On bases and the dimensions of twisted centralizer codes

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Abstract

Alahmadi et al. ["Twisted centralizer codes", Linear Algebra and its Applications 524 (2017) 235-249.] introduced the notion of twisted centralizer codes, $C_{F_q}(A, \gamma)$, defined as

$$C_{F_q}(A, \gamma) = \{ X \in F_q^{n \times n} : AX = \gamma XA \},$$

for $A \in F_q^{n \times n}$, and $\gamma \in F_q$. Moreover, Alahmadi et al. ["On the dimension of twisted centralizer codes", Finite Fields and Their Applications 48 (2017) 43-59.] also investigated the dimension of such codes and obtained upper and lower bounds for the dimension, and the exact value of the dimension only for cyclic or diagonalizable matrices $A$. Generalizing and sharpening Alahmadi et al.’s results, in this paper, we determine the exact value of the dimension as well as provide an algorithm to construct an explicit basis of the codes for any given matrix $A$.

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1 Introduction

In 2017, Alahmadi and his coauthors introduced a notion of twisted centralizer codes [2]. Let $A$ be an $n \times n$ matrix over a finite field $\mathbb{F}_q := GF(q)$ and $\gamma$ be an element in $\mathbb{F}_q$. The centralizer of $A$ twisted by $\gamma$, denoted by $C_{\mathbb{F}_q}(A, \gamma)$, is defined as

$$C_{\mathbb{F}_q}(A, \gamma) := \{ X \in \mathbb{F}_q^{n \times n} : AX = \gamma X A \}.$$ 

It is easy to see that $C_{\mathbb{F}_q}(A, \gamma)$ is a linear subspace of the matrix space $\mathbb{F}_q^{n \times n}$ over $\mathbb{F}_q$, and hence $C_{\mathbb{F}_q}(A, \gamma)$ is a linear code, called a twisted centralizer code, whose elements (i.e., codewords) are matrices, that can be viewed as vectors of length $n^2$, by reading them column-by-column. The notion of twisted centralizer codes is a generalization of centralizer codes [1], since a centralizer code is a twisted centralizer code, twisted by $1 = \gamma \in \mathbb{F}_q$.

Regarding twisted centralizer codes, so far, Alahmadi et al. ([2], [3]) have determined an upper and lower bound for the dimension, and obtained the exact values of the dimension only for cyclic or diagonalizable matrices $A$. Moreover, they [1] also state, "In general determining the dimension of such a code given $A$ is a non-trivial problem, and we were only able to give a spectral characterization of the dimension over an extension of the base field" [2]. Continuing and improving the results obtained by Alahmadi et al. ([2], [3]), the purpose of this paper is to provide the exact dimension of the codes, for any given matrix $A$. Moreover, we also provide an algorithm to construct a basis of the codes explicitly. The key idea to attack the problem is by observing twisted centralizer codes from the viewpoint of module theory, namely by transforming a twisted centralizer code into a polynomial module of homomorphisms. By taking this viewpoint, we solve the "non-trivial problem" in a more general setting, namely for arbitrary given $\gamma \in \mathbb{F}_q$, in an elementary way. Furthermore, and in contrast to Alahmadi et al. approach, the investigation of the proposed results keeps the underlying field being the base one; that is, lifting the problem over a field extension is not necessary.

The organization of the paper is as follows. In Section 2 we provide basic facts on some structural aspects of polynomial modules of homomorphisms. The main result is provided in Section 3. The paper is ended by concluding remarks in Section 4. We follow [4] and [6] for undefined terms

1 Namely, a twisted centralizer code with $\gamma = 1$.

2 [1], pp. 76
2 $\mathbb{F}_q[x]$-module of homomorphisms: basic facts

From now on $\mathbb{F}_q[x]$ is the polynomial ring over the finite field $\mathbb{F}_q$ of order $q$. In this section, we provide a decomposition of an $\mathbb{F}_q[x]$-module of homomorphisms. We do such a thing, since we can consider twisted centralizer codes as $\mathbb{F}_q[x]$-modules of homomorphisms, as we show in the next section (see Section 3). Our decomposition will make it easy to study the structure of twisted centralizer codes.

Projection and injection homomorphisms are our main decomposition tool. We will use a slightly abused notations of projection and injection. For an $\mathbb{F}_q[x]$-module with direct decomposition $M = N \oplus K$, the projection $\rho$ from $M$ on submodule $N$ along the submodule $K$, $\rho(x + y) = x$ for all $x \in N, y \in K$, can be considered as having the codomain $N$ or $M$. In a similar way, an injection $\iota$ from $N$ to $M$, $\iota(x) = x$ for all $x \in N$, can be considered as having the domain $N$ or $M$ with $\iota(y) = 0$ for all $y \in K$.

We notice that some facts are well known, but we mention here for the reader’s convenience. Let $N$ and $M$ be $\mathbb{F}_q[x]$-modules. Let $\text{Hom}_{\mathbb{F}_q[x]}(N, M)$ denote the set of $\mathbb{F}_q[x]$-module homomorphisms from $N$ to $M$. It is clear that $\text{Hom}_{\mathbb{F}_q[x]}(N, M)$ is also an $\mathbb{F}_q[x]$-module and we call it $\mathbb{F}_q[x]$-module of homomorphisms (from $N$ to $M$).

Let $A \in \mathbb{F}_q^{n \times n}$. Let us define an action

$$f(x) \cdot u := f(A)u, \quad \text{for every } f(x) \in \mathbb{F}_q[x] \quad \text{and } u \in \mathbb{F}_q^{n \times m}.$$ 

By this action, $\mathbb{F}_q^{n \times m}$ can be regarded as an $\mathbb{F}_q[x]$-module, which we call an $\mathbb{F}_q[x]$-module induced by $A$. Alahmadi et al. employed properties of $\mathbb{F}_q^n$ as an $\mathbb{F}_q[x]$-module to decompose a twisted centralizer code $C_{\mathbb{F}_q}(A, \gamma)$. Further, they noted that a centralizer code $C_{\mathbb{F}_q}(A, 1)$ is an $\mathbb{F}_q$-algebra and a twisted centralizer code $C_{\mathbb{F}_q}(A, \gamma)$ is an $C_{\mathbb{F}_q}(A, 1)$-module (see [3, pp. 44]). In Section 3, with the right choice of an action we will show that the twisted centralizer code $C_{\mathbb{F}_q}(A, \gamma)$ is an $\mathbb{F}_q[x]$-module of homomorphisms. This fact gives us materials to study the structures of twisted centralizer codes from module theory’s viewpoint.
It is well known that the polynomial ring $\mathbb{F}_q[x]$ is a principal ideal domain and every finitely generated torsion module over a principal ideal domain can be decomposed into its primary submodules. Further, any finitely generated primary module over a principal ideal domain can be decomposed as a direct sum of cyclic submodules. Hence, we will use those facts and the following two lemmas to derive the decomposition of an $\mathbb{F}_q[x]$-module of homomorphisms between two finitely generated torsion modules. The first lemma is a fact concerning a decomposition of modules. The proof can be directly derived from the decomposition form of both modules.

