q-Discrete Toda Molecule Equation

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Abstract

A q-discrete version of the two-dimensional Toda molecule equation is proposed through the direct method. Its solution, Bäcklund transformation and Lax pair are discussed. The reduction to the q-discrete cylindrical Toda molecule equation is also discussed.

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1. Introduction

The discovery of the quantum groups[1,2] gave rise to a new phase to the studies of integrable systems. It also shed light to the theory of so-called q-analog of the special functions, where the q-difference operator plays a similar role to the differential operator in the theory of the special functions[3-5]. On the other hand, it is known that the soliton equations have a close relationship with the special functions[6,7]. Hence it arises as an interesting problem to study “q-discrete integrable systems”.

In a previous paper[8], we have proposed the q-discrete version of the two-dimensional Toda lattice equation and shown that the q-discrete cylindrical Toda lattice equation is derived through a reduction. Moreover, we have shown that its solution is expressed by a determinant whose entries are the q-Bessel function. In this letter, we will discuss the q-discretization of the two-dimensional Toda molecule (2DTM) equation (finite, non-periodic two-dimensional Toda lattice)[9] and derive its solution, Bäcklund transformation and Lax pair. We also present the q-discrete cylindrical Toda molecule equation.

Let us first give a brief review of the 2DTM equation,

\[
\frac{\partial^2 u_N}{\partial x \partial y} = e^{u_{N-1}-u_N} - e^{u_N-u_{N+1}}, \quad N = 1, \cdots, M ,
\]

\[
u_0 = -\infty, \quad u_{M+1} = +\infty ,
\]

which is rewritten as

\[
\frac{\partial V_N}{\partial x} = V_N(J_N - J_{N+1}) ,
\]

\[
\frac{\partial J_N}{\partial y} = V_{N-1} - V_N ,
\]

\[
V_0 = V_M = 0 ,
\]

by a suitable variable transformation. Equations (2a)-(2c) are transformed to the bilinear form,
\[
\frac{\partial^2 \tau_N}{\partial x \partial y} \tau_N - \frac{\partial \tau_N}{\partial x} \frac{\partial \tau_N}{\partial y} = \tau_{N+1} \tau_{N-1},
\] (3)

through the dependent variable transformations,

\[
V_N = \frac{\tau_{N-1} \tau_{N+1}}{\tau_N^2}, \quad J_N = \frac{\partial}{\partial x} \log \frac{\tau_{N-1}}{\tau_N}.
\] (4)

The solution of the bilinear form (3) is given by the two-directional Wronskian,

\[
\tau_N = \begin{vmatrix}
  f & \partial_x f & \cdots & \partial_x^{N-1} f \\
  \partial_y f & \partial_x \partial_y f & \cdots & \partial_x^{N-1} \partial_y f \\
  \vdots & \vdots & \ddots & \vdots \\
  \partial_y^{N-1} f & \partial_x \partial_y^{N-1} f & \cdots & \partial_x^{N-1} \partial_y^{N-1} f 
\end{vmatrix}.
\] (5)

In (5), \( f \) is chosen to satisfy the boundary condition (2c) as

\[
f = \sum_{k=1}^{M} \phi_k(x) \chi_k(y),
\] (6)

where \( \phi_k(x) \) and \( \chi_k(y) \) are arbitrary functions in \( x \) and \( y \), respectively.

Our method relies on the fact that the bilinear forms of “generic” soliton equations are nothing but the identities of the determinants or the Pfaffians, or those derived from their reduction. In the present case, (3) is reduced to the Jacobi identity of the two-directional Wronskian. This fact can be a powerful guiding principle for the extension of the integrable systems. For example, usual discretization of soliton equations is performed based on the principle[10]. We will apply the idea to the q-discretization of the soliton equations.

