MOTIVIC & ARITHMETIC PROBABILITY OF A SEMISTABLE ELLIPTIC SURFACE WITH A WEIERSTRASS TORSION SECTION

JUN–YONG PARK

Abstract. We prove new sharp asymptotic for counting the semistable elliptic curves with two marked Weierstrass points at \( \infty \) and 0 and also the cases where 0 is a 2-torsion or a 3-torsion marked Weierstrass point over \( \mathbb{F}_p(t) \) by the bounded height of discriminant \( \Delta(X) \). We consider the motivic probabilities over any basefield \( K \) with \( \text{char}(K) \neq 2,3 \) of picking a nonsingular semistable elliptic surface over \( \mathbb{P}^1 \) with two marked Weierstrass sections at \( \infty \) and 0 such that marked Weierstrass section at 0 is 2-torsion or 3-torsion. In the end, we formulate an analogous heuristics on \( \mathbb{Z}_0(B) \) for the ratio of the semistable elliptic curves with a marked rational 2-torsion or 3-torsion Weierstrass point at 0 out of all semistable elliptic curves with a marked rational Weierstrass points at 0 over \( \mathbb{Q} \) by the bounded height of discriminant \( \Delta \) through the global fields analogy.

1. Motivic & Arithmetic probability of a semistable elliptic surface over \( \mathbb{P}^1 \) with a Weierstrass torsion section

It is well-known that any genus 1 algebraic curve can be embedded in \( \mathbb{P}^2 \) as a cubic hypersurface and a plane cubic having a single (nonsingular) rational Weierstrass point at \( \infty = [0 : 1 : 0] \) can be presented in the following affine generalized Weierstrass form \( \mathcal{W} \):

\[
\mathcal{W} : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6
\]

Where \( a_i \in \mathbb{Z} \). Let \( b_2 = a_1^2 + 4a_2 \), \( b_4 = 2a_4 + a_1a_3 \), \( b_6 = a_3^2 + 4a_6 \), \( b_8 = a_7^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_1^2 \) then such a curve defines a non-singular elliptic curve when the discriminant \( \Delta = -b_2b_8 - 8b_6^3 - 27b_3^2 + 9b_2b_4b_6 \neq 0 \). For an elliptic curve \( E \), every point on \( P \in E \) is a Weierstrass point as there always exists a function whose only pole is a double one at \( P \).

Recall that a pair \((E, p)\) is a stable elliptic curve if \( E \) is a nodal projective curve of arithmetic genus 1 and \( p \in E \) is a smooth point. Then, it is well–known by [DM] that \( \overline{\mathcal{M}}_{1,1} \) is a proper Deligne–Mumford stack of stable elliptic curves with a coarse moduli space \( \mathcal{M}_{1,1} \cong \mathbb{P}^1 \) parameterizing the \( j \)--invariants of elliptic curves. Denote \([\infty] \in \overline{\mathcal{M}}_{1,1} \) to be the unique point of \( \overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1} \). Notice that \( \overline{\mathcal{M}}_{1,1} \) comes equipped with a universal family \( p : \overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1} \). We consider the following definition of the weighted projective stack \( \mathcal{P}(\bar{\lambda}) \) for a more concrete description of \( \overline{\mathcal{M}}_{1,1} \):

**Definition 1.** Fix a tuple of nondecreasing positive integers \( \bar{\lambda} = (\lambda_0, \ldots, \lambda_N) \). The \( N \)-dimensional weighted projective stack \( \mathcal{P}(\bar{\lambda}) \) associated to the weight \( \bar{\lambda} \) is defined as a quotient stack

\[
\mathcal{P}(\bar{\lambda}) := \left[ (\mathbb{A}^{N+1}_{x_0, \ldots, x_N} \setminus 0) / \mathbb{G}_m \right]
\]

where \( \zeta \in \mathbb{G}_m \) acts by \( \zeta \cdot (x_0, \ldots, x_N) = (\zeta^\lambda_0 x_0, \ldots, \zeta^\lambda_N x_N) \). In this case, the degree of \( x_i \)'s are \( \lambda_i \)'s respectively. A line bundle \( \mathcal{O}_{\mathcal{P}(\bar{\lambda})}(m) \) is defined to be a line bundle associated to the sheaf of degree \( m \) homogeneous rational functions without poles on \( \mathbb{A}^{N+1}_{x_0, \ldots, x_N} \setminus 0 \).

When the characteristic of the base field \( K \) is not equal to 2 or 3, applying further change of variables to \( \mathcal{W} \) by the name of Tschirnhaus transformation to eliminate the middle degree terms, we have the short Weierstrass equation \( \mathcal{F}_0 : y^2 = x^3 + a_4 x + a_6 \) with \( \Delta = -16(a_4^3 + 27a_6^2) \). By using the short Weierstrass equation \( \mathcal{F}_0 \), [Hassett] shows that \( (\overline{\mathcal{M}}_{1,1})_K \cong [(\text{Spec} \ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \ldots \)
\[ \mathcal{P}_K(4, 6) \text{ where } \lambda \cdot a_i = \lambda^i a_i \text{ for } \lambda \in \mathbb{G}_m \text{ and } i = 4, 6. \] Thus, the \( a_i \)'s have degree \( i \) respectively. Similarly, we have the following various families of Weierstrass equations considered in \([\text{BH1, BH2}]\) for the families of the elliptic curves with the marked Weierstrass points at \( \infty \) and \((0, 0)\) which allows us to formulate the corresponding moduli stacks.

