On the Lie-formality of Poisson manifolds

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Abstract
Starting from the problem of describing cohomological invariants of Poisson manifolds we prove in a sense a “no-go” result: the differential graded Lie algebra of de Rham forms on a smooth Poisson manifold is formal.

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1 Introduction

The notion of formality was first introduced in the context of differential graded algebras in the beginning of 1970-ies. It was related to the investigations in the rational homotopy theory. In brief, to say that a DG algebra $A$ is formal would mean that $A$ is homotopy equivalent (as algebra) to its cohomology $H(A)$ (with zero differential), i.e. that there exists a sequence of quasi-isomorphisms of DG algebras, beginning at $A$ and ending at $H(A)$:

$$
\begin{array}{cccc}
A_1 & \leftarrow & A_2 & \leftarrow & \ldots & \leftarrow & A_n & \leftarrow & A_{n+1} = H(A) \\
A = A_0 & & & & & & & & \\
\end{array}
$$

Here quasi-isomorphism is a homomorphism of DG algebras, inducing an isomorphism of their cohomology.

More generally, formality is a particular case of the notion of homotopy equivalence of DG algebras: one says that algebras $A$ and $B$ are homotopy equivalent, if there exists a sequence of algebras and homomorphisms, similar to (1), connecting them. All homotopical constructions, applied to homotopy equivalent algebras give equivalent results. In particular, their (usual) cohomology are isomorphic as well as the DG-algebraic cohomology with coefficients in a suitable module.

However, the contrary is not true in general: two algebras with isomorphic cohomology are not necessarily homotopy equivalent. One should also note, that there can be no quasi-isomorphism $A \to B$ even if $A$ and $B$ are homotopy equivalent: it is not always possible to find homotopy inverse homomorphisms of DG-algebras for all the arrows, pointing to the left in the sequence (1). (Recall, that two chain maps $f : A \to B$ and $g : B \to A$ are called homotopy inverse to each other, if both their compositions $fg$ and $gf$ are homotopic to the corresponding identity maps.) On the contrary, it is always possible to find a homotopy inverse chain map for a quasi-isomorphism of chain complexes (i.e for a map inducing an isomorphism of cohomology), if the characteristic of the ground field is zero.

Using the last observation, one can define a suitable generalization of the morphism of DG-algebras, that would be homotopy invertible whenever it establishes an isomorphism of cohomology (in the characteristic zero case). It is the strong homotopical morphisms of algebras. In brief, such a morphism from $A$ to $B$ is a collection of linear maps $F = \{f_i\}_{i \geq 1}$, $f_i : A^\otimes i \to B$, where every map $f_n$ is a chain homotopy between zero and a certain combination of the maps $f_i$ with smaller indices. The first two equations of this series are

$$
\begin{align*}
  d_B f_1(a) &= f_1(d_A a), \\
  d_B f_2(a, b) + f_2(d_A a, b) - (-1)^{|a|} f_2(a, d_A b) &= f_1(ab) - f_1(a)f_1(b),
\end{align*}
$$

where $d_A$, $d_B$ are the differentials in $A$, $B$, and $a, b \in A$ are the arbitrary elements. These equations mean that $f_1$ is a chain map, and that $f_2$ is a chain homotopy, making $f_1^* : H(A) \to H(B)$ into a homomorphism of algebras.

In particular, every homomorphism $f$ of algebras can be represented as an $A_\infty$ morphism, by choosing $f_i = 0, \ i \geq 2$. 

\[1\]
As it was mentioned above, an important property of \(A_\infty\) maps is that they can be inverted up to a homotopy if \(f_1\) establishes an isomorphism in cohomology. For instance, every quasi-isomorphism of algebras can be inverted in the class of such maps. Starting from this observation one can prove that two DG-algebras \(A\) and \(B\) are homotopy-equivalent iff there exists an \(A_\infty\)-morphism \(F : A \to B\) with quasi-isomorphic \(f_1\). In this case, there will exist a homotopy inverse morphism \(G = F^{-1} : B \to A\). In particular, \(A\) is formal, iff there exists an \(A_\infty\)-quasi-isomorphism \(F : A \to H(A)\) (or \(G : H(A) \to A\)).

In analogy with associative algebras, there exists homotopy theory of differential graded Lie algebras (DGL-algebras). The analogue of \(A_\infty\) morphisms in this case is called \(L_\infty\). An \(L_\infty\)-map \(F : A \to B\) (where \(A\) and \(B\) are Lie algebras) consists of a series of linear mappings \(f_i : \wedge^i A \to B\) (where \(\wedge^n A\) denotes the \(n\)-th external power of the graded space \(A\)). These mappings are subject to some relations, analogous to the above (see §3 below). In particular, one has the following analog of (3)

\[
d_B f_2(a, b) - f_2(d_A a, b) + (-1)^{|a|} f_2(a, d_A b) = f_1(\{a, b\}_A) - \{f_1(a), f_1(b)\}_B, \tag{4}
\]

where \(\{,\}_A\), \(\{,\}_B\) are the commutators in \(A\) and \(B\). One can prove (see e.g. [1]), that, similarly to the DG-algebras case, any \(L_\infty\) map with a quasi-isomorphic first stage, can be homotopy inverted in the class of \(L_\infty\) morphisms. In particular, a DGL-algebra is formal, iff there exists an \(L_\infty\) morphism from its cohomology Lie algebra to itself, with quasi-isomorphic \(f_1\).

As in the case of associative algebras, homotopy equivalence of two DGLa’s implies homotopy equivalence of functorial constructions, that one applies to them. For example, their DGL cohomology with coefficients in a (differential graded) module are isomorphic; e.g. for a formal Lie algebra \(A\), one can calculate its Lie algebraic cohomology by substituting \(H(A)\) (with trivial differential) for \(A\) in the standard complex. This makes such cohomology calculable, since, for an arbitrary DG Lie algebra \(A\), \(H(A)\) is often of finite type (i.e. has finite dimensional homogeneous components).

In this note we prove the formality of the DG Lie algebra of de Rham forms on a Poisson manifold. This fact implies, that there are no cohomological invariants, associated to this algebra, except for its (usual) cohomology. Speaking somewhat loosely, this Lie algebra knows not more about the Poisson structure on the manifold, than its cohomology does.

The composition of this note is as follows. In the next section we recall the definition and the main properties of the DG Lie algebra of the differential forms on a Poisson manifold (the so-called Brylinski complex). We prove that the induced Lie bracket on the cohomology is trivial and deduce the formula (17), which is crucial for our proof of the formality.

In §3 we discuss the definitions and the basic properties of the \(L_\infty\) algebras and \(L_\infty\) morphisms. Finally, in §4 we prove the formality result for Brylinski complex of a Poisson manifold.

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2 Brylinski complex of a Poisson manifold

Let us consider a Poisson manifold \((M, \pi)\). Here \(\pi\) is a Poisson bivector, which in particular implies that the bracket on \(C(M)\) defined by the formula

\[
\{f, g\} = \pi(df, dg)
\]

endows this space with the Lie algebra structure. That is, this bracket is anti-commutative and the Jacobi identity holds:

\[
\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.
\]

The latter equation is equivalent to the condition that the Schouten square of \(\pi\) vanishes:

\[
\{\pi, \pi\} = 0.
\]

**Remark 1.** The Schouten bracket on polyvector fields is a unique graded bracket which coincides with the commutator of vector fields being restricted to the first graded component and obeying the Leibnitz rule.

In this section we collect several known and rather simple facts about the DGLA structure on the complex of de Rham forms on a Poisson manifold. We give it here mostly for the methodological reasons. Let

\[
0 \longrightarrow C(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \cdots
\]

be the de Rham complex of \(M\). The material of this section is organized as follows

- Firstly we recall the construction of the Koszul bracket on the space of 1-forms on a Poisson manifold and prove that it endows \(\Omega^1(M)\) with the lie algebra structure.
- Further we extend this bracket to the exterior algebra \(\bigwedge^\ast \Omega^1(M)\) and prove that it descends correctly to the de Rham complex.
- We prove that this generalized bracket behaves well with respect to the external differential and that the induced Lie algebra structure on the cohomology is trivial (i.e. the Lie bracket is equal to zero).

