Abstract

It is believed [6], [13, Section 4.1] that, among the coefficients entering Kontsevich’s formality quasi-isomorphism [12], there are irrational (possibly even transcendental) numbers. In this note, we prove the existence of a stable formality quasi-isomorphism for Hochschild cochains over rationals. In other words, we show that all the weights which enter Kontsevich’s construction [12] can be replaced by rational numbers in such a way that all the desired identities still hold. In doing this, we completely bypass Tamarkin’s approach [3], [9], [15].

1 Introduction

This paper is an answer to the question asked by several of my colleagues: “Can we deduce the existence of a formality quasi-isomorphism for Hochschild cochains of \( \mathbb{Q}[x_1, x_2, \ldots, x_d] \) starting from Kontsevich’s quasi-isomorphism [12]?” The usual folklore argument of rational homotopy theory cannot be directly applied here because the graded Lie algebra of polyvector fields is infinite dimensional. Besides, it would be interesting to get a construction whose output is a formality quasi-isomorphism for Hochschild cochains of \( \mathbb{Q}[x_1, x_2, \ldots, x_d] \) which admits a “graphical expansion” (as original Kontsevich’s quasi-isomorphism [12] does) with all weights being rational numbers. The goal of this paper is to produce such a construction.

Based on calculations performed in [6] by G. Felder and T. Willwacher, it is unlikely that coefficients entering Kontsevich’s formality quasi-isomorphism [12] are all rational. In addition, the combination of T. Willwacher’s theorem [16, Theorem 1.1] and the result [2, Theorem 6.2] indicate that the algebra of hypothetical “motivic avatars” of coefficients\(^1\) entering Kontsevich’s formality quasi-isomorphism [12] should be isomorphic to the algebra of motivic avatars of multiple zeta values [7], [8].

It is not surprising that the proof of the existence of the desired stable formality quasi-isomorphism is very similar to that of the existence of Drinfeld’s associator over rational numbers [5], [1, Section 4.1]. However, there is an important difference between Drinfeld associators and stable formality quasi-isomorphisms: unlike Drinfeld associators, stable formality quasi-isomorphisms have a nontrivial homotopy equivalence relation. This difference adds an additional “layer” to the proof. It should also be remarked that the author does not see what would be an analog of technical Lemma 4.1 for Drinfeld associators.

\(^1\)Two comments should be made about hypothetical “motivic avatars” of coefficients entering Kontsevich’s quasi-isomorphism [12]. First, in order to get an algebra one should consider integrals over strata of all dimensions. Second, the author does not see an obvious way to define “motivic avatars” for these coefficients/weights.
The paper is organized as follows. We conclude our introduction with notation and conventions used in this paper. In Section 2, we recall the Kajiura-Stasheff operad \( OC \) and the operad \( KGra \) which is assembled from graphs used in Kontsevich’s paper [12]. In this section, we also recall the definition of a stable formality quasi-isomorphism (SFQ). In Section 3, we formulate the main result (see Theorem 3.1) which implies the existence of an SFQ over rational numbers. Section 4 is devoted to various technical statements which will be used later in the proof of Theorem 3.1. Most of these statements are essentially borrowed from [2].

The proof of Theorem 3.1 goes by induction on the arities with respect to both colors involved in the game. In Section 5, we present all the necessary “building blocks” of this induction and finally, in Section 6, we assemble these “building blocks” into a proof of Theorem 3.1.

1.1 Notation and conventions

We mostly follow the notation and conventions of [2]. \( K \) is a field of characteristic zero and, in this note, \( K = \mathbb{R} \) or \( K = \mathbb{Q} \). The underlying symmetric monoidal category \( \mathcal{C} \) for our algebraic structures is either the category \( grVect_K \) of \( \mathbb{Z} \)-graded \( K \)-vector spaces or the category \( Ch_K \) of unbounded cochain complexes of \( K \)-vector spaces.

For a cochain complex \( V \) we denote by \( sV \) (resp. by \( s^{-1}V \)) the suspension (resp. the desuspension) of \( V \). In other words,

\[
(sV)^\bullet = V^{\bullet-1}, \quad (s^{-1}V)^\bullet = V^{\bullet+1}.
\]

For a homogeneous vector \( v \) in a graded vector space (or a cochain complex) \( V \), the notation \( |v| \) is reserved for the degree \( v \).

\( C^\bullet(A) \) denotes the Hochschild cochain complex of an associative algebra (or more generally an \( A_\infty \)-algebra) \( A \) with coefficients in \( A \). For a commutative ring \( R \) and an \( R \)-module \( V \) we denote by \( S_R(V) \) the symmetric algebra of \( V \) over \( R \). For a vector \( \xi \) in a (dg) Lie algebra \( \mathcal{L} \), the notation \( ad_\xi \) is reserved for the adjoint action of \( \xi \), i.e. \( ad_\xi(\eta) = [\xi, \eta] \).

Given an operad \( O \), we denote by \( o_i \) the elementary operadic insertions:

\[
o_i : O(n) \otimes O(k) \rightarrow O(n+k-1), \quad 1 \leq i \leq n.
\]

The symmetric group on \( n \) letters in denoted by \( S_n \) and the notation \( Sh_{p,q} \) is reserved for the set of \((p,q)\)-shuffles in \( S_{p+q} \). A graph is directed if each edge carries a chosen direction. A graph \( \Gamma \) with \( n \) vertices is called labeled if \( \Gamma \) is equipped with a bijection between the set of its vertices and the set \( \{1,2,\ldots,n\} \). In this paper we consider exclusively graphs without loops (i.e. cycles of length one).

We will freely use the conventions for colored (co)operads and colored pseudo- (co)operads from [2, Section 2]. For example, for a (colored) pseudo-cooperad \( \mathcal{C} \) and a (colored) pseudo-operad \( O \), the notation

\[
\text{Conv}(\mathcal{C}, O)
\]

is reserved for the convolution dg Lie algebra [2, Section 2.3], [4, Section 4], [14].

For most of colored (co)operads and pseudo- (co)operads used in this paper the ordinal of colors will have only two element \( c \) and \( o \) for which we set \( c < o \). Just as in [2], solid edges of colored planar trees are the edges which carry the color \( c \) and dashed edges are the edges which carry the color \( o \).
For a two colored operad $\mathcal{O}$, the notation $\mathcal{O}(n,k)^c$ (resp. $\mathcal{O}(n,k)^o$) is reserved for the space of “operations” with $n$ inputs of the color $c$, $k$ inputs of the color $o$, and with the output carrying the color $c$ (resp. $o$).

Using “arity” we can equip the convolution Lie algebra $\text{Conv} (\mathcal{C}, \mathcal{O})$ with the natural descending filtration

$$\text{Conv} (\mathcal{C}, \mathcal{O}) = \mathcal{F}_{-1} \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_0 \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_1 \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \ldots,$$

where

$$\mathcal{F}_m \text{Conv} (\mathcal{C}, \mathcal{O}) = \{ f \in \text{Conv} (\mathcal{C}, \mathcal{O}) \mid f\big|_{\mathcal{C}(q)} = 0 \ \forall \ \text{corollas} \ q \text{ satisfying } |q| \leq m \},$$

where $|q|$ is the total number of incoming edges of the corolla $q$.

This filtration is compatible with the Lie bracket and $\text{Conv} (\mathcal{C}, \mathcal{O})$ is complete with respect to this filtration. Namely,

$$\text{Conv} (\mathcal{C}, \mathcal{O}) = \lim_{m} \text{Conv} (\mathcal{C}, \mathcal{O}) / \mathcal{F}_m \text{Conv} (\mathcal{C}, \mathcal{O}). \quad (1.2)$$

We also introduce an additional descending filtration $\mathcal{F}_\chi$ on the convolution Lie algebra $\text{Conv} (\mathcal{C}, \mathcal{O})$ for each color $\chi$:

$$\text{Conv} (\mathcal{C}, \mathcal{O}) = \mathcal{F}_{-1}^\chi \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_0^\chi \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_1^\chi \text{Conv} (\mathcal{C}, \mathcal{O}) \supset \ldots,$$

where

$$\mathcal{F}_m^\chi \text{Conv} (\mathcal{C}, \mathcal{O}) = \{ f \in \text{Conv} (\mathcal{C}, \mathcal{O}) \mid f\big|_{\mathcal{C}(q)} = 0 \ \forall \ \text{corollas} \ q \text{ satisfying } \sharp_\chi (q) \leq m \},$$

where $\sharp_\chi (q)$ is the number of incoming edges of the corolla $q$ which carry the color $\chi$.

The filtration (1.3) is compatible with the Lie bracket on $\text{Conv} (\mathcal{C}, \mathcal{O})$ and $\text{Conv} (\mathcal{C}, \mathcal{O})$ is complete with respect to this filtration.

We denote by $\Lambda$ the endomorphism operad of the 1-dimensional vector space $s^{-1}K$ placed in degree $-1$

$$\Lambda = \text{End}_{s^{-1}K}. \quad (1.4)$$

In other words,

$$\Lambda(n) = s^{1-n} \text{sgn}_n,$$

where $\text{sgn}_n$ is the sign representation for the symmetric group $S_n$. We observe that the collection $\Lambda$ is also naturally a cooperad.

For a dg operad (resp. a dg cooperad) $P$ in we denote by $\Lambda P$ the dg operad (resp. the dg cooperad) which is obtained from $P$ via tensoring with $\Lambda$, i.e.

$$\Lambda P(n) = s^{1-n} P(n) \otimes \text{sgn}_n. \quad (1.5)$$

**Anniversary note:** If you have never met Volodya Rubtsov then I strongly suggest you visit him in the City of Angers in France. You will meet a person who radiates an amazing amount of generosity and charm! I would like to greet Volodya with his 60-th anniversary,
and wish him health, new brilliant mathematical ideas, as well as many days filled with a positive spirit.

