EVEN WALKS AND ESTIMATES OF HIGH MOMENTS
OF LARGE WIGNER RANDOM MATRICES

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Abstract

We revisit the problem of estimates of the moments $m_{2s}^{(n)} = \mathbb{E}\{\text{Tr} A_n^{2s}\}$ of random $n \times n$ matrices of the Wigner ensemble by using the approach elaborated by Ya. Sinai and A. Soshnikov and further developed by A. Ruzmaikina. We continue to investigate the structure of closed even walks $w_{2s}$ and their graphs $g(w_{2s})$ that arise in these studies. One of the key problems here is related with the graphs $g(w_{2s})$ that have at least one vertex $\beta$ that is the tail of a large number of edges. This situation occurs when the corresponding Dyck path (or equivalently, the plane rooted tree) has a vertex of large degree; in the opposite case this can happen when the self-intersection degree of $\beta$ is large. We show that there exists one more possibility; it is given by the case when $w_{2s}$ has a large number of open instants of self-intersections, or more precisely, a large total number of the instants of broken tree structure. Basing on this observation, we modify the technique mentioned above and prove the estimates of the moments $m_{2s_n}^{(n)}$ in the limit $s_n, n \to \infty$ when $s_n = O(n^{2/3})$.

1 Introduction

Random matrices of infinite dimensions represent a very rich and interesting subject of studies that relates various branches of mathematical physics, analysis, combinatorics and many others.

The spectral theory of large random matrices started half a century ago by E. Wigner is still a source of interesting and challenging problems. An important part of these problems concerns the universality conjecture for the local spectral properties of ensembles of large real symmetric (or hermitian) matrices (see e.g. the monograph [8]).

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One of the examples of such ensembles is given by the Wigner ensemble of \( n \times n \) real symmetric random matrices of the form

\[
(A_n)_{ij} = \frac{1}{\sqrt{n}} a_{ij},
\]

where \( \{a_{ij}, 1 \leq i \leq j \leq n\} \) are jointly independent random variables such that the following conditions are verified

\[
E\{a_{ij}\} = 0, \quad \text{and} \quad E\{a_{ij}^2\} = 1,
\]

where \( E\{\cdot\} \) denotes the mathematical expectation. E. Wigner proved that the normalized moments of \( A_n \) converge in the limit \( n \to \infty \)

\[
\lim_{n \to \infty} \frac{1}{n} E\{\text{Tr} A_n^p\} = \begin{cases} 
\frac{(2s)!}{s!(s+1)!}, & \text{if } p = 2s, \\
0, & \text{if } p = 2s + 1
\end{cases}
\]

under conditions that all moments of all \( a_{ij} \) exist and the probability distribution of random variables \( a_{ij} \) is symmetric [14, 15].

To study the moments (1.3), E. Wigner has interpreted the trace of the product

\[
E\{\text{Tr} A_n^p\} = \sum_{i_0, i_1, \ldots, i_p = 1}^n E\{A_{i_0,i_1} \cdots A_{i_p,i_0}\}
\]

as the weighted sum over all possible sequences \( I_p = (i_0, i_1, \ldots, i_{p-1}, i_0) \) [14]. The set of these sequences can be separated into classes of equivalence that in the case of even \( p = 2s \) can be labelled by simple non-negative walks of 2s steps \( \theta_{2s} \) that start and end at zero. These walks are known as the Dyck paths and the Catalan number \( C(s) = (2s)!/s!(s+1)! \) standing in (1.3) represents the number of all Dyck paths with 2s steps. Obviously, \( E\{\text{Tr} A_n^{2s+1}\} = 0 \) due to the symmetric distribution of random variables \( a_{ij} \) and it is sufficient to consider the even moments \( m_{2s}^{(n)} = E\{\text{Tr} A_n^{2s}\} \) only.

The method proposed by E. Wigner has been used as the starting point in the studies of the asymptotic behavior of variables \( m_{2s}^{(n)} \) in the limit when \( s \) goes to infinity at the same time as \( n \) does [1, 3, 4]. In particular, it was shown in the beginning of 80-s that the estimate

\[
m_{2s}^{(n)} \leq \frac{n(2s)!}{s!(s+1)!} (1 + o(1)) \quad \text{as} \quad s, n \to \infty
\]

is true when \( s = s_n = o(n^{1/6}) \) as \( n \to \infty \) provided random variables \( a_{ij} \) are bounded with probability 1 and their probability distribution is symmetric [3]. The method of [3] uses certain encoding of the paths \( I_{2s} \) additionally to the Dyck paths representation.
More precisely, the graph theory approach has been developed in [3], where, in particular, the Dyck paths $\theta_{2s}$ are considered as the canonical runs over the plane rooted trees $T_s = T(\theta_{2s})$ of $s + 1$ vertices.

Regarding the problem of estimates of high moments of large random matrices, it is important to determine the maximally possible rate of $s_n$ such that (1.5) is still valid. Here a breakthrough step was made in paper [10], where (1.5) was shown to be true for all $s_n = o(n^{1/2})$, $n \to \infty$ in the case when the moments of random variables $a_{ij}$ are of the sub-Gaussian form and the probability distribution of $a_{ij}$ is symmetric. Also the Central Limit Theorem for the random variable $4^{-s} \left( m_{2s}^{(n)} - E\{m_{2s}^{(n)}\} \right)$ was proved in this limit. It was shown that the limiting expressions for the corresponding correlation functions do not depend on the particular values of the moments of $a_{ij}$. This was a first step toward the proof of the universality conjecture for the Wigner ensembles. It should be stressed that in these studies the estimate (1.5) represents a key result that is crucial for the proof of the Central Limit Theorem.

The proof of [10] is based on the Wigner’s approach added by an important notion of the self-intersection of the sequence $I_{2s}$; in [10] these sequences are called the paths of $2s$ steps. In certain sense, the self-intersections of the path from the class $\theta_{2s}$ correspond to gluing of the vertices of the corresponding tree $T(\theta_{2s})$. Then the self-intersection degree of a vertex can be determined as a number of arrivals to this vertex by the edges of this tree. In [10], the set of all paths has been separated into the classes of equivalence according to the number of vertices of self-intersections and the self-intersection degrees of these vertices. This gives the tools to control the number of paths in the limit $s, n \to \infty$.

This method was modified in the subsequent paper [11] to prove that (1.5) is valid in the limit $s_n, n \to \infty$, $s_n = o(n^{2/3})$. Here the family of self-intersections has been further classified and the notion of the open vertex of two-fold self-intersection has been considered for the first time. The next in turn step has been made in [12] to show that an estimate of the form (1.5) is true in the limit $n \to \infty$, $s_n = O(n^{2/3})$ under the same conditions on the probability distribution of $a_{ij}$ as in [10].

The technique of [10, 11, 12] was further developed in paper [9], where the case of arbitrary distributed random variables $a_{ij}$ with symmetric law of polynomial decay was considered. It was indicated in [9] that certain estimate of [12] was not established in the full extent and a way was proposed to complete the proof. However, the way proposed in [9] is partly correct and the proof of corresponding estimate suffers in its own turn from a serious gap.

Let us briefly explain the problem. Following [11, 12], it was assumed in [9] that for the paths with typical $\theta_{2s}$ the presence of vertices with large number of steps ”out” is possible only when there is a sufficiently large number of steps ”in”; here the steps ”in” and ”out” are meant to be the steps that correspond to the ascending steps of $\theta_{2s}$. 
However, this is not the case. The walk can arrive at a given vertex $\beta$ by the steps that correspond to the descend parts of the Dyck path $\theta_{2s}$ and bring to $\beta$ the edges that do not belong the ascending steps "in". We call these edges the imported ones and refer to the corresponding arriving steps as to the imported cells. At the end of this paper, we present examples of paths with a number of imported cells and edges.

We see that the description of the family of paths $\{I_{2s}\}$ of (1.4) should be improved. This is the main subject of the present work. We will see that the studies require a number of generalizations of the notions introduced in [11]. In particular, the notion of the open vertex of two-fold self-intersection should be generalized for the case of self-intersections of any degree. Also the further analysis of the open instants of self-intersections leads us to the notion of the instant of broken tree structure. It plays even more important role with respect to the estimates of the moments $m_{2s}^{(n)}$ than that played by the open vertices of self-intersections. Clearly, these new circumstances require essential modifications of the technique proposed by [9, 12, 11].

The paper is organized as follows. In Section 2 we repeat some of the definitions of notions introduced [10, 11, 12] and [9] and formulate generalizations of them we need. Then we present our main result about the primary and imported cells. Namely, we prove that the number of imported cells at given vertex $\beta$ is determined by the self-intersection degree of $\beta$ and by the total number of broken tree instants of the path. We show that the number of the instants of broken tree structure is bounded by the number of open arrival instants and this finally gives us a tool to control the number of paths that have vertices of large degree.

The descriptive part given by Section 2 is followed by Section 3, where the principles of construction of the set of walks and corresponding estimates are described. Here we mainly follow the lines of [11, 12] with necessary modifications and specifications. In Section 4 we prove estimates of the form (1.5) of the averaged traces $\mathbb{E}\{\text{Tr} (A_n)^{2s_n}\}$. Our main result concern the case when the matrix entries are given by bounded random variables $a_{ij}$.

The reason for this restriction is two-fold. First, this helps us not to overload the paper and to present clearer the key points of the technique used. From another side, the description of classes of paths and walks in terms of vertices of self-intersections does not fit very well the problem of counting the number of multiple edges in the graphs of these paths and walks. As a result, the technique described can be applied to wider classes of random variables $a_{ij}$ while the optimal conditions are far to be reached. At the end of the paper we prove one of the possible results in this direction, where we require that a finite number of moments exist, $\mathbb{E}|a_{ij}|^{q} < \infty$, $q \leq q_0$.

Finally, in Section 5 we consider several examples of the walks with primary and imported cells and show why the estimates of [9, 12] do not work in the corresponding cases.
2 Even closed walks

As we mentioned above, it is natural to consider (1.4) as the sum of weights

\[ E\left\{ \text{Tr} A_{n}^{2s}\right\} = \frac{1}{n^s} \sum_{s_{1}, \ldots, s_{n}}^{n} E\{a_{i_0, i_1} \cdots a_{i_{2s-1}, i_0}\} = \frac{1}{n^s} \sum_{I_{2s} \in I_{2s}(n)} Q(I_{2s}), \quad (2.1) \]

where the sequence \( I_{2s} \) can be regarded a trajectory of \( 2s \) steps

\[ I_{2s} = (i_0, i_1, \ldots, i_{2s-1}, i_0), \quad i_t \in \{1, \ldots, n\} \quad (2.2) \]

and \( I_{2s}(n) \) denotes the set of all such trajectories; the weight \( Q(I_{2s}) \) is given by the average of the product of corresponding random variables \( a_{ij} \). Here and below we omit subscripts in \( s_{n} \) when they are not necessary. In papers [10, 11, 12] the sequences \( I_{2s} \) are referred as to the paths, so we keep this terminology in the present paper.

In the present section we study the paths \( I_{2s} \) and their graphs and describe partitions of the set \( I_{2s}(n) \) into the classes of equivalence according to the self-intersection properties of \( I_{2s} \). To do this, we give necessary definitions based on those of [9, 10, 11, 12] and consider their generalizations. Then we prove our main technical result about the paths with primary and imported cells.

2.1 Paths, walks and graphs of walks

In (2.2), it is convenient to consider the subscripts of \( i_t \) as the instants of the discrete time. Introducing variable \( 0 \leq t \leq 2s \), we write that \( I_{2s}(t) = i_t \). We will also say that the couple \( (t - 1, t) \) with \( 1 \leq t \leq 2s \) represents the step number \( t \) of the path \( I_{2s} \).

We determine the set of vertices visited by the path \( I_{2s} \) up to the instant \( t \)

\[ \mathcal{U}(I_{2s}; t) = \{I_{2s}(t'), \ 0 \leq t' \leq t\} \]

and denote by \(|\mathcal{U}(I_{2s}; t)|\) its cardinality.

Regarding a particular path \( I_{2s} \), one can introduce corresponding closed walk \( w(I_{2s}; t) \), \( 0 \leq t \leq 2s \) that is given by a sequence of \( 2s \) labels (say, numbers from \( (1, \ldots, n) \) or letters). Also we can determine the minimal closed walk \( w_{\text{min}}(t) = w_{\text{min}}(I_{2s}; t) \) constructed from \( I_{2s} \) by the following recurrence rules:

1) at the origin of time, \( w_{\text{min}}(0) = 1 \);

2) if \( I_{2s}(t+1) \notin \mathcal{U}(I_{2s}; t) \), then \( w_{\text{min}}(t+1) = |\mathcal{U}(I_{2s}; t)| + 1 \);

if there exists such \( t' \leq t \) that \( I_{2s}(t+1) = I_{2s}(t') \), then \( w_{\text{min}}(t+1) = w_{\text{min}}(t') \).

One can interpret \( w_{\text{min}}(t) \) as a path of \( 2s \) steps, where the number of each new label is given by the number of different labels used before increased by one. The following sequences

\[ I_{8} = (5, 2, 1, 5, 7, 3, 1, 5), \quad w_{\text{min}}(I_{8}) = (1, 2, 3, 1, 4, 5, 3, 1). \]
We see that our studies concern the even closed paths and even closed walks only. Then, there exists an instant \( t \) such that the number of times that the walk \( w \) passes the edge \([s, t] \) up to the instant \( t \) is the instant of the first arrival to \( \beta \), we say that the vertex \( \beta \) and the edge \((\alpha, \beta)\) are created at the instant \( t \). In general, the graph \( g(w_{2s}) \) is a multigraph because the couple \((\alpha, \beta)\) can be connected by several edges of \( E(g_{2s}) \); these could be oriented as \((\alpha, \beta)\) or \((\beta, \alpha)\). We will denote by \(|\alpha, \beta|\) the number of non-oriented edges \([\alpha, \beta] \). We can easily pass to the graph \( \hat{g} = (V, \hat{E}) \), where the set of non-oriented edges \( \hat{E} \) contains the couples \{\( \alpha, \beta \)\} such that \( \alpha \) and \( \beta \) are joined by elements of \( E \). In this case we denote the corresponding element of \( \hat{E} \) by \( [\alpha, \beta] \).

Obviously, \(|\mathcal{E}(g_{2s})| = 2s\). We will say that the number of non-oriented edges \(|\alpha, \beta|\) determines the number of times that the walk \( w_{2s} \) passes the edge \([\alpha, \beta] \).

We denote by \( m_w(\alpha, \beta; t) \) the multiplicity of the non-oriented edge \([\alpha, \beta] \), or in other words, the number of times that the walk \( w \) passes the edge \([\alpha, \beta] \) up to the instant \( t \), \( 1 \leq t \leq 2s \):

\[
m_w(\alpha, \beta; t) = \# \{ t' \in [1, t] : (w(t' - 1), w(t')) = (\alpha, \beta) \text{ or } (w(t' - 1), w(t')) = (\beta, \alpha) \}.
\]

Certainly, this number depends on the walk \( w_{2s} \) but we will omit the subscripts \( w \).

As we have seen from (2.1), the paths and the walks we consider are closed by definition, that is \( w_{2s}(2s) = w_{2s}(0) \). There is another important restriction for the paths and walks we consider. It follows from the fact that the probability distribution of \( a_{ij} \) is symmetric:

- the weight \( Q(I_{2s}) \) is non-zero if and only if each edge from \( \hat{E}(w_{2s}) \) is passed by \( w_{2s} \) an even number of times.

In this case we will say that the path \( I_{2s} \) and the corresponding walk \( w(I_{2s}) \) are even. We see that our studies concern the even closed paths and even closed walks only. Then
$g_{2s}$ is always a multigraph and the following equality holds

$$m_w(\alpha, \beta; 2s) = 0 \pmod{2}. \tag{2.3}$$

It should be noted that the requirement (2.3) concerns the case when random matrices $A_n$ are real symmetric. In the case of hermitian matrices corresponding condition is more restrictive and requires that each edge is passed an even number of times in the way that the numbers of "there" and "back" steps are equal. We will call such walks as the double-even walks. It is easy to see that all definitions and statements of the present section do not change when switching from the even to the double-even walks. In the present paper, we do not consider the case of hermitian matrices in details.

We denote by $W_{2s}$ the set of all possible minimal even closed walks of $2s$ steps. The next subsections are devoted to the further classification of its elements.

### 2.2 Closed and non-closed instants of self-intersections

Given $w_{2s} \in W_{2s}$, we say that the instant of time $t$ with $w(t) = \beta$ is marked if the walk has passed the edge $[\alpha, \beta]$ with $\alpha = w(t-1)$ an odd number of times during the time interval $[0, t]$, $t \leq 2s$;

$$m_w(\alpha, \beta; t) = 1 \pmod{2}, \quad \alpha = w(t-1), \ \beta = w(t).$$

Figure 1: A graph $g(w_{16})$ of the walk $w_{16}$ with two open and one closed instants of self-intersections

In this case we will also say that the step $(t-1, t)$ and the corresponding oriented edge $(\alpha, \beta) = (w(t-1), w(t)) \in E$ are marked. Other instants of time are referred to as the non-marked ones.
On Figure 1 we present an example of a minimal walk where the marked instants and corresponding edges are given in boldface.

Given a vertex $\beta$ of the graph $g_{2s}$, we determine the set of all marked steps $(t_i, t_i + 1)$ such that $w_{2s}(t_i) = \beta$. The set of corresponding edges of the form $(\beta, \gamma_i) = (w_{2s}(t_i), w_{2s}(t_i + 1)) \in E$ is called the exit cluster of $\beta$; we denote it by $D_e(\beta, w_{2s}) = D_e(\beta)$. The cardinality $|D_e(\beta)| = \deg_e(\beta)$ is called the exit degree of $\beta$. If $|D_e(\beta)| = 0$, then we will say that the exit cluster of $\beta$ is empty.

Each even closed walk $w_{2s}$ generates a binary sequence $\theta_{2s} = \theta(w_{2s})$ of $2s$ elements $0$ and $1$ that correspond to non-marked and marked instants, respectively. It is clear that $\theta_{2s}$ represents path of $2s$ steps known as the Dyck path [13]. We denote by $\Theta_{2s}$ the set of all Dyck paths of $2s$ steps.

