Abstract

We show that the equivalence between several possible characterizations of Frobenius algebras, and of symmetric Frobenius algebras, carries over from the category of vector spaces to more general monoidal categories. For Frobenius algebras, the appropriate setting is the one of rigid monoidal categories, and for symmetric Frobenius algebras it is the one of sovereign monoidal categories. We also discuss some properties of Nakayama automorphisms.

Mathematics Subject Classification (2000): 16B50, 18D10, 18D35
1 Introduction

Frobenius algebras in monoidal categories play a significant role in diverse contexts. Illustrative examples are the study of weak Morita equivalences of tensor categories [Mil1], certain correspondences of ribbon categories which give e.g. rise to the notion of trivializability of a ribbon category [FRS1, FRS2], the computation of correlation functions in conformal quantum field theory [FRS1, FRS2, SFR], the analysis of braided crossed G-categories [Mi1], the theory of subfactors and of extensions of C*-algebras [LR, EP], invariants of three-dimensional membranes [Lau], reconstruction theorems for modular tensor categories [Pfe], and a categorical version of Militaru’s D-equation [BS].

In the classical case of algebras in the category of vector spaces over a field or commutative ring, several equivalent characterizations of Frobenius algebras are in use, see e.g. [CR, Thm. 61.3]. The most common ones are via the existence of an isomorphism between an algebra and its dual as modules, or via the existence of a non-degenerate invariant bilinear form. A more recent description is via the existence of a coalgebra structure with appropriate compatibility properties [Ab, Qu]. Given such a characterization, there will be an analogous notion of Frobenius algebra in other categories that are monoidal and are equipped with sufficiently much additional structure. In view of the applications mentioned above, it is important to know whether the equivalence between different possible definitions persists in this more general situation.

Here we establish the equivalence between categorical versions of the three characterizations of Frobenius algebras just quoted, for the case that the monoidal category considered is also rigid, i.e. has left and right dualities. Analogous, and more extensive, results have been obtained for Frobenius algebras in any category of bimodules over a ring (Frobenius extensions) in [Kad], and for Frobenius monads, i.e. Frobenius algebras in categories of endofunctors [Law], in [Str], for commutative Frobenius algebras in compact closed categories some of the results can also be found in [Str]. In the vector space case there also exist other characterizations of Frobenius algebras, such as via ideals and their annihilators (see e.g. Theorem 16.40 of [Lam]); the study of their categorical versions is beyond the scope of this note.

Besides being rigid monoidal, no other properties are assumed for the categories in which these issues are studied. For instance, they need not be linear or abelian and need not have direct sums; but of course, in many applications they do have additional structure, like being ribbon categories [Tu] or (multi-)fusion categories [ENO]. In any rigid monoidal category there is an abundance of Frobenius algebras: for any object X, the objects X ⊗ X and X ⊗ X carry a natural structure of Frobenius algebra, with the structural morphisms being expressible through the evaluation and coevaluation morphisms. Examples for Frobenius algebras of this type are star-autonomous monoidal categories, regarded as objects in the monoidal bicategory Cat of small categories, see [Str, Cor. 3.3]. These Frobenius algebras are in fact all Morita equivalent to the tensor unit. A generic source for more general Frobenius algebras is provided by monoidal categories with nontrivial Picard group; Frobenius algebras that correspond to subgroups of the Picard group can be classified with the help of abelian group cohomology [FRS2].

We also discuss the equivalence between categorical versions of the notion of symmetric Frobenius algebra. For formulating these concepts, we require 1 that the categories in question are rigid monoidal and in addition sovereign, i.e. that the left- and right-duality functors coincide. Afterwards we introduce the notion of Nakayama automorphisms and study some of their properties. We show e.g. that a Frobenius algebra is symmetric iff its Nakayama automorphisms are inner automorphisms. We can then finally expose the relation between any two Frobenius structures on an algebra in a rigid monoidal category.

The three notions of Frobenius algebra are presented in Section 2, and their equivalence is proven in Section 3. Section 4 is devoted to an analogous discussion of symmetric Frobenius algebras. Nakayama automorphisms are studied in Section 5.

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1 The requirements that the categories are rigid, respectively sovereign, are not strictly necessary. For details see the Remarks 6 and 12 below.

2 Many, but not all rigid monoidal categories can be embedded as a full subcategory in a category of bimodules over a ring, in which case one is in the situation studied in [Kad]. Frobenius algebras not covered by this setting are obtained when the category does not possess all the generic properties that such full subcategories inherit from the category of bimodules. An example for a rigid monoidal category which is additive, but not abelian, is the category of finitely generated projective modules over a unital commutative ring, see e.g. [Tu, p. 25]. For an example which does not admit direct sums, see e.g. [Tu, p. 29]. And a rigid monoidal category that is not even preadditive, with the morphism sets not possessing any structure beyond being sets, is the two-dimensional cobordism category, whose objects are oriented one-manifolds and whose morphisms are cobordisms; a Frobenius algebra in this category is given by the circle, with the structural morphisms (product, coproduct, unit, counit) just being the elementary cobordisms [Lau].
2 Frobenius algebras

Algebras and coalgebras in monoidal categories

Let \( \mathcal{C} = (\mathcal{C}, \otimes, 1) \) be a monoidal category. Without loss of generality we assume \( \mathcal{C} \) to be strict. A (unital, associative) algebra (or monoid) \( A = (A, m, \eta) \) in \( \mathcal{C} \) consists of an object \( A \in \text{Obj}(\mathcal{C}) \) and morphisms \( m \in \text{Hom}(A \otimes A, A) \) and \( \eta \in \text{Hom}(1, A) \) satisfying \( m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m) \) and \( m \circ (\eta \otimes \text{id}_A) = \text{id}_A = m \circ (\text{id}_A \otimes \eta) \).

