STRICT CONVEXITY OF THE FREE ENERGY OF THE CANONICAL
ENSEMBLE UNDER DECAY OF CORRELATIONS

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Abstract. We consider a one-dimensional lattice system of unbounded, real-valued spins. We allow arbitrary strong, attractive, nearest-neighbor interaction. We show that the free energy of the canonical ensemble converges uniformly in \( C^2 \) to the free energy of the grand canonical ensembles. The error estimates are quantitative. A direct consequence is that the free energy of the canonical ensemble is uniformly strictly convex for large systems. Another consequence is a quantitative local Cramér theorem which yields the strict convexity of the coarse-grained Hamiltonian. With small adaptations, the argument could be generalized to systems with finite-range interactions on a graph, as long as the degree of the graph is uniformly bounded and the associated grand canonical ensemble has uniform decay of correlations.

1. Introduction

The broader scope of this article is the study of phase transitions. Phase transitions are one of the most interesting and most studied physical phenomena. A phase transition occurs if a microscopic change in a parameter leads to a fundamental change in one or more properties of the underlying physical system. The most well-known phase transition is when water becomes ice. A lot of physical and non-physical systems and mathematical models have phase transitions. For example, liquid to gas phase transitions are known as vaporization. Solid to liquid phase transitions are known as melting. Solid to gas phase transitions are known as sublimation. More examples are the phase transition in the 2-d Ising model (see for example [Sel16]), the Erdős-Renyi phase transition in random graphs (see for example [ER60], [ER61] or [KS13]) or phase transitions in social networks (see for example [FFH07]).

We are interested in studying a one-dimensional lattice systems of unbounded real-valued spins. The system consists of a finite number of sites \( i \in \Lambda \subset \mathbb{Z} \) on the lattice \( \mathbb{Z} \). For convenience, we assume that the set \( \Lambda \) is given by \( \{1, \ldots, K\} \). At each site \( i \in \Lambda \) there is a spin \( x_i \). In the Ising model the spins can take on the value 0 or 1. In this article, we consider real-valued spins \( x_i \in \mathbb{R} \). A configuration of the lattice system is given by a vector \( x \in \mathbb{R}^K \). The energy of a configuration \( x \) is given by the Hamiltonian \( H : \mathbb{R}^K \to \mathbb{R} \) of the system. For the detailed definition of the Hamiltonian \( H \) we refer to Section 2. We consider arbitrary strong, attractive, nearest-neighbor interaction. Because we will use ideas and heuristics from thermodynamics, it is sometimes helpful to think of the spin \( x_i \) as the temperature at the lattice point \( i \).

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We consider two ensembles of the lattice system. The first ensemble is the grand-canonical ensemble which is given by the Gibbs measure

$$
\mu^\sigma(dx) = \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^{K} x_i - H(x) \right) dx.
$$

Here, $Z$ is a generic normalization constant making the measure $\mu^\sigma$ a probability measure. The constant $\sigma \in \mathbb{R}$ is interpreted as an external field. The second ensemble is the canonical ensemble. It emerges from the grand-canonical ensemble by conditioning on the mean spin

$$
m = \frac{1}{K} \sum_{i=1}^{K} x_i.
$$

The canonical ensemble is given by the probability measure

$$
\mu_m(dx) = \mu^\sigma \left( dx \mid \frac{1}{K} \sum_{i=1}^{K} x_i = m \right) = \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{K} \sum_{i=1}^{K} x_i = m \right\}} \exp(-H(x))L^{K-1}(dx),
$$

where $L^{K-1}$ denotes the $K - 1$-dimensional Hausdorff measure.

The grand-canonical ensemble has a phase transition on the two-dimensional lattice (see for example [Pei36]). However, on the one-dimensional lattice the grand-canonical ensemble does not have a phase transition if the interaction decays fast enough (see for example [Isi25, Dob68, Dob74, Rue68, MN14]). There are many different notions of phase transition. In this work, we mean with having no phase transition that the infinite-volume Gibbs measure of the system is unique. It is a natural question if the canonical ensemble $\mu_m$ also does not have a phase transition on the one-dimensional lattice. There are known examples where the grand canonical ensemble has no phase transition but the canonical ensemble has (see for example [SS96, BCR02, BCK03]).

If the spins are $\{0,1\}$-valued spins there is no phase transition for the canonical ensemble on a one dimensional lattice with nearest-neighbor interaction. The authors could not find a proof of that statement in the literature but it follows from a result by Cancrini, Martinelli and Roberto [CMR02]. There, a logarithmic Sobolev inequality is deduced for the canonical ensemble on lattices of arbitrary dimension, provided the grand canonical ensemble satisfies a mixing condition. The mixing condition used in [CMR02] is that the grand canonical ensemble has an exponential decay of correlation that is uniform in the external field $\sigma$. This hypothesis is satisfied if the underlying lattice is one-dimensional. In our article we will use a similar mixing condition.

Up to the authors knowledge the question is still open in the case of real-valued spins. We conjecture that this is true i.e. the infinite-volume Gibbs measure of the canonical ensemble should be unique. A first step toward verifying this conjecture is to study the equivalence of the grand-canonical and canonical ensemble. Equivalence of ensembles is an indicator that the canonical ensemble also does not have a phase transition. If the canonical ensemble behaves similar as the grand canonical ensemble then one hopes that the property of not having a phase transition transfers as well. The equivalence of ensembles in one-dimensional lattice system was deduced by Dobrushin [DT77] for discrete (or bounded) spin values or by Georgii [Geo95] for quadratic Hamiltonians. However, our case where the spin values are
unbounded and the Hamiltonian is not quadratic seems to be open.

There are many different notions of equivalence of ensembles. We only consider the most simple type, namely the equivalence of thermodynamic quantities (see for example [Ada06]). Equivalence of ensemble then means that the free energy of the grand canonical ensemble converges to the free energy of the canonical ensemble if the system size goes to infinity (for more details see Section 2 below).

In the main result of this article, namely Theorem 2.3, we show that the grand canonical and canonical ensemble are equivalent. In fact, we show that free energies converge uniformly in $C^2$ as the system size goes to infinity. The rate of convergence in Theorem 2.3 is explicit. We therefore extend and refine the results of Dobrushin [DT77] and Georgii [Geo95].

Our argument is quite general and it should be possible to use it to deduce the same result for more general situations. In the argument we do not use the fact that the lattice is one-dimensional. Instead, we only use the fact that the grand canonical ensemble on a onedimensional lattice always has an uniform exponential decay of correlations (see for example [MN14] and [Zeg96]). Under the assumption of an uniform exponential decay of correlation, one should be able to use similar calculations to deduce the local Cramér theorem for spin systems on arbitrary graphs, as long as the degree is uniformly bounded and the interaction is finite range. However, we chose to consider the one-dimensional lattice because less notational burden makes it easier to explain the ideas of the argument and present the calculations.

A simple consequence of Theorem 2.3 is that the free energy of the canonical ensemble is uniformly strictly convex and quadratic (see Corollary 2.4) for large enough systems. Strict convexity of the free energy rules out phase coexistence which corresponds flat parts in the free energy. The most prominent example of phase coexistence is that under ordinary pressure water and ice can coexist at 0 degree Celsius. We want to point out that for the strict convexity of the free energy we do not have to go to the infinite-volume limit of the system. It applies to large but finite systems. To show that the free energy of the canonical ensemble is strictly convex in the infinite-volume limit ordinary equivalence of ensembles would suffice.

Closely related to the free energy $A_{ce}$ of the canonical ensemble is the notion of the coarse-grained Hamiltonian $\tilde{H}$ (cf. is [10] and [GOVW09]). As in [GOVW09], we derive from Theorem 2.3 a local Cramér theorem (see Theorem 2.6). The local Cramér theorem shows that the coarse-grained Hamiltonian converges in $C^2$ to the Legendre transform of the free energy of the grand canonical ensemble. It is a direct consequence of the $C^2$-local Cramér theorem that the coarse-grained Hamiltonian $\tilde{H}$ is also uniformly strictly convex for large enough system size $|\Lambda|$ (cf. Corollary 2.7).

The coarse-grained Hamiltonian $\tilde{H}$ plays an important role when studying the Kawasaki dynamics. The Kawasaki dynamics is natural drift diffusion process on our lattice system that conserves the mean spin of the system. The canonical ensemble is the stationary and ergodic distribution of the Kawasaki dynamics. The strict convexity of $\tilde{H}$ is a central ingredient for deducing a uniform logarithmic Sobolev inequality (LSI) for the canonical ensemble via the two-scale approach [GOVW09]. For the canonical ensemble and unbounded real-valued spins, it is known that the uniform LSI holds for no-interactions (see [LPY02, Cha03, GOVW09]).
or weak interaction (see [Men11]). It is not known if the uniform LSI holds in the case of strong interactions.

The LSI characterizes the speed of convergence of the Kawasaki dynamics to the canonical ensemble. With the equivalence of dynamic and static phase transitions (see [MN14] or [YOS03], a uniform LSI would also yield the absence of a phase transition and verify our conjecture (i.e. the existence of a unique infinite-volume Gibbs measure)). Additionally, a uniform LSI is one if the main ingredients when deducing hydrodynamic limit of the Kawasaki dynamic via the two scale approach (see next paragraph). The uniform LSI for the canonical ensemble with no interaction is a well-known result (see for example [Cha03, LPY02, GOVW09]). For weak interaction the uniform LSI was deduced in [Men11]. The question if the canonical ensemble satisfies a uniform LSI for strong nearest-neighbor interaction is still open. For discrete $\{0,1\}$-valued spin system the answer is yes (see [CMR02]). The authors believe that this should also be the case for real-valued spins.

The strict convexity of the coarse-grained Hamiltonian also plays a crucial role when deducing the hydrodynamic limit of the Kawasaki dynamic. The hydrodynamic limit is a type of law of large numbers for processes. It states that under the correct scaling the Kawasaki dynamics (which is a stochastic process) converges to the solution of a non-linear heat equation (which is deterministic). It is conjectured by H.T. Yau that the hydrodynamic limit also holds for strong finite-range interactions on a one-dimensional lattice. So far, this conjecture is still wide open. By providing the strict convexity of the coarse-grained Hamiltonian this article provides an important cornerstone to tackle this problem with the help of the two-scale approach (see [GOVW09]).

Let us now make some comments on how we deduce the $C^2$-equivalence of ensembles. The motivation for our approach comes from the proof of the local Cramèr theorem in [GOVW09] and [Men11]. By using Cramèr’s trick of an exponential shift it suffices to show $C^2$-bounds on the density of a sum of random variables $X_i$ (see also Proposition 3.6 below). Those desired bounds were derived [GOVW09] and [Men11] via a local central limit theorem (clt) for independent random variables. Our situation is a lot more subtle: Instead of deducing a local clt for independent random variables we would have to deduce a local clt for dependent random variables. At this point one could hope to use existing methods to deduce the local clt. Let us mention for example the approach of Dobrushin [Dob74], the approach of Bender [Ben73] or the approach of Wang and Woodroofe [WW90]. Unfortunately this does not help. All methods— at least the ones that are known to the authors— use the following principle (see also [DM16]):

\[
\text{integral clt} \ + \ \text{regularity} \ \Rightarrow \ \text{local clt}.
\]

The first ingredient, namely the integral clt for the dependent random variables $X_i$ is relatively easy to deduce. There are a lot of methods available. Let us mention for example Stein’s method (see for example [CGS11]), methods that are based on mixing, or methods that are based on Donsker’s theorem (see for example [Dur10]). Deducing the second ingredient is tricky, not to mention that Dobrushin [Dob74] carried out that step only for discrete or bounded random variables.
All in all, using this approach has two fundamental problems. The first one is that we need not only to control the density itself but also the first and second derivative. As a consequence one would need very detailed information about the regularity of the density. We also believe that showing the right regularity is as hard as directly deducing the local central limit theorem (as we do in this work). The second problem of using the principle from above is: In order to deduce Theorem 2.3 one needs a quantitative local central limit theorem. When using the principle from above to deduce quantitative rates of convergence, one gets a suboptimal rate of convergence. To deduce Theorem 2.3 one has to apply the principle from above three times getting worse and worse estimates on the convergence rate. One needs a lot of luck that the remaining convergence rate is still good enough.

We decided not to use the principle mentioned above. Instead we generalize a well-known method for proofing the local clt for independent random variables to dependent ones. We generalize the method that is based on characteristic functions and Fourier inversion (see [Fel71] and [GOVW09]). Calculations get quite evolved and lengthy. We chose not to deduce the local clt for dependent random variables in this work. Instead, we only deduce bounds that are needed to deduce Theorem 2.3. Those bounds are stated in Proposition 3.6 below. However, it is possible to use our calculations as a guideline to improve the estimates of Proposition 3.6 and deduce a quantitative, local clt for dependent random variables. When doing so, one would have to substitute some of our arguments that use the specific structure of our lattice model. We use the following special structure:

• Exponential decay of correlations (see Lemma 3.5).
• The interaction has finite range \( R \). More precisely, we use that two spins \( x_i \) and \( x_j \) become independent if \( |i - j| > R \) and one conditions on the spin values \( (x_k)_{i<k<j} \) between them (see Section 2).
• The Hamiltonian is quadratic. More precisely, we use the following consequence. For all \( i \in \Lambda \) the conditional variances \( \text{var}(X_i|X_j, |j - i| \leq l) \) is bounded from above and below uniformly in the values \( X_j, |j - i| \leq l \) (see Section 2).
• Higher moments of \( X_i \) conditioned on \( X_j, j \neq i \) are uniformly controlled by lower moments. This fact is used to show that the characteristic functions of \( X_i \) conditioned on \( X_j, j \neq i \) have a uniform decay (see Lemma 3.2 and Lemma 3.4).

As we mentioned before, we proof the \( C^2 \)-local Cramér theorem (see Theorem 2.6) by generalizing the argument of [GOVW09] for independent random variables to dependent random variables. This adds a lot more complexity to the task. We overcome the technical challenges of considering dependent random variables by using two strategies. The first strategy is to induce artificial independence by conditioning on even or odd random variables. The second strategy is to handle dependencies as a perturbation. We morally treat large blocks as single sites of a coarse-grained system. Because there is a big distance between the blocks, the blocks are only weakly dependent. We then can control the error term by using the decay of correlations. For more details we refer to the comments after Proposition 3.6 and at the beginning of Section 4.

Let us shortly discuss possible generalizations of our main result. We expect that one can generalize our method with only slight modifications to the following situations:

• instead of nearest-neighbor interaction to finite range interaction.
• instead of exponential decay of correlations to sufficiently fast algebraic decay.
• instead of a 1d lattice to any lattice or graph with bounded degree, as long as the
grand canonical ensemble $\mu_\sigma$ has sufficient decay of correlations, uniformly in the
system size and the external field $\sigma$.
• instead of attractive interaction to repulsive and mixed interactions, as long as the
estimate
$$\var_{\mu_\sigma} \left[ \sum_{i=1}^K X_i \right] \geq CK$$
is satisfied. For attractive interaction this estimate is deduced in Lemma 3.2.

More challenging, it would be very interesting to study the local Cramér theorem for the
following changes:

• In the case of the grand canonical ensemble on a one-dimensional lattice, there is no phase
transition if the interaction $M_{ij}$ decays algebraically faster than $(1 + |i - j|)^{2+\epsilon}$. The
decay condition is sharp (see for example [MN14] and references therein). It would
be very interesting to know if the $C^2$-local Cramér theorem (see Theorem 2.6) also
holds in this case or a stronger algebraic decay is needed.
• In our model we need a quadratic single-site potential. Inspired from [FM14], it is nat-
ural to ask if the local Cramér theorem also holds for super-quadratic or polynomially
increasing single-site potentials.
• Inspired by [Dob74] or [Geo95] it would be interesting to study more general interac-
tions than pairwise-quadratic interaction.

We conclude the introduction by giving a short overview over the remaining article. In
Section 2 we introduce the precise setting and formulate the main results. In Section 3 we
deduce the main results of this article except of Proposition 3.6. The main computations are
done in Section 4, where we state the proof of Proposition 3.6.

CONVENTIONS AND NOTATION

• The symbol $T_{(k)}$ denotes the term that is given by the line $(k)$.
• We denote with $0 < C < \infty$ a generic uniform constant. This means that the actual
value of $C$ might change from line to line or even within a line.
• With uniform we mean that a statement holds uniform in the system size $\Lambda$, the mean
spin $m$ and the external field $s$.
• $a \lesssim b$ denotes that there is a uniform constant $C$ such that $a \leq Cb$.
• $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$.
• $\mathcal{L}^k$ denotes the $k$-dimensional Hausdorff measure.
• $Z$ is a generic normalization constant. It denotes the partition function of a measure.

2. SETTING AND MAIN RESULTS

We start with explaining the details of the model that is studied. For convenience we consider
the sublattice $\{1, \ldots, K\} \subset \mathbb{Z}$. The Hamiltonian $H : \mathbb{R}^K \to \mathbb{R}$ of the system is defined as

$$H(x) = \sum_{i=1}^K \psi(x_i) + s_i x_i - J x_i x_{i+1}. \quad (2)$$

We make the following assumptions:
• The single-site potential $\psi : \mathbb{R} \to \mathbb{R}$ can be written as
\[
\psi(z) = \frac{1}{2} z^2 + \psi_b(z),
\]
where the function $\psi_b : \mathbb{R} \to \mathbb{R}$ satisfies
\[
|\psi_b|_\infty + |\psi'_b|_\infty + |\psi''_b|_\infty < \infty.
\]
• The numbers $s = (s_i) \in \mathbb{R}^K$ can be arbitrary. They model the interaction of the system with an external field or the boundary.
• The number $J \in \mathbb{R}$ can be arbitrary. It models the strength of the interaction. The interaction is attractive if $J > 0$. The interaction is repulsive if $J < 0$.

Now, let us turn to the first main result of this article, namely the equivalence of ensembles (see Theorem 2.3 from below). The grand canonical ensemble (gce) $\mu^\sigma$ is a probability measure on $\mathbb{R}^K$ given by the Lebesgue density
\[
\mu^\sigma(dx) := \frac{1}{Z} \exp \left( \sum_{i=1}^K \sigma x_i - H(x) \right) dx.
\]
The free energy of the gce $\mu^\sigma$ is given by (cf. [1] and [GOVW09])
\[
A_{gce}(\sigma) := \mathcal{H}_K := \frac{1}{K} \ln \int_{\mathbb{R}^K} \exp \left( \sigma \sum_{i=1}^K x_i - H(x) \right) dx.
\]
We observe that $A_{gce}$ is uniformly strictly convex. More precisely, it holds:

**Lemma 2.1.** Let $(X_1, X_2, \cdots, X_K)$ be a real-valued random variable distributed according to the gce $\mu^\sigma$. Assume that
\[
\text{var} \left( \sum_{i=1}^K X_i \right) \gtrsim K.
\]
Then the free energy $A_{gce}$ of the gce $\mu^\sigma$ is uniformly strictly convex in the sense that for all $\sigma \in \mathbb{R}$
\[
\frac{1}{C} \leq \text{Hess} A_{gce}(\sigma) \leq C.
\]
The proof of Lemma 2.1 is given in Section 3. The core ingredient of the argument is the logarithmic Sobolev Inequality (LSI).

Let us turn to the canonical ensemble (ce) $\mu_m$. It emerges from the gce by conditioning (i.e. fixing) on the mean spin (or mean temperature)
\[
m = \frac{1}{K} \sum_{i=1}^K x_i.
\]
The ce is given by the probability measure
\[
\mu_m(dx) = \mu^\sigma \left( dx \mid \frac{1}{K} \sum_{i=1}^K x_i = m \right) = \frac{1}{Z} \mathbb{1} \{ \frac{1}{K} \sum_{i=1}^K x_i = m \} \exp(-H(x)) \mathcal{L}^{K-1}(dx),
\]
where \( \mathcal{L}^{N-1} \) denotes the \( N-1 \)-dimensional Hausdorff measure. The free energy of the ce \( \mu_m \) is given by

\[
A_{ce}(\sigma) = \frac{1}{K} \ln \int \frac{1}{\mathcal{L}^{K-1}} \exp \left( \sigma \sum_{i=1}^{K} x_i - H(x) \right) \, dx.
\]

Equivalence of ensembles can only hold if we relate the external field \( \sigma \) of the gce \( \mu^\sigma \) and the mean spin \( m \) of the ce \( \mu_m \) in the following way.

**Assumption 2.2.** We then choose \( \sigma = \sigma(m) \in \mathbb{R} \) and \( m = m(\sigma) \in \mathbb{R} \) such that the following relation is satisfied:

\[
\frac{d}{d\sigma} A_{gce}(\sigma) = m. \tag{6}
\]

By the strict convexity of \( A_{gce} \) (see Lemma 2.1) there exists for any \( m \in \mathbb{R} \) a unique \( \sigma = \sigma(m) \in \mathbb{R} \) that satisfies the relation (6) or vice versa.

Now, let us formulate our first main result.

**Theorem 2.3 (Equivlance of ensembles).** Let \((X_1, X_2, \cdots, X_K)\) be a real-valued random variables distributed according to

\[
\mu^\sigma(dx) := \frac{1}{Z} \exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right) dx
\]

Assume that

\[
\text{var} \left( \sum_{i=1}^{K} X_i \right) \gtrsim K.
\]

Then it holds that

\[
\lim_{K \to \infty} |A_{gce} - A_{ce}|_{C^2} = 0
\]

where the convergence is uniform in the mean spin \( m \) and the external field \( s \). More precisely, given a constant \( \varepsilon > 0 \), it holds that

\[
\sup_{\sigma \in \mathbb{R}} |A_{gce}(\sigma) - A_{ce}(\sigma)| \lesssim \frac{1}{K} \tag{7},
\]

\[
\sup_{\sigma \in \mathbb{R}} \left| \frac{d}{d\sigma} A_{gce}(\sigma) - \frac{d}{d\sigma} A_{ce}(\sigma) \right| \lesssim \frac{1}{K^{1-\varepsilon}} \tag{8},
\]

\[
\sup_{\sigma \in \mathbb{R}} \left| \frac{d^2}{d\sigma^2} A_{gce}(\sigma) - \frac{d^2}{d\sigma^2} A_{ce}(\sigma) \right| \lesssim \frac{1}{K^{\frac{1}{2}-\varepsilon}}. \tag{9}
\]

We want to note that Theorem 2.3 contains quantitative error bounds and measures the equivalence of the free energies in \( C^2 \). We give the proof of Theorem 2.3 in Section 3. The additional assumption (5) is not very restrictive. For example, it is automatically satisfied if the interaction is attractive (see Lemma 3.2 below).

A direct consequence of Lemma 2.1 and Theorem 2.3 is that the free energy \( A_{ce} \) is uniformly strictly convex for large enough systems.
Corollary 2.4. There is a uniform constant $0 < C < \infty$ and an integer $K_0 \in \mathbb{N}$ such that for all $K \geq K_0$ and all $\sigma \in \mathbb{R}$
\[
\frac{1}{C} \leq \text{Hess} \ A_{ce}(\sigma) \leq C.
\]
Let us turn to the second main result of this article, the local Cramér theorem. For that purpose let us introduce $\mathcal{H}_K$ which denotes the Legendre transform of the free energy $A_{gce}$ (also denoted by $\widehat{\mathcal{H}}_K$) i.e.
\[
\mathcal{H}_K(m) = \sup_{\sigma \in \mathbb{R}} \left( \sigma m - \widehat{\mathcal{H}}_K(\sigma) \right).
\]
It follows from elementary observations that $\mathcal{H}_K$ is uniformly strictly convex.

