GEODESICS WITH ONE SELF-INTERSECTION, AND OTHER STORIES

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Abstract. In this note we show that for any hyperbolic surface $S$, the number of geodesics of length bounded above by $L$ in the mapping class group orbit of a fixed closed geodesic $\gamma$ with a single double point is asymptotic to $L^{d_{\text{dim(Teichmuller space of } S)}}$. Since closed geodesics with one double point fall into a finite number of $M\wr \Delta(S)$ orbits, we get the same asymptotic estimate for the number of such geodesics of length bounded by $L$. We also use our (elementary) methods to do a more precise study of geodesics with a single double point on a punctured torus, including an extension of McShane’s identity to such geodesics.

In the second part of the paper we study the question of when a covering of the boundary of an oriented surface $S$ can be extended to a covering of the surface $S$ itself, we obtain a complete answer to that question, and also to the question of when we can further require the extension to be a regular covering of $S$.

We also analyze the question (first raised by K. Bou-Rabee) of the minimal index of a subgroup in a surface group which does not contain a given element. We show that we have a linear bound for the index of an arbitrary subgroup, a cubic bound for the index of a normal subgroup, but also poly-log bounds for each fixed level in the lower central series (using elementary arithmetic considerations) – the results hold for free groups and fundamental groups of closed surfaces.

Introduction

In this paper we consider a number of at first apparently unrelated questions.

First, we study the set of closed geodesics with a single self-intersection (Section 1). We show that any such geodesic is contained in an embedded pair of pants (Theorem 1) and use that observation together with estimates on the length of the geodesic in terms of the lengths of the boundary components of the pair of pants (Theorem 2) and results of M. Mirzakhani to show that the number of geodesics with a single self-intersection and length bounded by $L$ on a hyperbolic surface...
$S$ is \emph{asymptotic} to $L$ raised to the dimension of the Teichmuller space of $S$. As a side observation we find that on any hyperbolic surface any curve with one self-intersection has to have length at least $2 \arccosh 3$ (this is attained for a thrice punctured sphere.

In the case of the punctured torus, we succeed in extending McShane’s identity to the set of all geodesics with one self-intersection, thus:

$$\sum \left(1 - \sqrt{1 - \left(\frac{6}{l(y)}\right)^2}\right) = 2,$$

where the sum is taken over all the geodesics with a single self-intersection (and does not depend on the hyperbolic structure).

One approach to understanding self-intersecting curves is to lift to a cover where the curve is simple (that this is always possible is the subject of G. Peter Scott’s classic paper \cite{22}. We show that for geodesics with a single self-intersection a four-fold covering is always sufficient (see Section 3), but the method of proof raises the question on when a given covering of a collection of curves on a surface can be extended to a covering of the whole surface. We study these questions in Sections 4 and 6.

Finally, we are led to the following related question: It is well-known that free groups and fundamental groups of closed surfaces are residually finite. It is reasonable to ask, given an element $g$ of one of these groups $G$, what the index of a subgroup (normal or otherwise) of $G$ not containing $g$ is. We are able to obtain a number of upper bounds, as follows:

For free and surface groups, given an element $g$ of length $n$, there is a subgroup of index $O(n)$ not containing $g$. This is Theorem 14, which is proved by considering coverings.

For free groups, there is a normal subgroup of index exponential in $n$ which contains no element of length $n$. For surface groups, we are only able to bound the index by exponential of $O(n^2)$. This is the content of Theorems 16 and 34. Theorem 16 is the result for free groups, and uses the results of Lubotzky-Phillips-Sarnak on expansion of Cayley graphs of special linear groups over finite fields. Theorem 34 concerns surface groups, and uses the result of Baumslag that surface groups are residually free (this can be made quantitative, as was pointed out to the author by Henry Wilton).

For free and surface groups there is a normal subgroup of index bounded by $O(n^3)$ which does not contain $g$. This uses arithmetic representations of both free and surface groups and a little number theory. (Theorems 19, 24)
If $g$ lies at the $k$-th level of the lower central series, for both free and surface groups we get a bound $O(\log^k n)$ for the index of the normal subgroup not containing $g$. This is the content of Theorems 20 and 33. The former is proved by constructing unipotent representations, the second follows from the former via the residual freeness approach. The idea of considering the lower central series comes from the (highly recommended) paper of J. Malestein and A. Putman [14].

The above result may be put in context by Theorem 17, which states that (for free groups) the average index of a subgroup not containing a given element is smaller than 3 – the proof gives the same result for normal subgroups, but the rank has to be at least four in the normal case. The argument uses the results of the author on distribution of homology classes in free groups, but the argument can be easily tweaked to work for surface groups.

1. Geodesics with a single intersection on general hyperbolic surfaces

Consider a hyperbolic surface $S$ and a geodesic $\gamma$ with exactly one double point $p$. Let $\gamma_1$ and $\gamma_2$ be the two simple loops into which $\gamma$ is decomposed by $p$, so that in $\pi_1(S,p)$ we can write $\gamma = \gamma_1\gamma_2^{-1}$. It is easy to see that $\gamma_3 = (\gamma_1\gamma_2)^{-1}$ is freely homotopic to a simple curve, and indeed:

**Theorem 1.** The geodesic $\gamma$ is contained in an embedded pair of pants whose boundary components are freely homotopic to $\gamma_1, \gamma_2, \gamma_3$.

