Existence and uniqueness of nonnegative solutions to the stochastic porous media equation

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August 2, 2018

Abstract. One proves that the stochastic porous media equation in 3-D has a unique nonnegative solution for nonnegative initial data in $H^{-1}(\Omega)$ if
the nonlinearity is monotone and has polynomial growth.

AMS subject Classification 2000: 76S05, 60H15.

∗Supported by the CEEX Project 05 of Romanian Minister of Research.
†Supported by the research program “Equazioni di Kolmogorov” from the Italian “Ministero della Ricerca Scientifica e Tecnologica”
‡Supported by the DFG-Research Group 399, the SFB-701, the BIBOS-Research Center, the INTAS project 99-559, the RFBR project 04–01–00748, the Russian–Japanese Grant 05-01-02941-JF, the DFG Grant 436 RUS 113/343/0(R).
Key words: Porous media equation, Stochastic PDEs, Yosida approximation.

1 Introduction

Let $\Omega$ be an open bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. We consider the linear operator $\Delta$ in $L^2(\Omega)$ defined on $H^2(\Omega) \cap H^1_0(\Omega)$. It is well known that $-\Delta$ is self-adjoint positive and anti-compact. So, there exists a complete orthonormal system $\{e_k\}$ in $L^2(\Omega)$ of eigenfunctions of $-\Delta$. In fact we have $e_k \in \bigcap_{p \geq 1} L^p(\Omega)$ for all $k \in \mathbb{N}$. We denote by $\{\lambda_k\}$ the corresponding sequence of eigenvalues, $\Delta e_k = -\lambda_k e_k$, $k \in \mathbb{N}$.

We shall consider a cylindrical Wiener process in $L^2(\Omega)$ of the following form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k, \quad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. To be more specific, we shall assume that $1 \leq n \leq 3$.

In this work we consider the stochastic partial differential equation,

$$\begin{cases}
  dX(t) - \Delta \beta(X(t))dt = XdW(t), & t \geq 0, \\
  \beta(X(t)) = 0, & on \partial \Omega, \quad t \geq 0, \\
  X(0, x) = x.
\end{cases} \tag{1.1}$$

Here $\beta$ is a continuous, differentiable, monotonically increasing function on $\mathbb{R}$ which satisfies the following conditions,

$$\begin{cases}
  |\beta'(r)| \leq \alpha_1 |r|^{m-1} + \alpha_2, & \forall \ r \in \mathbb{R}, \\
  j(r) := \int_0^r \beta(s)ds \geq \alpha_3 |r|^{m+1} + \alpha_4 r^2, & \forall \ r \in \mathbb{R},
\end{cases} \tag{1.2}$$

where $\alpha_i > 0$, $i = 1, 2, 3, 4$ and $1 \leq m$. We note that since $\beta$ is increasing, the mean value theorem implies that

$$r \beta(r) \geq j(r), \quad r \geq 0. \tag{1.3}$$
Equation (1.1) with additive noise was recently studied in [4], [5], [7], [8], [9], see also [3]. In particular, in [7] was given an existence result under similar conditions on $\beta$. Here we consider a multiplicative noise (of a special form, but it would be possible to consider a more general noise $f(X)dW(t)$ with $f(0) = 0$), which is needed in order to ensure positivity of solutions.

As was shown in [12] existence and uniqueness of solutions follow by the general results in [12] (see also [13] for generalizations). In this paper we present an alternative proof, based on the Yosida approximation of $-\Delta \beta$, and prove the positivity of solutions for nonnegative initial data $x$.

As in deterministic case the Sobolev space $H^{-1}(\mathcal{O})$ is natural for studying equation (1.1). Equation (1.1) can be written in the abstract form

\[
\begin{cases}
  dX(t) + AX(t) = \sigma(X(t))dW(t), & t \geq 0, \\
  X(0) = x,
\end{cases}
\]  

where the operator $A: D(A) \subset H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$ is defined by

\[
\begin{cases}
  Ax = -\Delta \beta(x), & x \in D(A), \\
  D(A) = \{ x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \beta(x) \in H_0^1(\mathcal{O}) \},
\end{cases}
\]

and where

\[
\sigma(X)dW(t) = \sum_{k=1}^{\infty} \mu_k X e_k d\beta_k(t), \quad X \in H^{-1}(\mathcal{O}).
\]

To give a rigorous sense to this noise term we first note that since $n \leq 3$, by Sobolev embedding it follows that

\[
\sup_{k \in \mathbb{N}} \frac{1}{\lambda_k} |e_k|_\infty < \infty. \tag{1.7}
\]

Furthermore, throughout this paper we shall assume that

\[
\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 =: C < \infty. \tag{1.8}
\]

(1.8) implies for some constant $c_1 > 0$

\[
\sum_{k=1}^{\infty} \mu_k^2 |xe_k|_{-1}^2 \leq c_1 \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 |x|_{-1}^2 \leq c_1 C |x|_{-1}^2, \quad \forall x \in H^{-1}(\mathcal{O}), \quad \tag{1.9}
\]
because $|xe_k|^2 \leq c_1 \lambda_k^2 |x|^2$ by an elementary calculation, since $n \leq 3$ and due to (1.7).

Defining

$$\sigma(x)h := \sum_{k=1}^{\infty} \mu_k(h, e_k)xe_k, \quad x \in H^{-1}(\mathcal{O}), \quad h \in L^2(\mathcal{O}).$$

(1.10)

we obtain by (1.9) that $\sigma(x) \in L_2(L^2(\mathcal{O}), H^{-1}(\mathcal{O}))$. Considering $(\beta_k)_{k \in \mathbb{N}}$ as a cylindrical Wiener process on $L^2(\mathcal{O})$, it follows that (1.6) is well defined. Note that since $\sigma$ is linear we have that $x \to \sigma(x)$ is Lipschitz from $H^{-1}(\mathcal{O})$ to $L_2(L^2(\mathcal{O}), H^{-1}(\mathcal{O}))$ (in particular [11], [12], [13] really apply).

The plan of the paper is the following: main results are stated in §2 and proofs are given in §3.

The following notations will be used throughout in the following.

(i) $H^1_0(\mathcal{O}), H^2(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$ endowed with their usual norms denoted by $| \cdot |_{H^1_0(\mathcal{O})}$ and $| \cdot |_{H^2(\mathcal{O})}$ respectively.

(ii) $H$ is the space $H^{-1}(\mathcal{O})$ (the dual of $H^1_0(\mathcal{O})$) endowed with the norm

$$|x|_H = |x|_{-1} = |-\Delta^{-1}x|_{H^1_0(\mathcal{O})}.$$

(Here $(-\Delta)^{-1}x = y$ is the solution to Dirichlet problem $-\Delta y = x$ in $\mathcal{O}, \ y \in H^1_0(\mathcal{O})$). The scalar product in $H$ is

$$\langle x, z \rangle_{-1} = \int_\mathcal{O} (-\Delta)^{-1}x z d\xi, \quad \forall \ x, z \in H^1_0(\mathcal{O}).$$

(iii) The scalar product and the norm in $L^2(\mathcal{O})$ will be denoted by $(\cdot, \cdot)$ and $| \cdot |_2$, respectively and the norm in $L^p(\mathcal{O}), 1 \leq p \leq \infty$ by $| \cdot |_p$.

(iv) For two Hilbert spaces $H_1, H_2$ the space of Hilbert-Schmidt operators from $H_1$ to $H_2$ is denoted by $L^2(H_1, H_2)$.

2 The main result

To begin with let us define the solution concept we shall work with. Formally, a solution to (1.1) (equivalently (1.4)) might be an $H$–valued continuous
adapted process such that \( X, AX \in C_W([0,T]; L^2(\Omega; H)) \) and

\[
X(t) = x - \int_0^t AX(s)ds + \int_0^t \sigma(X(s))dW(s), \quad t \in [0,T].
\]  

(2.1)

(By \( C_W([0,T]; L^2(\Omega; H)) \) we mean the Banach space of all the processes \( X \) in \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \( H \) which are adapted and mean square continuous, endowed with the norm

\[
\|X\|^2_{C_W([0,T]; L^2(\Omega; H))} := \sup_{t \in [0,T]} \mathbb{E}|X(t)|^2_H.
\]

Spaces \( L^p([0,T]; L^2(\Omega; H)), p \in [1, \infty], \) are defined similarly.)

