Models of flavour with discrete symmetries

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Abstract. We briefly review some recent developments in theoretical models of fermion masses, mixings and CP violation with discrete non-Abelian symmetries. Then, we explain the main ideas of a recently proposed Minimal $S_3$–invariant Extension of the Standard Model and its application to a unified analysis of masses, mixings and CP violation in the leptonic and quark sectors as well as the explicit computation of the $V_{PMNS}$ and $V_{CKM}$ mixing matrices.

Keywords: Flavor symmetries; Quark and lepton masses and mixings; Neutrino masses and mixings

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INTRODUCTION

In the last six or seven years, great advances have been made in the experimental knowledge of flavour physics, fermion masses and mixings and CP violation. These advances initiated a huge upsurge of theoretical activity aimed at uncovering the nature of this new physics. In the next section of this paper I will very briefly outline some recent theoretical developments on models of flavour with discrete non-Abelian symmetries in which the participation of the mexican community of particles and fields is visible. Section 3 is devoted to a brief explanation of the recently proposed Minimal $S_3$–invariant Extension of the Standard Model \[33\]. The paper ends with a short summary and some conclusions.

MODELS OF FLAVOUR WITH DISCRETE SYMMETRIES

The history of models for the quark mass matrices may possibly be traced back to Weinberg’s\[1\] observation that the Gatto, Sartori, Tonini\[2\] relation for the Cabbibo angle may be expressed as a relation between the Cabbibo angle and the quark masses of the first two generations,

$$V_{us} \approx \sqrt{\frac{m_d}{m_s}},$$

and that mass matrices of the form

$$M_u = \begin{pmatrix} m_u & 0 \\ 0 & m_c \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & p \\ p & q \end{pmatrix}$$

can account for the approximate equality \[1\]. As a consequence, in the early approaches to the problem of quark masses and mixings, it was natural to postulate that some entries
in the Yukawa matrix were equal to zero, the so-called “texture zeroes” [3, 4], thereby reducing the number of free parameters of the theory. Since then many approaches have been developed in the context of different theoretical and phenomenological models.

In the Standard $\left[ SU_C(3) \times SU_L(2) \times U_Y(1) \right]$ Model of the strong, weak and electromagnetic interactions, it is the Higgs mechanism that provides a theoretically consistent framework to generate masses for gauge bosons and fermions - the latter acquire masses after spontaneous breaking of the $SU(2)$ gauge symmetry, through the Yukawa couplings and the vacuum expectation value of the neutral Higgs field. However, this framework can neither predict the values of fermion masses nor interpret the observed hierarchy of their spectra. Hence, the three charged lepton masses, the six quark masses as well as the four parameters in the quark mixing matrix are free parameters of the Standard Model. As a straightforward consequence of the symmetry structure of the Standard Model, the renormalizable Yukawa couplings do not allow neutrino masses, although they can be introduced through the addition of non-renormalizable, higher-dimensional operators, presumably originating in physics beyond the Standard Model.

In the late 70’s and early 80’s, there already were a few promising indications for theoretical structures beyond the Standard Model which addressed the fermion mass problem. For simple symmetry breaking schemes, grand unification can relate quark and lepton masses. The most promising of such relations is the equality $m_b = m_\tau$, a result which applies at the Grand Unified Theory (GUT) scale. Radiative corrections are dominated by QCD interactions which increase the bottom quark mass in fair agreement with experiment. A fairly straightforward supersymmetric generalization of such GUT relations, which involves a new family symmetry, also provides the Georgi-Jarlskog relations between the down quarks and charged leptons of the first two generations. Supersymmetry also enables the gauge couplings to meet at the GUT scale to give a self-consistent unification picture [5-9].

The past ten years have seen great advances in the experimental and theoretical knowledge of flavour physics, CP-violation and fermion masses. In 1999, direct CP-violation in the Kaon system was established through the NA48 (CERN) and KTeV (FNAL) collaborations. In this decade, huge experimental efforts have been made to further explore CP-violation and the quark-flavour sector of the Standard Model. The main actor in these studies has been the B-meson system. In 2001, CP-violating effects were discovered and measured in the B-meson system by the BaBar [10] and Belle [11] Collaboration. A detailed investigation was also made of some benchmark, rare decay modes such as $B_d^0 \to J/\psi K_S$, $B_d^0 \to \phi K_S$ and $B_d^0 \to \pi^+ \pi^-$ and many others, for a recent review see R. Fleischer [12]. As of May 2006, it can be said that all existing data on CP-violation and rare decays in the quark sector can be described by the Standard Model within the theoretical and experimental uncertainties. The recent discovery and measurement of flavour conversion of solar [13, 14, 15, 16], atmospheric [17, 18], reactor [19, 20] and acelerator [21, 22] neutrinos have conclusively established that neutrinos have non-vanishing mass and they mix among themselves much like the quarks, thereby providing the first evidence of new physics beyond the Standard Model. The difference of the squared neutrino masses and the mixing angles in the lepton mixing matrix, $U_{PMNS}$, were determined, but neutrino oscillation data are insensitive to the absolute value of neutrino masses and also to the fundamental issue of whether neutrinos are Dirac or Majorana particles. Upper bounds on neutrino masses were provided by the searches that probe the neutrino
mass values at rest: beta decay experiments \cite{23}, neutrinoless double beta decay \cite{24} and precision cosmology \cite{25}.

These recent experimental advances triggered an enormous theoretical activity attempting to uncover the nature of this new physics. This includes further developments of the already existing mechanisms and theories such as GUT’s and Supersymmetric Grand Unified Theories (SUSY GUTs) and the appearance of new ideas and approaches implemented at a variety of different energy scales.

