Complete reducibility and subgroups of exceptional algebraic groups

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Abstract

This survey article has two components. The first part gives a gentle introduction to Serre’s notion of $G$-complete reducibility, where $G$ is a connected reductive algebraic group defined over an algebraically closed field. The second part concerns consequences of this theory when $G$ is simple of exceptional type, specifically its role in elucidating the subgroup structure of $G$. The latter subject has a history going back about sixty years. We give an overview of what is known, up to the present day. We also take the opportunity to offer several corrections to the literature.
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Introduction

Let $G$ be an affine algebraic group over an algebraically closed field $k$ of characteristic $p \geq 0$. In this article we will only be interested in the case that $G$ is of finite type, smooth, connected and reductive. Except in one or two places, the subgroups of $G$ that we consider in this article will be smooth, and so, as in common in the literature, we think of $G$ as a variety as per [44], rather than adopting the scheme-theoretic language of [77]. In any case, we may assume that $G$ is a subgroup of some $GL(V)$ defined by the vanishing of some polynomials in the entries—more specifically a radical ideal.

The idea of $G$-completely reducibility is due to J.-P. Serre [92]. It generalises the property of a finite-dimensional representation $\rho : H \to G = GL(V)$ of a group $H$ being completely reducible, by restating the definition in terms of the relationship between the subgroup $\rho(H) \subseteq G$ and the parabolic subgroups of $G$. In this way, the same property can be formulated when $G$ is any connected reductive algebraic group and $H$ is one of its subgroups.

The notion of $G$-complete reducibility appears in many unexpected places, by virtue of the links it offers between representation theory, group theory, algebraic geometry and geometric invariant theory. For the purposes of this article, it offers the cleanest language to talk about the subgroup structure of $G$.

Unless otherwise mentioned, we only consider closed subgroups of $G$—those cut out from $G$ by polynomial equations. A linear algebraic group is unipotent if it is isomorphic to a subgroup of the upper unitriangular matrices in $GL(V)$ for some $V$. As the rank of $G$ grows, one sees that it quickly becomes impossible to say very much about the unipotent subgroups of $G$, in much the same way that finite $p$-groups become unmanageable. The same problem arises when $H$ is allowed to have a non-trivial normal connected unipotent subgroup. Therefore we focus on those subgroups $H$ of $G$ which do not contain a non-trivial normal connected unipotent subgroup; in other words $H$ is reductive.

Next, a subgroup $H \subseteq GL(V)$ is the same thing as a faithful representation of $H$. Since we do not want to consider the representation theory of all finite groups, we will assume $H$ is connected. Further still, a connected reductive group $H$ takes the form $H = D(H) \cdot R(H)$ where $R(H)$ is a central torus of $H$ and $D(H)$ is the semisimple derived subgroup of $H$. It would be unenlightening to enumerate all tori in $G$ which commute with $D(H)$, and so:

we assume $H$ is semisimple.

This is still not enough in general to attempt a classification, unless $p = 0$. For, in positive characteristic a non-trivial semisimple group has a rich theory of indecomposable modules and it seems hopeless to classify all possibilities; therefore one cannot classify all the subgroups of $GL(V)$ with $\dim V$ unbounded.

On a more sanguine note: since irreducible representations of $H$ are classified by their highest weight, the completely reducible representations correspond to lists of such weights. If $p = 0$, all representations are completely reducible and Weyl’s dimension formula gives the dimensions of the irreducible factors. In this case one can consider the problem of listing the semisimple subgroups of $G = GL(V)$ to be solved. In positive characteristic, understanding subgroups acting completely reducibly means understanding the dimensions of irreducible modules. This is a hard problem. The most recent progress on this problem is due to Williamson and his collaborators—for example, see [87]—but regretfully there is no clear description for the dimensions of the irreducible $H$-modules in all characteristics, unless the rank of $H$ is small.
But now suppose we bound the rank of $G$; more specifically that we assume $G$ is simple of exceptional type, so that the rank of $G$ is at most 8. Then there is a realistic prospect of understanding of the poset of conjugacy classes of semisimple subgroups of $G$ in all characteristics: any semisimple subgroup $H \subseteq G$ also has rank at most 8.

**Main Problem.** Let $G$ be a simple algebraic group of exceptional type. Describe the poset of conjugacy classes of semisimple subgroups of $G$.

Such a project naturally divides along lines prescribed by $G$-complete reducibility, which we explain in §1.4. With this in mind, our purpose is twofold:

(i) We introduce $G$-complete reducibility and the links it provides between representation theory, geometric invariant theory and group theory. We will be light on technical details, but aim to impart the flavour of the techniques and the most important results.

(ii) We discuss the current state of affairs in describing semisimple subgroups of the exceptional algebraic groups. We start with a historical overview and then collate the principal results from across the literature. We also correct some errors and omissions that have arisen in this study. Significantly, we update the table in [97]; see Table 2.3.

**Prerequisite knowledge**

This is an article about linear algebraic groups over algebraically closed fields and so a healthy knowledge of such groups would be appropriate. Most of the results here do not require a scheme-theoretic background, and so one of [44, 94, 71, 13] would suffice. Our use of representation theory will not frequently stray far from the classification of irreducible modules, which can be found in these same references. Occasionally a discussion of cohomology takes us into the world of [49]; though we will typically content ourselves with pointing out references to deeper material when appropriate. At points, knowledge of the theory of finite-dimensional complex Lie algebras would be helpful, such as [34].

**Notation**

We will mostly introduce relevant notation as we need it. Dynkin diagrams and conventions on roots will be as in [16]. We always use left actions; so conjugation will be written $^g h = ghg^{-1}$. In keeping with this, a group $G$ that decomposes into a semidirect product of a normal subgroup $N$ with a complement $H$ will written $N \rtimes H \cong G = NH$, since then $(n, h)(m, k) = (n^h m, h k)$ maps to $n^h m h k$ under this isomorphism.

As mentioned, our algebraic groups are all varieties over algebraically closed fields, and can be thought of as subgroups of some ambient group $\text{GL}(V)$ for some $V$. The identity component of an algebraic group $G$ is denoted $G^o$, and the component group $G/G^o$ is always finite since our groups are of finite type.

**Structure of the paper**

The structure of the paper is as follows. In Part I, §§1.1–1.2 motivate the theory of $G$-complete reducibility as the natural generalisation of the representation-theoretic notion. This includes a sufficient overview of reductive algebraic groups to state the fundamental definition (Definition 1.3) and discuss the most useful tools for our applications. In §1.4
we describe a strategy for classifying subgroups of reductive groups, which arises naturally out of the dichotomy between $G$-cr and non-$G$-cr subgroups.

In Part II we turn to the particular problem of understanding semisimple subgroups of exceptional simple algebraic groups. We begin with a brief historical overview of results in characteristic zero (essentially due to Dynkin) and their translation to positive characteristic (largely due to Seitz and his collaborators), before delving into the current state of the art, and the application of the strategy described in Part I. Finally, in §2.4 we discuss further ongoing research directions in the area.
Part I. *G*-complete reducibility

1.1 Complete reducibility

Almost the first result one encounters in group representation theory is Maschke’s Theorem: Given a finite group $H$ and a $\mathbb{C}H$-module $V$—or if $V$ has finite dimension, equivalently a representation $H \to \text{GL}(V)$—every $H$-submodule (i.e. $H$-stable subspace) $U \subseteq V$ admits an $H$-stable complement, so that $V \cong U \oplus U'$ as $H$-modules. The proof proceeds by averaging the projection map $\phi : V \to U$ over $H$ to form the $H$-module homomorphism

$$\phi_0 : V \to U, \quad v \mapsto \frac{1}{|H|} \sum_{h \in H} h\phi(h^{-1}v)$$

whose kernel is then the required submodule $U'$. An identical proof works over any ring $k$ in which the order $|H|$ is invertible, hence Maschke’s theorem holds whenever $\text{char } k$ is sufficiently large relative to $H$. The analogous result (a consequence of the Peter–Weyl theorem) holds for unitary representations of connected compact topological groups, in particular for compact real Lie groups. Then Weyl’s unitary trick gives us the same result for reductive complex Lie groups and for semisimple (complex and real) Lie algebras. For a direct approach to semisimple complex Lie algebras, one can also follow [34, C.15].

These are some examples of natural module categories which are completely reducible or semisimple. When such a module is of finite dimension over a field, it can be repeatedly decomposed into submodules until we reach a direct sum of irreducible modules, i.e. those without any proper, non-zero submodules. In such a situation, understanding the category amounts to understanding the irreducible modules. Complete reducibility is certainly not ubiquitous, however. A common example is the following representation of the group of integers under addition:

$$\mathbb{Z} \to \text{GL}_2(\mathbb{C}); \quad n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

whose image stabilises a unique 1-dimensional subspace, spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that admits no complementary $\mathbb{Z}$-submodule. Performing a reduction modulo a prime $p$ produces a cyclic group of order $p$ acting on a module over a field of characteristic $p$, which again fails to be completely reducible.

This formulation of complete reducibility places the focus on the acting group $H$. One could equally decide to fix the target of the representation and ask:

*Given a finite-dimensional vector space $V$, which subgroups $H$ of $G = \text{GL}(V)$ is $V$ a completely reducible $H$-module?*

*Note that the condition of Maschke’s theorem is violated, since the characteristic $p$ of the field divides $|\mathbb{Z}/p| = p.$*
This is now a question about the subgroup structure of $G$. It may also be more tractable, since $H$ is either trivial or has a non-trivial module of dimension at most $\dim V$, which puts limitations on $H$. Moreover, this reformulation can be put in purely group-theoretic terms. Recall that a parabolic subgroup $P$ of $GL(V)$ is the stabiliser of a flag of subspaces

$$V = V_0 \supset V_1 \supset \cdots \supset V_r = \{0\}.$$  

(See for example [71, Prop. 12.13].) One sees from this that the maximal (proper) parabolic subgroups are stabilisers of (proper, non-zero) subspaces, so a subgroup $H \subseteq GL(V)$ acts irreducibly if and only if $H$ is contained in no proper parabolic subgroup, and $H$ is completely reducible if and only if: whenever $H$ stabilises a flag of subspaces $V_i$ as above, $H$ stabilises an opposite flag

$$V = W_0 \supset W_1 \supset \cdots \supset W_r = \{0\},$$

where $V = V_i \oplus W_{r-i}$ as $H$-modules for all $i$. In other words, whenever $H$ is contained in a parabolic subgroup of $GL(V)$, it is also contained in an opposite parabolic subgroup $P^-$ of $G$. The subgroup of $P$ acting trivially on each quotient $V_i/V_{i+1}$ is upper unitriangular according to an appropriate basis, and is therefore a unipotent group. Indeed this turns out to be the largest normal connected unipotent subgroup of $P$, i.e. its unipotent radical $\mathcal{R}_u(P)$. Furthermore the intersection $L := P \cap P^-$ turns out to be a reductive complement to $\mathcal{R}_u(P)$ in $P$, in other words a Levi factor. In $GL(V)$, Levi factors of parabolic subgroups are stabilisers of direct sum decompositions of $V$, and are therefore products of smaller general linear groups acting on the irreducible summands. By change of basis, one sees that Levi factors of $P$ are all conjugate (even by elements of $\mathcal{R}_u(P)$), so that a choice of basis in which the $V_i$ and $W_i$ are spanned by standard basis vectors leads to the following picture:

\begin{align*}
\begin{pmatrix}
* & * & \cdots & * \\
0 & * & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & *
\end{pmatrix} & \quad \mathcal{R}_u(P) & \quad \begin{pmatrix}
I_{n_1} & * & \cdots & * \\
0 & I_{n_2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & I_{n_r}
\end{pmatrix} & \quad \begin{pmatrix}
* & 0 & \cdots & 0 \\
0 & * & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & *
\end{pmatrix}
\end{align*}

1.2 Reductive algebraic groups

The above discussion generalises at once to other groups with an appropriate notion of parabolic subgroup and Levi factor; in particular when $G$ is a reductive algebraic group. The formal definition in this case is that a subgroup $P$ of $G$ is parabolic if $G/P$ is a projective variety. However, a more useful characterisation for us can be given using the structure theory of reductive groups, which we now outline.

Among connected linear algebraic groups, the simple objects are by definition those which are non-abelian and have no proper, connected normal subgroups. Such groups are determined up to isogeny—i.e. a homomorphism with a finite kernel—by the field $k$ and one of the Dynkin diagrams below, which are divided into those of classical type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$); and those of exceptional type $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$.  

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In addition to these simple groups, we mention the additive group of the field \( k \), often denoted \( \mathbb{G}_a \), and the multiplicative group \( k^* \), denoted \( \mathbb{G}_m \). Linear algebraic groups with a subnormal series whose successive quotients all isomorphic to \( \mathbb{G}_a \) coincide with the set of connected unipotent groups. Linear algebraic groups with a composition series whose quotients are isomorphic to \( \mathbb{G}_m \) are in fact a direct product \( \mathbb{G}_m^r \) for some \( r \) and are called tori. It is then a theorem that a connected soluble group is an extension of a torus by a unipotent group.\(^*\)

Tori play an important role in the structure theory of reductive groups. All maximal tori in a linear algebraic group are conjugate to one another, and their dimension is called the rank of the group. This is the number \( n \) of nodes in the Dynkin diagram when \( G \) is simple.

The simply-connected groups of type \( A_n-D_n \) are respectively \( \text{SL}_{n+1} \), \( \text{Spin}_{2n+1} \), \( \text{Sp}_{2n} \) and \( \text{Spin}_{2n} \), and from these one obtains the others as quotients with finite kernels. For our purposes there is often no harm in working with \( \text{SL}_{n+1} \), \( \text{SO}_n \) and \( \text{Sp}_{2n} \) which have more accessible descriptions in terms of the natural module and its orthogonal or symplectic form.

A linear algebraic group \( G \) contains a unique maximal connected normal soluble subgroup \( \mathcal{R}(G) \), the radical of \( G \), containing the unique maximal connected normal unipotent subgroup \( \mathcal{R}_u(G) \) of \( G \), the unipotent radical. Then \( G \) is called:

- reductive if \( \mathcal{R}_u(G) \) is trivial; this is equivalent to \( G^0 \) being an almost-direct product of simple algebraic groups and a torus.

- semisimple if \( G \) is connected and \( \mathcal{R}(G) \) is trivial; this is equivalent to \( G \) being an almost-direct product of simple algebraic groups.

\(^*\)All these statements are heavily dependent on the condition that \( k \) is algebraically closed.
For any linear algebraic group $G$, the quotient $G/\mathcal{R}_u(G)$ is reductive and if $G$ is connected then $G/\mathcal{R}(G)$ is semisimple.

For $G$ a linear algebraic group, we confine ourselves to discussing finite-dimensional rational representations of $G$, which can be identified with homomorphisms $G \to \text{GL}(V)$ of algebraic groups. One easy case to describe is when $G = \mathbb{G}_m$. The irreducible representations of $\mathbb{G}_m$ are 1-dimensional; if $V = \langle v \rangle$ is one such, then $x \cdot v = x^r v$ for some $r \in \mathbb{Z}$ and so corresponds with a homomorphism $x \mapsto x^r$ of $\mathbb{G}_m$ to $\mathbb{G}_m$. Even better, $\mathbb{G}_m$ always acts completely reducibly, so that a representation up to isomorphism is identified with a list of integers. More generally if $G = T$ is a torus $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$, then an irreducible representation $V$ is still 1-dimensional, determined up to isomorphism by the action

$$(x_1, \ldots, x_s) \cdot v = x_1^{r_1} \cdots x_s^{r_s} v,$$

which identifies with an element $\lambda \in \text{Hom}(T, \mathbb{G}_m) =: X(T)$; we say $\lambda$ is a weight of $T$. (We often identify $\lambda$ with the corresponding 1-dimensional representation.) Since $T$ acts completely reducibly, a representation for $T$ is simply a list of its weights.

