu\rho\sigma} := \frac{1}{2} (T^\mu_{\rho\sigma} + T^\rho_{\mu\sigma} + T^\sigma_{\mu\rho}), \quad (2.2)$$

satisfying $S_{\mu\rho\sigma} = -S_{\rho\mu\sigma}$. When one takes variation of the Einstein-Cartan action with torsion, one usually regards the metric tensor $g_{\mu\nu}$ and the contorsion tensor $S_{\mu\nu\rho}$ (or equivalently, the torsion tensor $T_{\mu\nu\rho}$) as independent variables. This seems unnatural, however, because these two tensors are different in nature from each other: In fact, the metric tensor plays the role of a gravitational potential, while the contorsion tensor is treated as a quantity like a force. It is thus desirable to find a potential for torsion, namely a set of fields whose first-order derivatives define the torsion.

For this purpose, let us introduce a linear transformation

$$\tilde{e}_A = t^\mu_A \tilde{\partial}_\mu, \quad \theta^A = t^A_\mu dx^\mu, \quad (2.3)$$

where $\tilde{\partial}_\mu$ is the natural basis and $\tilde{e}_A$ an arbitrary one with $\theta^A$ being the 1-form dual to $\tilde{e}_A \ (A = 0 \sim 3)$. Here $t^A_\mu$ and its inverse $t^\mu_A$ are some functions of coordinates satisfying the condition that the determinant $t = \det(t^A_\mu)$ never vanishes. Obviously they satisfy

$$t^\mu_A t^A_\nu = \delta^\mu_\nu, \quad t^A_\mu t^\mu_B = \delta^A_B. \quad (2.4)$$
On this new basis the metric tensor becomes
\[ g_{AB} = \bar{e}^A \cdot \bar{e}^B = t^\mu_A t^\nu_B g_{\mu\nu}. \] (2.5)

Here we do not assume that \( g_{AB} \) coincides with the Minkowski metric \( \eta_{AB} = \text{diag}(+1, -1, -1, -1) \). Accordingly, the field \( t^A_\mu \) is not a tetrad in the ordinary sense.

Now we define connection coefficients with respect to the bases \( \partial_\mu \) and \( e^A_\mu \) as follows:
\[ \nabla \partial_\nu = \Gamma^\mu_{\nu\lambda} \partial_\lambda, \quad \nabla e^B_\mu = \theta^A_{\mu B} e^A_\mu \] (2.6)
with the connection 1-forms \( \Gamma^\mu_{\nu\lambda} \) and \( \theta^A_{\mu B} \) being
\[ \Gamma^\mu_{\nu\lambda} := \Gamma^\mu_{\nu\lambda} dx^\lambda = \Gamma^\mu_{\nu A} \theta^A, \] \[ \theta^A_{\mu B} := \theta^A_{\mu B} dx^\lambda = \theta^A_{\mu C} \theta^C, \] (2.7) (2.8)
respectively. The two connection coefficients are related to each other by
\[ \theta^A_{\mu B} = t^A_\mu t^\nu_B t^\lambda_C \Gamma^\mu_{\nu\lambda} - t^\nu_B t^\lambda_C \partial_\lambda t^A_\nu. \] (2.9)

In conformity with this, the total covariant derivative of \( t^A_\mu \) with respect to indices \( A \) and \( \mu \) is vanishing:
\[ \begin{align*}
\mathcal{D}_\lambda t^A_\nu &:= \partial_\lambda t^A_\nu - \Gamma^\mu_{\nu\lambda} t^A_\mu + \theta^A_{\mu B} t^B_\nu = 0, \\
\mathcal{D}_\lambda t^\nu_B &:= \partial_\lambda t^\nu_B + \Gamma^\mu_{\nu\lambda} t^\nu_B - \theta^A_{\mu B} t^A_\mu = 0.
\end{align*} \] (2.10)

