THE EFFECTIVE REPRODUCTION NUMBER: CONVEXITY, CONCAVITY AND INVARIANCE

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Abstract. Motivated by the question of optimal vaccine allocation strategies in heterogeneous population for epidemic models, we study various properties of the effective reproduction number. In the simplest case, given a fixed, non-negative matrix $K$, this corresponds mathematically to the study of the spectral radius $R_e(\eta)$ of the matrix product Diag$(\eta)K$, as a function of $\eta \in \mathbb{R}_+^n$. The matrix $K$ and the vector $\eta$ can be interpreted as a next-generation operator and a vaccination strategy. This can be generalized in an infinite dimensional case where the matrix $K$ is replaced by a positive integral compact operator, which is composed with a multiplication by a non-negative function $\eta$.

We give sufficient conditions for the function $R_e$ to be convex or a concave. Eventually, we provide equivalence properties on models which ensure that the function $R_e$ is unchanged.

1. Introduction

1.1. The mathematical question. For $p \in [1, +\infty]$, we consider the Lebesgue space $L^p$, with its usual norm $\|\cdot\|_p$, on a $\sigma$-finite measure space $(\Omega, \mathcal{F}, \mu)$. We denote by $\|\cdot\|_{L^p}$ the operator norm on the Banach space of bounded operators from $L^p$ to $L^p$. For a bounded operator $T$ on $L^p$, we denote by $\rho(T) = \lim_{n \to \infty} \|T^n\|_p^{1/n}$ its spectral radius. We recall that an operator $T$ on $L^p$ is positive if $T(L^p_+ \subseteq L^p_+$, where $L^p_+$ denotes the set of non-negative functions in $L^p$. For $h \in L^\infty_+$, let $M_h : f \mapsto hf$ denote the bounded operator on $L^p$.

According to the Krein-Rutman theorem, if $T$ is a positive compact operator on $L^p$ such that $\rho(T)$ is positive, then $\rho(T)$ is also an eigenvalue. For such an operator we define the map $R_e[T]$ on $L^\infty_+$ by:

\[ R_e[T](h) = \rho(T M_h). \]  

(1)

By homogeneity of the spectral radius, the study of the map $R_e[T]$, it is enough to consider this map only on the subset $\Delta \subset L^\infty_+$ of non-negative measurable functions bounded by 1. Our aim is to provide sufficient conditions on $T$ for the map $R_e[T]$ to be convex or concave on $\Delta$. We briefly explain in the next section how this question is related to the optimal vaccination problem in epidemic models.

1.2. The epidemic motivation. In finite metapopulation models, the population is divided into $N \geq 2$ different sub-populations; this amounts to considering the discrete state space $\Omega_d = \{1, \ldots, N\}$. Following [21], the entry $K_{ij}$ of the so-called next-generation matrix $K$ is equal to the expected number of secondary infections for people in subgroup $i$ resulting from a single randomly selected non-vaccinated infectious person in subgroup $j$. The matrix $K$ has non-negative entries, and represents the compact positive operator $T$. Let $\eta \in \Delta = [0, 1]^N$ represent a vaccination strategy, that is, $\eta_i$ is the fraction of non-vaccinated individuals in the $i$th sub-population; thus $\eta_i = 0$ when the $i$th sub-population is fully vaccinated, and 1 when it is not vaccinated at all — this seemingly unnatural convention is in particular motivated by the simple form of Equation (1).
So, the strategy $\mathbb{1} \in \Delta$, with all its entries equal to 1, corresponds to an entirely non-vaccinated population.

The effective reproduction number $R_e[K](\eta)$ associated to the vaccination strategy $\eta$ is then the spectral radius of the matrix $K \cdot \text{Diag}(\eta)$:

$$R_e[K](\eta) = \rho(K \cdot \text{Diag}(\eta)),$$

where $\text{Diag}(\eta)$ is the diagonal matrix with diagonal entries $\eta$. It may be interpreted as the mean number of infections coming from a typical case in the SIS model (where “S” and “I” stand for susceptible and infected). In particular, we denote by $R_0 = R_e[K](\mathbb{1})$ the so-called basic reproduction number associated to the metapopulation epidemiological model, see Lajmanovich and Yorke [25]. Let us mention that in this model if $R_0 \leq 1$, then there is no endemic equilibrium (i.e., the epidemic vanishes asymptotically), whereas if $R_0 > 1$, there exists at least one non-trivial endemic equilibrium (which means that the epidemic is persistent). With the interpretation of the function $R_e$ in mind, it is then very natural to minimize it under a constraint on the cost of the vaccination strategies $\eta$. This constrained optimization problem appears in most of the literature for designing efficient vaccination strategies for multiple epidemic situation (SIS/SIR/SEIR) [6, 10, 14, 15, 21, 28, 31, 38]. Note that in some of these references, the effective reproduction number is defined as the spectral radius of the matrix $\text{Diag}(\eta) \cdot K$. Since the eigenvalues of $\text{Diag}(\eta) \cdot K$ are exactly the eigenvalues of the matrix $K \cdot \text{Diag}(\eta)$, this actually defines the same function $R_e[K]$.

Given the importance of convexity to solve optimization problems efficiently, it is natural to look for conditions on the matrix $K$ that imply convexity or concavity for the map $R_e[K]$ defined by (2). Those properties can be useful to design vaccination strategies in the best possible way; see the companion papers [9, 12].

1.3. The finite dimensional case. In their investigation of the behavior of the map $R_e[K]$ defined in (2), Hill and Longini conjectured in [21] sufficient spectral conditions to get either concavity or convexity. More precisely, guided by explicit examples, they state that $R_e[K]$ should be convex if all the eigenvalues of $K$ are non negative real numbers, and that it should be concave if all eigenvalues are real, with only one positive eigenvalue.

Our first series of results show that, while this conjecture cannot hold in full generality – see Section 4.1 – it is true under an additional symmetry hypothesis. Recall that a matrix $K$ is called diagonally symmetrizable if there exist positive numbers $(d_1, \ldots, d_N)$ such that for all $i, j$, $d_i K_{ij} = d_j K_{ji}$. Such a matrix is diagonalizable with real eigenvalues according to the spectral theorem for symmetric matrices. The following result, which appears below in the text as Theorem 4.1, settles the conjecture for diagonally symmetrizable matrices. Let us mention that the eigenvalue $\lambda_1$ in the theorem below is non-negative and is equal to the spectral radius of $K$, that is, $\lambda_1 = R_e[K](\mathbb{1}) = R_0$, thanks to the Perron-Frobenius theory. We consider the function $R_e = R_e[K]$ defined on $[0, 1]^N$.

**Theorem 1.1.** Let $K$ be an $N \times N$ matrix with non-negative entries. Suppose that $K$ is diagonally symmetrizable with eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_N$.

(i) If $\lambda_N \geq 0$, then the function $R_e$ is convex.

(ii) If $\lambda_2 \leq 0$, then the function $R_e$ is concave.

Note that the case (i) appears already in Cairns [6]; see Section 4.1 below for a detailed comparison with existing results. This completes results on log-convexity of the map $R_e[K]$ given in [16, 18]. Notice also that if $K$ and $K'$ are diagonally similar up to transposition, they define the same function $R_e$; see [11] for more results in this direction. Eventually, the concavity of the map $R_e[K]$ implies that $K$ has a unique irreducible component in its Perron-Frobenius diagonalization as shown in Lemma 5.10 below.

1.4. The general case. We now give our main result in the setting of Section 1.1. We give in Definition 4.2 an extension to the notion of “diagonally symmetrizable” for compact operators. For
example, according to Proposition 4.9, if $T'$ is a self-adjoint compact operator on $L^2$ and $f, g$ are two non-negative measurable functions defined on $\Omega$ bounded and bounded away from 0, then the operator $T = M_f T'M_g$ is a compact diagonally symmetrizable on $L^2$. In particular, Corollary 4.8 states that diagonally symmetrizable compact operators on $L^p$, with $p \in [1, +\infty)$, have a real spectrum.

For a compact operator $T$, let $p(T)$ (resp. $n(T)$) denote the number of eigenvalues with positive (resp. negative) real part taking into account their (algebraic) multiplicity. Then, we obtain the following result given in Theorem 4.10 below.

**Theorem 1.2** (Convexity/Concavity of $R_e$). Let $T$ be a positive compact diagonally symmetrizable operator on $L^p$ with $p \in [1, +\infty)$. We consider the function $R_e = R_e[T]$ defined on $\Delta$.

(i) If $n(T) = 0$, then the function $R_e$ is convex.

(ii) If $p(T) = 1$, then the function $R_e$ is concave.

The proof of the concavity property relies on the explicit expression of the second derivative of $R_e[T]$ when $T$ is self-adjoint and the uses of the Sylvester’s inertia theorem.

The concavity property of $R_e[T]$ implies a strong structural property on the operator $T$. In order to establish this result, we present in Section 5 an atomic decomposition of the space $\Omega$ related to the operator $T$ following [34]. In particular, we extend the notion of quasi-irreducible operator to the non self-adjoint case and say an operator is monatomic if it has only one non-trivial irreducible component; see Definition 5.5 in Section 5.2. If $T$ is a positive compact operator on $L^p$ for some $p \in [1, +\infty)$ with $R_0 = R_e[T](\mathbb{1}) > 0$, where $\mathbb{1} \in \Delta$ is the constant function equal to 1, then we have the following properties:

(i) If $R_e[T]$ is concave, then $T$ is monatomic according to Lemma 5.10.

(ii) If $p(T) = 1$, then $R_0$ is simple and the only eigenvalue in $\mathbb{R}_+$, and thus $T$ is monatomic according to Lemma 5.9.

(iii) More generally, using the decomposition of a reducible operator from Lemma 5.3, we get that if $\text{Spec}(T) \subset \mathbb{R}_- \cup \{R_0\}$ and $T$ is a diagonally symmetrizable operator, then the function $R_e$ is the maximum of $m$ concave functions which are non-zero on $m$ pairwise disjoint subsets of $\Delta$, where $m$ is the (algebraic) multiplicity of $R_0$.

