ON THE SPHERICAL TWISTS ON 3-CALABI-YAU CATEGORIES FROM MARKED SURFACES

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Abstract. We are interested in the 3-Calabi-Yau categories \( \mathcal{D} \) arising from quivers with potential associated to a triangulated marked surface \( S \) (without punctures). We prove that the spherical twist group ST of \( \mathcal{D} \), which is a subgroup of its auto-equivalence group generated by spherical twists, is isomorphic to a subgroup (generated by braid twists) of the mapping class group of the decorated marked surface \( S_\Delta \). Here \( S_\Delta \) is the surface obtained from \( S \) by decorating with a set of decorated points, where the number of points equals the number of triangles in any triangulations of \( S \). For instance, when \( S \) is an annulus, the result implies the faithfulness of the spherical twist group actions, in the sense that ST is isomorphic to the affine braid group of type \( \tilde{A} \). One application is that this faithfulness completes the description of the spaces of stability conditions on \( \mathcal{D} \) in the case of \( \tilde{A} \). Other applications include geometric realizations of Amiot’s quotient for cluster categories and of simple-projective duality for Ginzburg dg algebras.

Key words: Calabi-Yau categories, spherical twists, quivers with potential, braid group, cluster theory

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Date: July 4, 2014.
1. Introduction

1.1. Calabi-Yau (CY) categories from mirror symmetry. We are interested in a class of 3-Calabi-Yau categories $D$ arises from (homological) mirror symmetry. These 3-CY categories are not only interested in mathematics ([22] and [30]), but also in the string theory ([11], cf. [4]). On the symplectic geometry side, the category $D$ (of type $A$) was first studied by Khovanov-Seidel (KS) [22]. They showed that there is a faithful braid group action on $D$. Moreover, when realizing $D$ as the subcategory of the derived Fukaya category of the Milnor fibre of a simple singularities of type $A$, such a braid group is generated by the (higher) Dehn twists along certain Langragian spheres. On the algebraic geometry side, Seidel-Thomas [30] studied the mirror counterpart of [22] (also in type $A$). They showed that $D$ can be realized as a subcategory of the bounded derived category of coherent sheaves of the mirror variety with a faithful braid group action. Recently, Smith [31] showed that if $D$ is coming from triangulations of marked surfaces $S$, then it also can be embedded into some derived Fukaya category. This class of cases are the ones we will study. The focus is on the spherical twist group $ST = ST_D$, a subgroup of the auto-equivalence group of $D$ generated by Khovanov-Seidel-Thomas (KST) spherical twists. The aim is to generalize the KST result, that $ST$ is ‘faithful’, in the sense that $ST$ is isomorphic to the classical (type $A$) braid group (and in general, isomorphic to a subgroup of certain mapping class group). We need to restrict ourself in the case when marked surfaces are unpunctured. In the twin paper [20], we will make an effort to attack the problem when the marked surfaces are punctured.

Note that these spherical twist group $ST$ acts freely on the space $\text{Stab}^o D$ of Bridgeland’s stability condition of $D$. This is one of our main motivations to study such a group. In fact, Bridgeland-Smith (BS) [4] recently showed that the quotient (orbifold) $\text{Stab}^o D/\text{Aut}^o$ is isomorphism to the moduli space $\text{Quad}(S)$ of meromorphic quadratic differentials with simple zeroes on the marked surfaces $S$, where $\text{Aut}^o D$ is the extension of the (tagged) mapping class group of $S$ on top of $ST$. And one would expect that the faithfulness of spherical twist group actions will imply the simply connectedness of $\text{Stab}^o D$. For instance, this implication holds for the (3-CY) Dynkin case (see [26]); also, such faithfulness (and its implication of simply connectedness) was proved by Brav-Thomas [2] for the 2-CY Dynkin case and by Ishii-Ueda-Uehara [15] for the 2-CY affine type $\tilde{A}$ case.

Our main result says that $ST$ is isomorphic to a subgroup of the mapping class group of some surface. In the (3-CY) affine type $\tilde{A}$ case, it is exactly the faithfulness, in the sense that $ST$ is isomorphic to the affine braid group of type $\tilde{A}$. (In the type $A$ case, this is due to KST). And we will study the implication of simply connectedness in this case (3-CY affine $\tilde{A}$). In the sequel, we will study this implication for other cases.

1.2. Cluster theory and quivers with potential. Quiver mutation was invented by Fomin-Zelevinsky (FZ) around 2000, as the combinatorial aspect of cluster algebras. Later, mutation was developed by Derksen-Weyman-Zelevinsky (DWZ) for quivers with potential.
The first (additive) categorification of cluster algebras (with certain associated acyclic quivers) was due to Buan-Marsh-Reineke-Reiten-Todorov, via representation of the corresponding quivers. Amiot introduced the generalized cluster categories via Ginzburg dg algebras for quivers with potential. In her construction, the cluster category $\mathcal{C}(\Gamma)$ is defined by the following short exact sequence of triangulated categories

$$0 \to \mathcal{D}_{fd}(\Gamma) \to \text{per }\Gamma \xrightarrow{\pi} \mathcal{C}(\Gamma) \to 0,$$

(1.1)

where $\Gamma = \Gamma(Q, W)$ is the Ginzburg dg algebra of the quiver with potential and $\text{per }\Gamma$ (resp. $\mathcal{D}_{fd}(\Gamma)$) are the perfect (resp. finite-dimensional) derived category of $\Gamma$. Here, $\mathcal{D}_{fd}(\Gamma)$ is the 3-CY category we mentioned above.

There is an exchange graph associated to each of the categories in (1.1), namely:

- the reachable hearts/t-structures in $\mathcal{D}_{fd}(\Gamma)$ as vertices and simple tilting as edges for the exchange graph $\text{EG}^{s}(\mathcal{D}_{fd}(\Gamma))$;
- the reachable silting sets in $\text{per }\Gamma$ as vertices and mutation as edges for the silting exchange graph $\text{SEG}^{s}(\text{per}(\Gamma))$;
- the cluster tilting sets in $\mathcal{C}(\Gamma)$ as vertices and mutation as edges for the cluster exchange graph $\text{CEG}(\mathcal{C}(\Gamma))$.

They play a crucial role in studying cluster algebras, quantum dilogarithm identities and stability conditions (cf. e.g. [26]). By simple-projective duality, there is a canonical isomorphism (9.3) between the first two graphs. Moreover, they are coverings of the third (see (9.5) and cf. [19]) by the spherical twist group action we mentioned above.

1.3. Triangulations of marked surfaces. A geometric aspect of cluster theory was explored by Fomin-Shapiro-Thurston (FST). They constructed a quiver $Q_T$ for each (tagged) triangulation $T$ of a marked surface $S$ and showed that flipping triangulations corresponds to FZ mutation of quivers. Here, the marked surface $S$ is a surface with marked points on its boundaries and punctures in its interior. Further, Labardini-Fragoso gave a rigid potential $W_T$ for each FST quiver $Q_T$, which is the unique ‘good’ (rigid, to be precise) one (cf. [12]), that is compatible with DWZ mutation. Then one can construct the Ginzburg dg algebra $\Gamma_T = \Gamma(Q_T, W_T)$ and the associated categories, as in (1.1).

For each, triangulation $T$, one can construct a cluster category $\mathcal{C}(\Gamma_T)$. Moreover, there is an unique way to identify all them compatibly (cf., e.g. [28, Section 2.2]). In other words, the cluster category $\mathcal{C}_T$ can be regarded independent of the choice of $T$. Denote it by $\mathcal{C}_S$ by abuse of notation. $S$ provides a good geometric model for $\mathcal{C}_S$ in the sense that (due to [6] and [28])

- there is a bijection $\rho$ between the (simple) open arcs in $S$ and rigid indecomposables in $\mathcal{C}_S$;
- $\rho$ induces a canonical bijection ([10]) between the exchange graph $\text{EG}(S)$ of $S$ (whose vertices are the triangulations and whose edges correspond to flips) and the cluster exchange graph $\text{CEG}(\mathcal{C}_S)$.
- the intersection numbers between arcs in $S$ correspond to the dimensions of Ext^1 between the corresponding objects in $\mathcal{C}_S$.

We would like to construct a geometric model for the perfect/finite-dimensional derived category as well.
In this paper, we will deal the case when $S$ is unpunctured and introduce a new surface from $S$ by decorating it with a set $\triangle$ of ‘decorated’ points as a geometric model for these categories. The number of points in $\triangle$ equals the number of triangles in any triangulation of $S$. This decorating idea already appeared in various contexts (e.g. Krammer [21] and Gaiotto-Moore-Neitzke [11]). These points could be branched while constructing a double cover of $S$ (cf. Figure 21); they might also correspond to the simple zeroes of some quadratic differentials (cf. Figure 10). Further, when considering the mapping class group of $S_\triangle$, these decorated points are serving as punctures in topology; however, we reserve the terminology ‘punctures’ for the FST setting of marked surfaces.

Denote such a surface by $S_\triangle$ and called it the decorated marked surface. A triangulation of $S_\triangle$ is a maximal collection of simple open arcs that divides $S_\triangle$ into triangles such that each triangle contains exactly one decorated point. One important feature of $S_\triangle$ is that flipping a triangulations has directions (cf. Section 3.2). Then we obtain a list of correspondences, as shown in Table 1. Simple closed arcs, i.e. the simple arcs connecting decorated points, play the crucial role in the construction/proof of these correspondences. In the story of BS, they may correspond to stable objects (with respect to certain stability conditions) and finite-length trajectories (with respect to certain quadratic differentials). In particular, we have (cf. Remark 9.14)

- the simple-projective duality between hearts and silting sets (cf. (9.3)) corresponds to the graph duality (for triangulations);

| Geometric side | Categorical side |
|----------------|-----------------|
| Braid twists   | $\cong$         |
| Spherical twists |
| Simple closed arcs in $S_\triangle$ | $1-1$ up to $[1]$ |
| Spherical obj. in $D_{fd}(\Gamma_T)$ |
| Dual Tri. with Whitehead moves | Hearts with simple tilting |
| Dual Tri. with Whitehead moves | Hearts with simple tilting |
| Sim.-proj. dual |
| Reachable open arcs in $S_\triangle$ | $1-1$ |
| Reachable ind. in per $\Gamma_T$ |
| Triangulations with flips | Silting with mutation |
| S_\triangle | per $\Gamma_T$ |
| $C_S$ | Quotient map |
| Open arcs in $S$ | $1-1$ |
| Rigid ind. in $C_S$ |
| Triangulations with flips | Cluster tilting with mutation |
• Amiot’s quotient map \( \pi \) in (1.1) corresponds to the map \( F \) from the set of open arcs in \( S_\Delta \) to the set of open arcs in \( S \), (induced by the forgetful map \( F: S_\Delta \to S \));
• the shift functor on \( \text{per} \Gamma_S \) and \( \mathcal{C}_S \) correspond to the (universal) rotation on \( S_\Delta \) and \( S \), respectively.

One of the motivations for giving these correspondence is to study the existence of maximal green mutation sequences (cf. Remark 9.18) that is not only interested mathematically but physically (see comments in [16] and [5]).

1.4. Geometrical representations of braid groups. The geometric definition of the braid group \( \text{Br}(A_n) \) is the mapping class group \( \text{MCG}(D) \) of a disk \( D \) with \( n + 1 \) decorated points (serving as topological punctures), whose generators are the braid twists (cf. Definition 3.6 and Figure 5). The original proof ([22]) of the faithfulness of the spherical twist group (of type A) is based on proving a double grading formula between intersection numbers (between closed arcs) and dimensions of full \( \text{Hom} \) (between the corresponding spherical objects). To generalize this, the key observation is that the decomposition [22, Figure 11] of the disk \( D \) should really be thought as a triangulation, i.e. Figure 21. In fact, Birman-Hilden double cover could be constructed from the triangulation of a decorated disk (cf. Figure 12). Then a natural generalization of the braid group for a decorated marked surface \( S_\Delta \) would be the subgroup of its mapping class group generated by braid twists along simple closed arcs. That is our braid twist group \( \text{BT}(S_\Delta) \).

To prove that \( \text{BT}(S_\Delta) \) is isomorphic to the corresponding spherical twist group \( \text{ST}(\Gamma) \), we construct the perfect dg modules of the Koszul dual of Ginzburg dg algebra \( \Gamma \) from closed arcs. This is a differential version of KS construction (of ‘projective resolutions’). Further, instead of following their remarkable but very complicated double grading formula, we use another intersection formula (6.4) (between open and closed arcs). In this way, we obtain a proof that can be generalized to all marked surface without punctures.

For the marked surfaces with punctures, our model, as well as KS double grading formula, fails. This is explained in Remark 4.6. As mentioned there, we need to lift a braid twist to a Dehn twist, in the King-Qiu twisted surfaces ([20]) to have a chance proving the corresponding statement, that the Dehn twist group is isomorphic to the spherical twist group. Note that their twist surfaces are a modification of the branched double cover of \( S_\Delta \). In fact, in the unpunctured case, no modification is needed (cf. Section 7.3) and the Dehn twist group is isomorphic to the braid twist group due to Birman-Hilden. Thus, a corollary of our main result (Corollary 7.11) says that the spherical twist group is isomorphic to the Dehn twist group of the double cover, of the corresponding marked surface branching at those decorated points.

1.5. Plan of the paper. In Section 2 we review the relative background. Then, in Section 3 and Section 4, we introduce and study the braid twist groups of decorated marked surfaces.

In Section 5, we construct the string model, that associates a dg module \( \tilde{X}_\eta \) (up to shifts) in the 3-CY category \( D_{\text{fd}}(\Gamma_T) \) to a closed arc \( \eta \) in the decorated marked surface
In Section 6, we show that such a dg module is always spherical. In Section 7, we prove the main result of the paper (Theorem 7.7, cf. Remark 9.10):

**Theorem.** Suppose $S$ is a marked surface without punctures. Let $S\triangle$ be a decorated marked surface of $S$ and $T$ a triangulation of $S\triangle$. Denote by $\Gamma_T$ the Ginzburg dg algebra associated to $T$. Then there is a canonical isomorphism

$$\iota : BT(T) \rightarrow ST(\Gamma_T),$$

sending braid twists in the mapping class group of $S\triangle$ to spherical twists in the autoequivalence group of $D_{fd}(\Gamma_T)$.

Note that, we in fact exclude two special cases in Theorem 7.7. However, they are dealt with separately in Section 8, which will complete the theorem above. In particular, in the case when $S$ is an annulus, we obtain the faithfulness of the spherical twist group action (as affine braid group, see Theorem 8.4).

In Section 9, we give a geometric realization of simple-projective duality and Amiot’s quotient, by constructing a bijection (Theorem 9.13) between reachable rigid indecomposables in $\text{per } \Gamma$ and open arcs in $S\triangle$. Further, we will show that shift in $\text{per } \Gamma$ corresponds to universal rotation on $S\triangle$.

In Section 10, we discuss the stability conditions and prove (Theorem 10.4) that the space of stability condition is simply connected in the CY-3 affine $\tilde{A}$ case.

In Section 11, we discuss some further studies.

**Acknowledgements.** This work was inspired during joint working with Alastair King on the twin paper [20]. I would like to thank Tom Bridgeland and Ivan Smith for explaining their beautiful paper [4]; Alastair King, Yu Zhou, Dong Yang and Bernhard Keller for useful discussion; Luis Paris and Dan Margalit for sharing their expertise in topology.

2. Preliminaries

2.1. Quivers with potential and Ginzburg algebras. Fix an algebraically closed field $k$ and all categories are $k$-linear. Denote by $\Gamma = \Gamma(Q,W)$ the Ginzburg dg algebra (of degree 3) associated to a quiver with potential $(Q,W)$, which is constructed as follows (cf. [18]):

- Let $Q^3$ be the graded quiver whose vertex set is $Q_0$ and whose arrows are:
  - the arrows in $Q_1$ with degree 0;
  - an arrow $a^* : j \rightarrow i$ with degree $-1$ for each arrow $a : i \rightarrow j$ in $Q_1$;
  - a loop $e^*_i : i \rightarrow i$ with degree $-2$ for each vertex $i$ in $Q_0$.

The underlying graded algebra of $\Gamma(Q,W)$ is the completion of the graded path algebra $kQ^3$ in the category of graded vector spaces with respect to the ideal generated by the arrow of $Q^3$.

- The differential of $\Gamma(Q,W)$ is the unique continuous linear endomorphism, homogeneous of degree 1, which satisfies the Leibniz rule and takes the following values:
  - $d_a = 0$ for any $a \in Q_1$,
  - $d a^* = \partial_a W$ for any $a \in Q_1$ and
  - $d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]$. 

Example 2.1. Let $Q$ be a 3-cycle with edges $a, b, c$ and the potential $W = abc$. Then the (graded) quiver $Q^3$ is

$Q^3$: 

and the (non-trivial) differentials are

$$d(a^*) = bc, \ d(b^*) = ca, \ d(c^*) = ab, \ d(e_1) = cc^* - bb^*, \ d(e_2) = bb^* - a^*a, \ d(e_3) = aa^* - c^*c.$$ (2.2)

In this paper, the quivers with potential we are considering are rigid (and hence non-degenerated), which basically means that they behave nicely under mutation, in the sense of DWZ. For details about these notions, see, e.g. [18] and [12].

2.2. The 3-Calabi-Yau categories. A triangulated category $\mathcal{D}$ is called $N$-Calabi-Yau ($N$-CY) if, for any objects $X, X'$ in $\mathcal{D}$ we have a natural isomorphism

$$\mathcal{S} : \text{Hom}^\bullet_{\mathcal{D}}(X, X') \overset{\sim}{\rightarrow} \text{Hom}^\bullet_{\mathcal{D}}(X', X)^{\vee}[N].$$ (2.3)

Note that the graded dual of a graded $k$-vector space $V = \oplus_{k \in \mathbb{Z}} V_k[k]$ is

$$V^{\vee} = \bigoplus_{k \in \mathbb{Z}} V_k^*[−k].$$

Further, an object $S$ is $N$-spherical if $\text{Hom}^\bullet(S, S) = k \oplus k[−N]$ and (2.3) holds functorially for $X = S$ and $X'$ in $\mathcal{D}$.

Denote by $\mathcal{D}_{fd}(\Gamma)$ the finite-dimensional derived category of $\Gamma$. It is well-known that this is a 3-CY category. We also know that (see, e.g. [17]) $\mathcal{D}_{fd}(\Gamma)$ admits a canonical heart $\mathcal{H}_\Gamma$ generated by simple $\Gamma$-modules $S_e$, for $e \in Q_0$, each of which is 3-spherical. Recall that the twist functor $\phi$ of a spherical object $S$ is defined by

$$\phi_S(X) = \text{Cone} (S \otimes \text{Hom}^\bullet(S, X) \rightarrow X)$$ (2.4)

with inverse

$$\phi_S^{-1}(X) = \text{Cone} (X \rightarrow S \otimes \text{Hom}^\bullet(X, S)^{\vee}) [−1]$$

Denote by ST($\Gamma$) the spherical twist group of $\mathcal{D}_{fd}(\Gamma)$ in $\text{Aut} \mathcal{D}_{fd}(\Gamma)$, generated by $\{\phi_{S_e} \mid e \in Q_0\}$. By [30, Lemma 2.11], we have the formula

$$\phi_{\psi(S)} = \psi \circ \phi_S \circ \psi^{-1}$$ (2.5)

for any spherical object $S$ and $\psi \in \text{Aut} \mathcal{D}_{fd}(\Gamma)$.

Denote by Sph($\Gamma$) the set of reachable spherical objects in $\mathcal{D}_{fd}(\Gamma)$, that is,

$$\text{Sph}(\Gamma) = \text{ST}(\Gamma) \cdot \text{Sim} \mathcal{H}_\Gamma,$$ (2.6)

where Sim $\mathcal{H}$ denotes the set of simples of an abelian category $\mathcal{H}$. See Lemma 9.2 for the justification of the terminology ‘reachable’ here.
Definition 2.2. Two elements $\psi$ and $\psi'$ in $\text{Aut} \mathcal{D}_{fd}(\Gamma)$ are isotopy, denote by $\psi \sim \psi'$, if $\psi^{-1} \circ \psi'$ acts trivially on $\text{Sph}(\Gamma)$. In this paper, we will only consider the auto-equivalences up to isotopy, i.e. we will consider $\text{ST}(\Gamma)$ as a subgroup of $\text{Aut} \mathcal{D}_{fd}(\Gamma)/\sim$.

Remark 2.3. We have the following observations.

