Thermodynamic equivalence of two-dimensional imperfect attractive Fermi and repulsive Bose gases

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(Dated: March 15, 2018)

We consider two-dimensional imperfect attractive Fermi and repulsive Bose gases consisting of spinless point particles whose total interparticle interaction energy is represented by \( a N^2 / 2V \) with \( a = - a_F \leq 0 \) for fermions, and \( a = a_B \geq 0 \) for bosons. We show that in spite of the attraction the thermodynamics of \( d = 2 \) imperfect Fermi gas remains well defined for \( 0 \leq a_F \leq a_0 = \hbar^2 / 2\pi m \), and is exactly the same as the one of the repulsive imperfect Bose gas with \( a_B = a_0 - a_F \). In particular, for \( a_F = a_0 \) one observes the thermodynamic equivalence of the attractive imperfect Fermi gas and the ideal Bose gas.

PACS numbers: 05.30.Fk, 05.30.Jp, 67.10.Fj

I. INTRODUCTION

The fundamental and subtle problem of the existence of thermodynamics for interacting quantum gases has always been of utmost interest and has huge literature, see e.g., [3–12] and references therein. In the simplest case, if the point quantum particles, whether fermions or bosons, repel each other there is no danger of a collapse and provided the thermodynamic limit is well defined the thermodynamic description exists. On the other hand, if the particles attract each other the situation becomes different. For bosons the answer is simple: the pure attraction between bosons rules out the existence of stable equilibrium and in this sense leads the collapse of the system. For fermions, however, the Pauli principle, which already for the ideal Fermi gas leads to the effective repulsion, can play a decisive role and counterbalance the interparticle attraction.

In the following we give an answer to these questions by considering the model of the so-called imperfect, spinless quantum gases [3–12]. In the occupation number representation the Hamiltonian of the imperfect Fermi gas has the following form

\[
H_{F, \text{imp}} = \sum_k \frac{\hbar^2 k^2}{2m} n_k - a_F \frac{N^2}{2V}, \quad a_F \geq 0,
\]

where \( V \) denotes the volume of the system, \( N = \sum n_k \), \( n_k \) is the occupation number of one-particle state with momentum \( \hbar k \), \( n_k = 0, 1 \). The coupling constant \( a_F \) measures the strength of the mean-field attractive potential energy \(-a_F N^2 / 2V\). Analogous expression holds for the Hamiltonian of the repulsive imperfect Bose gas with the coupling constant \( a_B \) replaced by \(-a_B < 0\) and \( n_k = 0, 1, \ldots \infty \). The model of imperfect quantum gases has been succesfully applied to repulsive bosons [3–12] where, inter alia, the Bose-Einstein condensation and the Casimir forces were discussed. In this paper we discuss both the attractive imperfect Fermi gas and the repulsive imperfect Bose gas and show their thermodynamic equivalence in two dimensions.

II. THE IMPERFECTION FERMII GAS

We work in the grand canonical ensemble parametrized by temperature \( T \), chemical potential \( \mu \), volume \( V \), and assume periodic boundary conditions. The evaluation of the grand canonical partition function

\[
\Xi_{F, \text{imp}}(T, V, \mu) = \sum_{N=0}^{\infty} \sum_{\{n_k\}} \exp \left[ -\beta \left( \sum_k n_k (\epsilon_k - \mu) - \frac{a_F N^2}{2V} \right) \right],
\]

where \( \beta = 1/k_B T \), is stymied by the presence of the factor \( \exp \left( \frac{a_F N^2}{2V} \right) \) which calls into question the very existence of \( \Xi_{F, \text{imp}}(T, V, \mu) \). The ground state energy of \( d \)-dimensional ideal Fermi gas at fixed volume behaves as \( E_0 \sim N^{1 + \frac{d}{2}} \) which hints at the possibility that for \( d \leq 2 \) the grand canonical partition function may exist.
In order to resolve this issue we use the identity
\[
\exp \left( \frac{\beta a_F}{2V} N^2 \right) = \left( \frac{V}{2\pi \beta a_F} \right)^{\frac{d}{2}} \int_{-\infty}^{\infty} dq \exp \left( -\frac{V q^2}{2\beta a_F} - Nq \right)
\]  
and rewrite \( \Xi(T, V, \mu) \) as
\[
\Xi_{F, \text{imp}}(T, V, \mu) = \left( \frac{V}{2\pi \beta a_F} \right)^{\frac{d}{2}} \int_{-\infty}^{\infty} dq \exp \left( -\frac{V q^2}{2\beta a_F} \right) \Xi_{F, \text{id}} \left( T, V, \mu - \frac{q}{\beta} \right),
\]
where \( \Xi_{F, \text{id}}(T, V, \mu) \) denotes the grand canonical partition function of an ideal Fermi gas. For large \( V \) Eqs. 11 can be represented as
\[
\Xi_{F, \text{imp}}(T, V, \mu) = \left( \frac{V}{2\pi \beta a_F} \right)^{\frac{d}{2}} \int_{-\infty}^{\infty} dq \exp \left[ -V \varphi_F(q; T, \mu) \right],
\]
with
\[
\varphi_F(q; T, \mu) = \frac{1}{\lambda^d} \left[ \frac{\lambda^{d-2} a_0}{2} (q - \beta \mu)^2 - \frac{1}{\Gamma(\frac{d}{2} + 1)} \int_0^\infty dx \frac{x^{\frac{d}{2}}}{1 + e^{\mu-x}} \right],
\]