Eventually, by considering a general positive compact operator $T$ on $L^p$ for some $p \in [1, +\infty)$, following [34], we provide in Corollary 5.4 the decomposition $R_e[T]$ on the irreducible atoms:

$$R_e[T] = \max_{i \in I} R_e[T_i],$$

where $T_i(\cdot) = 1_{\Omega_i} T (\mathbb{1}_{\Omega_i} \cdot)$ with $(\Omega_i, i \in I)$ the at most countable collection of irreducible atoms in $\Omega$ associated to $T$.

1.5. **Structure of the paper.** After recalling the mathematical framework in Section 2, we discuss invariance properties of $R_e$ in Section 3. The convexity properties of $R_e$ and the related conjecture of Hill and Longini are discussed in Section 4. Finally, the case of reducible operators is treated in Section 5, using the Frobenius decomposition from [34].

2. **Setting, notations and previous results**

2.1. **Spaces, operators, spectra.** All metric spaces $(\mathcal{S}, d)$ are endowed with their Borel $\sigma$-field denoted by $\mathcal{B}(\mathcal{S})$. The set $\mathcal{K}$ of compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance $d_H$ is a metric space, and the function rad from $\mathcal{K}$ to $\mathbb{R}_+$ defined by $\text{rad}(K) = \max\{|\lambda|, \lambda \in K\}$ is Lipschitz continuous from $(\mathcal{K}, d_H)$ to $\mathbb{R}$ endowed with its usual Euclidean distance.

Let $(\Omega, \mathcal{F}, \mu)$ be a measured space, with $\mu$ a $\sigma$-finite (positive and non-zero) measure. For $f$ and $g$ real-valued functions defined on $\Omega$, we may write $(f, g)$ or $\int_{\Omega} f g \, d\mu$ for $\int_{\Omega} f(x)g(x) \mu(dx)$ whenever the latter is meaningful. For $p \in [1, +\infty)$, we denote by $L^p = L^p(\mu) = L^p(\Omega, \mu)$ the space of real-valued measurable functions $g$ defined on $\Omega$ such that $\|g\|_p = (\int |g|^p \, d\mu)^{1/p}$ (with
the convention that \( \| g \|_\infty \) is the \( \mu \)-essential supremum of \( |g| \) is finite, where functions which agree \( \mu \)-a.e. are identified. We denote by \( L^p_+ \) the subset of \( L^p \) of non-negative functions. We define \( \Delta \) as the subset of \( L^\infty \) of \([0,1]\)-valued measurable functions defined on \( \Omega \). We denote by \( \mathbb{1} \) (resp. \( 0 \)) the constant function on \( \Omega \) equal to 1 (resp. 0).

Let \( (E, \| \cdot \|) \) be a complex Banach space. We denote by \( \| \cdot \|_E \) the operator norm on \( \mathcal{L}(E) \) the Banach algebra of bounded operators. The spectrum \( \text{Spec}(T) \) of \( T \in \mathcal{L}(E) \) is the set of \( \lambda \in \mathbb{C} \) such that \( T - \lambda \text{Id} \) does not have a bounded inverse operator, where \( \text{Id} \) is the identity operator on \( E \). Recall that \( \text{Spec}(T) \) is a compact subset of \( \mathbb{C} \), and that the spectral radius of \( T \) is given by:

\[
(3) \quad \rho(T) = \text{rad}(\text{Spec}(T)) = \lim_{n \to \infty} \| T^n \|_E^{1/n}.
\]

The element \( \lambda \in \text{Spec}(T) \) is an eigenvalue if there exists \( x \in E \) such that \( Tx = \lambda x \) and \( x \neq 0 \).

Following [24], we define the (algebraic) multiplicity of \( \lambda \in \mathbb{C} \) by:

\[
m(\lambda, T) = \dim \left( \bigcup_{k \in \mathbb{N}^*} \ker(T - \lambda \text{Id})^k \right),
\]

so that \( \lambda \) is an eigenvalue if \( m(\lambda, T) \geq 1 \). We say the eigenvalue \( \lambda \) of \( T \) is simple if \( m(\lambda, T) = 1 \).

If \( E \) is also an algebra of functions, for \( g \in E \), we denote by \( M_g \) the multiplication operator (possibly unbounded) defined by \( M_g(h) = gh \) for all \( h \in E \); if furthermore \( g \) is the indicator function of a set \( A \), we simply write \( M_A \) for \( M_{1_A} \).

2.2. Invariance and continuity of the spectrum for compact operators. We collect some known results on the spectrum and multiplicity of eigenvalues related to compact operators. Let \( (E, \| \cdot \|) \) be a complex Banach space. Let \( A \in \mathcal{L}(E) \). We denote by \( A^\top \) the adjoint of \( A \). A sequence \( (A_n, n \in \mathbb{N}) \) of elements of \( \mathcal{L}(E) \) converges strongly to \( A \in \mathcal{L}(E) \) if \( \lim_{n \to \infty} \| A_n x - A x \| = 0 \) for all \( x \in E \).

Following [1], a set of operators \( A \subset \mathcal{L}(E) \) is collectively compact if the set \( \{Ax : A \in A, \| x \| \leq 1\} \) is relatively compact. Recall that the spectrum of a compact operator is finite or countable and has at most one accumulation point, which is 0. Furthermore, 0 belongs to the spectrum of compact operators in infinite dimension.

We refer to [33] for an introduction to Banach lattices and positive operators; we shall only consider the real Banach lattices \( L^p = L^p(\Omega, \mu) \) for \( p \in [1, +\infty] \) on a measured space \( (\Omega, \mathcal{F}, \mu) \) with a \( \sigma \)-finite positive non-zero measure, as well as their complex extension. (Recall that the norm of an operator on \( L^p \) or its natural complex extension is the same, see [17, Corollary 1.3]). A bounded operator \( A \) on \( L^p \) is positive if \( A(L^p_+ \subset L^p_+). \)

We say that two complex Banach spaces \( (E, \| \cdot \|) \) and \( (E', \| \cdot \|') \) are compatible if \( (E \cap E', \| \cdot \| +\| \cdot \|') \) is a Banach space, and \( E \cap E' \) is dense in \( E \) and in \( E' \). Given two compatible spaces \( E \) and \( E' \), two operators \( A \in \mathcal{L}(E) \) and \( A' \in \mathcal{L}(E') \) are said to be consistent if, with \( E'' = E \cap E' \), \( A(E'') \subset E'' \), \( A'(E'') \subset E'' \) and \( Ax = A'x \) for all \( x \in E'' \).

**Lemma 2.1** (Spectral properties). Let \( A, B \) be elements of \( \mathcal{L}(E) \).

(i) If \( E \) is a Banach lattice, and if \( A, B \) and \( A - B \) are positive operators, then we have:

\[
(4) \quad \rho(A) \geq \rho(B).
\]

(ii) If \( A \) is compact, then \( A^\top, AB \) and \( BA \) are compact and we have:

\[
(5) \quad \text{Spec}(A) = \text{Spec}(A^\top) \quad \text{and} \quad m(\lambda, A) = m(\lambda, A^\top) \quad \text{for} \ \lambda \in \mathbb{C}^*,
\]

\[
(6) \quad \text{Spec}(AB) = \text{Spec}(BA) \quad \text{and} \quad m(\lambda, AB) = m(\lambda, BA) \quad \text{for} \ \lambda \in \mathbb{C}^*,
\]

and in particular:

\[
(7) \quad \rho(AB) = \rho(BA).
\]
(iii) Let \((E', \| \cdot \|')\) be a complex Banach space and \(A' \in \mathcal{L}(E')\) such \((E, \| \cdot \|)\) and \((E', \| \cdot \|')\) are compatible, and \(A\) and \(A'\) are consistent. If \(A\) and \(A'\) are compact, then we have:

\[
\text{Spec}(A) = \text{Spec}(A') \quad \text{and} \quad m(\lambda, A) = m(\lambda, A') \quad \text{for} \ \lambda \in \mathbb{C}^*.
\]

(iv) Let \((A_n, n \in \mathbb{N})\) be a collectively compact sequence which converges strongly to \(A\). Then, we have \(\lim_{n \to \infty} \text{Spec}(A_n) = \text{Spec}(A)\) in \((\mathcal{K}, d_{\mathcal{K}})\), \(\lim_{n \to \rho} \rho(A_n) = \rho(A)\) and for \(\lambda \in \text{Spec}(A) \cap \mathbb{C}^*, r > 0\) such that \(\lambda' \in \text{Spec}(A)\) and \(|\lambda - \lambda'| \leq r\) implies \(\lambda = \lambda'\), and all \(n\) large enough:

\[
m(\lambda, A) = \sum_{\lambda' \in \text{Spec}(A_n), |\lambda - \lambda'| \leq r} m(\lambda', A_n).
\]

Proof. Property (i) can be found in [27, Theorem 4.2]. Property (iii) is in [7, Theorem 4.2.15].

Equation (5) from Property (ii) can be deduced from from24, Theorem p. 20]. Using [24, Proposition p. 25], we get the second part of (6) and \(\text{Spec}(AB) \cap \mathbb{C}^* = \text{Spec}(BA) \cap \mathbb{C}^*\), and thus (7) holds. To get the first part of (6), we only need to consider if 0 belongs to the spectrum or not. We first consider the infinite dimensional case: as \(A\) is compact, we get that \(AB\) and \(BA\) are compact, thus 0 belongs to their spectrum. We then consider the finite dimensional case: as \(\det(AB) = \det(A)\det(B) = \det(BA)\), where \(A\) and \(B\) denote also the matrix of the corresponding operator in a given base, we get that 0 belongs to the spectrum of \(AB\) if and only if it belongs to the spectrum of \(BA\).

We eventually check Property (iv). We deduce from [1, Theorems 4.8 and 4.16] (see also (d), (g) [take care that \(d(\lambda, K)\) therein is the algebraic multiplicity of \(\lambda\) for the compact operator \(K\) and not the geometric multiplicity] and (e) in [2, Section 3]) that \(\lim_{n \to \infty} \text{Spec}(A_n) = \text{Spec}(A)\) and (9). Then use that the function \(\text{rad}\) is continuous to deduce the convergence of the spectral radius from the convergence of the spectra.

