The Carathéodory topology for multiply connected domains II

Mark Comerford
University of Rhode Island

Follow this and additional works at: https://digitalcommons.uri.edu/math_facpubs

Citation/Publisher Attribution
Comerford, Mark. "The Carathéodory topology for multiply connected domains II." Central European Journal of Mathematics 12, 5 (2014): 721-741. doi: 10.2478/s11533-013-0365-y.

This Article is brought to you by the University of Rhode Island. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of DigitalCommons@URI. For more information, please contact digitalcommons-group@uri.edu. For permission to reuse copyrighted content, contact the author directly.
The Carathéodory topology for multiply connected domains II

Keywords
Bounded family of pointed domains; Carathéodory Topology; Meridians

Creative Commons License

This work is licensed under a Creative Commons Attribution-Noncommercial-No Derivative Works 3.0 License.

This article is available at DigitalCommons@URI: https://digitalcommons.uri.edu/math_facpubs/101
The Carathéodory topology for multiply connected domains II

Mark Comerford

Received 9 April 2012; accepted 15 March 2013

Abstract: We continue our exposition concerning the Carathéodory topology for multiply connected domains which we began in [Comerford M., The Carathéodory topology for multiply connected domains I, Cent. Eur. J. Math., 2013, 11(2), 322–340] by introducing the notion of boundedness for a family of pointed domains of the same connectivity. The limit of a convergent sequence of $n$-connected domains which is bounded in this sense is again $n$-connected and will satisfy the same bounds. We prove a result which establishes several equivalent conditions for boundedness. This allows us to extend the notions of convergence and equicontinuity to families of functions defined on varying domains.

MSC: 30C75, 30C20, 30C45

Keywords: Carathéodory Topology • Meridians • Bounded family of pointed domains

© Versita Sp. z o.o.

4. Bounded families

We resume our discussion begun in [6] concerning the behaviour of pointed domains with respect to limits in the Carathéodory topology. Recall that a pointed domain $(U, u)$ consists of a domain $U \subset \mathbb{C}$ and a point $u$ of $U$ (referred to as the base point). A sequence $\{(U_m, u_m)\}_{m=1}^{\infty}$ converges to $(U, u)$ in the Carathéodory topology as $m$ tends to infinity if

(i) $u_m \to u$ in the spherical topology,

(ii) for all compact sets $K \subset U$, $K \subset U_m$ for all but finitely many $m$,

(iii) for any connected (spherically) open set $N$ containing $u$, if $N \subset U_m$ for infinitely many $m$, then $N \subset U$.

There is also the degenerate case where $U = \{u\}$ where condition (ii) is omitted ($U$ has no interior of which we can take compact subsets) while condition (iii) becomes

(iii) for any connected open set $N$ containing $u$, $N$ is contained in at most finitely many of the sets $U_m$.

* E-mail: mcomerford@math.uri.edu
In [6] we reviewed the well-known fact that this can also be understood in terms of the convergence of the complements \( \mathbb{D} \setminus U_m \) in the Hausdorff topology (with respect to the spherical metric on \( \mathbb{D} \)) and the Carathéodory kernel (Theorem 1.1). We also have the very useful fact that, under reasonable circumstances, Carathéodory convergence is equivalent to convergence of suitably normalized universal covering maps on compact subsets of the unit disc (Theorem 1.2).

One of the issues when working with this topology is that the conformal type and the connectivity of the domain need not be preserved under limits taken with respect to this topology. Recall that a domain of finite connectivity is called non-degenerate if none of the components of the complement is a point. In Figure 6 below which is the same as Figure 2 in the first part of this paper we have a sequence of non-degenerate pointed domains of connectivity 6 whose limit is a degenerate pointed domain of connectivity 4.

**Figure 6.**

We observed that this pathological behavior can be completely understood in terms of the behaviour of certain hyperbolic geodesics associated with the domains in question. However, in the previous paper we did not give a rigorous justification of this observation and this justification is thus the main goal of this paper (see Theorem 4.15).

These geodesics are known as **meridians** of the domain. Given a hyperbolic domain \( U \subset \mathbb{D} \) and closed disjoint sets \( E, F \) neither of which is a point and for which \( \mathbb{D} \setminus U = E \cup F \), then a simple closed geodesic which separates \( E \) and \( F \) and whose hyperbolic length is as short as possible is called a meridian of \( U \). The existence of such geodesics was proved in [4, Theorem 1.4] and is re-stated (in slightly simplified form) in [6, Theorem 1.6]. Such geodesics are not in general uniquely defined [4, Theorem 1.6], but if at least one of the sets \( E, F \) is connected, then there is just one meridian which separates \( E \) and \( F \) ([4, Theorem 1.7] restated in [6] as Theorem 1.8) and in this case we call such meridians **principal meridians** of \( U \).

In order to begin to understand how we can prevent the pathologies referred to above, we first needed to see how meridians behaved under limits in the Carathéodory topology. The main result in [6] was Theorem 1.9 where we showed that for a sequence \( (U_m, u_m) \) of multiply connected pointed domains which converged to a (hyperbolic) pointed domain \( (U, u) \), any simple closed geodesic in \( U \) was a uniform limit in the appropriate sense of simple closed geodesics of the approximating domains and that the hyperbolic lengths of these geodesics and their hyperbolic distances from the base points also converged to those for the limit geodesic. Furthermore, in Theorem 1.11 of that paper, we showed that, given a separation \( E, F \) of the complement of the limit domain \( U \) as above, we could find a meridian of \( U \) which separated these sets and was the uniform limit of meridians of the approximating domains (note that we did not say that every meridian of \( U \) could be approximated in this way). Finally, using these results, we were able to prove a version of Theorem 1.2 for finitely connected pointed domains which used suitably normalized Riemann mappings to standard domains (in this case annuli from which concentric slits had been removed in the multiply connected case) instead of universal covering maps — see [6, Theorem 3.7] for details.

Having given a resume of the first part of this paper, we continue our exposition by giving the definition of a bounded family of non-degenerate pointed domains of the same connectivity. Essentially, a bounded family is precompact in the
sense that it is stable with respect to limits in the Carathéodory topology so that none of the pathologies in the example above can occur.

**Definition 4.1.**
Let \( n \geq 1 \) and let \( \mathcal{U} = \{(U_0, u_0)\}_{i=1}^4 \) be a family of pointed domains where every domain \( U_0 \) is an \( n \)-connected non-degenerate subdomain of \( \mathbb{C} \). We say that \( \mathcal{U} \) is **bounded** if every sequence in \( \mathcal{U} \) which is convergent in the Carathéodory topology has a limit which is a non-degenerate \( n \)-connected pointed domain and we write \( \text{pt} \subset \mathcal{U} \subset \mathbb{C} \). Otherwise we say that \( \mathcal{U} \) is **unbounded**.

It is desirable to be able to specify the boundedness of a family of pointed domains of the same connectivity in a quantitative way, the idea being that the larger the bound, the closer the members of the family are allowed to get to becoming degenerate (which includes having lower connectivity). This leads to the notion of the **Carathéodory bound** which we first define for an individual pointed domain and then for a family. Theorem 4.15 which is the main result of this part of the paper gives several equivalent conditions for boundedness, including finiteness of the Carathéodory bound.

As we will see, it turns out that one only needs to consider the principal meridians in order to ensure bounded behaviour. However, sometimes one is interested in the other meridians as well, which leads us to also define the **extended Carathéodory bound**. Recall that at the start of [4, Section 3] it was observed that a non-degenerate \( n \)-connected hyperbolic domain \( U \) has \( E(n) = 2^{n-1} - 1 \) meridians which separate the complement of \( U \) in distinct ways and \( P(n) = \min \{ n, E(n) \} \) distinct principal meridians ([4, Proposition 3.1] restated in [6] as Proposition 1.12). Note that these formulae are valid even in the simply connected case where there are no meridians, while if the domain is degenerate and some of the complementary components are points, then there will be fewer than \( P(n) \) principal meridians (again see [4, Proposition 3.1] restated in [6] as Proposition 1.12). We remind the reader of the convention introduced on p. 329 of [6] that for a non-degenerate pointed domain \((U, u)\) of connectivity \( n \) with an extended system \( \Gamma = \{ \gamma^i : 1 \leq i \leq E(n) \} \) of \( E(n) \) meridians, we always number the meridians of \( \Gamma \) so that the first \( P(n) \) are the principal ones. Recall that, for a curve \( \gamma \) in a hyperbolic domain \( U \) with hyperbolic metric \( \rho(\cdot, \cdot) \), we let \( \ell(\gamma) \) denote the hyperbolic length of \( \gamma \) while, if \( u \) is a point in \( U \), we let \( \rho(u, \gamma) \) denote the hyperbolic distance in \( U \) from \( u \) to \( \gamma \).

**Definition 4.2.**
Let \((U, u)\) be a hyperbolic pointed domain and let \( \Gamma = \{ \gamma^1, \gamma^2, \ldots, \gamma^\ell \} \) be a collection of curves in \( U \). We define the length and distance, \( \mathcal{L}(\Gamma) \) and \( \mathcal{D}(\Gamma) \) of \( \Gamma \) by

\[
\mathcal{L}(\Gamma) = \max_{1 \leq i \leq \ell} |\log \ell(\gamma^i)|, \quad \mathcal{D}(\Gamma) = \max_{1 \leq i \leq \ell} \rho(u, \gamma^i).
\]

**Definition 4.3.**
Let \((U, u)\) be an \( n \)-connected non-degenerate pointed domain. If \( n \geq 2 \), let \( \Gamma = \{ \gamma^i : 1 \leq i \leq E(n) \} \) be an extended system of meridians for \( U \) where for each \( 1 \leq i \leq E(n) \), \( \ell^i, d^i \) denote the hyperbolic length and hyperbolic distance respectively to the base point of each \( \gamma^i \). We define the length and extended length, \( \mathcal{L}(U), \mathcal{L}_E(U) \) of \( U \) by

\[
\mathcal{L}(U) = \max_{1 \leq i \leq \ell} |\log \ell^i|, \quad \mathcal{L}_E(U) = \max_{1 \leq i \leq \ell} |\log d^i|
\]

and the distance \( \mathcal{D}(U, u) \) of \((U, u)\) and extended distance \( \mathcal{D}_E(\Gamma) \) of \( \Gamma \) by

\[
\mathcal{D}(U, u) = \max_{1 \leq i \leq \ell} d^i, \quad \mathcal{D}_E(\Gamma) = \max_{1 \leq i \leq \ell} d^i.
\]

If \( n = 1 \) and \( U \) is simply connected, we define each of the quantities \( \mathcal{L}(U), \mathcal{L}_E(U), \mathcal{D}(U, u) \) to be zero.

Note that by Theorem 1.8 on the uniqueness of principal meridians the length and distance can be defined for \((U, u)\) rather than just \( \Gamma \). Also, as by Theorem 1.6 the lengths do not depend on the choice of system of meridians, we can say the same about the extended length of \((U, u)\).
The extended distances will in general depend on the choice of system, so it is of interest whether there is a system for which these distances are as small as possible. We postpone the proof of the following result until after the statement of Theorem 4.15.

**Lemma 4.4.**
For a pointed non-degenerate domain \((U, u)\) of finite connectivity \(n \geq 2\), there exists an extended system \(\Gamma = \{\gamma^i : 1 \leq i \leq E(n)\}\) of meridians for which the distances \(d^i, 1 \leq i \leq E(n)\), are as small as possible in the sense that for each \(1 \leq i \leq E(n)\) the distance of \(\gamma^i\) from the base point \(u\) is as small as possible among all meridians which separate \(\mathbb{C} \setminus U\) in the same way as \(\gamma^i\).

**Definition 4.5.**
Let \((U, u)\) be a non-degenerate pointed domain of finite connectivity \(n \geq 1\). For \(n \geq 2\), an extended system of meridians \(\Gamma = \{\gamma^i : 1 \leq i \leq E(n)\}\) as above is called **maximally close**. In this case we define the **extended distance** \(D_E((U, u))\) of \((U, u)\) by

\[
D_E((U, u)) = \max_{1 \leq i \leq E(n)} d^i,
\]

where \(d^i, 1 \leq i \leq E(n)\), are the distances of a maximally close system \(\Gamma\). For \(n = 1\) where \(U\) is simply connected, we define \(D_E((U, u))\) to be zero.

We observe here that using a different system other than a maximally close one to calculate \(D_E((U, u))\) will not give an answer that is very different. If \(\gamma\) and \(\gamma'\) are two meridians which are in the same homology class but different homotopy classes, then by [4, Lemma 2.9], these two curves must intersect. If \(\gamma\) and \(\gamma'\) have the same length \(l\) and distances \(d\) and \(d'\) respectively, then it is a simple calculation to check that

\[
d' \leq d + \frac{l}{2}.
\]

Thus if \(\Gamma\) and \(\Gamma'\) are two systems of meridians for \((U, u)\), we have

\[
D_E(\Gamma') \leq D_E(\Gamma) + \frac{e \cdot E(\Gamma)}{2}.
\]

The other remark worth making here is that even a maximally close system of meridians is not in general unique. This can be seen in Figure 7 below where both meridians are equally close to the base point but in different homotopy classes, see [4, Theorem 1.6] for a rigorous justification as to why these meridians are distinct.

