A UNIFORM QUANTITATIVE MANIN–MUMFORD THEOREM
FOR CURVES OVER FUNCTION FIELDS

NICOLE LOOPER, JOSEPH SILVERMAN, AND ROBERT WILMS

Abstract. We prove that any smooth projective geometrically connected
non-isotrivial curve of genus $g \geq 2$ over a one-dimensional function field of
any characteristic has at most $16g^2 + 32g + 124$ torsion points for any Abel–
Jacobi embedding of the curve into its Jacobian. The proof uses Zhang’s
admissible pairing on curves, the arithmetic Hodge index theorem over func-
tion fields, and the metrized graph analogue of Elkies’ lower bound for the
Green function. More generally, we prove an explicit Bogomolov-type result
bounding the number of geometric points of small Néron–Tate height on the
curve embedded into its Jacobian.

1. Introduction

Around 1963 Manin and Mumford independently conjectured that for any alge-
braically closed field $K$ of characteristic 0, any smooth projective curve $X$ of genus
$g \geq 2$ defined over $K$ and any divisor $D \in \text{Div}^1(X)$ of degree 1 on $X$, one has
$$\# j_D(X(K)) \cap J(K)_{\text{tors}} < \infty,$$
where $j_D \colon X \to J, P \mapsto P - D$ denotes the Abel–Jacobi embedding of $X$ into
its Jacobian $J = \text{Pic}^0(X)$ associated to $D$. This was proved by Raynaud [18,
Théorème I] in 1983. There are several further questions and aspects one then
naturally considers.

Positive characteristic. One might ask whether an analogue of this conjecture is true
in positive characteristic. Since for $K = \mathbb{F}_p$ every $K$-point of a curve over $K$ is a
torsion point, the naive analogue is certainly not true. Scanlon [19, Proposition 4.4],
and later Pink and Roessler [17, Theorem 3.6] found and proved a suitable ana-
logue of the Manin–Mumford conjecture in positive characteristic, using separate
methods.

Uniformity. It is reasonable to ask whether the number of torsion points contained
in $j_D(X(K))$ is uniformly bounded only in terms of the genus $g$. This has recently
been proven by Kühlne [12, Theorem 2] in characteristic 0.

Quantitative Bounds. It would be interesting to have an explicit upper bound for
the number of torsion points in $j_D(X(K))$. If the curve $X$ is defined over a number
field $F$ and $J$ has complex multiplication, Coleman [8, Theorem A] obtained the
explicit bound $pq$ for a prime number $p \geq 5$ depending on the ramification of
$K/\mathbb{Q}$ and the reduction type of $X$. Buium [3, Theorem A] obtained the bound
$p^{g^2/2}(p(2g - 2) + 6g)!$ without the condition of complex multiplication.

In this paper we address all three of these questions for non-isotrivial curves over
the function field $K = k(B)$ of any smooth projective connected curve $B$ over any

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algebraically closed field $k$. A curve $X$ over $K$ is called isotrivial if there is a finite field extension $K’$ of $K$ and a curve $C$ defined over $k$ such that $X \otimes_K K’ \cong C \otimes_k K’$.

**Theorem 1.1.** Let $k$ be any algebraically closed field, $B$ any smooth projective connected curve over $k$ and $K = k(B)$ its function field. For any smooth projective geometrically connected non-isotrivial curve $X$ of genus $g \geq 2$ defined over $K$ and any Abel–Jacobi embedding

$$j_D: X \to J, \quad x \mapsto x - D$$

of $X$ into its Jacobian $J = \text{Pic}^0(X)$, where $D \in \text{Div}^1(X)$ is any divisor of degree 1, the number of torsion points in $j_D(X(K))$ is uniformly bounded by

$$\#j_D(X(K)) \cap J(k)_{\text{tors}} \leq c(g) \leq 16g^2 + 32g + 124,$$

where $c(2) = 76$, $c(3) = 231$ and $c(g) = \left\lfloor \frac{16g^4 + 37g^2 - 28g - 1}{(g - 1)^2} \right\rfloor$ for $g \geq 4$.

Moreover, if $X$ has everywhere potentially good reduction, we may replace $c(g)$ by 1. If only $J$ has everywhere potentially good reduction, we may replace $c(g)$ by $c^{\text{tr}}(g)$ with $c^{\text{tr}}(2) = 11$ and $c^{\text{tr}}(g) = \left\lfloor \frac{4g^3 - 4g^2 + 2g + 1}{g - 1} \right\rfloor \leq 4g^2 + 3$ for $g \geq 3$. If char $k = 0$ or $X$ is hyperelliptic, we may replace $c(g)$ and $c^{\text{tr}}(g)$ by $\left\lfloor \frac{c(g) + 1}{2} \right\rfloor$ and $\left\lfloor \frac{c^{\text{tr}}(g) + 1}{2} \right\rfloor$ for $g \geq 3$.

We will deduce this theorem from the following more general result on the geometric Bogomolov conjecture, which bounds the number of geometric points $P \in X(K)$ of small Néron–Tate height $h_{\text{NT}}(j_D(P))$ on the Jacobian $J$.

**Theorem 1.2.** Let $k$ be any algebraically closed field, $B$ any smooth projective connected curve over $k$ and $K = k(B)$ its function field. Further, let $g \geq 2$ be an integer and $\epsilon \in \left[0, \frac{1}{4g^2 - 1}\right)$ a real number. For any smooth projective geometrically connected non-isotrivial curve $X$ of genus $g$ defined over $K$ and any divisor $D \in \text{Div}^1(X)$ of degree 1,

$$\#\{P \in X(K) \mid h_{\text{NT}}(j_D(P)) \leq \epsilon \omega_a^2\} \leq c(g, \epsilon),$$

where $h_{\text{NT}}$ denotes the Néron–Tate height on the Jacobian $J = \text{Pic}^0(X)$, $\omega_a^2$ is the self-pairing of the admissible dualizing sheaf $\omega_a$ on $X$, and $c(2, \epsilon) = \left\lfloor \frac{75}{1 - 12\epsilon} \right\rfloor + 1$, $c(3, \epsilon) = \left\lfloor \frac{3920}{1 - 12\epsilon} \right\rfloor + 1$ and $c(g, \epsilon) = \left\lfloor \frac{16g^4 + 37g^2 - 28g - 2}{(g - 1)^2(1 - 12\epsilon - 1)} \right\rfloor + 1$ for $g \geq 4$.

Moreover, if $X$ has everywhere potentially good reduction, we may replace $c(g, \epsilon)$ by 1. If only $J$ has everywhere potentially good reduction, we may replace $c(g, \epsilon)$ by $c^{\text{tr}}(g, \epsilon)$ with $c^{\text{tr}}(2, \epsilon) = \left\lfloor \frac{25}{1 - 12\epsilon} \right\rfloor + 1$ and $c^{\text{tr}}(g, \epsilon) = \left\lfloor \frac{2g^3 - 4g^2 + 2g + 2}{(g - 1)(1 - 12\epsilon + 1)} \right\rfloor + 1$ for $g \geq 3$. If char $k = 0$ or $X$ is hyperelliptic, we may replace $c(g, \epsilon)$ and $c^{\text{tr}}(g, \epsilon)$ by $\left\lfloor \frac{c(g, \epsilon) + 1}{2} \right\rfloor$ and $\left\lfloor \frac{c^{\text{tr}}(g, \epsilon) + 1}{2} \right\rfloor$ for $g \geq 3$.

The proof makes heavy use of Zhang’s admissible pairing introduced in [23]. If $F$ denotes the divisor of points $P_1, \ldots, P_s \in X(K)$, which are mapped to points of small Néron–Tate height by $j_D$, then the non-negativity of the Néron–Tate height of $s \omega - (2g - 2)F$ gives a bound of the admissible self-intersection $\omega_a^2$ in terms of the Green functions on the metrized reduction graphs on the pairs of the points of $F$. By an application of the arithmetic Hodge index theorem on $X^2$, we can bound $\omega_a^2$ from below by the Zhang invariant $\varphi$ of the metrized reduction graphs. We also
obtain an upper bound for the sum of the Green functions in the pairs of points of $F$ in terms of $\varphi$ through the metrized graph analogue of Elkies’ bound for the Green function and an estimate of the supremum of the Green function on metrized graphs. We may summarize these bounds by

$$\frac{\max(2,g-1)}{2g+1} \varphi(X) \leq \omega_a^2 \leq -\frac{4(g-1)g}{s(s-1)(1-4(g^2-1)\epsilon)} \sum_{j \neq k} g_v(R_v(P_j), R_v(P_k)) \leq \frac{4(g-1)g'c(g)}{(s-1)(1-4(g^2-1)\epsilon)} \varphi(X),$$

where $\varphi(X) = \sum_{v \in |B|} \varphi(\Gamma_v(X))$ is the sum of Zhang’s invariant $\varphi$ for all metrized reduction graphs $\Gamma_v(X)$ of $X$ at the closed points $v \in |B|$. $g_v$ denotes the Green function on $\Gamma_v(X)$ and $R_v : X(K) \to \Gamma_v(X)$ is the reduction map. The constant $c'(g)$ is explicitly given in Lemma 2.2. From this bound it is possible to deduce a bound of $s$ in terms of $g$ and $\epsilon$ independent of $K$.