Lemma 2.5. For any $m \in \mathbb{R}$
\[
\mathcal{H}_K(m) = \sigma(m)m - \widehat{\mathcal{H}}_K(\sigma(m)).
\]
Additionally, under the same assumptions as in Theorem 2.3 it holds that $\mathcal{H}_K$ is uniformly strictly convex in the sense that for all $\sigma \in \mathbb{R}$
\[
\frac{1}{C} \leq \text{Hess} \ \mathcal{H}_K(m) \leq C.
\]
We give the proof of Lemma 2.5 in Section 3.

The coarse-grained Hamiltonian $\tilde{H} : \mathbb{R} \rightarrow \mathbb{R}$ is defined as
\[
\tilde{H}(m) = -\frac{1}{K} \ln \int \{ \frac{1}{K} \sum_{i=1}^{K} x_i = m \} \exp(-H(x)) \mathcal{L}^{K-1}(dx).
\]
Hence, we can rewrite the free energy of the ce as
\[
A_{ce}(\sigma) = \sigma m - \tilde{H}(m). \tag{10}
\]
It follows that the difference of the free energies $A_{gce}$ and $A_{ce}$ can be expressed as
\[
A_{gce}(\sigma) - A_{ce}(\sigma) = A_{gce}(\sigma) - \sigma m + \tilde{H}(m) \\
= \mathcal{H}_K(\sigma) - \sigma m + \tilde{H}(m) \\
= \tilde{H}(m) - \mathcal{H}_K(m) \tag{11}
\]
From Theorem 2.3 we deduce the following local Cramér theorem.

Theorem 2.6 (C2-local Cramér theorem). Let $(X_1, X_2, \cdots, X_K)$ be a real-valued random variables distributed according to
\[
\mu^\sigma(dx) := \frac{1}{Z} \exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right) \ dx
\]
Assume that
\[
\text{var} \left( \sum_{i=1}^{K} X_i \right) \geq K.
\]
Then it holds that
\[
\lim_{K \to \infty} \left| \tilde{H}(m) - \mathcal{H}_K(m) \right|_{C^2} = 0,
\]
where the convergence is uniform in the mean spin \( m \) and the external field \( s \). More precisely, given a constant \( \varepsilon > 0 \), it holds that

\[
\sup_{m \in \mathbb{R}} |\bar{H}(m) - H_K(m)| \lesssim \frac{1}{K} \quad (12)
\]

\[
\sup_{m \in \mathbb{R}} \left| \frac{d}{dm} \bar{H}(m) - \frac{d}{dm} H_K(m) \right| \lesssim \frac{1}{K^{1-\varepsilon}} \quad (13)
\]

\[
\sup_{m \in \mathbb{R}} \left| \frac{d^2}{dm^2} \bar{H}(m) - \frac{d^2}{dm^2} H_M(m) \right| \lesssim \frac{1}{K^{2-\varepsilon}} \quad (14)
\]

Theorem 2.6 is an extension of the local Cramér theorems that were deduced in Proposition 31 in [GOVW09], Theorem 4 in [Men11] and [MO13]. The proof of Theorem 2.6 is stated in Section 3. The main ingredient is Theorem 2.3.

An important consequence of Lemma 2.5 and of Theorem 2.6 is that for large enough systems the coarse-grained Hamiltonian \( \bar{H} \) is uniformly strictly convex.

**Corollary 2.7.** Under the assumptions of Theorem 2.6 there is a positive integer \( K_0 \) such that for all \( K \geq K_0 \) the coarse-grained Hamiltonian \( \bar{H} : \mathbb{R} \to \mathbb{R} \) is uniformly strictly convex. More precisely, there is a uniform constant \( 0 < C < \infty \) such that for all \( m \in \mathbb{R} \)

\[
\frac{1}{C} \leq \text{Hess} \bar{H}(m) \leq C.
\]

3. Proof of the main results

In this section we prove the main results of this article.

**Assumption 3.1.** From now on we assume that \( X = (X_1, X_2, \ldots, X_K) \) is a real-valued random vector distributed according to

\[
\mu^\sigma(dx) := \frac{1}{Z} \exp \left( \sum_{i=1}^K \sigma x_i - H(x) \right) dx
\]

Be begin with simple auxiliary Lemma.

**Lemma 3.2.** There exists a uniform constant \( C \) such that

\[
\text{var} \left( \sum_{i=1}^K X_i \right) \leq CK
\]

Moreover, if \( J \) in (2) is nonnegative, then the condition (5) in Theorem 2.3 is satisfied.

**Proof of Lemma 3.2.** For the proof of the upper bounds, one can apply a result of [MN14] which proved the logarithmic Sobolev inequality (LSI) for \( \mu^\sigma \). More precisely, we have

\[
\text{var} \left( \sum_{i=1}^K X_i \right) \lesssim \frac{1}{\rho} \int \left| \nabla \left( \sum_{i=1}^K X_i \right) \right|^2 d\mu \lesssim \frac{1}{\rho} K
\]

where \( \rho > 0 \) is a constant independent of \( m \).
Proof of the lower bound relies on the result of [Men11]. In Lemma 9, Menz proved that independent of $t$, there is a constant $0 < C < \infty$ such that
\[
\frac{1}{C} \leq \text{var}_\nu (W) \leq C
\]
where $W$ is a random variable distributed according to the distribution
\[
\nu(dz) = \frac{1}{Z} \exp (-\psi(z) - tz) dz
\]
Observe that the random variable $E[X_i | X_j : j \neq i]$ has the Lebesgue density
\[
\mu(dx_i | \bar{x}_i) = \frac{1}{Z} \exp (-\psi(dx_i) - (-Jx_{i-1} - Jx_{i+1} + s_i - \sigma) x_i) dx_i
\]
which implies
\[
\text{var}_\mu (X_i) = \int \text{var}_{\mu(dx_i | \bar{x}_i)} (X_i) \bar{\mu}(d\bar{x}_i) + \text{var}_{\mu(dx_i | \bar{x}_i)} \left( \int x_i \mu(dx_i | \bar{x}_i) \right)
\]
\[
\geq \int \text{var}_{\mu(dx_i | \bar{x}_i)} (X_i) \bar{\mu}(d\bar{x}_i)
\]
\[
\geq \int \frac{1}{C} \bar{\mu}(d\bar{x}_i) \geq \frac{1}{C}
\]
(15)
On the other hand, Menz and Nittka (cf. [MN14]) proved that for $J \geq 0$, we have
\[
\text{cov} (X_i, X_j) \geq 0
\]
(16)
Then it follows using (15) and (16) that
\[
\text{var} \left( \sum_{i=1}^{K} X_i \right) = \sum_{i=1}^{K} \text{var}(X_i) + \sum_{i \neq j} \text{cov}(X_i, X_j) \geq \frac{1}{C} K
\]
\[
\square
\]
This Lemma shows that the variance of the mean spin of the gce $\mu^\sigma$ is well behaved. Now we are ready to give proofs of Lemma 2.1 and Lemma 2.5.

**Proof of Lemma 2.1.** It is a direct consequence of Lemma 3.2. Indeed, we have
\[
\frac{d}{d\sigma} A_{\text{gce}}(\sigma) = \frac{d}{d\sigma} \left( \frac{1}{K} \ln \int_{\mathbb{R}^K} \exp \left( \sigma \sum_{i=1}^{K} x_i - H(x) \right) dx \right)
\]
\[
= \frac{1}{K} \int_{\mathbb{R}^K} \sum_{i=1}^{K} x_i \exp \left( \sigma \sum_{i=1}^{K} x_i - H(x) \right) dx
\]
\[
= \frac{1}{K} \int_{\mathbb{R}^K} \sum_{i=1}^{K} x_i \mu^\sigma(dx)
\]
\[
= \frac{1}{K} \mathbb{E} \left[ \sum_{i=1}^{K} X_i \right]
\]
and taking the derivative with respect to $\sigma$ again, we obtain
\[
\frac{d^2}{d\sigma^2} A_{gce}(\sigma) = \frac{1}{K} \int_{\mathbb{R}^K} \sum_{i=1}^K x_i \left( \sum_{j=1}^K (x_j - \mathbb{E}[X_j]) \right) d\mu(dx)
\]
\[
= \frac{1}{K} \mathbb{E} \left[ \sum_{i=1}^K X_i \sum_{j=1}^K (X_j - \mathbb{E}[X_j]) \right]
\]
\[
= \frac{1}{K} \text{var} \left( \sum_{i=1}^K X_i \right)
\]

Therefore, we conclude from Lemma 3.2 that there is a constant $C > 0$ with
\[
\frac{1}{C} \leq \text{Hess} A_{gce}(\sigma) \leq C.
\]

\[\square\]

Lemma 2.5 also follows from similar argument.

**Proof of Lemma 2.5.** Since $\mathcal{H}_K(m)$ is the Legendre transform of the strict convex function $\widehat{\mathcal{H}}_K(\sigma)$, there exists a unique $\sigma = \sigma(m)$ such that
\[
\mathcal{H}_K(m) = \sigma(m)m - \widehat{\mathcal{H}}_K(\sigma(m))
\]
Moreover, for each $m$, $\sigma(m)$ satisfies
\[
\frac{d}{d\sigma} \left( \sigma m - \widehat{\mathcal{H}}_K(\sigma) \right) = 0
\]
or, equivalently,
\[
m = \frac{d}{d\sigma} \widehat{\mathcal{H}}_K(\sigma)
\]
\[
= \frac{1}{K} \int_{\mathbb{R}^K} \left( \sum_{i=1}^K x_i \right) \exp \left( \sum_{i=1}^K \sigma x_i - H(x) \right) dx
\]
\[
= \frac{1}{K} \mathbb{E} \left[ \sum_{i=1}^K X_i \right]
\]
\[
= \frac{1}{K} \sum_{i=1}^K m_i \quad (17)
\]

Then it follows that
\[
\frac{d}{dm} \mathcal{H}_K(m)
\]
\[
= \frac{d}{dm} \left( \sigma(m)m - \widehat{\mathcal{H}}_K(\sigma(m)) \right)
\]
\[
= \frac{d\sigma(m)}{dm}m + \sigma(m) - \frac{d}{d\sigma} \widehat{\mathcal{H}}_K(\sigma) \cdot \frac{d\sigma}{dm}c
\]

= \frac{d\sigma}{dm} m + \sigma - \frac{1}{K} \int_{\mathbb{R}^K} \sum_{i=1}^{K} x_i \exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right) dx \cdot \frac{d\sigma}{dm} \\
= \frac{d\sigma}{dm} m + \sigma - \frac{1}{K} \mathbb{E} \left[ \sum_{i=1}^{K} X_i \right] \frac{d\sigma}{dm} \\
= \sigma

Note that
\[ \frac{d}{d\sigma} \frac{1}{m} = 1 \]

Let us now turn to the proof of Theorem 2.3. We need some more auxiliary results. The first one is Cramér’s trick of exponential shift of measures.

**Lemma 3.3.** It holds that
\[ g_{K,\sigma}(0) = \exp \left( K A (\sigma) - K A_{gce} (\sigma) \right) \]
(18)

Here, \( g_{K,\sigma} \) denotes the distribution of
\[ \frac{1}{\sqrt{K}} \sum_{i=1}^{K} (X_i - m) \]

**Proof of Lemma 3.3.** The lemma follows from direct computation:
\[ K H_K (m) - K \bar{H} (m) \]
\[ = K \sigma(m) m - \ln \int_{\mathbb{R}^K} \exp \left( \sum_{i=1}^{K} \sigma(m) x_i - H(x) \right) dx \]
\[ + \ln \int_{\left\{ \frac{1}{K} \sum_{i=1}^{K} x_i = m \right\}} \exp (-H(x)) \mathcal{L}^{K-1}(dx) \]

and thus Lemma 3.2 implies there exists a constant \( C > 0 \) with
\[ \frac{1}{C} \leq \text{Hess} \mathcal{H}_K (m) = \frac{d\sigma}{dm} = \left( \frac{dm}{d\sigma} \right)^{-1} \leq C \]

\[ \square \]
\[= \ln \int_{\frac{1}{K} \sum_{i=1}^{K} x_i = m} \exp(K \sigma(m)m - H(x)) \mathcal{L}^{K-1}(dx)
- \ln \int_{\mathbb{R}^K} \exp \left( \sum_{i=1}^{K} \sigma(m)x_i - H(x) \right) dx
= \ln \frac{\int_{\frac{1}{\sqrt{K}} \sum_{i=1}^{K} (x_i - m) = 0} \exp \left( \sigma(m) \sum_{i=1}^{K} x_i - H(x) \right) \mathcal{L}^{K-1}(dx)}{\int_{\mathbb{R}^K} \exp \left( \sum_{i=1}^{K} \sigma(m)x_i - H(x) \right) dx}
= \ln g_{K,m}(0)\]

Next, we need the following direct consequence of Lemma 3.2.

**Lemma 3.4.** Assume that the single-site potential \(\psi\) satisfies (3) and (4). Then for any finite set \(A_i \subset \{1, 2, \cdots, K\}\) and \(k = 1, 2, \cdots, 6\), we have
\[
\left| \mathbb{E} \left[ \prod_{i_1 \in A_1} \cdots \prod_{i_k \in A_k} (X_{i_1} - m_{i_1}) \cdots (X_{i_k} - m_{i_k}) \right] \right| \lesssim \prod_{i} |A_i| = |A_k| \quad (19)
\]

**Proof of Lemma 3.4.** Using the arithmetic-geometric mean inequality we get
\[
\left| \mathbb{E} \left[ \prod_{i_1 \in A_1} \cdots \prod_{i_k \in A_k} (X_{i_1} - m_{i_1}) \cdots (X_{i_k} - m_{i_k}) \right] \right|
\leq \sum_{i_1 \in A_1} \cdots \sum_{i_k \in A_k} \mathbb{E} \left[ |(X_{i_1} - m_{i_1}) \cdots (X_{i_k} - m_{i_k})| \right]
\leq \sum_{i_1 \in A_1} \cdots \sum_{i_k \in A_k} \mathbb{E} \left[ \frac{1}{k} |(X_{i_1} - m_{i_1})^k + \cdots + \frac{1}{k} |(X_{i_k} - m_{i_k})|^k | \right].
\]

The lemma now follows from the observation that because the gce \(\mu^\sigma\) satisfies a uniform Poincaré inequality it holds that for all \(i \in \{1, 2, \cdots, K\}\) and \(k \in \mathbb{N}\)
\[
\mathbb{E} \left[ |(X_i - m_i)|^k \right] \leq C(k),
\]
where the constant \(C(k)\) only depends on \(k\). \(\square\)

The next auxiliary lemma states that on a one dimensional lattice with nearest-neighbor interaction, the gce has uniform exponential decay of correlations.

**Lemma 3.5.** For all functions \(f, g : \mathbb{R}^K \to \mathbb{R}\) we have
\[
|\text{cov}(f(X), g(X))| \lesssim \left( \int |\nabla f|^2 \mu^\sigma \right)^\frac{1}{2} \left( \int |\nabla g|^2 \mu^\sigma \right)^\frac{1}{2} \exp(-C \text{dist}(\text{supp } f, \text{supp } g)),
\]
where \(\text{supp } f\) and \(\text{supp } g\) denotes the support of the function \(f\) and \(g\) respectively.

**Proof of Lemma 3.2** See [MN14]. \(\square\)

Now, we get to the core estimated needed for the proof of Theorem 2.3 and of Theorem 2.6.
Proposition 3.6. For all \( \alpha > 0 \) and \( \beta > \frac{1}{2} \), there exists a positive constant \( C > 0 \) independent of the mean spin \( m \) and the external field \( s \) such that

\[
\frac{1}{C} \leq g_{K,m}(0) \leq C \quad (20)
\]

\[
\left| \frac{d}{d\sigma}g_{K,m}(0) \right| \lesssim K^\alpha \quad (21)
\]

\[
\left| \frac{d^2}{d\sigma^2}g_{K,m}(0) \right| \lesssim K^\beta \quad (22)
\]

The statement of Proposition 3.6 should be compared to Proposition 31 in [GOVW09] or Proposition 3.1 in [MO13]. The main difference is that in our situation the random variables \( X_1, \ldots, X_K \) are dependent. This also makes the proof of Proposition 3.6 a lot harder.

The estimates of Proposition 3.6 are motivated from deducing a quantitative local central limit theorem for the properly normalized sum of the random variables \( X_1, \ldots, X_K \). For example for iid random variables \( X_1, \ldots, X_K \), the estimate (20) is a weaker version of the quantitative local clt estimate

\[
\left| g_{K,m}(0) - \frac{1}{\sqrt{2\pi}} \right| \lesssim \frac{1}{\sqrt{K}}.
\]

The last inequality states that the density of the normalized sum at point 0 converges to the density of the normal distribution. As we mentioned in the introduction, we believe that one could strengthen the estimates of Proposition 3.6 to get a local central limit theorem for dependent random variables. However, we choose to derive weaker bounds instead because they are sufficiently strong for deducing our main results (see Theorem 2.3 and Theorem 2.6). Deducing those weaker estimates is already quite subtle and challenging.

We deduce Proposition 3.6 in Section 4. There, we also comment on how to overcome the problem of considering dependent random variables and not independent ones.

Now, we are prepared for the proof of Theorem 2.3.

Proof of Theorem 2.3. Theorem 2.3 follows from the Cramér’s representation (18) and Proposition 3.6. Indeed we have

\[
A_{ce}(\sigma) - A_{gce}(\sigma) = \frac{1}{K} \ln g_{K,m}(0) \quad (23)
\]

and thus we have (7) by using (20).

To establish (8) and (9), we choose \( \alpha = \varepsilon \) small enough and let \( \beta = \frac{1}{2} + \varepsilon \). We then take the derivative with respect to \( \sigma \) in (23) to obtain

\[
\frac{d}{d\sigma}A_{ce}(\sigma) - \frac{d}{d\sigma}A_{gce}(\sigma) = \frac{1}{K} \frac{1}{g_{K,m}(0)} \frac{dg_{K,m}(0)}{d\sigma} \quad (24)
\]

Combined with (20) and (21), we get

\[
\left| \frac{d}{d\sigma}A_{ce}(\sigma) - \frac{d}{d\sigma}A_{gce}(\sigma) \right| \lesssim \frac{1}{K} K^\alpha = \frac{1}{K^{1-\varepsilon}}
\]

as desired.
Lastly, we differentiate (24) again and get
\[
\frac{d^2}{d\sigma^2} A_{ce}(\sigma) - \frac{d^2}{d\sigma^2} A_{gce}(\sigma) = -\frac{1}{K} \left( \frac{1}{g_{K,m}(0)} \right)^2 \left( \frac{dg_{K,m}(0)}{d\sigma} \right)^2 + \frac{1}{K} \frac{1}{g_{K,m}(0)} \frac{d^2 g_{K,m}(0)}{d\sigma^2}
\]
Then a combination of (20), (21) and (22) yield the desired inequality
\[
\left| \frac{d^2}{d\sigma^2} A_{ce}(\sigma) - \frac{d^2}{d\sigma^2} A_{gce}(\sigma) \right| \lesssim \frac{1}{K^{1-2\alpha}} + \frac{1}{K^{1-\beta}} \lesssim \frac{1}{K^{1-\varepsilon}}.
\]
\[\Box\]

Let us now turn to the proof of Theorem 2.6.

Proof of Theorem 2.6. Recall the difference of the free energies $A_{gce}$ and $A_{ce}$ given in (11)
\[
A_{gce}(\sigma) - A_{ce}(\sigma) = \bar{H}(m) - H_K(m).
\]
The first inequality (12) follows from (7) in Theorem 2.3. To prove (13) and (14), we derive the following auxiliary result.

\[
\frac{d}{d\sigma} \mathbb{E} [f(X)] = \frac{d}{d\sigma} \int f(x) \frac{\exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right)}{\int \exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right) dx} \, dx
\]
\[
\quad + \int f(x) \left( \sum_{i=1}^{K} (x_i - m_i) \right) \frac{\exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right)}{\int \exp \left( \sum_{i=1}^{K} \sigma x_i - H(x) \right) dx} \, dx
\]
\[
= \mathbb{E} \left[ \frac{d}{d\sigma} f(X) \right] + \mathbb{E} \left[ f(X) \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right]
\]
where $m_i$ is defined by
\[
m_i := \mathbb{E} [X_i]
\]
In particular, we have
\[
\frac{d}{d\sigma} m_k = \mathbb{E} \left[ x_k \sum_{i=1}^{K} (X_i - m_i) \right]
\]
and from (17) and Lemma 3.2
\[
\frac{dm}{d\sigma} = \frac{1}{K} \sum_{k=1}^{K} \frac{d}{d\sigma} m_k
\]
\[
= \frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[ X_k \sum_{i=1}^{K} (X_i - m_i) \right]
\]
\[
= \frac{1}{K} \mathbb{E} \left[ \left( \sum_{i=1}^{K} (X_i - m_i) \right)^2 \right]$

\[ \frac{1}{K} \text{var} \left( \sum_{i=1}^{K} X_i \right) \sim 1 \]  

(25)

Then a direct computation yields

\[ \frac{d}{dm} (H_K(m) - \bar{H}(m)) = \frac{d}{dm} (A_{ce}(\sigma) - A_{gce}(\sigma)) \]

\[ = \frac{d}{d\sigma} (A_{ce}(\sigma) - A_{gce}(\sigma)) \frac{dm}{d\sigma} \]

\[ = \frac{d}{d\sigma} (A_{ce}(\sigma) - A_{gce}(\sigma)) \left( \frac{dm}{d\sigma} \right)^{-1} \]

(26)

and thus (8) and (25) implies, as desired,

\[ \left| \frac{d}{dm} (H_K(m) - \bar{H}(m)) \right| \lesssim \frac{1}{K^{1/\epsilon}} \]

Before we move on to the proof of (14), we deduce one more auxiliary result:

\[ \frac{d}{d\sigma} \left( \frac{d\sigma}{dm} \right) \]

\[ = \frac{d}{d\sigma} \left( \frac{K}{\mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right]} \right) \]

\[ = - \frac{K}{\mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right]} \frac{d}{d\sigma} \left( \mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right] \right) \]

\[ = - \frac{K}{\text{var} \left( \sum_{i=1}^{K} X_i \right)} \frac{d}{d\sigma} \left( \mathbb{E} \left[ \sum_{i=1}^{K} (X_i - m_i) \right] \right) \]

where

\[ \frac{d}{d\sigma} \left( \mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right] \right) \]

\[ = \mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right] \]

\[ = \mathbb{E} \left[ \sum_{i=1}^{K} (X_i - m_i) \right]^3 \]