**Lemma 2.1.** Let $N$, $M$ be two $\mathbb{F}_q[x]$-modules having decompositions

$$N = N_1 \oplus \cdots \oplus N_n, \quad \text{and} \quad M = M_1 \oplus \cdots \oplus M_m.$$ 

Let $\rho_i$ be the projection from $N$ on $N_i$ along $\bigoplus_{\nu=1, \nu \neq i}^n N_\nu$ for $i = 1, 2, \ldots, n$, and $\iota_j$ be the injection from $M_j$ to $M$ for $j = 1, 2, \ldots, m$. Then

$$\text{Hom}_{\mathbb{F}_q[x]}(N, M) = \bigoplus_{i=1}^n \bigoplus_{j=1}^m \iota_j \text{Hom}_{\mathbb{F}_q[x]}(N_i, M_j) \rho_i.$$ 

The second lemma is a fact concerning the order of modules. For any finitely generated torsion $\mathbb{F}_q[x]$-module $M$, we denote $o(M)$ as an order of $M$.

**Lemma 2.2.** Let $N$ and $M$ be torsion $\mathbb{F}_q[x]$-modules where $\gcd(o(N), o(M)) = 1$. Then

$$\text{Hom}_{\mathbb{F}_q[x]}(N, M) = \{0\}.$$ 

**Proof.** Let $o(N) = h$ and $o(M) = g$. Let $\theta \in \text{Hom}_{\mathbb{F}_q[x]}(N, M)$ and $u \in N$. Because $\gcd(h, g) = 1$, so there exist $a, b \in \mathbb{F}_q[x]$ such that $ah + bg = 1$. Consider that $\theta(u) = \theta(1 \cdot u) = \theta((ah + bg) \cdot u) = \theta((ah) \cdot u) + bg \cdot \theta(u)$. Because $h = o(N)$, so $h \cdot u = 0$. Because $g = o(M)$, so $g \cdot \theta(u) = 0$. This implies $\theta(u) = 0$ for every $u \in N$. Hence $\theta = 0$. Therefore $\text{Hom}_{\mathbb{F}_q[x]}(N, M) = \{0\}$. □

As mentioned before, any finitely generated $\mathbb{F}_q[x]$-module can be decomposed as the direct sum of its primary submodules with order prime/irreducible power. Combining this fact and the above two lemmas we obtain the following corollary.

**Corollary 2.3.** Let $N$ be an $\mathbb{F}_q[x]$-module having primary decomposition

$$N = N_{p_1} \oplus \cdots \oplus N_{p_n}.$$
with order \( o(N_{p_i}) = p_i^{\nu_i} \) for some \( p_i, i = 1, \ldots, n \) different irreducible monic elements in \( \mathbb{F}_q[x] \) and \( M \) be an \( \mathbb{F}_q[x] \)-module having primary decomposition

\[
M = M_{r_1} \oplus \cdots \oplus M_{r_m}
\]

with order \( o(M_{r_j}) = r_j^{\mu_j} \) for some distinct irreducible monic elements \( r_j \) in \( \mathbb{F}_q[x] \), \( j = 1, \ldots, m \). Let \( \rho_{p_i} \) be the projection from \( N \) on \( N_{p_i} \) along \( \bigoplus_{\tau=1, \tau \neq i}^n N_{p_{\tau}} \) and \( \iota_{r_j} \) be the injection from \( M_{r_j} \) to \( M \). It follows then

\[
\text{Hom}_{\mathbb{F}_q[x]}(N, M) = \bigoplus_{i=1}^k \iota_{s_i} \text{Hom}_{\mathbb{F}_q[x]}(N_{s_i}, M_{s_i}) \rho_{s_i}
\]

where \( \{s_1, \ldots, s_k\} = \{p_1, \ldots, p_n\} \cap \{r_1, \ldots, r_m\} \).

The above corollary uses the fact that \( \text{Hom}_{\mathbb{F}_q[x]}(N_{p_i}, M_{r_j}) = \{0\} \) when \( \gcd(p_i, r_j) = 1 \).

Next, let us look at the case when the modules \( N \) and \( M \) are primary modules whose order are not relatively prime. Regarding the primary cyclic decomposition (see, e.g., [6]), we know that primary submodules can be decomposed as a direct sums of cyclic submodules. Combining this fact and Lemma 2.4 we obtain the following corollary.

**Corollary 2.4.** Let \( p \) be an irreducible element in \( \mathbb{F}_q[x] \) and \( N, M \) be primary \( \mathbb{F}_q[x] \)-modules having cyclic decompositions as

\[
N = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle
\]

where \( o(u_i) = p^{a_i} \) for every \( i = 1, \ldots, k \), \( a_1 \geq a_2 \geq \cdots \geq a_k \), and

\[
M = \langle v_1 \rangle \oplus \cdots \oplus \langle v_\ell \rangle
\]

where \( o(v_j) = p^{b_j} \) for every \( j = 1, \ldots, \ell \) and \( b_1 \geq b_2 \geq \cdots \geq b_\ell \). Let \( \iota_j \) be the injection from \( \langle v_j \rangle \) to \( M \) for every \( j = 1, \ldots, \ell \) and \( \rho_i \) be the projection from \( N \) on \( \langle u_i \rangle \) along \( \bigoplus_{\nu=1, \nu \neq i}^n \langle u_\nu \rangle \) for every \( i = 1, \ldots, k \). Then \( \text{Hom}_{\mathbb{F}_q[x]}(N, M) = \bigoplus_{i=1}^k \bigoplus_{j=1}^\ell \iota_j \text{Hom}_{\mathbb{F}_q[x]}(\langle u_i \rangle, \langle v_j \rangle) \rho_i \).