However, such a concept of solution might fail to exist for equation (1.1) and so we shall confine to a weaker one inspired by [7] and [11].

**Definition 2.1** An \( H \)-valued continuous \( \mathcal{F}_t \)-adapted process \( X \) is called a solution to (1.1) on \([0,T]\) if \( X \in L^{m+1}\times(0,T)\times\mathcal{O}) \) and

\[
(X(t), e_j) = (x, e_j) + \int_0^t \int_\Theta \beta(X(s))\Delta e_j d\xi ds
\]

\[
+ \sum_{k=1}^\infty \mu_k \int_0^t (X(s)e_k, e_j)d\beta_k(s), \quad \forall \ j \in \mathbb{N}, \ t \in [0,T].
\]  

(2.2)

Taking into account that \( -\Delta e_j = \lambda_j e_j \) in \( \Theta \) we may equivalently write (2.2) as follows

\[
\langle X(t), e_j \rangle_{-1} = \langle x, e_j \rangle_{-1} - \int_0^t \int_\Theta \beta(X(s))e_j d\xi ds
\]

\[
+ \sum_{k=1}^\infty \mu_k \int_0^t \langle X(s)e_k, e_j \rangle_{-1} d\beta_k(s), \quad \forall \ j \in \mathbb{N},
\]

i.e.

\[
d\langle X(t), e_j \rangle_{-1} + \langle \beta(X(t)), e_j \rangle dt = \sum_{k=1}^\infty \mu_k \langle X(s)e_k, e_j \rangle_{-1} d\beta_k(s).
\]

Recalling (1.6) we see that

\[
\sum_{k=1}^\infty \mu_k \langle X(t)e_k, e_j \rangle d\beta_k(t) = (\sigma(X(t))dW(t), e_j), \quad j \in \mathbb{N}.
\]
We also note that since by assumption (1.2), $\beta(X) \in L^{\frac{m+1}{m}}((0,T) \times \Omega \times \mathcal{O})$, the integral arising in the right hand side of (2.2) makes sense because $e_j \in C^\infty(\mathcal{O})$ for all $j \in \mathbb{N}$. Of course, one might derive a vector valued version of Definition 2.1 as in [7]. Now we are ready to formulate the main results.

**Theorem 2.2** Assume that (1.2) and (1.8) hold. Then for each $x \in H^{-1}(\mathcal{O})$ there is a unique solution $X$ to (1.1). Moreover, if $x \in L^p(\mathcal{O})$ is non-negative a.e. on $\mathcal{O}$ where $p \geq \max\{m+1,4\}$ is a natural number then $X \in L^\infty W(0,T;L^p(\Omega;L^p(\mathcal{O})))$ and $X \geq 0$ a.e. on $(0,\infty) \times \mathcal{O}$, $\mathbb{P}$-a.s. If $x \in H^{-1}(\mathcal{O})$ is such that $x \geq 0$, i.e. $x$ is a positive measure, then $X(t) \geq 0$ for all $t \geq 0$, $\mathbb{P}$-a.s.

The positivity of the solution $X$ to (1.1) will be proven below by choosing an appropriate Lyapunov function.

**3 Proof of Theorem 2.2**

We mention that in our estimates in the sequel constants may change from line to line though we do not express this in our notation.

We recall that the operator $A$, defined by (1.5), is maximal monotone in $H$ (see e.g. [6]). Then we consider the Yosida approximation

$$A_\varepsilon(x) = \frac{1}{\varepsilon} (x - J_\varepsilon(x)) = A(1 + \varepsilon A)^{-1}(x), \quad \varepsilon > 0, \ x \in H,$$

where $J_\varepsilon(x) = (1 + \varepsilon A)^{-1}(x)$. The operator $A_\varepsilon$ is monotone and Lipschitzian on $H$. Then, by (1.9) it follows by standard existence theory for stochastic equations in the Hilbert spaces (see e.g. [10]) that the approximating equation

$$\begin{cases}
    dX_\varepsilon(t) + A_\varepsilon X_\varepsilon(t)dt = \sigma(X_\varepsilon(t))dW(t), & t \geq 0, \\
    X_\varepsilon(0) = x,
\end{cases}$$

(3.1)

has a unique solution $X_\varepsilon \in C_W([0,T];L^2(\Omega;H))$ such that $X_\varepsilon \in C([0,T];H)$, $\mathbb{P}$-a.s. with $A_\varepsilon X_\varepsilon \in C_W([0,T];L^2(\Omega;H))$. 

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By Itô’s formula we have
\[
\frac{1}{2} d|X_\varepsilon(t)|^2_{-1} + \langle A_\varepsilon X_\varepsilon(t), X_\varepsilon(t) \rangle_{-1} dt \\
= \langle \sigma(X_\varepsilon(t))dW(t), X_\varepsilon(t) \rangle_{-1} + \frac{1}{2} \sum_{k=1}^{\infty} \mu_k^2 |X_\varepsilon(t)e_k|^2_{-1} dt.
\] (3.2)

This yields (see (1.9))
\[
\frac{1}{2} \mathbb{E}|X_\varepsilon(t)|^2_{-1} + \mathbb{E} \int_0^t \langle A_\varepsilon X_\varepsilon(s), X_\varepsilon(s) \rangle_{-1} ds \\
\leq \frac{1}{2} |x|^2_{-1} + C \mathbb{E} \int_0^t |X_\varepsilon(s)|^2_{-1} ds
\]
and therefore
\[
\frac{1}{2} \mathbb{E}|X_\varepsilon(t)|^2_{-1} + \mathbb{E} \int_0^t \langle A_\varepsilon X_\varepsilon(s), X_\varepsilon(s) \rangle_{-1} ds \leq C |x|^2_{-1}, \quad \forall \varepsilon > 0. \tag{3.3}
\]

We set \(Y_\varepsilon(t) = J_\varepsilon(X_\varepsilon(t))\) (see (3.1)). Then
\[
\frac{1}{2} \mathbb{E}|X_\varepsilon(t)|^2_{-1} + \mathbb{E} \int_0^t \int_0^s j(Y_\varepsilon(s)) ds d\xi \\
+ \frac{1}{\varepsilon} \mathbb{E} \int_0^t |X_\varepsilon(s) - Y_\varepsilon(s)|^2_{-1} ds \leq C|x|^2_{-1}, \quad \forall \varepsilon > 0.
\] (3.4)

(Here we have used the equality
\[
\langle A_\varepsilon x, x \rangle_{-1} = \langle AJ_\varepsilon x, J_\varepsilon x \rangle_{-1} + \frac{1}{\varepsilon} |x - J_\varepsilon(x)|^2_{-1},
\]
and (1.3).)