On the other hand, we have

\[ \left| \mathbb{E} \left[ \left( \sum_{i=1}^{K} (X_i - m_i) \right)^3 \right] \right| \]
\begin{align*}
&\lesssim \sum_{i\leq j\leq k} |\mathbb{E} [(X_i - m_i) (X_j - m_j) (X_k - m_k)]| \\
&\lesssim \sum_{j=1}^{K} \sum_{s=0}^{K-j} \sum_{t=0}^{K-j} |\mathbb{E} [(X_{j-s} - m_{j-s}) (X_j - m_j) (X_{j+t} - m_{j+t})]| \\
&\leq \sum_{j=1}^{K} \sum_{s=0}^{K} \sum_{t=0}^{K} |\mathbb{E} [(X_{j-s} - m_{j-s}) (X_j - m_j) (X_{j+t} - m_{j+t})]| \\
&\quad + \sum_{j=1}^{K} \sum_{s=0}^{K} \sum_{t=0}^{K} |\mathbb{E} [(X_{j-s} - m_{j-s}) (X_j - m_j) (X_{j+t} - m_{j+t})]| \quad (27) \\
&\leq \sum_{j=1}^{K} \sum_{s=0}^{K} \sum_{t=0}^{K} \exp (-Cs) \quad (28)
\end{align*}

Then Lemma 3.5 implies
\begin{align*}
T_{(27)} &= \sum_{j=1}^{K} \sum_{s=0}^{K-j} \sum_{t=0}^{K-j} |\text{cov} ((X_j - m_j) (X_{j+t} - m_{j+t}), X_{j-s} - m_{j-s})| \\
&\lesssim \sum_{j=1}^{K} \sum_{s=0}^{K} \sum_{t=0}^{K} \exp (-Cs) \\
&= K \sum_{s=0}^{K} \exp (-Cs) \lesssim K
\end{align*}

By symmetry we also have
\begin{align*}
T_{(28)} \lesssim K
\end{align*}

and thus we conclude that
\begin{align*}
\left| \frac{d}{d\sigma} \left( \frac{d}{dm} \right) \left( \mathcal{H}_K(m) - \bar{H}(m) \right) \right| &= \left| \frac{K}{\text{var}\left( \sum_{i=1}^{K} X_i \right)^2} \left| \frac{d}{d\sigma} \left( \mathbb{E} \left[ \sum_{k=1}^{K} X_k \left( \sum_{i=1}^{K} (X_i - m_i) \right) \right] \right) \right| \right| \\
&\lesssim \frac{K}{K^2} = 1 \quad (29)
\end{align*}

Now we turn to proof of (14). We differentiate (26) to obtain
\begin{align*}
\frac{d^2}{dm^2} (\mathcal{H}_K(m) - \bar{H}(m)) &= \frac{d}{dm} \left( \frac{d}{d\sigma} \left( A_{ce}(\sigma) - A_{gce}(\sigma) \right) \frac{d\sigma}{dm} \right) \\
&= \frac{d}{d\sigma} \left( \frac{d}{d\sigma} \left( A_{ce}(\sigma) - A_{gce}(\sigma) \right) \frac{d\sigma}{dm} \right) \frac{d\sigma}{dm} \\
&= \frac{d^2}{d\sigma^2} (A_{ce}(\sigma) - A_{gce}(\sigma)) \left( \frac{d\sigma}{dm} \right)^2 + \frac{d}{d\sigma} (A_{ce}(\sigma) - A_{gce}(\sigma)) \frac{d\sigma}{dm} \frac{d\sigma}{dm} \\
&= \frac{d^2}{d\sigma^2} (A_{ce}(\sigma) - A_{gce}(\sigma)) \left( \frac{d\sigma}{dm} \right)^2 + \frac{d}{d\sigma} (A_{ce}(\sigma) - A_{gce}(\sigma)) \frac{d\sigma}{dm} \frac{d\sigma}{dm}
\end{align*}

Then a combination of (8), (9), (25) and (29) yields
\begin{align*}
\left| \frac{d^2}{dm^2} (\mathcal{H}_K(m) - \bar{H}(m)) \right| \lesssim \frac{1}{K^{2-\varepsilon}}
\end{align*}

and this finishes the proof of Theorem 2.6. \qed
4. Proof of Proposition 3.6

The proof of Proposition 3.6 represents the core of our argument. Before turning to the proof, let us motivate and explain our approach in more detail. We will especially emphasize on how the problem of considering dependent and not independent random variables \( X \) is solved.

As we mentioned before, the argument is inspired from deducing local central limit theorems via the Fourier inversion method (see [Fel71] or [MO13]). The main idea of this method is to write the density of the random variable

\[
Z = \frac{1}{\sqrt{K}} \sum_{i=1}^{K} X_i
\]

by Fourier inversion as an integral involving the characteristic function (see (42) below)

\[
\varphi_Z(\xi) = \mathbb{E} [\exp(-i\xi Z)].
\]

The next step is to split up the integral into an inner integral over the interval \(|\xi| \leq \delta\sqrt{K}\) and an outer integral over the interval \(|\xi| \geq \delta\sqrt{K}\). The outer integral usually is an error term and the main contribution comes from the inner integral.

The big advantage of considering independent random variables \( X_i \) is that the characteristic function \( \varphi_Z \) becomes a product of the characteristic functions \( \varphi_{X_i} \) i.e.

\[
\varphi_Z(\xi) = \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} \sum_{i=1}^{K} X_i \right) \right] = \prod_{i=1}^{K} \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} X_i \right) \right] = \prod_{i=1}^{K} \varphi_{X_i} \left( \frac{\xi}{\sqrt{K}} \right).
\]

Then the outer integral is small because each characteristic function \( \varphi_{X_i} < 1 \) is small and decays at least of the order \(|\xi|^{-1}\). For the inner integral, one applies a Taylor expansion onto the functions \( \ln \varphi_{X_i} \) and gets the correct contribution due to the normalization of the random variables. This strategy would yield the desired estimate (20) in the case of independent random variables (see also Section 3 in [MO13]).

For deducing the estimates (21) and (22) in the case of independent random variables one proceeds in a similar way. The obtained integral representation is split up into an inner and an outer integral. One shows that the outer integral is small by using decay of the characteristic functions. The inner integral is estimated again by Taylor expansion. However, the situation becomes more subtle when considering dependent random variables. The obtained integral representation involves several new terms that look like covariances i.e. they are covariances if \( \xi = 0 \). Therefore, we have to be a lot more careful when applying this strategy.

The following observation helps a lot when considering dependent random variables: Because we only consider nearest-neighbor interaction, the even random variables \( X_{\text{even}} \) become independent if we condition on the values of the odd random variables \( X_{\text{odd}} \). Additionally, because our Hamiltonian is quadratic the variances of the conditioned random variables \( X_{\text{even}} | X_{\text{odd}} \).
are uniformly bounded from above and from below (see proof of Lemma 4.1). Using this observation the outer integrals can be estimated in a straight-forward manner. We condition on the odd random variables $X_{\text{odd}}$. By conditional independence we get that the conditional characteristic function becomes a product i.e.

$$
\varphi_Z(\xi) = \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} \sum_{l=1}^{K} X_l \right) \right] 
= \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} \sum_{l,\text{odd}} X_l \right) \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} \sum_{k,\text{even}} X_k \right) | X_l, l \text{ odd} \right] \right] 
= \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} \sum_{l,\text{odd}} X_l \right) \prod_{k,\text{even}} \mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} X_k \right) | X_l, l \text{ odd} \right] \right].
$$

Because the variances of the conditional random variables $X_{\text{even}}|X_{\text{odd}}$ are controlled uniformly in the conditioned values $X_{\text{odd}}$ we have that the conditional characteristic functions

$$
\mathbb{E} \left[ \exp \left( -i\xi \frac{1}{\sqrt{K}} X_k \right) | X_l, l \text{ odd} \right]
$$

decay uniformly (see Lemma 4.1 below). Over-simplifying the argument, this yields the correct bounds on the outer integrals.

The situation for the inner integrals is more tricky and one has to proceed differently for the estimate (20) and for the estimates (21) and (22). Let us first consider the argument for (20). In the inner integral, we condition on the odd random variables $X_{\text{odd}}$. We use the conditional independence and the control on the conditional variances to do a Taylor expansion just for the characteristic functions of the conditional random variables $X_{\text{even}}|X_{\text{odd}}$. Then we show that this suffices to get the desired estimate of (20).

Let us turn to (21) and (22) and explain how the inner integrals are estimated there. As mentioned above, the Taylor expansion becomes a lot more tricky than for (20). For each additional derivative, the argument becomes more and more elaborate. The reason is that whenever calculating the inner and outer integral one ends up with more and more error terms. The first step of the argument is to carefully group those terms such that certain terms chancel and other terms become covariance-like. This means that those terms are a covariance if $\xi = 0$. However, if $\xi \neq 0$ they are not covariances and cannot be estimated by the decay of correlations. We are able to estimate those terms using the following idea: For each error term, we partition the sites of the lattice system into blocks (see Figure 1 and Figure 2 below). Then we carry out a multivariate Taylor expansion. Let’s say we expand the function $F(\xi_1, \xi_2)$ and after expanding we set $\xi_1 = \xi$ and $\xi_2 = \xi$. The variable $\xi_1$ corresponds to the sites within the block and the variable $\xi_2$ corresponds to terms outside of the block. The Taylor expansion with respect to $\xi_2$ can be controlled with the help of decay of correlations. The Taylor expansion with respect to $\xi_1$ cannot be controlled by decay of correlations. It gets worse if the block size is getting large. However, if one chooses the block size carefully, the error terms are not growing too fast and one still is able to control the Taylor expansion of $F(\xi_1, \xi_2)$. 
The proof is organized in the following way. In Section 4.1 we deduce the estimates for the conditional characteristic functions. In Section 4.2 we deduce the estimate (20). In Section 4.3 we deduce the estimate (21) and in Section 4.4 we verify the estimate (22).

4.1. **Auxiliary estimates.** In this section we provide some auxiliary estimates that are needed in the proof of Proposition 3.6. More precisely, we need to estimate the decay of certain characteristic functions (see Lemma 4.1 below).

Let us introduce the auxiliary sets (cf. Figure 1)

\[ E_l^1 := \{ k \mid |k - l| \leq L \} \]

\[ E_l^2 := \{ k \mid |k - n| \leq L \} \]

and (cf. Figure 2)

\[ F_{n,l}^{1,l} := \{ k \mid |k - n| \leq L \text{ or } |k - l| \leq L \} \]  \hspace{1cm} (30)

\[ F_{n,l}^{2,l} := \{ k \mid |k - n| > L \text{ and } |k - l| > L \} \]  \hspace{1cm} (31)

where \( L \ll K \) is a positive integer that will be chosen later.

**Lemma 4.1.** There exists a positive constant \( C > 0 \) such that

\[ \left| \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] \right| \lesssim \left( 1 + \frac{\xi^3}{\sqrt{K}} \right) \exp \left( -C \xi^2 \right) \]  \hspace{1cm} (32)

\[ \left| \mathbb{E} \left[ \exp \left( i \sum_{k \in E_l^2} (X_k - m_k) \frac{\xi}{\sqrt{K}} \right) \mid \mathcal{F}_l \right] \right| \lesssim \left( 1 + \frac{\xi^3}{\sqrt{K}} \right) \exp \left( -C \xi^2 \right) \]  \hspace{1cm} (33)

\[ \left| \mathbb{E} \left[ \exp \left( i \sum_{k \in F_{n,l}^{1,l}} (X_k - m_k) \frac{\xi}{\sqrt{K}} \right) \mid \mathcal{G}_{n,l} \right] \right| \lesssim \left( 1 + \frac{\xi^3}{\sqrt{K}} \right) \exp \left( -C \xi^2 \right) \]  \hspace{1cm} (34)

where \( \mathcal{F}_l \) and \( \mathcal{G}_{n,l} \) denotes the sigma algebra defined by

\[ \mathcal{F}_l := \sigma (X_k, |k - l| \leq L) \]

\[ \mathcal{G}_{n,l} := \sigma (X_k, |k - n| \leq L \text{ or } |k - l| \leq L) \]

**Proof of Lemma 4.1.** We first deduce (32). Let us consider the conditional expectation with respect to \( \{ X_j \mid j : \text{even} \} \). In the case of nearest-neighbor interaction, the conditional Lebesgue density \( \mu(dx_1 dx_3 \cdots \mid x_j, j : \text{even}) \) can be written as

\[
\mu(dx_1 dx_3 \cdots \mid x_j, j : \text{even}) = \frac{1}{Z} \exp \left( \sum_{i: \text{odd}} \sigma x_i - \psi(x_i) - s_i x_i - (J x_{i-1} + J x_{i+1}) x_i \right) \]

\[
= \prod_{i: \text{odd}} \frac{1}{Z} \exp \left( \sigma x_i - \psi(x_i) - s_i x_i + (J x_{i-1} + J x_{i+1}) x_i \right)
\]

which implies that \( \{ \mathbb{E} [X_i \mid X_j, j : \text{even}] \mid i : \text{odd} \} \) are independent and has conditional Lebesgue density

\[
\mu (dx_i | x_2, x_4, \cdots) = \frac{1}{Z} \exp \left( (\sigma - s_i + J x_{i-1} + J x_{i+1}) x_i - \psi(x_i) \right)
\]  \hspace{1cm} (35)
For each $i$, let us denote $m_{i,2}$ to be the conditional expectation
\[ m_{i,2} := \mathbb{E} [X_i \mid X_j, j : \text{even}] \]
From the observation above, we get the product structure of conditional expectation
\[
\mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right]
= \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k:\text{even}} (X_k - m_k) \xi \right) \right] \times \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{i:\text{odd}} (X_i - m_i) \xi \right) \mid X_j, j : \text{even} \right]
= \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k:\text{even}} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{i:\text{odd}} (m_{i,2} - m_i) \xi \right) \right]
\times \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{i:\text{odd}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right]
= \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k:\text{even}} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{i:\text{odd}} (m_{i,2} - m_i) \xi \right) \right]
\times \prod_{i:\text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \tag{36}
\]
Let $h_i$ denote the complex valued function
\[ h_i(\xi) := -\ln \left( \mathbb{E} \left[ \exp \left( i (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right) \]
Equivalently, denote
\[ F_i(\xi) := \exp \left( -h_i(\xi) \right) \]
\[ = \mathbb{E} \left[ \exp \left( i (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \]
Differentiating both sides, we have
\[ F_i'(\xi) = -h_i'(\xi) \exp \left( -h_i(\xi) \right) \]
\[ = i \mathbb{E} \left[ (X_i - m_{i,2}) \exp \left( i (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \]
\[ F_i''(\xi) = -h_i''(\xi) \exp \left( -h_i(\xi) \right) + h_i'(\xi)^2 \exp \left( -h_i(\xi) \right) \]
\[ = -\mathbb{E} \left[ (X_i - m_{i,2})^2 \exp \left( i (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \]
\[ F_i'''(\xi) \]
\[ = -h_i'''(\xi) \exp \left( -h_i(\xi) \right) + 3h_i''(\xi)h_i'(\xi) \exp \left( -h_i(\xi) \right) - (h_i'(\xi))^3 \exp \left( -h_i(\xi) \right) \]
\[ = -i \mathbb{E} \left[ (X_i - m_{i,2})^3 \exp \left( i (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \tag{37} \]
which implies $h_i(0) = h_i'(0) = 0$ and
\[ h_i''(0) = \mathbb{E} \left[ (X_i - m_{i,2})^2 \mid X_j, j : \text{even} \right] \]
Before we proceed, let us note that analogue of Lemma 3.4 holds for the conditional expectation. Precisely, it holds that

$$E \left[ |X_i - m_{i,2}|^k \mid X_j, j : \text{even} \right] \lesssim 1 \quad \text{for } k = 1, 2, \cdots 5$$  \hfill (38)

The proof for (38) is same as that of Lemma 3.4. Therefore, (38) implies

$$|F'_i(\xi)| \leq E \left[ |X_i - m_{i,2}| \mid X_j, j : \text{even} \right] \lesssim 1$$

Here, the constant does not depend on $s_k, \sigma, J$ and $i$. In particular for $|\xi|$ small enough, we have

$$\frac{1}{2} \leq |F_i(\xi)| = |\exp (h_i(\xi))| \leq \frac{3}{2}$$

Inserting this into (37), we obtain

$$\left| h''''_i(\xi) \right| = \left| 3h''_i(\xi)h'_i(\xi) - (h'_i(\xi))^3 \right| \lesssim |\xi|^3$$

where the last inequality follows from (38). We thus have for $|\xi|$ small,

$$\left| h_i(\xi) - \frac{1}{2} s_{i,2}^2 \xi^2 \right| \lesssim |\xi|^3$$

where $s_{i,2}^2$ denotes the conditional variance of $X_i$ with respect to sigma algebra generated by random variables $X_2, X_4, \cdots$ given by

$$s_{i,2}^2 = E \left[ (X_i - m_{i,2})^2 \mid X_j, j : \text{even} \right]$$

Summing up for all odd $i$’s, we have

$$\left| \sum_{i:\text{odd}} h_i \left( \frac{\xi}{\sqrt{K}} \right) - \sum_{i:\text{odd}} \frac{1}{2K} s_{i,2}^2 \xi^2 \right| \lesssim \frac{1}{\sqrt{K}} |\xi|^3$$

In particular for $\left| \frac{\xi}{\sqrt{K}} \right| \leq \delta$ and $\delta > 0$ small enough,

$$\Re \left( \sum_{i:\text{odd}} h_i \left( \frac{\xi}{\sqrt{K}} \right) \right) \geq \sum_{i:\text{odd}} \frac{s_{i,2}^2}{4K} \xi^2$$

Note that for odd $i$, (35) and criterion of Menz (cf. Lemma 9 in [Men11]) implies $s_{i,2}^2$ is uniformly bounded above and below. Therefore, there exists a positive constant $C$ such that

$$\frac{1}{C} \leq \sum_{i:\text{odd}} \frac{1}{2K} s_{i,2}^2 \leq C$$  \hfill (39)
On the other hand, Lipschitz continuity of complex function \( y \mapsto \exp(y) \in \mathbb{C} \) on \( \Re y \leq -\sum_{i: \text{odd}} \frac{s_i^2}{4K} \xi^2 \) yields

\[
\left| \exp \left( -\sum_{i: \text{odd}} h_i \left( \frac{\xi}{\sqrt{K}} \right) \right) - \exp \left( -\sum_{i: \text{odd}} \frac{s_i^2}{4K} \xi^2 \right) \right| \lesssim \frac{1}{\sqrt{K}} |\xi|^3 \exp \left( -\sum_{i: \text{odd}} \frac{s_i^2}{4K} \xi^2 \right) \tag{40}
\]

In particular, (39) implies

\[
\left| \exp \left( -\sum_{i: \text{odd}} h_i \left( \frac{\xi}{\sqrt{K}} \right) \right) \right| \lesssim \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-CL)
\]

where \( C > 0 \) is a positive constant. Inserting this into (36) gives the desired inequality (32).

Let us now address the inequality (33). The proof for this case is almost identical to (32).

Let us introduce the auxiliary sets

\[
E_1^l := \{ k \mid |k - l| \leq L \} I \\
E_2^l := \{ k \mid |k - l| > L \}
\]

We get

\[
\mathbb{E} \left[ \exp \left( \sum_{i \in E_2^l} (X_k - m_k) \frac{\xi}{\sqrt{K}} \right) \mid \mathcal{F}_l \right]
\]

\[
= \mathbb{E} \left[ \exp \left( \sum_{k \in E_2^l; k: \text{even}} (X_k - m_k) \frac{\xi}{\sqrt{K}} + i \sum_{i \in E_2^l; i: \text{odd}} (\mathbb{E} [X_i | \mathcal{F}_l] - m_i) \frac{\xi}{\sqrt{K}} \right) \right]
\]

\[
\times \mathbb{E} \left[ \exp \left( \sum_{i \in E_2^l; i: \text{odd}} (X_i - \mathbb{E} [X_i | \mathcal{F}_l]) \frac{\xi}{\sqrt{K}} \right) \mid \mathcal{F}_l \right] \mid \mathcal{F}_l \tag{41}
\]

where \( \mathcal{F}_l := \sigma \left( X_k, k \in E_1^l \text{ or } k: \text{even} \right) \). Then one can easily prove the analogue of (40) under the assumption that \( L \ll K \):

\[
\mathbb{E} \left[ \exp \left( \sum_{i \in E_2^l; i: \text{odd}} (X_i - \mathbb{E} [X_i | \mathcal{F}_l]) \frac{\xi}{\sqrt{K}} \right) \mid \mathcal{F}_l \right] - \exp \left( -\sum_{i \in E_2^l; i: \text{odd}} \frac{s_i^2}{2K} \xi^2 \right),
\]

\[
\lesssim \frac{1}{\sqrt{K}} |\xi|^3 \exp \left( -\sum_{i \in E_2^l; i: \text{odd}} \frac{s_i^2}{4K} \xi^2 \right)
\]

where \( s_{i: \text{odd}}^2 = \mathbb{E} \left[ (X_i - \mathbb{E} [X_i | \mathcal{F}_l])^2 \right] \mid \mathcal{F}_l \right) \). Moreover, for \( L \ll K \) and \( K \) large enough, there exists a uniform constant \( C > 0 \) with

\[
\frac{1}{C} \leq \sum_{i \in E_2^l; i: \text{odd}} \frac{s_i^2}{4K} \leq C
\]
This implies, as desired,
\[
\mathbb{E}\left[ \exp \left( i \sum_{i \in E^1_\text{odd}} (X_i - \mathbb{E}[X_i | \mathcal{F}^1_i]) \xi \frac{\xi}{\sqrt{K}} \right) \right] \lesssim \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp \left( - \sum_{i \in E^1_\text{odd}} \frac{s_i^2 \xi^2}{4K} \right)
\]
\[
\lesssim \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp (-C \xi^2)
\]
In particular, with (41),
\[
\mathbb{E}\left[ \exp \left( i \sum_{k \in E^1_\text{odd}} (X_k - m_k) \xi \frac{\xi}{\sqrt{K}} \right) \right] \lesssim \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp (-C \xi^2)
\]
which proves (33). The inequality (34) is deduced by the same type of argument.

