Global Dirichlet Heat Kernel Estimates for Symmetric Lévy Processes in Half-space

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Abstract

In this paper, we derive explicit sharp two-sided estimates for the Dirichlet heat kernels of a large class of symmetric (but not necessarily rotationally symmetric) Lévy processes on half spaces for all \( t > 0 \). These Lévy processes may or may not have Gaussian component. When Lévy density is comparable to a decreasing function with damping exponent \( \beta \), our estimate is explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent \( \beta \) of Lévy density.

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1 Introduction

Classical Dirichlet heat kernel is the fundamental solution of the heat equation in an open set with zero boundary values. Except for a few special cases, explicit form of the Dirichlet heat kernel is impossible to obtain. Thus the best thing we can hope for is to establish sharp two-sided estimates of Dirichlet heat kernels. See [19] for upper bound estimates and [26] for the lower bound estimate for Dirichlet heat kernels of diffusions in bounded \( C^{1,1} \) domains.

The generator of a discontinuous Lévy process is an integro-differential operator and so it is a non-local operator. Dirichlet heat kernels (if they exist) of the generators of discontinuous Lévy processes on an open set \( D \) are the transition densities of such Lévy processes killed upon leaving \( D \). Due to this connection, obtaining sharp estimates on Dirichlet is a fundamental problem both in probability theory and in analysis.

Before [9], sharp two-sided estimates for the Dirichlet heat kernel of any non-local operator in open sets are unknown. Jointly with R. Song, in [9] for the fractional Laplacian \( \Delta^{\alpha/2} := -(\Delta)^{\alpha/2} \)

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with zero exterior condition, we succeeded in establishing sharp two-sided estimates in any $C^{1,1}$ open set $D$ and over any finite time interval (see [2] for an extension to non-smooth open sets). When $D$ is bounded, one can easily deduce large time heat kernel estimates from short time estimates by a spectral analysis. The approach developed in [2] provides a road map for establishing sharp two-sided heat kernel estimates of other discontinuous processes in open subsets of $\mathbb{R}^d$ (see [10, 11, 13, 14, 15, 22]). In [18, 12, 13], sharp two-sided estimates for the Dirichlet heat kernels $p_D(t, x, y)$ of $\Delta^{\alpha/2}$ and of $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ are obtained for all $t > 0$ in two classes of unbounded open sets: half-space-like $C^{1,1}$ open sets and exterior open sets. Since the estimates in [18, 12, 13] hold for all $t > 0$, they are called global Dirichlet heat kernel estimates. An important question in this direction is for how general discontinuous Lévy processes one can prove sharp two-sided global Dirichlet heat kernel estimates in unbounded open subsets of $\mathbb{R}^d$.

We conjectured in [12, (1.9)] that, when $D$ is a half-space-like $C^{1,1}$ open set, the following two-sided estimates hold for a large class of symmetric Lévy processes which may not be isotropic and may have damped Lévy processes such as relativistic stable processes. We remark here that under condition (1.2) excludes damped Lévy processes such as relativistic stable processes. We remark here that under condition (1.2), it follows as a special case from [16] that the transition density $p(t, x)$ of the isotropically symmetric unimodal Lévy process has the following two-sided estimates:

$$c^{-1} \left( \Phi^{-1}(t) ∧ \frac{t}{|x|^d \Phi(|x|)} \right) \leq p(t, x) \leq c \left( \Phi^{-1}(t) ∧ \frac{t}{|x|^d \Phi(|x|)} \right)$$

for all $t > 0$ and $x ∈ \mathbb{R}^d$. (1.4)

In this paper, we mainly focus on estimate (1.1) when $D$ is a half space and we prove that (1.1) holds for a large class of symmetric Lévy processes which may not be isotropic and may have damped Lévy kernel. Moreover, our symmetric Lévy processes may or may not have Gaussian component.
Once the global Dirichlet heat kernel estimates for upper half space and short time heat kernel estimates on $C^{1,1}$ open sets are obtained, one can then use the "push inward" method introduced in [18] to extend the results to half-space-like $C^{1,1}$ open sets. See Remark 7.2. See [14, 15] for recent results on short time Dirichlet heat kernel estimates for symmetric Lévy processes in $C^{1,1}$ open sets. Note that, for all symmetric Lévy process in $\mathbb{R}$, except compound Poisson, the survival probability $P_x(\zeta > t)$ of its subprocess in half line $(0, \infty)$ is comparable to

$$1 \land \frac{\max_{0 \leq y \leq 1/x} 1/\sqrt{\Psi(y)}}{\sqrt{t}},$$

where $x \to \Psi(|x|)$ is its characteristic exponent (see [24, Theorem 4.6] and [4, Theorem 2.6]). This fact, which is used several times in this paper, is essential in our approach.

We now give more details on the main results of this paper. In this paper, $d \geq 1$ and $X = (X_t, P_x)_{t \geq 0, x \in \mathbb{R}^d}$ is a symmetric discontinuous Lévy process (but possibly with Gaussian component) on $\mathbb{R}^d$ with Lévy exponent $\Psi(\xi)$ and Lévy density $J$ where $P_x(X_0 = x) = 1$. That is, $X$ is a right continuous symmetric process having independent stationary increments with

$$\mathbb{E}_x \left[ e^{i\xi \cdot (X_t - X_0)} \right] = e^{-t\Psi(\xi)} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d. \quad (1.5)$$

It is known that

$$\Psi(\xi) = \sum_{i,j=1}^d a_{ij} \xi_i \xi_j + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) J(y) dy \quad \text{for } \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d,$$

where $A = (a_{ij})$ is a constant, symmetric, non-negative definite matrix and $J$ is a symmetric non-negative function on $\mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d}(1 \land |z|^2) J(z) dz < \infty$.

When

$$\int_{\mathbb{R}^d} \exp(-t\Psi(\xi)) d\xi < \infty \quad \text{for } t > 0, \quad (\text{Exp}_L)$$

the transition density $p(t,x,y) = p(t,y-x)$ of $X$ exists as a bounded continuous function for each fixed $t > 0$, and it is given by

$$p(t, x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t\Psi(\xi)} d\xi, \quad t > 0.$$

Moreover,

$$p(t, x) \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} d\xi = p(t, 0) < \infty. \quad (1.6)$$

Note that condition $(\text{Exp}_L)$ always holds if $\|A\| > 0$ where

$$\|A\| := \sup_{|\xi| \leq 1} \sum_{i,j=1}^d a_{ij} \xi_i \xi_j.$$

Let

$$\Psi^*(r) := \sup_{|z| \leq r} \Psi(z) \quad (1.7)$$
and use $\Phi$ to denote the non-decreasing function
\[ \Phi(r) = \frac{1}{\Psi^*(1/r)}, \quad r > 0. \] (1.8)

The right continuous inverse function of $\Phi$ will be denoted by the usual notation $\Phi^{-1}(r)$. Define
\[ \Psi_1^*(r) := \sup_{s \in (-r,r)} \Psi((0,s)). \] We consider the following condition: there exists a constant $c \geq 1$ such that
\[ \Psi^*(r) \leq c \Psi_1^*(r) \quad \text{for all } r > 0. \] (Comp)

Condition (Comp) is a mild assumption that is satisfied by a large class of symmetric Lévy processes, see Lemma 2.10. Under assumptions (ExpL) and (Comp), we derive in Lemma 2.11 a useful upper bound estimate for Dirichlet heat kernels.

In general, the explicit estimates of the transition density $p(t,y)$ in $\mathbb{R}^d$ depend heavily on the corresponding Lévy measure and Gaussian component (see [7, 16]). On the other hand, scale-invariant parabolic Harnack inequality holds with the explicit scaling in terms of Lévy exponent for a large class of symmetric Lévy processes (see [16] Theorem 4.12, [7] Theorem 4.11) and our Corollary 5.2). Motivated by this, we first develop a rather general version of Dirichlet heat kernel upper bound estimate in Proposition 3.5 under the assumption that parabolic Harnack inequality PHI($\Phi$) and (UJS) hold. See Section 3 for the definition of PHI($\Phi$). We say (UJS) holds if there exists a positive constant $c$ such that for every $y \in \mathbb{R}^d$,
\[ J(y) \leq \frac{c}{r^d} \int_{B(0,r)} J(y - z)dz \quad \text{whenever } r \leq |y|/2. \] (UJS)

Note that (UJS) is very mild assumption in our setting. In fact, (UJS) always holds if $J(x) \asymp j(|x|)$ for some non-increasing function $j$ (see [6] page 1070)). Moreover, if $J$ is continuous on $\mathbb{R}^d \setminus \{0\}$, then PHI($\Phi$) implies (UJS). In fact, using (3.1) below instead of [6, (2.10)], this follows from the proof of [6, Proposition 4.1].

Assume in addition that for every $t > 0$, $x \to p(t,x)$ is weakly radially decreasing in the following sense: there exist constants $c > 0$ and $C_1, C_2 > 0$ such that
\[ p(t,x) \leq c p(C_1 t, C_2 y) \quad \text{for } t \in (0, \infty) \text{ and } |x| \geq |y| > 0. \] (HKC)

We remark here that the same assumption with $C_1 = 1$ for small $t$ was made in [14]. Then our Dirichlet heat kernel upper bound estimate obtained in Propositions 3.5 yields the desired upper bound estimate in (1.1). Moreover, we show that this assumption on $p(t,x)$ together with condition (Dec) (see Sections 4 below) and the upper bound of $p_D(t,x)$ imply a very useful lower bound of $p_D(t,x)$; see Theorem 4.4.

Jointly with T. Kumagai, in [16, 7, 8] we have established two-sided sharp heat kernel estimates for a large class of symmetric Markov processes. In Sections 5-7 we assume the jumping kernels of our Lévy process satisfy the assumptions of [7, 8, 16], that is, conditions (UJS), (5.4) and (5.5) of this paper. Then all the aforementioned conditions (ExpL), (Comp), PHI($\Phi$), (Dec) and (HKC) are satisfied. Using the two-sided heat kernel estimates for symmetric Markov processes on $\mathbb{R}^d$ from [7, 8, 16] (see Theorem 5.3) and our lower bound estimates for Dirichlet heat kernels in Theorem 4.4, we obtain two-sided global Dirichlet heat kernel estimates (7.4), essentially prove the conjecture (1.1) for such symmetric Lévy processes and for $D = \mathbb{H}$. See Remark 7.2(i) for
details. Furthermore, our estimates are explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent $\beta$ of Lévy density; see Theorem 7.1.

In this paper, we use the following notations. For any two positive functions $f$ and $g$, $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1} g \leq f \leq c g$ on their common domain of definition. For any open set $V$, we denote by $\delta_V(x)$ the distance of a point $x$ to the boundary of $V$, i.e., $\delta_V(x) = \text{dist}(x, \partial V)$. We sometimes write point $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ as $(\tilde{z}, z_d)$ with $\tilde{z} \in \mathbb{R}^{d-1}$. We denote $\mathbb{H} := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ the upper half space. For a set $W$ in $\mathbb{R}^d$, $\overline{W}$ and $|W|$ denotes the closure and the Lebesgue measure of $W$ in $\mathbb{R}^d$, respectively. Throughout the rest of this paper, the positive constants $a_0, a_1, M_1, C_1$, $i = 0, 1, 2, \ldots$, can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \ldots)$, $i = 0, 1, 2, \ldots$, denote generic constants depending on $a, b, c, \ldots$, whose exact values are unimportant. They start anew in each statement and each proof. The dependence of the constants on the dimension $d \geq 1$ may not be mentioned explicitly.

## 2 Setup and preliminary estimates

Let $X$ be a symmetric Lévy process on $\mathbb{R}^d$ with Lévy exponent $\Psi(z)$ and Lévy density $J(z)$. Recall the definition of the non-decreasing functions $\Psi^*(r)$ and $\Phi(r)$ from (1.7) and (1.8), respectively. We emphasize that the Lévy process $X$ does not need to be rotationally symmetric. The following is known and true for any negative definite function (see [21, Lemma 1]).

**Lemma 2.1** For every $t > 0$ and $\lambda \geq 1$,

$$1 \leq \frac{\Phi(\lambda t)}{\Phi(t)} \leq 2(1 + \lambda^2).$$

For an open set $D$, denote by $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time of $D$.

**Theorem 2.2** There exists a constant $c = c(d) > 0$ such that

$$\mathbb{P}_0(|X_t| > r) \leq c t / \Phi(r) \quad \text{for } (t, r) \in (0, \infty) \times (0, \infty).$$

Consequently, there exists $\varepsilon_1 = \varepsilon_1(d) > 0$ such that for all $r > 0$,

$$\mathbb{P}_0(\tau_{B(0, r/2)} > \varepsilon_1 \Phi(r)) \geq 1/2. \quad (2.1)$$

To prove the above theorem, define, for $t, \lambda > 0$,

$$h_t(\lambda) := \int_{\mathbb{R}^d} (1 - e^{-t \Psi(\lambda \xi)}) e^{-|\xi|^2/4} d\xi.$$ 

**Lemma 2.3** There exists a constant $c = c(d)$ such that $h_t(\lambda) \leq c t \Psi^*(\lambda)$ for every $t > 0$.

**Proof.** It follows from the elementary inequality $1 - e^{-x} \leq x$ for $x > 0$ that

$$h_t(\lambda) \leq t \int_{\mathbb{R}^d} \Psi(\lambda \xi)e^{-|\xi|^2/4} d\xi \leq |B(0, 1)|t \int_0^{\infty} \Psi^*(\lambda r)r^{d-1}e^{-r^2/4} dr.$$
By Lemma 2.1

\[ h_t(\lambda) \leq 2|B(0,1)|t\Psi^*(\lambda) \int_0^\infty (1 + r^2)e^{-r^2/4}dr. \]

Therefore the lemma is proved with \( c = 2|B(0,1)|\int_0^\infty (1 + r^2)e^{-r^2/4}dr. \)

We now give the proof of Theorem 2.2.