Now we fix \(X \in C_W([0, T]; L^2(\Omega, H))\) and we consider the equation
\[
\begin{cases}
    d\tilde{X}_\varepsilon(t) + A_\varepsilon\tilde{X}_\varepsilon(t)dt = \sigma(X(t))dW(t), \quad t \geq 0, \\
    \tilde{X}_\varepsilon(0) = x.
\end{cases}
\] (3.5)
Equivalently,
\[
\begin{aligned}
&d\tilde{X}_\varepsilon(t) - \Delta \beta(\tilde{Y}_\varepsilon(t))dt = \sigma(X(t))dW(t), \quad t \geq 0, \\
&\tilde{X}_\varepsilon(0) = x,
\end{aligned}
\]
(3.6)
where
\[\tilde{Y}_\varepsilon = (1 + \varepsilon A)^{-1} \tilde{X}_\varepsilon.\]

On the other hand, for equation (3.5) we have the same estimates as for (3.1). In fact by Itô’s formula we get (see (3.4))
\[
\begin{aligned}
\mathbb{E}\left|\tilde{X}_\varepsilon(t)\right|_{-1}^2 &+ \mathbb{E}\int_0^t \int_{\mathcal{O}} j(\tilde{Y}_\varepsilon(s))d\sigma dsd\xi \\
&+ \frac{1}{\varepsilon} \mathbb{E}\int_0^t |\tilde{X}_\varepsilon(s) - \tilde{Y}_\varepsilon(s)|_{-1}^2 ds \\
&\leq C|x|_{-1}^2 + C\varepsilon \mathbb{E}\int_0^t |X(s)|_{-1}^2 ds,
\end{aligned}
\]
(3.7)
(where we have used (1.2) to estimate \(\mathbb{E}\int_0^t \|\sigma(X(s))\|^2_{L^2_{\mathcal{L}(\mathcal{H})}}ds\)). By virtue of assumption (1.2) this implies that
\[
\mathbb{E}\int_0^T \int_{\mathcal{O}} |\beta(\tilde{Y}_\varepsilon(s))|^{-\frac{m+1}{m}} dsd\xi \leq C(|x|_{-1}^2 + 1), \quad \varepsilon > 0,
\]
(because \(|\beta(r)| \leq \tilde{\alpha}_1 |r|^m + \tilde{\alpha}_2, \tilde{\alpha}_1 \geq 0\), and so along a subsequence, we have
\[
\beta(\tilde{Y}_\varepsilon) \rightarrow \eta \quad \text{weakly in } L^{\frac{m+1}{m}}((0, T) \times \Omega \times \mathcal{O}).
\]
(3.8)

On the other hand, we have by (3.6) that for \(t \in [0, T]\)
\[
\langle \tilde{X}_\varepsilon(t), e \rangle_{-1} + \int_0^t \int_{\mathcal{O}} \beta(\tilde{Y}_\varepsilon(s))e dsd\xi = \langle x, e \rangle_{-1} + \int_0^t \langle \sigma(X(s))dW(s), e \rangle_{-1} ds,
\]
for all \(e \in L^{m+1}(\mathcal{O})\). We note that by (3.7) there exists \(X^* \in L^2_{\mathcal{W}}([0, T]; L^2(\Omega; H))\) such that
\[
\tilde{X}_\varepsilon \rightarrow X^* \quad \text{weakly in } L^2_{\mathcal{W}}([0, T]; L^2(\Omega; H))
\]
(3.9)
and by (3.7) and (1.2) we obtain that also
\[
\tilde{Y}_\varepsilon \rightarrow X^* \quad \text{weakly in } L^2_{\mathcal{W}}([0, T]; L^2(\Omega; H)) \cap L^{m+1}(\Omega \times (0, T) \times \mathcal{O}).
\]
(3.10)
Hence along a subsequence \(\varepsilon \rightarrow 0\)
\[
\mathbb{E}\langle \tilde{X}_\varepsilon(t), e \rangle_{-1} \rightarrow \mathbb{E}\langle X^*(t), e \rangle_{-1} \quad \text{weakly in } L^2(0, T).
\]
Then letting $\varepsilon$ tend to 0 we get for a.e. $t \in [0, T]$

$$
\langle X^*(t), e \rangle_{-1} = \langle x, e \rangle_{-1} - \int_0^t \int_\mathcal{O} \eta(s) e d\xi ds + \int_0^t \langle \sigma(X(s))dW(s), e \rangle_{-1} ds.
$$

(3.11)

Taking into account (3.9)-(3.10), to conclude the proof of existence it suffices to show that

$$
\eta(t, \xi, \omega) = \beta(X^*(t, \xi, \omega)) \quad \text{a.e. } (\omega, t, \xi) \in \Omega \times (0, T) \times \mathcal{O}.
$$

(3.12)

Indeed, in such a case we may take in (3.11) $e = \Delta_j e$ for $j \in \mathbb{N}$.

To this end we consider the operator

$$
F: L^m(\Omega \times (0, T) \times \mathcal{O}) \to L^{m+1}(\Omega \times (0, T) \times \mathcal{O}) = (L^m(\Omega \times (0, T) \times \mathcal{O}))',
$$

defined by

$$
(Fx)(t, \xi, \omega) = \beta(x(t, \xi, \omega)) \quad \text{a.e. } (\omega, t, \xi) \in \Omega \times (0, T) \times \mathcal{O}.
$$

This operator is maximal monotone and more precisely, it is the subgradient of the convex function $\Phi: L^{m+1}(\Omega \times (0, T) \times \mathcal{O}) \to \mathbb{R}$ defined as,

$$
\Phi(x) = \mathbb{E} \int_0^T \int_\mathcal{O} j(x(t, \xi)) dtd\xi.
$$

For each $Z \in L^{m+1}(\Omega \times (0, T) \times \mathcal{O})$ we have

$$
\Phi(\tilde{Y}_\varepsilon) - \Phi(Z) \leq \mathbb{E} \int_0^T \int_\mathcal{O} \beta(\tilde{Y}_\varepsilon(t, \xi)) (\tilde{Y}_\varepsilon(t, \xi) - Z(t, \xi)) dtd\xi.
$$

Letting $\varepsilon$ tend to 0 we have by (3.8), (3.9), (3.10) and by the weak lower semicontinuity of $\Phi$

$$
\Phi(X^*) - \Phi(Z) \leq \liminf_{\varepsilon \to 0} \mathbb{E} \int_0^T \int_\mathcal{O} \beta(\tilde{Y}_\varepsilon(t, \xi)) \tilde{Y}_\varepsilon(t, \xi) dtd\xi - \mathbb{E} \int_0^T \int_\mathcal{O} \eta Z dtd\xi.
$$

To prove (3.12) by the uniqueness of the subgradient it suffices to show that

$$
\liminf_{\varepsilon \to 0} \mathbb{E} \int_0^T \int_\mathcal{O} \beta(\tilde{Y}_\varepsilon(t, \xi)) \tilde{Y}_\varepsilon(t, \xi) dtd\xi \leq \mathbb{E} \int_0^T \int_\mathcal{O} \eta X^* dtd\xi.
$$

(3.13)
To this end we come back to equation (3.6) and note that by Itô’s formula we have
\[
\frac{1}{2} E |\tilde{X}_\varepsilon(t)|^2 - 1 + \frac{1}{2} \sum_{k=1}^{\infty} E \int_0^t \mu_k^2 |X(s)e_k|^2 ds.
\]
Equivalently,
\[
\frac{1}{2} E |\tilde{X}_\varepsilon(t)|^2 - 1 + E \int_0^t \int_0^t \beta(\tilde{Y}_\varepsilon(s)) \tilde{X}_\varepsilon(s) dsd\xi
\]
\[+ E \int_0^t \int_0^t \beta(\tilde{Y}_\varepsilon(s))(\tilde{X}_\varepsilon(s) - \tilde{Y}_\varepsilon(s)) dsd\xi \]
\[= \frac{1}{2} |x|^2 - 1 + \frac{1}{2} \sum_{k=1}^{\infty} E \int_0^t \mu_k^2 |X(s)e_k|^2 ds.
\]
By (3.9)-(3.10) we have
\[
\int_0^t \beta(\tilde{Y}_\varepsilon(s))(\tilde{X}_\varepsilon(s) - \tilde{Y}_\varepsilon(s)) d\xi = \langle A_x \tilde{X}_\varepsilon(s), \tilde{X}_\varepsilon(s) - J_x(\tilde{X}_\varepsilon(s)) \rangle - \varepsilon |A_x \tilde{X}_\varepsilon(s)|^2.
\]
Fix \( \varphi \in L^\infty(0, T), \varphi \geq 0. \) Then \( \varphi X^* \in L^2_W(0, T; L^2(\Omega; H)). \) Thus by (3.9)-(3.10)
\[
E \int_0^T \varphi(t) |X^*(t)|^2 - 1 dt = \lim_{\varepsilon \to 0} E \int_0^T \langle X^*(t), X_\varepsilon(t) \rangle - 1 \varphi(t) dt
\]
\[\leq \left( E \int_0^T \varphi(t) |X^*(t)|^2 - 1 dt \right)^{1/2} \lim_{\varepsilon \to 0} \left( E \int_0^T \varphi(t) |X_\varepsilon(t)|^2 - 1 dt \right)^{1/2}.
\]
Hence simplifying we obtain
\[
E \int_0^T \varphi(t) |X^*(t)|^2 - 1 dt \leq \lim_{\varepsilon \to 0} E \int_0^T \varphi(t) |X_\varepsilon(t)|^2 - 1 dt.
\]
Hence (3.14), Fatou’s Lemma (see also (1.3)) and the arbitrariness of $\varphi$ implies that for a.e. $t \in [0, T]$ we obtain that
\[
\liminf_{\varepsilon \to 0} \mathbb{E} \int_0^t \int_\varnothing \beta(\tilde{Y}_\varepsilon(s)) \tilde{Y}_\varepsilon(s) ds d\xi + \frac{1}{2} \mathbb{E}|X^*(t)|^2_{-1} \\
\leq \frac{1}{2} |x|_{-1}^2 + \frac{1}{2} \sum_{k=1}^\infty \mathbb{E} \int_0^t \mu_k^2 |X(s)e_k|_{-1}^2 ds.
\]
(3.15)

