PICARD GROUPS OF SOME QUOT SCHEMES

CHANDRANANDAN GANGOPADHYAY AND RONNIE SEBASTIAN

Abstract. Let $C$ be a smooth projective curve over the field of complex numbers $\mathbb{C}$ of genus $g(C) > 0$. Let $E$ be a locally free sheaf on $C$ of rank $r$ and degree $e$. Let $Q := \text{Quot}_{C/\mathbb{C}}(E, k, d)$ denote the Quot scheme of quotients of $E$ of rank $k$ and degree $d$. For $k > 0$ and $d \gg 0$ we compute the Picard group of $Q$.

1. Introduction

Let $C$ be a smooth projective curve over the field of complex numbers $\mathbb{C}$. We shall denote the genus of $C$ by $g(C)$. Throughout this article we shall assume that $g(C) \geq 1$. Let $E$ be a locally free sheaf on $C$ of rank $r$ and degree $e$. Throughout this article

\begin{equation}
Q := \text{Quot}_{C/\mathbb{C}}(E, k, d)
\end{equation}

will denote the Quot scheme of quotients of $E$ of rank $k$ and degree $d$.

Stromme proved that $Q_{\mathbb{P}^1/\mathbb{C}}(O_{\mathbb{P}^1}^{\oplus n}, k, d)$ is a smooth projective variety and computed its Picard group and nef cone. In [Jow12], the author computes the effective cone of $Q_{\mathbb{P}^1/\mathbb{C}}(O_{\mathbb{P}^1}^{\oplus n}, k, d)$. In [Ito17], the author studies the birational geometry of $Q_{\mathbb{P}^1/\mathbb{C}}(O_{\mathbb{P}^1}^{\oplus n}, k, d)$. When $E$ is trivial and $g(C) \geq 1$, the space $Q$ is studied in [BDW96] and it is proved that when $d \gg 0$ it is irreducible and generically smooth. For $g(C) \geq 1$ and $E$ trivial, the divisor class group of $Q$ was computed in [HO10] under the assumption $d \gg 0$. When $g(C) \geq 1$, it was proved in [PR03] that $Q$ is irreducible and generically smooth when $d \gg 0$. See also [Gol19], [CCH21], [CCH22] for similar results on other variations of this Quot scheme. We use this as a starting point to further investigate the space $Q$ when $d \gg 0$ and compute its Picard group. In the case when $k = r - 1$ we have that $Q$ is a projective bundle over the Jacobian of $C$ for $d \gg 0$ (Theorem 3.3) and as a result its Picard group can be computed easily (Corollary 3.5). In Theorem 6.3 we show that if $d \gg 0$ then $Q$ is an integral variety which is normal, a local complete intersection and locally factorial. We compute the Picard group of $Q$ in the following cases.

Theorem 1.2 (Theorem 7.17). Let $k \leq r - 2$. Assume one of the following two holds

$\bullet$ $k \geq 2$ and $g(C) \geq 3$, or
$\bullet$ $k \geq 3$ and $g(C) = 2$.

Then for $d \gg 0$ we have

\[\text{Pic}(Q) \cong \text{Pic}^0(C) \times \mathbb{Z} \times \mathbb{Z}.\]

2010 Mathematics Subject Classification. 14J60.
Key words and phrases. Quot Scheme.
Note that we have a natural determinant map
\[ \text{det} : \mathcal{Q} \longrightarrow \text{Pic}^d(C) \]
which sends a quotient \([E \longrightarrow F] \mapsto \text{det}(F)\). In Theorem 6.3 we show that \text{det} is a flat map when \(d \gg 0\). For \([L] \in \text{Pic}^d(C)\) let \(\mathcal{Q}_L\) be the scheme theoretic fiber of \text{det} over \([L]\). We prove the following analogous results for \(\mathcal{Q}_L\).

**Theorem 1.3** (Theorem 8.7). Let \(k \geq 2, g(C) \geq 2\). Let \(d \gg 0\). Then \(\mathcal{Q}_L\) is a local complete intersection, integral, normal and locally factorial scheme.

**Theorem 1.4** (Theorem 8.9). Let \(k \leq r - 2\). Assume one of the following two holds
- \(k \geq 2\) and \(g(C) \geq 3\), or
- \(k \geq 3\) and \(g(C) = 2\).

Let \(d \gg 0\). Then \(\text{Pic}(\mathcal{Q}_L) \cong \mathbb{Z} \times \mathbb{Z}\).

When \(k = 1\) the above results can be improved to the case \(g(C) \geq 1\). In Theorem 9.1 we show that if \(d \gg 0\) then \(\text{Pic}(\mathcal{Q}) \cong \text{Pic}(\text{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}\) and \(\text{Pic}(\mathcal{Q}_L) \cong \mathbb{Z} \times \mathbb{Z}\).

We say a few words about how the above results are proved. By a very large open subset we mean an open set whose complement has codimension \(\geq 3\). Let \(\mathbb{H}\) be a closed point such that \(\mathcal{Q} \setminus \{y\}\) is empty then \(\mathbb{H}\) is nonempty. Note that we have a natural determinant map \(\text{det} : \mathcal{Q} \longrightarrow \text{Pic}^d(C)\).

2. **Preliminaries**

For a locally closed subset \(Z \subset X\) we shall refer to \(\text{dim}(X) - \text{dim}(Z)\) as the codimension of \(Z\) in \(X\). For a morphism \(f : X \longrightarrow Y\) and a closed point \(y \in Y\) we denote by \(X_y\) the fiber over \(y\).

**Lemma 2.1.** Let \(f : X \longrightarrow Y\) be a dominant morphism of integral schemes of finite type over a field \(k\). Let \(U \subset X\) be an open subset such that nonempty fibers of \(f|_U\) have constant dimension. Let \(Z := X \setminus U\).

1. If \(\text{dim}(X) - \text{dim}(Z) > \text{dim}(Y)\) then the dimension of nonempty fibers of \(f\) is constant.
2. Let \(t_0 \geq 0\) be an integer and assume \(\text{dim}(X) - \text{dim}(Z) > \text{dim}(Y) + t_0\). Let \(y \in Y\) be a closed point such that \(Z_y\) is nonempty. Then \(\text{dim}(X_y) - \text{dim}(Z_y) > t_0\).

**Proof.** Let \(y \in Y\) be a closed point such that \(X_y\) is nonempty. Note that \(X_y = U_y \sqcup Z_y\). If \(U_y\) is empty then
\[
\text{dim}(Z) \geq \text{dim}(Z_y) = \text{dim}(X_y) \geq \text{dim}(X) - \text{dim}(Y).
\]
This contradicts the hypothesis that $\dim(X) - \dim(Z) > \dim(Y)$. Thus, $U_y$ is nonempty. Since $f|_U$ has constant fiber dimension, it follows that $\dim(U_y) = \dim(U) - \dim(Y)$, see [Har77, Chapter 2, Exercise 3.22(b), (c)]. Since $X$ is integral, it follows that $\dim(U_y) = \dim(X) - \dim(Y)$. As $\dim(X) = \dim(Z) > \dim(Y)$ it follows that $\dim(Z_y) < \dim(X) - \dim(Y)$. It follows that

$$\dim(X_y) = \max\{\dim(U_y), \dim(Z_y)\} = \dim(X) - \dim(Y).$$

This proves (1).

Let $y \in Y$ be a closed point such that $Z_y$ is nonempty. Then $X_y$ is nonempty and so by the previous part we get that $\dim(X_y) = \dim(X) - \dim(Y)$. As $\dim(X) - \dim(Y) > \dim(Y) + t_0$ it follows that $\dim(Z_y) < \dim(X) - \dim(Y) - t_0 = \dim(X_y) - t_0$. This proves (2) and completes the proof of the Lemma.

Lemma 2.2. Let $f : X \to Y$ be a morphism of irreducible schemes of finite type over a field $k$ which is surjective on closed points. Let $Y' \subset Y$ be a closed subset. Then $\dim(X) - \dim(f^{-1}(Y')) \leq \dim(Y) - \dim(Y').$

Proof. Since it suffices to consider reduced schemes, we look at the map $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$. Thus, we may assume that $X$ and $Y$ are integral schemes. Let $Y'' \subset Y'$ be an irreducible component such that $\dim(Y'') = \dim(Y')$. Let $Z''$ be an irreducible component of $f^{-1}(Y'')$ which surjects onto $Y''$. By [Har77, Chapter 2, Exercise 3.22(a)] we have $\dim(X) - \dim(Z'') \leq \dim(Y) - \dim(Y'')$. As $Z'' \subset f^{-1}(Y')$ it follows that

$$\dim(X) - \dim(f^{-1}(Y')) \leq \dim(X) - \dim(Z'') \leq \dim(Y) - \dim(Y'') = \dim(Y) - \dim(Y').$$

This completes the proof of the Lemma.

Recall the space $Q$ from (1.1). Let

$$(2.3) \quad p_1 : C \times Q \to C \quad p_2 : C \times Q \to Q$$

denote the projections. Let

$$(2.4) \quad 0 \to K \to p_1^*E \to F \to 0$$

denote the universal quotient on $C \times Q$. The sheaf $K$ is locally free and so $p_1^*\det(E) \otimes (\wedge^{r-k} K)^{-1}$ is a line bundle on $C \times Q$ which is flat over $Q$. Using this we define the determinant map as

$$(2.5) \quad \det : Q \to \text{Pic}^d C,$$

which has the following pointwise description. Let $[q : E \to F] \in Q$ be a closed point. We denote the kernel of $q$ by $K$, so that there is a short exact sequence

$$\begin{equation}
(2.6) \quad 0 \to K \to E \xrightarrow{q} F \to 0.
\end{equation}$$

Then

$$\det(q) := \det(E) \otimes \det(K)^{-1} = \det(F).$$

Next we describe the differential of this map $\det$. 

Lemma 2.7. The differential of the map det (2.5) at the point \( q \) is the composite
\[
\Hom(K, F) \xrightarrow{-\delta} \Ext^1(F, F) \xrightarrow{\tr} H^1(C, \mathcal{O}_C),
\]
where the first map is obtained by applying \( \Hom(-, F) \) to (2.6) and the second map is the trace.

Proof. Let \( p_C : C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \to C \) denote the projection. Let \( \iota : C \hookrightarrow C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \) denote the reduced subscheme.

Given a vector \( v \in \Hom(K, F) \) it corresponds to an element in the Zariski tangent space at \( q \in \mathcal{Q} \), and so it corresponds to a short exact sequence on \( C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \)
\[
0 \to \tilde{K} \to p_C^*E \to \tilde{F} \to 0,
\]
whose restriction to \( C \) gives the sequence (2.6). Moreover, \( \tilde{F} \) is flat over \( \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \). Consider the line bundle \( \det(\tilde{F}) \) on \( C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \). Tensoring this line bundle with the short exact sequence
\[
0 \to (\epsilon) \to \mathbb{C}[\epsilon]/(\epsilon^2) \to \mathbb{C} \to 0
\]
gives the short exact sequence of sheaves on \( C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \)
\[
0 \to \iota_*\det(F) \to \det(\tilde{F}) \to \iota_*\det(F) \to 0.
\]
Using the definition of the differential of the map \( \det \) it is clear that
\[
d_\det_q(v) = \text{extension class of (2.9)} \in H^1(C, \mathcal{O}_C).
\]

Tensoring (2.8) with \( \tilde{F} \) gives a short exact sequence
\[
0 \to \iota_*F \to \tilde{F} \to \iota_*F \to 0.
\]
One checks easily, using the discussion before [HL10, Lemma 2.2.6], that the above extension, and in particular the sheaf \( \tilde{F} \), is obtained by taking the pushout of the sequence (2.6) along the map \( -v \). That is, the extension class of (2.11) in \( \Ext^1(F, F) \) is precisely \( -\delta(v) \).

