MINIMAL FAITHFUL REPRESENTATIONS OF THE FREE
2-STEP NILPOTENT LIE ALGEBRA OF THE RANK \( r \)

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Abstract. Given a finite dimensional Lie algebra \( g \), let \( z(g) \) denote the center of \( g \) and let \( \mu(g) \) be the minimal possible dimension for a faithful representation of \( g \). In this paper we obtain \( \mu(L_{r,2}) \), where \( L_{r,k} \) is the free \( k \)-step nilpotent Lie algebra of rank \( r \). In particular we prove that

\[
\mu(L_{r,2}) = \left\lceil \sqrt{2r(r-1)} \right\rceil + 2 \quad \text{for} \quad r \geq 4.
\]

It turns out that \( \mu(L_{r,2}) \sim \mu(z(L_{r,2})) \sim 2\sqrt{\dim L_{r,2}} \) (as \( r \to \infty \)) and we present some evidence that this could be true for \( L_{r,k} \) for any \( k \), this is considerably lower than the known bounds for \( \mu(L_{r,k}) \), which are (for fixed \( k \)) polynomial in \( \dim L_{r,k} \).

1. Introduction and main results

We fix throughout a field \( K \) of characteristic zero and all vector spaces considered in this paper are assumed to be finite dimensional over \( K \).

Ado’s Theorem states that for any Lie algebra \( g \) there exists a faithful (finite dimensional) representation of \( g \). Even though there are different proofs of Ado’s Theorem (see for instance [5, 12, 16, 27]), they do not usually yield faithful representations having low dimension compared to \( \dim g \), and obtaining efficient algorithms for producing faithful representations of small dimension is an active field of research, see for instance [7, 12].

The situation is similar for polycyclic groups, and in particular for finitely generated torsion-free nilpotent groups (\( \tau \)-groups): the works of Auslander [1], and Jennings for \( \tau \)-groups [15], show that these groups can be embedded into some group of matrices over the integers, but as in the case of Lie algebras, it is difficult to provide embeddings of low dimension (compared to the Hirsch length of the given group). Obtaining algorithms towards this end is an important problem (see [13, 18, 21]) and, in fact, the interest in low dimensional faithful representations also applies to other type of groups and algebras (see for instance [2, 17, 26]). This problem for \( \tau \)-groups is very closely related to that for nilpotent Lie algebras, mainly by the exp and log maps, and many ideas are borrowed from each other (see [13]). A classical reference for this is the book of Segal [25].

The interest in low dimensional faithful representations has many motivations. For instance, there are very few classes of groups for which the isomorphism problem is known to be solvable and it is acknowledged as a remarkable case the solution obtained by Grunewald and Segal [14] for

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the class of $\tau$-groups, which depend of having faithful representations of the groups. It is clear here the importance of having algorithms that provide low dimensional faithful representations. For Lie algebras, in addition to the connection already mentioned with groups, Milnor and Auslander related the problem of determining whether a given group is the fundamental group of a compact complete affinely-flat manifold with that of finding Lie algebras $\mathfrak{g}$ admitting faithful representations of dimension less than or equal to $\dim \mathfrak{g}$. These Lie algebras, in turn, yield Lie groups admitting a left-invariant affine structures. Many details about this can be found in [10].

This leads to consider, for a given Lie algebra $\mathfrak{g}$, the invariant

$$
\mu(\mathfrak{g}) = \min \{ \dim V : (\pi, V) \text{ is a faithful representation of } \mathfrak{g} \}
$$

which is, in general, very difficult to compute. Only for a few families of Lie algebra the value of $\mu$ is known, for example: semisimple [8], nilpotent of dimension less than or equal to 6 [21], direct sum of abelian plus Heisenberg Lie algebras [23, 24], current Heisenberg Lie algebras [10].

While it is known that $\mu(\mathfrak{g}) \leq (c+\dim \mathfrak{g})$ for a $c$-step nilpotent Lie algebra $\mathfrak{g}$ (see [12]), as far as we know all the evidence indicates that $\mu(\mathfrak{g}) \leq K \dim \mathfrak{g}$ for some constant $K$: to the best of our knowledge, it is not known whether there is a family of Lie algebras $\mathfrak{g}_n$ such that $\dim \mathfrak{g}_n \to \infty$ and $\mu(\mathfrak{g}_n) = O(\dim \mathfrak{g}_n)$, $c > 1$; and it is not known whether $\mu(\mathfrak{g})$ is bounded above by a polynomial in $\dim \mathfrak{g}$.

In this paper we consider the free 2-step nilpotent Lie algebra of rank $r$ $\mathcal{L}_{r,2} = \mathbb{K}^r \oplus \bigwedge^2 \mathbb{K}^r$ and we prove the following theorem.

**Theorem.** Let $\mathcal{L}_{r,2}$ be the free 2-step nilpotent Lie algebra of rank $r$. Then

$$
\mu(\mathcal{L}_{r,2}) = \begin{cases} 
\left\lceil \sqrt{2r(r-1)} \right\rceil + 2, & \text{if } r \geq 4; \\
2r - 1, & \text{if } r = 2, 3, 4, 5.
\end{cases}
$$

That is

| $r$   | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\mu(\mathcal{L}_{r,2})$ | 3  | 5  | 7  | 9  | 10 | 12 | 13 | 14 | 15 | 16 | 17 | 19 | 20 | 22 | 23 | 24 |

This theorem says that

$$
\mu(\mathcal{L}_{r,2}) \sim \sqrt{2r} \\
\sim 2\sqrt{\dim \mathcal{L}_{r,2}} \\
\sim \mu(\mathfrak{z}(\mathcal{L}_{r,2})),
$$

where $\mathfrak{z}(\mathcal{L}_{r,2}) = \bigwedge^2 \mathbb{K}^r$ is the center of $\mathcal{L}_{r,2}$.

The value $\mu(\mathcal{L}_{r,2})$ is surprisingly low for us, note that it follows at once that $\mu(\mathcal{L}_{r,2}) \geq \mu(\mathfrak{z}(\mathcal{L}_{r,2})) = \left\lceil \sqrt{2r(r-1)} - 1 \right\rceil$ (any faithful representation of an abelian Lie algebra of dimension $n$ has dimension greater than or equal to $\lceil 2\sqrt{n-1} \rceil$, see [24]). Consequently, it was a very hard task for us proving that there actually existed faithful representations having the necessary dimension.
1.1. Some words about the proof and some evidence about higher nilpotency degrees. It is natural to look for faithful matrix representations

$$\pi : \mathcal{L}_{r,2} \to \mathfrak{gl}(a + p + b, \mathbb{K}),$$

for some $a, p, b \in \mathbb{N}$, so that

$$\pi(\mathcal{L}_{r,2}) \subset \begin{bmatrix} 0 & * & * \\ a & 0 & b \\ p & 0 & 0 \end{bmatrix} \quad \pi(\wedge^2 \mathbb{K}^r) \subset \begin{bmatrix} 0 & 0 & * \\ a & 0 & 0 \\ p & 0 & 0 \end{bmatrix}.$$

(1.1)

