Cumulative Games:
Who is the current player?

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Abstract

Combinatorial Game Theory (CGT) is a branch of game theory that has developed almost independently from Economic Game Theory (EGT), and is concerned with deep mathematical properties of 2-player 0-sum games that are defined over various combinatorial structures. The aim of this work is to lay foundations to bridging the conceptual and technical gaps between CGT and EGT, here interpreted as multiplayer Extensive Form Games, so they can be treated within a unified framework. More specifically, we introduce a class of \(n\)-player, general-sum games, called Cumulative Games, that can be analyzed by both CGT and EGT tools. We show how two of the most fundamental definitions of CGT—the outcome function, and the disjunctive sum operator—naturally extend to the class of Cumulative Games. The outcome function allows for an efficient equilibrium computation under certain restrictions, and the disjunctive sum operator lets us define a partial order over games, according to the advantage that a certain player has. Finally, we show that any Extensive Form Game can be written as a Cumulative Game.

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1 Prologue

Consider a game, where 2 players, Alice and Bob, alternate in removing identical objects, say pebbles, from a common heap, given some restriction on the available actions, and given some ‘winning condition’. Usually, the players compete in achieving a certain goal, such as removing the last pebble, or grabbing the largest number of pebbles, and the final result is usually sensitive to who starts. Such games are well studied in an elegant niche of classical game theory, called Combinatorial Game Theory (CGT). Here we will de-emphasize the usual appeal of patterns and combinations, and instead lay a foundation for a bridge to mainstream game theory, i.e. Economic Game Theory (EGT), via multiplayer Extensive Form Games. The main contribution of this paper is conceptual, which makes it somewhat different from many theoretical papers. Most of the space is dedicated to exploring various definitions, that enable us to reconcile diverging concepts and modelling assumptions in CGT and EGT, and now under one umbrella. In particular, we bridge the facts that finite games in EGT are rooted trees, i.e. have an initial state, and that in CGT states are not a-priori coupled with a current player. We justify our definitions with examples, and we provide theorems with short and straight-forward proofs. In order to illustrate the basic concepts and motivate the later theory building, we begin by investigating some concrete situations.

Imagine that, at their turn, a player, Alice or Bob, must take either 2 or 3 pebbles from a single heap. The ultimate goal of the game will vary. Later we will increase the number of players to $n \geq 2$.

Normal play. In this example a player who at their turn cannot move loses. Thus, if Alice start from a heap of size 4, she should remove 3 pebbles.

Misère play. In this example a player who at their turn cannot move wins. Thus, if Alice start from a heap of size 4, she should remove 2 pebbles.

Scoring play. In this example the player with a larger number of pebbles when the game ends, wins. Or more precisely, one of the players, say Alice, is the maximizer, and Bob is the minimizer; a common score is updated during play by the number of pebbles they remove. If Alice (Bob) collects 2 pebbles then the score increases (decreases) by 2, etc. Hence, if Alice starts from a heap of size 4, she should remove 3 pebbles, the game ends, and the final score is 3. If they remove 2 pebbles each, the game ends in a draw with a total score of $2 - 2 = 0$. 

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Squirrel play. This is self-interest cumulative play. Here, there is no winner, but each squirrel attempts to gather the largest possible number of pebbles (nuts) for themselves. Thus, in our example, the first squirrel (Alice) should remove 3 pebbles, and the final utilities will be \((3, 0)\). This is not a zero-sum game; all partial cumulations, and in particular the final cumulations, are ordered pairs of nonnegative integers. If Alice starts instead from a heap of size 7, she should collect 2 instead of 3 pebbles. Why? (See Figure 1.) And this holds form Normal play and scoring play too, but in misère play you will lose whatever you play from a heap of size 7.

Auction play. This game will be revisited in Example 4. An auctioneer has set up the following 2-player auction: the starting position consists of 4 bidding-pebbles and a pair of initial bids, and the players may increase their bids by collecting either 2 or 3 bidding-pebbles. Each player has a utility function of the form: 0 utility if they do not win the auction, and otherwise the utility is \(4 - \text{‘their accumulated bid’}, \) i.e. in case Alice wins the auction, \(4 - (\text{her initial bid + all her play bids})\). (If no player wins the auction, then both players get utility 0). If the initial bids are \((0, 0)\) then Alice, playing first, should bid 3 to win the auction, and the utilities will be \((1, 0)\). However, if the initial bid is \((1, 0)\), then Alice should bid 2, because a bid of 3 would (in spite of winning the auction) give utility \(4 - 3 - 1 = 0\), whereas a bid of 2 suffices to win the auction, and her utility will be \(4 - 2 - 1 = 1\). Therefore the best-play bid, from a heap of size 4, may depend on the initial bids (or current cumulations), and, as we will discuss further, such situations cannot happen in the 4 first examples.

Wealth play. This is again Normal play, but where the players can remove any number of pebbles that does not exceed their current cumulation. Suppose that the heap is 3, and the current cumulation is \((2, 2)\). Then the first player loses. If the heap is 3, and the current cumulation is \((2, 1)\), then Alice will win if she starts by removing 1 pebble, but she loses if she starts by removing 2 pebbles. Suppose next that the heap is of size 6, and the initial cumulation is \((1, 1)\). If Alice starts, then the next position is \(5, (2, 1)\), followed by Bob, playing to \(4, (2, 2)\). Now, Alice loses if she removes 2, but she wins if she removes 1. (Indeed, this is a 0-sum game).

We believe the squirrel, auction and wealth play situations are new to combinatorial games’ study. Observe that, in auction play, although no part of the ruleset depends of the current cumulation (here initial bid), play in
Pure Subgame Perfect Equilibrium (PSPE, every player selects the action maximizing her utility, in every subtree of the game) depends on the current cumulation. Before continuing, the reader may wish to justify that the first four examples do not exhibit such behavior; in fact, as we will see, the first four examples (generalized) have smaller complexity due to the fact that their utility functions have simpler formulations. We emphasize that, in the auction play example, we chose the most direct utility function to justify the goal of that game.

Figure 1: The picture illustrates the self-interest game where the squirrels remove 2 or 3 nuts, and where the initial heap size is 7. In each node we show the current cumulation for each player. The cumulations at the leaf levels are also the utilities of the players. The thick edges mark the Pure Subgame Perfect Equilibrium of this game. The starting player has a small advantage, by adapting non-greedy play.

The squirrel play example is obviously not zero-sum play, and although very simple, we believe that the general class of such self-interest games has not yet been studied in the literature of Subtraction Games (or elsewhere). This is probably due to the fact that the combinatorial game tradition is rooted in recreational games, and most recreational games are (though of as) 0-sum games; although the classical recreational game Monopoly has ingredients of self-interest squirrel play, it is more of a type of wealth play, because the ranges of available actions depend on accumulated amounts of money. Of course Monopoly is not a combinatorial game, because it has
random moves and hidden cards, but the analogy is close enough to motivate a natural class of wealth games in our discussions.

All six examples may be included to a super class of combinatorial games. When we enlarge combinatorial games in this manner, we wish to have definitions match the existing literature of game theory at large, and in particular we will adapt to the conventions of Extensive Form Games. This is a natural way forward, since all combinatorial games, with a given starting player, are Extensive Form Games. The class of Extensive Form Games is much larger than that, but, in fact, they do not encompass combinatorial game theory for other reasons, that we will explain in more detail as we go along.

The auction play example shows that when we vary the utility function, we may end up in situations where a PSPE action depends on the history of the game, i.e. in our setting a player’s ‘current bid’ (also called called current cumulation), although the ruleset itself has no history dependency, apart from the move function (which is here alternating play).

In simpler cases, such as the first 4 examples, one may ignore the current cumulations while computing the various PSPEs (and adding initial cumulations only after the PSPE computation). In the sixth example, wealth play, then obviously one cannot ignore the current cumulation while computing even the available moves. In the two last examples, there is no meaning to assigning ‘a game value’ to a heap size alone, whereas in the four first examples, essentially, this is the correct approach. Our main interest in this paper will be to study the area between the four first examples and the auction example. We will do this by developing a ruleset large enough to encompass any Extensive Form Game, and small enough to not obscure the main direction, and distinctions we wish to address.

Before we move on, let us challenge the reader with a sample game, where the game is a composition of games, namely one game of each kind in the above sample games. Say Alice and Bob play the composite game $A + B + C + D + E + F$, where all game components are played on a heap of size 4, with subtraction set $\{2, 3\}$ and initial cumulation $(0, 0)$, except for the game $F$, which is played on a heap of size 4, with initial cumulation $(1, 1)$. The ‘+’ signs indicate that, when the game ends, we will add the component utilities. Hence the players seek to maximize the sum of the utilities of the component games.

Let us specify the various utility functions:

$A$: a player with the last move wins one point, and the other player loses one point;

$B$: a player with the last move loses one point, and the other player wins
one point;

C: Alice gets $p$ points, and Bob gets $-p$ points, if the final score is $p$;

D: a player gets $p$ points if they end up with $p$ pebbles in their own pocket;

E: the player who wins the auction gets $4$—‘winning bid’ points and the other player gets $0$ points, and in case of no winner, both players get $0$ points;

F: a player who cannot move, in any component of the composite game, loses one point, and the other player wins a point.\footnote{This convention will be neatly encoded in our utility functions to come, by using the idea: when there is no move in any component, then specifically there is no move in $F$.}

The total utility for each player when all games have ended is the sum of the utilities in the component games. How does best play change if wealth play instead is worth $-10$ and $10$ points respectively (and the other games stay the same)?

Such hybrid games have not yet been discussed in the (CGT) literature. The general line of thinking has been that, in order to play games together, in a so-called disjunctive sum, one needs to fix a ‘winning condition’ or ‘scoring convention’ or something similar.\footnote{For appealing game comparison, one usually restricts the set of games further.} Our new model makes such distinctions obsolete, and allows for various generalizations of these initial ideas, and note that this generalization is possible because of a merging of concepts from previously `different' game theories.

\section{Introduction}

Since the introduction of Chess-like games and their recursive solution by von Neumann, noncooperative game theory has developed in two almost independent directions. Mainstream work in economics started to consider games with multiple players and general utility functions (EGT), and generalized game values by considering subgame perfect equilibrium. Meanwhile, combinatorial game theory (CGT) continued to explore (2-player, 0-sum) games, with complete and perfect information, as abstract mathematical entities, by defining various operations such as game addition and game comparison, which take into account games’ underlying combinatorial structure. Another interesting feature of CGT is that it deals with games that people can actually (and often do) play for recreation. In fact, both the Milnor theory
(on positional games), and the breakthrough via Normal play theory [7, 1], took inspiration from endgame studies in the classical eastern game of Go, and CGT disjunctive sum theory was born in these ways (we now know that Milnor’s positional games are a subclass of Normal play games where infinitesimal games have been excluded; see [18] for a discussion). In this context one should mention that Berlekamp has fused CGT with EGT by playing out endgames of combinatorial games together with coupon stacks, to estimate their temperatures (see [3, 2] for more on these topics), but this topic leads towards a different direction than this work.

In both strands of the literature much work is dedicated to understand the outcome of a game when players play rationally, but often using different mathematical tools and even more different terminology. In this paper our purpose is to start building bridges between CGT and “economic” (or mainstream) game theory—in particular Extensive Form Games and sub-game perfect equilibrium.

We study a multiplayer, general-sum extension of the classical Subtraction Games [1] (a variation of Nim), called Cumulative Games.

A zero-sum variation of Cumulative Games is known as Cumulative Subtraction (CS) [5], introduced in Stewart’s Ph.D. Thesis [31] (also published in [32]), and with more recent work in [5]. Thus, from a CGT perspective, we are proposing a much delayed jump from zero-sum to general-sum games in the spirit of the historical jump in EGT, mastered first by von Neuman [28] (zero-sum) and then by von Neuman and Morgenstern [29] (general-sum), and others. From an EGT point of view, we propose a method to analyze economic style generalized combinatorial games using concepts from CGT, such as the outcome function and disjunctive sum play [33]. In CGT, games can be naturally added and compared [33] and we show how these ideas carry over into our more general setting.

2.1 The three layers of a Cumulative Game Form

To be able to apply both CGT and EGT concepts, an n-player Cumulative Game Form (CGF) will be defined in three layers, to be specified in the various settings. To begin with, a ruleset is a set function, which is defined on an infinite heap space, as in layer 1. It specifies what the n players can do, what actions they can take on the heaps, and what consequently happens to individual cumulations. Moreover, a turn function is given.

1. A \((d \times n)\)-dimensional heap space (CGT game space) is an infinite set of \(d\)-tuples of finite heaps, where each heap, in every tuple, memorizes an \(n\)-tuple of player cumulations on that heap.
2. A heap position (CGT game position) fixes a position from the heap space, and it applies to any starting player;

3. A grounded position (EGT game state). This is a heap position, together with a specified starting player.

Note that, on the two first layers, a game cannot yet be played; although sometimes, the ruleset alone, which requires only layer 1 is called ‘a game’. The heap positions on layer 2 are sometimes called game components, as they are prepared for so-called disjunctive sum play together with other game components. The important concept of a CGT outcome function is defined on layer 2. Layer 3 games cannot be played in a disjunctive sum, but a disjunctive sum game is of course grounded when play starts.

On layer 3, a game can be played, but without any incentive. For a Game Form to be a game, a utility function is required. Various incentives will be given by utility functions, and player maximization of individual utilities is the purpose of a game. As we will see in Section 5, a grounded position, together with player utilities, is an (EGT) Extensive Form Game. Here, we wish to emphasize the distinction between the three first layers, and in particular the often overlooked gap between layers 2 and 3. When a utility function is added to a CGF, we call it a Cumulative Game.

2.2 Subtraction Games

The models we develop in this paper are motivated by the classical CGT Subtraction Games. The overarching idea is that there is a heap of pebbles and two players, who alternate removing pebbles given some subtraction set $S \subset \mathbb{N} = \{1, 2, \ldots\}$. If the size of the heap is $x \in \mathbb{N}_0 = \{0, 1, \ldots\}$, then the current player acts by removing $a \in S$ pebbles, and leaves the position $x - a \geq 0$ for the opponent. If there is no such $a \in S$, then the game ends, and the result is determined, by some prescribed convention, for example, as is common in CGT, a player who cannot move loses.\footnote{This is the Normal play convention; see also the Section 10.} Apart from the winning condition, our games are all based on this simple idea. To the authors’ best knowledge, a general-sum variation of Subtraction Games (or any other combinatorial game) have not been studied before in the literature.