For a coherent sheaf \( G \), consider the trace map \( \tr : \Ext^1(G, G) \to H^1(C, \mathcal{O}_C) \). An element \( v \in \Ext^1(G, G) \) corresponds to a short exact sequence
\[
0 \to G \to \tilde{G} \to G \to 0,
\]
on \( C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \) such that \( \tilde{G} \) is flat over \( \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \). The image \( \tr(v) \) in \( H^1(C, \mathcal{O}_C) \) corresponds to the extension class obtained by tensoring (2.8) with the line bundle \( \det(\tilde{G}) \) on \( C \times \Spec(\mathbb{C}[\epsilon]/(\epsilon^2)) \). When \( G \) is locally free this can be seen using a Cech description, for example, see [Nit09]. The general case reduces to the locally free case using the discussion in [HL10, §10.1.2]. In particular, we can apply this discussion by taking \( G = F \). We get that \( \tr(-\delta(v)) \) is the extension class obtained by tensoring \( \det(\tilde{F}) \) in (2.11) with (2.8). But note that we obtained (2.9) also by tensoring \( \det(\tilde{F}) \) with (2.8). This proves that
\[
d_\det_q(v) = \tr(-\delta(v))
\]
and completes the proof of the Lemma. We also refer the reader to [HL10, Theorem 4.5.3], where a similar result is proved for the moduli of stable bundles. \( \square \)
3. Quot Schemes $\text{Quot}_{C/C}(E, r - 1, d)$

Recall that for a sheaf $G$ on $C$ we define $\mu_{\min}(G)$ as
$$\min\{\mu(F) \mid F \text{ is a quotient of } G \text{ of positive rank}\}.$$  

In this section we describe the Quot scheme $\text{Quot}_{C/C}(E, r - 1, d)$ which parametrizes quotients of $E$ of rank $(r - 1)$ and degree $d > 2g - 2 + e - \mu_{\min}(E)$. Let

$$\rho_1 : C \times \text{Pic}^{e-d}(C) \to C, \quad \rho_2 : C \times \text{Pic}^{e-d}(C) \to \text{Pic}^{e-d}(C)$$

be the projections. Let $L$ be a Poincare bundle on $C \times \text{Pic}^{e-d}(C)$. Define
$$E := \rho_2^* [\rho_1^* E \otimes L^\vee].$$

**Lemma 3.2.** Assume $d > 2g - 2 + e - \mu_{\min}(E)$. Then $E$ is a vector bundle on $\text{Pic}^{e-d}(C)$ of rank $rd - (r - 1)e - r(g - 1)$.

**Proof.** Let $K_C$ denote the canonical bundle of $C$. For any $L \in \text{Pic}^{e-d}(C)$, we claim
$$H^1(C, E \otimes L^\vee) = H^0(C, E^\vee \otimes L \otimes K_C)^\vee = 0.$$  

This is because a nonzero section of $H^0(C, E^\vee \otimes L \otimes K_C)$ corresponds to a nonzero map $E \to L \otimes K_C$ which cannot exist since by assumption $\mu_{\min}(E) > \text{deg}(L \otimes K_C) = e - d + 2g - 2$. Therefore by Grauert’s theorem $E$ is a vector bundle of rank $h^0(C, E \otimes L^\vee)$ which by Riemann-Roch is $rd - (r - 1)e - r(g - 1)$. 

Let $\pi : \mathbb{P}(E^\vee) \to \text{Pic}^{e-d}(C)$ be the projective bundle associated to $E^\vee$. Here we use the notation in [Har77], that is, for a vector space $V$, $\mathbb{P}(V)$ denotes the space of hyperplanes in $V$. Thus, $\mathbb{P}(V^\vee)$ denotes the space of lines in $V$. Recall that we have the map
$$Q_{C/C}(E, r - 1, d) \to \text{Pic}^{e-d}(C)$$

which sends a quotient $[E \to F \to 0]$ to its kernel.

**Theorem 3.3.** Assume $d > 2g - 2 + e - \mu_{\min}(E)$. We have an isomorphism of schemes over $\text{Pic}^{e-d}(C)$
$$\mathbb{P}(E^\vee) \sim \sim Q_{C/C}(E, r - 1, d).$$

In particular, under the above assumption on $d$, the space $Q_{C/C}(E, r - 1, d)$ is smooth.

**Proof.** Let
$$\sigma_1 : C \times \mathbb{P}(E^\vee) \to C, \quad \sigma_2 : C \times \mathbb{P}(E^\vee) \to \mathbb{P}(E^\vee)$$

be the projections. We define the map $\mathbb{P}(E^\vee) \to Q_{C/C}(E, r - 1, d)$ by producing a quotient on $C \times \mathbb{P}(E^\vee)$.

Recall the maps $\rho_i$ from (3.1). By adjunction we have a natural map on $C \times \text{Pic}^{e-d}(C)$
$$\rho_2^* E \otimes L \to \rho_1^* E.$$  

Pulling this morphism back to $C \times \mathbb{P}(E^\vee)$ we get a map
$$(\text{Id}_C \times \pi)^*[\rho_2^* E \otimes L] = (\pi \circ \sigma_2)^* E \otimes (\text{Id}_C \times \pi)^* L \to \sigma_1^* E.$$  

We also have the morphism of sheaves on $\mathbb{P}(E^\vee)$
$$O(-1) \hookrightarrow \pi^* E.$$
Pulling this back to $C \times \mathbb{P}(\mathcal{E}^\vee)$ we get a composed map of sheaves on $C \times \mathbb{P}(\mathcal{E}^\vee)$

$$\sigma_2^* \mathcal{O}(1) \otimes (\text{Id}_C \times \mu)^* \mathcal{L} \longrightarrow (\mu \circ \sigma_2)^* \mathcal{E} \otimes (\text{Id}_C \times \mu)^* \mathcal{L} \longrightarrow \sigma_1^* \mathcal{E}.$$  

As $\sigma_2^* \mathcal{O}(1) \otimes (\text{Id}_C \times \mu)^* \mathcal{L}$ is a line bundle and $C \times \mathbb{P}(\mathcal{E}^\vee)$ is smooth, it easily follows that (3.4) is an inclusion as it is nonzero. By the previous lemma, a point $x \in \mathbb{P}(\mathcal{E}^\vee)$ corresponds to a pair $(L, \phi : L \to \mathcal{E})$ where $L$ is a line bundle of degree $e - d$ and $\phi$ is a nonzero homomorphism of sheaves, up to scalar multiplication. The inclusion (3.4) restricted to $C \times x$ is nothing but the nonzero homomorphism $\phi$. Therefore we get that the cokernel of (3.4), which we denote $\mathcal{F}$, is flat over $\mathbb{P}(\mathcal{E}^\vee)$, and the restriction $\mathcal{F}|_{C \times x}$ has rank $r - 1$ and degree $d$. Thus, $\mathcal{F}$ defines a map $\phi : \mathbb{P}(\mathcal{E}^\vee) \to \mathcal{Q}_{C/C}(E, r - 1, d)$. It is easily checked that this map is bijective on closed points.

Let point $x = [E \to F \to 0]$ be a point in $\mathcal{Q}_{C/C}(E, r - 1, d)$. Let $L$ be the kernel. Then we have an exact sequence

$$\text{Ext}^1(L, L) \longrightarrow \text{Ext}^1(L, E) \longrightarrow \text{Ext}^1(L, F) \longrightarrow 0.$$  

From the proof of Lemma 3.2 it follows that $\text{Ext}^1(L, E) = 0$. Hence $\text{Ext}^1(L, F) = 0$. Therefore $\mathcal{Q}_{C/C}(E, r - 1, d)$ is smooth at $x$ [HL10, Proposition 2.2.8]. As $\phi$ is bijective on closed points, it follows it is an isomorphism.

**Corollary 3.5.** Assume $d > 2g - 2 + e - \mu_{\text{min}}(E)$. Then $\mathcal{Q}_{C/C}(E, r - 1, d)$ is a smooth projective variety and $\text{Pic}(\mathcal{Q}_{C/C}(E, r - 1, d)) \cong \text{Pic}(\text{Pic}^0(C)) \times \mathbb{Z}$.

When $E$ is the trivial bundle, Theorem 3.3 is proved in [BDW96, Corollary 4.23].

4. THE GOOD LOCUS FOR TORSION FREE QUOTIENTS

The following Lemma is an easy consequence of [PR03, Lemma 6.1].

**Lemma 4.1.** Let $k$ be an integer. There is a number $\mu_0(E, k)$, which depends only on $E$ and $k$, such that for all torsion free sheaves $F$ with $\text{rk}(F) \leq k$ and $\mu_{\text{min}}(F) \geq \mu_0(E, k)$ we have $H^1(\mathcal{E}^\vee \otimes F) = 0$.

**Proof.** When $F$ is stable and $\text{rk}(F) \leq k$, it follows using [PR03, Lemma 6.1], that there is $\mu_0(E, k)$ such that if $\text{rk}(F) \leq k$ and $\mu(F) \geq \mu_0(E, k)$ then $H^1(\mathcal{E}^\vee \otimes F) = 0$.

Next let $F$ be semistable (see Remark following [PR03, Lemma 6.1]). Take a Jordan-Hölder filtration for $F$ and let $G$ be a graded piece of this filtration. As $\text{rk}(G) \leq k$ and $\mu(G) = \mu(F) \geq \mu_0(E, k)$ it follows from the stable case that $H^1(\mathcal{E}^\vee \otimes G) = 0$. From this it easily follows that if $F$ is semistable, $\text{rk}(F) \leq k$ and $\mu(F) \geq \mu_0(E, k)$ then $H^1(\mathcal{E}^\vee \otimes F) = 0$.

Now let $F$ be a locally free sheaf with $\text{rk}(F) \leq k$ and let

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_i = F$$

be its Harder-Narasimhan filtration. Each graded piece is semistable with slope

$$\mu(F_i/F_{i-1}) \geq \mu_0(E, k) = \mu_{\text{min}}(F).$$

Thus, if $\mu_{\text{min}}(F) \geq \mu_0(E, k)$ then $\mu_{\text{min}}(F_i/F_{i-1}) \geq \mu_0(E, k)$ and so from the semistable case it follows that $H^1(\mathcal{E}^\vee \otimes (F_i/F_{i-1})) = 0$. Again it follows that $H^1(\mathcal{E}^\vee \otimes F) = 0$. This proves the lemma. \[\Box\]
Let $G$ be a locally free sheaf on $C$ and let $k$ be an integer. Define

$$d_k(G) := \min \{ d \mid \exists \text{quotient } G \to F \text{ with } \deg(F) = d, \rk(F) = k \}. \tag{4.2}$$

Remark 4.3. We recall some results from [PR03] (see [PR03, Lemma 6.1, Proposition 6.1, Theorem 6.4] and the remarks following these). There is an integer $\alpha(E, k)$ such that when $d \geq \alpha(E, k)$, the following three assertions hold:

1. If $F$ is a stable bundle of rank $k$ and degree $d$, then $E^\vee \otimes F$ is globally generated.
2. $Q$ is irreducible and generically smooth of dimension $rd - ek - k(r - k)(g - 1)$.
3. For the general quotient $E \to F$, with $F$ having rank $k$ and degree $d$, we have the sheaf $F$ is torsion free and stable.

\[ \square \]

Definition 4.4. Let $a, b$ be integers. Let $\Quot_{C/C}(E, a, b)$ be the Quot scheme parametrizing quotients of $E$ of rank $a$ and degree $b$. For a locally closed subset $A \subset \Quot_{C/C}(E, a, b)$ define the following locally closed subsets of $A$.

$$A_g := \{ [E \to F] \in A \mid H^1(E^\vee \otimes F) = 0 \}$$
$$A_b := A \setminus A_g$$
$$A_{\text{tf}} := \{ [E \to F] \in A \mid F \text{ is torsion free} \}$$
$$A_{g, \text{tf}} := A_{\text{tf}} \cap A_g$$
$$A_{b, \text{tf}} := A_{\text{tf}} \cap A_b$$

In particular, we get subsets $Q_{g, \text{tf}}$, $Q_{b, \text{tf}}$.

