THE BOGOMOLOV-TIAN-TODOROV THEOREM OF CYCLIC $A_{\infty}$-ALGEBRAS

JUNWU TU

Abstract. Let $A$ be a finite-dimensional smooth unital cyclic $A_{\infty}$-algebra. Assume furthermore that $A$ satisfies the Hodge-to-de-Rham degeneration property. In this short note, we prove the non-commutative analogue of the Bogomolov-Tian-Todorov theorem: the deformation functor associated with the differential graded Lie algebra of Hochschild cochains of $A$ is smooth. Furthermore, the deformation functor associated with the DGLA of cyclic Hochschild cochains of $A$ is also smooth.

1. The non-commutative Bogomolov-Tian-Todorov Theorem

Let $X$ be a Calabi-Yau manifold, i.e. a compact complex manifold with trivial canonical bundle. It is a classical result of Bogomolov-Tian-Todorov [11][12] that the formal deformation functor associated with the differential graded Lie algebra (DGLA)

$$\mathfrak{g}_X := \bigoplus_{p,q} A^{0,q}(\Lambda^p T_X)$$

of the Dolbeault resolution of holomorphic poly-vector fields is smooth. In order to prove this, the key observation was the existence of a BV operator $\Delta : \Lambda^* T_X \to \Lambda^{*-1} T_X$, which “trivializes” the Lie bracket by the Tian-Todorov identity

$$[\alpha, \beta] = \Delta(\alpha \wedge \beta) - \Delta(\alpha) \wedge \beta - (-1)^{||\alpha||} \alpha \wedge \Delta(\beta).$$

With the above formula, the smoothness of the deformation functor follows easily from the classical $\partial\bar{\partial}$-Lemma in Hodge theory.

Following Kontsevich-Soibelman [8] and Katzarkov-Kontsevich-Pantev [7], one can formulate the compactness, smoothness, and the Calabi-Yau property purely in terms of the differential graded category of coherent sheaves on $X$. Thus, a natural question is whether the analogues of the Bogomolov-Tian-Todorov’s theorem holds for any smooth and proper Calabi-Yau categories. This question might have been a folklore theorem for experts in the field. The purpose of this note is to fill some of the missing details in the literature.

A large class of dg categories of interests is compactly generated by a single object. For this reason, instead of considering formal deformations of dg categories (whatever that means), we shall consider deformations of $A_{\infty}$-algebras which is also much more tractable. To state the non-commutative version of the Bogomolov-Tian-Todorov theorem precisely, we first fix some notations and conventions. Throughout the paper, we use the homological degree of chain complexes. If $A$ is a chain complex, its suspension is denoted by $sA$ with $(sA)_n := A_{n-1}$. For

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Institute of Mathematical Sciences, ShanghaiTech University, Shanghai, China, 201210. E-mail: tujw.at.shanghaitech.edu.cn.
a unital $A_\infty$-algebra $A$, denote by $C^{-\bullet}(A)$ ($C_*(A)$) its reduced Hochschild cochain complex (chain complex respectively). The minus sign is due to that we use homological degree. Let $A$ be a cyclic unital $A_\infty$-algebra, denote by $C^\lambda(A) \subset C^{-\bullet}(A)$ the sub-complex consisting of cyclic cochains with respect to the pairing on $A$.

**Theorem 1.1.** Let $A$ be a $\mathbb{Z}/2\mathbb{Z}$-graded, finite-dimensional smooth unital cyclic $A_\infty$-algebra. Assume furthermore that $A$ satisfies the Hodge-to-de-Rham degeneration property. Then we have

(A.) The deformation functor $\text{Def}_{sC^{-\bullet}(A)}$ associated with the DGLA $sC^{-\bullet}(A)$ of Hochschild cochains is smooth.

(B.) The deformation functor $\text{Def}_{sC^\lambda(A)}$ associated with the DGLA $sC^\lambda(A)$ of cyclic Hochschild cochains is also smooth.

(C.) The natural transformation $f : \text{Def}_{sC^\lambda(A)} \rightarrow \text{Def}_{sC^{-\bullet}(A)}$ associated with the canonical inclusion map $sC^\lambda(A) \hookrightarrow sC^{-\bullet}(A)$ is smooth. In particular, every deformation of the $A_\infty$ structure of $A$ lifts to a deformation of the cyclic $A_\infty$ structure of $A$.

**Remark 1.1.** The assumption of the Hodge-to-de-Rham degeneration property automatically holds for any $\mathbb{Z}$-graded smooth and proper $A_\infty$-algebra by Kaledin [6]. In the general $\mathbb{Z}/2\mathbb{Z}$-graded case, this remains an open conjecture by [8] [7]. Part (A.) of the above Theorem was proved by Isamu Iwanari [5] with a different method.