The lower bound $\omega_a^2 \geq \frac{g-1}{2g+1}c(g)$ of the admissible self-intersection number $\omega_a^2$ was already obtained in the case of number fields by the third author in [21, Theorem 1.2]. It can be proven over function fields by exactly the same arguments. Since the arguments are widely dispersed over the literature and are often only formulated for number fields, however, we decided to include a more self-contained proof here.

Outline. In Section 2 we recall the theory of polarized metrized graphs. We give the definitions of the required invariants and we bound the Green function of a polarized metrized graph in terms of Zhang’s invariant $\varphi$. We recall the notion of adelic metrics in Section 3. In Sections 4 and 5 we discuss the admissible adelic metrics for line bundles on the curve $X$ and on its Jacobian $J$ as well as one of their relations. In Section 6 we deduce the lower bound for the admissible self-intersection number $\omega_a^2$ from the arithmetic Hodge index theorem on $X^2$. Finally in the last section, we give the proof of Theorems 1.1 and 1.2.

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2. Polarized metrized graphs

In this section we discuss the notion of polarized metrized graphs as introduced by Zhang [23] and Chinburg–Rumely [5], see also [2]. In particular, we are interested in the supremum of the canonical Green function of a polarized metrized graph and we will estimate it in terms of Zhang’s $\varphi$-invariant.

A metrized graph is a compact and connected metric space $\Gamma$, which is locally around any point $p \in \Gamma$ isometric to the star-shaped set

$$S(n_p, r_p) = \{ z \in \mathbb{C} \mid z = te^{2\pi ik/n_p} \text{ for some } 0 \leq t \leq r_p \text{ and } k \in \mathbb{Z} \},$$

for some integer $n_p \geq 1$, the valency of $p$, and some real number $r_p > 0$, or to a point, in which case we set $n_p = 0$. A vertex set for $\Gamma$ is a finite set $V \subseteq \Gamma$, such
that \( \{ p \in \Gamma \mid n_p \neq 2 \} \subseteq V \) and the closure of any connected component of \( \Gamma \setminus V \) intersects \( V \) in exactly two points. We fix a vertex set \( V \) and we write \( e_1, \ldots, e_r \) for the connected components of \( \Gamma \setminus V \), which are just open line segments. Let \( \ell(e) \) be the length of any line segment \( e \) and write

\[
\delta(\Gamma) = \sum_{i=1}^{r} \ell(e_i)
\]

for the total length of \( \Gamma \).

We would like to define the tangent space of \( \Gamma \) at a point \( p \in \Gamma \). By a path in \( \Gamma \) we mean a length-preserving continuous map \( \gamma: [0, l] \to \Gamma \) for some \( l > 0 \). We say that two paths \( \gamma_1, \gamma_2 \) are equivalent, written as \( \gamma_1 \sim \gamma_2 \), if there exists an \( \epsilon > 0 \) with \( \gamma_1(t) = \gamma_2(t) \) for all \( 0 \leq t < \epsilon \). The tangent space \( T_p(\Gamma) \) of \( \Gamma \) at \( p \) is given by

\[
T_p(\Gamma) = \{ \gamma \text{ path in } \Gamma \text{ with } \gamma(0) = p \} / \sim.
\]

For \( v \in T_p(\Gamma) \), represented by a path \( \gamma \), the one-sided directional derivative at \( p \) is defined by

\[
d_v f(p) = \lim_{t \to 0^+} \frac{f(\gamma(t)) - f(p)}{t}
\]

if the limit exists. For the functions

\[
Zh(\Gamma) = \{ f: \Gamma \to \mathbb{R} \mid f \text{ continuous, piecewise } C^2 \text{ and } d_v f(p) \text{ exists for all } p \in \Gamma, v \in T_p(\Gamma) \}
\]

we recall Zhang’s definition \([23, \text{Appendix}]\) of the Laplace operator

\[
\Delta(f) = f'' dx + \sum_{p \in \Gamma} \left( \sum_{v \in T_p(\Gamma)} d_v f(p) \right) \delta_p.
\]

Here, \( \delta_p \) is the Dirac measure at \( p \) and \( \Delta(f) \) is considered as a measure on \( \Gamma \).

For any measure \( \mu \) of total mass 1 on \( \Gamma \) there exists a unique continuous and symmetric function \( g_\mu: \Gamma \times \Gamma \to \mathbb{R} \), called the Green function associated to \( \mu \), such that \( g_\mu(x, \cdot) \in Zh(\Gamma) \) is characterized by

\[
\Delta_g g_\mu(x, y) = \delta_x(y) - \mu(y) \quad \text{and} \quad \int_\Gamma g_\mu(x, y) \mu(y) = 0
\]

for all \( x \in \Gamma \). Further, \( r(p, q) = g_{\delta_p}(q, q) \) is called the resistance function and measures the effective resistance between \( p \) and \( q \) if we consider \( \Gamma \) as a network, where \( \ell(e) \) is the resistance along any edge \( e \).

By a divisor on \( \Gamma \) we mean any finite formal sum \( D = \sum_{p \in \Gamma} m_p p \) for some integers \( m_p \in \mathbb{Z} \). A polarization on \( \Gamma \) is an effective divisor \( K = \sum_{p \in \Gamma} (n_p + m_p - 2)p \) with \( m_p \geq 0 \). We call \((\Gamma, K)\) a polarized metrized graph of genus \( g = \frac{1}{2} \deg K + 1 \). In the following we fix a polarization \( K \) of degree \( \deg K \neq -2 \) on \( \Gamma \). For any function \( f: \Gamma \to \mathbb{R} \) and any divisor \( D = \sum_{p \in \Gamma} m_p p \) on \( \Gamma \) we write \( f(D) = \sum_{p \in \Gamma} m_p f(p) \).

By \([23, \text{Theorem 3.2}]\) there exists a unique measure \( \mu_K \) of volume 1 on \( \Gamma \) such that

\[
(2.1) \quad c(\Gamma, K) = g_{\mu_K}(x, K) + g_{\mu_K}(x, x)
\]
is independent of $x \in \Gamma$. We call $g = g_{\mu K}$ the canonical Green function associated to $(\Gamma, K)$. Let us recall the following invariants of $(\Gamma, K)$: The $\epsilon$-invariant

$$\epsilon(\Gamma, K) = \int_{\Gamma} g(x, x)((2g - 2)\mu_K + \delta_K)$$

deefined by Zhang in [23] Theorem 4.4 and the $\varphi$-invariant

$$\varphi(\Gamma, K) = -\frac{1}{4}\delta(\Gamma) + \frac{1}{4} \int_{\Gamma} g(x, x)((10g + 2)\mu_K - \delta_K)$$

deefined by Zhang in [26] Theorem 1.3.1. The following lemma shows how the constant $c(\Gamma, K)$ can be expressed by these invariants.

**Lemma 2.1.** Let $(\Gamma, K)$ be any polarized metrized graph. The invariant $c(\Gamma, K)$ can be expressed by

$$c(\Gamma, K) = \frac{1}{12g}(4\varphi(\Gamma, K) + \delta(\Gamma) + \epsilon(\Gamma, K)).$$

**Proof.** If we integrate Equation (2.1) with respect to $\mu$, we obtain by the definition of the Green function $g$ that

$$c(\Gamma, K) = \int_{\Gamma} g(x, x)\mu_K.$$ 

Hence, by the defining equations for the invariants $\epsilon(\Gamma, K)$ and $\varphi(\Gamma, K)$ we get

$$\epsilon(\Gamma, K) = (2g - 2)c(\Gamma, K) + \int_{\Gamma} g(x, x)\delta_K,$$

$$4\varphi(\Gamma, K) + \delta(\Gamma) = (10g + 2)c(\Gamma, K) - \int_{\Gamma} g(x, x)\delta_K.$$ 

Taking the sum of both equations yields

$$12gc(\Gamma, K) = 4\varphi(\Gamma, K) + \delta(\Gamma) + \epsilon(\Gamma, K),$$

which proves the lemma after dividing by $12g$ on both sides. \[\square\]