**Proposition 2.** When the characteristic of the field \( K \) is not equal to 2 or 3, we can formulate the moduli stacks of stable elliptic curves with the marked Weierstrass points at \( \infty \) and \((0, 0)\) as follows.

1. The family of all elliptic curves with the marked Weierstrass points at \( \infty \) and \((0, 0)\) over \( K \) is

\[
F_1 : y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x
\]

\[ \Delta = -16a_2^3 a_3^2 + 16a_2^2 a_3^3 - 64a_4^3 - 27a_3^4 + 56a_2 a_4 a_3^2 \]

which implies that the moduli stack of stable elliptic curves with the marked Weierstrass points at \( \infty \) and \((0, 0)\) is isomorphic to

\[
(\mathcal{M}_{1,2})_K \cong [(\text{Spec } K[a_2, a_3, a_4] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4)
\]

2. The family of all elliptic curves with the marked Weierstrass points at \( \infty \) and the torsion point of order 2 at \((0, 0)\) over \( K \) is

\[
F_1(2) : y^2 = x^3 + a_2 x^2 + a_4 x
\]

\[ \Delta = 16a_2^3 a_3^2 - 64a_4^3 \]

which implies that the moduli stack of stable elliptic curves with the marked Weierstrass points at \( \infty \) and the torsion point of order 2 at \((0, 0)\) is isomorphic to

\[
(\mathcal{M}_{1,2(2)})_K \cong [(\text{Spec } K[a_2, a_4] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(2, 4)
\]

3. The family of all elliptic curves with the marked Weierstrass points at \( \infty \) and the torsion point of order 3 at \((0, 0)\) over \( K \) is

\[
F_1(3) : y^2 + a_1 xy + a_3 y = x^3
\]

\[ \Delta = a_1^3 a_3^2 - 27a_3^4 \]

which implies that the moduli stack of stable elliptic curves with the marked Weierstrass points at \( \infty \) and the torsion point of order 3 at \((0, 0)\) is isomorphic to

\[
(\mathcal{M}_{1,2(3)})_K \cong [(\text{Spec } K[a_1, a_3] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(1, 3)
\]

Where \( \lambda \cdot a_i = \lambda^i a_i \text{ for } \lambda \in \mathbb{G}_m \text{ and } i = 1, 2, 3, 4. \) Thus, the \( a_i \)'s have degree \( i \) respectively.

We quickly recall what we have learned in \([\text{HP}]\) regarding the motive & the arithmetic of the moduli stack \( \mathcal{L}_{1,12n} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,1} \cong \mathcal{P}(4, 6)) \) of nonsingular semistable elliptic fibrations over \( \mathbb{P}^1 \), also known as semistable elliptic surfaces, with 12\( n \) nodal singular fibers and a Weierstrass section as in \([\text{HP, FS}]\) by the series of the following results.

We first consider the moduli stack \( \mathcal{L}_{1,12n} \) of stable elliptic fibrations over \( \mathbb{P}^1 \) with 12\( n \) nodal singular fibers and a Weierstrass section.

**Proposition 3** (Proposition 9 of \([\text{HP}]\)). The moduli stack \( \mathcal{L}_{1,12n} \) of stable elliptic fibrations over \( \mathbb{P}^1 \) with 12\( n \) nodal singular fibers and a Weierstrass section is the Deligne–Mumford stack \( \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,1}) \) parameterizing morphisms \( f : \mathbb{P}^1 \to \mathcal{M}_{1,1} \) such that \( f^* \mathcal{O}_{\mathcal{P}(4,6)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n) \).

We show the equivalence of \( K \)-points between the two moduli functors where \( K \) is any field of characteristic neither 2 nor 3.
Theorem 5 (Theorem 1 of [HP] and Corollary 5 of [PS]). If \( \text{char}(K) \) does not divide \( a \) or \( b \), then the class \([\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))] \) in \( K_0(\text{Stck}_K) \) is equivalent to

\[ \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) = L^{(a+b)n+1} - L^{(a+b)n-1}. \]

If \( \text{char}(\mathbb{F}_q) \) does not divide \( a \) or \( b \), then

\[ \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))) = q^{(a+b)n+1} - q^{(a+b)n-1}. \]

And we apply this to the case of \( \overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6) \) to acquire the following.

