Uniqueness of Solutions to Nonlinear Schrödinger Equations from their Zeros

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Abstract
We show novel types of uniqueness and rigidity results for Schrödinger equations in either the nonlinear case or in the presence of a complex-valued potential. As our main result we obtain that the trivial solution $u = 0$ is the only solution for which the assumptions $u(t = 0)|_D = 0, u(t = T)|_D = 0$ hold, where $D \subset \mathbb{R}^d$ are certain subsets of codimension one. In particular, $D$ is discrete for dimension $d = 1$. Our main theorem can be seen as a nonlinear analogue of discrete Fourier uniqueness pairs such as the celebrated Radchenko–Viazovska formula in [21], and the uniqueness result of the second author and M. Sousa for powers of integers [22]. As an additional application, we deduce rigidity results for solutions to some semilinear elliptic equations from their zeros.

Contents

1 Introduction ............................................. 2
2 Main Results ............................................. 4
  2.1 Main Results for Rapidly Accumulating Zeros for Complex Potential and General Nonlinearities in $d = 1$ .......................................... 4
  2.2 Main Results for Slowly Accumulating Zeros for Cubic NLS for $d \geq 1$ ............ 5
  2.3 Consequences for a Related Semilinear Elliptic Problem ........................... 7
  2.4 Generalizations of the Main Results ............................................. 9
3 Preliminaries and Basic Estimates .................................. 10
  3.1 Conventions ........................................... 10
  3.2 Function Spaces ......................................... 10
  3.3 Strichartz Pairs and Estimates ............................. 11

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1 Introduction

Let \( f \in L^2(\mathbb{R}) \). The problem of determining necessary and sufficient conditions on sets \( A, B \) so that \( f|_A = \hat{f}|_B = 0 \) implies \( f \equiv 0 \) has been extensively studied in connection with the celebrated Heisenberg uncertainty principle. If a pair of sets \((A, B)\) possesses the property stated above, possibly under additional assumptions on the functions \( f \), we call it a Fourier uniqueness pair.

For instance, \((A, B) = (\mathbb{R} \setminus (-1/2, 1/2), \mathbb{Z})\) is seen, due to the Shannon–Whittaker interpolation formula for band-limited functions, to be a Fourier uniqueness pair. In the celebrated work [21], Radchenko and Viazovska recently proved that, restricted to the class of Schwartz functions, \((A, B) = (\sqrt{n}, \sqrt{n})_{n \in \mathbb{N}}\) is a Fourier uniqueness pair. After the Radchenko–Viazovska breakthrough, several other results came about giving necessary and sufficient conditions on the pair \((A, B)\) so that the Fourier uniqueness property holds, for which we refer the reader to [15, 23, 24] and the references therein.

While the Fourier transform is a rather distinguished and highly symmetric operator, it is now a natural problem to explore the validity of such uniqueness results for more general operators which are known to enjoy similar uncertainty properties. Indeed, in the recent work [7] by F. Gonçalves and the second author, the first step connecting Fourier uniqueness pairs to other operators has been taken. There, the authors consider the free Schrödinger equation

\[
\begin{aligned}
\frac{i}{\partial_t}u + \Delta u &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}; \\
\quad u(x, 0) &= u_0(x) \quad \text{on } \mathbb{R}.
\end{aligned}
\]  

(1.1)

Among other properties, the authors prove that, for certain discrete subsets of the \((x, t)\)–plane, if a solution \( u \) vanishes at the points of that set, and \( u_0 \) belongs to a fast-decaying class of functions, then \( u \equiv 0 \).

The result relies heavily on the explicit representation of the free evolution of (1.1) given by \( u = e^{it \Delta} u_0 = \frac{1}{\sqrt{4t \pi i}} e^{-|x|^2/4it} \ast u_0 \) which allows to resemble several aspects of the previously mentioned rigidity results for the Fourier transform. An important question immediately raised is whether such uniqueness results extend for...
the evolution in the presence of a given potential or for nonlinear interactions. Indeed, already in the linear case of the Schrödinger equation with a (possibly complex-valued) potential

\[
\begin{aligned}
  i\partial_t u + \Delta u + V u &= 0 \quad \text{in } \mathbb{R} \times [0, T]; \\
  u(x, 0) &= u_0(x) \quad \text{on } \mathbb{R},
\end{aligned}
\]

(1.2)

the techniques used by Gonçalves and the second author [7], which mainly rely on relating the evolution of the free Schrödinger equation to suitable Fourier transforms, completely fall apart, and it seems that new techniques are needed then.

The main purpose of this work is to provide first examples of discrete uniqueness sets for the Schrödinger equation, either in the case of a potential or in the nonlinear case. Before we introduce our main results in Section 2, however, we summarize some previous progress on the question of unique continuation for the Schrödinger equation.

Indeed, our starting point will be a result on unique continuation for the Schrödinger equation by Kenig, Ponce and Vega [14], where the authors proved the following. Let \( u_1, u_2 \in C([0, T]; H^k(\mathbb{R}^d)) \), \( s \geq \max\{d/2+, 2\} \), be solutions to

\[
i\partial_t u + \Delta u + F(u, \bar{u}) = 0 \text{ in } \mathbb{R}^d \times [0, T],
\]

(1.3)

where \( F \in C^{[s]+1}(\mathbb{C}^2; \mathbb{C}) \) satisfies \( |\nabla F(u, \bar{u})| \lesssim (|u|^{p_1-1} + |u|^{p_2-1}) \), \( p_1, p_2 > 1 \). If

\[
u_1(x, 0) = u_2(x, 0), \quad u_1(x, T) = u_2(x, T), \quad \forall x \notin \Gamma + y_0,
\]

where \( \Gamma \) is a convex cone contained in a half space, then \( u_1 \equiv u_2 \). One of the key steps in their proof is a lemma—which we shall use in this paper; see Lemma 5.1 for more details—which roughly states that exponential decay is preserved for times \( t \in (0, T) \), as long as we have exponential decay of the solution at two times \( t = 0, T \).

After the Kenig–Ponce–Vega breakthrough, several other advances in the problem of determining sharper conditions on \( u_1 - u_2 \) for uniqueness have been made, such as [16], the recent survey [17] for non-local operators and the references therein. In particular, in the series of papers [3–5], Escauriaza–Kenig–Ponce–Vega obtained a sharp form of Hardy’s uncertainty principle for the Schrödinger evolution: suppose \( u \in C([0, T]; L^2(\mathbb{R}^d)) \) is a solution to (1.2), and suppose additionally that there are two constants \( \alpha, \beta > 0 \) with \( \frac{\alpha \beta}{4T} < 1 \) such that

\[
\|e^{it|\cdot|^2/\beta^2} u(0)\|_2 + \|e^{it|\cdot|^2/\alpha^2} u(T)\|_2 < +\infty.
\]

(1.4)

If the potential \( V \) satisfies \( \lim_{R \to \infty} \|V\|_{L^1([0,T];L^\infty(\mathbb{R}^d \setminus B_R))} = 0 \), then one has \( u \equiv 0 \). They also proved that the conditions on \( \alpha, \beta \) are sharp, as in the case of \( \alpha \beta = 4T \) there is a potential satisfying the condition above and a non-zero solution to the Schrödinger equation (1.2) satisfying (1.4).

As a consequence of their results for the potential case (1.2), they obtain that if two solutions \( u_1, u_2 \in C([0, T]; H^k(\mathbb{R}^d)) \), \( k > d/2 \), of (1.3), with \( F \in C^k \), \( F(0) = 0 \), then

\[
u_1 \equiv u_2.
\]
\[ \partial_t F(0) = \partial_{\tilde{\tau}} F(0) = 0, \]
\[ e^{\frac{|x|^2}{\beta^2}} (u_1(0) - u_2(0)), e^{\frac{|x|^2}{\alpha^2}} (u_1(T) - u_2(T)) \in L^2, \]
where \( \alpha \beta < 4T \), then \( u_1 \equiv u_2 \).

## 2 Main Results

As highlighted before, the results above deal with decay properties of solutions to Schrödinger equations on large spatial sets, in particular sets with zero codimension. Our main results are concerned with knowledge of the solution on small spatial sets with codimension one. In particular, in the one-dimensional case, we obtain rigidity already from zeros (or decay) of the solution imposed on a discrete set.

We first consider the one dimensional case, where we show a robust rigidity result for the Schrödinger equation in the presence of a complex potential (Theorem 1) and for general nonlinearities (Theorem 1). The theorem, however, only applies in dimension \( d = 1 \) and we have to assume rapidly accumulating zeros (or decay) at infinity. This is the content of Section 2.1. Then, in Section 2.2, we state our main rigidity results Theorem 2 \((d = 1)\) and Theorem 3 \((d \geq 2)\) for the cubic NLS assuming only a sparser set of zeros. In Section 2.3 we present a novel rigidity result (Corollary 2) for certain semilinear elliptic PDEs as a corollary from the main results from Section 2.2. Finally, in Section 2.4 we give an outlook how to extend the main theorems of Section 2.2 to more general power nonlinearities for the Schrödinger equation.

### 2.1 Main Results for Rapidly Accumulating Zeros for Complex Potential and General Nonlinearities in \( d = 1 \)

In dimension \( d = 1 \), we are able to prove, through basic estimates on the linear unitary group \( \{e^{it\partial_x^2}\}_{t \in \mathbb{R}} \) and some cleaner estimates, the following result on uniqueness under quite weak regularity conditions.

**Theorem 1** \( \text{Let } u \in C([0, T]): L^2(\mathbb{R}) \text{ be a strong solution of the initial value problem associated with the Schrödinger equation with complex-valued potential:} \)

\[
\begin{aligned}
&i \partial_t u = -\partial_x^2 u + Vu \quad \text{in } \mathbb{R} \times [0, T], \\
&u(x, 0) = u_0(x) \quad \text{on } \mathbb{R},
\end{aligned}
\]

\[ \text{where we assume that the potential satisfies } V \in L^1([0, T]): L^\infty(\mathbb{R}) \cap L^\infty([0, T]) : L^2((1 + |x|)^2 dx) \text{ and } \lim_{R \to \infty} \| V \|_{L^1([0, T], L^\infty(\mathbb{R}\setminus BR))} = 0. \]

Suppose \( u_0, u(T) \in L^1((1 + |x|)dx) \) and that there are constants \( c_1, c_2, \delta > 0 \) so that, for some \( \alpha \in (0, \frac{1}{2}) \), we have

\[
|u_0(\pm c_1 \log(1 + n)^\alpha)| + |u(\pm c_2 \log(1 + n)^\alpha, T)| \leq n^{-\delta}, \forall n \geq 0. \tag{2.2}
\]
Then, we have $u \equiv 0$.

**Remark 2.1** The assumptions on the solution $u_0, u(T) \in L^1(\{0, T\} : L^\infty(\mathbb{R})) \cap L^\infty(\{0, T\} : L^2(\{0, T\} : (1 + |x|)dx))$ will guarantee that $u_0, u(T) \in L^1(\mathbb{R})$ (see already the proof of Theorem 1 in Section 5.1). Thus, the pointwise conditions on $u_0$ and $u(T)$ in (2.2) are well-defined.

The proof of Theorem 1, given in Section 5.1, departs from the Duhamel formula for the solution $u$. Then, we use the two dispersive estimates associated to $e^{it\partial_x^2}$ (see already Lemma 3.1) in different regimes in the Duhamel formula. Under the assumptions of Theorem 1, we will show that we subsequently obtain sub-Gaussian decay of the solution at two different times. By the results [3–5] of Escauriaza–Kenig–Ponce–Vega highlighted above, we must have that each solution with those properties is identically zero.

As a consequence of Theorem 1, we are able to prove the following corollary, which asserts that if two solutions of (1.3), which are of class $H^s, s > 1/2$, possess some mild spatial decay and nearly coincide at many points, then they must be the same.

**Corollary 1** Let $u, v \in C([0, T] : H^s(\mathbb{R}) \cap L^2(x^2 dx)), s > 1/2$, be two solutions of the initial value problem associated with the Schrödinger equation with a nonlinear term:

$$\begin{cases}
i \partial_t w = -\partial_x^2 w + F(w, \bar{w}) & \text{in } \mathbb{R} \times [0, T], \\
w(x, 0) = w_0(x) & \text{on } \mathbb{R},
\end{cases}$$

(2.3)

where $F \in C^1(\mathbb{C}^2 : \mathbb{C})$ satisfies

$$|\nabla F(w, \bar{w})| \leq C|w|^{p-1}$$

(2.4)

for some $p > 2$. Suppose that there are constants $c_1, c_2, \delta > 0$ so that, for some $\alpha \in (0, \frac{1}{2})$, we have

$$|(u_0 - v_0)(\pm c_1 \log(1 + n)^\alpha) + (u - v)(\pm c_2 \log(1 + n)^\alpha, T)| \leq n^{-\delta}, \forall n \geq 0$$

as well as $u_0 - v_0, u(T) - v(T) \in L^1((1 + |z|)dz)$. Then, we have $u \equiv v$.

Notice that Corollary 1 is much easier in the case of $u, v \in C([0, T], H^1(\mathbb{R}))$, as then the fundamental theorem of calculus gives us the sub-Gaussian estimates in a cheap way. This is, in part, the reason why that result is only stated in one dimension: for higher dimensions, the “natural” condition would be $s > d/2$, in which case a Poincaré inequality argument—see the proof of Lemma 4.9 for more details—would also give us the result almost instantly.

### 2.2 Main Results for Slowly Accumulating Zeros for Cubic NLS for $d \geq 1$

Notice that the conditions on the set where $u$ is “small” in Theorem 1—though discrete—are quite dense, in the sense that, given regularity, the Poincaré inequal-
ity argument as before already gives the result by previous estimates. On the other hand, if the nodes are “sparser” (i.e. \( \alpha \in (0, 1) \) instead of \( \alpha \in (0, \frac{1}{2}) \) as in Corollary 1), then sub-Gaussian decay cannot be obtained as described above. We will address this question for the cubic NLS

\[
\begin{aligned}
  i \partial_t u &= -\Delta u - \lambda |u|^2 u \quad \text{in } [0, T] \times \mathbb{R}^d, \\
  u(x, 0) &= u_0(x) \quad \text{on } \mathbb{R}^d,
\end{aligned}
\]  

(2.5)

for fixed \( \lambda \in \mathbb{C} \). We first state our result for dimension \( d = 1 \).

