Estimates for vector-valued intrinsic square functions and their commutators on certain weighted amalgam spaces

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Received: 3 March 2021 / Accepted: 5 June 2022 / Published online: 12 July 2022
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Abstract
In this paper, we first introduce some new kinds of weighted amalgam spaces. Then we deal with the vector-valued intrinsic square functions, which were introduced recently by Wilson. In his fundamental work, Wilson established strong-type and weak-type estimates for vector-valued intrinsic square functions on weighted Lebesgue spaces. The goal of this paper is to extend his results to these weighted amalgam spaces. Moreover, we define vector-valued analogues of commutators with $\text{BMO}(\mathbb{R}^n)$ functions, and obtain the mapping properties of vector-valued commutators on the weighted amalgam spaces as well. In the endpoint case, we also establish the weighted weak $L \log L$-type estimates for vector-valued commutators in the setting of weighted Lebesgue spaces.

Keywords Vector-valued intrinsic square functions · Weighted amalgam spaces · Vector-valued commutators · Muckenhoupt weights · Orlicz spaces

Mathematics Subject Classification 42B25 · 42B35 · 46E30 · 47B47

In memory of Li Xue.

Communicated by Daniel Pellegrino.

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1 Introduction

The intrinsic square functions were first introduced by Wilson in [23, 24]; they are defined as follows. We equip the $n$-dimensional Euclidean space $\mathbb{R}^n$ with the Euclidean norm $| \cdot |$ and the Lebesgue measure $dx$. For $0 < \gamma \leq 1$, let $C_\gamma$ be the family of functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\varphi$’s support is contained in $\{ x \in \mathbb{R}^n : |x| \leq 1 \}$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 0$, and for all $x, x' \in \mathbb{R}^n$,

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\gamma.$$  

For $(y, t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define

$$A_\gamma(f)(y, t) := \sup_{\varphi \in C_\gamma} |\varphi_t * f(y)| = \sup_{\varphi \in C_\gamma} \left| \int_{\mathbb{R}^n} \varphi_t(y-z)f(z) \, dz \right|,$$

where $\varphi_t$ denotes the usual $L^1$ dilation of $\varphi : \varphi_t(y) = t^{-n}\varphi(y/t)$. Then we define the intrinsic square function of $f$ (of order $\gamma$) by the formula

$$S_\gamma(f)(x) := \left( \int \int_{\Gamma(x)} \left[ A_\gamma(f)(y, t) \right]^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}.$$

This new function is independent of any particular kernel, and it dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations, one can see more details in [23, 24]. In this paper, we will consider the vector-valued extension of the scalar operator $S_\gamma$. Let $f = (f_1, f_2, \ldots)$ be a sequence of locally integrable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $0 < \gamma \leq 1$, Wilson [24] also defined the following vector-valued intrinsic square function of $f$ by

$$S_\gamma(f)(x) := \left( \sum_{j=1}^{\infty} |S_\gamma(f_j)(x)|^2 \right)^{1/2}.$$  

A weight is any positive measurable function $w$ which is locally integrable on $\mathbb{R}^n$. In [24], Wilson has established the following two results (weighted strong-type and weak-type estimates).

For the case of $1 < p < \infty$,

**Theorem 1.1** [24] Let $0 < \gamma \leq 1$, $1 < p < \infty$ and $w \in A_p$ (Muckenhoupt weight class, see Sect. 2). Then there exists a constant $C > 0$ independent of $f = (f_1, f_2, \ldots)$ such that
\[
\left\| \left( \sum_{j=1}^{\infty} |S_\gamma(f_j)|^2 \right)^{1/2} \right\|_{L^p_w} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p_w}.
\]

For the case of \( p = 1 \),

**Theorem 1.2** [24] Let \( 0 < \gamma \leq 1 \). Then for any given weight \( w \) and \( \lambda > 0 \), there exists a constant \( C > 0 \) independent of \( \mathbf{f} = (f_1, f_2, \ldots) \) and \( \lambda > 0 \) such that

\[
\left\{ x \in \mathbb{R}^n : \left( \sum_{j=1}^{\infty} |S_\gamma(f_j)(x)|^2 \right)^{1/2} > \lambda \right\}
\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} M(w)(x) \, dx,
\]

where \( M \) denotes the standard Hardy–Littlewood maximal operator.

If we take \( w \in A_1 \), then \( M(w)(x) \leq C \cdot w(x) \) for a.e. \( x \in \mathbb{R}^n \) by the definition of \( A_1 \) weight (see Sect. 2). Hence, as a straightforward consequence of Theorem 1.2, we obtain

**Theorem 1.3** Let \( 0 < \gamma \leq 1 \) and \( w \in A_1 \). Then there exists a constant \( C > 0 \) independent of \( \mathbf{f} = (f_1, f_2, \ldots) \) such that

\[
\left\| \left( \sum_{j=1}^{\infty} |S_\gamma(f_j)|^2 \right)^{1/2} \right\|_{W^{1,1}} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^1_w}.
\]

Let \( b \) be a locally integrable function on \( \mathbb{R}^n \) and \( 0 < \gamma \leq 1 \), the commutators generated by \( b \) and intrinsic square function \( S_\gamma \) are defined by the author as follows (see [20]).

\[
[b, S_\gamma](f)(x) := \left( \int_{\mathbb{R}^n} \sup_{y \in \mathcal{G}_\gamma} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_{y - z} f(z) \, dz \right|^2 \frac{dy}{r^{n+1}} \right)^{1/2}.
\]

(3)

In this paper, we will consider the vector-valued analogues of these commutator operators. Let \( \mathbf{f} = (f_1, f_2, \ldots) \) be a sequence of locally integrable functions on \( \mathbb{R}^n \). For any \( x \in \mathbb{R}^n \) and \( 0 < \gamma \leq 1 \), in the same way, we can define the commutators for vector-valued intrinsic square function of \( \mathbf{f} \) as
\[ [b, S_j](f)(x) := \left( \sum_{j=1}^{\infty} |[b, S_j](f_j)(x)|^2 \right)^{1/2}. \]  

(4)

For any \( r > 0 \) and \( y \in \mathbb{R}^n \), let \( B(y, r) = \{ x \in \mathbb{R}^n : |x - y| < r \} \) denote the open ball centered at \( y \) with radius \( r \), \( B(y, r)^c \) denote its complement and \( |B(y, r)| \) be the Lebesgue measure of the ball \( B(y, r) \). We also use the notation \( \chi_{B(y, r)} \) for the characteristic function of \( B(y, r) \). Let \( 1 \leq p, q, \alpha \leq \infty \). We define the amalgam space \( (L^p, L^q)^\alpha(\mathbb{R}^n) \) of \( L^p(\mathbb{R}^n) \) and \( L^q(\mathbb{R}^n) \) as the set of all measurable functions \( f \) satisfying \( f \in L^p_{loc}(\mathbb{R}^n) \) and \( \| f \|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} < \infty \), where

\[
\| f \|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ B(y, r)^{1/2} f \cdot \chi_{B(y, r)} \|L^p(\mathbb{R}^n) \right]^q dy \right\}^{\frac{1}{q}} = \sup_{r > 0} \left\{ \| B(y, r)^{1/2} f \cdot \chi_{B(y, r)} \|L^p(\mathbb{R}^n) \right\}^{\frac{1}{q}} \]

with the usual modification when \( p = \infty \) or \( q = \infty \). This amalgam space was originally introduced by Fofana in [7]. As proved in [7] the space \( (L^p, L^q)^\alpha(\mathbb{R}^n) \) is non-trivial if and only if \( p \leq \alpha \leq q \); thus in the remaining of this paper we will always assume that the condition \( p \leq \alpha \leq q \) is fulfilled. Note that

- For \( 1 \leq p \leq \alpha \leq q \leq \infty \), one can easily see that \( (L^p, L^q)^\alpha(\mathbb{R}^n) \subseteq (L^p, L^q)(\mathbb{R}^n) \), where \( (L^p, L^q)(\mathbb{R}^n) \) is the Wiener amalgam space defined by (see [8, 10] for more information)

\[
(L^p, L^q)(\mathbb{R}^n) := \left\{ f : \| f \|_{(L^p, L^q)(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left[ \| f \cdot \chi_{B(y, r)} \|L^p(\mathbb{R}^n) \right]^q dy \right)^{\frac{1}{q}} < \infty \right\};
\]

- If \( 1 \leq p < \alpha \) and \( q = \infty \), then \( (L^p, L^q)^\alpha(\mathbb{R}^n) \) is just the classical Morrey space \( L^{p, \kappa}(\mathbb{R}^n) \) defined by (with \( \kappa = 1 - p/\alpha \), see [13])

\[
L^{p, \kappa}(\mathbb{R}^n) := \left\{ f : \| f \|_{L^{p, \kappa}(\mathbb{R}^n)} = \sup_{y \in \mathbb{R}^n, r > 0} \left( \frac{1}{|B(y, r)^\kappa B(y, r)|^p} \int_{B(y, r)} |f(x)|^p \ dx \right)^{\frac{1}{p}} < \infty \right\};
\]

- If \( p = \alpha \) and \( q = \infty \), then \( (L^p, L^q)^\alpha(\mathbb{R}^n) \) reduces to the usual Lebesgue space \( L^p(\mathbb{R}^n) \).

In [5] (see also [4, 6]), Feuto considered a weighted version of the amalgam space \( (L^p, L^q)^\alpha(w) \). Let \( 1 \leq p \leq \alpha \leq q \leq \infty \) and \( w \) be a weight on \( \mathbb{R}^n \). We denote by \( (L^p, L^q)^\alpha(w) \) the weighted Fofana space, the space of all locally integrable functions \( f \) satisfying \( \| f \|_{(L^p, L^q)^\alpha(w)} < \infty \), where
\[\|f\|_{(L, L^q)^e}(w) := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^\frac{1}{p} \| f \cdot \mathcal{Z}_{B(y,r)} \|_{L^q_w} \right]^q \, dy \right\}^{1/q}\]

\[= \sup_{r > 0} \left\| w(B(y, r))^\frac{1}{p} \| f \cdot \mathcal{Z}_{B(y,r)} \|_{L^q_w} \right\|_{L^q(\mathbb{R}^n)}^{1/q},\]

with the usual modification when \( q = \infty \) and \( w(B(y, r)) = \int_{B(y,r)} w(x) \, dx \) is the weighted measure of \( B(y, r) \). Then for \( 1 \leq p \leq \alpha \leq q \leq \infty \), we know that \((L, L^q)^e(w)\) becomes a Banach function space with respect to the norm \( \| \cdot \|_{(L, L^q)^e}(w) \). Furthermore, we denote by \((WL, L^q)^e(w)\) the weighted weak amalgam space of all measurable functions \( f \) for which (see [5])

\[\|f\|_{(WL, L^q)^e}(w) := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^\frac{1}{p} \| f \cdot \mathcal{Z}_{B(y,r)} \|_{WL^q_w} \right]^q \, dy \right\}^{1/q}\]

\[= \sup_{r > 0} \left\| w(B(y, r))^\frac{1}{p} \| f \cdot \mathcal{Z}_{B(y,r)} \|_{WL^q_w} \right\|_{L^q(\mathbb{R}^n)} < \infty.\]

Note that

- If \( 1 \leq p < \alpha \) and \( q = \infty \), then \((L, L^q)^e(w)\) is just the weighted Morrey space \( L^{p, \kappa}(w)\) defined by (with \( \kappa = 1 - p/\alpha \), see [12])

\[L^{p, \kappa}(w) := \left\{ f : \| f \|_{L^{p, \kappa}(w)} = \sup_{y \in \mathbb{R}^n, r > 0} \left( \frac{1}{w(B(y, r))^{\kappa}} \int_{B(y, r)} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty \right\},\]

and \((WL, L^q)^e(w)\) is just the weighted weak Morrey space \( WL^{p, \kappa}(w)\) defined by (with \( \kappa = 1 - p/\alpha \))

\[WL^{p, \kappa}(w) := \left\{ f : \| f \|_{WL^{p, \kappa}(w)} = \sup_{y \in \mathbb{R}^n, r > 0} \sup_{\lambda > 0} \frac{1}{w(B(y, r))^{\kappa/p}} \left[ w\left( \{ x \in B(y, r) : |f(x)| > \lambda \} \right) \right]^{1/p} < \infty \right\};\]

- If \( p = \alpha \) and \( q = \infty \), then \((L, L^q)^e(w)\) reduces to the weighted Lebesgue space \( L^p_w(\mathbb{R}^n)\), and \((WL, L^q)^e(w)\) reduces to the weighted weak Lebesgue space \( WL^p_w(\mathbb{R}^n)\).

The main purpose of this paper is twofold. We first define some new kinds of weighted amalgam spaces, and then we are going to prove that vector-valued intrinsic square functions (2) and associated vector-valued commutators (4) which are known to be bounded on weighted Lebesgue spaces, are also bounded on these new weighted spaces under appropriate conditions.
Throughout this paper, \( f = (f_1, f_2, \ldots) \) will always stand for a sequence of locally integrable functions, the letter \( C \) always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. We also use \( A \approx B \) to denote the equivalence of \( A \) and \( B \); that is, there exist two positive constants \( C_1, C_2 \) independent of \( A \) and \( B \) such that \( C_1 A \leq B \leq C_2 A \).