Corollary 6 (Corollary 1.2 of [HP] and Corollary 5 of [PS]). If \( \text{char}(K) \neq 2, 3 \), then

\[ [\mathcal{L}_{1,12n}] = L^{10n+1} - L^{10n-1}. \]

If \( \text{char}(\mathbb{F}_q) \neq 2, 3 \),

\[ \#_q(\mathcal{L}_{1,12n}) = q^{10n+1} - q^{10n-1}. \]

This implies that the number of isomorphism classes of \( \mathbb{F}_q \)-points of \( \mathcal{L}_{1,12n} \) is \( |\mathcal{L}_{1,12n}(\mathbb{F}_q)| = 2 \cdot (q^{10n+1} - q^{10n-1}) \) (see Remark 19 of [HP]). Since a semistable elliptic surface \( f : X \to \mathbb{P}^1_{\mathbb{F}_q} \) is a semistable elliptic curve over \( \mathbb{F}^1_{\mathbb{F}_q} \), we acquire the following count of \( Z_{\mathbb{F}_q}(\mathcal{B}) \) by bounding the height of discriminant \( \Delta(X) \) when \( q \) is not divisible by 6:

Theorem 7 (Theorem 3 of [HP]). The counting of semistable elliptic curves over \( \mathbb{F}^1_{\mathbb{F}_q} \) by \( \text{ht}(\Delta(X)) = q^{12n} \leq \mathcal{B} \) satisfies the following inequality:

\[ Z_{\mathbb{F}_q}(\mathcal{B}) \leq 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left( \mathcal{B}^{\frac{5}{2}} - 1 \right) \]

which is an equality when \( \mathcal{B} = q^{12n} \) for some \( n \in \mathbb{N} \) implying that the acquired upper bound is a sharp asymptotic with the lower order term of zeroth order (i.e. constant).

The motivic identities and the arithmetic invariants of the \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\bar{\lambda})) \) for varying \( \bar{\lambda} \) is the following Grothendieck motive classes.

Theorem 8 (Theorem 2 of [HP2]). If \( \text{char}(K) \) does not divide \( \lambda_i \) for all \( 0 \leq i \leq N \) of \( \bar{\lambda} = (\lambda_0, \ldots, \lambda_N) \) with \( |\bar{\lambda}| := \sum_{i=0}^N \lambda_i \). The Grothendieck class of the Hom stack \([\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\bar{\lambda}))]\) in the Grothendieck ring of \( K \)-stacks \( K_0(\text{Stck}_K) \) is equivalent to

\[ [\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\bar{\lambda}))] = \left( \sum_{i=0}^N L^i \right) \cdot \left( L^{|\bar{\lambda}|n} - L^{\bar{\lambda}|n-N} \right) . \]
We now consider the moduli stack $\mathcal{L}_{1,12n,2}$ of stable elliptic fibrations over $\mathbb{P}^1$ with $12n$ nodal singular fibers and two marked Weierstrass sections at $\infty$ and $0$.

**Proposition 9.** The moduli stack $\mathcal{L}_{1,12n,2}$ of stable elliptic fibrations over $\mathbb{P}^1$ with $12n$ nodal singular fibers and two marked Weierstrass sections at $\infty$ and $0$ is the Deligne–Mumford stack $\text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2})$ parameterizing morphisms $f : \mathbb{P}^1 \to \mathcal{M}_{1,2}$ such that $f^* \mathcal{O}_{\mathbb{P}^1(2,3,4)}(1) \cong \mathcal{O}_{\mathbb{P}^1(n)}$.

**Proof.** By the definition of the universal family $p$, any stable elliptic fibration $f : Y \to \mathbb{P}^1$ with two marked Weierstrass sections at $\infty$ and $0$ comes from a morphism $\varphi_f : \mathbb{P}^1 \to \mathcal{M}_{1,2}$ and vice versa. As this correspondence also works in families, we can formulate the moduli of stable elliptic fibrations with two marked Weierstrass sections at $\infty$ and $0$ as $\text{Hom}(\mathbb{P}^1, \mathcal{M}_{1,2})$. Observe that $\mathcal{M}_{1,2} \cong \mathcal{P}(2,3,4)$ by the Weierstrass equation $\mathcal{F}_1$. Above discussion shows that $\mathcal{L}_{1,12n,2} \cong \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2})$. As $\mathcal{M}_{1,2}$ is Deligne–Mumford, the Hom stack $\text{Hom}(\mathbb{P}^1, \mathcal{M}_{1,2})$ is Deligne–Mumford by Olsson. And since $\deg f^* \mathcal{O}_{\mathbb{P}^1(2,3,4)}(1) = n$ is an open condition, $\text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2})$ is an open substack of $\text{Hom}(\mathbb{P}^1, \mathcal{M}_{1,2})$. □

We apply the Proposition 11 of [HP] to case of the moduli $\mathcal{L}_{1,12n,2} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2} \cong \mathcal{P}(2,3,4))$ and acquire the following Grothendieck motive class for the moduli $\mathcal{L}_{1,12n,2}$ of semistable elliptic surfaces with two Weierstrass sections at $\infty$ and $0$.

**Corollary 10.** If $\text{char}(K) \neq 2, 3$, then

$$|\mathcal{L}_{1,12n,2}| = \mathbb{L}^{9n+2} + \mathbb{L}^{9n+1} - \mathbb{L}^{9n-1} - \mathbb{L}^{9n-2}.$$  

Over $\text{char}(\mathbb{F}_q) \neq 2, 3$,

$$\#_q(\mathcal{L}_{1,12n,2}) = q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}.$$  

Similarly, we consider the moduli stack $\mathcal{L}_{1,12n,2(2)} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(2)} \cong \mathcal{P}(2,4))$ of stable elliptic surfaces with two marked Weierstrass sections at $\infty$ and $0$ such that the section at $0$ is 2-torsion as well as the 3-torsion case with $\mathcal{L}_{1,12n,2(3)} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(3)} \cong \mathcal{P}(1,3))$.

