Metric Results on Sumsets and Cartesian Products of Classes of Diophantine Sets

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Abstract. Erdős proved that any real number can be written as a sum, and a product, of two Liouville numbers. Motivated by these results, we study sumsets of classes of real numbers with prescribed (or bounded) irrationality exponents. We show that such sumsets turn out to be large in general, indeed almost every real number with respect to Lebesgue measure can be written as the sum of two numbers with sufficiently large prescribed irrationality exponents. In fact the Hausdorff dimension of the complement is small, and the result remains true if we impose considerably refined conditions on the orders of rational approximation (“exact approximation” with respect to an approximation function). As an application, we show that in many cases the Hausdorff dimension of Cartesian products of sets with prescribed irrationality exponent exceeds the expected dimension, that is the sum of the single Hausdorff dimensions. We also address their packing dimensions. Similar results hold when restricting to classical missing digit Cantor sets, relative to its natural Cantor measure. In particular, we prove that the subset of numbers with prescribed large irrationality exponent has full packing dimension, i.e. the same packing dimension as the entire Cantor set. This complements some results on the Hausdorff dimension of these sets, which is an extensively studied topic in Diophantine approximation. Our proofs are based on ideas of Erdős, but vastly extend them.

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1. Hausdorff Dimensions and Cartesian Products

Since we aim to establish metrical results, we start by recalling some fundamental metric concepts. Hausdorff measure and Hausdorff dimension are widely used concepts in measuring the size of a set. We will throughout denote by $\dim(A)$ the Hausdorff dimension of the set $A$. We will sporadically deal with the packing dimension $\dim_P(A)$ of a $A \subseteq \mathbb{R}^n$ as well. We omit the exact definitions and refer to Falconer [19]. We remind that the Hausdorff dimension of a set never exceeds its packing dimension.

We investigate Hausdorff dimensions of sumsets and Cartesian products of certain Euclidean sets. Let us focus on Cartesian products for now. This topic has been addressed for various classes of sets, see for example, Besicovitch and Moran [5], Eggleston [17], Marstrand [26], Xiao [39], Larman [23], Wegmann [37]. A fundamental property of Hausdorff dimension proved by Marstrand [26] is that for any measurable sets $A, B$ when taking their Cartesian product we have

$$\dim(A \times B) \geq \dim A + \dim B.$$  (1)

In general, we do not have equality in (1). However, in many interesting situations, the equality does hold, for example, for products of classical fractals like the Cantor middle-third set [19]. Critria on the sets $A, B$ that imply that the equality holds can be found as well in [19]. An upper bound due to Tricot [36] for the left hand side of (1) involving the packing dimension is contained in Theorem 1.1 below.

**Theorem 1.1 (Tricot).** For any measurable sets $A, B$ in $\mathbb{R}^n$, we have

$$\dim(A \times B) \leq \dim(A) + \dim_P(B).$$

Hence, if $A_1, \ldots, A_n$ are measurable subsets of $\mathbb{R}$, then

$$\dim(A_1 \times A_2 \times \cdots \times A_n) \leq n - 1 + \min_{1 \leq i \leq n} \dim(A_i).$$

See also Bishop and Peres [6] for refinements. For completeness we also want to mention the estimates

$$\dim_P(A) + \dim_P(B) \geq \dim_P(A \times B) \geq \dim(A) + \dim_P(B)$$  (2)

due to Tricot [36] as well, which we will not use as frequently in this paper. One purpose of this paper is to find sets that naturally occur in Diophantine approximation, whose Cartesian products are of Hausdorff dimension strictly larger than the sum of the single dimensions, i.e. there is no equality in (1). On the way, we will encompass sumsets and study their properties. An important tool to achieve this goal is the following rather elementary property of Hausdorff measures and dimensions, applied in suitable contexts.

**Proposition 1.2.** Let $A \subseteq \mathbb{R}^m$ be a measurable set and $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be Lipschitz. Then $\dim \phi(A) \leq \dim(A)$. More generally, for any $s \geq 0$ writing $H_s$ for the $s$-dimensional Hausdorff measure, we have $H_s(\phi(A)) \ll_{s,m,n} H_s(A)$. 
See [19, Proposition 2.2, Corollary 2.4] and also [20, Proposition 2.2] for a more general version. In Proposition 1.2, and the sequel, we use Vinogradov’s notation $A \ll B$ which means $A \leq c(.)B$, that is $A$ does not exceed $B$ by more than some multiplicative constant depending on the subscript variables only, with an absolute constant if no subscript occurs. As usual, we shall also use $A \asymp B$ as short notation for $A \ll B \ll A$.

2. Sumsets and Cartesian Products of the Set of Liouville Numbers

Even though the deepest results of the paper appear in Sect. 3, we prefer to start our investigation with Cartesian products of Liouville numbers, where the historical context and motivation can be presented more naturally.

Recall that $\xi \in \mathbb{R}\setminus\mathbb{Q}$ is called Liouville number if the inequality

$$\left|\xi - \frac{p}{q}\right| < q^{-N}$$

has solutions in rational numbers $p/q$ for arbitrarily large $N$. Let us denote the set of Liouville numbers by $\mathcal{L}$. The Hausdorff dimension of $\mathcal{L}$ equals to 0. But, on the other hand, it is co-meager, i.e. its complement $\mathbb{R}\setminus\mathcal{L}$ is of first category. See Chapter 2 of Oxtoby’s book [28] for short proofs of both results. For refined further measure theoretic results on $\mathcal{L}$ when considering general Hausdorff $f$-measures, we refer to Olsen and Renfro [27] and Bugeaud et al. [10].

A well-known result of Erdős [18] that motivates the investigations in this paper claims that every real number can be written as a sum (or product if $\neq 0$) of two Liouville numbers. Erdős gave two proofs. One is based on the mentioned fact that $\mathcal{L}$ is co-meager. Indeed, consequently the set $\mathcal{L} \cap \mathcal{L}_\xi$ with $\mathcal{L}_\xi = \{\xi - x : x \in \mathcal{L}\}$ is co-meager as well for any $\xi \in \mathbb{R}$, thus non-empty. Now any pair $(y, \xi - y)$ with $y$ in the intersection $\mathcal{L} \cap \mathcal{L}_\xi$ consists of Liouville numbers that by construction sum up to a given $\xi$. The argument can be widely extended, see Rieger [31], Schwarz [35], Burger [12,13] and Senthil Kumar et al. [34]. For the second proof, Erdős effectively constructs Liouville numbers $x, y$ with the property that $x + y = \xi$ for any given $\xi \in \mathbb{R}$. Let us recall this proof as well. Suppose $\xi$ has decimal expansion $\xi = 0.d_1d_2\ldots$, $0 \leq d_j \leq 9$. Define $b_j = j!$. Let $x$ be the number with the same base 10 digits of $\xi$ for indices from $b_{2j} + 1$ to $b_{2j+1}$, and 0 otherwise, and let $y$ be the number having the digits of $\xi$ in the remaining intervals from $b_{2j+1} + 1$ to $b_{2j}$ and 0 otherwise. Then $x + y = \xi$. On the other hand, $x, y$ are both Liouville numbers. Indeed, the rational numbers obtained from cutting off the decimal expansion of $x$ and $y$ after positions of the form $b_{2j+1}$ and $b_{2j}$ respectively, will be very good rational approximations to $x$ and $y$, respectively. (One thing unnoticed by Erdős is that $x$ or $y$ could potentially be rational. However, the method is flexible enough to overcome this problem by a short variation argument.)
Now observe the following consequence of Erdős’ result when combined with Proposition 1.2: Since the map
\[ \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R} \]
\[ (x, y) \mapsto x + y \]
is Lipschitz continuous and surjective, the product set \( \mathcal{L} \times \mathcal{L} \) has Hausdorff dimension at least 1, even though \( \mathcal{L} \) has Hausdorff dimension 0. In fact
\[ \dim(\mathcal{L} \times \mathcal{L}) = 1, \tag{3} \]
since the reverse bound follows from Theorem 1.1. Though (3) is an easy implication of Erdős’ result, the author did not find this fact explicitly in the literature. We should remark that (3) applies, by the same argument, to any subset \( L \) of \( \mathcal{L} \) with the property that \( L + L = \mathbb{R} \). As pointed out to me by Sidney A. Morris, it has recently been proved that there is an abundance of these sets, see [14] for details. In the sequel, we write \( A^n \) for the \( n \)-fold Cartesian product \( A \times A \times \cdots \times A \) of a set \( A \). We use a similar idea to generalise (3) to the \( n \)-fold product \( \mathcal{L}^n \). The packing dimension of \( \mathcal{L}^n \) will also be calculated.

**Theorem 2.1.** For any integer \( n \geq 1 \), the set \( \mathcal{L}^n \) has Hausdorff dimension \( n - 1 \) and packing dimension \( n \).

The proof of the theorem is not difficult. As Erdős, we can provide two different proofs. The first shorter one is essentially a special case of [31] or [35], the latter constructive proof prepares the reader for the more complicated constructions in the proofs of our results stated in Sect. 3.

**Proof.** We need only to show the identity for Hausdorff dimension, the claim on packing dimension then follows from Theorem 1.1 and the fact that \( \dim(\mathcal{L}) = 0 \) via
\[ n = \dim(\mathcal{L}^{n+1}) \leq \dim(\mathcal{L}) + \dim_p(\mathcal{L}^n) = \dim_p(\mathcal{L}^n) \leq n. \tag{4} \]

The above inequalities clearly also imply \( \dim(\mathcal{L}^n) \leq n - 1 \) for \( n \geq 1 \). For the lower bound we give two proofs again, each showing in a different way that the Lipschitz map
\[ \Psi : \mathcal{L}^n \rightarrow \mathbb{R}^{n-1} \]
\[ (x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0 + x_1, x_0 + x_2, \ldots, x_0 + x_{n-1}), \]
is surjective. Then by Proposition 1.2 we obtain \( \dim(\mathcal{L}^n) \geq n - 1 \), as desired.

First we see that for any real vector \( \xi = (\xi_1, \ldots, \xi_{n-1}) \) the intersection
\[ \mathcal{F} := \bigcap_{i=1}^{n-1} \mathcal{L}_{\xi_i} \cap \mathcal{L}, \quad \text{where} \quad \mathcal{L}_{\xi} := \{ \xi - \ell : \ell \in \mathcal{L} \}, \]
is co-meager since every set in the intersection is co-meagre. In particular \( \mathcal{F} \) is non-empty. Now it is again easy to check that any element \( \xi \in \mathcal{F} \) induces a vector \( \ell \in \mathcal{L}^n \) with \( \Psi(\ell) = \xi \).
We sketch a second, constructive proof. Let \( \xi_1, \ldots, \xi_{n-1} \in [0, 1) \) be arbitrary with decimal expansions \( \xi_i = 0.d_{i,1}d_{i,2} \ldots \) and \( 0 \leq d_{i,j} \leq 9 \). We use a similar argument to that of Erdős. Let \( b_j = j! \) for \( j \geq 1 \) and partition \( \mathbb{N} \) into intervals of the form \( I_0 = \{1\} \) and \( I_j = \{b_j + 1, b_j + 1, \ldots, b_{j+1}\} \) for \( j \geq 1 \). Define \( x_0 \) as follows. For every \( i \in \{1, 2, \ldots, n - 1\} \), if \( j \equiv i \mod n \) then take the decimal digits in places \( \ell \in I_j \) of \( x_0 \) to be those \( d_{i,\ell} \) of \( \xi_i \) in this interval. For \( j \equiv 0 \mod n \), we define the decimal digits in places \( \ell \in I_j \) as 0. Then \( x_0 \) is well-defined. Now, we claim \( x_0 \) for \( f \) open maps \( x \), that some arbitrary with decimal expansions \( \xi \). Define \( I \) within \( 0 \uparrow \) last position in \( I \), we may write \( |x_0 - p_j/q_j| \leq 10^{-b_{j+1}} \) since the decimal digits of \( x_0 \) in \( I_j \) are 0. As \( b_{j+1}/b_j \) tends to infinity this clearly implies \( x_0 \in \mathcal{L} \). For \( x \) with \( i > 0 \), we similarly cut off after the last decimal digit in the respective intervals \( I_{j-1} \) with \( j \equiv i \mod n \) to obtain some rational \( k_j/10^{b_j} \), and do the same for \( \xi_i \) to obtain some rational \( t_j/10^{b_j} \). Then let \( p_j/q_j = (t_j - k_j)/10^{b_j} \). Since by construction \( x_0 \) and \( \xi_i \) have the same digits within \( I_j \), we may write \( x_i = \xi_i - x_0 = p_j/q_j + v_{j,1} - v_{j,2} \) where the decimal expansions of the real numbers \( v_{j,1} \) resp. \( v_{j,2} \) are obtained from taking digit 0 up to last position in \( I_j \) and reading the digital expansion of \( \xi_i \) resp. \( x_0 \) onwards. Hence we again readily verify

\[
|x_i - p_j/q_j| = |v_{j,1} - v_{j,2}| \leq |v_{j,1}| + |v_{j,2}| \leq 2 \cdot 10^{-b_{j+1}}, \quad 1 \leq i \leq n - 1,
\]

and we conclude as for \( i = 0 \) above that \( x_i \in \mathcal{L} \) (again, we can easily exclude that some \( x_i \in \mathbb{Q} \) by a minor variation in our choice of the \( b_j \)).