- The twist functor is an invariant of shifting, i.e. $\phi_S = \phi_S[1]$ for any spherical object $S$. Thus the twist functor can be defined on $\text{Sph}(\Gamma)/[1]$.
- Clearly, for any $\phi \in \text{ST}(\Gamma)$ and $X \in \text{Sph}(\Gamma)$, $\phi(X)$ is still in $\text{Sph}(\Gamma)$.
- By (2.5), $\text{ST}(\Gamma)$ is the subgroup of $\text{Aut} \mathcal{D}_{fd}(\Gamma)$ generated by all $\phi_X$ for $X \in \text{Sph}(\Gamma)$ (cf. [19]).

2.3. Triangulations of marked surfaces. Throughout the paper, $S$ denotes a marked surface without punctures in the sense of [10], that is, a connected compact surface with a fixed orientation with a finite set $M$ of marked point on the (non-empty) boundary $\partial S$ satisfying that each connected component of $\partial S$ contains at least one marked point. Up to homeomorphism, $S$ is determined by the following data

- the genus $g$;
- the number $|\partial S|$ of boundary components;
- the integer partition of $|M|$ into $|\partial S|$ parts describing the number of marked points on its boundary.

As in [10, p5], we will exclude the case when there is no triangulation or there is no arcs in the triangulation. In other wards, we require $n \geq 1$ in (2.7).

An (open) arc in $S$ is a curve (up to homotopy) that connects two marked points in $M$. The intersection number is defined to be $$\text{Int}(\gamma_1, \gamma_2) = \min\{|\gamma'_1 \cap \gamma'_2 \cap (S - M)| \mid \gamma_i \sim \gamma'_i\}.$$ An (ideal) triangulation $T$ of $S$ is a maximal collection of compatible simple arcs. Here, compatible means any two arcs in $T$ that do not intersect. Moreover, it is well-known that any triangulation $T$ of $S$ consists of

$$n = 6g + 3|\partial S| + |M| - 6 \quad (2.7)$$

(simple) arcs and divides $S$ into

$$N = \frac{2n + |M|}{3} \quad (2.8)$$

triangles. Denote by $\text{EG}(S)$ the exchange graph of triangulations of $S$, that is, the unoriented graph whose vertices are triangulation of $S$ and whose edges correspond to flips (see the lower pictures in Figure 3 for a flip). It is known that $\text{EG}(S)$ is connected. If $S$ is an $(n + 3)$-gon, then $\text{EG}(S)$ is the associahedron of dimension $n$ (cf. Figure 1).

Let $S$ be a marked surface and $T$ a triangulation of $S$. Then there is an associated quiver $Q_T$ with a potential $W_T$, constructed as follows (See, e.g. [12] or [28] for the precise definition):

- the vertices of $Q_T$ are (indexed by) the arcs in $T$;
- for each triangle $T'$ in $T$, there are three arrows between the corresponding vertices as shown in Figure 2;
3. **Triangulations of decorated marked surfaces**

3.1. **Decorated marked surfaces.** Recall that any triangulation of $S$ consists of $\aleph$ triangles, where $\aleph$ is given by the formula (2.8).

**Definition 3.1.** The *decorated marked surface* $S_\Delta$ is a marked surface $S$ together with a fixed set $\Delta$ of $\aleph$ ‘decorated’ points (in the interior of $S$, where $\aleph$ is defined in (2.8)), which serve as punctures. Moreover,

- a **closed arc** in $S_\Delta$ is (the isotopy class of) a curve in $S_\Delta - \Delta$ that connects two decorated points in $\Delta$. Usually, we will only consider closed arcs connecting different decorated points. The case when the endpoints of a closed arc coincide will only be treated in Appendix B. Denote by $\text{CA}(S_\Delta)$ the set of simple closed arcs that connect different decorated points;
- an **open arc** in $S_\Delta$ (or $S$) is (the isotopy class of) a curve in $S_\Delta - \Delta$ that connects two marked points in $M$.

The *intersection numbers* between arcs in $S_\Delta$ are defined as follows:
I. For an open arc $\gamma$ and an arc $\eta$ (open or closed), their intersection number is the geometric intersection number in $S_\triangle - M$:
\[
\text{Int}(\gamma, \eta) = \min \{|\gamma' \cap \eta' \cap (S_\triangle - M)| \mid \gamma' \sim \gamma, \eta' \sim \eta\}.
\]

II. For two closed arcs $\alpha \neq \beta$ in $\text{CA}(S_\triangle)$, their intersection number is an half integer in $\frac{1}{2}\mathbb{Z}$ and defined as follows (following [22]):
\[
\text{Int}(\alpha, \beta) = \frac{1}{2} \text{Int}_\triangle(\alpha, \beta) + \text{Int}_{S_\triangle}(\alpha, \beta),
\]
where
\[
\text{Int}_L(\alpha, \beta) = \min \{|\beta \cap \eta' \cap L| \mid \alpha' \sim \alpha, \beta' \sim \beta\}, \quad (3.1)
\]
for $L = \triangle, S_\triangle - \triangle$. Further, by convention, $\text{Int}(\alpha, \alpha) = 1$ for any closed arc $\alpha$ in $\text{CA}(S_\triangle)$.

3.2. Triangulations and flips (after Krammer).

**Definition 3.2.** A triangulation $T$ of $S_\triangle$ is a maximal collection of open arcs such that
- for any $\gamma_1, \gamma_2 \in T$, $\text{Int}(\gamma_1, \gamma_2) = 0$;
- $T$ is compatible with $\triangle$ in the sense that the open arcs in $T$ divide $S_\triangle$ into $\aleph$ triangles, each of which contains exactly one point in $\triangle$.

Let $T$ be a triangulation of $S_\triangle$ (consisting of $n$ open arcs). The dual triangulation $T^*$ of $T$ is the collection of $n$ closed arcs in $\text{CA}(S_\triangle)$, such that every closed arc only intersects one open arc in $T$ and with intersection one. See the left picture of Figure 17 for an example. More precisely, for $\gamma$ in $T$, the corresponding closed arc in $T^*$ is the unique open arc $s$, that is contained in the quadrilateral $A$ with diagonal $\gamma$, connecting the two decorated points in $A$ and intersecting $\gamma$ only once (cf. left picture of Figure 12). We will call $s$ and $\gamma$ the dual of each other, with respect to $T$ (or $T^*$).

There is a canonical map, the forgetful map
\[
F: S_\triangle \to S,
\]
forgetting the decorated points. Clearly, $F$ induces a map from the set of open arcs in $S_\Delta$ to the set of open arcs in $S$. And the image of a triangulation $T$ is a triangulation $T = F(T)$. The (FST) quiver $Q_T$ associated to $T$ is defined to be the (FST) quiver $Q_T$ that associated to $T = F(T)$.

There is the notion of (forward/backward) flip of triangulations of $S_\triangle$ (after [21] and cf. [20]).

**Definition 3.3.** Let $\gamma$ be an open arc in a triangulation $T$ of $S_\Delta$. The arc $\gamma^\sharp = \gamma^\sharp(T)$ is the arc obtained from $\gamma$ by anticlockwise moving its endpoints along the quadrilateral in $T$ whose diagonal is $\gamma$ (cf. upper pictures of Figure 3), to the next marked points. The forward flip of a triangulation $T$ of $S_\Delta$ at $\gamma \in T$ is the triangulation $T^\sharp_\gamma$ obtained from $T$ by replacing the arc $\gamma$ with $\gamma^\sharp$.

Similarly, we can define arc $\gamma^\flat = \gamma^\flat(T)$ is the arc obtained from $\gamma$ by clockwise moving its endpoints and the backward flip $T^\flat_\gamma$ of $T$ at $\gamma \in T$ is the triangulation $T^\flat_\gamma$ obtained from $T$ by replacing the arc $\gamma$ with $\gamma^\flat$. 
Clearly, these two flips are inverse to each other. Also note that under the forgetful map $F$, a forward/backward flip in $S_\Delta$ becomes a normal flip (without direction) in $S$, cf. Figure 3.

**Definition 3.4.** The exchange graph $EG(S_\Delta)$ is the oriented graph whose vertices are triangulations of $S_\Delta$ and whose edges correspond to forward flips between them.

**Remark 3.5.** Recall that $\pi_1 EG(S)$ is generated by squares and pentagons ([10, Theorem 3.10]). By [21], forward flips also satisfy the square and pentagon relations (cf. Figure 4). we believe that so is $\pi_1 EG(S_\Delta)$.

### 3.3. The braid twists.

The mapping class group $MCG(S_\Delta)$ is the group of isotopy classes of homeomorphisms of $S_\Delta$, where all homeomorphisms and isotopies are required to

- fix $\partial S_\Delta (\supset M)$ pointwise;
- fix the decorated points set $\Delta$ (but allow to permute points in it).

Note that the mapping class group $MCG(S)$ of $S$ will require only the first condition and thus there is a canonical map

$$F_* : MCG(S_\Delta) \to MCG(S)$$

induced by the forgetful map $F$. As $MCG(S)$ is generated by Dehn twists along simple closed curves (that misses the decorated points), $F_*$ is clearly surjective.

For any closed arc $\eta \in CA(S_\Delta)$, there is the (positive) *braid twist* $B_\eta \in MCG(S_\Delta)$ along $\eta$, which is shown in Figure 5. Further, there is the following well-known formula

$$B_{\Psi(\eta)} = \Psi \circ B_\eta \circ \Psi^{-1},$$

for any $\Psi \in MCG(S_\Delta)$.

**Definition 3.6.** The *braid twist group* $BT(S_\Delta)$ of the decorated marked surface $S_\Delta$ is the subgroup of $MCG(S_\Delta)$ generated by the braid twists $B_\eta$ for $\eta \in CA(S_\Delta)$.
Figure 4. The pentagon relation for forward flips

Figure 5. The Braid twist

**Example 3.7.** If $\text{Int}(\alpha, \beta) = \frac{1}{2}$, there is a closed arc $\eta$ (cf. Figure 6) such that

$$\eta = B_\alpha(\beta) = B_\beta^{-1}(\alpha), \quad \alpha = B_\beta(\eta) = B_{\eta}^{-1}(\beta), \quad \beta = B_\eta(\alpha) = B_{\alpha}^{-1}(\eta). \quad (3.4)$$

Note that $\eta$ is the closed arc such that the interior of the triangle formed by $\alpha, \beta, \eta$ is contractible. In fact, there is exactly one more such closed arc (dashed arc in Figure 6), namely

$$\eta' = B_\alpha^{-1}(\beta) = B_\beta(\alpha),$$

satisfying the triangle formed by $\alpha, \beta$ and which is contractible.

We have the following straightforward observation:
Lemma 3.8. Let \( \gamma \) be a open arc in \( T \) and \( s \) be its dual in \( T^* \). Then in the triangulation \( T^* \), the dual of \( \gamma^b \) is still \( s \). Moreover, let \( T^i_\eta \) and \( T^b_\eta \) be the two flips of \( T \) at \( \gamma \). Then
\[
\gamma^b = B_s(\gamma^b), \quad T^b_\gamma = B_s(T^i_\gamma).
\]

Proof. The first claim follows from the upper pictures in Figure 3 and the equations follow from Figure 7. \( \square \)

As a consequence, we obtain a map between exchange graphs.

Lemma 3.9. As graphs, we have the following surjective map induced by the forgetful map \( F \):
\[
F_* : \text{EG}(S_\Delta)/\text{BT}(S_\Delta) \to \text{EG}(S). \tag{3.5}
\]

Proof. Recall that there is a canonical surjection \( F_* : \text{MCG}(S_\Delta) \to \text{MCG}(S) \) in (3.2). By definition, it is straightforward to see that
\[
\text{BT}(S_\Delta) \subset \ker F_* \tag{3.6}
\]
Thus, \( F \) induces a quotient map \( F_* : \text{EG}(S_\Delta)/\text{BT}(S_\Delta) \to \text{EG}(S) \) between sets. Next, by definition, \( \text{EG}(S_\Delta) \) is a \((n,n)\)-regular graph (that is, every vertex has \( n \) arrows in and \( n \) arrow out) and \( \text{EG}(S) \) is a \( n \)-regular graph. Moreover, the \( F_* \) preserves the edges (cf. Figure 3), in the sense that the forward and backward flips of a triangulation \( T \) at some closed \( \gamma \) both become the flip of \( T = F(T) \) at \( F(\gamma) \). Thus, \( F_* \) is a map between graphs. Since they are both regular graphs, we deduce that \( F \) is surjective. \( \square \)
Remark 3.10. In fact, if we take any connected component $EG^\chi(S_{\triangle})$ of $EG(S_{\triangle})$, then $F_\ast$ induces an isomorphism

$$F_\ast: EG^\chi(S_{\triangle})/BT(S_{\triangle}) \cong EG(S)$$

since $EG(S)$ is connected and both graphs are $n$-regular.

3.4. The initial triangulation.

Remark 3.11. Due to technique reason, we will exclude two special cases for the moment:

I). $S$ is an annulus with one marked point on each of its boundary components;

II). the torus with only boundary component and one marked point.

These cases will be discuss independently in Section 8.

Lemma 3.12. There exists a triangulation $T$ of $S_{\triangle}$ such that any two triangles share at most one edge. In other words, the quiver $Q_T$ has no double arrows.

Proof. The second statement, which is equivalent to the first one, follows from Proposition 7.13, noticing that we have exclude the two special cases (a torus with one marked point and an annulus with two marked points). □

Notations 3.13. We will fix such a triangulation $T_0$ such that its image $T_0 = F(T_0)$ (a triangulation of $S$) satisfies the condition in Lemma 3.12. Let

- $T_0 = \{\gamma_1, \ldots, \gamma_n\}$
- $T_0^\ast = \{s_1, \ldots, s_n\}$

where $s_i$ is the dual of $\gamma_i$ with respect to $T_0$. Denote by $EG^\circ(S_{\triangle})$ the connected component of $EG(S_{\triangle})$ that contains $T_0$.

We say a curve is in a minimal position with respect to $T_0$, if it has minimal intersections with (arcs in) $T_0$. Let $\text{Int}(T_0, \eta) = \sum_{i=1}^n \text{Int}(\gamma_i, \eta)$. Then a representative $\eta$ is in a minimal position if it intersects $T_0$ exactly $\text{Int}(T_0, \eta)$ times. We will repeatedly use induction on $\text{Int}(T_0, \eta)$ later and the next lemma will be used repeatedly.

Lemma 3.14. Suppose that a closed arc $\eta$ in $CA(S_{\triangle})$ that is not a closed arc $s$ in $T_0^\ast$. Then there are two closed arcs $\alpha, \beta$ in $CA(S_{\triangle})$ such that

$$\text{Int}(\alpha, \beta) = \frac{1}{2} \quad \text{and} \quad \eta = B_\alpha(\beta), \quad (3.7)$$

$$\text{Int}(T_0, \eta) = \text{Int}(T_0, \alpha) + \text{Int}(T_0, \beta). \quad (3.8)$$

Proof. Recall that we require any two triangles in $T_0$ share at most one edge. Thus if $\eta$ only intersects two triangles of $T_0$, then $\eta = s_j \in T_0$ for some $j$ and the lemma follows directly. Now suppose that $\eta$ intersects at least three triangles of $T_0$. Then one of the decorated points in these triangles is not the endpoints of $\eta$. Denote the triangle by $\Lambda_0$ with the decorated point $Z_0$ inside. Choose a representative of $\eta$, also denote by $\eta$ when there is no confusion, such that it is in a minimal position with respect to $T_0$. One can draw a line segment $l$ from $Z_0$ to some point $Y$ of $\eta$ within $\Lambda_0$ such that $l$ doesn’t intersect $\eta$ except at the endpoints (cf. Figure 8). Let $Z_1$ and $Z_2$ be the endpoints of $\eta$ such that $l$ is in the left side when pass from $Z_2$ to $Z_1$. Consider two closed arcs $\alpha$ and $\beta$ which are isotopy to $l \cup \eta |_{Z_1Y}$ and $l \cup \eta |_{Z_2Y}$ respectively (cf. Figure 9). Clearly,
we have \( \text{Int}(\alpha, \beta) = \frac{1}{2} \). Noticing that the triangle formed by \( \alpha, \beta, \eta \) is contractible, we deduce that \( \eta \) is one of \( B_\alpha(\beta) \) and \( B_\alpha^{-1}(\beta) \). By the choice of \( Z_1 \) and \( Z_2 \), we know that \( \eta \) is \( B_\alpha(\beta) \). Since \( \eta \) is in a minimal position (with respect to \( T_0 \)), so are \( \alpha \) and \( \beta \). Thus, we have (3.8) as required.

\[ \square \]

**Figure 8.** The line segment \( l \)

**Figure 9.** Decompose \( \eta \)

### 4. On the Braid Twist Groups

#### 4.1. Generators

Recall that we have the braid twist group for \( S_\Delta \). Now we define the braid twist group for \( T_0 \).

**Definition 4.1.** Let \( T \) be a triangulation of \( S_\Delta \). The braid twist group \( BT(T) \) of the triangulation \( T \) is the subgroup of \( \text{MCG}(S_\Delta) \) generated by the braid twists \( B_s \) for the closed arcs \( s \) in \( T^* \).

In fact, these two groups are the same.

**Lemma 4.2.** \( BT(S_\Delta) = BT(T_0) \).

**Proof.** Use induction on \( \text{Int}(T_0, \eta) \) to show that \( B_\eta \) is in \( BT(T_0) \). If so, then \( BT(S_\Delta) \subset BT(T_0) \). Clearly, \( BT(S_\Delta) \supseteq BT(T_0) \) and therefore the lemma follows.

If \( \text{Int}(T_0, \eta) = 1 \), then \( \eta \in T_0^* \) and the claim is trivial. Suppose that the claim holds for any \( \eta' \) with \( \text{Int}(T_0, \eta') < m \). Consider a closed arc \( \eta \in CA(S_\Delta) \) with \( \text{Int}(T_0, \eta) = m \). Applying Lemma 3.14, we have \( \eta = B_\alpha(\beta) \) for some \( \alpha, \beta \) with (3.8). By inductive assumption, \( B_\alpha \) and \( B_\beta \) are in \( BT(T_0) \). By formula (3.3), we have

\[
B_\eta = B_{B_\alpha(\beta)} = B_\alpha \circ B_\beta \circ B_\alpha^{-1} \in BT(T_0),
\]

which completes the proof. \( \square \)
Proposition 4.3. $BT(S_\Delta) = BT(T)$ for any $T \in EG(S_\Delta)$.

Proof. First, if $T_1$ and $T_2$ are related by a flip, then their dual graph are related by a Whitehead move, with respect to the corresponding closed arc $\eta$ (which is unchanged during the flip), see Figure 10. Notice that the changes of closed arcs in $T_1^*$ are given by the braid twist $B^{\pm 1}_\eta$. Then by (3.3) it is straightforward to see that $BT(T_1) = BT(T_2)$. By Lemma 4.2, the proposition holds for any $T \in EG^o(S_\Delta)$.

As for $T$ in other connected component of $EG(S_\Delta)$, we can always find one triangulation in that component satisfying the condition in Lemma 3.12. Then Lemma 4.2 would apply to that triangulation and thus the proposition holds for any $T \in EG(S_\Delta)$. □

![Figure 10. The Whitehead move, as the flip of the dual triangulations (red)](image)

Besides, the closed arcs are ‘reachable’, in the following sense.

Proposition 4.4. Let $T \in EG(S_\Delta)$. For any $\eta \in CA(S_\Delta)$, there exists $s \in T^*$ and $b \in BT(S_\Delta)$ such that $\eta = b(s)$, i.e.

$$CA(S_\Delta) = BT(S_\Delta) \cdot T^*.$$

Proof. Consider the case when $T = T_0$ first. Then this follows easily by induction on $\text{Int}(T_0, \eta)$, using Lemma 3.14. Second, by the Whitehead move (cf. Figure 10), if $T_1$ and $T_2$ are related by a flip, then

$$BT(S_\Delta) \cdot T_1^* = BT(S_\Delta) \cdot T_2^*.$$

Therefore the proposition holds for $T \in EG^o(S_\Delta)$. Finally, as in the last paragraph of the proof of Proposition 4.3, we deduce that the proposition holds for any $T \in EG(S_\Delta)$.