**Proposition 11.** The moduli stack $\mathcal{L}_{1,12n,2(2)}$ of stable elliptic fibrations over $\mathbb{P}^1$ with $12n$ nodal singular fibers and two marked Weierstrass sections at $\infty$ and $0$ such that the section at $0$ is 2-torsion is the Deligne–Mumford stack $\text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(2)} \cong \mathcal{P}(2,4))$ parameterizing morphisms $f : \mathbb{P}^1 \to \mathcal{M}_{1,2(2)}$ such that $f^* \mathcal{O}_{\mathbb{P}^1(2,4)}(1) \cong \mathcal{O}_{\mathbb{P}^1(n)}$. Similarly, the 3-torsion case applies with $\mathcal{L}_{1,12n,2(3)} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(3)} \cong \mathcal{P}(1,3))$.

**Proof.** Without the loss of the generality, we show the 2-torsion case with $\mathcal{L}_{1,12n,2(2)} := \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(2)} \cong \mathcal{P}(2,4))$. By the definition of the universal family $p$, any stable elliptic fibration $f : Y \to \mathbb{P}^1$ with two marked Weierstrass sections at $\infty$ and $0$ such that the section at $0$ is 2-torsion comes from a morphism $\varphi_f : \mathbb{P}^1 \to \mathcal{M}_{1,2(2)}$ and vice versa. As this correspondence also works in families, we can formulate the moduli of stable elliptic fibrations with two marked Weierstrass sections at $\infty$ and $0$ such that the section at $0$ is 2-torsion as Hom$(\mathbb{P}^1, \mathcal{M}_{1,2(2)})$. Observe that $\mathcal{M}_{1,2(2)} \cong \mathcal{P}(2,4)$ by the Weierstrass equation $\mathcal{F}_1(2)$ and its coarse map is $c : \mathcal{M}_{1,2(2)} \to \mathcal{M}_{1,2(2)} \cong \mathbb{P}^1$, so that $c$ can be identified with $c : \mathbb{P}(2,4) \to \mathbb{P}^1$ where $c(x,y) = [x^6 : y^3] \in \mathbb{P}^1$ for any $(x,y) \in \mathbb{P}(2,4) \cong [(\mathbb{A}^2_x \setminus \{0\})/\mathbb{G}_m]$. Since each coordinate function of $\mathbb{P}^1$ lifts to degree 12 functions on $\mathbb{P}(2,4)$, we conclude that $c^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}(2,4)}(12)$. This implies that $\deg(c \circ \varphi_f) = 12 \deg \varphi_f$ where $\deg \varphi_f := \deg \varphi_f^* \mathcal{O}_{\mathbb{P}(2,4)}(1)$. Note that the discriminant divisor $\Delta$ of $f$ can be recovered by pulling back $\infty \in \mathbb{P}^1$ via $c \circ \varphi_f$. Above discussion shows that $\mathcal{L}_{1,12n,2(2)} \cong \text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(2)})$. As $\mathcal{M}_{1,2(2)}$ is Deligne–Mumford, the Hom stack $\text{Hom}(\mathbb{P}^1, \mathcal{M}_{1,2(2)})$ is Deligne–Mumford by Olsson. And since $\deg f^* \mathcal{O}_{\mathbb{P}(2,4)}(1) = n$ is an open condition, $\text{Hom}_n(\mathbb{P}^1, \mathcal{M}_{1,2(2)})$ is an open substack of $\text{Hom}(\mathbb{P}^1, \mathcal{M}_{1,2(2)})$.
We apply the Proposition 11 of [HP] to case of the moduli $\mathcal{L}_{1,12n,2(2)} \cong \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2(2)} \cong \mathcal{P}(2,4))$ and now have the following Grothendieck motive class for the moduli $\mathcal{L}_{1,12n,2(2)}$ of semistable elliptic surfaces with two marked Weierstrass sections at $\infty$ and 0 such that the section at 0 is 2-torsion as well as the 3-torsion case with $\mathcal{L}_{1,12n,2(3)} := \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2(3)} \cong \mathcal{P}(1,3))$.

**Corollary 12.** If $\text{char}(K) \neq 2, 3$, then

$$[\mathcal{L}_{1,12n,2(2)}] = \mathbb{L}^{6n+1} - \mathbb{L}^{6n-1},$$

$$[\mathcal{L}_{1,12n,2(3)}] = \mathbb{L}^{4n+1} - \mathbb{L}^{4n-1}.$$  

Over $\text{char}(\mathbb{F}_q) \neq 2, 3$,

$$#_q(\mathcal{L}_{1,12n,2(2)}) = q^{6n+1} - q^{6n-1},$$

$$#_q(\mathcal{L}_{1,12n,2(3)}) = q^{4n+1} - q^{4n-1}.$$  

The motivic arithmetic invariants allow us to consider the following fractions from probability perspective. Similar fractions where elliptic curves with certain features over $\mathbb{Q}$ ordered by height of the invariants of discriminants or conductors have been considered in the past under the notion of densities where one takes the limit with respect to height. In our case, there is no need for the limiting procedure as we have explicit exact counts in terms of the polynomials giving us the exact probability which is free of base field as well since we have the same Grothendieck motive classes for any fields $K$ of $\text{char}(K) \neq 2, 3$.

