Entanglement Entropy, Current, and Chemical Potential

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We analytically and exactly compute the entanglement and Rényi entropies of the Dirac fermions on 2 dimensional torus in the presence of the constant current and/or chemical potential. The entropies are periodic in current, which also plays the role of a ‘beat frequency,’ in the small temperature limit. In the large radius limit, the dependence on current and/or chemical potential vanishes at least quadratically as a function of a sub-system size over a total system size. The entropies depend on chemical potential at zero temperature and are useful to probe energy spectra. Furthermore, current and chemical potential dependent parts of the Mutual (Rényi) information are shown to be independent of the separation between the disjoint sub-systems.

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Entanglement is at the heart of the quantum theories, encompassing the quantum mechanics, quantum field theories, quantum gravity and quantum information science [1]. Entanglement entropy [2][6] and its extension, Rényi entropy [7], can be used to measure the quantum information encoded in a quantum state. The entropies have been useful to probe quantum critical phenomena [12], to classify topological states of matter that can not be distinguished by symmetries [13][14], and to prove the irreversibility of the renormalization group in 3 dimensional field theories that do not have other ways to do so [15]. Recently, Rényi entropy has been measured in systems of interacting delocalized particles using quantum interference of many body systems [16].

Quantum fields can be manipulated by the background gauge fields, such as electric and magnetic fields. In the quantum world, gauge potentials are more useful. The time and space component of the 1+1 dimensional gauge potential are called a chemical potential and current, respectively. They are our primary physical objects.

Direct computations of the entanglement entropy on quantum field theories are known to be difficult. Nevertheless there have been progresses in 1+1 dimensions [8][11]. The entanglement entropy for quantum systems with finite chemical potential (or finite charge density) at zero temperature has been studied previously and indicated that it does not depend on chemical potential for free fermions [17][18] and for infinite field theory systems with a single interval [19]. The entanglement entropy with a current was also considered in a rather different context [20]. These scatter results are certainly interesting and deserve to be organized in a unified frame. Furthermore, exact and analytic results of the entropies for realistic physical systems are valuable resources for gaining insights on the nature of quantum entanglement.

In this letter, we provide a broader and complete picture of the entanglement entropy (and Rényi entropy) for the field theory systems with Dirac fermions in the presence of current, $J$, and/or chemical potential, $\mu$ on 2 dimensional torus. We analytically and exactly compute the entropies at the large radius limit as well as at the small temperature limit. We uncover the following four novel and interesting features. First, in the small temperature limit, the entropies are periodic in the current $J$, which also plays the role of a ‘beat frequency’ when a modulus parameter $\tau_1$ is dialed. The entropies for the periodic and anti-periodic fermions show distinctive features: the latter vanishes, while the former has non-zero contributions, at zero temperature.

Second, in the large radius limit, the dependence of the entropies on the chemical potential and current vanishes as fast as $O(\ell_t/L)^2$, where $\ell_t$ is the size of the sub-systems that we measure the entropies inside the total system with a size $2\pi L$. This supports a recent claim that the entanglement entropy of an interval in an infinite system is independent of chemical potential $\mu$ [19]. We further generalize the claim for multiple intervals and for the systems with the current $J$.

Third, in the zero temperature limit, the entropies are shown to be able to probe the energy spectra of the quantum system through chemical potential. The entropies develop non-vanishing contributions when chemical potential coincides with one of the energy levels of the Dirac fermions. This is a non-trivial generalization of earlier results [17][18].

Finally, we compute the current and chemical potential dependences of the mutual (Rényi) information between two sub-systems, which are the same as those of the entropies. Surprisingly, the information only depends on the sub-systems sizes independent of the separation of the disjoint sub-systems.

A. Partition function

We construct the partition function of 1+1 dimensional Dirac fermion in the presence of current $J$ and chemical potential $\mu$ based on previous studies found in [21][22]. In particular we use the equivalence between the twisted boundary condition and the presence of the background gauge fields to build up the partition function for the Dirac fermion in 2-dimensions.
Consider the action for Dirac fermion $\psi$

$$S = \frac{1}{2\pi} \int d^2x \; i \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu) \psi,$$  

where $\mu = 0, 1$ are the time and space coordinates with $\gamma^0 = \sigma_1, \gamma^1 = -i\sigma_2$ in terms of Pauli matrices and in matrix form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and constant background gauge fields $A_0 = \mu, A_1 = J$ that are identified as chemical potential and current.