4.2. **Proof of (20) in Proposition 3.6.** An application of the inverse Fourier transform with (17) yields the representation
\[
2\pi g_{K,m}(0) = \int \mathbb{E}\left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^K (X_k - m) \xi \right) \right] d\xi
\]
\[
= \int \mathbb{E}\left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^K (X_k - m_k) \xi \right) \right] d\xi
\]
(42)
Let us divide the integral (42) into two parts
\[
\int \mathbb{E}\left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^K (X_k - m_k) \xi \right) \right] d\xi
\]
\[
= \int \left\{ |(1/\sqrt{K})\xi| \leq \delta \right\} \mathbb{E}\left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^K (X_k - m_k) \xi \right) \right] d\xi
\]
\[
+ \int \left\{ |(1/\sqrt{K})\xi| > \delta \right\} \mathbb{E}\left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^K (X_k - m_k) \xi \right) \right] d\xi
\]
=: (a) + (b) (43)

**Argument for (a) in (43):** The estimate (40) gives
\[
\left| (a) - \int \left\{ |(1/\sqrt{K})\xi| \leq \delta \right\} \mathbb{E}\left[ \hat{e}(\xi) \exp \left( - \sum_{i \text{ odd}} \frac{s_i^2 \xi^2}{2K} \right) \right] d\xi \right|
\]
\[
\lesssim \int \left\{ |(1/\sqrt{K})\xi| \leq \delta \right\} \frac{1}{\sqrt{K}} |\xi|^3 \exp \left( - \sum_{i \text{ odd}} \frac{s_i^2 \xi^2}{4K} \right) d\xi
\]
\[
\lesssim \frac{1}{\sqrt{K}} \int_{\mathbb{R}} |\xi|^3 \exp (-C \xi^2) d\xi
\]
\[
\lesssim \frac{1}{\sqrt{K}}
\]
(44)
where \( \hat{e}(\xi) \) is defined to be
\[
\hat{e}(\xi) := \exp \left( i \frac{1}{\sqrt{K}} \sum_{k: \text{even}} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{i: \text{odd}} (m_{i,2} - m_i) \xi \right)
\]

By Fubini’s theorem, we have
\[
\int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \mathbb{E} \left[ \hat{e}(\xi) \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) \right] d\xi
= \mathbb{E} \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \hat{e}(\xi) \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) d\xi
\]

We claim that (45) is uniformly bounded below and above by positive constants. To prove this, we divide the integral as follows
\[
\mathbb{E} \left[ \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \hat{e}(\xi) \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) d\xi \right]
= \mathbb{E} \left[ \int_{\|X\| > \delta} \hat{e}(\xi) \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) d\xi \right]
\]

Our aim is to prove that the whole integral given by (46) is uniformly bounded above and below while the outer integral (47) is relatively small. Let us start with (46). The upper bound follows from (39) that
\[
|T_{\text{odd}}| \leq \mathbb{E} \left[ \int_{\|X\| > \delta} \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) d\xi \right] \leq \mathbb{E} \left[ \int_{\|X\| > \delta} \exp \left( - \frac{C}{2} \xi^2 \right) d\xi \right] \leq 1
\]

Next, we compute the integral inside the expectation directly to get the uniform lower bound
\[
\int_{\mathbb{R}} \hat{e}(\xi) \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \xi^2 \right) d\xi
= \int_{\mathbb{R}} \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \left( \xi - i \sum_{i: \text{odd}} \frac{s_{i,2}}{s_{i,1,2}} \left( \sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i) \right) \right)^2 \right) d\xi
\]

\[
\times \exp \left( - \frac{(\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i))^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right) d\xi
= \int_{\mathbb{R}} \exp \left( - \sum_{i: \text{odd}} \frac{s_{i,2}^2}{2K} \left( \xi - i \sum_{i: \text{odd}} \frac{s_{i,2}}{s_{i,1,2}} \left( \sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i) \right) \right)^2 \right) d\xi
\]

\[
\times \exp \left( - \frac{(\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i))^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right)
\]

(48)
Note that $\frac{\sqrt{R}}{\sum_{i: \text{odd}} s_{i,2}^2} (\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)) \in \mathbb{R}$. Then complex contour integration implies

$$T_{48} = \int_{\mathbb{R}} \exp \left( - \sum_{i: \text{odd}} s_{i,2}^2 \xi^2 \right) d\xi \times \exp \left( - \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right)$$

$$= \sqrt{\frac{2K\pi}{\sum_{i: \text{odd}} s_{i,2}^2}} \exp \left( - \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right)$$

Due to (39) we have

$$\mathbb{E} \left[ \frac{2K\pi}{\sum_{i: \text{odd}} s_{i,2}^2} \exp \left( - \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right) \right]$$

$$\geq \mathbb{E} \left[ \exp \left( - \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right) \right]$$

Then Jensen’s inequality yield

$$\mathbb{E} \left[ \exp \left( - \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right) \right]$$

$$\geq \exp \left( - \mathbb{E} \left[ \frac{\sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2}{2 \sum_{i: \text{odd}} s_{i,2}^2} \right] \right)$$

$$\geq \exp \left( - \mathbb{E} \left[ \frac{C}{K} \left( \sum_{k: \text{even}} (X_k - m_k) + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2 \right) \right] \right)$$

$$\geq \exp \left( - \mathbb{E} \left[ \frac{2C}{K} \left( \sum_{k: \text{even}} (X_k - m_k)^2 + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2 \right) \right] \right)$$

Finally, it follows from Lemma 9 in [Men11] and Lemma 3.2 that

$$\exp \left( - \mathbb{E} \left[ \frac{2C}{K} \left( \sum_{k: \text{even}} (X_k - m_k)^2 + \sum_{i: \text{odd}} (m_{i,2} - m_i)^2 \right) \right] \right)$$

$$= \exp \left( - \frac{2C}{K} \text{var} \left( \sum_{k: \text{even}} X_k \right) - \frac{2C}{K} \text{var} \left( \sum_{i: \text{odd}} m_{i,2} \right) \right)$$

$$\geq C$$

as desired.
On the other hand, the integral (47) is uniformly bounded above by
\[ |T^{(47)}| \leq E \left[ \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \exp \left( -\sum_{i:odd} \frac{s_i^2}{2K} \xi^2 \right) d\xi \right] \]
\[ \leq E \left[ \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \exp \left( -C \xi^2 \right) d\xi \right] \]
\[ \lesssim \frac{1}{\sqrt{K}} \] (50)

Therefore a combination of (49) and (50) implies that there exists a positive constant \( C > 0 \) with
\[ E \left[ \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \hat{e}(\xi) \exp \left( -\sum_{i:odd} \frac{s_i^2}{2K} \xi^2 \right) d\xi \right] \in \left( \frac{1}{C}, C \right) \]
for \( K \) large enough. Inserting this into (44) leads
\[ \frac{1}{C} \leq (a) \leq C \] (51)
for large enough positive integer \( K \).

**Argument for (b) in (43):** The main ingredients for this part are Lemma 3.4 and (47) in [MO13]. From the observation (35), we have in particular for any \( \delta > 0 \), there exists \( \lambda < 1 \) with
\[ |E[\exp (i (X_l - m_{l,2}) \xi) \mid X, j \geq even)]| \leq \lambda \quad \text{for all } |\xi| \geq \delta \] (52)
and
\[ |E[\exp (i (X_l - m_{l,2}) \xi) \mid X, j \geq even]| \lesssim |\xi|^{-1} \] (53)

Consider the conditional expectation with respect to \( \{ X_j \mid j \geq \text{even} \} \):
\[ \left| \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \right| \]
\[ = \left| \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k:even} (X_k - m_k) \xi \right) \right] \right| \]
\[ \times E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{i:odd} (X_i - m_i) \xi \right) \mid X, j \geq even \right] d\xi \]
\[ \leq \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \left| E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k:even} (X_k - m_k) \xi \right) \mid X, j \geq even \right] \right| d\xi \]

We apply (52) (on \( K/2 \) factors) and (53) (on the remaining \( K/2 \) factors) to obtain
\[ \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \left| E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{i:odd} (X_i - m_i) \xi \right) \mid X, j \geq even \right] \right| d\xi \]
\[ \lesssim \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \lambda^{K^2/2} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^2 d\xi \]
\[ \lesssim \int_{\{|(1/\sqrt{K})\xi| > \delta\}} K \lambda^{K/2 - 2} \frac{1}{K + \xi^2} d\xi \]
\[ \lesssim \int_{\{|(1/\sqrt{K})\xi| > \delta\}} K \lambda^{K/2 - 2} \frac{1}{1 + \xi^2} d\xi \]
\[ \leq K \lambda^{K/2 - 2} \int_{\mathbb{R}} \frac{1}{1 + \xi^2} d\xi \]
\[ \lesssim K \lambda^{K/2 - 2} \lambda < 1 \ll \frac{1}{\sqrt{K}} \]

That is,
\[ \left|(b)\right| \lesssim \frac{1}{\sqrt{K}} \quad (54) \]

Thus (43), (51) and (54) imply there exists a positive constant \( C > 0 \) such that
\[ \frac{1}{C} \leq g_{K,m}(0) \leq C \]
for \( K > 0 \) large enough, as desired.

\[ \Box \]

4.3. **Proof of (21) in Proposition 3.6** Let us turn to (21). Similar to (20), we will divide the integral (42) into inner part and outer part. More precisely, write
\[
2\pi \frac{d}{d\sigma} g_{K,m}(0)
= \frac{d}{d\sigma} \int_{\mathbb{R}} \mathbb{E} \left[ \exp\left( -i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi
= \frac{d}{d\sigma} \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \mathbb{E} \left[ \exp\left( -i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi
+ \frac{d}{d\sigma} \int_{\{|(1/\sqrt{K})\xi| > \delta\}} \mathbb{E} \left[ \exp\left( -i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \quad (55)
\]

**Argument for (55)**: To begin with, introduce for every site \( l \) the variable \( \sigma_l \). Then we have
\[
\frac{d}{d\sigma} \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \mathbb{E} \left[ \exp\left( -i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi
= \sum_l \left[ \frac{d}{d\sigma_l} \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \mathbb{E} \left[ \exp\left( -i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \right] \bigg|_{\sigma_l = \sigma} \quad (57)
\]

Let us introduce the auxiliary sets
\[ E_1^l := \{ k \mid |k - l| \leq L \} \quad (58) \]
\[ E_2^l := \{ k \mid |k - l| > L \}. \quad (59) \]
For each \( l \) we define the function \( F_l(\xi_1, \xi_2) \) by

\[
F_l(\xi_1, \xi_2) := \mathbb{E} \left[ \exp \left( i \sum_{i \in E_{l1}^l} (X_i - m_i) \xi_1 + i \sum_{j \in E_{l2}^l} (X_j - m_j) \xi_2 \right) \right]
\]

where \( E_{l_1}^l := \{ k : |k - l| \leq L \} \), \( E_{l_2}^l := \{ k : |k - l| > L \} \) and \( L = K^c \ll K \). (see Figure 1)

In view of (57), it is enough to prove

\[
\left| \int \{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \} d\sigma_l F_l \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \right|_{\sigma_l = \sigma} \lesssim \frac{1}{K^{1-\alpha}}
\]

for any given \( \alpha \in (0, \frac{1}{2}) \). Note that for \( j = 1, 2 \)

\[
d \frac{d}{d\sigma_l} F_l(0, 0) = d \frac{d}{d\xi_1} d \frac{d}{d\sigma_l} F_l(0, 0) = d \frac{d}{d\xi_j} d \frac{d}{d\sigma_l} F_l(0, 0) = 0.
\]

Let us carry out a Taylor expansion to the second order in the variable \( \xi_1 \), i.e.

\[
d \frac{d}{d\sigma_l} F_l(\xi_1, \xi_2) = d \frac{d}{d\sigma_l} F_l(0, \xi_2) + d \frac{d}{d\xi_1} d \frac{d}{d\sigma_l} F_l(0, \xi_2) \xi_1 + \frac{1}{2} d^2 \frac{d}{d\xi_1^2} d \frac{d}{d\sigma_l} F_l(\tilde{\xi}_1, \xi_2) \xi_1^2.
\]

(60)

Then we can use (60) to split the inner integral accordingly:

\[
\int \{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \} d\sigma_l F_l \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) d\xi
\]

\[
= \int \{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \} d\sigma_l F_l \left( 0, \frac{\xi}{\sqrt{K}} \right) d\xi
\]

\[
+ \int \{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \} d\sigma_l F_l \left( 0, \frac{\xi}{\sqrt{K}} \right) \frac{\xi}{\sqrt{K}} d\xi
\]

\[
+ \frac{1}{2} \int \{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \} d^2 \frac{d}{d\xi_1^2} d \frac{d}{d\sigma_l} F_l \left( \frac{\tilde{\xi}_1}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi
\]

\[
=: (a) + (b) + (c)
\]

(61)

Before we investigate each term, we derive the following auxiliary computation

\[
d \frac{d}{d\sigma_l} \mathbb{E} [f(X)]
\]
\[
\begin{align*}
&= \frac{d}{d\sigma_l} \int f(x) \exp \left( \sum_{k=1}^{K} \sigma_k x_k - H(x) \right) \\
&\quad \cdot \frac{\exp \left( \sum_{k=1}^{K} \sigma_k x_k - H(x) \right)}{\int \exp \left( \sum_{k=1}^{K} \sigma_k x_k - H(x) \right) dx} dx \\
&= \int (x_l - m_l) f(x) \exp \left( \sum_{k=1}^{K} \sigma_k x_k - H(x) \right) \frac{df(x)}{d\sigma_l} dx + \frac{df(x)}{d\sigma_l} dx
\end{align*}
\]

\[
= \mathbb{E} [(X_l - m_l) f(X)] + \mathbb{E} \left[ \frac{df}{d\sigma_l}(X) \right]
\]

(62)

In particular,

\[
\frac{d}{d\sigma_l} m_k = \mathbb{E} [X_k] = \mathbb{E} [(X_l - m_l) X_k]
\]

(63)

Now let us begin with the term (a) in (61). The integrand becomes

\[
\frac{d}{d\sigma_l} F_l(\xi_1, \xi_2)
\]

\[
= \mathbb{E} [(X_l - m_l) e(\xi_1, \xi_2)]
\]

\[
+ \mathbb{E} \left[ -i \xi_1 \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in E^l_1} X_i \right) \right] e(\xi_1, \xi_2) \right]
\]

\[
+ \mathbb{E} \left[ -i \xi_2 \mathbb{E} \left[ (X_l - m_l) \left( \sum_{j \in E^l_2} X_j \right) \right] e(\xi_1, \xi_2) \right]
\]

where \(e(\xi_1, \xi_2)\) denotes the exponential term

\[
e(\xi_1, \xi_2) := \exp \left( i \sum_{i \in E^l_1} (X_l - m_l) \xi_1 + i \sum_{j \in E^l_2} (X_j - m_j) \xi_2 \right)
\]

Plugging in \(\xi_1 = 0\) and \(\xi_2 = \frac{\xi}{\sqrt{K}}\) into the formula above gives

\[
\frac{d}{d\sigma_l} F_l \left( 0, \frac{\xi}{\sqrt{K}} \right)
\]

\[
= \mathbb{E} \left[ (X_l - m_l) \exp \left( i \sum_{j \in E^l_2} (X_j - m_j) \frac{\xi}{\sqrt{K}} \right) \right]
\]

\[
+ \mathbb{E} \left[ -i \frac{\xi}{\sqrt{K}} \mathbb{E} \left[ (X_l - m_l) \left( \sum_{j \in E^l_2} X_j \right) \right] \exp \left( i \sum_{j \in E^l_2} (X_j - m_j) \frac{\xi}{\sqrt{K}} \right) \right]
\]

\[
= \mathbb{E} \left[ (X_l - m_l) \exp \left( i \sum_{j \in E^l_2} (X_j - m_j) \frac{\xi}{\sqrt{K}} \right) \right]
\]

\[
- i \frac{\xi}{\sqrt{K}} \mathbb{E} \left[ (X_l - m_l) \left( \sum_{j \in E^l_2} X_j \right) \right] \mathbb{E} \left[ \exp \left( i \sum_{j \in E^l_2} (X_j - m_j) \frac{\xi}{\sqrt{K}} \right) \right]
\]

(64)

(65)
We have by using Lemma 3.5 that
\[
| T_{64} | = \left| \text{cov} \left( X_t, \exp \left( i \sum_{j \in E_t} (X_j - m_j) \frac{\xi}{\sqrt{K}} \right) \right) \right|
\]
\[
\lesssim \left( (K - L) \left| \frac{i \xi}{\sqrt{K}} \right|^2 \right)^{\frac{1}{2}} \exp (-CL)
\]
\[
\lesssim |\xi| \exp (-CL)
\]
(66)
and
\[
| T_{65} | \leq \left| \frac{\xi}{\sqrt{K}} \right| \left| \text{cov} \left( X_t, \left( \sum_{j_1 \in E_{t1}} X_{j_1} \right) \right) \right|
\]
\[
\lesssim \left| \frac{\xi}{\sqrt{K}} \right| (K - L)^{\frac{1}{2}} \exp (-CL)
\]
\[
\lesssim |\xi| \exp (-CL)
\]
(67)
As desired, inserting (66) and (67) into (a) in (61) leads
\[
| (a) | \lesssim \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} |\xi| \exp (-CL) \, d\xi
\]
\[
\lesssim K \exp (-CL) \lesssim \frac{1}{K^{1-\alpha}}
\]
(68)

We now address the third term (c) in (61). The second term (b) in (61) will be treated later. A direct computation using (62) and (63) yields
\[
\frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_t} F_t \left( \tilde{\xi}_{1,t}, \xi_2 \right)
\]
\[
= \frac{d}{d\sigma_t} \frac{d^2}{d\xi_1^2} F_t \left( \tilde{\xi}_{1,t}, \xi_2 \right)
\]
\[
= \frac{d}{d\sigma_t} \mathbb{E} \left[ \left( i \sum_{i \in E_{1t}} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,t}, \xi_2 \right) \right]
\]
\[
= \mathbb{E} \left[ (X_t - m_t) \left( i \sum_{i \in E_{1t}} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,t}, \xi_2 \right) \right]
\]
\[
+ \mathbb{E} \left[ 2 \left( i \sum_{i_1 \in E_{1t}} (X_{i_1} - m_{i_1}) \right) \cdot (-i) \mathbb{E} \left[ (X_t - m_t) \sum_{i_2 \in E_{2t}} X_{i_2} \right] e \left( \tilde{\xi}_{1,t}, \xi_2 \right) \right]
\]
\[
+ \mathbb{E} \left[ \left( i \sum_{i_1 \in E_{1t}} (X_{i_1} - m_{i_1}) \right)^2 (-i\tilde{\xi}_{1,t}) \mathbb{E} \left[ (X_t - m_t) \sum_{i_2 \in E_{2t}} X_{i_2} \right] e \left( \tilde{\xi}_{1,t}, \xi_2 \right) \right]
\]
(69)
\[ + \mathbb{E} \left[ \left( i \sum_{i \in E_1} (X_i - m_i) \right)^2 \right] (-i \xi_2) \mathbb{E} \left[ (X_l - m_l) \sum_{j \in E_2} X_j \right] e \left( \tilde{\xi}_{1, l}, \xi_2 \right) \] (72)

where \( e \left( \tilde{\xi}_{1, l}, \xi_2 \right) \) denotes
\[ e \left( \tilde{\xi}_{1, l}, \xi_2 \right) := \exp \left( i \sum_{i \in E_1} (X_i - m_i) \tilde{\xi}_{1, l} + i \sum_{j \in E_2} (X_j - m_j) \xi_2 \right) \]

Let us insert \( (\xi_1, \xi_2) = \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \). To address the terms above, we use the same trick given in the proof of (20). Let us begin with the first term (69).

\[ T_{69} = \mathbb{E} \left[ (X_l - m_l) \left( i \sum_{i \in E_1} (X_i - m_i) \right)^2 \exp \left( i \sum_{i_2 \in E_1} (X_i - m_i) \tilde{\xi}_{1, l} \right) \right] \times \mathbb{E} \left[ \exp \left( i \sum_{j \in E_2} (X_j - m_j) \xi_2 \right) \right] \]

where \( \mathcal{F}_l = \sigma \left( X_k, k \in E_1 \right) \). Then (33) implies
\[ |T_{69}| \lesssim \mathbb{E} \left[ |X_l - m_l| \left( \sum_{i \in E_1} (X_i - m_i) \right)^2 \right] \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \]

On the other hand, (19) in Corollary 3.4 yields
\[ \mathbb{E} \left[ |X_l - m_l| \left( \sum_{i \in E_1} (X_i - m_i) \right)^2 \right] \lesssim L^2 \]

Thus we conclude that
\[ |T_{69}| \lesssim L^2 \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \] (73)

Similarly, one gets
\[ |T_{70}| \lesssim \left| \mathbb{E} \left[ (X_l - m_l) \sum_{i_2 \in E_1} X_{i_2} \right] \mathbb{E} \left[ \sum_{i_1 \in E_1} |X_{i_1} - m_{i_1}| \right] \right| \times \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \]

\[ = \mathbb{E} \left[ (X_l - m_l) \sum_{i_2 \in E_1} (X_{i_2} - m_{i_2}) \right] \mathbb{E} \left[ \sum_{i_1 \in E_1} |X_{i_1} - m_{i_1}| \right] \times \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \]
\[ L^2 \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \tag{74} \]

and

\[
\left| T_{\text{74}} \right| \lesssim \frac{|\xi|}{\sqrt{K}} \left| \mathbb{E} \left[ (X_l - m_l) \sum_{j \in E'_2} X_j \right] \right| \left| \mathbb{E} \left[ \left( \sum_{i \in E'_1} (X_{i'i} - m_{i'i}) \right)^2 \right] \right| \\
\times \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \\
\lesssim \frac{L^3}{\sqrt{K}} |\xi| \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \tag{75} \]

To deduce an appropriate estimate for (72), we apply the exponential decay of correlations used in (67)

\[
\left| \mathbb{E} \left[ (X_l - m_l) \sum_{j \in E'_2} X_j \right] \right| = \left| \text{cov} \left( X_l, \sum_{j \in E'_2} X_j \right) \right| \lesssim \sqrt{K} \exp(-CL) 
\]

This bound combined with (19) and (33) leads

\[
\left| T_{\text{72}} \right| \lesssim \frac{|\xi|}{\sqrt{K}} \left| \mathbb{E} \left[ (X_l - m_l) \sum_{j \in E'_2} X_j \right] \right| \left| \mathbb{E} \left[ \left( \sum_{i \in E'_1} (X_{i'i} - m_{i'i}) \right)^2 \right] \right| \\
\times \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \\
\lesssim L^2 \exp(-CL) |\xi| \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \tag{76} \]

Combining the estimates (73), (74), (75), and (76), we conclude that (c) in (61) is bounded above by

\[
|c| \lesssim \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} L^2 \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \\
+ \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{L^3}{\sqrt{K}} |\xi| \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \\
+ \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \left( L^2 \exp(-CL) \\
\times |\xi| \left( 1 + \frac{1}{\sqrt{K}} |\xi|^3 \right) \exp(-C \xi^2) \left( \frac{\xi}{\sqrt{K}} \right)^2 \right) d\xi \\
\lesssim \frac{L^2}{K} + \frac{L^3}{K^{3/2}} + \frac{L^2}{K} \exp(-CL) \\
\lesssim \frac{1}{K^{1-\alpha}} \tag{77} \]
It remains to deal with the second term \( b \) in (61). Let us carry out a Taylor expansion in the second variable again to get
\[
\frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(0, \xi_2) \xi_1 = \frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(0, 0) \xi_1 + \frac{d}{d\xi_2} \frac{d}{d\sigma_1} \frac{d}{d\xi_1} F_l(0, \xi_2, \xi_1) \xi_1 \xi_2
\]
(78)

Here, the first term in (78) vanishes because
\[
\frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(0, 0) = \frac{d}{d\sigma_1} \frac{d}{d\xi_1} F_l(0, 0) = \frac{d}{d\sigma_1} \mathbb{E} \left[ \sum_{i \in E_1} (X_i - m_i) \right] = \frac{d}{d\sigma_1} 0 = 0
\]
and thus we may write
\[
(b) = \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(0, \xi) \frac{\xi}{\sqrt{K}} d\xi = \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d}{d\xi_2} \frac{d}{d\sigma_1} \frac{d}{d\xi_1} F_l(0, \frac{\xi_1}{\sqrt{K}} \frac{\xi_2}{\sqrt{K}})^2 d\xi
\]

