New Five Dimensional Black Holes Classified by Horizon Geometry, and a Bianchi VI Braneworld

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Abstract

We introduce two new families of solutions to the vacuum Einstein equations with negative cosmological constant in 5 dimensions. These solutions are static black holes whose horizons are modelled on the 3-geometries nilgeometry and solvegeometry. Thus the horizons (and the exterior spacetimes) can be foliated by compact 3-manifolds that are neither spherical, toroidal, hyperbolic, nor product manifolds, and therefore are of a topological type not previously encountered in black hole solutions. As an application, we use the solvegeometry solutions to construct Bianchi VI,\textsubscript{\textminus}1 braneworld cosmologies.

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I Introduction

In 1972, Hawking [1] proved the first black hole topology theorem, which stated that the smooth event horizon of a stationary black hole obeying the dominant energy condition was spherical. Hawking followed this with a topology theorem for apparent horizons [2], while Gannon [3] was able to remove the assumption of stasis in favour of a certain regularity condition. Much later, Chruściel and Wald [4] were able to apply the active topological censorship theorem to obtain a new topology theorem for black holes in asymptotically flat spacetimes that did not rely on smoothness of the event horizon.

Nonetheless, within the last decade, it has become clear that under appropriate circumstances the event horizon of a 4-dimensional black hole can be a compact Riemann surface of arbitrary genus. Examples are provided by the various black holes discovered in locally anti-de Sitter backgrounds [5, 6, 7]. These black holes avoid Hawking’s theorem because they do not obey the dominant energy condition. Hawking’s basic argument still applies in the absence of this condition, but leads only to a lower bound on the area of the horizon, not to a restriction on the topology [8]. Topological censorship arguments also apply, and they lead to a genus inequality stating that the genus of the horizon or, more generally, the sum of the genera of all components of the horizon, cannot exceed the genus of scri [9]. Thus, black holes with non-zero genus horizons have boundaries-at-infinity (scri) that do not have the usual topology $S^2 \times R$ for anti-de Sitter space.

Topological censorship constraints on spacetime topology have a surprising relevance to the case of 5-dimensional spacetime, where they resolve a puzzle that would otherwise plague the AdS/CFT correspondence [12, 13]. Now in 5 dimensions, the event horizon of a stationary black hole is a ruled hypersurface foliated by a compact orientable 3-dimensional Riemannian manifold, dragged along the null generators. Thurston’s famous geometrization conjecture, if correct, would provide a classification of such 3-manifolds, and hence of horizon topologies in 5 dimensions. The conjecture asserts that 3-dimensional compact manifolds can be decomposed by cutting along certain embedded 2-spheres and incompressible tori in a unique way such that the resulting pieces are each covered by 3

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3From this, one recovers the result of Chruściel and Wald as the special case in which the genus of scri is zero. Then the genus inequality asserts that the components of the horizon must be spherical.

There are also so-called temporarily toroidal black holes that occur in numerical simulations of gravitational collapse of asymptotically flat initial data. These horizons cut certain spacelike hypersurfaces in a torus, but cut later hypersurfaces in a sphere. The phenomenon is a consequence both of the nature of the collapse and of the choice of spacelike hypersurfaces. For the explanation of these horizons in the context of topological censorship, see [9]. Moreover, the discussion in [9] leads one to conclude that toroidal horizons of this nature will always contain “crossover points” where generators of the horizon begin, and so will not be smooth, thus circumventing Hawking’s theorem.
exactly one “model geometry.” We define this notion more precisely in Section 2.1.
Thurston has shown that there are eight model geometries. The model geometries admit
homogeneous metrics and are very nearly in correspondence with the nine homogeneous
geometries of Bianchi. Descriptions of the model geometries and Thurston’s work can
be found in Thurston’s book \[14\] and the article by Scott \[15\]. The correspondence with
Bianchi models is detailed in \[16\].

Then a natural question is whether static black hole horizons in 5 dimensions can be
built from arbitrary compact 3-manifolds. A less ambitious problem is to begin with the
Thurston model geometries themselves, and ask, “For each Thurston model geometry, can
one find a static 5-dimensional Einstein manifold with a black hole whose event horizon is
foliated by a compact 3-manifold modelled on the given model geometry?” In the present
work we explore this issue.

The question is of interest from several points of view. First, if it proves difficult to
construct solutions for various horizon topologies, then this may indicate that there are
new constraints on horizon topology. Such constraints may even descend to 4 dimensions.
Second, in 4 and 5 dimensions, the search for \( g > 0 \) black holes has led to spacetimes
that exhibit unexpected and remarkable properties, including negative mass solutions
that are not nakedly singular, being either black holes or geodesically complete. This has
led Myers and Horowitz to conjecture a new positive energy theorem for these solutions
in which negative but bounded mass is permitted \[10, 11\]. It would be interesting to
determine whether similar behaviour occurs when more general topologies are permitted.
Third, present ideas in dimensional reduction suggest that our cosmos may be a 3-brane
evolving in a 5-dimensional spacetime. As we will see below, in pursuing the above
question, we will be led to homogeneous but non-FRW braneworld cosmologies.

Before one can address such issues, it is important to have available exact solutions
from which to develop intuition and against which to test hypotheses. Therefore, in the
present article, we focus primarily on the question of whether we can obtain solutions
with horizons modelled on the Thurston geometries. We outline our approach to this in
Section 2.1. For the 5 “untwisted” model geometries, such black holes are easy to obtain
and have already appeared in the literature; we briefly discuss these cases in Section 2.2.
In Sections 2.3 and 2.4, we give two new families of black hole solutions whose horizons are
modelled on the Sol and Nil 3-geometries, respectively. These solutions admit topologies
not previously encountered in black holes. We give explicit constructions of some of these
topologies. The case of a horizon modelled on the \( \tilde{SL}(2, \mathbb{R}) \) Thurston geometry remains
open. We illustrate one application of our new solutions: from the solvegeometry black
hole we find Bianchi VI\(_{-1}\) braneworld cosmologies in Section 3.
Throughout this article, early roman indices are abstract indices. Middle roman indices label elements of a triad of basis vectors and indicate components of tensors with respect to that basis. We assume all manifolds are Hausdorff.