**Koszul bracket** on 1-forms is given by the formula:

\[
\{\alpha, \beta\} = -d\pi(\alpha, \beta) + \mathcal{L}_{\pi^\#\alpha}\beta - \mathcal{L}_{\pi^\#\beta}\alpha
\]

where \(\mathcal{L}\) denotes the Lie derivative. On exact forms \(df, dg\) it gives

\[
\{df, dg\} = d\{f, g\}
\]

where on the right-hand side we use the standard Poisson bracket of functions on Poisson manifold.
Let us recall that there is a Lie algebroid structure on $T^*(M)$ with the anchor map
\[ \pi^\# : T^*(M) \to T(M). \]
The important property of a Lie algebroid consists in a differential module structure on $\Gamma(T^*(M))$. Indeed, one has
\[ \{ \alpha, f \beta \} = f \{ \alpha, \beta \} + (\mathcal{L}_{\pi^\# f})\beta. \]  
(6)

Lemma 1.
\[ [\pi^\# \alpha, \pi^\# \beta] = \pi^\# \{ \alpha, \beta \} \]  
(7)

Proof For $\alpha, \beta$ exact the statement follows from the standard fact in Poisson geometry that the commutator of Hamiltonian vector fields is a Hamiltonian vector field corresponding to the Poisson bracket of Hamiltonians
\[ [X_f, X_g] = X_{\{f,g\}} \]
due to the fact that $\pi^\# df = X_f$. Then, locally each 1-form can be represented as a sum of expressions of the type $f dg$, and it is sufficient to prove the claim for $\alpha = f_1 df_2$, $\beta = g_1 dg_2$ due to the local nature of this relation. One has
\[ [\pi^\# f_1 df_2, \pi^\# g_1 dg_2] = f_1 g_1 [\pi^\# df_2, \pi^\# dg_2] + f_1 (\mathcal{L}_{\pi^\# \alpha g_1}) \pi^\# \beta - g_1 (\mathcal{L}_{\pi^\# \beta f_1}) \pi^\# \alpha \]
\[ \pi^\# \{ f_1 df_2, g_1 dg_2 \} = \pi^\# (f_1 g_1 \{ df_2, dg_2 \}) + f_1 (\mathcal{L}_{\pi^\# \alpha g_1}) \beta - g_1 (\mathcal{L}_{\pi^\# \beta f_1}) \alpha \]
\[ = f_1 g_1 \pi^\# \{ df_2, dg_2 \} + f_1 (\mathcal{L}_{\pi^\# \alpha g_1}) \pi^\# \beta - g_1 (\mathcal{L}_{\pi^\# \beta f_1}) \pi^\# \alpha \]

Then returning to the formula (7) for exact 1-forms one obtains the general statement
\[ [X_f, X_g] = X_{\{f,g\}} \]

Lemma 2. The bracket [3] satisfies the Jacobi identity.
Proof We proceed as in the previous lemma starting from consideration of exact 1-forms. The Jacobi identity for forms $\alpha = df$, $\beta = dg$, $\gamma = dh$ is equivalent to the Jacobi identity for the bracket of functions:
\[ J(df, dg, dh) = \{ \{ df, dg \} dh \} + \{ \{ dg, dh \} df \} + \{ \{ dh, df \} dg \} = dJ(f, g, h) \]
where on the right-hand side the Jacobi expression is taken with respect to the Poisson bracket on functions. To finish the proof one needs to show that the Jacobi expression $J(\alpha, \beta, \gamma)$ is tri-linear subject to multiplication by functions. Indeed,
\[ J(\alpha, \beta, f \gamma) = \{ \{ \alpha, \beta \} f \gamma \} + \{ \{ \beta, f \gamma \} \alpha \} + \{ \{ f \gamma, \alpha \} \beta \} \]
\[ = f \{ \alpha, \beta \}, \gamma \} + (\mathcal{L}_{\pi^\# \{ \alpha, \beta \} f}) \gamma + f \{ \{ \beta, \gamma \}, \alpha \} - (\mathcal{L}_{\pi^\# \alpha f}) \{ \beta, \gamma \} \]
\[ + (\mathcal{L}_{\pi^\# f}) \{ \gamma, \alpha \} - (\mathcal{L}_{\pi^\# \alpha (\mathcal{L}_{\pi^\# f})}) \gamma + f \{ \{ \gamma, \alpha \}, \beta \} - (\mathcal{L}_{\pi^\# \beta f}) \{ \gamma, \alpha \} \]
\[ = f J(\alpha, \beta, \gamma) \]
where we used the property [6] and the result of lemma [4] \[ \square \]
In fact the Lie bracket (5) can be extended to the graded Lie bracket on the de Rham complex. To do this we first observe, that it can be extended to the exterior algebra of 1-forms $\bigwedge\Gamma^{1}(M)$ as follows. Let us introduce the grading $\deg\omega = d = |\omega| - 1$ for $\omega \in \bigwedge^{d+1}\Gamma^{1}(M)$. Further we denote by $d_{i}$ the degree $\deg\omega_{i}$. Now one can use the following formula to define the sought extension of (5)

$$\{\omega_{1}, \omega_{2} \wedge \omega_{3}\} = \{\omega_{1}, \omega_{2}\} \wedge \omega_{3} + (-1)^{d_{i}(d_{2} + 1)}\omega_{2} \wedge \{\omega_{1}, \omega_{3}\}$$

(8)

Imposing the graded antisymmetry

$$\{\omega_{1}, \omega_{2}\} = -(-1)^{d_{1}d_{2}}\{\omega_{2}, \omega_{1}\}$$

one obtains

$$\{\omega_{1} \wedge \omega_{2}, \omega_{3}\} = (-1)^{d_{2}(d_{3} + 1)}\{\omega_{1}, \omega_{3}\} \wedge \omega_{2} + \omega_{1} \wedge \{\omega_{2}, \omega_{3}\}$$

This bracket satisfies the graded Jacobi identity:

$$0 = J_{gr}(\omega_{1}, \omega_{2}, \omega_{3})$$

$$= (-1)^{d_{1}d_{3}}\{\{\omega_{1}, \omega_{2}\}, \omega_{3}\} + (-1)^{d_{2}d_{1}}\{\{\omega_{2}, \omega_{3}\}, \omega_{1}\} + (-1)^{d_{3}d_{2}}\{\{\omega_{3}, \omega_{1}\}, \omega_{2}\}.\]$$

We can proceed by induction on the degree of $\omega_{3}$ due to the symmetry of the Jacobi relation. In fact,

$$J(\omega_{1}, \omega_{2}, \omega_{3} \wedge \alpha) = (-1)^{d_{1}}J(\omega_{1}, \omega_{2}, \omega_{3}) \wedge \alpha + (-1)^{d_{2}(d_{3} + 1)}\omega_{3} \wedge J(\omega_{1}, \omega_{2}, \alpha)$$

and hence it reduces to the Jacobi identity for lower order elements.

In fact the bracket on the exterior algebra $\bigwedge^{n}\Gamma^{1}(M)$ descends correctly to the de Rham complex with the same grading. To verify this it is sufficient to prove the next formula:

$$\{\omega_{1}, f\omega_{2} \wedge \omega_{3}\} = \{\omega_{1}, \omega_{2} \wedge f\omega_{3}\}.\]$$

(9)

We shall restrict ourselves to the decomposable case $\omega_{1} = \alpha_{1} \wedge \ldots \wedge \alpha_{k}$ and $\omega_{2} = \beta_{1} \wedge \ldots \wedge \beta_{l}$ and demonstrate that the following equation holds for all $i$ and $j$:

$$\{\alpha_{1} \wedge \ldots \wedge \alpha_{k}, \beta_{1} \wedge \ldots \wedge f\beta_{i} \wedge \ldots \wedge \beta_{l}\} = \{\alpha_{1} \wedge \ldots \wedge \alpha_{k}, \beta_{1} \wedge \ldots \wedge f\beta_{j} \wedge \ldots \wedge \beta_{l}\}$$

(10)

Consider the left-hand side of (10)

$$\sum_{m}(-1)^{m+i}\{\alpha_{m}, f\beta_{i}\} \wedge \alpha_{1} \wedge \ldots \wedge \alpha_{k} \wedge \beta_{1} \wedge \ldots \wedge \beta_{l}$$

$$+ f\sum_{m,k\neq i}(-1)^{m+k}\{\alpha_{m}, \beta_{k}\} \wedge \alpha_{1} \wedge \ldots \wedge \alpha_{k} \wedge \beta_{1} \wedge \ldots \wedge \beta_{l}$$

$$= \sum_{m}(-1)^{m+i}\mathcal{L}_{n\#\alpha_{m}}f\beta_{i} \wedge \alpha_{1} \wedge \ldots \wedge \alpha_{k} \wedge \beta_{1} \wedge \ldots \wedge \beta_{l}$$

$$+ f\sum_{m}(-1)^{m+i}\{\alpha_{m}, \beta_{i}\} \wedge \alpha_{1} \wedge \ldots \wedge \alpha_{k} \wedge \beta_{1} \wedge \ldots \wedge \beta_{l}$$

$$+ f\sum_{m,k\neq i}(-1)^{m+k}\{\alpha_{m}, \beta_{k}\} \wedge \alpha_{1} \wedge \ldots \wedge \alpha_{k} \wedge \beta_{1} \wedge \ldots \wedge \beta_{l}$$

$$= f\{\omega_{1}, \omega_{2}\} + \mathcal{L}_{n\#\omega_{1}}f \wedge \omega_{2};$$

(11)
\[ \mathcal{L}_{\pi^*\omega_1} f = \sum_m (-1)^m (\mathcal{L}_{\pi^*\alpha_m} f) \alpha_1 \wedge \ldots \wedge \widehat{\alpha_m} \wedge \ldots \wedge \alpha_k. \]  

(12)

Now it is enough to notice that the resulting formula does not depend on \( i \).