Let us first focus on the term \( \frac{d}{d\xi_2} \frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(0, \frac{\xi_1}{\sqrt{K}}) \) in the integrand. We directly differentiate \( F_l(\xi_1, \xi_2) \) to obtain
\[
\frac{d}{d\xi_2} \frac{d}{d\xi_1} \frac{d}{d\sigma_1} F_l(\xi_1, \xi_2)
\]
\[
= \frac{d}{d\sigma_1} \mathbb{E} \left[ \left( \sum_{i \in E_1} (X_i - m_i) \right) \left( \sum_{j \in E_2} (X_j - m_j) \right) e(\xi_1, \xi_2) \right]
\]
(79)
\[
+ \mathbb{E} \left[ \left( \sum_{i \in E_1} (X_i - m_i) \right) \left( \sum_{j \in E_2} (X_j - m_j) \right) e(\xi_1, \xi_2) \right]
\]
(80)
\[
+ \mathbb{E} \left[ \left( \sum_{i \in E_1} (X_i - m_i) \right) \left( \sum_{i \in E_1} (X_i - m_i) \right) e(\xi_1, \xi_2) \right]
\]
(81)
\[
+ \mathbb{E} \left[ \left( \sum_{i \in E_1} (X_i - m_i) \right) \left( \sum_{j \in E_2} (X_j - m_j) \right) \right. \left. e(\xi_1, \xi_2) \right]
\]
(82)
Using (84) and definition of covariances, we can combine (79) and (80) as follows:

\[
+ \mathbb{E} \left[ \left( \sum_{i \in E_1} (X_i - m_i) \right) \left( \sum_{j_1 \in E_2} (X_{j_1} - m_{j_1}) \right) \right] \times (-i \xi_2) \mathbb{E} \left[ (X_l - m_l) \sum_{j_2 \in E_2} X_{j_2} \right] e(\xi_1, \xi_2)
\]  

(83)

Now put \((\xi_1, \xi_2) = \left(0, \frac{\tilde{\xi}_l}{\sqrt{K}} \right)\) and denote \(\tilde{e}_2\) to be

\[\tilde{e}_2 = e(0, \frac{\tilde{\xi}_l}{\sqrt{K}}) = \exp \left(i \sum_{j \in E_2} (X_j - m_j) \frac{\tilde{\xi}_l}{\sqrt{K}} \right)\]

Note that

\[
\mathbb{E} \left[ (X_l - m_l) \sum_{k \in E_1} X_k \right] = \mathbb{E} \left[ (X_l - m_l) \sum_{k \in E_1} (X_k - m_k) \right]
\]  

(84)

Using (84) and definition of covariances, we can combine (79) and (80) as follows:

\[
T_{79} + T_{80}
\]

\[
= - \sum_{i: |i-l| \leq L/2} \text{cov} \left( (X_l - m_l) (X_i - m_i), \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right) \]

\[
- \sum_{i: L/2 \leq |i-l| \leq L} \mathbb{E} \left[ (X_l - m_l) (X_i - m_i) \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right]
\]

\[
+ \sum_{i: L/2 \leq |i-l| \leq L} \mathbb{E} \left[ (X_l - m_l) X_i \right] \mathbb{E} \left[ \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right]
\]

\[
= - \sum_{i: |i-l| \leq L/2} \text{cov} \left( (X_l - m_l) (X_i - m_i), \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right)
\]  

(85)

\[
- \sum_{i: L/2 \leq |i-l| \leq L} \text{cov} \left( X_l, (X_i - m_i) \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right)
\]  

(86)

\[
+ \sum_{i: L/2 \leq |i-l| \leq L} \text{cov} \left( X_l, X_i \right) \mathbb{E} \left[ \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right]
\]  

(87)

Again, we use Lemma 3.5 and (19) in Lemma 3.4 to conclude that for \(|i-l| \leq L/2\), it holds

\[
\left| \text{cov} \left( (X_l - m_l) (X_i - m_i), \left( \sum_{j \in E_2} (X_j - m_j) \right) \tilde{e}_2 \right) \right| \lesssim \left( \mathbb{E} \left[ (X_l - m_l)^2 + (X_i - m_i)^2 \right] \right)^{\frac{1}{2}}
\]
\[
\left( \frac{X}{\sqrt{K}} \sum_{j_{1} \in E_{2}^{i}} \sum_{j_{2} \in E_{2}^{i}} (X_{j_{2}} - m_{j_{2}}) \right) \left( i \frac{\xi}{\sqrt{K}} \sum_{j_{2} \in E_{2}^{i}} (X_{j_{2}} - m_{j_{2}}) \right) \right) \frac{1}{2} \exp (-CL/2)
\]

\[\lesssim (K + K^{2} \xi^{2}) \frac{1}{2} \exp (-CL/2)\]

\[\lesssim K (1 + \left| \xi \right|) \exp (-CL/2)\]

while for the case when \(L/2 \leq \left| i - l \right| \leq L\), similar computation yields

\[
\left| \text{cov} \left( X_{l}, (X_{i} - m_{i}) \sum_{j \in E_{2}^{i}} (X_{j} - m_{j}) \right) \right|
\]

\[\lesssim (K^{2} + K + K^{2} \xi^{2}) \frac{1}{2} \exp (-CL/2) \lesssim K (1 + \left| \xi \right|) \exp (-CL/2)\]

and

\[|\text{cov} (X_{l}, X_{i})| \lesssim \exp (-CL/2)\]

Inserting these estimates into (85), (86), and (87), it follows that

\[
\left| T_{85}^{[80]} + T_{80}^{[80]} \right| \lesssim KL (1 + \left| \xi \right|) \exp (-CL/2) \quad (88)
\]

The fourth term \(T_{82}^{[82]}\) vanishes when \((\xi_1, \xi_2) = \left(0, \frac{\xi}{\sqrt{K}}\right)\). On the other hand, one can prove the exponential decay of \(T_{81}^{[81]}\) and \(T_{83}^{[83]}\) as follows:

\[
\left| T_{81}^{[81]} \right| = \left| \text{cov} \left( X_{l}, \sum_{j \in E_{2}^{i}} X_{j} \right) \right| \left| \text{cov} \left( \sum_{i \in E_{1}^{i}} (X_{j} - m_{j}) \right) \right| \left| \text{cov} \left( \sum_{i \in E_{1}^{i}} (X_{j} - m_{j}) \right) \right| \lesssim K \exp (-CL) L \quad (89)
\]

and

\[
\left| T_{83}^{[83]} \right| = \left| \text{cov} \left( \sum_{j \in E_{2}^{i}} (X_{j} - m_{j}) \right) \sum_{i \in E_{1}^{i}} (X_{i} - m_{i}) \right| \left| \text{cov} \left( \sum_{i \in E_{1}^{i}} (X_{j} - m_{j}) \right) \sum_{j \in E_{2}^{i}} (X_{j} - m_{j}) \right| \lesssim \sqrt{K} \exp (-CL) L \frac{|\xi|}{\sqrt{K}} \quad (90)
\]

To sum up, (88), (89) and (90) yield the upper bound of (b) in (61)

\[
|b| \lesssim \int_{\{|(1/\sqrt{K})| \leq \delta\}} KL (1 + |\xi|) \exp (-CL/2) \left( \frac{|\xi|}{\sqrt{K}} \right)^{2} d\xi
\]
\[
+ \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \sqrt{K} L \exp (-CL) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \\
+ \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} KL \exp (-CL) |\xi| \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi
\]

\[
L = K^{\varepsilon} \geq \frac{1}{K^{1-\alpha}} \tag{91}
\]

A combination of (68), (91), and (77) leads the desired estimate for inner integral in (55). Precisely, we have proven that for any \( \alpha > 0, \)
\[
\left| \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d}{d\sigma} F_l \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) d\xi \right| \lesssim \frac{1}{K^{1-\alpha}}.
\]

Thus, putting this into (57) gives
\[
\left| \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d}{d\sigma} \E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \right| \lesssim K^\alpha \tag{92}
\]

**Argument for (56)**: The main ingredients for this part are (52) and (53) with Lemma 3.5 given in [MO13]. From the observation (35), we have in particular
\[
\left| \frac{d}{d\sigma} \E \left[ \exp \left( i (X_l - m_{l,2}) \xi \right) | X_j, j : even \right] \right| \lesssim (1 + |\xi|) |\xi|^3 \tag{93}
\]

Let us move on to the proof. Note it holds
\[
\frac{d}{d\sigma} \E \left[ \exp \left( i (X_l - m_{l,2}) \xi \right) | X_j, j : even \right] = \E \left[ \frac{d}{d\sigma} \left( \sum_{s=1}^{K} (X_s - m_s) f(X) \right) + \E \left[ \frac{d}{d\sigma} f(X) \right] \right] \tag{95}
\]

In particular, putting \( f(X) = X_k \) in (95), we obtain
\[
\frac{d}{d\sigma} m_k = \frac{d}{d\sigma} \E [X_k] = \E \left[ \sum_{s=1}^{K} (X_s - m_s) \right] X_k \tag{94}
\]

Let us again consider the conditional expectation of \( X_i \)'s with respect to \( X_2, X_4, \ldots \). Recall the conditional mean \( m_{i,2} \) is given by
\[
m_{i,2} = \E [X_i | X_j, j : even]
\]

Then the integrand in (94) becomes
\[
\frac{d}{d\sigma} \left( \E \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k: even} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{i: odd} (m_{i,2} - m_i) \xi \right) \right] \right)
\]
\[
\times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right)
\]

\[
= \left( \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right) \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right)
\]

\[
+ \mathbb{E} \left[ \frac{d}{d\sigma} (\hat{e}(\xi)) \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right)
\]

\[
+ \mathbb{E} \left[ \hat{e}(\xi) \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right) \right]
\]

(96)

(97)

(98)

where

\[
\hat{e}(\xi) = \exp \left( \frac{i}{\sqrt{K}} \sum_{k: \text{even}} (X_k - m_k) \xi + \frac{i}{\sqrt{K}} \sum_{i: \text{odd}} (m_{i,2} - m_i) \xi \right)
\]

First, we have

\[
|T_{[96]}| \leq E \left[ \left| \sum_{n=1}^{K} (X_n - m_n) \right| \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right]
\]

\[
\lesssim \mathbb{E} \left[ \left| \sum_{n=1}^{K} (X_n - m_n) \right| \lambda^{\frac{K}{2} - 2} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^2 \right]
\]

\[
\lesssim K \lambda^{\frac{K}{2} - 2} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^2 \lesssim K^2 \lambda^{\frac{K}{2} - 2} \frac{1}{K + \xi^2} \lesssim K^2 \lambda^{\frac{K}{2} - 2} \frac{1}{1 + \xi^2}
\]

(99)

where the second inequality follows from applying (52) on \( \frac{K}{2} - 2 \) factors and (53) on the remaining 2 factors.

For the second term (97), we note that

\[
\frac{d}{d\sigma} m_{i,2}
\]

\[
= \frac{d}{d\sigma} E [X_l \mid X_j, j : \text{even}]
\]

\[
= \frac{d}{d\sigma} \int x_l \frac{1}{Z} \exp (\sigma x_l - \psi(x_l) - s_l x_l + J x_l (x_{l+1} + x_{l-1})) dx_l
\]

\[
= \int (x_l - m_{l,2}) x_l \frac{1}{Z} \exp (\sigma x_l - \psi(x_l) - s_l x_l + J x_l (x_{l+1} + x_{l-1})) dx_l
\]
\[
\begin{align*}
&= \mathbb{E}[(X_l - m_{l,2}) X_l \mid X_j, j : \text{even}] \\
&= \mathbb{E}[(X_l - m_{l,2})^2 \mid X_j, j : \text{even}] \\
&= \kappa^2_{l,2} \lesssim 1
\end{align*}
\]

Therefore the partial derivative inside the second term \([97]\) satisfies
\[
\left| \frac{d}{d\sigma} \left( \exp \left( i \frac{1}{\sqrt{K}} \sum_{k \text{even}} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{l \text{odd}} (m_{l,2} - m_l) \xi \right) \right) \right|
\]
\[
= \left| \left( -i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) X_k \right] \right.ight.
\]
\[
+ i \frac{1}{\sqrt{K}} \xi \sum_{l \text{odd}} \mathbb{E} \left[ (X_l - m_{l,2})^2 \mid X_j, j : \text{even} \right] \left| \right| \hat{e} (\xi) | \\
\lesssim \frac{1}{\sqrt{K}} |\xi| \left( K^2 + \frac{K}{2} \right) \\
\lesssim K^{3/2} |\xi| \\
\end{align*}
\]

Inserting the estimate \((100)\) into \((97)\) followed by applying \((52)\) on \(\frac{K}{2} - 3\) factors and \((53)\) on the remaining 3 factors leads
\[
|T_{\xi}^{(97)}| \lesssim K^{3/2} |\xi| \lambda^{\frac{K}{2} - 3} \left( \frac{1}{1 + \left( \frac{1}{\sqrt{K}} \right) |\xi|} \right)^3 \\
\lesssim K^3 \lambda^{\frac{K}{2} - 3} |\xi| \frac{1}{\sqrt{K} + |\xi|} \K^2 + \xi^2 \lesssim K^3 \lambda^{\frac{K}{2} - 3} \frac{1}{1 + |\xi|} \quad (101)
\]

Lastly, in \((98)\), we apply \((52)\), \((53)\) and \((93)\) again to obtain
\[
|T_{\xi}^{(98)}| \lesssim K \left( 1 + \left| \frac{\xi}{\sqrt{K}} \right| \right) \left| \frac{\xi}{\sqrt{K}} \right| \lambda^{\frac{K}{2} - 5} \left( \frac{1}{1 + \left( \frac{1}{\sqrt{K}} \right) \xi} \right)^5 \\
\lesssim K^2 \lambda^{\frac{K}{2} - 5} \frac{1}{1 + \xi^2} \quad (102)
\]

Overall, combining \((99)\), \((101)\), and \((102)\) into \((94)\) leads, as desired,
\[
\left| \int_{\{(\xi > \delta) \}} \frac{d}{d\sigma} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \right|
\]
\[
\lesssim \int_{\{(\xi > \delta) \}} K^2 \lambda^{\frac{K}{2} - 2} \frac{1}{1 + \xi^2} + K^3 \lambda^{\frac{K}{2} - 3} \frac{1}{1 + \xi^2} + K^2 \lambda^{\frac{K}{2} - 5} \frac{1}{1 + \xi^2} d\xi \\
\lesssim K^2 \lambda^{\frac{K}{2} - 2} + K^3 \lambda^{\frac{K}{2} - 3} + K^2 \lambda^{\frac{K}{2} - 5} \lambda^{<1} \lesssim \frac{1}{\sqrt{K}} \quad (103)
\]

Note that \((92)\) and \((103)\) yields \((21)\) for some \(\alpha \in (0, \frac{1}{2})\).
4.4. Proof of (22) in Proposition 3.6. Let us address the last inequality (22). As before, we start with dividing the integral into inner and outer part.

\[
2\pi \frac{d^2}{d\sigma^2} g_{K,m}(0) \\
= \frac{d^2}{d\sigma^2} \int_{\mathbb{R}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \\
= \frac{d^2}{d\sigma^2} \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi 
\]  
\[+ \frac{d^2}{d\sigma^2} \int_{\{ |(1/\sqrt{K})\xi| > \delta \}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi. \] (104)

we introduce the auxiliary sets of sites (cf. Figure 2)

\[
F_{1,n,l} := \{ k \mid |k - n| \leq L \text{ or } |k - l| \leq L \} \quad F_{2,n,l} := \{ k \mid |k - n| > L \text{ and } |k - l| > L \}
\]

where \( L = K^\varepsilon \ll K \).

**Argument for (104):** We introduce for every site \( l \) the variable \( \sigma_l \) and define

\[
G_{n,l}(\xi_1, \xi_2) := \mathbb{E} \left[ \exp \left( i \sum_{i \in F_{1,n,l}} (X_i - m_i) \xi_1 + i \sum_{j \in F_{2,n,l}} (X_j - m_j) \xi_2 \right) \right].
\]

Note that

\[
\frac{d^2}{d\sigma^2} \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \\
= \frac{d^2}{d\sigma^2} \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) d\xi \\
= \sum_{l=1}^{K} \sum_{n=1}^{K} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \bigg|_{\sigma_k = \sigma} d\xi \\
= \sum_{l=1}^{K} \sum_{n=1}^{K} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \bigg|_{\sigma_k = \sigma} d\xi
\]
Our aim is to prove that given $\beta > \frac{1}{2}$, there is a small $\varepsilon > 0$ such that we have

$$\left| \sum_{n=1}^{K} \sum_{l=1}^{K} \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) d\xi \right| \lesssim K^\beta \quad (106)$$

To establish this, we take the 2nd order Taylor expansion to get

$$\int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) d\xi = \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( 0, \frac{\xi}{\sqrt{K}} \right) d\xi + \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \frac{d^2}{d\sigma_1^2} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\hat{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi =: (a) + (b) + (c) \quad (107)$$

Starting with the integrand in (a), we may replace $E_l$ with $F_{n,l}$ from the previous proof of (21) so that we get

$$\int \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} (\xi_1, \xi_2) = \int \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} (\xi_1, \xi_2) e(\xi_1, \xi_2) \quad (108)$$

where in this case, $e(\xi_1, \xi_2)$ indicates

$$e(\xi_1, \xi_2) := \exp \left( \sum_{i \in F_{1,n}^l} (X_i - m_i) \xi_1 + \sum_{j \in F_{2,n}^l} (X_j - m_j) \xi_2 \right) \quad (109)$$

It is easy to observe that

$$T_{\{109\}} = 0 \quad (111)$$

when $(\xi_1, \xi_2) = \left( 0, \frac{\xi}{\sqrt{K}} \right)$. So, let us begin with computing (108).

$$\frac{d}{d\sigma_n} \mathbb{E} [(X_l - m_l) e(\xi_1, \xi_2)] = \mathbb{E} [(X_n - m_n) (X_l - m_l) e(\xi_1, \xi_2)] + \mathbb{E} [-\mathbb{E} [(X_n - m_n) X_l] e(\xi_1, \xi_2)] \quad (110)$$
\[ + E \left( X_l - m_l \right) (-i\xi_1) E \left( X_n - m_n \right) \sum_{i \in F_1^{n,l}} X_i \right) e(\xi_1, \xi_2) \]
\[ + E \left( X_l - m_l \right) (-i\xi_2) E \left( X_n - m_n \right) \sum_{j \in F_2^{n,l}} X_j \right) e(\xi_1, \xi_2) \]
\[ =: (a_{1,1}) + (a_{1,2}) + (a_{1,3}) + (a_{1,4}) \]

Letting \((\xi_1, \xi_2) = \left( 0, \frac{\xi}{\sqrt{K}} \right)\) and using Lemma 3.5, we obtain
\[ |(a_{1,1}) + (a_{1,2})| \]
\[ = \left| E \left( X_n - m_n \right) (X_l - m_l) e(0, \frac{\xi}{\sqrt{K}}) \right| - E \left[ E \left( X_n - m_n \right) (X_l - m_l) e(0, \frac{\xi}{\sqrt{K}}) \right] \]
\[ = \left| \text{cov} \left( (X_n - m_n) (X_l - m_l), e(0, \frac{\xi}{\sqrt{K}}) \right) \right| \]
\[ \lesssim |\xi| \exp(-CL), \]

\((a_{1,3}) = 0,\)

and
\[ |(a_{1,4})| \leq \left| \frac{\xi}{\sqrt{K}} \right| \left| \text{cov} \left( X_n, \sum_{j \in F_2^{n,l}} X_j \right) \right| E |X_l - m_l| \]
\[ \lesssim \exp(-CL) \]

Therefore we conclude
\[ |T^{108}_{108}| \lesssim (1 + |\xi|) \exp(-CL) \quad (112) \]

Next, we consider the term \(110\)
\[ \frac{d}{d\sigma} E \left[ -i\xi_2 E \left( X_l - m_l \right) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right] e(\xi_1, \xi_2) \]
\[ = -i\xi_2 \frac{d}{d\sigma} \left( E \left( X_l - m_l \right) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right) E \left[ e(\xi_1, \xi_2) \right] \]
\[ = -i\xi_2 \frac{d}{d\sigma} \left( E \left( X_l - m_l \right) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right) \frac{d}{d\sigma} E \left[ e(\xi_1, \xi_2) \right] \quad (113) \]
\[ - i\xi_2 E \left( X_l - m_l \right) \left( \sum_{j \in F_2^{n,l}} X_j \right) \frac{d}{d\sigma} \left( E \left[ e(\xi_1, \xi_2) \right] \right) \quad (114) \]
Then (113) and (114) can be written as

\[
T_{113} = -i\xi_2 E \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right] E \left[ e(\xi_1,\xi_2) \right] \\
- i\xi_2 E \left[ -E[(X_n - m_n) X_l] \left( \sum_{j \in F_2^{n,l}} X_j \right) E \left[ e(\xi_1,\xi_2) \right] \right] \\
=: (a_{3,1}) + (a_{3,2})
\]

while

\[
T_{114} = -i\xi_2 E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right] E \left[ (X_n - m_n) e(\xi_1,\xi_2) \right] \\
- i\xi_2 E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_k \right) \right] \\
\times E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_j \right) \right] E \left[ e(\xi_1,\xi_2) \right] \\
- i\xi_2 E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_{j1} \right) \right] \\
\times E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} X_{j2} \right) \right] E \left[ e(\xi_1,\xi_2) \right] \\
=: (a_{3,3}) + (a_{3,4}) + (a_{3,5})
\]

We now plug in \((\xi_1,\xi_2) = (0, \frac{\xi}{\sqrt{K}})\) and apply Lemma 3.5 to obtain

\[
|(a_{3,1}) + (a_{3,2})| = \left| \frac{\xi}{\sqrt{K}} \right| \left| \text{cov} \left( (X_n - m_n) (X_l - m_l), \sum_{j \in F_2^{n,l}} X_j \right) \right| \left| E \left[ e \left( 0, \frac{\xi}{\sqrt{K}} \right) \right] \right| \\
\lesssim |\xi| \exp(-CL),
\]

\[
|(a_{3,3})| \leq \left| \text{cov} \left( X_l, \sum_{j \in F_2^{n,l}} X_j \right) \right| |X_n - m_n| \lesssim \sqrt{K} \exp(-CL),
\]

\[
(a_{3,4}) = 0,
\]

\[
(a_{3,5}) = 0.
\]
and
\[
|a_{3,5}| \leq \left( \frac{\xi}{\sqrt{K}} \right)^2 \left| \text{cov} \left( X_1, \sum_{j_1 \in F_2^{n,l}} X_{j_1} \right) \right| \left| \text{cov} \left( X_n, \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right) \right| \\
\lesssim \xi^2 \exp(-2CL)
\]

A combination of these bounds yield
\[
|T_{110}| \lesssim \left( |\xi| + \sqrt{K} \right) \exp(-CL) + \xi^2 \exp(-2CL) \\
\lesssim \left( \sqrt{K} + |\xi| + \xi^2 \right) \exp(-CL)
\]

To sum up, we provided the upper bounds of (108), (109) and (110) given by (112), (111) and (115) respectively. Thus we conclude that (a) in (107) is bounded by
\[
\left| \int \{(1/\sqrt{K})|\xi| \leq \delta \} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( 0, \frac{\xi}{\sqrt{K}} \right) d\xi \right| \\
\lesssim \int \{(1/\sqrt{K})|\xi| \leq \delta \} \left( \sqrt{K} + |\xi| + \xi^2 \right) \exp(-CL) d\xi \\
L_{=K^*}^{K^*} \ll \frac{1}{K^{2-\beta}}
\]

which fits to our claim (106).