We complete this section with an example of compatible Banach spaces. According to [7, Problem 2.2.9 p. 49], the spaces \(L^p(\mu)\) are compatible for all \(p \in [1, +\infty)\). We shall use the following slightly more general result. We recall that two \(\sigma\)-finite measures on \((\Omega, \mathcal{F})\), say \(\mu\) and \(\nu\), are mutually absolutely continuous if for \(A \in \mathcal{F}\), we have \(\mu(A) = 0 \iff \nu(A) = 0\). Thanks to the Radon-Nikodym theorem, the \(\sigma\)-finite measures \(\mu\) and \(\nu\) are mutually absolutely continuous if and only if there exists a positive finite measurable function \(h\) such that \(d\nu = h \, d\mu\).

**Lemma 2.2** (Compatibility of \(L^p\) spaces). Let \(\mu\) and \(\nu\) be two \(\sigma\)-finite measures on \((\Omega, \mathcal{F})\) which are mutually absolutely continuous, and let \(p, r \in [1, +\infty)\). Then, the spaces \(L^p(\mu)\) and \(L^r(\nu)\) are compatible.

**Proof.** First note that a property is true \(\mu\)-a.e. if and only if it is true \(\nu\)-a.e. since \(\mu\) and \(\nu\) are mutually absolutely continuous. Hence, we shall simply write that the property is true a.e. in this case.

Let us prove that \(L^p(\mu) \cap L^r(\nu)\) is dense in \(L^p(\mu)\). Let \(f \in L^r(\nu)\) such that \(f > 0\) a.e. For any \(g \in L^p_+(\mu)\) note that the non-decreasing sequence \((\min(g, nf), n \in \mathbb{N})\) of elements of \(L^p(\mu) \cap L^r(\nu)\) converges towards \(g\) a.e.; and so, it converges in \(L^p(\mu)\) according to the dominated convergence theorem. This gives \(L^p(\mu) \cap L^r(\nu)\) is dense in \(L^p(\mu)\) and in \(L^r(\nu)\) by symmetry.

To prove that \(L^p(\mu) \cap L^r(\nu)\) is complete (with respect to the norm given by the sum of the norms in \(L^p(\mu)\) and \(L^r(\nu)\)), it is enough to check that if a sequence \((h_n, n \in \mathbb{N})\) converges to \(g\) in \(L^p(\mu)\) and to \(f\) in \(L^r(\nu)\), then \(g = f\) a.e. This is immediate: for such a sequence, one can extract a sub-sequence which converges to \(g\) a.e. and to \(f\) a.e.

2.3. **The effective reproduction number** \(R_e\). For \(p \in [1, +\infty)\) and \(\eta \in \Delta\) the multiplication operator \(M_\eta\) is bounded, and if \(T\) is a compact operator on \(L^p\) then so is \(TM_\eta\). Following [10] where only integral operators where considered, and keeping similar notations, we define the
reproduction number associated to the positive compact operator $T$ (on $L^p$ for some $p \in [1, +\infty)$) as its spectral radius:

(10) \[ R_0[T] = \rho(T), \]

the effective spectrum function $\text{Spec}[T]$ from $\Delta$ to $\mathcal{K}$ by:

(11) \[ \text{Spec}[T](\eta) = \text{Spec}(TM_\eta), \]

and the effective reproduction number function $R_e[T] = \text{rad} \circ \text{Spec}[T]$ from $\Delta$ to $\mathbb{R}_+$ by:

(12) \[ R_e[T](\eta) = \text{rad}(\text{Spec}(TM_\eta)) = \rho(TM_\eta). \]

Take care that:

\[ \text{Spec}(T) = \text{Spec}[T](\mathbb{1}) \quad \text{and} \quad R_0[T] = R_e[T](\mathbb{1}). \]

When there is no risk of confusion on the positive compact operator $T$, we simply write $R_e$ and $R_0$ for the function $R_e[T]$ and the number $R_0[T]$. We have the following immediate properties for the function $R_e[T]$ (use Lemma 2.1 (i) for the third property).

**Proposition 2.3** (Elementary properties of $R_e$). The function $R_e = R_e[T]$, where $T$ is a positive compact operator on $L^p$ with $p \in [1, +\infty)$ satisfies the following properties:

(i) $R_e(\eta_1) = R_e(\eta_2)$ if $\eta_1 = \eta_2$, $\mu$ a.s., and $\eta_1, \eta_2 \in \Delta$,
(ii) $R_e(\mathbb{0}) = 0$ and $R_e(\mathbb{1}) = R_0$,
(iii) $R_e(\eta_1) \leq R_e(\eta_2)$ for all $\eta_1, \eta_2 \in \Delta$ such that $\eta_1 \leq \eta_2$,
(iv) $R_e(\lambda \eta) = \lambda R_e(\eta)$, for all $\eta \in \Delta$ and $\lambda \in [0, 1]$.

We shall use the following continuity property of the spectrum; see also [10, Proposition 3.6] for stronger results when considering integral operators and the weak topology on $\Delta$.

**Lemma 2.4** (Continuity of the spectrum). Let $T$ be a compact operator on $L^p$ with $p \in [1, +\infty)$. Let $(v_n, n \in \mathbb{N})$ and $(w_n, n \in \mathbb{N})$ be two bounded sequences in $L^\infty$ which converge respectively to $v_\infty$ and $w_\infty$, and let $T_n = M_{v_n} TM_{w_n}$. Then for any $\eta \in \Delta$, as $n$ goes to infinity, we have that:

(i) $\text{Spec}[T_n](\eta)$ converges to $\text{Spec}[T_\infty](\eta)$ in $\mathcal{K}$,
(ii) $R_e[T_n](\eta)$ converges to $R_e[T_\infty](\eta)$ in $\mathbb{R}$,
(iii) for any $\lambda \in \text{Spec}(T_\infty M_\eta) \cap \mathbb{C}^*$ and any $r > 0$ such that $\lambda' \in \text{Spec}(T_\infty M_\eta)$ and $|\lambda - \lambda'| \leq r$ implies $\lambda = \lambda'$, then for all $n$ large enough:

\[ m(\lambda, T_\infty M_\eta) = \sum_{\lambda' \in \text{Spec}(T_\infty M_\eta), |\lambda - \lambda'| \leq r} m(\lambda', T_n M_\eta). \]

**Proof.** Set $T_n' = TM_{v_n w_n}$ for $n \in \bar{\mathbb{N}}$, where $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. Using Lemma 2.1 (ii) for the second equality, we have that for $n \in \mathbb{N}$:

\[ \text{Spec}[T_n](\eta) = \text{Spec}(M_{v_n} TM_{w_n}) = \text{Spec}(TM_{v_n w_n}) = \text{Spec}(T_n'), \]

and similarly for the multiplicity. Notice the set of functions $\Delta' = \{\eta v_n w_n : \eta \in \Delta$ and $n \in \mathbb{N}\}$ is bounded in $L^\infty$ and thus the set of multiplication operators $\{M_h : h \in \Delta'\}$ is bounded in $L(L^p)$. We deduce from [1, Proposition 4.2] that the set $\{TM_h : h \in \Delta'\}$ is collectively compact. In particular, the sequence $(T_n', n \in \mathbb{N})$ is collectively compact.

Let $h \in L^p$, we have $\|T_n' h - T_\infty' h\|_p \leq \|T\|_{L^p} \|v_\infty w_\infty - v_n w_n\|_p$. Then, use dominated convergence to get that $\lim_{n \to +\infty} \|v_\infty w_\infty - v_n w_n\|_p = 0$. This implies that the sequence $(T_n', n \in \mathbb{N})$ converges strongly to $T_\infty'$. Then use Lemma 2.1 (iv) to conclude. \hfill \Box

**Remark 2.5** (On integral operators). Consider the positive integral operator defined by:

(14) \[ T_k(g)(x) = \int_{\Omega} k(x, y) g(y) \, d\mu(y), \]
where k is a kernel on \( \Omega \), that is a non-negative measurable function defined on \( \Omega \times \Omega \). Under the hypothesis that k has a finite double norm in \( L^p \) for some \( p \in [1, +\infty) \), that is:

\[
\| k \|_{p,q} = \int_{\Omega} \left( \int_{\Omega} |k(x,y)|^q \mu(dy) \right)^{p/q} \mu(dx)
\]

is finite with \( q = p/(p-1) \), the operator \( T_k \) is compact if \( p > 1 \) and \( T_k^2 \) compact if \( p = 1 \); see [19, p. 293]. When \( p > 1 \), one gets stronger results on the continuity of the function \( R_e[T_k] \); see Theorem 3.5 and Proposition 3.6 in [10] (where \( R_e[T_k] \) is denoted \( R_e[k] \) therein).

We conclude this section with a remark on the definition of the operator \( M_f T M_g \) when \( T \) is a positive operator and \( f \) and \( g \) are non-negative measurable function.

**Remark 2.6 (On \( M_f T M_g \)).** Let \( T \) be a positive compact operator on \( L^p(\mu) \) for some \( p \in [1, +\infty) \) and \( f, g \) be non-negative measurable functions defined on \( \Omega \). If the functions \( f, g \) are bounded, then the operator \( M_f T M_g \) is a positive compact operator on \( L^p(\mu) \). Motivated by Example 4.5, we shall however be interested in considering possibly unbounded functions \( f \) and \( g \). In this case, the operator \( T M_g \) is a positive compact operator from \( E = L^p((1+q)\mu) \) to \( L^p(\mu) \), and thus \( M_f T M_g \) is a positive compact operator from \( E \) to \( E' = L^p((1+f)^{-1}\mu) \). Let \( r \in [1, +\infty) \) and \( \nu \) a \( \sigma \)-finite measure mutually absolutely continuous with \( \mu \). Taking \( F = E \) or \( E' \), and using the compatibility between \( F \) and \( E'(\nu) \) given by Lemma 2.2, we deduce that there exists at most a unique continuous extension of \( M_f T M_g \) as a bounded operator on \( L^r(\nu) \), which we shall still denote by \( M_f T M_g \). By construction, this extension, when it exists, is also positive. However, let us stress that it is not compact a priori.

## 3. Spectrum-preserving transformations

In this section, we consider a measured space \((\Omega, \mathcal{F}, \mu)\) with \( \mu \) a non-zero \( \sigma \)-finite measure, and we discuss two operations on the positive compact operator \( T \) which leave invariant the functions \( \text{Spec}[T] \) and \( R_e[T] \) defined on \( \Delta \). Recall the discussion on the operator \( M_f T M_g \) from Remark 2.6.