II  Model 3-Geometries and Black Hole Solutions

II.1  Preliminaries

A model geometry is defined as a pair \((X, G)\), where \(X\) is a connected and simply connected \(n\)-manifold and

\(\begin{enumerate}
\item \(G\) is a Lie group of diffeomorphisms acting transitively on \(X\) with compact point stabilisers.
\item \(G\) is maximal; i.e., \(G\) is not a proper subgroup of a larger group \(H\) acting in the required way on \(X\), and
\item there is a subgroup \(\Gamma \leq G\) acting on \(X\) as a covering group, such that the quotient \(M\) is a compact \(n\)-manifold.
\end{enumerate}\)

Any \(M\) meeting the requirements of point \((iii)\) is said to be modelled on \((X, G)\), and is called an \((X, G)\) manifold. Note that point \((i)\) implies that \(X\) admits a complete, homogeneous, Riemannian metric invariant by \(G\) (cf. propositions 3.4.10 and 3.4.11 of [14]), a fact that we will exploit.

There are 8 model 3-geometries ([14], Theorem 3.8.4). They are denoted as spherical (or elliptical), hyperbolic, Euclidean (or flat), \(\mathbb{S}^2 \times \mathbb{S}^1\), \(\mathbb{H}^2 \times \mathbb{S}^1\), nilgeometry, solvegeometry, and \(\widetilde{\text{SL}}(2, \mathbb{R})\). While manifolds modelled on the first five geometries are familiar, manifolds modelled on last 3, the so-called “twisted product” cases, may not be to many readers. Sections 2.3 and 2.4 therefore contain explicit constructions of compact Sol- and Nil-manifolds, respectively. Further detail can be found in [14, 15]. We will not deal with \(\widetilde{\text{SL}}(2, \mathbb{R})\) manifolds in what follows, except for brief remarks in the Discussion section.

Our strategy is first to find 5-dimensional spacetimes \((M, g_{\alpha\beta})\) foliated by spacelike 3-surfaces \(X_{t,r}\) that carry one of the 3-geometries \((X, G)\). We assume a product topology, for several reasons. This assumption will not only lead to relatively simple Einstein equations, but will also ensure we do not come up against the known topological censorship

\footnote{The stabiliser of \(p \in X\) is the isotropy group at \(p\).}
constraints, which can forbid some non-product topologies. Moreover, we seek space-
times with a null hypersurface as inner boundary, representing an event horizon, and
by the product topology assumption this boundary will also be foliated by the relevant
3-geometry. We further assume that

(i) the spacetime is a static Einstein manifold whose

(ii) time-symmetric hypersurfaces are foliated by homogeneous 3-surfaces generated by
isometries of the spacetime.

We have chosen to impose assumption (ii) here because it considerably simplifies the sec-
ond part of our task, which is to determine compact topologies for the surfaces \(X_{t,r}\). Each
model geometry \((X, G)\) admits a homogeneous metric invariant by \(G\), so this assump-
tion implies that \(G\) is a subgroup of the spacetime isometry group, with orbits that are
3-surfaces foliating spacetime, such that the induced metric on the orbits is \(G\)-invariant.
We then seek to identify spacetime points that are related by isometries in \(G\) acting freely
and properly discontinuously. Since the identifications are by isometries, the quotient
metric will be smooth. Our task will be to choose the isometries to have cocompact
action on each leaf.5

Under the above conditions, one can write the metric in the form

\[
\begin{aligned}
ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + h_{ij}(r)\omega^i\omega^j ,
\end{aligned}
\]

(II.1)

where the \(\omega^i\) are invariant 1-forms and \(h_{ab}(r) = h_{ij}(r)\omega^i_a\omega^j_b\) is the metric induced on the
t = const., r = const. surfaces. Our invariant 1-forms are those found in Ryan and
Shepley [17]. The horizon is the zero set of \(V(r)\). The requirement that (II.1) be an
Einstein metric can be written in dimension \(n\) as

\[
R_{ab} = \frac{2\Lambda}{n-2}g_{ab} ,
\]

(II.2)

where \(R_{ab}\) is the Ricci tensor and \(\Lambda\) is the cosmological constant.

5An action of \(\Gamma\) on a locally compact space \(X\) is properly discontinuous iff for every compact subset
\(C \subseteq X\) the set \(\{\gamma \in \Gamma : \gamma C \cap C \neq \emptyset\}\) is finite. It is free if \(\gamma x \neq x\) whenever \(\gamma \in \Gamma\) is not the identity.
If \(\Gamma\) acts properly discontinuously on \(X\), it is called a discrete group of transformations of \(X\), or more
simply a discrete group. If a discrete group \(\Gamma\) acts freely on \(X\), then \(X/\Gamma\) will be a (Hausdorff) manifold.
If \(X/\Gamma\) is compact, the action of \(\Gamma\) (or \(\Gamma\) itself) is called cocompact.
II.2 Constant Curvature Horizons and Product Horizons

Here we catalogue solutions whose horizon topologies are modelled on 3-geometries of type spherical, flat, hyperbolic, $\mathbb{H}^2 \times S^1$ (where $\mathbb{H}^2$ is 2-dimensional hyperbolic space), and $S^2 \times S^1$.