Analogously to Equation (2.1), Baker and Rumely proved in [2] Theorem 14.1 that there also exists a unique measure $\mu_0$ of total mass 1 on $\Gamma$ such that

$$\tau(\Gamma) := g_{\mu_0}(x, x)$$

is a constant depending only on the metrized graph $\Gamma$. They show that it can be expressed by

$$\tau(\Gamma) = g_{\mu_0}(x, y) + \frac{1}{2}\varphi(x, y)$$

for any points $x, y \in \Gamma$. De Jong [9] Proposition 9.2] further proved the relation

$$\tau(\Gamma) = \frac{1}{12}(\delta(\Gamma) + 4\varphi(\Gamma, K) - 2\epsilon(\Gamma, K)).$$

Moreover, Baker and Rumely have proven the following analogue of Elkies’ lower bound for the Green function in [2] Proposition 13.7]

$$\sum_{j \neq k}^s g_{\mu}(x_j, x_k) \geq -s \cdot \sup_{x \in \Gamma} g_{\mu}(x, x),$$

(2.4)
where \( \mu \) is any measure of volume 1 on \( \Gamma \) and \( x_1, \ldots, x_s \in \Gamma \) are any points. For \( s = 2 \) and \( \mu = \mu_0 \) this implies \( g_{\mu_0}(x, y) \geq -\tau(\Gamma) \). In combination with Equations (2.2) and (2.3) we get

\[
(2.5) \quad r(x, y) \leq 4\tau(\Gamma) = \frac{1}{3}(\delta(\Gamma) + 4\varphi(\Gamma, K) - 2\varepsilon(\Gamma, K))
\]

for any points \( x, y \in \Gamma \).

Next, we note that we have

\[
(2.6) \quad \delta(\Gamma) \leq c_c(\varphi(\Gamma, K)
\]

with \( c_c(2) = 27, c_c(3) = 288/17 \) and \( c_c(g) = \frac{2g(7g+5)}{(g-1)^2} \) for \( g \geq 4 \) due to Çinkir [6, Theorem 2.11], respectively [7] for \( g = 3 \). If \( \Gamma \) is a tree, we may even replace \( c_c(g) \) by \( c_c'(g) = g/(2g - 2) \). The following Lemma gives a bound for the canonical Green function in terms of the \( \varphi \)-invariant.

**Lemma 2.2.** For any polarized metrized graph \((\Gamma, K)\) and any \( x, y \in \Gamma \),

\[
g(x, y) \leq \frac{(4g - 3)\delta(\Gamma) + (16g - 12)\varphi(\Gamma, K) - (8g - 3)\varepsilon(\Gamma, K)}{12g} \leq c'(g)\varphi(\Gamma, K),
\]

where \( c'(2) = 15/4, c'(3) = 140/51 \) and \( c'(g) = \frac{8g^4 + 18g^2 - 13g - 1}{2g(2g+1)(g-1)^2} \) for \( g \geq 4 \). If \( \Gamma \) is a tree, we may replace \( c'(g) \) by \( c''(g) = \frac{2g^2 - 3g + 2}{2g(2g - 2)} \).

**Proof.** By construction we have \( \sup_{y \in \Gamma} g(x, y) = g(x, x) \). Hence, it is enough to bound \( \sup_{x \in \Gamma} g(x, x) \). Due to Moriwaki [15, Lemma 4.1] we have

\[
g(x, x) = c(\Gamma, K) - g(x, x) = c(\Gamma, K) + \frac{r(x, K) - \varepsilon(\Gamma, K)}{2g}.
\]

As \( K \) is effective of degree \( 2g - 2 \), an application of the bound (2.5) yields

\[
\sup_{x, y \in \Gamma} g(x, y) \leq c(\Gamma, K) + \frac{(2g - 2)\delta(\Gamma) + (8g - 8)\varphi(\Gamma, K) - (4g - 1)\varepsilon(\Gamma, K)}{6g}.
\]

Now the first inequality in the lemma follows by Lemma 2.1.

To bound this expression in terms of \( \varphi(\Gamma, K) \) the direct way would be to use Çinkir’s inequality (2.6) and \( \varepsilon(\Gamma, K) \geq 0 \). To obtain a slightly better bound, we also recall from Çinkir’s work [6, Theorem 2.13] that the invariant

\[
\lambda(\Gamma, K) = \frac{g-1}{8g(2g+1)} \varphi(\Gamma, K) + \frac{3}{12} (\varepsilon(\Gamma, K) + \delta(\Gamma))
\]

satisfies \( (8g + 4)\lambda(\Gamma, K) \geq 9\delta(\Gamma) \). This can be rewritten as

\[-(2g + 1)\varepsilon(\Gamma, K) \leq 2(2g+1)\varphi(\Gamma, K) - (g - 1)\delta(\Gamma).
\]

Using this, we compute the estimate

\[
\frac{(4g - 3)\delta(\Gamma) + (16g - 12)\varphi(\Gamma, K) - (8g - 3)\varepsilon(\Gamma, K)}{12g}
\]

\[
\leq \frac{(3g - 2)\delta(\Gamma) + (16g^2 - 10g - 2)\varphi(\Gamma, K)}{4g(2g + 1)}
\]

\[
\leq \frac{(3g - 2)c_c(g) + 16g^2 - 10g - 2}{4g(2g + 1)} \varphi(\Gamma, K).
\]

If we denote the coefficient on \( \varphi(\Gamma, K) \) in the last expression by \( c'(g) \), we can explicitly write it as \( c'(2) = 15/4, c'(3) = 140/51 \) and \( c'(g) = \frac{8g^4 + 18g^2 - 13g - 1}{2g(2g+1)(g-1)^2} \) for
\[ g \geq 4. \text{ If } \Gamma \text{ is a tree, then we have } 2\varphi(\Gamma, K) = \delta(\Gamma) + \epsilon(\Gamma, K) \text{ by } [10] \text{ Equation (1.4)}, \]

as the Jacobian \( \text{Jac}(\Gamma) \) of a tree \( \Gamma \) is trivial. We compute

\[
\frac{(4g - 3)\delta(\Gamma) + (16g - 12)\varphi(\Gamma, K) - (8g - 3)\epsilon(\Gamma, K)}{2g} \leq \frac{(2g - 1)\epsilon(\Gamma) - 1}{2g} \varphi(\Gamma, K) = \frac{2g^2 - 3g + 2}{2g(2g - 2)} \varphi(\Gamma, K).
\]

\[ \Box \]

We would like to apply the lemma to obtain an explicit lower bound for the sum of the canonical Green functions in pairs of different points in a given set. Let \( x_1, \ldots, x_s \in \Gamma \) be any points on the metrized graph \( \Gamma \). Combining the bound in \([2, 4]\) for \( \mu = \mu_K \) with Lemma 2.2 we obtain

\[
(2.7) \quad \sum_{j \neq k} g(x_j, x_k) \geq -sc'(g)\varphi(\Gamma, K).
\]

3. Adelic metrics

We recall the notion of adelic metrics on line bundle in this section. We refer to \([2, 4]\) Section 1 and \([4]\) Section 2 for additional details. Let \( K = k(B) \) be the function field of a smooth projective connected curve \( B \) defined over an algebraically closed field \( k \). Let \( Y \) be a smooth projective variety over \( K \) and \( \mathcal{L} \) any line bundle on \( Y \). For any closed point \( v \in |B| \) we denote \( \overline{K}_v \) for the algebraic closure of the completion of \( K \) with respect to \( v \). This is a valuation field and we denote its valuation ring by \( \mathcal{O}_{\overline{K}_v} \). We write \( Y_v \) and \( \mathcal{L}_v \) for the pullbacks of \( Y \) and \( \mathcal{L} \) induced by the embedding \( K \to \overline{K}_v \).

By a metric \( \| \cdot \| \) on \( \mathcal{L}_v \) we mean a collection of \( K \)-norms \( \| \cdot \|_y \) on \( y^*\mathcal{L} \) for every \( y \in Y(\overline{K}_v) \). An important example is the model metric \( \| \cdot \|_{\mathcal{L}_v} \) associated to any projective flat model \( (\overline{Y}, \mathcal{L}_v) \) of \( (Y, \mathcal{L}_v^{\text{ad}}) \) over \( \mathcal{O}_{\overline{K}_v} \) for any integer \( e > 0 \), which is given by

\[
\| l \|_{\mathcal{L}_v} = \inf_{a \in \mathcal{O}_{\overline{K}_v}} \left\{ |a|^{1/e} \mid l \in a\overline{y}^*\mathcal{L}_v \right\}
\]

for any \( y \in Y(\overline{K}_v) \), where \( \overline{y} \in Y(\mathcal{O}_{\overline{K}_v}) \) denotes its unique extension. We call a metric \( \| \cdot \| \) on \( \mathcal{L}_v \) continuous and bounded if there exists a model \( (\overline{Y}, \mathcal{L}_v) \), such that \( \log \| \cdot \|_{\mathcal{L}_v} \) is continuous and bounded.