### 2 Main results

#### 2.1 Notations and preliminaries

A weight \( w \) is said to belong to the Muckenhoupt’s class \( A_p \) for \( 1 < p < \infty \), if there exists a constant \( C > 0 \) such that

\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{1/p} \left( \frac{1}{|B|} \int_B w(x)^{-p'/p} \, dx \right)^{1/p'} \leq C
\]

for every ball \( B \subset \mathbb{R}^n \), where \( p' \) is the dual of \( p \) such that \( 1/p + 1/p' = 1 \). The class \( A_1 \) is defined replacing the above inequality by

\[
\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \text{ess inf}_{x \in B} w(x)
\]

for every ball \( B \subset \mathbb{R}^n \). We also define \( A_{\infty} = \bigcup_{1 \leq p < \infty} A_p \). For some \( t > 0 \), the notation \( tB \) stands for the ball with the same center as \( B = B(y, r_B) \) and with radius \( t \cdot r_B \). It is well known that if \( w \in A_p \) with \( 1 \leq p < \infty \) (or \( w \in A_{\infty} \)), then \( w \) satisfies the **doubling condition**; that is, for any ball \( B \) in \( \mathbb{R}^n \), there exists an absolute constant \( C > 0 \) such that (see [9])

\[
w(2B) \leq C w(B). \tag{7}
\]

When \( w \) satisfies this **doubling condition** (7), we denote \( w \in \Delta_2 \) for brevity. In general, for \( w \in A_1 \) and any \( l \in \mathbb{Z}_+ \), there exists an absolute constant \( C > 0 \) such that (see [9])

\[
w(2^lB) \leq C \cdot 2^{ln} w(B). \tag{8}
\]

Moreover, if \( w \in A_{\infty} \), then for any ball \( B \) in \( \mathbb{R}^n \) and any measurable subset \( E \) of a ball \( B \), there exists a number \( \delta > 0 \) independent of \( E \) and \( B \) such that (see [9])

\[
\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{\delta}. \tag{9}
\]

Equivalently, we could define the above notions with cubes instead of balls. Hence we shall use these two different definitions appropriate to calculations. Given a weight \( w \) on \( \mathbb{R}^n \), as usual, the weighted Lebesgue space \( L^p_w(\mathbb{R}^n) \) for \( 1 \leq p < \infty \) is defined as the set of all functions \( f \) such that
\[ \|f\|_{L^p_w} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty. \]

We also denote by \( WL^p_w(\mathbb{R}^n)(1 \leq p < \infty) \) the weighted weak Lebesgue space consisting of all measurable functions \( f \) such that
\[
\|f\|_{WL^p_w} := \sup_{\lambda > 0} \lambda \cdot \left[ w\left( \left\{ x \in \mathbb{R}^n : |f(x)| > \lambda \right\} \right) \right]^{1/p} < \infty.
\]

We next recall some basic definitions and facts about Orlicz spaces needed for the proofs of the main results. For further information on the subject, one can see [16].

A function \( A \) is called a Young function if it is continuous, nonnegative, convex and strictly increasing on \([0, +\infty)\) with \( A(0) = 0 \) and \( A(t) \to +\infty \) as \( t \to +\infty \). Given a Young function \( A \), we define the \( A \)-average of a function \( f \) over a ball \( B \) by means of the following Luxemburg norm:
\[
\|f\|_{A,B} := \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B A\left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

When \( A(t) = t^p \), \( 1 \leq p < \infty \), it is easy to see that
\[
\|f\|_{A,B} = \left( \frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p};
\]
that is, the Luxemburg norm coincides with the normalized \( L^p \) norm. Given a Young function \( A \), we use \( \bar{A} \) to denote the complementary Young function associated to \( A \). Then the following generalized Hölder’s inequality holds for any given ball \( B \):
\[
\frac{1}{|B|} \int_B |f(x) \cdot g(x)| \, dx \leq 2 \|f\|_{A,B} \|g\|_{\bar{A},B}.
\]

In particular, when \( A(t) = t \cdot (1 + \log^+ t) \), we know that its complementary Young function is \( \bar{A}(t) \approx \exp(t) - 1 \) (see [14, 15]). In this situation, we denote
\[
\|f\|_{L_{\log L,B}} = \|f\|_{A,B}; \quad \|g\|_{\exp L,B} = \|g\|_{\bar{A},B}.
\]

So we have
\[
\frac{1}{|B|} \int_B |f(x) \cdot g(x)| \, dx \leq 2 \|f\|_{L_{\log L,B}} \|g\|_{\exp L,B} \tag{10}
\]

A locally integrable function \( b \) on \( \mathbb{R}^n \) is said to be in \( \text{BMO}(\mathbb{R}^n) \), if
\[
\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx < \infty,
\]
where
\[ b_B := \frac{1}{|B|} \int_B b(y) \, dy. \]

The BMO norm of \( b \) is defined by
\[
\|b\|_* := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx.
\]

### 2.2 Weighted amalgam spaces

Let us begin with the definitions of the weighted amalgam spaces with Lebesgue measure in (5) and (6) replaced by weighted measure.

**Definition 2.1** Let \( 1 \leq p \leq q \leq \infty \), and let \( w, \mu \) be two weights on \( \mathbb{R}^n \). We denote by \((L^p, L^q)^2(w; \mu)\) the weighted amalgam space, the space of all locally integrable functions \( f \) with finite norm
\[
\|f\|_{(L^p, L^q)^2(w; \mu)} := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^{1/p - 1/q} \|f \cdot \chi_{B(y, r)}\|_{L^p_w} \right]^q \mu(y) \, dy \right\}^{1/q}
\]
\[
= \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^{1/p - 1/q} \|f \cdot \chi_{B(y, r)}\|_{L^p_w} \right]^q \mu(y) \, dy \right\}^{1/q}
\]
with the usual modification when \( q = \infty \). Then we can see that the space \((L^p, L^q)^2(w; \mu)\) equipped with the norm \( \| \cdot \|_{(L^p, L^q)^2(w; \mu)} \) is a Banach function space. Furthermore, we denote by \((WL^p, L^q)^2(w; \mu)\) the weighted weak amalgam space of all measurable functions \( f \) for which
\[
\|f\|_{(WL^p, L^q)^2(w; \mu)} := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^{1/p - 1/q} \|f \cdot \chi_{B(y, r)}\|_{WL^p_w} \right]^q \mu(y) \, dy \right\}^{1/q}
\]
\[
= \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ w(B(y, r))^{1/p - 1/q} \|f \cdot \chi_{B(y, r)}\|_{WL^p_w} \right]^q \mu(y) \, dy \right\}^{1/q}
\]
with the usual modification when \( q = \infty \).

Recently, in [21, 22], we have established the strong type and weak type estimates for vector-valued intrinsic square functions on some Morrey-type spaces. Inspired by the works mentioned above, it is natural to discuss the boundedness properties in the context of weighted amalgam spaces. We will show that vector-valued intrinsic square function is bounded on \((L^p, L^q)^2(w; \mu)\), and is bounded from \((L^1, L^q)^2(w; \mu)\) into \((WL^1, L^q)^2(w; \mu)\). Our first two results in this paper can be formulated as follows. For the case of \( p > 1 \),

**Theorem 2.2** Let \( 0 < \gamma \leq 1 \). Assume that \( 1 < p \leq q \leq \infty \), \( w \in A_p \) and \( \mu \in \Delta_2 \). Then there is a constant \( C > 0 \) such that
holds for all $f = (f_1, f_2, \ldots) \in (L^p, L^q)^x(w; \mu)$. For the case of $p = 1$,

**Theorem 2.3** Let $0 < \gamma \leq 1$. Assume that $1 \leq z < q \leq \infty$, $w \in A_1$ and $\mu \in \Delta_2$. Then there is a constant $C > 0$ such that

$$\left\| \left( \sum_{j=1}^{\infty} |S_j(f_j)|^2 \right)^{1/2} \right\|_{(L^p, L^q)^x(w; \mu)} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^x(w; \mu)}$$

holds for all $f = (f_1, f_2, \ldots) \in (L^1, L^q)^x(w; \mu)$.

Here $f = (f_1, f_2, \ldots) \in (L^p, L^q)^x(w; \mu)$ means that

$$\left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \in (L^p, L^q)^x(w; \mu), \quad 1 \leq p < \infty.$$

For the strong type estimate of vector-valued commutator (4) defined above on the weighted amalgam spaces, we will prove

**Theorem 2.4** Let $0 < \gamma \leq 1$ and $b \in \text{BMO}(\mathbb{R}^n)$. Assume that $1 < p \leq z < q \leq \infty$, $w \in A_p$ and $\mu \in \Delta_2$. Then there is a constant $C > 0$ such that

$$\left\| \left( \sum_{j=1}^{\infty} |[b, S_j](f_j)|^2 \right)^{1/2} \right\|_{(L^p, L^q)^x(w; \mu)} \leq C \|b\| \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^x(w; \mu)}$$

holds for all $f = (f_1, f_2, \ldots) \in (L^p, L^q)^x(w; \mu)$.

To obtain endpoint estimate for the vector-valued commutator (4), we first need to define the weighted $A$-average of a function $f$ over a ball $B$ by means of the weighted Luxemburg norm; that is, given a Young function $A$ and $w \in A_\infty$, we define (see [16, 25])

$$\|f\|_{A(w), B} := \inf \left\{ \sigma > 0 : \frac{1}{w(B)} \int_B A \left( \frac{|f(x)|}{\sigma} \right) \cdot w(x) \, dx \leq 1 \right\}.$$ 

When $A(t) = t$, this norm is denoted by $\| \cdot \|_{L(w), B}$, when $A(t) = t \cdot (1 + \log^+ t)$ and $\log^+ t = \max\{\log t, 0\}$, this norm is also denoted by $\| \cdot \|_{L \log L(w), B}$. The complementary Young function of $t \cdot (1 + \log^+ t)$ is $\exp(t) - 1$ with mean Luxemburg norm denoted by $\| \cdot \|_{\exp L(w), B}$. For $w \in A_\infty$ and for every ball $B$ in $\mathbb{R}^n$, we can also show the weighted version of (10). Namely, the following generalized Hölder’s inequality in the weighted setting.
\[
\frac{1}{w(B)} \int_B |f(x) \cdot g(x)| w(x) \, dx \leq C \|f\|_{L^1(B_0, B)} \|w\|_{L^\infty(B_0, B)} \tag{11}
\]
is valid (see [25] for instance). Now we introduce new weighted spaces of \( L \log L \) type as follows.

**Definition 2.5** Let \( 1 \leq \alpha \leq q \leq \infty \), and let \( w, \mu \) be two weights on \( \mathbb{R}^n \). We denote by \((L \log L, L^q)^\#(w; \mu)\) the weighted amalgam space of \( L \log L \) type, the space of all locally integrable functions \( f \) defined on \( \mathbb{R}^n \) with finite norm \( \|f\|_{(L \log L, L^q)^\#(w; \mu)} \),

\[
(L \log L, L^q)^\#(w; \mu) := \left\{ f \in L^1_{\text{loc}}(w) : \|f\|_{(L \log L, L^q)^\#(w; \mu)} < \infty \right\},
\]

where

\[
\|f\|_{(L \log L, L^q)^\#(w; \mu)} := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[w(B(y, r))^{1/2} \|f\|_{L \log L(B(y, r))} \right]^2 \mu(y) \, dy \right\}^{1/q} = \sup_{r > 0} \left\| w(B(y, r))^{1/2} \|f\|_{L \log L(B(y, r))} \right\|_{L^q_w}.
\]

Note that \( t \leq t \cdot (1 + \log^+ t) \) for all \( t > 0 \), then for any ball \( B(y, r) \subset \mathbb{R}^n \) and \( w \in A_\infty \), we have \( \|f\|_{L(w, B(y, r))} \leq \|f\|_{L \log L(B(y, r))} \) by definition, i.e., the inequality

\[
\|f\|_{L(w, B(y, r))} = \frac{1}{w(B(y, r))} \int_{B(y, r)} |f(x)| \cdot w(x) \, dx \leq \|f\|_{L \log L(w, B(y, r))} \tag{12}
\]
holds for any ball \( B(y, r) \subset \mathbb{R}^n \). Hence, for \( 1 \leq \alpha \leq q \leq \infty \), we can further see the following inclusion:

\[
(L \log L, L^q)^\#(w; \mu) \subset (L^1, L^q)^\#(w; \mu).
\]

For the endpoint estimate of commutators generated by \( \text{BMO}(\mathbb{R}^n) \) function and vector-valued intrinsic square function, we will also prove the following weak-type \( L \log L \) inequality in the context of weighted amalgam spaces.

**Theorem 2.6** Let \( 0 < \gamma \leq 1 \) and \( b \in \text{BMO}(\mathbb{R}^n) \). Assume that \( 1 \leq \alpha < q \leq \infty \), \( w \in A_1 \) and \( \mu \in A_2 \), then for any given \( \sigma > 0 \) and any ball \( B(y, r) \subset \mathbb{R}^n \) with \( y \in \mathbb{R}^n \), \( r > 0 \), there is a constant \( C > 0 \) independent of \( B(y, r) \) and \( \sigma > 0 \) such that

\[
\left\| \frac{w(B(y, r))^{1/2} - 1}{w(B(y, r))} \cdot w \left( \left\{ x \in B(y, r) : \left( \sum_{j=1}^\infty \| [b, S_\gamma] \hat{f}_j(x) \|^2 \right)^{1/2} > \sigma \right\} \right) \right\|_{L^q_w} \leq C \cdot \|f\|_{(L \log L, L^q)^\#(w; \mu)},
\]

where
\[ \Phi(t) = t \cdot (1 + \log^+ t), \quad \|f(x)\|_{\ell^2} = \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2}, \]

and the norm \( \| \cdot \|_{L^q} \) is taken with respect to the variable \( y \), i.e.,

\[
\left\| w(B(y,r))^{1/2-1/q} \cdot w\left( \left\{ x \in B(y,r) : \left( \sum_{j=1}^{\infty} |[b,S_y](f_j)(x)|^2 \right)^{1/2} > \sigma \right\} \right) \right\|_{L^q_w} = \left\{ \int_{\mathbb{R}^n} w(B(y,r))^{1/2-1/q} \cdot w\left( \left\{ x \in B(y,r) : \left( \sum_{j=1}^{\infty} |[b,S_y](f_j)(x)|^2 \right)^{1/2} > \sigma \right\} \right) \right\}^{1/q} \mu(y) dy. \]

**Remark 2.7** From the above definitions and Theorem 2.6, we can roughly say that the vector-valued commutator (4) is bounded from \((L \log L, L^q)(w; \mu)\) into \((WL^1, L^q)(w; \mu)\) whenever \(1 \leq \alpha < q \leq \infty\), \(w \in A_1\) and \(\mu \in \Delta_2\).

