A Stronger Vision for Roman Domination in Graphs

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Abstract

Based on the history that the Emperor Constantine decreed that any undefended place (with no legions) of the Roman Empire must be protected by a “stronger” neighbor place (having two legions), a graph theoretical model called Roman domination in graphs was described. A Roman dominating function for a graph \(G = (V, E)\), is a function \(f : V \rightarrow \{0, 1, 2\}\) such that every vertex \(v\) with \(f(v) = 0\) has at least a neighbor \(w\) in \(G\) for which \(f(w) = 2\). The Roman domination number of a graph is the minimum weight, \(\sum_{v \in V} f(v)\), of a Roman dominating function. In this work we introduce a variant of Roman domination, which we call strong Roman domination, where we approach the problem of a Roman domination-type defensive strategy under multiple simultaneous attacks and begin with the study of several mathematical properties of this variant. Specifically, we show that the decision problem regarding the computation of the strong Roman domination number is NP-complete, even when restricted to bipartite graphs. We also give some realizability results for this parameter and show some bounds for it.

Keywords: Domination; Roman domination; Roman domination number; strong Roman domination.

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1 Introduction

The concept of Roman domination in graphs was introduced by Cockayne et al. \([6]\), according to some connections with historical problems of defending the Roman Empire described in 1999 by Steward \([18]\). Namely, given a graph \(G = (V, E)\), a function \(f : V \rightarrow \{0, 1, 2\}\) is a Roman dominating function for \(G\), if for every vertex \(v\) with \(f(v) = 0\) there exists at least a neighbor \(u\) of \(v\), for which \(f(u) = 2\). The weight of a Roman dominating function is given by \(w(f) = f(V) = \sum_{u \in V} f(u)\). The minimum weight among all the Roman dominating functions for \(G\) is called the Roman domination
number of $G$ and is denoted by $\gamma_R(G)$. A Roman dominating function for $G$ of minimum weight is called a $\gamma_R(G)$-function. After this seminal work [6], several investigations have been focused into obtaining properties of this invariant. A few good examples of this are for instance [9, 10, 12, 13, 19].

On the other hand, in order to generalize or improve some particular property of the Roman domination in its standard presentation, some other kind of variants of Roman domination have been introduced and frequently studied. Those variants are frequently related to modifying the conditions in which the vertices are dominated or to adding an extra property to the Roman domination property itself. For instance we remark here variants like the following ones: independent Roman domination [1, 5], edge Roman domination [17], weak Roman domination [7, 14], total Roman domination [16], signed Roman domination [2], Roman $k$-domination [11, 15] and distance Roman domination [3] among others. On the other hand, an interesting version regarding the defense of the “Roman Empire” against multiple attacks was described in [13]. In this article we propose a new version of Roman domination in which we deal with multiple attacks also.

To begin with our work, we first introduce the terminology and notation we shall use throughout the exposition. Unless stated on the contrary, other notation and terminology not explicitly given here could be find in [4]. Let $G = (V, E)$ be an undirected finite graph without loops and multiple edges. The set of vertices and edges of the graph $G$ are denoted by $V(G)$ and $E(G)$, respectively (or $V$ and $E$ for short). The order $n = |V|$ of a graph is the number of vertices of the graph and the size, $m = |E|$, correspond to the number of edges. By $u \sim v$ we mean that $u, v$ are adjacent, i.e., $uv \in E$. For a non-empty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors $v$ has in $X$, or equivalently, $N_X(v) = \{u \in X : u \sim v\}$. In the case $X = V$, we use only $N(v)$, instead of $N_V(v)$, which is also called the open neighborhood of the vertex $v \in V$. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. For any vertex $v$, the cardinality of $N(v)$ is the degree of $v$, denoted by $\delta(v)$. The minimum and maximum degrees of $G$ are the minimum and maximum of the degrees of any vertex of $G$ and are denoted by $\delta$ and $\Delta$, respectively. Given a set of vertices $S \subseteq V$, the open neighborhood of $S$ is the set $N(S) = \{w \in V(G) \setminus S : w \sim v, \text{ for some } v \in S\}$. The closed neighborhood of $S$ is $N[S] = N(S) \cup S$. An universal vertex of $G$ is a vertex which is adjacent to every other vertex of $G$.

A $uv$-path in $G$, joining the (end) vertices $u, v \in V$, is a finite alternating sequence: $u_0 = u, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of different vertices and edges, beginning with the vertex $u$ and ending with the vertex $v$, so that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \ldots, k$. The number of edges in a path is called the length of the path. The length of a shortest $uv$-path is the distance between the vertices $u$ and $v$, and it is denoted by $d(u, v)$. The maximum among all the distances between two vertices in a graph $G$ is denoted by $Diam(G)$, the diameter of $G$. A cycle is a $uu$-path. The length of a shortest cycle in the graph, if any, is called the girth of the graph. Otherwise, the girth of $G$ is not finite.

The set of vertices $D \subseteq V$ is a dominating set if every vertex $v$ not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of any dominating set of $G$ is the domination number of $G$ and is denoted by $\gamma(G)$. A dominating set $D$ in $G$ with $|D| = \gamma(G)$ is called a $\gamma(G)$-set. Notice that a graph having a universal vertex has domination number equal to one.

Let $f$ be a Roman dominating function for $G$ and let $V(G) = \{B_0, B_1, B_2\}$ be the sets of vertices of $G$ induced by $f$, where $B_i = \{v \in V : f(v) = i\}$, for all $i \in \{0, 1, 2\}$. It is clear that for any Roman dominating function $f$ for a graph $G$ of order $n$, we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$. Usually a Roman dominating function $f$ is denoted by $(B_0, B_1, B_2)$. In [6] was proved
that for any graph $G$, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ and in this paper the authors called as Roman graphs to those graphs $G$ satisfying that $\gamma_R(G) = 2\gamma(G)$. Note that if $G$ is not a connected graph with connected components $C_1, C_2, ..., C_t$, then $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(C_i)$. Therefore, from now on we will only consider connected graphs.