**Theorem 2** Let \( u \in C([0, T]; L^2(\mathbb{R})) \) be a strong solution to the initial value problem (2.5) for \( d = 1 \).

Suppose additionally that \( u_0, u(T) \in L^2(x^2 \, dx) \), and that the solution satisfies

\[
  u_0(\pm c_1 \log(1 + n)^\alpha) = u(\pm c_2 \log(1 + n)^\alpha, T) = 0, \quad \forall \ n \geq 0,
\]

(2.6)

for some \( \alpha \in (0, 1) \) and some \( c_1, c_2 > 0 \). Then \( u \equiv 0 \).

**Remark 2.2** By local smoothing of the Schrödinger equation (see already Lemma 4.1), the assumptions \( u_0, u(T) \in L^2(x^2 \, dx) \) will guarantee \( u_0, u(T) \in H^1(\mathbb{R}) \) and are thus sufficient to make sense of the pointwise conditions on \( u_0 \) and \( u(T) \) in (2.6).

**Remark 2.3** The seemingly arithmetic nature of the zero sets in the statement of Theorem 2 does not actually play a major role in its proof. As a matter of fact, the main property needed is a control on the gap between consecutive roots. That is, it would be enough to have \( u_0 \) and \( u(T) \) vanish at sequences of points \( \{\lambda_n\}_{n \in \mathbb{Z}} \) and \( \{\gamma_n\}_{n \in \mathbb{Z}} \), respectively, which satisfy

\[
\begin{align*}
  \lim_{n \to \pm \infty} \lambda_n &= \lim_{n \to \pm \infty} \gamma_n = \pm \infty, \\
  |\lambda_{n+1} - \lambda_n| &\leq C e^{-c|\lambda_n|^\beta}, \\
  |\gamma_{n+1} - \gamma_n| &\leq C e^{-c|\gamma_n|^\beta'}, \quad \forall \ n \in \mathbb{Z},
\end{align*}
\]

where \( \beta, \beta' > 1 \) and \( C, c > 0 \) are two universal constants. The choice of \( \pm c_i \log(1 + n)^\alpha \), \( \alpha \in (0, 1) \), in the statement of Theorem 2 is mostly for aesthetic reasons.

The strategy in the proof of Theorem 2, given in Section 5.2, resembles some of the strategy in [22]. First, one obtains an exponential decay estimate for the solution at two different times through the fundamental theorem of calculus (see Lemma 4.7). We then use the result on propagation of exponential decay by Kenig, Ponce and Vega (see Lemma 5.1), in order to conclude that the same exponential decay estimate holds inside the time interval. We then use Lemma 4.4 in order to conclude that the solution is, in fact, analytic in a small strip of the complex plane. By a result on the zeros of analytic functions in a strip (Lemma 4.6), which is based on a theorem by Szegő, we impose constraints that are violated if \( z_n = c_2 \log(1 + n)^\alpha \) are our sequence of zeros. This yields a contradiction, which concludes the proof.

Our one dimensional result also generalizes to higher dimensions.
Theorem 3  Let \( u \in C([0, T]; H^s(\mathbb{R}^d)) \), where \( s > d/2 - 1 \) be a strong solution to (2.5), where \( d \geq 2 \). Suppose additionally that \( u_0, u(T) \in L^2(|x|^{2k}dx) \), for some \( k \in \mathbb{N}, k \geq \frac{d}{2} \), and that the solution satisfies

\[
u_0|_{c_1 \log(1+n)^\alpha \mathbb{S}^{d-1}} = u(T)|_{c_2 \log(1+n)^\alpha \mathbb{S}^{d-1}} \equiv 0, \quad \forall n \geq 0,
\]

in the sense of Sobolev traces for some \( \alpha \in (0, 1) \) and some \( c_1, c_2 > 0 \). Then \( u \equiv 0 \).

Remark 2.4  As before, by local smoothing of the Schrödinger equation (see Lemma 4.2), the assumptions \( u_0, u(T) \in L^2(|x|^{2k}dx) \) guarantee that \( u_0, u(T) \in H^1(\mathbb{R}^d) \) and are thus sufficient to make sense of the condition (2.7) in the sense of Sobolev traces.

The main strategy in the proof of Theorem 3, given in Section 6, is the same as that of the proof of Theorem 2. The main difference here is that the analogues to Lemma 4.7 and Lemma 4.4 are slightly more technical and somewhat weaker at times. At the end of the proof, however, we must use a slightly different approach: in order to conclude that the solution is identically zero, we restrict the initial (or final) datum to lines, and consider the restriction as an analytic function. By the structure of the zero set, we may find zeros that grow similarly to \( c_2 \log(1+n)^\alpha \) for this new one-dimensional function, and thus the final part of the argument follows as well.

2.3 Consequences for a Related Semilinear Elliptic Problem

As a consequence of the proof before, we notice the following. Fix \( d \geq 2 \), and let \( w \in H^s(\mathbb{R}^d) \cap L^2(|x|^{2k}dx) \), \( k \in \mathbb{N}, s > d/2 - 1, k \geq s \) be a solution to the following nonlinear scalar field equation:

\[
\Delta w - w + \lambda w^3 = 0,
\]

where we fix \( \lambda \in \mathbb{C}, \text{Re}(\lambda) > 0 \). Let \( u(x, t) = e^{it} w(x) \). It is immediate to see that \( u \) is a standing wave solution to the cubic NLS (2.5). Moreover, by the decay properties of \( w \), we see that \( u \in C([0, T]; H^s(\mathbb{R}^d)) \), \( s > d/2 - 1 \) satisfies the hypothesis of Theorem 3. Thus, we have the following Corollary as a direct consequence of that result:

Corollary 2  Let \( w \in H^s(\mathbb{R}^d) \cap L^2(|x|^{2k}dx) \), \( s > d/2 - 1, k \in \mathbb{N}, k \geq d/2 \) be a solution to (2.8), with \( \text{Re}(\lambda) > 0 \). Suppose that

\[
w(c \log(n + 1)^\alpha \xi) = 0, \quad \forall n \geq 0, \quad \forall \xi \in \mathbb{S}^{d-1}.
\]

Then \( w \equiv 0 \).

As far as we know, this result is novel in the range \( \alpha \in [3/4, 1) \). For \( \alpha \in (0, 3/4) \), the result follows from the following result of Meshkov [18]: If \( v \) is a solution to \( \Delta v + V \cdot v = 0 \) in \( \mathbb{R}^d \) with \( V \in L^\infty \), and moreover \( |v(x)| \lesssim e^{-c|x|^{3/3+\varepsilon}} \), for some \( c > 0 \) and some \( \varepsilon > 0 \), then \( v \equiv 0 \).
In our case, we take $V = \lambda w^2 - 1$ and $v = w$. As $w$ satisfies (2.8), it also satisfies, as noted above, (2.5). Referring to Lemma 4.2 in Section 4, we have $w \in H^{k,k}(\mathbb{R}^d)$, $k \in \mathbb{N}$, $k \geq d/2$ (see already Section 3.2 for the definition of the space $H^{k,k}(\mathbb{R}^d)$). Now, already referring to the argument with Poincaré’s inequality in Lemma 4.9, we obtain that $|v(x)| \lesssim e^{-c|x|^{3/4}}$, as long as $\alpha \in (0, 3/4)$. One concludes then that $v$ and $V$ satisfy the hypotheses of Meshkov’s result above, and thus $v \equiv 0$.

One may, on the other hand, wonder whether Corollary 2 is sharp in any sense. We believe that this is not the case, and the reason is twofold. First, the proof above was a mere application of Theorem 3 for the nonlinear Schrödinger equation, without taking too much into account the additional rigidity that equation (2.8) possesses. We believe that a method similar to that of [22], coupled with the Poincaré inequality trick from before, should lead to an improved version of that result. As this would already escape the scope of this manuscript, we plan to revisit this question in a future work.

Second, if one additionally imposes the condition that the solutions to (2.8) are radial, then much more can be said about the structure of zeros. Indeed, let $\lambda > 0$ in what follows. If $w = w(|x|)$ is twice differentiable (as a radial function) on the positive real line, equation (2.8) becomes

$$w'' + \frac{d - 1}{r}w' + (\lambda w^2 - 1) \cdot w = 0, \forall r > 0. \quad (2.9)$$

Equivalently, this may be rewritten as

$$(r^{d-1}w')' + r^{d-1}(\lambda w^2 - 1) \cdot w = 0, \forall r > 0. \quad (2.10)$$

Suppose that $w \in L^\infty(\mathbb{R}^d)$. Let then $|\lambda||w|^2 - 1 = C$. Fix $\tilde{w}$ a non-zero solution to the equation

$$(r^{d-1}\tilde{w})' + C \cdot r^{d-1}\tilde{w} = 0, \forall r > 0.$$ 

If we let $u(r) = r^{d-1} \tilde{w} \left(\frac{1}{\sqrt{C}} r\right)$, we see that $u$ is a solution to Bessel’s equation

$$u'' + \left(1 - \frac{\nu}{r^2}\right) u = 0,$$

where $\nu = \frac{(d-1)(d-3)}{4}$. Let $\{r_k\}_{k \in \mathbb{N}}$ be the (increasing) sequence of zeros of $w$. By the Sturm–Picone comparison principle, as $w$ is not a constant, there is a sequence $\{r_k^1\}_{k \geq 1}$ of zeros of $\tilde{w}$ so that $r_k < r_k^1 < r_{k+1}$, $\forall k \geq 1$. By considering the sequence $\{\sqrt{C}r_k^1\}$, we see that this is a sequence of zeros of the Bessel function $u$ as above.

Fix an open interval $I$ of length $\pi$ on the real line, with the center of $I$ being sufficiently large in absolute value. We claim that there is at most one zero of $u$ in $I$. Indeed, suppose not. That is, suppose we have two zeros $z_1, z_2$ of $u$ in that interval. We then apply Sturm–Picone comparison once more: as $r \in I$ is sufficiently large in

[Springer]
absolute value, we see that $1 - \frac{\nu}{r^2} < 1$, and thus any solution of the equation

$$v'' + v = 0$$

has a zero in $(z_1, z_2) \subset I$. If $I = (a, a + \pi)$, then the function $v(t) = \sin(t - a)$ has no zeros in $I$, a contradiction which proves the claim.

Therefore, the difference between (sufficiently large) consecutive zeros of $u$ is at least $\geq \pi$, and thus concentrates at most like $\pi \mathbb{Z}$ near infinity. This shows that $c \log(1 + n)^\alpha$ in Corollary 2 may be replaced by $(\pi - \epsilon)n$, at least in the radial case.

Notice that, for the argument above, we did not need decay of $w$. Indeed, by using decay of the form $w \to 0$ in the argument above, one is able to obtain that any radial solution to (2.8) that decays at infinity does not change sign for $r > 0$ sufficiently large. We do not know whether this kind of behaviour can be extended to the non-radial setting. We do not investigate this question further here, however, as it would escape the scope of the manuscript, but we remark that this would be an interesting further question to pursue.

### 2.4 Generalizations of the Main Results

In spite of the fact that this paper focuses on the question of uniqueness from zeros of the equation (2.5), one may generalize Theorem 2 and Theorem 3 to equations of the form

$$\begin{cases}
    i \partial_t u = -\Delta u - \lambda |u|^{2k} u & \text{in } [0, T] \times \mathbb{R}^d, \\
    u(x, 0) = u_0(x) & \text{on } \mathbb{R}^d,
\end{cases}$$

(2.11)

where $k \in \mathbb{N}$ is a positive integer. *Mutatis mutandis*, all propagation of regularity and analyticity results adapt with minor modifications in the exponents and techniques to this context. The final part of the proof itself can be adapted almost verbatim. In particular, the fact that we have restricted ourselves to the cubic case was due to the simpler technical aspect.

It is interesting to compare Theorem 1, Theorem 2 and Theorem 3 with the results in [7]. In the latter manuscript, the authors obtained uniqueness for the free Schrödinger equation (1.1) with zero sets behaving like $n^\alpha$, for some $\alpha > 0$ in a specific range. The zero sets in the assumptions of Theorem 1, Theorem 2 and Theorem 3 are significantly denser than that, and the reason for that is in the non-linear/potential effect.

First of all, while the free equation (1.1) may be explicitly given as a Fourier transform of the initial datum times a complex Gaussian, such a representation is not available in the non-linear case. Moreover, the same estimates which allow for the bootstrap argument in [22], which are implicitly used in [7], are seen to *fail* to improve decay in the non-linear case of Theorem 2 and Theorem 3. This has then to be compensated through a denser zero set, in order to enforce decay from the beginning. Similarly, in the case of Theorem 1, the decay needed for the bootstrap argument of [22] could be achieved by further assumptions on the potential. This, on the other hand, would immediately impose much more restrictive conditions on Corollary 1, a
reason for which we decided to leave the question of obtaining a sparser zero set in Theorem 1 under additional decay on the potential to a future work.

This paper is structured as follows: in Section 3, we lay out the background knowledge and preliminaries. In Section 4 we prove certain smoothing and self-improving effects for the cubic Schrödinger equation based on our assumptions of the initial data and the solution at later times. Then, we show the main one-dimensional results in Section 5, and the higher-dimensional Theorem 3 in Section 6.