**Proof of Theorem 2.2** For \( r > 0 \) and \( t > 0 \) define \( k_t(r) = \mathbb{P}_0(|X_t| > \sqrt{r}) \). Fix \( t \in (0, \infty) \) and let \( K_t \) be the Laplace transform of \( k_t \); that is, \( K_t(\lambda) = \int_0^\infty \mathbb{P}_0(|X_t| > \sqrt{r})e^{-\lambda r}dr \). Then

\[ K_t(\lambda) = \mathbb{E}_0 \left[ \int_0^{[X_t]^2} e^{-\lambda r}dr \right] = \lambda^{-1} \mathbb{E}_0 \left[ 1 - e^{-\lambda[X_t]^2} \right], \quad \lambda > 0. \tag{2.2} \]

Since \( e^{-|z|^2} = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot z} e^{-|\xi|^2/4}d\xi \), we have by [1.5] that for \( \lambda > 0 \)

\[ \mathbb{E}_0 \left[ e^{-\lambda[X_t]^2} \right] = (4\pi)^{-d/2} \int_{\mathbb{R}^d} \mathbb{E}_0 \left[ e^{i\sqrt{\lambda} \xi \cdot X_t} \right] e^{-|\xi|^2/4}d\xi = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\lambda \Psi(\sqrt{\lambda} \xi)}e^{-|\xi|^2/4}d\xi. \]

Thus

\[ \mathbb{E}_0 \left[ 1 - e^{-\lambda[X_t]^2} \right] = (4\pi)^{-d/2} \int_{\mathbb{R}^d} (1 - e^{-\lambda \Psi(\sqrt{\lambda} \xi)})e^{-|\xi|^2/4}d\xi. \]

We now have by (2.2)

\[ K_t(\lambda) = \lambda^{-1} (4\pi)^{-d/2} \int_{\mathbb{R}^d} \left( 1 - e^{-\lambda \Psi(\sqrt{\lambda} \xi)} \right)e^{-|\xi|^2/4}d\xi = \lambda^{-1} (4\pi)^{-d/2} h_t(\sqrt{\lambda}). \]

On the other hand, since \( s \to k_t(s) \) is decreasing, for any \( r > 0 \)

\[ K_t(r^{-1}) = \int_0^\infty e^{-s/r}k_t(s)ds \geq \int_{r/2}^\infty e^{-s/r}k_t(s)ds \geq \int_{r/2}^r e^{-s/r}k_t(r)ds = rk_t(r)(e^{-1/2} - e^{-1}). \]

Thus

\[ k_t(r) \leq \frac{r^{-1}K_t(r^{-1})}{e^{-1/2} - e^{-1}} = c_1 h_t(r^{-1/2}), \quad r > 0. \tag{2.3} \]

This together with Lemma 2.3 yields

\[ \mathbb{P}_0 (|X_t| > r) = k_t \left( r^2 \right) \leq c_1 h_t(r^{-1}) \leq c_2 t \Psi^*(r^{-1}) = c_2 t / \Phi(r), \quad r > 0. \]

Since the Lévy process \( X \) is conservative, the above implies by [11] Lemma 3.8] that for every \( t, r > 0 \),

\[ \mathbb{P}_0 (\tau_{B(0,2r)} \leq t) = \mathbb{P}_0 \left( \sup_{s \leq t} |X_s| > 2r \right) \leq 2c_2 t / \Phi(r). \]

Thus \( \mathbb{P}_0 (\tau_{B(0,r/2)} \leq \varepsilon_1 \Phi(r)) \leq 2c_2 \varepsilon_1 \Phi(r) / \Phi(r/2), \) which by Lemma 2.1 is no larger than \( 20c_2 \varepsilon_1 \). Taking \( \varepsilon_1 = 1/(40c_2) \) proves the theorem. \[ \square \]
Recall that $J$ is the Lévy density of $X$, which gives rise to a Lévy system for $X$ describing the jumps of $X$. For any $x \in \mathbb{R}^d$, stopping time $S$ (with respect to the filtration of $X$), and nonnegative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \geq 0$ we have
\[
\mathbb{E}_x \left[ \sum_{s \leq S} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^S \left( \int_{\mathbb{R}^d} f(s, X_s, y)J(X_s - y)dy \right) ds \right]
\]  
(2.4)

(e.g., see [16, Appendix A]).

The following is a special case of [21, Corollary 1], whose upper bound will be used in the next lemma.

**Lemma 2.4** For every $r > 0$,
\[
\frac{1}{2\Phi(r)} \leq \frac{\|A\|}{r^2} + \int_{\mathbb{R}^d} J(z) \left( 1 \wedge \frac{|z|^2}{r^2} \right) dz \leq \frac{8(1 + 2d)}{\Phi(r)}.
\]

**Lemma 2.5** For every $a \in (0, 1)$, there exists $c = c(a) > 0$ so that for any $r > 0$ and any open set $U$ with $U \subset B(0, r)$,
\[
\mathbb{P}_x (X_{\tau_U} \in B(0, r)^c) \leq \frac{c \Phi(r)}{\Phi(r)} \mathbb{E}_x [\tau_U], \quad x \in U \cap B(0, ar).
\]

**Proof.** Using Lemma 2.4, the proof of the lemma is rather routine (see [23, Lemma 4.10]). In fact, this lemma is proved in [21, Lemma 3 and Corollary 1] for $a = 1/2$. The proof for general $a$ is similar. But for reader’s convenience, we spell out the details here.

Recall that $C_b^2(\mathbb{R}^d)$, the space of bounded $C^2$ functions, is in the domain of the generator $L$ of $X$ and for every $\varepsilon > 0$,
\[
Lg(x) = \sum_{i,j=1}^d a_{ij} \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d} (g(x + y) - g(x) - (\nabla g(x) \cdot y) \mathbb{1}_{B(0,\varepsilon)}(y))J(y)dy \quad \text{for } g \in C_b^2(\mathbb{R}^d)
\]
(see [25, Section 4.1]). By Dynkin’s formula (see, for instance, [20, (5.8)]), we have that for every $g$ in $C_b^2(\mathbb{R}^d)$ with $g(x) = 0$,
\[
\mathbb{E}_x [g(X_{\tau_U})] = \mathbb{E}_x \left[ \int_0^{\tau_U} Lg(X_t)dt \right], \quad x \in U.
\]  
(2.5)

Take a radial function $g$ in $C_b^2(\mathbb{R}^d)$ such that $0 \leq g \leq 1$,
\[
g(y) = \begin{cases} 0 & \text{when } |y| < a, \\
1 & \text{when } |y| \geq 1. \end{cases}
\]

For every $r > 0$, define $g_r(y) = g(\frac{y}{r})$ and let $c_1 = c_1(a) := \sup_{y \in \mathbb{R}^d} \sum_{i,j} |\frac{\partial^2 g}{\partial y_i \partial y_j}|$. Then $0 \leq g_r \leq 1$,
\[
g_r(y) = \begin{cases} 0 & \text{when } |y| < ar, \\
1 & \text{when } r \leq |y|. \end{cases}
\]
Combining Lemma 2.3 and Proposition 2.4, we have

\[ \sup_{y \in \mathbb{R}^d} \sum_{i,j=1}^d \left| \frac{\partial^2}{\partial y_i \partial y_j} g_r(y) \right| = c_1 r^{-2}. \]

By Lemma 2.4,

\[
\sup_{z \in \mathbb{R}^d} |Lg_r(z)| = \sup_{z \in \mathbb{R}^d} \left| \sum_{i,j=1}^d a_{i,j} \frac{\partial^2 g_r(z)}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d} \left( g_r(z+y) - g_r(z) - (\nabla g_r(z) \cdot y) \mathbf{1}_{B(0,r)}(y) \right) J(y) dy \right| \\
\leq \frac{c_2\|A\|}{r^2} + c_2 \sup_{z \in \mathbb{R}^d} \left( \int_{\{y \leq r\}} |g_r(z+y) - g_r(z) - (\nabla g_r(z) \cdot y)| J(y) dy + \int_{\{r < |y|\}} J(y) dy \right) \\
\leq \frac{c_2\|A\|}{r^2} + c_3 \left( \frac{1}{r^2} \int_{\{|y| \leq r\}} |y|^2 J(y) dy + \int_{\{r < |y|\}} J(y) dy \right) \leq \frac{c_4}{\Phi(r)}. \quad (2.6)
\]

When \( U \subseteq B(0,r) \) for some \( r > 0 \), we get, by combining (2.5) and (2.6), that for any \( x \in U \cap B(0,ar) \),

\[ \mathbb{P}_x(X_{\tau_U} \in B(0,r)^c) \leq \mathbb{E}_x[g_r(X_{\tau_U})] \leq \frac{c_4}{\Phi(r)} \mathbb{E}_x[\tau_U]. \]

\[ \square \]

Note that for \( d \)-th coordinate \( X^d_t \) of \( X_t = (X^1_t, \ldots, X^d_t) \) is a Lévy process with

\[ \mathbb{E}_x \left[ e^{\eta (X^d_t - X^d_0)} \right] = \mathbb{E}_{(0,x)} \left[ e^{i(\tilde{\eta}, \eta) \cdot (X^d_t - X^d_0)} \right] = e^{-r\Psi((\tilde{\eta}, \eta))} \quad \text{for every } x \in \mathbb{R} \text{ and } \eta \in \mathbb{R}. \]

That is, \( X^d_t \) is a 1-dimensional symmetric Lévy process with Lévy exponent \( \Psi_1(\eta) := \Psi((\tilde{\eta}, \eta)) \).

Throughout this paper we let \( \Psi^*_1(\cdot) := \sup_{z \in (-r,r)} \Psi_1(z) \) and use \( \Phi \) to denote the increasing function

\[ \Phi_1(r) = \frac{1}{\Psi^*_1(r^{-1})}, \quad r > 0. \]

Clearly

\[ \Psi^*_1(r) \leq \Psi^*_1(\cdot) \quad \text{and} \quad \Phi(r) \leq \Phi_1(r). \]

Since \( \tau_\mathbb{H} := \inf\{t > 0 : X^d_t \neq 0\} \), by [4] Proposition 2.6 (see also [24] Theorems 3.1 and 4.6) all symmetric Lévy processes, except compound Poisson processes, enjoy the following estimates of the survival probability on \( \mathbb{H} \).

**Lemma 2.6** Suppose that \( \Psi_1 \) is unbounded, then there exists \( C = C(\Psi_1, d) > 0 \) such that

\[ C^{-1} \left( \frac{\Phi_1(\delta_{\mathbb{H}}(x))}{t} \wedge 1 \right) \leq \mathbb{P}_x(\tau_\mathbb{H} > t) \leq C \left( \frac{\Phi_1(\delta_{\mathbb{H}}(x))}{t} \wedge 1 \right). \]

Let

\[ \tau^1_r := \inf \left\{ t > 0 : X^d_t \notin (0,r) \right\} \]

Combining [4] Lemma 2.3 and Proposition 2.4 we have
**Lemma 2.7** Suppose that $\Psi^1$ is unbounded, then there exists $c = c(\Psi, d) > 0$ such that for any $r \in (0, \infty)$ and

$$E((0, x)[r^2] \leq c \Phi_1(r)^{1/2} \Phi_1(\delta_{(0, r)}(x))^{1/2} \quad \text{for } x \in (0, r).$$

Recall that, when (ExpL) holds, the transition density $p(t, x, y)$ of $X$ exists as a bounded continuous function. In this case, for an open set $D$ we define

$$p_D(t, x, y) := p(t, x, y) - E_x[p(t - \tau_D, X_{\tau_D}, y) : \tau_D < t] \quad \text{for } t > 0, x, y \in D \quad (2.7)$$

Using the strong Markov property of $X$, it is easy to verify that $p_D(t, x, y)$ is the transition density for $X^D$, the subprocess of $X$ killed upon leaving an open set $D$.

**Lemma 2.8** Suppose (ExpL) holds. Then for every $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,

$$p_H(t, x, y) \leq 3C^2 p(t/3, 0) \left( \sqrt{\frac{\Phi_1(\delta_H(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi_1(\delta_H(y))}{t}} \wedge 1 \right)$$

where $C$ is the constant in Lemma 2.6.