where \( a_0 = \hbar^2 / 2\pi m \) and \( \lambda \) is the thermal de Broglie wavelength, \( \beta a_0 = \lambda^2 \). Thus the problem of the existence and evaluation of the series in Eq. 2 has been rephrased as the question of the existence of the integral in Eq. 5. Once the conditions for its existence are settled the bulk thermodynamics can be extracted from it via the method of steepest descent.

The parameter \( a_0 \) turns out to play a prominent role in our analysis and for this reason its presence in Eq. 6 is explicitly exposed. The existence of the integral in Eq. 5 depends on the behavior of function \( \varphi_F(q; T, \mu) \) for large \( |q| \). When \( q \to -\infty \) one has \( \varphi_F(q; T, \mu)/q^2 \sim a_0/2a_F \lambda^2 \) and this regime poses no problem. On the other hand for \( q \to \infty \) one has \( \varphi_F(q; T, \mu)/q^2 \sim -q^{\frac{d}{2}-1}/\lambda^d \Gamma(\frac{d}{2} + 2) + a_0/2a_F \lambda^2 \). Thus for \( d > 2 \) the grand canonical partition function ceases to exist (note that this observation refers to fermions with purely attractive interactions). For \( d = 2 \), when \( q \to \infty \) one finds \( \varphi_F(q; T, \mu)/q^2 \sim (a_0 - a_F)/2a_F \lambda^2 \) which delimits the allowed values of the coupling constant \( a_F \) to the range \( 0 \leq a_F \leq a_0 \). For the boundary value \( a_F = a_0 \) one has \( \varphi_F(q; T, \mu) \to -\frac{a_0}{\lambda \sqrt{2}} q \) which means that in this particular case only negative values of chemical potential are allowed. This limitation will be shown to have a simple physical interpretation. Note that in two dimensions parameter \( a_0 \) has a straightforward interpretation: the ground state energy per particle \( e_0 = n(a_0 - a_F)/2 \), where \( n \) is the density, becomes negative for \( a_F > a_0 \).

### III. THERMODYNAMICS OF THE TWO-DIMENSIONAL IMPERFECT FERMI GAS

The thermodynamic limit of the grand canonical potential density \( \lim_{V \to \infty} \omega_{F, \text{imp}}(T, V, \mu) / V = \omega_F(T, \mu) = -p_F(T, \mu) \) can be calculated using the method of steepest descent. It gives
\[
\omega_F(T, \mu) = k_B T \varphi_F(q_0(T, \mu); T, \mu),
\]
where \( q_0(T, \mu) \) minimizes \( \varphi_F(q; T, \mu) \) and fulfills the equation
\[
q_0 = \beta \mu + \frac{a_F}{a_0} \ln (1 + e^{q_0}).
\]
The number density \( n_F(T, \mu) = -\left( \frac{\partial \omega_F}{\partial \mu} \right)_T \) is thus related to \( q_0 \) via
\[
q_0(T, \mu) = \beta (\mu + a_F n_F(T, \mu))
\]
while the pressure \( p_F(T, \mu) \) is
\[
p_F(T, \mu) = -\frac{a_F}{2} n_F^2(T, \mu) + p_{F, \text{id}}(T, \mu + a_F n_F(T, \mu)),
\]
where \( p_{F, \text{id}}(T, \mu) \) denotes the pressure of the ideal Fermi gas
\[
p_{F, \text{id}}(T, \mu) = \frac{k_B T}{\lambda^2} \int_0^\infty dx \ln \left( 1 + e^{\beta \mu - x} \right).
\]
The density \( n_F(T, \mu) \) is obtained by solving Eq. (8) for \( q_0(T, \mu) \) and inserting the result into Eq. (9)

\[
n_F(T, \mu) = \frac{1}{\lambda^2} \ln \left( 1 + e^{q_0(T, \mu)} \right),
\]

see Fig. 1. The equation of state, \( p_F(T, n) \) takes the form

\[
p_F(T, n) = -\frac{a_F}{2} n^2 + p_{F, id}(T, n),
\]

where

\[
p_{F, id}(T, n) = -\frac{k_B T}{\lambda^2} \sum_{r=1}^{\infty} \frac{(1 - e^{n\lambda^2 r})^r}{r^2}.
\]

