Null vectors of the $W_3$ Algebra

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**ABSTRACT**

We construct $W_3$ null vectors of a restricted class explicitly in two different forms. The method we use is an extension of that of Bauer et al. in the Virasoro case. Our results are analogous to the formulae of Benoit and St. Aubin for the Virasoro null vectors. We derive in the Virasoro case some alternative formulae for the same null vectors involving only the $L_{-1}$ and $L_{-2}$ modes of the Virasoro algebra.
1 Introduction

A crucial step in the study of rational conformal field theories is to understand degenerate representations of the Virasoro algebra and its extensions. Much of the analysis to date relies on free field representations of these algebras to calculate the embedding structure of null vectors, determine the unitary representations and calculate the correlation functions of the corresponding rational models [1]. These methods only give implicit expressions for the null vectors. More recently however, explicit and surprisingly elegant formulae for a subset of the null vectors were found by Benoit and St-Aubin [2]. These formulae were explained by Bauer et al. [3] and extended by Kent [4]. These approaches only utilised information intrinsic to the Virasoro algebra. It is the aim of this letter to extend the above work to W-algebras.

We start with a brief review of the results of [3] for the Virasoro algebra representations. In this paper the formulae for null vectors of the type $(1, p)$ and $(q, 1)$ are obtained by considering the operator product expansion (ope) of a null vector in one representation with the highest weight in another. Surprisingly, we are able to derive new formulae for the Virasoro null states involving only the two generating modes $L_{-1}$ and $L_{-2}$ by expanding the operator product about a different point.

We find that to extend these results to other chiral algebras we need to consider several such opes simultaneously. We concentrate on the case of $W_3$ for simplicity, deriving explicit formulae for a corresponding subset of null vectors.

2 Null vectors in the Virasoro theory

The generators of infinitesimal conformal transformations satisfy the Virasoro algebra

$$[L_m, L_n] = (c/12)m(m^2 - 1)\delta_{m+n,0} + (m-n)L_{m+n}. \quad (2.1)$$

An irreducible highest weight representation of the Virasoro algebra with highest weight state $|h\rangle$ such that $L_m|h\rangle = h\delta_{m,0}|h\rangle$, $m \geq 0$, is uniquely determined by the values of $c$ and $h$. Whenever $h, c$ satisfy

$$c = c(t) = 13 - 6t - 6/t, \quad h = h_{p,q}(t) = (1/4t)((pt - q)^2 - (t - 1)^2). \quad (2.2)$$

for some $t$ and for $p, q \in \mathbb{Z}$ there is a null state at level $pq$ which we denote by $O_{p,q}|h_{p,q}\rangle$, which is itself a highest weight state. This state and all states formed by acting on it with modes of $L$ are orthogonal to all the states in the representation; we call these null states. We denote the primary field corresponding to the states $|h_{p,q}\rangle$ by $\phi_{p,q}$ where

$$\lim_{z \to 0} \phi_{p,q}(z)|0\rangle = |h_{p,q}\rangle. \quad (2.3)$$

The field corresponding to a state $L_{-p}|\phi\rangle$ for any state $\phi$ we write as $\tilde{L}_{-p}\phi(z)$, so that a basis for the fields in the theory is given by the fields $\tilde{L}_{-p_1} \ldots \tilde{L}_{-p_n}\phi(z)$ where $\phi(z)$ runs over the primary fields.

It is natural to require that the three point function of two primary fields and a null descendant of a third primary field vanish. By considering this Feigin and Fuchs gave
the condition for the coupling of three primary fields not to vanish [5], when one of the representations has a null state. Using their formula we find that the three point function

$$\langle h_{q,1}|\hat{O}_{1,2}\phi_{1,2}(z)|h_{q,0}\rangle = 0.$$  

(2.4)

In particular, if we consider the restriction of the operator product expansion of the null field at level 2 in the representation (1, 2) with the primary field \(\phi_{q,0}\) to the representation \(h_{q,1}\), then the coefficient of the highest weight itself must vanish. So, if the ope does not vanish identically, the leading term must be a null highest weight state in the \((q, 1)\) representation. In ref. [3] Bauer et al. found that is indeed the case and that this provides an explicit form for the null state.

