Vector bundles of finite rank on complete intersections of finite codimension in ind-Grassmannians

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Abstract

In this article we establish an analogue of the Barth–Van de Ven–Tyurin–Sato theorem. We prove that a finite rank vector bundle on a complete intersection of finite codimension in a linear ind-Grassmannian is isomorphic to a direct sum of line bundles.

1. Introduction

The Barth–Van de Ven–Tyurin–Sato theorem claims that any finite rank vector bundle on the infinite complex projective space $\mathbb{P}^\infty$ is isomorphic to a direct sum of line bundles. For rank two bundles this was established by Barth and Van de Ven in [1], and for finite rank bundle it was proved by Tyurin in [10] and Sato in [6]. In particular, the Barth–Van de Ven–Tyurin–Sato theorem holds for linear ind-Grassmannians and their linear sections [2, 9, 14, 8].

In this work we will extend these results on the case of a complete intersection in linear ind-Grassmannians. The ground field in this work is $\mathbb{C}$.

First, we recall the definition of linear ind-varieties in general and linear ind-Grassmannians studied by I. Penkov and A. Tikhomirov in [14, 8, 7] in particular.

Definition 1. An ind-variety $X = \lim_{\to} X_m$ is the direct limit of a chain of embeddings:

$$X := \{X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} \ldots \xleftarrow{\phi_{m-1}} X_m \xleftarrow{\phi_m} \ldots\},$$

where $X_m$ is a smooth algebraic variety for any $m \geq 1$.

Definition 2. A vector bundle $E$ of rank $r > 0$ on $X$ is the inverse limit

$$E = \lim_{\leftarrow} E_m$$

of an inverse system of vector bundles $\{E_m\}_{m \geq 1}$ of rank $r$ on $X$ (i.e., a system of vector bundles $E_m$ with fixed isomorphisms $E_m \cong \phi_m^* E_{m+1}$).
In particular, the structure sheaf $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_m}$ of an ind-variety $X$ is well defined. By the Picard group $\text{Pic} X$ we understand the group of isomorphism classes of line bundles on $X$.

**Definition 3.** An ind-variety $X$ is called linear, if a line bundle $\mathcal{O}_X(1) = \varprojlim \mathcal{O}_{X_m}(1)$ is defined on it, where for each $m \geq 1$ the line bundle $\mathcal{O}_{X_m}(1)$ is ample on $X_m$.

### 1. Linear ind-Grassmannians and complete intersections in them

For integers $m \geq 1$, $n_m$ and $k_m$ satisfying $1 \leq k_m \leq n_m$ consider the vector space $V^{n_m}$ of dimension $n_m$ and the Grassmannian $G(k_m, n_m)$ of $k_m$-dimensional vector subspaces in $V^{n_m}$. Consider as well the Plücker embedding of $G(k_m, n_m)$: $G(k_m, n_m) \rightarrow \mathbb{P}^{N_m-1} = \mathbb{P}(\Lambda^{n_m}_m)$, where $N_m = \binom{n_m}{k_m}$.

We define the ind-Grassmannian $G : = G(\infty)$ as the direct limit $\varinjlim G(k_m, n_m)$ of a chain of embeddings:

$$G(k_1, n_1) \xrightarrow{\phi_1} G(k_2, n_2) \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} G(k_m, n_m) \xrightarrow{\phi_m} \cdots,$$

with conditions

$$\lim_{m \rightarrow \infty} k_m = \lim_{m \rightarrow \infty} (n_m - k_m) = \infty.$$

Further on we assume that the ind-Grassmannian $G$ is linear, i.e., it has a line bundle $\mathcal{O}_G(1) = \varprojlim \mathcal{O}_{G(k_m, n_m)}(1)$, where the class of the line bundle $\mathcal{O}_{G(k_m, n_m)}(1)$ generates $\text{Pic}(G(k_m, n_m))$ for all $m \geq 1$.