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2 Reminder of OC, KGra and stable formality quasi-isomorphisms

The definition of a stable formality quasi-isomorphism (SFQ) [2, Section 5] is based on the 2-colored operads OC and KGra. The 2-colored dg operad OC governs open-closed homotopy algebras introduced in [10] by H. Kajiura and J. Stasheff and the 2-colored operad KGra is “assembled” from graphs used in Kontsevich’s paper [12].

2.1 The Kajiura-Stasheff operad OC

As an operad in the category grVect of graded vector spaces, OC is freely generated by the 2-colored collection oc with the following spaces:

\[
\text{oc}(n, 0) = s^{3-2n}K, \quad n \geq 2, \\
\text{oc}(0, k) = s^{2-k} \text{sgn}_k \otimes K[S_k], \quad k \geq 2, \\
\text{oc}(n, k) = s^{2-2n-k} \text{sgn}_k \otimes K[S_k], \quad n \geq 1, \quad k \geq 0,
\]

where \(\text{sgn}_k\) is the sign representation of \(S_k\). The remaining spaces of the collection oc are zero.

Following [2, Section 4], we represent generators of OC in \(\text{oc}(n, 0)\) by non-planar labeled corollas with \(n\) solid incoming edges (see figure 2.1). We represent generators of OC in \(\text{oc}(0, k)\) by planar labeled corollas with \(k\) dashed incoming edges (see figure 2.2). Finally, we use labeled 2-colored corollas with a planar structure given only on the dashed edges to represent generators of OC in \(\text{oc}(n, k)\) (see figure 2.3). Applying element \(\sigma \in S_k\) to the labeled corolla \(t^n_k\) depicted on figure 2.2 we get a basis for the vector space \(\text{oc}(0, k)\). Similarly, applying elements of \((\text{id}, \sigma) \in S_n \times S_k\) to the labeled corolla depicted on figure 2.3 we get a basis for the vector space \(\text{oc}(n, k)\).
The corollas $t^c_n$, $t^o_k$ and $t^o_{n,k}$ carry the following degrees:

$$|t^c_n| = 3 - 2n \quad n \geq 2,$$

$$|t^o_k| = 2 - k \quad k \geq 2,$$

$$|t^o_{n,k}| = 2 - 2n - k \quad n \geq 1, \quad k \geq 0.$$  

(2.4)  

(2.5)  

(2.6)

The differential $\mathcal{D}$ on $\mathcal{OC}$ is defined by the following equations\footnote{For a nice pictorial definition of this differential on $\mathcal{OC}$ we refer the reader to [2, Section 4.1].}

$$\mathcal{D}(t^c_n) := - \sum_{p=2}^{n-1} \sum_{\tau \in \text{Sh}_{p, n-p}} (\tau, id)(t^c_{n-p+1} \circ_{1,c} t^c_p).$$  

(2.7)

$$\mathcal{D}(t^o_k) := - \sum_{p=0}^{k-2} \sum_{q=p+2}^{k} (-1)^{p+(k-q)(q-p)} t^o_{k-q+p+1} \circ_{p+1,o} t^o_{q-p}.$$  

(2.8)

$$\mathcal{D}(t^o_{n,k}) := (-1)^k \sum_{p=2}^{n} \sum_{\tau \in \text{Sh}_{p, n-p}} (\tau, id)(t^o_{n-p+1,k} \circ_{1,c} t^c_p) - \sum_{r=0}^{n} \sum_{\sigma \in \text{Sh}_{r, n-r}} (-1)^{p+(k-q)(q-p)} (\sigma, id)(t^o_{r,k-q+p+1} \circ_{p+1,o} t^o_{n-r,q-p}).$$  

(2.9)

Since the right hand sides of equations (2.7), (2.8), and (2.9) are quadratic in generators, we conclude that the collection $s^{-1} \mathcal{oc}$ is a pseudo-cooperad and

$$\mathcal{OC} = \text{Cobar}(s^{-1} \mathcal{oc})$$

where $\mathcal{oc}^\vee$ is the 2-colored coaugmented cooperad obtained from $s^{-1} \mathcal{oc}$ via “adjoining the counit”.

Finally, we recall that algebras over $\mathcal{OC}$ (a.k.a. open-closed homotopy algebras) are pairs of cochain complexes $(\mathcal{V}, \mathcal{A})$ with the following data\footnote{Recall that introducing a $\Lambda\text{Lie}_\infty$-structure on $\mathcal{V}$ is equivalent to introducing an $L_\infty$-structure on $s^{-1} \mathcal{V}$.}:

- A $\Lambda\text{Lie}_\infty$-structure on $\mathcal{V}$,
- an $A_\infty$-structure on $\mathcal{A}$, and
- a $\Lambda\text{Lie}_\infty$-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of $\mathcal{A}$.

### 2.2 The operad $d\text{Gra}$ and its 2-colored version $K\text{Gra}$

To define the operad $d\text{Gra}$, we introduce a collection of auxiliary sets $\{\text{dgra}(n)\}_{n \geq 1}$. An element of $\text{dgra}_n$ is a directed labelled graph $\Gamma$ with $n$ vertices and with the additional piece of data: the set of edges of $\Gamma$ is equipped with a total order. An example of an element in $\text{dgra}_5$ is shown on figure 2.4. Here, we use roman numerals to specify a total order on a set of edges. For example, the roman numerals on figure 2.4 indicate that $(3,1) < (3,2) < (2,3) < (2,2)$. 

3
Next, we introduce a collection of graded vector spaces \( \{d^{\text{Gra}}(n)\}_{n \geq 1} \). The space \( d^{\text{Gra}}(n) \) is spanned by elements of \( d^{\text{gra}}_n \), modulo the relation \( \Gamma^\sigma = (-1)^{|\sigma|} \Gamma \), where the graphs \( \Gamma^\sigma \) and \( \Gamma \) correspond to the same directed labelled graph but differ only by permutation \( \sigma \) of edges. We also declare that the degree of a graph \( \Gamma \) in \( d^{\text{Gra}}(n) \) equals \( -e(\Gamma) \), where \( e(\Gamma) \) is the number of edges in \( \Gamma \). For example, the graph \( \Gamma \) on figure 2.4 has 4 edges. Thus its degree is \( -4 \).

Following [16], the collection \( \{d^{\text{Gra}}(n)\}_{n \geq 1} \) forms an operad in the category of graded vector spaces. The symmetric group \( S_n \) acts on \( d^{\text{Gra}}(n) \) in the obvious way by rearranging labels and the operadic multiplications are defined in terms of natural operations of erasing vertices and attaching edges to vertices.

The operad \( d^{\text{Gra}} \) upgrades naturally to a 2-colored operad \( K^{\text{Gra}} \) whose spaces are finite linear combinations of graphs used by M. Kontsevich in [12].

For \( K^{\text{Gra}} \), we declare that \( K^{\text{Gra}}(n, k)^c = 0 \) whenever \( k \geq 1 \).

For the space \( K^{\text{Gra}}(n, 0)^c \) \( (n \geq 0) \) we have

\[
K^{\text{Gra}}(n, 0)^c = d^{\text{Gra}}(n). 
\]

(2.10)

Finally, to define the space \( K^{\text{Gra}}(n, k)^o \) we introduce the auxiliary set \( d^{\text{gra}}_{n,k} \). An element of the set \( d^{\text{gra}}_{n,k} \) is a directed labelled graph \( \Gamma \) with \( n \) vertices of color \( c \), \( k \) vertices of color \( o \), and with the following data: the set of edges of \( \Gamma \) is equipped with a total order. In addition, we require that each graph \( \Gamma \in d^{\text{gra}}_{n,k} \) has no edges originating from any vertex with color \( o \).

**Example 2.1** Figure 2.5 shows an example of a graph in \( d^{\text{gra}}_{2,3} \). Black (resp. white) vertices carry the color \( c \) (resp. \( o \)). We use separate labels for vertices of color \( c \) and vertices of color \( o \). For example \( 2_c \) denotes the vertex of color \( c \) with label 2 and \( 3_o \) denotes the vertex of color \( o \) with label 3.

![Example 2.1](image)

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The space \( K^{\text{Gra}}(n, k)^o \) is spanned by elements of \( d^{\text{gra}}_{n,k} \), modulo the relation \( \Gamma^\sigma = (-1)^{|\sigma|}\Gamma \), where the graphs \( \Gamma^\sigma \) and \( \Gamma \) correspond to the same directed labelled graph but differ only by permutation \( \sigma \) of edges. As above, we declare that the degree of a graph \( \Gamma \) in \( K^{\text{Gra}}(n, k)^o \) equals \( -e(\Gamma) \), where \( e(\Gamma) \) is the total number of edge of \( \Gamma \).
The operadic structure on the resulting 2-colored collection $K\text{Gra}$ is defined in the similar way to that on $d\text{Gra}$. For more details, we refer the reader to [2, Section 3].

Several vectors of the operad $K\text{Gra}$ will play a special role in our consideration. For this reason, we will reserve separate symbols for these vectors:

$$\Gamma_{\bullet \bullet} = \begin{array}{c} 1 \\ \circ \end{array} + \begin{array}{c} 2 \\ \circ \end{array}$$  \quad (2.11)
$$\Gamma_{\bullet \circ} = \begin{array}{c} 1 \\ \circ \end{array} \quad \Gamma_{\circ \circ} = \begin{array}{c} 2 \\ \circ \end{array}$$  \quad (2.12)

and the series of “brooms” for $k \geq 0$ depicted on figure 2.6.