For integers $0 < k'' < k < r$ define constants

$$C_1(E, k, k'') := k''(r - k'') - d_{k''}(E)r + (k - k'')(r - k) - d_k(E)(r - k'')$$
$$C_2(E, k, k'') := -ek - k(r - k)(g - 1) - C_1(E, k, k'')$$
$$C_3(E, k) := \min_{k'' < k} \{ C_2(E, k, k'') \}. \tag{4.6}$$

Let $t_0$ be a positive integer. Define

$$\beta(E, k, t_0) := \max \{ (r - 1)\mu_0(E, r - 1), r^2\mu_0(E, r - 1) + t_0 - C_3(E, k), \alpha(E, k), 1 \}. \tag{4.7}$$

Remark 4.8. From the definition it is clear that $\beta(E, k, t_0) \geq \alpha(E, k)$ for all integers $t_0 \geq 1$, if $t_1 \geq t_0 \geq 1$ then $\beta(E, k, t_1) \geq \beta(E, k, t_0)$ and $\beta(E, k, t_0) \geq 1$ for all positive integers $t_0$. To define the constants $C_1, C_2, C_3$ we need that $r \geq 3$. Note that if $r = 2$, then the only possible value for $k$ is 1, which equals $r - 1$. This case has been dealt with in the previous section. Thus, from now on we may assume that $r \geq 3$. These constants will play a role while computing dimensions of some subsets of $\Quot_{C/C}(E, k, d)$. We emphasize that these constants are independent of $d$.

Lemma 4.9. Fix positive integers $t_0$ and $k$ such that $k < r$. Let $d \geq \beta(E, k, t_0)$. Let $S$ be an irreducible component of $Q_{b, \text{tf}}$. Then $\dim(Q) - \dim(S) > t_0$ and so also $\dim(Q) - \dim(Q_{b, \text{tf}}) > t_0$. 
Proof. We give \( S \) the reduced subscheme structure so that \( S \) is an integral scheme. Let \( q \in S \) be a closed point corresponding to a quotient \( E \rightarrow F \). If \( F \) is semistable, then using \( d \geq \beta(E, k, t_0) \geq (r - 1)\mu_0(E, r - 1) \) (note that as \( \beta(E, k, t_0) > 0 \) we have \( d > 0 \)) we get
\[
\mu(F) = \mu_{\min}(F) = \frac{d}{k} \geq \frac{d}{r - 1} \geq \mu_0(E, r - 1).
\]
It follows from Lemma 4.1 that \( q \in \mathcal{Q}^H_{\mathbb{g}} \), which is a contradiction as \( q \in \mathcal{Q}^H_{\mathbb{b}} \). Thus, \( F \) is not semistable.

Let \( p_1 : C \times S \rightarrow C \) denote the projection. Consider the pullback of the universal quotient from \( C \times Q \) to \( C \times S \) and denote it
\[
p_1^* E \rightarrow F.
\]
From [HL10, Theorem 2.3.2] (existence of relative Harder-Narasimhan filtration) it follows that there is a dense open subset \( U \subset S \) and a filtration
\[
0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_l = F
\]
such that \( F_i/F_{i-1} \) is flat over \( U \) and for each closed point \( u \in U \), the sheaf \( F_{i,u}/F_{i-1,u} \) is semistable. Consider the quotient \( p_1^* E \rightarrow F_l \rightarrow F_l/F_{l-1} \). Denote the kernel by \( S \) so that we have an exact sequence
\[
0 \rightarrow S \rightarrow p_1^* E \rightarrow F_l/F_{l-1} \rightarrow 0
\]
on \( C \times U \). Let us denote
\[
F'' := F_l/F_{l-1}, \quad F' := F_{l-1}.
\]
With this notation we have a commutative diagram of short exact sequences on \( C \times U \)
\[
\begin{array}{ccc}
0 & \rightarrow & S \\
& \downarrow & \downarrow \\
p_1^* E & \rightarrow & F'' \\
& \downarrow & \downarrow \\
0 & \rightarrow & F' \\
0 & \rightarrow & F \\
0 & \rightarrow & F'' \\
\end{array}
\]
In particular, we observe that the map \( E \rightarrow F_u \) can be obtained as the pushout of the short exact sequence \( 0 \rightarrow S_u \rightarrow E \rightarrow F''_u \rightarrow 0 \) along the map \( S_u \rightarrow F'_u \).

For a closed point \( u \in U \) define
\[
k'' := \text{rk}(F''_u), \quad d'' := \text{deg}(F''_u).
\]
Then
\[
\text{rk}(F'_u) = k - k'', \quad \text{deg}(F'_u) = d - d''.
\]
The top row of (4.10) defines a map
\[
\theta : U \rightarrow \text{Quot}_{C/C}(E, k'', d'').
\]
For ease of notation let us denote \( A := \text{Quot}_{C/C}(E, k'', d'') \). Let \( S_1 \) denote the universal kernel bundle on \( C \times A \). Then \((\text{Id}_C \times \theta)^* S_1 = S\). The left vertical arrow of (4.10) defines a
We will find a lower bound for $d$. Again using \((4.12)\) \(\dim(A)\) as the pushout of the short exact sequence $0 \to S \to E \to F \to 0$. This shows that $\theta$ is injective on closed points. We claim that the map $\tilde{\theta}$ is injective on closed points. Let $u_1, u_2 \in U$ be such that $\tilde{\theta}(u_1) = \tilde{\theta}(u_2)$. Then $\theta(u_1) = \theta(u_2)$. It follows that the quotients $E \to F_{u_1}$ and $E \to F_{u_2}$ are the same, that is, $S_u = S_u$. Since $\tilde{\theta}(u_1) = \tilde{\theta}(u_2)$ it follows that the quotients $S_u \to F_u$ and $S_u \to F_u$ are the same. We observed after (4.10), that the quotient $E \to F_u$ is obtained as the pushout of the short exact sequence $0 \to S_{u_1} \to E \to F_u \to 0$ along the map $S_u \to F_u$. From this it follows that the quotients $E \to F_u$ are the same. Thus, the map $\tilde{\theta}$ is injective on closed points.

Let us compute the dimension of $\text{Quot}_{C/A}(S_1, k - k'', d - d'')$. Consider a quotient $[E \to F''']$ which corresponds to a closed point in $A$. Let $S_{F'''}$ denote the kernel. It has rank $r - k''$. The fiber of $\pi$ over $[E \to F''']$ is the Quot scheme $\text{Quot}_{C/C}(S_{F''}, k - k'', d - d'')$. Recall from (4.2) the integer $d_{k-k''}(S_{F''})$, which is the smallest possible degree among all quotients of $S_{F''}$ of rank $k - k''$. Thus, if the fiber is nonempty then we have that

$$d - d'' \geq d_{k-k''}(S_{F''}).$$

By [PR03, Theorem 4.1] it follows that, if the fiber is nonempty then

\[
\dim(\text{Quot}_{C/C}(S_{F''}, k - k'', d - d'')) \leq (k - k'')(r - k) + (d - d'' - d_{k-k''}(S_{F''}))(r - k'').
\]

We will find a lower bound for $d_{k-k''}(S_{F''})$. Let $S_{F''} \to G$ be a quotient such that $\deg(G) = d_{k-k''}(S_{F''})$. Then we can form the pushout $G$ which sits in the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & S_{F''} \\
\downarrow & & \downarrow \\
0 & \to & G
\end{array}
\quad \begin{array}{ccc}
E & \to & F'' \\
\downarrow & & \downarrow \\
\tilde{G} & \to & F''
\end{array}
\quad \begin{array}{ccc}
& & 0 \\
& & \downarrow \\
& & 0
\end{array}
\]

Since $\tilde{G}$ is a quotient of $E$ of rank $k$, it follows that

$$\deg(\tilde{G}) = d'' + d_{k-k''}(S_{F''}) \geq d_k(E).$$

This shows that $d_{k-k''}(S_{F''}) \geq d_k(E) - d''$. Combining this with (4.11) yields

\[
\dim(\text{Quot}_{C/C}(S_{F''}, k - k'', d - d'')) \leq (k - k'')(r - k) + (d - d_k(E))(r - k'').
\]

Again using [PR03, Theorem 4.1] it follows that

\[
\dim(A) = \dim(\text{Quot}_{C/C}(E, k'', d'')) \leq k''(r - k'') + (d'' - d_{k''}(E))r.
\]
Combining (4.12) and (4.13), using (4.6) and injectivity of \( \tilde{\theta} \) we get that

\[
\dim(U) \leq \text{Quot}_{\mathcal{C} \times A/A}(S_1, k - k'', d - d'') \\
\leq k''(r - k'') + (d'' - d_k(E))r + (k - k'')(r - k) + (d - d_k(E))(r - k'') \\
= C_1(E, k, k'') + d(r - k'') + d''r
\]

From this, Remark 4.3(2) and (4.6) it follows that

\[
\dim(Q) - \dim(U) \geq C_2(E, k, k'') + dk'' - d''r. \tag{4.14}
\]

We claim that

\[
\frac{C_2(E, k, k'') + dk'' - d''r}{r} > t_0.
\]

But this yields

\[
\frac{C_3(E, k) + d - t_0}{r^2} \leq \frac{C_2(E, k, k'') + d - t_0}{r^2} < \frac{C_2(E, k, k'') + dk'' - t_0}{rk''} \leq \frac{d''}{k''}. \tag{4.15}
\]

Let \( u \in U \) be a closed point. Then \( \mu_{\text{min}}(F_u) = d''/k'' \). By the assumption on \( d \) we have that

\[
d \geq \beta(E, k, t_0) \geq r^2 \mu_0(E, r - 1) + t_0 - C_3(E, k).
\]

Using this and (4.15) gives

\[
\mu_0(E, r - 1) \leq \frac{C_3(E, k) + d - t_0}{r^2} < \frac{d''}{k''} = \mu_{\text{min}}(F_u).
\]

It follows from Lemma 4.1 that \( H^1(E^\vee \otimes F_u) = 0 \), that is, \( u \in Q^{tf}_b \). But this is a contradiction as \( U \subset Q^{tf}_b \). Thus, it follows from (4.14) that

\[
\dim(Q) - \dim(S) \geq C_2(E, k, k'') + dk'' - d''r > t_0.
\]

This completes the proof of the Lemma.

\[\square\]

5. Locus of Quotients which are not Torsion Free

For a sheaf \( F \), denote the torsion subsheaf of \( F \) by \( \text{Tor}(F) \). For an integer \( i \geq 1 \) define the locally closed subset

\[
Z_i := \{ [q : E \to F] \in Q | \deg(\text{Tor}(F)) = i \}.
\]

We now estimate the dimension of \( Z_i \) and \( (Z_i)_b \) (recall the definition of \( (Z_i)_b \) from (4.5)).

**Lemma 5.2.** With notation as above we have

1. Assume that \( d - i \geq \alpha(E, k) \) (see Remark 4.3). Then \( Z_i \) is irreducible and \( \dim(Z_i) = \dim(Q) - ki \). Moreover, \( \bar{Z}_i \supset \bigcup_{j \geq i} Z_j \).
2. Let \( t_1 \) be a positive integer. If \( d - i \geq \beta(E, k, t_1) \) (see (4.7) for definition of \( \beta \)) then \( \dim(Z_i) - \dim((Z_i)_b) > t_1 \).
3. If \( d - i \geq \beta(E, k, t_1) \) then \( \dim(Q) - \dim((Z_i)_b) > t_1 + ki \).
Proof. Consider the Quot scheme Quot\(_{C/C}(E,k,d-i)\). For ease of notation we denote \(A = \text{Quot}_{C/C}(E,k,d-i)\). Let
\[
0 \to \mathcal{I} \to p_1^*E \to \mathcal{F} \to 0
\]
be the universal quotient on \(C \times A\). Consider the relative Quot scheme
\[
(5.3) \quad \text{Quot}_{C \times A/A}(\mathcal{I},0,i) \xrightarrow{\pi} A.
\]
There is a map
\[
(5.4) \quad \text{Quot}_{C \times A/A}(\mathcal{I},0,i) \xrightarrow{\pi'} Q
\]
whose image consists of precisely those quotients \([E \to F]\) for which \(\text{deg}(\text{Tor}(F)) \geq i\).

