Quantized space-time and its influences on some physical problems

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Abstract: Based on the idea of quantized space-time of Snyder, we derive new generalized uncertainty principle and new modified density of states. Accordingly we discuss the influences of the modified density of states on some physical quantities and laws. In addition we analyzed the exact solution of the harmonic oscillator in Snyder’s quantized space-time.

Key words: quantized space-time, generalized uncertainty principle, Stefan-Boltzmann law

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1 Introduction

Recently a great interest has been devoted to the study of the generalized uncertainty principle (GUP)\cite{1–14}. The main consequence of the GUP is the existence of a minimal length scale of the order of the Planck length, which can be deduced in string theory and other theories of quantum gravity\cite{15–22}. Although at present no this kind of minimal length is probed experimentally, from the form of the GUP or the general commutation relation it should exist theoretically. The minimal length can provide a natural ultraviolet (UV) cut-off. The GUP can also influence the structure of phase space and modify the usual density of states to a different form with a weighted factor. With the modified density of states one can calculate its effects on cosmological constant and black body radiation\cite{7} and the entanglement entropy of black holes by means of statistical mechanics\cite{23}.

As mentioned above the existence of minimum length can be the result of generalized uncertainty principle. In fact one can also consider the problem in an opposite direction. To deal with the trouble in quantum field theory, Snyder\cite{24} proposed an suggestion that the usual four dimensional space-time may not be continuous but discrete or quantized. This means that there is a smallest unit of length in space-time and the space-time should be noncommutative. Based on the assumption of existence of a minimal length and Lorentz invariance, Snyder introduced some operators for position, momentum and angular momentum and obtained a sequence of commutators between them. It shows that the usual commutation relation $[\mathbf{x}, \mathbf{p}] = i$ and $[\mathbf{x}, \mathbf{x}] = 0$ no more exist. According to the commutators obtained by Snyder we can also derive a new GUP. Besides one can also obtain the modified density of states different from the one obtained from the usual GUP. Thus one can recalculate the influences of the new modified density of states on the physical quantities and laws, such as the cosmological constant, black body radiation, the entanglement entropy of black holes and so on. Particularly the minimal length in Snyder’s model give a natural UV cutoff in the calculation of entanglement entropy, thus the UV divergence caused by the infinite density near the horizon can be removed.

The paper is arranged as follows. In the next section we first introduce Snyder’s quantized space-time model and derive the GUP and modified density of states. In the third section, we calculate its influences on the cosmological constant and specially on the Stefan-Boltzmann laws. In addition we simply analyze the exact solution of the harmonic oscillator in Snyder’s quantized space-time. We shall give some concluding remarks in the final section.$(c = \hbar = G = k_B = 1)$

2 Snyder’s quantized space-time and GUP

Snyder developed the quantized space-time which is invariant under Lorentz transformation, namely the quadratic form $-\eta^{ij} = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$ should be invariant under Lorentz transformation. The $\eta_{b\mu}$, ($b = 0, 1, 2, 3, 4$) are the homogeneous projective coordinates of a real 4-dimensional space of constant curvature. Thus the space-time operator $\hat{x}_{\mu}$, ($\mu = 0, 1, 2, 3$) can be defined as

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\[ \hat{x}_0 = i\alpha (\eta_1 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_1}) \quad \hat{x}_1 = i\alpha (\eta_1 \frac{\partial}{\partial \eta_3} - \eta_3 \frac{\partial}{\partial \eta_1}) \quad \hat{x}_2 = i\alpha (\eta_2 \frac{\partial}{\partial \eta_3} - \eta_3 \frac{\partial}{\partial \eta_2}) \quad \hat{x}_3 = i\alpha (\eta_3 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_3}) \] (1)

where \( \alpha \) is the minimum length.