Regardless of the energy scales at which those theoretical models are built, the mechanisms for fermion mass generation and flavour mixing can roughly be classified into four different types:

1. Texture zeroes,
2. Family or flavour symmetries
3. Radiative mechanisms and
4. Seesaw mechanisms

These mechanisms are not disjoint but rather they are related and in many cases they complement and support each other. In the last six or seven years, important theoretical advances have been made in the understanding of these four mechanisms. The following points should be stressed.

1. Phenomenologically, some striking progress has been made with the help of texture zeroes and flavour symmetries in specifying the quantitative relationship between flavour mixing angles and quark or lepton mass ratios \cite{26, 27} and \cite{40, 41}.

2. After all the recent developments, the seesaw mechanism with large scale of the B-L violations still looks as the most appealing and natural mechanism of neutrino mass generation. At the same time, it is not excluded that some more complicated version of this mechanism is realized \cite{28}.

3. Gran Unification plus supersymmetry in some form still looks like the most plausible scenario of physics which naturally embeds the seesaw mechanism.

4. At the same time, it seems now clear that the “seesaw GUT” scenario does not provide a complete understanding of the neutrino masses and mixings as well as the quark masses and mixings or, in other words, the flavour structure of the mass matrices. Some new physics on top of this scenario seems essential. In this connection, two important questions arise:

- the possible existence of new symmetries that show up mainly or only in the lepton sector
- the need to understand the relation between quarks and leptons and the picture of flavour physics and CP violation in a unified way. The corresponding phenomenology is very rich.

These two issues point to the need of simpler models.

5. The search for simpler models starts by first constructing a low energy theory with the Standard Model and a discrete non-Abelian flavour symmetry group $\tilde{G}_F$ and then showing the possible embeddings of this theory into a GUT, like $SO(10)$ or $SU(5)$. The discrete symmetry will therefore be a subgroup of $SO(3)_f$ or $SU(3)_f$. Models in which the discrete non-Abelian flavour symmetry is only broken at low energies became very
The search for an adequate discrete group has concentrated on the smallest subgroups of $SO(3)$ or $SU(3)$ that have at least one singlet and one doublet irreducible representations to accommodate the fermions in each family [39].

To end this section, I will very briefly outline some recent developments on these questions in which the participation of the Mexican community of particles and fields is visible.

**Flavour permutational symmetry and Fritzsch textures**

The non-Abelian flavour permutational symmetry $S_{3L} \otimes S_{3R}$ and its explicit sequential breaking according to $S_{3L} \otimes S_{3R} \supset S_{3\text{diag}} \supset S_{2\text{diag}}$ was used by A. Mondragón and E. Rodríguez-Jáuregui [40, 41] to characterize the quark mass matrices, $M_u$ and $M_d$, with a texture of the same modified Fritzsch type. In a symmetry adapted basis, different patterns for breaking the permutational symmetry give rise to quark mass matrices which differ in the ratio $Z^{1/2} = M_{23}/M_{22}$ and are labeled in terms of the irreducible representations of an auxiliary $S_2$ group. After analytically diagonalizing the mass matrices, these authors derive explicit, exact expressions for the elements of the quark mixing matrix, $V_{\text{CKM}}$, the Jarlskog invariant, $J$, and the three inner angles, $\alpha$, $\beta$ and $\gamma$ of the unitarity triangle as functions of the quark mass ratios and only two free parameters, the symmetry breaking parameter $Z^{1/2}$ and one CP-violating phase $\Phi$. The numerical values of these parameters which characterize the experimentally preferred symmetry breaking pattern $Z^{1/2} = 9/2 \sqrt{2}$ and $\Phi = 90^\circ$, were extracted from a $\chi^2$ fit of the theoretical expressions for the moduli, $|V_{\text{th}}|$, to the experimentally determined values of the moduli of the elements of the quark mixing matrix $|V_{\text{exp}}^{\text{CKM}}|$. The agreement between theory and experiment, which initially was fairly good, improved as the experimental determination of the elements of the mixing matrix and the inner angles of the unitarity triangle improved [42]. The phase equivalence of $V_{\text{th}}^{\text{CKM}}$ and the mixing matrix $V_{\text{PDG}}^{\text{CKM}}$ in the standard parametrization advocated by the Particle Data Group allowed to translate those results into explicit exact expressions for the three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ and the CP-violating phase $\delta_{13}$ in terms of the four quark mass ratios and the symmetry breaking parameters $Z^{1/2}$ and $\Phi$ [41].

The main point in these results is simply that the hierarchy of quark masses and the texture of quark mass matrices are enough to determine, at least partly, some important features of the quark flavour mixing. In this sense, it was established that a scheme in which the two quark mass matrices, $M_u$ and $M_d$, have the same modified Fritzsch texture with the same value of the symmetry breaking parameter has some predictive power for the flavour mixing angles and CP-violating phase.

**Models of flavour with continuous symmetry**

There is a large variety of possible candidates for supersymmetric models of new physics beyond the Standard Model based in $N = 1$ SUSY with commuting GUT’s and
family symmetry groups, $G_{GUT} \otimes G_f$. This is so because there are many possible candidate GUT’s and family symmetry groups $G_f$. The model dependence does not end there since the details of the symmetry breaking vacuum plays a crucial role in specifying the model and determining the masses and mixing angles, for a recent review see S.F. King [43]. G. G. Ross and L. Velasco-Sevilla chose the largest family symmetry group, $SU(3)$, consistent with $SO(10)$ GUT’s and with additional Abelian family symmetries chosen to restrict the allowed Yukawa couplings. In a series of interesting papers [44, 45, 46] they explored the phenomenological implications of their model and were able to find a symmetry breaking scheme in which the observed hierarchical quark masses and mixings are described together with the hierarchy of charged lepton masses and a hierarchical structure for the neutrino masses. The significant differences between quark and lepton mixings are explained as due to the seesaw mechanism. Given the very large underlying symmetry, the fermion masses are heavily constrained. This $SO(10) \otimes SU(3)_f$ model provides a consistent description of the known masses and mixings of quarks and leptons. In the quark sector, the presence of CP violating phases is necessary, not only to reproduce CP violating processes, but also to reproduce the observed masses and mixings. In this model, the spontaneous breaking of CP in the flavour sector naturally solves the supersymmetric CP problem and the SUSY flavour problem, although flavour changing processes must occur at a level close to current experimental bounds. Motivated by the fact that leptogenesis is a very attractive candidate for explaining the large baryon asymmetry observed in the universe L. Velasco-Sevilla [46] also explored the very interesting possible connections between low energy CP violating phases appearing in the lepton mixing matrix and those phases relevant for leptogenesis.

