Packing colorings of subcubic outerplanar graphs

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Abstract

Given a graph $G$ and a nondecreasing sequence $S = (s_1,\ldots,s_k)$ of positive integers, the mapping $c : V(G) \to \{1,\ldots,k\}$ is called an $S$-packing coloring of $G$ if for any two distinct vertices $x$ and $y$ in $c^{-1}(i)$, the distance between $x$ and $y$ is greater than $s_i$. The smallest integer $k$ such that there exists a $(1,2,\ldots,k)$-packing coloring of a graph $G$ is called the packing chromatic number of $G$, denoted $\chi_p(G)$. The question of boundedness of the packing chromatic number in the class of subcubic (planar) graphs was investigated in several earlier papers; recently it was established that the invariant is unbounded in the class of all subcubic graphs. In this paper, we prove that the packing chromatic number of any 2-connected bipartite subcubic outerplanar graph is bounded by 7. Furthermore, we prove that every subcubic outerplanar graph has a $(1,2,2,2)$-packing coloring, and there exists a subcubic outerplanar graph that does not admit $(1,2,2,3)$-packing coloring. A similar dichotomy is shown for bipartite outerplanar graphs: every such graph admits an $S$-packing coloring for $S = (1,3,\ldots,3)$, where 3 appears $\Delta$ times ($\Delta$ being the maximum degree of vertices), and if one of the integers 3 is replaced by 4 in the sequence $S$, there exist outerplanar bipartite graphs that do not admit an $S$-packing coloring.

Keywords: outerplanar graph; packing chromatic number; cubic graph; coloring; packing.

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1 Introduction

The $S$-packing chromatic number was introduced a decade ago in [16] with motivation coming from the frequency assignment problem. Roughly the idea of the concept is to generalize the classical coloring by involving the distance between vertices and allowing larger color values only for vertices that are more distant. Nevertheless, the problem attracted many discrete mathematicians as it brings appealing combinatorial and computational challenges.

A number of papers considered packing coloring of different infinite grids and lattices [6, 12, 13, 15, 22], where the most interesting development is about the infinite square grid; we mention only the recent paper on the topic [4] where it was shown that $13 \leq \chi_{\rho}(\mathbb{Z} \times \mathbb{Z}) \leq 15$, which is the latest refinement of the known bounds (initial bounds were presented already in the seminal paper [16]). Fiala and Golovach have shown that the decision version of the packing chromatic number is NP-complete even in the class of trees [11]. Packing coloring of some other classes of graphs, such as the distance graphs [10, 20, 23], hypercubes [24], subdivision graphs of subcubic graphs [9, 14], and still other classes of graphs [1, 17, 19] was also studied.

One of the main questions in this area is concerning graphs with bounded maximum degree $\Delta$, in particular, subcubic graphs (i.e., graphs with $\Delta = 3$). For graphs with maximum degree $\Delta$, where $\Delta \geq 4$, already the infinite $\Delta$-regular tree serves as an example showing that in this class of graphs the packing chromatic number is unbounded (in fact, Sloper proved this in the context of so-called eccentric colorings, but his result implies the same for the packing coloring [21]). On the other hand, the question whether in subcubic graphs the packing chromatic number is bounded was much more intriguing. It was posed in the seminal paper [16], and then investigated in several papers [7, 8, 14] using different approaches. Recently, Balogh, Kostochka and Liu [2] have provided a negative answer to the question. Moreover, they proved that for every fixed $k$ and $g \geq 2k + 2$, almost every $n$-vertex cubic graph of girth at least $g$ has the packing chromatic number greater than $k$. An explicit infinite family of subcubic graphs with unbounded packing chromatic number was then presented in [5].

As the question was answered in negative for all graphs with bounded maximum degree 3, it becomes interesting for some subclasses of subcubic graphs. In particular, already in [12] it was asked, whether there is an upper bound for the packing chromatic number of all planar cubic graphs, and this question was repeated in [5]. Very recently, packing chromatic number of subcubic outerplanar graphs was considered [15]. The upper bounds obtained in the paper involve the number of (internal) faces of the plane embedding of an outerplanar graph; for instance, it is proven that if $G$ is a 2-connected subcubic outerplanar graph with $r$ internal faces, then $\chi_{\rho}(G) \leq 17 \cdot 6^{3r} - 2$. The question of boundedness of the packing chromatic number in subcubic outerplanar graphs thus seems widely open. In this paper, we prove that, quite surprisingly, only 7 color suffice if we restrict to bipartite 2-connected case.

In the following section we fix the notation. In Section 3 we prove the following theorem, our main result.
Theorem 1. Let $G$ be a 2-connected bipartite subcubic outerplanar graph. Then $\chi_\rho(G) \leq 7$.

We follow in Section 4 with the result about bipartite outerplanar graphs (i.e., no restriction to 2-connectedness and arbitrary maximum degree).

Theorem 2. Let $G$ be a bipartite outerplanar graph. Let $S = (1, 3, \ldots, 3)$ be the sequence containing one time the integer 1 and $k$ times the integer 3, $k \geq 3$. If $\Delta(G) \leq k$, then $G$ is $S$-packing colorable.

The result is complemented by an example showing that for $S = (1, 3, \ldots, 3, 4)$, where 3 appears $\Delta(G) - 1$ times, there exists a bipartite outerplanar graph that does not admit an $S$-packing coloring.

In Section 5, subcubic outerplanar graphs are considered (extending the consideration of Theorem 1 to the non-bipartite case), and we prove:

Theorem 3. If $G$ is a subcubic outerplanar graph, then $G$ is $(1, 2, 2, 2)$-packing colorable.

The result is complemented by an example showing that there exist subcubic outerplanar graphs, which are not $(1, 2, 2, 3)$-packing colorable.

In the final section we present some open problems.

2 Notation

A path between vertices $a$ and $b$ in a graph $G$ will be called also an $a,b$-path. The length of a shortest $a,b$-path is the distance $d_G(a,b)$ between $a$ and $b$ in $G$ (we also write $d(a,b)$ if the graph is understood from the context). An $i$-packing $G$ is a set of vertices $A$ such that for any two distinct vertices $x, y \in A$ we have $d_G(x,y) > i$. Clearly, a 1-packing coincides with an independent set.