**Proof.** Since by (1.6)

$$\sup_{z, w \in \mathbb{H}} p_H(t/3, z, w) \leq \sup_{z \in \mathbb{H}} p(t/3, z) = p(t/3, 0),$$

using the semigroup property and symmetry we have

$$p_H(t, x, y) = \int_{\mathbb{H}} \int_{\mathbb{H}} p_H(t/3, x, z)p_H(t/3, z, w)p_H(t/3, w, y)dzdw$$

$$\leq p(t/3, 0) P_x(\tau_H > t/3) P_y(\tau_H > t/3).$$

Now the lemma follows from Lemma 2.6 \qed

Using (2.4), the proof of next lemma is the same as the one in [15, Lemma 3.1] so it is omitted.

**Lemma 2.9** Suppose (ExpL) holds. Suppose that $U_1, U_3, E$ are open subsets of $\mathbb{R}^d$, with $U_1, U_3 \subset E$ and $\text{dist}(U_1, U_3) > 0$. Let $U_2 := E \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for every $t > 0$ we have

$$p_E(t, x, y) \leq P_x \left( X_{\tau_{U_1}} \in U_2 \right) \cdot \sup_{s \in U_2} p_E(s, z, y)$$

$$+ \int_0^t P_x(\tau_{U_1} > s) P_y(\tau_E > t - s) ds \cdot \sup_{u \in U_1,z \in U_3} J(u - z). \quad (2.8)$$

Recall condition (Comp) from the Introduction. The next lemma says that it is a mild assumption.
Lemma 2.10 Suppose there are a non-negative function \( j \) on \((0, \infty)\) and \( a \geq 0, c_i \geq 1, i = 1, 2, \) such that
\[
c_1^{-1}a|y|^2 \leq \sum_{i,j=1}^d a_{i,j}y_i y_j \leq c_1 a|y|^2 \quad \text{and} \quad c_1^{-1}j(|y|/c_2) \leq J(y) \leq c_1 j(c_2|y|) \quad \text{for all } y \in \mathbb{R}^d, \tag{2.9}
\]
Then \( c^{-1}\Psi^*_1(r) \leq \Psi^*(r) \leq c\Psi^*_1(r), \) and so \((\text{Comp})\) holds.

Proof. Let
\[
\phi(|\xi|) = a|\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))j(|y|)dy.
\]
By a change of variables, (2.9) implies that
\[
\Psi(\xi) \leq c_1 \left( a|\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))j(c_2|y|)dy \right)
\leq c_3 \left( a|c_2^{-1}\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(c_2^{-1}\xi \cdot z))j(|z|)dz \right) = c_3 \phi(|\xi|/c_2).
\]
and
\[
\Psi(\xi) \geq c_1^{-1} \left( a|\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))j(|y|/c_2)dy \right) \geq c_5 \phi(c_2|\xi|).
\]
Thus by Lemma 2.11 which holds for any negative definite function, \( \Psi(\xi) \propto \phi(|\xi|) \) for every \( \xi \in \mathbb{R}^d \) and \( \Psi_1(z) \propto \phi(|z|) \) for every \( z \in \mathbb{R}. \) These clearly imply that \( \Psi^*(r) \propto \sup_{x \leq r} \phi(s) \propto \Psi^*_1(r). \) \( \square \)

Using Lemma 2.9 we can obtain the following upper bound of \( p_H(t, x, y). \)

Lemma 2.11 Suppose \((\text{ExpL})\) and \((\text{Comp})\) hold. For each \( a > 0, \) there exists a constant \( c = c(a, \Psi) > 0 \) such that for every \((t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H} \) with \( a\Phi^{-1}(t) \leq |x - y|, \)
\[
p_H(t, x, y) \leq c\left( \frac{\Phi(\delta_H(x))}{t} \wedge 1 \right) \left( \sup_{(s, z) \leq t, |s - x| \leq |z - y| \leq |s - y|} p_H(s, z, y) \right)
+ \left( \sqrt{t\Phi(\delta_H(y))} \wedge t \right) \sup_{|w| \geq |x - y|/2} J(w). \tag{2.10}
\]

Proof. If \( \delta_H(x) > a\Phi^{-1}(t)/(24), \) by Lemma 2.1
\[
\sqrt{\frac{\Phi(\delta_H(x))}{t}} \geq \sqrt{\frac{\Phi(a\Phi^{-1}(t)/(24))}{\Phi(a\Phi^{-1}(t))}} \geq \sqrt{\frac{1 - a^2}{2a^2 + (24)^2}}.
\]
Thus (2.10) is clear.

We now assume \( \delta_H(x) \leq a\Phi^{-1}(t)/(24) \leq |x - y|/(24) \) and let \( x_0 = (\bar{x}, 0), U_1 := B(x_0, a\Phi^{-1}(t)/(12)) \cap \mathbb{H}, U_3 := \{ z \in \mathbb{H} : |z - x| > |x - y|/2 \} \) and \( U_2 := \mathbb{H} \setminus (U_1 \cup U_3). \) Recall that \( X_t^d \) is the \( d \)-th coordinate process of \( X \) with Lévy exponent \( \Psi_1(\eta) = \Psi((0, \eta)). \) Clearly,
\[
\tau_{U_1} \leq \inf \left\{ t > 0 : X_t^d \notin (0, a\Phi^{-1}(t)/12) \right\} =: \tau_1^d.
\]
Applying Lemma 2.7 on the interval \((0, a\Phi^{-1}(t)/12)\) and assumption \((\text{Comp})\), and noting Lemma 2.11 we have
\[
E_x[\tau_{U_1}] \leq E_{\delta_{U}(x)}^{X^d}[\tau_{U}] \leq c_1 \sqrt{t\Phi(\delta_{U}(x))}.
\] (2.11)

Since \(|z - x| > 2^{-1}|x - y| \geq a2^{-1}\Phi^{-1}(t)\) for \(z \in U_3\), we have for \(u \in U_1\) and \(z \in U_3\),
\[
|u - z| \geq |z - x| - |x_0 - x| - |x_0 - u| \geq \frac{1}{2}|x - y| - 6^{-1}a\Phi^{-1}(t) \geq \frac{1}{3}|x - y|.
\]

Thus, \(U_1 \cap U_3 = \emptyset\) and, by the monotonicity of \(j\)
\[
\sup_{u \in U_1, z \in U_3} J(u - z) \leq \sup_{(u, z) : |u - z| \geq \frac{1}{4}|x - y|} J(u - z) \leq \sup_{w : |w| \geq \frac{1}{4}|x - y|} J(w). \tag{2.12}
\]

Since for \(z \in U_2\)
\[
\frac{3}{2}|x - y| \geq |x - y| + |x - z| \geq |z - y| \geq |x - y| - |x - z| \geq \frac{|x - y|}{2} \geq a2^{-1}\Phi^{-1}(t),
\]
we have
\[
\sup_{s \leq t, z \in U_2} p_{\delta_{U}}(s, z, y) \leq \sup_{s \leq t, \frac{|x - y|}{2} \leq |z - y| \leq \frac{3|x - y|}{4}} p_{\delta_{U}}(s, z, y). \tag{2.13}
\]

Moreover, by Lemma 2.6 and \((\text{Comp})\)
\[
\int_{0}^{t} P_x(\tau_{U_1} > s) P_y(\tau_{\delta_{U}} > t - s) ds \leq \int_{0}^{t} P_x(\tau_{\delta_{U}} > s) P_y(\tau_{\delta_{U}} > t - s) ds
\]
\[
\leq c_3 \sqrt{\Phi(\delta_{U}(x))} \int_{0}^{t} s^{-1/2} \left( \sqrt{\frac{\Phi(\delta_{U}(y))}{t - s}} \wedge 1 \right) ds
\]
\[
\leq c_4 \sqrt{\Phi(\delta_{U}(x))} \left( \sqrt{\Phi(\delta_{U}(y))} \wedge \sqrt{t} \right)
\]

Applying this and (2.8), (2.11), (2.12) and (2.13), we obtain,
\[
p_{\delta_{U}}(t, x, y) \leq c_5 \int_{0}^{t} P_x(\tau_{U_1} > s) P_y(\tau_{\delta_{U}} > t - s) ds \sup_{w : |w| \geq \frac{1}{4}|x - y|} J(w)
\]
\[
+ c_5 P_x(X_{\tau_{U_1}} \in U_2) \sup_{s \leq t, z \in U_2} p(s, z, y)
\]
\[
\leq c_6 \sqrt{\Phi(\delta_{U}(x))} \left( \sqrt{\Phi(\delta_{U}(y))} \wedge \sqrt{t} \right) \sup_{w : |w| \geq \frac{1}{4}|x - y|} J(w)
\]
\[
+ c_5 P_x(X_{\tau_{U_1}} \in U_2) \sup_{s \leq t, \frac{|x - y|}{2} \leq |z - y| \leq \frac{3|x - y|}{4}} p(s, z, y).
\]

Finally, applying Lemma 2.5 and then (2.11), we have
\[
P_x(X_{\tau_{U_1}} \in U_2) \leq P_x(X_{\tau_{U_1}} \in B(x_0, a\Phi^{-1}(t)/(12))) \leq \frac{c_7}{t} E_x[\tau_{U_1}] \leq c_8 t^{-1/2} \sqrt{\Phi(\delta_{U}(x))}.
\]

Thus we have proved (2.10). \(\square\)
3 Consequences of parabolic Harnack inequality

Let $Z_s := (V_s, X_s)$ be the space-time process of $X$, where $V_s = V_0 - s$. The law of the space-time process $s \mapsto Z_s$ starting from $(t, x)$ will be denoted as $\mathbb{P}(t, x)$.

**Definition 3.1** A non-negative Borel measurable function $h(t, x)$ on $[0, \infty) \times \mathbb{R}^d$ is said to be parabolic (or caloric) on $(a, b] \times B(x_0, r)$ if for every relatively compact open subset $U$ of $(a, b] \times B(x_0, r)$, $h(t, x) = \mathbb{E}_{(t, x)}[h(Z_{\tau^U})]$ for every $(t, x) \in U \cap ([0, \infty) \times \mathbb{R}^d)$, where $\tau^U := \inf\{s > 0 : Z_s \notin U\}$.

It follows from the strong Markov property of $X$ and (2.7), $(t, x) \mapsto p_D(t, x, y)$ is parabolic on $(0, \infty) \times D$ for every $y \in D$.

Throughout this section, we assume the following (scale-invariant) parabolic Harnack inequality $\text{PHI}(\Phi)$ holds for $X$. For every $\delta \in (0, 1)$, there exists $c = c(d, \delta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $R > 0$ and every non-negative function $u$ on $[0, \infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4\delta \Phi(R)] \times B(x_0, R)$,

$$
\sup_{(t_1, y_1) \in Q_+} u(t_1, y_1) \leq c \inf_{(t_2, y_2) \in Q_+} u(t_2, y_2),
$$

where $Q_+ = (t_0 + \delta \Phi(R), t_0 + 2\delta \Phi(R)] \times B(x_0, R/2)$ and $Q_- = [t_0 + 3\delta \Phi(R), t_0 + 4\delta \Phi(R)] \times B(x_0, R/2)$.

In fact, in this paper we only need $\text{PHI}(\Phi)$ for $p_D(t, x, y)$.

$\text{PHI}(\Phi)$ implies that

$$
p(t, 0) \leq c_1 \inf_{\Phi^{-1}(t) \geq |z|} p(3t, 0, z) \leq c_2 (\Phi^{-1}(t))^{-d} \int_{B(0, \Phi^{-1}(t))} p(3t, 0, z) dz \leq c_3 (\Phi^{-1}(t))^{-d}. \quad (3.1)
$$

Under the assumptions $\text{PHI}(\Phi)$ and (UJS), we can derive an interior lower bound for $p_D(t, x, y)$ for all $t > 0$; see Propositions 3.3 and 3.4. Similar bound for $t \leq T$ was obtained in [15] for subordinate Brownian motions with Gaussian component. In this section, we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$.

**Lemma 3.2** Suppose $\text{PHI}(\Phi)$ holds. For any positive constant $a, b$, there exists $c = c(a, b, \Psi) > 0$ such that for all $z \in \mathbb{R}^d$ and $t > 0$,

$$
\inf_{y \in B(z, a\Phi^{-1}(t)/2)} \mathbb{P}_y (\tau_{B(z, a\Phi^{-1}(t))} > bt) \geq c.
$$

**Proof.** Note that, by Lemma 2.4

$$
bt = b\Phi(\Phi^{-1}(t)) \leq 2b(1 + a^{-2})\Phi(a\Phi^{-1}(t)). \quad (3.2)
$$

Thus, by (3.2), it suffices to prove the lemma for $2b(1 + a^{-2}) > \varepsilon_1$. Applying $\text{PHI}(\Phi)$ at most $2 + [2b(1 + a^{-2})/\varepsilon_1]$ times, we conclude that there exists $c_1 = c_1(a, b) > 0$ such that for every $w, y \in B(z, a\Phi^{-1}(t)/2),

$$
c_1 p_{B(z, a\Phi^{-1}(t))}(\varepsilon_1 \Phi(a\Phi^{-1}(t)), z, w) \leq p_{B(z, a\Phi^{-1}(t))}(2b(1 + a^{-2})\Phi(a\Phi^{-1}(t)), y, w).
$$
Thus using (2.1) and (3.2), we have for every $y \in B(z, a\Phi^{-1}(t)/2)$,
\[
\mathbb{P}_y \left( \tau_{B(z,a\Phi^{-1}(t))} > bt \right) \geq \mathbb{P}_y \left( \tau_{B(z,a\Phi^{-1}(t))} > 2b(1 + a^{-2})\Phi(a\Phi^{-1}(t)) \right)
\geq \int_{B(z,a\Phi^{-1}(t))} p_{B(z,a\Phi^{-1}(t))}(2b(1 + a^{-2})\Phi(a\Phi^{-1}(t)), y, w)dw
\geq c_1 \int_{B(z,a\Phi^{-1}(t)/2)} p_{B(z,a\Phi^{-1}(t)/2)}(\varepsilon_1 \Phi(a\Phi^{-1}(t)), z, w)dw
\geq c_1 \mathbb{P}_z(\tau_{B(z,a\Phi^{-1}(t)/2)} > \varepsilon_1 \Phi(a\Phi^{-1}(t))) \geq c_1/2.
\]
This proves the lemma. 