Consider an $n$-dimensional warped product spacetime

$$(M, g_{ab}) = (\overline{M}, \overline{g}_{ab}) \times f^2 (\tilde{M}, \tilde{g}_{ab}) \ ,$$

with metric $\overline{g}_{ab} \oplus f^2 \tilde{g}_{ab}$, $f : \overline{M} \to \mathbb{R}$, where $(\overline{M}, \overline{g}_{ab})$ is a spacetime of dimension $\overline{n}$ and $(\tilde{M}, \tilde{g}_{ab})$ is a Riemannian manifold of dimension $\tilde{n}$, so $\dim(M) = n = \overline{n} + \tilde{n}$. We use bars to denote $(\overline{M}, \overline{g}_{ab})$ quantities, tildes to denote $(\tilde{M}, \tilde{g}_{ab})$ quantities, and no adornment to denote $(M, g_{ab})$ quantities in what follows. The Ricci curvature of such a product is then the direct sum of two terms:

$$R_{ab} = \left[ R_{ab} - \frac{\tilde{n}}{f(r)} \nabla_a \nabla_b f(r) \right]$$

$$\oplus \left[ \tilde{R}_{ab} - \tilde{g}_{ab} \left( f(r) \Delta f(r) + (\tilde{n} - 1) \tilde{g}^{ab} \nabla_a f(r) \nabla_b f(r) \right) \right] \ ,$$

where $\nabla_a$ and $\Delta$ are respectively the covariant derivative and d’Alembertian on $(\overline{M}, \overline{g}_{ab})$. When $g_{ab}$ is an Einstein metric, $\tilde{R}_{ab}$ will satisfy an equation of the form $\tilde{R}_{ab} + \tilde{g}_{ab}(\ldots) = 0$. When $\tilde{n} > 2$, the contracted Bianchi identity on $(\tilde{M}, \tilde{g}_{ab})$ implies that the coefficient of $\tilde{g}_{ab}$ must be constant and therefore $\tilde{g}_{ab}$ will necessarily be an Einstein metric on $\tilde{M}$.

Taking $i, j \in \overline{n}, \ldots, n - 1$ and specializing to $\overline{n} = 2$, we take the line elements of these metrics to be

$$ds^2 = d\tilde{s}^2 + f^2(r) dr^2 \ ,$$

$$d\tilde{s}^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} \ ,$$

$$d\tilde{s}^2 = \tilde{g}_{ij} dx^i dx^j \ .$$

By using (II.5) (II.7) to expand (II.4) and substituting the results into the Einstein equations (II.2), we obtain

$$\frac{1}{2} V''(r) + \frac{(n - 2)}{2} V'(r) \frac{f'(r)}{f(r)} = -\frac{2}{n - 2} \Lambda \ ,$$

$$f''(r) = 0 \ ,$$

$$\frac{1}{f^2(r)} \tilde{R}_{ij} = \left[ (n - 3) V(r) \left( \frac{f'(r)}{f(r)} \right)^2 + V'(r) \frac{f'(r)}{f(r)} + \frac{2}{n - 2} \Lambda \right] \tilde{g}_{ij} \ .$$
Solving these equations, we find (up to constant rescalings and translations of the \( r \) coordinate) two classes of solutions, depending on the integration of \( f''(r) \). The first is

\[
\begin{align*}
  f(r) &= r^2, \\
  V(r) &= \frac{2\Lambda}{(n-1)(n-2)} r^2 + k - \frac{2M}{r^{n-3}}, \\
  \tilde{R}_{ij} &= (n-3)k \delta_{ij},
\end{align*}
\]

where \( k \) and \( M \) are constants, and \((\tilde{M}, \tilde{g}_{ab})\) is an Einstein \((n-2)\)-manifold of scalar curvature \((n-2)(n-3)k\). In the \( n = 5 \) case, the solutions of (II.13) are the constant curvature metrics in dimension 3:

\[
dS^2 = \begin{cases} 
  d\theta^2 + \sin^2 \theta (d\phi^2 + \sin^2 \phi \, d\xi^2), & k = 1, \\
  d\theta^2 + d\phi^2 + d\xi^2, & k = 0, \\
  d\theta^2 + \sinh^2 \theta (d\phi^2 + \sin^2 \phi \, d\xi^2), & k = -1.
\end{cases}
\]

These solutions appear in Birmingham [18], and include higher-dimensional versions of the Schwarzschild solution (\( \Lambda = 0, k = 1 \)), Kottler (also called “AdS-Schwarzschild,” \( \Lambda < 0, k = 1 \)), “hyperbolic Kottler” (\( \Lambda < 0, k = -1 \)) [3, 6], and Lemos (\( \Lambda < 0, k = 0 \)) [7] solutions. (When \( n > 5 \), there are solutions of (II.13) that are not constant curvature metrics; cf. [26] for an \( n = 6 \) metric foliated by \( S^3 \times \mathbb{S}^2 \)).

For Kottler and Schwarzschild solutions, the \( t = \text{const.}, \ r = \text{const.} \) surfaces will carry spherical geometry. They can have the topology of \( S^{n-2} = \mathbb{S}^\tilde{n} \) or its quotient by a discrete group of isometries, such as the lens spaces or, for \( \tilde{n} = 3 \), Poincaré dodecahedral space (see Theorem 4.4.14 of [14] for a classification of all possibilities when \( \tilde{n} = 3 \)). For the hyperbolic Kottler solutions, the surfaces \( t = \text{const.}, \ r = \text{const.} \) can be compact hyperbolic \( \tilde{n} \)-manifolds. In the Lemos case, these surfaces are closed flat manifolds. These are quotients of \( \mathbb{T}^2 \times \mathbb{R} \) by discrete groups; e.g., \( \mathbb{T}^3 \).

\(^6\)Starting from the Kottler solution, one can obtain both the hyperbolic and the Lemos solutions without further reference to the field equations. For example, in the \( n = 5 \) case, to obtain hyperbolic solutions, simply make the replacements \( t \to it, \ r \to ir, \ \theta \to i\theta, \) and \( M \to iM \) in the Kottler solution. To obtain the Lemos solutions, we follow Witten [19], who worked with a Euclidean version of Lemos’s metric. In the Kottler solution, in arbitrary dimension, let \( r = (M/\mu)^{1/(n-1)} \rho, \ t = (\mu/M)^{1/(n-1)} \tau \). Then the metric takes the form

\[
-W(\rho) d\tau^2 + d\rho^2/W(\rho) + (M/\mu)^{2/(n-1)} \rho^2 d\Omega^2 \quad \text{where} \quad W(\rho) = (\rho/\ell)^2 + k(\mu/M)^{2/(n-1)} - 2\mu/\rho^{n-3}, \quad \ell^2 = (n-1)(n-2)/(2\Lambda), \quad \text{and} \quad d\Omega^2 \text{ is the round sphere metric.}
\]

Then take \( M \to \infty \). Then \( W(\rho) \to (\rho/\ell)^2 - 2\mu/\rho^{n-3} \) and the radius of curvature of the induced metric \((M/\mu)^{2/(n-1)} \rho^2 d\Omega^2 \) on the \( \rho = \text{const.} \), \( \tau = \text{const.} \) submanifolds becomes infinite, so it goes over to the flat metric. Then the full metric takes the Lemos form with mass parameter \( \mu \). One might therefore think of the Lemos metric as being separated from the Kottler ones by an infinite mass gap.