For a nondecreasing sequence $S = (s_1, \ldots, s_k)$ of positive integers, the mapping $c : V(G) \rightarrow \{1, \ldots, k\}$ is an $S$-packing coloring of $G$ if for every $i \in [k]$ the set $c^{-1}(i)$ is an $s_i$-packing. If there exists an $S$-packing coloring of $G$, we say that $G$ is $S$-packing colorable. If the sequence $S = [k]$ for some positive integer $k$, we omit $S$ in the definition, and say only that $G$ is packing colorable (as usual, we set $[k] = \{1, \ldots, k\}$). The smallest integer $k$ such that $G$ is packing colorable is the packing chromatic number of $G$, denoted $\chi_\rho(G)$. When we say that the packing coloring condition holds for a set $A \subseteq V(G)$ we mean that each set $A \cap c^{-1}(i)$, for all $i \geq 1$, is an $i$-packing in $G$. If $A \subseteq V(G)$, then by $G[A]$ we denote the subgraph of $G$ induced by $A$.

An outerplanar graph is a graph that has a planar drawing in which all vertices belong to the outer face of the drawing. Each time an outerplanar graph will be considered, a drawing in which all vertices belong to the outer face of the drawing will be fixed. Let $G$ be an outerplanar graph. The outer cycle of $G$ corresponds to the cycle induced by the edges of the outer face.

For an outerplanar graph $G$, we denote by $T_G$ the weak dual of $G$, i.e., the graph whose vertex set is the set of all inner faces of $G$, and $E(T_G) =$
\{\alpha \beta | \alpha \text{ and } \beta \text{ share a common edge}\}. For \(\alpha \in V(T_G)\), we denote by \(C(\alpha)\), the (chordless) cycle in \(G\) that corresponds to the face \(\alpha\).

Let \(G\) be a 2-connected outerplanar graph (we consider such graphs in Section 3). Note that in this case \(T_G\) is a tree. Let \(\omega_0\) be a vertex of \(T_G\) of minimum eccentricity. We consider \(T_G\) as a rooted tree with \(\omega_0\) as the root. The notions of parent, child and descendant should be clear in this context. We can also define the depth of a vertex \(\beta \in V(T_G)\), denoted by \(p(\beta)\), as \(d_{T_G}(\beta, \omega_0)\). The depth of \(T_G\), denoted by \(p(T_G)\), is the maximum value of \(p(\beta)\), for \(\beta \in V(T_G)\).

3 Packing coloring of 2-connected bipartite subcubic outerplanar graphs

In this section we prove that a 2-connected bipartite subcubic outerplanar graph \(G\) has packing chromatic number bounded by 7, i.e., \(\chi^p(G) \leq 7\). At the end of the section we add the result, which shows that this is best possible.

The proof of the main theorem has two steps. In the first step we determine a subset \(A\) of \(V(G)\), and present a coloring \(f\) of the vertices of \(A\) by using only the colors from \(\{1, 2, 3\}\) (such that colors in vertices of \(A\) respect the condition of the packing coloring). The remaining set \(B = V(G) \setminus A\) will be called the set of big vertices, and so any vertex in \(B\) is called a big vertex. The big vertices will be colored in the second step by using only the colors from \(\{4, 5, 6, 7\}\). That is, we will extend \(f\) from \(A\) to all vertices of \(V(G)\). We will prove that \(f : V(G) \rightarrow \{1, \ldots, 7\}\) has four special properties, which will be helpful in proving that \(f\) is a packing coloring of \(G\) (that is, for any two distinct vertices \(u, v \in V(G)\), \(f(u) = f(v)\) implies \(d_G(u, v) > f(u)\)). As is often the case, additional (technical) conditions, which are not needed in the result, are helpful in the proof.

Proof of Theorem 1.

Let \(G\) be a 2-connected bipartite subcubic outerplanar graph. Let \(\omega_0\) be a vertex of \(T_G\) of minimum eccentricity; we consider \(T_G\) to be a rooted tree with \(\omega_0\) as its root.

In the proof, we construct a packing coloring \(f\) of \(G\), which satisfies the following additional properties.

(i) Any vertex with color from \(\{2, \ldots, 7\}\) has all its neighbors colored by color 1.

(ii) Any face \(\alpha\) of \(G\) contains exactly one big vertex if \(|V(\alpha)| \geq 6\) and at most one big vertex if \(|V(\alpha)| = 4\).

(iii) Any big vertex is at distance at least 4 of any other big vertex.

(iv) Any vertex with color from \(\{6, 7\}\) is at distance at least 6 of any vertex of color from \(\{6, 7\}\).
Step 1. In this first step of the proof, we determine the set $A$, and color the vertices of $A$ by using only the colors from $\{1, 2, 3\}$. (Note that $B = V(G) \setminus A$ is a set that will contains only big vertices.) In this step, properties (i), (ii) and (iii) will already be verified (while property (iv) will be verified in the second step, when we will assign colors to the vertices from $B$).

The proof uses the structure of the tree $T_G$. We consider the faces in a BFS order by starting with the face $\omega_0$. In each facial cycle we will repeatedly use the pattern $1, 2, 1, 3$; by this we mean that vertices along the cycle will follow in the order $1, 2, 1, 3, \ldots$ or $1, 3, 1, 2, \ldots$, which also applies when the length of the cycle is not divisible by 4 (in which case, we omit the last few numbers). If the length of the corresponding cycle is greater than 4, one of the vertices is determined as a big vertex and is not yet colored, and for the rest of the cycle we use the pattern $1, 2, 1, 3$ (by repeating it an appropriate number of times). In the 4-cycle we may either use only the colors from $\{1, 2, 3\}$ or one of the vertices is big, which depends on the type of the used coloring (described soon).

Clearly, the vertices of $\omega_0$ can be colored by using the above pattern. In particular, if $p(T_G) = 0$, then $G$ is the cycle $C(\omega_0)$, and the described pattern satisfies properties (i), (ii), (iii) and (iv).

Now, by following a BFS order of $T_G$, consider a face $\alpha \in T_G$, where $\omega \in T_G$ is the parent of $\alpha$. By the construction, a big vertex $u$ of $\omega$ is already determined (including the possibility that $\omega$ has no big vertices, which may happen when the size of $C(\omega)$ is 4), and other vertices of $\omega$ are colored by colors $\{1, 2, 3\}$ by repeatedly using the pattern $1, 2, 1, 3$.