In this case, it is necessary that $ab \geq r(r - 1)/2 = \dim \wedge^2 \mathbb{K}^r$. We prove that given $a, b \in \mathbb{N}$ such that $a + b$ takes the minimal possible value subject to $ab \geq r(r - 1)/2$, such representation $\pi$ is possible for $p = 2$ (and impossible for $p = 1$). In fact, it turns out that given $a, b \in \mathbb{N}$, with $ab \geq r(r - 1)/2$, then any random injective map $\pi_0 : \mathbb{K}^r \to \text{Hom}(\mathbb{K}^2, \mathbb{K}^a) \oplus \text{Hom}(\mathbb{K}^b, \mathbb{K}^2)$ extends to a faithful representation $\pi : \mathcal{L}_{r,2} \to \text{End}(\mathbb{K}^a \oplus \mathbb{K}^2 \oplus \mathbb{K}^c)$ as in (1.1). We think that this random property of minimal representations could provide a new perspective for constructing low-dimensional representations. In particular, it seems to us that this is a general pattern for the free $k$-step nilpotent Lie algebra on $r$ generators $\mathcal{L}_{r,k}$. More precisely, we think that the following claim is true: given $a_0, a_k \in \mathbb{N}$ such that $a_0a_k \geq \dim \mathfrak{z}(\mathcal{L}_{r,k})$, then there are $a_1, \ldots, a_{k-1} \in \mathbb{N}$, very low compared to $\max\{a_0, a_k\}$, such that any random injective map $\pi_0 : \mathbb{K}^r \to \bigoplus_{i=1}^{k} \text{Hom}(\mathbb{K}^{a_i}, \mathbb{K}^{a_{i-1}})$ extends to a faithful representation $\pi : \mathcal{L}_{r,k} \to \text{End}(\bigoplus_{i=0}^{k} \mathbb{K}^{a_i})$. Here, ‘very low compared to’ means that, eventually, $\mu(\mathcal{L}_{r,k}) \sim \mu(\mathfrak{z}(\mathcal{L}_{r,k})) \sim 2\sqrt{r^2}$. This is considerably lower than the upper bound $\mu(n) \leq (\dim n)^k + 1$ for a $k$-step nilpotent Lie algebra $n$ (see [22]). Note also that the Lo and Ostheimer algorithm produces a representation of dimension $1 + r + r^2 + \cdots + r^k$ for the free nilpotent group of rank $r$ and class $k$ (see [18], Prop. 6.1). For low ranks, our computer experiments show that

$$r = 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$$

$$\mu(\mathcal{L}_{r,3}) \leq 6 \quad 9 \quad 14 \quad 18 \quad 22 \quad 27 \quad 32 \quad 37 \quad 43$$

$$\text{2nd row} \quad \frac{2\sqrt{r^3/3}}{2} = 1.84 \quad 1.50 \quad 1.52 \quad 1.40 \quad 1.30 \quad 1.26 \quad 1.23 \quad 1.18 \quad 1.18$$

$$\mu(\mathcal{L}_{r,4}) \leq 8 \quad 15 \quad 23 \quad 34 \quad 47 \quad 62 \quad 79 \quad 101 \quad 122$$

$$\text{4th row} \quad \frac{4\sqrt{r^4/4}}{2} = 2.00 \quad 1.67 \quad 1.44 \quad 1.36 \quad 1.30 \quad 1.26 \quad 1.24 \quad 1.24 \quad 1.22$$

The paper is organized as follows. In §2 we give some basic results about $\mathcal{L}_{r,2}$. In §3 we prove the lower bound for $\mu(\mathcal{L}_{r,2})$ for all $r \in \mathbb{N}$. Our proof is basically by induction and requires some previous technical results. Even though the lower bound is so close to $\mu(\mathfrak{z}(\mathcal{L}_{r,2}))$ our proof turns out to be laborious and even the basic case $r = 5$ in the induction argument requires
a considerable amount of work. Finally, in §4 we prove that the proposed value for \( \mu(L_{r,2}) \) is actually attained by a difficult existence argument: we would be very interested in an explicit map describing a representation of \( L_{r,2} \) of dimension \( \mu(L_{r,2}) \).

2. Preliminaries

Let \( g \) be a Lie algebra and let \( V \) be a vector space. A representation \((\pi, V)\) of \( g \) on \( V \) is a Lie homomorphism \( \pi : g \rightarrow gl(V) \) and we say that
\[\begin{align*}
(1) & \quad (\pi, V) \text{ is faithful if } \pi \text{ is an injective,} \\
(2) & \quad (\pi, V) \text{ is a nilrepresentation if } \pi(X) \text{ is a nilpotent endomorphism for all } X \in g.
\end{align*}\]

Let \( \mu_{\text{nil}}(g) = \min \{ \dim V : (\pi, V) \text{ is a faithful nilrepresentation of } g \} \).

We know that if \( g \) is nilpotent and the center \( z(g) \) of \( g \) is contained in \( [g, g] \), then
\[
\mu_{\text{nil}}(g) = \mu(g) \tag{2.1}
\]
see [10, Theorem 2.4].

Given \( r \in \mathbb{N} \), the free 2-step nilpotent Lie algebra of rank \( r \) is the vector space \( L_{r,2} = \mathbb{K}^r \oplus \bigwedge^2 \mathbb{K}^r \) equipped with the Lie algebra structure
\[
[X, Y] = X \wedge Y, \quad X, Y \in \mathbb{K}^r.
\]
For example, \( L_{2,2} \) is the Heisenberg Lie algebra of dimension 3.

The Lie algebra \( L_{r,2} \) has dimension \( r + \frac{r(r-1)}{2} \) and possesses the following universal property: if \( h \) is a Lie algebra and \( f : \mathbb{K}^r \rightarrow h \) is a linear map satisfying
\[
[f(\mathbb{K}^r), [f(\mathbb{K}^r), f(\mathbb{K}^r)]] = 0,
\]
then there is a unique extension \( \bar{f} : L_{r,2} \rightarrow h \) of \( f \) to a homomorphism of Lie algebras.

Since \( \mathfrak{z}(L_{r,2}) = [L_{r,2}, L_{r,2}] \), it follows from (2.1) that \( \mu_{\text{nil}}(L_{r,2}) = \mu(L_{r,2}) \).

Some additional properties of \( L_{r,2} \) are stated in the following proposition.

**Proposition 2.1.** If \( a \) is a proper Lie subalgebra of \( L_{r,2} \) then \( a = a_1 \oplus a_2 \) with \( a_1 \) and \( a_2 \) subalgebras, \( a_2 \subset \bigwedge^2 \mathbb{K}^r \), such that \( a_1 \) is either zero, 1-dimensional or isomorphic to a free 2-step nilpotent Lie algebra of rank less than \( r \). In particular, if the center of \( a \) is not contained in \( \bigwedge^2 \mathbb{K}^r \), then \( a \) is abelian and \( \dim \left( a/(a \cap \bigwedge^2 \mathbb{K}^r) \right) = 1 \).

**Proof.** Let \( \{A_1, \ldots, A_k\} \) be a basis of a linear complement of \( a \cap \bigwedge^2 \mathbb{K}^r \) in \( a \). If \( a_1 \) is the Lie subalgebra of \( a \) generated by \( \{A_1, \ldots, A_k\} \), then \( a_1 \simeq L_{k,2} \) and, since \( a \) is proper, \( k < r \). If \( a_2 \) is a linear complement of \( a_1 \) in \( a \) we have \( a = a_1 \oplus a_2 \) and \( a_2 \subset \bigwedge^2 \mathbb{K}^r \). If the center of \( a \) is not contained in \( \bigwedge^2 \mathbb{K}^r \), then \( k = 1 \), for if \( k \geq 2 \) (or \( k = 0 \)) then the center of \( a \) is \( [a_1, a_1] \cap a_2 \subset \bigwedge^2 \mathbb{K}^r \). \( \square \)
We know that $\mathcal{L}_{r,2}$ is the nilradical of a parabolic subalgebra of the semisimple Lie algebra of rank $r$ of type $B$, where $\mathbb{K}^r$ corresponds to the set of short positive roots $\epsilon_i$, $i = 1, \ldots, r$, and $\wedge^2 \mathbb{K}^r$ corresponds to the set of positive roots $\epsilon_i + \epsilon_j$. This provides a standard faithful representation $\pi_0 : \mathcal{L}_{r,2} \to \mathfrak{gl}(2r+1, \mathbb{K})$ of dimension $2r+1$. More precisely, if $\{X_1, \ldots, X_r\}$ is a basis of $\mathbb{K}^r$ and, for $1 \leq i < j \leq r$, $Z_{ij} = [X_i, X_j]$, then

$$\pi_0 \left( \sum_{i=1}^{r} x_i X_i + \sum_{i<j}^{r} z_{ij} Z_{ij} \right) = \begin{bmatrix} 0 & z_{12} & \cdots & z_{1r} & 0 \\ z_{21} & 0 & \cdots & 0 & -z_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{r1} & 0 & \cdots & 0 & -z_{1r} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{for all } x_i, x_j, z_{ij} \in \mathbb{K}.$$  

This shows that $\mu(\mathcal{L}_{r,2}) \leq 2r + 1$.

