Practicalities of Renormalizing Quantum Field Theories

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Abstract. We review the techniques used to renormalize quantum field theories at several loop orders. This includes the techniques to systematically extract the infinities in a Feynman integral and the implementation of the algorithm within computer algebra. To illustrate the method we discuss the renormalization of $\phi^4$ theory and QCD including the application of the critical point large $N$ technique as a check on the anomalous dimensions. The renormalization of non-local operators in QCD is also discussed including the derivation of the two loop correction to the Gribov mass gap equation in the Landau gauge.

1 Introduction

Our theoretical understanding of quantum phenomena in particle physics and condensed matter is guided by the underlying quantum field theory. This is formulated in terms of a Lagrangian of fields which have symmetries motivated by experimental observation. For instance, the electromagnetic field governing light leads us to the gauge principle and gauge field theories, and in particular quantum electrodynamics. The properties of this field theory have been generalized to the current theory describing all elementary particles which is the standard model. In order to determine predictions from these quantum field theories requires one to develop the loop expansion of perturbation theory where the interactions of the fields are represented by Feynman diagrams. One of the main properties of such diagrams is that when one evaluates them they are divergent. However, as has been established for a long time there is a systematic and mathematical procedure for handling the resulting infinities which goes under the general title of renormalization. For example, a comprehensive survey of the area is provided in [1]. The topic is by nature a technical one and if one wishes to extract meaningful and accurate predictions from a quantum field theory, one needs to be able to perform the renormalization at a large number of loops. In this article we review some of
the practicalities of achieving this in several field theories which are of main interest. These are scalar $\phi^4$ theory which is relevant to phase transitions in various spacetime dimensions and quantum chromodynamics (QCD) which is the quantum field theory describing the quarks and gluons associated with the strong nuclear force. Our aim is to address the basics in the first part of the article where we discuss various techniques for evaluating the complicated and divergent Feynman integrals which arise in renormalizable field theories, and the way they are handled in practice by using computer algebra and symbolic manipulation packages. Since the renormalization is a complicated procedure we also indicate the rudiments of the important task of how one actually verifies that a calculation is in fact correct. The second part of the article discusses recent problems of interest which apply the techniques discussed in the first part and illustrate their application from a practical point of view. These will primarily be in the context of QCD where, for instance, its three loop renormalization is discussed in a particular non-linear gauge. We will also consider the problem of renormalizing operators which have a particular degree of non-locality.

The article is organized as follows. We outline the main issues concerning renormalization in section 2 before discussing various calculational techniques in section 3. Section 4 surveys the main checks one has on the derivation of the renormalization group functions which lead on to more specific checks for QCD in the large $N_f$ limit in section 5. We discuss more recent aspects of renormalization of QCD in section 6, before examining the issue of how one can treat a particular class of non-local operators in section 7. Finally, various concluding remarks are provided in section 8.

2 Statement of problem

We commence by summarizing from a general point of view the main issues underlying the renormalization of a quantum field theory and the main terminology of the subject used in this article. At the outset it is worth doing this with several basic field theories in mind but initially we will focus on the scalar quartic interaction, $\phi^4$, in four spacetime dimensions. Later we will consider theories with gauge symmetries and in particular QCD. However, the general remarks and comments will apply equally to all field theories which are renormalizable in four or other dimensions. For $\phi^4$ theory the basic Lagrangian is

$$L = \frac{1}{2} (\partial \phi_o)^2 + \frac{g_o}{4!} \phi_o^4 \tag{2.1}$$

where the subscript $o$ denotes that the field $\phi$ and coupling constant $g$ are bare quantities. This observation is essentially the foundation of the problem of the need for renormalization. If one were to naively compute quantum corrections to any Green’s function with (2.1) as the starting point, then the resulting Feynman diagrams would be infinite in four dimensions. This is not surprising as one is using a (local) Lagrangian involving quantum fields defined at the same spacetime point. In other words the variables, parameters or fields of the initial Lagrangian can be regarded as being insufficient or not the correct variables in which to define the problem. To circumvent this, one defines a new set of (renormalized) variables by scaling the original variables with a (multiplicative) factor, known as the renormalization constant. For (2.1) we define these formally as $\phi_o = \sqrt{Z_\phi} \phi$ and $g_o = Z_g g$
which leads to the Lagrangian in terms of renormalized variables

\[ L = \frac{1}{2} Z_{\phi} (\partial \phi)^2 + \frac{g}{4!} Z_{g}^2 Z_{g} \phi^4. \]  

(2.2)

This establishes the framework for renormalization. If one subsequently computes the previously infinite Green’s functions with these, as yet undetermined renormalization constants, then by choosing their value appropriately the infinities can be absorbed into the renormalization constants, \( Z_{\phi} \) and \( Z_{g} \). As the quantum field theory is renormalizable in four spacetime dimensions then the choice one makes for say the divergent 2-point and 4-point Green’s functions of \( \phi^4 \) theory, means that all other \( n \)-point functions are finite. Non-renormalizable theories require additional operators in the Lagrangian over and above the original ones to retain finiteness. By contrast a superrenormalizable theory either does not require all the available renormalization constants or these actually have a finite number of terms when expanded in a coupling constant expansion. Whilst this is the general procedure, one ordinarily fixes the renormalization constants order by order in a coupling constant or perturbative expansion. Though one can still follow this prescription non-perturbatively such as in a lattice regularization.

Whilst this is the overall essence of renormalization, there are clearly several technical issues to be addressed which we now briefly discuss. The choice in redefining the variables may seem ad hoc and has little connection with the real physical world. Indeed one may not only absorb the infinities of the Green’s functions but also an arbitrary finite part. How one practically removes the divergences is known as the renormalization scheme. Although there are a large number of such schemes the most widely used is the modified minimal subtraction scheme denoted by \( \overline{\text{MS}} \).

To ensure that the same physics emerges independently of how the infinities are absorbed, the information contained in the renormalization constants are encoded in renormalization group functions such as \( \gamma_{\phi}(g) \) and \( \beta(g) \) which appear in the renormalization group equation. These functions determine properties of the quantum theory. For instance, in field theories underlying condensed matter problems the critical points relating to the phase transitions are given by the non-trivial values of the coupling constant, \( g_c \), where \( \beta(g_c) = 0 \). In particle physics applications, for example, the solution of the differential equation defined by the \( \beta \)-function determines how the running coupling constant depends on the renormalization scale, which is defined later. For instance, in QCD since the \( \beta \)-function is negative for small values of the coupling constant then the theory is asymptotically free, \([2]\). In other words the (confined) quarks and gluons behave at very large energies as if they were free particles.

In alluding to infinities in general terms so far, for practical calculations one must have a mathematical way of handling them and determining their nature. Moreover, for theories with a classical symmetry which is assumed to be preserved in the quantum theory, the procedure for quantifying or regularizing the infinities must respect the symmetries of the theory. In this context we briefly mention the algebraic renormalization technology developed primarily to determine the general consequences for the renormalization of gauge theories but also for theories with supersymmetry. For example, see \([3]\). In essence algebraic renormalization determines the form of the renormalization constants consistent with the Slavnov-Taylor identities. Two of the most widely used regularization procedures are lattice regularization and dimensional regularization. In the former one replaces or approximates...
the continuum of spacetime by a discrete set of points regularly spaced a distance $a$ apart\(^1\). The presence of $a$ is a feature of any regularization which is that every regularization introduces an arbitrary scale. For lattice regularization one wishes to take $a$ as small as possible in order to be as close as possible to the continuum. Though in practical (and financial) terms this is very costly from the point of view of computer running time. However, the point is that whatever physical quantity one is computing the result has to be independent of the arbitrary regularization scale.

The same feature is true in the other regularization we consider. By contrast, one retains a continuum spacetime but changes the dimension away from four by an infinitesimal amount by defining the spacetime dimension as $d = 4 - 2\epsilon$ where $\epsilon \ll 1$ and $\epsilon > 0$. In this regularization the infinities in Green’s functions manifest themselves as poles in $\epsilon$ and after the poles are removed by some criterion in some scheme, then the $\epsilon \rightarrow 0$ limit can be taken in a non-singular way. However, in changing the spacetime dimension an arbitrary scale is introduced by requiring that the coupling constant remains dimensionless in $d$-dimensions. We now take

$$g_o = \mu^2 Z_g$$

where $\mu$ is the arbitrary renormalization scale. Since the Green’s function after renormalization, $\Gamma^{(n)}(\mu, p, g, \ldots)$, will necessarily depend on $\mu$ in the combination $p^2/\mu^2$ where $p$ is a momentum, we can quantify the essence of the renormalization group as

$$\frac{d}{d\mu} \Gamma^{(n)}(\mu, p, g, \ldots) = 0$$

so that results are independent of the arbitrary scale $\mu$.