**Theorem 13.** If $\text{char}(K) \neq 2, 3$, the motivic probability over $K$ of picking a nonsingular semistable elliptic surface with two marked Weierstrass sections at $\infty$ and 0 such that the section at 0 is 2-torsion out of all the nonsingular semistable elliptic surfaces with two marked Weierstrass sections at $\infty$ and 0 is

$$\frac{[\mathcal{L}_{1,12n,2(2)}]}{[\mathcal{L}_{1,12n,2}]} = \frac{\mathbb{L}^{6n+1} - \mathbb{L}^{6n-1}}{\mathbb{L}^{9n+2} + \mathbb{L}^{9n+1} - \mathbb{L}^{9n-1} - \mathbb{L}^{9n-2}} = \frac{1}{\mathbb{L}^{3n+1} + \mathbb{L}^{3n} + \mathbb{L}^{3n-1}}.$$  

We have the corresponding arithmetic probability over $\text{char}(\mathbb{F}_q) \neq 2, 3$,

$$\frac{#_q(\mathcal{L}_{1,12n,2(2)})}{#_q(\mathcal{L}_{1,12n,2})} = \frac{q^{6n+1} - q^{6n-1}}{q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}} = \frac{1}{q^{3n+1} + q^{3n} + q^{3n-1}}.$$  

Geometrically, as $\mathcal{P}(2,4)$ is embedded in $\mathcal{P}(2,3,4)$ this fraction computes the probability of a rational curve $\mathbb{P}^1 \to \mathcal{P}(2,3,4)$ landing inside $\mathcal{P}(2,4)$.

**Theorem 14.** If $\text{char}(K) \neq 2, 3$, the motivic probability over $K$ of picking a nonsingular semistable elliptic surface with two marked Weierstrass sections at $\infty$ and 0 such that the section at 0 is 3-torsion out of all the nonsingular semistable elliptic surfaces with two marked Weierstrass sections at $\infty$ and 0 is

$$\frac{[\mathcal{L}_{1,12n,2(3)}]}{[\mathcal{L}_{1,12n,3}]} = \frac{\mathbb{L}^{4n+1} - \mathbb{L}^{4n-1}}{\mathbb{L}^{9n+2} + \mathbb{L}^{9n+1} - \mathbb{L}^{9n-1} - \mathbb{L}^{9n-2}} = \frac{1}{\mathbb{L}^{5n+1} + \mathbb{L}^{5n} + \mathbb{L}^{5n-1}}.$$  

We have the corresponding arithmetic probability over $\text{char}(\mathbb{F}_q) \neq 2, 3$,

$$\frac{#_q(\mathcal{L}_{1,12n,2(3)})}{#_q(\mathcal{L}_{1,12n,3})} = \frac{q^{4n+1} - q^{4n-1}}{q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}} = \frac{1}{q^{5n+1} + q^{5n} + q^{5n-1}}.$$  

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2. Counting semistable elliptic curves with a Weierstrass torsion point over global fields by $\Delta$

Through the global fields analogy, we compute $\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B})$ which is the counting function of the semistable elliptic surfaces with $12n$ nodal singular fibers and two Weierstrass sections as well as $\mathcal{Z}_{2(2),\mathbb{F}_q(t)}(\mathcal{B})$ for the 2-torsion Weierstrass section case and $\mathcal{Z}_{2(3),\mathbb{F}_q(t)}(\mathcal{B})$ for the 3-torsion Weierstrass section case at $(0,0)$. We explicitly compute $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ by the arithmetic invariant $|\mathcal{L}_{1,12n,2}(\mathbb{F}_q)|$ in the function field setting. An analogous object in the number field setting is the global fields the counting of semistable elliptic curves over $\mathbb{Q}$, which is the counting of semistable elliptic curves over $\mathbb{Q}$. In the end, we formulate a heuristic that for both of the global fields the countings $\mathcal{Z}_K(\mathcal{B})$ will match with one another.

As the generic point of $\mathbb{P}_{\mathbb{F}_q}^1$ (the base of semistable elliptic fibrations) is indeed Spec of a rational function field of one variable $t$ over $\mathbb{F}_q$, one could think of a semistable elliptic surface $X$ over $\mathbb{P}_q^1$ as the choice of a model for semistable elliptic curves $E$ over $K = \mathbb{F}_q(t)$ or equivalently over $\mathcal{O}_K = \mathbb{F}_q[t]$ by clearing the denominators. On the number field, the analogy would be the semistable elliptic curves $E$ with the squarefree conductor $N = p_1 \cdots p_{\mu}$ over $\mathbb{Q}$ or equivalently over $\mathcal{O}_K = \mathbb{Z}$ as relative curves over a Dedekind scheme by the minimal integral Weierstrass model of an elliptic curve. In order to draw the analogy, we need to fix an affine chart $\mathbb{A}_{\mathbb{F}_q}^1 \subset \mathbb{P}_{\mathbb{F}_q}^1$ and its corresponding ring of functions $\mathbb{F}_q[t]$, since $\mathbb{F}_q[t]$ could come from any affine chart of $\mathbb{P}_{\mathbb{F}_q}^1$, whereas the ring of integers for the number field $K$ is canonically determined. We denote $\infty \in \mathbb{P}_{\mathbb{F}_q}^1$ to be the unique point not in the chosen affine chart.