As noticed above, the main step on the surjectivity of \( \Psi \) can be considered a special case of Rieger [31] or Schwarz [35]. Both show that for any continuous open maps \( f_1, \ldots, f_r \) on \((0, 1)\), there is \( \xi \in \mathcal{L} \) with all \( f_j(\xi) \) again in \( \mathcal{L} \) (according to [35] we may even take countably many \( f_j \)). Taking \( r = n - 1 \) and \( f_j(x) = \xi_j - x \) for \( 1 \leq j \leq n - 1 \) yields the surjectivity of \( \Psi \). The proof in [35] uses the same method as our first proof.

We want to briefly discuss two variants of Theorem 2.1. First, let \( C_{b,W} \) denote the classical missing digit Cantor sets consisting of numbers that can be written

\[
\sum_{i=1}^{\infty} a_i b^{-i}, \quad a_i \in W,
\]

where \( b \geq 3 \) is an integer and \( W \subseteq \{0, 1, \ldots, b - 1\} \). Then, for \( n = 2 \), upon minor modifications our argument implies

\[
\dim((\mathcal{L} \cap C_{b,W}) \times (\mathcal{L} \cap C_{b,W})) = \dim(C_{b,W}) = \frac{\log |W|}{\log b}, \quad (5)
\]
and 
\[ \dim_P((L \cap C_{b,W}) \times (L \cap C_{b,W})) = 2 \dim_P(C_{b,W}) = \frac{2 \log |W|}{\log b}. \]

See Theorem 3.5 below for a generalisation and further comments. However, for \( n \geq 3 \) the corresponding claims are unclear.

**Problem 1.** For \( n \geq 3 \), do we have 
\[ \dim((L \cap C_{b,W})^n) = (n-1) \frac{\log |W|}{\log b}, \]
and
\[ \dim_P((L \cap C_{b,W})^n) = n \log |W| / \log b? \]

Secondly, for any \( m \geq 1 \), a very similar idea applies to the set \( L_m \) of \( m \)-dimensional Liouville vectors, defined similarly as the classical Liouville numbers \( L_1 = L \) by imposing that 
\[ |p/q - \xi| < q^{-N} \]
has infinitely many solutions in rational vectors \( p/q = (p_1/q, \ldots, p_m/q) \in \mathbb{Q}^m \), for all \( N \). The same digit construction as in the proof of Theorem 2.1 simultaneously applied to all components \( \xi_1, \ldots, \xi_m \) of \( \xi \) (i.e. with the same interval choices simultaneously) readily yields that the map
\[ L_m \mapsto (\mathbb{R}^m)^{n-1}, \]
\[ (x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0 + x_1, x_0 + x_2, \ldots, x_0 + x_{n-1}), \]
is surjective and therefore \( \dim(L_m^n) \geq m(n-1) \). Again Theorem 1.1 gives the reverse estimate by using a consequence of a well-known result by Jarník [21] that \( \dim(L_m) = 0 \).

### 3. Sumsets and Cartesian Products of Sets of Diophantine Numbers with Restricted Irrationality Exponents

#### 3.1. Definitions

In this section we are concerned with Cartesian products of sets of numbers which are approximable up to a given order by rational numbers. For a real number \( \xi \), we consider its irrationality exponent \( \mu(\xi) \), defined as the supremum of numbers \( \mu \) for which the inequality
\[ |\xi - p/q| \leq q^{-\mu} \]  
has infinitely many solutions in rational numbers \( p/q \). Then \( \mu(\xi) = 1 \) for \( \xi \in \mathbb{Q} \), and by the theory of continued fractions (or Dirichlet’s theorem), \( \mu(\xi) \geq 2 \) for all \( \xi \in \mathbb{R}\setminus\mathbb{Q} \). Liouville numbers are precisely those \( \xi \) with \( \mu(\xi) = \infty \), the complement is sometimes referred to as Diophantine numbers. Further define \( \theta_b(\xi) \) like \( \mu(\xi) \) above but where we restrict the approximating rationals \( p/q \) in (6) to integral powers \( q = b^N \) of some fixed integer base \( b \geq 2 \). This corresponds to \( v_b(\xi) + 1 \) with exponent \( v_b \) as defined in [4]. We also remark that the exponent \( \mu(\xi) \) was denoted by \( v_1(\xi) + 1 \) in [4]. Clearly \( \mu(\xi) \geq \sup_{b \geq 2} \theta_b(\xi) \). Moreover, \( \theta_b(\xi) \geq 1 \) for any \( \xi \in \mathbb{R} \) and \( b \geq 2 \), with equality if \( \xi \in \mathbb{Q} \). We define level sets for both exponents.
Definition 1. Let $$\mathcal{W}_{\lambda;\mu} = \{ \xi \in \mathbb{R} : \lambda \leq \mu(\xi) \leq \mu \}, \quad 2 \leq \lambda \leq \mu \leq \infty,$$
and let $$\mathcal{W}_\mu = \mathcal{W}_{\lambda;\mu} = \{ \xi \in \mathbb{R} : \mu(\xi) = \lambda \}, \quad \lambda \in [2, \infty].$$

Further for $$1 \leq \lambda \leq \mu \leq \infty$$ define $$\mathcal{V}_{\lambda;\mu}(b)$$ and $$\mathcal{V}_{\lambda}(b)$$ accordingly with respect to the exponent $$\theta_b(\xi)$$ in place of $$\mu(\xi).$$

Any number in $$\mathcal{V}_{\lambda;\mu}(b)$$ with $$\lambda > 1$$ has arbitrarily long consecutive 0 and/or $$(b-1)$$ digit strings in its base $$b$$ expansion. Clearly the sets $$\mathcal{W}_{\lambda;\mu}$$ and $$\mathcal{V}_{\lambda;\mu}(b)$$ become larger as $$\lambda$$ decreases and as $$\mu$$ increases. Moreover $$\mathcal{V}_{\lambda;\mu}(b) \subseteq \mathcal{W}_{\lambda;\infty}$$ for $$\mu \geq \lambda.$$ However, notice that $$\mathcal{V}_{\lambda;\mu}(b) \not\subseteq \mathcal{W}_{\lambda;\mu}.$$ The set $$\mathcal{W}_{\infty}$$ is nothing but the set of Liouville numbers treated in Sect. 2. The union of the sets $$\mathcal{W}_\lambda$$ over $$\lambda > 2,$$ that is the set of all numbers with $$\mu(\xi) > 2,$$ is commonly referred to as the set of very well approximable numbers.

The remainder of the paper is driven by the following two questions that extend the results of Erdős [18] and their consequences recalled in Sect. 2.

Problem 2. What can we say (metrically) about sumsets $$\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$$ and $$\mathcal{V}_{\lambda_0}(b) + \mathcal{V}_{\lambda_1}(b)$$ for given $$\lambda_0, \lambda_1$$?

Problem 3. Determine the Hausdorff and packing dimensions of Cartesian product sets $$\prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i}$$ and $$\prod_{i=0}^{n-1} \mathcal{V}_{\lambda_i}(b)$$ for given real numbers $$\lambda_0, \ldots, \lambda_{n-1}.$$
The crucial claim is the lower bound \( n - 1 \) in (7). In fact by Borosh and Fraenkel [7] (see also Amou and Bugeaud [4]), we have the Hausdorff dimension formula

\[
\dim(\mathcal{V}_{\lambda, \mu, (b)}) = \frac{1}{\lambda}, \quad \mu \geq \lambda \geq 1.
\]  

(10)

Combined with (1) and Theorem 1.1, we then deduce all other assertions of (7). From (7), (10) we further derive (8) and (9). As to the packing dimension, from Theorem 1.1 upon introducing another variable \( \lambda_n = \infty \), we get

\[
n \geq \dim_P \left( \prod_{i=0}^{n-1} \mathcal{V}_{\lambda_i, (b)} \right) \geq \dim \left( \prod_{i=0}^{n} \mathcal{V}_{\lambda_i, (b)} \right) - \dim(\mathcal{V}_{\lambda_n, (b)}) \geq n - 0 = n,
\]

where we used (7) in dimension \( n + 1 \) and (10) in the last inequality. We notice that the case \( n = 2 \) constitutes the fact for any integer \( b \geq 2 \) and any \( \lambda \geq 1 \), we have

\[
\dim_P(\mathcal{V}_{\lambda, (b)}) = 1.
\]  

(11)

This formula may not come as a surprise to experts (see for example (14) below), however we believe it is a new result worth being mentioned.

We formulate a conjecture corresponding to Theorem 3.1 for the sets with unrestricted rationals.

**Conjecture 1.** For any \( \lambda_0, \ldots, \lambda_{n-1} \) in \( [2, \infty] \), we have

\[
n - 1 + \frac{2}{\max_{0 \leq i \leq n-1} \lambda_i} \geq \dim \left( \prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i} \right) \geq \max \left\{ n - 1, 2 \sum_{i=1}^{n} \lambda_i^{-1} \right\}.
\]  

(12)

In particular, if all \( \lambda_i \) are large enough compared to \( n \) we have

\[
\dim \left( \prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i} \right) > \sum_{i=0}^{n-1} \dim(\mathcal{W}_{\lambda_i}),
\]

and for every \( n \geq 1 \), with the limit understood as in Theorem 3.1, we have

\[
\lim_{\max \lambda_i \to \infty} \dim \left( \prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i} \right) = n - 1, \quad \dim_P \left( \prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i} \right) = n.
\]

Unfortunately, as remarked above there is no inclusion between sets \( \mathcal{W}_\lambda \) and \( \mathcal{V}_{\lambda, (b)} \) that would imply the claims via Theorem 3.1. The validity of the lower bound \( n - 1 \) in (12) is again the key problem. Similar to the remarks below Theorem 3.1, the remaining claims would again follow via the special case \( n = 1 \) of Jarník’s formula [21]:

\[
\dim(\mathcal{W}_{\lambda, \mu}) = \frac{2}{\lambda}, \quad 2 \leq \lambda \leq \mu \leq \infty.
\]  

(13)

Since for \( \mu < \infty \) the sets \( \mathcal{W}_{\lambda, \mu} \) in question are of first category as they lie in the complement of the set of Liouville numbers \( \mathcal{L} = \mathcal{W}_\infty \), we cannot apply
topological arguments similar to the unconstructive proof of \( \mathcal{L} + \mathcal{L} = \mathbb{R} \) by Erdős [18] recalled in Sect. 2. Indeed all proofs of partial results below have constructive character, and rely on similar ideas as Erdős’ digit based proof explained in Sect. 2. For completeness we also state the analogue of (11) for unrestricted rationals which reads

\[
\dim_{\mathcal{P}}(\mathcal{W}_{\lambda, \mu}) = 1, \quad 2 \leq \lambda \leq \mu \leq \infty. \tag{14}
\]

This is a direct consequence of more general results by Marnat [25]. We refer to Theorem 3.6 and Corollary 7 below for variants of (14) not implied by Marnat [25].

Our first result supporting Conjecture 1 is that similar to Liouville numbers, certain product sets of \( \mathcal{W}_{\lambda_i; \mu_i} \) have indeed Hausdorff dimension at least \( n - 1 \). This is the main substance of Theorem 3.2, where we also add other bounds for completeness.

**Theorem 3.2.** Let \( n \geq 2 \) be an integer. Let \( \lambda_0, \ldots, \lambda_{n-1} \) and \( \mu_0, \ldots, \mu_{n-1} \) be real numbers or infinity satisfying \( 2 \leq \lambda_i \leq \mu_i \leq \infty \). Suppose

\[
\mu_i > \frac{\Lambda}{\lambda_i - 1} + 1, \quad 0 \leq i \leq n - 1,
\]

where \( \Lambda = \lambda_0 \lambda_1 \cdots \lambda_{n-1} \). Then we have

\[
n - 1 + \frac{2}{\max_{0 \leq i \leq n-1} \lambda_i} \geq \dim \left( \prod_{i=0}^{n-1} \mathcal{W}_{\lambda_i; \mu_i} \right) \geq \max \left\{ n - 1, 2 \sum_{i=0}^{n-1} \lambda_i^{-1} \right\}. \tag{16}
\]

If \( \Lambda = \infty \) then the right hand sides in (15) are interpreted naturally in limits, and formal equalities \( \infty = \infty \) suffice for the claim, however this is of minor interest in view of Theorem 2.1. We emphasize that if otherwise \( \Lambda < \infty \) (in fact one \( \lambda_i = \infty \) is allowed) then all \( \mu_i \) can be effectively bounded, therefore the result is not covered by Theorem 2.1. If some \( \lambda_j \) is much larger than all other \( \lambda_i, \; i \neq j \), it may happen that the bound for \( \mu_j \) in (15) is smaller than \( \lambda_j \). Then for this index we can take \( \mu_j = \lambda_j \) and thus consider the set \( \mathcal{W}_{\lambda_j} \) replacing \( \mathcal{W}_{\lambda_j; \mu_j} \) in the Cartesian product. Theorem 3.2, as well as Theorem 3.3 below, contradicts the conjectured equality in [33, Conjecture 2.5] of the author, therefore the conditional implication in [33, Corollary 2.6] is very open.