Taking inspiration from the theory of Lie groups, one can construct a Lie algebra $g = \text{Lie}(G)$ from a linear algebraic group $G$ which affords a representation of $G$ through an adjoint action. Since maximal tori in $G$ are conjugate, the collection of weights of $T$ on $\text{Lie}(G)$ does not depend on the choice of $T$. The zero weight-space is $\text{Lie}(T)$, and the non-zero weights are called roots and form the root system which is denoted $\Phi := \Phi(G, T)$. A reductive group $G$ has a maximal connected soluble subgroup $B$ called a Borel subgroup which by Borel’s fixed point theorem is unique up to conjugacy in $G$. A Borel subgroup decomposes as $B = U S$ where $U = \mathcal{R}_u(B)$ (a maximal connected unipotent subgroup of $G$) and $S$ is a maximal torus of $G$. When $S = T \subseteq B$, the set of roots $\Phi(B, T)$ contains exactly half the elements of $\Phi(G, T)$ and defines a positive system $\Phi^+ \subset \Phi$. Moreover, $\Phi$ has a base of simple roots $\Delta$: this is a minimal subset of $\Phi^+$ from which all elements of $\Phi^+$ can be obtained as non-negative integer sums. We call $|\Delta|$ the semisimple rank of $G$.

One can define a Killing form on the $\mathbb{R}$-linear span of the roots; the resulting lengths of the simple roots and angles between them determine the Dynkin diagram of $G$.

For use later, we note here that the Killing form and $\Phi^+$ together determine a set of dominant weights, $X(T)^+$. If $\lambda \in X(T)^+$ is a dominant weight then we identify $\lambda$ with the corresponding 1-dimensional $T$-module. Composing with the map $B \to B/U \cong T$, we get a $B$-module $\lambda$. One can define an induced module $\Pi^\lambda := \text{Ind}^G_B(\lambda)$ in the category of rational $G$-modules. Since $G/B$ is projective, it turns out that this is has finite dimension, and is a reduction modulo $p$ of the irreducible representation $L_C(\lambda)$ for the complex Lie algebra $g_C$; so the weights of $\text{Ind}^G_B(\lambda)$ are given by Weyl’s character formula. If we define $L(\lambda)$ to be the socle of $\text{Ind}^G_B(\lambda)$ then it turns out to be simple, and $\lambda$ is the highest weight relative to the Killing form. These are all the irreducible modules: [49, II.2.4].

Parabolic subgroups $P$ of $G$ can be now be characterised as follows: $P$ is any subgroup containing a Borel subgroup. Fixing $B \subseteq P$ it turns out that $\Phi(P, T)$ is an enlargement of $\Phi(B, T)$ obtained by choosing simple roots $\Delta' \subseteq \Delta$ and taking the smallest additively-closed subset of $\Phi(G, T)$ containing $\Phi(B, T)$ and $-\Delta'$. If $r$ is the semisimple rank of $G$, then there are $2^r$ non-conjugate parabolic subgroups containing a given Borel subgroup $B$; these correspond to the possible subsets of nodes in the Dynkin diagram. The remaining parabolic subgroups are all conjugate to one of these under the action of $G$.

Just as we saw in $\text{GL}(V)$, a parabolic subgroup admits a Levi decomposition $P = \mathcal{R}_u(P) \rtimes L$, where $L$ is a reductive group called a Levi factor of $P$; all such Levi factors

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*Another reduction modulo $p$ gives the Weyl module $V(\lambda)$, which has $L(\lambda)$ as its head.*
are conjugate by elements of $\mathfrak{R}_u(P)$. If one insists $T \subseteq L$ then $L$ is unique. Agreeing $T \subseteq L$, the Levi decomposition can be seen at the level of roots. If $P$ corresponds to $\Delta'$ then the roots arising as sums from $\Delta \cup -\Delta'$ give those of $\Phi(L,T)$; their complement in $\Phi(P,T)$ are the roots in $\mathfrak{R}_u(P)$.

**Example 1.1.** Let $G$ be simple of type $F_4$. Elementary root system combinatorics (see e.g. [16]) tell us that $G$ has 48 roots (hence 24 positive roots). If we pick the two middle nodes of the Dynkin diagram of $G$ (cf. page 7), the corresponding roots and their negatives generate a subsystem of type $B_2$, which has 8 roots. So in the corresponding parabolic subgroup $P = \mathfrak{R}_u(P)L$ of $G$, the derived subgroup of $L$ will be simple of type $B_2$, which has dimension 10 (8 roots, plus a 2-dimensional maximal torus). The centre of $L$ will be a 2-dimensional torus since $L$ contains a maximal torus of $G$, which has rank 4. Finally, counting positive roots in $G$ and $L$, the unipotent radical $\mathfrak{R}_u(P)$ has dimension $24 - 4 = 20$. In fact, much of the structure of $Q$ as an $L$-group can also be quickly deduced from the root system. We will return to this in §1.4.3.

The following result [11, Thm. 2.5] is fundamental in our study and will henceforth be called the Borel–Tits theorem.

**Theorem 1.2** (Borel–Tits). *Let $G$ be a connected reductive algebraic group, and let $U$ be a unipotent subgroup of $G$. Then there exists a (canonically-defined) parabolic subgroup $P$ of $G$ such that $U \subseteq \mathfrak{R}_u(P)$.*

This implies in particular that a maximal subgroup of a reductive group $G$ is either reductive or parabolic. And a maximal connected subgroup of a semisimple group $G$ is either semisimple or parabolic.

### 1.3 $G$-complete reducibility

At last, we come to the central definition.

**Definition 1.3** ([92, p. 19], [93, §3.2.1]). *Let $G$ be a connected reductive algebraic group. A subgroup $H \subseteq G$ is called $G$-completely reducible ($G$-cr) if, whenever $H$ is contained in a parabolic subgroup $P$ of $G$, there exists a Levi factor $L$ of $P$ with $H \subseteq L$. We call $H$ $G$-irreducible ($G$-irr) if $H$ is contained in no proper parabolic subgroup of $G$; it is $G$-reducible if it is not $G$-irr. Lastly $H$ is $G$-indecomposable if $H$ is in no proper Levi subgroup of $G$ and $G$-decomposable otherwise.*

**Remark 1.4.** If $H$ is any subgroup of $G$, not necessarily closed, then $H$ and its Zariski closure are contained in precisely the same Zariski-closed subgroups of $G$, in particular, the same parabolic subgroups and Levi factors thereof. Therefore $H$ is $G$-cr if and only if its Zariski closure is; thus it does us no harm to work only with closed subgroups.

Many general results in the representation theory of groups can be viewed as cases of statements about $G$-complete reducibility, when $G$ is specialised to $GL(V)$. We now collect some of these.

#### 1.3.1 Characteristic criteria for complete reducibility

Jantzen proved in [48] that if $H$ is connected reductive over $k$ and $V$ is a representation for $H$ with $\dim V \leq p$ then $V$ is completely reducible. (This bound was improved by McNinch in [73].) A more general statement is:
Theorem 1.5. Let $G$ be a reductive algebraic group over $k$ of characteristic $p \geq 0$ and let $H$ be a connected subgroup of $G$. If $H$ is $G$-cr then $H$ is reductive. Conversely, suppose $p = 0$ or $p$ is bigger than the highest rank of a simple factor of $G$. Then if $H$ is reductive, $H$ is $G$-cr.

The implication ‘$G$-cr $\Rightarrow$ reductive’ follows from the Borel–Tits theorem: one can put $H$ in a parabolic subgroup $P$ such that $R_u(H) \subseteq R_u(P)$. Thus if $R_u(H)$ is non-trivial then $H$ is in no Levi subgroup of $P$.

The ‘reductive $\Rightarrow G$-cr’ direction can be deduced from Jantzen’s or McNinch’s result if $G$ is simple and classical. The exceptional case, tackled in §57, relies on a careful study of the 1-cohomology arising from the conjugation action of subgroups of parabolics on unipotent radicals. We discuss this technique in more detail in §2.3, where we explain how one can find non-$G$-cr subgroups, when they exist.

Given the result for $G$ simple, the general case follows in short order, since parabolic subgroups and Levi factors in $G$ are commuting products of the central torus $Z(G)^\circ$ with parabolic and Levi subgroups of the simple factors of $G$.

One of Serre’s initial motivations for studying $G$-complete reducibility was in studying complete reducibility of representations. Specifically, the starting point for Serre’s lectures [92] is the observation of Chevalley [21] that, over a field of characteristic 0, tensor products of completely reducible modules for an arbitrary group are again completely reducible. The corresponding statement is false in positive characteristic, however Serre observed [90] that it does hold when the characteristic is large relative to the dimension of the modules in question, and then began asking various questions of the form: If a tensor product (or symmetric power, or alternating power) of modules is completely reducible, must the initial module(s) also be completely reducible? Or conversely? The answer [91] depends (quite naturally) on certain congruence conditions on the characteristic. See §2.4.1 for more on this.

In a related vein, in [93, §5.2], for a reductive group $G$ with $G$-module $V$, Serre defines an invariant $n(V)$ in terms of the weights of $G$ on $V$; he then proves that if the characteristic is larger than $n(V)$, and the identity component of the kernel of $G \to \text{GL}(V)$ is a torus, then $H \subseteq G$ is $G$-cr if and only if $V$ is a completely reducible $H$-module [93, Thm. 5.4]. Thus for sufficiently large characteristic, $G$-complete reducibility can indeed be detected on the level of $G$-modules. When $V = \text{Lie}(G)$ is the adjoint module we have $n(V) = 2h_G - 2$, where $h_G$ is the Coxeter number of $G$. Note that this bound is typically much larger than that given in Theorem 1.5.

1.3.2 Equivalence with strong reductivity

A major result in the subject generalises the idea that a completely reducible module is the direct sum of a unique list of simple ones—the Jordan–Hölder theorem implies this statement for modules. Instead of focusing on the simple summands of a module, we focus on the Levi subgroup stabilising the decomposition.

Theorem 1.6 ([5, Cor. 3.6]). For a subgroup $H$ of a reductive algebraic group $G$, the following are equivalent.

(i) $H$ is $G$-cr;

(ii) $H$ is $C_G(S)$-irr for some maximal torus $S$ of $C_G(H)$;
(iii) for every parabolic subgroup $P$ of $G$ which is minimal with respect to containing $H$, the subgroup $H$ is $L$-irr for some Levi subgroup $L$ of $P$;

(iv) there exists a parabolic subgroup $P$ of $G$ which is minimal with respect to containing $H$, such that $H$ is $L$-irr for some Levi subgroup $L$ of $P$.

Note that since $S$ is a torus, $C_G(S)$ is a Levi subgroup. Then if $H$ is $C_G(S)$-irr for some maximal torus $S$ of $C_G(H)$, it is in fact $C_G(S)$-irr for all such $S$, since these are all $C_G(H)$-conjugate to one another.

Subgroups satisfying (ii) were termed strongly reductive by Richardson [86, Def. 16.1]. Richardson studied strongly reductive subgroups from a geometric viewpoint, centred around the following result.

**Theorem 1.7** ([86, §16]). Suppose that $H$ is topologically finitely generated by elements $h_1, \ldots, h_n \in G$. Then $H$ is $G$-cr (resp. $G$-irr) if and only if the orbit $G \cdot (h_1, \ldots, h_r)$ is Zariski closed in $G^n$ (resp. a stable point of $G^n$).

**Remarks 1.8.**

(i) This theorem is a lynchpin of the results in [5] and subsequent work. It relies on some geometry and geometric invariant theory, which we omit; we point the reader to [86] and [5].

(ii) In geometric invariant theory, a stable point of a $G$-variety is a point whose $G$-orbit is closed and whose stabiliser is a finite extension of the kernel of the action. Thus when $G$ acts on $G^n$ by simultaneous conjugation, a stable point is one whose orbit is closed, and whose centraliser is a finite extension of the centre $Z(G)$.

(iii) In fact, $H$ need not be finitely generated here—one need only pick a sufficiently large $n$-tuple so that $H$ and the elements of the $n$-tuple generate the same associative subalgebra of $\text{Mat}_{n\times n}(k)$ in some faithful representation of $G$; this is always possible for dimension reasons. Such an $n$-tuple is called a generic tuple [7, Def. 2.5].

A highlight of what can be proved through this approach is the following theorem.

**Theorem 1.9** ([5]). Let $G$ be a reductive algebraic group over the algebraically closed field $k$.

(i) If $N \triangleleft H$ and $H$ is $G$-cr then $N$ is $G$-cr.

(ii) If $H$ is $G$-cr then $N_G(H)$ and $C_G(H)$ are $G$-cr (in particular, $H$ is $G$-cr if and only if $N_G(H)$ is $G$-cr). More generally, if

$$HC_G(H) \subseteq K \subseteq N_G(H)$$

then $K$ is $G$-cr.

Part (i) here generalises Clifford’s theorem in representation theory: a completely reducible module for a group remains completely reducible upon restriction to a normal subgroup. Part (ii) gives a converse to this, and inspires the following question:

If $H$ is a commuting product $H = AB$, where $A$ and $B$ are $G$-cr, must $H$ also be $G$-cr?

By (ii) the answer is yes if $A = C_G(B)$ and $B = C_G(A)$. The answer is also positive when the characteristic is large enough:
Theorem 1.10 ([6, Thm. 1.3]). Suppose $A$ and $B$ are commuting $G$-cr subgroups of $G$ such that either $p = 0$, $p > 3$; or $p = 3$ and $G$ has no simple factors of exceptional type. Then $AB$ is also $G$-cr.

The proof in [6] involves some case-by-case arguments depending on the Lie type of $G$, although a uniform argument is also possible if one is willing to relax the bound on the characteristic [76, Prop. 40].

Remark 1.11. To date, we are not aware of a reductive group $G$ and a pair of $G$-cr subgroups $A$ and $B$ (connected or otherwise) whose product $AB$ is non-$G$-cr when $p = 3$. It is therefore plausible that the above theorem holds whenever $p \neq 2$. Such examples do however exist when $p = 2$, cf. [6, Ex. 5.3].

One powerful feature of $G$-complete reducibility is that one can allow the group $G$ to change under various constructions. For instance, in [5] it is shown that if $G = G_1 \times G_2$ is a direct product of reductive groups then $H \subseteq G$ is $G$-cr or $G$-irr if and only if both images under projection to a factor $G_i$ are $G_i$-cr (resp. $G_i$-irr). Similarly, taking a quotient by a normal subgroup $N$ sends $G$-cr subgroups to $(G/N)$-cr subgroups, and the converse also holds if $N^\circ$ is a torus.

The following result is useful in what follows. It can be applied equally when $S$ is a subgroup generated by a semisimple element, or a sub-torus of $G$. (Recall a group $S$ is linearly reductive if all its representations are completely reducible; in positive characteristic this means $S^\circ$ is a torus and the order of the finite group $|S/S^\circ|$ is coprime to $p$.)

Theorem 1.12. Let $G$ be reductive and let $S \subseteq G$ be linearly reductive. Then $S$ is $G$-cr, and if $H = C_G(S)^\circ$ then a subgroup of $H$ is $H$-cr if and only if it is $G$-cr.

In the particular case that $S$ is a torus, $C_G(S) = C_G(S)^\circ$ is a Levi subgroup of $G$; this is one of the ingredients in Theorem 1.6 and was first proved in [93, Prop. 3.2]. The following corollary further justifies our focus on semisimple subgroups:

Corollary 1.13. Let $G$ be reductive and $H \subseteq G$ a subgroup, and let $S \subseteq C_G(H)$ be linearly reductive. Then $H$ is $G$-cr if and only if $HS$ is $G$-cr. In particular, a connected reductive subgroup $H$ of $G$ is $G$-cr if and only if its (semisimple) derived subgroup $\mathcal{D}(H)$ is $G$-cr.

Proof. In characteristic 0, the subgroup $H$ is $G$-cr if and only if it is reductive (Theorem 1.5), which is the case if and only if $HS$ is reductive.

In positive characteristic, using Theorem 1.12 we can replace $G$ with $C_G(S)$ so that $S \subseteq Z(G)$. Now $S$ is a normal subgroup whose identity component is a torus, so by the above discussion, the subgroups $H$ and $HS$ are each $G$-cr if and only if $HS/S$ is $(G/S)$-cr.

