Hadron Vacuum Polarization from application of DSEs and analytical confinement

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The hadronic vacuum polarization function $\Pi_h$ for two light flavors is conventionally defined through the vacuum expectation of current-current correlator such that $\Pi_h^{\mu\nu}(x) = \sum_q \langle 0 | j^a_q(0) j^b_q(x) | 0 \rangle$ where the sum runs over all quark flavors. It is also an alternative name for the part of the photon self-energy $\Pi$ to QCD, noting that the methods based on utilization of Nakanishi Integral Representation are successfully applied in approximate solution for the function $\Pi_h$ in the formalism of Dyson-Schwinger equations and provide the first, albeit very approximate solution for the function $\Pi_h$ in the timelike domain of Minkowski space. To fill this gap in our knowledge, we extend the use of Nakanishi Integral Representation into the formalism of Dyson-Schwinger equations and provide an analytical continuation to the Minkowski space. To fill this gap in our knowledge, we extend the use of Nakanishi Integral Representation into the formalism of Dyson-Schwinger equations and provide the first, albeit very approximate solution for the function $\Pi_h$ in the timelike domain of Minkowski space.

I. INTRODUCTION

The hadronic vacuum polarization function $\Pi_h(x)$ is conventionally defined through the vacuum expectation of current-current correlator such that $\Pi_h^{\mu\nu}(x) = \sum_q \langle 0 | j^a_q(0) j^b_q(x) | 0 \rangle$ where the sum runs over all quark flavors. It is also an alternative name for the part of the photon self-energy $\Pi(x)$ due to the quark loops. Using the continuous functional formalism it can be precisely defined as double differentiation of generating functional $\Gamma[\phi_{SM}]$ with respect to the photon fields $A$:

$$\Pi(x - y)^{\mu\nu} = \frac{\delta^2 \Gamma[\phi_{SM}]}{\delta A^\mu(x) \delta A^\nu(y)} |_{\phi_{SM} = 0} - \ldots, \quad (1.1)$$

where $\phi_{SM}$ stand for whole known ensemble of Standard Model fields and where the dots stands for the inverse of the free photon propagator. Using a standard routine $[1, 2]$ one can derive for the hadronic part of the Fourier transform of $\Pi_h^{\mu\nu}$ a well known expression

$$\Pi_h^{\mu\nu}(s) = -ie^2 N_c \sum_q e_q Tr \int \frac{d^4 k}{(2\pi)^4} \Gamma_q^{\mu}(k - q, k) S_q(k) \gamma^\nu S_q(k - q), \quad (1.2)$$

where the photon momentum satisfies $s = q^2$ and the trace is taken in Dirac space and $\Gamma_q^{\mu}$ is the dressed quark-photon proper vertex, $S_q$ is the dressed quark propagator, both functions satisfy their own Dyson-Schwinger equations, solutions of them in Minkowski space will be the subject of presented paper.

Together with the leptonic polarization function and loops containing gauge bosons $W, Z$, the function $\Pi_h$ completes the (inverse) photon propagator $[1, 1]$. In the spacelike domain of momenta the polarization function is responsible for a smooth and slow increase of the running QED charge. However, for positive $s$ the complexity of hadronic polarization $\Pi_h$ causes measurable interference effect in the fine structure constant $\alpha_{QED}$.

It is an experimental fact, that heavier quark $q$ is a larger quantum fluctuations in the function $\Pi_h$ one gets. Thus at the so called B-factories like BABAR $[3]$, BESS and BELLE one can easily see an enhancement in muon pair production at vicinity of bottomonium energies $s \simeq M_\Upsilon^2$ of colliding pair $e^+ e^-$, the effect for strangeonium $\phi$ meson energy $[4]$ gets substantially smaller, while the precise KLOE2 experiment observed such effect bellow 1 GeV energy $[5]$ only very recently. Photon polarization function offers a great amount of physical information and in the timelike domain of momenta, it is measured with continuously improved accuracy for many reasons. Needless to say, most of nonperturbative methods available in a market deal with the metric of Euclidean space, thus being almost blind when trying to look on the timelike domain of Minkowski space.