Given $\beta \in V(g_{2s})$, let us denote by $1 \leq t_1^{(\beta)} < \ldots < t_N^{(\beta)} \leq 2s - 1$ the marked instants of time such that $w_{2s}(t_j^{(\beta)}) = \beta$. We call $t_j^{(\beta)}$, $1 \leq j \leq N$ the marked arrival instants at $\beta$. The non-marked arrival instants at $\beta$ are defined in obvious manner. We will also say that the step $(t_i^{(\beta)} - 1, t_i^{(\beta)})$ and the corresponding edge $e(t_i^{(\beta)}) \in E$ are the arrival step at $\beta$ and the arrival edge at $\beta$, respectively. If $N = 2$, then the corresponding vertex is called the vertex of simple self-intersection [10]. If $N = k$, then we say that $\beta$ is the vertex of $k$-fold self-intersection and that the self-intersection degree of $\beta$ is equal to $k$; we denote the self-intersection degree of $\beta$ by $\kappa(\beta) = \kappa_{w_{2s}}(\beta)$.

As it is mentioned in [12], one has to consider the origin of time $t = 0$ as the marked instant of time. This is needed to include the walks of the form $(1, 2, 3, 1, 2, 3, 1)$ with $\beta = \{1\}$ and only one marked arrival $t_1^{(1)} = 3$ into the family of walks with self-intersections. Let us note that such ”hidden” marked instants of time can differ from $t = 0$ and can be numerous. For example, this happens each time when the walk returns at its origin with all of the existing edges closed. Summing up, we accept that $\kappa(\alpha) \geq 1$ for any $\alpha \in V(g_{2s})$.

The following definition generalizes the notion of the open vertex of (simple) self-intersection introduced in [11] and used in [9] and [12].

**Definition 2.1.** The instant $t$ is called the non-closed (or open) arrival instant at the vertex $\beta \in V(g(w_{2s}))$, if the step $(t - 1, t)$ with $\beta = w_{2s}(t)$ is marked and if there exists at least one non-oriented edge $[\beta, \gamma] \in \tilde{E}$ attached to $\beta$ that is passed an odd number of times during the time interval $[0, t - 1]$;

$$m_w(\beta, \gamma; t - 1) = 1(\mod 2).$$

In this case we say that the edge $[\beta, \gamma]$ of the graph $g(w_{2s})$ is open up to the arrival instant $t = t^{(\beta)}$, or more briefly that this edge is $t$-open. The instant $t$ can be also called the open instant of self-intersection. Correspondingly, one can define the $t$-open vertex $\beta$ of self-intersection of the walk $w_{2s}$.
Remarks

1. Definition 2.1 remains valid in the case when \( \gamma \) coincides with \( \beta = \omega_2s(t) \), more precisely in the case when there exists another marked instant \( t' < t \) such that \( \beta = \omega_2s(t' - 1) = \omega_2s(t') \) and the graph \( g(\omega_2s) \) has a loop at the vertex \( \beta \). Definition 2.1 is also valid in the case when \( \beta = \omega_2s(0) = \omega_2s(t) \).

2. Both of the definitions of the open vertex of self-intersection [11] and the open arrival instant are based on the following property: the walk, when arrived at \( \beta \) at such a marked instant, has more than one possibility to continue its way with the non-marked step. In the opposite case, when the only one continuation with the non-marked step is possible, we say that the arrival instant is closed. A vertex of a walk can change the property to be closed or open several times during the run of the walk. On Figure 1 we present an example of the walk where two instants of self-intersection are open (these are \( t = 4 \) and \( t = 8 \)) and one instant of self-intersection \( t = 15 \) is closed.

Regarding the simple self-intersections only, we see that the definition 2.1 coincides with the definition of the open vertex of simple self-intersection formulated first in [11, 12] and then used in [9]. However, the definition of the open vertex of simple self-intersection presented in [9] slightly differs from that of [11]. In [9], the vertex \( \beta \) of the simple self-intersection at the marked instant \( t \) is called the open one when the edge \((\alpha, \beta)\) of the first arrival to \( \beta \) is not used again up to the instant \( t \), while in [11] only returns in the direction \((\beta, \alpha)\) are prohibited. Looking at the Figure 1, we see that \( \beta = \omega_{16}(2) \) is the open vertex of the self-intersection at the instant \( t = 8 \) according to the definition of [11], and is not the open vertex according to the definition of [9]. However, as we will see later, this slight divergence in definitions does not alter much the estimates one obtains (see subsection 3.4).

2.3 Primary and imported cells

Given a walk \( \omega_2s \), let us consider the following procedure of reduction that we denote by \( \mathcal{P} \): find an instant of time \( 1 \leq t < 2s \) such that the step \((t-1, t)\) is marked and \( \omega_2s(t - 1) = \omega_2s(t + 1) \); if it exists, consider a new walk \( \omega'_{2s-2} = \mathcal{P}(\omega_2s) \) determined by a sequence

\[
\omega'_{2s} = (\omega_2s(0), \omega_2s(1), \ldots, \omega_2s(t - 1), \omega_2s(t + 1), \ldots, \omega_{2s}(2s)).
\]

Performing this procedure once more, we get \( \omega''_{2s} = \mathcal{P}(\omega'_{2s}) \). We repeat this procedure as many times as it is possible and denote by \( \overline{W}(\omega_2s) \) the walk obtained as a result when all of the reductions are performed. Let us note that \( \overline{W}(\omega_2s) \) is again a walk that can be transformed into the minimal one by renumbering the values of \( \overline{W}(t), \ t \geq 1 \). Then one can construct the graph \( \overline{g}(\omega_2s) = g(\overline{W}(\omega_2s)) \) as it is done in subsection 2.1.
We accept the point of view when the graph $\bar{g}(w_{2s})$ is considered as a sub-graph of $g(w_{2s})$

$$\mathcal{V}(\bar{g}(w_{2s})) \subseteq \mathcal{V}(g(w_{2s})), \quad \mathcal{E}(\bar{g}(w_{2s})) \subseteq \mathcal{E}(g(w_{2s}))$$

and assume that the edges of $\bar{g}(w_{2s})$ are ordered according to the order of the edges of $g(w_{2s})$.

**Definition 2.2.** Given a walk $w_{2s}$, we consider a vertex of its graph $\beta \in \mathcal{V}(g(w_{2s}))$ and refer to the marked arrival edges $(\alpha, \beta) \in \mathcal{E}(g(w_{2s}))$ as to the primary cells of $w_{2s}$ at $\beta$. If $\beta \in \mathcal{V}(\bar{g}(\bar{W}))$ with $\bar{W} = \bar{W}(w_{2s})$, then we call the non-marked arrival edges $(\alpha', \beta) \in \mathcal{E}(g(\bar{W}))$ the imported cells of $w_{2s}$ at $\beta$.

As we will see later, in order to control the exit degree of a vertex of a walk with typical $\theta$, one needs to take into account the number of imported cells at this vertex. The main observation here is that the presence of the imported cells is closely related with the breaks of the tree structure performed by the walk. To formulate the rigorous statement, we need to introduce the notion of the instant of broken tree structure that will play the crucial role in our studies.

**Definition 2.3.** Any walk $\bar{W} = \bar{W}(w_{2s})$ contains at least one instant $\bar{\eta}$ such that the step $(\bar{\eta} - 1, \bar{\eta})$ is marked and the step $(\bar{\eta}, \bar{\eta} + 1)$ is not. We call such an instant $\bar{\eta}$ the instant of broken tree structure (or the BTS-instant of time) of the walk $\bar{W}$. Passing back to the non-reduced walk $w_{2s}$, we consider the edge $e(\eta)$ that corresponds to the edge $e(\bar{\eta}) \in \mathcal{E}(\bar{W})$ and refer to the instant $\eta$ as the BTS-instant of the walk $w_{2s}$.

![Figure 2](image)

**Figure 2:** The graph $g(\bar{W})$ of the reduced walk $\bar{W} = \bar{W}(w_{16})$

**Examples.** Let us consider the Figure 2, where we present the graph of the walk $\bar{W}_8 = \bar{W}(w_{16})$ obtained from the walk $w_{16}$ given on Figure 1 as a result of four reduc-
tions $\mathcal{P}$. Then each non-marked edge of the graph $g(\bar{W})$ represents an imported cell with respect to the vertices of $g(w_{2s})$. One can make the following observations.

- The vertex $\beta \in \mathcal{V}(g)$ has one non-marked edge attached to it, so there is one imported cell at $\beta$ of $g(w_{16})$; this is given by the edge $e(5)$. There are two primary cells at $\beta \in \mathcal{V}(g)$ given by the edges $e(2)$ and $e(8)$. The root vertex $\varrho \in \mathcal{V}(g)$ has one imported cell $e(14)$ and the vertex $\varepsilon \in \mathcal{V}(g)$ has two primary cells, $e(7)$ and $e(15)$, and no imported cells.

- The walk $\bar{W}_s$ has only one BTS-instant, $\bar{\eta} = 4$ because the edge $e(4)$ is marked and the edge $e(5)$ is not. The same is true for the non-reduced walk $w_{16}$ where $\eta = \bar{\eta} = 4$. However, it can happen that in the non-reduced walk $w_{2s}$ the marked BTS-instant $\eta$ is separated from the corresponding non-marked instant by a tree-type sub-walk that is to be removed during the reduction procedure. We will not give the rigorous definition of the tree-type sub-walk because we do not use it here in the full extent.

- Returning to the graph of $w_{16}$ depicted on Figure 1, we can explain the term "imported cell" on the example of the vertex $\beta = w_{16}(2)$. The exit cluster of this vertex consists of two edges, $e(3)$ and $e(6)$. Regarding the tree $T(w_{16})$, we see that these edges have different parents in $T(w_{16})$; the edge of the tree that corresponds to $e(3)$ has the origin determined by the end of the edge corresponding to $e(2)$ while the edge $e(6)$ is imported at $\beta$ by the edge $e(5)$ from another part of the tree $T(w_{16})$. We give more examples of the walks with primary and imported cells at the end of this paper.

Now we will present a proposition that trivially follows from the Definitions 2.1 and 2.3. However, it plays so important role in our studies that we formulate it as the separate statement.

**Lemma 2.1.** If the arrival instant $\tau$ is the BTS-instant of the walk $w_{2s}$, then $\tau$ is the open instant of self-intersection of $w_{2s}$.

Given a vertex $\beta \in \mathcal{V}(w_{2s})$, we refer to the BTS-instants $\eta_i$ such that $w_{2s}(\eta_i) = \beta$ as to the $\beta$-local BTS-instants. All other BTS-instants are referred to as the $\beta$-remote BTS-instants. Now we can formulate the main result of this section.

**Lemma 2.2.** Let $K_{w_{2s}}^{(\beta)}$ be the number of all $\beta$-remote BTS-instants of the walk $w_{2s}$. Then the number of all imported cells at $\beta$ denoted by $J_{w_{2s}}(\beta)$ is bounded as follows

$$J_{w_{2s}}(\beta) \leq K_{w_{2s}}^{(\beta)} + \kappa_{w_{2s}}(\beta). \quad (2.4)$$

**Proof.** Let us consider the reduced walk $\bar{W} = \bar{W}(w_{2s})$ and a vertex $\beta \in \mathcal{V}(g(\bar{W}))$. We introduce the function $\Lambda_\beta(t; \bar{W})$ determined as the number of $t$-open edges attached to $\beta$;

$$\Lambda_\beta(t; \bar{W}) = \# \{ i : m(\alpha_i, \beta; t) = 1 \text{(mod 2)} \}$$

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and consider how this function changes its values at the instants of time when $\tilde{W}$ arrives at $\beta$. The following considerations show that this value can be changed by $0, +2$ and $-2$ only.

I. The case when the value of $\Lambda_\beta$ stays unchanged is possible in two situations.

   a) The first situation happens when the walk $\tilde{W}$ leaves $\beta$ by a non-marked edge and arrives at $\beta$ by a marked edge. Then the corresponding cell is the primary one and we do not care about it.

   b) The second situation occurs when the walk $\tilde{W}$ leaves $\beta$ by a marked step $(x, x+1)$ and arrives at $\beta$ by a non-marked step $(y-1, y)$. Then the interval of time $[x+1, y-1]$ contains at least one BTS-instant of time. This is the first instant $\tau$ when the non-marked step follows immediately after the marked one. It is clear that $w_{2s}(\tau) \neq \beta$ and therefore $\tau$ is the $\beta$-remote BTS-instant.

   It should be noted that another such interval $[x'+1, y'-1]$ contains another $\eta'$ that obviously differs from $\eta$; $\eta' \neq \eta$. This is because the any couple of such time intervals $[x+1, y-1], [x'+1, y'-1]$ has an empty intersection. Then each imported cell of the type (Ib) has at least one corresponding $\beta$-remote BTS-instant and the sets of the BTS-instants that correspond to different intervals do not intersect.

II. Let us consider the arrival instants at $\beta$ when the value of $\Lambda_\beta$ is changed.

   a) The change by $+2$ takes place when the walk $\bar{W}$ leaves $\beta$ by a marked edge and arrives at $\beta$ by a marked edge also.

   b) The change by $-2$ occurs in the opposite case when $\hat{W}$ leaves $\beta$ with the help of the non-marked edge and arrives at $\beta$ by a non-marked edge.

   During the whole walk, these two different passages occur the same number of times. This is because $\Lambda_\beta = 0$ at the end of the even closed walk $\bar{W}$. Taking into account that the number of changes by $+2$ is bounded by the self-intersection degree $\kappa_{\bar{W}}(\beta)$, we conclude that the number of imported cells of this kind is not greater than $\kappa_{\bar{W}}(\beta)$.

   To complete the proof, we have to pass back from the reduced walk $\bar{W}(w_{2s})$ to the original $w_{2s}$. Since the number of imported cells of $w_{2s}$ and the number of BTS-instants of $w_{2s}$ are uniquely determined by $\bar{W}(w_{2s})$, and

   \[ \kappa_{\bar{W}}(\beta) \leq \kappa_{w_{2s}}(\beta), \]

   then (2.4) follows provided $\beta$ belongs to $g(\bar{W})$ as well as to $g(w_{2s})$. If $\beta \notin \mathcal{V}(g(\bar{W}))$, then $J_{w_{2s}}(\beta) = 0$ and (2.4) obviously holds. Lemma 2.2 is proved.

\textbf{Corollary of Lemma 2.2.} Given a vertex $\beta$ of the graph of $w_{2s}$, the number of primary and imported cells $L(\beta)$ at $\beta$ is bounded

\[ L(\beta) \leq 2\kappa(\beta) + K, \quad (2.5) \]

where $K$ is the total number of the BTS-instants performed by the walk $w_{2s}$. 

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Proof. The number of the primary cells at \(\beta\) is given by the \(\kappa(\beta)\). The number of the imported cells \(J(\beta)\) is bounded by the sum \(\kappa(\beta) + K(\backslash \beta)\), where \(K(\backslash \beta) \leq K\). Then (2.5) follows. \(\diamondsuit\)

Let us note that relations (2.4) and (2.5) reflect that non-local character of the imported cells that can arise at \(\beta\) due to the BTS-instants that can happen rather "far" from \(\beta\) at any remote part of the walk. In Section 4 we show that these cells can "bring" to \(\beta\) the edges that originally do not belong to the corresponding vertex in the initial tree \(T_s\). This explains the use of the term imported cells for the corresponding edges (or instants of time).

The results we have formulated and proved in this section are sufficient to obtain the estimates we need for the high moments of large random matrices (see Section 4). So, at this stage we terminate the study of the fairly rich and interesting subject given by the family of even closed walks.

3 Estimates of the set of paths

In the previous section, we have described the properties of the even closed paths \(I_{2s}\) based on the notion of the vertices and the instants of self-intersections. In the present section we describe the procedure of construction of the set of paths and present corresponding estimates. Here we mostly follow the scheme of [10] and [11] added by considerations of the vertices of open self-intersections that are not necessary simple.

3.1 Classes of equivalence

To describe the constructions procedure, we first complete and summarize the description of the even walk paths \(I_{2s}\) started in the Section 2. There a way to separate the set of all paths \(I_{2s}(n)\) into classes of equivalence \(C(w_{2s})\) was considered.

Each class of equivalence \(C(w_{2s})\) is uniquely determined by an element \(w_{2s} \in W_{2s}\). It is clear that the weights of the equivalent paths are also equal; if \(I_{2s} \sim I'_{2s}\), then \(Q(I_{2s}) = Q(I'_{2s}) = Q(w_{2s})\). Therefore we can rewrite (2.1) in the form

\[
E \left\{ \text{Tr} A_n^{2s} \right\} = \sum_{w_{2s} \in W_{2s}} |C(w_{2s})| \cdot Q(w_{2s}) \tag{3.1}
\]

Clearly, the cardinality of the class \(C(w_{2s})\) is determined by the number of vertices of the graph \(g(w_{2s}); |C(w_{2s})| = n(n-1) \cdots (n-|V(g_{2s})|+1)\).

Regarding \(g(w_{2s})\), we determine the partition of \(V(g)\) into subsets \(N_2, \ldots, N_s\); if \(\kappa(\alpha) = k\), then \(\alpha \in N_k\). We denote by \(\nu_k\) the cardinality of \(N_k\),

\[
\nu_k = |N_k|, \quad \nu_k \geq 2.
\]
Denoting by $N_1$ the subset of of vertices $\alpha$ of $g(w_{2s})$ with $\kappa(\alpha) = 1$, we get obvious equality $s = \sum_{k=1}^s k\nu_k$, where $\nu_1 = |N_1|$.

Given a walk $w_{2s}$ that has $\nu_k$ vertices of $k$-fold self-intersections, $2 \leq k \leq s$, we say that it is of the type $\tilde{\nu}_s = \tilde{\nu}(w_{2s}) = (\nu_1, \nu_2, \ldots, \nu_s)$. Then we can separate the set of all walks $W_{2s}$ into classes of equivalence. We say that two walks $w_{2s}$ and $w'_{2s}$ are equivalent, $w_{2s} \sim w'_{2s}$ if their types of self-intersections coincide, $\tilde{\nu}(w_{2s}) = \tilde{\nu}(w'_{2s})$. The number

$$|\tilde{\nu}_s|_1 = \sum_{k=2}^s (k - 1)\nu_k$$

(3.2)

determines the cardinality of $V(w_{2s})$; namely, $|V(w_{2s})| = s + 1 - |\tilde{\nu}_s|_1$. We consider the rearrangement of (3.1) according to the classes of equivalence described above in the next subsection.