Since $G$ is linear the embeddings

$$\{ \ldots \xleftarrow{\phi_m^{-1}} G(k_m, n_m) \xrightarrow{\phi_m} G(k_{m+1}, n_{m+1}) \xleftarrow{\phi_{m+1}} \ldots \}$$

can be extended to linear embeddings of Plücker spaces

$$\{ \ldots \xleftarrow{\phi_m^{-1}} \mathbb{P}^{N_{m-1}} \xrightarrow{\phi_m} \mathbb{P}^{N_{m+1}-1} \xleftarrow{\phi_{m+1}} \ldots \}.$$

Next, for $l \geq 1$ and $d_1, d_2, \ldots, d_l \geq 1$ consider the linear ind-subvariety $X$ of the linear ind-Grassmannian $G$ that is the direct limit $X = \varinjlim X_m$ of the following chain of embeddings

$$X := \{ X_1 \xleftarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots \xleftarrow{\phi_m^{-1}} X_m \xrightarrow{\phi_m} \cdots \}.$$  

(4)

Here $X_m$ is the intersection of the Grassmannian $G(k_m, n_m) \subset \mathbb{P}^{N_m-1}$ with $l$ hypersurfaces $Y_{1,m}, Y_{2,m}, \ldots, Y_{l,m} \subset \mathbb{P}^{N_m-1}$ of fixed degrees $\deg Y_{i,m} = d_i$, $i = 1, \ldots, l$, $m \geq 1$:

$$X_m = G(k_m, n_m) \cap \bigcap_{i=1}^l Y_{i,m}, \quad \text{codim}_{G(k_m, n_m)} X_m = l.$$  

(5)

**Definition 4.** The constructed ind-variety $X$ is called a complete intersection of codimension $l$ in the linear ind-Grassmannian $G$. 


In other words, the ind-variety $X$ is an intersection of the ind-Grassmannian $G$ with ind-hypersurfaces $Y_1, Y_2, \ldots, Y_l$ in the linear ind-space $P^\infty = \lim_{\rightarrow} P^{N_m-1}$:

$$X = G \cap \bigcap_{i=1}^l Y_i, \ Y_i = \lim\rightarrow Y_{i,m}, \ i = 1, \ldots, l, \ m \geq 1.$$ (6)

For $i = 1, \ldots, l$ number $\deg Y_i := \deg Y_{i,m} = d_i$, $m \geq 1$ is called the degree of the ind-hyperuspace $Y_i$ in $P^\infty$.

The main result of this work is the following theorem.

**Theorem 1.** Any vector bundle of finite rank on a complete intersection $X \subset G$ of finite codimension is isomorphic to a direct sum of line bundles.

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2. Preliminary notions and the idea of proof

To explain the idea of proof of Theorem 1 we will need to give some definitions and recall the main results of articles [12] and [13].

**Definition 5.** Let $X$ be a projective variety with an ample line bundle $O_X(1)$. A **projective space** in $X$ is a subvariety $M \simeq \mathbb{P}^r$ in $X$ such that $O_X(1)|_M \simeq O_{\mathbb{P}^r}(1)$.

In the case $\dim(M) = 1$ we call $M$ a **projective line** or just a **line** in $X$.

Using Definition 5 we can give the definition of projective subspace in an ind-variety $X$.

**Definition 6.** A **projective space** in $X$ is a variety $M \simeq \mathbb{P}^r$ in $X$, such that $O_X(1)|_M \simeq O_{\mathbb{P}^r}(1)$.

**Definition 7.** A **path** $p_n(x, y)$ of length $n$ on an ind-variety $X$ connecting points $x$ and $y$, is a collection of points $x = x_0, x_1, \ldots, x_n = y$ in $X$ and a collection of projective lines $l_0, \ldots, l_{n-1}$ in $X$ such that $x_i, x_{i+1} \in l_i$.

The variety of all length $n$ paths connecting $x$ and $y$ is denoted by $P_n(x, y)$.

**Definition 8.** Linear ind-variety $X$ is called **1-connected**, if for any two points $x, y \in X$ there exists a path connecting $x$ with $y$.

**Definition 9.** We will say that a vector bundle $E$ on $X$ is **trivial on lines** if for any projective line $l$ on $X$ the restriction $E|_l$ is trivial.

**Definition 10.** Let $E$ be a rank $r$ bundle on a linear variety $X$. The **splitting type** of the bundle $E$ on a projective line $l \subset X$ is a collection of numbers $r_i > 0$ and $a_i \in \mathbb{Z}$, $i = 1, \ldots, s$ such that

$$E|_l \cong r_1 O_{\mathbb{P}^1}(a_1) \oplus r_2 O_{\mathbb{P}^1}(a_2) \oplus \ldots r_s O_{\mathbb{P}^1}(a_s)$$

where $a_1 > a_2 > \ldots > a_s$, $\sum_{i=1}^s r_i = r$.

A bundle $E$ is called **uniform**, if its restriction to all projective lines has the same splitting type.
We will need to use several results of articles [12] and [13] on complete intersections $X \subset G$, that we recall now for convenience.