### 3 Proofs of Theorems 2.2 and 2.3

**Proof of Theorem 2.2** Let \(1 < p \leq \alpha < q \leq \infty\) and \(\sum_{j=1}^{\infty} |f_j|^2 \in (L^p, L^q)\) with \(w \in A_p\) and \(\mu \in \Delta_2\). For an arbitrary point \(y \in \mathbb{R}^n\) and \(r > 0\), we set \(B = B(y,r)\) for the ball centered at \(y\) and of radius \(r\), \(2B = B(y,2r)\). We represent \(f_j\) as

\[ f_j = f_j \cdot 1_{2B} + f_j \cdot 1_{(2B)^c} := f_j^0 + f_j^\infty, \]

where \(1_{2B}\) denotes the characteristic function of \(2B = B(y,2r), j = 1, 2, \ldots\). Then we write

\[
w(B(y,r))^{1/2-1/p-1/q} \left\| \left( \sum_{j=1}^{\infty} |S_y(f_j)|^2 \right)^{1/2} \cdot 1_{B(y,r)} \right\|_{L^q_w} = w(B(y,r))^{1/2-1/p-1/q} \left( \int_{B(y,r)} \left( \sum_{j=1}^{\infty} |S_y(f_j)(x)|^2 \right)^{p/2} w(x) \, dx \right)^{1/p} \leq \\
w(B(y,r))^{1/2-1/p-1/q} \left( \int_{B(y,r)} \left( \sum_{j=1}^{\infty} |S_y(f_j^0)(x)|^2 \right)^{p/2} w(x) \, dx \right)^{1/p} + w(B(y,r))^{1/2-1/p-1/q} \left( \int_{B(y,r)} \left( \sum_{j=1}^{\infty} |S_y(f_j^\infty)(x)|^2 \right)^{p/2} w(x) \, dx \right)^{1/p} := I_1(y,r) + I_2(y,r). \]

Below we will give the estimates of \(I_1(y,r)\) and \(I_2(y,r)\), respectively. By the weighted \(L^p\) boundedness of vector-valued intrinsic square function (see Theorem 1.1), we have
\[ I_1(y, r) \leq w(B(y, r))^{1/(2 - 1/p - 1/q)} \left\| \left( \sum_{j=1}^{\infty} |s_j(f_j^0)|^2 \right)^{1/2} \right\|_{L_w^p} \]

\[ \leq C \cdot w(B(y, r))^{1/(2 - 1/p - 1/q)} \left( \int_{B(y, 2r)} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{p/2} w(x) \, dx \right)^{1/p} \]

\[ = C \cdot w(B(y, 2r))^{1/(2 - 1/p - 1/q)} \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \chi_{B(y, 2r)} \right\|_{L_w^p} \times \frac{w(B(y, r))^{1/(2 - 1/p - 1/q)}}{w(B(y, 2r))^{1/(2 - 1/p - 1/q)}}. \] 

Moreover, since \( 1/(2 - 1/p - 1/q) < 0 \) and \( w \in A_p \) with \( 1 < p < \infty \), then by doubling inequality (7), we obtain

\[ \frac{w(B(y, r))^{1/(2 - 1/p - 1/q)}}{w(B(y, 2r))^{1/(2 - 1/p - 1/q)} \leq C. \] 

Substituting the above inequality (15) into (14) yields the inequality,

\[ I_1(y, r) \leq C \cdot w(B(y, 2r))^{1/(2 - 1/p - 1/q)} \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \chi_{B(y, 2r)} \right\|_{L_w^p}. \] 

As for the second term \( I_2(y, r) \), note that the family \( C_t \) is uniformly bounded with respect to the \( L^\infty \)-norm, then for any given \( \varphi \in C_{\gamma}, \, 0 < \gamma < 1, \, j = 1, 2, \ldots, \) and \( (\xi, t) \in \Gamma(x) \) with \( x \in B(y, r) \), we have

\[
\left| \int_{B^n} \varphi_t(\xi - z) f_j^\infty(z) \, dz \right| = \left| \int_{B(y, 2r)^c} \varphi_t(\xi - z) f_j(z) \, dz \right| \\
\leq C \cdot t^{-n} \int_{B(y, 2r)^c} \left| f_j(z) \right| \, dz \\
\leq C \cdot t^{-n} \sum_{l=1}^{\infty} \int_{[B(y, 2^{l+1}r) \setminus B(y, 2^l r)] \cap \{z : |z - z_t| \leq t\}} \left| f_j(z) \right| \, dz.
\]

Since \( |\xi - z| \leq t \) and \( (\xi, t) \in \Gamma(x) \), then one has \( |x - z| \leq |x - \xi| + |\xi - z| \leq 2r \). Hence, for any \( x \in B(y, r) \) and \( z \in B(y, 2^{l+1}r) \setminus B(y, 2^l r) \), a direct computation shows that

\[ 2t \geq |x - z| \geq |z - y| - |x - y| \geq 2^{l-1} r. \] 

Therefore, by using the inequalities (17) and (18) derived above, together with Minkowski’s inequality for integrals, we can deduce that
\[
S_{\gamma}(f_{j}^{\infty})(x) = \left( \int \int_{\Gamma(x) \in \mathbb{C}} \left| \int_{\mathbb{R}^n} \varphi_{r}(\zeta - z) f_{j}^{\infty}(z) \, dz \right|^2 \frac{d\zeta dr}{r^{n+1}} \right)^{1/2}
\]$

\[
\leq C \left( \int_{2^{-2}r}^{\infty} \int_{|x - \xi| < r} \left| \sum_{l=1}^{\infty} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} |f_{j}(z)| \, dz \right|^2 \frac{d\xi dr}{r^{n+1}} \right)^{1/2}
\]

\[
\leq C \sum_{l=1}^{\infty} \left( \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} |f_{j}(z)| \, dz \right) \left( \int_{2^{-2}r}^{\infty} \frac{dr}{r^{2n+1}} \right)^{1/2}
\]

\[
\leq C \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} |f_{j}(z)| \, dz.
\]

Then by duality and Cauchy–Schwarz inequality, we get the following pointwise estimate for any \( x \in B(y, r) \).

\[
\left( \sum_{j=1}^{\infty} \left| S_{\gamma}(f_{j}^{\infty})(x) \right|^2 \right)^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^{\infty} \left( \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} |f_{j}(z)| \, dz \right)^2 \right)^{1/2}
\]

\[
\leq C \sup_{\sum_{j=1}^{\infty} |\zeta_j|^2 \leq 1} \left( \sum_{j=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} |f_{j}(z)| \, dz \cdot |\zeta_j| \right)
\]

\[
\leq C \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} \sup_{\sum_{j=1}^{\infty} |\zeta_j|^2 \leq 1} \left( \sum_{j=1}^{\infty} |f_{j}(z)| \cdot |\zeta_j| \right) \, dz
\]

\[
\leq C \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} \left( \sum_{j=1}^{\infty} |f_{j}(z)|^2 \right)^{1/2} \, dz.
\]

From this pointwise estimate, it follows that

\[
I_2(y, r) \leq C \cdot w(B(y, r))^{1/2 - 1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r) \setminus B(y,2^l r)} \left( \sum_{j=1}^{\infty} |f_{j}(z)|^2 \right)^{1/2} \, dz.
\]

Applying Hölder’s inequality and \( A_p \) condition on \( w \), we get
\[
\frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \\
\leq \frac{1}{|B(y, 2^{l+1}r)|} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \\
\left( \int_{B(y, 2^{l+1}r)} w(z)^{-p'/p} \, dz \right)^{1/p'} \\
\leq C \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \cdot w(B(y, 2^{l+1}r))^{-1/p}.
\]

Hence,
\[
I_2(y, r) \leq C \cdot w(B(y, r))^{1/\alpha - 1/q} \\
\times \sum_{l=1}^{\infty} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \cdot w(B(y, 2^{l+1}r))^{-1/p} \\
= C \sum_{l=1}^{\infty} w(B(y, 2^{l+1}r))^{1/\alpha - 1/p - 1/q} \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \mathcal{X}_{B(y, 2^{l+1}r)} \right\|_{L^p_\mu} \\
\times \frac{w(B(y, r))^{1/\alpha - 1/q}}{w(B(y, 2^{l+1}r))^{1/\alpha - 1/q}}.
\]

(20)

Notice that \( w \in A_q \subset A_\infty \) for \( 1 \leq p < \infty \), then by using inequality (9) with exponent \( \delta > 0 \), we can see that
\[
\sum_{l=1}^{\infty} \frac{w(B(y, r))^{1/\alpha - 1/q}}{w(B(y, 2^{l+1}r))^{1/\alpha - 1/q}} \leq C \sum_{l=1}^{\infty} \left( \frac{|B(y, r)|}{|B(y, 2^{l+1}r)|} \right)^{\delta(1/\alpha - 1/q)} \\
= C \sum_{l=1}^{\infty} \left( \frac{1}{2(l+1)^n} \right)^{\delta(1/\alpha - 1/q)} \leq C.
\]

(21)

Here the exponent \( \delta(1/\alpha - 1/q) \) is positive by the assumption \( \alpha < q \), which guarantees that the last series is convergent. Moreover, it should be noticed that (21) holds independent of \( y \in \mathbb{R}^n \). Therefore by taking the \( L^q_\mu \)-norm of both sides of (13)(with respect to the variable \( y \)), and then using Minkowski’s inequality, (16) and (20), we obtain
\[ \left\| w(B(y, r))^{1/2-1/p-1/q} \left( \sum_{j=1}^{\infty} |S_j(f_j)|^2 \right)^{1/2} \cdot \mathcal{Z}_{B(y,r)} \right\|_{L_w^p} \leq \left\| I_1(y, r) \right\|_{L_w^q} + \left\| I_2(y, r) \right\|_{L_w^q} \]

\[ \leq C \left\| w(B(y, 2r))^{1/2-1/p-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \mathcal{Z}_{B(y,2r)} \right\|_{L_w^p} + C \sum_{l=1}^{\infty} \left\| w(B(y, 2^{l+1}r))^{1/2-1/p-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \mathcal{Z}_{B(y,2^{l+1}r)} \right\|_{L_w^p} \times \frac{w(B(y, r))^{1/2-1/p}}{w(B(y, 2^{l+1}r))^{1/2-1/q}} \right\|_{L_w^q}. \]

Furthermore, in view of (21), the above expression is dominated by

\[ C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p,L^q)^\omega(w;\mu)} + C \sum_{l=1}^{\infty} \left( \frac{1}{2^{(l+1)n}} \right)^{\delta(1/2-1/q)} \left\| w(B(y, 2^{l+1}r))^{1/2-1/p-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \mathcal{Z}_{B(y,2^{l+1}r)} \right\|_{L_w^p} \times \sum_{l=1}^{\infty} \left( \frac{1}{2^{(l+1)n}} \right)^{\delta(1/2-1/q)} \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p,L^q)^\omega(w;\mu)}. \]

Thus, by taking the supremum over all \( r > 0 \), we complete the proof of Theorem 2.2.

**Proof of Theorem 2.3** Let \( p = 1, \ 1 \leq q < \infty \) and \( \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \in (L^1,L^q)^\omega(w;\mu) \) with \( w \in A_1 \) and \( \mu \in \Delta_2 \). For an arbitrary ball \( B = B(y, r) \subset \mathbb{R}^n \) with \( y \in \mathbb{R}^n \) and \( r > 0 \), we represent \( f_j \) as

\[ f_j = f_j \cdot \chi_{2B} + f_j \cdot \chi_{(2B)^c} := f_j^0 + f_j^\infty, \ j = 1, 2, \ldots; \]

then one can write
\[ w(B(y, r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| S_j(f_j) \right|^2 \right)^{1/2} \cdot \chi_{B(y,r)} \right\|_{WL_1} \leq 2 \cdot w(B(y, r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| S_j(f_j^0) \right|^2 \right)^{1/2} \cdot \chi_{B(y,r)} \right\|_{WL_1} \]
\[ + 2 \cdot w(B(y, r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| S_j(f_j^\infty) \right|^2 \right)^{1/2} \cdot \chi_{B(y,r)} \right\|_{WL_1} \]
\[ := I'_1(y, r) + I'_2(y, r). \]

Let us first consider the term \( I'_1(y, r) \). By the weighted weak \((1, 1)\) boundedness of vector-valued intrinsic square function (see Theorem 1.3), we get
\[ I'_1(y, r) \leq 2 \cdot w(B(y, r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| S_j(f_j^0) \right|^2 \right)^{1/2} \cdot \chi_{B(y,2r)} \right\|_{L_1} \]
\[ \leq C \cdot w(B(y, r))^{1/2 - 1/q} \left( \int_{B(y,2r)} \left( \sum_{j=1}^{\infty} \left| f_j(x) \right|^2 \right)^{1/2} w(x) \, dx \right) \]
\[ = C \cdot w(B(y, 2r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \cdot \chi_{B(y,2r)} \right\|_{L_1} \]
\[ \times \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2r))^{1/2 - 1/q}}. \]

Moreover, since \( 1/\alpha - 1 - 1/q < 0 \) and \( w \in A_1 \), then we apply doubling inequality (7) to obtain that
\[ \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2r))^{1/2 - 1/q}} \leq C. \]

Substituting the above inequality (24) into (23), we thus obtain
\[ I'_1(y, r) \leq C \cdot w(B(y, 2r))^{1/2 - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \cdot \chi_{B(y,2r)} \right\|_{L_1}. \]

As for the second term \( I'_2(y, r) \), it follows directly from Chebyshev’s inequality and the pointwise estimate (19) that
\[ I'_2(y, r) \leq 2 \cdot w(B(y, r))^{1/q-1/q} \int_{B(y, r)} \left( \sum_{j=1}^{\infty} \left| S_{j, i} (f_j^\infty)(x) \right| \right)^2 \frac{1}{w(x)} \, dx \]

\[ \leq C \cdot w(B(y, r))^{1/q-1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} \, dz. \]