We conclude from Theorem 3.2 that for Cartesian products of general sets \( \mathcal{W}_{\lambda; \mu} \) there is no equality in (1). For simplicity we restrict to all \( \lambda_i \) being equal.

**Corollary 1.** For \( n \geq 2 \) an integer. Let \( \lambda, \mu \) be real numbers. If \( \lambda > 2n/(n-1) \) and \( \mu > (\lambda^n + \lambda - 1)/(\lambda - 1) \), we have

\[
\dim \left( \mathcal{W}_{\lambda; \mu}^n \right) > n \cdot \dim(\mathcal{W}_{\lambda; \mu}).
\]
Another corollary to Theorem 3.2 that contains Theorem 2.1 as a special case, with limits understood as in Theorem 3.1 again, reads as follows.

**Corollary 2.** Let \( n, \lambda_i, \mu_i \) be as in Theorem 3.2. Then

\[
\lim_{\max \lambda_i \to \infty} \dim (W_{\lambda_i; \mu_i}^n) = n - 1.
\]

If we assume \( \mu_i > (\max_{0 \leq i \leq n-1} \lambda_i) \cdot \Lambda / (\lambda_i - 1) + 1 \) for \( 0 \leq i \leq n-1 \), then

\[
\lim_{\max \lambda_i \to \infty} \dim_P \left( \prod_{i=0}^{n-1} W_{\lambda_i; \mu_i} \right) = n.
\]

In particular, if real numbers \( \mu \geq \lambda \geq 2 \) satisfy \( \mu > (\lambda^{n+1} + \lambda - 1) / (\lambda - 1) \), then

\[
\lim_{\lambda \to \infty} \dim_P (W_{\lambda; \mu}^n) = n.
\]

**Proof.** The first claim is a direct consequence of Theorem 3.2. For the second, we introduce another variable \( \lambda_n = \max_{0 \leq i \leq n-1} \lambda_i \) and let \( \mu_n = \infty \). Then by assumption (15) and thus (16) hold in dimension \( n + 1 \) for \( \lambda_0, \ldots, \lambda_n \), and hence by Theorem 1.1

\[
n \geq \dim_P \left( \prod_{i=0}^{n-1} W_{\lambda_i; \mu_i} \right) \geq \dim \left( \prod_{i=0}^{n-1} W_{\lambda_i; \mu_i} \right) - \dim(W_{\lambda_n; \mu_n}) \geq n - \frac{2}{\lambda_n}
\]

\[
= n - \frac{2}{\max_{1 \leq i \leq n-1} \lambda_i}.
\]

The claims follow, and the last assertion is just the special case \( \lambda := \lambda_0 = \cdots = \lambda_{n-1} \).

To continue with our second result towards Conjecture 1, we restrict ourselves to \( n = 2 \) and consider the sets of precise order of approximation. For the sequel fix

\[
\rho := \frac{5 + \sqrt{17}}{2} = 4.5615\ldots.
\]

**Theorem 3.3.** Let \( \lambda_0, \lambda_1 \) be real numbers (or infinity) satisfying

\[
\min \{ \lambda_0, \lambda_1 \} > \rho.
\]

Then we have

\[
1 + \frac{2}{\max \{ \lambda_0, \lambda_1 \}} \geq \dim(W_{\lambda_0} \times W_{\lambda_1}) \geq 1 = \max \left\{ 1, \frac{2}{\lambda_0} + \frac{2}{\lambda_1} \right\}.
\]

In particular, for every \( \lambda \in (\rho, \infty] \) we have

\[
\dim(W_{\lambda} \times W_{\lambda}) \geq 1 > 2 \dim(W_{\lambda}).
\]
For \( n = 2 \) and \( \lambda_0 = \lambda_1 = \lambda \), Conjecture 1 remains open only for \( \lambda \in (4, 4.5615 \ldots] \). Unfortunately, our method fails when \( n \geq 3 \). Still Theorem 3.3 admits the conclusion that the Hausdorff dimension of any \( n \)-fold Cartesian product is comparable with \( n \).

**Corollary 3.** Let \( n \geq 1 \) be an integer and \( \lambda_0, \ldots, \lambda_{n-1} \in [2, \infty] \). Then we have

\[
\dim \left( \prod_{i=0}^{n-1} W_{\lambda_i} \right) \geq \frac{2}{\rho} \cdot (n-1), \quad \dim_P \left( \prod_{i=0}^{n-1} W_{\lambda_i} \right) \geq \frac{2}{\rho} \cdot (n-2) + 1. \tag{18}
\]

If \( \min \lambda_i > \rho \), then we have the stronger bounds

\[
\dim \left( \prod_{i=0}^{n-1} W_{\lambda_i} \right) \geq \lfloor n/2 \rfloor, \quad \dim_P \left( \prod_{i=0}^{n-1} W_{\lambda_i} \right) \geq \lfloor (n+1)/2 \rfloor. \tag{19}
\]

**Proof.** The left inequality of (19) follows from Theorem 3.3 via recursive application of (1) and Theorem 3.3. The right is deduced from the left with aid of (2) and (14) via

\[
\dim_P \left( \prod_{i=0}^{n-1} W_{\lambda_i} \right) \geq \dim \left( \prod_{i=0}^{n-2} W_{\lambda_i} \right) + \dim_P (W_{\lambda_{n-1}}) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+1}{2} \right\rfloor.
\]

For (18), let us partition \( \lambda_0, \ldots, \lambda_{n-1} \) into pairs \((\lambda_i, \lambda_j)\) with exponents both larger resp. both smaller than \( \rho \), with possibly up to two remaining indices. By Theorem 3.3 we can estimate the Hausdorff dimension of twofold products of such pairs from below by 2 resp. \( 4/\rho < 2 \). It is easily checked that the worst case is that there is precisely one \( i \) with \( \lambda_i > \rho \), thus no pair of the first kind exists. Then (1) easily leads to the claimed left bound, the right is again inferred with (2) and (14) as above. \( \square \)

The following corollary emphasizes that for large \( \lambda_i \), a big improvement of the lower bound 1 in Theorem 3.3 cannot be made.

**Corollary 4.** With the definitions analogous to Theorem 3.1, we have

\[
\lim_{\min \{\lambda_0, \lambda_1\} \to \infty} \dim (W_{\lambda_0} \times W_{\lambda_1}) = 1.
\]

It would be desirable to deduce the limit 2 for the packing dimension, however Theorem 3.3 seems to be insufficient for this conclusion, even if \( \lambda_0 = \lambda_1 \). We remark that combination of Theorem 3.3 and Theorem 1.1 leads to a new proof of formula (14), for \( \lambda > \rho \). We refer to Theorem 3.6 below for a considerable generalisation. Theorem 3.3 can easily be deduced from the following claims on sumsets.

**Theorem 3.4.** Let \( \lambda_0, \lambda_1 \) be real numbers satisfying (17). Then

\[
W_{\lambda_0} + W_{\lambda_1} \supseteq W_2, \tag{20}
\]
i.e. the sumset $\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$ contains any irrational real number that is not very well approximable. Further, writing $\lambda = \min \{\lambda_0, \lambda_1\}$, its complement satisfies

$$\dim( (\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1})^c ) \leq \frac{2(2\lambda - 1)}{\lambda^2 - \lambda} < 1.$$  \hspace{1cm} (21)

In particular, we have

$$\lim_{\min \{\lambda_0, \lambda_1\} \to \infty} \dim((\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1})^c) = 0.$$  \hspace{1cm} (22)

Remark 1. For any $\lambda_0, \lambda_1$ the sumset $\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$ has either full or zero Lebesgue measure, since this is true for any set invariant under rational translation. See also [15]. However, it is in general very unclear which of the two cases occurs.

The basic idea in the proof of Theorem 3.4 is to consider the base 5 expansion (can be replaced by any integer $b \geq 2$) of any not very well approximable $\xi$, and to suitably manipulate the digits in a sequence of intervals to form two numbers that sum up to $\xi$ and have the desired irrationality exponents. We notice that if $\lambda_0 \neq \lambda_1$, then the sum set $\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$ does not contain any rational number since $\mu(\xi) = \mu(p/q - \xi)$, hence its complement is non-empty. Moreover, it follows from Petruska [30, Theorem 1] that if $\lambda_0 < \lambda_1 = \infty$ then $\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$ does not contain any so called strong Liouville number (see [3]). Hence the sumset has uncountable complement. Nevertheless, our result (22) suggests that the complement is always very small in a metrical sense.

Conjecture 2. For any $\lambda_0, \lambda_1 \in [2, \infty]$, the set $(\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1})^c$ has Hausdorff dimension 0.

For completeness, we state the following consequence of Theorem 3.4 first noticed by Chalebgwa and Morris, indeed a special case of [15, Theorem 4.3].

Corollary 5 (Chalebgwa, Morris). Let $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in (\rho, \infty)$. Then

$$\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1} + \mathcal{W}_{\lambda_2} + \mathcal{W}_{\lambda_3} = \mathbb{R}.$$ 

Proof. By (20) the sets $A := \mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}$ and $B := \mathcal{W}_{\lambda_2} + \mathcal{W}_{\lambda_3}$ both have full Lebesgue measure. Hence a short argument yields $A + B = \mathbb{R}$, see [15] for the details. \hfill $\square$

Similar claims can be derived from the underlying argument, for example the mixed sum-product set $(\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1}) \cdot (\mathcal{W}_{\lambda_2} + \mathcal{W}_{\lambda_3})$ equals $\mathbb{R}$ if $\lambda_0 = \lambda_1$ or $\lambda_2 = \lambda_3$, and $\mathbb{R} \setminus \{0\}$ otherwise. While two sets are insufficient for the conclusion of Corollary 5 as soon as $\lambda_0 \neq \lambda_1$ by the above remarks, we believe that three sets always suffice.

Problem 4. Is it true that for any $\lambda_0, \lambda_1, \lambda_2$ all at least 2 (or large enough) we have

$$\mathcal{W}_{\lambda_0} + \mathcal{W}_{\lambda_1} + \mathcal{W}_{\lambda_2} = \mathbb{R} ?$$
The answer is affirmative in the special case that (17) holds and $\lambda_2 = 2$ (after relabelling if necessary), by the argument of the proof of Corollary 5. Recalling another result by Erdős from [18] that any real number can be written as a product of two Liouville numbers, it is natural to ask if the product sets of Diophantine numbers with prescribed irrationality exponents are typically large as well.

**Problem 5.** Is there an analogue of Theorem 3.4 for the product sets $\mathcal{W}_{\lambda_0} \cdot \mathcal{W}_{\lambda_1}$?

Any such set has either 0 or full Lebesgue measure, as in Remark 1.

### 3.3. Generalisations

In this section, we state several extensions of Theorem 3.3 and Theorem 3.4 to more general settings. First, we point out that the proof of Theorem 3.4 below implies that the smaller sumsets $(\mathcal{W}_{\lambda_0} \cap \mathcal{V}_{\lambda_0,(b)}) + (\mathcal{W}_{\lambda_1} \cap \mathcal{V}_{\lambda_1,(b)})$ still contain the set $\mathcal{W}_2$ of not very well approximable numbers, and consequently

$$\dim((\mathcal{W}_{\lambda_0} \cap \mathcal{V}_{\lambda_0,(b)}) \times (\mathcal{W}_{\lambda_1} \cap \mathcal{V}_{\lambda_1,(b)})) \geq 1,$$

upon assumption (17). We derive a variant of (11) by an analogous proof.

**Corollary 6.** For any integer $b \geq 2$ and any $\lambda > \rho$, the set $\mathcal{W}_{\lambda} \cap \mathcal{V}_{\lambda,(b)}$ has packing dimension 1.

For $\lambda > \rho$, Corollary 6 refines the observation that $\mathcal{W}_{\lambda} \cap \mathcal{V}_{\lambda,(b)}$ is uncountable for any $\lambda > 2$ due to Amou and Bugeaud [4, Theorem 5].

Next, certain variants of Theorems 3.3 and 3.4 can be proved. We first present an application to the classical Cantor sets $C_{b,W}$ that have already been discussed in Sect. 2.

**Theorem 3.5.** Assume $\lambda_0, \lambda_1$ satisfy (17). Let $K = C_{b,W}$ with $b \geq 3$. If $0 \in W$, then

$$(\mathcal{W}_{\lambda_0} \cap K) + (\mathcal{W}_{\lambda_1} \cap K) \supseteq \mathcal{W}_2 \cap \mathcal{V}_{1,(b)} \cap K,$$

i.e. the sumset contains any number in $K$ that is neither very well approximable in the usual, nor in the $b$-ary setting. Regardless if $0 \in W$ or not, as a result we have

$$\dim((\mathcal{W}_{\lambda_0} \cap K) \times (\mathcal{W}_{\lambda_1} \cap K)) \geq \dim(K) = \frac{\log |W|}{\log b}.$$

Consequently, for any $\lambda > \rho$ we have

$$\dim_P(\mathcal{W}_{\lambda} \cap K) = \frac{\log |W|}{\log b} = \dim_P(K).$$

Note that (25) generalises (5). It has been known that $\mathcal{W}_{\lambda} \cap K$ is uncountable for any $\lambda \geq 2$ due to Bugeaud [9] (see also [24,32]), so (26) refines this claim for large parameters $\lambda$. The outline of the proof of (24) is the same as that of Theorem 3.3. However, there are some adjustments to be done, which in particular for our proof to work, force the sets $\mathcal{V}_{1,(b)}$ to enter in the right
hand side of the inclusion. The implications (25), (26) use metrical results from [24,38]. Therefore we present a detailed proof of Theorem 3.5 in Sect. 4. We remark that despite intense investigation, the Hausdorff dimensions of the sets \( \mathcal{W}_\lambda \cap K \) remain unknown. For partial results and further references we refer to Levesley et al. [24], Bugeaud and Durand [11] and the recent preprint by Yu [40].