2.3 Bridging the Fields

We think of an $n$-player combinatorial game position in terms of an $n$-tuple of EG states, one for each starting player, $p^0 \in [n]$. Here, we already encoun-
tered an important distinction between EG and CG, a combinatorial game position is defined with respect to any starting player (layer 2), whereas an EG state has assigned a given player to start (layer 3). For an EG there is a priori no immediate/obvious way to attach another starting player to a game state. For a combinatorial game, it is required that any player may start from any given game position. For example, if we view a given Chess position (without asking the question “who is to play?”) the position may often apply to any player as starting player (apart for special situations, such as if one of the player’s King is in check). A typical CGT-question for a given 2-player combinatorial game position is: “Would you prefer to start, or going second?” There are several more or less subtle distinctions between the two fields (that share the same heading “Game Theory”). In this study we will assume perfect and complete information, thus remaining close to standard CGT concepts.

2.4 Motivation from CGT

Before moving forward, we want to briefly explain the importance of two key concepts from CGT that this paper aims to extend: the outcome function and the disjunctive sum operator. Both concepts are reviewed, through the (original) setting of Normal play, at last, in Section 10.

The outcome function An outcome function assigns a value to every position in the game (i.e., it operates on layer 2), which essentially describes who wins under optimal play from this position. Therefore it is similar to the pure subgame perfect equilibrium in EGT, except that in CGT it turns out to hold much more information about the game, as we explain next.

Disjunctive sum An interesting feature of combinatorial games is that they can be added. Intuitively, playing the game $G + H$ means that the current player must make a move in exactly one of those games (consider for example a situation where $G$ is some Nim position, and $H$ is a Chess opening position). Under the Normal play convention, a player that cannot move in any of these games loses the game. One important question is which classes of games are closed under the disjunctive sum operation.

Partial order The two concepts above induces a partial relation over zero-sum games within a given class, where $G \geq H$ if the outcome of $G + X$ is never worse for the maximizing player than the outcome of $H + X$. For Normal play this definition turns out to be particularly powerful: every two games can be
compared constructively, by essentially playing them out together. Normal play games constitute a group structure, where the negative of a game \( G \) is the game \( -G \), where the two players have swapped positions. The main theorem of Normal play says that \( G \geq H \) if and only if player Left (the maximizer) wins the composite game \( G + (-H) \) playing second. Moreover this result implies a bijection of Normal play games with the four outcome classes (see Section 10), a unique feature within combinatorial games.\(^4\)

On a note, some subclasses of Normal play games have even stronger properties. For example, every impartial (symmetric) game \( G \), however complicated, is equivalent (under the above relation) to some single-pile Nim game of size \( \ell_G \). This means not only that the same player wins in \( G \) and in \( \ell_G \), but that we may replace the game \( G \) in every possible context (e.g. \( G + \text{Chess} \)) with the Nim pile \( \ell_G \), and the results would stay the same. The foundation for this result is the classical Sprague and Grundy theory \([35, 11]\) where all games are impartial, and it is not hard to see that it can be extended to all Normal play games.

**Cumulative play** An intuitive idea of cumulative play is that “you get what you take\(^5\), and you put it in your own pocket, perhaps for later use. This idea is simpler than a typical zero-sum setting \([5]\), where the players accumulated scores must be compared to access the utility of play. In a purely self-interest setting no comparison is required. In a self-interest setting, it is more natural to allow for several players, whereas in a zero-sum setting two players is the generic setting. Accumulation of points can have various other interpretations, as is discussed in auction play from the Prologue, which is revisited in Example 4, where ‘cumulations’ are interpreted as bids in an English auction type setting, and where we discover that rational play might depend on cumulations, even when the ruleset does not depends on them. In a more economic setting one would most likely want to study Cumulative Games, where action sets depend on accumulations.

\(^4\)Although, this simple and elegant idea fails for other universes of games, it has still been demonstrated to lay a foundation for larger classes of CGT, namely, first Guaranteed Scoring games \([18, 17]\), and then for Absolute CGT \([20, 21]\), that includes various misère play settings. By generalizing to general sum games (such as self interest), however, the Normal play foundation seems to break, mainly because games cease to be purely competitive, and the concept of a Normal play order embedding, as paved the way forward towards Absolute CGT, does not appear to make much sense in the larger class we study here. The big open problem is what to do instead to have interesting subclasses of games with constructive game comparison, if possible at all. (Another possible direction would be to weaken the overlaying idea that games comparison must include the “for all \( X \)” part, if this can be justified by interesting applications.)
as in wealth play in the Prologue. But we will not enter that territory in this paper. Here, we instead focus on how far we can extend the Cumulative Subtraction/scoring-play games from [5], and still maintain a tractable (or dynamic) outcome function, that essentially only takes as input the heap (and not the cumulation). The prospects of comparing cumulative games in a broader economic setting is an ultimate goal of bridging CGT and EGT. This work lays a foundation for such studies.

The previous player  A general remark, before we start; all through this study, the player about to move, the current player (the starting player), here denoted \( \tilde{p} \), will be defined via a previous player \( p \), and this terminology is standard in CGT literature, where the setting is mostly alternating play.\(^6\) The turn function will be generalized further, by adapting conventions from EGT; in either case the analysis will depend on the notion of a previous player. As we discussed above, EGT does not require the notion of a previous player. In the hybrid, we develop here, this distinction becomes obsolete.

2.5 Outline and contribution

In Section 3, we define a simple first class of 2-player general sum combinatorial games called Cumulative Subtraction, by using the three layered structure mentioned above. We analyze this class using both EGT and CGT tools, and in particular, we introduce the CGT-inspired outcome function, and demonstrate its usefulness. Thereby most of the ideas of this paper (except for disjunctive sum) are already present in this section in a greatly simplified form.

The next sections are dedicated to generalizing these ideas. In Section 4 we present our general class of Cumulative Games, followed by some more background on Extensive Form Games in Section 5. We demonstrate how Cumulative Games can capture both classical zero-sum Subtraction Games and our newer class Cumulative Subtraction, as well as many other variations.

Section 6 generalizes the notion of a CGT-type outcome function, by adapting to extensive form terminology, whenever applicable. Here, we study the conditions under which it is well defined and efficient.

\(^5\)In cyclic play.
\(^6\)The current player is traditionally called the “next player”, which is a bit unfortunate, because in a current position, the previous player and the next player would be the same player, which is not what is intended. Hence, we will not adapt to that convention here.
In Section 7 we return to the most important concept from CGT, which is disjunctive sum. We show that the disjunctive sum operation can be naturally defined on Cumulative Games thanks to the three-layered structure and various closure properties, and thereby inducing a CGT-type partial order over games. This, we hope, would be a cornerstone in extending Sprague’s & Grundy’s, Milnor’s and Conway’s et al. classical discoveries from 0-sum games to more general classes.

We close the loop back to EGT in Section 8, by showing that every Extensive Form Game is strategically equivalent to some Cumulative Game. We conclude with discussion some key issues and future directions in Section 9.

3 Warm up: Cumulative Subtraction Games

We will next define a variation of the subtraction game based on scores. As we reserve the term “scoring game” exclusively for zero-sum games, we name the new class of games by Cumulative Subtraction (CS) (although this term is also used in the 0-sum setting in [5]). We start with 2-player games, and generalize it later in Section 4. For 2-player Cumulative Subtraction, we call the players by player 1 and player 2.

We build onto the outline with the three layers described in the introduction.

Definition 1 (2-player Cumulative Subtraction). A one heap 2-player Cumulative Subtraction Form is defined in three layers. The ruleset $R$ is a given function, $S : \mathbb{N}_0 \times \{1, 2\} \to 2^{\mathbb{N}}$, where $S(0, p) = \emptyset$, together with update rules of cumulations, as specified in (1).

1. The heap space is of the form $\mathbb{N}_0 \times \mathbb{R}^2$;
2. A heap position on the ruleset $R$ is a tuple $(x, (C_1, C_2))$, where $x \in \mathbb{N}_0$ is the heap size, and where $C_i \in \mathbb{R}$ is the accumulated size of player $i$’s pocket;
3. A grounded position is a tuple $(x, (C_1^0, C_2^0), p)$, where $x$ is a heap size, $C_1^0, C_2^0$ are initial cumulations for the respective players, and where $p$ is the previous player.

In every non-terminal grounded position $(x, (C_1, C_2), p)$, the current player $\tilde{p}$ chooses an action $a \in S(x, p)$, and if player 1 is the current player, the

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7In Section 10, the CGT tradition is emphasized by calling Player 1 by Left and Player 2 by Right.
new position is \((x - a, (C_1 + a, C_2), 1)\), and in case of player 2 the current player, the new position is \((x - a, (C_1, C_2 + a), 2)\). That is, player \(\tilde{p}\) takes a pebbles from the heap to her own pocket, and passes the position forward to the other player. A grounded position is identified with its sets of options, which, assuming that player 1 is the current player, is

\[
(x, (C_1, C_2), 2) = \{(x - a, (C_1 + a, C_2), 1) \mid a \in S(x, 2)\},
\]

and similar for player 2. The game ends when the players have reached a terminal position \((x^T, (C^T_1, C^T_2), p) = \emptyset\), that is, whenever \(S(x^T, p) = \emptyset\).

Observe that Definition 1 assures that every game will end, but it does not specify who wins, or what are the players’ utilities. This allows for several variations which we discuss below. We use the superscript ‘\(T\)’, whenever we wish to emphasize that a position, heap size, cumulation etc. is terminal.

### 3.1 From a CGT-type interpretation

As we stated at the beginning of this section, our CS game is inspired by the traditional CGT-type Subtraction Game with Normal play rules (see Section 10). Closer to our approach is the more recent zero-sum scoring variation in [5], where each player \(i\) aims to maximize the utility \(C^T_i - C^T_{-i}\), the difference of cumulations when the game ends. To align the CGT model with the EGT one, we will introduce a self-interest variant (squirrel-play in the Prologue), where each player simply aims to maximize their own terminal cumulation. Even though the players remove objects from a common heap, there is often a clear distinction.

### 3.2 Towards an EGT-type interpretation

In mainstream game theory, each player aims to maximize their own utility. We define the self-interest utility of each player \(i\) in a terminal state \((x, (C^T_1, C^T_2), p)\) as equal to \(C^T_i\) (more general utility functions will be introduced in Section 4). Therefore, every ruleset \(S\) together with a grounded position \((x, (C_1, C_2), p)\), and a utility function, induces an Extensive Form Game. To decide on the game solution, we apply the pure subgame perfect equilibrium (PSPE) solution concept. We provide a detailed definition in Section 5, but for now it is sufficient to recall that PSPE means that every player selects the action maximizing her utility, in every subtree of the game. See Figure 2 for a self-interest interpretation of Cumulative Subtraction. The PSPE is unique because we assume that players have well defined
preferences for the other players. For example, in case of two players, they may be (i) antagonistic, that is, in facing equal utility, they aim to minimize the utility of the opponent, or they may be (ii) friendly, and this is when they maximize the other player’s utility in case of indifference. This is the first variant of a combinatorial game (that we are aware of) that is not zero-sum. In case of zero-sum games, instead of PSPE, we will refer to the simpler, but equivalent notion of optimal play.

3.3 The outcome function

A key concept in CGT is the outcome function. The outcome function depends only on the ruleset, and assigns an ‘optimal play value’ to every (ungrounded) game position. That is, it operates on layers 1 and 2 of the game, and had not yet any obvious relevance to (typically layer 3) EGT contexts. Notably, in symmetric zero-sum games, the outcome function captures the advantage to the starting player in every position (but it is indifferent to whether this is player 1 or player 2).
Outcome functions were originally defined for 2-player Normal play CGT (see Section 10); it is essentially a pair of (von-Neumann [28]) minimax algorithms, and first used in the setting of scoring combinatorial games by Milnor [25]. It was reconsidered in Stewart’s PhD thesis [31] for symmetric Subtraction Games, and simplified in [5]. See also [9] for a generalization of Milnor type games, and [18, 17] for interesting connections of scoring combinatorial games with Normal play games. Another reference is [13] for scoring games where all play sequences have the same parity. Most of this work was done in the context of the disjunctive sum theory, which we review in Section 7, where we also prove that our ruleset satisfies important properties of combinatorial games such as additive closure.

**Property 1** (Symmetric Rules). Rules are symmetric if, for all positions, the move options are independent of player. In this case, we write \( S(x, p) = S(x) \).

The outcome function of the symmetric zero-sum variation was recently defined in [5]. It is particularly appealing in its concise one-line definition.

**Definition 2** (Outcome, Symmetric, Zero-sum [5]). The outcome of a heap of size \( x \) is \( o_{zs}(x) = \max(\{-o_{zs}(x-a) + a \mid a \in S(x)\} \cup \{0\}) \).

Note that the outcome is \( o_{zs}(x_T) = 0 \) if \( S(x_T) = \emptyset \).

**Proposition 1** (Symmetric, Zero-sum, [5]). Consider a ruleset \( S \) in symmetric zero-sum Cumulative Subtraction. At any grounded position \( (x, (0, 0), 2) \), suppose that \( (x_T, (C_{T1}^T, C_{T2}^T)) \) is a terminal position under optimal play. Then, the utility is \( o_{zs}(x) = C_{T1}^T - C_{T2}^T \geq 0 \).

*Proof.* This is immediate by Definition 2. \( \square \)

Although the rules of game are more natural for self-interest than zero-sum ditto, the self-interest outcome function will require some extra consideration, namely the requirement of a tie-break rule for each player, in case of indifference.

The outcome at every position \( x \) should specify a value for both players, in the case that each of them starts, and thus, in the general case, we will need an \( n \times n \) matrix for the outcome \( o_{si}(x) \) (indeed this is what we do later in Section 4). For now we can use simpler notation.

---

8In CGT, the common term for symmetric is “impartial”; but these games are not ‘impartial’ in the economic sense, so we avoid this traditional CGT terminology here.

9Outcome functions have been defined for other scoring games under various names in the literature, e.g. [25, 12, 9, 33, 18, 17].
Figure 3: To the left: symmetric cumulative subtraction with \( S = \{2, 3\} \) represented as a DAG played from a heap of size 7. The middle picture displays the zero-sum outcomes, and to the right, the general-sum outcomes. The double edges indicate which child determines the parent’s value.