To compute the partition function, we consider a torus with the modular parameter $\tau = \tau_1 + i\tau_2$. Thus the space of coordinate $\zeta = \frac{1}{2\pi}(s + it)$ is identified as $\zeta \equiv \zeta + 1 \equiv \zeta + \tau$. Now $s = x$ with the circumference $2\pi L$ (with $L = 1$ in this subsection), while $t$ is the Euclidean time with periodicity $2\pi \tau_2 = \beta = 1/T$. We decompose the Dirac field

$$\psi = \left( \psi_-, \psi_+ \right),$$

and consider twisted boundary conditions

$$\psi_\pm (t, s) = e^{-2\pi i a \tau_2} \psi_\pm (t, s + 2\pi) = e^{-2\pi i b \tau_2} \psi_\pm (t + 2\pi \tau_2, s + 2\pi \tau_1).$$

There exists an equivalent description that has the periodic Dirac fermion with the following flat gauge connection $\tilde{A}_\mu$ on a torus

$$\tilde{A} = \tilde{A}_\mu dx^\mu = ads + \frac{b - a \tau_1}{\tau_2} dt.$$  

Thus in this equivalent description, one can identify the chemical potential and current

$$a = \tilde{J}, \quad b = \tau_1 \tilde{J} + i \tau_2 \mu.$$  

Partition function is a trace of the Hilbert space constructed with the twisted periodic boundary condition for $s \sim s + 2\pi$ along with the Euclidean time evolution $t \sim t + 2\pi \tau_2$ represented by the operator $e^{-2\pi \tau_2 H}$, where $H$ is Hamiltonian. The latter also induces the space translation $s \sim s - 2\pi \tau_1$ represented by the operator $e^{-2\pi i \tau_1 P}$ (with momentum operator $P$) together with phase rotation due to the presence of fermion $e^{-2\pi i (b - 1/2)F_A}$, where $F_A = \frac{1}{2\pi} \int ds (\psi_1^\dagger \psi_- - \psi_-^\dagger \psi_-)$ is a fermion number. The partition function with the twisted boundary condition is

$$Z_{[a, b]} = \int \{ \Theta \} e^{-\frac{i}{\beta} \int d^2x \; (\tilde{A}_\mu + iA_\mu + i\tilde{A}_\mu) \psi},$$

where the Dirac field has the periodic boundary condition. In particular, the mode expansion of the fermion field depends not only on the twisted boundary condition, but also on the presence of current $J$. For example,

$$\psi_- = \sum_{r \in \mathbb{Z} + a} \psi_r (t) e^{i\tau s} \rightarrow \sum_{r \in \mathbb{Z} + a + J} \psi_r (t) e^{i\tau s}.$$

Taking this into account, the partition function is

$$Z_{[a, b]} = \int \{ \Theta \} e^{-\frac{i}{\beta} \int d^2x \; (\tilde{A}_\mu + iA_\mu + i\tilde{A}_\mu) \psi},$$

where $Z_{[a, b]}$ has two distinct contributions, one is through the modification of the Hilbert space. The periodicity of temporal direction and the corresponding thermal boundary condition are not modified.

### B. Entanglement Entropy with Current and Chemical Potential

The entanglement entropy for a single interval of length $\ell_t = u - v$ can be computed using the replica trick in terms of the $n$-copies of the correlation function $C_k = \langle \sigma_k (u) \sigma_{-k} (v) \rangle$ with $k = -(n - 1)/2, \cdots, (n - 1)/2$, where $\sigma_{\pm k}$ are the $k$-th twist operators with conformal dimension $\frac{\ell_t}{2(n+1)}$. The entanglement entropy for a single interval of length $\ell_t = u - v$

$$S = \frac{1}{n^2} \sum_{k=-(n-1)/2}^{(n-1)/2} C_k,$$

where

$$C_k = \int d^2x \; \langle \psi_\mu (x) \psi^\dagger_\mu (y) \rangle \delta (x - y).$$

In the presence of current $J$ and chemical potential $\mu$, the partition function has a more general form

$$Z_{[a, b]} = \int \{ \Theta \} e^{-\frac{i}{\beta} \int d^2x \; (\tilde{A}_\mu + iA_\mu + i\tilde{A}_\mu) \psi},$$

where

$$\psi_- = \sum_{r \in \mathbb{Z}^+} \psi_r (t) e^{i\tau s} \rightarrow \sum_{r \in \mathbb{Z}^+ + \tau J} \psi_r (t) e^{i\tau s}.$$