**Proof.** Since $\gamma_1, \gamma_2, \gamma_3$ admit disjoint simple representatives, it is standard (see, eg, Freedman, Hass, Scott[7]) that the geodesic representative are disjoint, and obviously bound a pair of pants. Since a pair of pants is a convex surface, it follows that $\gamma = \gamma_1\gamma_2^{-1}$ is contained in it. $\Box$

Theorem 1 establishes a bijective correspondence between closed geodesics with a single double point and embedded pairs of pants.

The next observation is:

**Theorem 2.** Let $P$ be a three-holed sphere with boundary components $a, b, c$. Let $\gamma$ be a closed geodesic in $P$ freely homotopic to $a^{-1}b$. Then, the length of $\gamma$ satisfies:

$$(1) \quad \cosh \ell(\gamma)/2 = 2 \cosh \ell(a)/2 \cosh \ell(b)/2 + \cosh \ell(c)/2.$$

**Proof.** By the Cayley-Hamilton Theorem,

$$A^{-1} = \text{tr}(A)I - A.$$ 

So,

$$A^{-1}B = \text{tr}(A)B - AB,$$
and so
\[(2) \quad \text{tr}(A^{-1}B) = \text{tr}(A) \text{tr}(B) - \text{tr}(AB) = \text{tr}(A) \text{tr}(B) - \text{tr}(C).\]

The result follows from this, the relationship between trace and the translation length and the not-quite-obvious fact that we can take \(A, B, C\) with traces negative (this follows from deep results of W. Goldman [8], but in this case we need simply to have one case easy to compute – the thrice-punctured sphere does nicely). \(\square\)

**Corollary 3.** A closed geodesic with a single double point on a hyperbolic surface \(S\) is no shorter than
\[2 \arccosh 3 = 2 \log(3 + 2 \sqrt{2}) \approx 3.52549\]
Such a geodesic is exactly of length
\[2 \arccosh 3\] if and only if \(S\) is a three-cusped sphere.

**Proof.** Since \(\cosh(x)\) is monotonic for \(x \geq 0\), the result follows immediately from Eq. (1). \(\square\)

We are only interested in the case where \(\ell(a), \ell(b), \ell(c)\) are large, in which case we can write
\[\ell(\gamma)/2 \approx \log(\exp((\ell(a) + \ell(b))/2) + \exp(\ell(c)/2)).\]

Now, the boundary of the pair of pants can be viewed as a multicurve on the surface \(S\) of length \(l(a, b, c) = \ell(a) + \ell(b) + \ell(c)\). Further, the three-component multicurves which bound pairs of pants fall into a finite number of mapping class group orbits.

Letting \(l(a, b) = \ell(a) + \ell(b)\), we see that
\[\ell(\gamma)/2 \approx \log(\exp(l(a, b)/2) + \exp(\ell(c)/2)).\]

Let \(l_1 = \max(l(a, b)/2), \ell(c)/2)\), and let \(l(a, b, c)/2 = (k + 1)l_1\), where \(k \leq 1\). Then,
\[l(a, b)/2 < \log(\exp(l(a, b)/2) + \exp(\ell(c)/2)) < l(a, b)/2 + \log(2).\]

So, for large \(l(a, b)\), we see that
\[\ell(\gamma)/l(a, b, c) \approx 1/(k + 1).\]

From this, and the results of Rivin ([20]) and Mirzakhani ([16], see also [21]) we see that the order of growth of the number of geodesics with a single double point is the same as the order of growth of an orbit of a multicurve. By the results of Mirzakhani [16] on the equidistribution of the mapping class orbit of a multicurve in measured lamination space, we get:

**Theorem 4.** The number of the geodesics of length not exceeding \(L\) in the mapping class group orbit of a geodesic with a single double point on \(S\) is asymptotic to
\[L^{\text{dimension of Teichmuller space of } S}.\]

**Corollary 5.** The number of geodesics with a single double point of length not exceeding \(L\) is asymptotic to
\[L^{\text{dimension of Teichmuller space of } S}.\]
Proof. The result is immediate from Theorem 4 and the observation that there is a finite number of mapping class orbits of geodesics with a bounded number of self-intersections. □

2. Punctured tori

The results of the previous section have particularly a particularly nice form when the surface \( S \) is a once-punctured torus. A punctured torus \( T \), can be cut along a simple closed geodesic \( \gamma \) into a sphere with two boundary components (each of length \( \ell(\gamma) \) and one cusp. This means that to each simple geodesic \( \gamma \) we can associate two geodesics \( \gamma_1 \) and \( \gamma_2 \) both of which have a single self-intersection, are of the same length, and if the translation corresponding to \( \gamma \) is \( A \) and that corresponding to \( \gamma_1 \) is \( B \), then, from Eq. (2),

\[
\text{tr}(B) = 3 \text{tr}(A).
\]

If \( \ell(\gamma) \) is large, Eq. (3) implies that \( \ell(\gamma_1) \approx \ell(\gamma) + 2 \log 3 \). Set \( N_0(L, T) \) be the number of simple geodesics of length bounded above by \( L \) on the punctured torus \( T \), and let \( N_1(L, T) \) be the number of geodesics with a single double point on the same torus, then.

