From information theory and thermodynamic considerations a universal bound on the relaxation time \( \tau \) of a perturbed system is inferred, \( \tau \geq \hbar/\pi T \), where \( T \) is the system’s temperature. We prove that black holes comply with the bound; in fact they actually saturate it. Thus, when judged by their relaxation properties, black holes are the most extreme objects in nature, having the maximum relaxation rate which is allowed by quantum theory.

\[
\frac{\Delta S}{\tau} \leq \pi \Delta \mathcal{E}/\hbar ,
\] 

where \( \tau \) is the characteristic timescale for this dynamical process (the relaxation time required for the perturbed system to return to a quiescent state).

Taking cognizance of the second-law of thermodynamics, one obtains from Eq. (2) 

\[
\tau_{\text{min}} = \hbar/\pi T ,
\] 

where \( T \) is the system’s temperature. Thus, according to quantum theory, a thermodynamic system has (at least) one perturbation mode whose relaxation time is \((\pi T/\hbar)^{-1}\), or larger. This mode dominates the late-time relaxation dynamics of the system, and determines the characteristic timescale for generic perturbations to decay [18].

Typically for laboratory sized systems \( \tau T \) is many orders of magnitude larger than \( \hbar \). For example, the relaxation timescale of a gas composed of particles of mass \( m \) is of the order of \( R/c_s \), where \( R \) is the characteristic dimension of the system, and \( c_s \sim (T/m)^{1/2} \) is the sound velocity. Thus, \( \tau T \sim R(Tm)^{1/2} \sim 10^{11}\hbar \gg \hbar \) for room temperatures and \( R \sim 1 m \). One therefore wonders whether there are thermodynamic systems in nature whose relaxation times are of the same order of magnitude as the minimal relaxation time, \( \tau_{\text{min}} \), allowed by quantum theory?

The following argument will guide our search for physical systems that come close to challenge the dynamical bound, Eq. (3): The characteristic relaxation time of a thermodynamic system is bounded from below by the time it takes for the perturbation to propagate along the system. Thus, the relaxation time is bounded by the
characteristic size, $R$, of the system. This fact motivates a shift of interest to systems whose temperature is of the same order of magnitude (or smaller) as the reciprocal of the characteristic size of the system, $\frac{1}{T}$. 

The thermodynamic temperature of a Kerr black hole is given by the Bekenstein-Hawking temperature \[ T_{BH} = \frac{\hbar}{4 \pi (r_{+} + a^2)} \], where $r_{\pm} = M + (M^2 - a^2)^{1/2}$ are the black-hole outer and inner horizons, and $M$ and $a$ are the black-hole mass and angular momentum per unit mass, respectively. One therefore finds that a spherically symmetric Schwarzschild black hole (with $a = 0$) satisfies the relation $T_{BH} \sim \frac{\hbar}{r_{+}}$, and may therefore come close to saturate the relaxation bound, Eq. (3). Moreover, it seems that rotating Kerr black holes may even break the relaxation bound, since their temperatures are characterized by $T_{BH} \ll \frac{\hbar}{r_{+}}$ in the extremal limit $r_{-} \to r_{+}$ ($T_{BH} \to 0$). We shall now show that black holes conform to the relaxation bound; in fact they actually saturate it.

**Black-hole relaxation.**—The statement that black holes have no hair was introduced by Wheeler [19] in the early 1970’s. The various no-hair theorems state that the external field of a dynamically formed black hole (or a perturbed black hole) relaxes to a Kerr-Newman spacetime, characterized solely by three parameters: the black-hole mass, charge, and angular momentum. This implies that perturbation fields left outside the black hole would either be radiated away to infinity, or be swallowed by the black hole.

This relaxation phase in the dynamics of perturbed black holes is characterized by ‘quasinormal ringing’, damped oscillations with a discrete spectrum (see e.g. [20] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [21–24]. Being the characteristic ‘sound’ of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (especially, the mass and angular momentum of the black hole). This has motivated a flurry of research during the last four decades aiming to compute the quasinormal mode (QNM) spectrum of various types of black-hole spacetimes [20].

In addition, the highly damped black-hole resonances [25] have been the subject of much recent attention, with the hope that these classical frequencies may shed some light on the elusive theory of quantum gravity (see e.g. [26], and references therein). Furthermore, the anti-de Sitter/conformal field theory (AdS/CFT) conjecture [27] has led to an intensive study of black-hole QNM in asymptotically AdS spacetimes in the last few years [28–39]. According to the AdS/CFT correspondence, a large static black hole in AdS corresponds to an approximately thermal state in CFT. Perturbing the black hole corresponds to perturbing this thermal state, and the decay of the perturbation (characterized by QNM ring-down) describes the return to thermal equilibrium.

The dynamics of black-hole perturbations is governed by the Regge-Wheeler equation [40] in the case of a Schwarzschild black hole, and by the Teukolsky equation [41] for the Kerr black hole. The black hole QNMs correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [42]. Such boundary conditions single out a discrete set of black-hole resonances $\{ \omega_n \}$ (assuming a time dependence of the form $e^{-i\omega t}$).