**Theorem 2 ([12]).** Let $X$ be a complete intersection of $G(n, 2n)$ embedded by Plücker with a collection of hypersurfaces of degrees $d_1, \ldots, d_l : X = G(n, 2n) \cap \bigcap_{i=1}^l Y_i$. If $2 \sum_i (d_i + 1) \leq \left[ \frac{n}{2} \right]$, then the variety $P_n(u,v)$ of length $n$ paths connecting any two points $u, v$ in $X$ is non-empty and connected.

We will also need a corollary of this theorem for the case of complete intersection in $G(k,n)$. We will assume $k \leq \left[ \frac{n}{2} \right]$.

**Corollary 1.** Let $X$ be a complete intersection of $G(k,n)$ embedded by Plücker with a collection of hypersurfaces of degrees $d_1, \ldots, d_l : X = G(k,n) \cap \bigcap_{i=1}^l Y_i$. If $2 \sum_i (d_i + 1) \leq \left[ \frac{k}{2} \right] \leq \left[ \frac{n}{2} \right]$ then the variety $P_k(u,v)$ of length $k$ paths connecting any two points $u,v$ in $X$ is non-empty and connected.

**Proof.** We will assume that points $u$ and $v$ in $G(k,n)$ are generic $^1$; in the case when the points are not generic the proof goes in the same way the proof of Theorem 2, see [12].

Let $U$ and $V$ be the $k$-dimensional spaces corresponding to the points $u$ and $v$ of $G(k,n)$. To construct a path $p_k(u,v)$ connecting $u$ and $v$ consider the Grassmannian $G(k,2k) = G(k,U \oplus V)$ of $k$-dimensional spaces in the $2k$-dimensional vector space $U \oplus V$. It is easy to prove that any path $p_k(u,v)$ of length $k$ in the Grassmannian $G(k,n)$ is contained in the Grassmannian $G(k,U \oplus V)$. So all length $k$ paths connecting $u$ and $v$ in $X$ are contained in $G(k,U \oplus V)$.

Consider now the intersection $G(k,U \oplus V) \cap \bigcap_{i=1}^l Y_i \subset X$. By Theorem 2 the variety of paths $P_k(u,v)$ of the intersection $G(k,U \oplus V) \cap \bigcap_{i=1}^l Y_i$ is non-empty and connected. It follows that the variety of paths $P_k(u,v)$ in $X$ is non-empty and connected as well.

**Proposition 1 ([13]).** Consider the Segre embedding of $\mathbb{P}^l \times \mathbb{P}^m$ in $\mathbb{P}^{(l+1)(m+1)−1}$. Choose natural numbers $k$, $d$ such that $2kd < \min(l,m)$. Then for any variety $Y$ in $\mathbb{P}^{(l+1)(m+1)−1}$ whose irreducible components have codimension at most $k$ and degree at most $d$ the variety $Y \cap \mathbb{P}^l \times \mathbb{P}^m$ is $1$-connected.

**Lemma 1 ([13]).** Let $Y_1, \ldots, Y_l$ be hypersurfaces of degrees $d_1, \ldots, d_l$ in $\mathbb{P}^n$. Let $Y = Y_1 \cap \ldots \cap Y_l$ and let $\mathbb{P}^k \subset Y$ be a $k$-dimensional projective subspace. There exists a projective subspace $\mathbb{P}^{k+1} \subset Y$ containing $\mathbb{P}^k$ if the following holds:

$$n-k-1 > \sum_{i=1}^l \left( \frac{d_i + k}{d_i - 1} \right).$$

**Theorem 3 ([13]).** Any finite rank vector bundle $E$ on $X$ is uniform.

Finally we list the steps in our proof of Theorem 1.

$^1$ i.e., the intersection of the corresponding $k$-planes is $0$. 
• We prove first that any finite rank vector bundle $E$ contains a flag of subbundles $0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_s = E$ such that each quotient $F_i/F_{i-1}$ is a bundle trivial on lines twisted by $O(a_i)$ for $1 \leq i \leq s$.

• Next we prove that every finite rank bundle on $X$ trivial on lines is trivial.

• Finally, using Kodaira vanishing theorem we prove that the bundle $E$ splits as a sum $\oplus_i F_i/F_{i-1}$.

3. Constructing a flag of subbundles in $E$

In this section we will construct a flag of subbundles in a rank $r$ vector bundle $E$ on a complete intersection $X \subset G$ of finite codimension.

