Nonlinear effect on quantum control for two-level systems

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Abstract

The traditional quantum control theory focuses on linear quantum systems. Here we show the effect of nonlinearity on the quantum control of a two-level system, we find that the nonlinearity can change the controllability of the quantum system. Furthermore, we demonstrate that the Lyapunov control can be used to overcome this uncontrollability induced by the nonlinear effect.

Quantum control theory is about the application of classical and modern control strategy to quantum systems. It has generated increasing interest in the last few years due to its potential applications in metrology [1, 2], communications [3, 4] and other technologies [5–9], as well as its theoretical interest. As the effective combination of control theory and quantum mechanics, the quantum control theory is not trivial for several reasons. For a classical control, feedback is a key factor in the control design, and there has been a strong emphasis on the robust control of linear systems. The quantum system in the feedback control, on the other hand, cannot usually be modelled as linear control systems, except when both the system and the controller as well as their interaction are linear. In fact for many quantum systems, the nonlinear effects cannot be negligible1, and in some cases they dominate the dynamics of the quantum system. Moreover, the feedback control requires measurement of an observable and returns the measured result as a control back to the quantum system. This renders the dynamics of the quantum system both nonlinear and stochastic [10]. In special cases the resulting evolution can be mapped into a linear classical system driven by Gaussian noise, and consequently the optimal control problem can be solved by classical control theory. However, most control problems for such a quantum system cannot be solved in this way. Therefore, a study on nonlinear effects in quantum control theory is highly desired.

Lyapunov functions, originally used in control to analyse the stability of the control system, have formed the basis for new control design. Several papers have been published recently to discuss the application of the Lyapunov control to quantum systems [11–17]. Although the basic mathematical formulism is well established, many questions remain when one uses the Schrödinger equation or the master equation to describe the dynamics of the quantum system, for example the nonlinear effect in the quantum control.

In this paper, we shall address this issue by analysing a two-level system with nonlinear effect. Before studying the nonlinear effect in the quantum control on the two-level system, we recall that a linear two-level system is controllable by two independent parameters; then we show that nonlinear interactions may turn the controllable two-level system into uncontrollable one, and this uncontrollability can be overcome by Lyapunov control.

Consider a two-level system described by

\[ H = \frac{R}{2} \sigma_z + \frac{v}{2} \sigma_x, \]

where \( \sigma_z \) and \( \sigma_x \) are Pauli matrices. This model was proposed to describe the tunnelling of the quantum system in a double-well potential. In this model, \( v \) is the coupling constant of the two wells. \( R \) denotes the energy difference between the two levels. We first show that this system is controllable by manipulating the two independent parameters \( R \) and \( v \). The controllability requires that all initial states in the Hilbert space \( \mathcal{H}_2 \) of the system can evolve to an arbitrary pure target state. This requirement for the initial state can be partially lifted by

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1 By nonlinear we mean that the Heisenberg equation is nonlinear; hence, under the mean field approximation, the equation of motion of the system is nonlinear.
where 0 \leq \phi \leq 2\pi. All parameters are dimensionless. (a) and (b) (c) and (d) are the same but show the results from the opposite direction. \( \vec{p} = (p_x, p_y, p_z) \) is the Bloch vector, and \( \theta = \phi = 0 \).

requiring that the Hamiltonian \( H \) is unchanged up to \( R \) and \( v \) under the following unitary transformation [18]:

\[
F = \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
-\sin \theta e^{i\phi} & \cos \theta
\end{pmatrix},
\]

where 0 \leq \theta \leq \pi. By unchanged we mean \( H' = H(R', v') = \hat{F} H \hat{F}^\dagger \), namely the transformation \( F(\theta, \phi) \) changes the control parameter in the Hamiltonian \( H \) only. The proof is straightforward. Consider the Schrödinger equation

\[
\frac{i\hbar}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle,
\]

where \( |\psi(t)\rangle \) is the wavefunction of the system. By the time-independent transformation \( F(\theta, \phi) \), \( |\psi(t)\rangle \rightarrow F|\psi(t)\rangle \) we find

\[
\frac{i\hbar}{\partial t} (F|\psi(t)\rangle) = H(F|\psi(t)\rangle),
\]

and

\[
\frac{i\hbar}{\partial t} |\psi(t)\rangle = \hat{F} H \hat{F}^\dagger |\psi(t)\rangle.
\]

