PHASE TRANSITIONS AND PERCOLATION AT CRITICALITY IN PLANAR ENHANCED RANDOM CONNECTION MODELS

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ABSTRACT. We study phase transition and percolation at criticality for three planar random graph models, viz., the homogeneous and inhomogeneous enhanced random connection models (RCM) and the Poisson stick model. These models are built on a homogeneous Poisson point process \( \mathcal{P}_\lambda \) in \( \mathbb{R}^2 \) of intensity \( \lambda \). In the homogeneous RCM, the vertices at \( x, y \) are connected with probability \( g(|x-y|) \), independent of everything else, where \( g : [0, \infty) \rightarrow [0, 1] \) and \(| \cdot |\) is the Euclidean norm. In the inhomogeneous version of the model, points of \( \mathcal{P}_\lambda \) are endowed with weights that are non-negative independent random variables with distribution \( P(W > w) = w^{-\beta}1_{[1, \infty)}(w), \beta > 0 \). Vertices located at \( x, y \) with weights \( W_x, W_y \) are connected with probability \( 1 - \exp \left( -\frac{W_x W_y}{|x-y|^\alpha} \right), \eta, \alpha > 0 \), independent of all else. The edges of the graph are viewed as straight line segments starting and ending at points of \( \mathcal{P}_\lambda \). A path in the graph is a continuous curve that is a subset of the collection of all these line segments. The Poisson stick model consists of line segments of independent random lengths and orientation with the mid point of each line located at a distinct point of \( \mathcal{P}_\lambda \). Intersecting lines then form a path in the graph. A graph is said to percolate if there is an infinite connected component or path. We derive conditions for the existence of a phase transition and show that under some additional conditions that there is no percolation at criticality.

Key words and phrases. Random Geometric Graphs, Random Connection Model, Enhanced Random Connection Model, Percolation, Phase transition, Stochastic geometry.

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1. Introduction and Main Results

The study of random graphs started with the pioneering work by Erdős and Réyni [11], [12] and Gilbert [14] on the Erdős-Rényi model. The random graph in the Erdős-Rényi model is constructed on a set of \( n \) vertices, for some \( n \in \mathbb{N} \) with an edge drawn between any two pairs of nodes independently with probability \( p \in [0, 1] \). Detailed work on the Erdős-Rényi graph can be found in [5], [37], [19]. The Bernoulli lattice percolation model on \( \mathbb{Z}^d \) is an extensively studied random graph model [20], [21], [16] where the geometry of the underlying space plays an important role. The vertex set is \( \mathbb{Z}^d \) with an edge between points at Euclidean distance one with probability \( p \) independent of other edges. The above geometric model was extended to the continuum by considering a point process in \( \mathbb{R}^d \) with edges between points that are within a Euclidean distance \( r > 0 \). Such a model has found wide application in modeling ad-hoc wireless networks and sensor networks. This model is called the Gilbert disk model [15] or the random geometric graph (RGG). The questions of interest in such applications are percolation, connectivity and coverage, details of which can be found in [13] and [17]. Rigorous theoretical analysis of the percolation problem in such graphs can be found in [24] while the monograph [26] carries a detailed compilation of the important results on the topic in the sparse, thermodynamic and connectivity regimes. When each point of the underlying point process has an independent subset of \( \mathbb{R}^d \) associated with it, then the union of all such sets is what forms the germ-grain model which is of much interest in stochastic geometry [32].

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The main goal of this paper is to derive conditions under which three planar network models exhibit a phase transition and show that under some additional conditions the percolation function is continuous. All these three models are constructed over a homogeneous Poisson point process denoted $\mathcal{P}_\lambda$ in $\mathbb{R}^2$ with intensity parameter $\lambda$ for some $\lambda > 0$. A phase transition refers to the abrupt emergence of an infinite component in the graph, in which case we say that the graph percolates. A phase transition is said to occur if there exists a critical value $\lambda_c \in (0, \infty)$ of $\lambda$ such that for $\lambda > \lambda_c$ the random graph under consideration percolates and for $\lambda < \lambda_c$ the random graph does not percolate. It can be shown using ergodicity that for $\lambda > \lambda_c$, there is an infinite component with probability one. In many of percolation models in $\mathbb{R}^d$ it can also be shown that there is a unique infinite component by adapting the Burton-Keane argument [16], [24]. The percolation function refers to the probability that a typical vertex in the graph is part the infinite component. Percolation is equivalent to the percolation function being positive. The continuity of the percolation function is a problem of much interest in the random graph literature. See for instance, [10] for a new proof showing absence of phase transition at criticality in the Bernoulli bond percolation model on $\mathbb{Z}^d$ with $d = 2$. The problem remain open for instance in $d = 3$.

The random connection model (RCM) is a generalization of the RGG and a continuum version of the long range percolation on lattices [27]. It was studied in the context of wireless networks where communication between nodes depend on the distance between the nodes as well as the interference coming from transmissions from other nodes in the network [23], [24]. In the random connection model we consider, the vertex set will be a homogeneous Poisson point process denoted $\mathcal{P}_\lambda$ in $\mathbb{R}^d$ with intensity parameter $\lambda$ for some $\lambda > 0$. An undirected edge denoted $\{x, y\}$ exists between vertices located at $x, y$ with probability $g(|x - y|)$ independent of everything else, where $g : [0, \infty) \to [0, 1]$ is non-increasing. We denote this graph by $G_\lambda$. [27] showed that a phase transition occurs in $G_\lambda$ if and only if the connection function satisfies $0 < \int_{\mathbb{R}^d} g(|x|) \, dx < \infty$.

The first model that we consider is the enhanced RCM. Consider the RCM on the plane ($d = 2$) and view each edge $\{x, y\}$ in the RCM as a straight line segment denoted by $\overline{xy}$. For any two edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$ in the RCM that intersect, we say the vertex $x_1$ is direct neighbour of $x_2$, $x_3$ is direct neighbour of $x_4$ and the vertices $x_1, x_2$ are indirect neighbours of $x_3, x_4$ and vice versa. We will refer to the resulting graph as the enhanced random connection model (eRCM) and denote it by $G_\lambda^e$. It will be more useful to think of the eRCM as enhancing the available paths in the network rather than introducing additional edges as can be seen from the following applications. Intersecting edges along a path in the RCM allow for switching from one path (in the original graph) to another. The eRCM can be considered as a model for a road or a pipeline network where connections are made locally and intersecting roads or pipelines allow for the traffic or the fluid to switch paths. The above could also be used as a model for thin slab of porous media where connections between nodes resemble pipes and crossing of these pipes allow the fluid to flow from one pipe to another. An alternate model for road networks was studied in [3], [2]. The construction of an optimal road network by using the trade-off between a measure of shortness of route and normalized network length for a one parameter family of proximity graphs is studied in [3]. In [2] the author introduces scale invariant spatial networks whose primitives are the routes between points on the plane. The problems of interest are the existence and uniqueness of infinite geodesics, continuity of routes as a function of end points and the number of routes between distant sets on the plane.

In this context it would be more appropriate to consider an inhomogeneous version of the eRCM model where each vertex is endowed with a weight that is indicative of the size, importance of a city or town. In the basic inhomogeneous model we consider, an edge is formed between vertices located at $x, y \in \mathbb{R}^2$ endowed with random weights $W_x, W_y$ with probability

$$g(x, y) = 1 - \exp \left( -\frac{\eta W_x W_y}{|x - y|^\alpha} \right)$$ (1.1)
independent of everything else. Here \( \eta, \alpha \) are positive constants and the weights are independent and satisfy 
\[
P(W > w) = w^{-\beta} 1_{[1,\infty)}(w) \text{ for some } \beta > 0.
\]
The graph thus obtained is then enhanced in the same manner as described above to obtain the inhomogeneous eRCM (ieRCM). We will denote the random graph obtained in the inhomogeneous RCM and the enhanced inhomogenous RCM by \( H_\lambda, H^e_\lambda \) respectively. Percolation properties for inhomogeneous random connection model with this type of inhomogeniety has been studied for long range percolation model on lattice points by Deprez, Hazra and Wüthrich in [8] and in the continuum for fixed intensity \( \lambda \) by Deprez and Wüthrich in [9]. Phase transition is expressed in terms of the parameter \( \eta \) instead of \( \lambda \) but a simple scaling argument will show that these two are equivalent. In both these models a phase transition occurs for \( d = 1 \) only if \( \alpha \beta > 2 \) and \( 1 < \alpha < 2 \) and for \( d \geq 2 \) only if \( \alpha > d \) and \( \alpha \beta > 2d \). For all \( d \) the percolation function has been shown to be continuous only under the condition that \( \alpha \beta > 2d \) and \( \alpha \in (d, 2d) \). For \( d \geq 2 \) the case when \( \min\{\alpha, \alpha \beta\} > 2d \) is open.

The third model of planar graph we consider is the Poisson stick model which is an example of a model that satisfies the axiomatic conditions of so called scale invariant spatial networks mentioned above. This model which was introduced in [30] consists of sticks of independent random lengths whose mid points are located at points of \( \mathcal{P}_\lambda \) with each stick having a random independent orientation. The sticks were assumed to have bounded lengths with half-length density \( h \). Two points in \( \mathcal{P}_\lambda \) are neighbors in the resulting graph provided the corresponding sticks intersect. A phase transition was shown to occur in such a graph. The Poisson stick graph appears to be a natural model for a network structure formed by silicon nanowires and carbon and other nanotubes on the surface of substrates. Percolation, conductance and many other significant properties of these nanowire networks are studied in [28], [4], [25], [33], [18]. In this paper we consider the Poisson stick model with stick-length distribution having unbounded support and study existence of phase transition and the continuity of the percolation function.

1.1. Notations. We gather much of the notations we need here for easy reference. We define the notations with reference to the RCM and eRCM. However, they carry over to the ieRCM and the Poisson stick models in the obvious way. Let \( C(x) \) be the connected component containing \( x \in \mathcal{P}_\lambda \) in \( G_\lambda \) and \( C^e(x) \) be the connected component containing \( x \in \mathcal{P}_\lambda \) in \( G^e_\lambda \). Without loss of generality we assume that there is a vertex at the origin \( O \), that is, we consider the process \( \mathcal{P}_\lambda \) under the Palm measure \( P^o \), the probability distribution conditioned on a point being at origin. The distribution of the vertices other than \( O \) in \( \mathcal{P}_\lambda \) under \( P^o \) is the same as that of \( \mathcal{P}_\lambda \cup \{O\} \) under the original \( P \). Let \( C := C(O), C^e := C^e(O) \) and define the percolation probabilities for \( G_\lambda, G^e_\lambda \) as
\[
\theta(\lambda) := P^o(|C| = \infty) \text{ and } \theta^e(\lambda) := P^o(|C^e| = \infty) .
\]

The percolation thresholds denoted by \( \lambda_c, \lambda^e_c \) for the graphs \( G_\lambda \) and \( G^e_\lambda \) respectively are defined as,
\[
\lambda_c := \inf\{\lambda > 0 : \theta(\lambda) > 0\} \text{ and } \lambda^e_c := \inf\{\lambda > 0 : \theta^e(\lambda) > 0\} .
\]
Similarly let \( \tilde{\lambda}_c, \tilde{\lambda}_c^e, \lambda_{PS} \) be the percolation thresholds for the random graphs \( H_\lambda, H^e_\lambda \) and \( PS_\lambda \) respectively.

For any connected region \( D \subset \mathbb{R}^2 \) an event \( E \) is said to be \( D\)-measurable provided the occurrence or otherwise of \( E \) is independent of the points of \( \mathcal{P}_\lambda \) that fall outside \( D \).

Since we work with paths in the graph, as mentioned earlier we shall often view the enhanced models as providing additional paths in the original graphs rather than adding edges. In this view, edges in the graphs \( G_\lambda, H_\lambda \) are straight line segments joining the vertices of \( \mathcal{P}_\lambda \). Given any of the graphs \( G^e_\lambda, H^e_\lambda \) or \( PS_\lambda \) and \( x, y \in \mathbb{R}^2 \) we say that there is a path from \( x \) to \( y \) if there exists a closed continuous curve from \( x \) to \( y \) contained entirely in \( \bigcup_{i=1}^n e_i \) for some edges \( e_1, e_2, \ldots, e_n \) in the case of \( G_\lambda, H_\lambda \) and sticks in the case of \( PS_\lambda \). Paths thus need not start or end at vertices in the graph.
Figure 1. The green curve is a left-right crossing of the box.

A path is said to cross a box $[a, b] \times [c, d]$ if there exists a path completely contained within the box with end points on opposite sides. We shall refer to these paths as crossings (see Figure 1).

**Crossing events:** For $s > 0$ and $\rho > 1$ let $LR_s(\rho)$ be the event that there exists a crossing along the longer side of the rectangle $[0, \rho s] \times [0, s]$ and $TD_s(\rho)$ be the event that there exists a crossing along the shorter side of the rectangle $[0, \rho s] \times [0, s]$. $C_s(\rho) := P(LR_s(\rho))$.