Up to an asymptotically large spacelike momentum the function $\Pi_h$ is not calculable from perturbation theory. Historically, the first nonperturbative extraction of the function $\Pi_h(s)$ come from the $e^+ e^- \rightarrow hadrons$ experiments due to work of Cabbibo $[6]$. The method is based on unitarity and analyticity arguments and it does not rely on the underlying QCD/QED dynamics at all. Using nonperturbative methods like lattice QCD and the functional approach of Dyson-Schwinger equations $[7]$ the function $\Pi_h$ has been obtained at the Euclidean (spacelike) domain of momenta. In order to understand how QCD resonances emerge in the polarization function $\Pi_h$ and what is the amount of non-resonance background there, one could employ nonperturbative methods which can naturally provide an analytical continuation to the Minkowski space. To fill this gap in our knowledge, we extend the use of Nakanishi Integral Representation into the formalism of Dyson-Schwinger equations and provide the first, albeit very approximate solution for the function $\Pi_h$ in the entire domain Minkowski space momenta. It is the first application to QCD, noting that the methods based on utilization of Nakanishi Integral Representation are successfully applied in
quantum models without confinement for many years [8–16]. Encouraging results for the electromagnetic form factors were obtained [17] within the formalism as well. Here we offer generalization to strong coupling QCD showing also that it requires non-trivial minimization of unphysical effects, which would otherwise lead to an unwanted analytical behavior of the hadron vacuum polarization function $\Pi_h$.

In the next section a minimal system of QCD&QED Dyson-Schwinger equations is presented. In the Section III, the Gauge Technique is reviewed for the quark propagator satisfying Nakanishi Integral Representation and resulting formula for function $\Pi_h$ is presented there as well. The minimization technique necessary to get the numerical solution and results are presented in the last Section IV.

II. EXPRESSING THE HADRON VACUUM POLARIZATION IN MINKOWSKI SPACE

The Dyson-Schwinger equations are an infinite system of quantum equations of motions for Green’s functions and when solved exactly the whole provide the full information about theory. Continuous formalism of Dyson-Schwinger equations has found its most important applications in evaluations of hadronic properties within the use of QCD degrees of freedom: quark and gluon fields. Bethe-Salpeter equation is part of the system and traditional tool for calculation of meson masses [18–22], electromagnetic form-factors [23–25] and meson transition form factors [26, 27].

Here we begin with the QCD part of the model and restrict to two flavors QCD $q = u, d$ in the isospin (equal mass) limit. The solution for the quark propagator can be represented by two scalar functions:

$$S(q) = \hat{\rho} S_v(q) + 1 S_u(q) = [A(q) \hat{\rho} - B(q)]^{-1}, \quad (2.1)$$

where the inverse of $A$ is the renormalization wave function, while the (renormalization invariant) quark dynamical mass function is conventionally defined as $M = B/A$.

In this paper, we begin with the simplest symmetry preserving truncation of the equations system - the Ladder-Rainbow Approximation. Thus the quark-antiquark scattering kernel completes the model, written for arbitrary linear gauge $\xi$ it is choosen:

$$V(q) = \gamma_\mu \times \gamma_\nu \left( -g^{\mu\nu} V_v(q) - \frac{4g^2}{3} \xi \frac{L^{\mu\nu}(q)}{q^2} \right), \quad (2.2)$$

$$V_v(q) = \frac{c_V (m_q^2 - \Lambda_g^2)}{(q^2 - m_q^2 + i\epsilon)(q^2 - \Lambda_g^2 + i\epsilon)}, \quad (2.3)$$

$$L^{\mu\nu}(q) = q^\mu q^\nu / q^2, \quad (2.4)$$

for which the quark Dyson-Schwinger equation can be written as

$$S^{-1}(q) = \hat{\rho} - m_q - \Sigma(q),$$

$$\Sigma(q) = \int d^4k \frac{1}{(2\pi)^4} \gamma_\mu S(k) \gamma_\nu V_v(k - q)$$

$$- \frac{4g^2}{3} \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) \gamma_\nu L^{\mu\nu}(k - q) \frac{k^2}{(k - q)^2}. \quad (2.5)$$

where, as explained in further text, the numerical values of five parameters appearing in Eqs. (2.2), (2.3) are determined by the pion properties: e.g. by the pion mass and the pion decay constant with further requirement that the vacuum hadron polarization function $\Pi_h(s)$ has a cut at the timelike positive axis of $s$, and neither poles or branch points and associated cuts are not allowed everywhere else. The analytical form of kernel (2.2) is one of key ingredients for compliance with desired analyticity of the function $\Pi_h$.