4.2. The braid groups. In this subsection, we recall the braid groups (a.k.a. Artin groups) for (affine) Dynkin diagrams and its relation with braid twist groups. We will restrict ourself to the case of (affine) type $A$ and $D$ since

- type E does not admit a faithful geometric representation (cf. [32]);
- type E quiver does not arise from triangulations of marked surfaces.
Suppose that $Q$ (which is quiver or a diagram in this case) is of Dynkin type in \((4.1)\) as follows first:
\[
\begin{align*}
A_n: & \quad 1 \quad 2 \quad \cdots \quad n \\
D_n: & \quad 1 \quad 2 \quad \cdots \quad n-1 \quad \cdots \quad n-2 \quad \cdots \quad n
\end{align*}
\]
Denote by $\text{Br}(Q)$ the braid group associated to $Q$, with generated by $b = \{ b_i \mid i \in Q_0 \}$ and the relations
\[
\begin{align*}
b_j b_i b_j = b_i b_j b_i, & \quad \text{there is exactly an arrow between } i \text{ and } j \text{ in } Q, \\
b_i b_j = b_j b_i, & \quad \text{otherwise.}
\end{align*}
\]
Recall, e.g., from [23], that the quasi-center of $\text{Br}(Q)$ is the subgroup of elements $\varsigma(Q)$ in $\text{Br}(Q)$ satisfying $\varsigma(Q) \cdot b \cdot \varsigma(Q)^{-1} = b$, where $b$ is the standard generating set of $\text{Br}(Q)$, and that this subgroup is an infinite cyclic group generated by a special element $\tilde{z}$ of $\text{Br}(Q)$, called \textit{fundamental element}. The center $Z(\text{Br}(Q))$ of $\text{Br}(Q)$ is an infinite cyclic group. The $z_Q = \varsigma(Q)$ generates $Z(\text{Br}(Q))$ if $Q$ is of type $D_n$ for even $n$ and $z_Q = \varsigma(Q)^2$ generates $Z(\text{Br}(Q))$ if $Q$ is of type $D_n$ for odd $n$ or type $A$. By [23, Proposition 2.8], we have
\[
\varsigma(A_n)^2 = \left( \prod_{j=1}^{n} b_j \right)^{n+1}, \\
\varsigma(D_{2k}) = \left( \prod_{j=1}^{2k} b_j \right)^{2k-1}, \quad \varsigma(D_{2k+1})^2 = \left( \prod_{j=1}^{2k+1} b_j \right)^{4k}.
\]
For every simple closed curve $C$ in a surface $X$, there is the notion of the (positive) \textit{Dehn twist} $D_C \in \text{MCG}(X)$ along $C$, which is shown in Figure 11. The precise definition can be found in [9, Section 3.1] and the definition can be extended to the case when $C$ is a boundary component of $X$. It is known that in the type $A$,
\begin{itemize}
\item $\text{Br}(A_n)$ can be identified with the mapping class group of $\text{MCG}(D)$, where $D$ is a disk with $n+1$ punctures,
\item the central generator $z_{A_n}$ is the Dehn twist $D_{\partial D}$.
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{DehnTwist.png}
\caption{The Dehn twist}
\end{figure}
Further, we can extend the definition of braid group to the case when $Q$ is of the affine Dynkin type in (4.2) as follows:

$$
\begin{align*}
\tilde{A}_n &: 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \\
0 &
\end{align*}
$$

(4.2)

and

$$
\begin{align*}
\tilde{D}_n &: 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-3 \\
0 &
\end{align*}
$$

It is known that in the affine type $\tilde{A}$ (cf. [8]),

- $\text{Br}(\tilde{A}_n)$ can be identified with a subgroup of the mapping class group $\text{MCG}(A)$ where $A$ is an annulus with $n$ punctures,
- the center of $\text{Br}(A_n)$ is trivial.

Note that:

- the type of a marked surface means the mutation-equivalent type of the quivers with potential associated to its triangulations.
- the braid group is defined for a diagram while the braid twist group is defined for a mutation-equivalent type of the quivers with potential.

Our philosophy is that the braid twist group is the generalization of the braid group, due to the following reasons.

**Example 4.5.** The braid twist group is (canonically) isomorphic to the braid group of type $A$ or $\tilde{A}$:

1. **$A$.** $S$ is a $(n + 3)$-gon and $S_\Delta$ is essentially a disk with $n + 1$ decorated points (serve as punctures). Thus, say by choosing the triangulation $T$ as in the lower picture of Figure 21, we know that $BT(S_\Delta) = BT(T)$ is canonical isomorphic to the braid group $\text{Br}(A_n)$.

2. **$\tilde{A}$.** $S$ is an annulus with $p$ and $q$ marked points on its two boundary components respectively. Then $S_\Delta$ is essentially an annulus with $p$ and $q$ decorated points on its two boundaries (serve as punctures), respectively. Thus, say by choosing the triangulation $T$ consisting of arcs that does not connecting the marked points in the same boundary components (e.g. the blue triangulation in the left picture of Figure 23), we know that $BT(S_\Delta) = BT(T)$ is canonical isomorphic to the braid group $\text{Br}(\tilde{A}_n)$ for $n = p + q$ by [8].

**Remark 4.6.** As for the marked surface with punctures, say when $S$ is a polygon with one or two punctures, the associated quivers with potential (from any tagged triangulation) is mutation-equivalent of type $D$ or $\tilde{D}$. However, the corresponding braid twist group (of $S_\Delta$) is not of type $D$ or $\tilde{D}$. Then one should not expect to generalize (directly) the results of this paper in the punctured case.

To solve this issue, one needs to construct another surface, the twisted surface $\Sigma_T$ (with respect to a tagged triangulation $T$ of $S$), so that the braid twist on $S_\Delta$ is replaced by the Dehn twist on $\Sigma_T$. See [20] for detailed construction of twisted surfaces...
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It is conjectured in [20] that the Dehn twist group DTG(Σ_T) of Σ_T, as a subgroup of the mapping class group MCG(Σ_T), is isomorphic to the corresponding spherical twist group for any surface S.

In the unpunctured case, the twisted surface Σ_T is, in fact, the branched double cover of S_Δ, branching at the points in Δ. Moreover, there is a nice correspondence between braid twists and Dehn twist (cf. Section 7.3). Thus the conjecture above follows from Theorem 7.7.

However, the point is that, in the punctured case, while our construction that links spherical twists and braid twists fails, the construction in [20] that links spherical twists and Dehn twists still has chance. Note that by [24], such a Dehn twist group DTG(Σ_T) is indeed isomorphic to the braid group of type D, when S is a once-punctured polygon. And we do expect the corresponding spherical twist group is also isomorphic to the braid group of type D.

4.3. Center of the braid twist groups. Let Z^{BT}_0 be the center of BT(S_Δ) and BT*(S_Δ) = BT(S_Δ)/Z^{BT}_0. By the previous discussion (Example 4.5), we have the following:

- If S_Δ is a polygon, then BT(S_Δ) ≅ Br(A_n) and Z^{BT}_0 is the infinite cyclic group generated by D_{∂S_Δ}.
- If S_Δ is an annuli, then BT(S_Δ) ≅ Br(Â_n) and Z^{BT}_0 = 1 ([8]).

We will show that Z^{BT}_0 = 1 holds for the rest of the cases. Denote the boundary components of S_Δ by ∂_j, 1 ≤ j ≤ |∂S|.

Lemma 4.7. By cutting along the (initial) closed arcs in T^*_0, S_Δ will be divided into m annuli A_i, 1 ≤ i ≤ m, such that ∂_i is a boundary component of A_i.

Proof. For each boundary segment γ ⊂ ∂S_Δ that is in a triangle T in T_0 with decorated point Z, denote by γ* its dual, which is the simple arc in T (unique up to isotopy) connecting Z and the midpoint of γ. Call the union of T^*_0 and arcs γ* as above (for all segment γ in S_Δ) the full dual of T_0. Denote it by T^*_0, see Figure 13 for example.

Then the surface S_Δ - T^*_0 obtained from S_Δ by cutting along all arcs in T^*_0 satisfies the following:

- it consists of |M| connected components, each of which contains exactly one marked point in M;

Figure 12. The braid twist versus Dehn twist
Figure 13. The full dual of a triangulation

- each component is a disk, since it can be obtained by gluing many quadrilaterals (cf. the shaded area in Figure 13) long some segment containing the marked point in that component.

Further, by gluing back long the arcs dual to boundary segments in $S_{\Delta}$, we deduce that surface $S_{\Delta} - T^*_0$ obtained from $S_{\Delta}$ by cutting along all arcs in $T^*_0$ satisfies the following:

- it consists of $|\partial|$ connected components;
- each component $A_i$ is an annulus, such that one of the boundary components of $A_i$ is a boundary component of $S_{\Delta}$.

Thus, the lemma follows. □

Proposition 4.8. If $S_{\Delta}$ is neither a polygon nor an annulus, then $Z_0^{BT} = 1$.

Proof. Denote by $D(\partial S_{\Delta})$ the subgroup of $\text{MCG}(S_{\Delta})$ generated by the Dehn twist $\{D_{\partial_i}\}$ of its boundary components. We claim that $Z_0^{BT} \subset D(\partial S_{\Delta})$. Let $z \in Z_0^{BT}$ and so $z \circ B_\eta = B_\eta \circ z$ for any $\eta \in CA(S_{\Delta})$. Then by (3.3) we have

$$B_z(\eta) = z \circ B_\eta \circ z^{-1} = B_\eta.$$ (4.3)

Thus $z(\eta) = \eta$ for any $\eta \in CA(S_{\Delta})$, which in particular implies $z$ preserves $\Delta$ pointwise (note that $|\Delta| = 8 \geq 3$ in our situation) and $T^*_0$. By Lemma 4.7, cutting along closed arcs in $T^*_0$ divides $S_{\Delta}$ into $m$ annuli $A_i$, such that $\partial_i$ is a boundary component of $A_i$. Since $z$ preserves all such closed arcs, it can be realized as composition of some element $z_i \in \text{MCG}(A_i)$ (where the order of the composition does not matters). Note that $\text{MCG}(A_i)$ is generated by $D_{\partial_i}$, which implies $z \subset D(\partial S_{\Delta})$. Thus the claim holds.

Note that we also have $D(\partial S)$, the subgroup of $\text{MCG}(S)$ generated by the Dehn twist along its boundary components. Clearly, $F_* (D(\partial S_{\Delta})) = D(\partial S)$, which sends $D_{\partial_i}$ to itself. Since $S$ is not a polygon, $\{D_{\partial_i}\}$ are non-trivial in both $\text{MCG}(S_{\Delta})$ and $\text{MCG}(S)$. Further, since $S$ is not an annulus, $\{D_{\partial_i}\}$ are distinct and commutes with each other. Therefore, $F_* : D(\partial S_{\Delta}) \to D(\partial S)$ is an isomorphism.

Now, by (3.6), $F_*(Z_0^{BT}) = 1$ in $\text{MCG}(S)$ and we deduce that $Z_0^{BT} = 1$ in $\text{MCG}(S_{\Delta})$. □
5. From closed arcs to perfect objects

5.1. The Koszul dual. Let $\Gamma_T = \Gamma(Q_T, W_T)$ be the Ginzburg dg algebra from a triangulation $T$. Recall that there is a canonical heart $\mathcal{H}_T$ in $D_{fd}(\Gamma_T)$ and let

$$S_T = \bigoplus_{S \in \mathcal{H}_T} S$$

to be the direct sum of the simples in $\mathcal{H}_T$. Consider the (dg) endomorphism algebra

$$\mathcal{E}_T = R\text{Hom}(S_T, S_T). \quad (5.1)$$

By [17], we have the following derived equivalence:

$$D_{fd}(\Gamma_T) \xrightarrow{R\text{Hom}_{\Gamma_T}(S_T, ?)} \mathcal{E}_T, \quad (5.2)$$

We will identify these two categories when there is no confusion. In particular, $\{S\}_{S \in \mathcal{H}_T}$ in $D_{fd}(\Gamma_T)$ becomes (indecomposable) projectives in $\mathcal{E}_T$.

**Definition 5.1.** Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$ and

$$S = \bigoplus_{S \in \text{Sim} \mathcal{H}} S.$$ 

The Ext-quiver $Q(\mathcal{H})$ is the (positively) graded quiver whose vertices are the simples of $\mathcal{H}$ and whose graded edges correspond to a basis of $\text{End}^\bullet(S, S)$.

**Example 5.2.** The Ext-quiver of the canonical heart (in the corresponding 3-CY category) of the quiver with potential in Example 2.1 is shown as follows.

Moreover, the differentials in (2.2) become the following relations:

$$\text{Hom}^1(S_{i-1}, S_i) \otimes \text{Hom}^1(S_i, S_{i+1}) \cong \text{Hom}^2(S_{i-1}, S_{i+1}),$$

$$\text{Hom}^k(S_i, S_{i+k}) \otimes \text{Hom}^{3-k}(S_{i+k}, S_i) \cong \text{Hom}^3(S_i, S_i), \quad (5.4)$$

for $i = 1, 2, 3$ and $k = 0, 1, 2, 3$, where $S_{i+3} = S_i$ for $i \in \mathbb{Z}$.

**Notations 5.3.** Recall that we have fix an initial triangulation $\mathbf{T}_0$.

- We will write $\Gamma_0$ for $\Gamma_{\mathbf{T}_0}$. Similar for $\mathcal{E}_{\mathbf{T}_0}$, $\mathcal{H}_{\mathbf{T}_0}$ and so on.
- Let $\Gamma_i = e_i \Gamma_0$ be the indecomposable projective summands of $\Gamma_0$.
- Let $S_1, \ldots, S_n$ be the simples in $\mathcal{H}_0$, which corresponds to the projectives above.

Under the derived equivalence (5.2), $S_i$ becomes the (projective) summand of the silting objects $\mathcal{E}_0$ in $\text{per} \mathcal{E}_0$. 

$$\begin{align*}
\begin{array}{c}
\circ \quad 3 \\
\downarrow \quad \circ \\
S_2 \\
\downarrow \quad \circ \\
\circ \quad 1 \\
\downarrow \quad 2 \\
\circ \quad S_1 \\
\downarrow \quad 2 \\
\circ \quad S_3 \\
\downarrow \quad 3 \\
\circ
\end{array}
\end{align*} \quad (5.3)$$
5.2. The m-perfect dg modules. We consider a special class of perfect dg modules (cf. minimal perfect dg modules in Plamondon’s Ph.D thesis [25]).

**Definition 5.4.** A dg $E_0$-module $X$ is m-perfect if its underlying graded module (denoted by $|X|$) is of the form

$$|X| = \bigoplus_{k=1}^{l} X_k,$$

where each $X_k$ is a finite direct sum of shifted copies of direct summands of $E_0$ (i.e. copies of $S_j$), and if its differential is of the form $d_X = d_0 + d_1$, where $d_0$ is the direct sum of the differential of the $X_k$ and $d_1$ (the non-trivial part), as a degree 1 map from $X$ to itself, is a strictly upper triangular matrix whose entries are in the ideal of $E_0$ generated by the arrows (in $Q(H_0)$).

Clearly, any m-perfect dg $E_0$-module $X$ is in $\text{per } E_0$. Note that the arrows in $Q(H_0)$ might not be a ‘minimal generating’ set, but what we construct later will be ‘minimal’ in some sense.

The following lemma shows that it is easy to calculate the Hom’s from the silting object $\Gamma_0 \in \text{per } \Gamma_0$ to m-perfect dg $E_0$-modules in $\text{per } E_0$.

**Lemma 5.5.** Let $X$ be a m-perfect dg $E_0$-module in $\text{per } E_0$. Then $\dim \text{Hom}^j(\Gamma, X)$ equals the number of summands of $|X|$ that is isomorphic to $E_i[j]$ (or $S_i[j]$).

**Proof.** We have the following two facts. First, the projective-simple duality:

$$\text{Hom}^j(\Gamma, S_k[l]) = \delta_{ik} \cdot \delta_{jl} \cdot k, \quad 1 \leq i, k \leq n; \forall j, l \in \mathbb{Z}. \quad (5.6)$$

Second, since $d_1$ is generated by the arrows in $Q(H_0)$, we have

$$f_i(d_1) = 0, \quad \text{for } f_i = \text{Hom}^\bullet(\Gamma, ?), \quad 1 \leq i \leq n.$$

Now, $X$ with $(5.5)$ fits into a triangle

$$X' \rightarrow X \rightarrow X_1 \xrightarrow{d'} X'[1]$$

where $X'$ is a dg module with $|X'| = \bigoplus_{k=1}^{l} X_k$ and $d'$ is a truncation of $d_1$. Applying $f_i$ to this triangle, we see that

$$\dim \text{Hom}^\bullet(\Gamma, X) = \dim \text{Hom}^\bullet(\Gamma, X_1) + \dim \text{Hom}^\bullet(\Gamma, X').$$

Inductively, we deduce that

$$\dim \text{Hom}^\bullet(\Gamma, X) = \sum_{k=1}^{l} \dim \text{Hom}^\bullet(\Gamma, X_k),$$

which implies the lemma by $(5.6)$. \qed

As a corollary, a uniqueness criterion of m-perfect dg $E_0$-modules is given.

**Corollary 5.6.** Let $X$ be a m-perfect dg $E_0$-modules in $\text{per } E_0$. Then there is a unique form $(5.5)$ for $X$, up to permute the index and isomorphism.
Remark 5.7. Another way to see Corollary 5.6, is that $|X|$ actually provides the simple filtration of $X$ in $D_{fd}(\Gamma_0)$, with respect to the canonical heart $\Gamma_0$. Recall that the simple filtration of an object $E$ in a triangulated category $\mathcal{D}$, with respect to a given heart $\mathcal{H}$, is obtained as follows: first take the Harder-Narashimhan filtration (see [19, (2.3)]) of $E$ with factors $H_j \in \mathcal{H}[k_j]$ and $k_1 > \cdots > k_m$; then take union of the Jordan-Hölder(/simple) filtrations of $H_j$ in $\mathcal{H}[k_j]$.

5.3. The string model. Let $\eta$ be a closed arc in $S_\Delta$ such that it is in a minimal position with respect to $T_0$. This is equivalent to say that, there is no digon shown as in Figure 14. One can associate a $m$-perfect dg $E_0$-mod $X_{\eta}$ (uniquely up to shift) as follows.

- its underlying graded module $|X_{\eta}|$ has the form as in (5.5).
- Let the endpoints of $\eta$ are $Z$ and $Z'$. Suppose that from $Z$ to $Z'$, $\eta$ intersects $T_0$ at $V_1, \ldots, V_m$ accordingly, where $V_i$ is in the arc $\gamma_{j_i} \in T_0$ for $1 \leq i \leq m$ and some $1 \leq j_i \leq n$ (cf. Figure 15). Note that since when choose $\eta$ in a minimal position with respect to $T_0$, $j_i$ are independent of the choice of $\eta$ (but the isotopy class of $\eta$).
- Each line segment $V_iV_{i+1}$ in $\eta$ induces a graded arrow $a_i$ between $V_i$ and $V_{i+1}$ (clockwise within the corresponding triangle). See Figure 16 for how the following maps in (5.1) induces the graded arrow $a$ between $V$ and $W$:

$$\text{Hom}^1(S_1, S_2) \simeq \text{Hom}^2(S_2, S_1) \simeq \text{Hom}^3(S_3, S_3) = k.$$
respectively. Then we obtain a string quiver $H_\eta$, whose vertices are $V_i$'s and whose (graded) arrows are those induced arrows.

$$H_\eta: \quad V_1\xrightarrow{a_1} V_2\xrightarrow{a_2} \cdots \xrightarrow{a_{m-1}} V_m$$

Clearly, there is an obvious morphism from $H_\eta$ to $Q(H_{T_0})$.

- Each intersection $V_i$ corresponds to a summand $S_j[\delta_i]$ in some $X_{\delta_i}$ for some integer $\delta_i$. So we have

$$\bigoplus_{i=1}^m S_j[\delta_i].$$

- For the arrow $a_i$, we have two cases:

1°. If its orientation is $V_i \to V_{i+1}$, then the degree $\delta$'s satisfy

$$\delta_{i+1} = \delta_i + \deg a_i - 1. \quad (5.7)$$

Moreover, the map $S_{j_i} \to S_{j_{i+1}}[\deg a_i]$, cf. (5.1), induces a degree 1 map from

$$d_{a_i}: S_{j_i}[\delta_i] \xrightarrow{1} S_{j_{i+1}}[\delta_{i+1}].$$

2°. If its orientation is $V_i \leftarrow V_{i+1}$, then the degree $\delta$'s satisfy

$$\delta_i = \delta_{i+1} + \deg a_i - 1. \quad (5.8)$$

Moreover, the map $S_{j_{i+1}} \to S_{j_i}[\deg a_i]$, cf. (5.1), induces a degree 1 map from

$$d_{a_i}: S_{j_{i+1}}[\delta_{i+1}] \xrightarrow{1} S_{j_i}[\delta_i].$$

- Finally, the non-trivial part of the differential $d_{X_\eta}$ is

$$d_1 = \sum_{i=1}^{m-1} d_{a_i}.$$

**Lemma 5.8.** $X_\eta$ defined above is a perfect dg $E_0$-module.

**Figure 16.** Inducing graded arrows
Proof. We only need to check $d_i^2 = 0$, or equivalently $d_{a_i-1} d_{a_i} = 0$ for any $i$. If so, it can be considered as a complex of $\mathfrak{e}_0$-modules and hence in the derived category of the dg algebra $\mathfrak{e}_0$. Then it is a $m$-perfect dg $\mathfrak{e}_0$-module (by construction) and hence a perfect dg $\mathfrak{e}_0$-module.