Another application of \( A_1 \) condition on \( w \) leads to that

\[ \frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} \, dz \]

\[ \leq C \frac{1}{w(B(y, 2^{l+1}r))} \cdot \text{ess inf} \int_{z \in B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} \, dz \]

\[ \leq C \frac{1}{w(B(y, 2^{l+1}r))} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} w(z) \, dz \right). \]

Consequently,

\[ I'_2(y, r) \leq C \cdot w(B(y, r))^{1/q-1/q} \]

\[ \times \sum_{l=1}^{\infty} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} w(z) \, dz \right) \cdot w(B(y, 2^{l+1}r))^{-1} \]

\[ = C \sum_{l=1}^{\infty} w(B(y, 2^{l+1}r))^{1/q-1/q} \left\| \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \right\|_{L^1_w} \left\| z_{B(y, 2^{l+1}r)} \right\|_{L^1_w} \]

\[ \times \frac{w(B(y, r))^{1/q-1/q}}{w(B(y, 2^{l+1}r))^{1/q-1/q}}. \]

Therefore by taking the \( L^p_w \)-norm of both sides of (22) (with respect to the variable \( y \)), and then using Minkowski’s inequality, (25) and (26), we compute
\[ \left\| w(B(y, r))^{1/x-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot Z_{B(y, r)} \right\|_{L_w^1} \leq \left\| I_1'(y, r) \right\|_{L^q_w} + \left\| I_2'(y, r) \right\|_{L^q_w} \]

\[ \leq C \left\| w(B(y, 2r))^{1/x-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot Z_{B(y, 2r)} \right\|_{L_w^1} + C \sum_{l=1}^{\infty} \left\| w(B(y, 2^{l+1}r))^{1/x-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot Z_{B(y, 2^{l+1}r)} \right\|_{L_w^1} \]

\[ \times \frac{w(B(y, r))^{1/x-1/q}}{w(B(y, 2^{l+1}r))^{1/x-1/q}} \left\| L_w^{1/2} \right\| \]

Moreover, in view of the estimate (21), the above expression is bounded by

\[ C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^1, L^q)^2(w; \mu)} + C \sum_{l=1}^{\infty} \left( \frac{1}{2^{(l+1)n}} \right)^{\delta(1/x-1/q)} \]

\[ \left\| w(B(y, 2^{l+1}r))^{1/x-1/q} \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot Z_{B(y, 2^{l+1}r)} \right\|_{L_w^1} \left\| L_w^{1/2} \right\| \]

\[ \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^1, L^q)^2(w; \mu)} + C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^1, L^q)^2(w; \mu)} \]

\[ \times \sum_{l=1}^{\infty} \left( \frac{1}{2^{(l+1)n}} \right)^{\delta(1/x-1/q)} \]

\[ \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^1, L^q)^2(w; \mu)} \]

We end the proof by taking the supremum over all \( r > 0 \). \qed

### 4 Proof of Theorem 2.4

Given a real-valued function \( b \in \text{BMO}(\mathbb{R}^n) \), we will follow the idea developed in [1, 2] and denote \( F(\xi) = e^{i\xi[b(x)-b(c)]} \), \( \xi \in \mathbb{C} \). By the analyticity of \( F(\xi) \) on \( \mathbb{C} \) and the Cauchy integral formula, we first compute.
\[ b(x) - b(z) = F'(0) = \frac{1}{2\pi} \int_{|\xi|=1} \frac{F(\xi)}{\xi^2} \, d\xi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi |b(x) - b(z)|} \cdot e^{-i\theta} \, d\theta. \]

Thus, for any \( \varphi \in C_\gamma, 0 < \gamma \leq 1 \) and \( j \in \mathbb{Z}^+ \), we obtain

\[
\left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_j(y - z) f_j(z) \, dz \right| 
= \left| \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\mathbb{R}^n} \varphi_j(y - z) e^{-i\phi b(z)} f_j(z) \, dz \right) e^{i\phi b(x)} \cdot e^{-i\theta} \, d\theta \right| 
\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{\varphi \in C_\gamma} \left| \int_{\mathbb{R}^n} \varphi_j(y - z) e^{-i\phi b(z)} f_j(z) \, dz \right| e^{i\cos \theta \cdot b(x)} \, d\theta 
\leq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}_j(e^{-i\phi b} \cdot f_j)(y, t) \cdot e^{i\cos \theta \cdot b(x)} \, d\theta.
\]

From this, it follows that

\[
\left| [b, S_j](f_j)(x) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{S}_j(e^{-i\phi b} \cdot f_j)(x) \cdot e^{i\cos \theta \cdot b(x)} \, d\theta.
\]

Moreover, by using standard duality argument and Cauchy–Schwarz inequality, we compute

\[
\left( \sum_{j=1}^{\infty} \left| [b, S_j](f_j)(x) \right|^2 \right)^{1/2} 
\leq \frac{1}{2\pi} \left( \sum_{j=1}^{\infty} \left| \int_0^{2\pi} \mathcal{S}_j(e^{-i\phi b} \cdot f_j)(x) \cdot e^{i\cos \theta \cdot b(x)} \, d\theta \right|^2 \right)^{1/2} 
= \frac{1}{2\pi} \left( \sup_{\sum_{j=1}^{\infty} |\xi_j|^2 \leq 1} \left| \sum_{j=1}^{\infty} \left( \int_0^{2\pi} \mathcal{S}_j(e^{-i\phi b} \cdot f_j)(x) \cdot e^{i\cos \theta \cdot b(x)} \, d\theta \cdot \xi_j \right) \right| \right)^{1/2} 
\leq \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{\infty} \mathcal{S}_j(e^{-i\phi b} \cdot f_j)(x)^2 \cdot e^{i\cos \theta \cdot b(x)} \, d\theta \right)^{1/2}. \quad (27)
\]

Therefore, applying the \( L^p \)-boundedness of vector-valued intrinsic square function (see Theorem 1.1), and the same method as proving Theorem 1 in [2], we can also
show the following result (for the convenience of the reader, we give some details in Appendix).

**Theorem 4.1** Let $0 < \gamma \leq 1$, $1 < p < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ independent of $f = (f_1, f_2, \ldots)$ and $b$ such that

$$
\left\| \left( \sum_{j=1}^{\infty} |[b, S_j](f_j)|^2 \right)^{1/2} \right\|_{L^p_w} \leq C \|b\|_* \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p_w}
$$

holds for all $f = (f_1, f_2, \ldots) \in L^p_w(\mathbb{R}^n)$, provided that $b \in \text{BMO}(\mathbb{R}^n)$.

To prove our main theorem in this section, we also need the following lemma about BMO($\mathbb{R}^n$) functions.

**Lemma 4.2** Let $b$ be a function in BMO($\mathbb{R}^n$). Then

(i) For every ball $B$ in $\mathbb{R}^n$ and for all $l \in \mathbb{Z}^+$,

$$
|b_{2^{l+1}B} - b_B| \leq C \cdot (l + 1) \|b\|_*.
$$

(ii) For every ball $B$ in $\mathbb{R}^n$ and for all $w \in A_p$ with $1 \leq p < \infty$,

$$
\left( \int_B |b(x) - b_B|^p w(x) \, dx \right)^{1/p} \leq C \|b\|_* \cdot w(B)^{1/p}.
$$

**Proof** For the proof of (i), we refer the reader to [19]. For the proof of (ii), we refer the reader to [20].

**Proof of Theorem 2.4** Let $1 < p \leq \alpha < q \leq \infty$ and $(\sum_{j=1}^{\infty} |f_j|^2)^{1/2} \in (L^p, L^q)^2(w; \mu)$ with $w \in A_p$ and $\mu \in \Delta_2$. For each fixed ball $B = B(y, r) \subset \mathbb{R}^n$, as before, we represent $f_j$ as $f_j = f_j^0 + f_j^{\infty}$, where $f_j^0 = f_j \cdot \chi_{2B}$ and $2B = B(y, 2r) \subset \mathbb{R}^n$, $j = 1, 2, \ldots$. Then we write
\[
\begin{align*}
    w(B(y, r))^{1/2 - 1/p - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (f_j) \right|^2 \right)^{1/2} \cdot \chi_{B(y, r)} \right\|_{L^p_w}
    &= w(B(y, r))^{1/2 - 1/p - 1/q} \left( \int_{B(y, r)} \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (f_j) \right|^2 \right)^{p/2} w(x) \, dx \right)^{1/p}
    \leq w(B(y, r))^{1/2 - 1/p - 1/q} \left( \int_{B(y, r)} \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (f_j^0) (x) \right|^2 \right)^{p/2} w(x) \, dx \right)^{1/p}
    + w(B(y, r))^{1/2 - 1/p - 1/q} \left( \int_{B(y, r)} \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (f_j^\infty) (x) \right|^2 \right)^{p/2} w(x) \, dx \right)^{1/p}
    := J_1(y, r) + J_2(y, r).
\end{align*}
\]

By using Theorem 4.1 and the inequality (15), we obtain

\[
\begin{align*}
    J_1(y, r) &\leq w(B(y, r))^{1/2 - 1/p - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (f_j^0) \right|^2 \right)^{1/2} \right\|_{L^p_w}
    \leq C ||b||_* \cdot w(B(y, r))^{1/2 - 1/p - 1/q} \left( \int_{B(y, 2r)} \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{p/2} w(x) \, dx \right)^{1/p}
    = C ||b||_* \cdot w(B(y, 2r))^{1/2 - 1/p - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \cdot \chi_{B(y, 2r)} \right\|_{L^p_w}
    \times \frac{w(B(y, r))^{1/2 - 1/p - 1/q}}{w(B(y, 2r))^{1/2 - 1/p - 1/q}}
    \leq C ||b||_* \cdot w(B(y, 2r))^{1/2 - 1/p - 1/q} \left\| \left( \sum_{j=1}^{\infty} \left| f_j \right|^2 \right)^{1/2} \cdot \chi_{B(y, 2r)} \right\|_{L^p_w}.
\end{align*}
\]

Let us now turn to the estimate of \( J_2(y, r) \). For any given \( \varphi \in C_\gamma \), \( 0 < \gamma \leq 1 \), \( j = 1, 2, \ldots \), and \( (\zeta, t) \in \Gamma(x) \) with \( x \in B(y, r) \), we have
\[
\sup_{\varphi \in C_{\gamma}} \left| \int_{\mathbb{R}^{n}} [b(x) - b(z)] \varphi_{t}(\xi - z)f_{j}^{\infty}(z) \, dz \right|
\]
\[
\leq |b(x) - b_{B(y,r)}| \cdot \sup_{\varphi \in C_{\gamma}} \left| \int_{\mathbb{R}^{n}} \varphi_{t}(\xi - z)f_{j}^{\infty}(z) \, dz \right|
\]
\[
+ \sup_{\varphi \in C_{\gamma}} \left| \int_{\mathbb{R}^{n}} [b_{B(y,r)} - b(z)] \varphi_{t}(\xi - z)f_{j}^{\infty}(z) \, dz \right|.
\]

By definition, we can see that
\[
||[b, S_{\gamma}](f_{j}^{\infty})(x)|| \leq |b(x) - b_{B(y,r)}| \cdot S_{\gamma}(f_{j}^{\infty})(x) + S_{\gamma}[|b_{B(y,r)} - b|f_{j}^{\infty}](x).
\]

From this and Minkowski’ inequality for series, we further obtain that for any \( x \in B(y, r) \),
\[
\left( \sum_{j=1}^{\infty} \left| [b, S_{\gamma}](f_{j}^{\infty})(x) \right|^{2} \right)^{1/2} \leq |b(x) - b_{B(y,r)}| \left( \sum_{j=1}^{\infty} \left| S_{\gamma}(f_{j}^{\infty})(x) \right|^{2} \right)^{1/2}
\]
\[
+ \left( \sum_{j=1}^{\infty} \left| S_{\gamma}[|b_{B(y,r)} - b|f_{j}^{\infty}](x) \right|^{2} \right)^{1/2}.
\]

Fix \( x \in B(y, r) \), the following estimate is known from (19):
\[
\left( \sum_{j=1}^{\infty} \left| S_{\gamma}(f_{j}^{\infty})(x) \right|^{2} \right)^{1/2} \leq C \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} \left( \sum_{j=1}^{\infty} \left| f_{j}(z) \right|^{2} \right)^{1/2} \, dz. \quad (30)
\]

Using the same procedure as in the proof of (19), for any \( \varphi \in C_{\gamma}, \ 0 < \gamma \leq 1, \ j = 1, 2, \ldots, \) and \( (\xi, t) \in \Gamma(x) \) with \( x \in B(y, r) \), we can also show that
\[
\left\| \int_{\mathbb{R}^{n}} [b_{B(y,r)} - b(z)] \varphi_{t}(\xi - z)f_{j}^{\infty}(z) \, dz \right\|
\]
\[
= \left\| \int_{B(y, 2r)} [b_{B(y,r)} - b(z)] \varphi_{t}(\xi - z)f_{j}(z) \, dz \right\|
\]
\[
\leq C \cdot \Gamma^{n} \int_{B(y, 2r)^{c} \cap \{z: |\xi - z| \leq t\}} \left| b(z) - b_{B(y,r)} \right| \left| f_{j}(z) \right| \, dz
\]
\[
\leq C \cdot \Gamma^{n} \sum_{l=1}^{\infty} \int_{[B(y, 2^{j+1}r), B(y, 2^{j}r)] \cap \{z: |\xi - z| \leq t\}} \left| b(z) - b_{B(y,r)} \right| \left| f_{j}(z) \right| \, dz.
\]

