Fluctuations of eigenvalues and second order Poincaré inequalities

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Abstract Linear statistics of eigenvalues in many familiar classes of random matrices are known to obey gaussian central limit theorems. The proofs of such results are usually rather difficult, involving hard computations specific to the model in question. In this article we attempt to formulate a unified technique for deriving such results via relatively soft arguments. In the process, we introduce a notion of ‘second order Poincaré inequalities’: just as ordinary Poincaré inequalities give variance bounds, second order Poincaré inequalities give central limit theorems. The proof of the main result employs Stein’s method of normal approximation. A number of examples are worked out, some of which are new. One of the new results is a CLT for the spectrum of gaussian Toeplitz matrices.

Keywords Central limit theorem · Random matrices · Linear statistics of eigenvalues · Poincaré inequality · Wigner matrix · Wishart matrix · Toeplitz matrix

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1 Introduction

Suppose $A_n$ is an $n \times n$ matrix with real or complex entries and eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated by multiplicities. A linear statistic of the eigenvalues of $A_n$ is a function of the
form $\sum_{i=1}^{n} f(\lambda_i)$, where $f$ is some fixed function. Central limit theorems for linear statistics of eigenvalues of large dimensional random matrices have received considerable attention in recent years. A very curious feature that makes these results unusual and interesting is that they usually do not require normalization, i.e., one does not have to divide by $\sqrt{n}$; only centering is enough. Moreover, they have important applications in statistics and other applied areas (see e.g., the recent survey by Johnstone [35]).

The literature around the topic is quite large. To the best of our knowledge, the investigation of central limit theorems for linear statistics of eigenvalues of large dimensional random matrices began with the work of Jonsson [36] on Wishart matrices. The key idea is to express $\sum \lambda_i$ as

$$\sum \lambda_i = \text{Tr}(A_n^k) = \sum_{i_1, i_2, \ldots, i_k} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1},$$

where $A_n$ is an $n \times n$ Wishart matrix, and then apply the method of moments to show that this is gaussian in the large $n$ limit. In fact, Jonsson proves the joint convergence of the law of $(\text{Tr}(A_n), \text{Tr}(A_n^2), \ldots, \text{Tr}(A_n^p))$ to a multivariate normal distribution (where $p$ is fixed).

A similar study for Wigner matrices was carried out by Sinai and Soshnikov [46, 47]. A deep and difficult aspect of the Sinai–Soshnikov results is that they get central limit theorems for $\text{Tr}(f(A_n))$, where $p_n$ is allowed to grow at the rate $o(n^{2/3})$, instead of remaining fixed. They also get CLTs for $\text{Tr}(f(A_n))$ for analytic $f$.

Incidentally, for gaussian Wigner matrices, the best available results are due to Johansson [34], who characterized a large (but not exhaustive) class of functions for which the CLT holds. In fact, Johansson proved a general result for linear statistics of eigenvalues of random matrices whose entries have a joint density with respect to Lebesgue measure of the form $Z_n^{-1} \exp(-n \text{Tr} V(A))$, where $V$ is a polynomial function and $Z_n$ is the normalizing constant. These models are widely studied in the physics literature. Johansson’s proof relies on a delicate analysis of the joint density of the eigenvalues, which is explicitly known for this class of matrices.

Another important contribution is the work of Diaconis and Evans [21], who proved similar results for random unitary matrices. Again, the basic approach relies on the method of moments, but the computations require new ideas because of the lack of independence between the matrix entries. However, as shown in [20, 21], strikingly exact computations are possible in this case by invoking some deep connections between symmetric function theory and the unitary group.

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An alternative approach, based on Stieltjes transforms, has been developed in Bai and Yao [5] and Bai and Silverstein [6]. This approach has its roots in the semi-rigorous works of Girko [24] and Khorunzhy et al. [38].

Yet another line of attack, via stochastic calculus, was initiated in the work of Cabanal-Duvillard [14]. The ideas were used by Guionnet [26] to prove central limit theorems for certain band matrix models. Far reaching results for a very general class of band matrix models were later obtained using combinatorial techniques by Anderson and Zeitouni [1].

Other influential ideas, sometimes at varying levels of rigor, come from the papers of Costin and Lebowitz [19], Boutet de Monvel et al. [12], Johansson [33], Keating