On one hand, since $\eta$ is in a minimal position, with respect to $T_0$ and any two triangles in $T_0$ share at most one edge, we deduce that for any $i$, $V_{i-1}, V_i$ and $V_{i+1}$ are not in a single triangle of $T_0$.

On the other hand, if $d_{a_i-1} d_{a_i} \neq 0$, or

$$\text{Hom}_1(S_{i-1}[\delta_i], S_i[\delta_i]) \otimes \text{Hom}_1(S_i[\delta_i], S_{i+1}[\delta_{i+1}]) \cong \text{Hom}_2(S_{i-1}[\delta_i], S_{i+1}[\delta_{i+1}]),$$

then the equation above must be in the form of one of the equations in (5.4). This requires that $S_{i-1}, S_i$ and $S_{i+1}$ are in a 3-cycle in $Q_{T_0}$, or $V_i, V_i$ and $V_{i+1}$ are in a single triangle of $T_0$.

In sum, we deduce that $d_{a_i-1} d_{a_i} = 0$ as required.  

Remark 5.9. When specifying one of the shifts $\delta_i, 1 \leq i \leq m$, all other $\delta_i$ are determined by (5.7) or (5.8) inductively. Thus, $X_\eta$ is well-defined up to shifts. Denote the map by

$\tilde{X} : \text{CA}(S_\Delta) \to \text{per} \mathfrak{e}_0/[1],$

$\eta \mapsto \tilde{X}(\eta).$

We will use the convention that $X_\eta$ will be a representative in the shift orbits $\tilde{X}(\eta)$ and the notation $X[\mathbb{Z}]$ means the shift orbit that contains $X$.

Example 5.10. By construction, $\tilde{X}(s_i) = S_i[\mathbb{Z}]$, where $s_i$ are the ‘initial’ closed arcs in $T_0^*$ and $S_i$ are the simples in the canonical heart $\mathcal{H}_0$. Let us have a look at some non-trivial case. Take an initial triangulation of a 6-gon as shown in the left picture in Figure 17. The Ext-quiver of $\mathcal{H}_0$ is isomorphic to the one in Example 5.2. Then we have

$$\tilde{X}(\eta_1) = \text{Cone}(S_1 \to S_2[1])[\mathbb{Z}],$$

$$\tilde{X}(\eta_2) = \text{Cone}(X \to S_3[3])[\mathbb{Z}],$$

where

$X = \text{Cone}(S_1[-2] \to S_3).$

Here, the maps in the Cone are the unique map (up to scaling) between the corresponding objects.

We finish this subsection with showing the preparation of injectivity of $\tilde{X}$ in (5.9).

Lemma 5.11. If $X_\eta[\mathbb{Z}] = S_i[\mathbb{Z}]$ for some initial closed arc $s_i \in T_0^*$, then $\eta = s_i$.

Proof. By Corollary 5.6, $|X_\eta|$ has a unique form, which consists of only one copy of some shifts of $S_i$. Then, by Lemma 5.5, the condition implies that $\text{Int}(T_0, \eta) = \text{Int}(\gamma_i, \eta) = 1$, or $\eta = s_i$. □
6. Reachable spherical objects

6.1. The canonical triangles.

Proposition 6.1. Let $\alpha, \beta$ be two closed arcs in $\text{CA}(S_\Delta)$ such that $\text{Int}(\alpha, \beta) = \frac{1}{2}$ and $\eta = B_\alpha(\beta)$. Then there are representatives $X_? \in \tilde{X}(?)$ for $? = \alpha, \beta, \eta$ such that there is a non-trivial triangle

$$X_\beta \rightarrow X_\eta \rightarrow X_\alpha \rightarrow X_\beta[1].$$

(6.1)

Proof. Let $Z_1$ and $Z_2$ be the endpoint other than $Z_0$ of $\alpha$ and $\beta$ respectively. Suppose that from $Z_0$ to $Z_1$, $\alpha$ intersects $T_0$ at $V_1, \ldots, V_m$ accordingly, where $V_i$ is in the arc $\gamma_{j_i} \in T_0$ for $1 \leq i \leq m$ and some $1 \leq j_i \leq n$; suppose that from $Z_0$ to $Z_2$, $\beta$ intersects $T_0$ at $W_1, \ldots, W_l$ accordingly, where $W_i$ is in the arc $\gamma_{k_i} \in T_0$ for $1 \leq i \leq l$ and some $1 \leq k_i \leq n$.

First consider the three cases (as shown in Figure 18) satisfying

$$\text{Int}(T_0, \eta) = \text{Int}(T_0, \alpha) + \text{Int}(T_0, \beta).$$

Then from $Z_1$ to $Z_2$, $\eta$ intersects $T_0$ at $V_m, \ldots, V_1, W_1, \ldots, W_l$ accordingly. Let $a : V_1 \rightarrow W_1$ be the arrow in the string quiver $H_\eta$. Choose the following representatives in $\tilde{X}(?)$:

- $X_\eta$ such that $S_{j_1}[0]$ that corresponds to $V_1$ is in $|X_\eta|$;
- $X_\alpha$ such that $S_{j_1}[0]$ that corresponds to $V_1$ is in $|X_\alpha|$;
- $X_\beta$ such that $S_{j_1}[0]$ that corresponds to $V_1$ is in $|X_\beta|$.

Figure 17. An initial triangulation and two closed arcs in a 6-gon

Figure 18. The cases for $V_1$ and $W_1$
• $X_\beta$ such that $S_k\deg a - 1$ that corresponds to $W_1$ is in $|X_\beta|$.

By construction, $\text{Ext}^1(S_{j_1}, S_{k_1}\deg a - 1)$ induces a map from $X_\alpha$ to $X_\beta[1]$ such that

$$X_\eta[1] = \text{Cone}(X_\alpha \rightarrow X_\beta[1]),$$

(6.2)
i.e. (6.1). Clearly, this triangle is non-trivial.

Next consider the remaining case (cf. top picture of Figure 20) satisfying

$$\text{Int}(T_0, \eta) < \text{Int}(T_0, \alpha) + \text{Int}(T_0, \beta).$$

Suppose that $\kappa + 1 = i$ is the smallest integer such that $V_i$ and $W_i$ are not in the same arc $\gamma$ of $T_0$ (or equivalently, the smallest integer such that $j_i \neq k_i$). Then from $Z_1$ to $Z_2$, $\eta$ intersects $T_0$ at $V_m, \ldots, V_{\kappa+1}, W_{\kappa+1}, \ldots, W_l$ accordingly (see Figure 20 for the possible cases, up to mirror). Let us discuss the first case for demonstration while the other cases are similar (cf. Remark 6.2).

Let $Z$ be the decorated point in the triangle that contains the line segment $V_{\kappa+1}W_{\kappa+1}$ in $\eta$. Consider the following closed arcs (cf. the left picture of Figure 19):

• $\eta_0$ from $Z_0$ to $Z$ intersects $T_0$ at $V_1, \ldots, V_{\kappa}$ accordingly;
• $\eta_1$ from $Z_1$ to $Z$ intersects $T_0$ at $V_m, \ldots, V_{\kappa+1}$ accordingly;
• $\eta_2$ from $Z_2$ to $Z$ intersects $T_0$ at $W_l, \ldots, W_{\kappa+1}$ accordingly.

By the discussion above (i.e. (6.2)), there exists the representative $X_\eta, t = 0, 1, 2$ such that there are representatives $X_\alpha, X_\beta, X_\eta$ satisfying the following:

$$X_\alpha = \text{Cone}(X_{\eta_0} \rightarrow X_{\eta_1}[1]),$$

$$X_\beta[1] = \text{Cone}(X_{\eta_2} \rightarrow X_{\eta_0}[1]),$$

$$X_\eta[1] = \text{Cone}(X_{\eta_2} \rightarrow X_{\eta_2}[2]).$$

Moreover, the maps above are induced by the following $\text{Ext}^i$’s (with relation)

$$\text{Ext}^1(S_{k_{\kappa+1}}, S_{k_\kappa}) \otimes \text{Ext}^1(S_{j_k}, S_{j_{\kappa+1}}) \simeq \text{Ext}^2(S_{k_{\kappa+1}}, S_{j_{\kappa+1}}),$$

(6.3)

where $S_{k_\kappa} = S_{j_k}$. Thus, we obtain the commutative diagram of triangles as shown in Figure 19 by Octahedral Axiom, which completes the proof.

**Figure 19.** The three new closed arcs

\[\square\]

**Remark 6.2.** The differences between the cases in Figure 20 are:

1°. in the corresponding Figure 19, the position of the decorated point $Z$ could be different, e.g. $\alpha$ might intersect $\eta_2$ or $\beta$ might intersect $\eta_1$;
Figure 20. The cases for $V_\kappa$ and $W_\kappa$

2°. in the extreme case that $Z$ could coincide with $Z_1$ or $Z_2$;
3°. the composition of maps in (6.3) might be different.

6.2. The key proposition. By construction (that $\eta$ is in a minimal position with respect to $T_0$) and Lemma 5.5, we have the following consequence.

Lemma 6.3.

$$\dim \text{Hom}^\bullet(\Gamma_i, X_\eta) = \text{Int}(\gamma_i, \eta), \quad (6.4)$$

$$\dim \text{Hom}^\bullet(\Gamma_0, X_\eta) = \sum_{i=1}^n \dim \text{Hom}^\bullet(\Gamma_i, X_\eta) = \text{Int}(T_0, \eta). \quad (6.5)$$

Proposition 6.4. Let $\eta_k, k = 1, 2$ be two closed arcs in $\text{CA}(S_\Delta)$ and $\bar{X}(\eta_k) = X_k[Z]$ for some representative $X_k$.

1°. $X_{\eta_1}$ and $X_{\eta_2}$ are in $\text{Sph}(\Gamma_0)$.
2°. If $\text{Int}(\eta_1, \eta_2) = 0$ (in $S_\Delta$, so they share no endpoints in $\Delta$), then

$$\text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = 0. \quad (6.6)$$

3°. If $\text{Int}(\eta_1, \eta_2) = \frac{1}{2}$, then

$$\dim \text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = 1. \quad (6.7)$$

Proof. See Appendix B
In the proof of 1° of Proposition 6.4, we actually have shown that, if \( \alpha, \beta \in \text{CA}(S_\Delta) \) with \( \text{Int}(\alpha, \beta) = \frac{1}{2} \) and \( \eta = B_\alpha(\beta) \), then there exists representatives \( X_\alpha, X_\beta \) and \( X_\eta \) respectively satisfying (B.3). Equivalently, we can state this as follows.

**Corollary 6.5.** Let \( \alpha, \beta \in \text{CA}(S_\Delta) \) with \( \text{Int}(\alpha, \beta) = \frac{1}{2} \), Then

\[
\tilde{X}(B_\alpha(\beta)) = \phi_{\bar{X}(\alpha)}(\tilde{X}(\beta)).
\]

(6.8)

7. **Braid twists versus spherical twists**

7.1. **Two twist group actions.** We start with a generalized version of Corollary 6.5.

**Lemma 7.1.** For any \( s \in T_0 \) and \( \eta \in \text{CA}(S_\Delta) \), we have

\[
\phi_{\bar{X}(s)}(\tilde{X}(\eta)) = \tilde{X}(B_s(\eta)),
\]

where \( \varepsilon \in \{ \pm 1 \} \).

**Proof.** Without lose of generality, we only deal the case for \( \varepsilon = 1 \). Use induction on \( \text{Int}(T_0, \eta) \) starting with the trivial case when \( \text{Int}(T_0, \eta) = 1 \), or equivalently, \( \eta \in T_0 \). Now, for the inductive step, consider \( \eta \) with \( \text{Int}(T_0, \eta) = m \) while the lemma holds for any \( \eta' \) with \( \text{Int}(T_0, \eta') < m \). Applying Lemma 3.14, we have \( \eta = B_\alpha(\beta) \) for some \( \alpha, \beta \) with \( \text{Int}(\alpha, \beta) = \frac{1}{2} \) (3.8). Twisted by \( B_s \), we have \( \text{Int}(B_s(\alpha), B_s(\beta)) = \frac{1}{2} \) and \( B_s(\eta) = B_{B_s(\alpha)}(B_s(\beta)) \). By (6.8), we have

\[
\tilde{X}(B_s(\eta)) = \phi_{\bar{X}(B_s(\alpha))}(\tilde{X}(B_s(\beta))). \tag{7.2}
\]

By inductive assumption,

\[
\phi_{\bar{X}(s)}(\tilde{X}(\alpha)) = \tilde{X}(B_s(\alpha)), \quad \phi_{\bar{X}(s)}(\tilde{X}(\beta)) = \tilde{X}(B_s(\beta)). \tag{7.3}
\]

So

\[
\phi_{\bar{X}(s)}(\tilde{X}(\eta)) = \phi_{\bar{X}(s)}(\phi_{\bar{X}(\alpha)}(\tilde{X}(\beta)))
= \phi_{\bar{X}(s)} \circ \phi_{\bar{X}(\alpha)} \circ \phi_{\bar{X}(s)}^{-1}(\phi_{\bar{X}(s)}(\tilde{X}(\beta)))
= \phi_{\bar{X}(s)}(\tilde{X}(\alpha))(\phi_{\bar{X}(s)}(\tilde{X}(\beta)))
= \phi_{\bar{X}(s)}(\tilde{X}(\beta))(\tilde{X}(B_s(\beta)))
= \tilde{X}(B_s(\eta)),
\]

where the first equality follows from (6.8), the third equality follows from (2.5), the fourth equality follows from (7.3) and the last equality follows from (7.2), which completes the proof. \(\square\)

**Remark 7.2.** Let \( Z_0^{ST} = \text{ST}(\Gamma_0) \cap \mathbb{Z}[1] \) and

\[
\text{ST}_*(\Gamma_0) = \text{ST}(\Gamma_0)/Z_0^{ST} \subset \text{Aut} \circ D_{fd}(\Gamma_0)/\mathbb{Z}[1].
\]

Note that \( \text{ST}_*(\Gamma_0) \) also acts on \( \text{Sph}(\Gamma_0)[1] \). By [5, Theorem 4.4], \( Z_0^{ST} = 1 \) unless \( S \) is a polygon, in which case, \( Z_0^{ST} = \mathbb{Z}[n + 3] \).
Recall that the initial triangulation consists of closed arcs $s_i$, whose braid twists $b_i = B_{s_i}$ generate $BT(T_0) = BT(S_\triangle)$ by Lemma 4.2. Moreover, the canonical heart $\mathcal{H}_0$ in $D_{fd}(\Gamma_0)$ has simples $S_i$ satisfying $S_i[\mathbb{Z}] = \tilde{X}(s_i)$, whose spherical twists $\phi_i = \phi_{S_i}$ generate $BT(S_\triangle)$.

**Proposition 7.3.** There is a canonical group homomorphism

$$\iota: BT(T_0) \to ST_\ast(\Gamma_0),$$

(7.4)

sending the generator $b_i$ to the generator $\phi_i$.

**Proof.** Consider the case when $S$ is not a polygon first. We only need to prove that, if

$$b = b_{i_1}^{\varepsilon_1} \circ \cdots \circ b_{i_k}^{\varepsilon_k}$$

(7.5)

equals 1 in $\text{MCG}(S_\triangle)$, for some $i_j \in \{1, \ldots, n\}, \varepsilon_j \in \{\pm 1\}, 1 \leq j \leq k$ and $k \in \mathbb{N}$, then

$$\phi = \phi_{i_1}^{\varepsilon_1} \circ \cdots \circ \phi_{i_k}^{\varepsilon_k}$$

(7.6)

equals 1 in $\text{Aut}^\circ D_{fd}(\Gamma_0)$.

First, $b = 1$ implies $b(s_i) = s_i$ for any $1 \leq i \leq n$. By (repeatedly using) Lemma 7.1, we have

$$\tilde{X}(b(s_i)) = \phi \left( \tilde{X}(s_i) \right).$$

Thus, $S_i[\mathbb{Z}] = \tilde{X}(s_i) = \phi(S_i[\mathbb{Z}])$, i.e. $\phi(S_i) = S_i[t_i]$ for some integer $t_i$. Since $\phi$ is an equivalence, we deduce that all $t_i$ must be the same. Therefore $\phi = [t]$ for some integer $t$. However, we have $\phi \in Z_{ST}^\ast = 1$ in this case, which implies $t = 0$ and $\phi = 1$ in $\text{Aut}^\circ D_{fd}(\Gamma_0)$, as required.

In the case when $S_\triangle$ is a polygon, $b = 1$ still implies $\phi = [t]$ for some $t \in \mathbb{Z}$ and thus the proposition holds too. \qed

A consequence of the existence of $\iota$ is that the braid twist group actions $BT(S_\triangle)$ on $CA(S_\triangle)$ is compatible with the spherical twist group actions $ST_\ast(\Gamma_0)$ on $\text{Sph}(\Gamma_0)/[1]$, under the map $\tilde{X}$ in (5.9). More precisely, we have the commutative diagram below, where the commutativity is in the sense of (7.9) in the following corollary.

$$\begin{array}{ccc}
BT(S_\triangle) & \xrightarrow{\iota} & ST_\ast(\Gamma_0) \\
\bigcirc & & \bigcirc \\
CA(S_\triangle) & \xrightarrow{\tilde{X}} & \text{Sph}(\Gamma_0)/[1]
\end{array}$$

(7.7)

**Corollary 7.4.** For any $b \in BT(S_\triangle)$ and $\eta \in CA(S_\triangle)$, we have

$$\iota(B_\eta^\varepsilon) = \phi_{\tilde{X}(\eta)}^\varepsilon, \quad \varepsilon \in \{\pm 1\}$$

(7.8)

$$\tilde{X}(b(\eta)) = \iota(b) \left( \tilde{X}(\eta) \right).$$

(7.9)

**Proof.** Again, we will only deal the case when $\varepsilon = 1$. By Proposition 4.4, $\eta = b(s_j)$ for some $s_j \in \mathbb{T}$ and $b \in BT(S_\triangle)$ with the form (7.5). Let $\phi$ as in (7.6) and by (repeatedly using) (7.1), we have

$$\tilde{X}(\eta) = \tilde{X}(b(s_j)) = \phi(\tilde{X}(s_j)) = \phi(S_j).$$
Then using formulae (3.3), (2.5), the equality above and the fact $\iota$ is a group homomorphism, we have

$$\iota(B_\eta) = \iota(B_{b(s)}) = \iota\left(b^\varepsilon_1 \circ \cdots \circ b^\varepsilon_k \circ B_{s_j} \circ b^{-\varepsilon_1} \circ \cdots \circ b^{-\varepsilon_k}\right) = \iota(b^\varepsilon_1) \circ \cdots \circ \iota(b^\varepsilon_k) \circ \iota(b_j) \circ \iota(b^{-\varepsilon_1}) \circ \cdots \circ \iota(b^{-\varepsilon_k}) = \phi^\varepsilon_1 \circ \cdots \circ \phi^\varepsilon_k \circ \phi_j \circ \phi^{-\varepsilon_1} \circ \cdots \circ \phi^{-\varepsilon_k} = \phi \circ \phi_j \circ \phi^{-1} = \phi \phi(S_j) = \phi\tilde{X}(\eta),$$

i.e. (7.8). A similar calculation, noticing that $\iota$ is a group homomorphism, we obtain (7.9) as the generalized version of (7.1).

When specifying $b = B_\varepsilon$ in (7.9) and using (7.8), we see that (7.1) holds for any $s, \eta \in \text{CA}(S_\Delta)$.

**Corollary 7.5.** (7.1) holds for any $s, \eta \in \text{CA}(S_\Delta)$.

Now, we are ready to prove the main theorem of this paper.

### 7.2. The main result

We start to show that $\tilde{X}$ is bijective.