One point of clarification needs to be made in the context of defining (dimensional) regularization which is that infinities fall into two classes. Those which are ultraviolet arising from divergences in Feynman integrals at large momenta and those which are infrared coming from divergences in integrals at low loop momenta. In massless theories dimensional regularization regularizes both types of infinities so that it is never clear where the poles in $\epsilon$ originate from. However, infrared infinities can be regularized by a non-zero mass which clearly acts as a low momentum cutoff. This mass can either be present in the original Lagrangian or introduced by hand. In the latter case when the theory is renormalized it can be smoothly set to zero. However, in gauge and supersymmetric theories such extra masses can break the symmetries and if such regularizing masses are introduced, care is required to preserve the symmetry.

We close this section by briefly summarizing several renormalization schemes which are in common use. We have alluded to $\overline{\text{MS}}$ already and we note that the original minimal subtraction scheme, MS, on which it is based requires the removal of the poles in $\epsilon$ only. (Though it is worth noting that MS and $\overline{\text{MS}}$ are not solely tied to dimensional regularization. One can minimally subtract divergences in lattice regularization.) The $\overline{\text{MS}}$ scheme is a modification of MS in that a common finite term, $4\pi e^{-\bar{\gamma}}$ where $\bar{\gamma}$ is the Euler-Mascheroni constant, is removed in addition to the poles as it was observed that in practical calculations that the convergence of perturbative expansions was improved, $\text{H}$.

These two schemes fall into the class of mass independent renormalization schemes. By contrast mass dependent

\(^1\)This is known as a regular or square lattice which is the most widely used. However, lattice regularization is not restricted to square lattices. One can define triangular or random lattices.
schemes are those where, aside from the poles in $\epsilon$ being removed certain finite parts are also removed which involve masses of the fields introduced by putting the external particle masses on-shell or setting a Green’s function to a particular mass independent number at some value of the renormalization scale. Examples of such mass dependent schemes are MOM, [5], on-shell and the modified regularization (RI$'$), [6], schemes. In lattice regularization RI$'$ is one of the main choices as it is constructed in such a way as to minimize computer calculation time. In a similar way to requiring that physical quantities are independent of the renormalization scale introduced through the regularization, they also have to be independent of the renormalization scheme. However, it is possible to convert between different renormalization schemes. Indeed the $\overline{\text{MS}}$ scheme is used as the reference in this context primarily because one can compute much further in the coupling constant expansion than in a mass dependent renormalization scheme. Therefore, on the assumption that with more terms the series is more accurate, it should be closer to the real physical value of the quantity being computed. So, for example, in lattice computations which are non-perturbative but computed in the RI$'$ scheme, one has to convert the results to the $\overline{\text{MS}}$ scheme and match the large momentum behaviour to the perturbative $\overline{\text{MS}}$ result at whichever perturbative order is available. Thus the larger the number of terms in the coupling constant expansion that are available then this will reduce the error in the final numerical estimate. For certain problems this has been achieved at three and four loops. [7] [8].

3 Calculational methods

Having discussed the method of renormalization in general terms, we now detail some of the techniques used to extract regularized infinities from Feynman integrals. We will concentrate on the application of dimensional regularization to massless and massive Feynman diagrams. Throughout we will regard the integrals as expanded near four dimensions with $d = 4 - 2\epsilon$. First, we recall a simple but powerful technique to treat one loop diagrams which is based on a simple integral and attributed to Feynman. Within a Feynman graph one has products of propagators and the basic idea is to write these as an integral using

$$\frac{1}{ab} = \int_0^1 \frac{dx}{(ax + (1-x)b)^2} \quad (3.1)$$

where notionally $a = k^2 - m_1^2$ and $b = (k-p)^2 - m_2^2$ with $k$ regarded as an internal momentum and $p$ as external. With the two momenta within one factor, one can use Lorentz symmetry before performing the $k$-integration. This leaves a function of the Feynman parameter, $x$, which at one loop can be related to known functions. For example, when the masses $m_1$ and $m_2$ are equal the integral can be written in terms of a hypergeometric function. Whilst this is a powerful approach at one loop, it ceases to be practical at higher loops especially when there are more than one mass and external momenta. In practice one chooses a method of calculating the Feynman integral which is tailored to the overall (renormalization) problem of interest.

We will focus on one of these and consider the situation where the field theory involves particles of one non-zero mass and is renormalizable. Then to renormalize the divergent Green’s functions and to extract the infinities, one expands the set of (scalar) Feynman integrals in powers of $p^2$ where $p$ is the external momentum. This method is known as the vacuum bubble expansion. In regarding all integrals
as scalars, we have assumed that one has first decomposed Lorentz tensor integrals into their scalar pieces. To focus on the particulars of the vacuum bubble expansion we consider the simple example of a self energy type integral in a 2-point function which gives

\[
\int \frac{1}{[k^2 - m^2][(k - p)^2 - m^2]} = \int \frac{1}{[k^2 - m^2]^2} - \int \frac{p^2}{[k^2 - m^2]^3} + \frac{4}{d} \int \frac{k^2 p^2}{[k^2 - m^2]^4} + O((p^2)^2) \ .
\] (3.2)

In the final term the \((kp)^2\) numerator factor has been simplified by using Lorentz symmetry. The expansion truncates due to the renormalizability condition which states that since the theory is renormalizable the \(O((p^2)^2)\) terms are finite since otherwise one would require a fourth derivative 2-point term which is forbidden by renormalizability. The resulting integrals on the right hand side of (3.2) are simple vacuum loops which can be evaluated. More appropriately the method can be easily extended to higher loops. For instance, in four dimensions three loop single mass scale vacuum bubbles are known to their finite part and at two loops, three mass scale vacuum graphs are also known to their finite part with respect to \(\epsilon\). See, for example, [9]. One of the benefits of this algorithm is that it can be implemented in computer algebra in an automatic way. Also the method is applicable to higher leg Green’s functions if one wants to renormalize them too. However, in certain problems where the finite part is required exactly in, say, scattering problems, the more recent technique of [10] is appropriate. There each propagator is represented by a Mellin-Barnes integral and contour integration used to evaluate the two loop 4-point functions, for example, to their finite parts. Though currently results have yet to be derived for the physically relevant case of all possible massive propagator combinations.

Next, we note that one distinct advantage of the massive vacuum bubble expansion is that there are no infrared problems as the inherent mass provides a natural infrared regulator. For theories where there are massless fields it may seem that the vacuum bubble approach is inapplicable. However, [11], one can manufacture a fictitious mass \(\bar{\mu}\) which acts as an infrared regulator via the identity

\[
\frac{1}{(k - p)^2} = \frac{1}{[k^2 - \bar{\mu}^2]} + \frac{[2kp - p^2 - \bar{\mu}^2]}{(k - p)^2[k^2 - \bar{\mu}^2]} .
\] (3.3)

Within a Feynman diagram this identity can be used repeatedly with the truncation criterion based on Weinberg’s theorem, [12], for the overall finiteness of a Feynman integral. Like the completely massive case this algorithm can be implemented automatically in computer algebra. Though one ought to be aware of the potential problem of breaking an inherent symmetry of the theory when a non-zero \(\bar{\mu}\) is introduced. For non-abelian gauge theories this is discussed in [11].

Another equally powerful approach for massless field theories, such as QCD, is the use of integration by parts based on Weinberg’s theorem, [12], for the overall finiteness of a Feynman integral. The completely massive case this algorithm can be implemented automatically in computer algebra. Though one ought to be aware of the potential problem of breaking an inherent symmetry of the theory when a non-zero \(\bar{\mu}\) is introduced. For non-abelian gauge theories this is discussed in [11].

Another equally powerful approach for massless field theories, such as QCD, is the use of integration by parts based on the identity, in \(d\)-dimensions, [13],

\[
0 = \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} [k^\mu I(p, k, ....)]
\] (3.4)

where \(I(p, k, ....)\) is the integrand derived from the propagators and vertices. Although such identities can be used for massive integrals, the power lies in the observation that the differentiation can introduce numerator propagator factors to
simplify the momentum topology. By momentum topology we mean that diagram which represents all the scalar propagators. This is not necessarily the same as the actual Feynman diagram topology itself as the cancellation of a denominator factor means that that line would be omitted in the momentum topology. This cancellation is one of the principles which underlies the MINCER algorithm. [14]. This is a package for evaluating massless three loop 2-point functions in dimensional regularization to their finite parts. Moreover, it has been encoded in the symbolic manipulation language FORM, [15, 16]. Essentially at three loops there are fourteen basic integration topologies each with its own integration by parts routine. Though these momentum topologies are not all independent. Whilst primarily used for 2-point functions, MINCER can be applied to 3-point functions when one of the external momenta is set to zero or nullified. However, in this case one must ensure that the nullification does not introduce spurious infrared singularities such as $\int k_1^2 (k_2^2)^2$ which result in poles in $\epsilon$ which being infrared in nature, cannot be distinguished from ultraviolet ones. Despite this technicality there are methods for handling nullification known as infrared rearrangement, [17]. Though this has not been implemented automatically in computer algebra. The main application of MINCER is to the renormalization of four dimensional gauge theories at three loops and in particular QCD.