We consider the following three possibilities for the position of the big vertex in $\alpha$ (and the coloring of the vertices of $V(C(\alpha)) \cap A$).
• **0-position**: suppose that the big vertex $u$ of $\omega$ coincides with one of the two vertices in $V(C(\alpha)) \cap V(C(\omega))$. In this case, the big vertex of $\alpha$ is also determined, notably, $u \in B \cap V(C(\alpha))$. Let $u'$ be the other vertex in $V(C(\alpha)) \cap V(C(\omega))$. (See Fig. 1.) The corresponding **0-coloring** of the vertices of $C(\alpha)$ is obtained by starting with a neighbor of $u$, and repeatedly using the pattern 1, 2, 1, 3 along the cycle, taking care only that the neighbors of $u'$ get different colors (note that their colors are $f(u)$, 2 and 3, respectively).

• **1-position**: suppose that the big vertex of $\omega$ is not one of the vertices in $V(C(\alpha)) \cap V(C(\omega))$, but is adjacent to one of the two vertices in $V(C(\alpha)) \cap V(C(\omega))$. Let us denote by $u'$ the neighbor of $u$ in $V(C(\alpha)) \cap V(C(\omega))$, and let the other vertex of $V(C(\alpha)) \cap V(C(\omega))$ be called $v'$. Clearly, $f(u') = 1$,
and \( f(z) \in \{2, 3\} \). Now, if \( C(\alpha) \) has 4 vertices, then we use only colors 1, 2 or 1, 3 taking care that the neighbors of \( u' \) in \( C(\alpha) \) get distinct colors. Otherwise, we define the big vertex \( v \) of \( \alpha \) as the vertex at distance 2 from the vertex \( z \), different from a neighbor of \( u' \). (See Fig. 3.) The corresponding 2-coloring of the vertices of \( C(\alpha) \) is obtained by starting with \( u' \), and repeatedly using the pattern 1, 2, 1, 3 along the cycle, taking care only that the neighbors of \( u' \) get different colors in \( \{2, 3\} \).

Clearly, for any of the three colorings (0-coloring, 1-coloring, 2-coloring) the packing coloring condition holds for the vertices of the corresponding cycle that belong to \( A \), and properties (i), (ii) and (iii) extend from any vertices colored so far also to the vertices of \( \alpha \). We derive the following observation.

**Lemma 4.** Let \( A \) be the set constructed in Step 1 with \( B = V(G) \setminus A \), and let \( f \) the coloring of vertices of \( A \) by using colors \( \{1, 2, 3\} \) as described above. Then \( f \) and \( B \) satisfy the properties (i), (ii) and (iii), and for any two distinct vertices \( x \) and \( y \) in \( A \) such that \( f(x) = f(y) \) we have \( d_G(x, y) > f(x) \).

**Step 2.** In this step we need to determine the \( f \)-values of big vertices, and prove that the property (iv) holds for these vertices and that the packing coloring condition holds for the set \( B \). We determine the colors of big vertices following a BFS order on \( T_G \); we start by determining the possible color of the big vertex of \( \omega_0 \). If such a vertex exists (i.e., \( |V(C(\omega_0))| > 4 \)), then we color it by 4.

A big vertex \( x \) that belongs to a face \( \beta \) will be called a big vertex arising from \( \alpha \) if the following two conditions are true:
i) $\beta$ is a descendant of $\alpha$ with respect to $T_G$;

ii) $x$ is at distance at most two from $C(\alpha)$.

A step of the coloring construction consists of dealing with a face $\alpha$ that comes next in the chosen BFS order (beginning with the face $\omega_0$), and to color all the big vertices arising from $\alpha$. Let $\omega$ be the parent of the face $\alpha$ (if it exists). In a step of the coloring construction, it is supposed that both the big vertex of $\omega$ and the big vertices arising from $\omega$ are already colored. Consequently, the big vertex of $\alpha$ is already colored, because it arises from $\omega$. However, in the case $\alpha$ is $\omega_0$, only the big vertex of $\omega_0$ is already colored.

We distinguish three cases with respect to the type of the position of the big vertex (0-position, 1-position, 2-position) and the corresponding coloring, which was used to color $\alpha$. We will call the colors in $\{6, 7\}$ very big.

**Case 1.** $\alpha$ is in 0-position, see Fig. 1.

Consider the face $\omega$ and its descendants with respect to $T_G$, and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from $\alpha$). By property (i), all big vertices are at even distance from vertex $u$, and they are at distance at least 4 from $u$ by property (iii). We distinguish two kinds of big vertices that were already colored, notably those that are at shorter distance from $u$ than from $u'$, and those that are at shorter distance from $u'$ than from $u$. Those that are closer to $u$ than to $u'$ are at distance at least 8 from big vertices that arise from $\alpha$. (As shown in Fig. 1, big vertices arising from $\alpha$ belong to faces $\beta_1, \ldots, \beta_k$, which are children of $\alpha$ with respect to $T_G$, or to their children.) On the other hand, there can be big vertices, which are at distance 3 from $u'$ (and 4 from $u$), and have already been colored. More precisely, by using property (iii), and the fact that the neighbor of $u'$ that is not in $\alpha$ has at most two other neighbors, we find that there can be at most two such big vertices, which are at distance 3 from $u'$. If they indeed exists, we denote them by $r$ and $s$, and note that $r, s$ and $u$ are pairwise at distance 4.

Next we analyze possible positions of vertices that arise from $\alpha$. For every vertex $a$ in $C(\alpha)$, which is at even distance from vertex $u$, there can be a big vertex $a'$ that arises from $\alpha$ such that there is a $a, a'$-path of length 2 outside $\alpha$ (there is at most one such vertex by property (iii)). On the other hand, for every vertex $b$ in $C(\alpha)$, which is at odd distance from $u$, there can be a big vertex $b'$ that arises from $\alpha$, which is adjacent to $b$ (with the exception of the neighbors of $u$ in $C(\alpha)$, which cannot be adjacent to another big vertex due to property (iii)). The situation when all these big vertices exist is described in Fig. 4.

