NEUMANN CONDITIONS FOR THE HIGHER ORDER $s$-FRACTIONAL LAPLACIAN $(-\Delta)^s u$ WITH $s > 1$

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Abstract. In this paper we study a variational Neumann problem for the higher order $s$-fractional Laplacian, with $s > 1$. In the process we introduce some nonlocal Neumann boundary conditions that appear in a natural way from a Gauss-like integration formula.

1. Introduction and results

In this paper we introduce a natural Neumann problem for the higher order fractional Laplacian $(-\Delta)^s u$, $s > 1$.

Let us recall that when $0 < s < 1$ the operator is usually defined, for smooth functions, by means of the following principal value

$$(-\Delta)^s u(x) := c_{N,s} \, P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \quad 0 < s < 1.$$  \hspace{1cm} (1.1)

Here,

$$c_{N,s} := \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(-s)},$$  \hspace{1cm} (1.2)

is a normalized constant. See for example [9, 21, 23]. It is well-known that for functions, say, in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ this operator has an equivalent definition via the Fourier transform that is also valid when $s > 1$. More precisely,

$$\hat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \xi \in \mathbb{R}^N, \ s > 0.$$  \hspace{1cm} (1.3)

From now on, for the sake of simplicity we will consider here the higher order fractional Laplacian with $s = 1 + \sigma$, $0 < \sigma < 1$, so that $s \in (1,2)$. Following the expression given in (1.1), in this case for $u$ smooth, we can also define the operator as

$$(-\Delta)^s u(x) = c_{N,\sigma} \, P.V. \int_{\mathbb{R}^N} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x-y|^{N+2\sigma}} \, dy, \quad 1 < s < 2,$$  \hspace{1cm} (1.4)

where $c_{N,\sigma}$ is the normalization constant given in (1.2). If $0 < s < 1$ there are many results regarding existence, regularity and qualitative properties of solutions of nonlocal problems that involve the operator $(-\Delta)^s$ (see [7, 8, 24, 27, 36, 37] and the references therein; this list of publications is far from being complete). The study of the non local higher order operator, compared to the better understood lower order non local operator (i.e. $s \in (0,1)$) has not been entirely developed yet.

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In the higher order case, for example, the lack in general of a maximum principle introduces some new difficulties. Some results on this subject, like existence and representation of solutions, integration by parts, regularity, best Sobolev constants, maximum principles, Pohozaev identities and spectral results among others can be found in the list of papers [11, 18, 22, 27, 29, 33, 38, 43] or in the corresponding bibliography of each of them.

For what concerns the Neumann problem for the fractional Laplacian \((-\Delta)^s\), in the case \(s \in (0,1)\) and in other similar \(s\)-nonlocal operators, different approaches have been developed in the literature; see for instance [5, 6, 11, 14, 15, 16, 17, 25, 28, 34, 39]. The reader may find a comparison between some of these different models in [25]. We also notice here that all the Neumann conditions presented in the previous works regarding \((-\Delta)^s\), \(0 < s < 1\), are easily seen to approach the classical one \(\partial_\nu u\) when \(s \to 1\). Nevertheless the one presented in [25] by S. Dipierro, X. Ros-Oton and E. Valdinoci allows us to work in a variational framework and, as the authors describe in Section 2 of the aforementioned paper [25], it also has a natural probabilistic interpretation. To be more precise, the authors introduce and study the existence and uniqueness of solutions of the following Neumann problem for the fractional \((s \in (0,1))\) Laplacian

\[
\begin{aligned}
\left\{ \begin{array}{ll}
(-\Delta)^s u = f(x) & \text{in } \Omega \\
N_s u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega},
\end{array} \right.
\end{aligned}
\]

where \(f, g\) are appropriate problem data. Here, the operator \(N_s v\) denotes the nonlocal normal derivative defined, for smooth functions, by

\[
N_s v(x) := c_{N,s} \int_{\Omega} \frac{v(x) - v(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.
\]

This condition can be seen as the natural one to have the associated Gauss and Green formulas that allow to use a variational approach in the analysis of problem (1.5) similar to the local Neumann problem \(-\Delta u = f(x)\) in \(\Omega\), with \(\partial_\nu u = g\) on \(\partial \Omega\).

In the case of higher order operators even in the local case the situation is more involved in general as one can see in, for example, [3, 12, 13, 41]. In particular in [4], by using a Biharmonic Green Formula, the authors define the Neumann problem for the biharmonic operator \(\Delta^2 u\) and the natural boundary Neumann that, in dimension \(N = 2\), rises in the study of the bending of free plates. As far as we know the problem of establishing a reasonable Neumann condition associated to \((-\Delta)^s u\), \(s > 1\) has not been developed yet. Therefore, the aim of this work is to introduce a Neumann problem for the higher order fractional \(s\)-Laplacian, \(s \in (1,2)\), and to study the problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
(-\Delta)^s u = f(x) & \text{in } \Omega, \quad 1 < s < 2 \\
s\text{-Neumann conditions } u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega}.
\end{array} \right.
\end{aligned}
\]

Here, and throughout the paper, \(\Omega\) denotes a smooth bounded domain and our approach is to look for a variational formulation of the problem. Using a similar integration by parts as in the lower order case, \(0 < s < 1\), we can see that for a smooth function \(u\) one has

\[
\int_{\Omega} (-\Delta)^s u \, dx = - \int_{\mathbb{R}^N \setminus \Omega} N_s(-\Delta u) \, dx,
\]

where

\[
N_s(-\Delta u)(x) = (-\Delta)^s_{\Omega}(-\Delta u)(x) = \int_{\Omega} \frac{(-\Delta u)(x) - (-\Delta u)(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.
\]
However, in order to obtain a Green formula seeking a variational formulation of the problem, it will be necessary to split this last condition in two parts. Following this path and via a non local Green Formula type, we are lead to define two non local operators $N^1_\sigma, N^2_\sigma$, that will play the role of the $s$-Neumann conditions for our problem. More precisely, we will study the following

$$
\begin{cases}
(-\Delta)^s u = f(x) & \text{in } \Omega, \\
N^1_\sigma u = g_1 & \text{in } \mathbb{R}^N \setminus \Omega =: \mathcal{C}\Omega \\
N^2_\sigma u = g_2 & \text{on } \partial \Omega,
\end{cases}
$$

where $f$, $g_1$ and $g_2$ satisfy some suitable hypotheses that we will specify below and $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain (unless we specify something different as, for example, in Lemma 2.2 below). The definition of the operators $N^i_\sigma$, $i = 1, 2$ for suitable $v \in \mathcal{S}(\mathbb{R}^N)$ will come in a natural way from the integration by parts formula stated below in Theorem 2.7 as follows

$$(P)\quad N^1_\sigma v(x) := -\text{div}(-\Delta)^s_{\mathbb{R}^N \setminus \Omega}(\nabla v)(x), \quad x \in \mathbb{R}^N \setminus \Omega,$$

and

$$(1.7)\quad N^2_\sigma v(x) := (-\Delta)^s_{\mathbb{R}^N \setminus \Omega}(\nabla v)(x) \cdot \nu, \quad x \in \partial \Omega,$$

where $\nu$ is the outer unit normal field to $\partial \Omega$. Also, $(-\Delta)^s_A w$ denotes the regional fractional Laplacian that, for an open set $A \subset \mathbb{R}^N$ and regular functions $w$, is defined by

$$(1.9)\quad (-\Delta)^s_A w(x) := c_{N,\sigma} \lim_{\epsilon \to 0^+} \int_{A \setminus B_\epsilon(x)} \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy, \quad x \in \mathbb{R}^N \setminus \partial A,$$

where $c_{N,\sigma}$ is defined in (1.2).

We give now some remarks about the regional operator (see also [31] and the references therein). First of all we notice that, as we will see, the operator may not be pointwise well defined for $x \in \partial A$. For a detailed explanation under which conditions the pointwise definition up to the boundary can be considered see, for instance, [31] Theorem 5.3. Nonetheless, we observe that the principal value in the previous definition is not needed when $x \in \mathbb{R}^N \setminus \overline{A}$ if $w$ is sufficiently regular, say for instance $w \in \mathcal{S}(\mathbb{R}^N)$. The same is true if $x \in A$ and $\sigma < 1/2$. However if $x \in A$ and $\sigma \geq 1/2$, even if $w \in \mathcal{S}(\mathbb{R}^N)$, the principal value is required. In fact, if $x \in A$ denoting by $\rho(x) = \text{dist}(x, \partial A)$ and $B_x = B_{\rho(x)}(x)$ then

$$(1.10)\quad (-\Delta)^s_A w(x) = c_{N,\sigma} \left( \int_{A \setminus B_x} \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy + \lim_{\epsilon \to 0} \int_{B_\epsilon \setminus B_{\epsilon/2}(x)} \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy \right).$$

Using now that $B_x \setminus B_{\epsilon/2}(x)$ is a symmetric domain around $x$ it follows that

$$\lim_{\epsilon \to 0} \int_{B_x \setminus B_{\epsilon/2}(x)} \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy = \lim_{\epsilon \to 0} \int_{B_x \setminus B_{\epsilon/2}(x)} \frac{w(x) - w(y) - \nabla w(x)(x-y)}{|x-y|^{N+2\sigma}} dy.$$ 

Since the previous integral is absolutely convergent for example if $w \in C^{1,1}$, from (1.10) we get that, when $\sigma \geq 1/2$,

$$(1.11)\quad (-\Delta)^s_A w(x) = c_{N,\sigma} \begin{cases}
\int_{A \setminus B_x} \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy + \int_{B_x} \frac{w(x) - w(y) - \nabla w(x)(x-y)}{|x-y|^{N+2\sigma}} dy, & \text{if } x \in A, \\
\int_A \frac{w(x) - w(y)}{|x-y|^{N+2\sigma}} dy, & \text{if } x \in \mathbb{R}^N \setminus \overline{A}.
\end{cases}$$

Nevertheless, according to Theorem B in [35] the operator defined by (1.8) can be understood in the trace sense. In this way will be considered hereafter.

Before announcing the main result of this work we introduce the following notation and definitions:

**Definition 1.1.** By $\mathcal{P}_1(\mathbb{R}^N)$ we denote the vector space of all polynomials of degree one with real coefficients, that is,

$$\mathcal{P}_1(\mathbb{R}^N) = \{ p(x) : \mathbb{R}^N \to \mathbb{R} \mid p(x) = c_0 + (c, x), \text{ with } c_0 \in \mathbb{R} \text{ and } c, x \in \mathbb{R}^N \},$$

where $(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ represents the Euclidean scalar product in $\mathbb{R}^N$.

We define also $\dot{H}^s(\Omega)$ as the class of functions given by

$$\dot{H}^s(\Omega) = \{ u : \mathbb{R}^N \to \mathbb{R} \mid \text{u weakly differentiable, so that } D(u) < \infty \},$$

where

$$D(u) := \sqrt{\iint_{Q(\Omega)} \frac{\|\nabla u(x) - \nabla u(y)\|^2}{|x - y|^{N + 2\sigma}} dxdy},$$

and

$$Q(\Omega) := \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2.$$

Notice that $\mathcal{P}_1(\mathbb{R}^N) \subset \dot{H}^s(\Omega)$.

Next we will define the class of admissible data.

Let $g_1 \in L^1(\mathbb{R}^N \setminus \Omega, |x|^2 dx) \cap L^1(\mathbb{R}^N \setminus \Omega)$. Associated to $g_1$ we consider the positive measure in $\mathbb{R}^N$, absolutely continuous with respect to Lebesgue measure, defined by

$$d\mu_{g_1} = (\chi_\Omega + |g_1|\chi_{\mathbb{R}^N \setminus \Omega})dx,$$

and the class of functions

$$H^{s,0}_{g_1}(\Omega) := \{ u \in \dot{H}^s(\Omega) : \int_{\mathbb{R}^N} up d\mu_{g_1} = 0, \forall p \in \mathcal{P}_1(\mathbb{R}^N) \}.$$

For the associated measure $d\mu_{g_1}$ we consider the following Rayleigh quotient

$$\lambda_1(g_1) = \inf_{u \in H^{s,0}_{g_1} \setminus \{0\}} \frac{\iint_{Q(\Omega)} \frac{\|\nabla u(x) - \nabla u(y)\|^2}{|x - y|^{N + 2\sigma}} dxdy}{\iint_{\mathbb{R}^N} u^2 d\mu_{g_1}}.$$

**Definition 1.2.** (($A(f,g_1,g_2)$ assumptions). We say that $(f, g_1, g_2)$ is an admissible data triplet if

1. $f \in L^2(\Omega)$
2. $g_1 \in L^1(\mathbb{R}^N \setminus \Omega, |x|^2 dx) \cap L^1(\mathbb{R}^N \setminus \Omega)$ and the corresponding measure $d\mu_{g_1}$ satisfy that the spectral value $\lambda_1(g_1)$ defined by (1.15) is strictly positive.
3. $g_2 \in L^2(\partial\Omega)$.

As a direct consequence of the definition, given an admissible $g_1$ we have that

$$\int_{\mathbb{R}^N} u^2 d\mu_{g_1} < +\infty, \text{ for all } u \in H^{s,0}_{g_1}(\Omega).$$

Also, by the hypotheses on integrability of $g_1$, one has

$$\int_{\mathbb{R}^N} p^2 d\mu_{g_1} < +\infty, \text{ for all } p \in \mathcal{P}_1(\mathbb{R}^N).$$
Example 1.3. Every function \( g_1 \) such that \( g_1 \in L^p(\mathbb{R}^N \setminus \Omega) \) with \( p > \frac{N}{2} \) and with compact support satisfies condition (2) in Definition 1.2 (see Lemma 4.3 below).

Now we are ready to state the main result of the paper:

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded \( C^{1,1} \) domain and let us suppose that the assumptions \((A_{f,g_1,g_2})\) hold. Then, the problem \((P)\) has a weak solution (in the sense of Definition 4.1), if and only if the following compatibility condition holds

\[
(1.16) \quad \int_{\Omega} fp \, dx + \int_{\mathbb{R}^N \setminus \Omega} g_1 p \, dx + \int_{\partial \Omega} g_2 p \, dS = 0, \quad \text{for all} \ p \in P_1(\mathbb{R}^N).
\]

Moreover, if \((1.16)\) holds, the solution is unique up to an affine function \( p \in P_1(\mathbb{R}^N) \).