**Figure 7.**
We are now in a position to define the Carathéodory bound for a suitable family of domains. For a set $K \subset \overline{\mathbb{C}}$, let us denote the spherical diameter of $K$ by $\text{diam}^s K$. Recall also that for a domain $U \subset \mathbb{C}$ and some point $z \in U$, we denote the spherical distance from $z$ to $\partial U$ by $\delta_U^s(z)$ or just $\delta^s(z)$ if the domain involved is clear from the context.

**Definition 4.6.**
Let $n \geq 1$ and let $(U, u)$ be $n$-connected and non-degenerate. We define the Carathéodory bound or Carathéodory norm and the extended Carathéodory bound or extended Carathéodory norm, $|(U, u)|$ and $|(U, u)|_E$, respectively, of $(U, u)$ by

$$|(U, u)| = |\log \left( \delta^s(u) \text{diam}^s(\overline{\mathbb{C}} \setminus U) \right)| + L((U, u)) + \mathcal{D}(U, u),$$

$$|(U, u)|_E = |\log \left( \delta^s(u) \text{diam}^s(\overline{\mathbb{C}} \setminus U) \right)| + L_E((U, u)) + \mathcal{D}_E(U, u),$$

where the terms $\mathcal{L}(U, u)$, $\mathcal{D}(U, u)$, $\mathcal{L}_E(U, u)$, $\mathcal{D}_E(U, u)$ are all zero if $n = 1$.

Let $n \geq 1$ and let $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ be a family of pointed non-degenerate $n$-connected domains. We define the Carathéodory bound or Carathéodory norm and the extended Carathéodory bound or extended Carathéodory norm of $\mathcal{U}$, $\|\mathcal{U}\|$ and $\|\mathcal{U}\|_E$ by

$$\|\mathcal{U}\| = \sup_{\alpha} |(U_\alpha, u_\alpha)|,$$

$$\|\mathcal{U}\|_E = \sup_{\alpha} |(U_\alpha, u_\alpha)|_E.$$

One can think of the three terms above as each preventing a different way in which a Carathéodory limit of non-degenerate $n$-connected domains can fail to be another non-degenerate $n$-connected domain. The first term prevents any Carathéodory limit from simply being a point or $\overline{\mathbb{C}}$ with one point removed (recall that the spherical diameter of all of $\overline{\mathbb{C}}$ is $\pi/2$ and in particular finite so that the only way for this term to become unbounded is for one or both of the quantities $\delta^s(u)$, $\text{diam}^s(\overline{\mathbb{C}} \setminus U)$ to tend to zero) and also ensures that the basepoints $\{u_\alpha\}_{\alpha \in A}$ are bounded away from the boundaries of $U_\alpha$. The second term prevents componentary complements of merging or becoming points as in Figure 6 while the third term prevents components of the complement from being ‘engulfed’ by other components, again as in Figure 6.

Before we can state our theorem, we need more definitions relating to families of functions defined on varying domains.

**Definition 4.7.**
Let $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ be a family of pointed hyperbolic domains in $\overline{\mathbb{C}}$. We say $\mathcal{U}$ is hyperbolically non-degenerate or simply non-degenerate if the limit of any convergent sequence in $\mathcal{U}$ is another hyperbolic domain.

Note that a bounded family of non-degenerate pointed domains of the same connectivity is clearly hyperbolically non-degenerate. The following is immediate.

**Lemma 4.8.**
If $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ is a non-degenerate family of pointed hyperbolic domains, then $\left| \log \left( \left| \delta^s_{U_\alpha}(u_\alpha) \text{diam}^s(\overline{\mathbb{C}} \setminus U_\alpha) \right| \right) \right|$ are uniformly bounded in $\alpha$.

The converse of this is easily seen to be false, for example by considering the sequence $\{(A(0, 1/m, m), 1)\}_{m=2}^\infty$.

**Definition 4.9.**
Let $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ be a family of hyperbolic pointed domains and let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family of analytic functions with each $f_\alpha$ defined on $U_\alpha$. We say $\mathcal{F}$ is normal on $\mathcal{U}$ if every convergent sequence $\{\{(U_\alpha, u_\alpha)\}_{\alpha \in A}\}_{\beta = 1}^\infty$ in $\mathcal{U}$ has a limit $(U, u)$, where $U \neq \{u\}$, and we can find a subsequence for which the corresponding functions $f_{\alpha_\beta}$ converge uniformly on compact subsets of $U$ (in the sense given in Definition 3.1).
Definition 4.10.
Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in A} \) be a non-degenerate family of hyperbolic pointed domains and let \( \mathcal{F} = \{ f_\alpha \}_{\alpha \in A} \) be a family of analytic functions with each \( f_\alpha \) defined on \( U_\alpha \).

We say that \( \mathcal{F} \) is equicontinuous on compact subsets of \( \mathcal{U} \) if for any hyperbolic distance \( R > 0 \) there exists \( M \geq 0 \) depending on \( R \) such that, for each \( \alpha \in A \), \( f_\alpha \) is \( M \)-Lipschitz with respect to the spherical metric within hyperbolic distance \( \leq R \) of \( u_\alpha \) in \( U_\alpha \). In this case we write \( \mathcal{F} \preceq \mathcal{U} \).

We say that \( \mathcal{F} \) is bi-equicontinuous on compact subsets of \( \mathcal{U} \) if each \( f_\alpha \) is a (locally injective) covering map onto its image and for each hyperbolic radius \( R > 0 \) there exists \( K(R) \geq 1 \) such that, for each \( \alpha \in A \), \( f_\alpha \) is \( K \)-bi-Lipschitz and locally \( K \)-bi-Lipschitz with respect to the spherical metric within hyperbolic distance \( \leq R \) of \( u_\alpha \) in \( U_\alpha \). In this case we write \( \mathcal{F} \asymp \mathcal{U} \).

Note that if all the domains in our family are the same, then equicontinuity on compact subsets corresponds with the standard definition for a family of analytic functions on some domain (where we use the spherical rather than the Euclidean topology).

Proposition 4.11.
Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in A} \) be a non-degenerate family of hyperbolic pointed domains and let \( \mathcal{F} = \{ f_\alpha \}_{\alpha \in A} \) be a family of analytic functions with each \( f_\alpha \) defined on \( U_\alpha \). Then \( \mathcal{F} \) is normal on \( \mathcal{U} \) if and only if it is equicontinuous on compact subsets of \( \mathcal{U} \).

Proof. If \( \mathcal{F} \) is normal on \( \mathcal{U} \), it follows easily using the non-degeneracy of \( \mathcal{U} \) and Theorem 1.2 that \( \mathcal{F} \preceq \mathcal{U} \). The other direction follows easily from (i) of Carathéodory convergence, the compactness of \( \overline{\mathbb{C}} \) and a standard diagonalization argument as in the proof of the Arzelà–Ascoli theorem.

Definition 4.12.
Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in A} \) and \( \mathcal{V} = \{ (V_\alpha, v_\alpha) \}_{\alpha \in A} \) be families indexed by the same set \( A \). We say that a family \( \mathcal{F} = \{ f_\alpha \}_{\alpha \in A} \) maps \( \mathcal{U} \) to \( \mathcal{V} \) if, for each \( \alpha \), \( f_\alpha(U_\alpha) \subset V_\alpha \) and \( f_\alpha(u_\alpha) = v_\alpha \) and we write \( \mathcal{F} : \mathcal{U} \to \mathcal{V} \). By convention, if in addition \( f_\alpha \) is a covering map, we will require that \( f_\alpha(U_\alpha) = V_\alpha \).

Hyperbolic non-degeneracy and bi-equicontinuity are related by the following result whose proof we will also postpone until after the statement of Theorem 4.15. Recall that we use the notation \( (D, 0) \) for the constant family consisting of the unit disc and the point 0.

Theorem 4.13.
Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in A} \) be a family of pointed hyperbolic domains and let \( \Pi = \{ \pi_\alpha \}_{\alpha \in A} \) denote the family of normalized covering maps from \( (D, 0) \) to \( \mathcal{U} \). Then \( \mathcal{U} \) is non-degenerate if and only if \( \Pi \asymp (D, 0) \).

Recall that by Theorem 3.3, an \( n \)-connected domain \( U \) with \( n \geq 2 \) is conformally equivalent to a standard domain, i.e. an annulus from which some concentric slits have been removed. This domain is unique up to rotation and to specifying which of the components correspond to the closed unit disc and the unbounded complementary component of the standard domain. Given two multiply connected domains \( U, V \) of the same connectivity \( n \geq 2 \) and a conformal mapping \( f : U \to V \), we can then say that two components \( K^1, K^2 \) of \( \overline{\mathbb{C}} \setminus U \) and \( L^1, L^2 \) of \( \overline{\mathbb{C}} \setminus V \) correspond under \( f \) if, for a Riemann mapping \( \varphi \) from \( V \) to a standard domain for which \( L^1, L^2 \) correspond to \( \overline{D} \) and the unbounded complementary component, \( \varphi \circ f \) is a Riemann map which gives the same correspondence for \( K^1, K^2 \) (respectively).

Such a standard domain is specified uniquely by \( 3n - 5 \) real numbers. A pointed \( n \)-connected domain is then conformally equivalent to a pointed standard domain which is described using \( 3n - 3 \) numbers. However, as the pointed standard domain for an \( n \)-connected domain is unique only up to rotation, this allows us to eliminate one more parameter so that it is specified uniquely using \( 3n - 4 \) real numbers (where, for example, we insist that the basepoint of the standard domain lie on the positive real axis).
We consider the set $\mathcal{D}_n$ of all ordered 4-tuples $(U, u, K^1, K^2)$, where $U$ is a non-degenerate $n$-connected domain, $u \in U$ and $K^1, K^2$ are two distinct components of the complement $\mathbb{C} \setminus U$ (in the language of Riemann surfaces, we are essentially considering $n$-connected domains where we mark one point of the domain and two components of the ideal boundary). A sequence $\{((U_n, u_n, K^1_n, K^2_n))\}_{n=1}^{\infty}$ is said to converge to a limit $(U, u, K^1, K^2) \in \mathcal{D}_n$ if the pointed domains $(U_n, u_n)$ converge in the Carathéodory topology to $(U, u)$ and any limit in the Hausdorff topology of the sets $K^1_n$ is contained in $K^1$ while also any limit in the Hausdorff topology of the sets $K^2_n$ is contained in $K^2$. We can then use this notion of convergence to endow $\mathcal{D}_n$ with a corresponding topology. Specifically, we say a subset $\mathcal{U}$ of $\mathcal{D}_n$ is open if each point of $\mathcal{U}$ is not a limit of a sequence of points in $\mathcal{D}_n \setminus \mathcal{U}$ in the sense defined above. Conformal equivalence in $\mathcal{D}_n$ is an equivalence of pointed domains which also preserves the labelling of complementary components using Theorem 3.3 as described above which is then a well-defined equivalence relation.

The quotient of $\mathcal{D}_n$ by this relation gives rise to a moduli space $\mathcal{M}_n$, which we then endow with the quotient topology arising from the topology defined on $\mathcal{D}_n$. As we have seen, non-degenerate $n$-connected pointed slit domains may be identified with a subset $\mathcal{B}_n$ of $\mathbb{R}^{3n-4}$ which is easily seen to be open and which we endow with the topology of $\mathbb{R}^{3n-4}$. The canonical injection takes points in $\mathcal{B}_n$ to pointed slit domains and if we choose the first labelled complementary component of each slit domain to be 0 and the second the unbounded complementary component, then this gives us an injection from $\mathcal{B}_n$ to $\mathcal{D}_n$ which is easily seen to be continuous. These points in $\mathcal{D}_n$ are then in turn mapped to points in the moduli space $\mathcal{M}_n$ by the above equivalence relation.

Denoting this composition by $i_n$, it follows from the uniqueness part of Theorem 3.3 that $i_n$ is locally injective ($i_n$ is not in general globally injective as rotating all the slits in a standard domain by angle $2\pi r$ gives us the same standard domain again). Also, $i_n$ is surjective from the existence part of Theorem 3.3. Using conformal mappings to standard domains which preserves the above labelling of complementary components and applying Theorem 3.7 shows that sets of equivalence classes in $\mathcal{M}_n$ for which the numbers associated with the corresponding standard domains lie in an open subset of $\mathcal{B}_n$ are themselves open (note that the notion of labelling and convergence in $\mathcal{D}_n$ is compatible with that used at the start of the proof of this theorem). Hence $i_n$ is open (and $\mathcal{M}_n$ is Hausdorff as a topological space).

The map $i_n$ is then locally a homeomorphism on compact subsets of $\mathcal{B}_n$. This has the benefit that we can identify the topology of $\mathcal{B}_n$ on suitable relatively compact subsets with that on corresponding relatively compact subsets of $\mathcal{M}_n$ and thus consider $\mathcal{M}_n$ as a $(3n-4)$-dimensional manifold (the author gratefully acknowledges the help of Adam Epstein, Vaibhav Gadre and Jeremy Kahn with the above discussion). This allows us to make the following definition.