In light of these simple tricks, it is surprising the Lemos and hyperbolic Kottler solutions were not discovered until over 75 years after Kottler’s work (a class of toroidal horizons was reported in 1979 [21]).
The second class of solutions is given by

\begin{align*}
f(r) &= 1, \\
V(r) &= -\frac{2}{3}\Lambda r^2 + C, \\
\tilde{R}_{ij} &= \frac{2}{n-2}\Lambda \delta_{ij}.
\end{align*}

These solutions are products of a 2-dimensional Einstein spacetime with an Einstein \((n-2)\)-manifold, with no warping. They have recently appeared in [24] in connection with the near-horizon approximation. This class contains the Nariai solution as the \(n = 4, \Lambda > 0\) case. These solutions may also be thought of as cosmological analogues of the vacuum case of the Bertotti-Robinson solutions. They admit the same topologies as the Birmingham solutions, but do not have a scri (essentially because various metric coefficients are not \(\sim r^2\)). That is, they admit a notion of conformal infinity sufficient to define the event horizon as the boundary of the region in causal contact with it, but conformal infinity has codimension > 1, and so is not a boundary-at-infinity.

Other analogues of the Nariai and Bertotti-Robinson solutions can be obtained by again choosing the warping function \(f = 1\) in (II.4), but this time we take \(\pi = \text{dim}(\mathcal{M}) = 3\). Then the Ricci curvature decomposes into a direct sum of the Ricci curvature on a 3-dimensional manifold and the Ricci curvature on an \((n-3)\)-dimensional one. Thus, to obtain a 5-dimensional solution with negative cosmological constant and the horizon modelled on the 3-geometry \(H_2 \times S^1\), we merely take the BTZ black hole as our 3-manifold and any constant (negative) curvature \(g > 1\) Riemann surface as the other factor. The metric is

\[ ds^2 = -\left( -\frac{\Lambda}{3} r^2 - M \right) dt^2 + \frac{dr^2}{-\frac{\Lambda}{3} r^2 - M} + r^2 d\xi^2 + \frac{3}{(-2\Lambda)} \left( d\theta^2 + \sinh^2 \theta d\phi \right), \tag{II.18} \]

where \(\Lambda < 0\) and so we have a horizon whenever the BTZ mass constant \(M\) is positive.

To obtain a black hole of horizon topology \(S^2 \times S^1\), one might try replacing \(\theta\) by \(i\theta\), with the effect that \(\sinh \theta\) is replaced by \(\sin \theta\) and \(\Lambda\) must now be > 0 in (II.18). Letting \(\mu = -M\), we obtain that the coefficient of \(dt^2\) is > 0, and that of \(dr^2\) is < 0, iff \(r^2 < 3\mu/\Lambda\), so this is not a black hole exterior solution. However, an example of a black hole that

\begin{footnote}
Four years after Lemos’s paper, the 5-dimensional Lemos metric first appears in [22], as a dimensional reduction of a 10-dimensional 3-brane metric. The charged Lemos metric appeared in [23]. Both these results are independent rediscoveries/generalizations; neither seem to have been aware of [7]. Many subsequent string theory papers trace this metric back only as far as [22].
\end{footnote}
admits this topology (with zero cosmological constant) is the black string obtained by appending a \(d\xi^2\) term to the 4-dimensional Schwarzschild solution.

II.3 Solvegeometry Black Holes

We turn now to the new solutions with twisted product horizons. The first case is that of solvegeometry.

The 3-manifold \(X\) upon which solvegeometry manifolds are modelled is the Lie group \(\text{Sol}\), described by the semidirect product \(\mathbb{R}^2 \rtimes \mathbb{R}\) with the multiplication given by

\[
((x, y), z) \cdot ((x', y'), z') = ((x + e^{-z}x', y + e^z y'), z + z').
\] (II.19)

The \(\text{Sol}\)-invariant 1-form fields \(\omega^1 = e^z dx, \omega^2 = e^{-z} dy,\) and \(\omega^3 = dz\) can be used to construct a (left) invariant metric on \(\text{Sol}\) of Bianchi type VI\(\text{−}1\) \([16]\). The identity component of the isometry group for \(\text{Sol}\) is just \(\text{Sol}\) itself.

We have found the following family of solutions of the Einstein equation (II.2) with cosmological constant \(\Lambda < 0\):

\[
d s^2 = -\left(\frac{-2\Lambda}{9}r^2 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{2\Lambda r^2 - 2M} + \frac{3}{\Lambda} d\tilde{s}^2,
\] (II.20)

\[
d \tilde{s}^2 = r^2 (e^{2z} dx^2 + e^{-2z} dy^2) + dz^2.
\] (II.21)

For any positive value of the integration constant \(M\), there is one horizon, located at \(r = (9M/(-\Lambda))^{1/3}\). We may regard \(d\tilde{s}^2\) as giving the induced metric on \(r = \text{const.}, t = \text{const.}\) surfaces. This induced metric is of Bianchi type VI\(\text{−}1\) and so is \(\text{Sol}\)-invariant. We can extend the action of \(\text{Sol}\) on these surfaces to an action of \(\text{Sol}\) on the spacetime, such that the orbits are these surfaces.

Because we may take these orbits to be copies of \(\mathbb{R}^3\), these solutions can be regarded as “black 3-branes.” To construct black holes, we now compactify the orbits, making them compact 3-manifolds modelled on solvegeometry. We start with a \(2 \times 2\) symmetric matrix \(M\) other than the identity, having unit determinant, positive trace, and integer entries. These conditions are not overly restrictive, and amount to finding the solutions of \(ab = 1 + c^2\) in positive integers; \textit{e.g.}, \[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\] and \[
\begin{bmatrix}
2 & 3 \\
3 & 5
\end{bmatrix}
\] are examples. Any such matrix preserves the lattice \(\mathbb{Z}^2 \subseteq \mathbb{R}^2\).