![Fig. 2.6: Edges are ordered in this way (1c, 1o) < (1c, 2o) < ... < (1c, ko)](image)

Note that the graph $\Gamma_{0}^{br} \in K\text{Gra}(1, 0)^{o}$ consists of a single black vertex labeled by 1 and it has no edges.

### 2.3 Stable formality quasi-isomorphisms

We recall from [2] that

**Definition 2.1** A stable formality quasi-isomorphism (SFQ) is a morphism of 2-colored operads in the category of cochain complexes

$$F : OC \rightarrow K\text{Gra}$$  \quad (2.13)

satisfying the following “boundary conditions”:

$$F(t_{n}^{c}) = \begin{cases} \Gamma_{\bullet \bullet} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$  \quad (2.14)
$$F(t_{2}^{o}) = \Gamma_{\circ \circ},$$  \quad (2.15)

and

$$F(t_{1,k}^{o}) = \frac{1}{k!} \Gamma_{k}^{br},$$  \quad (2.16)

where $t_{n}^{c}$, $t_{k}^{o}$, and $t_{n,k}^{o}$ are corollas depicted on figures 2.1, 2.2, 2.3, respectively.

Following [2, Section 5.1], we identify SFQs with MC elements $\alpha$ of the graded Lie algebra

$$\text{Conv}(s^{-1} \alpha, K\text{Gra})$$  \quad (2.17)

subject to the three conditions:

$$\alpha(s^{-1} t_{n}^{c}) = \begin{cases} \Gamma_{\bullet \bullet} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$  \quad (2.18)
\( \alpha(s^{-1} t^o_s) = \Gamma_oo, \) \hspace{1cm} (2.19)

and

\( \alpha(s^{-1} t^o_{1,k}) = \frac{1}{k!} \Gamma^\text{br}_k \), \hspace{1cm} (2.20)

We should remark that, since all vectors in \( KGra(0,k)^o \) have degree zero, we have

\( \alpha(s^{-1} t^e_k) = 0 \), \hspace{1cm} (2.21)

for all \( k \geq 3 \) and for all degree 1 elements \( \alpha \) in (2.17).

### 2.4 Kontsevich’s SFQ

The first example of a stable formality quasi-isomorphism (over \( \mathbb{R} \)) was constructed in [12] by M. Kontsevich.

In paper [12], M. Kontsevich did not use the language of operads. However this language is very convenient for defining the action of the graph complex [11, Section 5] on formality quasi-isomorphisms and for describing the set of homotopy classes of formality quasi-isomorphisms in the “stable setting” [2].

Let us briefly recall here Kontsevich’s construction of a particular example of an SFQ.

For a pair of integers \((n,k)\), \(n \geq 0, k \geq 0\) satisfying the inequality \(2n + k \geq 2\), we denote by \( \text{Conf}_{n,k} \) the configuration space of \(n\) labeled points in the upper half plane and \(k\) labeled points on the real line:

\[
\text{Conf}_{n,k} := \{(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_k) \mid p_i \in \mathbb{C}, \ \text{Im}(p_i) > 0, \ q_j \in \mathbb{R}, \ p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2, \ q_{j_1} \neq q_{j_2} \text{ for } j_1 \neq j_2, \}.
\] (2.22)

Let us denote by \( G^{(1)} \) the 2-dimensional connected Lie group of the following transformations of the complex plane:

\[
G^{(1)} := \{z \mapsto az + b \mid a, b \in \mathbb{R}, \ a > 0\}.
\] (2.23)

The condition \(2n + k \geq 2\) guarantees that the diagonal action of \( G^{(1)} \) on \( \text{Conf}_{n,k} \) is free and hence the quotient

\[
C_{n,k} := \frac{\text{Conf}_{n,k}}{G^{(1)}}
\] (2.24)

is smooth real manifold of dimension \(2n + k - 2\).

We denote by \( \overline{C}_{n,k} \) the compactification of \( C_{n,k} \) constructed by M. Kontsevich in [12, Section 5]. \( \overline{C}_{n,k} \) comes with an involved stratification which is described in great detail in loc. cit.

Let \( \Gamma \) be a graph in \( \text{dgra}_{n,k} \) and \( e \) be an edge of \( \Gamma \) which originates at the black\(^4\) vertex with label \(i\). To such an edge \(e\), we assign a 1-form \(d\varphi_e\) on \(\text{Conf}_{n,k}\) by the following rule:

- if the edge \(e\) terminates at the black vertex with label \(j\) then
  \[
d\varphi_e := d\text{Arg}(p_j - p_i) - d\text{Arg}(p_j - \bar{p}_i),
\]

- if the edge \(e\) terminates at the white vertex with label \(j\) then
  \[
d\varphi_e := d\text{Arg}(q_j - p_i) - d\text{Arg}(q_j - \bar{p}_i).
\]

\(^4\)Let us recall that, by definition of \(\text{dgra}_{n,k}\) every edge \(e\) of \(\Gamma \in \text{dgra}_{n,k}\) should originate at a vertex with color \(\epsilon\) (i.e. black vertex).
It is easy to see that $d\varphi$ descends to a 1-form on $C_{n,k}$. Furthermore, $d\varphi$ extends to a smooth 1-form on Kontsevich’s compactification $\overline{C}_{n,k}$ of $C_{n,k}$.

Using the embedding

$$\text{Conf}_{n,k} \subset C^n \times \mathbb{R}^k$$

we equip the manifold $\text{Conf}_{n,k}$ with the natural orientation which descends to $C_{n,k}$ and extends to $\overline{C}_{n,k}$.

To every element $\Gamma \in \text{dgra}_{n,k}$, we assign the following weight:

$$W_\Gamma := \frac{1}{(2\pi)^{2n+k-2}} \int_{\overline{C}_{n,k}^+} \bigwedge_{e \in E(\Gamma)} d\varphi_e,$$  \hspace{1cm} \text{(2.25)}

where $\overline{C}_{n,k}^+$ is a connected component of $\overline{C}_{n,k}$ which is the closure of the subspace of configurations singled out by the condition

$$q_1 < q_2 < \cdots < q_k,$$

and the order of 1-forms in $\bigwedge_{e \in E(\Gamma)} d\varphi_e$ agrees with the total order on the set of edges of $\Gamma$.

It is clear that the weight $W_\Gamma$ for $\Gamma \in \text{dgra}_{n,k}$ is non-zero only if the total number of edges of $\Gamma$ equals $2n + k - 2$.

Let us observe that the set $\text{dgra}_{n,k}$ carries an obvious equivalence relation: two elements $\Gamma$ and $\Gamma'$ in $\text{dgra}_{n,k}$ are equivalent if and only if they have the same underlying labeled (colored) graph. We denote by

$$[\text{dgra}_{n,k}]$$

the set of corresponding equivalence classes.

Finally we define a degree 1 element $\alpha^K \in \text{Conv}(s^{-1}\mathfrak{oc},\text{KGra})$ by setting

$$\alpha^K(s^{-1}t^\circ_{n}) := \begin{cases} \Gamma \bullet\circ \bullet & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \hspace{1cm} \text{(2.26)}$$

and

$$\alpha^K(s^{-1}t^\circ_{2}) := \Gamma_\circ \circ,$$ \hspace{1cm} \text{(2.27)}

$$\alpha^K(s^{-1}t^\circ_{n,k}) := \sum_{z \in [\text{dgra}_{n,k}]} W_\Gamma z \Gamma_z,$$ \hspace{1cm} \text{(2.28)}

where $\Gamma_z$ is any representative of the equivalence class $z \in [\text{dgra}_{n,k}]$.

We observe that, since the vector

$$W_\Gamma \Gamma \in \mathbb{K}(\text{dgra}_{n,k})$$

depends only on the equivalence class of $\Gamma$, the right hand side of (2.28) does not depend on the choice of representatives $\Gamma_z$.

A direct computation shows that the weights of the “brooms” depicted on figure 2.6 are given by the formula

$$W_{1_{k}}^{br} = \frac{1}{k!},$$

Hence $\alpha^K$ satisfies all the required “boundary conditions” (2.18), (2.19), (2.20).
Following the line of arguments of [12, Section 6.4] one can show that $\alpha^K$ satisfies the MC equation
\[
[\alpha^K, \alpha^K] = 0
\]
in $\text{Conv}(s^{-1}\mathfrak{oc}, \text{Kgra})$. Thus $\alpha^K$ gives us an SFQ.

**Remark 2.2** The weight of a graph $\Gamma \in \text{dgra}_{n,k}$ defined in [12, Section 6.2] comes with additional factors. These factors are absent in (2.25) because our identification between polyvector fields and “functions” on the odd cotangent bundle is different from the one used by M. Kontsevich in [12, Section 6.3].

### 2.4.1 On connectedness condition for SFQs

Let $\alpha^K$ be the MC element of (2.17) constructed above, $n \geq 1$ and $\Gamma$ be a disconnected graph in $\text{dgra}_{n,k}$ with $2n + k - 2$ edges.

It is not hard to see that, for every point $P \in C_{n,k}$, the restrictions of 1-forms \[
\{d\varphi_e\}_{e \in E(\Gamma)}
\]
do not span the cotangent space $T^*_P C_{n,k}$.

Therefore, for every $n \geq 1$, the linear combination \[
\alpha^K(s^{-1}t^e_{n,k})
\]
involves only connected graphs.

In this paper we assume that degree zero vectors \[
\xi \in \text{Conv}(s^{-1}\mathfrak{oc}, \text{Kgra})
\]
entering the definition of homotopy equivalence [2, Definition 5.2] satisfy the additional condition: *all graphs in the linear combinations* \[
\xi(s^{-1}t^e_{n,k})
\]
are connected.