**SUSYGUT Models with discrete flavour symmetry**

As noted above, in supersymmetric Grand Unified models of flavour with a non-Abelian continuous family group, such as $SO(10) \otimes SU(3)_f$ or $SU(5) \otimes SU(2)_f$, the phenomenological success depends crucially on the details of the symmetry breaking vacuum and its alignment.

For instance, neutrino mixing angle relations such as the bimaximal mixings of the left handed neutrinos is achieved only if the Yukawa couplings involving different families are related in some special way. The condition for the required equalities of Yukawa couplings to emerge is that the several scalar fields which break the family symmetry, called flavons, have their vacuum expectation values carefully aligned (or misaligned) along special directions in family space. Then, if these flavons appear in the effective operators responsible for the Yukawa couplings, the relations between the Yukawa couplings may be due to the particular alignment of the flavons responsible for that particular operator.

In an interesting series of papers, A. Aranda, C.D. Carone and R.F. Lebed [47, 48, 49] showed that the physics of vacuum alignment simplifies if the continuous family symmetry $SU(2)_f$ is replaced by the discrete non-Abelian family symmetry $T' \otimes Z_3$ in the SUSYGUT model of flavour $SU(5) \otimes SU(2)_f$ proposed by Romanino, Barbieri and Hall [50, 51, 52]. The group $T'$ is the group of proper rotations that leave a
regular tetrahedron invariant in the $SU(2)$ double covering of $SO(3)$. It has singlet, doublet and triplet irreducible representations with the multiplication rule $2 \otimes 2 = 1 \oplus 3$, which is a requisite to reproduce the phenomenologically succesful mass textures derived from GUT $SU(5) \otimes SU(2)_f$. The extra Abelian $Z_3$ factor in $G_f = T' \otimes Z_3$ is included in order to obtain the minimal extension needed to reproduce the $SU(2)$ model textures and satisfy discrete anomaly cancellation conditions. The flavons have non-trivial transformation properties under the GUT $SU(5)$ symmetry and the up-type and down-type quark mass textures are accordingly modified. Additionally, in the lepton sector, the rich representation structure of $T'$ allows for the neutrinos to be placed in different reps than the charged leptons, which, in this model is the origin of different hierarchies in the two sectors. The symmetry breaking pattern is $T' \otimes Z_3 \rightarrow Z_3^{diag} \rightarrow$ nothing. The light neutrino masses are generated through the seesaw mechanism. Three generations of right-handed neutrinos are introduced with the assignements $2^0 \oplus 1^{-+}$. This assignement leads to Dirac and Majorana mass matrices that allow the introduction of flavons that do not contribute at all to the charged fermion mass matrices. In this way, mass matrices with a modified Fritzsch texture are generated for the $u$ and $d$–type quarks, and for the charged leptons while the light Majorana neutrino mass matrix has a texture that naturally leads to the bimaximal $U_{PMNS}$ lepton mixing matrix.

Some further advantages of using a finite, discrete family symmetry are the following:

1. The breaking of a discrete symmetry does not lead to unwanted massless Goldstone bosons, unlike continuous symmetries
2. If this breaking is only spontaneous, it might produce domain walls \cite{53} which can be a serious problem. However, it can be solved by either invoking low scale inflation or embedding the discrete symmetry group into a continous group \cite{54} as is the case for $T' \otimes Z_3 \subset SU(2)$.
3. In the context of SUSY, discrete gauge symmetries do not give rise to excessive flavour changing neutral currents (FCNC) as is the case for continous symmetries.

**Type II seesaw and $S_3 \otimes U(1)_{e-\mu-\tau}$ symmetry**

One of the first phenomenologically succesful models for reproducing the bimaximal mixing among the neutrinos was presented by R.N. Mohapatra, A. Pérez-Lorenzana and C. Pires \cite{55}. This model is an extension of the Standard Model where the bimaximal mixing pattern among the neutrinos naturally arises via the type II seesaw mechanism. The model does not include right handed neutrinos, the lepton content of the SM is left unaltered but the Higgs sector is modified. The $SU_L(2)$ content of the Higgs sector consists of three doublets, two triplets with $Y = 2$ and a charged isosinglet with $Y = +2$. The model has a global $S_3 \otimes U(1)_{e-\mu-\tau}$ flavour symmetry. The charged $\mu$ and $\tau$ fields are in doublet representations, while the $e$ field is in a singlet representation of $S_3$. The pattern of $SU_L(2)$ Higgs doublet vacuum expectation values leads to a diagonal mass matrix for the charged leptons while the additional Higgs triplet acquires naturally small vacuum expectation values due to the type II see saw mechanism. At tree level, the $\nu_\mu$ and $\nu_e$ masses are degenerate, but the presence of the global $L_e - L_\mu - L_\tau$ and $S_3$ symmetry leads naturally to the desired mass splittings among neutrinos at the one
loop level. The resulting neutrino masses have an inverted hierarchy $|m_1| \geq |m_2| >> |m_3|$. There is a well known difficulty of this very interesting model to fit the large angle solution of the solar neutrino problem. Indeed, barring cancellations between the perturbations, these must be very small in order to obtain a $\Delta m^2_{\odot}$ close to the best fit value, but then, the value of $\sin^2 2\theta$ comes out too close to unity in disagreement with the best global fits of solar data $[56]$.