The outcome is defined with respect to player 1 as starting player, thus \( p = 2 \) (by symmetry, the analogous result when player 2 starts is just reversing players’ utilities). However, to simplify notation, for symmetric rulesets we write \( \mathcal{S}(x) = \mathcal{S}(x, 1) = \mathcal{S}(x, 2) \).

**Definition 3** (Outcome, Symmetric, Self-Interest, Antagonistic). Consider a ruleset \( \mathcal{S} \) in a self-interest Cumulative Subtraction game, under antagonistic play. The symmetric self-interest outcome of a heap of size \( x \) is \( o_{si}(x) = (0, 0) \) if \( \mathcal{S}(x) = \emptyset \), and otherwise

\[
o_{si}^1(x) = \max\{o_{si}^2(x - a) + a \mid a \in \mathcal{S}(x)\},
\]

\[
o_{si}^2(x) = o_{si}^1(x - a'),
\]

where

\[
a' = \arg\min_{a \in \mathcal{S}'(x)} o_{si}^1(x - a),
\]

and where \( \mathcal{S}'(x) \subseteq \mathcal{S}(x) \) denotes player 1’s set of indifference actions in perfect play. The symmetric self-interest *outcome* of heap size \( x \) is the pair \( o_{si}(x) = (o_{si}^1(x), o_{si}^2(x)) \).

When play is self-interest *friendly* in case of indifference, instead of \( \arg\min \) in (10), we choose \( \arg\max \), and otherwise the definition is the same. See Observation 6 in Section 9, for interesting discussions and conjectures on these variations of games and outcome functions.
Property 2 (Fixed Rules). If there are subsets of the natural numbers $S_1, S_2 \subseteq \mathbb{N}$ such that for all $x \in \mathbb{N}_0$, for $p \in \{1, 2\}$, $S(x, p) = S_p \cap \{0, \ldots, x\}$, then the ruleset $S$ is fixed. In case of fixed rulesets, we abuse notation and denote, in case of symmetry, $S = S_p$, and otherwise $S = (S_1, S_2)$.

Example 1. Consider the fixed ruleset $S = \{2, 3\}$. The presentation in extensive form (of a particular grounded position, i.e. layer 3) is in Figure 2.

All heap sizes and the moves between them (i.e., layer 1 up to $x = 7$) are in Figure 3(a). Note that the tree does not include cumulations. The computed outcome function under both zero-sum and self-interest variants is displayed in Figures 3(b) and (c), respectively.

Note that the representation is smaller than the naïve representation in Figure 2, and yet it still allows for computing the PSPE. The realized outcome in both variations is $7 \mapsto 5 \mapsto 2 \mapsto 0$.

Both these representations are nice, but due to the layers 1 and 2, it is much more convenient to present the outcomes in an easily extendable table. Next, we display the zero-sum and self-interest outcomes, for the fixed symmetric subtraction game $S = \{2, 3\}$. (For this ruleset, the self-interest outcomes do not depend on the particular tie-breaking rule.)

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $o_{zs}(x)$ | $0$ | $0$ | $2$ | $3$ | $3$ | $1$ | $0$ | $1$ |
| $o_{si}(x)$ | $(0,0)$ | $(0,0)$ | $(2,0)$ | $(3,0)$ | $(3,0)$ | $(3,2)$ | $(3,3)$ | $(4,3)$ |

For example $o_{zs}(6) = \max\{-o_{zs}(3) + 3, -o_{zs}(4) + 2\} = 0$, and similarly $o_{si}(6) = \tilde{o}_{si}(4) + (3,0) = (3,3)$, if $\tilde{o}$ indicates swapped entries. This can also be useful when the game tree/DAG becomes complicated (exponential/quadratic growth compared to linear). In fact, these type of ‘outcome tables’ inspired this work; we wanted to explore how far we can extend the games, while still having a ‘one-line table computation’ of outcomes. And note that these type of tables prints all outcomes in the spirit of layers 1 and 2 in the game representation, not just the ones that belong to a particular grounded (layer 3) game, as in the DAG representation or, as is the case in the extensive form representation, where the tree only concerns computation of the value at the root, for a given starting player. For example, if we later want to display further outcomes, we will require $o(6)$, which is missing in the tree representations.

\[\text{In the setting of symmetric (impartial) Subtraction Games, fixed rules have also been called Vector Subtraction Games [10], and Invariant Games [8].}\]
The general (not necessarily symmetric) zero-sum outcome function has a very similar expression as Definition 2. The only difference is that the outcome function now requires a move flag corresponding to the previous player index of the ruleset and we may no longer assume that player 1 starts.

**Definition 4** (Outcome, Zero-sum [5]). Given a ruleset $S$ in symmetric zero-sum Cumulative Subtraction, the outcome of a heap of size $x$ is $o_{zs}(x) = (o_{zs}(x, 2), o_{zs}(x, 1))$, where

$$o_{zs}(x, 2) = \max\{o_{zs}(x - a, 1) + a \mid a \in S(x, 2)\}$$

and

$$o_{zs}(x, 1) = \min\{o_{zs}(x - a, 2) + a \mid a \in S(x, 1)\}.$$

Next, we define the (not necessarily symmetric) self-interest variation (in antagonistic play). Again, although the self-interest utility function seems more natural for cumulative games, the possibility of indifference situations requires extra care. The outcome is succinctly represented by a $2 \times 2$ matrix, because, similar to Definition 4, it is important to distinguish ‘who is the current player?’.

**Definition 5** (Outcome, Self-interest, Antagonistic). Consider a ruleset $S$ in a self-interest Cumulative Subtraction game, under antagonistic play. The self-interest outcome of a heap of size $x$, for $p \in \{1, 2\}$, is $o_{si}(x, p) = (0, 0)$ if $S(x, p) = \emptyset$, and otherwise

$$o_{si}^1(x, 2) = \max\{o_{si}^2(x - a, 1) + a \mid a \in S(x, 2)\},$$

$$o_{si}^2(x, 2) = o_{si}^1(x - a', 2),$$

where

$$a' = \arg\min_{a \in S'(x, 2)} o_{si}^1(x - a, 2),$$

and where $S'(x, 2) \subseteq S(x, 2)$ denotes player 1’s set of indifference actions in perfect play;

$$o_{si}^2(x, 1) = \max\{o_{si}^1(x - a, 2) + a \mid a \in S(x, 1)\},$$

$$o_{si}^1(x, 1) = o_{si}^2(x - a', 1),$$

where

$$a' = \arg\min_{a \in S'(x, 1)} o_{si}^2(x - a, 1),$$
and where $S'(x,1) \subseteq S(x,1)$ denotes player 2’s set of indifference actions in perfect play. The self-interest outcome of heap size $x$ is the matrix

$$o_{si}(x) = \begin{bmatrix} o^1_{si}(x,2) & o^2_{si}(x,2) \\ o^1_{si}(x,1) & o^2_{si}(x,1) \end{bmatrix}$$

The $i^{th}$ row of this matrix concerns player $i$ as starting player, and the $j^{th}$ column concerns player $j$’s game value, depending on who starts.

**Example 2.** Let us display the initial zero-sum and self-interest outcomes, for the fixed Cumulative Subtraction game $S = (\{2,3\}, \{1,4\})$:

| $S$   | $x$   | 0 | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-------|-------|---|-----|-----|-----|-----|-----|-----|-----|
| \{2,3\} | $o_{zs}(x,2)$ | 0 | 0   | 2   | 3   | 2   | 3   | 4   | $-1$ |
| \{1,4\} | $o_{zs}(x,1)$ | 0 | $-1$ | $-1$ | 1   | $-4$ | $-4$ | $-2$ | $-1$ |
| \{2,3\} | $o_{si}(x,2)$ | (0,0) | (0,0) | (2,0) | (3,0) | (3,1) | (4,1) | (5,1) | (3,4) |
| \{1,4\} | $o_{si}(x,1)$ | (0,0) | (0,1) | (0,1) | (2,1) | (0,4) | (0,4) | (2,4) | (3,4) |

That is, player 1 subtracts 2 or 3, whereas player 2 subtracts 1 or 4. So far, the self-interest outcomes do not depend on the particular tie-breaking rule (because initially there are no ties). Let us make some immediate reflections on these initial values, and indeed, by below Proposition 2, the rows in the table represent game values of the corresponding grounded positions, whenever $(C^0_1, C^0_2) = (0,0)$. For the zero-sum variation it is noted in [5] that all symmetric outcomes are nonnegative, but this will no longer hold in the non-symmetric case (for example $o_{zs}(3,1) = 1$ and $o_{zs}(7,2) = -1$), and moreover, now a player may get a zero-sum outcome larger than the maximum of their subtraction set, e.g. $x = 6$, which is not possible in case of symmetry. We postpone study of combinatorial properties and asymptotic of non-symmetric games, similar to and generalizing [5], and/or encourage other researchers to take on the many interesting topics, emerging from the definitions in this section. In Section 9, we guide the reader into some interesting problems related to this section.

We conclude with an intermediate step towards general Cumulative Games.

**Proposition 2** (Self-interest Cumulative Subtraction). Fix a previous player $p \in \{1,2\}$. At any grounded position $(x, (C^0_1, C^0_2),2)$, let $(x', (C'_1, C'_2))$ be the final position under PSPE play. Then, $o_{si}(x,p) = (C'_1 - C^0_1, C'_2 - C^0_2)$. That is, the utilities in PSPE play, when player $\bar{p}$ starts, are $C'_1 - C^0_1$ and $C'_2 - C^0_2$ for player 1 and 2 respectively.
This will be proved in a more general setting in Section 4 (Theorems 6 and 8).

Remark 1. Greedy play means to play the largest action possible, and sacrifice means to play some smaller action. E.g. in the PSPE play sequence above there is one sacrifice, in the first move from a heap of size 7. The necessity of sacrifices in certain situations was observed in [31] via a certain periodicity conjecture, which was solved in [5].

In this study, we de-emphasize such combinatorial aspects of solutions; instead, we wish to convey that for generalized optimal/PSPE play evaluations, the convenient means is recurrence via the outcome function. Before we move on, we review the complexity of a grounded Subtraction Game position.

3.4 Layer 3 complexity

In general Extensive Form Games, computing a PSPE requires traversing the entire game tree by backward induction. In the worst case, the size of the game tree may be exponential in its height. In the case of Cumulative Subtraction, we can merge identical nodes (e.g. in Figure 2 node [0;(5,2)] appears twice). Since in a tree rooted in \(x\), with a fixed initial cumulation, there are at most \(x\) different cumulation values for each player, we can construct a DAG representation with \(x^2\) nodes, which gives us an upper bound of \(O(x^2)\) on the computation time of a PSPE, for every game starting from a particular grounded position.\(^{11}\)

In some cases the outcome function allows us to compute the PSPEs even more efficiently.

**Proposition 3.** Consider any ruleset \(S\) with \(\sup_x |S(x)| \leq M\) for some constant \(M\). Computing all outcomes \(o_{si}(x,p)\) for all \(x \leq \hat{x}\), can be done in time \(O(\hat{x})\). In particular, a PSPE of any Extensive Form Game grounded in \((\hat{x}, (C_1^0, C_2^0), p)\) can be computed in time \(O(\hat{x})\).

**Proof.** The first part follows directly from Definition 5, as we can compute the outcome function via dynamic programming.

For PSPE computation from a particular grounded position, we compute the outcome \(o_{si}(x,p) = (C_1, C_2)\). By Proposition 2, we get the terminal utilities of both players as \(u_1 = C_1 + C_1^0\) and \(u_2 = C_2 + C_2^0\).

\(^{11}\)In general, we do not want to fix an initial cumulation, since we are interested in the outcomes on all heap sizes; the concept of initial cumulation belongs on layer 3, rather than the CGT-type layers 1 and 2.
4  Cumulative Games

In Section 3 we considered a specific class of self-interest combinatorial games, that are based on collecting pebbles, and showed that equilibria are efficiently computed, as expressed via the CGT outcome function.

Our purpose in this section is to define a more general class of games and extend Proposition 2. (Later, in Section 8, we show it is general enough to capture all Extensive Form Games.)

We first lay out the general definitions of *Cumulative Games*, and in particular we generalize the outcome function.

It is quite obvious that it is not possible to compute PSPE in every game in time that is subexponential in height, which means that a naive generalization of Propositions 2 and 3 could not apply to all games.

Indeed, in Example 4, we follow up on the auction game example in the Prologue, for which the proposition does not continue to hold.

This section is in preparation for Section 5, where we develop the required tools to reason about strategy profiles (in the EGT sense) for general Cumulative Games, and in Section 6, where we define a general variant of the outcome function (which attributes an $n \times n$ matrix of values to every position). We identify a property of Cumulative Games, that is a sufficient condition for a generalization of Proposition 2—see Theorem 6.

4.1 Cumulative Game Form

Let us give the general definitions for Cumulative Games. The general setting involves $n \in \mathbb{N}$ players, $d \in \mathbb{N}$ heaps (we can think about each heap as a different type of pebbles), reward functions, and a generalized turn function. We adapt the notion of a ruleset to the three layers of the Cumulative Game Form, and then at last we apply the utilities.

**Definition 6** (Cumulative Game Form). An $n$-player *Cumulative Game Form* on $d$ heaps of pebbles $F = (n, d, R, \Omega, \gamma)$, is a ruleset $R$ defined on a heap space $\Omega$, together with a turn function $\gamma: \Omega \times [n] \to [n]$, which specifies the current player.\footnote{As a shorthand, when $\omega$ is understood, we sometimes write $\gamma(\omega, p) = \tilde{p}$.}

There are three layers in a game form:

1. The heap space, $\Omega = (\mathbb{N}_0 \times \mathbb{R}^n)^d$;
2. The heap position, $\omega \in \Omega$;
3. The grounded position, $(\omega, p) \in \Omega \times [n]$, where $p$ is the previous player.
The ruleset is the pair \( R = (A, r) \), where

(i) the action-set \( A : \Omega \times [n] \to 2^{\mathbb{Z}^d} \) specifies the set of allowed actions, given a grounded position, possibly the empty set,\(^{13}\)

(ii) the reward \( r : \Omega \times [n] \times \mathbb{Z}^d \to \mathbb{R}^{n \times d} \) depends on a grounded position \( \omega \) and an action \( a \in A(\omega, p) \), and it is added to cumulations during play as defined in (11).

