Dynamical Mass Generation in a Finite-Temperature Abelian Gauge Theory

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Abstract

We write down the gap equation for the fermion self-energy in a finite-temperature abelian gauge theory in three dimensions. The instantaneous approximation is relaxed, momentum-dependent fermion and photon self-energies are considered, and the corresponding Schwinger-Dyson equation is solved numerically. The relation between the zero-momentum and zero-temperature fermion self-energy and the critical temperature \( T_c \), above which there is no dynamical mass generation, is then studied. We also investigate the effect which the number of fermion flavours \( N_f \) has on the results, and we give the phase diagram of the theory with respect to \( T \) and \( N_f \).

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1 INTRODUCTION

In this paper we study dynamical fermion mass generation in a three-dimensional abelian gauge theory at finite temperature. The interest in phenomena associated with finite-temperature strongly coupled systems lies not only in their non-trivial field theoretical interpretation but also in the fact that in some cases specific comparisons between experimental and theoretical results can be made. Dynamical fermion mass generation is such a phenomenon whose role in such a priori unrelated effects as chiral symmetry breaking in QCD or superconductivity in solid state systems gives it a particular importance. In the latter case, given the fact that certain copper oxides exhibit almost two-dimensional high-temperature superconductivity, the study of 2 + 1 field theories can be particularly instructive.

Perturbation theory on its own can unfortunately do little in exploring the critical behavior of such systems. One has to use therefore non-perturbative techniques like effective potentials and Schwinger-Dyson (S-D) equations, which, although they cannot guarantee the precision of their quantitative results, provide interesting qualitative insights.

We proceed therefore by writing down the S-D equation for the fermion self-energy in the real-time formalism and then study its behaviour with temperature. In particular, of considerable interest is the quantity \( r = 2\Sigma(0)/k_B T_c \), where \( \Sigma(0) \) is the fermion self-energy at zero temperature and momentum and \( T_c \) is the critical temperature above which there is no dynamical mass generation. The identification of \( \Delta = 2\Sigma(0) \) with the superconductor gap makes the model in principle experimentally testable \[1\].

Similar investigations have been hitherto limited by the complexity of the
problem, in that the fermionic and photonic two-point functions which appear in the gap equation depend not on only one variable—as in the zero-temperature case—but on two independent ones. Responsible for that is the preferred reference frame (heat bath) associated with the temperature, which breaks Lorenz invariance. Thus the quantities of interest depend independently on energy and three-momentum.

Popular approaches so far use drastic approximations such as taking the fermion self-energy to be momentum independent and truncating the energy dependence of the photon self-energies, either by considering them as functions only of the four-momentum squared [2], or by taking the energy to be zero (the “instantaneous approximation”) [3]–[4].

The study presented here is the first attempt to relax simultaneously all these approximations by numerically solving the problem and keeping the correct momentum dependence of the self-energies. This allows us to explore the critical behaviour of the theory with respect to temperature and number of fermion flavours. However, due to the complexity of the equations, we neglect their imaginary parts, even though they could play an important role [5]. Moreover, we consider only the one-fermion-loop contribution to the photon polarization diagrams, an approximation justified for a large number of fermion flavours. We also choose to truncate the infinite S-D-equation hierarchy by replacing the full photon-fermion vertex by the bare one. Being aware of the severity of this approximation in relation to gauge-invariance and wave-function renormalization [6], we plan to make use of a more suitable truncation of the S-D hierarchy in a future publication.
2 THE GAP EQUATION

The Lagrangian of the three-dimensional abelian gauge theory under study is a variation of the usual $QED$ one with massless fermions:

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_a (i\partial - \tau_3 A) \psi_a$$  \hspace{1cm} (1)

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\alpha = 1, ..., N_f$ with $N_f$ the number of fermion flavours, the representation of the gamma matrices is four dimensional (a possible choice is $\gamma_0 = \text{diag}(i\sigma_3, -i\sigma_3)$, $\gamma_1 = \text{diag}(i\sigma_1, -i\sigma_1)$, $\gamma_2 = \text{diag}(i\sigma_2, -i\sigma_2)$, where $\sigma_i$ are the usual Pauli matrices) and

$$\tau_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Dynamical mass generation in this case is parity conserving, which could be energetically preferred to a parity non-conserving one [7]. We will refer to the corresponding model as $\tau_3 - QED$, as it has been used in [1] in connection with high-temperature superconductors.

