Quantization of the Superstring with
Manifest U(5) Super-Poincaré Invariance

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The superstring is quantized in a manner which manifestly preserves a U(5) subgroup of the (Wick-rotated) ten-dimensional super-Poincaré invariance. This description of the superstring contains critical N=2 worldsheet superconformal invariance and is a natural covariantization of the U(4)-invariant light-cone Green-Schwarz description.

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1. Introduction

Although the uncompactified superstring is invariant under SO(9,1) super-Poincaré transformations, there is no quantizable formalism for the superstring where all of these invariances can be made manifest. In the conventional Ramond-Neveu-Schwarz (RNS) formalism, the bosonic SO(9,1) Poincaré invariances can be made manifest for computations involving external Neveu-Schwarz states. For computations involving external Ramond states, the necessity of bosonizing the ten $\psi$ fields breaks the SO(9,1) invariance to U(5) (after Wick-rotating from SO(9,1) to SO(10)). Furthermore, none of the spacetime-supersymmetries are manifest in the RNS formalism.

In the light-cone Green-Schwarz formalism for the superstring, only a U(4) subgroup of the super-Poincaré invariances can be made manifest. Although this U(4) subgroup includes some spacetime supersymmetries, the light-cone Green-Schwarz formalism is completely gauge-fixed (i.e. has no worldsheet conformal or superconformal invariances) which leads to technical difficulties when computing scattering amplitudes. There also exists a classical covariant version of the Green-Schwarz formalism which has manifest SO(9,1) super-Poincaré invariance, but it has not yet been quantized in a manner suitable for defining physical states or computing scattering amplitudes.

Over the last five years, a new formalism for the superstring has been developed which is a hybrid between the RNS and Green-Schwarz formalisms. It contains some manifest spacetime-supersymmetries, and is related to the conventional RNS formalism by a redefinition of the worldsheet variables. The formalism contains critical N=2 worldsheet superconformal invariance, and amplitudes can be computed using standard N=2 string methods or using N=4 topological string methods. In previous papers, it was shown how to perform a field-redefinition from RNS variables to Green-Schwarz-like variables which preserves manifest SO(3,1) super-Poincaré invariance or manifest SO(5,1) super-Poincaré invariance. In both these cases, four of the sixteen spacetime supersymmetries are manifest.

In this paper, it will be shown how to perform a field-redefinition of the RNS worldsheet variables which manifestly preserves six of the sixteen spacetime-supersymmetries. After Wick-rotating to SO(10), this formalism contains manifest U(5) super-Poincaré

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2 Although physical states in light-cone gauge can be described in an SO(8)-invariant manner, amplitude computations require that four of the eight $\theta$'s carry spin-0 and the other four carry spin-1, breaking SO(8) down to U(4).
invariance. (Before Wick-rotating, there is a corresponding non-compact subgroup of SO(9,1) containing 25 bosonic generators and 6 fermionic generators which is manifest.) So this formalism contains the same manifest bosonic symmetries as the RNS formalism in the presence of Ramond states, but it also contains six manifest spacetime supersymmetries. Furthermore, one can define the physical state conditions in a manner which is manifestly invariant under these symmetries.

It is interesting to note that the U(5) transformations (or their Minkowski counterpart) preserve up to rescaling a pure spinor, i.e. a bosonic complex spinor \( u^A \) satisfying \( u^A \sigma^m_{AB} u^B = 0 \) for \( m = 0 \) to 9. Pure spinors have been suggested by other authors to play a useful role in describing \( d = 10 \) super-Yang-Mills and supergravity in harmonic superspace\(^4\).

