New inequalities for $F$-convex functions pertaining generalized fractional integrals

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Abstract. In this paper, the authors, utilizing $F$-convex functions which are defined by B. Samet, establish some new Hermite-Hadamard type inequalities via generalized fractional integrals. Some special cases of our main results recaptured the well-known earlier works.

1. Introduction

Let $f : I \subseteq R \rightarrow R$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. If $f$ is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [17]:

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1}$$

Both inequalities in (1) hold in the reversed direction if $f$ is concave.

Over the last decade, this classical double inequality has been improved and generalized in a number of ways, see [5, 7, 8, 13, 18], [23]–[25] and the references therein. Also, many types of convexities have been defined, such as quasi–convex in [6], pseudo–convex in [14], strongly convex in [20], $\varepsilon$–convex in [11], $s$–convex in [10], $h$–convex in [28], etc. Recently, Samet in [21], has defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ satisfying the following axioms:

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(A1) If \( e_i \in L^1(0,1), i = 1, 2, 3 \), then for every \( \lambda \in [0,1] \), we have
\[
\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda) dt = F \left( \int_0^1 e_1(t) dt, \int_0^1 e_2(t) dt, \int_0^1 e_3(t) dt, \lambda \right);
\]

(A2) For every \( u \in L^1(0,1), w \in L^\infty(0,1) \) and \( (z_1, z_2) \in \mathbb{R}^2 \), we have
\[
\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t) dt = T_{F,w} \left( \int_0^1 w(t)u(t) dt, z_1, z_2 \right),
\]
where \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a function that depends on \( (F, w) \), and it is nondecreasing with respect to the first variable;

(A3) For any \( (w, e_1, e_2, e_3) \in \mathbb{R}^4, e_4 \in [0,1] \), we have
\[
w F(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,
\]
where \( L_w \in \mathbb{R} \) is a constant that depends only on \( w \).

**Definition 1.** Let \( f : [a, b] \to \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b \), be a given function. We say that \( f \) is a convex function with respect to some \( F \in \mathcal{F} \) (or \( F \)-convex function), if and only if:
\[
F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a, b] \times [a, b] \times [0,1].
\]

**Remark 1.** 1) Let \( \varepsilon \geq 0 \), and let \( f : [a, b] \to \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b \), be an \( \varepsilon \)-convex function, see [11], that is
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad (x, y, t) \in [a, b] \times [a, b] \times [0,1].
\]
Define the functions \( F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R} \) by
\[
(2) \quad F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1-e_4) e_3 - \varepsilon
\]
and \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
(3) \quad T_{F,w}(e_1, e_2, e_3) = e_1 - \left( \int_0^1 tw(t) dt \right) e_2 - \left( \int_0^1 (1-t)w(t) dt \right) e_3 - \varepsilon.
\]
For
\[
(4) \quad L_w = (1-w)\varepsilon,
\]
it is clear that \( F \in \mathcal{F} \) and
\[
F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,
\]
that is \( f \) is an \( F \)-convex function. Particularly, taking \( \varepsilon = 0 \), we show that if \( f \) is a convex function then \( f \) is an \( F \)-convex function with respect to \( F \) defined above.
2) Let $h : J \rightarrow [0, +\infty)$ be a given function which is not identical to $0$, where $J$ is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \rightarrow [0, +\infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an $h$-convex function, see [28], that is
\[
 f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]
Define the functions $F : \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by
\[
 (5) \quad F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3
\]
and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by
\[
 (6) \quad T_{F,w}(e_1, e_2, e_3) = e_1 - \left( \int_0^1 h(t)w(t)dt \right) e_2 - \left( \int_0^1 h(1-t)w(t)dt \right) e_3.
\]
For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and
\[
 F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,
\]
that is, $f$ is an $F$–convex function.