The defensive strategy of Roman domination is based in the fact that every place in which there is established a Roman legion (a label 1 in the Roman dominating function) is able to protect itself under external attacks; and that every unsecured place (a label 0) must have at least a stronger neighbor (a label 2). In that way, if an unsecured place (a label 0) is attacked, then a stronger neighbor could send one of its two legions in order to defend the weak neighbor vertex (label 0) from the attack. Two examples of Roman dominating functions are depicted in Figure 1.

![Figure 1: Two Roman dominating functions.](image)

Although these two functions (Figure 1) satisfy the conditions to be Roman dominating functions, they correspond to very different real situations. The unique strong place (2) in the left hand side graph must defend up to 12 weak places from possible external attacks. However, in the right hand side graph, the task of defending the unsecured places is divided into several strong places.

This observation has led us to pose the following question: how many weak places may defend a strong place having two legions? Taking into account that the strong place must leave one of its legions to defend itself, the situation depicted on the left hand side graph of Figure 1 seems to be a not efficient defensive strategy: the Roman domination strategy fails against a “multiple attack” situation. If several simultaneous attacks to weak places are developed, then the only stronger place will be not able to defend its neighbors efficiently. With this motivation in mind, we introduce the concept of strong Roman dominating function as follows. For our purposes, we consider that a strong place should be able to defend itself and, at least half of its weak neighbors.

Consider a graph $G$ of order $n$ and maximum degree $\Delta$. Let $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta+1}{2} \rceil \}$ be a function that labels the vertices of $G$. Let $B_j = \{v \in V : f(v) = j\}$ for $j = 0, 1$ and let $B_2 = V \setminus (B_0 \cup B_1) = \{v \in V : f(v) \geq 2\}$. Then, $f$ is a strong Roman dominating function (from now on SRDF for short) for $G$, if for every $v \in B_0$ there exists a vertex $w \in B_2$ such that $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap B_0| \rceil$. In Figure 2 we show a strong Roman dominating function for each of the graphs shown in Figure 1. The minimum weight, $w(f) = f(V) = \sum_{u \in V} f(u)$, over all the strong Roman dominating functions for $G$, is called the strong Roman domination number of $G$ and we denote it by $\gamma_{SR}(G)$. An SRDF of minimum weight is called a $\gamma_{SR}(G)$-function.
The relationship between the vertices having label two (2) in a Roman dominating function, and those ones having labels with value greater than one in a strong Roman dominating function is not exactly clear, as we can observe throughout the following examples. For instance, a minimum Roman dominating function is shown on the left hand side of Figure 3. However, if we modify the labels with value two (2) to labels with value four (4), then a strong Roman dominating function is obtained, but it has not minimum weight. Now, the left hand side of Figure 4 shows a minimum strong Roman dominating function. Nevertheless, if the vertices with label three (3) are changed to a label with value two (2), then a Roman dominating function is obtained, but it has not minimum weight.
We also can notice that if a graph \( G \) has maximum degree \( \Delta \leq 2 \) and \( w \in B_2 \) for some \( \gamma_{\text{STR}}(G) \)-function, then it follows that \( f(w) = 1 + \left\lceil \frac{1}{2}|N(w) \cap B_0| \right\rceil \leq 1 + \left\lceil \frac{1}{2}\delta(w) \right\rceil \leq 1 + \left\lceil \frac{1}{2}\Delta \right\rceil = 2 \). With this fact in mind, it is straightforward to observe the following result.

**Remark 1.** If \( G \) is a graph of maximum degree \( \Delta \leq 2 \), then it follows that \( \gamma_{R}(G) = \gamma_{\text{STR}}(G) \).

Based on the remark above, from now on, in this work we focus mainly on graphs with maximum degree \( \Delta \geq 3 \).

## 2 Realizability and primary results

First of all we observe that a strong Roman dominating defensive strategy needs, in general, more legions than a Roman dominating one, so the advantage is not to safe resources but to design a stronger empire against external attacks. Under the strong Roman dominating strategy, any strong vertex must be able to defend itself and at least one half of its weak neighbors. The goal is then to deal with this situation by using as few resources (legions) as possible.

In this sense, our first result gives the minimum number of legions which are needed to protect the Roman empire fortifications under the strong Roman domination strategy.

**Proposition 2.** Let \( G \) a graph of order \( n \). Then

\[
\gamma_{\text{STR}}(G) \geq \left\lceil \frac{n + 1}{2} \right\rceil.
\]

**Proof.** Let \( f \) be a \( \gamma_{\text{STR}}(G) \)-function. Let us define

\[
B_1 = \{ w \in V(G) : f(w) = 1 \}, \quad B_2 = \{ w \in V(G) : f(w) \geq 2 \}, \quad B_0 = \{ w \in V(G) : f(w) = 0 \},
\]

\[
B_0^1 = \{ w \in B_0 : |N(w) \cap B_2| = 1 \}, \quad B_0^2 = B_0 \setminus B_0^1.
\]

Previous sets are pairwise disjoint and verify that \( n = |B_1| + |B_2| + |B_0^1| + |B_0^2| = |B_1| + |B_2| + |B_0| \).

So,

\[
\gamma_{\text{STR}}(G) = \sum_{v \in B_1} f(v) + \sum_{v \in B_2} f(v) \\
\geq |B_1| + |B_2| + \left\lfloor \frac{1}{2}|B_0^1| \right\rfloor + \sum_{w \in B_0^2} \left\lfloor \frac{1}{2}|N(w) \cap B_2| \right\rfloor \\
\geq |B_1| + |B_2| + \frac{1}{2}|B_0^1| + |B_0^2| \\
= n - |B_0| + \frac{1}{2}|B_0^1| + |B_0^2| \\
= n - \frac{1}{2}|B_0^1| \\
\geq n - \frac{n - 1}{2} \quad \text{(since} \ |B_0^1| \leq n - 1) \\
= \frac{n + 1}{2}.
\]

Therefore, the result follows, since \( \gamma_{\text{STR}}(G) \) is an integer number. \qed
The bound above is sharp as we can observe in the next remark.

**Remark 3.** If $G$ is a graph of order $n$ and maximum degree $n - 1$, then $\gamma_{Str}(G) = \lceil \frac{n+1}{2} \rceil$.