The paper is organized as follows: in Section 2 we present the integration by parts formula that shows the key point in order to understand the variational structure of the problem \((P)\). In Section 3 we give some preliminaries related to the functional framework associated to problem \((P)\) and we introduce the proper notion of solution that will be used along this work. Section 4 deals with the proof of Theorem 1.4. In Section 5 we give the complete description of the structure of the eigenvalues and eigenfunctions of \((P)\). Finally, in Section 6 we briefly comment other problems and results related with the one studied here.

Throughout the paper, generic fixed numerical constants will be denoted by \( C \), in some cases with a subscript and/or a superscript, and will be allowed to vary within a single line or formula.

### 2. Computations in \( \mathbb{R}^N \) and a Motivation of the Problem \((P)\)

The main objective of this section is to prove a new integration by parts formula associated to \((-\Delta)^s, \, 1 < s < 2\). In the sequel by \( Q(\Omega) \), we mean

\[
Q(\Omega) := \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2.
\]

First of all we need the following result that allows us to write the fractional operator in a divergence form.

**Proposition 2.1.** Given \( u \in S(\mathbb{R}^N) \) and \( s = m + \sigma \) with \( m \in \mathbb{N} \) and \( \sigma \in (0, 1) \). The operator \((-\Delta)^s u\) can be expressed in one of the following ways

\[
(-\Delta)^s u = -\Delta^m((-\Delta)^\sigma u) = (-\Delta)^\sigma(-\Delta^m u) = -\text{div}(-\Delta)^{m-\frac{1}{2}}(-\Delta)^\frac{m-1}{2}(-\Delta)^m u,
\]

if \( m \) is odd, or

\[
(-\Delta)^s u = -\Delta^m((-\Delta)^\sigma u) = (-\Delta)^\sigma(-\Delta^m u) = (-\Delta)^\frac{m}{2}(-\Delta)^\sigma(-\Delta)^\frac{m}{2} u,
\]

if \( m \) is even.

**Proof.** It is sufficient to use the Fourier transform \( \mathcal{F}(\cdot) \) and the multiplicative semigroup property. We prove the last equality in \((2.1)\). The others follow in the same way.

\[
(2.2) \quad -\text{div}(-\Delta)^{m-\frac{1}{2}}(-\Delta)^\sigma(-\Delta)^{m-\frac{1}{2}} \nabla u = -\sum_{j=1}^N \partial_j(-\Delta)^{m-\frac{1}{2}}(-\Delta)^\sigma(-\Delta)^{m-\frac{1}{2}} \partial_j u.
\]
Using Fourier transform in (2.2), we obtain

\[
(2.3) \quad \mathcal{F} \left(-\text{div}(\Delta)^{m-1}_\sigma \nabla u\right) = -\sum_{j=1}^{N} |i\xi_j|^m |\xi|^2 \partial_x^2 \mathcal{F} u
\]

Recalling (1.3), from (2.3) we deduce

\[
(-\Delta)^{\sigma} u := \mathcal{F}^{-1} \left(|\xi|^{2(m+\sigma)} \mathcal{F} u\right) = -\text{div}(\Delta)^{m-1}_\sigma (\Delta)^{\sigma} \nabla u.
\]

2.1. Integration by parts formula. In this section we prove different integration formulas that will be essential to define a variational formulation of the Neumann boundary conditions.

To simplify the next results, recalling (1.9), for \( u \in \mathcal{S}(\mathbb{R}^N) \) and \( \Omega \) a smooth domain, we can write

\[
(-\Delta)^{\sigma} u = (-\Delta)^{\sigma}_\Omega u + (-\Delta)^{\sigma}_{\mathbb{R}^N \setminus \Omega} u, \ a.e.
\]

The operators \((-\Delta)^{\sigma}_\Omega u\) and \((-\Delta)^{\sigma}_{\mathbb{R}^N \setminus \Omega} u\) are the regional \(\sigma\)-Laplace for \(\Omega\) and \(\mathbb{R}^N \setminus \Omega\) respectively. We refer for instance to [21, 30, 35] and the references therein for the properties of the regional fractional laplacian.

For the reader convenience we include the following result that will be used in the next calculations.

**Lemma 2.2.** Let \( \Omega \) be a \( C^{1,1} \) domain that could be unbounded such that its boundary, \( \partial \Omega \), is a compact set. Then for all \( u \in \mathcal{S}(\mathbb{R}^N) \),

\[
\int_\Omega (-\Delta)^{\sigma}_\Omega u(x) dx = 0.
\]

**Proof.** Assume that \( \Omega \) is bounded; if \( 0 < \sigma < \frac{1}{2} \) the result is obvious given that the function

\[
G(x, y) = \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} \in L^1(\Omega \times \Omega),
\]

and \( G(x, y) = -G(y, x) \).

We consider now the case \( \frac{1}{2} \leq \sigma < 1 \) in which the principal value is present. Consider,

\[
f_\epsilon(x) = \int_{\Omega \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} dy, \quad \epsilon > 0, \ x \in \Omega.
\]

If we are able to find \( h(x) \in L^1(\Omega) \) such that \( |f_\epsilon(x)| \leq h(x), \ x \in \Omega \) then the result follows by the Dominated Convergence Theorem; indeed

\[
\int_\Omega (-\Delta)^{\sigma}_\Omega u(x) dx = \int_\Omega \lim_{\epsilon \to 0^+} f_\epsilon(x) dx = \lim_{\epsilon \to 0^-} \int_{\Omega \times \Omega \setminus \{(x, y) : |x - y| < \epsilon\}} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} dxdy = 0
\]

by the antisymmetry, as above.

To find a function \( h(x) \in L^1(\Omega) \) majoring the \( f_\epsilon(x) \) family, fix \( x \in \Omega \). Define \( \rho(x) = \text{dist}(x, \partial \Omega) \), \( B_\rho = B_{\rho(x)}(x) \) and consider first the case \( 0 < \epsilon < \rho(x) \). Then

\[
f_\epsilon(x) = \int_{\Omega \setminus B_\rho} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} dy + \int_{B_\rho \setminus B_\rho(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} dy.
\]
Now by antisymmetry we find that

\[
\left| \int_{B_r \setminus B(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} \, dy \right| \leq \int_{B_r} \left| \frac{u(x) - u(y) - \nabla u(x)(x - y)}{|x - y|^{N+2\sigma}} \right| \, dy
\]

where the last term has a quadratic cancelation and becomes a term in \(L^1(\Omega)\). Finally, we estimate the first term as follows. Take \(R = 2\text{diam}(\Omega)\)

\[
\left| \int_{\Omega \setminus B(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} \, dy \right| \leq \int_{\Omega \setminus B(x)} \left| \frac{u(x) - u(y)}{|x - y|^{N+2\sigma}} \right| \, dy \leq C_1 \int_{B_R(x) \setminus B(x)} \frac{dy}{|x - y|^{N+2\sigma-1}} \leq C_2 \int_{\rho(x)}^R \frac{dt}{t^{2\sigma}}.
\]

The case \(\epsilon \geq \rho(x)\) is simpler since then \(\Omega \setminus B(x) \supset \Omega \setminus B(x)\). Summarizing,

\[
|f_r(x)| \leq \begin{cases} O(1), & 0 < \sigma < \frac{1}{2} \\ -\log \rho(x) + O(1), & \sigma = \frac{1}{2} \\ \frac{1}{\rho(x)^{2\sigma-1}} + O(1), & \frac{1}{2} < \sigma < 1. \end{cases}
\]

If \(\Omega\) is unbounded, inside of a ball containing the boundary we reproduce the same calculations that in the bounded case and outside we take into account the decay of the kernel, that is

\[
|(-\Delta)\Omega u(x)| \leq \frac{C}{|x|^{N+2\sigma}}.
\]

Then we apply again the Dominated Convergence Theorem to conclude.

Now we can establish the following

**Proposition 2.3.** Let \(u \in S(\mathbb{R}^N)\), \(s = 1 + \sigma\) and \(\Omega \subseteq \mathbb{R}^N\) be a smooth domain, possibly unbounded, with compact boundary. Then

\[
\int_{\Omega} (-\Delta)^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} N_\sigma(-\Delta u) \, dx,
\]

where

\[
N_\sigma(-\Delta u)(x) = (-\Delta)^s\Omega(-\Delta u)(x) = \int_{\Omega} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x - y|^{N+2\sigma}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.
\]

**Proof.** For \(u \in S(\mathbb{R}^N)\) we note that \((-\Delta)^s u\) is well defined in all \(\mathbb{R}^N\) and actually there exists a positive constant \(C = C(N, \sigma, \|\Delta u\|_{L^\infty(\mathbb{R}^N)})\), such that \(|(-\Delta)^s u| \leq C\). By direct computations we obtain

\[
\begin{align*}
\int_{\Omega} (-\Delta)^s u \, dx &= c_{N, \sigma} \int_{\Omega} \int_{\mathbb{R}^N} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
&= c_{N, \sigma} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
&= c_{N, \sigma} \int_{\mathbb{R}^N \setminus \Omega} \int_{\Omega} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x - y|^{N+2\sigma}} \, dx \, dy \\
&= -\int_{\mathbb{R}^N \setminus \Omega} N_\sigma(-\Delta u) \, dy,
\end{align*}
\]

where in (2.4) we have use Lemma 2.2 that gives

\[
\int_{\Omega} P.V. \int_{\Omega} \frac{(-\Delta u(x)) - (-\Delta u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx = 0.
\]
We now show some calculation rules that will be needed later.

**Lemma 2.4.** Let \( u \in S(\mathbb{R}^N) \) and \( \Omega \subseteq \mathbb{R}^N \) be a smooth domain with compact boundary. Then, for every \( 0 < \sigma < 1 \) we have

\[
(i) \quad \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^\sigma_{\Omega} u \, dx = - \int_{\Omega} (-\Delta)^\sigma_{\mathbb{R}^N \setminus \Omega} u \, dx,
\]

\[
(ii) \quad \int_{\Omega} (-\Delta)^s u = - \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^s u, \quad s = 1 + \sigma.
\]

**Proof.** To prove \((i)\) it is sufficient to apply Fubini’s theorem and \((ii)\) follows by using Proposition 2.3 and Lemma 2.2. \(\square\)

Thanks to Lemma 2.4 we have the following result that will be needed to prove the main theorem of the present work.

**Proposition 2.5.** Let \( p \in \mathcal{P}_1(\mathbb{R}^N) \) and let \( u \in S(\mathbb{R}^N) \) be such that

\[
(2.5) \quad \int_{\mathbb{R}^N \setminus \Omega} |\text{div}((-\Delta)^\sigma_{\Omega} \nabla u(x)) \, p(x)| \, dx < +\infty.
\]

Then

\[
\int_{\Omega} p(-\Delta)^s u \, dx = - \int_{\mathbb{R}^N \setminus \Omega} p \mathcal{N}^1_\sigma u \, dx - \int_{\partial \Omega} p \mathcal{N}^2_\sigma u \, dS.
\]

**Proof.** If \( p \in \mathcal{P}_1(\mathbb{R}^N) \) and \( u \in S(\mathbb{R}^N) \) then

\[
(2.6) \quad \int_{\Omega} p(-\Delta)^s u \, dx = \int_{\Omega} -\text{div}((-\Delta)^s \nabla u) \, p \, dx
\]

\[
= \int_{\Omega} -\text{div}((-\Delta)^\sigma_{\Omega} \nabla u) \, p \, dx + \int_{\Omega} -\text{div}((-\Delta)^\sigma_{\mathbb{R}^N \setminus \Omega} \nabla u) \, p \, dx
\]

\[
=: I_1 + I_2.
\]

By the divergence theorem we have that

\[
I_1 = \int_{\Omega} -\text{div}((-\Delta)^\sigma_{\Omega} \nabla u) \, p \, dx
\]

\[
= \int_{\Omega} (-\Delta)^\sigma_{\Omega} \nabla u \cdot \nabla p \, dx - \int_{\partial \Omega} p(-\Delta)^\sigma_{\Omega} \nabla u \cdot \nu \, dS,
\]

where \( \nu \) denotes the unit outer normal field to the boundary \( \partial \Omega \). Since \( \nabla p \) is a constant vector, then using \((i)\) of Lemma 2.4 we obtain that

\[
(2.7) \quad I_1 = - \int_{\partial \Omega} p(-\Delta)^\sigma_{\Omega} \nabla u \cdot \nu \, dS.
\]

By divergence theorem

\[
I_2 := \int_{\Omega} -\text{div}((-\Delta)^\sigma_{\mathbb{R}^N \setminus \Omega} \nabla u) \, p \, dx
\]

\[
= \int_{\Omega} (-\Delta)^\sigma_{\mathbb{R}^N \setminus \Omega} \nabla u \cdot \nabla p \, dx - \int_{\partial \Omega} p(-\Delta)^\sigma_{\mathbb{R}^N \setminus \Omega} \nabla u \cdot \nu \, dS.
\]
Recalling that $\nabla p$ is a constant vector, using (i) of Lemma 2.4 we obtain

$$I_2 = - \int_{C_\Omega} (-\Delta)^\sigma_\Omega \nabla u \cdot \nabla p \, dx - \int_{\partial_\Omega} p (-\Delta)^\sigma_{R^N \setminus \Omega} \nabla u \cdot \nu \, dS. \tag{2.8}$$

Using (2.7) and (2.8) together with divergence theorem, we deduce

$$I_1 + I_2 = - \int_{\partial_\Omega} p (-\Delta)^\sigma_\Omega \nabla u \cdot \nu \, dS - \int_{C_\Omega} (-\Delta)^\sigma_\Omega \nabla u \cdot \nabla p \, dx - \int_{\partial_\Omega} p (-\Delta)^\sigma_{R^N \setminus \Omega} \nabla u \cdot \nu \, dS = - \int_{\partial_\Omega} p (-\Delta)^\sigma_\Omega \nabla u \cdot \nu \, dS + \int_{R^N \setminus \Omega} \div((-\Delta)^\sigma_\Omega \nabla u) \, p \, dx - \int_{\partial_\Omega} p (-\Delta)^\sigma_{R^N \setminus \Omega} \nabla u \cdot (-\nu) \, ds - \int_{\partial_\Omega} p (-\Delta)^\sigma_{R^N \setminus \Omega} \nabla u \cdot \nu \, dS = - \int_{R^N \setminus \Omega} \div((-\Delta)^\sigma_\Omega \nabla u) \, p \, dx - \int_{\partial_\Omega} p (-\Delta)^\sigma_{R^N \setminus \Omega} \nabla u \cdot \nu \, dS,$$

where $(-\nu)$ denotes the unit inner normal field to the boundary $\partial_\Omega$. We point out that in the previous computations, the divergence theorem (see for example [42, Theorem 6.3.4]) can be used using a truncation argument together with (2.5).