**Theorem 6.**

\[
N_0(L, T) \sim N_1(L, T)/2.
\]

**Remark 7.** The same result holds when instead of a punctured torus we consider a torus with a geodesic boundary component.

In addition, we easily deduce an analogy of McShane’s identity ([15]) for curves with a single self-intersection. Recall that McShane’s identity states that on a punctured torus

\[
\sum \frac{1}{e^\ell(\gamma) + 1} = \frac{1}{2},
\]

where the sum is taken over all the simple geodesics on the punctured torus. McShane’s identity can be rewritten by using the trace \( t(\gamma) \) of the hyperbolic translation corresponding to \( \gamma \) as follows:

\[
\sum \left( 1 - \sqrt{1 - \left( \frac{2}{t(\gamma)} \right)^2} \right) = 1.
\]

Using this last form and Eq. (3), we obtain the “McShane’s identity for self-intersection”

\[
\sum \left( 1 - \sqrt{1 - \left( \frac{6}{t(\gamma)} \right)^2} \right) = 2,
\]
where the sum is taken over all geodesics with a single double point.

Remark 8. Combinatorial results on geodesics with a single self-intersection on a punctured torus have been obtained by D. Crisp and W. Moran in [4].

3. Removing intersections by covering

It is a celebrated result of G. Peter Scott [22] that for any hyperbolic surface \( S \) and any closed geodesic \( \gamma \), there is a finite cover \( \pi \tilde{S} \to S \), such that the lift \( \tilde{\gamma} \) to \( \tilde{S} \) is simple (non-self-intersecting). Since for any \( k \), there is a finite number of mapping classes of geodesics on \( S \) with no more than \( k \) self-intersections, and the minimal degree of \( \pi \) corresponding to a curve \( \gamma \) is invariant under the mapping class group, it follows that there exists some bound \( d_\gamma(k) \) so that one can “desingularize” any curve with up to \( k \) self-intersections by going to a cover of degree at most \( k \). Unfortunately, Scott’s argument appears to give no such bound.

The question of providing good bounds for \( d_\gamma(k) \) is, as far as I can say, wide open. Here we will attempt to start the ball rolling by giving sharp bounds for \( d_\gamma(1) \).

Our first observation is that if \( S \) is a three-holed sphere, then \( d_\gamma(1) = 2 \). To prove this we consider (without loss of generality) the case where \( S \) is a thrice cusped sphere (the quotient of the hyperbolic plane by \( \Gamma(2) \)). The proof then is contained in the diagram below. As can easily be seen, the lifts of two of the boundary components (say, \( A \) and \( B \)) are connected, and the lift of \( C \) has two connected components.

Now, suppose that \( \gamma \) is contained in a closed surface \( T \). Since \( \gamma \) has a three-holed sphere neighborhood \( S \), if the covering map described above extends to all of \( T \), then \( d_T(1) = 2 \). However, there are obviously examples where the map does not extend (if when \( T\setminus S \) has three connected components, or, more generally, when the boundary one of the components of \( T\setminus S \) has one connected component, the lift of which is connected – since the Euler characteristic of a surface with one boundary component is odd, such a surface is not a double cover. It is easy to see that in all other cases the double cover does extend). It is, however, clear, that a further double cover removes the obstruction, and we obtain the following result:

**Theorem 9.** For any oriented hyperbolic surface \( S \) and a geodesic \( \gamma \subset S \) with a single double point, there is a four-fold cover of \( S \) where \( \gamma \) lifts to a simple curve.

Remark 10. The result is not sharp for some surfaces with boundary (for example, the thrice punctured sphere). The result is vacuously true for the 2-sphere and the 2-torus equipped with metrics of constant curvature, since all the geodesics for those metrics are simple.
4. Extending covering spaces

The results of the previous section suggest the following question:

Question 11. Given an oriented surface $S$ with boundary $\partial S$, and a covering map of 1-manifolds $\pi : \tilde{C} \to \partial S$ the fibers of which have constant cardinality $n$. When does $\pi$ extend to a covering map $\Pi : \tilde{S} \to S$, where $\partial \tilde{S} \cong \tilde{C}$?

It turns out that Question 11 has a complete answer – Theorem 12 below (due, essentially to D. Husemoller [10]). First, we note that to every degree $n$ covering map $\sigma : X \to Y$ we can associate a permutation representation $\Sigma : \pi_1(Y) \to S_n$. Further, two coverings $\sigma_1$ and $\sigma_2$ are equivalent if and only if the associated representations $\Sigma_1$ and $\Sigma_2$ are conjugate (see, for example, [9][Chapter 1] for the details). This means that the boundary covering map $\pi$ is represented by a collection of $k = |\pi_0(\partial S)|$ conjugacy classes $\Gamma_1, \ldots, \Gamma_k$ in $S_n$, each of which is the conjugacy class of the image of the generator of the fundamental group of the corresponding component under the associated permutation representation.