It is clear that the maximum number of legions which are necessary to defend a graph $G$, under the strong Roman domination strategy, is the order of the graph. In this sense, as usual when a new parameter is introduced, it would be interesting to know if there exist graphs of order $n$, achieving all the possible suitable values for the strong Roman domination number. That is, in concordance with Proposition 2, all the integer numbers in the interval $\{ \lceil \frac{n+1}{2} \rceil, ..., n \}$. Equivalently, we should deal with the problem of realization for the strong Roman domination. That is, given two positive integers $n, p$ such that $\lceil \frac{n+1}{2} \rceil \leq p \leq n$: Is there a graph of order $n$ and strong Roman domination number equal to $p$? Next we partially solve this problem. To do so we need to introduce the following family of graphs.

Let $F$ be the family of graphs of order at least five obtained in the following way. We begin with a star graph $S_{1, n-q-1}$, $0 \leq q \leq \lfloor \frac{n-1}{2} \rfloor$. Then, to obtain a graph $G_{n,q} \in F$ of order $n$, we subdivide $q$ of its edges. Next we first study the strong Roman domination number of graphs of the family $F$.

**Lemma 4.** For any graph $G_{n,q} \in F$, with $n \geq 4$ and $q \leq \lfloor \frac{n-1}{2} \rfloor$,

$$\gamma_{Str}(G_{n,q}) = 1 + q + \lceil \frac{n-q-1}{2} \rceil.$$  

**Proof.** Let $v$ be the central vertex of the star $S_{1, n-q-1}$, used to generate the graph $G_{n,q}$. We consider the following sets of vertices of $G_{n,q}$: $X_1 = \{ x \in N(v) : \delta(x) = 1 \}$, $X_2 = \{ x \in N(v) : \delta(x) = 2 \}$ and $Y = V(G) - X_1 - X_2 - \{ v \}$. It is clear that $|X_2| = |Y|$.

Let $h$ be a function on $V(G_{n,q})$ defined as follows. For any $u \in V(G_{n,q})$,

$$h(u) = \begin{cases} 0, & \text{if } u \in X_1 \cup X_2, \\ 1, & \text{if } u \in Y, \\ 1 + \lceil \frac{n-q-1}{2} \rceil, & \text{if } u = v. \end{cases}$$

It is straightforward to observe that $h$ is a strong Roman dominating function for $G_{n,q}$. Thus, $\gamma_{Str}(G_{n,q}) \leq 1 + q + \lceil \frac{n-q-1}{2} \rceil$.

On the other hand, let $f$ be a $\gamma_{Str}(G_{n,q})$-function such the number of vertices with label one (1) is the minimum possible. We consider the following situations.

**Situation I:** If there exists $x \in X_1$ with $f(x) \neq 0$, then we consider a function $f'$ on $G_{n,q}$ defined as follows. For any $u \in V(G_{n,q})$,

$$f'(u) = \begin{cases} 0, & \text{if } u = x, \\ f(u) + f(x), & \text{if } u = v, \\ f(u), & \text{otherwise}. \end{cases}$$
Hence, we have that
\[
\begin{align*}
f'(V(G^{n,q})) &= \sum_{u \in V(G^{n,q})} f'(u) \\
&= f'(v) + f'(x) + \sum_{u \in V(G^{n,q}) - \{x,v\}} f'(u) \\
&= f(v) + f(x) + \sum_{u \in V(G^{n,q}) - \{x,v\}} f(u) \\
&= f(V(G^{n,q})).
\end{align*}
\]
Thus, \( f' \) is also a \( \gamma_{StR}(G^{n,q}) \)-function.

**Situation II:** We suppose now there exists \( x' \in X_2 \) with \( f(x') \neq 0 \). Hence, let \( y' \in Y \) such that \( y' \sim x' \). If \( f(x') = 1 \), then \( f(y') = 1 \) and we can find a \( \gamma_{StR}(G^{n,q}) \)-function with less vertices having label one (1), a contradiction. Thus, \( f(x') \geq 2 \), which leads to \( f(y') = 0 \), otherwise we can find a function on \( G^{n,q} \) of weight shorter than that of \( f \), which is a contradiction. We consider now a function \( f'' \) on \( G^{n,q} \) defined as follows. For any \( u \in V(G^{n,q}) \),
\[
f''(u) = \begin{cases} 
0, & \text{if } u = x', \\
 f(u) + f(x') - 1, & \text{if } u = v, \\
1, & \text{if } u = y', \\
 f(u), & \text{otherwise}.
\end{cases}
\]
In this case we obtain that
\[
\begin{align*}
f''(V(G^{n,q})) &= \sum_{u \in V(G^{n,q})} f''(u) \\
&= f''(v) + f''(x') + f''(y') + \sum_{u \in V(G^{n,q}) - \{x',y',v\}} f''(u) \\
&= f(v) + f(x') - 1 + 1 + \sum_{u \in V(G^{n,q}) - \{x',y',v\}} f(u) \\
&= f(V(G^{n,q})).
\end{align*}
\]
Thus, \( f'' \) is also a \( \gamma_{StR}(G^{n,q}) \)-function.

As a consequence of the situations described above, we obtain that there exists always a \( \gamma_{StR}(G^{n,q}) \)-function \( g \) such that for any \( x \in X_1 \cup X_2 \) it follows \( g(x) = 0 \), which leads to \( g(y) \geq 1 \) for every \( y \in Y \). We consider now such a function \( g \) and we obtain that
\[
\begin{align*}
\gamma_{StR}(G^{n,q}) &= g(V(G^{n,q})) \\
&= \sum_{u \in V(G^{n,q})} g(u) \\
&\geq g(Y) + g(X_1 \cup X_2) + g(v) \\
&\geq |Y| + 1 + \left\lceil \frac{|X_1| + |X_2|}{2} \right\rceil \\
&= 1 + q + \left\lceil \frac{n - q - 1}{2} \right\rceil.
\end{align*}
\]
Therefore, the result follows. □

Once studied the strong Roman domination number of graphs of the family \( \mathcal{F} \), we are able to present our realizability result on \( \gamma_{\text{StR}}(G) \).

**Theorem 5.** Let \( n, p \) be two positive integers such that \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq p \leq \left\lfloor \frac{3n+3}{4} \right\rfloor \). Then, there exist a graph \( G \) with \( n \) vertices such that \( \gamma_{\text{StR}}(G) = p \).

