Quantum state transfer in graphs with tails

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Abstract

We consider quantum state transfer on finite graphs which are attached to infinite paths. The finite graph represents an operational quantum system for performing useful quantum information tasks. In contrast, the infinite paths represent external infinite-dimensional systems which have limited (but nontrivial) interaction with the finite quantum system. We show that perfect state transfer can surprisingly still occur on the finite graph even in the presence of the infinite tails. Our techniques are based on a decoupling theorem for eventually-free Jacobi matrices, equitable partitions, and standard Lie theoretic arguments. Through these methods, we rehabilitate the notion of a dark subspace which had been so far viewed in an unflattering light.

1 Introduction

Continuous-time quantum walk on graphs has been fundamental to quantum information and computation. Bose [2] studied quantum communication protocols on quantum spin chains using continuous-time quantum walk (see the comprehensive survey by Kay [21]). On the algorithmic side, Farhi et al. [9] designed a quantum algorithm (with provable speedup) for a famous Boolean formula evaluation problem. A notable feature of their algorithm is that it employs continuous-time quantum walk on finite graphs which are attached to infinite paths (or tails). This idea was used subsequently by Childs [4] to show that continuous-time quantum walk is a universal model for quantum computation.
In this work, we explore quantum state transfer (which was the original problem studied by Bose [2]) on finite graphs which are attached to infinite paths (as considered by Farhi et al. [9]). We view the finite graph, on one hand, to represent our internal quantum system for performing basic quantum information tasks. On the other hand, we view the attached infinite paths as a model of an external infinite-dimensional quantum system which has limited (but potentially destructive) interaction with our system. The main question which we investigate here is if our quantum protocols can still work in the presence of this infinite-dimensional system. We provide an affirmative answer to this question.

Our main result shows that, under modest conditions, if perfect state transfer occurs in a finite graph, then perfect state transfer can still occur in the finite graph even when infinite paths are attached. The main technique we employ is based on a decoupling theorem for eventually-free Jacobi matrices as stated by Golinskii [17]. This technique shows that the notion of a dark subspace is useful as a tool to construct protected computational subspace. This is in contrast to previous works which tried to mitigate the existence of these dark subspaces [23].

A second technique we use involves basic representation theory of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$. This allows us to construct perfect state transfer in the Cartesian square of Krawtchouk chains (or quotients of the $n$-cube) and the $n$-cube itself when both are attached to an infinite tail. These constructions exhibit perfect state transfer between multiple pairs of quantum states on the same graph. Through these, we establish connections between the dark subspaces and the walk modules of the Terwilliger algebra of the $n$-cube (see [28]).

Finally, we utilize the dual-rail encoding useful for heralded perfect state transfer in quantum spin chains (see [3, 21]). This method exploits the presence of an anti-symmetric subspace in our graphs with tails. We view this method in the context of a Cartesian product a perfect state transfer graph $G$ with $K_2$. This allows us to protect any graph with perfect state transfer from an infinite tail probe by simply “doubling” the finite graph.

2 Preliminaries

We review some basic notation and terminology which will be used throughout. The set of integers is denoted $\mathbb{Z}$ while $\mathbb{Z}^+$ denotes the set of positive integers.

The all-one $n \times m$ matrix is denoted $J_{n,m}$, for $n, m \geq 1$; we simply use
$J_n$ if $n = m$. The identity matrix of order $n$ is denoted $I_n$. Whenever the dimensions are clear from context, we omit them for brevity. We use $1_S$ to denote the characteristic vector of a subset $S$. The unit vectors are denoted $e_j$ or $e(j)$ (with an implied dimension). The unit matrix $E_{j,k}$ (which is $e_j e_k^T$) is zero everywhere except at entry $(j,k)$ where it is 1 (again with implied dimensions).

We adopt standard asymptotic notation where $o(f_n)$ denote functions $g_n$ for which $g_n/f_n \to 0$, $O(f_n)$ denote functions $g_n$ for which $g_n/f_n$ is bounded from above by a constant, as $n \to \infty$; see Janson et al. [20].

2.1 Graphs

The graphs we study are mainly simple and undirected (unless stated otherwise). For a graph $G$, we denote its vertex and edge sets as $V(G)$ and $E(G)$, respectively. The adjacency matrix of $G$ is a matrix $A(G)$ whose $(j,k)$ entry is 1 if $(j,k) \in E(G)$ and 0 otherwise. The neighborhood of a vertex $u$ in $G$ is denoted $N_G(u) = \{ v \in V(G) : (u,v) \in E(G) \}$. The degree $\deg(u)$ of vertex $u$ is the cardinality of $N_G(u)$. We use standard notation to denote common families of graphs: $K_n$ for complete graphs (cliques), $K_n$ for the empty graphs (cocliques), $P_n$ for paths, $Q_n$ for the binary $n$-cubes. See Godsil and Royle [16] for further background on graph theory.

Given two graphs $G$ and $H$, the join $G + H$ is the graph obtained by taking a disjoint union of $G$ and $H$ and then adding all edges $(g,h)$ for every $g \in G$ and $h \in H$. The adjacency matrix of $G + H$ is given by

$$A(G + H) = \begin{pmatrix} A(G) & J_{n,m} \\ J_{m,n} & A(H) \end{pmatrix}$$

where $G$ has $n$ vertices and $H$ has $m$ vertices. The cone of a graph $G$ is given by $\hat{G} := K_1 + G$. By extension, the $m$-cone of $G$ is given by $\hat{K}_m + G$.

The one-sum of two graphs $G$ and $H$ at a vertex $u$ is obtained by identifying a vertex of $G$ with a vertex of $H$; thus, $V(G) \cap V(H) = \{ u \}$. The adjacency matrix of this one-sum is given by

$$\begin{pmatrix} A(G \setminus u) & 1_{N_G(u)} & 0 \\ 1_{N_G(u)}^T & 0 & 1_{N_H(u)}^T \\ 0 & 1_{N_H(u)} & A(H \setminus u) \end{pmatrix}.$$

Let $G$ be a graph on $n$ vertices and let $\mathcal{Y} = \{ H_i(r_i) : i = 1, \ldots, n \}$ be a collection of rooted graphs (where $r_i$ is a distinguished root vertex in $H_i$). The rooted product of $G$ with $\mathcal{Y}$, denoted $G^\mathcal{Y}$, is a graph obtained from the
one-sum of $G$ with $H_i$ by identifying vertex $i$ of $G$ with vertex $r_i$ in $H_i$, for each $i=1,\ldots,n$; see Godsil and McKay [14].

