On Dubrovin Topological Field Theories

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I show that the new topological field theories recently associated by Dubrovin with each Coxeter group may be all obtained in a simple way by a “restriction” of the standard $ADE$ solutions. I then study the Chebichev specializations of these topological algebras, examine how the Coxeter graphs and matrices reappear in the dual algebra and mention the intriguing connection with the operator product algebra of conformal field theories. A direct understanding of the occurrence of Coxeter groups in that context is highly desirable.

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1. Restriction

In a topological field theory (TFT) the genus zero 3-point correlation functions $C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle$ are sufficient to reconstruct all the others. They are functions of deformation parameters $t_l$ and in this article, we shall consider TFT’s with a finite number of fields and parameters, $i, j, k, l \in \{0, 1, \cdots, n\}$. Then the $C$’s must satisfy a set of conditions [1–2] (the “Witten-Dijkgraaf-Verlinde-Verlinde” equations):

\begin{itemize}
  \item $\exists F(t.)$ such that $C_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}$ \hfill (1.1a)
  \item $\eta_{ij} = C_{0ij}$ is independent of the $t$’s and invertible : $\eta_{ij} \eta^{jk} = \delta_i^k$ \hfill (1.1b)
  \item $C_{ij}^k = \eta^{kl}C_{ijl}$ are the structure constants of an associative algebra $A$ i.e.
    \begin{equation}
    C_{ij}^k C_{kl}^m = C_{il}^k C_{kj}^m \hfill (1.1c)
    \end{equation}
  \item $F(t.)$ is a quasi-homogeneous function of the $t$’s. \hfill (1.1d)
\end{itemize}

The function $F(t_0, t_1, \cdots, t_n)$ is the free energy of the theory. It encodes all the information about the $C$’s.

Dubrovin [3] has reinterpreted these conditions in a geometric, coordinate invariant way, as defining a Frobenius manifold. More recently [4], he has shown how to associate a Frobenius manifold, hence a solution to (1.1), with each finite Coxeter group. Coxeter groups are linear groups generated by reflections in a real Euclidean space $V$. The finite Coxeter groups are classified [5–6] : in addition to the Weyl groups of the simple Lie algebras, $A_p$, $B_p$, $C_p$, (the two latter Coxeter groups being identical), $D_p$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, there are the groups $H_3$ and $H_4$ of reflections of the regular icosahedron and of a regular 4-dimensional polytope, and the infinite series $I_2(k)$ of the symmetry groups of the regular $k$-gons in the plane [3 4]. In Dubrovin’s work, the homogeneity degrees of the variables $t_i$ and of $F$ are respectively $1 – (d_i – 2)/h$ and $2 + 2/h$ where $h$ is the Coxeter number of $G$ and $d_i$ are the degrees of the $G$ invariant polynomials in the coordinates of $V$ ($d_0 = 2, \ldots, d_n = h$). Moreover, Dubrovin has proved a unicity theorem asserting that his solutions are the only ones with these assignments of degrees, and he conjectures that they are the only solutions with $F$ polynomial and homogeneity degrees satisfying

\begin{equation}
0 < \text{degree } (F) – 2 \leq \text{degree } (t.) \leq 1 . \hfill (1.2)
\end{equation}
The solutions of type $ADE$ that Dubrovin finds reproduce what is already well known as the minimal TFT’s, obtained by twisting and perturbing the minimal $N = 2$ superconformal theories (for explicit expressions of their free energies, see [2][7] and further references therein).

The purpose of this note is to show that the other solutions may be obtained from the latter by a “restriction”, thus obtaining in a simple and explicit way their free energy $F$, and to point to some properties of these additional solutions and connections with other problems. This restriction amounts to setting some of the $t$ parameters of an $ADE$ solution to zero, $t_i = 0$ if $i \notin I$, and to concentrating on the $C_{ijk}$, $i,j,k \in I \subset \{0, \cdots, n\}$. To prove that such a restriction still yields a solution to (1.1), the only thing to check is that the restricted $C$’s form a subalgebra, namely

$$C_{ij}^k(t)|_{t_l = 0, l \notin I} = 0 \quad \forall i, j \in I, \forall k \notin I.$$  

(1.3)

In other words, searching for restrictions amounts to looking for subalgebras of some specialization (for some $t_i = 0, i \notin I$) of the $ADE$ topological algebras.