Let us address the third term (c) in (107). From a direct computation we deduce
\[
\frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \tilde{\xi}_{1,l}, \xi_2 \right) \\
= \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} \frac{d^2}{d\xi_1^2} G_{n,l} \left( \tilde{\xi}_{1,l}, \xi_2 \right) \\
= \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} \mathbb{E} \left[ \left( i \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right] \\
= \frac{d}{d\sigma_n} \mathbb{E} \left[ (X_l - m_l) \left( i \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right] \\
+ \frac{d}{d\sigma_n} \mathbb{E} \left[ 2 \left( i \sum_{i_1 \in F_1^{n,l}} (X_{i_1} - m_{i_1}) \right) \right. \\
\left. \times (-i) \mathbb{E} \left[ (X_l - m_l) \sum_{i_2 \in F_1^{n,l}} X_{i_2} \right] e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right]
\]

which fits to our claim (106).
\[ + \frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i_1 \in F_{n,l}^1} (X_{i_1} - m_{i_1}) \right)^2 \right] \]
\[ \times \left( -i\tilde{\xi}_{1,l} \right) \mathbb{E} \left[ (X_l - m_l) \sum_{i_2 \in F_{n,l}^1} X_{i_2} \right] e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \tag{118} \]
\[ + \frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i \in F_{n,l}^1} (X_i - m_i) \right)^2 \right] \]
\[ \times \left( -i\tilde{\xi}_2 \right) \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{n,l}^2} X_j \right] e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \tag{119} \]

We will investigate term by term. To begin with, (116) is

\[
\frac{d}{d\sigma_n} \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in F_{n,l}^1} (X_i - m_i) \right)^2 e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \right] = -\mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{i \in F_{n,l}^1} (X_i - m_i) \right)^2 e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \right] \\
- \mathbb{E} \left[ (X_n - m_n) X_l \left( \sum_{i \in F_{n,l}^1} (X_i - m_i) \right)^2 e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \right] \\
- \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i_1 \in F_{n,l}^1} (X_{i_1} - m_{i_1}) \right) \right] \\
\times \left( -\mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_{n,l}^1} X_{i_2} \right] e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \right) \\
- \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i_1 \in F_{n,l}^1} (X_{i_1} - m_{i_1}) \right)^2 \right] \\
\times \left( -i\tilde{\xi}_{1,l} \right) \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_{n,l}^1} X_{i_2} \right] e\left( \tilde{\xi}_{1,l}, \xi_2 \right) \]
- \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 \right] \\
\times (\xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j \in F_2^{n,l}} X_j \right] e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \\
= (c_{1,1}) + (c_{1,2}) + (c_{1,3}) + (c_{1,4}) + (c_{1,5})

and (117) is

\frac{d}{d\sigma_n} \mathbb{E} \left[ 2 \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \right] \\
\times \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \\
= 2 \frac{d}{d\sigma_n} \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right] \\
+ 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right]

where

T_{(120)} \\
= 2 \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right] \\
+ 2 \mathbb{E} \left[ -\mathbb{E} \left[ (X_n - m_n) X_l \right] \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right] \\
= (c_{2,1}) + (c_{2,2})

and

T_{(121)} \\
= 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right] \\
+ 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ -\mathbb{E} \left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right] \\
+ 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \\
\times \mathbb{E} \left[ \sum_{i \in F_1^{n,l}} (X_i - m_i) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)} \right]
\[
\times \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right] \\
\times (\sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2})) \\
\times \left( -i \tilde{\xi}_{1,l} \right) \mathbb{E} \left[ (X_n - m_n) \sum_{i_3 \in F_{1,l}^{n,l}} X_{i_3} \right] e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

\[
+ 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \\
\times \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right] \\
\times (\sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2})) \\
\times \left( -i \xi_2 \right) \mathbb{E} \left[ (X_n - m_n) \sum_{j \in F_{1,l}^{n,l}} X_j \right] e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

\[
= (c_{2,3}) + (c_{2,4}) + (c_{2,5}) + (c_{2,6})
\]

To address these terms generated by (116) and (117), we will divide into two cases.

**Case 1.** \( |n - l| > 2L \)

Note first that (\( c_{1,1} \)) is

\[
(c_{1,1}) \\
= -\mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \right] \\
\times \left( \sum_{i_1: |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) + \sum_{i_2: |i_2 - n| \leq L} (X_{i_2} - m_{i_2}) \right)^2 e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

\[
= -\mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \right] \left( \sum_{i_1: |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) \right)^2 e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

\[
- \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \right] \left( \sum_{i_2: |i_2 - l| \leq L} (X_{i_2} - m_{i_2}) \right)^2 e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

\[
- 2 \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \right] \\
\times \left( \sum_{i_1: |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) \right) \left( \sum_{i_2: |i_2 - l| \leq L} (X_{i_2} - m_{i_2}) \right) e^{\left( \tilde{\xi}_{1,l}, \xi_2 \right)}
\]

(122)
For the terms $T_{122}$ and $T_{123}$, we do the Taylor expansion to get
\begin{align*}
T_{122} &= -\mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{i_1 : |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) \right)^2 e(0, \xi_2) \right] \\
& \quad - i \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{i_1 : |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) \right)^2 \right. \\
& \quad \left. \times \left( \sum_{i_2 \in F_1} (X_{i_2} - m_{i_2}) e\left( \hat{\xi}_{1,l}, \xi_2 \right) \right) \tilde{\xi}_{1,l} \right] 
\tag{125}
\end{align*}
where $\hat{\xi}_{1,l}$ is a real number between 0 and $\tilde{\xi}_{1,l}$, in particular,

$$\left| \hat{\xi}_{1,l} \right| \leq \tilde{\xi}_{1,l} \leq |\xi|$$

Note that for $i_1$ with $|i_1 - l| \leq L$, we have

$$|i_1 - l| \geq |n - l| - |i_1 - n| > 2L - L = L$$

Thus we can apply Lemma 3.5 with $(\xi_1, \xi_2) = \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right)$ to deduce exponential decay of (125):

$$|T_{125}| \lesssim \left| \tilde{\xi}_{l} \sqrt{K} \right| (1 + |\xi|^3 \sqrt{K}) \exp(-C\xi^2)$$

To address the second term (126), we consider the conditional expectation with respect to $\mathcal{G}_{n,l} := \sigma (X_k, k \in F_1)$.

$$T_{126} = \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{i_1 : |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) \right)^2 \right. \\
& \quad \left. \times \left( \sum_{i_2 \in F_1} (X_{i_2} - m_{i_2}) e\left( \hat{\xi}_{l}, \xi_2 \right) \right) \tilde{\xi}_l \right] \sqrt{K} \tag{126}
$$

We have by using (34) in Lemma 4.1 and (19) in Lemma 3.4,

$$|T_{126}| \lesssim \left| \hat{\xi}_{l} \sqrt{K} \right| \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)$$
\[ \times \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \left( \sum_{i_1: |i_1 - n| \leq L} (X_{i_1} - m_{i_1}) \right)^2 \left( \sum_{i_2 \in F_{n,l}} (X_{i_2} - m_{i_2}) \right) \right] \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C \xi^2) \]

and thus
\[ |T_{127}| \lesssim L^2 (1 + |\xi|) \exp (-CL) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C \xi^2) \]

By symmetry, we also have the same bound for \( T_{123} \). Before taking (124) into account, let us split \((c_{2,3})\).

\[ (c_{2,3}) = 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i_1: |i_1 - n| \leq L} X_{i_1} \right] \times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i_2 \in F_{n,l}} (X_{i_2} - m_{i_2}) \right) e^{\tilde{\xi}_{1,l}, \tilde{\xi}_2} \right] \tag{127} \]

\[ + 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i_2 \in F_{n,l}} (X_{i_2} - m_{i_2}) \right) e^{\tilde{\xi}_{1,l}, \tilde{\xi}_2} \right] \tag{128} \]

Again, for \(|i_1 - n| \leq L\), we have \(|i_1 - l| > L\) so that Lemma 3.5 and (19) yield the exponential decay of the first term (127):

\[ |T_{127}| = 2 \text{cov} \left( \sum_{i_1: |i_1 - n| \leq L} X_{i_1} \right) \left( \sum_{i_2 \in F_{n,l}} (X_{i_2} - m_{i_2}) \right) e^{\tilde{\xi}_{1,l}, \tilde{\xi}_2} \]
\[ \lesssim \sqrt{L} \exp (-CL) \sum_{i_2 \in F_{1}} \mathbb{E} |(X_n - m_n) (X_{i_2} - m_{i_2})| \]
\[ \lesssim L^2 \exp (-CL) \]
\[ \lesssim L^2 \exp (-CL) \]

On the other hand,

\[ T_{128} = 2 \mathbb{E} \left[ (X_n - m_n) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \]
At this point, we will apply the same trick which is used in proving the upper bound of (122). Taylor expansion with respect to the first variable gives

\[ T(129) = 2 \mathbb{E} \left[ \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \]

\[ \times \mathbb{E} \left[ \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - l| \leq L} (X_{i_2} - m_{i_2}) \right) e(0, \xi_2) \right] \]  \tag{131}

\[ + 2i \mathbb{E} \left[ \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} X_{k_1} \right] \]

\[ \times \mathbb{E} \left[ \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - l| \leq L} (X_{i_2} - m_{i_2}) \right) \right. \]

\[ \times \left. \left( \sum_{i_3 \in F_{1, l}^n} (X_{i_3} - m_{i_3}) \right) e\left( \hat{\xi}_1, \xi_2 \right) \right] \]  \tag{132}

For \((\xi_1, \xi_2) = \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right)\) we easily deduce

\[ |T(131)| \]

\[ = 2 \left| \mathbb{E} \left[ \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \right| \]

\[ \times \left| \text{cov} \left( \left( X_n - m_n \right), \left( \sum_{i_2: |i_2 - l| \leq L} (X_{i_2} - m_{i_2}) \right) e\left( 0, \frac{\xi}{\sqrt{K}} \right) \right) \right| \]

\[ \lesssim L (L + L^2 \xi^2)^{\frac{1}{2}} \exp(-CL) \]

\[ \lesssim L^2 (1 + |\xi|) \exp(-CL) \]

and by (34) and (19), we have

\[ |T(132)| \]
\[ \begin{align*}
\xi & \leq \left| \mathbb{E} \left[ \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \right| \\
& \times \left| \mathbb{E} \left[ \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - n| \leq L} \left( X_{i_2} - m_{i_2} \right) \right) \right] \right| \\
& \times \left| \left( \sum_{i_3 \in F_{n,l}^1} \left( X_{i_3} - m_{i_3} \right) \right) e \left( \frac{\tilde{\xi}_l}{\sqrt{K}}, 0 \right) \mathbb{E} \left[ e \left( 0, \frac{\xi}{\sqrt{K}} \right) \left| \mathcal{G}_{n,l} \right. \right] \right| \frac{\tilde{\xi}_l}{\sqrt{K}} \\
& \leq \left| \mathbb{E} \left[ \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} X_{i_1} \right] \right| \\
& \times \left| \frac{\tilde{\xi}_l}{\sqrt{K}} \left( 1 + \frac{\xi^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right) \right| \\
& \times \left| \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - n| \leq L} \left( X_{i_2} - m_{i_2} \right) \right) \left( \sum_{i_3 \in F_{n,l}^1} \left( X_{i_3} - m_{i_3} \right) \right) \right| \\
& \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{\xi^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\end{align*} \]

Now we combine (130) with (124) and carry out Taylor expansion in the first variable \( \xi_{1,l} \).

\[ T_{(130)} + T_{(124)} = 2 \text{ cov} \left( \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} \left( X_{i_1} - m_{i_1} \right), \right. \]

\[ \left. \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - n| \leq L} \left( X_{i_2} - m_{i_2} \right) \right) e \left( \tilde{\xi}_{1,l}, \tilde{\xi}_2 \right) \right) \]

\[ = 2 \text{ cov} \left( \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} \left( X_{i_1} - m_{i_1} \right), \right. \]

\[ \left. \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - n| \leq L} \left( X_{i_2} - m_{i_2} \right) \right) e \left( 0, \tilde{\xi}_2 \right) \right) \] (133)

\[ + 2i \text{ cov} \left( \left( X_l - m_l \right) \sum_{i_1: |i_1 - l| \leq L} \left( X_{i_1} - m_{i_1} \right), \right. \]

\[ \left. \left( X_n - m_n \right) \left( \sum_{i_2: |i_2 - n| \leq L} \left( X_{i_2} - m_{i_2} \right) \right) \right) \]
\[
\sum_{i_3 \in F_{l,t}^n} (X_{i_3} - m_{i_3}) \left( \sum_{i_3} (X_{i_3} - m_{i_3}) \right) e^{\hat{\xi}_{1,t}^{l,t}} \xi_{1,t}^{l,t}
\] (134)

To address the first term (133), we divide it into two parts again to get

\[
T_{133} = 2 \text{cov} \left[ (X_l - m_l) \sum_{i_1: |i_1 - l| \leq L/2} (X_{i_1} - m_{i_1}),
\right.
\]

\[
(X_n - m_n) \left( \sum_{i_2: |i_2 - n| \leq L} (X_{i_2} - m_{i_2}) \right) e^{0, \xi_2}
\]
(135)

\[
+ 2 \text{cov} \left[ (X_l - m_l) \sum_{i_1: L/2 \leq |i_1 - l| \leq L} (X_{i_1} - m_{i_1}),
\right.
\]

\[
(X_n - m_n) \left( \sum_{i_2: |i_2 - n| \leq L} (X_{i_2} - m_{i_2}) \right) e^{0, \xi_2}
\]
(136)

Note that for \((i_1, i_2)\) with \(|i_1 - l| \leq \frac{L}{2}\) and \(|i_2 - n| \leq L\), we have

\[
|i_1 - i_2| \geq |l - n| - |i_1 - l| - |i_2 - n| \geq 2L - \frac{L}{2} - L \geq \frac{L}{2}
\]

Therefore exponential decay of correlations implies

\[
|T_{135}| \lesssim (L^2 + L)^{\frac{1}{2}} (L^2 + L + L^2 \xi^2)^{\frac{1}{2}} \exp \left( -CL/2 \right)
\]
\[
\lesssim L^2 (1 + |\xi|) \exp \left( -CL/2 \right)
\]

On the other hand, by definition of covariances, we may write (136) as

\[
T_{136} = 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i_1: L/2 \leq |i_1 - l| \leq L} (X_{i_1} - m_{i_1})
\right.
\]

\[
\times (X_n - m_n) \left( \sum_{i_2: |i_2 - n| \leq L} (X_{i_2} - m_{i_2}) \right) e^{0, \xi_2}
\]
(137)

\[
- 2 \mathbb{E} \left[ (X_l - m_l) \sum_{i_1: L/2 \leq |i_1 - l| \leq L} (X_{i_1} - m_{i_1})
\right.
\]

\[
\times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i_2: |i_2 - n| \leq L} (X_{i_2} - m_{i_2}) \right) e^{0, \xi_2} \right]
\]
(138)

and we have

\[
|T_{136}| = 2 \text{cov} \left[ X_l, \sum_{i_1: L/2 \leq |i_1 - l| \leq L} (X_{i_1} - m_{i_1}) (X_n - m_n) \right]
\]
\[
\times \left( \sum_{i_2: \ |i_2-n| \leq L} (X_{i_2} - m_{i_2}) e(0, \xi_2) \right) \right| \lesssim (L^4 + L^3 + L^4 \xi_2^{1/2}) \exp(-CL/2) \\
\lesssim L^2 (1 + |\xi|) \exp(-CL/2)
\]

with
\[
|T_{138}| = \left| \text{cov} \left( X_t, \sum_{i_1: \ L/2 \leq |i_1-l| \leq L} (X_{i_1} - m_{i_1}) \right) \times E \left( X_n - m_n \right) \left( \sum_{i_2: \ |i_2-n| \leq L} (X_{i_2} - m_{i_2}) \right) \right| \\
\lesssim \sqrt{L} \exp(-CL/2) L \\
\lesssim L^2 \exp(-CL/2)
\]

The bound of the term \([134]\) could be deduced by the almost identical argument given for \([132]\) so that we conclude
\[
|T_{134}| \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)
\]

To sum up all the results we have proven so far, we got
\[
|(c_{1,1}) + (c_{2,3})| \\
\lesssim L^2 (1 + |\xi|) \exp(-CL/2) + L^2 (1 + |\xi|) \exp(-CL) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)
\]

\[
\lesssim L^2 (1 + |\xi|) \exp(-CL/2) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)
\]

Let us turn our attention to \((c_{1,2})\). Lemma 3.5 with (19) yield
\[
|(c_{1,2})| \lesssim \left| E \left[ (X_n - m_n) X_t \right] \right| \left| \sum_{k \in F_{1,t}} (X_k - m_k) \right|^{2} \exp(-CL) L^2
\]

To address \((c_{1,3})\), we combine this with \((c_{2,4})\).
\[
|(c_{1,3}) + (c_{2,4})| = 2 \left| E \left[ (X_n - m_n) \left( \sum_{i_1 \in F_{1,t}} X_{i_1} \right) \right] \right| \\
\times \left| \text{cov} \left( X_{i_1} - m_{i_1} \sum_{i_2 \in F_{1,t}} X_{i_2}, e \left( \xi_{1,t}, \xi_{2} \right) \right) \right|
\]
Then \( (19) \) implies
\[
2 \left| \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i_1 \in F_1^{n,l}} X_{i_1} \right) \right] \right| \lesssim L \tag{139}
\]
so let us take a look at the term
\[
\text{cov} \left( (X_l - m_l) \sum_{i_2 \in F_1^{n,l}} X_{i_2}, e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right) = \text{cov} \left( (X_l - m_l) \sum_{i_2 \in F_1^{n,l}} X_{i_2}, 0, \xi_2 \right) \tag{140}
\]
\[
+ i \text{cov} \left( (X_l - m_l) \sum_{i_2 \in F_1^{n,l}} X_{i_2}, \sum_{i_3 \in F_1^{n,l}} X_{i_3} e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right) \tilde{\xi}_{1,l} \tag{141}
\]
For \( (140) \), we divide into three parts
\[
T_{140} = \sum_{i_2 \in F_1^{n,l}, |i_2 - l| \leq L/2} \text{cov} \left( (X_l - m_l) X_{i_2}, e(0, \xi_2) \right) \tag{142}
\]
\[
+ \sum_{i_2 \in F_1^{n,l}, |i_2 - l| > L/2} \mathbb{E} \left[ (X_l - m_l) X_{i_2} e(0, \xi_2) \right] \tag{143}
\]
\[
- \sum_{i_2 \in F_1^{n,l}, |i_2 - l| > L/2} \mathbb{E} \left[ (X_l - m_l) X_{i_2} \right] \mathbb{E} \left[ e(0, \xi_2) \right] \tag{144}
\]
to obtain
\[
|T_{142}| \lesssim L |\xi| \exp (-C \xi^2),
\]
\[
|T_{143}| = \sum_{i_2 \in F_1^{n,l}, |i_2 - l| > L/2} \text{cov} \left( (X_l - m_l) X_{i_2}, e(0, \xi_2) \right) \right| \lesssim L |\xi| \exp (-C \xi^2),
\]
and
\[
|T_{144}| = \sum_{i_2 \in F_1^{n,l}, |i_2 - l| > L/2} \text{cov} \left( X_l, X_{i_2} \right) \mathbb{E} \left[ e(0, \xi_2) \right] \right| \lesssim L \exp (-C \xi^2)
\]
Moreover, an analogous argument given for the term \( (132) \) can be applied to \( (141) \) to obtain
\[
|T_{141}| \lesssim L^2 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C \xi^2)
\]
Combined with (139), we have obtained
\[
|c_{1,3} + c_{2,4}| \lesssim L^2 (1 + |\xi|) \exp \left( -CL/2 \right) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\]

The upper bound for the terms \((c_{1,4}, c_{2,5})\) can also be proved using (34) in Lemma 4.1.
Indeed,
\[
|c_{1,4}| = \frac{|\xi|}{\sqrt{K}} \left| \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_{1,l}^n} X_{i_2} \right] \right|
\times \left| \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i_1 \in F_{1,l}^n} (X_{i_1} - m_{i_1}) \right)^2 e^{\left( \hat{\xi}_{l,2}, \xi_{1,2} \right)} \right] \right|
\lesssim \frac{|\xi|}{\sqrt{K}} L \left| \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i_1 \in F_{1,l}^n} (X_{i_1} - m_{i_1}) \right)^2 \right] \right|
\times e^{\left( \hat{\xi}/\sqrt{K}, 0 \right)} \mathbb{E} \left[ \left( 0, \frac{\xi}{\sqrt{K}} \right) | G_{n,l} \right] \]
\lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\]

Similar argument leads
\[
|c_{1,5}| \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\]
Furthermore, we also have
\[
|c_{1,5}| = \frac{|\xi|}{\sqrt{K}} \left| \mathbb{E} \left[ (X_n - m_n) \sum_{j \in F_{2,l}^n} X_j \right] \right|
\times \left| \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in F_{1,l}^n} (X_i - m_i) \right)^2 e^{\left( \hat{\xi}/\sqrt{K}, \xi/\sqrt{K} \right)} \right] \right|
\lesssim \frac{|\xi|}{\sqrt{K}} \text{cov} \left( X_n, \sum_{j \in F_{2,l}^n} X_j \right) \left| \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in F_{1,l}^n} (X_i - m_i) \right)^2 \right] \right|
\lesssim L^2 |\xi| \exp \left( -CL \right)
\]
and same argument leads
\[
|c_{2,6}| \lesssim L^2 |\xi| \exp \left( -CL \right)
\]
Thus, it remains to address \((c_{2,1})\) and \((c_{2,2})\). We divide \((c_{2,1})\) into two parts

\[
(c_{2,1}) = 2 \sum_{i_1: |i_1-n| \leq L} \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) X_{i_1} \right]
\]

\[
\times \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^n} (X_{i_2} - m_{i_2}) e^{\left( \hat{\xi}_{1,l}, \xi_2 \right)} \right] \quad (145)
\]

\[
+ 2 \sum_{i_1: |i_1-l| \leq L} \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) X_{i_1} \right]
\]

\[
\times \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^n} (X_{i_2} - m_{i_2}) e^{\left( \hat{\xi}_{1,l}, \xi_2 \right)} \right] \quad (146)
\]

to see

\[
|T_{145}| \leq 2 \sum_{i_1: |i_1-n| \leq L} |\text{cov} (X_l, (X_n - m_n) X_{i_1})| \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^n} (X_{i_2} - m_{i_2}) \right] \quad (145)
\]

\[
\lesssim \exp \left( -CL \right) L
\]

and by symmetry

\[
|T_{146}| \lesssim \exp \left( -CL \right) L
\]