Collecting (2.6) and (2.9), using the definitions (1.7) and (1.8), we conclude the proof. \qed

**Remark 2.6.** We notice that from Proposition 2.3 and Proposition 2.5 it is clear that

$$\int_{R^N \setminus \Omega} N_\sigma(-\Delta u) \, dx = \int_{R^N \setminus \Omega} N_\sigma^1 u \, dx + \int_{\partial_\Omega} N_\sigma^2 u \, dS, \text{ for every } u \in \mathcal{S}(R^N) \tag{2.10}$$

in the hypotheses of Proposition 2.5; that is, (2.10) is the splitting of $N_\sigma(-\Delta u)$ in the two parts that will be needed for a variational formulation of the corresponding Neumann problem.

We conclude this section obtaining a natural Neumann condition for the s-Laplacian with $s > 1$. Roughly speaking, in the higher order case, to describe an appropriate weak formulation of our problem, we have to use two (non local) Neumann conditions. Our candidates are given in equations (1.7) and (1.8). Thus, although Proposition 2.3 suggests the use of $N_\sigma(-\Delta u)$ as the Neumann condition for problem (1.2), we rather split it into $N_\sigma^1 u$ and $N_\sigma^2 u$ via the equation (2.10). The fact that this is the right splitting follows from the following proposition.

**Proposition 2.7.** Let $u \in \mathcal{S}(R^N)$ be such that

$$\int_{R^N \setminus \Omega} \abs{\div((-\Delta)^\sigma_\Omega \nabla u)} \, dx < +\infty, \tag{2.11}$$

and set $s = 1 + 2\sigma$, $0 < \sigma < 1$. Then, for $v \in \mathcal{S}(R^N)$, we have

$$\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\nabla u(x) - \nabla u(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy \quad = \int_{\Omega} v (-\Delta)^s u \, dx + \int_{R^N \setminus \Omega} v N_\sigma^1 u \, dx + \int_{\partial_\Omega} v N_\sigma^2 u \, dS. \tag{2.12}$$

**Proof.** Since $u$ is regular, a similar argument as in Lemma 2.2 shows that

$$\int_A P.V. \int_A \frac{(\nabla v(x) + \nabla v(y)) \cdot (\nabla u(x)) - (\nabla u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx = 0,$$
for any open set $\mathcal{A} \subset \mathbb{R}^N$. Therefore we have

\begin{equation}
\frac{1}{2} \int_{Q(\Omega)} \frac{(\nabla u(x) - \nabla u(y))(\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy = \int_{\Omega} \nabla v(x) P.V. \int_{\mathbb{R}^N} \frac{(\nabla u(x) - \nabla u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
+ \int_{\mathbb{R}^N \setminus \Omega} \nabla v(x) \frac{(\nabla u(x) - \nabla u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx.
\end{equation}

In each term of the r.h.s of (2.13) we use the divergence theorem. Therefore we get the following identity

\begin{equation}
c_{N,\sigma} \int_{\Omega} \nabla v(x) P.V. \int_{\mathbb{R}^N} \frac{(\nabla u(x) - \nabla u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
= \int_{\Omega} (- \text{div} ((-\Delta)^{\sigma} \nabla u(x))) \, v(x) \, dx + \int_{\partial \Omega} v(x) ((-\Delta)^{\sigma} \nabla u(x)) \cdot \nu \, dS,
\end{equation}

where $\nu$ denotes the unit outer normal field to the boundary $\partial \Omega$ and

\begin{equation}
c_{N,\sigma} \int_{\mathbb{R}^N \setminus \Omega} \nabla v(x) \int_{\Omega} \frac{(\nabla u(x) - \nabla u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
= \int_{\mathbb{R}^N \setminus \Omega} (- \text{div} ((-\Delta)^{\sigma} \nabla u)) \, v(x) \, dx - \int_{\partial \Omega} v(x) ((-\Delta)^{\sigma} \nabla u) \cdot \nu \, dS.
\end{equation}

Thus, by Proposition 2.1 putting together (2.14) and (2.15), from (2.13) we obtain that

\begin{equation}
\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\nabla u(x) - \nabla u(y))(\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy \\
= \int_{\Omega} (- \text{div} ((-\Delta)^{\sigma} \nabla u)) \, v(x) \, dx + \int_{\mathbb{R}^N \setminus \Omega} (- \text{div} ((-\Delta)^{\sigma} \nabla u)) \, v(x) \, dx \\
+ \int_{\partial \Omega} v(x) ((-\Delta)^{\sigma} \nabla u) \cdot \nu \, dS \\
= \int_{\Omega} v(-\Delta)^{\sigma} u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v N_{\sigma}^1 u \, dx + \int_{\partial \Omega} v N_{\sigma}^2 u \, dS,
\end{equation}

concluding the proof. \hfill \Box

2.2. Some considerations about condition (2.11). Let us point out here that the integrability condition (2.11) in Proposition 2.1 is not needed when $0 < \sigma < 1/2$, for in this case one always has $\text{div} ((-\Delta)^{\sigma} (\nabla u)(x)) \in L^1(\mathbb{R}^N \setminus \Omega)$. To see this observe that for a function $u \in \mathcal{S}(\mathbb{R}^N)$ a simple computation shows that

\begin{equation}
\text{div} ((-\Delta)^{\sigma} (\nabla u)(x)) = c_{N,\sigma} \int_{\Omega} \frac{\Delta u(x) - (N + 2\sigma)(\nabla u(x) - \nabla u(y)) \cdot (x-y)}{|x - y|^{N+2\sigma}} \, dy
\end{equation}

We will use the following result whose proof is implicit in the proof of Lemma 2.2

**Lemma 2.8.** Let $\Omega$ be a $C^{1,1}$ domain such that its boundary, $\partial \Omega$, is a compact set and let $0 < \alpha < 1$. Then

\[ \int_{\mathbb{R}^N \setminus \Omega} \int_{\Omega} \frac{1}{|x - y|^{N+\alpha}} \, dy \, dx < C_{N,\Omega}. \]
Using this and the fact that
\[ | \text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) | \leq C \int_{\Omega} \frac{1}{|x-y|^{N+2\sigma}}, \]
we deduce our statement.

However, when \( 1/2 \leq \sigma < 1 \) we do not have in general that \( \text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) \in L^1(\mathbb{R}^N \setminus \Omega) \) as the following counterexample shows.

**Counterexample:** Let \( \Omega \) denote the unit ball centered at the origin in \( \mathbb{R}^N \). For \( R \) large, define the function \( u \) in the Schwartz class as follows
\[ u(x) = \begin{cases} \frac{1}{2} |x|^2, & \text{if } |x| \leq R \\ 0, & \text{if } |x| \geq 2R, \end{cases} \]
and \( u \in C^\infty \) everywhere. Then, formula (2.17) gives for this \( u \) and \( 1 < |x| < R \),
\[ \text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) = c_{N,\sigma} \int_{\Omega} \frac{-2\sigma}{|x-y|^{N+2\sigma}} \, dy. \]
This function is clearly not integrable in \( B_R(0) \setminus \Omega \) for \( 2\sigma \geq 1 \).

Therefore the extra hypothesis in Proposition 2.7 is necessary to justify our computations.

It is worth pointing out also that the integrability condition (2.11) is only needed in a local sense. More precisely, if \( \Omega \subset B_R(0) \) then we always have for \( u \in S(\mathbb{R}^N) \) that \( \text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) \in L^1(\mathbb{R}^N \setminus B_{2R}(0)) \). In fact we have the following stronger estimate

**Lemma 2.9.** Assume as before that \( \Omega \subset B_R(0) \). Then for every \( u \in S(\mathbb{R}^N) \) and every polynomial \( p \in \mathcal{P}_1(\mathbb{R}^N) \) we have
\[ \int_{\mathbb{R}^N \setminus B_{2R}(0)} |\text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) p(x)| \, dx < +\infty. \]

**Proof.** To see this, we use the expression given by (2.17). Since \( |\Delta u(x)| |p(x)| < C \), \( |x-y| \sim |x| \) for \( |y| < R \) and \( |x| > 2R \), and \( \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^2} \frac{p(x)}{|p(x)|} \leq C \frac{|p(x)|}{|x|} \leq C' \), for \( |x| \) large, we have
\[ |\text{div} \left( (-\Delta)^\sigma_{\Omega}(\nabla u)(x) \right) p(x)| \leq C'' \frac{1}{|x|^{N+2\sigma}}, \quad |x| > 2R. \]
This finishes the proof. \( \square \)

With all the above, we conclude that condition (2.5) in Proposition 2.5 is always granted when \( 0 < \sigma < 1/2 \) and is equivalent to condition (2.11) when \( 1/2 \leq \sigma < 1 \).

### 3. The functional setting of the problem

We recall that a function \( u \) is weakly differentiable in \( \mathbb{R}^N \) if there exists a vector field \( \vec{U} : \mathbb{R}^N \to \mathbb{R}^N \) such that
- \( u, [\vec{U}] \in L^1_{\text{loc}}(\mathbb{R}^N) \)
- for every smooth vector field \( \vec{F} \) of compact support we have
\[ \int u(x) \text{div}\vec{F}(x) \, dx = - \int \vec{U}(x) \cdot \vec{F}(x) \, dx. \]
We write $\overrightarrow{U} = \nabla u$. If $\overrightarrow{U} = (U^1, U^2, \ldots, U^N)$, then the $n$'th component $U^j$ is denoted by $\partial_j u$ and satisfies
\[
\int \partial_j u \, \varphi \, dx = - \int u \, \partial_j \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).
\]

We now define the appropriate functional space to solve the Neumann problem.

**Definition 3.1.** Given $g_1$ as in the assumptions $A_{(f,g_1,g_2)}$, we define the space
\[
H^s_{N(g_1,g_2)}(\Omega) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u \text{ weakly differentiable } \right\},
\]
where
\[
\|u\|_{H^s_{N(g_1,g_2)}(\Omega)} = \sqrt{\int_{\Omega} u^2 \, dx + \int_{\partial \Omega} \frac{\left| \nabla u(x) - \nabla u(y) \right|^2}{|x - y|^{N+2\sigma}} \, dxdy + \int_{\Omega} |g_1|^2 \, dx}.
\]

Notice that we have the formal function space identity
\[
H^s_{N(g_1,g_2)}(\Omega) = \tilde{H}^s(\Omega) \cap L^2(\mathbb{R}^N, d\mu_{g_1}),
\]
with $\tilde{H}^s(\Omega)$ and $d\mu_{g_1}$ defined in *(1.12)* and *(1.13)* respectively.

**Remark 3.2.** Even though the space $H^s_{N(g_1,g_2)}(\Omega)$ does not depend on the boundary data $g_2$, we prefer to include $g_2$ as a subscript in the notation in order to keep in mind both Neumann conditions in problem *(2)*.

Let us prove the following

**Proposition 3.3.** The space $H^s_{N(g_1,g_2)}(\Omega)$ is a Hilbert space, with the inner product given by
\[
(u,v)_{H^s_{N(g_1,g_2)}(\Omega)} = \int_{\mathbb{R}^N} uv \, d\mu_{g_1} + \int_{\partial \Omega} \frac{(\nabla u(x) - \nabla u(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dxdy.
\]

Clearly,
\[
(\cdot, \cdot)_{H^s_{N(g_1,g_2)}(\Omega)} : H^s_{N(g_1,g_2)}(\Omega) \times H^s_{N(g_1,g_2)}(\Omega) \to \mathbb{R},
\]
is a bilinear form defined over the reals. Moreover, if $\|u\|_{H^s_{N(g_1,g_2)}(\Omega)} = \sqrt{(u,u)_{H^s_{N(g_1,g_2)}(\Omega)}} = 0$, we have on the one hand that $D(u) = 0$ and this says that $u$ coincides a.e. with a polynomial of degree 1. Since, on the other hand, $\int_{\Omega} u^2 \, dx = 0$ we conclude that $u$ must be 0 a.e. Hence, we only need to show that $H^s_{N(g_1,g_2)}(\Omega)$ is complete.

Before proving that $H^s_{N(g_1,g_2)}(\Omega)$ is complete we will state some technical results that will be needed. We will denote by $\bar{v}_A$ the average integral value of $v$ on $A$, that is, $\bar{v}_A = \frac{1}{|A|} \int_A v$.

**Lemma 3.4.** There exists a constant $C = C(N, |\Omega|)$ so that, for every given a ball $B$ with $\Omega \subset B$, and $v : \mathbb{R}^N \to \mathbb{R}$ weakly differentiable with $|\nabla v| \in L^2(B)$, one has
\[
\int_B \left( v(x) - \bar{v}_\Omega \right)^2 \, dx \leq C |B|^{1 + \frac{\sigma}{N}} \int_B |\nabla v(x)|^2 \, dx.
\]

**Corollary 3.5.** With the same hypotheses and notation of Lemma 3.4, we have
\[
\frac{1}{2} \int_B |v(x)|^2 \, dx \leq C |B|^{1 + \frac{\sigma}{N}} \int_B |\nabla v(x)|^2 \, dx + \left( \int_B v(y) \, dy \right)^2.
\]
Proof. Use simply the numerical inequality \((b - a)^2 \geq \frac{1}{2}b^2 - a^2\).