Taking this into account, the partition function is

$$Z_{[a, b]} = \int \{ \Theta \} e^{-\frac{i}{\beta} \int d^2x \; (\tilde{A}_\mu + iA_\mu + i\tilde{A}_\mu) \psi},$$

where

$$\psi_- = \sum_{r \in \mathbb{Z}^+} \psi_r (t) e^{i\tau s} \rightarrow \sum_{r \in \mathbb{Z}^+ + \tau J} \psi_r (t) e^{i\tau s}.$$

Note that the current has two distinct contributions, one is through the modification of the Hilbert space. The periodicity of temporal direction and the corresponding thermal boundary condition are not modified.
The correlation function factorizes as $C_k = C_k^0 \times C_k^{\mu J}$

$$
C_k^0 = \frac{2\pi \eta(\tau)^3}{\theta_1^{[1/2]}(\tau)} \frac{2k^2/n^2}{}, \quad (12)
$$

$$
C_k^{\mu J} = \frac{\theta_1^{[1/2-a-J]}(\frac{k\ell}{nL} + \tau_1 J + i\tau_2 \mu(\tau))^2}{\theta_1^{[1/2-a-J]}(\frac{k\ell}{nL} - \tau_1 J + i\tau_2 \mu(\tau))}. \quad (13)
$$

The entanglement entropy becomes a sum $S = S^0 + S^{\mu J}$. $C_k^0$ is independent of $\mu$ and $J$, and so is the $S^0$. Thus we focus on $C_k^{\mu J}$ and $S^{\mu J}$. The generalizations with multiple intervals are straightforward by using $\ell_i = \sum_{a=1}^{\mathcal{N}} (u_a - v_a)$. (See e.g. [18]).

**Entanglement entropy with chemical potential**

**Chemical potential at zero temperature limit:** Let us consider a more familiar case with a non-zero chemical potential $\mu$ (with $\tau_1 = J = 0$). It has been shown that the entanglement entropy at zero temperature is independent of a finite chemical potential [17][18]. Here we take a more refined limit, $\beta \to \infty$, $\mu - N/2 \to 0$ keeping $\beta(\mu - N/2) \to const.$ for integer $N$, and see the chemical potential dependence even in the zero temperature limit.

For NS-sector ($a = 1/2$), the Rényi entropy $S_n^\mu$ is

$$
S_n^\mu = \frac{1}{1-n} \left[ \sum_{k=-(n-1)/2}^{(n-1)/2} \log \left| \frac{\theta_3(k \ell/n + \frac{i\mu}{2n} | \beta)}{\theta_3(k \ell/n | \beta)} \right|^2 \right]. \quad (14)
$$

Using the product representation of Jacobi theta function $\vartheta_3(z|\tau) = \prod_{m=1}^\infty (1 - q^m)(1 + iy^{m-\frac{1}{2}})(1 + y^{-1}q^{m-\frac{1}{2}})$, we compute the Rényi entropy at low temperature limit [24]

$$
S_n^\mu = \frac{2}{n-1} \sum_{l=1}^\infty \frac{(-1)^{l-1} \cosh(l\beta \mu)}{\sinh(l\beta \mu)} \left[ n - \frac{\sin(l\ell_1/2L)}{\sinh(l\ell_1/2L)} \right]. \quad (15)
$$

The computation can be found in [28] in the appendix.

The limit $n \to 1$ gives the entanglement entropy

$$
S^\mu = \sum_{l=1}^\infty \frac{(-1)^{l-1} \cosh(l\beta \mu)}{\sinh(l\ell_1/2L)} \left[ 1 - \frac{\ell_1}{2L} \cot\left(\frac{\ell_1}{2L}\right) \right]. \quad (16)
$$

These results are valid for $e^{-\beta - \beta^2/2} < 1$ and $e^{\beta - \beta^2/2} < 1$ and thus for $-1/2 < \mu < 1/2$. Thus the entropies vanish at zero temperature. Nevertheless, this conclusion is not valid for $\mu = \pm 1/2$. We carefully look into the values.