In addition there are another two groups of operators:
\[ \hat{p}_0 = \frac{\hbar}{2\pi} (\eta_0 / \eta_1) \quad \hat{p}_1 = \frac{\hbar}{2\pi} (\eta_1 / \eta_3) \quad \hat{p}_2 = \frac{\hbar}{2\pi} (\eta_2 / \eta_3) \] (2)

and
\[ \hat{L}_i = i(\eta_0 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_0}) \quad \hat{L}_j = i(\eta_1 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_1}) \quad \hat{M}_1 = i(\eta_2 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_2}) \quad \hat{M}_2 = i(\eta_3 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_3}) \quad (3) \]

We know that Lorentz group has six generators which can be recorded as \( M_{ij} (i,j = 1,2,3) \) and \( M_{0i} \). The three generators for rotation \( L_i = \frac{\hbar}{2\pi} \epsilon_{ijk} M_{jk} \) and the other three ones for boost can be described as \( M_i = M_{0i} \). It is easy to get the commutator below:
\[ [\hat{x}_i, \hat{x}_j] = i\alpha^2 \hat{M}_{ij} \quad [\hat{x}_0, \hat{x}_i] = i\alpha^2 \hat{M}_{0i} \] (4)

Obviously if one takes the limit \( \alpha \to 0 \), the quantized and noncommutative space-time will turn into the usual continuous and commutative space-time. In fact what we really care about is the commutators below:
\[ [\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \alpha^2 \hat{\delta}_i \hat{\delta}_j) \] (5)

This relation reminds us of the more general commutator from GUP, which is[4]
\[ [\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \lambda \delta_{ij} \hat{p}^2 + \lambda' \hat{p} \hat{p}_j) \] (6)

One can easily find out that the commutator Eq.(5) from quantized space-time is a special case of the commutators Eq.(6) from GUP with \( \lambda = 0 \) and \( \lambda' = \alpha^2 \).

In general it is known that for any pair of observables \( A \) and \( B \), the uncertainty relation
\[ \Delta A \Delta B \geq \frac{1}{2} |[A, B]| \] (7)

In view of \( \Delta A = A - \hat{A} \) and \( \Delta \hat{A} = \hat{A} - A \), we can obtain from Eq. (5) that
\[ \Delta x_i \Delta p_j \geq \frac{1}{2} (\delta_{ij} + \alpha^2 \Delta p_i \Delta p_j + \gamma) \] (8)

where \( \gamma \) is positive and dependent on the expectation value of \( p_i \). We can name the GUP above as Snyder’s GUP. If considering the \( i = j \) case only, the formula above will turns into
\[ \Delta x_i \Delta p_i \geq \frac{1}{2} [1 + \alpha^2 (\Delta p_i)^2 + \gamma] \] (9)

or
\[ \Delta x_i \geq \frac{1}{2} \left( \frac{1 + \gamma}{\Delta p_i} + a^2 \Delta p_i \right) \] (10)

which is similar to the frequently used GUP. Obviously when setting \( \gamma = 0 \), it will give a minimal uncertainty length \( \Delta x = \alpha \).

According to the usual Heisenberg uncertainty principle, one can obtain the D dimensional phase space volume
\[ d^{D}x d^{D}p \] (11)

Upon quantization, the corresponding number of quantum states per momentum space volume is
\[ \frac{d^{D}x d^{D}p}{(2\pi)^{D}} \] (12)

Considering the commutator Eq.(6) from GUP, the number of quantum states changes to[6]
\[ \frac{d^{D}x d^{D}p}{(2\pi)^{D}(1 + \lambda p^2)^{D-1}[1 + (\lambda + \lambda') p^2]^{1-\frac{\lambda}{2}}}(13) \]

Thus the number of quantum states for quantized space-time should be
\[ \frac{d^{D}x d^{D}p}{(2\pi)^{D}(1 + a^2 p^2)^{D/2}} \] (14)

In fact its counterpart
\[ \frac{d^{D}x d^{D}p}{(2\pi)^{D}(1 + \lambda p^2)^{D}} \] (15)

which corresponds to \( \lambda' = 0 \) in Eq.(13) is often considered by physicists. The difference between the two forms lies in the weighted factor, more precisely, the exponent there. The exponent in Eq.(15) is dimension-dependent, whereas the one in Eq.(14) is a constant 1/2. Thus when the two forms of number of quantum states are used in quantum field theory, one can deduce that Eq.(15) can remove the divergence more effectively, specially, the higher the space dimension is, the weaker the divergence is. Be that as it may, the Eq.(14) should be employed if the space-time is really discrete as Snyder’s proposition. One can use the formula to recalculate many quantities, like black body radiation, cosmological constant, et.al.
3 The influences on some physical problems