**Theorem 7.6.** The map $\tilde{X}$ in (5.9) induces a bijection

$$\tilde{X}: \text{CA}(S_\Delta) \overset{1-1}{\rightarrow} \text{Sph}(\Gamma_0)/[1].$$

**Proof.** First, we prove the injectivity. Suppose $\tilde{X}(\eta) = \tilde{X}(\eta')$ for $\eta, \eta' \in \text{CA}(S_\Delta)$. Let $\eta = b(s_i)$ for some $b \in \text{BT}(S_\Delta)$ and initial closed arc $s_i \in T_0$. Then by (7.9), we have

$$S_i[Z] = \tilde{X}(s_i) = \tilde{X}(b^{-1}(\eta)) = \iota(b^{-1}\left(\tilde{X}(\eta)\right)) = \iota(b^{-1}\left(\tilde{X}(\eta')\right)) = \tilde{X}(b^{-1}(\eta')).$$

By Lemma 5.11, $s_i = b^{-1}(\eta')$ or $\eta = \eta'$ as required.

Second, we prove the surjectivity. Let $\eta$ be a closed arc in $\text{CA}(S)$ and $\tilde{X}(\eta) = X_\eta[Z]$ for some representative $X_\eta$. We only need to show that $X_\eta$ is in $\text{Sph}(\Gamma_0)$. Use induction on $I = \text{Int}(T_0, \eta)$. If $I = 1$, then $\eta$ is some $s_i \in T_0$ and $X_\eta = S_i[\delta]$ for some integer $\delta$, which is in $\text{Sph}(\Gamma_0)$. Now suppose that the claim is true for $I \leq r$ for some $r \geq 1$ and consider the case when $I = r + 1$. Apply Lemma 3.14, we find $\alpha$ and $\beta$ with $\text{Int}(\alpha, \beta) = \frac{1}{2}$ and (3.4). By Corollary 6.5, we have representatives $X_\alpha$ and $X_\beta$ with (B.3). By inductive assumption, we know that $X_\alpha$ and $X_\beta$ are in $\text{Sph}(\Gamma_0)$. On the other hand, we have $\phi\tilde{X}_\alpha \in \text{ST}(\Gamma_0)$ by (2.5) and the theorem follows from (2.6). \qed

We proceed to show that the bijectivity above implies the isomorphism between twisted groups.

**Theorem 7.7.** Let $S$ be a unpunctured marked surface and $T_0$ be a triangulation of $S_\Delta$ such that the corresponding FST’quiver has no double arrows. Then there is a canonical isomorphism

$$\iota: \text{BT}(T_0) \rightarrow \text{ST}(\Gamma_0),$$

sending the generator $b_i$ to the generator $\phi_i$, where $\Gamma_0$ is the Ginzburg dg algebra associated to $T_0$. 


Proof. When $S$ is a polygon, this follows from [22] and [30]. Now suppose $S$ is not a polygon. Then $ST_0(G_0) = ST(G_0)$. So we have the homomorphism $\iota$ in (7.4) already, which is clearly surjective. Thus we only need to show it is injective.

Let $b \in BT(S_\Delta)$ with $\iota(b) = 1$ in $ST(G_0)$. By (7.9), we have
\[
\tilde{X}(b(\eta)) = \iota(b) \left( \tilde{X}(\eta) \right) = \tilde{X}(\eta),
\]
which implies $b(\eta) = \eta$ by Theorem 7.6, for any closed arc $\eta$. By (4.3), this implies $b \circ B_\eta = B_\eta \circ b$ and thus $b$ is the center $Z_0^{BT}$ of $BT(S_\Delta)$. But $Z_0^{BT} = 1$ in this case. So $b = 1$ and $\iota$ is injective. □

Remark 7.8. The proof in Theorem 7.7 actually shows that $\iota$ induce an isomorphism $BT_*(T_0) \cong ST_*(G_0)$, which is equivalent to (7.10) when $S$ is not a polygon. When $S_\Delta$ is a polygon, we have $Z_0^{BT} = ZD_{\partial S_\Delta}$, $\iota(D_0^{-1}) = [n + 3]$ and the following commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & ZD_{\partial S_\Delta} \\
\cong & & \cong \\
0 & \longrightarrow & Z[n + 3] \\
\end{array}
\begin{array}{ccc}
BT(S_\Delta) & \longrightarrow & BT_*(S_\Delta) \\
\iota & \cong & \iota \\
ST(G_0) & \longrightarrow & ST_*(G_0) \\
\end{array}
\begin{array}{ccc}
& & 0.
\end{array}
\]
Moreover, we can get rid of the star of $ST_*(G_0)$ in (7.7).

7.3. On the Dehn twist groups.

Definition 7.9. Denote by $\Sigma_\Delta S$ the branching double cover of $S_\Delta$, branching at the points in $\Delta$ with the projection $p: \Sigma_\Delta S \rightarrow S_\Delta$.

Further, we will treat points in $p^{-1}(\Delta) \subset \Sigma_\Delta S$ as usual points instead of punctures.

The intersection number on $\Sigma_\Delta S$ would be the normal (geometric) intersection number as in (3.1). By [22, Section 3a-3c], we have the following:

- A closed arc $\eta$ in $CA(S_\Delta)$ lifts to a (unoriented) simple closed curve $p^{-1}(\eta)$ on $\Sigma_\Delta S$.
- The lifting map $p^{-1}$ is an injective map from $CA(S_\Delta)$ to the set of simple closed curves in $\Sigma_\Delta S$.
- We have $\text{Int}(p^{-1}(\eta_1), p^{-1}(\eta_2)) = 2 \text{Int}(\eta_1, \eta_2)$.
- There is a homomorphism $p^*: \text{MCG}(S_\Delta) \rightarrow \text{MCG}(\Sigma_\Delta S)$, send the braid twist $B_\eta$ to the Dehn twist $D_{p^{-1}(\eta)}$, for any $\eta \in CA(S_\Delta)$ (cf. Figure 12).

Let $CC(\Sigma_\Delta S) = p^{-1}(CA(S_\Delta))$, which is a set of simple closed curves on $\Sigma_\Delta S$. The Dehn twist group $DTG(\Sigma_\Delta S)$ of $\Sigma_\Delta S$ is the subgroup of $\text{MCG}(\Sigma_\Delta S)$ generated by $\{D_C \mid C \in CC(\Sigma_\Delta S)\}$.

Recall the following famous result from topology (cf. e.g. [9]).
Theorem 7.10 (Birman-Hilden). The homomorphism $p^*$ is injective on $BT(S_\Delta)$, i.e. $p^*: BT(S_\Delta) \cong DTG(\Sigma^\Delta)$.

Combining Birman-Hilden Theorem above and Theorem 7.7, we have the following corollary, which is possible to be generalized to unpunctured case (cf. [20]).

Corollary 7.11. There is a canonical isomorphism $DTG(\Sigma^\Delta) \cong ST(\Gamma_0)$.

8. Special cases

In this section, we first deal with the two special cases in Remark 3.11. Then we discuss the affine type A case in more details.

8.1. The Kronecker case. We first demonstrate the special case I. Note that in case I, all triangulations of $S$ or $S_\Delta$ looks the same, cf. Figure 22. Choose any triangulation $T_0$ of $S_\Delta$ as the initial triangulation. Keep all the notations as above.

The dynamic of proof here is the reverse comparing to the previous cases: we will show the relation between the twist groups first; then the relations between closed arcs and spherical objects.

First, we claim that (7.10) also holds in this case.

Proposition 8.1. Let $S$ be an annulus with two marked points and $T_0$ a triangulation of $S_\Delta$. There is a canonical isomorphism

$$\iota_0: BT(T_0) \rightarrow ST(\Gamma_0),$$

sending the generator $b_i$ to the generator $\phi_i$, where $\Gamma_0$ is the Ginzburg dg algebra associated to $T_0$. Moreover, these two groups are free groups of rank two (and with generators $b_i$ resp. $\phi_i$).
Figure 22. The Kronecker case

Proof. For the first statement, use the faithfulness of affine quiver $Q'$:

\[
\begin{array}{c}
\text{3'} \\
\downarrow \\
\text{2'} \\
\downarrow \\
\text{1'}
\end{array}
\]

of type $\widetilde{A}_{1,2}$. Let $\Gamma'$ be the corresponding Ginzburg algebra and $D_{\text{fd}}(\Gamma')$ the associated 3-CY derived category with canonical heart $\mathcal{H}'$. Denote by $X'_i$ the corresponding simples in $\mathcal{H}'$. Since $Q'$ arises from some triangulation of an annulus with three marked points, consider the corresponding decorated marked surface $S'_\Delta$ with triangulation $T'$ (left picture in Figure 23) and dual triangulation $T'^* = \{s'_1, s'_2, s'_3\}$. By [19, Proposition 5.2], the backward tilt $H = (\mathcal{H}')_{X_3}'$ of $\mathcal{H}'$ has simples

\[X_1 = X'_1, X_2 = \phi_{X_3}(X'_2) \text{ and } X_3 = X'_3[-1],\]

which corresponds to closed arcs $t_1 = s'_1, t_2 = B_{s'_1}(s'_2)$ and $t_3 = s'_3$ in $S'_\Delta$. Note that the quiver with potential associated to $\mathcal{H}$ consists of the quiver

\[
\begin{array}{c}
\text{3} \\
\downarrow \\
\text{2} \\
\downarrow \\
\text{1}
\end{array}
\]

a 3-cycle potential. Further more, it corresponds to the triangulation $T$, as shown in the right picture in Figure 23. By Theorem 7.7, we have the corresponding isomorphism (7.10) for $S'_\Delta$. Thus, the group generated by $\phi_{X_1}$ and $\phi_{X_2}$ is isomorphic to the group generated by $B_{t_1}$ and $B_{t_2}$. On the other hand, we have the following two facts:

- the subcategory $D_0$ of $D_{\text{fd}}(\Gamma')$ generated by $X_1$ and $X_2$ is equivalent to the 3-CY category for a Kronecker quiver;
- there is a subsurface $(S_\Delta)_0$ of $S'_\Delta$, with inherited triangulation from $T$ (whose dual consists of $t_1$ and $t_2$), that is isomorphic to any triangulation of an annulus with two marked points.

Therefore, by identifying $D_{\text{fd}}(\Gamma_0)$ with $D_0$ and $S_\Delta$ with $(S_\Delta)_0$, we have

\[ST(\Gamma_0) \cong \langle \phi_{X_1}, \phi_{X_2} \rangle \cong \langle B_{t_1}, B_{t_2} \rangle \cong BT(T_0),\]
which implies the first claim.

By Theorem 7.10, $\text{BT}(T_0)$ is isomorphic to a subgroup of the mapping class group of the branched double cover $\Sigma_{S_D}^\Delta$ of $S_D$, generated by two Dehn twists $D_{C_i}$, $i = 1, 2$, where the curve $C_i$ is the lifting of $s_i$ in $\Sigma_{S_D}^\Delta$. Moreover, $\text{Int}(C_1, C_2) = 2 \text{Int}(s_1, s_2) = 2$. By [14], $\text{BT}(T_0)$ is a free group with generators $b_i = B_{s_i}$, $i = 1, 2$, which implies the second claim. □

A direct corollary is as follows.

**Corollary 8.2.** There is a bijection

$\tilde{X}: \text{CA}(S_D^\Delta) \overset{i_!}{\rightarrow} \text{Sph}(\Gamma_0)/[1]$,

sending $\eta = \Psi(s_i)$ to $i(\Psi)(S_i)[\mathbb{Z}]$, for any $\Psi \in \text{BT}(T_0)$ and $i = 1, 2$. In particular, (7.8) and (7.9) also hold.

8.2. The one marked point torus case. In this section, we give the statement the special case II. The tricky point is that we need to choose the correct potential. Namely, the rigid one instead of the other non-degenerated and non-rigid one (see [12] for details).

Then using the same proof (for the first claim) in Proposition 8.1, (i.e. consider a torus with one boundary component and two marked points for example) we have the following result.

**Proposition 8.3.** Let $S$ be a torus with one marked point and $T_0$ a triangulation of $S_D$. There is a canonical isomorphism

$i: \text{BT}(T_0) \rightarrow \text{ST}(\Gamma_0)$, \hspace{1cm} (8.2)

sending the generator $b_i$ to the generator $\phi_i$, where $\Gamma_0$ is the Ginzburg dg algebra associated to $T_0$. Further, Corollary 8.2 holds in this case too.

8.3. Example: annulus case (or affine type A). Recall that $D_{fd}(\Gamma_T)$ is the 3-CY category associated to a triangulation $T$ of $S$ with spherical twist group $\text{ST}(\Gamma_T)$. When $S$ is an annulus, Theorem 7.7, (together with Proposition 8.1) can be stated as follows.
**Theorem 8.4.** Let $S$ be an annulus and $T$ be a triangulation of $S$ with associated Ginzburg dg algebra $\Gamma_T$. Suppose there are $p$ and $q$ marked points on the two boundary components of $S$, respectively. Then the spherical twist group $\text{ST}(\Gamma_T)$ is (canonically) isomorphic to the braid group $\text{Br}(A_{p,q})$ of affine type $\tilde{A}_{p,q}$.

**Proof.** The case $p = q = 1$ is Proposition 8.1, noticing that the braid group $\text{Br}(A_{1,1})$ is a rank 2 free group. The other follows from Theorem 7.7, noticing that $\text{BT}(S_\Delta)$ is (canonically) isomorphic to $\text{Br}(A_{p,q})$ by the geometric description of the affine braid group in [8]. □

9. Geometric realizations

We give geometric realizations for the followings in this section:

- reachable rigid indecomposable in the perfect category as reachable simple open arcs in the decorated marked surface;
- the simple-projective duality for Ginzburg dg algebras as graph duality between dual-triangulations and triangulations;
- Amiot’s triangulated quotient that defines the cluster category as forgetful map from decorated marked surface to the original marked surface;
- the shift functor for the silting sets in the perfect category as (total) rotation in the marked mapping class group of decorated marked surface.

9.1. The triangulated quotient. Recall that $\Gamma_0$ is the Ginzburg dg algebra of the quiver with potential $(Q_{T_0}, W_{T_0})$ associated to the initial triangulation $T_0$ (we can treat the special cases equally in this section). There are two more categories associated to $\Gamma_0$. The first one is the perfect derived category $\text{per} \Gamma$. In fact, the 3-CY category $D_{fd}(\Gamma_0)$ is a full subcategory of $\text{per} \Gamma_0$. The second one is the cluster category $\mathcal{C}(\Gamma_0)$, which is the quotient $\text{per} \Gamma_0 / D_{fd}$. Denote the quotient map by $\pi$ and there is a short exact sequence of triangulated categories

\[ 0 \to D_{fd}(\Gamma_0) \to \text{per} \Gamma_0 \xrightarrow{\pi} \mathcal{C}(\Gamma_0) \to 0. \]

We will consider various exchange graphs of each of these categories and show the relation with exchange graph of triangulations.

9.2. The exchange graph of hearts. A torsion pair in an abelian category $\mathcal{C}$ is a pair of full subcategories $\langle F, T \rangle$ of $\mathcal{C}$, such that $\text{Hom}(T, F) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \to E^T \to E \to E^F \to 0$ for some objects $E^T \in T$ and $E^F \in F$.

A t-structure on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ such that, if one defines $\mathcal{P}^\perp = \{ G \in \mathcal{D} : \text{Hom}_\mathcal{D}(F, G) = 0, \forall F \in \mathcal{P} \}$, then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. A t-structure $\mathcal{P}$ is bounded if $\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j]$. The heart of a t-structure $\mathcal{P}$ is the full subcategory

\[ \mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}, \]

which determines $\mathcal{P}$ uniquely.
Recall, e.g. from [19, Section 3], that we can forward/backward tilt a heart \( H \) to get a new one, with respect to any torsion pair in \( H \) in the sense of Happel-Reiten-Smalø. Further, all forward/backward tilts with respect to torsion pairs in \( H \), correspond one-to-one to all hearts between \( H \) and \( H[\pm 1] \). Here, the partial order between hearts is in the sense of King-Qiu [19] that
\[
H_1 \leq H_2
\]
if and only if \( P_2 \subset P_1 \) for the corresponding t-structure. Note that \( H \leq H[1] \) for any \( H \).

In particular there is a special kind of tilting which is called simple tilting (cf.[19, Definition 3.6]), with respect to a rigid simple of a heart. We denote by \( H^\#_S \) and \( H^{\flat}_S \), respectively, the simple forward/backward tilts of a heart \( H \), with respect to a simple \( S \).

**Definition 9.1.** The *exchange graph* of a triangulated category \( D \) to be the oriented graph whose vertices are all hearts in \( D \) and whose edges correspond to simple forward tiltings between them.

Denote by \( \text{EG}^\circ(D_{fd}(\Gamma_0)) \) the principal component of the exchange graph \( \text{EG}(D_{fd}(\Gamma_0)) \), that is the connected component containing \( H_0 \).

**Lemma 9.2.** \( \text{Sph}(\Gamma_0) = \bigcup_{H \in \text{EG}^\circ(D_{fd}(\Gamma_0))} \text{Sim} H \).

**Proof.** By the tilting formulæ in [19, Proposition 5.2 and Remark 7.2] (cf. [20, Appendix A.1]), we know that the simples in the tilts of a heart \( H \) is of the form \( \phi^S_X \), where \( S \) and \( X \) are the simples in \( H \). Thus, by induction, we deduce that
\[
\text{Sim} H \subset \text{Sph}(\Gamma_0), \quad \forall H \in \text{EG}^\circ(D_{fd}(\Gamma_0)).
\]
On the other hand (see [18] and cf. [19, Corollary 8.4]), two backward/forward tilting on a heart \( H \) with respect to the same simple (up to shift) is equivalent to apply the twist/inverse of the twist along the simple on \( H \). Thus, we deduce that \( \text{ST}(\Gamma_0) \cdot H_0 \subset \text{EG}^\circ(D_{fd}(\Gamma_0)) \), which implies
\[
\text{Sph}(\Gamma_0) = \bigcup_{H \in \text{ST}(\Gamma_0) \cdot H_0} \text{Sim} H \subset \bigcup_{H \in \text{EG}^\circ(D_{fd}(\Gamma_0))} \text{Sim} H
\]
that completes the proof. \( \square \)

**9.3. The silting/cluster (tilting) exchange graph.** A silting set \( P \) in a category \( D \) is an Ext^\leq 0-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that \( \text{Ext}^i(P,T) = 0 \) for any \( P,T \in P \) and integer \( i > 0 \).

The *forward mutation* \( \mu_P^\# \) at an element \( P \in P \) is another silting set \( P^\#_P \), obtained from \( P \) by replacing \( P \) with
\[
P^\# = \text{Cone} \left( P \to \bigoplus_{T \in P-P} \text{Irr}(P,T)^* \otimes T \right),
\] (9.1)
where \( \text{Irr}(X,Y) \) is a space of irreducible maps \( X \to Y \), in the additive subcategory \( \text{Add} \bigoplus_{T \in P} T \) of \( C \). The *backward mutation* \( \mu_P^{\flat} \) at an element \( P \in P \) is another silting
set $P^\flat_P$, obtained from $P$ by replacing $P$ with

$$P^\flat = \text{Cone} \left( \bigoplus_{T \in \mathcal{P} - \{P\}} \text{Irr}(T, P) \otimes T \to P \right)[-1].$$

(9.2)

**Definition 9.3.** The **silting exchange graph** $\text{SEG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ to be the oriented graph whose vertices are all silting sets in $\mathcal{D}$ and whose edges correspond to forward mutations between them.

Note that $\Gamma_0$, consider as a set of its indecomposable summands, is a silting set in $\text{per} \Gamma_0$. Denote by $\text{SEG}^\circ(\text{per} \Gamma_0)$ the principal component of the exchange graph $\text{SEG}(\text{per} \Gamma_0)$, that is the connected component containing $\Gamma_0$.

A **cluster tilting set** $P$ in a category $\mathcal{C}$ is an $\text{Ext}^1$-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\text{Ext}^1(P, T) = 0$ for any $P, T \in \mathcal{P}$. We will only consider this structure on the cluster categories, which is 2-CY. The **forward mutation** $\mu^\flat_P$ at an element $P \in \mathcal{P}$ is another silting set $P^\flat_P$, obtained from $P$ by replacing $P$ with $P^\flat$ in (9.1). Similarly, we have the **backward mutation** $\mu^\sharp_P$ using formula (9.2). In fact, since $\mathcal{C}$ is 2-CY, we have $\mu^\flat_P P = \mu^\sharp_P P$ and we will denote the mutation by $\mu_P$.

**Definition 9.4.** The **cluster exchange graph** $\text{CEG}(\mathcal{C})$ is the (unoriented) graph whose vertices are cluster tilting sets and whose edges correspond to the mutations.

We will write $\text{CEG}(\Gamma_0)$ for $\text{CEG}(\mathcal{C}(\Gamma_0))$.