We first count the all possible two marked Weierstrass sections as the sample space.

**Theorem 15** (Computation of $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$). The counting of semistable elliptic curves over $\mathbb{P}_{\mathbb{F}_q}^1$ with two marked Weierstrass points at $\infty$ and 0 by $ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$ satisfies the following inequality:

\[
\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^2 - 1) + \frac{(q^7 - q^6)}{(q^6 - 1)} \cdot (\mathcal{B}^2 - 1)
\]

which is an equality when $\mathcal{B} = q^{12n}$ for some $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp asymptotic with the leading term of order $O\left(\mathcal{B}^2\right)$ and the lower order terms of order $O\left(\mathcal{B}^{3/2}\right)$ and zeroth order (i.e. constant).

**Proof.** Note that $\overline{\mathcal{M}}_{1,2} \cong \mathcal{P}(2,3,4)$ has the substack $\mathcal{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{L}_{1,12n,2}$ with discriminant degree $12n$ is $|\mathcal{L}_{1,12n,2}(\mathbb{F}_q)| = (q^{bn+2} + q^{bn+1} - q^{bn-2} - q^{bn-3}) + (q^{bn+1} - q^{bn-1})$ by Corollary [11] and [HP2 Proposition 30]. We can explicitly compute the bounds for $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,
\[ Z_{1,2,F_q(t)}(B) = \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} |L_{1,12n,2}(F_q)| = \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) \]

\[ = (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} q^{6n} \]

\[ \leq (q^2 + q^1 - q^{-1} - q^{-2}) \left( q^9 + \cdots + q^{9 \log_B \frac{B}{12}} \right) + (q^1 - q^{-1}) \left( q^6 + \cdots + q^{6 \log_B \frac{B}{12}} \right) \]

\[ = (q^2 + q^1 - q^{-1} - q^{-2}) \cdot \frac{q^9(B^\frac{1}{2} - 1)}{(q^9 - 1)} + (q^1 - q^{-1}) \frac{q^6(B^\frac{1}{2} - 1)}{(q^6 - 1)} \]

\[ = \frac{(q^{11} + q^{10} - q^8 - q^9)}{(q^9 - 1)} \cdot (B^\frac{1}{2} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (B^\frac{1}{2} - 1) \]

On the third line of the equations above, inequality becomes an equality if and only if \( n := \frac{\log_B \frac{B}{12}}{12} \in \mathbb{N} \), i.e., \( B = q^{12n} \) for some \( n \in \mathbb{N} \). This implies that the acquired upper bound on \( Z_{1,2,F_q(t)}(B) \) is a sharp asymptotic of with terms of order \( O \left( B^\frac{1}{2} \right) \), \( O \left( B^\frac{1}{2} \right) \) and zeroth order.

Next, we count the two sections with one 2-torsion section at 0 case.

**Theorem 16** (Computation of \( Z_{1,2,F_q(t)}(B) \)). The counting of semistable elliptic curves over \( \mathbb{P}^1_{\mathbb{F}_q} \) with two marked Weierstrass points at \( \infty \) and 0 where 0 is 2-torsion by \( \text{ht}(\Delta(X)) = q^{12n} \leq B \) satisfies the following inequality:

\[ Z_{1,2,F_q(t)}(B) \leq 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (B^\frac{1}{2} - 1) \]

which is an equality when \( B = q^{12n} \) for some \( n \in \mathbb{N} \) implying that the acquired upper bound is a sharp asymptotic with the leading term of order \( O \left( B^\frac{1}{2} \right) \) and the lower order term of zeroth order (i.e. constant).

**Proof.** Knowing the number of \( F_q \)-isomorphism classes of semistable elliptic surfaces with two marked Weierstrass sections at \( \infty \) and 0 such that the section at 0 is 2-torsion of discriminant degree 12n over \( \mathbb{F}_q \) is \( |L_{1,12n,2}(F_q)| = 2 \cdot (q^{6n+1} - q^{6n-1}) \) by Corollary [12] and [HP2 Proposition 30], we can explicitly compute the bounds for \( Z_{1,2,F_q(t)}(B) \) as the following,

\[ Z_{1,2,F_q(t)}(B) = \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} |L_{1,12n,2}(F_q)| = \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} 2 \cdot (q^{6n+1} - q^{6n-1}) \]

\[ = 2 \cdot (q^1 - q^{-1}) \cdot \sum_{n=1}^{\lfloor \log_B \frac{B}{12} \rfloor} q^{6n} \leq 2 \cdot (q^1 - q^{-1}) \left( q^6 + \cdots + q^{6 \log_B \frac{B}{12}} \right) \]

\[ = 2 \cdot (q^1 - q^{-1}) \frac{q^6(B^\frac{1}{2} - 1)}{(q^6 - 1)} = 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (B^\frac{1}{2} - 1) \]