We will freely use the graphical notation for morphisms of strict monoidal categories as described e.g. in [JS, Ma, Kas] and [FrFRS, FjFRS]. Thus we write

\[
\begin{align*}
\text{id}_U &= U \\
\text{id}_U &= U \\
\end{align*}
\]

for identity morphisms, general morphisms \( f \in \text{Hom}(U, V) \), and for composition and tensor product of morphisms of \( \mathcal{C} \) (all pictures are to be read from bottom to top), as well as

\[
\begin{align*}
m &= \begin{array}{c} A \\ A \end{array} \\
\eta &= \begin{array}{c} A \\ A \end{array}
\end{align*}
\]

for the product and unit morphisms of an algebra \( A \) (note that the morphism \( \text{id}_1 \) is ‘invisible’, owing to strictness of \( \mathcal{C} \)). The defining properties of \( A \) then read

\[
\begin{align*}
\text{id}_U &= U \\
\text{id}_U &= U \\
\end{align*}
\]

\[
\begin{align*}
m &= \begin{array}{c} A \\ A \end{array} \\
\eta &= \begin{array}{c} A \\ A \end{array}
\end{align*}
\]

(3)

We will also need the dual notion of a (coassociative, counital) coalgebra in \( \mathcal{C} \). This is a triple \( (C, \Delta, \varepsilon) \) consisting of an object \( C \) and morphisms

\[
\begin{align*}
\Delta &= \begin{array}{c} C \\ C \end{array} \\
\varepsilon &= \begin{array}{c} C \\ C \end{array}
\end{align*}
\]

satisfying

\[
\begin{align*}
\text{id}_C &= C \\
\text{id}_C &= C \\
\end{align*}
\]

(5)

For more details, see e.g. appendix A of [FrFRS].

Three definitions of Frobenius algebras

The classical characterizations of Frobenius algebras mentioned in the introduction require the existence of some extra structure – an isomorphism of \( A \)-modules, a bilinear form, or a coalgebra structure – for a given algebra. In the Definitions 1, 3 and 4 below we prefer to specify instead a choice of the relevant structure explicitly; this will
allow us to present our arguments in a somewhat more direct manner. A formulation in terms of the existence of the extra structures is given in Definition 1 after the equivalence of the three definitions has been established.

There is one notion of Frobenius algebra in a category $\mathcal{C}$ that does not require any further structure on $\mathcal{C}$ beyond what is needed to define algebras, i.e. monoidality:

**Definition 1.** A $(\Delta, \varepsilon)$-Frobenius structure on an algebra $(A, m, \eta)$ in a monoidal category $\mathcal{C}$ is a pair $(\Delta, \varepsilon)$ of morphisms such that $(A, \Delta, \varepsilon)$ is a coalgebra and the coproduct $\Delta$ is a morphism of $A$-bimodules.

Here the bimodule structures on $A$ and on $A \otimes A$ are the obvious ones for which the left and right representation morphisms are furnished by the product $m$. In pictures, the bimodule morphism property of $\Delta$ reads

\[
\begin{align*}
A & \xrightarrow{\Delta} A \otimes A \xrightarrow{m} A \\
& = A
\end{align*}
\]

Note that, unlike in the case of bialgebras (which can be defined in any braided monoidal category), neither the coproduct $\Delta$ nor the counit $\varepsilon$ is an algebra morphism.

**Remark 2.** The characterization of Frobenius algebras given in Definition 1 is e.g. used in [FRS1, Mü1, BS] and implicitly in [KO]. Besides requiring nothing else than monoidality of $\mathcal{C}$, it has proved to be convenient for performing graphical calculations, and has been excessively used for this purpose in e.g. [FRS2, FRS3}. Also, it is this definition that can readily be generalized to so-called non-compact Frobenius algebras, which have recently been discussed in the context of string topology [GLSUX, CM]. Alternative descriptions close to the one in Definition 1 have been discussed, and been shown to be equivalent to it, in [Lau].

The Definition 1 has been given in [Ab] for the category of vector spaces over a commutative ring, and in [Stri] for compact closed (and thus in particular symmetric monoidal) categories. More precisely, in [Ab, Stri] it is further assumed that $A$ is commutative; as a consequence, the requirement that the coproduct is a bimodule morphism is equivalent to requiring that it is a morphism of left (or right) modules. A similar definition, again for vector spaces, is given in [Qu, app. A.3], where in addition to the bimodule morphism property of $\Delta$ it is required that the morphism $\varepsilon \circ m$ is symmetric; in [Qu] the resulting structure is called an ambialgebra. In the present setting no extra properties are imposed; thus in particular there is no need that the category $\mathcal{C}$ is symmetric, nor even braided.

The next definition generalizes the familiar one in terms of a bilinear form. To formulate it, we will have to assume that $\mathcal{C}$ is in addition rigid, i.e. has left- and right-duality endofunctors. We denote the left and right dual of an object $U$ by $\dual{U}$ and $U^\vee$, respectively, and the corresponding evaluation and coevaluation morphisms by

\[
\begin{align*}
b_U &= U \quad & d_U &= U^\vee \\
\dual{b}_U &= \dual{U} \quad & \dual{d}_U &= U^\vee
\end{align*}
\]

In the sequel we will refer to a morphism in $\text{Hom}(A \otimes A, 1)$ as a pairing on $A$.

**Definition 3.** A $\kappa$-Frobenius structure on an algebra $(A, m, \eta)$ in a rigid monoidal category $\mathcal{C}$ is a pairing $\kappa \in \text{Hom}(A \otimes A, 1)$ on $A$ that is invariant, i.e. satisfies

\[
k \circ (m \otimes \text{id}_A) = k \circ (\text{id}_A \otimes m),
\]

and that is non-degenerate in the sense that

\[
(id_A \otimes k) \circ (\hat{b}_A \otimes \text{id}_A) \in \text{Hom}(A, \dual{A})
\]

is an isomorphism.

\[\text{In our conventions we follow [FRS1]; in most of the literature, what we refer to as a left duality is called a right duality, and vice versa.}\]
In pictures, denoting the pairing $\kappa$ by
\[
\kappa =: A \xrightarrow{\cdot} A
\] (10)
the invariance property reads
\[
\begin{array}{c}
A \xrightarrow{\cdot} A \\
\end{array} = \begin{array}{c}
A \xrightarrow{\cdot} A \xrightarrow{\cdot} A \xrightarrow{\cdot} A \\
\end{array}
\] (11)
while the isomorphism featuring in the non-degeneracy property is depicted as
\[
\forall A
\begin{array}{c}
\Phi_{\kappa,l} \xrightarrow{=} \Phi_{\kappa,r}
\end{array}
\] (12)
We note that instead of $\Phi_{\kappa,l}$ one may as well use the morphism
\[
\Phi_{\kappa,r} := (\kappa \otimes id_A) \circ (id_A \otimes b_A) = \begin{array}{c}
\xrightarrow{\cdot} \xrightarrow{\cdot}
\end{array} \in Hom(A, A^\vee)
\] (13)
Indeed, we have
\[
d_A \circ (\Phi_{\kappa,r} \otimes id_A) = \kappa = \tilde{d}_A \circ (id_A \otimes \Phi_{\kappa,l})
\] (14)
and a similar identity involving the coevaluation morphisms $b_A$ and $\tilde{b}_A$; as a consequence the invertibility of $\Phi_{\kappa,r}$ is equivalent to the invertibility of $\Phi_{\kappa,l}$.