2. Review of Spacetime Supersymmetry in the RNS Formalism

As shown in \([3,4]\), the \( N = 1 \) description of the RNS superstring can be embedded into a critical \( N = 2 \) string by defining the twisted \( \hat{c} = 2 \) \( N = 2 \) superconformal generators as:

\[
T = T_{RNS},
\]

\[
G^+ = \gamma \psi^m \partial x_m + c \left( \frac{1}{2} \partial x^m \partial x_m + \frac{1}{2} \psi^m \partial \psi_m - \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \gamma \partial \beta - b \partial c \right) - \gamma^2 b + \partial^2 c + \partial (c \xi \eta),
\]

\[
G^- = b,
\]

\[
J = cb + \eta \xi,
\]

where \( T_{RNS} = T_m + T_g \) is the \( c = 0 \) stress-tensor of the original RNS matter and ghost fields, and the \( [\beta, \gamma] \) super-reparameterization ghosts have been bosonized \([7]\) as \( [\beta = e^{-\phi} \partial \xi, \gamma = \eta e^\phi] \). Note that the zero mode of \( G^+ \) is the \( N = 1 \) BRST operator and \( J \) is related to the usual ghost current \( j_{\text{ghost}} = cb - \gamma \beta \) by \( J = j_{\text{ghost}} - j_{\text{picture}} \) where \( j_{\text{picture}} \) is the picture-current defined as \( j_{\text{picture}} = - \partial \phi + \xi \eta \).

It was shown in \([3]\) that \( N = 2 \) scattering amplitudes computed using \( N = 2 \) vertex operators reproduce the usual \( N = 1 \) scattering amplitudes computed using RNS vertex operators. The advantage of treating the RNS superstring as an \( N = 2 \) string theory is that the \( N = 2 \) generators, unlike the original \( N = 1 \) generators, have no square-root cuts with the spacetime-supersymmetry generators.
In the $-\frac{1}{2}$ picture, the spacetime-supersymmetry generator of the RNS superstring is

$$q_A^- = \oint e^{-\frac{i}{2}\phi} S^A$$

(2.2)

where $S^A = e^{\frac{i}{2} (\pm \tau_1 \pm \cdots \pm \tau_5)}$ is the sixteen-component Weyl spin-field constructed by bosonizing $\psi^m$ as $\psi^{2a-2} \pm i\psi^{2a-1} = e^{\pm i\tau^a}$ for $a = 1$ to 5. The index $A$ will sometimes be denoted using $U(5)$ notation as $\pm \pm \pm \pm \pm$ where the five choices of $\pm$ correspond to the five choices of $\pm$ in the definition of $S_A$. $A$ describes a Weyl spinor when there are an odd number of $+$’s and an anti-Weyl spinor when there are an even number of $+$’s. In the above bosonization formula, we have Wick-rotated the spacetime-signature to Euclidean space so that $\tau^a$ are real chiral bosons satisfying the OPE $\tau^a(y)\tau^b(z) \to -\log(y-z)$.

The algebra generated by the $q_A^-$’s is

$$\{q_A^-, q_B^-\} = \sigma_m^{AB} \oint e^{-\phi} \psi^m$$

(2.3)

using the OPE’s of the chiral bosons $\phi$ and $\tau^a$ where $\sigma_m^{AB}$ is the symmetric $16 \times 16$ Pauli matrix in ten dimensions. (2.3) is related to the standard supersymmetry algebra $\{q^A, q^B\} = \sigma_m^{AB} P^m$ by picture-changing since $\oint \{Q, \xi\} e^{-\phi} \psi^m = \oint \partial x^m$ is the momentum operator $P^m$ where $\{Q, \xi\} = e^{\phi} \psi^m \partial x_m + ...$ is the picture-changing operator. But if one wants the supersymmetry algebra to close off-shell, one cannot use picture-changing since the off-shell states are not independent of the location of the picture-changing operator.

One can also define the spacetime-supersymmetry generators in the $+\frac{1}{2}$ picture as

$$q_+ = (Q, \xi) q_-^A = \oint (e^{\frac{i}{2}\phi} \eta S^A + e^{\frac{i}{2}\phi} \partial x^m \sigma_m^{AB} S^B)$$

(2.4)

where $\tilde{S}^B$ is an anti-Weyl spin-field. Although $\{q_+^A, q_-^B\} = \sigma_{m}^{AB} \oint \partial x^m$ without picture-changing, this has twice as many supersymmetry generators as desired, so it is not the $N = 1 D = 10$ supersymmetry algebra. (It is also not the $N = 2 D = 10$ supersymmetry algebra since $\{q_+^A, q_-^B\}$ does not vanish.) So one needs to find a subset of the $q_A^-$’s and $q_+^B$’s which generates at least a subset of the $N = 1 D = 10$ supersymmetry algebra without using picture-changing.