For the next two results, $D$ is an arbitrary nonempty open set.

**Proposition 3.3** Suppose (ExpL) and PHI(Φ) hold. Let $a > 0$ be a constant. There exists $c = c(a) > 0$ such that
\[
p_D(t, x, y) \geq c(\Phi^{-1}(t))^{-d}
\tag{3.3}
\]
for every $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \land \delta_D(y) \geq a\Phi^{-1}(t) \geq 4|y - x|.$

**Proof.** We fix $(t, x, y) \in (0, \infty) \times D \times D$ satisfying $\delta_D(x) \land \delta_D(y) \geq a\Phi^{-1}(t) \geq 4|y - x|.$ Note that $|y - x| \leq a\Phi^{-1}(t)/4$ and that
\[
B(x, a\Phi^{-1}(t)/4) \subset B(y, a\Phi^{-1}(t)/2) \subset B(y, a\Phi^{-1}(t)) \subset D.
\]
So by the symmetry of $p_D$, PHI(Φ) and Lemma 2.1 there exists $c_1 = c_1(a) > 0$ such that
\[
c_1 p_D(t/2, x, w) \leq p_D(t, x, y) \quad \text{for every } w \in B(x, a\Phi^{-1}(t)/4).
\]
This together with Lemma 3.2 yields that
\[
p_D(t, x, y) \geq \frac{c_1}{|B(x, a\Phi^{-1}(t)/4)|} \int_{B(x, a\Phi^{-1}(t)/4)} p_D(t/2, x, w)dw
\geq c_2(\Phi^{-1}(t))^{-d} \int_{B(x, a\Phi^{-1}(t)/4)} p_{B(x,a\Phi^{-1}(t)/4)}(t/2, x, w)dw
= c_2(\Phi^{-1}(t))^{-d} \mathbb{P}_x(\tau_{B(x,a\Phi^{-1}(t)/4)} > t/2) \geq c_3(\Phi^{-1}(t))^{-d},
\]
where $c_i > 0$ for $i = 2, 3$. 

Recall the condition (UJS) from the Introduction.

**Proposition 3.4** Suppose (ExpL), PHI(Φ) and (UJS) hold. For every $a > 0$, there exists a constant $c = c(a) > 0$ such that $p_D(t, x, y) \geq c t J(x - y)$ for every $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \land \delta_D(y) \geq a\Phi^{-1}(t)$ and $a\Phi^{-1}(t) \leq 4|y - x|.$
Proof. By Lemma 3.2, starting at \( z \in B(y, (12)^{-1}a\Phi^{-1}(t)) \), with probability at least \( c_1 = c_1(a) > 0 \) the process \( X \) does not move more than \((18)^{-1}a\Phi^{-1}(t)\) by time \( t \). Thus, using the strong Markov property and the Lévy system in (2.1), we obtain

\[
\mathbb{P}_x \left( X^D_t \in B(y, 6^{-1}a\Phi^{-1}(t)) \right) \\
\geq c_1 \mathbb{P}_x(X^D_{t \wedge \tau_{B(y,(18)^{-1}a\Phi^{-1}(t))}} \in B(y, (12)^{-1}a\Phi^{-1}(t)) \) and \( t \wedge \tau_{B(x,(18)^{-1}a\Phi^{-1}(t))} \) is a jumping time \\
= c_1 \mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B(x,(18)^{-1}a\Phi^{-1}(t))}} \int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J(X_s - u) du ds \right].
\]

By (UJS), we obtain

\[
\mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B(x,(18)^{-1}a\Phi^{-1}(t))}} \int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J(X_s - u) du ds \right] \\
= \mathbb{E}_x \left[ \int_0^t \int_{B(y,(12)^{-1}a\Phi^{-1}(t))} J(X_s^{B(x,(18)^{-1}a\Phi^{-1}(t))} - u) du ds \right] \\
\geq c_2 \Phi^{-1}(t) d \int_0^t \mathbb{E}_x \left[ J(X_s^{B(x,(18)^{-1}a\Phi^{-1}(t))} - y) \right] ds \\
\geq c_2 \Phi^{-1}(t) d \int_0^{t/2} \int_{B(x,(72)^{-1}a\Phi^{-1}(t/2))} J(w - y) p_{B(x,(18)^{-1}a\Phi^{-1}(t))}(s,x,w) dw ds.
\]

Since, for \( t/2 < s < t \) and \( w \in B(x,(72)^{-1}a\Phi^{-1}(t/2)) \)

\[
\delta_{B(x,(18)^{-1}a\Phi^{-1}(t))}(w) \geq (18)^{-1}a\Phi^{-1}(t) - (72)^{-1}a\Phi^{-1}(t/2) \geq 2^{-1}(18)^{-1}a\Phi^{-1}(s)
\]

and

\[
|x - y| < (72)^{-1}a\Phi^{-1}(t/2) \leq 4^{-1}(18)^{-1}a\Phi^{-1}(s),
\]

we have by Lemma 3.3 that for \( t/2 < s < t \) and \( w \in B(x,(72)^{-1}a\Phi^{-1}(t/2)) \),

\[
p_{B(x,(18)^{-1}a\Phi^{-1}(t))}(s,x,w) \geq c_3 (\Phi^{-1}(s))^{-d} \geq c_3 (\Phi^{-1}(t))^{-d}.
\]

Combining (3.4), (3.5) with (3.6) and applying (UJS) again, we get

\[
\mathbb{P}_x \left( X^D_t \in B(y, 6^{-1}a\Phi^{-1}(t)) \right) \geq c_4 t \int_{B(x,(72)^{-1}a\Phi^{-1}(t/2))} J(w - y) dw \\
\geq c_5 t (\Phi^{-1}(t/2))^{-d} J(x - y) \geq c_6 t (\Phi^{-1}(t))^{-d} J(x - y).
\]

In the last inequality we have used Lemma 2.1. The proposition now follows from the Chapman-Kolmogorov equation along with (3.4), (3.5) and Proposition 3.3. Indeed,

\[
p_D(t,x,y) = \int_D p_D(t/2,x,z)p_D(t/2,z,y) dz \\
\geq \int_{B(y,a\Phi^{-1}(t/2)/6)} p_D(t/2,x,z)p_D(t/2,z,y) dz \\
\geq c_7 (\Phi^{-1}(t/2))^{-d} \mathbb{P}_x \left( X^D_{t/2} \in B(y,a\Phi^{-1}(t/2)/6) \right) \\
\geq c_6 c_7 t J(x - y).
\]
We now apply Lemma 2.11 to get the following heat kernel upper bound.

Proposition 3.5 Suppose (ExpL), (Comp), PHI(Φ) and (UJS) hold. Then there exists a constant \( c > 0 \) such that for every \( (t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H} \)

\[
p_{\mathbb{H}}(t, x, y) \leq c \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(y)) \right)^{\land 1} \sup_{|w| \geq |x-y|/6} p(t, w).
\]

Proof. By Lemma 2.8 and 3.1,

\[
p_{\mathbb{H}}(t, x, y) \leq c_1 (\Phi^{-1}(t))^{-d} \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(y)) \right)^{\land 1}.
\]

If \( \Phi^{-1}(t) \geq |x-y| \), by Proposition 3.3, \( p(t, x-y) \geq c_2 (\Phi^{-1}(t))^{-d} \). Thus

\[
p_{\mathbb{H}}(t, x, y) \leq c_3 \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(y)) \right)^{\land 1} p(t, x-y). \tag{3.8}
\]

We extend the definition of \( p(t, w) \) by setting \( p(t, w) = 0 \) for \( t < 0 \) and \( w \in \mathbb{R}^d \). For each fixed \( x, y \in \mathbb{R}^d \) and \( t > 0 \) with \( |x-y| > 8r \), one can easily check that \( (s, w) \mapsto p(s, w-y) \) is a parabolic function in \( (-\infty, \infty) \times B(x, 2r) \). Suppose \( \Phi^{-1}(t) \leq |x-y| \) and let \( (s, z) \) with \( s \leq t \) and \( |x-y|/2 \leq |z-y| \leq 3|x-y|/2 \). By PHI(Φ), there is a constant \( c_4 \geq 1 \) so that for every \( t > 0 \),

\[
\sup_{s \leq t} p(s, z-y) \leq c_4 p(t, z-y).
\]

Hence we have

\[
\sup_{s \leq t, |x-y|/2 \leq |z-y| \leq 3|x-y|/2} p(s, z-y) \leq c_4 \sup_{|x-y|/2 \leq |z-y| \leq 3|x-y|/2} p(t, z-y) = c_4 \sup_{|x-y|/2 \leq |z| \leq 3|x-y|/2} p(t, z). \tag{3.9}
\]

Using this and Lemma 2.11 and Proposition 3.4, we have for every \( (t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H} \) with \( \Phi^{-1}(t) \leq |x-y| \),

\[
p_{\mathbb{H}}(t, x, y) \\
\leq c_5 \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \left( \sup_{|z| \geq |x-y|/2} p(t, z) + \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(y)) \right)^{\land 1} \sup_{|w| \geq |x-y|/3} J(w) \right) \\
\leq c_6 \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \left( \sup_{|z| \geq |x-y|/2} p(t, z) + \sup_{|w| \geq |x-y|/3} p(t, w) \right) \\
\leq 2c_6 \left( \sqrt{t} \Phi(\delta_{\mathbb{H}}(x)) \right)^{\land 1} \sup_{|w| \geq |x-y|/3} p(t, w).
\]
In view of (3.8), the last inequality holds in fact holds for all \((t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}\). Thus we have by an analogy of (3.9) that for every \((t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}\) with \(|x - y| \geq \Phi^{-1}(t)\),

\[
\begin{aligned}
\sup_{s \leq t, \frac{|x - u|}{2} \leq |z - y| \leq \frac{|x - y|}{2}} p_{\mathbb{H}}(s, z, y) &\leq c_7 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \sup_{|z - y| \geq |x - y|/2} \sup_{|w| \geq |x - y|/3} p(t, w) \\
&\leq c_8 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \sup_{|w| \geq |x - y|/6} p(t, w).
\end{aligned}
\]  

(3.10)

Therefore by Lemma 2.11 Proposition 3.4 and (3.10),

\[
p_{\mathbb{H}}(t, x, y) \leq c_9 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left( \sup_{s \leq t, \frac{|x - u|}{2} \leq |z - y| \leq \frac{|x - y|}{2}} p_{\mathbb{H}}(s, z, y) + \left( \sqrt{\Phi(\delta_{\mathbb{H}}(y))} \wedge t \right) \sup_{|w| \geq |x - y|/3} J(w) \right) \\
\leq c_{10} \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \left( \sup_{|w| \geq |x - y|/6} p(t, w) + \sup_{|w| \geq |x - y|/3} p(t, w) \right) \\
\leq 2c_{10} \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) \sup_{|w| \geq |x - y|/6} p(t, w).
\]

\[\square\]

4 Condition (HKC) and its consequence

Under the condition (HKC), clearly we have the following by Proposition 3.5.

Theorem 4.1 Suppose that conditions (HKC), (ExpL), (Comp), PHI(\(\Phi\)) and (UJS) hold. Then there exists a constant \(C_3 > 0\) such that for every \((t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}\)

\[
p_{\mathbb{H}}(t, x, y) \leq C_3 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1 \right) p(C_1 t, 6^{-1} C_2 \mathbb{H}(x - y)).
\]

We introduce one more condition.

\[
\lim_{M \to \infty} \sup_{r > 0} \frac{\Psi^{*}(r)}{\Psi^{*}(Mr)} = 0. \tag{Dec}
\]

(Dec) implies

\[
\lim_{M \to \infty} \sup_{t > 0} \frac{t}{\Phi(M \Phi^{-1}(t))} = \lim_{M \to \infty} \sup_{t > 0} t \Psi^{*}(M^{-1} \Psi^{* -1}(1/t)) = 0. \tag{4.1}
\]

Lemma 4.2 Suppose that (Dec) holds. Then for each fixed \(c > 0\) the function

\[
H_c(M) := e^{-d} \sup_{t > 0} P(|X_t| > cM \Phi^{-1}(t))
\]

vanishes at \(\infty\), that is, \(\lim_{M \to \infty} H_c(M) = 0\).
Proof. By Theorem \[2.2\] we have

\[
\sup_{t>0} P \left( |X_t| > cM\Phi^{-1}(t) \right) \leq c_1 \sup_{t>0} \frac{t}{\Phi(cM\Phi^{-1}(t))},
\]

which goes to zero as \( M \to \infty \) by \[4.1\]. □

For the remainder of this section, we assume that conditions (Dec), (HKC), (ExpL), (Comp), \( \text{PHI}(\Phi) \) and (UJS) hold and discuss some lower bound estimates of \( p_H(t, x, y) \) under these conditions. We first note that by (Comp) and Lemma \[2.6\] there exists \( C_0 > 0 \) such that

\[
C_0^{-1} \left( \frac{\Phi(\delta_H(x))}{t} \right) \leq P_x(\tau_H > t) \leq C_0 \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1.
\]

(4.2)

We denotes by \( e_d \) the unit vector in the positive direction of the \( x_d \)-axis in \( \mathbb{R}^d \).