We now determine a generator of an \( \mathbb{F}_q[x] \)-module of homomorphisms between two cyclic \( \mathbb{F}_q[x] \)-modules which is obviously cyclic. For that purpose we first define the following. For \( f, g \in \mathbb{F}_q[x] \), define \( \min\{f, g\} := \begin{cases} f, & \text{if } \deg f \leq \deg g, \\ g, & \text{if } \deg f > \deg g. \end{cases} \)
Lemma 2.5. Let \( \langle u \rangle \) and \( \langle v \rangle \) be cyclic \( \mathbb{F}_q[x] \)-modules where \( \gcd(o(u), o(v)) = \min\{o(u), o(v)\} \). Then we have
\[
\text{Hom}_{\mathbb{F}_q[x]}(\langle u \rangle, \langle v \rangle) = \langle \theta \rangle,
\]
where \( \theta(u) = \frac{o(v)}{\gcd(o(u), o(v))} \cdot v \).

Proof. Let \( \gcd(o(u), o(v)) = d \) and \( o(v) = dr \). Consider \( \theta \in \text{Hom}_{\mathbb{F}_q[x]}(\langle u \rangle, \langle v \rangle) \) which maps \( u \) to \( r \cdot v \). Let \( \varphi \in \text{Hom}_{\mathbb{F}_q[x]}(\langle u \rangle, \langle v \rangle) \), \( \varphi(u) = p \cdot v \) for some \( p \in \mathbb{F}_q[x] \). Then \( d(p \cdot v) = d \cdot \varphi(u) = 0 \). This implies \( o(v) \mid dp \), and hence \( r \mid p \), which is equivalent to \( p = rs \) for some \( s \in \mathbb{F}_q[x] \). Hence \( \varphi(u) = (sr) \cdot v = s \cdot \theta(u) \) and \( \varphi \in \langle \theta \rangle \). Therefore \( \text{Hom}_{\mathbb{F}_q[x]}(\langle u \rangle, \langle v \rangle) = \langle \theta \rangle \). ■

Let us observe an example.

Example 2.1. Let \( N = \mathbb{F}_3^3 \) be an \( \mathbb{F}_3 \)-module induced by
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
Then its primary cyclic decomposition is \( N = \langle [1 0 0]^t \rangle \oplus \langle [0 0 1]^t \rangle \) where \( o([1 0 0]^t) = x \) and \( o([0 0 1]^t) = x^2 \). Let \( M = \mathbb{F}_3^2 \) be an \( \mathbb{F}_3 \)-module induced by
\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]
Then its primary cyclic decomposition is \( M = \langle [1 0]^t \rangle \) where \( o([1 0]^t) = x^2 \). Then
\[
\text{Hom}_{\mathbb{F}_3[x]}(N, M) = \iota_1 \text{Hom}_{\mathbb{F}_3[x]}(\langle [1 0 0]^t \rangle, \langle [1 0]^t \rangle) \rho_1 \oplus \iota_1 \text{Hom}_{\mathbb{F}_3[x]}(\langle [0 0 1]^t \rangle, \langle [1 0]^t \rangle) \rho_2
\]
\[
= \langle \begin{bmatrix}0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rangle \oplus \langle \begin{bmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rangle.
\]

From the explanation above we identify that the orders of primary submodules as well as the orders of cyclic submodules in their decomposition play an important role on determining the structure of \( \mathbb{F}_q[x] \)-module of homomorphisms. The multiset of orders of cyclic submodules in the decomposition of primary submodules of a module \( M \) is called the elementary divisors of \( M \) and denoted by \( \text{ElemDiv}(M) \) [6].
3 Bases for and dimensions of twisted centralizer codes

In this section we determine a basis for and the dimension of a twisted centralizer code. We show first, that a twisted centralizer code is nothing but an $\mathbb{F}_q[x]$-module of homomorphisms. As a consequence, we can decompose a twisted centralizer code as we did in the previous section. At the end, we determine a primary cyclic decomposition of a twisted centralizer code, and then extract a basis for and the dimension of it explicitly.

We divide it into two cases: twisted centralizer codes twisted by $\gamma = 0$, and twisted by $\gamma \neq 0$. To obtain the result for twisted centralizer codes twisted by $\gamma \neq 0$, we consider first twisted polynomials and derive their properties.

3.1 Twisted centralizer codes with $\gamma = 0$

We obtain a theorem regarding the centralizer code of $A$ which is twisted by 0, namely

$$C_{\mathbb{F}_q[x]}(A, 0) = \{ X \in \mathbb{F}_q^{n \times n} : AX = 0 \}.$$ 

**Theorem 3.1.** Let a primary cyclic decomposition of $\mathbb{F}_q^n$ as $\mathbb{F}_q[x]$-module induced by $A$ be

$$\mathbb{F}_q^n = \bigoplus_{p_1} \left[ \langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,f(1)} \rangle \right] \oplus \cdots \oplus \left[ \langle v_{m,1} \rangle \oplus \cdots \oplus \langle v_{m,f(m)} \rangle \right]$$

where $p_1 = x$ and $o(v_{k,j}) = p_k^{s(k,j)}$ for every $k = 1, \ldots, m$ and $j = 1, \ldots, l(k)$. Then

$$C_{\mathbb{F}_q[x]}(A, 0) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l(1)} \langle E_i^{(1)}_{x^{s(1,j)-1}v_{1,j}} \rangle$$

where $E_i^{(1)}_{x^{s(1,j)-1}v_{1,j}}$ is in $\mathbb{F}_q^{n \times n}$ with zero columns except the $i$-th column which is equal to $x^{s(1,j)-1}v_{1,j}$.

**Proof.** Obviously, $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l(1)} \langle E_i^{(1)}_{x^{s(1,j)-1}v_{1,j}} \rangle \subseteq C_{\mathbb{F}_q[x]}(A, 0)$. Now let $X \in C_{\mathbb{F}_q[x]}(A, 0)$. Because $\mathbb{F}_q^{n \times n} = \bigoplus_{i=1}^{n} \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{l(k)} \langle E_i^{(k)}_{v_{k,j}} \rangle$, so $X$ can be stated as $X = \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{j=1}^{l(k)} f_{i,k,j} E_i^{(k)}_{v_{k,j}}$ where $f_{i,k,j} \in \mathbb{F}_q[x]$. Since $X \in C_{\mathbb{F}_q[x]}(A, 0)$, we get $0 = AX = \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{j=1}^{l(k)} f_{i,k,j} x^{s(k,j)} E_i^{(k)}_{v_{k,j}}$. This implies $p_k^{s(k,j)} | x f_{i,k,j}$. Because $\gcd(x, p_k) = 1$ for $k \neq 1$, so $p_k^{s(k,j)} | f_{i,k,j}$. Hence $f_{i,k,j} E_i^{(k)}_{v_{k,j}} = 0$, for $k \neq 1$. For $k = 1,
Then as a vector space, the dimension of a centralizer code of $A$ determined.