1.3.3 Separability, reductive pairs

The property of subgroups being ‘separable’ often interacts with statements about $G$-complete reducibility. A subgroup (scheme) $H$ of $G$ is said to be separable if the scheme-theoretic centraliser $C_G(H)$ is smooth. Avoiding schemes, the statement that $H$ is separable is equivalent to saying the Lie algebra $\text{Lie}(C_G(H))$ of the group variety $C_G(H)$ is equal to the fixed-point space $C_{\text{Lie}(G)}(H)$; the latter always contains the former but can be larger in general. If $h \in G^n$ is a topological generating tuple (or generic tuple) for $H$, and $G$ acts on $G^n$ by simultaneous conjugation, the statement is also equivalent to saying
the orbit map \( G \to G \cdot h \) is a separable morphism \([13, \text{Prop. 6.7}]\), whence the terminology. In \( \text{GL}(V) \), all subgroup (schemes) are separable, cf. \([40, \text{Lemma 3.5}]\), and non-separable subgroups are always a low-characteristic phenomenon. Indeed, the main result of \([40]\) states that all (closed) subgroup schemes are separable if and only if the characteristic is 0 or pretty good for \( G \). The latter is a very mild condition, for instance if \( G \) is simple of exceptional type then a prime \( p \) is pretty good for \( G \) unless \( p = 2 \) or 3, or \( G = E_8 \) and \( p = 5 \).

Next, a pair \((G, H)\) of reductive groups is called a reductive pair \([85]\) if \( \text{Lie}(H) \) is an \( H \)-module direct summand of \( \text{Lie}(G) \). Again, \((G, H)\) is always a reductive pair if the underlying characteristic is large relative to \( G \) (since, for instance, \( \text{Lie}(G) \) is a completely reducible module for all reductive subgroups in sufficiently large characteristic).

The particular application to complete reducibility is as follows.

**Theorem 1.14** ([5, Thm. 3.35, Cor. 3.36]). *Suppose that \((G, M)\) is a reductive pair, and let \( H \subseteq M \) be a separable subgroup of \( G \). If \( H \) is \( G \)-cr, it is also \( M \)-cr.*

In particular, if \((\text{GL}(V), M)\) is a reductive pair and \( H \subseteq M \) acts completely reducibly on \( V \), then \( H \) is \( M \)-cr.

The proof in op. cit. is geometric, following \([85]\), show that that under the given hypotheses, the separable orbit \( O \) of \( G \) on \( G^n \) under simultaneous conjugacy splits into finitely-many Zariski-closed \( M \)-orbits in \( O \cap M^n \). So if \( O \) is closed in \( G^n \), then the \( M \)-orbits on \( O \cap M^n \) are closed in \( M^n \), which in turn implies that \( H \) is \( M \)-cr.

### 1.3.4 \( G \)-complete reducibility in classical groups

In this section, we write \( G = \text{Cl}(V) \) to mean that either \( G \subseteq \text{SL}(V) \) is the identity component of the stabiliser of a form on \( V \), which is either identically zero; or non-degenerate, whether alternating or quadratic. Then parabolic subgroups of \( G \) are the stabilisers in \( G \) of flags of subspaces where the relevant form vanishes if non-zero, i.e. totally isotropic and totally singular subspaces, respectively—see \([71, \text{Prop. 12.13}]\). Two parabolic subgroups corresponding to flags \((V_i)_{i=1,\ldots,r}\) and \((W_i)_{i=1,\ldots,s}\) are opposite if \( i = s \) and \( V \) is an orthogonal direct sum \( V_i \perp W_i^{\perp} \) for each \( i \), where \( W_i^{\perp} \) is the annihilator of \( W_i \) relative to the alternating or symmetric form on \( V \). Hence we can establish when a subgroup \( H \) of \( G \) is \( G \)-cr by knowledge of the action of \( H \) on \( V \) (together with its form).

When \( G = \text{SL}(V) \) we have seen that \( G \)-complete reducibility coincides with \( V \) being completely reducible. By Theorem 1.12 this extends to classical groups \( G = \text{Cl}(V) \) in characteristic not 2, since \( G \) is then the centraliser in \( \text{GL}(V) \) of an involutory outer automorphism.* Therefore, unless the characteristic is 2, complete reducibility in classical groups coincides with the usual notion, and so classifying the semisimple subgroups of \( \text{Cl}(V) \) (up to isogeny) amounts to knowing the dimensions of irreducible modules for semisimple groups and the Schur indicator on those modules.

The condition \( \text{char } k = p \neq 2 \) is in fact necessary to make these assertions. If \( p \neq 2 \) then recall that a quadratic form \( q \) gives rise to a symmetric bilinear form \( B \) on \( V \) via

\[
B(v, w) := \frac{1}{2} (q(v + w) - q(v) - q(w)),
\]

and \( q \) can be recovered from \( B \) via \( q(x) = B(x, x) \). If \( p = 2 \) then \( B(v, w) := q(v+w) - q(v) - q(w) \) defines a symmetric bilinear form, but \( B(x, x) = 0 \) so \( q \) can no longer be recovered.

*As an alternative proof, when \( p \neq 2 \) one can show that \((\text{GL}(V), G)\) is a reductive pair; and since every subgroup is separable in \( \text{GL}(V) \), Theorem 1.14 tells us that every \( \text{GL}(V) \)-cr subgroup of \( G \) is \( G \)-cr.
If $B$ is non-degenerate and $B(v, w) \neq 0$ then $B$ is also non-degenerate on $\langle v, w \rangle^\perp$ and so $V$ must have even dimension. Thus the bilinear form on the natural module for $\text{SO}_{2n+1}$ has a 1-dimensional radical, spanned by a non-singular vector for $q$. For convenience, we define the \textit{natural irreducible module} for $\text{Cl}(V)$ to be the largest non-trivial irreducible quotient of $V$; this is $V$ itself unless $p = 2$ and $G = \text{SO}(V) \cong \text{SO}_{2n+1}$, in which case it is the $2n$-dimensional quotient by the radical.

The following example is attributed to Liebeck in \cite[Example 3.45]{liebeck}.

**Example 1.15.** Let char $k = 2$ and let $H$ be a group preserving a symplectic (resp. orthogonal) form on an irreducible module $W$. Set $V = W \perp W$ as an orthogonal direct sum. Then $V$ has a unique non-zero totally isotropic (resp. totally singular) $H$-submodule. Thus $H$ is $\text{GL}(V)$-cr but not $\text{Cl}(V)$-cr.

**Proof.** By Schur’s lemma, every proper non-zero $H$-submodule of $V$ identifies with the image of a diagonal embedding $W \to V$ by $w \mapsto (aw, bw)$ for $[a : b] \in \mathbb{P}^1(k)$. If $B$ is an $H$-invariant non-degenerate alternating form on $W$ then $B((aw, bw), (aw, bw)) = (a^2 + b^2)B(w, u) = (a + b)^2B(w, u)$. This is zero if and only if $a = b$. The orthogonal case is similar. \hfill \Box

### 1.3.5 $G$-irreducibility in classical groups

By the characterisation of parabolic subgroups above, a subgroup of $G = \text{Cl}(V)$ is $G$-irr if and only if it preserves no proper non-zero totally isotropic (or totally singular) subspace. In more detail:

**Proposition 1.16.** Let $G$ be a simple algebraic group of classical type with $V$ the natural irreducible $G$-module, and let $H$ be a subgroup of $G$. Then $H$ is $G$-irr if and only if one of the following holds:

(i) $G$ has type $A_n$ and $V$ is an irreducible $H$-module;

(ii) $G$ has type $B_n$, $C_n$ or $D_n$ and $V = V_1 \perp \ldots \perp V_k$ as $H$-modules, where the $V_i$ are non-degenerate, irreducible and pairwise inequivalent;

(iii) $p = 2$, $G$ has type $D_n$ and $H$ fixes a non-singular vector $v \in V$, such that $H$ is $G_v$-irr in the point stabiliser $G_v$ and does not lie in a subgroup of $G_v$ of type $D_{n-1}$.

(See \cite[Prop. 3.1]{liebeck} for a full proof.)

**Remarks 1.17.**

(a) In case (iii), the bilinear form $B$ on $V$ preserved by $G \cong \text{SO}_{2n}$ is alternating when char $k = 2$, hence every 1-space is isotropic. Now $G_v$ preserves the space $\langle v \rangle^\perp$ of codimension 1 in $V$, and as $v$ is nonsingular, $q$ is non-degenerate on this space, so $G_v \cong \text{SO}_{2n+1}$ is simple of type $B_{n-1}$. For more details, see for instance \cite[Prop. 4.1.7]{liebeck}.

(b) This proposition can be applied to any simple group of type $A-D$, such as $\text{PSp}_{2n}$ or the half-spin groups of type $D_n$. For if $G$ is simple and classical then $G$ is related to some $\text{Cl}(V)$ by a quotient or an extension by a finite central subgroup (scheme)—or if $G$ is a half-spin group, then one of each. If $Z$ is such a finite central subgroup then as $G$ is connected and reductive, $Z$ is contained in all maximal tori, hence in all parabolic subgroups. It follows that a subgroup $H$ of $G$ is $G$-irr if and only if its preimage $HZ$ or quotient $HZ/Z$ is $G$-irr. Furthermore if char $k = 2$ then...
taking the quotient by the 1-dimensional radical of the bilinear form induces an exceptional isogeny \( \psi : \text{SO}_{2n+1} \to \text{Sp}_{2n} \). In this case, \( \psi \) is bijective and it follows that \( H \subseteq \text{SO}_{2n+1} \) is \( \text{SO}_{2n+1} \)-cr if and only if \( \psi(H) \) is \( \text{Sp}_{2n} \)-cr.

1.4 A strategy for classifying semisimple subgroups

We remind the reader that we wish to tackle the following:

**Main Problem.** Let \( G \) be a simple algebraic group of exceptional type. Describe the poset of conjugacy classes of semisimple subgroups of \( G \).

Just as in representation theory, where one may begin by studying irreducible modules for a given object (immediately yielding the completely reducible modules) and then considering extensions, one can stratify the search for subgroups of \( G \), beginning with \( G \)-cr subgroups and building up from these to non-\( G \)-cr subgroups.

Let us suppose the maximal connected subgroups of simple groups up to a certain rank (such as 8) have been classified, and suppose that \( G \) is one of these simple groups. Let \( H \subseteq G \) be semisimple. If \( H \) lies in a maximal connected subgroup \( M \) of \( G \) and if \( M \) is reductive then \( M = \mathcal{Z}(M) \cdot \mathcal{D}(M) \) and as \( H \) is perfect, we have \( H \subseteq \mathcal{D}(M) \). Since \( \mathcal{D}(M) \) is a central product of simple groups of smaller dimension, we hope to know \( H \) by induction on dimension. The problem is that one may have \( H \subset M \) where \( M \) is not reductive; this means \( M = \mathcal{P} \) is a parabolic subgroup by the Borel–Tits theorem. So we would also like to know the semisimple subgroups of \( \mathcal{P} = QL \). Since \( L \) is reductive of the same rank as \( G \) we might again assume we know its semisimple subgroups. But it remains to find those semisimple subgroups of \( \mathcal{P} \) which are not conjugate to subgroups of \( L \); in other words, the non-\( G \)-cr subgroups. Of course, Theorem 1.5 tells us that such subgroups do not exist if \( \text{char } k \) or large enough relative to the root system of \( G \).

1.4.1 \( G \)-cr subgroups

Suppose for now we wish to list the conjugacy classes of \( G \)-cr semisimple subgroups of \( G \). Theorem 1.6 reduces our task to finding those which are \( L \)-irr, as \( L \) varies over all Levi subgroups of \( G \). Let \( \{ L_1, \ldots, L_s \} \) be a list of representatives of the \( G \)-conjugacy classes of Levi subgroups. Then we can find all conjugacy classes of semisimple \( G \)-cr subgroups at least once, by listing the \( L_i \)-conjugacy classes of \( L_i \)-irr subgroups. The full analogue of the Jordan–Hölder theorem given below says that this will give each \( G \)-class of \( G \)-cr subgroups exactly once. It can be proven in various ways (cf. [19, Props. 2.8.2, 2.8.3], [68, Lemma 3.26]) but the cleanest is via geometric invariant theory [8, Thm. 5.8].

**Lemma 1.18.** Let \( H \) be a subgroup of the reductive group \( G \), let \( P \) be minimal among parabolic subgroups of \( G \) containing \( H \) and let \( \pi : P \to P/\mathcal{R}_u(P) = L \) be the natural projection to a Levi factor \( L \). Then \( \pi(H) \) is \( L \)-irr, and the \( G \)-conjugacy classes of \( L \) and \( \pi(H) \) are uniquely determined by the \( G \)-conjugacy class of \( H \).

Moreover, \( H \) is \( G \)-cr if and only if \( H \) and \( \pi(H) \) are \( \mathcal{R}_u(P) \)-conjugate.

As \( L \) contains a maximal torus, the group \( N_G(L) \) is an extension of \( L \) by a finite group: the part of the Weyl group of \( G \) which stabilises the root system of \( L \). Thus \( N_G(L) \) may induce non-trivial conjugacy between some simple factors of \( L \). Such conjugacy is easy to

*more details on p. 17.
describe—cf. [71, Corollary 12.11]. In light of this, the key issue is to find the \( \mathcal{D}(L) \)-classes of \( \mathcal{D}(L) \)-irr semisimple subgroups for each \( G \)-class of Levi subgroup \( L \); this includes the case that \( L = G \).

**Definition 1.19.** For a connected reductive group \( G \), let \( \text{Irr}(G) \) denote the poset of \( G \)-classes of connected \( G \)-irr subgroups of \( G \) under inclusion.

Suppose \([H] \in \text{Irr}(G)\). It follows from the Borel–Tits theorem that \( H \) is reductive. Since it cannot centralise a non-central torus of \( G \), we get:

**Lemma 1.20 ([63, Lemma 2.1], [5, Cor. 3.18]).** Suppose \( G \) is semisimple and let \( H \) be a \( G \)-irr connected subgroup. Then \( H \) is semisimple and \( C_G(H) \) is finite and linearly reductive.

**Remark 1.21.** Work of the third author and Liebeck in [64] classifies the finite subgroups which can occur as centralisers of semisimple \( G \)-irr subgroups for any simple algebraic group \( G \).

As \( H \) is not contained in any proper parabolic subgroup of \( G \), it must be contained in some semisimple maximal connected subgroup \( M \) of \( G \). Furthermore, \( H \) is \( M \)-irr, as any parabolic subgroup of \( M \) is contained in a parabolic subgroup of \( G \), again by the Borel–Tits theorem. When \( M \) is simple of classical type, one may determine \( \text{Irr}(M) \) at once by use of Proposition 1.16; if \( M \) is simple of exceptional type, then induction on the dimension of \( G \) yields \( \text{Irr}(M) \). For general semisimple \( M \), knowledge of \( \text{Irr}(M_i) \) for each simple factor \( M_i \) of \( M \) yields \( \text{Irr}(M) \) (more on this shortly). Now, it can happen that an \( M \)-irr subgroup \( H \) lies in a proper parabolic subgroup of \( G \). We call \([H] \in \text{Irr}(M)\) a candidate and aim to decide which candidates are actually \( G \)-irr. Furthermore, a \( G \)-irr subgroup can be contained in more than one semisimple maximal connected subgroup \( M \). So two candidates \( H_1 \subseteq M_1 \) and \( H_2 \subseteq M_2 \) could be in the same \( G \)-class and we wish to detect this.

Returning to the issue of finding \( \text{Irr}(M) \) when \( M \) is not simple, the parabolic subgroups of \( M \) are the products of parabolic subgroups of its factors, so any \( M \)-irr subgroup needs to project to an \( M_i \)-irr subgroup of each simple factor \( M_i \); [103, Lemma 3.6]. As a partial converse, if for each \( i \) we have an \( M_i \)-irr subgroup \( H_i \), then \( H_1 \ldots H_r \subseteq M \) is \( M \)-irr; however, these do not quite exhaust all the \( M \)-irr subgroups. To complete the list, one must also discuss diagonal subgroups: Whenever \( \{M_i\} \) are one or more simple factors of \( M \) having the same Lie type, let \( M \) be the corresponding simply-connected simple group. Then \( M \) admits a homomorphism into \( M \) with non-trivial projection to each simple factor \( M_i \). By definition, a diagonal subgroup is the commuting product of images of such homomorphisms; this will be \( M \)-irr if it has non-trivial projection in every simple factor of \( M \).