In previous papers, it was shown how to do this and preserve manifest $SO(3, 1)$ super-Poincaré invariance [2] or manifest $SO(5, 1)$ super-Poincaré invariance [3]. It will now be shown how to choose a subset of the supersymmetry generators which preserves $U(5)$ invariance (after Wick rotation).
3. U(5)-Invariant Description of the Superstring

Consider the five spacetime-supersymmetry generators defined by

\[ q^a = \oint e^{-\frac{1}{2}\phi} S^a \]  \hspace{1cm} (3.1)

where \( S^a \) contains one + sign and four − signs in its exponential. These five generators rotate into each other under the \( SU(5) \times U(1) \) subgroup of \( SO(10) \) rotations which rotate \( x^{2a-2} + ix^{2a-1} \) into each other. (There is a Minkowski-space version of this subgroup, but it is non-compact and has no standard name.) For later convenience, we shall define

\[ X^a = x^{2a-2} + ix^{2a-1}, \quad \bar{X}^a = x^{2a-2} - ix^{2a-1}, \]  \hspace{1cm} (3.2)

which satisfy the OPE \( X^a(y)\bar{X}^b(z) \rightarrow -2\delta^{ab}\log |y - z| \) and which transform respectively as a \((5,1)\) and \((\bar{5},-1)\) under the \( SU(5) \times U(1) \) subgroup.

Note that \( \{q^a, q^b\} = 0 \), so it trivially generates the correct supersymmetry algebra without having to resort to picture-changing. One can still introduce one more spacetime-supersymmetry generator without spoiling this property. This generator will be defined in the \( +\frac{1}{2} \) picture as

\[ q^+ = \oint (e^{\frac{3}{2}\phi} b\eta S^{+++++} + \frac{1}{2} e^{\frac{1}{2}\phi} \bar{S}\partial X^a) \]  \hspace{1cm} (3.3)

where \( \bar{S}^{\bar{a}} \) is defined to contain four + signs and one − signs in its exponential. It is easy to check that \( \{q^+, q^a\} = \frac{1}{2} \oint \partial X^a \) and \( \{q^+, q^+\} = 0 \), so one reproduces a subset of the desired supersymmetry algebra without having to use picture-changing. So six of the spacetime-supersymmetries can be chosen to close off-shell, and they transform covariantly as a \((5, -3/2)\) and \((1, 5/2)\) under the \( SU(5) \times U(1) \) subgroup of Lorentz transformations.

To make these supersymmetries manifest, one needs to find a field-redefinition from the RNS worldsheet variables to Green-Schwarz-like variables which transform simply under the spacetime-supersymmetry transformations. First, one should define \( \theta^{\bar{a}} \) and \( \theta^{-} \) variables which satisfy \( \{q^a, \theta^{\bar{b}}\} = \delta^{ab} \) and \( \{q^+, \theta^{-}\} = 1 \). This can be done by defining

\[ \theta^a = e^{\frac{1}{2}\phi} \bar{S}^{\bar{a}}, \quad \theta^{-} = c\xi e^{-\frac{3}{2}\phi} \bar{S}^{---}. \]

One then defines the conjugate momentum to these \( \theta \) variables as

\[ p^a = e^{-\frac{1}{2}\phi} S^a, \quad p_+ = b\eta e^{\frac{3}{2}\phi} S^{+++++}. \]
Note that $\theta^\alpha$ and $\theta^+$ have zero conformal weight, transform under $SU(5) \times U(1)$ as $(5, 3/2)$ and $(1, -5/2)$, and satisfy the free-field OPE’s

$$\bar{\theta}^a(y)p^b(z) \to (y-z)^{-1}\delta^{ab}, \quad \theta^-(y)p^+(z) \to (y-z)^{-1}.$$ 

In terms of these variables, the spacetime-supersymmetry generators of $(3.1)$ and $(3.3)$ take the simple form

$$q^a = \oint p_a, \quad q^+ = \oint (p^+ + \frac{1}{2} \theta^a \partial X^a).$$  \hfill (3.4)