These restrictions come in two classes. The first is associated with symmetries of some solution of $ADE$ type. Suppose that the free energy of an $ADE$ solution is left invariant by some group of transformations of its arguments $t_l \rightarrow \omega_l t_l$. It turns out that the symmetry groups are either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. The restriction consists in a projection on the invariant sector, $I = \{j | \omega_j = 1\}$. Clearly condition (1.3) is fulfilled since $C_{ij}^k(t)|_{t_l = 0, l \notin I}$ is not invariant under the action of the symmetry group. This procedure yields the $B_n \equiv C_n$, $F_4$ and $G_2$ solutions. The other class of restriction does not seem to be associated with any symmetry; this is the way the $H_3$, $H_4$ and $I_2(k)$ solutions will be obtained.

In the following, instead of labelling the $t$ parameters of the $ADE$ solutions in a consecutive way, from 0 to $n$, we index them by the degree $-2$ of the associated invariant polynomial : $d_i = i + 2$. The two labellings coincide in the $A$ cases. Through the restriction, we shall see that the $t$’s of the non $ADE$ solutions inherit a labelling consistent with this convention.

- Consider first the free energy of the $A_{2n+1}$ case. It is known to be invariant under the following $\mathbb{Z}_2$ action on the $t$ parameters : $t_k \rightarrow (-1)^k t_k$. It is thus consistent to let all $t$’s of odd index vanish. Then

$$F_{B_{n+1}}(t_0, t_2, \cdots, t_{2n}) = F_{A_{2n+1}}(t_0, 0, t_2, \cdots, 0, t_{2n}).$$  

(1.4)
In particular

\[ F_{B_2}(t_0, t_2) = \frac{t_0^2 t_2}{2} + \frac{t_2^5}{60} \]

\[ F_{B_3}(t_0, t_2) = \frac{t_0^2 t_4}{2} + \frac{t_0 t_2^2}{2} + \frac{t_2^3 t_4}{6} + \frac{t_2^2 t_4^3}{6} + \frac{t_4^7}{210}. \]

Notice that this procedure of restricting oneself to the even \( t \)'s resembles that yielding the \( D_{n+2} \) case from the \( A_{2n+1} \) one \[2, 4\]. In the latter case, however, this is accompanied by the introduction of a new parameter \( t_n \) and one is really dealing with an orbifold of \( A \) rather than a restriction.

- The \( E_6 \) free energy \( F_{E_6}(t_0, t_3, t_4, t_6, t_7, t_{10}) \) is invariant under the same \( \mathbb{Z}_2 \) action \( t_k \to (-1)^k t_k \). The \( F_4 \) solution may thus be obtained from the \( E_6 \) one:

\[ F_{F_4}(t_0, t_4, t_6, t_{10}) = F_{E_6}(t_0, 0, t_4, t_6, 0, t_{10}) \]

\[ = t_0 t_4 t_6 + \frac{t_0^2 t_6}{2} + \frac{6}{12} t_2^4 t_{10} + \frac{t_6^4 t_{10}}{60} + \frac{t_6^4 t_{10}}{252} + \frac{t_{10}^{13}}{185328}. \] (1.6)