A circuit around $S$ in the region $T \setminus S$, $S \subset T \subset \mathbb{R}^2$ where both $S, T$ are connected, is a path that starts and ends at the same point and is entirely contained inside $T \setminus S$ but the end points of the edges that contains the path can include vertices outside $T \setminus S$. Let $B_s := [-s, s]^2$, $A_{s,t} := B_t \setminus B_s$. $A_s$ be the event that there exists a circuit in the annulus $A_{s,2s}$.

**One arm events:** Let $S$ be a connected measurable subsets of $\mathbb{R}^2$. For $A, B \subset S$ with $A, B$ connected and $A \cap B = \phi$, in the graph $G^e_\lambda$

$$A \xrightarrow{S \xrightarrow{B}}: \text{the event that there exists a path from some point in } A \text{ to some point in } B \text{ entirely confined in } S.$$ 

Similarly for connected subsets $C, D, Q \subset \mathbb{R}^2$ such that $C \subset D \subset Q$

$$C \xrightarrow{Q \xrightarrow{\partial D}}: \text{the event that there exists a path from some point in } C \text{ to some point in } D^c \text{ entirely confined in } Q.$$ 

1.2. Main Results. We are now ready to state our main results. Our first result is on the existence of a phase transition in the three models described earlier. Penrose [27] showed that there is a non-trivial phase transition in the RCM, that is, $0 < \lambda_c < \infty$ under the condition $0 < \int_0^\infty r^2g(r)\,dr < \infty$. We now prove a similar result for the eRCM albeit under a stronger restriction on $g$. The condition required in the case of the ieRCM is $\alpha > 2, \alpha \beta > 4$ which is same as the one in for the iRCM as derived in [9]. [30] showed the existence of a phase transition in the Poisson stick model under the assumption that the stick length distribution has bounded support, a result which we extend to sticks of unbounded lengths.

**Theorem 1.1.** A phase transition occurs in the

(i) $eRCM \ G^e_\lambda$ if the connection function $g$ satisfies $0 < \int_0^\infty r^2g(r)\,dr < \infty$.

(ii) $ieRCM \ H^e_\lambda$ with the connection function of the form (1.1) if $\alpha > 2$ and $\alpha \beta > 4$. 
Theorem 1.2. Let $l$ be the length intersecting a box as the dimensions of the box diverge. This arises from having to derive an estimate on the longest edge/stick RSF lemma for all the three models are proved under a stronger condition on the connection function (half length density in case of Poisson stick model). A similar result about occupied and vacant crossings was proved for Boolean model on $\mathbb{R}^2$ by Roy [29]. The RSW results in this article are analogous to those in [35] for the percolation model on Poisson-Voronoi tessellations in $\mathbb{R}^2$. RSW results, continuity of critical parameter and sharpness of phase transition for the Boolean model with unbounded radius distribution has been studied by Ahlberg et. al. in [11]. For the continuum percolation model with random ellipses on the plane, percolation and connectivity behavior of the vacant and covered set has been studied by Teixeira et. al. in [36].

Our next result establishes a RSW lemma which is one of the most useful result in planar percolation models. It states that if the probability of crossing a square is uniformly bounded away from zero then so is the probability of crossing a rectangle along the longer side. We demonstrate its utility by establishing that percolation does not occur in the case of the Boolean model on $\mathbb{R}^2$ independently by Russo [34]. A similar result about occupied and vacant crossings was proved for Boolean model on $\mathbb{R}^2$ by Roy [29]. The RSW results in this article are analogous to those in [35] for the percolation model on Poisson-Voronoi tessellations in $\mathbb{R}^2$. RSW results, continuity of critical parameter and sharpness of phase transition for the Boolean model with unbounded radius distribution has been studied by Ahlberg et. al. in [11].

We prove the RSW lemma under the condition that the connection function is of the form $g(r) = O(r^{-c})$ as $r \to \infty$, $g(x, y) = 1 - \exp \left( -\frac{nW_{xy}}{|x-y|^\alpha} \right)$ and the half length density $h$ satisfies $h(l) = O(l^{-c})$ as $l \to \infty$, for the eRCM, the iERCM and the Poisson stick models respectively. By Theorem 1.1 a phase transition occurs under the above assumptions in the eRCM, iERCM, PS$\lambda$ provided $c > 3$, $\min\{\alpha, \frac{\alpha \beta}{2} \} > 2$ and $c > 2$ respectively. The RSW lemma for all the three models are proved under a stronger condition on the connection function (half length density in case of Poisson stick model). This arises from having to derive an estimate on the longest edge/stick length intersecting a box as the dimensions of the box diverge.

Theorem 1.2. Suppose the following conditions hold.

(I) In the eRCM $G^e_\lambda$, the connection function $g$ satisfies $g(r) = O(r^{-c})$ as $r \to \infty$ with $c > 3$.

(II) In the iERCM $H^e_\lambda$, the connection function $g$ is of the form $O(1)$ with $\min\{\alpha, \alpha \beta \} > 4$.

(III) In the graph $PS_\lambda$ with half length density $h$ satisfies $h(l) = O(l^{-c})$ with $c > 3$.

Then the following three conclusions hold for all the three graphs $G^e_\lambda$, $H^e_\lambda$ and $PS_\lambda$:

(i) If $\inf_{s>0} C_s(1) > 0$ then for any $\rho > 0$, $\inf_{s>0} C_s(\rho) > 0$.

(ii) If $\lim_{s \to \infty} C_s(1) = 1$ then for any $\rho > 0$, $\lim_{s \to \infty} C_s(\rho) = 1$.

(iii) The set of parameters $\lambda$ for which percolation occurs is an open set.

2. Proofs

In what follows $c_0, c_1, c_2, \cdots$ and $C_1, C_2, \cdots$ will denote constants whose values will change from place to place. $| \cdot |$ will be used to refer to the Euclidean norm, the cardinality of a set as well as the Lebesgue measure.

The condition for the existence of an infinite component for large intensities is obtained by comparing the enhanced model with the usual non-enhanced version. For the other side we will bound the component containing the origin by a sub-critical branching process in case of the eRCM that dies out with probability one. For the iERCM we evaluate the probability of a self avoiding path of length $n$ and then show that the probability that there is such a path converges to zero as $n \to \infty$. In order to show that the percolation function is continuous, we first derive a RSW Lemma which is interesting in its own right. We do this by adapting the technique developed in Tassion [35]. We then use a renormalization technique similar to the one used in Daniels [6] to show that the parameter set over which percolation occurs is open. In this case we will prove the results in detail for the eRCM. Much of the proof carries over to the other two models for which we will provide only the necessary details.
2.1. Proof of Theorem 1.1. It is clear from the definition that $G^e_\lambda$ percolates if $G_\lambda$ does. So we have $\lambda^e_\lambda \leq \lambda_\lambda$. From Theorem 1 in Penrose [27] we know that $\lambda_\lambda \in (0, \infty)$ if $\int_0^\infty r^2 g(r) \ dr \in (0, \infty)$. Since $g(r) \in [0, 1]$, $\int_0^\infty r^2 g(r) \ dr \in (0, \infty)$ implies $\int_0^\infty g(r) \ dr \in (0, \infty)$. It follows from the above observations that $\lambda^e_\lambda < \infty$.

We now show that $\lambda^e_\lambda > 0$. To do this we assume without loss of generality that there is a vertex at the origin, that is, we consider the process $\mathcal{P}_\lambda$ under the Palm measure $F^o$ and show that the component containing the origin is finite with probability one for all $\lambda$ sufficiently small. To do this we construct an exploration branching process from $G^e_\lambda$ such that any vertex that is in the component containing the origin is present in the branching process. Thus if the branching process dies out almost surely then the component containing the origin will be finite with probability one. We start our exploration from the origin $O$.

Let $\sigma^{(0)}_1 = \{x^{(1)}_1, x^{(2)}_1, \ldots, x^{(N^{(0)}_1)}_1\}$ be the points of a PPP $\mathcal{P}^{(0)}_1$ on $\mathbb{R}^2$ of intensity $\lambda g(|\cdot|)$. $N^{(0)}_1$ has the same distribution as the degree of $O$ in $G_\lambda$ under $F^o$. Let $\mathcal{F}^{(0)}_1 = \sigma(C^{(0)}_1)$. The points of $C^{(0)}_1$ are direct neighbours of $O$.

We now enumerate the indirect neighbours of $O$ conditional on $\mathcal{F}^{(0)}_1$. These are points from which emanate edges that intersects some edge from $O$ to some point in $\sigma^{(0)}_1$. If $N^{(0)}_1 = 0$, set $C^{(1)}_1 = \phi$, and the exploration process terminates. Otherwise let $\mathcal{P}^{(1)}_1$ be an independent PPP in $\mathbb{R}^2$ of intensity $\lambda(1 - g(|\cdot|)\Pi_{x \in C^{(0)}_1}(1 - g(|\cdot - x|))$.

The point process $\mathcal{P}^{(1)}_1$ is the collection of possible indirect neighbours of $O$ and thus we do not want any of these points to have an edge to any of the points in $\{O\} \cup C^{(0)}_1$. We need to thin this collection suitably to obtain the set of indirect neighbours of $O$. Denote by $\overline{ab}$ the line segment joining the points $a$ and $b$. For each pair of distinct (unordered) points $\{y, z\} \subset \mathcal{P}^{(1)}_1$ such that $\overline{yz}$ intersects at least one of the line segments $Ox^{(i)}_1$, $i = 1, 2, \ldots, N^{(0)}_1$, mark both the points $y, z$ red with probability $g(|y - z|)$. Note that points may get marked red more than once. The decision to mark a pair red is independent of choices made with other pairs of nodes. Let $C^{(1)}_1$ be the set of points marked red and $C_1 := C^{(0)}_1 \cup C^{(1)}_1$. Let $\mathcal{F}_1 := \sigma(C^{(0)}_1 \cup C^{(1)}_1)$, $N^{(1)}_1 = |C^{(1)}_1|$, and $Z_1 = N^{(0)}_1 + N^{(1)}_1$. Relabeling the points in $C_1$, we write $C_1 = \{y^{(1)}_1, y^{(2)}_1, \ldots, y^{(Z_1)}_1\}$.

To generate the points in the next generation conditional on $\mathcal{F}_1$ and $N^{(1)}_1 > 0$, consider first an independent Poisson point process $\mathcal{P}^{(01)}_2$ of intensity

$$\lambda(1 - g(|\cdot|)) \left(1 - \Pi_{x \in C^{(0)}_1}(1 - g(|\cdot - x|))\right).$$

These are the direct neighbours of points in $C^{(0)}_1$ that are not direct neighbours of $O$. We denote this collection of points by $C^{(01)}_2$. If $N^{(1)}_1 > 0$ then consider an independent Poisson point process $\mathcal{P}^{(02)}_2$ of intensity

$$\lambda(1 - g(|\cdot|)) \Pi_{x \in C^{(0)}_1 \cup C^{(01)}_2}(1 - g(|\cdot - x|)) \left(1 - \Pi_{x \in C^{(1)}_1}(1 - g(|\cdot - x|))\right).$$

These are the potential direct neighbours of points in $C^{(1)}_1$. They must have an edge to some point in $C^{(1)}_1$ but none to any of the points in $O \cup C^{(0)}_1 \cup C^{(01)}_2$. We need to discount for the indirect neighbours of $O$. We consider all unordered pairs $\{v, w\} \subset \mathcal{P}^{(02)}_2$ such that $\overline{vw}$ intersects one of the edges $Ox^{(i)}_1$, $i = 1, 2, \ldots, N^{(0)}_1$ and mark both the points blue with probability $g(|v - w|)$ independent of everything else. For all unordered pairs $\{v, w\}$, $v \in \mathcal{P}^{(02)}_2$ and $w \in C^{(1)}_1$ such that $\overline{vw}$ intersects one of the edges $Ox^{(i)}_1$, $i = 1, 2, \ldots, N^{(0)}_1$, mark the point $v$ blue with probability $g(|v - w|)$ independent of everything else. The direct neighbours of $C^{(1)}_1$ that have not already been explored are then the points of $\mathcal{P}^{(02)}_2$ that are not marked blue. We denote this collection of points by $C^{(02)}_2$ which are the unexplored direct neighbours of points in $C^{(1)}_1$. 
Suppose that Proposition 2.2.
The proof of Theorem 1.1 follows immediately from the following two propositions.

where

we obtain

We have

By Proposition 2.2, \( X_n \) being a non-negative super-martingale converges almost surely. From Proposition 2.1 we have \( \mu \in (0, 1) \) for all \( \lambda > 0 \) sufficiently small. Since \( Z_n \in \mathbb{N} \) and \( \mu^n \to 0 \) as \( n \to \infty \) for \( \mu \in (0, 1) \) it follows that the sequence \( Z_n \) must converge to zero almost surely for all \( \lambda \) sufficiently small. It remains to prove the two propositions.

Proof of Proposition 2.1. Recall that \( Z_1 = N_1^{(0)} + N_1^{(1)} \), the sum of the direct and indirect neighbors of the origin.

