Realistic Quark and Lepton Masses
Through SO(10) Symmetry

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Abstract
In a recent paper a model of quark and lepton masses was proposed. Without any family symmetries almost all the qualitative and quantitative features of the quark and lepton masses and Kobayashi-Maskawa mixing angles are explained, primarily as consequences of various aspects of SO(10) symmetry. Here the model is discussed in much greater detail. The threefold mass hierarchy as well as the relations \( m_\tau \simeq m_0 \), \( m_\mu \simeq 3m_0 \), \( m_e \simeq \frac{1}{3} m_0 \), \( m_0 \ll m_d \), \( m_0 \ll m_b \), \( \tan \theta_C \simeq \sqrt{m_d/m_s}, \) \( V_{cb} \ll \sqrt{m_b/m_s} \) and \( V_{ub} \sim V_{us} V_{cb} \) follow from a simple Yukawa structure at the unification scale. The model also gives definite predictions for \( \tan \beta \), the neutrino mixing angles, and proton decay branching ratios. The \( (\nu_\mu - \nu_\tau) \) mixing angle is typically large, \( \tan \beta \) is close to either \( m_t/m_b \) or \( m_c/m_s \), and proton decay is in the observable range, but there is a group theoretical suppression factor in the rate.

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1 Introduction

In a recent paper\textsuperscript{1} we proposed a model of quark and lepton masses based upon the gauge group $SO(10)$. In that paper it was shown how most of the features of the fermion masses and mixings can be understood to be consequences of various aspects of $SO(10)$ invariance rather than of some family symmetry or arbitrarily imposed texture.

In this paper we go into greater detail. In Section 2 the basic ideas of the model are explained and its structure is reviewed in detail. The results of numerical fits are presented in Section 3. Section 4 deals with the predictions of the model for $\tan\beta$ and the neutrino mixing angles. Predictions for proton-decay branching ratios are also briefly discussed there (for details see Ref. (2)). In Section 5 various technical issues relating to the Higgs sector, symmetry breaking, and the naturalness of the gauge hierarchy are examined. Finally, in Section 6 certain alternative possibilities for variant models are discussed. Various technical details are summarized in three Appendices.

2 Review of the Model

(a) The Root Model

The model we shall study is based on $SO(10)$, and though supersymmetry is not essential for its account of quark and lepton masses it shall be assumed because of the gauge hierarchy. The “matter” fields (quarks and leptons) are contained in three spinors ($\mathbf{16}_i$, $i = 1, 2, 3$), which are the “families”, and a real (in the group theory sense) set of additional representations. These latter consist of a pair of family and anti-family ($\mathbf{16} + \overline{16}$), and a pair of vectors
In the “long version” of the model there is also a pair of adjoints $(45 + 45')$. The Higgs multiplets which couple to the matter are in a vector $(10_H)$, two adjoints $(45_H + 45_H')$, and a pair of spinors $(16_H + \overline{16}_H)$. The short version of the model has two sets of Yukawa terms in the superpotential, $W_{\text{spinor}}$ and $W_{\text{vector}}$. The long version has an additional set of terms, $W_{\text{adjoint}}$. These are given by

$$W_{\text{spinor}} = M(16\overline{16}) + \sum_i b_i (16, \overline{16}) 45_H + \sum_i a_i (16, 16) 10_H,$$

$$W_{\text{vector}} = d(10, 10') 45_H + \sum_i c_i (16, 10) 16_H + \sum_i c'_i (16, 10') 16_H,$$

$$W_{\text{adjoint}} = f(45, 45') 45_H + \sum_i e_i (16, 45) \overline{16}_H + \sum_i e'_i (16, 45') \overline{16}_H.$$

Note that all three pieces of the Yukawa superpotential have the same basic structure. In each there is one term (the first) which couples together the pair of extra representations and gives them superheavy mass, and two terms (the second and third) which couple these extra representations to the ordinary families. What is not allowed is a direct mass term $\sum_i g_{ij} (16, 16) 10_H$, which would, unless there were fine tuning, tend to give comparable masses to all the generations. All these three pieces, in other words, have a kind of “see-saw” structure in which the ordinary families get masses through their mixing with the extra fields.

The dominant contribution to the fermion mass matrices is assumed to come from $W_{\text{spinor}}$. (As will be seen in Section 5, this is most simply explained as being a consequence of the condition $\langle \overline{5}(10_H) \rangle \ll \langle \overline{5}(16_H) \rangle$. Here and throughout the notation $p(q)$ refers to a $p$ of $SU(5)$ contained in a $q$ of $SO(10)$.) Diagrammatically, this contribution comes from the tree graph.
shown in Fig. 1. Its approximate form can be read off from that figure.

\[ W_0 \cong \sum_{ij} a_i b_j \frac{\langle 10_H \rangle \langle 45_H \rangle}{M} (16, 16_j). \]  

(4)

Defining \( \hat{a}_i = a_i/a, \hat{b}_i = b_i/b \) \((a = |\vec{a}|, b = |\vec{b}|)\); writing the VEV of the \( 45_H \) as \( \langle 45_H \rangle = \Omega Q \), where \( \Omega \), like \( M \), is of order \( M_{\text{GUT}} \), and where \( Q \) is a linear combination of \( \text{SO}(10) \) generators; and defining \( T \equiv \frac{M}{M} \); one has

\[ W_0 \cong aT \langle 10_H \rangle \sum_{ij} \left[ \hat{a}_i \hat{b}_j Q_{(16)_j} \right] (16, 16_j). \]  

(5)

Then for charge-(\( \frac{2}{3} \)) quarks the mass matrix, \( U_{ij} \) is given by

\[ U_{ij} \cong aTv \sum_{ij} \left[ \hat{a}_i \hat{b}_j Q_u + \hat{a}_j \hat{b}_i Q_u^c \right] u_i^c u_j, \]  

(6)

with similar expressions for the mass matrices of the charge-(\( \frac{-1}{3} \)) quarks \( (D_{ij}) \), charged leptons \( (L_{ij}) \), and the Dirac mass matrix of the neutrinos \( (N_{ij}) \). One can choose, without loss of generality, the axes in family space so that \( \hat{b}_i = (0, 0, 1) \) and \( \hat{a}_i = (0, \sin \theta, \cos \theta) \). The quantity \( Q \) is a linear combination of \( \text{SO}(10) \) generators, with in general complex coefficients since \( 45_H \) is a chiral superfield. There is a two-dimensional space of such generators that commute with \( SU(3)_c \times SU(2)_L \times U(1)_Y \). We can thus choose to write

\[ Q = 2I_{3R} + \frac{6}{5}\epsilon (Y/2), \]  

(7)

where \( I_{3R} \) is the third generator of \( SU(2)_R \). Then Eq. (6) and its analogues take the form

\[ U_0 = aTv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_u \sin \theta / N_u \\ 0 & Q_u \sin \theta / N_u^c & (Q_u^c + Q_u) \cos \theta / N_u N_u^c \end{pmatrix}, \]  

(8)
\[ D_0 = a T v' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_d \sin \theta/N_d \\ 0 & Q_{d'} \sin \theta/N_{d'} & (Q_{d'} + Q_d) \cos \theta/N_{d'}N_d \end{pmatrix}, \]  
\tag{9}

\[ L_0 = a T v' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_{l'} \sin \theta/N_{l'} \\ 0 & Q_{l} \sin \theta/N_l & (Q_l + Q_{l'}) \cos \theta/N_{l'}N_{l'} \end{pmatrix}, \]  
\tag{10}

\[ N_0 = a T v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_{\nu} \sin \theta/N_{\nu} \\ 0 & Q_{\nu'} \sin \theta/N_{\nu'} & (Q_{\nu'} + Q_{\nu}) \cos \theta/N_{\nu'}N_{\nu} \end{pmatrix}, \]  
\tag{11}

where
\[
Q_u = Q_d = \frac{1}{3} \epsilon \\
Q_{u'} = -1 - \frac{4}{5} \epsilon \\
Q_{d'} = 1 + \frac{2}{5} \epsilon \\
Q_{l'} = -\frac{3}{5} \epsilon \\
Q_{l} = 1 + \frac{6}{5} \epsilon \\
Q_{\nu'} = -1 
\]  
\tag{12}

The factors \( N_f \equiv \sqrt{1 + T^2 |Q_f|^2} \) come from doing the algebra exactly rather than evaluating the lowest order graph in Fig. 1. [The linear combination of \(16\)'s which has a Dirac mass with \(\overline{16}\) and is thus superheavy is clearly seen from Eq. (1) and the definition of \(T\) to be \((16 + T Q_{16} \overline{16})/\sqrt{1 + T^2 |Q_{16}|^2}\). So the superheavy \(u\), for example, is \((u_{16} + T Q_u u_{16})/\sqrt{1 + T^2 |Q_u|^2}\), etc. Finding the orthogonal (light) linear combinations and writing the term \(a_{16,1610_H}\) which appears in Eq. (1) in terms of them, one obtains the exact expressions in Eqs. (8) – (11).]

The striking feature of Eqs. (8) – (11) is that the mass matrices are rank 2. This is a consequence of ‘factorization’; that is, that the mass matrices do not come from a Yukawa coupling coefficient that is a \textit{matrix} in family space.
but from a product of Yukawa coefficients that are vectors in family space. The rank 2 comes directly from the fact that \( \sum_i (a_i \mathbf{16}, 10_H) \) involves just two distinct linear combinations of the light generations, namely \( \hat{a}_i \mathbf{16} \), and \( \mathbf{16} \). In other words, the fact that there are two heavy generations is a direct consequence of the fact that Yukawa terms are bilinear in matter fields. (It is amusing to note that in this context if there were less than three generations then there would be no light family such as makes up the ordinary matter of our world. Perhaps this is a partial answer to the famous question of Rabi. Without the \( \mu \) and the \( \tau \) one would not have the electron!)

A second striking feature is that these equations provide an explanation of the puzzling fact that the minimal \( SU(5) \) prediction \( m_0^b \cong m_0^\tau \) works so well while the corresponding predictions for the lighter generations fail badly. (Here and throughout, the superscript \( ^0 \) refers to a quantity evaluated at the GUT scale.) \( m_0^b \cong m_0^\tau \) is a consequence of the fact that \( D_{33} \cong L_{33} \), which follows from the relation \( Q_{dc} + Q_d = Q_{t^+} + Q_{l^-} \). This relation, in turn, is implied by the fact that both \( d \) and \( l \) get mass from the same Higgs field, \( H' \), so that \( Q_{dc} + Q_d + Q_{H'} = Q_{t^+} + Q_{l^-} + Q_{H'} = 0 \). On the other hand, \( m_0^\mu \cong -L_{32}L_{23} \) and \( m_0^s \cong -D_{32}D_{23} \), so that \( m_0^\mu / m_0^s \cong L_{32}L_{23} / D_{32}D_{23} = Q_{dc}Q_{d} / Q_{H'}, \) which is a group-theoretical factor of order unity but not equal to unity in general.

Another way to understand the fact that \( m_0^b \cong m_0^\tau \) while \( m_0^s \neq m_0^\mu \) is from the group-theoretical structure of the term \( \mathbf{16}_i \mathbf{16}_j \mathbf{10}_H \mathbf{45}_H \). (Cf. Eq. (4).) The \( \mathbf{10}_H \times \mathbf{45}_H \) can be contracted into a \( \mathbf{10}, \mathbf{120}, \) or \( \mathbf{320} \) of \( SO(10) \). \( \mathbf{320} \) is not contained in \( \mathbf{16} \times \mathbf{16} \) and so does not couple. If \( i = j = 3 \) then the contraction \( \mathbf{120} \), which must couple antisymmetrically in flavor, is also not allowed. Hence only the contraction into a \( \mathbf{10} \) couples to \( \mathbf{16}_3 \mathbf{16}_3 \), and
therefore the minimal SU(5) relation $D_{33} = L_{33}$ holds. But for $(i, j) = (2, 3)$ or $(3, 2)$ the antisymmetric contraction into a 120 also couples, and thus the minimal SU(5) relation does not hold for the second generation.

Three more facts are explained if the assumption is made that $|\epsilon| \ll 1$, i.e. that $\langle 45_H \rangle$ points approximately in the $I_{3R}$ direction. First, since $|m_0^s|^2 \sim \frac{1}{5} \epsilon \sin^2 \theta$, one has for $\theta \sim 1$ that $V_{cb}^0 \sim 2 \left( \frac{m_0^s}{m_0^b} \right)$, which is a true relation.

A further consequence of the structure of Eq. (4) is that the Higgsino-mediated proton-decay amplitude is proportional to $\epsilon$. Thus the smallness of $\epsilon$ helps solve the problem of excessively rapid proton decay which tends to afflict supersymmetric grand unified models, especially those with large $\tan \beta$. (It should be noted that many SUSY GUT models based on SO(10) give $\tan \beta \simeq \frac{m_t^0}{m_b^0}$.) The branching ratio $Br[(p \rightarrow K^+\nu \mu)/(p \rightarrow \pi^+\nu \mu)]$ is a calculable quantity in the model, but it differs from that of minimal SUSY SU(5). Similarly, the charged lepton mode $p \rightarrow K^0\mu^+$ which could become significant in the large $\tan \beta$ scenario has a different branching ratio than in
$SU(5)$. These distinctions could serve as testing grounds for the high scale flavor structure.