**Proof.** Let a primary cyclic decomposition of $\mathbb{F}_q^n$ be given by

$$\mathbb{F}_q^n = \left( \langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,l(1)} \rangle \right) \oplus \cdots \oplus \left( \langle v_{n,1} \rangle \oplus \cdots \oplus \langle v_{n,l(m)} \rangle \right),$$

where $p_1 = x$ and $o(v_{k,j}) = p_k^{s(k,j)}$ for every $k = 1, \ldots, m$ and $j = 1, \ldots, l(k)$. By Theorem 3.1, $\ker(A) = \bigoplus_{j=1}^{l(1)} \langle x^{s(1,j)}-1 \rangle v_{1,j} \rangle$. Then $k_0 = l(1)$. Again, by Theorem 3.1, $C_{\mathbb{F}_q[x]}(A, 0) = \bigoplus_{j=1}^{l(1)} \langle x^{s(1,j)}-1 \rangle v_{1,j} \rangle$. Then $\dim C_{\mathbb{F}_q[x]}(A, 0) = nl(1) = nk_0$.

## 3.2 Twisted polynomial

We consider twisted polynomials and derive their properties. The following definition is adapted from twisted characteristic polynomial in [3].

**Definition 3.1.** Let $0 \neq \gamma \in \mathbb{F}_q, f(x) \in \mathbb{F}_q[x]$, and $\deg(f)$ be the degree of $f$. The $\gamma$-twisted polynomial of $f(x)$, denote by $f^\gamma(x)$, is defined as $f^\gamma(x) := \gamma^{\deg(f)} f(x/\gamma)$.

**Remark 3.1.** Observe that $f^{1/\gamma}(x) = \gamma^{-\deg(f)} f(\gamma x)$. Moreover, $f^{1/\gamma}$ and $f^\gamma$ are monic if and only if $f$ is monic.

Since $\deg(f(x)) = \deg(f(\gamma x)) = \deg(f(x/\gamma))$, we also have

$$\deg(f) = \deg(f^\gamma) = \deg(f^{1/\gamma}). \quad (1)$$

**Proposition 3.3.** Let $f(x) \in \mathbb{F}_q[x]$. Then the following statements are equivalent.

(1) $f(x) = g(x)h(x)$. 


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(2) \( f^\gamma(x) = g^\gamma(x)h^\gamma(x) \).

(3) \( f^{1/\gamma}(x) = g^{1/\gamma}(x)h^{1/\gamma}(x) \).

Proof.

((1) \Rightarrow (2)) \( f^\gamma(x) = \gamma^{\deg(f)}f(x/\gamma) = \gamma^{\deg(gh)}g(x/\gamma)h(x/\gamma) = g^\gamma(x)h^\gamma(x) \).

((2) \Rightarrow (3)) \( f^{1/\gamma}(x) = \gamma^{-2\deg(f)}f^\gamma(\gamma^2x) = \gamma^{-2\deg(g)}g^\gamma(\gamma^2x)\gamma^{-2\deg(h)}h^\gamma(\gamma^2x) = g^{1/\gamma}(x)h^{1/\gamma}(x) \).

((3) \Rightarrow (1)) \( f(x) = \gamma^{\deg(f)}f^{1/\gamma}(x/\gamma) = \gamma^{\deg(gh)}g^{1/\gamma}(x/\gamma)h^{1/\gamma}(x/\gamma) = g(x)h(x) \). \( \Box \)

As a direct result, we obtain the following Lemma.

Lemma 3.4. Let \( f(x) \in \mathbb{F}_q[x] \). Then the following statements are equivalent.

(1) \( f(x) \) is irreducible.

(2) \( f^\gamma(x) \) is irreducible.

(3) \( f^{1/\gamma}(x) \) is irreducible.

By Equation (1) and Proposition 3.3 we obtain the following lemma.

Lemma 3.5. Let \( f, g \in \mathbb{F}_q[x] \). Then the following statements are equivalent.

(1) \( \gcd(f, g) = h \).

(2) \( \gcd(f^\gamma, g^\gamma) = h^\gamma \).

(3) \( \gcd(f^{1/\gamma}, g^{1/\gamma}) = h^{1/\gamma} \).

Let \( A \in \mathbb{F}_q^{n \times n} \). Observe that \( f(A) = \gamma^{-\deg(f)}f^\gamma(\gamma A) \). This implies \( f(A) = 0 \) if and only if \( f^\gamma(\gamma A) = 0 \). Then we have the lemma below.

Lemma 3.6. Let \( m_A(x) \) be the minimal polynomial of \( A \). Then the minimal polynomial of \( \gamma A \) is \( m_A^\gamma(x) \).
3.3 Twisted centralizer codes with $\gamma \neq 0$

In this section we determine a basis for and the dimension of a twisted centralizer code by examining relations between twisted centralizer codes and collection of module homomorphisms.

Let $S \in \mathbb{F}_q^{n \times n}$. Let $[S]F_q^n$ denote $F_q^n$ as the $F_q[x]$-module is induced by $S$. The following is the key theorem that opens a way to utilize the structure of $F_q[x]$-module of homomorphisms discussed in Section 2 to obtain a basis of a twisted centralizer code and hence its dimension.

**Theorem 3.7.** Let $A \in \mathbb{F}_q^{n \times n}$ and $\gamma \neq 0$. Then $\mathcal{C}(A, \gamma) = \text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n)$.

**Proof.** Let $X \in \mathcal{C}(A, \gamma)$. Let $f(x), g(x) \in F[x]$, and $u, v \in [\gamma A]F_q^n$. Then

$$X(f(x) \cdot u + g(x) \cdot v) = X(f(\gamma A)u + g(\gamma A)v)$$
$$= Xf(\gamma A)u + Xg(\gamma A)v$$
$$= f(A)Xu + g(A)Xv$$
$$= f(x) \cdot (Xu) + g(x) \cdot (Xv).$$

Hence $X \in \text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n)$. It follows that $\mathcal{C}(A, \gamma) \subseteq \text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n)$.

Let $X \in \text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n)$. Then

$$\gamma X Au = X(\gamma Au) = X(x \cdot u) = x \cdot (Xu) = A(Xu) = AXu \quad \text{for all} \quad u \in [\gamma A]F_q^n.$$

Thus $\gamma X A = AX$ which implies $X \in \mathcal{C}(A, \gamma)$. Hence $\text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n) \subseteq \mathcal{C}(A, \gamma)$. We conclude that $\mathcal{C}(A, \gamma) = \text{Hom}_{F_q[x]}([\gamma A]F_q^n, [A]F_q^n)$. ■

Note that for the case $\gamma = 1$, Theorem 3.7 says that the centralizer code $\mathcal{C}(A, 1)$ is equal to the $F_q[x]$-module of endomorphisms on $[A]F_q^n$.