Another good motivation for the use of the kernel (2.2) is that its generalization is very straightforward. Actually, within the method of Nakanishi Integral Representation the whole formal derivations presented in this paper remains valid for a large class of possibly considered interactions. To begin with the simplest, we avoid further integrations and stay with two poles in the kernel characterized by two constant masses $\mu_q$ and $\Lambda_q$ respectively. Nonetheless, kindred Bethe-Salpeter equation (BSE) models [19, 20, 28] turned out to be successful in description of ground and excited states of pions and charmonia.

Let us clarify, that the method based on utilization of Nakanishi Integral Representation we employ here can hardly compete with impressive amount of achievements already made in the Euclidean space, either obtained in the Rainbow-Ladder Approximation [18, 19, 29] or calculated with even more sophisticated truncation [32]. Instead of, the main goal here, is to provide the first reliable form of generalized quark spectral functions. Within use of them, the function $\Pi_h$ will be obtained in the entire domain of the Minkowski space momentum for the first time.
Depending on values of parameters appearing in the equation (2.5), we can get many curious solutions. In order to extract solution, which is consistent with QCD dynamic one needs to reproduce correct hadron properties, e.g. properties of lightes meson - the pion-. The meson bound states in the vacuum are described by (BSE) which explicitly depends on the dressed momentum dependent quark propagator, determined by the quark equation (2.5). For the sacle of consistency, the BSE and the Eq. (2.5) must use the identical kernel. For this purpose we solve the pion BSE, which reads:

\[
\Gamma(p, p) = i \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S_\mu(k) \Gamma(p, k) S(k) \gamma_\mu \{ -g^{\mu\nu} V_\nu(p - k) - 4/3g^2 L^\mu_\nu(p - k) \},
\]

(2.6)

where \( P \) is the total momentum of meson satisfying \( P^2 = M^2 \), \( M = 140 \text{MeV} \) for the ground state and the arguments in the quark propagator are \( k_\pm = k \pm P/2 \). The pion BSE vertex function reads

\[
\Gamma(p, p) = \gamma_5 [\Gamma_A(p, p) + \beta \Gamma_B(p, p) + \gamma_4 \Gamma_C(p, p) + \beta \gamma_5 \Gamma_D(p, p)],
\]

(2.7)

where \( \Gamma_X \) are four contributing scalar functions.

The equation (2.6) has been solved by method of iterations described in the papers [19, 20, 28]. Here, however the numerical search complicates, since we intend to get simultaneous solution of the quark gap equation (2.5) in Minkowski space as well. There are nontrivial subtleties, which follows from a search of Minkowski solution of the Eq. (2.5) and we describe them lately in the section (2.5).

### A. Gauge Technique entry

To evaluate Eq. (1.2) one needs to know the solution for the Abelian gauge vertex \( \Gamma^\mu \). The best would be to solve the Dyson-Schwinger equation for this vertex as well, however to do this in Minkowski space is recently impossible. To accomplish this we appreciate the fact that \( U(1) \) electromagnetic symmetry is unbroken in the Nature and employ the Gauge Technique. This allows us to close the system of DSEs by construction of the quark-photon vertex in a minimal gauge invariant manner. We are going to miss some of transverse components in this way (in fact, a large amount of expected \( \omega \) and \( \rho \) meson poles), but we should get reliable result in off resonance region.