Next, we turn towards generalisations with respect to the order of approximation. By minor adjustments of our proofs below, we can replace the sets \( \mathcal{W}_\lambda \) in Theorems 3.3 and 3.4 by a larger class of sets on which we impose refined approximation conditions. Concretely, we consider sets \( \text{Exact}(\Phi) \) in the following definition.

**Definition 2.** For \( \Phi : \mathbb{N} \to (0, \infty) \) any function and \( C \in (0,1) \), let \( \mathcal{W}_{\Phi,C} \subseteq \mathbb{R} \) be the set of real numbers \( \xi \) that satisfy the properties that

\[
\left| \xi - \frac{p}{q} \right| \leq \Phi(q)
\]

has infinitely many solutions \( p/q \), whereas the estimate

\[
\left| \xi - \frac{p}{q} \right| > C \cdot \Phi(q)
\]

holds for all except possibly finitely many rational numbers \( p/q \). Define the set of numbers with “exact approximation of order \( \Phi \)” associated to \( \Phi \) via

\[
\text{Exact}(\Phi) = \bigcap_{C \in (0,1)} \mathcal{W}_{\Phi,C}.
\]

The Hausdorff dimension of \( \text{Exact}(\Phi) \) was studied by Bugeaud in a series of papers, starting with [8]. See also the forthcoming work [22]. We now generalise Theorems 3.3 and 3.4 at once and in particular determine their packing dimension.

**Theorem 3.6.** For \( i = 0, 1 \), assume \( \lambda_i \) is a real number satisfying (17) and let \( \Phi_i \) be functions as in Definition 2 that satisfy \( \Phi_i(q) \leq q^{-\lambda_i} \) for all large \( q \geq q_0 \). Then we have

\[
\text{Exact}(\Phi_0) + \text{Exact}(\Phi_1) \supseteq \mathcal{W}_2.
\]

In particular, the sumset \( \text{Exact}(\Phi_0) + \text{Exact}(\Phi_1) \) has full Lebesgue measure, and consequently for \( \Phi = \Phi_i \) as above we have

\[
\dim_P(\text{Exact}(\Phi)) = 1.
\]

According generalisations of Theorem 3.1 and Corollary 6 can be formulated as well. On the other hand, if an according variant of Theorem 3.5 holds is not quite clear, see the remarks below its proof at the end of the paper. As indicated in Sect. 3.2, formula (28) is independent from the results in [25]. Indeed, due to their deduction from the variational principle [16], the proofs in
[25] ideally admit the full dimension result only for the larger sets $\mathcal{W}_{\Phi,C}$ with some fixed $C \in (0,1)$.

We finally study generalisations to simultaneous approximation. We only explicitly state an extension of Theorem 3.4, however the other results of Sect. 3.2 can be modified accordingly. Let $\mathcal{W}_{\lambda}(m)$ be the set of real vectors $\xi = (\xi_1, \ldots, \xi_m)$ simultaneously approximable of order precisely $\lambda$, i.e. so that $\mu_m(\xi) = \lambda$ where $\mu_m(\xi)$ is the supremum of $\lambda$ so that $|\max_{1 \leq k \leq m} \xi_k - p_k/q| < q^{-\lambda}$ infinitely often. Note that $\mathcal{W}_{\lambda} = \mathcal{W}_{\lambda}(1)$. Then $\mu_m(\xi) \geq 1 + 1/m$ for any irrational $\xi \in \mathbb{R}^m$ by Dirichlet’s Theorem, and $\xi$ is called very well approximable if the inequality is strict. As for $m = 1$, the latter vectors form a Lebesgue nullset in $\mathbb{R}^m$, see [21].

**Theorem 3.7.** There exists a decreasing sequence of real numbers

$$\frac{5 + \sqrt{17}}{2} = \rho = \rho_1 > \rho_2 > \rho_3 > \cdots$$

with limit $\gamma := (3 + \sqrt{5})/2 = 2.6180 \ldots$ and so that $\min\{\lambda_0, \lambda_1\} > \rho_m$ implies

$$\mathcal{W}_{\lambda_0}(m) + \mathcal{W}_{\lambda_1}(m) \supseteq \mathcal{W}_{1+1/m}(m), \quad m \geq 1,$n

i.e. the sumset contains any irrational real $m$-vector that is not very well approximable. Further, writing $\lambda = \min\{\lambda_0, \lambda_1\}$, its complement satisfies

$$\dim\left(\left(\mathcal{W}_{\lambda_0}(m) + \mathcal{W}_{\lambda_1}(m)\right)^c\right) \leq \frac{(m+1)(2\lambda - 1)}{\lambda^2 - \lambda} < m. \quad (29)$$

In particular, we have

$$\lim_{\min\{\lambda_0, \lambda_1\} \to \infty} \dim\left(\left(\mathcal{W}_{\lambda_0}(m) + \mathcal{W}_{\lambda_1}(m)\right)^c\right) = 0. \quad (30)$$

The extension to the accordingly defined sets Exact$^{(m)}(\Phi_i)$, $i = 0,1$, holds again, for functions satisfying $\Phi_i(t) = \Phi_i^{(m)}(t) < t^{-\rho_m - \epsilon}$ for some $\epsilon > 0$ and $t \geq t_0$. By Theorem 1.1, this implies

**Corollary 7.** For $\Phi$ as above, the sets Exact$^{(m)}(\Phi)$ have full packing dimension $m$.

Similar to the special case $m = 1$ in (28), this result is again not covered by [16,25]. It may be compared with a non metrical claim by Jarník [21, Satz 5] that Exact$^{(m)}(\Phi)$ is uncountable for any reasonable function $\Phi$. See also Akhunzhanov [1] and Akhunzhanov and Moshchevitin [2] for refinements.

4. Proofs of Results in Sect. 3.2

The principal idea of the proofs from Sect. 3.2 is similar to Theorem 2.1. Again we define very elementary Lipschitz maps from the corresponding product sets
into an Euclidean space with codimension 1, with large image. The next elementary lemma guarantees that these images still have full Lebesgue measure relative to the corresponding dimension.

**Lemma 4.1.** If $A_1, \ldots, A_k$ are subsets of $\mathbb{R}^n$, then $\dim(\prod (A_i \cup Q)) = \dim(\prod A_i)$.

**Proof.** We only show the claim for two factors, i.e. $\dim((A \cup Q) \times (B \cup Q)) = \dim(A \times B)$. The general case works very similarly. Clearly $\dim((A \cup Q) \times (B \cup Q)) \geq \dim(A \times B)$ by monotonicity of Hausdorff dimension. For the reverse estimate, the difference set $((A \cup Q) \times (B \cup Q)) \setminus (A \times B)$ is contained in $(A \times Q) \cup (Q \times B) \cup Q^2$. Clearly $\dim(Q^2) = 0$ and $A \times Q$ and $B \times Q$ are countable unions of translates of $A, B$ respectively, thus their dimensions are bounded by $\dim(A)$ and $\dim(B)$, respectively. Hence by (1) their union has dimension $\max\{\dim(A), \dim(B)\} \leq \dim(A) + \dim(B) \leq \dim(A \times B)$. Thus adding the parts $A \times Q, Q \times B$ and $Q^2$ to $A \times B$ does not increase the Hausdorff dimension. \(\square\)

Notice we only used the property that $Q$ is countable in the proof. We also use continued fractions in the proofs. The next proposition recalls a relation between the growth of the denominators of the convergents and order of approximation.

**Proposition 4.2.** Let $\xi$ be an irrational real number and denote by $p_k/q_k$ its continued fraction convergents. Let $\tau_k$ be the real numbers defined by

$$|\xi - \frac{p_k}{q_k}| = q_k^{-\tau_k}.$$ 

Then $\tau_k \geq 2$ for any $k \geq 1$ and we have $q_{k+1} \approx q_k^{\tau_k-1}$. In other words

$$q_{k+1} \approx |q_k \xi - p_k|^{-1}.$$ 

**Proof.** It is well-known from the theory of continued fractions (see Perron [29]) that

$$\frac{1}{2q_{k+1}} < \frac{1}{q_k + q_{k+1}} < |q_k \xi - p_k| < \frac{1}{q_{k+1}}.$$ 

Hence we get $q_{k+1} \approx |q_k \xi - p_k|^{-1}$, consequently $q_{k+1} \approx q_k^{\tau_k-1}$. \(\square\)

Complementary to Proposition 4.2, we require Legendre’s Theorem on continued fractions that tells us that every good approximating rational is a convergent, see Perron [29].

**Theorem 4.3** (Legendre). If $\xi \in \mathbb{R}$ and $p/q$ is rational and satisfies $|p/q - \xi| < q^{-2}/2$, then $p/q$ is a convergent to $\xi$.

We will first prove Theorem 3.2.
Proof of Theorem 3.2 Let $\lambda_i \geq 2$ for $1 \leq i \leq n$ be fixed throughout. The upper bound in (16) follows from (13) and Theorem 1.1. We prove the lower bound. As mentioned above, by (1) and Jarník’s formula (13) we have

$$\dim \left( \prod_{i=0}^{n-1} W_{\lambda_i;\mu_i} \right) \geq \sum_{i=0}^{n-1} \dim(W_{\lambda_i;\mu_i}) = 2 \sum_{i=0}^{n-1} \lambda_i^{-1}.$$  

To derive the lower bound $n - 1$, we follow a similar idea as in the proof of Theorem 2.1 above. We show that for every $\lambda_i, \mu_i$ as in the theorem, the map $\Psi_1 : \prod_{i=0}^{n-1} (W_{\lambda_i;\mu_i} \cup \mathbb{Q}) \mapsto \mathbb{R}^{n-1},$

$$\Psi_1 : \prod_{i=0}^{n-1} (W_{\lambda_i;\mu_i} \cup \mathbb{Q}) \mapsto \mathbb{R}^{n-1},
\begin{pmatrix} x_0, x_1, \ldots, x_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0 + x_1, x_0 + x_2, \ldots, x_0 + x_{n-1} \end{pmatrix},$$  

(31)
is surjective. Then by Proposition 1.2, the domain set of $\Psi_1$ has Hausdorff dimension at least $n - 1$ as well, and finally by Lemma 4.1 when we remove $\mathbb{Q}$ from each factor in the domain of the map, the lower bound $n - 1$ remains valid.

Construction of the preimage: We may restrict the image to real vectors within $[0,1)^{n-1}$, so let $(\xi_1, \ldots, \xi_{n-1}) \in [0,1)^{n-1}$ be arbitrary. Write

$$\xi_i = \sum_{\ell \geq 1} d_{i,\ell} \frac{1}{5^\ell}, \quad d_{i,\ell} \in \{0,1,2,3,4\}, \quad 1 \leq i \leq n - 1,$$

for their base 5 expansions. We construct a preimage under $\Psi_1$ in our product set for given $\lambda_i, \mu_i$ in the theorem. Partition the positive integers in interval sets $(I_j)_{j \geq 1}$ according to the following recursion. Let $I_0 = \{1,2\}$ and write $g_0 = 1$ and $h_0 = 2$ for the interval ends. Define

$$I_j = \{g_j, g_j + 1, \ldots, h_j\}$$

for $j \geq 1$, where $g_j, h_j$ are recursively given by

$$g_j = h_{j-1} + 1, \quad h_j = \lceil \lambda_i h_{j-1} \rceil,$$

(32)

where we take $i$ the residue class of $j$ modulo $n$ in the representation system $\{0,1,\ldots, n-1\}$. Thereby we obtain

$$\frac{h_j}{g_j} = \lambda_i + o(1), \quad \frac{g_{j+1}}{g_j} = \lambda_i + o(1),$$

as $j \to \infty$, with $i = i(j)$ as above, that is two consecutive right (and left) interval endpoints roughly differ by a multiplicative factor among our numbers $\lambda_0, \ldots, \lambda_{n-1}$ depending on the index. In case some $\lambda_i = \infty$, instead we let the according quotients $h_j/g_j$ tend to infinity as $j \to \infty$, and the arguments below remain valid. We construct the base 5 expansion of a real number

$$x_0^* = \sum_{\ell \geq 1} d_{0,\ell}^* \frac{1}{5^\ell}, \quad d_{0,\ell}^* \in \{0,1,2,3,4\},$$
as follows. For \( 1 \leq i \leq n - 1 \), let the digits \( d_{0,\ell}^* \) equal the base 5 digit of \( \xi_i \) in intervals \( I_j \) with \( j \equiv i \mod n \), i.e. \( d_{0,\ell}^* = d_{i,\ell} \) for \( i \equiv j \mod n \), where \( j \) is the index with \( \ell \in I_j \). Finally, put the base 5 digit zero for digits in \( I_j \) with \( j \equiv 0 \mod n \), i.e. \( d_{0,\ell}^* = 0 \) if \( \ell \in I_j \) with \( j \equiv 0 \mod n \). Then \( x_0^* \) is well-defined. It will be convenient to let \( \xi_0 := x_0^* \). We slightly alter \( x_0^* \) at the interval endpoints to derive another real number \( x_0 \) via

\[
x_0 = x_0^* - \sum_{k \geq 1} \frac{a_k}{5^{h_k}}, \quad a_k \in \{0, 1, 2, 3, 4\},
\]

with concrete choice of \( a_k \) to be defined below (in fact it suffices to consider \( a_k \in \{0, 1, 2\} \), or alternatively \( a_k \in \{-1, 0, 1\} \)). Further define

\[
x_i := \xi_i - x_0, \quad 1 \leq i \leq n - 1.
\]