Thus we conclude

\[
|c_{2,1}| \lesssim L \exp \left( -CL \right)
\]

On the other hand we have

\[
|c_{2,2}| = 2 |\mathbb{E} \left[ (X_n - m_n) X_l \right]| \mathbb{E} \left[ \sum_{i_1 \in F_{1,l}^n} X_{i_1} \right] \quad (146)
\]

\[
\times \left| \mathbb{E} \left[ \sum_{i_2 \in F_{1,l}^n} (X_{i_2} - m_{i_2}) e^{\left( \hat{\xi}_{1,l}, \xi_2 \right)} \right] \right| \quad (146)
\]

\[
\lesssim \exp \left( -CL \right) L^2
\]

Overall we have proven that

\[
|T_{116} + T_{117}| \lesssim L^2 \left( 1 + |\xi| \right) \exp \left( -CL/2 \right) + (L + L^2) \exp \left( -CL \right) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\]

\[
\lesssim L^2 \left( 1 + |\xi| \right) \exp \left( -CL/2 \right) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp \left( -C\xi^2 \right)
\]

Case 2. \(|n - l| \leq 2L\)
Except for the terms \((c_{1,5})\) and \((c_{2,6})\), we will apply \((34)\) in Lemma 4.1 followed by \((19)\). For example, we have
\[
|\langle c_{1,1} \rangle| = \left| \mathbb{E} \left[ (X_n - m_n)(X_l - m_l) \left( \sum_{i \in \mathcal{F}_n^l} (X_i - m_i) \right)^2 \right] \right|
\times e \left( \frac{\bar{\xi}_l}{\sqrt{K}}, 0 \right) \mathbb{E} \left[ e \left( 0, \frac{\xi}{\sqrt{K}} \right) | \mathcal{G}_{n,l} \right] \left| \sum_{i \in \mathcal{F}_n^l} (X_i - m_i) \right|^2
\leq \mathbb{E} \left[ (X_n - m_n)(X_l - m_l) \left( \sum_{i \in \mathcal{F}_n^l} (X_i - m_i) \right)^2 \right]
\times \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2)
\lesssim L^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2)
\]
Similar computations yield
\[
|\langle c_{1,2} \rangle|, |\langle c_{1,3} \rangle|, |\langle c_{2,1} \rangle|, \ldots, |\langle c_{2,4} \rangle| \lesssim L^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2)
\]
and
\[
|\langle c_{1,4} \rangle|, |\langle c_{2,5} \rangle| \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2)
\]
Lastly, we may take the results from Case 1 to conclude
\[
|\langle c_{1,5} \rangle|, |\langle c_{2,6} \rangle| \lesssim L^2 |\xi| \exp (-CL)
\]
Therefore we have
\[
|T_{(116)} + T_{(117)}| \lesssim L^2 |\xi| \exp (-CL) + L^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp (-C\xi^2)
\]
and this ends Case 2.
Let us move on to \((118)\)
\[
\frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i \in \mathcal{F}_n^l} (X_i - m_i) \right)^2 \right]
\]
\begin{align*}
&\times \left(-i\xi_{1,l}\right) \mathbb{E} \left[ (X_l - m_l) \sum_{i_2 \in F_{1,l}^{n,l}} X_{i_2} \right] e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \\
&= i\xi_{1,l} \frac{d}{d \sigma_n} \left[ \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \right] \\
&\quad \times \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right)^2 e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \right] 
\end{align*}

\begin{align*}
&+ i\xi_{1,l} \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \\
&\quad \times \frac{d}{d \sigma_n} \left[ \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,l}^{n,l}} (X_{n,l} - m_{n,l}) \right)^2 e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \right] \right] 
\end{align*}

where

\begin{align*}
T^{(147)} & = i\xi_{1,l} \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \\
&\quad \times \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right)^2 e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \right] \\
&+ i\xi_{1,l} \mathbb{E} \left[ -\mathbb{E} [(X_n - m_n) X_l] \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \\
&\quad \times \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right)^2 e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \right] \\
&=: (c_{3,1}) + (c_{3,2})
\end{align*}

and

\begin{align*}
T^{(148)} & = i\xi_{1,l} \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,l}^{n,l}} X_{i_1} \right] \\
&\quad \times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i_2 \in F_{1,l}^{n,l}} (X_{i_2} - m_{i_2}) \right)^2 e^{\left(\tilde{\xi}_{1,l}, \xi_2\right)} \right]
\end{align*}
\[
+ i \tilde{\xi}_{1,l} \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,n,l}} X_{i_1} \right] \\
\times \mathbb{E} \left[ 2 \left( \sum_{i_2 \in F_{1,n,l}} (X_{i_2} - m_{i_2}) \right) \times \left( -\mathbb{E} \left[ (X_n - m_n) \sum_{i_3 \in F_{1,n,l}} X_{i_3} \right] e^{(\tilde{\xi}_{1,l}, \xi_2)} \right) \right] \\
+ i \tilde{\xi}_{1,l} \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,n,l}} X_{i_1} \right] \\
\times \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,n,l}} (X_{i_2} - m_{i_2}) \right)^2 \right] \\
\times (-i \xi_{1,l}) \mathbb{E} \left[ (X_n - m_n) \sum_{i_3 \in F_{1,n,l}} X_{i_3} \right] e^{(\tilde{\xi}_{1,l}, \xi_2)} \\
+ i \tilde{\xi}_{1,l} \mathbb{E} \left[ (X_l - m_l) \sum_{i_1 \in F_{1,n,l}} X_{i_1} \right] \\
\times \mathbb{E} \left[ \left( \sum_{i_2 \in F_{1,n,l}} (X_{i_2} - m_{i_2}) \right)^2 \right] \\
\times (-i \xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j \in F_{2,n,l}} X_j \right] e^{(\tilde{\xi}_{1,l}, \xi_2)} \\
:= (c_{3,3}) + (c_{3,4}) + (c_{3,5}) + (c_{3,6})
\]

Arguing as before, (34) and (19) imply

\[
|c_{3,1}|, \ldots, |c_{3,4}| \lesssim L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^2}{\sqrt{K}} \right) \exp(-C\xi^2)
\]

and

\[
|c_{3,5}| \lesssim L^4 \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)
\]
for \((\xi_1, \xi_2) = (\frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}})\). On the other hand, Lemma 3.5 and (19) yield

\[
|(c_{3,6})| \lesssim L^3 \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \sqrt{K} \exp(-CL)
\]

Therefore we conclude

\[
|T_{\ref{118}}| \lesssim L^3 \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \sqrt{K} \exp(-CL) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2)
\]

+ \[ L^4 \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2) \]

Lastly, we address the term (119)

\[
\frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i \in F_{1,n}^l} (X_i - m_i) \right)^2 \right]
\times (-i\xi_2) \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2,n}^l} X_j \right] e \left( \tilde{\xi}_{1,l}, \xi_2 \right)
= \xi_2 \frac{d}{d\sigma_n} \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2,n}^l} X_j \right]
\times \mathbb{E} \left[ \left( \sum_{i \in F_{1,n}^l} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right]
\]

\[ + \xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2,n}^l} X_j \right]
\times \frac{d}{d\sigma_n} \mathbb{E} \left[ \left( \sum_{i \in F_{1,n}^l} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right]
\]

Direct computation yields

\[
T_{\ref{119}} = i\xi_2 \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \sum_{j \in F_{2,n}^l} X_j \right]
\times \mathbb{E} \left[ \left( \sum_{i \in F_{1,n}^l} (X_i - m_i) \right)^2 e \left( \tilde{\xi}_{1,l}, \xi_2 \right) \right]
\]
\[ + i \xi_2 \mathbb{E} \left[ \mathbb{E} \left[ (X_n - m_n) \sum_{j \in F_2^{n,l}} X_j \right] \right] \]
\[ \times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 e^{i \xi_1, \xi_2} \right] \]
\[ =: (c_{4,1}) + (c_{4,2}) \]

and

\[ T_{150} = i \xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_2^{n,l}} X_j \right] \]
\[ \times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 e^{i \xi_1, \xi_2} \right] \]
\[ + i \xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_2^{n,l}} X_j \right] \]
\[ \times \mathbb{E} \left[ 2 \left( \sum_{i_1 \in F_1^{n,l}} (X_{i_1} - m_{i_1}) \right) \right. \]
\[ \left. \times \left( \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_1^{n,l}} X_{i_2} \right] e^{i \xi_1, \xi_2} \right) \right] \]
\[ + i \xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_2^{n,l}} X_j \right] \]
\[ \times \mathbb{E} \left[ \left( \sum_{i_1 \in F_1^{n,l}} (X_{i_1} - m_{i_1}) \right)^2 \right. \]
\[ \left. \times (-i \xi_1, \xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_1^{n,l}} X_{i_2} \right] e^{i \xi_1, \xi_2} \right] \]
\[ + i \xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j_1 \in F_2^{n,l}} X_{j_1} \right] \]
\[ \times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_k - m_k) \right)^2 \right] \times (-i\xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j_2 \in F_2^{n,l}} X_{j_2} e^{i (\tilde{\xi}_{1,l}, \xi_2)} \right] \]

\[ =: (c_{4,3}) + (c_{4,4}) + (c_{4,5}) + (c_{4,6}) \]

Put \((\xi_1, \xi_2) = \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right)\) and we combine \((c_{4,1})\) and \((c_{4,2})\). Usual argument using Lemma 3.5 and (19) yield

\[ |(c_{4,1}) + (c_{4,2})| = \frac{|\xi|}{\sqrt{K}} \left| \operatorname{cov} \left( (X_I - m_I) (X_n - m_n), \sum_{j \in F_2^{n,l}} X_j \right) \right| \times \left| \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right)^2 e^{i (\tilde{\xi}_{1,l}, \xi_2)} \right] \right| \]

\[ \lesssim |\xi| L^2 \exp(-CL) \]

Similarly, we have

\[ |(c_{4,3})|, |(c_{4,4})| \lesssim |\xi| L^2 \exp(-CL), \]

\[ |(c_{4,5})| \lesssim \left( \frac{|\xi|}{\sqrt{K}} \right)^2 L^3 \sqrt{K} \exp(-CL) \]

and

\[ |(c_{4,6})| \lesssim \xi^2 L^2 \exp(-2CL) \]

Therefore we deduce

\[ |T_{149}| \lesssim |\xi| L^2 \exp(-CL) + \frac{L^3}{\sqrt{K}} \xi^2 \exp(-CL) + \xi^2 L^2 \exp(-2CL) \]

To conclude, we sum up all the bounds we have proven so far. That is, for \(|n - l| > 2L\), we have

\[ \left| \frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\tilde{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \right| \]

\[ \lesssim L^2 \left( 1 + |\xi| \right) \exp(-CL/2) + \frac{L^3}{\sqrt{K}} \xi^2 \exp(-CL) + L^2 \xi^2 \exp(-2CL) \]

\[ + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|^2}{\sqrt{K}} \right) \exp(-C\xi^2) + L^4 \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \left( 1 + \frac{|\xi|^3}{\sqrt{K}} \right) \exp(-C\xi^2) \]

so that

\[ \left| \int_{\{|(1/\sqrt{K})\xi| \leq \delta\}} \frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\tilde{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \right| \]
\[ \lesssim L^2 \left( \sqrt{K} + K \right) \exp\left( -CL/2 \right) + L^3 K \exp\left( -CL \right) + L^2 K^{3/2} \exp\left( -2CL \right) + \frac{L^3}{K^{3/2}} + \frac{L^4}{K^2} \]

and for \(|n - l| \leq 2L|\),

\[
\left| \frac{d^2}{d\xi^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\tilde{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \right| \lesssim L^2 |\xi| \exp\left( -CL \right) + \frac{L^3}{\sqrt{K}} \xi^2 \exp\left( -CL \right) + L^2 \xi^2 \exp\left( -2CL \right) + L^3 \frac{|\xi|}{\sqrt{K}} \left( 1 + \frac{|\xi|}{\sqrt{K}} \right) \exp\left( -C\xi^2 \right)
\]

and thus by similar computation as above, we have

\[
\left| \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d^2}{d\xi^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\tilde{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \right| \lesssim L^{K^2} \frac{L^3}{K^{3/2}}
\]

Summing up for all pairs \((n, l)\), we obtain as desired,

\[
\sum_{n=1}^{K} \sum_{l=1}^{K} \int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \left| \frac{d^2}{d\xi^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\tilde{\xi}}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \left( \frac{\xi}{\sqrt{K}} \right)^2 d\xi \right| \lesssim K^2 \frac{L^3}{K^{3/2}} + KL^2 \frac{L^3}{K}
\]

The last step concerning the inner integral is to address the term \((b)\) in (107)

\[
\int_{\{ |(1/\sqrt{K})\xi| \leq \delta \}} \frac{d}{d\xi_1} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}(0, \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}}) d\xi_1 d\sigma_n d\sigma_l
\]

As we did in (78), we carry out a Taylor expansion with respect to the second variable in \(G_{n,l}\).

\[
\frac{d}{d\xi_1} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}(0, \xi_2) \xi_1
\]
\[
\frac{d \xi_1}{d \xi_2} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} G_{n,l}(0,0) \xi_1 + \frac{d \xi_2}{d \xi_1} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} G_{n,l}(0,0) = \frac{d \xi_1}{d \xi_2} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] = 0
\]

The first term in (151) vanishes because

\[
\frac{d \xi_1}{d \xi_2} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} G_{n,l}(0,0) = \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] = 0
\]

Then we directly differentiate the second term in (151)

\[
\frac{d \xi_1}{d \xi_2} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} G_{n,l}(\xi_1, \xi_2) = \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] \left( \sum_{j \in F_{1,l}} (X_j - m_j) \right) e(\xi_1, \xi_2)
\]

\[
\frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] \left( \sum_{j \in F_{1,l}} (X_j - m_j) \right) e(\xi_1, \xi_2)
\]

\[
+ \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] \left( \sum_{j \in F_{1,l}} (X_j - m_j) \right) e(\xi_1, \xi_2)
\]

\[
+ \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} \frac{d \xi_1}{d \sigma} E \left[ \sum_{i \in F_{1,l}} (X_i - m_i) \right] \left( \sum_{j \in F_{1,l}} (X_j - m_j) \right) e(\xi_1, \xi_2)
\]

Let us begin with (152) and (153)
\[ T_{152} = \frac{d}{d\sigma_n} \mathbb{E} \left[ (X_l - m_l) \left( i \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( i \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e(\xi_1, \xi_2) \right] \]

\[ = -\mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \right] \times \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e(\xi_1, \xi_2) \]

\[ + \mathbb{E} \left[ (X_n - m_n) X_l \right] \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e(\xi_1, \xi_2) \]

\[ + \mathbb{E} \left[ (X_l - m_l) \mathbb{E} \left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e(\xi_1, \xi_2) \right] \]

\[ - \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i_1 \in F_1^{n,l}} (X_{i_1} - m_{i_1}) \right) \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) \right] \times (-i\xi_1) \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_1^{n,l}} X_{i_2} \right] e(\xi_1, \xi_2) \]

\[ - \mathbb{E} \left[ (X_l - m_l) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) \right] \times (-i\xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] e(\xi_1, \xi_2) \]

\[ =: (b_{1,1}) + (b_{1,2}) + (b_{1,3}) + (b_{1,4}) + (b_{1,5}) + (b_{1,6}) \]

and

\[ T_{153} = \frac{d}{d\sigma_n} \left( \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n,l}} X_i \right] \right) \mathbb{E} \left[ \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e(\xi_1, \xi_2) \right] \]

(157)
\[
+ \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] \frac{d}{d\sigma_n} \left( \mathbb{E} \left[ \left( \sum_{j \in F_2^{n.i}} (X_j - m_j) \right) e (\xi_1, \xi_2) \right] \right) \quad (158)
\]

where

\[
T \quad (157) = \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] \mathbb{E} \left[ \left( \sum_{j \in F_2^{n.i}} (X_j - m_j) \right) e (\xi_1, \xi_2) \right] + \mathbb{E} \left[ -\mathbb{E} [(X_n - m_n) X_l] \sum_{i \in F_1^{n.i}} X_i \right] \mathbb{E} \left[ \left( \sum_{j \in F_2^{n.i}} (X_j - m_j) \right) e (\xi_1, \xi_2) \right]
\]

\[
=: (b_{2,1}) + (b_{2,2})
\]

with

\[
T \quad (158) = \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] \mathbb{E} \left[ (X_n - m_n) \left( \sum_{j \in F_2^{n.i}} (X_j - m_j) \right) e (\xi_1, \xi_2) \right] + \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] \mathbb{E} \left[ -\mathbb{E} [(X_n - m_n) \sum_{j \in F_2^{n.i}} X_j] e (\xi_1, \xi_2) \right] + \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] 
\]

\[
\times \mathbb{E} \left[ \left( \sum_{j \in F_2^{n.i}} (X_j - m_j) \right) (-i\xi_1) \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_1^{n.i}} X_{i_2} \right] e (\xi_1, \xi_2) \right] + \mathbb{E} \left[ (X_l - m_l) \sum_{i \in F_1^{n.i}} X_i \right] 
\]

\[
\times \mathbb{E} \left[ \left( \sum_{j_1 \in F_2^{n.i}} (X_{j_1} - m_{j_1}) \right) (-i\xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j_2 \in F_2^{n.i}} X_{j_2} \right] e (\xi_1, \xi_2) \right] =: (b_{2,3}) + (b_{2,4}) + (b_{2,5}) + (b_{2,6})
\]

Put \((\xi_1, \xi_2) = (0, \frac{\xi}{\sqrt{K}})\) and we combine \((b_{1,1}), (b_{1,2}), (b_{2,1})\) and \((b_{2,2})\) to get

\[
|(b_{1,1}) + (b_{1,2}) + (b_{2,1}) + (b_{2,2})|
\]
Moreover, we also have by Lemma 3.5 and (19) that

\[ E_{n,l} := E[(X_n - m_n)(X_l - m_l)] \]

Moreover, we also have by Lemma 3.5 and (19) that

\[
\begin{align*}
&= E \left[ (X_n - m_n)(X_l - m_l) - E_{n,l} \right] \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \\
&\quad \times \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right) - E \left[ \sum_{i \in F_1^{n,l}} (X_i - m_i) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \bigg] \\
&\leq \sum_{i \in F_1^{n,l}} \left| \text{cov} \left( ((X_n - m_n)(X_l - m_l) - E_{n,l})(X_i - m_i), \sum_{j \in F_2^{n,l}} (X_j - m_j) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right) \right| \\
&\quad + \sum_{i \in F_1^{n,l}} \left| \text{cov} \left( ((X_n - m_n)(X_l - m_l) - E_{n,l}), (X_i - m_i) \sum_{j \in F_2^{n,l}} (X_j - m_j) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right) \right| \\
&\leq L \left( K + K^2 \xi^2 \right)^{\frac{1}{2}} \exp \left( -CL/2 \right) + L \left( K^2 + K + K^2 \xi^2 \right)^{\frac{1}{2}} \exp \left( -CL/2 \right) \\
&\leq KL \left( 1 + |\xi| \right) \exp \left( -CL/2 \right)
\end{align*}
\]

where \( E_{n,l} \) denotes

Moreover, we also have by Lemma 3.5 and (19) that

\[
|b_{1,3}| = E \left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] \| E \left[ (X_l - m_l) \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \| \\
= E \left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] \left| \text{cov} \left( X_l, \left( \sum_{j \in F_2^{n,l}} (X_j - m_j) \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right) \right| \\
\leq L \left( K + K^2 \xi^2 \right)^{\frac{1}{2}} \exp \left( -CL \right) \\
\leq KL \left( 1 + |\xi| \right) \exp \left( -CL \right),
\]

\[
|b_{1,4}| = E \left[ (X_n - m_n) \sum_{j \in F_2^{n,l}} X_j \right] \| E \left[ (X_l - m_l) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \| \\
= E \left[ (X_n - m_n) \sum_{j \in F_2^{n,l}} X_j \right] \left| \text{cov} \left( X_l, \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right) \right| \\
\leq L \left( K + K^2 \xi^2 \right)^{\frac{1}{2}} \exp \left( -CL \right) \\
\leq KL \left( 1 + |\xi| \right) \exp \left( -CL \right),
\]
\[
\begin{align*}
\| \text{cov} \left( X_n, \sum_{j \in F_{2}^{n,l}} X_j \right) \| & = \text{E} \left[ \left( X_l - m_l \right) \left( \sum_{i \in F_{1}^{n,l}} X_i - m_i \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \\
& \leq L \sqrt{K} \exp(-CL),
\end{align*}
\]

\( (b_{1.5}) = 0, \)

\[
\begin{align*}
|\text{E} \left[ \left( X_l - m_l \right) \sum_{i \in F_{1}^{n,l}} X_i - \left( X_l - m_l \right) \sum_{j \in F_{2}^{n,l}} X_j \right] | & = \frac{\| \xi \|}{\sqrt{K}} \left| \text{E} \left[ \left( X_n - m_n \right) \sum_{j \in F_{2}^{n,l}} X_j \right] \right| \\
& \times \left| \text{E} \left[ \left( X_l - m_l \right) \left( \sum_{i \in F_{1}^{n,l}} X_i - m_i \right) \left( \sum_{j \in F_{2}^{n,l}} X_j - m_j \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \right| \\
& = \frac{\| \xi \|}{\sqrt{K}} \text{cov} \left( X_n, \sum_{j \in F_{2}^{n,l}} X_j \right) \\
& \times \left| \text{E} \left[ \left( X_l - m_l \right) \left( \sum_{i \in F_{1}^{n,l}} X_i - m_i \right) \left( \sum_{j \in F_{2}^{n,l}} X_j - m_j \right) e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \right| \\
& \leq KL \| \xi \| \exp(-CL),
\end{align*}
\]

\( |\text{E} \left[ e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] | \leq L \sqrt{K} \exp(-CL), \)

\( (b_{2.4}) = 0, \)

\[
\begin{align*}
\left| \text{E} \left[ \left( X_n - m_n \right) \sum_{j \in F_{2}^{n,l}} X_j \right] \right| & = \text{E} \left[ \left( X_l - m_l \right) \sum_{i \in F_{1}^{n,l}} X_i \right] \left| \text{E} \left[ e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \right| \\
& = \text{cov} \left( X_n, \sum_{j \in F_{2}^{n,l}} X_j \right) \left| \text{E} \left[ \left( X_l - m_l \right) \sum_{i \in F_{1}^{n,l}} X_i \right] \right| \left| \text{E} \left[ e \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right] \right| \\
& \leq L \sqrt{K} \exp(-CL),
\end{align*}
\]