Finally, let us recall that the walk $w_{2s}$ generates a Dyck path $\theta_{2s} = \theta(w_{2s})$ that is in one-by-one correspondence with a sequence of $s$ marked instants $\Xi_s = (\xi_1, \ldots, \xi_s)$ such that $\xi_1 < \xi_2 < \ldots < \xi_s$ and $\xi_j \in \{1, 2s - 1\}$. As before, here it is convenient to consider the subscript of $\xi_j$ as a sort of the discrete "time" we denote by $\tau$, $1 \leq \tau \leq s$. Sometimes we will refer to $\tau$ as to the instants of $\tau$-time, or more simple as to the $\tau$-instants. Given a value of $\tau$, we say that $\xi_\tau = \Xi_s(\tau)$.

Remembering the definition of the vertex $\alpha$ of $k$-fold self-intersection of the walk $w_{2s}$, we see that it is determined by an ordered $k$-plet of variables $\tau^{(1)} < \ldots < \tau^{(k)}$ such that the marked instants $\xi_{\tau^{(1)}} < \ldots < \xi_{\tau^{(k)}}$ indicate the marked arrival instants at the vertex $\alpha$ determined by the first arrival $\alpha = w_{2s}(\xi_{\tau^{(1)}})$. We also observe that the elements $\alpha_j \in N_k$, $1 \leq j \leq \nu_k$ are naturally ordered in the chronological way. Therefore the walk $w_{2s}$ generates an ordered set of ordered $k$-plets that we denote by $T^{(k)}(w_{2s})$.

More generally, we denote the ordered set of $\nu_k$ ordered $k$-plets by

$$T_s^{(k)}(\nu_k) = \{(\tau_j^{(1)}(k), \ldots, \tau_j^{(k)}(k)), 1 \leq j \leq \nu_k\},$$

where all values $\tau_j^{(l)}(k)$ are distinct. In what follows, we omit variable $k$ in $\tau_j^{(l)}(k)$ when regarding such $k$-plets. Also, we will use denotations $(\tau', \tau'')$ for the $\tau$-instants of simple self-intersection.

The last observation concerns the fact that each instant of self-intersection can by characterized by one more property - is the corresponding arrival instant open or not. So, the variable $\tau_j^{(l)}$, $l \geq 2$ should be assigned by one more subscript that in the binary way indicates the openness of $\xi_{\tau_j^{(l)}}$. An obvious but important remark is that the openness of $\xi_{\tau_j^{(l)}}$ depends on the whole pre-history of walk $w_{2s}$; that is on its sub-walk determined by the time interval $[1, \xi_{\tau_j^{(l)}} - 1]$. 

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3.2 Cover the set of walks

Let us recall that the walk $w_{2s}$ of 2s steps is determined as a sequence of $2s + 1$ symbols (labels or numbers). To determine a particular walk, one starts with the initial (root) label and then indicates the symbols that appear at the subsequent instants of time $t$, $1 \leq t \leq 2s$.

In the case of closed even walk one needs less information. The first condition implies relation $w_{2s}(2s) = w_{2s}(0)$ and the second says that, for instance, if $w_{2s}$ has no self-intersections, then it is sufficient to indicate the values of $w_{2s}(t)$ at $s$ instants of time. Indeed, elementary reasoning shows that this choice of values corresponds to the choice of one of the Dyck paths from the set $\Theta_{2s}$. Then the choice of labels at $s$ instants of time corresponds to the choice of the marked instants. The values of $w_{2s}(t)$ at each non-marked instant of time are uniquely determined by the rule that the non-marked step $(\beta, \alpha)$ closes the last marked arrival $(\alpha, \beta)$. This arrival is unique in the case when $w_{2s}$ has no self-intersections.

The rule described above uniquely determines the even walk by the knowledge of the Dyck path $\theta_{2s}$ and the partition $\bar{N}_s = (N_1, N_2, \ldots, N_s)$. We call this rule the canonical run and refer to such a walk as to the tree-like walk.

The situation becomes more complicated in the case when we want to determine the set of all even walks with a number of self-intersections. The simplest example is given by the following two walks of $2s = 8$ steps

$$\tilde{w}_8 = (1, 2, 3, 4, 2, 4, 3, 2, 1) \quad \text{and} \quad \check{w}_8 = (1, 2, 3, 4, 2, 3, 4, 2, 1)$$

that have the same Dyck path $\theta_8 = (1, 1, 1, 1, 0, 0, 0, 0)$ and the same partitions with $N_1 = \{3, 4\}$ and $N_2 = \{2\}$. The walk $\tilde{w}_8$ is the tree-like walk but $\check{w}_8$ is not of the tree-like structure.

We denote the set of minimal even walks that have the same $\theta_{2s}$ and $\bar{N}_s$ by $W(\theta_{2s}, \bar{N}_s)$ and denote by $\bar{N}_s^{(r)}$ with $r, 0 \leq r \leq \nu_2$ open vertices of simple self-intersections.

It is argued in [11] that given $\theta_{2s}$, the cardinality of $W(\theta_{2s}, \bar{N}_s)$ is bounded as follows;

$$|W(\theta_{2s}, \bar{N}_s^{(r)})| \leq W(\tilde{w}_8, r) = 3^r \prod_{k=3}^{\nu_2} (2k)^{k\nu_2}. \quad (3.3)$$

Let us prove that (3.3) is true. To do this, we will need the following simple statement.

**Lemma 3.1.** Consider a vertex $\beta$ with the self-intersection degree $\kappa(\beta) = k$. The total number of non-marked edges of the form $(\beta, \alpha_i)$ is equal to $k$ and at any instant of time $1 \leq t \leq 2s$, the number of $t$-open marked edges attached to $\beta$ is bounded by $2k$. 

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Proof. To prove the first part of this lemma, we denote by $\text{In}_m$ and $\text{In}_{nm}$ the numbers of marked and non-marked enters at $\beta$ and by $\text{Out}_m$ and $\text{Out}_{nm}$ the numbers of marked and non-marked exits from $\beta$, respectively.

The walk is closed and therefore number of enters at $\beta$ is equal to the number of exits from $\beta$. Since the walk is even, then the number of the marked edges attached to $\beta$ is equal to the number of non-marked edges attached to $\beta$. Corresponding equalities,

$$\text{In}_m + \text{In}_{nm} = \text{Out}_m + \text{Out}_{nm}, \quad \text{and} \quad \text{In}_m + \text{Out}_m = \text{In}_{nm} + \text{Out}_{nm}$$

result in $2(\text{In}_m - \text{Out}_{nm}) = 0$. Then $\text{Out}_{nm} = k$.

Given $t$, we consider the numbers $\text{In}_m(t)$, $\text{In}_{nm}(t)$, $\text{Out}_m(t)$, and $\text{Out}_{nm}(t)$ that count the corresponding numbers of edges during the time interval $[0, t - 1]$. If $t$ is the instant of the exit from $\beta$, then

$$\text{In}_m(t) + \text{In}_{nm}(t) = \text{Out}_m(t) + \text{Out}_{nm}(t) + \varphi, \quad (3.4)$$

where $\varphi = 0$ in the cases when $\beta$ is the root vertex and $\varphi = 1$ otherwise. The choice of edges to close by the non-marked exit edge is bounded by the number of marked edges that enter and leave $\beta$, that is by the number $\text{In}_m + \text{Out}_m - \text{In}_{nm} - \text{Out}_{nm}$. Taking into account (3.4), we get inequality

$$\text{In}_m + \text{Out}_m - \text{In}_{nm} - \text{Out}_{nm} \leq 2(\text{In}_m - \text{Out}_{nm}) - \varphi \leq 2 \text{In}_m - \varphi \leq 2k.$$

Lemma 3.1 is proved. \diamond

Lemma 3.1 says that the multiplicative contribution of each vertex $\beta$ to the estimate (3.3) is bounded by $(2\kappa(\beta) - \varphi)$!! Let us recall that in the case when the root vertex $\rho$ has $l$ marked edges $(\gamma_i, \rho)$ attached, then we accept that $\kappa(\rho) = l + 1$ because of the first marked arrival instant at $\rho$ that is hidden.

In the case of $\kappa(\beta) = 2$ the bound $4 - \varphi$ can be immediately improved. If the second arrival instant at $\beta$ is closed, then there is only one possibility to continue the walk at the non-marked instant. If the second arrival instant in open, then we have not more than three possibility to continue the run at the non-marked departure from $\beta$ in the case when $\beta \neq \rho$ and not more than two possibilities in the case when $\beta = \rho$. This terminates the proof of (3.3).

In fact, the estimate (3.3) can be further improved by the analysis of the number of BTS-instants at the vertices of high self-intersection degree. This goes out of the frameworks of the present paper.

### 3.3 The choice of instants of self-intersections

In this subsection we estimate the number of possibilities to choose $\nu_k$ instants of $k$-fold self-intersections that is obviously bounded by the number of possibilities to choose the
set of $k$-plets $T_s^{(k)}(\nu_k)$. We denote the latter number by $|T_s^{(k)}(\nu_k)|$. In this subsection we do not do the distinction between the open and the closed arrival instants.

**Lemma 3.2.** Given any $\theta_{2s}$, the number of possibilities the number of possibilities to choose $\nu_2$ instants of simple self-intersections is bounded by

$$|T_s^{(2)}(\nu_2)| \leq \frac{1}{\nu_2!} \left( \frac{s^2}{2} \right)^{\nu_2}. \quad (3.5)$$

If $k \geq 3$, then the number of possibilities the number of possibilities to choose $\nu_k$ instants of $k$-fold self-intersections is bounded by

$$|T_s^{(k)}(\nu_k)| \leq \frac{1}{\nu_k!} \left( \frac{s^k}{(k-1)!} \right)^{\nu_k}. \quad (3.6)$$

**Proof.** Let us start with the proof of (3.5). Here we mostly repeat the computations of [11]. Regarding the second arrival instant at the vertex $\beta_j$ of simple self-intersection, we have to point out its position $\tau_{j(2)}$ among the $s$ marked instants, $1 \leq j \leq \nu_2$. Denoting for simplicity $\tau_{j(2)} = l_j$, we can write that

$$|T_s^{(2)}(\nu_2)| \leq \sum_{1 \leq l_1 < l_2 < \cdots < l_{\nu_2} \leq s} (l_1 - 1) + (l_2 - 3) + (l_3 - 5) + \cdots + (l_{\nu_2} - 2\nu_2 + 1), \quad (3.7)$$

where $(x)_+$ is equal to $x$ if $x \geq 0$ and to zero otherwise. Indeed, the first factor of (3.7) reflects the fact that at the $\tau$-instant $\tau_{1(2)} = l_1$ we dispose of $l_1 - 1$ marked instants of $\Xi(\theta_{2s})$ to choose the vertex $\beta_1$ of this self-intersection. In other words, we have to choose the first $\tau$-instant $\tau_{1(1)}$ that creates $\beta_1 = w_{2s}(\xi_{1(1)})$. Regarding the second arrival $\tau$-instant of the next simple self-intersection $\tau_{2(2)} = l_2$, we can choose among $l_2 - 1 - 2$ marked instants to point out the vertex of the second self-intersection. Here by subtraction of 2, we avoid the instants of the first self-intersection already used. We proceed like this up to $\nu_2$-th self-intersection.

Changing variables $l_i - 1 = \hat{l}_i$, we derive from (3.7) that

$$|T_s^{(2)}(\nu_2)| \leq \sum_{1 \leq \hat{l}_1 < \cdots < \hat{l}_{\nu_2} \leq s-1} \hat{l}_1 \hat{l}_2 \cdots \hat{l}_{\nu_2} \leq \frac{1}{\nu_2!} \sum_{1 \leq \hat{l}_1, \ldots, \hat{l}_{\nu_2} \leq s-1} \hat{l}_1 \hat{l}_2 \cdots \hat{l}_{\nu_2} \leq \frac{1}{\nu_2!} \left( \sum_{i=1}^{s-1} \hat{l}_i \right)^{\nu_2}.$$ 

Then (3.5) follows.
To prove (3.6), let us consider first the case of $\nu_k = 1$. To determine a vertex $\alpha$ of $k$-fold self-intersection, we have to point out $k$ marked arrival instants at this vertex. Denoting corresponding $\tau$-instants by $\tau^{(i)}$, $1 \leq i \leq k$ we see that

$$|T_s^{(k)}(1)| \leq \sum_{1 \leq \tau^{(1)} < \cdots < \tau^{(k)} \leq s} 1.$$ 

This relation gives an obvious estimate

$$|T_s^{(k)}(1)| \leq \binom{s}{k} = \frac{s(s-1) \cdots (s-k+1)}{k!} \leq \frac{s^k}{k!}.$$ 

To study the case of $\nu_k \geq 2$ vertices $\alpha_j$, $1 \leq j \leq \nu_k$ of $k$-fold self-intersections with $k \geq 3$, we use the estimate of $T_s^{(k)}(1)$ with $k$ replaced by $k - 1$. Namely, pointing out the last arrival $\tau$-instant $\tau^{(k)}$, we can write that

$$|T_s^{(k)}(1)| = \sum_{\tau^{(k)} = k}^s |T_s^{(k-1)}(1)| \leq \sum_{\tau^{(k)} = k}^s \binom{\tau^{(k)} - 1}{k - 1} \leq \frac{s^{k-1}}{(k - 1)!} \sum_{\tau^{(k)} = k}^s 1 \leq \frac{s^k}{(k - 1)!}, \quad (3.8)$$

This estimate is slightly worse than (3.7) but it simplifies the study of the general case of $\nu_k \geq 1$.

Let us denote the last arrival instants at $\nu_k$ vertices $\tau_1^{(k)} < \cdots < \tau_{\nu_k}^{(k)}$. Using the representation described in (3.8) and ignoring the fact that some of the marked instants are already in use, we get inequality

$$|T_s^{(k)}(\nu_k)| \leq \sum_{1 < \tau_1^{(k)} < \cdots < \tau_{\nu_k}^{(k)} \leq s} |T_s^{(k-1)}(1)||T_s^{(k-1)}(1)| \cdots |T_s^{(k-1)}(1)|$$

$$\leq \left(\frac{s^{k-1}}{(k - 1)!}\right)^{\nu_k} \sum_{1 < \tau_1^{(k)} < \cdots < \tau_{\nu_k}^{(k)} \leq s} 1 \leq \frac{1}{\nu_k!} \left(\frac{s^k}{(k - 1)!}\right)^{\nu_k}, \quad (3.9)$$

that gives (3.6). Lemma 3.2 is proved. ∎

Let us note that our estimate of $T_s^{(k)}(1)$ and that of (3.9) slightly differ from the estimates of $T_s^{(k)}(\nu_k)$ presented in [11, 12] and used in [9]. In these works the vertex $\alpha$ of $k$-fold self-intersection has been considered as the vertex, where the first two arrivals at $\alpha$ produce a simple self-intersection and then the remaining $k - 2$ arrivals increase
the self-intersection degree $\kappa(\alpha)$ up to $k$. Our representation (3.8) can be viewed as the backward procedure of the choice that starts from the last arrival instant $\tau^{(k)}$ at $\alpha$ and ends with the first one $\tau^{(1)}$. Evidently, these two descriptions coincide in the case of the simple self-intersections (3.7). Our approach is in more agreement with the scheme of (3.7). Also the backward scheme of the estimates seems to be the most convenient in the studies of vertices with one or more open instants of self-intersections that we carry out in the next subsections.

### 3.4 Open and closed simple self-intersections

Lemma 3.2 gives the following estimate for the total number of possibilities

$$\prod_{k=2}^s |T^{(k)}(\nu_k)| \leq \frac{1}{\nu_2!} \left( \frac{s^2}{2} \right)^{\nu_2} \cdot \prod_{k=3}^s \frac{1}{\nu_k!} \left( \frac{s^k}{(k-1)!} \right)^{\nu_k}. \tag{3.10}$$

However, this estimate cannot be used directly in this form. The reason is in the presence of the factor $3^r$ in the estimate (3.3) that increases too much the product of the right-hand sides of these estimates. This obstacle was treated first in the paper [10], where it was shown that the choice of the vertex of open self-intersection is much less than that of the vertex of closed simple self-intersection given by $s^2/2$ as $s \to \infty$.

In the present subsection we describe the procedure of construction of the open vertices of simple self-intersections by formalizing the arguments of [10] and slightly modifying them to take into account two different types of open self-intersections.

Let us recall that given $\theta_{2s}$, we can construct a walk with vertex of (simple) self-intersection by pointing out a couple of marked instants $(\xi', \xi''')$ from the set $\Xi(\theta_{2s})$ or equivalently by pointing out the couple of $\tau$-instants $(\tau', \tau''')$. This means that we choose first $\tau'''$ and then we point out the vertex of the self-intersection $\alpha = w_{2s}(\xi''')$ by choosing $\tau' < \tau'''$.

Aiming the construction of the open vertex of simple self-intersection, we want to be sure that $\alpha$ is such that there exists an edge $e'$ attached to $\alpha$ that is $(\xi'''-1)$-open. The vertex $\alpha$ can be the end (or the head) of $e'$ and then we have the open self-intersection of the first type we call the E-open self-intersection; if $\alpha$ represents the start (or the tail) of $e'$, then the self-intersections is of the second type, or briefly of the S-type.

An important remark is to be made here. As we have seen, the openness of the edge depends on the particular run of the walk. For example, regarding the Dyck path $\theta_{12} = (1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0)$ with two self-intersections determined by the couples of instants $t$ of time (1, 4) and (2, 8), we see that the instant $t'_{2} = 8$ is the open or the closed instant of self-intersection in dependence on the value of $w_{12}$ at the non-marked instant $t = 5$.

However, the number of choices of the edge $e'$ does not depend on the particular...
run of the walk and is bounded by the number of \((t'_2 - 1)\)-open edges. This number is not greater than the value \(\theta_{2s}(t'')\).

Taking into account the previous reasoning, we describe the following procedure of construction of the set of walks with closed and open simple self-intersections.

1. Given \(\theta_{2s}\), we choose among \(s\) \(\tau\)-instants the positions of \(\nu_2\) second arrivals \(\tau''_1 < \ldots < \tau''_{\nu_2}\) at vertices of simple self-intersections. Among them, we choose \(\nu_2 - r_E - r_S\) instants that will be the instants of closed self-intersections, and \(r_E\) and \(r_S\) instants of E-open and S-open self-intersections, respectively. We denote \(r = r_E + r_S\). With this information in hands, we start the run of the walk.