In summary up to this point, the simple structure of $W_{\text{spinon}}$ together with the assumption that $Q \sim I_{3R}$ has explained or contributed to explaining six facts. And the single group-theoretical assumption about the direction of $Q$ has played a role in four of these explanations, a striking economy. It should be noted that the direction $I_{3R}$ is not an arbitrary one but a point of higher symmetry, so that certain simple superpotentials can give an adjoint Higgs a VEV in that direction. For example, a superpotential with the terms $W = S^3 + S^2 + SA^2 + A^2$, where $A$ is a 45 and $S$ is a 54 has solutions for $A$ in the $I_{3R}$ and $B - L$ directions. And a superpotential with terms $W = A^2 + (A^2)^2 + A^4$ has solutions in the $I_{3R}$, $B - L$ and $X$ directions. ($X$ is the $SU(5)$-singlet generator of $SO(10)$.) It is also interesting that the Dimopoulos-Wilczek mechanism, which appears to be the only viable way to make the gauge hierarchy natural in $SO(10)$, requires there to exist an adjoint VEV in the $B - L$ direction.

At this point, more needs to be said about the factors $N_f \equiv \sqrt{1 + T^2|Q_f|^2}$. Note that for $T$ small, which corresponds to small mixing between the 16, and the 16, these factors become very close to unity and can be ignored. [There cannot be too small, however, since $m_t^0 \approx aTv\cos\theta$, and therefore $a \approx 1/T$. For $a \gg 1$ perturbation theory would break down above the GUT scale.] Even with $T$ of order unity the $N_f$ become simple in the limit $|\epsilon| \ll 1$, where $N_{u,d,l} = 1 + O(\epsilon^2)$, and $N_{c,d,l} = \sqrt{1 + T^2} + O(\epsilon)$. Effectively, then, there is for small $\epsilon$ only a single parameter, $N \equiv \sqrt{1 + T^2} \sim 1$, introduced by these mixing factors. This parameter, $N$, plays no significant role. For without it
there would be three parameters ($\epsilon$, $\theta$, and $\tan \beta$) to fit the five mass ratios and one mixing angle of the second and third generations. That means there would be three quantitative “predictions”, which are (for the small $\epsilon$ case)

\[m_{\tau}^0 \simeq m_b^0, \quad m_{\mu}^0 \simeq 3m_s^0, \quad m_{\ell_{e}}^0/m_{\ell_{t}}^0 \simeq m_s^0/m_b^0\] (about which more later). But one easily sees that the factors of $1/N$ appearing in the (32) and (33) elements of all the mass matrices approximately cancel out in all these relations and thus leave them unaffected. As for the two qualitative “predictions”, namely that $V_{cb}$ and the ratios of second to third generation masses are of order $\epsilon$ and hence small, these are clearly unaffected by the presence of $N$, which is of order unity.

The simple structure of $W_{\text{spinor}}$ (Eq. (1)), as we have just seen, explains in a highly economical fashion most of the features of the masses and mixings of the two heavy generations, and indeed the very fact that there are two heavy generations. Two facts about the heavy generations are not explained. The first is that $m_t^0 \gg m_b^0$, which is just equivalent in this model to the statement that $\tan \beta \gg 1$. An explanation of this must rely on an understanding of the Higgs sector. If we regard this model as a model of the Yukawa sector, this question is beyond its scope. The second unexplained fact is the failure of the “proportionality” relation of $SO(10)$,

\[m_{c}/m_{t}^0 = m_s^0/m_b^0\]

which is not significantly broken by any effects discussed so far. Different mechanisms for breaking this bad relation lead to different versions of the model, see subsections (c) and (d) below.

(b) The first generation: $W_{\text{vector}}$

The masses and mixings of the first generation arise from the next layer
of the model, given by $W_{\text{vector}}$. As before, in the case where the mixing of the $16_i$ with the $10 + 10'$ is small the contribution of $W_{\text{vector}}$ to the light quark and lepton masses can be read off from a simple graph, shown in Fig. 2. At first glance it might seem that many new parameters are introduced by adding $W_{\text{vector}}$. (In fact, nine: the six Yukawa couplings, $c_i$ and $c'_i$, and three VEVs.) But the important point is that in the limit of small mixing Fig. 2 gives a flavor-antisymmetric contribution to $D_{ij}$ and $L_{ij}$. (There is no contribution to $U_{ij}$ as $10$ and $10'$ do not contain charge-$(-\frac{2}{3})$ quarks.) The flavor-antisymmetry is obvious from Fig. 2, for under the interchange $10 \leftrightarrow 10'$ the Yukawa coupling $10 \tilde{45}H 10'$ changes sign because the adjoint of $SO(10)$ is an antisymmetric tensor, while the flavor indices $i$ and $j$ are interchanged. Moreover, if it is assumed that the $\tilde{45}H$ acquires an $SU(5)$-invariant VEV, the contributions to $D_{ij}$ and $L_{ij}$ will be equal (up to a sign since $\delta D = \delta L^T$). Thus in the small mixing limit (we will return to the more general case later) there are really only three new parameters introduced which we will call $c_{ij}$, $(i \neq j)$, where

$$c_{ij} = (c_i c'_j - c_j c'_i) \frac{M\langle 1(16_H) \rangle}{a b d \langle \tilde{45}_H \rangle \langle 45_H \rangle \langle 5(10_H) \rangle}.$$  \hspace{1cm} (13)

It should be noted that if the $c_{ij}$ are all comparable then $c_{23}$ is small compared to the other contributions to the terms in which it appears ($D_{23}$, $D_{32}$, $L_{23}$, and $L_{32}$) and can therefore be neglected.

The mass matrices then take the form (putting in also the values of the $Q_f$ from Eq. (12))
Several more features of the quark and lepton mass spectrum are explained by this form. First, that $U_{ij}$ is still rank 2 corresponds to the fact that $m_0^u/m_0^t$ ($\approx 10^{-5}$) is much smaller than $m_0^d/m_0^b$ ($\approx 10^{-3}$) and $m_0^e/m_0^\tau$ ($\approx 0.3 \times 10^{-3}$). Second, the antisymmetry of the contributions to the first row and column of $D_{ij}$ implies that one has effectively the form 

$$D = aT v' \begin{pmatrix} 0 & -c_{12} & -c_{13}/N_d \\ c_{12} & 0 & \left(\frac{1}{5} \epsilon \sin \theta - c_{23}/N_d\right) \\ c_{13}/N_{dc} & \left((1 + \frac{2}{5} \epsilon) \sin \theta + c_{23}\right)/N_{dc} & \left(1 + \frac{3}{5} \epsilon\right) \cos \theta/N_{dc}\tilde{N}_d \end{pmatrix},$$

(15)
usual $\sqrt{m_0^2/m_0^0} \approx 0.07$. Thus this “prediction” works better here than in the usual scenarios.)

A third consequence of the forms given in Eqs. (15) and (16) is that

$$\det D = \det L.$$  \hspace{1cm} (18)

Note that, remarkably, this is true for any values of $\theta$, $\epsilon$, and $c_{ij}$, if $T$ is small enough that the factors $N_f$ can be taken to be one. And for any $T$, this equality holds for small $\epsilon$. From this follows the well-known Georgi-Jarlskog relation$^6$ $m_e^0/m_d^0 \cong (m_\mu^0/m_s^0)^{-1} \cong \frac{1}{3}$.

Finally, if all the $c_{ij}$ are comparable then $V_{ub} \sim V_{us} V_{cb}$, which is the true order of magnitude statement which underlies the Wolfenstein parameterization.$^{15}$ (For $V_{ub} \sim D_{31}/D_{33} \sim aT v'(c_{13}/m_b^0)$, $V_{us} \sim aT v'(c_{12}/m_h^0)$, and $V_{cb} \sim m_s^0/m_b^0$, the last being both empirically true and a consequence in the model of $\theta \sim 1$ as noted earlier.)

(c) The proportionality relation: The short model

The structure described so far, given in Eqs. (1) and (2), gives a satisfactory account of all the features of the quark and lepton spectrum (the threefold hierarchy, the mixing angles, and the mass ratios) with one glaring exception. The ratio $m_e^0/m_t^0$ is seemingly predicted to be equal to $m_s^0/m_b^0$, whereas empirically it is about one-fifth of that. The most obvious explanation of this anomaly would be that another additive contribution to $U_{ij}$ exists which happens to approximately cancel $U_{23}$. It is plausible that this might happen without notably changing any of the other successful features of the model since $U_{23}$ is a small element and also does not affect the mixing
angles. This is an attractive possibility. But it suffers from the apparent difficulty that the new contribution to $U_{ij}$ must leave it very nearly rank 2 in order not to produce a value of $m_u$ that is too large. We have found a way to do this, which is discussed in Section 6(b). This idea, however, has certain drawbacks, discussed in Section 6(b), that make it less attractive than the ideas we shall now present. However, we cannot discount the possibility that a more elegant solution along these lines may exist.

A remarkable fact about the model described so far, containing only the Yukawa couplings in Eqs. (1) and (2), is that it already contains, without any further additions, a multiplicative correction to the mass matrices that can break the bad proportionality relation while leaving the other good relations largely intact.

We have emphasized that the forms of the mass matrices given in Eqs. (14)–(17) are valid for small mixing. The exact expressions for the $D$, $L$, and $N$ matrices including all mixing effects are

$$D = aTv'(I+\Delta_d')^{-\frac{1}{2}} \left( \begin{array}{ccc} 0 & -c_{12} & -c_{13}/N_d \\ c_{12} & 0 & \left(\frac{4}{5}e \sin \theta - c_{23}\right)/N_d \\ c_{13}/N_d & \left((1 + \frac{2}{5}e) \sin \theta + c_{23}\right)/N_d & \left((1 + \frac{2}{3}e) \cos \theta / N_d\right) \end{array} \right),$$

(19)

$$L = aTv' \left( \begin{array}{ccc} 0 & c_{12} & c_{13}/N_l^- \\ -c_{12} & 0 & \left(-\frac{3}{5}e \sin \theta + c_{23}\right)/N_l^- \\ -c_{13}/N_l^+ & \left((1 + \frac{2}{5}e) \sin \theta - c_{23}\right)/N_l^+ & \left((1 + \frac{2}{3}e) \cos \theta / N_l^+ N_l^-\right) \end{array} \right) (I+\Delta_l')^{-\frac{1}{2}},$$

(20)
\[
N = aT\nu \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{3}{5} \epsilon \sin \theta / N_{\nu} \\
0 & -\sin \theta / N_{\nu} & -(1 + \frac{4}{5} \epsilon) \cos \theta / N_{\nu} N_{\nu},
\end{pmatrix} (I + \Delta_{\nu}^T)^{-\frac{1}{2}},
\] (21)

where
\[
(\Delta_{\nu})_{ij} = C^C_i C_j + C^C_i C_j',
\] (22)

and where
\[
(C_1, C_2, C_3) \equiv \left| \frac{1(16_H)}{d(45_H)} \right| (c_1, c_2, c_3/N_{\nu}),
\] (23)

with a similar expression for \(C_j'\) in terms of the \(c_j'\). The expression for \(\Delta_{l-}\) is of the same form with \(N_{l-}\) instead of \(N_{d-}\), and \(\Delta_{\nu} = \Delta_{l-}\) by \(SU(2)_L\). See Appendix A for the derivation. The expression for \(U_{ij}\) given in Eq. (14) is exact as it stands.

The matrices \(\Delta_f\) characterize the mixing between the \(16_i\) and the \(10 + 10'\), just as \(TQ_{16_i}\) characterizes the amount of mixing between the \(16_i\) and the \(16\). The factors \((I + \Delta_f)^{-\frac{1}{2}}\) are completely analogous to the factors of \((1 + T^2 |Q_f|^2)^{-\frac{1}{2}} \equiv N_f^{-1}\). For small mixing these factors are close to the identity and can be neglected. Even if the \(\Delta_f\) are not small, most of the quantitative and qualitative successes of the model are only slightly affected. Obviously the fact that \(m_u \approx 0\) and the threefold hierarchy are among these. Moreover, if \(N_f \approx 1\) then for arbitrarily large \(\Delta_f\) the relation \(\det D \equiv \det L\) continues to hold, since then \(\Delta_{d-} \approx \Delta_{l-}^T\).

Consider a case in which
\[
(I + \Delta_{d-})^{-\frac{1}{2}} \approx (I + \Delta_{l-})^{-\frac{1}{2}} \approx \begin{pmatrix} 1 \\ \delta \end{pmatrix},
\] (24)
where $\delta < 1$. From Eqs. (19) and (20) it can be seen that the main effect of this factor is to suppress the $\tau$ and $b$ masses by a factor of approximately $\delta$. This suppression can be understood simply and intuitively in the following way. $\delta \ll 1$ arises, as can be seen from Eqs. (22) and (23), from a large value of $c_3$ or $c_3'$ compared to $d\langle 4\tilde{5}_H\rangle /\langle 1(16_H)\rangle$. From Eq. (2) one sees that there is a superheavy mass term $5(10)[(d\langle 4\tilde{5}_H\rangle) \cdot 5(10') + (c_3\langle 1(16_H)\rangle) \cdot 5(16)]$. Large $c_3$, therefore, corresponds to the superheavy linear combination being approximately $5(16)$ and the orthogonal, light linear combination that is the third generation being approximately $5(10')$. That means that the term $a \cos \theta \cdot 5(16) \cdot 10(16) \cdot 5(16)$ in Eq. (1) gives a contribution mostly to the superheavy $5$ rather than to the third generation $5$ which contains $b^c$ and $\tau^-$. Thus the masses of $b$ and $\tau$ are suppressed by a factor of approximately $\delta$.