2. q-Discrete 2DTM Equation

We propose a system given by

\[
\delta_{q^\alpha,x} V_N(x,y) = V_N(q^\alpha x,y) J_N(x,q^\beta y) - V_N(x,y) J_{N+1}(x,y),
\] (7a)

\[
\delta_{q^\beta,y} J_N(x,y) = V_{N-1}(q^\alpha x, y) - V_N(x,y),
\] (7b)

\[
V_0(x,y) = V_M(x,y) = 0,
\] (7c)
where $\delta_{q^\alpha,x}$ and $\delta_{q^\beta,y}$ are the q-difference operators defined by

$$
\delta_{q^\alpha,x} f(x) = \frac{f(x) - f(q^\alpha x)}{(1 - q)x}, \quad \delta_{q^\beta,y} f(y) = \frac{f(y) - f(q^\beta y)}{(1 - q)y}.
$$

The operators $\delta_{q^\alpha,x}$ and $\delta_{q^\beta,y}$ tend to $\alpha \frac{\partial}{\partial x}$ and $\beta \frac{\partial}{\partial y}$ in the limit $q \to 1$, respectively. In this limit, (7a)-(7c) are reduced to the 2DTM equation (2a)-(2c). We call eqs. (7a)-(7c) the q-discrete 2DTM equation.

Equations (7a)-(7c) are transformed into the bilinear form,

$$
\delta_{q^\alpha,x} \delta_{q^\beta,y} \tau_N(x, y) \cdot \tau_N(x, y) - \delta_{q^\alpha,x} \tau_N(x, y) \delta_{q^\beta,y} \tau_N(x, y) = \tau_{N+1}(x, y) \tau_{N-1}(q^\alpha x, q^\beta y),
$$

through the dependent variable transformations,

$$
J_N(x, y) = \frac{1}{(1 - q)x} \left\{ \frac{\tau_{N-1}(x, y) \tau_N(q^\alpha x, y)}{\tau_{N-1}(q^\alpha x, y) \tau_N(x, y)} - 1 \right\},
$$

$$
V_N(x, y) = \frac{\tau_{N+1}(x, y) \tau_{N-1}(x, q^\beta y)}{\tau_N(x, y) \tau_N(x, q^\beta y)}.
$$

The solution of the bilinear form (9) is given by the two-directional Wronski-type determinant,

$$
\tau_N(x, y) = \begin{vmatrix}
    f(x, y) & \delta_{q^\alpha,x} f(x, y) & \cdots & \delta_{q^\alpha,x}^{N-1} f(x, y) \\
    \delta_{q^\beta,y} f(x, y) & \delta_{q^\alpha,x} \delta_{q^\beta,y} f(x, y) & \cdots & \delta_{q^\alpha,x}^{N-1} \delta_{q^\beta,y} f(x, y) \\
    \vdots & \vdots & \ddots & \vdots \\
    \delta_{q^\beta,y}^{N-1} f(x, y) & \delta_{q^\alpha,x} \delta_{q^\beta,y}^{N-1} f(x, y) & \cdots & \delta_{q^\alpha,x}^{N-1} \delta_{q^\beta,y}^{N-1} f(x, y)
\end{vmatrix},
$$

where $f(x, y)$ is chosen to satisfy the boundary condition (7c) as (6).

Let us prove that (11) really gives the solution of the bilinear form (9). We have for example,
\[
\tau_N(q^{-\alpha} x, y) = \\
\begin{vmatrix}
  f(q^{-\alpha} x, y) & \delta_q^{\alpha, x} f(q^{-\alpha} x, y) & \cdots & \delta_q^{N-1} f(q^{-\alpha} x, y) \\
  \delta_q^{\beta, y} f(q^{-\alpha} x, y) & \delta_q^{\alpha, x} \delta_q^{\beta, y} f(q^{-\alpha} x, y) & \cdots & \delta_q^{N-1} \delta_q^{\beta, y} f(q^{-\alpha} x, y) \\
  \vdots & \vdots & \ddots & \vdots \\
  \delta_q^{N-1} f(q^{-\alpha} x, y) & \delta_q^{\alpha, x} \delta_q^{N-1} f(q^{-\alpha} x, y) & \cdots & \delta_q^{N-1} \delta_q^{N-1} f(q^{-\alpha} x, y) \\
  f(x, y) & \delta_q^{\alpha, x} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{\beta, y} f(q^{-\alpha} x, y) \\
  \delta_q^{\beta, y} f(x, y) & \delta_q^{\alpha, x} \delta_q^{\beta, y} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{\beta, y} f(q^{-\alpha} x, y) \\
  \vdots & \vdots & \ddots & \vdots \\
  \delta_q^{N-1} f(x, y) & \delta_q^{\alpha, x} \delta_q^{N-1} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{N-1} f(q^{-\alpha} x, y) \\
\end{vmatrix}
\]  \qquad (12)