The original RNS variables consisted of $[x^m, \psi^m, b, c, \beta, \gamma]$ and, besides $x^m$, we have defined twelve fermionic variables. So there remain two chiral bosons, $\rho$ and $\sigma$, which need to be defined. Requiring that they have no singularity with the $\theta$‘s or $p$‘s implies that they are given by

$$\partial\sigma = i(bc + \xi\eta), \quad \partial\rho = bc - \xi\eta + 3\partial\phi + \sum_{a=1}^5 \psi_{2a-2}\psi_{2a-1}.$$ 

These chiral bosons satisfy the free-field OPE’s

$$\sigma(y)\sigma(z) \to -2\log(y - z), \quad \rho(y)\rho(z) \to -2\log(y - z)$$ \hfill (3.5)

and the conformal weight of $e^{im\sigma + n\rho}$ is $m^2 - m - n^2 - n$. Furthermore, $e^{im\sigma + n\rho}$ transforms under $SU(5) \times U(1)$ as a $(1, 5n)$ representation. The fact that $\rho$ appears without a factor of $i$ in exponentials implies that it is a ‘time-like’ chiral boson similar to $\phi$ of the RNS formalism. GSO-projected RNS variables get mapped under this field-redefinition to exponentials with both $m$ and $n$ integer or both $m$ and $n$ semi-integer.

Since the new variables obey free-field OPE’s, the worldsheet action is still quadratic and is

$$S = \frac{1}{2\pi} \int d^2 \sigma \left[ \frac{1}{2} \partial X^a \partial \bar{X}^\alpha + p^a \partial \theta^\alpha + p^+ \partial \theta^- + \bar{p}^a \partial \bar{\theta}^\alpha + \bar{p}^+ \partial \bar{\theta}^- \right]$$ \hfill (3.6)

where $[\theta^\alpha, \bar{\theta}^\alpha, \bar{p}^a, \bar{p}^+]$ are right-moving variables defined in the same way as their left-moving counterparts. We have not tried to write the worldsheet action for $[\rho, \sigma]$ and $[\bar{\rho}, \bar{\sigma}]$ because of the usual problems with actions for chiral bosons. It is straightforward to write the twisted $N = 2$ superconformal generators of $(2.1)$ in terms of these new variables, and one finds

$$T = \frac{1}{2} \partial X^a \partial \bar{X}^\alpha + \partial^a \partial \theta^\alpha + p^+ \partial \theta^- + \frac{1}{4} (\partial \sigma \partial \sigma + \partial \rho \partial \rho) + \frac{1}{2} \partial^2 (\rho - i\sigma),$$ \hfill (3.7)
\[ G^+ = e^{\frac{i}{2}(\rho+i\sigma)} [d^a (\partial \bar{X}^\bar{a} + \theta^- \partial \bar{\theta}^\bar{a}) - \partial \rho \partial \theta^- - \frac{3}{2} \partial^2 \theta^-] + e^{\frac{i}{2}(3\rho+i\sigma)} (d)^5, \]

\[ G^- = e^{-\frac{i}{2}(\rho+i\sigma)} p^+, \]

\[ J = i\partial \sigma, \]

where \((d)^5 = \frac{1}{120} \epsilon_{abcd} d^a d^b d^c d^d d^e\) and

\[ d^a = \rho^a - \frac{1}{2} \theta^- \partial X^a. \] (3.8)

Note that \(d^a\) and \(\partial \bar{X} + \theta^- \partial \bar{\theta}^\bar{a}\) commute with the spacetime supersymmetry generators \(q^a\) and \(q^+\), and since \(T\) can be rewritten as

\[ T = \frac{1}{2} \partial X^a (\partial \bar{X}^\bar{a} + \theta^- \partial \bar{\theta}^\bar{a}) + d^a \partial \bar{\theta}^\bar{a} + p^+ \partial \theta^- + \frac{1}{4} (\partial \sigma \partial \sigma + \partial \rho \partial \rho) + \frac{1}{2} \partial^2 (\rho - i\sigma), \] (3.9)

the \(N=2\) constraints are manifestly spacetime-supersymmetric.