We call a collection of metrics \( \| \cdot \| = \{ \| \cdot \|_v \mid v \in |B| \} \) an adelic metric on \( \mathcal{L} \) if \( \| \cdot \|_v \) is a continuous and bounded metric on \( \mathcal{L}_v \) for all \( v \in |B| \) and there is an open non-empty subset \( U \subseteq B \) and a model \( (\overline{Y}, \mathcal{L}) \) of \( (Y, \mathcal{L}) \) over \( U \) such that \( \| \cdot \|_v = \| \cdot \|_{\mathcal{L}_v} \) for every \( v \in U \), where \( \mathcal{L}_v \) denotes the pullback of \( \mathcal{L} \) along \( \text{Spec}(\mathcal{O}_{\overline{K}_v}) \to U \). A pair \( \mathcal{L} = (\mathcal{L}, \| \cdot \|) \) of a line bundle \( \mathcal{L} \) and an adelic metric \( \| \cdot \| \) on \( \mathcal{L} \) is called an adelic line bundle. An isometry between two adelic line bundles \( \mathcal{L}_1 = (\mathcal{L}_1, \| \cdot \|_1) \) and \( \mathcal{L}_2 = (\mathcal{L}_2, \| \cdot \|_2) \) is an isomorphism of line bundles \( \mathcal{L}_1 \cong \mathcal{L}_2 \), which induces an isometry between the metrized line bundles \( (\mathcal{L}_1, \| \cdot \|_1, e) \) and \( (\mathcal{L}_{c_1}, c_1 \| \cdot \|_1, e) \) on \( Y_v \) for all \( v \in |B| \), where the \( c_v \)’s are any constants such that \( \prod_{v \in |B|} c_v = 1 \). In particular, there is up to isometry a unique way to consider a real number \( r \in \mathbb{R} \) as the trivial bundle \( \mathcal{O}_Y \) equipped with its canonical metric multiplied by a constant such that \( \prod_{v \in |B|} \| 1 \|_v = e^{-r} \).
We call an adelic line bundle \( \mathcal{L} \) nef if there is a sequence of models \((\widetilde{Y}_n, \mathcal{L}_n)\) with \(\mathcal{L}_n\) nef on \(\widetilde{Y}_n\), such that \(\log ||\mathcal{L}_n||_{v}\) converges to 0 uniformly in \(Y(\mathcal{K}_v)\) for all \(v \in |B|\). Furthermore, we call an adelic line bundle \(\mathcal{L}\) integrable if there exist nef adelic line bundles \(\mathcal{L}_1\) and \(\mathcal{L}_2\) and an isometry \(\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}\). In what follows we will use the additive notation \(\mathcal{L}_1 + \mathcal{L}_2 := \mathcal{L}_1 \otimes \mathcal{L}_2\) and \(n\mathcal{L} := \mathcal{L}^{\otimes n}\) for \(n \in \mathbb{Z}\).

Next we define intersection numbers of integrable line bundles. Let \(Z \subseteq Y\) be a subvariety of dimension \(d = \dim Z\) and let \(\mathcal{L}_0, \ldots, \mathcal{L}_d\) be integrable line bundles on \(Y\). We choose local sections \(l_0, \ldots, l_d\) of \(\mathcal{L}_0, \ldots, \mathcal{L}_d\) such that their common zero locus does not intersect \(Z\). Fix a \(v \in |B|\). If the metric \(\|\cdot\|_v\) on \(\mathcal{L}_i\) is induced by a model \((\widetilde{Y}_v, \mathcal{L}_{i,v})\) of \((Y, \mathcal{L}_i^{\otimes e})\) for all \(i\) and some \(e > 0\), we define the local intersection number at \(v\) by the usual intersection number on \(\widetilde{Y}_v\)

\[
\left(\text{div}(l_0) \cdots \text{div}(l_d) \cdot [Z]\right)_v = \text{div}^{\hat{\cdot}}_{\hat{v}}(l_0|_{\hat{Y}_{v,c}}) \cdots \text{div}^{\hat{\cdot}}_{\hat{v}}(l_d|_{\hat{Y}_{v,c}}) \cdot [\hat{Z}_v]/e^{d+1},
\]

where \(\hat{Z}_v\) denotes the Zariski closure of \(Z_v\) in \(\hat{Y}_v\). In general, we assume the local intersection number \(\left(\text{div}(l_0) \cdots \text{div}(l_d) \cdot [Z]\right)_v\) at \(v\) varies continuously with respect to the metrics on the line bundles. As the metric on \(\mathcal{L}_i\) at \(v\) is the limit of metrics induced by models for all \(v \in |B|\) and all \(i\), we can define the global intersection number by

\[
\mathcal{L}_0 \cdots \mathcal{L}_d \cdot Z = \sum_{v \in |B|} \left(\text{div}(l_0) \cdots \text{div}(l_d) \cdot [Z]\right)_v.
\]

If \(\dim Z = 0\) we also write \(\deg(\mathcal{L}_0|_Z) = \mathcal{L}_0 \cdot Z\) and if \(Z = Y\) we write \(\mathcal{L}_0 \cdots \mathcal{L}_d = \mathcal{L}_0 \cdots \mathcal{L}_d \cdot Z\) by way of abbreviation.

If moreover \(\text{div}(l_d)\) is a prime divisor on \(Y\) such that \(l_i|_{\text{div}(l_d)} \neq 0\) for all \(i < d\), the intersection number can be computed recursively by the formula

\[
(3.1)
\mathcal{L}_0 \cdots \mathcal{L}_d = \mathcal{L}_0|_{\text{div}(l_d)} \cdots \mathcal{L}_{d-1}|_{\text{div}(l_d)} - \sum_{v \in |B|} \int_{Y(\mathcal{K}_v)} \log ||l_d||_v c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_{d-1}),
\]

where the integral is defined as follows: First, let each adelic line bundle \(\mathcal{L}_i\) be induced by a model \((\widetilde{Y}, \mathcal{L}_i)\) of \((Y, \mathcal{L}_i^{\otimes e})\). We write \(\tilde{l}_d\) for the section of \(\mathcal{L}_d\) extending \(l_d^{\otimes e}\). Then \(V = \text{div}(\tilde{l}_d) - e \cdot \text{div}(l_d)\) is a Weil divisor \(V = \sum_{v \in |B|} V_v\) supported on the closed fibers of \(\widetilde{Y}\) and we set

\[
\int_{Y(\mathcal{K}_v)} \log ||\tilde{l}_d||_v c_1(\tilde{L}_0) \cdots c_1(\tilde{L}_{d-1}) = c_1(\tilde{L}_0) \cdots c_1(\tilde{L}_{d-1})|_{V_v}/e^{d+1}.
\]

In general the integral is defined by continuity and taking limits.

Let \(f : Y \to S\) be any proper smooth morphism of smooth projective varieties over \(K\) and let \(\mathcal{L}_0, \ldots, \mathcal{L}_d\) be integrable line bundles on \(S\). Then we have a projection formula

\[
(3.2)
\mathcal{L}_0 \cdots \mathcal{L}_d \cdot f_*(Z) = f^*\mathcal{L}_0 \cdots f^*\mathcal{L}_d \cdot Z.
\]

For model metrics this formula follows from the projection formula in classical intersection theory, see for example [11, Example 2.4.3]. In general it follows by taking limits. If \(s = \dim S\) is the dimension of \(S\) and \(\mathcal{M}_0, \ldots, \mathcal{M}_s\) are integrable
line bundles on $S$, we obtain another projection formula
\begin{equation}
(3.3) \quad f^*\mathcal{M}_0 \cdots f^*\mathcal{M}_s \cdot \mathcal{L}_1 \cdots \mathcal{L}_{d-s} = \mathcal{M}_0 \cdots \mathcal{M}_s \cdot (c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d-s}))[f],
\end{equation}
which one can similarly prove first for model metrics via the classical projection formula, and in the general case by taking limits. Here, $c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_n)[f]$ denotes the multidegree of the generic fiber of $f$ with respect to the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

4. Admissible metrics on the curve

In this section we recall Zhang’s admissible pairing on curves introduced in \[23\]. We use the notation as in Theorem 1.1, where we additionally assume that $X$ has semistable reduction over $B$ and that $X(K)$ is non-empty.