For practical purpose we will use un-amputated vertex, which relates the propagators and the proper gauge vertex in usual way:

\[
\Lambda^\mu(p, l) = S(p) \Gamma^\mu(p, l) S(l)
\]

(2.8)

and solve the Ward-Takahashi identity, which reads

\[
(p - l)^\mu \Lambda^\mu(p, l) = S(p) - S(l)
\]

(2.9)

The Gauge Technique has been introduced in Ref. [33] and it represents gauge covariant tool for solution of Dyson-Schwinger equation in the entire domain of momenta. It consists of writing a solution for the vector un-amputated vertex in the form

\[
\Lambda^\mu(p, q) = \int_\Gamma dx \rho(x) \frac{1}{p - x} \gamma_\mu \frac{1}{\not{q} - x},
\]

(2.10)

where one assumes there exists a generalized spectral representation for the quark propagator

\[
S(p) = \int_\Gamma dx \frac{\rho(x)}{(p - x)}. \tag{2.11}
\]

In this paper we generalized Gauge Technique, which instead of spectral representation allows to use the Hilbert transformation

\[
S_q(p^2) = \int_\infty^\infty da \frac{f p \sigma_s(a) + \sigma_s(a)}{p^2 - a + i\epsilon}. \tag{2.12}
\]

The procedure is relatively straightforward and the Gauge technique solution for the quark-photon vertex reads:

\[
\Lambda^\mu(p, q) = \int_{-\infty}^{\infty} d\omega \frac{\sigma_s(\omega)[\gamma_\mu \omega + \not{p} \gamma_\mu \not{q}]}{(p^2 - \omega + i\epsilon)(q^2 - \omega + i\epsilon)} + \int_{-\infty}^{\infty} d\omega \frac{\sigma_s(\omega)[\not{p} \gamma_\mu + \gamma_\mu \not{q}]}{(p^2 - \omega + i\epsilon)(q^2 - \omega + i\epsilon)} + \Lambda^\mu_q(p, q), \tag{2.13}
\]

\[
\Lambda^\mu_q(p, q) = \int_{-\infty}^{\infty} d\omega \frac{\sigma_s(\omega)[\gamma_\mu \omega + \not{q} \gamma_\mu \not{p}]}{(p^2 - \omega + i\epsilon)(q^2 - \omega + i\epsilon)} + \Lambda^\mu_q(p, q).
\]
where the transverse piece satisfies $Q \Gamma_T(p, q) = 0$ and where $Q$ is photon momentum. Inclusion of the transverse components $\Lambda_T^\mu(p, q)$ requires the solution of Dyson-Schwinger equation for the quark-photon vertex and is subject of recent study [34]. By converting the momentum space gap equation for the vertex into a new but equivalent integro-differential equation for the so called Nakamichi weights, it turns out to be feasible task, which could provide the solution in the entire Minkowski space. Leaving this important calculation for a future work, we take $\Lambda_T^\mu(p, q) = 0$ in this paper for purpose of simplicity.

The Eq. (2.12) should be regarded as the generalization of Lehmann representation, with two properties in absence: neither of function $\sigma_\tau$ or $\sigma_s$ is positive definite and the position of branch point is not assumed in advance. The introduction of negative cuts in relations (2.12) and (2.15) could be regarded as an auxiliary step, which when missing, the solution of Dyson-Schwinger equation for the quark propagator would be hardly achievable in practice, if possible at all. Anticipate here the solution: the negative cut is gradually vanishing as it is subject of minimization.

Furthermore, no prohibited acausal behavior is observed as a solution. There is no evidence for negative branch point associated with appearance of pathological singularities like tachyonic poles.

In fact, we do not expect and we actually do not get the quark propagator pole within the timelike axis as well. The observed absence of the real pole in the propagator is the analytical realization of confinement mechanism in presented framework. At last but not at least, let us make a technical note: the so called Wick rotation contour can be used for purpose of analytical continuation into the Euclidean space. This fact will be silently used during the derivation.

Using the Gauge technique (2.13), the relation (1.2) necessary to evaluate reads

\[ \hat{\Pi}_h(q) = \Sigma_f \Sigma_{c,d} \int \frac{d^4k}{(2\pi)^4} \Lambda^\mu(p - q, q) \gamma_\mu, \]  

which within the use of on-shell renormalization prescription and after long but rather straightforward calculation gives the desired result:

\[ \hat{\Pi}(q) = q^2 \Pi(q); \]
\[ \Pi_h(q) = \Sigma_{f=u,d} \frac{e^2 N_c}{4\pi^2} 8q^2 \int_0^1 dx X(x) S_v^a(a) \]  

where the argument of the function $S_v$ (see Eq. 2.1) is $a = x(1-x)q^2$ and

\[ X(x) = 4x^4 + 3x^3 - x^2. \]  

The expression (2.15) shows an elusive way how the dressed quark propagator appears in the hadron polarization function $\Pi_h$. As the Eq. (2.15) is deduced from gauge covariant consideration, this term should be always presented in any other meaningful approximation.