On the other hand, by (3.11) we see via Itô’s formula (applied to the right hand side of (3.11), since the left hand side might not be continuous in $t$) that for all $j \in \mathbb{N}$ and a.e. $t \in [0, T]$,
\[
\frac{1}{2} \mathbb{E} |\langle X^*(t), e_j \rangle|_{-1}^2 + \mathbb{E} \int_0^t \langle \eta_s, e_j \rangle \langle X^*(s), e_j \rangle_{-1} ds \\
= \frac{1}{2} \langle x, e_j \rangle_{-1}^2 + \frac{1}{2} \mathbb{E} \sum_{k=1}^\infty \mu_k^2 \int_0^t \langle X(s)e_k, e_j \rangle^2 ds
\]
and dividing by $|e_j|_{-1}^2$ and summing over $j$ we obtain
\[
\frac{1}{2} \mathbb{E}|X^*(t)|^2_{-1} + \mathbb{E} \int_0^t \int_\varnothing \eta(s)X^*(s) ds d\xi \\
= \frac{1}{2} |x|_{-1}^2 + \frac{1}{2} \sum_{k=1}^\infty \mu_k^2 \mathbb{E} \int_0^t |X(s)e_k|_{-1}^2 ds
\]
(3.16)

We note that the integral in the left hand side makes sense since by (3.4), $X^* \in L^{m+1}((0, T) \times \varnothing \times \varnothing)$ while $\eta \in L^{m+1+1}((0, T) \times \varnothing \times \varnothing)$.

Comparing (3.15) and (3.16) we infer that
\[
\liminf_{\varepsilon \to 0} \mathbb{E} \int_0^T \int_\varnothing \beta(\tilde{Y}_\varepsilon(t)) \tilde{Y}_\varepsilon(t) dtd\xi \leq \mathbb{E} \int_0^T \int_\varnothing \eta(t)X^*(t) dtd\xi,
\]
as claimed. A formal problem arises, however, because $X^*(t)$ as constructed before might not be $H$-continuous. However, arguing as in [11], [12] we may replace it by an $H$-continuous version defined by
\[
\tilde{X}^*(t) = x + \int_0^t \Delta \eta(s) ds + \int_0^t \sigma(X(s))dW(s).
\]
It follows that $X^* = \tilde{X}^*$ a.e. and that $\tilde{X}^*$ is also an $\mathcal{F}_t$-adapted process. Moreover, the Itô formula from ([11], Theorem I-3-2]) holds. Hence $\tilde{X}^* \in C_W([0,T]; L^2(\Omega; H)) \cap L^{m+1}((0,T) \times \Omega \times \mathcal{O})$ is a solution (in the sense of Definition 2.1) to
\[
\begin{align*}
&\quad dX^* + AX^* dt = \sigma(X) dW \\
&X^*(0) = x.
\end{align*}
\]

**Uniqueness.** Let $X^*_1, X^*_2$ be two solutions to equation (1.1) for $X = X_i, \ i = 1, 2$. We have (see (2.2))
\[d\langle X^*_1 - X^*_2, e_j \rangle_{-1} + \int_0^t (\beta(X^*_1) - \beta(X^*_2)) e_j d\xi dt = \sum_{k=1}^\infty \mu_k \langle (X_1 - X_2) e_k, e_j \rangle_{-1} d\beta_k.
\]
By Itô’s formula we obtain
\[
\begin{align*}
\frac{1}{2} \mathbb{E} |X^*_1(t) - X^*_2(t), e_j\rangle_{-1}|^2
\quad &+ \mathbb{E} \int_0^t (\beta(X^*_1(s)) - \beta(X^*_2(s)), e_j) \langle X^*_1(s) - X^*_2(s), e_j \rangle_{-1} ds \\
&= \frac{1}{2} \mathbb{E} \int_0^t \sum_{k=1}^\infty \mu_k^2 \langle (X_1(s) - X_2(s)) e_k, e_j \rangle_{-1}^2 ds
\end{align*}
\]
Dividing by $|e_j|^2_{-1}$ and summing over $j$ we see that
\[
\frac{1}{2} \mathbb{E} |X^*_1(t) - X^*_2(t)|^2_{-1} + \mathbb{E} \int_0^t (\beta(X^*_1) - \beta(X^*_2), X^*_1(s) - X^*_2(s) ds
\]
\[
= \frac{1}{2} \mathbb{E} \int_0^t \sum_{j,k=1}^\infty \mu_k^2 \langle (X_1(s) - X_2(s)) e_k, |e_j|_{-1}^{-1} e_j \rangle_{-1}^2 ds.
\]
Hence (see (1.9))
\[
\mathbb{E} |X^*_1(t) - X^*_2(t)|^2_{-1} \leq CE \int_0^t |X_1(s) - X_2(s)|^2_{-1} ds, \quad \forall \ t \in [0,T]
\]
Now we shall use the latter inequality to prove existence of a unique solution
\[ X \in C_W([0, T]; L^2(\Omega; H)) \cap L^{m+1}((0, T) \times \Omega \times \mathcal{O}) \]
to equation (1.1). Indeed the operator \( X \rightarrow X^* \) is a contraction on the space
\( C_W([0, T]; L^2(\Omega; H)) \) if \( T \) is sufficiently small and so, we have existence (and
uniqueness) for \( T > 0 \) small. By a standard unique continuation argument
it follows existence and uniqueness on an arbitrary interval \([0, T]\).

**Positivity.** We shall assume now that \( x \in L^p(\mathcal{O}) \), where \( p \geq \max\{m + 1, 4\} \), and \( x(\xi) \geq 0 \) a.e. in \( \mathcal{O} \). We shall prove that
\[ X \geq 0 \quad \text{a.e. in } (0, T) \times \mathcal{O} \times \Omega. \quad (3.19) \]
We shall first assume in addition that \( \beta \) is strictly monotone, i.e.
\[ (\beta(r) - \beta(\bar{r}))(r - \bar{r}) \geq \alpha(r - \bar{r})^2, \quad \forall \ r, \bar{r} \in \mathbb{R}, \quad (3.20) \]
where \( \alpha > 0 \). Below we shall use the following lemma.