**Proof of Lemma 3.4.** The proof of (3.3) is standard. First we observe that, from Jensen’s inequality, we have

\[
\left(v(x) - \int_{\Omega} v\right)^2 = \left(\int_{\Omega} (v(x) - v(y))dy\right)^2 \leq \int_{\Omega} (v(x) - v(y))^2 dy.
\]

Integrating both sides with respect to \(dx\) on \(B\), and using the identity

\[
v(x) - v(y) = \int_0^1 \nabla v(tx + (1 - t)y) \cdot (x - y)dt, \quad \text{a.e.} \quad x, y \in \mathbb{R}^N,
\]

and Jensen’s again, we have

\[
\int_B \left(v(x) - \int_{\Omega} v\right)^2 dx \leq \int_B \int_0^1 |\nabla v(tx + (1 - t)y) \cdot (x - y)|^2 dt dy dx.
\]

By Fubini and the change of variables \(x \rightarrow z = tx + (1 - t)y\), we obtain that

\[
J := \int_B \int_\Omega \int_0^1 |\nabla v(z)|^2 \left(\frac{|z - y|}{t}\right)^2 \chi_B \left(\frac{z - y}{t} + y\right) dz \frac{dt}{t^N} dy dx
\]

where we have used that \(\Omega \subseteq B\). We observe now that if the ball \(B\) has radius \(R\) and both, \(y\) and \(\frac{z - y}{t} + y\) are in \(B\), then \(|\frac{z - y}{t}| < 2R\), which forces \(t\) to be bigger than \(\frac{|z - y|}{2R}\). Thus,

\[
J \leq \int_B |\nabla v(z)|^2 \int_\Omega \int_{\left|\frac{z - y}{t}\right|^2 < 2} |z - y|^2 \frac{dt}{t^N+2} dy dz
\]

\[
\leq \frac{(2R)^{N+1}}{N+1} \int_B |\nabla v(z)|^2 \int_\Omega |z - y|^{N-1} dy dz,
\]

Finally, using that \(\int_\Omega \frac{1}{|z - y|^{N-1}} dy \leq C(|\Omega|\Omega)^{1/N}\), we conclude the lemma.

Now we prove the following

**Lemma 3.6.** If \(u \in \dot{H}^s(\Omega)\) then \(|\nabla u| \in L^2(B)\) for every ball \(B\). Moreover, if \(\Omega \subset B\) one has the estimate

\[
(3.5) \quad \int_B |\nabla u(x) - \int_{\Omega} \nabla u| \, dx \leq C(N, \sigma) |B|^{1+\frac{2\sigma}{N}} \int_B \int_{\Omega} |\nabla u(x) - \nabla u(y)|^2 |x - y|^{N+2\sigma} \, dx dy
\]

As an easy consequence we obtain the following inequality

**Corollary 3.7.** There exists a positive constant \(C = C(N) > 0\) such that for \(u \in \dot{H}^s(\Omega)\) and every ball \(B \supset \Omega\),

\[
\frac{1}{2} \int_B |\nabla u(x)|^2 dx \leq \frac{C(N, \sigma)}{|\Omega|} |B|^{1+\frac{2\sigma}{N}} D^2(u) + \left(\int_{\Omega} \nabla u(y) dy\right)^2,
\]

where \(D(u)\) was given in (1.12). In particular, if \(u \in \dot{H}^s(\Omega)\) then \(u \in H^1_{\text{loc}}(\mathbb{R}^N)\).
Proof of Lemma 3.6. To simplify the notation, set
\[ \overrightarrow{b} = (b^1, \ldots, b^N) := \left( \int_{\Omega} \partial_1 u, \ldots, \int_{\Omega} \partial_N u \right) = \int_{\Omega} \nabla u(y) dy. \]
As in the proof of Lemma 3.4, we easily get
\[ \left| \nabla u(x) - \overrightarrow{b} \right|^2 dx \leq \Omega \int_B \int_{\Omega} \left| \nabla u(x) - \nabla u(y) \right|^2 dy dx. \]
Therefore, if \( B \) is a ball that contains \( \Omega \) and has radius \( R \), we obtain
\[ \Omega \left| \nabla u(x) - \nabla u(y) \right|^2 dx \leq (2R)^{N+2\sigma} \Omega \int_B \int_{\Omega} \left| \nabla u(x) - \nabla u(y) \right|^2 \frac{\left| x - y \right|^{N+2\sigma}}{dx dy}, \]
as stated. \( \square \)

Proof of Proposition 3.3. As we pointed out above, we only need to show that \( H_{N(g_1,g_2)}^s(\Omega) \) is complete. To that end, consider a Cauchy sequence \( \{u_k\}_k \) in our space. We proceed in several steps:

**Step 1:** There exists a function \( u^* \) such that
\[ \lim_{k \to \infty} \left( \int_{\Omega} |u^* - u_k|^2 dx + \int_{\Omega} |u^* - u_k|^2 |g_1| dx \right) = 0. \]
This comes simply from the fact that \( \{u_k\}_k \) is a Cauchy sequence in \( L^2(\mathbb{R}^N, d\mu_{g_1}) \). Since, in particular,
\[ \lim_{k \to \infty} \int_{\Omega \cup \{g_1 > 1/m \}} |u^* - u_k|^2 dx = 0, \]
for all \( m \in \mathbb{N} \), there exists a subsequence that converges pointwise to \( u^* \) in the set \( \Omega \cup \{g_1 \neq 0\} \) a.e. with respect to Lebesgue measure.

**Step 2:** There exists a vector field \( \overrightarrow{U} : \mathbb{R}^N \to \mathbb{R}^N \) such that for every ball \( B \subset \mathbb{R}^N \) we have
\[ \lim_{k \to \infty} \int_{B} |\overrightarrow{\nabla u_k} - \overrightarrow{U}|^2 dx = 0. \]
The idea here is to prove that the sequence of vector fields \( \{\nabla u_k\}_k \) is a Cauchy sequence in \( [L^2(B)]^N \). By using (3.5) and putting as above
\[ \overrightarrow{b}_k = (b^1_k, \ldots, b^N_k) := \int_{\Omega} \nabla u_k(y) dy, \]
we find that the sequence of vector fields \( \{\nabla u_k - \overrightarrow{b}_k\}_k \) is a Cauchy sequence in \( [L^2(B)]^N \) for every ball \( B \subset \mathbb{R}^N \) and, hence, there exists a vector field \( \overrightarrow{U}_0 = (U_0^1, \ldots, U_0^N) \) so that
\[ \lim_{k \to \infty} \int_{B} |\overrightarrow{\nabla u_k}(x) - \overrightarrow{b}_k - \overrightarrow{U}_0(x)|^2 dx = 0, \quad \forall B \] ball.

Let us prove that the sequence of vectors \( \{\overrightarrow{b}_k\}_k \) has a limit. To see it, we observe that if \( \varphi \) is a smooth bump function supported in \( \Omega \) with \( \int_{\Omega} \varphi dx = 1 \) we have
\[ \lim_{k \to \infty} \int_{\Omega} \left( \partial_j u_k - b^j_k \right) \varphi = \int_{\Omega} U_0^j \varphi, \quad j = 1, \ldots, N. \]
Since \( \int \left( \partial_j u_k - b_j^k \right) \varphi \, dx = - \int u_k \partial_j \varphi \, dx - b_j^k \) and \( u_k \to u^* \) as \( k \to \infty \) in \( L^2(\Omega, dx) \), we have that there exists the limit
\[
b_j^0 := \lim_{k \to \infty} b_j^k = - \int \left( u^* \partial_j \varphi + U_j^0 \varphi \right) \, dx.
\]
If we set \( b_0 = (b_0, \ldots, b_0) \) then \( \overline{U} = \overline{U}_0 + \overline{b}_0 \) represents the vector field sought in 3.7.

**Step 3:** From Corollaries 3.5 and 3.7 we have that the family \( \{u_k\}_k \) is a Cauchy sequence on \( L^2(B, dx) \) for every ball \( B \subset \mathbb{R}^N \). In particular, there exists a function \( u \) defined on all \( \mathbb{R}^N \) so that
\[
\lim_{k \to \infty} u_k = u \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^N), \quad \text{as} \quad k \to \infty.
\]
Since, from Step 2, we also have
\[
\nabla u_k \to \nabla \overline{U} \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^N), \quad \text{as} \quad k \to \infty,
\]
we conclude that \( \overline{U} = \nabla u \). Obviously we also have that \( u = u^* \) a.e. on the set \( \{x \in \mathbb{R}^N : g_1(x) \neq 0\} \).

We collect now all the information to prove that the function \( u \) is indeed the limit of the sequence \( \{u_k\}_k \) in the norm of \( H^{s}_{N(1, g_2)}(\Omega) \). First, we have from (3.11) and Fatou’s Lemma that
\[
\lim_{k \to \infty} D^2(u - u_k) = \lim_{k \to \infty} \int_{Q(\Omega)} \left| \frac{(\overline{U}(x) - \nabla u_k(x)) - (\overline{U}(y) - \nabla u_k(y))}{|x - y|^{N+2s}} \right|^2 \, dxdy = 0.
\]
This, together with (3.6) and the above observation on \( u^* \) gives
\[
\lim_{k \to \infty} \|u - u_k\|_{H^{s}_{N(1, g_2)}} = \lim_{k \to \infty} \left( \int |u - u_k| d\mu_{g_1} + D^2(u - u_k) \right) = \lim_{k \to \infty} \left( \int |u^* - u_k| d\mu_{g_1} + D^2(u - u_k) \right) = 0.
\]
This finishes the proof of Proposition 3.3. \( \square \)

### 4. Existence of solutions to (P). The proof of Theorem 1.3

We start defining the following weak formulation for the problem (P). We have

**Definition 4.1.** Assume that \( f \in L^2(\Omega) \), \( g_1 \in L^1(\mathbb{R}^N \setminus \Omega, |x|^2 \, dx) \cap L^1(\mathbb{R}^N \setminus \Omega) \), and \( g_2 \in L^2(\partial\Omega) \). Then \( u \in H^{s}_{N(1, g_2)}(\Omega) \) is a weak solution to (P) if and only if
\[
\frac{c_{N,s}}{2} \int_{Q(\Omega)} \frac{(|\nabla u(x) - \nabla v(y)| \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2s}} \, dxdy = \int_{\Omega} f v \, dx + \int_{\mathbb{R}^N \setminus \Omega} g_1 v \, dx + \int_{\partial\Omega} g_2 v \, dS,
\]
for all \( v \in H^{s}_{N(1, g_2)}(\Omega) \).

**Remark 4.2.** We point out that if \( u, v \in H^{s}_{N(1, g_2)}(\Omega) \), each term in (4.1) is well defined. In particular since \( v \in H^1(\Omega) \), we deduce that \( v \) has a trace \( Tv \) on \( \partial\Omega \), in \( L^2(\partial\Omega) \). The regularity of \( g_2 \) can also be sharpened according to the trace theory, that is, it is sufficient to require that \( g_2 \in L^{q}(\partial\Omega) \) with \( q = 2(N - 1)/N < 2 \) (see [20]). Moreover, by Sobolev inequality, see [21], since
Let us denote by $\mathcal{H}^s$ the quotient space with respect to this equivalence relation, that is

$$\mathcal{H}^s := \frac{H^s_{\mathcal{N}(g_1,g_2)}(\Omega)}{\mathcal{P}_1(\mathbb{R}^N)} = \left\{ [u], \ u \in H^s_{\mathcal{N}(g_1,g_2)}(\Omega) \right\},$$

where, given $u \in H^s_{\mathcal{N}(g_1,g_2)}(\Omega)$

$$[u] = \{ v \in H^s_{\mathcal{N}(g_1,g_2)}(\Omega) : v \sim u \} := \{ u + p : u \in H^s_{\mathcal{N}(g_1,g_2)}(\Omega), \ p \in \mathcal{P}_1(\mathbb{R}^N) \} \subseteq H^s_{\mathcal{N}(g_1,g_2)}(\Omega).$$

It is well known that

$$\| [u] \|_{\mathcal{H}^s}^2 = \inf_{p \in \mathcal{P}_1(\mathbb{R}^N)} \| u - p \|_{H^s_{\mathcal{N}(g_1,g_2)}(\Omega)}^2.$$ 

By the Hilbert projection theorem it is clear that the previous infimum is attained, that is there exists $\tilde{p} \in \mathcal{P}_1(\mathbb{R}^N)$ such that

$$\| [u] \|_{\mathcal{H}^s}^2 = \| u - \tilde{p} \|_{H^s_{\mathcal{N}(g_1,g_2)}(\Omega)}^2.$$ 

Moreover $v := u - \tilde{p} \in H^s_{\mathcal{N}(g_1,g_2)}^0(\Omega)$ where

$$(4.4) \quad H^s_{\mathcal{N}(g_1,g_2)}^0(\Omega) := \left\{ u \in H^s_{\mathcal{N}(g_1,g_2)}(\Omega) : \int_{\mathbb{R}^N} u p d\mu_{g_1} = 0, \ \forall p \in \mathcal{P}_1(\mathbb{R}^N) \right\},$$

where $d\mu_{g_1}$ was defined in (1.13). We notice that $H^s_{\mathcal{N}(g_1,g_2)}^0(\Omega) = H^s_{g_1}(\Omega) \cap L^2(\mathbb{R}^N, d\mu_{g_1})$ is a closed subspace of $H^s_{\mathcal{N}(g_1,g_2)}(\Omega)$. Let us define

$$\| [u] \|_{\mathcal{H}^s}^2 = \int_{Q(\Omega)} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x - y|^{N+2\sigma}} \, dx \, dy,$$

that is a norm in $\mathcal{H}^s$. In fact for $p \in \mathcal{P}_1(\mathbb{R}^N)$, we have that $\| u + p \|_{\mathcal{H}^s} = 0$ implies that $u \in \mathcal{P}_1(\mathbb{R}^N)$, that is the zero function in $\mathcal{H}^s$.
Lemma 4.3. Let us suppose \( g_1 \in L^p(\mathbb{R}^N \setminus \Omega) \), with \( p > N/2 \), then \( \lambda_1(g_1) > 0 \), where \( \lambda_1(g_1) \) is defined in (1.15). As a consequence, the norm \( \| \cdot \|_{\mathcal{H}^s} \) is equivalent to the norm \( \| \cdot \|_{\ast} \), that is there exists a positive constant \( C = C(N, \sigma, \Omega) \) such that

\[
(4.5) \quad \frac{1}{C} \|[u]\|_{\ast} \leq \|[u]\|_{\mathcal{H}^s} \leq C\|[u]\|_{\ast}, \quad \text{for all } [u] \in \mathcal{H}^s.
\]

Proof. The fact that \( g_1 \) verifies that \( g_1 \in L^1(\mathbb{R}^N \setminus \Omega, |x|^2 \, dx) \cap L^1(\mathbb{R}^N \setminus \Omega) \) easily follows by a simple application of the Hölder inequality. Moreover since \( H \), we deduce that \( H^{s,0}_g(\Omega) = H^{s,0}_g(N_{(g_1,g_2)})(\Omega) \). Show that \( \lambda_1(g_1) > 0 \) it is equivalent to obtain, for every \( v \in H^{s,0}_g(\Omega) = H^{s,0}_g(N_{(g_1,g_2)})(\Omega) \), the following Poincaré-type inequality

\[
(4.6) \quad \int \Omega v^2 \, dx + \int \mathcal{C}_\Omega |g_1|v^2 \, dx \leq C(N, \sigma, \Omega) \iint_{Q(\Omega)} \frac{\nabla v(x) - \nabla v(y)}{|x - y|^{N+2\sigma}} \, dxdy.
\]

Observe that if (4.6) is true, then the second inequality of (4.5) is also valid. Indeed if we consider \( [u] \in \mathcal{H}^s \) and

\[
(4.7) \quad w := u - \tilde{p} \in H^{s,0}_g(N_{(g_1,g_2)})(\Omega), \quad \tilde{p} \in \mathcal{P}_1(\mathbb{R}^N),
\]

the function where the infimum in the norm is attained, by (4.6) it will follow that

\[
\|[u]\|_{\mathcal{H}^s}^2 = \|[w]\|_{H^{s,0}_g(N_{(g_1,g_2)})(\Omega)}^2 = \int \Omega w^2 \, dx + \iint_{Q(\Omega)} \frac{\nabla w(x) - \nabla w(y)}{|x - y|^{N+2\sigma}} \, dxdy + \int \mathcal{C}_\Omega |g_1|w^2 \, dx 
\leq C(N, \sigma, \Omega) \iint_{Q(\Omega)} \frac{\nabla u(x) - \nabla u(y)}{|x - y|^{N+2\sigma}} \, dxdy \leq C(N, \sigma, \Omega)\|[u]\|_{\ast},
\]

as wanted.