In deriving the second determinant from the first, we have subtracted \((k + 1)\)-th column multiplied by \((1 - q)q^{-\alpha} x\) from \(k\)-th column for \(k = 1, \cdots N - 1\), to confine the shift of the independent variable \(x\) to the most right column of the determinant. Moreover, we note that \(\delta_q^{N-1} f(q^{-\alpha} x, y)\) means \(\delta_q^{N-1} f(x, y)|_{x \rightarrow q^{-\alpha} x}\). Multiplying \(N\)-th column by \((1 - q)q^{-\alpha} x\) and adding \((N - 1)\)-th column to \(N\)-th column in the second determinant of (12), we get

\[
(1 - q)q^{-\alpha} x \quad \tau_N(q^{-\alpha} x, y) \\
= \\
\begin{vmatrix}
  f(x, y) & \delta_q^{\alpha, x} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{\beta, y} f(x, y) \\
  \delta_q^{\beta, y} f(x, y) & \delta_q^{\alpha, x} \delta_q^{\beta, y} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{\beta, y} f(q^{-\alpha} x, y) \\
  \vdots & \vdots & \ddots & \vdots \\
  \delta_q^{N-1} f(x, y) & \delta_q^{\alpha, x} \delta_q^{N-1} f(x, y) & \cdots & \delta_q^{N-1} \delta_q^{N-1} f(q^{-\alpha} x, y) \\
\end{vmatrix}
\]  \qquad (13)

Similarly, we obtain

\[
(1 - q)^2 q^{-(\alpha + \beta)} x y \quad \tau_N(q^{-\alpha} x, q^{-\beta} y) \\
= \\
\begin{vmatrix}
  f(x, y) & \cdots & \delta_q^{N-2} f(x, y) & \delta_q^{N-2} f(q^{-\alpha} x, y) \\
  \delta_q^{\alpha, x} f(x, y) & \cdots & \delta_q^{N-2} \delta_q^{\beta, y} f(x, y) & \delta_q^{N-2} \delta_q^{\beta, y} f(q^{-\alpha} x, y) \\
  \vdots & \ddots & \vdots & \vdots \\
  \delta_q^{N-2} f(x, q^{-\beta} y) & \cdots & \delta_q^{N-2} \delta_q^{N-2} f(x, q^{-\beta} y) & \delta_q^{N-2} \delta_q^{N-2} f(q^{-\alpha} x, q^{-\beta} y) \\
\end{vmatrix}
\]  \qquad (14)

Applying Jacobi’s identity on (14) with \(N\) replaced by \(N + 1\), we obtain

\[
\tau_N(q^{-\alpha} x, q^{-\beta} y) \tau_N(x, y) - \tau_N(q^{-\alpha} x, y) \tau_N(x, q^{-\beta} y) \\
= (1 - q)^2 q^{-(\alpha + \beta)} x y \tau_{N+1}(q^{-\alpha} x, q^{-\beta} y) \tau_N(x, y), \qquad (15)
\]
which is nothing but the bilinear form (9) with \(x\) and \(y\) replaced by \(q^{-\alpha}x\) and \(q^{-\beta}y\), respectively. Thus we have proved that (11) gives the solution of (9).