A minimal $S_3$--invariant extension of the Standard Model

The discovery of neutrino masses and mixings added ten new parameters to the already long list of free parameters in the Standard Model and made evident the urgent need of a systematic and unified treatment of all fermions in the theory. These two facts, taken together, pointed to the necessity and convenience of eliminating parameters and systematizing the observed hierarchies of masses and mixings as well as the presence or absence of CP violating phases by means of a flavour or family symmetry under which the families transform in a non-trivial fashion. As explained above, such a flavour symmetry might be a continuous or, more economically, a finite group.

In a recent paper, J. Kubo, A. Mondragón, M. Mondragón and E. Rodríguez-Jáuregui $[33]$ argued that such a flavour symmetry, unbroken at the Fermi scale, is the permutational symmetry of three objects, $S_3$, and introduced a Minimal $S_3$--invariant Extension of the Standard Model. In this model, $S_3$ is imposed as a fundamental symmetry in the matter sector which is only spontaneously broken together with the electroweak gauge symmetry. This assumption leads to extend the concept of flavour and generations to the Higgs sector. Hence, going to the irreducible representations of $S_3$, the model has one Higgs $SU(2)_L$ doublet in the $S_3$--singlet representation plus two more Higgs $SU(2)_L$ doublets which can only belong to the two components of the $S_3$--doublet representation. The fermion content of the Standard Model is left unaltered. In this way, all the matter fields - Higgs, quarks and lepton fields including the right-handed neutrino fields - belong to the three dimensional representation $1_s \oplus 2$ of the permutation group $S_3$. The leptonic sector is further constrained by an Abelian $Z_2$ symmetry. A defined structure of the Yukawa couplings is obtained which permits the calculation of mass and mixing matrices for quarks and leptons in a unified way. The Majorana neutrinos acquire mass via the type I seesaw mechanism. In a recent paper, O. Felix, A. Mondragón, M. Mondragón and E. Peinado $[57]$ reparametrized the mass matrices of charged leptons and neutrinos in terms of the respective mass eigenvalues and derived explicit analytic and exact expressions in closed form for the mixing angles appearing in the $U^\text{PMNS}$ matrix as functions of the masses of charged leptons and neutrinos and one Majorana phase $\Phi_\nu$. The $U^\text{PMNS}$ matrix has also one Dirac phase which has its origin in the charged lepton mass matrix. The numerical values of the mixing angles $\theta_{13}$ and $\theta_{23}$ are determined by the mass of charged leptons only in very good agreement with the best fit experimental values. The solar mixing angle $\theta_{12}$ is almost insensitive to the values of the masses of the charged leptons, but its experimental value allows the determination of the neutrino mass spectrum which has an inverted hierarchy with the values $|m_{\nu_2}| = 0.0507$ eV, $|m_{\nu_1}| = 0.0499$ eV and $|m_{\nu_3}| =$
0.0193 eV. A complete and detailed discussion of the Majorana phases of the neutrino mixing matrix in this model is given in J. Kubo [58]. A numerical analysis of the quark mass matrices and the $V_{\text{CKM}}$ matrix gives one set of parameters that are consistent with the experimental values given by the Particle Data Group [59]. A slightly less sketchy explanation of this model is given in the next section.

**A MINIMAL $S_3$– INvariant Extension of the Standard Model**

Recently, a minimal $S_3$–invariant extension of the Standard Model was suggested in [33], in this section I will explain in a slightly more detailed fashion the main ideas of this model and some recent results on neutrino masses and mixings.

*Symmetric Lagrangian and fermions masses*. In the Standard Model analogous fermions in different generations have completely identical couplings to all gauge bosons of the strong, weak and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to permutations of the left and right fermionic fields.

The six possible permutations of three objects $(f_1, f_2, f_3)$ are elements of the permutation group $S_3$. This is the discrete, non-Abelian group with the smallest number of elements. The three-dimensional real representation is not an irreducible representation of $S_3$, it can be decomposed into the direct sum of a doublet and a singlet, $1_s \oplus 2$. The direct product of two doublets may be decomposed into the direct sum of two singlets and one doublet, $2 \otimes 2 = 1_s \oplus 1_A + 2$. The antisymmetric singlet is not invariant under $S_3$.

Since the Standard Model has only one Higgs $SU(2)_L$ doublet, which can only be an $S_3$ singlet, it can only give mass to the quark or charged lepton in the $S_3$ singlet representation, one in each family, without breaking the $S_3$ symmetry. Therefore, in order to impose $S_3$ as a fundamental symmetry, unbroken at the Fermi scale, we are led to extend the concept of flavour and generations to the Higgs sector of the theory. Hence, going to the irreducible representations of $S_3$, we add to the Higgs $SU(2)_L$ doublet in the $S_3$–singlet representation, two more $SU(2)_L$ doublet in the $S_3$–doublet representation. In this way, all the quark, lepton and Higgs fields, $Q^T = (u_L, d_L), u_R, d_R, L^T = (\nu_L, e_L), e_R, \nu_R$ and $H$, are in reducible representations $1_s \oplus 2$. The most general renormalizable Yukawa interactions are given by

$$\mathcal{L}_Y = \mathcal{L}_{Y_D} + \mathcal{L}_{Y_U} + \mathcal{L}_{Y_E} + \mathcal{L}_{Y_{\nu}},$$

where

$$\mathcal{L}_{Y_{D,E}} = -Y_1^d \overline{Q}_1 H_S d_{IR} - Y_3^d \overline{Q}_3 H_S d_{sR}$$