In the most complex case when both vertices $r$ and $s$ exist and are already colored, vertices $u, r$ and $s$ are pairwise at distance 4. By property (iv), exactly one of these three vertices is colored by a very big color. We consider two subcases, depending on the color of vertices $u, r$ and $s$. We can assume, without loss of generality, that if the color of $u$ is very big, then it is 6 and otherwise it is 4; we can also assume without loss of generality that in the case $f(u) = 4$, we have $\{f(r), f(s)\} = \{5, 7\}$. This is described in the next table.
Subcase 1.a. We present three patterns that define the coloring of the big vertices with respect to different lengths $n$ of the cycle $C(\alpha)$. The patterns give $f$-values of the big vertices following their presentation in Fig. 4 (from left to right); in the case when some of the big vertices that arise from $\alpha$ do not exist, we simply skip the corresponding values in the pattern. Note that the numbers between vertical bars are to be repeated $k$ times (case $k = 0$ included).

| Pattern for length $n = 4k + 8$: | $f(u)$ | a | b |
|-----------------------------------|--------|---|---|
| $\{f(r), f(s)\}$                 | 7      | 6 | 4 | 5 |
|                                   | 5      | 4 | 5 | 6 |

| Pattern for length $n = 8k + 6$: | $f(u)$ | a | b |
|-----------------------------------|--------|---|---|
| $\{f(r), f(s)\}$                 | 5      | 4 | 5 | 6 |
|                                   | 5      | 4 | 5 | 6 |

| Pattern for length $n = 8k + 10$: | $f(u)$ | a | b |
|-----------------------------------|--------|---|---|
| $\{f(r), f(s)\}$                 | 5      | 4 | 5 | 6 |
|                                   | 7      | 4 | 5 | 6 |

Note that because $f(u) = 4$, the first and the last two values in the patterns could not be 4, and because $\{f(r), f(s)\} = \{5, 7\}$, the last two values in the patterns could not be 7. Also note that two vertices that correspond to two successive values in the upper row of the pattern are at distance 6, two vertices that correspond to two successive values in the lower row of the pattern are at distance 4, and two vertices that correspond to two consecutive values in the patterns (one in the upper and the other in the lower row) are at distance 4.

In each of the patterns, one can check that the property (iv) holds, and that for any two identical numbers, the distance between the corresponding vertices in $G$ is bigger than this number (i.e., the packing coloring condition). Note that for $k = 0$ one needs to omit the numbers that are between the vertical bars in
each pattern, which covers the lengths of cycles \( n \), where \( n \in \{6, 8, 10\} \). For \( n = 4 \), there is only one big vertex that arises from \( \alpha \), and we color it by 5.

**Subcase 1.b.** We present patterns that define the coloring of the big vertices in the case \( f(u) = 6 \), and \( \{f(r), f(s)\} = \{4, 5\} \). Note that it implies that the first and the last four values in the patterns cannot be 6. Again numbers between vertical bars are to be repeated \( k \) times (including \( k = 0 \), where we omit the numbers between vertical bars):

- Pattern for \( n = 4k + 8 \):
  
  \[
  \begin{array}{cccc}
  5 & 7 & 6 & 4 \\
  4 & 5 & 4 & 5 \\
  \end{array}
  \]

- Pattern for \( n = 4k + 10 \):
  
  \[
  \begin{array}{cccc}
  5 & 5 & 4 & 6 \\
  4 & 7 & 4 & 7 \\
  5 & 4 & 5 & 5 \\
  \end{array}
  \]

- Pattern for \( n = 6 \):
  
  \[
  \begin{array}{cc}
  4 & 4 \\
  5 & 4 \\
  \end{array}
  \]

For \( n = 4 \) we can use color 4 for the only big vertex that possibly arises from \( \alpha \).

**Case 2.** \( \alpha \) is in 1-position, see Fig. 2.

Consider the face \( \omega \) and its descendants with respect to \( T_G \), and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from \( \alpha \)). Also, as face \( \alpha \) is in 1-position, there can be a big vertex arising from \( \alpha \), which is at distance 2 from \( C(\omega) \). If this vertex exists, we denote it by \( p \). By definition, \( p \) is a big vertex arising from \( \omega \), hence it is already colored.

By property (iii), the neighbors of \( v' \), different from \( v \), are not in \( B \), and there can be at most one big vertex different from \( v \) at distance 4 from \( p \), which is already colored. If such a vertex exists, we denote it by \( r \). All other big vertices that are already colored are at distance at least 8 from the vertices arising from \( \alpha \).

Similarly as in Case 1 (respecting the distance from \( v \) and using property (iii)), we find the position of big vertices that possibly arise from \( \alpha \), and depict it in Fig. 5. Note that some of the big vertices indicated in this figure may not exist in \( G \).

Since, by property (iii), the vertices \( v, p \) and \( r \) (see Fig. 5) are at pairwise mutual distance 4, exactly one of them has a very big color. Hence we consider three subcases, depending on the values of the vertices \( v, p \) and \( r \). We can assume, without loss of generality, that if the color of \( v, p \) or \( r \) is very big, then it is 6; also, we may assume that \( f(v) = 4 \), if \( v \) is not very big. (Also, in the case \( r \) or \( p \) do not exist and no vertex among \( v, p \) and \( r \) is very big, it is possible to deal with this situation by considering that a non existing vertex among \( v, p \) and \( r \) is very big.) The different cases are described in the following table:
Subcase 2.a. We present patterns that define the coloring of the big vertices with respect to different lengths $n$ of the cycle $C(\alpha)$. The patterns give $f$-values of the big vertices following their presentation in Fig. 5 as in Case 1, if some of the big vertices that arise from $\alpha$ do not exist, we simply skip the corresponding values in the pattern. Note that the numbers between vertical bars are to be repeated $k$ times (case $k = 0$ included). Note also that the last value in the patterns represents $f(p) = 4$.

Pattern for $n = 4k + 8$: 4 5 7 6 4 5 7

Pattern for $n = 4k + 10$: 4 5 7 6 4 5 4 5

Note that because $f(v) = 6$, the first four and the last three values (including $p$) in the patterns could not be 6. In each of the patterns, one can check that the property (iv) holds.