Recall the locus \(A^{\text{tf}}\) from (4.5). One checks easily that
\[
(5.5) \quad \pi^{i-1}(Z_i) = \pi^{-1}(A^{\text{tf}}).
\]
In fact, one easily checks that \(\pi' : \pi^{-1}(A^{\text{tf}}) \to Z_i\) is a bijection on points and so they have the same dimension. As \(d-i \geq \alpha(E,k)\), by Remark 4.3(2), it follows that \(A\) is irreducible of dimension
\[
\dim(A) = r(d-i) - ek - k(r-k)(g-1).
\]
By Remark 4.3(3), it follows that \(A^{\text{tf}}\) is a dense open subset of \(A\). If \([E \to F] \in A\) is a quotient, let \(S_F\) denote the kernel. The fiber of \(\pi\) over this point is the Quot scheme \(\text{Quot}_{C/C}(S_F,0,i)\), which is irreducible and has dimension \((r-k)i\). From this it follows that \(\text{Quot}_{C \times A/A}(\mathcal{I},0,i)\) is irreducible of dimension \(\dim(Q) - ki\). Thus, the open set \(\pi^{-1}(A^{\text{tf}})\) also has the same dimension and is irreducible. As this open subset dominates \(Z_i\), the claim about the irreducibility and dimension of \(Z_i\) follows. We have already observed that the image of \(\pi'\) is the locus \(\bigcup_{j \geq i} Z_j\). As \(\pi^{-1}(A^{\text{tf}})\) is dense in \(\text{Quot}_{C \times A/A}(\mathcal{I},0,i)\), the proof of (1) is complete.

To prove the second assertion, note that
\[
H^1(E^\vee \otimes F) = H^1(E^\vee \otimes (F/\text{Tor}(F))).
\]
One checks easily that
\[
(5.6) \quad \pi'^{-1}((Z_i)_b) = \pi^{-1}(A^{\text{tf}}_b).
\]
As \(\pi\) has constant fiber dimension, we see
\[
\dim(A^{\text{tf}}) - \dim(A^{\text{tf}}_b) = \dim(\pi^{-1}(A^{\text{tf}})) - \dim(\pi^{-1}(A^{\text{tf}}_b)).
\]
By applying Lemma 2.2 to the map \(\pi'\), and using (5.5) and (5.6), we get
\[
\dim(A^{\text{tf}}) - \dim(A^{\text{tf}}_b) = \dim(\pi^{-1}(A^{\text{tf}})) - \dim(\pi^{-1}(A^{\text{tf}}_b)) = \dim(\pi'^{-1}(Z_i)) - \dim(\pi'^{-1}((Z_i)_b)) \leq \dim(Z_i) - \dim((Z_i)_b).
\]
As \(d-i \geq \beta(E,k,t_1) \geq \alpha(E,k)\) it follows from Remark 4.3(2) and (3) that \(\text{Quot}_{C/C}(E,k,d-i)\) is irreducible and so \(\dim(\text{Quot}_{C/C}(E,k,d-i)) = \dim(\text{Quot}_{C/C}(E,k,d-i)^{\text{tf}})\). By Lemma 4.9 it follows that
\[
\dim(A^{\text{tf}}) - \dim(A^{\text{tf}}_b) = \dim(\text{Quot}_{C/C}(E,k,d-i)^{\text{tf}}) - \dim(\text{Quot}_{C/C}(E,k,d-i)^{\text{tf}}_b) > t_1.
\]
This proves that \(\dim(Z_i) - \dim((Z_i)_b) > t_1\). This proves (2).

Assertion (3) of the Lemma follows easily using the first two. \(\square\)
6. Flatness of det

We begin by showing that when \( d \gg 0 \), \( Q \) is a local complete intersection. This result seems well known to experts (see [BDW96, Theorem 1.6] and the paragraph following it); however, we include it as we could not find a precise reference.

**Lemma 6.1.** Let \( d \geq \alpha(E, k) \). Then \( Q \) is a local complete intersection scheme. In particular, it is Cohen-Macaulay.

**Proof.** By Remark 4.3(2), \( Q \) is irreducible and so \( \dim_q(Q) \) is independent of the closed point \( q \in Q \). Let \( F \) denote the universal quotient and let \( K \) denote the universal kernel on \( C \times Q \). For a closed point \( q \in Q \) we shall denote the restrictions of these sheaves to \( C \times q \) by \( K_q \) and \( F_q \). The sheaf \( K \) is locally free on \( C \times Q \). It follows that \( K_q \otimes F \) is flat over \( Q \), and so the Euler characteristic of \( K_q \otimes F_q \) is constant, call it \( \chi \). As \( Q^f \) is nonempty, let \( q \in Q^f \) be a closed point. As \( h^1(K_q \otimes F_q) = 0 \), it follows from [HL10, Proposition 2.2.8] that
\[
\dim_q(Q) = h^0(K_q \otimes F_q) = h^0(K_q \otimes F_q) - h^1(K_q \otimes F_q) = \chi.
\]
Let \( t \in Q \) be a closed point. We already observed that \( \dim_t(Q) \) is independent of the closed point \( t \in Q \) and so is equal to \( \chi \). It follows that for all closed points \( t \in Q \) we have
\[
\dim_t(Q) = \chi = h^0(K_t \otimes F_t) - h^1(K_t \otimes F_t).
\]
By [HL10, Proposition 2.2.8] it follows that the space \( Q \) is a local complete intersection at any closed point and so is also Cohen-Macaulay. \( \square \)

**Lemma 6.2.** Fix a positive integer \( t_0 \). Let \( i_0 \) be the smallest integer such that \( ki_0 > g(C) + t_0 \). If \( d \geq \beta(E, k, g(C) + t_0) + i_0 \) then \( \dim(Q) - \dim(Q_b) > g(C) + t_0 \).

**Proof.** First observe that we can write
\[
Q = Q^f \sqcup \bigcup_{i \geq 1} Z_i.
\]
Only finitely many indices \( i \) appear. In fact, \( i \) can be at most \( d - d_k(E) \), see (4.2). In view of this we get
\[
Q_b = Q_b^f \sqcup \bigcup_{i \geq 1} (Z_i)_b.
\]
By Lemma 4.9, since \( d \geq \beta(E, k, g(C) + t_0) \) we have
\[
\dim(Q) - \dim(Q^f_b) > g(C) + t_0.
\]
If \( 1 \leq i \leq i_0 \) then \( d - i \geq d - i_0 \geq \beta(E, k, g(C) + t_0) \), and so by Lemma 5.2(3) we get
\[
\dim(Q) - \dim((Z_i)_b) > g(C) + t_0 + ki.
\]
By Lemma 5.2(1) we also get that \( \bar{Z}_{i_0} \supset \bigcup_{j \geq i_0} Z_j \). For \( j \geq i_0 \),
\[
\dim((Z_j)_b) \leq \dim(Z_j) \leq \dim(Z_{i_0}) = \dim(Q) - ki_0.
\]
This shows that for \( j \geq i_0 \) we have
\[
\dim(Q) - \dim((Z_j)_b) \geq ki_0 > g(C) + t_0.
\]
Combining these shows that \( \dim(Q) - \dim(Q_b) > g(C) + t_0 \). This completes the proof of the Lemma. \( \square \)
Theorem 6.3. Recall the map $\det$ defined in (2.5).

1. Let $n_0$ be the smallest integer such that $k n_0 > g(C) + 1$. Let $d \geq \beta(E, k, g(C) + 1) + n_0$. Then $\det : Q \to \text{Pic}^d(C)$ is a flat map. Further, $Q$ is an integral and normal variety.

2. Let $n_1$ be the smallest integer such that $k n_1 > g(C) + 3$. Let $d \geq \beta(E, k, g(C) + 3) + n_1$. Then $Q$ is locally factorial.

Proof. Let $q \in Q$ be a closed point and let $K$ denote the kernel of the quotient $q$. Then we have a short exact sequence

$$0 \to K \to E \to F \to 0.$$ 

Applying $\text{Hom}(-, F)$ and using Lemma 2.7 we get the following diagram, in which the top row is exact.

$$\begin{array}{cccccc}
\text{Hom}(K, F) & \to & \text{Ext}^1(F, F) & \to & \text{Ext}^1(E, F) & \to \text{Ext}^1(K, F) \to 0 \\
& & & & \Big\downarrow & \\
& & & & \text{tr} & \\
& & & & & \text{H}^1(C, \mathcal{O}_C) \\
\end{array}$$

If $H^1(E^\vee \otimes F) = 0$ then we make the following two observations. First observe that it follows that $H^1(K^\vee \otimes F) = 0$, which shows that $Q_g$ is contained in the smooth locus of $Q$, by [HL10, Proposition 2.2.8]. Second observe that the map $\text{Hom}(K, F) \to \text{Ext}^1(F, F)$ will be surjective. As $\text{Ext}^1(F, F) \to \text{H}^1(C, \mathcal{O}_C)$ is surjective, it follows that if $H^1(E^\vee \otimes F) = 0$ then the diagonal map in the above diagram is surjective. However, the diagonal map is precisely the differential of det at the point $q$. As $Q_g$ and $\text{Pic}^d(C)$ are smooth, it follows that the restriction of det to $Q_g$ is a smooth morphism and so also flat and dominant.

Assume $d \geq \beta(E, k, g(C) + 1) + n_0$. Applying Lemma 6.2 we get

$$\dim(Q) - \dim(Q_b) > g(C) + 1.$$ 

We observed in Lemma 6.1 that $Q$ is a Cohen-Macaulay scheme and so it satisfies Serre’s condition $S_2$. The open subset $Q_g$ is smooth. As $Q_b = Q \setminus Q_g$, it follows that $Q$ satisfies Serre’s condition $R_1$. Thus, $Q$ is an integral and normal variety.

In view of Lemma 6.1 and [Mat86, Theorem 23.1] or [Stk, Tag 00R4], to prove the first assertion of the theorem, it suffices to show that the fibers of det have constant dimension. Applying Lemma 2.1(1), by taking $U$ to be the open subset $Q_g$, we get that det is flat. This proves (1).

Now we prove (2). Assume $d \geq \beta(E, k, g(C) + 3) + n_1$. Applying Lemma 6.2 we get

$$\dim(Q) - \dim(Q_b) > g(C) + 3.$$ 

This implies that the singular locus has codimension 4 or more. Now we use a result of Grothendieck which states that if $R$ is a local ring that is a complete intersection in which the singular locus has codimension 4 or more, then $R$ is a UFD. We refer the reader to [Gro05], [Cal94], [AH20, Theorem 1.4]. This implies that $Q$ is locally factorial. The proof of the theorem is now complete. \qed
7. Locus of stable quotients and Picard group of $Q$

7.1. In this section we will be using two Quot schemes. Thus, it is worth recalling that $Q$ denotes the Quot scheme $\text{Quot}_{\mathbb{C}/\mathbb{C}}(E, k, d)$. We begin by explaining a result from [Bho99] that we need. Assume one of the following two holds

- $k \geq 2$ and $g(C) \geq 3$, or
- $k \geq 3$ and $g(C) = 2$.