Given a graph $G=(V,E)$, a vertex partition $\pi=(V_j)$ of $G$ is called equitable if $V=\bigcup_j V_j$ is a disjoint union, each induced subgraph $G[V_j]$ is $d_j$-regular for some $d_j$, and each induced bipartite subgraph $G[V_j,V_k]$ is $(d_{jk},d_{kj})$-biregular for some $d_{jk}$ and $d_{kj}$. We call each $V_j$ a cell of the equitable partition $\pi$. The partition matrix of $\pi$ is given by

$$Q=(q_{uj})$$

where $q_{uj}=\mathbb{1}_{\{u\in V_j\}} \cdot |V_j|^{-1/2}$. Then, the quotient graph $G/\pi$ is a weighted graph whose adjacency matrix is $A(G/\pi)=Q^\dagger A(G)Q$. For a fixed vertex $u$ of $G$, the equitable partition relative to $u$ is an equitable partition $\pi$ of $G$ where $\{u\}$ is a cell of $\pi$. In the case where $A(G/\pi)$ is a symmetric tridiagonal matrix (or Jacobi matrix) we say $\pi$ forms a distance partition relative to $u$.

### 2.2 Linear operators

We will allow our graphs to be infinite with a countable vertex set (see [24, 25, 15]). A main example is the infinite path $P_\infty$ with $\mathbb{Z}^+$ as vertices and with edges connecting consecutive positive integers. The adjacency matrix of $P_\infty$ is the free Jacobi matrix $J_0$ given by

$$J_0 = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
1 & 0 & 1 & \cdots \\
0 & 1 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}.$$

A Jacobi matrix $J$ is called eventually-free if $J_0$ can be obtained from $J$ by the removal of the first $k$ rows and $k$ columns, for some finite $k$. In this case, we also say $J$ is an extension of $J_0$ (see Golinskii [17]).

Let $\{G_n\}$ be a family of infinite graphs where each $G_n$ is parameterized by a finite subgraph of size $n$. As an example, consider the infinite lollipop graph $L_n$ which is the one-sum of the complete graph $K_n$ with the infinite path $P_\infty$ (see Figure 1). We say $G$ is locally finite if $\text{deg}(u)$ is finite for each $u \in V$. Let $\text{deg}(G) = \sup\{\text{deg}(u) : u \in V\}$.

We consider the complex separable Hilbert space $\ell^2(V)$ which is equipped with the inner product $\langle x,y \rangle = \sum_{u \in V} x_u \overline{y_u}$ for $x,y \in \mathbb{C}^V$ and an induced norm $\|x\|_2 = \sqrt{\langle x,x \rangle}$. Recall $\ell^2(V)$ consists of vectors $x$ with $\|x\|_2 < \infty$. A complete orthonormal system for $\ell^2(V)$ is given by the standard basis $\{e_v : v \in V\}$.

The adjacency matrix $A$ of $G$ is defined on the basis vectors as $Ae_v = \sum_{u \in V} a_{u,v} e_u$ (the sum converges if $G$ is locally finite); or, $\langle e_u, Ae_v \rangle = a_{u,v}$. If
deg(G) < ∞, then the adjacency matrix A is a bounded self-adjoint operator (see Mohar and Woess [25]). We also say that the adjacency operator of G admits a matrix representation relative to the standard basis (see [1], section 26).

The spectrum \( \sigma(A) \) of A is the set of all \( \lambda \in \mathbb{C} \) for which \( \lambda I - A \) is not invertible. The point spectrum \( \sigma_p(A) \) is the set of all \( \lambda \in \mathbb{C} \) for which \( \lambda I - A \) is not one-to-one. Any element of \( \sigma_p(A) \) is called an eigenvalue of A. The continuous spectrum \( \sigma_c(A) \) is the set of all \( \lambda \in \mathbb{C} \) for which \( \lambda I - A \) is a one-to-one mapping of \( H \) onto a dense proper subspace of \( H \). If A is bounded and self-adjoint, then \( \sigma(A) \subset \mathbb{R} \) and \( \sigma(A) = \sigma_p(A) \cup \sigma_c(A) \); that is, A has no residual spectrum. Our primary sources for functional analysis are [1, 27].

**Theorem 1.** (Spectral theorem) If A is a bounded self-adjoint operator on a complex Hilbert space \( H \), then there exists a unique resolution of the identity \( E \) on the Borel subsets of \( \sigma(A) \) which satisfies

\[
A = \int_{\sigma(A)} \lambda \, dE(\lambda).
\]

Moreover, if \( f \) is a bounded Borel function on \( \sigma(A) \), then there is a linear operator \( f(A) \) where

\[
\langle f(A)x, y \rangle = \int_{\sigma(A)} f(\lambda) dE_{x,y}(\lambda)
\]

for every \( x, y \in H \), where \( E_{x,y}(\omega) = \langle E(\omega)x, y \rangle \) is a complex measure over the Borel subsets of \( \sigma(A) \).

Based on (1), it is customary to adopt the notation

\[
f(A) = \int_{\sigma(A)} f \, dE.
\]

It is often convenient to assume the resolution \( E \) of the identity is defined over all Borel subsets of \( \mathbb{C} \) by letting \( E(\omega) = 0 \) if \( \omega \cap \sigma(A) = \emptyset \).

The following result is the basis of our main technique.

**Theorem 2.** (see [1], Theorem 3, section 40) Let \( \mathcal{H} \) be a complex separable Hilbert space and let \( A \) be a bounded self-adjoint operator on \( \mathcal{H} \). Let \( W_k \) (\( k = 1, 2, \ldots, m \)) be pairwise orthogonal invariant subspaces of \( A \); that is, \( \mathcal{H} = \bigoplus_{k=1}^{m} W_k \) and \( AW_k \subset W_k \), for every \( k \). Let \( P_k \) be the projection
operator on $W_k$ and $A_k$ be the restriction of $A$ to $W_k$. Then, for every $\psi \in \mathcal{H}$, we have

$$A\psi = \sum_{k=1}^{m} A_k P_k \psi.$$ 

### 2.3 Quantum walk

Let $G = (V, E)$ be an infinite graph whose adjacency operator $A$ is a bounded self-adjoint linear operator on $\ell^2(V)$. A continuous-time quantum walk on $G$ is given by the unitary operator

$$e^{-itA} = \int_{\sigma(A)} e^{-it\lambda} \, dE_\lambda.$$ 

We say that a graph $G$ has perfect state transfer (adopting Bose [2]) between vertices $u$ and $v$ if there is a time $\tau$ so that

$$|\langle e_v, e^{-i\tau A} e_u \rangle| = 1.$$ 

A family $\{G_n\}$ of graphs has asymptotically efficient perfect state transfer (see Chen et al. [8]) between vertices $u$ and $v$ if there is a time $\tau = n^{O(1)}$ so that

$$|\langle e_v, e^{-i\tau A(G_n)} e_u \rangle| = 1 - o_n(1).$$ 

### 3 Conical illusion

We begin by examining a small yet illustrative example of an infinite graph. A vertex $u$ in a graph $G_n$ is called sedentary (see Godsil [13]) if for all time $t$, we have

$$|\langle e_u, e^{-itA(G_n)} e_u \rangle| = 1 - o_n(1).$$ 

It is known that the (finite) clique $K_n$ is sedentary at any vertex. To see this, note that the spectral decomposition of $A(K_n)$ is given by $A(K_n) = (n-1)J/n - (I - J/n)$, which shows that

$$e^{-itA(K_n)} = e^{-it(n-1)}J/n + e^{it}(I - J/n).$$ 

This immediately yields

$$|\langle e_u, e^{-itA(K_n)} e_u \rangle| = |e^{-it(n-1)}/n + e^{it}(1 - 1/n)| = 1 - o_n(1).$$ 