- In a similar way, the \( D_4 \) free energy written as

\[ F_{D_4} = \frac{t_0^3 t_4}{2} + t_0 (t_3^2 - t_{2}^2) + \frac{t_0^3}{6} (t_2^2 - t_{2}^2) + t_4 (\frac{t_2^3}{6} + \frac{t_2 t_{10}^2}{2}) + \frac{t_{10}^7}{210} \] (1.7)

is invariant under the \( \mathbb{Z}_3 \) symmetry \( t_0 \to t_0, t_4 \to t_4, t_2 \pm t_2^2 \to \exp \pm 2i\pi/3 (t_2 \pm t_2^2) \). Projecting again onto the invariant sector, the \( G_2 \) solution is obtained from the \( D_4 \) one,

\[ F_{G_2}(t_0, t_4) = F_{D_4}(t_0, t_2 = 0, t_4^2 = 0). \] (1.8)

 Explicitly

\[ F_{G_2}(t_0, t_4) = \frac{t_0^3 t_4}{2} + \frac{t_{10}^7}{210}. \] (1.9)

We now come to the second class of restrictions.

- Let us consider the \( E_8 \) solution \( F(t_0, t_6, t_{10}, t_{12}, t_{16}, t_{18}, t_{22}, t_{28}) \) and let \( t_6 = t_{12} = t_{16} = t_{22} = 0 \). It is a tedious but straightforward exercise to check on the explicit expression given in \[4\] that the terms in \( \frac{\partial^2 F}{\partial t_i \partial t_j} \), \( i, j \in I = \{0, 10, 18, 28\} \) that contain a \( t_k \notin I \) are at least quadratic in the \( t_l, l \notin I \), thus proving \([1, 3]\). Only 12 of the original 140 terms of \( F_{E_8} \) survive! and we obtain in this way the \( H_4 \) solution of Dubrovin

\[ F_{H_4}(t_0, t_{10}, t_{18}, t_{28}) = t_0 t_{10} t_{18} + \frac{t_0^2 t_{28}}{2} + t_{18}^2 \frac{t_{10}^{19}}{1539000} \]

\[ + \frac{t_{10}^3}{6} \left( \frac{t_{10}^{13}}{1800} + \frac{t_{18} t_{28}}{20} + \frac{3 t_{18}^2 t_{28}}{10} \right) + \frac{t_{28}^{31}}{245764125000} \]

\[ + t_{10}^2 \left( \frac{t_{10}^{11}}{4950} + \frac{t_{18} t_{28}}{20} + \frac{t_{10} t_{28}}{6} \right) + t_{18} t_{10}^2 \left( \frac{t_{10}^{19}}{360} + \frac{t_{18} t_{28}^3}{6} \right) \]. (1.10)
In a similar way, the $H_3$ solution may be obtained from the $D_6$ one. It was shown in [7] that the $D_6$ solution is also related by slight changes of parametrization to a $SU(3)$ related solution. The restriction that yields the $H_3$ case is simpler to express in terms of the latter. One finds, denoting with boldface $t$’s and $\tau$ the $D_6$ parameters in the notations of [7] (and with $a^4 = -4$)

$$
F_{H_3}(t_0, t_4, t_8) = F_{SU(3)2}(t_{00} = t_0, t_{10} = 0, t_{01} = 0, t_{20} = t_4, t_{11} = 0, t_{02} = t_8) = a^7 F_{D_6}(t_0 = 1, t_1 = 0, t_2 = -\frac{1}{a^3} t_4, t_3 = 0, t_4 = -\frac{1}{a} t_8, \tau = -\frac{1}{2a} t_4)
$$

$$
= t_0^2 t_8 + \frac{t_0 t_4^2}{2} + \frac{t_4^2 t_8}{6} + \frac{t_4^2 t_8^5}{20} + \frac{t_8^{11}}{3960} .
$$