Next we note that for $U, V \in \text{Obj}(C)$ we can associate to any $f \in \text{Hom}(U, V^\vee)$ a morphism
\[
f^\wedge := (\tilde{d}_V \otimes id_{U^\vee}) \circ (id_V \otimes f \otimes id_{U^\vee}) \circ (id_V \otimes b_V) \in \text{Hom}(V, U^\vee).
\] (15)
The morphism $f^\wedge$ is an isomorphism iff $f$ is, in which case its inverse is given by $(f^\wedge)^{-1} = (d_U \otimes id_V) \circ (id_U \otimes f^{-1} \otimes id_V \otimes id_{U^\vee}) \circ (id_{U^\vee} \otimes b_V)$ (this shows e.g. that an object $U$ of $C$ is isomorphic to $^\vee U$ iff it is isomorphic to $U^\vee$). Using the notation (15), the equality
\[
\Phi_{\kappa,r} = (\Phi_{\kappa,l})^\wedge
\] (16)
is equivalent to (14). For convenience we also note some other immediate consequences of (14) and its analogue for $b_A$ and $\tilde{b}_A$: we have $d_A \circ (id_{A^\vee} \otimes \Phi_{\kappa,l}^{-1}) = \tilde{d}_A \circ (\Phi_{\kappa,r}^{-1} \otimes id_A) \in \text{Hom}(A^\vee \otimes A, 1)$, as well as
\[
(\Phi_{\kappa,l} \otimes \Phi_{\kappa,r}^{-1}) \circ b_A = \tilde{b}_A \quad \text{and} \quad d_A \circ (\Phi_{\kappa,r} \otimes \Phi_{\kappa,l}^{-1}) = \tilde{d}_A.
\] (17)
Let us display the latter identities also pictorially:
Next we note that the left dual $\bar{\gamma}A$ of $A$ is naturally a left module over $A$, while the right dual $A^\vee$ is naturally a right $A$-module. The corresponding representation morphisms $\rho \in \text{Hom}(A \otimes \bar{\gamma}A, \bar{\gamma}A)$ and $\eta \in \text{Hom}(A^\vee \otimes A, A^\vee)$ are given by

$$\rho = \left\{ \begin{array}{c}
\bar{\gamma}A \\
A \end{array} \right\}$$

$$\eta = \left\{ \begin{array}{c}
A \\
A^\vee \end{array} \right\}$$

It is therefore natural to wonder whether the isomorphisms (12) and (13) are compatible with the $A$-module structures of $A$, $\bar{\gamma}A$ and $A^\vee$. This leads to the

**Definition 4.** A $\Phi_\rho$-Frobenius structure on an algebra $A = (A, m, \eta)$ in a rigid monoidal category $C$ is a left-module isomorphism $\Phi_\rho \in \text{Hom}(A, \bar{\gamma}A)$ between the left $A$-modules $A = (A, m)$ and $\bar{\gamma}A = (\bar{\gamma}A, \rho)$. We hasten to remark that $\bar{\gamma}A$ is of course not preferred over $A^\vee$. Indeed we have

**Lemma 5.** An algebra $A = (A, m, \eta)$ in a rigid monoidal category $C$ is isomorphic to $(\bar{\gamma}A, \rho)$ as a left $A$-module iff $(A, m)$ is isomorphic to $A^\vee = (A^\vee, \eta)$ as a right $A$-module.

**Proof.** That $\Phi_\rho \in \text{Hom}(A, \bar{\gamma}A)$ is a morphism of left $A$-modules means that $\Phi_\rho \circ m = \rho \circ (id_A \otimes \Phi_\rho)$, which in turn is equivalent to the equality

$$\Phi_\rho \circ m = \rho \circ (id_A \otimes \Phi_\rho)$$

between morphisms in $\text{Hom}(A \otimes A \otimes A, 1)$. Now given $\Phi_\rho$, consider $\Phi_\rho \wedge := \Phi_\rho \wedge \in \text{Hom}(A, A^\vee)$ defined according to the prescription (15). $\Phi_\rho \wedge$ is invertible if and only if $\Phi_\rho$ is. And the equality

$$\Phi_\rho \wedge = \Phi_\rho$$

is equivalent to $\Phi_\rho \wedge$ being a morphism of right $A$-modules.

Now notice that expressing $\Phi_\rho \wedge$ in terms of $\Phi_\rho$ and invoking the defining properties of the evaluation and coevaluation morphisms, one sees that the left hand side of (21) is equal to the right hand side of (20), while the right hand side of (21) is equal to the left hand side of (20). Thus validity of (21) is equivalent to validity of (20).

**Remark 6.** In the considerations above, the existence of left and right dual objects and of the corresponding (co)evaluation morphisms is only needed for the particular object $A$ under study. Still we prefer to assume the stronger condition that the category $C$ is rigid, since the presence of such structures for $A$ cannot be expected unless they form a part of suitable left- and right-duality endofunctors of $C$.

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4 In contrast, there is no natural right $A$-module structure on $\bar{\gamma}A$ unless the double dual $\bar{\gamma} \bar{\gamma}A$ is equal to $A$, and similarly for $A^\vee$. 
Remark 7. The fact that $\Phi_{\rho}^{-1} \in \text{Hom}(\mathcal{A}, \mathcal{A})$ is a morphism of left $\mathcal{A}$-modules is equivalent to

$$
\Phi_{\rho}^{-1} = \Phi_{\rho}^{-1} \circ \tilde{b}_{A} \circ \tilde{b}_{A} \in \text{Hom}(\mathcal{A}, \mathcal{A})
$$

(22)

As a consequence, the morphism

$$
e := (\Phi_{\kappa_{\Delta}}^{-1} \otimes id_{A}) \circ \tilde{b}_{A} \in \text{Hom}(1, \mathcal{A} \otimes \mathcal{A})
$$

(23)

is invariant in the sense that $(m \otimes id_{A}) \circ (id_{A} \otimes e) = (id_{A} \otimes m) \circ (e \otimes id_{A})$. If $A$ is such that $e$ in addition satisfies $m \circ e = \eta$, then $e$ is an idempotent with respect to the convolution product $\heartsuit$ on $\text{Hom}(1, \mathcal{A} \otimes \mathcal{A})$ that is given by

$$
\heartsuit(g, h) := g \otimes h
$$

(24)

for $f, g \in \text{Hom}(1, \mathcal{A} \otimes \mathcal{A})$. In fact, $e$ is then a separability idempotent for the algebra $A$.