The commutator of the basis \( e^A_\mu \) and the exterior derivative of the 1-form \( \theta^A \) are respectively given by

\[ [\bar{e}^B_\mu, \bar{e}^C_\nu] = f^A_{\mu B} \bar{e}^A_\nu, \quad d\theta^A = -\frac{1}{2} f^A_{\mu B} \theta^B \wedge \theta^C, \] (2.11)
where \( f^A_{\mu B} \), which are called the Ricci rotation coefficients in the usual tetrad case, are given by
\[ f^A_{\mu B} = (t^\mu_B t^\nu_C - t^\mu_C t^\nu_B) \partial_\nu t^A_\mu. \] (2.12)
On the basis \( e_A \), the torsion 2-form \( T^A \) and the curvature 2-form \( R^A_B \) are defined by

\[
T^A := d\theta^A + \theta^A_B \wedge \theta^B = -\frac{1}{2} T^A_{BC} \theta^B \wedge \theta^C, \tag{2.13}
\]

\[
R^A_B := d\theta^A_B + \theta^A_C \wedge \theta^C_B = \frac{1}{2} R^A_{BCD} \theta^C \wedge \theta^D, \tag{2.14}
\]

where the torsion tensor and the curvature tensor are respectively given by

\[
T^A_{BC} = t_A^{\mu} t_B^\nu t_C^\lambda \theta_{BC}^{\mu\nu\lambda} = -\frac{1}{2} T^A_{BCD} \theta^C \wedge \theta^D, \tag{2.15}
\]

\[
R^A_{BCD} = t_A^{\mu} t_B^\nu t_C^\rho t_D^\sigma \theta_{BC}^{\mu\nu\rho\sigma}. \tag{2.16}
\]

We notice that since \( g_{AB} \neq \eta_{AB} \) by assumption, \( \theta_{ABC} \neq -\theta_{BAC} \), and therefore that \( \theta_{ABC} \) is not a spin connection. In order to compare with the torsionless case and for convenience of later use, let us separate the curvature tensor \( R^\mu_{\nu\rho\sigma} \) into torsionless and torsion parts,

\[
R^\mu_{\nu\rho\sigma} = \tilde{R}^\mu_{\nu\rho\sigma} + \tilde{R}^\mu_{\nu\rho\sigma}, \tag{2.17}
\]

where \( \tilde{R}^\mu_{\nu\rho\sigma} \) is the curvature tensor for the symmetric connection,

\[
\tilde{R}^\mu_{\nu\rho\sigma} = \partial_\rho \tilde{\Gamma}^\mu_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^\mu_{\nu\rho} + \tilde{\Gamma}^\mu_{\lambda\rho} \tilde{\Gamma}^\lambda_{\nu\sigma} - \tilde{\Gamma}^\mu_{\lambda\sigma} \tilde{\Gamma}^\lambda_{\nu\rho}, \tag{2.18}
\]

and \( \tilde{R}^\mu_{\nu\rho\sigma} \) is the torsion part of the curvature tensor defined by

\[
\tilde{R}^\mu_{\nu\rho\sigma} := \tilde{\nabla}_\rho S^\mu_{\nu\sigma} - \tilde{\nabla}_\sigma S^\mu_{\nu\rho} + S^\mu_{\lambda\rho} S^\lambda_{\nu\sigma} - S^\mu_{\lambda\sigma} S^\lambda_{\nu\rho}. \tag{2.19}
\]

Here \( \tilde{\nabla}_\rho \) denotes the covariant derivative with respect to \( \tilde{\Gamma}^\mu_{\nu\rho} \).

Until now we have not yet imposed any restriction on \( t^A_\mu \) and \( \theta^A_{BC} \). From now on, however, we suppose that the following two conditions are satisfied:

(i) The metric condition

\[
\mathcal{D}_\lambda g_{\mu\nu} = 0. \tag{2.20}
\]

(ii) The symmetric condition

\[
\theta^A_{BC} = \theta^A_{CB} \quad \text{or equivalently} \quad \theta^A_B \wedge \theta^B = 0. \tag{2.21}
\]