**Definition 4.14.** Let $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha=1}^{A}$ be a family of $n$-connected non-degenerate pointed domains where $n \geq 2$. We say $\mathcal{U}$ is bounded in moduli space if for each pointed domain $(U_\alpha, u_\alpha)$ and any choice of distinct components $K^1_\alpha, K^2_\alpha$ of $\mathbb{C} \setminus U_\alpha$ which gives us an embedding into $\mathcal{D}_n$, the image of this family in $\mathcal{M}_n$ forms a set which is the image under $i_n$ of a precompact set in $\mathcal{B}_n$.

We remark that the subset of $\mathcal{M}_n$ which arises for such a family must also be precompact. We can also at this point say a certain amount about the shape of $\mathcal{M}_n$. Suppose for example we have a point of $\mathcal{M}_n$ which corresponds to a standard domain where none of the slits lie on the same circle about 0. We can then twist these $n-2$ slits relative to the base point (and thus each other) without making the Carathéodory norm become unbounded. Hence we have $n-2$ independent directions in which $\mathcal{M}_n$ is ‘cylinder-like’. On the other hand, we have $2n-2$ remaining independent directions (corresponding to the position of the base point, the radii of the circles on which the slits are, the outer radius of the slit domain and the differences rather than the averages of the angles for the end of the slits) for each of which we can make the Carathéodory norm tend to infinity. Note that this situation where the slits lie on different circles is obviously generic and it is easy to see that the above argument can be modified to obtain the same number of cylinder-like directions even when some of the slits lie on the same circle about 0.

Finally, we remark that the distances and lengths of a domain of connectivity $n \geq 3$ are specified by $2n$ numbers. Adding this to the $n-2$ cylinder-like directions gives $3n-2$ which is greater than the dimension of $\mathcal{M}_n$. However, it should be borne in mind that for domains of connectivity greater than 2 these parameters are not independent. For example, altering the distance of one meridian from the base point will likely alter the distances to the other meridians. Note, however, that if $n = 2$, then we just have $2P(2) = 2(2-1) = 2 = 3(2) - 4$ numbers so there is no redundancy in this case.
Before stating the main result of this paper, we remind the reader that one condition is said to imply another up to constants if the constants for the first condition imply non-trivial bounds on those for the second.

**Theorem 4.15.**

Let $\mathcal{U} = \{(U_0, u_0)\}_{\alpha \in \mathcal{A}}$ be a family of pointed $n$-connected non-degenerate domains. If $n \geq 2$, then the following are equivalent:

1. $\mathcal{U} \subset \mathbb{C}$
2. $\|\mathcal{U}\|_E < \infty$
3. $\|\mathcal{U}\| < \infty$
4. there exist constants $\delta_1, \delta_2 > 0$ and closed curves $\gamma_\alpha, \alpha \in \mathcal{A}$, $1 \leq i \leq P(n)$, such that if $K^i_\alpha, 1 \leq i \leq n$, are the components of $\mathbb{C} \setminus U_0$, we have the following:
   a) $\gamma_\alpha$ is a curve in $U_\alpha$ and $u_\alpha$ lies on each $\gamma_\alpha$,
   b) $\gamma_\alpha$ separates $K^i_\alpha$ from the other components of $\mathbb{C} \setminus U_\alpha$,
   c) any point on $\gamma_\alpha$ is at least spherical distance $\delta_1$ away from $\mathbb{C} \setminus U_\alpha$,
   d) the spherical diameters of the sets $K^i_\alpha, 1 \leq i \leq n$, are at least $\delta_2$.
5. if for each $\alpha$ we let $(A^\alpha, a_\alpha)$ be a pointed standard domain for $(U_0, u_0)$ (where we allow any choice of which components of $\mathbb{C} \setminus U_0$ correspond to the closed unit disc and unbounded complementary component of $A^\alpha$), then the corresponding family $\mathcal{A} = \{(A^\alpha, a_\alpha)\}_{\alpha \in \mathcal{A}}$ satisfies $\|\mathcal{A}\| < \infty$ and the family of inverse Riemann maps $\psi_\alpha$ gives a univalent family $\Psi$ with $\Psi: \mathcal{A} \to \mathcal{U}$ and $\Psi \circ \mathcal{U}$.
6. $\mathcal{U}$ is hyperbolically non-degenerate and bounded in moduli space.

Furthermore, conditions 2, 3, 4, and 5 are equivalent up to constants. If $n = 1$, part b) of condition 4 is vacuous while a) and c) become the single condition $\delta^1_\alpha(u_\alpha) \geq \delta_1$. For 5., the standard pointed domains are all $(\mathbb{D}, 0)$ while condition 6. is omitted (hyperbolic non-degeneracy is equivalent to boundedness in the simply connected case while all simply connected pointed domains are equivalent under a conformal map which preserves base points).

**Proof of Lemma 4.4.** The conclusion is trivial for $n = 2$ as the equator of a conformal annulus is unique and so we can assume $n \geq 3$. Again for convenience, let us assume that $U \subset \mathbb{C}$ so that we can use homology to characterize how simple closed curves separate $\mathbb{C} \setminus U$.

Let $2 \leq i \leq n - 2$, select integers $1 \leq j_1 < j_2 < \ldots < j_i \leq n$ and let $K^i_1, K^i_2, \ldots, K^i_n$ be (distinct) components of $\mathbb{C} \setminus U$. In view of [6, Theorem 1.3], selecting components in this way is essentially equivalent to choosing a homology class of a simple closed curve in $U$ where we separate these components of the complement of $U$ from the remaining ones and in view of Theorem 1.6, we know that these homology classes must contain meridians. Note that we do not need to consider the cases where $i = 1$ or $n - 1$ as these are the cases where we have principal meridians in which case the result is trivially true. We then set

$$d = \inf_{\gamma} \rho(u, \gamma),$$

where the infimum is taken over all those meridians which separate the components $K^i_1, K^i_2, \ldots, K^i_n$ from the rest of $\mathbb{C} \setminus U$.

Let $\{\gamma_m\}$ be a sequence of meridians which are in this homology class and for which the distances $d_m = \rho(u, \gamma_m)$ converge to $d$.

Now let $\pi: \mathbb{D} \to U$, with $\pi(u) = 0$, $\pi'(u) > 0$, be the normalized covering map as in Theorem 1.2 and for each $m$ let $\sigma_m$ be a lift of $\gamma_m$ which is at hyperbolic distance $d_m$ from 0 in $\mathbb{D}$. Next let $\eta_m$ be a full segment of $\gamma_m$ of length $\ell_m = \ell_\gamma(\gamma_m)$ which is at distance $d_m$ from 0 so that $\pi(\eta_m) = \gamma_m$ and $\pi$ is injective on $\eta_m$ except at the endpoints. Note that by Theorem 1.6 again, all the segments $\eta_m$ will have the same hyperbolic length.

Passing to a subsequence if necessary, we can choose our segments $\eta_m$ so that they converge to a segment $\eta$ of another geodesic $\sigma$ which is at distance $d$ from 0. If we now let $\gamma = \pi(\eta)$, then it is clear that $\gamma$ is a closed geodesic in $U$ and it follows easily from this convergence and looking at the derivative $\pi'$ at the endpoints of these segments that $\gamma$ must be a smooth curve without any ‘corners’.

It also follows from the uniform convergence of the segments $\eta_m$ to $\eta$ that $\gamma_m$ is homotopic to $\gamma$ for $m$ large enough. By [10, Theorem 7.2.5, p. 129], $\gamma_m = \gamma$ for $m$ large enough and it follows that $\gamma$ must be simple and a meridian in the desired
homology class. Repeating this argument for each choice of subset of components of $\overline{C} \setminus U$ then gives the required extended system of meridians.

**Proof of Theorem 4.13.** The direction where $\mathcal{U}$ is non-degenerate follows by Theorem 1.2. Suppose now that $\Pi \rightharpoonup (\mathbb{D}, 0)$, let $\{(U_{n}, u_{n})\}_{n=1}^{\infty}$ be a sequence which converges in the Carathéodory topology to a limit $(U, u)$ and suppose for the sake of contradiction that $U$ is not a hyperbolic domain. As $\Pi \rightharpoonup (\mathbb{D}, 0)$, by Proposition 4.11, $\Pi$ is normal on $(\mathbb{D}, 0)$. Again as $\Pi \rightharpoonup (\mathbb{D}, 0)$, it follows from Bloch’s Theorem (e.g. [7, p. 293, Chapter XII, Theorem 1.4]) that $U$ cannot be a point. By postcomposing as usual with a suitable Möbius transformation, we can then assume without loss of generality that $U$ is either $\mathbb{C} \setminus \{0\}$ or all of $\mathbb{C}$.

If $U = \mathbb{C} \setminus \{0\}$, by the normality of $\Pi$, on passing to a subsequence if needed, we can clearly assume that the covering maps $\pi_{n_{0}}$ converge to a limit function $\pi$. By the bi-equicontinuity of $\Pi$, $\pi$ must be locally injective. Using Rouche’s theorem and local compactness as observed by Epstein in the remarks preceding the proof of [8, Lemma 6, p. 15], it follows that $\pi(\mathbb{D}) = \mathbb{C} \setminus \{0\}$.

Now $e^{\sigma}$ maps $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$ and, by the monodromy theorem, we can then find a branch of $\pi^{-1} \circ e^{\sigma}$ which is defined on all of $\mathbb{C}$. However, this gives us an entire function whose range is a subset of $\mathbb{D}$ which contradicts Liouville’s theorem. Finally, the argument where $U = \mathbb{C}$ is then a simpler version of this.

**Proof of Theorem 4.15.** For $1. \Rightarrow 2.$, if the first term in the Carathéodory bound for $\mathcal{U}$ was unbounded, using the Hausdorff version of Carathéodory convergence, we could then find a sequence which tended to either a point (which could possibly be infinity as we defined Carathéodory convergence using convergence of the base points in the spherical and not the Euclidean topology) or $\overline{C}$ with one point removed, both of which would contradict the boundedness of $\mathcal{U}$. The required bounds on the lengths and distances of the pointed domains of $\mathcal{U}$ follow again from the boundedness of $\mathcal{U}$ and Corollary 1.13.

$2. \Rightarrow 3.$ is immediate.

For $3. \Rightarrow 4.$, for each $\alpha$ and each $1 \leq i \leq P(n)$, let $\eta_{\alpha}^{i}$ be the curve obtained by adding to the principal meridian $\gamma_{\alpha}^{i}$ the hyperbolic segment which connects $u_{\alpha}$ to the closest point on $\gamma_{\alpha}^{i}$ in the hyperbolic metric and which is then traversed in both directions (note that we can easily modify $\eta_{\alpha}^{i}$ slightly if we wish to ensure that it is a simple closed curve). The curve $u_{\alpha}$ then lies on $\eta_{\alpha}^{i}$ and it is clear that this curve separates $K_{\alpha}^{i}$ from the other components of $\overline{C} \setminus U_{\alpha}$ and so we have a) and b).

We will need to apply the estimates of Lemma 3.4 on the hyperbolic metric in a uniform manner which will be independent of the choice of pointed domain in $\mathcal{U}$ and of the point at which we wish to estimate the hyperbolic metric for that domain. In particular we want to find $C > 0$ and $\delta > 0$ such that for any $\alpha$ and any $x \in \partial U_{\alpha}$, these estimates hold for this $C$ for all $w \in U_{\alpha}$ within spherical distance $\delta$ of $x$.

In view of our remarks after the statement of Lemma 3.4, we will be able to do this if we can prove the claim that we can find $\delta > 0$ independent of $\alpha$ such that if $z_{\alpha}^{j}$ is any point in $\partial U_{\alpha}$, we can find two other points $z_{\alpha}^{j}$ and $z_{\alpha}^{j}$ also in $\partial U_{\alpha}$ so that these three points are separated by at least distance $\delta$ from each other in the spherical metric.

So suppose this fails. This means that we can either find a sequence in $\mathcal{U}$ which tends to $\overline{C}$ with either one or two points removed. As usual, without loss of generality, we can assume such a sequence tends either to $\mathbb{C}$ or to a punctured plane in which case each of the corresponding domains contains an annulus which separates the components of the complement and whose modulus tends to infinity.

The first case is excluded in view of the first term of the Carathéodory bound. In the second case, the equator of such an annulus has very small hyperbolic length and by Theorem 1.6 separates the complement of the domain in the same way as a meridian whose hyperbolic length is at least as small. If the connectivity $n$ of all the domains in $\mathcal{U}$ is 2 or 3, then this is impossible in view of the second term in the Carathéodory norm as here all meridians are principal.

On the other hand, if all the domains of $\mathcal{U}$ have higher connectivity and thus have meridians which are not principal, it follows from [4, Theorem 2.8] that the principal meridians do not intersect any other meridians of a domain (including other principal meridians). Additionally, a meridian which is not principal must contain principal meridians in both of its complementary components. By the collar lemma [10, Lemma 7.7.1, p. 148], we can then find a principal meridian whose distance to this very short meridian becomes unbounded and this is impossible in view of the third term in the Carathéodory bound. The claim then follows.
The first term of the Carathéodory bound implies that the distances $\delta^\alpha(u, u_i)$ to the boundary are uniformly bounded below away from 0 and the second and third terms together imply that the hyperbolic lengths of the curves $\gamma^\alpha_i$ are uniformly bounded above. As we have shown the lower estimate for the hyperbolic metric from Lemma 3.4 is uniform, and as the improper integral

$$\int_0^{1/2} \frac{1}{x \log(1/x)} \, dx$$

diverges, we can deduce that we can find $\delta_1$ not depending on $\alpha$ such that the curves $\gamma^\alpha_i$ are at least (spherical) distance $\delta_1$ away from the complement of $U_i$ as desired and so we have shown c).