We show that the clique is still sedentary even in the presence of an infinite tail (see Figure 1).
Figure 1: The infinite lollipop graph \( L_n \). All shaded vertices are sedentary.

**Proposition 3.** For \( n \geq 2 \), let \( L_n \) be the infinite graph that is the one-sum of the clique \( K_n \) and the infinite path \( P_\infty \) at vertex \( u \). Then, any vertex \( v \neq u \) of \( K_n \) in \( L_n \) is sedentary.

**Proof.** Following Golinskii [17], we label the vertices of \( K_n \) with \( 1, 2, \ldots, n \), where \( n \) is the attachment vertex, and the vertices of \( P_\infty \) with the integers \( n + 1, n + 2, \ldots \). The adjacency matrix of \( L_n \) is a bounded self-adjoint operator in \( \ell_2(\mathbb{Z}^+) \). Relative to the standard basis \( \{ e_k : k \in \mathbb{Z}^+ \} \), we have

\[
A(L_n) = \begin{pmatrix}
A(K_n) & E_{n,1} \\
E_{1,n} & J_0
\end{pmatrix}
\]

where \( J_0 \) is the free Jacobi matrix. Now, define a new basis \( \{ \tilde{e}_k : k \in \mathbb{Z}^+ \} \) where \( \tilde{e}_k = e_k \), for \( k = n, n + 1, \ldots \), and

\[
\tilde{e}_{n-1} = \frac{1}{\sqrt{n-1}} \sum_{k=1}^{n-1} e_k
\]

and

\[
\tilde{e}_k = \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n-1} \zeta_{n-1}^{(j-1)k} e_j \quad (k = 1, \ldots, n-2)
\]

where \( \zeta_p = e^{2\pi i/p} \) denotes the principal \( p \)-th root of unity. The action of \( A(L_n) \) on the new basis is given by

\[
U^{-1} A(L_n) U = \begin{pmatrix}
-I_{n-2} & O \\
O & \tilde{J}_0
\end{pmatrix}
\]

where \( \tilde{J}_0 \) is an eventually-free Jacobi matrix (of rank two) given by

\[
\tilde{J}_0 = \begin{pmatrix}
n-2 & \sqrt{n-1} & O \\
\sqrt{n-1} & 0 & e_1^T \\
O & e_1 & J_0
\end{pmatrix}.
\]
Here, $U$ is the unitary operator which maps $e_k$ to $\tilde{e}_k$ for every $k = 1, 2, \ldots$. Thus, $W_0 = \text{span}\{\tilde{e}_1, \ldots, \tilde{e}_{n-2}\}$ and $W_\infty = \text{span}\{\tilde{e}_{n-1}, \tilde{e}_n, \ldots\}$ are orthogonal invariant subspaces which span the entire space.

Finally, to see why any vertex $j \in \{1, \ldots, n - 1\}$ in $K_n$ is sedentary, observe that
\[
|\langle e_j, e^{-itA(L_n)}e_j \rangle| = 1 - o_n(1) \text{ for all } t.
\]

The next result shows that even with an arbitrary but finite number of infinite tails the clique is still sedentary. But, we make a slight shift in our approach (which sheds light on the title of this section); see Figure 2.

**Theorem 4.** Let $n$ and $m$ be positive integers where $n \geq 2$ and $m \geq 1$. Let $H$ be the infinite graph obtained from the join $\overline{K}_m + K_n$ by attaching separate infinite paths to each vertex of the coclique $\overline{K}_m$. Then, any vertex of the clique $K_n$ is sedentary.

**Proof.** The quotient of $H$ is the clique $K_n$ attached to a single infinite path where each vertex of $K_n$ is connected to the first vertex of $P_\infty$ by an edge of weight $\sqrt{m}$. The proof now proceeds as in Proposition 3 without major changes.

Observe that Proposition 3 is a corollary of Theorem 4 when $m = 1$.

It seems curious to investigate sedentariness as it is the opposite of quantum state transfer. Yet, as we will see, the above results reveal a dark subspace which holds interesting properties.
4 Dark subspace transport over cones

We explore properties of a finite invariant subspace of graphs with tails. First, observe that the clique $K_n$ is a cone over a smaller clique, that is, $K_n = K_1 + K_{n-1}$. Now, we consider the cone $\hat{G}_n := K_1 + G_n$ of an arbitrary $(n,d)$-regular graph $G_n$. The following result shows that the one-sum of $\hat{G}_n$ with $P_\infty$ at the conical vertex inherits the state transfer properties of $G_n$.

**Theorem 5.** Let $G_n$ be a connected $d$-regular graph which admits efficient perfect state transfer between vertices $u$ and $v$ at time $\tau$. Let $H$ be the infinite graph that is the one-sum of the cone $\hat{G}_n$ and the infinite path $P_\infty$ at the conical vertex. Then, $H$ has asymptotically efficient perfect state transfer between $u$ and $v$ at time $\tau$.