Cases $k = 0$, where the values between vertical bars are omitted, cover $n \in \{8, 10\}$, while pattern for length $n = 6$ is: 5 4

Subcase 2.b. We present patterns that define the coloring of the big vertices with respect to different lengths $n$ of the cycle $C(\alpha)$. Note that the numbers between vertical bars are to be repeated $k$ times (case $k = 0$ included). The last value in the patterns represents $f(p) = 6$.

Pattern for $n = 4k + 6$: 6 5 7 4 5 6

Pattern for $n = 4k + 8$: 7 5 6 4 7 5 4 6
Note that \( f(v) = 4 \) implies that patterns must avoid to have the first two values equal to 4. Cases \( n \in \{6, 8\} \) are covered by the above patterns with \( k = 0 \) (i.e., removing the values between vertical bars).

**Subcase 2.c.** The following two patterns define the coloring of the big vertices (numbers between vertical bars are to be repeated \( k \) times, with \( k = 0 \) included, covering \( n = 6 \) and \( n = 8 \)). The last value in the patterns presents \( f(p) = 5 \).

Pattern for \( n = 4k + 6 \):

\[
\begin{array}{c|c|c|c}
7 & 6 & 4 & 7 \\
5 & 4 & 5 & 5 \\
\end{array}
\]

Pattern for \( n = 4k + 8 \):

\[
\begin{array}{c|c|c|c|c|c|c}
7 & 6 & 4 & 7 & 5 & 4 & 5 \\
5 & 4 & 7 & 5 & 5 & 5 & 5 \\
\end{array}
\]

In the above patterns, in particular, the first two values are not 4, as \( f(v) = 4 \), and the last three values are not 6, as \( f(r) = 6 \).

Note that if the length \( n \) of \( C(\alpha) \) is 4, there are no big vertices arising from \( \alpha \), and this case is trivially resolved.

**Case 3.** \( \alpha \) is in 2-position, see Fig. 8.

Again we consider the face \( \omega \) and its descendants with respect to \( T_G \), and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from \( \alpha \)). Also, by definition of 2-position of face \( \alpha \), \( u \) and \( v \) are already colored. There can be at most one additional vertex, which is already colored and is at distance 6 to a big vertex arising from \( \alpha \) that is not yet colored; if this vertex exists, we denote it by \( r \), see Fig. 8.

\[\text{Figure 6: Case 3 (} \alpha \text{ is in 2-position).}\]

Similarly as in Case 1 (respecting the distance from \( v \) and using property (iii)), we find the possible positions of big vertices that arise from \( \alpha \), and depict them in Fig. 8. Note that some of the big vertices indicated in this figure may not exist in \( G \).

Since the vertices \( u \), \( v \) and \( r \) (see Figs. 3 and 6) are at pairwise mutual distance 4, exactly one of them has a very big color. Hence we consider three subcases, depending on which vertex among \( u \), \( v \) and \( r \) is very big. We can
assume, without loss of generality, that if the color of \( u, v \) or \( r \) is very big, then
it is 6; also, we may assume that \( f(v) = 4 \), if \( v \) is not very big. (Also, in the case
\( r \) does not exist and no vertex among \( u \) and \( v \) is very big, it possible to deal
with this situation by considering that \( r \) is very big.) The cases are described
in the following table:

| Subcase | a | b | c |
|---------|---|---|---|
| \( f(u) \) | 4 | 6 | 5 |
| \( f(v) \) | 6 | 4 | 4 |
| \( f(r) \) | 5 | 5 | 6 |

**Subcase 3.a.** The following two patterns define the coloring of the big
vertices (numbers between vertical bars are to be repeated \( k \) times, including
\( k = 0 \)).

Pattern for \( n = 4k + 8 \): \[
\begin{array}{cccc}
7 & 6 & 7 & 4 \\
5 & 4 & 5 & 5
\end{array}
\]

Pattern for \( n = 4k + 10 \): \[
\begin{array}{cccc}
7 & 6 & 5 & 5 \\
5 & 4 & 7 & 4
\end{array}
\]

Note that \( f(v) = 6 \) and \( f(u) = 4 \) enforce the last four numbers of the
patterns being different from 6, and the first two numbers different from 4 and
6.

For \( n = 6 \), there can be at most one big vertex arising from \( \alpha \), and we can
color it by 5. For \( n = 4 \) there are no big vertices arising from \( \alpha \).

**Subcase 3.b.** The following two patterns define the coloring of the big
vertices (numbers between vertical bars are to be repeated \( k \) times, including
\( k = 0 \)).

Pattern for \( n = 4k + 8 \): \[
\begin{array}{cccc}
4 & 5 & 7 & 6 \\
5 & 4 & 5 & 7
\end{array}
\]

Pattern for \( n = 4k + 10 \): \[
\begin{array}{cccc}
5 & 7 & 6 & 7 \\
4 & 5 & 4 & 5
\end{array}
\]

Note that \( f(u) = 6 \) and \( f(v) = 4 \) enforce the first four numbers of the
patterns being different from 6, and the last two numbers different from 4.

For \( n = 6 \), there can be at most one big vertex arising from \( \alpha \), and we can
color it by 5. For \( n = 4 \) there are no big vertices arising from \( \alpha \).

**Subcase 3.c.** Finally, the following two patterns define the coloring of the
big vertices (numbers between vertical bars are to be repeated \( k \) times, \( k = 0 
\) included).

Pattern for \( n = 4k + 6 \): \[
\begin{array}{cccc}
7 & 6 & 7 \\
4 & 5 & 7
\end{array}
\]
Pattern for \( n = 4k + 8 \):

\[
\begin{array}{cccc}
7 & 6 & 4 & 5 \\
4 & 5 & & 7 \\
\end{array}
\]

Note that \( f(r) = 6 \) implies that the patterns must not use color 6 in the first two values. Also, \( f(u) = 5 \) implies that the first two values in the patterns are not 5: the last two values are not 4 because of \( f(v) = 4 \).

For \( n = 4 \) there are no big vertices arising from \( \alpha \).

It is straightforward to see that in all of the above patterns property (iv) is obeyed, and that the resulting coloring \( f \) is a packing coloring that uses 7 colors. This completes the proof.

We complete this section by showing that the upper bound 7 for the packing coloring of 2-connected bipartite outerplanar graphs is sharp.