Let $d \geq \alpha(E, k)$. Fix a closed point $P \in C$. For a closed point $q \in Q$, let $[E \longrightarrow \mathcal{F}_q]$ denote the quotient corresponding to this closed point. We may choose $n \gg 0$ such that for all $q \in Q^\text{tf}$ we have $H^1(C, \mathcal{F}_q(nP)) = 0$ and $\mathcal{F}_q(nP)$ is globally generated. As $d \geq \alpha(E, k)$, by Remark 4.3, it follows that $Q^\text{tf}$ is irreducible, and so $h^0(C, \mathcal{F}_q(nP))$ is independent of $q$. Let

$$N := h^0(C, \mathcal{F}_q(nP))$$

and consider the Quot scheme $\text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn)$. Let $\mathcal{G}'$ denote the universal quotient on $C \times \text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn)$. Let $R \subset \text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn)$ be the open subset containing closed points $[x : \mathcal{O}_C^\oplus N \longrightarrow \mathcal{G}'_x]$ such that $\mathcal{G}'_x$ is torsion free, $H^1(C, \mathcal{G}'_x) = 0$ and the quotient map $\mathcal{O}_C^\oplus N \longrightarrow \mathcal{G}'_x$ induces an isomorphism $\mathbb{C}^N \cong H^0(C, \mathcal{G}'_x)$. This is the space $R$ in [Bho99, page 246, Proposition 1.2], see [Bho99, page 246, Notation 1.1]. The space $R$ is a smooth equidimensional scheme. Let $R^s$ (respectively, $R^ss$) denote the open subset of $R$ consisting of closed points $x$ for which $\mathcal{G}'_x$ is stable (respectively, semistable). In [Bho99, page 246, Proposition 1.2] it is proved that $\dim(R) - \dim(R \setminus R^s) \geq 2$.

Let

$$p_1 : C \times \text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn) \longrightarrow C$$

$$p_2 : C \times \text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn) \longrightarrow \text{Quot}_{\mathbb{C}/\mathbb{C}}(\mathcal{O}_C^\oplus N, k, d + kn)$$

denote the projections. Let

$$\mathcal{G} := \mathcal{G}' \otimes p_1^*(\mathcal{O}_C(-nP)).$$

Let $R' \subset R$ be the open subset containing closed points $x$ for which $H^1(C, E^\vee \otimes \mathcal{G}_x) = 0$. By Cohomology and Base change theorem it follows that $p_2^*(p_1^* E^\vee \otimes \mathcal{G})$ is locally free on $R'$. The fiber over a point $x \in R'$ is isomorphic to the vector space $\text{Hom}(E, \mathcal{G}_x)$. Consider the projective bundle

$$\mathbb{P}(p_2^*(p_1^* E^\vee \otimes \mathcal{G})^\vee) \overset{\Theta}{\longrightarrow} R'.$$

The fiber of $\Theta$ over a point $x \in R'$ is the space of lines in the vector space $\text{Hom}(E, \mathcal{G}_x)$. For ease of notation we denote $\mathbb{P}(p_2^*(p_1^* E^\vee \otimes \mathcal{G})^\vee)$ by $\mathbb{P}$. Denote the projection maps from $C \times \mathbb{P}$ by

$$p'_1 : C \times \mathbb{P} \longrightarrow C,$$

$$p'_2 : C \times \mathbb{P} \longrightarrow \mathbb{P}.$$

Consider the following Cartesian square

$$\begin{array}{ccc}
C \times \mathbb{P} & \overset{\Theta}{\longrightarrow} & C \times R' \\
p'_2 & & \downarrow p_2 \\
\mathbb{P} & \overset{\Theta}{\longrightarrow} & R' \\
\end{array}$$
Let $O(1)$ denote the tautological line bundle on $\mathbb{P}$. Then we have a map of sheaves on $C \times \mathbb{P}$

$$p_1^*E \rightarrow \tilde{\Theta}^*G \otimes p_2^*O(1).$$

(7.4)

A closed point $v \in \mathbb{P}$ corresponds to the closed point $\Theta(v) \in R'$ and a line spanned by some $w_v \in \text{Hom}(E, G_{\Theta(v)})$. The restriction of (7.4) to $C \times \mathbb{P}$ gives the map $w_v : E \rightarrow G_{\Theta(v)}$. Let $U \subset \mathbb{P}$ denote the open subset parametrizing points $v$ such that $w_v$ is surjective. On $C \times U$ we have a surjection

$$p_1^*E \rightarrow \tilde{\Theta}^*G \otimes p_2^*O(1).$$

(7.5)

This defines a morphism

$$\Psi : U \rightarrow Q^{{\text{tf}}}. $$

(7.6)

Lemma 7.7. $\Psi$ is surjective on closed points.

Proof. Let $[q : E \rightarrow F_q] \in Q^{{\text{tf}}}$ be a closed point. By our choice of $n$ and $N$ (see (7.2)), we have that $F_q(nP)$ is globally generated and $N = h^0(C, F_q(nP))$. Therefore, by choosing a basis for $H^0(C, F_q(nP))$ we get a surjection $[O_C^N \rightarrow F_q(nP)]$. Now it follows easily that $\Psi$ is surjective on closed points. $\square$

Now further assume $d \geq \max\{\alpha(E, k), k\mu_0(E, k)\}$. By Lemma 4.1 we have $H^1(C, E^\vee \otimes G_x) = 0$ for $x \in R^s$. Thus, we have inclusions of open sets $R^s \subset R^s \subset R$. Let $P^s \subset \mathbb{P}$ denote the inverse image of $R^s$ under the map $\Theta$. Similarly, let $U^s \subset U$ denote the inverse image of $R^s$ under the restriction of $\Theta$ to $U$. Let

$$Q^s := \{[E \rightarrow F] \in Q | F \text{ is stable}\}. $$

(7.8)

As $d \geq k\mu_0(E, k)$, by Lemma 4.1 we have $H^1(C, E^\vee \otimes F) = 0$ for $[E \rightarrow F] \in Q^s$. It follows that $Q^s \subset Q^{{\text{tf}}}$. It is easily checked that

$$\Psi^{-1}(Q^s) = U^s. $$

(7.9)

The group $\text{PGL}(N)$ acts freely on $P^s$ and leaves the open subset $U^s$ invariant. Consider the trivial action of $\text{PGL}(N)$ on $Q^s$. Then the restriction $\Psi : U^s \rightarrow Q^s$ is $\text{PGL}(N)$-equivariant. It is clear that the restriction of the map $\Theta : P^s \rightarrow R^s$ is also $\text{PGL}(N)$-equivariant. Let $M^s_{k,d+kn}$ (respectively, $M^s_{k,d+kn}$) denote the moduli space of stable (respectively, semistable) bundles of rank $k$ and degree $d + kn$. Then $M^s_{k,d+kn}$ is the GIT quotient

$$\psi : R^s \rightarrow R^s/\text{PGL}(N) = M^s_{k,d+kn}. $$

Let $p_C : C \times Q \rightarrow C$ denote the projection and let $p_C^*: E \rightarrow F$ denote the universal quotient on $C \times Q$. The sheaf $p_C^*O_C(nP) \otimes F$ on $C \times Q$ defines a morphism $Q^s \rightarrow M^s_{k,d+kn}$. One easily checks that we have the following commutative diagram, in which all arrows are surjective on closed points

$$U^s \xrightarrow{\Psi} Q^s \xrightarrow{\theta} M^s_{k,d+kn}. $$

(7.10)
The map $\psi$ is a principal $\text{PGL}(N)$-bundle. For a closed point $x \in R^s$, the points in the fiber $\Theta^{-1}_x(x)$ are in bijection with the points in the fiber $\theta^{-1}(\psi(x))$. Here we use the stability of the quotient sheaf to assert that no two distinct points in the fiber $\Theta^{-1}_x(x)$ map to the same point in the fiber $\theta^{-1}(\psi(x))$. The natural map from $U^*$ to the Cartesian product of $\psi$ and $\theta$ is a bijective map of smooth varieties and hence an isomorphism. This shows that the above diagram is Cartesian.

In this section we shall compute the Picard group of $Q$ when $d \gg 0$. As we saw in Theorem 6.3, $Q$ is locally factorial and so the Picard group is isomorphic to the divisor class group. Let $CH^1(Q)$ denote the divisor class group of $Q$. We shall first show that $CH^1(Q) \isom CH^1(Q^s)$ and then use the diagram (7.10) to compute $CH^1(Q^s)$.

In the following Lemma we shall use the fact that $U$ is irreducible. This is easily seen as follows. The moduli space $M_{s,k,k+d_n}$ is an integral scheme. It easily follows that $R^s$ is irreducible as $M_{s,k,k+d_n}$ is the GIT quotient $R^s//\text{PGL}(N)$. By [Bho99, Proposition 1.2] we have that $\dim(R) - \dim(R\setminus R^s) \geq 2$. As $R$ is equidimensional, it follows that $R$ is irreducible. As $R$ is smooth it follows that $R$ is an integral scheme and so is $R'$.

It follows that $U$ is integral.

**Lemma 7.11.** Assume one of the following two holds

- $k \geq 2$ and $g(C) \geq 3$, or
- $k \geq 3$ and $g(C) = 2$.

Also assume $d \geq \max\{\alpha(E,k) + 1, k\mu_0(E,k), \beta(E,k,1)\}$. Then the map $CH^1(Q) \to CH^1(Q^s)$ is an isomorphism.

**Proof.** Recall the definition of $Z_1$ from (5.1) and observe that $Q^{\text{tf}} = Q \setminus Z_1$. Taking $i = 1$ in Lemma 5.2(1) we get $\dim(Q) - \dim(Z_1) \geq k$. Since $k \geq 2$, it follows that $CH^1(Q) = CH^1(Q^{\text{tf}})$.

By Lemma 4.9 it follows that

$$\dim(Q^{\text{tf}}) - \dim(Q_b^{\text{tf}}) = \dim(Q) - \dim(Q_b^{\text{tf}}) > 1.$$  

Observe that $Q_g^{\text{tf}} = Q^{\text{tf}} \setminus Q_b^{\text{tf}}$. It follows that $CH^1(Q^{\text{tf}}) = CH^1(Q_g^{\text{tf}})$.

We had observed earlier that $Q^s \subset Q_g^{\text{tf}}$. To prove the Lemma it suffices to show that

$$\dim(Q_g^{\text{tf}}) - \dim(Q_g^{\text{tf}} \setminus Q^s) > 1.$$  

We will now show this.

As $d \geq k\mu_0(E,k)$, by Lemma 4.1 we have $H^1(C,E^\vee \otimes G_x) = 0$ for $x \in R^s$. We have already checked above, see (7.9), that $\Psi^{-1}(Q^s) = U^s$.

As the map $\Theta$ is flat and $U$ is integral, it follows using Lemma 2.2 (applied to the map $\Psi : U \to Q_g^{\text{tf}}$) that

$$2 \leq \dim(R') - \dim(R' \setminus R^s) = \dim(U) - \dim(U \setminus U^s) \leq \dim(Q_g^{\text{tf}}) - \dim(Q_g^{\text{tf}} \setminus Q^s).$$

This completes the proof of the Lemma. \qed

**Lemma 7.12.** Let $r - k \geq 2$. Let $d \geq \max\{\alpha(E,k), k\mu_0(E,k) + k\}$. The natural map $CH^1(P^s) \to CH^1(U^s)$ is an isomorphism.