Before doing this formally we will describe the main idea. Chose a point $x \in X$ and consider the fibre $E_x$ of $E$ over $x$. According to Grothendieck’s theorem ([4] Theorem 2.1.1), for any projective line $l$ passing through $x$ the restriction $E|_l$ has a canonical flag of subbundles $0 = F_0 \subset F_1 \subset \ldots \subset F_s = E|_l$.

Hence, we get in $E_x$ a flag of subspaces that we denote by $F(x, l)$. A priori the flag $F(x, l)$ might depend on the choice of $l$ passing through $x$ but we will show that this is not the case.

Let $B_m(x)$ be the base of the family of lines on $X_m$ passing through $x$, considered as a reduced scheme. We will show that the map from $B_m(x)$ to the space of flags $E_x$ that associates to each line $l \subset B_m(x)$ the flag $F(x, l)$ is a morphism for all $m$. After that we will apply the following theorem.

**Theorem 4.** For any $d$ there exists a number $M := M(d)$ such that for any $m > M$ any morphism from $B_m(x)$ to a projective variety of dimension less than $d$ is constant.

3.1. Proof of Theorem 4

We will start by analysing $B_m(x)$. Recall that for $i = 1, \ldots, l$ the number $d_i = \deg Y_{i,m}$, $m \geq 1$ denotes the degree of the hypersurface $Y_{i,m}$. Denote by $\mathbb{P}^{N_m-2}_x$ the projectivised tangent space to the Plücker space $\mathbb{P}^{N_m-1}$ at point $x$.

**Proposition 2.** $B_m(x)$ is given by the intersection of $\mathbb{P}^{k_m-1} \times \mathbb{P}^{n_m-k_m-1}$ with $\sum_i d_i$ hypersurfaces in $\mathbb{P}^{N_m-2}_x$ of degrees no more than $\max_i d_i$.

**Proof:** The base of the family of lines passing through $x$ on $G(k_m, n_m)$ is $\mathbb{P}^{k_m-1} \times \mathbb{P}^{n_m-k_m-1}$. So it is enough to prove that for any $i$ the space of lines on $Y_{i,m}$ passing through $x$ is the intersection of $d_i$ hypersurfaces in $\mathbb{P}^{N_m-2}_x$.

Let us introduce homogeneous coordinates $(z_0 : z_1 : \ldots)$ on $\mathbb{P}^{N_m-1}_x$. Let $x = (1 : 0 : \ldots : 0)$, then $(z_1 : z_2 : \ldots)$ are homogeneous coordinates on $\mathbb{P}^{N_m-2}_x$. Let us write the equation of the hypersurface $Y_{i,m}$ in the form

$$F = F_{d_1}(z_1 : z_2 : \ldots) + z_0 F_{d_1-1}(z_1 : z_2 : \ldots) + \ldots + z_0^{d_1-1} F_1(z_1 : z_2 : \ldots) = 0.$$
Then the variety of lines on $Y_{i,m}$ passing through $x$ is given in $\mathbb{P}_x^{N_m-2}$ by a system of $d_i$ equations

$$F_{d_i}(z_1 : z_2 : ...) = F_{d_i-1}(z_1 : z_2 : ...) = ... = 0$$

of degrees $d_i, d_i - 1, ..., 1$ correspondingly. This proves our claim.

\[\square\]

**Proof of Theorem 4.** Note that for any $d$ there is a number $M$ such that for any $m > M$ the following two conditions hold:

1) The variety $B_m(x)$ is 1-connected. This follows from Proposition 1 together with Proposition 2.

2) Any projective line on $B_m(x)$ is contained in a projective subspaces $\mathbb{P}^d \subset B_m(x)$ of dimension $d$. This follows from Lemma 1.

From conditions 1) and 2) it follows that any morphism from $B_m(x)$ to any variety of dimension less than $d$ is a morphism to a point. Indeed any line in $B_m(x)$ is mapped to a point since it is contained in some $\mathbb{P}^d$ (which in its turn has to be mapped to a point by [11], section II, § 7, exercise 7.3a). Since any two points in $B_m(x)$ are connected by a chain of lines, the whole variety $B_m(x)$ is mapped to a point as well.

\[\square\]

### 3.2. A standard lemma

Further on we will need the following standard fact.