**Lemma 3.1** Let \( y \in D(A) \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) Lipschitz and increasing. Then
\[ \langle \nabla \beta(y), \nabla g(y) \rangle_{\mathbb{R}^n} \geq 0, \quad \text{a.e. on } \mathcal{O}. \]

**Proof.** First note that by definition of \( D(A) \) we have that \( y, \beta(y) \in H_0^1(\mathcal{O}) \). Using a Dirac sequence we can find mollifiers \( g_k \in C^1(\mathbb{R}), \ g_k' \geq 0, \ k \in \mathbb{N}, \) such that
\[ \nabla g(y) = \lim_{k \rightarrow \infty} g_k'(y) \nabla y \quad \text{in } L^2(\mathcal{O}). \]
So, it suffices to prove that
\[ \langle \nabla \beta(y), \nabla g(y) \rangle_{\mathbb{R}^n} \geq 0, \quad \text{a.e. on } \mathcal{O}. \]
But
\[ \langle \nabla \beta(y), \nabla y \rangle_{\mathbb{R}^n} = \langle \nabla \beta(y), \nabla \beta^{-1}(y) \rangle_{\mathbb{R}^n}. \]
Since \( \beta \) is strictly monotone, \( \beta^{-1} \) is Lipschitz, so applying the above mollifier
argument with \( \beta^{-1} \) replacing \( g \), we prove the assertion. \( \square \)

We shall use the approximating equation (3.1) whose solution \( X_\epsilon \) is
weakly convergent to \( X \) in \( L^2_W(\Omega; L^2(0, T; H)). \) Namely, we have for \( Y_\epsilon(t) := J_\epsilon(X_\epsilon(t)), \ t \geq 0, \)
\[ dX_\epsilon(t) - \Delta \beta(Y_\epsilon(t)) dt = \sigma(X_\epsilon(t)) dW(t), \quad t \geq 0. \quad (3.21) \]
We note that equation (3.1) can be equivalently written as
\[
\begin{aligned}
&\left\{
\begin{array}{l}
dX_\varepsilon(t) + \frac{1}{\varepsilon} X_\varepsilon(t)dt = \frac{1}{\varepsilon} J_\varepsilon(X_\varepsilon(t))dt + \sigma(X_\varepsilon(t))dW(t), \quad t \geq 0, \\
X_\varepsilon(0) = x,
\end{array}
\right.
\end{aligned}
\tag{3.22}
\]

Fix \(x \in H\) and set
\[
y = J_\varepsilon(x) = (1 - \varepsilon \Delta \beta)^{-1}x,
\]
i.e.
\[
y - \varepsilon \Delta \beta(y) = x \tag{3.23}
\]
Then \(y \in D(A)\). Since \(\beta\) is strictly monotone, \(\beta^{-1}\) is Lipschitz. Therefore, since \(\beta(y) \in H^1_0(\mathcal{O})\), also \(y \in H^1_0(\mathcal{O})\). Now assume \(x \in L^p(\mathcal{O})\). By multiplying both sides of (3.23) by \(\frac{y^{p-1}}{1 + \lambda |y|^{p-2}}\) and integrating over \(\mathcal{O}\) we get by Lemma 3.1
\[
\int_{\mathcal{O}} \frac{y^p}{1 + \lambda |y|^{p-2}} d\xi \leq \int_{\mathcal{O}} \frac{y^{p-1}x}{1 + \lambda |y|^{p-2}} d\xi.
\]
Then, letting \(\lambda \to 0\) we find the estimate
\[
|y|^p \leq \int_{\mathcal{O}} y^{p-1}xd\xi \leq |y|^{p-1} |x|_p. \tag{3.24}
\]
Hence
\[
|J_\varepsilon(x)|_p \leq |x|_p, \quad \forall \ x \in L^p(\mathcal{O}), \tag{3.25}
\]
and therefore,
\[
|A_\varepsilon(x)|_p = \frac{1}{\varepsilon} |x - J_\varepsilon(x)|_p \leq \frac{2}{\varepsilon} |x|_p, \quad \forall \ x \in L^p(\mathcal{O}).
\]
(3.23) and (3.25) imply that \(J_\varepsilon\) is continuous from \(L^p(\mathcal{O})\) into itself.

**Lemma 3.2** For each \(x \in L^2(\mathcal{O})\) equation (3.22) has a unique solution \(X_\varepsilon \in C_W([0,T]; L^2(\Omega; L^2(\mathcal{O})))\).

**Proof.** Let us first prove that \(J_\varepsilon = (1 - \varepsilon \Delta \beta)^{-1}\) is Lipschitz continuous in \(L^2(\mathcal{O})\). Indeed, by the equation
\[
J_\varepsilon(x) - \varepsilon \Delta \beta(J_\varepsilon(x)) = x, \quad \text{in} \ \mathcal{O},
\]
(taking into account that $\beta(J_\varepsilon(x)) \in H^1_0(\partial)$) we have for $x, \bar{x} \in L^2(\partial)$

$$\int_\partial (J_\varepsilon(x) - J_\varepsilon(\bar{x})) (\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))) d\xi$$

$$+ \varepsilon \int_\partial |\nabla(\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x})))|^2 d\xi \leq \int_\partial (x - \bar{x})(\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))) d\xi.$$

This yields, recalling (3.20)

$$\alpha |J_\varepsilon(x) - J_\varepsilon(\bar{x})|^2 + \varepsilon |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\partial)} \leq |x - \bar{x}|_2 |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|_2.$$  

On the other hand, by the Poincaré inequality there exists $C > 0$ such that

$$|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2 \leq C |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\partial)}.$$  

Therefore

$$\alpha |J_\varepsilon(x) - J_\varepsilon(\bar{x})|^2 + \varepsilon |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\partial)} \leq \frac{C}{2\varepsilon} |x - \bar{x}|_2^2 + \frac{\varepsilon}{2C} |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2,$$

and consequently

$$\alpha |J_\varepsilon(x) - J_\varepsilon(\bar{x})|^2 + \varepsilon |\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\partial)} \leq \frac{C}{2\varepsilon} |x - \bar{x}|_2^2.$$  

So, $J_\varepsilon$ is Lipschitz continuous in $L^2(\partial)$ as claimed. Consequently $A_\varepsilon = \frac{1}{\varepsilon} (1 - J_\varepsilon)$ is Lipschitz continuous in $L^2(\partial)$ as well. Moreover, since

$$\|\sigma(x)\|_{L^2(\Omega; L^2(\partial))} \leq \sum_{k=1}^{\infty} \nu_k^2 |x e_k|^2 \leq \sum_{k=1}^{\infty} \nu_k^2 |e_k|^2_{L^\infty(\partial)} \|x\|_2^2 \leq C \sum_{k=1}^{\infty} \nu_k^2 \lambda_k^2 |x|^2$$

we infer by standard existence theory for stochastic PDEs that for each $x \in L^2(\partial)$ equation (3.22) has a unique solution in $X_\varepsilon \in C_W([0, T]; L^2(\Omega; L^2(\partial)))$ (see e.g. [10]).

For $R > 0$ define

$$K_R := \{ X \in L^\infty_W(0, T; L^p(\Omega \times \partial)) : e^{-4\alpha t} \mathbb{E}|X(t)|_p^p \leq R^p \text{ for a.e. } t \in [0, T] \}$$
Lemma 3.3 Let $T > 0$ and $x \in L^p(\mathcal{O})$. Then for the solution $X_\varepsilon$ of (3.1) (or equivalently (3.22)) we have $X_\varepsilon \in L^\infty_W(0, T; L^p(\Omega \times \mathcal{O}))$ and $X_\varepsilon$ is bounded in $L^\infty_W(0, T; L^p(\Omega \times \mathcal{O}))$.