2. At the instant \(\tau''_1\) we have choose a vertex of the first self-intersection. Then the following estimates are valid:

- if \((\tau'_1, \tau''_1)\) is prescribed to be the closed self-intersection, then there is not more that \(\tau''_1 - 1\) possibilities to choose \(\tau'_1\);
- if \((\tau'_1, \tau''_1)\) is prescribed to be the E-open self-intersection, we choose the vertex among the heads of the edges that are \((\xi_{\tau''_1} - 1)\)-open; the number of such edges is not greater than \(\theta_{2s}(\xi_{\tau''_1})\);
- if \((\tau'_1, \tau''_1)\) is prescribed to be the S-open self-intersection, we choose the vertex among the tails of the \((\xi_{\tau''_1} - 1)\)-open edges; again, there is not more than \(\theta_{2s}(\xi_{\tau''_1})\) of such edges.

3. To continue the run if the walk at the instant \(\xi_{\tau''_1} + 1\), we look whether it is marked or not. If it is marked and is not the instant of self-intersection, we produce a new vertex; if it is the instant of self-intersection, we return to the paragraph 2 given above.

4. If the instant \(\xi_{\tau''_1} + 1\) is non-marked, and the self-intersection \((\tau'_1, \tau''_1)\) is the closed one, then vertex \(w_{2s}(\xi_{\tau''_1} + 1)\) is uniquely determined. If \((\tau'_1, \tau''_1)\) is open, then we consider all possibilities to choose the vertex \(w_{2s}(\xi_{\tau''_1} + 1)\); this produces several sub-walks that correspond to different runs. In the previous subsection we have seen that there are not more than three possible runs in the case of simple open self-intersection. Then we proceed with the next step \(\xi_{\tau''_1} + 2\) till we meet the second arrival instant \(\xi_{\tau''_2}\) of the second self-intersection. At this stage we redirect ourselves to the paragraph 2.

We continue the procedure described until the last vertex of the self-intersection \((\tau'_{\nu_2}, \tau''_{\nu_2})\) is determined. In this construction, we take into account that the choice of vertices of \(j\)-th closed self-intersection is bounded by \(l_j - 2j - 1\), where \(l_j\) is the position of \(\tau''_j\) in \(\Xi(\theta_{2s})\).
Having all the steps of the construction performed, we obtain not more than $3^{r_E+r_S}$ different runs, and whatever run is chosen, there is not more than

$$(l_1 - 1) \cdots (l_{\nu_2} - 2\nu_2 - 1) \cdot M_\theta^r, \quad M_\theta = \max_t \theta_{2s}(t)$$

possibilities to choose the instants and vertices of the self-intersections. We see that in the case of walks with simple self-intersections, we have not more than

$$\sum_{1 \leq i_1 < \cdots < i_s \leq s} \sum_{(\nu_2-r,r_E,r_S) \subset (1,2,\ldots,s)} (l_i - 1) \cdots (l_{\nu_2-r} - 2(\nu_2 - r) - 1) \cdot M_\theta^r,$$

possibilities to choose the vertices of simple self-intersection of the types indicated, where the second sum corresponds to the choice of $\nu_2 - r, r_E$ and $r_S$ vertices mentioned above. This gives the estimate

$$\frac{1}{\nu_2!} \cdot \frac{\nu_2!}{(\nu_2 - r)! \cdot r_E! \cdot r_S!} \left(\frac{s^2}{2}\right)^{\nu_2-r} \cdot (sM_\theta)^r.$$

Summing over all possible values of $r_E$ and $r_S$, we arrive at the proof of the following statement.

**Lemma 3.3.** Given $\theta_{2s}$, the number of possibilities $|T_s^{(2)}(\nu_2;r)|$ to construct $\nu_2$ vertices of simple self-intersections with $r$ open ones is bounded as follows:

$$|T_s^{(2)}(\nu_2;r)| \leq \frac{1}{(\nu_2 - r)! \cdot r!} \cdot (2sM_\theta)^r \left(\frac{s^2}{2}\right)^{\nu_2-r}, \quad 0 \leq r \leq \nu_2. \quad (3.11)$$

In Section 4, we use (3.10) modified with the help of (3.11).

### 3.5 Triple self-intersections with open arrival instants

Repeating the main lines of the previous subsection, it is easy to estimate the number of possibilities to create a walk with one vertex of triple self-intersection that has one or two open arrival instants. Let us denote the arrival instants at this vertex by $\tau^{(1)} < \tau^{(2)} < \tau^{(3)}$. We set up the value of $\tau^{(3)}$ and consider the following three possible scenarios that can happen.

1) The first situation is given by the case when the second arrival instant $\tau^{(2)}$ is open and $\tau^{(3)}$ is closed. In this case we can choose the position for $\tau^{(2)}$ among $s$ marked instants of $\Xi(\theta_{2s})$ with the only restriction that $\tau^{(2)} < \tau^{(3)}$.

We start the run of the walk $w_{2s}$ and at the instant $\tau^{(2)}$ we indicate an edge $e$ that already exists in the graph of the sub-walk and that is $(\xi_{\tau^{(2)}} - 1)$-open. This edge can be
chosen from the set of cardinality less or equal to $\theta_{2s}(\xi_{\tau(3)})$ that is bounded by $M_\theta$. We set $w_{2s}(\xi_{\tau(2)}) = \beta$ to be equal to the head or to the tail of $e$ and continue the run of the walk remembering that at the $\tau$-instant $\tau^{(3)}$ the walk has to visit the vertex $\beta$ already pointed out: $w_{2s}(\xi_{\tau(3)}) = \beta$. The walk is constructed and the number of possibilities to choose the $\tau$-instants $\tau^{(1)}$ and $\tau^{(2)}$ is bounded by $\tau^{(3)} M_\theta$.

ii) Let us consider the case when the arrival instant $\tau^{(2)}$ is closed and $\tau^{(3)}$ is open. Remembering that the value of $\tau^{(3)}$ is already set, we start the run of the walk. At the instant $\tau^{(3)}$, we choose the vertex $\beta$ that is the head or the tail of an edge that is $(\xi_{\tau(3)} - 1)$-open and set $w_{2s}(\xi_{\tau(3)}) = \beta$. Then the $\tau$-instant $\tau^{(2)}$ is determined in the way that $\beta = w_{2s}(\xi_{\tau(2)})$ and the choice of $\tau^{(2)}$ is bounded by $M_\theta$. It remains to attribute to $\tau^{(1)}$ any value such that $\tau^{(1)} < \tau^{(2)}$ and we are done. Again we see that the number of possibilities to create a couple $\tau^{(1)}$ and $\tau^{(2)}$ is bounded by $\tau^{(3)} M_\theta$.

iii) Finally, let us construct a walk with triple self-intersection such that the both of the instants $\tau^{(2)}$ and $\tau^{(3)}$ are the open ones. We still assume that $\tau^{(3)}$ is already determined. We start the run of the walk and the question now is to choose two vertices $\beta_1$ and $\beta_2$ that already exist at the instant $\xi(\tau^{(3)}) - 1$. There are two different cases.

Let us assume that $\beta_2$ is such that the $(\xi_{\tau(3)} - 1)$-open edge is attached it. Then the choice of $\tau^{(2)}$ is possible from the set of cardinality less or equal to $\theta(\xi_{\tau(3)})$. Since the arrival instant $\xi(\tau^{(2)})$ is open, the same concerns the choice of the instant $\tau^{(1)}$. Then the number of possibilities to created such a vertex is bounded by $M_\theta^3$.

Now let us consider the second possibility when the $(\xi_{\tau(3)} - 1)$-open edge is attached not to $w_{2s}(\xi_{\tau(2)})$ but to $w_{2s}(\xi_{\tau(1)})$. In this case we have no restrictions for the choice of $\tau^{(2)}$ but the choice of $\tau^{(1)}$ is restricted to the set of cardinality less than $\theta(\xi_{\tau(2)})$. This gives the estimate of the number of choices by $\tau^{(3)} M_\theta$.

Regarding the cases (i)-(iii) and summing over all possible values of $\tau^{(3)}$, we conclude that in the case of triple self-intersection the number of choices to construct a vertex with one or two open arrival instants is bounded by $\max \left\{ s M_\theta^2; s^2 M_\theta/2 \right\} \leq s^2 M_\theta$.

Now it is not difficult to estimate the number of possibilities to choose the values of $T_s^{(3)}(\nu_3; r_3)$ with given number $r_3$ of open arrival instants at the vertices of triple self-intersections.

**Lemma 3.4** Given $\theta_{2s}$, the number of possibilities $|T_s^{(3)}(\nu_3; r_3)|$ to construct $\nu_3$ vertices of triple self-intersections with the total number of $r_3$ open arrival instants is bounded by the following expression:

$$|T_s^{(3)}(\nu_3; r_3)| \leq \frac{1}{\nu_3!} \left( \frac{2\nu_3}{r_3} \right) \cdot \left( 4 s^2 M_\theta \right)^{r_3/2} \left( \frac{s^3}{2t} \right)^{\nu_3 - (r_3/2)}, \quad 0 \leq r_3 \leq 2\nu_3, \quad (3.12)$$
where we denoted

\[ \langle r_3/2 \rangle = \begin{cases} 
    l, & \text{if } r_3 = 2l, \\
    l + 1, & \text{if } r_3 = 2l + 1.
\end{cases} \]

**Proof.** First we point out \( r_3 \) arrival instants among \( 2\nu_3 \) ones; this produces the factor \( \binom{2\nu_3}{r_3} \). Then we set the values of the last arrival \( \tau \)-instants \( \tau_1^{(3)} < \ldots < \tau_{\nu_3}^{(3)} \).

The vertex \( \alpha' \) that has at least one open arrival instant can be constructed by not more than \( 4sM_\theta \) choices of the values of the arrival \( \tau \)-instants \( \tau_{i(1)}^{(3)} \) and \( \tau_{i(2)}^{(3)} \), where the factor 4 estimates the choice of the head or the tail of the open edges. The number of such vertices \( \alpha' \) is not less than \( \langle r_3/2 \rangle \).

The two first arrival instants \( \tau_{i(1)}^{(3)} \) and \( \tau_{i(2)}^{(3)} \) at the vertices that have no open arrival instants can be chosen in not more than \( s^3/2 \) ways.

Taking into account these observations and summing over the last arrival \( \tau \)-instants, we repeat the computations of (3.9) and obtain (3.12). Lemma is proved. \( \diamond \)

### 3.6 Counting the walks with BTS-instants

In the previous subsections we have described the construction of all walks with the same \( \theta_2 \), that have closed and open instants of simple and triple self-intersections. This is important for the estimate of number of walks because the instants of the open self-intersections leave a certain freedom for the walk to continue its run at the non-marked instants of time. This run can differ from the canonical run of the walk according to the tree structure dictated by the corresponding \( \theta_2 \). So, the open self-intersection represent a potential possibility to choose the canonical run or the non-canonical run at the non-marked instants of time. The arrival instants followed by the non-canonical continuation are the BTS-instants considered in Section 2.

The BTS-instants are important when counting the number of imported cells at one or another vertex of the walk. So, let us inverse somehow the point of view of the previous subsections and look at the set of walks that have a certain number \( R \) of BTS-instants. We assume that \( R = \rho_2 + \rho_3 + \ldots + \rho_s \), where \( \rho_k \) is the number of BTS-instants that happen in the vertices \( \beta \) such that \( \kappa(\beta) = k \).

As usual, let us consider first the vertices of simple self-intersections. Supposing that the walk \( w_{2s} \) has \( \rho_2 \) BTS-instants, we conclude that there are at least \( \rho_2 \) open instants of self-intersections in \( w_{2s} \). Then this walk belongs to the class \( (\nu_2, r_2) \) with \( r_2 \geq \rho_2 \).

If \( w_{2s} \) is such that \( \rho_3 > 0 \), then there is a number of vertices of triple self-intersections that contain at least one open instant of self-intersection. Clearly, the number of such vertices is greater than \( \langle \rho_3/2 \rangle \).
In what follows, we will need the estimates of choices of vertices with \( \kappa = 2 \) and \( \kappa = 3 \) that have at least one open instant of self-intersection. In contrast, for the vertices with \( \kappa \geq 4 \) we take into account their number only. So, if \( w_{2s} \) is such that \( \rho_k > 0 \) with \( k \geq 4 \), we observe that this walk contains at least one vertex of \( k \)-fold self-intersection.

At this point we terminate the study of such a rich and interesting subject of even walks and pass to the estimates of high moments of large random matrices.

4 Estimates of the moments of random matrices

In this section we use the results of our studies of even walks and prove the estimates of the moments (1.2) in the asymptotic regime \( n \to \infty, s^3 = O(n^2) \). We are restricted to the simplest case when the random matrix entries are given by bounded random variables. Therefore our results represent a particular case of statements formulated for the sub-gaussian random variables [12] and random variables having a number of moments finite [9]. However, it should be stressed that the arguments presented in [9] and [12] are not sufficient to get the full proof of the estimates even in the simplest case considered here. In the next subsection we address this question in more details.

Another important thing to say is that the results of [9, 11, 12] rely strongly on the assumption that the following property of the Dyck paths is true

\[
B(\lambda) = \lim_{s \to \infty} \frac{1}{C^{(s)}} \sum_{\theta_{2s} \in \Theta_{2s}} \exp \left( \frac{\lambda M^{(s)}_{\theta}}{\sqrt{s}} \right) < +\infty, \quad \lambda > 0, \quad (4.1)
\]

where \( \Theta_{2s} \) is the set of all Dyck paths of \( 2s \) steps, \( C^{(s)} = \frac{(2s)!}{s!(s + 1)!} \), and \( M^{(s)}_{\theta} = \max_{0 \leq t \leq 2s} \theta_{2s}(t) \). This estimate is closely related with the corresponding property of the normalized Brownian excursion that is proved to be true (see [2] and references therein). This is because the limiting distribution of the random variable \( M^{(s)}_{\theta}/\sqrt{s} \) considered on the probability space generated by \( \Theta_{2s} \) coincides with that given by this half-plane Brownian excursion. We did not find any explicit reference where (4.1) would be proved, but it is widely believed to be true. So, we also prove our statements under the hypothesis (4.1).

4.1 Main estimate and the scheme of the proof

The main result of the present section is given by the following statement.
Theorem 4.1 Consider the ensemble of real symmetric matrices $A_n$ (1.1) whose elements

$$(A_n)_{ij} = \frac{1}{\sqrt{n}} a_{ij}$$

are given by a family $A_n = \{a_{i,j}, 1 \leq i \leq j \leq n\}$ of jointly independent identically distributed random variables that have symmetric distribution. We assume that there exists such a constant $U \geq 1$ that $a_{i,j}$ are bounded with probability 1,

$$\sup_{1 \leq i \leq j \leq n} |a_{i,j}| \leq U \quad (4.2)$$

and we denote the moments of $a$ by $V_{2m}$ with $V_2 = 1$;

$$\mathbf{E}\{a_{ij}^2\} = 1, \quad \mathbf{E}\{a_{ij}^{2m}\} = V_{2m}. \quad (4.3)$$

If $s_n^3 = \mu n^2$ with $\mu > 0$, then in the $n \to \infty$ the estimate

$$\frac{\sqrt{\pi \mu}}{4s_n} \cdot \mathbf{E} \left\{ \operatorname{Tr} (A_n)^{2 s_n} \right\} \leq B(6\mu^{1/2}) \exp\{C\mu\} \quad (4.4)$$

is true with a constant $C$ that does not depend on $n$ and on $V_{2m}, m \geq 2$.

Here we denoted by $[x], x > 0$ the largest integer not greater than $x$. Not to overload the formulas, we will omit this sign when no confusion can arise. Then one can rewrite (4.4) in the following form similar to (1.5)

$$\frac{s_n! (s_n + 1)!}{n(2s_n)!} \cdot \mathbf{E} \left\{ \operatorname{Tr} (A_n)^{2 s_n} \right\} \leq B(6\mu^{1/2}) \exp\{C\mu\} \quad (4.4')$$

that is asymptotically equivalent to (4.4) due to the Stirling formula.

Remarks.

1. We denote the limiting transition $s_n, n \to \infty$ such that $s_n^3 = \mu n^2$ by $(s_n, n)_{\mu} \to \infty$ or simply by $(s, n)_{\mu} \to \infty$. It can be easily seen from the proof of Theorem 4.1 that the estimate (4.4) is also true in the limit $s_n, n \to \infty$ such that $s_n^3/n^2 \to \mu$. More generally, one can prove (4.4) in the limit when $s_n^3 = O(n^2)$ with $\mu = \limsup_{n \to \infty} s_n^3/n^2 \geq 0$.

2. Theorem 4.1 is also true for the ensemble of hermitian random matrices. In this case the closed even walks that give non-zero contribution to $\mathbf{E}\{\operatorname{Tr} A_n^{2 s_n}\}$ are to be replaced by the closed double-even walks. This could lower the estimate (4.4); in particular, one could replace the coefficient 6 in $B(6\mu^{1/2})$ of (4.4).

3. One can compare these results with the non-asymptotic estimates of the moments of real symmetric (or hermitian) random matrices of the form (4.2) whose elements have joint Gaussian distribution. Corresponding ensembles are represented by the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Unitary Ensemble (GUE) (see e.g. [8]).
It is proved in [6] that the moments of GUE \( m_{2k}^{(n)} \) are bounded by (cf. (1.5))
\[
\frac{n(2s)!}{s!(s+1)!} \left( 1 + \frac{\alpha k(k-1)(k+1)}{n^2} \right)
\]
for all values of \( k, n \) such that \( k^3/n^2 \leq \chi \), where \( \alpha > (12 - \chi)^{-1} \) with \( \chi < 12 \). Then in this case the estimate (4.4') can be replaced by more explicit inequality, say
\[
\frac{s!(s+1)!}{n(2s)!} m_{2s}^{(n)} \leq \left( 1 + \frac{\mu}{11 - \mu} \right), \quad \mu < 11.
\]

For more result on the non-asymptotic estimates of the moments of random matrices, see e.g. [7].

**Scheme of the proof of Theorem 4.1.** We mainly follow the scheme proposed in paper [9] that slightly modifies the approach of [10, 11, 12] and gives more detailed account on the computations involved.

The strategy is to consider the natural representation of \( \text{Tr} A_n^{2s} \) as the sum over the set \( \mathcal{I}_{2s}(n) \) of all possible paths \( I_{2s} \) (cf. 2.1) and split this sum into four sub-sums according to the properties of the graphs of the paths \( I_{2s} \in \mathcal{I}_{2s}(n) \).