\(7\)One can understand the absence of “unwarped” product solutions \(M^3 \times S^2\) generally. By (II.17), the presence of the \(S^2\) factor implies \(\Lambda > 0\). But recent work of Ida \([22]\) shows that \(\overline{M}\) will be a black hole only if \(\Lambda < 0\), a contradiction.
Now, $M$ is defined to have orthogonal eigenvectors and distinct, real, reciprocal eigenvectors which we may write as $e^{\pm a}$, $a \neq 0$. Align the $x$- and $y$-axes in $\mathbb{R}^3$ to lie along the two eigenspaces. Then we may define

$$\psi(\mathbf{x}, y, z) = (M \cdot (x, y), z + a) = (e^{-a}x, e^{a}y, z + a).$$

From (II.19), $\psi$ is the action of the element $(0, 0, a)$ of Sol. Finally, define two independent translations $T_{1.2} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x', y', z)$, both preserving $z$, and preserving the same lattice as $M$ does (be careful to note that this lattice is the set of points with integer coordinates in the original basis, not the eigenvector basis, a source of some confusion in a similar discussion on p.389 of [14]). By (II.19), these are also Sol transformations. It is now an easy exercise to show that the Sol subgroup generated by $(0, 0, a)$ and the two translations is discrete (meaning that it acts properly discontinuously) and acts freely. Therefore, the quotient of $\mathbb{R}^3$ by this subgroup is a Sol manifold. With the natural projection $(x, y, z) \mapsto z \in S^1$, it has the structure of a torus bundle over the circle, and so is compact.

Finally, the Sol solutions can be extended through the horizon in the usual way, but there will a curvature singularity at $r = 0$ when $M \neq 0$; $R_{abcd}R_{abcd} \sim M^2/r^6$ there. The Hawking temperature, computed from the regularity of the Euclideanized metric, is $(\Lambda^2 M/24\pi^3)^{1/3}$, and so vanishes as $M \to 0$.

### II.4 Nilgeometry Black Holes

For $M > 0$, the $\Lambda < 0$ Einstein metric

$$ds^2 = -\left(\frac{-2\Lambda}{11}r^2 - \frac{2M}{r^{5/3}}\right)dt^2 + \frac{dr^2}{\frac{-2\Lambda}{11}r^2 + \frac{2M}{r^{5/3}}} + r^{4/3}(dx^2 + dy^2) + r^{8/3}\left(dz - \sqrt{-\frac{4\Lambda}{9}}xdy\right)^2$$

has horizon located at $r = (11M/(\Lambda))^{3/11}$. This horizon (and the exterior spacetime) can be foliated by 3-manifolds modelled on nilgeometry.

For nilgeometry, we take the manifold $X$ to be the Heisenberg group, which is denoted Nil (it’s a nilpotent Lie group) and consists of all $3 \times 3$ upper triangular matrices of the

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8Another viewpoint is obtained by first identifying points in $\mathbb{R}^3$ under the two translations, to obtain a “toroidal cylinder.” Then $\psi$ induces an automorphism $\phi$ of the torus, which can be used to glue the two toroidal ends of the cylinder together. This construction is called the mapping torus of $\phi$. 

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Figure 1: Identifications of $\mathbb{R}^3$ for a compact three-manifold modelled on solvegeometry, using $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ in (II.22). The element $(0, 0, a)$ of Sol maps rectangles on the left-hand face to parallelograms on the right, preserving the lattice of pairs of integers. The faces are each periodically identified, becoming tori.

We can identify $(x, y, z) \in \mathbb{R}^3$ with the above matrix, giving us the multiplication $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$. We can determine a (left) invariant metric on $\mathbb{R}^3$ under the action of Nil on itself by picking a metric arbitrarily at a point and using invariance. If we pick $ds^2 = dx^2 + dy^2 + dz^2$ at the origin, the resulting invariant metric is

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$  \hspace{1cm} (II.24)

As pointed out in [16], this metric is of Bianchi type II. The rescalings $x \mapsto r^{2/3}x$, $y \mapsto r^{2/3}y$, $z \mapsto r^{4/3}z$ give the metric induced by (II.23) on each of the $t = \text{const.}$, $r = \text{const.}$ surfaces.

As before, the metric (II.23) can be taken to describe a black 3-brane with horizon $X$, but once again we can use isometries to compactify $X$ and obtain a black hole. A standard example of a compact 3-manifold with geometric structure modelled on Nil can be constructed by taking the quotient of Nil by the subgroup $\Gamma$ of Nil consisting of matrices
with only integer entries. If \( a, b, c \in \mathbb{Z} \), then
\[
\begin{bmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & x + a & z + c + ay \\
0 & 1 & y + b \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

Thus, two points \((x, y, z)\) and \((x', y', z')\) are identified in \(\text{Nil}/\Gamma\) iff \(x' - x = a\), \(y' - y = b\), and \(z' - z = ay + c\) for any triple of integers \((a, b, c)\). Choosing \(a = 0\), we see that we should identify points within planes of constant \(x\), turning these planes into tori. The resulting “toroidal cylinder” can be compactified using the \(a \neq 0\) identifications. To identify the torus at \(x\) with the torus at \(x' = x + a\), for \(a\) a non-zero integer, we take lines \(z = k = \text{const}\), on the \(yz\)-plane (covering the torus) at \(x\) and identify them with \(z' = k + ay\) at \(x'\). The identifications are shown in Figure 2. As with Sol, it is an easy exercise to show that \(\Gamma\) acts freely and properly discontinuously. The resulting manifold \(\text{Nil}/\Gamma\) is a circle bundle over the torus.