3 Equivalence of the three notions of Frobenius algebra

In this section we establish that in rigid monoidal categories the three Definitions 1, 3 and 4 actually describe one and the same concept.

Proposition 8. In a rigid monoidal category $\mathcal{C}$ the notions of $(\Delta, \varepsilon)$-Frobenius structure and of $\kappa$-Frobenius structure on an algebra $(A, m, \eta)$ are equivalent.

More concretely:

(i) If $(A, m, \eta, \Delta, \varepsilon)$ is an algebra with $(\Delta, \varepsilon)$-Frobenius structure, then $(A, m, \eta, \kappa_{\varepsilon})$ with

$$
\kappa_{\varepsilon} := \varepsilon \circ m
$$

(25)

is an algebra with $\kappa$-Frobenius structure.

(ii) If $(A, m, \eta, \kappa)$ is an algebra with $\kappa$-Frobenius structure, then $(A, m, \eta, \Delta_{\kappa}, \varepsilon_{\kappa})$ with

$$
\Delta_{\kappa} := (id_{A} \otimes m) \circ (id_{A} \otimes \Phi_{\kappa_{\Delta}}^{-1} \otimes id_{A}) \circ (b_{A} \otimes id_{A}) \quad \text{and} \quad \varepsilon_{\kappa} := \kappa \circ (id_{A} \otimes \eta)
$$

(26)

is an algebra with $(\Delta, \varepsilon)$-Frobenius structure.

Proof. (i) Let $(A, m, \eta, \Delta, \varepsilon)$ be an algebra with $(\Delta, \varepsilon)$-Frobenius structure in the category $\mathcal{C}$ and define the pairing $\kappa_{\varepsilon} \in \text{Hom}(A \otimes A, 1)$ by (25). Then the morphisms

$$
\Phi_{\kappa_{\varepsilon, 1}} = (id_{\mathcal{A}} \otimes \kappa_{\varepsilon}) \circ \tilde{b}_{A} \otimes id_{A}) \quad \text{and} \quad \Phi_{\kappa_{\varepsilon, 3}} = (\kappa_{\varepsilon} \otimes id_{\mathcal{A}}) \circ (id_{A} \otimes b_{A})
$$

(27)

are isomorphisms, with inverses given by the morphisms $(id_{\mathcal{A}} \otimes \tilde{b}_{A}) \circ ((\Delta \circ \eta) \otimes id_{A}) \in \text{Hom}(\mathcal{A}, \mathcal{A})$ and $(id_{\mathcal{A}} \otimes id_{A}) \circ (id_{\mathcal{A}} \otimes (\Delta \circ \eta)) \in \text{Hom}(\mathcal{A}, \mathcal{A})$, respectively. Thus $\kappa_{\varepsilon}$ is non-degenerate. That $\kappa_{\varepsilon}$ is invariant is an immediate consequence of the associativity of the product $m$.

Thus $(A, m, \eta, \kappa_{\varepsilon})$ is an algebra with $\kappa$-Frobenius structure.

(ii) Let $(A, m, \eta, \kappa)$ be an algebra with $\kappa$-Frobenius structure in $\mathcal{C}$ and define $\Delta_{\kappa}$ and $\varepsilon_{\kappa}$ by (26).
To see that $\Delta_\kappa$ is a coassociative coproduct, first notice that with the help of invariance of $\kappa$, the product $m$ can be rewritten as

$$m = \Phi^{-1}_{\kappa,r} \circ \Phi_{\kappa,r} \circ m = (-1)$$

Together with (18) this implies that two alternative descriptions of $\Delta_\kappa$ are

$$\Delta_\kappa \equiv (-1)$$

Using the first two descriptions we can write

$$\Delta_\kappa \circ m = (\Phi^{-1}_{\kappa,r} \circ \Phi_{\kappa,r} \circ m) \circ (\Phi^{-1}_{\kappa,l} \circ \Phi_{\kappa,l})$$

Thus coassociativity of $\Delta_\kappa$ follows immediately from associativity of $m$.

Next we verify that $\varepsilon_\kappa$ is a counit for $\Delta_\kappa$. First note that, by invariance of $\kappa$, an alternative description of $\varepsilon_\kappa$ is $\varepsilon_\kappa = \kappa \circ (\eta \otimes id_A)$. Thus together with (14) one obtains

$$\varepsilon_\kappa = d_A \circ (id_A \otimes (\Phi_{\kappa,1} \otimes \eta)) = d_A \circ ((\Phi_{\kappa,r} \otimes \eta) \otimes id_A).$$

Combining the first of these expressions for $\varepsilon_\kappa$ with the third description of $\Delta_\kappa$ in (29) one immediately arrives at $(id_A \otimes \varepsilon_\kappa) \circ \Delta_\kappa = id_A$, while the second expression together with the first in (29) yields $(\varepsilon_\kappa \otimes id_A) \circ \Delta_\kappa = id_A$.

To derive the equalities (6) is now easy. Using the first of the descriptions (29) for $\Delta_\kappa$ and associativity, one sees that

$$\Delta_\kappa \circ m = (id_A \otimes m) \circ (\Phi_{\kappa,r} \circ \Phi_{\kappa,l}) \circ \Delta_\kappa$$

while

$$\Delta_\kappa \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta_\kappa)$$

follows by using instead the third of those descriptions and again associativity.

With the informations gathered so far at hand, it is straightforward to verify also

**Proposition 9.** In a rigid monoidal category $\mathcal{C}$ the notions of $\kappa$-Frobenius structure and of $\Phi_{\rho}$-Frobenius structure on an algebra $(A,m,\eta)$ are equivalent.

More specifically, for any algebra $A$ in $\mathcal{C}$ the following holds:

(i) There exists a non-degenerate pairing on $A$ iff $A$ is isomorphic to $\mathcal{V}A$ as an object of $\mathcal{C}$.

(ii) There exists an invariant pairing on $A$ iff there exists a morphism from $A$ to $\mathcal{V}A$ that is a morphism of left $A$-modules.
Proof. Given a morphism \( \varphi \in \Hom(A, \mathcal{V}A) \), we can define a pairing \( \kappa_\varphi \) on \( A \) by

\[
\kappa_\varphi := \tilde{d}_A \circ (id_A \otimes \varphi). \tag{34}
\]

Conversely, given a pairing \( \kappa \) on \( A \) we can define a morphism \( \psi \in \Hom(A, \mathcal{V}A) \) by \( \psi := \Phi_{\kappa,1} \) as in (12). Obviously the two operations are inverse to each other, in the sense that

\[
\Phi_{\kappa,1} \equiv \varphi. \tag{35}
\]

(i) Thus, given an isomorphism \( \varphi \in \Hom(A, \mathcal{V}A) \), the morphism \( \Phi_{\kappa,1} \) associated to the pairing \( \kappa_\varphi \) is invertible, and hence \( \kappa_\varphi \) is non-degenerate. Conversely, given a non-degenerate pairing \( \kappa \) on \( A \), the morphism \( \varphi := \Phi_{\kappa,1} \in \Hom(A, \mathcal{V}A) \) is an isomorphism.