Hence, for any \( x \in B(y, r), \) by using the inequalities (31) and (18) together with Minkowski’s inequality for integrals, we can deduce
Therefore, by duality and Cauchy–Schwarz inequality, we get

\[
S_j \left( [b_{B(y,r)} - b]f]_j^\infty \right) (x) = \left( \int \int \sup_{\Gamma(z)} \left| \int \Phi(\xi - z) f_\xi^j \, d\xi \right|^2 \, d\xi \, dr \right)^{1/2}
\]

\[
\leq C \left( \int_{2^{j+1} - r}^{\infty} \int_{|x - \xi| < r} \left| \sum_{l=1}^{C_20} \int_{Y_3/2^j}^{r} \left( b(z) - b_{B(y,r)}(z) \right) \leq C \sum_{l=1}^{C_20} \left( b(z) - b_{B(y,r)}(z) \right) \right| \, dz \right)^{1/2}
\]

\[
\leq C \sum_{l=1}^{C_20} \left( b(z) - b_{B(y,r)}(z) \right) \right| \, dz \right)^{1/2}
\]

\[
\leq C \sum_{l=1}^{C_20} \left( b(z) - b_{B(y,r)}(z) \right) \right| \, dz \right)^{1/2}
\]

Therefore, by duality and Cauchy–Schwarz inequality, we get

\[
\left( \sum_{j=1}^{\infty} \left| S_j \left( [b_{B(y,r)} - b]f]_j^\infty \right) (x) \right|^2 \right)^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^{\infty} \left| \sum_{l=1}^{C_20} \left( b(z) - b_{B(y,r)}(z) \right) \right| \, dz \right)^{1/2}
\]

\[
\leq C \sum_{j=1}^{\infty} \left| \left( b(z) - b_{B(y,r)}(z) \right) \right| \, dz \right)^{1/2}
\]

Consequently, from the above two pointwise estimates (30) and (32), it follows that
\[ J_2(y, r) \leq C \cdot w(B(y, r))^{1/2 - 1/p - 1/q} \left( \int_{B(y, r)} |b(x) - b_{B(y, r)}|^p w(x) \, dx \right)^{1/p} \]

\[
\times \left( \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1} r)|} \int_{B(y, 2^{l+1} r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \right)
\]

\[
+ C \cdot w(B(y, r))^{1/2 - 1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1} r)|} \int_{B(y, 2^{l+1} r)} |b_{B(y, 2^{l+1} r)} - b_{B(y, r)}| \cdot \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz
\]

\[
+ C \cdot w(B(y, r))^{1/2 - 1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1} r)|} \int_{B(y, 2^{l+1} r)} |b(z) - b_{B(y, 2^{l+1} r)}| \cdot \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz
\]

\[
:= J_3(y, r) + J_4(y, r) + J_5(y, r).
\]

Below we will give the estimates of \( J_3(y, r) \), \( J_4(y, r) \) and \( J_5(y, r) \), respectively. Using (ii) of Lemma 4.2, Hölder’s inequality and the \( A_p \) condition on \( w \), we obtain

\[
J_3(y, r) \leq C||b||_* \cdot w(B(y, r))^{1/2 - 1/q} \]

\[
\times \sum_{l=1}^{\infty} \left( \frac{1}{|B(y, 2^{l+1} r)|} \int_{B(y, 2^{l+1} r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \right)
\]

\[
\leq C||b||_* \cdot w(B(y, r))^{1/2 - 1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1} r)|} \left( \int_{B(y, 2^{l+1} r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \times \left( \int_{B(y, 2^{l+1} r)} w(z)^{-p'/p} \, dz \right)^{1/p'}
\]

\[
\leq C||b||_* \cdot w(B(y, r))^{1/2 - 1/q} \sum_{l=1}^{\infty} \left( \int_{B(y, 2^{l+1} r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \cdot w(B(y, 2^{l+1} r))^{-1/p}.
\]

On the other hand, applying (i) of Lemma 4.2, Hölder’s inequality and the \( A_p \) condition on \( w \), we can deduce that
\[ J_4(y, r) \leq C \|b\|_* \cdot w(B(y, r))^{1/q} \times \sum_{l=1}^{\infty} \frac{(l + 1)}{|B(y, 2^{l+1}r)|} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \]

\[ \leq C \|b\|_* \cdot w(B(y, r))^{1/q} \sum_{l=1}^{\infty} \frac{(l + 1)}{|B(y, 2^{l+1}r)|} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \times \left( \int_{B(y, 2^{l+1}r)} w(z)^{-p'/p} \, dz \right)^{1/p'} \]

\[ \leq C \|b\|_* \cdot w(B(y, r))^{1/q} \times \sum_{l=1}^{\infty} (l + 1) \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \times w(B(y, 2^{l+1}r))^{-1/p}. \]

It remains to estimate the last term \( J_5(y, r) \). An application of Hölder’s inequality gives us that

\[ J_5(y, r) \leq C \cdot w(B(y, r))^{1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1}r)|} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \]

\[ \times \left( \int_{B(y, 2^{l+1}r)} |b(z) - b_{B(y, 2^{l+1}r)}|^{p'} w(z)^{-p'/p} \, dz \right)^{1/p'}. \]

If we denote \( v(z) = w(z)^{-p'/p} \), then we know that the weight \( v(z) \) belongs to \( A_{p'} \) whenever \( w \in A_p \) (see [3, 9]). From this fact together with \((ii)\) of Lemma 4.2 and the \( A_p \) condition, it follows that

\[ \left( \int_{B(y, 2^{l+1}r)} |b(z) - b_{B(y, 2^{l+1}r)}|^{p'} v(z) \, dz \right)^{1/p'} \]

\[ \leq C \|b\|_* \cdot v(B(y, 2^{l+1}r))^{1/p'} \]

\[ = C \|b\|_* \cdot \left( \int_{B(y, 2^{l+1}r)} w(z)^{-p'/p} \, dz \right)^{1/p'} \]

\[ \leq C \|b\|_* \cdot \frac{|B(y, 2^{l+1}r)|}{w(B(y, 2^{l+1}r))^{1/p}}. \]

Therefore, in view of (33), we can see that

\[ J_5(y, r) \leq C \|b\|_* \cdot w(B(y, r))^{1/q} \]

\[ \times \sum_{l=1}^{\infty} \left( \int_{B(y, 2^{l+1}r)} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{p/2} w(z) \, dz \right)^{1/p} \cdot w(B(y, 2^{l+1}r))^{-1/p}. \]

Summarizing the above discussions, we conclude that
Moreover, by using the estimate (35), we have that the last expression is bounded by

\[ J_2(y, r) \leq C\|b\|_s \cdot w(B(y, r))^{1/2 - 1/q} \]

\[
= C\|b\|_s \sum_{l=1}^{\infty} (1 + 1) \cdot \left( \int_{B(y, 2^{l+1}r)} \left| \sum_{j=1}^{\infty} |f_j(z)|^2 \right| w(z) \, dz \right)^{1/p} \cdot w(B(y, 2^{l+1}r))^{-1/p} \\
\times (l + 1) \cdot \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2^{l+1}r))^{1/2 - 1/q}}.
\]

(34)

Notice that when \( w \in A_p \) with \( 1 \leq p < \infty \), one has \( w \in A_{\infty} \). Thus, by using inequality (9) with exponent \( \delta^* > 0 \) together with our assumption that \( \alpha < q \), we obtain

\[
\sum_{l=1}^{\infty} (1 + 1) \cdot \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2^{l+1}r))^{1/2 - 1/q}} \leq C \sum_{l=1}^{\infty} (1 + 1) \cdot \left( \frac{|B(y, r)|}{|B(y, 2^{l+1}r)|} \right)^{\delta^*(1/2 - 1/q)} \\
= C \sum_{l=1}^{\infty} (1 + 1) \cdot \left( \frac{1}{2^{(l+1)q}} \right)^{\delta^*(1/2 - 1/q)} \\
\leq C,
\]

(35)

where the last series is convergent since the exponent \( \delta^*(1/2 - 1/q) \) is positive. Therefore by taking the \( L^p_B \)-norm of both sides of (28)(with respect to the variable \( y \)), and then using Minkowski’s inequality, (29) and (34), we can get

\[
\left\| w(B(y, r))^{1/2 - 1/p - 1/q} \cdot \left( \sum_{j=1}^{\infty} |[b, S_{y_j}](f_j)|^2 \right)^{1/2} \cdot \lambda_{B(y, r)} \right\|_{L^p_B} \\
\leq \| J_1(y, r) \|_{L^p_B} + \| J_2(y, r) \|_{L^p_B} \\
\leq C\|b\|_s \left\| w(B(y, 2r))^{1/2 - 1/p - 1/q} \cdot \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \lambda_{B(y, 2r)} \right\|_{L^p_B} \\
\leq C\|b\|_s \sum_{l=1}^{\infty} \left\| w(B(y, 2^{l+1}r))^{1/2 - 1/p - 1/q} \cdot \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \cdot \lambda_{B(y, 2^{l+1}r)} \right\|_{L^p_B} \\
\times (l + 1) \cdot \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2^{l+1}r))^{1/2 - 1/q}}.
\]

Moreover, by using the estimate (35), we have that the last expression is bounded by
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\[ C\|b\|_s \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^3(w; \mu)} + C\|b\|_s \sum_{l=1}^{\infty} (l + 1) \cdot \left( \frac{1}{2(l+1)^n} \right)^{\delta'(1/\alpha - 1/q)} \]

\[ \left\| w(B(y, 2^{l+1}r))^{1/\alpha - 1/p - 1/q} \right\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \left\| (L^p, L^q)^3(w; \mu) \right\|_{L^p_v} \]

\[ \leq C\|b\|_s \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^3(w; \mu)} + C\|b\|_s \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^3(w; \mu)} \]

\[ \times \sum_{l=1}^{\infty} (l + 1) \cdot \left( \frac{1}{2(l+1)^n} \right)^{\delta'(1/\alpha - 1/q)} \]

\[ \leq C\|b\|_s \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{(L^p, L^q)^3(w; \mu)}. \]

Thus, by taking the supremum over all \( r > 0 \), we complete the proof of Theorem 2.4.

\[ \Box \]

5 Proof of Theorem 2.6

To show Theorem 2.6, we first give the following endpoint estimate for vector-valued commutator (4) in the weighted Lebesgue space \( L^1_w(\mathbb{R}^n) \). This result was already proved in [22], but we shall repeat the argument here for completeness.

**Theorem 5.1** Let \( 0 < \gamma \leq 1 \), \( w \in A_1 \) and \( b \in \text{BMO}(\mathbb{R}^n) \). Then for any given \( \sigma > 0 \), there exists a constant \( C > 0 \) independent of \( f = (f_1, f_2, \ldots) \) and \( \sigma > 0 \) such that

\[ w \left( \left\{ x \in \mathbb{R}^n : \left( \sum_{j=1}^{\infty} |[b, S^\gamma_j](f_j)(x)|^2 \right)^{1/2} > \sigma \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\|f(x)\|_{L^2}}{\sigma} \right) \cdot w(x) \, dx, \]

(36)

where \( \Phi(t) = t \cdot (1 + \log^+ t) \) and \( \|f(x)\|_{L^2} = \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2}. \)

**Proof** Assume that \( \|f(x)\|_{L^2} \) is locally integrable. Inspired by the works in [14, 15, 25], for any fixed \( \sigma > 0 \), we apply the Calderón–Zygmund decomposition of \( f = (f_1, f_2, \ldots) \) at height \( \sigma \) to obtain a collection of disjoint non-overlapping dyadic cubes \( \{Q_i\} \) such that the following property holds (see [15, 18])

\[ \sigma < \frac{1}{|Q_i|} \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \, dy \leq 2^n \cdot \sigma, \]

(37)

where \( Q_i = Q(c_i, \ell_i) \) denotes the cube centered at \( c_i \) with side length \( \ell_i \) and all cubes...
are assumed to have their sides parallel to the coordinate axes. If we set \( E = \bigcup_i Q_i \), then
\[
\left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \leq \sigma, \quad \text{a.e. } y \in \mathbb{R}^n \setminus E.
\]

Now we proceed to construct vector-valued version of the Calderón–Zygmund decomposition. Define two vector-valued functions \( g = (g_1, g_2, \ldots) \) and \( h = (h_1, h_2, \ldots) \) as follows:

\[
g_j(x) := \begin{cases} f_j(x) & \text{if } x \in E^c, \\ \frac{1}{|Q_i|} \int_{Q_i} f_j(y) \, dy & \text{if } x \in Q_i, \end{cases}
\]

and

\[
h_j(x) := f_j(x) - g_j(x) = \sum_i h_{ij}(x), \quad j = 1, 2, \ldots,
\]

where \( h_{ij}(x) = h_j(x) \cdot \chi_{Q_i}(x) \). Then one has

\[
\left( \sum_{j=1}^{\infty} |g_j(x)|^2 \right)^{1/2} \leq C \cdot \sigma, \quad \text{a.e. } x \in \mathbb{R}^n,
\]

and

\[
f = g + h := (g_1 + h_1, g_2 + h_2, \ldots).
\]

Obviously, \( h_{ij} \) is supported on \( Q_i \), \( i, j = 1, 2, \ldots \),

\[
\int_{\mathbb{R}^n} h_{ij}(x) \, dx = 0, \quad \text{and} \quad \|h_{ij}\|_{L^1} = \int_{\mathbb{R}^n} |h_{ij}(x)| \, dx \leq 2 \int_{Q_i} |f_j(x)| \, dx
\]

according to the above decomposition. By (39) and Minkowski’s inequality,

\[
\left( \sum_{j=1}^{\infty} |[b, S_{y_j}](f_j(x))|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{\infty} |[b, S_{y_j}](g_j(x))|^2 \right)^{1/2} + \left( \sum_{j=1}^{\infty} |[b, S_{y_j}](h_j(x))|^2 \right)^{1/2}.
\]

Then we can write
\[
\begin{align*}
w\left( \left\{ x \in \mathbb{R}^n : \left( \sum_{j=1}^{\infty} \left| [b, S_j] (f_j)(x) \right|^2 \right)^{1/2} > \sigma \right\} \right) \\
\leq w\left( \left\{ x \in \mathbb{R}^n : \left( \sum_{j=1}^{\infty} \left| [b, S_j] (g_j)(x) \right|^2 \right)^{1/2} > \sigma/2 \right\} \right) \\
+ w\left( \left\{ x \in \mathbb{R}^n : \left( \sum_{j=1}^{\infty} \left| [b, S_j] (h_j)(x) \right|^2 \right)^{1/2} > \sigma/2 \right\} \right) \\
:= K_1 + K_2.
\end{align*}
\]