Finally to prove d) we first note that from above the complementary components are all distance at least $2\delta_i$ apart. Thus if one of the components $K^\alpha_i$ of the set $\mathbb{C} \setminus U_i$ had arbitrarily small spherical diameter, then $U_i$ would contain a (round) annulus of very large modulus separating $K^\alpha_i$ from the rest of $\mathbb{C} \setminus U_i$. Next we define the index $j_i$ to be $i$ when the connectivity $n$ is $\geq 3$ and 1 when $n = 2$ and we have a family of pointed annuli. If we then consider the principal meridian $\gamma^\alpha_i$, then $\gamma^\alpha_i$ separates $K^\alpha_i$ from the rest of $\mathbb{C} \setminus U_i$ as does the equator of this annulus.

By Theorem 1.6, the length $\ell^\alpha_i$ of $\gamma^\alpha_i$ would then be very small. However, the third term in the Carathéodory bound shows that the lengths $\ell^\alpha_i$ are uniformly bounded below and so we must have $\delta_2 > 0$ independent of $\alpha$ for which all the components $K^\alpha_i$ of $\mathbb{C} \setminus U_i$ have (spherical) diameter at least $\delta_2$ as required.

To show 4. $\Rightarrow$ 1, suppose we have $\delta_1, \delta_2 > 0$ and curves $\gamma^\alpha_i$ as above and suppose we have a sequence which converges in the Carathéodory topology which we will label $\{(U_i, u_i)\}_{i=1}^\infty$. Let the limit of this sequence be $(U, u)$. By conditions a) and c) of 4., $\delta^\alpha_U(u, u_i) \geq \delta_1$ for every $\alpha$. It then follows easily using (iii) of Carathéodory convergence that $U \neq \{u\}$.

Now take a subsequence $U_{i_k}$ so that the complementary components of each $U_{i_k}$ converge in the Hausdorff topology. By 4., b) and c), the Hausdorff limit must consist of exactly $n$ components $K^1, K^2, \ldots, K^n$ which are at least (spherical) distance $2\delta_i$ apart and by 4. d) each of these must have diameter at least $\delta_2$.

By relabelling if necessary, we can ensure that for each $i$ the sets $K_{i_k}$ converge in the Hausdorff topology to $K^i$. As a Carathéodory limit of $n$-connected domains, $U$ has connectivity $n' \leq n$ and we need to show $U$ has connectivity exactly $n$. This will be done by establishing a one-to-one correspondence between the sets $K^i$ and the components of $\mathbb{C} \setminus U$.

We first show that for every $i$ we can find a point of $\partial K^i$ which is in $\partial U$. Fix $1 \leq i \leq n$. Again define the index $j_i$ to be $i$ when the connectivity $n$ is $\geq 3$ and 1 when $n = 2$ and we have a family of pointed annuli. The curve $\gamma^\alpha_{i_k}$ separates $K^\alpha_{i_k}$ from the rest of $\mathbb{C} \setminus U_{i_k}$. By joining the closest points on this curve and $K^\alpha_{i_k}$ with a line segment, we can make a curve $\tilde{\eta}^\alpha_{i_k}$ which joins $u_{i_k}$ with a point on $\partial K^\alpha_{i_k}$.

Now pass to a further subsequence if needed so that these curves converge in the Hausdorff topology to a continuum $\tilde{\eta}^h$ which by (i) of Carathéodory convergence then clearly joins $u$ to $K^h$. By making this continuum a little smaller if needed, we can assume that it meets $K^i$ only at its endpoint, and so terminates in a line segment of (spherical) length $\geq \delta_1$ at some point $\tilde{z} \in \partial K^i$ which is then at least distance $2\delta_i$ away from the sets $K^j$, $j \neq i$.

From (iii) of Carathéodory convergence it then follows that any compact subset of $\tilde{\eta}^h \setminus \{\tilde{z}\}$ will lie in $U$ whence $\tilde{\eta}^h \setminus \{\tilde{z}\} \subset U$. The point $\tilde{z}$ can then be approximated by points in $U$ and since $\tilde{z} \in K^i$, it follows from the Hausdorff convergence of the sets $K^\alpha_{i_k}$ to $K^i$ and (ii) of Carathéodory convergence that $\tilde{z}$ cannot be in $U$. Thus $\tilde{z} \in \partial K^i \cap \partial U$ as we want.

By Lemma 2.2, if $z \in \partial U$, then $z$ must meet $K^i$ for some $i$ and since the sets $K^i$ are at least distance $2\delta_i$ apart, there can be only one such $i$. Now if $U$ had connectivity $< n$, since from above each of the sets $\partial K^i$ and hence $K^i$ meets $U$, we could find a component $L$ of $\mathbb{C} \setminus U$ and two points $z_1, z_2$ of $\partial L \subset \partial U$, which were in contained in sets $K^{i_1}, K^{i_2}$ respectively with $i_1 \neq i_2$. Now for each $1 \leq i \leq n$, let $G^i = \partial L \cap K^i$. The sets $G^i$ are each clearly closed and from above they give a non-trivial separation of $\partial L$. However, by the corollary to [13, Theorem 14.4, p. 124], $\partial L$ is connected and with this contradiction we see that $U$ must be $n$-connected.

Now, for each $1 \leq i \leq n$, let $L^i$ be the component of $\mathbb{C} \setminus U$ whose boundary meets $K^i$. As $K^i$ is a connected subset of $\mathbb{C} \setminus U$, we must have that $K^i \subset L^i$. It then follows from above that the sets $L^i$ will have spherical diameter $\geq \delta_i$. Thus the Carathéodory limit $(U, u)$ is a non-degenerate $n$-connected pointed domain and so $U$ is bounded. We still need to show that 4. $\Rightarrow$ 2. holds up to constants. However, the bound on the first term of the extended Carathéodory bounds
follows directly from what we have just proved while the bound on the other two terms follows from what we have just proved combined with Corollary 1.13.

We now show $3 \implies 5$, up to constants. So suppose $3$ holds. It follows from $3 \implies 1$, Theorem 3.7 and (ii) of Carathéodory convergence applied to the pointed standard domains that for the points $a_\alpha = \varphi_\alpha(U_\alpha)$, the quantity $\log \delta_{U_\alpha}^0(a_\alpha)$ is uniformly bounded in $\alpha$ and that this bound is uniform with respect to $\|U\|$.

Note that, for $n \geq 2$, on examining the start of the proof of Theorem 3.7, we see that by passing to a subsequence if necessary, we can label the complementary components of a limit pointed domain and the corresponding choice of standard domain to be compatible with those of the approximating pointed domains as specified in the statement of Theorem 3.7. Now there are only $n(n-1)$ ways of assigning which complementary components of a limit domain are mapped to the unit disc or the unbounded complementary component of a standard domain. Hence again by Theorem 3.7 and (ii) of Carathéodory convergence applied to the pointed standard domains, the bound on $\log \delta_{U_\alpha}^0(a_\alpha)$ will not depend on our choice of standard domain for each $U_\alpha$.

Since for $n \geq 2$ all standard domains avoid $\mathbb{R}$, and for $n = 1$, they avoid $\mathbb{C} \setminus \mathbb{D}$, the quantity $\log \text{diam}^\theta(C \setminus \Lambda^\infty)$ is also uniformly bounded in $\alpha$ and must also be uniform with respect to $\|U\|$. Thus the first term of the Carathéodory bound is uniformly bounded for the family $A$ and this bound is uniform with respect to $\|U\|$. By Lemma 3.5, principal meridians are conformally invariant and using this and the conformal invariance of hyperbolic length, the second and third terms for each $|A^{\infty}, a_\alpha|$ are the same as those for $|U_\alpha, u_\alpha|$ and with this we have shown $\|A\| < \infty$ and that this bound is uniform with respect to $\|U\|$.

We still need to show that $\Psi \asymp A$ and that the estimates on $\Psi$ are uniform with respect to $\|U\|$. For each $\alpha$ let $\chi_\alpha$ be the unique normalized covering map from $[D, 0]$ to $|A^{\infty}, a_\alpha|$ as in Theorem 1.2. As in the proof of Theorem 3.7, the composition $\pi_\alpha = \psi_\alpha \circ \chi_\alpha$ is then the corresponding normalized covering map for $(U_\alpha, u_\alpha)$.

Now let $(\{U_{\alpha_n}, u_{\alpha_n}\})_{n=1}^\infty$ be any sequence in $U$ with corresponding pointed standard domains $(\{A^{\infty}, a_{\alpha_n}\})_{n=1}^\infty$. Using $3 \implies 1$, once more, by selecting a subsequence if necessary, we can ensure that both sequences converge to non-degenerate $n$-connected pointed domains $(U, u)$ and $(A, a)$ respectively and where of course $A$ must be a standard domain.

By Theorem 1.2, if we let $\pi$ and $\chi$ be the normalized covering maps for $U$ and $A$ respectively, then $\pi_{\alpha_n} \to \pi$ and $\chi_{\alpha_n} \to \chi$ locally uniformly on $D$. It then follows that for each hyperbolic radius $R \geq 0$, we can find $K = K(R) > 1$ not depending on $\alpha$ such that within hyperbolic distance $R$ of $a_\alpha$ in $A^{\infty}$, $\psi_\alpha$ is locally bi-Lipschitz with constant $K$. The same argument shows that $K$ is uniform with respect to $R$ and the $\|U\|$ and so we have shown $3 \implies 5$, up to constants.

To show $5 \implies 3$, up to constants, as $\|A\| < \infty$ and $\Psi \asymp A$, the quantities $\delta_{U_\alpha}^0(a_\alpha)$ are obviously uniformly bounded below on applying the Koebe one-quarter theorem. Again for the same reasons as before, the second and third terms for each $|U_\alpha, u_\alpha|$ are the same as those for $|A_{\alpha_n}, a_{\alpha_n}|$.

Suppose now we could find a sequence $\{U_{\alpha_n}, u_{\alpha_n}\}_{n=1}^\infty$ with limit $(\mathbb{C} \setminus \{v\}, u)$ for some points $u, v \in \mathbb{C}$ with $u \neq v$ (note that we cannot have $u = v$ as we have already ruled out the possibility that such a Carathéodory limit could be a point). As usual, for convenience we can assume that $v = \infty$ so that our sequence converges to $(\mathbb{C}, u)$. Now let $\varphi_{\alpha_n}$ be the Riemann map from each $U_{\alpha_n}$ to the corresponding standard domain. By (ii) of Carathéodory convergence, $3 \implies 1$, applied to $\|A\| < \infty$ and Montel’s theorem, these Riemann mappings would clearly give a (classical) normal family on any bounded open subset of $\mathbb{C}$. Any limit function would then be an entire function which was either constant or a non-constant entire function which avoided either $\mathbb{D}$ (for $n \geq 2$) or $\mathbb{C} \setminus \mathbb{D}$ (for $n = 1$). The case of a constant limit function would violate the bi-equicontinuity of $\Psi$ while the non-constant case would violate Picard’s theorem on the range of a non-constant entire function (e.g. p. 319 of Lang’s book [11]). Thus the quantity $\log \text{diam}^\theta(C \setminus U_\alpha)$ is uniformly bounded and this bound is uniform with respect to the bounds on $\|U\|$ for $A$ and the bi-equicontinuity of $\Psi$.

Finally, we show $5 \implies 6$. Suppose first $5$ holds. From $5 \implies 1$, we have that $U$ is hyperbolically non-degenerate. By $5$, $\|A\| < \infty$ (where for $n \geq 2$ we are free to make any choice of which complementary components correspond to $\mathbb{D}$ and the unbounded complementary component of the standard domain) and so we can assume that $A$ is the image of a bounded set in $\mathbb{B}_{\alpha_n} \subset \mathbb{R}^{2\alpha_n}$ (note that we will in general have to rotate these standard domains so that the base points lie on the positive real axis). $3 \implies 1$, for $A$ shows that this set must be bounded away from $\partial \mathbb{B}_{\alpha_n}$ and so $U$ must be bounded in moduli space. For the other direction, if $6$ holds, then by Lemma 4.8, as $U$ is non-degenerate, the first term in the Carathéodory bound for the pointed domains of $U$ is uniformly bounded. Since $U$ is bounded in moduli space, if we label two of the complementary components of each $U_\alpha$ in any way we like, the vectors associated with the corresponding
family of standard domains can be chosen to form a precompact set in \( \mathcal{B}_\alpha \) whence the corresponding standard domains are precompact in \( \mathcal{D}_\alpha \). This shows that the resulting family \( \mathcal{A} \) of standard domains is bounded. The uniform bounds on the other two terms of the Carathéodory bounds for the domains of \( \mathcal{U} \) then follow from Lemma 3.5 on the conformal invariance of principal meridians and 5. then follows from 3. \( \Rightarrow \) 5. for \( \mathcal{U} \). With this the proof is complete. \( \square \)

Theorem 4.15 has a useful consequence. Recall that we say that an annulus \( A \) separates a set \( F \subset \mathbb{C} \) if \( A \) does not meet \( F \) but each of the two complementary components of \( A \) does meet \( F \). Recall also that a closed set \( F \subset \mathbb{C} \) is called \( K \)-uniformly perfect if the moduli of (round) annuli which separate \( F \) are uniformly bounded above by \( K \) (see [14]). For hyperbolic plane domains \( \Omega \), whether or not the boundary \( \partial \Omega \) is uniformly perfect can also be characterized in terms of the hyperbolic metric or, equivalently, the hyperbolic density [2, Corollary 1].