(ii) If a pairing on \( A \) is of the form \( \kappa = \kappa_{\varphi} \), then the statement that it is invariant is precisely the equality (20). Hence if \( \varphi \equiv \varphi_\rho \) is a left module morphism, then \( \kappa_\varphi \) is invariant. Conversely, given an invariant pairing \( \kappa \) on \( A \), the morphism \( \psi := \Phi_{\kappa,1} \) satisfies (20), and hence it is a morphism of left \( A \)-modules.

Together, the validity of (i) and (ii) implies that if \( (A, m, \eta, \Phi) \) is an algebra with \( \Phi_p \)-Frobenius structure, then the pairing \( \kappa_\Phi \) is a \( \kappa \)-Frobenius structure on \( A \). And conversely, if \( (A, m, \eta, \kappa) \) is an algebra with \( \kappa \)-Frobenius structure, then \( \psi = \Phi_{\kappa,1} \) is invertible, because \( \kappa \) is non-degenerate, and hence it is a \( \Phi_p \)-Frobenius structure on \( A \).

We are now in a position to state

**Definition 10.** A *Frobenius algebra* in a rigid monoidal category \( C \) is an algebra in \( C \) for which the following three equivalent conditions are satisfied:

(i) There exists a \( (\Delta, \varepsilon) \)-Frobenius structure on \( A \).

(ii) There exists a \( \kappa \)-Frobenius structure on \( A \).

(iii) There exists a \( \Phi_p \)-Frobenius structure on \( A \).

### 4 Symmetric Frobenius algebras

The classical notion of symmetric Frobenius algebra can be formulated in the categorical setting if the monoidal category \( C \) is sovereign. This means that \( C \) is rigid and that the left- and right-duality endofunctors are equal: That is, one has \( \mathcal{V}U = U^\vee \) for all objects \( U \), as well as \( \mathcal{V}f = f^\vee \), i.e.

\[
(id_{\mathcal{V}U} \otimes \tilde{d}_V) \circ (id_{\mathcal{V}U} \otimes f \otimes id_{\mathcal{V}V}) \circ (b_U \otimes id_{\mathcal{V}V})
\]

\[
= (d_V \otimes id_{\mathcal{V}V}) \circ (id_{\mathcal{V}V} \otimes f \otimes id_U) \circ (id_{\mathcal{V}V} \otimes b_U) \in \Hom(\mathcal{V}V, U^\vee), \tag{36}
\]

for all objects \( U, V \) and all morphisms \( f \in \Hom(U, V) \).

**Remark 11.** As one particular aspect of sovereignty, we note that when applied to the left and right (co)evaluations, the equality (36) amounts to

\[
\mathcal{V}^\vee = \mathcal{V} \quad \text{and} \quad U^\vee = U \tag{37}
\]

These equalities may be written as \( g^\vee = \mathcal{V}g \) and as \( (\mathcal{V}g)^\vee = \mathcal{V}(g^\vee) \) for \( g = id_{\mathcal{V}U} \equiv id_{U^\vee} \), respectively, where for \( f \in \Hom(U, \mathcal{V}V) \) the morphism \( \mathcal{V}f \) is the one obtained from \( f \) according to the prescription (15), while the morphism \( \mathcal{V}f \in \Hom(V, \mathcal{V}U) \) is defined by

\[
\mathcal{V}f := (id_{\mathcal{V}U} \otimes d_V) \circ (id_{\mathcal{V}U} \otimes f \otimes id_V) \circ (b_U \otimes id_V) \tag{38}
\]

for any \( f \in \Hom(U, V^\vee) \).

With the help of (37) it is easy to check that, for \( \mathcal{V}V \) equal to \( V^\vee \), the sovereignty relation (36) is equivalent to having

\[
\mathcal{V}f = f^\vee \tag{39}
\]

for all \( f \in \Hom(U, V^\vee) \). Also note that the operations \( ?^\vee \) and \( ?^\wedge \) are defined only on morphisms, but not on objects, and that \( ?^\vee ?^\wedge \) is the identity mapping.
The formulation (39) of the sovereignty relation is often quite convenient. As an illustration, it allows one to quickly obtain the following ‘opposite’ version of the second identity in (18):

\[
\Phi_{\nu, l} \Phi_{\nu, 1}^{-1} = \Phi_{\nu, r} \Phi_{\nu, 1}^{-1} = A \quad (40)
\]

Remark 12. As a matter of fact, for introducing the notion of symmetric Frobenius algebra it is already sufficient that \( A \) and \( A^\vee \) are equal as objects of \( C \). Similarly, all the results below remain true if one has \( A = A^\vee \) as well as \( \forall f = f^\vee \) for just a few particular morphisms \( f \), like for the product \( m \) of \( A \) and the morphisms \( \Phi_{\kappa, l}, \nu \in \text{Hom}(A, A^\vee) \) defined in (12). However, the only situation known to us in which these weaker conditions are satisfied naturally is that \( C \) is indeed sovereign. (Compare the analogous comments on rigidity in Remark 6.)

We start with the notion of symmetric Frobenius algebra that is used e.g. in [FRS1, Mi].

Definition 13. A symmetric \((\Delta, \varepsilon)\)-Frobenius structure on an algebra \( A = (A, m, \eta) \) in a sovereign monoidal category \( C \) is a \((\Delta, \varepsilon)\)-Frobenius structure \((\Delta, \varepsilon)\) for which the endomorphism

\[
\Phi_{\nu, l} := (d_A \otimes id_A) \circ (\Delta \circ \eta \circ \varepsilon \circ m) \circ (\tilde{b}_A \otimes id_A)
\]

of \( A \) equals \( id_A \).