In order to apply the results in Section 2 to a twisted centralizer code $\mathcal{C}(A, \gamma)$, we need to identify the primary and cyclic decomposition of $[A]F_q^n$ and $[\gamma A]F_q^n$. The following two corollaries show the relation between orders of $[A]F_q^n$ and $[\gamma A]F_q^n$, between the primary and cyclic decompositions of $[A]F_q^n$ and $[\gamma A]F_q^n$, as well as between $\text{ElemDiv}([A]F_q^n)$ and $\text{ElemDiv}([\gamma A]F_q^n)$.

**Corollary 3.8.** Let $A \in \mathbb{F}_q^{n \times n}$ and $\gamma \neq 0$. Let $[A]F_q^n$ has an order $p_1^{s_1}p_2^{s_2}\cdots p_m^{s_m}$ where $p_1, \ldots, p_m$ are distinct irreducible monic elements in $F_q[x]$ and $s_1, \ldots, s_m \in \mathbb{N}$. Then $[\gamma A]F_q^n$ has an order $(p_1^{\gamma})^{s_1} \cdot (p_2^{\gamma})^{s_2} \cdots (p_m^{\gamma})^{s_m}$.
Thus, the action of $f$ from Lemma 3.6, we obtain an order of $[\gamma A]F_q^n$ is

$$m_\gamma A(x) = m_A(x) = (p_1^{s_1}p_2^{s_2} \cdots p_m^{s_m})^\gamma.$$  

Using Corollary 3.3, we conclude that

$$o([\gamma A]F_q^n) = m_\gamma A(x) = (p_1^{\gamma s_1}p_2^{\gamma s_2} \cdots p_m^{\gamma s_m}).$$

Now, let $v \in F_q^n$ and $f(x) \in F_q[x]$. Considering $F_q^n$ as the $[A]F_q^n$ module, the action of $f(x)$ on $v$ produces

$$f(x) \cdot v = f(A)v.$$  

Meanwhile, considering $F_q^n$ as the $[\gamma A]F_q^n$ module, the action of $f^\gamma(x)$ on $v$ produces

$$f^\gamma(x) \cdot v = \gamma^{\deg(f)}f \left( \frac{x}{\gamma} \right) \cdot v = \gamma^{\deg(f)}\left( \frac{\gamma A}{\gamma} \right)v = \gamma^{\deg(f)}f(A)v.$$  

Thus, the action of $f(x)$ on $v$ in the context of $[A]F_q^n$ module will produce a scalar multiple of the action of $f^\gamma(x)$ on $v$ in the context of $[\gamma A]F_q^n$ module. By utilizing this property we obtain the following.

**Corollary 3.9.** If one of the primary submodules of $[A]F_q^n$ is $([A]F_q^n)_{p_k}$ with a cyclic decomposition

$$([A]F_q^n)_{p_k} = \langle v_{k,1} \rangle \oplus \cdots \oplus \langle v_{k,l(k)} \rangle$$

where $o(v_{k,i}) = (p_k)^{s(k,i)}$ and $s_k = s(k,1) \geq s(k,2) \geq \cdots \geq s(k,l(k))$, then one of primary submodules of $[\gamma A]F_q^n$ is $([\gamma A]F_q^n)_{p_k^\gamma} = ([A]F_q^n)_{p_k}$ with a cyclic decomposition

$$([\gamma A]F_q^n)_{p_k^\gamma} = \langle v_{k,1} \rangle \oplus \cdots \oplus \langle v_{k,l(k)} \rangle$$

where $o(v_{k,i}) = (p_k^\gamma)^{s(k,i)}$ and $s_k = s(k,1) \geq s(k,2) \geq \cdots \geq s(k,l(k))$.

**Proof.** Let $C = \langle v_{k,j} \rangle$ be a cyclic submodule of $[A]F_q^n$ with $o(v_{k,j}) = (p_k)^{s(k,j)}$. It suffices to show that $C$ is also the cyclic submodule of $[\gamma A]F_q^n$ generated by $v_{k,j}$ with $o(v_{k,j}) = (p_k^\gamma)^{s(k,j)}$. Let $f(x) \in F_q[x]$. Then there exists $g(x) \in F_q[x]$ such that $f(x) = g^\gamma(x)$. Considering (2), the action of $f(x)$ on $v_{k,j}$, in view of $[\gamma A]F_q^n$ module, implies

$$f(x) \cdot v_{k,j} = \gamma^{\deg(g)}g(A)v_{k,j} \in C.$$
Conversely, by applying (2), any element of $C$ is an element of the cyclic submodule of $[\gamma A] F_{q}^{n}$ generated by $v_{k,j}$. Thus $C$ is the cyclic submodule of $[\gamma A] F_{q}^{n}$ generated by $v_{k,j}$. Applying (2) once more, we obtain the order of $v_{k,j}$ as an element of $[\gamma A] F_{q}^{n}$ is $o(v_{k,j}) = (p_{k}^{\gamma})^{s(k,j)}$.

The above corollary implies that the primary and cyclic decompositions of $[A] F_{q}^{n}$ and $[\gamma A] F_{q}^{n}$ are the same, even though their orders are twisted. Moreover,

$$
\text{ElemDiv}([\gamma A] F_{q}^{n}) = \{ f^{T}(x) : f(x) \in \text{ElemDiv}([A] F_{q}^{n}) \}.
$$

As a direct implication of Theorem 3.7 and Lemma 2.1, we have the following corollary.