Lemma 4.3 There exist \( a_1 > 0 \) and \( M_1 > 4a_1 \) such that for every \( x \in \mathbb{H} \) and \( t > 0 \) we have

\[
\int_{\{u \in \mathbb{H} \cap B(\xi_x(t), M_1 \Phi^{-1}(t)) : \Phi(\delta_H(u)) > a_1t\}} p_H(t, x, u)du \geq 4^{-1} C_0^{-1} \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1
\]

where \( \xi_x(t) := x + a_1\Phi^{-1}(t)e_d \) and \( C_0 \) is the constant in \[4.2\].

Proof. By Theorem \[4.1\] and a change of variable, for every \( t > 0 \) and \( x \in \mathbb{H} \),

\[
\int_{\{u \in \mathbb{H} : \Phi(\delta_H(u)) \leq at\}} p_H(t, x, u)du \\
\leq C_3 \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1 \int_{\{u \in \mathbb{H} : \Phi(\delta_H(u)) \leq at\}} p(C_1 t, 6^{-1} C_2 (x - u))du \\
\leq C_3 \sqrt{a} \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1 \int_{\{u \in \mathbb{H} : \Phi(\delta_H(u)) \leq at\}} p(C_1 t, 6^{-1} C_2 (x - u))du \\
\leq C_3 \sqrt{a} \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1 \int_{\mathbb{H}} p(C_1 t, 6^{-1} C_2 (x - u))du \\
= C_3 (6/C_2)^d \sqrt{a} \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1 \int_{\mathbb{H}} p(C_1 t, w)dw = C_3 (6/C_2)^d \sqrt{a} \left( \frac{\Phi(\delta_H(x))}{t} \right) \land 1.
\]

(4.3)

Choose \( a_1 > 0 \) small so that \( C_3 (6/C_2)^d \sqrt{a_1} \leq (8C_0)^{-1} \) where \( C_0 \) is the constant in \[4.2\].

For \( x \in \mathbb{H} \), we let \( \xi_x(t) := x + a_1\Phi^{-1}(t)e_d \). For every \( t > 0 \), \( M \geq 2a_1 \) and \( u \in \mathbb{H} \cap B(\xi_x(t), M\Phi^{-1}(t)) \), we have

\[
|x - u| \geq |\xi_x(t) - u| - |x - \xi_x(t)| \geq |\xi_x(t) - u| - a_1\Phi^{-1}(t) \geq (1 - \frac{a_1}{M})|\xi_x(t) - u| \geq \frac{1}{2}|\xi_x(t) - u|
\]
Thus using Theorem 4.1 and condition (HKC), by a change of variable we have for every \( t > 0 \) and \( M \geq 2a_1 \),

\[
\int_{\mathbb{H} \cap B(\xi_x(t), M\Phi^{-1}(t))} p_{\mathbb{H}}(t, x, u)du \\
\leq C_3 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \int_{\mathbb{H} \cap B(\xi_x(t), M\Phi^{-1}(t))} \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(u))}{t}} \land 1 \right) p(C_1 t, 6^{-1} C_2^2 (x - u))du \\
\leq C_3 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \int_{\mathbb{H} \cap B(\xi_x(t), M\Phi^{-1}(t))} p(C_1^2 t, (12)^{-1} C_2^2 \Phi(\xi_x(t) - u))du \\
\leq C_3 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \int_{B(0, M\Phi^{-1}(t))} p(C_1^2 t, (12)^{-1} C_2^2 u)du \\
= C_3 ((12)^{-1} C_2^2)^d \left( \int_{B(0, (12)^{-1} C_2^2 M\Phi^{-1}(t))} p(C_1^2 t, v)dv \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \\
\leq C_3 H_{(12)^{-1} C_2^2}(M) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right). \tag{4.4}
\]

By Lemma 4.2, we can choose \( M_1 > 4a_1 \) large so that \( C_3 H_{(12)^{-1} C_2^2}(M_1) < 8^{-1} \cdot C_0^{-1} \). Then by (4.2), (4.3), (4.4) and our choice of \( a_1 \) and \( M_1 \), we conclude that

\[
\int_{\{u \in \mathbb{H} \cap B(\xi_x(t), M_1\Phi^{-1}(t)) : \Phi(\delta_{\mathbb{H}}(u)) > a_1 t\}} p_{\mathbb{H}}(t, x, u)du \\
= \int_{\mathbb{H}} p_{\mathbb{H}}(t, x, u)du - \int_{\mathbb{H} \cap B(\xi_x(t), M_1\Phi^{-1}(t))} p_{\mathbb{H}}(t, x, u)du - \int_{\{u \in \mathbb{H} : \Phi(\delta_{\mathbb{H}}(u)) \leq a_1 t\}} p_{\mathbb{H}}(t, x, u)du \\
\geq 4^{-1} \cdot C_0^{-1} \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right). 
\]

For \( x \in \mathbb{H} \) and \( t > 0 \), let \( \xi_x(t) := x + a_1 \Phi^{-1}(t) e_d \) and define

\[
\mathcal{B}(x, t) := \{ z \in \mathbb{H} \cap B(\xi_x(t), M_1\Phi^{-1}(t)) : \Phi(\delta_{\mathbb{H}}(z)) > a_1 t \}. \tag{4.5}
\]

**Theorem 4.4** There exist \( c_1, c_2 > 0 \) such that for all \((t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}, \)

\[
p_{\mathbb{H}}(t, x, y) \geq c_1 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \land 1 \right) \left( \inf_{(u, v) \in \mathcal{B}(x, t) \times \mathcal{B}(y, t)} p_{\mathbb{H}}(t/3, u, v) \right) \tag{4.6}
\]

\[
\geq c_2 \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \land 1 \right) \left( \sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \land 1 \right) \times \\
\times \begin{cases} 
\inf_{(u, v) : 2M_1\Phi^{-1}(t) \leq |u - v| \leq |x - y|/2} p_{\mathbb{H}}(t/3, u, v) & \text{if } |x - y| > 4M_1\Phi^{-1}(t), \\
\Phi^{-1}(t) & \text{if } |x - y| \leq 4M_1\Phi^{-1}(t). 
\end{cases} \tag{4.7}
\]
Proof. By Chapman-Kolmogorov equation,
\[
p_H(t, x, y) \geq \int_{B(x, t)} \int_{B(y, t)} p_H(t/3, x, u)p_H(t/3, u, v)p_H(t/3, v, y)dudv
\]
\[
\geq \left( \inf_{(u, v) \in B(x, t) \times B(y, t)} p_H(t/3, u, v) \right) \int_{B(x, t)} \int_{B(y, t)} p_H(t/3, x, u)p_H(t/3, v, y)dudv.
\]
Thus (4.6) follows from Lemma 4.3.

Observe that for \((u, v) \in B(x, t) \times B(y, t)\),
\[
|\xi_x(t) - \xi_y(t)| = |x - y|, \quad \delta^H(u) \land \delta^H(v) \geq a_1 \Phi^{-1}(t), \quad (4.8)
\]
and
\[
|x - y| - 2M_1 \Phi^{-1}(t) \leq |u - v| \leq |x - y| + |u - \xi_x(t)| + |v - \xi_y(t)| \leq |x - y| + 2M_1 \Phi^{-1}(t). \quad (4.9)
\]

When \(|x - y| > 4M_1 \Phi^{-1}(t)\), we have by (4.9) that for \((u, v) \in B(x, t) \times B(y, t)\),
\[
|x - y|/2 \leq |u - v| \leq 3|x - y|/2
\]
and so \(2M_1 \Phi^{-1}(t) \leq |u - v|\). Thus, for \(|x - y| > 4M_1 \Phi^{-1}(t)\),
\[
\inf_{(u, v) \in B(x, t) \times B(y, t)} p_H(t/3, u, v) \geq \inf_{(u, v); 2M_1 \Phi^{-1}(t) \leq |u - v| \leq 3|x - y|/2} p_H(t/3, u, v). \quad (4.10)
\]

When \(|x - y| \leq 4M_1 \Phi^{-1}(t)\), by (4.9) \(|u - v| \leq 6M_1 \Phi^{-1}(t)\) for \((u, v) \in B(x, t) \times B(y, t)\). Thus using (4.8) and PHI(\Phi) (at most \(2 + 12[M_1/a_1]\) times) and Lemma 2.1 and Proposition 6.2 we get
\[
p_H(t/3, u, v) \geq c_1p_H(t/6, u, u) \geq c_2\Phi^{-1}(t) \quad \text{for every } (u, v) \in B(x, t) \times B(y, t). \quad (4.11)
\]
(4.7) now follows from (4.6), (4.10) and (4.11). \(\Box\)

5 Heat kernel upper bound estimates in half spaces

In this section, we consider a large class of symmetric Lévy processes with concrete condition on the Lévy densities. Under these conditions, we can check that conditions (Dec), (ExpL), (Comp), (HKC) and PHI(\Phi) all hold. Thus we can apply Proposition 3.5 and Theorem 4.4 to establish sharp two-sided estimates of the transition density of such Lévy processes in half spaces.

Suppose that \(\psi_1\) is an increasing function on \([0, \infty)\) with \(\psi_1(r) = 1\) for \(0 < r \leq 1\) and there are constants \(a_2 \geq a > 0\), \(\gamma_2 \geq \gamma_1 > 0\) and \(\beta \in [0, \infty]\) so that
\[
a_1e^{\gamma_1r^\beta} \leq \psi_1(r) \leq a_2e^{\gamma_2r^\beta} \quad \text{for every } 1 < r < \infty. \quad (5.1)
\]

Suppose that \(\phi_1\) is a strictly increasing function on \([0, \infty)\) with \(\phi_1(0) = 0\), \(\phi_1(1) = 1\) and there exist constants \(0 < a_3 < a_4\) and \(0 < \beta_1 \leq \beta_2 < 2\) so that
\[
a_3\left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi_1(R)}{\phi_1(r)} \leq a_4\left(\frac{R}{r}\right)^{\beta_2} \quad \text{for every } 0 < r < R < \infty. \quad (5.2)
\]
Since $0 < \beta_1 \leq \beta_2 < 2$, (5.2) implies that
\[
\int_0^r s \frac{1}{\phi_1(s)} ds \leq \frac{r^2}{\phi_1(r)} \quad \text{for } r > 0.
\] (5.3)

Throughout the remainder of this paper, we assume that (UJS) holds and that there are constants $\gamma \geq 1$, $\kappa_1$, $\kappa_2$ and $a_0 \geq 0$ such that
\[
\gamma^{-1} a_0 |\xi|^2 \leq \sum_{i,j=1}^d a_{i,j} \xi_i \xi_j \leq \gamma a_0 |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^d,
\] (5.4)

and
\[
\gamma^{-1} \frac{1}{|x|^d \phi_1(|x|)} \leq J(x) \leq \gamma \frac{1}{|x|^d \phi_1(|x|)} \quad \text{for } x \in \mathbb{R}^d.
\] (5.5)

Note that (UJS) holds if $\kappa_1 = \kappa_2$ in (5.5).

Recall $\Phi$ is the function defined in (1.8). The next lemma gives explicit relation between $\Phi$ and $\phi_1$.