These properties are related with the value of \( |\bar{\nu}| (3.2) \) and the maximal exit degree of the graph \( g(I_{2s}) = g_{2s} \)
\[
\Delta(I_{2s}) = \max_{\alpha \in V(g_{2s})} \deg_e(\alpha),
\]
where \( \deg_e(\alpha) = |D_e(\alpha)| \). Namely, we use the same partition of \( \text{Tr} A_n^{2s} \) as in [9, 12] given by relation
\[
\mathbb{E} \left\{ \text{Tr} A_n^{2s} \right\} = \sum_{l=1}^{4} Z_{2s}^{(l)},
\]
where

- \( Z_{2s}^{(1)} \) is the sum over the set \( \mathcal{I}_{2s}^{(1)} \subset \mathcal{I}_{2s}(n) \) of all possible paths \( I_{2s} \) such that \( |\bar{\nu}(I_{2s})|_1 \leq C_0 s^2/n \) and there is no edges in \( \mathcal{E}(I_{2s}) \) passed by \( I_{2s} \) more than two times;

- \( Z_{2s}^{(2)} \) is the sum over the set \( \mathcal{I}_{2s}^{(2)} \) of all the paths \( I_{2s} \) such that \( |\bar{\nu}(I_{2s})|_1 \leq C_0 s^2/n \) and \( \Delta(I_{2s}) \leq s^{1/2-\epsilon} \), \( \epsilon > 0 \) and there exists at least one edge of \( \mathcal{E}(I_{2s}) \) passed by \( I_{2s} \) more than two times;

- \( Z_{2s}^{(3)} \) is the sum over the set \( \mathcal{I}_{2s}^{(3)} \) of all the paths \( I_{2s} \) such that \( |\bar{\nu}(I_{2s})|_1 \leq C_0 s^2/n \) and \( \Delta(I_{2s}) \geq s^{1/2-\epsilon} \);

- \( Z_{2s}^{(4)} \) is a sum over the set \( \mathcal{I}_{2s}^{(4)} \) of all the paths \( I_{2s} \) such that \( |\bar{\nu}(I_{2s})|_1 \geq C_0 s^2/n \).
The constant $C$ of (4.4) is chosen according the condition $C > C_0 + 36$ where $C_0$ is determined in the proof of the estimate of $Z_{2s}^{(4)}$; it is sufficient to take $C_0 > 2eC_1^2U^4$, where $C_1 = \sup_{k \geq 1} \frac{2k}{((k-1)!)^{1/k}}$. The appropriate value of $\epsilon$ will be determined in the estimate of $Z_{2s}^{(3)}$; it is sufficient to choose $0 < \epsilon < 1/6$.

In the limit $(s, n)_\mu \to \infty$, the sub-sum $Z_{2s}^{(1)}$ contributes to (4.5) as a non-vanishing term, while other three sub-sums are of the order $o(1)$. In the following four subsections we consider these sub-sums one by one. Let us explain the use of the results of Sections 2 and 3 in the proof of the estimates of $Z_{2s}^{(i)}$.

In Section 3 we presented the three types of estimates; first we estimated

A) the number of walks with given $\theta_{2s}$ and $\bar{\nu}_s$;
then we specified these estimates and studied

B) the number of walks with a number of simple open self-intersections; finally, we estimated

C) the number of walks that have a number of triple self-intersections with open arrival instants.

In the present section we use these estimates completed in some cases by the information about

D) walks with self-intersections that produce factors $V_{2m}, m \geq 2$.

Namely,

- to estimate $Z_{2s}^{(1)}$, we use mostly parts (A) and (B) described above; here our computations repeat almost word-by-word those of [11, 12];

- to estimate $Z_{2s}^{(2)}$, we use the parts (A), (B), and (D); the structure of simple and triple self-intersection that produce the factors $V_{2m}$ is studied in more details than it is done in [9, 12];

- to estimate $Z_{2s}^{(3)}$, we use the items (A), (B), (C), and (D); this is the most complicated part of the present section that involves the results of Section 2; here we use the new ingredient of the structure of even closed walk we called the primary and imported cells; this makes our arguments and computations essentially different from those of [12] and [9];

- to estimate $Z_{2s}^{(4)}$, the part (A) is sufficient; here we slightly modify the reasoning of [9] by adding some missing elements of the proof.

Let us start to perform the program presented.
4.2 Estimate of $Z_{2s}^{(1)}$

Taking into account observations and results of Section 3, we can write that

$$Z_{2s}^{(1)} \leq \sum_{\theta \in \Theta_{2s}} \sum_{\sigma = 0}^{C_{0s}^2/n} \sum_{\nu: |\bar{\nu}|_1 = \sigma} \frac{n(n - 1) \cdots (n - |\mathcal{V}(g_{2s})| + 1)}{n^{s}} \times$$

$$\sum_{r=0}^{\nu_2} |T_s^{(2)}(\nu_2; r; \theta_{2s})| \cdot \prod_{k=3}^{s} |T^{(k)}(\nu_k; \theta_{2s})| \cdot W_s(\bar{\nu}_s; r),$$  \hspace{1cm} (4.6)

where $|T_s^{(2)}(\nu_2; r)|$ represents an estimate of the number of possibilities to point out $\nu_2$ vertices of simple self-intersections such that $r$ self-intersections are non-closed (3.11), the variables $|T^{(k)}(\nu_k; \theta_{2s})|$ with $k \geq 3$ are as in (3.6) and $W_s(\bar{\nu}_s; r)$ is given by (3.3).

Remembering that $|\mathcal{V}(g_{2s})| = s + 1 - |\bar{\nu}|_1$ and that $\sigma = \sum_{k=2}^{s}(k - 1)\nu_k$, we can write the following inequality

$$n \frac{(n - 1) \cdots (n - s + \sigma)}{n^{s-\sigma}} \cdot \frac{1}{n^{\sigma}} \leq n \exp \left\{ - \frac{s^2}{2n} + \frac{s\sigma}{n} \right\} \cdot \prod_{k=2}^{s} \frac{1}{n(k-1)\nu_k}.$$  \hspace{1cm} (4.7)

In this computation we have used the following elementary result.

**Lemma 4.1** ([10]). If $s < n$, then for any positive natural $\sigma$ the following estimate holds

$$\prod_{k=1}^{s-\sigma} \left(1 - \frac{k}{n}\right) \leq \exp \left\{ - \frac{s^2}{2n} \right\} \exp \left\{ \frac{s\sigma}{n} \right\}.$$  \hspace{1cm} (4.8)

*Proof.* The proof mainly repeats the one of [10]. We present it for completeness. Elementary computations show that

$$\prod_{k=1}^{s-\sigma} \left(1 - \frac{k}{n}\right) = \exp \left\{ \sum_{k=1}^{s-\sigma} \log \left(1 - \frac{k}{n}\right) \right\} = \exp \left\{ - \sum_{k=1}^{s-\sigma} \left( \sum_{j=1}^{\infty} \frac{k^j}{jn^j} \right) \right\}$$

$$\leq \exp \left\{ - \sum_{k=1}^{s-\sigma} \frac{k}{n} \right\} \leq \exp \left\{ - \frac{(s - \sigma)^2}{2n} \right\} \leq \exp \left\{ - \frac{s^2}{2n} + \frac{s\sigma}{n} \right\}.$$

Lemma is proved. ⋄

Taking into account results (3.6) and (3.11) of Lemmas 3.2 and Lemma 3.3 and using the estimate of $W_n$ (3.3), we derive from (4.6) with the help of (4.7) the following
inequality

\[ Z^{(1)}_{2s} \leq n \exp \left\{ C_0 \frac{s^3}{n^2} - \frac{s^2}{2n} \right\} \sum_{\theta \in \Theta_{2s}} \sum_{\sigma=0}^{C_0 s^2/n} \sum_{\nu: |\nu|_1=\sigma} 1 \nu_2 ! \left( \frac{s^2 + 6sM_\theta}{n} \right)^{\nu_2} \times \prod_{k=3}^{s} \frac{1}{\nu_k !} \left( \frac{(2k)^k s^k}{(k-1)! n^{k-1}} \right)^{\nu_k}. \]

Passing to the sum over \( \nu_i \geq 0, i \geq 2 \) without any restriction, we can write that

\[ Z^{(1)}_{2s} \leq n \exp \{ C_0 \mu \} \sum_{\theta \in \Theta_{2s}} \exp \left\{ \frac{6M_\theta \sqrt{s}}{\sqrt{s}} + 36 \mu + \sum_{k \geq 4} \frac{(C_1 s)_k}{n^{k-1}} \right\}, \quad (4.10) \]

where \( C_1 = \sup_{k \geq 3} 2k/((k-1)!)^{1/k} \). It is easy to see that the Stirling formula implies relation

\[ n |\Theta_{2s}| = n C^{(s)} = \frac{n(2s)!}{s! (s+1)!} = \frac{4^s}{\sqrt{\pi \mu}} (1 + o(1)), \quad (s, n)_\mu \to \infty. \]

Multiplying and dividing the right-hand side of (4.10) by \( C^{(s)} \) and using (4.1), we conclude that

\[ \frac{\sqrt{\pi \mu}}{4^s} \cdot Z^{(1)}_{2s} \leq B(6\mu^{1/2}) \cdot \exp \{ C \mu \}, \quad (s, n)_\mu \to \infty, \quad (4.11) \]

where \( C > C_0 + 36 \). This is the estimate for \( Z^{(1)}_{2s} \) we need.

4.3 Estimate of \( Z^{(2)}_{2s} \)

In the present subsection we study the walks whose weight contains at least one factor \( V_{2m} \) with \( m \geq 2 \). The first important observation here is that it is sufficient to consider in details the multiple edges attached to the vertices of the self-intersection degree \( \kappa \leq 3 \) only. The contribution of the multiple edges attached to other vertices will be replaced by the bound \( U \) with the appropriate degree.

Also we have to say that here one meets the following general inconvenience of the method: the classes of equivalence of the paths and the walks are characterized by the set \( \tilde{\nu} \) that determines the multiplicities of the vertices of self-intersections. However, this description does not take explicitly into account the multiplicities of the edges. Sometimes this makes the study of the classes of walks whose weight contains the factors \( V_{2m}, m \geq 2 \) rather cumbersome.
4.3.1 Simple self-intersections of the first kind

Let us consider first the vertices attached to the edges that produce the factors $V_4$. Assume that among $\nu_2$ vertices of simple self-intersections there are $r_2$ open vertices of self-intersection. We assume also that among $\nu_2 - r_2$ vertices of simple self-intersections there are $q_2$ vertices $\beta$ such the following condition is verified: the second arrival to $\beta$ at the marked instant is performed along the edge oriented in the same direction as the edge corresponding to the first arrival to $\beta$. We denote the edge of this first arrival by $(\alpha, \beta)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The marked part of $g(w_{18})$ with the simple self-intersections of different types}
\end{figure}

We say that these $q_2$ simple self-intersections are of the type one or I-type simple self-intersections. On Figure 3 we present the marked edges of the graph of the walk that has one simple self-intersection $(t', t'') = (2, 16)$ of the I-type.

Let us formulate the following elementary proposition.

**Lemma 4.2.** ([10]) Given the position of the second arrival instant $\tau''$ of simple self-intersection of the first kind, there exist not more than $\Delta^{(\alpha)}$ possibilities to choose the first arrival $\tau'$-instant $\tau'$, where

$$\Delta^{(\alpha)} = \deg_e(\alpha), \quad \alpha = w_{2s}(\xi_{t''} - 1).$$

**Proof.** The proof immediately follows from the observation that the vertex of the simple I-type self-intersection $(\tau', \tau'')$ is such that the edge of first arrival belongs to the exit cluster $\beta \in D_e(\alpha)$ of the sub-walk $w_{[1, \xi_{t''}]}$ of $w_{2s}$. \(\Box\)
Corollary of Lemma 4.2. If one is restricted with the class of walks with given $\bar{\nu}$ and bounded maximal exit degree

$$\Delta(I_{2s}) = \max_{\alpha \in V(g_{2s})} \deg_e(\alpha) \leq d,$$

then the number of possibilities to construct such a walk of $r_2$ open self-intersections with $q_2$ of the I-type self-intersections is bounded by

$$|T(\nu_2, r_2, q_2)| \cdot \prod_{k=3}^s |T_s^{(k)}(\nu_k)| \cdot W_s(\bar{\nu}; r),$$

(4.12)

where

$$|T(\nu_2; r_2; q_2)| \leq \frac{(2sM\theta)^{r_2} (sd)^{q_2}}{r_2! q_2! (\nu_2 - r_2 - q_2)!}$$

(4.13)

Proof. Relation (4.13) corresponds to the choice of the vertices of simple self-intersections. Then (4.12) obviously follows after the reasoning similar to that used in Section 3.

4.3.2 Simple self-intersections of the second kind and chains of them

The case of simple self-intersections of the first kind is described in [9, 10]. Now let us consider other possibilities to construct the walks whose weight contains the factors $V_4$.

First let us assume that among remaining $\nu_2 - r - q_2$ vertices there are $p_2$ vertices $\gamma$ that verify the following property: the second arrival at $\gamma$ at the marked instant is performed along the edge $e = (\delta, \gamma)$ such that the marked edge $(\gamma, \delta) = e$ already exists in $g(w_{2s})$. We say that these simple self-intersections are of the type two, or II-type simple self-intersections. On Figure 4 we give an example of the walk $w_{2s}$ with $p_2 = 2$ and $q = 1$. We show there the marked edges of the graph $g(w_{2s})$.

To construct the self-intersection of II-type, the walk has to close the edge $e$. The direction of this closure indicates two different ways to create the II-type self-intersection.

In the first case the closure of $e = (\gamma, \delta)$ is performed in the inverse direction $(\gamma, \delta)$. Then the marked edge $e$ can be created after the marked arrival at $\delta$. We see that the vertex $\delta$ is by itself the vertex of a self-intersection. When arrived at $\delta$, the walk has to decide about the choice of the next vertex; there is not more than three marked edges that arrive at $\gamma$ and the number of choices of the vertex $\gamma$ is bounded by $3$. Let us recall that we consider the edges attached to vertices with $\kappa \leq 3$ only. Therefore the simple self-intersection enters into the sub-sum we study with the factor of the order $\text{Const} \cdot \frac{s^2}{2^n} \cdot \frac{3}{n} = O(s^2/n^2)$.

Another possibility is given by the closure of the edge $e = (\gamma, \delta)$ in the direction $(\gamma, \delta)$. In this case the walk has to perform an open self-intersection before this closure.
and such a II-type self-intersection contributes by the factor of the order $M_{\theta}/n$, where $M_{\theta}$ estimates the number of choices of $\tau''$. We do not give the rigorous proofs of the statements presented above.

Now let us consider the situation when $p_{2}$ vertices of simple self-intersections of the second type are organized into several chains of neighboring self-intersections as it is shown on Figure 4. Regarding one group only that contains $l_{1}$ elements in the chain, we see that there is less than $3s^{2}$ possibilities to produce the last self-intersection at the vertex $\gamma^{(1)}$ where $w_{2s}$ arrives at the second time at the instant $t$. To produce the second element of this chain, the next in turn instant of self-intersection is to be chosen from the exit cluster of $w(t)$. Therefore the number of possibilities to perform this is not greater than $\sup_{t\geq 0}\deg_{w}(w(t)) = \Delta(I_{2s}) = \Delta$. Then the total number of possibilities to produce such a chain is bounded by $3s^{2}\Delta^{l_{1}-1} \leq s^{2}\Delta^{l_{1}}$, where we assumed for simplicity that $\Delta \geq 3$.

If there are $v$ chains of $l_{i}$ elements with $l_{1} + \ldots + l_{v} = p$, then the number of possibilities is bounded by $s^{2v}\Delta^{p}$ and the number of vertices in the corresponding graph $g(w_{2s})$ is bounded by $s - |\vec{v}|_{1} - v$. This means that such a configuration enters into the sum with the factor

$$\frac{1}{v!} \left( \frac{s^{2}}{n} \right)^{v} \sum_{l_{1} + \ldots + l_{v} = p - v} \left( \frac{\Delta}{n} \right)^{l_{1}} \cdots \left( \frac{\Delta}{n} \right)^{l_{v}} \leq \frac{1}{v!} \left( \frac{s^{2}}{n} \right)^{v} \left( \sum_{l_{1} \geq 1} \frac{\Delta}{n} \right)^{l_{1}} \leq \frac{1}{v!} \left( \frac{2s^{2}\Delta}{n^{2}} \right)^{v}.$$  

Here we have taken into account that $\Delta/n \leq s^{1/2}/n = o(1)$ as $n \to \infty$.

4.3.3 Triple self-intersections and factors $V_{2j}, j \geq 2$

Let us consider first the vertices of the triple self-intersections that can be seen at the edges that produce the factor $V_{4}$. Regarding the simple self-intersections of the type one, we see that one can add an arrival edge at $\beta$ and keep the factor $V_{4}$ with no changes. If there are $q_{3}$ vertices $\beta$ with $\kappa(\beta) = 3$ of this type, then there is not more than $(s^{2}\Delta)^{q_{3}}/q_{3}!$ possibilities to choose the instants to create such a group. This group enters $Z_{2s}^{(2)}$ with the factor $n^{-2q_{3}}$. Some of the vertices of the chains described above can be also the vertices of triple self-intersection. Then each of the factors $(\Delta/n)^{l_{i}}$ given above should be replaced by

$$\left( \frac{\Delta}{n} \right)^{l_{i}} \sum_{u_{i}=0}^{l_{i}} \binom{l_{i} + 1}{u_{i}} \left( \frac{s}{n} \right)^{u_{i}} = \left( \frac{\Delta}{n} \right)^{l_{i}} \left( 1 + \frac{s}{n} \right)^{l_{i}} = \left( \frac{\Delta}{n}(1 + o(1)) \right)^{l_{i}}.$$  

The same concerns each of the $v$ factors $s^{2}/n$.

Let us pass to factors $V_{6}$. To consider these, we have to study the vertices of triple self-intersections or the mixed cases given by the vertices of simple self-intersections of
type two and/or the vertices of triple self-intersections. Slightly modifying the previous reasonings, it is easy to see that \((V_6)^Q\) enters with the factor bounded in these two cases by

\[
\frac{1}{Q!} \left( \frac{s \Delta^2}{n^2} + \frac{s \Delta^3}{n^3} (1 + o(1)) \right)^Q.
\]

The first factor takes into account either the edges whose ends are the vertices of simple self-intersections of the types I and II or the triple self-intersections of the type one; the second one corresponds to the triple self-intersection of the type two and the chains of them. Both of these types an be determined by straight analogy with the types of simple self-intersections. We do not present the details here.