The U(4)-invariant light-cone Green-Schwarz description is recovered from this \(N=2\) superconformal field theory by using the gauge invariances associated with \(T\) and \(J\) to set \(\partial \bar{X}^\bar{1} = \rho + i\sigma = 0\), and the gauge invariances associated with \(G^+\) and \(G^-\) to set \(\theta^\bar{1} = \theta^- = 0\). Imposing the \(N=2\) constraints in this gauge fixes \(X^1, p^1, p^+\) and \(\sigma\), leaving the light-cone degrees of freedom \([X^j, X^j, \theta^j, p^j]\) where \(j = 2\) to 5.

4. Physical State Conditions

As described in [3], physical states of the superstring can be described by U(1)-neutral vertex operators, \(\Phi\), which satisfy

\[ G_0^+ \tilde{G}_0^+ \Phi = 0, \quad \Phi \sim \Phi + G_0^+ \Omega + \tilde{G}_0^+ \tilde{\Omega} \] (4.1)

where \(G_0^+\) is the zero mode of the \(G^+\) generator (which is the BRST operator in RNS variables) and \(\tilde{G}_0^+\) is the zero mode of the generator \(\tilde{G}^+ = [e^{\int J^+} G^-]\). In terms of the RNS variables, \(\tilde{G}^+ = \eta\), and in terms of the Green-Schwarz-like variables,

\[ \tilde{G}^+ = e^{-\frac{i}{2}(\rho-i\sigma)} p^+. \] (4.2)

These U(1)-neutral vertex operators \(\Phi\) can be related to the usual physical RNS vertex operators \(V_{RNS}\) by

\[ V_{RNS} = \tilde{G}_0^+ \Phi = \eta_0 \Phi; \quad \Phi = \xi_0 V_{RNS}. \] (4.3)
Note that (4.1) and (4.3) imply that $V_{RNS}$ is annihilated by the BRST operator and is in the “small” Hilbert space, i.e. it is independent of the $\xi$ zero mode.

In the RNS formalism, each physical state is represented infinitely many times in the BRST cohomology because of picture-changing. This infinite degeneracy is usually removed by fixing all bosons to be in the zero picture and all fermions to be in the $-\frac{1}{2}$ picture. But such a choice is not manifestly spacetime-supersymmetric since the generators of (3.4) carry both $\frac{1}{2}$ and $-\frac{1}{2}$ picture.

An alternative solution for removing the infinite degeneracy is to require that all states (both bosons and fermions) carry either $-1$ or $0$ $\rho$-charge, i.e. their vertex operator is proportional to $e^{n\rho}$ where $n = -1$ or $0$. This solution is manifestly spacetime-supersymmetric since the generators of (3.4) carry zero $\rho$ charge. It will now be shown that this solution assigns a unique representative to each physical state. (A similar solution was used in [8] for the superstring with manifest SO(3,1) super-Poincaré invariance.)

Physical vertex operators $V_{RNS}$ of the superstring must be independent of the $\xi$ zero mode, i.e. they must be annihilated by $\oint \eta$ as well as by $Q$. As mentioned before, there are infinitely many such vertex operators for each physical state since one can use the picture-raising operator, $Z = \{Q, \xi\}$, or picture-lowering operator, $Y = c\partial \xi e^{-2\phi}$, to move from one such vertex operator to another one. However, for each physical state, there is a unique vertex operator $\hat{V}$ which is annihilated by $1 - Z$ where $Z$ is the picture-raising operator. If $V$ is any physical vertex operator representing this state, then $\hat{V}$ is given (up to an overall multiplicative factor) by

$$\hat{V} = V + \sum_{n=0}^{\infty} Y^n V + \sum_{n=0}^{\infty} Z^n V. \quad (4.4)$$