**Lemma 3.1.** Let \( T \) be a positive compact operator on \( L^p \) for some \( p \in [1, +\infty) \) and \( h \) be a measurable non-negative function defined on \( \Omega \).

(i) If \( M_h T \) and \( T M_h \) are positive compact operators (respectively on \( L^r \) and \( L^s \) with \( r, s \in [1, +\infty) \) possibly distinct), then we have:

\[
\text{Spec}[M_h T] = \text{Spec}[M_h T M_{\{h>0\}}] = \text{Spec}[M_{\{h>0\}} T M_h] = \text{Spec}[T M_h],
\]

\[
R_e[M_h T] = R_e[M_h T M_{\{h>0\}}] = R_e[M_{\{h>0\}} T M_h] = R_e[T M_h].
\]

(ii) If \( h \) is positive and if \( M_h T M_{1/h} \) is a positive compact operators (on some \( L^r \) with \( p, r \in [1, +\infty) \) possibly distinct), then we have:

\[
\text{Spec}[T] = \text{Spec}[M_h T M_{1/h}] \quad \text{and} \quad R_e[T] = R_e[M_h T M_{1/h}].
\]

(iii) The adjoint operator \( T^\top \) is a positive compact operator on \( L^q \), with \( q = p/(p-1) \) and we have:

\[
\text{Spec}[T^\top] = \text{Spec}[T] \quad \text{and} \quad R_e[T^\top] = R_e[T].
\]

Let us stress that the compactness hypothesis of \( T \) can be removed in the statement of (i). Even if (ii) is a consequence of (i), we state it separately since (ii) and (iii) describe two modifications of \( T \) that leave the functions \( R_e \) and \( \text{Spec} \) invariant. See Remark 5.2 and Lemma 4.7 for other transformations on the operators which leaves the functions \( R_e \) and \( \text{Spec} \) invariant. See also [11] for further results in the finite dimensional case.

**Proof.** Since \( R_e = \text{rad} \circ \text{Spec} \), we only need to prove (i)-(iii) for the function \( \text{Spec} \). We give a detailed proof of (ii) and leave the proof of (i), which is very similar, to the reader. We first assume that \( T \) is a positive compact operator on \( L^p \), and \( h \) and \( 1/h \) are bounded. The operators
$TM_{\eta}$ and $M_{h}TM_{\eta/h}$ and the multiplication operators $M_{h}$ and $M_{1/h}$ are bounded operators on $L^{p}$. We have, using that $TM_{\eta/h}$ is compact and (6) for the second equality:

$$\text{Spec}(TM_{\eta}) = \text{Spec}(TM_{\eta/h} M_{h}) = \text{Spec}(M_{h}TM_{\eta/h}).$$

Since $\eta \in \Delta$ is arbitrary, this gives that $\text{Spec}[T] = \text{Spec}[M_{h}TM_{1/h}]$.

In the general case, we use an approximation scheme. Assume that $T$ and $T' = M_{h}TM_{1/h}$ are positive compact operators (respectively on $L^{p}$ and $L'$ with $p, r \in [1, +\infty)$) and $h$ is a positive function. For $n \in \mathbb{N}^{*}$, set:

$$v_{n} = \mathbb{I}_{\{n > h > 1/n\}} \quad \text{and} \quad h_{n} = n^{-1} \wedge (h \wedge n).$$

Notice that $T_{n} = M_{v_{n}}TM_{v_{n}}$ and $T''_{n} = M_{v_{n}h_{n}}TM_{v_{n}/h_{n}}$ are positive compact operators on $L^{p}$. Let $\eta \in \Delta$. From the first part of the proof, we get that:

$$\text{Spec}(T_{n}M_{\eta}) = \text{Spec}(T''_{n}M_{\eta}).$$

Consider also the positive compact operator on $L'$ defined by $T'_{n} = M_{v_{n}}T'M_{v_{n}}$. Since the sequence $(v_{n}, n \in \mathbb{N}^{*})$ converges in $L^{\infty}$ to $\mathbb{1}$, we deduce from Lemma 2.4 that:

$$\lim_{n \rightarrow \infty} \text{Spec}(T_{n}M_{\eta}) = \text{Spec}(TM_{\eta}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{Spec}(T'_{n}M_{\eta}) = \text{Spec}(T'M_{\eta}).$$

Since $L^{p}$ and $L'$ are compatible, see Lemma 2.2 and the compact operators $T_{n}'$ and $T''_{n}$ are consistent, we deduce from Lemma 2.1 (iii) that:

$$\text{Spec}(T''_{n}M_{\eta}) = \text{Spec}(T'_{n}M_{\eta}).$$

In conclusion, we obtain that $\text{Spec}(TM_{\eta}) = \text{Spec}(T'M_{\eta})$ and thus $\text{Spec}[T] = \text{Spec}[T']$.

We now prove (iii). Notice that $TM_{\eta}$ and $(TM_{\eta})^{\top}$ are compact operator. We have:

$$\text{Spec}(TM_{\eta}) = \text{Spec}(M_{h}TM_{\eta}^{\top}) = \text{Spec}((TM_{\eta}^{\top})^{\top}) = \text{Spec}(TM_{\eta}),$$

where we used (6) for the first equality, and (5) for the second. Since this is true for any $\eta \in \Delta$, this gives $\text{Spec}[T^{\top}] = \text{Spec}[T]$. \hfill \Box

**Remark 3.2 (Multiplicity of the eigenvalues).** Following closely the proof of Lemma 3.1 (ii), we also get under the assumption of Lemma 3.1 (ii) that:

$$m(\lambda, T) = m(\lambda, M_{h}TM_{1/h}) \quad \text{for all } \lambda \in \mathbb{C}^{*}.$$  

4. **Sufficient conditions for convexity or concavity of $R_{e}$**

4.1. **A conjecture from Hill and Longini.** Recall that, in the metapopulation framework with $N$ groups, the effective reproduction number is equal to the spectral radius of the matrix $K \cdot \text{Diag}(\eta)$, where the next-generation matrix $K$ is a $N \times N$ matrix with non-negative entries and $\eta \in \Delta = [0, 1]^{N}$ is the vaccination strategy giving the proportion of non-vaccinated people in each groups. Hill and Longini conjecture in [21] conditions on the spectrum of the next-generation matrix that should imply convexity or concavity of the effective reproduction number. The conjecture states that the function $R_{e}[K]$ is:

(i) convex when $\text{Spec}(K) \subset \mathbb{R}_{+}$,

(ii) concave when $\text{Spec}(K) \setminus \{R_{0}\} \subset \mathbb{R}_{-}$.

It turns out that the conjecture cannot be true without additional assumptions on the matrix $K$. Indeed, consider the following next-generation matrix:

$$K = \begin{pmatrix}
16 & 12 & 11 \\
1 & 12 & 12 \\
8 & 1 & 1
\end{pmatrix}.$$  

Its eigenvalues are approximately equal to 24.8, 2.9 and 1.3. Since $R_{e}$ is homogeneous, the function is entirely determined by the value it takes on the plane $\{\eta : \eta_{1} + \eta_{2} + \eta_{3} = 1/3\}$. The graph
Figure 1. Counter-example of the Hill-Longini conjecture. The plan of strategies $P = \{\eta : \eta_1 + \eta_2 + \eta_3 = 1/3\}$ is represented as a gray surface. The triangulated surface corresponds to the graph of $\eta \mapsto R_e[K](\eta)$ restricted to $P$.

of the function $R_e$ restricted to this set has been represented in Figure 1(a). The view clearly shows the saddle nature of the surface. Hence, the Hill-Longini conjecture (i) is contradicted in its original formulation.

In the same manner, the eigenvalues of the following next-generation matrix:

$$K = \begin{pmatrix} 9 & 13 & 14 \\ 18 & 6 & 5 \\ 1 & 6 & 6 \end{pmatrix}$$

are approximately equal to 26.3, −1.4 and −3.9. Thus, $K$ satisfies the condition that should imply the concavity of the effective reproduction number in the Hill-Longini conjecture (ii). However, as we can see in Figure 1(b), the function $R_e$ is neither convex nor concave.

Despite these counter-examples, the Hill-Longini conjecture is indeed true when making further assumption on the next-generation matrix. Let $M$ be a square real matrix. The matrix $M$ is \textit{diagonally similar} to a matrix $M'$ if there exists a non singular real diagonal matrix $D$ such that $M = D \cdot M' \cdot D^{-1}$. The matrix $M$ is said to be \textit{diagonally symmetrizable} or simply \textit{symmetrizable} if it is diagonally similar to a symmetric matrix, or, equivalently, if $M$ admits a decomposition $M = D \cdot S$ (or $M = S \cdot D$), where $D$ is a diagonal matrix with positive diagonal entries and $S$ is a symmetric matrix. If a matrix $M$ is diagonally symmetrizable, then its eigenvalues are real since similar matrices share the same spectrum. We obtain the following result when the next-generation matrix is symmetrizable as a particular case of Theorem 4.10 below.

**Theorem 4.1.** Let $K$ be a diagonally symmetrizable $N \times N$ matrix with non-negative entries, and consider the function $R_e = R_e[K]$ defined on $\Delta = [0, 1]^N$.

(i) If $\text{Spec}(K) \subset \mathbb{R}_+$, then the function $R_e$ is convex.

(ii) If $R_0 = R_0[K]$ is a simple eigenvalue of $K$ and $\text{Spec}(K) \subset \mathbb{R}_- \cup \{R_0\}$, then the function $R_e$ is concave.

The first point (i) has been proved by Cairns in [6]. In [18], Friedland obtained that, if the next-generation matrix $K$ is not singular and if its inverse is an M-matrix (i.e., its non-diagonal coefficients are non-positive), then $R_e[K]$ is convex. Friedland’s condition does not imply that $K$
is symmetrizable nor that Spec(K) ⊂ R_+. On the other hand, the following matrix is symmetric definite positive (and thus R_σ is convex) but its inverse is not an M-matrix:

\[
K = \begin{pmatrix}
3 & 2 & 0 \\
2 & 2 & 1 \\
0 & 1 & 4
\end{pmatrix}
\]

with inverse \( K^{-1} = \begin{pmatrix}
1.4 & -1.6 & 0.4 \\
-1.6 & 2.4 & -0.6 \\
0.4 & -0.6 & 0.4
\end{pmatrix} \).