**Lemma 5.1** When $\beta = 0$,
\[
\Phi(r) \asymp \begin{cases} 
\phi_1(r) & \text{for } r \in [0, 1), \\
\phi_1(r) & \text{for } r \geq 1;
\end{cases}
\] (5.6)

while for $\beta \in (0, \infty]$,
\[
\Phi(r) \asymp \begin{cases} 
\phi_1(r) & \text{for } r \in [0, 1), \\
\frac{1}{r^2} & \text{for } r \geq 1.
\end{cases}
\] (5.7)

**Proof.** By Lemma 2.4 and (5.4),
\[
\frac{1}{\Phi(r)} \asymp \frac{a_0}{r^2} + \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|^2}{r^2}\right) J(z) dz.
\]

Thus, by (5.3) and (5.6)
\[
c_0^{-1} \left( \frac{a_0}{r^2} + r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-\kappa_2 s^\beta} ds + \int_r^\infty \frac{1}{s \phi_1(s)} e^{-\kappa_2 s^\beta} ds \right) \leq \frac{1}{\Phi(r)} \leq c_0 \left( \frac{a_0}{r^2} + r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-\kappa_1 s^\beta} ds + \int_r^\infty \frac{1}{s \phi_1(s)} e^{-\kappa_1 s^\beta} ds \right).
\] (5.8)

When $\beta = 0$, it follows from (5.3) and (5.8) that
\[
\frac{1}{\Phi(r)} \asymp \frac{a_0}{r^2} + r^{-2} \int_0^r \frac{s}{\phi_1(s)} ds + \int_r^\infty \frac{1}{s \phi_1(s)} ds \asymp \frac{a_0}{r^2} + \frac{1}{\phi_1(r)} \quad \text{for } r > 0.
\] (5.9)

Note that taking $R = 1$ and $r = 1$ in (5.2), we have
\[
\phi_1(r) \geq a_4^{-1} r^{\beta_2} \geq a_4^{-1} r^{2} \quad \text{for } r \in [0, 1] \quad \text{and} \quad \phi_1(R) \leq a_4 R^{\beta_2} \leq a_4 R^2 \quad \text{for } R \geq 1.
\] (5.10)
This together with (5.9) establishes (5.6).

When \( r \geq 1 \) and \( \beta > 0 \),
\[
\int_r^\infty s^{-\beta_1+1} e^{-s^{\beta}} ds \leq c_1 \int_r^\infty s^{-3} ds \leq c_1 r^{-2}/2.
\]

Thus by (5.2), for \( \beta > 0 \) and \( r \geq 1 \),
\[
r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-s^{\beta}} ds + \int_r^\infty \frac{1}{s\phi_1(s)} e^{-s^{\beta}} ds \\
\leq r^{-2} \int_0^\infty \frac{s}{\phi_1(s)} e^{-s^{\beta}} ds + \int_r^\infty \frac{1}{s\phi_1(s)} e^{-s^{\beta}} ds \leq c_2 r^{-2},
\]
while
\[
r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-s^{2\gamma s^\beta}} ds + \int_r^\infty \frac{1}{s\phi_1(s)} e^{-s^{2\gamma s^\beta}} ds \geq r^{-2} \int_0^1 \frac{s}{\phi_1(s)} e^{-s^{2\gamma s^\beta}} ds \geq c_3 r^{-2}.
\]

By (5.3), for \( r \leq 1 \) and \( \beta > 0 \),
\[
r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-s^{\beta}} ds + \int_r^\infty \frac{1}{s\phi_1(s)} e^{-s^{\beta}} ds \\
\leq r^{-2} \int_0^r \frac{1}{s\phi_1(s)} ds + \int_r^1 \frac{1}{s\phi_1(s)} ds + \int_1^\infty e^{-s^{\beta}} ds \\
\leq \frac{c_4}{\phi_1(r)} + c_4 \leq \frac{c_5}{\phi_1(r)}
\]
and
\[
r^{-2} \int_0^r \frac{s}{\phi_1(s)} e^{-s^{2\gamma s^\beta}} ds + \int_r^\infty \frac{1}{s\phi_1(s)} e^{-s^{2\gamma s^\beta}} ds \geq e^{-s^{2\gamma s^\beta}} r^{-2} \int_0^r \frac{s}{\phi_1(s)} ds \geq \frac{c_6}{\phi_1(r)}.
\]

These combined with (5.8) and (5.10) immediately yield (5.7).

\( \text{Corollary 5.2} \)

The conditions (Dec), (ExpL) and (Comp) hold.

Since we have assumed (UJS), (5.4) and (5.5), our Lévy process \( X \) belongs to a subclass of the processes considered in [16, 17, 4, 8]. Therefore \( p(t, x, y) \) is Hölder continuous on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) and for every open set \( D \), transition density \( p_D(t, x, y) \) for the killed process \( X_D \) is Hölder continuous on \( (0, \infty) \times D \times D \). Define
\[
p^\epsilon(t, r) = t^{-d/2} \exp(-r^2/t).
\]
(5.11)
Recall that \( a_0 \) is the ellipticity constant in (5.4). For each \( a, T > 0 \), we define a function \( h_{a,T}(t, r) \) on \( (t, r) \in (0, T] \times [0, \infty) \) as
\[
h_{a,T}(t, r) := \begin{cases} 
    a_0 p^\epsilon(t, ar) + (\Phi^{-1}(t)^{-d} \wedge (t_j(ar))) & \text{if } \beta \in [0, 1] \text{ or } r \in [0, 1], \\
    t \exp \left( -a \left( r \left( \log \frac{Tr}{t} \right)^{(\beta-1)/\beta} \wedge r^\beta \right) \right) & \text{if } \beta \in (1, \infty) \text{ with } r \geq 1, \\
    (t/(Tr))^{ar} & \text{if } \beta = \infty \text{ with } r \geq 1.
\end{cases}
\]

(5.12)
and, for each $a, T > 0$, define a function $k_{a,T}(t, r)$ on $(t, r) \in [T, \infty) \times [0, \infty)$ as

$$k_{a,T}(t, r) := \begin{cases} 
\Phi^{-1}(t)^{-d} \land (a_0 p^c(t, ar)) + t \cdot j(ar) & \text{if } \beta = 0, \\
(t^{-d/2} \exp \left( -a r^\beta \land \frac{r^2 T}{t} \right))^{-1} & \text{if } \beta \in (0, 1], \\
(t^{-d/2} \exp \left( -a r \left( 1 + \log^+ \frac{r T}{t} \right)^{(\beta - 1)/\beta} \land \frac{r T}{t} \right))^{-1} & \text{if } \beta \in (1, \infty), \\
(t^{-d/2} \exp \left( -a r \left( 1 + \log^+ \frac{r T}{t} \right) \land \frac{r^2 T}{t} \right))^{-1} & \text{if } \beta = \infty.
\end{cases} \quad (5.13)$$

Note that $r \to h_{a,T}(t, r)$ and $r \to k_{a,T}(t, r)$ are decreasing.

**Theorem 5.3** The parabolic Harnack inequality $\Phi(\Phi)$ holds. Moreover, for each positive constant $T$, there are positive constants $c_i$, $i = 1, \ldots, 6$, which depend on the ellipticity constant $a_0$ of $(5.3)$, such that

$$c_2^{-1} h_{c_1, T}(t, |x|) \leq p(t, x) \leq c_2 h_{c_3, T}(t, |x|) \quad \text{for every } (t, x) \in (0, T] \times \mathbb{R}^d,$$

and

$$c_4^{-1} h_{c_5, T}(t, |x|) \leq p(t, x) \leq c_4 h_{c_6, T}(t, |x|) \quad \text{for every } (t, x) \in [T, \infty) \times \mathbb{R}^d.$$

In particular, the condition (HKC) holds.

The above two-sided estimates on $p(t, x, y)$ follow from [16, Theorem 1.2] and [7, Theorems 1.2 and 1.4] when $a_0 = 0$, and from [17, 18] when $a_0 > 0$. Note that even though in [16, Theorem 1.2] and [7, Theorems 1.2 and 1.4] two-sided estimates for $p(t, x, y)$ are stated separately for the cases $0 < t \leq 1$ and $t > 1$, the constant 1 does not play any special role. In fact, for example for $T < 1$ one can easily check

$$c_3^{-1} h_{c_2, 1}(t, r) \leq h_{c_1, T}(t, r) \leq c_3 h_{c_2, 1}(t, r) \quad \text{on } t < T,$$

and the two-sided estimates for $p(t, x)$ hold for the cases $0 < t \leq T$ and $t > T$, and can be stated in the above way.

**Remark 5.4** We remark here that in [7, Theorems 1.2(2.b)], the $|\log \frac{|x-y|}{t}|$ term should replaced by $1 + \log^+ \frac{|x-y|}{t}$. In the proof of [7, Theorems 1.2(2.b)], the case that $|x-y| \gg t$ when $\beta \in (1, \infty)$ missed to be considered. Once taking into account of that missed case, One can see from [7] that $(5.13)$ is the correct form. See the statement and the proof of Proposition 6.7 below for the lower bound.

We now present the main result of this section.

**Theorem 5.5** There exist $c_1, c_2 > 0$ such that for all $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H},$

$$p_{\mathbb{H}}(t, x, y) \leq c_1 \left( \frac{\Phi(\Phi)(x)}{t} \land 1 \right) \left( \frac{\Phi(\Phi)(y)}{t} \land 1 \right) \begin{cases} 
h_{c_2, 1}(t, |x-y|/6) & \text{if } t \in (0, 1), \\
k_{c_2, 1}(t, |x-y|/6) & \text{if } t \in [1, \infty). 
\end{cases}$$
Proposition 6.2
Let \( \immediate \text{from Propositions 3.3 and 3.4, Lemma 5.1 and condition (5.5).} \)

(\text{Corollary 6.3)}
Suppose \( \text{for every} \ n \ \text{and} \ \text{particular, when} \ c \ \text{for every} \)

Theorem 6.1
Let \( \immediate \text{through several propositions. The following proposition follows} \)

6 Interior lower bound estimates

Proof. Since \( r \rightarrow h_{a,T}(t,r) \) and \( r \rightarrow k_{a,T}(t,r) \) are decreasing, by Theorem \( 5.3 \)

\[
\sup_{w:|w| \geq \frac{|x-y|}{6}} p(t,w) \leq c_1 \begin{cases} 
\sup_{w:|w-x| \leq |w|} h_{c_1,1}(t,|w|) & \text{if } t \in (0,1), \\
\sup_{w:|w-x| \leq |w|} k_{c_2,1}(t,|w|) & \text{if } t \in [1,\infty), \\
C_1(t,|x-y|/6) & \text{if } t \in (0,1), \\
k_{c_2,1}(t,|x-y|/6) & \text{if } t \in [1,\infty). 
\end{cases}
\]

This together with Proposition 3.5 proves the theorem. \( \square \)

6 Interior lower bound estimates

In this section, we derive following preliminary lower bound estimates on \( p_H(t,x,y) \). Recall that we have assumed (UJS), \( 5.4 \) and \( 5.5 \).

Theorem 6.1 Let \( a, T \) be positive constants. There exist \( c = c(a, \beta_1, \beta_2, \beta, T) > 0 \) and \( C_4 = C_4(a, \beta_1, \beta_2, \beta, T) > 0 \) such that

\[
p_H(t,x,y) \geq c \begin{cases} 
h_{C_4,T}(t,|x-y|) & \text{if } t \in (0,T), \\
C_4,T(t,|x-y|) & \text{if } t \in [T,\infty), 
\end{cases}
\]

for every \( (t,x,y) \in (0,\infty) \times \mathbb{H} \times \mathbb{H} \) with \( \delta_H(x) \wedge \delta_H(y) \geq a \Phi^{-1}(t) \).

We will prove this theorem through several propositions. The following proposition follows immediately from Propositions 3.3 and 3.4, Lemma 5.1 and condition (5.5).

Proposition 6.2 Let \( D \) be an open subset of \( \mathbb{R}^d \). For every \( a > 0 \), there exists a constant \( c = c(a) > 0 \) so that

\[
p_D(t,x,y) \geq c ((\Phi^{-1}(t))^{-d} \wedge t j(|x-y|))
\]

for every \( (t,x,y) \in (0,\infty) \times D \times D \) with \( \delta_D(x) \wedge \delta_D(y) \geq a \Phi^{-1}(t) \).

Proposition 6.2 yields the interior lower bound for \( p_D(t,x,y) \) and \( p(t,x,y) \) for the case \( \beta = 0 \) and \( a_0 = 0 \). Proposition 6.2 also yield the interior lower bound for \( p_D(t,x,y) \) and \( p(t,x,y) \) for the case \( \beta \in (0,1), t \leq T \) and \( a_0 = 0 \). As a direct consequence of Proposition 3.4, we have

Corollary 6.3 Suppose \( \beta \in (0,\infty) \). For every \( a,T,C_* > 0 \), there exist \( c_1, c_2 > 0 \) so that

\[
p_H(t,x,y) \geq c_1 t e^{-c_2|x-y|}\beta
\]

for every \( (t,x,y) \in [T,\infty) \times \mathbb{H} \times \mathbb{H} \) with \( \delta_H(x) \wedge \delta_H(y) \geq a \Phi^{-1}(t) \) and \( |x-y| \geq C_* \Phi^{-1}(t) \). In particular, when \( 0 < \beta \leq 1 \), for every \( a,T,C_* > 0 \), there exist \( c_1, c_2 > 0 \) such that

\[
p_H(t,x,y) \geq c_1 t e^{-c_2|x-y|}\beta \quad \text{when} \quad |x-y|^{2-\beta} \geq t/C_* \delta_H(x) \wedge \delta_H(y) \geq a \Phi^{-1}(t) \text{ and } t \geq T.
\]
The last assertion in Corollary 5.3 holds because \( \Phi^{-1}(t) \asymp t^{1/2} \) for \( t \geq T \) (by Lemma 5.1), and for \( t \geq T \) and \( x, y \) with \( |x - y|^{2-\beta} \geq t/C_* \), one has \( |x - y|^2 \geq ct \) where \( c = (T/C_*)^{1/(2-\beta)}C_*^{-1} \).