**Example 1.22.** If \( M = M_1 M_2 \) where the factors have the same type then \( M \) has a proper diagonal subgroup \( H \), with simply-connected simple group \( \hat{M} \). Then the compositions \( M \to H \to M_i \) are isogenies. It follows that up to \( M \)-conjugacy, these are the composition of powers of Frobenius maps \( F \) (or their square roots in some very special cases) with automorphisms which induce a symmetry of the Dynkin diagram of \( \hat{M} \). For instance, if \( M \) has type \( A_1 A_1 \) then, since the \( A_1 \) Dynkin diagram has no symmetries, \( H \) corresponds to a pair of integers \((r, s)\) and the map \( \hat{M} \to H \) is \( x \to (F^r(x), F^s(x)) \). Since \( H \cong F(H) \) we may assume \( rs = 0 \). For brevity, we use the notation \( A_1 \leftrightarrow A_1 A_1 \) via \((1^r, 1^s)\) to describe these diagonal subgroups. See [105, Chapters 2,11] for further discussion and notation.
To recap, our recipe is now to iterate through the semisimple maximal connected subgroups $M$ of $G$, collecting candidates $H$. We throw away all those candidates which fall into a parabolic subgroup, and then reconstruct the poset $\text{Irr}(G)$ by identifying conjugacy amongst the remaining candidates.

The passage from classifying $G$-irr subgroups to all $G$-cr subgroups is now easy using Lemma 1.18. Suppose $H$ is $\mathcal{D}(L)$-irr. By the above remarks we can write down the $\mathcal{D}(L)$-classes of $\mathcal{D}(L)$-irr subgroups; then from the lemma we need only establish conjugacy amongst those classes by examining the action of stabiliser of $L$ in the Weyl group of $G$.

**Remark 1.23.** A related representation-theoretic question is to classify triples $(G,H,V)$ where $H \subseteq G$ and $V$ is an irreducible $G$-module which remains irreducible as an $H$-module. This property is strictly stronger than $G$-irreducibility, and has its own extensive literature, cf. [17] and the references therein.

### 1.4.2 Non-$G$-cr subgroups

We now turn our attention to non-$G$-cr semisimple subgroups $H$. Then $H \subseteq P$ for some proper parabolic subgroup $P = QL$ with $Q = \mathcal{R}_u(P)$ and we may assume $P$ is minimal subject to containing $H$. Let $\bar{H}$ denote the image of $H$ in $L$ under the projection $\pi : P \rightarrow L$. Then by Lemma 1.18 $\bar{H}$ is $L$-irr (hence $G$-cr) and is not $\mathcal{R}_u(P)$-conjugate to $H$. We may assume from the previous section that we know $\bar{H}$ up to conjugacy. To make further progress, we use non-abelian cohomology, whose techniques are similar to those employed in Galois cohomology.

Firstly, either $\pi : H \rightarrow \bar{H}$ is an isomorphism of algebraic groups or a very special situation occurs:

(i) $p = 2$;

(ii) $H$ has a simple factor $H_0 \cong SO_{2n+1}$, such that:

(a) the scheme-theoretic intersection $H_0 \cap Q$ is non-trivial, being an infinitesimal normal subgroup isomorphic to $\alpha_2^{2n}$ and corresponding to the ideal of short root spaces in the Lie algebra $\mathfrak{g}$; 

(b) the image of $H_0$ in $L$ is isomorphic to $Sp_{2n}$.

Suppose for now that this special situation does not hold. Then $H \cap Q = 1$ as group schemes, which means $H$ is a scheme-theoretic complement to $Q$ in the semidirect product $QH$. Any element of $H$ can thus be written uniquely as $\gamma(h)h$ with $h \in \bar{H}$, for some regular map of varieties $\gamma : \bar{H} \rightarrow Q$. The definition of the semidirect product implies $\gamma$ satisfies a 1-cocycle condition; namely:

$$\gamma(gh) = \gamma(g)(\gamma(h)).$$

If $H'$ is another complement to $Q$ corresponding to $\gamma'$ then $H$ is $Q$-conjugate to $H'$ if and only if $\gamma$ is related to $\gamma'$ via a coboundary: in other words the $Q$-conjugacy classes of complements are given by classes $[\gamma] \in H^1(\bar{H}, Q)$. We leave the description of the precise relationship of $\gamma$ and $\gamma'$ to [98, §2], but note that since $Q$ is typically non-abelian, the set $H^1(\bar{H}, Q)$ does not admit the structure of a group—rather, it is only a pointed set, having a distinguished element corresponding to the class of the trivial cocycle. In

---

*this is the scheme-theoretic kernel of $H \rightarrow \bar{H}$; the derivative $\text{Lie}(H) \rightarrow \text{Lie}(\bar{H})$ kills the ideal of short roots of $\text{Lie}(SO_{2n+1})$; see [12] for the theory surrounding this map, or one of [81, Lem. 2.2], [106], [32] for more concrete treatments.
contrast, when $Q$ has the structure of an $\hat{H}$-module—i.e. $Q$ is a vector group on which $\hat{H}$ acts linearly—then both $Q$ and $H^1(\hat{H}, Q)$ are naturally $k$-vector spaces. In the latter case it can be shown that $H^1(\hat{H}, Q)$ is isomorphic to the first right-derived functor of the fixed point functor $H^0(\hat{H}, ?)$ in the category of rational $G$-modules. For more on this last point, see [49, 1.4].

**Example 1.24.** By way of illustration, we list the semisimple subgroups of $G = \text{SL}_3$. There are four parabolic subgroups of $G$ up to conjugacy, respectively stabilising flags with submodule dimensions $(3)$, $(2, 1)$, $(1, 2)$ and $(1, 1, 1)$. The first is $G$ itself, the last is a Borel subgroup (whose only reductive subgroups are tori) and the other two have $G$-conjugate Levi factors $\text{GL}_2$, one of which can be described as the image of the embedding

$$\text{GL}_2 \to \text{SL}_3; A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}. $$

Of course $G$ is $G$-irr; and if $L$ is a Levi subgroup isomorphic to $\text{GL}_2$ then $\mathcal{D}(L) \cong \text{SL}_2$, which has no proper semisimple subgroups. If $p \neq 2$, then there is one further $G$-irr subgroup $\text{PGL}_2$, embedded via the irreducible adjoint action on its Lie algebra. These are all the $G$-cr semisimple subgroups.

Suppose $H \subseteq G$ is non-$G$-cr. By rank considerations, $\hat{H}$ is isomorphic to $\text{SL}_2$, and it is easy to check that the conjugation action of $\hat{H}$ on $Q$ gives it the structure of the natural module $L(1)$. This means $H^1(\hat{H}, Q) = 0$, which rules out the existence of non-$G$-cr subgroups unless $p = 2$. If $p = 2$, though, we find we are in the special situation of (ii) above with $n = 1$. Indeed, the action of $\text{SL}_2$ on its Lie algebra and its dual are both non-completely reducible. (Both actions are indecomposable with two composition factors. The first is isomorphic to the Weyl module $V(2)$ which has $L(2)$ in the head and $L(0)$ in its socle; we denote this $L(2)/L(0)$. The second is upside down: $H^0(2) \cong L(0)/L(2)$.) These are in fact all the semisimple subgroups.

### 1.4.3 Abelian and non-abelian cohomology

Suppose $G$ is a connected reductive group acting on a $G$-module $V$. To mount a proper investigation into $H^1(G, V)$, a scheme-theoretic treatment such as [49] is essential. This is not least because one can make use of the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(G/N, H^j(N, V)) \Rightarrow H^{i+j}(G, V)$$

for calculations, where $N$ is a normal subgroup scheme of $G$. In this framework, $N$ is allowed to be an *infinitesimal subgroup scheme*, the most important example being the Frobenius kernel $G_1 := \ker(G \to F(G))$ of $G$. At the level of points, $G_1 = \{1\}$, but $\text{Lie}(G_1) = \text{Lie}(G)$ has far more structure. We leave the interested reader to pursue this further, but give some references: The first general investigation of $H^1(G, V)$ using the LHS spectral sequence applied to $G_1 \triangleleft G$ is probably that of Jantzen in [47], which connects $H^1(G, L(\lambda))$ with the structure of the Weyl module $V(\lambda)$. Other relevant papers are too numerous to mention, but some highlights are [22], [9], [10], [79].

On the understanding that $H^1(\hat{H}, V)$ has been well-studied for $V$ an $\hat{H}$-module, let us return to the calculation of $H^1(\hat{H}, Q)$, where $P = QL$ is a parabolic subgroup of a reductive algebraic group $G$. The fact that $Q$ is connected, smooth and unipotent means that it admits a filtration

$$Q = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_n = 1$$

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for some $n$, such that $Q_i \triangleleft Q$ and the subquotients $Q_i/Q_{i+1}$ admit the structure of $\bar{H}$-modules. The statement for general connected unipotent groups $Q$ can be found in [98, Thm. 3.3.5] and [75, Thm. C], but one can be more explicit here since $Q$ has a filtration by subgroups $Q_i$ generated by root subgroups of $G$. Following [3], suppose $P$ and $L$ are a standard parabolic and Levi subgroup, corresponding to a subset $I$ of the simple roots $\Delta$ (which can be identified with nodes of the Dynkin diagram of $G$). Then $P$ is generated by a maximal torus and root subgroups $U_\alpha$ where $\alpha$ runs through positive roots, as well as negative roots in $I$. Expressing each root $\alpha$ uniquely as

$$\alpha = \left( \sum_{\alpha_i \in I} c_i \alpha_i \right) + \left( \sum_{\alpha_j \in \Delta \setminus I} d_j \alpha_j \right),$$

the roots in $P$ are those with $\sum d_i \geq 0$; the roots occurring in $L$ are those with $\sum d_i = 0$, and those in $Q$ have $\sum d_i > 0$. The quantity $\sum d_i$ is called the level of the root, and $\sum d_j \alpha_j$ is called its shape. For each $i > 0$, we let $Q_i$ be the subgroup generated by root subgroups of level $i$; then the Chevalley commutator relations imply that each $Q_i$ is normal in $Q$. Furthermore, from knowledge of the root system of $G$, say by reference to [16], one can write down explicitly the representations $Q_i/Q_{i+1}$ as Weyl modules $V(\lambda)$ for the Levi subgroup $L$.

**Example 1.25.** Recall the Dynkin diagram of $G_2$ is \[\begin{array}{c|c|c|c} 1 & 2 & 3 \hline 1 & - & 1 \end{array}\]. The nodes represent the two simple roots which we denote by lists of coefficients, $\alpha_1 = 10$ and $\alpha_2 = 01$. The remaining positive roots, in order of height, are 11, 21, 31, 32, giving the roots in a Borel subgroup. Let $P$ be the standard parabolic containing the negative of $\alpha_2$, i.e. $-01$. Then a Levi $L$ of $P$ has roots $\pm 01$, and $\mathcal{R}_u(P)$ has roots of three levels $\{11,10\}, \{21\}$ and $\{31,31\}$, which one checks induce modules $L(1), L(0), L(1)$ for the Levi subgroup $\mathcal{P}(L) \cong SL_2$.

Once we understand these modules $Q_i/Q_{i+1}$, by working out restrictions of the $V(\lambda)$ to $\bar{H}$ and from knowledge of the cohomology of irreducible modules for $\bar{H}$, one can, as in [99], establish a prototype $\nabla := \bigoplus H^1(\bar{H}, Q_i/Q_{i+1})$ for $H^1(\bar{H}, Q)$. Indeed one can define a partial map $\nabla \to H^1(\bar{H}, Q)$, which turns out to be surjective. Understanding $H^1(\bar{H}, Q)$ then comes down to two issues:

(i) When is $H^1(\bar{H}, Q/Q_i) \to H^1(\bar{H}, Q)$ not injective?

(ii) When is $H^1(\bar{H}, Q/Q_i) \to H^1(\bar{H}, Q/Q_{i+1})$ not defined?

Both questions are answered by looking at the maps in a certain ‘long’ exact sequence of non-abelian cohomology. Let

$$1 \to R \to Q \to S \to 1$$

be maps of $\bar{H}$-groups with $R$ contained the centre of $Q$. Then there is an exact sequence

$$1 \to H^0(\bar{H}, R) \to H^0(\bar{H}, Q) \to H^0(\bar{H}, S) \xrightarrow{\delta_{\bar{H}}} H^1(\bar{H}, R) \to H^1(\bar{H}, Q) \to H^1(\bar{H}, S) \xrightarrow{\Delta_{\bar{H}}} H^2(\bar{H}, R). \quad (1.1)$$

Question (i) asks whether $\delta_{\bar{H}}$ is non-zero. This happens precisely when cocycle classes in $H^1(\bar{H}, R)$ fuse inside $H^1(\bar{H}, Q)$ due to conjugacy induced by the fixed points $S^H$. Question (ii) asks whether $\Delta_{\bar{H}}$ is non-zero. If so then cocycles in $H^1(\bar{H}, S)$ are obstructed from lifting to cocycles in $H^1(\bar{H}, Q)$. In particular, this only happens when $H^2(\bar{H}, R) \neq 0$.

\*Here, an exact sequence of pointed sets means only that the image of each map is the preimage of the distinguished element under the next.
Applying this exact sequence inductively allows us to calculate $H^1(\bar{H}, Q)$ completely. The matter is easy when the maps $\delta_{\bar{H}}$ and $\Delta_{\bar{H}}$ are forced to be zero. However, this is often not the case and one must resort to explicit computations with cocycles; this is the approach taken in [99].
Part II. Subgroup structure of exceptional algebraic groups

2.1 Maximal subgroups

Lie and Dynkin

The question of classifying maximal sub-objects of Lie type objects dates back to work of Sophus Lie [55]. Taking inspiration from Galois’s work on univariate polynomials, op. cit. develops ‘continuous transformation groups’—now Lie groups—with a view to classifying differential equations in terms of symmetries among their solutions. Since one builds up the group actions from primitive actions, which correspond to maximal subgroups, Lie was motivated to describe such subgroups. The same problem for finite groups was not to be posed until a paper of Aschbacher and Scott rather later, so Lie concentrated on connected subgroups of connected Lie groups. In that case the exp and log maps show the question is equivalent to finding maximal subalgebras $m$ of (real) Lie algebras $g$ and Lie solved the problem when $\dim g \leq 3$. Otherwise the question lay dormant for another fifty years.

Using the Killing–Cartan–Weyl classification of finite-dimensional complex simple Lie algebras, E. Dynkin solved Lie’s problem over the complex numbers to great surprise. We give a quick example—stolen from Seitz’s excellent tribute in [33]—to indicate the nature of his results. As is well-known, the complex 3-dimensional Lie algebra $\mathfrak{sl}_2$ has a unique irreducible representation of every degree up to isomorphism. Each of these amounts to an embedding of $\mathfrak{sl}_2$ into $\mathfrak{so}_{2n-1}$ or $\mathfrak{sp}_{2n}$ where $n$ is the degree of the representation. Dynkin showed that for $n \geq 2$, the image of each of these embeddings is a maximal subalgebra, with precisely one exception: when $n = 7$ and the exceptional Lie algebra of type $G_2$ has a self-dual 7-dimensional module, it occurs as a (maximal) subalgebra of $\mathfrak{so}_7$ and in turn contains the irreducible $\mathfrak{sl}_2$ as a maximal subalgebra. There is a remarkably short list of such situations. Dynkin in effect classified the maximal subalgebras of the classical Lie algebras $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ by classifying non-maximal ones which nevertheless act irreducibly on the natural modules for those algebras. A key ingredient in Dynkin’s work was detailed information on the representations of these Lie algebras, developed by Weyl and others, in terms of the weights for their Cartan subalgebras.