Let \( \mathcal{P}_{\lambda}^3 = \{(x,y,z) : x,y,z \in \mathcal{P}_\lambda, \text{ distinct}\} \). For \( (x,y,z) \in \mathcal{P}_{\lambda,\neq}^3 \) define the indicator function

We have \( N_1^{(1)} \leq \sum_{(x,y,z) \in \mathcal{P}_{\lambda,\neq}^3} h_1(x,y,z) \). The reason for this inequality is that an edge \( \{y,z\} \) may intersect more than one edge emanating from the origin. Further two edges intersecting an edge emanating from the origin may be of the form \( \{y,z_1\}, \{y,z_2\} \). We also don’t discount for the vertices \( y, z \) not being connected to the origin. Taking
expectations and using the Campbell-Mecke formula we obtain

$$E^o \left[ N_1^{(1)} \right] \leq E^o \left[ \sum_{(x,y,z) \in P_{\lambda, \phi}} h_1(x,y,z) \right] = \lambda^3 \int_{\mathbb{R}^2} \int_{\{ |y| > 0 \}} \int_{D(O,x,y)} g(|x|) g(|y-z|) \, dz \, dy \, dx,$$

where $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2), dx = dx_1 dx_2$ etc. $D(O, x, y) := \{ z \in \mathbb{R}^2 : \overline{xy} \text{ intersects } \overline{Ox} \}$ is the unbounded region $AOLA'$ in Figure 2 where we have taken $x = (\ell, 0)$.

Without loss of generality consider a fixed $x$ at a distance $\ell$ from $O$ on positive real axis. Changing the variables $y, z$ to $u, v, r, \theta$ according to $y = (u + v \cos \theta, v \sin \theta), z = (u - (r - v) \cos \theta, -(r - v) \sin \theta)$ and noting that the determinant of the Jacobian satisfies $|J^{-1}| = r \sin \theta$ we can rewrite the right hand side of (2.1) to obtain

$$E^o[N_1^{(1)}] \leq \lambda^3 \int_0^2 \int_0^{2\pi} \int_0^\ell \int_0^{\pi} \int_0^r g(r) r \sin \theta dv \, dr \, d\theta \, du \, d\phi \, d\ell
\int_0^\infty r^2 g(r) \, dr = 4\pi \lambda^3 \left( \int_0^\infty r^2 g(r) \, dr \right)^2.

(2.2)

By the Campbell-Mecke formula we have

$$E[N_1^{(0)}] = \lambda \int_{\mathbb{R}^2} g(|x|) \, dx.
\int_0^\infty r^2 g(r) \, dr < \infty.
(2.3)

Both assertions of the proposition now follow from (2.2) and (2.3) and the fact that $Z_1 = N_1^{(0)} + N_1^{(1)}$ provided

Proof of Proposition 2.2. Recall that $Z_0 = 1$. Consequently $E[X_1 | \mathcal{F}_0] = E[Z_1] / \mu = 1$. For $k \geq 2$, the elements of the set $C_{k-1}^{(1)}$, the direct descendants of points in $C_{k-1}$ can be thought of as follows. Given $\mathcal{F}_k$, if $Z_k \geq 1$ let $C_{k-1} = \{ y_{k-1}^{(1)}, y_{k-1}^{(2)}, \ldots, y_{k-1}^{(Z_k)} \}$ be the set of vertices in generation $k-1$. For each $y_{k-1}^{(i)}$, form the sets $C_k^{(0)}, C_k^{(1)}$ in the same manner as we obtained $C_1^{(0)}, C_1^{(1)}$ respectively for the origin. Let $M_k^{(0)} = |C_k^{(0)}|$ and $M_k^{(1)} = |C_k^{(1)}|$. 

\[ \text{Figure 2. } D(O, x, y) \text{ the unbounded region } AOLA' \]
It is clear from the construction that
\begin{equation}
M_k^{(ij)} = N_1^{(j)}, \quad j = 0, 1, \quad i = 1, 2, \ldots, Z_{k-1}.
\end{equation}

A further thinning of the points in \( \bigcup_{i=1}^{Z_{k-1}} C_k^{(ij)} \) to discount for duplication and direct, indirect neighbors of earlier generations then yields the sets \( c_k^{(j)} \) for \( j = 0, 1 \). Thus
\begin{equation}
N_k^{(j)} \leq \sum_{i=1}^{Z_{k-1}} M_k^{(ij)}, \quad j = 0, 1.
\end{equation}

It follows from (2.4), (2.5) that
\begin{equation}
E[Z_k|F_{k-1}] = E[N_1^{(0)} + N_1^{(1)}|F_{k-1}]
\leq \sum_{i=1}^{Z_{k-1}} E[M_k^{(0)} + M_k^{(1)}|F_{k-1}]
= Z_{k-1}E[N_1^{(0)} + N_1^{(1)}] = Z_{k-1} \mu.
\end{equation}

This proves Proposition 2.2 and completes the proof of Theorem 1.1. \( \square \)

2.2. Proof of Theorem 1.2 (i) for eRCM. Consider the graph \( G_\lambda^e \) with connection function \( g(r) = O(r^{-c}) \) as \( r \to \infty \) where \( c > 4 \) is arbitrary. Since much of the proof follows the ideas in Tassion [35] and we borrow results from this paper we will keep the notations close to those in that paper. A key ingredient in the proof is the following result on the length of the longest edge in \( G_\lambda^e \) which allows us to localize the analysis.

Proposition 2.3. For any \( s > 0 \) let \( M_s \) be the length of the longest edge in \( G_\lambda^e \) intersecting the box \( B_s = [-s, s]^2 \). Suppose that the connection function \( g \) satisfies \( g(r) = O(r^{-c}) \) as \( r \to \infty \). Then for any \( c > 4, \ t > 0 \) and \( \tau > \frac{2}{c-2} \) we have \( P(M_s > s^\tau) \to 0 \) as \( s \to \infty. \)

Proof of Proposition 2.3. Fix \( c > 4, \ t > 0. \) Let \( B(O, s) := \{ x \in \mathbb{R}^2 : |x| \leq s \} \) be the ball of radius \( s \) centered at the origin. Recall that for any two points \( x, y \in \mathbb{R}^2, XY \) denotes the line segment joining \( x \) and \( y. \) Define the events \( D_s(l) = \{ M_s > l \}, \)
and
\( O_{ts}(\tau) = \{ X \in P_\lambda : \text{there is an edge of length longer than } s^\tau \text{ incident on } X \text{ in } G_\lambda \} \)
and
\( \tilde{O}_{ts}(\tau) = \{(X,Y) \in P_\lambda^2 : \text{there is an edge in } G_\lambda \text{ joining } X, Y \text{ of length longer than } s^\tau, \text{XY intersects } B(O, \sqrt{2ts}) \}. \)

\begin{equation}
P(D_s(s^\tau)) \leq E\left[ \sum_{X,Y \in P_\lambda} 1_{\{XY \text{ intersects } B_s \}} 1_{\{|X-Y| \geq s^\tau\}} \right]
\leq E\left[ \sum_{X \in P_\lambda \cap B(O, \sqrt{2ts})} 1_{\{X \in O_{ts}(\tau)\}} \right] + E\left[ \sum_{X,Y \in P_\lambda \cap B(O, \sqrt{2ts})^c} 1_{\{(X,Y) \in \tilde{O}_{ts}\}} \right]
\end{equation}
The Campbell-Mecke formula applied to the first term on the right hand side of the last inequality in (2.7) yields

\[ E \left[ \sum_{X \in \mathcal{P}_\lambda \cap B(O, \sqrt{2}t_s)} 1_{\{X \in O_{ts}(\tau)\}} \right] = C\lambda(ts)^2 P^o(O \in O_{ts}(\tau)) \]

\[ = C\lambda(ts)^2 \left( 1 - P^o(\text{none of the edges incident on } O \text{ is of length } \geq s^\tau) \right) \]

\[ = C\lambda(ts)^2 \left( 1 - \exp \left\{ -\lambda \int_{B(O,s^\tau)^c} \right. \right) \]

\[ \leq C(\lambda ts)^2 \int_{B(O,s^\tau)^c} g(|x|) dx = C_1 s^2 \int_{s^\tau}^\infty rg(r) dr \leq C_2 s^{2-\tau(c-2)}, \]

(2.8)

where we have used the fact that the points of \( \mathcal{P}_\lambda \) from which there is incident on \( O \) an edge that is of length longer than \( s^\tau \) is a Poisson point process of intensity \( \lambda g(|x|)1\{x \in B(O,s^\tau)^c\} \), the inequality \( 1 - e^{-y} \leq y \) and the assumption on \( g \). Similarly we can bound the second term on the right hand side in the last inequality in (2.7) as follows (see Figure 3).

\[ E \left[ \sum_{X,Y \in \mathcal{P}_\lambda \cap B(O, \sqrt{2}t_s)^c} 1_{\{X,Y \in O_{ts}\}} \right] = \lambda^2 \int_{B(O,\sqrt{2}t_s)^c} \int_{D_x \cap B(x,s^\tau)^c} g(|x-y|) \ dy \ dx, \]

(2.9)
where $D_s$ is the unbounded region $ARS'R'A'$ as shown in Figure 3. Changing to polar coordinates and using the obvious bounds for the range of the $y$-variable we can bound the expression on the right in (2.9) by

$$C_3 \int_{\sqrt{2s}}^{\infty} \int_{\phi=0}^{2\pi} R d\phi dR \int_{\sqrt{r^2-s^2}}^{\infty} r g(r) dr \leq C_4 \int_{\sqrt{2s}}^{\infty} (s^t + \sqrt{R^2 - 2t^2s^2})^{2-c} R dR$$

$$= C_4 \int_{ts}^{\infty} s^t R dR + C_4 \int_{\sqrt{2t^2s^2} + s^2}^{\infty} (\sqrt{R^2 - 2t^2s^2})^{2-c} R dR$$

(2.10)

$$= C_5 s^{-\tau(c-4)} + C_6 \int_{s^2}^{\infty} u^{3-c} du = C_7 s^{-\tau(c-4)},$$

where we have used the assumption that $g(r) \leq C r^{-c}$. Substituting from (2.8) and (2.10) in (2.7) we obtain

$$P(D_{ts}(s^t)) \leq C_2 s^{2-\tau(c-2)} + C_7 s^{-\tau(c-4)}.$$

Hence $P(D_{ts}(s^t)) \to 0$ as $s \to \infty$, since $\tau > \frac{2}{c-2}$ and $c > 4$.

The following corollary gives us the precise form in which we will be using Proposition 2.3

**Corollary 2.4.** For the graph $G_\lambda$ with the connection function $g$ satisfying $g(r) = O(r^{-c})$ as $r \to \infty$ let $L_{ts}(\tau)$ be the event that there exists an edge of length longer than $s^t$ intersecting the annulus $A_{2ts,4ts}$. Then for any $c > 4$, $t > 0$ and $\tau > \frac{2}{c-2}$ we have $P(L_{ts}(\tau)) \to 0$ as $s \to \infty$.

Proposition 2.5 is a restatement of the result for the case $\rho = 2$. We will complete the proof of the first assertion in Theorem 1.2 using this proposition followed by the proof of the proposition.

By assumption we have for some $c_0 > 0$

$$C_s(1) \geq c_0 \text{ for all } s \geq 1.$$  

(2.11)

**Proposition 2.5.** Suppose (2.11) holds for the graph $G_\lambda$ with the connection function $g$ satisfying $g(r) = O(r^{-c})$ as $r \to \infty$ for some $c > 4$. Then $\inf_{s \geq 1} C_s(2) > 0$.

Let $s \geq 1$. Assuming that Proposition 2.5 holds, it suffices to prove the result for $\rho > 2$. We need to build a left to right crossing in $[0, \rho s] \times [0, s]$. Observe that

$$[0, \rho s] \times [0, s] \subset \bigcup_{j=0}^{n_\rho} \left( (js, 0) + [0, 2s] \times [0, s] \right),$$

where $n_\rho \leq \rho + 1$. Let $F_s(\rho)$ be the event that there exists left to right crossing in $(js, 0) + [0, 2s] \times [0, s]$ for all $j = 0, 1, 2, \ldots, n_\rho$ and top to down crossing in $(js, 0) + [0, s] \times [0, s]$ for all $j = 1, 2, \ldots, n_\rho$. From (2.12) we have $F_s(\rho) \subset LR_s(\rho)$ (see Figure 4). Using this inclusion and applying the FKG inequality we obtain

$$C_s(\rho) \geq P[LR_s(2)]^{n_\rho-1} P[TD_s(1)]^{n_\rho}.$$

The assertion in Theorem 1.2 now follows from (2.11) and Proposition 2.5.

It remains to prove Proposition 2.5. The proof is derived from the next proposition that follows from a geometric construction. Recall that $A_s$ is the event that there exists a circuit in the annulus $A_{s,2s}$.
Proposition 2.6. Suppose the conditions given in Proposition 2.5 hold. Then there exists constants $c_2 > 0$, $C > 4$ and an increasing sequence of scales $\{s_n\}_{n \geq 1}$ satisfying $4s_n \leq s_{n+1} \leq Cs_n$ such that $P[A_{s_n}] \geq c_2$ for all $n \geq 1$.