First, let us recall the notion of the metrized reduction graph. Let $\pi : X' \to B$ be the minimal regular model of $X$ over $B$. The polarized metrized reduction graph $\Gamma_v(X)$ of $X$ at $v \in |B|$ is defined as follows:

- The underlying graph of $\Gamma_v(X)$ is the dual graph of the fiber $X_v = \pi^{-1}(v)$:
  - Its vertex set $V$ consists of the irreducible components of $X_v$,
  - Its edge set $E$ consists of the nodes in $X_v$, such that any edge $e$ connects the two vertices corresponding to the irreducible components meeting in the node corresponding to $e$.
- The metric is obtained by assigning to each edge $e \in E$ its length $\ell(e) = 1$.
- The polarization of $\Gamma_v(X)$ is given by
  \[ K_v = \sum_{p \in V} (n_p + g_p - 2)p, \]
  where $n_p$ denotes the valency of the vertex $p$ and $g_p$ denotes the genus of the normalization of the irreducible component corresponding to $p \in V$.

The polarized metrized graph $\Gamma_v(X) = (\Gamma_v(X), K_v)$ has genus $g$. We write $g_v$ for the canonical Green function associated to $\Gamma_v(X)$. There is a canonical map $R_v : X(K) \to \Gamma_v(X)$ given as follows: We can uniquely extend any point $P \in X(K)$ to a section $\bar{P} : B \to X'$ of $\pi$ and we set $R_v(P)$ to be the vertex of $\Gamma_v(X)$ corresponding to the irreducible component of $X_v$, where $\bar{P}$ intersects $X_v$.

Zhang defined canonical admissible adelic metrics on the canonical bundle $\omega$ on $X$, on the bundle $\mathcal{O}_X(D)$ associated to any divisor $D$ on $X$ and on the diagonal bundle $\mathcal{O}_{X^2}(\Delta)$ on $X^2$. If $1_\Delta$ denotes the canonical section on $\mathcal{O}_{X^2}(\Delta)$, the metric satisfies
\begin{equation}
(4.1) \quad \log \|1_\Delta(P, Q)\|_v = -i_v(P, Q) - g_v(R_v(P), R_v(Q)),
\end{equation}
where $P, Q \in X(K)$ are different $K$-rational points and $i_v(P, Q)$ is the usual intersection index on $X_v$ of the unique sections $\bar{P}$ and $\bar{Q}$ of $\pi$ extending $P$ and $Q$.

In a similar way we may obtain $\log \|1_\Delta(P, Q)\|_v$ for any different geometric points $P, Q \in X(K)$ by repeating the above construction for $X \otimes_K K'$ instead of $X$, where $K'/K$ denotes a finite field extension such that $P, Q \in X(K')$ and dividing the resulting number by $[K' : K]$ afterwards. The canonical admissible metric on $\omega$ is chosen such that the canonical isomorphism $s^*\mathcal{O}_{X^2}(\Delta) \cong \omega^\otimes -1$ is an isometry, where $s : X \to X^2, x \mapsto (x, x)$ denotes the diagonal embedding. Similarly, the admissible metric on $\mathcal{O}_X(P)$ for a point $P$ is given such that the canonical isomorphism $s_P^*\mathcal{O}_{X^2}(\Delta) \cong \mathcal{O}_X(P)$ is an isometry, where $s_P : X \to X^2, x \mapsto (x, P)$ is the embedding associated to $P$. We let $\omega_a$, $D_a$ and $\Delta_a$ denote the canonical admissible
adelic line bundles \( \omega, \mathcal{O}_X(D) \) and \( \mathcal{O}_{X^2}(\Delta) \). According to [26] Proposition 3.5.1 these are integrable line bundles.

In general we call an adelic line bundle \( \mathcal{L}_a = \mathcal{T} \) on \( X \) admissible if its metric differs only by a constant from the metric on \( D_a \), where \( D \) is a divisor on \( X \) satisfying \( \mathcal{O}_X(D) \cong \mathcal{L} \). The admissible pairing \( (\mathcal{L}_a, \mathcal{M}_a) \) of two admissible line bundles \( \mathcal{L}_a, \mathcal{M}_a \) on \( X \) is defined to be the intersection number \( \mathcal{L}_a \cdot \mathcal{M}_a \) of the integrable line bundles. For two different points \( P, Q \in X(K) \) we deduce from Equation (4.1), that

\[
(P_a, Q_a) = \sum_{v \in B} (i_v(P, Q) + g_v(R_v(P), R_v(Q))).
\]

Moreover, Zhang obtained in [23, Theorem 4.4] for the self-intersection number of \( \omega_a \)

\[
\omega_a^2 = (\omega_a, \omega_a) = \omega_{X/B}^2 - \sum_{v \in B} \epsilon(\Gamma_v(X)),
\]

where \( \omega_{X/B} \) denotes the usual self-intersection number of the relative dualizing sheaf \( \omega_{X/B} \).

We also recall the adjunction formula [23 Theorem 4.2]

\[
(\omega_a, P_a) = -(P_a, P_a)
\]

for any point \( P \in X(K) \). This can be checked by applying formula (3.1) to the intersection product \(-\Delta_a \cdot \Delta_a \cdot P_1^* P_a\), where \( p_1 : X^2 \to X \) denotes the projection to the first factor. Indeed, applying (3.1) with respect to the canonical section \( 1_\Delta \in H^0(X^2, \mathcal{O}_{X^2}(\Delta)) \) yields the left hand side and applying \( 4.1 \) with respect to the canonical section \( p_1^* 1_R \in H^0(X^2, p_1^* \mathcal{O}_X(P)) \) yields the right hand side. In both cases, the integral in (3.1) vanishes, as worked out in [26 Section 3.5], see also Section [5] for similar computations. If \( \mathcal{L}_a \) is an admissible line bundle on \( X \) with \( \deg \mathcal{L} = 0 \), then \( \mathcal{L} \) defines a point in the Jacobian variety \( \text{Pic}^0(X) \) of \( X \) and due to [23 (5.4)] its Néron–Tate height is equal to

\[
h_{NT}(\mathcal{L}) = -(\mathcal{L}_a, \mathcal{L}_a).
\]

\[5\] Admissible metrics on the Jacobian variety

Continuing the notation from the last section, we discuss admissible metrics on the Jacobian variety \( J = \text{Pic}^0(X) \) of \( X \). We will compare them to the admissible metrics on the curve. We refer to [24 Section 2] and [22] for more details and to [13 Section 1] for the definition of the Jacobian variety.

We fix a point \( P \in X(K) \) and we denote \( \Theta \) for the divisor given by the image of the map

\[
X^{g-1} \to J, \quad (P_1, \ldots, P_{g-1}) \mapsto [P_1 + \cdots + P_{g-1} - (g-1)P],
\]

which is ample by [13 Theorem 6.6]. Hence, we get an ample and symmetric line bundle \( \mathcal{L} = \mathcal{O}_J(\Theta) \otimes [-1]^* \mathcal{O}_J(\Theta) \) on \( J \), where in general \( [n] : J \to J \) denotes the multiplication-by-\( n \) morphism on \( J \) for any \( n \in \mathbb{Z} \). Thus, we can choose an isomorphism \( \phi : \mathcal{L} \cong [2]^* \mathcal{L} \). Let \( (J, \mathcal{L}) \) be a model of \( (J, \mathcal{L}) \) over \( B \). As \( \mathcal{L} \) is ample, we may assume that \( \mathcal{L} \) is nef. There exists an open non-empty subset \( U \subseteq B \), such that the maps [2] and \( \phi \) extend to maps [2]: \( \tilde{J}_U \to \tilde{J}_U \) and \( \phi_U : \tilde{\mathcal{L}}_U^\otimes 4 \to [2]^* \tilde{\mathcal{L}}_U \) on the base changes \( \tilde{J}_U \) and \( \tilde{\mathcal{L}}_U \) of \( \tilde{J} \) and \( \tilde{\mathcal{L}} \) along the embedding \( U \to B \). We
set \((\tilde{J}_0, \tilde{E}_0) = (\tilde{J}, \tilde{E})\) and inductively, we construct models \((\tilde{J}_n, \tilde{E}_n)\) of \((J, \mathcal{L}^{\otimes 4^n})\), such that \([2]: J \to J\) extends to a map \(f_n: \tilde{J}_{n+1} \to \tilde{J}_n\) and \(\tilde{E}_{n+1} = f_n^*\tilde{E}_n\). Then the sequence of metrics on \(\mathcal{L}\) associated to these diagrams, we compute that \(\tilde{\pi}\) on \(X\) bounded for all \(\nu\). As \(\nu\) is motivated by a similar result in \([14, Lemme 4.10.2]\). Let \(U\) be the universal bundle which is trivial on \(\{P\} \times J\). According to \([14, Corollaire 2.5]\) we have \(\det Rq\mathcal{U} \cong \mathcal{O}_J(-\Theta)\), where \(q: X \times J \to J\) denotes the projection to the second factor. We let \(f' = [-1] \circ f\) and we consider the following diagrams

\[
\begin{array}{ccc}
X \times X^2 & \xrightarrow{id \times f} & X \times J \\
p_{23} \downarrow & & \downarrow q \\
X^2 & \xrightarrow{f} & J
\end{array}
\]

where in general \(p_{ij}: X^3 \to X^2\) denotes the projection to the \(i\)-th and \(j\)-th factors. By the base changes associated to these diagrams, we compute that

\[
f^*\mathcal{L} \cong f^*\mathcal{O}_J(-\Theta)^{\otimes -1} \otimes f^*[\text{-}1]^*\mathcal{O}_J(-\Theta)^{\otimes -1}
\]

\[
\cong f^*(\det Rq\mathcal{U})^{\otimes -1} \otimes f'^*(\det Rq\mathcal{U})^{\otimes -1}
\]

\[
\cong (\det Rp_{23*}(id \times f)^*\mathcal{U})^{\otimes -1} \otimes (\det Rp_{23*}(id \times f')^*\mathcal{U})^{\otimes -1}.
\]