III. RESULTS, HVP CONSTRAINED BY THE PION PROPERTIES AND VICE VERSA

To get the solution for the function $\Pi_h$ one just needs to substitute the quark propagator $S$ into the expression (2.15) and integrate over the variable $x$, which was done numerically. To get the propagator $S$ we converted the Eq. (2.16) into the integral equations for the Nakamichi weight functions $\sigma_\tau$ and $\sigma_s$ and solve it by the method of iterations. These new equations, called Unitary equations in the paper [10], provide stable numerically convergent solution for Nakamichi weights.

The quark propagator is then substituted into the BSE (2.6) in order to identify the pion bound state. For this purpose we accommodate the method developed for kindred model described in the paper [19, 20]. The method works in its Euclidean approximation, it requires the quark propagator evaluated at complex value of momenta, which is quite easily achieved within the use of the integral representation (2.12). As a consequence of the use of the Nakamichi Integral Representation, the solution of BSE stays completely real in isospin limit considered here.

To get the correct physical picture within the model one should reproduce not only static properties of the pion i.e. pion mass $m_\pi = 140 MeV$, pion decay constant $f_\pi \simeq 95 MeV$, two photon decay width of neutral pion, etc., but also one needs to reproduce desired analytical properties of continuous form factors, i.e. the function $\Pi_h$ in our case. As follows from standard unitarity arguments, it implies that the imaginary part of the vacuum hadron polarization function should be vanishing for the spacelike arguments. It requires to add the Eq. (2.15) into the coupled set of DSE and BSE and solve the whole system simultaneously together.
FIG. 1: The function $\Pi_h/C$ obtained via Gauge Technique with the constant $C$ defined as $C = -40\alpha/(9\pi)$. Two distinct curves represent -up to the scale- identical solution of Dyson-Schwinger equation. The one has $m_\pi = 140 MeV$ (solid one) and rescaled one corresponds with $m_\pi = 210 MeV$.

Alternatively, one can infer from the Eq. (2.15) solution, that the requirement $\Im \Pi(s) \rightarrow 0; s < 0$ is equivalent to similar condition for the quark propagator, which we rather take in the form

$$\sum_s |\sigma_v(s)| = \min$$

for some some subdomain where $s < 0$. Note for clarity, that experience teach us, that when imposing a lower cutoff on the quark propagator Nakanishi weight function then one does not get any solution of Unitary equations in QCD at all.

In practice, it means that when solving the coupled system of equations for propagators and mesons, one also need to minimize a certain functional of $\sigma_v(s); s < 0$ by a search of optimal choice of the truncation of Dyson-Schwinger system. In our simple Ladder-Rainbow model it is equivalent to search for an optimal parameters $m_g, \Lambda_g, c_V$ and at last but not at least, the biproduct $g\xi$ should be a part of the game. This minimization procedure has been embedded into the iterations cycles of solution of both Eqs. the BSE (2.6) and the DSE (2.5) as well. This is not an easy task, but it is feasible with recent computer facilities.

The numerical search was performed providing the rate $\sigma_v(\pm 1 GeV)/\sigma_v(1 GeV) \approx 10^{-4}$ was achieved.

The parameters $c_V/(4\pi)^2 = 1.8$ and $g^2 \xi/(4\pi)^2 = 0.17$ and $m_\pi^2/\Lambda_g^2 = 2/7.5, m_\pi/m_g = 1.38/\sqrt{2}$ provide the physical pion mass and the correct value of the pion decay constant. Resulting function $\Pi_h$ is depicted in figures 1 and 2 respectively. In the absence of transverse components the function does not exhibit usual $\rho$ meson peak but rather small bump positioned numerically at $\sqrt{s} = 500 MeV$. Assuming for any reason, that the observed cusp/bump could be positioned at physical mass of $\rho$ and $\omega$ meson, one can find another solution of the system by simple rescaling. In this case one gets slightly wrong pion mass $m_\pi = 210$. We do not know wether the global minimum was actually achieved, we assume there exist further solutions, for which the condition (3.1) is reasonably satisfied as well, however due to the computer time consumption, they are difficult to find.