The presence of the $\tau_3$ matrix is responsible for the fact that this model does not have chiral symmetries that would be broken by a non-zero local order parameter like $< \bar{\psi}_L \psi_R >$, which is associated with mass generation. Therefore, dynamical mass generation in this three-dimensional theory can occur at finite non-zero temperatures without breaking any global symmetries and therefore problems with the Mermin-Wagner theorem [8] are avoided. On the other hand, this will not influence the form of the gap equation, which is the same as for usual $QED$.

The corresponding S-D formalism for the two-point fermion function in the rest frame of the heat bath gives the following gap equation in the real-time for-
malism for the fermion self-energy $\Sigma$, for external momentum $p_\mu = (p_0, \vec{p})$ and loop momentum $k_\mu = (k_0, \vec{k})$ in Euclidean space:

$$\Sigma(p_0, |\vec{p}|) = \frac{\alpha}{N_f} \int \frac{d^3k}{(2\pi)^3} D_\beta(k_0, |\vec{k}|) S_\beta(p_0 - k_0, |\vec{p} - \vec{k}|)$$  \hspace{1cm} (2)

where $D_\beta$ and $S_\beta$ are the photon and fermion propagators, with the subscript $\beta$ indicating their temperature dependence, and $\alpha = e^2 N_f$ is the dimensionful coupling of the superrenormalizable theory under study. It can be considered as an effective ultra-violet cut-off of the model \cite{9}. In order to simplify the notation we write the first argument of the various functions symbolically as $p_0$ instead of the more accurate $\sqrt{-p_0^2}$. Since we are working at the one-loop level, an approach justified for large $N_f$, we do not expect field-doubling problems associated with the real-time formalism to play any direct role \cite{10}.

A bare photon-fermion vertex is used for simplicity, since a more involved vertex would lead us to a system of coupled integral equations. The propagators $D_\beta$ and $S_\beta$ in the Landau gauge at temperature $T$ are given by

$$D_\beta(k_0, |\vec{k}|) = \sum_{P=L,T} \frac{1}{k^2 + \Pi_P(k_0, |\vec{k}|)} + \frac{2\pi \delta(k^2 + \Pi_P(k_0, |\vec{k}|))}{e^{\beta |k_0|} - 1}$$

$$S_\beta(k_0, |\vec{k}|) = \left(\frac{1}{k^2 + \Sigma^2(k_0, |\vec{k}|)} - \frac{2\pi \delta(k^2 + \Sigma^2(k_0, |\vec{k}|))}{e^{\beta |k_0|} + 1}\right) \Sigma(k_0, |\vec{k}|)$$  \hspace{1cm} (3)

where $k^2 = k_\mu k_\mu$, $\beta = 1/k_B T$, we sum over the longitudinal and transverse photon polarizations $P = L, T$, we have dropped the $\vec{k}$ term of the fermion propagator since it disappears after the momentum integration, and we have suppressed the imaginary $i$'s since they appear on both sides of the gap equation.
The photon polarization functions $\Pi_{L,T}$ appearing above are given by

\[
\Pi_L(k_0, |\vec{k}|) = \frac{\alpha k}{8} + \Pi_{1k} \\
\Pi_T(k_0, |\vec{k}|) = \frac{\alpha k}{8} - \Pi_{1k} + \Pi_{2k},
\]

with

\[
\Pi_{1k} = \frac{2\alpha k^2}{\pi k^2} \int \frac{d|\vec{p}|}{e^{\beta |\vec{p}|} + 1} \left( 1 - \left( \frac{B_k + D_k}{2k^2} \right)^{1/2} \right) \\
\Pi_{2k} = \frac{2\alpha}{\pi} \int \frac{d|\vec{p}|}{e^{\beta |\vec{p}|} + 1} \left( 1 - \left( \frac{k^2(B_k + D_k)}{2D_k^2} \right)^{1/2} \right) \\
B_k = k^2 - 4\vec{p}^2 \\
D_k = (B_k^2 + 16k_0^2\vec{p}^2)^{1/2}
\]  