**Corollary 4.16.**

Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in \mathcal{A}} \) be a family of non-degenerate pointed domains each of which has finite connectivity such that either

(i) the hyperbolic lengths for all the meridians in an extended system for each \( U_\alpha \) are uniformly bounded below away from zero, or

(ii) the hyperbolic lengths for all the principal meridians of each \( U_\alpha \) are uniformly bounded below away from zero while the distances for these principal meridians are uniformly bounded above.

Then there exists \( K_2 > 0 \) independent of \( \alpha \) for which the boundaries \( \partial U_\alpha \) are \( K_2 \)-uniformly perfect. Equivalently, there exists \( K_2 \geq 1 \) such that if \( \rho_{U_\alpha}(\cdot, \cdot) \) denotes the hyperbolic metric on \( U_\alpha \), then

\[
\frac{1}{K_2} \frac{|dz|}{\delta_{U_\alpha}^2(z)} \leq \rho_{U_\alpha}(z) \leq K_2 \frac{|dz|}{\delta_{U_\alpha}^2(z)}.
\]

In particular, the above holds if \( \mathcal{U} \) is a bounded family.

**Proof.** In the first case, if we could find round annuli of arbitrarily large moduli which separated the complements of the domains \( U_\alpha \), the equators of these annuli would have arbitrarily short hyperbolic length. By Theorem 1.6, we could then find arbitrarily short meridians, which is impossible in view of the lower bound on the hyperbolic lengths of the meridians in the statement.

For the second case, again it follows that the hyperbolic lengths of all the meridians of each \( U_\alpha \) (and not merely the principal meridians) are uniformly bounded below away from zero. To see this, note that by the collar lemma, if we could find arbitrarily short meridians, then the same argument as used in 3. \( \Rightarrow \) 4. of the last result would imply that the distances for \( \mathcal{U} \) were unbounded. The argument for the existence of a uniform upper bound for the moduli of annuli which separate the boundaries of these domains is then the same as before.

The above condition on the hyperbolic metric is usually given in terms of the Euclidean distance to the boundary [2, Corollary 1] and it is valid everywhere in the domain instead of just near the boundary. To see that we also have the above version using the spherical metric, we observe that the Euclidean and spherical metrics are equivalent on the closed unit disc, while for \( |z| > 1 \) we can as usual apply the transformation \( z \mapsto 1/z \) which is an isometry of the spherical metric. \( \square \)

The above result is essentially a statement about the behaviour near the boundary of the hyperbolic density and it can of course be stated in these terms. Note that it is essential that we consider either the lengths and distances or just the extended lengths in the above statement as the following example shows. For each \( m \geq 3 \), let \( U_m \) be the quadruply connected domain obtained by removing the two small closed discs \( \overline{D(3/4m, 3/4m)} \), \( \overline{D(-5/4m, 3/4m)} \) and two large closed discs \( \overline{D(5m/4, 3m/4)} \), \( \overline{D(-5m/4, 3m/4)} \) from \( \mathbb{C} \) and consider the sequence of pointed domains \( \{ (U_m, 1) \}_{m=3}^\infty \) (note that these discs are chosen so that each \( U_m \) is invariant under the transformation \( z \mapsto 1/z \)).

By rescaling in turn about 0 and \( \infty \), we can apply the estimates of Lemma 3.4 in a uniform manner to deduce that the lengths of the principal meridians are uniformly bounded above in \( m \) while it follows from the collar lemma and the fact that conformal annuli of large modulus must contain round annuli of large modulus that they are also uniformly
bounded below away from 0 in \(m\). If we then let \(\gamma_a\) be the meridian which separates \(\overline{D}(5/4m, 3/4m) \cup \overline{D}(−5/4m, 3/4m)\) from \(\overline{D}(3m/4, 3m/4) \cup \overline{D}(−3m/4, 3m/4)\), then it follows by comparing the length of this meridian with that of the unit circle using Theorem 1.6 that the length of this meridian clearly tends to 0 as \(m\) tends to infinity. Note also that, again by the collar lemma, the width of the collar associated with this meridian and thus the distance from it to the principal meridians also tends to infinity.

The point here is that the lack of uniform bounds on the moduli of annuli which separate the complements of these domains is seen only in terms either of the behaviour of a meridian which is not a principal meridian for any of the pointed domains in this sequence or of the fact that the two pairs of principal meridians for these domains are getting farther apart. We conclude this section with a few more general remarks. First, it is obvious in the case of annuli (and also trivially for discs), that one can find a bound on the minimum dilatation required for a quasiconformal mapping from one domain of a bounded family to another. It is also probably true that this remains the case for families of domains of higher connectivity. It therefore seems plausible that there might be another way of defining boundedness in terms of an appropriate Teichmüller space.

It is also worth remarking that, as observed earlier, the set \(B_n \subset \mathbb{R}^{2n-1}\) ‘unrolls’ some of the directions in \(\mathcal{M}_n\) where we get ‘cylinder-like’ geometry. This leads to the question of whether \(B_n\) is actually contractible and how close it therefore comes to being a realization of such a Teichmüller space. We hope to investigate these and other questions in subsequent papers.

In complex dynamics, one often has to construct a quasiconformal map between two different Riemann surfaces. However, one often has to do so in such a way as to preserve dynamics and so one is usually given certain identifications of the boundary components which come from these dynamics. A good example is the Sullivan straightening theorem or, more critically, the non-autonomous version where one considers a polynomial-like mapping sequence instead of just one polynomial-like map. In particular, one is dealing with an infinite sequence of fundamental annuli and one has to be much more careful about how to identify these annuli so as to preserve the dynamics and ensure that we get a sequence of mappings with bounded quasiconformal dilatation and suitable equicontinuity properties which serves as a (non-autonomous) conjugacy between the polynomial-like sequence and a suitable sequence of polynomials. In order to avoid excessively cumbersome statements and proofs and produce results which preserve as closely as possible the spirit of the original classical theorems, one needs to make extensive use of the notions in this paper (see [5] for details).

5. Families of functions

In this section we present some theorems which illustrate how the results we have proved allow us to show how the definitions we gave of normality, equicontinuity and local uniform convergence are natural extensions of the standard ones. We also present some further useful results concerning bounded families of pointed domains.

From now on let us assume that \(U = \{(U_a, u_a)\}_{a \in A}\) and \(V = \{(V_a, v_a)\}_{a \in A}\) are two families of pointed domains indexed by the same set \(A\), and let \(A = \{(A^a, a)\}_{a \in A}\) denote a family of standard pointed domains for \(U\).

If we let \(\mathcal{F} = \{f_a\}_{a \in A}\), \(\mathcal{G} = \{g_a\}_{a \in A}\) be families defined on \(U, V\) respectively (in the sense given in Definition 4.9) with \(\mathcal{F} : U \to V\) (as in Definition 4.12), then we obtain the family of compositions \(\{g_a \circ f_a\}_{a \in A}\) which is a family defined on \(\mathcal{F}\) and which we denote in the obvious way by \(\mathcal{G} \circ \mathcal{F}\). Similarly, if the members of \(\mathcal{F}\) are all univalent, then the family of inverse functions will be defined on \(V\) and we will denote it by \(\mathcal{F}^{-1}\).

The following is immediate in view of the Schwarz lemma for the hyperbolic metric.

**Proposition 5.1.**

Let \(U, V\) be two hyperbolically non-degenerate families and let \(\mathcal{F}\) be a family of functions which maps \(U\) to \(V\).

1. If \(\mathcal{F} \circ \mathcal{G}\) and \(\mathcal{G} \circ \mathcal{F}\), then \(\mathcal{G} \circ \mathcal{F} \prec \mathcal{G}\).
2. If \(\mathcal{F} \circ \mathcal{G}\) and \(\mathcal{G} \circ \mathcal{F}\), then \(\mathcal{F} \circ \mathcal{G} \prec \mathcal{F}\).
3. If \(\mathcal{F}\) is univalent and \(\mathcal{F} \circ \mathcal{G}\), then \(\mathcal{F}^{-1} \circ \mathcal{G}\).
Let $\Pi = \{\pi_a\}_{a \in A}$ denote the family of normalized covering maps from $\mathbb{D}, 0$ to $\mathbb{U}$. The following is an immediate consequence of Theorem 4.13 as any bounded family is automatically hyperbolically non-degenerate.

**Proposition 5.2.**
If $\text{pt} \subset \mathbb{U} \subset \mathbb{C}$, then $\Pi \bowtie (\mathbb{D}, 0)$.

The converse of this result is false. To see this consider the sequence $\{(U_m, 0)\}_{m=1}^\infty$ where $U_m = \mathbb{D} \setminus \mathbb{D}(1-2/m, 1/m)$. By Theorem 1.2 again, the normalized covering maps clearly converge to the identity and so give a bi-equicontinuous family on $(\mathbb{D}, 0)$. However, the limit of the doubly connected pointed domains $(U_m, 0)$ is $(\mathbb{D}, 0)$ and, as $\mathbb{D}$ is simply connected, this sequence is not bounded. Nevertheless, for Riemann maps and simply connected domains, the converse is true as can be seen immediately from Theorem 4.15.

From Theorem 4.13 and Proposition 5.1, we immediately get the following.

**Corollary 5.3 (bounded families and equicontinuity).**
Let $\mathbb{U}$ and $\mathbb{V}$ be families of pointed hyperbolic domains with $\mathbb{U}$ hyperbolically non-degenerate and let $\mathcal{F} : \mathbb{U} \to \mathbb{V}$ be a family of covering maps. Then $\mathbb{V}$ is hyperbolically non-degenerate if and only if $\mathcal{F} \bowtie \mathbb{U}$.

In view of the counterexample above, even if $\mathbb{U}$ is bounded and $\mathcal{F} : \mathbb{U} \to \mathbb{V}$ with $\mathcal{F} \bowtie \mathbb{U}$, we cannot say that $\mathbb{V}$ is bounded. However, we do have the following.

**Theorem 5.4 (bi-equicontinuity and boundedness).**
Let $d \geq 1$, let $\mathbb{U}$ be bounded and let $\mathcal{F} : \mathbb{U} \to \mathbb{V}$ be a family of covering mappings with $\mathcal{F} \bowtie \mathbb{U}$ such that the degrees of the mappings of $\mathcal{F}$ are uniformly bounded above by $d$ and all the domains in $\mathbb{V}$ have the same finite connectivity. Then $\mathbb{V}$ is also bounded. Further $\|\mathbb{V}\|$ is uniformly bounded with respect to $\|\mathbb{U}\|$ and $d$.

In view of what we said above, the assumption on the boundedness of the degrees is certainly necessary and we proceed by first proving the following technical lemma. The main reason we require this lemma is that, although a lifting of a meridian under a covering map of finite degree must be a simple closed geodesic, there is no guarantee that it will still be a meridian. For two subsets $E, F$ of $C$, let us use $\text{dist}(E, F)$ to denote the Euclidean distance from $E$ to $F$.

**Lemma 5.5.**
Let $E$ be a hyperbolic domain, suppose $\mathbb{C} \setminus E = E \cup F$ where $E$ and $F$ are closed disjoint non-empty subsets of $\mathbb{C}$ with $\infty \in F$ and let $\gamma$ be a curve in $E$ which separates $E$ and $F$. Let $r, R$ be two positive real numbers such that $\text{dist}(\gamma, \partial E) \geq r$, $\ell(\gamma) \leq R$ and $\gamma \subset \mathbb{D}[0, R]$. Then we can find a curve $\tilde{y}$ in $\mathbb{U}$ with $n(y, z) = 1$ for all $z \in E$, $n(y, z) = 0$ for all $z \in F$ and whose hyperbolic length $\ell(y)$ is uniformly bounded above in $r$ and $R$.

**Proof.** We begin by defining an enlarged version $\tilde{E}$ of $E$ by setting $\tilde{E} = \{z : \text{dist}(z, E) \leq r/4\}$. Now cover the plane with a mesh of closed squares of side length $r/4\sqrt{2}$ whose boundaries are oriented positively. Let $\Gamma$ be the cycle

$$
\Gamma = \sum_i Q_i,
$$

where the sum ranges over all those squares $Q_i$ which meet $\tilde{E}$. Note that the number of such squares is uniformly bounded above in $r$ and $R$, a crude bound being $(2R/r + 1)^2$.

By cancelling those segments which are on the boundary of more than one square, we can see that $n(\Gamma, z) = 1$ for every $z \in \tilde{E}$ and thus for every $z \in E$, even if $z$ lies on the edge or corner of a square, while $n(\Gamma, z) = 0$ for every $z \in F$.