An element \( \omega \in \Omega \) is \( \omega = ( (x_1, C_1), \ldots, (x_d, C_d) ) \), where

(iii) \( x = (x_1, \ldots, x_d) \in \mathbb{N}_0^d \) is a \( d \)-tuple of heaps. We will sometimes refer to the set of all heap positions as \( X = X(\Omega) \). See Section 6.

(iv) the cumulation vector for each heap \( h \in [d] \) is \( C_h \in \mathbb{R}^n \).

A grounded position \( (\omega, p) \) is terminal if \( A(\omega, p) = \emptyset \), and otherwise, a grounded position \( (\omega, p) \) is identified with its set of options: for all players \( p \in [n] \), for all \( \omega \in \Omega \), with \( \omega = (x, C) \),

\[
(\omega, p) = \{ (\omega^{(a)}, \gamma(\omega, p)) \mid a \in A \},
\]

where \( \omega^{(a)} = (x + a, C + r(\omega, p, a)) \).

The cumulation \( C \) is an \( n \times d \) matrix, which represents the accumulated memory of the game, since it started, and where the entry at \((i, h)\) is player \( i \)'s cumulation on heap \( h \). Row \( i \) is the current cumulation for player \( i \) on each heap, whereas column \( h \) is the current cumulation for each player on heap \( h \). If, given \( \omega \), we wish to extract the cumulation vector, then we write \( C = C(\omega) \), and we may abuse notation as \( \omega = (x, C) \).

**Property 3** (Feasible CGF). A CGF is feasible if every grounded game terminates.

A CGF does not yet specify players utilities. The utilities are only realized at terminal positions, i.e. when the current player cannot move on any heap. Therefore, all CGFs will be assumed feasible.

As with Extensive Form Games, the game form alone dictates how the game can be played but not what players should do. Intuitively, players

---

\(^{13}\)If \( A(\omega, p) = \emptyset \), then \((\omega, p)\) is a terminal grounded position as in item 3) with respect to current player \( \hat{p} \) and cyclic move order say. This does not imply that \( \omega \) is a ‘terminal position’ at layer 2 because some player \( \sim\hat{\hat{p}} \) might have a non-empty action set (in fact this latter notion is not defined unless it holds for all players).
want to maximize their individual utilities, where the utility of each player depends on the particular terminal position \( \omega^T \), where player \( \tilde{p} = \gamma(\omega^T, p) \) cannot move. We omit the superscript ‘T’ as in terminal in the definition of the utility map.

**Definition 7 (Utility Map).** Consider a CGF on a short ruleset. A utility map for player \( i \) on heap \( h \), is a function \( u_{i,h} : \mathbb{R}^n \times [n] \to \mathbb{R}^n \), which maps a terminal cumulation on heap \( h \) to player \( i \)’s utility. Player \( i \)’s utility at any terminal grounded position \((\omega, p)\) is

\[
u_i(C, \tilde{p}) = \sum_{h \in [d]} u_{i,h}(C_{i,h}, \tilde{p}), \tag{12}\]

where \( C = C(\omega) \). Let \( u(C, \tilde{p}) = (u_1(C, \tilde{p})), \ldots, u_n(C, \tilde{p}) \).

**Observation 1.** The utility functions are sensitive to ‘who is to play?’, and this is useful, for example when utility should simulate Normal play ending, on all heaps.\(^\text{14}\) The reward function is the correct means, whenever we wish to simulate normal (or misère) play on individual heaps. See also Example 3.

**Definition 8 (Cumulative Game).** A feasible Cumulative Game Form \( F \) together with a utility map \( u \) induces a Cumulative Game \((F, u)\).

For a given previous player, we may want to be specific, and refer to a grounded Cumulative Game, etc.

Every grounded Cumulative Game is an Extensive Form Game (see Section 8), and so we can talk about specific strategy profiles etc.

The game is encoded succinctly in Definitions 6, 7 and 8, but let us spell out the flow of play.

1. A game, that is a grounded position \((\omega^0, p^0)\) on a ruleset \( R \), is forwarded by the previous player \( p^0 \), together with an initial \( d \)-tuple of heap positions \( \omega^0 = ((x_1^0, C_1^0), \ldots, (x_d^0, C_d^0)) \).

2. At each stage of play, the current player \( \gamma(\omega, p) \) moves from a grounded position \((\omega, p)\), where \( \omega = ((x_1, C_1), \ldots, (x_d, C_d)) \), to \((\omega(a), \gamma(\omega, p))\) by taking an action \( a \in A(\omega, p) \).

   (a) The heap sizes update as \( x \leftarrow x + a \).

   (b) The cumulation matrix updates as \( C \leftarrow C + r(\omega, p, a) \).

3. When the game ends because the current player cannot move, then the utility, for each player \( i \), is realized as in (12).

\(^{14}\)We have even more motivation in the Normal play embedded Guaranteed Scoring play [18, 17] and Absolute CGT [20, 21].
4.2 Continued game examples

Our definitions allow a variety of games, and combinations of game situations to appear in one and the same compound game. In order to stretch our preconceived ideas a bit on which games ‘fit together’ let us expand even further on the game example in the Prologue, and then encode it succinctly as a Cumulation Game.

Example 3. Let us introduce a third player, Charlie, to the Alice/Bob gamester hangout in the Prologue. And say that he plays as Bob and Alice, but using instead the subtraction set \{1, 4\}, except for heap \(F\), where all players have the same move options. Charlie has an additional option to add one pebble each to heaps \(C\) and \(F\), and this does not cost him anything. All six heaps \(A, B, C, D, E, F\) have size 4, and the play order is cyclic, Alice-Bob-Charlie-Alice and so on. Except, there is a twist to the turn function. If Bob collects exactly 5 pebbles, then he gets an extra turn (wherupon the game continues in cyclic order).

The utility functions require an update, when we shift from two to three players:

\[A: \text{a player with the last move wins one point, and the other players lose one point each;}
\]

\[B: \text{a player with the last move loses one point, and the other players win one point each;}
\]

\[C: \text{Alice and Charlie get } p \text{ points each, whereas Bob gets } -p \text{ points, if the final score is } p;
\]

\[D: \text{a player gets } p \text{ points if they end up with } p \text{ pebbles in their own pocket;}
\]

\[E: \text{the player who wins the auction gets } 4-\text{‘winning bid’ points and the other players get 0 points, and in case of no winner, all players get 0 points;}
\]

\[F: \text{a player who cannot move, in any component of the composite game, loses one point, and the other players win a point each.}
\]

Let us run through a play example, to see how they might play this game, succinctly encoded as a Cumulation Game. We have \(n = 3\), \(d = 6\), and the heap space is \(\Omega = (\mathbb{N}_0 \times \mathbb{R}^3)^6\). The heap position is

\[\omega^0 = ((4, 0), (4, 0), (4, 0), (4, 0), (4, 0), (4, 1))\]
The game rules are $R = (A, r)$, where $A$ is the action-set that specifies the possible actions at each grounded position, and where $r$ is the $3 \times 6$ reward matrix depending on the taken action, and in some cases in combination with the current heap size (but not on the cumulation). The action set $A$ has various elements:

(i) The action sets are fixed subtraction sets on heaps $A$ to $E$, $\{-2, -3\}$ for Bob and Alice, and $\{-1, -4\}$ for Charlie;

(ii) For each player $i$, the action set is $[C_i] \cap [x]$ on heap $F$

(iii) Charlie has the possibility of the action $a = (0, 0, 1, 0, 0, 1)$

The utilities and rewards are as follows.

(i) There are identity utilities on heaps $A$-$D$. On heap $E$, $u_{i,5} = 4 - C_{i,5}$, if $C_{i,5} > C_{j,5}$, for all $j \neq i$, and otherwise $u_{i,5} = 0$;

(a) On heap $A$ the reward is 0, unless $-a = x$, in which case $r_{i,1} = 1$ if $\tilde{p} = i$, and otherwise $r_{i,1} = -1$.

(b) On heap $B$ the reward is 0, unless $-a = x$, in which case $r_{i,1} = -1$ if $\tilde{p} = i$, and otherwise $r_{i,1} = 1$.

(c) On heap $C$, the reward is $r_3 = (a, -a, a)$ if the action is $a$ and $\tilde{p} = \text{Bob}$, and if $\tilde{p} = \text{Alice or Charlie}$, the reward is $r_3 = (-a, a, -a)$; all other reward entries are 0.

(d) Heap $D$ has identity rewards

(e) Heap $E$ has identity rewards.

(ii) Heap $F$ has reward $r_{i,6} = 0$, for all taken actions, and the utility is $u_{i,6} = -1$ if $i = \tilde{p}$ and otherwise $u_{i,6} = 1$.

(iii) Charlie has the possibility of the action $a = (1, 1, 0, 0, 0, 1)$, which adds one pebble on heap 1 and one pebble on heap 2, with reward 0.\footnote{Note, the twist this gives to normal and misère play; the termination of a single heap is not terminal for the composite game; it can be repeated, and each time rewarded or penalized, respectively.}

Note, that although there are moves that add pebbles to some heaps, all play sequences terminate.\footnote{If more game components are added, then we cannot assure that these local rules lead to termination; it has to be checked again.} The turn function $\gamma$ is cyclic, unless Bob gets cumulation $\sum C_{2,j} = 5$ and $\tilde{p}=2$, in which case $\gamma(\omega, 2) = 2$, i.e. Bob must carry on playing.
Suppose that Alice starts, the game, so the grounded heap position is \( (\omega^0, 3) \). She takes 3 from heap \( A \) and then Bob takes 3 from heap \( B \). Next Charlie takes 4 from heap \( C \). Thus we arrived at the position
\[
\omega^1 = ((2, 0), (1, 0), (0, (4, -4, 4)), (4, 0), (4, 1)),
\]
with Alice to move, and there are no more moves on heap 3. Next, Alice removes a pebble from heap \( F \), and Bob removes 3 pebbles from heap \( D \), while Charlie adds one pebble each on heaps 1 and 2. Thus, the new position is:
\[
\omega^2 = ((3, 0), (2, 0), (0, (4, -4, 4)), (1, (0, 3, 0)), (4, 0), (3, (2, 1, 1)))
\]
Now, Alice takes 3 from \( A \), Bob takes 2 from \( B \), and Charlie takes 1 from \( F \), and we arrive at:
\[
\omega^3 = ((0, (1, -1, -1)), (0, (1, -1, 1)), (0, (4, -4, 4)), (1, (0, 3, 0)), (4, 0), (2, (2, 1, 2)))
\]
Next, Alice plays in the auction by taking 2 pebbles, and Bob responds in \( F \), while Charlie adds pebbles to the two first heaps:
\[
\omega^4 = ((1, (1, -1, -1)), (1, (1, -1, 1)), (0, (4, -4, 4)), (1, (0, 3, 0)), (2, (2, 0, 0), (1, (2, 2, 2)))
\]
Finally, Alice removes the last pebble from \( F \), Bob plays in the auction and Charlie takes the last self-interest pebble:
\[
\omega^5 = ((1, (1, -1, -1)), (1, (1, -1, 1)), (0, (4, -4, 4)), (0, (0, 3, 1)), (0, (2, 2, 0), (0, (2, 2, 3))).
\]
This is a terminal position, since \( \bar{p} = \) Alice cannot move in \( A \) and \( B \), and all the other heaps are empty, and Alice can only remove pebbles (not add). (Observe that Bob never was required to carry on playing, because his accumulated number of points never approached 5.) Let us compute the player utilities. Alice gets \( u_1 = 1 + 1 + 4 - 1 = 5 \), where the \(-1\) is due to her losing the Normal play in the composite game, as triggered by component \( F \). Similarly, \( u_2 = -1 - 1 - 4 + 3 + 1 = -2 \), and \( u_3 = -1 + 1 + 4 + 1 + 1 = 6 \). Bob did not do so well: where is his first mistake?

Although this game is finite played alone (due to the cyclic turn function) it has an infinite game tree, and cannot be added in a disjunctive sum as defined in Section 7. (However, if we modify Charlie’s extra action to instead \( a = (0, 0, 1, 0, 0, -1) \), then the composite game is finite.)

Recall 2-player Cumulative Subtraction.
Observation 2. Both self-interest and zero-sum Cumulative Subtraction from Section 3, are simple special cases of a Cumulative Game. Both of them have the same Cumulative Game form, where there are two players, \( n = 2 \); a single heap: \( d = 1 \), \( \Omega = \mathbb{N} \times \mathbb{R}^2 \); the action set \( \mathcal{A} \) is the negation of \( \mathcal{S} \); the reward is the identity function: \( r(\omega, 2, a) = (a, 0) \) and \( r(\omega, 1, a) = (0, a) \); and the turn function is alternating: \( \gamma(\omega, p) = 1 - p \). Recall that, in the self-interest variation, \( u_i(\omega, p) = C_i \) whereas in the zero-sum variation, \( u_i(\omega, p) = C_i - C_{-i} \).

In view of our results to come we will require utility to be identity, i.e. for all players \( i \), \( u_i(C, \bar{p}) = C_i \). But this was not the case in Definition 1 for zero-sum symmetric Cumulative Subtraction. The situation has a simple remedy, via the reward function (and as will be apparent we will not require rewards to be identity, but they must be cumulation independent).

Zero-sum Cumulative Subtraction does not have identity utility when stated as above, but we can define an equivalent game with identity utilities, pushing the zero-sum interaction into the reward function. Let us model zero-sum Cumulative Subtraction as a Cumulative Game with self-interest utility.

**Proposition 4.** For all \( \omega \) with \( a(\omega) \in \mathcal{A} \), set \( r(\omega, 2, a) = (-a, a) \) and \( r(\omega, 1, a) = (a, -a) \). If utilities are identity, i.e. \( u_i(C, \bar{p}) = C_i \), then any sequence of play gives the same utility as zero-sum Cumulative Subtraction, as reviewed in Observation 2, and in particular, the optimal play strategies are identical.

**Proof.** Obvious.

Note that the cumulation part of a heap position is not used in modelling Cumulative Subtraction as self-interest. This should be put into contrast with the motivating examples, Example 4 and (in particular) Example 5 in Section 4.3. Let us introduce a central terminology in this context.