To show (4.6) let us suppose, by contradiction, that there exists, up to a renormalization, a sequence \( \{v_k\} \subset H^{s,0}_g(N_{(g_1,g_2)})(\Omega) \) such that

\[
(4.8) \quad \int \Omega v_k^2 \, dx + \int \mathbb{R}^N \setminus \Omega |g_1|v_k^2 \, dx = 1 \quad \text{and} \quad \iint_{Q(\Omega)} \frac{\nabla v_k(x) - \nabla v_k(y)}{|x - y|^{N+2\sigma}} \, dxdy < \frac{1}{k}.
\]

First of all, we will show that actually

\[
(4.9) \quad \int \Omega v_k^2 \, dx + \int \Omega |\nabla v_k|^2 \, dx + \int \Omega \int \frac{|\nabla v_k(x) - \nabla v_k(y)|}{|x - y|^{N+2\sigma}} \, dxdy := \|v_k\|_{W^{s,2}(\Omega)}^2 < C.
\]

In fact, by contradiction, let us suppose that there exists a subsequence that we still denote by \( \{v_k\} \), such that

\[
(4.10) \quad \rho_k := \int \Omega |\nabla v_k|^2 \, dx \to +\infty.
\]

Defining \( z_k = v_k/\rho_k \), since \( \|\nabla z_k\|_{L^2(\Omega)} = 1 \), from (4.8) is clear that

\[
(4.11) \quad \int \Omega z_k^2 \, dx + \int \Omega |\nabla z_k|^2 \, dx + \int \Omega \int \frac{|\nabla z_k(x) - \nabla z_k(y)|}{|x - y|^{N+2\sigma}} \, dxdy < C,
\]
that is, \( \|z_k\|^2_{W^{s,2}(\Omega)} \leq C \) so \( z_k \rightharpoonup z^* \) in \( W^{s,2}(\Omega) \). In particular, by Rellich’s theorem we have that, up to subsequence \( z_k \rightharpoonup z^* \). Moreover by (4.8) and (4.10)-(4.11) it follows that
\[
C \geq \int_\Omega z_k^2 \, dx + \int_\Omega \int_\Omega \frac{|\nabla z_k(x) - \nabla z_k(y)|^2}{|x-y|^{N+2\sigma}} \, dxdy
\]
\[
= \frac{1}{\rho_k^2} \left( \int_\Omega v_k^2 \, dx + \int_\Omega \int_\Omega \frac{|\nabla v_k(x) - \nabla v_k(y)|^2}{|x-y|^{N+2\sigma}} \, dxdy \right) \rightarrow 0, \quad k \rightarrow \infty.
\]
So that in particular \( z^* = 0 \). On the other hand using the functional compact embeddings theorem [21 Theorem 7.1], we have that (up to subsequence)
\[
1 = \lim_{k \rightarrow +\infty} \int_\Omega |\nabla z_k|^2 \, dx = \int_\Omega |\nabla z^*|^2 \, dx,
\]
which is a contradiction and therefore (4.9) follows. Thus, from (4.9) in particular we infer that \( \{v_k\} \) is bounded in \( W^{s,2}(\Omega) \), so, up to subsequence, \( v_k \rightharpoonup v \) in \( W^{s,2}(\Omega) \) and \( v_k \rightarrow v \) in \( L^2(\Omega) \). Moreover by Lemma 3.6 Lemma 3.4 and from that fact that \( |\nabla g|_{L^2(\Omega)} \leq C \) we get that
\[
\int_B v_k^2 \, dx + \int_B |v_k|^2 \, dx \leq C,
\]
where \( B \) is a ball centered at the origin with \( \Omega \subset B \) and \( C = C(N, |B|, |\Omega|) \) is a positive constant. That is, \( \|v_k\|^2_{H^1(B)} \leq C \), so that \( v_k \rightarrow v \) in \( L^q \), \( q < 2N/(N-2) \). Hence, using the fact that \( g_1 \in L^p_c(\mathbb{R}^N \setminus \Omega), p > N/2 \) we can pass to the limit in (4.8) getting that
\[
\int_\Omega v^2 \, dx + \int_{\mathbb{R}^N \setminus \Omega} |g_1|^2 \, dx = 1.
\]
By the lower semicontinuity of the norm w.r.t. the weak convergence, form (4.8) is also clear that
\[
\iint_{Q(\Omega)} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x-y|^{N+2\sigma}} \, dxdy = 0.
\]
So that \( v \in \mathcal{P}_1(\mathbb{R}^N) \) which, by (4.12), clearly implies a contradiction with the fact that \( v \in H^{s,0}_{\mathcal{N}(g_1,g_2)}(\Omega) \).

To conclude the proof of Lemma let us mention that the first inequality of (4.3) is obviously true because \( \|u\|^2_{\mathcal{H}^s} = \|w\|^2_{H^{s,0}_{\mathcal{N}(g_1,g_2)}(\Omega)} \) where \( w \) was given in (4.7). \( \Box \)

Next we will emphasize that \( J \) is well defined in \( \mathcal{H}^s \). In fact if \( f, g_1 \) and \( g_2 \) satisfies de compatibility condition (1.16) and \( u \sim v \) then
\[
J(u) = J(v) = J(u - p).
\]
Therefore we can establish now the following

**Theorem 4.4.** Assume that \( (A_{(f,g_1,g_2)}) \) holds and let \( J : \mathcal{H}^s \rightarrow \mathbb{R} \) be the functional defined in (4.12). If \( f, g_1 \) and \( g_2 \) satisfying the compatibility condition (1.16) then

1. \( J \) has a unique minimum in \( \mathcal{H}^s \).
2. Every critical point of \( J \) is in fact a weak solution to the problem (P) modulo a polynomial in \( \mathcal{P}_1(\mathbb{R}^N) \).
**Lemma 4.5** (Necessary condition). Let us suppose that $(A_{f,g_1,g_2})$ hold and let $u$ be a weak solution to (1.2). Then (1.14) is satisfied. That is,
\[
\int_{\Omega} fp \, dx + \int_{\mathbb{R}^N \setminus \Omega} g_1 p \, dx + \int_{\partial \Omega} g_2 p \, dS = 0, \quad \text{for all } p \in \mathcal{P}_1(\mathbb{R}^N).
\]

**Proof.** First of all, it is easy to check (see also Remark 4.2) that the functional $J(u)$ is well defined in $\mathcal{H}^s$ that is, it is enough to prove that
\[
J(|u|) < \infty.
\]
By abuse of notation, taking into account (4.13) we will write $J(u)$ instead of $J(|u|)$. To obtain (4.14) it is sufficient to point out that we have
\[
\left| \int_{\Omega} fu \, dx \right| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{H}^s}.
\]
Moreover by the Cauchy-Schwarz inequality,
\[
\left| \int_{\mathbb{R}^N \setminus \Omega} u g_1 \, dx \right| \leq \int_{\mathbb{R}^N \setminus \Omega} |g_1| \left| \frac{s}{2} \right| |u| \, dx \leq \left( \int_{\mathbb{R}^N \setminus \Omega} |g_1| \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \leq C\|u\|_{\mathcal{H}^s}.
\]
On the other hand, by using the Hölder and trace inequality and the Poincaré-Wintinger inequality given in Corollary 3.7 we get that
\[
\left| \int_{\partial \Omega} g_2 u \, dS \right| \leq \|g_2\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C\|g_2\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leq C(1+\|u\|_{\mathcal{H}^s}).
\]
By the previous computations and the fact that $\lambda_1(g_1) > 0$, (see (4.5)-(4.7)) we also deduce that $J$ is coercive in $\mathcal{H}^s$, that is
\[
J(u) \geq C_1\|u\|_{\mathcal{H}^s}^2 - C_2\|u\|_{\mathcal{H}^s} - C_3,
\]
for some positive constants $C_1, C_2, C_3$. As $J$ is continuous, convex and coercive then, (1) is an elementary consequence of the classical minimization results.

To obtain (2) let us consider $[u], [v] \in \mathcal{H}^s$ and $t \in \mathbb{R}$. We deduce that
\[
\lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} = \frac{c_{N,s}}{2} \int_{Q(\Omega)} (\nabla u(x) - \nabla u(y)) \cdot (\nabla v(x) - \nabla v(y)) \frac{dx \, dy}{|x - y|^{N+2s}} - \int_{\Omega} fv \, dx
\]
\[
- \int_{\mathbb{R}^N \setminus \Omega} g_1 v \, dx - \int_{\partial \Omega} g_2 v \, dS.
\]
In fact to get (4.15), we observe that the first term on the r.h.s. of (4.2) can be view as a bilinear form and the other terms are linear. From (4.15) we obtain the conclusion, that is
\[
J'(u)[v] = \frac{c_{N,s}}{2} \int_{Q(\Omega)} (\nabla u(x) - \nabla u(y)) \cdot (\nabla v(x) - \nabla v(y)) \frac{dx \, dy}{|x - y|^{N+2s}} - \int_{\Omega} fv \, dx
\]
\[
- \int_{\mathbb{R}^N \setminus \Omega} g_1 v \, dx - \int_{\partial \Omega} g_2 v \, dS,
\]
for all $[u], [v] \in \mathcal{H}^s$ and therefore, a critical point of $J$ is in fact a weak solution (in the sense of Definition 4.1) to (1.2) modulo first degree polynomials. \hfill \square

We next show a lemma useful to obtain the proof of Theorem 1.4 because show that the compatibility condition is a necessary condition for the existence of a solution to (1.2):

**Lemma 4.5** (Necessary condition). Let us suppose that $(A_{f,g_1,g_2})$ hold and let $u$ be a weak solution to (1.2). Then (1.14) is satisfied. That is,
\[
\int_{\Omega} fp \, dx + \int_{\mathbb{R}^N \setminus \Omega} g_1 p \, dx + \int_{\partial \Omega} g_2 p \, dS = 0, \quad \text{for all } p \in \mathcal{P}_1(\mathbb{R}^N).
\]
Proof. It is sufficient to observe that \( P(x) \subset H^s_{N_{g_1,g_2}}(\Omega) \). Therefore using \( p \in P(x) \) as a test function in (4.1), and taking into account that \( \nabla p(x) \) is a constant function we conclude. □

Proof of Theorem 1.4: By Lemma 4.5 it is clear that if there exists a weak solution \( u \in H^s_{N_{g_1,g_2}}(\Omega) \) to (P), then (1.16) is obtained. On the contrary if (1.16) is true, then by Theorem 4.4 there exists \( u \in \mathcal{H}^s \) a solution of (P). The solution is unique up to a polynomial \( p \in P(\mathbb{R}^N) \). □

The next lemma will be useful in order to prove the right uniqueness result for weak solutions to (P) and to analyze the spectral properties of the Neumann Problem (see Section 5). We notice here that this result is the equivalent of [25, Lemma 3.8] for the Neumann problem associated to the fractional Laplacian operator of order \( 0 < s < 1 \).

Lemma 4.6. Let assume that \((A(f,g_1,g_2))\) hold and let \( u \) be a weak solution to
\[
\begin{align*}
(-\Delta)^s u &= f(x) \quad \text{in } \Omega \\
N^1_\sigma u &= g_1 \quad \text{in } \mathbb{R}^N \setminus \Omega \\
N^2_\sigma u &= g_2 \quad \text{on } \partial \Omega,
\end{align*}
\]
with \( f, g_1, g_2 \) non negative functions. Then
\[ u \in P_1(\mathbb{R}^N). \]

Proof. Taking \( P_1(\mathbb{R}^N) \ni p \equiv 1 \) as a test function we get
\[
\int_\Omega f \, dx + \int_{\mathbb{R}^N \setminus \Omega} g_1 \, dx + \int_{\partial \Omega} g_2 \, dS = 0,
\]
and thus, since \( f, g_1, g_2 \) are non negative, we deduce that \( f = 0 \) a.e. in \( \Omega \), \( g_1 = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \) and \( g_2 = 0 \) a.e. (w.r.t the measure \( S \) of the boundary) on \( \partial \Omega \). Therefore considering now \( v = u \) as a test function we get
\[
\int_{Q(\Omega)} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x - y|^{N+2\sigma}} \, dxdy = 0,
\]
that is \( \nabla u(x) \) is constant in \( \mathbb{R}^N \). Thus \( u \in P_1(\mathbb{R}^N) \). □

Next we will analyze the existence of the resonant problem with a different approach that, in particular, will be useful to study the spectrum of the Neumann problem (P) in the next section. This is the approach done in [25] for \( 0 < s < 1 \).

We start by considering the problem (4.3) with homogeneous Neumann condition, namely we set \( g_1 = 0 \) in \( \mathbb{R}^N \setminus \Omega \) and \( g_2 = 0 \) on \( \partial \Omega \). We also assume that \( f \not\equiv 0 \), since otherwise the result holds considering the trivial solution.

We call, to be short, \( H^s_{N,0}(\Omega) \), the space \( H^s_{N_{g_1,g_2}}(\Omega) \) with homogeneous Neumann conditions \( g_1 = g_2 = 0 \) in the problem (P).