**Reduction** The next crucial property of the considered construction that makes \( \Omega^*(M) \) a DGLA is relation between the graded Lie algebra structure on \( \Omega^*(M) \) and the de Rham differential, exactly it is that \( d \) differentiates the Lie bracket introduced above.

**Lemma 3.**

\[ d\{\omega_1, \omega_2\} = \{d\omega_1, \omega_2\} + (-1)^{d_1} \{\omega_1, d\omega_2\} \]

(13)

**Proof** We proceed by induction on the degree of \( \omega_2 \). We take a decomposable element \( \omega_2 \wedge \omega_3 \) and suppose that

\[
\begin{align*}
d\{\omega_1, \omega_2\} &= \{d\omega_1, \omega_2\} + (-1)^{d_1} \{\omega_1, d\omega_2\}; \\
d\{\omega_1, \omega_3\} &= \{d\omega_1, \omega_3\} + (-1)^{d_1} \{\omega_1, d\omega_3\}.
\end{align*}
\]

Then one needs to prove that

\[ d\{\omega_1, \omega_2 \wedge \omega_3\} = \{d\omega_1, \omega_2 \wedge \omega_3\} + (-1)^{d_1} \{\omega_1, d(\omega_2 \wedge \omega_3)\}. \]

On the left-hand side we have:

\[
\begin{align*}
d\left(\{\omega_1, \omega_2\} \wedge \omega_3 + (-1)^{d_1(d_2+1)} \omega_2 \wedge \{\omega_1, \omega_3\}\right) \\
= \{d\omega_1, \omega_2\} \wedge \omega_3 + (-1)^{d_1} \{\omega_1, d\omega_2\} \wedge \omega_3 + (-1)^{d_1+1} \{\omega_1, \omega_2\} \wedge d\omega_3 \\
+ (-1)^{d_1(d_2+1)} d\omega_2 \wedge \{\omega_1, \omega_3\} + (-1)^{(d_1+1)(d_2+1)} (\omega_2 \wedge \{d\omega_1, \omega_3\} + (-1)^{d_1} \omega_2 \wedge \{\omega_1, d\omega_3\}).
\end{align*}
\]

On the right-hand side we have:

\[
\begin{align*}
\{d\omega_1, \omega_2\} \wedge \omega_3 + (-1)^{(d_1+1)(d_2+1)} \omega_2 \wedge \{d\omega_1, \omega_3\} \\
+ (-1)^{d_1} \{\omega_1, d\omega_2\} \wedge \omega_3 + (-1)^{d_1(d_2+2)} d\omega_2 \wedge \{\omega_1, \omega_3\} \\
+ (-1)^{d_2+1} \{\omega_1, \omega_2\} \wedge d\omega_3 + (-1)^{d_2+1+d_1(d_2+1)} \omega_2 \wedge \{\omega_1, d\omega_3\}.
\end{align*}
\]

Comparing both sides one obtains the lemma \( \square \)

**Lemma 4.** The Lie algebra structure on \( \Omega^*(M) \) can be pushed down to its de Rham cohomology

**Proof** Let \( \omega_1, \omega_2 \) be closed then \( \{\omega_1, \omega_2\} \) is also closed; let \( \omega_1 \) be closed and \( \omega_2 \) - exact, then \( \{\omega_1, \omega_2\} \) is exact due to (13) \( \square \)
Lemma 5. The induced Lie algebra structure on $H^*(M)$ of a Poisson manifold is trivial.

Proof. It is obvious for 1-forms. Indeed, let $\alpha, \beta$ be closed: $d\alpha = 0, d\beta = 0$. Then

$$\mathcal{L}_{\pi^#\alpha} \beta = d_{\pi^#\alpha} \beta = d\pi(\alpha, \beta)$$

and

$$\{\alpha, \beta\} = d\pi(\alpha, \beta) \tag{14}$$

so, it is exact.

Let us next consider closed forms $\alpha, \omega$ of degrees 1 and $k$ respectively. There is a simplification of the formula (8) for this case:

$$\{\alpha, \omega\} = \mathcal{L}_{\pi^#\alpha} \omega = d(\tilde{\pi}^#\alpha) \tag{15}$$

Indeed, on both sides one has a differentiation and the equality fulfills when $\omega$ is a 1-form.

Now observe that the right hand side of the last formula can be written down in the form

$$\{\omega_1, \omega_2\} = d\tilde{\pi}(\omega_1, \omega_2), \tag{15}$$

where $\tilde{\pi}$ is a bilinear operation, sending two forms $\omega_1, |\omega_1| = k$ and $\omega_2, |\omega_2| = l$ to a $(k+l-2)$-form

$$\tilde{\pi}(\omega_1, \omega_2) = \sum_{p,q} \sum_{1 \leq i \leq k, 1 \leq j \leq l} (-1)^{k+j-1} \pi(\alpha_i, \beta_j) \wedge \alpha_i^p \wedge \ldots \wedge \beta_j^q \wedge \ldots \wedge \alpha_k^p \wedge \ldots \wedge \beta_l^q, \tag{16}$$

where, as usually, $\wedge$ over an element means that this element is omitted and

$$\omega_1 = \sum_p \alpha_1^p \wedge \ldots \wedge \alpha_k^p,$$

$$\omega_2 = \sum_q \beta_1^q \wedge \ldots \wedge \beta_l^q.$$ 

As before, $\alpha_i^p, \beta_j^q$ are 1-forms. Observe, that

$$\tilde{\pi}(\omega_1, \omega_2) = \pi \vdash (\omega_1 \wedge \omega_2) - (\pi \vdash \omega_1) \wedge \omega_2 - (-1)^{\omega_1} \omega_1 \wedge (\pi \vdash \omega_2),$$

where $\pi \vdash$ denotes the internal multiplication of a differential form by the bivector $\pi$. In particular, it follows that $\tilde{\pi}$ is a bilinear with respect to the multiplication by functions and differentiation with respect to either argument (which can also be proved independently directly from the formula (16)).

We shall prove, that the formula (15) holds in the general case, i.e that $\omega_1$, $\omega_2$ are closed forms, then their Poisson bracket is an exact form, defined by formulas (15), (16).

To this end, let us observe that the right hand side of the formula (16) (and consequently that of (15)) is well-defined, that is it doesn’t depend on the way one decomposes the forms $\alpha, \beta$ into the wedge-product of 1-forms. Second, it is enough to prove formula (15) locally, i.e. on an arbitrary open domain in the manifold. Indeed, the expressions on
both its sides are well-defined everywhere on the manifold and don’t depend on the choice of the decompositions of \( \alpha, \beta \) into 1-forms. Finally, recall, that locally all closed forms are exact, and hence they are representable as the sum of products of closed (and even exact) forms (one has \( \omega = d\omega' = d(\sum_{i=1}^{n} f_i dx^i \wedge \ldots \wedge dx^n) = \sum_i df_i dx^i \wedge \ldots \wedge dx^n) \). Now the conclusion follows from the formula (14). □

Remark 2. In effect, one can prove the following formula (pointed out by I. Nikonov):

\[
\{\omega_1, \omega_2\} = d\tilde{\pi}(\omega_1, \omega_2) - \tilde{\pi}(d\omega_1, \omega_2) - (-1)^{|\omega_1|} \tilde{\pi}(\omega_1, d\omega_2).
\] (17)

Here \( \omega_1, \omega_2 \) are arbitrary de Rham forms on the manifold and \( \tilde{\pi} \) is the operation, defined above by the formula (16).

In order to prove (17) just observe, that its right hand side is a skew-symmetric bilinear form of degree \(-1\) on the de Rham complex of the Poisson manifold, verifying the Leibnitz rule with respect to the multiplication of the forms (i.e. the formula (9) holds for \( \{, \} \) replaced with the expression from (17)), and differentiated by the de Rham differential (i.e. verifying the formula (13)). Finally, observe that it coincides with the bracket \( \{, \} \) when both forms have degree 1.

3 \( L_\infty \)-algebras and \( L_\infty \)-morphisms

In this section we recall the basic definitions and constructions of the homotopy Lie algebras (\( L_\infty \)-algebras) and \( L_\infty \) morphisms, which are crucial for our treatment of the formality (see section 1). First, let us recall the definition of \( L_\infty \)-algebras. In what follows, we use the notation and signs from the paper [2].