(4)

where the subscript $k$ is just a reminder that the quantities are momentum dependent. It is worth noting that these are calculated via a one-loop massless fermion diagram, using the fermion propagator of Eq.3. The quantities $\Pi_{1,2k}$ are the finite-temperature contributions to the photon polarization. They provide the thermal screening responsible for the softening of the infrared behaviour of the theory, since at small loop momenta they take values on the order of what is usually referred to as “plasmon mass” squared $\omega_P^2 = \frac{\alpha \ln 2}{\pi \beta}$ [3]. The one-loop approximation is justified for large $N_f$. However, the masslessness of the fermions in this calculation could introduce in principle consistency problems with Eq.2 which describes fermion mass generation. We will return to this issue later.

We are thus confronted with a three-dimensional non-linear integral equation for a function of two variables. An analytical study of the full problem seems an impossible task, so one is led to a computer simulation. Before proceeding to a numerical solution however, we can try to somewhat simplify the equation by noting that the delta function appearing in the photon propagator gives a negligible contribution. In fact, this function has two roots, one at very large momenta -of
order $\alpha$- where the fermion self-energy, a decreasing function with momentum, is vanishingly small, and one at very low momenta where the integrand is also small. Several authors make the approximation of dropping this delta-function \cite{4,12,13}, and we will also adopt it. The S-D equation therefore takes the following form

\[
\Sigma(p_0, |\vec{p}|) = \frac{\alpha}{N_f} \int \frac{d|\vec{k}|d\theta}{(2\pi)^3} \frac{\Sigma(p_0 - k_0, |\vec{p} - \vec{k}|)}{(p - k)^2 + \Sigma^2(p_0 - k_0, |\vec{p} - \vec{k}|)} \times \sum_{P=L,T} \frac{1}{k^2 + \Pi_P(k_0, |\vec{k}|)}
\]

\[
- \frac{\alpha}{N_f} \int \frac{|\vec{k}|d|\vec{k}|d\theta}{(2\pi)^2} \frac{\Sigma(E, |\vec{p} - \vec{k}|)}{2E(e^{\beta E} + 1)} \times \sum_{\epsilon=1,-1} \sum_{P=L,T} \frac{1}{(p_0 - \epsilon E)^2 + \vec{k}^2 + \Pi_P(p_0 - \epsilon E, |\vec{k}|)}
\]

where we sum over the photon polarizations $P = L, T$ and over the two roots of the delta function by introducing $\epsilon = 1, -1$, and $E$ is given by the relation $E^2 = |\vec{p} - \vec{k}|^2 + \Sigma^2(E, |\vec{p} - \vec{k}|)$.

We note that the definition of $E$ involves the fermion self-energy, which we are trying to solve for, calculated at $p_0 = E$. Therefore, we make the approximation $E \approx \sqrt{|\vec{p} - \vec{k}|^2 + \Sigma^2(0, 0)}$, which is valid in the limits $|\vec{p} - \vec{k}|^2 \gg \Sigma^2(0, 0)$ or $|\vec{p} - \vec{k}|^2 \ll \Sigma^2(0, 0)$. Another issue related to $E$ is that, for small external momenta, we have to calculate the photon polarization functions at regions which, if continued to Minkowski space, would make these functions imaginary \cite{2}. This is due to the fact that, whereas we study fermion mass generation, the polarization functions were calculated for massless fermions. In order to use consistently massless fermions in connection with the photon polarization, we should really calculate the functions $\Pi_P$ at energies $p_0 \pm |\vec{p} - \vec{k}|$ instead of $p_0 \pm E$. We checked that both prescriptions give similar results.