The condition (i) implies that \( \tilde{\Gamma}^\mu_{\nu\rho} \) is nothing but the Christoffel symbol. Using the differential operators \( d_C := t^\mu_C \partial_\mu \) and \( \mathcal{D}_C := t^\mu_C \mathcal{D}_\mu \), we see from (2.5), (2.10) and (2.20) that

\[
\mathcal{D}_C g_{AB} = 0. \tag{2.22}
\]
The condition (ii) can be regarded as a generalization of absolute parallelism, because we have $\theta_{ABC} = 0$ when $g_{AB} = \eta_{AB}$. Following the procedure to obtain the Christoffel symbol $\Gamma^\rho_\mu_\sigma$ from $g_{\mu\nu}$, we can express $\theta^A_{BC}$ as

$$\theta^A_{BC} = \frac{1}{2} g^{AD} (d_B g_{CD} + d_C g_{BD} - d_D g_{BC})$$  (2.23)

by means of (2.21) and (2.22).

Like (2.1), $\theta^A_{BC}$ can be written as

$$\theta^A_{BC} = \hat{\theta}^A_{BC} + S^A_{BC},$$  (2.24)

where $\hat{\theta}^A_{BC}$ are obtained by using $\hat{\Gamma}^\mu_\rho_\sigma$ instead of $\Gamma^\mu_\rho_\sigma$ in (2.9). It should be noted that $\hat{\theta}^A_{BC} \neq \hat{\theta}^A_{CB}$ unless the torsion vanishes identically. $S^A_{BC}$ are the components of the contorsion tensor with respect to the basis $\vec{e}_A$, being related to $S^\mu_\nu_\lambda$ by

$$S^A_{BC} = t^A_\mu t^\nu_B t^\lambda_C S^\mu_\nu_\lambda.$$  (2.25)

Applying (2.21) to (2.13) and (2.15), we get

$$T^A = d\theta^A$$  and  $$T^A_{BC} = f^A_{BC}.$$  (2.26)

According to (2.12), the latter equation of (2.26) means that the torsion tensor is expressed by first-order derivatives of $t^A_\mu$. For example, the torsion tensor $T^\mu_\nu_\lambda$ is given by

$$T^\mu_\nu_\lambda = t^\mu_A (\partial_\lambda t^A_\nu - \partial_\nu t^A_\lambda).$$  (2.27)

We call the set of sixteen functions $t^A_\mu$ “the torsion potential”, since it is changed like a set of covariant vectors under coordinate transformations and its first-order derivatives define the torsion and hence the contorsion tensor.

### 3. Field equations

In this section we give field equations for the metric tensor and the torsion potential. Let us start from the Lagrangian

$$\mathcal{L}_G = -\frac{1}{2\kappa} \sqrt{-g} R = -\frac{1}{2\kappa} \sqrt{-g} \left( \tilde{R} + \tilde{R} \right),$$  (3.1)
where $\tilde{R}$ is the Riemannian scalar curvature made of the metric tensor $g_{\mu\nu}$, and $\bar{R}$ is the torsion part of the scalar curvature which contains $g_{\mu\nu}$ and $t^A_\mu$. Here $\kappa = 8\pi G$ is the Einstein gravitational constant. According to (2.19), we have

$$\bar{R} = 2 \tilde{\nabla}_\mu S^\mu_\rho - S^\mu_\rho S_\mu^\sigma - S^\mu_\rho S_\rho^\sigma S^\sigma_\mu. \quad (3.2)$$

Taking variation of (3.1) with respect to $g_{\mu\nu}$, we get

$$\tilde{\mathcal{G}}^{\mu\nu} = S^\mu_\rho S_\rho^\sigma - S^\mu_\rho S_\mu^\sigma - \frac{1}{2} g^{\mu\nu} S^\lambda_\rho S_\lambda^\sigma - \frac{1}{2} g^{\mu\nu} S^\lambda_\rho S^\rho_\sigma S^\sigma_\lambda. \quad (3.3)$$

where $\tilde{\mathcal{G}}^{\mu\nu}$ is the torsion-free Einstein tensor. The right-hand side of (3.3) is the energy-momentum tensor of the torsion potential $t^A_\mu$, which now plays the role of a matter field. If $S^\mu_\rho_\sigma$ vanishes, (3.3) returns to the vacuum Einstein equation. The contraction of (3.3) with $g_{\mu\nu}$ gives

$$\tilde{\mathcal{R}} = S^\mu_\rho S_\rho^\sigma - S^\mu_\rho S_\mu^\sigma = 2 \tilde{\nabla}_\mu S^\mu_\nu - \bar{R}, \quad \text{(3.4)}$$

where we have used (3.2) in the second equation.