A standard chaining argument give the following Gaussian lower bound. The proof is similar to the one of [7, Theorem 5.4].

**Proposition 6.4** Suppose \( \beta \in (0, \infty) \). For every \( C_*, a, T > 0 \), there exist constants \( c_1, c_2 > 0 \) such that

\[
p_{\mathbb{H}}(t, x, y) \geq c_1t^{-d/2}\exp\left(-\frac{c_2|x-y|^2}{t}\right)
\]

for every \((t, x, y) \in [T, \infty) \times \mathbb{H} \times \mathbb{H}\) with \( \delta_{\mathbb{H}}(x) \land \delta_{\mathbb{H}}(y) \geq a\Phi^{-1}(t) \) and \( C_*|x-y| \leq t/T \).

**Proof.** By considering \( t/T \) instead of \( t \), without loss of generality we assume \( T = 1 \). Fix a constant \( C_* > 0 \) and let \( R := |x-y| \). When \( t \geq 1 \geq R \), by Proposition 6.2 and (5.7), \( p_D(t, x, y) \geq c_1\Phi^{-1}(t)^{-d} \geq c_2t^{-d/2} \). When \( t \geq R^2 \geq 1 \), note that \( R^2 \leq c_3\Phi(r) \) for some \( c_3 > 0 \). Thus in view of Lemma 2.1 by applying the parabolic Harnack inequality \( \Phi(\Phi) \) at most \( 3(1 + 16a^2)c_3 \) times, we have from Proposition 3.3 that \( p_D(t, x, y) \geq c_4\Phi^{-1}(t)^{-d} \geq c_5t^{-d/2} \). Hence we only need to consider the case \( 1 \lor (C_*R) \leq t \leq R^2 \) (so \( C_* \leq 1 \)), which we now assume. By (6.2), there exist a constant \( c_0 \in (0, 1) \) such that

\[
c_0^{-1}\sqrt{s} \geq \Phi^{-1}(s) \geq c_0\sqrt{s} \quad \text{for every } s \geq 2^{-1}(C_*)^2.
\]

Thus \( \delta_{\mathbb{H}}(x) \land \delta_{\mathbb{H}}(y) \geq ac_0\sqrt{t} \).

Let \( n \) be the smallest positive integer so that \( t/n \geq (R/n)^2 \). Then

\[
1 \leq R^2/t \leq n < 1 + R^2/t \leq 2R^2/t \quad \text{and} \quad 2(R/n)^2 \geq t/n \geq (R/n)^2.
\]

Since \( t \geq C_*R \), by (6.1)

\[
\frac{t}{n} \geq \frac{t}{1 + R^2/t} = \frac{t^2}{t + R^2} \geq 2^{-1}\left(\frac{t}{R}\right)^2 \geq 2^{-1}(C_*)^2.
\]

Let \( x = x_0, x_1, \cdots, x_n = y \) be the points equally spaced on the line segment connecting \( x \) to \( y \) so that \( |x_i - x_{i+1}| = R/n \) for \( i = 0, \cdots, n - 1 \). Set \( B_i := B(x_i, 2^{-1}ac_0R/n) \). Since \( t/n \geq (R/n)^2 \) (by (6.1)) and \( t/n \geq 2^{-1}(C_*)^2 \) (by (6.2)), we have for every \((y_i, y_{i+1}) \in B_i \times B_{i+1}\),

\[
\delta_{\mathbb{H}}(y_i) \land \delta_{\mathbb{H}}(y_{i+1}) \geq ac_0\sqrt{t} - 2^{-1}ac_0R/n \geq 2^{-1}ac_0\sqrt{t/n} \geq 2^{-1}ac_0^2\Phi^{-1}(t/n)
\]

and

\[
4|y_i - y_{i+1}| \leq 4(1 + 2^{-1}ac_0)R/n \leq 4(1 + 2^{-1}ac_0)\sqrt{t/n} \leq 4(c_0^{-1} + 2^{-1}a)\Phi^{-1}(t/n).
\]

By Proposition 3.3 and applying \( \Phi(\Phi) \) at most \( N \) times, where \( N \) depends only on \( a, \Phi \) and \( C_* \) (or by Proposition 6.2), we have

\[
p_{\mathbb{H}}(t/n, y_i, y_{i+1}) \geq c_6(t/n)^{-d/2} \quad \text{for every } (y_i, y_{i+1}) \in B_i \times B_{i+1}.
\]
Using (6.3) and then (6.1), we have
\[
p_{\Phi}(t,x,y) \geq \int_{B_1} \cdots \int_{B_{n-1}} p_{\Phi}(t/n,x,y_1) \cdots p_{\Phi}(t/n,y_{n-1},y) dy_1 \cdots dy_{n-1}
\geq c_0 (t/n)^{-d/2} \prod_{i=1}^{n-1} \left( c_7 (t/n)^{-d/2} (R/n)^d \right) \geq c_8 (t/n)^{-d/2} (c_2^{-d/2})^{n-1}
\geq c_8 (t/n)^{-d/2} \exp(-c_9 n) \geq c_{10} t^{-d/2} \exp\left(-\frac{c_{11} |x-y|^2}{t}\right),
\]
(6.4)

Proposition 6.5 Suppose \(a_0, a > 0\). There are positive constants \(c_1\) and \(c_2\) so that
\[
p_{\Phi}(t,x,y) \geq c_1 \Phi^{-1}(t)^{-d} \wedge \left( t^{-d/2} \exp\left(-\frac{c_2 |x-y|^2}{t}\right) + tj(|x-y|) \right)
\]
for every \((t,x,y) \in [0,\infty) \times \mathbb{H} \times \mathbb{H}\) with \(\delta_{\Phi}(x) \wedge \delta_{\Phi}(y) \geq a \Phi^{-1}(t)\) if either \(\beta \in [0,1]\) or \(|x-y| \leq 1\).

Proof. We first consider following five cases: (1) \(t \geq 1\) and \(|x-y| \leq 1\) when \(\beta \in [0,\infty]\), (2) \(t \geq 1\) and \(x,y \in \mathbb{H}\) when \(\beta = 0\), (3) \(|x-y|^2 - \beta \geq t \geq 1\) when \(\beta \in (0,1]\), (4) \(t \leq 1\) and \(|x-y| \geq 1\) when \(\beta \in (0,1]\) (5) \(|x-y|^2 \leq t \leq 1\) when \(\beta \in [0,\infty]\).

Using the condition (5.5), we see that for these five cases it holds that for every \(c_1 > 0\) there is \(c_2 > 0\) such that
\[
t^{-d/2} \exp\left(-\frac{c_1 |x-y|^2}{t}\right) \leq c_2 tj(|x-y|).
\]
(6.5)

Hence by Propositions 6.2 and 6.3 and (5.6)–(5.7), it suffices to consider the case when \(t \leq |x-y|^2 \leq 1\), which we will assume for the remainder of the proof.

By (5.6)–(5.7), there is a constant \(c_1 \in (0,1/2)\) so that \(c_1 r^2 \leq \Phi(r) \leq r^2/c_1\). Set \(R = |x-y|\). Let \(n\) to be the smallest integer so that \(t/n \geq c_1^{-1} (R/n)^2\). Observe that \(R^2/c_1 t \leq n \leq 2R^2/c_1 t\). Let \(x_0 = x, x_1, \ldots, x_n = y\) be the evenly spaced points on be the line segment connecting \(x\) to \(y\) so that \(|x_i - x_{i+1}| = R/n\). Let \(B_i = B_i(x_i, aR/4n)\). Then for every we have for every \((y_i,y_{i+1}) \in B_i \times B_{i+1},\)
\[
\delta_{\delta_{\Phi}}(y_i) \wedge \delta_{\delta_{\Phi}}(y_{i+1}) \geq a \sqrt{c_1 t} - aR/2n \geq a \sqrt{c_1 t}/2 \geq 2^{-1} a c_1 \Phi^{-1}(t)
\]
and
\[
4 |y_i - y_{i+1}| \leq 4(1+a)R/n \leq 8 \sqrt{c_1 t}/n \leq 8 \Phi^{-1}(t/n).
\]
By Proposition 3.3 and applying PHI(\(\Phi\)) at most finite many times (or by Proposition 6.2), we have
\[
p_{\Phi}(t/n,y_i,y_{i+1}) \geq c_3 (t/n)^{-d/2} \text{ for every } (y_i,y_{i+1}) \in B_i \times B_{i+1}.
\]
(6.6)

Using (6.6) and then (6.1), by the same argument in (6.5) we have
\[
p_{\Phi}(t,x,y) \geq c_4 t^{-d/2} \exp\left(-\frac{c_5 |x-y|^2}{t}\right).
\]
This together with Proposition 6.2 gives the desired lower bound interior estimate for \(t \leq |x-y|^2 \leq 1\). This completes the proof of the proposition. \(\square\)
Proposition 6.2 Corollary 6.3 and Propositions 6.4, 6.5 give the desired interior lower bound stated in Theorem 6.1 for $p_{\mathbb{H}}(t, x, y)$ when $\beta \in [0, 1]$. We now consider the case $\beta = \infty$ and the case $\beta \in (1, \infty)$ separately.

**Proposition 6.6** Suppose that $T, a > 0$ and $\beta = \infty$. Then, there exist constants $c_i = c_i(a, T) > 0$, $i = 1, 2$, such that for any $x, y$ in $\mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq a \Phi^{-1}(t)$, we have

$$p_{\mathbb{H}}(t, x, y) \geq c_1 \left(\frac{t}{T|x-y|}\right)^{c_2|x-y|} \text{ when } |x-y| \geq 1 \vee (t/T). \quad (6.7)$$

**Proof.** By considering $t/T$ instead of $t$, without loss of generality we assume $T = 1$. We let $R_1 := |x-y| \geq 1$. We define $k$ as the integer satisfying $(4 \leq) 4R_1 > k < 4R_1 + 1 < 5R_1$ and $r_t := 2^{-1}a\Phi^{-1}(t)$. Let $x = x_0, x_1, \ldots, x_k = y$ be the points equally spaced on the line segment connecting $x$ to $y$ so that $|x_i - x_{i+1}| = R_1/k$ for $i = 0, \ldots, k - 1$ and $B_i := B(x_i, r_t)$, with $i = 0, 1, 2, \ldots, k$.

Since $4R_1 \leq k$, for each $y_i \in B_i$ we have

$$|y_i - y_{i+1}| \leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \leq \frac{1}{8} + \frac{R_1}{k} + \frac{1}{8} < \frac{1}{2}. \quad (6.8)$$

Moreover, $\delta_{\mathbb{H}}(y_i) \geq \delta_{\mathbb{H}}(x_i) - |y_i - x_i| > r_t > r_{t/k}$ and $t/k \leq R_1/k \leq 1/4$.

Thus, by Proposition 6.2 and (6.3), there are constants $c_i = c_i(a) > 0$, $i = 1, 2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$ we have

$$p_{\mathbb{H}}(t/k, y_i, y_{i+1}) \geq c_1 \left(\phi_1^{-1}(t/k)^{-d/2} \wedge \frac{t/k}{|y_i - y_{i+1}|^{d\phi_1(|y_i - y_{i+1}|)}}\right) \geq c_2 \left(\phi_1^{-1}(t/k)^{-d/2} \wedge t/k\right) = c_2 t/k. \quad (6.9)$$

Observe that $4R_1 \leq k < 2(k-1) < 8R_1$, $\phi_1^{-1}(t/k) \geq a_3^{1/\beta_1}(t/k)^{1/\beta_1}$ and $r_t \geq r_{t/k}$. Thus, from (6.9) we obtain

$$p_{\mathbb{H}}(t, x, y) \geq \int_{B_1} \cdots \int_{B_{k+1}} p_{\mathbb{H}}(t/k, x, y_1) \cdots p_{\mathbb{H}}(t/k, y_{k+1}, y) dy_{k+1} \cdots dy_1 \geq (c_2 t/k)^k \Pi_{i=1}^{k-1} |B_i| \geq (c_2 t/k)^k c_{3}^{k-1}(t/k)^{d(k-1)/\beta_1} \geq c_4 (c_3 t/k)^{c_5 k} \geq c_7 (c_8 t/R_1)^{c_9 R_1} \geq c_{10} (t/R_1)^{c_11 R_1}. \quad \Box$$

**Proposition 6.7** Suppose that $T > 0$, $a > 0$ and $\beta \in (1, \infty)$. Then, there exist constants $c_i = c_i(a, \beta, T) > 0$, $i = 1, 2$ such that for any $x, y$ in $\mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq a \Phi^{-1}(t)$ we have

$$p_{\mathbb{H}}(t, x, y) \geq c_1 t \exp \left(-c_2 \left(\left|\frac{T|x-y|}{t}\right|^{\frac{\beta+1}{\beta}} \wedge (|x-y|)^\beta\right)\right) \text{ if } t \leq T, |x-y| > 1,$$

and

$$p_{\mathbb{H}}(t, x, y) \geq c_1 t^{-d/2} \exp \left(-c_2 \left(\left|\frac{T|x-y|}{t}\right|^{\frac{\beta+1}{\beta}} \wedge (|x-y|)^\beta\right)\right) \text{ if } t > T, |x-y| > t/T.$$
Proof. Without loss of generality we assume \( T = 1 \). We fix \( a > 0 \), and we let \( R_1 := |x - y| \).

