A GRADIENT ESTIMATE FOR ALL POSITIVE SOLUTIONS OF THE
CONJUGATE HEAT EQUATION UNDER RICCI FLOW

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Abstract. We establish a point-wise gradient estimate for all positive solutions of the conjugate heat equation. This contrasts to Perelman’s point-wise gradient estimate which works mainly for the fundamental solution rather than all solutions. Like Perelman’s estimate, the most general form of our gradient estimate does not require any curvature assumption. Moreover, assuming only lower bound on the Ricci curvature, we also prove a localized gradient estimate similar to the Li-Yau estimate for the linear Schrödinger heat equation. The main difference with the linear case is that no assumptions on the derivatives of the potential (scalar curvature) are needed.

A generalization of Perelman’s W-entropy is defined in both the Ricci flow and fixed metric case. We also find a new family of heat kernel estimates.

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1. Introduction

In the paper [P], Perelman discovers a monotonicity formula for the $W$ entropy of positive solutions of the conjugate heat equation.

\[
\begin{aligned}
\Delta u - Ru + \partial_t u &= 0 \\
\partial_t g &= -2Ric.
\end{aligned}
\]

(1.1)

Here $u = u(x,t)$ with $x \in M$, a compact manifold and $t \in (0,T)$, $T > 0$; $\Delta$ is the Laplace-Beltrami operator under the metric $g$ and $Ric$ is the Ricci curvature tensor. Here and though out it is assumed the metric $g$ is smooth in the region $M \times (0,T)$ with $T > 0$, unless stated otherwise. Moreover, he shows that this formula implies a point-wise gradient estimate for the fundamental solution of the conjugate heat equation (Corollary 9.3 [P]).
Namely, let $u$ be the fundamental solution of (1.1) in $M \times (0, T)$ and $f$ be the function such that $u = (4\pi\tau)^{-n/2}e^{-f}$ with $\tau = T - t$. Then

$$|\tau(2\Delta f - |\nabla f|^2 + R) + f - n|u \leq 0$$

in $M \times (0, T)$. This formula can be regarded as a generalization of the Li-Yau-Hamilton gradient estimate for the heat equation. By now it is clear that the importance of Perelman’s monotonicity formula and gradient estimate can hardly be overstated. See for example [CCGGIIKLLN], [CZ], [KL] and [MT]. However, there is one place where some improvement is still desirable, namely the gradient estimate does not apply to all positive solutions to the conjugate heat equation. For instance, for the Ricci flat manifold $S^1 \times S^1$. The constant 1 is a solution to the conjugate heat equation. Clearly it does not satisfy Perelman’s gradient estimate stated above. Whether a Perelman type gradient estimate exists for all positive solutions of the conjugate heat equation is a question circulating for a few years.

The main goal of this paper is to establish a gradient estimate that works for all positive solutions of the conjugate heat equation. Like Perelman’s estimate for the fundamental solution, the most general form of the new gradient estimate does not require any curvature assumption. Moreover, assuming only lower bound on the Ricci curvature, it also has a local version which appears similar to the Li-Yau estimate for the linear heat equation. An immediate consequence of the gradient estimate is a classical Harnack inequality for positive solutions of the conjugate heat equation.

We also introduce a generalization of Perelman’s $W$-entropy and prove its monotonicity. Specializing to the fixed metric case, we prove a family of gradient estimates for the fundamental solution of the heat equation. This family includes the Li-Yau estimate and Perelman’s estimate (specialized to the heat equation cf [NT]) as special cases.

The rest of the paper is organized as follows. The results concerning the conjugate heat equation under Ricci flow is given in sections 2-3. In section 4 we set up a generalization of Perelman’s $W$-entropy. In Section 5, we present the results for the linear heat equation in the fixed metric case. Some useful calculations which are around in various papers and preprints are collected in the appendix.

2. NEW GRADIENT ESTIMATE AND HARNACK INEQUALITY FOR POSITIVE SOLUTIONS TO THE CONJUGATE HEAT EQUATION

The main result of this section is

**Theorem 2.1.** Suppose $g(t)$ evolve by the Ricci flow, that is, $\frac{\partial g}{\partial t} = -2\text{Ric}$ on a closed manifold $M$ for $t \in [0, T)$, and $u : M \times [0, T) \rightarrow (0, \infty)$ be a positive $C^{2,1}$ solution to the conjugate heat equation $\Box^* u = -\Delta u - u_t + Ru = 0$. Let $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$ and $\tau = T - t$. Then:

(i) if the scalar curvature $R \geq 0$, then for all $t \in (0, T)$ and all points,

$$2\Delta f - |\nabla f|^2 + R \leq \frac{2n}{\tau};$$

(ii) without assuming the non-negativity of $R$, then for $t \in [\frac{T}{2}, T)$ and all points,

$$2\Delta f - |\nabla f|^2 + R \leq \frac{3n}{\tau}.$$
Remark 2.1. Since \( f = -\ln u - \frac{n}{2} \ln(4\pi \tau) \), if we replace \( f \) by \( u \) accordingly, then we get

\[
\frac{\|\nabla u\|^2}{u^2} - \frac{2u_r}{u} - R \leq \frac{2n}{\tau}, \quad \text{if } R \geq 0;
\]
\[
\frac{\|\nabla u\|^2}{u^2} - \frac{2u_r}{u} - R \leq \frac{3n}{\tau}, \quad \text{if } R \text{ changes sign and } t \geq T/2.
\]

It is similar to the Li-Yau gradient estimate for the heat equation on manifolds with non-negative Ricci curvature, i.e.

\[
\frac{\|\nabla u\|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}
\]

for positive solutions of \( \Delta u - \partial_t u = 0 \).

Remark 2.2. Some related gradient estimates with various dependence on the Ricci and other curvatures can be found in [G] and [N3].

Proof. of Theorem 2.1.

By a standard approximation argument, we can assume without loss of generality that \( g = g(t) \) is smooth in the closed time interval \([0, T]\) and that \( u \) is strictly positive everywhere.

(i) By standard computation (one can consult various sources for more details ([CK] and [T] e.g.),

\[
\left( \frac{\partial}{\partial t} + \triangle \right) (\triangle f) = \triangle \frac{\partial f}{\partial t} + 2 \langle \text{Ric}, \text{Hess}(f) \rangle + \triangle (\triangle f)
\]
\[
= \triangle \left( -\triangle f + |\nabla f|^2 - R + \frac{n}{2\tau} \right) + 2 \langle \text{Ric}, \text{Hess}(f) \rangle
\]
\[
+ \triangle (\triangle f)
\]
\[
= 2 \langle \text{Ric}, \text{Hess}(f) \rangle + \triangle \left( |\nabla f|^2 - R \right).
\]

(2.4)

Also using the evolution equation of \( g \),

\[
\left( \frac{\partial}{\partial t} + \triangle \right) |\nabla f|^2 = 2 \text{Ric}(\nabla f, \nabla f) + 2 \langle \nabla f, \nabla \frac{\partial f}{\partial t} \rangle + \triangle |\nabla f|^2
\]
\[
= 2 \text{Ric}(\nabla f, \nabla f) + 2 \langle \nabla f, \nabla (-\triangle f + |\nabla f|^2 - R) \rangle
\]
\[
+ \triangle |\nabla f|^2.
\]

(2.5)

Notice also

\[
(\frac{\partial}{\partial t} + \triangle) R = 2\triangle R + 2|Ric|^2.
\]

(2.6)

Combining these three expressions, we deduce

\[
\left( \frac{\partial}{\partial t} + \triangle \right) (2\triangle f - |\nabla f|^2 + R)
\]
\[
= 4 \langle \text{Ric}, \text{Hess}(f) \rangle + \triangle |\nabla f|^2 - 2 \text{Ric}(\nabla f, \nabla f)
\]
\[
- 2 \langle \nabla f, \nabla (-\triangle f + |\nabla f|^2 - R) \rangle + 2|Ric|^2.
\]

(2.7)

Denote \( q(x, t) = 2\triangle f - |\nabla f|^2 + R \).
By Bochner's identity,
\[ \Delta|\nabla f|^2 = 2|f_{ij}|^2 + 2\nabla f \nabla (\Delta f) + 2R_{ij}f_if_j, \]
the above equation becomes
\[
\left( \frac{\partial}{\partial t} + \Delta \right) q = 4R_{ij}f_{ij} + (2|f_{ij}|^2 + 2\nabla f \nabla (\Delta f) + 2R_{ij}f_if_j) - 2R_{ij}f_if_j \\
- 2\nabla f \nabla (-\Delta f + |\nabla f|^2 - R) + 2R_{ij}^2 \\
= 4R_{ij}f_{ij} + 2|f_{ij}|^2 + 2R_{ij}^2 + 2\nabla f \nabla (2\Delta f - |\nabla f|^2 + R) \\
= 2|R_{ij} + f_{ij}|^2 + 2\nabla f \nabla q
\]
that is,
\[
(2.8) \quad \left( \frac{\partial}{\partial t} + \Delta \right) q - 2\nabla f \nabla q = 2|R_{ij} + f_{ij}|^2 \geq \frac{2}{n}(R + \Delta f)^2.
\]
Since
\[ q = 2\Delta f - |\nabla f|^2 + R = 2(\Delta f + R) - |\nabla f|^2 - R, \]
and hence
\[ R + \Delta f = \frac{1}{2}(q + |\nabla f|^2 + R), \]
we have
\[
(2.9) \quad \left( \frac{\partial}{\partial t} + \Delta \right) q - 2\nabla f \nabla q \geq \frac{1}{2n}(q + |\nabla f|^2 + R)^2.
\]
By direct computation, we also have, for any \( \epsilon > 0 \)
\[
(2.10) \quad \left( \frac{\partial}{\partial t} + \Delta \right) \frac{2n}{T - t + \epsilon} - 2\nabla f \nabla \left( \frac{2n}{T - t + \epsilon} \right) = \frac{1}{2n}(\frac{2n}{T - t + \epsilon})^2.
\]
Combine the above two expressions, we get
\[
\left( \frac{\partial}{\partial t} + \Delta \right) \left( q - \frac{2n}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{2n}{T - t + \epsilon} \right) \\
\geq \frac{1}{2n} \left( q + \frac{2n}{T - t + \epsilon} + |\nabla f|^2 + R \right) \left( q - \frac{2n}{T - t + \epsilon} + |\nabla f|^2 + R \right) .
\]
We deal with the above inequality in two cases:

**Case 1.** at a point \((x, t), q + \frac{2n}{T - t + \epsilon} + |\nabla f|^2 + R \leq 0\), then also
\[ q - \frac{2n}{T - t + \epsilon} + |\nabla f|^2 + R \leq 0 \]
thus,
\[
(2.12) \quad \left( \frac{\partial}{\partial t} + \Delta \right) \left( q - \frac{2n}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{2n}{T - t + \epsilon} \right) \geq 0.
\]
Case 2. at a point \((x,t)\), \(q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R > 0\), then the inequality \((2.11)\) can be transformed to

\[
\begin{align*}
\left(\frac{\partial}{\partial t} + \triangle\right) & (q - \frac{2n}{T-t+\epsilon}) - 2\nabla f \nabla (q - \frac{2n}{T-t+\epsilon}) \\
&- \frac{1}{2n} \left( q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R \right) \left( q - \frac{2n}{T-t+\epsilon} \right) \\
&\geq \frac{1}{2n} (|\nabla f|^2 + R) \left( q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R \right) \geq 0.
\end{align*}
\]