Dealing with the Lie algebras of exceptional type required Dynkin to adopt a different, more exhaustive and algorithmic approach. He first showed how to produce all the semisimple subalgebras of $g = \text{Lie}(G)$ containing a given Cartan subalgebra $\mathfrak{h}$; so called regular subalgebras. Since root spaces are 1-dimensional, it follows that such a subalgebra will be $\mathfrak{h}$ together with a sum of root spaces corresponding to a subset of roots $\Phi' \subset \Phi$. The non-degeneracy of the Killing form implies that $\Phi = -\Phi$ and Dynkin showed that one can
find everything by iteratively extending the Dynkin diagram (adding a node corresponding to the negative of the highest long root) and then deleting some nodes.

**Example 2.26.** Let \( \Phi \) be an irreducible root system of type \( F_4 \) with roots labelled as in the following diagram.

Then the highest long root is \( \alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \). The only simple root one can add to \(-\alpha_0\) and still get a root is \( \alpha_1 \). Therefore, the extended Dynkin diagram is:

The maximal subalgebras of maximal rank correspond to deleting a node of this extended diagram corresponding to a simple root with prime coefficient in the expression of \( \alpha_0 \). So in our case that is \( \alpha_1, \alpha_2 \) and \( \alpha_4 \). Removing \( \alpha_1 \) leaves a Dynkin diagram of type \( A_1C_3 \), removing \( \alpha_2 \) leaves one of type \( A_2A_2 \) and removing \( \alpha_4 \) leaves one of type \( B_4 \).

The non-semisimple subalgebras were described by a theorem of Morozov* and the maximal ones are the maximal parabolics. It remained to find those maximal subalgebras which do not contain a Cartan subalgebra; so-called \( S \)-subalgebras. Dynkin tackles those of type \( \mathfrak{sl}_2 = \langle e, f, h \rangle \) first, associating to each class of these under the adjoint action of \( G \) a Dynkin diagram with a label of 0, 1 or 2 above each node determining the conjugacy class of \( h \). It turns out there is a unique conjugacy class of \( \mathfrak{sl}_2 \)-subalgebras such that \( h \) is a regular element; i.e. such that the centraliser \( g_h \) of \( h \) in \( g \) is as small as possible, i.e. is a Cartan subalgebra. The corresponding Dynkin diagram for this class of subalgebras has a 2 above each node and it is usually maximal. From there, if \( \mathfrak{sl}_3 \) is a subalgebra of \( g \), then one can look to build it up from its own regular \( \mathfrak{sl}_2 \).

There are many reasons to want to extend this theory to positive characteristic—not least because one can use relationships between algebraic groups and their points over finite fields to say something about finite groups, for example in furtherance of the Aschbacher–Scott programme. Over several important monographs of Seitz, Liebeck–Seitz and Testerman, Dynkin’s classification is extended to describe the maximal subgroups of simple algebraic groups over algebraically closed fields of positive characteristic.

Unfortunately, there is no space to do anything else but state the main result in case when \( G \) is exceptional. In the following, conditions such as \( p \geq 13 \) also include the case \( p = 0 \).

**Theorem 2.27** ([61, Cor. 2], [30, Thm. 1]). Let \( G \) be a simple algebraic group of exceptional type in characteristic \( p \) and let \( M \) be maximal among connected subgroups of \( G \). Then \( M \) is either parabolic or is \( G \)-conjugate to precisely one subgroup \( H \) in Table 2.1, where each \( H \) denotes one \( G \)-conjugacy class of subgroups.

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*the precursor of Borel–Tits’ Theorem 1.2.
Table 2.1: The reductive maximal connected subgroups of exceptional algebraic groups.

| G  | H                        |
|----|--------------------------|
| G_2| \tilde{A}_2, \tilde{A}_2 (p = 3), A_1 \tilde{A}_1, A_1 (p \geq 7) |
| F_4| B_1, C_4 (p = 2), A_1 C_3 (p \neq 2), A_1 G_2 (p \neq 2), \tilde{A}_2 A_2, G_2 (p = 7), A_1 (p \geq 13) |
| E_6| A_1 A_5, \tilde{A}_2^3, F_4, C_4 (p \neq 2), A_2 G_2, G_2 (p \neq 7); 2 classes, A_2 (p \geq 5); 2 classes |
| E_7| A_1 D_6, \tilde{A}_2 A_5, A_7, G_2 C_3, A_1 F_4, A_1 G_2 (p \neq 2), A_2 (p \geq 5), A_1 A_1 (p \geq 5), A_1 (p \geq 17), A_1 (p \geq 19) |
| E_8| D_8, A_1 E_7, \tilde{A}_2 E_6, A_8, \tilde{A}_2^3, G_2 F_4, F_4 (p = 3), B_2 (p \geq 5), A_1 A_2 (p \geq 5), A_1 (p \geq 31), A_1 (p \geq 29), A_1 (p \geq 23) |

Remarks 2.28.

(i) As discussed in §1.2, the classes of parabolic subgroups are in bijection with subsets of the simple roots \( \Delta \), and the maximal ones correspond to subsets of size \( |\Delta| - 1 \).

(ii) In the caption of Table 2.1 we use the phrase reductive maximal connected. By this mean reductive subgroups maximal among connected subgroups. Similarly, in the caption to the next table we say reductive maximal positive-dimensional to mean reductive subgroups that are maximal among positive-dimensional subgroups.

(iii) The subgroups of maximal rank can be enumerated using the Borel–de-Siebenthal algorithm. This is a more general version of Dynkin’s procedure for regular subalgebras, and includes some extra cases where Dynkin diagram has an edge of multiplicity \( p = \text{char } k \). For example if \( G \) is of type \( F_4 \), then there is a maximal regular subgroup of type \( C_4 \). See [71, §13.2] for a complete explanation.

(iv) When \( G = E_6 \) there are two classes of maximal subgroups of type \( G_2 \) (\( p \geq 7 \)) and \( A_2 \) (\( p \geq 5 \)). The graph automorphism of \( G \) exchanges these two classes.

(v) The maximal subgroup of type \( F_4 \) when \( G = E_8 \) and \( p = 3 \) was overlooked in [89] and subsequently missed in [61]. This was rectified by Craven and the second two authors; more information can be found in [30].

It is also natural to ask about non-connected maximal subgroups. For example, finite subgroups of \( E_8 \) remain unclassified. See §2.4.3 for a brief description of the latest developments. However one can successfully weaken ‘connected’ to ‘positive-dimensional’:

**Theorem 2.29** ([61, Cor. 2]). Let \( G \) be a simple algebraic group of exceptional type in characteristic \( p \). Let \( M \) be a positive-dimensional maximal subgroup of \( G \). Then \( M \) is either parabolic or \( G \)-conjugate to precisely one subgroup \( H \) in Table 2.2, where each isomorphism type of \( H \) denotes one \( G \)-conjugacy class of subgroups.

**Remark 2.30.** The subgroup \( A_2 G_2 < E_6 \) is a maximal subgroup and its presence above corrects a small mistake in [61, Table 1]. In loc. cit. it is claimed that \( N_{E_6}(A_2 G_2) = (A_2 G_2)_2 \) with the outer involution acting as a graph automorphism of the \( A_2 \) factor. This is not possible as the action of \( A_2 G_2 \) on the 27-dimensional \( E_6 \) module \( V_{27} \) is not self-dual. Instead, it is the graph automorphism of \( E_6 \) which induces the outer involution of \( A_2 G_2 \).
There are three key things we need to determine $\text{Irr}(G)$ (Definition 1.19) for $G$ a simple exceptional algebraic group.

(i) Determine the semisimple maximal connected subgroups of $G$;

(ii) Decide whether a candidate subgroup $^*$ is $G$-irr;

(iii) Decide whether two $G$-irr candidate subgroups are $G$-conjugate.

Since $G$ is a simple exceptional algebraic group, (i) is immediate from Theorem 2.27. We consider (ii) and (iii) in the next two sections.

### 2.2 The connected $G$-irreducible subgroups

In light of §1.4.1 there are three key things we need to determine $\text{Irr}(G)$ for $G$ a simple exceptional algebraic group.

Let $H$ be an $M$-irr connected subgroup of $G$, where $M$ is maximal semisimple. We need to decide whether or not $H$ is $G$-irr.

#### 2.2.1 Testing candidate subgroups

If a candidate $H$ is in fact contained in a parabolic subgroup $P = QL$ of $G$ then we can consider the image $\pi(H)$ in a Levi factor. The action of $H$ and $\pi(H)$ on $G$-modules may differ but their composition factors will always match. *Thus one method of proving that $H$ is $G$-irr is to show that no subgroups of any proper Levi subgroup have matching composition factors on some $G$-module. For instance, every semisimple subgroup of a proper Levi subgroup has a trivial composition factor on the adjoint module for $G$, so a candidate is $G$-irr if it has no trivial composition factors in its adjoint action.

$^*$Recall a candidate subgroup is an $M$-irr subgroup from a semisimple maximal connected subgroup $M$. See [103, Lemma 3.8] for the precise definition of match.

| $G$ | $H$ |
|-----|-----|
| $G_2$ | $\bar{A}_2.2, \bar{A}_2.2 \ (p = 3), \bar{A}_1.1, A_1 \ (p \geq 7)$ |
| $F_4$ | $B_4, D_4.S_3, C_4 \ (p = 2), \bar{D}_4.S_3 \ (p = 2), \bar{A}_1C_3 \ (p \neq 2)$, $A_1G_2 \ (p \neq 2), (A_2\bar{A}_2).2, G_2 \ (p = 7), A_1 \ (p \geq 13)$ |
| $E_6$ | $\bar{A}_1A_5, (\bar{A}_2^3).S_3, (D_4T_2).S_3, T_6.W(E_6), F_4, C_4 \ (p \neq 2), A_2G_2, G_2 \ (p \neq 7), A_2.2 \ (p \geq 5)$ |
| $E_7$ | $\bar{A}_1D_6, (\bar{A}_2A_5).2, A_7.2, (\bar{A}_1^3D_4).S_3, (A_1^7).\text{PSL}(3)(2), (E_6T_1).2, T_7.W(E_7), G_2C_3, A_1F_4, (2^2 \times D_4).S_3, A_1G_2 \ (p \neq 2), A_2.2 \ (p \geq 5), A_1A_1 \ (p \geq 5), A_1 \ (p \geq 17), A_1 \ (p \geq 19)$ |
| $E_8$ | $D_8, \bar{A}_1E_7, (\bar{A}_2E_6).2, A_8.2, (\bar{A}_2^4).4, (D_4^2).S_3 \times 2, (\bar{A}_2^4).\text{GL}(2)(3), (\bar{A}_2^8).\text{AGL}(3)(2), T_8.W(E_8), G_2F_4, A_1(G_2^2).2 \ (p \neq 2), F_4 \ (p = 3), B_2 \ (p \geq 5), A_1A_2 \ (p \geq 5), A_1 \ (p \geq 31), A_1 \ (p \geq 29), A_1 \ (p \geq 23), A_1 \times S_5 \ (p \geq 7)$ |
Proving candidates are not $G$-irr

If $H$ is $G$-cr then $H$ is contained in some Levi subgroup $L$ of $G$ by Lemma 1.18 and thus $C_G(H)$ will contain the non-trivial torus $Z(L)$. In this case, it is often easy to find a non-trivial torus commuting with $H$ and thus conclude that $H$ is not $G$-irr.

The complicated cases occur when $H$ is non-$G$-cr. These cases are relatively rare: [105, Cor. 3] classifies the non-$G$-cr connected subgroups which are $M$-irr for every (and at least one) reductive maximal connected subgroup they are contained in. We have two main methods used for dealing with a candidate $H \subset M$ we suspect is non-$G$-cr; the full details can be found in [103, 104]. In short, we either

1. directly show that $H$ is contained in a parabolic subgroup $P$; or
2. find a non-$G$-cr subgroup $Z \subset P$ and show that $Z$ is contained in $M$ and conjugate to $H$.

One way to implement (1) is to exhibit a non-zero fixed point of $H$ on the adjoint module $L(G)$. By [89, Lemma 1.3], this places $H$ in either a proper maximal-rank subgroup or a proper parabolic subgroup of $G$, and one can use representation theory to prove that $H$ is not contained in a proper maximal-rank subgroup. Another way is to find a unipotent subgroup of $G$ normalised by $H$, since the Borel–Tits theorem then places $H$ in a proper parabolic subgroup. This is used in [103, Lemmas 7.9, 7.13], where calculations in Magma are used to construct an ad-nilpotent subalgebra $S \subset L(G)$ stabilised by a ‘large enough’ finite subgroup $H(q) < H$. It then follows that $H$ also stabilises $S$, and exponentiating $S$ yields a unipotent subgroup normalised by $H$.

Method (2) is implemented in [103, Lemmas 6.3, 7.4]. Here one starts with a candidate $H$ contained in a semisimple maximal connected subgroup $M$ of maximal rank. One constructs the relevant non-$G$-cr subgroup $Z$ according to the recipe in 1.4.2, and then shows that $Z \subset M$. To establish the latter, one proves that any group acting with the same composition factors as $Z$ on the adjoint module of $G$ fixes a non-zero element of $L(G)$. One then shows that $Z$ does not fix any non-zero nilpotent element; thus it fixes a non-zero semisimple element and is contained in a maximal rank subgroup $M'$, again by [89, Lemma 1.3]. This part is rather technical and requires the full classification of stabilisers of nilpotent elements and their structure, as found in [62]. It is however then possible to identify that $M' = M$ and show that $Z$ is conjugate to $H$.

2.2.2 $G$-conjugacy

Once we know that two isomorphic candidates $H_1$ and $H_2$ are $G$-irr, we must check whether they are $G$-conjugate. One easy test is to check whether their composition factors on various $G$-modules agree. If so then it turns out that, with a single exception, the two candidates are in fact $G$-conjugate. The exception occurs when $G = E_8$, $p \neq 3$ and $H_1$ and $H_2$ are diagonally embedded in $A_2^r \subset D_4^r \subset D_8 \subset E_8$ via $(10, 10[r])$ and $(10, 01[r])$ with $r \neq 0$. For more detail see [105, Cor. 1] and its proof.

If $H_1$ and $H_2$ are not simple then it is usually straightforward to show they are conjugate when they have the same composition factors, by considering the centraliser of one of the simple factors.

Example 2.31. Let $G = E_8$ and let $M_1$ be the maximal-rank subgroup of type $A_1E_7$, and $M_2$ the maximal-rank subgroup of type $D_8$. Take $H_1 = A_1^2D_6 \subset M_1$ and $H_2 = A_1^2D_6 \subset D_8 = M_2$. Then $H_1$ and $H_2$ are $G$-irr and $G$-conjugate. Indeed, taking $Y$ to be one of the
Now allow us to concentrate connected subgroups of type $A$, where $\tilde{G}$ is a composition factor, hence it is $A$-irr. Now let $G = E_8$ and $p \neq 2$. Take $H = B_4 \subset A_8$, with $H$ acting irreducibly on the natural 9-dimensional module for $A_8$. Then $H$ is the centraliser in $G$ of an involution $t$ in the disconnected subgroup $A_8.2$. By calculating the trace of this involution on the adjoint module for $G$ and using [59, Prop. 1.2], we find that $C_G(t) = D_8$ and hence $X \subset D_8$. Similar calculations are carried out in [57, pp. 56–68].

### 2.2.3 Main results

We now present results classifying the $G$-cr semisimple subgroups of exceptional algebraic groups. Suppose $p$ is large enough so that all subgroups of a given type are $G$-cr. Then simple $G$-cr subgroups were classified in [57, 54]. The $G$-irr subgroups of type $A_1$ were studied for $G$ of exceptional type except $E_8$ in [1], and $G$-irr subgroups of $G = G_2, F_4$ are classified in [96], [99], respectively. The reductions in §1.4.1 now allow us to concentrate on semisimple, $G$-irr subgroups.

**Theorem 2.33.** Let $G$ be a simple algebraic group of exceptional type and let $H$ be a $G$-irr connected subgroup of $G$. Then $H$ is $\text{Aut}(G)$-conjugate to exactly one subgroup in Tables [105, §11, Tables 1–5] and each subgroup in the tables is $G$-irr.

