Further applications of scheme for reducing numerical viscosity: 3D hypersonic flow

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Abstract. A Total Variation Diminishing (TVD) scheme implemented in a non structured finite volume formulation for solving the 3D Euler equations is presented. To simultaneously, achieve adequate accuracy in smooth flows, high resolution at flow discontinuities and to avoid spurious oscillations, different flux limiter functions are applied in a wave-to-wave basis. By appropriately using compressive limiter functions with waves of the family two to four (linear waves), and diffusive limiter functions with waves of the family one and five (non linear waves), important reductions of the numerical viscosity can be achieved. This sort of adaptive scheme has satisfactorily been applied to the shock tube problem and to the slip interface between two parallel flows. It has also been used when computing the flow over a spherically blunted bi-conic body at free stream Mach numbers varying from five to fifteen. The results obtained with the blunted body are presented here, and they show that the new adaptive scheme besides being less diffusive, it does not lose robustness regarding other TVD methods.

1. Introduction

When solving the fluid mechanic equations numerically using finite volume techniques, the necessity of computing convective fluxes arises. Traditionally, polynomial functions were used to approach the variations of variables. This approach gives excellent results when the variables undergo smooth variations but it has serious difficulties if the solution contains discontinuities. In these cases, the schemes that use second or higher order approximations present inconveniences during the convergence process and the solutions have oscillations next to the discontinuity. On the other hand, the schemes that use first order approximations generate solutions without oscillations but, the discontinuities may poorly be resolved (scheme highly diffusive).

To deal with this problem, flux limiter functions were built as linear combinations of first and second order approximations [1,2]. These schemes are known as TVD although, strictly speaking, the TVD condition has formally only been demonstrated for scalar convection equations. If in the linear combination, the first order approach has more weight than that of second order, the resultant scheme is diffusive and reciprocally, if the second order approach has more weight the scheme is compressive. When extending these concepts from systems of linear equations to non linear, the certainty that they continue being TVD is lost [3], and it becomes necessary to take decisions about as how to carry out such extension. By instance:

- To apply only one limiter function or so many as it is the number of equations.
- To apply the limiter functions according to the spectral decomposition of related variable jumps [3].
- What variable must be kept in mind for the calculation of the scalar limiter function.

For the more robust schemes the number of limiter functions is equal to the number of equations; and in addition, a spectral decomposition is used [3].

For the three-dimensional Euler equations system, the spectral decomposition makes that three lineally degenerate families of waves appear [4]. Discontinuities associated with these eigenvectors

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are very difficult to solve exactly except for schemes that use higher compressive limiter functions, however, these schemes are not very robust in solving discontinuities associated with the non linear wave families [2].

In some papers, looking for the best way to capture contact discontinuities without the characteristic smearing linked to degenerate linear waves, different schemes were proposed. One of the most popular was introduced by Harten [5] using an Artificial Compression Method (ACM) that renders the local slope of the minmod limiter steeper; however, if the field becomes genuinely nonlinear the limiter needed should not be so compressive [6]. The ACM implemented on superbee limiter functions [7,8] was not utilized in solving the gasdynamics equations [6], probably because it was recognized that it was not a high-resolution gasdynamics shock-capturing technique [3]. The ACM was also implemented on minmod limiter functions to solve the scalar wave and Burgers equations [6], and in second order central difference schemes for Euler equations [9]. More recent ACM applications are given in Ref. [10].

In this work an alternative scheme is presented, which has the capacity to solve satisfactorily discontinuities associated with lineally degenerate wave families and without losing robustness in dealing with gasdynamics shock waves. This new scheme was implemented in a three-dimensional computational code to solve the Euler equations using a non-structured finite volume technique.

2. Description of the scheme
The three-dimensional Euler equations can be written as:

$$\frac{\partial U}{\partial t} + \nabla \cdot F = 0$$

(1)

$U$ is the vector of conservative variables, and $F$ is the 3D vectorial flow.

The temporal change of the conservative variables can be expressed as:

$$U^{n+1} = U^n - \frac{\Delta t}{Vol} \sum_{faces} F^* \cdot n_i \cdot A_i$$

(2)

where the flux of the conservative variables “$F$” has been replaced by the numerical flux tensor “$F^*$”.