More explicitly, the null state at level 2 in the representation \(h_{1,2}\) is

$$|\chi\rangle = \left[L_{-2} - tL_{-1}^2\right]|h_{1,2}\rangle.$$  

(2.5)

Using contour manipulations it is possible to write the ope of a descendant state of an arbitrary field \(\phi\) of conformal weight \(\Delta_{\phi}\) with a highest weight state \(|h\rangle\) in two ways,

$$\hat{L}_{-p}\phi(z)|h\rangle = \left[\sum_{s=0}^{\infty} \left(p - 2s\right) z^s L_{-p-s} + (-1/z)^p (h(p - 1) + zL_{-1} - z \partial)\right] \phi(z)|h\rangle,$$  

(2.6)

or alternatively,

$$\phi h(z) \hat{L}_{-p} |\phi\rangle = \left[L_{-p} - z^{-p+1} \partial + (p - 1)hz^{-p}\right] \phi h(z)|\phi\rangle.$$  

(2.7)

Using (2.6) Bauer et al. re-wrote the operator product expansion of the field \(\chi(z)\) with \(\phi_{q,0}\) in terms of the ope of \(\phi_{1,2}(z)\) with \(\phi_{q,0}\) as

$$\chi(z)|h_{q,0}\rangle = \left[\sum_{s=0}^{\infty} z^s L_{-2-s} + (1/z)^2 (h_{q,0} + zL_{-1} - z \partial) - t \partial^2\right] \phi(z)|h_{q,0}\rangle.$$  

(2.8)

Expanding the two opes as

$$\phi_{1,2}(z)|h_{q,0}\rangle = \sum_{n=0}^{\infty} \sum_{y=0}^{n-2} z^{n-y} f_n, \quad \chi(z)|h_{q,0}\rangle = \sum_{n=0}^{\infty} z^{n-y-2} \tilde{f}_n,$$  

(2.9)

where \(y = -h_{q,1} + h_{1,2} + h_{q,0}\), we find from (2.8) that

$$\tilde{f}_n = -t(n - q) f_n + \sum_{i>0} L_{-i} f_{n-i}.$$  

(2.10)

We see that \(\tilde{f}_0 = 0\) as required, and that setting \(\tilde{f}_i, 0 < i < q\) to zero we can recursively solve for \(f_i, i < q\). We obtain an equation for \(\tilde{f}_q\), which yields the Benoit-St. Aubin form for the null state at level \(q\),

$$O_{q,1}|q, 1\rangle = \tilde{f}_q = \prod_{i=1}^{r} \frac{L_{-m_i}}{\prod_{j=1}^{r} (\sum_{j=1}^{l} m_j)(\sum_{j=1}^{i} m_j - q)} |q, 1\rangle.$$  

(2.11)
Here we note that if instead we use the form (2.7) and consider \( \phi_{q,0}(z)|\chi\rangle \), we obtain an alternative expression for this null vector in terms of \( L_{-1}, L_{-2} \) only,

\[
\mathcal{O}_{q,1}|q, 1\rangle = \sum_{(m_1, \ldots, m_r), \sum m_i = q} \frac{\prod_{i=1}^r \alpha_{N_i}^m \tilde{L}_{-m_i}}{\prod_{i=1}^r t(\sum_{j=1}^r m_j)(\sum_{j=1}^r m_j - q)} |q, 1\rangle
\]

where the numerator is ordered from right to left, \( N_i = \sum_{j=1}^r m_i \) and we have used

\[
\tilde{L}_{-1} = L_{-1}, \quad \tilde{L}_{-2} = L_{-2} - tL_{-1}^2
\]

\[
\alpha_1^N = 1 + t(1 + q - 2N), \quad \alpha_2^N = 1
\]