**The free cocommutative coalgebra of a graded space**

Let \( L \) be a graded vector space. Consider the external algebra, associated to \( L \):

\[
\Lambda^* L = \bigoplus_{i \geq 1} L^\otimes i \left/ \left\{ a \otimes b + (-1)^{|a||b|} b \otimes a \right\} \right.. 
\] (18)

Here and below \(|a|\) will denote the degree of an element \( a \) in \( L \). Recall, that one can introduce double grading on \( \Lambda^* L \), putting bideg\((a_1 \wedge a_2 \wedge \cdots \wedge a_n) = (\sum |a_i|, n)\), \( a_i \in L \). For an element of bi-degree \((m, n)\) we define its total degree to be equal to the difference \( n - m \). We shall denote the second degree of an element \( \alpha \) by \(|\alpha|'\) (in contrast with its first degree, denoted by \(|\alpha|\)).

One uses the following formula as the definition of comultiplication \( \nabla \) in \( \Lambda^* L \) (\( a_i \in L \) are arbitrary elements):

\[
\nabla(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in Sh_{k,l}} (-1)^{\sigma + kl} a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)} \otimes a_{\sigma(k+1)} \wedge \cdots \wedge a_{\sigma(n)}.
\] (19)

Here \( Sh_{k,l} \) stands for the subset of all \((k, l)\)-shuffles in the group of permutations of \( k + l \) elements, that is of the permutations, preserving the order of the first \( k \) and the last \( l \)
elements. And the sign $(-1)^{\sigma}$ of the shuffling $\sigma$, is determined by the following rule: every time we change the order of two elements $a_i$ and $a_j$ in the sequence $a_1, \ldots, a_n$, we multiply it by $|a_i||a_j| + 1$. Equipped with this comultiplication $\wedge^* L$ is the free cocommutative coalgebra, (without counit) (co)generated by $L$.

Clearly, the comultiplication $\nabla$ respects the double grading and hence the total grading in $\wedge^* L$, so that the latter space becomes a (bi)graded cocommutative coalgebra.

Now one can introduce bi-degrees for all the maps from $\Lambda^* L$ to itself, as (minus) the difference of degrees of a (bi-homogeneous) element and its image. Of course, this is not always correctly defined (i.e. it may depend on the element). We would say that the map is (bi-)homogeneous, if it has bi-degree. As in the case of elements, we denote the second degree of a map $f$ by $|f|'$, reserving the symbol $|f|$ for the first (usual) degree of the map $f$.

As before, one defines the total degree of a map as the difference of its first and second degrees. Observe, that a map, which is homogeneous with respect to this grading, is not necessarily bihomogeneous (while the contrary is true).

**Graded coderivatives**  
One can give the following important definition

**Definition 1.** Let $(C, \nabla_C)$ be a bigraded coalgebra with comultiplication $\nabla_C$. A bi-homogeneous map $D : C \to C$ of bidegree $(|D|, |D|')$ is called (bi-homogeneous) coderivative, if the following diagram is commutative up to a sign, depending on the bidegrees of elements:

$$
\begin{array}{ccc}
C & \xrightarrow{\nabla_C} & C \otimes C \\
D \downarrow & & \downarrow D \otimes 1 + 1 \otimes D \\
C & \xrightarrow{\nabla_C} & C \otimes C.
\end{array}
$$

Or, in more explicit terms,

$$
\nabla(D\alpha) = D\alpha(1) \otimes \alpha(2) + (-1)^{|D||\alpha(1)| + |D'||\alpha(1)'|} \alpha(1) \otimes D\alpha(2),
$$

for all $\alpha$ in $C$. Here we use the Sweedler’s notation for comultiplication: $\nabla(\alpha) = \alpha(1) \otimes \alpha(2)$.

**Proposition 1.** Let $D_1$ and $D_2$ be two bi-homogeneous coderivatives of $\Lambda^* V$. Then their bi-graded commutator

$$
[D_1, D_2] = D_1D_2 - (-1)^{|D_1||D_2| + |D_1'||D_2'|} D_2D_1
$$

is also a bi-homogeneous coderivative. Its bi-degree is equal to the (element-wise) sum of the bi-degrees of $D_1$ and $D_2$.

**Proof.** Evident. □

By the virtue of freeness of the coalgebra $\wedge^* L$, any coderivative on it is uniquely determined by its “values on cogenerators”, that is by a collection of linear mappings $l_n : \wedge^n L \to L$ (see [3]). If the total degree of $l$ is equal to $p$, we conclude, that the $n$-th map in this set, $l_n$, sends elements $a_1 \wedge \cdots \wedge a_n \in \wedge^n L$ to the elements of degree
\[ |a_1| + \cdots + |a_n| - n + p + 1 \] in \( L \), due to the introduced grading in \( \land^* L \). And the converse is also true: if \( \{ f_k : \Lambda^k L \to \Lambda^1 L \} \) is an arbitrary collection of linear maps, such that the first degree of \( f_k \) is equal to \( m_k \), then one can in a unique way extend \( \{ f_k \} \) to a coderivative \( F : \Lambda^* V \to \Lambda^* V \), equal to the sum of bi-homogeneous codervatives of bidegrees \( (m_k, k - 1) \). (Here the phrase “\( F \) extends \( f \)” means that \( F|_{\land^1 V} = f_i \).) This is an immediate consequence of freeness of \( \Lambda^* V \) (see Sweedler’s book). More explicitly, one uses following formula as the definition of coderivative \( F \):

\[
F(a_1 \land \cdots \land a_n) = \sum_{k<n} \sum_{\sigma \in \Sigma_n} (-1)^\sigma \frac{1}{k!(n-k)!} f_k(a_\sigma(1) \land \cdots \land a_\sigma(k)) \land a_\sigma(k+1) \land \cdots \land a_\sigma(n) \tag{22}
\]

Here the sign \((-1)^\sigma\) is defined as above.

One can check, that the commutator of two bi-homogeneous coderivatives \( \Phi \) and \( \Psi \) of bidegrees \( (|\phi|, k - 1) \) and \((|\psi|, l - 1)\), extending the maps \( \varphi : \Lambda^k L \to \Lambda^1 L \) and \( \psi : \Lambda^l L \to \Lambda^1 L \) coincides with the extension of the map \([\varphi, \psi] : \Lambda^{k+l-1} L \to \Lambda^1 L\), given by the formula

\[
[\varphi, \psi](a_1 \land \cdots \land a_{k+l-1}) = \sum_{\sigma \in \Sigma_{k+l-1}} \frac{1}{l!(k-1)!} \varphi(\sigma(a_1) \land \cdots \land a_{\sigma(l)} \land a_{\sigma(l+1)} \land \cdots \land a_{\sigma(k+l-1)}) - (-1)^\epsilon \sum_{\sigma \in \Sigma_{k+l-1}} \frac{1}{(l-1)!k!} \psi(\varphi(a_\sigma(1) \land \cdots \land a_\sigma(k)) \land a_{\sigma(k+1)} \land \cdots \land a_{\sigma(k+l-1)}) \tag{23}
\]

where \( \epsilon = |\varphi||\psi| + (k - 1)(l - 1) \).

The space of (bi-graded) coderivatives of \( \land^* L \), endowed with the commutator \([, ]\), turns into a bi-graded Lie algebra. However, one can turn it into the graded Lie algebra with respect to the total degree simply by changing slightly the sign in \([\dots]\), namely

\[
\{D_1, D_2\} = (-1)^{|D_1||D_2'|}[D_1, D_2]. \tag{24}
\]

An alternative way of looking at this formula is as follows. Consider a “deformation” \( \tilde{D} \) of a bihomogeneous map \( D : \land^* L \to \land^* L \), defined by the formula \( \tilde{D}(\alpha) = (-1)^{|D||\alpha|'} D(\alpha) \). Then \( \overline{D_1 D_2} = (-1)^{|D_1||D_2|'} \tilde{D}_1 \tilde{D}_2 \). Indeed,

\[
\tilde{D}_1(\tilde{D}_2(\alpha)) = (-1)^{|D_2||\alpha|'} D_1(D_2(\alpha)) = (-1)^{|D_1||D_2|'+|\alpha|'} D_1(D_2(\alpha)) = (-1)^{|D_1||D_2|'} \overline{D_1 D_2}(\alpha).
\]

Observe, that if \( D \) is a coderivative, then \( \tilde{D} \) is not a bi-graded coderivative for the given diagonal \( \nabla \) (the signs rule is violated). In effect, it is a graded coderivative (i.e. a coderivative with respect to the total degree), for a “deformed” comultiplication \( \nabla \), given by the formula \( \nabla(\alpha) = (-1)^{|\alpha(1)||\alpha(2)|'} \alpha(1) \otimes \alpha(2) \). This is verified
by a direct computation:

$$\nabla(D(\alpha)) = (-1)^{|\alpha|'}\nabla(D(\alpha)) =$$

$$= (-1)^{|\alpha|'}\left( (-1)^{|D|+|\alpha(1)|}D(\alpha(1)) \otimes \alpha(2) + (-1)^{|D|+|\alpha(1)|+|\alpha(2)|'}D(\alpha(1)) \otimes D(\alpha(2)) \right)$$

$$= (-1)^{|\alpha|'}\left( (-1)^{|D|+|\alpha(1)|+|\alpha(2)|'}D(\alpha(1)) \otimes \alpha(2) + (-1)^{|D|+|\alpha(1)|+|\alpha(2)|'}\alpha(1) \otimes D(\alpha(2)) \right)$$

$$= (-1)^{|\alpha|'}\left( (D \otimes 1 + (-1)^{|D|+|\alpha|'}\alpha(1) \otimes D) \nabla(\alpha) \right).$$