Given this emphasis on automatic computer algebra we make some specific remarks. The need for such machinery can be gauged from the fact that when one increases the loop order of a calculation, the number of Feynman diagrams increases almost exponentially. As examples we note that the recent full three loop renormalization of QCD in the maximal abelian gauge required the evaluation of 37322 diagrams, [18]. Also the four loop QCD $\beta$-function of [19] in a linear covariant gauge required of the order of 50000 diagrams. Clearly computers are necessary to not only implement the computational algorithm, such as MINCER or the vacuum bubble method, but also to handle the sum of the individual results. Packages such as MINCER, which have been optimized, are essential to having as short a computation time as possible. Indeed a publicly available four loop MINCER would be equally as useful. For such automatic computations the Feynman diagrams themselves need to be generated electronically and QGRAF, [20], has been developed specifically for this purpose. It has various output formats which can be readily converted to the notation used by MINCER before applying the algorithm itself. The implementation of the renormalization procedure can also be performed automatically without the need for the traditional method of subtractions. This method determines the absolute divergence of a diagram by subtracting all subgraph divergences. Instead in the automatic approach, [21], the Green’s function is computed as a function of the bare parameters. Then the counterterms are introduced naturally (and equivalently to the subtraction method) by rescaling by the perturbatively expanded renormalization constants $g_d = Z_g g$ and so on. Once the counterterms have been implemented at a particular loop order, the divergence remaining is then that associated with the renormalization constant of that particular Green’s function.

Finally, we briefly comment on other recent techniques of evaluating Feynman diagrams. One which is also appropriate to n-point functions at two loops for non-zero external momenta is the differential equation method of [22]. The basic idea is to derive a complete set of differential equations at a particular loop order for
relevant momentum topologies. These are then solved with a basic set of master integrals as the boundary conditions which have to be evaluated by direct methods. For completely massless integrals in problems where conformal symmetry is present, such as at a fixed point, the method of uniqueness [23, 24], is also powerful. Essentially when the sum of powers of three momenta in a loop integral satisfy a particular condition depending on the spacetime dimension, then the integral can simply be replaced by products of related propagators and Euler Γ-functions of the original propagator exponents. For large $N_f$ methods, which will be discussed later, this has proved to be extremely powerful in computing information on the renormalization group functions beyond the leading large $N_f$ order and to all orders in perturbation theory.

4 Checks

Performing a renormalization even of a simple quantum field theory to several loop orders can lead to the computation of a large number of Feynman diagrams which are determined by one or other of the methods previously discussed. However, one natural question arises in the ultimate derivation of the renormalization group functions and that is whether the results are correct. For major loop calculations in four dimensional gauge theories to check by repeating the work using independent computer algebra programmes will not only be time consuming but make a large demand in both human and computer resources, which could be used for other problems. The exception to this is the situation where one is extending existing anomalous dimensions and $\beta$-functions to the next loop order. This requires the counterterms which depend on the finite part of the Green’s functions which were renormalized in the previous loop calculation but which are not ordinarily computed to construct the established renormalization constants. Therefore, one invariably has to reconstruct the previous calculation prior to tackling the new loop order. Aside from this check we now address one standard way of assessing whether the calculation one has performed is correct. By this we mean that it satisfies a set of internal consistency checks at the very least and others tailored to the problem in hand. First, the renormalization group provides a clear insight into the structure of the renormalization constants. To illustrate this, if we work from the $\overline{\text{MS}}$ $\beta$-function and anomalous dimension $\gamma_\phi(g)$ given by

$$\beta(g) = (d - 4)g + Ag^2 + Bg^3 + Cg^4 + O(g^5)$$

$$\gamma_\phi(g) = ag + bg^2 + cg^3 + O(g^4)$$

in $d = 4 - 2\epsilon$ dimensions where $g$ is a generic coupling constant then the corresponding renormalization constants must be of the form

$$Z_g = 1 - \frac{Ag}{(d-4)} + \left(\frac{A^2}{(d-4)^2} - \frac{B}{2(d-4)}\right)g^2$$

$$+ \left[ -\frac{A^3}{(d-4)^3} + \frac{7AB}{6(d-4)^2} - \frac{C}{3(d-4)} \right]g^3 + O(g^4)$$

$$Z_\phi = 1 + \frac{ag}{(d-4)} + \left(\frac{(a^2 - aA)}{2(d-4)^2} + \frac{b}{2(d-4)}\right)g^2$$

$$+ \left[ \frac{(2aA^2 - 3a^2A + a^3)}{6(d-4)^3} + \frac{(3ab - 2aB - 2bA)}{6(d-4)^2} - \frac{c}{3(d-4)} \right]g^3$$

$$+ O(g^4)$$

(4.2)
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...to three loops. Clearly the residues of the simple poles are in a one to one correspondence with the coefficients of the renormalization group functions since the MS scheme is used. However, the residues of the other poles depend only on the structure of the previous loop calculations. Therefore, in a new loop calculation these poles are already predetermined and must emerge from the new one loop calculation. This is important in automatic calculations since one determines the full renormalization constant without using the subtraction method. Another internal check is that provided by symmetries of the original theory. Ignoring the technicalities produced by anomalies, in a gauge theory certain renormalization group functions are independent of the choice of gauge in mass independent renormalization schemes. Therefore, working in arbitrary covariant gauges means that the gauge parameter must be absent in the final result. By contrast, when renormalization group functions which depend on the gauge parameter, \( \alpha \), are calculated, the residues of the triple and double poles of the renormalization constants will depend on \( \alpha \) and also be constrained by the conditions above. Briefly, in supersymmetric theories similar conditions emerge. For example, when supersymmetry is unbroken, the anomalous dimensions of the fields in the same supermultiplet have to be equal when the component field version of the Lagrangian is considered. This is a stringent check not only on the renormalization procedure but also on the regularization which must preserve the supersymmetry. Whilst this ensures the renormalization constants are checked to an extent, the residues of the simple poles are not. However, there are partial checks available in some theories from the large \( N_f \) methods developed for scalar field theories. 

This can be illustrated in the case of \( O(N) \) \( \phi^4 \) theory where the scalar field is a vector in \( O(N) \). In MS in \( d \)-dimensions the \( \beta \)-function has the form

\[
\beta(g) = \frac{(d - 4)g}{2} + \left[ N + 8 \right] \frac{g^2}{6} - \left[ 3N + 14 \right] \frac{g^3}{6} + \left[ 33N^2 + 922N + 2960 + 96(5N + 22)\zeta(3) \right] \frac{g^4}{432} + O(g^5)
\]

(4.3)

where \( \zeta(n) \) is the Riemann zeta function. In \( d \)-dimensions there is a non-trivial fixed point of the \( \beta \)-function, \( g_c \), defined by \( \beta(g_c) = 0 \) known as the Wilson-Fisher fixed point. Clearly \( g_c \) will be a function of \( d \) and \( N \). Though near four dimensions with \( d = 4 - 2\epsilon \) we have \( g_c = g_c(\epsilon, N) \), which can be expanded in powers of \( 1/N \) as \( N \to \infty \) giving

\[
g_c = \frac{6\epsilon}{N} + \left[ -48\epsilon + 108\epsilon^2 - 99\epsilon^3 + O(\epsilon^4) \right] \frac{1}{N^2} + O \left( \frac{1}{N^3} \right).
\]

(4.4)

Evaluating the renormalization group invariant universal critical exponent \( \beta'(g_c) \) in the same limit gives

\[
\beta'(g_c) = -\epsilon + \left[ 18\epsilon^2 - 33\epsilon^3 - \frac{5}{2}\epsilon^4 + O(\epsilon^5) \right] \frac{1}{N} + O \left( \frac{1}{N^2} \right).
\]

(4.5)

which encodes the information of the original \( \beta \)-function, albeit in a different way. However, if one can compute \( \beta'(g_c) \) in \( d \)-dimensions order by order in the \( 1/N \) expansion then the coefficients in the polynomials of \( N \) in (4.3) can be read off directly. In \[24, 25, 26, 27\] the \( d \)-dimensional large \( N \) technique was developed for the \( O(N) \) non-linear \( \sigma \) model in \( d = 2 + \epsilon \) dimensions and the first three terms in the \( 1/N \) series for the exponents \( \eta, \nu \) and \( \omega \) have been determined in \( d \)-dimensions.
As this model is in the same universality class as $O(N) \phi^4$ theory at the Wilson-Fisher fixed point then one can partially check off the known coefficients in the corresponding renormalization group functions. For instance, writing the leading large $N$ wave function anomalous dimension as

$$\gamma(g) = \sum_{r=1}^{\infty} \left( c_r N^2 + d_r N + e_{r} \right) N^{r-2} g^{r+1}$$