First of all we observe that, by the Riesz theorem, given \( h \in L^2(\Omega) \), since the functional
\[
v \to \int_\Omega hv \, dx, \quad v \in H^s_{N,0}(\Omega)
\]
is linear and continuous in \( H^s_{N,0}(\Omega) \), there exists a unique function \( w \in H^s_{N,0}(\Omega) \) such that
\[
\int_\Omega wv \, dx + \frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\nabla w(x) - \nabla w(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dxdy = \int_\Omega hv \, dx,
\]
for all $v \in H_{N,0}^s(\Omega)$, with $\mathcal{N}_\sigma^1 w(x) = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$ and $\mathcal{N}_\sigma^2 w(x) = 0$ on $\partial \Omega$. Therefore we will define the inverse operator

$$K : L^2(\Omega) \rightarrow H_{N,0}^s(\Omega)$$

with $w$ the solution to (4.17). We can also define the restriction operator $\hat{K}$ as

$$\hat{K} h = Kh|_{\Omega},$$

and readily follows that $\hat{K} : L^2(\Omega) \rightarrow H_{N,0}^s(\Omega) \subseteq L^2(\Omega)$.

Notice that we can use the Fredholm alternative, given that $\hat{K}$ is compact. Indeed, taking $w$ as a test function in (4.17) we have that $\|w\|_{H_{N,0}^s(\Omega)} \leq C\|h\|_{L^2(\Omega)}$. Therefore taking a sequence $\{h_n\}_{n\in\mathbb{N}}$ bounded in $L^2(\Omega)$, we obtain that the sequence $w_n = \hat{K} h_n$ is bounded in $H_{N,0}^s(\Omega)$ as well, that is

$$\|w_n\|_{H_{N,0}^s(\Omega)} \leq C,$$

for some constant $C$ that does not depend on $n$. In particular from (3.1) it follows that

$$\int_{\Omega} w_n^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{\|
abla w_n(x) - \nabla w_n(y)\|^2}{|x - y|^{N+2\sigma}} \, dx \, dy \leq C.$$

As we did in the proof of Lemma 4.3 the previous inequality implies that $\|w_n\|_{W^{s,2}(\Omega)} < C$ ($s = 1+\sigma$) so, since $W^{s,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we deduce that, up to subsequences, $\{w_n\}$ converges in $L^2(\Omega)$ as wanted.

Moreover the operator $\hat{K}$ is self-adjoint. Indeed, taking $h_1, h_2 \in C^\infty_c(\Omega)$ and using the weak formulation (4.17), for every $v \in H_{N,0}^s(\Omega)$ we get that

$$\int_{\Omega} v K h_1 \, dx + \frac{c_{N,s}}{2} \int_{Q(\Omega)} \frac{(\nabla K h_1(x) - \nabla K h_1(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy = \int_{\Omega} h_1 v \, dx$$

and

$$\int_{\Omega} v K h_2 \, dx + \frac{c_{N,s}}{2} \int_{Q(\Omega)} \frac{(\nabla K h_2(x) - \nabla K h_2(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy = \int_{\Omega} h_2 v \, dx.$$

Using $v = K h_2$ as test function in (4.20) and $v = K h_1$ as test function in (4.21), by (4.18), we deduce

$$\int_{\Omega} h_1 \hat{K} h_2 \, dx = \int_{\Omega} h_2 \hat{K} h_1 \, dx.$$

Then by a density argument, (4.22) holds for $h_1, h_2 \in L^2(\Omega)$ so this implies that $\hat{K}$ is self-adjoint. To conclude the proof in the homogeneous case we will show that

$$\text{Ker}(\text{Id} - \hat{K}) = P_1(\mathbb{R}^N),$$

that is, the Kernel of the operator $\text{Id} - \hat{K}$ is the space of affine functions given in Definition 1.1. Let $p \in P_1(\mathbb{R}^N)$, since $\nabla p$ is constant, firstly is clear that and observe that

$$\int_{\Omega} pv \, dx + \frac{c_{N,s}}{2} \int_{Q(\Omega)} \frac{(\nabla p(x) - \nabla p(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy = \int_{\Omega} pv \, dx.$$
Moreover, using the definitions (1.7) and (1.8), it is also true that \( \mathcal{N}^1 \sigma p(x) = 0 \) and \( \mathcal{N}^2 \sigma p(x) = 0 \). Therefore \( Kp(x) = p(x) \) in \( \mathbb{R}^N \) and hence \( \widehat{K}p(x) = p(x) \) in \( \Omega \). This shows that

\[
\mathcal{P}_1(\mathbb{R}^N) \subset \text{Ker}(Id - \widehat{K}).
\]

The reverse inclusion is also true. In fact, taking now \( w \in \text{Ker}(Id - \widehat{K}) \subseteq L^2(\Omega) \), that is, \( w = Kw = Kw \) in \( \Omega \), by the definition of \( K \) we have that

\[
\begin{align*}
\hat{\Omega}(Kw) v dx + \frac{\varepsilon_{N, \sigma}}{2} \int_{Q(\Omega)} & (\nabla Kw(x) - \nabla Kw(y)) \cdot (\nabla v(x) - \nabla v(y)) |x - y|^{N+2\sigma} dx dy \\
= & \int_{\Omega} \nabla w \cdot \nabla v dx, \quad \forall v \in H^{s, \sigma}(\Omega).
\end{align*}
\]

Then taking \( v = w \) as a test function in (4.24) we get

\[
\begin{align*}
\int_{Q(\Omega)} & (\nabla w(x) - \nabla w(y))^2 |x - y|^{N+2\sigma} dx dy = 0,
\end{align*}
\]

which in particular implies that \( w \) is a affine function, that is, \( w \in \mathcal{P}_1(\mathbb{R}^N) \) as wanted.

Once we have proved (4.23) applying the Fredholm alternative we obtain

\[
\text{Im}(Id - \widehat{K}) = \text{Ker}(Id - \widehat{K})^\perp = \mathcal{P}_1(\mathbb{R}^N)^\perp,
\]

that is

\[
\text{Im}(Id - \widehat{K}) = \left\{ f \in L^2(\Omega) : (f, p)_{L^2(\Omega)} = 0, \; p \in \mathcal{P}_1(\mathbb{R}^N)^\perp \right\},
\]

where by \((\cdot, \cdot)_{L^2(\Omega)}\) we denote the classical inner product in \( L^2(\Omega) \). By Theorem 4.3 we have that

(4.25) the homogeneous problem \([\mathcal{P}]\) has a solution if and only if \( f \in \mathcal{P}_1(\mathbb{R}^N)^\perp \).

We can obtain again the same result by using the previous arguments:

Consider \( f \in \mathcal{P}_1(\mathbb{R}^N)^\perp = \text{Im}(Id - \widehat{K}) \). Then there exists \( h \in L^2(\Omega) \) such that

\[
(4.26) \quad f = h - \widehat{K}h.
\]

If we set \( u = Kh \), then by construction, for every \( v \in H^{s, \sigma}(\Omega) \), we get

\[
(4.27) \quad \int_\Omega uv \, dx + \frac{\varepsilon_{N, \sigma}}{2} \int_{Q(\Omega)} (\nabla u(x) - \nabla u(y)) \cdot (\nabla v(x) - \nabla v(y)) |x - y|^{N+2\sigma} dx dy = \int_\Omega hv \, dx,
\]

with

\[
\mathcal{N}^1 \sigma u(x) = 0, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \quad \text{and} \quad \mathcal{N}^2 \sigma u(x) = 0, \quad x \in \partial \Omega.
\]

Since \( u = Kh = \widehat{K}h \) in \( \Omega \), from (4.26) and (4.27) it follows that

\[
(4.28) \quad \begin{cases} (-\Delta)^s u = f(x) \quad \text{in } \Omega \\ \mathcal{N}^1 \sigma u = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ \mathcal{N}^2 \sigma u = 0 \quad \text{on } \partial \Omega, \end{cases}
\]

in the weak sense. Thus, \( u \) is the desired solution. On the other hand if \( u \in H^{s, \sigma}(\Omega) \) is a weak solution of (4.28), then we have

\[
(-\Delta)^s u + u = f + u, \quad \text{in } \Omega,
\]
that is, by the definition of $K$, one has $u = K(u + f)$ in $\mathbb{R}^N$ and then $u = \hat{K}(u + f)$ in $\Omega$. We deduce that 

$$(I - \hat{K})(u + f) = f, \quad \text{in } \Omega.$$ 

Then $f$ belongs to $\text{Im}(I - \hat{K})$ and, therefore, it is such that $(f, p)_{L^2(\Omega)} = 0$ for all functions $p \in \mathcal{P}_1(\mathbb{R}^N)$ as wanted.

This says that the nonhomogeneous case of problem (P) can be solved if we have an additional condition of the data, that is, if there exists $\psi$ sufficiently smooth such that

$$N_1^s(\psi) = g_1 \quad \text{in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad N_2^s(\psi) = g_2 \quad \text{on } \partial \Omega.$$ 

If this is the case, then for $f$, $g_1$ and $g_2$ admissible data, we have that

$$\int_{\Omega} fp \, dx + \int_{\mathbb{R}^N \setminus \Omega} N_1^s(\psi) p \, dx + \int_{\partial \Omega} N_2^s(\psi) p \, dS = 0, \quad \text{for all } p \in \mathcal{P}_1(\mathbb{R}^N),$$ 

By Proposition 2.5 we obtain

$$\int_{\Omega} (f - (-\Delta)^s \psi) p \, dx = 0, \quad \text{for all } p \in \mathcal{P}_1(\mathbb{R}^N).$$ 

Thus, by (4.25) and (4.29), there exists a weak solution $\hat{u}$ to

$$\left\{ \begin{array}{l} (-\Delta)^s \hat{u} = \hat{f}(x) \quad \text{in } \Omega \\ N_1^s \hat{u} = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ N_2^s \hat{u} = 0 \quad \text{on } \partial \Omega, \end{array} \right.$$ 

where

$$\hat{f} = f - (-\Delta)^s \psi \in \text{Im}(I - \hat{K}).$$

Therefore, defining $u := \hat{u} + \psi$ we get that $u \in H_{N, g_1, g_2}^s(\Omega)$ is a weak solution to

$$\left\{ \begin{array}{l} (-\Delta)^s u = f(x) \quad \text{in } \Omega \\ N_1^s u = g_1 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ N_2^s u = g_2 \quad \text{on } \partial \Omega. \end{array} \right.$$ 

In both cases, homogeneous and non-homogeneous, the uniqueness up to a function $p \in \mathcal{P}_1(\mathbb{R}^N)$, follows easily by contradiction using Lemma 4.6.

5. Spectral theory

We will develop now the spectral theory associated to problem (P) using some general results established for compact operators. More precisely the complete description of the structure of the eigenvalues and eigenfunctions are given in the following

**Theorem 5.1.** Let $\Omega \subset \mathbb{R}^N$ be a regular bounded domain. Then there exist a nondecreasing sequence $\{\lambda_i\} \geq 0$ and a sequence of functions $u_i : \mathbb{R}^N \to \mathbb{R}$ such that

$$\left\{ \begin{array}{l} (-\Delta)^s u_i = \lambda_i u_i \quad \text{in } \Omega \\ N_1^s u_i = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ N_2^s u_i = 0 \quad \text{on } \partial \Omega, \end{array} \right.$$ 

Moreover the functions $u_i|_{\Omega}$ form a complete orthogonal system in the space $L^2(\Omega)$. 
Proof. First of all we define the set

\[ L^{2,0}(\Omega) := \{ u \in L^2(\Omega) : \int_{\Omega} u \, p \, dx = 0, \forall p \in P_1(\mathbb{R}^N) \}, \]

that contains the set \( H^{s,0}_{N,0}(\Omega) \) defined in (4.3) for \( g_1 = g_2 = 0 \). Let us now consider the linear operator \( T : L^{2,0}(\Omega) \rightarrow H^{s,0}_{N,0}(\Omega) \), such that \( T(f) := u \) where \( u \) is the (unique) solution of the problem

\[
\begin{align*}
(-\Delta)^s u &= f & \text{in } \Omega \\
N_1^2 u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \\
N_2^2 u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

given in Theorem 1.4, recalling that \( f \) satisfies (1.16). We observe that the uniqueness come from the fact that \( L^{2,0}(\Omega) \) is a closed subspace of \( L^2(\Omega) \) and \( L^{2,0}(\Omega) = P_1(\mathbb{R}^N)^\perp \). As in the proof of Theorem 1.4 we define the restriction operator \( T \) as

\[ \overset{\circ}{T} f = T f|_{\Omega} \]

and therefore

\[ \overset{\circ}{T} : L^{2,0}(\Omega) \rightarrow L^{2,0}(\Omega). \]

With this notation is clear that a function \( u_i \) is a solution of problem \((P_i)\) if and only if

\[ u_i = T(\lambda_i u_i) = \lambda_i T(u_i), \]

therefore it is possible to transform the question of the solvability of \((P_i)\), in the investigation of the eigenvalues and eigenfunctions of the operator \( T \). In order to use the well-know theory that establish the spectral properties of the operator, we will prove that \( \overset{\circ}{T} \) is compact, self-adjoint and positive in the Hilbert space \( L^{2,0}(\Omega) \). Indeed using the weak formulation (4.1), for every \( f_1, f_2 \in L^{2,0}(\Omega) \) and \( v, \varphi, \in H^{s,0}_{N,0}(\Omega) \) it follows

\[
\begin{align*}
\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\nabla T f_1(x) - \nabla T f_1(y)) \cdot (\nabla v(x) - \nabla v(y))}{|x - y|^{N + 2\sigma}} \, dx dy &= \int_{\Omega} f_1 v \, dx, \\
\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\nabla T f_2(x) - \nabla T f_2(y)) \cdot (\nabla \varphi(x) - \nabla \varphi(y))}{|x - y|^{N + 2\sigma}} \, dx dy &= \int_{\Omega} f_2 \varphi \, dx.
\end{align*}
\]