The resulting propagator function, its inverse given by function $A, B$ as well as the quark dynamical function $M = B/A$ are shown in figures 3, 4, 5 and 6 respectively. Both propagator Nakanishi weights change the sign, the quark propagator does not have a real pole as well as we do not see evidence for branch point singularity (exhibiting as a cusp). All these properties are in beautiful accordance with the confinement of quarks in standard model vacuum.

At given stage there are many open questions to be answered. For instance I have found that the cut in the spacelike domain of momenta is minimal (but not trivial) if the gauge is fixed such that $4\xi \frac{g}{g\xi} \approx 10 \pm 3$. This is a curious observation and it is hard to look for any interpretation at given stage of our study.
FIG. 2: The

§omenta.

FIG. 3: The quark propagator function $S_\nu$

FIG. 4: Typical look known from the Euclidean studies: The quark mass function in the spacelike domain of momenta.
IV. CONCLUSIONS AND PROSPECTS

We completed a computation of the hadron vacuum polarization function $\Pi_h(q^2)$ in two flavor QCD. All required elements are determined by the solution of QCD’s Dyson-Schwinger equations obtained in the rainbow-ladder truncation, the leading order in a systematic and symmetry preserving approximation scheme. Using a single interaction kernel, the model provides correct pion properties as well as it shows up a cusp at $\omega/\rho$ mass, albeit for $140 MeV$ heavy pion the value of the peak maximum is located $250 MeV$ lower then the vector meson mass observed in experiments.

The novel analysis technique we employed made possible to compute $\Pi_h(q^2)$ on the entire domain of spacelike as well as of timelike momenta for the first time. Our prediction agree with other methods in the spacelike domain, while it miss a large amount of $\rho/\omega$ peak well known from the standard treatment based on use of experimental data on hadroproduction in electron-positron annihilation. The solution for transverse vertices in the entire domain of Minkowski space is one of the main future aims (34), which could be achieved in the formalism of Dyson-Schwinger equations.

At given stage there are many open questions necessary to answer. The method suffers by need of nontrivial and demanding minimization of an auxiliary introduced cut in the quark propagator function. Including Yang-Mills part more seriously into the game and getting the similar solution for the gluon propagator is challenging and still opened task for years (35) as well.
Appendix A: Rainbow ladder quark self-energy in arbitrary linear gauge

The Dyson-Schwinger equation for the quark propagator can be converted into the Unitary equations for Nakanishi weights, by comparing of imaginary and real parts of assumed integral representation (2.11), i.e.

\[
\sigma_{v,s}(p^2) = -\frac{3S_{v,s}(p^2)}{\pi} \quad \Re S_{v,s}(p^2) = P. \int ds \frac{\sigma_{v,s}(x)}{p^2 - x},
\]

and by the integral representation for the inverse of the propagator, which is readily derivable from quark gap equation (2.5). After the renormalization it reads

\[
S^{-1} = \not{p} - m(\mu) - \Sigma(p),
\]

where the self-energy functions \( \Sigma = \Sigma^V + \Sigma^\xi \) are evaluated in details in this Appendix.

Let us start with unrenormalized self-energy (hence index 0), which comes from the product of a gauge term and the quark propagator expressed through the Hilbert transformation. The first line in (2.5) reads

\[
\omega\xi
\]

where we have employed the Hilbert transformation (2.12) and label

\[
\gamma
\]

After a standard treatment and a little algebra it can be written into the following form

\[
\Sigma^0_\xi(q) = -i\xi g^2 \int d^4k \int d^4k E \int do \frac{4(1 - x)\sigma_v(o)k.q k + \sigma_v(o)(-2 - x) k + \sigma_v(o) + 4(1 - x)x^2\sigma_v(o)q^2}{D^3 D^2} \gamma^\mu \gamma^\nu (k - q)^\mu (k - q)^\nu (k^2 - o + i\epsilon)^2.
\]

After a standard treatment and a little algebra it can be written into the following form

\[
\Sigma^0(0) = \int d^4x \int d^4k \int d\sigma_v(o)(1 - x) x^2\sigma_v(o)q^2 \gamma^\mu \gamma^\nu (k^2 - o + i\epsilon)^2,
\]

where the denominator \( D = -k^2 - q^2(1 - x)x - o\epsilon \) is strictly negative, noting the Wick rotation is working for positive as well as for the negative variable \( o \).