**Proof.** It suffices to show that $\dim(P^s) - \dim(P^s \setminus U^s) \geq 2$. Let $[x : \mathcal{O}_{P^s}^N \to F]$ be a quotient corresponding to a closed point $x \in R^s$. It suffices to show that $\dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus U^s) \geq 2$ for every closed point $x \in R^s$. We now show this.
The space $\Theta^{-1}(x)$ is the space $\mathbb{P}(\text{Hom}(E,F)^\vee)$ parametrizing lines in the vector space $\text{Hom}(E,F)$. Let $c \in C$ be a closed point. As $F$ is stable, note $\mu^{\text{min}}(F(-c)) = \mu(F) - 1$. As $d \geq k\mu_0(E,k) + k$, it follows that

$$\mu^{\text{min}}(F(-c)) = \mu(F) - 1 = \frac{d - k}{k} \geq \mu_0(E,k).$$

Let $p_i$ denote the projections from $C \times C$. Let $\Delta$ denote the diagonal in $C \times C$. Consider the short exact sequence of sheaves on $C \times C$ given by

$$0 \longrightarrow p_1^*(E^\vee \otimes F)(-\Delta) \longrightarrow p_1^*(E^\vee \otimes F) \longrightarrow \Delta^*(E^\vee \otimes F) \longrightarrow 0.$$

By Lemma 4.1 we have $H^1(E^\vee \otimes F(-c)) = 0$. Applying $p_{2*}$ to the above, we get that the sheaf

$$\mathcal{V} := p_{2*}(p_1^*(E^\vee \otimes F)(-\Delta)),$$

which is locally free on $C$, sits in a short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \text{Hom}(E,F) \otimes \mathcal{O}_C \longrightarrow E^\vee \otimes F \longrightarrow 0.$$

The restriction of the above sequence to a closed point $c \in C$ gives the short exact sequence of vector spaces

$$(7.13) \quad 0 \longrightarrow \text{Hom}(E,F(-c)) \longrightarrow \text{Hom}(E,F) \longrightarrow \text{Hom}(E|_c,F|_c) \longrightarrow 0.$$

Consider the closed subset $\mathbb{P}(\mathcal{V}^\vee) \subset \mathbb{P}(\text{Hom}(E,F)^\vee) \times C$. Let $T \subset \mathbb{P}(\text{Hom}(E,F)^\vee)$ denote the image of $\mathbb{P}(\mathcal{V}^\vee)$ under the projection map

$$\mathbb{P}(\text{Hom}(E,F)^\vee) \times C \longrightarrow \mathbb{P}(\text{Hom}(E,F)^\vee).$$

Then $T$ is a closed subset and set theoretically it is the union

$$T = \bigcup_{c \in C} \mathbb{P}(\text{Hom}(E,F(-c))^{\vee}).$$

As $r - k \geq 2$ we have $rk \geq (k + 2)k > 2$. Therefore,

$$(7.14) \quad \text{dim}(\mathbb{P}(\text{Hom}(E,F)^\vee)) - \text{dim}(T) \geq \text{dim}(\mathbb{P}(\text{Hom}(E,F)^\vee)) - \text{dim}(\mathcal{V}^\vee) = rk - 1 \geq 2.$$

Let $V$ denote the open set $\mathbb{P}(\text{Hom}(E,F)^\vee) \setminus T$. Let $\mathcal{O}(1)$ denote the restriction of the tautological bundle on $\mathbb{P}(\text{Hom}(E,F)^\vee)$ to $V$. Let $p_C$ denote the projection from $C \times V$ to $C$ and let $p_V$ denote the projection to $V$. Consider the canonical map of sheaves on $C \times V$

$$(7.15) \quad p_C^*(E \otimes F^\vee) \longrightarrow \text{Hom}(E,F)^\vee \otimes \mathcal{O}_{C \times V} \longrightarrow p_V^*\mathcal{O}(1).$$

Let $\varphi \neq 0$ be an element in $\text{Hom}(E,F)$ such that the line $[\varphi]$ it defines is in $V$. The dual of equation (7.15) restricted to $C \times [\varphi]$ is described as follows. This restriction maps

$$\mathbb{C} \longrightarrow \mathbb{C}[\varphi] \otimes \mathcal{O}_C \longrightarrow E^\vee \otimes F.$$

The second map is precisely the global section corresponding to the map $\varphi$. For a point $c \in C$, the map (7.15) restricted to $(c,[\varphi])$ is adjoint to the map $E|_c \xrightarrow{\varphi|_c} F|_c$. As $[\varphi] \in V$, it follows that the map $E|_c \xrightarrow{\varphi|_c} F|_c$ is nonzero, and so it follows that the restriction of (7.15) to $(c,[\varphi])$ is nonzero, that is, $E|_c \otimes F|_c^\vee \longrightarrow \mathbb{C}$ is nonzero and hence surjective. This proves
that the map \((7.15)\) is surjective. This defines a map \(C \times V \xrightarrow{\kappa} \mathbb{P}(E \otimes F^\vee)\) which sits in a commutative diagram

\[
\begin{array}{ccc}
C \times V & \xrightarrow{\kappa} & \mathbb{P}(E \otimes F^\vee) \\
\downarrow & & \downarrow \pi \\
C & &
\end{array}
\]

The restriction of the map \(\kappa\) over a point \(c \in C\) is the composite map below, where the second arrow is obtained using \((7.13)\)

\[
V \longrightarrow \mathbb{P}(\text{Hom}(E, F)^\vee) \setminus \mathbb{P}(\text{Hom}(E, F(-c))^\vee) \longrightarrow \mathbb{P}(\text{Hom}(E|_c, F|_c)^\vee).
\]

The second arrow is a surjective flat map and the first arrow is an open immersion. It follows that the composite is a flat map and hence has constant fiber dimension. It follows that the map \(\kappa\) has constant fiber dimension, and so using [Mat86, Theorem 23.1] or [Stk, Tag 00R4] we see that \(\kappa\) is a flat map. Consider the canonical map

\[
\pi^* E \longrightarrow \pi^* F \otimes \mathcal{O}_{\mathbb{P}(E \otimes F^\vee)}(1)
\]

on \(\mathbb{P}(E \otimes F^\vee)\) and let \(Z\) denote the support of the cokernel. The set \(Z \cap \pi^{-1}(c)\) is precisely the locus of non-surjective maps in \(\mathbb{P}(E|_c \otimes F|_c)^\vee\). By [ACGH85, Chapter II, §2, page 67] we have that the codimension of \(Z \cap \pi^{-1}(c)\) in \(\mathbb{P}(E|_c \otimes F|_c)^\vee\) is \(r - k + 1\). It follows that the codimension of \(\kappa^{-1}(Z)\) in \(C \times V\) is \(r - k + 1\) and the codimension of \(p_V(\kappa^{-1}(Z))\) in \(V\) is at least \(r - k \geq 2\). The set \(V \setminus p_V(\kappa^{-1}(Z))\) is precisely the locus of points in \(\mathbb{P}(\text{Hom}(E, F)^\vee)\) corresponding to maps which are surjective. The locus of points in \(\mathbb{P}(\text{Hom}(E, F)^\vee)\) corresponding to non-surjective maps \(E \longrightarrow F\) is the set \(T \cup p_V(\kappa^{-1}(Z))\), which has codimension at least 2. This proves that \(\dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus U^o) \geq 2\), which completes the proof of the Lemma.

**Remark 7.16.** The proof of Lemma 7.12 also shows the following. Let \(k = 1\) and \(r \geq 3\) so that \(k \leq r - 2\). Let \(d \geq \max\{\alpha(E, 1), \mu_0(E, 1) + 1\}\). Let \(L\) be a line bundle on \(C\) of degree \(d\). Then the closed subset in \(\mathbb{P}(\text{Hom}(E, L)^\vee)\) consisting of non-surjective maps has codimension \(\geq 2\).

**Theorem 7.17.** Let \(r - k \geq 2\). Assume one of the following two holds

- \(k \geq 2\) and \(g(C) \geq 3\), or
- \(k \geq 3\) and \(g(C) = 2\).

Let \(n_1\) be the smallest integer such that \(kn_1 > g(C) + 3\). Assume

\[d \geq \max\{\alpha(E, k) + 1, k\mu_0(E, k) + k, \beta(E, k, g(C) + 3) + n_1\}.
\]

Then

\[\text{Pic}(\mathcal{Q}) \cong \text{Pic}(M^s_{k,d+kn}) \times \mathbb{Z} \cong \text{Pic}(\mathcal{Q}^0) \times \mathbb{Z} \times \mathbb{Z}.
\]

**Proof.** We saw in Theorem 6.3 that \(\mathcal{Q}\) is an integral variety which is normal and locally factorial. So the Picard group is isomorphic to the divisor class group. By Lemma 7.11 it is enough to show that

\[\text{Pic}(\mathcal{Q}^s) \cong \text{Pic}(M^s_{k,d+kn}) \times \mathbb{Z}.
\]
Recall that we have the following diagram (7.10), which we checked is Cartesian:

\[
\begin{array}{ccc}
U^s & \xrightarrow{\Psi} & Q^s \\
\downarrow{\Theta_{U^s}} & & \downarrow{\theta} \\
R^s & \xrightarrow{\psi} & M^s_{k,d+kn}.
\end{array}
\]

Recall from §7.1 that we had fixed a closed point \( P \in C \). Note that for any \( [x : O^n_C \to F(nP)] \in R^s \), the fibre \( \Theta_{U^p}^{-1}(x) \cong \theta^{-1}([F]) \). In the proof of Lemma 7.12 we proved that \( \dim(\Theta^{-1}(x)) - \dim(\Theta^{-1}(x) \setminus U^s) \geq 2 \) for every closed point \( x \in R^s \). It follows that \( \Theta_{U^p}^{-1}(x) = \Theta^{-1}(x) \cap U^s \) is an open subset of projective space (that is, \( \Theta^{-1}(x) \)) whose complement has codimension \( \geq 2 \). Thus,

\[
Z = \text{Pic}(\Theta^{-1}(x)) = \text{Pic}(\Theta_{U^p}^{-1}(x)) = \text{Pic}(\theta^{-1}([F])).
\]

Therefore we have the restriction map

\[
\text{res} : \text{Pic}(Q) \cong \text{Pic}(Q^s) \to \text{Pic}(\theta^{-1}([F])) \cong Z.
\]

We claim this map is nontrivial. Let \( L \) be a very ample line bundle on \( Q \). If \( \text{res}(L) \) were trivial, it would follow that \( \text{res}(L) \) is trivial and very ample, which is a contradiction as \( \theta^{-1}([F]) \cong \Theta^{-1}(x) \) is an open subset of a projective space whose complement has codimension \( \geq 2 \). Thus, the image of \( \text{res} \) is isomorphic to a copy of \( Z \). We will show that the kernel of \( \text{res} \) is isomorphic to \( \text{Pic}(M^s_{k,d+kn}) \).

Let \( L \in \text{Pic}(Q^s) \) be such that \( \text{res}(L) \) is trivial. We need to show that \( L \) is isomorphic to the pullback of some line bundle on \( \text{Pic}(M^s_{k,d+kn}) \). Consider the pullback \( \Psi^*L \). Since \( \Psi \) is \( \text{PGL}(N) \)-invariant, this line bundle carries a \( \text{PGL}(N) \)-linearization. By Lemma 7.12, the complement of \( U^s \) in \( P^s \) has codimension \( \geq 2 \). Therefore, both \( L \) and this \( \text{PGL}(N) \)-linearization extend uniquely to \( P^s \). Let us denote this extension of \( \Psi^*L \) to \( P^s \) by \( L' \) and the linearization on \( \text{PGL}(N) \times P^s \) by \( \alpha' : m_{ps}P \to p_{ps}L \), where \( m_{ps} : \text{PGL}(N) \times P^s \to P^s \) is the multiplication map and \( p_{ps} : \text{PGL}(N) \times P^s \to P^s \) is the second projection. Since \( \Theta : P^s \to R^s \) is a projective bundle, \( L' \cong O(n) \otimes \Theta^*L'' \) for some \( L'' \in \text{Pic}(R^s) \) and for some \( n \). However, since the fibers of \( \Theta \) and \( \theta \) are isomorphic, the condition \( \text{res}(L) \) is trivial implies that \( n = 0 \), that is, \( L' \cong \Theta^*L'' \). Now note that since the map \( P^s \to R^s \) is \( \text{PGL}(N) \)-equivariant we have a commutative diagram

\[
\begin{array}{ccc}
\text{PGL}(N) \times P^s & \xrightarrow{m_{ps}} & P^s \\
\downarrow{\text{Id} \times \Theta} & & \downarrow{\Theta} \\
\text{PGL}(N) \times R^s & \xrightarrow{m_{Rs}} & R^s
\end{array}
\]