(i) If \( 1 \leq R_1 \leq 3 \) and \( t \leq 1 \), the proposition holds by virtue of Proposition \ref{prop:6.2}

(ii) If \( R_1 (\log(R_1/t))^{(\beta - 1)/\beta} \geq (R_1)^\beta \) (when \( t \leq 1 \)), the proposition holds also by virtue of Proposition \ref{prop:6.2}

(iii) If \( t > 1 \) and \( 3t \geq R_1 \geq t \), the proposition holds by virtue of Proposition \ref{prop:6.4}

(iv) We now assume \((t, R_1) \in ((0, 1] \times (3, \infty)) \cup ((1, \infty) \times (3t, \infty))\) and \( R_1 (\log(R_1/t))^{(\beta - 1)/\beta} < (R_1)^\beta \), which is equivalent to \( R_1 \exp\{-(R_1)^\beta\} < t \). Note that \( R_1/t > 3 \).

Let \( k \geq 2 \) be a positive integer such that

\[
1 < R_1 \left( \log \frac{R_1}{t} \right)^{-1/\beta} \leq k < R_1 \left( \log \frac{R_1}{t} \right)^{-1/\beta} + 1 < 2R_1 \left( \log \frac{R_1}{t} \right)^{-1/\beta}.
\]  

(6.10)

We define \( r_t := (2^{-1}a\Phi^{-1}(t/R_1)) \wedge ((6)^{-1}(\log(R_1/t))^{1/\beta}) \). Then, by (6.10) we have

\[
(2^{-1}a\Phi^{-1}(t/R_1)) \wedge \frac{R_1}{6k} \leq r_t \leq \frac{1}{6} \left( \log \frac{R_1}{t} \right)^{1/\beta} < \frac{R_1}{3k}.
\]  

(6.11)

Let \( x = x_0, x_1, \ldots, x_k = y \) be the points equally spaced on the line segment connecting \( x \) to \( y \) so that \( |x_i - x_{i+1}| = R_1/k \) for \( i = 0, \ldots, k - 1 \) and \( B_i := B(x_i, r_t) \), with \( i = 0, 1, 2, \ldots, k \). Then, \( \delta_H(y_i) \geq 2^{-1}a\Phi^{-1}(t) > 2^{-1}a\Phi^{-1}(t/k) \) for every \( y_i \in B_i \). Note that from (6.11) we obtain

\[
\frac{R_1}{3k} \leq |x_i - x_{i+1}| - 2r_t \leq |y_i - y_{i+1}| \leq |x_i - x_{i+1}| + 2r_t \leq \frac{5R_1}{3k}
\]  

(6.12)

for every \((y_i, y_{i+1}) \in B_i \times B_{i+1}\). We also observe that, by (6.10)

\[
\frac{t}{k} \leq \frac{t}{R_1} (\log(R_1/t))^{1/\beta} \leq \sup_{s \geq 3} s^{-1} (\log s)^{1/\beta} < \infty
\]

and

\[
\frac{R_1}{2k} \geq \frac{1}{4} (\log(R_1/t))^{1/\beta} \geq \frac{1}{2} (\log 3)^{1/\beta} > 0.
\]

Thus, using Proposition \ref{prop:6.2} along with (6.10) and (6.12) we obtain

\[
\begin{align*}
\mathbb{P}_H(t/k, y_i, y_{i+1}) &\geq c_1 \frac{t}{k} j(|y_i - y_{i+1}|) \geq c_2 \frac{t}{k} (R_1/k)^{-(d-\beta_2)} e^{-c_3(R_1/k)^\beta} \\
&\geq c_4 \frac{t}{R_1} \left( \frac{k}{2R_1} \right)^{d+\beta_2-1} e^{-c_3(R_1/k)^\beta} \geq c_4 \frac{t}{R_1} \left( \log \frac{R_1}{t} \right)^{-\frac{d+\beta_2-1}{\beta}} \left( \frac{t}{R_1} \right)^{c_3} \geq c_4 \left( \frac{t}{R_1} \right)^{c_5}.
\end{align*}
\]  

(6.13)

Since the definition of \( r_t \) yields

\[
r_t \geq c_6 \left( (t/R_1)^{(\beta_2 \wedge \beta)^{-1}} \wedge (\log(R_1/t))^{1/\beta} \right) \geq c_7 (t/R_1)^{(\beta_2 \wedge \beta)^{-1}}.
\]
by using (6.10), (6.13) and the semigroup property we conclude that

\[ p_H(t, x, y) \geq \int_{B_1} \cdots \int_{B_{k-1}} p_H(t/k, x, y_1) \cdots p_H(t/k, y_{k-1}, y) dy_1 \cdots dy_{k-1} \]

\[ \geq \frac{k^{k-1}}{c_4} \left( \frac{t}{R_1^k} \right)^{c_5 k + (\beta_2 \wedge \beta)^{-1} (k-1)} \]

\[ \geq c_8 \exp(-c_9 k \log(R_1/t)) \]

\[ \geq c_8 \exp\left( -c_9 \left( R_1 \log \left( R_1/t \right)^{-1/\beta} + 1 \right) \log(R_1/t) \right) \]

\[ \geq c_8 \exp\left( -2c_9 \left( R_1 \log \left( R_1/t \right)^{\frac{\beta-1}{\beta}} \right) \right) \]

\[ \geq c_8 \begin{cases} t \exp\left( -2c_9 \left( R_1 \log \left( R_1/t \right)^{\frac{\beta-1}{\beta}} \right) \right) & \text{if } (t, R_1) \in (0, 1] \times (3, \infty) \\ t^{-d/2} \exp\left( -2c_9 \left( R_1 \left( 1 + \log \left( R_1/t \right)^{\frac{\beta-1}{\beta}} \right) \right) \right) & \text{if } (t, R_1) \in (1, \infty) \times (3t, \infty). \end{cases} \]

Propositions 6.6-6.7 together with Proposition 6.2 and Propositions 6.4-6.5 yield the interior lower bound estimates of Theorem 6.1 for \( \beta \in (1, \infty] \).

**Remark 6.8** Assume that \( D \) is an connected open set with the following property: there exist \( \lambda_1 \in [1, \infty) \) and \( \lambda_2 \in (0, 1] \) such that for every \( r \leq 1 \) and \( x, y \in D \) with \( \delta_D(x) \wedge \delta_D(y) \geq r \) there exists in \( D \) a length parameterized rectifiable curve \( l \) connecting \( x \) to \( y \) with the length \( |l| \) of \( l \) less than or equal to \( \lambda_1 |x - y| \) and \( \delta_D(l(u)) \geq \lambda_2 r \) for \( u \in [0, |l|] \).

Under this assumption, we can also prove Theorem 6.1 on such \( D \) with minor modifications. We omit the details here; see [22, Section 3] for the case \( t < T \) and \( \phi(r) = r^\alpha \).

7 Two-sided heat kernel estimates

In this section we prove the two-sided estimates of \( p_H(t, x, y) \) under conditions (5.4) and (5.5).

**Theorem 7.1** Suppose (UJS), (5.4), and (5.5) hold. There exist \( c_1, c_2, c_3 > 0 \) such that for all \( (t, x, y) \in (0, \infty) \times H \times H \),

\[ p_H(t, x, y) \leq c_1 \left( \sqrt{\frac{\Phi(\delta_H(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi(\delta_H(y))}{t}} \wedge 1 \right) \times \begin{cases} h_{c_1,1}(t, |x - y|/6) & \text{if } t \in (0, 1), \\ k_{c_1,1}(t, |x - y|/6) & \text{if } t \in [1, \infty), \end{cases} \]

and

\[ p_H(t, x, y) \geq c_1^{-1} \left( \sqrt{\frac{\Phi(\delta_H(x))}{t}} \wedge 1 \right) \left( \sqrt{\frac{\Phi(\delta_H(y))}{t}} \wedge 1 \right) \times \begin{cases} h_{c_1,1}(t, 3|x - y|/2) & \text{if } t \in (0, 1), \\ k_{c_1,1}(t, 3|x - y|/2) & \text{if } t \in [1, \infty). \end{cases} \]

**Proof.** By Theorem 5.3, we only need to show the lower bound of \( p_H(t, x, y) \). For this, we will apply Theorem 6.1 with \( T = 1/3 \).
Since \( r \to h_{a,T}(t, r) \) and \( r \to k_{a,T}(t, r) \) are decreasing, we have by Theorem 6.1 that for \( |x-y| > 4M_1\Phi^{-1}(t) \),

\[
\inf_{(u,v):2M_1\Phi^{-1}(t)\leq|u-v|\leq|3|x-y|/2} p_{\Xi}(t/3, u,v) \geq c_1 \left\{ \begin{array}{ll}
h_{C_{4.1/3}}(t/3, 3|x-y|/2) & \text{if } t \in (0, 1), \\
k_{C_{4.1/3}}(t, 3|x-y|/2) & \text{if } t \in [1, \infty), \\
\end{array} \right.
\]

\[
\geq c_2 \left\{ \begin{array}{ll}
h_{c_{4.1}}(t, 3|x-y|/2) & \text{if } t \in (0, 1), \\
k_{c_{4.1}}(t, 3|x-y|/2) & \text{if } t \in [1, \infty). \\
\end{array} \right. \tag{7.1}
\]

When \( 6M_1\Phi^{-1}(t) \geq r \) and \( t \geq 1 \), by (5.7), we have \( c_8M_1t^{1/2} \geq r \). Thus on \( 6M_1\Phi^{-1}(t) \geq r \) and \( t \geq 1 \)

\[
k_{C_{4.1/3}}(t/3, r) \geq c_4 \left\{ \begin{array}{ll}
\Phi^{-1}(t)^{-d} t^{-d/2} \exp \left\{ -C_{4}(r^\beta \wedge 3^2) \right\} & \text{if } \beta = 0, \\
t^{-d/2} \exp \left\{ -C_{4}\left( (r + \log \frac{t}{r}) \wedge 3^2 \right) \right\} & \text{if } \beta \in (0, 1], \\
t^{-d/2} \exp \left\{ -C_{4}\left( (r \left( 1 + \log \frac{t}{r} \right) \wedge 3^2 \right) \right\} & \text{if } \beta \in (1, \infty), \\
\end{array} \right.
\]

\[
\geq c_5\Phi^{-1}(t)^{-d}. \tag{7.2}
\]

So by (7.2) and Theorem 6.1 for \( |x-y| \leq 4M_1\Phi^{-1}(t) \),

\[
\Phi^{-1}(t) \geq c_6 \left\{ \begin{array}{ll}
h_{c_{7.1}}(t, 3|x-y|/2) & \text{if } t \in (0, 1), \\
k_{c_{7.1}}(t, 3|x-y|/2) & \text{if } t \in [1, \infty). \\
\end{array} \right. \tag{7.3}
\]

Combining (4.7), (4.11) and (7.3), we conclude that

\[
p_{\Xi}(t, x, y) \geq c_8 \left( \sqrt{\Phi(\delta_H(x))} \wedge 1 \right) \left( \sqrt{\Phi(\delta_H(y))} \wedge 1 \right) \left\{ \begin{array}{ll}
h_{c_{9.1}}(t, 3|x-y|/2) & \text{if } t \in (0, 1), \\
k_{c_{9.1}}(t, 3|x-y|/2) & \text{if } t \in [1, \infty). \\
\end{array} \right.
\]

\[
\square
\]

**Remark 7.2** (i) In view of Theorem 5.3, we can restate Theorem 7.1 as follows. There are positive constants \( c_i, 1 \leq i \leq 5 \), so that

\[
\frac{1}{c_1} \left( \frac{1}{\sqrt{t} \Psi(1/\delta_D(x))} \wedge 1 \right) \left( \frac{1}{\sqrt{t} \Psi(1/\delta_D(y))} \wedge 1 \right) p(c_2t, c_3(y-x)) \leq p_D(t, x, y)
\]

\[
\leq c_1 \left( \frac{1}{\sqrt{t} \Psi(1/\delta_D(x))} \wedge 1 \right) \left( \frac{1}{\sqrt{t} \Psi(1/\delta_D(y))} \wedge 1 \right) p(c_4t, c_5(y-x)) \tag{7.4}
\]

where \( p(t, x) \) is the transition density of \( X \). This essentially confirms the conjecture (1.1) for this class of symmetric Lévy processes and for \( D = \mathbb{H} \).

(ii) Recently sharp two-sided Dirichlet heat kernel estimates have been established in [15, 14] for a large class of symmetric Lévy processes in \( C^{1,1} \) open sets for \( t \leq 1 \). The Lévy process considered in [15, 14] satisfy the conditions (5.4), (6.5) and (UJS) of this paper. Now assume \( X \) is a symmetric Lévy process considered [15, 14]. Then using the “push inward” method of [18] (see [12] for its
use in relativistic stable processes case) and the short time heat kernel estimates in [15, 14], we can obtain global sharp two-sided Dirichlet heat kernel estimates on half-space-like $C^{1,1}$ open sets from the the Dirichlet heat kernel estimates established in this paper on half-spaces. We leave the details to the interested reader.

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