The Nil solutions can be extended through the horizon but, for \(M \neq 0\), \(r = 0\) is a curvature singularity; \(R^{abcd}R_{abcd} \sim M^2/r^{22/3}\) there. The Hawking temperature is \((11\Lambda^{8/3}M)^{3/11}/6\pi\), and so vanishes as \(M \to 0\).

\[\]
III A Bianchi VI−1 Braneworld

In this section, we will construct a simple “braneworld” cosmology by embedding a time-like hypersurface in each of two solvegeometry spacetimes. The braneworld construction involves cutting each spacetime along the hypersurface and gluing together one piece from each spacetime along the boundaries resulting from the cutting. One obtains by this process a spacetime with a singular hypersurface of induced metric $h_{ab}$ and extrinsic curvature $K_{ab}$ and a solvegeometry metric “in the bulk” (i.e., everywhere else). The induced metric in the present case can be read off from (A.11) and (II.21):

$$h_{ab} dx^a dx^b = -d\tau^2 + f^2(\tau) (e^{2z} dx^2 + e^{-2z} dy^2) + dz^2,$$

(III.1)

where the function $f(\tau)$ will be determined from junction conditions. In some approaches, the bulk metric may differ on either side of this hypersurface, because of different values of $M$. We will allow this, and will distinguish the metric potential $V$ on the two sides as $V^+$ and $V^-$, resp. In any case, the hypersurface will be a singularity of the curvature, so by the junction conditions [27] it has a surface energy density tensor $S_{\mu\nu}$. For simplicity, we take $S_{\mu\nu}$ to be diagonal in the coordinate system of the Appendix with eigenvalues $-\rho, p_x, p_y, \text{and } p_z$. The conservation law (A.16) gives

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial}{\partial z} \left(p - \frac{1}{2}\rho\right) = 0,$$

(III.2)

$$\frac{\partial}{\partial \tau} \left(\rho f^2(\tau)\right) = -p \frac{\partial}{\partial \tau} \left(f^2(\tau)\right).$$

(III.3)

We now compute from (A.13, A.14, A.15) the junction conditions in the case that the bulk spacetime is comprised of solvegeometry pieces. We get

$$_{uu} \text{ component : } \frac{d}{d\tau} \left[ \sqrt{V^+(f(\tau)) + f'^2(\tau)} - \sqrt{V^-(f(\tau)) + f'^2(\tau)} \right]
= \frac{8\pi}{3} (2\rho + p_x + p_y + p_z) f'(\tau)
= \frac{8\pi}{3} (2p_x + p_y - p_z + \rho)
= \frac{8\pi}{3} (2p_y - p_z - p_x + \rho)
= 2p_x - p_y - p_z + \rho
= 0 \quad (\text{III.4})$$

$$_{xx} \text{ component : } \sqrt{V^+(f(\tau)) + f'^2(\tau)} + \sqrt{V^-(f(\tau)) + f'^2(\tau)}
= \frac{8\pi}{3} (2p_x - p_y - p_z + \rho)
= \frac{8\pi}{3} (2p_y - p_z - p_x + \rho)
= \frac{8\pi}{3} (2p_z - p_x - p_y + \rho)
= 0 \quad (\text{III.5})$$

$$_{yy} \text{ component : } \sqrt{V^+(f(\tau)) + f'^2(\tau)} + \sqrt{V^-(f(\tau)) + f'^2(\tau)}
= \frac{8\pi}{3} (2p_y - p_z - p_x + \rho)
= \frac{8\pi}{3} (2p_z - p_x - p_y + \rho)
= 0 \quad (\text{III.6})$$

$$_{zz} \text{ component : } 0 = 2p_z - p_x - p_y + \rho \quad (\text{III.7})$$
From the spatial component equations, we obtain

\[ p := p_x = p_y = p_z + \frac{1}{2} \rho, \quad \text{(III.8)} \]

\[ \sqrt{V^+(f(\tau)) + f'^2(\tau)} + \sqrt{V^-(f(\tau)) + f'^2(\tau)} = 4\pi \rho f(\tau). \quad \text{(III.9)} \]

It can be useful to re-express this last equation in the form

\[ \sqrt{V^\pm(f(\tau)) + f'^2(\tau)} = \pm \frac{(V^+(f(\tau)) - V^-(f(\tau)))}{8\pi \rho f(\tau)} + 2\pi \rho f(\tau). \quad \text{(III.10)} \]

In particular, (III.10) can be used to cast (III.4) in the form

\[ \frac{d}{d\tau} \left[ \frac{V^+(f(\tau)) - V^-(f(\tau))}{4\pi \rho f(\tau)} \right] = 8\pi \left( p + \frac{1}{2} \rho \right) f'(\tau), \quad \text{(III.11)} \]

where we have also used (III.8) to simplify the right-hand side.

We now analyze (III.9) and (III.11), beginning with the latter. By inserting the explicit form of \( V(a) \) for the spherically symmetric black hole into the left-hand side, (III.11) becomes

\[ 8\pi \left( p + \frac{1}{2} \rho \right) f'(\tau) = \frac{d}{d\tau} \left( \frac{-\Delta M}{2\pi \rho f^3(\tau)} \right) = \frac{-\Delta M}{\pi} \frac{p}{\rho^2 f^3(\tau)} f'(\tau), \quad \text{(III.12)} \]

where in the last step we used the energy conservation law (III.3), and we have defined \( \Delta M := M^+ - M^- \). This equation leads to three possibilities:

(i) We are in the steady state \( f'(\tau) = 0 \). Steady state solutions arise even in standard Friedmann cosmology, and are associated with non-uniqueness of solutions.

(ii) We have the equation of state \( \rho = -2p \), whence \( \Delta M = 0 \) and we have reflection symmetry in the “bulk” about the braneworld. We can now easily integrate the conservation law to get the result \( \rho \propto 1/f(\tau) \). This model obeys the Weak and Dominant Energy Conditions, but not the Strong Energy Condition.