$$-Y_3^d \left[ \overline{Q}_I (\sigma_1) I_J H_{1d_{IR}} - \overline{Q}_I (\sigma_3) I_J H_{2d_{IR}} \right]$$

$$-Y_4^d \overline{Q}_s H_{1d_{IR}} - Y_5^d \overline{Q}_l H_{1d_{sR}} + \text{h.c.},$$

$$\mathcal{L}_{Y_{U,\nu}} = -Y_1^u \overline{Q}_l (i\sigma_2) H_S^* u_{IR} - Y_3^u \overline{Q}_3 (i\sigma_2) H_S^* u_{sR}$$
\[-Y_u^I \left[ \overline{\sigma}_I(i\sigma_2)H^*_I u_{JR} - \overline{\sigma}_I(i\sigma_2)H^*_I u_{JR} \right] - Y_u^I \overline{\sigma}_3(i\sigma_2)H^*_I u_{JR} - Y_u^I \overline{\sigma}_I(i\sigma_2)H^*_I u_{sR} + \text{h.c.}, \tag{5}\]

The fields in the $S_3$-doublets carry capital indices $I$ and $J$, which run from 1 to 2 and the singlets are denoted by the subscript $s$.

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos

\[\mathcal{L}_M = -M_1 \nu^T_{IR} C \nu_{IR} - M_3 \nu^T_{3R} C \nu_{3R}. \tag{6}\]

Due to the presence of three Higgs fields, the Higgs potential $V_H(H_S, H_D)$ is more complicated than that of the Standard Model. This potential was analyzed by Pakvasa and Sugawara [60], see also Kubo [61], who found that in addition to the $S_3$ symmetry, it has a permutational symmetry $S_2$: $H_1 \leftrightarrow H_2$, which is not a subgroup of the flavour group $S_3$ and an Abelian discrete symmetry that will be used for selection rules of the Yukawa couplings in the leptonic sector. Here, we will assume that the vacuum respects the accidental $S_2$ symmetry of the Higgs potential and $\langle H_1 \rangle = \langle H_2 \rangle$.

With these assumptions, the Yukawa interactions, eqs. (4)-(5) yield mass matrices, for all fermions in the theory, of the general form

\[M = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}. \tag{7}\]

The Majorana masses for the left neutrinos $\nu_L$ will be obtained from the see-saw mechanism. The corresponding mass matrix is given by

\[M_\nu = M_{\nu_D} \tilde{M}^{-1} (M_{\nu_D})^T \tag{8}\]

where $\tilde{M} = \text{diag}(M_1, M_1, M_3)$.

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry $S_3$.

The mass matrices are diagonalized by bi-unitary transformations as

\[U_{d(u,e)}^T M_{d(u,e)} U_{d(u,e)R} = \text{diag}(m_{d(u,e)}, m_{s(c,\mu)}, m_{p(t,\tau)}), \]
\[U_{\nu}^T M_\nu U_{\nu} = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}). \tag{9}\]

The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The mixing matrices are, by definition,

\[V_{CKM} = U_{uL}^T U_{dL}, \quad V_{PMNS} = U_{eL}^T U_{\nu}. \tag{10}\]
Leptonic sector and $Z_2$ symmetry.

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian $Z_2$ symmetry. A set of charge assignments of $Z_2$, compatible with the experimental data on masses and mixings in the leptonic sector is given in Table 1.

| $H_S$, $\nu_{3R}$ | $H_1$, $L_3$, $L_I$, $e_{3R}$, $e_{IR}$, $\nu_{IR}$ |
|-------------------|--------------------------------------------------|

Table I. $Z_2$ assignment in the leptonic sector.

These $Z_2$ assignments forbid certain Yukawa couplings,

$$Y^e_1 = Y^e_3 = Y^\nu_1 = Y^\nu_3 = 0.$$  \hspace{1cm} (11)

Therefore, the corresponding entries in the mass matrices vanish, i.e., $\mu^e_1 = \mu^e_3 = 0$ and $\mu^\nu_1 = \mu^\nu_3 = 0$.

The mass matrix of the charged leptons

The mass matrix of the charged leptons takes the form

$$M_e = m_e \left( \begin{array}{ccc} \bar{\mu}_2 & \bar{\mu}_2 & \bar{\mu}_5 \\ \bar{\mu}_2 & -\bar{\mu}_2 & \bar{\mu}_5 \\ \bar{\mu}_4 & \bar{\mu}_4 & 0 \end{array} \right).$$  \hspace{1cm} (12)

The unitary matrix $U_{eL}$ that enters in the definition of the mixing matrix, $U_{PMNS}$, is calculated from

$$U_{eL}^\dagger M_e M_e^\dagger U_{eL} = \text{diag}(m_e^2, m_\mu^2, m_\tau^2),$$  \hspace{1cm} (13)

The entries in the mass matrix squared, $M_e M_e^\dagger$, may readily be expressed in terms of the mass eigenvalues $(m_e^2, m_\mu^2, m_\tau^2)$. Then, the matrix $U_{eL}$ may be expressed in terms of the charged lepton masses and one Dirac phase,

$$U_{eL} \approx \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_e^2}{m_\mu^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \\ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_\mu^2}{m_\tau^2}} \end{array} \right) \right) \right).$$  \hspace{1cm} (14)
The mass matrix of the neutrinos

According with the $Z_2$ selection rule eq. (11), $\mu_1^{\nu_D} = \mu_5^{\nu} = 0$ in (12). Then, the mass matrix for the left-handed Majorana neutrinos obtained from the see-saw mechanism takes the form

$$M_\nu = M_{\nu_D} \tilde{M}^{-1} (M_{\nu_D})^T$$

as in the case of the charged leptons, the matrix $M_{\nu_D}$ has been reparametrized in terms of its eigenvalues, the complex neutrino masses.