### 3 Null vectors in the \( W_3 \) algebra

The \( W_3 \) algebra is the simplest of the non-linear extended chiral algebras, and was first written down by Zamolodchikov [6]. We give it here in a normalisation which will be useful later on. The generators are \( L_m, Q_m \) satisfying (2.1) and

\[
[L_m, Q_n] = (2m-n)Q_{m+n},
\]

\[
[Q_m, Q_n] = \frac{(22 + 5c)}{48} \frac{c}{3 \cdot 5!} (m^2 - 4)(m^2 - 1)m \delta_{m+n} + \frac{1}{3} (m-n) \Lambda_{m+n} + \frac{(22 + 5c)(m-n)}{48} \frac{30}{2} (2m^2 - mn + 2n^2 - 8) L_{m+n},
\]

where \( |\Lambda\rangle = (L_{-2} L_{-2} - (3/5) L_{-4})|0\rangle \). This is related to the usual normalisation by \( Q = \sqrt{\frac{22+5c}{48}} W \). The representation theory of the \( W_3 \) algebra can be developed in analogy with that of the Virasoro algebra. A \( W_3 \) highest weight vector \( |h, q\rangle \) satisfies

\[
L_m|h, q\rangle = \delta_{m,0} h|h, q\rangle, \quad Q_m|h, q\rangle = \delta_{m,0} q|h, q\rangle, \quad m \geq 0.
\]

We can parametrise the weights of a \( W \)-highest weight vector as follows [7],

\[
h = \frac{1}{3} (x^2 + xy + y^2 - 3a^2), \quad w = \frac{1}{27} (x-y)(2x+y)(x+2y),
\]

where we define \( a, \alpha_{\pm} \) by

\[
c = 2 - 24a^2, \quad \alpha_{\pm}^2 - \alpha_{\pm} a - 1 = 0.
\]

The determinant formula [8] indicates which representations have null vectors and at what levels they occur. Let \( \lambda^i \) be the two fundamental weights of \( A_2 \), and form the vector \( \mu = x\lambda^1 + y\lambda^2 \). If \( \mu \cdot e = r \alpha_+ + s \alpha_- \) for some root \( e \) of \( A_2 \) then there is a null state at level \( rs \). For generic \( c \), a representation has zero, one or two independent null vectors. Doubly degenerate primary fields are given by those representations with two independent null vectors. Such representations have

\[
x = a\alpha - c/\alpha, \quad y = b\alpha - d/\alpha.
\]

for integer \( a, b, c, d \), and we denote them by \( \phi_{a,b,c,d} \), or simply \( (ab; cd) \).
The generalisations of equations (2.6,2.7) are

\[ \hat{Q}_p \phi(z) |h, q \rangle = (-1)^{p+3}z^{-p} \left[ \frac{(p-1)(p-2)}{2}q - z(p-2)\hat{Q}_{-1} - z^2(p-1)\hat{Q}_{-2} \right] \phi(z) |h, q \rangle 
\]

\[ - \frac{(1/z)^p}{z} \left[ z\hat{Q}_{-1} + z^2\hat{Q}_{-2} \right] \phi(z) |h, q \rangle 
\]

\[ + \sum_{s=0}^{\infty} \left( p - 3 - s \right) [z^s \hat{Q}_{-p-s} \phi(z) |h, q \rangle] , \tag{3.6} \]

and

\[ \phi_{h,q}(z) \hat{Q}_p \phi = \phi_{h,q}(z) \left[ z^{-p+2}(p-1)\hat{Q}_{-2} - z^{-p+1}(p-2)\hat{Q}_{-1} \right] |\phi \rangle \tag{3.7} \]

\[ + \left[ Q_p + \frac{(p-1)(p-2)}{2}qz^{-p} + z^{-p+1}(p-2)\hat{Q}_{-1} - z^{-p+2}(p-1)\hat{Q}_{-2} \right] \phi_{h,q}(z) |\phi \rangle . \]

If we wish to generalise the construction of null vectors in the previous section by fusing a single null state with a highest weight state, then we are still left with the unknown opes \( \hat{Q}_{-1}\phi_{h,q}(z) |\phi \rangle , \hat{Q}_{-2}\phi_{h,q}(z) |\phi \rangle \). In the Virasoro case we did indeed get such opes, but were able to use \( L_{-1}\phi(z) = \partial \phi(z) \). Here we have no such geometrical interpretation.