For this purpose, we consider $\beta(\mu - 1/2) = M \to const.$ in the limit $\beta \to \infty$ and $\mu \to 1/2$. A modified expansion $\vartheta_3(z|\tau) = (1+y^{-1}q^{\frac{1}{2}}) \prod_{m=1}^\infty (1 - q^m)(1 + iy^{m-\frac{1}{2}})(1 + y^{-1}q^{m-\frac{1}{2}})$ is useful. While the contribution of the front factor $(1+y^{-1}q^{\frac{1}{2}})$ to the entanglement entropy exists for general value for $M$, we consider $M < 1$ here for simplicity. Then the entanglement entropy $S^\mu$ becomes

$$
2 \sum_{l=1}^\infty \frac{(-1)^{l-1}}{l} \left[ e^{-\beta M} + e^{-\frac{i\beta}{2}} \right] \left[ 1 - \frac{\ell_1}{2L} \cot\left(\frac{\ell_1}{2L}\right) \right]. \quad (17)
$$

Here $\mu \to 1/2$ and $|y_1 q^{1/2}| < 1$ and $|y_1^{-1} q^{-1/2}| < 1$ are used, and the result is valid for $-1/2 < \mu < 3/2$. The first term in the square parenthesis is non-zero at zero temperature limit. We identify the $\mu = 1/2$ as one of the energy level of the fermion at finite radius. More generally, there are non-zero contributions when $\beta(\mu - \frac{2N+1}{2}) = const.$ for $\beta \to \infty$ and $\mu \to \frac{2N+1}{2}$ where $N$, which is identified as the energy levels of the particles in a compactified circle with an anti-periodic boundary condition.

Similarly, the entanglement entropy for Dirac fermions with a periodic boundary condition picks up non-zero contributions when $\mu \to N$ at zero temperature. Combining them, the entanglement entropy reveals non-trivial contributions depending on the chemical potential when

$$
\beta\left(\mu - \frac{N}{2}\right) = const. , \quad (18)
$$

for $\beta \to \infty$ and $\mu \to N/2$ with integer $N$, which is identified as the energy levels of the particles in a compactified circle. We expect this will happen generically, providing a useful way to probe the energy levels of a given system.

**Chemical potential dependence in the large radius limit:** It has been argued that the entanglement entropy is independent of chemical potential for a single interval in the infinitely long space [19]. In this section we support and generalize the claim by evaluating the entanglement entropy by taking the limit $\frac{T}{L} \to 0$, which can be considered as an infinite space limit or a limit of small systems size. Our result is also valid for multiple intervals.

We focus on the fermion in NS-NS sector with an anti-periodic boundary condition in both time and space circles. We compute the Rényi entropy (14) that can be found in [28] in the appendix.

$$
S_n^\mu = \frac{(n+1) \ell_1^2}{12nL^2} \sum_{m=1}^\infty \left( \cosh(Im - \frac{m}{2}) + \cosh(Im + \frac{m}{2}) \right)^2 + O(\ell_1^4/L^4), \quad (19)
$$

where $\ell_1^2 \ll 1$ for a general $\beta$. Taking $n = 1$ provides the entanglement entropy. The entropies vanish at least as $\ell_1^2/L^2$ as the size approaches infinite space limit. The same is true for the periodic sector.

**Entanglement entropy with current**

Let us consider the twisted boundary condition (2) in the presence of the current $J$. The mode expansion of fermion has the form

$$
\psi_\pm = \sum_{\ell \in \mathbb{Z}+a+J} \psi_{\ell}(t)e^{i\ell s}. \quad (20)
$$

Current changes the periodicity of a compact fermion and thus produces distinctive physical effects on the entanglement entropy. For example, let us fix $\tau_1 = 0$, $a = 0$ and $\mu = 0$ for simplicity and increase the current, from $J = 0$...
to $J = 1/2$, in the zero temperature limit. The net effect is the modification of the boundary condition in the spatial circle from periodic to anti-periodic. Explicitly,

$$
S = \begin{cases}
\sum_{l=1}^{\infty} \frac{2(-1)^{l-1}}{l \sinh(l/2)} \left[ 1 - \frac{l \ell}{2L} \cot\left( \frac{l \ell}{2L} \right) \right], & J = 0, \\
\sum_{l=1}^{\infty} \frac{2(-1)^{l-1}}{l \tanh(l/2)} \left[ 1 - \frac{l \ell}{2L} \cot\left( \frac{l \ell}{2L} \right) \right], & J = 1/2.
\end{cases}
$$

Increasing the current $J$ from $J = 0$ to $J = 1/2$ brings visible effects in the entanglement entropy, from a zero value to a non-zero value.