3.1 The cosmological constant

According to quantum field theory, the cosmological constant should be obtained by summing over the zero-point fluctuation energies of harmonic oscillators, each of which corresponds to a particular particle momentum state. If the dispersion relation is the usual one $E^2 = p^2 + m^2$ (if modified dispersion relation is considered, the discussion below should be modified correspondingly), we assume that the zero-point energy of each oscillator is of the form

$$\frac{\omega}{2} = \frac{1}{2} \sqrt{p^2 + m^2}$$  \hfill (16)

Thus the cosmological constant should be

$$\Lambda(m) = \frac{1}{2} \int \frac{d^3p}{(1+a^2p^2)^{1/2}} \sqrt{p^2 + m^2}$$

$$= 2\pi \int_0^\infty \frac{p^2 dp}{(1+a^2p^2)^{1/2}} \sqrt{p^2 + m^2}$$  \hfill (17)

Obviously the integral is power-law divergent. The weighted factor can only weaken the divergence but cannot cancel it. However, because of the existence of a minimal length $a$, the momentum cannot be integrated to infinity. The minimal length means there should be some $p_{\text{max}}$ above which one cannot probe and observe. Taking the $p_{\text{max}} \sim 1/a$, for the $m = 0$ case one can obtain a finite result

$$\Lambda = \frac{2\pi(2-\sqrt{2})}{3a^4} \sim \frac{2\pi(2-\sqrt{2})}{3} p_{\text{max}}$$  \hfill (18)


3.2 The Stefan-Boltzmann law

We discuss the black-body radiation and consider the radiation field as photon gas. In general the quantum states with momentum from $p \sim p + dp$ in volume $V$ is

$$V \frac{d^3p}{\pi^2} dp = \frac{V}{\pi^2} \omega^3 d\omega$$  \hfill (19)

here we have considered the spin degeneracy of photons and $\varepsilon = \omega = p$. The average quantum number should be

$$\frac{V}{\pi^2} \frac{\omega^3 d\omega}{e^{\varepsilon/T} - 1}$$  \hfill (20)

The internal energy of the photon gas is

$$U(\omega,T) d\omega = \frac{V}{\pi^2} \frac{\omega^3 d\omega}{e^{\varepsilon/T} - 1}$$  \hfill (21)

which is Planck formula. Integrating the equation above one can obtain

$$U = \frac{V}{\pi^2} \int_0^\infty \frac{\omega^3 d\omega}{e^{\varepsilon/T} - 1} = \frac{V T^4}{\pi^2} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{\pi^2 V}{15} T^4$$  \hfill (22)

which can give the usual Stefan-Boltzmann law.

Considering the quantized space-time, the density of states is modified by Eq.(14). Thus the internal energy of the photon gas should be

$$U = \frac{V}{\pi^2} \int_0^\infty \frac{\omega^3 d\omega}{(1+a^2 T^2 \omega^2)^{1/2}(e^{\varepsilon/T} - 1)}$$

$$= \frac{V T^4}{\pi^2} \int_0^\infty \frac{1 - \frac{1}{2}(a T x)^2 + \frac{3}{8}(a T x)^4 - \frac{5}{16}(a T x)^6 + ...}{e^x - 1} x^3 dx$$

$$= \frac{V T^4}{\pi^2} \left[ \int_0^\infty x^3 dx - \frac{a^2 T^2}{2} \int_0^\infty x^5 dx + \frac{3 a^4 T^4}{8} \int_0^\infty x^7 dx - ... \right]$$  \hfill (23)

It is know that

$$\int_0^\infty \frac{x^\alpha - 1}{e^x - 1} dx = \Gamma(\alpha) \zeta(\alpha)$$  \hfill (24)

where $\Gamma(\alpha)$ and $\zeta(\alpha)$ are the Gamma function and Riemann zeta function respectively. Because the minimal length $a$ should be very small, the series expansion at $a = 0$ is appropriate. Thus Eq.(23) can be calculated exactly

$$U = \frac{V T^4}{\pi^2} \left( \frac{\pi^4}{15} - \frac{4 \pi^6 a^2 T^2}{63} + \frac{3 \pi^8 a^4 T^4}{15} - ... \right)$$

$$= \frac{\pi^2 V T^4}{15} \left( 1 - \frac{60 \pi^2 a^2 T^2}{63} + \frac{3 \pi^4 a^4 T^4}{63} - ... \right)$$  \hfill (25)