Now any of the remaining segments avoids $\tilde{E}$ and so must be at least distance $r/4$ from $E$ (note that this positive lower bound on the distance of $\Gamma$ to $E$ was precisely the reason why we introduced the enlarged set $\tilde{E}$). On the other hand,
each of these segments adjoins a square which does meet \( \overline{E} \) and so must be at most distance \( r/2 \) from \( E \) and thus at least distance \( r/2 \) from \( \overline{\gamma} \) and distance \( 3r/2 \) from \( F \).

It is not hard to see by checking cases that no endpoint of a segment in \( \Gamma \) which remains after cancellation can be a meeting point of exactly one or exactly three segments. From this, a relatively straightforward argument by induction on the number of segments in \( \Gamma \) shows we can express it as a sum of simple closed curves

\[
\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n.
\]

Note that the number \( n \) of such curves is again uniformly bounded above in terms of \( r \) and \( R \).

If \( n = 1 \) and \( \Gamma \) consists of only one curve, then the result follows on using the upper bound on the hyperbolic metric in Lemma 3.4 (note that this upper bound holds everywhere and not just within a certain distance of the boundary of the domain as is the case for the lower bound). The main work, then, is in making use of the curve \( \overline{\gamma} \) given in the statement to join these potentially disjoint curves together without increasing the hyperbolic length too much.

If any two curves in \( \Gamma \) cross each other, they must do so at a point were four segments come together. Hence, by making our curves so that we always turn left or right where four segments meet (i.e. by not going straight ahead at such points), we can ensure that for any two such curves, each must lie entirely inside or outside the other with the exception of at most finitely many points (these being points where four segments meet). Further, by removing from \( \Gamma \) any of these curves which lie entirely inside another (including the case above where the two curves may meet at a finite number of points), it follows from the Jordan curve theorem we can still ensure that the winding number of \( \Gamma \) about points of \( \overline{E} \) and thus about points of \( E \) is 1 (it follows easily from the fact that the boundaries of all our original squares were oriented positively that this winding number remains positive). If we further remove any curves which do not contain any points of \( E \), we can ensure that each curve \( \Gamma_i \) contains points of \( E \) in its bounded complementary component without altering the fact that the winding number of \( \Gamma \) about points of \( E \) is 1.

Again by the Jordan curve theorem, any curve \( \Gamma_i \), \( 1 \leq i \leq n \), of \( \Gamma \) separates the plane into two complementary components, only one of which can meet \( \overline{\gamma} \) which is connected and does not meet \( \Gamma_i \). As \( \infty \in F \), \( E \) and thus \( \Gamma_i \) lies in a bounded component of \( \overline{C} \setminus \overline{\gamma} \) so that any path connecting \( \Gamma_i \) to infinity must meet \( \overline{\gamma} \). Hence, as a connected set, \( \overline{\gamma} \) must lie in the unbounded component of \( \overline{C} \setminus \Gamma_i \). Thus both \( \Gamma_i \) and its bounded complementary component lie in the same bounded component of \( \overline{C} \setminus \overline{\gamma} \). Since \( \overline{\gamma} \) separates \( E \) and \( F \) while \( \Gamma_i \) contains points of \( E \), it follows that there can be no points of \( F \) inside \( \Gamma_i \). Hence the winding number of \( \Gamma \) about points of \( F \) after removal from \( \Gamma \) of any curve which lies inside another or contains no points of \( E \) is still zero and our cycle is still in the correct homology class.

Pick \( 1 \leq i \leq n \), let \( a \) and \( b \) be the two points on \( \overline{\gamma} \) and \( \Gamma_i \), respectively which are as close as possible and let \( L_i \) denote the line segment joining \( a \) and \( b \). From above, as a point on \( \overline{\gamma} \), \( a \) lies outside \( \Gamma_j \) for every \( 1 \leq j \leq n \) while none of the curves \( \Gamma_j \), \( 1 \leq j \leq n \), is contained inside another. Thus if \( L_i \) meets some \( \Gamma_j \) it must enter \( \Gamma_j \) and subsequently leave it (we can ignore the case where \( L_i \) meets \( \Gamma_j \) but does not penetrate inside it). Since the curves of \( \Gamma \) completely enclose \( E \) (as can be seen from above using winding numbers), by replacing those parts of \( L_i \) (if any) which lie inside one of the other curves \( \Gamma_j \), \( j \neq i \), with parts of segments of \( \Gamma_j \), we can obtain a curve \( \tilde{L}_i \) joining \( a \) to \( b \) with \( \text{dist}(\tilde{L}_i, E) \geq r/4 \). Note that as the number of squares enclosed by \( \Gamma \) is uniformly bounded, so is the number of such replacements we need to carry out and thus the amount of Euclidean length we add to \( L_i \) (in fact, it is not too hard to show we can do this in such a way that the parts of \( \Gamma_j \) which are used in this manner can be chosen so as not to overlap).

Next, we examine the distances of points on \( \tilde{L}_i \) to \( F \). As the points \( a \) and \( b \) are as close as possible, it follows that, except for the endpoint \( a \), the line segment \( L_i \) must lie entirely in the same bounded complementary component of \( \overline{\gamma} \). As noted above, for each \( 1 \leq j \leq n \), the curve \( \Gamma_j \) and its bounded complementary component lie in the same bounded complementary component of \( \overline{\gamma} \). Hence the same is true of \( \tilde{L}_i \) and since \( \Gamma_i \) contains points of \( E \) while \( \overline{\gamma} \) separates \( E \) and \( F \) and is distance \( \geq r \) from \( F \), any point on \( L_i \) must then be at least distance \( r \) from \( F \).

Since \( \overline{\gamma} \) is within distance \( R \) of \( 0 \), so must be the set \( E \) which it contains as well as the sets \( \overline{E} \) and hence \( \Gamma \). It then follows that each of the line segments \( L_i \), above has Euclidean length at most \( 2R \) and as the length of \( \Gamma \) is uniformly bounded, the length of the modified segments \( \tilde{L}_i \) is also uniformly bounded above and this bound is uniform in \( r \) and \( R \).

Now using the upper bound on the hyperbolic metric in Lemma 3.4 (which, as we observed earlier, holds throughout the domain \( U \)), the hyperbolic length of \( \overline{\gamma} \), of the curves \( \Gamma_i \) and of the curves \( \tilde{L}_i \) above are all bounded uniformly in terms
of \( r \) and \( R \). If we now connect the curves \( \Gamma, \) of \( \Gamma \) using the curves \( \tilde{\gamma} \) and pieces of \( \tilde{y} \) both of which we traverse in both directions (so that the new curve we obtain is homologous in \( U \) to \( \Gamma \)), then the fact that, as we have already seen, the number \( n \) of such curves \( \Gamma, \) is bounded in terms of \( r \) and \( R \) shows that we can obtain the desired curve \( y \) of the statement.

Before proving Theorem 5.4, we need one more lemma which is a generalization of part of [6, Lemma 3.6].

**Lemma 5.6.**
Let \( U, V \subset \mathbb{C} \) be hyperbolic domains of finite connectivity with \( U \) being non-degenerate and let \( f: U \to V \) be a covering map of some finite degree \( d \geq 1 \). Then \( V \) is non-degenerate.

**Proof.** Suppose for the sake of contradiction that \( V \) was degenerate and that some of the complementary components of \( V \) were points. Given \( \varepsilon > 0 \) small we could then find a simple closed curve \( \gamma \) in \( V \) which was homotopically non-trivial in \( V \) and whose hyperbolic length in \( U \) was less than \( \varepsilon \). If we then let \( \eta \) be a lift of \( \gamma \) to \( U \) using \( f \), then \( \eta \) is a simple closed curve in \( U \) which must also be homotopically non-trivial in \( U \) and whose hyperbolic length is less than \( d \varepsilon \).

As \( \eta \) is a simple closed curve, by the Jordan curve theorem, it has two complementary components, both of which must contain components of \( \mathbb{C} \setminus U \) and, as \( U \) is non-degenerate, the spherical length of \( \eta \) cannot be arbitrarily low. It then follows from Lemma 3.4 that the hyperbolic length of \( \eta \) also cannot be arbitrarily low and with this contradiction we see that \( V \) is non-degenerate as desired.

**Proof of Theorem 5.4.** By Lemma 5.6 the domains of \( V \) must be non-degenerate. We will show that \( \| V \|_E < \infty \) and then appeal to Theorem 4.15. By the bi-equicontinuity of \( \mathcal{F} \), the infinitesimal ratios \( |d^2f_\alpha(z)|/|dz|^2 \) (i.e. the expansion of \( f_\alpha \) in with respect to the spherical metric in cotangent space) are bounded below at \( z = u_\alpha \). By pre- and postcomposing with \( 1/z \) which preserves the spherical metric, we can assume that \( u_\alpha \) and \( v_\alpha = f_\alpha(u_\alpha) \) both lie in the closed unit disc \( \mathbb{D} \). The fact that \( \delta^\mathbb{D}(v_\alpha) \) is bounded below then follows immediately from Blach’s theorem (e.g. [7, p. 293, Chapter XII, Theorem 1.4]).

For the lower bound on the spherical diameters \( \text{diam}^\mathbb{S}(\mathbb{C} \setminus V_\alpha) \), since the quantities \( \delta^\mathbb{S}_u(u_\alpha), \delta^\mathbb{S}_v(v_\alpha) \) are bounded below, we can pre- and postcompose with suitable uniformly bi-Lipschitz Möbius transformations to assume without loss of generality that \( U_\alpha, V_\alpha \) are both subdomains of \( \mathbb{C} \) and that \( u_\alpha = v_\alpha = 0 \). The bi-equicontinuity of \( \mathcal{F} \) then implies that the absolute values of the (Euclidean) derivatives \( |f'_\alpha(u_\alpha)| \) are bounded above. The desired conclusion then follows from the corresponding lower bound on the spherical diameters \( \text{diam}^\mathbb{S}(\mathbb{C} \setminus U_\alpha) \) for the domains of \( \mathbb{U} \) and on applying the Koebe one-quarter theorem to a suitable inverse branch of each \( f_\alpha \) on the disc \( D(v_\alpha, \delta^\mathbb{S}_v(v_\alpha)) \) where \( \delta^\mathbb{S}_v(v_\alpha) \) is the Euclidean distance from \( v_\alpha \) to the boundary \( \partial V_\alpha \).

We now turn to examining the (extended) lengths of \( V \) and we first establish a positive lower bound for the lengths of the extended meridians. Let \( y_\alpha \) be a meridian in one of the domains \( V_\alpha \) of \( V \) and let \( \eta_\alpha \) be a lifting of \( y_\alpha \) which lies in \( U_\alpha \). Then, since \( f_\alpha \) is of degree \( \leq d \) as a covering map, \( \eta_\alpha \) is a simple closed hyperbolic geodesic in \( U_\alpha \) which has length at most \( d\ell(y_\alpha) \). By Theorem 1.6, we can find meridians in the homology class of \( \eta_\alpha \) which separate the complement of \( U_\alpha \) in the same way as \( \eta_\alpha \) and the length of any such meridians is also at most \( d\ell(y_\alpha) \). It then follows from the fact that \( \| U \|_E < \infty \) that the lengths of the extended meridians of \( V \) must be bounded below. It is worth remarking at this point that this is where we need the bound \( d \) on the degrees of the covering maps \( f_\alpha \). If one examines the counterexample after Proposition 5.2, one sees that all the covering maps (with the exception of that for the limit domain) have infinite degree.

We now show that the extended lengths of \( V \) are also bounded above. Again, by the uniform lower bound on the quantities \( \delta^\mathbb{S}(u_\alpha), \delta^\mathbb{S}(v_\alpha) \), we can assume that the domains \( U_\alpha, V_\alpha \) are subdomains of \( \mathbb{C} \) with \( u_\alpha = v_\alpha = 0 \) for each \( \alpha \). Again let \( y_\alpha \) be a meridian of \( V_\alpha \) which separates \( \mathbb{C} \setminus V_\alpha \) into disjoint non-empty closed sets \( E_\alpha, F_\alpha \) with \( \infty \in F_\alpha \). If we once again let \( \eta_\alpha \) be a lifting of \( y_\alpha \), then \( \eta_\alpha \) is a simple closed geodesic in \( U_\alpha \), and by Theorem 1.6 we can find a meridian \( \tilde{\eta}_\alpha \) in the homology class of \( \eta_\alpha \). Using Cauchy’s theorem, if we let \( \tilde{y}_\alpha \) be the curve \( f_\alpha(\tilde{\eta}_\alpha) \) (which may possibly be traversed more than once), then one can check that the winding number of \( \tilde{\eta}_\alpha \) about points of \( E_\alpha \) is non-zero while the winding number about points of \( F_\alpha \) is zero. Hence \( \tilde{y}_\alpha \) separates \( E_\alpha \) and \( F_\alpha \).