Since \( H(R', v') = \hat{F} H(R, v) \hat{F}^\dagger \), we claim that there exists a one-to-one correspondence in sets \(|\psi(t)\rangle\) and \(|\psi(t)\rangle\). Therefore, if \( |\psi(t)\rangle \) covers all (pure) states in \( \mathcal{H}, \{ |\psi(t)\rangle \} \) is a convex set of all possible (pure) states for the two-level system. This observation tells us that if the system initially prepared in state \( |e\rangle \) can be controlled to evolve to an arbitrary target state \( |g\rangle \) by the Hamiltonian \( H' = H(R', v') = \hat{F} H(R, v) \hat{F}^\dagger \), the system is controllable. This is exactly the case as shown in figure 1, where we plot the accessible states represented by the Bloch vector \( \vec{p} = (p_x, p_y, p_z) \). The Bloch vector is connected to an arbitrary state \( |\psi(t)\rangle = a(t)|e\rangle + b(t)|g\rangle \) of the two-level system through

\[
\rho = |\psi(t)\rangle \langle \psi(t)| = \frac{1}{2} + \frac{1}{2} \vec{p} \cdot \vec{\sigma},
\]

with

\[
p_z = 2a(t)^2 - 1, \quad p_x = a^*(t)b(t) + b^*(t)a(t),
\]

and

\[
p_y = i(a(t)b^*(t) - b(t)a^*(t)).
\]

We find from figure 1 that by varying the parameters \( R \) and \( v \), the two-level system indeed can evolve to an arbitrary target pure state, provided the evolution time is long enough and there is a wide range of parameters \( R \) and \( v \) to manipulate. This controllability can be found by observing the accessible states which can be represented by the Bloch vectors. When the accessible states occupy every point on the Bloch sphere, we say the system is controllable. It is worth addressing that the Hamiltonian in equation (1) becomes

\[
H' = \frac{R}{2} \sigma_z + \frac{v}{2} \sigma_x + \frac{u}{2} \sigma_y
\]

after the unitary transformation \( F(\theta, \phi) \), where \( R' \) (or \( v' \)) is a function of \( R, v, \theta, \) and \( \phi \). \( u' \) in general is not zero. At first sight, \( H \) does not satisfy the condition \( H(R', v') = \hat{F} H \hat{F}^\dagger \); then the two-level system driven by the Hamiltonian equation (1) is uncontrollable. This is not the case, however, because there are only two independent parameters \( (R, v) \) in the Hamiltonian; hence, \( u' = R \cos \theta \sin \theta \sin \phi - \frac{\sqrt{2}}{2} \sin^2 \theta \sin 2\phi \) is not independent. Therefore, the term with \( u' \) in \( H' \) plays no role in the control on the two-level system, and for details, we refer the readers to [19, 20].

Now we study the effect of nonlinearity on the controllability of the system. For this goal, we consider a nonlinear two-level model,

\[
H_{nl} = \frac{R}{2} \sigma_z - \frac{C}{2} (|\psi\rangle \langle \sigma_z | \psi\rangle \sigma_z + v \sigma_x),
\]

where \( |\psi\rangle = |\psi(t)\rangle = (a(t)|e\rangle + b(t)|g\rangle \) and the parameter \( C \) characterizes the nonlinear interaction strength, and the other parameters have the same notations as in equation (1). This model can be used to describe the tunnelling of Bose–Einstein condensates in a double-well potential and was widely used to study the self-trapping and tunnelling in those systems [21, 22].