**Definition 2.2.** Let $a, p, b$ be natural numbers. We say that a representation of $\pi : \mathcal{L}_{r,2} \to \mathfrak{gl}(V)$ is of type $(a, p, b)$ if there is a basis $B = \{u_1, \ldots, u_a, v_1, \ldots, v_p, w_1, \ldots, w_b\}$ of $V$ such that the corresponding matrix representation $\pi_B$ associated to $\pi$ satisfies

$$\pi_B(X) = \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for all } X \in \mathcal{L}_{r,2}.$$  

(2.2)

**Remark 2.3.** The standard representation $\pi_0$ is a faithful representation of type $(r, 1, r)$, but it is not of least dimension among all faithful representations of type $(a, 1, b)$. Indeed, let $\pi_1$ be the representation defined, for $X = \sum_{i=1}^{r} x_i X_i + \sum_{i<j}^{r} z_{ij} Z_{ij}$, by

$$\pi_1(X) = \begin{bmatrix} 0 & z_{12} & \cdots & z_{1r} & 0 \\ z_{21} & z_{22} & \cdots & 0 & -z_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{r1} & 0 & \cdots & 0 & -z_{1r} \\ 0 & x_r & \cdots & x_2 & x_1 \end{bmatrix} \quad \text{for all } x_i, x_j, z_{ij} \in \mathbb{K}.$$
It is not difficult to see that \((\pi_1, \mathbb{K}^{2r-1})\) is faithful of type \((r - 1, 1, r - 1)\). We will show next that \(2r - 1\) is the least possible dimension for a faithful representations of \(L_{r,2}\) of type \((a, 1, b)\).

**Proposition 2.4.** Let \(a, p, b \in \mathbb{N}\) and let \(\pi : L_{r,2} \to \mathfrak{gl}(V)\) be a faithful representation of type \((a, p, b)\). Then

\[
r \leq p \min(a, b) + 1 \quad \text{and} \quad \frac{r(r - 1)}{2} \leq ab.
\]

In particular, if \(p = 1\) then the minimal possible dimension of \(V\) is \(2r - 1\).

**Proof.** Let \(B\) be a basis of \(V\) as in Definition 2.2 and let \(\pi_B\) be the corresponding matrix representation. It is clear that for any \(Z \in [L_{r,2}, L_{r,2}]\), the non-zero entries of \(\pi_B(Z)\) are contained in the upper-right block of (2.2). Since \(\pi\) is faithful and \(\dim[L_{r,2}, L_{r,2}] = \frac{r(r - 1)}{2}\), it follows that \(\frac{r(r - 1)}{2} \leq ab\).

In order to prove that \(r \leq p \min(a, b) + 1\) let us fix a set \(\{X_1, \ldots, X_r\}\) of generators of \(L_{r,2}\) and assume \(b \geq a\). Let

\[
M = \text{span}_\mathbb{K}\{\pi_B(X_1), \ldots, \pi_B(X_r)\}.
\]

If \(r \geq pa + 2\), after a Gaussian elimination process, we may find a basis \(\{M_1, \ldots, M_r\}\) of \(M\) such that \(M_{r - 1}\) and \(M_r\) have the following block structure

\[
\begin{array}{ccc}
0 & 0 & * \\
0 & 0 & * \\
0 & 0 & 0
\end{array}
\]

Hence \([M_{r - 1}, M_n] = 0\) and this contradicts the fact that \(\pi\) is faithful. Finally, if \(p = 1\) then \(\dim V = a + b + 1 \geq 2 \min(a, b) + 1 \geq 2r - 1. \)

\(\square\)

### 3. The lower bound for \(\mu(L_{r,2})\)

In this section we will prove that

\[
(3.1) \quad \mu(L_{r,2}) \geq \begin{cases} 
\sqrt{2r(r - 1)} + 2, & \text{if } r \geq 4; \\
2r - 1, & \text{if } r = 2, 3, 4, 5.
\end{cases}
\]

The cases \(r = 2, 3\) are well known. On the one hand, \(L_{2,2}\) is a Heisenberg Lie algebra and \(\mu(L_{2,2}) = 3\) follows from [3]. On the other hand, \(\dim L_{3,2} = 6\) and \(L_{3,2}\) is isomorphic to the Lie algebra labeled as \(L_{6,26}\) in the paper [24], and it follows that \(\mu(L_{3,2}) = 5\).

For the rest of this section we assume \(r \geq 4\). The main tool used in the proof of (3.1) is the following result which is a particular instance of [11, Theorem 2.3].
Theorem 3.1. Let $V$ be a vector space and let $0 \neq n_2 \subset n_1$ be a chain of vector subspaces of $\text{End}(V)$. Then there exist natural numbers $s_1 \geq s_2 > 0$, a linearly independent set $\{v_1, \ldots, v_{s_1}\} \subset V$ and a family of non-zero subspaces $n_{k,j} \subset \text{End}(V)$, $1 \leq j \leq s_k$ and $k = 1, 2$, such that:

$$
n_1 = n_{1,1} \oplus n_{1,2} \oplus \cdots \oplus n_{1,s_2} \oplus \cdots \oplus n_{1,s_1}
$$

and

$$(1) \ A \in n_{1,j} \text{ and } Av_j = 0 \implies A = 0 \text{ for } j = 1, \ldots, s_1.$$

$$(2) \ n_{1,j}v_i = 0 \text{ for } 1 \leq i < j \leq s_1.$$

$$(3) \ n_{k,j}V \subseteq n_{k,i}v_i \text{ for } 1 \leq i < j \leq s_k \text{ and } k = 1, 2.$$

$$(4) \text{ If in addition } n_1 \text{ consists of nilpotent operators and } [n_1, n_2] = 0 \text{ then } n_{1,1}v_1 \cap \text{span}_k\{v_1, \ldots, v_{s_2}\} = 0.$$

Remark 3.2. It follows from (1) that $\dim n_{k,j} = \dim(n_{k,j}v_j)$ for $1 \leq j \leq s_k$ and $k = 1, 2$. This, combined with (3), implies $\dim n_{k,j} \leq \dim(n_{k,i})$ for $1 \leq i < j \leq s_k$ and $k = 1, 2$.

Remark 3.3. This theorem is useful for us since it allows to argue inductively as follows. In the context of Theorem 3.1, let us assume that $n_1$ is a Lie subalgebra of $\text{gl}(V)$. For any $2 \leq j_0 \leq s_2$, let

$$\bar{n} = \bigoplus_{j=j_0}^{s_1} n_{1,j} \quad \text{and} \quad V' = \text{span}_k\{n_{1,1}v_1 \cup \{v_{j_0}, \ldots, v_{s_2}\}\}.$$

We claim that

(i) $\bar{n}$ is a Lie subalgebra of $n_1$ and it preserves $V'$.