Now we consider the line bundle

\[
\mathcal{M} = p_{123*}\mathcal{O}_{X^3}(\Delta)^{\otimes g - 1} \otimes p_{13*}\mathcal{O}_{X^2}(\Delta)^{\otimes g - 1} \otimes \omega^{\otimes 1}.
\]

on \(X^3\), where \(p: X^3 \to X\) denotes the projection to the first factor. It has degree 0 on every fiber of \(p_{23}\) and

\[
(id \times f)(P_1, P_2, P_3) = (P_1, [\mathcal{M}|_{X \times \{P_2, P_3\}}])
\]

As \(\mathcal{U}\) is the universal bundle which is trivial on \(\{P\} \times J\), we obtain

\[
(id \times f)^*\mathcal{U} \cong \mathcal{M} \otimes p_{23*}\mathcal{S}_p\mathcal{M}^{\otimes -1}, \quad (id \times f')^*\mathcal{U} \cong \mathcal{M}^{\otimes -1} \otimes p_{23*}\mathcal{S}_p\mathcal{M},
\]

for all \(x \in J(K)\).

The following lemma gives a comparison of the admissible metrics obtained on the curve and on the Jacobian variety.

**Lemma 5.1.** Consider the map

\[
f: X^2 \to J, \quad (P_1, P_2) \mapsto [(g - 1)(P_1 + P_2) - \omega]
\]

and write \(p_i: X^2 \to X\) for the projection to the \(i\)-th factor. There is an isometry

\[
f^*\mathcal{L}_J \cong (g - 1)(g + 1)(p_1^*\omega_a + p_2^*\omega_a) - 2(g - 1)^2 \Delta_a - \omega_a^2.
\]

In particular, the adelic metrized line bundle on the right hand side is nef.

**Proof.** We first show, that both line bundles are isomorphic. This part of the proof is motivated by a similar result in \([14, Lemme 4.10.2]\). Let \(\mathcal{U}\) be the universal line bundle on \(X \times J\), which is trivial on \(\{P\} \times J\). According to \([14, Corollaire 2.5]\) we have \(\det Rq\mathcal{U} \cong \mathcal{O}_J(-\Theta)\), where \(q: X \times J \to J\) denotes the projection to the second factor. We let \(f' = [-1] \circ f\) and we consider the following diagrams

\[
\begin{array}{ccc}
\begin{array}{c}
X \times X^2 \\
p_{23}
\end{array}
& \xrightarrow{id \times f} & \begin{array}{c}
X \times J \\
q
\end{array} \\
p_{23} \downarrow & & \downarrow q \\
\begin{array}{c}
X^2 \\
f
\end{array} & \xrightarrow{f'} & \begin{array}{c}
J \\
\end{array}
\end{array}
\]
where \( s_P : X^2 \to X^3 \), \((P_1, P_2) \mapsto (P, P_1, P_2)\) is the section of \( p_{23} \) associated to \( P \).

We recall from [14, Section 1] the following two rules for calculating the determinant: First,

\[
\det R_{p_{23}}(L \otimes p_{23}^* M) \cong \det R_{p_{23}}(L \otimes M^{\otimes X_{p_{23}}}(L)),
\]

where \( L \) is any line bundle on \( X^3 \), \( M \) is any line bundle on \( X^2 \) and \( \chi_{p_{23}} \) denotes the relative Euler characteristic of the family \( p_{23} : X^3 \to X^2 \). Note that \( \chi_{p_{23}}(M) = \chi_{p_{23}}(M \otimes \omega_{p_{23}}^{-1}) = 1 - g \), as \( M \) has degree 0 on every fiber of \( p_{23} \). Secondly, for any section \( s : X^2 \to X^3 \) of \( p_{23} \) we have

\[
\det R_{p_{23}}(L \otimes \mathcal{O}_{X^3}(s)) \cong \det R_{p_{23}}(L \otimes s^*(L \otimes \mathcal{O}_{X^3}(s))).
\]

We define the sections

\[
\begin{align*}
s_1 : X^2 &\to X^3, \quad (P_1, P_2) \mapsto (P_1, P_1, P_2), \\
s_2 : X^2 &\to X^3, \quad (P_1, P_2) \mapsto (P_2, P_1, P_2)
\end{align*}
\]

of the projection \( p_{23} \). We have \( s_1^* \mathcal{O}_{X^3}(s_2) = s_2^* \mathcal{O}_{X^3}(s_1) = \mathcal{O}_{X^3}(\Delta) \), as well as \( s_1^* \mathcal{O}_{X^3}(s_i) = p_1^* \omega_{p_{23}}^{\otimes -1} \) and \( p \circ s_i = p_i \). It follows that

\[
\begin{align*}
\det R_{p_{23}}((id \times f)^* \mathcal{U}) &\cong \det R_{p_{23}}(s_1^* \mathcal{O}_{X^3}(s_2) \otimes \mathcal{O}_{X^3}(s_2) \otimes p^*_\omega \omega_{p_{23}}^{\otimes -1}) \\
&\cong \det R_{p_{23}}(s_1^* \mathcal{O}_{X^3}(s_2) \otimes \mathcal{O}_{X^3}(s_2) \otimes p^*_\omega \omega_{p_{23}}^{\otimes -1}) \\
&\cong (p_1^* \omega \otimes p_2^* \omega)^{\otimes -(g+2)(g-1)/2} \otimes \mathcal{O}_{X^2}(\Delta)^{\otimes (g-1)^2} \otimes s_1^* \mathcal{O}_{X^2}(\omega_{p_{23}} \otimes \omega_{p_{23}}^{-1})
\end{align*}
\]

where in the last line we used that \( \det R_{p_{23}}(p^*_\omega \omega_{p_{23}}^{\otimes 1-g}) \cong p_{1,\omega}^{\otimes 1-g} \cong \mathcal{O}_{X^2} \) by the base change

\[
\begin{array}{ccc}
X^3 & \xrightarrow{p} & X \\
p_{23} & \downarrow & \pi_1 \\
X^2 & \xrightarrow{\pi_2} & \text{Spec}(K).
\end{array}
\]

In a similar way we get

\[
\det R_{p_{23}}((id \times f')^* \mathcal{U}) \cong (p_1^* \omega \otimes p_2^* \omega)^{\otimes -(g+1)(g-1)/2} \otimes \mathcal{O}_{X^2}(\Delta)^{\otimes (g-1)^2} \otimes s_2^* \mathcal{O}_{X^2}(\omega_{p_{23}} \otimes \omega_{p_{23}}^{-1}).
\]

Putting everything together results in

\[
f^* \mathcal{L} \cong (p_1^* \omega \otimes p_2^* \omega)^{\otimes (g+1)(g-1)} \otimes \mathcal{O}_{X^2}(\Delta)^{\otimes -(g-1)^2}
\]

as desired.