Taking variation of (3.1) with respect to the torsion potential $t^A_\mu$, we get another field equation

$$S^\mu_\rho S^\rho_\sigma t^\sigma_A + S^\mu_\nu t^\rho_A S_\rho^\sigma - S^\mu_\nu S_\rho^\sigma t^\rho_A - \tilde{\nabla}_\rho S^\mu_\nu t^\nu_A + S^\mu_\nu \tilde{\nabla}_\rho t^\rho_A + \tilde{\nabla}_\nu S^\mu_\rho t^\nu_A + S^\nu_\rho S^\rho_\sigma t^\sigma_A - S^\nu_\rho \tilde{\nabla}_\nu t^\rho_A = 0. \quad (3.5)$$

The contraction of (3.3) with $t^A_\mu$ gives

$$\tilde{\nabla}_\mu S^\mu_\nu = 0. \quad (3.6)$$

Applying (3.6) to (3.4), we have

$$\bar{R} = \tilde{R} + \bar{R} = 0. \quad (3.7)$$

Further, with the help of (3.4) Einstein equation (3.3) can be rewritten as

$$\tilde{\mathcal{R}}^{\mu\nu} = S^\mu_\rho S_\rho^\sigma - S^\mu_\rho S_\sigma^\nu. \quad (3.8)$$
We remark that the equation (3.5) is a tensor equation under coordinate transformations, while it is not invariant under local linear transformations which we shall discuss in the next section. We rewrite (3.5) and (3.6) on the basis $\vec{e}_A$:

\[
\begin{align*}
\mathcal{D}_B S^{AB}_C - \mathcal{D}_C S^{AB}_B + \hat{\theta}^B_{CD} S^{AD}_B + \hat{\theta}^A_{BC} S^{BD}_D &= 0, \\
\mathcal{D}_B S^{BD}_D &= 0,
\end{align*}
\]

where $\mathcal{D}_B$ is the total covariant derivative operator using $\Gamma^{\mu\nu B}$ and $\theta^{ABC}$. The equation (3.9) is not a tensor equation because it contains connection $\theta^{ABC}$, while (3.10) is really a tensor one since its connection terms are canceled out with each other when the indices $A$ and $C$ are contracted in (3.9).

4. Local linear transformations of the torsion potential

In this section we consider properties of the torsion potential under local linear transformations. Let $L^A_B$ be some transformation functions which define

\[
\begin{align*}
\hat{t}^A_\mu &= L^A_B t^B_\mu, & \hat{t}^{BA}_{\mu} &= L^{-1}^A_B t^{A}_{\mu}.
\end{align*}
\]

The 1-form $\theta^A$ and the connection $\theta^{A}_{B}$ are then transformed like

\[
\begin{align*}
\hat{\theta}^A &= L^A_B \theta^B, \\
\hat{\theta}^A_{B} &= L^A_C L^{-1} D_B \theta^C_D + L^A_C dL^{-1} B.
\end{align*}
\]

Here $\hat{\theta}^A_{B} (= \hat{\theta}^{ABC} \hat{\theta}^C)$ is a new connection, which we require to satisfy the condition (ii); namely,

\[
\hat{\theta}^A_{BC} = \hat{\theta}^A_{CB} \quad \text{or equivalently} \quad \hat{\theta}^A_{B} \wedge \hat{\theta}^B = 0.
\]