By the unitary transformation \( F(\theta, \phi) \), the Hamiltonian \( H_{nl} \) is transformed into

\[
H'_{nl} = \hat{F} H_{nl} \hat{F}^\dagger,
\]

\[
H'_{nl} = \left( \frac{R_{nl}}{2} \cos^2 \phi - v \cos \theta \sin \theta \cos \phi - \frac{R_{nl}}{2} \sin^2 \theta \right) \sigma_z
\]

\[
+ \left( \frac{v}{2} \cos^2 \phi + R_{nl} \cos \theta \sin \theta \cos \phi - \frac{v}{2} \sin^2 \theta \sin 2\phi \right) \sigma_x
\]

\[
+ R_{nl} \cos \theta \sin \theta \sin \phi - \frac{\sqrt{2}}{2} \sin^2 \theta \sin 2\phi \right) \sigma_y,
\]

where \( R_{nl} = R - C \langle \sigma_z | \psi \rangle \). We have performed extensive numerical simulations for the Schrödinger equation \( \frac{i\hbar}{\partial t} |\psi\rangle = H'_{nl} |\psi\rangle \) with \( \theta = \phi = 0 \); the selected results are presented in figure 2. Two observations can be made from figure 2. (1) The nonlinearity affects the controllability of the two-level system, regardless of how small the nonlinear coupling constant \( C \) is. (2) The larger the nonlinear term is, the smaller the set of the accessible states. Further numerical simulation shows that \( \theta \) and \( \phi \) (determining the initial state) can change the accessible
where $H_0$ is the free evolution Hamiltonian and $H_1$ is the control Hamiltonian. The general control task we consider can be formulated as, given a target state $|\psi_d(t)\rangle$, we wish to apply a certain control field $f(t)$ to the system that modifies its dynamics such that $|\psi(t)\rangle \rightarrow |\psi_d(t)\rangle$ as $t \rightarrow \infty$. Since the free Hamiltonian can in general not be turned off, it is natural to assume $|\psi_d(t)\rangle$ to be time dependent and satisfies

$$i\hbar \frac{\partial}{\partial t} |\psi_d(t)\rangle = H_0 |\psi_d(t)\rangle.$$  

Since the evolution of both $|\psi(t)\rangle$ and $|\psi_d(t)\rangle$ are unitary in our case, we can define a function

$$V(|\psi_d(t)\rangle, |\psi(t)\rangle) = 1 - |\langle\psi_d(t)|\psi(t)\rangle|^2$$

(8) to measure the distance between the resulting and target states. Clearly $V \geq 0$ with equality only if $|\psi_d(t)\rangle = |\psi(t)\rangle$. Taking derivative of $V$ with respect to time $t$, we have ($|\psi_d\rangle = |\psi_d(t)\rangle$ and $|\psi\rangle = |\psi(t)\rangle$ hereafter)

$$\dot{V} = -2f(t) \text{Im}(|\langle\psi_d|H_1|\psi\rangle|\langle\psi|\psi_d\rangle),$$

(9) where Im(... denotes the imaginary part of (...). So when we choose

$$f(t) = \kappa \text{Im}(|\langle\psi_d|H_1|\psi\rangle|\langle\psi|\psi_d\rangle)$$

with a rate $\kappa > 0$, we have $\dot{V} \leq 0$. Therefore, $V$ is a Lyapunov function for the following dynamical system:

$$\frac{\partial}{\partial t} |\psi\rangle = H_0 |\psi\rangle,$$

$$\frac{\partial}{\partial t} |\psi_d\rangle = H_0 |\psi_d\rangle,$$

$$f(t) = \kappa \text{Im}(|\langle\psi_d|H_1|\psi\rangle|\langle\psi|\psi_d\rangle).$$

(10)

For a two-level system, $|\psi_d\rangle$ always can be written as

$$|\psi_d\rangle = c(t)|e\rangle + d(t)|g\rangle$$

with $|c(t)|^2 + |d(t)|^2 = 1$, and

$$|\psi\rangle = a(t)|e\rangle + b(t)|g\rangle$$

with $|a(t)|^2 + |b(t)|^2 = 1$. When the control Hamiltonian takes $H_1 = \frac{\sigma_z}{2}$, it is easy to find the Lyapunov control $f(t) = -2\kappa \text{Im}(ac^* + bd^*)(ac^* + bd^*)$. In the following, we shall focus on the control Hamiltonian $H_1 = \frac{1}{2}(\psi|\sigma_z|\psi)\sigma_z$, which yields the Lyapunov control