**Proof.** We begin with the simplest cases \( n \in \{1, 2, 3, 4\} \). If \( n = 1 \), then \( \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{3n+3}{4} \right\rfloor = 1 \). So, it must be \( p = 1 \) and the only possibility for \( G \) is the singleton graph \( K_1 \). If \( n = 2 \), then \( \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{3n+3}{4} \right\rfloor = 2 \). Thus, \( p = 2 \) and a possibility for \( G \) in this case is the complete graph \( K_2 \). If \( n = 3 \), then \( \left\lfloor \frac{n+1}{2} \right\rfloor = 2 \leq p \leq \left\lfloor \frac{3n+3}{4} \right\rfloor = 3 \). Now, if \( p = 2 \), then a possible graph would be the path \( P_3 \). Also, if \( p = 3 \), then a possible graph would be a graph with two components isomorphic to \( K_2 \) and \( K_1 \). If \( n = 4 \), then \( \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{3n+3}{4} \right\rfloor = 3 \). Thus, it must happen \( p = 3 \) and a possibility for a graph \( G \) would be the star graph \( S_{1,3} \).

From now on, we assume \( n \geq 5 \) and we will give a graph \( G \) of order \( n \) with \( \gamma_{\text{StR}}(G) = p \). To this end, we consider a graph \( G^{n,q} \in \mathcal{F} \) with \( q \) satisfying the following equalities: \( n = r + 2q + 1 \) and \( p = 1 + q + \left\lfloor \frac{r+q}{2} \right\rfloor \), in terms of \( n, p, r \) where \( q = |\{w \in N(v) : \delta(w) = 2\}|, r = |\{u \in N(v) : \delta(u) = 1\}| \) and \( v \) is the central vertex of the star \( S_{1,r+q} \) used to generate \( G^{n,q} \).

Notice that, in such a case, by Lemma 4 we have that

\[
\gamma_{\text{StR}}(G^{n,q}) = 1 + q + \left\lfloor \frac{r+q}{2} \right\rfloor = 1 + p - 1 - \left\lfloor \frac{r+q}{2} \right\rfloor + \left\lfloor \frac{r+q}{2} \right\rfloor = p.
\]

To complete the proof, we just need to show now that exists such a graph \( G^{n,q} \in \mathcal{F} \), with \( n = r + 2q + 1 \) and \( p = 1 + q + \left\lfloor \frac{r+q}{2} \right\rfloor \). We consider the following cases.

Case 1: \( n \) even. Thus, \( r \) is odd, which means \( r \geq 1 \), and we have the following possibilities.

Case 1.1: \( q \) odd. So, \( q \geq 1 \) and \( r + q \) is even. Thus, \( \left\lfloor \frac{r+q}{2} \right\rfloor = \frac{r+q}{2} \) and the equality \( p = 1 + q + \left\lfloor \frac{r+q}{2} \right\rfloor \) becomes \( 2p = 2 + 2q + r + q \). By solving the linear system

\[
\begin{cases}
  r + 2q = n - 1 \\
  r + 3q = 2p - 2
\end{cases}
\]

with respect to the variables \( r, q \) we obtain that \( r = 3n - 4p + 1 \) and \( q = 2p - n - 1 \). Since \( r \geq 1 \) and \( q \geq 1 \), we obtain that \( \left\lfloor \frac{n+2}{2} \right\rfloor \leq p \leq \left\lfloor \frac{3n}{4} \right\rfloor \).

Case 1.2: \( q \) even. So, \( q \geq 0 \) and \( r + q \) is odd. Thus, \( \left\lfloor \frac{r+q}{2} \right\rfloor = \frac{r+q+1}{2} \) and, by an analogous procedure to the one above, we obtain that \( r = 3n - 4p + 3 \) and \( q = 2p - n - 2 \). Since \( r \geq 1 \) and \( q \geq 0 \), we have \( \left\lfloor \frac{n+2}{2} \right\rfloor \leq p \leq \left\lfloor \frac{3n+2}{4} \right\rfloor \).

Case 2: \( n \) odd. Thus, \( r \) is even, which means \( r \geq 0 \), and we have the following possibilities which are similar to the Cases 1.1 and 1.2.

Case 2.1: \( q \) odd. So, \( q \geq 1 \), \( r + q \) is odd and it follows \( r = 3n - 4p + 3 \) and \( q = 2p - n - 2 \). Since \( r \geq 0 \) and \( q \geq 1 \), we have \( \left\lfloor \frac{n+3}{2} \right\rfloor \leq p \leq \left\lfloor \frac{3n+3}{4} \right\rfloor \).
Case 2.2: $q$ even. So, $q \geq 0$, $r + q$ is even and it similarly follows $r = 3n - 4p + 1$ and $q = 2p - n - 1$. Since $r \geq 0$ and $q \geq 0$, we obtain that $\left\lceil \frac{n+1}{2} \right\rceil \leq p \leq \left\lfloor \frac{3n+3}{4} \right\rfloor$.

Therefore, by using these values for $r$ and $q$ we can construct a graph $G^{n,q}$ of order $n$ and $\gamma_{StR}(G) = p$ and this makes sense for every $p \in \left\{ \left\lceil \frac{n+1}{2} \right\rceil, \ldots, \left\lfloor \frac{3n+3}{4} \right\rfloor \right\}$ which completes the proof. □

The result above immediately brings up another question. Is it the case that $\gamma_{StR}(G) \leq \left\lfloor \frac{3n+3}{4} \right\rfloor$ for every nontrivial connected graph $G$ of order $n$?

3 The complexity of the strong Roman domination problem

In this section we deal with the following decision problem.

| STRONG ROMAN DOMINATION PROBLEM |
|----------------------------------|
| INSTANCE: A non-trivial graph $G$ and a positive integer $r$ |
| PROBLEM: Deciding whether the strong Roman domination number of $G$ is less than $r$ |

For our purposes of studying the complexity of the STRONG ROMAN DOMINATION PROBLEM (SRD-Problem for short) we will use the following variation of the 3-SAT problem which was proved to be NP-complete in [8]. Let $F$ be a boolean formula with set of variables $U$ and set of clauses $C$. The clause-variable graph of $F$ is defined as the graph $G_F$ with vertex set $V = U \cup C$ and edge set $E = \{(v, c) : v \in V, c \in C$ and $v \in c\}$.