In a different context, Bajnok, Palla and Takacs [9] considered the Toda theory of which the \( W_3 \) algebra is a symmetry. They found a field in classical \( A_2 \) Toda theory which satisfies a third order differential equation of motion\(^1\). When they attempted to quantise the field they found that this equation implied the presence of two independent null vectors at levels one and two, and that the four point functions of this field satisfied a second order differential equation, as suggested in ref. [7].

This work indicates that the key to understanding the fusion and operator product expansions of \( WA_n \) primary fields in more generality is to consider the null vectors and their descendents simultaneously. We found that given a ‘completely degenerate’ \( WA_n \) primary field (in the notation of [11]) one can solve not only for the allowed fusions of this field but also (under some conditions) for the whole ope with another W-primary field. We intend to discuss these topics in a later work [12]. In this letter we shall restrict ourselves to the construction of null vectors of the \( WA_2 \) algebra, in which case the completely degenerate representations are the doubly degenerate representations (\( ab; cd \)).

We consider the fusion

\[ (21;11) \times (01; ss') \rightarrow (11; ss') . \tag{3.8} \]

Using (3.6), we can reexpress the ope of some descendant \( W\phi_{(21;11)}(z) |01; ss' \rangle \) at level \( N \) in terms of the basis opes

\[ P(m,n) = (\hat{Q}_{-1})^m(\hat{Q}_{-2})^n\phi_{21;11}(z) |01; ss' \rangle , \tag{3.9} \]

where \( m + 2n \leq N \).

\(^1\)This is the analogue of the Liouville theory equation in the footnote on p. 357 in ref. [10]
Using the equations (2.6) and (3.6), we see that the opes of all the null states can be related to the subset

$$\mathcal{N}(m, n, i) = (\hat{Q}_{-1})^m (\hat{Q}_{-2})^n \hat{O}_{21,11} \phi_{21,11} |01; ss'\rangle, \quad (3.10)$$

where $\mathcal{O}_{-1}[21;11]$, $\mathcal{O}_{-2}[21;11]$ are the two highest weight vectors in the representation $(21;11)$ at level one and two respectively. Even the $\mathcal{N}(m, n, i)$ are not fully independent, being related by further identities arising from the embedded null vectors at levels four, five and six. From now on we shall denote the state $|21;11\rangle$ by $|\phi\rangle$.

If we count the equations arising from the null vectors (3.10), and the unknown opes $P(m,n)$, then we see that at level one there is only one null vector, while there are two basis opes $P(0,0)$, $P(1,0)$. Up to level two there are three null vectors to consider and four basis opes. However, up to level three, there are six null vectors to consider and six basis opes, and this balance persists to all higher levels (see [12]).

The two highest weight null vectors at levels one and two are

$$\mathcal{O}_{-1}|\phi\rangle = [\alpha Q_{-1} - \frac{5\alpha^2 - 3}{6} L_{-1}]|\phi\rangle \quad (3.11)$$
$$\mathcal{O}_{-2}|\phi\rangle = [L_{-1}^2(7\alpha^4 - 24\alpha^2 + 9) - 12L_{-1}Q_{-1}\alpha(-4\alpha^2 + 3) + 24L_{-2}\alpha^2(-2\alpha^4 + 3\alpha^2 - 1) + 36Q_{-1}\alpha^2 - 6Q_{-2}\alpha(12\alpha^4 - 13\alpha^2 + 3)]|\phi\rangle.$$  