**Corollary 3.10.** (1) Let $\rho_{p_{i}^{\gamma}}$ be the projection from $[\gamma A] F_{q}^{n}$ to $([\gamma A] F_{q}^{n}) p_{i}^{\gamma}$ and $\iota_{p_{k}}$ be the injection from $([A] F_{q}^{n}) p_{k}$ to $[A] F_{q}^{n}$. Then

$$
C(A, \gamma) = \text{Hom}_{F_{q}[x]}([\gamma A] F_{q}^{n}, [A] F_{q}^{n}) = \bigoplus_{i=1}^{m} \bigoplus_{k=1}^{m} \iota_{p_{k}} \text{Hom}_{F_{q}[x]}(([\gamma A] F_{q}^{n}) p_{i}^{\gamma}, ([A] F_{q}^{n}) p_{k}) \rho_{p_{i}^{\gamma}}.
$$

(2) Let $\rho_{(p_{i}^{\gamma})}^{(c)}$ be the projection from $([\gamma A] F_{q}^{n}) p_{i}^{\gamma}$ to $\langle v_{i,c} \rangle$ and $\iota_{(p_{k,d})}$ be the injection from $\langle v_{k,d} \rangle$ to $([A] F_{q}^{n}) p_{k}$. Then

$$
\text{Hom}_{F_{q}[x]}(([\gamma A] F_{q}^{n}) p_{i}^{\gamma}, ([A] F_{q}^{n}) p_{k}) = \bigoplus_{c=1}^{l(i)} \bigoplus_{d=1}^{l(k)} \iota_{(p_{k,d})} \text{Hom}_{F_{q}[x]}((\langle v_{i,c} \rangle, \langle v_{k,d} \rangle)) \rho_{(p_{i}^{\gamma})}^{(c)}.
$$

(3) As a vector space over $F_{q}$,

$$
\dim \text{Hom}_{F_{q}[x]}((\langle v_{i,c} \rangle, \langle v_{k,d} \rangle)) = \deg \gcd(p_{i}^{\gamma}, p_{k}) \min\{s(i, c), s(k, d)\}.
$$

(4) As a vector space over $F_{q}$,

$$
\dim \text{Hom}_{F_{q}[x]}(([\gamma A] F_{q}^{n}) p_{i}^{\gamma}, ([A] F_{q}^{n}) p_{k}) = \sum_{c=1}^{l(i)} \sum_{d=1}^{l(k)} \dim \text{Hom}_{F_{q}[x]}((\langle v_{i,c} \rangle, \langle v_{k,d} \rangle))
$$

Statement in the Corollary 3.10 (3) can be considered as the degree of the greatest common divisor of orders of the involved cyclic submodules. Combining this observation, Corollary 3.10 (2), and Equation (3) we obtain the following theorem.

**Theorem 3.11.** As a vector space over $F_{q}$,

$$
\dim C(A, \gamma) = \sum_{f,g \in \text{ElemDiv}([A] F_{q}^{n})} \deg \gcd(f^{\gamma}, g).
$$

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Example 3.1. Let $F = \mathbb{F}_5$ and $A \in F^{11 \times 11}$ having primary cyclic decomposition

\[ [A]F^{11} = \left[ \langle v_{1,1} \rangle \oplus \langle v_{1,2} \rangle \right] \oplus \left[ \langle v_{2,1} \rangle \oplus \langle v_{2,2} \rangle \right] \oplus \left[ \langle v_{3,1} \rangle \right] \]

where $o(v_{1,1}) = x^2$, $o(v_{1,2}) = x$, $o(v_{2,1}) = (x^2 + x + 1)^2$, $o(v_{2,1}) = x^2 + x + 1$, and $o(v_{3,1}) = x^2 + 2x + 4$. Let us calculate the $\dim C(A, 2)$. First, we have

\[ \text{ElemDiv}([A]F_q^n) = \{ x^2, x, (x^2 + x + 1)^2, x^2 + x + 1, x^2 + 2x + 4 \}, \text{ and} \]

\[ \text{ElemDiv}(2A)F_{11} = \{ x^2, x, (x^2 + 2x + 4)^2, x^2 + 2x + 4, x^2 + 4x + 1 \}. \]

Then applying Theorem 3.11 we obtain

\[ \dim C(A, 2) = 2 + 1 + 1 + 2 + 2 = 9. \]

\[ \diamond \]

Let us consider the case where $A$ is a cyclic matrix. In this case, all the associated primary submodules are cyclic and its minimal polynomial is equal to the characteristic polynomial. Further, the collection of its elementary divisors is in fact a set (not a multiset) in which every two elements are relatively prime. Similar facts are obtained for the twisted matrix $\gamma A$. Hence, if $g$ is an elementary divisor of $A$, then there is at most one elementary divisor $f$ such that $f^\gamma$ and $g$ are not relatively prime. As a result

\[ \prod_{f \in \text{ElemDiv}([A]F_q^n)} \gcd(f^\gamma, g) = \gcd(m_A(x)^\gamma, g). \]

And so

\[ \prod_{f, g \in \text{ElemDiv}([A]F_q^n)} \gcd(f^\gamma, g) = \prod_{g \in \text{ElemDiv}([A]F_q^n)} \gcd(m_A(x)^\gamma, g) = \gcd(m_A^\gamma(x), m_A(x)). \]

Applying Theorem 3.11 we can identify

\[ \dim C(A, \gamma) \sum_{f, g \in \text{ElemDiv}([A]F_q^n)} \deg \gcd(f^\gamma, g) = \deg \gcd(m_A^\gamma, m_A). \]

Thus we obtain [3, Theorem 4.2], i.e., if $A$ is cyclic, the dimension of $C(A, \gamma)$ is equal to the degree of greatest common divisors of the characteristic polynomials of $A$ and $\gamma A$. 13
As a result of the above discussions and results, we obtain a procedure to construct a basis of a twisted centralizer code $C(A, \gamma)$. For this purpose, it is necessary to translate Lemma 2.5 in a vector space context [6, p.166].

**Theorem 3.12.** As a vector space over $\mathbb{F}_q$, a basis for $\text{Hom}_{\mathbb{F}_q[x]}(\langle v_{i,c} \rangle, \langle v_{k,d} \rangle) \neq \{0\}$ is

$$\{\theta, A\theta, A^2\theta, \ldots, A^{\deg p_k \min\{s(i,c),s(k,d)\}-1}\theta\},$$

where $\theta(v_{i,c}) = p_k^{\max\{s(k,d)-s(i,c),0\}} (A)v_{k,d}$.

**Algorithm 3.13.** The following algorithm produces a basis for $C(A, \gamma)$.

1. Given $A \in \mathbb{F}_q^n$ and $\gamma \in \mathbb{F}_q, \gamma \neq 0$.

2. Construct a primary cyclic decomposition of $[A]_{\mathbb{F}_q^n}$. Let it be

$$[A]_{\mathbb{F}_q^n} = \left[\langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,l(1)} \rangle\right] \oplus \cdots \oplus \left[\langle v_{m,1} \rangle \oplus \cdots \oplus \langle v_{m,l(m)} \rangle\right]$$

where $p_1, \ldots, p_m$ are different irreducible monic elements in $\mathbb{F}_q[x]$ and $\phi(v_{k,i}) = p_k^{s(k,i)}$ where $s_k = s(k,1) \geq s(k,2) \geq \cdots \geq s(k,l(k))$ for every $k = 1, \ldots, m$.