Lemma 6. [8] The problem of deciding whether a boolean formula $F$ is satisfiable is NP-complete, even if

- every variable occurs exactly once negatively and once or twice positively,
- every clause contains two or three distinct variables,
- every clause with three distinct variables contains at least one negative literal, and
- $G_F$ is planar.

In [8], the problem above was called 1-Negative Planar 3-SAT. If there are two equal clauses in a boolean formula $F$, then we can reduce such a formula to another one having all its clauses unique and this does not change the veracity of the formula. Also, we can consider that the number of clauses in the boolean formula is greater than or equal to the number of variables. Therefore, from now on, we will consider a boolean formula on $n$ variables and $m$ pairwise different clauses, with $m \geq n$, and satisfying the conditions of Lemma 6. Such a boolean formula will be represented by $F_3$.

To prove that SRD-Problem is NP-complete, we present a reduction from 1-Negative Planar 3-SAT. The outline of the procedure behind this reduction is the following. We begin with an instance of 1-Negative Planar 3-SAT, that is a boolean formula $F_3$. We consider a planar embedding of its clause-variable graph $G_{F_3}$, and replace each variable vertex of $G_{F_3}$ by a variable gadget, and each
clause vertex of $G_{F_3}$ by a clause gadget. Hence, we identify the vertices of the variable gadgets and the vertices of the clause gadgets in its “corresponding” way and, in this sense, we obtain a planar graph $G_{F_3}$ which we will use as our instance of SRD-Problem.

We consider the following variable gadgets and clause gadgets. Let $X = \{a_1, a_2, ..., a_n\}$ (the variables) and $C = \{C_1, C_2, ..., C_m\}$ (the clauses) be an arbitrary instance of 1-Negative Planar 3-SAT with $m \geq n$. The literals are denoted by $a_i$ (for positive) or $\overline{a_i}$ (for negative). Every clause $C_i$, is represented by a vertex denoted by $c_i$. Every variable $a_i$ is represented by a complete bipartite graph $H_i \cong K_{2,3}$, with partite sets $A_i = \{a_i, \overline{a_i}\}$ (each one for the corresponding literals of $a_i$) and $B_i = \{x_i, y_i, z_i\}$. To construct our graph $G_{F_3}$, we add the edge $a_i\overline{a_i}$, and the two or three edges connecting each clause vertex $c_i$ with the vertices corresponding to the literals in the clause $C_i$. In order to be used while proving our results, since $m \geq n$, we consider a partition of the vertex set of $G_{F_3}$ into $n$ sets $S_i = V(H_i) \cup \{c_j\}$, where either $a_i \in C_j$ or $\overline{a_i} \in C_j$ and a set $Y = V(G_{F_3}) - (\bigcup_{i=1}^n S_i)$, given by those vertices $c_i$ not belonging to any $B_i$ (notice that this $Y$ could be empty, in the case $m = n$). Also notice that $|S_i| = 6$ for every $i \in \{1, ..., n\}$. We must point out that a very similar construction was already used by Paul A. Dreyer [9] in his Ph. D. thesis, to study the complexity of the standard Roman domination.

We will prove that a boolean formula $F_3$ on $n$ variables and $m$ clauses, being an instance of 1-Negative Planar 3-SAT, has a satisfying truth assignment if and only if the graph $G_{F_3}$ satisfies $\gamma_{SR}(G_{F_3}) = 4n$. Notice that $G_{F_3}$ has order $m + 5n$ and size at most $3m + 7n$. Moreover, every vertex $c_i$ has degree two or three (since every clause has two or three literals), every vertex $a_i$ has degree five or six, and every $\overline{a_i}$ has degree five (since each variable appears in $F_3$ exactly once negatively and once or twice positively). Figure 5 shows an example for the case $F_3 = (a_1 \lor a_2 \lor \overline{a_3}) \land (\overline{a_1} \lor a_3) \land (a_1 \lor \overline{a_2} \lor a_3)$. Next we observe some other properties of the graph $G_{F_3}$.

![Figure 5: The graph $G_{F_3}$ where $F_3 = (a_1 \lor a_2 \lor \overline{a_3}) \land (\overline{a_1} \lor a_3) \land (a_1 \lor \overline{a_2} \lor a_3)$.

Remark 7. $G_{F_3}$ is planar.

Proof. Since all the clauses of $F_3$ are distinct, not any copy of the complete bipartite graph $K_{3,3}$ is a subgraph of $G_{F_3}$. On the other hand, not any subgraph of $G_{F_3}$ is isomorphic to the complete graph $K_5$. Therefore, by the Kuratowski’s theorem is obtained that $G_{F_3}$ is planar. □

Remark 8. For every subgraph of $G_{F_3}$ induced by $S_i$, with $i \in \{1, ..., n\}$ and every $\gamma_{SR}(G_{F_3})$-function $f$, it follows $f(S_i) \geq 4$. 

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Proof. Notice that the vertices of the set $B_i \subset S_i$ are pairwise independent and they have the same shared neighbors (the two vertices of $A_i$). Let us suppose that $f(S_j) \leq 3$ for some $j \in \{1, \ldots, n\}$. Since $|S_j| \geq 6$, at least three vertices of $S_j$ have label zero ($0$) by $f$ and two of them cannot be both in $A_j$. As a consequence, at least one vertex of $B_j$ has label zero ($0$) and at least one vertex $u \in \{a_j, \overline{a}_j\} = A_j$, has label at least two ($2$) by $f$. In this sense, let $t \in \{3, 4, 5\}$ be the number of vertices in $S_j$ with label zero ($0$) by $f$. So, we have that $f(u) \geq \left\lceil \frac{t}{2} \right\rceil + 1$. Therefore, it follows that

$$f(S_j) \geq f(u) + f(A_j - \{u\}) + f(B_j) \geq \left\lceil \frac{t}{2} \right\rceil + 1 + 5 - t = 6 - \left\lceil \frac{t}{2} \right\rceil \geq 4,$$

a contradiction. Therefore, $f(S_i) \geq 4$ for every $i \in \{1, \ldots, n\}$. \hfill $\square$

Since, any boolean formula $F_3$ has $n$ variables, the result above leads to the following corollary.