(ii) If $\bar{n}$ is nilpotent with center $\bar{j}(\bar{n})$ contained in $n_2$ then $\bar{n}$ acts on $V'$ faithfully.

First we prove (i). Since $j_0 \geq 2$, item (3) implies that $V'$ is preserved by $\bar{n}$. Let us show that $\bar{n}$ is a Lie subalgebra of $n_1$. Given $X, Y \in \bar{n}$, since $n_1$ is a Lie subalgebra of $\text{gl}(V)$ we have

$$[X, Y] = \sum_{i=1}^{s_1} X_i \quad \text{with} \quad X_i \in n_{1,i}.$$

We must show that $X_i = 0$ for all $i < j_0$. Let $j_1$ be the least $i < j_0$ with $X_{j_1} \neq 0$ (if there is any). Then

$$0 = XY(v_{j_1}) - YX(v_{j_1}) \quad \text{by definition of } \bar{n} \text{ and } (2)$$

$$= [X, Y](v_{j_1})$$

$$= \sum_{i=1}^{s_1} X_i(v_{j_1}) \quad X_i \in n_{1,i}$$

$$= X_{j_1}(v_{j_1}) \quad \text{by } (2)$$

which is a contradiction to (4).

We now prove (ii). Since $\bar{n}$ is nilpotent it suffices to see that its center acts faithfully on $V'$. It follows from item (1) that $\bigoplus_{j=j_0}^{s_2} n_{2,j}$ acts faithfully
on $V'$ since $\{v_{j_0}, \ldots, v_{s_2}\} \subset V'$. Therefore, $\mathfrak{z}(\mathfrak{n}) \subset \mathfrak{n}_2 \cap \mathfrak{n}$ acts faithfully on $V'$. This concludes the remark.

Let us fix a faithful nilrepresentation $(\pi, V)$ of $\mathcal{L}_{r,2}$. Let

$$\mathfrak{n}_1 = \pi(\mathcal{L}_{r,2}) \quad \text{and} \quad \mathfrak{n}_2 = \pi(\mathfrak{z}(\mathcal{L}_{r,2})).$$

It is clear that $\mathfrak{n}_1$ is a nilpotent Lie subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms which is isomorphic to $\mathcal{L}_{r,2}$. Thus

$$\dim \mathfrak{n}_1 = r + \frac{r(r-1)}{2} \quad \text{and} \quad \dim \mathfrak{n}_2 = \frac{r(r-1)}{2}.$$

Now the subspaces $\mathfrak{n}_{k,i}$ obtained by applying Theorem 3.1 to the chain $\mathfrak{n}_2 \subset \mathfrak{n}_1$ in this particular case have some additional properties.

**Proposition 3.4.** Let $(\pi, V)$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ as above, and let $s_1 \geq s_2 > 0$, $\{v_1, \ldots, v_{s_1}\} \subset V$, and $\mathfrak{n}_{k,j} \subset \text{End}(V)$ ($1 \leq j \leq s_k$, $k = 1,2$), be the output obtained by applying Theorem 3.1 to the chain $\mathfrak{n}_2 \subset \mathfrak{n}_1$. Then

$$\dim (\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) \leq \dim (\mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2}) + r - 1.$$

In the case

$$\dim (\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) = \dim (\mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2}) + r - 1,$$

there is a Lie subalgebra $\mathfrak{m} \subset \mathcal{L}_{r,2}$, with $\mathfrak{m} \simeq \mathcal{L}_{r-1,2}$, such that $(\pi, V)$ contains a faithful subrepresentation of $\mathfrak{m}$ of type $(\dim \mathfrak{n}_{1,1} - 1,1,s_2-1)$. In particular

$$\dim V \geq \begin{cases} 2r - 2 & \text{for } r \geq 6 \\ 2r - 1, & \text{for } r = 4, 5, \end{cases}$$

and thus (5.1) holds for $r = 4, 5$.

**Proof.** It is clear that

$$\dim (\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) - \dim (\mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2})$$

$$< \dim (\mathfrak{n}_{1,1} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) - \dim (\mathfrak{n}_{2,1} \oplus \cdots \oplus \mathfrak{n}_{2,s_2})$$

$$= r$$

and thus we have the first part of the proposition.

We now assume that

$$\dim (\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) = \dim (\mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2}) + r - 1,$$

and in particular $s_1 \geq 2$. Also $s_2 \geq 2$, otherwise $s_2 = 1$ and it follows, from Remark 3.3, that $\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}$ is an abelian Lie subalgebra with $(\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) \cap \mathfrak{n}_2 = 0$. Since $\mathfrak{n}_1 \simeq \mathcal{L}_{r,2}$, it follows (see Proposition 2.1) that $\dim (\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}) = 1$ and hence $r = 2$ a contradiction (recall that $r \geq 4$).

From now on we assume $s_1, s_2 \geq 2$. Let $\mathfrak{m}_0$ be a linear complement of $\mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2}$ in $\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1}$, that is

$$\mathfrak{n}_{1,2} \oplus \cdots \oplus \mathfrak{n}_{1,s_1} = \mathfrak{m}_0 \oplus \mathfrak{n}_{2,2} \oplus \cdots \oplus \mathfrak{n}_{2,s_2},$$

and let $\mathfrak{m}$ be the Lie subalgebra of $\mathfrak{n}_1$ generated $\mathfrak{m}_0$. Since $\dim \mathfrak{m}_0 = r - 1$, Proposition 2.1 implies $\mathfrak{m} \simeq \mathcal{L}_{r-1,2}$ and, it follows from Theorem 3.1 item (5) that

$$\dim \mathfrak{m} V \subset \mathfrak{n}_{1,1} v_1.$$
From Remark 3.3 we know that $n_{1,2} + \cdots + n_{1,s_1}$ is a Lie algebra, containing $m \simeq \mathcal{L}_{r-1,2}$, acting faithfully on

$$V' = \text{span}_k \left( n_{1,1}v_1 \cup \{v_2, \ldots, v_{s_2} \} \right).$$

We claim that the faithful representation of $\mathcal{L}_{r-1,2}$ given by the action of $m$ on $V'$ is of type $(n_{1,1} - 1, 1, s_2 - 1)$.

Let $n_{i,1} = \dim n_{i,1}$, $i = 1, 2$. It follows from (3.2) that $n_{1,1} = 1 + n_{2,1}$. Let $\{A_1, \ldots, A_{n_{2,1}}, A_{n_{1,1}} \}$ be a basis of $n_{1,1}$ with $A_i \in n_{2,1}$ for $i = 1, \ldots, n_{2,1}$. Let

$$B_1 = \{A_1v_1, \ldots, A_{n_{2,1}}v_1 \},$$

$$B_2 = \{A_{n_{1,1}}v_1 \},$$

$$B_3 = \{v_2, \ldots, v_{s_2} \},$$

and let $V'_i = \text{span}_k (B_i)$, $i = 1, 2, 3$. It follows from Theorem 3.1 item (1) and (3) that $B_1 \cup B_2 \cup B_3$ is linearly independent and thus

$$V' = V'_1 \oplus V'_2 \oplus V'_3.$$ 

In order to show that this representation of $\mathcal{L}_{r-1,2}$, we must show that, given $A \in m$, we have

(i) $Av \in \text{span}_k (B_1 \cup B_2)$ for all $v \in V'$,

(ii) $AA_{n_{1,1}}v_1 \in \text{span}_k (B_1)$, and

(iii) $Av = 0$ for all $v \in \text{span}_k (B_1)$.