Similarly, instead of using the full directed graph complex $\text{dfGC}$ (see [16, Appendix K] or [2, Section 6.1]) we will only use its “connected part” $\text{dfGC}_{\text{conn}}$.

Thus we will deal exclusively with SFQs whose MC elements $\alpha$ satisfy the additional condition:

**Condition 2.1** For every $n \geq 1$ and $k \geq 0$ the linear combination \[
\alpha(s^{-1}t^e_{n,k})
\]
\*involves only connected graphs.*

By the argument given above, Kontsevich’s SFQ satisfies this condition.
3 The main theorem

Thanks to boundary conditions (2.18), (2.19), (2.20) and identity (2.21), we know that 
\( \alpha^K(s^{-1} t_n) \) and \( \alpha^K(s^{-1} t_{k'}) \) and \( \alpha^K(s^{-1} t_{1,k}) \) have rational coefficients for all \( n \geq 2, k' \geq 2 \) and \( k \geq 0 \).

The main result of this note is the following theorem:

**Theorem 3.1** There exists a sequence of MC elements
\[
\{\alpha_m\}_{m \geq 1}
\]
in the dg Lie algebra (2.17) (with real coefficients) which satisfy the following properties:

- Each \( \alpha_m \) satisfies boundary conditions\(^5\) (2.18), (2.19), and (2.20),
- \( \alpha_1 := \alpha^K \),
- \( \alpha_m - \alpha_{m-1} \in \mathcal{F}^\ast_{m-1} \text{Conv}(s^{-1} \mathcal{O}, \text{KGra}) \), and
- \( \alpha_m(s^{-1} t_{n,k}) \) has rational coefficients for all \( n \leq m \).

Since the filtration \( \mathcal{F}^\ast \text{Conv}(s^{-1} \mathcal{O}, \text{KGra}) \) on \( \text{Conv}(s^{-1} \mathcal{O}, \text{KGra}) \) is complete, Theorem 3.1 implies that

**Corollary 3.1** There exists a stable formality quasi-isomorphism defined over rational numbers. □

A proof of Theorem 3.1 is given in Section 6 at the end of the paper. Before proceeding to a construction of the desired sequence of MC elements, we present some required technical statements. Most of these statements are essentially borrowed from [2].

4 Technical statements

4.1 The operator \( \partial^{\text{Hoch}} \) and its properties

Just as in [2], vectors \( c \in \text{KGra}(n,k)^o \) singled out by the following properties will play a special role:

**Property 4.1** All white vertices in each graph of the linear combination \( c \) have valency one.

**Property 4.2** For every \( \sigma \in S_k \) we have
\[
(id, \sigma)(c) = (-1)^{|\sigma|} c.
\]

For example, the “brooms” \( \Gamma_k^{br} \) depicted on figure 2.6 obviously satisfy these properties.

We denote by \( \Pi\text{KGra}(n,k)^o \) the subspace of all vectors in \( \text{KGra}(n,k)^o \) satisfying Properties 4.1 and 4.2
\[
\Pi\text{KGra}(n,k)^o := \{ c \in \text{KGra}(n,k)^o \mid c \text{ satisfies Properties 4.1 and 4.2} \}.
\]

\(^5\)i.e. each MC element \( \alpha_m \) corresponds to an SFQ.
For every vector $c \in \text{KGr}a(n, k)^o$, we denote by $\Pi_1(c)$ the linear combination of graphs in $\text{dgra}_{n,k}$ which is obtained from $c$ by retaining only graphs whose all white vertices are univalent. The assignment

$$c \mapsto \Pi_1(c)$$ (4.3)

is obviously a linear projection from $\text{KGr}a(n, k)^o$ onto the the subspace of vectors in $\text{KGr}a(n, k)^o$ which satisfy Property 4.1.

Composing $\Pi_1$ with the alternation operator

$$\text{Alt}^o = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} (\text{id}, \sigma) : \text{KGr}a(n, k)^o \to \text{KGr}a(n, k)^o$$ (4.4)

we get a canonical projection

$$\Pi := \text{Alt}^o \circ \Pi_1 : \text{KGr}a(n, k)^o \to \Pi \text{KGr}a(n, k)^o.$$ (4.5)

Let us recall from [2] that, for every given $n$, the direct sum

$$\bigoplus_{k \geq 0} \text{KGr}a(n, k)^o$$ (4.6)

carries the degree 0 endomorphism $\partial^\text{Hoch}$ defined by the formula

$$\partial^\text{Hoch}(\gamma) = \Gamma_{o,o} \circ_{2,\sigma} \gamma + \sum_{i=1}^{k} (-1)^i \gamma \circ_{i,\sigma} \Gamma_{o,o} + (-1)^{k+1} \Gamma_{o,o} \circ_{1,\sigma} \gamma,$$ (4.7)

$$\gamma \in \text{KGr}a(n, k)^o.$$

A direct computation shows that

$$\partial^\text{Hoch} \circ \partial^\text{Hoch} = 0$$

and hence the direct sum (4.6) can be turned into a cochain complex by redefining the degrees of terms $\text{KGr}a(n, k)^o$ in the obvious way.

We claim that

**Claim 4.1** For every vector $\gamma \in \text{KGr}a(n, k)^o$

$$\partial^\text{Hoch}(\Pi(\gamma)) = \Pi(\partial^\text{Hoch} \gamma) = 0,$$ (4.8)

$\Pi(\gamma)$ belongs to the image of $\partial^\text{Hoch}$ if and only if $\Pi(\gamma) = 0$. Furthermore, for every vector $c \in \ker(\partial^\text{Hoch})$ the difference

$$c - \Pi(c)$$

belongs to the image of $\partial^\text{Hoch}$.

**Proof.** The proof of equations (4.8) is straightforward. The remaining two statements follow directly from Proposition A.1 in [2, Appendix A]. □
Fig. 4.1: The black vertex with label 1 is a pike but the black vertex with label 3 is not a pike.

4.2 The operator $\partial$ and its properties

Let us recall that a pike is a univalent black vertex $v$ of $\Gamma \in \mathrm{dgra}_{n,k}$ whose adjacent edge terminates at $v$. An example of a graph with a pike is shown on figure 4.1.

Let us consider the direct sum of spaces of invariants

$$\bigoplus_{n,k} \left( \Pi \mathrm{KGr}(n,k)^{\circ} \right)^{S_n} \tag{4.9}$$

and introduce the following families of cycles

$$\tau_{n,i} = (i, i+1, i+2, \ldots, n-1, n) \in S_n, \quad (4.10)$$

$$\sigma_{k,i} = (i, i-1, \ldots, 2, 1) \in S_k. \quad (4.11)$$

Next, we recall the following endomorphism of (4.9) from [2, Appendix B]

$$\partial(\gamma) = k \sum_{i=1}^{n+1} (\tau_{n+1,i}, \text{id})(\gamma \circ_{1,i} \Gamma_{0}^{\mathrm{br}}), \quad \gamma \in \left( \Pi \mathrm{KGr}(n,k)^{\circ} \right)^{S_n}. \tag{4.12}$$

Notice that, since the graph $\Gamma_{0}^{\mathrm{br}}$ consists of a single black vertex and has no edges, the insertion $\circ_{1,i}$ of $\Gamma_{0}^{\mathrm{br}}$ replaces the white vertex with label 1 by a black vertex with label $n+1$ and shifts the labels on the remaining white vertices down by 1.

Since the linear combination $\gamma \in \Pi \mathrm{KGr}(n,k)^{\circ}$ is anti-symmetric with respect to permutations of labels on white vertices, we have

$$\partial^2 = 0. \tag{4.13}$$

Thus, shifting the degrees of summands in (4.9) in an appropriate way, one can turn (4.9) into a cochain complex with $\partial$ being the differential. This cochain complex was analyzed in [2, Appendix B].

For our purposes, we need another endomorphism $\partial^*$ of (4.9) which we will now define. Let $\gamma$ be a vector in $\left( \Pi \mathrm{KGr}(n,k)^{\circ} \right)^{S_n}$. To compute $\partial^*(\gamma)$ we follow these steps:

- First, we omit in $\gamma$ all graphs for which the black vertex with label 1 is not a pike. We denote the resulting linear combination in $\mathrm{KGr}(n,k)^{\circ}$ by $\gamma'$.

- Second, we replace the black vertex with label 1 in each graph of $\gamma'$ by a white vertex and shift all labels on black vertices down by 1. We assign label 1 to this additional white vertex and shift the labels of the remaining white vertices up by 1. We denote the resulting linear combination in $\mathrm{KGr}(n-1,k+1)^{\circ}$ by $\gamma''$. 

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Finally, we set
\[ \delta^*(\gamma) = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{k+1} (\text{id}, \sigma_{k+1,i})(\gamma'''). \] (4.14)

Note that the linear combination \( \gamma' \) is invariant with respect to the action of the group \( S_{\{2,3,...,n\}} \times \{\text{id}\} \). Hence, the linear combination \( \delta^*(\gamma) \) is \( S_{n-1} \)-invariant. Furthermore, \( \delta^*(\gamma) \) obviously satisfies Properties 4.1 and 4.2.

Lemma B.3 from [2, Appendix B] says that for every \( \gamma \in (\Pi_{\text{KGr}}(n,k)^o)^{S_n} \) we have
\[ \delta \delta^*(\gamma) + \delta^* \delta(\gamma) = k \gamma + \sum_{r \geq 1} r \gamma_r, \] (4.15)
where \( \gamma_r \) is obtained from the linear combination \( \gamma \) by retaining only the graphs with exactly \( r \) pikes.