2. BV$_\infty$-algebra structure on $C^{-\bullet}(A)$

To prove the above Theorem 1.1, one follows the same idea as in the proof of Bogomolov-Tian-Todorov’s Theorem. However, the key identity Equation 1 fails to hold. It only holds up to homotopy. Also, the cup product on $C^{-\bullet}(A)$ is only commutative up to homotopy. Thus it is natural to work with a homotopy version of the underlying algebraic structures.

In this section, we first exhibit a homotopy BV algebra structure, or BV$_\infty$-algebra structure on $C^{-\bullet}(A)$. The definition of BV$_\infty$-algebras used in this paper is from the article [3]. In fact, it was argued in Loc. Cit. that combining a TCFT structure defined by [1] [8] and the formality of the operad BV, one easily deduces the existence of a BV$_\infty$-algebra structure on $C^{-\bullet}(A)$ with $A$ as in Theorem 1.1. However, to make such structure useful in order to deduce Theorem 1.1, one needs to say a bit more about this BV$_\infty$ structure. For example, its underlying Lie$_\infty$ algebra is in fact given by the differential graded Lie algebra $(C^{-\bullet}(A), \delta, [-,-]_C)$. For this purpose, we need to use a construction of Tamarkin in his proof of the Deligne’s conjecture [10]. We introduce the following notations:

- **Lie** — The Lie operad.
- **Lie$_\infty$** — The homotopy Lie operad.
- **$E_2$** — The operad whose representation gives Gerstenhaber algebras.
- **$G_\infty$** — The homotopy $E_2$ operad.
- **BV** — The operad whose representation gives BV algebras.
- **BV$_\infty$** — The homotopy BV operad.
- **B$_\infty$** — The brace operad [4, Section 5.2].
- **$\mathcal{F}$** — The operad defined by Tamarkin in [10, Section 6].
- **$C_{comb}^s(FD)$** — The operad of black-and-white ribbon trees defined by Kontsevich-Soibelman [8, Section 11.6], see also Wahl-Westerland [14, Section 2]. This operad gives a combinatorial model for the framed little disk operad.
• $\mathcal{X}$ — A cofibrant replacement of $C^\text{comb}_*(FD)$.

For an operad $O$, denote by $O\{1\}$ its shifted version so that an $O\{1\}$-algebra structure on a chain complex $A$ is equivalent to an $O$-algebra structure on $sA$. The endomorphism operad of a chain complex is denoted by $\text{End}(-)$.

The starting point to construct a $\text{BV}_\infty$ structure on $C^{*-}(A)$ is that the operad $C^\text{comb}_*(FD)$ naturally acts on $C^{*-}(A)$:

\[(2)\]  

$C^\text{comb}_*(FD) \to \text{End}\left(C^{*-}(A)\right)$.

We refer to Kontsevich-Soibelman [8] and Wahl-Westerland [14] for details of this action. Here we illustrate this action with a few examples. Indeed, the following black-and-white ribbon tree

\[\begin{array}{c}
\times \\
\cdots
\end{array}\]

inside $C^\text{comb}_*(FD)(1)$ gives the pull-back of the Connes operator $B : C_*(A) \to C_*(A)$ under the isomorphism $C^{*-}(A) \cong (C_*(A))^\vee$. Denote it by $\Delta : C^{*-}(A) \to C^{*-}(A)$.

There are also binary operators associated with trees in $C^\text{comb}_*(FD)(2)$. For example, consider the following two black-and-white ribbon trees:

\[T = \begin{array}{c}
\times \\
\cdots
\end{array}\quad R = \begin{array}{c}
\times \\
\cdots
\end{array}\]

The graph $T$ gives the familiar cup product on $C^{*-}(A)$, while the graph $R$ gives the first brace operator $\circ$ on $C^{*-}(A)$ whose commutator (after shift) is the Gerstenhaber Lie bracket $[-,-]_G$.