(2.13)

Defining a potential term by

\[
V = V(x,t) = \begin{cases} 
0; & \text{if } q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R \leq 0 \text{ at } (x,t) \\
\frac{1}{2n} (q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R); & \text{if } q + \frac{2n}{T-t+\epsilon} + |\nabla f|^2 + R \geq 0 \text{ at } (x,t).
\end{cases}
\]

We know \(V\) is continuous; further, by the above two cases, we conclude

\[
\begin{align*}
\left(\frac{\partial}{\partial t} + \triangle\right) & (q - \frac{2n}{T-t+\epsilon}) - 2\nabla f \nabla (q - \frac{2n}{T-t+\epsilon}) - V(q - \frac{2n}{T-t+\epsilon}) \geq 0.
\end{align*}
\]

(2.15)

Since we assumed that the Ricci flow is smooth in \([0,T]\) and that \(u(x,t)\) is a positive \(C^{2,1}\) solution to the conjugate heat equation, thus

\[
q = 2\Delta f - |\nabla f|^2 + R = \frac{|\nabla u|^2}{u^2} - \frac{2\Delta u}{u} + R
\]

is bounded for \(t \in [0,T]\). If we choose \(\epsilon\) sufficiently small, then \(q(x,T) \leq \frac{2n}{\epsilon}\), thus by the maximum principle([CK], e.g.), for all \(t \in [0,T]\), \(q(x,t) \leq \frac{2n}{T-t+\epsilon}\). Let \(\epsilon \to 0\), we have for all \(t \in [0,T]\),

\[
q(x,t) \leq \frac{2n}{T-t}.
\]

(2.16)

Recall \(q = 2\Delta f - |\nabla f|^2 + R\), \(\tau = T-t\), then we have

\[
2\Delta f - |\nabla f|^2 + R \leq \frac{2n}{\tau}.
\]

(2.17)

Further, \(f = -\ln u - \frac{n}{2}(4\pi \tau)\), then the above yields

\[
\frac{|\nabla u|^2}{u^2} - \frac{2u\tau}{u} - R \leq \frac{2n}{\tau}.
\]

(2.18)

Proof of (ii). Next we prove the gradient estimate without the non-negativity assumption for the scalar curvature \(R\). Let \(c \geq 2n\) be a constant to be determined later; denote

\[
B = |\nabla f|^2 + R.
\]
Similar to the inequality (2.11), we also have,

\[
\left( \frac{\partial}{\partial t} + \triangle \right) \left( q - \frac{c}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{c}{T - t + \epsilon} \right) \geq \frac{1}{2n} (q + B)^2 - \frac{c}{(T - t + \epsilon)^2} \\
= \frac{1}{2n}[(q + B)^2 - \frac{c^2}{(T - t + \epsilon)^2} + \frac{c^2}{(T - t + \epsilon)^2} - \frac{2cn}{(T - t + \epsilon)^2}] \\
= \frac{1}{2n}[(q - \frac{c}{T - t + \epsilon} + B)(q + \frac{c}{T - t + \epsilon} + B) + \frac{c(c - 2n)}{(T - t + \epsilon)^2}] .
\]

We deal with the previous inequality at a given point \((x, t)\) in three cases:

**Case 1.** \(B \geq 0\), and \(q - \frac{c}{T - t + \epsilon} + B \leq 0\), then also

\[
q - \frac{c}{T - t + \epsilon} + B \leq 0
\]

thus,

\[
\left( \frac{\partial}{\partial t} + \triangle \right) \left( q - \frac{c}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{c}{T - t + \epsilon} \right) \geq 0 .
\]

**Case 2.** \(B \geq 0\), and \(q + \frac{c}{T - t + \epsilon} + B > 0\), then the inequality (2.19) can be changed to

\[
\left( \frac{\partial}{\partial t} + \triangle \right) \left( q - \frac{c}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{c}{T - t + \epsilon} \right) - \frac{1}{2n} \left( q + \frac{c}{T - t + \epsilon} + B \right) \left( q - \frac{c}{T - t + \epsilon} \right) \\
\geq \frac{1}{2n} B \left( q + \frac{c}{T - t + \epsilon} + B \right) \geq 0 .
\]

**Case 3.** \(B \leq 0\), then the inequality (2.19) can be changed to

\[
\left( \frac{\partial}{\partial t} + \triangle \right) \left( q - \frac{c}{T - t + \epsilon} \right) - 2\nabla f \nabla \left( q - \frac{c}{T - t + \epsilon} \right) \\
\geq \frac{1}{2n} \left( q + \frac{c}{T - t + \epsilon} + B \right) \left( q - \frac{c}{T - t + \epsilon} \right) + \frac{1}{2n} B \left( \frac{2Bc}{T - t + \epsilon} + \frac{c(c - 2n)}{(T - t + \epsilon)^2} \right) .
\]

To continue, we need the following estimate of the scalar curvature \(R\) under the Ricci flow i.e.

\[
R \geq -\frac{n}{2(t + \epsilon)}
\]

for some \(\epsilon > 0\) depending on the initial value of \(R\)(it comes from the weak minimum principle for a differential inequality \(\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n} R^2\) ([CK], e.g.); thus

\[
B = |\nabla f|^2 + R \geq R \geq -\frac{n}{2(t + \epsilon)} \geq \frac{n}{2(T - t + \epsilon)}
\]
for $t \geq \frac{T}{2}$ because $t \geq \frac{T}{2} \Rightarrow t \geq T-t \Rightarrow t+\epsilon \geq T-t+\epsilon \Rightarrow \frac{1}{t+\epsilon} \leq \frac{1}{T-t+\epsilon}$; thus

\[
\frac{1}{2n} \left( \frac{2Bc}{T-t+\epsilon} + \frac{c(c-2n)}{(T-t+\epsilon)^2} \right) \geq \frac{1}{2n} \left( \frac{n}{2(T-t+\epsilon)} \frac{2c}{T-t+\epsilon} + \frac{c(c-2n)}{(T-t+\epsilon)^2} \right) = \frac{1}{2n} \left( \frac{c(c-3n)}{(T-t+\epsilon)^2} \right).
\]

Therefore

\[
\left( \frac{\partial}{\partial t} + \triangle \right) \left( q - \frac{2n}{T-t+\epsilon} \right) - 2\nabla f \nabla \left( q - \frac{2n}{T-t+\epsilon} \right) - V \left( q - \frac{2n}{T-t+\epsilon} \right) \geq 0
\]

where $V = V(x,t)$ is a continuous function defined by

\[
V = \begin{cases} 
0; & \text{if } B \geq 0, q + \frac{2n}{T-t+\epsilon} + B \leq 0 \text{ at } (x,t) \\
\frac{1}{2n} \left( q + \frac{2n}{T-t+\epsilon} + B \right); & \text{if } B \geq 0, q + \frac{2n}{T-t+\epsilon} + B > 0 \text{ at } (x,t) \\
\frac{1}{2n} \left( q + \frac{2n}{T-t+\epsilon} + 2B \right); & \text{if } B < 0 \text{ at } (x,t).
\end{cases}
\]

Follow the similar argument for the inequality (2.1), by the maximum principle again, we have

\[
2\triangle f - |\nabla f|^2 + R \leq \frac{3n}{\tau} \quad \text{and} \quad \frac{|\nabla u|^2}{u^2} - \frac{2u_{rr}}{u} - R \leq \frac{3n}{\tau}, \quad t \geq T/2.
\]

An immediate consequence of the above theorem is:

**Corollary 1** (Harnack Inequality). Given a smooth Ricci flow on a closed manifold $M$, let $u : M \times [0,T) \to (0,\infty)$ be a positive $C^{2,1}$ solution to the conjugate heat equation.

(a). Suppose the scalar curvature $R \geq 0$ for $t \in [0,T)$. Then for any two points $(x,t_1)$, $(y,t_2)$ in $M \times (0,T)$ such that $t_1 < t_2$, it holds

\[
u(y,t_2) \leq u(x,t_1) \left( \frac{T_1}{T_2} \right)^n \exp \left[ \int_0^1 \left[ 4R(s)^2 + (\tau_1 - \tau_2)^2 R \right] ds \right].
\]

Here $\tau_i = T - t_i$, $i = 1, 2$, and $\gamma(s) : [0,1] \to M$ is a smooth curve from $x$ to $y$. 

\[
\square
\]
(b). Without assuming the nonnegativity of the scalar curvature $R$, then for $t_2 > t_1 \geq T/2$, it holds
\begin{equation}
(2.31) \quad u(y, t_2) \leq u(x, t_1) \left(\frac{\tau_1}{\tau_2}\right)^{3n/2} \exp \left[\int_0^1 \frac{4|\gamma'(s)|^2 + (\tau_1 - \tau_2)^2 R}{2(\tau_1 - \tau_2)} ds\right].
\end{equation}

Proof. We will only prove (a) since the proof of (b) is similar. Denote $\tau(s) := \tau_2 + (1 - s)(\tau_1 - \tau_2)$, $0 \leq \tau_2 < \tau_1 \leq T$, define
$$
\ell(s) := \ln u(\gamma(s), T - \tau(s))
$$
where $\ell(0) = \ln u(x, t_1)$, $\ell(1) = \ln u(y, t_2)$. By direct computation,
$$
\frac{\partial \ell(s)}{\partial s} = \frac{u_s}{u} = \frac{\nabla u \cdot \partial \gamma}{u} - \frac{u_x(\tau_1 - \tau_2)}{u} = (\tau_1 - \tau_2) \left(\frac{\nabla u}{\sqrt{2u}} \frac{\sqrt{2} \gamma'(s)}{1 - \tau_2} - \frac{u_x}{u}\right)
$$
$$
\leq (\tau_1 - \tau_2) \left(\frac{2|\gamma'(s)|^2}{(\tau_1 - \tau_2)^2} + \frac{|\nabla u|^2}{2u^2} - \frac{u_x}{u}\right)
$$
$$
(2.32) \quad = \frac{2|\gamma'(s)|^2}{(\tau_1 - \tau_2)} + \frac{\tau_1 - \tau_2}{2} \left(\frac{|\nabla u|^2}{u^2} - \frac{2u_x}{u}\right).
$$
By our gradient estimate, if $R \geq 0$, then
$$
\frac{|\nabla u|^2}{u^2} - \frac{2u_x}{u} \leq R + \frac{2n}{\tau}
$$
where $\tau = \tau_2 + (1 - s)(\tau_1 - \tau_2)$. Therefore
\begin{equation}
(2.33) \quad \frac{\partial \ell(s)}{\partial s} \leq \frac{2|\gamma'(s)|^2}{(\tau_1 - \tau_2)} + \frac{\tau_1 - \tau_2}{2} \left(R + \frac{2n}{\tau}\right).
\end{equation}
Integrating with respect to $s$ on $[0, 1]$, we have
\begin{equation}
(2.34) \quad \ell(1) - \ell(0) \leq \frac{2 \int_0^1 |\gamma'(s)|^2 ds}{(\tau_1 - \tau_2)} + \frac{(\tau_1 - \tau_2)}{2} \int_0^1 R ds + n \ln \frac{\tau_1}{\tau_2}.
\end{equation}
Recall $\ell(0) = \ln u(x, t_1)$, $\ell(1) = \ln u(y, t_2)$, then
\begin{equation}
(2.35) \quad \ln \frac{u(y, t_2)}{u(x, t_1)} \leq \frac{\int_0^1 [4|\gamma'(s)|^2 + (\tau_1 - \tau_2)^2 R] ds}{2(\tau_1 - \tau_2)} + n \ln \left(\frac{\tau_1}{\tau_2}\right).
\end{equation}
Therefore, given any two points $(x, t_1)$, $(y, t_2)$ in the space-time, we have
\begin{equation}
(2.36) \quad u(y, t_2) \leq u(x, t_1) \left(\frac{\tau_1}{\tau_2}\right)^n \exp \left[\frac{\int_0^1 [4|\gamma'(s)|^2 + (\tau_1 - \tau_2)^2 R]}{2(\tau_1 - \tau_2)} ds\right].
\end{equation}
3. Localized version of the Gradient Estimate in section 2