This was proved in a sequence of papers [103, 104, 105]. The tables are lengthy and so we do not reproduce them here. The last of these papers describes the poset structure of $\text{Irr}(G)$: [105, §11, Tables 1A–5A]. A detailed explanation of the tables can be found at the start of §11 of loc. cit.. Furthermore, the tables provide the composition factors of each subgroup in $\text{Irr}(G)$ on both the minimal and adjoint module.

**Example 2.34.** When $G = G_2$, the reductive maximal connected subgroups of $G$ are:

$$A_1 \tilde{A}_1, \quad A_2, \quad \tilde{A}_2(p = 3), \quad \text{and} \quad A_1(p \geq 7),$$

where $\tilde{A}_1$ and $\tilde{A}_2$ denote subgroups whose roots coincide with short roots of $G$. There are no proper irreducible connected subgroups of $A_1$ so it requires no further consideration.

Take $M_1 = A_1 A_1$. Since the factors have the same type, there are diagonal $M$-irr connected subgroups of type $A_1$. As in Example 1.22, these are determined by non-negative integers $r, s$ with $rs = 0$ and we use the notation $H^{r,s}$ for such a subgroup. So $\text{Irr}(M_1) = \{H^{r,s} \mid rs = 0\}$. We now need to decide whether the members of $\text{Irr}(M_1)$ are $G$-irr. If $(r, s) \neq (0, 0)$ or $p > 3$ then $H^{r,s}$ acts on the adjoint module for $G$ without trivial composition factors, hence it is $G$-irr. When $p = 3$, the composition factors $H^{0,0}$ on $L(G)$ do not match those of a Levi subgroup and so is also $G$-irr. However, when $p = 2$, $H^{0,0}$ is contained in an $A_1$-parabolic subgroup and is non-$G$-cr (appearing in Theorem 2.40). To see this, it suffices to demonstrate that $H^{0,0}$ stabilises a 1-space on the irreducible 6-dimensional module $L_G(10)$, since by [65, Thm. B], $G$ is transitive on 1-spaces of $L_G(10)$ and the stabiliser of such a 1-space is a long root parabolic subgroup.

Now let $M_2 = A_2$. Applying Proposition 1.16, we find a single candidate, $H$, of type $A_1$ with $p \neq 2$, acting irreducibly on the adjoint 3-dimensional module $L(2)$. We must
check that $H$ is $G$-irr. In fact, one can show $H$ is conjugate to $H^{0,0}$ and thus $G$-irr. To see this, note that $N_G(M_2) = M_2(t)$, with $t$ an involution inducing a graph automorphism on $M_2$ [37, Table 4.3.1]. As $L(2)$ is self-dual, one concludes that $H$ centralises $t$ and hence $H \subset C_G(t) = M_1$. The only subgroups of type $A_1$ in $M$ are $A_1$, $A_1$ and $H^{r,s}$. Since $p \neq 2$, the composition factors of the action of these subgroups on $V_G(10)$ distinguish them and we conclude that $H$ is conjugate to $H^{0,0}$.

The same method applies to $\tilde{A}_2$ when $p = 3$ and one finds the single candidate subgroup $H$ of type $A_1$ is $G$-irr and conjugate to $H^{1,0}$.

We present this classification in Figure 2.1, with a straight line depicting containment. This gives a small flavour of the additional information in [105].

![Diagram](image)

**Figure 2.1**: The poset of $G_2$-irr connected subgroups.

**Remark 2.35.** Further information about the $G$-reducible semisimple $G$-cr subgroups can be found in [68]. In particular, the semisimple $G$-reducible $G$-cr connected subgroups of $G = F_4$ are classified and written down explicitly in [69, §6.4], correcting [99, Corollary 5]. For each reducible $G$-cr subgroup $H$, the socle series of the action of $H$ on $L(G)$ is computed together with the centraliser $C_G(H)$ (which is another $G$-cr subgroup by Theorem 1.9).

### 2.3 Non-$G$-completely reducible subgroups

The method in §1.4.2 has been applied to classify the non-$G$-cr semisimple subgroups of exceptional algebraic groups $G$ in many cases. The following is a more precise version of [57, Thm. 1]. It generalises Theorem 1.5 for exceptional groups by ruling out certain non-$G$-cr subgroups of certain types even when $p \leq \text{rank}(G)$.

**Theorem 2.36.** Let $G$ be a simple algebraic group of exceptional type in characteristic $p$. Let $X$ be an irreducible root system and let $N(X, G)$ be the set of primes defined by Table 2.3, e.g. $N(B_2, E_8) = \{2, 5\}$.

If $H$ is a connected reductive subgroup of $G$ and $p \notin N(X, G)$ whenever $H$ has a simple factor of type $X$, then $H$ is $G$-cr. Conversely, whenever $p \in N(X, G)$, there exists a non-$G$-cr simple subgroup $H$ of type $X$.

**Remark 2.37.** This corrects [97, Thm. 1], which had claimed the existence of non-$G$-cr subgroups $H$ of type $G_2$ when $p = 2$ and $G$ is of type $F_4$ or $E_6$. We discuss this further in
Table 2.3: Values of \( N(X,G) \).

| \( X \) | \( G = E_8 \) | \( E_7 \) | \( E_6 \) | \( F_4 \) | \( G_2 \) |
|---|---|---|---|---|---|
| \( A_1 \) | \( \leq 7 \) | \( \leq 7 \) | \( \leq 5 \) | \( \leq 3 \) | 2 |
| \( A_2 \) | \leq 3 | \leq 3 | \leq 3 | 3 |
| \( B_2 \) | 5 | 2 | 2 | 2 |
| \( G_2 \) | 7 | 3 | 2 | 7 | 2 |
| \( A_3 \) | 2 | 2 |
| \( B_3 \) | 2 | 2 | 2 | 2 |
| \( C_3 \) | 3 |
| \( B_4 \) | 2 |
| \( C_4, D_4 \) | 2 | 2 |

Example 2.39.

The result “\( p \not\in N(G,X) \Rightarrow \) all simple subgroups of type \( X \) are \( G \)-cr” is largely proven in [57]. The strategy is to show that for each parabolic subgroup \( P = QL \) and each \( L \)-irr subgroup \( \hat{X} < L \), the levels of \( Q \) restricted to \( \hat{X} \) have trivial 1-cohomology. In loc. cit. and [99] this is accomplished by determining all modules \( V \) for which \( H^1(\hat{X},V) \neq 0 \) such that \( \dim V \) is small enough for it to potentially appear as an \( \mathcal{D}(L) \)-composition factor in the filtration of \( Q \). For \( G \) of type \( E_8 \), the largest dimension of a such a \( \mathcal{D}(L) \)-composition factor is 64, when \( \mathcal{D}(L) \) is of type \( D_7 \), which makes this a tractable problem. (See [97, Lemma 5.1] for more information on the upper bounds on \( \dim V \).)

Example 2.38. Let us use results from §1.3.2 to exhibit a non-\( G \)-cr subgroup of type \( A_1 \) in \( G = E_6 \) when \( p = 5 \). Then \( G \) has a Levi subgroup \( L \) with \( \mathcal{D}(L) \) of type \( A_5 \). It follows from Theorem 1.12 that any non-\( L \)-cr subgroup is non-\( G \)-cr. So it suffices to find a non-\( L \)-cr subgroup of type \( A_1 \), which is equivalent to finding an indecomposable, reducible 6-dimensional module for \( SL_2 \). To that end, if \( V \cong L(1) \) is the natural module for \( SL_2 \), then the \( p \)-th symmetric power \( S^p(V) \) is indecomposable of dimension \( p + 1 \) with two composition factors, \( L(p) \) and \( L(0) \).

The following is a delicate case of showing that every simple subgroup of type \( X \) is \( G \)-cr when \( p \not\in N(X,G) \); this corrects an error in [99].

Example 2.39. Let \( G = F_4 \) and \( p = 2 \). We show that every subgroup of type \( G_2 \) is \( G \)-cr, and note that a similar but easier argument applies for \( G = E_6 \). The only proper Levi subgroups with a subgroup of type \( G_2 \) are those with derived subgroup \( B_3 \) or \( C_3 \). Write \( P = QL \) where \( \mathcal{D}(L) = B_3 \) and let \( H \) be the \( \mathcal{D}(L) \)-irr subgroup of type \( G_2 \). In [99, Lemma 4.4.3] it is claimed that \( H^1(G_2,Q) = k \), in turn using a claim that an element of \( (Q/Q_2)^H \) lifts to an element of \( Q^H \). However this is false. To see this, note that \( Q^H \subset C := C_G(H) \), which is \( G \)-cr and has rank 1. By the Borel–de-Siebenthal algorithm, \( H \subset \mathcal{D}(L) \) centralises the subgroup \( \hat{A}_1 \) of \( G \). Thus \( C \) has type \( A_1 \), and has no 2-dimensional unipotent subgroup, so \( \dim Q^H \leq 1 \). By restricting the \( \mathcal{D}(L) \)-action on \( Q_2 \) to \( H \), we find \( Q_2 \downarrow H \cong V(1,0) = L(1,0)/L(0,0) \), where \( V(1,0) \) is the Weyl module of high weight \((1,0)\) with the trivial module \( L(0,0) \) in its socle. So \( Q^H_2 \cong k \). As \( Q^H_2 \subset Q^H \), we have equality by comparing dimension. Therefore, no non-trivial element of \( (Q/Q_2)^H \) lifts to an element of \( Q^H \). One can now use the exact sequence \( (1.1) \) to see that \( Q \)-conjugacy kills the non-trivial elements of \( H^1(H,Q_2) \), and so \( H^1(H,Q) = 0 \).

Use of the graph morphism of \( G \) now allows one to conclude that the \( C_3 \)-parabolic also contains no non-\( G \)-cr subgroups of type \( G_2 \).
Moving on, let us assume \( p \in N(X,G) \). Then there exist non-\( G \)-cr semisimple subgroups with a factor of type \( H \), and we wish to classify them all. The case \( G = G_2 \) was settled by the second author in [96]. Since the classification is unusually short, we present it here. Recall that \( G_2 \) has semisimple maximal connected subgroups of type \( A_1 \overline{A}_1 \) and \( A_2 \) (see Theorem 2.27).

**Theorem 2.40.** Let \( G \) be a simple algebraic group of type \( G_2 \) in characteristic \( p \), and let \( H \) be a non-\( G \)-cr semisimple subgroup. Then \( p = 2 \), \( H \) has type \( A_1 \overline{A}_1 \) and is \( G \)-conjugate to precisely one of \( Z_1 \) and \( Z_2 \) below.

(i) \( Z_1 \subset A_1 \overline{A}_1 \) embedded diagonally via \( x \mapsto (x,x) \);
(ii) \( Z_2 \subset A_2 \) embedded via \( V(2) \cong L(2)/L(0) \).

This theorem exhibits an interesting feature: every non-\( G \)-cr subgroup has a proper reductive overgroup in \( G \). This is not true in general, and we consider semisimple subgroups with no proper reductive overgroups further in §2.3.1.

For the next result, recall that \( p \) is a good prime for \( G_2, F_4, E_6, E_7 \) if \( p > 3 \) and for \( E_8 \) if \( p > 5 \). The non-\( G \)-cr semisimple subgroups of \( F_4 \) were extensively studied by the second author in [99]; however the paper contains a number of errors. These are systemically dealt with in [35], from which we distil a headline result, presented together with the classification of non-\( G \)-cr semisimple subgroups in good characteristic due to the first and third authors.

**Theorem 2.41.** Let \( G \) be a simple algebraic group of exceptional type in characteristic \( p > 0 \) and let \( H \) be a non-\( G \)-cr semisimple subgroup of \( G \). Then one of the following holds:

(i) \( (G,p) = (G_2,2) \) and \( H \) is a subgroup \( Z_1 \) or \( Z_2 \) from Theorem 2.40;
(ii) \( (G,p) = (F_4,3) \) and \( H \) has type \( A_2, A_1 A_1 \) or \( *A_1 \);
(iii) \( (G,p) = (F_2,2) \) and \( H \) has type \( B_3, B_2, *A_1 B_2, *A_1 A_2 \) or \( *A_1^n \) with \( n \leq 3 \);
(iv) \( (G,p) = (E_6,5) \) and \( H \) has type \( A_1^2 \) or \( A_1 \);
(v) \( (G,p) = (E_7,5) \) and \( H \) has type \( A_2 A_1, A_1^2 \) or \( A_1 \);
(vi) \( (G,p) = (E_7,7) \) and \( H \) has type \( G_2 \) or \( A_1 \);
(vii) \( (G,p) = (E_8,7) \) and \( H \) has type \( A_1 G_2, A_1^2 \) or \( A_1 \);
(viii) \( G \) has type \( E_6, E_7 \) or \( E_8 \) and \( p \) is bad for \( G \).

Conversely, there is a non-\( G \)-cr semisimple subgroup of each type listed and infinitely many conjugacy classes for those marked *.

Case (viii) remains the subject of ongoing work. The particular case where \( H \) is semisimple with all factor of rank at least 3 will be the subject of a paper by the first and third authors. (The case where each simple factor has rank at least 2 is work in progress also.)

**Remark 2.42.** When \( p \) is good for \( G \), a non-\( G \)-cr semisimple subgroup \( H \) is \( G \)-conjugate to precisely one subgroup in [68, Tables 11–17] and conversely, each subgroup in those tables is non-\( G \)-cr. Furthermore, [68] also provides the connected centraliser of each subgroup and the action on minimal and adjoint modules.
2.3.1 Semisimple subgroups with no proper reductive overgroups

To describe the poset of connected reductive groups, one needs to describe the maximal elements. If $H$ is one such, then either it has a non-trivial central torus $S$, so $H = C_G(S)$ is a Levi subgroup; or $H$ is semisimple and non-$G$-cr.

**Example 2.43.** Let $(G, p) = (E_7, 7)$. We exhibit a non-$G$-cr subgroup of type $G_2$ which is maximal among proper reductive subgroups of $G$. In fact, this is unique up to conjugacy (see [68, §6.1]). Let $P = QL$ be parabolic subgroup of $G$, where the derived subgroup $D(L)$ has type $E_6$. This has a maximal subgroup of type $F_4$ which itself has a maximal subgroup $\bar{H}$ of type $G_2$ when $p = 7$ (see Theorem 2.27). This subgroup $\bar{H}$. The subgroup $\bar{H}$ turns out to be $D(L)$-irr ([103, Thm. 1]). Now the unipotent radical $Q$ is abelian, a 27-dimensional module for $D(L)$, which implies either $Q \cong L_{E_6}(\lambda_1)$ or its dual $L_{E_6}(\lambda_6)$. Moreover, $Q \downarrow \bar{H} = L(20) \oplus L(00)$, and $H^1(\bar{H}, L(20))$ is 1-dimensional. This implies $H^1(\bar{H}, Q) \cong k$. Furthermore, the torus $Z(L)$ acts by scalars on $Q$, inducing conjugacy among the non-zero elements of $H^1(\bar{H}, Q)$ and it follows that there is a unique $G$-conjugacy class $[\bar{H}]$ of non-$G$-cr subgroups of type $G_2$ complementing $Q$ in $Q\bar{H}$.

In [68, §10] it is proved that if $V$ is the 56-dimensional module for $E_7$ then $V \downarrow H = T(20)^2$ where $T(20) = L(00)/L(20)/L(00)$ is a tilting module of high weight 20. Any reductive overgroup of $H$ therefore acts on $V$ either indecomposably or with indecomposable summands of dimension $\text{dim} T(20) = 28$. Inspecting the maximal subgroups $G$ and their actions on $V$, we see the only plausible maximal reductive connected overgroup has type $A_7$. But the only non-trivial 8-dimensional $G_2$ modules are Frobenius twists of $L(10) \oplus L(00)$, and so any $G_2$ subgroup of $A_7$ is contained in a Levi factor $A_6$. But these act on $V$ with four indecomposable summands, which rules out $A_7$ as a reductive overgroup of $H$.

There are several more instances of non-$G$-cr semisimple subgroups with no proper reductive overgroup, which are thus maximal amongst connected reductive subgroups of $G$. The following partial result begins to extend Theorem 2.27 towards describing the classes of maximal connected reductive subgroups.