4.3 An auxiliary lemma

To construct the MC element \( \alpha_2 \) in sequence (3.1), we need the following lemma:

**Lemma 4.1** If \( \alpha^K \) is the MC element of \( \text{Conv}(s^{-1} \circ \circ \circ, \text{KGr}) \) defined by equations (2.26), (2.27), and (2.28) then
\[ \Pi(\alpha^K(s^{-1} t_{2,k})) = 0 \] (4.16)
for every \( k \geq 0 \).

Proof. Since the case \( k = 0 \) is already established in [12, Subsection 7.3.1.1], we proceed by induction on \( k \).

Let us observe that for every \( k \geq 0 \), the vector \( \alpha^K(s^{-1} t_{2,k}) \) in \( \text{KGr}(2,k)^o \) has degree \( -2 - k \). Therefore each graph in the linear combination \( \alpha^K(s^{-1} t_{2,k}) \) has \( k + 2 \) edges and hence every graph in \( \alpha^K(s^{-1} t_{2,k}) \) does not have a pike.

Next we assume, by induction, that equation (4.16) holds for every \( k < q \). Moreover, since every graph in \( \Pi(\alpha(s^{-1} t_{2,q})) \) has \( q + 2 \) edges, \( \Pi(\alpha(s^{-1} t_{2,q})) \) must be a linear combination of graphs shown on figure 4.2.

![Fig. 4.2: Here \{i_1, i_2, \ldots, i_p\} \cup \{j_1, j_2, \ldots, j_{q-p}\} is a partition of the set \{1, 2, \ldots, q\}](image)

Using the above inductive assumption, boundary conditions (2.18), (2.19), (2.20) and identity (2.21), we see that \( \mu \circ (\alpha^K s^{-1} \otimes \alpha^K s^{-1}) \circ D(s^{-1} t_{3,q-1}^o) \) is the following sum
\[ (-1)^{q-1} \alpha^K(s^{-1} t_{2,q-1}^o) \circ_{1,c} \Gamma \ldots + \] (4.17)
\[ \Gamma_n^0 \circ_2 \circ_2 \alpha^K(s^{-1} t^0_{3,q-2}) + \sum_{p=1}^{q-2} (-1)^p \alpha^K(s^{-1} t^0_{3,q-2}) \circ_{p,0} \Gamma_n^0 + (-1)^{q-1} \Gamma_n^0 \circ_1 \circ_2 \alpha^K(s^{-1} t^0_{3,q-2}) \quad (4.18) \]

\[ + \sum_{p=1}^{q} (-1)^p \left( \alpha^K(s^{-1} t^0_{q-2}) \circ_{p,0} \Gamma^0_{0} + \text{cyclic. perm. in } \{1, 2, 3\} \right) + \sum_{p=1}^{q-1} \sum_{k=1}^{q-k} \frac{(-1)^{p+k(q-p-k)}}{k!} \alpha^K(s^{-1} t^0_{2,q-k}) \circ_{p,0} \Gamma^0_k \quad (4.19) \]

\[ + \sum_{p=1}^{q-1} \sum_{k=1}^{q-k} \frac{(-1)^{p+k(q-p-k)}}{(q-k)!} \Gamma^0_{q-k} \circ_{p,0} \alpha^K(s^{-1} t^0_{2,k}) \quad (4.20) \]

Since \( \alpha^K \) is a MC element in \( \text{Conv}(s^{-1} \mathfrak{a}, \mathfrak{KGr}) \),

\[ \mu \circ (\alpha^K s^{-1} \otimes \alpha^K s^{-1}) \circ D(s^{-1} t^0_{3,q-2}) = 0. \]

Hence, applying the projection \( \Pi \) (4.5) to the sum of expressions (4.17), (4.18), (4.19), (4.20), and (4.21), we must get zero.

Expression (4.18) belongs to the image of \( \partial^{\text{Hoch}} \) (4.7). Hence, due to (4.8), this expression may be discarded. Applying \( \Pi \) to expression (4.17) we get zero by the inductive assumption.

Retaining only graphs with pikes in the linear combination

\[ \sum_{p=1}^{q} (-1)^p \left( \Pi(\alpha^K(s^{-1} t^0_{q-2}) \circ_{p,0} \Gamma^0_{0} + \text{cyclic. perm. in } \{1, 2, 3\} \right) \quad (4.22) \]

we get the sum

\[ (-q) \left( \Pi(\alpha^K(s^{-1} t^0_{q-2})) \right) \circ_{1,0} \Gamma^0_{0} + \text{cyclic. perm. in } \{1, 2, 3\} \]. \quad (4.23) \]

Finally, it is easy to see that expressions (4.20) and (4.21) do not involve graphs with pikes.

Thus we conclude that the sum in (4.23) must be zero. Since the linear combination \( \Pi(\alpha^K(s^{-1} t^0_{q-2})) \) does not involve graphs with pikes, the latter conclusion implies that

\[ \Pi(\alpha^K(s^{-1} t^0_{q-2})) = 0. \quad (4.24) \]

Lemma 4.1 is proved. \( \square \)

**Remark 4.2** We would like to observe that, for large values of \( q \), it is hard to prove equation (4.24) by direct computations of integrals entering Kontsevich’s formula. Of course, at the end of the day, equation (4.24) is a consequence of the Stokes theorem for the compactified configuration spaces \( \mathcal{C}_{n,k} \).
5 Correcting Kontsevich’s MC element $\alpha^K$ step by step

5.1 Construction of the MC element $\alpha_2$

Evaluating the left hand side of the MC equation for $\alpha^K$ on $s^{-1} t^e_{2,k}$ and using boundary conditions (2.18), (2.19), (2.20) together with identity (2.21), we deduce that

$$\partial^{Hoch} \alpha^K(s^{-1} t^e_{2,k-1}) = -\frac{(-1)^k}{k!} \Gamma_{k} \circ_1 \Gamma_{\bullet \bullet} +$$

\begin{equation}
+ \sum_{0 \leq p \leq q \leq k} \frac{(-1)^{p+(k-q)(q-p)}}{(k-q+p+1)! (q-p)!} (\Gamma_{k-q+p+1} \circ_{p+1} \Gamma_{q-p} + (\sigma_{12}, \text{id})(\Gamma_{k-q+p+1} \circ_{p+1} \Gamma_{q-p})) ,
\end{equation}

where $\sigma_{12}$ is the transposition $(1,2)$ and $\partial^{Hoch}$ is defined in (4.7).

We observe that for every triple $n, k, e$ the set $\text{dgra}_{n,k}$ contains finitely many graphs with $e$ edges. Hence, for every fixed $k \geq 1$, equation (5.1) is a finite size inhomogeneous linear system of equations on coefficients in $\alpha^K(s^{-1} t^e_{2,k-1})$. Furthermore, since all coefficients in the right hand side of (5.1) are rational, and the system admits a solution over reals, it also admits a solution over rationals.

Let us denote by $\beta_{2,k-1}$ a degree $-k-1$ element in $\text{KGr}(2, k-1)^o$ which solves (5.1), i.e.

$$\partial^{Hoch} \beta_{2,k-1} = -\frac{(-1)^k}{k!} \Gamma_{k} \circ_1 \Gamma_{\bullet \bullet} +$$

\begin{equation}
+ \sum_{0 \leq p \leq q \leq k} \frac{(-1)^{p+(k-q)(q-p)}}{(k-q+p+1)! (q-p)!} (\Gamma_{k-q+p+1} \circ_{p+1} \Gamma_{q-p} + (\sigma_{12}, \text{id})(\Gamma_{k-q+p+1} \circ_{p+1} \Gamma_{q-p})) .
\end{equation}

For every $k \geq 1$ we choose a solution $\beta_{2,k-1}$ of (5.2) which satisfies the additional property

$$\Pi(\beta_{2,k-1}) = 0 .$$

This is possible because $\Pi$ projects vectors of $\text{KGr}(n,k)^o$ onto cocycles of the differential $\partial^{Hoch}$.

Due to Lemma 4.1 and equations (5.1), (5.2), (5.3), the difference

$$\alpha^K(s^{-1} t^e_{2,k}) - \beta_{2,k}$$

is $\partial^{Hoch}$-closed and satisfies the additional condition

$$\Pi(\alpha^K(s^{-1} t^e_{2,k}) - \beta_{2,k}) = 0$$

for every $k \geq 0$.

Therefore, Claim 4.1 implies that for every $k \geq 1$ there exists a vector $\xi_{2,k-1} \in \text{KGr}(2, k-1)^o$ of degree $-2-k$ such that

$$\alpha^K(s^{-1} t^e_{2,k}) - \beta_{2,k} = \partial^{Hoch}(\xi_{2,k-1}) .$$

So we define a degree zero vector $\xi_2$ in $\text{Conv}(s^{-1} \text{oc}, \text{KGr})$ by setting

\begin{align}
\xi_2(s^{-1} t^e_{2,k}) &= \xi_{2,k} , \quad \forall \ k \geq 0 , \\
\xi_2(s^{-1} t^e_{n,k}) &= 0 , \quad \forall \ n \neq 2 , \ k \geq 0 , \\
\xi_2(s^{-1} t^e_{k'}) &= 0 , \quad \forall \ k' \geq 2 , \\
\xi_2(s^{-1} t^e_{n'}) &= 0 , \quad \forall \ n' \geq 2 .
\end{align}

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Finally, we set
\[ \alpha_2 := \exp(\xi_2, \] \alpha^K. \] (5.8)

Since \( \xi_2 \) satisfies the conditions of [2, Proposition 5.1], the new MC element \( \alpha_2 \) satisfies boundary conditions (2.18), (2.19), (2.20). Therefore, the difference
\[ \alpha_2 - \alpha^K \]
belongs to \( \mathcal{F}_1 \text{Conv}(s^{-1} \text{oc}, \text{KGra}) \).