(4.6)

then with $U_{6,2} = \sum_{n>m>0} (-1)^{n-m} \frac{\eta^n}{n^m m^2}$ as the $(3, 4)$ torus knot number, 28, one finds at ten loops and $O(1/N^3)$ that

$$\omega_9 = [1560674304 \zeta(10) - 12534896640 \zeta(9) + 11070010560 \zeta(8)$$

$$+ 17320181176 \zeta(7) \zeta(3) + 581961984 \zeta(4) - 3411394560 \zeta(1) \zeta(5) \zeta(3)$$

$$- 2684240640 \zeta(6) + 209534976 \zeta^2(5) - 1567752192 \zeta(5) \zeta(4)$$

$$+ 1754664960 \zeta(5) \zeta(3) - 975533568 \zeta(5) - 9289728 \zeta(4) \zeta^2(3)$$

$$+ 1310201856 \zeta(4) \zeta(3) + 1636615872 \zeta(4) - 137158656 \zeta^2(3)$$

$$- 1708996608 \zeta^3(3) + 294403968 \zeta(3)$$

$$- 89800704 U_{6,2} - 3413504331950396973056.$$ 

(4.7)

This requires not only knowledge of $\eta$ at $O(1/N^3)$ but also $\omega$ at $O(1/N^2)$ as this encodes the value of $g_c$ required for $\eta = \gamma(g_c)$. By way of illustration as to the form of such large $N$ exponents we note that with

$$\omega = \mu - 2 + \sum_{i=1}^{\infty} \frac{\omega_i}{N^i}$$

(4.8)

then

$$\omega_1 = (2\mu - 1)^2 \eta_1$$

(4.9)

and, 27,

$$\omega_2 = \left[ - \frac{4(\mu^2 - 5\mu + 5)(2\mu - 3)^2(\mu - 1)^2}{(\mu - 2)^3(\mu - 3)} \left( \Phi(\mu) + \Psi^2(\mu) \right)\right.$$ \n
$$- \frac{16\mu(2\mu - 3)^2}{(\mu - 2)^3(\mu - 3)^2 \eta_1},$$

$$- \frac{3(4\mu^5 - 48\mu^4 + 241\mu^3 - 549\mu^2 + 566\mu - 216)(\mu - 1)^2 \hat{\Theta}(\mu)}{2(\mu - 2)^3(\mu - 3)},$$

$$- [16\mu^{10} - 240\mu^9 + 1608\mu^8 - 6316\mu^7 + 15861\mu^6$$

$$- 25804\mu^5 + 26111\mu^4 - 14508\mu^3 + 2756\mu^2$$

$$+ 672\mu - 144)]/[(\mu - 2)^4(\mu - 3)^2] \hat{\Phi}(\mu)$$

$$+ [144\mu^{14} - 2816\mu^{13} + 24792\mu^{12} - 130032\mu^{11} + 452961\mu^{10}$$

$$- 1105060\mu^9 + 1936168\mu^8 - 2447910\mu^7 + 2194071\mu^6$$

$$- 1320318\mu^5 + 460364\mu^4 - 43444\mu^3 - 26280\mu^2$$

$$+ 8208\mu - 864]/[2(\mu - 3)(\mu - 1)(\mu - 2)^5(\mu - 3)^2 \mu] \eta_1^2$$

(4.10)

where $\eta_1 = -4\Gamma(2\mu - 2)/[\Gamma(2 - \mu) \Gamma(\mu - 1) \Gamma(\mu - 2) \Gamma(\mu + 1)]$ and we have set $d = \mu/2$. The expression for $\omega_2$ contains derivatives of the $\Gamma$-function which are
denoted by
\begin{align*}
\tilde{\Psi}(\mu) &= \psi(2\mu - 3) + \psi(3 - \mu) - \psi(\mu - 1) - \psi(1) \\
\tilde{\Theta}(\mu) &= \psi'(\mu - 1) - \psi'(1) \\
\tilde{\Phi}(\mu) &= \psi'(2\mu - 3) - \psi'(3 - \mu) - \psi'(\mu - 1) + \psi'(1).
\end{align*}

(4.11)

Whilst $O(N)$ $\phi^4$ theory has been examined to several orders in $1/N$, the same technique has been developed for QCD in $[29, 30]$. There the expansion is in terms of the number of quark flavours, $N_f$, and not $N_c$ which is the number of colours. The latter $1/N_c$ expansion deals with the structure of QCD from a completely different point of view.

5 Large $N_f$ QCD

Having outlined the critical point approach for $O(N)$ $\phi^4$ theory, we now concentrate on how one develops the same formalism for QCD in the large $N_f$ limit. Examining the $d$-dimensional MS $\beta$-function for QCD, one finds that there is an equivalent Wilson-Fisher fixed point which can be accessed in powers of $1/N_f$ as a function of $\epsilon$. However, to compute critical exponents in QCD in $d$-dimensions and relate them to the associated anomalous dimensions in perturbation theory at whatever order they are available, one does not compute with the QCD Lagrangian itself. Instead one exploits the properties of the $d$-dimensional fixed point by realising that at this fixed point QCD is in the same universality class as the non-abelian Thirring model (NATM), $[31]$. This is a four-fermi theory which is renormalizable in two dimensions and plays the role of the non-linear $\sigma$ model in the previous $\phi^4$ theory example. The Lagrangian of the NATM is

$$L^{\text{NATM}} = i\bar{\psi}^I \not{D} \psi^I + \frac{\lambda^2}{2} \left( \bar{\psi}^{iJ} T^a_{ij} \gamma^\mu \psi^{jI} \right)^2.$$ (5.1)

where $\lambda$ is the coupling constant of the NATM and is dimensionless in two dimensions. Introducing an auxiliary spin-1 field the Lagrangian can be written as

$$L^{\text{NATM}} = i\bar{\psi}^I \not{D} \psi^I + A^a_{\mu} \bar{\psi}^{iI} T^a_{ij} \gamma^\mu \psi^{jI} - \frac{A^a_{\mu} A^a_\mu}{2\hat{\lambda}}.$$ (5.2)

Clearly there is no field strength term in the non-abelian Thirring model and from a critical point of view this is due to the fact that at this infrared stable fixed point that operator is irrelevant. The critical behaviour is driven by the common quark gluon vertex when one compares with the QCD Lagrangian

$$L^{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^a{\mu\nu} - \frac{1}{2\hat{\alpha}} (\partial^a A^a_{\mu})^2 - \bar{\psi}^a \not{D}_{\mu} \psi^a + i\bar{\psi}^{iI} \not{D} \psi^{iI}.$$ (5.3)

where in this section we use $\hat{\alpha}$ as the covariant gauge parameter with $\hat{\alpha} = 0$ corresponding to the Landau gauge. Within the large $N_f$ critical point formalism the triple and quartic gluon vertices are embedded inside quark loops with respectively three or four external auxiliary spin-1 fields, $[31]$.

As an example of the large $N_f$ critical point formalism, the $O(1/N_f)$ correction to the $\beta$-function can be computed by considering the critical point structure of the gluon 2-point function with the insertion of the operator $G^a_{\mu\nu} G^a{\mu\nu}$, $[29]$. The anomalous dimension of this operator will be related to the critical exponent of the associated coupling constant which is clearly the QCD $\beta$-function when expanded in powers of $\epsilon$ near four dimensions since $\omega = -\beta'(g_c)$. At criticality the propagators of the fields of the non-abelian Thirring model or QCD Lagrangian obey asymptotic
scaling forms where the canonical exponents are derived from the dimensionality of each term in the $d$-dimensional Lagrangian. Thus we have, in the Landau gauge, \cite{29, 30},

$$
\psi(k) \sim \frac{A_k}{(k^2)^{\mu - \alpha}}, \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu - \beta}} \left[ \eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right]
$$

(5.4)

where $A$ and $B$ are momentum independent amplitudes and

$$
\alpha = \mu - 1 + \frac{1}{2}\eta, \quad \beta = 1 - \eta - \chi.
$$

(5.5)

The exponents $\eta$ and $\chi$ are respectively the quark anomalous dimension and the quark gluon vertex anomalous dimension at criticality. In large $N_f$ there are four diagrams contributing to the gluon 2-point function with the operator inserted which are illustrated in Figure 1. The three loop diagram is included since it is of the same order in large $N_f$ as the two loop ones because in the large $N_f$ counting one regards the amplitudes as $A = O(1)$ and $B = O(1/N_f)$. These diagrams are infinite but can be regularized by shifting $\beta$ to $\beta - \Delta$ where $\Delta$ is the regularizing parameter, \cite{24, 25, 26}. It plays a role akin to $\epsilon$ in dimensional regularization and it is important to appreciate that within the critical point large $N_f$ method $\epsilon$ is not the regularization. The calculations are performed in fixed $d$-dimensions. After subtraction of the poles in $\Delta$, using the renormalization procedure of \cite{32}, the scaling behaviour of the remaining finite Green’s function determines $\omega$ and we have, \cite{29},

$$
\omega = (\mu - 2) - [(2\mu - 3)(\mu - 3)C_F
- \frac{(4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14)C_A}{4(2\mu - 1)(\mu - 1)}] \frac{\eta_0}{T(R)N_f}
$$