**Property 4 (Ruleset Cumulation Independence).** A ruleset that does not depend on cumulations is *cumulation independent*.

Notice that, rewards and actions, but not utility is included in the notion of a ruleset. (Obviously utility depends on the (terminal) cumulation.) We have arrived at the motivating example for the main results in Section 6.
4.3 Auction play

We discuss the variation of a Cumulative Subtraction game from the Prologue, where the PSPE strategy depends on the cumulation vector. We use the same ruleset as in the previous examples. The variation is in the utility function.

In below Example 4, we illustrate that:

1. For a modest and economic style generalization of Cumulative Subtraction, we cannot hope for a general outcome representation via a single table, as in the setting of our various examples on Subtraction Games. We can still define an outcome function, but succinctness of input size may get lost (because there is a history dependency in form of a cumulation matrix). In worst case, one can use the standard PSPE algorithm to compute the \( n \) values that comprise the outcome.

2. The second purpose hints at bridges between classical Subtraction Games from CGT to a classical concept in EGT (similar to so-called English auctions).

We review a variation of the one heap Auction Play example form the Prologue in the style of the new notation. This example has a symmetric utility function, which does not depend on the previous player, and hence we write \( u_i(C) = u_i(C, \cdot) \) for the utility of player \( i \).

**Example 4 (Auction Play).** Consider Cumulative Subtraction as in Definition 1, with a fixed symmetric ruleset \( S = \{2, 3\} \), announced by an auctioneer, and with identity rewards. The starting heap size is \( x^0 \in \mathbb{N} \), and player 1 starts bidding on a single item of value \( v \geq x^0 \). Each player has offered an initial bid, corresponding to an initial cumulation vector \( C^0 \). During play they can increase their bids according to their actions: their bids increment with the sizes of their subtractions.

For each player \( i \in \{1, 2\} \) the utility depends on the final cumulation vector \( C \), and it is of the form \( u_i(C) = v - C^i \) if \( C^i > C^{-i} \), and otherwise \( u_i(C) = 0 \). That is, the players desire to win the auction, but with the smallest possible margin, because they pay their bid. (Similar to “chicken game” this game has an element of win-loss, but is still not a zero-sum game).

From a heap of size \( x^0 = 3 \), with initial bid vector \( C^0 = (0, 0) \) or \( C^0 = (0, 1) \), player 1 wins the item in a single turn by removing 2 pebbles, so this is equilibrium play. However, if the initial cumulation is \( C^0 = (0, 2) \), then, in equilibrium play, player 1 must remove 3, and wins with a lower utility.
Thus, Player 1’s PSPE strategy depends on the initial cumulation, although the rules do not depend on cumulations (!).

The following example complements Example 4. It is a main motivation for this study, a context that will be developed further in Section 6, where we prove main results.

**Example 5 (Modified Auction Play).** Interestingly enough, Example 4 can be rewritten with identity utility, if we elaborate the rewards to mimic the situation. It is non-intuitive, but certainly doable for simple games like this. Let

1. \( r_1((3, (0, 0)), 1, 2) = 2 \),
2. \( r_1((3, (0, 0)), 1, 3) = 1 \),
3. \( r_1((3, (0, 1)), 1, 2) = 2 \),
4. \( r_1((3, (0, 2)), 1, 2) = 0 \),
5. \( r_1((3, (0, 2)), 1, 3) = 1 \),

and so on. Player 1 will use exactly the same strategy as in the previous example if we set self-interest utility, i.e. \( u_1(C) = C \). This, however, requires that we let the reward depend, not only on the action taken (and perhaps the heap size), but also on the cumulation. This is the second condition, that we will disallow in a (heap size) dynamic computation of the outcome (in equilibrium). Note for example the distinction between items 3. and 4. In item 3, Player 1 wins and her utility is 2, whereas in item 4, nobody wins and her utility is 0. In the latter case it is beneficial to play instead as in item 5., which indeed coincides with Example 4.

Of course, for generic Cumulative Games, where rules might depend in a complicated way on the player cumulations, we should not hope for more efficient computation of game values/outcomes than what is given by a generic non-efficient PSPE computation (i.e. exponential in the depth of the game tree). And indeed, the ‘fix’ of the PSPE cumulation dependency, via the reward function, to obtain identity utility, only made things worse, because now even the rules of game depend on cumulations. Of course, we know already that sometimes this ‘fix’ gives a neat outcome, via Proposition ??.

There is a fine distinguishing line somewhere between these examples and Proposition 4, and to make a formal treatment of this “fine line”, we will use definitions from main stream game theory via the Extensive Form Games.
Recall the table approach: if one would be interested instead in the PSPE outcome, i.e. the vectors of game values of for a different initial state (with different heap size or cumulation) the backward induction had to start all over again. The idea of an efficient outcome function is that a computation of game value vectors for a given heap size is universally valid, within the same ruleset.

The distinction, following our examples, will be almost self explanatory: cumulation independency is indeed a central property for appealing outcome functions. But we require a solid framework built on the theory of Extensive Form Games. We define an outcome function for the large class of Cumulative Games and then we restrict the class somewhat to demonstrate when the ‘one-table approach’ of Example 8 still applies.

5 Extensive Form Games

In Section 4 we defined a generalization of Cumulative Subtraction Games, and we aim to prove that it can capture all Extensive Form Games. To this end, we begin by defining Extensive Form Games in a modular way.

**Definition 9.** An Extensive Form Game is a tuple $G = (F, U)$, where $F = ([n], S, T, s_0, \delta, g)$ is the game form with

- $[n]$ is a set of $n$ players.
- $S$ is a finite set of states.
- $T \subseteq S$ is a set of terminal states.
- $s_0 \in S$ is an initial state.
- $\delta : S \to [n]$ is a turn function.$^{17}$
- $g : S \to 2^S$ is the game function.

And where $U = (U_1, \ldots, U_n)$, where $U_i : T \to \mathbb{R}, i \in [n]$ is the utility function of player $i$.

---

$^{17}$We assign players to terminal states, even though no action is possible. Intuitively, one reason is that the most important theory in CGT concerns Normal play, where by definition a player who cannot move loses. In any CGT situation, the basic terminating issue is: what happens when the current player cannot move? In misère play they win, and in scoring play a score will be assigned, depending on who is to play. The question in the title, ‘who is the current player?”, is fundamental to the study of cumulative/combinatorial games, not just to assign the starting player, but perhaps even more so, to punish or reward a terminal player.
Intuitively, player $i = \delta(s)$ is playing in state $s \in S$, and should select the next state from $g(s) \subseteq S$. We assume that there are no cycles in $g$, and $g(s) = \emptyset$ if and only if $s \in T$. We denote by $S_i \subseteq S$ all states such that $\delta(s) = i$. A state $s'$ is a descendant of state $s$ if there is a path in $g$ from $s$ to $s'$. Without loss of generality, we assume that each game state in $S$ is a descendant of $s^0$.

**Strategies and strategy profiles** Consider an Extensive Form Game $G = (F, U)$, with $F = (N, S, T, s_0, \delta, g)$. A player $i$ strategy is a function $\varsigma_i : S_i \setminus T \rightarrow S$, such that $\varsigma_i(s) \in g(s)$. That is, a unique action (next state) is selected in every state such that $\delta(s) = i$. A strategy profile is a vector $\varsigma = (\varsigma_1, \ldots, \varsigma_n)$. Consider any $s \in S$. We denote by $\varsigma|_s$ the restriction of $\varsigma$ on the subgame $G|_s$, the subgame of $G$ rooted by $s$. Denote by $\Sigma(G)$ the set of all strategies in $G$.

**Terminal maps and utility maps** Given a game $G$, and a strategy profile $\varsigma \in \Sigma(G)^n$, the terminal map $\tau_\varsigma : S \rightarrow T$ maps any state $s$ to the terminal that is reached when players start from state $s$ and follow their strategies in $\varsigma$ (in the subgame $G|_s$ of $G$). We may omit the parameter $\varsigma$ when clear from the context. Similarly, the utility map $\mu_i : \Sigma(G)^n \times S \rightarrow \mathbb{R}$ maps any state to the utility of player $i$. That is, $\mu_i(\varsigma, s) = U_i(\tau_\varsigma(s))$ is the utility to player $i$ in game $G|_s$ under strategy profile $\varsigma$.

**Observation 3.** For any profile $\varsigma$, the terminal state $\tau_\varsigma(s)$ is constant for any $s$ along the unique path the profile defines from $s^0$ to $\tau_\varsigma(s^0)$, and thus so is $\mu_i(\varsigma, s)$, for all $i \in [n]$.

**Definition 10.** Consider a game $G$. A strategy profile $\varsigma = \varsigma^*$ is a pure subgame perfect Nash equilibrium (PSPE) if, for all $s \in S \setminus T$, for all $i \in [n]$, for any alternative strategy $\varsigma'_i$, $$\mu_i(\varsigma, s) \geq \mu_i((\varsigma_{-i}, \varsigma'_i), s).$$

A game is generic if a player is never indifferent between two terminals $\tau$ and $\tau'$ unless $U(\tau) = U(\tau')$. Any game can be made generic by specifying some tie-breaking rule (say, lexicographic) in case of indifference.

Generic games are known to have a unique PSPE utility or game value (an $n$-tuple of real values),\textsuperscript{18} which can be found by backward induction.

\textsuperscript{18}There may be several PSPEs leading to the same value.
**Definition 11.** The game value of an Extensive Form Game $G = (N, S, T, s_0, \delta, g, U)$ is $v(G) = (\mu_1(\varsigma^*, s_0), \ldots, \mu_n(\varsigma^*, s_0))$.

In the context of grounded Cumulative Games, the value $v(G)$ will be referred to as the grounded value. See also Definition 13 to come.

### 5.1 Cumulative Games’ strategy profiles

A strategy profile $\sigma = \sigma(R)$ for a ruleset $R$ is an infinite set of extensive form strategy profiles $\varsigma_{\omega,p}$, one for each grounded position $(\omega, p)$, that is consistent, in the following sense. For every $(\omega', p')$ that is a descendant of $(\omega, p)$, the strategy $\varsigma_{\omega', p'}$ coincides with $\varsigma_{\omega, p \mid \omega', p'}$. Equivalently, for any $\omega \in \Omega$, and any $p \in [n]$, $\sigma(\omega, p)$ selects an action from $a \in A(\omega, p)$.

**Definition 12 (Cumulative Map).** Consider a ruleset $R$, with a given strategy profile $\sigma$. Then $c(\sigma, \omega, p) = C(\tau_{\sigma}(\omega, p))$ is the cumulative map of the grounded position $(\omega, p)$. Let

$$(\omega, p) = (\omega^0, p^0), (\omega^1, p^1), (\omega^2, p^2), \ldots, (\omega^k, p^k) = \tau_{\sigma}(\omega)$$

be the sequence of grounded positions and actions from $\omega$ in profile $\sigma$. Then, for each heap $h$,

$$c_h(\sigma, \omega, p) = C_h(\omega, p) + \sum_{i=1}^{k-1} r_h(\omega^i, p^i, a^{i+1}, \delta(\omega^i, p^i)),$$

and $c(\sigma, \omega, p) = \sum_{h \in [d]} c_h(\sigma, \omega, p)$.

A PSPE in a Cumulative Game $(R, \gamma, \omega, p, u)$ is just a PSPE in the induced Extensive Form Game, $G(R, \gamma, \omega, p, u)$; we assume that game $G$ is generic unless otherwise stated, and thus each player has a unique preference order of the other players. Since this is a property inherited from EG, we strengthen it somewhat by assuring a global uniformity (layer 1).

**Property 5.** A ruleset is generic if each player has a tie breaking rule (preference order of the other players), which is independent of starting position and initial player.

**Definition 13.** The grounded game value of a grounded Cumulative Game $(R, \gamma, \omega, p, u)$ is the game value of $G(R, \gamma, \omega, p, u)$, $v(G) \in \mathbb{R}^n$.

---

19In fact it is an uncountable set since the cumulation vectors are vectors of real numbers.
Although this definition gives a valid and important notion for a combinatorial game, it is not what is usually called a ‘game value’. The reason for this is better understood when we first develop the outcome function, which outputs a vector of grounded game values. Even though this stronger notion, which does not depend on the previous player, it is usually not sufficient to describe (perfect play in) a setting of combinatorial games, namely one of the most important properties is how a combinatorial game behaves together with other games in the same class, using the operation of disjunctive sum. For the discussion to proceed, one needs to define the notion of outcome function. While we do this, we will also prove some important properties of the function, while restricting the class of all Cumulative Games appropriately.

6 Game values and the outcome function

We generalize the CGT-outcome function e.g. [1] in our setting for Cumulative Games to include general-sum games, and to allow for any prescribed strategy profile (not just the one in PSPE). The below results hold for the multi heap situation, but, for simplicity, we choose to state and prove everything for the one-heap case, to emphasize the ideas (before the technology).

**Definition 14 (σ-outcome).** Consider a given profile σ for a Cumulative Game form \((R, \gamma)\). Let \(o_\sigma : \Omega \times [n] \to \mathbb{R}^n\). For any grounded terminal position \((\omega, p) \in T\), set \(o_\sigma(\omega, p) := 0\). For any non-terminal \((\omega, p)\), with \(a = \sigma(\omega) \in A(\omega)\), set

\[
o_\sigma(\omega, p) := o_\sigma(\omega(a), \gamma(\omega, p)) + r(\omega, p, a),
\]

The σ-outcome is an \(n \times n\)-matrix, denoted \(o_\sigma(\omega)\), one row vector for each previous player \(p\).

Similarly, column vector \(i\) of \(o_\sigma(\omega)\), \(o'_{i\sigma}(\omega)\), describes player \(i\)’s possible results given the profile \(\sigma\).

Note that, in Definition 14, the notation \(\omega(a)\), is used in the sense of being an option of \(\omega\), but without mention of the cumulation. This is intentionally (alas with some abuse of notation) while the next result clarifies the connection.