Note that by the Frobenius relations (6) the equality \( \Phi_{\nu, l} = id_A \) is equivalent to the equality \( \Phi_{\kappa, l} = \Phi_{\kappa, r, l} \) between the two morphisms from \( A \) to \( A \otimes A \) that we introduced in (27) and which, owing to the Frobenius relations, are isomorphisms, and are related to the morphism (41) by \( \Phi_{\nu, l} = \Phi_{\kappa, l}^{-1} \circ \Phi_{\kappa, l} \). Pictorially this identity reads

\[
\Phi_{\kappa, l} = \quad = \quad = \Phi_{\kappa, r, l} . \quad (42)
\]

Definition 14. A symmetric \(\kappa\)-Frobenius structure on an algebra \( A = (A, m, \eta) \) in a sovereign monoidal category \( C \) is a \(\kappa\)-Frobenius structure \(\kappa \in \text{Hom}(A \otimes A, 1) \) on \( A \) that is symmetric in the sense that the equality

\[
d_A \circ (id_A \otimes \kappa \otimes id_A) \circ (\tilde{b}_A \otimes id_A \otimes id_A) = \kappa
\]

holds.

Note that on the left hand side of (43), the evaluation \( d_A \in \text{Hom}(A^\vee \otimes A, 1) \) is composed with a morphism in \( \text{Hom}(A \otimes A, A^\vee \otimes A) \), which in the present context is the reason why we need \( A = A^\vee \). Pictorially, (43) reads

\[
\quad = \quad (44)
\]

By the defining properties of the (co)evaluation, this relation is equivalent to

\[
A = A \quad (45)
\]
Next note that when \( A \) has a \( \Phi_\rho \)-Frobenius structure and the equality \( \check{\check{A}} = A^\vee \) between left and right dual objects holds, then \( A \) is isomorphic to \( \check{\check{A}} = A^\vee \) both as a left and as a right module. Accordingly the following definition is natural.

**Definition 15.** A symmetric \( \Phi_\rho \)-Frobenius structure on an algebra \( A = (A, m, \eta) \) in a sovereign monoidal category \( C \) is an isomorphism \( \Phi_\rho \) from \( A \) to \( \check{\check{A}} = A^\vee \) that is both a morphism of left \( A \)-modules and a morphism of right \( A \)-modules (and thus a morphism of \( A \)-bimodules).

In terms of the notations used in the proof of Lemma 5, that a \( \Phi_\rho \)-Frobenius structure is symmetric means that we have \( \Phi_\rho = \Phi_\rho \), or equivalently, that the isomorphism \( \Phi_\rho \) satisfies

\[
\Phi_\rho \circ \phi = \phi \circ \Phi_\rho \quad \text{(46)}
\]

**Proposition 16.** In a sovereign monoidal category \( C \) the notions of symmetric \((\Delta, \varepsilon)\)-Frobenius structure, symmetric \( \kappa \)-Frobenius structure and symmetric \( \Phi_\rho \)-Frobenius structure are equivalent.

**Proof.** Recalling the relations between the characteristic morphisms of \((\Delta, \varepsilon)\)-Frobenius structures, \( \kappa \)-Frobenius structures and \( \Phi_\rho \)-Frobenius structures the assertion is close to a tautology. Let us nonetheless write out the proof.

(i) Given a symmetric \((\Delta, \varepsilon)\)-Frobenius structure, define the pairing \( \kappa_\varepsilon \) as in (25). For \( \kappa = \kappa_\varepsilon \), the symmetry property \((44)\) is satisfied because it is nothing but (up to composition with appropriate (co)evaluations) the equality \((42)\). Conversely, given a symmetric \( \kappa \)-Frobenius structure, define the counit \( \varepsilon_\kappa \) as in \((26)\). Then the symmetry property \((42)\) is satisfied because owing to \( \varepsilon_\kappa \circ m = \kappa \) it is nothing but (up to composition with appropriate (co)evaluations) the equality \((44)\).

Thus the notions of symmetric \((\Delta, \varepsilon)\)-Frobenius structure and of symmetric \( \kappa \)-Frobenius structure are equivalent.

(ii) Given a symmetric \( \Phi_\rho \)-Frobenius structure, define the pairing \( \kappa_{\Phi_\rho} \) as in (34) with \( \phi = \Phi_\rho \). For \( \kappa = \kappa_{\Phi_\rho} \), the symmetry property \((44)\) is satisfied because after use of the defining property of \( \check{\check{b}}_A \) and \( \check{\check{d}}_A \) it is nothing but the equality \((46)\). Conversely, given a symmetric \( \kappa \)-Frobenius structure, define the morphism \( \Phi_\rho := \Phi_{\kappa,1} \) as in \((12)\). Then the symmetry property \((46)\) is satisfied because, again after use of the duality axioms, it is nothing but the equality \((44)\).

Thus the notions of symmetric \( \Phi_\rho \)-Frobenius structure and of symmetric \( \kappa \)-Frobenius structure are equivalent.

In analogy with Definition 10 we now give

**Definition 17.** A symmetric Frobenius algebra in a sovereign monoidal category \( C \) is an algebra in \( C \) for which the following three equivalent conditions are satisfied:

(i) There exists a symmetric \((\Delta, \varepsilon)\)-Frobenius structure on \( A \).

(ii) There exists a symmetric \( \kappa \)-Frobenius structure on \( A \).

(iii) There exists a symmetric \( \Phi_\rho \)-Frobenius structure on \( A \).

It is worth pointing out that none of these structures requires \( C \) to be braided. If \( C \) does have a braiding, then it can of course be used to reformulate the property of \( \varepsilon, \kappa \) and \( \Phi_\rho \) of being symmetric in a manner that resembles more closely the customary description in the vector space case.

5 Nakayama automorphisms

Given an algebra \( A \) in a sovereign monoidal category \( C \) and a pairing \( \kappa \) on \( A \), we call an endomorphism \( \check{\Omega} \equiv \check{\Omega}_\kappa \) a Nakayama morphism iff \( d_A \circ (id_A \otimes \kappa \otimes id_A) \circ (b_A \otimes id_A \otimes id_A) = \kappa \circ (\check{\check{\Omega}} \otimes id_A) \) (see e.g. [Lam, §16E] for the classical case). Pictorially, the defining relation for \( \check{\Omega} \) reads

\[
\check{\Omega} \circ \phi = \phi \circ \check{\Omega} \quad \text{(47)}
\]
Defining $\Phi_{\kappa,1} \in \text{Hom}(A, {^\vee}A)$ and $\Phi_{\kappa,r} \in \text{Hom}(A, A^r)$ as in (12) and (13), one can rewrite this relation as $\Phi_{\kappa,r} \circ \mathcal{U} = \Phi_{\kappa,1}$. Now if $A$ is a Frobenius algebra with $\kappa$-Frobenius structure $\kappa$, then $\Phi_{\kappa,r}$ is invertible, so that

$$\mathcal{U} = \Phi_{\kappa,r}^{-1} \circ \Phi_{\kappa,1},$$

and in particular $\mathcal{U}$ is an automorphism of $A$ as an object of $C$. (Note that the right hand side of (48) may also be written as $\Phi_{\kappa,r}^{-1} \circ \lbrack (\Phi_{\kappa,r}) \rbrack$; $\mathcal{U}$ belongs therefore to the class of automorphisms $\mathcal{V}_A$ that are used in [FS] for assigning a Frobenius-Schur indicator to $A$.)