Proof. Fix $s \geq 1$. For $\alpha, \beta \in [-s/2, s/2]$, $\alpha < \beta$, let $\mathcal{H}_s(\alpha, \beta)$ be the event (see Figure 5) that there exists a path in the box $B_{s/2}$ from left to $\{s/2\} \times [\alpha, \beta]$. For $\alpha \in [0, s/2]$, define the event $\chi_s(\alpha) := \chi^+_s(\alpha) \cap \chi^-_s(\alpha) \cap LR(s, \alpha)$, where $\chi^+_s(\alpha)$ is the event that there exists a path from $\{-s/2\} \times [-s/2, -\alpha]$ to $\{-s/2\} \times [\alpha, s/2]$ in $B_{s/2}$, $\chi^-_s(\alpha)$ is the event that there exists a path from $\{s/2\} \times [-s/2, -\alpha]$ to $\{s/2\} \times [\alpha, s/2]$ in $B_{s/2}$, and $LR(s, \alpha)$ is the event that there exists a path that intersects the paths for the events $\chi^+_s(\alpha), \chi^-_s(\alpha)$ in $B_{s/2}$ (see Figure 6).

Given the assumption (2.11) that the probability of crossing boxes $C_s(1)$ is uniformly bounded away from zero, the following two Lemmas from Tassion [35] provide lower bounds for probabilities of certain paths that will allow us to glue them together to construct paths with desired properties. These Lemmas are true for any planar percolation model which in conjunction with a result such as Corollary 2.4 allows us to derive the RSW result.

Lemma 2.7. If for some $c_0 > 0$, $\inf_{s \geq 1} C_s(1) \geq c_0$, then for all $s \geq 1$ there exists $\alpha_s \in [0, s/4]$ and $c_1 > 0$ such that,
By (2.19) and Lemma 2.8(i) we have
\begin{equation}
P(\chi_s(\alpha)) \geq c_1,
\end{equation}
and
\begin{equation}
P(\chi_s(\alpha)) - P(\chi_s(\alpha/2)) \geq c_0/4.
\end{equation}

\textbf{Lemma 2.8.} Let \( \alpha_s \) be as in Lemma 2.7 Then the following two statements are true.

i) There exists \( c_2 > 0 \) such that whenever \( \alpha_s \leq 2\alpha_{\frac{3}{4} s} \) for some \( s \geq 2 \), then
\begin{equation}
P(A_s) \geq c_2.
\end{equation}

ii) Let \( s \geq 1 \). If \( P(A_s) \geq c_2 \) and \( \alpha_t \leq s \) for some \( t \geq 4s \), then there exists \( c_3 > 0 \) such that
\begin{equation}
P(A_t) \geq c_3.
\end{equation}

Let \( c_3 \) be as in Lemma 2.8(ii) and \( c_0 > 0 \) be as in (2.11). Since \( c_0, c_3 \in (0, 1) \) and \( c > 4 \), we can and do choose \( C_1 > 16 \) such that,
\begin{equation}
\left( 1 - \frac{c_3}{2} \right) \left( \log_5 \frac{C_1}{2} \right)^{-1} < \frac{c_0}{4}
\end{equation}
and \( \tau \in (\frac{2}{c-2}, 1) \). By Corollary 2.4 there exists \( s_0 \geq 1 \) such that
\begin{equation}
P(L_{\frac{5}{2} s_i}(\tau)) \leq \frac{c_3}{2} \quad \text{for all } i = 2, 3, \ldots, \left( \log_5 \frac{C_1}{2} \right) \text{ and } s \geq s_0.
\end{equation}

Let \( \alpha_s \) be as in Lemma 2.7 Since \( \alpha_s < s \) there must exist a \( s_1 > s_0 \) such that
\begin{equation}
\alpha_{s_1} \leq 2\alpha_{\frac{3}{4} s_1}.
\end{equation}

By (2.19) and Lemma 2.8(ii) we have
\begin{equation}
P(A_{s_1}) \geq c_2.
\end{equation}

Having found \( s_1 \) the next task is to find \( s_2 \). This is done using the two steps described in the following Lemma.

\textbf{Lemma 2.9.} Let \( C_1 \) satisfy (2.17) and \( s_0 \geq 1 \) be such that (2.18) holds. If \( P(A_s) \geq c_2 \) for any \( s \geq s_0 \), then there exists \( s' \in [4s, C_1 s] \) such that \( \alpha_{s'} \geq s \). Further, there exists a \( C'_1 \) and \( s_2 \in [s', C'_1 s'] \) such that \( \alpha_{s_2} \leq 2\alpha_{\frac{3}{4} s_2} \).

We now complete the proof of Proposition 2.6 By Lemma 2.9 and (2.20) there exists \( s'_1 \in [4s_1, C_1 s_1] \) such that \( \alpha_{s'_1} \geq s_1 \). Consequently by the second assertion Lemma 2.9 there exists a \( C'_1 \) and \( s_2 \in [s'_1, C'_1 s'_1] \) such that \( \alpha_{s_2} \leq 2\alpha_{\frac{3}{4} s_2} \). An application of Lemma 2.8(ii) now yields
\begin{equation}
P(A_{s_2}) \geq c_2.
\end{equation}

Observe that \( 4s_1 \leq s'_1 \leq s_2 \leq C'_1 s'_1 \leq C_1 C'_1 s_1 \). Setting \( C = C_1 C'_1 \) and iterating this procedure we obtain the desired sequence \( \{ s_n \}_{n \geq 1} \). This proves the Proposition 2.6 except for Lemma 2.9

\textbf{Proof of Lemma 2.9} The first part of the proof of Lemma 2.9 is similar to Lemma 3.2 in [35]. Suppose that for some \( s \geq s_0 \) and \( c_2 > 0 \), \( P(A_s) \geq c_2 \). Suppose if possible \( \alpha_t < s \) for all \( t \in [4s, C_1 s] \). If we take \( t = C_1 s \), then
this yields \( \alpha_{C_i} < s < \frac{C_i}{2} \). It follows by Lemma 2.7(ii) that
\[
(2.22) \quad P\left( \mathcal{H}(0, s) \right) - P\left( \mathcal{H}\left(s, \frac{C_i s}{2}\right) \right) \geq c_0 \left( \frac{C_i}{2} \right).
\]

We will now derive a contradiction to (2.22). Note that \( \frac{C_i}{2} \) is \([4s, C_i]\) for \( i = 2, 3, \ldots, \lfloor \log_5 \frac{C_i}{2} \rfloor \). Since \( P(A_s) \geq c_2 \), taking \( t = \frac{C_i}{2} \) we have by Lemma 2.8(ii)
\[
(2.23) \quad P\left( A_{\frac{C_i}{2}} \right) \geq c_3.
\]
Fix \( \tau < 1 \) be such that (2.18) holds for all \( s \geq s_0 \). Combining (2.23) and (2.18) we can write for \( i = 2, 3, \ldots, \lfloor \log_5 \frac{C_i}{2} \rfloor \),
\[
(2.24) \quad P\left( A_{\frac{C_i}{2}} \cap L_{\frac{C_i}{2}}^c (\tau) \right) \geq c_3.
\]
Consider \( \mathcal{E}_s \) to be the event that there exists a circuit in \( A_{\frac{C_i}{2}} \). Observe that if \( A_{\frac{C_i}{2}} \cap L_{\frac{C_i}{2}}^c (\tau) \) occurs for some \( i = 2, 3, \ldots, \lfloor \log_5 \frac{C_i}{2} \rfloor \), then \( \mathcal{E}_s \) will also occur. Hence
\[
(2.25) \quad P(\mathcal{E}_s) \geq P\left( \bigcup_{i=2}^{\lfloor \log_5 \frac{C_i}{2} \rfloor} \left( A_{\frac{C_i}{2}} \cap L_{\frac{C_i}{2}}^c (\tau) \right) \right).
\]
In \( G_{\lambda}^c \) for any \( A_{2s, 4s} \) measurable event \( E \), the event \( E \cap L_\tau (\tau)^c \) is measurable with respect to \( \mathcal{P}_\lambda \) restricted to the region \( A_{5s} \) for \( \tau < 1 \). Indeed, if \( E \) is an event measurable w.r.t. \( A_{2s, 4s} \) and if the event \( L_\tau (\tau)^c \) occurs then there is no edge that intersects \( A_{2s, 4s} \) and has at least one end vertex out side \( A_{2s-	au, 4s+	au} \). Hence \( E \cap L_\tau (\tau)^c \) depends on \( \mathcal{P}_\lambda \) restricted to \( A_{2s-	au, 4s+	au} \) and \( A_{2s-	au, 4s+	au} \) \( A_{5s} \) since \( \tau < 1 \). It follows that \( E \cap L_\tau (\tau)^c \) depends on \( \mathcal{P}_\lambda \) restricted to \( A_{5s} \). Consequently the events \( A_{\frac{C_i}{2}} \cap L_{\frac{C_i}{2}}^c (\tau) \) for \( i = 2, 3, \ldots, \lfloor \log_5 C_i \rfloor \) are independent. Using this fact in (2.25) and by substituting from (2.24) and (2.17) yields
\[
(2.26) \quad P(\mathcal{E}_s) \leq \prod_{i=2}^{\lfloor \log_5 \frac{C_i}{2} \rfloor} P\left( (A_{\frac{C_i}{2}} \cap L_{\frac{C_i}{2}}^c (\tau))^c \right).
\]
In the square \( B_{C_1, 2} + (-C_1, 2, 0) = [-C_1, 2, 0] \times [C_1, 2, C_1, 2] \) consider the following two events, \( LH \) is the event that there is a path from left to \( \{0\} \times [0, s] \) in \( B_{C_1, 2} + (-C_1, 2, 0) \) and \( UH \) is the event that there is a path from left to \( \{0\} \times [s, C_1, 2] \) in \( B_{C_1, 2} + (-C_1, 2, 0) \). Observe that when \( LH \cap UH^c \) occurs, there cannot exists a circuit in \( A_{C_1, 2} \) around \( B_s \), that is \( LH \cap UH^c \subset \mathcal{E}^c \). This observation together with (2.26) and translation invariance yields
\[
(2.27) \quad \frac{c_0}{4} > P\left( LH \cap UH^c \right) \geq P(LH) - P(UH) = P\left( \mathcal{H}(0, s) \right) - P\left( \mathcal{H}(s, \frac{C_1 s}{2}) \right),
\]
which contradicts (2.22) and hence the assumption that \( \alpha t < s \) for all \( t \in [4s, C_1] \). So there must exists some \( s' \in [4s, C_1] \), such that \( \alpha s' \geq s \). This proves the first assertion in Lemma 2.9

From the first part of this lemma there exists \( s' \in [4s, C_1] \) such that
\[
(2.28) \quad \alpha s' \geq s \geq \frac{s'}{C_1}.
\]
We shall prove the second part as well by contradiction. Suppose if possible $\alpha_t > 2\alpha_{2t/3}$ for all $t \geq s'$. By iterating this inequality we obtain

$$\alpha_{(\frac{3}{2})^k s'} > 2\alpha_{(\frac{3}{2})^{k-1} s'} > 2^k \alpha_{s'} \geq 2^k \frac{s'}{C_1},$$

for all $k \geq 1$, where the last inequality follows from (2.28). Since $\alpha_s \leq \frac{s}{4}$ for all $s \geq 1$, we have for all $k \geq 1$ that

$$\alpha_{(\frac{3}{2})^k s'} \leq \frac{1}{4} \left(\frac{3}{2}\right)^k s'.$$

The inequalities (2.29) and (2.30) implies that for all $k \geq 1$,

$$\frac{1}{4} \left(\frac{3}{2}\right)^k > \frac{2^k}{C_1},$$

which contradicts the fact that $C_1 < \infty$. Hence the statement that $\alpha_t > 2\alpha_{2t/3}$ for all $t \geq s'$ is not true. In particular $\alpha_{(\frac{3}{2})^k s'} > 2\alpha_{(\frac{3}{2})^{k-1} s'}$ is not true for all $k \geq 1$.

Let $k^* := \min \{k \in \mathbb{N} : \alpha_{(\frac{3}{2})^k s'} \leq 2\alpha_{(\frac{3}{2})^{k-1} s'} \}$. By definition of $k^*$

$$\alpha_{(\frac{3}{2})^{k^*} s'} \leq \frac{1}{4} \left(\frac{3}{2}\right)^{k^*} s'.$$

Again by definition of $k^*$ and the argument leading to (2.31) we have $\left(\frac{3}{2}\right)^{k^*} s' < \frac{C_1}{4}$, which implies that $k^* \leq \lfloor \log_{\frac{3}{2}} \frac{C_1}{4} \rfloor + 1$.

Let $s_2 := (\frac{3}{2})^{k^*} s'$. Observe that $s_2 \geq s'$ and $s_2 \leq (\frac{3}{2})^{\lfloor \log_{\frac{3}{2}} \frac{C_1}{4} \rfloor + 1} s'$. Let $C'_1 = (\frac{3}{2})^{\lfloor \log_{\frac{3}{2}} \frac{C_1}{4} \rfloor + 1}$. Thus we have found $s_2 \in [s', C'_1 s']$ such that $\alpha_{s_2} \leq 2\alpha_{(\frac{3}{2}) s_2}$.