Theorem 12. A covering $\pi : \tilde{C} \to C = \partial S$ extends to a covering of the surface $S$ if and only if the following conditions hold:

1. $S$ is a planar surface (that is, the genus of $S$ is zero and there exists a collection $\{\sigma_i\}_{i=1}^k$ of elements of the symmetric group $S_n$ with $\sigma_i \in \Gamma_i$ such that $\sigma_1 \sigma_2 \ldots \sigma_k = e$, where $e \in S_n$ is the identity.
2. $S$ is not a planar surface, and the sum of the parities of $\Gamma_1, \ldots, \Gamma_k$ vanishes.

Proof. In the planar case, the fundamental group of $S$ is freely generated by the generators $\gamma, \ldots, \gamma_{k-1}$ fundamental groups of (any) $k - 1$ of the boundary components. All $k$ boundary components satisfy $\gamma_1 \ldots \gamma_k = e$, whence the result in this case.

In the nonplanar case, let us first consider the case where $k = 1$. The generator $\gamma$ of the single boundary component is then a product of $g$ commutators (where $g$ is the genus of the surface, and so $\Sigma(\gamma)$ is in the commutator subgroup of $S_n$, which is the alternating group $A_n$, so the class $\Sigma(\gamma)$ has to be even. On the other hand, it is a result of O. Ore [18] that any even permutation is a commutator, $\alpha \beta \alpha^{-1} \beta^{-1}$ and thus sending some pair of handle generators to $\alpha$ and $\beta$ respectively and the other generators of $\pi_1(S)$ to $e$ defines the requisite homorphism of $\pi_1(S)$ to $S_n$.

If $k > 1$, the surface $S$ is a connected sum of a surface of genus $g > 0$ (by assumption) and a planar surface with $k$ boundary components. Let $\gamma$ be the “connected summing” circle. By the planar case, there is no obstruction to defining $\Pi$ on the planar side (since $\gamma$ is not part of the original data). However, $\Sigma(\gamma)$ will be the inverse of the product of elements $\sigma_i \in \Gamma_i$ and so its parity will be the sum of the
parities of $\Gamma_i$. To extend the cover to the non-planar side of the connected sum, it is necessary and sufficient for this sum to be even. □

Some remarks are in order. The first one concerns the planar case of Theorem 12. It is not immediately obvious how one might be able to figure out whether given some conjugacy classes in the symmetric group, there are representatives of these classes which multiply out to the identity. Luckily, there is the following result of Frobenius (see [23][p. 69])

**Theorem 13.** Let $C_1, \ldots, C_k$ be conjugacy classes in a finite group $G$. The number $n$ of solutions to the equation $g_1g_2\ldots g_k = e$, where $g_i \in C_i$ is given by

$$n = \frac{1}{|G|} \left| C_1 \right| \ldots \left| C_k \right| \sum_{\chi} \frac{\chi(x_1)\ldots \chi(x_k)}{\chi(1)^{k-2}},$$

where $x_k \in C_k$ and the sum is over all the complex irreducible characters of $G$.

Special cases of the planar case are considered in [6]; enumeration questions for covers are considered in a number of papers by A. Mednykh – see [11] and references therein.

The second remark is on Ore’s result that every element of $A_n$ is even. This result was strengthened by E. Bertram in [1] and, independently and much later, by H. Cejtín and the author in [3] (the second argument has the virtue of being completely algorithmic, the first, aside from being 30 years earlier, proves a stronger result) to the statement that every even permutation $\sigma$ is the product of two $n$-cycles (Bertram actually shows that it is the product of two $l$ cycles for any $l \geq (M(\sigma) + C(\sigma))/2$, where $M(\sigma)$ is the number of elements moved by $\sigma$ while $C(\sigma)$ is the number of cycles in the cycle decomposition of $\sigma$.

The significance of this to coverings is that we have a very simple way of constructing a covering of a surface with one boundary component with specified cycle structure of the covering of the component, as follows.

First, the proof of Theorem 12 shows that the construction reduces to the case where $g = 1$, so that we are constructing a covering of a torus with a single perforation.

Suppose now that the permutation can be written as $\sigma \tau \sigma^{-1} \tau^{-1}$, where $\sigma$ is an $n$-cycle. This means that the “standard” generators of the punctured torus group go to $\sigma$ and $\tau$, respectively. To construct the cover, then, take the standard square fundamental domain $D$ for the torus (the puncture is at the vertices of the square), then arrange $n$ of these fundamental domains in a row, and then a strip, by gluing the rightmost edge to the leftmost edge. Then, for each $i$, the upper edge of the $i$-th domain from the left ($D_i$) is glued to the lower edge of $D_{\tau(i)}$. 

In an upcoming joint paper with Manfred Droste we extend the results of this section to infinite covers.

5. Quantifying residual finiteness

Khalid Bou-Rabee in [2] has analyzed the following question: given a residually finite group $G$ and an element $g \in G$, how high an index subgroup $H < G$ must one take so that $g \notin H$, in terms of the word-length of $g$. Bou-Rabee answers the questions for important classes of groups, including arithmetic lattices and nilpotent groups. Here we wish to point out that for surface groups (including free groups) we have the following bound:

**Theorem 14.** Given an element $g \in G$ of word length $l(g)$, there is a subgroup $H$ of $G$ of index of order $O(l(g))$, such that $g \notin H$.