(iii) We have a rather complicated equation of state, typical of braneworlds, which we express below in the implicit form

13
\[
f(\tau) = \left(\frac{-p\Delta M}{2\pi\rho^2(p + \rho/2)}\right)^{1/3}.
\] (III.13)

For case (\(ii\)) above, the converse also holds: \(\Delta M = 0 \Rightarrow \rho = -2p\) (unless \(f'(\tau) = 0\)). This restriction on the equation of state has no analogue in the more standard scenario of reflection symmetry about a braneworld in a Kottler black hole bulk. \(^{[10]}\)

Finally, we use the explicit form of \(V(a)\) to recast (III.9) in the form “kinetic + potential energy = constant:”

\[
\frac{1}{2}f^2(\tau) + U(f(\tau)) = 0,
\] (III.14)

\[
U(A) = \frac{1}{2} \left[1 - \left(\frac{\rho}{\rho_c}\right)^2\right] \left(\frac{A}{\ell}\right)^2 - \frac{(M^+ + M^-)}{2A} - \frac{(\Delta M)^2\ell^2}{128A^4} \left(\frac{\rho_c}{\rho}\right)^2,
\] (III.15)

where \(\ell^2 := 9/(-2\Lambda)\) and \(\rho_c := 2/(\pi\ell)\). The energy density \(\rho\) enters (III.13) as a square, as it does in FRW braneworld cosmology, but unlike in standard cosmology.

As it is not our intention to explore the resulting cosmology in depth in the present work, we limit ourselves to a few observations and assume, from here on, reflection symmetry \(\Delta M = 0\). As already noted, \(\rho \propto 1/f(\tau)\) in this case, so we write

\[
\rho(\tau) = \sigma \rho_c \ell / f(\tau),
\] (III.16)

where \(\sigma\) is a constant and use this to rewrite (III.14, III.15) as

\[
f'^2(\tau) = \sigma^2 - \left(\frac{f(\tau)}{\ell}\right)^2 + \left(\frac{M^+ + M^-}{f(\tau)}\right).
\] (III.17)

Somewhat remarkably, this is precisely the usual Friedmann equation for a negative spatial curvature (or flat, if \(\sigma = 0\)), matter dominated, FRW cosmology with constant mass density \(3(M^+ + M^-)/8\pi\), zero pressure, and cosmological constant \(-3/\ell^2 < 0\).

If \(M^+ + M^- = 0\), two kinds of solutions are well-known. There is the steady state solution \(f(\tau) = \sigma \ell\) and a family of solutions

\[
f(\tau) = \sigma \ell \sin \frac{\tau - \tau_0}{\ell},
\] (III.18)

\(^{[10]}\)We thank Shinji Mukohyama for bringing this to our attention, and for suggesting that this could be a consequence of our assumption of a static bulk. His suggestion is that, since there is (apparently) no suitable analogue of the Birkhoff theorem for solvegeometry black holes, the requirement of a static bulk is non-trivial, and so can lead to non-trivial constraints on the braneworld. \(^{[8]}\).
where we should consider only one half-cycle, corresponding to the birth of the universe at a singularity, its expansion to a maximum, and its subsequent recollapse. At the maximum, the square root of the right-hand side of [11.17] is not Lipschitz. The uniqueness theorem for solutions therefore fails \[29\], so that a cosmos expanding to a maximum size can then remain in the steady state for an arbitrary time before recollapsing. When \( M^+ + M^- \neq 0 \), these qualitative details persist, though in this case the time coordinate \( \tau \) diverges on approach to the singularity.

IV Discussion

We have been able to give new solutions of the 5-dimensional Einstein equations. These solutions can be regarded as black branes with horizons given by solvegeometry and nilgeometry, resp., but standard techniques permit the horizons to be “spatially compactified,” turning the new solutions into black holes. As a result, static 5-dimensional black holes are now known with horizons modelled on all but one of the Thurston model geometries.

The case of a horizon modelled on the \( \widetilde{\text{SL}}(2, \mathbb{R}) \) geometry remains open. \( \widetilde{\text{SL}}(2, \mathbb{R}) \) is the universal cover of the group of \( 2 \times 2 \) matrices with determinant 1, and this model geometry corresponds to Bianchi type VIII. \( \text{SL}(2, \mathbb{R}) \) has the structure of a line bundle over the hyperbolic plane \( \mathbb{H}^2 \), but is not isometric to the \( \mathbb{H}^2 \times \mathbb{R} \) case. In this case, the field equations are somewhat more complicated than the others and simple ansätze such as those we applied above do not work. At this stage, it is too early to say whether there is a fundamental obstruction or whether the difficulty is that our ansätze have been too naïve in this case. The issue remains under active consideration.

The new solutions raise several questions: are they subject to the Gregory-Laflamme instability \[30, 31\] of black strings, can horizons of completely arbitrary topology be constructed in 5 dimensions, and do any of these solutions have an interpretation in terms of compactifications of fundamental physics in higher dimensions? They also raise the following issue of black hole uniqueness. Assume specific fall-off behaviour near infinity, say that of locally asymptotically anti-de Sitter spacetimes. Will this fall-off behaviour and reasonable causality and energy conditions in the bulk spacetime preclude the existence of a black hole horizon modelled on Nil or Sol? This is a next logical step beyond the product spacetimes studied in this article. Now the fundamental group of the domain of outer communications of this spacetime would be non-trivial, and then from topological censorship there comes the constraint that the fundamental group of scri must map onto it, but for hyperbolic scri this seems easy to arrange, and so spacetimes of this nature

\[11\] This “hesitant universe” scenario is a well-known property of the Friedmann equation. EW wishes to thank Connell McCluskey for a discussion on this point.
remain a possibility. To explore this issue, it may prove useful to consider whether there is a higher-dimensional analogue of Hawking’s early approach [1, 2] to the question of black hole topology in 4 dimensions.