Observe that \( w \in A_1 \subset A_2 \). Applying Chebyshev’s inequality and Theorem 4.1, we obtain

\[
K_1 \leq \frac{4}{\sigma^2} \cdot \left\| \left( \sum_{j=1}^{\infty} \left| [b, S_j] (g_j) \right|^2 \right)^{1/2} \right\|_{L^2_w}^2 \\
\leq \frac{C}{\sigma^2} \cdot \left\| \left( \sum_{j=1}^{\infty} \left| g_j \right|^2 \right)^{1/2} \right\|_{L^2_w}^2.
\]

Moreover, in view of (38), one has

\[
\left\| \left( \sum_{j=1}^{\infty} \left| g_j \right|^2 \right)^{1/2} \right\|_{L^2_w}^2 \leq C \cdot \sigma \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \left| g_j(x) \right|^2 \right)^{1/2} w(x) \, dx \\
\leq C \cdot \sigma \left( \int_{E} \left( \sum_{j=1}^{\infty} \left| f_j(x) \right|^2 \right)^{1/2} w(x) \, dx + \int_{\bigcup Q} \left( \sum_{j=1}^{\infty} \left| g_j(x) \right|^2 \right)^{1/2} w(x) \, dx \right).
\]

Recall that \( g_j(x) = \frac{1}{|Q|} \int_{Q_j} f_j(y) \, dy \) when \( x \in Q_i \). As before, by using duality and Cauchy–Schwarz inequality, we can see the following estimate is valid for all \( x \in Q_i \).

\[
\left( \sum_{j=1}^{\infty} \left| g_j(x) \right|^2 \right)^{1/2} \leq \frac{1}{|Q_i|} \int_{Q_i} \left( \sum_{j=1}^{\infty} \left| f_j(y) \right|^2 \right)^{1/2} \, dy. \tag{40}
\]

This estimate (40) along with the \( A_1 \) condition yields
\[ \left\| \left( \sum_{j=1}^{\infty} |g_j|^2 \right)^{1/2} \right\|_{L^2_w}^2 \leq C \cdot \sigma \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} w(x) \, dx \right) \]

\[ + \sum_{i} \frac{w(Q_i)}{|Q_i|} \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \, dy \]

\[ \leq C \cdot \sigma \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} w(x) \, dx \right) + \sum_{i} \text{ess inf} w(y) \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \, dy \]

\[ \leq C \cdot \sigma \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} w(x) \, dx \right) + \int_{\bigcup Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} w(y) \, dy \]

\[ \leq C \cdot \sigma \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} w(x) \, dx. \]

(41)

So we have

\[ K_1 \leq C \int_{\mathbb{R}^n} \frac{\|f(x)\|_{L^2}}{\sigma} \cdot w(x) \, dx \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\|f(x)\|_{L^2}}{\sigma} \right) \cdot w(x) \, dx, \]

where the last inequality is due to \( t \leq \Phi(t) = t \cdot (1 + \log^+ t) \) for any \( t > 0 \). To deal with the other term \( K_2 \), let \( Q_i^* = 2\sqrt{n}Q_i \) be the cube concentric with \( Q_i \) such that \( \ell(Q_i^*) = (2\sqrt{n})\ell(Q_i) \). Then we can further decompose \( K_2 \) as follows.

\[ K_2 \leq w \left( \left\{ x \in \bigcup_i Q_i^* : \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (h_j)(x) \right|^2 \right)^{1/2} > \sigma/2 \right\} \right) \]

\[ + w \left( \left\{ x \not\in \bigcup_i Q_i^* : \left( \sum_{j=1}^{\infty} \left| [b, \mathcal{S}_j] (h_j)(x) \right|^2 \right)^{1/2} > \sigma/2 \right\} \right) \]

\[ := K_3 + K_4. \]

Since \( w \in A_1 \), then by the inequality (7), we can get

\[ K_3 \leq \sum_i w(Q_i^*) \leq C \sum_i w(Q_i). \]

Furthermore, it follows from the inequality (37) and the \( A_1 \) condition that
\[ K_3 \leq C \sum_i \frac{1}{\sigma} \cdot \text{ess inf}_{y \in Q_i} w(y) \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} dy \]
\[
\leq \frac{C}{\sigma} \sum_i \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} w(y) dy \leq \frac{C}{\sigma} \int_{\bigcup_j Q_j} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} w(y) dy 
\]
\[
\leq C \int_{\mathbb{R}^n} \frac{\|f(y)\|_\ell}{\sigma} \cdot w(y) dy \leq C \int_{\mathbb{R}^n} \Phi\left( \frac{\|f(y)\|_\ell}{\sigma} \right) \cdot w(y) dy ,
\]

where the last inequality is also due to \( t \leq \Phi(t) \) for any \( t > 0 \). Arguing as in the proof of Theorem 2.4, for any given \( x \in \mathbb{R}^n \), \( (y, t) \in \Gamma(x) \) and for \( j = 1, 2, \ldots \), we also find that

\[
\sup_{\phi \in C_j} \left| \int_{\mathbb{R}^n} \left[ b(x) - b(z) \right] \phi(y - z) \sum_i h_{ij}(z) \ dz \right| 
\leq \sup_{\phi \in C_j} \left| \sum_i \left[ b(x) - b_{Q_i} \right] \int_{\mathbb{R}^n} \phi(y - z) h_{ij}(z) \ dz \right|
\]
\[\quad + \sup_{\phi \in C_j} \left| \int_{\mathbb{R}^n} \phi(y - z) \sum_i \left[ b_{Q_i} - b(z) \right] h_{ij}(z) \ dz \right|
\]
\[
\leq \sum_i \left| b(x) - b_{Q_i} \right| \cdot \sup_{\phi \in C_j} \left| \int_{\mathbb{R}^n} \phi(y - z) h_{ij}(z) \ dz \right|
\]
\[\quad + \sup_{\phi \in C_j} \left| \int_{\mathbb{R}^n} \phi(y - z) \sum_i \left[ b_{Q_i} - b(z) \right] h_{ij}(z) \ dz \right| .
\]

Hence, by definition, we have that for any given \( x \in \mathbb{R}^n \) and \( j \in \mathbb{Z}^+ \),

\[
\left| \left[ b, S_j \right](h_j)(x) \right| \leq \sum_i \left| b(x) - b_{Q_i} \right| \cdot S_j(h_{ij})(x) + S_j\left( \sum_i \left[ b_{Q_i} - b \right] h_{ij} \right)(x). \tag{42}
\]

On the other hand, by duality argument and Cauchy–Schwarz inequality, we can see the following vector-valued form of Minkowski’s inequality is true for any real numbers \( v_{ij} \in \mathbb{R}, i, j = 1, 2, \ldots \).

\[
\left( \sum_j \left| \sum_i v_{ij} \right|^2 \right)^{1/2} \leq \sum_i \left( \sum_j |v_{ij}|^2 \right)^{1/2} . \tag{43}
\]

In view of the estimates (42) and (43), we get
\[
\left( \sum_{j=1}^{\infty} |[b, S_j](h_{ij})(x)|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{\infty} \left| \sum_i |b(x) - b_{Q_i}| \cdot S_j(h_{ij})(x) \right|^2 \right)^{1/2} \\
+ \left( \sum_{j=1}^{\infty} S_j \left( \sum_i |b_{Q_i} - b| h_{ij}(x) \right)^2 \right)^{1/2} \\
\leq \sum_i |b(x) - b_{Q_i}| \cdot \left( \sum_{j=1}^{\infty} |S_j(h_{ij})(x)|^2 \right)^{1/2} \\
+ \left( \sum_{j=1}^{\infty} S_j \left( \sum_i |b_{Q_i} - b| h_{ij}(x) \right)^2 \right)^{1/2}.
\]

Therefore, the term $K_4$ can be divided into two parts as follows:

\[
K_4 \leq w \left( \left\{ x \notin \bigcup_i Q_i^c : \sum_i |b(x) - b_{Q_i}| \cdot \left( \sum_{j=1}^{\infty} |S_j(h_{ij})(x)|^2 \right) > \sigma/4 \right\} \right) + w \left( \left\{ x \notin \bigcup_i Q_i^c : \left( \sum_{j=1}^{\infty} S_j \left( \sum_i |b_{Q_i} - b| h_{ij}(x) \right)^2 \right) > \sigma/4 \right\} \right)
\]

\[
:= K_5 + K_6.
\]

It follows directly from the Chebyshev’s inequality that

\[
K_5 \leq \frac{4}{\sigma} \int_{R^n \setminus \bigcup_i Q_i} \sum_i |b(x) - b_{Q_i}| \cdot \left( \sum_{j=1}^{\infty} |S_j(h_{ij})(x)|^2 \right)^{1/2} w(x) \, dx \\
\leq \frac{4}{\sigma} \sum_i \left( \int_{\mathbb{R}^n} |b(x) - b_{Q_i}| \cdot \left( \sum_{j=1}^{\infty} |S_j(h_{ij})(x)|^2 \right)^{1/2} w(x) \, dx \right).
\]

Denote by $c_i$ the center of $Q_i$. For any $\varphi \in C_c$, $0 < \gamma \leq 1$, by the cancellation condition of $h_{ij}$ over $Q_i$, we obtain that for any $(y, t) \in \Gamma(x)$ and for $i, j = 1, 2, \ldots$,

\[
|\langle \varphi_i \ast h_{ij}(y) \rangle| = \left| \int_{Q_i} \left[ \varphi_i(y - z) - \varphi_i(y - c_i) \right] h_{ij}(z) \, dz \right| \\
\leq \int_{Q_i} \int_{\{z : |z - y| \leq t\}} \frac{|z - c_i|^\gamma}{t^{n+\gamma}} |h_{ij}(z)| \, dz \\
\leq C \cdot \frac{\ell(Q_i)^7}{t^{n+\gamma}} \int_{Q_i} \int_{\{z : |z - y| \leq t\}} |h_{ij}(z)| \, dz.
\]

In addition, for any $z \in Q_i$ and $x \in (Q_i)^c$, we have $|z - c_i| < \frac{|x - c_i|}{2}$. Thus, for all $(y, t) \in \Gamma(x)$ and $|z - y| \leq t$ with $z \in Q_i$, it is easy to see that
\[ t + t \geq |x - y| + |y - z| \geq |x - z| \geq |x - c_i| - |z - c_i| \geq \frac{|x - c_i|}{2}. \tag{45} \]

Hence, for any \( x \in (Q_i^+)^c \), by using the above inequalities (44) and (45) along with the fact that \( \|h_{ij}\|_{L^1} \leq 2 \int_{Q_i} |f_j(x)| \, dx \), we obtain that for any \( i, j = 1, 2, \ldots \),

\[
|S_j(h_{ij})(x)| = \left( \int_{\Gamma(x)} \left[ \sup_{\varphi \in C_0} |(\varphi_r * h_{ij})(y)| \right]^2 \, dy \, dr \right)^{1/2} \]

\[
\leq C \cdot \ell(Q_i)^7 \left( \int_{Q_i} |h_{ij}(z)| \, dz \right) \left( \int_{|x-c_i|^{n+1}} \int_{|y-x| < t} \frac{dy \, dr}{t^{2(n+\gamma)+n+1}} \right)^{1/2} \]

\[
\leq C \cdot \ell(Q_i)^7 \left( \int_{Q_i} |h_{ij}(z)| \, dz \right) \left( \int_{|x-c_i|^{n+1}} \frac{dr}{t^{2(n+\gamma)+1}} \right)^{1/2} \]

\[
\leq C \cdot \frac{\ell(Q_i)^7}{|x-c_i|^{n+\gamma}} \left( \int_{Q_i} |f_j(z)| \, dz \right). \]

Furthermore, by duality and Cauchy–Schwarz inequality again, one has

\[
\left( \sum_{j=1}^{\infty} |S_j(h_{ij})(x)|^2 \right)^{1/2} \leq C \cdot \frac{\ell(Q_i)^7}{|x-c_i|^{n+\gamma}} \times \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right) \, dz. \]

Since \( Q_i^+ = 2\sqrt{n}Q_i \supseteq 2Q_i \), then \( (Q_i^+)^c \subset (2Q_i)^c \). This fact together with the pointwise estimate derived above yields
\[
K_5 \leq \frac{C}{\sigma} \sum_i \left( \ell(Q_i)^\gamma \int_{Q_i} \left( \sum_{j=1}^\infty |f_j(z)|^2 \right)^{1/2} \, dz \times \int_{(Q_i)^c} |b(x) - b_{Q_i}| \cdot \frac{w(x)}{|x - c_i|^{n+\gamma}} \, dx \right)
\]
\[
\leq \frac{C}{\sigma} \sum_i \left( \ell(Q_i)^\gamma \int_{Q_i} \left( \sum_{j=1}^\infty |f_j(z)|^2 \right)^{1/2} \, dz \times \int_{(2Q_i)^c} |b(x) - b_{Q_i}| \cdot \frac{w(x)}{|x - c_i|^{n+\gamma}} \, dx \right)
\]
\[
\leq \frac{C}{\sigma} \sum_i \left( \ell(Q_i)^\gamma \int_{Q_i} \left( \sum_{j=1}^\infty |f_j(z)|^2 \right)^{1/2} \, dz \right.
\]
\[
\times \sum_{l=1}^\infty \int_{2^{l+1}Q_i \setminus 2^lQ_i} |b(x) - b_{2^{l+1}Q_i}| \cdot \frac{w(x)}{|x - c_i|^{n+\gamma}} \, dx 
\]
\[
+ \frac{C}{\sigma} \sum_i \left( \ell(Q_i)^\gamma \int_{Q_i} \left( \sum_{j=1}^\infty |f_j(z)|^2 \right)^{1/2} \, dz \right.
\]
\[
\times \sum_{l=1}^\infty \int_{2^{l+1}Q_i \setminus 2^lQ_i} |b_{2^{l+1}Q_i} - b_{Q_i}| \cdot \frac{w(x)}{|x - c_i|^{n+\gamma}} \, dx 
\]
\[
:= I + II.
\]