Property (i) follows from (3.3).

Property (ii) is obtained as follows

$$AA_{n_{1,1}}v_1 = A_{n_{1,1}}Av_1 + [A, A_{n_{1,1}}]v_1$$

$$= [A, A_{n_{1,1}}]v_1 \quad \text{by Theorem 3.1 item (2)}$$

$$\in n_{2}v_1$$

$$\subseteq n_{2,1}v_1 = V'_1 \quad \text{by Theorem 3.1 item (2)}.$$

Finally, Theorem 3.1 item (2) implies $AA_jv_1 = A_jAv_1 = 0$ for $j = 1, \ldots, n_{2,1}$, and this proves (iii).

This shows that the action of $m$ on $V'$ corresponds to a faithful representation of $\mathcal{L}_{r-1,2}$ of type $(n_{1,1} - 1, 1, s_2 - 1)$, and hence, by Proposition 2.4 we obtain that $\dim V' \geq 2r - 3$. But $\dim V > \dim V'$ since $v_1 \notin V'$ (see Theorem 3.1 item (3)), and hence $\dim V > 2r - 2$. This completes the proof for $r \geq 6$.

Now assume $r = 5$, in this case, $m \simeq \mathcal{L}_{4,2}$, we have proved $\dim V \geq 8$, $\dim V' \geq 7$ and we must prove that $\dim V \geq 9$.

Assume, if possible that $\dim V = 8$. This implies

$$\dim V = \dim V' + 1, \quad \dim V' = n_{1,1} + s_2 - 1 = 7.$$

In particular $B = B_1 \cup B_2 \cup B_3 \cup \{v_1\}$ is a basis of $V$.

Since $V'$ is a faithful representation of type $(n_{1,1} - 1, 1, s_2 - 1)$ of $m$, Proposition 2.4 implies

$$(n_{1,1} - 1)(s_2 - 1) \geq 6 \quad \text{and} \quad n_{1,1} - 1, s_2 - 1 \geq 3.$$
Thus, the only possibility is \( n_{1,1} = s_2 = 4 \). That is, both \( B_1 \) and \( B_3 \), have 3 elements, that is

\[
B = \{ A_1 v_1, A_2 v_1, A_3 v_1 \} \cup \{ A_4 v_1 \} \cup \{ v_2, v_3, v_4 \} \cup \{ v_1 \}.
\]

Let \( \{ X_1, \ldots, X_5 \} \) be any set of generators of \( n_1 \) with \( \{ X_1, \ldots, X_4 \} \subset m \) and \( X_5 = A_4 \). If we denote by \( \tilde{X} \) the matrix corresponding to the action of \( X \) on \( V \) associated to the basis \( B \), we know that

\[
\begin{pmatrix}
\tilde{X}_1, \ldots, \tilde{X}_4 &=& \begin{pmatrix}
0 & 0 & 0 & 0 & e_1 \\
0 & 0 & 0 & 0 & e_2 \\
0 & 0 & 0 & 0 & e_3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\tilde{X}_5 &=& \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

By definition of \( B_1 \), we know that \( \text{span}_K \{ X_j A_1 v_1 : j = 1, \ldots, 4 \} = \text{span}_K (B_1) \). Therefore, after a Gaussian elimination process, we can redefine \( \{ X_1, \ldots, X_5 \} \) so that (we do not change \( B_1, B_3 \), but we may need to permute \( B_3 \))

\[
\tilde{X}_i = \begin{pmatrix}
0 & 0 & 0 & e_i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \ i = 1, 2, 3,
\]

(here \( \{ e_1, e_2, e_3 \} \) are the canonical vectors of \( K^3 \)) and

\[
\begin{pmatrix}
\tilde{X}_4 &=& \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_1 & 0 \\
0 & 0 & 0 & e_2 & 0 \\
0 & 0 & 0 & e_3 & 0
\end{pmatrix}, \\
\tilde{X}_5 &=& \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Moreover:

1. Replacing \( X_k \) by \( X_k + t_1 [X_1, X_4] + t_2 [X_2, X_4] + t_3 [X_3, X_4] \), for some appropriate \( t_1, t_2, t_3 \in K \), we can assume, for all \( k \), that

\[
(\tilde{X}_k)_{1,5} = (\tilde{X}_k)_{2,5} = (\tilde{X}_k)_{3,5} = 0.
\]

2. Replacing \( v_3 \) by \( v_3 + t_1 v_2 \) and \( v_4 \) by \( v_4 + t_2 v_2 \) for some appropriate \( t_1, t_2 \in K \), we can assume, without changing the properties already obtained, that

\[
(\tilde{X}_4)_{4,6} = (\tilde{X}_4)_{4,7} = 0.
\]

3. Since \( \{ [X_1, X_2], [X_1, X_3], [X_2, X_3] \} \) is linearly independent, it is necessary that these three 2-coordinates vectors

\[
(\tilde{X}_1)_{4,6}, (\tilde{X}_1)_{4,7}, (\tilde{X}_2)_{4,6}, (\tilde{X}_2)_{4,7}, (\tilde{X}_3)_{4,6}, (\tilde{X}_3)_{4,7}
\]

span a 2-dimensional space. We may assume that the first two vectors do (this may require to permute \( B_1 \) in order to keep property (3.4)). In this case, replacing \( v_3 \) by \( (\tilde{X}_1)_{4,6} v_3 + (\tilde{X}_2)_{4,6} v_4 \) and \( v_4 \) by \( (\tilde{X}_1)_{4,7} v_3 + (\tilde{X}_3)_{4,7} v_4 \), we may assume

\[
(\tilde{X}_1)_{4,6}, (\tilde{X}_1)_{4,7} = (1, 0) \quad \text{and} \quad (\tilde{X}_2)_{4,6}, (\tilde{X}_2)_{4,7} = (0, 1).
\]
(4) Since $\tilde{X}_5$ is nilpotent, the equation $[[\tilde{X}_k, \tilde{X}_4], \tilde{X}_5] = 0$ for $k = 1, 2, 3$ implies that the first three columns and the 5th row of $\tilde{X}_5$ are zero. Similarly the equation $[[\tilde{X}_1, \tilde{X}_2], \tilde{X}_5] = 0$ implies that the 6th and 7th rows of $\tilde{X}_5$ are zero.

(5) The equation $[[\tilde{X}_1, \tilde{X}_5], \tilde{X}_1] = 0$ implies $(\tilde{X}_5)^4 = 0$ and the equation $[[\tilde{X}_1, \tilde{X}_5], \tilde{X}_5] = 0$ implies that the last row of $\tilde{X}_5$ is zero.

At this point we have $[X_4, X_5] = 0$, a contradiction, and thus dim $V \geq 9$.  

The argument is similar for $r = 4$ but easier. □

**Corollary 3.5.** If $(\pi, V)$ is a faithful nilrepresentation of $L_{r,2}$ and $r \geq 6$, then

$$\text{dim } V \geq \left\lceil \sqrt{2r(r-1)} \right\rceil + 2.$$  

**Proof.** Let $v_1 \in V$ as in Theorem 3.1 and let $\phi$ be the linear map $\phi : L_{r,2} \to V$, $\phi(X) = \pi(X)(v_1)$.