**Remark 9.5** (Connectedness). As $\mathcal{C}(\Gamma_0)$ arises from a (tagged) triangulation of marked surfaces with boundaries, the cluster exchange graph $\text{CEG}(\Gamma_0)$ is connected (due to [6] for the unpunctured case and [28] for the punctured case). Note that for the cluster categories arise from (tagged) triangulations of marked surfaces without boundaries, the corresponding cluster exchange graphs are not necessarily connected.

**Remark 9.6.** For each cluster tilting set $P$, denote by $Q_P$ the Gabriel quiver of $\text{add} \bigoplus_{P \in \mathcal{P}} P$. Then mutation on cluster tilting sets become FZ mutation on the corresponding Gabriel quivers.

**9.4. The relations between various exchange graphs.** We list the known relations between the exchange graphs mentioned before.

(a). There is a canonical isomorphism

$$\text{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0)) \cong \text{SEG}^\circ(\text{per} \Gamma_0),$$

(9.3)

where the canonical heart $\mathcal{H}_0$ corresponds to $\Gamma_0$ (due to Keller-Nicolás). Moreover (by Kellers Morita Theorem), if a heart $\mathcal{H}$ corresponds to a silting set $\mathcal{P} = \{P_i\}_{i=1}^n$ under (9.3), then its simples can be labeled as $\{X_i\}_{i=1}^n$ such that

$$\text{Hom}^*(P_i, X_j) = \delta_{ij} k.$$

(9.4)

(b). The quotient map $\pi: \text{per} \Gamma_0 \to \mathcal{C}(\Gamma_0)$ induces an isomorphism

$$\pi_*: \text{SEG}^\circ(\text{per} \Gamma_0)/\text{ST}(\Gamma_0) \cong \text{CEG}(\Gamma_0)$$

(9.5)

(due to Keller-Nicolás). Here, we consider a 2-cycle in the quotient graph $\text{SEG}^\circ(\text{per} \Gamma_0)/\text{ST}(\Gamma_0)$ as an unoriented edge, cf. [19, Section 9]. Denote the image of $\Gamma_0$ by $\mathbf{P}_0$. 
(c). A fundamental domain for \( \text{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0))/\text{ST} \) is the full subgraph
\[
\text{EG}^\circ(\mathcal{H}_0) := \{ \mathcal{H} \in \text{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0)) \mid \mathcal{H}_0 \leq \mathcal{H} \leq \mathcal{H}_0[1] \}
\]
in \( \text{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0)) \) (cf. [19]). In particular, \( \text{EG}^\circ(\mathcal{H}_0) \cong \text{CEG}(\Gamma_0) \) as unoriented graphs. Denote \( \text{SEG}^\circ(\Gamma_0) \) to be the full subgraph of \( \text{SEG}^\circ(\text{per } \Gamma_0) \) that corresponds to \( \text{EG}^\circ(\mathcal{H}_0) \) under the isomorphism in (a). So in particular, \( \text{SEG}^\circ(\Gamma_0) \cong \text{CEG}(\Gamma_0) \).

(d). There is a canonical isomorphism
\[
\wp : \text{EG}(\mathbf{S}) \cong \text{CEG}(\Gamma_0)
\]
such that \( \wp_0 \) corresponds to the initial triangulation \( T_0 = \mathcal{T}_0 \).

(e). The forgetful map \( F : \mathbf{S}_\triangle \to \mathbf{S} \) induces an isomorphism (cf. Remark 3.10)
\[
F_* : \text{EG}^\circ(\mathbf{S}_\triangle)/\text{BT}(\mathbf{S}_\triangle) \cong \text{EG}(\mathbf{S}).
\]
Recall that \( \text{EG}(\mathbf{S}) \) is the exchange graph of (triangulations of) \( \mathbf{S} \) (which is well-known to be connected) and \( \text{EG}^\circ(\mathbf{S}_\triangle) \) the connected component of the exchange graph \( \text{EG}(\mathbf{S}_\triangle) \) that contains \( T_0 \).

Using (7.4), we could extend the isomorphism in (d) above as follows.

**Proposition 9.7.** There is a canonical isomorphism (between graphs)
\[
\wp : \text{EG}^\circ(\mathbf{S}_\triangle) \cong \text{SEG}^\circ(\text{per } \Gamma_0)
\]
sending \( \Gamma_0 \) to \( T_0 \). Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{BT}(\mathbf{S}_\triangle) & \overset{\iota}{\longrightarrow} & \text{ST}(\Gamma_0) \\
\downarrow & & \downarrow \\
\text{EG}^\circ(\mathbf{S}_\triangle) & \overset{\wp}{\longrightarrow} & \text{SEG}^\circ(\text{per } \Gamma_0) \\
\downarrow F_* & & \downarrow \pi_* \\
\text{EG}(\mathbf{S}) & \overset{\wp}{\longrightarrow} & \text{CEG}(\Gamma_0)
\end{array}
\]

where the upper commutativity means \( \wp \circ \Psi(T) = \iota(\Psi)(\wp(T)) \) for any \( T \in \text{EG}^\circ(\mathbf{S}_\triangle) \) and \( \Psi \in \text{BT}(\mathbf{S}_\triangle) \).

**Proof.** Combine (b) and (d) above, we have \( \text{SEG}^\circ(\Gamma_0) \cong \text{EG}(\mathbf{S}) \). In particular, \( \text{EG}(\mathbf{S}) \) inherits the orientation of \( \text{SEG}^\circ(\Gamma_0) \). Note that \( \text{SEG}^\circ(\Gamma_0) \) has an unique source \( \mathcal{H}_0 \) and an unique sink \( \mathcal{H}_0[1] \). Lifting \( \text{EG}(\mathbf{S}) \) to \( \text{EG}^\circ(\mathbf{S}_\triangle) \) with respect to such an orientation, such that \( T_0 \) (corresponds to \( \mathcal{H}_0 \)) become \( T_0 \), we obtain a fundamental domain \( \text{EG}^\circ(T_0) \) in \( \text{EG}^\circ(\mathbf{S}_\triangle) \) for \( \text{EG}^\circ(\mathbf{S}_\triangle)/\text{BT}(\mathbf{S}_\triangle) \), which is isomorphic to \( \text{EG}^\circ(\mathcal{H}_0) \).

Next, we claim \( \text{BT}(\mathbf{S}_\triangle) \) and \( \text{ST}(\Gamma_0) \) act freely on \( \text{EG}^\circ(\mathbf{S}_\triangle) \) and \( \text{SEG}^\circ(\text{per } \Gamma_0) \), respectively. If so, by isomorphism \( \iota \) in (7.4), we can extend \( \text{EG}^\circ(T_0) \cong \text{EG}^\circ(\mathcal{H}_0) \) to the required isomorphism.

For \( \text{BT}(\mathbf{S}_\triangle) \), the freeness follows from the Alexander method. A triangulation of \( \mathbf{S}_\triangle \) divides \( \mathbf{S}_\triangle \) into once-punctured triangles/disks); hence if \( \Psi \in \text{BT}(\mathbf{S}_\triangle) \) preserves any triangulation, then \( \Psi = 1 \) in \( \text{MCG}(\mathbf{S}_\triangle) \). For \( \text{ST}(\Gamma_0) \), the freeness follows from the definition of isotopy of auto-equivalences. If \( \psi \in \text{ST}(\Gamma_0) \) preserves any heart, then it
preserves all tilts from this heart, i.e. preserves $\text{EG}^\circ (\mathcal{D}_{fd}(\Gamma_0))$ (and thus their simples). By Lemma 9.2, this implies $\psi = 1$ in $\text{Aut}^\circ (\mathcal{D}_{fd}(\Gamma_0))$. \qed

**Remark 9.8.** For the dual triangulations in $S_\Delta$, the flip becomes the Whitehead move, as shown in (10). Then the (principal component of) exchange graph of dual triangulations of $S_\Delta$ (with Whitehead moves) is isomorphic to shift quotient of $\text{EG}^\circ (\mathcal{D}_{fd}(\Gamma_0))$, which is, in fact, induced by the bijection $\tilde{X}$ between closed arcs and shift orbits of spherical objects in Theorem 7.6. See Corollary 9.9 for a precise statement.

### 9.5. Open arcs versus reachable rigid indecomposables

This subsection is devoted to construct a bijection between open arcs and ‘silting summands’, which induces the isomorphism $\varphi$ in Proposition 9.7.

Let $\text{OA}(S)$ be the set of simple open arcs in $S$ and $\text{RR}(\mathcal{C}(\Gamma_0))$ be the set of reachable rigid indecomposables in $\mathcal{C}(\Gamma_0)$, that is

$$\text{RR}(\mathcal{C}(\Gamma_0)) = \bigcup_{P \in \text{CEG}(\Gamma_0)} P.$$  

Then there is a bijection ([6] and [28])

$$\varphi: \text{OA}(S) \to \text{RR}(\mathcal{C}(\Gamma_0)),$$  (9.10)

which induces the isomorphism $\varphi$ in (9.6), i.e. $\varphi = \varphi_*$, in the sense that

$$\varphi(T) = \{ \rho(\nu) \mid \nu \in T \}.$$  

We proceed to construct the analogue bijection for $S_\Delta$ and per $\Gamma_0$. First, we give a precise statement for Remark 9.8.

Let $T$ be a triangulation in $\text{EG}^\circ (S_\Delta)$ consisting of open arcs $\{\nu_i\}_{i=1}^n$ and its dual $T^*$ consists of corresponding closed arcs $\{\eta_i\}_{i=1}^n$. As

$$\pi_* \circ \varphi(T) = \rho_* \circ F_*(T)$$

in (9.9), let $P = \varphi(T)$ consists of indecomposables $\{P_i\}_{i=1}^n$, such that $\pi(P_i) = \rho \circ F(\nu)$. Equivalently, $P_i$ is the unique element in the intersection

$$\varphi(T) \cap (\pi^{-1} \circ \rho \circ F(\nu))$$  (9.11)

for $\nu = \nu_i$. Furthermore, under the isomorphism (9.3), let $\mathcal{H}$ with simples $\{X_i\}_{i=1}^n$ be the heart corresponding to $P$ so we obtain a diagram (9.12).

$$\begin{array}{ccc}
T = \{\nu_i\} & \stackrel{\varphi}{\longrightarrow} & P = \{P_i\} \\
\text{graph dual} & & \text{proj.-sim. dual} \\
T^* = \{\eta_i\} & \stackrel{\tilde{X}}{\longrightarrow} & \text{Sim } \mathcal{H} = \{X_i\}.
\end{array}$$  (9.12)

We have the following, which says the graph duality (between triangulation and its dual) corresponds to the projective-simple duality.

**Corollary 9.9.** Under the notation above, we have $\tilde{X}(\eta_i) = X_i[\mathbb{Z}]$ for $i = 1, \ldots, n$. 


Proof. Use induction, on the minimal step of flips from $T$ to $T_0$, starting with the case when $T = T_0$. Then $P_i = \Gamma_i = e_i\Gamma_0$, $\eta_i = s_i$ and $X_i = S_i$. The claim follows from construction of $\tilde{X}$. The inductive step, assume that the claim holds for a triangulation $T_1$ and consider the flip $T_2$ of it with respect to an open arc $\nu$ (whose dual is $\alpha$ in $T_1$). Without lose of generality, assume the flip is forward. Then we have the following:

- The flip becomes the Whitehead move of the dual triangulation $T_1$ (see Figure 10). The corresponding closed arc $\eta$ is unchange (from $T_1$ to $T_2$), unless there are arrows from $\eta^*$ to $\nu$ in the quiver $Q_{T_1}$, and $\eta$ becomes $B_{T_1}^{-1}(\eta)$.

- The flip becomes the (forward) mutation of the corresponding silting set and (forward) simple tilting of the corresponding heart. Let $S$ be the simple corresponding to $\nu$ and $H_i$ be the hearts corresponding to $T_i$. By the tilting formulae in [19, Proposition 5.2 and Remark 7.2], a simple $X$ is unchange (from $H_1$ to $H_2$) up to shift, unless

- $\operatorname{Ext}^1(X, S) \neq 0$,
- or equivalently, there are arrows from $\eta^*$ to $\nu$ in the quiver $Q_{T_1}$, and $X$ becomes $\phi_S^{-1}(X)$.

By inductive assumption, we have $\tilde{X}(\alpha) = S[\mathbb{Z}]$ and $\tilde{X}(\eta) = X[\mathbb{Z}]$. Thus, by Corollary 7.4,

$$\tilde{X}(B_{\alpha}^{-1}(\eta)) = \nu(B_{\alpha}^{-1})\tilde{X}(\eta) = \phi_S^{-1}(X)[\mathbb{Z}],$$

which completes the induction. \qed

Remark 9.10. In fact, the triangulation $T_0$ in Theorem 7.7 can be chosen arbitrarily. This is because, under a flip of triangulation or the tilt of a heart, the formulae of generators changing for $\text{BT}(T)$ and $\text{ST}(T)$ do match (cf. the conjugate formulae for braid/spherical twists and formulae of how simple/closed arc changes mentioned above in Corollary 9.9). Thus, as $\text{BT}(T) \to \text{ST}(\Gamma_T)$ holds for $T = T_0$, it holds for any $T$ by induction.

Second, we define the reachable simples open arcs and reachable rigid indecomposables. Note that in the case for $S$ and $\mathcal{C}(\Gamma_0)$, all such open arcs or indecomposables are reachable, which is not in this case.

Definition 9.11. Denote by $\text{OA}^0(\mathbb{S}_\Delta)$ is the set of reachable simple open arcs in $\mathbb{S}_\Delta$, that is

$$\text{OA}^0(\mathbb{S}_\Delta) = \bigcup_{T \in \text{EG}^0(\mathbb{S}_\Delta)} T,$$

where a triangulation $T$ here is considered to be a set of open arcs. Denote by $\text{RR}(\mathcal{C}(\Gamma_0))$ is the set of reachable rigid indecomposables in $\mathcal{C}(\Gamma_0)$, that is

$$\text{RR}(\text{per } \Gamma_0) = \bigcup_{P \in \text{SEG}^0(\text{per } \Gamma_0)} P.$$

Third, we prepare with a lemma. Let $\text{SEG}^0(\mathbb{T}_0) = \phi^{-1}(\text{SEG}^0(\Gamma_0))$, which is a full subgraph of $\text{EG}^0(\mathbb{S}_\Delta)$ (cf. (b) in Section 9.4 for the definition of $\text{SEG}^0(\Gamma_0)$). Then it is a fundamental domain for $\text{EG}^0(\mathbb{S}_\Delta)/\text{BT}$ and we have $F_*: \text{EG}^0(\mathbb{T}_0) \cong \text{EG}(\mathbb{S})$.

Lemma 9.12. Let $\nu_i \in T_i$ and $T_i \in \text{EG}^0(\mathbb{T}_0)$ for $i = 1, 2$. If $F(\nu_1) = F(\nu_2)$, then $\nu_1 = \nu_2$. 

**Proof.** Let \( \nu = F(\nu_1) \) and \( T_i = F_i(T_i) \) that contain \( \nu \). Consider the surface \( S \setminus \nu \), which is obtained from \( S \) by cutting along \( \nu \) (see [28, Figure 17 and 18] for the procedure of cutting). Denote by \( \text{EG}_\nu(S) \) the full subgraph of \( \text{EG}(S) \) consisting of triangulations that contains \( \nu \). We have \( \text{EG}_\nu(S) \cong \text{EG}(S \setminus \nu) \), which is connected. Thus, there is a path \( p \) in \( \text{EG}_\nu(S) \) connecting \( T_1 \) and \( T_2 \), which lifts, via \( F_{\nu}^{-1} \), to a path \( \bar{p} \) in \( \text{EG}_\nu(T_0) \) connecting \( \bar{T}_1 \) and \( \bar{T}_2 \). Notice that any triangulation in \( \bar{p} \) contains \( \nu \), or equivalently, \( \nu \) remains unchanged during these flips. Thus, by looking at the lifted flips in \( \bar{p} \), we deduce that \( \nu_1 \) in \( T_1 \) corresponds to \( \nu_2 \) in \( T_2 \) unchanged, as required. \( \square \)

**Theorem 9.13.** There is a canonical bijection

\[
\rho: \text{OA}^\circ(S_\Delta) \to \text{RR}(\text{per } \Gamma_0)
\]

sending initial arcs \( \gamma_i \in T_0 \) to \( \Gamma_0 \) and fitting into the following commutative diagram:

\[
\begin{array}{ccc}
\text{OA}^\circ(S_\Delta) & \xrightarrow{\rho} & \text{RR}(\text{per } \Gamma_0) \\
F \downarrow & & \downarrow \pi \\
\text{OA}(S) & \xrightarrow{\rho} & \text{RR}(\text{C}(\Gamma_0))
\end{array}
\]  

Further, it induces the isomorphism \( \varphi \) in (9.8), i.e. \( \varphi = \rho_* \) in the sense that

\[
\varphi(T) = \{ \rho(\nu) \mid \nu \in T \}.
\]

**Proof.** Consider a pair \((\nu, T)\), where \( \nu \) is an open arc in a triangulation \( T \) of \( S_\Delta \). Define \( \rho(\nu, T) \) to be the element in the silting set \( \varphi(T) \) whose image under \( \pi \) in \( \text{C}(\Gamma_0) \) is \( \rho \circ F(\nu) \). That is, (9.11) to be precise. Note that for any \( \Psi \in \text{BT}(S_\Delta) \), we have

\[
\rho(\Psi(\nu), \Psi(T)) = (\rho \circ \Psi(T)) \cap L(\Psi(\nu)) = (\rho \circ \Psi(T)) \cap L(\nu) = (\iota(\Psi) \circ \rho(\nu, T)) \cap L(\nu) = \iota(\Psi)(\rho(\nu, T)) = \iota(\Psi)(\rho(\nu, T)),
\]

where \( L = \pi^{-1} \circ \rho \circ F \) satisfying \( L \circ \Psi = L \circ \iota(\Psi) \circ L \) and the third equality follows from the commutativity of the upper square in (9.9).

To finish the proof, we only need to show that \( \rho(\nu, T) \) is independent of \( T \), or equivalently,

\[
\rho(\nu, T_1) = \rho(\nu, T_2)
\]

for any \( T_1 \) and \( T_2 \) containing \( \nu \). If so, \( \rho(\nu): = \rho(\nu, T) \) clearly satisfies all the required conditions.

First, consider the case when \( T_1 \) and \( T_2 \) are related by a flip. In such a case, they only differ by one close arc and so does the corresponding silting set \( \varphi(T_1) \). As \( \nu \in T_1 \cap T_2 \), the flip is not with respect to \( \nu \). Thus both \( \rho(\nu, T_1) \) and \( \rho(\nu, T_2) \) are in \( \varphi(T_1) \cap \varphi(T_2) \), which implies the claim (9.15).

Second, consider the case when \( T_1 \) and \( T_2 \) are both in the fundamental domain \( \text{EG}^\circ(T_0) \). Then \( F_*: \text{EG}^\circ(T_0) \cong \text{EG}(S) \) as unoriented graphs. Let \( \text{EG}_\nu(S) \) be the
full subgraph of \( \text{EG}(S) \), consisting of triangulations that contains \( \nu = F(\nu) \). By the first case above, it is sufficient to show that \( T_1 \) and \( T_2 \) are connected by a path in \( \text{EG}^0(T_0) \) such that any triangulation in this path contains \( \nu \). This is equivalent to \( T_1 = F_s(T_1) \) and \( T_2 = F_s(T_2) \) are connected in \( \text{EG}_\nu(S) \). Consider the cut surface \( S \setminus \nu \) as in Lemma 9.12. We have \( \text{EG}_\nu(S) \cong \text{EG}(S \setminus \nu) \), which is connected. Thus, the claim (9.15) holds in this situation.

Third, consider the case when \( T_2 = \Psi(T_1) \) and \( \Psi(\nu) = \nu \) for some \( \Psi \in \text{BT}(S_\Delta) \). By the former condition, we have \( \rho(\nu, T_2) = \iota(\Psi)(\rho(\nu, T_1)) \). What is left to prove in this case is that the later condition implies \( \iota(\Psi) \) preserves \( P = \rho(\nu, T_1) \). Consider the cut surface \( S \setminus \nu \) again, which inherits all the decorated points and \( (S \setminus \nu)_\Delta \) inherits a triangulation \( T_1 \setminus \nu \). Since \( \Psi \) preserves \( \nu \), it is actually in \( \text{BT}(S \setminus \nu) \), or \( \text{BT}(T_1 \setminus \nu) \) by Proposition 4.3. As \( \text{BT}(T_1 \setminus \nu) \) is generated by \( B_\eta \), for the closed arc \( \eta \in (T_1 \setminus \nu)^* \) dual to some open arc \( \gamma \in T_1 \setminus \nu \), we only need to show that \( \iota(\Psi) \) preserves \( P \). Consider \( \rho(\gamma, T_1) \) in the silting set \( \varphi(T_1) \) and the corresponding simple \( X \) in the corresponding heart. Then, by (9.4),

\[
\text{Hom}^*(P, X) = 0
\]

and, by Corollary 9.9, \( \tilde{X}(\eta) = X[\mathbb{Z}] \). Together with (7.8), we have

\[
\iota(B_\eta)(P) = \phi_{\tilde{X}(\eta)}(P) = \phi_X(P) = P,
\]

as required.