Samet in [21], established the following Hermite–Hadamard type inequalities using the new convexity concept:

**Theorem 1.** Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an $F$-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a, b]$. Then
\[
 F\left( f\left( \frac{a+b}{2} \right) , \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{2} \right) \leq 0,
\]
\[
 T_{F,1}\left( \frac{1}{b-a} \int_a^b f(x)dx, f(a), f(b) \right) \leq 0.
\]

**Definition 2.** Let $f \in L^1[a, b]$. The Riemann–Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ are defined by
\[
 J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]
and
\[
 J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,
\]
respectively. Here, $\Gamma(\alpha)$ is the Gamma function and
\[
 J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).
\]

**Definition 3.** Let $f \in L^1[a, b]$. Then $k$–fractional integrals of order $\alpha$, $k > 0$ are defined by
\[
 I^\alpha_{a+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\alpha-k-1} f(t)dt, \quad x > a,
\]
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and

$$I_{b^+}^\alpha, k f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma_k(\cdot)$ stands for the $k$-gamma function. For $k = 1$, the $k$-fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the $k$-fractional integrals yield classical integrals. For more details, see [9, 12, 15, 19].

It is remarkable that Sarikaya et al. in [26], first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

**Theorem 2.** Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2},$$

with $\alpha > 0$.

Budak et al. in [1], prove the following Hermite-Haddamard type inequalities for $F$-convex functions via fractional integrals:

**Theorem 3.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on $I^0$, $a, b \in I^0$, $a < b$. If $f$ is $F$-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have

$$F\left( f \left( \frac{a+b}{2} \right), \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha} J_{a^+}^\alpha f(b), \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha} J_{b^-}^\alpha f(a), \frac{1}{2} \right)$$

$$+ \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F, w} \left( \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)], f(a) + f(b), f(a) + f(b) \right)$$

$$+ \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \alpha t^{\alpha - 1}$.

For other papers involving $F$-convex functions, see [1]-[4], [16, 27].

Now we summarize the generalized fractional integrals defined by Sarikaya and Erteğrul in [22].

Let’s define a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty,$$
\[
\frac{1}{A_1} \leq \frac{\varphi(v)}{\phi(u)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2,
\]
(12)

\[
\frac{\varphi(u)}{u^2} \leq A_2 \frac{\varphi(v)}{v^2} \quad \text{for} \quad v \leq u,
\]
(13)

\[
\left| \frac{\varphi(u)}{u^2} - \frac{\varphi(v)}{v^2} \right| \leq A_3 |u - v| \frac{\varphi(u)}{u^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2,
\]
(14)

where \( A_1, A_2, A_3 > 0 \) are independent of \( u, v > 0 \). If \( \varphi(u)u^\alpha \) is increasing for some \( \alpha \geq 0 \) and \( \frac{\varphi(u)}{u^\beta} \) is decreasing for some \( \beta \geq 0 \), then \( \varphi \) satisfies the above conditions.

The following left-sided and right–sided generalized fractional integral operators are defined respectively, as follows:

\[
a^{+}_I \varphi f(x) = \int_a^x \frac{\varphi(x - t)f(t)dt}{x - t}, \quad x > a,
\]
(15)

\[
b^{-}_I \varphi f(x) = \int_x^b \frac{\varphi(t - x)f(t)dt}{t - x}, \quad x < b.
\]
(16)

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, \( k \)–Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarikaya and Ertuğral in [22], establish the following Hermite–Hadamard inequality and lemmas for the generalized fractional integral operators:

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\) with \( a < b \), then the following inequalities for fractional integral operators hold:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2\Psi(1)} \left[ a^{+}_I \varphi f(b) + b^{-}_I \varphi f(a) \right] \leq \frac{f(a) + f(b)}{2},
\]
(17)

where the mapping \( \Lambda : [0, 1] \to \mathbb{R} \) is defined by

\[
\Psi(x) = \int_0^x \frac{\varphi((b - a)t)}{t} dt.
\]

Budak et al. prove the following Hermite Hadamard type inequalities for \( F \)-convex functions.

**Theorem 5** ([4]). Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^c \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping on \( I^c \), \( a, b \in I^c \), \( a < b \). If \( f \) is \( F \)-convex on \([a, b]\) for some \( F \in \mathcal{F} \), then we have

\[
F\left(f\left(\frac{a + b}{2}\right), \frac{1}{\Psi(1)} a^{+}_I \varphi f(b), \frac{1}{\Psi(1)} b^{-}_I \varphi f(a), \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0,
\]
and
\[ T_{F,w} \left( \frac{1}{\Phi(1)} \left[ a + I_{\varphi} f(b) + (a + b) I_{\varphi} f(a) \right], f(a) + f(b), f(a) + f(b) \right) \]
\[ \quad + \int_{0}^{1} L_{w(t)} dt \leq 0, \]
where \( w(t) = \frac{\varphi((b-a)t)}{t\Phi(1)} \).