Since the perturbation field can fall into the black hole or radiate to infinity, the perturbation decays and the corresponding QNM frequencies are complex. It turns out that there exist an infinite number of quasinormal modes, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [43,44]. The mode with the smallest imaginary part (known as the fundamental mode) gives the dynamical timescale $\tau$ for generic perturbations to decay (the relaxation time required for the perturbed black hole to return to a quiescent state). Namely, $\tau = \frac{\omega_I}{\pi}$, where $\omega_I$ denotes the imaginary part of the fundamental, least damped black-hole resonance.

Taking cognizance of the relaxation bound Eq. (3), one finds an upper bound on the black-hole fundamental quasinormal frequency,\[ \omega_I \leq \frac{\pi T_{BH}}{\hbar} . \] (4)

This relation implies that every black hole should have (at least) one QNM resonance whose imaginary part conform to the upper bound Eq. (4), and which determines the characteristic relaxation timescale of the perturbed black hole [45].

**Analytical and numerical confirmation of the bound.**—We now confirm the validity of the universal relaxation bound, starting with the ‘canonical’ Kerr black holes. Figure 1 displays the quantity $\frac{\hbar \omega_I}{\pi T_{BH}}$ for the least damped (fundamental) quasinormal frequencies of rotating Kerr black holes [43,46–48]. In contrast with laboratory sized systems, black holes are found to have relaxation times which are of the same order of magnitude as the minimally allowed one, $\tau_{\text{min}}$. Specifically, one finds (see Fig. 1) that the fundamental black-hole resonances are characterized by $\frac{\hbar \omega_I}{\pi T_{BH}} \lesssim 1$ [49]. In Fig. 2 we depict similar results for Kerr-Newman black holes [50]. Remarkably, extremal black holes (which are characterized by $T_{BH} \to 0$) saturate the dynamical relaxation bound. Namely, their relaxation times are infinitely long, in accord with the third-law of thermodynamics.

It is of great interest to check the validity of the upper bound, Eq. (4), for other black-hole spacetimes. The endpoint of a charged and non-rotating gravitational collapse is described by the Reissner-Nordström spacetime. The black-hole formation is followed by a relaxation
phase, which describes the decay of two types of perturbation fields: coupled gravitational-electromagnetic perturbations, and charged perturbation fields. One finds [51] that the fundamental QNM frequency of a charged scalar field is characterized by \( \omega_I = \pi T_{BH}/\bar{\hbar} \) in the \( eQ/T_{BH} \gg 1 \) limit, where \( e \) is the charge coupling constant. Thus, a charged Reissner-Nordström black hole saturates the relaxation bound.

The QNM of a D-dimensional Schwarzschild black hole can be calculated analytically in the limit of a large angular index \( \ell \gg 1 \) [52]. Using the results of [52], we find

\[
\frac{\hbar \omega_I}{\pi T_{BH}} = \left( \frac{2}{D-1} \right)^{1/(D-3)} \frac{2}{\sqrt{D-1}},
\]

which is a monotonic decreasing function of \( D \), with a maximum of \( \hbar \omega_I/\pi T_{BH} = 4/(3\sqrt{3}) < 1 \) at \( D=4 \). Thus, D-dimensional Schwarzschild black holes conform to the upper bound Eq. (4).

Furthermore, D-dimensional Schwarzschild-de Sitter black holes [53] also conform to the upper bound Eq. (4), with the desired property of saturating it in the extremal limit, \( T_{BH} \to 0 \). Large black holes in asymptotically AdS spacetime (large compared to the AdS radius) have the interesting property that for odd parity gravitational perturbations, the imaginary parts of the QNM frequencies scale like the inverse of the black-hole temperature (in this regime, \( T_{BH} \sim r_+^{1/2} \)) [31,39]. Hence, these modes are particularly long lived and conform to the relaxation bound Eq. (4). In the small AdS black-hole regime, the imaginary parts of the quasinormal frequencies scale with \( r_+^{1/2} \) [31]. Thus, these black-hole resonances also conform to the relaxation bound.

**Summary.** — In this Letter we have derived a universal bound on relaxation times of perturbed thermodynamic systems, \( \tau \geq \hbar/\pi T \) [54]. The relaxation bound is a direct consequence of quantum information theory and thermodynamic considerations.

We conjecture that a relation of this form could serve as a quantitative way to express the third-law of thermodynamics. Namely, one cannot reach a temperature \( T \) in a timescale shorter than \( \hbar/\pi T \) (which indeed goes to infinity in the limit of absolute zero of temperature, in accord with the third-law).

Remarkably, black holes comply with the dynamical bound; in fact they have relaxation times which are of the same order of magnitude as \( \tau_{\text{min}} \), the minimal relaxation time allowed by quantum theory. Moreover, extremal black holes (in the \( T_{BH} \to 0 \) limit) actually attain the bound— their relaxation time is infinitely long. Since black holes saturate the fundamental bound, we conclude that when judged by their relaxation properties, black holes are the most extreme objects in nature.

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