Thus Friedland’s condition and Property (i) in Theorem 4.1 are not comparable. Note that if K is diagonally symmetrizable and its inverse is an M-matrix, then the eigenvalues of K are actually non-negative thanks to [3, Chapter 6 Theorem 2.3] and one can apply Theorem 4.1 (i) to recover Friedland’s result in this case.

4.2. Generalization to compact operators. In this section, we give the analogue of Theorem 4.1 for positive compact operators instead of matrices. First, we proceed with some definitions. By analogy with the matrix case, we introduce the notion of diagonally symmetrizable operators.

**Definition 4.2** (Diagonally symmetrizable operator). A compact operator \( T \) on \( L^p(\mu) \), with \( p \in [1, +\infty) \), is called diagonally symmetrizable if there exists a \( \sigma \)-finite measure \( \mu' \) mutually absolutely continuous with respect to \( \mu \), and a compact self-adjoint operator \( T' \) on \( L^2(\mu') \) such that \( T \) and \( T' \) are consistent.

**Remark 4.3** (Diagonally symmetrizable operator in finite dimension). Let us check that Definition 4.2 coincide with the definition of diagonally symmetrizable matrices in finite dimension. Let \( \Omega \) be a finite set, say \( \{1, \ldots, n\} \), and without loss of generality assume that the measure \( \mu \) as well as \( \mu' \), which can be seen as vectors of \( \mathbb{R}^n \), have positive entries. The sets \( L^p(\mu) \) and \( L^2(\mu') \) are all equal to \( \mathbb{R}^n \) and \( T = T' \) can be represented by a matrix, say \( M \), in the canonical base of \( \mathbb{R}^n \). Let \( D \) be the diagonal matrix with diagonal entries \( \mu' \). Then \( T' \) being self-adjoint in \( L^2(\mu') \) is equivalent to \( DM \) being symmetric, and thus the matrix \( M \) is diagonally symmetrizable (in the sense of the previous section).

We give an example of diagonally symmetrizable integral operator motivated by the epidemiological framework of Example 4.5 below. Recall from Remark 2.5 that a kernel \( k \) on \( \Omega \) is a non-negative measurable function defined on \( \Omega \times \Omega \).

**Proposition 4.4** (Diagonally symmetrizable integral operators). Let \( p \in (1, +\infty) \) and \( q \) its conjugate, \( k \) be a symmetric kernel on \( \Omega^2 \), and \( f, g \) be two positive measurable functions on \( \Omega \) such that:

\[
\int_{\Omega} f(x)^p \left( \int_{\Omega} k(x,y) g(y)^q \mu(dy) \right)^{p/q} d\mu(x) < +\infty,
\]

\[
\int_{\Omega^2} f(x)g(x) k^2(x,y) f(y)g(y) \mu(dx) \mu(dy) < +\infty.
\]

Then, the integral operator \( T : u \mapsto (x \mapsto \int_{\Omega} f(x) k(x,y) g(y) u(y) \mu(dy)) \) on \( L^p(\mu) \) is compact positive and diagonally symmetrizable.

**Proof.** The measure \( d\mu' = (g/f) \, d\mu \) is \( \sigma \)-finite and mutually absolutely continuous with respect to \( \mu \). Consider the integral operator \( T' : u \mapsto (x \mapsto \int_{\Omega} f(x) k(x,y) f(y) u(y) \mu'(dy)) \). The integrability assumptions ensure that \( T \) is compact on \( L^p(\mu) \) and \( T' \) is compact (and in fact, Hilbert-Schmidt) on \( L^2(\mu') \); see Remark 2.5. According to Lemma 2.2, the Banach spaces \( L^p(\mu) \) and \( L^2(\mu') \) are compatible. Since the operators \( T \) and \( T' \) are defined by the same kernel formula on their respective space, they are consistent. Finally since the compact operator \( T' \) is clearly self-adjoint on \( L^2(\mu') \). This implies that \( T \) is diagonally symmetrizable. \( \square \)
Example 4.5 (Epidemics on graphon). Consider the SIS model on graphon introduced in [8, Example 1.3]. In this example, the next-generation operator is an integral operator as defined in Remark 2.5 associated to the kernel $k$ given by $k(x,y) = \beta(x) W(x,y) \theta(y)/\gamma(y)$ where $\beta(x)$ represents the susceptibility, $\theta(x)$ the infectiousness and $\gamma(x)$ the recovery rate of the individuals with trait $x$, and $W$ corresponds to the graph of the contacts within the population. More precisely, for $x,y \in \Omega$, the quantity $W(x,y) \in [0,1]$ represents the density of contacts between individuals with traits $x$ and $y$ and is equal to $W(y,x)$ by construction. We deduce from Proposition 4.4 that if $\beta \in L^p(\mu)$ and $\theta/\gamma \in L^q(\mu)$ with $p \in (1, +\infty)$ and $q$ its conjugate, then the integral operator $T_k$ with kernel $k$ defined by (14) is diagonally symmetrizable.

Remark 4.6 (Related notions). An operator $T$ on a Hilbert space is classically called symmetrizable if there exists a positive bounded self-adjoint operator $H$ such that $HT$ is self-adjoint: this notion is discussed for example in [20, 32, 36]. Our definition is closer in spirit to [26], where symmetrizability is discussed for operators on Banach spaces with respect to a scalar product. In the matrix case our setting is a bit more restrictive than generally symmetrizability, since we symmetrize by a diagonal matrix with positive terms. In the general case the conditions are not comparable, since we do not impose any upper nor lower bound assumption on the density $d\nu/d\mu$.

We complete Section 3 with an other example of operators having the same effective spectrum.

Lemma 4.7. Let $T$ be a diagonally symmetrizable compact operator on $L^p(\mu)$, with $p \in [1, +\infty)$, and let $T'$ be the associated self-adjoint operator from Definition 4.2. Then, we have that on $\Delta$:

$$\text{Spec}[T] = \text{Spec}[T'], \quad R_e[T] = R_e[T'] \quad \text{and} \quad m(\lambda, T) = m(\lambda, T') \quad \text{for } \lambda \in \mathbb{C}^*.$$  

Proof. Let $\mu'$ be the measure from Definition 4.2. Recall that the Banach spaces $L^p(\mu)$ and on $L^2(\mu')$ are compatible thanks to Lemma 2.2. Let $\eta \in \Delta$. Since $M_\eta$ is bounded (both on $L^p(\mu)$ and $L^2(\mu')$), the operators $TM_\eta$ and $TM_\eta$, acting respectively on $L^p(\mu)$ and $L^2(\mu')$, are both compact. Since $T$ and $T'$ are consistent, the operators $TM_\eta$ and $T'M_\eta$ are also consistent. Then use Lemma 2.1 (iii) to conclude. \hfill $\Box$

The next corollary is immediate as the spectrum of a self-adjoint operator, say $T'$, is real and its spectral radius is zero if and only if $T' = 0$.

Corollary 4.8. Let $T$ be a compact operator on $L^p(\mu)$, with $p \in [1, +\infty)$. If $T$ is diagonally symmetrizable, then its spectrum is real, and $T$ cannot be quasi-nilpotent: $R_0(T) = 0$ if and only if $T = 0$.

For a compact operator $T$, let $p(T)$ and $n(T)$ denote respectively the number of its eigenvalues with positive and negative real part taking into account their (algebraic) multiplicity:

$$p(T) = \sum_{\Re(\lambda) > 0} m(\lambda, T) \quad \text{and} \quad n(T) = \sum_{\Re(\lambda) < 0} m(\lambda, T).$$

We now give a consequence of Sylvester’s inertia theorem [5, Theorem 6.1].

Proposition 4.9 (Sylvester). Let $T$ be a compact diagonally symmetrizable operator on $L^p(\mu)$, with $p \in [1, +\infty)$. Let $f, g$ be positive bounded measurable functions defined on $\Omega$ which are also bounded away from 0. Then the compact operator $M_f TM_g$ on $L^p(\mu)$ is diagonally symmetrizable with the same inertia as $T$:

$$p(T) = p(M_f TM_g) \quad \text{and} \quad n(T) = n(M_f TM_g).$$

Proof. First note that if $h$ is a positive bounded and bounded away from 0, then for any $r \in [1, +\infty)$ and $\sigma$-finite non-zero measure $\nu$, the multiplication operator $M_h$ is bounded with bounded inverse on $L^r(\nu)$. In particular, the operator $\overline{T} = M_f TM_g$ is a compact operator on $L^p(\mu)$ as $T$ is compact.
Let \( T' \) be the compact self-adjoint operator on \( L^2(\mu') \) associated to \( T \) from Definition 4.2. The measure \( \mu' = (g/f) \, d\mu' \) is \( \sigma \)-finite and mutually absolutely continuous with respect to \( \mu' \) and \( \mu \) also. The mapping \( \Phi = M_{\sqrt{Tg}} \) is an isometry between the Hilbert spaces \( L^2(\mu') \) and \( L^2(\mu) \).

We now define the operator \( \tilde{T}' \) on \( L^2(\mu') \) by:

\[
\tilde{T}' = \Phi \circ (M_{\sqrt{Tg}} T' M_{\sqrt{Tg}}) \circ \Phi^{-1}.
\]

Since \( T' \) is compact, the operator \( \tilde{T}' \) is also compact. Since \( f \) and \( g \) are bounded and bounded away from 0, the sets \( L^p(\mu) \cap L^2(\mu') \) and \( L^p(\mu) \cap L^2(\tilde{\mu}) \) are equal. Since \( T \) and \( T' \) coincide on this set, so do \( M_{\sqrt{Tg}} T M_{\sqrt{Tg}} = \tilde{T} \) and \( \tilde{T}' \). The operator \( M_{\sqrt{Tg}} T' M_{\sqrt{Tg}} \) is bounded and symmetric on \( L^2(\mu') \), and therefore self-adjoint. Since \( \Phi \) is an isometry, we deduce that \( \tilde{T}' \) is self-adjoint on \( L^2(\mu') \). Therefore the operator \( \tilde{T} = M_{\sqrt{Tg}} T M_{\sqrt{Tg}} \) on \( L^p(\mu) \) is diagonally symmetrizable.