In this section we prove a localized version of the previous gradient estimate. Here we apply Li-Yau’s idea of using certain cut-off functions to the new equations derived in the last section. However the computation is more complicated for two reasons. One is that the metric is also evolving. The other is that the equations coming from the last section have a more complex structure.

**Theorem 3.1.** Let $M$ be a compact Riemannian manifold equipped with a family of Riemannian metrics evolving under Ricci flow, that is, $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$. Given $x_0 \in M$ and $r_0 > 0$, let $u$ be a smooth positive solution to the conjugate heat equation $\square u = -\triangle u - u_t + Ru = 0$ in the cube $Q_{r_0,T} := \{(x,t) | d(x,x_0,t) \leq r_0, 0 < t \leq T\}$ and $\tau = T - t$. Suppose $\text{Ric} \geq -K$ throughout $Q_{r_0,T}$ for some positive constant $K$. Then for $(x,t)$ in the half cube $Q_{\frac{r_0}{4},\frac{T}{2}} := \{(x,t) | d(x,x_0,t) \leq \frac{r_0}{2}, 0 < t \leq \frac{T}{2}\}$, we have

$$\frac{\lvert \nabla u \rvert^2}{u^2} - \frac{2u_t}{u} - R \leq cK + \frac{c}{T} + \frac{c}{r_0^2}$$

where $c > 0$ is a constant depending only on the dimension $n$.

**Proof.** As before let $f$ be a function defined by $u = e^{-f} \frac{1}{(4\pi T)^{n/2}}$ and

$q = 2\triangle f - \lvert \nabla f \rvert^2 + R = \frac{\lvert \nabla u \rvert^2}{u^2} - \frac{2u_t}{u} - R.$

From inequality (2.9) in the last section, we have

$$\triangle q - q_t - 2\nabla f \nabla q \geq \frac{1}{2n} (q + \lvert \nabla f \rvert^2 + R)^2.$$

For the fixed point $x_0$ in $M$, let $\varphi(x,t)$ be a smooth cut-off function (mollifier) with support in the cube

$Q_{r_0,T} := \{(x,t) | x \in M, d(x,x_0,t) \leq r_0, 0 < t \leq T\}$

possessing the following properties:

1. $\varphi = \varphi(d(x,x_0,t),t) \equiv \psi(r(x,t))\eta(t), r(x,t) = d(x,x_0,t); \frac{\partial \psi}{\partial r} \leq 0, \frac{\partial \eta}{\partial t} \leq 0, \tau = T - t$;

2. $\varphi(x,t) \equiv 1$ in $Q_{\frac{r_0}{2},\frac{T}{2}} := \{(x,t) | d(x,x_0,t) \leq \frac{r_0}{2}, 0 < t \leq \frac{T}{2}\}$;

3. $\lvert \frac{\partial \psi}{\psi^a} \rvert \leq \frac{c(n,a)}{r_0}, \lvert \frac{\partial \varphi}{\varphi^a} \rvert \leq \frac{c(n,a)}{r_0^2}$, for some $c(n,a)$, $0 < a < 1$;

4. $\lvert \frac{\partial \eta}{\sqrt{\eta}} \rvert \leq \frac{c}{T'}$, for some $c$ depending on $n$.

Now we focus on the product $(\varphi q)(x,t)$. Since $\varphi$ has support in $Q_{r_0,T}$, we can assume $\varphi q$ reaches its maximum at some point $(y,s) \in Q_{r_0,T}$. If $q(y,s) = 2\triangle f(y,s) - \lvert \nabla f(y,s) \rvert^2 + R(y,s)$ is negative, then the theorem is trivially true. Thus we can assume $q(y,s) \geq 0$. By direct computation,

$$\triangle (\varphi q) - (\varphi q)_t - 2\nabla f \nabla (\varphi q) - 2\frac{\nabla \varphi}{\varphi} \nabla (\varphi q)$$
\[ = \varphi (\Delta q - q_r - 2 \nabla f \nabla q) + (\Delta \varphi) q - 2 \frac{\| \nabla \varphi \|}{\varphi} q - q \varphi_r - 2q \nabla f \nabla \varphi \]

\[ \geq \frac{\varphi}{2n} (q + |\nabla f|^2 + R)^2 + (\Delta \varphi) q - 2 \frac{\| \nabla \varphi \|}{\varphi} q - q \varphi_r - 2q \nabla f \nabla \varphi. \]

At the point \((y, s)\) where the maximum value for \(\varphi q\) is attained, there hold

1. \(\Delta (\varphi q)(y, s) \leq 0\)
2. \(\nabla (\varphi q)(y, s) = 0\)
3. \((\varphi q)_r(y, s) \geq 0\).

The last inequality comes from the fact that \((\varphi q)|_{t=T} = 0\) since \(\varphi|_{t=T} = 0\), \(\varphi q\) can only take its maximum for \(t \in [0, T)\). We have also borrowed the idea of Calabi as used in [LY] to circumvent the possibility that \((y, s)\) is in the cut locus of \(g(s)\).

Thus at the point \((y, s)\), inequality (3.3) becomes

\[ \frac{\varphi}{2n} (q + |\nabla f|^2 + R)^2 (y, s) \leq \varphi q + 2q \nabla f \nabla \varphi + 2 \frac{\| \nabla \varphi \|}{\varphi} q - (\Delta \varphi) q = (I) + (II) + (III) + (IV). \]

We estimate each term on the right-hand side by the following:

\((I) = \psi_r \eta(\tau) q + \psi \eta_r q = \psi_r r \eta(\tau) q + \psi \eta_r q\).

From the lower bound assumption on the Ricci curvature \(Ric \geq -K\), we have (see [CK] e.g.)

\[ \frac{\partial r}{\partial t} = -\frac{\partial r}{\partial \tau} \geq -Kr. \]

By construction of \(\psi\), we have \(\psi_r \leq 0\). Therefore,

\[ \psi_r \frac{\partial r}{\partial \tau} \leq Kr |\psi_r| = Kr \frac{|\psi_r|}{\sqrt{\psi}} \sqrt{\psi} \leq Kr \frac{c}{\sqrt{r_0}} \sqrt{\psi} \leq cK \sqrt{\psi}. \]

This shows

\[ (I) \leq cK \sqrt{\psi} \eta q + \psi \eta_r q \leq cK \sqrt{\varphi} q + \psi \frac{|\eta_r|}{\sqrt{\eta}} \sqrt{\eta} q \leq cK \sqrt{\varphi} q + \sqrt{\varphi} \frac{c}{\sqrt{r_0}} \sqrt{\eta q}. \]

Recall that \(\varphi = \psi \eta\), for a parameter \(\epsilon\) to be chosen later, we have

\[ (I) \leq cK \sqrt{\varphi} q + \frac{c}{T} \sqrt{\varphi} q \leq \frac{c}{\epsilon} K^2 + \frac{c}{T^2} + 2 \epsilon \varphi q^2. \]

\((II) \leq 2|\nabla f| |\nabla \varphi| q \leq 2|\nabla f| \frac{|\nabla \varphi|}{\sqrt{\varphi}} \sqrt{\varphi} q \]

\[ \leq \frac{1}{\epsilon} 4 |\nabla f|^2 \frac{|\nabla \varphi|^2}{\varphi} + \epsilon \varphi q^2 \]

\[ = \frac{1}{\epsilon} 4 \sqrt{\varphi} |\nabla f|^2 \frac{|\nabla \varphi|^2}{\varphi^2} + \epsilon \varphi q^2 \]

\[ \leq \epsilon \varphi |\nabla f|^4 + \frac{1}{\epsilon^3 \sqrt[4]{r_0}} + \epsilon \varphi q^2. \]
Notice we assumed \( R \) satisfies (3.11)
\[
\text{(III)} = 2 |\nabla \varphi|^2 \sqrt{\varphi} q \leq 2 |\nabla \varphi|^{3/4} \sqrt{\varphi} q \leq \epsilon \varphi q^2 + \frac{1}{\epsilon} \frac{c}{r_0^2}.
\]

\( (IV) = - (\Delta \varphi) q = - \left( \partial_r \varphi + (n-1) \frac{\partial_r \varphi}{r} + \partial_r \varphi \log \sqrt{g} \right) q \)
\[
\leq \frac{|\partial_r \varphi|}{\sqrt{\varphi}} \sqrt{\varphi} q + (n-1) \frac{\partial_r \varphi}{r} q + \frac{|\partial_r \varphi|}{\sqrt{\varphi}} \sqrt{K} \sqrt{\varphi} q
\]
\[
\leq \frac{1}{\epsilon} \left( \frac{\partial_r \varphi}{\sqrt{\varphi}} \right)^2 + \epsilon \varphi q^2 + (n-1) \frac{\partial_r \varphi}{r} q + \frac{K}{\epsilon} \left( \frac{|\partial_r \varphi|}{\sqrt{\varphi}} \right)^2 + \epsilon \varphi q^2
\]
\[
\leq 2 \epsilon \varphi q^2 + \frac{c}{r_0^2} + \frac{c K}{r_0^2} + (n-1) \frac{\partial_r \varphi}{r} q.
\]