Notice that for \( 1 \leq i \leq n - 1 \) and given large \( j \) with \( j \equiv i \mod n \), we may write

\[
x_i = \sum_{\ell \geq 1} d_{i,\ell} \frac{5^{\ell}}{5^{h_{j-1}}} - \sum_{\ell \geq 1} d_{0,\ell}^* \frac{5^{\ell}}{5^{h_{j-1}}} + \sum_{k \geq 1} \frac{a_k}{5^{h_k}}
\]

\[
= \sum_{\ell = 1}^{h_{j-1}} \frac{d_{i,\ell} - d_{0,\ell}^*}{5^{\ell}} + \sum_{k = 1}^{h_{j-1}} \frac{a_k}{5^{h_k}} + \sum_{\ell > h_j} \frac{d_{i,\ell} - d_{0,\ell}^*}{5^{\ell}} + \sum_{k \geq j} \frac{a_k}{5^{h_k}}
\]

where we used that in the missing middle range at positions \( \ell \in I_j = [h_{j-1} + 1, h_j] \cap \mathbb{Z} \) the digits \( d_{i,\ell} \) and \( d_{0,\ell}^* \) are equal by construction, so the difference vanishes. For given \( i, j \) with \( i \equiv j \mod n \), adding the first two finite sums gives a rational number

\[
\frac{p_j}{q_j} = \frac{p_{i,j}}{q_{i,j}} = \frac{e_{i,j}}{5^{h_{j-1}}}, \quad e_{i,j} = \sum_{\ell = 1}^{h_{j-1}} 5^{h_{j-1} - \ell} (d_{i,\ell} - d_{i,\ell}^*) + \sum_{k = 1}^{h_{j-1} - h_k} 5^{h_{j-1} - h_k} a_k \in \mathbb{Z}.
\]

We emphasize that throughout we will assume \( j \equiv i \mod n \) whenever we just write \( p_j/q_j \). We remark that the \( p_{i,j}/q_{i,j} \) are essentially obtained by chopping off the base 5 expansion of \( x_i \) after position \( h_{j-1} \). Notice that

\[
e_{i,j} \equiv d_{i,h_{j-1}} - d_{i,h_{j-1}}^* + a_{j-1} \mod 5
\]

obtained from the last terms of the two sums, since all other involved expressions are divisible by 5. Thus it is possible to choose \( a_{j-1} \in \{0, 1, \ldots, 4\} \) in each step so that \( 5 \nmid e_{i,j} \) by just avoiding the residue class \( -(d_{i,h_{j-1}} - d_{i,h_{j-1}}^*) \mod 5 \). Then all \( p_{i,j}/q_{i,j} = e_{i,j}/5^{h_{j-1}} \) are reduced (note that for different \( i \) we consider different \( a_{j-1} \) by the restriction \( i \equiv j \mod n \), so we have only one condition for each \( j \)). The remainder term from the remaining two infinite sums can obviously be estimated by

\[
\left| \frac{x_i - p_{i,j}}{q_{i,j}} \right| = \left| \sum_{\ell > h_j} \frac{d_{i,\ell} - d_{0,\ell}^*}{5^{\ell}} + \sum_{k \geq j} \frac{a_k}{5^{h_k}} \right| \ll 5^{-h_j}.
\]
For \( i = 0 \), there is a small twist. Since \( d_{0,\ell}^i = 0 \) if \( \ell \in I_j \) with \( j \equiv 0 \mod n \), we have for any \( j \equiv 0 \mod n \) that
\[
x_0 = \sum_{\ell=1}^{h_j-1} \frac{d_{0,\ell}^i}{5^\ell} - \sum_{k=1}^{j-1} \frac{a_k}{5^{h_k}} + \sum_{\ell > h_j} \frac{d_{0,\ell}^i}{5^\ell} - \sum_{k \geq j} \frac{a_k}{5^{h_k}}.
\]

Now again taking the rational number obtained from subtracting the second sum the first, we get rationals \( p_{0,j}/q_{0,j} = e_{0,j}/5^{h_j-1} \). By an analogous argument excluding the residue class of \( d_{0,h_j-1}^i \) mod 5 for \( a_{j-1} \), we can assume that all \( p_{0,j}/q_{0,j} = e_{0,j}/5^{h_j-1} \) are already in reduced form. The analogue of the remainder term estimate (34) holds for similar reasons for \( i = 0 \) as well.

To finish the proof, we need to show the following claim.

**Claim:** The numbers \( x_i \) for \( 0 \leq i \leq n-1 \) all lie in the prescribed \( \mathbb{W}_{\lambda_i,\mu_i} \cup \mathbb{Q} \), i.e. they take irrationality exponents in the corresponding intervals \( [\lambda_i, \mu_i] \) unless they are rational.

**Proof of the claim.** Let \( 0 \leq i \leq n-1 \) be fixed. We easily check from the above that that \( \mu(x_i) \geq \lambda_i \). Indeed, considering \( p_{i,j}/q_{i,j} \) constructed above for \( i \equiv j \mod n \), from (34) (which we observed also holds for \( i = 0 \)) and \( q_{i,j} = 5^{h_j-1} \) we have for any \( j \)
\[
|x_i - p_{i,j}/q_{i,j}| \ll 5^{-h_j} \ll 5^{-g_{j+1}} \ll 5^{-\lambda_i g_j} \ll q_{i,j}^{-\lambda_i}.
\]

This means \( \mu(x_i) \geq \lambda_i \) unless the approximations are ultimately constant and equal to \( x_i \), thus \( x_i \in \mathbb{Q} \).

We need to show the inequality \( \mu(x_i) \leq \Lambda/(\lambda_i - 1) + 1 \) for \( 0 \leq i \leq n-1 \). Write \( \nu_i = \Lambda/(\lambda_i - 1) + 1 \) for simplicity. Assume the contrary that \( \mu(x_i) > \nu_i \) for some \( i \). Then for some \( \mu > \nu_i \) the inequality
\[
|x_i - p/q| < q^{-\mu}
\]
has infinitely many rational solutions \( p/q \). We consider \( i \) fixed in the sequel. We distinguish two cases: \( p/q \) can be among the \( p_j/q_j \) defined above, with the convention \( i \equiv j \mod n \) as explained above, or distinct from them.

**Case 1:** The rational in (36) satisfies \( p/q = p_j/q_j \) for some \( j \equiv i \mod n \). Since we have observed that \( p_j/q_j \) with \( p_j = e_{i,j}, q_j = 5^{h_j-1} \) for some \( j \equiv i \mod n \) are reduced, we can restrict to \( p = e_{i,j}, q = 5^{h_j-1} \). Hence it remains to be checked that the reverse inequality to (35) holds, i.e.
\[
|x_i - p_j/q_j| \gg 5^{-h_j},
\]
for any \( j \equiv i \mod n \). However, if \( 1 \leq i \leq n-1 \), we have
\[
|x_i - p_j/q_j| = \sum_{\ell > h_j} \frac{d_{i,\ell} - d_{0,\ell}^i}{5^\ell} + \sum_{k \geq j} \frac{a_k}{5^{h_k}},
\]
and similarly for \( i = 0 \). In any case, the first sum is fixed and the second has dominating term \( a_j/5^{h_j} \approx 5^{-h_j} \), so it is clear that at most one integer choice \( a_j \) may contradict (37). So far, we have only excluded one residue class modulo 5 above for any \( a_k \). Thus, upon avoiding another residue class for \( a_j \) if necessary, clearly a suitable choice of the \( a_j \in \{0, 1, 2, 3, 4\} \) remains possible.

Case 2: Now assume infinitely many \( p/q \) with property (36) are distinct from all \( p_j/q_j \), where we assume \( j \equiv i \mod n \). Let \( \lambda = \lambda_i \). First we settle that any such \( p/q \) as in (36) must satisfy
\[
q_u^{\lambda-1} \ll q \ll q_u^{1/(\mu-1)},
\]
for some \( u \equiv i \mod n \). We can take \( u \) to be the unique integer satisfying \( u \equiv i \mod n \) and so that \( q_u < q \leq q_{u+n} \). In (35) we have noticed that \( |x_i - \frac{p_u}{q_u}| \ll q_u^{-\lambda} \). Since \( \lambda > 2 \), clearly the reduced fraction \( p_u/q_u \) is a convergent in the continued fraction expansion of \( x_1 \) by Legendre’s Theorem 4.3. Then Proposition 4.2 gives that the next convergent denominator is \( \gg q_u^{\lambda-1} \) and since \( p/q \) is clearly also a convergent to \( x_i \) by (36) and Legendre’s Theorem and as \( q > q_u \), we infer \( q \gg q_u^{\lambda-1} \), the left estimate in (38). The right is induced very similarly using the assumption that \( p/q \) satisfies \( |x_i - p/q| < q^{-\mu} \). Indeed, by Proposition 4.2, the subsequent convergent has denominator \( \gg q_u^{\lambda-1} \). Since \( p_{u+n}/q_{u+n} \) is another reduced convergent to \( x_1 \) and \( q_{u+n} \geq q \) and \( p_{u+n}/q_{u+n} \neq p/q \) by assumption of Case 2, we derive \( q_{u+n} \gg q_u^{\lambda-1} \), equivalent to the right bound in (38).

Now from (38) we see that
\[
q_{u+n} \gg q_u^{(\lambda-1)(\mu-1)}. \tag{38}
\]

On the other hand (32) implies \( q_{u+n} \gg q_u^\Lambda \). Hence \( \Lambda \geq (\lambda-1)(\mu-1) - \varepsilon_1 \), which by choice of \( \mu > \nu_1 \) is however false for \( \varepsilon_1 \) small enough in view of assumption (15). Thus we have derived the desired contradiction to (36) in both cases and our claim is proved. We conclude that the dimension of our Cartesian product set is at least \( n - 1 \). \( \square \)

If all \( \lambda_i = \infty \), we have already proved the claim within the proof of Theorem 2.1. We prove Theorem 3.1 in a similar way. A notable twist occurs in Case 2. To rule out the existence of putative good approximations of the form \( p/b^N \) different from the \( p_u/q_u \), we cannot use the argument above since we do not assume (15) any longer. To find a way around this obstruction, we will employ the following easy lemma.

Lemma 4.4. Let \( b \geq 2 \) be an integer. Let \( \mathcal{T} \subseteq \mathbb{R}^{n-1} \) be the set of real vectors \((\xi_1, \ldots, \xi_{n-1})\) for which any \( \xi_i \) for \( 1 \leq i \leq n-1 \), as well as any \( \xi_i - \xi_t \) for every index pair \( 1 \leq i < t \leq n-1 \), all lie in \( \mathcal{V}_{1,(b)} \). Then \( \mathcal{T} \) has full \((n-1)\)-dimensional Lebesgue measure. In fact the complement set \( \mathbb{R}^{n-1} \setminus \mathcal{T} \) is of Hausdorff dimension \( n - 2 \).