It is easily seen by substituting Eq. (24) into Eqs. (19) and (20) that the masses of $b$ and $\tau$ are multiplied approximately by $\delta$ while the masses of $s$ and $\mu$ are not much affected. Thus the ratio $m_s^0/m_b^0$ is enhanced relative to $m_c^0/m_t^0$ by a factor of $\delta^{-1}$. Hence $\delta$ should be about $\frac{1}{5}$. At the same time, the near cancellation between the (23) mixing angles in the up and down sectors that is responsible for the smallness of $V_{cb}$ is not disturbed, since the ratio $D_{32}/D_{33}$ is left unchanged.

What this shows is that the factors $(I + \Delta)^{-\frac{1}{2}}$ in Eqs. (19) and (20) allow a fit to be made to $(m_c^0/m_t^0)/(m_s^0/m_b^0)$ without greatly disturbing the qualitative and quantitative successes of the model. However, there are certain prices to be paid for this method of breaking the proportionality relation. First, $\theta$ must be somewhat small. Since $V_{cb} \approx \frac{2}{5} \epsilon \tan \theta$ and $m_s^0/m_b^0 \approx \frac{1}{5} \epsilon \sin^2 \theta/\delta$, it follows that $\theta \approx 2\delta^2 m_s^0/m_b^0 V_{cb} \sim \delta \sim \frac{1}{5}$. There is noth-
ing in principle wrong with this, except that as $\theta$ is the angle between two supposedly unrelated vectors, $\vec{a}$ and $\vec{b}$, it might have been expected to be closer to unity. Second, the smallness of $\left(\frac{m_0^c}{m_0^b}\right)/\left(\frac{m_0^s}{m_0^b}\right)$ is not so much explained as fit, and, in particular, a special choice of the form of $(I + \Delta d)^{-\frac{1}{2}}$ has to be made. (Cf. Eq. (24).) This choice amounts to the statement that the vector $c_i$ (or $c'_i$) points nearly in the ‘3’ direction. There is then a kind of preferred direction in family space, in spite of our having eschewed family symmetry. (Though, on the other hand, the ‘small numbers’ involved in this fortuitous alignment are only of order $\frac{1}{5}$, whereas the intergenerational hierarchies being explained involve ratios of $10^{-2}$ to $10^{-5}$.) A third cost is that the exactness of the relation $m_0^b \cong m_0^\tau$ is lost. For example, the simple form given in Eq. (24) would lead (for small $\epsilon$ and $c_{23}$) to $m_0^b \cong \delta(aTv')$ and $m_0^\tau \cong \sqrt{\delta^2 \cos^2 \theta + \sin^2 \theta (aTv')}$. Since both $\delta$ and $\theta$ are of order $\frac{1}{5}$, $m_0^b/m_0^\tau$ can deviate from unity by as much as 40%. As we shall see in Section 3, the form of $\Delta d$ can be chosen to fit $m_0^b/m_0^\tau$. But now it can only be claimed as a “prediction” that the masses of $b$ and $\tau$ are equal in a rough sense. Finally, because the directions of $c_i$ and $c'_i$ are not arbitrary but must be chosen to give $\Delta d$ the required form, it turns out that there is a hierarchy among the $c_{ij}$. In particular, $c_{23}$ is somewhat larger than $c_{12}$ and $c_{13}$, and therefore, while still small compared to the terms in which it enters, it is not so small as to be negligible. In fact, the Georgi-Jarlskog factors of 3 are significantly affected. (See Section 3.)

(d) The long version of the model

While all of the costs of fitting $\left(\frac{m_0^c}{m_0^l}\right)/\left(\frac{m_0^s}{m_0^b}\right)$ that had to be paid in
the short version of the model are relatively minor, they do somewhat take away from the cleanness of the model and its explanatory power. There is, however, another very beautiful way to break the proportionality relation. This involves adding the terms $W_{\text{adjoint}}$ given in Eq. (3) to the superpotential. To some extent this reduces the economy of the model, though as emphasized earlier the overall structure of $W_{\text{adjoint}}$ is parallel to that of $W_{\text{spinor}}$ and $W_{\text{vector}}$. Moreover, by adding these terms one is enabled not merely to fit but to explain the smallness of $(m^0_\chi/m^0_\nu)/(m^0_\nu/m^0_\mu)$ in a rather elegant way, as will now be explained. Finally, this explanation is achieved in such a way that the other positive features of the model are virtually unaffected. None of the costs that have to be paid in the short version have to be paid here. This then is a much cleaner version of the model.

If one assumes that $\langle 5(\overline{16}_H) \rangle = 0$ (this will be seen to be desirable on other grounds in Section 5) there is no additive contribution to the mass matrices (analogous to the $c_{ij}$) coming from $W_{\text{adjoint}}$. $W_{\text{adjoint}}$ does introduce, however, new multiplicative corrections of the form $(I + \Delta_f)^{-\frac{1}{2}}$, where $f = u^c$, $u$, $d$, and $l^+$, that reflect the mixing of the $10(16_i)$ with the $10(45)$ and $10(45')$. This mixing comes from the following terms contained in $W_{\text{adjoint}}$:

$$
\overline{10}(45) \left[ (f \langle 45_H \rangle) \cdot 10(45') + \sum_i (e_i \langle 1(\overline{16}_H) \rangle) \cdot 10(16_i) \right] \\
+ \overline{10}(45') \left[ (f \langle 45_H \rangle) \cdot 10(45) + \sum_i (e'_i \langle 1(\overline{16}_H) \rangle) \cdot 10(16_i) \right].
$$

(25)

The effect of this is to introduce factors of the form $(I + \Delta_f)^{-\frac{1}{2}}$ into the mass matrices $U$, $D$, and $L$ wherever they multiply an $SU(5)$ $10$ of fermions. For example, the matrix $U_{ij}$ now takes the form
\[ U = a T v (I + \Delta_{u^c})^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} \epsilon \sin \theta / N_u \\ 0 & -(1 + \frac{4}{5} \epsilon) \sin \theta / N_u & -(1 + \frac{3}{5} \epsilon) \cos \theta / N_u \end{pmatrix} (I + \Delta_u)^{-\frac{1}{2}}. \] (26)

The matrices \( \Delta_{u^c}, \Delta_u, \Delta_d, \) and \( \Delta_{l^+} \) all have the form

\[ (\Delta_f)_{ij} = E_i^* E_j + E_i'^* E_j', \] (27)

where

\[ (E_1, E_2, E_3) \equiv \left\{ \frac{\{1(\mathbf{16}_H)\}}{f} \Omega Q_f(\mathbf{45}) \right\}(e_1, e_2, e_3 / N_f), \] (28)

and similarly for the definition of \( E_i' \). (The expressions for \( \Delta_{d^c}, \Delta_{l^-}, \) and \( \Delta_\nu \) have already been given in Eqs. (22) and (23).)

A point of crucial importance is that the expression for \( \Delta_f, f = u^c, u, d, \) and \( l^+ \), given in Eqs. (27) and (28) has a factor of \( Q_f(\mathbf{45}) \) in the denominator. This simply comes from the VEV of the \( \mathbf{45}_H \) that appears in Eq. (25). Note that this adjoint is the same one that appears in \( W_{\text{spinor}} \) (rather than the \( \mathbf{45}_H \) that appears in \( W_{\text{vector}} \) and has its VEV in the direction \( Q \). But it can be seen from Eq. (3) that the generator \( Q \) in \( W_{\text{adjoint}} \) is not acting on fermions that are in a \( \mathbf{16} \), as in \( W_{\text{spinor}} \), but on fermions that are in a \( \mathbf{45} \). What is the difference? Using the relation \( I_{3R} = -\frac{1}{10} X + \frac{3}{5} \left( \frac{Y}{2} \right) \), one can rewrite Eq. (7) as

\[ Q = \left( -\frac{4}{5} \right) X + \left( \frac{6}{5} (1 + \epsilon) \right) \frac{Y}{2}. \] (29)

The \( SU(5) \)-singlet generator \( X \) (as we have normalized it) takes the value 1 on the representation \( \mathbf{10}(\mathbf{16}) \), but -4 on the representation \( \mathbf{10}(\mathbf{45}) \). Thus

\[ Q_f(\mathbf{45}) = Q_f(\mathbf{16}) + 1. \] (30)
Thus the charges that enter into Eq. (28) are

\[
\begin{align*}
Q_{u(45)} &= Q_{d(45)} = 1 + \frac{4}{5}\epsilon \\
Q_{u^c(45)} &= -\frac{4}{5}\epsilon \\
Q_{t^c(45)} &= 2 + \frac{6}{5}\epsilon.
\end{align*}
\]

Note the important fact that for $|\epsilon| \ll 1$ the only field in the 45 that has a small charge is the $u^c$. The remarkable consequence of this is that the elements of $\Delta_{u^c}$ are of order $\frac{1}{\epsilon}$ (see Eqs. (25), (26), and (29)), while all the other $\Delta_f$ have elements that are not enhanced in this way. In this long version of the model, then, the assumption can be made that all the $\Delta_f$ are less than unity so that the factors $(I + \Delta_f)^{-\frac{1}{2}}$ are unimportant, except for the matrix $\Delta_{u^c}$, which has large elements of order $\frac{1}{\epsilon^2}$.

Because of the foregoing only the matrix $U$ gets substantially affected by the presence of $W_{\text{adjoint}}$. Most of the successful relations of the model are then essentially unaffected, since they follow from the forms of $D$ and $L$. The quantities that depend strongly on the form of $U$ are $m_u$, $m_{c}/m_t$, and the mixing angles. But, as the factor $(I + \Delta_{u^c})^{-\frac{1}{2}}$ does not alter the fact that $U$ is rank 2, the smallness of $m_u$ is still explained. And because the factor $(I + \Delta_{u^c})^{-\frac{1}{2}}$ involves a transformation of the right-handed quarks, the mixing angles are not strongly affected. One might expect that the cancellation between the contributions from the up and down sectors that makes $V_{cb}$ small would be seriously disrupted if $(I + \Delta_{u^c})^{-\frac{1}{2}}$ were very different from the identity matrix. It turns out, however, that $V_{cb}$ remains of order $\epsilon$. In fact (see Appendix B for the derivation) $V_{cb}^0 \approx \frac{2}{5}\epsilon \sin \theta \cos \theta \left[ 1 + \frac{N}{2} \tan \theta \frac{(E \times E')_2}{(E \times E')_3} \right]$.

From these considerations it is seen that the only prediction of the model that is substantially altered by the addition of the term $W_{\text{adjoint}}$ is the pro-
portionality relation for $m_c^0/m_t^0$. At first glance one might expect that the factor $(I + \Delta_{\text{uc}})^{-\frac{1}{2}}$ would be just as likely to increase as to decrease the ratio $m_c^0/m_t^0$, for arbitrary values of the parameters in $\Delta_{\text{uc}}$. But, remarkably, this proves not to be the case. In Appendix B we give an explicit proof which shows that if the elements of $\Delta_{\text{uc}}$ are large then $m_c^0/m_t^0$ will be suppressed except for special directions in parameter space. (Interestingly, this is a consequence of the fact that $\Delta_{\text{uc}}$ is a rank 2 matrix, as can be seen from Eq. (27). The ratio $(m_c^0/m_t^0)/(m_s^0/m_b^0)$ in this case is given by

$$\frac{(m_c^0/m_t^0)}{(m_s^0/m_b^0)} \sim \frac{|\vec{E} \times \vec{E}'|}{|\vec{E} \times \vec{E}'|_3^2} \sqrt{|E_1|^2 + |E'|^2} \simeq O(\epsilon),$$

from which it is clear that $(m_c^0/m_t^0)$ is suppressed relative to $(m_s^0/m_b^0)$ for $|E_i| \gg 1$ (i.e., $\epsilon \ll 1$, Cf. Eq. (28)). See Appendix B for a derivation. If it were an arbitrary matrix of rank one or rank three, then $m_c^0/m_t^0$ would with equal likelihood be suppressed or enhanced. This also is shown in Appendix B. These statements have also been checked by numerical tests.) Of course, if the elements of $\Delta_{\text{uc}}$ are small, $m_c^0/m_t^0$ will be only slightly affected one way or the other. In particular, since the elements of $\Delta_{\text{uc}}$ are of order $\frac{1}{\epsilon}$ the ratio $m_c^0/m_t^0$ is suppressed by a factor of order $\epsilon$ as shown above (see Appendix B), which is what is needed to agree with experiment.

The surprising conclusion is that the addition of the terms $W_{\text{adjoint}}$ to the superpotential for generic values of the parameters suppresses the ratio $m_c^0/m_t^0$ by order $\epsilon$, while having only a minor effect otherwise. The violation of the proportionality relation is therefore not merely fit in some arbitrary way, but explained in a group-theoretical way. In this version of the model, the relation $|\epsilon| \ll 1$ plays a role in explaining no less than five facts! Such
economy of explanation is not something that can be contrived.