One can similarly define a unique U(1)-neutral vertex operator $\hat{\Phi}$ for any physical state. This vertex operator $\hat{\Phi}$ is defined by

$$(G_0^+ - \tilde{G}_0^+) \hat{\Phi} = 0, \quad \hat{\Phi} \sim \hat{\Phi} + (G_0^+ - \tilde{G}_0^+) \hat{\Omega}, \quad (4.5)$$

and can be obtained by hitting $\hat{V}$ of (4.4) with $\xi_0$. In terms of the Green-Schwarz-like variables, $\hat{\Phi}$ can be expanded as $\hat{\Phi} = \sum_{n=-\infty}^{\infty} \hat{\Phi}_n$ where $\hat{\Phi}_n$ carries $\rho$-charge $n$. It will now be shown that, when $\hat{\Phi}$ satisfies (4.5), all components of $\hat{\Phi}$ can be determined in terms of $\hat{\Phi}_0$ and $\hat{\Phi}_{-1}$.
Using the formulas of (3.7) and (1.2),
\[
G^+ = \bar{G}^+ = -e^{\frac{1}{2}(i\sigma - \rho)}p^+ + e^{\frac{1}{2}(i\sigma + \rho)}[a^a(\partial X^a + \theta^- \partial \theta^a) - \partial \rho \partial \theta^- - \frac{3}{2} \partial^2 \theta^-] + e^{\frac{1}{2}(i\sigma + 3\rho)}(d)\],
\]
so one can write
\[
G_0^+ - \bar{G}_0^+ = a_{-\frac{1}{2}} + a_{\frac{1}{2}} + a_{3}
\]
where \(a_{-\frac{1}{2}}\) is the zero mode of \(-e^{\frac{1}{2}(i\sigma - \rho)}p^+\), \(a_{\frac{1}{2}}\) is the zero mode of \(e^{\frac{1}{2}(i\sigma + \rho)}[a^a(\partial X^a + \theta^- \partial \theta^a) - \partial \rho \partial \theta^- - \frac{3}{2} \partial^2 \theta^-]\), and \(a_{3}\) is the zero mode of \(e^{\frac{1}{2}(i\sigma + 3\rho)}(d)\). It will be useful to note that the cohomologies of \(a_{-\frac{1}{2}}\) and \(a_{\frac{1}{2}}\) are trivial. This is easy to show since \(a_{-\frac{1}{2}}F = 0\) implies that \(F = a_{-\frac{1}{2}}(- e^{-\frac{1}{2}(i\sigma + \rho)}\theta^F)\) and \(a_{\frac{1}{2}}F = 0\) implies that \(F = a_{\frac{1}{2}}(e^{-\frac{1}{2}(i\sigma + 3\rho)}(\theta)^5F)\)
\[
\text{where } (\theta)^5 = \frac{1}{120}e^{abcde}\theta^a\theta^b\theta^c\theta^d\theta^e.
\]
Using \((G_0^+ - \bar{G}_0^+)\hat{\Phi} = 0\), one can show that
\[
a_{\frac{1}{2}}\hat{\Phi}_{-2} = -a_{\frac{1}{2}}\hat{\Phi}_{-1} - a_{-\frac{1}{2}}\hat{\Phi}_0, \quad a_{-\frac{1}{2}}\hat{\Phi}_1 = -a_{\frac{1}{2}}\hat{\Phi}_0 - a_{\frac{3}{2}}\hat{\Phi}_{-1}.
\]
These equations are invariant under the gauge transformations coming from \(\delta \hat{\Phi} = (G_0^+ - \bar{G}_0^+)\hat{\Omega}\),
\[
\delta \hat{\Phi}_{-2} = a_{\frac{3}{2}}\hat{\Omega}_{-\frac{1}{2}}, \quad \delta \hat{\Phi}_{-3} = a_{\frac{1}{2}}\hat{\Omega}_{-\frac{3}{2}}, \quad \delta \hat{\Phi}_{-4} = a_{-\frac{1}{2}}\hat{\Omega}_{-\frac{7}{2}},
\]
\[
\delta \hat{\Phi}_1 = a_{-\frac{1}{2}}\hat{\Omega}_{\frac{1}{2}}, \quad \delta \hat{\Phi}_2 = a_{\frac{1}{2}}\hat{\Omega}_{\frac{3}{2}}, \quad \delta \hat{\Phi}_3 = a_{\frac{3}{2}}\hat{\Omega}_{\frac{7}{2}}
\]
where \(\hat{\Omega} = \sum_{n=\text{odd}} \hat{\Omega}_n\). Since the cohomologies of \(a_{\frac{3}{2}}\) and \(a_{-\frac{1}{2}}\) are trivial, one can use (4.7) to express \(\hat{\Phi}_{-2}\) and \(\hat{\Phi}_1\) in terms of \(\hat{\Phi}_{-1}\) and \(\hat{\Phi}_0\). Similarly, one can show that \(\hat{\Phi}_n\) for all \(n\) can be expressed in terms of \(\hat{\Phi}_{-1}\) and \(\hat{\Phi}_0\).