(5.6)

where $\eta_0 = (2\mu - 1)(2\mu - 2)/[4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)]$. The factors deriving from the group theory are defined by

$$
T^aT^a = C_F I, \quad f^{acd}f^{bcd} = C_A \delta^{ab}, \quad \text{Tr}(T^aT^b) = T_F \delta^{ab}.
$$

(5.7)
Consequently one can verify that the known leading $1/N_f$ coefficients agree with the \( \overline{\text{MS}} \) four loop $\beta$-function of [19]. Moreover, the unknown higher loop coefficients at $O(1/N_f)$ are encoded in (5.6). For instance, if we write the leading large $N_f$ part of the QCD $\beta$-function as

\[
\beta(g) = \beta_0 g^2 + \sum_{n=1}^{\infty} a_{n+1}[T_F N_f]^{n} g^{n+2}
\]  

(5.8)

then at five loops we have

\[
a_5 = [(288\zeta(3) + 214)C_F + (480\zeta(3) - 229)C_A]/31104.
\]  

(5.9)

For $O(N)$ $\phi^4$ theory, it was possible with the critical point method to go to several orders beyond the leading $1/N$ order. For QCD the same is possible, in principle, but one has the added complication of the Lorentz structure associated with fermions and the fact that the structure of the quark gluon vertex is such that the method of uniqueness cannot be applied immediately. Instead one has to use integration by parts and other methods to produce vertices which satisfy the uniqueness condition before that integration technique can be used. Nevertheless the quark anomalous dimension and quark mass anomalous dimensions have been determined at $O(1/N_f^2)$. For the quark anomalous dimension in arbitrary covariant gauge we have,[33]

\[
\eta_1 = \frac{[(2\mu - 1)(\mu - 2) + \hat{\alpha}_i C_F \eta_0]}{[(2\mu - 1)(\mu - 2) T_F]} \quad (5.10)
\]

and

\[
\eta_2 = \left[ \begin{array}{c}
- \mu(2\mu^2 + \mu \hat{\alpha} - 5\mu + 2)(\mu - 1) \left[ \overline{\Phi}^2(\mu) + \Phi(\mu) \right] C_A \\
+ (8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)((2\mu - 1)(\mu - 2) + \mu \hat{\alpha}) \\
\times \frac{\Phi(\mu) C_A}{2(2\mu - 1)(\mu - 2)} \\
+ 3\mu(\mu - 1)[\mu \hat{\alpha} C_A + 2(2\mu^2 + \mu \hat{\alpha} - 5\mu + 2)C_F] \hat{\Theta}(\mu) \\
- [(32\mu^7 \hat{\alpha} - 96\mu^6 \hat{\alpha}^2 - 224\mu^5 \hat{\alpha}^3 + 912\mu^4 \hat{\alpha}^4 - 4\mu^3 \hat{\alpha}^5 - 8\mu^2 \hat{\alpha}^6) \\
+ 704\mu^9 \hat{\alpha}^2 - 3360\mu^8 \hat{\alpha}^3 + 16\mu^7 \hat{\alpha}^4 + 278\mu^6 \hat{\alpha}^5 - 1124\mu^5 \hat{\alpha}^6 \\
+ 6240\mu^4 - 19\mu^3 \hat{\alpha}^2 + 84\mu^2 \hat{\alpha}^3 - 6292\mu^3 \hat{\alpha}^4 \\
+ 6\mu^2 \hat{\alpha}^5 + 230\mu^3 \hat{\alpha}^6 - 222\mu^2 \hat{\alpha}^7 + 3416\mu^4 \hat{\alpha}^8 - 48\mu^5 \hat{\alpha}^9 \\
- 4\mu \hat{\alpha} - 908\mu + 88)C_A \hat{\alpha} \\
- 8(4\mu^5 + 4\mu^4 \hat{\alpha} - 32\mu^3 - 13\mu^2 \hat{\alpha} + 75\mu^3 \hat{\alpha}^2 + 8\mu^2 \hat{\alpha} - 70\mu^2 - 2\mu \hat{\alpha} \\
+ 3\mu - 6)(2\mu - 1)(2\mu - 3)(\mu - 2)C_F \\
+ 1/4(2\mu - 1)(2\mu - 3)(\mu - 2) \eta_0 \left( \frac{C_F \eta_0}{2(2\mu - 1)(\mu - 2) T_F} \right)
\end{array} \right] (5.11)
\]

where

\[
\eta = \sum_{i=1}^{\infty} \frac{\bar{\eta}_i}{N_f^i}.
\]  

(5.12)

The expression for the quark mass anomalous dimension is independent of $\hat{\alpha}$ and in the same notation,

\[
\eta_{\bar{\psi}} = - \frac{2C_F \eta_0}{(\mu - 2) T_F}.
\]  

(5.13)
\[
\eta_{\tilde{\phi}\psi, 2} = - \left[ 2 \left( 3\tilde{\Theta}(\mu) + \frac{(4\mu^3 - 13\mu^2 + 9\mu - 3)}{\mu^2(\mu - 1)^2} \right) C_F + \right. \\
+ \left. \left( \frac{(8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)\tilde{\Psi}(\mu)}{2\mu(\mu - 1)(2\mu - 1)(2\mu - 3)(\mu - 2)} \right) \\
- \left. \frac{[16\mu^6 - 128\mu^5 + 480\mu^4 - 900\mu^3 + 831\mu^2}{4\mu(2\mu - 1)(2\mu - 3)(\mu - 1)^2(\mu - 2)] \right) \\
- \frac{\tilde{\Psi}(\mu) - \Phi(\mu)}{(\mu - 1)(2\mu - 1)(\mu - 2)T_F^2} C_A \right] \left( \frac{\mu(\mu - 1)(2\mu - 1)C_F\eta_0}{(2\mu - 1)(\mu - 2)^2T_F^2} \right). \tag{5.14}
\]

Having focused on the operator associated with the coupling constant of QCD, one can also repeat the same analysis for the analogous operator of the non-abelian Thirring model which is the dimension two gluon mass operator. Whilst it clearly is not a gauge invariant operator its anomalous dimension has an interesting structure. Inserting \( \frac{1}{2}A^a_\mu A^{a\mu} \) into the same Green’s function as \( G^{a \mu \nu} \), the \( O(1/N_f) \) exponent in the Landau gauge is, \[34\],
\[
\eta_{\tilde{\phi}^2} = \frac{- C_A\eta_0^0}{4(\mu - 2)T_F N_f} + O\left( \frac{1}{N_f^2} \right). \tag{5.15}
\]

Interestingly this can be rewritten as
\[
\gamma_{\tilde{\phi}^2}(g_c) = \gamma_{\tilde{\phi}}(g_c) + \gamma_c(g_c) \tag{5.16}
\]
in all dimensions \( d \). It turns out that this is a general property of this operator in the Landau gauge. In \[35\] it was shown that in QCD
\[
\gamma_{\tilde{\phi}^2}(g) = \gamma_{\tilde{\phi}}(g) + \gamma_c(g) \tag{5.17}
\]
to all orders in perturbation theory having first been shown at three loops in \( \overline{\text{MS}} \) by explicit computation. \[39\]. The explicit four loop value is also now available in \( SU(N_c) \). \[37\]. Although this operator is gauge variant, it has been the subject of intense study in recent years as an effective gluon mass term. See, for example, \[38\]. Moreover, it has been studied in other gauges such as the maximal abelian gauge (MAG), \[39, 40, 18\]. In this latter gauge the gauge field is written in terms of its diagonal (centre) and off-diagonal fields
\[
A^A_\mu T^A = A^a_\mu T^a + A^i_\mu T^i \tag{5.18}
\]
where \( [T^i, T^j] = 0, T^i \in \{\text{centre}\}, 1 \leq i \leq N^d_A, 1 \leq a \leq N^d_A \) and \( 1 \leq A \leq N_A \) with \( N^d_A \) and \( N^o_A \) the respective dimensions of the centre of the group and its complement. (Here and in the next section we use the index \( A \) to denote the whole colour group, using the notation usually used in discussing the maximal abelian gauge.) Then in this gauge it turns out that the off-diagonal BRST invariant mass operator
\[
O = \frac{1}{4} A^a_\mu A^{a\mu} + \alpha e^a e^a \tag{5.19}
\]
satisfies a Slavnov-Taylor identity, similar to that for the analogous operator in the Landau gauge, which is. \[39\],
\[
\gamma_O(g) = \gamma_{\tilde{\phi}^2}(g) - \gamma_c(g) \tag{5.20}
\]
where \( e^a \) is the ghost in the centre of the colour group. Whilst this identity has been established on general grounds, for practical calculations, such as studying the condensation of the operator \( O \) of \( \text{5.19} \), one needs the explicit values of the...
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anomalous dimensions. This requires the application of much of the earlier general
discussion on renormalization and the maximal abelian gauge provides a compre-
hensive example to illustrate these general remarks.