3 Preliminaries and Basic Estimates

3.1 Conventions

For the Fourier transform we use the convention \( \hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx \). We will also denote generic constants with \( C > 0 \) which may become larger from line to line and which may depend on the dimension \( d \), on any Hölder, Strichartz, Besov exponents, on \( \lambda \) as in (2.5), and on the fixed constants \( c_1, c_2 > 0 \) from the statements of the theorems. Similarly, we use \( a \lesssim b \) if \( a \leq Cb \) for a generic constant \( C \).

3.2 Function Spaces

We denote the standard Lebesgue spaces with \( L^p(\mathbb{R}^d) \), the weighted Lebesgue spaces for a weight \( w(x) \) with \( L^p(w(x)dx) = L^p(\mathbb{R}^d, w(x)dx) \), and the Sobolev spaces with \( W^{k,p}(\mathbb{R}^d) \) and the special case \( H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d) \). For \( k \in \mathbb{Z}_{\geq 0} \) we define \( H^{k,k}(\mathbb{R}^d) \) as the completion of the space of Schwartz functions under the norm \( \| u \|_{H^{k,k}}^2 := \sum_{|\alpha|+|\beta| \leq k} \| x^\alpha \partial_x^\beta u \|_{L^2(\mathbb{R}^d)}^2 \) for multiindices \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^d \). We will also say \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \) for all \( i = 1, \ldots, d \) and we will use the convention that \( \alpha < \beta \) means that \( \alpha \neq \beta \). We shall also use the notations \( \| \cdot \|_p = \| \cdot \|_{L^p(\mathbb{R}^d)} = \| \cdot \|_{L^p} \) interchangeably. Similarly, for mixed spatial and temporal norms we also use the following notations \( \| \cdot \|_{L^q([0,T]:L^r(\mathbb{R}^d))} = \| \cdot \|_{L^q_tL^r_x} = \| \cdot \|_{L^q L^r} \).

We will also make use of Besov spaces. To do so, we fix \( \eta \in C^\infty_c(\mathbb{R}^d) \) with \( \eta(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \eta(\xi) = 0 \) for \( |\xi| \geq 2 \). For \( j \in \mathbb{Z} \) we also define \( \psi_j(\xi) := \eta(\xi 2^{-j}) - \eta(\xi 2^{-j+1}) \). Now, we define the Besov norm

\[
\| u \|_{B_{p,q}^s(\mathbb{R}^d)} := \| \mathcal{F}^{-1}(\eta \hat{u}) \|_{L^p(\mathbb{R}^d)} + \left( \sum_{j=1}^{\infty} 2^{sj} \| \mathcal{F}^{-1}(\psi_j \hat{u}) \|_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{q}} \quad \text{if } q < \infty
\]

\[
\sup_{j \geq 1} 2^{j} \| \mathcal{F}^{-1}(\psi_j \hat{u}) \|_{L^p(\mathbb{R}^d)} \quad \text{if } q = \infty
\]

(3.1)

and denote with \( B_{p,q}^s(\mathbb{R}^d) \) the space of tempered distribution \( u \in \mathcal{S}'(\mathbb{R}^d) \) with \( \| u \|_{B_{p,q}^s(\mathbb{R}^d)} < \infty \). We will mainly follow the convention that \( p, q, r \in [1, \infty], s \in \mathbb{R}_{\geq 0}, \) and \( k \in \mathbb{Z}_{\geq 0} \).
3.3 Strichartz Pairs and Estimates

We recall that a pair \((q, r)\) of exponents is called admissible if \(2 \leq q, r \leq \infty\), \(\frac{2}{q} + \frac{d}{r} = \frac{d}{2}\) and \((q, r, d) \neq (2, \infty, 2)\). Then, for any such pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\), the following Strichartz estimates (e.g. [25, Theorem 2.3]) hold

\[
\|e^{it\Delta}u_0\|_{L^q L^r} \lesssim \|u_0\|_{L^2} \tag{3.2}
\]

\[
\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{L^q L^r} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}, \tag{3.3}
\]

where \((\tilde{q}', \tilde{r}')\) are the Hölder conjugates of \((\tilde{q}, \tilde{r})\).

3.4 Notation and Basic Estimates

**Lemma 3.1** For \(f \in L^1(\langle x \rangle dx)\) denote by \(v(x, t) = e^{it\partial_x^2} f(x)\) the solution of the free Schrödinger equation

\[
\begin{align*}
    &i\partial_t v = -\partial_x^2 v, \quad \text{in } \mathbb{R} \times \mathbb{R}, \\
    &v(x, 0) = f(x) \quad \text{in } \mathbb{R}.
\end{align*} \tag{3.4}
\]

Then, for any \(t > 0\), the following estimate holds:

\[
|e^{it\partial_x^2} f(x) - e^{it\partial_x^2} f(y)| \leq C \min \left\{ \frac{\max(\{|x|, |y|\}|x-y|}{t^{3/2}} \| (1 + |z|) f(z) \|_1, \frac{1}{t^{1/2}} \| f \|_1 \right\}. \tag{3.5}
\]

**Proof** Let \(x, y \in \mathbb{R}\) be two arbitrary points. We remark the following formula for the solution of the free Schrödinger equation in one dimension:

\[
e^{it\partial_x^2} f(z) = \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} e^{i\frac{(z-y)^2}{4t}} f(y) dy.
\]

Then we have the following comparison estimate:

\[
|e^{it\partial_x^2} f(x) - e^{it\partial_x^2} f(y)| \leq \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} \left| e^{i\frac{(x-z)^2}{4t}} - e^{i\frac{(y-z)^2}{4t}} \right| |f(z)| dz \\
\leq \frac{C}{t^{3/2}} \left( |x^2 - y^2| \| f \|_1 + |x - y| \| z f(z) \|_1 \right) \\
\leq \frac{C \max(\{|x|, |y|\}|x-y|}{t^{3/2}} \| (1 + |z|) f(z) \|_1,
\]

for some absolute constant \(C > 0\). On the other hand, by just using the triangle inequality, we have

\[
\| e^{it\partial_x^2} f \|_\infty \leq \frac{C}{t^{1/2}} \| f \|_1.
\]
for some possibly different absolute constant $C > 0$. This finishes the proof of the lemma.

3.5 The Operator $\Gamma$

We now move on to analyse how regularity and decay relate to each other in the context of the cubic NLS. Indeed, we recall from [10] the operators

$$\Gamma^\beta_t u(t) = e^{i|x|^2/4t} (2it)^{|eta|} \partial_x^\beta (e^{-i|x|^2/4t} u(t)) = (x + 2it\partial_x)^\beta u(t)$$

for a multiindex $\beta \in \mathbb{Z}^d_{\geq 0}$. It is straightforward to check that the operator $\Gamma^\beta_t$ commutes with the free Schrödinger operator:

$$[\Gamma^\beta_t, -i\partial_t - \Delta] = 0,$$

and satisfies

$$\Gamma^\beta_t f = e^{it\Delta} x^\beta e^{-it\Delta} f \text{ and } \Gamma^\beta_t e^{is\Delta} f = e^{is\Delta} \Gamma^\beta_t f.$$ (3.8)

Moreover, for $|\alpha| = |\beta| = 1$ we have the commutator relation

$$[\Gamma^\alpha_t, x^\beta] = 2it\delta_{\alpha\beta},$$ (3.9)

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$. In particular, for general multiindices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, a direct induction argument shows

$$[x^\beta, \Gamma^\alpha_t] = \sum_{\tilde{\alpha} < \alpha, \tilde{\beta} < \beta} C_1(\tilde{\alpha}, \tilde{\beta}, \alpha, \beta, t)x^{\tilde{\beta}} \Gamma^\tilde{\alpha}_t, \quad [x^\beta, \Gamma^\alpha_t] = \sum_{\tilde{\alpha} < \alpha, \tilde{\beta} < \beta} C_2(\tilde{\alpha}, \tilde{\beta}, \alpha, \beta, t) \Gamma^\tilde{\alpha}_t x^{\tilde{\beta}}$$

(3.10)

for suitable $C_1$ and $C_2$.

3.6 The Cubic NLS and Well-posedness

We now recall standard local well-posedness of (2.5) in the subcritical regime. We begin with the case of $d = 1$ for $L^2(\mathbb{R})$ initial data.

**Proposition 3.2** Let $u_0 \in L^2(\mathbb{R})$. Then there exist a $T^* (\|u_0\|_{L^2(\mathbb{R})}) > 0$ and a unique strong solution $u \in C([0, T^*); L^2(\mathbb{R})) \cap L^4_{loc}([0, T^*); L^\infty(\mathbb{R}))$ to (2.5). Moreover, $u \in L^q([0, T]; L^r(\mathbb{R}))$ for any Strichartz pair $(q, r)$ and every $T < T^*$.

In the case $d \geq 2$, the critical regularity obtained from scaling is

$$s_c = \frac{d}{2} - 1$$

(3.11)
and we have the following subcritical well-posedness.

**Proposition 3.3** Let \( d \geq 2 \) and \( u_0 \in H^s(\mathbb{R}^d) \) for \( s = s_c + \epsilon \) and some \( \epsilon \in (0, 1) \). Then there exist a \( T^*(\|u_0\|_{H^s}) > 0 \) and a unique strong solution \( u \in C([0, T^*]; H^s(\mathbb{R}^d)) \cap L^\gamma_{loc}([0, T^*]; B^{s}_{\rho, 2}(\mathbb{R}^d)) \) to (2.5), where \( \gamma := \frac{8}{d-2s} \) and \( \rho := \frac{4d}{d+2s} \). Moreover,

\[
\begin{align*}
u(t) = e^{i(t-T)\Delta}u_0 + i\lambda \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau) d\tau. \tag{3.12}
\end{align*}
\]

**4 Self-improving of Decay and Regularity for the Cubic NLS**

We begin this section with two lemmas that guarantee that solutions to the cubic Schrödinger equation are smooth and have decay measured by weighted Sobolev spaces, if decay is only known at two different times.

**4.1 Decay and Regularity in Weighted Sobolev Spaces**

The results in this subsection may be considered folklore, see [8–10] and [2, Chapter 5.6] for similar results. We have however decided to lay out the details in order to keep the exposition self-contained, aiming at being a unified reference for those results. We hope that, by doing so, this also improves the readability of the paper. We begin with the result for the case \( d = 1 \).

**Lemma 4.1** Let \( u_0 \in L^2(\mathbb{R}) \) and let \( u \in C([0, T]; L^2(\mathbb{R})) \cap L^4([0, T]; L^\infty(\mathbb{R})) \) for \( T < T^*(\|u_0\|_{L^2(\mathbb{R})}) \) be the unique strong solution to the cubic Schrödinger equation (2.5) as defined in Proposition 3.2.

If in addition \( u_0, u(T) \in L^2(x^{2k} dx) \) for some \( k \in \mathbb{N} \), then \( u \in C([0, T]; H^k(\mathbb{R}) \cap L^2((x)^{2k} dx)) \).

In higher dimensions, we have the following result, where we recall that \( s_c = \frac{d}{2} - 1 \) from (3.11).

**Lemma 4.2** Let \( d \geq 2 \) and \( u_0 \in H^s(\mathbb{R}^d) \) for some \( s = s_c + \epsilon \) and \( \epsilon \in (0, 1) \). Let \( u \in C([0, T]; H^s(\mathbb{R}^d)) \cap L^\gamma([0, T]; B^{s}_{\rho, 2}(\mathbb{R}^d)) \) for \( T < T^*(\|u_0\|_{H^s(\mathbb{R})}) \) be the unique strong solution to the cubic Schrödinger equation (2.5) as defined in Proposition 3.3.

If in addition \( u_0, u(T) \in L^2((x)^{2k} dx) \) for some \( k \in \mathbb{N}, k \geq s \), then \( u \in C([0, T]; H^k(\mathbb{R}) \cap L^2((x)^{2k} dx)) \).
Proof of Lemmata 4.1 and 4.2 We shall break the proof into several smaller steps.

Step 1: Proving $\Gamma^\alpha u \in C([0, T] : L^2(\mathbb{R}^d))$ for $|\alpha| = k$. Let $u_{0,n}$ be a sequence of Schwartz functions with $u_{0,n} \to u_0$ in $L^2((x)^{2k}dx) \cap H^s(\mathbb{R}^d)$ as $n \to \infty$. Let $u_n$ be the sequence of corresponding solutions to (3.12), i.e.

$$u_n(t, x) = e^{it\Delta}u_{0,n}(x) + i\lambda \int_0^t e^{i(t-\tau)\Delta} |u_n|^2 u_n(\tau, x)d\tau.$$  \hspace{1cm} (4.1)

For $|\alpha| = k$ we apply $\Gamma^\alpha$ on both sides and obtain

$$\Gamma^\alpha u_n = e^{it\Delta} x^\alpha u_{0,n} + i\lambda \int_0^t e^{i(t-\tau)\Delta} \Gamma^\alpha (|u_n|^2 u_n)(\tau, x)d\tau$$ \hspace{1cm} (4.2)

in view of $\Gamma^\alpha e^{it\Delta} = e^{it\Delta} x^\alpha$.