**Corollary 9.** For any boolean formula $F_3$ on $n$ variables and $m$ clauses with $m \geq n$, being an instance of 1-Negative Planar 3-SAT, $\gamma_{SR}(G_{F_3}) \geq 4n$.

Now we present the main result of this section.

**Theorem 10.** Let $F_3$ be a formula on $n$ variables and $m$ clauses, being an instance of 1-Negative Planar 3-SAT. Then $\gamma_{SR}(G_{F_3}) = 4n$ if and only if $F_3$ is satisfiable.

**Proof.** We assume that $F_3$ has satisfying truth assignment. That is, for any variable $a_i$, we have either $a_i$ or $\overline{a}_i$ has assigned the value True. We will define a function $g$ on $V(G_{F_3})$ in the following way. If $a_i$ has the value True, then we define $g(a_i) = 4$. On the contrary, if $a_i$ has the value False, then we define $g(\overline{a}_i) = 4$. For any other vertex $w \in V(G_{F_3})$, we define $g(w) = 0$. Since the definition of this function is based on the satisfying truth assignment of $F_3$, it is straightforward to observe that the function $g$ is a strong Roman dominating function of weight $w(g) = 4n$. Thus, $\gamma_{SR}(G_{F_3}) \leq 4n$ and, by Corollary 9, we have that $\gamma_{SR}(G_{F_3}) = 4n$.

On the other hand, we assume that $\gamma_{SR}(G_{F_3}) = 4n$. Since every vertex $a_i$ has degree five or six, and every $\overline{a}_i$ has degree five (each variable appears in $F_3$ exactly once negatively and once or twice positively), there exists a $\gamma_{SR}(G_{F_3})$-function $h$ such that for every $i \in \{1, \ldots, n\}$, either $h(a_i) = 4$ or $h(\overline{a}_i) = 4$, and any other vertex of $G_{F_3}$ has label zero ($0$) by $h$. Now, if $h(a_i) = 4$, then we set $a_i$ as True. On the contrary, if $h(\overline{a}_i) = 4$, then we set $\overline{a}_i$ as False. Since every vertex $c_j$ corresponding to a clause (notice that it has label zero) is adjacent to at least one vertex with label four ($4$), it follows that the clause is satisfied. Therefore, the formula $F_3$ has a satisfying truth assignment. \hfill $\square$

As a consequence of the theorem above and Remark 7 we have the following result, which completes the proof of the NP-completeness of the SRD-Problem.

**Corollary 11.** STRONG ROMAN DOMINATION PROBLEM is NP-complete, even when restricted to planar graphs.
4 Bounds on the strong Roman domination number

According to the NP-completeness of the SRD-Problem, it is therefore desirable to bound the value of the strong Roman domination number of graphs as tight as possible. In this sense, from now on, we present a few good lower and upper bounds for this invariant in terms of several parameters of the graph.

**Proposition 12.** Let $G$ be a graph on $n$ vertices and with maximum degree $\Delta$. Then

$$\gamma(G) \leq \gamma_{R}(G) \leq \gamma_{Str}(G) \leq \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil \right) \gamma(G).$$

*Proof.* Let $f$ be a minimum strong Roman dominating function for $G$. We define $\tilde{f}$ as follows. For any $v \in V(G)$,

$$\tilde{f}(v) = \begin{cases} 2, & \text{if } f(v) \geq 2, \\ f(v), & \text{otherwise}. \end{cases}$$

It is straightforward to observe that $\tilde{f}$ is a Roman dominating function. Therefore, $\gamma_{R}(G) \leq w(\tilde{f}) \leq w(f) = \gamma_{Str}(G)$.

Now, let $D$ be a dominating set of minimum cardinality and let us define $h(v)$ as follows. For any $v \in V(G)$,

$$h(v) = \begin{cases} 1 + \left\lceil \frac{\Delta}{2} \right\rceil, & \text{if } v \in D, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, $h$ is a strong Roman dominating function for $G$ and, as a consequence, $\gamma_{Str}(G) \leq w(f) = (1 + \left\lceil \frac{\Delta}{2} \right\rceil) \gamma(G)$. \qed

**Proposition 13.** Let $G$ be a graph on $n$ vertices and with maximum degree $\Delta$. Then

$$\gamma_{Str}(G) \leq n - \left\lfloor \frac{\Delta}{2} \right\rfloor.$$

*Proof.* Let $w \in V$ such that $\delta(w) = \Delta$. We define a function $f$ as follows. For any $v \in V(G)$,

$$f(v) = \begin{cases} 1 + \left\lceil \frac{\Delta}{2} \right\rceil, & \text{if } v = w, \\ 0, & \text{if } v \in N(w), \\ 1, & \text{otherwise}. \end{cases}$$

Clearly, $f$ is a strong Roman function for $G$. Therefore, $\gamma_{Str}(G) \leq w(f) = 1 + \left\lceil \frac{\Delta}{2} \right\rceil + n - \Delta - 1 = n - \left\lfloor \frac{\Delta}{2} \right\rfloor$. \qed

The two propositions above give two different upper bounds on $\gamma_{Str}(G)$ which are involving the maximum degree $\Delta$ of the graph. So, an interesting question regarding this could be: Can we compare them with respect to its efficiency? As we can observe in the next two example of graphs, such a question could be not completely clearly addressed.

Notice that the left hand side graph of Figure 6 satisfies that $\left(1 + \left\lceil \frac{\Delta}{2} \right\rceil \right) \gamma(G) = 6 < 7 = n - \left\lfloor \frac{\Delta}{2} \right\rfloor$. Nevertheless, the right hand side graph shown in Figure 6 carries out that $n - \left\lfloor \frac{\Delta}{2} \right\rfloor = 4 < 6 = \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil \right) \gamma(G)$.