From this diagram it follows that we have an isomorphism of sheaves

\[
(Id \times \Theta)^*m^s_{Rs}L'' \cong m^s_{ps}\Theta^*L'' \xrightarrow{\sim} p^s_{Rs}\Theta^*L'' \cong (Id \times \Theta)^*p^s_{Rs}L''.
\]

where the middle isomorphism is given by the linearization \( \alpha' \). Since \( \text{Id} \times \Theta \) is a projective bundle, applying \((\text{Id} \times \Theta)_*\) to this composition of isomorphisms we get a linearization

\[
\alpha'' : m^s_{Rs}L'' \xrightarrow{\sim} p^s_{Rs}L''
\]
of \( L'' \) such that \((\text{Id} \times \Theta)^* \alpha'' = \alpha'\). Now recall that the map \( \psi \) is a principal \( \text{PGL}(N) \)-bundle. By [HL10, Theorem 4.2.14] we get that there exists \( L'' \in \text{Pic}(M_{k,d+kn}) \) such that \( \psi^* L'' \cong L'' \) and the induced \( \text{PGL}(N) \) linearization is \( \alpha'' \). Therefore we get that

\[
\Psi^* \theta^* L'' \cong \Theta^* \psi^* L'' \cong \Theta^* L'' \cong L' \cong \Psi^* L
\]

and also the induced \( \text{PGL}(N) \)-linearizations are also the same. Since the diagram (7.10) is Cartesian, the map \( \Psi \) is a principal \( \text{PGL}(N) \)-bundle. Hence by [HL10, Theorem 4.2.16] we get that \( \theta^* L'' \cong L' \). This completes the proof of the first equality in the statement of the Theorem. The second equality follows from [DN89, Theorem A, Theorem C] and from the fact that

\[
\dim(M_{k,d+kn}) - \dim(M_{k,d+kn} \setminus M_{k,d+kn}^s) \geq 2.
\]

One way to see this inequality is to apply [Bho99, Proposition 1.2 (3)] and Lemma 2.2 to the GIT quotient \( R^{ss} \to M_{k,d+kn} \).

8. Fibers of \( \text{det} \)

Let \( L \) be a line bundle on \( C \) of degree \( d \) and let \( Q_L \) denote the scheme theoretic fiber \( \text{det}^{-1}(L) \). As a corollary of Theorem 6.3 we have the following Proposition.

**Proposition 8.1.** Let \( n_1 \) be the smallest integer such that \( kn_1 > g(C) + 3 \). Let \( d \geq \beta(E,k,g(C) + 3) + n_1 \). Then \( Q_L \) is a local complete intersection scheme which is equidimensional, normal and locally factorial.

**Proof.** We use Theorem 6.3 and Lemma 6.1. As \( Q \) is a local complete intersection scheme, \( \text{Pic}^d(C) \) is smooth and the map det is flat, it follows using [Avr77, (1.9.2)] (see also [BH93, Remark 2.3.5] and [Stk, Tag 09Q2]) that \( Q_L \) is a local complete intersection scheme and so also Cohen-Macaulay. As \( Q \) is irreducible, flatness of det also implies that \( Q_L \) is equidimensional.

We observed in the proof of Theorem 6.3 that the restriction of det to the open subset \( Q_g \) is a smooth morphism. It follows that \( Q_L \cap Q_g \) is contained in the smooth locus of \( Q_L \). The singular locus of \( Q_L \) is thus contained in \( Q_L \cap Q_b \). As \( d \geq \beta(E,k,g(C) + 3) + n_1 \), applying Lemma 6.2 we get

\[
\dim(Q_L) - \dim(Q_L \cap Q_b) > g(C) + 3.
\]

By Lemma 2.1(2) it follows that

\[
\dim(Q_L) - \dim(Q_L \cap Q_b) > 3.
\]

It follows that the singular locus of \( Q_L \) has codimension 4 or more. This proves that \( Q_L \) is normal, that is, it is the disjoint union of finitely many normal varieties, all of the same dimension. Using Grothendieck’s theorem (see [AH20, Theorem 1.4]) it follows that \( Q_L \) is locally factorial. \( \square \)

Next we want to find conditions under which \( Q_L \) becomes irreducible. We use the notation used in Lemma 5.2. In the proof of the next Lemma we will use the following fact. Let \( X \to S \) be a projective morphism of schemes with relative ample line bundle \( \mathcal{O}(1) \). Let \( \mathcal{S} \) be a coherent sheaf on \( X \). Let \( P(n) \) denote the constant polynomial defined by \( P(n) = 1 \) for all \( n \). Then the relative Quot scheme \( \text{Quot}_{X/S}(\mathcal{S}, P) \) is isomorphic to \( \mathbb{P}(\mathcal{S}) \to X \).
Lemma 8.3. Let \( n_0 \) be the smallest integer such that \( kn_0 > g(C) + 1 \). Let \( n_1 \) be the smallest integer such that \( kn_1 > g(C) + 3 \). Let

\[
d \geq \max\{\beta(E, k, g(C) + 1) + n_0 + 1, \beta(E, k, g(C) + 3) + n_1\}.
\]

Then \( Q^1_L \) is dense in \( Q_L \).

Proof. Recall the relative Quot scheme in equation (5.3). We are interested in the case \( i = 1 \), that is, the relative Quot scheme \( \text{Quot}_{C \times A/A}(S, 0, 1) \), where \( A \) is the Quot scheme \( \text{Quot}_{C/C}(E, k, d - 1) \). For ease of notation we denote by \( B \) the scheme \( \text{Quot}_{C \times A/A}(S, 0, 1) \).

Recall the map \( \pi : B \rightarrow A \) from (5.3). On \( C \times B \) we have a quotient

\[(8.4) (\text{Id}_C \times \pi)^* \mathcal{S} \rightarrow \mathcal{T},\]

such that \( \mathcal{T} \) is flat over \( B \). Using \( \mathcal{T} \) we get the determinant map

\[
\det_B : B \rightarrow \text{Pic}^1(C).
\]

This map has the following pointwise description. A closed point \( b \in B \) gives rise to the closed point \( \pi(b) \in A \), which corresponds to a short exact sequence on \( C \)

\[
0 \rightarrow S_F \rightarrow E \rightarrow F \rightarrow 0,
\]

where \( F \) is of rank \( k \) and degree \( d - 1 \) on \( C \). The restriction of the universal quotient (8.4) to the point \( b \) is a torsion quotient on \( C \)

\[
S_F \rightarrow M,
\]

such that \( \text{length}(M) = 1 \). Let \( c = \text{Supp}(M) \). Then \( \det_B(b) = \mathcal{O}_C(c) \). Consider the natural embedding (recall that \( g(C) > 0 \)) \( \iota : C \hookrightarrow \text{Pic}^1(C) \) given by \( c \mapsto \mathcal{O}_C(c) \). It is clear that the image of \( B \) is the image of \( \iota \). Next we want to show that \( B \) is an integral scheme.

As \( d - 1 \geq \alpha(E, k) \), it follows from Lemma 6.1 that \( A \) is a local complete intersection. By Theorem 6.3(1) it follows that \( A \) is integral. As \( i = 1 \), using the fact stated before this Lemma, it is easily checked that \( B \) is the projective bundle \( \mathbb{P}(\mathcal{S}) \rightarrow C \times A \). It follows that \( B \) is integral and a local complete intersection and so Cohen-Macaulay. As \( B \) is integral, the map \( \det_B \) factors through the map \( \iota \), that is, we have a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\det_B} & \text{Pic}^1(C) \\
\downarrow{\det_T} & & \downarrow{\iota} \\
C & & \\
\end{array}
\]

Let \( \det_A : A \rightarrow \text{Pic}^{d-1}(C) \) denote the determinant map for the Quot scheme \( A \). This is flat due to Theorem 6.3(1). Consider the map

\[
\begin{array}{ccc}
B & \xrightarrow{\det_B} & \text{Pic}^d(C) \\
\downarrow{\det_T} & & \\
C \times \text{Pic}^{d-1}(C) & \xrightarrow{(\det_T, \det_A \circ \pi)} & \text{Pic}^d(C).
\end{array}
\]

The second map is given by \( (c, M) \mapsto M \otimes \mathcal{O}_C(c) \). It is easily checked that both maps have constant fiber dimension. In view of [Mat86, Theorem 23.1] it follows that both maps are
flat and so the composite $\det_B$ is also flat. Recall the map $\pi'$ from (5.4). It is clear that we have a commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\pi'} & Q \\
\downarrow{\det_B} & & \downarrow{\det} \\
\Pic^d(C) & & 
\end{array}
$$

Recall the definition of $Z_1$, see (5.1). We saw in the proof of Lemma 5.2 that $\pi'(B) = \overline{Z}_1$. Let

$$
\overline{Z}_{1,L} := \{[q : E \to F] \in \overline{Z}_1 | \det(F) = L\}.
$$

Let $B_L := \det_B^{-1}(L)$ denote the scheme theoretic fiber over $L$. Then it is clear that $\pi'(B_L) = \overline{Z}_{1,L}$. Thus, it follows that $\dim(\overline{Z}_{1,L}) \leq \dim(B_L)$. In the proof of Lemma 5.2 (after equation (5.5)) we had remarked that there is an open set $U \subset B$ such that $\pi'$ is injective on points of $U$. It is easily checked that this open set $U$ meets all fibers $B_L$. Thus, $\pi'$ is also injective on the subset $U \cap B_L$. Thus, it follows that $\dim(\overline{Z}_{1,L}) \geq \dim(U \cap B_L)$. Since $\det_B$ is flat, the fibers are equidimensional and so it follows that every open set of $B_L$ has the same dimension as $B_L$. Combining these we get

$$
(8.5) \quad \dim(\overline{Z}_{1,L}) = \dim(B_L) = \dim(Q) - k - g = \dim(Q_L) - k.
$$

As $k \geq 1$, and all irreducible components of $Q_L$ have the same dimension, it follows that $Q_L \setminus \overline{Z}_{1,L} = Q^f_L$ is dense in $Q_L$. \hfill \Box

The above Lemma implies that irreducibility of $Q_L$ is equivalent to the irreducibility of the open subset $Q^f_L$. Let

$$
Q^f_{g,L} := Q^f_g \cap Q_L.
$$

Combining Proposition 8.1 and Lemma 8.3 we get the following.

**Lemma 8.6.** Let $n_0$ be the smallest integer such that $kn_0 > g(C) + 1$. Let $n_1$ be the smallest integer such that $kn_1 > g(C) + 3$. Let

$$
d \geq \max\{\beta(E, k; g(C) + 1) + n_0 + 1, \beta(E, k; g(C) + 3) + n_1\}.
$$

Then $Q^f_{g,L}$ is dense in $Q^f_L$.

Proof. As all components of $Q_L$ have the same dimension, the same holds for the open subset $Q^f_L$. Note that

$$
Q^f_L \setminus Q^f_{g,L} = Q^f_L \cap Q^f_0.
$$

The Lemma follows using (8.2). \hfill \Box

Combining the above results we have the following.

**Theorem 8.7.** Let $k \geq 2$, $g(C) \geq 2$. Let $n_0$ be the smallest integer such that $kn_0 > g(C) + 1$. Let $n_1$ be the smallest integer such that $kn_1 > g(C) + 3$. Let

$$
d \geq \max\{\beta(E, k; g(C) + 1) + n_0 + 1, \beta(E, k; g(C) + 3) + n_1\}.
$$

Then $Q_L$ is a local complete intersection scheme which is also integral, normal and locally factorial.
Proof. The Theorem follows using Proposition 8.1 once we show that $Q_L$ is irreducible. In view of Lemma 8.3 and Lemma 8.6, it suffices to show that $Q^\text{tf}_{g,L}$ is irreducible.