Taking into account \(q_3\) vertices of triple self-intersections described above and assuming that \(v\) chains of edges are constructed with the help of \(p_3\) vertices of triple self-intersections, we can write that the contribution of the vertices of triple self-intersections that produce moments \(V_{2j}\) with \(j \geq 2\) is given by the factor bounded by

\[
\frac{1}{(q_3 + p_3 + Q)!} \left( \frac{s^2 \Delta^4}{n^4} V_8 + \frac{s^2 \Delta^2}{n^2} V_{10}(1 + o(1)) + \frac{s \Delta^3}{n^3} V_{10} \right)^{q_3 + p_3 + Q}.
\]

Here the factor \(1 + o(1)\) corresponds to the chains of vertices of triple self-intersections of the type two.

The factors \(V_8\) can arise due to the presence of triple self-intersections of the special type similar to the type two of the simple and triple self-intersections. It is not hard to see that the factor \((V_8)^P\) enters together with the number estimated by expression \(\frac{1}{P!}(s \Delta^4/n^4)^P\).

Gathering the observations of this subsection and using the computations of the previous subsection, we conclude that

\[
Z_{2s}^{(2)} \leq n \sum_{\theta \in \Theta_{2s}} \sum_{\sigma=0}^{C_0 s^2/n} \sum_{\nu_3} \sum_{\nu_2} \sum_{\nu_2 - r_2} \sum_{p_2 + q_2 = 0} \sum_{Q_3 + P + \nu_3'} \exp \left\{ -\frac{s^2}{2n} + C_0 \mu \right\} \times
\]

\[
\frac{1}{(\nu_2 - p_2 - q_2 - r_2)!} \left( \frac{s^2}{2n} \right)^{\nu_2 - p_2 - q_2 - r_2} \cdot \frac{1}{r_2!} \left( \frac{3s M_0}{n} \right)^{r_2} \cdot \frac{1}{q_2!} \left( \frac{s \Delta}{n} V_4 \right)^{q_2} \times
\]

\[
\frac{1}{p_2!} \cdot \frac{1}{v!} \left( \frac{2s^2 \Delta}{n^2} V_4 \right)^{v} \cdot \frac{1}{Q_3!} \left( \frac{s \Delta}{n^2} V_4 + \frac{2s \Delta^2}{n^2} V_6 + \frac{s \Delta^3}{n^4} V_6 \right)^{Q_3} \times
\]

\[
\frac{1}{P!} \left( \frac{s^3}{n^4} \right)^P V^P_{1s} \cdot I_{[1,s]}(p_2 + q_2 + P + Q_3) \cdot \frac{1}{\nu_3!} \left( \frac{36s^3}{n^3} \right)^{\nu_3} \cdot \frac{1}{\nu_k!} \prod_{k=4}^{s} \frac{C_{2k}^k U_{2k} k^k}{n^{k-1}} \right)^{\nu_k},  \quad (4.14)
\]
where we have replaced all factors \( 1 + o(1) \) by 2 and have denoted by \( I_B(\cdot) \) the indicator function

\[
I_B(x) = \begin{cases} 
1, & \text{if } x \in B, \\
0, & \text{if } x \notin B 
\end{cases}
\]

and by \( \nu_3' \) the number of vertices from \( N_3 \) that are not included into the subsets considered above. Although we could use inequality \( V_2m \leq U^{2m} \) in (4.14), we prefer to keep the factors \( V_2m \) to indicate clearly the origin of the corresponding factors.

Remembering that \( \Delta \leq s^{1/2-\epsilon} \) with \( \epsilon > 0 \), we repeat the computations of the previous subsection that lead to (4.10) and deduce from (4.14) inequality

\[
\sqrt{\frac{\pi \mu}{4^s}} Z^{(2)}_m \leq \exp\left\{ \left( C_0 + 36 \right) \mu \cdot B(6\mu^{1/2}) \cdot \left( \exp\left\{ \frac{\sqrt{\mu}}{\sqrt{s^{\epsilon}}} V_4(1 + o(1)) \right\} - 1 \right) \right\}.
\]

Then

\[
\sqrt{\frac{\pi \mu}{4^s}} Z^{(2)}_m = o(1), \quad \text{as } (s,n) \mu \to \infty.
\]  

This estimate shows that the paths \( I^{(2)}_m \) do not contribute to \( m^{(n)}_m \) in the limit we consider.

### 4.4 Estimate of \( Z^{(3)} \)

We have seen in the previous subsection that the presence of \( V_2m \) in the weight is related mainly with the I-type simple self-intersections and therefore with the exit degree of a vertex. The exit degree of \( \beta \) is determined as the cardinality of the exit cluster \( D_e(\beta) \) defined in subsection 2.2. In the present subsection we concentrate on the classes of walks such that their maximal exit degree is large; \( \Delta = \max_{\beta} |D_e(\beta)| \geq s^{1/2-\epsilon} \).

Given \( \theta_{2s} \in \Theta_{2s} \), we determine the canonical walk \( w^{(0)}_{2s} = w(\theta_{2s}) \) as the walk without self-intersections constructed with the help of \( \theta_{2s} \). Clearly, the graph \( g(w^{(0)}_{2s}) \) represents a rooted half-plane tree of \( s \) edges \( T_s = T(w^{(0)}_{2s}) = T(\theta_{2s}) \) introduced in Section 1. We determine the vertices and the exit clusters of the tree \( T_s \) in the obvious way.

#### 4.4.1 Exit sub-clusters and \( \mathcal{L} \)-property of the Dyck paths

Given a walk \( w_{2s} \), we consider a vertex \( \beta \) of \( g(w_{2s}) \) and denote by \( 0 < \zeta_1 < \ldots < \zeta_L < 2s \) the arrival instants at \( \beta \) that represent either primary or imported cells. We determine a partition of the exit cluster \( D_e(\beta) \) into subsets \( D^{(l)}(\beta), 1 \leq l \leq L \), where the elements of \( D^{(l)}_\beta \) are given by the marked edges created during the time interval \( (\zeta_l, \zeta_{l+1}) \) with \( \zeta_{L+1} \equiv 2s \). If there is no such marked edges, we say that the corresponding subset \( D^{(l)}(\beta) \) is empty. The following statement is a simple consequence of the definition of the primary and imported cells.
Lemma 4.3 Consider a walk \( w_{2s} \) and its Dyck path \( \theta(w_{2s}) \) with the corresponding tree \( T(\theta) \). Then the edges of the same subset \( D^{(l)}(\beta) \) correspond the edges of the tree \( T(\theta) \) that belong to the same exit cluster of \( T(\theta) \). Denoting by \( L' \) the number of all such exit clusters of \( T(\theta) \) that correspond to \( D^{(l)}(\beta), 1 \leq l \leq L \), we have

\[
L' \leq L.
\]

(4.16)

Proof. Let us denote by \( t_1^{(l)} = t_1 \) and \( t_2^{(l)} = t_2 \) the instants when the first and the last elements of \( D^{(l)}(\beta) \) are created. According to the definition of \( \zeta_i \), the time interval \([\zeta_l + 1, \zeta_{l+1} - 1]\) contains the non-marked arrival instants at \( \beta \) only, if they exist, and these arrivals do not represent the imported cells. Then the sub-walk \( \tilde{W} = w_{[t_1, t_2-1]} \) starts and ends at \( \beta \) and is of the tree-type structure. This means that after a series of reductions described in subsection 2.3 this sub-walk can be reduced to the empty sub-walk. Not to overload this paper, we do not present corresponding rigorous definitions and proofs.

Since \( \tilde{W} \) is of the tree-type structure, then the edges of \( D^{(l)}(\beta) \) correspond to the children of the same parent in \( T(\theta) \). It can happen that the edges of \( T(\theta) \) that correspond to different clusters \( D^{(l)}(\beta) \) and \( D^{(l')}(\beta) \) have the same parent. Lemma 4.3 is proved. \( \diamond \)

Now let us introduce an important characterization of the Dyck paths that represent a simplified version of the property proposed in [12] and used in [9].

**Definition 4.1.** We say that the Dyck path \( \theta \in \Theta_{2s} \) verifies the \( L(m) \)-property, if there exists \( \alpha \in V(T(\theta_{2s})) \) such that \( \deg_e(\alpha) \geq m \). We denote by \( \Theta_{2s}^{(m)} \) the subset of Dyck paths that verify this property.

Let us explain the use of the \( L \)-property in the estimate of \( Z_{2s}^{(3)} \). The set of walks involved in \( Z_{2s}^{(3)} \) is characterized by the fact that the graph of each of these walks contains at least one vertex \( \beta \) such that its exit degree \( D_e(\beta) \) is greater than \( d \geq s^{1/2-\epsilon} \). We assume \( \beta_0 \) to be the first vertex of this kind in the chronological order. Let us denote by \( N \) the self-intersections degree of \( \beta_0 \) and by \( K \) the total number of the BTS-instants performed by the walk. Then one can observe that some of the Dyck paths \( \theta \) cannot be used to construct the walks of the type \((d, N, K)\) determined.

Indeed, it follows from the corollary of Lemma 2.1 that the number \( L \) of primary and imported cells at \( \beta \) is bounded by \( 2N + K \). According to (4.16), the number of parents \( L' = L'_\beta \) in the corresponding tree \( T \) is also bounded by \( 2N + K \). If \( \theta' \notin \Theta_{2s}^{(d/(2N+K))} \), then any tree \( T(\theta') \) has no vertices with the exit degree greater than or equal to \( d/(2N + K) \). Then obviously \( \deg_e(\beta_0) \) is strictly less than \( d \).
In [12] it is argued that the subset of Dyck paths $\Theta_{2s}^{(m)}$ has an exponentially bounded cardinality with respect to $C^{(s)} = |\Theta_{2s}|$:

$$|\Theta_{2s}^{(m)}| \leq a^s C^{(s)} \exp\{-C_2 m\},$$  \hfill (4.17)

where $a = 1$, $b = 2$ and $C_2$ is a constant. In [5] (4.17) is proved with $a = 2$, $b = 1$ and $C_2 = \log(4/3)$.

### 4.4.2 General expression to estimate $Z_{2s}^{(3)}$

To estimate the sum $Z_{2s}^{(3)}$, we determine the values of variables $d, N, K$ and consider the paths such there exists a vertex $\beta_0$ of the corresponding graph of the exit degree $\deg_e(\beta_0) = d$ with the self-intersection degree $\kappa(\beta_0) = N$; the sum over $\theta$ is restricted to the subset $\Theta_{2s}^{(d/(2N+K))}$ and according to Lemma 2.1, the structure of the path is such that the corresponding walk has $R$ open instants of self-intersection with $R \geq K$ and $R = \rho_2 + \ldots + \rho_s$, where $\rho_k$ is the number of open arrival instants at the vertices of self-intersection degree $k$.

Repeating the reasoning of the subsection 4.3 leading to the estimate (4.16) and taking into account the arguments of the previous subsection, we conclude that $Z_{2s}^{(3)}$ is bounded by the following expression

$$Z_{2s}^{(3)} \leq n \exp \left\{ C_0 \frac{s^3}{n^2} - \frac{s^2}{2n} \right\} \sum_{\theta \in \Theta_{2s}} \sum_{d \geq s^{1/2-\epsilon}} \sum_{N=1}^{s} \sum_{K \geq 0} I_{\Theta_{2s}^{(d/(2N+K))}}(\theta)$$

$$\times \frac{C_0 s^2}{n} \sum_{\sigma=0}^{N} \sum_{R=K}^{\sigma} \sum_{r_2 + \ldots + r_3 = R}^{r_2 \geq 0} \frac{1}{r_2!} \cdot \left( \frac{6 s M_{\theta}}{n} \right)^{r_2}$$

$$\times \frac{1}{(\nu_2 - r_2)!} \left( \frac{s^2}{2n} + \frac{sdV_3}{n} (1 + o(1)) \right)^{\nu_2 - r_2} \cdot \frac{1}{(\nu_3 - \langle r_3/2 \rangle)!} \left( \frac{36 V_{12} s^3}{n^2} \right)^{\nu_3 - \langle r_3/2 \rangle}$$

$$\times \frac{1}{\langle r_3/2 \rangle!} \left( \frac{72 V_{12} s^2 M_{\theta}}{n^2} \right)^{\langle r_3/2 \rangle} \cdot \prod_{k=4}^{s} \left( \frac{(k-1)\nu_k}{\nu_k} \right) \cdot \frac{1}{\nu_k!} \left( \frac{C_k U_{2k} s^k}{n^{k-1}} \right)^{\nu_k}. \hfill (4.18)$$

In this expression the sum $\sum_{\theta}^{(N)}$ is taken over the sets $\theta$ such that $\nu_N \geq 1$. The term $\binom{(k-1)\nu_k}{\nu_k}$ stands for the choice of $r_k$ BTS-instants among $(k-1)\nu_k$ arrival instants at the corresponding vertices; here we assume that $\binom{l}{0} = \delta_{l,0}$ for any integer $l \geq 0$. The factor $72 = 2 \cdot 36$ gives the estimate of the corresponding choices for the vertices of triple self-intersections multiplied by the estimate that comes from the corresponding part of $W_n$. 

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Let us explain the presence of the factors $V_{12}$ in (4.18). The last product corresponds to the vertices with $\kappa \geq 4$. Regarding the vertex $\beta$ of the triple self-intersection, we denote three marked arrival edges at $\beta$ by $(\alpha_i, \beta)$, $i = 1, 2, 3$. The vertices $\alpha_i$ can coincide between them. However, we see that the vertex $\beta$ produces that maximal weight in the case when all $\alpha_i$ are distinct and each of $\alpha_i$ is the head of two marked edges $(\beta, \alpha_i)$. This gives the factor $V_3^4 \leq V_{12}$. Here we have taken into account the agreement that $\kappa(\alpha) \geq 1$, $\alpha \in V(g_s)$.

Remembering that $s = \mu^{1/3} n^{2/3}$, we can write equality

$$U^{2k} C_1^k \frac{s^{k-1}}{n^{k-1}} = U^{2k} C_1^k \frac{n^{2/3}}{n^{(k-1)/3}} = \delta_n H_n^{k-4} \frac{1}{n^{(k-1)/12}},$$

where

$$\delta_n = U^8 C_1^4 \mu^{4/3} n^{-1/12} \quad \text{and} \quad H_n = U^2 C_1 \mu^{1/3} n^{-1/12}. \quad (4.19a)$$

Summing a part of (4.18) over all possible values of $r_i$, $i \geq 4$, we obtain with the help of multinomial theorem that

$$\sum_{r_4 + \ldots + r_s = \rho_4} \prod_{k=4}^{s} \frac{1}{r_k!} ((k-1)\nu_k)^{r_k} \frac{1}{\nu_k!} \left( \frac{C_k U^{2k} s^k}{n^{k-1}} \right)^{\nu_k} = \frac{\sigma_4^{\rho_4}}{\rho_4!} \cdot \frac{1}{n^{\sigma_4/12}} \prod_{k=4}^{s} \frac{\delta_k^{\nu_k}}{\nu_k!} \cdot H_n^{(k-4)\nu_k}, \quad (4.19b)$$

where we denoted $\sigma_4 = \sum_{k \geq 4} (k-1)\nu_k$. Here we have used the obvious estimate

$$\binom{(k-1)\nu_k}{r_k} \leq \frac{1}{r_k!} ((k-1)\nu_k)^{r_k}.$$

Now we are ready to estimate the right-hand side of (4.18). To better explain the principle and the estimates, we split our considerations into three parts. In the first one we consider sub-sum given by the right-hand side of (4.18) with $\sigma_4 = 0$ and $r_3 = 0$. We denote this sub-sum by $\tilde{Z}_{2s}^{(3)}$. When estimating $\tilde{Z}_{2s}^{(3)}$, we illustrate the main tools of the present subsection.

Then we denote the sub-sum of (4.18) with $\sigma_4 = 0$ and $r_3 \geq 0$ by $\tilde{Z}_{2s}^{(3)}$ and the sub-sum of (4.18) with $\sigma_4 \geq 1$ by $\hat{Z}_{2s}^{(3)}$. Obviously,

$$Z_{2s}^{(3)} \leq \tilde{Z}_{2s}^{(3)} + \hat{Z}_{2s}^{(3)}.$$

We show that $\tilde{Z}_{2s}^{(3)} = o(1)$ and $\hat{Z}_{2s}^{(3)} = o(1)$ as $(s, n)_\mu \to \infty$ in the second and the third parts of the proof. This implies the conclusion that $Z_{2s}^{(3)} = o(1)$ as $(s, n)_\mu \to \infty$. 

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4.4.3 The basic case of $\sigma_4 = 0$ and $r_3 = 0$

Relation $\sigma_4 = 0$ implies equality $\rho_4 = 0$. Also we observe that the sum over $N$ runs from 1 to 3. Remembering that the corresponding sum is denoted by $\tilde{Z}^{(3)}$ and taking into account that $r_2 = R$, we can write the following estimate

$$\tilde{Z}^{(3)} \leq \sum_{d \geq s^{1/2-\epsilon}} \sum_{\sigma=0}^{C_0 s^2/n} \sum_{\nu_2=0}^{\nu_2} \sum_{K=0}^{\theta=\Theta_2 s} \sum_{r_2=K}^{1/r_2!} \left( \frac{6sM_\theta}{n} \right)^{r_2} \cdot \frac{1}{(\nu_2 - r_2)!} \left( \frac{s^2}{2n} \right)^{\nu_2 - r_2} \times \frac{1}{\nu_3!} \left( \frac{36V_1 s^3}{n^2} \right)^{\nu_3} \cdot \exp \left\{ \frac{C_0 s^3}{n^2} - \frac{s^2}{2n} \right\} \cdot \exp \left\{ \frac{2sd}{n} V_4 - \frac{C_2 d}{6 + K} \right\} \cdot I_{\Theta_2 s/(\nu_3 + K)}(\theta).$$

Let us denote $X = s^2/(2n)$ and $\Phi = 6sM_\theta/n$ and consider the sum over $r_2$:

$$S^{(K)}_{\nu_2}(X, \Phi) = \sum_{r_2=K}^{\nu_2} \frac{1}{(\nu_2 - r_2)!} X^{\nu_2 - r_2} \cdot \frac{1}{r_2!} \Phi^{r_2} = \frac{1}{\nu_2!} \sum_{r_2=K}^{\nu_2} \left( \frac{\nu_2}{r_2} \right) X^{\nu_2 - r_2} \Phi^{r_2}.$$

Multiplying and dividing by $h^K$, we conclude that if $h > 1$, then

$$S^{(K)}_{\nu_2}(X, \Phi) = \frac{1}{h^K \nu_2!} \left( \Phi^{\nu_2} h^K + \ldots + \left( \frac{\nu_2}{K} \right) X^{\nu_2 - K} \Phi^K h^K \right) \leq \frac{1}{h^K \nu_2!} \frac{(X + h\Phi)^{\nu_2}}{\nu_2!}. \quad (4.21)$$

This inequality illustrates the principle we use to estimate $\tilde{Z}^{(3)}$. Regarding the last line of (4.20), we see the factor $\exp\{-s^2/(2n)\}$ that normalises the sum of the powers of $s^2/n$ diverging in the limit $(s, n)_\mu \to \infty$. This sum is in certain sense not complete because of the presence of powers of asymptotically bounded factors $6sM_\theta/n$, and this makes possible to use the corresponding exponentially decaying factor $h^{-K}$.