On physical grounds, we expect that increasing temperature ought to have decorrelating effects, which, after a certain point, should make the order parameter

\footnote{We assume of course that $\Sigma(0, |\vec{k}|) \approx \Sigma(\Sigma(0, 0), |\vec{k}|)$, since in the infrared, non-perturbative, region our formalism does not have the required accuracy. We checked however that our final results do not depend on this detail.}
\[< \bar{\psi}_L \psi_R >, \text{ and the fermion self-energy associated with it, vanish. In fact, we see that the effect of finite temperature, both directly \textit{via} the negative second term of Eq.}\text{5, and indirectly \textit{via} the photon polarization functions is to reduce the integrand for a given self-energy. It is therefore expected that, for large enough temperatures, the self-energy will not find enough support from the right-hand side and dynamical mass generation will be impossible, i.e. the only solution to the gap equation will be the trivial one. We proceed now to verify -and quantify- this expectation.}

3 \textbf{NUMERICAL RESULTS}

In order to attack the problem numerically, we have to discretize our external and loop momentum space having in mind the characteristic energy scales of the model as well as the expected behaviour of the relevant quantities. The coupling \(\alpha\) sets an effective UV-cut-off, as has been seen in previous zero-temperature studies \cite{9}. At small loop momenta, two quantities compete for the role of a physical IR-cut-off: the fermion self-energy \(\Sigma\) and \(k_B T\), the energy associated with the finite temperature. The latter would become in effect only when both external and loop-momenta, as well as the fermion self-energy, are small. Problems related to the ambiguity of this cut-off due to the non-analyticities of the photon polarization functions are discussed in Ref.[2] as well as in Ref.[11]. This does not influence our results since in our study the fermion self-energy finally prevails as an IR-cut-off, as we will see that convergence of the algorithm is lost before \(\Sigma\) becomes less important than \(k_B T\). The gap equation is therefore both IR- and UV- finite, and we will use IR and UV cut-offs only for numerical reasons. Noting that \(\Sigma\) is a function decreasing with momentum, and that the expected hierarchy of scales is large (three orders of magnitude or more), we discretize \(p_0^2, \|\vec{p}\|^2, k_0^2, \|\vec{k}\|^2\) according to \(\log_{10}(\Lambda^2_{IR}) + \frac{i}{n}\log_{10}(\Lambda^2_{UV}/\Lambda^2_{IR})\)
and the angle $\theta$ according to $2\pi i/n$, where $i = 1, \ldots, n$. We do not take the points $\theta = 0, \pi$ as integration points, since the kernel has an integrable singularity there.

We thus have a lattice with five dimensions, two coming from the external and three from the loop momenta. Furthermore, we are always careful that $\Lambda_{IR} < k_B T, \Sigma(0,0)$ (by $\Sigma(0,0)$ of course we mean here and in the following $\Sigma(\Lambda_{IR}, \Lambda_{IR})$), and $\Lambda_{UV} > \alpha$, and check the dependence of our results on the particular values of the cut-offs and the size of the momentum lattice, which we took to be $8^5, 12^5$ and $16^5$. Even though the quantities $\Sigma(0,0)/\alpha$ and $k_B T_c/\alpha$ vary with different choices of cut-offs or relaxation-speed parameter, the ratio $r = 2\Sigma(0,0)/k_B T_c$, remains pretty stable.

The numerical relaxation method employed to solve the gap equation consists of inserting an initial “input” configuration for the fermion self-energy $\Sigma$ to (5), taking the “output” configuration as the new “input”, and then iterating the equation until it is satisfied to a good accuracy. In particular, we consider that the algorithm has converged when the mean difference as well as the standard deviation of the points of the “input” and “output” functions is less than 10%. This is about the best accuracy our algorithm can achieve at zero temperature and $N_f = 1$. It should be noted that in all cases studied, for sufficiently small values for the temperature and fermion flavors, the mean difference of the input and output configurations converges much faster to zero than the standard deviation between them. This allows us to distinguish the deviation caused by numerical error from the deviation caused by a possible overall fall of the solution towards zero. One can then be confident that the solution found is stable and not converging to the trivial one. In order to avoid convergence to unstable, oscillating solutions corresponding to small $\Sigma(0,0)$, we take the original configuration to be flat and equal to $\alpha$. Initial configura-
tions falling smoothly with momentum only decrease the convergence time and are therefore preferable, since they are closer to the final solution, and they give the same results.