We now discuss a reduction of the q-2DTM equation. Putting \(xy = r^2\) and \(\alpha = \beta = 2\), and imposing the condition that \(\tau_N(x, y)\) depends only on \(r\), we find that the bilinear form (9) and its solution (11) are reduced to

\[
\left( \frac{1}{r} \delta_{q,r} + q \delta_{q,r}^2 \right) \tau_N(r) \cdot \tau_N(r) - \left\{ \delta_{q,r} \tau_N(r) \right\}^2 = \tau_{N+1}(r) \tau_{N-1}(q^2r) ,
\]

and

\[
\tau_N(r) = q^{-\frac{N(N-1)(N-2)}{2}} r^{-N(N-1)}
\times
\begin{vmatrix}
  f(r) & r \delta_{q,r} f(r) & \cdots & (r \delta_{q,r})^{N-1} f(r) \\
  r \delta_{q,r} f(r) & (r \delta_{q,r})^2 f(r) & \cdots & (r \delta_{q,r})^N f(r) \\
  \vdots & \vdots & \ddots & \vdots \\
  (r \delta_{q,r})^{N-1} f(r) & (r \delta_{q,r})^N f(r) & \cdots & (r \delta_{q,r})^{2N-2} f(r)
\end{vmatrix},
\]

respectively. Equation (16) tends to the cylindrical Toda molecule (cTM) equation[11],

\[
\left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) \tau_N(r) \cdot \tau_N(r) - \left\{ \frac{\partial \tau_N(r)}{\partial r} \right\}^2 = \tau_{N+1}(r) \tau_{N-1}(r) ,
\]

in the limit \(q \to 1\), and hence we call (16) the q-cTM equation. Note that (16) is transformed to

\[
\delta_{q,r} V_N(r) = q J_N(qr) V_N(qr) - J_{N+1}(r) V_N(r) ,
\]

\[
(q \delta_{q,r} + \frac{1}{r}) J_N(r) = V_N(r) - V_{N-1}(qr) ,
\]

\[
V_0(r) = V_M(r) = 0 ,
\]

through the dependent variable transformations,
fact is shown by the Plücker relation as follows. In other words, (21a) and (21b) are the identities for the determinants (22) and (23). This leads to another solution, which transforms a solution of the q-2DTM equation, to another solution,

$$V_N(r) = \frac{\tau_{N-1}(qr)\tau_{N+1}(r)}{\tau_N(r)\tau_N(qr)},$$  \hspace{1cm} (20a)  

$$J_N(r) = \frac{1}{(1-q)r} \left\{ \frac{\tau_N(qr)\tau_{N-1}(r)}{\tau_N(r)\tau_N(qr)} - 1 \right\}.$$  \hspace{1cm} (20b)  

3. Bäcklund Transformation and Lax Pair

By using the fact that the solution of the q-2DTM equation (7) is given by (11), we here propose the Bäcklund transformation. It is written by

$$\delta_{q^\beta,y} \tau_N(x,y) \cdot \tau'_N(x,y) - \tau_N(x,y) \delta_{q^\beta,y} \tau'_N(x,y)$$
$$= -\tau_{N+1}(x,y)\tau'_{N-1}(x,q^\beta y),$$  \hspace{1cm} (21a)  

$$\delta_{q^\alpha,x} \tau_N(x,y) \cdot \tau'_{N-1}(x,y) - \tau_N(x,y) \delta_{q^\alpha,x} \tau'_{N-1}(x,y)$$
$$= \tau_{N-1}(q^\alpha x,y)\tau'_N(x,y),$$  \hspace{1cm} (21b)  

which transforms a solution of the q-2DTM equation,

$$\tau_N(x,y) = \begin{vmatrix} f(x,y) & \delta_{q^\alpha,x} f(x,y) & \cdots & \delta_{q^{N-1},x} f(x,y) \\ \delta_{q^\beta,y} f(x,y) & \delta_{q^\alpha,x} \delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^{N-1},x} \delta_{q^\beta,y} f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^{N-1},y} f(x,y) & \delta_{q^\alpha,x} \delta_{q^{N-1},y} f(x,y) & \cdots & \delta_{q^{N-1},x} \delta_{q^{N-1},y} f(x,y) \end{vmatrix},$$  \hspace{1cm} (22)  