6 QCD in non-linear gauges

Given this interest in studying QCD in covariant gauges other than the cano-
ical linear ones, we now focus on the explicit renormalization in the MAG. First,
we define the gauge fixing for the maximal abelian gauge by recalling that for the
Landau gauge. With (5.18) the gauge fixing is given by

$$L_{gf}^{\text{Landau}} = \delta \bar{\delta} \left[ \frac{1}{2} A^A_\mu A^A_\mu + \frac{1}{2} \alpha \bar{c}^A c^A \right]. \tag{6.1}$$

By contrast, the maximal abelian gauge is such that one minimizes the quadratic
form in the off-diagonal sector only by fixing, \[39\],

$$L_{gf}^{\text{MAG}} = \delta \bar{\delta} \left[ \frac{1}{2} A^a_\mu A^a_\mu + \frac{1}{2} \alpha \bar{c}^a c^a \right] + \delta \left[ \bar{c}^i \partial^\mu A^i_\mu \right] \tag{6.2}$$

where again \(\delta\) and \(\bar{\delta}\) are the respective BRST and anti-BRST variations with

\[
\begin{align*}
\delta A^a_\mu &= - (\partial_\mu c^a + gf^{ajc} A^j_\mu c^c + gf^{abc} A^b_\mu c^c + gf^{abk} A^b_\mu c^k) \\
\delta c^a &= gf^{abk} b^k c^a + \frac{1}{2} f^{abc} b^c c^a \\
\delta c^a &= b^a, \delta b^a = 0 \\
\delta A^i_\mu &= - (\partial_\mu c^i + gf^{ibc} A^b_\mu c^c) \\
\delta c^i &= \frac{1}{2} gf^{ibc} c^b c^c \\
\delta b^i &= b^i, \delta b^i = 0 \\
\end{align*}
\tag{6.3}
\]

and

\[
\begin{align*}
\bar{\delta} A^a_\mu &= - (\partial_\mu \bar{c}^a + gf^{ajc} \bar{A}^j_\mu c^c + gf^{abc} \bar{A}^b_\mu c^c + gf^{abk} \bar{A}^b_\mu c^k) \\
\bar{\delta} c^a &= - b^a + gf^{abc} b^c c^a + gf^{abk} b^c c^k + gf^{abc} b^c c^k \\
\bar{\delta} c^a &= gf^{abk} \bar{c}^k b^c + \frac{1}{2} f^{abc} \bar{c}^c b^c \\
\bar{\delta} b^a &= - gf^{abc} \bar{b}^c c^a - gf^{abk} \bar{b}^c b^k + gf^{abk} \bar{b}^c b^k \\
\bar{\delta} A^i_\mu &= - (\partial_\mu \bar{c}^i + gf^{ibc} \bar{A}^b_\mu c^c) \ , \ \bar{\delta} c^i = - b^i + gf^{ibc} c^b c^c \\
\bar{\delta} b^i &= \frac{1}{2} gf^{ibc} c^b c^c, \ \bar{\delta} b^i = - gf^{ibc} \bar{b}^c c^c \ . \tag{6.4}
\end{align*}
\]

The final term in (6.2) is required to avoid a singular ghost propagator. The residual
gauge freedom in the diagonal sector is fixed by applying the usual Landau gauge.
Though one can introduce an extra gauge parameter, \(\bar{\alpha}\), for the inversion to obtain
a centre gluon propagator. Varying \(\delta\) and \(\bar{\delta}\) gives contributions to the interaction
Lagrangian with the remaining terms derived from \((G^A_{\mu\nu})^2\). Thus,

\[
L_{gf} = -\frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 - \frac{1}{2\alpha} (\partial^\mu A_\mu^i)^2 + \bar{c}^a \partial^\mu \partial_\mu c^a + \bar{c}^i \partial^\mu \partial_\mu c^i + \left[ f^{abk} A_\mu^a \phi^b \phi^k - f^{abc} A_\mu^a \phi^c \phi^b - \frac{1}{\alpha} f^{abk} \phi^b A_\mu^a \phi^k + f^{abc} \phi^b A_\mu^a \phi^c - f^{abk} \phi^b A_\mu^a \phi^c \phi^k \right] \right] g + \left[ f^{abcd} A_\mu^a A_\nu^b A_\lambda^c A_\sigma^d \right. \\
\left. - \frac{1}{2} f_{\alpha}^{abcd} A_{\mu}^{a} A_{\nu}^{b} A_{\lambda}^{c} A_{\sigma}^{d} - \frac{1}{2\alpha} f_{\alpha}^{abcd} A_{\mu}^{a} A_{\nu}^{b} A_{\lambda}^{c} A_{\sigma}^{d} + f_{\alpha}^{abcd} A_{\mu}^{a} A_{\nu}^{b} A_{\lambda}^{c} A_{\sigma}^{d} \right] g^2 \right)
\]

where \(f_{ABCD} = f^{ABij} f^{CDi}\). Although this is a much more involved Lagrangian than that which results in the Landau gauge where there are a handful of interactions, it can be shown that it contains the usual covariant gauge fixed Lagrangian.\[39, 40\]. This is crucial to formulating the renormalization of QCD in the gauge Lagrangian which has been established by the algebraic renormalization technology.\[39\]. Moreover, in the limit where the fields deriving from the centre of the colour group are set to zero one recovers the non-linear Curci-Ferrari gauge.\[41\]. In other words the off-diagonal sector of the maximal abelian gauge corresponds to QCD fixed in the Curci-Ferrari gauge. Clearly the Lagrangian includes quartic ghost interactions which are always a feature of a non-linear gauge fixing. Though they do not destroy the renormalizability of the maximal abelian gauge Lagrangian which has been established by the algebraic renormalization technology.\[39\][40]. This is crucial to formulating the renormalization of QCD in the maximal abelian gauge as it produces the Slavnov-Taylor identities originating from the BRST transformations. More importantly it determines the form of the renormalization of the fields and parameters as

\[
A_0^{a\mu} = \sqrt{Z_A} A^a_{\mu} \ , \ A_0^{i\mu} = \sqrt{Z_A} A^i_{\mu} \\
c_0^a = \sqrt{Z_c} c^a \ , \ c_0^i = \sqrt{Z_c} c^i \\
c_0^i = \sqrt{Z_c} c^i \ , \ c_0^a = \sqrt{Z_c} c^a \\
\psi_0 = \sqrt{Z_\psi} \psi \ , \ g_0 = \mu^2 g \ , \ Z_\alpha = Z_\alpha^{-1} Z_A \alpha \ , \ \bar{\alpha}_0 = Z_\alpha^{-1} Z_A \bar{\alpha} .
\]
coupling constant. Hence, the $\beta$-function can be determined without resort to the renormalization of a vertex function. Moreover, this simplification is similar to what happens in QCD fixed in the background field gauge, [42]. Though in practical calculations it reduces substantially the number of diagrams to be determined as well as computation time which can be significant at high loop orders. The key point is that the preliminary analysis from algebraic renormalization not only determines the structure of the renormalization constants consistent with the underlying symmetry, but also provides an efficient route for computing the anomalous dimensions themselves.