Since \( \| U \|_E < \infty \), it follows from Theorem 4.15 that the hyperbolic length of \( \tilde{\eta}_\alpha \) and the distance of this meridian to the basepoint \( u_\alpha \) are uniformly bounded above. Hence by the Schwarz lemma for the hyperbolic metric, the hyperbolic
lengths of the curves $\tilde{\gamma}_a$ and their distances from the base points $v_a$ in $V_o$ are also uniformly bounded above by some bound $R > 0$. Any point on $\tilde{\gamma}_a$ is then hyperbolic distance at most $3R/2$ away from $v_a = 0$, and since $\delta^\#(v_a)$ is uniformly bounded below and $\infty \in F_a \subset \overline{\mathbb{T}} \setminus V_o$, it follows from using the fact that the extended lengths of $\mathcal{V}$ are bounded below which allows us to apply Corollary 4.16 that we can make $R$ larger if needed so that the curves $\tilde{\gamma}_a$ all lie in $\overline{D(0, R)}$. Again by Corollary 4.16 and the fact that $\delta^\#(v_a)$ is uniformly bounded below, we can also find $r > 0$ such that for every $a$, $\text{dist}(\tilde{\gamma}_a, \partial V_o) \geq r$.

We thus have satisfied the hypotheses of the above lemma for the domain $V_o$, the sets $E_o, F_o$ and the curve $\tilde{\gamma}_a$. Applying the lemma then gives us a curve in the homology class of $\gamma_a$ whose hyperbolic length in $V_o$ is uniformly bounded above in $\alpha$. The desired uniform upper bound on the length of $\gamma_a$ is then immediate in view of Theorem 1.6.

Finally, we examine the extended distances of $\mathcal{V}$. As observed above, the hyperbolic distance from $u_\alpha$ to $\tilde{\eta}_a$ is uniformly bounded above and by [4, Lemma 2.9], $\eta_a$ and $\tilde{\eta}_a$ must intersect (which includes the possibility that they are the same curve). Again using $\|\U\|_E < \infty$, the hyperbolic distance from $u_\alpha$ to $\eta_a$ is uniformly bounded above and by the Schwarz lemma the hyperbolic distance from $v_\alpha$ to $\gamma_a$ is then also uniformly bounded above. 

\[ \square \]

6. Bounded containment

Instead of being bounded (in $\mathbb{T}$), a family of pointed domains can be bounded within another bounded family.

**Definition 6.1.**

Let $n' \geq 1$ and let $(U', u')$ and $(U, u)$ be pointed domains where $U'$ is non-degenerate with connectivity $n'$, $U$ is hyperbolic and $U' \subset U$. We say that $U'$ is \textit{bounded above} and \textit{below} or \textit{just bounded} in $U$ with constant $K \geq 1$ if

1. $U'$ is a subset of $U$ which lies within hyperbolic distance at most $K$ about $u$ in $U$;
2. $\delta^\#(u') \geq \delta^\#(u)/K$;
3. $L((U', u')) \leq K$;
4. $D((U', u')) \leq K$.

If $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ is a family of pointed hyperbolic domains, we say that another family $\mathcal{U}' = \{(U'_\alpha, u'_\alpha)\}_{\alpha \in A}$ of $n'$-connected non-degenerate pointed domains is \textit{bounded above and below} or \textit{just bounded} in $\mathcal{U}$ if we can find $K \geq 1$ such that for every $\alpha$, $U'_\alpha$ is bounded in $U_\alpha$ with this constant $K$. In this case we write $\mathcal{U} \subset \mathcal{U}' \subset \mathcal{U}$.

Note the similarities with the Carathéodory bound as given in Definition 4.6. Once again, if the domains of $\mathcal{U}'$ are simply connected, then conditions 3. and 4. are vacuously true.

We are also interested in boundedness for the degenerate case of a family of simple closed curves, such as meridians of a family of domains (this is useful in considering quasiconformal gluing problems, for example, when gluing the boundaries of certain annuli in order to prove a non-autonomous version of the Sullivan straightening theorem e.g. [5]).

**Definition 6.2.**

Let $\Gamma = \{(y_\alpha, z_\alpha)\}_{\alpha \in A}$ be a family of pointed simple smooth (i.e. $C^1$) closed curves where for each $\alpha$, $z_\alpha$ is a point on $y_\alpha$ and $\varphi_\alpha : \mathbb{T} \to \overline{\mathbb{T}}$ is a parametrization of $y_\alpha$. Let $\mathcal{U}$ be a family of pointed hyperbolic domains. We say $\Gamma$ is \textit{bounded above and below} or \textit{just bounded} in $\mathcal{U}$ with constant $K \geq 1$ if

1. for each $\alpha$, $y_\alpha$ is a subset of $U_\alpha$ which lies within hyperbolic distance at most $K$ about $u_\alpha$ in $U_\alpha$;
2. the mappings $\varphi_\alpha$ can be chosen so that the resulting family $\Phi$ is bi-equicontinuous on $\mathbb{T}$ in the sense that

\[
\frac{1}{K} \leq \frac{|\varphi'_\alpha(z)|}{(1 + |\varphi'_\alpha(z)|^2)\delta^\#_{U_\alpha}(z_\alpha)} \leq K, \quad z \in \mathbb{T}, \quad \alpha \in A.
\]
In this case, we write \( \text{pt} \subset \Gamma \subset \mathbb{C} \).

We say \( \Gamma \) is \textit{bounded above and below} or \textit{just bounded} in \( \mathbb{C} \) with constant \( K \geq 1 \) if the mappings \( \varphi_{\alpha} \) can be chosen so that the resulting family \( \Phi \) is bi-equicontinuous on \( \mathbb{T} \) in the sense that

\[
\frac{1}{K} \leq \frac{|\varphi'_{\alpha}(z)|}{1 + |\varphi_{\alpha}(z)|^2} \leq K, \quad z \in \mathbb{T}, \quad \alpha \in A.
\]

In this case, we write \( \text{pt} \subset \Gamma \subset \mathbb{C} \).

Using Theorems 4.15, 4.13, and Proposition 5.2, we immediately have the following.

\textbf{Proposition 6.3.}

Let \( \mathcal{U} = \{ (U_\alpha, u_\alpha) \}_{\alpha \in A} \) be a bounded family of \( n \)-connected domains, for each \( \alpha \) let \( \gamma_\alpha \) be a meridian of \( U_\alpha \), let \( z_\alpha \) be a point on \( \gamma_\alpha \) and let \( \Gamma \) be the resulting family. Then \( \text{pt} \subset \Gamma \subset \mathcal{U} \).

From now on, unless otherwise specified, we will use \( \mathcal{U}, \mathcal{V}, \mathcal{W} \) and \( \mathcal{W} \) to refer to families of non-degenerate finitely connected domains where all the domains in a given family have the same connectivity.

The next two results are natural consequences of the notion of boundedness.

\textbf{Theorem 6.4 (transitivity of boundedness).}

If \( \text{pt} \subset \mathcal{U} \subset \mathcal{V} \subset \mathcal{W} \) where \( \mathcal{W} \) is either bounded or all the domains of \( \mathcal{W} \) are \( \mathbb{C} \), then \( \text{pt} \subset \mathcal{U} \subset \mathcal{W} \). This includes the degenerate case where \( \mathcal{U} \) is a family of simple closed curves. The constants for the boundedness of \( \mathcal{U} \) in \( \mathcal{W} \) are uniformly bounded above in terms of those for \( \mathcal{U} \) in \( \mathcal{V} \) and \( \mathcal{V} \) in \( \mathcal{W} \).

\textbf{Proof. } Consider first the case when \( \mathcal{U} \) is a family of pointed domains and \( \mathcal{W} \) is bounded. The only thing which needs to be checked is 2. and by 1. and 2. for the boundedness of \( \mathcal{U} \) in \( \mathcal{V} \) and of \( \mathcal{V} \) in \( \mathcal{W} \), Corollary 4.16 and the Schwarz lemma for the hyperbolic metric

\[
\delta^w_{\gamma_\alpha}(u_\alpha) \sim \delta^w_{\gamma_\alpha}(v_\alpha) \sim \delta^w_{\gamma_\alpha}(w_\alpha) \sim \delta^w_{\gamma_\alpha}(u_\alpha).
\]

where as usual the subscripts denote the domains with respect to which the distance to the boundary is taken.

In the case where \( \mathcal{U} \) is a family of hyperbolic domains and all the domains of \( \mathcal{W} \) are \( \mathbb{C} \), we shall show that \( \| \mathcal{U} \| < \infty \) and then appeal to Theorem 4.15. By 1. and 2. for the boundedness of \( \mathcal{U} \) in \( \mathcal{V} \) combined with Corollary 4.16 and finally the bound for \( \delta^w_{\gamma_\alpha}(v_\alpha) \) arising from the fact that by Theorem 4.15 the boundedness of \( \mathcal{V} \) implies that \( \| \mathcal{V} \| < \infty \),

\[
\delta^w_{\gamma_\alpha}(u_\alpha) \sim \delta^w_{\gamma_\alpha}(v_\alpha) \sim \delta^w_{\gamma_\alpha}(w_\alpha) \sim 1,
\]

whence we have the correct bounds on the spherical distances from each \( u_\alpha \) to the boundary of \( U_\alpha \). As \( U_\alpha \subset V_\alpha \) for each \( \alpha \), the spherical diameters \( \text{diam}_D(\mathbb{C} \setminus U_\alpha) \) are bounded below away from 0 and so we have taken care of the first term in the Carathéodory bound. Finally, the bounds on the lengths of \( U_\alpha \) and on the hyperbolic distances in \( U_\alpha \) from the basepoints \( u_\alpha \) to the principal meridians for \( U_\alpha \) follow directly from 3. and 4. for the boundedness of \( \mathcal{U} \) in \( \mathcal{V} \).

If \( \mathcal{U} \) is a family of curves and \( \mathcal{W} \) is bounded, the result follows easily from Corollary 4.16 and the fact that if \( z \) is a point on the curve \( U_\alpha \), then, similarly to above,

\[
\delta^w_{\gamma_\alpha}(z) \sim \delta^w_{\gamma_\alpha}(u_\alpha) \sim \delta^w_{\gamma_\alpha}(v_\alpha) \sim \delta^w_{\gamma_\alpha}(w_\alpha) \sim \delta^w_{\gamma_\alpha}(u_\alpha) \sim \delta^w_{\gamma_\alpha}(z).
\]

For the final possibility, if \( \mathcal{U} \) is a family of curves and all the domains of \( \mathcal{W} \) are \( \mathbb{C} \), if \( z \in U_\alpha \), by the boundedness of \( \mathcal{U} \) in \( \mathcal{V} \), the boundedness of \( \mathcal{V} \) and again Corollary 4.16,

\[
\delta^w_{\gamma_\alpha}(z) \sim \delta^w_{\gamma_\alpha}(u_\alpha) \sim \delta^w_{\gamma_\alpha}(v_\alpha) \sim 1.
\]

This shows the curves \( U_\alpha \) also give a bi-equicontinuous family with respect to the spherical metric and with this the proof is now complete. \( \square \)
Theorem 6.5 (bi-equicontinuity and bounded containment).

Let \( \mathcal{U}, \mathcal{U}' \) be two families with \( \text{pt} \subset \mathcal{U} \subset \mathcal{U} \subset \mathbb{C} \) and let \( \mathcal{F} \) be a family of covering maps (for both the domains of \( \mathcal{U} \) and \( \mathcal{U}' \) of uniformly bounded degree with \( \mathcal{F} \to \mathcal{U} \)). Then if \( \mathcal{V}, \mathcal{V}' \) are the corresponding image families, we have \( \text{pt} \subset \mathcal{V} \subset \mathcal{V} \subset \mathbb{C} \) and the constant for this boundedness is uniform with respect to the boundedness of \( \mathcal{U}' \) in \( \mathcal{U} \) and the degree bound for the mappings of \( \mathcal{F} \). This includes the degenerate case where \( \mathcal{F} \) is univalent and \( \mathcal{U}' \) and \( \mathcal{V}' \) are families of simple closed curves.

Proof. For the non-degenerate case where we have families of pointed domains, the bound on the hyperbolic distance of points in \( \mathcal{V}'_0 \) from the basepoint \( \nu_0 \) for the larger domain \( \mathcal{V}' \) follows from the Schwarz lemma.

For the estimate on \( \delta_{\mathcal{V}'_0}(\nu'_0) \) we examine the inverse branches of the mappings \( f_\alpha \). Note that as before we can postcompose with \( 1/z \) which preserves spherical distances so that we can assume that the points \( \nu'_0 \) lie in the closed unit disc and it will thus suffice to look at the Euclidean distances \( \delta_{\mathcal{V}'_0}(\nu'_0), \delta_{\mathcal{V}_0}(\nu'_0) \) to the boundaries of \( \mathcal{V}'_0, \mathcal{V}_0 \) respectively.

Now apply the Koebe one-quarter theorem on the discs \( D(\nu'_0, \delta_{\mathcal{V}'_0}(\nu'_0)) \) and then the distortion theorem for univalent mappings (e.g. [3, Theorem 1.6, p.3]) to inverse branches of \( f_\alpha \) on the smaller discs \( D(\nu'_0, \delta_{\mathcal{V}_0}(\nu'_0)) \). It then follows that \( \delta_{\mathcal{V}_0}(\nu'_0) \) cannot be too small compared to \( \delta_{\mathcal{V}'_0}(\nu'_0) \) since otherwise this would force \( \delta_{\mathcal{V}_0}(u'_0) \) to be very small compared to \( \delta_{\mathcal{V}'_0}(u'_0) \) which would contradict the boundedness of \( \mathcal{U}' \) in \( \mathcal{U} \).