**Lemma 2.** Let $X, Y, Z$ be projective varieties and suppose that $Y$ is smooth. Suppose we have morphisms $p : X \to Y$ and $\pi : X \to Z$ such that the fibres of the morphism $p$ are contained in the fibres of the morphism $\pi$. Then there is a morphism $f : Y \to Z$ such that $f \circ p = \pi$.

\[\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow^\pi & & \downarrow^f \\
Z & \xleftarrow{\pi'} & Y \times Z
\end{array}\]

**Proof.** Consider the morphism $\phi : X \to Y \times Z$, $\phi(x) = (p(x), \pi(x))$ and denote by $p'$, $\pi'$ the projections of $Y \times Z$ on $Y$ and $Z$ correspondingly.

\[\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow^\pi & & \downarrow^{\pi'} \\
Z & \xleftarrow{\pi'} & Y \times Z
\end{array}\]

Note that the projection $\pi'_1 : \phi(X) \to Y$ is an isomorphism ([15], section II.4, Theorem 2) since by our assumptions the projection $p'$ is a bijective morphism and $Y$ is smooth. The desired morphism $f : Y \to Z$ is given by the composition $f = \pi' \circ p'^{-1}$.

\[\square\]
3.3. Constructing a flag of subbundles in \( E \) on \( X_m \)

Finally, we start to construct the flag of subbundles. Let \( B_m \) be the base of the family of lines on \( X_m \). Let us consider the following set

\[
\Gamma_m = \{(x, l) \in X_m \times B_m | x \in l\}
\]

as a reduced scheme. Denote by \( p_m : \Gamma_m \to X_m \) the morphism such that \( p_m(x, l) = x \).

Recall that \( E|_{X_m} = E_m \) and the rank of \( E_m \) is \( r \). The fibre at point \( x \in X_m \) is denoted by \( E_m(x) \).

According to Theorem 3 the bundle \( E_m \) is uniform. So for any line \( l \in B_m \) we have the following splitting

\[
E_m|_l = r_1 \mathcal{O}_{P^1}(a_1) \oplus \ldots \oplus r_s \mathcal{O}_{P^1}(a_s), \quad a_1 > a_2 > \ldots > a_s, \quad \sum_{i=1}^s r_i = r,
\]

where \( r_i \) and \( a_i \) do not depend on the choice of the line \( l \).

**Theorem 5.** Let \( X = \lim X_m \) be a complete intersection of codimension \( l \) in the linear ind-Grassmannian \( G \). For any \( m \geq 1 \) the bundle \( E_m \) on \( X_m \) has a flag of subbundles

\[
0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_s = E_m
\]

such that for any line \( l \in B_m \)

\[
F_i|_l = r_1 \mathcal{O}_{P^1}(a_1) \oplus \ldots \oplus r_i \mathcal{O}_{P^1}(a_i), \quad 1 \leq i \leq s.
\]  

(7)

**Proof.** It is sufficient to prove this statement for all \( m \) from Theorem 4 that are larger than \( M(2r) \). Let \( E_{(1)} := E_m \). Let

\[
\mathfrak{St}(r_1, E_{(1)}) = \bigcup_{x \in X_m} G(r_1, E_{(1)}(x))
\]

be the grassmannisation of the bundle \( E_{(1)} \) with its natural projection \( \varphi_1 : \mathfrak{St}(r_1, E_{(1)}) \to X_m \), where \( \varphi_1 : (x, r_1 \mathcal{O}_{P^1}(a_1)|_x) \to x \). Consider the morphism \( \pi_1 : \Gamma_m \to \mathfrak{St}(r_1, E_{(1)}) \) where \( \pi_1 : (x, l) \to (x, r_1 \mathcal{O}_{P^1}(a_1)|_x) \). Let us show that the fibres of the morphism \( p_m : (x, l) \to x \) are contained in the fibres of the morphism \( \pi_1 \). Indeed, for any point \( x \in X_m \) the fibre \( p_m^{-1}(x) \) of the morphism \( p_m \) is isomorphic to the base of the family of lines passing through \( x \) on \( X_m \). The morphism \( \pi_1 \) sends \( p_m^{-1}(x) \) to \( \mathfrak{St}(r_1, E_{(1)}(x)) \) and this map is a map to a point according to Theorem 4 since \( \dim \mathfrak{St}(r_1, E_{(1)}(x)) < 2r \).

Consider the following diagram:

\[
\begin{array}{ccc}
\Gamma_m & \xrightarrow{p_m} & X_m \\
\downarrow & & \downarrow \\
\mathfrak{St}(r_1, E_{(1)}) & \xrightarrow{\pi_1} & X_m
\end{array}
\]
The existence of the section $f_1$ of the projection $\varphi_1$ follows from Lemma 2 in which we set $X = \Gamma_m$, $Y = X_m$, $Z = \mathcal{G}r(r_1, E_{(1)})$, $p = p_m$, $\pi = \pi_1$.