Given that the maximal abelian gauge gluon mass operator does not renormalize independently, to have its explicit anomalous dimension requires only knowledge of the diagonal ghost anomalous dimension. Clearly given the large number of interactions in (6.3), a three loop renormalization can only proceed with an automatic computer algebra approach for which the MINCER algorithm is the ideal tool, using the FORM version, [16], and QGRAF, [20]. In completing the full three loop maximal abelian gauge renormalization of QCD, 37322 Feynman diagrams had to be considered, [13]. Though not all of these are non-zero. Some vanish trivially by the group theory structure of a diagram. For instance, where one has a self-energy insertion in a propagator line with a diagonal and off-diagonal field as external to this subgraph, then it will vanish due to

$$f^{ijk} = 0, \quad f^{ijc} = 0, \quad f^{ibc} \neq 0, \quad f^{abc} \neq 0. \quad (6.7)$$

The result of the renormalization procedure yields the anomalous dimensions, [13], which have the structure illustrated by the example,

$$\gamma_{O}(a) = \frac{1}{12 N_{A}^{3}} \left[ N_{A}^\alpha((- 3\alpha + 35)C_{A} - 16 T_{f} N_{f}) + N_{A}^d((- 6\alpha - 18)C_{A}) \right] a$$

$$\begin{aligned}
&+ \frac{1}{96 N_{A}^{2}} \left[ N_{A}^\alpha^2 \left((- 6\alpha^2 - 66\alpha + 898)C_{A}^2 - 560 C_{A} T_{f} N_{f}ight)
- 384 C_{F} T_{f} N_{f} \right] + N_{A}^d\left((- 54\alpha^2 - 354\alpha - 323)C_{A}^2 + 160 C_{A} T_{f} N_{f}\right)
+ \alpha N_{A}^\alpha \left((- 60\alpha^2 - 372\alpha + 510)C_{A}^3\right) a^2 \\
&+ \frac{1}{6912 N_{A}^{2}} \left[ N_{A}^{\alpha 3}((- 162\alpha^3 - 2727\alpha^2 - 2592\zeta(3)\alpha - 18036\alpha)
- 1944\zeta(3) + 302428)C_{A}^3 + (6912\alpha + 62208\zeta(3) - 356032)C_{A}^2 T_{f} N_{f}
+ (- 82944\zeta(3) - 79680)C_{A} C_{F} T_{f} N_{f} + 49408 C_{A} T_{f}^2 N_{f}^2
+ 13824 C_{F}^2 T_{f} N_{f} + 33792 C_{F} T_{f}^2 N_{f}^2\right)
+ N_{A}^\alpha\left((- 2754\alpha^3 + 648\alpha^2\zeta(3) - 28917\alpha^2 - 4212\alpha\zeta(3) - 69309\alpha
+ 37260\zeta(3) - 64544)C_{A}^3 + (25488\alpha + 103680\zeta(3)
- 13072)C_{A}^2 T_{f} N_{f} + (- 165888\zeta(3) + 155520)C_{A} C_{F} T_{f} N_{f}
+ 17920 C_{A} T_{f}^2 N_{f}^2\right)
+ N_{A}^d\left((- 7884\alpha^3 + 22680\alpha^2\zeta(3) - 84564\alpha^2
+ 97524\alpha\zeta(3) - 47142\alpha + 433836\zeta(3) - 56430)C_{A}^3
+ (25056\alpha - 124416\zeta(3) - 18144)C_{A}^2 T_{f} N_{f}\right)
+ N_{A}^\alpha\left((- 6480\alpha^3 + 34992\alpha^2\zeta(3) - 70092\alpha^2 + 8424\alpha\zeta(3)
+ 114912\alpha + 77112\zeta(3) - 161028)C_{A}^3\right] a^3 + O(a^4)
\end{aligned} \quad (6.8)
where \( a = g^2/(16\pi^2) \) is conventionally used as the coupling constant. To be confident that the anomalous dimensions are in fact correct, the standard internal renormalization group checks we discussed above have been shown to hold. Moreover, since the \( \beta \)-function is a gauge independent object then the cancellation of \( \alpha \) and the parameters \( N_A^d \) and \( N_A^o \), provided not only a strong check on the calculation but on the group theory FORM module which handles the consequences of splitting the gauge group into two sectors. Additional checks for this specific computation are provided by noting that the anomalous dimensions of the off-diagonal fields are correctly equivalent for all \( \alpha \) to those of QCD fixed in the non-linear Curci-Ferrari gauge, which was introduced in \([41]\), in the limit \( N_A^d/N_A^o \to 0 \) where the explicit results were given in \([43, 36]\).

7 Renormalization and non-locality

We now turn to more recent aspects of renormalization and that is the renormalization of quantum field theories where a non-locality is present. The general properties of renormalization theory as discussed, for example, in \([44]\), is based on the assumption that the Lagrangian is local. In other words there are no non-local interactions or operators. However, in certain field theories of interest, such as QCD, one encounters important operators which are non-local but whose properties require investigation. We now summarize the status of two such studies where although there is a degree of non-locality present, it falls into a class which does not prevent calculations from being performed. By this we mean that the non-local operators can be rewritten in terms of a finite number of local fields and operators. We refer to this as a localizable non-locality. By contrast, there are operators which, whilst being non-local, do not allow for a finite number of auxiliary fields to lead to a local Lagrangian. We term this a non-localizable non-locality.

The first example is that of the Gribov problem which relates to the impossibility of globally fixing the gauge in a non-abelian gauge theory. This was first pointed out by Gribov, \([45]\), who proceeded to construct a path integral to study the problem. There the non-abelian gauge field is restricted to the region defined by the first zero of the Faddeev-Popov operator. This region, known as the Gribov volume, is of finite size and characterised by the Gribov mass, \( \gamma \). The resulting path integral, \([45]\),

\[
Z = \int DA \delta(\partial^\mu A^a_\mu) \det (-\partial^\nu D^a_\nu) e^{-S} \tag{7.1}
\]

essentially leads to a non-local Lagrangian

\[
L = -\frac{1}{4} G^a_{\mu\nu} G^{a \mu\nu} + \frac{C_A \gamma^4}{2} A^a_\mu \frac{1}{\partial^\nu D_\nu} A^a_\nu - \frac{d N_A \gamma^4}{2g^2} \tag{7.2}
\]

The parameter \( \gamma \) is not independent and satisfies the Gribov gap equation which is, at one loop, \([45]\),

\[
1 = C_A \left[ \frac{5}{8} - \frac{3}{8} \ln \left( \frac{C_A \gamma^4}{\mu^4} \right) \right] a + O(a^2) \tag{7.3}
\]

Subsequently, in \([46, 47]\) Zwanziger managed to localize the original Gribov path integral by introducing a set of extra ghost fields \( \{ \phi^{ab}_\mu, \phi^{\bar{a} b}_\mu, \omega^{ab}_\mu, \omega^{a \bar{b}}_\mu \} \) where the last two fields are anti-commuting. This led to the Gribov-Zwanziger Lagrangian in the
Landau gauge, \[46, 47,\]

\[ L_{\text{GZ}} = L_{\text{QCD}} + \bar{\phi}^{ab\mu}(D_{\nu}\phi_{\mu})^{ab} - \bar{\phi}^{ab\mu}(D_{\nu}\omega_{\mu})^{ab} - g f^{abc}\omega_{\mu\nu}(D_{\nu}\omega_{\mu})^{ab} - \frac{\gamma^2}{\sqrt{2}}\left(f^{abc}A_{\mu}^{a}\phi_{\mu}^{bc} + f^{abc}A_{\mu}^{a}\bar{\phi}_{\mu}^{bc}\right) + \frac{dN_A\gamma^4}{2g^2} \] (7.4)

which clearly involves only a finite number of local interactions. Not only is the Lagrangian local, it is also renormalizable, \[48, 49,\] which implies that one can perform loop computations and study the implications the Gribov parameter has on the infrared structure of QCD. The renormalization structure of (7.4) is interesting in that in the Landau gauge the renormalization constants of the extra ghost fields are not independent with, \[48, 49,\]

\[ Z_{\phi} = Z_{\omega} = Z_c. \] (7.5)

Moreover, the renormalization of the Gribov parameter is not independent in the Landau gauge, satisfying, \[49,\]

\[ \gamma_\gamma(a) = \frac{1}{4}\left[\gamma_A(a) + \gamma_c(a)\right] \] (7.6)

which is similar to the renormalization of \(\frac{1}{4}A_{a\mu}^{a}A_{a\mu}^{a}\). Moreover, the quark, gluon, Faddeev-Popov ghost anomalous dimensions and the \(\beta\)-function are unaltered by the presence of \(\gamma\) and the extra Zwanziger ghost fields in \(\overline{\text{MS}}\). In a general linear covariant gauge one has the additional two loop \(\overline{\text{MS}}\) results for the anomalous dimensions that

\[ \gamma_\phi(a) = \gamma_\omega(a) = \gamma_c(a) \]

\[ \gamma_\gamma(a) = (16 T_F N_f - (35 + 3\alpha) C_A) \frac{a^4}{48} + (192 C_F T_F N_f + 280 C_A T_F N_f - (449 - 3\alpha)) \frac{a^2}{192} + O(a^3) \] (7.7)

which was deduced using Mincer.