This proves second part of Lemma 2.9 and completes the proof of Proposition 2.6.
Proof of Proposition 2.5 Let \( \{s_n\}_{n \geq 1} \) be the sequence of scales as in Proposition 2.6. For any \( s \geq 1 \) let \( k = k(s) \) be such that \( s_k \leq s < s_{k+1} \). Since \( s_{k+1} \leq C s_k \) with \( C \) as in Proposition 2.6 we have
\[
[0, 2s] \times [0, s_k) \subset \bigcup_{i=0}^{n_1} ((is_k, 0) + [0, 2s_k] \times [0, s_k]),
\]
where \( n_1 = \lfloor 2s/s_k \rfloor + 1 \leq 2C + 1 \).

Let \( F_s \) be the event (see Figure 7) that there is a left to right crossing in each of the rectangles \((is_k, 0) + [0, 2s_k] \times [0, s_k]\) for \( i = 0, 1, \ldots, n_1 \) and top to down crossing in each of the squares \((is_k, 0) + [0, s_k] \times [0, s_k]\) for \( i = 1, 2, \ldots, n_1 \). Clearly \( F_s \subset LR_s(2) \) and by Proposition 2.6 we have \( C_s(2) \geq P(A_{s_k}) \geq c_2 \). It follows by the FKG inequality that \( C_s(2) \geq c_2 n_1 + 1, C_0 n_1 > 0 \). This proves Proposition 2.5.

\[\square\]

2.3. Proof of Theorem 1.2 (ii) for eRCM. As in the proof of the first part it suffices to show the result for \( \rho = 2 \).

We first complete the proof using the following lemma which will be proved subsequently using techniques similar to that used to prove Lemma 2.9.

Lemma 2.10. Suppose that \( \lim_{s \to \infty} C_s(1) = 1 \). Then for any fixed \( \epsilon > 0 \) there exists \( \eta \in (0, \frac{1}{4}) \) such that for all \( s \) sufficiently large we have
\[
P[\text{there exists a circuit around } B_{\eta s} \text{ in the annulus } A_{\eta s, \frac{\pi}{4}}] > 1 - \epsilon.
\]

Let \( \eta > 0 \) be as in Lemma 2.10. Divide the side \( \{\frac{s}{2}\} \times [-\frac{s}{2}, \frac{s}{2}] \) into intervals labeled \( J_i(\eta, s), i = 1, 2, \ldots, k, k = \left\lfloor \frac{1}{2\eta s} \right\rfloor + 1 \), of length \( 2\eta s \) (except for one interval that is of length at most \( 2\eta s \)). For \( i = 1, 2, \ldots, k \), let \( \mathcal{H}(J_i(\eta, s)) \) be the event that there exists a path in the box \( B_{\frac{s}{2}} \) from left to \( J_i(\eta, s) \). Clearly \( LR_s(1) = \bigcup_{i=1}^{k} \mathcal{H}(J_i(\eta, s)) \). Using square root trick there exists \( \beta_s \in [-\frac{s}{2}, \frac{s}{2}] \) satisfying \(-\frac{1}{2} - \eta s \leq \beta_s \leq \frac{1}{2} - \eta s \) such that,
\[
P(\mathcal{H}(\beta_s - \eta s, \beta_s + \eta s)) = \max_{i} P(\mathcal{H}(J_i(\eta, s))) \geq 1 - \left(1 - P\left(\bigcup_{i=1}^{k} \mathcal{H}(J_i(\eta, s))\right)\right)^{\frac{1}{k}} = 1 - (1 - C_s(1))^{\frac{1}{k}}.
\]

Let \( R_s(\eta) \), \( A'_{\eta s, \frac{\pi}{4}} \) be the events that there exists a path from \( \{\frac{s}{2}\} \times [\beta_s - \eta s, \beta_s + \eta s] \) to right in \((s, 0) + B_{\frac{s}{2}} \) and there exists a circuit in \( (\frac{s}{2}, \beta_s) + A_{\eta s, \frac{\pi}{4}} \) respectively (see Figure 8). By translation and rotation invariance \( P(R_s(\eta)) = P(\mathcal{H}(\beta_s - \eta s, \beta_s + \eta s)) \). For any \( \epsilon > 0 \), by Lemma 2.10 we have \( P(A'_{\eta s, \frac{\pi}{4}}) > 1 - \epsilon \) for all \( s \) sufficiently large. Let \( T_s := \mathcal{H}(\beta_s - \eta s, \beta_s + \eta s) \cap R_s(\eta) \cap A'_{\eta s, \frac{\pi}{4}} \). By the FKG inequality and (2.34) we obtain
\[
C_s(2) \geq P(T_s) = P(\mathcal{H}(\beta_s - \eta s, \beta_s + \eta s) \cap R_s(\eta) \cap A'_{\eta s, \frac{\pi}{4}}) \geq \left[1 - (1 - C_s(1))^{\frac{1}{k}}\right]^2 (1 - \epsilon) \to (1 - \epsilon),
\]
as \( s \to \infty \). The result now follows since \( \epsilon > 0 \) is arbitrary.

\[\square\]

2.4. Proof of Lemma 2.10. Fix \( \epsilon, \tau \in (0, 1) \). Since \( \lim_{s \to \infty} C_s(1) = 1 \), there exists a \( s_0 > 0 \) such that \( \inf_{s \geq s_0} C_s(1) > 0 \) and hence by Theorem 1.2 (ii) \( \inf_{s \geq s_0} C_s(4) > 0 \). By the FKG inequality,
\[
c := \inf_{s \geq s_0} P(A_{s}) \geq \inf_{s \geq s_0} (C_s(4))^{4} > 0.
\]
Choose \( \eta \in (0, \frac{1}{4}) \) satisfying
\[
(2.36) \quad \left( 1 - \frac{c}{2} \right)^{\lfloor \log_5 \frac{1}{8\eta} \rfloor} < \epsilon.
\]
From 2.35 there is a \( s_1 > s_0 \) such that for all \( s \geq s_1 \) and \( i = 1, 2, \cdots, \lfloor \log_5 \frac{1}{8\eta} \rfloor \) we have
\[
(2.37) \quad P(A_{s_i}^{s_1} \eta_8) \geq c.
\]
Using Corollary 2.4 choose \( s_2 > s_1 \) such that for all \( s \geq s_2 \) we have
\[
(2.38) \quad P(L_{s_i}^{s_1} \eta_8 (\tau)) \leq \frac{c}{2}.
\]
Combining (2.37) and (2.38) we can write for \( i = 1, 2, \cdots, \lfloor \log_5 \frac{1}{8\eta} \rfloor \),
\[
(2.39) \quad P(A_{s_i}^{s_1} \eta_8 \cap L_{s_i}^{s_1} (\tau)) \geq \frac{c}{2}.
\]
Let \( \mathcal{E}_s \) be the event that there exists a circuit around \( B_{\eta_8} \) in the annulus \( A_{\eta_8} \frac{1}{2} \). Observe that if \( A_{s_i}^{s_1} \eta_8 \cap L_{s_i}^{s_1} \eta_8 (\tau) \) occurs for some \( i = 1, 2, \cdots, \lfloor \log_5 \frac{1}{8\eta} \rfloor \), then \( \mathcal{E}_s \) will also occur. By the same argument as in Lemma 2.9 appearing below (2.25), the events \( A_{s_i}^{s_1} \eta_8 \cap L_{s_i}^{s_1} \eta_8 (\tau), i = 1, 2, \cdots, \lfloor \log_5 \frac{1}{8\eta} \rfloor \) are independent. Using the above two observations along with (2.39) and (2.36) yields
\[
P(\mathcal{E}_s^c) \leq P\left( \bigcap_{i=1}^{\lfloor \log_5 \frac{1}{8\eta} \rfloor} (A_{s_i}^{s_1} \eta_8 \cap L_{s_i}^{s_1} \eta_8 (\tau))^c \right)
\]
\[
= \prod_{i=1}^{\lfloor \log_5 \frac{1}{8\eta} \rfloor} P\left( (A_{s_i}^{s_1} \eta_8 \cap L_{s_i}^{s_1} \eta_8 (\tau))^c \right) < \left( 1 - \frac{c}{2} \right)^{\lfloor \log_5 \frac{1}{8\eta} \rfloor} < \epsilon. \quad \square
\]

2.5. **Proof of Theorem 1.2 (iii)**. The proof follows by a renormalization argument that uses the RSW Lemma (Theorem 1.2 (ii)) and Proposition 2.3 on the length of the longest edge in the graph \( \hat{G}_n^{\lambda} \). Suppose \( \theta^e(\lambda) > 0 \), that is, the graph \( \hat{G}_n^{\lambda} \) percolates. For \( u < n \) and fixed \( \tau \in \left( \frac{2}{c-2}, 1 \right) \) define the events
\[
(2.40) \quad E(u, n) := \left\{ B_u \xrightarrow{(2n, 0) \cdots 4n, 0} B_u \right\} \quad \text{and} \quad \tilde{E}(u, n) := E(u, n) \cap A_u \cap \tilde{A}_u \cap \{ M_{4n} \leq n^\tau \},
\]
where $A_u$ (or $\tilde{A}_u$) be the event that there exists a circuit around $B_u ((4n, 0) + B_u)$ in the annulus $A_{u, 2u} ((4n, 0) + A_{u, 2u})$, $M_{4n} :=$ length of the longest edge in the box $(2n, 0) + B_{4n}$. We complete the proof using the following proposition, the proof of which shall be provided subsequently.

**Proposition 2.11.** For the random graph $G^c_{\lambda}$ with the connection function $g$ satisfying $g(r) = O(r^{-c})$ as $r \to \infty$ for $c > 4$, if $\theta^c(\lambda) > 0$ then there exists a sequence $\{u_n\}_{n \geq 1}$ satisfying $u_n \to \infty$ as $n \to \infty$ and $u_n < \frac{q_0}{2}$ such that

$$\lim_{n \to \infty} P \left( E(u_{2n}, n) \right) = 1.$$  

Let $u_n$ be as in Proposition 2.11. We define a coupled nearest neighbour bond percolation model on $4n\mathbb{Z}^2$. The edge $((0, 0), (4n, 0))$ is said to be open if $E(u_{2n}, n)$ occurs in $G^c_{\lambda}$. An edge that is not open is designated closed. For any two nearest neighbours $z_1, z_2 \in \mathbb{Z}^2$ we can define an open edge between $4nz_1, 4nz_2$ in an analogous manner. Otherwise the edge is said to be closed. Denote by $\tilde{G}_n$ the graph on the vertex set $4n\mathbb{Z}^2$ formed by the open edges. By translation and rotational invariance, the probability that any edge is open has probability $P(\tilde{E}(u_{2n}, n))$. The edge $((0, 0), (4n, 0))$ being open does not depend on the configuration of points of $\mathcal{P}_{\lambda}$ outside $(2n, 0) + B_{5n}$. Thus the status of the edge $((0, 0), (4n, 0))$ can influence that of at most forty neighbouring edges. By Theorem 0.0 by Liggett et.al [22] for finitely dependent nearest neighbour bond percolation model on $4n\mathbb{Z}^2$, $n \in \mathbb{N}$, there exists a constant $q_0 \in (0, 1)$ such that the random graph $\tilde{G}_n$ percolates whenever

$$P(\tilde{E}(u_{2n}, n)) > q_0.$$  

By the FKG inequality $P(A_u) \geq C_u(4)^4$. Hence by translational invariance, Theorem 1.2(ii), Proposition 2.11 and Proposition 2.3 we have

$$P(\tilde{E}(u_{2n}, n)) \to 1, \quad \text{as} \quad n \to \infty.$$  

For $n \in \mathbb{N}$ and $\lambda > 0$ define $f_n(\lambda) := P(\tilde{E}(u_{2n}, n))$. Let $X_h$ be a Poisson random variable with mean $100n^2h$. Then a simple coupling argument shows that $|f_n(\lambda + h) - f_n(\lambda)| \leq P(X_h \geq 1) \to 0$ as $h \to 0$. So $f_n$ is continuous.

Let $\Lambda := \{\lambda : \theta^c(\lambda) > 0\}$ be the set of parameters $\lambda > 0$ for which $G^c_{\lambda}$ percolates. Since $G^c_{\lambda}$ percolates if $\tilde{G}_n$ does we have from (2.42) that $\bigcup_n f_n^{-1}(q_0, 1) \subset \Lambda$. On the other hand if $\lambda \in \Lambda$, then by (2.43) there exists an $n_0 \in \mathbb{N}$ such that $\lambda \in f_{n_0}^{-1}(q_0, 1)$ and hence $\Lambda \subset \bigcup_n f_n^{-1}(q_0, 1)$. It follows that $\Lambda = \bigcup_n f_n^{-1}(q_0, 1)$. Since the functions $f_n$ are continuous, $\Lambda$ is an open set. This completes the proof of Theorem 1.2(iii). It remains to prove the Proposition 2.11.