3 Numerical Fits

(a) Long version

In this Section we shall present the details of our numerical fits. Let us first focus on the long version of the model. In this version, we can consistently set the multiplicative matrices $\Delta_u, \Delta_d, \Delta_d^c, \Delta_{l-}, \Delta_\nu$ and $\Delta_{l+}$ (Cf. Eqs. (19)-(21)) all to zero. The elements of $\Delta_{u_c}$ are enhanced by a factor $1/|\epsilon|^2$ and therefore $\Delta_{u_c}$ cannot be ignored (Cf. Eq. (26)). In this limit we have the following approximate analytic expressions for the mass ratios and mixing angles:

\[ m_0^t \simeq \frac{aT v |(\vec{E} \times \vec{E}')_3|}{N |\vec{E} \times \vec{E}'|} \]
\[ m_0^e/m_0^t \simeq \frac{\sqrt{5}/6 \epsilon \sin^2 \theta |\vec{E} \times \vec{E}'| \sqrt{(E_1^2 + E_2^2)}}{|(\vec{E} \times \vec{E}')_3|^2} \]
\[ m_0^b/m_0^t \simeq 0 \]  \hspace{1cm} (32)

\[ m_0^b \simeq \frac{aT v}{N} \]
\[ m_0^2/m_0^b \simeq \frac{N}{5} \epsilon \sin^2 \theta \]
\[ m_0^d m_0^s/m_0^b \simeq c_{12} N^2 (c_{12} \cos \theta - c_{13} \sin \theta) \]  \hspace{1cm} (33)
\[ m_0^\tau \simeq m_0^b, \; m_0^\mu \simeq 3 m_0^s, \; m_0^e \simeq \frac{1}{3} m_0^d \]  \hspace{1cm} (34)

\[ V_{ub}^0 \simeq c_{13} \]
\[ V_{cb}^0 \simeq \frac{2}{5} \epsilon \cos \theta \sin \theta \left[ 1 + \frac{N (\vec{E} \times \vec{E}')_2}{2 (\vec{E} \times \vec{E}')_3} \right] \]

\[ V_{us}^0 \simeq \sqrt{m_d^0/m_s^0} \left[ \cos \theta - \frac{c_{13}}{c_{12}} \sin \theta \right]^{-1/2} \]

(35)

Note that in this version, the angle \( \theta \) is of order one, as a result, there is a significant correction to the expression \( \tan \theta \simeq \sqrt{m_d^0/m_s^0} \). However, the ratio \( c_{13}/c_{12} \) is nearly equal to 0.5 in order to fit \( V_{ub} \), and with this value the correction factor in Eq. (25) is small for a wide range of the angle \( \theta \) as long as \( c_{13} \) and \( c_{12} \) have a relative negative sign. (For example, if \( c_{13}/c_{12} = -1/2 \), the correction factor in Eq. (25) is 1.04 corresponding to \( \theta = 60^0 \).) This fact is borne out in our numerical fits.

A good fit follows by choosing

\[ \epsilon = 0.15i, \quad \vec{e} = (1, 1, 0), \quad \vec{e}' = (1, -1, 0) \]

\[ T = 1, \quad \sin \theta = \cos \theta, \quad \frac{4f \Omega}{5 \langle 45_H \rangle} = 1.2 \]

\[ c_{12} = 0.0035, \quad c_{13} = -0.002, \quad c_{23} = 0.002 \]

(36)

The resulting mass eigenvalues are

\[ (m_u^0, m_c^0, m_t^0) = [0, 3.64 \times 10^{-3}, 0.708]aTv \]

\[ (m_d^0, m_s^0, m_b^0) = [8.54 \times 10^{-4}, 1.59 \times 10^{-2}, 0.709]aTv' \]

\[ (m_e^0, m_{\mu}^0, m_{\tau}^0) = [2.98 \times 10^{-4}, 4.53 \times 10^{-2}, 0.707]aTv \]

(37)

The quark mixing angles have the values

\[ V_{us}^0 = 0.213, \quad V_{ub}^0 = 0.002, \quad V_{cb}^0 = 0.030 \]

(38)

Since the input parameters are all complex in general, there is room for sufficient KM type CP violation. In the specific example given above, this
CP violation is somewhat small, but for other choices of input phases large enough CP violation can be obtained.

(b) Short version

Here the fit is somewhat nontrivial since the same set of vectors $c_i$ and $c'_i$ of Eq. (2) should generate the antisymmetric contribution $c_{ij}$ of Eq. (13) as well as the multiplicative factor $\Delta_{i\ell} = \Delta_{\ell i}$ given in Eq. (19)-(20) needed to correct the proportionality relation. As already noted, this can be achieved by choosing an approximate form of $\Delta_{i\ell}$ as given in Eq. (24). However, $m^0_b$ and $m^0_\tau$ are now only approximately equal for generic values of the model parameters, to within 40% or so. We shall choose the parameters so that $m^0_b \simeq m^0_\tau$ remains good to within about 10%. The angle $\theta$ in this case is of order $1/5$, so that the correction to the relation $\tan \theta_c \approx \sqrt{m^0_d/m^0_s}$ is small.

We will also need $T \ll 1$ so that $N \simeq 1$.

The following choice of parameters gives a good fit:

$$\vec{c} = (1, 0.7, 7), \quad \vec{c'} = (0.75, 1.5, 4.5) \times 10^{-3}$$

$$c_{ij} = (c_i c'_j - c'_i c_j), \quad \sin \theta = 0.22, \quad \epsilon = 0.45i, \quad T = 0.25 \quad (39)$$

The masses and mixing angles with this choice are

$$(m^0_u, m^0_s, m^0_t) = [0, 4.5 \times 10^{-3}, 1.0]aTv$$

$$(m^0_d, m^0_s, m^0_b) = [1.7 \times 10^{-4}, 4.2 \times 10^{-3}, 0.22]aTv'$$

$$(m^0_e, m^0_\mu, m^0_\tau) = [5.2 \times 10^{-5}, 1.2 \times 10^{-2}, 0.25]aTv' \quad (40)$$

$$V^0_{us} = 0.213, \quad V^0_{ub} = 0.0025, \quad V^0_{cb} = 0.0295 \quad (41)$$

Note that these numbers correspond to the ratios $m^0_\mu/m^0_s = 2.82, m^0_d/m^0_e =$
3.26, $m_c^0/m_t^0 = 1/225$, $m_s^0/m_b^0 = 1/52$, $(m_c^0/m_t^0)/(m_s^0/m_b^0) = 1/4.3$, all of which are in good agreement with data.

4 Predictions

(a) tan $\beta$

If one neglects all the factors of $(I + \Delta_f)^{-\frac{1}{2}}$, then $v/v' \approx m_t^0/m_b^0 \approx m_c^0/m_s^0$. Of course, it is the failure of the latter equality that requires one to assume that not all of the mixing matrices are trivial. In the short version of the model, the mass of the $b$ quark is suppressed by such effects while the masses of $s$, $c$, and $t$ are little affected. Thus $v/v' \approx m_c^0/m_s^0$ still holds but $m_t^0/m_b^0$ is larger. If one assumes that $\langle \overline{5}(16_H) \rangle \ll \langle \overline{5}(10_H) \rangle$ (see the discussion in Section 5(a) for why this is sensible) then $\tan \beta \approx v/v'$ and there follows the unusual prediction that $\tan \beta^0 \approx m_c^0/m_s^0$. After renormalization group corrections, $\tan \beta$ at weak scale is $\approx 18$ (corresponding to $m_t = 175 \text{ GeV}$.)

In the long version of the model, the masses of $b$ and $s$ are little affected by the mixing matrices, but both $m_c^0$ and $m_t^0$ are suppressed. (See Appendix B.) However, $m_t^0$ is only suppressed by a factor of order unity, while $m_c^0$ is suppressed by a factor of order $\epsilon$. Thus (again assuming $\tan \beta \approx v/v'$) one has the prediction $\tan \beta \approx m_t^0/m_b^0$. (This relation is unaffected by renormalization group running.)

(b) Neutrino mixing angles

The predictions for neutrinos will be discussed first in the long version of the model where they are simpler. In the long version it is assumed that all the $\Delta_f$ are somewhat less than unity so that the factors $(I + \Delta_f)^{-\frac{1}{2}}$ are
close to unity and negligible, except for $\Delta_{\nu^c}$, which is enhanced by the group-theoretical effect discussed above. But obviously the factor $(1 + \Delta_{\nu^c})^{-\frac{1}{2}}$ does not affect the neutrino mixing angles at all.

At first glance it might be thought that there could be no predictions for neutrinos in either version of this model, since the neutrino mass matrix, $M_{\nu}$, depends not only on the Dirac neutrino mass matrix, $N$, which is known (after fitting the charged fermion masses and mixing angles), but also on the superheavy, Majorana mass matrix, $M_R$, which is unrelated directly to the others and not known or predicted in this model.

$$M_{\nu} = N^T M_R^{-1} N.$$  \hspace{1cm} (42)

However, ignoring those effects which produce $m_u \neq 0$, the matrix $N$, like $U$, has rank 2, with vanishing first row and column in the basis in which we have worked. (See Eq. (11).) It is then clear that (in this limit) for any form of $M_R^{-1}$, the matrix $M_{\nu}$ also has vanishing first row and column. Thus, in diagonalizing $M_{\nu}$ no rotation in the 1-2 or 1-3 planes is necessary, and the mixing angles $V_{e\mu}$ and $V_{e\tau}$ come entirely from the diagonalization of $L$, which is a known matrix, just as $V_{us}$ and $V_{ub}$ come from the diagonalization of $D$. Since the form of $L$ is similar to that of $D$, there are relations between the lepton and quark mixing angles. In particular, since $L_{33} \cong D_{33}$ and $L_{31} = -D_{31}$, one has $V_{13}^{\text{lepton}} \cong -(V_{\text{KM}})_{13}$, or

$$V_{e\tau} \cong -V_{ub}.$$ \hspace{1cm} (43)

And for the same reason that $V_{us} \cong \sqrt{m_d^0/m_s^0}$,

$$V_{e\mu} \cong \sqrt{m_e^0/m_\mu^0}.$$ \hspace{1cm} (44)
It is likely that whatever effect makes $U$ a rank 3 matrix also makes $N$ rank 3. Just as these effects are likely to correct $\tan \theta_c$ by $O(m_u^0/m_c^0)$, one would expect that the corrections to $V_{e\mu}$ and $V_{e\tau}$ would be roughly of order $m_u^0/m_c^0 \approx 0.005$ and $m_u^0/m_t^0 \approx 10^{-5}$ respectively. But it is possible also that the effects that produce $m_u$ act differently on the neutrinos.

The mixing angle $V_{\mu\tau}$ cannot be predicted in the absence of any information about $M_R$. However, in certain limits interesting predictions arise. Simply multiplying out Eq. (42) one finds that

\[
(M_{\nu})_{33} = (M^{-1}_{R})_{33}(1 + \frac{3}{5} \epsilon) \sin^2 \theta/N^2 + 2(M^{-1}_{R})_{23} \frac{3}{5} \epsilon (1 + \frac{3}{5} \epsilon) \sin \theta \cos \theta/N
\]

\[
(M_{\nu})_{23} = (M_{\nu})_{32} = (M^{-1}_{R})_{33}(1 + \frac{3}{5} \epsilon) \sin \theta \cos \theta/N^2 + (M^{-1}_{R})_{23} \frac{3}{5} \epsilon \sin^2 \theta/N,
\]

\[
(M_{\nu})_{22} = (M^{-1}_{R})_{33} \sin^2 \theta/N^2.
\]

And, of course, $(M_{\nu})_{1i} = (M_{\nu})_{1i} = 0$. From these equations it follows that

\[
(M^{-1}_{R})_{33} > (M^{-1}_{R})_{23}, \quad \epsilon (M^{-1}_{R})_{22}
\]

\[
\Rightarrow \quad V^0_{\mu\tau} = -3V^0_{e\mu} \left[1 + O\left(\frac{(M^{-1}_{R})_{23}(M^{-1}_{R})_{33}(M^{-1}_{R})_{33}}{(M^{-1}_{R})_{33}} \tan \theta\right) + O(\epsilon)\right].
\]

Another interesting case is

\[
(M^{-1}_{R})_{23} > (\epsilon \tan \theta)^{-1}(M^{-1}_{R})_{33}, \quad \epsilon \tan \theta(M^{-1}_{R})_{22}
\]

\[
\Rightarrow \quad \frac{V^0_{\mu\tau}}{m(\nu_\mu)/m(\nu_\tau)} \approx \frac{1}{4} \theta, \quad \frac{m(\nu_\mu)/m(\nu_\tau)}{m(\nu_\mu)/m(\nu_\tau)} \approx \frac{1}{4}.
\]

In the short version of the model the neutrino predictions are complicated by the presence of the factors $(I + \Delta_{I-})^{-\frac{1}{2}} = (I + \Delta_{\nu})^{-\frac{1}{2}}$ in Eqs. (20) and (21).
These factors affect the mixing of the left-handed charged leptons. In the absence of these factors one would have the same predictions for neutrinos as in the long version.