The unitary matrix $U_\nu$ that brings $M_{\nu_D}$ to diagonal form is

$$U_\nu = \begin{pmatrix}
\sqrt{m_{\nu_2} - m_{\nu_3}} & \sqrt{m_{\nu_3} - m_{\nu_1}} & \sqrt{m_{\nu_1} - m_{\nu_2}} \\
0 & \sqrt{m_{\nu_3} - m_{\nu_1}} & \sqrt{m_{\nu_1} - m_{\nu_2}} \\
-\sqrt{m_{\nu_3} - m_{\nu_1}} & -\sqrt{m_{\nu_3} - m_{\nu_1}} & \sqrt{m_{\nu_1} - m_{\nu_2}} \\
\end{pmatrix}, \quad (15)$$

The unitarity of $U_\nu$ constrains its entries to be real. This condition fixes the phases $\phi_1$ and $\phi_2$ as

$$|m_{\nu_1}| \sin \phi_1 = |m_{\nu_2}| \sin \phi_2 = |m_{\nu_3}| \sin \phi_\nu = 0 \quad (17)$$

The only free parameter in $M_\nu$ and $U_\nu$, other than the real neutrino masses $|m_{\nu_1}|$, $|m_{\nu_2}|$ and $|m_{\nu_3}|$, is the phase $\phi_\nu$.

The neutrino mixing matrix

The neutrino mixing matrix $V_{PMNS}$, in the standard form advocated by the PDG, is obtained by taking the product $U_{eL}^T U_\nu$ and making an appropriate transformation of phases, $U_{PMNS}$ is, then equal to

$$K = \begin{pmatrix}
\frac{1}{\sqrt{2}} x \sin \eta + \frac{\sqrt{1-2x}}{\sqrt{1-x^2}} \cos \eta & \frac{1}{\sqrt{2}} x \cos \eta - \frac{\sqrt{1-2x}}{\sqrt{1-x^2}} \sin \eta & -\frac{1}{\sqrt{2}} x e^{-i\delta_x} \\
\frac{1}{\sqrt{2}} \sin \eta - \frac{x}{\sqrt{1-x^2}} \cos \eta e^{i\delta_x} & \frac{1}{\sqrt{2}} \sin \eta + \frac{x}{\sqrt{1-x^2}} \cos \eta e^{i\delta_x} & \frac{1}{\sqrt{2}} \sin \eta e^{i\delta_x} \\
\frac{1}{\sqrt{2}} \sin \eta - \frac{\sqrt{z}}{\sqrt{1+z}} \cos \eta e^{i\delta_x} & \frac{1}{\sqrt{2}} \sin \eta + \frac{\sqrt{z}}{\sqrt{1+z}} \cos \eta e^{i\delta_x} & \frac{1}{\sqrt{2}} \sin \eta e^{i\delta_x} \\
\end{pmatrix}, \quad (18)$$

where

$$\sin \eta = \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} - m_{\nu_1}} \quad x = \frac{m_\nu}{m_\mu}, \quad y = \frac{m_\mu^2 + m_\tau^2}{m_\mu^2}, \quad \text{and} \quad z = \left( \frac{m_\mu m_\mu}{m_\tau^2} \right)^2 \quad (19)$$
and $K = \text{diag}(1, e^{i\alpha}, e^{i\beta})$ is the diagonal matrix of the Majorana phases.

Explicit expressions for the mixing angles in terms of the lepton masses are obtained from a comparison of $U_{PMNS}^{th}$, eq.(18) with the standard parametrization advocated by the PDG\[59].

$$\tan \theta_{12} \approx \sqrt{\frac{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu} - |m_{\nu_3}| \cos \phi_{\nu}}{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu} + |m_{\nu_3}| \cos \phi_{\nu}}},$$

$$\sin \theta_{13} \approx \frac{1}{\sqrt{2} m_{\nu}} \times \frac{1}{\sqrt{1 - \left(\frac{m_e}{m_\mu}\right)^2}}, \text{ and } \sin \theta_{23} \approx -\frac{1}{\sqrt{2}} \sqrt{\frac{1}{1 - \frac{1}{2} \left(\frac{m_e}{m_\mu}\right)^2}},$$

Similarly, the Majorana phases are given by

$$\sin 2\alpha = \sin(\phi_1 - \phi_2) = \frac{|m_{\nu_3}| \sin \phi_{\nu}}{|m_{\nu_3}|^2} \times \left(\sqrt{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu} + |m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu}}\right)$$

$$\sin 2\beta = \sin(\phi_1 - \phi_{\nu}) = \frac{\sin \phi_{\nu}}{|m_{\nu_3}|} \left(|m_{\nu_3}| \sqrt{1 - \sin^2 \phi_{\nu} + |m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu}}\right),$$

A detailed discussion of the Majorana phases in the neutrino mixing matrix $U_{PMNS}$ obtained in our model is given in J. Kubo\[58].

**Neutrino masses and mixings**

In this model, $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ are determined by the masses of the charged leptons in very good agreement with the experimental values\[62, 63, 64, 65],

$$(\sin^2 \theta_{13})^{th} = 1.1 \times 10^{-5}, \quad (\sin^2 \theta_{13})^{exp} \leq 0.046,$$

and

$$(\sin^2 \theta_{23})^{th} = 0.49, \quad (\sin^2 \theta_{23})^{exp} = 0.5^{+0.06}_{-0.05}.$$}

In the present model, the experimental restriction $|\Delta m^2_{12}| < |\Delta m^2_{13}|$ implies an inverted neutrino mass spectrum, $|m_{\nu_3}| < |m_{\nu_1}| < |m_{\nu_2}|$\[33].