\( (b_{2.5}) = 0, \)
and

\[ |(b_{2,6})| = \frac{\| |\xi| \sqrt{K} \| E \left[ (X_n - m_n) \sum_{j_2 \in F_{2,n,l}^2} X_{j_2} \right] \| E \left[ (X_l - m_l) \sum_{i \in F_{1,n,l}^1} X_i \right]} \times \left| E \left[ \sum_{j_1 \in F_{2,n,l}^n} (X_{j_1} - m_{j_1}) \right] \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right| \]

\[ = \frac{\| |\xi| \sqrt{K} \| \text{cov} \left( X_n, \sum_{j_2 \in F_{2,n,l}^2} X_{j_2} \right) \| E \left[ (X_l - m_l) \sum_{i \in F_{1,n,l}^1} X_i \right]} \times \left| E \left[ \sum_{j_1 \in F_{2,n,l}^n} (X_{j_1} - m_{j_1}) \right] \left( 0, \frac{\tilde{\xi}}{\sqrt{K}} \right) \right| \]

\[ \lesssim KL \| \xi \| \exp(-CL) \]

Next, (154) is

\[ T_{(154)} = \frac{d}{d\sigma_n} \left( E \left[ (X_l - m_l) \sum_{k \in F_2} X_k \right] \right) E \left[ \left( \sum_{j \in F_1} (X_j - m_j) \right) e^{(\xi_1, \xi_2)} \right] \]

\[ + E \left[ (X_l - m_l) \sum_{k \in F_2} X_k \right] \frac{d}{d\sigma_n} \left( E \left[ \left( \sum_{j \in F_1} (X_j - m_j) \right) e^{(\xi_1, \xi_2)} \right] \right) \]

where

\[ T_{(159)} \]

\[ = E \left[ (X_n - m_n) (X_l - m_l) \sum_{j \in F_{2,n,l}^2} X_j \right] E \left[ \left( \sum_{i \in F_{1,n,l}^1} (X_i - m_i) \right) e^{(\xi_1, \xi_2)} \right] \]

\[ + E \left[ -E[(X_n - m_n) X_l] \sum_{j \in F_{2,n,l}^2} X_j \right] E \left[ \left( \sum_{i \in F_{1,n,l}^1} (X_i - m_i) \right) e^{(\xi_1, \xi_2)} \right] \]

\[ =: (b_{3,1}) + (b_{3,2}) \]

and

\[ T_{(160)} \]
\[
\begin{align*}
\mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2}^{n,l}} X_j \right] &= \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i \in F_{1}^{n,l}} (X_i - m_i) \right) e(\xi_1, \xi_2) \right] \\
+ \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2}^{n,l}} X_j \right] \mathbb{E} \left[ (X_n - m_n) \sum_{i \in F_{1}^{n,l}} X_i \right] e(\xi_1, \xi_2) \\
+ \mathbb{E} \left[ (X_l - m_l) \sum_{j \in F_{2}^{n,l}} X_j \right] \\
\times \mathbb{E} \left[ \left( \sum_{i_1 \in F_{1}^{n,l}} (X_{i_1} - m_{i_1}) \right) (-i_1 \xi_1) \mathbb{E} \left[ (X_n - m_n) \sum_{i_2 \in F_{1}^{n,l}} X_{i_2} \right] e(\xi_1, \xi_2) \right] \\
+ \mathbb{E} \left[ (X_l - m_l) \sum_{j_1 \in F_{2}^{n,l}} X_{j_1} \right] \\
\times \mathbb{E} \left[ \left( \sum_{i \in F_{1}^{n,l}} (X_i - m_i) \right) (-i_2 \xi_2) \mathbb{E} \left[ (X_n - m_n) \sum_{j_2 \in F_{2}^{n,l}} X_{j_2} \right] e(\xi_1, \xi_2) \right] =: (b_{3,3}) + (b_{3,4}) + (b_{3,5}) + (b_{3,6})
\end{align*}
\]

Again put \((\xi_1, \xi_2) = \left( 0, \frac{\xi}{\sqrt{K}} \right)\) to get

\[
|(b_{3,1}) + (b_{3,2})| = \left| \text{cov} \left( (X_l - m_l)(X_n - m_n), \sum_{j \in F_{2}^{n,l}} X_j \right) \right| \\
\times \left| \mathbb{E} \left[ \left( \sum_{i \in F_{1}^{n,l}} (X_i - m_i) \right) e\left( 0, \frac{\xi}{\sqrt{K}} \right) \right] \right| \\
\lesssim L\sqrt{K} \exp(-CL),
\]

\[
|(b_{3,3})| = \left| \text{cov} \left( X_l, \sum_{j \in F_{2}^{n,l}} X_j \right) \right| \left| \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i \in F_{1}^{n,l}} (X_i - m_i) \right) e\left( 0, \frac{\xi}{\sqrt{K}} \right) \right] \right| \\
\lesssim L\sqrt{K} \exp(-CL),
\]

\[
|(b_{3,4})| 
\]
\[
\begin{align*}
&= \left| \mathbb{E}\left[ (X_l - m_l) \sum_{j \in F_2^{n,l}} X_j \right] \right| \left| \mathbb{E}\left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] \right| \left| \mathbb{E}\left[ e\left(0, \frac{\xi}{\sqrt{K}}\right)\right] \right| \\
&= \text{cov}\left(X_l, \sum_{j \in F_2^{n,l}} X_j \right) \left| \mathbb{E}\left[ (X_n - m_n) \sum_{i \in F_1^{n,l}} X_i \right] \right| \left| \mathbb{E}\left[ e\left(0, \frac{\xi}{\sqrt{K}}\right)\right] \right| \\
&\lesssim L\sqrt{K} \exp(-CL),
\end{align*}
\]

Note that

\[
T(155) = i \xi_1 \frac{d}{d\sigma_n} \mathbb{E}\left[ \left( \sum_{k \in F_1} (X_k - m_k) \right) \left( \sum_{j \in F_2} (X_j - m_j) \right) \right] \mathbb{E}\left[ (X_l - m_l) \sum_{k \in F_1} X_k \right] e
\]

\[
= 0
\]

when \((\xi_1, \xi_2) = \left(0, \frac{\xi}{\sqrt{K}}\right)\). Let us move on to the term \((156)\).

\[
T(156) = i \xi_2 \frac{d}{d\sigma_n} \mathbb{E}\left[ (X_l - m_l) \sum_{j \in F_2^{n,l}} X_{j_2} \right]
\]
\[\times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) e^{(\xi_1, \xi_2)} \right] \tag{161}\]

\[+ i\xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] \times \frac{d}{d\sigma_n} \left[ \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) e^{(\xi_1, \xi_2)} \right] \right] \tag{162}\]

where

\[T_{161} = i\xi_2 \mathbb{E} \left[ (X_n - m_n) (X_l - m_l) \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] \times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) e^{(\xi_1, \xi_2)} \right] \]

\[+ i\xi_2 \mathbb{E} \left[ -\mathbb{E} \left[ (X_n - m_n) X_l \right]\sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] \times \mathbb{E} \left[ \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) e^{(\xi_1, \xi_2)} \right] \]

\[= : (b_{5,1}) + (b_{5,2}) \]

and

\[T_{162} = i\xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] \times \mathbb{E} \left[ (X_n - m_n) \left( \sum_{i \in F_1^{n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_2^{n,l}} (X_{j_1} - m_{j_1}) \right) e^{(\xi_1, \xi_2)} \right] \]

\[+ i\xi_2 \mathbb{E} \left[ (X_l - m_l) \sum_{j_2 \in F_2^{n,l}} X_{j_2} \right] \]
We insert \((\xi_1, \xi_2) = (0, \frac{\xi}{\sqrt{K}})\) and proceed as before to deduce

\[
| (b_{5,1}) + (b_{5,2}) | = \frac{|\xi|}{\sqrt{K}} \left| \text{cov} \left( (X_n - m_n)(X_l - m_l), \sum_{j_2 \in F_{2,n,l}} X_{j_2} \right) \right| \times \left| \mathbb{E} \left[ \left( \sum_{i \in F_{1,n,l}} (X_i - m_i) \right) \left( \sum_{j_1 \in F_{2,n,l}} (X_{j_1} - m_{j_1}) \right) e \left( 0, \frac{\xi}{\sqrt{K}} \right) \right] \right|
\]

We insert \((\xi_1, \xi_2) = (0, \frac{\xi}{\sqrt{K}})\) and proceed as before to deduce
\[ \lesssim KL |\xi| \exp (-CL) \]

and similar argument leads
\[ |(b_{5,3}), (b_{5,4}), (b_{5,5})| \lesssim KL |\xi| \exp (-CL), \]

\( (b_{5,6}) = 0 \)

and lastly,
\[ |(b_{5,7})| \]
\[ = \left( \frac{|\xi|}{\sqrt{K}} \right)^2 \operatorname{cov} \left( X_l, \sum_{j \in F_{2,n,l}} X_{j3} \right) \left| \mathbb{E} \left[ (X_n - m_n) \sum_{j_1 \in F_{n,l}^3} X_{j_1} \right] \right| \times \left| \mathbb{E} \left[ \left( \sum_{i \in F_{1,n,l}} (X_i - m_i) \right) \left( \sum_{j_2 \in F_{2,n,l}^3} (X_{j_2} - m_{j_2}) \right) e \left( 0, \frac{\xi}{\sqrt{K}} \right) \right] \right| \]
\[ \lesssim KL^2 |\xi|^2 \exp (-CL) \]

To conclude, we have proven that
\[ \left| \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \right| \]
\[ \lesssim KL \left( 1 + |\xi| \right) \exp (-CL/2) + KL \left( 1 + |\xi| \right) \exp (-CL) \]
\[ + \left( L\sqrt{K} + L^2 \sqrt{K} \xi^2 \right) \exp (-CL) \]
\[ \lesssim KL \left( 1 + |\xi| \right) \exp (-CL/2) + \left( L\sqrt{K} + L^2 \sqrt{K} \xi^2 \right) \exp (-CL) \]

so that we have
\[ \int_{|\xi/(1/\sqrt{K})| \leq \delta} \left| \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l} \left( \frac{\xi}{\sqrt{K}}, \frac{\xi}{\sqrt{K}} \right) \right| ^2 \frac{d\xi}{\xi^2} \]
\[ \lesssim KL \left( \sqrt{K} + \frac{L}{K} \right) \exp (-CL/2) + \left( KL + \frac{L^2}{K} \right) \exp (-CL) \]
\[ \lesssim KL \left( \sqrt{K} + \frac{L}{K} \right) \exp (-CL/2) + \left( KL + \frac{L^2}{K} \right) \exp (-CL) \]

and this finishes the proof for the inner integral \((104)\).

**Argument for \((105)\)**: In this part, we use \((53), (52)\) and \((93)\) again. In addition, by Lemma 3.5. in \([MO13]\), we have
\[ \left| \frac{d}{d\sigma} \frac{d}{d\sigma} E \left[ \exp \left( i X_l - m_l \right) \xi \right] \right| \leq \left( 1 + |\xi|^2 \right) |\xi|^3 \quad (163) \]

Let us move on to the outer integral \((105)\). The integrand in \((105)\) is
\[ \frac{d^2}{d\sigma^2} \mathbb{E} \left[ \exp \left( i X_l - m_l \right) \xi \right] \]
\[ = \frac{d^2}{d\sigma} \mathbb{E} \left[ \hat{e}(\xi, \xi) \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \left( X_i - m_i \right) \xi \right) \right] X_{j, even} \right] \]
\[
\begin{align*}
&= \frac{d}{d\sigma} \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right] \\
&\quad \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \\
&+ \frac{d}{d\sigma} \mathbb{E} \left[ \frac{d}{d\sigma} \left( \hat{e}(\xi) \right) \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \right] \\
&+ \frac{d}{d\sigma} \mathbb{E} \left[ \hat{e}(\xi) \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \right) \right] \\
&=: (A) + (B) + (C)
\end{align*}
\]

where

\[
\hat{e}(\xi) := \exp \left( i \frac{1}{\sqrt{K}} \sum_{k: \text{even}} (X_k - m_k) \xi + i \frac{1}{\sqrt{K}} \sum_{i: \text{odd}} (m_{i,2} - m_i) \xi \right)
\]

We begin with (A).

(A)

\[
\begin{align*}
&= \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right)^2 \hat{e}(\xi) \right] \\
&\quad \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \\
&+ \mathbb{E} \left[ \frac{d}{d\sigma} \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right] \\
&\quad \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \\
&+ \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \frac{d}{d\sigma} \left( \hat{e}(\xi) \right) \right] \\
&\quad \times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \\
&+ \mathbb{E} \left[ \hat{e}(\xi) \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \left| X_j, j: \text{even} \right. \right] \right) \right] \\
&=: (A_1) + (A_2) + (A_3) + (A_4)
\end{align*}
\]

Note that (19) implies

\[| (A_1) | \]
\[
\leq \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right)^2 \right] \\
\times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \\
\lesssim K^2 \mathbb{E} \left[ \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right]
\]

On the other hand, applying (52) \( K^2 \) - 2 times and (53) on the remaining 2 factors leads
\[
|(A_1)| \lesssim K^{2} \lambda K^{2} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^2 \\
\lesssim K^{3} \lambda K^{2} \frac{1}{K + \xi^2} \\
\lesssim K^{3} \lambda K^{2} \frac{1}{1 + \xi^2}
\]

Next, same computation yields
\[
|(A_2)| = \left| \mathbb{E} \left[ - \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right)^2 \hat{e}(\xi) \right] \right] \right| \\
\times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \\
\lesssim K^{3} \lambda K^{2} \frac{1}{1 + \xi^2}
\]

Recall that we have
\[
\frac{d}{d\sigma} m_{i,2} = s_{i,2}^2 = \mathbb{E} \left[ (X_i - m_{i,2})^2 \mid X_j, j : \text{even} \right]
\]

Then it follows from (19) and (39) combined with (52) \( K^2 \) - 3 times and (53) (3 times) that
\[
|(A_3)| = \left| \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right] \right| \\
\times \left( -i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) X_k \right] + i \frac{1}{\sqrt{K}} \xi \sum_{i: \text{odd}} s_{i,2}^2 \right) \\
\times \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \\
\lesssim K \left( \frac{|\xi|}{\sqrt{K}} + \frac{|\xi|}{\sqrt{K}} \right) \lambda K^{2} \frac{1}{1 + (1/\sqrt{K}) |\xi|} \\
\lesssim K^{4} \lambda K^{2} \frac{1}{\sqrt{K} + |\xi|} |\xi| \left( \frac{1}{\sqrt{K} + |\xi|} \right)^3
\]
Lastly, a combination of (19), (52), (53) and (93) yields

\[
|A_4| \lesssim K^2 \cdot K \left(1 + \left|\frac{\xi}{\sqrt{K}}\right|\right) \left|\frac{\xi}{\sqrt{K}}\right|^{\lambda - 7} \left(\frac{1}{1 + (1/\sqrt{K})|\xi|}\right)^6
\]

\[
\lesssim K^3 \lambda^{\lambda - 7} \frac{1}{1 + \xi^2}
\]

Let us address \((B)\). A direct computation yields

\[
(B) = \mathbb{E} \left[\left(\sum_{n=1}^{K} (X_n - m_n) \right) \frac{d}{d\sigma} \left(\hat{e}(\xi)\right) \right]
\]

\[
\times \prod_{i: \text{odd}} \mathbb{E} \left[\exp \left(i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi\right) \mid X_j, j : \text{even}\right]
\]

\[
+ \mathbb{E} \left[\frac{d^2}{d\sigma^2} \left(\hat{e}(\xi)\right) \right]
\]

\[
\times \prod_{i: \text{odd}} \mathbb{E} \left[\exp \left(i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi\right) \mid X_j, j : \text{even}\right]
\]

\[
+ \mathbb{E} \left[\frac{d}{d\sigma} \left(\hat{e}(\xi)\right) \right]
\]

\[
\times \frac{d}{d\sigma} \left(\prod_{i: \text{odd}} \mathbb{E} \left[\exp \left(i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi\right) \mid X_j, j : \text{even}\right]\right)
\]

\[
=: (B_1) + (B_2) + (B_3)
\]

Note that

\[
|(B_1)| = |(A_3)| \lesssim K^4 \lambda^{\lambda - 3} \frac{1}{1 + \xi^2}
\]

Before we compute \((B_2)\), we compute an auxiliary result

\[
\frac{d^2}{d\sigma^2} \left(\hat{e}(\xi)\right)
\]

\[
= \frac{d}{d\sigma} \left(\hat{e}(\xi)\right) \left(-i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[\left(\sum_{n=1}^{K} (X_n - m_n) \right) X_k\right] + i \frac{1}{\sqrt{K}} \xi \sum_{i: \text{odd}} s_{i,2}^2\right)
\]

\[
= \frac{d}{d\sigma} \left(\hat{e}(\xi)\right) \left(-i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[\left(\sum_{n=1}^{K} (X_n - m_n) \right) X_k\right] + i \frac{1}{\sqrt{K}} \xi \sum_{i: \text{odd}} s_{i,2}^2\right) + \hat{e}(\xi) \frac{d}{d\sigma} \left(\frac{d}{d\sigma} \left(\hat{e}(\xi)\right)\right)
\]

\[
(164)
\]

\[
= \hat{e}(\xi) \frac{d}{d\sigma} \left(\frac{d}{d\sigma} \left(\hat{e}(\xi)\right)\right)
\]

\[
(165)
\]
where
\[
T_{164} = \hat{e}(\xi) \left( -i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) X_k \right] + i \frac{1}{\sqrt{K}} \xi \sum_{i: \text{odd}} s_{i,2}^2 \right)^2
\]
and
\[
T_{165} = \hat{e}(\xi) \left( -i \frac{1}{\sqrt{K}} \xi \sum_{k=1}^{K} \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right)^3 \right] + i \frac{1}{\sqrt{K}} \xi \sum_{i: \text{odd}} t_{i,2} \right)
\]
Here, \( t_{i,2} \) denotes
\[
t_{i,2} = \frac{d}{d\sigma} s_{i,2} = \mathbb{E} \left[ (X_i - m_{i,2})^3 \mid X_j, j: \text{even} \right]
\]
and in particular by Lemma 3.2. in [MO13] we have
\[
|t_{i,2}| \lesssim 1
\]
Therefore we have
\[
\left| \frac{d^2}{d\sigma^2} (\hat{e}(\xi)) \right| \lesssim \left( \frac{|\xi|}{\sqrt{K}} K^2 + \frac{|\xi|}{\sqrt{K}} K \right)^2 + \left( \frac{|\xi|}{\sqrt{K}} K^3 + \frac{|\xi|}{\sqrt{K}} K \right)
\]
\[
\lesssim K^3 |\xi|^2 + K^{5/2} |\xi|
\]
Inserting (166) into (B_2) yields
\[
|(B_2)| \lesssim \left( K^3 |\xi|^2 + K^{5/2} |\xi| \right) \mathbb{E} \left[ \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j: \text{even} \right] \right]
\]
\[
\lesssim K^3 |\xi|^2 \lambda^{K/2-4} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^4 + K^{5/2} |\xi| \lambda^{K/2-3} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^3
\]
\[
\lesssim K^5 \lambda^{K/2-4} \frac{1}{1 + \xi^2} + K^4 \lambda^{K/2-3} \frac{1}{1 + \xi^2}
\]
The last term (B_3) follows from similar computation. Use (93) with (52) and (53) to conclude
\[
|(B_3)| \lesssim K \cdot \left( \frac{|\xi|}{\sqrt{K}} K^2 + \frac{|\xi|}{\sqrt{K}} K \right) \left( 1 + \left| \frac{\xi}{\sqrt{K}} \right| \right) \left| \frac{\xi}{\sqrt{K}} \right|^3 \lambda^{K/2-8} \left( \frac{1}{1 + (1/\sqrt{K}) |\xi|} \right)^7
\]
\[
\lesssim K^4 \lambda^{K/2-8} \frac{1}{1 + \xi^2}
\]
The term (C) is deduced by the same type of argument, namely
\[
(C) = \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right.
\]
\[
\times \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j: \text{even} \right] \right)
\]
\[
= \mathbb{E} \left[ \left( \sum_{n=1}^{K} (X_n - m_n) \right) \hat{e}(\xi) \right.
\]
\[
\times \left. \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j: \text{even} \right] \right) \right]
\]
\[ + \mathbb{E}\left[ \frac{d}{d\sigma} \left( \dot{e}(\xi) \frac{d}{d\sigma} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right) \right) \right] \\
+ \mathbb{E}\left[ \dot{e}(\xi) \frac{d^2}{d\sigma^2} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right) \right] \\
=: (C_1) + (C_2) + (C_3) \]

Note \((C_1) = (A_4), \) \((C_2) = (B_3).\) To address \((C_3),\) we have by using (163), (52) and (53)

\[
\left| \frac{d^2}{d\sigma^2} \left( \prod_{i: \text{odd}} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : \text{even} \right] \right) \right| \\
\lesssim \frac{K}{2} \left( 1 + \frac{\xi^2}{K} \right) \left| \frac{\xi}{\sqrt{K}} \right|^3 \lambda^{\frac{k}{2} - 8} \left( \frac{1}{1 + \left( \frac{1}{\sqrt{K}} \right) |\xi|} \right)^7 \\
+ \frac{K}{2} \left( \frac{K}{2} - 1 \right) \left( 1 + \frac{\xi^2}{K} \right) \left| \frac{\xi}{\sqrt{K}} \right|^3 \lambda^{\frac{k}{2} - 7} \left( \frac{1}{1 + \left( \frac{1}{\sqrt{K}} \right) |\xi|} \right)^6 \\
\lesssim K^2 \lambda^{\frac{k}{2} - 8} \frac{1}{1 + \xi^2} + K^3 \lambda^{\frac{k}{2} - 7} \frac{1}{1 + \xi^2}
\]

and thus we conclude \(|(C_3)| \lesssim K^2 \lambda^{\frac{k}{2} - 8} \frac{1}{1 + \xi^2} + K^3 \lambda^{\frac{k}{2} - 7} \frac{1}{1 + \xi^2} .\)

Overall, we have proven that

\[
\left| \frac{d^2}{d\sigma^2} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] \right| \\
\lesssim K^3 \lambda^{\frac{k}{2} - 2} \frac{1}{1 + \xi^2} + K^4 \lambda^{\frac{k}{2} - 3} \frac{1}{1 + \xi^2} + K^3 \lambda^{\frac{k}{2} - 7} \frac{1}{1 + \xi^2} \\
+ K^5 \lambda^{\frac{k}{2} - 4} \frac{1}{1 + \xi^2} + K^4 \lambda^{\frac{k}{2} - 3} \frac{1}{1 + \xi^2} + K^4 \lambda^{\frac{k}{2} - 8} \frac{1}{1 + \xi^2} \\
+ K^2 \lambda^{\frac{k}{2} - 8} \frac{1}{1 + \xi^2} + K^3 \lambda^{\frac{k}{2} - 7} \frac{1}{1 + \xi^2}
\]

\[\lambda^{\leq 1} \leq K^{2-\beta} \frac{1}{1 + \xi^2}\]

Therefore we have

\[
\left| \int_{\{|(1/\sqrt{K})| > \delta\}} \frac{d^2}{d\sigma^2} \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{K}} \sum_{k=1}^{K} (X_k - m_k) \xi \right) \right] d\xi \right| \\
\lesssim K^{2-\beta} \left( 1 + \xi^2 \right) \frac{1}{\xi^2} d\xi \\
\lesssim K^{2-\beta}
\]

and this finishes the proof of (22) in Proposition 3.6.
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