Taking the spatial $L^2$-norm of (4.2) together with the fact that $\|e^{it\Delta} f\|_{L^2} = \|f\|_{L^2}$ yields

$$\|\Gamma^\alpha u_n\|_{L^2}(t) \leq \|x^\alpha u_{0,n}\|_{L^2} + |\lambda| \int_0^t \|\Gamma^\alpha (|u_n|^2 u_n)\|_{L^2}(\tau)d\tau.$$ \hspace{1cm} (4.3)

We will now estimate $\Gamma^\alpha (|u_n|^2 u_n)$. To do so we set $\tilde{u}_n := e^{-\frac{1}{4\tau}} u_n$ such that

$$\Gamma^\alpha (|u_n|^2 u_n) = \Gamma^\alpha \left( e^{\frac{3}{4\tau}} |\tilde{u}_n|^2 \tilde{u}_n \right) = e^{\frac{3}{4\tau}} (2i \tau \partial_x)^\alpha \left( |\tilde{u}_n|^2 \tilde{u}_n \right).$$ \hspace{1cm} (4.4)

Now, using the Gagliardo–Nirenberg interpolation inequality we obtain

$$\|\Gamma^\alpha (|u_n|^2 u_n)\|_{L^2(\mathbb{R}^d)} \lesssim \tau^k \|\partial_x^\alpha (|\tilde{u}_n|^2 \tilde{u}_n)\|_{L^2(\mathbb{R}^d)} \lesssim \tau^k \|\partial_x^\alpha \tilde{u}_n\|_{L^2(\mathbb{R}^d)} \|\tilde{u}_n\|^2_{L^\infty(\mathbb{R}^d)} \lesssim \|\Gamma^\alpha u_n\|_{L^2(\mathbb{R}^d)} \|u\|^2_{L^\infty(\mathbb{R}^d)}. \hspace{1cm} (4.5)$$

Using the above estimate and applying Gronwall’s lemma to (4.3) gives

$$\|\Gamma^\alpha u_n\|_{L^2(\mathbb{R}^d)}(t) \leq \|x^\alpha u_{0,n}\|_{L^2(\mathbb{R}^d)} \exp \left( |\lambda| \int_0^t \|u_n\|^2_{L^\infty(\mathbb{R}^d)}(\tau) d\tau \right). \hspace{1cm} (4.6)$$

Since $u_{0,n} \to u_0$ in $L^2((x)^{2k}dx) \cap H^s(\mathbb{R}^d)$, we have $\|x^\alpha u_{0,n}\| \to \|x^\alpha u_0\|$ as $n \to \infty$. Thus, in order to obtain uniform control over $\|\Gamma^\alpha u_n\|_{L^2(\mathbb{R}^d)}$ we have to control

$$\int_0^t \|u_n\|^2_{L^\infty(\mathbb{R}^d)}(\tau) d\tau.$$

We first consider the case $d = 1$. In this case we have from well-posedness in Proposition 3.2 that $u_n \to u$ in $C([0, T]: L^2(\mathbb{R})) \cap L^4([0, T]: L^\infty(\mathbb{R}))$, which shows

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that
\[
\int_0^t \| u_n \|^2_{L^\infty(\mathbb{R})}(\tau) \, d\tau \leq t^{\frac{1}{2}} \| u_n \|^2_{L^4([0,T]: L^{\infty}(\mathbb{R}))},
\]
and the term on the right-hand side is uniformly bounded.

The case \( d \geq 2 \) is slightly more involved. We first use the Gagliardo–Nirenberg interpolation inequality estimate
\[
\| u_n \|_{L^\infty(\mathbb{R}^d)} \lesssim \| u_n \|_{W^{s,r}(\mathbb{R}^d)} \quad (4.7)
\]
for \( r = \frac{d}{s} = \frac{2d}{d-2+2\epsilon} \). The corresponding temporal Strichartz exponent is \( q = \frac{2}{1-\epsilon} \). Hence,
\[
\int_0^t \| u_n \|^2_{L^\infty(\mathbb{R}^d)}(\tau) \, d\tau = \| u_n \|^2_{L^2([0,T]: L^{\infty}(\mathbb{R}^d))} \lesssim \| u_n \|^2_{L^2([0,T]: W^{s,r}(\mathbb{R}^d))} \lesssim T^\epsilon \| u_n \|^2_{L^{\frac{2}{1-\epsilon}}([0,T]: W^{s,r}(\mathbb{R}^d))}. \quad (4.8)
\]

Using the embedding \( B_{r,2}^s(\mathbb{R}^d) \hookrightarrow W^{s,r}(\mathbb{R}^d), r \geq 2 \) (see e.g. [1, Theorem 6.4.5]) we then obtain from well-posedness in Proposition 3.3 the uniform control over \( \int_0^T \| u_n \|^2_{L^\infty(\mathbb{R}^d)}(\tau) \, d\tau \).

Thus, we have shown in both cases, \( d = 1 \) and \( d \geq 2 \), that the following a priori bound holds
\[
\sup_n \sup_{0 \leq t \leq T} \| \Gamma^\alpha_t u_n \|_{L^2(\mathbb{R}^d)}(t) \leq \| x^\alpha u_0 \|_{L^2(\mathbb{R}^d)} \exp \left( |\lambda| \int_0^T \| u \|^2_{L^\infty(\mathbb{R}^d)} \, d\tau \right). \quad (4.9)
\]

We will now show that \( (\Gamma^\alpha_t u_n)_n \) is a Cauchy sequence in \( C([0, T]: L^2(\mathbb{R}^d)) \). We begin by estimating
\[
\| \Gamma^\alpha_t u_n - \Gamma^\alpha_t u_m \|_{L^2(\mathbb{R}^d)} \leq \| x^\alpha (u_0_n - u_0,m) \|_{L^2(\mathbb{R}^d)} + \int_0^t \| \Gamma^\alpha_\tau (u_n |u_n|^2 - u_m |u_m|^2) \|_{L^2(\mathbb{R}^d)}(\tau) \, d\tau. \quad (4.10)
\]
In order to control the nonlinearity we use the following estimate which is similar to (4.5):

\[
\| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \| \| \leq \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) \leq \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right)
\]

\[
\cdot \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right)
\]

\[
\cdot \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right)
\]

\[
\cdot \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right)
\]

Inserting the above in the right-hand side of (4.10) and using the uniform bounds on \| u_n \|_{L^2(0, T; L^\infty(\mathbb{R}^d))}, \| u_n \|_{L^\infty(0, T; L^2(\mathbb{R}^d))}, \| \Gamma^\alpha_t u_n \|_{L^\infty(0, T; L^2(\mathbb{R}^d))}, \| \Gamma^\alpha_t u_n \|_{L^\infty(0, T; L^2(\mathbb{R}^d))} we obtain

\[
\begin{align*}
\| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 & \leq \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) \\
& + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) \\
& + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right) \\
& + \left( \left\| \left( \| \Gamma^\alpha_t (u_n | u_m) \|^2 - \| u_m \|^2 \right) \right\| \right)
\end{align*}
\]

for some \( \tilde{C} = \tilde{C}(\|u_n\|_{L^2(\mathbb{R}^d)}, \|u_m\|_{H^s}) > 0 \). Using Gronwall’s lemma, together with the fact that

\[
\begin{align*}
\sup_t \| x^\alpha (u_0, n | u_0, m) \|_{L^2(\mathbb{R}^d)} & \to 0, \\
\int_0^T \| u_n | u_m \|^2 \|_{L^\infty(\mathbb{R}^d), d\tau} & \to 0, \\
\int_0^T \| u_n | u_m \|^2 \|_{L^2(\mathbb{R}^d), d\tau} & \to 0
\end{align*}
\]

as \( n, m \to \infty \), shows that \( \Gamma^\alpha_t u_n \) is a Cauchy sequence in \( C([0, T]; L^2(\mathbb{R}^d)) \). As \( u_n \to u \) in \( C([0, T]; L^2(\mathbb{R}^d)) \), we conclude that \( \Gamma^\alpha_t u \) is a well-defined \( L^2(\mathbb{R}^d) \) function for each \( t \in [0, T] \) and indeed \( \Gamma^\alpha_t u_n \to \Gamma^\alpha_t u \) in \( C([0, T]; L^2(\mathbb{R}^d)) \) with the bound

\[
\begin{align*}
\sup_{0 \leq \tau \leq t} \| \Gamma^\alpha_t u \|^2_{L^2(\mathbb{R}^d)} & \leq \| x^\alpha u_0 \|^2_{L^2(\mathbb{R}^d)} \exp \left( | \lambda | \int_0^t \| u \|^2_{L^\infty(\mathbb{R}^d), d\tau} \right)
\end{align*}
\]

This concludes Step 1.

**Step 2: Showing that** \( u_0, u(T) \in H^{k,k}(\mathbb{R}^d) \). From Step 1 and the assumption of the lemma, we have \( \Gamma^\alpha_t u(T) \in L^2(\mathbb{R}^d) \) and \( x^\alpha u(T) \in L^2(\mathbb{R}^d) \) for \( |\alpha| \leq k \). In particular, this shows that \( \partial^\alpha \tilde{u}(T) \in L^2(\mathbb{R}^d) \) and \( x^\alpha \tilde{u}(T) \in L^2(\mathbb{R}^d) \) for \( |\alpha| \leq k \), where we recall that \( \tilde{u}(T) = e^{-i \| u \|^2_{\mathbb{R}^d}} u(T) \).

In order to continue, we will need an auxiliary interpolation result, which appears as Lemma 4.3 below. With Lemma 4.3 we now conclude that \( \tilde{u}(T) \in H^{k,k} \).
particular, this shows that \( \sum_{|\alpha|+|\beta| \leq k} \|x^\alpha \Gamma^\beta_T u(T)\|_{L^2(\mathbb{R}^d)} < \infty \). To conclude that \( u(T) \in H^{k,k}(\mathbb{R}^d) \), we expand the differentiation operator of order \( \beta \) as

\[
\partial^\alpha_x = \sum_{\beta+\gamma \leq \alpha} C(\alpha, \beta, T)x^\beta \Gamma^\gamma_T.
\]

Thus,

\[
\sum_{|\alpha|+|\beta| \leq k} \|x^\alpha \partial^\beta_x u(T)\|_{L^2(\mathbb{R}^d)} < \infty \tag{4.13}
\]

and \( u(T) \in H^{k,k}(\mathbb{R}^d) \subset H^k(\mathbb{R}^d) \). Completely analogously we have \( u_0 \in H^{k,k}(\mathbb{R}^d) \subset H^k(\mathbb{R}^d) \) and by a standard persistence of regularity result (analogous to Step 1 above, see e.g. [2, Chapter 5]) in \( H^k(\mathbb{R}^d) \), \( k \geq 0 \), we have \( u \in C([0, T]; H^k(\mathbb{R}^d)) \), where we used for \( d = 1, k \geq s \) for \( d \geq 2 \). Again, using the interpolation result from Lemma 4.3 for each \( t \in [0, T] \), we finally obtain \( u \in C([0, T]; H^k(\mathbb{R}^d) \cap L^2((x)^{2k} dx)) \). This concludes the proof.

As Lemma 4.3 was a crucial ingredient in the proof above, we will provide a proof for it below. The result can be originally found in a previous work of H. Triebel [26], where the author employs tools from the theory of interpolation spaces to prove such a theorem. In the present case, however, a much more direct proof is available, which we sketch below.

**Lemma 4.3** Let \( f \in L^2(\mathbb{R}^d) \) satisfy that \( \| |x|^j f\|_{L^2(\mathbb{R}^d)} + \| \partial^\alpha_x f\|_{L^2(\mathbb{R}^d)} < +\infty \) for \( j, |\alpha| \leq k \), for some positive integer \( k \geq 1 \). Then we have that

\[
\sum_{l+|\beta| \leq k} \| |x|^l \partial^\beta_x f\|_{L^2(\mathbb{R}^d)} \lesssim \sum_{j, |\alpha| \leq k} (\| |x|^j f\|_{L^2(\mathbb{R}^d)} + \| \partial^\alpha_x f\|_{L^2(\mathbb{R}^d)}).
\]

**Proof** (Sketch of proof) We argue by induction on \( k \). For \( k = 1 \), the result is tautological. For \( k = 2 \), we only need to prove that \( \| x_i \partial_{x_j} f\|_2 < +\infty \), for \( i, j \in \{1, \ldots, d\} \). But

\[
\| x_i \partial_{x_j} f\|_2^2 = \int_{\mathbb{R}^d} x_i^2 \partial_{x_j} f \bar{x}_j f \, dx
\]

\[
= -\int_{\mathbb{R}^d} x_i^2 \partial_{x_j} f \cdot f(x) \, dx - 2 \delta_{i,j} \int_{\mathbb{R}^d} x_i \partial_{x_j} f f(x) \, dx
\]

\[
\lesssim \| \partial_{x_j}^2 f\|_2 \| |x|^2 f\|_2 + \| |x|^j f\|_2 \| \partial_{x_j} f\|_2.
\]

This shows the result in that case.

Now, for \( k \geq 3 \), we suppose the result holds for \( j = 1, \ldots, k - 1 \) and fix \( \beta \) a multiindex of integers, \( l \in \mathbb{N} \) so that \( l + |\beta| = k \). Select \( i \in \{1, \ldots, d\} \) so that \( \beta_i > 0 \).
Let also $e_i$ denote the multiindex whose $j$-th coordinate is $\delta_{i,j}$. Then we may write

$$\| |x|^l \partial_x^\beta f\|_2 = \| |x|^l \partial_x^{\beta-e_i} (\partial_x^{e_i} f)\|_2.$$ 

By induction hypothesis applied to the function $g := \partial_x^{e_i} f$, we have that the latter term on the right-hand side above is bounded by

$$\sum_{j, |\alpha| \leq k-1} \left( \| |x|^j \partial_x^{e_i} f\|_2 + \| \partial_x^\alpha (\partial_x^{e_i} f)\|_2 \right).$$

Notice that the second terms in the summand are all included in $\sum_{|\alpha| \leq k} \| \partial_x^\alpha f\|_2$, and so we focus on the first term. Indeed, if $j \leq k-2$, we may use the induction hypothesis again, and we are thus only concerned with estimating $\| |x|^{k-1} \partial_x f\|_2$. Analogously as in the $k=2$ case, we have

$$\| |x|^{k-1} \partial_x f\|_2^2 \lesssim \int_{\mathbb{R}^d} |x|^{2k-3} |\partial_x f| |f| + \int_{\mathbb{R}^d} |x|^{2k-2} |\partial_x^2 f| |f|$$

$$\lesssim \| |x|^{k-2} \partial_x f\|_2 \| |x|^{k-1} f\|_2 + \| |x|^{k-2} \partial_x^2 f\|_2 \| |x|^k f\|_2.$$ 

We now use the same trick once more: we may write

$$\| |x|^{k-2} \partial_x^2 f\|_2 = \| |x|^{k-2} \partial_x (\partial_x f)\|_2,$$

and again, by the induction hypothesis applied to $g := \partial_x f$, we get that the term above is bounded by

$$\sum_{l, |\alpha| \leq k} \left( \| |x|^l \partial_x f\|_2 + \| \partial_x^\alpha f\|_2 \right).$$

Notice that the only term in the bound above that is not controlled by either induction or by the quantity in the statement is exactly $\| |x|^{k-1} \partial_x f\|_2$. For shortness, let

$$C_k(f) := \sum_{l, |\alpha| \leq k} \left( \| |x|^l f\|_2 + \| \partial_x^\alpha f\|_2 \right).$$

Putting together all we did so far, we obtain

$$\| |x|^{k-1} \partial_x f\|_2^2 \lesssim C_k^2 + C_k \left( \| |x|^{k-1} \partial_x f\|_2 + C_k \right).$$

This clearly implies that

$$\| |x|^{k-1} \partial_x f\|_2 \lesssim C_k,$$

and thus the induction closes, and we are done. \qed
4.2 Analyticity and the Flow of the Cubic NLS

In this subsection, we will prove some properties on analyticity and local well-posedness in some spaces of analytic function, as first considered by Hayashi and Saitoh [11, 12]. In particular, we will follow closely the strategy used in [13] by Hoshino and Ozawa, where the authors analyse the case of the quintic NLS in one dimension.