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities via generalized fractional integrals utilizing \( F \)-convex functions. Some special cases of our main results recaptured the well–known earlier works. At the end, a briefly conclusion will be given as well.

2. Main results

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via \( F \)-convex functions.

**Theorem 6.** Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^* \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping on \( I^* \), \( a, b \in I^* \), \( a < b \) and let \( F \) be linear with respect to the first three variables. If \( f \) is \( F \)-convex on \([a, b] \) for some \( F \in \mathcal{F} \), then we have

\[ F \left( f \left( \frac{a+b}{2} \right), \frac{1}{\Lambda(1)} \left( \frac{a+b}{2} \right) + I_{\varphi} f \left( b \right), \frac{1}{\Lambda(1)} \left( \frac{a+b}{2} \right) - I_{\varphi} f \left( a \right), \frac{1}{2} \right) \]
\[ + \int_{0}^{1} L_{w(t)} dt \leq 0, \]

and

\[ T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ \left( \frac{a+b}{2} \right) + I_{\varphi} f \left( b \right) + \left( \frac{a+b}{2} \right) - I_{\varphi} f \left( a \right) \right] \right), \]
\[ f(a) + f(b), f(a) + f(b) \]
\[ + \int_{0}^{1} L_{w(t)} dt \leq 0, \]

where \( w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)} \) and the function \( \Lambda : [0, 1] \to \mathbb{R} \) is defined by

\[ \Lambda(x) = \int_{0}^{x} \frac{\varphi \left( \frac{b-a}{2} t \right)}{t} dt. \]

**Proof.** Since \( f \) is \( F \)-convex, we have

\[ F \left( f \left( \frac{x+y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b]. \]
For
\[ x = \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \quad \text{and} \quad y = \left( \frac{2-t}{2} \right) a + \frac{t}{2} b, \]
we have
\[ F \left( f \left( \frac{a+b}{2} \right), f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right), f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b, \frac{1}{2} \right) \right) \leq 0. \]
for all \( t \in [0,1] \). Multiplying this inequality by \( w(t) = \frac{\varphi((b-a) t)}{t \Lambda(1)} \) and using axiom (A3), we get
\[ F \left( \varphi \left( \frac{(b-a) t}{t \Lambda(1)} \right) f \left( \frac{a+b}{2} \right), \varphi \left( \frac{(b-a) t}{t \Lambda(1)} \right) f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right), \right) \]
\[ \varphi \left( \frac{(b-a) t}{t \Lambda(1)} \right) f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b, \frac{1}{2} \right) + L w(t) \leq 0 \]
for all \( t \in (0,1) \). Integrating over \((0,1)\) with respect to the variable \( t \) and using axiom (A1), we obtain
\[ F \left( \frac{f \left( \frac{a+b}{2} \right)}{\Lambda(1)} \int_0^1 \varphi \left( \frac{(b-a) t}{t} \right) \frac{1}{\Lambda(1)} \int_0^1 \varphi \left( \frac{(b-a) t}{t} \right) f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right) dt, \right) \]
\[ \frac{1}{\Lambda(1)} \int_0^1 \varphi \left( \frac{(b-a) t}{t} \right) f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b, \frac{1}{2} \right) dt + \int_0^1 L w(t) dt \leq 0. \]
Using the facts that
\[ \int_0^1 \varphi \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right) dt = \int_{\frac{a+b}{2}}^{b} \varphi \left( \frac{b-x}{b-x} \right) f(x) dx = (\frac{a+b}{2})+I_\varphi f (b) \]
and
\[ \int_0^1 \varphi \left( \frac{(2-t)}{t} a + \frac{t}{2} b \right) dt = \int_{\frac{a+b}{2}}^{\frac{a+b}{2}} \varphi \left( \frac{x-a}{x-a} \right) f(x) dx = (\frac{a+b}{2})-I_\varphi f (a) , \]
we obtain
\[ F \left( f \left( \frac{a+b}{2} \right), \frac{1}{\Lambda(1)} (\frac{a+b}{2})+I_\varphi f (b), \frac{1}{\Lambda(1)} (\frac{a+b}{2})-I_\varphi f (a), \frac{1}{2} \right) \]
\[ + \int_0^1 L w(t) dt \leq 0, \]
which gives (18).