**Theorem 2.44.** Let $G$ be a simple algebraic group of exceptional type in characteristic $p$, and let $M$ be maximal among connected reductive subgroups of $G$. Then one of the following holds:

(i) $M$ is maximal among connected subgroups and is $G$-conjugate to precisely one subgroup in Table 2.1;

(ii) $M$ is a Levi subgroup, with $(G, \mathcal{O}(M)) = (E_6, D_5)$ or $(E_7, E_6)$;

(iii) $(G, p) = (F_4, 3)$ and $M$ is non-$G$-cr of type $A_1$;

(iv) $(G, p) = (F_4, 2)$ and $M$ is non-$G$-cr of type $B_2$ or $A_1A_1$;

(v) $(G, p) = (E_7, 7)$ and $M$ is non-$G$-cr of type $G_2$ (unique);

(vi) $G$ has type $E_6$, $E_7$ or $E_8$ and $p$ is bad for $G$.

**Remark 2.45.** Case (vi) will contain many conjugacy classes of subgroups. An interesting example is subgroup of type $A_2$ in $G = E_8$ which acts on the adjoint module with inde-

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*This follows, for example, by a dimension-shifting argument [19, II.2.1(4)] with the induced module $H^0(20) \cong L_{G_2}(00)/L_{G_2}(20)$. 

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composable summands of dimension 240 and 8, which already precludes its containment in a proper connected reductive subgroup.

2.4 Further directions and related problems

2.4.1 Hereditary subgroups

Recall from [93] and §1.3.1 that one of Serre’s reasons to formalise $G$-complete reducibility was for the study of converse theorems. An archetypal question [93, §5.3, Remarque] is:

When does the exterior square of a module being semisimple imply the original module is semisimple?

This can be translated into $G$-complete reducibility as follows. Let $H$ be a semisimple algebraic group, $V$ an $H$-module and suppose $\wedge^2(V)$ is semisimple. The action of $\text{SL}(V)$ on $\wedge^2(V)$ furnishes inclusions $\bar{H} \subseteq M \subseteq \text{SL}(\wedge^2(V))$, where $\bar{H}$ and $M$ are the images in $\text{SL}(\wedge^2(V))$ of the groups $H$ and $\text{SL}(V)$, respectively. Then the question above asks whether $H$ being $G$-cr implies that $H$ is $M$-cr.

**Definition 2.46.** Let $M$ be a connected subgroup of a reductive group $G$. We define $M$ to be $G$-ascending hereditary ($G$-ah) if, for all connected subgroups $H$ of $M$, if $H$ is $M$-cr then $H$ is $G$-cr. And $M$ is defined to be $G$-descending hereditary ($G$-dh) if, for all connected subgroups $H$ of $M$, if $H$ is $G$-cr then $H$ is $M$-cr. We define $M$ to be $G$-hereditary if it is both $G$-ah and $G$-dh.

Let $G$ be a simple algebraic group with subgroup $H \subset M$. Theorem 1.12 says that centralisers of linearly reductive subgroups of $G$, such as Levi subgroups, are $G$-hereditary. Theorem 1.14 gives criteria for a subgroup $M$ to be $G$-dh. We have also seen examples of non-hereditary subgroups. Theorem 2.40 shows that in $G = G_2$, the subgroup $H = Z_1$ is $M$-irreducible for $M = A_1 \tilde{A}_1$, but that $H$ is non-$G$-cr; so $M$ is non-$G$-ah. The following example shows non-$G$-dh subgroups also exist:

**Example 2.47.** Let $G = F_4$ and $p = 2$. Then $G$ contains subgroups $B_4$ and $C_4$, which respectively contain subgroups $D_4$ and $\tilde{D}_4$, and these in turn respectively contain simple subgroups $H_1$ and $H_2$ of type $G_2$. The simple module $L_{C_4}(\lambda_1) \downarrow H_1 = T(10)$, and the Weyl module $V_{B_4}(\lambda_1) \downarrow H_2 = T(10) \oplus 00$. These subgroups are swapped by the exceptional isogeny of $G$. By Proposition 1.16, $H_1$ is non-$C_4$-cr and thus $H_2$ is non-$B_4$-cr. However, in Example 2.39 we saw that every subgroup of type $G_2$ is $G$-cr. So $H_1$ and $H_2$ are examples of $G$-cr subgroups that are non-$M$-cr in some reductive maximal subgroup $M$ of $G$.

Given the plethora of results (§1.3.1, [5] and elsewhere) which guarantee $G$-hereditary behaviour, it is natural to ask how often such behaviour fails. Ongoing work of the first and third authors seeks more precise results in this vein. For instance, it is very uncommon for a subgroup $H$ in a reductive pair $(G, H)$ (cf. p. 13) to be non-$G$-dh, in fact based on empirical evidence we speculate:

**Conjecture 2.48.** Let $(G, H)$ be a reductive pair of algebraic groups in characteristic $p$. If $H$ is non-$G$-dh then $p = 2$.

2.4.2 Unipotent elements in exceptional algebraic groups

Much effort has been spent studying unipotent elements in algebraic groups. It is even non-trivial to show that there are finitely many of unipotent classes in the exceptional
groups. There is an extensive literature on this and the book [62] contains a comprehensive treatment. We mention those results most closely related to subgroup structure.

For many applications it is useful to understand how an embedding $H \to G$ fuses classes of unipotent elements. When $H$ is a reductive maximal connected subgroup this has been completely determined by Lawther in [53] (supplemented by [30] for the newly-discovered maximal subgroup of type $F_4$ in $E_8$ when $p = 3$).

One can flip this question and ask: given a unipotent element, what are its overgroups? For example, a regular unipotent element is one whose centraliser has the minimal possible dimension (the rank of $G$), and these are all $G$-conjugate. Overgroups of regular unipotent elements have been heavily studied, first of all by Saxl and Seitz in [88], classifying the maximal positive-dimensional reductive subgroups containing a regular unipotent element. Extending this to all positive-dimensional reductive subgroups containing a regular unipotent element was difficult. Suppose $H$ is a subgroup of $G$ containing a regular unipotent element. If $H$ is connected then Testerman–Zalesski [102, Thm 1.2] show that $H$ is $G$-irr. If instead $H$ is only assumed positive-dimensional, Malle–Testerman [72, Thm. 1] show that either $H$ is $G$-irr or $H^2$ is a torus; the latter case does in fact occur. These papers went much further, providing a full classification of the positive-dimensional reductive subgroups containing a regular unipotent element. Indicative of the narrative in Part I, Bate–Martin–Röhrle were able to produce a uniform proof of [102, Thm. 1.2] and [72, Thm. 1] (and further generalisations to disconnected groups, finite groups of Lie type and Lie algebras) without intricate case-by-case considerations. A key ingredient in their proof was the observation of Steinberg that regular unipotent elements normalise a unique Borel subgroup of $G$.

2.4.3 Finite subgroups and groups of Lie type

One of the main motivations for studying maximal and then reductive subgroups of exceptional algebraic groups was to deduce results for the exceptional finite groups of Lie type, see [60] for the work up to the early 2000s. Intimately related is the problem of understanding finite subgroups of the exceptional algebraic groups, whose study cannot employ any of the techniques requiring connectedness. In attempting to follow the strategy of §1.4, the primary difficulty is in classifying $G$-irr subgroups, since ad-hoc methods are required in place of uniform statements about representations of reductive groups. A result of Borovik [15] quickly reduces one to studying almost-simple finite subgroups, and the isomorphism types of simple subgroups have been enumerated by Cohen, Griess, Serre, Wales and others (e.g. [23, 15, 14]) over the complex numbers using character-theoretic methods, and by Liebeck and Seitz [59] in positive characteristic.

In positive characteristic, it is essential to understand generic subgroups, i.e. finite groups of Lie type $H(q)$ with embeddings $H(q) \to G$ which factor through an inclusion $H \to G$ of algebraic groups. The main result in this direction is due to Liebeck and Seitz [58], giving an explicit bound on $q$ (usually $q > 9$) ensuring that all embeddings $H(q) \to G$ arise in this fashion.

Further progress on non-generic subgroups was made by the first author [67] by comparing Brauer characters of finite simple groups with the Brauer traces of elements of exceptional algebraic groups $G$ on low-dimensional modules. The resulting tables limit the composition factors of simple groups acting on these modules. This carries sufficient information to rule out $G$-irr subgroups, for instance a subgroup fixing a vector on Lie$(G)$ lies in the corresponding stabiliser, which is often parabolic. One can also rule out the
existence of non-$G$-cr subgroups: For instance, if a subgroup is contained in a parabolic subgroup of some algebraic group $G$, then it normalises the unipotent radical, and the Lie algebra of this is a submodule of $\operatorname{Lie}(G)$. So if the subgroup has no composition factors on $\operatorname{Lie}(G)$ with non-zero cohomology, the subgroup cannot have non-zero cohomology in its action on the radical, so is $G$-cr. These techniques have been extended by Craven (e.g. [25]) to also consider the action of unipotent elements of finite groups.

For maximal subgroups of finite groups of Lie type, there has been considerable recent progress. The maximal subgroups of $^2B_2(q)$, $^2F_4(q)$, $^3D_4(q)$, $^2G_2(q)$ and $G_2(q)$ have already been classified by Cooperstein, Kleidman, Malle, Suzuki [24, 51, 70]. So the remaining cases were $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$. The maximal subgroups of these groups have been completely classified for some very small $q$ (e.g. $E_7(2)$ in [4]). The most recent work has led to a complete classification of the maximal subgroups of $F_4(q)$, $E_6(q)$ and $^2E_6(q)$ [26] and almost a complete classification for $E_7(q)$ [28]. This all builds on previous work [29, 27, 25]. The maximal subgroups of $E_8(q)$ are also a work in progress by Craven.

We round off our discussion here by mentioning one more way in which $G$-complete reducibility applies to finite groups of Lie type: Namely, through optimality. The details are somewhat technical (cf. [8, Def. 5.17]), but the upshot is: A non-$G$-cr subgroup $H$ is contained in a canonical parabolic subgroup of $G$, which is normalised by all automorphisms of $G$ which normalise $H$. Applying this to a Frobenius endomorphism $F$ of $G$, if $H$ is a non-$G$-cr finite subgroup of some group of Lie type $G(q) = G^F$, we get a method of constructing subgroups in between $H$ and $G(q)$, namely, $F$-fixed points of the corresponding canonical parabolic subgroup (cf. [66, Prop. 2.2] and [67, §4.1–4.2]).

### 2.4.4 Variations of complete reducibility

Serre’s definition 1.3 admits generalisations in various directions. If $G$ is equipped with a Frobenius endomorphism $\sigma$ and one considers only $\sigma$-stable parabolic and Levi subgroups, one arrives at so called ‘$\sigma$-complete reducibility’ [43]. In another direction, parabolic and Levi factors can be characterised as subsets of $G$ where along which certain morphisms $\mathbb{G}_m \to G$ have limits (see for instance [5, §2] for full details). This characterisation then extends to disconnected reductive groups $G$, and many results in $G$-complete reducibility generalise at once (cf. [5, §6]). The resulting geometric invariant theory is in fact the natural setting in which to derive the most general results, only a handful of which have been mentioned in the present article. Since one is now working with collections morphisms $\mathbb{G}_m \to G$, restricting these morphisms to land in a subgroup $K$ yields yet another generalisation, ‘$G$-complete reducibility with respect to $K$', and once again many natural results extend immediately [38, 2].

Ultimately, one can view complete reducibility as a property of the spherical building of $G$ [93]. Here, opposite parabolics are opposite simplices, and non-$G$-cr subgroups correspond to contractible subcomplexes; omitting all details, we simply mention that recasting the above results in terms of the building allows one to unite the various generalisations above and derive yet stronger statements, e.g. [39], and even extend to Euclidean buildings, Kac–Moody groups and other settings, see for instance [31] and [18, §4.3].

### 2.4.5 Structure of the Lie algebra of exceptional algebraic groups

Through the exponential and logarithm maps, classifying maximal connected subgroups of a complex algebraic group $G$ is equivalent to classifying maximal subalgebras of $\operatorname{Lie}(G)$
and indeed Dynkin’s original work is set in this context. It was noticed by Chevalley that any complex simple Lie algebra $g$ has a $\mathbb{Z}$-basis. This means there is an integral form $g_\mathbb{Z}$ from which one may build a Lie algebra $g_R := g_\mathbb{Z} \otimes \mathbb{Z} R$ over any ring $R$. In particular, we may take $R = k$ for $k$ an algebraically closed field of characteristic $p > 0$. There is typically more than one $\mathbb{Z}$-form available, leading to non-isomorphic Lie algebras over $k$; Chevalley’s recipe gives the simply-connected form, i.e. $g_k \cong \text{Lie}(G)$, where $G$ is the simply-connected algebraic group over $k$ of the same type ([45, Chap. VII]).

It natural to ask about maximal subalgebras of $g = g_k$ and more generally its subalgebra structure and their conjugacy under the adjoint action of $G$. Motivation for this question also arises from viewing $G$ as a scheme. In that context, one gets a much wider collection of subgroups, due to the presence of non-smooth subgroup schemes of $G$. At the most extreme end, a subgroup $H$ of $G$ is infinitesimal if its only $k$-point is the identity element; so $H$ is connected. The most natural non-trivial example of an infinitesimal subgroup is the first Frobenius kernel $G_1$ of $G$; one may view this as the functor from $k$-algebras to groups such such the $A$-points of $G_1$ applied to a $k$-algebra $A$ is the group $G_1(A) = \{ x \in G(A) \mid F(x) = 1 \}$, where $F$ is the (standard) Frobenius map on $G$.

Recall that a Lie algebra $g$ is restricted if it is equipped with a $[p]$-map $x \mapsto x^p$ which is $p$-semilinear in $k$ and satisfies $\text{ad}(x^p)(y) = \text{ad}(x)^p(y)$. A subalgebra $\mathfrak{h} \subseteq g$ is a $p$-subalgebra if it is closed under the $[p]$-map. There is an equivalence between $G_1$ and the Lie algebra $g$ in the following senses: Lie($G_1$) = $g$ is a restricted Lie algebra; any finite-dimensional restricted Lie algebra $\mathfrak{t}$ is Lie($K$) for a unique connected height-one group scheme $K$; under this correspondence, any $p$-subalgebra $\mathfrak{h}$ of $g$ maps to a unique subgroup $H$ of $G_1$; the finite-dimensional representation theory of $G_1$ is equivalent to the finite-dimensional restricted representation theory of $g$. For more on this, see the article of Brion from this volume.

Moving from the smooth subgroups of $G$ to the subalgebras of $g$ introduces many new and difficult problems. First, the classification of simple Lie algebras [84] is vastly more complicated than that of the algebraic groups, and is only complete when $p > 3$. Second, semisimple subalgebras are not the sums of simple Lie algebras, or even closely related to them [101, §3.3]. Third, it is not in general true that Theorem 1.2 has an analogue for $g$, and indeed maximal non-semisimple subalgebras of $G$ do not have to be parabolic—where a Lie subalgebra of $g$ is called parabolic if it is the Lie algebra of a parabolic subgroup. (Indeed, [101, p. 149] describes (all) irreducible representations of the soluble Heisenberg Lie algebras, most of which have dimensions divisible by $p$.)

Circumventing these problems in the case that $G$ is classical is wide open. But at least when $G$ is of exceptional type in good characteristic, these problems have been dealt with in [41], [83] and [82]. Roughly, the Liebeck–Seitz classification of maximal connected subgroups of $G$ holds, with certain exceptions. For example, for $p \geq 3$ the first Witt algebra $W_1 := \text{Der}(k[X]/X^p)$ is a simple Lie algebra of dimension $p$ and appears between $G$ and its regular $\mathfrak{sl}_p$ subalgebra whenever $p = h + 1$ and $h$ is the Coxeter number. There are some maximal semisimple Lie algebras when $G$ has type $E_7$ and $p = 5$ or 7, which have nothing to do with semisimple subgroups of $G$. We also point out that [82, Cor. 1.4] establishes an exact analogue of the Borel–Tits Theorem for exceptional Lie algebras in good characteristic.