**Proposition 5.** There exists a 2-connected bipartite subcubic outerplanar graph \( G \) such that \( \chi_\rho(G) \geq 7 \).

**Proof.** Let \( T \) be the infinite binary tree. Sloper [21] has proven that \( \chi_\rho(T) = 7 \). A consequence is that there exists a finite subcubic tree \( T \) such that \( \chi_\rho(T) = 7 \).

Let \( d \) be the depth of \( T \). Finally, let \( k = 2d + 2 \).

Let \( x \) and \( y \) be two adjacent vertices of degree 2 in a graph \( G \). Adding a \( k \)-cycle on \( x \) and \( y \) is an operation that consists of adding a path of \( k - 2 \) vertices to \( G \) and joining one endvertex of the path to \( x \) and the other endvertex to \( y \). Let \( G_1 \) be a cycle of order \( k \). Let \( G_{i+1}, i \geq 1 \), be the graph obtained from \( G_i \) by adding a \( k \)-cycle on every adjacent pair of vertices of degree 2 in \( G_i \), where we arrange these pairs in such a way that each vertex of degree 2 belongs to one adjacent pair; this can be done by adding \( k \)-cycles on adjacent vertices of degree 2 following the outer cycle of \( G_i \). In this way, in \( G_{i+1} \) there does not remain any vertex of degree 2 from \( V(G_i) \). Note that, by construction, \( G_i \) is a 2-connected bipartite subcubic outerplanar graph for every integer \( i \geq 1 \).

Let \( u \) be a vertex that belongs to \( G_1 \) in the construction of \( G_k \). Note that the set \( \{ v \in V(G_k) \mid d(u, v) \leq d \} \) induces a subcubic tree containing \( T \) as an induced subgraph. Thus, since every graph has a packing chromatic number larger or equal than the packing chromatic number of any of its (induced) subgraphs, we derive \( \chi_\rho(G_k) \geq 7 \). 

\[ \square \]

4 \((1, 3, \ldots, 3)\)-packing coloring of bipartite outerplanar graphs

In this section, we need to extend the definition of \( T_G \) in order to have an underlying tree even if the outerplanar graph is not 2-connected. For an outerplanar graph \( G \), let \( D = \{ u \in V(G) \mid u \notin C(v), v \in V(T_G) \} \) and let \( A = V(G) \setminus D \). Note that the graph with vertex set \( V(T_G) \cup D \) and edge set \( E(T_G) \cup E(G[D]) \) is a forest. We construct \( L_G \) from the forest with vertex set \( V(T_G) \cup D \) and with edge set \( E(T_G) \cup E(G[D]) \) as follows. First, for each bridge \( uv \) of \( G \) such that \( u \in A \) and \( v \in D \), we add an edge to \( L_G \) between \( v \) and an arbitrary face
\(\alpha\) containing \(u\). Second, for each bridge \(uv\) of \(G\) such that \(u \in A\) and \(v \in A\), we add an edge to \(L_G\) between an arbitrary face \(\alpha\) containing \(u\) and an arbitrary face \(\beta\) containing \(v\). Third, let \(G'\) be the graph obtain from \(G\) by removing the bridges. For a cut vertex \(u\) of \(G'\), let \(B_1, \ldots, B_k\) be the 2-connected components of \(G'\) containing \(u\) and \(\alpha_i(u)\) be a face chosen arbitrarily among the faces from \(B_i\) containing \(u\), \(1 \leq i \leq k\). For each cut vertex \(u\) and each integer \(i\) between 2 and \(k\), we add an edge to \(L_G\) between \(\alpha_1(u)\) and \(\alpha_i(u)\).

It is easily seen that the graph \(L_G\) is a tree for any outerplanar graph \(G\).

![Figure 7: The graph \(L_G\) for an outerplanar graph \(G\) (circle: vertex of \(G\); square: vertex of \(T_G\); line: edge of \(G\); dashed line: edge of \(L_G\)).](image)

First, we prove Theorem 2, which states that a bipartite outerplanar graph with maximum degree \(\Delta\) bounded by \(k\) is \((1, 3, \ldots, 3)\)-colorable, where 3 appears \(k\) times in the sequence, \(k \geq 3\).

**Proof of Theorem 2**

In the construction of the \(S\)-packing coloring, the vertices of color 1 will form an independent set and the vertices of color \(a_i\), \(i \in \{1, \ldots, k\}\) will form a 3-packing. The proof is by induction on the order of \(L_G\). If \(G\) has more than one vertex, vertices colored by 1 will present one sets of the bipartition of \(G\). For the basis of induction, let \(L_G\) contain only one vertex \(u\). If \(u \in D\), then it suffices to color \(u\) with color 1. If \(u \in V(T_G)\), then we have to color the cycle \(C(u)\). We can clearly color it with colors 1, \(a_1\), \(a_2\) and \(a_3\) by (repeatedly) using the pattern 1, \(a_1\), 1, \(a_2\), and using the color 1, \(a_3\) for the two last vertices when the length of the cycle is not divisible by four.

Now, consider a graph \(G\) such that \(L_G\) has order \(n + 1\). Let \(u\) be a leaf of \(L_G\) and let \(G'\) be the following graph:

\[
G' = \begin{cases} 
G - u & \text{if } u \in D \\
G - B(u) & \text{if } u \in V(T_G); 
\end{cases}
\]

where \(B(u)\) is the set of the vertices of \(V(G) \setminus D\), which belong to \(C(u)\) and to no other inner face of \(G\). By induction, we can color the vertices of \(G'\), since \(L_{G'}\) has order \(n\), and it suffices to extend the coloring of \(G'\) to the uncolored vertices of \(G\). Let \(v\) be the neighbor of \(u\) in \(L_{G'}\).

**Case 1** \(u \in D\).
Let \( v' \) denote the neighbor of \( u \) in \( C(v) \) if \( v \in T_G \), and \( v' = v \) if \( v \in D \).
If \( v' \) is not colored by 1, then we can color \( u \) by 1 and the induction is finished. Otherwise, when \( v' \) is colored by color 1, then note that \( v' \) has at most \( k - 1 \) colored neighbors, whose neighbors are all colored by 1. Therefore, we can color \( u \) by a color \( a_i \), which is not given to vertices of \( N_G(v') \), since other vertices of color \( a_i \) are at distance at least 4 from \( u \).