On the other hand, since $f$ is $F$–convex, we have

$$F\left( f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0, 1],$$

and

$$F\left( f\left(\frac{2-t}{2}a + \frac{t}{2}b\right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0, 1].$$

Using the linearity of $F$, we get

$$F\left( f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right), f(a) + f(b), f(a) + f(b), t \right) \leq 0,$$

for all $t \in [0, 1]$. Applying the axiom (A3) for $w(t) = \frac{\varphi\left(\frac{(b-a)t}{2}\right)}{t\Lambda(1)}$, we obtain

$$F\left( \varphi\left(\frac{(b-a)t}{2}\right) \right) \left[ f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right],$$

$$\varphi\left(\frac{(b-a)t}{2}\right) \frac{\varphi\left(\frac{(b-a)t}{2}\right)}{t\Lambda(1)} [f(a) + f(b)], \frac{\varphi\left(\frac{(b-a)t}{2}\right)}{t\Lambda(1)} [f(a) + f(b), t] + L_{w(t)} \leq 0,$$

for all $t \in (0, 1)$. Integrating over $(0, 1)$ and using axiom (A2), we have

$$T_{F,w}\left( \int_0^1 \varphi\left(\frac{(b-a)t}{2}\right) \left[ f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt,\right.$$

$$f(a) + f(b), f(a) + f(b) + \int_0^1 L_{w(t)} dt \leq 0,$$

that is

$$T_{F,w}\left( \frac{1}{\Lambda(1)} \left[ (a+b)^{\frac{1}{2}} I_{\varphi} f(b) + (a+b)^{-\frac{1}{2}} I_{\varphi} f(a) \right], f(a) + f(b), f(a) + f(b) \right)$$

$$+ \int_0^1 L_{w(t)} dt \leq 0.$$

The proof of Theorem 6 is completed. □
Remark 2. If we choose $\phi(t) = t$ in Theorem 6, then we have the following inequalities

\[
F \left( f \left( \frac{a+b}{2} \right), \frac{2}{b-a} \int_a^b f(t) dt, \frac{2}{b-a} \int_a^b \frac{a+b}{2} f(t) dt \right) \times \frac{1}{1} 
\]
(20)

\[
+ \int_0^1 L_w(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{2}{b-a} \int_a^b f(t) dt, f(a), f(b), f(b), f(b) \right) + \int_0^1 L_w(t) dt \leq 0,
\]

where $w(t) = 1$.

Remark 3. If we choose $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 6, then we have the following inequalities for Riemann-Liouville fractional integrals

\[
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_\alpha^{\alpha+1} f(b), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_\alpha^{\alpha+1} f(a) \right) \times \frac{1}{1} 
\]
(21)

\[
+ \int_0^1 L_w(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_\alpha^{\alpha+1} f(b) + J_\alpha^{\alpha+1} f(a) \right], f(a), f(b), f(b), f(b) \right) + \int_0^1 L_w(t) dt \leq 0,
\]

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

Corollary 1. If we take $\phi(t) = \frac{t^\frac{\alpha}{k \Gamma(k)}}{k \Gamma(k)}$ in Theorem 6, then we have the following inequalities for $k$–Riemann–Liouville fractional integrals

\[
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma_k(\alpha+k)}{(b-a)^\alpha} I_\alpha^{\alpha+1} f(b), \frac{2^\alpha \Gamma_k(\alpha+k)}{(b-a)^\alpha} I_\alpha^{\alpha+1} f(a) \right) \times \frac{1}{1} 
\]

\[
+ \int_0^1 L_w(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{2^\alpha \Gamma_k(\alpha+k)}{(b-a)^\alpha} \left[ I_\alpha^{\alpha+1} f(b) + I_\alpha^{\alpha+1} f(a) \right], f(a), f(b), f(b), f(b) \right) + \int_0^1 L_w(t) dt \leq 0,
\]

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].
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$$f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) dt \leq 0,$$

where $w(t) = \frac{a}{t} t^{\frac{n}{k} - 1}$.