**Lemma 5.** Consider a ruleset \(R\), and a given strategy profile \(\sigma\). For any grounded position \((\omega, p)\), the \(i\)th player’s σ-outcome is \(o'_{i\sigma}(\omega, p) = c_i(\sigma, \omega, p) - C_i(\omega, p)\).
Proof. Indeed, if \((\omega, p) \in T\), then by definition \(c_i(\sigma, \omega, p) - C_i(\omega, p) = 0 = o^i_\sigma(\omega, p)\). Otherwise, if \((\omega, p) \in (\Omega \times [n]) \setminus T\), then

\[
o^i_\sigma(\omega, p) = o^i_\sigma(\omega \leftarrow a), \gamma(\omega, p) + r_i(\omega, p, a) = (c_i(\sigma, \omega \leftarrow a), \gamma(\omega, p)) - C_i(\omega \leftarrow a), \gamma(\omega, p)) + r_i(x, p, a) = c_i(\sigma, \omega, p) - (C_i(\omega \leftarrow a), \gamma(\omega, p)) - r_i(x, p, a).
\]

For (13), we use the recursive definition of the outcome function, and for (14), we use the induction hypothesis. For (15), we use Observation 3. For (16), recall that change in \(C_i\) after action \(a\) is exactly the reward.

The outcome function is used in zero-sum combinatorial Subtraction Games to characterize the optimal outcomes \([1, 33]\). However it is also meaningful when considering self-interest extensive-form games. Next we show how the \(\sigma\)-outcome finds the vector of grounded game values in case of generic Cumulative Games with self-interest utility.

**Theorem 6.** Consider a generic Cumulative Game with self-interest utility, and a given strategy profile \(\sigma\). Then for any grounded position \((\omega, p) = (x, C, p)\), player i’s utility is, \(\mu_i(\sigma, (\omega, p)) = o^i_\sigma(\omega, p) + C_i(\omega, p)\). In particular, this holds for a strategy profile \(\sigma^*\) in PSPE.

Proof.

\[
\mu_i(\sigma, (\omega, p)) = c_i(\sigma, \omega, p) \quad \text{(by identity utility)}
\]

\[
= o^i_\sigma(\omega, p) + C_i(\omega, p), \quad \text{(by Lemma 5)}
\]

as required.

**Corollary 7.** \(v(G) = o_{\sigma^*}(\omega, p) + C\).

Although Theorem 6 finds the grounded game values via the outcome function, it is not yet evident whether a reasonable efficient algorithm (type the one-row approach in the tables in Section 2) could find these values. However, with appropriate restrictions, we can shed some light on this issue.

Recall Example 5. It points at an important property of rewards.

**Property 6.** If the reward does not depend on the cumulation, then the reward is *cumulation independent.*
That is, a player gets rewarded by the actions they take, possibly depending on the heap size, and not by the history of the game. Note that this notion assumes that actions are cumulation independent.

**Property 7** (Heap Size Dynamic Games). If the rewards are independent of the cumulations, we write \( r(\omega, p, a) = \rho(x, p, a) \), for \( x = x(\omega) \). Similarly, if the actions are independent of the cumulations, we write \( A(\omega, p) = A(x, p) \), and the turn function becomes instead \( \gamma(x, p) \). A game is heap size dynamic if the CGF, i.e. the ruleset (the actions and the rewards) and the turn function, are independent of the cumulations in the game positions.

Note that the variations of Cumulative Subtraction as defined in Section 3 are heap size dynamic. We state one more (common) restriction of a ruleset.

**Property 8** (Short Ruleset). If for any \( \omega \), and any player \( p \) the set \( A(\omega, p) \) (or \( A(x, p) \)) is finite, then the ruleset is short.

Note, that if the ruleset is short, then the sizes of \( A(x, p) \) might not be bounded in terms of the heap size \( x \), but, for all \( x \), \( |A(x, p)| < \infty \).

We state the recursive computation of the \( \sigma^* \)-outcome for a given generic ruleset with cumulation independent reward. The outcome function becomes particularly simple in the case of heap size dynamic games, and implies the existence of a dynamic programming algorithm to solve the game.

**Theorem 8** (Recursive Outcome). Consider a generic game with a short ruleset, a self-interest utility and cumulation independent rewards, and let \( o = o_{\sigma^*} \). For any grounded position \((\omega, p)\), if \( A(\omega, p) = \emptyset \), then \( o(\omega) = 0 \), and otherwise

\[
o^p(\omega, p) = \max_{a \in A(\omega, p)} \{ o^p(\omega(a), \gamma(x, p)) - \rho_p(x, p, a) \},
\]

and if \( i \neq p \), then \( o^i(\omega, p) = o^i(\omega(a'), \gamma(x, p)) - \rho_i(x, p, a') \), where \( a' \) is a generic maximizing action, i.e. an action that in case of indifference follows the preference order of player \( \gamma(x, p) \) in (17).

**Proof.** Combine Definition 7 with Theorem 6. Observe that, since the rewards are cumulation independent, then the max operator is well defined, and, because the ruleset is short, there are only finitely many actions available, for any given position \( \omega \). Indeed, even if the actions are cumulation dependent, we can find a required generic \( a' \), and similarly for \( \gamma(x) \).

\[20\]Such algorithms can also have theoretical importance, as was recently shown in [6], where dynamic programming approach lead to discovery of a simulation of a 2-player Normal play game via a one dimensional (diamond shaped) cellular automaton.
Observation 4 (Heap Size Dynamic Rules). If the Cumulative Game $G$ is heap size dynamic, then the complexity of finding the outcome, the vector of game values is linear in the input size which is bounded by $\max |A(x)| \cdot \text{rank}(G)$. In the case of a subtraction game $G = S(x)$, this is bounded by $x^2$. If actions depend on the cumulations, we cannot a priori say anything about the input size in terms of $x$, apart from bounding it by the number of game states, which is in general exponential in $\text{rank}(G)$.

7 Partially ordered heap monoids

In this section, we designate the use of letters $G, H$ to layer 2 games, that is (ungrounded) heap positions of Cumulative Games. Here, the notation $\tilde{p}$, denotes cyclic move order, i.e., for all $p \in [n]$, $\tilde{p} = p \pmod{n} + 1$, which is indeed the standard in CGT.

The real elegance of CGT starts with the notion of disjunctive sum, game comparison and the partial order induced by the outcome function $[1, 7, 33]$. For all this to make sense, the perhaps most important property is that of additive closure. For if we add two games in a well defined class of games, then their sum should remain in the same class. Traditional recreational rulesets mostly are not additively closed, e.g. Tic-tac-toe: if we add two Tic-tac-toe positions the resulting game does not (easily) correspond to another Tic-tac-toe position. On the other hand, rulesets composed in the context of CGT often satisfy closure properties, such as Nim, Domineering, Hackenbush, and many others.

Let us define disjunctive sum for an $n$-player Cumulative Game. Suppose that $G$ and $H$ are game positions. When player $\tilde{p}$ moves in the composite game $G + H$, they move in either $G$ or $H$, following the usual rules, and leaves the other game component as it is. For example, if player $p$ moved to the option $H'$ in game $H$, then the resulting game for the next player is $G + H'$, and so forth. Since, any disjunctive sum of game must be feasible, we impose here that every play sequence in a CGF must be finite. Thus, utility functions are well defined, independently of the sum played.

If the current player has no move in either game component $G$ or $H$, then the game is over, and utilities are computed as usual. This, ‘as usual’

\[\text{This is important, for addition of game positions including a state dependent turn function (as in EGT) might be very complicated (and probably not possible).}\]

\[\text{But not always. If not one can define a ruleset closure of all sums of positions in a given ruleset; see e.g. [30, 26]. This is not necessary in our study. Therefore, our class has potentially good properties for future theoretical work generalizing the current framework of CGT to include general-sum, and so on.}\]
depends of course on that, for any Cumulative Game positions $G$ and $H$, the composite game $G + H$ is a Cumulative Game position.

Let us therefore formally code the sum given individual Cumulative Games. We define the elements of the new game. (The number of players is the same, and the turn function is not part of a game position, so it need not be considered in the notion of disjunctive sum.)

1. Game positions: $\omega_{G+H} = \omega_G + \omega_H$.
2. Actions: $A_{G+H} = A_G + A_H$.
3. Rewards: $r_{G+H} = r_G + r_H$.
4. Utilities: $u_{G+H} = u_G + u_H$.

We must justify/define the “+” for each item.

The main reason why all these items are well defined is that we treat the game positions $G$ and $H$ as independent object, and this property is carried over as a direct consequence of the definitions of Cumulative Games. Notice that, if $G$ consists of $d$ heaps and $H$ consists of $d'$ heaps, then the (candidate) position $G + H$ has $d + d'$ heaps. Each heap has an independent cumulation vector, so the use of “+” is well defined for game positions. When a player chooses an action, it will be chosen according to $A_G$ or $A_H$, and this is a permitted action in $A_{G+H}$. The rewards follow the actions so their “+” is well defined. The utilities are added at the end of play as usual, so this “+” is well defined. Hence, the sum of two Cumulative Games is again a Cumulative Game.

Other properties of a class of combinatorial games are 1) Existence of neutral element 2) Closure under taking options, i.e. each option of a game is again a game 3) Closed under swapping game positions. For 1) take all $\omega = (\mathbf{x}, 0)$, such that $A(\omega, p) = \emptyset$, for all $p \in [n]$. 2) is direct by definition. For 3) again, directly by definition, any couple of players may swap roles, and the resulting game position is still a Cumulative Game.\(^{23}\)

Thus, Cumulative Games satisfy the most important closure properties for a class of Combinatorial Games. They are defined recursively, and they have outcome functions. What remains is to define an order of games under the disjunctive sum operation.

\(^{23}\)Recently [20, 21] two more useful properties have been added to the list. 4) Parental closure, i.e. any $n$ finite subsets of games as options for the respective players represent a game in the class 5) Dense, given any outcome matrix $M$ and any game $G$, there is another game $H$ such that $o(G + H) = M$. 37
To have a partially ordered heap monoid, one starts by defining the order of outcome matrices \( M \) and \( Q \) as \( M < Q \) if and only if all entries \( m_{i,j} \) and \( q_{i,j} \) represent the order \( m_{i,j} < q_{i,j} \). Equality is the obvious definition, and thus many matrices are incomparable. This defines a partial order of outcomes. We will at last state the partial order of games.

**Definition 15.** Let \( G \) and \( H \) be generic cumulative \( n \)-player games. Then player \( p \) weakly prefers game \( G \) to game \( H \), i.e. \( G \succeq_p H \) if, for all Cumulative Games \( X \), \( \omega_p(G + X) \trianglerighteq \omega_p(H + X) \). That is, for all starting players \( j \), \( \omega_p(G + X, j) \trianglerighteq \omega_p(H + X, j) \).

**Theorem 9.** The class of Cumulative Games is a partially ordered heap monoid, under disjunctive sum, with respect to player \( j \in [n] \), say. Similarly, the restriction to heap-size dynamic Cumulative Subtraction is a partially ordered heap monoid.

**Proof.** Definition 15 is well defined, because of the closure of the disjunctive sum operator: the outcome function takes a Cumulative Game as input. Similarly, the sum of two heap-size dynamic games is again a heap size dynamic game, and so, if we also restrict the “for all \( X \)” part in Definition 15, we have another partial order.

Similarly, one can have a subclass of Cumulative Games where the rewards are cumulation independent (but where actions may depend on cumulations), and this class would again satisfy all closure properties, and therefore define a partial order specific for this class of games. In CGT, usually, when one restricts the class of attention, then the partial order changes. This is mostly studied in the setting of Misère games (see [27, 34] for surveys). Restriction to subclasses of games can be important to obtain efficient reductions of games, to increase the sizes of the equivalence classes of games.

## 8 Strategic Equivalence of Games

Given a position \( \omega^0 \) and a previous player \( p^0 \), we define the set of all possible descendants under a given ruleset \( R \) as

\[
D_R(\omega^0, p^0) = \{ (\omega, p) \mid \text{there is an } R\text{-path from } (\omega^0, p^0) \text{ to } (\omega, p) \}
\]

For a given ruleset \( R \) and initial state \( s^0 = (\omega^0, p^0) \), let length(s) denote the number of actions from \( s^0 \) to \( s \in D_R(s^0) \).\(^{24}\) Let rank\((s^0)) =

\(^{24}\)We are abusing notation here a bit, the reason being Observation 5 to come. We are identifying states with paths, since they may not be uniquely identified otherwise in a Cumulative Game.
\[
\max \{ \text{length}(s) \mid s \in D_R(\omega^0, p^0), p^0 \in N \}.
\]

8.1 Every Cumulative Game is an Extensive Form Game

A ruleset \( R \) together with a game position \( \omega = (x, C) \) and a utility function \( u \) defines an \( n \)-tuple of Extensive Form Games. Every grounded Cumulative Game is an Extensive Form Game.

**Observation 5.** Any grounded Cumulative Game, with an initial state \( \omega^0 \in \Omega \) and a previous player \( p^0 \in [n] \), defines a unique extensive form \( F = ([n], S, T, s_0, \delta, g) \), where

1. \( S = D_R(\omega^0, p^0) \);
2. \( T = \{ s \in S \mid A(s, \gamma(s)) = \emptyset \} \);
3. \( s^0 = (\omega^0, p^0) \);
4. For any \( s = (\omega, p) \in S \), \( \delta(s) = \gamma(\omega, p) \);
5. For any \( s = (\omega, p) \in S \), \( g(s) = \{ (\omega'(a), \gamma(s)) : a \in A(s, \gamma(s)) \} \), where \( \omega'(a) = (x + a, C + r(\omega, p, a)) \).

Moreover, a grounded Cumulative Game together with a utility function \( u \) defines the Extensive Form Game \( G = (F, U) \) where

6. For any \( t = (x, C) \in T \) and \( i \in [n] \), \( U_i(t) = u_i(C, \delta(t)) \).

**Definition 16.** We denote by \( G(R, \gamma, \omega^0, p^0, u) \) the Extensive Form Game induced by the the respective grounded Cumulative Game \( (R, \gamma, \omega^0, p^0, u) \), i.e. \((\omega^0, p^0)\) under form \((R, \gamma)\) and utility \(u\).

8.2 Any Extensive Form Game is a one heap Cumulative Game

Since the move order is arbitrary in EG but cyclic in the heap games, we cannot hope to find a heap game for each Extensive Form Game. However, we can do almost as well by introducing equivalence classes of Extensive Form Games.