**Proof.** Suppose the adjacency matrix of $G_n$ is diagonalized by the unitary matrix $Z$, that is, $Z^\dagger A(G_n)Z = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of the eigenvalues and the columns of $Z$ are eigenvectors of $A(G_n)$. Let $A(G_n) = \sum_{k=1}^n \lambda_k z_k z_k^\dagger$ with $\lambda_n = d$ and $z_n = \frac{1}{\sqrt{n}} 1_n$. Similar to Proposition 3, the adjacency matrix of $H$ can be decomposed into

$$U^{-1} A(H) U = \begin{pmatrix} \Lambda_0 & O \\ O & \hat{J}_0 \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} Z & O \\ O & I \end{pmatrix}. \quad (3)$$

Here, $U$ is the unitary change of basis transformation where $\tilde{e}_j = U e_j$, $j = 1, 2, \ldots$, shows the new basis vectors in terms of the standard basis vectors. By the definition of $U$, notice that $\tilde{e}_j = e_j$ whenever $j \geq n+1$. Also, $\Lambda_0 = \text{diag}(\lambda_1, \ldots, \lambda_{n-1})$ is a diagonal matrix containing all non-principal eigenvalues (with multiplicities) of $G_n$. The eventually-free Jacobi matrix $\hat{J}_0$ is given by

$$\hat{J}_0 = \begin{pmatrix} d & \sqrt{n} & O \\ \sqrt{n} & 0 & e_1^T \\ O & e_1 & \hat{J}_0 \end{pmatrix}.$$ 

Let $P$ denote the projection operator onto the subspace span\{\tilde{e}_j : j \geq n\}. From (3), we have

$$\langle e_v, e^{-i\tau A(H)} e_u \rangle = \langle e_v, U \exp \left( -i\tau \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \hat{J}_0 \end{pmatrix} \right) U^{-1} e_u \rangle$$

$$= \langle e_v, U \begin{pmatrix} e^{-i\tau \Lambda_0} & 0 \\ 0 & e^{-i\tau \hat{J}_0} \end{pmatrix} U^{-1} e_u \rangle.$$
Figure 3: A cone over the cube $Q_n$ attached to an infinite tail. Asymptotically efficient perfect state transfer occurs between the shaded antipodal vertices with fidelity $1 - O(1/2^n)$ at time $(\pi/2)n$.

The last expression equals to

$$\langle e_v, \sum_{j=1}^{n-1} e^{-i\tau \lambda_j} z_j z_j^T e_u \rangle + \langle e_v, U P e^{-i\tau \tilde{J}_0} P U^{-1} e_u \rangle.$$ 

Therefore,

$$\langle e_v, e^{-i\tau A(H)} e_u \rangle = \langle e_v, \sum_{j=1}^{n-1} e^{-i\tau \lambda_j} z_j z_j^T e_u \rangle + \langle e_v, z_n \rangle \langle z_n, e^{-i\tau \tilde{J}_0} z_n \rangle \langle z_n, e_u \rangle$$

$$= \langle e_v, e^{-i\tau A(G_n)} e_u \rangle - \frac{e^{-i\tau d}}{n} + \frac{1}{n} \langle z_n | e^{-i\tau \tilde{J}_0} | z_n \rangle$$

$$= \langle e_v, e^{-i\tau A(G_n)} e_u \rangle + O(n^{-1}).$$

This shows $|\langle e_v, e^{-i\tau A(H)} e_u \rangle| = 1 - o_n(1)$. 

**Example 6.** The one-sum of the cone of the $n$-cube with $P_\infty$ at the conical vertex has asymptotically efficient perfect state transfer between any pair of antipodal vertices of $Q_n$ (see Figure 3). The fidelity of state transfer is $1 - O(1/2^n)$ at the usual time of $(\pi/2)n$.

We remark that Theorem 5 allows for any other quantum state transfer tasks (such as, fractional revival) provided the initial and target states have negligible overlap with the principal eigenvector of the graph.
5 Decoupling

Proposition 3 offers another useful insight. The eventually-free Jacobi matrix in the infinite lollipop graph has the quotient of the clique as its leading principal submatrix. We generalize this to graphs with well-behaved quotients (for example, distance-regular graphs). Our primary goal is still to understand the structure of the underlying protected (dark) subspaces.

Let \( G \) be a graph with adjacency matrix \( A \) and let \( v \) be one of its vertices. The walk matrix \( W_G(v) \) of \( G \) with respect to \( v \) is defined as

\[
W_G(v) = (e_v \quad Ae_v \quad A^2e_v \quad \ldots \quad A^{n-1}e_v).
\]

Thus, the columns of the walk matrix generate the Krylov subspace of \( A \) relative to the unit vector \( e_v \). The vertex \( v \) is called controllable if \( W_G(v) \) has full rank (see Godsil [12]).

The following result appeared in Golinskii [17]. We will provide an alternative proof which illuminates the structure of the dark (protected) subspace.

**Theorem 7.** Let \( G_n \) be a connected graph and let \( v \) be one of its vertices. Let \( H \) be the infinite graph that is the one-sum of a finite graph \( G_n \) with the infinite path \( P_\infty \) at vertex \( v \). Then, there is an invertible matrix \( U \) so that

\[
U^{-1}A(H)U = \begin{pmatrix}
S_v & 0 \\
0 & \tilde{J}_0
\end{pmatrix}
\]

where \( \dim(S_v) = n - \text{rank}(W_{G_n}(v)) \) and \( \tilde{J}_0 \) is an eventually-free Jacobi matrix.

Moreover, if \( G_n \) admits an equitable distance partition \( \pi \) relative to \( v \) so that \( A(G_n/\pi) \) is a Jacobi matrix, then the leading principal submatrix of \( \tilde{J}_0 \) is \( A(G_n/\pi) \).

**Proof.** For brevity, let \( A = A(G_n) \). First, we show that we can construct an extension \( \tilde{J}_0 \) of the free Jacobi matrix \( J_0 \). We apply the Lanczos algorithm for tridiagonalizing a symmetric matrix starting with \( e_v \) (see Horn and Johnson [19], 3.5P5, page 221). This algorithm will find an orthonormal basis for the Krylov subspace \( K = \{e_v, Ae_v, \ldots, A^{n-1}e_v\} \). Assume that \( m = \dim K \) and let \( \{\tilde{e}_0, \ldots, \tilde{e}_{m-1}\} \) be an orthonormal basis of \( K \), where \( \tilde{e}_0 = e_v \). Denote \( \tilde{e}_{-1} \) as the unit vector corresponding to the unique neighbor of \( v \) on the infinite path. Then,

\[
A\tilde{e}_j = \text{span}\{\tilde{e}_{j-1}, \tilde{e}_j, \tilde{e}_{j+1}\} \quad (j = 0, \ldots, m-2)
\]
and

\[ A\tilde{e}_{m-1} = \text{span}\{\tilde{e}_{m-2}, \tilde{e}_{m-1}\}. \]

This shows that the subspace spanned by \( \tilde{e}_0, \ldots, \tilde{e}_{m-1} \) and the standard basis \( \{e_w : w \in V(P_\infty)\} \) for \( P_\infty \) is an invariant subspace of \( H \); moreover, the restriction of \( H \) on this subspace forms an extension \( \tilde{J}_0 \) of \( J_0 \) (because of the tridiagonal structure imposed by the Krylov subspace).