**Proof of Proposition 2.11.** Since $\theta^c(\lambda) > 0$, there exists almost surely an infinite connected component in $G^c_{\lambda}$. Hence for any sequence $\{u_n\}_{n \geq 1}$ satisfying $u_n \to \infty$ as $n \to \infty$ such that

$$P(C \text{ intersects } B_{u_n}) \to 1 \quad \text{as} \quad n \to \infty,$$  

where $C$ is an infinite connected component in $G^c_{\lambda}$. Fix one such sequence for which $u_n < \frac{q_0}{2}$. An immediate consequence of (2.44) is that

$$P(B_{u_n} \leftrightarrow B_n) \to 1 \quad \text{as} \quad n \to \infty.$$  

Let $v_n, w_n$ be as in Lemma 2.12 below. Define the events

$$H_n := B_{u_{2n}} \leftrightarrow \{2n\} \times [v_{2n}, w_{2n}], \quad H'_n := (4n, 0) + B_{u_{2n}} \leftrightarrow \{2n\} \times [v_{2n}, w_{2n}].$$
For $n \in \mathbb{N}$ let $\hat{A}_n$ be the event that there exists a circuit around $(2n, \frac{1}{2}(v_{2n} + w_{2n})) + B_{\frac{1}{2}(w_{2n} - v_{2n})}$ within the annulus $(2n, \frac{1}{2}(v_{2n} + w_{2n})) + A_{\frac{1}{2}(w_{2n} - v_{2n})}(w_{2n} - v_{2n})$. By the Lemma 2.12 as $n \to \infty$

(2.46) $P(H_n) = P(H'_n) \to 1$ and $(2n, \frac{1}{2}(v_{2n} + w_{2n})) + A_{\frac{1}{2}(w_{2n} - v_{2n})}(w_{2n} - v_{2n}) \subset (2n, 0) + B_{2n}$.

By translation invariance, the FKG inequality and Theorem 1.2 we have

(2.47) \[ \lim_{n \to \infty} P(\hat{A}_n) = 1 \]

Observe that the paths that enable the events $H_n, H'_n$ must intersect the circuit in $(2n, \frac{1}{2}(v_{2n} + w_{2n})) + A_{\frac{1}{2}(w_{2n} - v_{2n})}(w_{2n} - v_{2n})$ (see Figure 9) and hence

(2.48) \[ H_n \cap H'_n \cap \hat{A}_n \subset \left\{ B_{u_{2n}}(2n, 0) + B_{\frac{1}{2}n}, (4n, 0) + B_{2n} \right\} \]

Proposition 2.11 now follows from (2.46)-(2.48).

**Lemma 2.12.** Consider the random graph $G^\kappa_n$ with the connection function $g$. If $6^\kappa(\lambda) > 0$ for any $k \in \mathbb{N}$ there exists sequence $\{v_n\}_{n \geq 1}, \{w_n\}_{n \geq 1}$ satisfying $[v_{2n} - \frac{1}{2}(w_{2n} - v_{2n}), w_{2n} + \frac{1}{2}(w_{2n} - v_{2n})] \subset [-2n, 2n]$ such that the following holds as $n \to \infty$

(2.49) $P(B_{u_{2n}} \xrightarrow{B_{2n}} \{2n\} \times [v_{2n}, w_{2n}]) \to 1$.

**Proof.** Consider the square $B_{2n}$. By rotational invariance the probability of having a path from $B_{u_{2n}}$ to any of the eight half intervals on the sides of the square $B_{2n}$ are same. In other words

(2.50) $P(B_{u_{2n}} \xrightarrow{B_{2n}} A_i) = P(B_{u_{2n}} \xrightarrow{B_{2n}} A_j)$

for $i, j \in \{1, 2, \ldots, 8\}$, where $A_1 = \{2n\} \times [0, 2n], A_2 = [0, 2n] \times \{2n\}, \ldots, A_8 = \{2n\} \times [-2n, 0]$ are the half intervals on the sides of $B_{2n}$. Applying square root trick and using (2.45) we have

(2.51) $P(B_{u_{2n}} \xrightarrow{B_{2n}} A_i) \geq 1 - \left(1 - P(B_{u_{2n}} \xrightarrow{B_{2n}} \partial B_{2n})\right)^{\frac{1}{8}} \to 1$,

as $n \to \infty$, for all $i = 1, \ldots, 8$. For any $n \in \mathbb{N}$ and $\theta \in [0, 2n]$

\[ \left\{ B_{u_{2n}} \xrightarrow{B_{2n}} \{2n\} \times [0, \theta] \right\} \cup \left\{ B_{u_{2n}} \xrightarrow{B_{2n}} \{2n\} \times [\theta, 2n] \right\} = \left\{ B_{u_{2n}} \xrightarrow{B_{2n}} A_1 \right\} , \]
For $\theta \in [0, 2n]$ define the functions

$$L_{2,n}(\theta) := P(\{ B_{u_{2n}} \stackrel{B_{2n}}{\longleftrightarrow} (2n) \times [0, \theta] \}), \quad U_{2,n}(\theta) := P(\{ B_{u_{2n}} \stackrel{B_{2n}}{\longleftrightarrow} (2n) \times [\theta, 2n] \}).$$

Observe that $U_{2,n}(0) > L_{2,n}(0) = 0$, $L_{2,n}(2n) > U_{2,n}(2n) = 0$, $L_{2,n}$ is non-decreasing and $U_{2,n}$ is non-increasing and $L_{2,n}, U_{2,n}$ are continuous. By the properties of $L_{2,n}, U_{2,n}$ there exists $\alpha_{2n} \in (0, 2n)$ such that $U_{2,n}(\theta) \geq L_{2,n}(\theta)$ for $\theta < \alpha_{2n}$, $L_{2,n}(\theta) \geq U_{2,n}(\theta)$ for $\theta > \alpha_{2n}$. Further from (2.51) and another application of square root trick we obtain

$$U_{2,n}(\alpha_{2n}) = L_{2,n}(\alpha_{2n}) \to 1. \tag{2.52}$$

Set $\beta_{2n} = 2n - \alpha_{2n} > 0$ and let $\gamma_{2n} := \frac{\alpha_{2n} \wedge \beta_{2n}}{16}$. Let $\{2n\} \times [0, \alpha_{2n}] = \bigcup_{j=1}^{k} I_{j}^{(n)}$ where $I_{j}^{(n)} = \{2n\} \times ([j-1] \gamma_{2n}, j \gamma_{2n}]$, for $j = 1, 2, \ldots, (k-1)$ and $I_{k}^{(n)} = \{2n\} \times ([k-1] \gamma_{2n}, k \gamma_{2n}]$, $k = \lfloor \frac{\alpha_{2n}}{\gamma_{2n}} \rfloor + 1$ (2.52) together with an application of square root trick yields

$$\max_{j \in [k]} \left\{ P\left( B_{u_{2n}} \stackrel{B_{2n}}{\longleftrightarrow} I_{j}^{(n)} \right) \right\} \to 1. \tag{2.53}$$

as $n \to \infty$. Let $t(n) = \arg \max_{j \in [k]} P\left( B_{u_{2n}} \stackrel{B_{2n}}{\longleftrightarrow} I_{j}^{(n)} \right)$. (2.49) follows by taking $[v_{2n}, w_{2n}] = I_{t(n)}$. \hfill \Box

### 2.6. Proof of Theorem 1.1 (ii)

The proof of non-trivial phase transition for iERCM is different than that of eRCM. With $d = 2$, the condition for a phase transition to occur in $H_\lambda$ in the statement of Theorem 3.2 (a2) in [D] reduces to $\alpha > 2, \alpha \beta > 4$. However, this condition is required only to show that $\tilde{\lambda} > 0$. Clearly, $H_\lambda^c$ percolates if $H_\lambda$ does and hence $\tilde{\lambda} < \infty$. We now show that $\tilde{\lambda} > 0$.

For any $n \in \mathbb{N}$ we will enumerate and segregate self-avoiding paths in the graph $H_\lambda^c$. To this end consider the random graph $H_\lambda$ on $\mathcal{P}_\lambda$ and let $E(H_\lambda)$ denote the edge set of the graph $H_\lambda$. Let $x_0 = O$ and let $x := (x_1, x_2, \ldots, x_n) \in \mathcal{P}_\lambda \setminus \emptyset$ be an ordered collection of $n$ distinct points in $\mathcal{P}_\lambda$. Define the sub-collection of indices

$$I(x) := \{ i \in \{n-1\} : \{x_{i-1}, x_i\}, \{x_i, x_{i+1}\} \in E(H_\lambda) \},$$

$$J(x) := \{ i, i+1 : i \in \{n-2\} : \{x_{i-1}, x_{i+1}\}, \{x_i, x_{i+2}\} \in E(H_\lambda), x_{i-1}x_i \cap x_{i+1}x_{i+2} = z_i \not\in \mathcal{P}_\lambda \}. \tag{2.54}$$

The last condition in the definition of $J(x)$ requires that the edges intersect at a point interior to both the edges. Suppose $I(x) = \{i_1, i_2, \ldots, i_k\}$ for some $0 \leq k \leq n - 1$. For $1 \leq j \leq k + 1$ define the blocks $B_j(x) := \{i_{j-1} \leq i \leq i_j\}$ where $i_0 := 0, i_{k+1} := n$. We call $P_x := \bigcup_{j=1}^{k+1} \bigcup_{j=1}^{k} B_j(x) = \{0, 1, 2, \ldots, n\}$ and it contains no loop. Note that all blocks have even cardinality.

For $0 \leq k \leq n - 1, B_k$ be the collection of all blocks $(B_1, \ldots, B_{k+1})$ such that $|B_j|$ is even and for some $1 \leq i_1 < i_2 < \cdots < i_k < n, B_j := \{i_{j-1} \leq i \leq i_j\}$ with $i_0 := 0, i_{k+1} := n$. Let $eB_j = \{i_{j-1} + 2r - 1 : r = 1, 2, \ldots, (i_j - i_{j-1} + 1)/2\}$ be the set of indices with an even ordering in $B_j$.

$$\tilde{\theta}^e(\lambda) \leq P^0(\text{there is a self-avoiding path on } n \text{ vertices in } H_\lambda^c) \tag{2.54}$$

$$\leq E^0 \left[ \sum_{x \in \mathcal{P}_\lambda \setminus \emptyset} 1\{P_x \text{ is a self-avoiding path}\} \right] \leq \sum_{k=0}^{n-1} \sum_{(B_1, \ldots, B_{k+1}) \in B_k} E^0 \left[ \sum_{x \in \mathcal{P}_\lambda \setminus \emptyset} \prod_{j=1}^{k+1} 1\{B_j(x) = B_j\} \right].$$
Using the inequality $1 - e^{-x} \leq x \wedge 1$ for $x \geq 0$ we can bound the expectation in 2.54 as follows.

$$
E^o \left[ \sum_{x \in \mathcal{P}_\lambda, \neq} \prod_{j=1}^{k+1} \mathbb{1} \{ B_j(x) = B_j \} \right] = E^o \left[ \sum_{x \in \mathcal{P}_\lambda, \neq} E^o \left[ \prod_{j=1}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \wedge 1 \mid \mathcal{P}_\lambda \right] \right] 
$$

$$
\leq E^o \left[ \sum_{x \in \mathcal{P}_\lambda, \neq} E^o \left[ \prod_{j=1}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \wedge 1 \mid \mathcal{P}_\lambda \right] \prod_{j \text{ even}}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \wedge 1 \mid \mathcal{P}_\lambda \right] \right].
$$

(2.55)

By applying the Cauchy-Schwarz inequality followed by using the independence of the weights in the alternating blocks, the conditional expectation on the right hand side of 2.55 can be bounded by

$$
\prod_{j \text{ odd}}^{k+1} \prod_{l \in e B_j} \left( E^o \left[ \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right)^2 \wedge 1 \mid \mathcal{P}_\lambda \right] \right)^{\frac{1}{2}} \prod_{j \text{ even}}^{k+1} \prod_{l \in e B_j} \left( E^o \left[ \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right)^2 \wedge 1 \mid \mathcal{P}_\lambda \right] \right)^{\frac{1}{2}}.
$$

(2.56)

By Lemma 4.3 of [7] we have

$$
E^o \left( \frac{W_1W_2}{t} \right)^2 \wedge 1 \leq 1 \{ t < 1 \} + c 1 \{ t \geq 1 \} \left( 1 + (\beta \vee 2) \log t \right) t^{-\left( \frac{2}{\beta - 2} \right)} =: h(t) \text{ (say)}.
$$

(2.57)

where $c = \left( 1 + 1 \{ \beta \neq 2 \} \frac{2}{\beta - 2} \right)^{\frac{1}{2}}$. Using the bound from 2.57 in 2.56 and substituting in the inequality 2.55 we obtain

$$
E^o \left[ \sum_{x \in \mathcal{P}_\lambda, \neq} \prod_{j=1}^{k+1} \mathbb{1} \{ B_j(x) = e B_j \} \right] \leq E^o \left[ \sum_{x \in \mathcal{P}_\lambda, \neq} \prod_{j \text{ odd}}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \prod_{j \text{ even}}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \wedge 1 \mid \mathcal{P}_\lambda \right] \right].
$$

(2.58)

Applying Campbell-Mecke formula the right hand side of (2.58) equals

$$
\lambda^\alpha \int \ldots \int \prod_{j \text{ odd}}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \prod_{j \text{ even}}^{k+1} \prod_{l \in e B_j} \eta_{W_{x_{l-1}}W_{x_l}} \left( \frac{|x_l - x_{l-1}|^\alpha}{|x_l - x_{l-1}|} \right) \prod_{l=1}^{n} dx_l.
$$

(2.59)