The mass $|m_{\nu_2}|$ assumes its minimal value when $\sin \phi_{\nu}$ vanishes, then

$$|m_{\nu_2}| \approx \sqrt{\frac{\Delta m^2_{13}}{\sin 2\theta_{12}}}. \quad (25)$$

Hence, we find

$$|m_{\nu_2}| \approx 0.0507 eV, \quad |m_{\nu_1}| \approx 0.0499 eV, \quad |m_{\nu_3}| \approx 0.0193 eV \quad (26)$$
where we used the values $\Delta m_{12}^2 = 2.2^{+0.37}_{-0.27} \times 10^{-3} \text{eV}^2$ and $\sin^2 \theta_{12} = 0.31^{+0.02}_{-0.03}$ taken from M. Maltoni et al. \cite{64,65} and G. L. Fogli et al. \cite{63}.

With those values for the neutrino masses we compute the effective electron neutrino mass $m_\beta$.

$$m_\beta = \left[ \sum_i |U_{ei}|^2 m_{\nu i}^2 \right]^{\frac{1}{2}} = 0.0502 \text{eV}, \quad (27)$$

well below the upper bound $m_\beta < 1.8 \text{eV}$ coming from the tritium $\beta$-decay experiments \cite{23,63,64}.

**The hadronic sector**

The $Z_2$ assignements in the hadronic sector are independent of those in the leptonic sector. Hence, in principle, it can be assumed that $Z_2$ is a good symmetry at a more fundamental level and verify that $Z_2$ is free from any quantum anomaly. However, if we give all quarks even parity as we did to the charged leptons, the Yukawa couplings $Y_1^{u,d}$ and $Y_3^{u,d}$ will be forbidden. Consequently, the squared mass matrix, of the $u-$quarks, $M_u M_u^\dagger$ would have a texture similar to the texture of the mass matrix of the Majorana neutrinos, given in (12), which would lead to a $U_\nu$ that produces large values of the quark mixing angles in disagreement with the small experimental values.

Therefore, to give one set of parameters that are consistent with the experimental values given by the Particle Data Group \cite{59}, and show that the model is phenomenologically viable, we proceeded under the assumption that $Z_2$ is explicitly broken in the hadronic sector. Since all the $S_3$ invariant Yukawa couplings are now allowed, the mass matrices for the quarks take the general form (7), where all the entries can be complex.

One can easily see that all the phases, except for those of $\mu_1^{u,d}$ and $\mu_3^{u,d}$, can be removed through an appropriate redefinition of the quark fields. Of course, only one of the four phases of $\mu_1^{u,d}$ and $\mu_3^{u,d}$ is observable in $V_{\text{CKM}}$. So, we assume that only $m_1^d$ is a complex number.

The gross structure of realistic mass matrices can be obtained, if $\mu_3^{u,d} \sim O(m_{t,b})$ and $\mu_1^{u,d} \sim O(m_{c,s})$ (to achieve realistic mass hierarchies), and the non-diagonal elements $\mu_4^{u,d}$ and $\mu_5^{u,d}$ along with $\mu_1^{u,d}$ can produce a realistic mixing among the quarks. There are 10 real parameters and one phase to produce six quark masses, three mixing angles and one CP-violating phase. The set of dimensionless parameters

\[
\begin{align*}
    m_1^u/m_0^u &= -0.000293, \\
    m_4^u/m_0^u &= 0.31, \\
    m_1^d/m_0^d &= 0.0004, \\
    m_4^d/m_0^d &= 0.283
\end{align*}
\]

yields the mass hierarchies

$$m_u/m_t = 1.33 \times 10^{-5}, \quad m_c/m_t = 2.99 \times 10^{-3},$$


\[ m_d/m_b = 1.31 \times 10^{-3}, \quad m_s/m_b = 1.17 \times 10^{-2}, \]  

(29)

where \( m_0^u = \mu_3^u \) and \( m_0^d = \text{Re}(\mu_3^d) \), and the mixing matrix becomes

\[
V_{\text{CKM}} = U_{\mu L}^\dagger U_{\mu L}^L = \begin{pmatrix}
0.968 + 0.117I & 0.198 + 0.0974I & -0.00253 - 0.00354I \\
-0.198 + 0.0969I & 0.968 - 0.115I & -0.0222 - 0.0376I \\
0.00211 + 0.00648I & 0.0179 - 0.0395I & 0.999 - 0.00206I
\end{pmatrix} \]  

(30)

The magnitudes of the elements are given by

\[ |V_{\text{CKM}}| = \begin{pmatrix}
0.975 & 0.221 & 0.00435 \\
0.221 & 0.974 & 0.0437 \\
0.00682 & 0.0434 & 0.999
\end{pmatrix}, \]  

(31)

which should be compared with the experimental values[59]

\[ |V_{\text{CKM}}^{\text{exp}}| = \begin{pmatrix}
0.9741 \text{ to } 0.9756 & 0.219 \text{ to } 0.226 & 0.0025 \text{ to } 0.0048 \\
0.219 \text{ to } 0.226 & 0.9732 \text{ to } 0.9748 & 0.038 \text{ to } 0.044 \\
0.004 \text{ to } 0.014 & 0.037 \text{ to } 0.044 & 0.9990 \text{ to } 0.9993
\end{pmatrix}. \]  

(32)

Note that the mixing matrix (30) is NOT in the standard parametrization. So, we give the invariant measure of CP-violations[67]

\[ J = \text{Im} [(V_{\text{CKM}})_{11}(V_{\text{CKM}})_{22}(V_{\text{CKM}}^*{}_{12})(V_{\text{CKM}}^*{}_{21})] = 2.5 \times 10^{-5} \]  

(33)

for the choice (28), which is slightly larger than the experimental value \((3.0 \pm 0.3) \times 10^{-5}\) (see [59] and also [68]). The angles of the unitarity triangle for \( V_{\text{CKM}} \) (30) are given by

\[ \phi_1 \simeq 22^\circ, \quad \phi_3 \simeq 38^\circ, \]  