Proof. Obviously, $K_R$ is a closed subset of $L^\infty_W(0, T; L^p(\Omega \times \mathcal{O}))$. Since by (3.22) $X_\varepsilon$ is a fixed point of the map

$$X \mapsto e^{-\frac{t}{\varepsilon}}x + \varepsilon \int_0^t e^{-\frac{(t-s)}{\varepsilon}} J_\varepsilon(X(s))ds + \int_0^t e^{-\frac{(t-s)}{\varepsilon}} \sigma(X(s))dW(s), \quad t \in [0, T],$$

obtained by iteration in $C_W(0, T; L^2(\Omega \times \mathcal{O}))$, it suffices to prove that this map leaves $K_R$ invariant for $R$ large enough. But for $X \in K_R$ we have by (3.25) for $t \geq 0$

$$\left(e^{-p\alpha t}E \left|e^{-\frac{t}{\varepsilon}}x + \varepsilon \int_0^t e^{-\frac{(t-s)}{\varepsilon}} J_\varepsilon(X(s))ds\right|^p\right)^{1/p} \leq e^{-\alpha t}e^{-\frac{t}{\varepsilon}}|x|_p + e^{-\alpha t}E \left(\int_0^t \varepsilon e^{-\frac{(t-s)}{\varepsilon}}|J_\varepsilon(X(s))|_p ds\right)^p \leq e^{-\frac{1}{\varepsilon}(1+\alpha)t}|x|_p + e^{-\alpha t}K\int_0^t \varepsilon e^{-\frac{(t-s)}{\varepsilon}} \sigma(X(s))dW(s)$$

$$\leq e^{-\frac{1}{\varepsilon}(1+\alpha)t}|x|_p + e^{-\alpha t}R \int_0^t e^{-\frac{(t-s)}{\varepsilon}} e^{s_1} \cdots e^{s_p} ds_1 \cdots ds_p$$

$$\leq e^{-\frac{1}{\varepsilon}(1+\alpha)t}|x|_p + e^{-\alpha t}R \int_0^t \varepsilon e^{-\frac{(t-s)}{\varepsilon}} e^{s} ds$$

$$\leq e^{-\frac{1}{\varepsilon}(1+\alpha)t}|x|_p + \frac{R}{1+\alpha \varepsilon}.$$ 

Now we set

$$Y(t) = \int_0^t e^{-\frac{(t-s)}{\varepsilon}} X(s)dW(s), \quad t \geq 0.$$ 

Then

$$\left\{ \begin{array}{l}
\quad dY(t) + \frac{1}{\varepsilon} Y(t)dt = \sigma(X(t))dW(t), \quad t \geq 0, \\
Y(0) = 0.
\end{array} \right.$$
Let $\lambda > 0$. Applying Itô’s formula to the function

$$\Psi_\lambda(y) := \frac{1}{p} |(1 + \lambda A_0)^{-1}y|_p^p, \quad y \in L^p(\mathcal{O}),$$

(see the beginning of the proof of the next lemma for a detailed justification) we obtain via Hölder’s inequality that

$$\mathbb{E}[\Psi_\lambda(Y(t))] + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \int_{\mathcal{O}} |(1 + \lambda A_0)^{-1}Y(s)|^p d\xi ds$$

$$= \frac{p - 1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} |(1 + \lambda A_0)^{-1}Y(s)|^{p-2}$$

$$\times |(1 + \lambda A_0)^{-1}(X(s)e_k)|^2 d\xi ds$$

$$\leq C \mathbb{E} \int_0^t |(1 + \lambda A_0)^{-1}Y(s)|_p^2 |X(s)|_p^2 ds$$

$$\leq \frac{1}{2\varepsilon} \mathbb{E} \int_0^t |(1 + \lambda A_0)^{-1}Y(s)|_p^p ds + \frac{9C^2\varepsilon}{8} \mathbb{E} \int_0^t |X(s)|_p^p ds$$

$$\leq \frac{1}{2\varepsilon} \mathbb{E} \int_0^t |(1 + \lambda A_0)^{-1}Y(s)|_p^p ds + \frac{9C^2\varepsilon(e^{4\alpha t} - 1)}{32\alpha} R^p.$$

Then letting $\lambda \to \infty$, we see by Fatou’s lemma that for a.e. $t \in [0, T]$ we have for $C_1$ independent of $\varepsilon$

$$e^{-4\alpha t} \mathbb{E}|Y(t)|_p^p \leq \frac{C_1\varepsilon}{\alpha} R^p, \quad \forall \ t \in [0, T].$$

This means that for $\alpha$ large enough and $R > 2|x|_p$ the map leaves $K_R$ invariant as claimed.

**Lemma 3.4** For $x \in L^p(\mathcal{O})$ we have

$$X_\varepsilon \to X \quad \text{strongly in } L^\infty_W(0, T; L^2(\Omega; H)),$$

$$X_\varepsilon \to X \quad \text{weakly in } L^\infty_W(0, T; L^p(\Omega; L^p(\mathcal{O}))),$$

where $X$ is the solution to (1.1).
Proof. By (3.4) and Lemma 3.3 we know that \{X_\varepsilon\} is bounded in 
\[ L^2_W(0, T; L^2(\Omega; H)) \cap L^\infty_W(0, T; L^p(\Omega; L^p(\mathcal{E}))) \]
Subtracting equations (1.1) and (3.1) we get via Itô’s formula and because \beta is increasing that
\[
\frac{1}{2} \mathbb{E}|X_\varepsilon(t) - X(t)|^2 - \frac{1}{2} + \mathbb{E} \int_0^t \int_{\mathcal{E}} (\beta((1 + \varepsilon A)^{-1} X) - \beta(X))(X_\varepsilon - X)dsd\xi 
\leq c \mathbb{E} \int_0^t |X_\varepsilon(s) - X(s)|^2 ds,
\]
and by Gronwall’s lemma we obtain
\[
\mathbb{E}|X_\varepsilon(t) - X(t)|^2 \leq C \mathbb{E} \int_0^1 \int_{\mathcal{E}} (\beta((1 + \varepsilon A)^{-1} X) - \beta(X))(X_\varepsilon - X)ds d\xi. \quad (3.26)
\]
On the other hand, it follows by (3.25) that
\[
\int_{\Omega \times [0,T] \times \mathcal{E}} |(1 + \varepsilon A)^{-1} X|^p d\omega dt d\xi \leq \int_{\Omega \times [0,T] \times \mathcal{E}} |X|^p d\omega dt d\xi,
\]
while for \varepsilon \to 0
\[
(1 + \varepsilon A)^{-1} X \to X \quad \text{in} \quad L^1(\mathcal{E})
\]
for \((\omega, t) \in \Omega \times [0, T]\) (which is a consequence of the fact that the operator \(A\) is \(m\)-accretive in \(L^1(\mathcal{E})\), cfr. [2]). Hence (at least along a subsequence)
\[
(1 + \varepsilon A)^{-1} X \to X \quad \text{a.e. on} \quad \Omega \times [0, T] \times \mathcal{E}.
\]
Hence
\[
(1 + \varepsilon A)^{-1} X \to X \quad \text{weakly in} \quad L^p(\Omega \times [0, T] \times \mathcal{E})
\]
as \varepsilon \to 0 and according to the above inequality this implies that for \varepsilon \to 0,
\[
|(1 + \varepsilon A)^{-1} X|_{L^p} \to |X|_{L^p}. \quad \text{Hence since} \quad L^p(\Omega \times [0, T] \times \mathcal{E}) \text{is uniformly convex,}
\]
\[
(1 + \varepsilon A)^{-1} X \to X \quad \text{strongly in} \quad L^p(\Omega \times [0, T] \times \mathcal{E}),
\]
see [2]. Next by assumption (1.2) we have
\[
|\beta((1 + \varepsilon A)^{-1} X) - \beta(X)|
\leq \int_0^1 \beta'(\lambda(1 + \varepsilon A)^{-1} X) + (1 - \lambda)X)(1 + \varepsilon A)^{-1} X - X|d\lambda 
\leq C \left( |(1 + \varepsilon A)^{-1} X|^{m-1} + |X|^{m-1} + 1 \right) |(1 + \varepsilon A)^{-1} X - X|.
\]
This yields, via Hölder’s inequality

\[
\left| \mathbb{E} \int_0^t \int_\Theta (\beta((1 + \varepsilon A)^{-1}X) - \beta(X))(X_\varepsilon - X)ds d\xi \right|
\]

\[
\leq C |X_\varepsilon - X|_{L^p(\Omega \times [0,T] \times \Theta)} |(1 + \varepsilon A)^{-1}X - X|_{L^p(\Omega \times [0,T] \times \Theta)}
\]

\[
\times \left( |(1 + \varepsilon A)^{-1}X|_{L^p(\Omega \times [0,T] \times \Theta)}^{m-1} + |X|_{L^p(\Omega \times [0,T] \times \Theta)}^{m-1} + 1 \right)
\]

\[
\leq C_1 |(1 + \varepsilon A)^{-1}X - X|_{L^p(\Omega \times [0,T] \times \Theta)} \to 0,
\]

because \(\{X_\varepsilon\}\) is bounded in \(L^p(\Omega \times [0,T] \times \Theta)\) and \((m - 1)^{\frac{p}{p-2}} \leq p\). Now the assertion follows by (3.26).