The coefficient of isothermal compressibility \( \chi_T(T, n) = \frac{1}{n} \left( \frac{\partial P}{\partial n} \right)_T \) of the imperfect Fermi gas has the following form

\[
\chi_T(T, n) = \left[ n^2(a_F - a_F) + \frac{n^2 a_0}{e^{n\lambda^2} - 1} \right]^{-1}.
\]

For \( a_F < a_0 \) it remains finite. However, for \( a_F = a_0 \) the compressibility becomes infinitely large for \( n \rightarrow \infty \) at fixed \( T \), or equivalently for \( \mu \rightarrow 0^- \) at fixed \( T \) (in this limit \( n_F \lambda^2 = -\ln (-\beta \mu) \)).

\[
\chi_T(T, n) = \left[ n^2(a_F - a_F) + \frac{n^2 a_0}{e^{n\lambda^2} - 1} \right]^{-1}.
\]

For the special case \( a_F = a_0 \), it follows from Eqs (8) and (11) that

\[
n_F(T, \mu) = -\frac{1}{\lambda^2} \ln \left( 1 - e^{\beta \mu} \right)
\]

and

\[
p_F(T, n) = -\frac{k_B T}{\lambda^2} \sum_{r=1}^{\infty} \frac{(1 - e^{n\lambda^2})^r}{r^2}.
\]

One notes that remarkable identities follow from the above formulas. They relate the imperfect Fermi gas at \( a_F = a_0 \) and the ideal Bose gas \( \text{id} \) (both defined for \( \mu < 0 \)) for which

\[
n_{B, id}(T, \mu) = -\frac{1}{\lambda^2} \ln \left( 1 - e^{\beta \mu} \right)
\]

and

\[
p_{B, id}(T, n) = -\frac{k_B T}{\lambda^2} \sum_{r=1}^{\infty} \frac{(1 - e^{n\lambda^2})^r}{r^2}.
\]

It follows from Eqs (10) and (13) that for \( a = a_0 \)

\[
n_F(T, \mu) = n_{B, id}(T, \mu)
\]

while from Eqs (17) and (19) one finds

\[
p_F(T, n) = p_{B, id}(T, n).
\]

The above equality can be checked using the integral representation of function \( p_{B, id}(T, \mu) \) in Eq. (19) and \( p_{F, id}(T, \mu) \) in Eq. (14). A straightforward calculation leads to

\[
\frac{\partial}{\partial n} [p_F(T, n) - p_{B, id}(T, n)] = 0
\]

from which Eq. (21) follows because \( [p_F(T, n) - p_{B, id}(T, n)]|_{n=0} = 0 \).

Thus according to Eqs (20) and (21) the two-dimensional imperfect Fermi gas at \( a = a_0 \) and the ideal Bose gas, both in the same state characterized by arbitrary \( T \) and \( \mu < 0 \) have the same densities \( n_F(T, \mu) = n_{B, id}(T, \mu) \) and pressures \( p_F(T, n) = p_{B, id}(T, n) \).

In order to put this remarkable equivalence into a broader perspective \( 14-19 \) we analyze the two-dimensional imperfect, spinless Bose gas with repulsive interactions characterized by the coupling constant \( a_B > 0 \)

\[
H_{B, imp} = \sum_k \frac{\hbar^2 k^2}{2m} n_k + \frac{a_B N^2}{2N}.
\]