Notice \( \varphi \equiv 1 \) in \( Q_{r_0} \), thus \( \partial_r \varphi = 0 \) for \( 0 \leq r \leq \frac{r_0}{2} \), we can just focus on \( r \geq \frac{r_0}{2}, \frac{1}{r} \leq \frac{2}{r_0} \), then (IV) can be estimated as
\[
\text{(IV)} \leq 2 \epsilon \varphi q^2 + \frac{c}{r_0^2} + \frac{c K}{r_0^2} + (n-1) \frac{2 \partial_r \varphi}{r_0} q
\]
\[
\leq 2 \epsilon \varphi q^2 + \frac{c}{r_0^2} + \frac{c K}{r_0^2} + \epsilon \varphi q^2 + \frac{4(n-1)^2}{r_0^2} |\partial_r \varphi|^2
\]
\[
\leq 3 \epsilon \varphi q^2 + \frac{c}{r_0^2} + \frac{c K}{r_0^2}.
\]

Combine (I)-(IV), we have
\[
\frac{\varphi}{2n} (q + |\nabla f|^2 + R)^2 (y, s) \leq 7 \epsilon \varphi q^2 + c R^2 + \frac{c}{T^2} + \frac{c}{r_0^2} + \frac{c K}{r_0^2} + \epsilon \varphi |\nabla f|^4
\]
\[
\leq 7 \epsilon \varphi q^2 + c R^2 + \frac{c}{T^2} + \frac{c}{r_0^2} + \epsilon \varphi |\nabla f|^4.
\]

Notice we assumed \( q(y, s) \geq 0 \), otherwise the theorem is trivially true,
\[
(q + |\nabla f|^2 + R)^2 (y, s) = (q + |\nabla f|^2 + R^+ - R^-)^2 (y, s)
\]
\[
\geq \frac{1}{2} (q + |\nabla f|^2 + R^+)^2 (y, s) - (R^-)^2 (y, s)
\]
\[
\geq \frac{1}{2} (q^2 + |\nabla f|^4) (y, s) - (\sup_{Q_{r_0} \times T} R^-)^2
\]
\[
\geq \frac{1}{2} (q^2 + |\nabla f|^4) (y, s) - n^2 K^2.
\]

Here we have used the inequalities \( 2(a-b)^2 \geq a^2 - 2b^2, (a+b)^2 \geq a^2 + b^2 \) for \( a, b \geq 0 \) and the lower bound assumption for the Ricci curvature \( Ric \geq -K \Rightarrow R \geq -nK \Rightarrow R^- \leq nK \) since \( R = -R^- \) if \( R < 0 \). Substituting into (3.11) and reorganizing, we have
\[
\frac{1}{4n} (\frac{1}{4n} - 7 \epsilon) \varphi q^2 (y, s) \leq (\epsilon - \frac{1}{4n}) \varphi |\nabla f|^4 + c K^2 + \frac{c}{T^2} + \frac{c}{r_0^2}.
\]
Take \( \epsilon \) such that \( 7\epsilon \leq \frac{1}{4n} \), then the above inequality becomes

\[
\varphi q^2(y, s) \leq cK^2 + \frac{c}{T^2} + \frac{c}{r_0^4}.
\]

By using inequality \( a_1^2 + a_2^2 + \ldots + a_n^2 \leq (a_1 + a_2 + \ldots + a_n)^2 \),

\[
(\varphi q)^2(y, s) \leq \varphi q^2(y, s) \leq (cK + \frac{c}{T} + \frac{c}{r_0^2})^2.
\]

Since if \((x, t) \in Q_{\frac{r_0}{2}, \frac{T}{2}}\), then \( \varphi(x, t) \equiv 1 \), thus for any \((x, t) \in Q_{\frac{r_0}{2}, \frac{T}{2}}\),

\[
q(x, t) = \varphi(x, t) q(x, t) \leq \max_{Q_{r_0, T}} (\varphi q)(x, t) = (\varphi q)(y, s)
\]

\[
\leq cK + \frac{c}{T} + \frac{c}{r_0^2}.
\]

Therefore we just proved that in \( Q_{\frac{r_0}{2}, \frac{T}{2}} \),

\[
q(x, t) \leq cK + \frac{c}{T} + \frac{c}{r_0^2}.
\]

If we bring back \( u \), recall \( q = 2\Delta f - |\nabla f|^2 + R \), \( f = -\ln u - \frac{n}{2} \ln(4\pi\tau) \), then we have

\[
\frac{|\nabla u|^2}{u^2} - \frac{2u_r}{u} - R \leq cK + \frac{c}{T} + \frac{c}{r_0^2}.
\]

\[\Box u = -u_t - \Box u + Ru = 0.\]

4. A Generalization of Perelman’s W-Entropy

In this section, we show that Perelman’s W-entropy and its monotonicity can be generalized to a wider class. It is an established fact that monotonicity formulas tend to provide useful information on the underlining equation. Therefore the more of them are found the better. For a related but different generalization of Perelman’s formula and its applications, please see a recent paper [Li].

Define a family of entropy formulas for the Ricci flow case by:

\[
W(g, f, \tau) := \int_M \left( \frac{a_1^2}{2\pi\tau}(R + |\nabla f|^2) + f - n \right) u \, dx
\]

where \( R \) is the scalar curvature, \( \tau = T - t > 0 \); \( u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \) is a positive solution to the following conjugate heat equation (4.2), satisfying \( \int u \, dx = 1 \),

\[
\Box u = -u_t - \Box u + Ru = 0.
\]

Notice that

\[
\begin{cases}
    f = -\ln u - \frac{n}{2} \ln(4\pi\tau) \\
    \nabla f = -\frac{\nabla u}{u} \ln(4\pi\tau) \\
    \Delta f = -\frac{\nabla u}{u} + |\nabla f|^2 \\
    \frac{\partial f}{\partial t} = -\frac{u_r}{u} + \frac{n}{2\tau}
\end{cases}
\]

we get the evolution equation for \( f \),

\[
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}.
\]
Now we come to the theorem of the section:

**Theorem 4.1.** Let \( g(t) \) evolve by the Ricci flow, that is, \( \frac{\partial g}{\partial t} = -2\text{Ric} \) on a closed manifold \( M \) for \( t \in [0,T) \), and \( u : M \times [0,T) \mapsto (0,\infty) \) with \( u = \frac{e^{-f}}{(4\pi t)^\frac{n}{2}} \) be a positive solution to the conjugate heat equation (4.2). For \( 0 \leq a^2 \leq 2\pi \), the functional defined in (4.1) is increasing according to

\[
\frac{\partial}{\partial t} W(g,f,\tau) \geq \frac{a^2}{2\pi} \int_M |\text{Ric} + \text{Hess}(f)|^2 u \, dx \geq 0.
\]

We start the proof with the following results by Perelman [P] in the form of lemmas from [T]. For completeness, the proofs are given in the Appendix.

**Lemma 4.1.** Let \( u,f,g,\tau \) defined as in Theorem 4.1, then the F-entropy defined by

\[
F(g,f) := \int (R + |\nabla f|^2) u \, dx
\]

is non-decreasing in \( t \) under

\[
\frac{\partial F(g,f)}{\partial t} = 2 \int |\text{Ric} + \text{Hess}(f)|^2 u \, dx
\]

**Lemma 4.2.** Let \( g,f,\tau \) defined as in Theorem 4.1, define

\[
P := \tau (2\triangle f - |\nabla f|^2 + R) + f - n
\]

then

\[
\Box^* P = -2\tau |\text{Ric} + \text{Hess}(f) - \frac{g}{2\tau}|^2 u.
\]

**Proof.** (of Theorem 4.1) Notice

\[
W(g,f,\tau) = \int_M \left( \frac{a^2}{2\pi} \tau (R + |\nabla f|^2) + f - n \right) u \, dx
\]

\[
= \frac{a^2}{2\pi} \int_M \left( \tau (R + |\nabla f|^2) + f - n \right) u \, dx
\]

\[
+ (1 - \frac{a^2}{2\pi}) \left( \int_M f u \, dx \right) - (1 - \frac{a^2}{2\pi}) n
\]

we split the derivative of \( W \) over the time \( t \) into two parts,

\[
\frac{\partial}{\partial t} W(g,f,\tau) = \frac{a^2}{2\pi} \frac{\partial}{\partial t} \left( \int_M \left( \tau (R + |\nabla f|^2) + f - n \right) u \, dx \right)
\]

\[
+ (1 - \frac{a^2}{2\pi}) \frac{\partial}{\partial t} \left( \int_M f u \, dx \right)
\]

\[
= \frac{a^2}{2\pi} \frac{\partial}{\partial t} \int_M P \, dx + (1 - \frac{a^2}{2\pi}) \frac{\partial}{\partial t} \left( \int_M f u \, dx \right).
\]

We compute for each term,

\[
\frac{a^2}{2\pi} \frac{\partial}{\partial t} \int_M P \, dx = \frac{a^2}{2\pi} \int_M (P_t \, dx + P \frac{\partial dx}{\partial t}) = \frac{a^2}{2\pi} \int_M (P_t \, dx + P(-R) \, dx)
\]

\[
= -\frac{a^2}{2\pi} \int_M \Box^* P \, dx - \frac{a^2}{2\pi} \int_M \Delta P \, dx = -\frac{a^2}{2\pi} \int_M \Box^* P \, dx
\]
the last equality comes from \( \int_M \Delta P\,dx = 0 \) for closed manifold \( M \). By Lemma 4.2, we have

\[
\frac{a^2}{2\pi} \frac{\partial}{\partial t} \int_M P\,dx = \frac{a^2\tau}{\pi} \int_M |Ric + Hess(f) - \frac{g}{2\tau}|^2 u\,dx \geq 0.
\]

It suffices to prove the non-negativity of \( \frac{\partial}{\partial t} \left( \int_M f u\,dx \right) \). Follow the direct computation,

\[
\frac{\partial}{\partial t} \left( \int_M f u\,dx \right) = \int_M f_t u\,dx + f u_t\,dx + f u \frac{\partial (dx)}{\partial t}
\]

\[
= \int_M (-\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}) u\,dx
\]

\[
+ \int_M (f(-\Delta u + Ru) - R fu)\,dx.
\]

Using integration by parts, we have

\[
\frac{\partial}{\partial t} \left( \int_M f u\,dx \right) = \int_M (-2\Delta f + |\nabla f|^2) u\,dx + \int_M (\frac{n}{2\tau} - R) u\,dx
\]

\[
= \int_M (\frac{\Delta u}{u} - |\nabla f|^2) u\,dx + \int_M (\frac{n}{2\tau} - R) u\,dx
\]

\[
= \int_M -|\nabla f|^2 u\,dx + \int_M (\frac{n}{2\tau} - R) u\,dx
\]

(4.8)

Now we turn to estimate of \( F(g,\tau) = \int_M (|\nabla f|^2 + R) u\,dx \).