Finally, I claim that one may set all $t$’s but $t_0$ and $t_n$ equal to zero in the $A_{n+1}$ free energy and get a consistent solution. This will be the $I_2(n+2)$ solution of [4]. To prove this, we recall that (i), in $F_{A_{n+1}}$, the variable $t_i$ appears in terms that are at most $(i + 3)$-linear in the $t$’s [2], and that (ii), $F$ is a quasihomogeneous function of degree $2 + 2/(n + 2)$ of the $t$’s if $t_i$ is assigned the degree $1 - i/h + 2$. The only terms in $F$ involving $t_0$ are $\frac{1}{2} t_0 \sum_{k=0}^{n} (t_k t_{n-k})$: they generate terms that satisfy (1.3). Let us now look at the others. They would violate (1.3) if and only if they are of the form $t_n^p t_k, p \geq 2, 1 \leq k \leq n - 1$. This, however, has degree $1 + (2p - k)/n + 2 = 2 + 2/n + 2$, whence $p = n - 4 + k$ and by the above observation (i), $p < k + 1$, which leads to the contradiction $k \geq n$. This completes the proof of the consistency of this restriction, leading to the $I_2(n+2)$ solution

$$
F_{I_2(n+2)}(t_0, t_n) = F_{A_{n+1}}(t_0, 0, \cdots, t_n) = \frac{t_0^2 t_n}{2} + \frac{t_n^{n+3}}{(n + 1)(n + 2)(n + 3)} .
$$

In all cases, the labelling of the $t$’s is such that these labels take their values in the “exponents” minus one of the corresponding Coxeter group $G$, i.e. the degrees minus two of a basis of the ring of invariant polynomials [5–6]: $d_i = i + 2$. The homogeneity degrees of the variables $t_i$ and of $F$ are respectively $1 - i/h$ and $2/h + 2$ where $h$ is the Coxeter number of $G$. In view of the unicity theorem of [4], this justifies that we have found the desired solutions.

1 There are unfortunate misprints in the last four lines of Appendix C of [7]: one should read $t_{11} = -a^3 t_3; t_{10} = 2 t_1; t_{00} = 2 a t_0.$
Also we check that the expressions (1.3), (1.8) and (1.12) are consistent with the well-known identifications of Coxeter groups $A_2 \equiv I_2(3)$, $B_2 \equiv I_2(4)$ and $G_2 \equiv I_2(6)$. In fact, all these expressions are defined up to a change of normalization of $F$ and of each $t$.

Note that all the restrictions of $ADE$ cases that we have found respect the $t$ of lowest label 0 — as expected — and highest $h - 2$, hence preserve the Coxeter number $h$. This is because they have to respect the pairing between 0 and $h - 2$ in the metric

$$\eta_{0i} = C_{00i} = \delta_{i, h - 2}.$$  

At this stage, it is not clear that we have exhausted all possibilities of restriction (with $0, h - 2 \in I$), although Dubrovin’s conjecture asserts so. We are going to prove it by specializing our algebras in a definite way, namely by letting all $t$ vanish but $t_{h - 2}$, the one with the smallest homogeneity (the least relevant coupling in physical terms). Recall that $t_0$ (which is of homogeneity degree 1) can appear at most quadratically in $F$ (of degree $< 3$), hence doesn’t appear in the $C$’s. In this specialization, the $C$’s depend only on $t_{h - 2}$ in an homogeneous way, hence $t_{h - 2}$ may be taken equal to 1 with no loss of generality. In the simplest case of the $A_{k+1}$ topological algebra, this specialization is known to reproduce the fusion algebra of $\hat{su}(2)_k$ that has a polynomial representation in terms of Chebishev polynomial. We thus call this specialization in general the Chebishev specialization.

2. The Chebishev specialization

This specialization to all $t_i = 0$ but $t_{h - 2} = 1$ of the $C$ algebra is known to enjoy many nice properties, in the $ADE$ cases [8][7]. After studying all the subalgebras of the $ADE$ algebras, we shall extend these properties to them and present a curious connection with operator product expansions (OPE) of conformal field theories (CFT).

2.1. The subalgebras of the $ADE$ algebras

We shall now prove that the only subalgebras containing the generators of smallest and largest label 0 and $h - 2$ of the $ADE$ algebras are those associated by the construction of the first section to the other Coxeter groups.