The proof is not deep and only uses standard measure theoretic arguments.
Proof. It is well-known and follows from (10) by a standard measure theoretic argument that the complement of $\mathcal{V}_{1,(b)}$ in $\mathbb{R}$ has Lebesgue measure 0 (is a nullset in $\mathbb{R}$). We can write the set $\mathcal{F}$ in the lemma as the intersection of $U = \mathcal{V}_{1,(b)}^{n-1}$ with the sets

$$U_{i,t} := \{(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} : \xi_i - \xi_t \in \mathcal{V}_{1,(b)}\}, \quad 1 \leq i < t \leq n - 1.$$ 

Clearly $U$ has full $(n - 1)$-dimensional Lebesgue measure as it is the Cartesian product of full measure sets. We show that every set $U_{i,t}$ has full measure as well. Indeed, upon relabelling, it can be identified with $\mathbb{R}^{n-3} \times V$ where $V \subseteq \mathbb{R}^2$ is given by $(\mathcal{V}_{1,(b)} + \mathbb{R}) \times \mathbb{R} = \{(x + y, y) : x \in \mathcal{V}_{1,(b)}, y \in \mathbb{R}\}$. Hence the set $V$ is the image of $\mathcal{V}_{1,(b)} \times \mathbb{R}$ under the Lipschitz map $(x, y) \mapsto (x + y, y)$. Since $\mathcal{V}_{1,(b)} \times \mathbb{R} \subseteq \mathbb{R}^2$ has full 2-dimensional Lebesgue measure again as a Cartesian product of full measure sets, the same applies to $V$ by Proposition 1.2. Again using that it can be written as Cartesian product of full measure sets, we obtain that any set $U_{i,t}$ has full $(n - 1)$-dimensional Hausdorff measure. Hence the intersection of the finitely many $U_{i,t}$ has full measure as well, and intersecting it with the full measure set $U$ again preserves the property. \hfill $\square$

Proof of Theorem 3.1 Let $b \geq 2$ be any integer. The main claim is the lower bound $n - 1$ in (7). We proceed as in the proof of Theorem 3.2, where for obvious reasons we work in base $b$ instead of 5 (very minor adaptions related to the choice of the $a_j$ have to be made when $b = 2$). We show that for any choices of $\lambda_i \geq 1$, the image of the map

$$\Psi_2 : \prod_{i=0}^{n-1} (\mathcal{V}_{\lambda_i,(b)} \cup \mathbb{Q}) \rightarrow \mathbb{R}^{n-1},$$

$$(x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0 + x_1, x_0 + x_2, \ldots, x_0 + x_{n-1}), \quad (39)$$

contains the set $\mathcal{F}$ from Lemma 4.4. Since the lemma claims that $\mathcal{F}$ has full $(n - 1)$-dimensional measure, then again we can conclude with Proposition 1.2 and Lemma 4.1.

We start with arbitrary $\xi \in \mathcal{F}$ and derive the preimage components $x_i$ by the same construction as in the proof of Theorem 3.2.

Define the rational approximations $p_j/q_j$ to $x_i$ identically as in Theorem 3.2 as well. They are of the desired form $p/b^N$, with $p = p_j$ and $N = h_j - 1$, and approximate $x_i$ of order $\lambda_i$, hence $\theta_b(x_i) \geq \lambda_i$, unless $x_i \in \mathbb{Q}$. We have to prove the reverse estimate $\theta_b(x_i) \leq \lambda_i$. Notice that in contrast to Theorem 3.2 we do not impose the assumption (15) here. However, it will follow from our setup and $\xi \in \mathcal{F}$ that no sufficiently good rational approximations with denominator an integer power of $b$ to $x_i$ can exist. We again distinguish the two cases $p/q = p_j/q_j$ and $p/q$ not of this form. The proof of Case 1 is immediate as in Theorem 3.2, however it is also implicitly contained in the general argument below. So again assume otherwise that $x_i$ satisfies $\theta_b(x_i) > \lambda_i$ for some
\[ i \in \{0, 1, \ldots, n - 1\} \text{ that we assume fixed. Then the estimate} \]
\[ \left| x_i - \frac{p}{b^N} \right| \leq b^{-\mu N} < b^{-\lambda_i N} \tag{40} \]
holds for some \( \mu > \lambda_i \) and infinitely many pairs of integers \( p, N \). For the moment, let us fix such large \( p, N \). Then \( x_i \) has a string of digits all equal to 0 or all equal to \( b - 1 \) at positions in the interval \( J = \{N + 1, N + 2, \ldots, \lfloor \mu N \rfloor \} \). Note that if \( N \) is large then \( J \) defined above cannot be contained in some \( I_j \) with \( j \equiv i \mod n \) because \( \max I_j / \min I_j = h_j / g_j = \lambda_i + o(1) \) as \( j \to \infty \) by construction and \( \mu = \max J / \min J - o(1) > \lambda_i \). More precisely, if we let \( \epsilon = (\mu - \lambda_i) / 2 > 0 \), it is clear that \( J \) intersects some \( I_{j_0} \) with \( j_0 \neq i \mod n \) in some interval \( J' = \{M, M + 1, \ldots, \lfloor M(1 + \epsilon) \rfloor \} \subseteq J \). Consider first \( i = 0 \).

Recall that by construction of \( x_0 \), it has locally the same digits as some \( \xi_j \) for some \( 1 \leq j \leq n - 1 \) within \( J' \). Hence for \( j \in \{1, 2, \ldots, n - 1\} \) the residue class of \( j_0 \) above modulo \( n \), the real number \( \xi_j \) has a pure 0 or \( b - 1 \) digit string in \( J' \). However, this means
\[ \left| \xi_j - \frac{p_0}{b^M} \right| < b^{-(1 + \epsilon) M} \]
for some integer \( p_0 \) and thus contradicts \( \xi_j \in \mathcal{V}_{1, (b)} \). Let us now consider \( i \neq 0 \).

Then, since \( x_0 \) has the same digits as \( \xi_j \) for again \( j \equiv j_0 \mod n \) within \( J' \subseteq I_{j_0} \), the fact that \( x_i = x_0 - \xi_i \) has zero digits in \( J' \) means that we may write
\[ x_i = \frac{p_1}{b^{M-1}} + (\xi_j - \xi_i) + E, \quad |E| \ll b^{-M(1+\epsilon)} \]
for an integer \( p_1 \), coming from certain base \( b \) digit differences between certain \( \xi_t \), at the positions up to \( M - 1 \). But this means
\[ \left| (\xi_j - \xi_i) - x_i + \frac{p_1}{b^{M-1}} \right| \ll b^{-M(1+\epsilon)} \]
On the other hand, estimate (40) and \( J' \subseteq J \) imply that we may write \( x_i = \frac{p_2}{b^M} + O(b^{-M(1+\epsilon)}) \) for an integer \( p_2 \). Combining and by triangle inequality we derive
\[ \left| (\xi_j - \xi_i) - \frac{p_2}{b^M} + \frac{p_1}{b^{M-1}} \right| = \left| (\xi_j - \xi_i) - \frac{p_3}{b^M} \right| \ll b^{-M(1+\epsilon)} \]
for some integer \( p_3 \), which finally again contradicts our assumption \( \xi \in \mathcal{V} \). Hence the hypothetical assumption \( \theta_b(x_i) > \lambda_i \) cannot occur, and the proof of the lower bound \( n - 1 \) for the Hausdorff dimension is finished. The remaining claims follow easily when taking (1) and (10) into account, as explained below the formulation of Theorem 3.1. \( \square \)

We remark that for \( n = 2 \) and \( \lambda_0 \neq \lambda_1 \), the map \( \Psi_2 \) when restricted to \( \prod_{i=0}^{n-1} \mathcal{V}_{\lambda_{i, (b)} = \mathcal{V}_{\lambda_0, (b)} \times \mathcal{V}_{\lambda_1, (b)} \) is not surjective. Indeed, any rational of the form \( p/b^N \) cannot be in the image of \( \Psi_2 \) because \( \theta_b(\xi) = \theta_b(p/b^N - \xi) \) is easily verified.

The proof of Theorem 3.4 and consequently Theorem 3.3 again uses similar ideas as Theorem 3.2. We first show how Theorem 3.3 can be inferred
from Theorem 3.4. Here we consider not very well approximable numbers in the one-dimensional image of our sum map.

**Deduction of Theorem 3.3 from Theorem 3.4** Anything apart from the lower bound 1 upon our assumption (17) on the $\lambda_i$, follows easily from (13) and Theorem 1.1 again. We can assume both $\lambda_i < \infty$, otherwise the claim follows directly from the upper bound being 1. However, by Theorem 3.4, the image of the Lipschitz map

$$\Psi_3: \left( W_{\lambda_0} \cup \mathbb{Q} \right) \times \left( W_{\lambda_1} \cup \mathbb{Q} \right) \rightarrow \mathbb{R},$$

$$(x_0, x_1) \mapsto x_0 + x_1,$$  \hspace{1cm} (41)

contains the set $W_2$ of not very well approximable numbers. As $W_2$ has full 1-dimensional Lebesgue measure, the claim follows from Proposition 1.2 and Lemma 4.1. □

Now we turn towards the proof of Theorem 3.4. We again use a similar construction as in the proof of Theorem 3.2. The core of the proof is that our restriction to $\xi$ that are not very well approximable in (20) will guarantee that there is no other good rational approximation to $x_i$, $i = 0, 1$ apart from the $p_j/q_j$ constructed in the proof of the lower bound. Indeed, with some trick (that only works for $n = 2$), we show that the existence of other putative good approximations would induce good rational approximations to $\xi$, which contradicts the hypothesis $\xi \in W_2$.

**Proof of Theorem 3.4** Start with arbitrary $\xi \in W_2 \cap [0, 1)$ with base 5 representation

$$\xi = \sum_{\ell \geq 1} d_\ell 5^{-\ell}, \quad d_\ell \in \{0, 1, 2, 3, 4\}.$$ 

To show (20), we construct $x_0, x_1$ that sum up to $\xi$ and have the prescribed irrationality exponents $\lambda_0$ and $\lambda_1$, respectively. Similar to the proof of Theorem 3.2, partition $\mathbb{N}$ into intervals $I_j = \{g_j, g_j + 1, \ldots, h_j\}$ with $g_{j+1} = h_j + 1$ and $h_j = \lceil \lambda_1 g_j \rceil$ for odd $j$ and $h_j = \lceil \lambda_0 g_j \rceil$ for even $j$. In particular $h_j/g_j = \lambda_1 + o(1)$ for odd $j$ and $h_j/g_j = \lambda_0 + o(1)$ for even $j$, as $j \rightarrow \infty$. We further repeat the construction of $x_0, x_1$ from that proof. Due to $n = 2$, the construction can be described in a slightly simpler way here, which we want to explain. Let $x_0^*$ be the real number that has the base 5 digits of $\xi$ in intervals $I_j$ for odd $j$ and 0 in intervals $I_j$ for even $j$, which agrees with the definition of $x_0^*$ from the proof of Theorem 3.2 when $n = 2$. Let vice versa $x_1^*$ be the real number with that has the digits of $\xi$ in intervals $I_j$ for even $j$ and zero in $I_j$ for odd $j$, which corresponds to $\xi - x_0^*$. This means if we write the base 5 expansions

$$x_0^* = \sum_{\ell \geq 1} e_\ell 5^{-\ell}, \quad x_1^* = \sum_{\ell \geq 1} f_\ell 5^{-\ell},$$

then $e_\ell = d_\ell$ if $\ell \in I_j$ for odd $j$ and $e_\ell = 0$ else if $\ell \in I_j$ for even $j$, and vice versa for $f_j$. Let $a_k \in \{0, 1, \ldots, 4\}$ for $k \geq 1$ to be chosen later on. We again
modify the digits at the interval endpoints by considering
\[ x_0 = x_0^* - \sum_{k \geq 1}^{h_{j-1}} \frac{a_k}{5^h_k}, \quad x_1 = x_1^* + \sum_{k \geq 1}^{j-1} \frac{a_k}{5^h_k}. \]

Then clearly \( x_0 + x_1 = x_0^* + x_1^* = \xi \). We need to show that \( \mu(x_i) = \lambda_i, i = 0, 1 \)
upon the lower bound assumed on \( \lambda_i \). First observe that when again splitting
the series for \( x_0^* \) and \( x_1^* \), for even \( j \geq 1 \) we may write
\[ x_0 = \sum_{\ell=1}^{h_{j-1}} \frac{e_\ell}{5^\ell} - \sum_{k=1}^{j-1} \frac{a_k}{5^h_k} + \sum_{\ell > h_j} \frac{e_\ell}{5^\ell} - \sum_{k \geq j} \frac{a_k}{5^h_k}, \]

and for odd \( j \) similarly we have
\[ x_1 = \sum_{\ell=1}^{h_{j-1}} \frac{f_\ell}{5^\ell} + \sum_{k=1}^{j-1} \frac{a_k}{5^h_k} + \sum_{\ell > h_j} \frac{f_\ell}{5^\ell} + \sum_{k \geq j} \frac{a_k}{5^h_k}. \] (42)

Here again we used the vanishing of the digits in intervals \( I_j = [h_{j-1} + 1, h_j] \)
for \( j \) with the stated parity. We again consider the rational numbers \( p_{i,j}/q_{i,j} = p_j/q_j, j \equiv i \mod 2 \), obtained from the respective two first finite sums in the
above representation of \( x_0, x_1 \). For example, for odd \( j \) related to \( x_1 \) this reads
\[ \frac{p_j}{q_j} = \frac{p_{1,j}}{q_{1,j}} = \sum_{\ell=1}^{h_{j-1}} \frac{f_\ell}{5^\ell} + \sum_{k=1}^{j-1} \frac{a_k}{5^h_k} = \sum_{\ell=1}^{h_{j-1}} \frac{f_\ell 5^{h_{j-1}-\ell}}{5^{h_{j-1}}} + \sum_{k=1}^{j-1} \frac{a_k 5^{h_{j-1}-h_k}}{5^{h_{j-1}}}. \]

Then essentially the same arguments from the proof of Theorem 3.2 show that
upon choosing \( a_{j-1} \) appropriately (avoiding the residue class \(-f_{h_{j-1}} \mod 5 \) for each odd \( j \), and \( e_{h_{j-1}} \) for even \( j \)), the rationals \( p_j/q_j \) are reduced with
denominator \( q_j = 5^{h_{j-1}} \) and the error from the remaining infinite sums can be estimated
\[ |x_1 - p_j/q_j| = \left| \sum_{\ell > h_j} \frac{f_\ell}{5^\ell} + \sum_{k \geq j} \frac{a_k}{5^h_k} \right| \ll 5^{-h_j}, \] (43)

and similarly for \( x_0 \), i.e. as in (34). Hence \( \mu(x_i) \geq \lambda_i \) for \( i = 0, 1 \) unless \( x_i \in \mathbb{Q} \),
can be derived as in the proof of Theorem 3.2. Again the main difficulty is to
show the converse \( \mu(x_i) \leq \lambda_i \), \( i = 0, 1 \). By symmetry, we restrict to \( i = 1 \), and
occasionally comment how to modify the arguments for \( i = 0 \).