Now, if $[A, B]_{\text{tot}}$ denotes the commutator of two maps $A, B : \wedge^* L \to \wedge^* L$ with respect to their total degree, then one has

$$[\vec{D}_1, \vec{D}_2]_{\text{tot}} = \vec{D}_1 \vec{D}_2 - \vec{D}_2 \vec{D}_1$$

$$= (-1)^{|D|_1|D|_2'}\vec{D}_1 \vec{D}_2 + (-1)^{|D|_2'||D|_1}\vec{D}_2 \vec{D}_1$$

$$= (-1)^{|D|_1|D|_2'}(\vec{D}_1 \vec{D}_2 + (-1)^{|D|_1|D|_2'+|D|_2'||D|_1} \vec{D}_2 \vec{D}_1) = (D_1, D_2) \quad (25)$$

$L_\infty$-algebras and morphisms

The definition of an $L_\infty$-algebra can now be formulated in a very concise way

**Definition 2.** One says, that the graded space $L$ is an $L_\infty$-algebra, if its free coalgebra $\wedge^* L$ is equipped with a degree-1 (i.e. its total degree should be equal to 1) bigraded coderivative $\vec{l}$ verifying the relation $\{l, l\} = 0$. Equivalently, in the view of equation (25), one can say, that the graded coderivative $\vec{l}$ verifies the equation $(\vec{l})^2 = 0$.

The equations, appearing in this definition, can be interpreted as an infinite collection of quadratic equations on the components $l_i$ of $l$. Here are the first three equations of this collection:

$$l_1(l_1(a)) = 0;$$

$$l_1(l_2(a \wedge b)) = l_2(l_1(a) \wedge b) - (-1)^{|a|}l_2(a \wedge l_1(b));$$

$$l_1(l_3(a \wedge b \wedge c)) + l_3(l_1(a) \wedge b \wedge c) + (-1)^{|a|}l_3(a \wedge l_1(b) \wedge c) + (-1)^{|a|+|b|}l_3(a \wedge b \wedge l_1(c))$$

$$= \frac{1}{2}\left( (-1)^{|a|+|c|}l_2(l_2(a \wedge b) \wedge c) + (-1)^{|a|(|b|+|c|)}l_2(l_2(b \wedge c) \wedge a) + (-1)^{|a|+|b|+|c|}l_2(l_2(c \wedge a) \wedge b) \right).$$

The first and the second of these equalities mean, that $l_1$ is a square-zero differential and $l_2$ - an anti-symmetric bracket on the space $L$, so that $l_1$ is a differentiation of $l_2$. The
third equation implies, that the Jacobi identity on $L$ holds up to a homotopy, moreover, the chain homotopy connecting it (the Jacobi formula) to zero is $l_3$.

On the whole, the equations, verified by the mappings $l_i$ can be written down in the following way:

$$
\sum_{\sigma \in \text{Sh}_{k,I-1}} (-1)^{\sigma + (k-1)n} l_i(l_k(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)}) \wedge \cdots \wedge a_{\sigma(n)}) = 0. 
$$

(26)

Or, in a more conceptual form,

$$
d(l_n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2} l_i \circ (l_{n-i} \times 1). 
$$

Here we have denoted the differential $l_1$ in $L$ (see above) by a more traditional letter $d$, $[x]$ denotes the integer part of the number $x$, $d(l_n) = d \circ l_n + (-1)^{n-2} l_n \circ d$, and $l_i \times 1 \overset{\text{def}}{=} (l_i \otimes 1) \circ \nabla$. In brief, $l_n$ is a chain homotopy connecting certain linear combination of $l_i$, $i < n$ with zero.

In particular, one sees from these formulas, that every differential graded Lie algebra is $L_\infty$-algebra. It is enough to put

$$
l_1 = d \quad - \text{differential in } L, \\
l_2 = [\cdot, \cdot] \quad - \text{the Lie bracket in } L, \text{ and} \\
l_i = 0, \quad i \geq 2. 
$$

One can define two different types of natural maps between $L_\infty$-algebras. First is the so-called strict $L_\infty$-maps, i.e. the maps, commuting with all the mappings $l_n$. The second, and the most important type is the strong homotopy morphisms or $L_\infty$-morphisms. By definition, an $L_\infty$-morphism from an $L_\infty$-algebra $L$ to an $L_\infty$-algebra $L'$ is a (total) degree-0 linear map $F : \wedge^* L \to \wedge^* L'$, commuting with comultiplications and (co)differentials $l$, $l'$ in $\wedge^* L$ and $\wedge^* L'$.

As before, due to the freeness of $\wedge^* L'$, any homomorphism to this coalgebra from a coalgebra $C$ is uniquely determined by its “zero stage”, i.e. by the linear map $C \to L$, which it defines. In our situation, this reduces to a collection of maps $f_n : \wedge^n L \to L'$, $n \geq 1$, each changing the (first) degree by $-(n-1)$. The condition that the total map, made up from all the $f_i$ commutes with the (co)differentials, takes the form of an infinite series of equations on the maps $f_i$. The first two equations of this series are as follows (c.f. (2), (3)):

$$
l'_1(f_1(a)) = f_1(l_1(a)), \\
l'_1(f_2(a \wedge b)) - f_2(l_1(a) \wedge b) - (-1)^{|a|} f_2(a \wedge l_1(b)) = f_1(l_2(a \wedge b)) - l'_2(f_1(a) \wedge f_1(b)). 
$$

(27)

(28)

The equation (27) shows that the map $f_1$ should commute with the differentials $l_1$, $l'_1$ in $L$, $L'$, and the equation (28) means that $f_2$ is the chain homotopy, which makes the
map $f_1$ a homomorphism of Lie algebras on the corresponding cohomology. The general equation from the definition of $L_\infty$-morphisms can be written down in the following form:

$$
\begin{align*}
df_n &= \sum_{k=2}^{n} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \pm l_k \circ (f_1 \times \cdots \times f_k) \\
&+ \sum_{k=1}^{n-1} \pm \frac{1}{2} f_k \circ (l_{n-k+1} \times 1).
\end{align*}
$$

Here, as above, we denote by $f_1 \times f_2$ the composition $(f_1 \otimes f_2) \circ \nabla$ (the “external product” of more than 2 terms is defined by induction).

The signs in this formula depend on the dimensions of the maps, and on their order. Since below we shall consider only the case when both $L_\infty$-algebras are in effect (graded, differential) Lie algebras, let us give the precise formulas for the equations, verified by an $L_\infty$-morphism just in this case:

$$
\begin{align*}
\begin{align*}
&\left.\right. \\
&d f_n (a_1 \wedge \cdots \wedge a_n) = \\
&\left.\right. = \sum_{i=1}^{n} (-1)^{n-1+\varepsilon_i} f (a_1 \wedge \cdots \wedge da_i \wedge \cdots \wedge a_n) \\
&+ \frac{1}{2} \sum_{i+j=k} \sum_{\sigma \in Sh_{i,j}} (-1)^{\sigma+i} \{ f_i (a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(i)}), f_j (a_{\sigma(i+1)} \wedge \cdots \wedge a_{\sigma(n)}) \} \\
&+ \frac{1}{2} \sum_{1 \leq i < j \leq n} (-1)^{(i-1)\varepsilon_i-1+(j-1)\varepsilon_j-1} f_{n-1} (\{a_i, a_j\} \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_n).
\end{align*}
\end{align*}
$$

(29)

As usually, the “hat” over an entry means that the entry is omitted in the formula, $Sh_{i,j}$ denotes the collection of all $(i, j)$-shuffles, the sign $(-1)^\sigma$ is defined as above and

$$
\varepsilon_i = -i + \sum_{k=1}^{i} |a_k|, \quad \varepsilon_i^j = \varepsilon_i - |a_j| + 1.
$$

4 Main theorem

The purpose of this section is to prove the formality of the DG Lie algebra, associated to a Poisson manifold. To this end we shall construct an $L_\infty$-quasi-isomorphism (i.e. an $L_\infty$-map with quasi-isomorphic first stage) between this Lie algebra and its cohomology Lie algebra. More accurately, in order to speak about the Lie algebras in this case, one should first change grading in $\Omega^\ast (\mathcal{M})$. In fact, the bracket, introduced in §2 sends a couple of elements of degrees $k$ and $l$ in $\Omega^\ast (\mathcal{M})$ to an element of degree $k+l-1$ and not $k+l$. So, we pass to the suspension $\Sigma \Omega^\ast (\mathcal{M})$ of $\Omega^\ast (\mathcal{M})$. Here for a graded space $V$, $\Sigma V$ is the graded space, defined by

$$
(\Sigma V)_i = V_{i+1}.
$$

(\text{In particular, } \Sigma \Omega^\ast (\mathcal{M})_{-1} = C^\infty (\mathcal{M})\text{.)} It is easy to check, that the new grading is respected by the Cartan bracket on $\Omega^\ast (\mathcal{M})$. Similarly, one should replace $H(\mathcal{M})$ by its suspension $\Sigma H(\mathcal{M})$.}
In order to prove the formality of the Lie algebra $\Sigma\Omega^*(\mathcal{M})$ for a Poisson manifold $\mathcal{M}$, we shall need some special properties of the free coalgebra, generated by $\Sigma\Omega(\mathcal{M})$. These properties result from the usual constructions, related to the differential forms on a smooth manifold, such as the Cartan calculus and external multiplication of forms. We have collected these results in a separate subsection.