Equation 2 allows the use of a locally aligned system of coordinates whose unit vector $i$ coincides with the normal to the face “$l$” of the cell, and the unit vectors $j$ and $k$ are tangential directions. To achieve second order accuracy the numerical flux at the interface between cells “$l$” and “$l+1$” in the direction normal to the face “$l$” [11], is calculated by:

$$f_{r=l/2}^* = \frac{f_i + f_{i+1}}{2} + \frac{1}{2} \sum_m \Phi^m_{r=l/2} \cdot r^m_{r=l/2}$$

(3)

where $f_i$ and $f_{i+1}$ are the physical fluxes normal to the face in each cells, $r^m_{r=l/2}$ is the $m$-th right eigenvector, and $\Phi^m_{r=l/2}$ is, in the original Harten-Yee scheme [12-14], defined as:

$$\Phi^m_{r=l/2} = g^m_r + g^m_{r+1} - r^m_{r+1} \cdot [g^m_{r+1} + r^m_{r+1} \cdot a^m_{r+1/2}]$$

(4)

being:
\[ g_i^m = \frac{S}{2} \max \left[ 0, \min \left( \left[ \frac{p_i^{m+1/2}}{\alpha_i^{m+1/2}}, S \frac{\lambda_i}{\alpha_i^{m+1/2}} \right], \left[ \frac{p_i^{m-1/2}}{\alpha_i^{m-1/2}} \right] \right) \right] \] (5)

\[ S = \text{sign} \left( \lambda_i^m \right) \] (6)

\[ \alpha_{i+1/2}^m = \left\{ \begin{array}{ll}
\frac{g_{i+1}^m - g_i^m}{\alpha_{i+1/2}^m} & \text{if} \quad \alpha_{i+1/2}^m \neq 0 \\
0 & \text{if} \quad \alpha_{i+1/2}^m = 0
\end{array} \right. \] (7)

where \( \alpha_{i+1/2}^m \) is the jump of the conserved variables across the interfaces between cells “i” and “i+1”, and \( \lambda_i^m \) is the m-th eigenvalue of the Jacobian matrix. Since the local Riemann problem is solved with rotated data, the eigensystem is calculated in the locally aligned coordinate frame.

The limiter function given in Eq. (5) is known as minmod [1-4]. The minmod selects the minimum possible value, so that the scheme is TVD. The other end is the limiter function “superbee” thatpondersthe contribution of the high order flux [2]. The implementation of the “superbee” function leads to an excessively compressive scheme which it is not very robust for general practical aerospace applications.

In the numerical solution of the three-dimensional Euler equations five wave families appear. If the five wave families are enumerated in correspondence with their speed, being one the slowest and five the faster, it can be demonstrated that for waves of the families two to four, the characteristic velocities at both sides of the discontinuity are the same and equal to the velocity discontinuity [2,3]. This property makes very difficult to solve these waves accurately, unless they are solved diffusely.

In this work, it is explored the possibility of implementing different limiter functions for different wave families. The objective is to improve the numerical resolution of the discontinuities associated with the families two to four using compressive limiter functions (superbee), and without losing robustness due mainly to the use of diffusive limiter functions (minmod) for the wave families one and five.

To introduce in the numerical fluxes calculations the limiter function superbee the Eq. (5) is replaced by the following expression:

\[ g_i^m = \begin{cases} 
0 & \text{if} \quad \alpha_{i+1/2}^m \cdot \alpha_{i-1/2}^m < 0 \\
\max \left[ 0, \min \left( 2 \cdot r, 1 \right), \min \left( r, 2 \right) \right] \frac{1}{2} \lambda_i^m \cdot \alpha_i^{m+1/2} & \text{if} \quad \alpha_{i+1/2}^m \cdot \alpha_{i-1/2}^m \geq 0 
\end{cases} \] (8)

being:

\[ r = \frac{\lambda_i^{m+1/2}}{\lambda_i^{m-1/2}} \] (9)

To improve the overall scheme robustness, the implementation of different limiter functions is carried out only in those cells interfaces where the greater relative intensities of the discontinuities in
central waves are registered, and using the conventional Harten-Yee TVD scheme in all other cases. Notice that the comparison among the intensity of the waves cannot be made using directly the coefficients of the spectral decomposition \( \alpha_{i+1/2}^m \) since these coefficients depend on the module assigned to each eigenvector.