Regarding the normalized sum over $\Theta_{2s}$ as a kind of the mathematical expectation $E\{\cdot\}$, we use elementary inequality $E\{fI_A\} \leq P(A)(Ef)^{1/2}$ and deduce from (4.20) with the help of (4.21) the following estimate;

$$\tilde{Z}^{(3)} \leq 3C_0^2 s^3 \cdot \exp \{C_0 \mu + 36V_1 \mu \} \cdot \left( B(12h\mu^{1/2}) \right)^{1/2} \cdot \frac{(2s)!}{s! (s + 1)!} \times \sum_{d \geq s^{1/2-\epsilon}} \sum_{K \geq 0} \frac{1}{h^K} \cdot \exp \left\{ - \frac{C_2 d}{6 + K} \right\} \cdot \exp \left\{ \frac{2sd}{n} V_4 \right\}.$$

We see that the problem of the estimate of $\tilde{Z}^{(3)}$ is reduced to the question about the maximum value of the expression

$$F(K) = \frac{2sd}{n} V_4 - \frac{C_2 d}{6 + K} - K \log h$$

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as a function of variable $K$. Function

$$f(h)(x) = \frac{2\mu^{1/3}dV_4}{n^{1/3}} - \frac{C_2 d}{6 + x} - x \log h, \quad x \geq 0$$

takes its maximum value at the point $x_0 = \sqrt{\frac{C_2 d}{\log h}} - 6$. This gives the estimate

$$f(h)(x) \leq 2\sqrt{d}\left(\frac{\mu^{1/3}d^{1/2}V_4}{n^{1/3}} - \sqrt{C_2 \log h}\right) + 6 \log h.$$

Remembering that $s^{1/2-\epsilon} \leq d \leq s$, we see that if $h = h_0 = \exp\left\{\frac{2+2\mu^{1/2}}{C_2}\right\}$, then

$$F(K) \leq f(h_0)_{\text{max}} = -2\left(n^{2/3}\mu^{1/3}\right)^{1/4-\epsilon/2} + 6 \log h_0.$$

Returning to (4.22), we conclude that the choice of sufficiently small positive $\epsilon$, say $\epsilon < 1/6$, leads to the estimate

$$\tilde{Z}_2(s) \leq 3C_0 s^5 \cdot \frac{(2s)!}{s!(s+1)!} \cdot \exp\{-\mu^{1/24}n^{1/9} + 6 \log h_0 + (C_0 + 36V_12)\mu\} \cdot \left(B(12h_0\mu^{1/2}) + 1\right).$$

Then obviously $\tilde{Z}_2(s) = o(1)$ in the limit $(s, n)_\mu \to \infty$.

### 4.4.4 Estimate of $\tilde{Z}_2(s)$

In the present subsection we consider (4.18) with $\sigma_4 = 0$ and $r_2 + r_3 = R$ denoted by $\tilde{Z}_2(s)$. To estimate this sub-sum, we use the same principle of the estimates as in the previous subsection: in a part of terms the infinitely increasing factor $X$ is replaced either by $\Phi = 6sM_\theta/n$ or by $\Psi = 72V_12s^2M_\theta/n^2$ that are asymptotically bounded. All that we need here is the following elementary computation, where we use (4.22):

$$\sum_{r_2 + r_3 = R}^{r_2 \leq \nu_2, r_3 \leq \nu_3} \frac{\tilde{X}^{\nu_2-r_2}}{(\nu_2 - r_2)!} \cdot \frac{\Phi^{r_2}}{r_2!} \cdot \frac{\Psi^{r_3/2}}{(r_3/2)!} \cdot \frac{Y^{\nu_3-(r_3/2)}}{(\nu_3 - (r_3/2))!}$$

$$\leq \left(\frac{\tilde{X} + h\Phi}{\nu_2}\right)^{\nu_2} \sum_{r_3=0}^{2\nu_1} \frac{1}{h^{R-r_3}} \cdot \frac{\Psi^{r_3/2}}{(r_3/2)!} \cdot \frac{Y^{\nu_3-(r_3/2)}}{(\nu_3 - (r_3/2))!}$$

$$\leq \left(\frac{2}{h^R}\right) \cdot \frac{(\tilde{X} + h\Phi)^{\nu_2}}{\nu_2!} \cdot \frac{(Y + \tilde{h}\Psi)^{\nu_3}}{\nu_3!}. \quad (4.23)$$
Regarding (4.18) and using denotation \( Y = 36V_{12}s^3/n^2 \), we get the following estimate:

\[
\tilde{Z}_{2s}^{(3)} \leq \frac{n(2s)!}{s!(s+1)!} \exp \left\{ C_0 \frac{s^3}{n^2} - \frac{s^2}{2n} \right\} \sum_{\theta \in \Theta_{2s}} \sum_{d \geq s^{1/2} - \epsilon} \sum_{N=1}^{s} \sum_{K=0}^{N} I_{\tilde{\theta}_{2s}^{(d/(2N+K))}}(\theta) \\
\times \sum_{\sigma=0}^{C_0 s^2/n} \sum_{\nu_2=2\nu_4=\sigma} \sum_{R \geq K} \sum_{r_2+r_3=R} \frac{1}{r_2!} \cdot \Phi^{r_2} \\
\times \frac{1}{(\nu_2 - r_2)!} \left( X + \frac{2sdV_4}{n} \right)^{\nu_2-r_2} \cdot \frac{\Phi^{r_2}}{r_2!} \cdot \frac{Y^{\nu_3-(r_3/2)}}{(\nu_3 - \langle r_3/2 \rangle)!} \cdot \frac{\Psi^{(r_3/2)}}{\langle r_3/2 \rangle!}.
\]

(4.24)

Applying (4.23) with \( \tilde{X} = X + 2sdV_4/n \) to the last two sums of (4.24) and repeating computations of the previous subsection, we can write that

\[
\tilde{Z}_{2s}^{(3)} \leq 6C_0 s^5 \cdot \exp \{-\mu^{1/4} n^{1/9} + 6 \log h_0 + (C_0 + 36V_{12}) \mu \cdot \left( B(24h_0 \mu^{1/2}) + 1 \right) \}.
\]

(4.25)

Here we have used the fact that \( \Psi = o(\Phi) \) as \( (s,n) \mu \rightarrow \infty \). Clearly, \( \tilde{Z}_{2s}^{(3)} = o(1) \) in the limit \( (s,n) \mu \rightarrow \infty \).

### 4.4.5 Estimate of \( \tilde{Z}_{2s}^{(3)} \)

In this subsection we obtain the estimate of the right-hand side of (4.18) in the case of \( \sigma_4 \geq 1 \). It differs from the previous case of \( \sigma_4 = 0 \) by the factor \( (k - 1) \nu_k \rho_k / \rho_k! \) that estimates the number of choices of \( \rho_k \) among \( \sigma_4 \) ones. Relation (4.19) shows that this factor is compensated because of the presence of the vertices with \( \kappa \geq 4 \) that produce the factor \( n^{-\sigma_4/12} \) of (4.19b).

Repeating the arguments and the computations of the previous subsection, we get the estimate

\[
\tilde{Z}_{2s}^{(3)} \leq n \sum_{d=s^{1/2} - \epsilon}^{s} \sum_{\sigma=0}^{C_0 s^2/n} \sum_{N=1}^{s} \sum_{K=0}^{\sigma} \sum_{r_2+r_3+\rho_4=R} \sum_{K=0}^{r_3} \sum_{k=1}^{N} \frac{(2s)!}{s!(s+1)!} \\
\times \frac{1}{(\nu_2 - r_2)!} \left( X + \frac{2sdV_4}{n} \right)^{\nu_2-r_2} \cdot \frac{\Phi^{r_2}}{r_2!} \cdot \frac{Y^{\nu_3-(r_3/2)}}{(\nu_3 - \langle r_3/2 \rangle)!} \cdot \frac{\Psi^{(r_3/2)}}{\langle r_3/2 \rangle!} \\
\times \frac{1}{n^{\sigma_4/12}} \cdot \frac{\sigma_4^{\rho_4}}{\rho_4!} \cdot \prod_{k=1}^{s} \frac{\delta_4^{\nu_k}}{\nu_k!} \cdot \frac{H^{(k-1)\nu_k}}{n} \\
\times \exp \left\{ C_0 \mu - \frac{s^2}{2n} \right\} \cdot \exp \left\{ -\frac{C_2d}{2N^2 + K} \right\} \cdot \left( \frac{U_{2N} C_1^N s^N}{n^N - 1} \right)^{I_4(N)}.
\]

(4.26)
where we denoted $N' = \max\{3, N\}$ and $I_4(N) = I_{(4, +\infty)}(N)$.

Let us describe the operations we perform to estimate the right-hand side of (4.24). First we estimate the sum over all possible sets $(\nu_4, \nu_5, \ldots, \nu_s)$ as follows;

$$
\sum_{\nu_4 + \ldots + \nu_s = \sigma_4 \geq 1} \prod_{k=4}^{s} \frac{\delta_{\nu_k}}{\nu_k!} \cdot H_{\nu_k}^{(k-4)\nu_k} \leq \exp \left\{ \delta_n \sum_{k \geq 4} H_{\nu_k}^{k-4} \right\},
$$

(4.27)

where the series over $k$ is obviously convergent. The last expression tends to 1 as $(s, n) \mu \to \infty$.

Next, using an analog of (4.23), we can write that

$$
\sum_{r_2 + r_3 = R - \rho_4} \Phi_{r_2} \cdot \tilde{X}_{r_2} \cdot Y_{r_3 - (r_3/2)} \cdot \Psi_{r_3/2} \leq \frac{2}{h^{R-\rho_4}} \cdot \frac{1}{\nu_2!} (\tilde{X} + h\Phi)^{\nu_2} \cdot \frac{1}{\nu_3!} (Y + h^2\Psi)^{\nu_3}.
$$

(4.28)

Finally, we observe that the following inequality is true;

$$
\sum_{R \geq K} \frac{1}{h^R} \sum_{\rho_4 = 0}^{R} \frac{(h\sigma_4)^{\rho_4}}{\rho_4!} \cdot \frac{1}{n^{\sigma_4/12}} \leq \frac{h}{h^K(h - 1)} \cdot e^{h\sigma_4}.
$$

(4.29)

Taking into account (4.27), (4.28), and (4.29), we derive from (4.26) inequality

$$
Z_{2_s}^{(3)} \leq \sum_{d = \sigma_4^{1/2} - \epsilon}^{s} \frac{n(2s)!}{s!(s + 1)!} \cdot \exp\{C_0\mu + 36V_{12}\mu^3\} \cdot \left( B(24h\mu^{1/2}) + 1 \right) \cdot \left( \frac{U_{2N}C_{1}^{N}S_{N}^{N}}{n^{N-1}} \right)^{I_4(N)}
$$

(4.30)

for all $n$ such that for $n \geq \exp\{12h\}$.

Similarly to the situation encountered in (4.22), we consider the following function of two variables

$$
F(N, K) = \frac{2sdV_4}{n} - \frac{C_2d}{2\max(3, N) + K} - K \log h - \frac{N - 3}{3} I_4(N) \log n.
$$

Denoting $N'' = (N - 3)I_{(4, +\infty)}(N)$, we see that the problem of estimate of $Z_{2_s}^{(3)}$ for large enough values of $n$ such that $\log n^{1/6} \geq \log h$ is reduced to the study of the maximal value of the function

$$
\tilde{F}(N'', K) = \frac{2sdV_4}{n} - \frac{C_2d}{6 + K + 2N''} - (K + 2N'') \log h.
$$
This maximum corresponds to the value \( f_{\text{max}} \) determined in the previous subsections and we can write that
\[
F(N, K) \leq \tilde{F}(N''', K) \leq -2 \left( n^{2/3} \mu^{1/3} \right)^{1/4-\epsilon/2} + 6 \log h_0
\]
provided \( \log n \geq 6 \log h_0 \).

Then we deduce from (4.30) inequality
\[
\tilde{Z}^{(3)}_{2s} \leq \frac{n(2s)!}{s!(s+1)!} \cdot \exp\{C_0 \mu + 36 V_{12} \mu^3\} \cdot \left( 24 h \mu^{1/2} + 1 \right)
\times \frac{4C_0 h_0 s^7}{h_0 - 1} \cdot \exp\left\{ -2 \left( n^{1/6-\epsilon/3} \mu^{1/12-\epsilon/6} \right) + 6 \log h_0 \right\}
\]
that holds for all \( n \) such that \( n \geq \exp\{12h_0\} \). We see that the choice of \( 0 < \epsilon < 1/6 \) makes the product of the last two terms of (4.28) vanishing in the limit \((s, n)_{\mu} \to \infty\). Then \( \tilde{Z}^{(3)}_{2s} = o(1) \) as \((s, n)_{\mu} \to \infty\) provided (4.1) holds.

This result together with (4.25) shows that the estimate of \( Z^{(3)}_{2s} \) is completed.

### 4.5 Estimate of \( Z^{(4)}_{2s} \)

In this part, we follow the general description of the walks, and give the estimate of \( Z^{(4)}_{2s} \) in the way that slightly differs from that presented in [9].

Remembering (3.10) and \( W_n \), we can write that
\[
Z^{(4)}_{2s} \leq \frac{(2s)!}{s!(s+1)!} \sum_{\sigma \geq C_0 s^2/n} \sum_{\nu_1 = \sigma} \frac{n(n-1) \cdots (n-s+\sigma)}{n^s} \prod_{k=2}^s \left( \frac{(2kU^2 s^k)}{k!} \right)^{\nu_k}
\]

The main difference between this sub-sum and the previous ones is that in (4.8) the factor \( \exp\{s\sigma\} \) is not bounded for \( \sigma \geq C_0 s^2/n \) and \( \exp\{-s^2/n\} \) cannot be used as the normalizing factor for the terms \((s^2/n)^{\sigma}/\sigma!\). In this case the last expression is bounded by itself. This is the main idea of the proof of the estimate of \( Z^{(4)}_{2s} \) proposed in [9]. We reconstruct this proof with slight modifications and corrections.

Denoting
\[
|\vec{v}|_2 = \sum_{k \geq 2} (k-2) \nu_k,
\]
we can write that
\[
\prod_{k=2}^s s^{k\nu_k} = s^{2\sigma - \chi}.
\]
Then
\[ Z_{2s}^{(4)} \leq \frac{n(2s)!}{\sigma!(s+1)!} \sum_{\sigma \geq C_0 s^2 n} \frac{s^{2\sigma}}{\sigma!} \frac{1}{s!} \frac{(n-1) \cdots (n-s+\sigma)}{n^{s-\sigma}} \cdot \frac{1}{n^\sigma} \]
\[ \times \frac{s^{2\sigma}}{s^\chi} \cdot \frac{1}{\nu_2! \nu_3! \cdots \nu_s!} \sum_{\chi=0}^\sigma \frac{(\sigma-\chi)!}{\chi!} \prod_{k=2}^s \left( C_1 U^2 \right)^{k\nu_k}. \]

Multiplying and dividing by \( \sigma! \) and by \( (\sigma - \chi)! \), we obtain inequality
\[ Z_{2s}^{(4)} \leq \frac{n(2s)!}{\sigma!(s+1)!} \sum_{\sigma \geq C_0 s^2 n} \frac{s^{2\sigma}}{\sigma!} \sum_{\chi=0}^\sigma \frac{(\sigma-\chi)!}{\chi!} \frac{1}{s^\chi} \frac{(n-1) \cdots (n-s+\sigma)}{n^{s-\sigma}} \cdot \frac{1}{n^\sigma} \]
\[ \times \frac{s^{2\sigma}}{s^\chi} \cdot \frac{1}{\nu_2! \nu_3! \cdots \nu_s!} \sum_{\chi=0}^\sigma \frac{(\sigma-\chi)!}{\chi!} \prod_{k=2}^s \left( C_1 U^2 \right)^{k\nu_k}. \]

Using the definition of \( |\bar{\nu}|_2 \), we can rewrite the previous estimate in the form
\[ Z_{2s}^{(4)} \leq \frac{n(2s)!}{\sigma!(s+1)!} \sum_{\sigma \geq C_0 s^2 n} \frac{s^{2\sigma}}{\sigma!} \sum_{\chi=0}^\sigma \frac{(\sigma-\chi)!}{\chi!} \frac{1}{s^\chi} \frac{(n-1) \cdots (n-s+\sigma)}{n^{s-\sigma}} \cdot \frac{1}{n^\sigma} \]
\[ \times \frac{s^{2\sigma}}{s^\chi} \cdot \frac{1}{\nu_2! \nu_3! \cdots \nu_s!} \sum_{\chi=0}^\sigma \frac{(\sigma-\chi)!}{\chi!} \prod_{k=2}^s \left( C_1 U^2 \right)^{k\nu_k}. \]

At this point we separate the sum over \( \sigma \) into two parts: the first sub-sum we denote by \( \hat{Z}_{2s}^{(4)} \) corresponds to the interval \( \frac{C_0 s^2}{n} \leq \sigma \leq \frac{s}{2C_1 U^2} \), the second denoted by \( \check{Z}_{2s}^{(4)} \) corresponds to the remaining part \( \sigma \geq \frac{s}{2C_1 U^2} \).