We now establish the following string of equalities:

\[
p(T) = p(\tilde{T}) = p(M_{\sqrt{Tg}} T' M_{\sqrt{Tg}}) = p(\tilde{T}') = p(\tilde{T}) = p(M_{\sqrt{Tg}} T M_{\sqrt{Tg}}),
\]

By Lemma 4.7, \( p(T) = p(T') \) and \( p(\tilde{T}') = p(\tilde{T}) \). Since \( M_{\sqrt{Tg}} \) is invertible in \( L^2(\mu') \) and \( T' \) is self-adjoint (thus with real eigenvalues), we get, using the generalization of Sylvester’s inertia theorem [5, Theorem 6.1] (the definition of inertia in that paper being consistent with the definition of \( p(\cdot) \) and \( n(\cdot) \), which can be checked using [5, Theorem 4.5(ii)]: \( p(T') = p(M_{\sqrt{Tg}} T' M_{\sqrt{Tg}}) \). Finally, since \( \Phi \) is an isometry, \( p(M_{\sqrt{Tg}} T' M_{\sqrt{Tg}}) = p(\tilde{T}') \), and (19) is justified.

The equalities are similar for the number of negative eigenvalues \( n(\cdot) \).

The following result is the analogue of Theorem 4.1 for positive compact operators. Note that if \( T \) is a positive compact operator with \( R_0[T] > 0 \), then \( R_0[T] \) is an eigenvalue of \( T \) thanks to the Krein-Rutman theorem, see [34, Corollary 9], and thus \( p(T) \geq 1 \).

**Theorem 4.10** (Convexity/Concavity of \( R_e \)). Let \( T \) be a positive compact diagonalyally symmetrizable operator on \( L^p(\mu) \), with \( p \in [1, +\infty] \). We consider the function \( R_e = R_e[T] \) defined on \( \Delta \).

(i) If \( n(T) = 0 \), then the function \( R_e \) is convex.

(ii) If \( p(T) = 1 \), then the function \( R_e \) is concave.

The proof for positive self-adjoint operator \( T \) is given in Section 4.3 for the convex case and in Section 4.4 for the concave case when \( T \) is compact. The extension to diagonally symmetrizable positive compact operators follows directly from Lemma 4.7.

**Remark 4.11** (Rank-one operator). The so-called configuration model occurs in finite dimension when the next-generation matrix has rank one. This corresponds to a classical mixing structure called the proportionate mixing introduced by [29] and used in many different epidemiological models. Motivated by the finite dimensional case, we consider a configuration kernel \( k \) defined by

\[
k = f \otimes g \quad \text{where} \quad (f \otimes g)(x, y) = f(x)g(y),
\]

with \( f \in L^p \) and \( g \in L^q \) for some \( p \in (1, +\infty) \) and \( q = p/(p - 1) \). We also suppose that \( \mu(fg > 0) > 0 \). Let \( T_k \) denote the integral operator with kernel \( k \), see Remark 2.5. According to Proposition 4.4, with \( k = I_{\{f>0\}} \otimes I_{\{g>0\}} \) and \( h \in \{f, g\} \) replaced by \( h + h' I_{\{h=0\}} \) for some positive function \( h' \in L^p(\mu) \cap L^q(\mu) \), we deduce that the integral operator \( T_k \) on \( L^p(\mu) \) is compact positive and diagonally symmetrizable. Since \( T_k \) is of rank one, we deduce from Theorem 4.10 that \( R_e[T_k] \) is convex and concave and thus linear. This can be checked directly as it is immediate to notice that:

\[
R_e[T_k](\eta) = \int_{\Omega} fg \eta \, d\mu.
\]

We shall provide in [12] a deeper study of configuration kernels in the context of epidemiology.
4.3. The convex case. The proof of Property (i) in Theorem 4.10 relies on an idea from [18] (see therein just before Theorem 4.3). Let $T$ be a self-adjoint operator on $L^2 = L^2(\mu)$ such that $\text{Spec}(T) \subset \mathbb{R}_+$. As $R_0[T] = 0$ implies $T = 0$ and thus $R_e[T] = 0$, we shall only consider the case $R_0[T] > 0$. Since $T$ is a self-adjoint positive semi-definite operator on $L^2$, there exists a self-adjoint positive semi-definite operator $Q$ on $L^2$ such that $Q^2 = T$. Thanks to (7), we have for $\eta \in \Delta$:

$$R_e[T](\eta) = \rho(T M_\eta) = \rho(Q^2 M_\eta) = \rho(Q M_\eta Q).$$

Since the self-adjoint operator $Q M_\eta Q$ on $L^2$ is also positive semi-definite, we deduce from the Courant-Fischer-Weyl min-max principle that:

$$R_e[T](\eta) = \rho(Q M_\eta Q) = \sup_{u \in L^2(\mu) \setminus \{0\}} \frac{\langle u, Q M_\eta Q u \rangle}{\langle u, u \rangle}.$$  

Since the map $\eta \mapsto \langle u, Q M_\eta Q u \rangle$ defined on $\Delta$ is linear, we deduce that $\eta \mapsto R_e[T](\eta)$ is convex as a supremum of linear functions.

4.4. The concave case. The proof of Property (ii) in Theorem 4.10 relies on a computation of the second derivative of the function $R_e$. Let $T$ be a positive compact self-adjoint operator on $L^2(\mu)$ such that $p(T) = 1$. Let $\Delta^*$ be the subset of $\Delta$ of the functions which are bounded away from 0. The set $\Delta^*$ is a dense convex subset of $\Delta$ (for the $L^2(\mu)$-convergence or simple convergence). The function $R_e = R_e[T]$ is continuous on $\Delta$, see Lemma 2.4 (indeed, with the notations therein, take $v_n = 1$, $w_n \in \Delta$ and notice that $R_e[T_n](1) = R_e[T](w_n)$ converges to $R_e[T_\infty](1) = R_e[T](w_\infty)$). So its suffice to prove that $R_e = R_e[T]$ is concave on $\Delta^*$. Let $\eta_0, \eta_1$ be elements of $\Delta^*$, and set $\eta_\alpha = (1 - \alpha) \eta_0 + \alpha \eta_1$ for $\alpha \in [0, 1]$ (which is also an element of $\Delta^*$). We write $T_\alpha = T M_{\eta_\alpha}$, so that $T_\alpha = T_0 + \alpha T M$, where $M = M_{\eta_1 - \eta_0}$ is the multiplication by $(\eta_1 - \eta_0)$ operator, and, with $R(\alpha) = R_e(\eta_\alpha)$:

$$R(\alpha) = \rho(T_\alpha) = \rho(T_0 + \alpha T M).$$

So, to prove that $R_e$ is concave on $\Delta^*$ (and thus on $\Delta$), it is enough to prove that $\alpha \mapsto R(\alpha)$ is concave on $(0, 1)$. Thanks to Sylvester's inertia theorem stated in Proposition 4.9 (with $f = 1$ and $g = \eta_0$), we also get that $p(T_\alpha) = p(T) = 1$. This implies that $R(\alpha)$ is positive and a simple eigenvalue.

We consider the following scalar product on $L^2(\mu)$ defined by $\langle u, v \rangle_\alpha = \langle u, \eta_\alpha v \rangle$. The operator $T_\alpha$ is self-adjoint and compact on $L^2(\eta_\alpha d\mu)$ with spectrum $\text{Spec}(T_\alpha)$ thanks to Lemma 2.1 (iii) and Lemma 2.2. Let $(\lambda_n, n \in I = [0, N])$, with $N \in \mathbb{N} \cup \{\infty\}$ be an enumeration of the non-zero eigenvalues of $T_\alpha$ with their multiplicity so that $\lambda_0 = R(\alpha) > 0$ and thus $\lambda_n < 0$ for $n \in I^* = I \setminus \{0\}$; and denote by $(u_n, n \in I)$ a corresponding sequence of orthogonal eigenvectors (in $L^2(\eta_\alpha d\mu)$). The functions $v_\alpha = u_0$ and $\phi_\alpha = \eta_\alpha u_0$ are the right and left-eigenvectors for $T_\alpha$ (seen as an operator on $L^2(\mu)$) associated to $R(\alpha)$.

We now follow [23] to get that $\alpha \mapsto R(\alpha) = \rho(T_0 + \alpha T M)$ is analytic and compute its second derivative. Let $\pi_\alpha$ be the projection on the $(\langle \cdot, \cdot \rangle_\alpha)$-orthogonal of $v_\alpha$, and define:

$$S_\alpha = (T_\alpha - R(\alpha))^{-1} \pi_\alpha.$$  

In other words, $S_\alpha$ maps $u_0$ to 0 and $u_i$ to $(\lambda_i - R(\alpha))^{-1} u_i$. Let $\alpha \in (0, 1)$ and $\varepsilon$ small enough so that $\alpha + \varepsilon \in [0, 1]$. We have:

$$T_{\alpha + \varepsilon} = T_\alpha + \varepsilon T M,$$

and thus $\|T_{\alpha + \varepsilon} - T_\alpha\|_{L^2(\eta_\alpha d\mu)} = O(\varepsilon)$. Using [23, Theorem 2.6] on the Banach space $L^2(\eta_\alpha d\mu)$, we get that:

$$R(\alpha + \varepsilon) = R(\alpha) + \varepsilon \langle v_\alpha, T M v_\alpha \rangle_\alpha - \varepsilon^2 \langle v_\alpha, T M S_\alpha T M v_\alpha \rangle_\alpha + O(\varepsilon^3).$$
Let $N_\alpha = M_{1/\eta_\alpha}M = MM_{1/\eta_\alpha}$ be the multiplication by $(\eta_1 - \eta_0)/\eta_\alpha$ bounded operator. Since $\alpha \mapsto R(\alpha)$ is analytic and $T$ is self-adjoint (with respect to $\langle \cdot, \cdot \rangle$), we get that:

$$R''(\alpha) = -2\langle v_\alpha, TMS_\alpha TMv_\alpha \rangle_{\alpha}$$

$$= -2\langle MT_\alpha v_\alpha, S_\alpha TMv_\alpha \rangle$$

$$= -2R(\alpha)\langle Mv_\alpha, S_\alpha TMv_\alpha \rangle$$

$$= -2R(\alpha)\langle N_\alpha v_\alpha, S_\alpha T_\alpha N_\alpha v_\alpha \rangle_{\alpha}.$$  

Since the kernel and the image of $T_\alpha$ are orthogonal (in $L^2(\eta_\alpha d\mu)$), and the latter is generated by $(u_n, n \in I)$, we have the decomposition $N_\alpha v_\alpha = g + \sum_{n \in I} a_n u_n$ with $g \in \text{Ker}(T_\alpha)$ and $a_n = \langle N_\alpha v_\alpha, u_n \rangle / \langle u_n, u_n \rangle$. This gives, with $I^* = I \setminus \{0\}$:

$$R''(\alpha) = 2R(\alpha)\sum_{n \in I^*} \frac{\lambda_n}{R(\alpha)} a_n^2 \langle u_n, u_n \rangle_{\alpha}. \quad (21)$$

Since $\lambda_n < 0$ for all $n \in I^*$, we deduce that $R''(\alpha) \leq 0$ and thus $\alpha \mapsto R(\alpha)$ is concave on $[0, 1]$. This implies that $R_\mu[T]$ is concave.