First consider the $A_{k+1}$ case; as recalled above, it is isomorphic to the $\hat{su}(2)_k$ fusion algebra, i.e.

$$\phi_i \phi_j = \sum_{l = |i - j|, |i - j| + 2, \ldots, \text{inf}(i + j, 2k - i - j)} \phi_l.$$  

(2.1)
Suppose that we have a subalgebra containing $\phi_0$, and let $i$ be the smallest non zero label such that $\phi_i$ belongs to the subalgebra. Consider

$$\phi_i \phi_i = \phi_0 + \cdots + \phi_{\text{inf}(2i, 2k - 2i)}.$$ (2.2)

Either $\text{inf}(2i, 2k - 2i) = 0$, i.e. $i = k$, and this is what we call the $I_2(k + 2)$ algebra. Or $\phi_2$ belongs to the sum (2.2), hence $i \leq 2$. If $i = 1$, we are back to the original $A_{k+1}$ algebra, whereas $i = 2$ corresponds to the subalgebra $\{\phi_0, \phi_2, \cdots, \phi_{2|k|}\}$; this subalgebra satisfies the extra requirement to contain the generator $\phi_k$ of largest Coxeter exponent only if $k$ is even: we then get the $B_{k+1}$ solution.

The case of the $D_{2p+2}$ algebra is discussed along similar lines. This algebra is known to have $2p + 2$ generators $\{\phi_0, \phi_2, \cdots, \phi_{4p}, \alpha\}$ satisfying the relations (2.1) together with

$$\alpha \phi_{2l} = (-1)^l \alpha$$
$$\alpha \alpha = \phi_0 - \phi_2 + \cdots + \phi_{4p}$$ (2.3)

[This should not be confused with the $D_{2p+2}$ extended algebra generated by the combinations $\Phi_l = \frac{1}{2}(\phi_{2l} + \phi_{4p-2l}), l = 0, 1 \cdots p-1$ and $\Phi_p^\pm = \frac{1}{2}(\phi_{2p} + \epsilon \alpha), \epsilon^2 = (-1)^p$: see for example [3]]. Beside the obvious subalgebras $B_{2p+1} = \{\phi_0, \phi_2, \cdots, \phi_{4p}\}$ and $I_2(4p+2) = \{\phi_0, \phi_{4p}\}$ already encountered, the unique other possibility is to generate the subalgebra by a combination of two generators of $D$ of same degree, namely $\frac{1}{2}(\phi_{2p} + \epsilon \alpha)$. This turns out to be consistent only for $p = 2$, $\epsilon^2 = 1$, and we have the algebra $H_3 = \{\phi_0, \frac{1}{2}(\phi_4 \pm \alpha), \phi_8\}$.

This possibility doesn’t exist for the $D_{2p+1}$ cases, for which we conclude that there is no other subalgebra.

Finally the three exceptional cases are handled case by case, with the anticipated result that the only subalgebras (other than $I_2$ and containing $\phi_{h-2}$) are those corresponding to $F_4$ and $H_4$, in $E_6$ and $E_8$ respectively.

2.2. The dual algebras

In [3], it was noticed that the $C$ matrices of the $ADE$ Chebishev specializations have, after an appropriate change of basis, the same eigenvectors as the $ADE$ Cartan matrices. That was studied more systematically in [7], where it was shown that the matrices $(C_i)_j^k = C_{ij}^k$ can be made normal (i.e. commuting with their transpose) by a diagonal change of basis,
and that their normal form $M_i$ may be written in terms of the orthonormalized eigenvectors $\psi^{(i)}_a$ as follows
\[
C_{ij}^k = \frac{\rho_i \rho_j}{\rho_k} M_{ij}^k
\]
\[
M_{ij}^k = \sum_a \frac{\psi^{(i)}_a \psi^{(j)}_a \psi^{(k)}_a}{\psi^{(0)}_a}.
\]
(2.4)

Here $\psi^{(i)}_a$ denotes the $a$-th component of the $i$-th eigenvector of the Cartan matrix, or equivalently, since we are still considering the simply laced $ADE$ cases, of the adjacency matrix $G_{ab}^b$ of the Dynkin diagram. Thus the label $a$ runs over the vertices of the Dynkin diagram, and the label $i$ over the Coxeter exponents (minus 1), in accordance with our previous conventions.