**Proof.** Let $F$ be a surface such that $\pi_1(F) = G$. It is obviously equivalent to construct a cover $\tilde{F}$ of the surface $F$ whose fundamental group is $H$, such that $g$ does not lift to $\tilde{F}$. There are two cases. The first is when the geodesic $\gamma(g)$ in the conjugacy class of $g$ is simple. In that case there are two further cases: the first arises when $\gamma(g)$ is homologically nontrivial. In that case, there is a geodesic $\beta$ transversely intersecting $\gamma$ in one point. Cutting $F$ along $\beta$ and then doubling gives us a double cover where $g$ does not lift. The second case is when $\gamma(g)$ bounds. In that case, cut along $\gamma(g)$ to obtain two surfaces with boundary. each of them admits a (connected) cover which restricts to a triple connected cover over $\gamma(g)$. Gluing along this cover, we obtain a cover of $F$ where $g$ does not lift.

The second case is when $g$ is not simple. In this case, an examination of G. P. Scott’s argument in [22] shows that there is a cover $\tilde{F}$ of $F$ of index linear in the word length of $g$ where the lift of $g$ is a simple. The first case analyzed above then completes the argument. \qed

The usual definition of residual finiteness is the following: a group $G$ is residually finite, if for every $g \in G$ there is a homomorphism $\psi_g : G \to H$, where $H$ is finite and such that $\psi_g(g) \neq e$. In other words, it postulates the existence of a normal subgroup of finite index (ker $\psi_g$) which does not contain $g$. Now, since every subgroup of index $k$ in an infinite group $G$ contains a normal subroup of index $k!$ (index in $G$, that is) the two points of view on residual finiteness are logically equivalent if we don’t care too much about the index. If we do, note that Theorem 14 gives us the following Corollary:

**Corollary 15.** Let $G$ be a surface group. Given an element $g \in G$ of word length $l(g)$, there is a normal subgroup $H$ of index at most $(c(l(g)))!$ which does not contain $g$.

**Corollary 15** can be improved considerably for free groups:
**Theorem 16.** Consider the free group on \( k \) letters \( F_k \) and let \( n > 1 \). There exists a normal subgroup \( H_n \) of \( F_k \) of index \( f(n) \) which contains no non-trivial elements of word length smaller than \( n \), where the index \( f(n) \) can be bounded by

\[
f(n) \leq c(2k - 1)^{3n/4}.
\]

for some constant \( c \).

**Proof.** We first note that if we have a homomorphism \( \phi \) of \( F_k = \langle a_1, \ldots, a_k \rangle \) onto a finite group \( H \) with the Cayley graph \( C_H \) of \( H \) with respect to the generating set \( \phi(a_1), \ldots, \phi(a_k) \), then no word in \( F_k \) shorter than the girth of \( C_H \) is in the kernel of \( \phi \).

We now use the following result of Lubotzky, Phillips, and Sarnak ([12], see [5] for an expository account): or \( p, q \) prime, with \( p \geq 5 \) and \( q \gg p \) and \( p \) a quadratic non-residue mod \( q \) there is a symmetric generating set \( S \) of the group \( \text{PSL}(2, q) \) of order \( p + 1 \) such that the Cayley graph of \( \text{PSL}(2, q) \) has girth no smaller than

\[
4 \log_p q - \log_p 4.
\]

It follows no element of \( F_{(p+1)/2} \) of length shorter than \( n(p, q) = 4 \log_p q - \log_p 4 \) is killed by the homomorphism \( \phi_q \) that sends the free generators of \( F_{(p+1)/2} \) and their inverses to \( S \). Since the order of \( \text{PSL}(2, q) \) has order \( m_q q(q^2 - 1)/2 \sim q^3/2 \), which we can write down as

\[
m_q = (4p^{n(p,q)})^{3/4} = 2^{3/2} n(p, q).
\]

If \( k \neq (p + 1)/2 \) for some prime \( p \), we can find a subgroup of small index in \( F_k \) which is a free group on \( (p + 1)/2 \) letters for some prime \( p \). Using Dirichlet’s theorem on primes in arithmetic progressions we can then find a suitable \( q \). \( \square \)

Theorem 14 can be combined with the results of [19] to obtain the following result:

**Theorem 17.** Consider the set \( B_N \) of all elements in the free group \( F_k \) having length no more than \( N \) in the generators. Then the average index of the subgroup not containing a given element over \( B_N \) is bounded above by a constant (which can be taken to be approximately 2.92).

**Proof Sketch.** Theorem 14 together with the results of [19] reduce the question to the same question, but with \( F_k \) replaced by \( \mathbb{Z}^k \). Consider an element \( x = (x_1, \ldots, x_k) \in \mathbb{Z}^k \). The element \( x \) is not contained in \( p \mathbb{Z} \times \mathbb{Z}^{k-1} \) if \( p \) does not divide \( x_1 \). The result now follows by the Lemma 18 below. \( \square \)

**Lemma 18.** For every \( n \) define \( p(n) \) to be the smallest prime which does not divide \( n \). Then the expectation of \( p(n) \) over all \( n < N \) converges to \( c = 2.902 \ldots \) as \( N \) tends to infinity.
Proof. For a fixed $p$, the probability that $p(n) = p$, for some $n < N$ is given by
\[ \frac{p - 1}{p} \prod_{q < p} q, \]
where the product is over all the primes smaller than $p$, and so the expectation of $p(n)$ is given by:
\[ E(p) = \sum_{p} (1 - p) / \prod_{q < p} q, \]
and the latter sum converges rapidly to 2.92005...