Aiming the estimate of \( \hat{Z}_{2s}^{(4)} \), we use the multinomial theorem in the form
\[ \sum_{\nu_2 + \cdots + \nu_s = \sigma - \chi} \frac{(\sigma - \chi)!}{\nu_2! \cdots \nu_s!} \frac{1}{2^{(k-2)\nu_k}} \prod_{k=2}^s \frac{1}{2(2s-2)\nu_k} \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{2s-2}\right)^{\sigma-\chi} \leq 2^{\sigma-\chi}. \]

Since \( 2C_1^2 U^4 > 1 \), we deduce from (4.32) with the help of the Stirling formula that

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\[
\hat{Z}_s^{(4)} \leq \frac{n(2s)!}{s!(s+1)!} \sum_{C_0 s^2/n \leq \sigma \leq s/(2C_1 U^2)} \frac{1}{\sigma!} \left( \frac{s^2}{n} \right)^\sigma \sum_{\chi=0}^\sigma (2C_1^2 U^4)^{\sigma-\chi}
\]

\[
\leq \frac{n(2s)!}{s!(s+1)!} \sum_{\sigma \geq C_0 s^2/n} \frac{(2C_1^2 U^4)^{\sigma+1}}{2C_1^2 U^4 - 1} \cdot \left( \frac{e s^2}{n \sigma} \right)^\sigma \cdot \frac{1 + o(1)}{\sqrt{2\pi \sigma}}
\]

\[
\leq \frac{n(2s)!}{s!(s+1)!} \cdot \frac{2C_1^2 U^4}{2C_1^2 U^4 - 1} \cdot \sum_{\sigma \geq C_0 s^2/n} \left( \frac{2C_1^2 U^4 e}{C_0} \right)^\sigma \cdot \frac{2}{\sqrt{\pi \sigma}}
\]

\[
\leq \frac{n(2s)!}{s!(s+1)!} \cdot \frac{2C_1^2 U^4}{2C_1^2 U^4 - 1} \cdot \frac{2}{\sqrt{\pi C_0 s^2/n}} \cdot \left( \frac{2C_1^2 U^4 e}{C_0} \right)^{C_0 s^2/n}.
\quad (4.33)
\]

The product of two last terms vanishes as \((s, n)_\mu \to \infty\) in the case when \(C_0 \geq 2 e C_1^2 U^4\).

Now it remains to consider the part that is complementary to \(\hat{Z}_s^{(4)}\); it is estimated by the following expression

\[
\hat{Z}_s^{(4)} \leq \frac{n(2s)!}{s!(s+1)!} \sum_{\sigma \geq s/(2C_1 U^2)} \frac{1}{\sigma!} \left( \frac{s^2}{n} \right)^\sigma \sum_{\chi=0}^\sigma \left( \frac{2C_1^2 U^4}{C_0} \right)^{s-\chi}
\]

\[
\times \sum_{\nu: |\nu|=\sigma, |\nu_1|=\chi} \frac{1}{\nu_1! \ldots \nu_s!} \prod_{k=2}^s (C_1 U^2)^{k \nu_k}.
\quad (4.34)
\]

Using the identity \(\prod_{k=2}^s (2C_1 U^2)^{(k-1)\nu_k} = (2C_1 U^2)^{\sigma}\), we can replace the product of the last four factors of (4.34) by

\[
\frac{1}{\sigma!} \left( \frac{2C_1 U^2 s^2}{n} \right)^{\sigma \chi} \cdot \frac{\sigma!}{s^\chi \cdot (\sigma-\chi)!} \cdot \frac{(\sigma-\chi)!}{\nu_2! \ldots \nu_s!} \prod_{k=2}^s \left( \frac{C_1 U^2}{2^{k-1}} \right)^{\nu_k}
\]

Using again the Stirling formula, remembering that \(\sigma \leq s\), and applying the multinomial theorem to the sum over \(\nu\), we get that

\[
\hat{Z}_s^{(4)} \leq \frac{n(2s)!}{s!(s+1)!} \sum_{\sigma \geq s/(2C_1 U^2)} \sqrt{\frac{\sigma}{2\pi}} \left( \frac{4C_1^3 U^6 \mu^{1/3} e}{n^{1/3}} \right)^\sigma.
\]

The last series is obviously \(o(1)\) as \((s, n)_\mu \to \infty\). The estimate of \(Z_s^{(4)}\) is completed.
5 More examples of the walks

Let us consider a walk $W_{18}^{(0)}$ that has a number of open simple self-intersections.

![Figure 4: The marked edges of the graph $g(W_{18}^{(0)})$ and the full graph $g(W_{18}^{(0)})$](image)

Basing on the example given on Figure 4, one can easily construct a sequence of walks of $2s$ steps such that their graphs contain a vertex with the exit degree that infinitely increases as $s \to \infty$; the self-intersection degree of this vertex remains bounded and the corresponding Dyck path $\theta_{2s}$ is such that its tree $T(\theta_{2s})$ has no vertices with large degree. Below we present one of the possible examples.

Regarding the walk $W_{2s}^{(0)}$ with $2s = 18$ steps depicted on Figure 4, we see that the marked instants are given by the instants of time $1, 2, 3, 5, 6, 8, 9, 11, 12$ and the vertex $\beta$ is the vertex of the self-intersection degree 1. This walk contains four BTS-instants given by $3, 5, 8, 11$. There are four imported cells at the vertex $\beta$ determined by the instants of time $4, 7, 10$ and $13$.

Let us modify this walk $W_{18}^{(0)}$ to $W^{(1)}$ by adding some "there-and-back" steps after each non-marked arrival $(\gamma_i, \beta)$, $i = 1, 2, 3$; let them be three each time. Then the graph of the walk is added by nine edges and we get the walk

$$W^{(1)} = (\alpha, \beta, \gamma_1, \alpha, \beta, \gamma_2, \beta, \varepsilon_1, \beta, \eta_1, \beta, \mu_1, \beta, \gamma_3, \gamma_2, \beta, \varepsilon_2, \beta, \eta_2, \beta, \mu_2, \beta, \delta, \gamma_3, \ldots)$$

three edges  three edges

The number of marked steps that leave $\beta$ denoted by $\nu_n$ in papers [12] and [9] and by $\deg_e(\beta)$ in the present paper is increased by 9 but the self-intersection degree of $\beta$ is still equal to one; $\kappa(\beta) = N = 1$. 

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Certainly, one can consider analogs of $W^{(1)}$ with more vertices of the type $\gamma_i$, say ten: $\gamma_1, \ldots, \gamma_Q$, $Q = 10$. If we add ten triplets of "there-and-back" edges with vertices $\varepsilon_i, \eta_i, \mu_i$ and pass them after each arrival to $\beta$ by non-marked steps $(\gamma_i, \gamma)$, respectively, then we get a walk $W^{(2)}(Q)$ with $\deg_e(\beta) = \nu_n = 3Q + Q + 1 = 41$ and still $\kappa(\beta) = 1$. If $Q \to \infty$, then $\deg_e(\beta)$ infinitely increases.

Regarding the walks $W^{(2)}(Q)$ with arbitrarily large $Q$, we see that the expression $\exp\{c \deg_e(\beta) s_n/n\}$ of (4.14 [9]) goes out of the control in the limit of large $s$ because it cannot be suppressed by the factor of the form $s^{N/N-1}$ with $N = \kappa(\beta)$, $\kappa(\beta) = 1$.

From another hand, the trees $T(W^{(2)}(Q))$ are such that for any value of $Q$ there is no vertices in $T(W^{(2)}(Q))$ with the exit degree greater than 5. This situation is possible because the groups of three edges are imported at $\beta$ from different parts of the tree $T$; freely expressed, these edges grow in $g(W^{(2)}(Q))$ from imported cells. As we have seen in Section 2, the presence of the imported cells is possible due to the presence of BTS-instants in the walk. Thus we conclude that the trees under consideration do not verify the $L(6)$-property (see subsection 4.1) and therefore there is no factors with the exponential estimates of the form (4.17) that would suppress the growth of (4.14 [9]). Therefore the proof of the estimate of $Z_{2s}^{(3)}$ presented in [9] is not correct. The same is true with respect to the proof of the Lemma 3 of [11].

6 Appendix

The main aim of the paper [9] was to extend the universality results of the papers [11, 12] to the more general case when the entries of the Wigner random matrices $a_{ij}$ (1.1), (1.2) are given by random variables with polynomially decaying probability distribution. We are going to prove the statement that shows that this is indeed the case. To do this, we need just a slight modification of the proof of Theorem 4.1.

In paper [9] the Wigner ensemble is considered, where the random variables $a_{ij}$ verify condition $P\{|a_{ij}| > x\} \leq x^{-18}$. However, to make inequality (4.7 [9]) true, one has to require more restrictive conditions, say with exponent 18 replaced by 36. In the present paper, we do not aim the optimal conditions for $a_{ij}$ and prove our statement in the frameworks of [9] and [12]; modifications of the proof of Theorem 4.1 necessitate also more conditions on $a_{ij}$ than those of [9] in the case of polynomially bounded random variables.

**Theorem 6.1** Let us consider the random matrix ensemble described in Theorem 4.1 with the bound (4.2) replaced by the following condition:

$$E|a_{ij}|^q < \infty$$

for all $q \leq q_0 = 76$. Then the estimate (4.4) is true in the limit $(s, n)_\mu \to \infty$, where the constants do not depend on particular values of the moments $Ea_{ij}^{2k}$, $k \geq 2$. 46
Proof. By the standard approach of the probability theory, we introduce the truncated random variables
\[ \hat{a}_{ij} = \begin{cases} a_{ij}, & \text{if } |a_{ij}| \leq U_n; \\ 0, & \text{if } |a_{ij}| > U_n, \end{cases} \] (6.2)
where \( U_n = n^\alpha \) with \( \alpha = 1/25 \). Then we consider the random matrices \( \hat{A}_{ij} = \hat{a}_{ij}/n^{1/2} \) and write down equality
\[ \mathbb{E} \left\{ \operatorname{Tr} \hat{A}_{n}^2 \right\} = \sum_{l=1}^{4} \hat{Z}_{2s}^{(l)}, \]
where \( \hat{Z}_{2s}^{(l)} \) are determined exactly as it is done in (4.5). We are going to show that these sub-sums admit the same asymptotic estimates as \( Z_{2s}^{(l)} \) of (4.5).

The first sub-sum \( \hat{Z}_{2s}^{(1)} \) is estimated as \( Z_{2s}^{(1)} \) with no changes.

To estimate \( \hat{Z}_{2s}^{(2)} \), we repeat the reasonings of subsection 4.3.1 that lead to inequality (4.14) and (4.15). The estimate \( \hat{Z}_{2s}^{(2)} = o(1) \) as \((s, n) \mu \to \infty \) is valid due to relations
\[ \frac{C_k U_{2k} s^k}{n^{k-1}} = C_k \mu^{k/3} \frac{n^{2k/3+2k/25}}{n^{k-1}} = C_k \mu^{k/3} n^{1-19k/75} \to 0, \quad n \to \infty \] (6.3)
for all \( k \geq 4 \).

To estimate \( \hat{Z}_{2s}^{(3)} \), we introduce a slight modification of the computations used to estimate \( Z_{2s}^{(3)} \). All that we need here is to redefine the variables \( \delta_n \) and \( H_n \) of (4.19). We rewrite (4.19a) in the form
\[ \frac{C_k U_{2k} n^{2k/3}}{n^{(k-1)/3}} = C_k \mu^{A/3} \frac{n^{8\alpha+2/3}}{n^{1-\beta}} \left( \frac{C_k \mu^{1/3} n^{2\alpha}}{n^{1-\beta/3}} \right)^{k-4} \frac{1}{n^{(k-1)/3}} = \hat{\delta}_n \hat{H}_n^{k-4} \frac{1}{n^{(k-1)/3}}. \] (6.4)

If \( 8\alpha + \frac{2}{3} \leq 1 - \beta \), then \( \hat{\delta}_n = O(1) \) and \( \hat{H}_n \to 0 \). The choice of \( \beta = 1/75 \) leads to the value \( \alpha = 1/25 \) imposed in (6.2). We see that the value \( \alpha_0 = 1/24 \) represents the lower bound for \( \alpha \) in the approach developed.

Then we can use the analogue of (4.19b) with \( n^{\sigma_k/12} \) replaced by \( n^{\sigma_k/225} \). All other computations that lead to the estimate of \( Z_{2s}^{(3)} \) can be repeated as they are.

The estimate of \( \hat{Z}_{2s}^{(4)} \) requires somehow more work. To estimate this sub-sum, let us prove the following auxiliary statement.

Lemma 6.1 Given any walk \( w_{2s} \) of the type \( \tilde{\nu} \), the weight \( Q(w_{2s}) \) (2.1) is bounded as follows;
\[ Q(w_{2s}) \leq \prod_{k=2}^{8} \left( V_{12} U_{n}^{2(k-2)} \right)^{\nu_k}. \] (6.4)
Proof. Let us consider a vertex $\gamma$ with $\kappa(\gamma) \geq 2$ of the multi-graph $g(w_{2s}) = (V, E)$ and color in certain color the first two marked arrival edges at $\gamma$ and their non-marked closures. Passing to another vertex with $\kappa \geq 2$, we repeat the same procedure and finally get $4 \sum_{k=2}^{s} \nu_k$ colored edges. Clearly, it remains $2\nu_1 + 2 \sum_{k=2}^{s} (k-2)\nu_k$ non-colored (grey) edges in $E(g)$. Let us remove from the graph $(V, E)$ all grey edges excepting the marked edges $e_j$ whose heads are the vertices of $N_1$; also we do not remove the closures of these edges $e_j$. We denote the remaining graph by $g^\circ = (V, E^\circ)$. The number of the edges removed from $g$ is greater or equal to $2 \sum_{k=2}^{s} (k-2)\nu_k$. When estimating the weight of $w_{2s}$, we replace corresponding random variables by non-random bounds $U_n$ when the removed marked edges end at the vertices with $\kappa \geq 2$. Then we can write that

$$Q(w_{2s}) \leq U_n^{2(k-2)\nu_k} \cdot Q^\circ(w_{2s}),$$

where $Q^\circ(w_{2s})$ represents the product of the mathematical expectations of the random variables associated with the edges of $E^\circ$. Obviously, we did not replace by $U_n$ those random variables that give factors $V_2 = 1$.

In the remaining (possibly non-connected) graph $g^\circ$ the set $V$ contains a subset $V^\circ$ such that if $\beta \in V^\circ$, then $\beta$ is the head of of two colored marked edges. Clearly, $|V^\circ| = \sum_{k=2}^{s} \nu_k$. Regarding a head $\beta \in V^\circ$ of two marked colored edges, we see that their tails $\alpha'$ and $\alpha''$ can be either equal or distinct.

Let us consider first the case when the tails are distinct, $\alpha' \neq \alpha''$. Then each of the multi-edges $(\alpha', \beta)$ and $(\alpha'', \beta)$ can produce factors $V_4$ or $V_6$ in dependence how many edges arrive at $\alpha'$ and $\alpha''$ from $\beta$. Then the contribution of the vertex $\beta$ to $Q^\circ(w_{2s})$ is bounded by $V_8^2 \leq V_{12}$.

Now let us consider the case when the tails are equal, $\alpha' = \alpha'' = \alpha$. Then the multi-edge $|\alpha, \beta|$ produces the factors equal to either $V_4$ or $V_6$ or $V_8$ in dependence of how many marked edges of $E^\circ$ arrive at $\alpha$ from $\beta$. This can be either one grey edge or two colored edges. Since $1 \leq V_4 \leq V_6 \leq V_8$, then the contribution of this vertex $\beta$ to $Q^\circ(w_{2s})$ is bounded by $V_8 \leq V_{12}$. Here we have used inequality $\kappa(\alpha) \geq 1$, $\alpha \in V(g_{2s})$.

Collecting the contributions of all vertices of $V^\circ$, we get the estimate

$$Q^\circ(w_{2s}) \leq \prod_{k=2}^{s} V_{12}^{\nu_k}.$$
\[
\hat{Z}_{2s}^{(4)} \leq \frac{n(2s)!}{s!(s+1)!} \sum_{\sigma \geq C_0 s^2/n} \sum_{\chi=0}^\sigma \frac{1}{\sigma!} \left( \frac{s^2}{n} \right)^\sigma \\
\times \sum_{\nu: |\nu|_1=\sigma, |\nu|_2=\chi} \frac{(\sigma-\chi)!}{\nu_2! \nu_3! \cdots \nu_s!} \prod_{k=2}^s \left( \frac{C_1 U_n^2 \sigma}{s} \right)^{(k-2)\nu_k} \left( \frac{C_1^2 V_{12}}{C_0} \right)^{\nu_k}. 
\]

(6.5)

Let us consider first the interval \( C_0 s^2/n \leq \sigma \leq s/(2C_1 U_n^2) \). Using the multinomial theorem and the Stirling formula, we get from (6.5) the following analog of the estimate (4.32);

\[
\hat{Z}_{2s} \leq \frac{n(2s)!}{s!(s+1)!} \sum_{\sigma \geq C_0 s^2/n \leq s/(2C_1 U_n^2)} \frac{1}{\sigma!} \left( \frac{s^2}{n} \right)^\sigma \sum_{\chi=0}^\sigma (2C_1^2 V_{12})^{\sigma-\chi} \\
\leq \frac{n(2s)!}{s!(s+1)!} \cdot \frac{2C_1^2 V_{12}}{2C_1^2 V_{12} - 1} \cdot \frac{2}{\sqrt{\pi C_0 s^2/n}} \cdot \left( \frac{2eC_1^2 V_{12}}{C_0} \right)^{C_0 s^2/n}.
\]

This expression is \( o(1) \) in the limit \((s,n)_\mu \to \infty\) provided \( C_0 \geq 2eC_1^2 V_{12} \).

To estimate the sub-sum \( \hat{Z}_{2s} \) that corresponds to the interval \( \sigma \geq s/(2C_1 U_n^2) \), we repeat word by word the computations presented at the end of Section 4 (see formulas (4.34) and (4.35)).

Summing up the arguments presented above, we see that the following analog of (4.4)

\[
\frac{\sqrt{\pi} \mu}{4^s} \mathbb{E} \left\{ \text{Tr} \hat{A}_n^{2s} \right\} \leq B(6\mu^{1/2}) \exp\{(36 + C_0)\mu\} 
\]

is valid in the limit \((s,n)_\mu \to \infty\).

By the standard arguments of the probability theory, we have

\[
P\{A_n \neq \hat{A}_n\} \leq 1 - (1 - n^{-q_0 \alpha} \mathbb{E}|a_{ij}|^{q_0})^{n(n+1)/2} = O(n^{-1-\delta}), \quad \delta > 0
\]
as \( n \to \infty \). Then by the Borel-Cantelli lemma,

\[
P\{A_n \neq \hat{A}_n \text{ infinitely often}\} = 0.
\]

(6.7)

Relations (6.6) and (6.7) complete the proof of Theorem 6.1.

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