We now evaluate the contribution to (2.59) from blocks of various sizes. Blocks of size four and higher yield a nice formula for the upper bound. However to see this one needs to compute the bound for a block of size six. We start with the simplest block of size two. The contribution from a block of the form $\{y_1, y_2\}$ is

$$
\int_{\mathbb{R}^2} h(\eta^{-1}|y_1 - y_2|^\alpha) dy_2 = \int_{\mathbb{R}^2} h(\eta^{-1}|x|^\alpha) dx = 2\pi \int_0^\infty r h(\eta^{-1} r^\alpha) dr.
$$

(2.60)

We now compute the contribution from the larger blocks. Consider the block $\{y_1, y_2, y_3, y_4\}$ where $\overline{y_3y_4}$ intersects $\overline{y_1y_2}$. We can bound from above the contribution from this block by using the same procedure as in the proof of Proposition 2.1 (see 2.2). For $a = (a_1, a_2), b = (b_1, b_2), \text{ let } H^+(a, b) := \left\{ (c_1, c_2) \in \mathbb{R}^2 \mid \frac{c_2 - a_2}{c_1 - a_1} \geq \frac{b_2 - a_2}{b_1 - a_1} \right\}$ be the set of points in $\mathbb{R}^2$ lying above the line joining $a$ and $b$. For $c \in H^+(a, b)$ let $D(a, b, c)$ be the region as defined in
the proof of Proposition 2.1. It is the set of all points for which the line segment to \( c \) intersects \( \tilde{a}\tilde{b} \). The contribution to (2.59) from block \( \{y_1, y_2, y_3, y_4\} \) equals

\[
\int_{\mathbb{R}^2} h(\eta^{-1}|y_1 - y_2|^{\alpha}) \, dy_2 \int_{\mathcal{H}^+ (y_1, y_2)} \int_{\mathcal{D} (y_1, y_2, y_3)} h(\eta^{-1}|y_4 - y_3|^{\alpha}) \, dy_4 \, dy_3
\]

(2.61) \quad \leq \quad \int_{\mathbb{R}^2} h(\eta^{-1}|y_1 - y_2|^{\alpha}) \, dy_2 \int_{\mathcal{H}^+ (y_1, y_2)} \int_{\mathcal{D} (y_1, y_2, y_3)} h(\eta^{-1}|y_4 - y_3|^{\alpha}) \, dy_4 \, dy_3 \quad \int_{\mathcal{H}^+ (y_5, y_4)} \int_{\mathcal{D} (y_3, y_4, y_5)} h(\eta^{-1}|y_6 - y_5|^{\alpha}) \, dy_6 \, dy_5

Consider the block \( \{y_1, y_2, \ldots, y_6\} \) where \( \overline{y_3y_4} \) intersects \( \overline{y_1y_2} \) and \( \overline{y_5y_6} \) intersects \( \overline{y_3y_4} \). The contribution from this block to (2.59) can be bounded above by applying the above procedure twice. This yields

\[
\int_{\mathbb{R}^2} h(\eta^{-1}|y_2 - y_1|^{\alpha}) \, dy_2 \int_{\mathcal{H}^+ (y_1, y_2)} \int_{\mathcal{D} (y_1, y_2, y_3)} h(\eta^{-1}|y_4 - y_3|^{\alpha}) \, dy_4 \, dy_3 \int_{\mathcal{H}^+ (y_5, y_4)} \int_{\mathcal{D} (y_3, y_4, y_5)} h(\eta^{-1}|y_6 - y_5|^{\alpha}) \, dy_6 \, dy_5
\]

(2.62)

By iterating the above procedure the contribution from the block of size \( 2m + 2 \) on the vertices \( \{y_1, y_2, \ldots, y_{2m+2}\} \) where the edge \( \overline{y_i, y_{i+1}} \) intersects \( \overline{y_{i+2}, y_{i+3}} \) for all \( i = 1, 2, \ldots, 2m - 1 \) can be bounded from above by

\[
2^{m+1} \pi \left( \int_0^\infty r^2 h(\eta^{-1} r^{\alpha}) \, dr \right) \left( \int_0^\infty r^3 h(\eta^{-1} r^{\alpha}) \, dr \right)^{m-1} \left( \int_0^\infty r^2 h(\eta^{-1} r^{\alpha}) \, dr \right).
\]

For a path with blocks \( B_1, B_2, \ldots, B_{k+1} \) let \( k_p := |\{j : |B_j| = 2p\}| \) and \( \bar{k} = \sum_{1 \leq j \leq n} |B_j| / 2 - 2 \). Since each vertex can be in at most two adjacent blocks we have that

(2.63) \quad \bar{k} \leq \sum_{j=1}^{k+1} \frac{|B_j|}{2} \leq n.

Using the upper bounds for the contributions from each block \( B_1, B_2, \ldots, B_{k+1} \) from (2.60)-(2.62) the expression in (2.59) can be bounded above by

(2.64) \quad \left( \prod_{m=0}^{n} (2^{m+1} \pi)^{k_m} \right) \left( \int_0^\infty r^2 h(\eta^{-1} r^{\alpha}) \, dr \right)^{k_1} \left( \int_0^\infty r^2 h(\eta^{-1} r^{\alpha}) \, dr \right) \left( \int_0^\infty r^3 h(\eta^{-1} r^{\alpha}) \, dr \right)^{2(k-k_1)} \left( \int_0^\infty r^3 h(\eta^{-1} r^{\alpha}) \, dr \right)^{\bar{k}}
Using [2.58, 2.59, 2.64] in 2.54 we obtain
\[(2.65) \quad \hat{\theta}^\alpha(\lambda) \leq \lambda^n \sum_{k=0}^{n-1} \sum_{(B_1, \ldots, B_{k+1}) \in B_k} \left( \prod_{m=0}^{k} (2^{m+1} \pi)^{km} \right) (C_1^n \wedge 1) \sum_{k=0}^{n-1} |B_k|,\]
where \( C_1 = \max \left\{ \int_0^\infty r^j h(\eta^{-1} r^\alpha) \, dr : j = 1, 2, 3 \right\}. \)

Since \( \sum_{m=0}^{k} (m+1)k_m \leq \sum_{m=0}^{k} 2m k_m \leq 2n \) and \( |B_k| = \binom{n-1}{k} \), there exists a constant \( C \) such that
\[\hat{\theta}^\alpha(\lambda) \leq (C\lambda)^n \to 0\]
as \( n \to \infty \) provided \( \int_0^\infty r^3 h(\eta^{-1} r^\alpha) \, dr < \infty \), which is true since \( \alpha > 2 \) and \( \alpha \beta > 4 \). \( \square \)

2.7. Proof of Theorem 1.2 for ierCM. The RSW results for the ierCM as enumerated in Theorem 1.2 follow in a manner identical to that for the eRCM once we prove the analog of Proposition 2.13 for the length of the longest edge in the graph \( H_x \). It thus suffices to prove the following proposition, the proof of which is identical to that of Proposition 2.3 with the obvious changes.

Proposition 2.13. Let \( \min\{\alpha, \alpha \beta\} > 4 \) and consider the graph \( H_x \) with connection function \( g \) satisfying defined as in (1.1). For any \( s > 0 \) let \( M_s \) be the length of the longest edge in \( H_x \) intersecting the box \( B_s = [-s, s]^2 \). Then for any \( t > 0 \) and \( \tau > \frac{2}{\min\{\alpha, \alpha \beta\} - 2} \) we have \( P(M_{ts} > s^\tau) \to 0 \) as \( s \to \infty \).

We shall use the following upper bound on the expected value of the connection function. The proof will be given later. We shall write \( E^{W_x, W_y} \) to denote the expectation with respect to the random weight \( W_x \) and \( W_y \).

Lemma 2.14. For \( \alpha, \beta, \eta > 0 \) and any \( x \in \mathbb{R}^2 \) with \( |x|^\alpha > \eta \) there exists constants \( C_1, C_2, C_3 \) such that the connection function given by (1.1) satisfies
\[(2.66) \quad E^{W_x, W_y}[g(O, x)] = \begin{cases} C_1|x|^{-\alpha} + C_2|x|^{-\alpha \beta} \log |x| + C_3|x|^{-\alpha \beta}, & \text{if } \beta \neq 1 \\ C_1|x|^{-\alpha}(\log |x|)^2 + C_2|x|^{-\alpha} \log |x| + C_3|x|^{-\alpha}, & \text{if } \beta = 1. \end{cases}\]

Proof of Proposition 2.13 Fix \( c > 4, t > 0 \). Let \( B(O, s) := \{x \in \mathbb{R}^2 : |x| \leq s\} \) be the ball of radius \( s \) centered at the origin. Define the events \( D_s(l) = \{M_s > l\}, \)
\( O_{ts}(\tau) = \{X \in \mathcal{P}_x : \text{there is an edge of length longer than } s^\tau \text{ incident on } X \in H_x\} \)
\( \tilde{O}_{ts}(\tau) = \{(X, Y) \in \mathcal{P}_x^2 : \text{there is an edge in } H_x \text{ joining } X, Y \text{ of length longer than } s^\tau, \overline{XY} \text{ intersects } B(O, \sqrt{2}ts)\}. \)

Recall that for any two points \( x, y \in \mathbb{R}^2, \overline{xy} \) denotes the line segment joining \( x \) and \( y \).
\[P\left( D_{ts}(s^\tau) \right) \leq E\left[ \sum_{X, Y \in \mathcal{P}_x} 1_{\{\overline{XY} \text{ intersects } B_{ts}\}} 1_{|X - Y| \geq s^\tau} \right] \]
\[(2.67) \leq E\left[ \sum_{X \in \mathcal{P}_x \cap B(O, \sqrt{2}ts)} 1_{\{X \in O_{ts}(\tau)\}} \right] + E\left[ \sum_{X, Y \in \mathcal{P}_x \cap B(O, \sqrt{2}ts)} 1_{(X, Y) \in \tilde{O}_{ts}} \right].\]
In what follows we shall write $E^{W_x}$ to denote the expectation with respect to the random weight $W_x$. By the Campbell-Mecke formula applied to the first term on the right hand side of the last inequality in (2.7) we obtain

\[
E \left[ \sum_{X \in \mathcal{P}_\lambda \cap B(O, \sqrt{2}ts)} 1_{\{X \in O_{ts}(\tau)\}} \right] = C\lambda(ts)^2 E^{W_0} \left[ P^\alpha(O \in O_{ts}(\tau) | W_o) \right] 
= C\lambda(ts)^2 E^{W_0} \left[ 1 - P^\alpha(\text{none of the edges incident on O is of length } s^\tau | W_o) \right] 
= C\lambda(ts)^2 E^{W_0} \left[ 1 - \exp \left\{ - \lambda \int_{B(O, s^\tau)^c} E^{W_x} [g(|x|) | W_o] \, dx \right\} \right] 
\leq C(\lambda ts)^2 E^{W_0} \left[ \int_{B(O, s^\tau)^c} E^{W_x} [g(|x|) | W_o] \, dx \right] 
\leq C(\lambda ts)^2 \int_{B(O, s^\tau)^c} E^{W_x}[g(|x|)|W_o] \, dx 
\times \left[ 1 - \exp \left( -\eta W_o W_x \right) \right] \, dx, 
\tag{2.68}
\]

where we have used the fact that conditional on the weight $W_o$ at the origin $O$, the points of $\mathcal{P}_\lambda$ from which there is incident on $O$ an edge that is of length longer than $s^\tau$ is a Poisson point process of intensity $\lambda E^{W_x} [g(|x|) | W_o] \{x \in B(O, s^\tau)^c\}$ and the inequality $1 - e^{-y} \leq y$. Using Lemma 2.14 and the fact that $\log r \leq r^\alpha$,

\[
\int_{B(O, s^\tau)^c} E^{W_x}[g(|x|)|W_o] \, dx = C_1 s^2 \int_{s^\tau}^{\infty} r \left( C_1 r^{-\alpha} + C_2 r^{-\alpha \beta} \log r + C_3 r^{-\alpha \beta} \right) \, dr 
\leq C_4 s^{2-\tau(\alpha-2)} + C_5 s^{2-\tau(\alpha \beta-2)} + C_6 s^{2-\tau(\alpha \beta-\epsilon-2)},
\tag{2.69}
\]

Similarly we can bound the second term on the right hand side in the last inequality in (2.67) as follows (see Figure 10). Let $\tilde{g}(x, y) := E[g(|x-y|), x, y \in \mathbb{R}^2$, where $g$ is as specified in (1.1). Observe that $\tilde{g}(x, y)$ depends on $x, y$ only via $|x-y|$ and so by an abuse of notation we will write $\tilde{g}(|x-y|)$ for $\tilde{g}(x, y)$.