The null vectors up to level 6 are then

$$\mathcal{N} = \{ \mathcal{O}_{-1}|\phi\rangle, Q_{-1}\mathcal{O}_{-1}|\phi\rangle, Q_{-2}\mathcal{O}_{-1}|\phi\rangle, Q_{-2}\mathcal{O}_{-2}|\phi\rangle, Q_{-1}\mathcal{O}_{-2}|\phi\rangle \}.$$  

Using these, and the six basis opes up to level 6,

$$\mathcal{P} = \{ P(0,0), P(0,1), P(1,0), P(2,0), P(3,0), P(1,1) \}, \quad (3.13)$$

we have a matrix operator equation,

$$M(z)\mathcal{P}(z) = \mathcal{N}(z), \quad (3.14)$$

where $M(z)$ is a $6 \times 6$ matrix whose entries are differential operators in $\partial/\partial z$ whose coefficients are $z$ dependent polynomials in the lowering modes $Q_{-p}, L_{-p}, p > 0$, and $\mathcal{P}(z), \mathcal{N}(z)$ are the opes of the fields corresponding to the states $\mathcal{P}, \mathcal{N}$ with the state $|01; ss'\rangle$. We can now impose the requirement that $\mathcal{N}(z)$ is null, which gives us a recursion relation for $\mathcal{P}$.

For calculational purposes, it is easiest to use two null vectors

$$\left(\alpha Q_{-1} - \frac{5\alpha^2 - 3}{6} L_{-1}\right)|\phi\rangle = \mathcal{O}_{-1}|\phi\rangle \quad (3.15)$$
$$\left(\alpha Q_{-2} - L_{-1}^2 + \frac{2\alpha^2}{3} L_{-2}\right)|\phi\rangle = \frac{((78\alpha^2 - 54)L_{-1} + 36\alpha Q_{-1})\mathcal{O}_{-1} - \mathcal{O}_{-2}}{36(2\alpha^2 - 1)(\alpha^2 - 1)|\phi\rangle}$$

to relate $P(m,n)$ to $P(0,0)$, and then use the null vector of ref. [9]

$$|\tilde{\phi}\rangle = \left(\alpha Q_{-3} - \alpha^{-2} L_{-1}^3 + L_{-1}L_{-2} + \frac{(\alpha^2 - 3)}{6} L_{-3}\right)|\phi\rangle \quad (3.16)$$

$$= (1/72\alpha(2\alpha^2 - 1)(\alpha^2 - 1)) \left\{ \left(30\alpha(\alpha^2 - 3)L_{-1} - 36\alpha^2 Q_{-1}\right)\mathcal{O}_{-2} 
+ (-24(2\alpha^2 - 1)(\alpha^2 - 1)\alpha^2 L_{-2} + 12\alpha(4\alpha^2 + 3)L_{-1}Q_{-1} + 36\alpha^2 Q_{-1} 
- 6\alpha(12\alpha^4 - 31\alpha^2 + 15)Q_{-2} + (7\alpha^4 + 132\alpha^2 - 99)L_{-1}^2\mathcal{O}_{-1}\right)|\phi\rangle \right.$$
where \( 
abla \rightarrow - \nabla \) and, as a descendent primary state, it is equal to the singular vector in the module and other algebras.

We defer a rigorous proof that these vectors are indeed null to our forthcoming paper [12], where

\[
\alpha_n = -\frac{1}{\alpha^2}n(n-s)(n+\alpha^2-s-s').
\]  

We would like to impose \( f_0 = |11; ss'\rangle, \tilde{f}_0 = 0 \) and \( \tilde{f}_n \sim 0, n > 0 \). We can in fact impose \( \tilde{f}_0 = 0, n < s, \) and solve for \( f_n, n < s, \) but we find that the coefficient of \( f_n \) vanishes explicitly at \( n = s \). Nonetheless, for \( n = s \) the right hand side of (3.18) is perfectly well-defined, and, as a descendent primary state, it is equal to the singular vector in the module generated from \( f_0 \).