Then we see that the transformation functions $L^A_B$ must satisfy the constraint

\[
\begin{align*}
d_B L^A_C = d_C L^A_B \quad \text{or} \quad L^B_C d_D L^{-1} A_B = L^B_D d_C L^{-1} A_B
\end{align*}
\]
with $d_B = t^\mu \partial_\mu$. Incidentally we remark that general coordinate transformations $J_\nu^\mu = (\partial x^\mu / \partial x^\nu)$ automatically satisfy the similar condition $\partial_\rho J_\nu^\mu = \partial_\mu J_\nu^\rho$. Under transformations (4.1) satisfying (4.5), the torsion and the curvature 2-forms transform as follows:

$$\hat{T}^A = L^A_B T^B, \quad (4.6)$$

$$\hat{R}^A_B = L^A_C L^{-1} D_B R^C_D. \quad (4.7)$$

Next, we consider properties of the torsion equation (3.9) under local linear transformations: The left-hand side of (3.9) is changed like

$$\hat{D}_B \hat{S}^{AC} - \hat{D}_C \hat{S}^{AB} + \hat{\theta}^{\ B}_{\ C} \hat{S}^{AD}_B + \hat{\theta}^{\ A}_{\ C} \hat{S}^{BD}_D - \hat{\theta}^{\ A}_{\ BD} \hat{S}^{BD}_C - \hat{\theta}^{\ B}_{\ CB} \hat{S}^{AD}_D$$

$$= L^A_B L^{-1} D_C \left( \hat{\theta}^{\ B}_{\ CD} \hat{S}^{BM}_D - \hat{\theta}^{\ M}_{\ DN} \hat{S}^{BN}_M - \hat{\theta}^{\ M}_{\ DM} \hat{S}^{BM}_D - \hat{\theta}^{\ B}_{\ MN} \hat{S}^{MN}_D \right)$$

$$+ d_D \left( L^A_B L^{-1} M_C \right) S^{BD}_M - d_M \left( L^A_B L^{-1} M_C \right) S^{BD}_D. \quad (4.8)$$

Thus, the torsion equation is invariant under (4.1), if and only if $L^A_B$ satisfy the constraint,

$$L^A_B d_D L^{-1} M_C = L^A_D d_B L^{-1} M_C. \quad (4.9)$$

We notice that contracting $A$ and $C$ in (4.9) gives (4.5).

In order to see the consequence of (4.9), let us consider infinitesimal transformations

$$L^A_B = \delta^A_B + \varepsilon^A_B \quad (4.10)$$

with $|\varepsilon^A_B| \ll 1$. If (4.10) satisfy the constraint (4.9), we have

$$d_B \varepsilon^A_C = 0, \quad (4.11)$$

which implies that $\varepsilon^A_B$ must be constant. Therefore we see that the invariance linear group of (3.9) is the global $GL(4, \mathbb{R})$. This situation is to be compared with that in new general relativity where the invariance linear group is the global Lorentz group.
5. Scalar model of the torsion potential

Now we consider as a simple example the torsion potential made of a scalar field. Namely, let us assume that the torsion potential takes the form

\[ t^A{}_{\mu} = \delta^A_{\mu} \varphi, \quad t_{\mu}{}^A = \delta^\mu_A \varphi^{-1} \]  

(5.1)

with \( \varphi \) being a nonvanishing scalar field. Due to Kronecker’s \( \delta^A_{\mu} \) in (5.1), the index \( A \) is assumed to undergo the transformation generated by \( L^A{}_{B} = \delta^A_{\nu} \delta^\nu_B \left( \partial x^\nu / \partial x'^\mu \right) \) whenever general coordinate transformations, \( x^\mu \rightarrow x'^\mu \), are made. The metric tensor \( g_{AB} \) is given by

\[ g_{AB} = \varphi^{-2} \delta^\mu_A \delta^\nu_B g_{\mu\nu} \]  

(5.2)

The torsion and the contorsion tensors are expressed by

\[ T_{\mu\nu\lambda} = g_{\mu\nu} \varphi^{-1} \partial_\lambda \varphi - g_{\mu\lambda} \varphi^{-1} \partial_\nu \varphi, \]  

(5.3)

\[ S^{\mu\nu}_{\lambda} = (g^{\mu\rho} \delta^\nu_{\lambda} - g^{\nu\rho} \delta^\mu_{\lambda}) \varphi^{-1} \partial_\rho \varphi. \]  