From Lemma 4.1 we have

\[
\frac{\partial F}{\partial t} = 2 \int_M |Ric + Hess(f)|^2 u\,dx = 2 \int_M \left( \sum_{i,j} |R_{ij} + f_{ij}|^2 \right) u\,dx
\]

\[
\geq 2 \int_M \left( \sum_{i=j} |R_{ij} + f_{ij}|^2 \right) u\,dx \geq 2 \int_M \left( \sum_{i=j} R_{ii} + \sum f_{ii} \right)^2 u\,dx
\]

\[
= \frac{2}{n} \int_M (R + \Delta f)^2 u\,dx.
\]

(4.9)

The last inequality comes from \( \sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + \ldots + a_n}{n} \) for \( a_i \geq 0 \). Also by Cauchy-Schwarz inequality, we have

\[
\int (R + \Delta f)\sqrt{u}\sqrt{u}\,dx \leq \left( \int (R + \Delta f)^2 u\,dx \right)^{\frac{1}{2}} \left( \int u\,dx \right)^{\frac{1}{2}}.
\]

(4.10)

Since \( \int u\,dx = 1 \), the above inequality can be simplified as

\[
\left( \int (R + \Delta f)u\,dx \right)^2 \leq \int (R + \Delta f)^2 u\,dx.
\]

(4.11)
Then the evolution of $F$ along the time $t$ would be estimated by

$$\frac{\partial F}{\partial t} \geq \frac{2}{n} \left( \int (R + \Delta f) u \, dx \right)^2 = \frac{2}{n} \left( \int (R + |\nabla f|^2) u \, dx \right)^2$$

(4.12)

due to the following equality in closed manifold $M$

$$\int_M (\Delta f - |\nabla f|^2) u \, dx = \int_M \left( -\frac{\Delta u}{u} + |\nabla f|^2 - |\nabla f|^2 \right) u \, dx = -\int_M \Delta u \, dx = 0$$

(4.13)

$$\Rightarrow \int_M (\Delta f) u \, dx = \int_M |\nabla f|^2 u \, dx.$$

From the definition $F = \int (R + |\nabla f|^2) u \, dx$, we get

$$\frac{\partial F}{\partial t} \geq \frac{2}{n} F^2 \geq 0.$$  

(4.14)

We claim

$$F(t) \leq \frac{n}{2(T-t)}.$$  

(4.15)

Here is the proof of the above claim,

$$\frac{dF}{dt} \geq \frac{2}{n} F^2 \Rightarrow \frac{dF}{F^2} \geq \frac{2}{n} \, dt \Rightarrow \int_t^T \frac{dF}{F^2} \geq \frac{2}{n} (T-t)$$

$$\Rightarrow -\left( \frac{1}{F(T)} - \frac{1}{F(t)} \right) \geq \frac{2}{n} (T-t) \Rightarrow \frac{1}{F(t)} \geq \frac{2}{n} (T-t) + \frac{1}{F(T)}.$$

If $F(T) > 0$, then $\frac{1}{F(T)} \geq \frac{2}{n} (T-t)$, that is, $F(t) \leq \frac{n}{2(T-t)}$;

If $F(T) \leq 0$, since $\frac{dF}{dt} \geq 0$, then $F(t) \leq 0 \leq \frac{n}{2(T-t)}$ for all $t \in [0, T)$, therefore,

$$F(t) = \int (R + |\nabla f|^2) u \, dx \leq \frac{n}{2(T-t)} = \frac{n}{2\tau}.$$

(4.16)

plugging into (4.18), we obtain

$$\frac{\partial}{\partial t} \left( \int_M f u \, dx \right) = \frac{n}{2\tau} - \int_M (|\nabla f|^2 + R) u \, dx \geq 0.$$

Thus we complete the proof of Theorem 4.1.

\[ \square \]

5. The Case for the Heat Equation Under a Fixed Metric

It is well known that gradient estimates and monotonicity formulas for the heat equation in the fixed metric case are important in their own right. The Li-Yau estimate is one of several examples. Recently Perelman’s W-entropy and gradient estimate for the Ricci flow were transformed to the case of the heat equation in the fixed metric case by Lei Ni [N1]. He also pointed out some useful geometric applications.

Here we will transplant some of the results in the previous sections to the heat equation case. More specifically, we will introduce a family of entropy formulas which include both
Perelman’s $W$-entropy (for the heat equation as defined in [N1]) and the 'Boltzmann-Shannon' entropy as considered in thermodynamics, information theory.

Let $(M,g)$ be a closed Riemannian $n$-manifold with the metric $g$ not evolving along time $t$, suppose $u$ is a positive solution to the heat equation (5.1) with $\int_M u dx = 1$,

\[(\triangle - \frac{\partial}{\partial t}) u = 0.\] (5.1)

Let $f : M \to R$ be smooth, defined by $f := - \ln u - \frac{n}{2} \ln(4\pi\tau)$, that is, $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$, where $\tau = \tau(t) > 0$ is a scale parameter with $\frac{d\tau}{dt} = 1$. We define a family of entropy formulas by:

\[W(f,\tau,a) := \int_M \left( \frac{a^2}{2\pi} |\nabla f|^2 + f - n + \frac{n}{2} \ln \frac{2\pi}{a^2} \right) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dx.\] (5.2)

The main results of the section are the next two theorems.

**Theorem 5.1.** Let $M$ be a closed Riemannian $n$-manifold with fixed metric $g$, and $u, f$ be defined as above satisfying $\int_M u dx = 1$ and $(\triangle - \frac{\partial}{\partial t}) u = 0$. The entropy $W(f,\tau,a)$ defined in (5.2) satisfies:

\[\frac{\partial W(f,\tau,a)}{\partial t} = - \frac{a^2}{2\pi} \int_M u |f_{ij} - \frac{g_{ij}}{2\tau}|^2 dx - \frac{a^2}{\pi} \int_M u R_{ij} f_i f_j dx \]

\[+ (1 - \frac{a^2}{2\pi}) \int_M u \left( \frac{|\nabla u|^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2\tau} \right) dx.\] (5.3)

Moreover, if $0 \leq a^2 \leq 2\pi$, $\tau = t$ and $M$ has non-negative Ricci curvature, then $W$ is monotone non-increasing in time $t$, that is,

\[\frac{\partial W(f,t,a)}{\partial t} \leq 0.\] (5.4)

**Remark 5.1.** If we choose $a^2 = 2\pi$ in $W(f,\tau,a)$, we recover Perelman’s $W$-entropy worked out in [N1] and its monotonicity as a special case.

\[W(f,\tau,\sqrt{2\pi}) = \int_M (\tau |\nabla f|^2 + f - n) u dx\] (5.5)

\[\frac{\partial W(f,\tau,\sqrt{2\pi})}{\partial t} = -2\tau \int_M u \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) dx \leq 0.\] (5.6)

**Remark 5.2.** If $a = 0$, $W(f,\tau,0)$ is related with 'Boltzmann-Shannon' entropy as considered in thermodynamics, information theory,

\[N = \int_M -u \ln u \, dx.\] (5.7)

The entropy $W(f,\tau,a)$ thus defined serves as a connection between 'Boltzmann(1870s)-Shannon(1940s)' entropy and Perelman’s $W$-entropy(2002). We are partially motivated by the logarithm Sobolev inequality of the Gross [Gr] as stated in [LL].

From this new entropy formula, we deduce the corresponding differential inequality for the fundamental solution to the heat equation.
Theorem 5.2. Let $M$ be a closed Riemannian $n$-manifold with fixed metric $g$ and non-negative Ricci curvature, $u$ be the fundamental solution, and $f$ be defined as $f := -\ln u - \frac{n}{2} \ln (4\pi t)$. Then for any constant $\alpha \geq 1$,

$$t \left( \alpha \triangle f - |\nabla f|^2 \right) + f - \alpha \frac{n}{2} \leq 0. \tag{5.8}$$

Remark 5.3. In particular, if $\alpha = 2$, then it becomes the following differential inequality proved in [N1].

$$t \left( 2 \triangle f - |\nabla f|^2 \right) + f - n \leq 0. \tag{5.9}$$

If we divide the left-hand side of the inequality $(5.8)$ by $\alpha t$, where $\alpha \geq 1$ and $t > 0$, we get $\triangle f - |\nabla f|^2 \leq n \frac{1}{2}$, let $\alpha \to \infty$, then we conclude that the inequality $(5.8)$ includes Li-Yau gradient estimate, that is, $\frac{\nabla u^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2t} \leq 0$ since $\triangle f = \frac{\nabla u^2}{u^2} - \frac{\triangle u}{u}$.

Remark 5.4. For $\alpha > 2$, the gradient estimate $(5.8)$ is an interpolation of Perelman’s gradient estimate cf. [N1] and Li-Yau estimate; however, for $1 \leq \alpha \leq 2$, the gradient estimate $(5.8)$ is new here, it can’t be directly obtained from Perelman’s gradient estimate and Li-Yau gradient estimate.

Remark 5.5. In the Euclidean Space $\mathbb{R}^n$, if $u$ is the fundamental solution to the heat equation, then $(5.8)$ becomes an equality.

We start the proof of Theorem 5.1 with the following lemmas. Some computation results can be directly found in [LY], [N1] or some other sources, we give the details here for completeness.

Lemma 5.1. Let $u$ be a positive solution to the heat equation $(5.1)$ in a closed Riemannian $n$-manifold $M$, and $f = -\ln u - \frac{n}{2} \ln (4\pi \tau)$, where $\tau = \tau(t) > 0$, $\frac{\partial \tau}{\partial t} = 1$. Then

$$\triangle f = f_t + \nabla f \cdot \nabla f + 2R_{ij}f_if_j. \tag{5.10}$$

Proof. By direct computation,

$$\triangle f = 2|f_{ij}|^2 + 2 \nabla f \cdot \nabla f + 2R_{ij}f_if_j.$$

Notice that

$$f_t = -\ln u - \frac{n}{2} \ln (4\pi \tau) \tag{5.12}$$

then,

$$\triangle f = f_t + |\nabla f|^2 + \frac{n}{2\tau}. \tag{5.13}$$
Thus,

\[
\Delta (\|\nabla f\|^2) = 2|f_{ij}|^2 + 2 \nabla f \nabla (f_t + |\nabla f|^2 + \frac{n}{2\tau}) + 2 R_{ij} f_i f_j
\]

\[
= 2|f_{ij}|^2 + 2 \nabla f \nabla f_t + 2 \nabla f \nabla (|\nabla f|^2) + 2 R_{ij} f_i f_j
\]

(5.14)

\[
= 2|f_{ij}|^2 + (\|\nabla f\|^2)_t + 2 \nabla f \nabla (|\nabla f|^2) + 2 R_{ij} f_i f_j
\]

therefore,

\[
(\Delta - \partial_t)(\|\nabla f\|^2) = 2|f_{ij}|^2 + 2 \nabla f \nabla (|\nabla f|^2) + 2 R_{ij} f_i f_j.
\]

\[\square\]

**Lemma 5.2.** Let \(u, f\) be defined as in Lemma 5.1 then

\[
(\Delta - \partial_t)(\triangle f) = 2|f_{ij}|^2 + 2 \nabla f \nabla (\triangle f) + 2 R_{ij} f_i f_j
\]