We obtain a primary cyclic decomposition of $[\gamma A]_{\mathbb{F}_q^n}$

$$[\gamma A]_{\mathbb{F}_q^n} = \left[\langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,l(1)} \rangle\right] \oplus \cdots \oplus \left[\langle v_{m,1} \rangle \oplus \cdots \oplus \langle v_{m,l(m)} \rangle\right]$$

where $p_1, \ldots, p_m$ are different irreducible monic elements in $\mathbb{F}_q[x]$ in point 2. above and $\phi(v_{k,i}) = (p_i^\gamma)^{s(k,i)}$ where $s_k = s(k,1) \geq s(k,2) \geq \cdots \geq s(k,l(k))$ for every $k = 1, \ldots, m$.

See Corollary 3.8 and Corollary 3.9

3. Set $TCCB = \emptyset$. ($TCCB$ stands for Twisted Centralizers Code’s Bases)

4. For $i$ from 1 to $m$ and $k$ from 1 to $m$.

   - Set $HPB = \emptyset$. ($HPB$ stands for Hom Primary’s Bases)
   - If $p_i^\gamma = p_k$.

     For $c$ from 1 to $l(i)$ and $d$ from 1 to $l(k)$. 

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$$\theta(v_{i,c}) = p_k^{\max\{s(k,d) - s(i,c)\}}(A)v_{k,d}$$ and \(\theta \in \text{Hom}_{F_q[x]}(\langle v_{i,c} \rangle, \langle v_{k,d} \rangle)\).

See Corollary 3.10 and Theorem 3.12

• Else, continue.
  See Lemma 2.2

• \(\text{TCCB} = \text{TCCB} \cup \text{HPB}\).
  See Corollary 3.10

5. Hence, \(\text{TCCB}\) is a basis for \(C(A, \gamma)\).

### 3.4 Examples

Now, let us look at the three examples below.

**Example 3.2.** In this example, we determine a basis for and the dimension of \(C(A, 1)\). Let \(F = \mathbb{F}_3\) and

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}.
\]

We obtain

1. \(m_A(x) = x^2 + x = x(x + 1)\), a product of irreducible polynomials \((x)\) and \((x + 1)\), and

2. \(\ker(A) = \text{span}\{[1 1 1 1]^t\} \) and \(\ker(A + I) = \text{span}\{[2 0 0 1]^t, [2 0 1 0]^t, [2 1 0 0]^t\}\).

Let \(v_{1,1} = [1 1 1 1]^t, v_{2,1} = [2 0 0 1]^t, v_{2,2} = [2 0 1 0]^t, \) and \(v_{2,3} = [2 1 0 0]^t\). Hence a primary cyclic decomposition of \([A]F^4\) is

\[
[A]F^4 = [\langle v_{1,1} \rangle] \oplus [\langle v_{2,1} \rangle \oplus \langle v_{2,2} \rangle \oplus \langle v_{2,3} \rangle]
\]

\([A]F^4_{(x)} \oplus [A]F^4_{(x+1)}\)
where \( o(v_{1,1}) = x \) and \( o(v_{2,1}) = o(v_{2,2}) = o(v_{2,3}) = x + 1 \). Therefore \( s(1,1) = 1, s(2,1) = s(2,2) = s(2,3) = 1 \). This implies

\[
\dim \text{End}_{F[x]}[A]F_4^x = \dim \text{Hom}_{F[x]}(\langle v_{1,1} \rangle, \langle v_{1,1} \rangle)
= \deg(x) \min\{s(1,1), s(1,1)\}
= 1, \quad \text{and}
\]

\[
\dim \text{End}_{F[x]}[A]F_4^{x+1} = \sum_{c=1}^{3} \sum_{d=1}^{3} \dim \text{Hom}_{F[x]}(\langle v_{2,c} \rangle, \langle v_{2,d} \rangle)
= \sum_{c=1}^{3} \sum_{d=1}^{3} \deg(x + 1) \min\{s(2,c), s(2,d)\}
= 9.
\]

Therefore \( \dim C(A, 1) = 10 \).

Now, we will determine a basis for \( C(A, 1) \).

The first element of the basis, say \( X_1 \), is the linear mapping which is the \( F[x] \)-endomorphism on the cyclic submodule \( \langle v_{1,1} \rangle \). Hence,

\[
X_1(v_{1,1}) = v_{1,1}, \quad X_1(v_{2,j}) = 0, \quad \text{for all } j = 1, 2, 3.
\]

The second element of the basis, say \( X_2 \), is the linear mapping which is the \( F[x] \)-endomorphism on the cyclic submodule \( \langle v_{2,1} \rangle \). The third element of the basis, \( X_3 \) is the linear mapping which is the \( F[x] \)-homomorphism from the cyclic submodule \( \langle v_{2,1} \rangle \) to \( \langle v_{2,2} \rangle \). That is

\[
X_3(v_{2,1}) = v_{2,2}, \quad X_3(v_{1,1}) = X_3(v_{2,2}) = X_3(v_{2,3}) = 0.
\]

Continuing that process we obtain \( \{X_1, X_2, \ldots, X_{10}\} \) a basis of \( C(A, 1) \). In this case
Example 3.3. In this example, we determine the dimension of and a basis for \( C(A,2) \). Let \( F = \mathbb{F}_5 \) and

\[
A = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Similar to Example 3.2, we obtain that a primary cyclic decomposition of \([A]^{F^4}\) is

\[
[A]^{F^4} = \underbrace{\langle v_{1,1} \rangle} + \underbrace{\langle v_{1,2} \rangle} + \underbrace{\langle v_{2,1} \rangle}
\]

where \( v_{1,1} = [1 \ 0 \ 0 \ 0]^t, v_{1,2} = [0 \ 1 \ 0 \ 0]^t, \) and \( v_{2,1} = [0 \ 0 \ 0 \ 1]^t, o(v_{1,1}) = o(v_{1,2}) = x + 2 \) and \( o(v_{2,1}) = (x + 4)^2 \), and \( s(1,1) = s(1,2) = 1 \) and \( s(2,1) = 2 \). The above decomposition is also a
primary cyclic decomposition of $[2A]F^4$, with $o(v_{1,1}) = o(v_{1,2}) = x + 4$ and $o(v_{2,1}) = (x + 3)^2$
and $s(1,1) = s(1,2) = 1$ and $s(2,1) = 2$.