Thus to go to the Euclidean space is what we only need here in order to integrate over the momenta, as the result we get

\[
\Sigma^0_\xi(q) = -\frac{\xi g^2}{(4\pi)^2} \int d^4x \int d\sigma_v(o)(1 - x) x^2\sigma_v(o)[c(d) + \ln (\Omega/\mu^2)]
\]

\[
+ \left( \sigma_v(o)(-2 - x) k + \sigma_v(o) \right)[c(d) + \ln (\Omega/\mu^2)] + \frac{(1 - x)x^2\sigma_v(o)q^2}{q^2(1 - x) - o + i\epsilon},
\]

where \( \Omega = q^2(1 - x) - o\epsilon \) and \( \mu \) is the spacelike renormalization scale ( \( \mu^2 < 0 \) in our metric convention).

The third term in the Eq. (A3) is UV finite and can be rewritten as

\[
-\frac{\xi g^2}{(4\pi)^2} \int d^4x \int d\sigma_v(o)(1 - x) x^2\sigma_v(o)q^2 = -\frac{\xi g^2}{(4\pi)^2} \int d^4x \int d\sigma_v(o)(1 - x) x^2\sigma_v(o)q^2 \gamma^\mu \gamma^\nu (k^2 - o + i\epsilon) \not{\gamma},
\]

where we have employed the Hilbert transformation (2.12) and label \( \omega = q^2(1 - x) \).

In addition to the usual Minimal Subtraction counter-terms we will sent also the following terms

\[
\delta Z_v = \int d^4x \int d\sigma_v(o)(1 + 2x)lnx = \frac{3g^2}{2(4\pi)^2} \int d\sigma_v(o)
\]

\[
\delta Z_m = -\int d^4x \int d\sigma_s(o)lnx = -\frac{g^2}{(4\pi)^2} \int d\sigma_s(o)
\]

into the renormalization constant \( Z_2(Z_v) \) and \( Z_3(Z_m) \).

For the first two terms in (A5) we thus have

\[
-\frac{\xi g^2}{(4\pi)^2} \int d^4x \int d\sigma_v(o)(1 - 2x)\sigma_v(o) + \sigma_v(o)lnq^2(1 - x) - o + i\epsilon
\]

where we have drop out all renormalization constants. Using per-parts integration and sending momentum independent boundary terms into renormalization constants again one finally gets for the rest of (A8):

\[
q^2 \frac{\xi g^2}{(4\pi)^2} \int d^4x \int d\sigma_v(o)(1 + x)\sigma_v(o) + x\sigma_v(o)lnq^2(1 - x) - o + i\epsilon
\]

(A9)
Summing this with (A10) one finally gets

\[ \Sigma_\xi(q) = q^2 \frac{\xi g^2}{(4\pi)^2} \int_0^1 dx \int_0^{q_0} ds \frac{-2x\sigma_v(o) \, \hat{g} + x\sigma_s(o)}{q^2(1-x) - o + i\epsilon} \]

\[ = q^2 \frac{\xi g^2}{(4\pi)^2} \int_0^1 dx \int_0^{q_0} ds \left[ -2\Sigma_v(\omega') \, \hat{g} + \Sigma_s(\omega') \right]. \tag{A11} \]

where we have employed the Hilbert transformation once again. To end, we make the substitution \( s = \omega' \) and arrive into the desired renormalized result:

\[ \Sigma_\xi(q) = -\frac{\xi g^2}{(4\pi)^2} \int_0^{q_0} ds \left[ 1 - \frac{s}{q^2} \right] \left[ q^2 \Sigma_v(s) - \Sigma_s(s) \right] + C_A \, \hat{g} + C_B \]

which keeps the lower integral boundary smaller then the upper one.