It remains to show that the leading \( m \times m \) principal submatrix of \( \tilde{J}_0 \) is \( A(G_n/\pi) \). By our assumption on \( \pi \), we may write \( A(G_n) \) in a tridiagonal block form:

\[
A(G_n) = \begin{pmatrix}
A_1 & B_1^T \\
B_1 & A_2 & B_2^T \\
& B_2 & A_3 & B_3^T \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & B_{m-1}^T \\
& & & & 1_{N_{G_n}(v)}
\end{pmatrix}.
\]

Here, the last row and column correspond to vertex \( v \). Notice that each \( A_i \) is the adjacency matrix of a regular subgraph of \( G_n \) and each \( B_i \) represents the incidence matrix between subgraphs \( A_i \) and \( A_{i+1} \), where \( B_i \) has constant row-sum and constant column-sum. Because of these properties, it is easy to see that the Lanczos algorithm, starting with \( e_v \) (last row and column), will construct a basis \( \{\tilde{e}_0 = e_v, \tilde{e}_1, \ldots, \tilde{e}_{m-1}\} \) so that the action of \( H \) on this basis is given by

\[
\begin{pmatrix}
d_1 & d_{1,2} \\
d_{2,1} & d_2 & d_{2,3} \\
& d_{3,2} & d_3 & d_{3,4} \\
& & \ddots & \ddots & \ddots \\
& & & d_{m-1} & 1 \\
& & & & |N(v)|
\end{pmatrix}
\]

where \( d_i \) is the degree of regular subgraph \( G[V_i] \) and \( d_{i,j} \) denotes the number of neighbors in \( V_j \) of each vertex in \( V_i \). After a standard transformation, the previous matrix is equal to the adjacency matrix of the symmetrized quotient of \( G_n/\pi \).

We refer to the finite invariant subspace of \( H \) in Theorem 7 as the protected (or dark) subspace of \( H \). Using the above theorem, we observe that a random graph offers no protection.
Corollary 8. Let $G_n = G(n, 1/2)$ be a random graph. Then, the one-sum of $\tilde{G}_n = K_1 + G_n$ with $P_\infty$ has no protected subspace almost surely.

Proof. This follows since a random graph $G(n, 1/2)$ is controllable relative to $1_n$ almost surely (see O’Rourke and Touri [26]).

6 Dark subspace transport via equitable partitions

Using Theorem 7, we explore the dark subspaces induced by regular subgraphs within the (equitable) distance partitions of graphs. Our goal is to show that state transfer occurs within each local subgraph under suitable conditions.

Given a regular graph $G$, consider the graph $P_m(G)$ obtained by connecting $m$ disjoint copies of $G$ in a series using the graph join operator. So, the adjacency matrix of this graph is $I_m \otimes A(G) + A(P_m) \otimes J$. We now generalize this series graph by allowing $m$ (possibly distinct) regular graphs, $G_0, G_1, \ldots, G_{m-1}$ and denote the resulting graph as $P_m(G_0, \ldots, G_{m-1})$.

Theorem 9. Let $G_n = P_m(\{u\}, G_1, \ldots, G_{m-1})$ be a series graph. Let $H$ be the infinite graph that is the one-sum of $G_n$ and the infinite path $P_\infty$ at $u$. If there is $G_j, j = 1, \ldots, m-1$, with perfect state transfer between its vertices $a$ and $b$ at time $\tau$, then $H$ has asymptotic perfect state transfer between $a$ and $b$ at time $\tau$.

Proof. We prove the claim by induction on $m$. The base case of $m = 2$ is simply Theorem 5. Next, we prove the case for $m = 3$ (and leave the full inductive argument as a straightforward exercise). Let $G_n = P_3(\{u\}, G_1, G_2)$ where $G_i$ is a $(n_i, d_i)$-regular graph. The adjacency matrix $A$ of $G_n$ is given by

$$A = \begin{pmatrix} A_2 & J & 0 \\ J & A_1 & 1_{V_1} \\ 0 & 1_{V_1}^T & 0 \end{pmatrix}.$$  

By Theorem 7, we know that

$$A = \begin{pmatrix} 3_u & 0 \\ 0 & J_0 \end{pmatrix}, \quad \text{where} \quad J_0 = \begin{pmatrix} d_2 & \sqrt{n_1 n_2} & 0 \\ \sqrt{n_1 n_2} & d_1 & \sqrt{n_1} \\ 0 & \sqrt{n_1} & 0 \end{pmatrix}.$$  

Suppose $z_1$ is an eigenvector of $A_1$ corresponding to a non-principal eigenvalue $\lambda_1$ (that is, $\lambda_1 < d_1$) and $z_2$ is an eigenvector of $A_2$ corresponding to
Figure 4: A complex Hermitian clique $K_6$ attached to the infinite path $P_\infty$. The shaded vertices have asymptotically efficient universal perfect state transfer (PST between every pair).

a non-principal eigenvalue $\lambda_2$ (that is, $\lambda_2 < d_2$). Then,

$$ (\lambda_1 - A) \begin{pmatrix} 0 \\ z_1 \\ 0 \end{pmatrix} = 0, \quad (\lambda_2 - A) \begin{pmatrix} z_2 \\ 0 \\ 0 \end{pmatrix} = 0. $$

This implies that $\mathfrak{Z}_u$ is block diagonal. Therefore, for some invertible matrix $U$, we have

$$ U^{-1}AU = \begin{pmatrix} \mathfrak{Z}_1 & O & O \\ O & \mathfrak{Z}_2 & O \\ O & O & \tilde{J}_0 \end{pmatrix} $$

So, asymptotic perfect state transfer occurs within each block (by a similar argument to Theorem 5).

**Example 10.** Let $\vec{K}_3$ be the complex oriented clique of order 3 whose adjacency matrix is

$$ \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}. $$

(4)

We consider the graph $H = P_3(K_1, \vec{K}_3, \vec{K}_3)$ (see Figure 4). By Theorem 7 and an observation in Cameron et al. [6], we may conclude that each local $\vec{K}_3$ has asymptotically efficient universal perfect state transfer.

7 Perfect state transfer in a dark subspace

In this section, we show that perfect state transfer (with unit fidelity) can be engineered to occur in graphs with tails. Our techniques are primarily
Figure 5: The fly-swatter graph: $P_3^{\otimes 2}$ attached to an infinite tail. Perfect state transfer occurs between $\frac{1}{\sqrt{2}}(e_{1,2} - e_{1,1})$ and $\frac{1}{\sqrt{2}}(e_{2,3} - e_{3,2})$ at time $\pi/\sqrt{2}$.

based on the representation theory of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ and the dual-rail encoding scheme in spin chains.