Remark 4.12. The same proof with obvious changes gives that if $T$ is a positive quasi-irreducible compact self-adjoint operator (see Section 5.2 for the precise definition of quasi-irreducible operator) such that $n(T) = 0$, then $R_\mu[T]$ is convex on $\Delta$. Then, using the decomposition of a compact operator on its irreducible atoms, see Section 5.1 and more precisely Lemma 5.3, and since the maximum of convex functions is convex (used in (26)), we can recover Theorem 4.10 (i).

5. The reproduction number and reducible positive compact operators

Following [34], we present in Section 5.1 the atomic decomposition of a positive compact operator $T$ on $L^p$ where $p \in [1, +\infty)$ and state a formula which “reduces” the effective reproduction function of $T$ on the whole space to the ones of the restriction of $T$ to each atoms (or irreducible components); see Corollary 5.4 below. Then, we consider the notion of quasi-irreducible and monatomic operators in Section 5.2, and provide some properties of monatomic operator and prove that if the effective reproduction number is concave then the operator is monatomic.

5.1. Atomic decomposition. Our presentation is a direct application of the Frobenius decomposition, see [22, 35] and [34] or the “super diagonal” form; see [13, Part II.2]. For convenience, we follow [34] for positive compact operator on $L^p(\mu)$ for some $p \in [1, +\infty)$, see also [4, Lemma 5.17] in the case of integral operators with symmetric kernel. We stress that the results in [34] are stated under the hypothesis that $\mu$ is a finite measure, but it is elementary to check the main results (Theorems 7 and 8 therein) also hold if the measure $\mu$ is $\sigma$-finite.

For $A, B \in F$, we write $A \subset B$ a.e. if $\mu(B^c \cap A) = 0$ and $A \subset B$ a.e. if $A \subset B$ a.e. and $B \subset A$ a.e.. Let $T$ be a positive compact operator on $L^p$ for some $p \in [1, +\infty)$. Let $f_0 \in L^p$ and $g_0 \in L^q$ be positive functions and consider the operator $T_0 = M_{g_0}TM_{f_0}$ from $L^\infty$ to $L^1$. We define the function $k_T$ on $F^2$ as, for $A, B \in F$:

$$k_T(B, A) = \int_B (T_0 1_A)(x) \mu(dx) = \langle 1_B, T_0 1_A \rangle. \quad (22)$$

It is clear from (22) that the family of sets $(B, A)$ such that $k_T(B, A) = 0$ does not depend on the choice of the positive functions $f_0 \in L^p$ and $g_0 \in L^q$. If the measure $\mu$ is finite, then one can take $f_0 = g_0 = 1$ and thus $T_0 = T$.

A set $A \in F$ is $T$-invariant, or simply invariant when there is no ambiguity on the operator $T$, if $k_T(A^c, A) = 0$. We recall that $I$ is a closed ideal of $L^p$ (with $p \in [1, +\infty)$) if and only if it is equal to $I_B = \{f \in L^p : f 1_B = 0\}$ for some measurable set $B \in F$, see Example 2 in [33,
Section III.1] or [37, Section III.2]. Notice that a set $A \in \mathcal{F}$ is invariant if and only if the ideal $\mathcal{I}_A$ is invariant for $T$, that is, $T(\mathcal{I}_A) \subset \mathcal{I}_A$.

A positive compact operator $T$ on $L^p$ is (ideal-)irreducible if the only closed invariant ideal are $\{0\}$ and $L^p$. Thus, the positive compact operator $T$ is irreducible if and only any $T$-invariant set $A$ is such that either $\mu(A) = 0$ or $\mu(A^c) = 0$. According to [30, Theorem 3] (see also [37, Section III.3] for an elementary presentation in $L^p$), if $T$ is an irreducible positive compact operator on $L^p$, then either $R_0[T] > 0$, or the situation is degenerate in the sense that $\Omega$ is an atom of $\mu$ (that is, for all $A \in \mathcal{F}$, we have either $\mu(A) = 0$ or $\mu(A^c) = 0$) and $T = 0$.

Let $\mathcal{A}$ be the set of $T$-invariant sets, and notice that $\mathcal{A}$ is stable by countable unions and countable intersections. Let $\mathcal{F}_{\text{inv}} = \sigma(\mathcal{A})$ be the $\sigma$-field generated by $\mathcal{A}$. Then, thanks to [34, Theorem 8], the operator $T$ restricted to an atom of $\mu$ in $\mathcal{F}_{\text{inv}}$ is irreducible. We shall only consider non degenerate atom, and say the atom (of $\mu$ in $\mathcal{F}_{\text{inv}}$) is non-zero if the restriction of the operator $T$ to this atom has a positive spectral radius. We denote by $(\Omega_i, i \in I)$ the at most countable (but possibly empty) collection of non-zero atoms of $\mu$ in $\mathcal{F}_{\text{inv}}$. Notice that the atoms are defined up to an a.e. equivalence and can be chosen to be pair-wise disjoint. For $i \in I$, we set:

$$T_i = M_{\Omega_i} T M_{\Omega_i},$$

which is a positive compact operator on $L^p$. Note that:

$$T \geq T' \quad \text{where} \quad T' = \sum_{i \in I} T_i.$$

We now give some properties of the Frobenius decomposition.

**Remark 5.1 (Properties of Frobenius decomposition).** We have, with $i \in I$:

(i) By definition of the non-zero atoms: $\mu(\Omega_i) > 0$ and $T$ restricted to $\Omega_i$ is irreducible with positive spectral radius, that is, $R_0[T_i] > 0$.

(ii) According to [34, Theorem 8], the spectral radius of $T_i$ is a simple eigenvalue of $T_i$: $m(R_0(T_i), T_i) = 1$.

(iii) According to [34, Theorem 7], for all $\lambda \in \mathbb{C}^*$, we have:

$$m(\lambda, T) = \sum_{j \in I} m(\lambda, T_j).$$

(iv) Consider the complement of the non-zero atoms, say $\Omega_0 = (\cup_{j \in I} \Omega_j)^c$ (with the convention that 0 does not belong to the set of indices $I$). Then, the restriction of $T$ to $\Omega_0$ is quasi-nilpotent, that is $R_0[T](1_{\Omega_0}) = 0$.

From those properties, we deduce the following elementary results.

(v) The cardinal of the set of indices $i \in I$ such that $R_0[T_i] = R_0[T]$ is exactly equal to the multiplicity of $R_0[T_i]$ for $T$, that is $m(R_0[T], T)$.

(vi) There exists at least one non-zero atom ($\sharp I > 1$) if and only if $R_0[T] > 0$.

(vii) The operator $T$ is quasi-nilpotent if and only if there is no non-zero atom ($\sharp I = 0$).

(viii) If $A \in \mathcal{F}$ invariant implies $A^c$ invariant (which is in particular the case if $T$ is self-adjoint and $p = 2$), then we have $T = \sum_{i \in I} T_i$ and thus the restriction of $T$ to $\Omega_0$ is zero (intuitively $T$ is block diagonal).

**Remark 5.2.** Assume that $T = T_k$ is an integral operator with kernel $k$ on $\Omega = [0, 1]$; see Remark 2.5. Then, the operators $T_i$ are integral operators with respective kernel $k_i$ given by $k_i(x, y) = 1_{\Omega_i}(x) k(x, y) 1_{\Omega_i}(y)$; and the operator $T' = T_{k'}$ is also an integral operator with kernel $k' = \sum_{i \in I} k_i$. We represent in Figure 2 (A) an example of a kernel $k$ with its atomic decomposition using a “nice” order on $\Omega$ (see [13, 22, 35] on the existence of such an order relation; intuitively the kernel is upper block triangular: the population on the “left” of an atom does not infect the population on the “right” of this atom) and in Figure 2 (B) the corresponding kernel $k'$. Notice that $k(\Omega_i, \Omega_j) = 0$ for $j$ “smaller” that $i$, where $k(A, B) = \int_{\Omega^2} 1_A(x) k(x, y) 1_B(y) \mu(dx) \mu(dy)$ is a consistent notation with (22).
Proof of Lemma 5.3. Let \( T \) be a positive compact operator on \( L^p \) for some \( p \in [1, +\infty) \). With the convention that \( \max_\emptyset = 0 \), we have for \( \eta \in \Delta \):

\[
R_e[T](\eta) = \max_{i \in I} R_e[T_i](\eta_i) = \max_{i \in I} R_e[T_i](\eta) = \max_{i \in I} R_e[T](\eta \mathbb{1}_{\Omega_i}),
\]

and more generally:

\[
m(\lambda, TM_\eta) = \sum_{i \in I} m(\lambda, T_i M_\eta) \quad \text{for all } \lambda \in \mathbb{C}^*.
\]

Before proving the lemma, we first state a direct consequence of (27), in the spirit of Section 3 on a spectrum-preserving transformation. Recall \( T' = \sum_{i \in I} T_i \) in (24).