Denote by $\mathcal{S}_{r_1}$ the tautological $r_1$-dimensional subbundle in $\varphi_1^*E_{(1)}$. Applying the functor $f_1^*$ to the monomorphism of bundles

$$f_1^* \tau_1 : \mathcal{S}_{r_1} \rightarrow \varphi_1^*E_{(1)},$$

we get the following monomorphism of bundles:

$$f_1^* \tau_1 : f_1^* \mathcal{S}_{r_1} \rightarrow E_{(1)}.$$

Denote $E_{(2)} = E_{(1)}/f_1^* \mathcal{S}_{r_1}$, and consider the grassmannisation

$$\mathcal{G}r(r_2, E_{(2)}) = \bigcup_{x \in X_m} G(r_2, E_{(2)}(x))$$

of the bundle $E_{(2)}$.

We get the following diagram

$$\begin{array}{ccc}
\Gamma_m & \xrightarrow{p_m} & X_m, \\
\pi_2 \downarrow & & \downarrow \\
\mathcal{G}r(r_2, E_{(2)}) & \xleftarrow{f_2} & \varphi_2
\end{array}$$

where the existence of the morphism $f_2$ follows from Lemma 2 in the same way as the existence of the morphism $f_1$. By the same considerations as before we get the embedding

$$f_2^* \tau_2 : f_2^* \mathcal{S}_{r_2} \rightarrow E_{(2)}.$$

Denote $E_{(3)} = E_{(2)}/f_2^* \mathcal{S}_{r_2}$.

Repeating our previous considerations we get a family of epimorphisms of bundles:

$$E_{(1)} \rightarrow E_{(2)} \rightarrow \ldots \rightarrow E_{(s+1)}.$$

We associate to it a family of subbundles in $E_{(1)}$:

$$0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_s = E_{(1)}$$

such that for any line $l \in B_m$

$$F_i|_l \cong r_1 \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus r_i \mathcal{O}_{\mathbb{P}^1}(a_i), \quad 1 \leq i \leq s.$$

\[\square\]

**Corollary 2.** Any bundle $F_i/F_{i-1}$ is a bundle trivial on lines twisted by the line bundle $\mathcal{O}_{X_m}(a_i)$. 

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Indeed, from the construction of the flag of subbundles $E_m$ it follows that the restriction of $F_i/F_{i-1}$ to any line $l \in B_m$ is equal to $r_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ for $1 \leq i \leq s$. In other words the bundle $F_i/F_{i-1} \otimes \mathcal{O}_{X_m}(-a_i)$ is trivial on lines.

**Remark 1.** Subbundles $F_i$ of the bundle $E_m$ in Theorem 5 are defined in a unique way. This follows from the fact that any vector bundle $E$ on a projective line $\mathbb{P}^1$ has a canonically defined filtration $0 = F_0 \subset F_1 \subset F_2 \subset ... \subset F_s = E_m$ such that $F_i/F_{i-1}$ is isomorphic to $r_i \mathcal{O}(a_i)$ where $a_1 > a_2 > ... > a_s$.

Using the fact that the flag of subbundles $0 = F_0 \subset F_1 \subset F_2 \subset ... \subset F_s$ constructed in Theorem 5 does not depend on $m$, and using the linearity of the ind-variety $X = \lim X_m$ we get the following corollary from Theorem 5.

**Theorem 6.** Let $X$ be a complete intersection in the linear ind-Grassmannian $G$ and let $E$ be a uniform bundle on $X$. Then there exists a flag of subbundles

$$0 = F_0 \subset F_1 \subset ... \subset F_s = E$$

such that any quotient bundle $F_i/F_{i-1}$ is a bundle trivial on lines twisted by a line bundle.

### 4. A criterion of triviality of bundles trivial on lines

The goal of this section is to prove Theorem 7 which gives a criterion for a bundle trivial on lines to be trivial.

Let $X$ be a normal projective variety and $E$ a be vector bundle on $X$. Let $Y$ be the Fano scheme of lines on $X$. Let $Z \subset X \times Y$ be the universal line. Denote the projections to $Y$ and to $X$ by $\pi$ and $p$ respectively. Note that $\pi : Z \rightarrow Y$ is a $\mathbb{P}^1$-bundle. Consider the scheme

$$Z_1 = Z \times_Y Z,$$

It parameterizes lines on $X$ with pair of points on them. Let $p_1, p_2 : Z_1 \rightarrow X$ be the compositions of the projections $p_1, p_2 : Z_1 \rightarrow Z$ with the map $p : Z \rightarrow X$. Further, we define inductively the variety

$$Z_{n+1} = Z_n \times_X Z_1,$$

with projections $p_1 : Z_1 \rightarrow X$ and $p_{n+2} : Z_{n+1} \rightarrow X$, using the following diagram:
Finally, for a point $x \in X$ we define

$$Z_n(x) := p_1^{-1}(x).$$

The projection $Z_n(x) \to X$ induced by the projection $p_{n+1} : Z_n \to X$ will be denoted by $f_{x,n}$.