Finally, consider the general case. Recall that \( \text{EG}^0(T_0) \) is a fundamental domain for \( \text{EG}^0(S_\Delta)/\text{BT} \). Let \( T_i = \Psi_i(T'_i) \) for some \( \Psi_i \in \text{BT}(S_\Delta) \) and \( T'_i \in \text{EG}^0(T_0) \), \( i = 1, 2 \). Let \( \nu'_i = \Psi_i(\nu) \) and then \( F(\nu'_i) = F(\nu) = F(\nu'_2) \), which implies

\[
\nu'_1 = \nu'_2 =: \nu',
\]

by Lemma 9.12. Since \( \nu' \in T'_1 \cap T'_2 \), by the second case above, we have

\[
\rho(\nu', T'_1) = \rho(\nu', T'_2).
\]

Since \( \Psi_2^{-1} \circ \Psi_1(\nu) = \Psi_2^{-1}(\nu') = \nu \), by the third case above, we have

\[
\rho(\nu, T_1) = \rho(\nu, \Psi_2^{-1} \circ \Psi_1(T_1)).
\]

Combining (9.16),(9.17) and formula (9.14), we have

\[
\rho(\nu, T_1) = \rho(\nu, \Psi_2^{-1} \circ \Psi_1(T_1)) = \iota(\Psi_2^{-1})(\iota(\Psi_2)(\rho(\nu', T'_1)) = \iota(\Psi_2^{-1})(\rho(\nu', T'_2)) = \rho(\nu, T_2),
\]

which finishes the proof.

\[\square\]

**Remark 9.14.** By Theorem 9.13, the forgetful map \( F: S_\Delta \to S \) is a ‘geometric realization’ of Amiot’s quotient map \( \pi: \Gamma_0 \to \mathcal{C}(\Gamma_0) \):

- the correspondence from open arcs to reachable rigid indecomposables \( (\rho \text{ and } \rho') \) commutes with them;
- the closed arcs get killed under \( F \), so do the spherical objects under \( \pi \).
9.6. Rotations in marked mapping class groups.

Definition 9.15. The marked mapping class group $\text{MMCG}(S)$ of a marked surface $S$ is the group of isotopy classes of homeomorphisms of $S$, where all homeomorphisms and isotopies are required to

- fix the set $M$ of marked points as a set.

Note that the boundaries are NOT required to be fixed pointwise. Therefore, for each boundary component $C \in \partial S$ with $m$ marked points, denote by $\xi_C$ the $m$-th root of the Dehn twist around $C$, that is, simultaneous (anticlockwise) rotation to the next marked point on $C$. Then the universal rotation $\xi$, as an element in $\text{MMCG}(S)$, is

$$\xi = \prod_{C \in \partial S} \rho_C.$$ 

Here, the product is over all connected components $C$ of $\partial S$.

Similar for the definition of marked mapping class group of $S_\triangle$ (requires fixing the set $M$ of marked points and the set $\triangle$ of decorated points as sets) and the universal rotation $\xi \in \text{MMCG}(S_\triangle)$.

Remark 9.16. For the punctured case, the marked mapping class group should evolve to the tagged mapping class group, cf. [5] and [4].

Recall that, under the bijection $\rho$ in (9.10), the action $\xi$ on $\text{OA}(S)$ corresponds to the shift (i.e. [1], or equivalently, the Auslander-Reiten translation $\tau$) on $\text{RR}(\mathcal{C}(\Gamma_0))$. In other words, we have (cf. [6] and [5])

$$\rho(\gamma)[1] = \rho(\xi(\gamma)),$$

for any $\gamma \in \text{OA}(S)$.

We prove the analogue result for $S_\triangle$.

Proposition 9.17. For any $\gamma \in \text{OA}^\circ(S_\triangle)$, we have

$$\rho(\gamma)[1] = \rho(\xi(\gamma)).$$

Proof. This follows exactly the same way as in [5, Lemma 3.5] (cf. [5, Figure 6]), noticing that the (forward/backward) mutation formulae for silting/cluster tilting coincide indeed. □
Note that, under the forgetful map, $\xi$ becomes $\xi'$; under Amiot’s quotient, $[1]$ in $\text{per} \Gamma_0$ becomes $[1]$ in $\mathcal{C}(\Gamma_0)$; and they are compatible (cf. the diagram blow).

\[
\begin{array}{c}
S_{\Delta} \circ \xi \\
\downarrow F
\end{array}
\begin{array}{c}
[1] \circ \text{per} \Gamma_0
\downarrow \pi
\end{array}
\begin{array}{c}
S \circ \xi \\
\downarrow \rho
\end{array}
\begin{array}{c}
[1] \circ \mathcal{C}(\Gamma_0)
\end{array}
\]

**Remark 9.18** (On the maximal green mutation sequence). Combining (9.8) and (9.3), we have $\text{EG}(\mathcal{D}_{fd}(\Gamma_0)) \cong \text{SEG}(\text{per} \Gamma_0) \cong \text{EG}(S_{\Delta})$. Further, Proposition 9.17 implies the shift (as graph automorphism) on the first two graphs corresponds to the universal rotation on the third one. This provide a geometric/combinatorial model for studying the existence of Keller’s (maximal) green mutation sequences.

More precisely, consider a quiver with potential $(Q,W)$ that corresponds to a heart $\mathcal{H}$ in $\mathcal{D}_{fd}(\Gamma_0)$. A maximal green mutation sequence for $(Q,W)$ corresponds (by [16] and cf. [27]) to a forward simple tilting sequence from $H$ to $H[1]$ in $\text{EG}(\mathcal{D}_{fd}(\Gamma_0))$. By the isomorphisms above, this is equivalent to a forward flip sequence from $T$ to its universal rotation $\xi(T)$, where $T$ is the triangulation that corresponds to $H$.

### 10. On the simply connectedness of stability spaces via quadratic differentials

**10.1. Stability conditions.** Let recall Bridgeland’s notion of stability conditions (cf. e.g. [4] or [26]).

**Definition 10.1.** A stability condition $\sigma = (Z,P)$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism (the central charge) $Z : K(\mathcal{D}) \to \mathbb{C}$ and full additive subcategories $P(\varphi) \subset \mathcal{D}$ for each $\varphi \in \mathbb{R}$, satisfying the following axioms:

- if $0 \neq E \in P(\varphi)$ then $Z(E) = m(E) \exp(\varphi \pi i)$ for some $m(E) \in \mathbb{R}_{>0}$;
- $P(\varphi + 1) = P(\varphi)[1]$, for all $\varphi \in \mathbb{R}$;
- if $\varphi_1 > \varphi_2$ and $A_i \in P(\varphi_i)$ then $\text{Hom}_D(A_1, A_2) = 0$;
- for each nonzero object $E \in \mathcal{D}$ there is a finite sequence of real numbers $\varphi_1 > \varphi_2 > ... > \varphi_m$ and a collection of triangles (the Harder-Narashimhan filtration)

\[
\begin{array}{cccccc}
0 = E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & ... & \rightarrow & E_{m-1} & \rightarrow & E_m = E
\end{array}
\]

with $A_j \in P(\varphi_j)$ for all $j$.

For any nonzero object $E \in \mathcal{D}$ with Harder-Narashimhan filtration above, define its upper phase to be $\Psi^+_{P}(E) = \varphi_1$ and lower phase to be $\Psi^-_{P}(E) = \varphi_m$. Note that, $P(\varphi)$ is abelian. Let $I$ be an interval in $\mathbb{R}$ and define

$$
P(I) = \{ E \in \mathcal{D} \mid \Psi^+_P(E) \in I \}.$$
Then for any $\varphi \in \mathbb{R}$, $P[\varphi, \infty)$ and $P(\varphi, \infty)$ are t-structures in $D$. Further, we say the heart of a stability condition $\sigma = (Z, P)$ on $D$ is $P[0, 1)$.

A important result by Bridgeland is the following.

**Theorem 10.2.** [4, Theorem 7.4] All stability conditions on a triangulated category $D$ form a space $\text{Stab}(D)$ that has the structure of a complex manifold, such that the forgetful map $\text{Stab}(D) \to \text{Hom}_Z(K(D), \mathbb{C})$ taking a stability condition to its central charge, is a local isomorphism.

We are interested in the stability space $\text{Stab}^\circ D_{fd}(\Gamma_0)$. Note that for the stability conditions on $D_{fd}(\Gamma_0)$ whose heart is the canonical heart $H_{\Gamma}$ form a half open half close n-cell $U_T$ in $\text{Stab} D_{fd}(\Gamma_0)$ (see [26]). Denote by $\text{Stab}^\circ D_{fd}(\Gamma_0)$ the connected component of $\text{Stab} D_{fd}(\Gamma_0)$ that contains $U_T$.

### 10.2. Quadratic differentials.

Denote by $\text{Quad}(S)$ is the moduli space of quadratic differentials on $S$, in the sense of [4, Section 6]. Recall the main result there as follows.

**Theorem 10.3.** [4, Theorem 11.2]

$$\text{Stab}^\circ D_{fd}(\Gamma_0)/\text{Aut}^\circ \cong \text{Quad}(S).$$

(10.2)

For our purpose, we prefer to deal the space $\text{Quad}(S)$ of quadratic differential on a fixed marked surface $S$ instead of the moduli space. These two spaces of quadratic differentials differ by the symmetry of the (marked) mapping class group $\text{MMCG}(S)$ (cf. Definition 9.15):

$$\text{Quad}(S) = \text{Quad}(S)/\text{MMCG}(S).$$

By [4, Theorem 9.9], there is the short exact sequence

$$1 \to \text{ST}(\Gamma_0) \to \text{Aut}^\circ D_{fd}(\Gamma_0) \to \text{MMCG}(S) \to 1$$

and therefore we have

$$\text{Stab}^\circ D_{fd}(\Gamma_0)/\text{ST} \cong \text{Quad}(S).$$

(10.4)

Thus there is a short exact sequence

$$1 \to \pi_1 \text{Stab}^\circ D_{fd}(\Gamma_0) \to \pi_1 \text{Quad}(S) \xrightarrow{\xi} \text{ST}(\Gamma_0) \to 1.$$  

(10.5)

### 10.3. On the simply connectedness.

In this subsection, let $S$ be an annulus with $p$ and $q$ marked points on its boundary components respectively.

Suppose $p \neq q$ first. It is straightforward to calculate $\text{MMCG}(S)$ in this case: it is generated by the two rotations along the two boundary components. More precisely, $\text{MCG}(S)$ is the infinite cyclic group generated by the Dehn twist $D_C$ along the only (up to isotopy) non-trivial simple closed curve in $S$. The two rotations are the $p$-th and $q$-th roots of $D_C$, denoted by $r_0$ and $r_1$, respectively. Then $\text{MMCG}(S)$ is the abelian group with generators $r_0$ and $r_1$ with relation $r_0^p = r_1^q$, which fits into the following short exact sequence

$$1 \to \mathbb{Z}\langle r_0 \rangle \to \text{MMCG}(S) \to \mathbb{Z}_q \langle r_1 \rangle \to 1.$$  

Besides $\xi = r_0 \cdot r_1$ is the universal rotation that corresponds to $[1]$.

Next, as shown in [4, Section 12.3],

$$\text{Quad}(S) \cong \text{Conf}^n(\mathbb{C}^*)/\mathbb{Z}_q,$$

(10.6)
where \( \text{Conf}^n(C^*) \) denotes the configuration space of \( n \) distinct points in \( C^* \) and \( \mathbb{Z}_q \) acts by multiplication by a \( q \)-th root of unity. By the description of \( \text{Br}(\tilde{A}_{p,q}) \) in [8], there is short exact sequence
\[
1 \rightarrow \text{Br}(\tilde{A}_{p,q}) \rightarrow \pi_1 \text{Conf}^n(C^*) \rightarrow \mathbb{Z} \rightarrow 1. \tag{10.7}
\]
As \( \text{Quad}(S) \) consists of differentials of the form
\[
\Theta(z) = \prod_{i=1}^{n} (z - z_i) \frac{d z^{e_2}}{z^{p+2}}, \quad z_i \in C^*, \quad z_i \neq z_j
\]
and considered modulo the action of \( C \) rescaling \( z \). Note that \( z_i \) corresponds to the decorated points in \( S_\Delta \), the rotation \( r_q \) becomes the \( \mathbb{Z}_q \) symmetry at the origin and the rotation \( r_p \) becomes the \( \mathbb{Z}_p \) symmetry at the infinity. Thus, combining the short exact sequences above plus a calculation for \( \pi_1 \) of (10.6), we have the commutative diagram (10.8), which implies the dashed short exact sequence.
\[
\begin{array}{cccccc}
1 & \rightarrow & \text{Br}(\tilde{A}_{p,q}) & \rightarrow & \pi_1 \text{Conf}^n(C^*) & \rightarrow & \mathbb{Z} & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1 \text{Conf}^n(C^*) & \rightarrow & \pi_1 \text{Quad}(S) & \rightarrow & \mathbb{Z}_q & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \text{MMCG}(S) & \rightarrow & \mathbb{Z}_q & \rightarrow & 1 \\
\end{array} \tag{10.8}
\]
Therefore we have \( \pi_1 \text{Quad}(S) = \text{Br}(\tilde{A}_{p,q}) \) and hence \( \pi_1 \text{Quad}(S) \cong \text{ST}(\Gamma_0) \) by Theorem 8.4. Further, by examining the generators, we deduce that the surjective map \( \pi \) in (10.5) gives the isomorphism above. Thus, \( \text{Stab}^\circ D_{fd}(\Gamma_0) \) is simply connected.

In the case when \( p = q \), \( \text{MMCG}(S_\Delta) \) contains one more \( \mathbb{Z}_2 \) symmetry. In the same way, we will have \( \pi_1 \text{Quad}(S) = \text{Br}(\tilde{A}_{p,q}) \) and simply connectedness.

**Theorem 10.4.** Let \( S \) be an annulus (without punctures) and \( D_{fd}(\Gamma_0) \) be the 3-CY category associated to some triangulation of \( S \). Then \( \text{Stab}^\circ D_{fd}(\Gamma_0) \) is the universal cover of \( \text{Conf}^n(C^*) \).

By [7, Theorem 2.7], the following is immediate from above.

**Corollary 10.5.** \( \text{Stab}^\circ D_{fd}(\Gamma_0) \) is contractible.

## 11. Further studies

### 11.1. Fundamental group of quadratic differentials

Let \( S \) be any unpunctured marked surface. We establish the relation between \( \text{Quad}(S) \) and the braid twist group
First choose a base point $\omega_0 \in \text{Quad}(S)$ s.t. it is generic and saddle-free and its WKB triangulation is the initial triangulation $T_0$ of $S$ (cf. [4, Section 1.4]). Up to isotopy, we can assume that the set $\Delta$ of decorated point on $S$ are chosen to be the set of zeroes of $\omega_0$. Then, a loop $p$ in $\pi_1(\text{Quad}(S),\omega_0)$ induces a homeomorphisms in $\text{MCG}(S_\Delta)$, by forgetting the structure of quadratic differentials and only remembering the position of the zeroes at each point in $p$. Thus we obtain a map

$$\pi_1(\text{Quad}(S),\omega_0) \to \text{MCG}(S_\Delta). \tag{11.1}$$

We believe that this map is injective and it is not hard to see the image is $\text{BT}(S_\Delta)$. As $\text{BT}(S_\Delta) \cong \text{ST}(\Gamma_0)$, this should imply $\pi$ in (10.5) is an isomorphism. Therefore, we have the following conjecture (which is being studied in [3]).

**Conjecture 11.1.** Let $S$ be an unpuncture marked surface. Then the $\text{Stab}^o \mathcal{D}_fd(\Gamma_0)$ is simply connected.

### 11.2. Intersection formulae

Note that interpreting the intersection formulae between open/open arcs as dimension of $\text{Hom}$ (or $\text{Ext}$)'s plays a crucial role in many proofs (e.g. the faithfulness of spherical twist group action of type A in [22] and the connectedness of cluster exchange graph in [28]). Although we avoid to prove the analogue formula in [22], it is still important and should hold.

**Conjecture 11.2.** Let $\alpha, \beta \in \text{CA}(S_\Delta)$. We have

$$\dim \text{Hom}^\bullet(\tilde{X}(\alpha),\tilde{X}(\beta)) = 2\text{Int}(\alpha,\beta). \tag{11.2}$$

Note that Proposition 6.4 shows that the conjecture holds if $\text{Int}(\alpha,\beta) < 1$.

**Observation 11.3.** Recall that $\Sigma^\Delta_S$ is the branched double cover of $S_\Delta$ and there is a bijection, by lifting, from $\text{CA}(S_\Delta)$ to $\text{CC}(\Sigma^\Delta_S)$, a set of (simply closed) curves on $\Sigma^\Delta_S$ (cf. Section 7.3). Then there is a bijection

$$X : \text{CC}(\Sigma^\Delta_S) \to \text{Sph}(\Gamma_0)$$

induced by the bijection in Theorem 7.6 and (11.2) becomes

$$\dim \text{Hom}^\bullet(X(C_1),X(C_2)) = \text{Int}(C_1,C_2). \tag{11.3}$$

The reason we are interested to make this observation is that (11.2) will fail in the punctured case while we believe the existence of the bijection $X$ above with formula (11.3), if we modify the branched double cover $\Sigma^\Delta_S$ to be the twisted surface (cf. [20]). Moreover, we have another conjectured formula.

**Conjecture 11.4.** Let $\gamma \in \text{OA}^o(S_\Delta)$ and $\eta \in \text{CA}(S_\Delta)$. We have

$$\dim \text{Hom}^\bullet(\rho(\gamma),\tilde{X}(\eta)) = \text{Int}(\gamma,\eta). \tag{11.4}$$

These two intersection formulae will be proved in [29].
Presentations of twist groups. We ends the paper with a conjecture concerning about the algebraic twist group. Let $T \in EG(S_{\Delta})$ and $T = F(T) \in EG(S)$ with associated quiver with potential $(Q_T, W_T)$. We have the following conjecture and comments.

Conjecture 11.5. If there is no double arrows in $Q_T$, then the braid twist group $BT(T)$ is canonical isomorphic to the algebraic twist group $AT(Q_T, W_T)$ (cf. Appendix A).

In other words, the conjecture above is saying that $BT(T)$ admits a presentation with generators $B_s$ for $s$ in $T^*$ and a set of relations, in the form of $3^\circ$ of Definition A.1, indexed by the triangles in $T$.

Note that when $S$ is a $(n + 3)$-gon, Conjecture 11.5 holds. To see this, on one hand, it is known that

$$Br(A_n) = MCG(S_{\Delta}) = BT(S_{\Delta}) = BT(T).$$

On the other hand, by Proposition A.3, $AT(Q, W) \cong Br(A_n)$ if $(Q, W)$ is mutation-equivalent to a quiver of type $A_n$. Then inductive, one can prove Conjecture 11.5 in this case. Similarly, when $S$ is a once punctured $n$-gon, Conjecture 11.5 holds by Proposition A.3.

Remark 11.6. Our philosophy is that each term in a potential $W$ contributes a relation in the spherical twist group $ST\Gamma(Q, W)$ (with generators indexed by $Q_0$).

**Appendix A. Algebraic twist group of quivers with potential**

Let $(Q, W)$ be a rigid quiver with potential such that there is no double arrow in $Q$ and $W$ is the sum of some cycles in $Q$.

**Definition A.1.** The algebraic twist group $AT(Q, W)$ of such a quiver with potential $(Q, W)$ is the group with generated by $\{t_i \mid i \in Q_0\}$ with the relations

1°. $t_i t_j = t_j t_i$ if there is no arrow between $i$ and $j$ in $Q$.

2°. $t_i t_j t_i = t_j t_i t_j$ if there is exactly one arrow between $i$ and $j$ in $Q$.

3°. $R_i = R_j$ for any $i, j$ (cyclic relation), if there is a cycle $Y: 1 \to 2 \to \cdots \to m \to 1$ in $Q$ (or a term in $W$ by definition), where $R_i = t_i t_{i+1} \cdots t_{2m+i-3}$ with convention $k = m + k$ here.