We recall that the Cartesian product of two graphs $G_1$ and $G_2$, denoted $G_1 \square G_2$, is a graph with vertex set $V(G_1) \times V(G_2)$ where the pairs $(a_1, a_2)$ and $(b_1, b_2)$ are adjacent if either $a_1 = b_1$ and $(a_2, b_2) \in E(G_2)$ or $(a_1, b_1) \in E(G_1)$ and $a_2 = b_2$. The adjacency matrix of $G_1 \square G_2$ is given by $A(G_1) \otimes I + I \otimes A(G_2)$. If the two graphs are equal, we use $G_2\square G_2$ to denote the Cartesian square.

7.1 Walk modules

The following example shows perfect pair state transfer (see Chen and Godsil [7]) on $P_3^{\otimes 2}$ in the presence of an infinite tail.

Example 11. Suppose the vertex set of $P_3$ is $\{1, 2, 3\}$ where 1 and 3 are the leaves. Let $H$ be the one-sum of $P_3^{\otimes 2}$ and $P_\infty$ at vertex $(1, 1)$ (see Figure 5). Let $e_{j,k}$ denote $e_j \otimes e_k$. We show there is perfect state transfer between the states $\frac{1}{\sqrt{2}}(e_{1,2} - e_{1,1})$ and $\frac{1}{\sqrt{2}}(e_{2,3} - e_{3,2})$ at time $\pi/\sqrt{2}$. Consider a new basis for $A(H)$. Let $\tilde{e}_k = e_k$, $k \geq 10$, correspond to vertices of $P_\infty$, and let

\begin{align*}
\tilde{e}_9 &= e_{1,1} \\
\tilde{e}_8 &= \frac{1}{\sqrt{3}}(e_{1,2} + e_{1,1}) \\
\tilde{e}_7 &= \frac{1}{\sqrt{6}}(e_{1,3} + 2e_{2,2} + e_{3,1}) \\
\tilde{e}_6 &= \frac{1}{\sqrt{2}}(e_{2,3} + e_{3,2}) \\
\tilde{e}_5 &= e_{3,3} \\
\tilde{e}_4 &= \frac{1}{\sqrt{2}}(e_{2,3} - e_{3,2}) \\
\tilde{e}_3 &= \frac{1}{\sqrt{2}}(e_{1,3} - e_{3,1}) \\
\tilde{e}_2 &= \frac{1}{\sqrt{2}}(e_{1,2} - e_{2,1}) \\
\tilde{e}_1 &= \frac{1}{\sqrt{3}}(e_{1,3} - e_{2,2} + e_{3,1})
\end{align*}
be the basis vectors for the finite graph. The subspaces \( W_0 = \text{span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \) and \( W_1 = \text{span}\{\tilde{e}_k : k \geq 5\} \) are orthogonal and invariant under \( A(H) \). Under this new basis, we have

\[
A(H) = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

where \( \tilde{J}_0 \) is an eventually-free Jacobi matrix. As the action of \( A(H) \) on the invariant subspace spanned by \( \{\tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \) is equivalent to the adjacency matrix of \( P_3 \), it follows that perfect state transfer occurs between \( \tilde{e}_2 \) and \( \tilde{e}_4 \) at time \( \pi/\sqrt{2} \).

We now generalize the above example. For the \( n \)-cube \( Q_n \), let \( \pi \) denote the equitable partition whose cells correspond to the set of vertices with the same Hamming weight. The quotient graph \( Q_n/\pi \) is a weighted path of length \( n + 1 \) (also known as the Krawtchouk chain; see Christandl et al. [5]).

As we will use some basic Lie theoretic arguments, we briefly recall some relevant terminology; see [18, 10] for more background on Lie theory. The Lie algebra \( \mathfrak{sl}_2 \mathbb{C} \) is the algebra of complex \( 2 \times 2 \) traceless matrices that is equipped with a bracket\(^1\) operation \([A, B] = AB - BA\). A representation of \( \mathfrak{sl}_2 \mathbb{C} \) is a homomorphism \( \rho : \mathfrak{sl}_2 \mathbb{C} \rightarrow \text{End}(V) \) for some vector space \( V \) of finite dimension (here, \( \text{End}(V) \) is equipped with the natural bracket).

**Theorem 12.** For any integer \( n \geq 2 \), let \( H_n \) be the infinite weighted graph that is the one-sum of the Cartesian square \((Q_n/\pi)^{\Box 2}\) and the infinite path \( P_\infty \) at vertex \((0_n, 0_n)\). Then, there exist multiple perfect state transfer on \( H_n \) at time \( \pi/2 \).

**Proof.** The Clebsch-Gordan decomposition (see [18], appendix C) states that if \( \rho_m \) is an irreducible representation of \( \mathfrak{sl}_2 \mathbb{C} \) of dimension \( m+1 \), then a tensor product of two irreducible representations can be decomposed into a direct sum of irreducible representations. In particular, for any \( n \geq 1 \):

\[
\rho_n(M) \otimes I + I \otimes \rho_n(M) \cong \bigoplus_{k=0}^{n} \rho_{2n-2k}(M), \quad \forall M \in \mathfrak{sl}_2 \mathbb{C}.
\]  

\(^1\)The Lie bracket is bilinear, anti-symmetric, and satisfies \([A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\).
Figure 6: The 3-cube attached to an infinite tail. Perfect state transfer occurs between $\frac{1}{\sqrt{3}}(e_{100} + \zeta e_{010} + \zeta^2 e_{001})$ and $\frac{1}{\sqrt{3}}(e_{011} + \zeta e_{101} + \zeta^2 e_{110})$ at time $\pi/2$, where $\zeta = \exp(2\pi i/3)$ is the primitive cube root of unity.

To use (5), we observe there exists $M \in \mathfrak{sl}_2 \mathbb{C}$ where $\rho(M) \cong A(Q_m/\pi)$, for every $m \geq 2$. Recall that a basis for $\mathfrak{sl}_2 \mathbb{C}$ is given by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If we let $M = X + Y$, then $\rho(M) = \rho(X) + \rho(Y)$. Under the basis of all eigenvectors of $\rho(H)$, we have

$$\rho(Y) = \begin{pmatrix} 0 & 1 & 0 & \vdots & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 1 \end{pmatrix}, \quad \rho(X) = \begin{pmatrix} 0 & m & 2(m-1) & \vdots & 0 & m \\ 0 & 0 & 3(m-2) & \vdots & 0 & 0 \end{pmatrix}.$$

This shows that $\rho(M) \cong A(Q_m/\pi)$ by a standard basis change. Hence, perfect state transfer occurs in each of the Krawtchouk chains in (5).

Next, we show that the $n$-cube itself can be “immunized” against the infinite tail. Recall that $\zeta_n = \exp(2\pi i/n)$ is the primitive $n$th root of unity. It will be convenient to identify the vertices of the $n$-cube $Q_n$ with subsets of $\{1, 2, \ldots, n\}$.