For the term I, it then follows from (ii) of Lemma 4.2 (consider \(2^{l+1}Q_i\) instead of \(B\), (8) and the assumption \(w \in A_1\) that
\[ I \leq \frac{C}{\sigma} \sum_i \left( \|\ell(Q_i)\|_1^\gamma \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \right) \times \sum_{l=1}^{\infty} \frac{1}{|2^{l-1}\ell(Q_i)|^{n+\gamma}} \int_{2^{l-1}Q_i} |b(x) - b_{2^{l-1}Q_i}| \cdot w(x) \, dx \]

\[ \leq \frac{C \cdot \|b\|_\sigma}{\sigma} \sum_i \left( \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \times \sum_{l=1}^{\infty} \frac{w(2^{l+1}Q_i)}{(2^{l-1})^{n+\gamma}|Q_i|} \right) \]

\[ \leq \frac{C \cdot \|b\|_\sigma}{\sigma} \sum_i \left( \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \times \sum_{l=1}^{\infty} \frac{1}{2^\gamma} \right) \]

\[ \leq \frac{C}{\sigma} \sum_i \text{ess inf}_{z \in Q_i} w(z) \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \leq \frac{C}{\sigma} \int_{\bigcup_i Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} w(z) \, dz \]

\[ \leq C \int_{\mathbb{R}^n} \frac{\|f(z)\|_\sigma}{\sigma} \cdot w(z) \, dz \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\|f(z)\|_\sigma}{\sigma} \right) \cdot w(z) \, dz. \]

For the term II, from (i) of Lemma 4.2 and (8) along with the assumption \( w \in A_1 \), it then follows that

\[ II \leq \frac{C \cdot \|b\|_\sigma}{\sigma} \sum_i \left( \|\ell(Q_i)\|_1^\gamma \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \times \sum_{l=1}^{\infty} (l + 1) \cdot \frac{w(2^{l+1}Q_i)}{(2^{l-1})^{n+\gamma}|Q_i|} \right) \]

\[ \leq \frac{C \cdot \|b\|_\sigma}{\sigma} \sum_i \left( \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \times \sum_{l=1}^{\infty} (l + 1) \cdot \frac{(2^{l+1})^n w(Q_i)}{(2^{l-1})^{n+\gamma}|Q_i|} \right) \]

\[ \leq \frac{C}{\sigma} \sum_i \left( \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz \times \sum_{l=1}^{\infty} \frac{l + 1}{2^\gamma} \right) \]

\[ \leq \frac{C}{\sigma} \sum_i \left( \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\|f(z)\|_\sigma}{\sigma} \right) \cdot w(z) \, dz. \]

On the other hand, by using the weighted weak-type \((1,1)\) estimate of vector-valued intrinsic square function (see Theorem 1.3) and (43), we have
\[
K_6 \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \left| \sum_i |b(x) - b_{Q_j}| \left| h_{ij}(x) \right| \right|^2 \right)^{1/2} \, w(x) \, dx \\
\leq \frac{C}{\sigma} \int_{\mathbb{R}^n} \sum_i |b(x) - b_{Q_i}| \left( \sum_{j=1}^{\infty} |h_j(x)|^2 \right)^{1/2} \, w(x) \, dx \\
= \frac{C}{\sigma} \sum_i \int_{Q_i} |b(x) - b_{Q_i}| \left( \sum_{j=1}^{\infty} |h_j(x)|^2 \right)^{1/2} \, w(x) \, dx \\
\leq \frac{C}{\sigma} \sum_i \int_{Q_i} |b(x) - b_{Q_i}| \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} \, w(x) \, dx \\
+ \frac{C}{\sigma} \sum_i \frac{1}{|Q_i|} \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \, dy \times \int_{Q_i} |b(x) - b_{Q_i}| \, w(x) \, dx \\
:= \text{III}+\text{IV}.
\]

For the term III, by the generalized Hölder’s inequality (11), we can deduce that

\[
\text{III} = \frac{C}{\sigma} \sum w(Q_i) \cdot \frac{1}{w(Q_i)} \int_{Q_i} |b(x) - b_{Q_i}| \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2} \, w(x) \, dx \\
\leq \frac{C}{\sigma} \sum w(Q_i) \cdot \|b - b_{Q_i}\|_{\exp L(w), Q_i} \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L \log L(w), Q_i} \\
\leq \frac{C \cdot \|b\|_*}{\sigma} \sum w(Q_i) \cdot \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L \log L(w), Q_i}.
\]

In the last inequality, we have used the well-known fact that (see [25])

\[
\|b - b_Q\|_{\exp L(w), Q} \leq C \|b\|_*, \quad \text{for any cube } Q \subset \mathbb{R}^n.
\]

It is equivalent to the inequality

\[
\frac{1}{w(Q)} \int_Q \exp \left( \frac{|b(y) - b_Q|}{c_0 \|b\|_*} \right) w(y) \, dy \leq C,
\]

which is just a corollary of the well-known John–Nirenberg’s inequality (see [11]) and the comparison property of \(A_1\) weights. Moreover, it can be shown that for every cube \(Q \subset \mathbb{R}^n\) and \(w \in A_1\) (see [16, 25]),

\[
\]
\[ \| f \|_{L^\infty L^{\log w}(Q)} \approx \inf_{\eta > 0} \left\{ \eta + \frac{\eta}{w(Q)} \int_Q \Phi \left( \frac{|f(y)|}{\eta} \right) \cdot w(y) \, dy \right\}. \quad (47) \]

Hence, it is concluded from (47) that

\[
\begin{align*}
\text{III} & \leq \frac{C \cdot \| b \|_*}{\sigma} \sum_i w(Q_i) \cdot \inf_{\eta > 0} \left\{ \eta + \frac{\eta}{w(Q_i)} \int_{Q_i} \Phi \left( \frac{\| f(y) \|_\infty}{\eta} \right) \cdot w(y) \, dy \right\} \\
& \leq \frac{C \cdot \| b \|_*}{\sigma} \sum_i w(Q_i) \cdot \left\{ \sigma + \frac{\sigma}{w(Q_i)} \int_{Q_i} \Phi \left( \frac{\| f(y) \|_\infty}{\sigma} \right) \cdot w(y) \, dy \right\} \\
& \leq C \left\{ \sum_i w(Q_i) + \sum_i \int_{Q_i} \Phi \left( \frac{\| f(y) \|_\infty}{\sigma} \right) \cdot w(y) \, dy \right\} \\
& \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\| f(y) \|_\infty}{\sigma} \right) \cdot w(y) \, dy.
\end{align*}
\]

For the term IV, by (ii) of Lemma 4.2 (consider \( Q_i \) instead of \( B \)) and the assumption \( w \in A_1 \), we conclude that

\[
\begin{align*}
\text{IV} & \leq \frac{C \cdot \| b \|_*}{\sigma} \sum_i \frac{w(Q_i)}{|Q_i|} \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} \, dy \\
& \leq \frac{C \cdot \| b \|_*}{\sigma} \sum_i \int_{Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} w(y) \, dy \\
& \leq \frac{C \cdot \| b \|_*}{\sigma} \int_{\bigcup Q_i} \left( \sum_{j=1}^{\infty} |f_j(y)|^2 \right)^{1/2} w(y) \, dy \\
& \leq C \int_{\mathbb{R}^n} \frac{\| f(y) \|_\infty}{\sigma} \cdot w(y) \, dy \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\| f(y) \|_\infty}{\sigma} \right) \cdot w(y) \, dy.
\end{align*}
\]

Summing up all the above estimates, we get the desired inequality (36). This ends the proof of Theorem 5.1.

**Proof of Theorem 2.6** For any fixed ball \( B = B(y, r) \) in \( \mathbb{R}^n \), as before, we represent \( f_j \) as \( f_j = f_j^0 + f_j^\infty \), where \( f_j^0 = f_j \cdot \chi_{2B} \) and \( f_j^\infty = f_j \cdot \chi_{(2B)^c} \), \( j = 1, 2, \ldots \). Then for any given \( \sigma > 0 \), one can write
Moreover, since we now turn to deal with the term

\[ J_1(y, r) + J_2(y, r). \]

By using Theorem 5.1, we obtain

\[
J_1'(y, r) \leq C \cdot w(B(y, r))^{1/2 - 1/q} \int_{B(y, r)} \Phi \left( \frac{\|f_0(x)\|_2}{\sigma} \right) \cdot w(x) \, dx
\]

\[
= C \cdot w(B(y, r))^{1/2 - 1/q} \int_{B(y, 2r)} \Phi \left( \frac{\|f(x)\|_2}{\sigma} \right) \cdot w(x) \, dx
\]

\[
= C \cdot \frac{w(B(y, r))^{1/2 - 1/q}}{w(B(y, 2r))^{1/2 - 1/q}} \cdot w(B(y, 2r))^{1/2 - 1/q} \int_{B(y, 2r)} \Phi \left( \frac{\|f(x)\|_2}{\sigma} \right) \cdot w(x) \, dx.
\]

Here we use the following notation:

\[
\|f_0(x)\|_2 := \left( \sum_{j=1}^{\infty} |f_j^0(x)|^2 \right)^{1/2} = \left( \sum_{j=1}^{\infty} |f_j(x) \cdot \chi_{2B}|^2 \right)^{1/2}.
\]

Moreover, since \( w \in A_1 \), then by inequalities (24) and (12), we have

\[
J_1'(y, r) \leq C \cdot \frac{w(B(y, 2r))^{1/2 - 1/q}}{w(B(y, 2r))^{1/2 - 1/q}} \int_{B(y, 2r)} \Phi \left( \frac{\|f(x)\|_2}{\sigma} \right) \cdot w(x) \, dx
\]

\[
\leq C \cdot w(B(y, 2r))^{1/2 - 1/q} \left\| \Phi \left( \frac{\|f(\cdot)\|_2}{\sigma} \right) \right\|_{L \log L(w, B(y, 2r))}.
\]
\[
\left( \sum_{j=1}^{\infty} \left| b_j \cdot S_j (f_j^\infty) (x) \right|^2 \right)^{1/2} \leq \left| b(x) - b_{B(y,r)} \right| \left( \sum_{j=1}^{\infty} \left| S_j (f_j^\infty) (x) \right|^2 \right)^{1/2} + \left( \sum_{j=1}^{\infty} \left| S_j \left( [b_{B(y,r)} - b] f_j^\infty \right) (x) \right|^2 \right)^{1/2}
\]

is valid. Thus, we can further decompose \( J_2'(y,r) \) as

\[
J_2'(y,r) \leq w(B(y,r))^{1/\alpha - 1 - 1/q} \cdot \frac{4}{\sigma} \cdot \mathcal{W} \left( \left\{ x \in B(y,r) : \left| b(x) - b_{B(y,r)} \right| \cdot \left( \sum_{j=1}^{\infty} \left| S_j (f_j^\infty) (x) \right|^2 \right)^{1/2} > \frac{\sigma}{4} \right\} \right)
\]

\[
+ w(B(y,r))^{1/\alpha - 1 - 1/q} \cdot \mathcal{W} \left( \left\{ x \in B(y,r) : \left( \sum_{j=1}^{\infty} \left| S_j \left( [b_{B(y,r)} - b] f_j^\infty \right) (x) \right|^2 \right)^{1/2} > \frac{\sigma}{4} \right\} \right)
\]

\[
:= J_3'(y,r) + J_4'(y,r).
\]

Applying Chebyshev’s inequality and previous pointwise estimate (19), together with (ii) of Lemma 4.2, we deduce that

\[
J_3'(y,r) \leq w(B(y,r))^{1/\alpha - 1 - 1/q} \cdot \frac{4}{\sigma} \cdot \mathcal{W} \left( \left\{ x \in B(y,r) : \left| b(x) - b_{B(y,r)} \right| \cdot \left( \sum_{j=1}^{\infty} \left| S_j (f_j^\infty) (x) \right|^2 \right)^{1/2} \right\} \right)
\]

\[
\leq C \cdot w(B(y,r))^{1/\alpha - 1 - 1/q} \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r)} \left\| \frac{f(z)}{\sigma} \right\|_{L^2} \, dz \times \frac{1}{w(B(y,r))} \int_{B(y,r)} \left| b(x) - b_{B(y,r)} \right| w(x) \, dx
\]

\[
\leq C \|b\|_s \sum_{l=1}^{\infty} \frac{1}{|B(y,2^{l+1}r)|} \int_{B(y,2^{l+1}r)} \left\| \frac{f(z)}{\sigma} \right\|_{L^2} \, dz \times w(B(y,r))^{1/\alpha - 1 - 1/q}.
\]

Furthermore, note that \( t \leq \Phi(t) = t \cdot (1 + \log^+ t) \) for any \( t > 0 \). This fact, together with previous estimate (12) and the \( A_1 \) condition, implies that
\[
J'_3(y, r) \leq C \|b\| \sum_{l=1}^{\infty} w(B(y, 2^{l+1}r)) \int_{B(y, 2^{l+1}r)} \frac{\|f(z)\|_2}{\sigma} \cdot w(z) \, dz \times w(B(y, r))^{1/x-1/q}
\]

\[
\leq C \|b\| \sum_{l=1}^{\infty} \Phi\left( \frac{\|f(z)\|_2}{\sigma} \right) \cdot w(z) \, dz \times w(B(y, r))^{1/x-1/q}
\]

\[
\leq C \|b\| \sum_{l=1}^{\infty} \left| \Phi\left( \frac{\|f(z)\|_2}{\sigma} \right) \right| \times w(B(y, r))^{1/x-1/q}.
\]