**Lemma 3.** Let $E$ be a vector bundle on $X$ trivial on lines. Then on $Z_1$ we have an isomorphism $p_1^*E \cong p_2^*E$.

**Proof.** First consider the vector bundle $p_2^*E$ on $Z$. Since $E$ is trivial on lines, it is trivial on all fibres of $\pi : Z \to Y$. Since the latter is a $\mathbb{P}^1$-bundle, it follows that $p^*E \cong \pi^*F$ for some vector bundle $F$ on $Y$. Now consider the diagram

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{q_1} & Z_1 \\
p_1 \downarrow & & \downarrow p_2 \\
X & \xleftarrow{f_{x,1}} & X
\end{array}
\]

We have

$$p_1^*E = q_1^*p^*E \cong q_1^*\pi^*F \cong q_2^*\pi^*F \cong q_2^*p^*E \cong p_2^*E,$$

which proves the Lemma. \hfill $\square$

The next step is the following

**Lemma 4.** If $E$ is trivial on lines then for each $n > 0$ the bundle $f_{x,n}^*E$ on $Z_n(x)$ is trivial.

**Proof.** We argue by induction on $n$. For $n = 1$ we have

$$f_{x,1}^*E = (p_2^*E)|_{Z_1(x)} \cong (p_1^*E)|_{Z_1(x)} \cong E_x \otimes \mathcal{O}_{Z_1(x)}$$

since the composition $Z_1(x) \subset Z_1 \xrightarrow{p_1} X$ factors through the point $x$. This justifies the base of the induction.

Now assume that the claim is true for some $n$. Then for $n + 1$ consider the diagram

\[
\begin{array}{ccc}
Z_{n+1}(x) & \xrightarrow{q_{n+1}} & Z_1 \\
f_{x,n+1} \downarrow & & \downarrow p_1 \\
Z_n(x) & \xleftarrow{f_{x,n+1}} & X
\end{array}
\]

We have

$$f_{x,n+1}^*E = q_{n+1}^*p_2^*E \cong q_{n+1}^*p_1^*E \cong q_n^*f_{x,n}^*E \cong q_n^*f_{x,n}^*E \cong q_n^*\mathcal{O}_{Z_n(x)} \cong \mathcal{O}_{Z_n+1(x)},$$

which proves the Lemma. \hfill $\square$
Now we can finish by the following argument

**Theorem 7.** Assume that $X$ is normal and for some $n > 0$ and some point $x \in X$ the map $f_{x,n} : Z_n(x) \to X$ is dominant and has connected fibers. Then any vector bundle on $X$ trivial on all lines is trivial.

**Proof.** Assume that $f_{x,n}$ is dominant and has connected fibers. Then $(f_{x,n})_* \mathcal{O}_{Z_n(x)} \cong \mathcal{O}_X$ since $X$ is normal. Hence, by projection formula we have

$$(f_{x,n})_* f_{x,n}^* E \cong E \otimes (f_{x,n})_* \mathcal{O}_{Z_n(x)} \cong E \otimes \mathcal{O}_X \cong E.$$

Finally, by Lemma 3 we have $f_{x,n}^* E \cong \mathcal{O}_{Z_n(x)}^{\oplus r}$, hence

$$(f_{x,n})_* f_{x,n}^* E \cong (f_{x,n})_* \mathcal{O}_{Z_n(x)}^{\oplus r} \cong \mathcal{O}_X^{\oplus r}.$$

Comparing these two equalities we see that $E$ is trivial.

\[\square\]

5. Splitting of the bundle $E$

To finish the proof of splitting of the bundle $E$ on the variety $X$ we apply Kodaira vanishing theorem [3].

**Theorem 8.** Let $X$ be a smooth projective variety and let $L$ be an ample line bundle on it. Then for any $q > 0$ we have $H^q(X, K_X \otimes L) = 0$.

Recall the following standard fact.