**Theorem 7.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping on $I^\circ$, $a, b \in I^\circ$, $a < b$ and let $F$ be linear with respect to the first three variables. If $f$ is $F$-convex on $[a, b]$ for some $F \in F$, then we have

$$F \left( f\left( \frac{a + b}{2} \right), \frac{1}{\Lambda(1)} b - I_\varphi f \left( \frac{a + b}{2} \right), \right),$$

(22)

$$\frac{1}{\Lambda(1)} a + I_\varphi f \left( \frac{a + b}{2} \right) + \int_0^1 L_w(t) dt \leq 0,$$

and

$$T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ a + I_\varphi f \left( \frac{a + b}{2} \right) + b - I_\varphi f \left( \frac{a + b}{2} \right) \right], \right),$$

(23)

$$f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) dt \leq 0,$$

where $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$.

*Kanıt.* Since $f$ is $F$-convex, we have

$$F \left( f\left( \frac{x + y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b].$$

For

$$x = \left( \frac{1 - t}{2} \right) a + \left( \frac{1 + t}{2} \right) b \quad \text{and} \quad y = \left( \frac{1 + t}{2} \right) a + \left( \frac{1 - t}{2} \right) b,$$

we have

$$F \left( f \left( \frac{a + b}{2} \right), f \left( \left( \frac{1 - t}{2} \right) a + \left( \frac{1 + t}{2} \right) b \right), \right),$$

$$f \left( \left( \frac{1 + t}{2} \right) a + \left( \frac{1 - t}{2} \right) b \right) \leq 0,$$

for all $t \in [0, 1]$. Multiplying this inequality by $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$ and using axiom (A3), we get

$$F \left( \frac{\varphi \left( \frac{b-a}{2} \right) t}{t\Lambda(1)} f \left( \frac{a + b}{2} \right), \frac{\varphi \left( \frac{b-a}{2} \right) t}{t\Lambda(1)} f \left( \left( \frac{1 - t}{2} \right) a + \left( \frac{1 + t}{2} \right) b \right), \right),$$

$$\frac{\varphi \left( \frac{b-a}{2} \right) t}{t\Lambda(1)} f \left( \left( \frac{1 + t}{2} \right) a + \left( \frac{1 - t}{2} \right) b \right) \frac{1}{2} + L_w(t) \leq 0,$$
for all $t \in (0,1)$. Integrating over $(0,1)$ with respect to the variable $t$ and using axiom (A1), we obtain

$$
F \left( \int_0^1 \varphi \left( \frac{(b-a)t}{2} \right) \frac{t}{2} \, dt \right),
$$

$$
\frac{1}{\Lambda(1)} \int_0^1 \varphi \left( \frac{(b-a)t}{2} \right) f \left( \left( \frac{1 \mp t}{2} \right) a + \left( \frac{1 \mp t}{2} \right) b \right) \, dt,
$$

$$
\frac{1}{\Lambda(1)} \int_0^1 \varphi \left( \frac{(b-a)t}{2} \right) f \left( \left( \frac{1}{2} + \frac{1}{2} t \right) a + \left( \frac{1}{2} - \frac{1}{2} t \right) b \right) \, dt, \frac{1}{2}
$$

$$
+ \int_0^1 L_{w(t)} \, dt \leq 0.
$$

Using the facts that

$$
\int_0^1 \varphi \left( \frac{(b-a)t}{2} \right) \frac{t}{2} \, dt = \int_{a+b}^{b} \varphi \left( \frac{x-a+b}{2} \right) \frac{x-a+b}{2} \, dx
$$

$$
= b - I_\varphi f \left( \frac{a+b}{2} \right),
$$

and

$$
\int_0^1 \varphi \left( \frac{(b-a)t}{2} \right) \frac{t}{2} \, dt = \int_{a}^{a+b} \varphi \left( \frac{a+b}{2} - x \right) \frac{a+b}{2} - x \, dx
$$

$$
= a + I_\varphi f \left( \frac{a+b}{2} \right),
$$

we obtain

$$
F \left( f \left( \frac{a+b}{2} \right), \frac{1}{\Lambda(1)} b - I_\varphi f \left( \frac{a+b}{2} \right), \frac{1}{\Lambda(1)} a + I_\varphi f \left( \frac{a+b}{2} \right), \frac{1}{2} \right)
$$

$$
+ \int_0^1 L_{w(t)} \, dt \leq 0,
$$

which gives (22).