In [7], it was further observed that it is interesting to also look at the dual algebra structure generated by the matrices $N_a$ of matrix elements
\[
N_{ab}^c = \sum_i \frac{\psi^{(i)}_a \psi^{(i)}_b \psi^{(i)}_c}{\psi^{(i)}_0}.
\]
(2.5)

This is again an associative algebra, with the matrix $N_0 = I$, the identity matrix. Eq. (2.5) assumes that there exists at least one vertex labelled 0 such that all the $\psi^{(i)}_0$ are non vanishing. If we may take such a point among the extremal vertices of the diagram, i.e. those connected with only one other vertex denoted $f$, then it is easy to see that $\psi^{(i)}_f / \psi^{(i)}_0 = \lambda^{(i)}$ is the $i$-th eigenvalue of the adjacency matrix $G$, hence $N_f = G$. In fact, one may always choose the point 0 as the end point of the long branch of the Dynkin diagram, except for the $D_{odd}$ cases, where one has to take it as the end point of one of the short branches.

To summarize, the dual algebra is generated by the identity matrix $N_0$ and the adjacency matrix of the Dynkin diagram $N_f = G$, all the matrices $N_a$ thus have integral entries and, depending on the case (and the choice of the vertex 0), these integers are or are not all non negative (cf.[7]).

It is now natural to wonder if these properties extend to the subalgebras labelled by other Coxeter groups that we have discussed (still in the Chebishev specialization). By a case by case analysis, one may convince oneself that
\[
\star \text{ for all the } B_n, F_4, G_2, H_3, H_4, I_2 \text{ cases, the } C \text{ matrices may again be brought to a normal form by a diagonal change of basis;}
\]
the normal form $M_i$ of $C_i$ has only non negative entries;

in contrast with the $ADE$ cases, the matrices of the dual algebra are no longer all with integral entries; the dual algebra, however, is generated by the identity matrix and the matrix $N_f$

\[
(N_f)_a^a = 0
\]

\[
(N_f)_a^b = 2 \cos \frac{\pi}{m(a,b)} \quad a \neq b ,
\]

where the integer $m(a,b) = m(b,a)$ takes a value different from 2 or 3 (hence $N_a^b \neq 0, 1$) only for one pair of vertices $(a,b)$; $m = 4$ for the $B$ and $F_4$, $m = 5$ for the $H_3$, $H_4$ cases (for $I_2$ see below).

In fact one recognizes in these $N$ matrices essentially the “Coxeter matrices” of the corresponding “Coxeter graphs”. Let us recall the definitions, following \[3\]. We consider an unoriented graph, the edges $(a,b)$ of which are labelled by integers $m(a,b) = m(b,a) \geq 3$. It is convenient to extend this function $m$ to all pairs of vertices of the graph : $m(a,a) = 1$, $m(a,b) = m(b,a) = 2$ if $a$ and $b$ are not neighbours on the graph. These integers are tabulated for the relevant graphs in Table I. To such a graph, we associate the Coxeter matrix

\[
\Gamma_{ab} = - \cos \frac{\pi}{m(a,b)} .
\]

For the Weyl-Coxeter groups, this is (up to a factor 2) the “normalized” form of the Cartan matrix

\[
C_{ab} = 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_b, \alpha_b \rangle} = \frac{|\alpha_a|}{|\alpha_b|} 2 \cos (\alpha_a, \alpha_b) = - \frac{|\alpha_a|}{|\alpha_b|} 2 \cos \left( \frac{\pi}{m(a,b)} \right) .
\]

Now the matrix $N_f$ of (2.6) reads

\[
N_f = 2(\mathbf{I} - \Gamma) .
\]
Table I: Coxeter graphs

The well known ADE diagrams are not represented here; neither is $G_2 = I_2(6)$; $m(a,b) = m(b,a) = 3$ unless otherwise specified above the edge $(a,b)$.