It is easy to check that

$$\ker \phi = n_{1,2} \oplus \cdots \oplus n_{1,s_1}$$

$$\ker \phi |_{\mathfrak{z}(L_{r,2})} = n_{2,2} \oplus \cdots \oplus n_{2,s_2}$$

$$\text{Im } \phi = n_{1,1} v_1$$

(3.5)  

It follows from Theorem 3.1 part (4) and (3.5)

$$\text{dim } V \geq \dim \text{Im } \phi + s_2.$$  

This implies

$$\text{dim } V + \dim \ker \phi \geq \dim L_{r,2} + s_2$$

and hence

(3.6)  

$$\dim V + \dim (n_{1,2} \oplus \cdots \oplus n_{1,s_1}) \geq \dim (n_{2,1} \oplus n_{2,2} \oplus \cdots \oplus n_{2,s_2}) + r + s_2.$$  

On the other hand, it follows from Proposition 3.4 that

(3.7)  

$$\dim (n_{1,2} \oplus \cdots \oplus n_{1,s_1}) \leq \dim (n_{2,2} \oplus \cdots \oplus n_{2,s_2}) + r - 1.$$  

We now consider two cases.

(A) If $\dim (n_{1,2} \oplus \cdots \oplus n_{1,s_1}) < \dim (n_{2,2} \oplus \cdots \oplus n_{2,s_2}) + r - 1$, then it follows from (3.6) and (3.7) that

(3.8)  

$$\dim V \geq \dim n_{2,1} + s_2 + 2.$$  

Since

$$\pi(\mathfrak{z}(L_{r,2})) = n_2 = \bigoplus_{j=1}^{s_2} n_{2,j}$$

and $\dim n_{2,j} \leq \dim n_{2,1}$ (see Remark 3.2) we have

$$s_2 \dim n_{2,1} \geq \frac{r(r-1)}{2}$$

and hence $\dim n_{2,1} + s_2 \geq \left\lceil \sqrt{\frac{r(r-1)}{2}} \right\rceil$. This, combined with (3.8), implies

$$\dim V \geq \left\lceil \sqrt{2r(r-1)} \right\rceil + 2.$$
(B) If \( \dim(n_{1,2} \oplus \cdots \oplus n_{1,s_1}) = \dim(n_{2,2} \oplus \cdots \oplus n_{2,s_2}) + r - 1 \), from Proposition 3.4

\[
\dim V \geq 2r - 2.
\]

Hence, if \( r \geq 6 \) we obtain

\[
\dim V \geq \left\lceil \sqrt{2r(r-1)} \right\rceil + 2.
\]

This completes the proof. \( \square \)

4. The Upper Bound for \( \mu(L_{r,2}) \)

Let \( n \in \mathbb{N} \) a fixed natural number. It is not difficult to see that the natural numbers \( a \geq b \) defined by

\[
a = \left\lceil \sqrt{n} \right\rceil \quad \text{and} \quad b = \begin{cases} a - 1, & \text{if } a(a-1) \geq n; \\ a, & \text{if } a(a-1) < n; \end{cases}
\]

satisfy

\[
ab \geq n
\]

\[
a + b = \left\lceil 2\sqrt{n} \right\rceil = \min\{c + d : c, d \in \mathbb{N} \text{ and } cd \geq n\}.
\]

We point out that \( a, b \) might not be the only pair satisfying (4.2), for instance if \( n = 26 \), then \( a, b = 6, 5 \) but \( a', b' = 7, 4 \) also work.

**Definition 4.1.** Given \( n \in \mathbb{N} \), we will say that the integer square roots of \( n \) are the numbers \( a \geq b \) defined in (4.1).

This section is devoted to prove the following theorem.

**Theorem 4.2.** Let \( r \in \mathbb{N} \) such that \( r \geq 2 \), and let \( a \geq b \) be the integer square roots of \( \binom{r}{2} \). Then there exists a faithful nilrepresentation of \( L_{r,2} \) of type \( (a, 2, b) \). In particular

\[
\mu(L_{r,2}) \leq a + b + 2 = \left\lceil \sqrt{2r(r-1)} \right\rceil + 2.
\]

The main idea to prove the above theorem is to show that, if \( a \geq b \) are the integer square roots of \( \binom{r}{2} \), then any “generic” assignment

\[
X_i \mapsto \begin{cases} 0 & \ast & 0 \\ 0 & 0 & \ast \\ 0 & 0 & 0 \end{cases}
\]

for \( i = 1, \ldots, r \), has the property that \( [X_i, X_j], 1 \leq i < j \leq r \) are mapped to a linearly independent set, and thus it produces a faithful representation of \( L_{r,2} \).
Indeed, since
\[ ab = \frac{n(n+1)}{2} + i_0 \text{ with } 0 \leq i_0 \leq n. \]
(we informally say that \( n \) and \( i_0 \) are the triangular representation of \( ab \)).

Let \( S_{a,b} \subset M_{a,2}^{n+1} \times M_{2,b}^n \) be the set of all of sequences of matrices
\[
A_1, \ldots, A_{i_0-1}, A_{i_0}, A_{i_0}', A_{i_0+1}, \ldots, A_n \in M_{a,2}
\]
\[
B_1, \ldots, B_n \in M_{2,b},
\]
such that the following products
\[
\begin{array}{c c c c c c c c c c c}
A_1B_1 \\
A_2B_1 & A_2B_2 \\
& \vdots & \ddots \\
A_{i_0}B_1 & A_{i_0}B_2 & \cdots & A_{i_0}B_{i_0} \\
A_{i_0}'B_1 & A_{i_0}'B_2 & \cdots & A_{i_0}'B_{i_0} \\
A_{i_0+1}B_1 & A_{i_0+1}B_2 & \cdots & A_{i_0+1}B_{i_0} & A_{i_0+1}B_{i_0+1} \\
\vdots & \vdots & \cdots & \vdots & \ddots \\
A_nB_1 & A_nB_2 & \cdots & A_nB_{i_0} & A_nB_{i_0} & \cdots & A_nB_n
\end{array}
\]
constitute a basis of \( M_{a,b} \). The question is whether \( S_{a,b} \) is not empty. This is partially answered in the following theorem.

**Theorem 4.3.** Let \( a, b \in \mathbb{N} \) and let \( n \) and \( i_0 \) be the triangular representation of \( ab \). Assume that \( a \) and \( b \) satisfy the following conditions:

1. \( a = b \) or \( a = b + 1 \) and
2. \( i_0 \leq b \) whenever \( a = b \),

then \( S_{a,b} \) is a non-empty Zariski open of \( M_{a,2}^{n+1} \times M_{2,b}^n \).

**Proof.** First we recall that the condition determining whether the set of matrices in (4.4) is linearly independent corresponds to showing that a given determinant is not zero. Therefore, we only need to show that \( S_{a,b} \neq \emptyset \).

It is not difficult to see that \( S_{a,b} \neq \emptyset \) when \( a, b \leq 4 \). Therefore, since \( a \geq b \), we can assume \( a \geq b \geq 4 \).

We start the proof by pointing out that
\[
b < n < 2b.
\]
Indeed, since \( ab = \frac{n(n+1)}{2} + i_0 \) with \( i_0 < n + 1 \), we have
\[
(n + 2)(n + 1) > 2ab \geq 2b^2 > (b + 2)(b + 1) \quad \text{(for } b \geq 4)\]
and hence \( b < n \). In addition, since \( ab = \frac{n(n+1)}{2} + i_0 \) with \( i_0 \geq 0 \), we have
\[
n(n + 1) \leq 2ab \leq 2(b + 1)b < 2(n + 1)b
\]
and hence \( n < 2b \).

We now proceed by induction on \( ab \), we must consider different cases.