Next we continue with another upper bound on the strong Roman domination number of graphs. The proof of such a bound uses a probabilistic approach, which is some kind of generalization of a similar result presented in [6].
**Proposition 14.** Let $G$ be a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$, such that $\left\lceil \frac{\Delta}{2} \right\rceil < \delta$. Then

$$\gamma_{\text{StR}}(G) \leq \frac{(1 + \left\lceil \frac{\Delta}{2} \right\rceil) n}{\delta + 1} \left( \ln \left( \frac{1 + \delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} \right) + 1 \right).$$

**Proof.** Given a set $A \subset V(G)$, we consider the set $B = V(G) - N[A]$. We denote by $A^c = \{v \in V(G) : v \notin A\}$. Notice that $B = A^c \cap N(A)^c = N[A]^c$. Given $v \in V(G)$, we define $p$ as the probability that $v$ would belong to $A$, $p = P[v \in A]$. We consider now $P[v \in B]$ as the probability of the event such that $v$ does not belong to $A$ and also, the neighbors of $v$ are not in $A$. That is, $P[v \in B] = P[v \in A^c \cap N(A)^c] = (1 - p)(1 - p)^{\delta(v)} = (1 - p)^{1+\delta(v)} \leq (1 - p)^{1+\delta}$. According to this, we can approximate the expected values of $|A|$ and $|B|$: $E[|A|] = np$ and $E[|B|] = n(1 - p)^{1+\delta(v)} \leq n(1 - p)^{1+\delta(G)} \leq ne^{-p(1+\delta)}$. Now, let us define a function $f$ on $G$ as follows. For any $v \in V(G)$,

$$f(v) = \begin{cases} 1 + \left\lceil \frac{\Delta}{2} \right\rceil, & \text{if } v \in A, \\ 0, & \text{if } v \in N(A), \\ 1, & \text{if } v \in B. \end{cases}$$

Thus, the expected value of $f(V)$ is:

$$E[f(V)] = \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) E[|A|] + E[|B|] \leq \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) np + ne^{-p(1+\delta)}.$$ 

The last expression attains its minimum value $(1 + \left\lceil \frac{\Delta}{2} \right\rceil) n - n(1+\delta)e^{-p(1+\delta)}$ if and only if $e^{-p(1+\delta)} = \frac{1 + \left\lceil \frac{\Delta}{2} \right\rceil}{1 + \delta + 1}$. Moreover, this last equality has solution $p$ such that $0 < p < 1$, if $\frac{1 + \left\lceil \frac{\Delta}{2} \right\rceil}{1 + \delta + 1} < 1$, which means $\frac{1 + \delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} > 1$. This leads to $\left\lceil \frac{\Delta}{2} \right\rceil < \delta$ and, as a consequence, the solution is $p = \frac{1}{1+\delta} \ln \left( \frac{1 + \delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} \right)$. Therefore, $\gamma_{\text{StR}}(G) \leq (1 + \left\lceil \frac{\Delta}{2} \right\rceil) \frac{n}{1+\delta} \ln \left( \frac{1 + \delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} \right) + (1 + \left\lceil \frac{\Delta}{2} \right\rceil) \frac{n}{1+\delta}$, which leads to the result.

The last proposition set a new upper bound on $\gamma_{\text{StR}}(G)$, which we can compare with some of the previous upper bounds. For instance, we could compare it with Proposition 13. That is, we look for those graphs $G$ where

$$\frac{(1 + \left\lceil \frac{\Delta}{2} \right\rceil) n}{\delta + 1} \left( \ln \left( \frac{1 + \delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} \right) + 1 \right) \leq n - \left\lceil \frac{\Delta}{2} \right\rceil.$$
As an example, we consider for instance those graphs $G$ of order $n$ such that $\left\lceil \frac{\Delta}{2} \right\rceil = \delta - 1$. Hence, some algebraic work leads to that the minimum degree of $G$ must satisfy $\delta \leq \sqrt{n+1}$. So, if $G$ is a graph as described above, then the bound of Proposition 14 is better than Proposition 13 if $\delta \leq \sqrt{n+1}$ and, if $\delta > \sqrt{n+1}$, then Proposition 13 is the best. Some similar conclusions could be obtained by using other different statements, which makes that the process of comparing all the bounds above is not exactly clear.

5 Structure of graphs $G$ with extremal values for $\gamma_{\text{StR}}(G)$

In this section we approach the structure of graphs with limit values of the strong Roman domination number. We begin with the case $\gamma_{\text{StR}}(G)$ equals to the order of $G$.

**Theorem 15.** Let $G$ be a graph or order $n$. Then $\gamma_{\text{StR}}(G) = n$ if and only if $\Delta(G) = 1$.

**Proof.** If $\Delta(G) = 1$, then it is straightforward to observe that $\gamma_{\text{StR}}(G) = n$. On the other hand, assume $\gamma_{\text{StR}}(G) = n$ and let us suppose that there is $w \in V(G)$ with $\delta(w) = \Delta(G) \geq 2$. Let $f$ be a function defined as follows. For any $v \in V(G)$,

$$f(v) = \begin{cases} 1 + \left\lceil \frac{\Delta}{2} \right\rceil, & \text{if } v = w, \\ 0, & \text{if } v \in N(w), \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $f$ is a strong Roman dominating function. Thus, $\gamma_{\text{StR}}(G) \leq w(f) = 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil + n - \Delta(G) - 1 = n - \left\lceil \frac{\Delta}{2} \right\rceil \leq n - 1$, a contradiction. Therefore, $\Delta(G) = 1$. \hfill $\square$

Since the only graphs with $\gamma_{\text{StR}}(G) = n$ are those ones isomorphic to the disjoint union of complete graphs $K_2$ or isolated vertices, it would be desirable to study the case when $\gamma_{\text{StR}}(G) = n - 1$. To do so, we begin with some useful lemmas.