to another solution,

$$\tau'_N(x,y) = \begin{vmatrix} \delta_{q^\alpha,x} f(x,y) & \delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^N,x} f(x,y) \\ \delta_{q^\alpha,x} \delta_{q^\beta,y} f(x,y) & \delta_{q^\beta,y} \delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^N,x} \delta_{q^\beta,y} f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\alpha,x} \delta_{q^{N-1},y} f(x,y) & \delta_{q^\beta,y} \delta_{q^{N-1},y} f(x,y) & \cdots & \delta_{q^N,x} \delta_{q^{N-1},y} f(x,y) \end{vmatrix}. \hspace{1cm} (23)$$  

In other words, (21a) and (21b) are the identities for the determinants (22) and (23). This fact is shown by the Plücker relation as follows.
Let us prove the second equation (21b). First, we introduce notations

\[ \tau_N(x, y) = |0, 1, \ldots, N - 1|, \quad (24) \]
\[ \tau'_N(x, y) = |1, 2, \ldots, N| . \quad (25) \]

Namely, the number "\(k\)" in (24) and (25) means a column vector

\[ \begin{bmatrix}
\delta^k_{q^\alpha, x} & f(x, y) \\
\delta^k_{q^\alpha, x} & \delta^k_{q^\beta, y} & f(x, y) \\
\vdots & \\
\delta^k_{q^\alpha, x} & \delta^{N-1}_{q^\beta, y} & f(x, y)
\end{bmatrix} \]

Then we have

\[ \tau_{N-1}(x, y) = |0, 1, \ldots, N - 2, \phi|, \quad (27) \]
\[ \tau_N(q^{-\alpha} x, y) = |0, 1, \ldots, N - 2, N - 1_{q^{-\alpha} x}|, \quad (28) \]
\[ \tau'_{N-1}(q^{-\alpha} x, y) = |1, 2, \ldots, N - 2, N - 1_{q^{-\alpha} x}, \phi|, \quad (29) \]

and

\[ (1 - q)q^{-\alpha} x \ \tau'_N(q^{-\alpha} x, y) = |1, 2, \cdots N - 1, N - 1_{q^{-\alpha} x}|, \quad (30) \]

where

\[ \begin{bmatrix}
\delta^{N-1}_{q^\alpha, x} f(q^{-\alpha} x, y) \\
\delta^{N-1}_{q^\alpha, x} & \delta^N_{q^\beta, y} f(q^{-\alpha} x, y) \\
\vdots & \\
\delta^{N-1}_{q^\alpha, x} & \delta^{N-1}_{q^\beta, y} f(q^{-\alpha} x, y)
\end{bmatrix} \]

and
\[
\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \bigg\}\bigg\}_N ,
\] (32)

which is inserted to equalize the size of the determinant.

We now consider an identity of \(2N \times 2N\) determinant,

\[
0 = \begin{vmatrix} 1 & \cdots & N - 2 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{vmatrix} N - 1 \begin{vmatrix} 1 & \cdots & N - 2 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{vmatrix} \begin{vmatrix} N - 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{vmatrix} \phi .
\] (33)

Applying the Laplace expansion to the right-hand side, we obtain an identity (the Plücker relation),

\[
0 = \begin{vmatrix} 0, 1, \cdots, N - 2, N - 1 & 1, \cdots, N - 2, N - 1, \phi \end{vmatrix} + \begin{vmatrix} 0, 1, \cdots, N - 2, N - 1, N - 1 & 1, \cdots, N - 2, N - 1, N - 1, \phi \end{vmatrix} - \begin{vmatrix} 1, \cdots, N - 2, N - 1 & 0, 1, \cdots, N - 2, \phi \end{vmatrix},
\] (34)

or equivalently,

\[
\tau_N(q^{-\alpha}x, y)\tau'_{N-1}(x, y) - \tau_N(x, y)\tau'_{N-1}(q^{-\alpha}x, y) = (1 - q)q^{-\alpha}x \tau_{N-1}(x, y)\tau'_{N}(q^{-\alpha}x, y) ,
\] (35)

which is nothing but (21b) with \(x\) replaced by \(q^{-\alpha}x\). Thus we have completed the proof. The first equation (21a) is proved in a similar way.