On the second line of the equations above, inequality becomes an equality if and only if \( n := \frac{\log_B \frac{B}{12}}{12} \in \mathbb{N} \), i.e., \( B = q^{12n} \) for some \( n \in \mathbb{N} \). This implies that the acquired upper bound on
$Z_{1,2(3),F_q(t)}(B)$ is a sharp asymptotic of with the leading term of order $O\left(B^{12}\right)$ and the lower order term of zeroth order (i.e. constant).

Next, we count the two sections with one 3-torsion section at 0 case.

**Theorem 17 (Computation of $Z_{1,2(3),F_q(t)}(B)$).** The counting of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with two marked Weierstrass points at $\infty$ and 0 where 0 is 3-torsion by $\text{ht}(\Delta(X)) = q^{12n} \leq B$ satisfies the following inequality:

$$Z_{1,2(3),F_q(t)}(B) \leq 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (B^{12} - 1)$$

which is an equality when $B = q^{12n}$ for some $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp asymptotic with the leading term of order $O\left(B^{12}\right)$ and the lower order term of zeroth order (i.e. constant).

**Proof.** Knowing the number of $\mathbb{F}_q$-isomorphism classes of semistable elliptic surfaces with two marked Weierstrass sections at $\infty$ and 0 such that the section at 0 is 3-torsion of discriminant degree $12n$ over $\mathbb{F}_q$ is $|\mathcal{L}_{1,12n,2(3)}(\mathbb{F}_q)| = q^{4n+1} - q^{4n-1}$ by Corollary [12] and [HP2, Proposition 30], we can explicitly compute the bounds for $Z_{1,2(3),F_q(t)}(B)$ as the following,

$$Z_{1,2(3),F_q(t)}(B) = \sum_{n=1}^{\left\lfloor \frac{\log q}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)| = \sum_{n=1}^{\left\lfloor \frac{\log q}{12} \right\rfloor} q^{4n+1} - q^{4n-1}$$

$$= (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{\log q}{12} \right\rfloor} q^{4n} \leq (q^1 - q^{-1}) (q^4 + \cdots + q^{4(\left\lfloor \frac{\log q}{12} \right\rfloor)})$$

$$= (q^1 - q^{-1}) q^4 \frac{(B^{12} - 1)}{(q^4 - 1)} = \left(\frac{q^5 - q^3}{q^4 - 1}\right) \cdot (B^{12} - 1)$$

On the second line of the equations above, inequality becomes an equality if and only if $n := \frac{\log q}{12} \in \mathbb{N}$, i.e., $B = q^{12n}$ for some $n \in \mathbb{N}$. This implies that the acquired upper bound on $Z_{1,2(3),F_q(t)}(B)$ is a sharp asymptotic of with the leading term of order $O\left(B^{12}\right)$ and the lower order term of zeroth order (i.e. constant).

Thus we have the following arithmetic probability for curves over global function fields $\mathbb{F}_q(t)$.

**Theorem 18.** If $\text{char}(\mathbb{F}_q) \neq 2, 3$, the arithmetic probability of picking a semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ by $\text{ht}(\Delta(X)) = q^{12n} \leq B$ with two marked Weierstrass points at $\infty$ and 0 where 0 is 2-torsion out of all the semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with two marked Weierstrass points at $\infty$ and 0 satisfies the following inequality:

$$\frac{Z_{1,2(3),F_q(t)}(B)}{Z_{1,2(3),F_q(t)}(B)} \leq \frac{2 \cdot (q^9 - 1) \cdot (q^7 - q^5) \cdot (B^{12} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (q^9 - 1) \cdot (B^{12} - 1) + (q^9 - 1) \cdot (q^7 - q^5) \cdot (B^{12} - 1)}$$

which is an equality when $B = q^{12n}$ for some $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp asymptotic with the leading term of order $O\left(B^{12}\right)$.
Proof.
\[
\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{2}(t)(B)}{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{3}(t)(B)} \leq \frac{2 \cdot (q^7 - q^5) \cdot (B_\frac{1}{4} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (B_\frac{1}{4} - 1) + (q^7 - q^5) \cdot (B_\frac{1}{4} - 1)}
\]
\[
= \frac{2 \cdot (q^7 - q^5) \cdot (B_\frac{3}{4} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (q^6 - 1) \cdot (B_\frac{3}{4} - 1) + (q^9 - 1) \cdot (q^7 - q^5) \cdot (B_\frac{3}{4} - 1)}
\]
\[
\]

Similar computation for \(\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{3}(t)(B)}{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{4}(t)(B)}\) can be done as well.