**Lemma 16.** If $\Delta(G) \geq 4$, then $\gamma_{\text{StR}}(G) \leq n - 2$.

**Proof.** Let $v$ be a vertex of maximum degree in $G$ and let $x_1, x_2, x_3, x_4 \in N(v)$. Hence, the function $f$ given by $f(v) = 3$, $f(u) = 0$ if $u \in \{x_1, x_2, x_3, x_4\}$ and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 5 + 3 = n - 2$. Thus $\gamma_{\text{StR}}(G) \leq n - 2$. \hfill $\square$

**Lemma 17.** If $\text{Diam}(G) \geq 5$, then $\gamma_{\text{StR}}(G) \leq n - 2$.

**Proof.** Let $x_1 x_2 \ldots x_6$ be a diametral path in $G$ with $x_1 \sim x_2 \sim \ldots \sim x_6$. Hence, the function $f$ given by $f(u) = 0$ if $u \in \{x_1, x_3, x_4, x_6\}$, $f(u) = 2$ if $u \in \{x_2, x_5\}$ and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 6 + 4 = n - 2$ then $\gamma_{\text{StR}}(G) \leq n - 2$. \hfill $\square$

**Lemma 18.** If there are two vertices $x, y$ in a graph $G$ of degree at least two such that $d_G(x, y) \geq 3$, then $\gamma_{\text{StR}}(G) \leq n - 2$.

**Proof.** Let $\{x_1, x_2\} \in N(x)$ and $\{y_1, y_2\} \in N(y)$. Since $d_G(x, y) \geq 3$, the vertices $x_1, x_2, y_1, y_2$ are different. Hence, the function $f$ given by $f(u) = 0$ if $u \in \{x_1, x_2, y_1, y_2\}$, $f(u) = 2$ if $u \in \{x, y\}$ and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 6 + 4 = n - 2$. So $\gamma_{\text{StR}}(G) \leq n - 2$. \hfill $\square$
Lemma 19. If there are two adjacent vertices $x, y$ of degree at least three in a graph $G$ and $N(x) \cap N(y) = \emptyset$, then $\gamma_{\text{STR}}(G) \leq n - 2$.

Proof. Let $\{x_1, x_2\} \subseteq N(x) - \{y\}$ and $\{y_1, y_2\} \subseteq N(y) - \{x\}$. Hence, the function $f$ given by $f(u) = 0$ if $u \in \{x_1, x_2, y_1, y_2\}$, $f(u) = 2$ if $u \in \{x, y\}$ and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 6 + 4 = n - 2$. Thus $\gamma_{\text{STR}}(G) \leq n - 2$. \qed

Lemma 20. If there are two non adjacent vertices $x, y$ of degree at least three in a graph $G$ and $|N(x) \cap N(y)| \leq 1$, then $\gamma_{\text{STR}}(G) \leq n - 2$.

Proof. Let $\{x_1, x_2\} \subseteq N(x)$ and $\{y_1, y_2\} \subseteq N(y)$. Then, the function $f$ given by $f(u) = 0$ if $u \in \{x_1, x_2, y_1, y_2\}$, $f(u) = 2$ if $u \in \{x, y\}$, and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 6 + 4 = n - 2$ then $\gamma_{\text{STR}}(G) \leq n - 2$. \qed

Once presented the results above, in this point we are able to present some necessary and/or sufficient conditions for a graph $G$ of order $n$ to satisfy $\gamma_{\text{STR}}(G) = n - 1$.

Theorem 21. Let $G$ be a graph or order $n$. If $\gamma_{\text{STR}}(G) = n - 1$, then one of the following situations is satisfied.

(i) $G \cong C_q$ or $G \cong P_q$ with $q = 3, 4, 5,$ or

(ii) $\Delta(G) = 3$, $\text{Diam}(G) \leq 4$, $n \leq 10$ and do not exist two vertices $x, y$ in $G$ such that either

• $x, y$ have degree at least two and $d_G(x, y) \geq 3$, or
• $x, y$ have degree at least three, $x \sim y$ and $N(x) \cap N(y) = \emptyset$, or
• $x, y$ have degree at least three, $x \not\sim y$ and $|N(x) \cap N(y)| \leq 1$.

Proof. We assume $\gamma_{\text{STR}}(G) = n - 1$. Hence, by Theorem 15 and Lemma 16, we have $1 < \Delta(G) \leq 3$. So, we have the following cases.

Case 1: $\Delta(G) = 2$. Hence $G \subseteq C_q$ or $G \subseteq P_q$ for some integer $q$. Now, by Lemma 17 follows that $\text{Diam}(G) \leq 4$. So $G \cong C_q$ or $G \cong P_q$ with $q = 3, 4, 5$.

Case 2: $\Delta(G) = 3$. If $\text{Diam}(G) \geq 5$, by the Lemma 17, we have $\gamma_{\text{STR}}(G) \leq n - 2$. So $\text{Diam}(G) \leq 4$. As a consequence, the maximum possible order of $G$ is ten ($n \leq 10$). We consider now any pair of different vertices $x, y$ of $G$.

If $x, y$ have degree at least two and $d_G(x, y) \geq 3$, then by Lemma 18 we have $\gamma_{\text{STR}}(G) \leq n - 2$, a contradiction. Also, if $x, y$ have degree at least three, $x \sim y$ and $N(x) \cap N(y) = \emptyset$, then by Lemma 19 we have $\gamma_{\text{STR}}(G) \leq n - 2$, a contradiction. Finally, if $x, y$ have degree at least three, $x \not\sim y$ and $|N(x) \cap N(y)| \leq 1$, then by Lemma 20, it follows $\gamma_{\text{STR}}(G) \leq n - 2$, a contradiction again. Therefore the proof is completed. \qed

Now, we give a characterization for the specific case of tress $T$ of order $n$ satisfying that $\gamma_{\text{STR}}(T) = n - 1$. To this end, we introduce the following family of four trees, which we shall denote by $T_4$.  

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Theorem 22. Let $T$ be a tree with $n$ vertices. Then $\gamma_{Str}(T) = n - 1$ if and only if $T \cong P_q$ with $q = 3, 4, 5$ or $T \in T_4$.

Proof. $\Leftarrow$ It is straightforward to observe that $\gamma_{Str}(T) = n - 1$ when $T \cong P_q$ for $q = 3, 4, 5$ or $T \in T_4$.

$\Rightarrow$ Assume $T$ is a tree with $n$ vertices and $\gamma_{Str}(T) = n - 1$. If $\Delta(T) = 2$, then by Theorem 21, $T \cong P_q$ with $q = 3, 4, 5$. Now, if $\Delta(T) = 3$, then $T$ verifies the conditions of Theorem 21 (ii). So, we notice that $n \leq 10$ and there cannot be two vertices of degree three in $T$. As a consequence, $T$ is isomorphic to one of the trees of $T_4$. 

\[\Box\]

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