Consider now the function

\[
\varphi(x) = \frac{1}{p} |x^{-}|_p^p.
\]

For any \(x \in L^p(\Theta)\), \(\varphi\) is Gâteaux differentiable and its differential \(D\varphi: L^p(\Theta) \to L^{p/(p-1)}(\Theta)\) is given by

\[
D\varphi(x) = -(x^{-})^{p-1},
\]

while the second Gâteaux derivative \(D^2\varphi(x) \in L(L^p(\Theta); L^{p/(p-1)}(\Theta))\) is given by

\[
(D^2\varphi(x)h, g) = (p-1) \int_\Theta h g |x^{-}|^{p-2}d\xi, \quad \forall \ h, g, x \in L^p(\Theta).
\]

**Lemma 3.5** Let \(n \leq 3\). For each \(x \in L^p(\Theta)\) we have

\[
\mathbb{E}[\varphi(X_\varepsilon(t))] + \mathbb{E} \int_0^t (A_\varepsilon X_\varepsilon(s), D\varphi(X_\varepsilon(s)))ds
\]

\[
= \varphi(x) + \frac{p-1}{2} \sum_{k=1}^\infty \mu_k^2 \mathbb{E} \int_0^t \int_\Theta |X_\varepsilon^{-}(s)e_k|^2 |X_\varepsilon^-(s)|^{p-2}dsd\xi. \tag{3.27}
\]

**Proof.** We note first that since \(X_\varepsilon \in L^\infty_W(0,T; L^p(\Omega; L^p(\Theta)))\) the above formula makes sense. Next we approximate \(\varphi\) by

\[
\varphi_\lambda(x) = \varphi((1 + \lambda A_0)^{-1}x), \quad A_0 = -\Delta, \quad D(A_0) = H^2(\Theta) \cap H^1_0(\Theta), \quad \lambda > 0.
\]
Since $\varphi \in C^2(C(\mathcal{O}))$ and $(1 + \lambda A_0)^{-1}$ is linear continuous from $L^2(\mathcal{O})$ to $C(\mathcal{O})$ (due to our assumption $n \leq 3$) we infer that $\varphi_\lambda \in C^2(L^2(\mathcal{O}))$ and its first order and second order differentials are given, respectively, by

$$D\varphi_\lambda(x) = D\varphi((1 + \lambda A_0)^{-1}x))(1 + \lambda A_0)^{-1},$$

$$(D^2\varphi_\lambda(x)h, k) = (D^2\varphi((1 + \lambda A_0)^{-1}x)((1 + \lambda A_0)^{-1}h, (1 + \lambda A_0)^{-1}k)$$

for $h, k \in L^2(\mathcal{O}), x \in L^2(\mathcal{O})$. Note that if $x \in L^p(\mathcal{O})$, then

$$D\varphi_\lambda(x) = -(1 + \lambda A_0)^{-1}((1 + \lambda A_0)^{-1}x)^p.$$

So, for $\lambda \to 0$ we have $\varphi_\lambda(x) \to \varphi(x)$ and $D\varphi_\lambda(x) \to D\varphi(x)$ in $L^p/(p-1)(\mathcal{O})$.

Next we write Itô’s formula for $\varphi_\lambda$ in the space $L^2(\mathcal{O})$ which makes sense by Lemma 3.2.

We get

$$E[\varphi_\lambda(X_\varepsilon(t))] + E\int_0^t (A_\varepsilon(X_\varepsilon(s)), D\varphi_\lambda(X_\varepsilon(s)))ds = \varphi_\lambda(x)$$

$$+ \frac{p-1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} |((1 + \lambda A_0)^{-1}(X_\varepsilon(s)e_k)|^2 |((1 + \lambda A_0)^{-1}X_\varepsilon(s))^p - 2|d\xi ds.$$

This yields

$$E[\varphi_\lambda(X_\varepsilon(t))] - E\int_0^t (1 + \lambda A_0)^{-1}(A_\varepsilon(X_\varepsilon(s)))((1 + \lambda A_0)^{-1}X_\varepsilon(s))^p d\xi ds$$

$$= \varphi_\lambda(x)$$

$$+ \frac{p-1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} |((1 + \lambda A_0)^{-1}X_\varepsilon(s))^p - 2|((1 + \lambda A_0)^{-1}(X_\varepsilon(s)e_k)|^2 d\xi ds.$$

(3.28)

We know that for $\lambda \to 0$, $(1 + \lambda A_0)^{-1}X_\varepsilon(s) \to X_\varepsilon(s)$ strongly in $L^p(\mathcal{O})$ a.e. in $\Omega \times (0,T)$ and

$$|(1 + \lambda A_0)^{-1}X_\varepsilon|_p \leq |X_\varepsilon|_p, \text{ a.e. in } \Omega \times (0,T).$$

Then by the Lebesgue dominated convergence theorem we have

$$\lim_{\lambda \to 0} (1 + \lambda A_0)^{-1}X_\varepsilon = X_\varepsilon \text{ strongly in } L^p(\Omega \times (0,T) \times \mathcal{O}).$$

(3.29)
Similarly, since $A_{\varepsilon}(X_{\varepsilon}) \in L^p(\Omega \times (0, T) \times \mathcal{O})$ we have for $\lambda \to 0$

$$(1 + \lambda A_0)^{-1}(A_{\varepsilon}(X_{\varepsilon})) \to A_{\varepsilon}(X_{\varepsilon}), \quad \text{strongly in } L^p(\Omega \times (0, T) \times \mathcal{O}).$$

and

$$((1 + \lambda A_0)^{-1}X_{\varepsilon})^- \to X_{\varepsilon}^-, \quad \text{strongly in } L^p(\Omega \times (0, T) \times \mathcal{O}).$$

This yields

$$\lim_{\lambda \to 0} \mathbb{E} \int_0^t \int_{\mathcal{O}} (1 + \lambda A_0)^{-1}(A_{\varepsilon}(X_{\varepsilon}(s)))((1 + \lambda A_0)^{-1}X_{\varepsilon}(s))^p d\xi ds$$

$$= \int_0^t \int_{\mathcal{O}} A_{\varepsilon}(X_{\varepsilon}(s))(X_{\varepsilon}^-(s)) d\xi ds.$$  \hspace{1cm} (3.30)

Then, if $x \in L^p(\mathcal{O})$ letting $\lambda \to 0$ in (3.28) we get (since by Fatou’s lemma $\mathbb{E}\varphi(X_{\varepsilon}(t)) \leq \lim \inf_{\lambda \to 0} \mathbb{E}\varphi(\lambda)(X_{\varepsilon}(t)), \forall t \geq 0$)

$$\mathbb{E}[\varphi(X_{\varepsilon}(t))] - \mathbb{E} \int_0^t \int_{\mathcal{O}} A_{\varepsilon}(X_{\varepsilon}(s))(X_{\varepsilon}^-(s))^p d\xi ds$$

$$= \varphi(x) + \frac{p-1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} |X_{\varepsilon}(s)e_k|^2 |X_{\varepsilon}^-(s)|^p d\xi ds,$$

and so (3.27) follows. □

We have by (3.27) and the definition of $Y_{\varepsilon}$ that for $x \in L^p(\mathcal{O}), x \geq 0$,

$$\mathbb{E}[\varphi(X_{\varepsilon}(t))] + \mathbb{E} \int_0^t \int_{\mathcal{O}} \Delta \beta(Y_{\varepsilon}(s))(X_{\varepsilon}^-(s))^p d\xi d\xi$$

$$= \frac{p-1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} |X_{\varepsilon}^-(s)e_k|^2 |X_{\varepsilon}^-(s)|^p d\xi ds$$

$$\leq C \mathbb{E} \int_0^t |X_{\varepsilon}^-(s)|^p ds.$$

(Recall that $A_{\varepsilon}(X_{\varepsilon}) = -\Delta \beta(Y_{\varepsilon})$.)