In the local coordinate system adopted for computing the numerical fluxes across each face, the corresponding eigenvectors are given by:

\[
\begin{bmatrix}
1 \\
u - a \\
v \\
w \\
H - u \cdot a \\
\end{bmatrix},
\begin{bmatrix}
1 \\
u \\
v \\
w \\
\frac{u^2 + v^2 + w^2}{2} \\
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
v \\
0 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
w \\
1 \\
\end{bmatrix},
\begin{bmatrix}
1 \\
u + a \\
v \\
w \\
H + u \cdot a \\
\end{bmatrix}
\]

where \( H \) is the stagnation enthalpy. It can be deduced from Eq. (10) that \( \alpha_{i+1/2}^1 \) and \( \alpha_{i+1/2}^5 \) measure the density jump in the waves 1, 2 and 5 respectively and that \( \alpha_{i+1/2}^3 \) and \( \alpha_{i+1/2}^4 \) measure the momentum jump in waves three and four. To compare these jumps it becomes necessary the select reference values for the density and velocity. Thus:

\[
I_1 = \frac{\alpha_{i+1/2}^1}{\rho_{ref}}, \quad I_2 = \frac{\alpha_{i+1/2}^2}{\rho_{ref}}, \quad I_3 = \frac{\alpha_{i+1/2}^3}{\rho_{ref} u_{ref}}, \quad I_4 = \frac{\alpha_{i+1/2}^4}{\rho_{ref} u_{ref}}, \quad I_5 = \frac{\alpha_{i+1/2}^5}{\rho_{ref}}
\]

In this investigation is taken as density reference \( \rho_{ref} = 0.5 (\rho_i + \rho_{i+1}) \) and as the velocity reference the average of the sound velocities at the cells \( u_{ref} = 0.5 (c_i + c_{i+1}) \).

Finally, if the maximum of \( I_1, I_5 \) is higher than the maximum of \( I_2, I_4 \), the conventional Harten-Yee TVD scheme is used; otherwise, the values of \( g_i^2, g_i^3, g_i^4 \) are calculated with the limiter function superbee and \( g_i^1, g_i^5 \) with the limiter function minmod.

### 3. Implementation

#### 3.1 The limiter functions evaluation

For the evaluation of \( g_i^n \) and \( g_i^{n+1} \) in Eq. (4), it is necessary to calculate the spectral decompositions of the conservative variables increments at the interfaces “i-1/2”, “i+1/2” and “i+3/2”. In the context of three-dimensional not structured meshes of tetrahedrons, the identification of the cells “i” and “i+1” is intuitive (they are two cells that share a face) but the determination of the points “i-1” and “i+2” is not direct. If two tetrahedrons that share a face are analyzed, the nodes not belonging to the common face can be used as representative points for “i-1” and “i+2”. Then, these points can be used as imaginary cells. In this work these ideas have been implemented, being the nodal values calculated as a pondered average of the conservative variables between all cells that are in contact with the nodes “i-1” and “i+2”. Such pondered average is given by [15]:

\[
U_{node k} = \frac{\sum_{i=1}^{n} U_{cell i} d_{GC cell i - node k}}{\sum_{i=1}^{n} d_{GC cell i - node k}} (12)
\]
where \( d_{GC_n \text{-node } k} \) is the distance that separates the gravity center of the cell \( i \) from the node \( k \), and \( n \) is the cell number in contact with the node \( k \).

3.2 Boundary conditions
The treatment of the boundary conditions is carried out through the imaginary cells technique [2,3]. Five different types of boundaries are considered: 1 – Subsonic inlet, 2 - Supersonic inlet, 3 – Subsonic exit, 4 - Supersonic exit, 5 - Non penetration (solid boundary and symmetry).

4. Test cases
In order to verify the accuracy and robustness of the adaptive scheme three test cases were simulated. The first one is the flow inside of a shock tube. This example was used to explore the capacity of the adaptive scheme which has been described, to model contact discontinuities; the flow does not have velocity regarding the contact discontinuity (discontinuity in waves of the family 2). The second test is the simulation of a slip interface layer between two flows with different velocities and densities, but equal pressures. This test was chosen to study the capacity of the scheme to solve flows in which the discontinuity is in the velocity tangent to a given interface (discontinuity in waves of the families 3 and 4). The third case is the flow around a spherically blunted bi-conic body at several supersonic Mach numbers. The object of this third case was to evaluate the robustness of the scheme to capture strong shock waves (discontinuity in waves of the families 1 and 5).

Although the main objective of this work is to present computer results pertaining to the third case, some brief comments on the other two cases are appropriate. A more detailed description of test cases one and two may be found in Ref [15].