### 4.1 Coderivatives in $\wedge^*\Sigma\Omega(\mathcal{M})$

Let $L = \Sigma\Omega(\mathcal{M})$ – the suspension of the algebra of de Rham forms for some manifold $\mathcal{M}$ (see above). We shall denote by $s$ the evident degree $-1$ isomorphism $\Omega(\mathcal{M}) \overset{s}{\to} \Sigma\Omega(\mathcal{M})$, in particular $s\alpha$ will denote the element of $\Sigma\Omega(\mathcal{M})$, corresponding to a form $\alpha \in \Omega(\mathcal{M})$.

Below we give few geometric examples of coderivatives of $\Lambda^*L$ in this case.

**Cartan algebra** Let $X$ be a vector field. We can associate to it two differentiations of $\Omega^*(\mathcal{M})$, Lie derivative and the internal multiplication by $X$. We shall denote them by $L_X$ and $i_X$ respectively. The degree of $L_X$ is equal to 0 and of $i_X$ — to $-1$.

We shall associate to $L_X$ and $i_X$ the maps $\tilde{L}_X, \tilde{i}_X : \Sigma\Omega(\mathcal{M}) \to \Sigma\Omega(\mathcal{M})$ as follows:

$$\tilde{L}_X(s\alpha) = sL_X(\alpha), \quad \tilde{i}_X(s\alpha) = -si_X(\alpha).$$

The degrees of these maps are also equal to 0 and $-1$.

These two maps can be extended to coderivatives of $\Lambda^*\Sigma\Omega(\mathcal{M})$ of bi-degrees $(0, 0)$ and $(0, -1)$ respectively (see the remark, following the proposition 1). We shall denote these coderivatives by $L_X, I_X$.

Similarly, the external differential $d : \Omega^*(\mathcal{M}) \to \Omega^{*+1}(\mathcal{M})$, gives rise to a map $\tilde{d} : \Sigma\Omega(\mathcal{M}) \to \Sigma\Omega(\mathcal{M})$, $\tilde{d}s\alpha = -sda\alpha$. The corresponding coderivative on $\Lambda^*\Sigma\Omega(\mathcal{M})$ is denoted by $D$, its bi-degree is $(0, 1)$.

**Proposition 2.** The maps $L_X, I_X$ and $D$ verify the same Cartan identities as the original maps on the level of $\Omega(\mathcal{M})$, i.e.

$$[L_X, L_Y] = L_{[X, Y]}, \quad [I_X, I_Y] = 0, \quad [L_X, I_Y] = I_{[X, Y]},$$
$$[D, L_X] = 0, \quad [D, I_X] = L_X, \quad \text{and} \quad D^2 = 0.$$

As above the symbol $[,]$ denotes the bi-graded commutator of the coderivatives. Observe, that similar formulae hold for the modified commutator $\{,\}$ (this is due to its definition).

**Proof.** Direct calculations with the help of the formula (23) and (22), which in the case of the maps $f : \Lambda^1V \to \Lambda^1V$ reduces to

$$\tilde{f}(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^n (-1)^{|a_i|(|a_1|+\cdots+|a_{i-1}|)+i-1} f(a_i) \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n$$

$$= \sum_{i=1}^n (-1)^{|f||a_1|+\cdots+|a_{i-1}|} a_1 \wedge \cdots \wedge f(a_i) \wedge \cdots \wedge a_n$$

$\square$
Remark 1. As a matter of fact, due to the discussion preceding Definition 2, one can replace all the considered maps by their “skewed versions” and from equation (25) it follows, that these “skewed” maps verify equations, similar to the equations of Proposition 1 but with the commutator \([,]\) substituted for \([,]\). The same remark is true for all the examples, that will follow. In effect, one could rewrite these examples and the rest of the paper with “skewed” operators substituted for the usual ones and with \([,]_{\text{tot}}\) instead of \([,]\).

Multiplication

Let \(\cdot: \Omega(\mathcal{M}) \otimes \Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{M})\), \(\alpha \otimes \beta \rightarrow \alpha \cdot \beta\) be the external multiplication of forms. It turns out, that it can be extended to a correctly defined map \(m : \Lambda^2 \Sigma \Omega(\mathcal{M}) \rightarrow \Lambda^1 \Sigma \Omega(\mathcal{M})\). Namely, put

\[
m(s\alpha \wedge s\beta) = (-1)^{|\alpha|} s(\alpha \cdot \beta)
\]

for all \(s\alpha, s\beta \in \Sigma \Omega(\mathcal{M})\).

Observe, that here \(|\alpha|\) denotes the non-suspended degree of \(\alpha\). One easily checks, that this definition is compatible with the commutator relations in \(\Lambda^2 \Sigma \Omega(\mathcal{M})\). Note, that the second degree of this map is equal to 1. So, we obtain a homogeneous coderivative \(M\) of \(\Lambda^* \Sigma \Omega(\mathcal{M})\) of bi-degree \((-1, 1)\).

Proposition 3. The map \(M\) commutes with \(I_x\), \(L_x\) and \(D\) (the commutator is understood in the bi-graded sense, i.e. all the bi-graded commutators, defined as above, vanish).

Proof. We shall check the statement solely for the maps \(I_x\). In the case of the maps \(L_x\) and \(D\) the proof is similar. In the view of the formula (25), it is enough to check that the maps \(m\) and \(i_x\) anti-commute (the sign \(\epsilon\) in this case is equal to 1).

We compute for all \(s\alpha, s\beta \in \Sigma \Omega(\mathcal{M})\):

\[
i_x(m(s\alpha \wedge s\beta)) = (-1)^{|\alpha|} i_x(s(\alpha \cdot \beta))
\]

\[
= (-1)^{|\alpha|+1} s i_x(\alpha \cdot \beta)
\]

\[
= (-1)^{|\alpha|+1} s (i_x(\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot i_x(\beta))
\]

But \(s(i_x(\alpha) \cdot \beta) = (-1)^{|\alpha|+1} m(s i_x(\alpha) \wedge s \beta) = (-1)^{|\alpha|} m(i_x(s\alpha) \wedge s\beta)\). Similarly \(s(\alpha \cdot i_x(\beta)) = (-1)^{|\alpha|} m(s\alpha \wedge s i_x(\beta)) = (-1)^{|\alpha|+1} m(s\alpha \wedge i_x(s\beta))\). Hence, we continue the equation (30):

\[
i_x(m(s\alpha \wedge s\beta)) = -m(i_x(s\alpha) \wedge s\beta) + (-1)^{|\alpha|} m(s\alpha \wedge i_x(s\beta))
\]

\[
= -(m(i_x(s\alpha) \wedge s\beta) + (-1)^{|\alpha|-1)(|\beta|-1)+1} m(i_x(s\beta) \wedge s\alpha)).
\]

Remark 2. As a matter of fact, in the proof of the proposition we used only the fact that \(i_x\) is a differentiation of the algebra \(\Omega(\mathcal{M})\). It is easy to check that this proposition holds for arbitrary differentiations of the de Rham algebra of a manifold. Namely: let \(\phi : \Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{M})\) be a degree \(k\) differentiation, i.e. \(\phi(\Omega^i(\mathcal{M})) \subseteq \Omega^{i+k}(\mathcal{M})\), and

\[
\phi(\alpha \cdot \beta) = \phi(\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot \phi(\beta).
\]

Define a degree-\(k\) map \(\tilde{\phi} : \Sigma \Omega(\mathcal{M}) \rightarrow \Sigma \Omega(\mathcal{M})\) as \(\tilde{\phi}(s\alpha) = (-1)^k s\phi(\alpha)\). Then the coderivative \(\Phi\), extending this map to \(\Lambda \Sigma \Omega(\mathcal{M})\), commutes with the map \(M\) above in the bi-graded sense.
The following definition is important for our proof of the formality.