First, we show that any cyclic relations in Definition A.1, that correspond to the same cycle $Y$, are equivalent to each other.

**Lemma A.2.** Let $m \geq 3$ and suppose that $t_1, t_2, \cdots, t_m$ satisfies the relations

$$\begin{align*}
    t_j t_i t_j &= t_i t_j t_i, & |j - i| = 1 \text{ or } \{i, j\} = \{1, m\}, \\
    t_i t_j &= t_j t_i, & \text{otherwise.}
\end{align*}$$

(A.1)

Let $k = m + k$ and $R_i = t_i t_{i+1} \cdots t_{2m+i-3}$. Then relation $R_i = R_j$ is equivalent to $R_1 = R_i$ for any $3 \leq i \leq m$.

**Proof.** By the relations in (A.1), it is straightforward to check the following

$$\begin{align*}
    t_i R_1 &= R_1 t_{i-2}, & i = 2, \cdots, m - 1, \\
    t_i R_{i+1} &= R_{i} t_{i-2}, & i = 3, \cdots, m.
\end{align*}$$
Then we have
\[ R_1 = R_i \iff R_1t_{i-2} = R_it_{i-2} \]
\[ \iff t_1R_1 = t_iR_{i+1} \]
\[ \iff R_1 = R_{i+1} \]
for any \( i = 2, \ldots, m-1 \), which implies the lemma. \( \Box \)

A consequence of Lemma A.2 is
\[ R_i = R_j \iff R_k = R_l \]
provided \( i \neq j \) and \( k \neq l \).

The following result was original in [20] for type \( A \) and \( D \), which is also independent obtained by Grant-Marsh for all Dynkin type.

**Proposition A.3.** If \( (Q, W) \) is mutation-equivalent to a Dynkin diagram \( Q \), then the algebraic twist group \( \text{AT}(Q, W) \) is isomorphic to the corresponding braid group \( \text{Br}(Q) \).

**Remark A.4.** We believe that the proposition above also holds for affine Dynkin case, as long as \( Q \) does not have double arrows. The point is, one should be able to define an algebraic twist group for a (good) quiver with potential, which provides a presentation of the corresponding spherical twist group (or/and Dehn twist group).

**Appendix B. Proof of Proposition 6.4**

**B.1. L-arcs.** A closed arcs is an \( L \)-arc if it is simple (no self intersection except at the endpoints) and its endpoints coincide. It is straightforward to see \( \text{Int}(T_0, \eta) \geq 2 \) for an \( L \)-arc \( \eta \) and the equality holds if and only if \( \eta \) is contained within two triangles of \( T_0 \) (which encloses one decorated point).

First thing to notice is that for an \( L \)-arc \( \eta \), one can associate a \( \mathfrak{m} \)-perfect dg \( \mathfrak{C}_0 \)-mod \( X_\eta \), well-defined up to shifts, as shown in Section 5.3. Note that, it is possible that different \( L \)-arcs correspond to the same \( \mathfrak{C}_0 \)-mod (up to shifts). Second, Proposition 6.1 can be generalized to the \( L \)-arc case, using the same proof, as follows.

**Proposition B.1.** Let \( \alpha, \beta \) be two closed arcs in \( \text{CA}(\mathbf{S}_\Delta) \) and \( \eta \) a closed arc or an \( L \)-arc. If these three arcs form a contractible triangle, then there are representatives \( X_? \) in \( \hat{X}(?) \) for \( ? = \alpha, \beta, \eta \) such that they form a non-trivial triangle (cf. (6.1), the order of them depends on their relative positions).

Third, we have the corresponding version of Lemma 3.14. Note that any \( L \)-arc will intersect at least two triangles in \( T_0 \).

**Lemma B.2.** Let \( \eta \) be an \( L \)-arc with base point \( Z \). Choose any triangle \( T_0 \) in \( T_0 \) with decorated point \( Z_0 \neq Z \), such that \( \eta \) intersects \( T_0 \). Then there are two closed arcs \( \alpha, \beta \) in \( \text{CA}(\mathbf{S}_\Delta) \) connecting \( Z \) and \( Z_0 \), such that \( \text{Int}(T_0, \eta) = \text{Int}(T_0, \alpha) + \text{Int}(T_0, \beta) \), and \( \alpha, \beta, \eta \) form a contractible triangle in \( \mathbf{S}_\Delta \).

**Remark B.3.** Note that, in the setting of this lemma (and Lemma B.2), the line segment \( l \), from \( Z_0 \) to some point \( Y \) in \( \eta \) (cf. Figure 8 and Figure 25), plays an important role (cf. Proof of Lemma B.2). We will also say \( \eta \) decomposes to \( \alpha \) and \( \beta \) (using \( Z_0 \) or using \( l \)) in this case.
B.2. Stronger statement and double induction. We will prove Proposition 6.4 with a generalization of $2\circ$:

2.5° Let $\eta_i$ be either an L-arc or in $\text{CA}(S_\Delta)$ with $\text{Int}(\eta_1, \eta_2) = 0$, then (6.6) holds.

Use double induction, the first on

$$I = \text{Int}(T_0, \eta_1) + \text{Int}(T_0, \eta_2).$$  \hfill (B.1)

Start with the trivial case when $I = 2$, which implies $\eta_k \in T_0^*$, since the only $\eta$ such that $\text{Int}(T_0, \eta) = 1$ are the arcs $s_i$ in $T_0^*$. Now suppose that the Proposition holds for any $\eta_k$ with $I \leq r$ and consider the case when $I = r + 1$.

First, let us prove 1° for $X_\eta$ (where $\eta = \eta_1$ or $\eta_2$). Apply Lemma 3.14 to $\eta$ and keep the notation there (so we have $\alpha$ and $\beta$). Then by Proposition 6.1, there is a non-trivial triangle

$$X_\beta \to X_\eta \to X_\alpha \to X_\beta[1].$$  \hfill (B.2)

By inductive assumption, the proposition holds for $(\alpha, \beta)$. So we have

$$\dim \text{Hom}^*(X_\alpha, X_\beta) = 1,$$

since $\text{Int}(\alpha, \beta) = \frac{1}{2}$. Then the triangle above is equivalent to

$$X_\eta = \phi_{X_\alpha}(X_\beta) = \phi_{X_\beta}^{-1}(X_\alpha),$$  \hfill (B.3)

where $\phi$ is the spherical twist functor in (2.4), which implies that $X_\eta$ is in $\text{Sph}(\Gamma_0)$.

Next, we prove 2.5° and 3°. Use the second induction on

$$\min\{\text{Int}(T_0, \eta_1), \text{Int}(T_0, \eta_2)\}.$$  \hfill (B.4)

Without lose of generality, suppose that

$$\text{Int}(T_0, \eta_1) \leq \text{Int}(T_0, \eta_2).$$  \hfill (B.4)
B.3. **Start step.** The starting case is when Int(\(T_0, \eta_1\)) = 1 (and for any \(I \geq 2\)), which implies that \(\eta_i = s_i\) for some \(i\). Note that Int(\(T_0, \eta_2\)) > 1 and as above, we can apply Lemma 3.14 and Proposition 6.1 to \(\eta = \eta_2\) to get (B.2), if \(\eta\) is in CA(\(S_\Delta\)); we will apply Lemma B.2 and Proposition B.1 to \(\eta = \eta_2\) if \(\eta\) is an L-arc. Without lose of generality, we will assume that in the L-arc case the \(\alpha, \beta\) are choose so that the triangle in Proposition B.1 is (B.2). Note that, it is clear here that the line segment \(l\) (cf. Remark B.3) for \(\eta\) will not intersect \(\eta_i = s_i\) in \(S_\Delta - \Delta\).

If \(Z_0\) is not an endpoint \(\eta_i\), then by inductive assumption the proposition holds for \((\eta_1, \alpha)\) and \((\eta_1, \beta)\). In the case 2.5\(^\circ\), we have \(\text{Int}(\eta_1, \alpha) = 0 = \text{Int}(\eta_1, \beta)\) and hence

\[
\text{Hom}^\bullet(X_{\eta_1}, X_\alpha) = 0 = \text{Hom}^\bullet(X_{\eta_1}, X_\beta).
\] (B.5)

Applying Hom(\(X_{\eta_1}, ?\)) to triangle (B.2), we obtain (6.6). In the case 3\(^\circ\), without lose generality suppose that \(Z_1\) is the common endpoint of \(\eta_1\) and \(\eta_2\). Then we have \(\text{Int}(\eta_1, \alpha) = \frac{1}{2}\) and \(\text{Int}(\eta_1, \beta) = 0\). Hence

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_\alpha) = 1 \quad \text{and} \quad \text{Hom}^\bullet(X_{\eta_1}, X_\beta) = 0.
\]

Applying Hom(\(X_{\eta_1}, ?\)) to triangle (B.2), we obtain (6.7).

If \(Z_0\) is an endpoint \(\eta_1\), a more careful analysis is needed.

In the case 2.5\(^\circ\), we will have

\[
\text{Int}(\eta_1, \alpha) = \frac{1}{2} = \text{Int}(\eta_1, \beta)
\]

and

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_\alpha) = 1 = \dim \text{Hom}^\bullet(X_{\eta_1}, X_\beta).
\] (B.6)

There are four cases (as shown in Figure 18 and the upper picture in Figure 20) for the positions of \(\alpha\) and \(\beta\) in the triangle \(\Lambda_0\) (cf. Figure 8). Since \(\text{Int}(\eta_1, \eta_2) = 0\), then there are, up to mirror, six cases for the positions of \(\eta_1\) in \(\Lambda_0\) as shown in Figure 26. We claim that in either case, when applying Hom(\(X_{\eta_1}, ?\)) to triangle (B.2), there will be an isomorphism (between one-dimensional spaces)

\[
\text{Hom}^t(X_{\eta_1}, \alpha) \cong \text{Hom}^t(X_{\eta_1}, \beta[1])
\] (B.7)

in the long exact sequence for some \(t \in Z\), which will imply (6.6) by (B.6). Let us check the first case in Figure 26 while the rest is similar. Let \(\gamma_i, i = 0, 1, 2\) be the edges of triangle \(\Lambda_0\) that intersect \(\eta_1, \alpha, \beta\) respectively. Then \(X_{\gamma_i} = S_{\gamma_i}\) in fact. By the proof of Proposition 6.1, we have the following:

- the map \(\text{Hom}^\bullet(X_{\eta_1}, X_\alpha)\) is induced by the map \(\text{Ext}^1(S_{\gamma_1}, S_{\gamma_2})\);
- the map \(\text{Hom}^\bullet(X_{\eta_1}, X_\beta)\) is induced by the map \(\text{Ext}^1(S_{\gamma_2}, S_{\gamma_3})\);
- the map \(\text{Hom}^\bullet(X_{\eta_1}, X_\beta)\) is induced by the map \(\text{Ext}^2(S_{\gamma_1}, S_{\gamma_3})\).

Then

\[
\text{Ext}^1(S_{\gamma_1}, S_{\gamma_2}) \otimes \text{Ext}^1(S_{\gamma_2}, S_{\gamma_3}) \cong \text{Ext}^2(S_{\gamma_1}, S_{\gamma_3})
\]

implies

\[
\text{Ext}^t(X_{\eta_1}, X_\alpha) \otimes \text{Ext}^1(X_\alpha, X_\beta) \cong \text{Ext}^{t+1}(X_{\eta_1}, X_\beta)
\]

for some \(t \in Z\). Thus the claim holds.

Similarly for the case 2.5\(^\circ\).

In the case 3\(^\circ\), without lose generality suppose that \(Z_1\) is the common endpoint of \(\eta_1\) and \(\eta_2\). Then we have \(\text{Int}(\eta_1, \beta) = \frac{1}{2}\) and \(\dim \text{Hom}^\bullet(X_{\eta_1}, X_\beta) = 1\). Note that now \(\alpha\)
and \( \eta_1 \) does not intersect in \( S_\Delta - \triangle \) but share both endpoints. If \( \alpha \) is in \( T_0 \), then we must have \( \alpha = \eta_1 \) and we have (6.7) already. Otherwise, apply Lemma 3.14 to \( \alpha \) and there are two closed arcs \( \alpha' \) and \( \beta' \) satisfying \( \text{Int}(\alpha', \beta') = \frac{1}{2} \) and \( \alpha = B_{\alpha'}(\beta') \). Note that we have

\[
\text{Int}(\eta_1, \alpha') = \frac{1}{2} = \text{Int}(\eta_1, \beta')
\]

in this case. By Proposition 6.1, we have a triangle

\[
X_{\beta'} \to X_\alpha \to X_{\alpha'} \to X_{\beta'}[1]. \tag{B.8}
\]

By inductive assumption, the proposition holds for \((\eta_1, \alpha')\) and \((\eta_1, \beta')\) and thus

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_{\alpha'}) = 1 = \dim \text{Hom}^\bullet(X_{\eta_1}, X_{\beta'}).
\]

for some representatives \( X_{\alpha'} \) and \( X_{\beta'} \). Then (B.8) implies \( \dim \text{Hom}^\bullet(X_{\eta_1}, X_\alpha) \) is either zero or two. If it is zero, then by triangle (B.2) we deduce that

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = \dim \text{Hom}^\bullet(X_{\eta_1}, X_{\beta}) = 1.
\]

If it is two, then

\[
\text{Hom}^\bullet(X_{\eta_1}, X_\alpha) \simeq \text{Hom}^\bullet(X_{\eta_1}, X_{\alpha'}) \oplus \text{Hom}^\bullet(X_{\eta_1}, X_{\beta'}).
\]

As above, by considering the position of \( \alpha', \beta \) and \( \eta_1 \), we can show that

\[
\text{Ext}^1(X_{\eta_1}, X_{\alpha'}) \otimes \text{Ext}^1(X_{\alpha'}, X_\beta) \simeq \text{Ext}^{t+1}(X_{\eta_1}, X_\beta)
\]

for some \( t \in \mathbb{Z} \). By triangle (B.2), this will imply that

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \leq \dim \text{Hom}^\bullet(X_{\eta_1}, X_\alpha) + \dim \text{Hom}^\bullet(X_{\eta_1}, X_{\beta}) - 2 = 1.
\]

Noticing that

\[
\dim \text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \equiv \dim \text{Hom}^\bullet(X_{\eta_1}, X_\alpha) + \dim \text{Hom}^\bullet(X_{\eta_1}, X_{\beta}) \equiv 1 \pmod{2},
\]

we have (6.7) as required.
B.4. **Inductive step.** To finish the proof, we only need to show that if $2.5^\circ$ and $3^\circ$ hold for $I = r + 1$ and $\text{Int}(T_0, \eta_1) \leq r_1$, then they hold for $I = r + 1$ and $\text{Int}(T_0, \eta_1) = r_1 + 1$ (recall that $I$ is defined in (B.1) and we assume (B.4)).

Consider $2^\circ$ and $3^\circ$ first. Apply Lemma 3.14 to $\eta_1$, we get the corresponding $\alpha, \beta$ and (B.8) for $\eta = \eta_1$. If the line segment $l$ (cf. Remark B.3) does not intersect $\eta_2$, then neither does $\alpha$ or $\beta$. Since $\eta_1$ and $\eta_2$ don’t share two endpoints, without lose of generality, suppose that the common endpoint of $\eta_1$ and $\beta$ is not an endpoint of $\eta_2$; consider

\[ \eta'_1 = B_\beta(\eta_1) = \alpha \quad \text{and} \quad \eta'_2 = B_\beta(\eta_2) \]

See Figure 27 for the two cases, where it is possible $Z' = Z''$. In the left case in Figure 27, we have $\eta_2 = \eta'_2$ in fact. As (B.3), we have

\[ X_{\eta_1} = \phi^{-1}_{X,\beta}(X_\alpha) = \phi^{-1}_{X,\beta}(X_{\eta'_1}) \quad \text{and} \quad X_{\eta_2} = \phi^{-1}_{X,\beta}(X_{\eta'_2}), \]

which implies

\[ \text{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \simeq \text{Hom}^\bullet(X_{\eta'_1}, X_{\eta'_2}) \quad \text{(B.9)} \]

Moreover, we have

\[
\begin{align*}
\text{Int}(T_0, \eta'_1) &= \text{Int}(T_0, \alpha) = \text{Int}(T_0, \eta_1) - \text{Int}(T_0, \beta); \\
\text{Int}(T_0, \eta'_2) &= \text{Int}(T_0, \beta(\eta_2)) \leq \text{Int}(T_0, \eta_2) + \text{Int}(T_0, \beta).
\end{align*}
\]

Thus $2^\circ$ and $3^\circ$ hold for $(\eta'_1, \eta'_2)$ by the inductive assumption, which implies that they also hold for $(\eta_1, \eta_2)$ by (B.9).

If the line segment $l$ (from decorated point $Z_0$, cf. Remark B.3) does intersect $\eta_2$, let $Y'$ be the intersection of $l$ with $\eta_2$ that is the nearest to $Z_0$. Then we can decompose $\eta_2$ to $\alpha'$ and $\beta'$, using the line segment $l' = YZ_0(\subset l)$. One of $\alpha'$ and $\beta'$ might be an L-arc but it does not matter. Then $X_{\eta_2}$ will sit in a triangle between $X_{\alpha'}$ and $X_{\beta'}$ with suitable choices of the shifts, by Proposition B.1. In case $2^\circ$, we have $\text{Int}(\eta_1, \alpha') = \text{Int}(\eta_1, \beta') = 0$ and using inductive assumption for them, we will have

\[ \text{Hom}^\bullet(X_{\eta_1}, X_{\alpha'}) = \text{Hom}^\bullet(X_{\eta_1}, X_{\beta'}) = 0. \]

which implies (6.6). Similarly, in case $3^\circ$, $\{\text{Int}(\eta_1, \alpha'), \text{Int}(\eta_1, \beta')\}$ equals $\{0, \frac{1}{2}\}$. As before, using inductive assumption for them will imply (6.7).
Finally, let us consider the case $2.5^\circ$, when \( \eta_1 \) or \( \eta_2 \) or both of them are L-arcs. Suppose they are both L-arcs. Let \( Z_i \) be the base point of \( \eta_i \). If both \( \eta_i \) are contained in two triangles, then the decorated points in these triangles must be \( Z_1 \) and \( Z_2 \). Moreover, \( \eta_i \) will encloses \( Z_{3-i} \) for \( i = 1, 2 \); so \( \text{Int}(\eta_1, \eta_2) > 0 \), which is a contradiction. Thus, at least one of \( \eta_i \) intersects a triangle with decorated point \( Z_0 \) satisfying \( Z_0 \neq Z_i \) for \( i = 1, 2 \). Now, we can apply Lemma B.2 to one of \( \eta_i \) (say \( \eta_2 \)), using \( Z_0 \) and some line segment \( l \), to decompose \( \eta_2 \) to \( \alpha \) and \( \beta \) such that \( l \) does not intersect \( \eta_1 \). The point here is we will have \( \text{Int}(\eta_1, \alpha) = \text{Int}(\eta_1, \beta) = 0 \). Now, apply the inductive to them as above, we will get (B.5) and hence (6.6). Lastly, suppose only one of \( \eta_i \) is an L-arc, say \( \eta_2 \). Again, apply Lemma B.2 to decompose \( \eta_2 \) to \( \alpha \) and \( \beta \) using \( Z_0 \) and \( l \). There are three cases:

- \( l \) intersect \( \eta_1 \). Then decompose \( \eta_1 \) to \( \alpha' \) and \( \beta' \) instead, using \( Z_0 \) and some line segment \( l' \subset l \). Then, we have
  
  \[ \text{Hom}^\bullet(X_{\eta_2}, X_{\alpha'}) = \text{Hom}^\bullet(X_{\eta_2}, X_{\beta'}) = 0. \]

  so (6.6) also follows from inductive assumption.

- \( l \) does not intersect \( \eta_1 \) and \( Z_0 \) is not the endpoints of \( \eta_1 \). Then we have (B.5) so (6.6) follows from inductive assumption.

- \( Z_0 \) is the endpoints of \( \eta_1 \) and \( l \) does not intersect \( \eta_1 \). As before, we have (B.5) so (6.6) follows from inductive assumption.

- \( l \) does not intersect \( \eta_1 \) and \( Z_0 \) is one of the endpoints of \( \eta_1 \). Without lose of generality, assume we are in the situation of the left picture of Figure 25 (with \( \alpha \) out he right) and we have a triangle (B.2) with \( \eta = \eta_2 \). Then in the triangle of \( Z_0 \), we will have sixes possibilities in Figure 26 (up to mirror). As before, we can show that \( \text{Hom}(X_{\eta_1}, ?) \) induces isomorphism (B.7), which implies (6.6) as required.

And we are done.

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