We note that the above argument does not work if we replace the words index of subgroup by index of normal subgroup, since in that case we don’t have the necessary control over the commutator subgroup. The bound given by Theorem 16 is certainly not good enough – we need a uniform estimate on the index of the normal subgroup of order $o(n^k)$.

We can get an estimate of the right type as follows:

**Theorem 19.** Let $w$ be represented by a word of length $n$ in $F_2$. There exists a prime $p \leq cn$ such that $w$ is not in the kernel of the homomorphism of $F_2$ to $\text{SL}_2(\mathbb{Z}/p\mathbb{Z})$ which maps the generators of $F_2$ to $g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ respectively.

**Proof.** First, consider $g$ and $h$ to be elements of $\text{SL}_2(\mathbb{Z})$. The matrix given by the word $w(g, h)$ has entries no larger than $2^n$ in absolute value. By the Chinese Remainder Theorem, if $p_1, \ldots, p_k$ are primes whose product exceeds $2^n$, any such matrix which is the identity modulo all of the $p_i$ is, in fact, the identity matrix, and thus the word $w$ is the trivial element of $F_2$ (since $g$ and $h$ lie in the principal congruence subgroup of level 2 in the modular group, and that principal congruence subgroup is free). By the prime number theorem, the product of all the primes not exceeding $m$ is asymptotic to $e^m$ for $m$ large, whence the result.

The method of proof of Theorem 19 is quite suggestive. For example, consider a word $w$ of length $n$ in $F_2 = \langle a, b \rangle$ where the sum of the exponents of all the terms is not zero. Then, by mapping both $a$ and $b$ to $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we see that the elements of $w(g, g)$ are no bigger than $n$, and so such a word is not in the kernel of a homomorphism of $F_2$ into a group of order at most $\log n$. If the exponent sum of $w$ is zero, but the individual exponent sums of $a$ and $b$ are not, we can modify the construction by sending $a$ to $g$ (as above) but sending $b$ to $g^2$. This will give us the same result up to an additive constant. If the individual exponent sums are zero, the method fails, but the element $w$ is in the commutator subgroup (so $w$ is in...
the kernel of every homomorphism of $F_2$ to an abelian group). However, we can replace the abelian group by a 2-step nilpotent group of $(3 \times 3)$ unipotent matrices. Using Lemma 22 below the proof of Theorem ?? shows that $w$ is in the second level of the lower central series of $F_2$, it is not in the kernel of a homomorphism to a group of order $\log^2(n)$, where $n$ is the length of $w$, and so on. We thus obtain the following statement which we believe sharp:

**Theorem 20.** For every element $w$ of length $n$ at the $k$-th level in the lower central series of $F_2$, there is a normal subgroup $H(w)$ of index $O(\log^k(n))$ which does not contain $w$. The subgroup $H(w)$ is the kernel of a homomorphism onto a $k$-step nilpotent subgroup represented by $(k + 1) \times (k + 1)$ unipotent matrices.

**Remark 21.** Since $F_n$ is a finite index subgroup of $F_2$ (for any $n \geq 2$) we could have replaced $F_2$ by $F_n$ in the statement of Theorem 20.

We have used the following lemma:

**Lemma 22.** Let $m_1, \ldots, m_k$ be $n \times n$ unipotent matrices. Let $M = w(m_1, \ldots, m_k)$, where $w$ is a word of length $m$. Then, the entries of $M$ grow no faster than $O(n^{m-1})$, where the implicit constant in the $O$-notation depends only on the matrices $m_1, \ldots, m_k$.

**Proof.** Write each $m_i$ as $m_i = I_n + m'_i$. The matrices $m'_1, \ldots, m'_k$ generate a nilpotent ideal $I$, where $I^n = 0$. This means that $w(m_1, \ldots, m_k)$ can be written as sum of $j$-fold products of $m'_i$, where $j$ ranges between 0 and $n - 1$. The number of $j$-fold products is at most $\binom{n}{j} = O(n^j)$, whence the result. \qed

The remaining question, then, is: how deeply in the lower central series can an element of word-length $n$ be? It is clear that a word of length $n$ cannot have depth greater than $n$, so $k = O(n)$. It has been proved by J. Malestein and A. Putman in [14] that $k = \Omega(\sqrt{n})$.

### 5.1. Surface Groups.

The first observation is that the proof of Theorem 19 does not depend on the fact that the group is free, but only on the existence of representations of the group into $SL(2, (O))$, where $(O)$ is the ring of integers in some algebraic number field – instead of the prime number theorem we then use Landau’s Prime Ideal Theorem (see, eg, [17]) to get exactly the same estimate. To show that every surface group admits such a representation we use a (stronger) observation of C. Machlachlan and A. Reid (see [13]):

**Observation 23.** The Picard Modular Group – $PSL(2, Z[i])$ – contains the fundamental group of the compact surface of genus 2, and thus the fundamental group of every compact surface.