The fermion and photon self-energies are needed inside the integrand on points outside the original momentum lattice. Their corresponding values are found by linear interpolation from the values of the quantities on the lattice. For large internal and external momenta, some arguments can fall also outside the lattice space. Even though we use some extrapolation to recover the functions with such arguments, the values of the self-energies for momenta near the UV-cut-off, even though small and insignificant for the final results, should be taken with caution. Moreover, in order to obtain a smooth solution we take the values of the output function at an external momentum lattice point \((i, j)\) to be the average of the integration result \(\Sigma_{\text{out}}\), i.e. \(\Sigma(i, j) = (\Sigma_{\text{out}}(i + 1, j) + \Sigma_{\text{out}}(i - 1, j) + \Sigma_{\text{out}}(i, j + 1) + \Sigma_{\text{out}}(i, j - 1))/4\). This is just one of several similar stabilizing procedures one could use, and we have checked that the results do not depend on the particular choice of such a procedure.

For \(T = 0\) and \(N_f = 2\) we get the solution shown in Fig. 1. It exhibits roughly the expected behaviour \(\Sigma(p) \approx \sin[\gamma \{\ln(p/\Sigma(0)) + \delta\}]\) for \(\Sigma(0) < p < \alpha\), with \(\gamma\) and \(\delta\) in principle functions of \(N_f\) \[3\]. We find that the general form of \(\Sigma(p_0, |\vec{p}|)\) does not change with various \(T\) or \(N_f\).

When the temperature exceeds some critical value \(T_c\), the algorithm does not converge, i.e. the deviation of the “output” from the “input” function oscillates enormously from iteration to iteration, both in magnitude and in sign, instead of decreasing monotonically and reaching the required 10% value. For these high temperatures, even though the algorithm does not converge, the “output” configuration
Figure 1: The self-energy $\Sigma(p_0, |\vec{p}|)$ at $T = 0$ and $N_f = 2$ for a $16^5$ lattice, plotted as a function of momenta $p_0$ and $|\vec{p}|$ in logarithmic scale. All quantities are scaled by the coupling $\alpha$.

For the self-energy after each iteration is tending on the average rapidly to zero (the higher the temperature, the more rapid the fall). Stating this differently, only a trivial input configuration seems to be able to satisfy the self-consistency of the gap equation. On the other hand, for temperatures just below critical, $\Sigma(0, 0)$ is still at about the same the order of magnitude as its zero-temperature value. This does not necessarily mean that the self-energy drops “immediately to zero” when the critical temperature is reached. It most probably means that we are slightly
Fermion flavours \( \Rightarrow \) \( N_f = 1 \) \( N_f = 2 \) \( N_f = 3 \)

| Lattice size \( n^5 \) | \( s_0 \) | \( t_c \) | \( r \) | \( s_0 \) | \( t_c \) | \( r \) | \( s_0 \) | \( t_c \) | \( r \) |
|------------------------|--------|--------|-----|--------|--------|-----|--------|--------|-----|
| 8^5                    | 15     | 3.8    | 7.9 | 3.1    | 0.6    | 10.3 | 0.9    | 0.17   | 10.6 |
| 12^5                   | 19     | 4.5    | 8.4 | 3.8    | 0.67   | 11.3 | 0.68   | 0.11   | 12.4 |
| 16^5                   | 21     | 4.8    | 8.7 | 2.4    | 0.4    | 12   | 0.23   | 0.035  | 13.1 |

Table 1: The quantities \( s_0 = 10^3 \times \Sigma(0,0)/\alpha \) at \( T = 0 \), \( t_c = 10^3 \times k_BT_c/\alpha \) and \( r = 2s_0/t_c \) for different numbers of fermion flavours and lattice sizes \( n^5 \) and for \( \Lambda_{UV}/\alpha = 0.1 \).