(34)

where the experimental values are: \( \phi_1 = 24^\circ \pm 4^\circ \) and \( \phi_3 = 59^\circ \pm 13^\circ \) [59]. The normalization masses \( m_0^u \) and \( m_0^d \) are fixed at

\[ m_0^u = 174 \text{ GeV}, \quad m_0^d = 1.8 \text{ GeV} \]  

(35)

for \( m_t = 174 \text{ GeV} \) and \( m_b = 3 \text{ GeV} \), yielding that \( m_u \simeq 2.3 \text{ MeV}, \quad m_c \simeq 0.52 \text{ GeV}, \quad m_d \simeq 3.9 \text{ MeV} \) and \( m_s = 0.035 \text{ GeV} \). Although these values cannot be directly compared with the running masses, because our calculation is of the tree level, it is nevertheless worthwhile to observe how close they are to [26]

\[
m_u(M_Z) = 0.9 - 2.9 \text{ MeV}, \quad m_d(M_Z) = 1.8 - 5.3 \text{ MeV}, \\
m_c(M_Z) = 0.53 - 0.68 \text{ GeV}, \quad m_s(M_Z) = 0.035 - 0.100 \text{ GeV}, \\
m_t(M_Z) = 168 - 180 \text{ GeV}, \quad m_b(M_Z) = 2.8 - 3.0 \text{ GeV}. \]  

(36)
CONCLUSIONS

The recent advances in the experimental knowledge of flavour physics, CP-violation and fermion masses and mixings triggered an enormous theoretical activity aimed to uncover the nature of this new physics. Important advances have been made in the further development of the already existing mechanisms and theories and the proposal of ingenious new ideas.

As an instance of the first approach, here I discussed the unified SUSY SO(10) theory with an additional $SU(3)_f$ flavour symmetry explored by G.G. Ross and L. Velasco-Sevilla [44, 45, 46]. The phenomenological success of this kind of theories with a continuous gauged family symmetry is achieved through the details of the symmetry breaking vacuum and elaborate mechanisms for its alignment. In this class of models, the physics of vacuum alignment simplifies if the continuum family symmetry is replaced by a discrete non-Abelian family symmetry as shown by A. Aranda, C.D. Carone and R.F. Lebed [47,48,49] who replaced the discrete non-Abelian group $T' \otimes Z_3$ for $SU(2)_f$ in the SUSY $SU(5) \times SU(2)$ unified theory of Barbieri, Romanino et al [50,51,52], and found that the reduction of the underlying continuous family symmetry to a discrete subgroup renders the desired vacuum alignment a generic property of such models.

As an example of the second approach, I discussed two extensions of the Standard Model in which the Higgs sector is modified and have an additional $S_3$ non-Abelian symmetry. They have this symmetry in common with the phenomenologically succesful efforts of A. Mondragón and E. Rodríguez-Jáuregui [40, 41] to uncover a flavour $S_3$ symmetry in the Fritzsch texture zeroes of the quark mass matrices and the $V_{CKM}$ phenomenology.

In the model proposed and discussed by R. N. Mohapatra, A. Pérez-Lorenzana and C.A. de S. Pires [55], there are no right handed neutrinos but additional Higgs triplets which acquire naturally small vacuum expectation values due to the type II see-saw mechanism. The presence of a global $S_3 \otimes U(1)_{\mu-\tau}$ symmetry leads naturally to the desired neutrino mass textures and generates the desired small splittings among neutrinos in fair agreement with experiment.

In the Minimal $S_3$-invariant Extension of the Standard Model proposed by J. Kubo, A. Mondragón, M. Mondragón and E. Rodríguez-Jáuregui [33], the concept of flavour and generations is extended to the Higgs sector by introducing three Higgs fields that are $SU(2)_L$ doublets in such away that all matter fields - lepton, quark and Higgs fields - belong to the three dimensional reducible $1 \oplus 2$ representation of the permutation group $S_3$. A well defined structure of the Yukawa couplings is obtained which permits the calculation of mass and mixing matrices for quarks and leptons in a unified way. A further reduction of redundant parameters is achieved in the leptonic sector by introducing a $Z_2$ symmetry. In this model, the Majorana neutrinos acquire mass via the type I see-saw mechanism. The flavour symmetry group $S_3 \otimes Z_2$ relates the mass spectrum and mixings. This allows the computation of the neutrino mixing matrix explicitly in terms of the masses of the charged leptons and neutrinos [57]. The magnitudes of the three neutrino mixing angles are determined by the interplay of the flavour $S_3 \times Z_2$ symmetry, the see-saw mechanism and the charged lepton mass hierarchy. It is also found that the lepton mixing matrix $V_{PMNS}$ has one Dirac CP-violating phase and two Majorana phases. The
numerical values of the $\theta_{13}$ and $\theta_{23}$ mixing angles are determined by the charged leptons only in very good agreement with experiment. The solar mixing angle $\theta_{12}$ is almost insensitive to the values of the masses of the charged leptons but its experimental value allows to fix the scale and origin of the neutrino mass spectrum which has an inverted hierarchy with the values $|m_{\nu_2}| = 0.0507 \text{ eV}$, $|m_{\nu_1}| = 0.0499 \text{ eV}$ and $|m_{\nu_3}| = 0.0193 \text{ eV}$.

In conclusion, a discernible trend is perceptible in the formulation of symmetry based models of flavour, fermion masses and mixings and CP violation. In a bottom up approach, the search for simpler models starts with the formulation of a phenomenologically successful low energy theory with a minimal extension of the Standard Model and a discrete, non-Abelian flavour or family group $\tilde{G}_f$, and then, showing the possible embeddings of this theory into an SO(10) or SU(5) GUT and a continuous flavour group $\tilde{G}_f \subset G_f$.

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