\[
E \left[ \sum_{X, Y \in \mathcal{P}_\lambda \cap B(O, \sqrt{2}ts)^c} 1_{\{X, Y \in O_{ts}^c\}} \right] \leq \chi^2 \int_{B(O, \sqrt{2}ts)^c} \int_{D_x \cap B(x, s^\tau)^c} E^{W_x, W_y}[g(x, y)] \, dy \, dx 
\tag{2.70}
\]

where $D_x$ is the unbounded region $ATS'T'A'$ as shown in Figure 10. Changing to polar coordinates as in the proof of Proposition 2.3 and using Lemma 2.14 we can bound the last expression in (2.70) by

\[
C_7 \int_{\sqrt{2}ts}^{2\pi} R \, d\phi \, dR \int_{s^\tau \sqrt{R^2 - 2t^2 s^2}}^{\infty} r \tilde{g}(r) \, dr \leq C_8 \int_{\sqrt{2}ts}^{\infty} \int_{(s^\tau \sqrt{R^2 - 2t^2 s^2})}^{\infty} r \left( r^{-\alpha} + r^{-\alpha \beta} \log r + r^{-\alpha \beta} \right) \, dr \, dR 
\tag{2.71}
\]

\[
\leq C_9 \int_{\sqrt{2}ts}^{\infty} \left( (s^\tau \sqrt{R^2 - 2t^2 s^2})^{2-\alpha} + (s^\tau \sqrt{R^2 - 2t^2 s^2})^{2-\alpha \beta} \right) \, dR.
\]
The integral of the first integrand on right hand side in (2.71) can be evaluated as follows.

\[
\int_{ts\sqrt{2}}^{\infty} \left( s^\tau \vee \sqrt{R^2 - 2t^2s^2} \right)^{2-\alpha} R \, dR = \int_{ts\sqrt{2}}^{\infty} s^{\tau(2-\alpha)} R \, dR + \int_{\sqrt{2}t^2s^2}^{\infty} (\sqrt{R^2 - 2t^2s^2})^{2-\alpha} R \, dR
\]

(2.72)

\[
= C_{10}s^{-\tau(\alpha-4)}.
\]

Similarly the second term on the right hand side in (2.71) can be evaluated to obtain

(2.73)

\[
\int_{ts\sqrt{2}}^{\infty} \left( s^\tau \vee \sqrt{R^2 - 2t^2s^2} \right)^{2-\alpha\beta} R \, dR = C_{11}s^{-\tau(\alpha\beta-4)}
\]

Substituting (2.72), (2.73) in (2.71) and using (2.68) and (2.71) in (2.67) we obtain

\[
P(D_{ts}(s^\tau)) \leq C_{12} \left( s^{2-\tau(\alpha-2)} + s^{2-\tau(\alpha\beta-2)} + s^{-\tau(\alpha-4)} + s^{-\tau(\alpha\beta-4)} \right) \to 0
\]

as \( s \to \infty \) since \( \tau > \frac{2}{\min\{\alpha,\alpha\beta\}-2} \) and \( \min\{\alpha,\alpha\beta\} > 4. \)

\[ \square \]

**Proof of Lemma 2.14** We will prove the result for the case \( \beta \neq 1 \). The proof for \( \beta = 1 \) follows with minor changes. Fix \( x \in \mathbb{R}^2 \) such that \( |x|^\alpha > \eta \). Then

\[
E^{W_0,W_x}[g(O,x)] = E^{W_0,W_x} \left[ 1 - \exp \left\{ - \frac{\eta W_0W_x}{|x|^\alpha} \right\} \right]
\]

\[
\leq E^{W_0,W_x} \left[ \frac{\eta W_0W_x}{|x|^\alpha} \wedge 1 \right]
\]

\[
= E^{W_0,W_x} \left[ \frac{\eta W_0W_x}{|x|^\alpha} ; W_0W_x < \frac{|x|^\alpha}{\eta} \right] + P \left( W_0W_x \geq \frac{|x|^\alpha}{\eta} \right).
\]

(2.74)
By our assumption on the distribution of the weights it is easy to see that the product $W_0W_x$ has a density given by

$$f(w) = \beta^2 w^{-\beta-1} \log w, \ w \geq 1.$$  

The first term on the right in (2.74) can be evaluated using the expression in (2.75) and the integration by parts formula to yield

$$E^{W_0,W_x} \left[ \frac{\eta W_0 W_x}{|x|^\alpha}; W_0 W_x < \frac{|x|^\alpha}{\eta} \right] = \frac{\eta}{|x|^\alpha} \int_1^{\infty} w f(w) \, dw$$

for some constants $c_1, c_2, c_3$. The second term on the right in (2.74) can be computed in a similar fashion.

$$P \left( W_0 W_x \geq \frac{|x|^\alpha}{\eta} \right) = \int_{|x|^\alpha/\eta}^{\infty} f(w) \, dw$$

for some constants $c_4, c_5$. (2.66) now follows by substituting from (2.76), (2.77) in (2.74).

2.8. **Proof of Theorem 1.1 (iii).** The proof of non-trivial phase transition for Poisson stick model is quite similar to that of eRCM. We first prove that $\lambda_{PS} < \infty$. Consider the graph $PS_\lambda$ with half length density $h$ satisfying $0 < \int_0^\infty h(l) \, dl < \infty$. Let $R_0 := \inf\{l : h(l) > 0\}$. Pick any $R_1 > R_0$ finite such that $\int_{R_1}^{R_2} h(l) \, dl > 0$. Consider the graph $PS_\lambda$ and set sticks of length greater than $2R_1$ to be equal to $2R_1$ to obtain the graph $\hat{PS}_\lambda$. We have from [30] that the critical threshold parameter $\lambda_{PS} < \infty$ for the truncated model. Since $PS_\lambda$ percolates if $PS_\lambda$ does, we have $\lambda_{PS} < \infty$.

To show that $\lambda_{PS} > 0$ we need to prove the analog of Propositions 2.1, 2.2 to state which we shall need to modify the notations used for the eRCM. Consider $P_\lambda$ under the Palm measure $P^\circ$. Thus we have a stick having mid point at the origin. We will construct an exploration branching process starting from $O$ and show that the branching process dies out almost surely for all values of $\lambda > 0$ sufficiently small. The graph in this case is simpler since we do not have secondary neighbours. Recall that $L_x$ denotes the stick with mid point at $x$. Call a point $x \in P_\lambda$ to be the neighbour of $O$ if $L_x$ intersects $L_O$. These neighbours will constitute the children of $O$ in the branching tree $(T)$, that is, the first generation. Let $N_1$ be the number of neighbours of $O$ and $\{x_1^{(1)}, x_2^{(1)}, \ldots, x_{N_1^{(1)}}^{(1)}\}$ be the set of neighbours of $O$ ordered by increasing distance from origin. If $N_1 = 0$ then the exploration process stops. Else we proceed to construct the second generation as follows. Conditional on the first generation, for $1 \leq i \leq N_1$ let $N_i^{(2)}$ be the number of sticks that intersects $L_{x_i^{(1)}}$ but not $L_O$ and the mid points of all such sticks constitute the children of $x_i^{(1)}$. Note that we are over counting here. The tree is then grown in a similar manner for the subsequent generations. The process terminates when all points in the tree have been explored.

Let $S_k$ denote the set of children in generation $k$ for the nodes in generation $k - 1$ and let $M_k = |S_k|$. Define the filtration $\mathcal{F}_k := \sigma\{ \bigcup_{j=1}^{k} S_j \}$. Let $F_k := \gamma_n\{ \bigcup_{j=1}^{k} S_j \}, k \geq 1$.

Define the process $Y_n := M_n/\mu^n$ where $\mu = E^\circ[M_1]$. The proof of Theorem 1.1 (iii) follows from the following two propositions using the same arguments as in case of the eRCM.
Proposition 2.15. Let $\mu = E^o[M_1]$ be the expected number of neighbours of the origin in the $PS_\lambda$ under the Palm measure $P^\infty$. If $\int l h(l) \, dl < \infty$ holds, then $\mu \in (0, \infty)$ and $\mu \to 0$ as $\lambda \to 0^+$.

Proposition 2.16. Suppose that $\int l h(l) \, dl < \infty$. Then the process $\{Y_n\}_{n \geq 0}$ is super-martingale with respect to the filtration $\{F_n\}_{n \geq 0}$.

The proof of Proposition 2.16 is same as that of Proposition 2.2 and so we omit it. It remains to prove Proposition 2.15. Proof of Proposition 2.15 We assume without loss of generality that the stick with midpoint at the origin $L_O$ is lying on the x-axis. Conditional on $L_O = l$ let $D(r, \theta, l)$ be the region enclosed by the parallelogram $ABB'A'$ of base length $2l$ and height $2r \sin \theta$ (see Figure 11). Using the Campbell-Mecke formula and changing to polar coordinates we can write

$$
\mu = E^o[M_1] = \sum_{X \in P_\lambda} 1_{\{L_X \text{ intersects } L_O\}} = C_1 \int_0^\infty \int_0^{\pi} \int_0^\infty h(r) \, dr \, d\theta \, dx \, D(r, \theta, l)
$$

$$
= C_1 \int_0^\infty h(l) \, dl \int_0^{\pi} h(r) \, d\theta \, dr \, |D(r, \theta, l)|
$$

$$
\leq C_2 \lambda \left[ \int_0^\infty l h(l) \, dl \right]^2 \to 0,
$$

as $\lambda \to 0$ provided $\int l h(l) \, dl < \infty$.

2.9. Proof of Theorem 1.2 for Poisson stick model. The proof follows along the same lines as that for the eRCM using the following Proposition on the length of the longest stick in $PS_\lambda$. Since $c > 3$ we can choose a $\tau < 1$ so that the conclusion of Proposition 2.17 holds, which gives us the condition under which the rest of the proof works.

Proposition 2.17. Consider the graph $PS_\lambda$ with stick’s half length density function $h$ satisfying $h(l) = O(l^{-c})$ as $l \to \infty$ for some $c > 3$. For any $s > 0$ let $M_s$ be the half length of the longest stick in $PS_\lambda$ intersecting the box $B_s = [-s, s]^2$. Then for any $t > 0$ and $\tau > \frac{2}{c-1}$ we have $P(M_{ts} > s^\tau) \to 0$ as $s \to \infty$. 
Proof of Proposition 2.17 Fix \(c > 3\), \(t > 0\) and \(\tau > \frac{2}{c-1}\). Since \(c > 3\) it should suffice to let \(\tau > \frac{2}{c-1}\). Recall that \(B(O, s)\) denotes the ball of radius \(s\) centered at the origin and for \(X \in \mathcal{P}_\lambda\), \(L_X\) denotes the stick with mid point at \(X\) distributed independently according to the probability density function \(h\). Define the events \(\tilde{D}_s(l) = \{M_s > l\}\), \(O_{ts}(\tau) = \{X \in \mathcal{P}_\lambda : L_X\text{ has half length longer than } s\tau\text{ and intersects } B(O, \sqrt{2}ts)\}\).

\[
P\left(\tilde{D}_{ts}(s^\tau)\right) \leq E\left[\sum_{X \in \mathcal{P}_\lambda} 1\{L_X \text{ intersects } B_{ts}\} 1\{L_X \geq s^\tau\}\right]
\]

(2.79)

By our assumption \(h(l) \leq Cl^{-c}\) for all \(l\) sufficiently large. The Campbell-Mecke formula applied to the first term on the right hand side of the last inequality in (2.79) yields

\[
(2.80) \quad E\left[\sum_{X \in \mathcal{P}_\lambda \cap B(O, ts\sqrt{2})} 1\{X \in O_{ts}(\tau)\}\right] = C_0 \lambda(ts)^2 P^O(L_O \geq s^\tau) = C_1 s^2 \int_{s^\tau}^\infty h(l) \, dl \leq C_2 s^{2-\tau(c-1)},
\]

for all \(s\) sufficiently large. For the second term in (2.79) using the assumption on \(h\), the Campbell-Mecke formula and the fact that \(|X\overline{Y}| \geq s^\tau \lor (R - \sqrt{2}ts)\) (see Figure 12) we obtain for all \(s\) sufficiently large
\[ E \left[ \sum_{X \in \mathcal{P} \cap B(O,t \sqrt{2})} 1_{\{X \in O_{ts}(\tau)\}} \right] \leq C \lambda \int_{t \sqrt{2}}^{s \sqrt{2}} R \, dR \int_{s \sqrt{2} \vee (R - \sqrt{2} t \sqrt{s})}^{\infty} h(l) \, dl = C_1 \int_{\sqrt{2} t \sqrt{s}}^{\infty} (s \tau \vee (R - \sqrt{2} t \sqrt{s}))^{1-c} R \, dR \\
= C_4 \int_{\sqrt{2} t \sqrt{s} + s \tau}^{\infty} s^{(1-c)} R \, dR + C_4 \int_{\sqrt{2} t \sqrt{s}}^{\infty} (R - \sqrt{2} t \sqrt{s})^{1-c} R \, dR \\
= C_5 s^{-\tau(c-3)} + C_6 s^{-(\tau(c-2)-1)} + C_7 \int_{s \tau}^{\infty} u^{1-c} (u + \sqrt{2} t \sqrt{s}) \, du \\
(2.81) \\
= C_8 s^{-\tau(c-3)} + C_9 s^{-(\tau(c-2)-1)}. \]

Substituting from (2.80) and (2.81) in (2.79) we obtain
\[ P(D_{ts}(s \tau)) \leq C_2 s^{-(\tau(c-1)-2)} + C_7 s^{-\tau(c-3)} + C_8 s^{-(\tau(c-2)-1)} \to 0, \]
as \( s \to \infty \), since \( \tau > \frac{2}{c-1} > \frac{1}{c-2} \) for \( c > 3 \).

\[ \square \]

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