The equations of (4.7) imply that \(\hat{\Phi}_0\) and \(\hat{\Phi}_{-1}\) satisfy
\[
a_{\frac{1}{2}}(a_{\frac{3}{2}}\hat{\Phi}_{-1} + a_{\frac{1}{2}}\hat{\Phi}_0) = 0, \quad a_{-\frac{1}{2}}(a_{\frac{3}{2}}\hat{\Phi}_0 + a_{\frac{1}{2}}\hat{\Phi}_{-1}) = 0.
\]
These equations are invariant under the gauge transformations
\[
\delta \hat{\Phi}_{-1} = a_{\frac{3}{2}}\hat{\Omega}_{-\frac{1}{2}} + a_{\frac{1}{2}}\hat{\Omega}_{-\frac{3}{2}} + a_{-\frac{1}{2}}\hat{\Omega}_{-\frac{7}{2}}, \quad \delta \hat{\Phi}_0 = a_{\frac{3}{2}}\hat{\Omega}_{\frac{1}{2}} + a_{\frac{1}{2}}\hat{\Omega}_{\frac{3}{2}} + a_{-\frac{1}{2}}\hat{\Omega}_{\frac{7}{2}}.
\]
So any physical state of the superstring is uniquely described by the fields \(\hat{\Phi}_0\) and \(\hat{\Phi}_{-1}\), which satisfy the equations of (4.9), and which are defined up to the gauge transformations of (4.10). It should be noted that \(\Phi = \hat{\Phi}_{-1} + \hat{\Phi}_0\) does not satisfy (4.1), and therefore is
not an acceptable physical vertex operator for computing scattering amplitudes. However, one can construct a physical vertex operator from $\hat{\Phi}_-1$ and $\hat{\Phi}_0$ as

$$
\Phi = a_2 \left[ (\theta)^5 e^{-\frac{i}{2} (i\sigma + 3\rho)} \hat{\Phi}_-1 \right] + \hat{\Phi}_-1 + \hat{\Phi}_0 - a_2 \left[ \theta e^{-\frac{i}{2} (i\sigma - \rho)} \hat{\Phi}_-1 \right], \quad (4.11)
$$

which, using (4.9) and (4.10), can be shown to satisfy (4.1).

It will now be shown that (4.9) and (4.10) correctly describe the physical degrees of freedom for the massless states of the open superstring. For massless states, the vertex operator must have conformal weight zero and zero momentum. Since it is $U(1)$-neutral, the most general such vertex operator with $\rho$-charge $-1$ or $0$ is

$$
\hat{\Phi}_-1 = e^{-\rho} B(x^m, \theta^a, \theta^-), \quad \hat{\Phi}_0 = C(x^m, \theta^a, \theta^-),
$$

where $B$ and $C$ are two superfields depending on the ten $x$’s and six $\theta$’s.

The equations of (4.9) imply that

$$
\nabla_b \nabla_c \nabla_d (2\nabla_{\bar{a}} \partial_a B + \nabla_{\bar{a}} C) = 0, \quad -\frac{1}{2} (\nabla^4)_{\bar{a}} \partial_{\bar{a}} B + (\nabla^5)_{\bar{a}} \nabla_{\bar{a}} C = 0 \quad (4.12)
$$

where $\nabla_\alpha = \partial/\partial \theta^\alpha$, $\nabla_{\bar{a}} = \partial/\partial \theta^{\bar{a}} + \theta^\alpha \partial_a$, $\partial_a = \partial/\partial x^a$, $\partial_{\bar{a}} = \partial/\partial \bar{x}^{\bar{a}}$, $(\nabla^4)_{\bar{a}} = \frac{1}{24} \epsilon_{abcde} \nabla_b \nabla_c \nabla_d \nabla_{\bar{e}}$, and $(\nabla^5) = \frac{1}{120} \epsilon_{abcde} \nabla_b \nabla_c \nabla_d \nabla_{\bar{e}}$. These equations are invariant under the gauge transformations