The recurrence relation which we obtain is of the same form as that in the Virasoro case. We can solve for \( \tilde{f}_n \) and arrive at the following formula which is analogous to that derived by Benoit and St-Aubin,

\[
\mathcal{O}_{1,s}|11; ss'\rangle = \sum_{p_i; \sum_i p_i = s} \frac{\prod_i \{\alpha Q_{-p_i} + [(p_i(\alpha^2 + 3) + 2(\alpha^2 - 2s - s'))/6 + n_i]L_{-p_i}\}|11; ss'\rangle}{\prod_i n_i(n_i - s)(n_i + \alpha^2 - s-s')\alpha^{-2}},
\]

where \( n_i = \sum_{j=1}^i p_j \) and the product in the numerator is ordered from right to left with increasing \( i \). Note that this formula is valid for arbitrary \( s' \), not just \( s' \in \mathbb{N} \).

It is also possible to derive an alternative expression for this null vector by swapping the order of the fields in the ope and using (3.7) rather than (3.6). This can be written

\[
\mathcal{O}_{1,s}|11; ss'\rangle = \sum_{\sum p_i = s} \frac{\prod_i \alpha^2 \left\{\Gamma_{p_i}^n\right\}}{\prod_i n_i(n_i - s)(n_i + \alpha^2 - s-s')}|11; ss'\rangle,
\]

where \( n_i \) is as above, as before the numerator is ordered from right to left, \( p_i = 1, 2, 3 \) and

\[
\begin{align*}
\Gamma_1^n &= \alpha Q_{-1} + \left\{-\frac{3}{\alpha^2}(n^2 + n[\frac{\alpha^2}{3} - 1 - \frac{2}{3}(2s + s')]\right\) \\
&+ \frac{(\alpha^2 + 1)}{2} + \frac{1}{3}(s-s') - \frac{1}{\alpha^2}[s+s'+1][s+1]L_{-1}, \\
\Gamma_2^n &= 2\alpha Q_{-2} - \frac{1}{\alpha^2}(3n - 2s - s' - 3)L_{-1}^2 - \left\{\frac{(2s + s' - 2\alpha^2)}{3} + 1 - n\right\}L_{-2}, \\
\Gamma_3^n &= \alpha Q_{-3} - \frac{1}{\alpha^2}L_{-1}^2 + L_{-1}L_{-2} + \left\{\frac{(\alpha^2 - 3)}{6}\right\}L_{-3}.
\end{align*}
\]

We defer a rigorous proof that these vectors are indeed null to our forthcoming paper [12], together with more discussion on the fusion of general representations of the \( W_3 \) algebra and other algebras.

The formulae for the related null vectors at level \( s \) in representations \((11; s's'), (ss'; 11), (s's; 11)\) are easily obtained from the expressions (3.20, 3.21) by making the substitutions \( \alpha \rightarrow -\alpha, \alpha \rightarrow -1/\alpha, \alpha \rightarrow 1/\alpha \) respectively.
4 Conclusions

We have shown how to obtain two different explicit formulae for a subset of null vectors in the representation theory of the Virasoro algebra and $W_3$ algebra. It is fairly clear how to generalise these results to the algebras $W_n$ for $n > 3$ for e.g. the fusion $(21\ldots1;11\ldots1) \times (01\ldots1;p1\ldots1) \rightarrow (11\ldots1;p1\ldots1)$.

One can also try applying the same techniques to opes involving fields corresponding to other degenerate representations. For example, in order to derive a matrix equation of the form (3.14) for the ope of the field $(31;11)$ with some other primary field, we need to go to level five before the null vectors and basis functions $P(m,n)$ balance in the required fashion. In [12], we shall address more rigorously the possibility of defining covariant opes, their uniqueness or otherwise, and the resulting fusion rules.

Finally it might prove interesting to try to reproduce these formulae from the explicit null vectors in the affine $su(3)$ representations, as the Virasoro null vectors have been reproduced from affine $su(2)$ null vectors by Ganchev and Petkova [13].

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