**Theorem 13.** For integer $n \geq 2$, let $H_n$ be the infinite graph that is the one-sum of the $n$-cube $Q_n$ and the infinite path $P_\infty$ at vertex $0_n$. Then, perfect state transfer occurs in $H_n$ between the states

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \zeta_n^{-1} e(\{j\}) \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \zeta_n^{-1} e([n] \setminus \{j\})$$
at time $\pi/2$

Proof. By Theorem 7, the infinite invariant subspace of $A = A(H)$ is the walk module $U_0$ generated by $e_0$ (which corresponds to the vertex $0_n$). Let $W_d$ be the set of vertices of $Q_n$ of Hamming weight $d$. Thus,

$$U_0 = \text{span}\{e(\emptyset), Ae(\emptyset), A^2e(\emptyset), \ldots\}$$

$$= \text{span}\{e_0, 1_{W_1}, 1_{W_2}, \ldots, 1_{W_n}\} \cup \{e_u : u \in V(P_\infty)\}.$$ 

The action of $A$ on $U_0$ is given by the eventually-free Jacobi matrix

$$A|_{U_0} = \begin{pmatrix}
    0 & n & & \\
    1 & 0 & n-1 & \\
    2 & & 0 & \\
    & \ddots & \ddots & \ddots \\
    n & 0 & 1 & \\
    1 & 0 & 1 & \ldots \\
    & \vdots & \ddots & \\
\end{pmatrix}.$$ 

Next, consider a state supported on the singleton subsets of $Q_n$ which is defined as

$$\psi_1 := \sum_{j=1}^{n} \zeta_n^{j-1} e(\{j\}).$$

and its walk module

$$U_1 = \text{span}\{\psi_1, A\psi_1, A^2\psi_1, \ldots\}.$$ 

It is clear that $U_1$ is an invariant subspace of $A$ and that $U_1 \subset U_0^\perp$. The latter follows as $A$ is self-adjoint and $\psi_1$ is orthogonal to $1_{W_1}$. We now show that $\dim(U_1) = n - 1$ and

$$A|_{U_1} \cong \begin{pmatrix}
    n - 2 & & \\
    1 & n - 3 & \\
    & 2 & \ddots \\
    & \vdots & \ddots \\
    & & 1 \\
\end{pmatrix}. \quad (6)$$
To see this, consider the set \( \{ \psi_1, \psi_2, \ldots, \psi_{n-1} \} \) of orthogonal basis vectors for \( U_1 \), where
\[
\psi_k = \sum_{S \subseteq \{1, 2, \ldots, n\}, |S| = k} \alpha(S) e(S), \quad \text{where} \quad \alpha(S) = \sum_{j \in S} \zeta_j^j.
\]
Then, for \( k = 1, \ldots, n - 2 \), we have
\[
\langle \psi_{k+1}, A \psi_k \rangle = k \quad \text{and} \quad \langle \psi_k, A \psi_{k+1} \rangle = n - k,
\]
which shows (4). Thus, it follows \( A|_{U_1} \cong A(Q_{n-2}/\pi) \), where \( \pi \) is the equitable partition whose cells are the vertices with the same Hamming weight. Hence, perfect state transfer occurs in \( U_1 \) from \( n^{-1/2} \psi_1 \) to \( n^{-1/2} \psi_{n-1} \) at time \( \pi/2 \) as antipodal perfect state transfer occurs in \( A(Q_{n-2}/\pi) \) at that time.

In the proof of Theorem [13] the infinite invariant subspace \( U_0 \) is related to the primary module of the Terwilliger algebra of the \( n \)-cube (see Terwilliger [28], Definition 7.2). Our dark subspaces correspond to the irreducible modules orthogonal to this primary module.

We offer an alternative proof of Theorem [13] which emphasizes the underlying connection with the Lie algebra \( \mathfrak{sl}_2 \mathbb{C} \).

**Theorem 14.** For integer \( n \geq 2 \), let \( H_n \) be the infinite graph that is the one-sum of the \( n \)-cube \( Q_n \) and the infinite path \( P_\infty \) at vertex 0. Then, multiple perfect state transfer occur in \( H_n \).

**Proof.** Given the \( n \)-cube \( Q_n \), recall the standard lowering \( L_n \) and raising \( R_n \) operators on the lattice of subsets of \( \{1, 2, \ldots, n\} \). For subsets \( A, B \subseteq \{1, 2, \ldots, n\} \), define \( (L_n)_{A,B} = 1_{\{|A|=|B|+1 \land B \subseteq A\}} \) and \( R_n = (L_n)^T \). Now, let \( H_n = R_n L_n - L_n R_n \). Notice that \( A(Q_n) = L_n + R_n \). Inductively, it can be shown that
\[
[R_n, L_n] = H_n, \quad [H_n, R_n] = 2R_n, \quad [H_n, L_n] = -2L_n.
\]
Thus, the homomorphism \( \rho : \mathfrak{sl}_2 \mathbb{C} \to \text{End}(V_n) \), with \( |V_n| = 2^n \), where \( \rho(H) = H_n, \rho(X) = R_n, \) and \( \rho(Y) = L_n \), is a representation of the Lie algebra \( \mathfrak{sl}_2 \mathbb{C} \) (where \( H, X, Y \in \text{Mat}_2(\mathbb{C}) \) are the standard basis of \( \mathfrak{sl}_2 \mathbb{C} \)). This allows us to employ the following standard Lie theoretic argument (see [18, 10]).
The eigenvalues of $H_n$ are $\lambda = -n, -n + 2, \ldots, n - 2, n$. Let $W_\lambda$ be the eigenspace of $H_n$ corresponding to eigenvalue $\lambda$. By the properties of $\rho$ as a representation of $\mathfrak{sl}_2 \mathbb{C}$, if $z \in W_\lambda$, then $R_nz \in W_{\lambda+2}$ (or 0 if $\lambda = n$) and $L_nz \in W_{\lambda-2}$ (or 0 if $\lambda = -n$). Now, let $z$ be an eigenvector in the maximum eigenspace $W_n$. Consider the invariant subspace $K_0$ spanned by the following set of orthogonal eigenvectors of $H_n$. Let $z_k = L_n^kz, k = 0, 1, \ldots, n$, and set $K_0 = \text{span}\{z, L_nz, L_n^2z, \ldots, L_n^nz\}$.