(5.15)

\[
(\Delta - \partial_t)(2 \triangle f - \|\nabla f\|^2) = 2|f_{ij}|^2 + 2 \nabla f \nabla (2 \triangle f - \|\nabla f\|^2) + 2 R_{ij} f_i f_j.
\]

(5.16)

**Proof.** By the result in Lemma 5.1 we have \(\triangle f = f_t + |\nabla f|^2 + \frac{n}{2\tau}\), then

\[
(\Delta - \partial_t)(\triangle f) = \triangle (\triangle f) - \partial_t (\triangle f) = \triangle (f_t + |\nabla f|^2 + \frac{n}{2\tau}) - \partial_t (\triangle f)
\]

\[
= \triangle (f_t) + \triangle (|\nabla f|^2) - \triangle (f_t) = \triangle (|\nabla f|^2)
\]

\[
= 2|f_{ij}|^2 + 2 \nabla f \nabla (\triangle f) + 2 R_{ij} f_i f_j.
\]

Combining (5.10) and (5.15), we have,

\[
(\Delta - \partial_t)(2 \triangle f - \|\nabla f\|^2) = 2|f_{ij}|^2 + 2 \nabla f \nabla (2 \triangle f - \|\nabla f\|^2) + 2 R_{ij} f_i f_j.
\]

\[\square\]

**Lemma 5.3.** Let \(u, f\) be defined as in Lemma 5.1 let \(F = \frac{a^2}{2\tau} |\nabla f|^2 + f\), then

\[
(\Delta - \partial_t)F = \frac{a^2}{2\pi} |f_{ij}|^2 + \frac{g_{ij}}{2\tau} + (1 - \frac{a^2}{2\pi})(|\nabla f|^2 + \frac{n}{2\tau})
\]

\[
+ \frac{a^2}{2\pi} \triangle f + \frac{a^2}{2\tau} \nabla f \nabla (|\nabla f|^2) + \frac{a^2}{2\pi} R_{ij} f_i f_j.
\]
Proof. Keep in mind that in local normal coordinates $\sum f_{ij}g_{ij} = \Delta f$, $\sum (g_{ij})^2 = n$ and the result in Lemma 5.1 we follow the direct computation,

$$\begin{align*}
(\Delta - \partial_t)F &= (\Delta - \partial_t)\left(\frac{a^2\tau}{2\pi} |\nabla f|^2 + f\right) \\
&= \frac{a^2\tau}{2\pi} (\Delta - \partial_t)(|\nabla f|^2) - \frac{a^2}{2\pi} |\nabla f|^2 + \Delta f - f_t \\
&= \frac{a^2\tau}{2\pi} (2(f_{ij})^2 + 2 \nabla f \nabla(|\nabla f|^2) + 2R_{ij} f_if_j) \\
&\quad - \frac{a^2}{2\pi} |\nabla f|^2 + |\nabla f|^2 + \frac{n}{2\tau} \\
&= \frac{a^2\tau}{\pi} |f_{ij} - \frac{g_{ij}}{2\tau}|^2 - \frac{a^2}{2\pi} \frac{n}{2\tau} + \frac{a^2}{\pi} \Delta f + \frac{a^2\tau}{\pi} \nabla f \nabla(|\nabla f|^2) \\
&\quad + \frac{a^2\tau}{\pi} R_{ij} f_if_j - \frac{a^2}{2\pi} |\nabla f|^2 + |\nabla f|^2 + \frac{n}{2\tau} \\
&= \frac{a^2\tau}{\pi} |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + (1 - \frac{a^2}{2\pi})(|\nabla f|^2 + \frac{n}{2\tau}) \\
&\quad + \frac{a^2}{\pi} \Delta f + \frac{a^2\tau}{\pi} \nabla f \nabla(|\nabla f|^2) + \frac{a^2\tau}{\pi} R_{ij} f_if_j.
\end{align*}$$

Now we turn to the

Proof. (of Theorem 5.1) We have,

$$\begin{align*}
W(f, \tau, a) &= \int_M \left(\frac{a^2\tau}{2\pi} |\nabla f|^2 + f\right) ud\tau - n + \frac{n}{2} \ln \frac{2\pi}{a^2} \\
&= \int_M Fu \, d\tau - n + \frac{n}{2} \ln \frac{2\pi}{a^2}
\end{align*}$$

then

$$\begin{align*}
\frac{\partial W}{\partial t} &= \int_M \frac{\partial}{\partial t}(Fu) \, d\tau - 0 \\
&= \int_M \frac{\partial}{\partial t}(Fu) \, d\tau - \int_M \Delta (Fu) \, d\tau \\
&= \int_M uF_t + (Fu_t - F\Delta u) - u\Delta F - 2 \nabla u \nabla F \, d\tau \\
&= \int_M -u(\Delta - \partial_t)F - 2 \nabla u \nabla F \, d\tau.
\end{align*}$$
By the result of Lemma 5.3, we get,
\[
\frac{\partial W}{\partial t} = \int_M \frac{a^2}{\pi} u |f_{ij} - g_{ij}|^2 - (1 - \frac{a^2}{2\pi}) u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) - \frac{a^2}{\pi} u \triangle f \, dx
\]
\[- \int_M \frac{a^2}{\pi} u \nabla f \nabla(|\nabla f|^2) + \frac{a^2}{\pi} u R_{ij} f_i f_j + 2 \nabla u \nabla F \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} - \frac{g_{ij}}{2\tau} \, dx - (1 - \frac{a^2}{2\pi}) \int_M u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} - \frac{g_{ij}}{2\tau} \, dx - (1 - \frac{a^2}{2\pi}) \int_M u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) \, dx
\]
\[- \int_M 2 \nabla u \frac{a^2}{2\pi} \nabla(|\nabla f|^2) + 2 \nabla u \nabla f \, dx.
\]
The last equality comes from \( \nabla F = \frac{a^2}{2\pi} \nabla(|\nabla f|^2) + \nabla f \). Also notice \( u \nabla f = -\nabla u \), there are two more terms canceled. The above becomes,
\[
\frac{\partial W}{\partial t} = \int_M \frac{a^2}{\pi} u |f_{ij} - g_{ij}|^2 \, dx - (1 - \frac{a^2}{2\pi}) \int_M u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) \, dx
\]
\[+ \int_M \frac{a^2}{2\pi} u \left( \frac{2\Delta u}{u} - 2|\nabla f|^2 \right) - \frac{a^2}{\pi} u R_{ij} f_i f_j + 2u|\nabla f|^2 \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} - \frac{g_{ij}}{2\tau} \, dx + (1 - \frac{a^2}{2\pi}) \int_M u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} - \frac{g_{ij}}{2\tau} \, dx + (1 - \frac{a^2}{2\pi}) \int_M u \left( |\nabla f|^2 + \frac{n}{2\pi} \right) \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} + \int_M \frac{a^2}{\pi} \Delta u + (1 - \frac{a^2}{2\pi}) \int_M 2u(|\nabla f|^2) \, dx.
\]
Reorganize the terms, and by the fact \( \int_M \Delta u \, dx = 0 \) for closed manifold \( M \), we have
\[
\frac{\partial W}{\partial t} = \int_M -\frac{a^2}{\pi} u |f_{ij} - g_{ij}|^2 \, dx - \frac{a^2}{\pi} u R_{ij} f_i f_j \, dx
\]
\[+ (1 - \frac{a^2}{2\pi}) \int_M u \left( \nabla f \cdot \nabla f + \frac{\Delta u}{u} - \frac{n}{2\tau} \right) \, dx + (1 + \frac{a^2}{2\pi}) \int_M \Delta u \, dx
\]
\[- \int_M \frac{a^2}{\pi} u f_{ij} - \frac{g_{ij}}{2\tau} \, dx - \frac{a^2}{\pi} u R_{ij} f_i f_j \, dx
\]
\[+ (1 - \frac{a^2}{2\pi}) \int_M u \left( \nabla u \cdot \nabla u + \frac{\Delta u}{u} - \frac{n}{2\tau} \right) \, dx
\]
therefore, for the positive solution \( u \) and non-negative Ricci curvature, all three integral terms are non-positive, the non-positivity of the last integral term is due to the Li-Yau [LY] gradient estimate \( \frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} - \frac{n}{2\tau} \leq 0 \), when \( \tau = t \).

Now we give the proof for Theorem 5.2 starting with the following lemma.

**Lemma 5.4.** Let \( u, f \) be defined as in Lemma 5.1, define \( P := t(2\Delta f - |\nabla f|^2)u \), then
\[
(\Delta - \partial_t)P = 2t u |f_{ij} - \frac{g_{ij}}{2t}|^2 + u(|\nabla f|^2 - \frac{n}{2t}) + 2t u R_{ij} f_i f_j.
\]
Proof. By direct computation, we have
\[
(\Delta - \partial_t)P = tu(\Delta - \partial_t)(2\Delta f - |\nabla f|^2) + t(2\Delta f - |\nabla f|^2) \Delta u
+ 2t \nabla(2\Delta f - |\nabla f|^2) \nabla u - (2\Delta f - |\nabla f|^2) u - t(2\Delta f - |\nabla f|^2) u_t.
\]
Since $\Delta u - u_t = 0$, two more terms are canceled,
\[
(\Delta - \partial_t)P = tu(\Delta - \partial_t)(2\Delta f - |\nabla f|^2) + 2t \nabla(2\Delta f - |\nabla f|^2) \nabla u
- (2\Delta f - |\nabla f|^2) u.
\]
Using the result in Lemma 5.2, we have
\[
(\Delta - \partial_t)P = tu\left[2|f_{ij}|^2 + 2\nabla f \nabla(2\Delta f - |\nabla f|^2) + 2R_{ij}f_if_j\right]
+ 2t \nabla u \nabla(2\Delta f - |\nabla f|^2) - (2\Delta f - |\nabla f|^2) u.
\]
By the fact $\nabla f = -\sum \nabla u_i$, $\nabla u = -u \nabla f$, the 2nd term and 4th term are canceled, then
\[
(\Delta - \partial_t)P = 2tu|f_{ij}|^2 + 2tuR_{ij}f_if_j - (2\Delta f - |\nabla f|^2) u.
\]
Keeping in mind that in local normal coordinate $\sum f_{ij}g_{ij} = \Delta f$, and $\sum g_{ij}^2 = n$, by completing the square,
\[
(\Delta - \partial_t)P = 2tu|f_{ij}|^2 - \frac{g_{ij}}{2t}^2 - 2tu\left(\frac{g_{ij}}{2t}\right)^2 + 2tu\frac{2f_{ij}g_{ij}}{2t}
+ 2tuR_{ij}f_if_j - (2\Delta f - |\nabla f|^2) u
= 2tu|f_{ij}|^2 - \frac{n}{2t}u + u(2\Delta f) + 2tuR_{ij}f_if_j - (2\Delta f - |\nabla f|^2) u
= 2tu|f_{ij}|^2 - \frac{n}{2t}u(\nabla f^2 - \frac{n}{2t}) + 2tuR_{ij}f_if_j.
\]
\[
\square
\]
Finally we are in a position to give