The first term is the usual Stefan-Boltzmann relation and the latter ones are correction terms. For the usual Stefan-Boltzmann law we know that there is a maximal frequency, $x = \omega_m/T \approx 2.82$, corresponding to the maximal internal energy. According to the modified expression of internal energy for the photon gas, we can also
find the maximal frequency $\omega_{max}$. Given different values of $aT$ one can find the maximal values of the modified expression. The Fig.1 shows that the bigger the value of $aT$ is, the smaller the values of the maximal frequency and the maximal internal energy are.

![Graph](image)

Fig. 1. For the case of $aT = 0.1$ the maximal value lies at $x = 2.74$, the case of $aT = 1$ at $x = 1.88$, and the case of $aT = 10$ at $x = 1.60$.

### 3.3 The exact solutions of harmonic oscillator

Snyder gives the expression for $x_i$ in the momentum representation.

$$x_i = i(\delta_{ij} + a^2 p^2_i \delta_{ij} + a^2 p_i p_j) \frac{\partial}{\partial p_j}$$  \hspace{1cm} (26)

In the one-dimensional case, it turns into

$$x = i(1 + a^2 p^2) \frac{\partial}{\partial p}$$  \hspace{1cm} (27)

which is nearly the same as the one introduced according to GUP[4–6], in which case the position operators is $x = i \left[ (1 + \beta p^2) \frac{\partial}{\partial p} + \gamma p \right]$. For the Hamiltonian

$$H = \frac{1}{2} m \omega^2 x + \frac{p^2}{2m},$$

Chang obtained an $\gamma$-independent exact solution[6]. Thus referring to the result of Chang, in the Snyder’s quantized space-time model we can deduce that

$$E_n = \sqrt{n + 1} \sqrt{\frac{1}{4} + \frac{a^2 m^2 \omega^2}{4} + (n^2 + n + \frac{1}{2}) \frac{a^2 m^2 \omega^2}{2}} $$  \hspace{1cm} (28)

Once the higher dimensional case is considered, the results must be different. The general position operator introduced according to GUP is

$$x_i = i(\delta_{ij} + \beta p^2 i \delta_{ij} + \beta' p_i p_j) \frac{\partial}{\partial p_j}$$  \hspace{1cm} (29)

The main difference lies at the second term. In Eq.(26) it is $p^2_i$ (no summation here) and in Eq.(29) it is $p^2$. Thus the two operators must correspond to different exact solutions in higher dimensional case. We will leave the question for further consideration.

### 4 Conclusion

According to the idea of quantized space-time of Snyder, we derive the generalized uncertainty principle and modified density of states. The density of states obtained from Snyder’s model is different from the ones from the usual GUP. The weighted factor in the modified density of states is $1/(1 + \alpha^2 p^2)^{1/2}$ with a constant exponent 1/2, whereas the one from usual GUP is $1/(1 + \lambda p^2)^D$ with a dimension-dependent exponent $D$. This difference leads to different modifications to the ordinary physical problems.

Based on the Snyder’s GUP we calculate the cosmological constant. The minimal length gives an natural ultraviolet cutoff, which makes the cosmological constant be proportional to $p^4_{\text{max}}$ and finite. The Stefan-Boltzmann laws in thermodynamics may be also modified because of the modified density of states. Except the usual $\sim T^4$ term some correction terms also exist. Considering the modified Stefan-Boltzmann laws, the rate of black holes radiation will be influenced and the evolution of the universe should also be modified. These problems will be discussed in another paper. At last we discussed the exact solution of harmonic oscillator in one-dimensional case. The result is nearly the same as the one obtained in general GUP. When higher-dimensional case is considered the exact solutions will be different due to different position operators. This problem is left for further consideration.

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