By definition, the action of $L_n$ on $K_0$ is given by $\langle z_j, L_nz_k \rangle = 1_{\{j=k+1\}}$, while the action of $R_n$ on $K_0$ is given by $\langle z_j, R_nz_k \rangle = k(n-k)1_{\{j=k-1\}}$. Therefore, the action of $L_n + R_n$ on $K_0$ is similar to $A(Q_n/\pi)$ (a Krawtchouk chain of length $n + 1$). Next, we repeat the argument on $K_0$ to extract additional Krawtchouk chains. This shows that the entire space $V_n$ can be decomposed into the invariant subspaces (or walk modules) as follows:

$$V_n = \bigoplus_{\ell \geq 0} K_\ell.$$

As the action of $L_n + R_n$ (or the adjacency operator of $Q_n$) on each invariant subspace $K_\ell$ is similar to $A(Q_d/\pi)$, for some $d \leq n$, perfect state transfer occur in all of these Krawtchouk chains.

7.2 Dual-Rail Encoding

We describe a simple idea for constructing perfect state transfer in graphs with tails based on the dual-rail encoding in spin chains (see Burgarth and Bose [3], and Kay [21] for a relevant survey). Recall that the notation $e(x)$ denotes the unit vector corresponding to vertex $x$.

**Theorem 15.** Let $G$ be a graph with perfect state transfer between vertices $u$ and $v$ at time $\tau$. Let $H$ be the rooted product $P_3^\mathcal{Y}$ where $\mathcal{Y} = \{G_0, P_\infty, G_1\}$ with $G_0$ and $G_1$ represent two identical copies of $G$. Then, $H$ has perfect state transfer between the states $\frac{1}{\sqrt{2}}(e(u_0) - e(u_1))$ and $\frac{1}{\sqrt{2}}(e(v_0) - e(v_1))$ at time $\tau$, where $u_0, v_0$ and $u_1, v_1$ are vertices which correspond to the copies of $u, v$ in $G_0, G_1$, respectively.

**Proof.** Let $(b, w)$ denote the vertices in the two copies of $G$, where $b \in \{0, 1\}$ and $w \in V(G)$. Let $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard two-dimensional unit vectors. We show there is perfect state transfer in $H$ between the states $\frac{1}{\sqrt{2}}(e_0 - e_1) \otimes e(u)$ and $\frac{1}{\sqrt{2}}(e_0 - e_1) \otimes e(v)$ at time $\tau$. 

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Figure 7: A rooted product $P_3^Y$ where $Y = \{\tilde{Q}_n, P_\infty, \tilde{Q}_n\}$ and $\tilde{Q}_n$ is a Krawtchouk chain (quotient of the $n$-cube). Perfect pair state transfer occurs between $\frac{1}{\sqrt{2}}(e(a_1) - e(b_1))$ and $\frac{1}{\sqrt{2}}(e(a_n) - e(b_n))$ at time $\pi/2$.

Let $q_0$ be the middle vertex of $P_3$ in the rooted product $H = P_3^{\{G_0, P_\infty, G_1\}}$. Also, let $r \in V(G)$ be the designated root vertex of $G$. Note that $A(H)$ is given by

$$A(H) = \begin{pmatrix} A(G) & O & e(r) \\ O & A(G) & e(r) \\ e(r)^T & e(r)^T & 0 & e_1^T \\ e_1 & 0 & J_0 \end{pmatrix}$$

where the third row and column correspond to vertex $q_0$. Consider the subspaces $W^-$ and $K$ defined as

$$W^- = \text{span} \left\{ \frac{1}{\sqrt{2}}(e_0 - e_1) \otimes e(w) : w \in V(G) \right\}$$

and

$$K = \text{span} \{ A(H)^m e(q_0) : m = 0, 1, \ldots \}.$$  

Claim 16. $W^-$ is orthogonal to $K$.

Proof. Let $W^+$ be a subspace defined as

$$W^+ = \text{span} \left\{ \frac{1}{\sqrt{2}}(e_0 + e_1) \otimes e(w) : w \in V(G) \right\}.$$  

Suppose the vertex set of $P_\infty$ is denoted as $\{q_n : n \geq 0\}$. Let $W^\infty$ be defined as $W^\infty = \text{span} \{e(q_n) : n \geq 0\}$. Notice that

$$A(H)e(q_0) = e(q_1) + \frac{1}{\sqrt{2}}(e_0 + e_1) \otimes e(r).$$

Therefore, we see that

$$K = W^+ + W^\infty.$$
Each vector in $W^-$ is orthogonal to any vector in $W^+$ since $\frac{1}{\sqrt{2}}(e_0 \pm e_1)$ form an orthogonal pair. Moreover, each vector in $W^-$ is orthogonal to any $e(q_n)$ since the vertices of $G_0, G_1$ and $P_\infty$ are disjoint from each other.

Claim 17. $W^-$ is an invariant subspace of $A(H)$ and the action of $A(H)$ restricted to $W^-$ is equivalent to $A(G)$.

Proof. The action of $A(H)$ on $\frac{1}{\sqrt{2}}(e_0 - e_1) \otimes e(w)$ is given by $I_2 \otimes A(G)$ (by the structure of the rooted product $H$). This shows $W^-$ is an invariant subspace of $A(H)$ and that the action of $A(H)$ is equivalent to $A(G)$.

The above two claims show that for some unitary matrix $U$ we have

$$U^{-1}A(H)U = \begin{pmatrix} A(G_0) & O \\ O & A(H)|_K \end{pmatrix}.$$ 

This shows that there is perfect state transfer between $\frac{1}{\sqrt{2}}(e_0 - e_1) \otimes |u\rangle$ and $\frac{1}{\sqrt{2}}(e_0 - e_1) \otimes |v\rangle$ at time $\tau$.

We remark that the construction used in Theorem 15 applies also to the graph $G \square K_2$ whose adjacency matrix is $I_2 \otimes A(G) + A(K_2) \otimes I$ (that is, trading the zero matrix for the identity matrix).

8 Conclusions

This work was motivated by the following question: can state transfer occur in a finite graph which is connected to an infinite graph? We show that the answer is affirmative in graphs with tails. As such, our work was influenced by Golinskii [17] who studied the spectra of graphs with tails. More surprisingly, we found that perfect state transfer is possible in graphs with tails. Our main observation is that state transfer can occur in the finite invariant subspace obtained from the decoupling theorem for eventually-free Jacobi matrices. It would be interesting to investigate state transfer (or lack thereof) in the infinite invariant subspace. In fact, it is possible to replace the infinite tail with an arbitrary (but suitably defined) finite graph. We leave this for future work.

We conclude with the following open questions.

1. The original motivation for this work is to investigate if spatial search works on infinite graphs. Although this question remains open, the corresponding problem for discrete-time quantum walk was studied by Konno et al. [22].
2. Are there deeper connections lurking between the dark subspaces studied here and the non-primary modules of the Terwilliger algebra?

3. Corollary 8 shows $\hat{G}(n, 1/2)$ has no dark subspace. But, what about $G(n, 1/2)$?

4. What information about the dark (protected) subspace is revealed by scattering (as employed by Farhi et al. [9])?

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