The matrix $\Delta_{l^-}$ is given by the expressions in Eqs. (22) and (23) with $N_d^c$ replaced by $N_l^c$. A satisfactory suppression of $m_{0b}$ (and thus of $(m_{0c}/m_{0t})/(m_{0s}/m_{0b})$) would be achieved if $c_1$, $c_2$, $c_1'$, and $c_2'$ were very small compared to one, while $c_3$ and/or $c_3'$ were larger than one. For simplicity of discussion let us henceforth ignore $c_i'$ and just assume that $c_3 > 1$. Then $(\Delta_{l^-})_{33} \cong |c_3|^2 > 1$ and all other elements of $(\Delta_{l^-})$ are small, which gives $(I + \Delta_{l^-})^{-\frac{1}{2}} \cong \text{diag}(1, 1, 1/\sqrt{1 + |c_3|^2})$. This has the effect of multiplying $L_{33}$ by $\frac{1}{\sqrt{1 + |c_3|^2}}$, and thus the rotation in the $(\mu_L, \tau_L)$-plane required to diagonalize $L$ is not approximately $\tan \theta$ but $\sqrt{1 + |c_3|^2} \tan \theta \cong \frac{m_{0c}/m_{0t}}{m_{0s}/m_{0b}} \tan \theta$, which numerically (see Section 3) is of order unity. The short version, then, typically predicts that $V_{\mu\tau}$ is large.

A second effect of the matrix $(I + \Delta_{l^-})^{-\frac{1}{2}}$ arises from the fact that $c_1$ and $c_2$ are not in general exactly zero. In fact, as was seen in Section 3, one can fit the relation $m_{0b} \cong m_{0\tau}$ by choosing $\text{Re}(c_2/m_3) \cong -\frac{1}{2} \tan \theta$. Now, the presence of a non-vanishing $c_1$, which would be natural to expect, leads to non-vanishing (12) and (13) elements of $(I + \Delta_{l^-})^{-\frac{3}{2}}$. These, in turn, contribute to $V_{e\mu}$ and $V_{e\tau}$. In fact, $V_{e\tau}$ can be quite large. Because of the presence of the unknown parameter $c_1$, there are not independent predictions for $V_{e\mu}$ and $V_{e\tau}$, but one prediction for $V_{e\mu}$ in terms of $V_{e\tau}$. For $V_{e\tau}$ small, the prediction for $V_{e\mu}$ just goes over to that for the long version of the model.
More generally
\[ V_{e\mu} \cong \sqrt{m_0^2/m_\mu^2} + O(V_{e\tau}^0). \] (48)

5 Technical Issues

(a) The Higgs sector: doublet-triplet splitting

In SO(10) SUSY GUTS the simplest way to naturally achieve the doublet-triplet splitting of the Higgs fields that is required for the gauge hierarchy is by the Dimopoulos-Wilczek mechanism.\(^{12}\) The essential idea involves an adjoint of Higgs \((A_1)\) whose VEV is in the \(B - L\) direction and a pair of fundamental Higgs multiplets \((T_1\) and \(T_2)\). Consider the following Higgs superpotential.

\[ W_{2/3} = \lambda T_1 A_1 T_2 + M_2(T_2)^2. \] (49)

Since the \(T_i\) are in \(10\) representations, each contains a \(5 + \bar{5}\) of \(SU(5)\) that we will denote \(5(T_i)\) and \(\bar{5}(T_i)\). Then one has the following mass matrix for those fields

\[ \begin{pmatrix} \bar{5}(T_1), \bar{5}(T_2) \end{pmatrix} \begin{pmatrix} 0 & \lambda \langle A_1 \rangle \\ -\lambda \langle A_1 \rangle & M_2 \end{pmatrix} \begin{pmatrix} 5(T_1) \\ 5(T_2) \end{pmatrix}. \] (50)

If \(\langle A_1 \rangle = a \cdot (B - L) + b \cdot (I_{3R})\), then, in an obvious notation, one has for the masses of the \(SU(2)_L\)-doublets and color-triplets contained in \(T_i\)

\[ W_{\text{mass}} = \begin{pmatrix} \bar{2}(T_1), \bar{2}(T_2) \end{pmatrix} \begin{pmatrix} 0 & \lambda b \\ -\lambda b & M_2 \end{pmatrix} \begin{pmatrix} 2(T_1) \\ 2(T_2) \end{pmatrix} + \begin{pmatrix} \bar{3}(T_1), \bar{3}(T_2) \end{pmatrix} \begin{pmatrix} 0 & \lambda a \\ -\lambda a & M_2 \end{pmatrix} \begin{pmatrix} 3(T_1) \\ 3(T_2) \end{pmatrix}. \] (51)

If \(a \approx M_{\text{GUT}} \approx M_2\), while \(b = 0\), there is a massless pair of doublets, \(2(T_1) + \bar{2}(T_1)\), that play the role of \(H + H'\) in the supersymmetric standard
model (SSM), while the other pair of doublets and all of the triplets become superheavy. Moreover, if one assumes that $T_2$ does not couple to light quarks and leptons the proton-decay amplitude coming from the exchange of the color-triplet Higgsinos, $\mathbf{3}(T_1) + \overline{\mathbf{3}}(T_1)$, is proportional to $M_2/(\lambda a)^2$. For the theory to be perturbative $\lambda$ cannot be large compared to unity, and so $\lambda a \lesssim M_{\text{GUT}}$. However, $M_2$ can be somewhat smaller than $M_{\text{GUT}}$, and this provides a means by which the proton lifetime can be made consistent with experiment.\(^8\) (In the minimal $SU(5)$ SUSY GUT it is well-known that the proton lifetime is only marginally consistent, if at all, with experiment.\(^9\))

In general, $b$ will not stay exactly zero if non-renormalizable terms (induced by gravity, for example) are taken into account, and, in fact, it is not a completely trivial matter to ensure that nonrenormalizable terms do not destabilize it. How small must $b$ remain to preserve the gauge hierarchy? As noted, the proton-decay amplitude is proportional to $M_2/(\lambda a)^2$, which must therefore be $\lesssim (10M_{\text{GUT}})^{-1}$. The mass of the light Higgs doublets, on the other hand, is given by $(\lambda b)^2/M_2$, and therefore can be written $m_H \sim \left(\frac{b}{a}\right)^2(10 M_{\text{GUT}})$. thus, it must be that $b/a \lesssim 10^{-7} \lesssim (M_{\text{GUT}}/M_{\text{Pl}})^2$. That this can naturally be achieved is shown in Ref. 16.

This simple picture of doublet-triplet splitting must be modified in the context of the present model, since in addition to the $\mathbf{10}_H$ appearing in $W_{\text{spinor}}$, which is to be identified with $T_1$, there is the $\mathbf{16}_H$ appearing in $W_{\text{vector}}$, which also must break $SU(2)_L \times U(1)_Y$ and must as a consequence partially contain a light $\overline{2}$. To be more exact, there is just a single light $\overline{2}$ that gets a Weak-scale VEV and is a linear combination of doublets in $\mathbf{10}_H$ and $\mathbf{16}_H$. This raises two issues: (1) Can the Dimopoulos-Wilczek mechanism
for doublet-triplet splitting still work?, and (2) does the mechanism for suppressing Higgsino-mediated proton-decay still operate?

In the long version of the model there is also a $\mathbf{16}_H$ that is required not to get an $SU(2)_L \times U(1)_Y$-breaking VEV. So issue (3) is whether this is natural. Finally, it is desirable that $\langle \mathbf{5}(\mathbf{16}_H) \rangle / \langle \mathbf{5}(\mathbf{10}_H) \rangle$ (which we shall define to be $\tan \gamma$) be small for two reasons. First, it would explain (see Eq. (13)) why the $c_{ij}$ are small, and, second, it would mean that the light Higgs, $H'$, of the SSM would be almost purely in the $\mathbf{10}_H$, so that the ratio $\langle \mathbf{5}(\mathbf{10}_H) \rangle / \langle \mathbf{5}(\mathbf{10}_H) \rangle$, which is predicted in the model, is just the empirically measurable parameter $\tan \beta$. Issue (4) is whether the smallness of $\tan \gamma$ can be simply and naturally achieved.

Let us denote $\mathbf{10}_H$, $\mathbf{16}_H$, and $\mathbf{16}_H$ by $T_1$, $C$, and $C$. A satisfactory generalization of Eq. (49) is

$$W_{2/3} = \lambda T_1 A_1 T_2 + M_2 (T_2)^2 + \rho T_1 C C + M C C. \quad (52)$$

which gives

$$W_{\text{mass}} = \begin{pmatrix} 0 & \lambda \langle A_1 \rangle & \rho \langle C \rangle \\ -\lambda \langle A_1 \rangle & M_2 & 0 \\ 0 & 0 & M_C \end{pmatrix} \begin{pmatrix} \mathbf{5}(T_1) \\ \mathbf{5}(T_2) \\ \mathbf{5}(C) \end{pmatrix}. \quad (53)$$

If $b = 0$ (that is, if $\langle A_1 \rangle = a(B - L)$), then the matrix for the doublets has one massless eigenvalue. The massless $2$ is purely in $T_1$, while the massless $2$ is a linear combination of $\mathbf{2}(T_1)$ and $\mathbf{2}(C)$:

$$\mathbf{2}_{\text{light}} = \cos \gamma \mathbf{2}(T_1) + \sin \gamma \mathbf{2}(C), \quad (54)$$

with

$$\tan \gamma = -\rho \langle C \rangle / M_C. \quad (55)$$
Since the orthogonal, superheavy doublet, $\mathbf{2}_{\text{heavy}} = -\sin \gamma \mathbf{2}(T_1) + \cos \gamma \mathbf{2}(C)$, must have vanishing VEV, it follows that $\langle \mathbf{2}(C) \rangle / \langle \mathbf{2}(T_1) \rangle \equiv \langle \mathbf{5}(16_H) \rangle / \langle \mathbf{5}(10_H) \rangle = \tan \gamma$. Thus by making the ratio $\rho \langle \mathbf{C} \rangle / M_C$ small one ensures that issue (4) raised above is satisfactorily resolved. One possibility, discussed later, is that the term $\rho T_1 \overline{C} C$ arises from a higher-dimension operator, so that $\rho \sim O(M_{\text{GUT}}/M_{\text{Pl}})$.

It is necessary, as before, that $b/a \lesssim 10^{-7}$ to preserve the gauge hierarchy (issue (1)), but it is no longer sufficient. It is also necessary that the lower left entry in Eq. (53), call it $x$, which connects $\mathbf{5}(C)$ to $\mathbf{5}(T_1)$ be extremely small. To be precise, it must be that $x \rho \langle \mathbf{C} \rangle / M_C \lesssim m_W$, or, in other words, $x \lesssim \cot \gamma m_W$. It should be noted that this would also automatically ensure that $\langle \mathbf{5}(16_H) \rangle \cong 0$, which resolves issue (3). For $\langle \mathbf{5}(16_H) \rangle / \langle \mathbf{5}(10_H) \rangle \equiv \langle \mathbf{2}(\overline{C}) \rangle / \langle \mathbf{2}(T_1) \rangle = x / M_C \lesssim m_W / \rho \langle \mathbf{C} \rangle \sim m_W / M_{\text{GUT}}$.

The remaining question (issue 2) is whether Higgsino-mediated proton decay can still be suppressed by making $M_2$ somewhat smaller than $M_{\text{GUT}}$. This is easily seen to be the case from the form of Eq. (53). If $M_2$ is set to zero, then the colored Higgsino $\mathbf{3}(T_1)$ (which is by assumption the only $\mathbf{3}$ to couple to the light quarks and leptons) only has a mass connecting it to $\mathbf{3}(T_2)$, which does not mix with the $\mathbf{3}(T_1)$ and $\mathbf{3}(C)$ that couple to the light quarks and leptons. Thus the Higgsino-mediated proton-decay amplitude is proportional to $M_2$ as in the simpler case of Eq. (49). Note also that there is a group theoretical suppression of order $\epsilon$ in the rate for proton decay.$^2$

In order for this simple scenario to work naturally, we have seen that several conditions must be satisfied. (a) As before, $b/a$ must be $\lesssim 10^{-7}$. (b) For $x$ to be sufficiently small the coefficient of any effective $T_1 \overline{C} C$ term must
be less than about $m_W/M_{\text{GUT}}$. And (c) the $T_1\overline{C}C$ term should have a small ($O(M_{\text{GUT}}/M_{\text{Pl}})$) coefficient. In the next subsection we will see that a realistic superpotential satisfying these criteria can be constructed.

(b) The Higgs sector: the breaking of $SO(10)$

The model of quarks and leptons requires the existence of adjoints with VEVs in the $I_{3R}$ and $X$ directions. We will denote these by $A_2$ and $A_3$, respectively. The Dimopoulos-Wilczek mechanism requires the existence of an adjoint whose VEV is in the $B-L$ direction, which we have been denoting $A_1$. (As above, we will denote the $\overline{16}_H$ and $16_H$ by $\overline{C}$ and $C$, and the two $10_H$ by $T_1$ and $T_2$.) A satisfactory form for the superpotential is

$$W_{\text{Higgs}} = \text{tr}(A_1)^4/M_{\text{Pl}} + M_{A_1}(A_1)^2 + \text{tr}(A_2)^4/M_{\text{Pl}} + M_{A_2}(A_2)^2 + M_{\overline{C}}\overline{C}C + M_{A_3}(A_3)^2 + \text{tr}(A_1A_2A_3) + T_1A_1T_2 + M_{T_2}(T_2)^2 + \frac{R}{M_{\text{Pl}}}T_1\overline{C}C.$$  

(56)

We have not written the dimensionless coefficients, which are assumed to be of order unity. This as a hybrid of the forms proposed in Ref. 16.