On the other hand, since $f$ is $F$–convex, we have

$$
F \left( \left( \frac{1 \mp t}{2} \right) a + \left( \frac{1 \mp t}{2} \right) b \right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0,1],
$$
and
\[ F \left( f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0, 1]. \]

Using the linearity of \( F \), we get
\[ F \left( f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right), f(a) + f(b), f(a) + f(b), t \right) \leq 0, \quad \forall t \in [0, 1]. \]

Applying the axiom (A3) for \( w(t) = \varphi \left( \frac{(b-a) t}{2} \right) t \Lambda(1) \), we obtain
\[ F \left( \varphi \left( \frac{(b-a) t}{2} \right) t \Lambda(1) \times \left[ f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right], \varphi \left( \frac{(b-a) t}{2} \right) t \Lambda(1) \right] \left[ f(a) + f(b) \right], t \right) + L_w(t) \leq 0, \]
for all \( t \in (0, 1) \). Integrating over \((0, 1)\) and using axiom (A2), we have
\[ T_{F,w} \left( \int_0^1 \varphi \left( \frac{(b-a) t}{2} \right) t \Lambda(1) \times \left[ f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right] dt, f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_w(t) dt \leq 0, \]
that is
\[ T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ a + I_{\varphi} f \left( \frac{a+b}{2} \right) + b - I_{\varphi} f \left( \frac{a+b}{2} \right) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_w(t) dt \leq 0. \]

The proof of Theorem 7 is completed. \( \square \)

**Remark 4.** If we take \( \varphi(t) = t \) in Theorem 7, then the inequalities (22) and (23) reduce to the inequalities (20) and (21).
Remark 5. If we take \( \varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)} \) in Theorem 7, then we have the following inequalities for Riemann-Liouville fractional integrals

\[
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_b^\alpha f \left( \frac{a+b}{2} \right), \right.
\]

\[
\frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_a^\alpha f \left( \frac{a+b}{2}, \frac{1}{2} \right) + \int_0^1 Lw(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_b^\alpha f \left( \frac{a+b}{2} \right) + J_a^\alpha f \left( \frac{a+b}{2} \right) \right], \right.
\]

\[
f(a) + f(b), f(a) + f(b) + \int_0^1 Lw(t) dt \leq 0,
\]

where \( w(t) = \alpha t^{\alpha-1} \) which is given by Budak et al. in [5].

Corollary 2. If we take \( \varphi(t) = \frac{t^\frac{\alpha}{k}}{\Gamma(\alpha)} \) in Theorem 7, then we have the following inequalities for \( k \)-Riemann–Liouville fractional integrals:

\[
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\frac{\alpha}{k} \Gamma_k(\alpha+k)}{(b-a)^\frac{\alpha}{k}} I_{b^-}^\alpha, \frac{1}{2} \right), \right.
\]

\[
\frac{2^\frac{\alpha}{k} \Gamma_k(\alpha+k)}{(b-a)^\frac{\alpha}{k}} I_{a^+}^\alpha, \frac{1}{2} + \int_0^1 Lw(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{2^\frac{\alpha}{k} \Gamma_k(\alpha+k)}{(b-a)^\frac{\alpha}{k}} \left[ I_{b^-}^\alpha + I_{a^+}^\alpha \right], \right.
\]

\[
f(a) + f(b), f(a) + f(b) + \int_0^1 Lw(t) dt \leq 0,
\]

where \( w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1} \).

Remark 6. One can obtain several results for convexity, \( \varepsilon \)-convexity, \( h \)-convexity, etc by special choice of the function \( F \) in Theorems 6 and 7.

3. Conclusion

In the development of this work, using the definition of \( F \)-convex functions some new Hermite-Hadamard type inequalities via generalized fractional integrals have been deduced. We also give several results capturing Riemann-Liouville fractional integrals and \( k \)-Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area.
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