This gives us the following version of Theorem 19:
**Theorem 24.** Let $w$ be represented by a word of length $n$ in the fundamental group $\Gamma_g$ of the surface of genus $g$. There exists a normal subgroup of order $O(n^3)$ which does not contain $w$.

The arithmetic method does not (at least not obviously) extend the proofs of Theorems 20 and 16 to the case of surface groups. To extend these results we note that we need only extend these results to the fundamental group $\Gamma_2$ of the surface of genus 2. This is so because of the following observations:

**Observation 25.** Every surface group $\Gamma_g$ has a subgroup of finite index of $\Gamma_2$ (in multiple ways, but we pick a fixed (eg, cyclic) covering for each $g$).

**Observation 26.** By word-hyperbolicity, there exists a constant $c$, such that every word of length $l$ in $\Gamma_g$ has word length between $cl$ and $l/c$ in $\Gamma_2$. In fact, it is not hard to show that for the cyclic cover, the constant $c$ can be taken to be $g-1$ -- it can be shown that with a more judicious choice of covering this can be improved to $O(\log g)$.

**Observation 27.** If $g \in \Gamma_g$ lies at the $k$-th level of the lower central series of $\Gamma_g$, then it lies in at most the $k$-th level of the lower central series of $\Gamma_2$. This is a more-or-less immediate corollary of the definition.

**Observation 28.** Let $H$ be a subgroup of finite index $k$ in $\Gamma_2$. Then $H \cap \Gamma_g$ is of index at most $k$ in $\Gamma_g$. This is a standard exercise.

To deal with $\Gamma_2$, we use the following method, suggested by Henry Wilton: Write $\Gamma_2$ as $\langle a, b, c, d \mid [a, b] = [c, d] \rangle$. There is a standard map of $r : \Gamma_2 \to F_2 = \langle a, b \rangle$, with $r(a) = r(c) = x$, and $r(b) = r(d) = y$. The retraction $r$ has the obvious property of not decreasing the lower central series depth, but it does have the unfortunate property of having a nontrivial kernel. However, this can be dealt with by using the following result, attributed by H. Wilton to G. Baumslag:

**Lemma 29** ([24][Lemma 4.13].) Let $F$ be a free group, let $z \in F$, $z \neq 1$, and let

$$g = a_0 z_i^{a_1} \ldots z_i^{a_n},$$

with $a_1, \ldots, a_n \in F$. Assume further that $n \geq 1$ and $[a_k, z] \neq 1$ for $0 < k < n$. Then, if for every $1 \leq k \leq n$ it is true that

$$|z_i| \geq |a_{k-1}| + |a_k| + |z|,$$

then $g$ does not represent the trivial word in $F$.

To use Lemma 29 we introduce the Dehn Twist automorphism $\phi$ of $\Gamma_2$, which is defined by $\phi(a) = a, \phi(b) = b, \phi(c) = c[a,b], \phi(d) = c[a,b]$, and recall the following easy fact:
Lemma 30. Consider an element $x$ in the fundamental group $G$ of a compact surface $S$. Then $[x, y] = 1$ if and only if there exists an element $w$ and integers $k, l$ such that $x = w^k$, $y = w^l$.

Proof. Represent $G$ as a Fuchsian group of isometries of $\mathbb{H}^2$. Then, $x$ and $y$ are isometries of $\mathbb{H}^2$. Since $S$ is a compact manifold, both $x$ and $y$ are hyperbolic elements, and since they commute, they have the same axis. Since the group $G$ is discrete, the translation distances of $x$ and $y$ are commensurate, and so $x = \gamma^m$ and $y = \gamma^n$ for some translation $\gamma$ (which is not necessarily in $G$.) However, by the obvious application of the Euclidean algorithm, $\beta = \gamma^{(m,n)} \in G$, and $x = \beta^{m/(m,n)}$, and $y = \beta^{n/(m,n)}$, so $k = m/(m,n), l = n/(m,n)$ and $w = \beta$ as stated.

Now we are ready to prove the key observation

Theorem 31. Let $g \in \Gamma_2$, $g \neq 1$. Then $r(\phi^{l(g)/4}(g)) \neq 1$, where $l(g)$ is the minimal length of $g$ in terms of the generating set $\{a, b, c, d\}$.