The only exception to our discussion is the $I_2(n+2)$ case, which is somehow degenerate: the dual algebra is generated by the two-by-two matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; the later matrix is a multiple of the Coxeter matrix $\begin{pmatrix} 0 & \cos \frac{\pi}{n+2} \\ \cos \frac{\pi}{n+2} & 0 \end{pmatrix}$ attached to the $I_2(n+2)$ graph, and our dual algebra does not “see” the factor $\cos \frac{\pi}{n+2}$.

To summarize, except maybe in the $I_2$ case, the dual algebra exhibits the most natural generalization of the property observed in the simply laced case, namely it is generated by the Coxeter matrices of the relevant diagrams. It is quite curious to see the emergence of these Coxeter matrices in this problem. Recall that the $A \cdots I$ Coxeter graphs solve the following problem: they are the unique graphs such that the associated Coxeter matrices (2.7) are positive definite, or equivalently, such that the matrix $N_f$ of (2.9) has all its eigenvalues smaller than 2. It would be quite interesting to see directly why this property is requested of the dual algebra.

2.3. A connection with OPE of CFT

We finally turn to a remarkable fact that gives another perspective to (the Chebishev specializations of) these topological algebras: their connection with operator product expansions in WZW or minimal conformal field theories. That the $C$ algebra of the $A_{k+1}$ case yields the fusion algebra of the $\widehat{su}(2)$ theories has been recalled above. This is already quite astonishing, since it relates an algebra associated with a “massive” topological field
theory (or its perturbed $N = 2$ partner) with a similar structure in a conformal theory. Moreover, this result has a non trivial extension to the other simply laced cases $D$ and $E$. This fact, already anticipated a long time ago by Pasquier [10], has been put in a quantitative form in a collaboration with V. Petkova and will be presented in detail elsewhere [11]. In short, the normalized form $M_i$ of the $C_i$ matrices yields the ratios of the structure constants of spinless operators in the non diagonal $D$ or $E$ WZW $\hat{su}(2)$ theories over the same structure constants evaluated in the $A$ theory.

One would like to summarize the findings of this section by stating that the subalgebras of the OPE of spinless fields of $\hat{su}(2)_k$ theories that contain both the identity and the field of largest label (twice the isospin) $i = k$ are classified by Coxeter groups. This is, unfortunately, not quite correct, as the spinless operators of the WZW or minimal theories do not form a closed algebra: the OPE of two such fields may involve non zero spin fields, and the relevant structure constants do not seem determined in a similar algebraic way. The more correct (and cumbersome) statement is therefore that the projection on spinless fields of the subalgebras of the OPE of $\hat{su}(2)$ theories that contain both the identity and the field of largest label $i = k$ are classified by Coxeter groups.

3. Discussion

The $ADE$ solutions are known to admit a “Landau-Ginsburg picture”, which means that the algebra $A$ with structure constants $C_{ij}^k$ has a polynomial representation [13–15]

$$A \equiv \frac{\mathcal{C}(x,y,\cdots)}{\mathcal{J}(\partial_x W, \partial_y W, \cdots)}$$

where $W(x, y, \cdots; t)$ is the “Landau-Ginsburg potential”, $\mathcal{C}(x, y, \cdots)$ is the ring of complex polynomials in the indeterminates $x, y, \cdots$ and $\mathcal{J}(\partial_x W, \cdots)$ is the ideal generated by the derivatives of $W$. In other words, the representation is provided by the multiplication of the polynomials $p_i(x, y, \cdots) = \partial W/\partial t_i$

$$p_i p_j = C_{ij}^k p_k \text{ mod } \partial_x W, \partial_y W, \cdots .$$

This approach is deeply related to the theory of singularities: $W(x, y, \cdots; t)$ is the versal deformation of the singularity $W(x, y, \cdots; 0)$ with the $t$’s the flat coordinates.