We state some definitions in order to prove the next Lemma. Indeed, first we define the class

$$A_2(r) = \left\{ \phi \in L^2(\mathbb{R}) : \| \phi \|_{A_2(r)} = \sum_{j \geq 0} \frac{r^j}{j!} \| x^j \phi \|_2 < +\infty \right\}. \tag{4.14}$$

By [20, Proposition 2] (see also [19, Remark 3]), this norm can be bounded as

$$\| e^{r|x|} \phi(x) \|_2 \leq \| \phi \|_{A_2(r)} \leq (1 + 2 \log 2)^{1/2} (1 + r|x|)^{1/2} e^{r|x|} \| \phi(x) \|_2.$$

Moreover, we define the spaces

$$A_{p,q}(r; T) = \left\{ u \in L^\infty_T L^2_x : \| u \|_{A_{p,q}(r; T)} = \sum_{j \geq 0} \frac{r^j}{j!} \| \Gamma^j u \|_{L^p_T L^q_x} < +\infty \right\}. \tag{4.15}$$

We state our next result in terms of persistence in this last space, which, as we shall see afterwards, may be translated into a result on analyticity of solutions of the cubic nonlinear Schrödinger equation (2.5).

**Lemma 4.4** Let $u_0 \in A_2(r)$, for some $r > 0$. Then there is $T = T(\| u_0 \|_{A_2(r)})$ such that the initial value problem (2.5) has a unique solution

$$u \in A_{\infty,2}(r; T) \cap A_{8,4}(r; T) =: X(r; T).$$

Moreover, the flow map $u_0 \mapsto u(t)$ is locally Lipschitz continuous from $A_2(r)$ to $A_{\infty,2}(r; T)$.

**Proof** We will use a Banach fixed-point argument. Indeed, we define the map

$$\Phi(u) = e^{it\Delta} u_0 + i \lambda \int_0^t e^{i(t-\tau)\Delta} (|u|^2 u)(\tau) \, d\tau,$$

and we wish to prove that this map is a contraction in some set

$$B(u_0) = \{ u \in X(r; T) : \| u \|_{X(r; T)} \leq 2C \| u_0 \|_{A_2(r)} \}.$$
In order to do so, we notice that

\[
\Gamma^j(\Phi(u)) = e^{it\Delta}(x^ju_0) + i\lambda \int_0^t e^{i(t-\tau)\Delta}(\Gamma^j_1(|u|^2u)(\tau)) \, d\tau, \tag{4.16}
\]

for \(j \in \mathbb{N}\) and moreover, using the properties of \(\Gamma\) from (3.6),

\[
\Gamma^j(|u|^2u) = \sum_{k_1+k_2+k_3=j} \frac{j!(-1)^k_3}{(k_1)!(k_2)!(k_3)!} \Gamma^k_1u \cdot \Gamma^k_2u \cdot \Gamma^k_3u. \tag{4.17}
\]

By using the usual Strichartz estimates from (3.2) and (3.3), we obtain that

\[
\|\Gamma^j(\Phi(u))\|_{L^p_TL^q_x} + \|\Gamma^j(\Phi(u))\|_{L^\infty_TL^1_x} \lesssim \|x^ju_0\|_2
\]

\[
+ \sum_{k_1+k_2+k_3=j} \frac{j!}{(k_1)!(k_2)!(k_3)!} \|\Gamma^k_1u \cdot \Gamma^k_2u \cdot \Gamma^k_3u\|_{L^p'_TL^q'_x},
\]

where \((p, q)\) are a Strichartz pair of exponents and \(p', q'\) are their respective Hölder conjugates. We choose \(p = 8, q = 4\). By Hölder’s inequality, we have

\[
\|\Gamma^j(\Phi(u))\|_{L^p_TL^q_x} + \|\Gamma^j(\Phi(u))\|_{L^\infty_TL^1_x} \lesssim \|x^ju_0\|_2
\]

\[
+ \sum_{k_1+k_2+k_3=j} \frac{j!}{(k_1)!(k_2)!(k_3)!} \|\Gamma^k_1u\|_{L^{24/7}TL^{1/7}_x} \cdot \|\Gamma^k_2u\|_{L^{24/7}TL^{1/7}_x} \cdot \|\Gamma^k_3u\|_{L^{24/7}TL^{1/7}_x}.
\]

\[
\lesssim \|x^ju_0\|_2
\]

\[
+ T^{\omega} \sum_{k_1+k_2+k_3=j} \frac{j!}{(k_1)!(k_2)!(k_3)!} \|\Gamma^k_1u\|_{L^{3}TL^{1/3}_x} \cdot \|\Gamma^k_2u\|_{L^{3}TL^{1/3}_x} \cdot \|\Gamma^k_3u\|_{L^{3}TL^{1/3}_x},
\]

for some positive constant \(\omega > 0\). By multiplying this estimate by \(\frac{r^j}{j!}\) and summing up on \(j \geq 0\), we obtain

\[
\|\Phi(u)\|_{X(r; T)} \leq C\|u_0\|_{A_2(r)} + CT^{\omega}\|u\|_{X(r; T)}^3.
\]

In the same way, for \(u, v \in B(u_0)\) we obtain

\[
\|\Phi(u) - \Phi(v)\|_{X(r; T)} \leq CT^{\omega}(\|v\|_{X(r; T)}^2 + \|u\|_{X(r; T)}^2)\|u - v\|_{X(r; T)}.
\]

Thus, for \(T \leq \frac{1}{C'\|u_0\|_{A_2(r)}}\), for \(C'\) sufficiently large, \(\Phi\) becomes a contraction in \(B(u_0)\), and thus we are able to conclude the claim. The claim about Lipschitz continuity follows in a similar manner. \(\square\)

In analogy to the lemmata 4.1 and 4.2, we present a higher-dimensional version of the last lemma, whose formulation is slightly weaker. In order to state it, we define
a generalization of the spaces $A_2(r)$ above, where regularity is included. Indeed, we define the class $A^k_2(r)$ as

$$A^k_2(r) = \left\{ \phi \in L^2(\mathbb{R}^d) : \| \phi \|_{A^k_2(r)} = \sum_{j \in (\mathbb{Z}_{\geq 0})^d} \frac{r^{\lfloor |j|/j!} \| x^j \phi \|_{2, k} < +\infty} \right\},$$

where we use the notation $\| f \|_{2, k} = \sum_{|\alpha| \leq k} \| \partial^\alpha_x f \|_2$ for a modification of the $H^k$ norm. Once more in analogy to the one dimensional case, we define the time-dependent space where we will prove that our solution lies as follows.

$$A^k_{\infty, 2}(r, T) = \left\{ u \in L^\infty_T L^2_x : \| u \|_{A^k_{\infty, 2}(r, T)} = \sum_{j \in (\mathbb{Z}_{\geq 0})^d} \frac{r^{\lfloor |j|/j!} \| \Gamma_1^j u \|_{2, k} L^\infty_T < +\infty} \right\}.$$

**Lemma 4.5** Let $k > \frac{d}{2}$ be a positive integer, and $u_0 \in A^k_2(r)$. Then there is $T = T(\| u_0 \|_{A^k_2(r)})$ such that the initial value problem (2.5) has a unique solution

$$u \in A^k_{\infty, 2}(r, T).$$

Moreover, the flow map $u_0 \mapsto u(t)$ is locally Lipschitz continuous from $A^k_2(r)$ to $A^k_{\infty, 2}(r, T)$.

**Proof** We will use, once more, a Banach fixed-point argument. Indeed, consider again the map $\Phi(u)$ from the proof of Lemma 4.4. We wish to prove that this map is a contraction in some set

$$B(u_0) = \{ u \in A^k_{2, \infty}(r, T) : \| u \|_{A^k_{(r, T)} \leq 2C \| u_0 \|_{A^k_2(r)} \}. $$

In order to do so, we recall (4.16) and (4.17), and notice that those are still valid in case $d \geq 2$ and $j, k_1, k_2, k_3$ are multiindices of integers. Instead of employing Strichartz estimates, we use the fact that, for $f, g, h \in S(\mathbb{R}^d)$, we have the following trilinear estimate

$$\| fgh \|_{2, k} \lesssim \| f \|_{2, k} \| g \|_{\infty} \| h \|_{\infty} + \| f \|_{\infty} \| g \|_{2, k} \| h \|_{\infty} + \| f \|_{\infty} \| g \|_{\infty} \| h \|_{2, k}. $$

Thus, we may estimate

$$\| \Gamma_j(\Phi(u))(t) \|_{2, k} \leq C \| x^j u_0 \|_2 + \sum_{k_1 + k_2 + k_3 = j} \frac{j!}{(k_1)!(k_2)!(k_3)!} \int_0^t \| \Gamma^{k_1} u \|_{\infty} \cdot \| \Gamma^{k_2} u \|_{\infty} \cdot \| \Gamma^{k_3} u \|_{2, k} \| ds$$

+ analogous terms.
By Sobolev’s inequality, we are able to bound each $L^\infty$ norm above by the $(2, k)$—norms, as long as $k > \frac{d}{2}$. Thus, we have

$$
\sup_{t \in [0, T]} \| \Gamma^j(\Phi(u)) \|_{2, k} \lesssim C \| x^j u \|_2 
$$

$$
+ T \sum_{k_1 + k_2 + k_3 = j} \frac{j!}{(k_1)!(k_2)!(k_3)!} \left( \sup_{t \in [0, T]} \| \Gamma^{k_1} u \|_{2, k} \right) \cdot \left( \sup_{t \in [0, T]} \| \Gamma^{k_2} u \|_{2, k} \right) \cdot \left( \sup_{t \in [0, T]} \| \Gamma^{k_3} u \|_{2, k} \right)
$$

the last factors by $r^{\lceil j \rceil} / j!$ and summing up in $j \in \mathbb{Z}_{\geq 0}^d$, we obtain

$$
\| \Phi(u) \|_{A^k(T, r)} \lesssim \| u_0 \|_{A^k(T)} + T \| u \|_{A^k(T, T)}^3.
$$

In the same way, for $u, v \in B(u_0)$ we obtain

$$
\| \Phi(u) - \Phi(v) \|_{A^k(T, r)} \leq C T (\| v \|_{A^k(T, T)}^2 + \| u \|_{A^k(T, T)}^2) \| u - v \|_{A^k(T, T)}.
$$

Thus, if $T \leq \frac{1}{C \| u_0 \|_{A^k(T)}^2}$, for $C'$ sufficiently large, $\Phi$ becomes a contraction in $B(u_0)$, and thus we are able to conclude the claims, similarly as in the proof of Lemma 4.4. $\Box$

From these results, we use the following argument: as

$$
\| u \|_{A^{k, \infty}_2(T, r)} = \sum_{j \in \mathbb{Z}_{\geq 0}^d} \frac{r^{\lceil j \rceil}}{j!} \left( \sup_{t \in [0, T]} \| \Gamma^j u \|_{2, k} \right),
$$

we get in particular that

$$
\sum_{j \in \mathbb{Z}_{\geq 0}^d} \frac{r^{\lceil j \rceil}}{j!} \| \Gamma^j u \|_2 = \sum_{j \in \mathbb{Z}_{\geq 0}^d} \frac{(|t|^r)^{\lceil j \rceil}}{j!} \| x^j e^{i|x|^2/4\tau} u(t) \|_2
$$

$$
= \sum_{j \in \mathbb{Z}_{\geq 0}^d} \frac{(2\pi r |t|)^{\lceil j \rceil}}{j!} \| x^j F(e^{i|x|^2/4\tau} u(t)) \|_2
$$

is uniformly bounded in $t \in [0, T]$. Just as noted before, one may prove that this directly implies that

$$
\| e^{2\pi r |t|(|x_1| + \ldots + |x_d|)} F(e^{i|x|^2/4\tau} u(t)) \|_2
$$

is uniformly bounded in $t \in [0, T]$. But a simple Fourier analysis argument shows that, if the equation above holds, then we have

$$
e^{i|x|^2/4\tau} u(t) \text{ is analytic in } S(r|t|), \tag{4.18}
$$
where we use the notation $|z|^2 = z_1^2 + \cdots + z_k^2$, and $S(a) = \{ z \in \mathbb{C}^d : |\text{Im}(z_j)| < a, \forall j \in \{1, \ldots, d\} \}$. In particular, it can also be shown that $e^{i|\zeta|^2/4t} u(t)$ is analytic and bounded in any strip $S(r')$, where $r' < r|t|$. 