The form $W(A) = \text{tr}(A^4)/M_{\text{Pl}} + M(A)^2$ has as possible solutions $A \propto X$, $B-L$, and $I_{3R}$, with $|\langle A \rangle| \sim \sqrt{M_{\text{Pl}}M}$. Thus $M_{A_1}$ and $M_{A_2}$ in Eq. (56) must be of order $M_{\text{GUT}}^2/M_{\text{Pl}}$.

The terms involving $A_3$, $\overline{C}$, and $C$ can be shown to have a solution in which the VEVs of these fields are in $SU(5)$-singlet directions. The magnitude of $\langle A_3 \rangle$ is determined by the $F_C$ and $\overline{F_C}$ equations to be $O(M_{\text{Pl}})$. This means that $SO(10)$ is broken to $SU(5) \times U(1)_X$ near the Planck scale. The fractional mass splittings within $SU(5)$ multiplets will then be $O(M_{\text{GUT}}/M_{\text{Pl}})$.
and many of the threshold corrections to $\sin^2 \theta_W$ will be suppressed by that small ratio. (See Ref. 16.) On the other hand, it is desirable that the VEV of $\mathcal{C}$ be of order $M_{\text{GUT}}$ because of the role it plays in $W_{\text{adjoint}}$. The scale of $\langle \mathcal{C} \rangle$ and $\langle C \rangle$ is determined by the $F_{A3}$ equation to be $O(\sqrt{M_{A3}/M_{\text{C}M_{\text{Pl}}}})$.

The term $\text{tr}(A_1A_2A_3)$, by linking the $A_1$, $A_2$, and $A_3$ sectors, ensures that there are no goldstone modes, while at the same time not destabilizing the VEVs of the $A_i$. (See the first paper of Ref. 16 for a discussion of this term.)

The terms involving $T_i$ have already been discussed in the last subsection. If $\langle R \rangle \sim M_{\text{GUT}}$, then $\tan \gamma \sim M_{\text{GUT}}/M_{\text{Pl}}$ as desired.

(c) Discrete symmetries

The essential core of the model of quark and lepton masses consists of the Yukawa terms given in Eqs. (1) – (3). In particular, the “root model” is defined by the terms in Eq. (1). The most important feature of that set of terms is that there is no direct $g_{ij} 16_i 16_j 10_H$ coupling. This structure can be explained by a very simple $Z_2$ parity under which the Higgs fields, $10_H \equiv T_1$ and $45_H \equiv A_2$, and the families, $16_i \equiv F_i$, are odd, while the extra real representations of matter fields, $16 \equiv F$ and $\overline{16} \equiv F^*$, are even.

The structure of the complete set of Yukawa terms given in Eqs. (1) – (3) can be ensured by a $Z_3 \times Z_2$ symmetry, where $Z_2$ is a matter parity under which “matter fields” are odd and Higgs fields are even, and $Z_3$ is a symmetry which acts on the fields as shown in Table I. (The other extra real representations of matter are denoted $10 \equiv T$, $10' \equiv T'$, $45 \equiv A$, and $45' \equiv A'$.)

This particular symmetry allows in addition to the terms in Eqs. (1) –
a few extra harmless terms (namely $\overline{F}FT_1$, $FT\overline{C}$, and $FT\overline{C}$).

A realistic Higgs superpotential (that is, one which completely breaks $SO(10)$, avoids goldstone modes, preserves the unification of couplings, naturally achieves doublet-triplet splitting and sufficient suppression of Higgsino-mediated proton decay, and gives adjoint VEVs in the desired directions) can be constructed. And symmetries can be found that ensure its structure and the stability of the gauge hierarchy against possible Planck-scale effects. This has been shown in Ref. 16.

It remains to show that symmetries can be found which render natural the full model, including both the Higgs and Yukawa parts of the superpotential. This is done in Appendix C. It should be emphasized that most of the technical difficulty of making the model natural has to do with the Higgs sector, and that the problem is not made significantly more difficult by the particular Yukawa structure assumed.

6 Variants of the Root Model

(a) A variant of the root model with $\epsilon \cong -\frac{5}{4}$

As explained in Section 2(a), the root model consisting of the terms in $W_{\text{spinor}}$ predicts that

$$\frac{m_\mu^0}{m_s^0} \cong \frac{Q_\ell + Q_{\ell^-}}{Q_{d^c} q_{d}} = 3 \left| \frac{1 + \frac{6}{5} \epsilon}{1 + \frac{2}{5} \epsilon} \right|^5.$$

There are two values of $\epsilon$ which give the Georgi-Jarlskog result $m_\mu^0/m_s^0 \cong 3$, namely $\epsilon \cong 0$ and $\epsilon \cong -\frac{5}{4}$. The first corresponds to $Q \cong I_{3R}$ and gives the model proposed in Ref. 1 and studied in detail in previous sections. The second value gives an interesting variant that we shall now discuss briefly.
With $\epsilon = -\frac{5}{4} + \delta$, $|\delta| \ll 1$, the mass matrices take the forms (see Eqs. (8) –(10))

\[
U_0 \simeq a T v \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & (\frac{1}{4} + \frac{1}{3} \delta) \sin \theta \\
0 & \frac{4}{5} \delta \sin \theta & (-\frac{1}{4} \pm \frac{2}{5} \delta) \cos \theta
\end{pmatrix},
\]

(58)

\[
D_0 \simeq a T v' \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & (\frac{1}{4} + \frac{1}{3} \delta) \sin \theta \\
0 & (\frac{1}{2} + \frac{2}{5} \delta) \sin \theta & (\frac{1}{4} + \frac{2}{3} \delta) \cos \theta
\end{pmatrix},
\]

(59)

\[
L_0 \simeq a T v' \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & (\frac{3}{4} - \frac{2}{5} \delta) \sin \theta \\
0 & (-\frac{1}{2} + \frac{6}{5} \delta) \sin \theta & (\frac{1}{4} + \frac{3}{5} \delta) \cos \theta
\end{pmatrix}.
\]

(60)

There is one immediately apparent explanatory success: $(m_0^c/m_0^t)/(m_0^s/m_0^b) = O(\delta)$. However, the contributions to $V_{cb}$ from the up and down sectors no longer nearly cancel and $V_{cb}^0 \approx \sqrt{m_0^s/m_0^b}$. The situation is thus the reverse of the model with $|\epsilon| \ll 1$. There the prediction for $V_{cb}^0$ is good and that for $(m_0^c/m_0^t)/(m_0^s/m_0^b)$ is bad and has to be corrected by another mechanism. Here $(m_0^c/m_0^t)/(m_0^s/m_0^b)$ works well, but $V_{cb}^0$ has to be corrected by some other mechanism. The question arises: How could $V_{cb}^0$ be corrected without disturbing the Georgi-Jarlskog relation? One possibility is the following.

All the mass matrices coming from $W_{\text{spinor}}$ have the form

\[
M_0 = m_0 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & Q \sin \theta \\
0 & Q \sin \theta & (Q^c + Q) \cos \theta
\end{pmatrix}.
\]

(61)

This means that

\[
\frac{m_\mu}{m_s} \approx \frac{m_\mu m_T}{m_s m_b} = \frac{\det L_0}{\det D_0} = \frac{Q_{l+} Q_{l-}}{Q_{d} Q_{d}},
\]

(62)

while

\[
V_{cb}^0 \approx \left( \frac{Q_{d} - Q_{u}}{Q_{d} + Q_{d}} \right) \tan \theta.
\]

(63)
Consider adding to each matrix a contribution

\[
\Delta M = m_0 \begin{pmatrix}
0 & 0 & 0 \\
0 & \Delta \cdot \sin^2 \theta & \Delta \cdot \sin \theta \cos \theta \\
0 & \Delta \cdot \sin \theta \cos \theta & \Delta \cdot \cos^2 \theta
\end{pmatrix},
\]

with \( \Delta \) the same at least for \( D \) and \( L \). Then \( D_{33} \) and \( L_{33} \) remain equal, and so, for small \( \theta \), \( m_b^0 \approx m^0 \) remains valid and \( \frac{m_b^0}{m_\tau^0} \approx \frac{\det_{23}(L)}{\det_{23}(D)} \) still holds. But \( \det_{23}(M_0 + \Delta M) = (\Delta \sin^2 \theta)((Q^c + Q) \cos \theta + \Delta \cos^2 \theta) - (Q \sin \theta + \Delta \sin \theta \cos \theta)(Q^c \sin \theta + \Delta \sin \theta \cos \theta) = -QQ^c \sin^2 \theta \), which is unaffected by the addition of \( \Delta M \)! Thus, the Georgi-Jarlskog relation remains good. On the other hand, for small \( \theta \),

\[
V^0_{cb} \approx \left[ \frac{Q_{d^c} + \Delta \cos \theta}{(Q_{d^c} + Q_d) + \Delta \cos \theta} - \frac{Q_{u^c} + \Delta \cos \theta}{(Q_{u^c} + Q_u) + \Delta \cos \theta} \right] \tan \theta,
\]

which is very greatly affected by the addition of \( \Delta M \), and can easily be made small.

How can such a \( \Delta M \) arise? In the original root model, defined by \( W_{\text{spinor}} \), the \( SU(2)_L \times U(1)_Y \)-breaking term connects \( \sum_i \hat{a}_i \mathbf{16}_i = \sin \theta \mathbf{16}_2 + \cos \theta \mathbf{16}_3 \) with \( \mathbf{16} \), which in turn mixes with \( \sum_i \hat{b}_i \mathbf{16}_i = \mathbf{16}_3 \). This gives the form of \( M_0 \). Clearly, if there were effectively a term of the form \( \sum_{ij} \hat{a}_i \hat{a}_j \mathbf{16}_i \mathbf{16}_j \) it would give a contribution of the form \( \Delta M \). Such a term can arise in a modified version of the root model. Imagine that instead of a simple vectorlike pair of family, there were two such pairs: \( \mathbf{16} + \mathbf{16} + \mathbf{16}' + \mathbf{16}' \). Consider the superpotential \( (M \left\{ \mathbf{16} \mathbf{16} + \sum_i b_i \mathbf{16} \mathbf{16}, 45_H \right\}) + (M \left\{ \mathbf{16} + \mathbf{16}' \mathbf{16}, 1_H \right\}) + g \left\{ \mathbf{16} \mathbf{16}' \mathbf{10}_H + h \mathbf{16}' \mathbf{16}' \mathbf{10}_H \right\} \). Clearly the \( \mathbf{16} \) and \( \mathbf{16}' \) mix with \( \sum_i \hat{b}_i \mathbf{16}_i \) and \( \sum_i \hat{a}_i \mathbf{16}_i \), respectively. Then the last two terms give, effectively, contributions of the form \( M_0 \) and \( \Delta M \).
This variant root model has several features which make it seem less attractive than the model with $\epsilon \cong 0$. In the latter model the choice of small $\epsilon$ explains three facts (the smallness of $V_{cb}$, the 2nd to 3rd generation hierarchy, and $m_{\mu}^0/m_{\tau}^0 \cong 3$) and also helps explain the suppression of Higgsino-mediated proton decay. (In the long version it also plays a crucial role in explaining the suppression of $m_{\mu}^0/m_{\tau}^0$.) In the variant root model the choice of $\epsilon$ explains only two facts (the smallness of $m_c^0/m_t^0$ and $m_{\mu}^0/m_{\tau}^0 \cong 3$). Secondly, the variant model is less economical by virtue of the introduction of the extra pair of family and anti-family, and involves a reintroduction of a family symmetry of sorts, since the $\overline{16} + 16'$ must be distinguished from the $\overline{16} + 16$. Finally, the point $\epsilon = -\frac{5}{4}$, unlike the point $\epsilon = 0$, does not seem to correspond to a group-theoretically interesting direction. Nevertheless, perhaps some of the ideas in this section are capable of further interesting development.

(b) A variant way to suppress $m_c^0/m_t^0$

In Sections 2(c) and (d) two ways to break the proportionality relation $m_c^0/m_t^0 = m_s^0/m_b^0$ based on a multiplicative correction to the mass matrices were described. They give the two versions of the model. Another possibility is a mechanism based on an additive correction to the mass matrices given in Eqs. (8) – (11). (We are now assuming again that $\epsilon \cong 0$.) Adding something to $D$ to increase $m_s^0/m_b^0$ without disturbing $V_{cb}^0$ or the Georgi-Jarlskog relation seems difficult if not impossible. On the other hand, a small additive correction to $U$ could approximately cancel off the small $U_{23}$ element without significantly affecting $U_{32}$ or $U_{33}$. Then $m_c^0/m_t^0$ would be suppressed without affecting $V_{cb}^0$ or any of the relations that come from the
forms of $D$ and $L$.