**Proposition 18.** Any Nakayama automorphism $\mathcal{U}$ of a Frobenius algebra $A$ in a sovereign monoidal category is a unital algebra morphism.

**Proof.** Any morphism $\omega$ that is an automorphism of $A$ as an associative algebra is automatically also unital, i.e. satisfies $\omega \circ \eta = \eta$.

It is therefore sufficient to show that $\mathcal{U}$ is compatible with the product of $A$. We demonstrate this property by showing that $\mathcal{U}^{-1} \circ m \circ (\mathcal{U} \otimes \text{id}_A) = m \circ (\text{id}_A \otimes \mathcal{U}^{-1})$. Consider the following chain of equalities:

$$\mathcal{U}^{-1} \circ m \circ (\mathcal{U} \otimes \text{id}_A) = \Phi_{\kappa,1}$$

The first of these follows by combining (48) with the fact that $\Phi_{\kappa,r}$ is a morphism of right $A$-modules and using the explicit form (19) of the right action of $A$ on $A^r$; the second equality is a consequence of $\Phi_{\kappa,r} = (\Phi_{\kappa,1})^{\text{r}}$; and the third expresses again the fact that $\Phi_{\kappa,r}$ is a morphism of right $A$-modules.

Next we apply the sovereignty relation (39) to the product that is contained in the morphism on the right hand side of (49), thereby obtaining the left hand side of the following equality:

$$\mathcal{U}^{-1} \circ m \circ (\mathcal{U} \otimes \text{id}_A) = \Phi_{\kappa,1}^{-1} \circ \rho \circ (\text{id}_A \otimes \Phi_{\kappa,r}).$$

Here $\rho$ the left action of $A$ on $A^r$ as in (19). Owing to the left-module morphism property of $\Phi_{\kappa,1}^{-1}$, the right hand side of (50) is, in turn, equal to $m \circ (\text{id}_A \otimes (\Phi_{\kappa,1}^{-1} \circ \Phi_{\kappa,r})) = m \circ (\text{id}_A \otimes \mathcal{U}^{-1})$. \[\Box\]

According to Definition (17) a Frobenius algebra $A$ is symmetric iff for some choice of $\kappa$ the morphism $\mathcal{U}_\kappa$ is the identity morphism. As we will see, this also means that $A$ is symmetric iff every Nakayama automorphism of $A$ is an inner automorphism. To derive this characterization, we need to introduce some further terminology. First notice that the set $A_\circ := \text{Hom}(1, A)$ has an associative product $m_\circ: A_\circ \times A_\circ \to A_\circ$ given by convolution, $m_\circ(a,b) := m \circ (a \otimes b)$; the unit $\eta$ of $A$ acts as an identity element, $m_\circ(a, \eta) = a = m_\circ(\eta, a)$. Owing to the associativity of $m$ the algebra $A$ carries the structure of a bimodule over $A_\circ$, with the left and right action of $a \in A_\circ$ given by

$$\ell_a := m \circ (a \otimes \text{id}_A) \quad \text{and} \quad r_a := m \circ (\text{id}_A \otimes a),$$

respectively. Note that the assignments $\ell, r: A_\circ \to \text{End}(A)$ mapping $a \in A_\circ$ to $\ell_a$ and $r_a$, respectively, are the analogues of the left and right regular representation of an ordinary algebra in a category of vector spaces – for
every $a \in A_\circ$ the action $\ell_a$ furnishes an endomorphism of $A$ as a right $A$-module, while $r_a$ furnishes an endomorphism of $A$ as a left $A$-module.

Of particular interest is the subset of $A_\circ$ consisting of morphisms that are invertible with respect to the product $m_\circ$. These form a group, the group of units of $A$, which we will denote by $A^\times_\circ$. For $g \in A^\times_\circ$, $\ell_g$ and $r_g$ are automorphisms of $A$ (as an object, and also as a right and left $A$-module, respectively), with inverses $\ell_{g^{-1}}$ and $r_{g^{-1}}$, respectively. Moreover, the composition

$$ad_g := \ell_{g^{-1}} \circ r_g = r_g \circ \ell_{g^{-1}}$$

with $g \in A^\times_\circ$ is an automorphism of $A$ as an algebra. An automorphism of $A$ is called an inner automorphism iff it is of this particular form; clearly, the inner automorphisms form a group under composition, with $ad_g \circ ad_h = ad_{m_\circ(h,g)}$ and $ad_g^{-1} = ad_{g^{-1}}$.

The module morphism properties of $r_a$ and $\ell_a$ imply that if $\kappa$ is an invariant pairing on $A$, then so are $\kappa \circ (id_A \otimes r_a)$ and $\kappa \circ (\ell_a \otimes id_A)$ for any $a \in A_\circ$. Moreover, for $g \in A^\times_\circ$ the composition of any automorphism of $A$ with $r_g$ or $\ell_g$ is again an automorphism; thus if $\kappa$ is a non-degenerate pairing, then so are $\kappa \circ (id_A \otimes r_g)$ and $\kappa \circ (\ell_g \otimes id_A)$. Conversely, we have

**Lemma 19.** Any two invariant non-degenerate pairings $\kappa$ and $\kappa'$ on an algebra $A$ in a rigid monoidal category differ by composition with an endomorphism of the form $id_A \otimes r_g$ for some $g \in A^\times_\circ$.

**Proof.** If the pairings $\kappa$ and $\kappa'$ are non-degenerate, then the corresponding morphisms $\Phi_{\kappa,1}$ and $\Phi_{\kappa',1}$ in $\text{Hom}(A, \nabla A)$ are isomorphisms, and hence $\Phi_{\kappa',1} = \Phi_{\kappa,1} \circ \sigma_1$ for some automorphism $\sigma_1$ of $A$. Equivalently, $\kappa$ and $\kappa'$ are related by

$$\kappa' = \kappa \circ (id_A \otimes \sigma_1).$$

(53)

$\sigma_1$ is even an automorphism of $A$ as a left $A$-module, because $\Phi_{\kappa,1}$ and $\Phi_{\kappa',1}$ are. Rewriting $\sigma_1$ identically as $\sigma_1 \circ m \circ (id_A \otimes \eta)$ it follows in particular that

$$\sigma_1 = m \circ [id_A \otimes (\sigma_1 \otimes \eta)] = r_{\sigma_1 \otimes \eta}.$$

(54)

Thus indeed $\kappa' = \kappa \circ (id_A \otimes r_g)$, with $g = \sigma_1 \otimes \eta$.