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[32] GRTensorII is a package which runs within MapleV. It is distinct from packages distributed with MapleV and must be obtained independently. The GRTensorII software is distributed freely on the World Wide Web from the address http://grtensor.phy.queensu.ca

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A Braneworld Formalism

In this appendix, we review braneworlds following the approach of [33, 34], wherein the cosmos is a singular hypersurface embedded in a five-dimensional spacetime. For any static metric of the form

$$ds^2 = -V(a) dt^2 + \frac{da^2}{V(a)} + d\sigma^2(a, x^k) \ ,$$  \hspace{1cm} \text{(A.1)}

where the coordinates $x^k$ parametrize the $t = \text{const.}, \ a = \text{const.}$ surfaces, consider an embedded timelike hypersurface parametrized by $(\tau, x^k)$ of the form

$$a = f(\tau) \ , \ \ t = g(\tau) \ .$$  \hspace{1cm} \text{(A.2)}

At each point of the hypersurface, the vector

$$N^a := \frac{f'(\tau)}{V(a)} \frac{\partial}{\partial t} + \frac{V(a) g'(\tau)}{\partial a} \ ,$$  \hspace{1cm} \text{(A.3)}

is normal to the hypersurface. We seek a vector field $n^a$ that is (i) normalized and spacelike: $g(n, n) = 1$, and (ii) tangent to an affinely parametrized geodesic congruence: $\nabla_n n = 0$, that (iii) meets the hypersurface orthogonally: $n^a|_0 \propto N^a$, where $v|_0$ denotes the restriction of $v$ to the hypersurface. From (i) and (ii), and choosing the sign so that $n^a$ points to increasing $a$-values, we conclude that

$$n^a = \frac{E}{V(a)} \frac{\partial}{\partial t} + \sqrt{V(a) + E^2} \frac{\partial}{\partial a} \ ,$$  \hspace{1cm} \text{(A.4)}

where $E = E(\tau, x^k)$ is a constant of the motion along each integral curve of $n^a$, but depends on the parameters $(\tau, x^k)$ of the point at which the integral curve meets the hypersurface. Condition (iii) determines $E$ such that the curve meets the hypersurface orthogonally, giving

$$E = f'(\tau) \sqrt{\frac{V(f(\tau))}{V^2(f(\tau))g'^2(\tau) - f'^2(\tau)}} \ .$$  \hspace{1cm} \text{(A.5)}

If we choose the parameter $\tau$ such that

$$g'^2(\tau) = \frac{f'^2(\tau) + V(f(\tau))}{V^2(f(\tau))} \ ,$$  \hspace{1cm} \text{(A.6)}

then we obtain that

$$E = f'(\tau) \ .$$  \hspace{1cm} \text{(A.7)}

We can read off \( n^1 := da/d\lambda \) from (A.4) and integrate it using the fact that \( E \) is constant along each integral curves of \( n^a \), and so can express the affine parameter along these geodesics as

\[
\lambda - \lambda_0 = \int_{f(\tau)}^{a} \frac{da'}{\sqrt{V(a')} + E^2} .
\]

We now have a Gaussian normal coordinate system in a neighbourhood of the hypersurface, wherein each point \( p \) is specified by coordinates \((\tau, \lambda, x^k)\) where \((\tau, x^k)\) are the parameter values at which the integral curve of \( n^a \) that passes through \( p \) meets the hypersurface, and \( \lambda \) is the value of the affine parameter along this geodesic at \( p \). That is, we can promote \( \tau \) to a function on a neighbourhood of the hypersurface such that

\[
\mathcal{L}_n \tau = n^a \nabla_a \tau = 0.
\]

Rather than explicitly solving for \( \tau \) in terms of \( t \) and \( a \), however, it suffices for our purposes to write the metric in the form

\[
ds^2 = -u \otimes u + d\lambda^2 + d\sigma^2(a(\tau, \lambda), x^k) ,
\]

where

\[
u_a = \frac{\sqrt{V(a)} + E^2 dt - \frac{E}{V(a)} da}{V(a)} ,
\]

so \( u^a := g^{ab} u_b \) is unit past-timelike and orthogonal to \( n^a \). Note that \([u, n] \neq 0\). By computing the restrictions of \( dt \) and \( da \) on the hypersurface, one can easily see that \( u_a = d\tau \) there, so the first fundamental form \( h_{ab} \) on the brane has line element

\[
ds^2 = h_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + d\sigma^2(f(\tau), x^k) ,
\]

This induced metric is not assumed to be governed by the 4-dimensional Einstein equation. Instead, the undetermined metric coefficients are fixed by applying junction conditions to the second fundamental form. If we denote the hypersurface as \( \Sigma \), then for any \( v^a, w^a \in T\Sigma \) and \( n^a \in (T\Sigma)^\perp \) we have the second fundamental form

\[
K(v, w) = v^a w^b \nabla_a n_b = \frac{1}{2} \left\{ \mathcal{L}_n (g(v, w)) + g(v, [w, n]) + g(w, [v, n]) - g(n, [v, w]) \right\} .
\]

Now the procedure is to cut each of two copies of the 5-dimensional spacetime (designated here the “+” and “−” copies, resp.) along a timelike hypersurface with fundamental form \((h^{\pm}_{ab}, K^{\pm}_{ab})\), throwing away one side of each spacetime to create two spacetimes-with-timelike-boundary, and gluing these along the boundary. The result is a 5 dimensional spacetime with a singular hypersurface where the glueing took place. This hypersurface
is the braneworld. According to the junction conditions, it is necessary to prescribe a surface energy density along the boundary in order to balance the difference in the extrinsic curvatures of the two boundaries that were glued together [27].

\[
K_{ab}^+ - K_{ab}^- = -8\pi \left( S_{ab} - \frac{1}{3} h_{ab} S \right),
\]

where \( S_{ab} \) is the surface energy tensor, \( S := h^{ab} S_{ab} \), and signs as superscripts distinguish the extrinsic curvatures on the two sides of the braneworld (we do not \textit{a priori} assume a \( \mathbb{Z}_2 \) symmetry of reflection through the braneworld). In order that \( \partial/\partial a \) match up across the braneworld, we must reverse it on one side, say the “−” side, since otherwise it would point outward from both sides of the braneworld. This leads to a ± sign in the spatial components of \( K_{ab} \), which are

\[
K_{ij}^\pm = \pm \frac{1}{2} n^a \partial_a h_{ij} = \frac{1}{2} \sqrt{V^\pm(f(\tau)) + f'^2(\tau)} \frac{\partial}{\partial a} h_{ij} \big|_{a = f(\tau)}.
\]

The time-time component, here denoted \( K_{uu} \), obeys

\[
K_{uu}^\pm = -\frac{1}{f'(\tau)} \frac{d}{d\tau} \sqrt{V^\pm(f(\tau)) + f'^2(\tau)}.
\]

The Gauss-Codazzi relations can be applied to the extrinsic curvature to yield identities given in [27], including the conservation law

\[
S^a_{\ b|a} = 0.
\]