This implies
\[
\dim C(A, 2) = \dim \text{Hom}_{F[x]}([2A]F^4_{(x+4)}, [A]F^4_{(x+4)}) = 2.
\]

Therefore \( \dim C(A, 2) = 2 \).

Now, we will determine \( \{X_1, X_2\} \) a basis for \( C(A, 2) \). The first element \( X_1 \) is a linear mapping which is an \( F[x] \)-homomorphism from \( \langle v_{1,1} \rangle \) the submodule of \( [2A]F^4 \) to \( \langle v_{2,1} \rangle \) the submodule of \( [A]F^4 \). That is
\[
X_1(v_{1,1}) = (A + 4I)(v_{2,1}) = [0, 0, 1, 0], \quad X_1(v_{1,2}) = X_1(v_{2,1}) = X_1(2Av_{1,2}) = 0.
\]

Similarly, we obtain \( X_2 \) is a linear mapping
\[
X_1(v_{1,2}) = (A + 4I)(v_{2,1}) = [0, 0, 1, 0], \quad X_1(v_{1,2}) = X_1(v_{2,1}) = X_1(2Av_{1,2}) = 0.
\]

Hence
\[
X_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
X_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

\[\Box\]

**Example 3.4.** In this example, we determine the dimension of and a basis for \( C(A, 2) \). Let \( F = \mathbb{F}_5 \) and
\[
A = \begin{bmatrix}
1 & 4 & 2 & 0 & 4 \\
1 & 1 & 3 & 2 & 0 \\
4 & 4 & 3 & 4 & 1 \\
2 & 4 & 4 & 3 & 3 \\
3 & 4 & 1 & 2 & 1
\end{bmatrix}.
\]

We obtain \( m_A(x) = x(x^2 + 2x + 3)(x^2 + 4x + 2) \) where \( x, (x^2 + 2x + 3) \) and \( (x^2 + 4x + 2) \) are irreducible in \( F[x] \). A primary cyclic decomposition of \( [A]F^5 \) is
\[
[A]F^5 = [\langle v_{1,1} \rangle] \oplus [\langle v_{2,1} \rangle] \oplus [\langle v_{3,1} \rangle].
\]
where $v_{1,1} = [1\ 1\ 4\ 3\ 3]^t$, $v_{2,1} = [1\ 0\ 1\ 1\ 1]^t$, and $v_{3,1} = [1\ 1\ 0\ 4\ 0]^t$, $o(v_{1,1}) = x$, $o(v_{2,1}) = x^2 + 2x + 3$, and $o(v_{3,1}) = x^2 + 4x + 2$, and $s(1, 1) = 1$, $s(2, 1) = 1$, and $s(3, 1) = 1$. Further

$$[2A]F^5 = \langle \langle v_{1,1} \rangle \rangle \oplus \langle \langle v_{2,1} \rangle \rangle \oplus \langle \langle v_{3,1} \rangle \rangle \oplus \langle \langle v_{2,1} \rangle \rangle_{(x^2 + 4x + 2)} \oplus \langle \langle v_{3,1} \rangle \rangle_{(x^2 + 3x + 3)}$$

where $o(v_{1,1}) = x$, $o(v_{2,1}) = x^2 + 4x + 2$, and $o(v_{3,1}) = x^2 + 3x + 3$, and $s(1, 1) = 1$, $s(2, 1) = 1$, and $s(3, 1) = 1$. This implies $\dim C(A, 2) = 1 + 2 = 3$. A basis of $C(A, 2)$ is of the form \{ $X_1, X_2, AX_2$ \}.

Now, we will determine $X_1$ and $X_2$. The first matrix $X_1$ is a linear mapping which is an $F[x]$-homomorphism from $\langle v_{1,1} \rangle$ the submodule of $[2A]F^5$ to $\langle v_{1,1} \rangle$ the submodule of $[A]F^5$. That is

$$X_1(v_{1,1}) = v_{1,1}, \ X_1(v_{2,1}) = X_1(2Av_{2,1}) = X_1(v_{3,1}) = X_1(2Av_{3,1}) = 0.$$

The second matrix $X_2$ is a linear mapping which is an $F[x]$-homomorphism from $\langle v_{2,1} \rangle$ the submodule of $[2A]F^5$ to $\langle v_{3,1} \rangle$ the submodule of $[A]F^5$. That is

$$X_2(v_{2,1}) = v_{3,1}, \ X_2(2Av_{2,1}) = Av_{3,1} = [0\ 0\ 4\ 3\ 0]^t, \ X_2(v_{1,1}) = X_2(v_{3,1}) = X_2(2Av_{3,1}) = 0.$$

Hence

$$X_1 = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \\ 4 & 0 & 2 & 4 & 0 \\ 3 & 0 & 4 & 3 & 0 \\ 3 & 0 & 4 & 3 & 0 \end{bmatrix}, \ X_2 = \begin{bmatrix} 0 & 3 & 4 & 3 & 4 \\ 0 & 3 & 4 & 3 & 4 \\ 2 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \text{and} \ AX_2 = \begin{bmatrix} 4 & 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 3 & 4 \\ 2 & 4 & 3 & 1 & 3 \\ 0 & 4 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\Diamond$

## 4 Concluding remarks

In this paper we have studied twisted centralizer codes and determined a basis and the dimension of the codes. Information about the dimension of a twisted centralizer code is very crucial from the viewpoint of coding theory, since the dimension of a linear code is one of the three important parameters of the code. By identifying a twisted centralizer code as an $\mathbb{F}_q[x]$—module of
homomorphisms, and also by using basic facts concerning structures of \( \mathbb{F}_q[x] \)-module of homomorphisms we have reached our goal. Our module theory approach has helped us in determining a basis for and the dimension of a twisted centralizer code algorithmically.

This paper can open a new viewpoint in dealing with generalized twisted centralizer codes defined in [5] by \( C(A, D) = \{ X \in \mathbb{F}_q^{n \times n} : AX = XAD \} \), where \( D \in \mathbb{F}_q^{n \times n} \). As an example, it is easy to identify that a generalized twisted centralizer code \( C(A, D) \) is nothing but a module of homomorphisms from \( \mathbb{F}_q^n \) as \( \mathbb{F}_q[x] \)-module induced by \( AD \) to \( \mathbb{F}_q^n \) as \( \mathbb{F}_q[x] \)-module induced by \( A \). By investigating some relationship between minimal polynomials of \( AD \) and \( A \) we should be able to identify some structures of generalized twisted centralizer codes. These results, which are in preparation, will be published in a separated paper.

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