**Lemma 2.1.** There is a morphism of operad $h : B_\infty \to C^\text{comb}_*(FD)$ defined by

\[
h(m_k) := T_k,\; k \geq 2 \\
h(m_{1,k}) := W_k,\; k \geq 1, \\
h(m_{p,k}) := 0,\; p \geq 2.
\]

Here $T_k$ and $W_k$ are given by the following black-and-white ribbon trees:

\[T_k := \begin{array}{c}
\times \\
\cdots
\end{array}\quad W_k := \begin{array}{c}
\times \\
\cdots
\end{array}\]

**Proof.** This is a straight-forward check. □

In [10], Tamarkin constructed morphisms $s : G_\infty \to \mathcal{F}$ and $t : \mathcal{F} \to B_\infty$. 3
Lemma 2.2. There exists a commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{t} & B_{\infty} \\
\downarrow & & \downarrow h \\
X & \xrightarrow{\sim} & C_{\ast}^{\text{comb}}(FD)
\end{array}
\]

Proof. This follows the lifting property since \(X \to C_{\ast}^{\text{comb}}(FD)\) is a trivial fibration, while \(F\) by construction is cofibrant. \(\square\)

The framed little disk operad is known to be formal with cohomology the \(BV\) operad, which implies that \(BV \cong C_{\ast}^{\text{comb}}(FD)\) in the homotopy category of differential graded operads. Since the operad \(X\) is cofibrant, and \(BV\) is fibrant (as any dg operad is fibrant), we obtain a morphism \(X \to BV\) such that the roof diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sim} & BV \\
C_{\ast}^{\text{comb}}(FD) & \xrightarrow{\sim} & BV
\end{array}
\]

represents an isomorphism in the homotopy category of differential graded operads.

Lemma 2.3. The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Lie}_{\infty}\{1\} & \xrightarrow{\sim} & G_{\infty} & \xrightarrow{\sim} & F & \xrightarrow{\sim} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{BV}_{\infty} & \xrightarrow{\sim} & BV & & & & \text{BV}
\end{array}
\]

Proof. It is clear that the left composition factors as

\[
\text{Lie}_{\infty}\{1\} \to \text{Lie}\{1\} \to BV.
\]

For the right composition, consider the following composition

\[
\text{Lie}_{\infty}\{1\} \to F \to X \to C_{\ast}^{\text{comb}}(FD).
\]

By Lemma 2.2 above, it is equal to

\[
\text{Lie}_{\infty}\{1\} \to F \to B_{\infty} \to C_{\ast}^{\text{comb}}(FD),
\]

which by [10] can be factored as

\[
\text{Lie}_{\infty}\{1\} \to \text{Lie}\{1\} \to B_{\infty} \to C_{\ast}^{\text{comb}}(FD).
\]

This shows that both compositions vanish on generators \(l_k\) of \(\text{Lie}_{\infty}\{1\}\) with \(k \geq 3\). For \(k = 2\), this is a direct check by definition. \(\square\)
By definition of $X$, the right vertical map in the above diagram is a trivial fibration since $BV$ is minimal. The left vertical map is a cofibration. Thus by the lifting property, we obtain a map $BV_\infty \to X$:

\[
\text{Lie}_\infty \{1\} \to G_\infty \to \mathcal{F} \to X
\]

We define a $BV_\infty$-algebra structure on $C^{-*}(A)$ via the composition:

(3) $\rho : BV_\infty \to X \to C^*_{\text{comb}}(FD) \to \text{End}(C^{-*}(A))$.

**Corollary 2.1.** Let $C^{-*}(A)$ be endowed with the $BV_\infty$-algebra structure defined by the Equation 3. Then its underlying $\text{Lie}_\infty$ structure on the suspension of $C^{-*}(A)$ is given by the differential graded Lie algebra $(sC^{-*}(A), \delta, [-, -]_G)$.

**Proof.** This follows from Lemma 2.2 and Tamarkin’s commutative diagram:

\[
\text{Lie}_\infty \{1\} \to G_\infty \to \mathcal{F} \\
\text{Lie} \{1\} \to B_\infty
\]

Consider a subset of generating operators of a $BV_\infty$ structure given by

\[
\{t^d_k | k \geq 1, d \geq 0\}
\]

which label a basis of the convolution between the Koszul dual cooperad of $\text{Lie}$ and that of the operad generated by the circle operator $\Delta$. We denote the sub-operad generated by $t^d_k$ in $BV_\infty$ by $q\text{Lie}_\infty \{1\}$. The notation is because that a $q\text{Lie}_\infty \{1\}$-algebra structure on a chain complex $V$ is equivalent to an $\text{Lie}_\infty$ structure on $sV [[u]]$ (with $u$ a degree 2 formal variable), which may be thought of as a “quantum” $\text{Lie}_\infty \{1\}$ structure.

**Corollary 2.2.** The induced $q\text{Lie}_\infty \{1\}$ structure on $C^{-*}(A)$ is of the form

(4) $t^0_1 = \delta, t^0_2 = [-, -]_G, t^d_k = 0 \forall k \geq 3$,

(5) $t^1_1 = \Delta, t^1_k(\alpha_1 \cdots \alpha_k) = 0, \forall \alpha_1, \cdots, \alpha_k \in \text{Ker} \Delta, \forall k \geq 2$,

(6) $t^d_k = 0, \forall d \geq 2$.