Finally, $g$ is invertible, with inverse $g^{-1} = \sigma_1^{-1} \circ \eta$.

Note that only ‘left-nondegeneracy’ of the pairings $\kappa$ and $\kappa'$ (and only left-rigidity of $C$) enters the proof. An analogous argument based on right-nondegeneracy (and right-rigidity) shows that the two invariant non-degenerate pairings are also related by $\kappa' = \kappa \circ (\sigma_r \otimes id_A)$ with $\sigma_r = r_{\sigma_r \otimes \eta}$ for some invertible right $A$-module morphism $\sigma_r \in \text{End}(A)$.

As a consequence of Lemma 19 we have

**Proposition 20.** Any two Nakayama automorphisms of a Frobenius algebra $A$ in a sovereign monoidal category $C$ differ by composition with an inner automorphism of $A$.

**Proof.** Recall from the proof of the lemma that

$$\Phi_{\kappa,1} = \Phi_{\kappa,1} \circ r_g$$

(55)

for some $g \in A^\times_\circ$. Together with (16) it then follows that $\Phi_{\kappa',1} = (\Phi_{\kappa,1})^\vee = r_{g\vee} \circ (\Phi_{\kappa,1})^\vee$ and thus $\Phi_{\kappa',1}^{-1} = \Phi_{\kappa,1}^{-1} \circ r_{g\vee^{-1}}$. Now by using sovereignty together with the second identity in (16) one can rewrite the latter equality as

$$\Phi_{\kappa',1}^{-1} = \Phi_{\kappa,1}^{-1} \circ r_{g^{-1}} =$$

(56)

\[
\begin{array}{c}
\xymatrix{A & A^\vee \\
\Phi_{\kappa,1}^{-1} & \Phi_{\kappa',1}^{-1} \\
\Phi_{\kappa,1} & \Phi_{\kappa',1} \\
\Phi_{\kappa,1}^{-1} & \Phi_{\kappa',1}^{-1} \\
A \\
A^\vee}
\end{array}
\]
Furthermore, since $\Phi_{\kappa,r}$ is a morphism of right $A$-modules, for any $h \in A$ one has
\[ \Phi_{\kappa,r} \circ \eta = (d_A \otimes id_A) \circ (\Phi_{\kappa,r} \otimes \ell_h \otimes id_A) \circ (id_A \otimes b_A), \quad (57) \]
and as a consequence we can rewrite (56) as $\Phi_{\kappa,r}^{-1} \circ \Phi_{\kappa,1} \circ (id_A \otimes \tau_h) \circ \Phi_{\kappa,1}^{-1}$. When combined with (48) and (55), we therefore conclude that the Nakayama automorphisms $\mathcal{U} \equiv \mathcal{U}_{\kappa}$ and $\mathcal{U}' \equiv \mathcal{U}_{\kappa'}$ are related by
\[ \mathcal{U}' = \Phi_{\kappa,r}^{-1} \circ \Phi_{\kappa,1}^{-1} \circ \Phi_{\kappa,1} \circ r_g^{-1} \circ \Phi_{\kappa,1} \circ r_g = \Phi_{\kappa,r}^{-1} \circ \Phi_{\kappa,1} \circ r_g^{-1} \circ r_g = \mathcal{U} \circ \text{ad}_g, \quad (58) \]
thus proving the claim.

It follows in particular that when selecting a $(\Delta, \varepsilon)$-Frobenius structure $\varepsilon$ on $A$ we can write any Nakayama automorphism $\mathcal{U}$ of $A$ in the form $\mathcal{U} = \mathcal{U}_{\kappa} \circ \text{ad}_g$ with some $g \in A^\times$. Thus combining the statements that $\mathcal{U}_{\kappa}$ is an algebra morphism (see the proposition in Section 4 of [Fu]) and that any inner automorphism of $A$ is an algebra morphism as well (see above) provides an alternative derivation of Proposition 18. Also note that by combining (58) with the results about the morphisms $g = \sigma_r \circ \eta$ and $h = \sigma_l \circ \eta$ obtained above one finds that $\mathcal{U}_{\kappa} \circ \ell_g = \ell_h \circ \mathcal{U}_{\kappa}$ which, in turn, by the algebra morphism property of $\mathcal{U}_{\kappa}$ implies that $h = \mathcal{U}_{\kappa} \circ g$.

Further, if there exists a $\kappa$-Frobenius structure $\kappa$ on $A$ such that the associated Nakayama automorphism $\mathcal{U}_{\kappa}$ is inner, say $\mathcal{U}_{\kappa} = \text{ad}_h$, then one has $\mathcal{U}_{\kappa'} = \text{id}_A$ for the $\kappa$-Frobenius structure $\kappa' := \kappa \circ (\ell_h^{-1} \otimes \text{id}_A)$ on $A$, and so $A$ is symmetric. Thus indeed a Frobenius algebra $A$ is symmetric iff every Nakayama automorphism of $A$ is inner.

To conclude, we combine Lemma 19, and the remarks preceding it, with the formulas (12) and (26) to arrive at

\begin{corollary}
Let $A = (A, m, \eta)$ be an algebra in a rigid monoidal category.
(i) Let $\varepsilon, \kappa, \Phi_\rho$ be $\varepsilon$-, $\kappa$- and $\Phi_\rho$-Frobenius structures on $A$, respectively, and let $g, h \in A^\times$. Then also $\varepsilon \circ r_g \circ \ell_h$, $\kappa \circ (\ell_g \otimes \eta_h)$ and $\Phi_\rho \circ r_g$ are $\varepsilon$-, $\kappa$- and $\Phi_\rho$-Frobenius structures on $A$, respectively.
(ii) Any two triples $(\varepsilon, \kappa, \Phi_\rho)$ and $(\varepsilon', \kappa', \Phi_\rho')$ of $\varepsilon$-, $\kappa$- and $\Phi_\rho$-Frobenius structures on $A$ are related as in (i) for some choice of $g, h \in A^\times$.
\end{corollary}

Acknowledgments:
We thank A.A. Davydov, V. Hinich, C. Schweigert and A. Stolin for helpful discussions, and L. Kadison and V. Ostrik for a correspondence. JF is partially supported by VR under project no. 621-2006-3343.
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