Assume conversely, for some \( \mu > \lambda_1 \) we have infinitely many \( p/q \) with
\[ |x_1 - p/q| \leq q^{-\mu}. \] (44)

Again we split into two cases according to the cases of rational approxima-
tions \( p/q \) being among \( p_j/q_j \) above for some odd \( j \), or not. Case 1: \( p/q = p_j/q_j \)
for some odd \( j \). Then again \( p_j/q_j \) are reduced and by the same argument as
in Theorem 3.2 satisfy \( |x_1 - p_j/q_j| \asymp 5^{-h_j} \asymp q_j^{-\lambda_1 + o(1)} \) as \( j \to \infty \) for a proper
choice of \( a_j \), so (44) is impossible.
Case 2: $p/q$ is not among $p_j/q_j$ above for odd $j$. We will show that (44) implies $\mu(\xi) > 2$, contradicting our hypothesis $\xi \in \mathbb{N}_2$. For given $p/q$, let again $u$ be the largest odd index with $5^{h_{u-1}} = q_u < q$. Clearly $q_u < q \leq q_{u+2}$. Very similarly as in the proof of Theorem 3.2, Legendre Theorem implies

$$q_u^{\lambda_1-1} \ll q \ll q_{u+2}^{1/(\mu-1)} < q_{u+2}^{1/(\lambda_1-1)}.$$  

Let $\Lambda = \lambda_0\lambda_1$. Now since $q_{u+2} \gg q_u^\Lambda$, and as by (34) we have $q_u = 5^{h_{u-1}} \gg 5^{g_u}$, we derive

$$5^{g_u(\lambda_1-1)} \ll q \ll 5^{\Lambda g_u/(\lambda_1-1)}.$$  

We remark that this implies $\lambda_0 \geq (\lambda_1-1)^2/\lambda_1$, however we will not require this conclusion.

The next important step is to construct good rational approximations to $\xi - x_1$. Since $u$ is odd, we may apply (42) for $j = u$, and when we split the third sum as

$$\sum_{\ell \geq 1} \frac{e_\ell}{5^\ell} = \sum_{\ell=1}^{h_{j+1}} \frac{f_\ell}{5^\ell} + \sum_{\ell > h_{j+1}} \frac{f_\ell}{5^\ell},$$

and take the first summand of the last sum to the front, we may rearrange

$$\xi - x_1 = \sum_{\ell \geq 1} \frac{d_\ell}{5^\ell} - \left(\sum_{\ell=1}^{h_u-1} \frac{f_\ell}{5^\ell} + \sum_{k=1}^{u-1} \frac{a_k}{5^{h_k}} + \sum_{\ell > h_u} \frac{f_\ell}{5^\ell} + \sum_{k \geq u} \frac{a_k}{5^{h_k}}\right)$$

$$= \left(\sum_{\ell=1}^{h_u} \frac{d_\ell}{5^\ell} + \sum_{\ell > h_u+1} \frac{d_\ell}{5^\ell} - \sum_{\ell=1}^{h_u-1} \frac{f_\ell}{5^\ell} - \sum_{\ell = h_u+1} \frac{f_\ell}{5^\ell} - \sum_{k \geq u+1} \frac{a_k}{5^{h_k}}\right)$$

$$+ \sum_{\ell > h_u+1} \frac{d_\ell}{5^\ell} - \sum_{\ell > h_u+1} \frac{f_\ell}{5^\ell} - \sum_{k \geq u+1} \frac{a_k}{5^{h_k}}.$$

But now by construction and since $u$ is odd, we have $d_\ell = f_\ell$ for $\ell \in I_{u+1} = [h_u+1, h_{u+1}]$. Hence the above expression simplifies to

$$\xi - x_1 = \left(\sum_{\ell=1}^{h_u} \frac{d_\ell}{5^\ell} - \sum_{\ell=1}^{h_{u-1}} \frac{f_\ell}{5^\ell} - \sum_{k=1}^{u} \frac{a_k}{5^{h_k}}\right) + \sum_{\ell > h_u+1} \frac{d_\ell}{5^\ell} - \sum_{\ell > h_u+1} \frac{f_\ell}{5^\ell} - \sum_{k \geq u+1} \frac{a_k}{5^{h_k}}.$$

The finite sums in the brackets add up to a rational number $r/s$ with denominator $s = 5^{h_u}$ (we may again assume it is reduced upon avoiding some $a_j$, however we do not require this here). The remaining sums are obviously of order $\ll 5^{-h_u} = 5^{-\Lambda g_u}$, thus

$$\left|\xi - x_1 - \frac{r}{s}\right| \ll 5^{-\Lambda g_u}.$$
Combining with (44) and $\mu > \lambda_1$ yields

$$\left| \xi - \left( \frac{p}{q} + \frac{r}{s} \right) \right| \leq \left| \xi - x_1 - \frac{r}{s} \right| + \left| x_1 - \frac{p}{q} \right| \ll \max\{q^{-\mu}, 5^{-\Lambda g_u}\}$$

$$\ll \max\{q^{-\lambda_1}, 5^{-\Lambda g_u}\}.$$  \hspace{1cm} (46)

We distinguish two cases. Firstly assume the right bound in the maximum is larger, i.e. $q > 5^{\lambda_0 g_u}$. Then the rational number $M/N := p/q + r/s = (ps + qr)/(qs)$ satisfies

$$\left| \xi - \frac{M}{N} \right| \ll 5^{-\Lambda g_u}.$$  

On the other hand, as $u$ is odd, its denominator can be estimated as

$$N = qs = 5^h u q \ll 5^{\lambda_1 g_u}.$$  \hspace{1cm} (47)

Thus we have

$$- \frac{\log |\xi - M/N|}{\log N} \geq \frac{\log 5 \cdot \Lambda g_u}{\log q + \log 5 \cdot \lambda_1 g_u}.$$  

The right hand side is obviously decreasing in $q$ and thus in view of the upper bound in (45) after some calculation we derive

$$- \frac{\log |\xi - M/N|}{\log N} \geq \frac{\Lambda}{\lambda_1 - 1} - \epsilon_1 = \frac{\lambda_0 (\lambda_1 - 1)}{\lambda_0 + \lambda_1 - 1} - \epsilon_1,$$  \hspace{1cm} (48)

with $\epsilon_1$ arbitrarily small for large enough $u$. The right hand side increases as a function of both $\lambda_0, \lambda_1$, hence a short calculation verifies that condition (17) suffices for the conclusion $\mu(\xi) > 2$ upon choosing $\epsilon_1$ small enough. Thus we have the desired contradiction to $\xi \in \mathcal{W}_2$. (We notice that if only finitely many $M/N$ would occur, then $M/N = \xi$ is ultimately constant thus $\xi \in \mathbb{Q}$, again contradicting our hypothesis $\xi \notin \mathcal{W}_2$.) A very similar argument applies for $i = 0$, we end up at (48) with interchanged $\lambda_0, \lambda_1$, and the same argument applies. Finally assume the left expression in (46) is larger, thus $q \leq 5^{\lambda_0 g_u}$. Then from (46), (47) we infer

$$- \frac{\log |\xi - M/N|}{\log N} \geq \frac{\lambda_1 \log q}{\log q + \log 5 \cdot \lambda_1 g_u}.$$  

Since the right hand side expression increases in $q$, by (45) we conclude

$$- \frac{\log |\xi - M/N|}{\log N} \geq \frac{\lambda_1 \cdot (\lambda_1 - 1)}{(\lambda_1 - 1) + \lambda_1} - \epsilon_2 = \frac{\lambda_1^2 - \lambda_1}{2 \lambda_1 - 1} - \epsilon_2.$$  \hspace{1cm} (49)

Hence again $\mu(\xi) > 2$ as soon as $\lambda_1 > \rho$ and $\epsilon_2$ is sufficiently small, again contradicting $\xi \in \mathcal{W}_2$. Similarly we require $\lambda_0 > \rho$ for the contradiction when we start with $i = 0$. Thus condition (17) guarantees the implication in any case.

Finally by a similar argument we show (21). If we assume $\mu(\xi) > \tilde{\mu}$ for any given $\tilde{\mu} \geq 2$ in place of $\mu(\xi) > 2$, then similarly as above Case 1 cannot
happen and in Case 2 we get a contradiction as soon as the right hand sides in (48), (49) exceed \( \tilde{\mu} \), and similarly for \( \lambda_0 \) as well. This means upon these assumptions we have

\[
W_{\lambda_0} + W_{\lambda_1} \supseteq W_{2;\tilde{\mu}},
\]

or equivalently the complement of the sumset is contained in \( W_{\tilde{\mu};\infty} \). The left estimate in (21) thus follows from Jarník’s formula (13) for the Hausdorff dimensions of the latter sets, and again using that the right hand sides in (48), (49) represent increasing functions. The upper bound 1 in (21) follows from the bound (17) on the \( \lambda_i \).

\[ \square \]

5. Proofs of Generalisations in Sect. 3.3

We first sketch how to extend Theorem 3.4 to simultaneous approximation.

Proof of Theorem 3.7 Starting with a vector \( \xi \in \mathbb{R}^m \), we may apply the construction from the proof of Theorem 3.4 coordinatewise with the same intervals \( I_j \) for each coordinate to obtain \( x_0, x_1 \) that sum up to \( \xi \). We verify \( \mu_m(x_i) = \lambda_i \) for \( i = 0, 1 \). The lower bounds \( \mu_m(x_i) \geq \lambda_i \) follow analogously as for \( m = 1 \), without restrictions on \( \lambda_i \). For the reverse estimates, Case 1 follows again very similarly. In Case 2, we first find a generalisation of Proposition 4.2 to general \( m \), derived from the one-dimensional case.

Proposition 5.1. Let \( \tau_k > 1 \). Let \( 1 < q_k < q_{k+1} \) be integers and assume

\[
\max_{1 \leq i \leq m} |\xi_i - p_{k,i}/q_k| < q_k^{-\tau_k}/2, \quad \max_{1 \leq i \leq m} |\xi_i - p_{k+1,i}/q_{k+1}| < q_{k+1}^{-1}/2
\]

hold for rational vectors \( p_k/q_k, p_{k+1}/q_{k+1} \) in reduced form. Then \( q_{k+1} \gg q_k^{\tau_k-1} \).

Proof. The deduction from Proposition 4.2 is obvious if for some \( i \) the pair \( (q_k, p_{k,i}) \) is coprime, but this need not be the case. Otherwise, fix some \( i \) with largest common divisor \( T = q_k^a > 1 \) for some \( a \in (0, 1] \) and consider the reduced fraction \( p^{i,0}/q^0 = (q_k/T)/(p_{k,i}/T) \) which is a convergent to \( \xi_i \) since \( \tau_k > 1 \). We see that \( q^0 = q_k^{1-a} \) and so

\[
|p^{i,0}/q^0 - \xi_i| = |p_{k,i}/q_k - \xi_i| < q_k^{-\tau_k} = (q^0)^{-\tau_k/(1-a)},
\]

thus by Proposition 4.2 the next convergent to \( \xi_i \) is of order

\[
\gg (q^0)^{\tau_k/(1-a)-1} = q_k^{(1-a)(\tau_k/(1-a)-1)} = q_k^{\tau_k-1+a} \geq q_k^{\tau_k-1}.
\]

Hence by Legendre Theorem 4.3 also \( q_{k+1} \gg q_k^{\tau_k-1} \). \( \square \)

From Proposition 5.1, we again conclude that (45) holds. Then we proceed precisely as in the one-dimensional setting. Since Lebesgue almost all \( \xi \in \mathbb{R}^m \)
are only simultaneously approximable of order $\mu_m(\xi) = 1 + \frac{1}{m}$, according to (48), (49) we eventually end up at the conditions
\[
\frac{\lambda_i (\lambda_{1-i} - 1)}{\lambda_i + \lambda_{1-i} - 1} > 1 + \frac{1}{m}, \quad \frac{\lambda_i^2 - \lambda_i}{2\lambda_i - 1} > 1 + \frac{1}{m}
\]
for $i = 0, 1$. Thus, for the analogous full measure result, as $m \to \infty$ the lower bound $\min \lambda_i > \gamma + o(1)$ suffices, giving rise to $\rho_m$ as in the theorem. Finally (29) follows similarly from a well-known generalisation of (13) to higher dimension with right hand side $(m+1)/\lambda$, also due to Jarník [21].

We explain the modifications to be made to obtain the more general Theorem 3.6.