Thus, since \( \overset{\circ}{T} f = T f \) in \( \Omega \), arguing as in equations (4.20) - (4.22), we conclude that \( T \) is self-adjoint in \( L^{2,0}(\Omega) \). Further, using again (4.4), it follows

\[ (\overset{\circ}{T} f, f)_{L^2(\Omega)} = (T f, f)_{L^2(\Omega)} = \frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{|(\nabla T f(x) - \nabla T f(y))|^2}{|x - y|^{N + 2\sigma}} \, dx dy \geq 0, \]

for every \( f \in L^{2,0}(\Omega) \). Moreover if \((\overset{\circ}{T} f, f)_{L^2(\Omega)} = 0\) then \( f = 0 \). Indeed if

\[ (\overset{\circ}{T} f, f)_{L^2(\Omega)} = \int_{Q(\Omega)} \frac{|(\nabla T f(x) - \nabla T f(y))|^2}{|x - y|^{N + 2\sigma}} \, dx dy = 0, \]

then \( T f \in P_1(\mathbb{R}^n) \) so that by (5.4) we deduce that \( f = 0 \) as wanted. That is, the operator \( \overset{\circ}{T} f \) is positive in \( L^{2,0}(\Omega) \). Finally we will show that \( T \) is compact in \( L^{2,0}(\Omega) \). In fact, from (5.4), with
To the sequence \( \{\lambda_i\}_{i \geq 2} \) we get that

\[
(5.5) \quad \left( \int_{\Omega} \int_{\Omega} \frac{\left| \nabla u(x) - \nabla u(y) \right|^2}{|x-y|^{N+2\sigma}} \, dx \, dy \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} \int_{\Omega} \frac{\left| \nabla u(x) - \nabla u(y) \right|^2}{|x-y|^{N+2\sigma}} \, dx \, dy \right)^{\frac{1}{2}} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.
\]

Since the Poincaré inequality given in (4.6) is clearly satisfied by every \( u \in H^{s,0}(\Omega) \), from (5.5) it follows that

\[
(5.6) \quad \left( \int_{\Omega} \int_{\Omega} \frac{\left| \nabla u(x) - \nabla u(y) \right|^2}{|x-y|^{N+2\sigma}} \, dx \, dy \right)^{\frac{1}{2}} \leq C\|f\|_{L^2(\Omega)}.
\]

Let us now consider \( \{f_n\} \) a bounded sequence in \( L^{2,0}(\Omega) \). By (4.6) and (5.6), repeating the arguments done to prove (4.9) in Lemma 4.3, we infer that \( \{u_n = Tf_n\} \) is also bounded in the space \( W^{s,2}(\Omega) \) (\( s = 1 + \sigma \)). Therefore since, in particular, the inclusion \( W^{s,2}(\Omega) \to L^2(\Omega) \) is compact, by subsequence, \( u_n \to u \) in \( L^2(\Omega) \).

Once we have proved that \( \overline{T} \) is compact, self-adjoint and a positive operator in the separable space \( L^{2,0}(\Omega) \) then (see for instance [30, Theorem 3.8]) the operator \( \overline{T} \) has a countable set of eigenvalues \( \{\mu_i\}_{i \geq 2} \), all of them being positive. In particular

\[ \mu_2 \geq \mu_3 \geq \ldots > 0, \text{ satisfying } \lim_{i \to \infty} \mu_i = 0. \]

To the sequence \( \{\mu_i\}_{i \geq 2} \) there corresponds a finite number of linearly independent eigenfunctions \( \{u_i\}_{i \geq 2} \) that form a complete orthonormal system in \( L^{2,0}(\Omega) \). Moreover, as we noticed in (5.3)

\[
(5.7) \quad \frac{c_{N,\sigma}}{2} \int_{\Omega} \left( \nabla u_i(x) - \nabla u_i(y) \right) \cdot \left( \nabla \varphi(x) - \nabla \varphi(y) \right) \frac{dx \, dy}{|x-y|^{N+2\sigma}} = \frac{1}{\mu_i} \int_{\Omega} u_i \varphi \, dx =: \lambda_i \int_{\Omega} u_i \varphi \, dx.
\]

Thus, by (5.7) we finally infer that

\[
\{\lambda_i := 1/\mu_i, u_i\}_{i \geq 2},
\]

form part of the suitable family of eigenfunctions and eigenvalues of \( (P_i) \) that we are looking for. To complete this family we observe that, by Lemma 4.6 it follows that \( \lambda_1 = 0 \) is an eigenvalue with eigenfunctions

\[
(5.8) \quad \{u_{1,0}(x) = 1, u_{1,1}(x) = x_1, \ldots u_{1,N}(x) = x_N\}.
\]

Therefore, up to a reordering, we have obtained the sequence of eigenvalues

\[ 0 = \lambda_1 < \lambda_2 \leq \ldots, \lim_{i \to \infty} \lambda_i = \infty, \]

and its corresponding eigenfunctions \( \{\{u_{1,j}\}_{j=0}^N, u_i\}_{i \geq 2} \) that are a complete orthogonal system in \( L^2(\Omega) \). Indeed, as we have seen above, the eigenfunctions \( \{u_i\}_{i \geq 2} \) are orthonormal w.r.t the \( L^2 \)-scalar product and moreover, each \( u_i \) is orthogonal to the subspace generated by the eigenfunctions (5.8), since the system \( \{u_i\}_{i \geq 2} \) belongs to \( L^{2,0}(\Omega) \). Finally, to show that the orthogonal system is maximal in \( L^2(\Omega) \), let us consider \( h \in L^2(\Omega) \) and we define

\[
\tilde{h} = h - h_1,
\]

where \( h_1 \) is the orthogonal projection (w.r.t. the \( L^2 \)-scalar product) of \( h \) in the subspace \( P_1(\Omega) \). Then \( \tilde{h} \in L^{2,0}(\Omega) \) and \( h_1 = b_0 + (b_j, x) \in P_1(\Omega) \), for some \( b_j \in \mathbb{R}, j = 0, \ldots, N \). Since \( \tilde{h} \in L^{2,0}(\Omega) \)
and \( \{u_i\}_{i \geq 2} \) forms a complete system in \( L^{2,0}(\Omega) \), then we obtain that

\[
\lim_{k \to \infty} \left\| \tilde{h} - \sum_{i=2}^{k} a_i u_i \right\|_{L^2(\Omega)} = 0,
\]

for some real numbers \( \{a_i\} \). Moreover,

\[
\tilde{h} = h - \sum_{j=0}^{N} a_{1,j} u_{1,j}.
\]

Thus, by (5.9), it follows that

\[
\lim_{k \to \infty} \left\| h - \sum_{j=0}^{N} a_{1,j} u_{1,j} - \sum_{i=2}^{k} a_i u_i \right\|_{L^2(\Omega)} = 0,
\]

as wanted. \( \square \)

6. FURTHER RESULTS AND PROBLEMS

In this final section we describe in an informal way some further results and interesting open problems related with what we have seen in the previous sections.

6.1. The Neumann problem for \((-\Delta)^s u\) in the case \( s > 2 \). In this subsection, using several integrations by parts (i.b.p., in short), we highlight the generalization in the higher-order case \( s > 2 \) of Proposition 2.7, which was basic to define the Neumann problem. We write \( s = m + \sigma \) and consider the case \( m \geq 2 \), even. The case \( m \geq 3 \) and odd can be obtained in the same way as in the proof of Proposition 2.7. Therefore we skip this case.

Case: \( m \geq 2 \), even. Let us define the natural Neumann conditions that come from the following non local higher-order integration by parts formula. For suitable \( v \in S(\mathbb{R}^N) \), we define

\[
\mathcal{N}_\sigma^1, (i-1) v(x):= \Delta^{i-1} (\Delta_\sigma^m v)(x), \quad x \in \partial \Omega \quad \text{and} \quad i = 1, 2, \ldots, m/2
\]

and

\[
\mathcal{N}_\sigma^2, (i-1) v(x):= \frac{\partial}{\partial \nu} \Delta^{i-1} (\Delta_\sigma^m v)(x), \quad x \in \partial \Omega \quad \text{and} \quad i = 1, 2, \ldots, m/2,
\]

with \( \nu \) denoting the unit outer normal to \( \partial \Omega \). With these definitions, for suitable \( u, v \in S(\mathbb{R}^N) \) (in particular with \( u \) satisfying similar hypotheses to (2.11)) it can be shown the following

\[
\frac{c_{N,\sigma}}{2} \mathcal{N}_\sigma \int_Q \left( \frac{\Delta_\sigma^m u(x) - \Delta_\sigma^m u(y)}{|x - y|^{N+2\sigma}} \right) dxdy
\]

\[
= \int_{\Omega} v (-\Delta)^s u dx + \int_{\mathbb{R}^N \setminus \Omega} \Delta_\sigma^m \mathcal{N}_\sigma (\Delta_\sigma^m u(x)) v(x) dx
\]

\[
+ \sum_{i=1}^{m/2} \int_{\partial \Omega} \frac{\partial}{\partial \nu} \left( \Delta_\sigma^{m-2i} v(x) \right) \mathcal{N}_\sigma^1, (i-1) u(x) dS - \sum_{i=1}^{m/2} \int_{\partial \Omega} \mathcal{N}_\sigma^2, (i-1) u(x) \Delta_\sigma^{m-2i} v(x) dS.
\]
In fact, roughly speaking, denoting by $\nu$ the unit outer normal field to the boundary $\partial \Omega$ and integrating twice by parts we obtain

\begin{equation}
\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))(\Delta_{m}^{\frac{m}{2}} v(x) - \Delta_{m}^{\frac{m}{2}} v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy
\end{equation}

\begin{align*}
&= c_{N,\sigma} \int_{\Omega} \Delta_{m}^{\frac{m}{2}} v(x) \int_{\mathbb{R}^{N}} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx \\
&+ c_{N,\sigma} \int_{\partial \Omega} \Delta_{m}^{\frac{m}{2}} v(x) \int_{\Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \, dx
\end{align*}

1st i.b.p.

\begin{align*}
&= - \int_{\mathbb{R}^{N}\setminus \Omega} \nabla(\Delta_{m}^{\frac{m-2}{2}} v(x)) \cdot \nabla \left( c_{N,\sigma} \int_{\mathbb{R}^{N}} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dx \\
&+ \int_{\partial \Omega} \frac{\partial}{\partial \nu} \left( \Delta_{m}^{\frac{m-2}{2}} v(x) \right) \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS
\end{align*}

2nd i.b.p.

\begin{align*}
&= \int_{\Omega} \frac{\Delta_{m}^{\frac{m-2}{2}} v(x) \Delta}{c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy} \, dx \\
&+ \int_{\partial \Omega} \frac{\partial}{\partial \nu} \left( \Delta_{m}^{\frac{m-2}{2}} v(x) \right) \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS \\
&- \int_{\partial \Omega} \Delta_{m}^{\frac{m-2}{2}} v(x) \frac{\partial}{\partial \nu} \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS.
\end{align*}

Then if we continue to integrate by parts, as we did in (6.3), after $m/2$ steps we get

\begin{align*}
\frac{c_{N,\sigma}}{2} \int_{Q(\Omega)} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))(\Delta_{m}^{\frac{m}{2}} v(x) - \Delta_{m}^{\frac{m}{2}} v(y))}{|x - y|^{N+2\sigma}} \, dx \, dy
\end{align*}

\begin{align*}
&\quad \vdots \\
&= mth \text{ i.b.p.} \\
&= \int_{\Omega} v(x) \Delta_{m}^{\frac{m}{2}} \left( c_{N,\sigma} \int_{\mathbb{R}^{N}} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dx \\
&\quad + \int_{\partial \Omega} v(x) \Delta_{m}^{\frac{m}{2}} \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS \\
&\quad + \sum_{i=1}^{m/2} \int_{\partial \Omega} \frac{\partial}{\partial \nu} \left( \Delta_{m}^{\frac{m-2i}{2}} v(x) \right) \Delta^{i-1} \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS \\
&\quad - \sum_{i=1}^{m/2} \int_{\partial \Omega} \Delta_{m}^{\frac{m-2i}{2}} v(x) \frac{\partial}{\partial \nu} \Delta^{i-1} \left( c_{N,\sigma} \int_{\mathbb{R}^{N}\setminus \Omega} \frac{(\Delta_{m}^{\frac{m}{2}} u(x) - \Delta_{m}^{\frac{m}{2}} u(y))}{|x - y|^{N+2\sigma}} \, dy \right) \, dS
\end{align*}

and then we obtain the conclusion using Proposition 2.1 and equations (1.6), (6.1) and (6.2).
6.2. A semilinear Neumann problem and some open questions. Consider the problem

\[ \begin{cases} d(-\Delta)^su + u = |u|^{p-1}u & x \in \Omega \\ N_s(u) = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.4} \]

where \(0 < s < 1\), \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain and the diffusion coefficient \(d\) is positive.

For the classical Laplacian this problem was deeply analyzed by Lin, Ni and Takagi in their classical paper [32] where the reader can also see the motivations of this model in the local case.

First of all we notice that \(v_0 = 0\) and \(v = \pm 1\) are the unique possible constant solutions of problem (6.4). Therefore we will be interested in finding non-trivial solutions. It is clear that if \(1 < p < 2^*_s\) this is equivalent to look for non-constant critical points of the energy functional,

\[ J_d(u) = \frac{d}{2} \int_{Q(\Omega)} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx, \tag{6.5} \]

where

\[ 2^*_s = \begin{cases} \frac{N + 2s}{N - 2s}, & N > 2s \\ +\infty, & N \leq 2s. \end{cases} \]

is the critical fractional exponent. Notice that \(J_d\) is well defined in

\[ H^s(\Omega) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \mid u \text{ measurable}, \int_{Q(\Omega)} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy < +\infty \right\}, \tag{6.6} \]

Therefore the main result is the following

**Theorem 6.1.** There exists a nontrivial nonconstant solution, \(u_d\), to problem (6.4), provided \(d\) is sufficiently small.

**Proof.** Following closely the arguments done in [32], we can use the Mountain-Pass Lemma by Ambrosetti-Rabinowitsch, [3], in order to find critical points of \(J_d\). Indeed, it is easy to check that the geometry of the Mountain-Pass Lemma holds for all \(d > 0\), that is, there exists \(\rho\) such that \(J_d(u) > 0\) for all \(0 < \|u\| \leq \rho\), \(J_d(u) > \beta > 0\), \(\|u\| = \rho\) and there exists \(v\), with \(\|v\| > \rho\) such that \(J_d(v) < 0\). Moreover any Palais-Smale sequence is bounded and by the Rellich Compactness Theorem admits a convergent subsequence. Let us assume now that

\[ \exists \phi_d \in H^s(\Omega), \text{ and } t_1, \bar{C} > 0 \text{ such that } J_d(t_1\phi_d) = 0 \text{ and } J_d(t\phi_d) < \bar{C}d^{N/2s}, 0 \leq t \leq t_1. \tag{6.7} \]

Then if \(\Gamma = \{ \gamma \in C([0,1],H^s(\Omega)) \mid \gamma(0) = 0, \gamma(1) = t_1\phi_d \}\), it follows that the minimax value

\[ c_d = \inf_{\gamma \in \Gamma} \max_{[0,1]} J_d(\gamma(t)) \geq \beta > 0 = J_d(0), \]

is a critical value of \(J_d\), that is, there exists a solution \(\tilde{u}\) such that \(J_d(\tilde{u}) = c_d\). Moreover since, by (6.7), taking \(d\) small enough,

\[ J_d(\tilde{u}) = c_d \leq \max_{[0,t_1]} J_d(t\phi_d) < \bar{C}d^{N/2s} \left( \frac{1}{2} - \frac{1}{p+1} \right) |\Omega| = J_d(1) = J_d(-1), \]

we conclude that in the set \(J_d^{-1}(c_d)\) there is some nonconstant critical point.