Since \( \mathcal{U}' \subset \mathcal{U} \) and \( \mathcal{U} \) is bounded, by Theorem 6.4, \( \mathcal{U}' \) is also bounded. The required bounds on the distances and lengths of both \( \mathcal{V} \) and \( \mathcal{V}' \) then follow from the bi-equicontinuity of \( \mathcal{F} \) and Theorem 5.4 (we remind the reader at this point of our standing hypotheses for the families \( \mathcal{V}, \mathcal{V}' \) which include that, for each family, all the domains in that family have the same connectivity). Finally, the proof in the case of families of simple closed curves is straightforward in view of Theorem 4.13 and again the Koebe one-quarter theorem applied to both the mappings \( f_\alpha \) and their inverses.

The following is a more or less direct consequence of the definition and Corollary 4.16.

Corollary 6.6.

Let \( \mathcal{V} \) be a bounded family of \( n \)-connected domains for some \( n \geq 1 \) and let \( \mathcal{U} \) be either a family of \( m \)-connected domains for some \( m \geq 1 \) or a family of curves. Then if \( \text{pt} \subset \mathcal{U} \subset \mathcal{V} \subset \mathbb{C} \) with constant \( K \), there exists \( \delta > 0 \) depending only on \( K, \|\mathcal{V}\| \) such that, for every \( \alpha \in \mathcal{A}, \mathcal{V}_0 \) contains a (spherical) \( \delta \)-neighbourhood of \( \mathcal{U}_0 \).

Sometimes it is useful to know whether we can get a new bounded family if we remove the closures of the domains for one bounded family from another and then make a suitable choice of base points. If the two original families are families of discs and one is bounded in the other, the answer is yes.

Theorem 6.7.

Let \( \mathcal{U} = \{\{\mathcal{U}_0, \alpha_0\}\}_{\alpha \in \mathcal{A}} \) and \( \mathcal{V} = \{\{\mathcal{V}_0, \nu_0\}\} \) be two families of pointed discs with \( \text{pt} \subset \mathcal{U} \subset \mathcal{V} \subset \mathbb{C} \). For each \( \alpha \), let \( \mathcal{A}_\alpha \) be the conformal annulus \( \mathcal{V}_0 \setminus \mathcal{U}_0 \), let \( \gamma_\alpha \) be the equator of \( \mathcal{A}_\alpha \) and let \( \alpha_\alpha \) be a point on \( \gamma_\alpha \). Then the family \( \mathcal{A} = \{\{\alpha_\alpha, \alpha_\alpha\}\}_{\alpha \in \mathcal{A}} \) is a bounded family of pointed annuli.

Proof. For each \( \alpha \), let \( \pi_\alpha \) be the unique normalized (inverse) Riemann mapping which sends \( \mathbb{D} \) to \( \mathcal{V}_0 \). By Proposition 5.2 these mappings form a family \( \Pi \) which is univalent and bi-equicontinuous on compact subsets of \( \mathbb{D} \). Now for each \( \alpha \) set \( \tilde{\mathcal{U}}_\alpha = \pi_\alpha^{-1}(\mathcal{U}_\alpha), \tilde{\mathcal{A}}_\alpha = \mathbb{D} \setminus \tilde{\mathcal{U}}_\alpha \). It follows from the fact that \( \text{pt} \subset \mathcal{U} \subset \mathcal{V} \subset \mathbb{C} \) that the moduli and hence the lengths of the annuli \( \tilde{\mathcal{A}}_\alpha \) are uniformly bounded above and below. Thus, by conformal invariance, the same is true of the moduli and lengths of the annuli \( \mathcal{A}_\alpha \). If for each \( \alpha \), we now pick an arbitrary point \( \alpha_\alpha \) on the equator \( \gamma_\alpha \) of \( \mathcal{A}_\alpha \), then it follows that the second and third terms in the Carathéodory bound of each \( \mathcal{A}_\alpha \) will be uniformly bounded.

To finish, we need to establish a uniform bound on the first term, namely on \( \delta_{\mathcal{V}_0}(\alpha_\alpha) \) and \( \text{diam}^{\mathbb{H}}(\mathbb{C} \setminus \mathcal{A}_\alpha) \). It follows from the above bounds on the moduli and from the usual estimates on the hyperbolic metric in Lemma 3.4 that the equators \( \gamma_\alpha \) of the annuli \( \mathcal{A}_\alpha \) must be bounded away from the boundary curves in terms of Euclidean distance. Since one of these boundary curves is simply the unit circle, it follows that they also must all lie inside a disc of some bounded hyperbolic radius about 0 in \( \mathbb{D} \). Again we can apply \( 1/z \) if needed so as to assume that the points \( \alpha_\alpha \) lie in \( \mathbb{D} \). This allows us to apply the Koebe one-quarter theorem and the bi-equicontinuity of \( \Pi \) to deduce that the equators of the annuli \( \mathcal{A}_\alpha \) will
also be uniformly bounded away from their boundary curves in terms of Euclidean and hence spherical distances. Since the unbounded complementary component of each annulus $A_\alpha$ is simply $\mathbb{C} \setminus V_\alpha$, the uniform lower bound on the spherical diameters of the complements $\mathbb{C} \setminus A_\alpha$ follows from Theorem 4.15 and the corresponding lower bound on the spherical diameters of the sets $\mathbb{C} \setminus V_\alpha$.

The reader might wonder if it is necessary to restrict ourselves to families of discs in the above statement. However, when one tries to consider domains of higher connectivity, problems can arise. Consider, for example the case in Figure 8 where for each $m \geq 1$

$$A_m = A(0, 2, 5), \quad D_m = \left\{ z : 3 < |z| < 4, \frac{1}{m} < \arg z < 2\pi - \frac{1}{m} \right\}.$$  

**Figure 8.**

It is then easy to see that the pointed annuli $(A_m, -3.5)$ and the pointed discs $(D_m, -3.5)$ both give rise to bounded families which we will call $A$ and $D$ respectively. It is then also obvious to see that $\text{pt} < D < A < C$.

Now remove the closure of the pointed disc $D_m$ from the annulus $A_m$ and let $U_m$ be the resulting triply connected region. Next let $\{u_m\}_{m=1}^\infty$ be a convergent sequence where for each $m$, $u_m \in U_m$ and let $U$ be the resulting family of pointed domains. Using the Hausdorff version of Carathéodory convergence, one can see that, depending on our choice of base points $u_m$, the Carathéodory limit for the pointed domains $(U_m, u_m)$ will be either a point or doubly connected, essentially because the discs $D_m$ will ‘close off’ at least one of the complementary components of $U_m$ as seen from the base point $u_m$. We thus have a family of triply connected domains where the only possibility for a limit is either a point or doubly connected and so $U$ cannot be bounded.

To conclude this paper, as an application we solve an extremal problem. Suppose once more we have a multiply connected domain $U$ whose complement can be expressed as the union of two disjoint non-empty closed subsets $E$ and $F$ neither of which is a point. We know from Theorem 1.6 that we can find a meridian which separates $E$ and $F$ and hence by the collar lemma there is a conformal annulus of some definite but bounded modulus which also separates $E$ and $F$.

Since neither $E$ nor $F$ is a point, any annulus which separates them must have uniformly bounded modulus. Now let $M < \infty$ be the supremum over the moduli of all such annuli and take a sequence of these pointed annuli $(A_m, a_m)$ with base points on their equators and whose moduli tend to $M$ as $m$ tends to infinity. The spherical diameters of the complementary components of these annuli must be bounded below since they contain $E$ and $F$. Hence the spherical diameters of their equators must also be bounded below away from 0 and using Corollary 4.16 (or Lemma 3.4), it follows easily that the spherical distances $\delta(z)$ of points on their equators (including the base points) to the boundary will be uniformly bounded below away from 0. By 4. $\Rightarrow$ 1. of Theorem 4.15, this gives us a bounded family and if we now take a subsequence if necessary so that the base points $a_m$ converge to some limit $a$ and the complementary components converge in the Hausdorff topology, then the domains $(A_m, a_m)$ converge in the Carathéodory topology to a pointed annulus $(A, a)$. By 2. of Theorem 1.9, the modulus of $A$ must then be $M$. 

740
We still need to show that $A$ separates $E$ and $F$. Note that, by (ii) of Carathéodory convergence, $A \cap (E \cup F) = \emptyset$. If we let $\gamma_n$ be the equator of each $A_n$ and $\gamma$ the equator of $A$, then by Theorem 1.9 the curves $\gamma_n$ converge uniformly to $\gamma$ and are thus eventually homotopic in $U$ to $\gamma$. Since each curve $\gamma_n$ separates $E$ and $F$, it then follows easily using winding numbers and Cauchy’s theorem that $\gamma$ and thus $A$ separates $E$ and $F$. We have thus proved the following.

**Theorem 6.8.**

Let $U$ be multiply connected and suppose $\overline{C \setminus U} = E \cup F$ where $E$ and $F$ are closed and disjoint and neither set is a point. Then we can find a conformal annulus $A$ in $U$ which separates $E$ and $F$ for which $\mod A = \sup_{\gamma_n} \mod B$ where the supremum is taken over all conformal annuli $B$ in $U$ which separate $E$ and $F$.

Essentially this is a variant of the extremal problems considered by Grötzsch, Teichmüller and Mori (e.g. [1, 12]) and more recently in the paper of Herron, Liu and Minda [9]. We remark that it is also possible to give a somewhat longer but more elementary proof of this result using Theorem 1.2 and we are indebted to Adam Epstein for pointing this out.

Although the maximum possible modulus is attained, there is still the question as to whether such an annulus is unique and the following counterexample shows this is not in general true. Let $E = \overline{D(-3,1)} \cup \overline{D(3,1)}$, let $F = \overline{D(0,5)}$ and let $U$ be the quadruply connected domain $\overline{C \setminus (E \cup F)}$.

From above there is a conformal annulus $A$ of maximum possible modulus in $U$ which separates $E$ and $F$ and let us suppose for the sake of contradiction that $A$ is unique. Now $U$ is symmetric under the transformation $z \mapsto -z$ and, by the assumed uniqueness, so is $A$. If we then let $\gamma$ be the equator of $A$, then by the symmetry of $A$ and the uniqueness of the equator of a conformal annulus, $\gamma$ must also in turn be symmetric under this transformation. If $z$ is any point on $\gamma$, then $\gamma$ must consist of two symmetric pieces, one from $z$ to $-z$ and the other from $-z$ to $z$. The change in argument from traversing either of these pieces will then be the same odd integer multiple of $\pi$ whence it follows that the winding number of $\gamma$ about $0$ will be non-zero. Using the Jordan curve theorem, $\gamma$ must enclose $\overline{D}$ and, by symmetry again, it follows that $\gamma$ encloses either just $\overline{D}$ or $\overline{D(-3,1)} \cup \overline{D(3,1)}$. In neither case does $\gamma$ separate $E$ and $F$ which is clearly impossible as $A$ must separate these sets and, with this contradiction, we have what we want.

**References**

[1] Ahlfors LV, Lectures on Quasiconformal Mappings, Van Nostrand Mathematical Studies, 10, Van Nostrand, Toronto, 1966
[2] Beardon A.F., Pommerenke Ch., The Poincaré metric of plane domains, J. London Math. Soc., 1978, 18(3), 475–483
[3] Carleson L., Gamelin T.W., Complex Dynamics, Universitext Tracts Math., Springer, New York, 1993
[4] Comerford M., Short separating geodesics for multiply connected domains, Cent. Eur. J. Math., 2011, 9(5), 984–996
[5] Comerford M., A straightening theorem for non-autonomous iteration, Comm. Appl. Nonlinear Anal., 2012, 19(2), 1–23
[6] Comerford M., The Carathéodory topology for multiply connected domains I, Cent. Eur. J. Math., 2013, 11(2), 322–340
[7] Conway J.B., Functions of One Complex Variable, Grad. Texts in Math., 11, Springer, New York–Heidelberg, 1972
[8] Epstein A.L., Towers of Finite Type Complex Analytic Maps, PhD thesis, CUNY Graduate School, 1993
[9] Herron D.A., Liu X.Y., Minda D., Ring domains with separating circles or separating annuli, J. Analyse Math., 1989, 53, 233–252
[10] Keen L., Lakic N., Hyperbolic Geometry from a Local Viewpoint, London Math. Soc. Stud. Texts, 68, Cambridge University Press, Cambridge, 2007
[11] Lang S., Complex Analysis, 3rd ed., Grad. Texts in Math., 103, Springer, New York, 1993
[12] McMullen C.T., Complex Dynamics and Renormalization, Ann. of Math. Stud., 135, Princeton University Press, Princeton, 1994
[13] Newman M.H.A., Elements of the Topology of Plane Sets of Points, 2nd ed., Cambridge University Press, Cambridge, 1961
[14] Pommerenke Ch., Uniformly perfect sets and the Poincaré metric, Arch. Math., 1979, 32(2), 192–199