**Definition 3.** Let $\phi : \Lambda^k \Sigma \Omega(\mathcal{M}) \to \Sigma \Omega(\mathcal{M})$ and $\psi : \Lambda^l \Sigma \Omega(\mathcal{M}) \to \Sigma \Omega(\mathcal{M})$ be two linear maps. We define map $\phi \cup \psi : \Lambda^{k+l} \Sigma \Omega(\mathcal{M}) \to \Sigma \Omega(\mathcal{M})$ by the following formula (here $a_1, \ldots, a_{k+l}$ are arbitrary elements of $\Sigma \Omega(\mathcal{M})$)

\[
\phi \cup \psi(a_1 \wedge \cdots \wedge a_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \Sigma_{k+l}} (-1)^{\sigma'} \phi(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)}) \cdot \psi(a_{\sigma(k+1)} \wedge \cdots \wedge a_{\sigma(k+l)}) \tag{31}
\]

where the sign $(-1)^{\sigma'} = (-1)^{\sigma + (|\psi|+1)(\sum_{i=1}^{k+l} |\alpha_{\sigma(i)}|) + (k-1)(l-1) + |\phi|}$ (the sign $(-1)^{\sigma}$ is defined above).

**Lemma 6.**

(i) The cup-product is (bi)graded-commutative, i.e.

\[
\phi \cup \psi = (-1)^{|\phi||\psi|+(k-1)(l-1)} \psi \cup \phi. \tag{32}
\]

(ii) For any differentiation $\delta : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$, the following formula holds

\[
[\tilde{\delta}, \phi \cup \psi] = [\tilde{\delta}, \phi] \cup \psi + (-1)^{|\phi||\psi|} \phi \cup [\tilde{\delta}, \psi], \tag{33}
\]

where $\tilde{\delta} : \Sigma \Omega(\mathcal{M}) \to \Sigma \Omega(\mathcal{M})$ is defined in remark 2 and commutator $[\cdot, \cdot]$ – by formula (28).

(iii) If $\alpha$, $\beta$, $\gamma$ and $\delta$ are differentiations of $\Omega(\mathcal{M})$, then

\[
[\tilde{\alpha} \cup \tilde{\beta}, \tilde{\gamma} \cup \tilde{\delta}] = \tilde{\alpha} \cup [\tilde{\beta}, \tilde{\gamma} \cup \tilde{\delta}] + (-1)^{|\beta|(|\gamma|+|\delta|)}[\tilde{\alpha}, \tilde{\gamma} \cup \tilde{\delta}] \cup \tilde{\beta} = \tilde{\alpha} \cup [\tilde{\beta}, \tilde{\gamma}] \cup \tilde{\delta} + (-1)^{|\alpha||\gamma|}[\tilde{\alpha} \cup \tilde{\gamma}] \cup [\tilde{\beta}, \tilde{\delta}] + (-1)^{|\beta|(|\gamma|+|\delta|)+|\alpha||\gamma|}[\tilde{\alpha} \cup \tilde{\gamma}] \cup [\tilde{\beta}, \tilde{\delta}] + (-1)^{|\beta|(|\gamma|+|\delta|)}[\tilde{\alpha}, \tilde{\gamma}] \cup \tilde{\delta} \cup \tilde{\beta}. \tag{34}
\]

**Proof.** Parts (i) and (ii) follow by direct inspection of formulae. Part (iii) follows from (ii) and the fact that the map $\tilde{\alpha} \cup \tilde{\beta}(sa \wedge -) : \Sigma \Omega(\mathcal{M}) \to \Sigma \Omega(\mathcal{M})$ (where $-$ stands for the argument and $sa \in \Sigma \Omega(\mathcal{M})$ is an arbitrary element) is generated by a degree $|\alpha| + |\beta| + |a| + 1$ differentiation

\[
\iota_a(\alpha \cup \beta)(x) \overset{\text{def}}{=} (-1)^{|\beta|+1(|\alpha|+1)+|a|} \alpha(a) \cdot \beta(x) + (-1)^{|a|+1(|x|+1)+(|\beta|+1)(|x|+1)+|a|+1} \alpha(x) \cdot \beta(a)
\]

of $\Omega(\mathcal{M})$. \hfill \Box
4.2 Proof of the main theorem

Let \( \pi = \sum_k X_k \wedge Y_k \) be a bivector on \( \mathcal{M} \). One easily checks that the map \( \tilde{\pi} = \sum_k \tilde{i}_X_k \cup \tilde{i}_Y_k \) is well-defined, i.e. doesn’t depend on the choice of \( X_k, Y_k \) in the presentation of \( \pi \). The following statement is evident.

Lemma 7. (i) Let \( sa, sb \in \Sigma\Omega(\mathcal{M}) \) be arbitrary elements, then

\[
\tilde{\pi}(sa \wedge sb) = (-1)^{|a|} s\tilde{\pi}(a, b),
\]

where \( \tilde{\pi} \) is the map defined in equation (16).

(ii) Let us denote the coderivation of \( \Lambda\Sigma\Omega(\mathcal{M}) \), induced by the map \( \tilde{\pi} = \pi \) by \( \Pi \). The following formula holds

\[
[\tilde{\pi}, \tilde{d}](sa \wedge sb) = s\{a, b\},
\]

where \( \{, \} \) is the Poisson bracket from the first paragraph.

Corollary 1. The map \( s\{\} \), defined as \( sa \wedge sb \mapsto s\{a, b\} \) is equal to

\[
s\{\} = \sum_k (\tilde{i}_X_k \cup \tilde{\mathcal{L}}_Y_k - \tilde{\mathcal{L}}_X_k \cup \tilde{i}_Y_k).
\]

Let us denote by \( e^\Pi \) the map

\[
1 + \Pi + \frac{1}{2} \Pi \circ \Pi + \frac{1}{6} \Pi \circ \Pi \circ \Pi + \ldots : \Lambda\Sigma\Omega(\mathcal{M}) \rightarrow \Lambda\Sigma\Omega(\mathcal{M})
\]

(35)

It is easy to check that this map is a homomorphism of coalgebras (in effect this follows just from the fact, that \( \Pi \) is a coderivative). The following formula is the principal result of this paper.

Theorem 1. If the bivector \( \pi \) is a Poisson bivector (i.e. its Schouten bracket with itself vanishes), then

\[
e^\Pi \circ D \circ e^\Pi = D + \{\},
\]

(36)

where \( \{\} \) is the extension to \( \Lambda\Sigma\Omega(\mathcal{M}) \) of the Poisson bracket.

Proof. Consider the map \( e^{t\Pi} \), where \( t \) is the formal parameter, \( t \in \mathbb{R} \) and the formal deformation \( e^{t\Pi} \circ D \circ e^{t\Pi} \) of the left hand side of the formula (36). We shall prove, that it coincides with the formal deformation \( D + t\{\} \) of the right hand side of the same formula, i.e. we are going to prove the formula

\[
e^{t\Pi} \circ D \circ e^{t\Pi} = D + t\{\}
\]

(36)

To this end it is enough to show, that the formal derivatives at zero \( \frac{\partial^n}{\partial t^n} |_{t=0} \) of the deformed maps on both sides are equal for all \( n \).
Indeed, the equality is evident when \( n = 0 \). If \( n = 1 \), we compute (we omit the composition signs, where it is possible):

\[
\frac{\partial}{\partial t}(e^{t\Pi} \circ D \circ e^{t\Pi}) = e^{t\Pi}(\Pi \circ D + D \circ \Pi)e^{t\Pi} = e^{t\Pi}[\Pi, D]e^{t\Pi} = e^{t\Pi}[\tilde{\Pi}, \tilde{D}]e^{t\Pi} = e^{t\Pi}\{\tilde{\Pi}, e^{t\Pi}\}.
\]

We have used the Lemma \( \ref{lemma} \) and the fact that in our case, the bidegrees of the maps being \((1, 0)\) and \((-2, 1)\), the bi-graded commutator coincides with the anti-commutator (the bi-degree of \( D \) is equal to \((1, 0)\) and that of \( \Pi \) to \((-1, -1)\)). So, \( \frac{\partial}{\partial t}|_{t=0} \) applied to both sides of the formula \( \ref{eq:formula} \) gives \( \{\tilde{\Pi}, e^{t\Pi}\} \).