4.1 Flow inside a shock tube (ST)
It is well known that an analytical solution for the flow inside a ST can be built considering three types of waves: shocks, expansion fans and contact discontinuities [2-4]. Figures 1 and 2 show analytical and computed solutions in terms of the density plotted as a function of \( x/L \) (being \( x \) the distance along the tube and \( L \) its total length). In Figure 1, the analytical solution is compared with the results obtained applying both, the adaptive TVD scheme here proposed and the conventional Harten-Yee TVD scheme. It can be appreciated that the simulation of the non linear waves is equally good for both schemes TVD, however, the capture of the contact discontinuity has been improved notably. In Figure 2, the results obtained applying the adaptive scheme is compared with those obtained using superbee limiter functions in all waves. It shall be remarked that the results obtained using the flux limiters superbee have a tendency to show oscillations, which are not present if the new scheme is used.

4.2 Slip surface
The second test case is depicted by an air layer flowing over another one. The velocities and densities in both flows are different, but the pressures are equal (no forces can be supported by the slip surface). The straightforward solution predicts the slip of a flow on another without interferences. However, due to the numerical viscosity, the computed solution produces an apparent mixture zone that gets wider downstream the end of an assumed splitter plane. The spreading of such unphysical mixing region can be interpreted as a measure of the accuracy of the numerical method. In Figure 3 is presented the absolute value of the percentage of error incurred in the velocity calculated at a lateral plane of the computational domain [15]. The results using the traditional Harten-Yee TVD scheme, are shown in the upper part, those obtained with the new scheme in the middle, and in the lower part, are shown the results calculated using superbee limiter functions in all waves.
Figure 1. Shock tube results. Comparison between analytical solution, Harten-Yee TVD and proposed scheme.

Figure 2. Shock tube results. Comparison between scheme using superbee limiter function for all waves and the new scheme.
As it can be deduced from Figure 3, the proposed adaptive scheme is notably less diffusive than the conventional Harten-Yee method, although more diffusive than the scheme using superbee in all five waves of the Euler equations.

4.3 Spherically blunted bi-conic body at supersonic speeds

It was analyzed the flow around the spherically blunted bi-conic configuration shown in Figure 4. The nose radius is of 0.15m, the aft part of the nose is conical with an angle of 10º and a length of 0.3m, and there after, there is another conical part with an angle of 20º and 0.2m of length. Two Mach numbers, 5 and 15 were considered for numerical simulations.

By means of these numerical studies, the following properties of the computer program were intended to be analyzed:

- The quality of the results describing the flow on the spherical nose.
- The ability for capturing accurately the shock wave between the first and second cone.
- The preservation of the axial symmetry of the flow despite using a 3D code.
- The potential of the new scheme for capturing strong shock waves. It is realized nevertheless, that the results at M = 15 can be questioned because real gas effects are not accounted for.

The mesh, shown in Figure 5, has 110000 cells and 23000 nodes. The boundary conditions are: (1) supersonic flow enters through the control volume front; (2) supersonic flow exits through the back plane and (3) non penetration on the body and in the planes y-x and z-x. The flow that enters is air and has a density of 1.225Kg/m³, and a pressure of 100000Pa. The free stream components in the directions of the z- and y-axis are null, and in the direction of the x-axis are 1690.3 m/s for M = 5, and 5070.9 m/s for M = 15.
In Figures 6 and 7 pressure distributions on the body for $M = 5$, are presented. Figure 6 shows the pressure distribution on the body, however due to the picture scale only the nose shock appears clearly defined. In Figure 7 the pressures plotted are of all the cells in contact with the body (from $0^\circ$ to $90^\circ$). It can clearly be perceived the axial symmetry of the results and the accurate capture of shock waves. The distributions of pressure agree satisfactorily if they are compared with results presented by other authors [16].

In Figures 8 and 9, the pressure distributions are plotted for $M = 15$. In Figure 8, again is shown the pressure distribution produced by the primary or nose shock. The resolution of the discontinuities is satisfactory, oscillations are not present and the axial symmetry of the flow is maintained appropriately.
It can be appreciated from Figures 7 and 9 that the spread of the pressure results, due to the cells arrangements on the body at each x-station, is very small compared with related local values.

Figure 10 shows the shock in front of the blunt nose of the body at $M = 15$ as it is determined by the new adaptive scheme and by the traditional Harten-Yee TVD technique. It can be observed that the two figures look very much alike. Consequently, the new scheme captures strong shock waves as well as it does the traditional Harten-Yee. Furthermore, Figure 11 which is a close up of the blunt body nose region, shows that the shock is captured between two grid cells only.
Figure 8. Pressure distribution over axial-symmetric body. $M = 15$. Primary shock.

Figure 9. Pressure distribution over axial-symmetric body. $M = 15$ external flow.