| Fermion flavours \( \Rightarrow \) | \( N_f = 1 \) | \( N_f = 2 \) | \( N_f = 3 \)
|------------------------|--------|--------|-----|--------|--------|-----|--------|--------|-----|
| \( \Lambda_{UV}/\alpha \) \( \downarrow \) | \( s_0 \) | \( t_c \) | \( r \) | \( s_0 \) | \( t_c \) | \( r \) | \( s_0 \) | \( t_c \) | \( r \) |
| \( 10^{-2} \)          | 9.3    | 2.3    | 8.1 | 0.9    | 0.17   | 10.6 | 0.1    | 0.016  | 12.5 |
| \( 10^{-1} \)          | 21     | 4.8    | 8.7 | 2.4    | 0.4    | 12   | 0.23   | 0.035  | 13.1 |
| \( 1 \)                | 23     | 5.1    | 9   | 3.3    | 0.54   | 12.2 | 0.5    | 0.074  | 13.5 |

Table 2: The quantities \( s_0 = 10^3 \times \Sigma(0,0)/\alpha \) at \( T = 0 \), \( t_c = 10^3 \times k_BT_c/\alpha \) and \( r = 2s_0/t_c \) for different values of flavours and ultra-violet cut-offs \( \Lambda_{UV} \), for a \( 16^5 \) lattice.

underestimating \( T_c \), since at temperatures just below the real critical value it is very difficult to find an input configuration irregular enough to achieve convergence to the solution, which in the neighbourhood of \( T_c \) is probably oscillating and on the average much smaller in magnitude.

We list in Table 1 the values of the fermion self-energy \( \Sigma(0,0) \) at zero momentum and temperature, of the critical temperature, both scaled by \( \alpha \), and of the ratio \( r \), for different values of the number of fermions \( N_f \) and lattice sizes. We
choose the UV-cut-off such that $\Lambda_{UV}/\alpha = 0.1$, so that not only it is smaller than the physical cut-off $\alpha$, but also truncates the integration before the uninteresting region where the self-energy is vanishingly small. Even though the $\Sigma$ and $T_c$ values do not follow a specific pattern with respect to the lattice size $n^5$, the $r$ ratio seems to be converging for larger $n$. In particular, the difference between the $r$-values for the $12^5$ and $16^5$ lattice is consistent with the 10% accuracy provided by the numerical algorithm. However, given the large scale hierarchies of the problem, the $n = 8$ case for instance is just indicative. Moreover, since we justified the one-loop calculations with the $1/N_f$ expansion, the small values of $N_f$, and especially the ones reported for $N_f = 1$, should be taken with caution. The values for $N_f = 2$ are the ones relevant to the high-$T_c$ superconductors. Furthermore, we see that for $N_f = 3$ the hierarchy between $\Sigma(0,0)$ and $\alpha$ is very large. The relatively small lattice size forced upon us by computer-power limitations might therefore explain the stronger dependence of $s_0$ and $t_c$ on the lattice size for three fermion flavors.

In Table 2 we present our results for a $16^5$ lattice for different ratios $\Lambda_{UV}/\alpha$. The ratio $r$ follows a similar pattern for various UV-cut-offs, which is expected from a super-renormalizable theory. Lower values for $\Lambda_{UV}$ would truncate the integration too early, since the self-energy would be non-negligible there. When $\Lambda_{UV}$ is taken to be larger than $\alpha$, the algorithm does not converge, showing that it is not easy to perform the integrations on momenta above the physical cut-off $\alpha$ with such small lattices. The reason is that numerical errors dominate the calculation at this region, since the self-energy, a function decreasing with increasing momenta, is very small there (typically $\Sigma(0,0)/\alpha$ is on the order of $10^{-2}$ and smaller).

We see that the ratio $r$ increases with increasing $N_f$. This behavior, as well as the overall magnitude of $r$, is consistent with the results of Ref. for the case
including retardation, when the photon polarization functions are approximated by
\[ \Pi_L \approx \Pi_T \approx \frac{ak}{8} + 2\omega_P^2 \quad \text{or} \quad \frac{ak}{8} + \omega_P^2, \]
as in [12]. Since in our treatment the exact one-loop photon polarizations for massless fermions are used, our results indicate that this is the most sensible approximation. The other approximations in [2] get smaller \( r \) ratios by increasing at the same time the value of the critical temperature by a factor as large as 2 or 3, which renders their validity questionable. An indication that the results in Tables 1 and 2 cannot be easily compared to the ones in [2] for various numbers of fermion flavors is the fact that in that paper for instance results for \( N_f = 4, 5 \) are reported, whereas in our case the behaviour of the theory with \( N_f \) renders mass generation with so many fermion flavors difficult or very small, which is in accordance with theoretical expectations. A more accurate study of the behaviour of the theory with \( N_f \) is enabled by the relaxing of several approximations used in [2], and is presented later in this paper.