Case 2: \( u \in V(T_G) \) and \( v \in D \).

Let \( u' \) be the neighbor of \( v \) in \( C(u) \). Firstly, if \( v \) is colored by 1, then, since \( \Delta(G) \leq k \), at most \( k - 1 \) neighbors of \( v \) are colored in \( G \), and we can color \( u' \) by a color \( a_i \) not used in other neighbors of \( v \). We can color the remaining uncolored vertices of \( C(u) \) by using the color 1 and three more colors (proceeding in the same way as in the coloring of a cycle described in the initial step of the induction). Secondly, if \( v \) is not colored by color 1, then we color \( u' \) with color 1. We can again extend the coloring to the remaining vertices of \( C(u) \) using color 1 and three more colors in an analogous way as in the initial step of the induction.

Case 3: \( u \in V(T_G) \) and \( v \in V(T_G) \).

Subcase 3.a \( |V(C(u)) \cap V(C(v))| = 0 \).

In this case, a vertex \( u' \in C(u) \) is adjacent to a vertex \( v' \in C(v) \). For the proof of this case, we follow the same steps as in Case 2, where the vertex \( v' \), defined in the previous sentence, plays the role of \( v \) in Case 2.

Subcase 3.b \( |V(C(u)) \cap V(C(v))| = 1 \).

Let \( \{ w \} = V(C(u)) \cap V(C(v)) \). Suppose \( w \) is colored by color 1. Since \( w \) is in the uncolored facial cycle \( C(u) \), at most \( k - 2 \) neighbors of \( w \) are colored so far in \( G \). Thus, we can give two (distinct) colors to the two neighbors of \( w \) in \( C(u) \). We can easily extend the coloring to the remaining uncolored vertices of \( C(u) \) using color 1 and three more colors. Now, if \( w \) is not colored with color 1, then every neighbor of \( w \) can be colored with color 1. It is again easy to color the remaining uncolored vertices of \( C(u) \) by using color 1 and three more colors, applying the pattern as in the basis of induction.

Subcase 3.c \( |V(C(u)) \cap V(C(v))| = 2 \).

Let \( \{ w_1, w_2 \} = V(C(u)) \cap V(C(v)) \). Since \( G \) is outerplanar, \( w_1 \) and \( w_2 \) are adjacent in \( G \). Without loss of generality suppose \( w_1 \) is colored by color 1. By induction hypothesis, \( w_2 \) is already colored. Let \( x \) be the neighbor of \( w_1 \) (which is not \( w_2 \)) in \( C(u) \). Since the vertex \( w_1 \) has at most \( k - 1 \) colored neighbors, we can color the vertex \( x \) by a color \( a_i \) that is not used in any of the neighbors of \( w_1 \). The remaining vertices of \( C(u) \) can be colored by using color 1 and three more colors in the same way as in the previous cases.
In the following proposition, we prove that Theorem 2 does not hold if in $S$ an integer 3 is replaced by an integer 4.

**Proposition 6.** There exists a bipartite outerplanar graph $G$ with $\Delta(G) \leq k$, which is not $S$-packing colorable for the list $S = (1, 3, \ldots, 3, 4)$ containing $k - 1$ times the integer 3.

**Proof.** Let $T$ be the complete $k$-ary tree of height 5 and root $r$, and suppose there exists an $S$-coloring $c$ of $T$, using $S = (1, 3, \ldots, 3, 4)$ as in the statement of the proposition. Note that there exists a vertex $x \in \{r\} \cup N(r)$ that is colored by 1. Since $x$ has $k$ neighbors, all must receive distinct colors, different from 1. In particular, there exists a neighbor $y$ of $x$ such that $c(y) = 4$. Let $z$ be any neighbor of $x$ different from $y$. Clearly, $z$ must be colored by one color '3', while all neighbors of $z$ are colored by 1. Finally, the neighbors of the vertices in $N(z)$ must receive all colors different from 1. In particular, there exists a vertex $u$ with $d(u, y) = 4$, such that $c(u) = 4$, which is a contradiction.

## 5 \(1, 2, 2, 2\)-packing coloring of subcubic outerplanar graphs

Recall that in this section we prove that a subcubic outerplanar graph $G$ is \(1, 2, 2, 2\)-packing colorable.

**Proof of Theorem 3.**

Let $G$ be a subcubic outerplanar graph and let $D$ and $L_G$ be defined as in Section 4. In this proof, the vertices of color 1 will form an independent set and the vertices of color $a_i, i \in \{1, 2, 3\}$ will form a 2-packing.

By induction on the order of $L_G$, we prove that there is an $S$-packing coloring of $G$. Suppose $L_G$ has only one vertex $u$. If $u \in D$, then it suffices to color $u$ with color 1. If $u \in V(T_G)$, then we have to color a cycle of order $n$. We color it with colors 1, $a_1$, $a_2$ and $a_3$ using the pattern 1, $a_1$, 1, $a_2$ and using the color $a_3$ for the last vertex if $n \equiv 1 \pmod{4}$, colors 1, $a_3$ for the two last vertices if $n \equiv 2 \pmod{4}$ or colors 1, $a_3$, $a_2$ for the three last vertices if $n \equiv 3 \pmod{4}$.

Now, consider a graph $G$ such that $L_G$ has order $n + 1$. Let $u$ be a leaf of $L_G$ and let $G'$ be the following graph:

$$G' = \begin{cases} G - u & \text{if } u \in D \\ G - B(u) & \text{if } u \in V(T_G) \end{cases}$$

where $B(u)$ is the set of the vertices of $V(G) \setminus D$, which belong to $C(u)$ and to no other inner face of $G$. By induction, we can color the vertices of $G'$, since $L_{G'}$ has order $n$, and it suffices to extend the coloring of $G'$ to the uncolored vertices of $G$. Let $v$ be the neighbor of $u$ in $L_{G'}$. 

Case 1 \( u \in D \).

Let \( v' \) denote the neighbor of \( u \) in \( C(v) \) if \( v \in T_G \) and \( v' = v \) if \( v \in D \). If \( v' \) is not colored by 1, then we can color \( u \) by 1 and the induction is finished. Otherwise (\( v' \) is colored by 1), since \( G \) is subcubic, \( v' \) has at most two colored neighbors. Therefore, we can color \( u \) by a color not given to vertices from \( N_G(v') \).