- **Case** \( a = b + 1 \) and \( 2b - i_0 < n \): Let
  \[
  \tilde{a} = b \text{ and } \tilde{b} = a - 2 = b - 1.
  \]
We want to apply the induction hypothesis to $\tilde{a}$ and $\tilde{b}$. Thus we need to check properties (1) and (2): since $\tilde{a} = \tilde{b} + 1$, condition (1) is satisfied and condition (2) is vacuous.

In order to continue, we need the triangular representation of $\tilde{a}\tilde{b}$. Since $n$ and $i_0$ are the triangular representation of $ab$ it follows that

$$\tilde{a}\tilde{b} = b(a - 2) = \frac{n(n + 1)}{2} + i_0 - 2b = \frac{n(n - 1)}{2} + n - (2b - i_0).$$

Let us denote $i_1 = 2b - i_0$. It follows from (4.5) $2b > n \geq i_0$, thus

$$0 < i_1;$$

and in this case we have assumed $i_1 < n$. Therefore $\tilde{n} = n - 1$ and $\tilde{i}_0 = n - i_1$ are the triangular representation of $\tilde{a}\tilde{b}$ since we have shown

$$0 < \tilde{i}_0 \leq \tilde{n}.$$

We are now in a position to apply the induction hypothesis to $\tilde{a}$ and $\tilde{b}$ and thus we obtain that $S_{\tilde{a},\tilde{b}} \neq \emptyset$. Therefore, we may choose sequences of matrices

$$\tilde{A}_1, \ldots, \tilde{A}_{i_0-1}, \tilde{A}_{i_0}, \tilde{A}_{i_0}', \tilde{A}_{i_0+1}, \ldots, \tilde{A}_{\tilde{n}} \in M_{\tilde{a},2}$$

$$\tilde{B}_1, \ldots, \tilde{B}_{\tilde{n}} \in M_{2,\tilde{b}},$$

such that the following products

$$\begin{align*}
\tilde{A}_1 \tilde{B}_1 & \\
\tilde{A}_2 \tilde{B}_1 & \tilde{A}_2 \tilde{B}_2 \\
\vdots & \vdots & \ddots \\
\tilde{A}_{i_0} \tilde{B}_1 & \tilde{A}_{i_0} \tilde{B}_2 & \ldots & \tilde{A}_{i_0} \tilde{B}_{i_0} \\
\tilde{A}_{i_0} \tilde{B}_1 & \tilde{A}_{i_0} \tilde{B}_2 & \ldots & \tilde{A}_{i_0} \tilde{B}_{i_0} \\
\tilde{A}_{i_0+1} \tilde{B}_1 & \tilde{A}_{i_0+1} \tilde{B}_2 & \ldots & \tilde{A}_{i_0+1} \tilde{B}_{i_0+1} \\
\vdots & \vdots & \vdots & \ddots \\
\tilde{A}_{\tilde{n}} \tilde{B}_1 & \tilde{A}_{\tilde{n}} \tilde{B}_2 & \ldots & \tilde{A}_{\tilde{n}} \tilde{B}_{i_0} & \tilde{A}_{\tilde{n}} \tilde{B}_{i_0} & \ldots & \tilde{A}_{\tilde{n}} \tilde{B}_{\tilde{n}}
\end{align*}$$

(4.6)

constitute a basis of $M_{\tilde{a},\tilde{b}}$. Moreover, since $S_{\tilde{a},\tilde{b}} \subset M_{\tilde{a},2}^{\tilde{n}+1} \times M_{2,\tilde{b}}^{\tilde{n}}$ is Zariski open, we may additionally require that

$$\text{“any subset of } b \text{ elements of the set of all the columns of the matrices } A_1, \ldots, A_{i_0}, A_{i_0}', \ldots, A_{\tilde{n}} \text{ be linearly independent.”}$$

Let

$$B_i = \begin{cases} 
\tilde{A}_{n-i}' & \text{if } i < i_1; \\
(\tilde{A}_{i_0}')^t & \text{if } i = i_1; \\
\tilde{A}_{n+1-i}^t & \text{if } i > i_1.
\end{cases}$$
Given $X \in M_{q_1-2,q_2}$, then $\tilde{X} \in M_{q_1,q_2}$ denotes the matrix $X$ with two null rows added at the bottom. Let

$$A_i = \begin{cases} 
\tilde{B}_{n-i}^t, & \text{if } i < i_1; \\
\left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right)^t, & \text{if } i = i_1; \\
\tilde{B}_{n+1-i}^t, & \text{if } i > i_1.
\end{cases}$$

for $i = 1, \ldots, n$, and let $A'_i = \left( \begin{array}{c}
0 \\
0 \\
1
\end{array} \right)^t \in M_{a_2}$.

Now, among all the products (4.4) we have that $A_1B_1$

| $A_1B_1$ | $A_2B_1$ | $A_2B_2$ |
|---------|---------|---------|
| $\vdots$ | $\vdots$ | $\ddots$ |
| $A_{i_1}B_1$ | $A_{i_1}B_2$ | $\ldots$ | $A_{i_1}B_{i_1-1}$ |
| $A_{i_1+1}B_1$ | $A_{i_1+1}B_2$ | $\ldots$ | $A_{i_1+1}B_{i_1}$ | $A_{i_1+1}B_{i_1+1}$ |
| $\vdots$ | $\vdots$ | $\ddots$ |
| $A_nB_1$ | $A_nB_2$ | $\ldots$ | $A_nB_{i_1-1}$ | $A_nB_{i_1}$ | $A_nB_{i_1+1}$ | $\ldots$ | $A_nB_n$

are linearly independent, as they are the widehat of the transpose of the products in (4.6), and each product has its two last rows equal to zero.

On the other hand, if $B_i = \left( \begin{array}{c}
\omega_i \\
\omega_i
\end{array} \right)$ we know that the submatrix consisting of the two last rows of each of the following matrices

$$A_iB_1 \ A_iB_2 \ \ldots \ A_iB_{i_1} \ A'_iB_1 \ A'_iB_2 \ \ldots \ A'_iB_{i_1} \ A'_iB_{i_0}$$

(here we have assumed $i_0 \geq i_1$ but it may happen $i_1 < i_0$) are, respectively, equal to

$$\left( \begin{array}{c}
v_1 \\
v_2 \\
v_1
\end{array} \right) \left( \begin{array}{c}
v_2 \\
v_2 \\
v_1
\end{array} \right) \ldots \left( \begin{array}{c}
v_1 \\
v_1 \\
v_1
\end{array} \right)$$

which are linearly independent by (4.7). Therefore, the set of products (4.4) is a basis of $M_{n,b}$ and this completes the induction step in this case.

**Case** $a = b+1$ and $2b-i_0 \geq n$: In this case, from (4.5) we obtain $i_0 < b$. Let

$$\tilde{a} = b \text{ and } \tilde{b} = a = b+1.$$ We want to apply the induction hypothesis to $\tilde{a}$ and $\tilde{b}$ and thus we need to check properties (1) and (2). Since $\tilde{a} = \tilde{b}$, condition (1) is satisfied and, in order to check condition (2) we need the triangular representation of $\tilde{a}\tilde{b}$. Since $n$ and $i_0$ are the triangular representation of $ab$ it follows that

$$\tilde{a}\tilde{b} = \frac{n(n+1)}{2} + i_0 - b = \frac{n(n-1)}{2} + n - (b - i_0).$$

If $i_1 = b - i_0$, we obtain

$$\tilde{n} = n - 1 \text{ and } \tilde{i}_0 = n - i_1.$$ Since

$$0 < \tilde{i}_0 \leq \tilde{n},$$
we have \( \tilde{i}_0 \) and \( \tilde{n} \) are the triangular representation of \( \tilde{a} \tilde{b} \).