**Proof.** Property (3) is proven in the previous Corollary 2.1. To prove Property (4) and (5), observe that the degree of the operator $t^d_1$ is equal to $2d + 2k - 3$. But, the top dimensional chains in the chain complex $C^*_{\text{comb}}(FD)(k)$ is equal to $2k - 1$, and
which implies the vanishing in (5). The operator $l^1_1 = \Delta$, since the black-and-white graph

\[ \begin{array}{c}
\end{array} \]

is the unique degree one graph representing the fundamental class in the homology group $H_1(C_{\text{comb}}^*(FD)(1)) \cong H_1(S^1)$. For $k \geq 2$, since the degree of the operator $l^1_k$ is $2k - 1$ which is top dimensional in the chain complex $C_{\text{comb}}^*(FD)(k)$. Thus $l^1_k$ is a linear combination of operators coming from black-and-white ribbon trees with $k$ white vertices, and of degree $2k - 1$. One can show (easy combinatorics) that any such tree must contain at least one white vertex of the form:

\[ \begin{array}{c}
\end{array} \]

If we input any $\alpha \in \text{Ker} \Delta$ at this type of white vertex, the result gives zero by definition of the action of ribbon trees on $C^{-\ast}(A)$ (by Equation 2).

\[ \square \]

3. Proof of Theorem 1.1

As explained in the previous section, via the composition in Equation 3, we have a BV$_\infty$ structure on $C^{-\ast}(A)$. Thus, we may form its bar-cobar resolution:

\[ \Omega BC^{-\ast}(A) \cong C^{-\ast}(A), \]

which yields a differential graded BV algebra homotopy equivalent to $C^{-\ast}(A)$. At this point, we use the following theorem due to Katzarkov-Kontsevich-Pantev [7] and Terilla [13].

**Theorem 3.1.** Let $S$ be a differential graded BV algebra. Assume that the spectral sequence associated with the complex $(S[[u]], d + u\Delta)$ endowed with the $u$-filtration is degenerate at the first page. Then both the DGLA’s $(S, d, [-, -])$ and $(S[[u]], d + u\Delta, [-, -])$ are homotopy abelian.

Since the degeneration of the spectral sequence is a homotopy invariant property (see [2]), we may use the above theorem to deduce the homotopy abelian property of the DGLA $sC^{-\ast}(A)$. Note that here it is essential that the BV$_\infty$ structure on $C^{-\ast}(A)$ extends the DGLA structure of $sC^{-\ast}(A)$ by Corollary 2.1.

Similarly, we may also restrict the BV$_\infty$ structure on $C^{-\ast}(A)$ to the sub-operad qLie$_\infty$ {1}. Then theorems above implies that the following Lie$_\infty$-algebra

\[ (sC^{-\ast}(A)[[u]], \delta + u\Delta, [-, -], u \cdot l^2_2, u \cdot l^k_1 (k \geq 3)) \]

is homotopy abelian. Here the structure maps $l^k_1$ are as in Corollary 2.2. The following lemma then finishes the proof of (B.) in Theorem 1.1, using the homotopy invariance of deformation functors.

**Lemma 3.1.** The canonical inclusion map

\[ \iota : sC^\Lambda(A) \rightarrow sC^{-\ast}(A)[[u]] \]

is a quasi-isomorphism of Lie$_\infty$-algebras.

**Proof.** This inclusion is a quasi-isomorphism is a classical result, see for example [9]. By Corollary 2.2, the higher brackets $l^k_1$ vanish on elements inside $\text{Ker} \Delta$ while we certainly have that $\text{Im} \iota \subset \text{Ker} \Delta$. This shows that

\[ l^k_1(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall \alpha_1, \ldots, \alpha_k \in sC^\Lambda(A), \]

implies that $\iota$ is a morphism of Lie$_\infty$-algebras. \[ \square \]
Denote by $\pi : sC^{-*}(A)[[u]] \to sC^{-*}(A)$ the projection map defined by setting $u = 0$. Part $(C.)$ of Theorem 1.1 easily follows from $(A.)$ and $(B.)$. Indeed, since both functors are smooth, it suffices to check that the inclusion map 

$$\pi \circ \iota : sC^\Lambda(A) \hookrightarrow sC^{-*}(A)$$

induces a surjection on the tangent space of the associated deformation functors. This is clear as $\iota$ is a quasi-isomorphism, and $\pi$ is a surjective on cohomology by the Hodge-to-de-Rham degeneration assumption.

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