Recall the notation from §7, in particular, the map $\Psi$ from (7.6). This sits in the following commutative diagram whose maps we describe next.

\begin{equation}
\begin{array}{ccc}
U & \xrightarrow{\Psi} & Q^\text{tf}_{g,L} \\
\downarrow & & \downarrow \\
R' & \xrightarrow{\Theta} & \text{Pic}^{d+kn}(C)
\end{array}
\end{equation}

The bottom horizontal map sends a closed point $[x: O_C^{\oplus N} \to F] \in R'$ to $\det(F)$. The right vertical map sends a closed point $[g: E \to F] \in Q^\text{tf}_{g,L}$ to $\det(F) \otimes O_C(knP)$. Let $L' := L \otimes O_C(knP)$.

The bottom horizontal map in (8.8) is a smooth morphism. This follows using Lemma 2.7 and the reason explained after (6.4) applied to the space $R'$. In particular, the morphism $R' \to \text{Pic}^{d+kn}(C)$ is flat. Thus, $R'_{L'}$ is a smooth equidimensional scheme. Using [Bho99, Corollary 1.3] we easily see that $R'_{L'}$ is irreducible. Taking the “fiber” of (8.8) over the point $[L'] \in \text{Pic}^{d+kn}(C)$ we get the following commutative diagram

\begin{equation}
\begin{array}{ccc}
U_{L'} & \xrightarrow{\Psi_{L'}} & Q^\text{tf}_{g,L} \\
\downarrow & & \downarrow \\
R'_{L'} & \xrightarrow{\Theta_{L'}} & [L']
\end{array}
\end{equation}

It follows that $U_{L'}$ is irreducible. By surjectivity of $\Psi$ on closed points we get that $\Psi_{L'}$ is also surjective on closed points. It follows that $Q^\text{tf}_{g,L}$ is irreducible. This completes the proof of the Theorem. \hfill \square

Let $M^s_{k,L}$ denote the moduli space of stable bundles of rank $k$ and determinant $L$.

Theorem 8.9. Let $r - k \geq 2$. Assume one of the following two holds

- $k \geq 2$ and $g(C) \geq 3$, or
- $k \geq 3$ and $g(C) = 2$.

Let $n_0$ be the smallest integer such that $kn_0 > g(C) + 1$. Let $n_1$ be the smallest integer such that $kn_1 > g(C) + 3$. Let

$$d \geq \max\{k\mu_0(E,k) + k, \beta(E,k,g(C) + 1) + n_0 + 1, \beta(E,k,g(C) + 3) + n_1\}.$$ 

We have isomorphisms

$$\text{Pic}(Q_L) \cong \text{Pic}(M^s_{k,L}) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}.$$ 

Proof. The proof is similar to Theorem 7.17 and so we only sketch it. From (8.2) and the fact that $Q^\text{tf}_L \setminus Q^\text{tf}_{g,L} = Q^\text{tf}_L \cap Q_b$ it follows that

$$\dim(Q_L) - \dim(Q_L \setminus Q^\text{tf}_{g,L}) \geq 2.$$
Now consider the diagram

\[
\begin{array}{ccc}
\mathbb{U}_L' & \xrightarrow{L'} & Q_{g,L}^{\text{tf}} \\
\Theta_{L'} & \downarrow & \\
R_{L'} & \rightarrow & [L']
\end{array}
\]

Just as in Lemma 7.11, using [Bho99, Corollary 1.3], and Lemma 2.2 we have

\[
\dim(Q_{g,L}^{\text{tf}}) - \dim(Q_{g,L}^{\text{tf}} \setminus Q_L^s) \geq 2.
\]

Therefore we get that

\[
\dim(Q_L^s) - \dim(Q_L \setminus Q_L^s) \geq 2.
\]

Since \(Q_L^s\) is locally factorial we have

\[
\text{Pic}(Q_L) \cong \text{Pic}(Q_L^s).
\]

Now we have the cartesian diagram

\[
\begin{array}{ccc}
\mathbb{U}_L & \xrightarrow{\psi} & Q_L^s \\
\Theta & \downarrow & \\
R_L & \rightarrow & M_{k,L}^s
\end{array}
\]

which we get by taking the fiber over \([L]\) of the diagram (7.10). The rest of the proof is the same as the proof of Theorem 7.17, by considering this diagram instead of (7.10). The second equality follows from [DN89, Theorem B].

\[\square\]

9. Quot Schemes \(\text{Quot}_{\mathbb{C}/\mathbb{C}}(E, 1, d)\)

In this section we consider the case \(k = 1\). We only sketch the proofs as they are similar to the earlier cases considered.

**Theorem 9.1.** Let \(k = 1\). Let \(d \geq \max\{\mu_0(E, 1) + 1, \beta(E, 1, g(C) + 3) + g(C) + 4\}\). Then

\[
\text{Pic}(Q) \cong \text{Pic}(\text{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}, \quad \text{Pic}(Q_L) \cong \mathbb{Z} \times \mathbb{Z}.
\]

**Proof.** We can apply Theorem 6.3 to conclude that \(Q\) is integral, normal and locally factorial. We claim that \(Q^{\text{tf}}\) is smooth. To see this, let

\[
0 \rightarrow S \rightarrow E \rightarrow L \rightarrow 0
\]

be a quotient. Applying \(\text{Hom}(-, L)\) we get a surjection \(\text{Ext}^1(E, L) \rightarrow \text{Ext}^1(S, L) \rightarrow 0\). By Lemma 4.1 it follows that \(\text{Ext}^1(E, L) = 0\). It easily follows that \(Q^{\text{tf}}\) is smooth.

Let

\[
\rho_1 : C \times \text{Pic}^d(C) \rightarrow C, \quad \rho_2 : C \times \text{Pic}^d(C) \rightarrow \text{Pic}^d(C)
\]

be the projections. Let \(\mathcal{L}\) be a Poincaré bundle on \(C \times \text{Pic}^d(C)\). Define

\[
\mathcal{E} := \rho_2^*[\rho_1^*E \otimes \mathcal{L}].
\]

Using Lemma 4.1 and cohomology and base change we easily conclude that \(\mathcal{E}\) is a locally free sheaf on \(\text{Pic}^d(C)\) such that the fibre over the point \([L] \in \text{Pic}^d(C)\) is isomorphic to \(\text{Hom}(E, L)\). Let \(\mathcal{W} \subset \mathbb{P}(\mathcal{E}^\vee)\) be the open subset consisting of points parametrizing surjective maps. Both
$\mathcal{W}$ and $\mathcal{Q}^{tf}$ are smooth. There is a map $\mathcal{W} \to \mathcal{Q}^{tf}$ which is bijective on points (and hence an isomorphism as both are smooth) and sits in a commutative diagram

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\sim} & \mathcal{Q}^{tf} \\
& \downarrow & \downarrow \text{det} \\
\text{Pic}^d(C) & \to & \\
\end{array}
$$

Using Remark 7.16 it follows that $\dim(\mathbb{P}(\mathcal{E}^\vee)) - \dim(\mathbb{P}(\mathcal{E}^\vee) \setminus \mathcal{W}) \geq 2$. Thus, it follows that $\text{Pic}(\mathcal{Q}^{tf}) \cong \text{Pic}(\mathcal{W}) \cong \text{Pic}(\mathbb{P}(\mathcal{E}^\vee)) \cong \text{Pic}(\text{Pic}^d(C)) \times \mathbb{Z}$. By Lemma 5.2, $\mathcal{Q} \setminus \mathcal{Q}^{tf} = \bar{Z}_1$ is irreducible of codimension 1 and so we have an exact sequence

$$0 \to \mathbb{Z} \to \text{Pic}(\mathcal{Q}) \to \text{Pic}(\mathcal{Q}^{tf}) \to 0.$$ 

It easily follows that we have an isomorphism

$$\text{Pic}(\mathcal{Q}) \cong \text{Pic}(\text{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}.$$ 

For $\mathcal{Q}_L$, we first show that $\mathcal{Q}_L$ is integral, normal and locally factorial. This is easily done using Proposition 8.1, Lemma 8.3 and using the fact that $\mathcal{Q}_L^{tf} \cong \mathcal{W}_L$. The rest of the proof follows in the same way as that of $\mathcal{Q}$, once we use the irreducibility of $\bar{Z}_1,L$ and the fact that it is of codimension 1, see (8.5). We remark that when $k = 1$, unlike in Theorem 8.7, we do not need to use [Bho99] and hence do not need the hypothesis that $g(C) \geq 2$. 

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of algebraic curves. Vol. I, volume 267 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985. doi:10.1007/978-1-4757-5323-3.

[AH20] Tigran Ananyan and Melvin Hochster. Strength conditions, small subalgebras, and Stillman bounds in degree $\leq 4$. *Trans. Amer. Math. Soc.*, 373(7):4757–4806, 2020, arXiv:1810.00413.pdf.

[Avr77] Luchezar L. Avramov. Homology of local flat extensions and complete intersection defects. *Math. Ann.*, 228(1):27–37, 1977. doi:10.1007/BF01360771.

[BDW96] Aaron Bertram, Georgios Daskalopoulos, and Richard Wentworth. Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians. *J. Amer. Math. Soc.*, 9(2):529–571, 1996. doi:10.1090/S0894-0347-96-00190-7.

[BH93] Winfried Bruns and Jürgen Herzog. Cohen-Macaulay rings, volume 39 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.

[Bho99] Usha N. Bhosle. Picard groups of the moduli spaces of vector bundles. *Math. Ann.*, 314(2):245–263, 1999. doi:10.1007/s002080050293.

[Cal94] Frederick W. Call. A theorem of Grothendieck using Picard groups for the algebraist. *Math. Scand.*, 74(2):161–183, 1994. doi:10.7146/math.scand.a-12487.

[CCH21] Daewoong Cheong, Insong Choe, and George H. Hitching. Isotropic Quot schemes of orthogonal bundles over a curve. *Internat. J. Math.*, 32(8):Paper No. 2150047, 36, 2021. doi:10.1142/S0129167X21500476.

[CCH22] Daewoong Cheong, Insong Choe, and George H. Hitching. Irreducibility of Lagrangian Quot schemes over an algebraic curve. *Math. Z.*, 300(2):1265–1289, 2022. doi:10.1007/s00209-021-02807-6.

[DN89] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989. doi:10.1007/BF01850655.

[Gol19] Thomas Goller. A weighted topological quantum field theory for Quot schemes on curves. *Math. Z.*, 293(3-4):1085–1120, 2019. doi:10.1007/s00209-018-2221-z.
[Gro05] Alexander Grothendieck. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), volume 4 of Documents Mathématiques (Paris) [Mathematical Documents (Paris)]. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HL10] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. doi:10.1017/CBO9780511711985.

[HO10] Rafael Hernández and Daniel Ortega. The divisor class group of a Quot scheme. Tbil. Math. J., 3:1–15, 2010. doi:10.32513/tbilisi/1528768854.

[Ito17] Atsushi Ito. On birational geometry of the space of parametrized rational curves in Grassmannians. Trans. Amer. Math. Soc., 369(9):6279–6301, 2017. doi:10.1090/tran/6840.

[Jow12] Shin-Yao Jow. The effective cone of the space of parametrized rational curves in a Grassmannian. Math. Z., 272(3-4):947–960, 2012. doi:10.1007/s00209-011-0966-8.

[Mat86] Hideyuki Matsumura. Commutative ring theory, volume 8 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[Nit09] Nitin Nitsure. Deformation theory for vector bundles. In Moduli spaces and vector bundles, volume 359 of London Math. Soc. Lecture Note Ser., pages 128–164. Cambridge Univ. Press, Cambridge, 2009.

[PR03] Mihnea Popa and Mike Roth. Stable maps and Quot schemes. Invent. Math., 152(3):625–663, 2003. doi:10.1007/s00222-002-0279-y.

[Stk] The Stacks Project. https://stacks.math.columbia.edu.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, PUNE, 411008, MAHARASHTRA, INDIA.

Email address: chandranandan@iiserpune.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI 400076, MAHARASHTRA, INDIA.

Email address: ronnie@math.iitb.ac.in