Now we show that this isomorphism is an isometry up to the constant \( \omega_2^2 \). To do this, we choose two geometric points \( P_1, P_2 \in X(K) \). On the one hand, we know by Equations [32] and [51] that

\[
\deg \left( f^* \mathcal{L}|_{(P_1, P_2)} \right) = \deg \left( \mathcal{L}|_{f(P_1, P_2)} \right) = h_{NT}((g - 1)(P_1 + P_2) - \omega).
\]
On the other hand, we may compute using Equations (4.4) and (4.5)
\[
\begin{align*}
    h_{\text{NT}}((g-1)(P_1 + P_2) - \omega) \\
    = -(g-1)(P_{1,a} + P_{2,a}) - \omega_a, (g-1)(P_{1,a} + P_{2,a}) - \omega_a) \\
    = (g-1)(g + 1)(\omega_a, P_{1,a} + P_{2,a}) - 2(g-1)^2(P_{1,a}, P_{2,a}) - \omega_a^2 \\
    = \deg(((g-1)(g + 1)(p_1^*\omega_a + p_2^*\omega_a) - 2(g-1)^2\Delta_a - \omega_a^2)_{(p_1, p_2)}).
\end{align*}
\]

Hence, we have shown that the above isomorphism induces an isometry
\[
f^*\hat{\mathcal{L}} \cong (g-1)(g + 1)(p_1^*\omega_a + p_2^*\omega_a) - 2(g-1)^2\Delta_a - \omega_a^2
\]
of line bundles on \(X^2\), as the degree of the adelic line bundles is the same at every geometric point in \(X^2(K)\). \(\square\)

6. An application of the arithmetic Hodge index theorem

The goal of this section is to deduce a lower bound for \(\omega_a^2\) as an application of the arithmetic Hodge index theorem over function fields by Carney [4, Theorem 3.1] for \(X^2\). This is motivated by the analogous result in the number field case found by the third author in [21, Theorem 1.2]. We retain the notation from the previous sections. We first recall the inequality part of the arithmetic Hodge index theorem over function fields for the variety \(X^2\). If \(\mathcal{N}\) and \(\mathcal{M}\) are integrable line bundles on \(X^2\) such that

(i) \(\mathcal{N}\) is nef,
(ii) \(\mathcal{N}\) is big and
(iii) their usual intersection number satisfies \(\mathcal{M} \cdot \mathcal{N} = 0\),

then \(\mathcal{N} \cdot \mathcal{M} \cdot \mathcal{N} \leq 0\).

**Proposition 6.1.** Any smooth projective geometrically connected curve \(X\) of genus \(g \geq 2\) defined over \(K\) satisfies
\[
\omega_a^2 \geq \frac{\max(2, g - 1)}{2g + 1} \sum_{v \in |B|} \varphi(\Gamma_v(X)).
\]

If \(\text{char } k = 0\) or \(X\) is hyperelliptic, we may replace \(\max(2, g - 1)\) by \(2(g - 1)\).

**Proof.** If \(\text{char } k = 0\), this follows from [26, Section 1.4] and if \(X\) is hyperelliptic, equality in fact holds, by [26, Corollary 1.3.3]. As all curves of genus \(g = 2\) are hyperelliptic, it remains to prove \(\omega_a^2 \geq \frac{g - 1}{2g + 1} \sum_{v \in |B|} \varphi(\Gamma_v(X))\) for \(g \geq 3\). To simplify the notation, we write \(\omega_{12} = p_1^*\omega + p_2^*\omega\), where \(p_i: X^2 \to X\) denotes the projection to the \(i\)-th factor. Further, we set \(\hat{\omega}_{12} = p_1^*\omega_a + p_2^*\omega_a\). We would like to apply the arithmetic Hodge index theorem to the integrable line bundles
\[
\mathcal{N} = (g + 1)\hat{\omega}_{12} - 2(g-1)\Delta_a - \frac{\omega_a^2}{g-1}, \quad \mathcal{M} = (g - 1)\Delta_a - \hat{\omega}_{12}
\]
on \(X^2\).

As \((g-1)\mathcal{N}\) is nef by Lemma 5.1, \(\mathcal{N}\) is also nef, such that condition 4 is satisfied. This also implies, that the underlying line bundle \(\mathcal{N}\) is nef on \(X^2\), such that we
have \( \text{vol}(\mathcal{N}) = N^2 \). We compute the volume explicitly

\[
\text{vol}(\mathcal{N}) = ((g+1)(\omega_{12}) - 2(g-1)O_{X^2}(\Delta))^2
\]

\[
= 2(g+1)^2p_1^*\omega \cdot p_2^*\omega - 4(g-1)^2\omega_{12} \cdot O_{X^2}(\Delta) + 4(g-1)^2O_{X^2}(\Delta)^2
\]

\[
= (4g - 1)^2(4g - 1 - 4g - 1) \deg \omega = 8g - 1 > 0.
\]

Hence, condition (ii) is also satisfied. To check condition (iii), we calculate

\[
\mathcal{M} \cdot \mathcal{N} = ((g+1)(\omega_{12}) - 2(g-1)O_{X^2}(\Delta)) \cdot ((g-1)O_{X^2}(\Delta) - \omega_{12})
\]

\[
= -2(g+1)p_1^*\omega \cdot p_2^*\omega + (g+3)(g-1)\omega_{12}O_{X^2}(\Delta) - 2(g-1)^2O_{X^2}(\Delta)^2
\]

\[
= (-4g+1)(g-1) + 2g + 3(g-1) + 2g - 1 \deg \omega = 0.
\]

Thus, the arithmetic Hodge index theorem implies that \( \mathcal{M} \cdot \mathcal{N} \leq 0 \). We now compute the left hand side to be

\[
(\mathcal{M} \cdot \mathcal{N}) = -2(g-1)^3\Delta_3^3 + (g+5)(g-1)^2\omega_{12} \cdot \Delta_a^2 - 2(g+2)(g-1)\Delta_{12} \cdot \Delta_a^2
\]

\[
+ (g+1)\omega_{12}^3 - \frac{\mathcal{M} \cdot \mathcal{M}}{g-1} \omega_a^2.
\]

By a similar computation as above we have

\[
\mathcal{M} \cdot \mathcal{N} = 2p_1^*\omega \cdot p_2^*\omega - 2(g-1)\omega_{12} \cdot O_{X^2}(\Delta) + (g-1)^2O_{X^2}(\Delta)^2 = -2(g-1)^3.
\]

By symmetry and by formula (3.3) we obtain

\[
\omega_{12}^3 = 6(p_1^*\omega_a \cdot p_1^*\omega_a \cdot p_2^*\omega_a) = 6 \cdot \omega_a^2 \cdot \deg \omega = 12(g-1)\omega_a^2.
\]

In the same way, we obtain \( p_1^*\omega_a \cdot p_2^*\omega_a \cdot \Delta_a = \omega_a^2 \). Let \( s: X \to X^2 \), \( x \mapsto (x,x) \) the diagonal embedding. Then we can compute by the recursion formula for the intersection number (3.1) and by the definition of the metric on \( \Delta_a \) in (4.1) that

\[
p_1^*\omega_a \cdot p_2^*\omega_a \cdot \Delta_a = s^*p_1^*\omega_a \cdot s^*p_2^*\omega_a + \sum_{v \in |B|} \int_{X^2(K_v)} g_v c_1(p_1^*\omega_a) c_1(p_2^*\omega_a) = \omega_a^2.
\]

where the integral vanishes by the definition of the Green function \( g_v \). See also [26, Section 3.5] where it is also explained in more detail why we can replace \(- \log ||1_{\Delta}||_v \) by \( g_v \). Hence, we can conclude using again symmetry

\[
\omega_{12}^2 \cdot \Delta_a = 2(p_1^*\omega_a \cdot p_1^*\omega_a \cdot \Delta_a + p_1^*\omega_a \cdot p_2^*\omega_a \cdot \Delta_a) = 4\omega_a^2.
\]

In a similar way, we obtain from (3.1) and (4.1) that

\[
p_1^*\omega_a \cdot \Delta_a = s^*p_1^*\omega_a \cdot s^*\Delta_a + \sum_{v \in |B|} \int_{X^2(K_v)} g_v c_1(p_1^*\omega_a) c_1(\Delta_a) = -\omega_a^2,
\]

where the integral vanishes as in [26, Lemma 3.5.2]. We conclude by symmetry that

\[
\omega_{12}^2 \cdot \Delta_a^2 = 2(p_1^*\omega_a \cdot \Delta_a \cdot \Delta_a) = -2\omega_a^2.
\]

Finally, we compute the self-intersection number \( \Delta_a^3 \) to be

\[
\Delta_a^3 = (s^*\Delta_a)^2 + \sum_{v \in |B|} \int_{X^2(K_v)} g_v c_1(\Delta_a)^2 = \omega_a^2 - \sum_{v \in |B|} \varphi(\Gamma_v(X)),
\]
where the last equality follows from [26, Lemma 3.5.4]. Substituting everything into Equation (6.1) yields

\[ 0 \geq \mathcal{M} \cdot \mathcal{M} \cdot \mathcal{N} = 2(g - 1)^3 \sum_{v \in |B|} \varphi(\Gamma_v(X)) - 2(g - 1)^2(2g + 1)\omega_a^2, \]

which is equivalent to the inequality we wanted.

\[ \square \]

7. PROOF OF THEOREMS 1.1 AND 1.2

In this section we prove Theorems 1.1 and 1.2. We use the same notation as in the theorems. By the semistable reduction theorem there exists a finite field extension $K'$ of $K$ such that $X_{K'} = X \otimes_K K'$ has semistable reduction over the normalization $B'$ of $B$ in $K'$. As $X_{K'}(\overline{K}) = X(\overline{K})$, we may assume that $X$ has semistable reduction over $B$. We write $\pi: X \to B$ for the minimal regular model of $X$ over $B$.