It is possible in general to construct the Lax pair from the Bäcklund transformation. Following the method developed by Hirota \textit{et.al.}\cite{12}, we derive the Lax pair for the q-2DTM equation (7) from (21). Introducing \(\psi\) by

\[
\tau'_N(x, y) = \tau_N(x, y)\psi_{N+1}(x, y) ,
\] (36)
we have from (21),

\[
\delta_{q^\beta,y}\psi_{N+1}(x,y) = V_N(x,y)\psi_N(x,q^\beta y) ,
\]

(37a)

\[
\delta_{q^\alpha,x}\psi_N(x,y) = -J_N(x,y)\psi_N(x,y) - \psi_{N+1}(x,y) .
\]

(37b)

Let us define two matrices \( L \) and \( R \) by

\[
L(x,y) = \begin{pmatrix}
0 & V_1(x,y) & 0 & 0 \\
& \ddots & \ddots & \ddots \\
& 0 & V_{M-1}(x,y) & 0
\end{pmatrix},
\]

(38)

\[
R(x,y) = -\begin{pmatrix}
J_1(x,y) & 1 \\
J_2(x,y) & 1 & 0 \\
& \ddots & \ddots \\
& 0 & J_{M-1}(x,y) & 1 \\
& & & J_M(x,y)
\end{pmatrix}.
\]

(39)

Then (37) are rewritten as

\[
\delta_{q^\beta,y}\Psi(x,y) = L(x,y)\Psi(x,q^\beta y) ,
\]

\[
\delta_{q^\alpha,x}\Psi(x,y) = R(x,y)\Psi(x,y) ,
\]

(40)

where

\[
\Psi(x,y) = \begin{pmatrix}
\psi_1(x,y) \\
\vdots \\
\psi_M(x,y)
\end{pmatrix}.
\]

(41)

The compatibility condition of the linear system (40) yields

\[
\delta_{q^\alpha,x}L(x,y) - \delta_{q^\beta,y}R(x,y) = R(x,y)L(x,y) - L(q^\alpha x,y)R(x,q^\beta y) ,
\]

(42)

which recovers the q-2DTM equation (7). Consequently, (38) and (39) give the Lax pair of the q-2DTM equation.
4. Concluding Remarks

In this letter, we have proposed the q-2DTM equation, its solution, Bäcklund transformation and Lax pair. There are several definitions of integrability for the continuous equations, such as the existence of N-soliton solution, that of an infinite number of conserved quantities or symmetries, Painlevé property and so on. For a given continuous equation, there are several ways to discretize it, depending on the definition to be taken. In our case, the guiding principle of discretization is to preserve integrability in such a sense that the determinant structure of solutions of the discretized equation is the same as that of the original continuous equation. For the discretization we have employed the bilinear formalism, since the determinant structure is clearly seen in it.

It was revealed that the time evolution of the solutions of the usual integrable system is subject to the infinite dimensional Lie algebra[13]. A question naturally arising is what the algebra describing the q-discrete system is. Is it the quantum group or some others? It may be an interesting and important problem to find the structure of the algebras underlying the q-2DTM equation.

Finally, we mention the possibility of q-discretization of the Painlevé equations. There is a close relationship between the Painlevé equations and Toda molecule equation. Okamoto has shown that the \( \tau \) functions of Painlevé equations satisfy the several types of Toda molecule equations[14]. In particular, the \( \tau \) function of the third Painlevé equation satisfies the cTM equation. Moreover, it has been shown that the Painlevé equations are reduced to the identities of determinants[15,16] for special values of the parameters in the equation. Since we have the q-cTM equation, it is natural to expect that we can perform the q-discretization of the third Painlevé equation. We will report on this subject in a forthcoming paper.

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