Motivated by this observation, we consider the entanglement entropy as a function of the current $J$ and a modulus parameter $\alpha = 2\pi \gamma_1$ for $\mu = 0$ at large radius limit as well as zero temperature limit.

**Anti-periodic fermion:** First we focus on the anti-periodic fermion with $a + J = 1/2$. In the small temperature limit, the entanglement entropy is

$$
S^J = 4 \sum_{l,m=1}^{\infty} \frac{(-1)^{l-1}}{e^{(m-1/2)\beta}} \cos\left[ \left( m - \frac{1}{2} \right) \alpha l \right] \cos(\alpha J l) \\
\times \left[ 1 - \frac{l \ell}{2L} \cot\left( \frac{l \ell}{2L} \right) \right].
$$

(21)

We note that the entanglement entropy is a periodic function of $J$ and $\alpha$. This is in contrast to the entanglement entropy dependence on chemical potential. Interestingly, it depends on $\alpha$ in two different ways. The term with $l = m = 1$ provides a dominant contribution

$$
S^J_{l=m=1} \propto 4 e^{-\frac{\beta}{2}} \cos\left( \frac{\alpha}{2} \right) \cos(\alpha J).
$$

(22)

When we dial the parameter $\alpha$ for a fixed $J$, the product of two cosine functions produce an ‘interference pattern.’ For $J < 1/2$, one observes an interference pattern with a ‘beat frequency’ $J/\pi$. These interference and beat frequency are expected to be present for general quantum systems. Rényi entropy has the same physical properties.

**Periodic fermion:** We also consider the the periodic fermion with $a + J = 0$. With similar computations using (11) and (13), we have

$$
S^J = 2 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \left[ 1 + \sum_{m=1}^{\infty} \frac{2 \cos(m \alpha l)}{\sinh(m \beta l)} \right] \cos(\alpha J l) \\
\times \left[ 1 - \frac{l \ell}{2L} \cot\left( \frac{l \ell}{2L} \right) \right].
$$

(23)

In the zero temperature limit, the entanglement entropy has a non-zero contribution with the explicit dependence of current $J$. This is drastically different from that of the anti-periodic fermion. The entanglement entropy depends on current only through the function of $\alpha J$. It becomes negative for $\alpha J = \pi$ because of the combination $(-1)^{l-1} \cos(\alpha J l)$. Of course, total entanglement entropy is positive if one takes into account another contribution that is independent of current and divergent.

As one dials the current $J$ for a fixed $\alpha$, the entanglement entropy changes from (21) (that vanishes at $T = 0$) to (23), which has a non-zero contribution. This effect is expected to be present in general quantum theories.

**Generalization with current and Chemical potential:** We also compute the small temperature and large radius limits of the entropies in the presence of both constant current and chemical potential. The results are consistent with various different limits that have been described in this letter. In particular, for both periodic and anti-periodic fermions at large radius limit, the terms of the entropies depending on current and chemical potential vanish as fast as $O(\ell^2)$ as $\ell/l \to 0$. These confirm the result of (19) and extend in the presence of current and multiple intervals.

### C. Mutual Information

Mutual (Rényi) information measure the entanglement between two intervals, $A$ and $B$ of length $\ell_A$ and $\ell_B$ separated by $\ell_C$. It is given by

$$
I_n(A,B) = S_n(A) + S_n(B) - S_n(A \cup B).
$$

(24)

This is free of UV divergences and finite. Mutual (Rényi) information shares the same dependences on the current and chemical potential as those of Rényi entropy and the entanglement entropy.

**Chemical potential dependent part of the Mutual Rényi information of NS-NS sector, with $J = \alpha = 0$ at the small temperature limit, has the following form

$$
I_n^\mu(A,B) = \frac{2}{n-1} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \frac{\cosh(l \beta \mu)}{\sinh(l \beta / 2)} \\
\times \left[ n - \frac{\sinh(\frac{\ell A}{2L})}{\sinh(\frac{\ell A}{2L})} - \frac{\sinh(\frac{\ell B}{2L})}{\sinh(\frac{\ell B}{2L})} + \frac{\sinh(\frac{\ell A+\ell B}{2L})}{\sinh(\frac{\ell A+\ell B}{2L})} \right].
$$

(25)

Note that the first line is the same as that of (15). In general, the functional dependences of current and chemical potential are the same as those of the entropies. Surprisingly, the current dependent part of mutual information is independent of the separation of the two intervals $A$ and $B$.