$$
\delta B = \nabla_{\bar{a}} \nabla_{\bar{b}} \omega^{\bar{a}\bar{b}} - 2\nabla_{\bar{a}} \partial_a \lambda, \quad \delta C = (\nabla^5) \lambda - \nabla_{\bar{a}} \Sigma \quad (4.13)
$$

which come from choosing the gauge parameters of (4.10) to be

$$
\hat{\Omega}_{-\frac{5}{2}} = e^{-\frac{i}{2} (i\sigma + 5\rho)} \epsilon_{abcde} \partial^{\bar{a}} \partial^{\bar{b}} \partial^{\bar{c}} \partial^{\bar{d}} \omega^{\bar{c}\bar{d}}, \quad \hat{\Omega}_{-\frac{5}{2}} = e^{-\frac{i}{2} (i\sigma + 3\rho)} \lambda, \quad \hat{\Omega}_{-\frac{3}{2}} = e^{-\frac{i}{2} (i\sigma - \rho)} \Sigma.
$$

Note that $\hat{\Omega}_{-\frac{5}{2}}$ has no contribution at the massless level.

Using the gauge transformations of (4.13), one can algebraically gauge $B$ and $C$ into the form

$$
B = -2(\theta^4)^a A_a + (\theta)^5 \Psi + \theta^-(\theta^4)^a \Psi_a - \theta^-(\theta^5) D, \quad (4.14)
$$

$$
C = \theta^- [\theta^a A_{\bar{a}} + \theta^2 \theta^\alpha \Psi_{\bar{a}b} + \theta^3 \theta^\alpha \theta^\beta N_{\bar{a}abc} + (\theta^4)^\alpha \xi_a + (\theta)^5 P]
$$

where $(\theta^4)^a = \frac{1}{24} \epsilon_{abcde} \theta^a \theta^\beta \theta^\gamma \theta^\delta \theta^\epsilon$ and all fields on the right-hand side of (4.14) are component fields depending only on $x^m$. There is still one remaining gauge parameter given by $\lambda = (\theta)^5 c$ and $\Sigma = \theta^- c$, which transforms the component fields of (4.14) as

$$
\delta A_a = \partial_a c, \quad \delta A_{\bar{a}} = \partial_{\bar{a}} c, \quad \delta D = 2\partial_a \partial_{\bar{a}} c.
$$
It is straightforward to check that the equations of (4.12) imply that

\[ D = \partial_a A_a + \partial_{\dot{a}} A_{\dot{a}}, \quad \partial_a D = 2\partial_b \partial_{\dot{b}} A_a, \quad \partial_{\dot{a}} D = 2\partial_b \partial_{\dot{b}} A_{\dot{a}}, \quad (4.15) \]

\[ 4\varepsilon_{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}} \partial_a \Psi_b = -\partial_{[\dot{c}} \Psi_{\dot{d}\dot{e}]}], \quad \partial_a \Psi = -4\partial_b \Psi_{\dot{a}\dot{b}}, \quad \partial_{\dot{a}} \Psi_a = 0, \]

\[ N_{\dot{c}\dot{d}\dot{e}} = -\frac{2}{3}\varepsilon_{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}} \partial_a A_b, \quad \xi_a = -2\partial_a \Psi, \quad P = 0. \]

The first line of (4.15) implies that \( A_a \) and \( A_{\dot{a}} \) can be combined into a ten-component vector \( A_m \) satisfying Maxwell’s equation \( \partial^m \partial_{[m} A_{n]} = 0 \). The second line of (4.15) implies that \( \Psi, \Psi_a \) and \( \Psi_{\dot{d}\dot{e}} \) can be combined into a sixteen-component anti-Weyl spinor \( \Psi^\dot{A} \) satisfying the Dirac equation \( \partial^m \sigma^A_m \bar{\Psi}^\dot{B} = 0 \). So the physical content of the massless states of the open superstring has been correctly described.

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