**Theorem 9.** Let $X$ be a complex projective variety and let $E$ be a vector bundle on $X$ with a subbundle $F$. Suppose that $H^1((E/F)^* \otimes F) = 0$ then $E \cong F \oplus E/F$.

Recall as well the formula for the canonical class of the Grassmannian $G(k, n)$:

$$K_{G(k,n)} = \mathcal{O}_{G(k,n)}(-n).$$

Recall the adjunction formula

**Theorem 10.** Let $X$ be a smooth variety that is a complete intersection of $G(k, n)$ with a finite collection of smooth hypersurfaces $Y_1, \ldots, Y_l$ of degrees $d_1, \ldots, d_l$ correspondingly, and let $d = \sum_{i=1}^k d_i$. Then we have $K_X = K_{G(k,n)} \otimes \mathcal{O}(d_1 + \ldots + d_l) = \mathcal{O}(d - n)$.

Recall finally that in the case when the number $l$ of hypersurfaces is less than $\dim G(k, n) - 2$ (i.e., $\dim X > 2$) by Lefschetz hyperplane theorem ([5], Theorem 3.1.17) we have $\text{Pic}(X) = \text{Pic}(G(k,n)) = \mathbb{Z}$.

**Corollary 3.** Consider on $X$ the bundle $L = \mathcal{O}(r)$. If $r > d - n$ then $H^1(L) = 0$. $H^1(\mathcal{O}_X(d - n + r)) = 0$ for all $r > 0$. In particular for $n \gg 1$, $d = 2$ we have $H^1(\mathcal{O}_X(a))$ for all $a > 0$. 

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Theorem 11. Let $X$ be an ind-variety and $E$ be a vector bundle on it. Let $0 = F_0 \subset F_1 \subset \ldots \subset F_s = E$ be a flag of subbundles such that each bundle $F_i/F_{i-1}$ is a twist of a bundle trivial on lines by a line bundle. Then $E = \bigoplus_i F_i/F_{i-1}$.

Proof. Let us show that there is $M \in \mathbb{Z}^+$ such that for any $m > M$ the restriction of the bundle $E$ on $X_m$ splits as a sum $\bigoplus_i F_i/F_{i-1}|_{X_m}$ of corresponding subbundles on $X_m$. Namely, chose $M$ such that $d - n_M < 0$. We will establish this splitting by induction in $i$. Suppose it is proven that $F_{i-1} = \sum_{1 \leq j \leq i-1} r_j \mathcal{O}(a_j)$, $a_j > a_i$. Then $F_i$ is an extension of the bundle $F_{i-1}$ by $r_i \mathcal{O}(a_i)$. To prove that $F_i$ splits it is enough to know that $H^1(r_i \mathcal{O}(-a_i) \otimes \sum_{1 \leq j \leq i-1} r_j \mathcal{O}(a_j)) = 0$.

The latter holds by Corollary 3 since $a_j - a_i > 0 > d - n_M$ for any $j < i$.

6. Proof of Theorem 1

In this section we prove Theorem 1.

Since the bundle $E$ is uniform we can apply Theorem 6. It follows that each quotient bundle $F_i/F_{i-1}$ for $1 \leq i \leq s$ is a twist of a bundle trivial on lines by a line bundle.

Set in Theorem 7 $n = k_m$ and apply it to $X = X_m$ (a complete intersection in $G(k_m, n_m)$). Then the fibre of the morphism $f_{x,k_m}$ over $y \in X_m$ is the space of paths of length $k_m$ on $X_m$ that start at $x$ and finish at $y$. In Corollary 1 set $X = X_m$ (recall that $X_m = G(k_m, n_m) \cap \bigcap_{i=1}^s Y_{i,m}$). Then for a sufficiently large $m$ such that $k_m$ and $n_m$ satisfy the inequality $2 \sum_i (d_i + 1) \leq \frac{k_m}{2} \leq \frac{n_m}{4}$ where $d_i = \deg Y_{i,m}$, we deduce that the space of paths $P_{k_m}(x,y)$ connecting $x$ with $y$ on $X_m$ is non-empty and connected. Hence the fibres of the morphism $f_{x,k_m}$ are non-empty and connected, in particular $f_{x,k_m}$ is dominant.

So the conditions of Theorem 7 hold for $X_m$. It follows that any vector bundle on $X$ trivial on lines is trivial. So each $F_i/F_{i-1}$ is a twist of a trivial bundle by a line bundle.

Finally, using Theorem 11 we deduce that the bundle $E$ is a direct sum $E = \bigoplus_i F_i/F_{i-1}$. This proves Theorem 1.

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