The values given for \( r \) could be overestimating this ratio, as a result of several effects. First, since \( \Sigma \) is momentum dependent, one could take the relevant value for the ratio \( r \) to be \( \Sigma(p = \Sigma(0)) \) instead of \( \Sigma(0) \). This could decrease \( r \) by 10–20\%, which is justified not only by the accuracy of our algorithm but also by our limited understanding of infrared dynamics. Second, as was already noted, at the temperature at which fluctuations destroy the convergence of our algorithm, \( \Sigma(0, 0) \) has still the same order of magnitude as its value at zero-temperature. This not only shows that the fermion self-energy falls pretty abruptly when the temperature reaches its critical value, but that we might be slightly underestimating \( T_c \). The self-energy might be non-zero and falling with temperature for somewhat higher temperature values by taking highly irregular shapes that our algorithm is unable to find. In addition, it might not be fair to expect the algorithm at zero temperature
and at temperatures close to the critical value to reach the same accuracy of 10%, which after all determines which value of critical temperature we report, since larger temperatures make the numerical error coming from the second term of the right-hand side of Eq. more important.

Figure 2: The fermion self-energy at zero momentum and zero temperature, scaled by $\alpha$ and on a logarithmic scale, with respect to $N_f$ for a $16^5$ lattice. We fit our results with the curve $\Sigma(0,0)/\alpha = \exp\left\{ -4.5/(N_c/N_f - 1)^{1/2} \right\}/6$, with $N_c = 4.35$. Values of $N_f$ larger than 3.4 are not considered, because then the self-energy falls below the IR-cut-off.

We found that for large values of $N_f$ the algorithm is also not converging. As in the critical temperature case, failure of convergence is accompanied with a
rapid decrease of the output configurations, which is a sign that the solution tends
to the trivial one. This indicates that there might be a critical value $N_c$ above which
dynamical mass generation is impossible. In particular, for a lattice size of $16^5$ we
found that $N_c = 3.4$ for $k_B T \approx \Sigma(0,0) \approx \Lambda_{IR} \approx 10^{-5} \times \alpha$. Larger values of $N_f$ not
only produce fermion self-energies tending below the IR-cut-off, but are also unable
to reach the required accuracy of 10%. This value is consistent with theoretical
expectations which give $N_c = 32/\pi^2$ at zero temperature \[9\].

In Fig. 2 we plot the fermion self-energy at zero momentum and zero tem-
perature as a function of $N_f$, and we fit it with the -phenomenological- curve
$\Sigma(0,0)/\alpha = \exp\{-4.5/(N_c/N_f - 1)^{1/2}\}/6$. From the fit a value $N_c = 4.35$ seems to
be close to our data. The non-analytic form of the fitted curve is needed not only
to describe the exponential fall-off of the self-energy but also its probable vanishing
at the critical value of $N_f$, and it exhibits the non-perturbative character of the
dynamics. However, the approach of the self-energy to $\Lambda_{IR}$ does not allow us to
draw firm conclusions on the exact vanishing of $\Sigma$ at a critical number of fermion
flavors. The deviation of this formula from theoretical expectations which give it as
$\Sigma(0,0)/\alpha \approx \exp\{-2\pi/(N_c/N_f - 1)^{1/2}\}$ with $N_c = 32/\pi^2$ \[9\] is most probably due
to the fact that the latter non-analytic formula is intended only for $N_f$ very close
to $N_c$, and is not expected to fit data with smaller $N_f$.