Furthermore, equation (5.6) implies that
\[ \alpha_2(s^{-1} t_{2,k}^o) = \beta_{2,k}. \]

Thus the desired MC element \( \alpha_2 \) is constructed.

5.2 Pikes in \( \alpha_{m-1}(s^{-1} t_{m,0}^o) \) can be “killed”

Let \( m \) be an integer \( \geq 3 \) and assume that the desired MC elements \( \alpha_{m'} \) in (3.1) are constructed for all \( m' \leq m - 1 \).

We need to show that \( \alpha_{m-1} \) is isomorphic to another MC element \( \alpha'_{m-1} \) for which all graphs in the linear combination \( \alpha'_{m-1}(s^{-1} t_{m,0}^o) \) do not have pikes and
\[ \alpha'_{m-1} - \alpha_{m-1} \in \mathcal{F}_1 \text{Conv}(s^{-1} \text{oc}, \text{KGra}) \).

Let us denote by \( \alpha^r_{m-1} \) the linear combination in \( (\text{KGra}(m, 0)^o)^S \) which is obtained from \( \alpha_{m-1}(s^{-1} t_{m,0}^o) \) by retaining only graphs with exactly \( r \) pikes. Since \( \alpha^r_{m-1} \) does not have white vertices at all, we conclude that \( \alpha^r_{m-1} \) belongs to the graded vector space (4.9) and
\[ \mathfrak{d}(\alpha^r_{m-1}) = 0, \]
where the operator \( \mathfrak{d} \) is defined in (4.12).

Hence, using (4.15), we deduce that for
\[ \chi_{m-1} = -\sum_{r \geq 1} \frac{1}{r} \mathfrak{d}^r(\alpha^r_{m-1}) \] (5.9)
the linear combination
\[ \alpha_{m-1}(s^{-1} t_{m,0}^o) + \mathfrak{d}(\chi_{m-1}) \] (5.10)
does not involve graphs with pikes.

Next, using \( \chi_{m-1} \) we define a degree zero vector
\[ \xi \in \text{Conv}(s^{-1} \text{oc}, \text{KGra}) \] (5.11)
by setting
\[ \xi(s^{-1} t_{m-1,1}^o) = \chi_{m-1}, \quad \xi(s^{-1} t_{n,1}^o) = \xi(s^{-1} t_{k,1}^o) = \xi(s^{-1} t_{n,k}^o) = 0 \] (5.12)
for all \( n, k \) and for all pairs \( (n, k) \neq (m - 1, 1) \).

Then, we act by \( \exp(\text{ad}_\xi) \) on the MC element \( \alpha_{m-1} \) and get a new MC element
\[ \alpha'_{m-1} := \exp(\text{ad}_\xi) \alpha_{m-1}. \] (5.13)
It is not hard to see that
\[
\exp(\text{ad}_\xi) \alpha_{m-1}(s^{-1} t_{m,0}^o) = \alpha_{m-1}(s^{-1} t_{m,0}^o) + [\xi, \alpha_{m-1}](s^{-1} t_{m,0}^o)
\]  
and
\[
[\xi, \alpha_{m-1}](s^{-1} t_{m,0}^o) = \mathfrak{d}(\chi_{m-1}).
\]

Therefore, due to the above observation, the linear combination \( \alpha'_{m-1}(s^{-1} t_{m,0}^o) \) does not involve pikes.

It is easy to see that, for every \( 1 \leq n < m - 1 \) and for every \( k \geq 0 \)
\[
(\alpha'_{m-1} - \alpha_{m-1})(s^{-1} t_{n,k}^o) = 0.
\]
It is also easy to see that for every \( k \neq 2 \)
\[
(\alpha'_{m-1} - \alpha_{m-1})(s^{-1} t_{n-1,k}^o) = 0. \tag{5.15}
\]

Finally, we observe that each graph in the linear combination \( \chi_{m-1} \) (5.9) has the univalent white vertex. Hence,
\[
\partial^{\text{Hoch}} \chi_{m-1} = 0
\]
and, using this identity, it is easy to deduce that
\[
(\alpha'_{m-1} - \alpha_{m-1})(s^{-1} t_{m-1,2}^o) = 0. \tag{5.16}
\]

Thus we proved the following technical statement:

**Claim 5.1** The SFQ corresponding to the MC element \( \alpha_{m-1} \) is homotopy equivalent to an SFQ which corresponds to a MC element \( \alpha'_{m-1} \) satisfying following properties:

- the linear combination \( \alpha'_{m-1}(s^{-1} t_{m,0}^o) \) does not involve graphs with pikes;
- the difference \( \alpha'_{m-1} - \alpha_{m-1} \) belongs to \( \mathcal{F}^\chi_{m-1} \text{Conv}(s^{-1} \mathcal{O}, \mathcal{K}\text{Gra}) \).

\[\Box\]

### 5.3 Adjusting \( \alpha_{m-1} \) by the action of \( dfGC_{\text{conn}} \)

Due to Claim 5.1 we may assume, without loss of generality, that \( \alpha_{m-1}(s^{-1} t_{m,0}^o) \) does not involve graphs with pikes.

Evaluating the left hand side of the MC equation for \( \alpha_{m-1} \) on \( s^{-1} t_{m+1,0}^o \) and using the usual boundary conditions on \( \alpha_{m-1} \) we get the following identity
\[
\sum_{\tau \in \text{Sh}_{2,m-1}} (\tau, \text{id})(\alpha_{m-1}(s^{-1} t_{m,0}^o) \circ_{1,c} \Gamma_{\tau,\tau}) \tag{5.17}
\]
\[
- \sum_{i=1}^{m+1} (\sigma_{m+1,i}, \text{id})(\Gamma_{1}^{\text{br}} \circ_{1,o} \alpha_{m-1}(s^{-1} t_{m,0}^o)) \tag{5.18}
\]
\[
- \sum_{i=1}^{m+1} (\tau_{m+1,i}, \text{id})(\alpha_{m-1}(s^{-1} t_{m,1}^o) \circ_{1,o} \Gamma_{0}^{\text{br}}) \tag{5.19}
\]
− \sum_{r=2}^{m-1} \sum_{\sigma \in Sh_r,m+1-r} (\sigma, id)(\alpha_m - 1(s^{-1} t_{r,1}^o) \circ_{1,\sigma} \alpha_m - 1(s^{-1} t_{m+1-r,0}^o)) = 0,

(5.20)

where \( \sigma_{m+1,i} \) is the cycle \((i, i - 1, \ldots, 2, 1) \in S_{m+1} \) and \( \tau_{m+1,i} \) is the cycle \((i, i + 1, \ldots, m, m + 1) \in S_{m+1} \).

Since the graph \( \Gamma^b_0 \in KGr(1, 0) \) is a single black vertex labeled by 1, the linear combination in line (5.19) involves only graphs with pikes.

Graphs with pikes do not appear in line (5.20) because all graphs in \( \alpha_m - 1(s^{-1} t_{n,k}^o) \) for \( n \geq 1 \) are connected and each graph in \( \alpha_m - 1(s^{-1} t_{m+1-r,0}^o) \) has at least 2 black vertices.

On the other hand, the linear combination \( \alpha_m - 1(s^{-1} t_{m,0}^o) \) does not involve graphs with pikes. Hence the sum in line (5.19) must cancel all the contributions with pikes coming from the insertion of \( \Gamma_{\bullet, \bullet} \) into vertices of graphs in \( \alpha_m - 1(s^{-1} t_{m,0}^o) \) in line (5.17).

Thus the first three lines combine into

\[-\Gamma^b_0 \circ_{1,\xi} [\Gamma_{\bullet, \bullet}, \alpha_m - 1(s^{-1} t_{m,0}^o)]\]

(5.21)

and the whole identity can be rewritten as

\[\Gamma^b_0 \circ_{1,\xi} [\Gamma_{\bullet, \bullet}, \alpha_m - 1(s^{-1} t_{m,0}^o)] = \]

(5.22)

\[-\sum_{r=2}^{m-1} \sum_{\sigma \in Sh_r,m+1-r} (\sigma, id)(\alpha_m - 1(s^{-1} t_{r,1}^o) \circ_{1,\sigma} \alpha_m - 1(s^{-1} t_{m+1-r,0}^o)) ,

\]

where \( \alpha_m - 1(s^{-1} t_{m,0}^o) \) is viewed as a degree 0 vector in \( dfGC_{conn} \).

By the inductive assumption, all coefficients in the right hand side of (5.22) are rational. Therefore, (5.22) is a finite size inhomogeneous linear system with rational right hand side.

Since this system admits a solution over reals, it also admits a solution over rationals. In other words, there exists a degree \( 2 - 2m \) vector

\[\beta_{m,0} \in (dGra_{conn}(m))^{S_m}(Q)\]

which also solves the equation

\[\Gamma^b_0 \circ_{1,\xi} [\Gamma_{\bullet, \bullet}, \beta_{m,0}] = \]

(5.23)

\[-\sum_{r=2}^{m-1} \sum_{\sigma \in Sh_r,m+1-r} (\sigma, id)(\alpha_m - 1(s^{-1} t_{r,1}^o) \circ_{1,\sigma} \alpha_m - 1(s^{-1} t_{m+1-r,0}^o)) = 0.

Therefore, the difference \( \alpha_m - 1(s^{-1} t_{m,0}^o) - \beta_{m,0} \) is a degree zero cocycle in \( F_{m-1} dfGC \).

Thus we proved the following statement:

**Claim 5.2** Let us suppose that an SFQ is given by a MC element \( \alpha_m - 1 \) which satisfies the following properties:

- for every \( n \leq m - 1 \) and \( k \geq 0 \) the linear combination \( \alpha_m - 1(s^{-1} t_{n,k}^o) \) has only rational coefficients,

- the linear combination \( \alpha_m - 1(s^{-1} t_{m,0}^o) \) does not involve graphs with pikes.