One may wonder whether the lemmata 4.4 and 4.5 can be improved, in the sense that, if $u_0$ is analytic in a strip $S(r)$ and possesses exponential decay, then a solution $u(t)$ to the IVP (2.5) is analytic in a slightly larger strip $S(r + \varepsilon(t))$, for some (positive) $\varepsilon(t)$ depending on time, and some—possibly large—time $t > 0$.

For the sake of comparison, let us first analyse the case of the free Schrödinger equation. If $v(x,t)$ is a solution to

$$
\begin{align*}
\begin{cases}
  i \partial_t v + \partial_x^2 v = 0, \\
  v(x, 0) = u_0(x),
\end{cases}
\end{align*}
$$

then we may write $v(x, t) = \frac{e^{it^2/4t}}{(4\pi t)^{1/2}} (e^{i(|\zeta|^2/4t)} u_0)(\frac{x}{\sqrt{4\pi t}})$. If $|u_0(x)| \leq C e^{-A|x|}$, then an argument with the Fourier transform and Morera’s theorem shows that $v(x, t)$ is analytic in a strip $S(2\pi^2 t A)$. If we suppose, for instance, that $u_0 \in L^2(S(r), dz)$, then Lemma 1 in Hayashi–Saitoh [11] shows that, in fact, $\hat{u}_0 \in L^2(e^{4\pi r|z|} dz)$, which directly implies that $v(x, t)$ is analytic in $S(r)$, for any $t > 0$. Thus, decay and (a certain level of) analyticity are seen to be preserved and, for large times, improve how analytic the solution to (4.19) is.

On the other hand, analyticity in such a strip is seen to be sharp, even if $u_0$ is analytic. Indeed, letting $f_0(x) = \frac{2}{e^{\pi x} + e^{-\pi x}}$, it is a classical result that $\hat{f}_0(\xi) = f_0(\xi)$. Let then $v(x,t)$ be a solution to (4.19) with initial data $u_0(x) = e^{-ix^2/4t_0} f_0(x)$. For time $t = t_0$, we know that the solution may be written as

$$
v(x, t_0) = \frac{e^{i x^2/4t_0}}{(4\pi i t_0)^{1/2}} f_0(x/(4\pi t_0)),
$$

which is seen to be analytic in $S(2\pi t_0)$ but not in a larger strip.

In the case of the cubic NLS, we see that this long-time analytic improvement is, in general, false. Indeed, one easily sees that

$$u(x, t) = e^{i \pi^2 t \sqrt{2\pi}} f_0(x)$$

is a solution to (2.5). As a solitary wave, it preserves, for any time $t \in \mathbb{R}$, the same properties of the initial datum $\sqrt{2\pi} f_0(x)$. In particular, $u(\cdot, t)$ decays as $e^{-\pi|x|}$ for any time, and is seen to be analytic in the strip $S(1/2)$, with poles at $\pm \frac{i}{2}$. This shows, in particular, that the nonlinearity induces a different long-time analytic behaviour in the dynamic of the equation, and that in general analyticity cannot be improved for the cubic NLS even if there is some exponential gain.

Finally, we state a result on how zeros of a function $f$, where $f$ is assumed to be analytic and bounded in a strip, may behave.
Lemma 4.6  Let $f$ be analytic and bounded in the strip $S(r)$. Let $\{z_n\}_{n \geq 0} = \{x_n + iy_n\}_{n \geq 0}$ denote an enumeration of the zeros of $f$, accounting for multiplicity. Then the following condition must be satisfied:

$$
\sum_{n \geq 0} \frac{e^{\pi x_n/(2r)} \cos(\pi y_n/(2r))}{((e^{\pi x_n/r} + 1)^2 - 4e^{\pi x_n/r} \cos^2(\pi y_n/(2r)))^{1/2}} < +\infty.
$$

Proof  We start by noticing that the map $\varphi(z) = \frac{e^{\frac{\pi z}{2r}} - 1}{e^{\frac{\pi z}{2r}} + 1}$ maps the strip $S(r)$ biholomorphically into the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, and moreover it takes the closure $\overline{S(r)}$ of the strip into the closure of the unit disc $\overline{D}$.

Thus, the function $f \circ \varphi^{-1} : D \to \mathbb{C}$ is holomorphic inside the unit disc $D$, and bounded up to the boundary $\overline{D}$. We now use a theorem by Szegő (see also [6, Chapter 2, Theorem 2.1]): if $g : D \to \mathbb{C}$ is a non-zero bounded analytic function, then the zeros $\{w_n\}_{n \geq 0}$, enumerated accounting for multiplicity, satisfy

$$
\sum_{n \geq 0} (1 - |w_n|) < +\infty.
$$

If $f \not\equiv 0$, then $f \circ \varphi^{-1}$ satisfies the conditions of this result. As we know that $w_n = \frac{e^{\frac{\pi z}{2r} - 1}}{e^{\frac{\pi z}{2r}} + 1}$ are the zeros of $f \circ \varphi^{-1}$, we must have then

$$
\sum_{n \geq 0} \left(1 - \left|\frac{e^{\frac{\pi z_n}{2r}} - 1}{e^{\frac{\pi z_n}{2r}} + 1}\right|\right) < +\infty.
$$

But using that

$$
\left|\frac{e^{\frac{\pi z}{2r}} - 1}{e^{\frac{\pi z}{2r}} + 1}\right| = \left(1 - 4e^{\pi x_n/(2r)} \cos(\pi y_n/(2r))\right) \left(e^{\pi x_n/r} + 2e^{\pi x_n/(2r)} \cos(\pi y_n/(2r)) + 1\right)^{1/2},
$$

and estimating $1 - (1 - \alpha)^{1/2} \geq \frac{\alpha}{4(1 - \alpha)^{1/2}}$ for $\alpha > 0$ small, we obtain that a necessary condition that the zeros of $f$ must satisfy is

$$
\sum_{n \geq 0} \frac{e^{\pi x_n/(2r)} \cos(\pi y_n/(2r))}{((e^{\pi x_n/r} + 1)^2 - 4(e^{\pi x_n/(2r)} \cos(\pi y_n/(2r)))^2)^{1/2}} < +\infty.
$$

This finishes the proof. \qed

Notice that, if the sequence of zeros satisfies $y_n = 0$, then this result has a much more pleasant formulation. Indeed, we may rewrite the condition then as

$$
\sum_{n \geq 0} e^{\pi x_n/(2r)} = \sum_{n \geq 0} \frac{1}{e^{\pi x_n/(2r)} - e^{-\pi x_n/(2r)}} < +\infty. \tag{4.20}
$$
This is, in fact, the form of this result that shall be useful to us when dealing with some specific sequences, such as the ones described in our main results.

### 4.3 Decay from Zeros

Finally, we address the question of how to obtain decay for a solution to (2.5) given the locations of its zeros.

We start with the one-dimensional case, where we have the following stronger result:

**Lemma 4.7** Let \( u \in C([0, T] : H^1(\mathbb{R})) \) be a strong solution to (2.5). Suppose that
\[
    u_0(\pm c_1 \log(1 + n)^\alpha) = u(\pm c_2 \log(1 + n)^\alpha, T) = 0, \forall n \geq 0,
\]
and some \( \alpha \in (0, 1] \). Then, for each \( N > 0 \), there is \( C_N > 0 \) so that
\[
    |u_0(x)| + |u(x, T)| \leq C_N e^{-N|x|}.
\]

**Proof** We start by noticing that, since \( u \in C([0, T] : H^1(\mathbb{R})) \), then we may use a simple comparison for both times \( t = 0, T \), using the fundamental theorem of calculus. Indeed, for \( t = 0 \) for instance, we have,
\[
    |u_0(x)| = |u_0(x) - u_0(c_1 \log(1 + n)^\alpha)| \leq \int_x^{c_1 \log(1+n)^\alpha} |u_0'(t)| \, dt \\
    \leq |c_1 \log(1 + n)^\alpha - c_1 \log(n)^\alpha|^{1/2} \|u_0'\|_2 \leq 2c_1^{1/2} \frac{\|u_0'\|_2}{(n + 1)^{1/2}} \\
    \leq 4c_1 e^{-|x|/(2c_1)} \|u_0'\|_2.
\]

Here, we have taken \( x \in (c_1 \log(n), c_1 \log(1 + n)) \), without loss of generality, with \( n \in \mathbb{N} \) sufficiently large. In particular, this shows exponential decay for the initial data and, by applying the same procedure to the data at time \( t = T \), we have
\[
    |u(x, T)| \leq 4c_2^{1/2} e^{-|x|/(2c_2)} \|\partial_x u(T)\|_2.
\]

By Lemma 4.1, this readily implies that \( u \in C([0, T] : H^k \cap L^2(x^{2k})) \), for any \( k \in \mathbb{N} \). In particular, we know that \( \partial^k_x u_0, \partial^k_x u(T) \in L^2(\mathbb{R}) \), \( \forall k \in \mathbb{N} \).

Before moving on, we divert our attention for a second towards the following property, which we state as a lemma:

**Lemma 4.8** Let \( f : \mathbb{R} \to \mathbb{C} \) be a (smooth) function with a sequence of zeros at \( \{x_n\}_{n \geq 0} \). Write \( f = u + iv \). We may find, for each \( k \geq 0 \), sequences \( \{a_n^{(k)}\}_{n \geq k}, \{b_n^{(k)}\}_{n \geq k} \) so that the following conditions are met:

1. \( \{a_n^{(0)}\}_{n \geq 0} = \{b_n^{(0)}\}_{n \geq 0} = \{x_n\}_{n \geq 0} \);
2. For each \( n \geq k \), we have \( a_n^{(k)} \in (a_{n-1}^{(k-1)}, a_n^{(k-1)}) \) and \( b_n^{(k)} \in (b_{n-1}^{(k-1)}, b_n^{(k-1)}) \).
(3) For each \( n \geq k \), we have \( \partial_x^k u(a_n^{(k)}) = \partial_x^k v(b_n^{(k)}) = 0 \).

The proof of this result follows by a simple inductive argument, using the mean value theorem, so we omit it. Notice that, in particular, if we start off with \( x_n = c \log(1+n)^\alpha \), with \( c \in \{c_1, c_2\} \), we may take \( f \) to be either \( u_0 \) or \( u(T) \). This implies that there are, for each \( k \geq 0 \), sequences

\[
\{a_n^{(k)}\}_{n \geq k}, \ {b_n^{(k)}\}_{n \geq k}, \ {c_n^{(k)}\}_{n \geq k}, \ {d_n^{(k)}\}_{n \geq k},
\]

so that

\[
\partial_x^k \text{Re}(u_0)(a_n^{(k)}) = \partial_x^k \text{Im}(u_0)(b_n^{(k)}) = \partial_x^k \text{Re}(u(T))(c_n^{(k)}) = \partial_x^k \text{Im}(u(T))(d_n^{(k)}) = 0, \ \forall \ n \geq k, \ k \geq 0.
\]

In particular, from the inductive construction of these sequences, we see that \( \eta_n^{(k)} \in (c \log(n-k), c \log(1+n)) \), where \( c \in \{c_1, c_2\} \) and \( \eta \in \{a, b, c, d\} \).

Now, in order to conclude the argument, notice that, by applying the argument, which we used to deduce decay above, to the \( k \)-th derivatives of the real and imaginary parts of \( u_0 \) and \( u(T) \), we have that, for some \( C(k) > 0 \),

\[
|\partial_x^k \text{Re}(u_0)(x)| + |\partial_x^k \text{Im}(u_0)(x)| \leq C(k) c_1^{1/2} e^{-|x|/(2c_1)} \|\partial_x^{k+1} u_0\|_2,
\]

and analogously,

\[
|\partial_x^k \text{Re}(u(T))(x)| + |\partial_x^k \text{Im}(u(T))(x)| \leq C(k) c_2^{1/2} e^{-|x|/(2c_2)} \|\partial_x^{k+1} u(T)\|_2.
\]

Suppose then \( y \in (a_n^{(k)}, a_n^{(k+1)}) \). Then

\[
|\partial_x^{k-1} u_0(y)| \leq \int_y^{a_n^{(k+1)}} |\partial_x^k \text{Re}(u_0)(x)| \, dx \leq \tilde{C}(k) c_1 e^{-|y|/c_1} \|\partial_x^{k+1} u_0\|_2.
\]

Analogous estimates hold for \( \partial_x^{k-1} \text{Im}(u_0), \partial_x^{k-1} \text{Re}(u(T)), \partial_x^{k-1} \text{Im}(u(T)) \), and we may use this process of improving backwards in order to obtain that, for each \( k \geq 0 \),

\[
|u_0(x)| \leq |\text{Re}(u_0)(x)| + |\text{Im}(u_0)(x)| \leq C'(k) c_1^{1/2} e^{-|x|/(2c_1)} \|\partial_x^{k+1} u_0\|_2,
\]

\[
|u(x, T)| \leq |\text{Re}(u(T))(x)| + |\text{Im}(u(T))(x)| \leq C'(k) c_2^{1/2} e^{-|x|/(2c_2)} \|\partial_x^{k+1} u(T)\|_2.
\]

By making \( k > 0 \) sufficiently large, we conclude the assertion, as desired. \( \square \)

In higher dimensions, we have a slightly worse version of the result above.

**Lemma 4.9** Let \( u \in C([0, T]; H^s(\mathbb{R}^d)) \), \( s > \frac{d}{2} - 1 \) be a strong solution to (2.5), where \( d \geq 2 \). If \( d = 2 \) or \( d = 3 \), assume additionally that \( s \geq 1 \). Suppose that

\[
u_0(\pm c_1 \log(1+n)^{\alpha} d^{d-1}) = u(\pm c_2 \log(1+n)^{\alpha} d^{d-1}, T) = 0, \ \forall \ n \geq 0,
\]

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where the identity is to be interpreted in the sense of Sobolev traces, for some $\alpha \in (0, 1]$. Then, for any $N \geq 0$ fixed real number, there is a positive constant $c_N > 0$ so that

$$|\partial^\alpha u_0(x)| + |\partial^\alpha u(x, T)| \lesssim N \cdot u_0, u(T) \cdot e^{-c_N|x|},$$

for all multiindices $\alpha$ so that $|\alpha| \leq N$.

