Bribing in First-Price Auctions: Corrigendum

Maciej H. Kotowski∗ Shiran Rachmilevitch†

February 15, 2013

Abstract

We clarify the sufficient condition for a trivial equilibrium to exist in the model of Rachmilevitch (2013).

Rachmilevitch (2013), henceforth R13, studies the following game. Two ex ante identical players are about to participate in an independent-private-value first-price, sealed bid auction for one indivisible object. After the risk-neutral players learn their valuations but prior to the actual auction, player 1 can offer a take-it-or-leave-it (TIOLI) bribe to his opponent in exchange for the opponent dropping out of the contest. If the offer is accepted, player 1 is the only bidder and obtains the item for free; otherwise, both players compete non-cooperatively in the auction as usual. This is called the first-price TIOLI game.1 R13 shows that under the restriction to continuous and monotonic bribing strategies for player 1, any equilibrium of this game must be trivial—the equilibrium bribing function employed by player 1, if it is continuous and non-decreasing, must be identically zero. In this note, we clarify the sufficient conditions under which a trivial equilibrium exists. These are less stringent than originally proposed.

Let $F$ denote the cumulative distribution function of players’ types (valuations). $F$ is atomless, has full support on $[0, 1]$, and its density is $f$. The following is Theorem 2 from R13.

**Theorem 2.** Suppose that $F$ is differentiable and that it satisfies $2F(t) + tf(t) \geq 1$ for all $t \in (0, 1]$, where $f = F'$. Then, the first-price TIOLI game has a trivial equilibrium.

---

∗Kennedy School of Government, Harvard University. E-mail: <maciej_kotowski@hks.harvard.edu>
†(Corresponding Author) Department of Economics, University of Haifa. E-mail: <shiranrach@gmail.com>, <shiranrach@econ.haifa.ac.il>

1See R13 for a detailed description of the game. The inspiration for this game comes from Eső and Schummer (2004), who study a second-price auction preceded by a bribing stage.
An unfortunate fact regarding this theorem is that it is vacuously true.

**Claim 1.** There does not exist a distribution \( F \) such that \( 2F(t) + tf(t) \geq 1 \) for all \( t \in (0, 1] \).

**Proof.** Suppose the contrary and choose \( 0 < \epsilon < 1/2 \). Since \( F \) is continuous and \( F(0) = 0 \), there exists \( \bar{t} > 0 \) such that for all \( t < \bar{t}, F(t) < \epsilon \). Thus, for all \( 0 < t < \bar{t}, 2\epsilon + tf(t) \geq 1 \). However, this implies that for \( t \in (0, \bar{t}) \), \( F(t) = \int_0^t f(x)dx \geq \int_0^t \frac{1-2\epsilon}{x}dx = \infty \), which is a contradiction. \( \square \)

Fortunately, Theorem 2’s conclusion is true under a relatively weak alternative condition. All that is required is that \( F \) is concave. The intuition is that when there is a high probability that bidders have low valuations, player 1 does not find it worthwhile to bribe player 2. This is the same intuition as initially proposed by R13.

**Theorem 2’.** If \( F \) is concave, the first-price TIOLI game has a trivial equilibrium.

To prove this theorem we first establish a useful lemma using a geometric argument.

**Lemma 1.** Suppose \( b \) and \( x \) are two positive numbers such that \( b + x \leq 1 \). Then

\[
F(b)x + [F(b + x) - F(b)]b \leq \int_0^{b+x} F(t)dt. \tag{1}
\]

**Proof.** We consider two cases. In case 1, suppose \( 0 \leq x \leq b \). We make our argument with reference to Figure 1a. In the figure, \( \int_0^{b+x} F(t)dt \) is the region below the thick curve, \( F(t) \), to the left of \( b + x \). The left-hand side of (1), \( F(b)x + [F(b + x) - F(b)]b \), equals the shaded region, or \( A + B + C + D \). Since \( F \) is concave and therefore \( F(x) \geq F(b + x) - F(b) \), it easily follows that \( Y \geq B \). Thus, it is sufficient to show that \( X \geq C \). By concavity of \( F \), \( D \geq C \). Finally,

\[
D = \int_b^{b+x} F(t) - F(b)dt = \int_x^{b+x} F(t)dt - \int_0^x F(t)dt = \int_0^x (F(t) - F(b))dt \leq \int_0^x F(t)dt = X.
\]

Hence, \( X \geq D \geq C \) as required.