By taking $n \to 1$ limit of (25), one computes the entanglement entropy to find the similar result as (10) with modification only on the dependence of the subsystem sizes. For the same length of the two intervals $\ell = \ell_A = \ell_B$, we get

$$
I^\mu = 2 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \cosh(l \beta \mu) \left[ 1 - \frac{\ell \ell}{L} \right].
$$

(26)

As a function of the sub-system size $\ell$, the dominant contribution ($i=1$) of the mutual information decreases, while that of the entanglement entropy increases.
In this letter, we provide a general formula for the entanglement and Rényi entropies of 2 dimensional Dirac fermions in the presence of current and chemical potential. We present several new results: the entropies are periodic function of chemical potential and current in the small temperature limit. The latter has also the role of beat frequency. Current can be used as a tuning parameter that derives the entropies, for example, from zero to a non-zero value. All these properties are expected to present in general quantum systems and have distinctive experimental signatures, that can be easily verifiable in experiments. Mutual (Rényi) information is especially relevant for this purpose because it is finite. It will be interesting to perform similar computations for a lattice model that can be readily applicable for available experiments [10].

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Appendix

Example Computations for a zero temperature limit in NS-sector: Here we derive the equation (16) of the latter. The entanglement entropy for the NS-sector is given by

$$S_{n=1}^\mu = \frac{1}{1-n} \left[ \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \vartheta_3 \left( \frac{k L}{n} \right) \frac{i \beta}{\pi} \right|^2 \right]_{n=1},$$

(S1)

where $\beta = 2\pi \tau_2$ and $\ell_l = u-v$. Using the product representation $\vartheta_3(z|\tau) = \prod_{m=1}^\infty (1-q^m)(1+y_1 q^{-m/2})(1+y_2^{-1} q^{m-1/2})$, with $y_1 = e^{-\beta \mu} + \frac{i}{2\pi} \frac{n L}{\vartheta_3}$, $y_2 = e^{-\beta}$, $q = e^{-\beta}$, we compute the entanglement entropy at low temperature limit, $\beta \to \infty$,

$$S_A = \frac{1}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

(S2)

$$= \frac{1}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \frac{2}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \frac{2}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= 2 \sum_{l=1}^\infty \left. \frac{1}{l} \cosh (l \beta) \left[ 1 - \pi \frac{L}{\cosh (l \beta/2)} \right] \right]$$

$$+ \cdots.$$ (S3)

Example Computations for a large radius limit in NS-sector: Here we derive the equation (19) of the latter. Using $\vartheta_3(z|\tau)$ along with the identification $z_1 = i \frac{2\mu}{\pi}$, $z_2 = i \frac{2\mu}{\pi} + \frac{k L}{\pi}$, $q = e^{-2\pi \beta}$. For $L/\ell_l \ll 1$, one has

$$\cos (i \beta \mu + 2\pi \frac{k L}{n} \ell_l) = \cos (\beta \mu) - 2\pi i \frac{k L}{n} \ell_l \sinh (\beta \mu) \left[ \frac{1}{2} (2\pi i \frac{k L}{n})^2 \cosh (\beta \mu) + \cdots \right],$$

(S3)

$$1 + q^{2m-1} + 2 \cos (2\pi z_2) q^{m-1/2} = 1 + q^{2m-1} + 2 \cos (2\pi z_1) q^{m-1/2} \left( 2\pi i \frac{k L}{n} \sinh (\beta \mu) + \frac{1}{2} (2\pi i \frac{k L}{n})^2 \cosh (\beta \mu) \right) + \cdots.$$ (S3)

Then the Rényi entropy has the form

$$S_n^\mu = \frac{1}{1-n} \left[ \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \vartheta_3 \left( \frac{k L}{n} \right) \frac{i \beta}{\pi} \right|^2 \right]_{n=1}$$

(S4)

$$= \frac{1}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \frac{1}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \frac{1}{1-n} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \frac{(n+1)^2 \ell_l^2}{24 n L^2} \left[ \sum_{m=1}^\infty \left( \prod_{m=1}^\infty (1-q^m)(1+y_2^{-1} q^{m-1/2}) \right) \right]_{n=1}$$

$$= \left( \frac{\ell_l}{L} \right)^4,$$

where we use the following approximation for $\frac{\ell_l}{L} \ll 1$ for a general temperature.