Consider an Extensive Form Game \( G \). We introduce a reduced form of \( G \), \( \text{red}(G) \) and a cyclic extension, \( \text{cyc}(G) \). The idea is that two games are *strategically equivalent* if we bypass each child with exactly one option. When there is no further bypass possible of \( G \), we call this game \( \text{red}(G) \). Reversely, we can adjoin a sequence of states (each state with exactly one
child) between a parent and a child, to _cycle-complete_ the game modulo the \( n \) players. For any parent-child \( s, s' \) such that \( \delta(s') = \delta(s) + k \), we connect \( s, s' \) using a path of \( k - 1 \) states \( s_1, \ldots, s_{k-1} \) instead of the single edge \( (s, s') \), and where \( \delta(s_i) = \delta(s) + i \), for all \( i \). When each child is cycle-completed, we call this game \( \text{cyc}(G) \). The terminal utilities remain the same, and note that terminal states cannot be bypassed.

**Theorem 10.** For each Extensive Form Game \( G = ([n], S, T, s_0, \delta, g, U) \) there exists a strategically equivalent grounded Cumulative Game with a single heap.

**Proof.** Enumerate all states in \( S \) from \( s_0 \) in preorder, so that every node precedes all of its descendents. Denote by \( q(s) \) the index of \( s \) in this order, and let \( Q = |S| \) be the maximal index. Since \( q \) is a one-to-one mapping \( q : S \to [Q] \), the function \( q^{-1}(z) \in S \) in well defined, and we set \( s(x) = q^{-1}(Q - x) \).

W.l.o.g., \( \delta(s') \) is the same for all \( s' \in g(s) \) (otherwise we can add dummy states).

A position \( \omega = (x, C) \) is **valid** if \( x + C_j = Q \) for all \( j \in N \). Let \( \Omega \) contain all valid positions.

Intuitively, every state \( s \) in the original game \( G \) corresponds to a valid grounded position with heap size \( x = Q - q(s) \), and where all players have the same cumulation \( C_j = q(s) \). We complete the definition of the ruleset \( R = ([n], d = 1, \Omega, A, r) \) as follows. Set \( A(x) := \{ x - y : s(y) \in g(s(x)) \} \).

The move order function \( \gamma \) is defined as \( \gamma(\omega, p) := \delta(s(x)) \) where \( \omega = (x, C) \).

The (identical reward, identity) reward function is \( r(\omega, a, p) := (a, a, \ldots, a) \).

This means that if players start from some valid position \( \omega = (x, C) \) then after action \( a \) that reaches state \( \omega = (x', C') \), we will have that \( C'_j = (Q - x) + a = (Q - x) + (x - x') = Q - x' \). Thus \( \omega' \) is also valid.

As we intended, every state \( s \in S \) in the original game \( G \), induces a unique valid position \( \omega_s = (x = Q - q(s), C = (g(s), q(s), \ldots, q(s))) \) in the new ruleset \( R \).

The initial previous player \( p_0 \) can be set arbitrarily, as \( \gamma \) essentially ignores it.

Finally, we define the self-interest utility function as \( u_i(C_i) := U_i(s(C_i)) \) if \( s(C_i) \in T \) and otherwise 0.

We claim that the game \( (R, \omega_{s_0}, p_0, u) \) is equivalent to \( G \). By induction, every move from \( s = (\omega, p) \in S \) to \( s' = (\omega', p') \in g(s) \) corresponds to a move from \( \omega_s \) to \( \omega_{s'} \) where \( p' = \gamma(\omega, p) \) plays action \( a = q(s') - q(s) = x - x' \). When players in \( G \) reach a terminal \( t \in T \) and earn \( U_i(t) \) each, the corresponding
player in \((R, \omega, p_0, u)\) gets \(u_i(C_i) = U_i(s(C_i)) = U_i(s(q(t))) = U_i(t)\), as required.

For example, if \(g(s) = \{s', s''\}\) and \(q(s) = 4, q(s') = 10, q(s'') = 13\), \(Q = 100\) then \(s, s', s''\) correspond to \(x = 96, x' = 90, x'' = 87\), respectively.

We have another proof of this result, slightly modified, namely where we impose a cyclic turn function on the one heap game, which is the standard in CGT.

**Theorem 11.** For each Extensive Form Game \(G = ([n], S, T, s_0, \delta, g, U)\) there exists a strategically equivalent grounded Cumulative Game with a single heap, and a cyclic turn function.

**Proof.** We construct a one heap game \(\Delta\), with \(x_0 = 0\) and \(C_0 = 0\), and we will let the \(n\) players increase the heap size, assuming they have sufficient budgets of pebbles. Study \(\text{cyc}(G)\). The, say \(a\), children of \(s_0\), the root of \(\text{cyc}(G)\), can be enumerated \(s_1, \ldots, s_a\). For each child \(s_i\) let the heap size be \(x_0 + i\), so \(A(s_0) = \{1, \ldots, a\}\), and the cumulation is updated trivially (no rewards).

Study an arbitrary non-terminal node \(s\) at length \(\ell\), and suppose that the heap sizes on level \(\ell - 1\) range between \(X\) and \(Y\), with \(X < Y\). We enumerate all children at depth \(\ell\), denoted say \(s_{\ell,1}, \ldots, s_{\ell,a}\), and let the actions be \(Y - X, Y - X + 1, \ldots, Y - X + \alpha\). These actions are applied to the heap sizes of the parents at depth \(\ell - 1\) in increasing order, and we may assume the parents were enumerated in non-decreasing order. Again, the rewards are trivial, unless the action is to a terminal state. In this way we obtain a 1-1 mapping of heap sizes with the original game states in the EG. It remains to assign the correct utilities, and this will be achieved by setting them to the corresponding rewards for each terminating action. Empty sets of actions are attached to the terminal heap sizes.

Consider a cyclic turn function. By combining Observation 5 with Theorem 11, a consequence is that any multi-heap game can be simulated by a single heap game. In fact, this is somewhat simpler than the generic case since a multi-heap game regarded as an Extensive Form Game is already cycle-completed.

**Corollary 12.** Consider a cyclic turn function. Each grounded multi-heap game has an equivalent one heap game. That is, their game trees and utilities are the same.
Proof. Combine Observation 5 with Theorem 10.

Observe that the projection of multiple heaps to one heap here is on layer 3, grounded positions, whereas the famous result that any multi-heap nim position is equivalent to a single heap is a layer 2 result. Equivalence on layer 2 is in general harder, and we will briefly return to this question in Section 7.

In fact any combinatorial game with cyclic (or alternating) turns, with a given starting player, is equivalent to a one heap Cumulative Game, since grounded combinatorial games are Extensive Form Games.

9 Discussion

Let us return to the simpler cases of analysis, concerning zero-sum and self-interest Subtraction Games. Any action, that is consistent with an outcome function as in Definitions 2 and 5, will be called an optimal-action, in a given context.

Observation 6 (Zero-sum versus Self-interest). One of the first general sum questions for Subtraction Games is: for fixed heap sizes, and fixed subtraction sets, when do Definitions 2 and 5 assign the same optimal-action sets? A first (probably correct) guess is that the antagonistic variation is much closer to the zero-sum setting, than the friendly variation. One interesting problem is to explore precisely how much closer that is. And we provide some intuition via some preliminary computations.

For subtraction sets of size 2, we have not yet detected any difference between zero-sum and self-interest antagonistic optimal-actions. But for the friendly variation, the first difference appears already on the subtraction set $S = \{3, 5\}$, at heap size 14, and where $o_{zs}(14) = 3$, obtained by subtracting 5 pebbles, but $o^1_{si}(14) - o^2_{si}(14) = 2$, which is obtained by subtracting 3 pebbles. For subtraction sets of size 3, we have detected the first difference of the antagonistic and zero-sum variations for the subtraction set $S = \{6, 13, 17\}$, at a heap of size 76. The optimal action is either 6 or 17, and $o^1_{si}(76) - o^2_{si}(76) = 4$, whereas $o_{zs}(76) = 5$. Moreover, the number of such critical two or three element subtraction sets with numbers weakly smaller than 20 is one for the antagonistic case, whereas in the friendly case we find altogether 493 cases. If we increase 20 to 30 we find 16 and 2081 cases respectively and by increasing 30 to 40, we find 68 and 5386 cases respectively. We conjecture that both these numbers grow towards infinity with $\max S$. Fix a subtraction set. We conjecture that the outcome discrepancy is bounded
(for either antagonistic or friendly tie breaking convention), and this would be a corollary of another conjecture, that the greedy action is eventually an optimal action for self-interest (independent of tie-breaking convention); this was proved for zero-sum games in [5]. On the other hand, we conjecture that the outcome discrepancy can be arbitrarily large when we let the subtraction set vary.

Let us mention some examples where Pareto efficiency is the interesting concept.

**Example 6** (Tragedy of the Common). In welfare economics, for general sum games, at the core of the heart is the notion of Pareto efficiency (PE). A Pareto efficient play sequence is such that it is impossible to reallocate the actions so as to make any one player better off without making another one worse off. Here, we need to respect that the number of actions for the starting (earlier) player is either the same as the other players or they have one more action.

Consider the 2-player symmetric CS game \( S = \{20, 31, 51\} \), playing from \( x = 100 \), with identity rewards and identity utilities. The unique play sequence in PSPE is for player 1 to take 51, and then player 2 takes 31. At this point, no further action is possible. Any other play from player 1 would give player 2 the opportunity to take 51, and so she would get at most 31. However, there is a solution, which is better for both players. It is when both players have agreed beforehand to take 20 in each move. Then all resources will be allocated, which implies Pareto efficiency.

Returning to the main example with \( S = \{2, 3\} \) with identity rewards and identity utilities. Then, playing from any heap size it is easy to see that the outcome is Pareto efficient, both for antagonistic and cooperative tie break rule.

The smallest symmetric subtraction game that is not Pareto efficient for many heap sizes is \( S = \{3, 7\} \), and the first starting position that fails is \( x = 30 \). The reason is that player 1 cannot afford to make a big sacrifice and play 3. Because then player 2 easily responds with 7, and now player 1 starts from a heap of size 20, which has game value (10, 10). Thus the utility would be 13, when they obtain 14 by playing 7 twice. But clearly a Pareto efficient cooperation would yield the utility (15, 15).

We propose a resolution to the tragedy of the common in the setting of Cumulative Games. By introducing a principal-agent situation, one can offer a solution to the tragedy of the common in Example 6, by letting the principal choose the reward functions, and the agents play as usual, given the

43
suggested reward function by the principal. So, when they play they seek to maximize the utility as defined here. However, there is a second stage of game, when the principal reveals their ‘true utilities’, and in the example they are simply their respective sum of their actions: set the rewards for each action to $r(s) = -s$. Then indeed the Pareto (efficient) solution is achieved. In this way, rewards can be seen as a way to correct incitements that go wrong because of perhaps too individualistic behavior.

**Memory** A big class of games in the Normal play theory is games with memory, with the specific meaning that the action set, may depend on previous actions (in full generality the action sets may depend on all history of a game). The results in this paper will still hold, but with more cumbersome notation, so we omit this class of games. The most famous such game in the Normal play theory is Fibonacci nim [36], which depends only on the most recent action, and it has recently been generalized to multi heap situations [22, 23]. Another game in this family that has deeper memory is Imitation nim [15]. Here we let any memory (including possible dependencies) be stored in a cumulation vector. In theory we could have memory be more general, but the point is to have some natural restrictions. We feel that, for an economic type game, the available actions may depend on the players’ cumulations/budgets/endowments etc, but ‘how’ they got to their present cumulations can be ignored.

**Cyclic games** A natural extension of the current work is to study Cumulative Games with cycles. One can define various results ‘in the limit’, using standard lim sup and lim inf maps of results along strategy profiles, and thus generalize the outcome functions.

**Philosophy** We mentioned one difference between EG and CG: it is given who starts in EG but not in CG. Another major distinction concerns the movability, and in particular ‘who gets to make the final move’, often it is useful to have more moves than your opponent, and this is the de facto standard that all CGT should be put in relation with. Extensive Form Games are often more utility oriented, and the notion of who moves last is rarely the most important issue. In fact, even for sequential games the standard assumption is that each player declare their strategies, and so the move ability is rarely mentioned. It suffices to know the players’ strategy profile, and then their utilities follow. So the underlying philosophy used to be quite different. Here we show that they need not be that different.
Can we have non-trivial game comparison for self-interest Cumulative Games? If one aspires big equivalence classes of games, in the spirit of Normal play theory, then it would have to be defined with respect to a given player \( p \), as we do in Definition 15. For this to be interesting, one would most likely have to restrict the class of games, and the first step would be to find a class of games such that the equivalence class of \( 0 \) is non-trivial. Indeed, a first step in this direction has been taken in that Normal play CGT is bridged with Scoring-play CGT in a useful restriction of the full class of Scoring Games [31] to the class of Guaranteed Scoring [18, 17] in which Normal play is order embedded; the full class has only the trivial neutral element, whereas for Guaranteed Scoring the equivalence class of neutral elements contain all 0s of Normal play. This recent Scoring-play development started with Ettinger’s seminal Ph.D thesis [9]; he extended the Milnor-type positional (nonnegative incentive scoring play) games to include so-called zugzwangs (where no player wants to start), a common concept in recreational play.

Solve wealth play Wealth play as introduced in the Prologue is a partizan Normal play 2-player game, and it is all small (if one player has a move, then the other player has a move) if we assume that all heaps start with cumulations at least \((1, 1)\) (and otherwise it is trivial). Hence, in the context of a disjunctive sum of games, the notion of atomic weights [1, 33] will readily arrive as a tool. Thus, a first step to characterize these games would be to determine heap sizes atomic weights as a function of the players current cumulations. Atomic weights are a rough measure of how many times you can afford to ‘pass’ (or wait) in any given component. Of course, if you lead by a certain amount in one component, then a move there is not urgent, but it could be more useful to accumulate more wealth (which here corresponds to ‘ability to wait’) in a neighboring game component, and this is the essence of atomic weigh play (roughly, if your atomic weight is two steps ahead, then you are safe for winning). We propose this game, as a step forward to develop an economic branch of classical CGT.

10 Normal play CGT, an overview

The Normal play convention in CGT means that the player with no available options loses the game. The most well known game played with this convention is Nim [4] (players take-away any number of pebbles from one heap out of many). However, the game of Nim is impartial (players have
the same options), and within the disjunctive sum theory, which is one of the fundaments of CGT, all impartial games are pairwise incomparable. Let us review the building blocks of the more interesting partizan game theory (players may have different options), by using examples of heap games, and more specifically partizan subtraction games.