$$f(t) = -\kappa m \text{Im}[(ac^* + bd^*)(ac^* - bd^*)],$$

$$m = |a|^2 - |b|^2,$$

(11)

where we omitted the argument $t$ of $a(t), b(t), c(t)$ and $d(t)$ to shorten the notations. Clearly, the Lyapunov control renders the dynamics of the quantum system nonlinear even if the control Hamiltonian is linear. For a set of coupled nonlinear differential equations, it is difficult to find an analytical solution. We have performed extensive numerical simulations for the dynamics of these nonlinear system; the numerical simulations show that the two-level system described by equation (1) with Lyapunov control $f(t)$ is controllable, namely an arbitrary pure state is accessible driven by

$$H_{ad} = H_0 + f(t)H_1,$$

(12)
problem under consideration. The reason is that we choose the nonlinear coefficient $C$ and the initial state. To overcome this uncontrollability induced by the nonlinear effect, we propose to use the Lyapunov control to manipulate the two-level system; the Lyapunov function for the control system is constructed and the dependence of accessible set of states, which can be reached in a short-time limit and within a small range of control parameters on the rate $\kappa$, is shown and discussed. This study suggests that quantum control that can induce nonlinear effect changes the controllability of the quantum system, and Lyapunov control is better in this case for manipulating a quantum system.

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References

[1] Wiseman H M 1995 Phys. Rev. Lett. 75 4587
[2] Berry D W, Wiseman H M and Breslin J K 2001 Phys. Rev. A 63 053804
[3] Jacobs K 2007 Quantum Inf. Comput. 7 127
[4] Geremia J M 2004 Phys. Rev. A 70 062303
[5] Ahn C, Doherty A C and Landahl A J 2002 Phys. Rev. A 65 042301
[6] Hopkins A, Jacobs K, Habib S and Schwab K 2003 Phys. Rev. B 68 235328
[7] Sarovar M, Ahn C, Jacobs K and Milburn G J 2004 Phys. Rev. A 69 052324
[8] Steck D A, Jacobs K, Mabuchi H, Bhattacharya T and Habib S 2004 Phys. Rev. Lett. 92 223004
[9] Armen M A, Au J K, Stockton J K, Doherty A C and Mabuchi H 2002 Phys. Rev. Lett. 89 133602
[10] Habib S, Jacobs K and Shimizu K 2006 Phys. Rev. Lett. 96 010403
[11] Vettori P 2002 Proc. of the MTNS Conference http://www.nd.edu/~mtns/papers/21350.pdf
[12] Ferrante A, Pavon M and Raccanelli G 2002 Proc. of the 41st IEEE Conf. on Decision and Control
[13] Grivopoulos S and Bamieh B 2003 Proc. of the 42nd IEEE Conf. on Decision and Control
[14] Mirrahimi M and Rouchon P 2004 Proc. of IFAC Symp. LOLCOS
[15] Mirrahimi M and Rouchon P 2004 Proc. of the Int. Symp. MTNS
[16] Mirrahimi M and Turinici G 2005 Automatica 41 1987
[17] Altini C 2007 Quantum Inf. Process. 6 9
[18] Wang X and Schirmer S 2008 arXiv:0801.0702
[19] Nie J, Fu H C and Yi X X 2008 arXiv:0805.0057 to be published as QIC
[20] Schirmer S G, Fu H and Solomon A I 2001 Phys. Rev. A 63 063410
[21] Fu H, Schirmer S G and Solomon A I 2001 J. Phys. A: Math. Gen. 34 1679
[22] Milburn G J, Corney J, Wright E M and Walls D F 1997 Phys. Rev. A 55 4318
[23] Smerzi A, Fantoni S, Giovanazzi S and Shenoy S R 1997 Phys. Rev. Lett. 79 4950
[24] Lasalle J and Lefschetz S 1961 Stability by Liapunov’s Direct Method with Applications (New York: Academic)