**Proof** The proof consists of two steps, one where one obtains exponential decay for both $u_0, u(T)$, and an interpolation argument in order to obtain exponential decay for higher derivatives.

**Step 1:** $u_0, u(T)$ are bounded by $e^{-c_0|x|}$, where $c_0 > 0$ depends on the dimension and on the parameters $c_1, c_2$. For this step, the idea is to use Poincaré’s inequality between spheres where zeros are located.

Indeed, for that we will need the following proposition:

**Proposition 4.10** Let $C(\Omega)$ denote the best constant in the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)}$$

for $u \in H^1_0(\Omega)$. If $\Omega = B_R \setminus B_r$, $r \in (0, R)$, then

$$C(\Omega) \lesssim_d (R^d - r^d)^{2/d}.$$ 

**Proof of the Proposition** It is a well-known fact that the best constant in Poincaré’s inequality in a domain $\Omega$ is $1/\lambda_1$, where $\lambda_1$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions on $\Omega$. Thus, we only need to estimate this eigenvalue.

In order to do that, we use the Faber–Krahn inequality, which says that

$$\lambda_1(\Omega) \geq \lambda_1(B),$$

where $B$ is a ball with $|B| = |\Omega|$. Thus, taking into account the eigenvalue of the Dirichlet problem for the ball, we have

$$\lambda_1(\Omega) \geq c_d \cdot \left(\frac{1}{|\Omega|}\right)^{2/d},$$

where $c_d = \pi \cdot \Gamma(d/2 + 1)^{2/d} \cdot j_{d/2-1}^2$, where $j_{m,1}$ denotes the first positive zero of the Bessel function $J_m$. Thus, we obtain the control

$$C(\Omega) \leq \frac{1}{c_d} |\Omega|^{2/d} = \tilde{c}_d (R^d - r^d)^{2/d}.$$ 

This ends the proof of the proposition. \qed
With this proposition, we use the Poincaré inequality above with \( u_0, u(T) \) in each of the annuli \( A_i, n = B_{e_i} \log((1+n)^\alpha) \setminus B_{e_i} \log(n)^\alpha, n \geq 1 \). We obtain that

\[
\|u_0\|_{L^2(A_{1,n})} + \|u(T)\|_{L^2(A_{2,n})} \\
\leq \log(n + 1)^{2\alpha - \frac{d - 1}{d}} (\log(n + 1)^\alpha - \log(n)^\alpha)^{2/d} \left( \|\nabla u_0\|_{L^2(A_{1,n})} + \|\nabla u(T)\|_{L^2(A_{2,n})} \right) \\
\leq \log(n + 1)^{2\alpha - \frac{d - 1}{d}} \left( \frac{1}{(n + 1)^{2/d}} \right) (\|\nabla u_0\|_{L^2(A_{1,n})} + \|\nabla u(T)\|_{L^2(A_{2,n})}).
\]

Reordering terms, we obtain that

\[
\frac{\|e^{2|x|/(dc_1)}\|_{L^2(A_{1,n})}}{(1 + |x|)^2} u_0 + \frac{\|e^{2|x|/(dc_2)}\|_{L^2(A_{2,n})}}{(1 + |x|)^2} u(T) \\
\lesssim \|\nabla u_0\|_{L^2} + \|\nabla u(T)\|_{L^2}.
\]

Summing over \( n \geq 0 \), we obtain that

\[
\frac{\|e^{2|x|/(dc_1)}\|_{L^2}}{(1 + |x|)^2} u_0 + \frac{\|e^{2|x|/(dc_2)}\|_{L^2}}{(1 + |x|)^2} u(T) \\
\lesssim \|\nabla u_0\|_{L^2} + \|\nabla u(T)\|_{L^2}.
\]

We shall now use the following proposition, whose proof is inspired in that of [23, Prop 3.1]:

**Proposition 4.11** Let \( f \in L^2(e^{2a|x|}dx), \nabla f \in L^\infty \). Then \( |f(x)| \leq Ce^{-\frac{a}{4}\log|x|}, \forall x \in \mathbb{R}^d, \) where \( C \) depends on \( \|f\|_{L^\infty}, \|\nabla f\|_{L^\infty}, \|f\|_{L^2(e^{a|x|}dx)} \).

**Proof** Without loss of generality, \( f \) is not identically zero. As \( \nabla f \in L^\infty \), we have that \( f \) is absolutely continuous so it suffices to consider large \( |x| \). A standard argument shows that \( \nabla f \in L^\infty \) and \( f \in L^2 \) imply that \( f(x) \to 0 \) as \( x \to \infty \). In particular, \( f \in L^\infty \). Now, for each \( x \in \mathbb{R}^d \), fix a ball \( B_x \) centered at \( x \), with radius \( r_x > 0 \) to be determined in a while. Then, for any \( y \in B_x \), we have

\[
|f(x) - f(y)| \leq \left( \sup_{z \in B_x} |\nabla f(z)| \right) |x - y| \leq 2r_x \|\nabla f\|_{L^\infty}.
\]

We then pick \( r_x = \frac{|f(x)|}{4\|\nabla f\|_{L^\infty}}, \) and thus \( |f(y)| \geq |f(x)|/2 \). Note also that \( \sup_x r_x < \infty \) as \( f \in L^\infty \). Moreover, if \( |x| \) is sufficiently large, then triangle’s inequality shows that \( |y| \geq |x|/2, \forall y \in B_x \). Therefore, for such \( x \),

\[
\|fe^{a|x|}\|_2^2 \geq \int_{B_x} |e^{a|x|}f(y)|^2 \, dy \geq r_x^d e^{a|x|} \|f(x)\|_2^2 \geq f \|f(x)\|^{d+2} e^{a|x|}.
\]

This readily implies that \( |f(x)| \lesssim f e^{-\frac{a}{4\log|x|}}, \forall |x| \gg 1 \). This finishes the proof. \( \square \)
We now employ Lemma 4.2 for \( u_0, u(T) \). This shows, in particular, that \( \nabla u_0, \nabla u(T) \in L^\infty \). By Proposition 4.11, we are able to finish Step 1.

**Step 2: Propagating decay for derivatives.** For this step, we use an inductive argument on the number \( N = |\alpha| \). In particular, if \( N = 0 \), we are done by Step 1.

Suppose our result holds for \( N \in \mathbb{N} \) fixed. By Lemma 4.2, we are able to deduce that derivatives of all orders are bounded. We use this fact, together with a Taylor expansion of order one in order to conclude.

Indeed, fixing \( \alpha \) a multiindex so that \( |\alpha| = N + 1 \), let \( j \in \{1, \ldots, d\} \) be so that \( \alpha_j > 0 \). In particular, we may perform a Taylor expansion of order one of the function \( \partial_x^{\alpha - e_j} u_0(x + te_j) \). In particular, we know that, since the derivative \( \partial_x^{\alpha + e_j} u_0 \) belongs to \( L^\infty(\mathbb{R}^d) \),

\[
|\partial_x^{\alpha - e_j} u_0(x + te_j) - \partial_x^{\alpha - e_j} u_0(x) - t\partial_x^{\alpha} u_0(x)| \lesssim \|\partial_x^{\alpha + e_j} u_0\|_{L^\infty} t^2.
\]

Thus, reordering terms,

\[
|\partial_x^{\alpha} u_0(x)| \lesssim_{u_0} t^{-1} \left( |\partial_x^{\alpha - e_j} u_0(x + te_j)| + |\partial_x^{\alpha - e_j} u_0(x)| \right) + t
\]

\[
\lesssim_{u_0} t^{-1} \left( e^{-cN|x+te_j|} + e^{-cN|x|} \right) + t,
\]

where we used the induction hypothesis in the last line. In particular, we may choose \( t = e^{-cN|x|} \) above. Routine computations then imply

\[
|\partial_x^{\alpha} u_0(x)| \lesssim_N e^{-\frac{cN}{2}|x|}.
\]

This closes the inductive argument, and we are done. \( \square \)

## 5 Proof of the One-dimensional Results

### 5.1 Proof of the Unique Continuation Results

**Proof of Theorem 1** By using the invariance of the Schrödinger equation and the associated properties of the potential (see, for instance, [5, p. 180]), we may suppose that \( T = 1 \).

We now use the Duhamel formula for representing the solution as

\[
u(x, t) = e^{i\alpha^2 t} u_0(x) + \int_0^t e^{i(t-\tau)\alpha^2} (V(\tau) u(\tau))(x) d\tau, \quad \forall t \in [0, 1].\] (5.1)
Step 1: $u_0, u(1)$ are $\frac{1}{2}$-Hölder continuous. Using (5.1) we estimate for some $\theta > 0$

$$|u(x, 1) - u(y, 1)| \leq |e^{i\partial_x^2/2} u_0(x) - e^{i\partial_x^2/2} u_0(y)|$$

$$+ \int_0^{1-\theta} |e^{(1-s)\partial_x^2} (V(s)u(s))(x) - e^{(1-s)\partial_x^2} (V(s)u(s))(y)|ds$$

$$+ \int_{1-\theta}^1 |e^{(1-s)\partial_x^2} (V(s)u(s))(x) - e^{(1-s)\partial_x^2} (V(s)u(s))(y)|ds.$$ 

For the first term we use the first bound in Lemma 3.1 to obtain

$$|e^{i\partial_x^2/2} u_0(x) - e^{i\partial_x^2/2} u_0(y)| \lesssim \max(|x|, |y|) \| (1 + |z|) u_0(z) \|_{L^1} |x - y|.$$ 

For the second term we again use the first bound in Lemma 3.1 and obtain

$$\int_0^{1-\theta} |e^{(1-s)\partial_x^2} (V(s)u(s))(x) - e^{(1-s)\partial_x^2} (V(s)u(s))(y)|ds$$

$$\lesssim \max(|x|, |y|)|x - y| \int_0^{1-\theta} \| (1 + |z|) V(z, s) u(z, s) \|_{L^1} ds$$

$$\lesssim \max(|x|, |y|)|x - y| \theta^{-\frac{1}{2}} \| Vu \|_{L^\infty_{(0, 1)} L^1((1+|z|)dz)}.$$ 

For the third term we use the second bound in Lemma 3.1 to obtain

$$\int_{1-\theta}^1 |e^{(1-s)\partial_x^2} (V(s)u(s))(x) - e^{(1-s)\partial_x^2} (V(s)u(s))(y)|ds \lesssim \int_{1-\theta}^1 \| V(s)u(s) \|_{L^1} ds$$

$$\lesssim \theta^{-\frac{1}{2}} \| Vu \|_{L^\infty_{(0, 1)} L^1}.$$ 

We first observe that

$$\| Vu \|_{L^\infty_{(0, 1)} L^1} \lesssim \| V \|_{L^\infty_{(0, 1)} L^2} \| u \|_{L^\infty_{(0, 1)} L^2} < \infty,$$

as well as

$$\| Vu \|_{L^1_{(0, 1)} L^1((1+|z|)dz)} \lesssim \| V \|_{L^\infty_{(0, 1)} L^2((1+|z|^2)dz)} \| u \|_{L^\infty_{(0, 1)} L^2} < \infty$$

in view of the assumptions of Theorem 1. Then, choosing $\theta = |x - y| \max(|x|, |y|)$ shows that

$$|u(x, 1) - u(y, 1)| \leq C (\| V \|_{L^\infty_{(0, 1)} L^2((1+|z|^2)dz)}, \| u \|_{L^\infty_{(0, 1)} L^2}) |x - y|^{\frac{1}{2}} \max(|x|, |y|)^{\frac{1}{2}}$$

(5.2)

which shows that $u(\cdot, 1)$ is $\frac{1}{2}$-Hölder continuous. Evolving backwards, we analogously obtain $\frac{1}{2}$-Hölder continuity for $u_0.$
Step 2: Decay estimate from the zeros. With the $C^{1/2}$-continuity at hand, we will make use of the pointwise assumptions. The idea is then to compare the solution at two different points for times $t = 0, 1$. In order to do so, we argue similarly to Step 1 and make use of Lemma 3.1. Indeed, suppose that $|u(\pm c_1 \log(1+n)^\alpha, 1)| \leq n^{-\delta}$, for some $\delta > 0$. Suppose that

$$x \in (c_1 \log(1+n)^\alpha, c_1 \log(2+n)^\alpha).$$

Then, as before, using the Duhamel formula for $t = 1$,

$$|u(x, 1)| \leq |u(x, 1) - u(c_1 \log(1+n)^\alpha, 1)| + n^{-\delta} \leq |e^{i\theta_0} u_0(x) - e^{i\theta_2} u_0(c_1 \log(1+n)^\alpha)|$$

$$+ \int_0^{1-\theta} |e^{(1-s)i\theta_2} (V(s)u(s))(x) - e^{(1-s)i\theta_2} (V(s)u(s))(c_1 \log(1+n)^\alpha)| \, ds$$

$$+ \int_{1-\theta}^1 |e^{(1-s)i\theta_2} (V(s)u(s))(x) - e^{(1-s)i\theta_2} (V(s)u(s))(c_1 \log(1+n)^\alpha)| \, ds + n^{-\delta}.$$ 

(5.3)