Theorem 19. If \(\text{char} \mathbb{F}_q \neq 2, 3\), the arithmetic probability of picking a semistable elliptic curves over \(\mathbb{P}^1_{\mathbb{F}_q}\) by \(ht(\Delta(X)) = q^{12n} \leq B\) with two marked Weierstrass points at \(\infty\) and \(0\) where \(0\) is 3-torsion out of all the semistable elliptic curves over \(\mathbb{P}^1_{\mathbb{F}_q}\) with two marked Weierstrass points at \(\infty\) and \(0\) satisfies the following inequality:

\[
\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{3}(t)(B)}{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{4}(t)(B)} \leq \frac{(q^9 - 1) \cdot (q^6 - 1) \cdot (q^5 - q^3) \cdot (B_\frac{1}{4} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (q^4 - 1) \cdot (q^4 - 1) \cdot (B_\frac{3}{4} - 1) + (q^9 - 1) \cdot (q^4 - 1) \cdot (q^7 - q^5) \cdot (B_\frac{3}{4} - 1)}
\]

which is an equality when \(B = q^{12n}\) for some \(n \in \mathbb{N}\) implying that the acquired upper bound is a sharp asymptotic with the leading term of order \(O \left( B^{-\frac{3}{12}} \right)\)

Proof.
\[
\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{3}(t)(B)}{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{4}(t)(B)} \leq \frac{(q^5 - q^3) \cdot (B_\frac{1}{4} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (q^6 - 1) + (q^7 - q^5) \cdot (B_\frac{1}{4} - 1)}
\]
\[
= \frac{(q^9 - 1) \cdot (q^6 - 1) \cdot (q^5 - q^3) \cdot (B_\frac{1}{4} - 1)}{(q^{11} + q^{10} - q^8 - q^7) \cdot (q^4 - 1) \cdot (B_\frac{3}{4} - 1) + (q^9 - 1) \cdot (q^4 - 1) \cdot (q^7 - q^5) \cdot (B_\frac{3}{4} - 1)}
\]

Lastly, we consider the global fields analogy, which says that global function field \(\mathbb{F}_q(t)\) and algebraic number field \(\mathbb{Q}\) are expected to share many properties. Thus, we formulate the following conjecture by passing the above sharp asymptotic through the global fields analogy:

Conjecture 20 (Heuristic on \(\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{2}(B)}{Z_{\mathbb{P}^1_{\mathbb{Q}}}(B)}\)). The probability \(\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{2}(B)}{Z_{\mathbb{P}^1_{\mathbb{Q}}}(B)}\) of picking semistable elliptic curves with a marked 2-torsion Weierstrass rational point at \((0, 0)\) out of all the semistable elliptic curves with a marked Weierstrass rational point at \((0, 0)\) over \(\mathbb{Z}\) by \(ht(\Delta) \leq B\) follows from the sharp asymptotic arithmetic probability on \(\frac{Z_{\mathbb{P}^1_{\mathbb{F}_q}}^{2}(B)}{Z_{\mathbb{P}^1_{\mathbb{Q}}}(B)}\) through the global fields analogy. Namely, \(Z_{\mathbb{Q}}(B)\) has the leading term of order \(O \left( B^{-\frac{1}{2}} \right)\).

And similarly for the 3-torsion case as well.
Conjecture 21 (Heuristic on $\mathcal{Z}_1(3)\mathbb{Q}(B)$). The probability $\frac{\mathcal{Z}_1(3)\mathbb{Q}(B)}{\mathcal{Z}_1(3)\mathbb{Z}(B)}$ of picking semistable elliptic curves with a marked 3-torsion Weierstrass rational point at $(0,0)$ out of all the semistable elliptic curves with a marked Weierstrass rational point at $(0,0)$ over $\mathbb{Z}$ by $ht(\Delta) \leq B$ follows from the sharp asymptotic arithmetic probability on $\frac{\mathcal{Z}_1(3)\mathbb{Q}(B)}{\mathcal{Z}_1(3)\mathbb{Z}(B)}$ through the global fields analogy. Namely, $\mathcal{Z}_1(3)\mathbb{Q}(B)$ has the leading term of order $O\left(B^{-\frac{1}{12}}\right)$.

Remark 22. The analogous counting problem has been considered over number field $\mathbb{Q}$ by the work of Robert Harron and Andrew Snowden [HS] where they were able to show that the order of the leading term for $\mathcal{Z}_{1,2}(2)\mathbb{Q}(B)$ is indeed $O\left(B^{\frac{1}{2}}\right)$ as well as the order of the leading term for $\mathcal{Z}_{1,2}(3)\mathbb{Q}(B)$ is indeed $O\left(B^{\frac{1}{3}}\right)$. While the orders of the lower order terms are still unknown over $\mathbb{Q}$, the matching correspondence in terms of the order of the leading term for the counting function $\mathcal{Z}_{1,2}(n)\mathbb{Q}(B)$ with $n = 2, 3$ over global fields $\mathbb{F}_q(t)$ and $\mathbb{Q}$ seems suggest that if one could count all the other corresponding counting functions for the rest of the twelve possibilities for the torsion subgroups classified as in [Mazur], one would acquire the order of the leading term to be equal to the ones found in [HS] with corresponding lower order terms that are difficult to acquire over number field $\mathbb{Q}$ in general. We will investigate this in future work.

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Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea

E-mail address: junepark@ibs.re.kr