One would like to consider the analogues for Lie algebras of the main problem addressed in this article. The following definition was given in [74] and developed in [7].

**Definition 2.49.** Let $g = \text{Lie}(G)$ for $G$ a reductive algebraic group. Then a subalgebra $\mathfrak{h}$ of $g$ is $G$-cr if whenever $\mathfrak{h}$ is in a parabolic subalgebra $\mathfrak{p} = \text{Lie}(P)$ of $G$, then $\mathfrak{h}$ is in a Levi
subalgebra \( \mathfrak{l} = \text{Lie}(L) \) of \( p \).

An analogue of Theorem 1.5 for Lie algebras (building on work in [42]) is given in [100, Thm. 1.3]:

**Theorem 2.50.** Let \( G \) be a connected reductive algebraic group in characteristic \( p \) with Lie algebra \( \mathfrak{g} \). Suppose \( \mathfrak{h} \) is a semisimple subalgebra of \( \mathfrak{g} \) and \( p > h \). Then \( \mathfrak{h} \) is \( G \)-cr.

In the case \( \mathfrak{h} \cong \mathfrak{sl}_2 \), the theorem interacts surprisingly closely with Kostant’s uniqueness result about the embeddings of nilpotent elements into \( \mathfrak{sl}_2 \)-subalgebras, which builds on the Jacobson–Morozov theorem [46, 78], which says that for any complex semisimple Lie algebra \( \mathfrak{g} = \text{Lie}(G) \), there is a surjective map

\[
\{\text{conjugacy classes of } \mathfrak{sl}_2\text{-triples}\} \longrightarrow \{\text{nilpotent orbits in } \mathfrak{g}\},
\]

where an \( \mathfrak{sl}_2 \)-triple is a triple \((e, h, f)\) satisfying \([h, e] = 2e, [h, f] = -2f, [e, f] = h\). The surjective map is induced by sending \((e, h, f)\) to the nilpotent element \( e \). So any such \( e \) can be embedded into some \( \mathfrak{sl}_2 \)-triple. In [52], Kostant showed that this can be done uniquely up to conjugacy by the centraliser \( G_e \) of \( e \); i.e. the map \((*)\) is actually a bijection.

Much work has been done on extending this important result into characteristic \( p > 0 \). We mention some critical contributions. In [80], Pommerening showed that under the mild restriction that \( p \) is a good prime, one can always find an \( \mathfrak{sl}_2 \)-subalgebra containing a given nilpotent element, but this may not be unique; in other words, the map \((*)\) is still surjective, but not necessarily injective. In [95] Springer and Steinberg prove that the uniqueness holds whenever \( p \geq 4h - 1 \) and in his book [20], Carter uses an argument due to Spaltenstein to reduce this bound to \( p > 3h - 3 \); both proofs make use of an exponentiation argument. One use of Theorem 2.50 is to prove the following.

**Theorem 2.51.** Let \( G \) be a connected reductive group in characteristic \( p > 2 \) with Lie algebra \( \mathfrak{g} \). Then \((*)\) is a bijection if and only if \( p > h \).

In fact, *op. cit.* also considers a map

\[
\{\text{conjugacy classes of } \mathfrak{sl}_2\text{-subalgebras}\} \rightarrow \{\text{nilpotent orbits in } \mathfrak{g}\},
\]

and when a bijection exists, realises it in a natural way. The equivalence of bijections \((*)\) and \((***)\) is easily seen in large enough characteristics by exponentiation, but there are quite a few characteristics where there exists a bijection \((***)\), but not \((*)\).

Further work has been done by Goodwin–Pengelly [36], characterising the subvariety of nilpotent elements where bijections \((*)\) and \((***)\) hold.
Bibliography

[1] Amende, Bonnie. 2005. *G-irreducible subgroups of type A1*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Oregon.

[2] Attenborough, Christopher, Bate, Michael, Gruchot, Maïke, Litterick, Alastair, and Röhrle, Gerhard. 2020. On relative complete reducibility. *Q. J. Math.*, 71(1), 321–334.

[3] Azad, H., Barry, M., and Seitz, G. 1990. On the structure of parabolic subgroups. *Comm. Algebra*, 18(2), 551–562.

[4] Ballantyne, John, Bates, Chris, and Rowley, Peter. 2015. The maximal subgroups of $E_7(2)$. *LMS J. Comput. Math.*, 18(1), 323–371.

[5] Bate, Michael, Martin, Benjamin, and Röhrle, Gerhard. 2005. A geometric approach to complete reducibility. *Invent. Math.*, 161(1), 177–218.

[6] Bate, Michael, Martin, Benjamin, and Röhrle, Gerhard. 2008. Complete reducibility and commuting subgroups. *J. Reine Angew. Math.*, 621, 213–235.

[7] Bate, Michael, Martin, Benjamin, Röhrle, Gerhard, and Tange, Rudolf. 2011. Complete reducibility and conjugacy classes of tuples in algebraic groups and Lie algebras. *Math. Z.*, 269(3-4), 809–832.

[8] Bate, Michael, Martin, Benjamin, Röhrle, Gerhard, and Tange, Rudolf. 2013. Closed orbits and uniform $S$-instability in geometric invariant theory. *Trans. Amer. Math. Soc.*, 365(7), 3643–3673.

[9] Bendel, C. P., Nakano, D. K., and Pillen, C. 2004. Extensions for Frobenius kernels. *J. Algebra*, 272(2), 476–511.

[10] Bendel, Christopher P., Nakano, Daniel K., Parshall, Brian J., Pillen, Cornelius, Scott, Leonard L., and Stewart, David. 2015. Bounding cohomology for finite groups and Frobenius kernels. *Algebr. Represent. Theory*, 18(3), 739–760.

[11] Borel, A., and Tits, J. 1971. Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I. *Invent. Math.*, 12, 95–104.

[12] Borel, A., and Tits, J. 1973. Homomorphismes “abstraits” de groupes algébriques simples. *Ann. of Math. (2)*, 97, 499–571.

[13] Borel, Armand. 1991. *Linear algebraic groups*. Second edn. Graduate Texts in Mathematics, vol. 126. New York: Springer-Verlag.

[14] Borovik, A. V. 1989a. Jordan subgroups of simple algebraic groups. *Algebra i Logika*, 28(2), 144–159, 244.
[32] Dowd, Michael F., and Sin, Peter. 1996. On representations of algebraic groups in characteristic two. *Comm. Algebra*, 24(8), 2597–2686.

[33] Dynkin, E. B. 2000. *Selected papers of E. B. Dynkin with commentary*. American Mathematical Society, Providence, RI; International Press, Cambridge, MA. Edited by A. A. Yushkevich, G. M. Seitz and A. L. Onishchik.

[34] Fulton, William, and Harris, Joe. 1991. *Representation theory*. Graduate Texts in Mathematics, vol. 129. New York: Springer-Verlag. A first course, Readings in Mathematics.

[35] Ganeshalingam, Vanthana, and Thomas, Adam R. On the non-completely reducible subgroups of $F_4$. in preparation.

[36] Goodwin, Simon M., and Pengelly, Rachel. On $\mathfrak{sl}_2$-triples for classical algebraic groups in positive characteristic. *Transform. Groups*, (to appear).

[37] Gorenstein, Daniel, Lyons, Richard, and Solomon, Ronald. 1998. *The classification of the finite simple groups. Number 3. Part I. Chapter A*. Mathematical Surveys and Monographs, vol. 40. American Mathematical Society, Providence, RI. Almost simple $K$-groups.

[38] Gruchot, Maike, Litterick, Alastair, and Röhrle, Gerhard. 2020. Relative complete reducibility and normalized subgroups. *Forum Math. Sigma*, 8, Paper No. e30, 32.

[39] Gruchot, Maike, Litterick, Alastair, and Röhrle, Gerhard. 2022. Complete reducibility: variations on a theme of Serre. *Manuscripta Math.*, 168(3-4), 439–451.

[40] Herpel, Sebastian. 2013. On the smoothness of centralizers in reductive groups. *Trans. Amer. Math. Soc.*, 365(7), 3753–3774.

[41] Herpel, Sebastian, and Stewart, David I. 2016a. Maximal subalgebras of Cartan type in the exceptional Lie algebras. *Selecta Math.*, 22(2).

[42] Herpel, Sebastian, and Stewart, David I. 2016b. On the smoothness of normalisers, the subalgebra structure of modular Lie algebras, and the cohomology of small representations. *Doc. Math.*, 21, 1–37.

[43] Herpel, Sebastian, Röhrle, Gerhard, and Gold, Daniel. 2011. Complete reducibility and Steinberg endomorphisms. *C. R. Math. Acad. Sci. Paris*, 349(5-6), 243–246.

[44] Humphreys, James E. 1975. *Linear algebraic groups*. New York: Springer-Verlag. Graduate Texts in Mathematics, No. 21.

[45] Humphreys, James E. 1978. *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, vol. 9. Springer-Verlag, New York-Berlin. Second printing, revised.

[46] Jacobson, Nathan. 1951. Completely reducible Lie algebras of linear transformations. *Proc. Amer. Math. Soc.*, 2, 105–113.

[47] Jantzen, J. C. 1991. First cohomology groups for classical Lie algebras. Pages 289–315 of: *Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991)*. Progr. Math., vol. 95. Basel: Birkhäuser.
[65] Liebeck, Martin W., Saxl, Jan, and Seitz, Gary M. 1996. Factorizations of simple algebraic groups. *Trans. Amer. Math. Soc.*, **348**(2), 799–822.

[66] Liebeck, Martin W., Martin, Benjamin M. S., and Shalev, Aner. 2005. On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function. *Duke Math. J.*, **128**(3), 541–557.

[67] Litterick, Alastair J. 2018. On non-generic finite subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, **253**(1207), v+156.

[68] Litterick, Alastair J., and Thomas, Adam R. 2018a. Complete reducibility in good characteristic. *Trans. Amer. Math. Soc.*, **370**(8), 5279–5340.

[69] Litterick, Alastair J., and Thomas, Adam R. 2018b. Reducible subgroups of exceptional algebraic groups. *Journal of Pure and Applied Algebra*.

[70] Malle, Gunter. 1991. The maximal subgroups of $^2F_4(q^2)$. *J. Algebra*, **139**(1), 52–69.

[71] Malle, Gunter, and Testerman, Donna. 2011. *Linear algebraic groups and finite groups of Lie type*. Cambridge Studies in Advanced Mathematics, vol. 133. Cambridge: Cambridge University Press.

[72] Malle, Gunter, and Testerman, Donna. 2021. Overgroups of regular unipotent elements in simple algebraic groups. *Trans. Amer. Math. Soc. Ser. B*, **8**, 788–822.

[73] McNinch, George J. 1998. Dimensional criteria for semisimplicity of representations. *Proc. London Math. Soc. (3)*, **76**(1), 95–149.

[74] McNinch, George J. 2007. Completely reducible Lie subalgebras. *Transformation Groups*, **12**(1), 127–135.

[75] McNinch, George J. 2014. Linearity for actions on vector groups. *J. Algebra*, **397**, 666–688.

[76] McNinch, George J., and Testerman, Donna M. 2007. Completely reducible SL(2)-homomorphisms. *Trans. Amer. Math. Soc.*, **359**(9), 4489–4510.

[77] Milne, J. S. 2017. *Algebraic groups. The theory of group schemes of finite type over a field*. Vol. 170. Cambridge: Cambridge University Press.

[78] Morozov, V. V. 1942. On a nilpotent element in a semi-simple Lie algebra. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, **36**, 83–86.

[79] Parker, Alison E. 2007. Higher extensions between modules for SL$_2$. *Adv. Math.*, **209**(1), 381–405.

[80] Pommerening, Klaus. 1980. Über die unipotenten Klassen reduktiver Gruppen. II. *J. Algebra*, **65**(2), 373–398.

[81] Prasad, Gopal, and Yu, Jiu-Kang. 2006. On quasi-reductive group schemes. *J. Algebraic Geom.*, **15**(3), 507–549. With an appendix by Brian Conrad.

[82] Premet, A., and Stewart, D. I. 2019. Classification of the maximal subalgebras of exceptional Lie algebras. *J. Amer. Math. Soc.*, to appear.
[83] Premet, Alexander. 2017. A modular analogue of Morozov’s theorem on maximal subalgebras of simple Lie algebras. *Adv. Math.*, 311, 833–884.

[84] Premet, Alexander, and Strade, Helmut. 2006. Classification of finite dimensional simple Lie algebras in prime characteristics. Pages 185–214 of: *Representations of algebraic groups, quantum groups, and Lie algebras*. Contemp. Math., vol. 413. Amer. Math. Soc., Providence, RI.

[85] Richardson, R. W. 1967. Conjugacy classes in Lie algebras and algebraic groups. *Ann. of Math. (2)*, 86, 1–15.

[86] Richardson, R. W. 1988. Conjugacy classes of n-tuples in Lie algebras and algebraic groups. *Duke Math. J.*, 57(1), 1–35.

[87] Riche, Simon, and Williamson, Geordie. 2021. A simple character formula. *Ann. H. Lebesgue*, 4, 503–535.

[88] Saxl, Jan, and Seitz, Gary M. 1997. Subgroups of algebraic groups containing regular unipotent elements. *J. London Math. Soc. (2)*, 55(2), 370–386.

[89] Seitz, Gary M. 1991. Maximal subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 90(441), iv+197.

[90] Serre, Jean-Pierre. 1994. Sur la semi-simplicité des produits tensoriels de représentations de groupes. *Invent. Math.*, 116(1-3), 513–530.

[91] Serre, Jean-Pierre. 1997. Semisimplicity and tensor products of group representations: converse theorems. *J. Algebra*, 194(2), 496–520. With an appendix by Walter Feit.

[92] Serre, Jean-Pierre. 1998. *Morsund lectures, University of Oregon*.

[93] Serre, Jean-Pierre. 2005. Complète réductibilité. *Astérisque*, Exp. No. 932, viii, 195–217. Séminaire Bourbaki. Vol. 2003/2004.

[94] Springer, T. A. 1998. *Linear algebraic groups*. Second edn. Progress in Mathematics, vol. 9. Boston, MA: Birkhäuser Boston Inc.

[95] Springer, T. A., and Steinberg, R. 1970. Conjugacy classes. Pages 167–266 of: *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*. Lecture Notes in Mathematics, Vol. 131. Springer, Berlin.

[96] Stewart, David I. 2010. The reductive subgroups of $G_2$. *J. Group Theory*, 13(1), 117–130.

[97] Stewart, David I. 2013a. Non-G-completely reducible subgroups of the exceptional algebraic groups. *International Mathematics Research Notices*.

[98] Stewart, David I. 2013b. On unipotent algebraic $G$-groups and 1-cohomology. *Trans. Amer. Math. Soc.*, 365(12), 6343–6365.

[99] Stewart, David I. 2013c. The reductive subgroups of $F_4$. *Mem. Amer. Math. Soc.*, 223(1049), vi+88.
[100] Stewart, David I., and Thomas, Adam R. 2018. The Jacobson-Morozov theorem and complete reducibility of Lie subalgebras. *Proc. Lond. Math. Soc. (3)*, **116**(1), 68–100.

[101] Strade, H. 2004. *Simple Lie algebras over fields of positive characteristic. I*. de Gruyter Expositions in Mathematics, vol. 38. Berlin: Walter de Gruyter & Co. Structure theory.

[102] Testerman, Donna, and Zalesski, Alexandre. 2013. Irreducibility in algebraic groups and regular unipotent elements. *Proc. Amer. Math. Soc.*, **141**(1), 13–28.

[103] Thomas, Adam R. 2015. Simple irreducible subgroups of exceptional algebraic groups. *J. Algebra*, **423**, 190–238.

[104] Thomas, Adam R. 2016. Irreducible $A_1$ subgroups of exceptional algebraic groups. *J. Algebra*, **447**, 240–296.

[105] Thomas, Adam R. 2020. The irreducible subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, **268**(1307), v+191.

[106] Vasiu, A. 2005. Normal, unipotent subgroup schemes of reductive groups. *C. R. Math. Acad. Sci. Paris*, **341**(2), 79–84.