On the other hand, applying the pointwise estimate (32) and Chebyshev’s inequality, we have

\[
J'_4(y, r) \leq w(B(y, r))^{1/x-1/q-1} \cdot \frac{4}{\sigma} \int_{B(y, r)} \left( \sum_{j=1}^{\infty} \left| S_j \left( \left| b_B(y, r) - b\right| f_j^\infty \right) \right|^2 \right)^{1/2} w(x) \, dx
\]

\[
\leq w(B(y, r))^{1/x-1/q} \cdot \frac{C}{\sigma}
\]

\[
\sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left| b(z) - b_B(y, r) \right| \cdot \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz
\]

\[
\leq w(B(y, r))^{1/x-1/q} \cdot \frac{C}{\sigma}
\]

\[
\sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left| b_B(y, 2^{l+1}r) - b_B(y, r) \right| \cdot \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz
\]

\[
+ w(B(y, r))^{1/x-1/q} \cdot \frac{C}{\sigma}
\]

\[
\sum_{l=1}^{\infty} \frac{1}{|B(y, 2^{l+1}r)|} \int_{B(y, 2^{l+1}r)} \left| b_B(y, 2^{l+1}r) - b_B(y, r) \right| \cdot \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{1/2} \, dz
\]

\[
:= J'_5(y, r) + J'_6(y, r).
\]

For the term \(J'_5(y, r)\), since \(w \in A_1\), it follows from the \(A_1\) condition and the inequality \(t \leq \Phi(t)\) that
\[ J_5'(y, r) \leq w(B(y, r))^{1/x-1/q} \]
\[ \times \frac{C}{\sigma} \sum_{l=1}^{\infty} \frac{1}{w(B(y, 2^{l+1}r))} \int_{B(y, 2^{l+1}r)} \left| b(z) - b_{B(y, 2^{l+1}r)} \right| \cdot \left( \sum_{j=1}^{\infty} \left| f_j(z) \right|^2 \right)^{1/2} w(z) \, dz \]
\[ \leq C \cdot w(B(y, r))^{1/x-1/q} \]
\[ \times \sum_{l=1}^{\infty} \frac{1}{w(B(y, 2^{l+1}r))} \int_{B(y, 2^{l+1}r)} \left| b(z) - b_{B(y, 2^{l+1}r)} \right| \cdot \Phi \left( \frac{\|f(z)\|_{L_2}}{\sigma} \right) w(z) \, dz. \]

Furthermore, we use the generalized Hölder’s inequality (11) and (46) to obtain
\[ J_5'(y, r) \leq C \cdot w(B(y, r))^{1/x-1/q} \]
\[ \times \sum_{l=1}^{\infty} \|b - b_{B(y, 2^{l+1}r)}\|_{\text{exp} L(w), B(y, 2^{l+1}r)} \left\| \Phi \left( \frac{\|f(z)\|_{L_2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \]
\[ \leq C\|b\|_\ast \sum_{l=1}^{\infty} \left\| \Phi \left( \frac{\|f(z)\|_{L_2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \times w(B(y, r))^{1/x-1/q}. \]

For the last term \( J_6'(y, r) \) we proceed as follows. Using (i) of Lemma 4.2 together with the \( A_1 \) condition on \( w \) and the inequality \( t \leq \Phi(t) \), we deduce that
\[ J_6'(y, r) \leq C \cdot w(B(y, r))^{1/x-1/q} \sum_{l=1}^{\infty} (l+1)\|b\|_\ast \cdot \frac{1}{\|B(y, 2^{l+1}r)\|_{B(y, 2^{l+1}r)}} \int_{B(y, 2^{l+1}r)} \|f(z)\|_{L_2} \sigma \, dz \]
\[ \leq C \cdot w(B(y, r))^{1/x-1/q} \]
\[ \sum_{l=1}^{\infty} (l+1)\|b\|_\ast \cdot \frac{1}{w(B(y, 2^{l+1}r))} \int_{B(y, 2^{l+1}r)} \frac{\|f(z)\|_{L_2}}{\sigma} \cdot w(z) \, dz \]
\[ \leq C\|b\|_\ast \cdot w(B(y, r))^{1/x-1/q} \]
\[ \sum_{l=1}^{\infty} \frac{(l+1)}{w(B(y, 2^{l+1}r))} \int_{B(y, 2^{l+1}r)} \Phi \left( \frac{\|f(z)\|_{L_2}}{\sigma} \right) \cdot w(z) \, dz \]
\[ \leq C\|b\|_\ast \sum_{l=1}^{\infty} (l+1) \cdot \left\| \Phi \left( \frac{\|f(z)\|_{L_2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \times w(B(y, r))^{1/x-1/q}, \]

where in the last inequality we have used (12). Summarizing the estimates derived above, we conclude that
\[ J_2^l(y, r) \leq C \|b\|_s \sum_{l=1}^{\infty} (l + 1) \cdot \left\| \Phi \left( \frac{\|f(\cdot)\|_{L^2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \times w(B(y, r))^{1/x-1/q} \\
\]

\[ = C \|b\|_s \sum_{l=1}^{\infty} w(B(y, 2^{l+1}r))^{1/x-1/q} \left\| \Phi \left( \frac{\|f(\cdot)\|_{L^2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \times (l + 1) \cdot \frac{w(B(y, r))^{1/x-1/q}}{w(B(y, 2^{l+1}r))^{1/x-1/q}}. \]

\[(50)\]

Therefore by taking the \(L^q_{\mu}\)-norm of both sides of (48) (with respect to the variable \(y\)), and then using Minkowski’s inequality, (49) and (50), we finally obtain

\[
\left\| w(B(y, r))^{1/x-1-1/q} \cdot \left( x \in B(y, r) : \left( \sum_{j=1}^{\infty} \left| [b, S_{\gamma_j}(f_j)](x) \right|^2 \right)^{1/2} > \sigma \right\|_{L^q_{\mu}} \leq \left\| J_1^l(y, r) \right\|_{L^q_{\mu}} + \left\| J_2^l(y, r) \right\|_{L^q_{\mu}} \\
\leq C \|b\|_s \left\| w(B(y, 2r))^{1/x-1/q} \left\| \Phi \left( \frac{\|f(\cdot)\|_{L^2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2r)} \right\|_{L^q_{\mu}} \\
+ C \|b\|_s \sum_{l=1}^{\infty} \left\| w(B(y, 2^{l+1}r))^{1/x-1/q} \left\| \Phi \left( \frac{\|f(\cdot)\|_{L^2}}{\sigma} \right) \right\|_{L \log L(w), B(y, 2^{l+1}r)} \right\|_{L^q_{\mu}} \times (l + 1) \cdot \frac{w(B(y, r))^{1/x-1/q}}{w(B(y, 2^{l+1}r))^{1/x-1/q}}. \]

Furthermore, by the estimate (35), we have that the last expression is dominated by
Proof of Theorem 4.1 Let us begin by recalling the John–Nirenberg inequality for the rate of growth of functions in BMO\((\mathbb{R}^n)\). There exist two positive constants \(C_1\) and \(C_2\), depending only on the dimension \(n\), such that for any cube \(Q\) in \(\mathbb{R}^n\) and any \(\lambda > 0\),

\[
\left| \left\{ x \in Q : |f(x) - f_Q| > \lambda \right\} \right| \leq C_1 |Q| \exp \left\{-\frac{C_2 \lambda}{\|b\|_*} \right\}.
\]

This result shows that in some sense logarithmic growth is the maximum possible for BMO functions (More specifically, we can take \(C_1 = \sqrt{2}\), \(C_2 = \log 2 / 2^{n+2}\), see [3, P.123–125]). For any fixed \(\eta > 0\), it is known that when \(b \in \text{BMO}(\mathbb{R}^n)\) with \(\|b\|_* < \min\{C_2 / \eta, C_2(p - 1) / \eta\}\), where \(C_2\) is the constant in the John–Nirenberg inequality, we have \(e^{\eta b(x)} \in A_p\) for \(1 < p < \infty\). By the elementary property of \(A_p\) weights, we know that there exists an \(\varepsilon > 0\) such that \(v_1(x) := w(x)^{1+\varepsilon} \in A_p\). Thus by Theorem 1.1, we can get

\[
\left\| \left( \sum_{j=1}^{\infty} |\mathcal{S}_j(h_j)|^2 \right)^{1/2} \right\|_{L^p_{1'}} \leq C' \left\| \left( \sum_{j=1}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^p_{1'}},
\]

for all \(h = (h_1, h_2, \ldots) \in L^p_{1'}(\mathbb{R}^n)\). Now we choose

Appendix

Proof of Theorem 4.1 Let us begin by recalling the John–Nirenberg inequality for the rate of growth of functions in BMO\((\mathbb{R}^n)\). There exist two positive constants \(C_1\) and \(C_2\), depending only on the dimension \(n\), such that for any cube \(Q\) in \(\mathbb{R}^n\) and any \(\lambda > 0\),

\[
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\[
\left\| \left( \sum_{j=1}^{\infty} |\mathcal{S}_j(h_j)|^2 \right)^{1/2} \right\|_{L^p_{1'}} \leq C' \left\| \left( \sum_{j=1}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^p_{1'}},
\]

for all \(h = (h_1, h_2, \ldots) \in L^p_{1'}(\mathbb{R}^n)\). Now we choose
\[ \eta := \frac{p(1+\varepsilon)}{\varepsilon}. \]

For such \( \eta > 0 \), we may assume that \( \|b\|_* < \min\{C_2/\eta, C_2(p - 1)/\eta\} \). Then for any \( \theta \in [0, 2\pi] \), we have \( \cos \theta \cdot b(x) \in \text{BMO}(\mathbb{R}^n) \), and

\[
\| \cos \theta \cdot b \|_* \leq \|b\|_* < \min\{C_2/\eta, C_2(p - 1)/\eta\},
\]

which implies that \( v_2(x) := e^{\eta \cos \theta b(x)} \in A_p \) for \( 1 < p < \infty \). This fact along with Theorem 1.1 gives us that

\[
\left\| \left( \sum_{j=1}^{\infty} |S_j(h_j)|^2 \right)^{1/2} \right\|_{L^p_{v_2}} \leq C'' \left\| \left( \sum_{j=1}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^p_v},
\]

for all \( h = (h_1, h_2, \ldots) \in L^p_v(\mathbb{R}^n) \). Applying the Stein–Weiss interpolation theorem with change of measures (see [17]) between (51) and (52), we can deduce that for any \( \theta \in [0, 2\pi] \),

\[
\left\| \left( \sum_{j=1}^{\infty} |S_j(h_j)|^2 \right)^{1/2} \right\|_{L^p_v} \leq C \left\| \left( \sum_{j=1}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^p_v},
\]

holds for all \( h = (h_1, h_2, \ldots) \in L^p_v(\mathbb{R}^n) \), where

\[
v(x) := w(x) \cdot e^{p \cos \theta b(x)} \quad \text{and} \quad C \leq (C')^{1/\tau} \cdot (C'')^{1/\tau}.
\]

For any \( \theta \in [0, 2\pi] \) and \( j \in \mathbb{Z}^+ \), we set

\[
g_{j, \theta}(x) := e^{-\varepsilon b(x)} \cdot f_j(x).
\]

When \( f = (f_1, f_2, \ldots) \in L^p_v(\mathbb{R}^n) \), it is easy to check that for \( \theta \in [0, 2\pi] \), \( (g_1, \theta, g_2, \theta, \ldots) \in L^p_v(\mathbb{R}^n) \) with \( v(x) = w(x) \cdot e^{p \cos \theta b(x)} \), and

\[
\left\| \left( \sum_{j=1}^{\infty} |g_{j, \theta}|^2 \right)^{1/2} \right\|_{L^p_v} = \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p_v}.
\]

Using Minkowski’s inequality, (27) and (53), we obtain
\[
\left\| \left( \sum_{j=1}^{\infty} |[b, S_j] (f_j)|^2 \right)^{1/2} \right\|_{L^p_w}
\leq \frac{1}{2\pi} \int_0^{2\pi} \left\| \left( \sum_{j=1}^{\infty} S_j(e^{-e^{ab}} \cdot f_j) \right)^{1/2} \cdot e^{e^{ab}b} \right\|_{L^p_w} \, d\theta
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \left\| \left( \sum_{j=1}^{\infty} |S_j(g_{j,\theta})|^2 \right)^{1/2} \right\|_{L^p_w} \, d\theta
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} C \left( \sum_{j=1}^{\infty} |g_{j,\theta}|^2 \right)^{1/2} \, d\theta
\]
\[
= C \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2},
\]
where in the last step we have used (54). The conclusion of Theorem 4.1 is proved here in the case \( \|b\|_* < \min\{C_2/\eta, C_2(p - 1)/\eta\} \) with \( \eta = p(1 + \epsilon)/\bar{\epsilon} \). For the general case, we set
\[
\tilde{b}(x) = \tilde{\eta} \cdot \frac{b(x)}{\|b\|_*},
\]
where \( \tilde{\eta} \) is chosen so that \( 0 < \tilde{\eta} < \min\{C_2/\eta, C_2(p - 1)/\eta\} \). Then
\[
\|\tilde{b}\|_* = \tilde{\eta} < \min\{C_2/\eta, C_2(p - 1)/\eta\}.
\]
From the previous proof, it actually follows that
\[
\left\| \left( \sum_{j=1}^{\infty} |[\tilde{b}, S_j] (f_j)|^2 \right)^{1/2} \right\|_{L^p_w} = \frac{\|b\|_*}{\tilde{\eta}} \left( \sum_{j=1}^{\infty} |[\tilde{b}, S_j] (f_j)|^2 \right)^{1/2} \left\| L^p_w \right\|
\]
\[
\leq C\|b\|_* \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \left\| L^p_w \right\|.
\]
Thus, the proof is complete. \( \square \)

Acknowledgements This work was done while the author was visiting Memorial University of Newfoundland in Canada. The author wishes to thank Prof. Jie Xiao for the invitation and the warm hospitality during his visit. The author would also like to express his deep gratitude to the referee for his/her careful reading, valuable comments and suggestions in the revision of the original manuscript. This work was supported by the Natural Science Foundation of China (No. XJEDU2020Y002)
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