Proof. (of theorem 5.2) Define
\[
H(f, t, a) := \left(t \left(1 + \frac{a^2}{2\pi}\right) \Delta f - t|\nabla f|^2 + f - \frac{n}{2}\left(1 + \frac{a^2}{2\pi}\right)\right) u.
\]
Reorganize the terms by using $\Delta f = -\frac{\Delta u_a}{a} + \frac{\nabla u_a^2}{a^2} = -\frac{\Delta u_a}{a} + |\nabla f|^2$,
\[
H = \left(\frac{a^2}{2\pi} t \Delta f + t(\Delta f - |\nabla f|^2) + f - \frac{n}{2}\left(1 + \frac{a^2}{2\pi}\right)\right) u
= \left(\frac{a^2t}{2\pi}(2\Delta f - |\nabla f|^2) - \frac{a^2}{2\pi}t(\Delta f - |\nabla f|^2) + t(\Delta f - |\nabla f|^2)\right) u
+ (f - \frac{n}{2})u - \frac{na^2}{4\pi}u.
\]
Combine the last two terms in the brackets,

\[ H = \left( \frac{a^2}{2\pi} (2\Delta f - |\nabla f|^2) + (1 - \frac{a^2}{2\pi}) t \left( \Delta - \frac{\Delta u}{u} \right) u + (f - \frac{n}{2}) u - \frac{na^2}{4\pi} u \right) \]

\[ = \frac{a^2}{2\pi} (2\Delta f - |\nabla f|^2) u - (1 - \frac{a^2}{2\pi}) t \Delta u + (f - \frac{n}{2}) u - \frac{na^2}{4\pi} u \]

\[ = \frac{a^2}{2\pi} P - (1 - \frac{a^2}{2\pi}) t \Delta u + (f - \frac{n}{2}) u - \frac{na^2}{4\pi} u \]

where \( P = t (2\Delta f - |\nabla f|^2) u \) as defined in Lemma 5.4, then,

\[ \begin{align*}
(\Delta - \partial_t)H &= \frac{a^2}{2\pi} (\Delta - \partial_t) P - (1 - \frac{a^2}{2\pi}) t (\Delta - \partial_t) (\Delta u) \\
&\quad + (1 - \frac{a^2}{2\pi}) \Delta u + (\Delta f - f_t) u + 2 \nabla f \nabla u.
\end{align*} \]

Notice that

\[ \Delta u - u_t = 0 \implies 0 = \Delta (\Delta u - u_t) = \Delta (\Delta u) - \Delta (u_t) \]

\[ \implies 0 = \Delta (\Delta u) - \partial_t (\Delta u) = (\Delta - \partial_t) (\Delta u) \]

which means, \( \Delta u \) is also a solution to the heat equation; also observed \( \Delta f = f_t + |\nabla f|^2 + \frac{n}{2t} \) and \( \nabla u = -u \nabla f \), then the above expression (5.21) can be simplified as

\[ (\Delta - \partial_t)H = \frac{a^2}{2\pi} (\Delta - \partial_t) P + (1 - \frac{a^2}{2\pi}) \Delta u + \left( |\nabla f|^2 + \frac{n}{2t} \right) u - 2 |\nabla f|^2 u \]

\[ = \frac{a^2}{2\pi} (\Delta - \partial_t) P + (1 - \frac{a^2}{2\pi}) \Delta u + \left( \frac{n}{2t} - |\nabla f|^2 \right) u. \]

By the result in Lemma 5.4 we have

\[ \begin{align*}
(\Delta - \partial_t)H &= \left( \frac{a^2}{\pi} \left( f_{ij} - \frac{g_{ij}}{2t} \right) |u|^2 + \frac{a^2}{2\pi} (|\nabla f|^2 - \frac{n}{2t}) u + \frac{a^2}{\pi} u R_{ij} f_i f_j \right) \\
&\quad + \left( 1 - \frac{a^2}{2\pi} \right) \frac{\Delta u}{u} u + \left( \frac{n}{2t} - \frac{|\nabla u|^2}{u^2} \right) u \\
&\quad + \left( \frac{a^2}{2\pi} - 1 \right) \left( \frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} - \frac{n}{2t} \right) u.
\end{align*} \]

If \( 0 \leq a^2 \leq 2\pi \), and \( M \) has non-negative Ricci curvature, then all three terms are non-negative,

\[ (\Delta - \partial_t)H(f, t, a) \geq 0. \]

We claim that,

\[ \lim_{t \to 0} H(f, t, a) \leq 0. \]

Therefore by the maximum principle, for any \( t \geq 0 \)

\[ H(f, t, a) \leq 0 \]
that is,

\[(5.26) \quad \left( t \left( 1 + \frac{a^2}{2\pi} \right) \Delta f - t |\nabla f|^2 + f - \frac{n}{2} \left( 1 + \frac{a^2}{2\pi} \right) \right) u \leq 0. \]

Recall that \(u\) is a positive solution to the heat equation, consequently

\[(5.27) \quad t \left( 1 + \frac{a^2}{2\pi} \right) \Delta f - t |\nabla f|^2 + f - \frac{n}{2} \left( 1 + \frac{a^2}{2\pi} \right) \leq 0. \]

Let \(\alpha = 1 + \frac{a^2}{2\pi}\), where \(0 \leq a^2 \leq 2\pi\), then for any \(1 \leq \alpha \leq 2\),

\[(5.28) \quad t \left( \alpha \Delta f - |\nabla f|^2 \right) + f - \frac{n}{2} \leq 0. \]

In particular, if \(\alpha = 2\), then it becomes the following differential inequality, which is one of the results proved in [N1]:

\[(5.29) \quad t \left( 2 \Delta f - |\nabla f|^2 \right) + f - n \leq 0. \]

To prove the case for \(\alpha > 2\), consider Li-Yau gradient estimate,

\[\frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} - \frac{n}{2t} \leq 0 \Rightarrow \Delta f - \frac{n}{2t} \leq 0 \Rightarrow t \left( \Delta f - \frac{n}{2t} \right) \leq 0 \]

\[\Rightarrow t \Delta f - \frac{n}{2} \leq 0 \Rightarrow (\alpha - 2) t \Delta f - \frac{n}{2} \leq 0 \]

\[(5.30) \quad \Rightarrow (\alpha - 2) t \Delta f - \alpha \frac{n}{2} + n \leq 0. \]

Combine the inequality (5.30) and one of the results proved in [N1], that is, the inequality (5.29), we obtain

\[(5.31) \quad t \left( \alpha \Delta f - |\nabla f|^2 \right) + f - \frac{n}{2} \leq 0 \quad \text{for any} \quad \alpha > 2. \]

This proves Theorem 5.2, except for the claim (5.25). The following gives the proof of (5.25):

Let \(h(y,t)\) be any positive smooth function with compact support, by Equation (5.20), we have

\[\int_M h(y,t) H(f,t,a) \, dy \]

\[= \int_M \left( \frac{a^2 t}{2\pi} \left( 2 \Delta f - |\nabla f|^2 \right) u - \left( 1 - a^2 \right) t \Delta u + \left( f - \frac{n}{2} \right) u - \frac{na^2}{4\pi} u \right) h \, dy \]

\[= \frac{a^2 t}{2\pi} \int_M \left( \Delta f - |\nabla f|^2 \right) u h \, dy - \left( 1 - a^2 \right) t \int_M h \Delta u \, dy \]

\[+ \frac{a^2 t}{2\pi} \int_M \left( f - \frac{n}{2} \right) u h \, dy + \int_M \left( f - \frac{n}{2} \right) u h \, dy. \]
Since $\triangle f = -\frac{\triangle u}{u} + \frac{\nabla u^2}{u^2}$, we have $(\triangle f - |\nabla f|^2)u = -\triangle u$, then

$$
\int_M h(y, t) H(f, t, u) dy
$$

$$
= \frac{a^2 t}{2\pi} \int_M - (\triangle u) \, h \, dy + \left( \frac{a^2}{2\pi} - 1 \right) t \int_M h \, \triangle u \, dy
$$

$$
+ \frac{a^2 t}{2\pi} \int_M \left( \frac{\nabla u^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2t} \right) u \, h \, dy + \int_M (f - \frac{n}{2})u \, h \, dy
$$

$$
= -t \int_M h \, \triangle u \, dy + \frac{a^2 t}{2\pi} \int_M \left( \frac{\nabla u^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2t} \right) u \, h \, dy + \int_M (f - \frac{n}{2})u \, h \, dy
$$

(5.32)

$$
= I + II + III.
$$

We estimate each term as $t \to 0$,

$$
(I) = -t \int_M h \, \triangle u \, dy = -t \int_M u \, \triangle h \, dy \to 0 \text{ as } t \to 0 \text{ (integration by parts)}
$$

$$
(II) = \frac{a^2 t}{2\pi} \int_M \left( \frac{\nabla u^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2t} \right) u \, h \, dy \leq 0 \text{ as } t \to 0
$$

due to Li-Yau gradient estimate $\frac{\nabla u^2}{u^2} - \frac{\triangle u}{u} - \frac{n}{2t} \leq 0$.

$$
(III) \leq 0 \text{ as } t \to 0 \text{ by the following argument. Let } x \text{ be a fixed point in } M, \text{ and } (y, t) \in M \times (0, T], \text{ by the asymptotic behavior of the fundamental solution } u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}} \text{ to the heat equation as } t \to 0, (\text{MP}),
$$

(5.33)

$$
u(x, y, t) \sim \frac{e^{-\frac{d^2(x, y)}{4t}}}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \tau^j u_j(x, y, t)
$$

where $d(x, y)$ is the distance function. By (5.33) we mean that there exists a suitable small $T > 0$ and a sequence $(u_j)_{j \in \mathbb{N}}$ with $u_j \in C^\infty(M \times M \times [0, T])$ such that

(5.34)

$$u(x, y, t) - \frac{e^{-\frac{d^2(x, y)}{4t}}}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{m} \tau^j u_j(x, y, t) = w_m(x, y, t)
$$

with

(5.35)

$$w_m(x, y, t) = O(t^{m+1-\frac{n}{2}})
$$

as $t \to 0$, uniformly for all $x, y \in M$. The function $u_0(x, y, 0)$ in (5.33) can be chosen so that $u_0(x, y, 0) = 1$. Therefore,

(5.36)

$$f = \frac{d^2(x, y)}{4t} + \ln \left(1 + tu_1 + t^2u_2 + \ldots + t^m u_m + O(t^{m+1-\frac{n}{2}})\right) \to \frac{d^2(x, y)}{4t} \text{ as } t \to 0.
Thus,
\[
\lim_{t \to 0} (III) = \lim_{t \to 0} \int_M \left( f - \frac{n}{2} \right) uh(y, t) \, dy = \lim_{t \to 0} \int_M \left( \frac{d^2(x, y)}{4 t} - \frac{n}{2} \right) uh(y, t) \, dy
\]
\[
= \lim_{t \to 0} \int_M \left( \frac{d^2(x, y)}{4 t} - \frac{n}{2} \right) e^{\frac{-d^2(x, y)}{4 t}} \left( 1 + tu_1 + t^2u_2 + \ldots + t^m u_m + O(t^{m+\frac{1}{2}}) \right) h(y, t) \, dy
\]
\[
= \lim_{t \to 0} \int_M \left( \frac{d^2(x, y)}{4 t} - \frac{n}{2} \right) e^{\frac{-d^2(x, y)}{4 t}} h(y, t) \, dy.
\]

It is easy to see that for any given \( \delta > 0 \), the integration of the above integrand in the domain \( d(x, y) \geq \delta \) converges to zero exponentially fast. Therefore
\[
\lim_{t \to 0} (III) = \lim_{t \to 0} \int_{d(x, y) \leq \delta} \left( \frac{d^2(x, y)}{4 t} - \frac{n}{2} \right) e^{-\frac{d^2(x, y)}{4 t}} h(y, t) \, dy.
\]

When \( \delta \) is sufficiently small, \( d(x, y) \) is sufficiently close to the Euclidean distance. After a standard approximation process using local normal coordinates, it is clear that
\[
\lim_{t \to 0} (III) = \lim_{t \to 0} \int_{R^n} \left( \frac{|x-y|^2}{4 t} - \frac{n}{2} \right) e^{-\frac{|x-y|^2}{4 t}} h_p(y) \, dy.
\]

Here \( h_p \) is the pull back of \( h(\cdot, 0) \) to the Euclidean space from the region \( d(x, y) \leq \delta \).