(5.4)

If we substitute \( S^{\mu\nu}_{\lambda} \) of (5.4) to the full equation (3.5), we will encounter a difficulty, because 16 equations cannot be satisfied by only one unknown function \( \varphi \). Thus, we will choose a different way: Namely, we directly substitute \( S^{\mu\nu}_{\lambda} \) to the Lagrangian (3.1). Then we have

\[ \mathcal{L}_G = -\frac{1}{2\kappa} \sqrt{-g} \left( \varphi R - 6 g^{\mu\nu} \varphi^{-2} \partial_\mu \varphi \partial_\nu \varphi \right). \]  

(5.5)

This is the Lagrangian for gravitational field coupled to a massless scalar field: In fact, if we put \( \psi = \ln \varphi \), the second term of (5.5) is just the kinetic term of the field \( \psi \).

Taking variation of (5.3) with respect to \( g_{\mu\nu} \) and \( \varphi \), we get

\[ G^{\mu\nu} = \frac{6}{\varphi^2} \left( g^{\mu\rho} g^{\nu\sigma} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \right) \partial_\rho \varphi \partial_\sigma \varphi, \]  

(5.6)

\[ \Box \varphi - \frac{1}{\varphi} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = 0, \]  

(5.7)
where \( \Box \) denotes the torsionless d’Alembertian operator defined by \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \).

From (5.6) and (5.7) it follows

\[
\hat{R} = \frac{6}{\varphi^2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi,
\]

(5.8)

\[
\Box \varphi - \frac{1}{6} \hat{R} \varphi = 0.
\]

(5.9)

If \( \hat{R} = 0 \), we get the massless scalar field equation

\[
\Box \varphi = 0.
\]

(5.10)

Incidentally we remark that substituting (5.4) to the contracted form (3.6) gives (5.9).

The form of (5.2) recalls us to conformal rescaling. Define a new “unphysical” metric \( \hat{g}_{\mu\nu} \) from the original “physical” metric by

\[
\hat{g}_{\mu\nu} = \varphi^{-2} g_{\mu\nu},
\]

(5.11)

then the Riemannian scalar curvature made of the new metric \( \hat{g}_{\mu\nu} \) is given by

\[
\hat{R} = \varphi^2 \left( \hat{R} - 6 g^{\mu\nu} \varphi^{-2} \partial_\mu \varphi \partial_\nu \varphi + 6 \Box \ln \varphi \right).
\]

(5.12)

Owing to this, the Lagrangian (5.5) can be rewritten as

\[
\mathcal{L}_G = -\frac{1}{2\kappa} \varphi^2 \sqrt{-\hat{g}} \hat{R},
\]

(5.13)

where a total divergence term is omitted. We see that the kinetic term of (5.5) is now absorbed into the scalar curvature \( \hat{R} \) made of \( \hat{g}_{\mu\nu} \), and the Lagrangian has the form of dilaton gravity without the derivative terms of \( \varphi \).

6. Summary

We have constructed a geometrical framework of the gravitational theory with torsion by introducing the torsion potential \( t^A{}_{\mu} \). The torsion acquires a dynamical
property, being expressed by first-order derivatives of the torsion potential. Starting from the Einstein-Cartan action with torsion, we have obtained field equations for the metric tensor and the torsion potential. The full equation for the torsion potential is not invariant under local linear transformations of the torsion potential. However, its contracted form is found to be invariant. We have shown that the invariance group of the full equation is the global general linear group. In this paper we have proposed the framework of the torsion potential only in source-free case, and given a simple example where the torsion potential is described by a scalar field. As a next step we shall study the possibility that the torsion potential is constructed from a vector field or a spinor field.

On the other hand, matter fields other than the torsion potential can be introduced into this framework as follows. Scalar and vector fields can be treated in the same manner as in general relativity. Spinor fields can be introduced with the help of the tetrad fields, which are different from the torsion potential $t^A_{\mu}$, assuming that the covariant differentiation for them is defined by the spin connection with torsion. Detailed consideration of matter couplings will be given elsewhere.

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