In Fig. 3 we proceed in a similar manner for a non-zero temperature case,
namely $k_B T_c/\alpha = 10^{-4}$. We fit the zero-momentum fermion self-energy with the
-phenomenological- curve $\Sigma(0,0)/\alpha = \exp\{-2.1/(N_c/N_f - 1)^{1/2}\}/15$. From the fit
a value $N_c = 3.2$ is favored. However, we loose convergence of our algorithm at
$N_f \approx 2.6$, which should be much closer to the real critical value $N_c$, according to
the criteria used so far. This deviation indicates, as in the zero-temperature case,
that the non-analytic functional form of the fit should not be used for values of $N_f$ far away from $N_c$. In this finite-temperature case, convergence of the numerical algorithm is lost before the self-energy falls below the IR-cut-off. Figures 2 and 3 are similar to the corresponding ones in [4], which also indicate the existence of a critical behaviour with $N_f$, even though our value for $N_c$ at zero temperature is closer to the theoretical expectations. The decrease of $N_c$ with increasing temperature is also expected, and is shown in more detail in the following figure, i.e. the phase diagram.

In Fig. 4 we plot the phase diagram of the theory with respect to $N_f$ and $k_B T$, where by the term “phase” we mean a situation where there is or there is no dynamical mass generation. The non-perturbative nature of our formalism allows us to consider also non-integer values of $N_f$. We were able to fit the critical line with the -phenomenological- curve $k_B T_c/\alpha = \exp \{-2.5N_f\}/17$. A similar exponential fall-off has already been seen in [4]. This functional form should not be used for values of $N_f$ larger than about 3, because then $T_c$ falls below the IR-cut-off. It is quite possible that when $N_f \approx 3.4$ the behavior of $k_B T$ with $N_f$ becomes here also non-analytic. Nevertheless, we are not able to explore this region, since for values of $N_f$ larger than about 3, $kT_c$ falls below the IR-cut-off.

4 CONCLUSIONS

We report results indicating that for temperatures below a finite critical value, dynamical mass generation in finite temperature $\tau_3 - QED$ in three dimensions is possible. We find a large ratio $r$ that not only exceeds the usual BCS prediction, but also the value measured in high-temperature superconductors, which is close to 8 [14], and indicates anyway that these are very strongly coupled systems. This
ratio, on the order of 10, is generally consistent with results of previous studies [4],
even though its precise behaviour with $N_f$ shows some differences. We also find
that for a number of fermion flavours larger than a critical value, of order 3, there
cannot be mass generation, or, even if there is any, it falls below any IR-cut-off
that we may set so it is impossible to study with the present algorithm. We are
then able to draw the phase diagram of the theory, with respect to temperature and
fermion flavours, which separates the regions where there is and where there is not
dynamical mass generation.

We were able to go beyond the instantaneous approximation and consider
momentum dependent fermion and photon self-energies by working with quantities
depending on two independent variables simultaneously. Previous results on this
context, relying on severe truncations and approximations, are thus put now on a
firmer basis. The increased lattice dimensionality that followed however allowed us
to use only a limited number of lattice cites in each dimension, even though larger
lattices would be better suited for a problem with such large scale hierarchies as the
present one, and they would possibly allow for a better accuracy than 10% in our
solutions.

It is also true that the specific truncation of the $S-D$ equation that we chose,
as well as the dropping of the imaginary parts of all the self-energies could influence
these results non-trivially. Moreover, the number of fermion flavours might not be
large enough to justify the $1/N_f$ expansion and the one-loop diagrams we consider.
We will try to return to these issues in a future publication.

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Figure 3: The self-energy at zero momentum and at $k_B T/\alpha = 10^{-4}$, scaled by $\alpha$ and on a logarithmic scale, with respect to $N_f$ for a $16^5$ lattice. We fit our results with the curve $\Sigma(0)/\alpha = \exp\{-2.1/(N_c/N_f - 1)^{1/2}\}/15$, with $N_c = 3.2$. The algorithm does not converge for values of $N_f$ larger than 2.6.
Figure 4: The phase diagram of the theory with respect to $k_B T$ on a logarithmic scale and $N_f$ for a $16^5$ lattice. We fit the critical line with the curve $k_B T/\alpha = \exp\{-2.5 N_f\}/17$. This functional form could lose its validity for $N_f$ larger than about 3, since then $T_c$ falls below the IR-cut-off.