We may use the first bound in Lemma 3.1 directly in the first term in the sum above. As $x \in (c_1 \log(1+n)^\alpha, c_1 \log(2+n)^\alpha)$, then $|c_1 \log(1+n)^\alpha| \leq 2|x|$ if $n \geq 1$. This implies that

$$|e^{i\theta_2} u_0(x) - e^{i\theta_2} u_0(c_1 \log(1+n)^\alpha)| \leq C|x||x - c_1 \log(1+n)^\alpha||1 + |z||u_0(z)||1$$

$$\leq C \cdot c_1 |x| \log(2+n)^\alpha - log(1+n)^\alpha ||(1 + |z||u_0(z)||1$$

$$\leq \tilde{C}|x|e^{-\frac{1}{2}|x|^{1/\alpha}} ||(1 + |z||u_0(z)||1.$$

Similarly to Step 1 we estimate

$$\int_0^{1-\theta} |e^{(1-s)i\theta_2} (V(s)u(s))(x) - e^{(1-s)i\theta_2} (V(s)u(s))(c_1 \log(1+n)^\alpha)| \, ds$$

$$+ \int_{1-\theta}^1 |e^{(1-s)i\theta_2} (V(s)u(s))(x) - e^{(1-s)i\theta_2} (V(s)u(s))(c_1 \log(1+n)^\alpha)| \, ds$$

$$\leq C|x||x - c_1 \log(1+n)^\alpha| \int_0^{1-\theta} \frac{|(1 + |z||V(z, s)u(z, s)||1}{(1-s)^{3/2}} \, ds$$

$$+ C \int_{1-\theta}^1 \frac{|V(z, s)u(z, s)||1}{(1-s)^{1/2}} \, ds \leq C \theta^{-1/2}|x|e^{-\frac{1}{2}|x|^{1/\alpha}} \|Vu\|_{L^\infty[0,1]} L^1((1+|z|)dz)$$

$$+ C \theta^{1/2}\|Vu\|_{L^\infty[0,1]} L^1(\mathbb{R}).$$

We want to make both estimates on the right-hand side above compatible, so we take

$$\theta = |x|e^{-\frac{1}{2}|x|^{1/\alpha}} \frac{\|Vu\|_{L^\infty[0,1]} L^1((1+|z|)dz)}{\|Vu\|_{L^\infty[0,1]} L^1(\mathbb{R})}.$$
are bounded by
\[ C|x|^{1/2}e^{-\frac{1}{2}|x|^{1/\alpha}}\|Vu\|^{1/2}_{L^\infty_{[0,1]}L^1((1+|z|)dz)}\|Vu\|_{L^\infty_{[0,1]}L^1(\mathbb{R})}. \]

On the other hand, by a simple use of Cauchy–Schwarz, this last term is bounded by
\[ C|x|^{1/2}e^{-\frac{1}{2}|x|^{1/\alpha}}\|u\|_{L^\infty_{[0,1]}L^2(\mathbb{R})}\|V\|_{L^\infty_{[0,1]}L^2((1+|z|)^2dz)}. \]

Finally, as \(|x| \leq c_1 \log(2 + n)^\alpha\) and
\[(2 + n)^{-\delta} = e^{-\delta \log(n+2)} = e^{-\frac{\delta (c_1 \log(n+2)^\alpha)}{c_1^\alpha}} \leq e^{-\frac{\delta}{c_1^\alpha}|x|^{1/\alpha}},\]
we may estimate then
\[ |u(x, 1)| \leq C|x|e^{-\omega|x|^{1/\alpha}} \left(1 + \|(1 + |z|)u_0\|_1 + \|u\|_{L^\infty_{[0,1]}L^2(\mathbb{R})}\|V\|_{L^\infty_{[0,1]}L^2((1+|z|)^2dz)}\right). \]

(5.4)

where \(\omega = \min(1/4, \delta/c_1^{1/\alpha})\). By using that the equation (5.3) is time-reversible, we obtain together with (5.4) that
\[ |u(x, 1)| + |u_0(x)| \leq K(u, V)|x|e^{-\tilde{\omega}|x|^{1/\alpha}}, \]
\[ (5.5) \]

where we let \(\tilde{\omega} = \min\{\omega, \delta/c_2^{1/\alpha}\}\), and we define the constant \(K(u, V)\) to be
\[ C \left(1 + \|(1 + |z|)u_0\|_1 + \|(1 + |z|)u(1)\|_1 + \|u\|_{L^\infty_{[0,1]}L^2(\mathbb{R})}\|V\|_{L^\infty_{[0,1]}L^2((1+|z|)^2dz)}\right). \]

We now simply invoke the result by Escauriaza–Kenig–Ponce–Vega [5], even in a weak form: if \(V\) satisfies the conditions of Theorem 1 and \(u_0, u(1) \in L^2(e^{A|x|^2}, \mathbb{R})\) for \(A \gg 1\) sufficiently large, then \(u \equiv 0\). By (5.5), we see that this condition is fulfilled as \(\alpha < \frac{1}{2}\), and thus we have finished the proof of the Theorem.

We are now able to use Theorem 1 in order to conclude Corollary 1 about unique continuation for solutions to Schrödinger equations.

Proof of Corollary 1 If \(u, v \in C([0, T] : H^s(\mathbb{R}) \cap L^2(x^2 dx))\) are solutions to (2.3), then \(u - v =: \tilde{w}\) satisfies (2.1) for the potential
\[ V = \frac{F(u, \overline{u}) - F(v, \overline{v})}{u - v}. \]

If \(F \in C^1(\mathbb{C}^2 : \mathbb{C})\), one may find by (2.4) that
\[ |V(x, t)| \leq C(|u(x, t)|^{p-1} + |v(x, t)|^{p-1}). \]

(5.6)
As $u, v \in L^\infty([0, T]: L^\infty(\mathbb{R}))$, the conditions on $u, v$ imply that $V \in L^1([0, T]: L^\infty(\mathbb{R})) \cap L^\infty([0, T]: L^2((1 + |x|)^2dx))$. By (5.6), $V$ is seen to satisfy the other conditions in Theorem 1, and thus we may apply that result directly, which implies that $w \equiv 0$, or equivalently $v \equiv u$.  

5.2 Proof of the One-dimensional Rigidity Result

**Proof of Corollary 2** We start off by noticing that from Lemma 4.1 from $k = 1$, the solution $u \in C([0, T]: H^1(\mathbb{R}))$. From Lemma 4.7, we have that

$$|u_0(x)| + |u(x, T)| \lesssim_N e^{-N|x|}, \forall N \geq 1.$$  

We then use the following result by Kenig–Ponce–Vega:

**Lemma 5.1 ([14, Lemma 2.1])** For any $T > 0$, there is $\varepsilon(T) > 0$ so that the following holds: if $\lambda \in \mathbb{R}^d$ is a vector, $V : \mathbb{R}^d \times [0, T] \to \mathbb{C}$ is a potential so that

$$\|V\|_{L^1([0, T]; L^\infty(\mathbb{R}))} \leq \varepsilon,$$

and $w \in C([0, T]: L^2(\mathbb{R}^d))$ is a solution to

$$i \partial_t w = -\Delta w + V \cdot w + F,$$

then

$$\sup_{t \in [0, T]} \|e^{\lambda \cdot x} w(t)\|_{L^2(\mathbb{R}^d)} \lesssim \|e^{\lambda \cdot x} w_0\|_{L^2(\mathbb{R}^d)} + \|e^{\lambda \cdot x} w(T)\|_{L^2(\mathbb{R}^d)} + \|e^{\lambda \cdot x} F(x, t)\|_{L^1([0, T]; L^2(\mathbb{R}^d))}.$$  

In our case, we shall apply this result with $V = \chi_{[R, 1]} B_R |u|^2$ and $F = \chi_{B_R} |u|^2 u$. We clearly see that $V$ satisfies the conditions of Lemma 5.1 if we pick $R > 0$ sufficiently large. Thus, by applying this result with both $N, -N$ with the estimate above, we have

$$\sup_{t \in [0, T]} \|e^{N|x|} u(t)\|_2 \leq C(\|e^{N|x|} u_0\|_2 + \|e^{N|x|} u(T)\|_2) + e^{NR} \|u_0\|^2_{L^2_T L^\infty_x} \|u\|_{L^\infty_T L^2_x}.$$  

In particular, we have that $e^{N|x|} u(\cdot, t) \in L^2$ for all $t \in [0, T]$. Now, we use Lemma 4.4 as well as (4.18) in order to conclude that there is a time

$$T' = T'(|e^{N|x|} u_0\|_2, \|e^{N|x|} u(T)\|_2, \|u_0\|_{H^1}, \|u(T)\|_{H^1})$$

so that for all $t$, $e^{ix^2/4} u(t + \tau)$ admits an analytic continuation in a strip of width $N \tau$, where $\tau < T'$.

In particular, if we take $t = T - T'/2$, $\tau = (T')/2$ we will obtain that $e^{ix^2/4} u(T)$ is analytic and bounded in a strip $S(r), r < N(T')/2$.  

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In order to conclude, we need only to invoke Lemma 4.6. In fact, for \( r < \frac{NT'}{2} \) as above, we know that (4.20) holds in particular for all real zeros of \( e^{\frac{i\chi^2}{4}} u(T) \). But then we should have
\[
\sum_{n \geq 0} \frac{1}{e^{c_2 \log(1+n)\alpha/(2r)} - e^{-c_2 \log(1+n)\alpha/(2r)}} < +\infty. \tag{5.7}
\]
But, in case \( \alpha < 1 \), we have that \( e^{c_2 \log(1+n)\alpha/(2r)} < e^{\log(1+n)} = n + 1 \) for all \( n \gg u, c_2 \). Thus, we should have, by (5.7),
\[
\sum_{n \geq 0} \frac{1}{n + 1} < +\infty.
\]
This is a clear contradiction. This contradiction arises from the fact that, in order to use Lemma 4.6, we must assume \( u(T) \neq 0 \). Thus, \( u(T) \equiv 0 \), and it follows at once that \( u_0 \equiv 0 \), as desired. \( \square \)

6 Proof of the Higher-dimensional Results

In this section, we will discuss how to prove a suitable version of the theorem above in higher dimensions.

Proof of Theorem 3 We note that for \( d = 2, 3 \), we first use Lemma 4.2 to guarantee that \( u_0 \) and \( u(T) \) are in \( H^1(\mathbb{R}^d) \). (For \( d \geq 4 \), this is already guaranteed as \( s > s_c = d/2 - 1 \).) Now, we employ Lemma 4.9 for \( N = 0 \) in order to conclude the exponential decay of \( u_0 \) and \( u(T) \). From that, we use Lemma 4.2, which guarantees that \( u_0, u(T) \in H^k(\mathbb{R}^d) \), for all \( k \in \mathbb{N} \).

The next step is to employ Lemma 4.5, together with a Corollary of the argument of Kenig–Ponce–Vega from Lemma 5.1. Indeed, we use an induction argument to prove that, for each \( N > 0 \), there is \( c_N > 0 \) so that \( e^{c_N |x|} \partial_x^\alpha u(t) \in L^2(\mathbb{R}^d) \), for all \( t \in [0, T] \) and for all \( \alpha \) multiindices so that \( |\alpha| \leq N \).

Indeed, for \( N = 0 \), the result follows in the same fashion as the argument in the proof of Theorem 2. Assuming it holds for some \( N \in \mathbb{N} \), a routine computation shows that any derivative \( \partial_x^\alpha u =: v_\alpha, |\alpha| = N + 1 \), satisfies an equation of the form
\[
i \partial_t v_\alpha = -\Delta v_\alpha + |u|^2 v_\alpha + H,
\]
for \( H \) a finite sum of products of derivatives of \( u \) and \( \overline{u} \). The key property of \( H \) is that each of the summands is a product of some lower order derivative of \( u \) with two other derivatives of \( u \) of order \( < N + 1 \). By picking \( \tilde{V} = \chi_{\mathbb{R}^d \setminus B_R} |u|^2 \) and \( F = H + \chi_{B_R} |u|^2 \), we see immediately that the conditions of Lemma 5.1 hold for some \( R > 0 \) sufficiently large. As we have proved in Lemma 4.9 that \( \partial_x^\alpha u_0, \partial_x^\alpha u(T) \in L^2(e^{c_{N+1} |x|}) \) for some \( c_{N+1} > 0 \), and the induction hypothesis shows that \( H \in L^1([0, T]; L^2(e^{c_N |x|})) \), we conclude by Lemma Lemma 5.1 that \( \sup_{t \in [0, T]} \| e^{c_{N+1} |x|} \partial_x^\alpha u(t) \|_2 < +\infty \), as desired.
We are now in a position to use Lemma 4.5. Indeed, choosing \( N = \lfloor d/2 \rfloor \) in the argument above, let \( C(u) = \sup \{ t \in [0,T] \, \|u(t)\|_{A^N_2(g N+1/2)} < +\infty \) for shortness. Fix the time \( T'(C(u)) \) given by Lemma 4.5. We pick \( t = T - \frac{T'(C(u))}{2} \), and from Lemma 4.5 itself and the comment thereafter we have that \( e^{i |z|^2/(2T'(C(u)))} u(T) =: v(T) \) is analytic and bounded in each strip \( S^d(r), \quad r < \pi c N+1 T'(C(u)) \).

In order to finish, we shall resort back to one-dimensional models. Indeed, if \( v(T) \) is as before, then we consider the one-dimensional functions \( v(T) \tilde{z}(z_1) = v(T)(z) \), where \( z_1 \) is the first coordinate of \( z \), and \( \tilde{z} \) denotes the projection of the remaining coordinates onto \( \mathbb{C}^{d-1} \). We fix any \( \tilde{z} \in \mathbb{R}^d \) for the rest of the argument.

By construction, \( v(T) \tilde{z} \) is bounded and analytic in any one (complex) dimensional strip \( S^1(r') \) as above. But, by construction once more, we have that \( v(T) \tilde{z} \) has, when restricted to \( z_i \in \mathbb{R} \), zeros of the form \( (c_\tilde{z}^2 \log(n + 1)^{2\alpha} - |\tilde{z}|^2)^{1/2} \), with \( n > 0 \) sufficiently large. But we may employ the same argument with Lemma 4.6 to this sequence as well, as it grows like \( c_\tilde{z} \log(1 + n)^\alpha \). This shows that \( v(T) \tilde{z} \) vanishes identically, and so does \( u(T) \tilde{z} \). As \( \tilde{z} \) was arbitrary, we conclude that \( u(T) \equiv 0 \), as desired. \( \square \)

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