For \( n = 2 \) similar calculations give

\[
\frac{\partial^2}{\partial t^2}(e^{t\Pi} \circ D \circ e^{t\Pi}) = \frac{\partial}{\partial t}(e^{t\Pi}\{\tilde{\Pi}, e^{t\Pi}\}) = e^{t\Pi}[\tilde{\Pi}, \{\tilde{\Pi}, e^{t\Pi}\}]e^{t\Pi}.
\]

Now we can use the formula \( \ref{eq:formula} \) and the Cartan relations verified by the Lie derivatives and the internal derivatives to show, that (up to a certain sign) the commutator \( [\tilde{\Pi}, s\{\}] \) is equal to

\[
\sum_{k,j}\{\tilde{i}X_k \cup \tilde{i}[y_k, x_j] \cup \tilde{i}y_j - \tilde{i}X_k \cup \tilde{i}X_j \cup \tilde{i}[y_k, y_j] - \tilde{i}[X_k, x_j] \cup \tilde{i}y_k \cup \tilde{i}y_j + \tilde{i}X_k \cup \tilde{i}y_k \cup \tilde{i}y_j\}
\]

(of course, some of the terms may cancel each other). This expression depends only on the Schouten bracket of \( \pi \) with itself. Actually, the Schouten bracket \( \{\pi, \pi\} \) is defined with the help of the commutation relations, similar to \( \ref{eq:formula} \), and one checks that it is equal to

\[
\sum_{k,j}\{X_k \wedge [Y_k, X_j] \wedge Y_j - X_k \wedge X_j \wedge [Y_k, Y_j] - [X_k, X_j] \wedge Y_k \wedge Y_j + X_j \wedge [X_k, Y_j] \wedge Y_k\},
\]

and the map, given by the formula \( \sum_m \tilde{i}X_m \cup \tilde{i}y_m \cup \tilde{i}Z_m \) depends only the tri-vector \( \sum_m X_m \wedge Y_m \wedge Z_m \).

From these observations and from the hypotheses that the Schouten bracket of \( \pi \) with itself is equal to 0, it follows, that \( \frac{\partial^3}{\partial t^3}(e^{t\Pi} \circ D \circ e^{t\Pi}) = 0 \). Consequently, one has \( \frac{\partial^3}{\partial t^3}(e^{t\Pi} \circ D \circ e^{t\Pi}) = \ldots = \frac{\partial^n}{\partial t^n}(e^{t\Pi} \circ D \circ e^{t\Pi}) = \ldots = 0 \).

**Remark 3.** It would, probably be more convenient and more insightful to use the “skewed” version of all the operators. Then the bi-graded commutator \( [\cdot, \cdot] \) should be replaced by \( [\cdot, \cdot]_{tot} \) (see remark \( \ref{total} \)). Consequently, formula \( \ref{eq:formula} \) would look like \( e^{t\Pi} \circ D \circ e^{-t\Pi} = D + \{\cdot, \cdot\} \), i.e. the operator \( e^{t\Pi} \) intertwines the standard \( L_\infty \)-structure on \( \Sigma\Omega(L) \) (that is the structure, induced by the DGL-algebra structure on it) with the trivial one (that is the structure, for which all the maps \( l_i \) are equal to 0, when \( i \geq 2 \)).
Corollaries and discussions

Corollary 2. The Lie algebra $\Omega(M)$ is formal.

Proof. We need to produce an $L_\infty$-quasi-isomorphism between $\Sigma \Omega(M)$ and $\Sigma H(M)$. Since all such maps are invertible, it is enough to find a morphism only in one direction. So, we shall construct an $L_\infty$-quasi-isomorphism from $\Sigma H(M)$ to $\Sigma \Omega(M)$.

Let us first rewrite (29), taking into account that both the differential and the Lie bracket vanish in $\Sigma H(M)$:

$$df_n(\alpha_1 \wedge \ldots \wedge \alpha_n) = \sum_{i \geq \lfloor \frac{n}{2} \rfloor, \sigma \in Sh_{i,n-i}} (-1)^{\sigma + (i-1)i} \{ f_i(\alpha_{\sigma(1)} \wedge \ldots \wedge \alpha_{\sigma(i)}), f_{n-i}(\alpha_{\sigma(i+1)} \wedge \ldots \wedge \alpha_{\sigma(n)}) \}. \quad (37)$$

for all $\alpha_i \in \Sigma H(M)$. Or, in brief $(D + \{ , \}) \circ F = 0$, where $F$ is the map of coalgebras $F : \wedge^* H(M) \to \wedge^* \Omega(M)$, assembled from $f_k$. It is the map $F$ that we shall construct.

To this end we first define the bottom stage of this morphism as a linear splitting of the projection $Z(M) \to H(M)$ from the space of closed forms to cohomology (we omit the suspension signs): $f_1([\alpha]) = \alpha$, where $\alpha$ is a closed form, representing the class $[\alpha]$. For instance, one can choose $\alpha$ to be the only harmonic form (with respect to a Riemannian structure) in the class $[\alpha]$. We shall have $df_1([\alpha]) = 0$, as prescribed by (37), and it is almost by definition, that $f_1$ is a quasi-isomorphism of chain complexes. But the commutator of two harmonic forms is not necessarily harmonic, so $f_1$ is not in general a homomorphism.

Now we can extend $f_1$ in a trivial way to the map of coalgebras $\wedge^* f_1 : \wedge^* \Omega(M) \to \wedge^* H(M)$. It is clear that $D \circ \wedge^* f_1 = 0$.

Let us put $F = e^{-\Pi} \circ \wedge^* f_1$. Then

$$(D + \{ , \}) \circ F = (D + \{ , \}) \circ e^{-\Pi} \circ \wedge^* f_1 = e^\Pi \circ D \circ \wedge^* f_1 = 0. \quad \square$$

One can visualize the first few stages of $F$. For instance,

$$f_2([\alpha] \wedge [\beta]) = \tilde{\pi}(f_1([\alpha]), f_1([\beta])), \quad (38)$$

$$f_3([\alpha_1] \wedge [\alpha_2] \wedge [\alpha_3]) = \sum_{\sigma \in \Sigma_3} \frac{1}{2} \tilde{\pi}(f_1([\alpha_{\sigma(1)}]), f_1([\alpha_{\sigma(2)}]), f_1([\alpha_{\sigma(3)}])). \quad (39)$$

In effect, these formulas can be found independently. For instance, it follows from (15) that

$$\{ f_1([\alpha]), f_1([\beta]) \} = \{ \alpha, \beta \} = d\bar{\pi}(\alpha, \beta).$$

(Here $\alpha, \beta$ denote the forms, representing classes $[\alpha], [\beta]$.) So we have

$$\{ f_1([\alpha]), f_1([\beta]) \} = df_2([\alpha] \wedge [\beta]),$$

as prescribed by (37).
Finally for this choice of \( f_1, f_2 \) and \( n = 3 \) \( (37) \) takes the following form:

\[
d f_3([\alpha] \wedge [\beta] \wedge [\gamma]) = \{ f_2([\alpha] \wedge [\beta]), f_1([\gamma]) \} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \{ f_2([\beta] \wedge [\gamma]), f_1([\alpha]) \}
+ (-1)^{|\beta|+|\gamma|} \{ f_2([\alpha] \wedge [\gamma]), f_1([\beta]) \}
= \{ \tilde{\pi}(\alpha, \beta), \gamma \} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \{ \tilde{\pi}(\beta, \gamma), \alpha \}
+ (-1)^{|\beta|+|\gamma|} \{ \tilde{\pi}(\alpha, \gamma), \beta \}
\]

\( (40) \)

(here \( \alpha, \beta, \gamma \) are some closed forms). Let us introduce a map \( \pi_3 \) by the following equation (for all forms \( \alpha, \beta, \gamma \)):

\[
\pi_3(\alpha, \beta, \gamma) = \{ \tilde{\pi}(\alpha, \beta), \gamma \} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \{ \tilde{\pi}(\beta, \gamma), \alpha \}
+ (-1)^{|\beta|+|\gamma|} \{ \tilde{\pi}(\alpha, \gamma), \beta \}.
\]

\( (41) \)

This map extends to all \( \Sigma \Omega(\mathcal{M}) \) the right hand side of \( (40) \). One can show, that \( \pi_3 \) verifies the following equation:

\[
\pi_3(\alpha_1 \wedge \alpha_2, \beta, \gamma) = \alpha_1 \wedge \pi_3(\alpha_2, \beta, \gamma) \pm \pi_3(\alpha_1, \beta, \gamma) \wedge \alpha_2
\pm \left( \{ \alpha_1, \beta \} \wedge \tilde{\pi}(\alpha_2, \gamma) + (-1)^{|\alpha_1|+|\beta|} \tilde{\pi}(\alpha_1, \beta) \wedge \{ \alpha_2, \gamma \} \right)
\pm \left( \{ \alpha_1, \gamma \} \wedge \tilde{\pi}(\alpha_2, \beta) + (-1)^{|\alpha_1|+|\gamma|} \tilde{\pi}(\alpha_1, \gamma) \wedge \{ \alpha_2, \beta \} \right).
\]

\( (42) \)

Observe, that if all the forms \( \alpha_1, \alpha_2, \beta, \gamma \) are closed then it follows from \( (15) \) that the last two lines on the right hand side of \( (42) \) consist of exact forms. Now, one can consider local coordinates and use the argument, similar to the first proof of \( (15) \) to show, that the following is true on the whole manifold

\[
\pi_3(\alpha_1, \alpha_2, \alpha_3) = d \sum_{\sigma \in \Sigma_3} \frac{1}{2} \tilde{\pi}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}),
\]

which means that \( (39) \) is a good choice of \( f_3 \). One can continue this process infinitely to find all the maps \( f_i, i \geq 4 \) in a straightforward way.

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