Within the partizan Normal play theory, there are many comparable games, and for example, if Left’s subtraction set is \{1, 4\}, whereas Right’s subtraction set is \{2, 3\}. Then, if \(G\) is the heap of 4 pebbles, and \(H\) is the heap of 3 pebbles, then, within the Normal play theory, Left prefers \(G\) before \(H\). As we will explain in the coming, this will follow, since Left wins the game “\(G - H\)” independently of who starts. We encourage the reader to ponder upon this observation, while we lay out the Normal play foundations in the spirit of this paper, staying close to the theory of Cumulative (Subtraction) Games.

10.1 Subtraction Games

In this section we consider a specific class of partizan games known as subtraction games, which are the basis for our general definition of Cumulation games in the main text.

Let the two players be Luise (Left) and Richard (Right).\(^{25}\) In some more generality, than in the Introduction, we let the subtraction set depend on the position presented by the previous player. Combinatorial games are defined in terms of their sets of move options [1], so we follow this approach. Since the two players alternate turns, we often refer to the players by the current player, i.e. the player who is about to take an action, and a previous player, i.e. the other player. We will adopt a convention: the previous player is called player \(p\), and the current player is player \(\bar{p}\).

Let \(\mathbb{N} = \{1, 2, \ldots\}\). The standard notation for subtraction games is (as we wrote in the Introduction) \(S \subset \mathbb{N}\), but for two reasons we will instead use sets of the form \(A \subset \mathbb{N}\). The first reason is the we will use the letter \(S\) for the game states in extensive form games, and the second reason is that in general Cumulation games will allow addition as well, so later we will have \(A \subset \mathbb{Z}\). Thus, we may think of the letter \(A\) as ‘addition’ or perhaps ‘action’.

From a notational point of view, we will depart slightly from CGT-traditions. Typically we let the previous player define the setting (instead of using the current player); the current position, the possible actions, the

\(^{25}\)In CGT the players are often called Left and Right. The naming is after Richard Guy, who did so much for the field of CGT, and his wife Luise.
outcome and so on will be labeled by the previous player.\footnote{In the much younger tradition of games with a Muller twist (a.k.a. blocking maneuver) this is indeed the standard; see e.g. \cite{muller-twist, laplant}.}

**Definition 17** (Subtraction Games, Normal play). A 2-player Subtraction Game is defined in three layers. A ruleset (a generalization of a subtraction set) is a function $S : \mathbb{N}_0 \times \{\text{Left, Right}\} \to 2^{\mathbb{N}}$, which takes as input a heap size and a (previous) player.\footnote{We say that a ruleset $S$ is valid if $s \leq x$ for any grounded position $(x, p)$ and $s \in S(x, p)$. We assume unless stated otherwise that all rulesets we consider are valid.}

1. The heap space is $\mathbb{N}_0$;
2. A heap position is an ordered pair $(x, S)$, where $x \in \mathbb{N}_0$ is the size of a heap of pebbles, and $S$ is a ruleset.
3. A grounded position is a triple $(x, S, p)$, where $(x, S)$ is a position, and where $p$ is the previous player.

A grounded position is identified with its sets of options, which is

$$(x, p) = \{(x - s, \tilde{p}) \mid s \in S(x, p)\}, \quad (18)$$

where $\tilde{p}$ is the current player. The game ends when the current player, player $\tilde{p}$, cannot move, i.e. $(x, S, p) = \emptyset$, and the previous player, player $p$, wins. A game is played by agreeing on an initial grounded position $(x^0, p^0)$, i.e. a starting position $x^0$ and a previous player $p^0$. Then the players take turns moving as prescribed (18), until one of the players cannot take any action, and thereby loses.

**Example 7.** Let $S = \{2, 3\}$ be a fixed and symmetric ruleset. An instance of a position is $(7, \{2, 3\})$, and the corresponding grounded position when Left starts is $(7, \{2, 3\}, \text{Right})$. Thus a move option is $(4, \{2, 3\}, \text{Left})$. From here Right has a winning move to $(1, \{2, 3\}, \text{Right}) = \emptyset$.\footnote{For fixed rulesets such as $S = \{2, 3\}$, for terminal positions, we abuse notation and write $(1, \{2, 3\}, \text{Right}) = \emptyset$ instead of $(1, \emptyset, \text{Right}) = \emptyset$.} Indeed, our notation reveals the winner, in the sense that if $(x, S, p) = \emptyset$, then player $p$ wins.

The way we think about the three layers in Definition 17, starts with that the function $S$ encodes the game. Likewise to (18), a heap position is identified with the set of grounded positions, one for each previous player,
and so \((x, S)\) is identified with the set \(\{(x, S, \text{Left}), (x, S, \text{Right})\}\). The heap space together with the ruleset \(S\) is identified with the set of heap positions, one for each heap size, i.e. the set \(\{(x, S) \mid x \in \mathbb{N}_0\}\).

For example, the picture to the left in Figure 4 (page 50) describes for the game tree identified with position \((7, \{2, 3\})\), which is a subtree of an infinite non-rooted game tree identified with the fixed ruleset \(S = \{2, 3\}\). The three layers of a game (as in Definition 17) are essential to this study, bridging combinatorial game theory (CGT) and classical game theory (EGT). Grounded game positions (layer 3), together with the Normal play winning condition, are extensive form games from EGT.

### 10.2 Outcome Functions

The reason we need to define the first two layers, is that some important concepts in CGT are defined explicitly on these layers. The first of these concepts is the outcome function (which describes the result of a perfect, or rational, play), which partitions games into equivalence classes. We begin by reviewing this function in its traditional setting.

We state the four outcome classes for classical CGT (assuming general partizan play). The definitions are written in general terms but using our definitions of subtraction games. For readers with EGT background, the recursive definition of the outcome function is similar to the backward induction solution of extensive form zero-sum games, where Left is the maximizing player. CGT assumes by convention that both players play their optimal strategies.

**Definition 18** (Classical CGT outcomes [1]). Consider a ruleset \(S\). The result (utility) of an acyclic Normal play grounded position, is either Left wins (result \(L\)) or Right wins (result \(R\)). By convention, we treat the two results as being ordered, with \(L > R\). Let

\[
o(x, \text{Right}) = \max\{o(Z, \text{Left}), R\},\]

where \(Z\) runs over all options of \((x, \text{Right})\), and let

\[
o(x, \text{Left}) = \min\{o(Z, \text{Right}), L\},\]

where \(Z\) runs over all options of \((x, \text{Left})\). The (perfect play) outcome at position \((x, S)\) is a pair \(o(x) = (o(x, \text{Right}), o(x, \text{Left}))\).\(^{29}\)

\(^{29}\)By convention, \(\max(\emptyset) = -\infty\) and \(\min(\emptyset) = \infty\).
Thus \( o(x, \text{Right}) \) is the result (either \( L \) or \( R \)) in perfect play if Left starts, and similar for Right. Note that the games are zero-sum, so in EGT (layer 3), the result corresponds to the usual value of a game, either \( L \) or \( R \). Thus there are four possible outcomes at the outset:

- The outcome \( \mathcal{L} \) means that Left wins independently of who starts, i.e., \( o(x) = (L, L) \);
- The outcome \( \mathcal{R} \) means that Right wins independently of who starts, i.e., \( o(x) = (R, R) \);
- The outcome \( \mathcal{P} \) (for “Previous”) means that the player who does not start wins, i.e., \( o(x) = (R, L) \);
- The outcome \( \mathcal{N} \) means that the player who starts wins, i.e., \( o(x) = (L, R) \).

Given a ruleset \( S \), we say that \( x \) is a \( \mathcal{Z} \) position if \( o(x) = \mathcal{Z} \in \{\mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{N}\} \). Note that symmetric games have only \( \mathcal{P} \) and \( \mathcal{N} \) positions.

As discussed in the main part of this paper, one of the main ideas of acyclic combinatorial rulesets, is that one can recursively compute the outcome class of each game position. The first example, is the case of impartial (symmetric) Normal play games: a game is a \( \mathcal{P} \) position if all its options are \( \mathcal{N} \) positions, and otherwise it is an \( \mathcal{N} \) position. Observe that this implies that each terminal position is a \( \mathcal{P} \) position.

Note that the total order of the results imply that the partizan play outcomes are partially ordered, with \( \mathcal{L} > \mathcal{N} > \mathcal{R} > \mathcal{P} \), but where \( \mathcal{N} \) and \( \mathcal{P} \) are incomparable.

**Example 8** (Outcomes of Normal play Subtraction). Consider \( S = \{2, 3\} \), as in Example 7. The initial outcomes are:

| \( x \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|---|
| \text{starting player} | \( o(x) \) | \( \mathcal{P} \) | \( \mathcal{P} \) | \( \mathcal{N} \) | \( \mathcal{N} \) | \( \mathcal{P} \) | \( \mathcal{P} \) | \( \mathcal{N} \) |
| Left | \( o(x, \text{Right}) \) | \( R \) | \( R \) | \( L \) | \( L \) | \( R \) | \( R \) | \( L \) |
| Right | \( o(x, \text{Left}) \) | \( L \) | \( L \) | \( R \) | \( R \) | \( L \) | \( L \) | \( R \) |

In Figure 4 we display optimal play from a heap of size 7, via the use of a directed acyclic graph representation (DAG). The outcome is \( \mathcal{N} \), and thus there must exist a winning move option (independently of who starts). By inspection, the unique winning subtraction is \( 7 - 2 = 5 \) (so Left picked the wrong option in Example 7). Note that the left DAG in Fig. 4 contains all the information on the ruleset, up to position 7.
Figure 4: The leftmost picture shows a DAG representation of the position 
(7, \{2, 3\}), and the picture to the right shows the corresponding outcomes.

Example 9 (Outcomes of a partizan game). Let \( S = (\{2, 3\}, \{1, 4\}) \) be a 
fixed partizan ruleset (Thus Left subtracts 2 or 3 and Right subtracts 1 or 
4). The initial outcomes are:

| Starting player | \( x \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Left            | \( o(x) \) | \( P \) | \( R \) | \( N \) | \( L \) | \( R \) | \( N \) | \( L \) | \( P \) |
| Right           | \( o(x, \text{Right}) \) | \( R \) | \( R \) | \( L \) | \( L \) | \( R \) | \( L \) | \( L \) | \( R \) |

Consider optimal play from a heap of size 7. The outcome is \( P \), because 
Left has options to \( L \) (7 − 2) and \( R \) (7 − 3), whereas Right has both options 
to \( L \) (7 − 1 and 7 − 4). However if we start from position \( x = 6 \), then Left 
wins regardless of who starts.

10.3 Disjunctive Sum

Let \( G_1 \) and \( G_2 \) be the subtraction games defined in Examples 7 and 9, 
respectively, both with heaps of size 7.

The disjunctive sum \( G + H \) is a combinatorial game on two heaps, \( G \) 
and \( H \) (and note that, in its literal form, it is not a subtraction game as 
per Definition 17). In every turn, the current player must make a move in 
exactly one of those games, following the rules: Right may remove 2 or 3 
pebbles from either heap, whereas Left may either remove 2 or 3 pebbles 
from the first heap, or 1 or 4 pebbles from the second. As usual, a player 
that has no available option loses the game. Note that \( G \) includes both the
ruleset and the position. Thus the disjunctive sum operation takes place at layer 2.

If we extend the class of Subtraction Games to games with multiple piles, then it is not necessarily closed under the disjunctive sum operation. But, of course, it is closed in the class of all Normal play games. Note that we can also add Subtraction Games to other combinatorial games such as Nim or Chomp, with the same Normal play rule. Open problem: for what subclasses of (Normal play) Subtraction Games are all composite games equivalent to a single heap game? This problem requires an explanation:

### 10.4 Games comparison

The importance of both the outcome function and the disjunctive sum operation comes into light when we want to compare games. Whether a game $G$ is “good” for player Left may depend on the context: a-priori, Left may have a winning strategy in $G$ but not in $G + X$ for some other game $X$.

**Definition 19.** Consider Normal play games $G$ and $H$. Then $G \geq_D H$ if $o(G + X) \geq o(H + X)$ for any Normal play game $X$.

It turns out that the outcome function contains the entire information on the game $G$ under any possible context! The $\geq_D$ relation is a partial order over all Normal play games. Any two games can be compared, and it assures that Left weakly prefers the game $G$ before the game $H$, in any situation that may occur, that is in any disjunctive sum play. Notice in this sense that outcome equivalence is a much weaker definition, and not sufficient to guide the players, for example in a play on several heaps (subtraction game).

The Fundamental Theorem of Normal play, is that the for all $X$ part of the definition disappears and game comparison simplifies to: $G \geq_D H$ if and only if Left wins the game $G - H = G + (-H)$ playing second, where $-H$ is the game $H$, where the players have swapped roles. This result uses that Normal play is a group structure, and that mimic is the way to prove that $G - G = 0$, i.e. that $G - G$ is a $P$-position. Game comparison in any other class of combinatorial games should at least be put in relation to this elegant result, that paved the way for modern CGT.

Let us finish off the example in the beginning of this section, with subtraction sets as in Example 9. So, $G$ is the heap of size 4, and $H$ is the heap of size 3, and thus, we wonder if Left wins $G - H = G + (-H)$ independently of who starts. Thus, we reverse the roles of the players in the game $H$, so Left (Right) has the subtraction set $\{2, 3\}$ ($\{1, 4\}$) in the game $-H$. If Left starts she can win by eliminating the $G$ component. Now Right must remove
one pebble from the heap of size 3, and so Left wins by removing the last two pebbles. If Right starts, then he can remove 2 or 3 from the heap of size 4, or he can remove 1 from the heap of size 3. If he removes 2, then Left wins by removing 1 more from the $G$ component. If he removes 3, then Left wins by eliminating the $H$ component. If he removes 1, then Left wins by eliminating this heap. The reader may fill in the missing last move(s). Hence, we know that in any Normal play situation, Left will prefer the heap of size 4 before the heap of size 3.

This served to illustrate an elegant and playful argument to prove an abstract mathematical entity, that distinguishes combinatorial games from other areas of mathematics. We are not yet aware of any similar result for the general class of Cumulative Games, and we believe that the general class must be much restricted to approach constructive game comparison, in for example a class of pure self-interest games.

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