Equipped with these properties one can compute the corrections to the one loop mass gap equation which Gribov originally derived. This corresponds to the horizon condition definition of \[47,\] which is equivalent to

\[ f^{abc}(A^{a\mu}(x)\phi_{\mu}^{bc}(x)) = \frac{dN_A\gamma^2}{\sqrt{2}g^2} \] (7.8)

which can be evaluated order by order in perturbation theory using the propagators, \[46, 47, 50,\]

\[ \langle A_{a}^{\mu}(p)A_{b}^{\mu}(-p)\rangle = -\frac{\delta^{ab}p^2}{(p^2)^2 + C_A\gamma^4}P_{\mu\nu}(p) \]

\[ \langle A_{a}^{\mu}(p)\phi_{\mu}^{bc}(-p)\rangle = -\frac{f^{abc\gamma^2}}{\sqrt{2}[p^2]^2 + C_A\gamma^4}P_{\mu\nu}(p) \]

\[ \langle \phi_{\mu}^{ab}(p)\phi_{\nu}^{cd}(-p)\rangle = -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abc}f^{cde}\gamma^4}{p^2(p^2)^2 + C_A\gamma^4}P_{\mu\nu}(p) \] (7.9)
where the gluon propagator is suppressed in the infrared. Hence, at two loops in MS, \[50\],

\[
1 = C_A \left[ \frac{5}{8} - \frac{3}{8} \ln \left( \frac{C_A \gamma^4}{\mu^4} \right) \right] a + \left[ C_A^2 \left( \frac{2017}{768} - \frac{11097}{2048} s_1 + \frac{95}{256} \zeta(2) - \frac{65}{48} \ln \left( \frac{C_A \gamma^4}{\mu^4} \right) \right)^2 + \frac{1137}{2560} \sqrt{5} \zeta(2) - \frac{205}{512} \pi^2 \right] a^2 + O(a^3) \tag{7.10}
\]

where \( s_2 = (2\sqrt{3}/9)\text{Cl}_2(2\pi/3) \). The one and two loop corrections are evaluated using the vacuum bubble approach discussed earlier. Though we note that the basic two loop multiscale vacuum bubble integral

\[
I(m_1^2, m_2^2, m_3^2) = \int \frac{1}{(k^2 - m_1^2)(l^2 - m_2^2)((k - l)^2 - m_3^2)} \tag{7.11}
\]

has to be determined for \( m_1^2 \in \{0, i\sqrt{C_A \gamma^2}, -i\sqrt{C_A \gamma^2}\} \). For instance, the quantities \( s_2 \) and \( \sqrt{5} \zeta(2) \) arise from the finite parts of \( I(i\sqrt{C_A \gamma^2}, i\sqrt{C_A \gamma^2}, i\sqrt{C_A \gamma^2}) \) and \( I(i\sqrt{C_A \gamma^2}, i\sqrt{C_A \gamma^2}, -i\sqrt{C_A \gamma^2}) \) respectively.

One interesting consequence of this gap equation is that it ensures the Kugo-Ojima confinement condition, \[51\], of Faddeev-Popov ghost propagator enhancement is preserved at two loops. If one considers the full ghost propagator to have the form

\[
G_c(p^2) = \frac{\delta^{ab}}{p^2[1 + u(p^2)]} \tag{7.12}
\]

where \( u(p^2) \) represents the radiative corrections, then the Kugo-Ojima confinement criterion is that \( u(0) = -1 \), \[51\]. Hence, the ghost propagator behaves as \( 1/(p^2)^2 \) as \( p^2 \to 0 \). As \( u(p^2) \) corresponds to the loop corrections of the ghost 2-point function, then computing \( u(p^2) \) in the vacuum bubble expansion to two loops one can check if the Kugo-Ojima criterion holds at this order. Using the Gribov-Zwanziger Lagrangian, \[73\], it transpires that the gap equation emerges as the \( u(0) + 1 \) term at \( O(p^2) \) so that \( u(0) = -1 \) precisely at two loops, \[50\].

Finally, we discuss a more recent study of non-locality in QCD, \[52\]. Earlier we considered the gauge variant dimension two operator \( \frac{1}{2} A_{\mu}^{a} A^{a \mu} \) given the current interest in it as a potential mass operator for the gluon. However, if one wishes to have a gauge invariant Lagrangian with gluon mass then this operator is excluded. Instead to preserve gauge invariance and insist on a mass operator, one has to allow for a non-local mass operator. In \[52\], it was pointed out that aside from the one usually associated with the (non-renormalizable) Stueckelberg term,

\[
\tilde{A}_{\mu}^{a} = \min_{(\nu)} \int \left( A_{\nu}^{\mu} \right)^2 \tag{7.13}
\]
there is another independent operator which is
\[ G_{\mu\nu}^a \frac{1}{D^2} G^{a\mu\nu} \] (7.14)
which can be added to the gauge fixed Lagrangian as
\[ L = L_{gf} - \frac{m^2}{4} G_{\mu\nu}^a \frac{1}{D^2} G^{a\mu\nu}. \] (7.15)
This is a localizable operator in the sense we defined earlier. Consequently, it can be localized by introducing the ghost fields \{\bar{B}_{\mu\nu}^a, \bar{H}_{\mu\nu}^a, H_{\mu\nu}^a, \bar{H}_{\mu\nu}^a\} where the last two are anti-commuting, to give
\[ L = L_{gf} + \frac{im}{4} (B_{\mu\nu}^a - \bar{B}_{\mu\nu}^a) G^{a\mu\nu} + \frac{1}{4} \bar{B}_{\mu\nu}^a (D^\sigma D^\sigma B^{a\mu\nu})^a - \frac{1}{4} \bar{H}_{\mu\nu}^a (D^\sigma D^\sigma H^{a\mu\nu})^a. \] (7.16)
This Lagrangian has been analysed by algebraic renormalization and whilst it is renormalizable, it is not multiplicatively renormalizable since new quartic ghost terms are generated through quantum corrections. Moreover, the localized dimension three operator \((B_{\mu\nu}^a - \bar{B}_{\mu\nu}^a) G^{a\mu\nu}\) operator itself mixed into the lower dimensional mass operators \((B_{\mu\nu}^a B^{a\mu\nu} - \bar{H}_{\mu\nu}^a H^{a\mu\nu})\) and \((B_{\mu\nu}^a - \bar{B}_{\mu\nu}^a)^2\). Given that (7.16) is renormalizable, the one loop anomalous dimensions have been determined. Clearly the addition of such an operator ought not to affect the established anomalous dimensions of the gluon, Faddeev-Popov ghost and quark fields and explicit calculations verify this. Essentially within the respective 2-point functions the contributions from the extra ghost fields cancel. Moreover,
\[ \gamma_{B}(a) = \gamma_{H}(a) = (\alpha - 3)C_A a + O(a^2) \] (7.17)
and the anomalous dimension of the mass operator itself can be deduced by inserting the gauge invariant equivalent operator \((B_{\mu\nu}^a - \bar{B}_{\mu\nu}^a) G^{a\mu\nu}\) into the \(B_{\mu\nu}^a\)-gluon 2-point function. The explicit MINCER calculation yields the result, \[ \gamma_{BG}(a) = - \left( \frac{11}{6} C_A - \frac{2}{3} T_F N_f \right) a^2 + O(a^3) \] (7.18)
which is not only independent of the gauge parameter as it ought to be, but is equivalent to the one loop QCD \(\beta\)-function. This is the same as the one loop anomalous dimension of higher dimensional operators with similar Lorentz structure. In other words the operators \(G_{\mu\nu}^a G^{a\mu\nu}, D_{\mu} G_{\nu\sigma}^a D^{\mu} G^{\nu\sigma}, D_{\mu} D_{\nu} G_{\sigma\rho}^a D^{\mu} D^{\nu} G^{\sigma\rho}\) and \(D_{\mu} D_{\nu} D_{\sigma} G_{\rho\theta}^a D^{\mu} D^{\nu} D^{\sigma} G^{\rho\theta}\) all have same one loop anomalous dimension as the non-local operator. We complete this section by remarking that it would be interesting to determine the higher loop corrections to (7.18).

8 Conclusions

We close with general remarks. First, in the initial part of the article we have reviewed the techniques and general structure associated with the renormalization of quantum field theories in the context of \(O(N)\) \(\phi^4\) theory and QCD. One of the interesting aspects of renormalization is the rich interplay between the abstract and mundane exercise of evaluating renormalization group functions and the implications such results have on the underlying physics. For example, the anomalous dimensions when evaluated at a critical point of the \(\beta\)-function lead to predictions for the scaling behaviour of Green’s functions which can be measured in experiments. However, from a technical point of view the critical exponents, when studied in the
large $N$ expansion, also actually complement the checking of the explicit perturbative results. For large loop order computations this plays a role in establishing the correctness of the result as well as providing new information on the as yet undetermined terms in the series at several orders in large $N$. This can be important due to the fact that such high order computations can presently only be undertaken with intense use of computer algebra and symbolic manipulation techniques. The latter part of the article dealt with the application of renormalization to more current computations including the renormalization of QCD in non-linear gauges. Although this follows the application of established techniques, such calculations do provide additional checks on the already determined three loop gauge independent renormalization group functions such as the $\beta$-function. Whilst much of the theory of renormalization is based on the assumption of locality of the initial Lagrangian, we have also touched on very recent calculations of operators which are non-local in structure. Although these fall in the class of localizable non-local operators, it is possible that such studies might open the possibility of studying problems in QCD where such operators are important in probing the infrared structure of the theory. This is more evident, for example, in the Gribov problem where the structure of the ghost propagators satisfies the Kugo-Ojima confinement condition at two loops.

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