Then there exists a degree 0 cocycle \( \gamma \in dfGC \) such that the new MC element \( \bar{\alpha}_m - 1 := \exp(ad_\gamma)(\alpha_m - 1) \) satisfies the properties:
• $\tilde{\alpha}_{m-1} - \alpha_{m-1} \in F_{m-1} \text{Conv}(s^{-1} \alpha, \text{KGr})$, in particular, for every $n \leq m - 1$ and $k \geq 0$ the linear combination $\tilde{\alpha}_{m-1}(s^{-1} t^o_{n,k})$ has rational coefficients;

• all coefficients in the linear combination $\tilde{\alpha}_{m-1}(s^{-1} t^o_{n,0})$ are also rational.

\[ \square \]

5.4 Replacing coefficients in $\alpha(s^{-1} t^o_{m,k})$ for $k \geq 1$ by rational numbers

Let $m$ be, as above, a fixed integer $\geq 3$. Our previous work shows that there exists a MC element $\alpha \in \text{Conv}(s^{-1} \alpha, \text{KGr})$ satisfying “boundary conditions” (2.18), (2.19), (2.20) and such that

• all coefficients in $\alpha(s^{-1} t^o_{n,k'})$ are rational for $n \leq m - 1$ and all $k' \geq 0$

• all coefficients in $\alpha(s^{-1} t^o_{n,0})$ are rational as well.

Let $k$ be a positive integer such that all coefficients in the linear combination $\alpha(s^{-1} t^o_{m,k'})$ are rational for $k' < k$. We will show that, playing with the homotopy equivalence, we can get a MC element $\alpha'$ such that

$$\alpha'(s^{-1} t^o_{n,k_1}) = \alpha(s^{-1} t^o_{n,k_1}) \quad \forall n \leq m - 1, \ k_1 \geq 0$$

$$\alpha'(s^{-1} t^o_{m,k'}) = \alpha(s^{-1} t^o_{m,k'}) \quad \forall k' < k$$

and the linear combination $\alpha'(s^{-1} t^o_{m,k})$ has rational coefficients.

Using the usual “boundary conditions” for $\alpha$ and the inequality $m \geq 3$, we evaluate

$$\mu(\alpha s^{-1} \otimes \alpha s^{-1}) D(t^o_{m,k}) = (5.24)$$

$$\partial_t^{\text{Hoch}} \alpha(s^{-1} t^o_{m,k}) + (-1)^k \sum_{\tau \in \text{Sh}_{2,n-2}} (\tau, \text{id})(\alpha(s^{-1} t^o_{m-1,k}) \circ_{1,\tau} \Gamma_{\bullet \bullet})$$

$$- \sum_{r=1}^{m-1} \sum_{r \in \text{Sh}_{r,m-r}} \sum_{0 \leq p \leq q \leq k} (-1)^{p+(k-q)(q-p)}(\sigma, \text{id})(\alpha(s^{-1} t^o_{r,p+k-q-1}) \circ_{p+1,\sigma} \alpha(s^{-1} t^o_{m-r,q-p})).$$

Since $\alpha$ is a MC element of $\text{Conv}(s^{-1} \alpha, \text{KGr})$, the left hand side of (5.24) is zero. So we conclude that the vector $\alpha(s^{-1} t^o_{m,k})$ satisfies the following cohomological equation

$$\partial_t^{\text{Hoch}} \alpha(s^{-1} t^o_{m,k}) = (5.25)$$

$$\sum_{r=1}^{m-1} \sum_{r \in \text{Sh}_{r,m-r}} \sum_{0 \leq p \leq q \leq k} (-1)^{p+(k-q)(q-p)}(\sigma, \text{id})(\alpha(s^{-1} t^o_{r,p+k-q-1}) \circ_{p+1,\sigma} \alpha(s^{-1} t^o_{m-r,q-p})$$

$$- (-1)^k \sum_{\tau \in \text{Sh}_{2,n-2}} (\tau, \text{id})(\alpha(s^{-1} t^o_{m-1,k}) \circ_{1,\tau} \Gamma_{\bullet \bullet}).$$

Since the subspace of vectors in $\text{KGr}(m, k)$ of a fixed degree is finite dimensional, equation (5.25) can be viewed as an inhomogeneous linear system with the right hand side defined over rationals. Thus, since the system has a solution over reals it must also have a solution over rationals.
Let us denote by $\beta_{m,k}$ a linear combination of degree $2 - 2m - k$ in $\text{KGr}_{m,k}$ with all rational coefficients and such that
\[
\partial_{\text{Hoch}} \beta_{m,k} = \sum_{r=1}^{m-1} \sum_{\sigma \in \text{Sh}_{r,m-r}} \sum_{0 \leq p \leq q \leq k} (-1)^{p+(k-q)(q-p)}(\sigma, \text{id})(\alpha(s^{-1} t_{r,p+k-q+1}^o) \circ_{p+1,o} \alpha(s^{-1} t_{m-r,q-p}^o))
\]

Equations (5.25) and (5.26) imply that the difference $\alpha(s^{-1} t_{m,k}^o) - \beta_{m,k}$ is $\partial_{\text{Hoch}}$-closed. Therefore, by Claim 4.1, there exist vectors
\[
\xi_{m,k-1} \in \left(\text{KGr}(m, k-1)\right)^{S_m}, \quad |\xi_{m,k-1}| = 2 - 2m - k, \quad k \geq 2
\]
such that
\[
\alpha(s^{-1} t_{m,k}^o) - \beta_{m,k} = \partial_{\text{Hoch}} \xi_{m,k-1} + \gamma_{m,k}
\]
where
\[
\gamma_{m,k} := \Pi(\alpha(s^{-1} t_{m,k}^o) - \beta_{m,k}) \in \left(\Pi \text{KGr}(m, k)\right)^{S_m}.
\]

Let $\xi(k)$ be a degree vector in $\text{Conv}(s^{-1} \circ \text{KGr})$ defined by the equations
\[
\xi(k)(s^{-1} t_{m,k+1}^o) = \xi_{m,k-1}, \quad \forall \; k \geq 1,
\]
\[
\xi(k)(s^{-1} t_{m,k}^o) = 0, \quad \forall \; n \neq m, k' \geq 0,
\]
\[
\xi(k)(s^{-1} t_{n,k}^o) = 0, \quad \forall \; n \neq m, k' \geq 2,
\]
\[
\xi(k)(s^{-1} t_{n,k}^o) = 0, \quad \forall \; n \geq 2.
\]

We consider an SFQ corresponding to the MC element
\[
\tilde{\alpha} := \exp(\text{ad} \xi(k)) \alpha.
\]

It is easy to see that
\[
(\tilde{\alpha} - \alpha)(s^{-1} t_{m,k}^o) = 0 \quad \forall \; n \leq m - 1, k \geq 0.
\]

Furthermore,
\[
(\tilde{\alpha} - \alpha)(s^{-1} t_{m,q}^o) = \begin{cases} -\partial_{\text{Hoch}} \xi(k)(s^{-1} t_{m,k-1}^o) & \text{if } q = k \\ 0 & \text{otherwise}. \end{cases}
\]

In particular,
\[
\tilde{\alpha}(s^{-1} t_{m,k'}^o) = \alpha(s^{-1} t_{m,k'}^o) \quad \forall \; k' < k
\]
and
\[
\tilde{\alpha}(s^{-1} t_{m,k}^o) = \gamma_{m,k} + \beta_{m,k}.
\]

Since $\tilde{\alpha}$ is a MC element in $\text{Conv}(s^{-1} \circ \text{KGr})$
\[
\mu \circ \tilde{\alpha} s^{-1} \otimes \tilde{\alpha} s^{-1}(\mathcal{D}t_{m+1,k-1}^o) = 0.
\]
Unfolding this equation, we get
\[
\partial(\gamma_{m,k}) = \partial^{\text{Hoch}}(\tilde{\alpha}(s^{-1}t_{m+1,k-2}^e))
\]
\[
+ \sum_{p=1}^{k} \sum_{j=1}^{m+1} (-1)^p(\tau_{m+1,j}, \text{id})((\beta_{m,k} \circ p, \text{O})\Gamma_{0}^{br}) + \ldots ,
\]
where \(\tau_{m+1,j}\) is the cycle \((i, i+1, i+2, \ldots, m, m+1)\) and \(\ldots\) is a sum of terms involving only \(\tilde{\alpha}(s^{-1}t_{n,k'}^e)\) for \(n < m\) or \(\tilde{\alpha}(s^{-1}t_{m,k'}^e)\) for \(k' < k\). In particular, all coefficients in the terms \(\ldots\) and in the sum
\[
\sum_{p=1}^{k} \sum_{j=1}^{m+1} (-1)^p(\tau_{m+1,j}, \text{id})((\beta_{m,k} \circ p, \text{O})\Gamma_{0}^{br})
\]
are rational.

Applying the projection \(\Pi (4.5)\) to both sides of (5.37) and using Claim 4.1, we conclude that coefficients of \(\gamma_{m,k}\) solve the following finite size linear system:
\[
\partial(\gamma_{m,k}) = \sum_{p=1}^{k} \sum_{j=1}^{m+1} (-1)^p\Pi(\tau_{m+1,j}, \text{id})((\beta_{m,k} \circ p, \text{O})\Gamma_{0}^{br}) + \Pi(\ldots)
\]
with the right hand side defined over rationals.