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2 The issue of subalgebras of the OPE was also addressed several years ago by Christe and Flume [12]; they didn’t, however, impose the condition that $\phi_k$ should be kept in the subalgebra.
Our construction of the other, non ADE, Coxeter solutions to (1.1) was based on the restriction procedure. This is a familiar idea in singularity theory. There [16], it is known that by quotient of an $A_{2n+1}$, $E_6$ or $D_4$ singularity by a discrete symmetry, one obtains a “boundary singularity” labelled by the $B$, $F_4$ or $G_2$ Dynkin diagram. Moreover, the $H_3$ and $I_2$ cases have also received a similar treatment [17]; it would be interesting to see if the present construction of $H_3$, $H_4$ and $I_2$ by restriction suggests other possibilities.

In fact the same procedure has also been encountered in a different context, that of modular invariant or sub-modular invariant partition functions $Z$. The partition function of a conformal field theory on a torus encodes its field content that must be consistent with (i) the operator product algebra, (ii) modular invariance. It may sometimes be possible to find restrictions of this field content that are still consistent with a closed operator product algebra but only satisfy invariance under a subgroup of the modular group. For example, it is known that modular invariant partition functions of $N = 2$ superconformal (or affine $\hat{su}(2)$) theories are classified according to an ADE scheme. One may verify that the fusion algebras of the $A_{2n+1}$ and $E_6$ cases admit a $\mathbb{Z}_2$ symmetry. Projecting on the even sector produces a partition function that is invariant under a subgroup of the modular group [18]. For example for $E_6$

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2,$$

that is modular invariant, gives rise to

$$Z_{F_4} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2$$

that is invariant under the subgroup $\hat{\Gamma}_0(2)$ of modular transformations \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} \pm 1 & * \\ 0 & \pm 1 \end{array} \right) \mod 2 \). Likewise, the modular invariant labelled $D_4$ leads under a $\mathbb{Z}_3$ quotient to a submodular invariant that we may label by $G_2$. As we have seen in the first section, the existence of a symmetry group is not required; the only condition is that the operator product algebra of the restricted theory be closed. We thus have additional cases corresponding to the $H_3$, $H_4$ and $I_2$ groups:

$$Z_{H_3} = |\chi_0 + \chi_8|^2 + |\chi_4|^2$$

$$Z_{H_4} = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2$$

$$Z_{I_2(n+2)} = |\chi_0|^2 + |\chi_n|^2$$

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For each of these submodular invariants, the level of the affine $\hat{su}(2)$ algebra is read off the highest label, resp. 8,28 and $n$. Just as in the end of last section, the role of the operators with non zero spin is not clear, and one might for example write as well $Z'_{H_3} = |\chi_0|^2 + |\chi_{8}|^2 + |\chi_{4}|^2$. This ambiguity notwithstanding, it appears that there is a connection between these submodular invariants and Coxeter groups.

In this context of modular invariants, we thus see that the $ADE$ and non $ADE$ situations are not exactly on the same footing. The former describe sound and consistent theories, whereas the latter describe some projection thereof. As far as we can see, this does not invalidate the TFT's associated with the latter, but only reminds us that these cannot be obtained by a simple twisting of a consistent, modular invariant, $N = 2$ theory.

Finally, I return to the most intriguing and challenging point already mentioned at the end of the previous section: the projections on spinless fields of consistent operator product algebras of $\hat{su}(2)$ theories containing the field of largest isospin are classified by Coxeter groups. If this could be established directly, independently of the $ADE$ classification of modular invariants, it might offer an alternative route to the latter. One would first classify the consistent operator algebras as $AB\cdots I$ and then retain among them only the $ADE$ that give rise to a modular invariant partition function.

So we are left with the two questions

☆ Why are Coxeter groups involved in the operator algebra ?
☆ What would be the generalization of this correspondence for higher rank current algebras ?

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