Let us make short digression here and remind that the dynamical symmetry breaking and massless pion are attributed to chiral limit \( m = 0 \) in QCD. This is a more complicated issue when one is dealing with spectral representation and as we have sent (irrespective of their UV finitness) scalar piece of the self-energy into the renormalized constant, we cannot use our scheme directly for the calculation in the chiral limit. To set the mass exactly to zero one must also require

\[ \int d\rho_s(o) = 0, \tag{A13} \]

which, at least at the formal level allows us to skip the mass renormalization at all. The sum rule condition (A13) could be explicitly used before the momentum integration in the chiral limit, otherwise we are facing the ambiguity \( c(d) \int \sigma_s = \infty \) and the result turns to be ordering dependent (not well defined). In this paper we will deal with the physical pion and we leave the question of solution in exact chiral limit \( m = 0 \) unanswered for future task.

For the combination of the vector interaction \( V_v \) with the spectral part of the quark propagator we get

\[ \Sigma_V = -i \int \frac{d^4k}{(2\pi)^4} \int d\rho_s(o) \frac{\gamma^\mu (k\sigma_v(o) + \sigma_s(o))\gamma_\mu}{k^2 - o + i\epsilon} \left[ \frac{c_V}{(k-q)^2 - m_y^2 + i\epsilon} - \frac{c_V}{(k-q)^2 - \Lambda_y^2 + i\epsilon} \right], \tag{A14} \]

which is the standard one loop expression integrated over the continuous mass \( o \) giving us the known result:

\[ \Sigma_V = c_v \int_0^1 dx \int_0^{q_0} ds \frac{2\hat{g}(1-x)\sigma_v(o) + 4\sigma_s(o)}{(4\pi)^2} \log \left( \frac{q^2(1-x) - o - m_y^2 \frac{1-x}{x} + i\epsilon}{q^2(1-x) - o - \Lambda_y^2 \frac{1-x}{x} + i\epsilon} \right). \tag{A15} \]

For numerical purpose it is suited to further proceed by per partes integration

\[ \Sigma_V = c_v \int_0^1 dx \int_0^{q_0} ds \frac{2\hat{g}(1-x/2)\sigma_v(o) - 4\sigma(o)}{q^2(1-x) - o - m_y^2 \frac{1-x}{x} + i\epsilon} (-q^2 + m_y^2 \frac{1-x}{x}) - (m_y \to \Lambda_y) \]

\[ = c_v \int_0^1 dx \int_0^{q_0} ds \frac{2\hat{g}(1-x/2)\sigma_v(\tilde{o}) - 4\sigma(\tilde{o})}{q^2(1-x) - o - m_y^2 \frac{1-x}{x} + i\epsilon} (-q^2 + m_y^2 \frac{1-x}{x}) - (m_y \to \Lambda_y), \tag{A16} \]

where the argument in the first term of the second line reads \( \tilde{o} = q^2(1-x) - m_y^2 \frac{1-x}{x} \), which can be seen by virtue of Hilbert transformation again.
The last step advantageous for numerical solution is the introduction of the following functional identities

\[ 1 = \int_{-\infty}^{\infty} da \delta(a - \hat{a}), \delta(f(x)) = \sum_i \delta(x - x_i) \frac{df(x_i)}{dx} \]  

(A17)

in the previous equation. Interchanging the order of integration and integrating over the variable \( x \) one gets

\[ \Sigma_V = \sum_{i=\pm} \frac{c_v}{(4\pi)^2} \int_{-\infty}^{\infty} da \frac{2 \delta(x_i - x_i/2) S_v(a) - 4x_i S_v(a)}{|sgn(-q^2 x_i^2 + m_g^2)|} \Theta(x_i) \Theta(1 - x_i) \Theta(D) - (m_g \rightarrow \Lambda_g), \]  

(A18)

where the roots are

\[ x_\pm = \frac{-\left(m_g^2 + q^2 - a\right) \pm \sqrt{D}}{-2q^2} \]

\[ D = (m_g^2 + q^2 - a)^2 - 4q^2m_g^2 \]  

(A19)

for the first term. The expression (A18) has been actually used in our numerical code.

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