There is only one difficulty with this idea: such an additive contribution to $U$ would in general make $U$ be rank 3 and, possibly, make $m_u$ too large. Since $\Delta U_{23} \cong -U_{0,23} \cong (m_s^0 m_t^0 / m_h^0) / \sin \theta \gg m_u^0$, the form of $\Delta U$ must be such that $\Delta U_{11} \ll \Delta U_{23}$.

One way to ensure that $\Delta U$ has a form that maintains the rank 2 nature of $U_0$ is by extending the root model ($W_{\text{spinor}}$) in a way analogous to what was described in Section 6(a). Introduce, as there, two pairs of family plus anti-family instead of one. Consider the superpotential $W'_{\text{spinor}} = (M \mathbf{T} 16 + \sum_i b_i \mathbf{T} 16 i 16 45_H) + (M' \mathbf{T} 16' + \sum_i a_i \mathbf{T} 16' i 16 1_H) + g \mathbf{16} \mathbf{16}' \mathbf{10}_H + \sum_i f_i \mathbf{16} i 16 \mathbf{T} 6_H \mathbf{T} 6_H$. The term $g \mathbf{16} \mathbf{16}' \mathbf{10}_H$ gives effectively a contribution of the same form as Eq. (4). The last term contributes to $U$ (if $\langle 1(\mathbf{T} 6_H) \rangle$ and $\langle 5(\mathbf{T} 6_H) \rangle$ are both non-zero), but not to $D$ or $L$. The sum of the last two terms effectively involves only two linear combinations of $\mathbf{16}$’s, and therefore the resulting total $U$ is still rank 2.

The first generation can still be given mass by adding $W_{\text{vector}}$. The present variant would dispense with need for $W_{\text{adjoint}}$, however. The main drawback compared to the long version of the model described in Section 2(d) is that the predictions of neutrino mixing angles are lost because of the extra parameters $f_i$.

**Appendix A**

Derivation of exact mass matrices

38
Consider the matrix
\[
\mathcal{M} = \begin{pmatrix} m_0 & M' \\ m & M \end{pmatrix},
\]
where \( M \) and \( M' \) contain elements of order \( M_{GUT} \), and \( m \) and \( m_0 \) contain only Weak scale entries. \( \mathcal{M} \) may be block-diagonalized as follows.

\[
U_R \mathcal{M} U_L^\dagger = \begin{pmatrix} (I + x x^\dagger)^{-\frac{1}{2}} (m_0 - M'M^{-1}m) & 0 \\ 0 & (M^\dagger M + M'^\dagger M')^{-\frac{1}{2}} \end{pmatrix},
\]
where
\[
U_R = \begin{pmatrix} (I + x x^\dagger)^{-\frac{1}{2}} & 0 \\ 0 & (M^\dagger M + M'^\dagger M')^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & -x \\ M'^\dagger & M^\dagger \end{pmatrix},
\]
and
\[
U_L = \begin{pmatrix} I & -(m_0^\dagger M' + m^\dagger M)(M^\dagger M + M'^\dagger M')^{-1} \\ (M^\dagger M + M'^\dagger M')^{-1}(M'^\dagger m_0 + M^\dagger m) & I \end{pmatrix}.
\]

Here \( x \equiv M'M^{-1} \). Terms of order \( (M_{Weak}/M_{GUT})^2 \) have been dropped. This see-saw form will now be applied to the case of the mass matrix of the charge-\((-\frac{1}{3})\) quarks in the short version of the model.

We will define the superheavy linear combination of 16’s to be \( 16' \equiv \frac{1}{\sqrt{1+7^2|Q|^2}}[16 + TQ16_3] \), and the light orthogonal combination to be \( 16_3' \equiv \frac{1}{\sqrt{1+7^2|Q|^2}}[-TQ16 + 16_3] \). Then

\[
\begin{align*}
16 &= \frac{1}{\sqrt{1+7^2|Q|^2}}[16' - TQ16_3], \\
16_3 &= \frac{1}{\sqrt{1+7^2|Q|^2}}[+TQ16' + 16_3'].
\end{align*}
\]

Define \( d_1, d_2, d_3', d', \overline{d}, g', \) and \( g' \) to be the charge-\((-\frac{1}{3})\) quarks in the 16, 16_2, 16_4', 16', \overline{16}, 10, \) and 10', respectively; and define \( d_1^c, d_2^c, d_3^c, d'^c, \overline{d}, g^c, \) and \( g'^c \) to be the charge-\((\frac{1}{3})\) antiquarks in the same representations. Then
by substituting Eq. (70) into Eqs. (1) and (2), and restricting attention to the down-type quarks, one gets the following $7 \times 7$ mass matrix

$$W_{\text{mass}} = (d_e^c, d_e^c, d_e^c, d_e^c, d_e^c, d_e^c, d_e^c) \cdot \left( \begin{array}{c} m_0 \\ m \\ M' \\ \frac{d_e}{d_e} \\ d_e \\ g_e \\ g_e' \end{array} \right),$$

(71)

where

$$m_0 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & -aT v'^2 Q_{d} c_0 / N_{d} & 0 \\ 0 & -aT v'^2(Q_{d} + Q_{d}) c_0 / N_{d} N_{d} & 0 \end{array} \right),$$

(72)

$$m = \left( \begin{array}{cccc} 0 & av' s_0 / N_{d} & av'(1 - T^2 Q_{d} Q_{d}) c_0 / N_{d} N_{d} & 0 \\ 0 & 0 & 0 & 0 \\ c_1 v' & c_2 v' & c_3 v' / N_{d} & 0 \\ c_1 v' & c_2 v' & c_3 v' / N_{d} & 0 \end{array} \right),$$

(73)

$$M' = \left( \begin{array}{cccc} 0 & 0 & c_1 v_R & c_1 v_R \\ 0 & 0 & c_2 v_R & c_2 v_R \\ av'(1 - T^2 Q_{d} Q_{d}) c_0 / N_{d} N_{d} & 0 & c_3 v_R / N_{d} & c_3 v_R / N_{d} \\ 0 & 0 & c_3 v_R / N_{d} & c_3 v_R / N_{d} \end{array} \right),$$

(74)

$$M = \left( \begin{array}{cccc} aT v'(Q_{d} + Q_{d}) c_0 / N_{d} N_{d} & MN_{d} & c_3 T Q_{d} v_R / N_{d} & c_3 T Q_{d} v_R / N_{d} \\ c_3 T Q_{d} v_R / N_{d} & 0 & 0 & 0 \\ c_3 T Q_{d} v_R / N_{d} & 0 & 0 & -d(\bar{15}_H) \\ c_3 T Q_{d} v_R / N_{d} & 0 & +d(\bar{15}_H) & 0 \end{array} \right),$$

(75)

and where $v_R \equiv \langle 1(16_H) \rangle$, $\bar{v}' \equiv \langle \bar{5}(16_H) \rangle$, $v' \equiv \langle \bar{5}(10_H) \rangle$, $c_0 \equiv \cos \theta$, and $s_0 \equiv \sin \theta$. Then by Eq. (57)

$$D_0 = m_0,$$

(76)
and
\[ D = (I + xx^\dagger)^{-\frac{1}{2}}(m_0 - M'M^{-1}m). \] (77)

Comparing with Eq. (19) one sees that
\[ \Delta_{dc} = xx^\dagger = (M'M^{-1})(M'M^{-1})^\dagger. \] (78)

Multiplying out this expression using Eqs. (74) and (75) (one can neglect the terms of order \( v' \) and \( \tilde{v}' \) in \( M' \) and \( M \)) gives the result in Eqs. (22) and (23). And multiplying out the expression \( M'M^{-1}m \) gives the flavor-antisymmetric piece in Eq. (13).

The same kind of calculation gives the mixing matrices in the long version of the model as well.

**Appendix B**

**Suppression of \( m_c/m_t \) in long version of model**

From Eq. (27) one has
\[ \Delta_{uc} = \bar{E}^* \bar{E}^T + \bar{E}'^* \bar{E}'^T. \] (79)

Define
\[ \bar{P} \equiv \bar{E} \times \bar{E}' \left[ 1 + \frac{\bar{E}'^2 + \bar{E}''^2}{|\bar{E} \times \bar{E}'|^2} \right]. \] (80)

So that
\[ P^2 \equiv |\bar{P}|^2 = |\bar{E}'|^2 + |\bar{E}''|^2 + |\bar{E} \times \bar{E}'|^2. \] (81)
We will assume that $|\vec{E}|$ and $|\vec{E}'|$ are large compared to one (in fact, of order $1/\epsilon$), so that there is a hierarchy: $P \sim E^2, E'^2 \gg E, E' \gg 1$. Further, define
\[
\vec{F} \equiv \hat{P}^* \times \vec{E}, \\
\vec{F}' \equiv \hat{P}^* \times \vec{E}'.
\] (82)

Then one can write an exact expression for the square of the mixing matrix $(I + \Delta_{uc})^{-1/2}$:
\[
(I + \Delta_{uc})^{-1} = \frac{1}{1 + |P|^2} \left[ I + \vec{F} \vec{F}^\dagger + \vec{F}' \vec{F}'^\dagger + \vec{P} \vec{P}^\dagger \right].
\] (83)

This expression is easily checked by going to a particular basis. Since it is in a rotationally invariant form, it is true in any basis.

One can write the full matrix $U$ as
\[
U \approx (I + \Delta_{uc})^{-1/2} U_0,
\] (84)

with
\[
U_0 \approx \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \eta \\
0 & \sin \theta & \cos \theta \end{pmatrix} = (\vec{0}, \vec{A}, \vec{B})
\] (85)

where
\[
\eta \equiv \frac{1}{5} \epsilon \sin \theta,
\] (86)

\[
\vec{A} \equiv \begin{pmatrix} 0 \\
0 \\
\sin \theta \end{pmatrix},
\] (87)

\[
\vec{B} \equiv \begin{pmatrix} 0 \\
0 \\
\eta \cos \theta \end{pmatrix}.
\] (88)

Then the matrix $U_0$ has eigenvalues
\[
m_{t,0} \approx 1, \\
m_{c,0} \approx -\eta \sin \theta = |\vec{A} \times \vec{B}|.
\] (89)
To find the eigenvalues of $U$ one considers

$$U^\dagger U = U_0^\dagger (I + \Delta_w)^{-1} U_0$$

$$= \frac{1}{1 + |P|^2} \begin{pmatrix} 0 \\ \vec{A}^\dagger \\ \vec{B}^\dagger \end{pmatrix} \cdot (I + \vec{F} \vec{F}^\dagger + \vec{F}^\dagger \vec{F} + \vec{P} \vec{P}^\dagger) \cdot (\vec{0}, \vec{A}, \vec{B}) \quad (90)$$

Then

$$(U^\dagger U)_{33} = \frac{1}{1 + |P|^2} \left(|\vec{A}^\dagger \cdot \vec{P}|^2 + |\vec{A}^\dagger \cdot \vec{F}|^2 + |\vec{A}^\dagger \cdot \vec{F}^\dagger|^2 + |\vec{A}|^2\right)$$

$$(U^\dagger U)_{22} = \frac{1}{1 + |P|^2} \left(|\vec{B}^\dagger \cdot \vec{P}|^2 + |\vec{B}^\dagger \cdot \vec{F}|^2 + |\vec{B}^\dagger \cdot \vec{F}^\dagger|^2 + |\vec{B}|^2\right)$$

$$(U^\dagger U)_{23} = \frac{1}{1 + |P|^2} (\vec{A}^\dagger \cdot \vec{P} \vec{P}^\dagger \cdot \vec{B} + \vec{A}^\dagger \cdot \vec{F} \vec{F}^\dagger \cdot \vec{B} + \vec{A}^\dagger \cdot \vec{F}^\dagger \vec{F} \cdot \vec{B} + \vec{A}^\dagger \cdot \vec{B}^\dagger \cdot \vec{B} + \vec{A}^\dagger \cdot \vec{B}) \quad (91)$$

The first row and column are zero in our approximation. One then has

$$m^2 / m^2_{t,0} \approx m^2_t \approx \text{tr}(U^\dagger U)$$

$$\approx \frac{1}{1 + |P|^2} \left(|\vec{A}^\dagger \cdot \vec{P}|^2 + |\vec{A}^\dagger \cdot \vec{F}|^2 + |\vec{A}^\dagger \cdot \vec{F}^\dagger|^2 + |\vec{A}|^2\right)$$

$$\approx \frac{1}{1 + |P|^2} \left(|\vec{B}^\dagger \cdot \vec{P}|^2 + |\vec{B}^\dagger \cdot \vec{F}|^2 + |\vec{B}^\dagger \cdot \vec{F}^\dagger|^2 + |\vec{B}|^2\right)$$

$$\approx \frac{\det_{23} U^\dagger U}{1 + |P|^2} \quad (92)$$

where we are keeping only the leading terms. And

$$m^2_c m^2_t \approx \det_{23} U^\dagger U \quad (93)$$

Keeping only the leading terms (recalling the hierarchy $P \ll F, F' \ll 1$) one has after some algebra

$$m^2_c m^2_t \approx \frac{1}{(1 + |P|^2)^2} \left[|\vec{A}^\dagger \cdot \vec{P} \vec{B}^\dagger \cdot \vec{F} - \vec{B}^\dagger \cdot \vec{P} \vec{A}^\dagger \cdot \vec{F}^\dagger|^2 + (F \rightarrow F')\right] \quad (94)$$