It is important to notice that a numerical scheme which uses only superbee flux limiters to simulate blunted bodies problem, introduces oscillations, produces negative pressures near discontinuities and it ends up being unstable. To emphasize this last point, two cases with different ways of introducing initial conditions were considered. In the first one, starting constant values typical of a free stream flow were used, and in the second one, the simulation was started from a “steady-state results” obtained after using the new adaptive scheme. In both cases, the scheme using superbee functions became unstable.
Figure 10. Shock wave over axial-symmetric body at M = 15. Right figure: Traditional Harten-Yee. Left figure: adaptive scheme.

Figure 11. Close up of the blunt body nose region. (1) $p/p_\infty > 272$; (2) $224 < p/p_\infty < 272$; (3) $176 < p/p_\infty < 224$; (4) $128 < p/p_\infty < 176$; (5) $80 < p/p_\infty < 128$; (6) $33 < p/p_\infty < 80$.
5. Conclusions
In previous test cases, it has been proven the capacity that the adaptive TVD scheme here described has to model contact discontinuities (ST test case), and to simulate interfaces between two flows with different velocities and densities (slip surface test case). Now, its potential to capture strong pressure discontinuities (shock waves) in supersonic flows has been tested.

After comparing results obtained with the proposed TVD scheme, with those obtained using the conventional TVD of Harten-Yee and with other solver that uses superbee functions as limiters for all numerical fluxes, the following conclusions can be written.

• From the Shock Tube test case: The adaptive TVD scheme diminishes the numerical viscosity with regards to the conventional Harten-Yee TVD and it does not introduce oscillations.

• From the Slippery Surface test case: The adaptive TVD scheme works more efficiently than the conventional Harten-Yee TVD. However, the scheme that only uses superbee limiters functions produces less numeric diffusion than the other two.

• From the Blunted Body test case: The adaptive TVD scheme proposed capture strong shocks as well as it does the conventional TVD of Harten-Yee. The much higher resolution of discontinuities (the shock in the nose region is captured between two grid cells, only), does not affect its robustness. No difficulties have ever been experimented in computing the flow around a blunted bi-conic body in the free stream Mach number range of 5 to 15. Attempts to apply the numerical scheme that only uses superbee limiters functions failed (generate oscillations, produces negative pressure and it finishes being unstable).

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References
[1] Sweeby P 1984 High resolution schemes using flux limiters for hyperbolic conservation laws SIAM Journal on Numerical Analysis 21 995-1011.
[2] Hirsch C 1992 Numerical Computation of Internal and External Flows, Vol.2 Computational Methods for Inviscid and Viscous Flows (London: John Wiley & Sons Ltd.).
[3] Toro E 1999 Riemann Solvers and Numerical Methods for Fluid Dynamics (Berlin: Springer-Verlag).
[4] Leveque R 1992 Numerical Methods for Conservation Law (Basel: Birkhäuser Verlag).
[5] Harten A 1978 The artificial compression method for computation of shocks and contact discontinuities: III self adjusting hybrid schemes Mathematics of Computation 32 363-99.
[6] Rider W 1992 The design of high-resolution upwind shock-capturing methods Los Alamos National Laboratory.
[7] Rider W and Woodruff S 1991 High-order solute tracking in two-phase thermal hydraulics LA-UR-91-2263 Los Alamos National Laboratory.
[8] RELAP5-3D© Code Development Team 2005 RELAP5-3D© code manual. Volume 1: code structure, system models, and solution methods Idaho National Engineering and Environmental Laboratory.
[9] Lie K and Noelle S 2003 On the artificial compression method for second-order nonoscillatory central difference schemes for systems of conservation laws SIAM Journal of Scientific Computation 24 1157-74.
[10] Lo S Blaisdell G and Lyrintzis A 2007 High-order shock capturing schemes for turbulence calculations AIAA Paper Nº2007-827.
[11] Udrea B 1999 An advanced implicit solver for MHD PhD Thesis, University of Washington.
[12] Harten A 1982 On to class of high resolution total-variation-stable finite difference schemes Technical Report NYU.

[13] Harten A 1983 High resolution schemes for hyperbolic conservation laws Journal of Computational Physics 49 250-7.

[14] Yee H Warming R and Harten A 1985 Implicit total variation diminishing (TVD) schemes for steady-state calculations Journal of Computational Physics 57 327-60.

[15] Falcinelli O Elaskar S and Tamagno J 2008 Reducing the numerical viscosity in non structured three-dimensional finite volumes computations Journal of Spacecraft and Rocket, AIAA 45 506-9.

[16] Pirolo J and Wardlaw Jr. A 1991 High temperature effects for missile type bodies using the Euler solver ZEUS AIAA Paper Nº91-3259.