We split the above integral to
\[
\lim_{t \to 0} (III) = \lim_{t \to 0} \int_{R^n} \left( \frac{|x-y|^2}{4 t} - \frac{n}{2} \right) e^{-\frac{|x-y|^2}{4 t}} h_p(x) \, dy
\]
\[
+ \lim_{t \to 0} \int_{R^n} \left( \frac{|x-y|^2}{4 t} - \frac{n}{2} \right) e^{-\frac{|x-y|^2}{4 t}} (h_p(y) - h_p(x)) \, dy.
\]

By a straight forward calculation, the second integral on the right hand side of the last identity converges to zero as \( t \to 0 \), since \( |h_p(y) - h_p(x)| \leq C|x-y| \). Hence
\[
\lim_{t \to 0} (III) = h_p(x) \lim_{t \to 0} \int_{R^n} \left( \frac{|x-y|^2}{4 t} - \frac{n}{2} \right) e^{-\frac{|x-y|^2}{4 t}} \, dy
\]
\[
= h_p(x) \left[ \lim_{t \to 0} \int_{R^n} \frac{|y|^2}{4 t} e^{-\frac{|y|^2}{4 t}} \, dy - \frac{n}{2} \right] = 0.
\]

The last step is by an integration as in an exercise in calculus.

Since all three terms in (5.32) are non-positive as \( t \to 0 \), we conclude for any positive smooth function \( h(y, t) \) with compact support,
\[
\int_M h(y, t) H(f, t, a) \, d\mu_t(y) \leq 0
\]
then for any small \( t \geq 0 \)
\[
H(f, t, a) \leq 0.
\]

This completes the proof of the claim [5.25].
6. Appendix

The material below, due to Perelman [P], can be found in several recent papers and books. They are given for completeness. Here we follow the presentation in [T].

Lemma 6.1. Let $u, f, g, \tau$ defined as in Theorem 4.1, then the $F$-functional defined by
\[
F(g, f) := \int (R + |\nabla f|^2) u \, dx
\]
is non-decreasing in $t$ under
\[
\frac{\partial F(g, f)}{\partial t} = 2 \int |\text{Ric} + \text{Hess}(f)|^2 u \, dx.
\]

Proof. For simplicity, we denote $\frac{\partial g}{\partial t} = h$, $\frac{\partial f}{\partial t} = k$. Keeping in mind that $\frac{\partial}{\partial t} g_{ij} = -h_{ij}$, we may calculate
\[
\frac{\partial |\nabla f|^2}{\partial t} = -h(\nabla f, \nabla f) + 2\langle \nabla k, \nabla f \rangle.
\]
Also
\[
\frac{\partial}{\partial t} R = -\langle \text{Ric}, h \rangle + \delta^2 h - \triangle (\text{tr} h)
\]
where $\delta$ is the divergence operator $\delta : \Gamma(\otimes^k T^*M) \to \Gamma(\otimes^{k-1} T^*M)$ defined by: $\delta(T) = -tr_{12} \nabla T$, here $tr_{12}$ means the trace over the first and second entries of $\nabla T$. Further,
\[
\frac{\partial}{\partial t} dx = \frac{1}{2}(\text{tr} h) dx
\]
where $dx$ is the volume form $dx \equiv \sqrt{\det(g_{ij})} dx^1 \wedge ... \wedge dx^n$.

Therefore,
\[
\frac{\partial}{\partial t} F(g, f) = \int [-h(\nabla f, \nabla f) + 2\langle \nabla k, \nabla f \rangle - \langle \text{Ric}, h \rangle + \delta^2 h - \triangle (\text{tr} h)] u dx
\]
\[
+ \int (R + |\nabla f|^2)[-k + \frac{1}{2}(\text{tr} h)] \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx.
\]
Three of these terms may be useful addressed by integrating by parts. First
\[
\int 2\langle \nabla k, \nabla f \rangle \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx = \int -2k(\triangle f - |\nabla f|^2) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx.
\]
Second,
\[
\int (\delta^2 h) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx = \int \langle \delta h, d(\frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} \rangle dx = \int \langle h, \nabla d(\frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} \rangle dx
\]
\[
= \int (h(\nabla f, \nabla f) - \langle \text{Hess}(f), h \rangle) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx.
\]
Third,
\[
\int -\triangle (\text{tr} h) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx = \int -\text{tr} h \triangle (\frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx
\]
\[
= \int (\triangle f - |\nabla f|^2) (\text{tr} h) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dx.
\]
Combining these calculations, we find that
\[
\frac{\partial}{\partial t} F(g, f) = \int \left[ -\langle Ric, h \rangle - \langle Hess(f), h \rangle + (\triangle f - |\nabla f|^2)(trh - 2k) \\
+ (R + |\nabla f|^2)[-k + \frac{1}{2}(trh)] \frac{e^{-f}}{(4\pi \tau)^n} \right] dx
\]
(6.9)\[
= \int \left[ (-Ric - Hess(f), h) + (2\triangle f - |\nabla f|^2 + R)(\frac{1}{2}trh - k) \right] u \, dx .
\]
Notice that under the Ricci flow given by
\[
\frac{\partial g}{\partial t} = -2Ric
\]
(6.10)\[
\frac{\partial f}{\partial t} = -\triangle f + |\nabla f|^2 - R
\]
(6.11)
which is equivalent to another decoupled system of equations by a proper transform of diffeomorphisms,
\[
\frac{\partial \hat{g}}{\partial t} = -2(Ric(\hat{g}) + Hess_\hat{g}(\hat{f}))
\]
(6.12)\[
\frac{\partial \hat{f}}{\partial t} = -\triangle \hat{g} f - R_{\hat{g}}
\]
(6.13)
substituting into (6.9), we have
\[
\frac{\partial}{\partial t} F(g, f) = 2 \int \left[ (-Ric - Hess(f), -Ric - Hess(f)) \right] u \, dx
\]
(6.14)\[
= 2 \int |Ric + Hess(f)|^2 u \, dx .
\]
□

Lemma 6.2. Let $g, f, \tau$ defined as in Theorem [4.1] define $P := [\tau(2\triangle f - |\nabla f|^2 + R) + f - n]u$, then
\[
\Box^* P = -2\tau|Ric + Hess(f) - \frac{g}{2\tau}|^2 u .
\]
(6.15)

Proof. Consider $P = \frac{P}{u} u$, we find that
\[
\Box^* P = \Box^* \left( \frac{P}{u} u \right) = \frac{P}{u} \Box^* u - u \left( \frac{\partial}{\partial t} + \triangle \right) \frac{P}{u} - 2\langle \nabla \frac{P}{u}, \nabla u \rangle .
\]
Since $\Box^* u = 0$, and $\nabla f = -\frac{\nabla u}{u}$, we have
\[
\frac{\Box^* P}{u} = - \left( \frac{\partial}{\partial t} + \triangle \right) \frac{P}{u} + 2\langle \nabla \frac{P}{u}, \nabla f \rangle
\]
(6.16)
for the first term on the right hand side,

\[- \left( \frac{\partial}{\partial t} + \triangle \right) \left( \frac{P}{u} \right) = - \left( \frac{\partial}{\partial t} + \triangle \right) [\tau (2\Delta f - |\nabla f|^2 + R) + f - n]
\]

\[= (2\Delta f - |\nabla f|^2 + R)
\]

\[- \tau \left( \frac{\partial}{\partial t} + \triangle \right) (2\Delta f - |\nabla f|^2 + R) - \left( \frac{\partial}{\partial t} + \triangle \right) f\]

using the evolution equation for $f$ in (4.4) on the final term, this reduces to

\[- \left( \frac{\partial}{\partial t} + \triangle \right) \left( \frac{P}{u} \right) = 2\Delta f - 2|\nabla f|^2 + 2R - \frac{n}{2\tau}
\]

(6.17)

Recall that

\[\left( \frac{\partial}{\partial t} + \triangle \right) (2\Delta f - |\nabla f|^2 + R)
\]

\[= 4\langle Ric, Hess(f) \rangle + \triangle|\nabla f|^2 - 2Ric(\nabla f, \nabla f)
\]

(6.18)

Also

\[2\langle \nabla \frac{P}{u}, \nabla f \rangle = 2\tau (\nabla (2\Delta f - |\nabla f|^2 + R), \nabla f) + 2|\nabla f|^2
\]

(6.19)

Combining the above expressions altogether, we find that

\[\Box^* \frac{P}{u} = 2\Delta f + 2R - \frac{n}{2\tau} - \tau (4\langle Ric, Hess(f) \rangle + 2|Ric|^2)
\]

(6.20)

\[+ \tau [-\triangle|\nabla f|^2 + 2Ric(\nabla f, \nabla f) + 2\langle \nabla f, \nabla (\Delta f) \rangle].
\]

The three terms in the square brackets simplified to $-2|Hess(f)|^2$, so

\[\Box^* \frac{P}{u} = 2\Delta f + 2R - \frac{n}{2\tau}
\]

\[- \tau [4\langle Ric, Hess(f) \rangle + 2|Ric|^2 + 2|Hess(f)|^2]
\]

\[= 2\Delta f + 2R - \frac{n}{2\tau} - 2\tau (|Ric + Hess(f)|)^2
\]

\[= 2\langle Ric + Hess(f), g_{ij} \rangle - \frac{g_{ij}}{2\tau} - 2\tau (|Ric + Hess(f)|)^2
\]

\[= -2\tau |Ric + Hess(f)| - \frac{g}{2\tau}.\]

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