To prove (6.7) let us consider

\[ \phi(x) = \begin{cases} (1 - |x|), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases} \]
and define \( \phi_d(x) = d^{-N/2s} \phi \left( \frac{x}{d^{1/2s}} \right) \). Assume without loss of generality that \( 0 \in \Omega \) and take \( d \) small enough in such a way that the ball of radius \( d^{1/2s} \) is contained in \( \Omega \).

We have by a direct calculation one can check that

\[
\frac{1}{2} \int \int_{Q(d^{1/2s})} \frac{|\phi_d(x) - \phi_d(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \leq C(N) d^{-N/2s(1+1/2s)}, \quad ||\phi_d||_q = C_q(N) d^{N/2s(1-q)}
\]

Therefore, for constants depending only on the dimension,

\[
g(t) := J_d(t\phi_d) = c_1\frac{1}{2}d^{-N/2} t^2 - c_2 \frac{1}{p+1}d^{-N/2} t^{p+1}.
\]

It is easy to check that there exists \( t_1 > 0 \) such that \( g(t_1) = 0 \). Moreover \( g \) verifies that its maximum for \( t > 0 \) is attained in \( t_0 = (\frac{c_2}{c_1})^{\frac{1}{p+1}} d^{N/2} < t_1 \), that is

\[
g(t) = J_d(t\phi_d) \leq J_d(t_0 \phi_d) \leq Cd^{N/2},
\]

and (6.7) follows as wanted. \( \square \)

**Remark 6.2.** The higher order Neumann semilinear problem can be studied in a similar way. To be precise we consider the case \( s = 1 + \sigma, \ 0 < \sigma < 1 \) and leave to the reader the details for the interval \( s > 2 \).

Consider the problem

\[
(P) \begin{cases}
d(-\Delta)^s u + u = |u|^{p-1} u & \text{in } \Omega, \quad 1 < s < 2 \\
\mathcal{N}_d^1 u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} =: C \overline{\Omega} \\
\mathcal{N}_d^2 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

for \( d > 0, s = 1 + \sigma, \ \sigma > 0 \) and \( 1 < p < 2^*_s \). The energy functional now is

\[
(6.8) \quad J_d(u) = \frac{d}{2} \int \int_{Q(\Omega)} \frac{\nabla u(x) - \nabla u(y)^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx.
\]

Observe that, as in the case \( s < 1 \), \( J_d \) verifies the geometrical and compactness hypotheses to apply the Mountain Pass Theorem so taking, for instance, \( \phi(x) = (1 - |x|^2)^{\frac{\sigma}{2}} \) it follows that

\[
J_d(\phi_d) \leq Cd^{N/2}.
\]

Therefore a similar argument as above shows that, for \( d \) small enough, the mountain pass critical point is nonconstant.

### 6.2.1. Some open questions.

Among others, the following questions seem to be open and interesting to solve.

1. Asymptotic behavior of the nonconstant solutions when \( d \to 0, \ 0 < s \). The local case \( s = 1 \) this problem was studied for positive solutions in the pioneering paper [32], where a concentration phenomenon appears in the point of maximum curvature of \( \partial \Omega \). As far as we know, this result should be new in the local case \( s = 2 \) or higher integer order.
2. Study of the critical case \( p = 2^* \) and the behavior of the nonconstant solutions. The local case, \( s = 1 \), was studied in [2].
6.3. A Neumann condition for the p-Laplacian operator (-Δ)_s^p in the standard nonlocal case 0 < s < 1. Using the variational approach for the higher order operator developed in Section 4 we define a Neumann problem for the nonlinear p-Laplacian nonlocal operator that, to the best of our knowledge, it has not been studied up to now. Throughout all this section, let us suppose p ∈ (1, ∞), s ∈ (0, 1) and let Ω ⊂ ℝ^N be a smooth bounded domain with N > sp. For smooth functions, we define the fractional p-Laplacian operator (-Δ)_s^p (see for instance [10, 19, 26] and the references therein), as

\[(6.9) \quad (-Δ)_s^p u(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.\]

Moreover for $\mathcal{S}(\mathbb{R}^N)$, define

\[(6.10) \quad \mathcal{N}_{s,p} v(x) := \int_{\Omega} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega},\]

namely the non local Neumann condition in the case of the non local p-Laplace operator. The equation (6.10) represents the counterpart in the non local case, of the the local Neumann condition $|\nabla u|^p \partial_j u$, i.e. the normal component of the flux across the boundary. Following the proofs of the Proposition 2.3 and of the Proposition 2.7 using definitions (6.10) and (6.9) it can be proved the following

**Theorem 6.3.** Let $u, v \in \mathcal{S}(\mathbb{R}^N)$. Then

\[\int_{\Omega} (-Δ)_s^p u \, dx = - \int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_{s,p} u \, dx\]

and

\[\frac{1}{2} \int_{Q(\Omega)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy\]

\[= \int_{\Omega} v (-Δ)_s^p u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_{s,p} u \, dx.\]

Theorem 6.3 suggests, as we did in Section 3, the idea of which should be the correct weak formulation of the nonlocal p-Neumann problem in the case 0 < s < 1, i.e. it gives the good candidate for the weak form of the p-Laplacian operator (6.9). Now we can give the following

**Definition 6.4.** Let $g \in L^1(\mathbb{R}^N \setminus \Omega)$, we set

\[W_{-s,p}^{\Omega}(\mathcal{N}(g)) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u \text{ measurable and } \|u\|_{W_{-s,p}^{\Omega}(\mathcal{N}(g))} < +\infty \right\},\]

where

\[(6.11) \quad \|u\|_{W_{-s,p}^{\Omega}(\mathcal{N}(g))} = \left( \int_{\Omega} |u|^p \, dx + \int_{Q(\Omega)} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy + \int_{\mathcal{C}(\Omega)} |g||u|^p \, dx \right)^{\frac{1}{p}}.\]

Following the ideas done in [25] Proposition 3.1 we can be proved the next

**Proposition 6.5.** $W_{-s,p}^{\Omega}(\mathcal{N}(g))$ is a reflexive Banach space.

**Proof.** We sketch the proof. We can readily check that (6.11) is a norm and, arguing as in [25] Proposition 3.1, that $W_{-s,p}^{\Omega}(\mathcal{N}(g))$ is a Banach space. To prove that it is reflexive let us define the
space $\mathcal{A} = L^p(Q(\Omega), dx dy) \times L^p(\Omega, dx) \times L^p(\mathbb{R}^N \setminus \Omega, |g| dx)$. By standard results, this product space is reflexive. Then the operator $T : W_{\mathcal{N}(g)}^{s,p}(\Omega) \to \mathcal{A}$ defined as

$$Tu = \left[ \frac{|u(x) - u(y)|}{|x - y|^{N+2s}} \chi_Q(\Omega)(x,y), u \chi_\Omega, u \chi_{\mathbb{R}^N \setminus \Omega} \right],$$

where $\chi_S(\cdot)$ denotes the characteristic function of a measurable set $S$, is an isometry from $W_{\mathcal{N}(g)}^{s,p}(\Omega)$ into $\mathcal{A}$ (the space $\mathcal{A}$ is also equipped with the norm in (6.11)). Thus, since $W_{\mathcal{N}(g)}^{s,p}(\Omega)$ is a Banach space, $T(W_{\mathcal{N}(g)}^{s,p}(\Omega))$ is a closed subspace of $\mathcal{A}$ so that, $T(W_{\mathcal{N}(g)}^{s,p}(\Omega))$ is reflexive and therefore, using the fact that $T$ is an isometry, $W_{\mathcal{N}(g)}^{s,p}(\Omega)$ as well.

Thanks to the previous result, we can use the variational arguments developed in, for instance [20], to get the existence and uniqueness result for the $(-\Delta)_p^s$ operator, $0 < s < 1$. That is, the following

**Theorem 6.6.** Let $\Omega \subset \mathbb{R}^N$ a bounded $C^1$ domain and let us suppose that $f \in L^p(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $g \in L^\infty_c(\mathbb{R}^N \setminus \Omega)$. Then, the problem

$$\begin{cases}
(-\Delta)_p^s u = f(x) & \text{in } \Omega \\
\mathcal{N}_{s,p} u = g & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}$$

has a weak solution, that is,

$$\frac{1}{2} \int_{Q(\Omega)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} f v dx + \int_{\mathbb{R}^N \setminus \Omega} g v \, dx, \forall v \in W_{\mathcal{N}(g)}^{s,p}(\Omega),$$

if and only if the following compatibility condition holds

$$\int_{\Omega} f \, dx + \int_{\mathbb{R}^N \setminus \Omega} g \, dx = 0. \tag{6.12}$$

Moreover, if (6.12) holds, the solution is unique up to a constant $c \in \mathbb{R}^N$.

The proof of the previous result can be done using the same minimization techniques developed in the proof Theorem 4.4 under hypothesis ($\mathcal{A}_{(f,g_1,g_2)}$) — Case A once we prove a Poincaré-type inequality in the Banach space

$$\tilde{W}_{\mathcal{N}(g)}^{s,p}(\Omega) := \{ u \in W_{\mathcal{N}(g)}^{s,p}(\Omega) : \int_{\Omega} u \, dx = 0 \}. \tag{6.13}$$

This inequality will allow us to affirm that the norm in $\tilde{W}_{\mathcal{N}(g)}^{s,p}(\Omega)$ defined as

$$\|u\|_{\tilde{W}_{\mathcal{N}(g)}^{s,p}(\Omega)} := \int_{Q(\Omega)} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy,$$

is equivalent to the one in $W_{\mathcal{N}(g)}^{s,p}(\Omega)$ given in (6.11).

**Lemma 6.7.** For every $v \in \tilde{W}_{\mathcal{N}(g)}^{s,p}(\Omega)$ if $g \in L^\infty_c(\mathbb{R}^N \setminus \Omega)$ then the following Poincaré-type inequality holds

$$\int_{\Omega} |v|^p \, dx + \int_{\mathbb{R}^N \setminus \Omega} |g||v|^p \, dx \leq C(N, s, \Omega) \int_{Q(\Omega)} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dxdy.$$
Proof. Following the proof of (4.6) let us suppose, by contradiction, that there exists, up to a renormalization, a sequence \( \{v_k\} \subset \tilde{W}^{s,p}(\Omega) \) such that

\[
(6.14) \quad \int_{\Omega} |v_k|^p \, dx + \int_{\mathbb{R}^N \setminus \Omega} |g||v_k|^p \, dx = 1 \quad \text{and} \quad \int_{Q(\Omega)} \frac{|v_k(x) - v_k(y)|^p}{|x - y|^{N+sp}} \, dxdy < \frac{1}{k}.
\]

Using now that the embedding of \( W^{s,p}(\Omega) \) is compact (see [21]), it follows that, up to subsequence, there exists \( v \in L^p(\Omega) \) such that

\[
(6.15) \quad v_k \to v, \quad \text{in} \quad L^p(\Omega).
\]

Moreover if we take a ball \( B \) centered at the origin with \( \Omega \subset B \) we get by elementary inequalities that

\[
(6.16) \quad \int_{B} |v_k|^p \, dx \leq C(s,p,B,N,\Omega) \int_{B} \int_{\Omega} \frac{|v_k(x) - v_k(y)|^p}{|x - y|^{N+sp}} \, dxdy + \int_{\Omega} |v_k|^p \, dx
\]

By (6.14)–(6.16) we deduce that for all \( \varepsilon > 0 \), there exists \( k \) such that for all \( m,k > k \)

\[
\int_{B} |v_k - v_m|^p \, dx < \varepsilon,
\]

namely in particular \( v_k \) is a Cauchy sequence in \( L^p(B) \) and therefore, up to a subsequence, \( v_k \) converges to some \( v \) in \( L^p(B) \) and a.e. in \( B \). Passing to the limit in (6.14), by the lower semicontinuity of the norm w.r.t. the weak convergence, on one hand we get that \( v \) must be a constant in \( \mathbb{R}^N \) and on the other hand that

\[
\int_{\Omega} |v|^p \, dx + \int_{\mathbb{R}^N \setminus \Omega} |g||v|^p \, dx = 1,
\]

that is a contradiction with (6.13). \( \square \)

Now we can give the

Proof of Theorem 6.6. For every \( u \in W^{s,p}(\Omega) \) we define the nonlinear functional

\[
J_p(u) := \frac{1}{p} \int_{Q(\Omega)} \frac{\nabla u(x) - \nabla u(y)}{|x - y|^{N+sp}} \, dxdy - \int_{\Omega} fu \, dx - \int_{\mathbb{R}^N \setminus \Omega} gu \, dx.
\]

We note that, by the compatibility condition (6.12), it follows that \( J(u) = J(u - \overline{u}) \), where

\[
\overline{u} = \frac{1}{|\Omega|} \int_{\Omega} u \, dx.
\]

Therefore \( J_p \) can be defined in the space \( \tilde{W}^{s,p}(\Omega) \). Following the proof of Theorem 4.4 and the Poincaré inequality given in Lemma 6.7 the result follows. \( \square \)

6.3.1. Some open questions. Among many other possible choices, we think that it would be very interesting to find a natural Neumann condition for the operator

\[
(-\Delta)^s_p u(x) := -\operatorname{div} \left( \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus \Omega \setminus B_\varepsilon(x)} \frac{|\nabla u(x) - \nabla u(y)|^{p-2} (\nabla u(x) - \nabla u(y))}{|x - y|^{N+sp}} \, dy \right), \quad x \in \mathbb{R}^N,
\]

where \( s = 1 + \sigma \).
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