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We therefore have, taking into account that $\Delta \beta(Y_{\varepsilon}) = \frac{1}{\varepsilon}(Y_{\varepsilon} - X_{\varepsilon})$,
\[
\frac{1}{p} \mathbb{E}|X_{\varepsilon}^{-}(t)|^p + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \int_{\mathcal{O}} (Y_{\varepsilon}(s) - X_{\varepsilon}(s))(X_{\varepsilon}^{-}(s))^p d\xi ds 
\leq C \mathbb{E} \int_0^t |X_{\varepsilon}^{-}(s)|^p ds. 
\] (3.31)
We have
\[
|Y_{\varepsilon}^{-}(t)|^p \leq \int_{\mathcal{O}} X_{\varepsilon}(t)(-Y_{\varepsilon}^{-}(t))^p d\xi, \quad \mathbb{P}\text{-a.s.,} \quad (3.32)
\] analogously to deriving (3.24), for $x \in L^p(\mathcal{O})$. To see this multiply (3.23) by $g(y)$ where
\[
g(y) : = -\frac{(y^-)^{p-1}}{1 + \lambda(y^-)^{p-2}},
\]
to get (after integration by parts) that
\[
\int_{\mathcal{O}} \frac{(y^-)^2}{1 + \lambda(y^-)^{p-2}} d\xi + \varepsilon \int_{\mathcal{O}} \langle \nabla \beta(y), \nabla g(y) \rangle_{\mathbb{R}^n} d\xi = \int_{\mathcal{O}} \frac{x^-(-y^-)^3}{1 + \lambda(y^-)^2} d\xi.
\]
Note that $g$ as a composition of two decreasing Lipschitz functions is Lipschitz and decreasing. So, we can apply Lemma 3.1 to obtain
\[
\int_{\mathcal{O}} \frac{(y^-)^4}{1 + \lambda(y^-)^2} d\xi \leq \int_{\mathcal{O}} \frac{x^-(-y^-)^3}{1 + \lambda(y^-)^2} d\xi
\]
and (3.32) follows by taking $\lambda \to \infty$. By (3.32) we have
\[
-|Y_{\varepsilon}^{-}(t)|^p \geq \int_{\mathcal{O}} (X_{\varepsilon}^{+}(t) - X_{\varepsilon}^{-}(t))(Y_{\varepsilon}^{-}(t))^p d\xi \geq - \int_{\mathcal{O}} X_{\varepsilon}^{-}(t)(Y_{\varepsilon}^{-}(t))^p d\xi
\]
and therefore $|Y_{\varepsilon}^{-}(t)|^p \leq |X_{\varepsilon}^{-}(t)|^p |Y_{\varepsilon}^{-}(t)|^p$. Hence $|Y_{\varepsilon}^{-}(t)|^p \leq |X_{\varepsilon}^{-}(t)|^p$ and so
\[
\int_{\mathcal{O}} Y_{\varepsilon}^{-}(t)(X_{\varepsilon}^{-}(t))^p d\xi \leq |X_{\varepsilon}^{-}(t)|^p |Y_{\varepsilon}^{-}(t)|^p \leq |X_{\varepsilon}^{-}(t)|^p.
\]
Inserting the latter into (3.31) and taking into account that $Y_{\varepsilon}X_{\varepsilon}^{-} \geq -Y_{\varepsilon}X_{\varepsilon}^{-}$ we see that $\mathbb{E}|X_{\varepsilon}^{-}(t)|^p = 0$, a.e. $t \geq 0$ i.e. $X_{\varepsilon}^{-}(t) = 0$ a.e. and therefore $X_{\varepsilon}(t) \geq 0$ a.e.. Taking into account Lemma 3.4 we infer that $X \geq 0$. This completes the proof in the case when $\beta$ is strictly monotone.
To treat the general case of $\beta$ satisfying (1.2) we shall associate to (1.4) the equation
\[
\begin{cases}
  dX^\lambda(t) + A^\lambda X^\lambda(t) = \sigma(X^\lambda(t))dW(t), & t \geq 0, \\
  X^\lambda(0) = x,
\end{cases}
\tag{3.33}
\]
where
\[A^\lambda(x) = -\Delta(\beta(x) + \lambda x), \quad \lambda > 0\]
and
\[D(A^\lambda) = \{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \beta(x) + \lambda x \in H^1_0(\mathcal{O})\}\]
According to the first part of the proof, for each $x \in L^p(\mathcal{O}), x \geq 0$ and $\lambda > 0$, equation (3.33) has a unique strong solution $X^\lambda$ which is nonnegative a.e. on $\Omega \times (0, T) \times \mathcal{O}$.

On the other hand, applying the Itô formula from [11, Theorem I 3.2] to the equation
\[
d(X^\lambda(t) - X(t)) + (A^\lambda X^\lambda(t) - AX(t))dt = (X^\lambda(t) - X(t))dW(t)
\]
where $X$ is the solution to (1.1), we get after some calculations that
\[
\frac{1}{2} \mathbb{E}|X^\lambda(t) - X(t)|^2_{-1} + \lambda \mathbb{E} \int_0^t \langle X^\lambda(s), X^\lambda(s) - X(s) \rangle_{-1} ds
\]
\[
\leq \frac{1}{2} \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E} \int_0^t |(X^\lambda(s) - X(s))e_k|^2_{-1} ds.
\]
This yields (see (1.9)), since
\[
\langle X^\lambda(s), X^\lambda(s) - X(s) \rangle_{-1} \geq \langle X(s), X^\lambda(s) - X(s) \rangle_{-1},
\]
\[
\mathbb{E}|X^\lambda(t) - X(t)|^2_{-1} \leq C \mathbb{E} \int_0^t |X^\lambda(s) - X(s)|^2_{-1} ds + \lambda^2 \mathbb{E} \int_0^t |X(s)|^2_{-1} ds.
\]
Since $X \in C_W([0, T]; L^2(\Omega, L^2(\mathcal{O})))$, we infer via Gronwall’s lemma that
\[
\lim_{\lambda \to 0} X^\lambda = X \quad \text{in} \quad C_W([0, T]; L^2(\Omega, L^2(\mathcal{O})))
\]
and so $X \geq 0$ a.e. in $\Omega \times (0, T) \times \mathcal{O}$ as claimed.

The final part of the assertion in Theorem 2.2 follows by the continuity of sample paths, since $L^p(\mathcal{O})$ is dense in $H^{-1}(\mathcal{O})$ and the continuity of solutions $X = X(t, x)$ with respect to the initial data $x$ (which follows via Itô’s formula in the proof of uniqueness). □
4 Concluding remarks

Assumption 1 ≤ n ≤ 3 is unnecessarily strong and was taken for convenience only. As a matter of fact, under suitable conditions of the form (1.8) we expect that Theorem 2.2 can be established for any dimension n. This will be the subject of a forthcoming paper.

2) Theorem 2.2 and its proof remain valid for time–dependent nonlinear functions β = β(t, x) where β is monotonically increasing in x, satisfies (1.2) uniformly with respect to t and is continuous in t.

3) One might speculate however that nonnegativity of X(t, x) for x ≥ 0 follows directly in $H^{-1}(\mathcal{O})$ by taking instead of $\varphi(x) = \frac{1}{p}|x^-|^p$ a suitable $C^2$-function on $H^{-1}(\mathcal{O})$ which is zero on the cone of positive $x \in H^{-1}(\mathcal{O})$ but so far we failed to find such a function.

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