Thus, we conclude that, there exists a degree \(2 - 2m - k\) vector
\[
\tilde{\gamma}_{m,k} \in (\Pi \text{KGra}(m, k)^{o})^{S_{m}}(\mathbb{Q})
\]
such that
\[
\partial(\tilde{\gamma}_{m,k} - \gamma_{m,k}) = 0 .
\]

Let us denote by \(\kappa_{m,k}^r\) (\(r \geq 0\)) the linear combination which is obtained from \(\tilde{\gamma}_{m,k} - \gamma_{m,k}\) by retaining only graphs with exactly \(r\) pikes. (In fact, since \(\partial(\tilde{\gamma}_{m,k} - \gamma_{m,k}) = 0\), the component \(\kappa_{m,k}^0 = 0\), i.e. each graph in the difference \(\tilde{\gamma}_{m,k} - \gamma_{m,k}\) has a pike.)

Then equation (4.15) implies that
\[
\tilde{\gamma}_{m,k} - \gamma_{m,k} = \partial(\kappa_{m,k}),
\]
where
\[
\kappa_{m,k} = \sum_{r \geq 0} \frac{1}{k + r} \partial^{*}\kappa_{m,k}^{r}.
\]

Using the vector \(\kappa_{m,k}\), we define the degree zero vector \(\psi(k) \in \text{Conv}(s^{-1}oc, \text{KGra})\) by setting
\[
\psi(k)(s^{-1}t_{m-1,k+1}^e) = \kappa_{m,k}, \quad \psi(k)(s^{-1}t_{n,k'}^e) = \psi(k)(s^{-1}t_{n_1}^c) = \psi(k)(s^{-1}t_{k_1}^o) = 0
\]
for all pairs \((n', k') \neq (m - 1, k + 1)\) and for all \(n_1 \geq 2\) and \(k_1 \geq 2\).

Next, we consider the new MC element
\[
\alpha' := \exp(\text{ad}_{\psi(k)})(\tilde{\alpha}) .
\]

It is obvious that for every \(n < m - 1\) and \(k' \geq 0\)
\[
\alpha'(s^{-1}t_{n,k'}^e) = \tilde{\alpha}(s^{-1}t_{n,k'}^e).
\]
Furthermore, since \( \partial^{\text{Hoch}}(k_{m,k}) = 0 \), we also have
\[
\alpha'(s^{-1}t^o_{m-1,q}) = \tilde{\alpha}(s^{-1}t^o_{m-1,q}), \quad \forall \, q \geq 0.
\]
(5.44)

It is not hard to see that, for \( q < k \),
\[
\alpha'(s^{-1}t^o_{m,q}) = \tilde{\alpha}(s^{-1}t^o_{m,q}),
\]
(5.45)
and finally, for \( t^o_{m,k} \), we have
\[
\alpha'(s^{-1}t^o_{m,k}) = \tilde{\alpha}(s^{-1}t^o_{m,k}) + \partial(\psi(k)(s^{-1}t^o_{m-1,k+1})) = \tilde{\alpha}(s^{-1}t^o_{m,k}) + \partial(\psi(k)),
\]
(5.46)
Combining the above observations with equations (5.35) and (5.41), we conclude that
\[
\alpha'(s^{-1}t^o_{n,k_1}) = \alpha(s^{-1}t^o_{n,k_1}) \quad \forall \, n \leq m - 1, \, k_1 \geq 0
\]
(5.47)
and
\[
\alpha'(s^{-1}t^o_{m,k'}) = \alpha(s^{-1}t^o_{m,k'}) \quad \forall \, k' < k
\]
and
\[
\alpha'(s^{-1}t^o_{m,k}) = \tilde{\gamma}_{m,k} + \beta_{m,k},
\]
i.e. all coefficients in \( \alpha'(s^{-1}t^o_{m,k}) \) are rational.

Thus we proved the following statement

**Proposition 5.1** Let \( m \) be an integer \( \geq 3 \) and \( \alpha \) be a MC element in \( \text{Conv}(s^{-1}o, \text{KGr}) \) satisfying “boundary conditions” (2.18), (2.19), (2.20) and such that

- all coefficients in \( \alpha(s^{-1}t^o_{n,k'}) \) are rational for \( n \leq m - 1 \) and all \( k' \geq 0 \)
- all coefficients in \( \alpha(s^{-1}t^o_{m,0}) \) are rational as well.

Then there exist sequences of degree zero vectors
\[
\{\xi(k)\}_{k \geq 1} \quad \text{and} \quad \{\psi(k)\}_{k \geq 1}
\]
in \( \text{Conv}(s^{-1}o, \text{KGr}) \) such that for every \( k \geq 1 \)
\[
\psi(k)(s^{-1}t^e_n) = \xi(k)(s^{-1}t^e_n) = \psi(k)(s^{-1}t^o_{1,q}) = \xi(k)(s^{-1}t^o_{1,q}) = 0 \quad \forall \, n \geq 2, \, q \geq 0,
\]
(5.48)
\[
\xi(k) \in \mathcal{F}_{m+k-2}\text{Conv}(s^{-1}o, \text{KGr}) \quad \text{and} \quad \psi(k) \in \mathcal{F}_{m+k-1}\text{Conv}(s^{-1}o, \text{KGr})
\]
(5.49)
and the MC elements
\[
\alpha^{(k)} := \exp(\text{ad}_{\psi(k)}) \exp(\text{ad}_{\xi(k)}) \exp(\text{ad}_{\psi(k-1)}) \exp(\text{ad}_{\xi(k-1)}) \ldots
\]
\[
\ldots \exp(\text{ad}_{\psi(2)}) \exp(\text{ad}_{\xi(2)}) \exp(\text{ad}_{\psi(1)}) \exp(\text{ad}_{\xi(1)})(\alpha)
\]
(5.50)

enjoy the following properties

- for every \( k \geq 0 \), the element \( \alpha^{(k)} \) satisfies the required boundary conditions (2.18), (2.19), and (2.20),
- for every \( k \geq 0 \),
\[
\alpha^{(k)}(s^{-1}t^o_{n,q}) = \alpha(s^{-1}t^o_{n,q}), \quad \forall \, n \leq m - 1, \, q \geq 0,
\]
(5.51)
• for every \( k \geq 1 \) and \( q < k \)
  \[
  \alpha^{(k)}(s^{-1} t_{m,q}^e) = \alpha^{(k-1)}(s^{-1} t_{m,q}^e),
  \]

• for every \( q \leq k \), all coefficients in \( \alpha^{(k)}(s^{-1} t_{m,q}^e) \) are rational.

\[\square\]

6 The proof of Theorem 3.1

We will now assemble all the statements presented in Section 5 into a proof of Theorem 3.1. The desired MC element \( \alpha_2 \) is constructed in Section 5.1. So let us assume that, for \( m \geq 3 \), the desired MC element \( \alpha_{m-1} \) is constructed.

According to Claim 5.1, there exists a MC element \( \alpha'_{m-1} \) in Conv\((s^{-1} \mathcal{G}, K\mathcal{G})\) such that

• it satisfies the required boundary conditions (2.18), (2.19), and (2.20),
• the linear combination \( \alpha'_{m-1}(s^{-1} t_{m,0}^e) \) does not involve graphs with pikes,
• for every \( n \leq m - 1 \) and every \( k \geq 0 \)
  \[
  \alpha'_{m-1}(s^{-1} t_{n,k}^e) = \alpha_{m-1}(s^{-1} t_{n,k}^e);
  \]
  in particular, all coefficients in \( \alpha'_{m-1}(s^{-1} t_{n,k}^e) \) are rational.

Next, replacing \( \alpha_{m-1} \) by \( \alpha'_{m-1} \) and using Claim 5.2, we conclude that there exists a MC element \( \tilde{\alpha}_{m-1} \) in Conv\((s^{-1} \mathcal{G}, K\mathcal{G})\) such that

• it satisfies the required boundary conditions (2.18), (2.19), and (2.20),
• for every \( n \leq m - 1 \) and every \( k \geq 0 \)
  \[
  \tilde{\alpha}_{m-1}(s^{-1} t_{n,k}^e) = \alpha_{m-1}(s^{-1} t_{n,k}^e);
  \]
  in particular, all coefficients in \( \tilde{\alpha}_{m-1}(s^{-1} t_{n,k}^e) \) are rational,
• all coefficients in the linear combination \( \tilde{\alpha}_{m-1}(s^{-1} t_{m,0}^e) \) are also rational.

Setting \( \alpha = \tilde{\alpha}_{m-1} \) and using Proposition 5.1, we conclude that there exists a sequence of MC elements

\[
\{ \alpha^{(k)} \}_{k \geq 0} \in \text{Conv}(s^{-1} \mathcal{G}, K\mathcal{G})
\]

such that

• \( \alpha^{(0)} = \tilde{\alpha}_{m-1} \),
• for every \( k \geq 0 \), the element \( \alpha^{(k)} \) satisfies the required boundary conditions (2.18), (2.19), and (2.20),
• for every \( k \geq 0 \),
  \[
  \alpha^{(k)}(s^{-1} t_{n,q}^e) = \tilde{\alpha}_{m-1}(s^{-1} t_{n,q}^e), \quad \forall \ n \leq m - 1, \quad q \geq 0,
  \]
• for every $k \geq 1$ and $q < k$

$$
\alpha^{(k)}(s^{-1}t_{m,q}^e) = \alpha^{(k-1)}(s^{-1}t_{m,q}^e),
$$

• for every $q \leq k$, all coefficients in $\alpha^{(k)}(s^{-1}t_{m,q}^e)$ are rational.

Thus, the desired MC element $\alpha_m$ in (3.1) is obtained as the limiting element$^6$ of the sequence (6.1).

Theorem 3.1 is proved. □

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$^6$Recall that the dg Lie algebra Conv($s^{-1}oc, KGra$) is complete with respect to the filtration $F^\bullet_{Conv}(s^{-1}oc, KGra)$. 
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