The crucial point is that the leading terms in the brackets, which are $O(P^4)$,
cancel leaving $O(P^2)$. Further manipulations give

$$m_c^2 m_t^2 = \frac{1}{(1+|P|^2)^2} \left[ |(A \times B)^\dagger \cdot (\tilde{P} \times \tilde{F})|^2 + (F \rightarrow F') \right]$$

$$= \frac{|P|^2}{(1+|P|^2)^2} \left[ |(A \times B)^\dagger \cdot \tilde{E}|^2 + (E \rightarrow E') \right]$$

$$= |\tilde{A} \times \tilde{B}|^2 \frac{|P|^2}{(1+|P|^2)^2} (|E_1|^2 + |E'_1|^2)$$

$$\cong m_{c,0}^2 m_{t,0}^2 \frac{|P|^2}{(1+|P|^2)^2} (|E_1|^2 + |E'_1|^2).$$

Or

$$\frac{m_c m_t}{m_{c,0} m_{t,0}} \cong \frac{|P|}{1+|P|^2} \sqrt{|E_1|^2 + |E'_1|^2},$$

(95)

and, since $\frac{m_c^2}{m_{t,0}^2} \cong \frac{|P|^2}{1+|P|^2}$, the suppression factor is given by

$$\frac{m_c/m_t}{m_{c,0}/m_{t,0}} \cong \frac{|P|}{|P_3|} \sqrt{|E_1|^2 + |E'_1|^2}. \quad (96)$$

Using the definition of $\tilde{P}$

$$\frac{m_c/m_t}{m_{c,0}/m_{t,0}} \cong \frac{|\tilde{E} \times \tilde{E}'|}{|\tilde{E} \times \tilde{E}'|^3} \sqrt{|E_1|^2 + |E'_1|^2}. \quad (97)$$

This is the desired result. Notice that for large $E$ and $E'$ this goes as $1/\min(E, E')$, unless $\tilde{E} \times \tilde{E}'$ happens to point nearly along the 3 direction. In other words, generically the ratio $m_c/m_t$ is suppressed. Since $E$, and $E'$ are of order $\frac{1}{\epsilon}$, $m_c/m_t$ is suppressed by a factor of $O(\epsilon)$.

**Cases of $\Delta = \text{rank 1 or rank 3}$**

The curious fact that $m_c/m_t$ is generally suppressed is a result of the fact that $\Delta_{u^c}$ is rank 2. We can show this by doing the analogous calculation for
the cases where $\Delta$ is rank 1 and rank 3. If $\Delta$ is rank 1 it can be written

$$\Delta = \vec{E}^* \vec{E}^T. \quad (99)$$

Then

$$(I + \Delta)^{-1} = I - \frac{1}{1 + |E|^2} \vec{E}^* \vec{E}^T. \quad (100)$$

Parallelling the calculation for the rank-2 case closely one finds

$$\frac{m_c/m_t}{m_{c,0}m_{t,0}} \approx \frac{\sqrt{1 + |E_1|^2} \sqrt{1 + |E|^2}}{1 + |E_1|^2 + |E_2|^2}. \quad (101)$$

This is not small unless $\vec{E}$ happens to be nearly in the 2 direction, which agrees with intuitive expectation.

If $\Delta$ is rank 3, then $(I + \Delta)^{-1}$ is just an arbitrary 3-by-3 matrix which we will call $M$. A calculation similar to the preceding two gives

$$\frac{m_c/m_t}{m_{c,0}m_{t,0}} \approx \frac{\sqrt{M_{33}M_{22} - M_{32}^2}}{M_{33}}. \quad (102)$$

Again, this is not small except for particular special choices of $\Delta$.

**A calculation of $V_{cb}^0$**

Let the transformation of the left-handed charge-$\left(\frac{2}{3}\right)$ quarks required to diagonalize $U$ be

$$V_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s^* \\ 0 & -s & c \end{pmatrix}, \quad (103)$$

and the matrix required to diagonalize $U_0$ be the same with $s \to s_0$ and $c \to c_0$. Similarly, let the transformation of the left-handed charge-$\left(-\frac{1}{3}\right)$ quarks
required to diagonalize $D$ be of the same form with $s \to s'$ and $c \to c'$. We take $c$, $c'$, and $c_0$ to be real and $s$, $s'$, and $s_0$ to be complex. Then

$$V_{cb}^0 \simeq sc' - s'c$$  \hspace{1cm} (104)

The primed quantities are easy to find: $s'/c' \simeq D_{32}/D_{33} \simeq (1 - \frac{1}{5} \epsilon) \tan \theta$ as can be seen from Eq. (15). (We ignore the effects of $c_{23}$ and the $N_f$.) This gives

$$s' \cong (\sin \theta(1 - \frac{1}{5} \epsilon_R \cos^2 \theta), -\frac{1}{5} \epsilon_I \sin \theta).$$  \hspace{1cm} (105)

And $s_0 \cong (U_0)_{32}/(U_0)_{33}$ which is given by the same expressions with $\epsilon \to -\epsilon$, as can be seen from Eq. (14). To find $s$ we can use the expressions derived for $U^\dagger U$ earlier in this Appendix:

$$\frac{2 |s| c}{c^2 - |s|^2} = \frac{2 |(U^\dagger U)_{32}|}{(U^\dagger U)_{33} - (U^\dagger U)_{22}},$$  \hspace{1cm} (106)

$$\arg s = \arg (U^\dagger U)_{32}. $$  \hspace{1cm} (107)

This gives, after multiplying out the expressions for the elements of $U^\dagger U$ given above,

$$\frac{2 |s| c}{c^2 - |s|^2} = \frac{2 |s_0|(c_0 + \text{Re}K)}{c_0^2 - |s_0|^2 + 2c_0\text{Re}K + O(\eta^2)},$$  \hspace{1cm} (108)

and

$$\arg s = \arg s_0 - \text{Im}K/c_0,$$  \hspace{1cm} (109)

where

$$K \equiv \eta \left[ \frac{P_3P_3^* + F_3F_3^* + F_3^*F_3^{**}}{|P_3|^2 + |F_3|^2 + |F_3^*|^2 + 1} \right].$$  \hspace{1cm} (110)
Using \( \eta \equiv \frac{1}{5} \epsilon \sin \theta \) and the assumption that \( P \gg F, F' \gg 1 \), one obtains

\[
K \simeq \frac{1}{5} \epsilon \sin \theta \left( \frac{P_2^*}{P_3^*} \right).
\]  

(111)

These expressions can be combined to give after straightforward algebra

\[
V_{cb}^0 \simeq \sin \theta \cos \theta \left( \frac{2}{5} \epsilon - \frac{K}{\cos \theta} \right),
\]

(112)
or

\[
V_{cb}^0 \simeq \frac{2}{5} \epsilon \sin \theta \cos \theta \left[ 1 - \frac{1}{2} \tan \theta \left( \frac{P_2}{P_3} \right)^* \right].
\]

(113)

Appendix C

In order to find a symmetry that makes the superpotential of Eq. (56) natural and stable it is necessary to introduce some singlet superfields. This can be shown as follows. With the terms tr\((A_1)^4\) and \((A_1)^2\), \(A_1\) cannot transform non-trivially except under a \(Z_2\). (Similarly for \(A_2\) and \(A_3\).) But it is crucial for the doublet-triplet splitting that \((T_2)^2\) be allowed, while \((T_1)^2\) is forbidden, which implies a non-trivial relative transformation of these two fields and hence of \(A_1\) because of the presence of the term \(T_1 A_1 T_2\). Thus the terms tr\((A_1)^4\) and \((A_1)^2\) must be replaced with some form that allows a non-trivial transformation of \(A_1\). The simplest possibility is to insert singlet superfields with non-trivial transformation properties. For example, \(\phi_1 \text{tr}(A_1)^4/M_{Pl}^2\). For similar reasons it is convenient to introduce singlets into other terms as well.

There is no point in trying to find the most elegant or simplest combination of symmetries that works. Rather here it will only be shown that some
symmetry can be found. The easiest way to do this is to restrict the search to a single $U(1)$ symmetry, and to introduce singlet fields where convenient to make a needed term allowed. No attempt has been made to economize on these singlets.

Consider, then, the following set of fields. Its $U(1)$ charge is given in parentheses after the name of the field. Adjoint Higgs: $A_1(-a_2 - a_3)$, $A_2(a_2)$, $A_3(a_3)$; Fundamental Higgs: $T_1(t_1)$, $T_2(t_2)$; Spinor Higgs: $\overline{C}(-c)$, $C(c - x)$; Singlet Higgs: $P(-t_1 - t_2 + a_2 + a_3)$, $Q(-2t_2/3)$, $R(r)$, $S(s)$, $X(x)$, $Y(y)$; Matter Spinors: $F_i(e)$, $F(-e - t_1)$, $\overline{F}(e + t_1 - s)$; Matter Fundamentals: $T(-a_3/2)$, $T'(a_3/2 - x)/2$; Matter Adjoints: $A(-a_2/2)$, $A'(-a_2/2)$. These charges are not all independent, but satisfy $c = (a_3 - a_2 + 2x + y)/4$, and $t_1 = -a_3/2 - 3a_2/2 + x - y/2 + s$. With these charge assignments the following terms are allowed:

$$W = \text{tr}(A_1)^4\phi_1/M_{Pl}^2 + (A_1)^2\tilde{\phi}_1$$
$$+ \text{tr}(A_2)^4\phi_2/M_{Pl}^2 + (A_2)^2\phi_2$$
$$+ X\overline{C}C + X\overline{C}C(A_3\phi_3)^2/M_{Pl}^2 + (A_3\phi_3)^2/M_3$$
$$+ \text{tr}(A_1A_2A_3)$$
$$+ T_1A_1T_2(P/M_{Pl}) + (T_2)^2(Q^3/M_{Pl}^2) + T_1\overline{C}C(R/M_{Pl})$$
$$+ [s(FF) + (FF_i)A_2 + (FF_i)T_1$$
$$+ Y(TT')A_3/M_{Pl} + (TF_i)C + (T'F_i)C$$
$$+ (AA')A_2 + (AF_i)\overline{C} + (A'F_i)\overline{C}].$$

All the terms have coefficients of order unity that are not shown. The VEVs of the fields are of the following orders of magnitude. $\langle A_1 \rangle$, $\langle A_2 \rangle$, $\langle \overline{C} \rangle$, $\langle C \rangle \sim M_{GUT}$, $\langle A_3 \rangle \sim M_{Pl}$, $\langle X \rangle$, $\langle Y \rangle$, $\langle P \rangle$, $\langle Q \rangle$, $\langle R \rangle$, $\langle S \rangle \sim M_{GUT}$, $\langle \phi_i \rangle \sim M_{Pl}$, and $\langle \tilde{\phi}_i \rangle \sim M_{GUT}/M_{Pl}$. (Thus $1/M_3$ must be of order $M_{GUT}^3/M_{Pl}^4$.)

The most dangerous terms for the hierarchy are those that lead effectively to $T_1CC$ or to linear terms for $A_1$. The lowest term involving $T_1CC$

48
is $T_1 CC[A_2 \phi_3 R\overline{Y}]/M_{Pl}^4$ (assuming a field $\overline{Y}$ with opposite quantum numbers to $Y$ exists). This gives effectively $(M_{\text{GUT}}/M_{Pl})^3 T_1 CC$, which is sufficiently suppressed. The lowest dimension terms linear in $A_1$ (that would destabilize $\langle A_1 \rangle$; note that $\text{tr}(A_1 A_2 A_3)$ does not) are $A_1 (X\overline{CC})(RS\overline{Y})/M_{Pl}^4$ and $(A_1 A_2 A_3) (X\overline{CC})/M_{Pl}^3$. The first is harmless, but the second gives effectively $(M_{\text{GUT}}^2/M_{Pl}^2) A_1$. Since the mass of $A_1$ is necessarily of order $M_{\text{GUT}}^2/M_{Pl}$, one has $\delta \langle A_1 \rangle/\langle A_1 \rangle \sim M_{\text{GUT}}/M_{Pl}$. What is needed is $10^{-7}$, so that this dangerous term must be assumed to have a dimensionless coefficient that is of order $10^{-4}$. All other potentially dangerous terms are sufficiently suppressed if one assumes that all terms are suppressed only by dimensionally appropriate powers of the Planck mass.

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Table I:

| Matter : $F_1$ | $F$ | $F'$ | $T$ | $T'$ | $A$ | $A'$ | Higgs : $A_2$ | $A_3$ | $T_1$ | $C$ | $C'$ |
|----------------|-----|-----|-----|-----|-----|-----|-------------|-----|-----|-----|-----|
| $Z_3$          | 1   | $z^2$ | $z$ | $z^2$ | $z$ | $z$ | $Z_3$       | $z$ | $z^2$ | $z$ | $z^2$ |

Figure Captions

**Fig.1:** The $W_{\text{spinor}}$ contribution to light fermion masses.

**Fig.2:** The $W_{\text{vector}}$ contribution to light fermion masses.
Fig. 1

Fig. 2