Corollary 5.4. Let \( T \) be a positive compact operator on \( L^p \) for some \( p \in [1, +\infty) \). We have:

\[
\text{Spec}[T] = \text{Spec}[T'] = \bigcup_{i \in I} \text{Spec}[T_i] \quad \text{and} \quad R_e[T] = R_e[T'] = \max_{i \in I} R_e[T_i].
\]

Proof of Lemma 5.3. Let \( T' \) be a positive compact operator on \( L^p \). Recall the kernel \( k_{T'} \) defined in (22). For \( A \in \mathcal{F} \), let \( m(\lambda, T', A) \) denote the multiplicity (possibly equal to 0) of the eigenvalue \( \lambda \in \mathbb{C}^* \) for the operator \( T' M_A \). A direct application of [34, Lemma 11] (which holds also if \( \mu \) is a \( \sigma \)-finite measure) gives that for \( A, B \in \mathcal{F} \) such that \( A \cap B = \emptyset \) a.e. and \( k_{T'}(B, A) = 0 \), we have for all \( \lambda \in \mathbb{C}^* \) that:

\[
m(\lambda, T', A \cup B) = m(\lambda, T', A) + m(\lambda, T', B),
\]

and thus

\[
R_e[T'](1_A + 1_B) = \max(R_e[T'](1_A), R_e[T'](1_B)).
\]

Let \( A, B \in \mathcal{F} \) be such that \( A \cap B = \emptyset \) a.e. and \( k_{T'}(B, A) = 0 \). Let \( \eta \in \Delta \). Clearly we have \( k_{TM_\eta}(B, A) \leq k_{T}(B, A) \) and thus \( k_{TM_\eta}(B, A) = 0 \). Use (28) to get that for \( \eta \in \Delta \) and \( \lambda \in \mathbb{C}^* \):

\[
m(\lambda, TM_\eta, A \cup B) = m(\lambda, TM_\eta, A) + m(\lambda, TM_\eta, B).
\]
Then, an immediate adaptation of the proof of [34, Theorem 7] gives that for all $\lambda \in \mathbb{C}^*$:

\begin{equation}
 m(\lambda, TM_\eta, \Omega) = \sum_{i \in I} m(\lambda, TM_\eta, \Omega_i).
\end{equation}

By definition of $m(\lambda, \cdot, \cdot)$, we get $R_{e}[T](\eta) = \max\{|\lambda| : m(\lambda, TM_\eta, \Omega) > 0\}$ and $R_{e}[TM_\Omega](\eta) = \max\{|\lambda| : m(\lambda, TM_\eta, \Omega_i) > 0\}$. This gives that:

\[ R_{e}[T](\eta) = \max_{i \in I} R_{e}[TM_\Omega](\eta_i). \]

To conclude, notice, using Lemma 3.1 (i) for the second equality, that:

\[ R_{e}[T](\eta_1) = R_{e}[TM_\Omega](\eta) = R_{e}[M_\Omega, TM_\Omega](\eta) = R_{e}[T_i](\eta) = R_{e}[T_i](\eta_i). \]

Similarly we deduce (27) from (30). \hfill \Box

5.2. Monatomic operators and applications. Following [4, Definition 2.11], a positive compact operator $T$ is quasi-irreducible if there exists a measurable set $\Omega_a \subset \Omega$ such that $\mu(\Omega_a) > 0$, $T = M_{\Omega_a} TM_{\Omega_a}$ and $T$ restricted to $\Omega_a$ is irreducible with positive spectral radius. The quasi-irreducible property is natural in the setting of positive compact self-adjoint operators; in a more general setting, one would still want to consider positive compact operator with only one irreducible component. This motivates the next definition. Recall the atomic decomposition of the previous section.

**Definition 5.5** (Monatomic operator). Let $T$ be a positive compact operator on $L^p$ with some $p \in [1, +\infty)$. The operator is monatomic if there exists a unique non-zero atom ($\sharp I = 1$).

In a sense, the operator $T$ is “truly reducible” when $\sharp I \geq 2$. We shall give in a forthcoming work other characterizations of monatomic operator.

**Remark 5.6** (Link between (quasi-)irreducible and monatomic operators). Irreducible positive compact operators with positive spectral radius and quasi-irreducible positive compact operators are monatomic, and we have $T = T_a$ where $T_a = M_{\Omega_a} TM_{\Omega_a}$ and $\Omega_a$ is the non-zero atom, with $\Omega_a = \Omega$ in the reducible case.
Remark 5.7 (Reducibility for integral operators). We consider an integral operator $T_k$ with kernel $k$, see Remark 2.5, and we say the kernel $k$ is irreducible, quasi-irreducible or monatomic whenever the integral operator $T_k$ satisfies the corresponding property. Then, the notion of irreducibility of a kernel depends only on its support. Indeed, provided that the measure $\mu$ is finite and the kernel so that all the operators are well defined and compact, the kernel $k$ is irreducible (resp. quasi-irreducible, resp. monatomic) if and only if the kernel $\mathbb{1}_{\{k>0\}}$ is irreducible (resp. quasi-irreducible, resp. monatomic). Furthermore, the corresponding integral operators have the same atoms.

We have represented in Figure 3(a) a monatomic kernel $k$ on $\Omega = [0, 1]$ and in Figure 3(b) the kernel $k_a$ (with $k_a(x, y) = 1_{\Omega_a}(x)k(x, y)1_{\Omega_a}(y))$ associated to the quasi-irreducible integral operator $T_a = M_{\Omega_a}T_kM_{\Omega_a}$; the set $\Omega = [0, 1]$ being “nicely ordered” so that the representation of the kernels are upper triangular. Using the epidemic interpretation of Remark 5.8 below, we also represented the subset $\Omega_1$ of the population infected by the non-zero atom $\Omega_a$.

Remark 5.8 (Epidemiological interpretation). In the infinite dimensional SIS model developed in [8], the space $(\Omega, \mathcal{F}, \mu)$ represents all the traits of the population with $\mu(dy)$ the infinitesimal size of the population with trait $y$. The next-generation operator is given by the integral operator $T_k$, see Equation (14), where the kernel $k = k/\gamma$ is defined in terms of a transmission rate kernel $k$ and a recovery rate function $\gamma$ by the formula $k(x, y) = k(x, y)/\gamma(y)$ and has a finite double norm in $L^p$ for some $p \in (1, +\infty)$; and the basic reproduction number $R_0 = R_0(T_k)$ is then the spectral radius of $T_k$. Intuitively, $k(x, y) > 0$ (resp. $= 0$) means that individuals with trait $y$ can (resp. cannot) infect individual with trait $x$.

When the integral operator $T_k$ is monatomic, with non-zero atom $\Omega_a$, then the population with trait in $\Omega_a$ can infect itself as well as the population with other distinct traits, say $\Omega_i$. The population with trait $\Omega_i$ can only infect itself (but not $\Omega_a$); and there is no persistent epidemic outside $\Omega_a \cup \Omega_i$. We shall see in a forthcoming paper that the set $\Omega_a \cup \Omega_i$ is characterized as the smallest invariant set containing the atom $\Omega_a$.

From Lemma 5.3, we deduce the following two results related to monatomic operators.

**Lemma 5.9**. Let $T$ be a positive compact operator on $L^p$ with some $p \in [1, +\infty)$, and set $R_0 = R_0[T]$. If the operator $T$ is monatomic then $R_0 > 0$ and $R_0$ is simple (i.e., $m(R_0, T) = 1$). If $R_0$ is simple and the only eigenvalue in $(0, +\infty)$, then the operator $T$ is monatomic.

**Proof.** Let $T$ be monatomic, so that there exists only one non-zero atom, say $\Omega_a$. Set $T_a = M_{\Omega_a}T(M_{\Omega_a})$. Since the restriction of $T_a$ (or $T$) to $\Omega_a$ is irreducible and non-zero, we deduce from [30, Theorem 3] that its spectral radius is positive, and thus $R_0[T_a] > 0$. Using Lemma 5.3, this implies that $R_0[T] = R_0[T_a] > 0$. According to [34, Theorem 8], we get that $R_0[T_a]$ is simple for $T_a$. Since according to (25) $m(\lambda, T_a) = m(\lambda, T_a)$ for all $\lambda \in \mathbb{C}^*$, we deduce that $R_0[T]$ is simple for $T$.

For the second part, if $T$ is not monatomic and $R_0[T] = 0$, we deduce that there exists at least two non-zero atoms, and thus $\not\exists \lambda \geq 2$ (if there is no non-zero atom, then $T$ would be quasi-nilpotent and $R_0[T] = 0$). The restrictions of $T$ to those non-zero atoms have positive spectral radius according to [30, Theorem 3] and thus at least one positive eigenvalue by the Krein-Rutman theorem. We deduce from (25) that $T$ has at least two positive eigenvalues (counting their multiplicity if they are equal). This gives the result by contraposition. \hfill $\square$

**Lemma 5.10.** Let $T$ be a positive compact operator on $L^p$ for some $p \in [1, +\infty)$ such that $R_0[T] > 0$. If the function $R_a[T]$ is concave on $\Delta$, then the operator $T$ is monatomic.

**Proof.** Since $R_0[T]$ is positive, we deduce that $T$ is not quasi-nilpotent. Suppose that $T$ is not monatomic. This means that the cardinal of the at most countable set $I$ in the decomposition (26) is at least $2$. So let $T_1$ and $T_2$ be two quasi-irreducible components of $T$, where we assume that $\{1, 2\} \subset I$. Let $\Omega_1$ and $\Omega_2$ denote their respective non-zero atoms. Without loss of generality, we can suppose that $R_0[T_2] \geq R_0[T_1] > 0$. Consider the strategies $\eta_1 = 1_{\Omega_1}$ and...
\[ \eta_2 = R_0[T_1] R_0[T_2]^{-1} \mathbb{1}_{\Omega_2} \] (which both belong to \( \Delta \)). For \( \theta \in [0,1] \), we deduce from (26) and the homogeneity of the spectral radius that \( R_e[T](\theta \eta_1 + (1 - \theta) \eta_2) = R_e[T_1] \max(\theta, 1 - \theta) \). Since \( \theta \mapsto \max(\theta, 1 - \theta) \) is not concave, we deduce that \( R_e[T] \) is not concave on \( \Delta \). \qed

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