Moreover, since in this case \( 2b - i_0 \geq n \), it follows that \( \tilde{i}_0 \leq b = \tilde{b} \). This shows that condition (2) is also satisfied and we can apply the induction hypothesis to \( \tilde{a} \) and \( \tilde{b} \).

Now the argument is analogous to that of the previous case. The main difference is that, in this case, if \( X \in M_{q-1,2} \), then \( \tilde{X} \in M_{\tilde{q},2} \) denotes the matrix \( X \) with one (instead of two) null row added at the bottom.

Since \( \emptyset \not= S_{a,b} \subset M_{\tilde{a},2}^{n+1} \times M_{\tilde{b},2}^{n} \) is Zariski open we may choose sequences of matrices

\[
\tilde{A}_1, \ldots, \tilde{A}_{i_0-1}, \tilde{A}_{i_0}, \tilde{A}_{i_0+1}, \ldots, \tilde{A}_{\tilde{n}} \in M_{\tilde{a},2}
\]

\[
\tilde{B}_1, \ldots, \tilde{B}_{\tilde{n}} \in M_{\tilde{b},2},
\]

such that (4.7) and such that the following products (4.6) constitute a basis of \( M_{\tilde{a},\tilde{b}} \); Let

\[
B_i = \begin{cases} 
\hat{A}_n, & \text{if } i < i_1; \\
(\hat{A}_n)_{i_1-i}, & \text{if } i = i_1; \\
\hat{A}_{n+1-i}, & \text{if } i > i_1;
\end{cases}
\]

let \( A'_{i_0} = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \end{pmatrix} \in M_{a,2} \) (note the difference with the previous case) and, for \( i = 1, \ldots, n \), let

\[
A_i = \begin{cases} 
\hat{B}_n, & \text{if } i < i_1; \\
(0 \ldots 0)_{i_1-i}, & \text{if } i = i_1; \\
\hat{B}_{n+1-i}, & \text{if } i > i_1.
\end{cases}
\]

Now, among all the products (4.4) we have that

\[
\begin{array}{cccccc}
A_1B_1 & A_2B_1 & A_2B_2 & \\
\vdots & \ddots & \ddots & \ddots \\
A_{i_1-1}B_1 & A_{i_1-1}B_2 & \ldots & A_{i_1-1}B_{i_1-1} & \\
A_{i_1+1}B_1 & A_{i_1+1}B_2 & \ldots & A_{i_1+1}B_{i_1-1} & A_{i_1+1}B_{i_0} & A_{i_1+1}B_{i_0+1} & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
A_nB_1 & A_nB_2 & \ldots & A_nB_{i_1-1} & A_nB_{i_1} & A_nB_{i_1+1} & \ldots & A_nB_n
\end{array}
\]

are linearly independent (as in the previous case) and each product has its last (instead of two last) row equal to zero. Finally, if \( \begin{pmatrix} v_i \\ w_i \end{pmatrix} \) is the submatrix consisting of the two last rows of \( B_i \), we know that the last row of each of the following matrices

\[
\begin{array}{cccccc}
A_1B_1 & A_1B_2 & \ldots & A_1B_{i_1} & \\
A'_{i_0}B_1 & A'_{i_0}B_2 & \ldots & A'_{i_0}B_{i_1} & A'_{i_0}B_{i_0}
\end{array}
\]

are, respectively, equal to

\[
\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \ldots \begin{pmatrix} v_{i_1} \\ w_{i_1} \end{pmatrix} \ldots \begin{pmatrix} v_{i_0} \\ w_{i_0} \end{pmatrix}
\]
which are linearly independent by (4.7). Therefore, the set of products (4.4)
is a basis of $M_{a,b}$ and this completes the induction step in this case.

- **Case $a = b$ and $i_0 < b$:** Let

  \[ \tilde{a} = b, \]
  \[ \tilde{b} = a - 1. \]

  Since $\tilde{a} = \tilde{b} + 1$, condition (2) is vacuous and we can apply the induction hypothesis. As in the previous case, if

  \[ i_1 = b - i_0, \]

  then $\tilde{n} = n - 1$ and $\tilde{i}_0 = n - i_1$ are the triangular representation of $\tilde{a}\tilde{b}$. Now the argument is the same as in the previous case.

- **Case $a = b = i_0$:** Again, let

  \[ \tilde{a} = b, \]
  \[ \tilde{b} = a - 1. \]

  Since $\tilde{a} = \tilde{b} + 1$, condition (2) is vacuous and we can apply the induction hypothesis.

  In contrast to the previous case, now $\tilde{n} = n$ and $\tilde{i}_0 = 0$ are the triangular representation of $\tilde{a}\tilde{b}$. If we take $i_1 = b$, then the argument is the same as in the case $a = b + 1$ and $2b - i_0 \geq n$. □

**Theorem 4.4.** Let $r \in \mathbb{N}$, $r \geq 2$, and let $a \geq b$ be the integer square roots of $(\frac{r}{2})$. Then there exist sequences of matrices

\[ X_1, \ldots, X_r \in M_{a,2}, \]
\[ Y_1, \ldots, Y_r \in M_{2,b}, \]

such that the following $(\frac{r}{2})$ matrices $Z_{i,j} = X_iY_j - X_jY_i$ for $1 \leq j < i \leq r$, are linearly independent in $M_{a,b}$.

**Proof.** Let $n = r - 1$ and let

\[ A_1, \ldots, A_{i_0-1}, A_{i_0}, A'_{i_0}, A_{i_0+1}, \ldots, A_n \in M_{a,2}, \]
\[ B_1, \ldots, B_n \in M_{2,b}, \]

be the sequences provided by Theorem 4.3. We ignore the matrix $A'_{i_0}$ and we rename the sequence $A_1, \ldots, A_n$ as $A_2, \ldots, A_{n+1}$. It follows from Theorem 4.3 that the set of matrices

\[ A_2B_1 \]
\[ A_3B_1 \quad A_3B_2 \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ A_{n+1}B_1 \quad A_{n+1}B_2 \quad \ldots \quad A_{n+1}B_n \]

is linearly independent. Now, we define $A_1 = 0$ and $B_r = 0$ and let

\[ X_i = \epsilon^i A_i \quad \text{and} \quad Y_i = B_i \quad \text{for} \quad i = 1, \ldots, r, \]

for some $\epsilon \neq 0$ to be defined later. For $1 \leq j < i \leq r$, we have

\[ C_{i,j} = \epsilon^i (A_iB_j - \epsilon^{i-j} A_jB_i). \]
Since $\epsilon \neq 0$, the set $\{C_{i,j}, \ 1 \leq j < i \leq r\}$ is linearly independent if and only if $\{A_iB_j - \epsilon^{j-i}A_jB_i, \ 1 \leq j < i \leq r\}$ is linearly independent. Since the set of matrices in (13) is linearly independent and linear independence is a non-vanishing polynomial condition on an infinite field, it follows that there exists $\epsilon \neq 0$ such that $\{C_{i,j}, \ 1 \leq j < i \leq r\}$ is linearly independent. \[\Box\]

**Proof of Theorem 4.2.** Let $a \geq b$ be the integer square roots of $(r^2)$. By Theorem 4.4 there exist sequences of matrices

$$X_1, \ldots, X_r \in M_{a,2}$$
$$Y_1, \ldots, Y_r \in M_{2,b}.$$

It is easy to check that the following $r$ matrices in $M_{a+b+2,a+b+2}$

$$\begin{bmatrix}
0 & X_1 & 0 \\
0 & 0 & Y_i \\
0 & 0 & 0
\end{bmatrix}$$

generates a Lie subalgebra isomorphic to $L_{r,2}$. \[\Box\]

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