**Proof Theorem 3.6** We define $x_0^*$ as in the proof of Theorem 3.4 and again choose integers $a_k$ to analogously infer $x_0, x_1$, however now do not restrict to $a_k \in \{0, 1, 2, 3, 4\}$ any longer. Assume for given $j$, we have already defined the initial elements $h_1, h_2, \ldots, h_{j-1}$ and $a_1, \ldots, a_{j-1}$, and the corresponding reduced $p_j/q_j$ with $q_j = 5^{h_{j-1}}$. We define the next $h_j = -[\log \Phi_i(q_j)/\log 5] + j$ with $i \in \{0, 1\}$ so that $i \equiv j \mod 2$. Then
\[
\Phi_i(q_j) \asymp 5^{-h_j+j}. \tag{50}
\]
If $i = 1$, we split (43) as
\[
|x_1 - \frac{p_j}{q_j}| = \left| \sum_{\ell > h_j} f_\ell \frac{a_j}{5^{h_j}} + \sum_{k \geq j} a_k \frac{a_j}{5^{h_j}} \right| = \left| \left( \sum_{\ell > h_j} f_\ell \frac{a_j}{5^{h_j}} \right) + \sum_{k > j} a_k \frac{a_j}{5^{h_k}} \right| \tag{51}
\]
and accordingly for $i = 0$. Let us for simplicity assume $i = 1$ from now on, the other case works analogously. We take the next $a_j$ so that the absolute value of the bracket expression
\[
E_j := \sum_{\ell > h_j} f_\ell \frac{a_j}{5^{h_j}}
\]
is at most $\Phi_i(q_j)$ and as close as possible to it, unless if the residue class condition modulo 5 for $p_{j+1}/q_{j+1}$ to be reduced in the next step (see proof of Theorem 3.4) fails, in which case we alter it by $\pm 1$ to get the next nearest number of this form smaller than $\Phi_i(q_j)$. It is clear that then
\[
0 \leq \Phi_i(q_j) - |E_j| \leq 2 \cdot 5^{-h_j}. \tag{52}
\]
First assume $a_j \geq 0$. Then clearly also $E_j \geq 0$ and (52) implies
\[
a_j/5^{h_j} \leq \Phi_i(q_j) - \sum_{\ell > h_j} f_\ell \frac{a_j}{5^{h_j}} \leq \Phi_i(q_j)
\]
hence by (50) we infer
\[
0 \leq a_j \leq 5^{h_j} \Phi_i(q_j) \ll 5^j.
\]
Thus, the remaining expression in (51) can be bounded by
\[
\left| \sum_{k \geq j} a_k \frac{h_k}{5^h_k} \right| \ll 5^{j-h_j+1} = o(5^{-h_j}), \tag{53}
\]
where for the last estimate we used that the sequence \((h_j)_{j \geq 1}\) grows exponentially. Indeed, the decay assumption on \(\Phi_1\) and (50) lead to
\[
5^{-h_j} \leq 5^{-h_j+j} \ll \Phi_1(q_j) \leq q_j^{-\lambda_i} \leq 5^{-h_j-1}\rho
\]
implying \(h_j/h_{j-1} > \rho - o(1) > 1\) for large \(j\), and hence the claim. From (51), (52) and (53), we infer that for large \(j\) we may write
\[
\left| x_1 - \frac{p_j}{q_j} \right| = \Phi_1(q_j) - R,
\]
for some remainder term \(0 \leq R \leq 3 \cdot 5^{-h_j}\). Thus, by choice of \(h_j\), we infer
\[
0 < 1 - \frac{|x_1 - p_j/q_j|}{\Phi_1(q_j)} = \frac{R}{\Phi_1(q_j)} \ll \frac{5^{-h_j}}{\Phi_1(q_j)} \ll 5^{-j}.
\]
The expression tends to 0 as \(j \to \infty\). Now assume \(a_j < 0\). Then \(E_j < 0\) and thus similarly (52) implies the analogous estimate
\[
0 \leq |a_j| = -a_j \leq \Phi_1(q_j) - \sum_{\ell > h_j} f_\ell \frac{5^\ell}{5^\ell} \leq \Phi_1(q_j)
\]
and the same argument as above applies. A very similar line of arguments applies to \(i = 0\). Finally, in Case 2 of other rationals \(p/q \neq p_j/q_j\), we get that
\[
|x_i - p/q|/\Phi_i(q) \to \infty
\]
from the same line of arguments as in Theorem 3.4 upon our assumptions on the \(\Phi_i\). Combining these facts yields \(x_i \in \text{Exact}(\Phi_i)\) for \(i = 0, 1\). The last claim (28) on packing dimension again follows via Theorem 1.1 by
\[
\dim_P(\text{Exact}(\Phi_1)) \geq \dim \left( \prod_{i=0}^{1} \text{Exact}(\Phi_i) \right) - \dim(\text{Exact}(\Phi_0))
\]
and choosing any \(\Phi_0\) of very fast decay so that the last term vanishes by (13).

We finally describe how to alter the construction of Theorem 3.4 for missing digit Cantor sets. The main obstacle is that we require \(p_j/q_j\) to be reduced for Case 1. We may not be able to choose integers \(a_j\) as before without leaving the Cantor set for either \(x_0\) or \(x_1\). To avoid this problem, we slightly redefine our intervals \(I_j\) and our restriction to \(V_{1,\ldots,2}\) in (24) enters. We keep the notation from the proof of Theorem 3.4 above.

**Proof of Theorem 3.5** Assume \(0 \in W\). Let \(\xi \in V_{1,\ldots,2} \cap \mathcal{K}\) be arbitrary and \(\varepsilon > 0\). Write \(\xi = \sum_{\ell \geq 1} d_\ell b^{-\ell}\) for its base \(b\) representation, with \(d_\ell \in W\).

The fact \(\xi \in V_{1,\ldots,2}\) guarantees that there are no long blocks of consecutive 0
digits in its base $b$ representation, more precisely in intervals $[N, (1 + \varepsilon)N]$ for any $\varepsilon > 0$ and $N \geq N_0(\varepsilon)$. Hence we can modify our sequences $g_j, h_j$ from the proof of Theorem 3.4 so that the quotients $h_j/g_j$ still tend to $\lambda_i$ with $i \in \{0, 1\}$ so that $i \equiv j \mod 2$, and additionally at the first position $\ell = g_j + 1 = h_j + 1$ of $I_{j+1}$ the digit is in $d_\ell \in W \setminus \{0\}$. Derive $x_0^*, x_1^*$ similar to Theorem 3.4 via

\[ x_0^* = \sum_{\ell \geq 1} \frac{e_\ell}{b^\ell}, \quad x_1^* = \sum_{\ell \geq 1} \frac{f_\ell}{b^\ell}, \]

where we let $e_\ell = d_\ell$ if $\ell \in I_j$ for odd $j$ and $e_\ell = 0$ otherwise, and vice versa for $f_\ell$. Now we simply let $x_0 = x_0^*, x_1 = x_1^*$, without twisting the digits at interval endpoints. Then for each even $j$ we have

\[ x_0 = \sum_{\ell = 1}^{h_j - 1} \frac{e_\ell}{b^\ell} + \sum_{\ell > h_j} \frac{e_\ell}{b^\ell}, \quad e_\ell \in W, \]

and for odd $j$ we have

\[ x_1 = \sum_{\ell = 1}^{h_j - 1} \frac{f_\ell}{b^\ell} + \sum_{\ell > h_j} \frac{f_\ell}{b^\ell}, \quad f_\ell \in W. \]

Notice that the digits at positions $h_j - 1 + 1, \ldots, h_j$ vanish. By construction, in the infinite sums for $j$ in question, at first position $\ell = h_j + 1$ the digit $e_\ell$ of $x_0$ resp. $f_\ell$ of $x_1$ is non-zero. Notice that $x_0 + x_1 = x_0^* + x_1^* = \xi$ and $x_i \in K$. We define $p_j/q_j$ for $j \equiv i \mod 2$ likewise as in Theorem 3.4 as the rational number obtained from the respective finite partial sums, in particular $q_j = b^{h_j - 1}$. By the above properties, it is obvious that for $i = 0, 1$ and $j \equiv i \mod 2$, as $j \to \infty$ we have

\[ |x_i - p_j/q_j| \asymp b^{-h_j} \asymp b^{-\lambda_i h_j - o(1)} \asymp q_j^{-\lambda_i + o(1)}, \quad (54) \]

in particular $\mu(x_i) \geq \lambda_i$. For the reverse inequalities that settle (24), we again assume the opposite, namely we have $p/q$ that approximate some $x_i$ of order larger than $\lambda_i$ and consider the cases $p/q = p_j/q_j$ and $p/q$ not of this form separately. The latter Case 2 can be handled precisely as in the proof of Theorem 3.4, here our assumption $\xi \not\in \mathbb{W}_2$ enters. In Case 1, here we have the problem that we lack the twist with the $a_j$ to guarantee that $x_i$ have non-zero digit at positions $h_j - 1$ when $j \equiv i \mod 2$, and that consequently we cannot deduce that $p_j/q_j$ are reduced in the stated form with denominator $b^{h_j - 1}$. A priori it may happen that $p_j/q_j$ is not reduced, and after reduction the approximation exponent may be larger than predicted.

To deal with this technical obstruction, we modify the argument as follows. Since $\theta_b(\xi) = 1$, given $\varepsilon > 0$, for each large $j \geq j_0(\varepsilon)$ at some slightly smaller position

\[ z_{j-1} \in Z_{j-1} := [h_{j-1}(1 - \varepsilon), h_{j-1}] \cap \mathbb{Z} \subseteq I_{j-1}, \]
the number $\xi$ must have a non-zero digit, i.e. $d_{z_j-1} \in W \setminus \{0\}$. Assume $z_j-1$ is largest possible with this property. By construction, for $i \in \{0,1\}$ with $i \equiv j \mod 2$, the number $x_i$ has the same non-zero digit $d_{z_j-1}$ as $\xi$ at position $z_j-1$, followed by a digit string of zeros up to (including) the last position of $I_j$. Write $r_j/s_j$ for the reduced fraction $p_j/q_j$. First assume $b$ is prime (more generally $(b,w) = 1$ for all $w \in W \setminus \{0\}$ suffices). Then the above property readily implies that the according fraction $p_j/q_j = p_j/b^{z_j-1}$ is almost reduced. More precisely, after reduction it has denominator

$$s_j = b^{z_j-1} > b^{(1-\varepsilon)h_j-1} = q_j^{1-\varepsilon}. \quad (55)$$

Combining (55) with (54) readily implies for $i = 0,1$ and all large $j \equiv i \mod 2$ that

$$|x_i - r_j/s_j| = |x_i - p_j/q_j| \gg q_j^{-\lambda_i+o(1)} \gg s_j^{(-\lambda_i+o(1))/(1-\varepsilon)} \gg s_j^{-\lambda_i-\varepsilon_j},$$

where $\epsilon_j > 0$ tend to 0 as $\varepsilon \to 0$ and $j \to \infty$. Since $\varepsilon$ can be taken arbitrarily small, we are done. If otherwise $b$ is composite, we possibly have to slightly alter $x_0,x_1$ by interchanging their base $b$ digits at certain places in the intervals $Z_{j-1}$ to guarantee $s_j > q_j^{1-\varepsilon}$ as in (55) for the rationals analogously obtained from the twisted numbers. We claim that a suitable choice of digit positions is possible. This can be seen by first observing that iterating the argument for given $\varepsilon > 0$, we actually find many positions in $Z_{j-1}$ where $\xi$ takes non-zero digits. Then considering variations of exchanging digits at some of these places while keeping others, we finally see that some arising fractions must be almost reduced again. We prefer to omit an exhaustive exposition of the slightly technical details. Moreover, the $x_i$ still sum up to $\xi$. Furthermore, the digit changes have only minor effect on the approximation quality $|x_i - r_j/s_j|$ since $|Z_{j-1}|$ is small compared to $|I_{j-1}|$. Finally the proof of Case 2 also remains essentially unaffected by these digital changes.

When $0 \in W$, the implication (25) follows from the observation that the right hand side in (24) still has full Hausdorff dimension $\dim(K)$. The latter claim is a consequence of the facts that almost all numbers in $C_{b,W}$ with respect to its natural Cantor measure (restricted Hausdorff measure of dimension $\log |W|/\log b$) satisfy both $\mu(\xi) = 2$, proved by Weiss [38], and $\theta_b(\xi) = 1$, shown by Levesley, Salp and Velani [24, Corollary 1]. The intersection clearly shares the same property. The general case of (25) follows from the special case and Proposition 1.2 via the rational shift map $x \to x - \min W/(b - 1)$ that preserves the irrationality exponent $\mu$ and maps $C_{b,W}$ to a missing digit Cantor set $C_{b,\tilde{W}}$ with $0 \in \tilde{W}$. For the last implication (26), very similar to Corollary 6 we let $\lambda_0 = \infty$, $\lambda_1 = \lambda$ in (25) and apply Theorem 1.1 and (13).

\[ \square \]

Due to the twist in the proof, we lose some flexibility regarding the order of approximation compared to Theorem 3.4. In particular, we cannot guarantee an analogue of Theorem 3.6 for Cantor sets. From [24, Corollary 1], some
weaker generalisation of Theorem 3.4 can still be deduced, however we prefer not to state it here.

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