For case 2, suppose \( 0 \leq b \leq x \). The situation is as in Figure 1b. Again it is sufficient to show that \( D \leq X \). This inequality follows since

\[
D = \int_x^{b+x} F(t) - F(x)dt = \int_0^b F(x+t) - F(x)dt \leq \int_0^b F(t)dt = X.
\]

\( \square \)
We can now prove Theorem 2' by adapting the argument from the proof of Theorem 2 in R13.

**Proof of Theorem 2'**. Consider the following strategy profile. Player 1 offers a bribe of zero independent of his type. If this bribe is rejected, he bids as in the one-shot, symmetric Bayesian Nash equilibrium (BNE) of the first-price auction. Irrespective of type, player 2 rejects a bribe of zero and post-rejection bids as in the symmetric BNE of the auction. Player 2 accepts the bribe of $b > 0$ if and only if his valuation $\theta_2 \leq b$. If player 2 rejects the bribe $b > 0$, he believes that player 1 is bidding $(\theta_2 - b)$, and he bids $(\theta_2 - b)^+$.² In this case, player 1 is prescribed his optimal bid in this post-rejection-of-$b$ information set (it is easy to show that such a best-response exists).

It is sufficient to verify that player 1 does not have a profitable deviation to a strictly positive bribe. Let $b > 0$ be the bribe offered by player 1 and let $x$ be player 1’s bid in the auction following the (possible) rejection of $b$ by player 2. Obviously, we can assume that $x \leq 1 - b$.³ Given the prescribed (off-equilibrium path) behavior of player 2, the expected

²The bid $r^+$ is identical to the bid $r$, except that it wins for sure if the competing bid is $r' \leq r$. See R13 for the details.

³If player 2 rejects the sure payoff $b$, then optimality dictates that he does not bid more than $\theta_2 - b$ in the auction; therefore, player 1 has no reason to bid strictly above $1 - b$. 
payoff of player 1 is
\[ \Pi(b, x|\theta_1) = F(b)(\theta_1 - b) + [F(b + x) - F(b)](\theta_1 - x). \]  
(2)

On the equilibrium path, the expected payoff of bidder 1 of type \( \theta_1 \) is \( \pi(\theta_1) = \int_0^{\theta_1} F(t)dt \). It is sufficient to verify that for all \( \theta_1 \) and for all \( 0 < b \leq \theta_1 \) and \( 0 \leq x \leq 1 - b \), it is the case that \( \Pi(b, x|\theta_1) \leq \pi(\theta_1) \). Let \( \psi(\theta_1) \equiv \pi(\theta_1) - \Pi(b, x|\theta_1) \). Note that \( \psi'(\theta_1) = F(\theta_1) - F(b + x) \), so \( \psi \) has a minimum at \( \theta_1 = b + x \). Also, \( \Pi(b, x|b + x) = F(b)x + [F(b + x) - F(b)]b \). By Lemma 1, \( \psi(b + x) \geq 0 \). Therefore, \( \psi \geq 0 \). Put differently, \( \pi(\theta_1) - \Pi(b, x|\theta_1) \geq 0 \) for all \( \theta_1 \).

\[ \Box \]

References

Eső, P. and Schummer, J. (2004). Bribing and signaling in second price auctions. *Games and Economic Behavior*, 47(2):299–324.

Rachmilevitch, S. (2013). Bribing in first-price auctions. *Games and Economic Behavior*, 77(1):214–228.