Let $0 \leq \epsilon < \frac{1}{2g - 1}$ and $F = \{P_1, \ldots, P_s\} \subseteq X(\overline{K})$ be any set of $s \geq 2$ geometric points, such that $h_{NT}(j_D(P_i)) \leq \epsilon\omega_a^2$ for any $1 \leq i \leq s$. If no such set exists, we have nothing to prove. Replacing $K$ by a finite field extension again, we can assume that $F \subseteq X(K)$. By the bilinearity of the pairing and by Equation (4.5) one checks directly that

\[
\begin{align*}
    h_{NT}(P_j + P_k - 2D) + h_{NT}(P_j - P_k) &= -\left(P_{j,a} + P_{k,a} - 2D, P_{j,a} + P_{k,a} - 2D\right) - \left(P_{j,a} - P_{k,a}, P_{j,a} - P_{k,a}\right) \\
    &= -2P_{j,a} - D_a - P_{j,a} - D_a - 2(P_{k,a} - D_a, P_{k,a} - D_a) \\
    &= 2h_{NT}(j_D(P_j)) + 2h_{NT}(j_D(P_k)),
\end{align*}
\]

for all $j, k \leq s$. Using the height bounds $h_{NT}(j_D(P_i)) \leq \epsilon\omega_a^2$ for all $i \leq s$ and $h_{NT}(P_j + P_k - 2D) \geq 0$, we deduce that

\[ 2(P_{j,a}, P_{k,a}) - (P_{j,a}, P_{j,a}) - (P_{k,a}, P_{k,a}) \leq 4\epsilon\omega_a^2. \]

Summing over all pairs of different points in $F$, we obtain

\[ (7.1) \quad - \sum_{j=1}^{s} (P_{j,a}, P_{j,a}) \leq -\frac{1}{s - 1} \sum_{j \neq k}^{s} (P_{j,a}, P_{k,a}) + 2s\epsilon\omega_a^2. \]

We also write $F$ for the divisor $F = P_1 + \cdots + P_s$. Again by Equation (4.5) we have

\[ 0 \leq h_{NT}(s\omega - (2g - 2)F) = -(s\omega_a - (2g - 2)F_a, s\omega_a - (2g - 2)F_a). \]

By bilinearity and symmetry of the pairing we may rewrite this inequality as

\[ \omega_a^2 \leq \frac{4(g - 1)}{s} \sum_{j=1}^{s} (\omega_a, P_{j,a}) - \frac{4(g - 1)^2}{s^2} \left( \sum_{j=1}^{s} (P_{j,a}, P_{j,a}) + \sum_{j \neq k} (P_{j,a}, P_{k,a}) \right), \]
As \((\omega_a, P_{j,a}) = -(P_{j,a}, P_{j,a})\) by the adjunction formula (14.1), we obtain by an application of (7.1) that
\[
\omega_a^2 \leq \left( -\frac{4(g-1)}{s} + \frac{4(g-1)^2}{s^2} \right) \sum_{j \neq k} (P_{j,a}, P_{k,a}) + \frac{4(g-1)}{s^2} \sum_{j \neq k} (P_{j,a}, P_{k,a}) + \frac{4(g-1)}{s} 2 \epsilon \omega_a^2
\]
\[
\omega_a^2 \leq \frac{4(g-1)}{s(s-1)} \sum_{j \neq k} (P_{j,a}, P_{k,a}) + 8(g-1) 2 \epsilon \omega_a^2
\]
\[
\omega_a^2 \leq \frac{4(g-1)}{s(s-1)} \sum_{j \neq k} (P_{j,a}, P_{k,a}) + 4(g^2-1) \epsilon \omega_a^2,
\]
where the last inequality follows from the fact that \(\frac{2-1+s}{s} \) is maximized at \(s = 2\) for \(s \geq 2\). Solving for \(\omega_a^2\) yields
\[
\omega_a^2 \leq -\frac{4(g-1)}{s(s-1)(4g^2-1)\epsilon} \sum_{j \neq k} (P_{j,a}, P_{k,a}).
\]

By Equation (4.2), we have
\[
(P_{j,a}, P_{k,a}) = \sum_{v \in [B]} (\varphi(P_{j,v}), P_{k,v}) + \frac{4(g-1)}{(s-1)(4g-1)\epsilon} \sum_{v \in [B]} \varphi(\varphi(P_{j,v}, P_{k,v})) \geq \sum_{v \in [B]} g_v(\varphi(P_{j,v}, P_{k,v}))
\]
for \(j \neq k\). Thus, we conclude using Proposition 6.1 and Equation (2.7) that
\[
\frac{4(g-1)}{2g+1} \sum_{v \in [B]} \varphi(\varphi(P_{j,v}(X))) \leq \omega_a^2 \leq \frac{4(g-1)}{(s-1)(4g-1)\epsilon} \sum_{v \in [B]} \varphi(\varphi(P_{j,v}(X))),
\]
with \(c'(g)\) as in Lemma 7.2.

If \(X\) has everywhere good reduction, we have \(\omega_a^2 = \omega_a^2\) by Equation (14.3). The Noether formula states that \(\omega_a^2 = 12 \deg \pi_* \omega_a^2\) in this case, and it has been shown by Parshin \[16\] Proposition 5 when \(\deg \pi > 0\) and by Szpiro \[20\] Theorem 1 when \(\deg \pi > 0\) that \(\deg \pi_* \omega_a^2 > 0\), as \(X\) is non-isotrivial. On the other hand we have \(\sum_{v \in [B]} \varphi(v) = 0\) in this case, contradicting the above inequality. Hence there is no set \(F\) as above if \(X\) has everywhere good reduction. Consequently, we obtain
\[
\# \{P \in X(\overline{K}) \mid h_{NT}(P) \leq \epsilon \omega_a^2\} \leq 1
\]
if \(X\) has everywhere good reduction.

If \(X\) does not have everywhere good reduction, we have \(\sum_{v \in [B]} \varphi(v) > 0\) by (2.6) and we deduce from the inequality above that
\[
(7.2) \quad s \leq \left\lfloor \frac{4(g-1)g(2g+1)c'(g)}{\max(2, g-1)(4g^2-1)\epsilon} \right\rfloor + 1 = c(g, \epsilon).
\]

We now compute this number \(c(g, \epsilon)\) explicitly. For \(g \leq 3\), one obtains
\[
c(2, \epsilon) = \left\lfloor \frac{75}{1 - 12\epsilon} \right\rfloor + 1, \quad c(3, \epsilon) = \left\lfloor \frac{3920}{17(1 - 32\epsilon)} \right\rfloor + 1.
\]
In general, \(c(g, \epsilon) = \left\lfloor \frac{16a^4+36a^2-26a-2}{(g-1)(4\epsilon)} \right\rfloor + 1\) for \(g \geq 4\).

As \(X\) and \(J\) have semistable reduction, they have good reduction at a place if they have potentially good reduction at that place. As already mentioned above, we have \(\# \{P \in X(\overline{K}) \mid h_{NT}(P) \leq \epsilon \omega_a^2\} \leq 1\) if \(X\) has everywhere good reduction. If \(J\) has everywhere good reduction, then every reduction graph \(\Gamma_{\varphi}(X)\) is a tree.
Hence by Lemma 2.2 we may replace \( c'(g) \) in (7.2) by \( c^{tr}(g) \), such that we obtain

\[
s \leq c^{tr}(g, \epsilon) = \left\lfloor \frac{4g^3 - 4g^2 + g + 2}{(g - 1)(1 - 4(g^2 - 1)\epsilon)} \right\rfloor + 1
\]

for \( g \geq 3 \). If char \( k = 0 \) or \( X \) is hyperelliptic, we may replace \( \max(2, g - 1) \) in (7.2) by \( 2g - 2 \) by Proposition 6.1. Thus, for \( g \geq 3 \) we get

\[
s \leq \left\lfloor \frac{c(g, \epsilon) + 1}{2} \right\rfloor
\]

if \( J \) has everywhere good reduction. This completes the proof of Theorem 1.2. As geometric torsion points \( x \in J(\mathbb{K})_{\text{tors}} \) have Néron--Tate height \( h_{NT}(x) = 0 \), we recover Theorem 1.1 by setting \( \epsilon = 0 \) and \( c(g) = c(g, \epsilon) \) and \( c^{tr}(g) = c^{tr}(g, \epsilon) \).

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Nicole Looper, Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912 USA

*Email address: nicole_looper@brown.edu*

Joe Silverman, Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912 USA

*Email address: jhs@math.brown.edu*

Robert Wilms, Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland

*Email address: robert.wilms@unibas.ch*