Case 2 \( u \in T_G \) and \( v \in D \).

Let \( u' \) be the neighbor of \( v \) in \( C(u) \). First, if \( v \) is colored by color 1, then, since \( \Delta(G) \leq 3 \), only at most two neighbors of \( v \) are colored in \( G \) and we can easily color \( u' \). We can color the remaining uncolored vertices of \( C(u) \) by the color 1 and the three remaining colors (by proceeding as in the coloring of a cycle described in the initialization step of the induction). Second, if \( v \) is not colored by color 1, then we color \( u' \) with color 1. Since \( u' \) is in \( C(u) \) which is uncolored, exactly one neighbor of \( u' \), namely \( v \), is colored in \( G \). Thus, we can give two different colors from \( \{a_1, a_2, a_3\} \) to the neighbor of \( u' \) in \( C(u) \). We now extend the coloring to the remaining vertices of \( C(u) \) using color 1 and the three remaining colors (by proceeding as in the coloring of a cycle described in the initialization step of the induction).

Case 3 \( u \in T_G \) and \( v \in T_G \).

Since \( G \) is subcubic, \( |V(C(u)) \cap V(C(v))| = 0 \) or \( |V(C(u)) \cap V(C(v))| = 2 \) (if \( |V(C(u)) \cap V(C(v))| = 1 \), then the common vertex should have degree at least 4).

Subcase 3.a \( |V(C(u)) \cap V(C(v))| = 0 \).

In this case, a vertex \( u' \in C(u) \) is adjacent to a vertex \( v' \in C(v) \). For the proof of this case, we follow the same steps as in Case 2, where the vertex \( v' \), defined in the previous sentence, plays the role of \( v \) in Case 2.

Subcase 3.b \( |V(C(u)) \cap V(C(v))| = 2 \).

Let \( \{w_1, w_2\} = V(C(u)) \cap V(C(v)) \). Since \( G \) is outerplanar, \( w_1 \) and \( w_2 \) are adjacent in \( G \). By induction hypothesis, \( w_1 \) and \( w_2 \) are already colored. Let \( x_1 \) be the neighbor of \( w_1 \) (which is not \( w_2 \)) in \( C(u) \) and let \( x_2 \) be the neighbor of \( w_2 \) (which is not \( w_1 \)) in \( C(u) \). If \( w_1 \) has no neighbor of color 1, then we recolor it with color 1. If after this, \( w_2 \) has no neighbor of color 1, then we recolor it with color 1. Now, it is easy to see that we can extend the coloring to \( C(u) \) in such a way that it is an \( S \)-packing coloring of \( G \), when \( |C(u)| \leq 5 \).

So, let \( |C(u)| > 5 \). We color the uncolored vertices of \( C(u) \) starting by coloring the vertex \( x_1 \). We color \( x_1 \) by using a color not given to \( w_1 \) and the colored neighbors of \( w_1 \). If \( w_2 \) is not colored by 1 we color \( x_2 \) by 1. Otherwise, we color \( x_2 \) by a color not given to \( w_2 \) and the colored neighbors of \( w_2 \). The remaining vertices of \( C(u) \) can be colored by the
pattern described in the initial step of the induction, alternating 1, $a_i$, 1, $a_j$, and eventually completing the coloring of the cycle with the third color $a_k$, depending on the length of $C(u)$.

Analogously as in the previous section, we prove that Theorem 3 does not hold if in $(1,2,2,2)$ an integer 2 is replaced by an integer 3.

**Proposition 7.** There exists a subcubic outerplanar graph, which is not $(1,2,2,3)$-packing colorable.

![Graphs $G$ and $G'$](image)

**Figure 8:** Graphs $G$ and $G'$ ($G'$ is not $(1,2,2,3)$-packing colorable).

**Proof.** Let $G$ be the graph depicted on the left-hand side of Fig. 8 and rounded by a rectangle, and let $c$ be a $(1,2,2,3)$-packing coloring of $G$, using colors 1, $2'$, $2''$ and 3, the meaning of which should be clear. Note that at most two vertices of $G$ can receive color 1 (because $G$ consists of two triangles) and at most one vertex of $G$ can receive color 3 (because diam$(G)$ = 3). This implies that two vertices of $G$ must receive the color of a 2-packing, and we may assume without loss of generality that two vertices of $G$ receive color $2'$.

Since only $w$ and $z$ are at distance 3 among all pairs of vertices in $G$, we infer that $c(w) = 2' = c(z)$. Now, consider the graph $G'$, obtained from two copies of $G$ by adding an edge between vertices of degree 2, one from each copy of $G$, see Fig. 8. By viewing at $G$ as a subgraph of $G'$, already colored as described above, we derive that in the only possible $(1,2,2,3)$-packing coloring of the vertices of $V(G') \setminus V(G)$, color $2''$ must be given to the vertices $z'$ and $w'$. Now, since not both vertices $x'$ and $y'$ can receive color 1, one of these two vertices, say $x'$, receives color 3 or color $2'$. Clearly, $c(x') \neq 2'$, because $d(x', z) = 2$ and $c(z) = 2'$. Finally, if $c(x') = 3$, this implies that $\{c(x), c(y)\} = \{1, 2''\}$, but this contradicts $c(z') = 2''$, since $z'$ is at distance 2 from both $x$ and $y$. This contradiction implies that $G'$ is not $(1,2,2,3)$-packing colorable, and it is clear that $G'$ is a subcubic outerplanar graph. 

□
6 Concluding remarks

Theorem 1 gives a partial (affirmative) answer to the question posed in several papers concerning the boundedness of the packing chromatic number in the class of planar subcubic graphs. Instead of repeating the question, we propose two problems that lie between Theorem 1 and this question. In one of them, we consider non-bipartite extension of the theorem, and in the other we replace outerplanar graphs by planar graphs.

**Question 1.** Is the packing chromatic number bounded in the class of 2-connected outerplanar subcubic graphs?

**Question 2.** Is the packing chromatic number bounded in the class of 2-connected bipartite planar subcubic graphs?

While we do not dare to suggest what is the answer to the above questions, we strongly believe that Theorem 1 could be extended from the 2-connected case to all bipartite outerplanar subcubic graphs.

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