Proof. Let $w(g)$ be a shortest word in $a, b, c, d$ representing $g$. First, write $w(g) = L_1 R_1 \ldots L_k R_k$, where $L_i$ are blocks of $as$ and $bs$ and $R_i$ are blocks of $bs$ and $cs$. Further, let $z_1 = [a, b]$ and $z_2 = [c, d]$ (these represent the same element $z$ of $\Gamma_2$, but we think of them as words for now.) Now, apply the following rewriting process (see Algorithm 5.1) to $w(g)$:

First, if any $L_i$ is a power of $z_1$, replace it by the same power of $z_2$. Second, if any $R_j$ is a power of $z_2$, replace it by the same power of $z_1$. Then repeat the two steps until neither can be applied. Note that this process will terminate eventually, since each steps reduces the total number of blocks (in fact, it reduces both the number of $L$ blocks and the number of $R$ blocks). Call the resulting word $w_0(g)$ ($|w_0(g)| = |w(g)|$, since $w(g)$ was assumed minimal). By abuse of notation, let $w_0(g) = L_1 R_1 \ldots L_k R_k$ (where the $k$ might be different from the $k$ in $w(g)$). Note that applying the automorphism $\phi^n$ to $g$ replaces each occurrence of a block $R_j$ in $w_0$ by $z_1^{-m} R_j z_1^m$, and hence applying $r \circ \phi^n$ to $g$ maps $g$ to the word $w_0(g) = L_1 z_1^{-m} r(R_1) z_1^m L_2 z_1^{-m} r(R_2) z_1^m \ldots L_k z_1^{-m} r(R_k) z_1^m$. By construction of $w_0$ and Lemma 30, the hypotheses of Lemma 29 hold, as long as $4(m-1) \geq |w_0(g)|$.

Lemma 32. The length of $r(\phi^{l(g)/4}(g))$ is bounded above by $l(g)^2 + l(g)$.

Proof. Computation.

As a corollary of Theorem 31 and Lemma 32, we have the following extensions of Theorems 20 and 16 respectively:

Theorem 33. For every element $w$ of length $n$ at the $k$-th level in the lower central series of the fundamental group of a closed surface of genus $g$ there is a normal subgroup $H(w)$ of index $O(\log^k(n))$ which does not contain $w$. 
Algorithm 1 Rewriting Algorithm

1: loop
2: if \( w(g) = z_1^n \) then
3: Return \( w(g) \).
4: else if Any \( L_i \) is a power of \( z_1 \), so that \( L_i = z_1^{m_i} \) then
5: Replace \( L_i \) by \( z_1^{m_i} \).
6: else if any \( R_j \) is a power of \( z_2 \), so that \( R_j = z_2^{n_j} \) then
7: Replace \( R_j \) by \( z_2^{n_j} \).
8: end if
9: end loop

Theorem 34. Consider the fundamental group \( \Gamma_k \) of a surface of genus \( g \), and let \( n > 1 \). There exists a normal subgroup \( H_n \) of \( \Gamma_k \) of index \( f(n) \) which contains no non-trivial elements of word length smaller than \( n \), where the index \( f(n) \) can be bounded by

\[
f(n) = O(g^{O(n^2)}).
\]

I do not expect that the bound in the statement of Theorem 34 is close to sharp (but it is a bound).

6. Regular coverings

D. Futer asked whether the results of the previous section had analogues when the covering given by \( \Pi \) was additionally required to be regular. This seems to be a hard question in general. For example, in the case where \( S \) is a planar surface with \( k \) boundary components, we have the following result:

Theorem 35. In order for a covering of the boundary of \( S \) to extend to a regular covering of \( S \), it is necessary and sufficient that, in addition to the requirements of Theorem 12 (part 1), there must be a subgroup \( G < S_n \) with \( |G| = n \) and \( G \) is generated by \( \gamma_1, \ldots, \gamma_k \), where \( \gamma_i \in \Gamma_i \), for \( i = 1, \ldots, k \).

As far as the author knows, there is no particularly efficient way of deciding whether the condition of Theorem 35 is satisfied.

Here is a more satisfactory (in not very positive) result:

Theorem 36. Let \( S \) be a surface with one boundary component. There does not exist a nontrivial covering of finite index \( \Pi : \tilde{S} \to S \) where \( \tilde{S} \) also has one boundary component.

Proof. Let the degree of the covering be \( n \). If \( \tilde{S} \) has one boundary component, the generator of the \( \pi_1(\partial S) \) gives rise to the cyclic group \( \mathbb{Z}/n\mathbb{Z} \), which is a subgroup of the deck group of \( \Pi \). Since the deck group has order \( n \) (by regularity), the
covering is cyclic (so that the deck group is, in fact, \( \mathbb{Z}/n\mathbb{Z} \)). A cyclic group is abelian, and since the generator of \( \pi_1(\partial S) \) is a product of commutators, it is killed by the map \( \Sigma \). But this contradicts the statement of the first sentence of this proof (that this same element generates the entire deck group).

\[ \square \]

The proof also shows the following:

**Theorem 37.** Let \( \Pi : \tilde{S} \to S \), where \( S \) has a single boundary component, be an abelian regular covering of degree \( n \). Then \( \tilde{S} \) has \( n \) boundary components.

We can combine our results in the following omnibus theorem:

**Theorem 38.** Let \( S \) be a surface, whose boundary has \( k \) connected components. Let the conjugacy classes of the \( k \) coverings be \( \Gamma_1, \ldots, \Gamma_k \). In order for a constant cardinality \( n \) covering of \( \partial S \) to extend to a regular covering of \( S \) it is necessary and sufficient that there be elements \( \gamma_1 \in \Gamma_1, \ldots, \gamma_k \in \Gamma_k \) such that \( \gamma_1, \ldots, \gamma_k \) generate an order \( n \) subgroup of \( S_n \) and \( \gamma_1 \cdots \gamma_k = e \).

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