As far as the existence of thermodynamics is concerned there is no upper bound on the coupling constant \( a_B \). The repulsive imperfect Bose gas has been intensely discussed in the literature, see e.g., \( 4, 14, 21 \). For \( d > 2 \)
it shows Bose-Einstein condensation taking place for 
\( \mu > a_B \zeta (d/2) \lambda^{-d} \) with the critical indices belonging to
the mean-spherical model universality class. The formalism used in [11, 12] to evaluate the partition function
is analogous to the one employed here with the identity
in Eq. (3) taking the role of the Hubbard-Stratonovich transformation [12] used in [11, 12]. Thus we only quote the
relevant formulas. Analogously to the Fermi case, the
grand canonical free energy density \( \omega_B(T, \mu) \) can be
obtained as the minimum of

\[
\varphi_B(s; T, \mu) = -\frac{1}{\lambda^2} \left[ \frac{a_0}{2a_B} (s - \beta \mu)^2 + g_2(\exp(s)) \right]
\]

with respect to variable \( s \), where \( g_2(z) \) is the Bose function, for details see [11, 12]. The thermodynamics follows
from the following set of equations

\[
\omega_B(T, \mu) = -p_B(T, \mu) = k_B T \varphi_B(s_0(T, \mu); T, \mu)
\]

where the parameter \( s_0(T, \mu) \) minimizing the function
\( \varphi_B(s; T, \mu) \) solves the equation

\[
s_0 - \beta \mu = \frac{a_B}{a_0} \ln (1 - e^{-s_0}). \tag{26}
\]

The density \( n_B(T, \mu) \) is related to \( s_0(T, \mu) \) via

\[
n_B(T, \mu) = \beta(\mu - a_B n_B(T, \mu)) \tag{27}
\]

and can be rewritten as

\[
n_B(T, \mu) \lambda^2 = -\ln \left( 1 - e^{-s_0(T, \mu)} \right) \tag{28}
\]

which gives \( p_B(T, n) \) in the following form

\[
p_B(T, n) = \frac{a_B}{2} n^2 + \frac{k_B T}{\lambda^2} g_2 \left( 1 - e^{-n \lambda^2} \right). \tag{29}
\]

The above set of equation Eqs (25-29) can be confronted with the corresponding set Eqs (7-11) for
the attractive imperfect Fermi gas. In order to find the relation
between the imperfect attractive Fermi and the imperfect repulsive Bose gas [4, 14, 20] we rewrite Eq. (12) in the
following form

\[
n_F \lambda^2 = -\ln \left( 1 - e^{-\zeta_0} \right) \tag{30}
\]

where the parameter \( \zeta_0 \) is defined via

\[
\zeta_0 = -\ln \left( 1 + e^{-\beta \mu} \right). \tag{31}
\]

It follows from Eqs (31) and (5) that

\[
\zeta_0 - \beta \mu = \left( 1 - \frac{a_F}{a_0} \right) \ln \left( 1 - e^{-\zeta_0} \right) \tag{32}
\]

and thus the parameter \( \zeta_0 \) fulfills the same equation as
parameter \( s_0 \), see Eq. (20), provided

\[
a_B = a_0 - a_F. \tag{33}
\]

In other words, if the above relation holds then the fermion and boson densities are identical:
\( n_F(T, \mu) = n_B(T, \mu) \). Similarly, it follows from Eqs (10), (13), (19), (20) that

\[
p_F(T, n) = \frac{n^2}{2} (a_0 - a_F + p_{B,id}(T, n)) = \frac{n^2}{2} (a_0 - a_F - a_B) + p_B(T, n) \tag{34}
\]

and thus when the relation (33) is fulfilled one has
\( p_F(T, n) = p_B(T, n) \). Thus for each value of the coupling constant \( a_F \leq a_0 \) there exists a repulsive imperfect Bose
gas characterized by the coupling constant \( a_B = a_0 - a_F \); it has the same density and pressure as the Fermi gas.
In particular, when \( a_F = a_0 \) one has \( a_B = 0 \) and the previously proven result \( n_F(T, n) = n_B(id)(T, n) \) and
\( p_F(T, n) = p_B(id)(T, n) \), see Eqs (20, 21) is recovered. In this case the attractive mean field suppresses the
effect of the Fermi statistics and makes the pressure of the Fermi gas approach zero for \( T \rightarrow 0 \) at
any density \( n \). On the other hand, for \( a_F = 0 \) one has \( p_{F,id}(T, n) = a_0 n^2/2 + p_{B,id}(T, n) \). These are special
cases of the general equivalence condition in Eq. (33).

To summarize, we have shown that thermodynamics exists for the two-dimensional attractive imperfect
Fermi gas provided the coupling constant \( a_F \) measuring the strength of the attractive energy is not too large, \( 0 \leq a_F \leq a_0 \). No thermodynamics exists for three and
higher dimensional imperfect attractive Fermi systems. We showed that in two dimensions the density and pressure
of the attractive Fermi gas are the same as those of the repulsive imperfect Bose gas provided the coupling constant \( a_B \) measuring the strength of repulsion fulfills the relation \( a_B = a_0 - a_F \). Needless to say, the above
derived equivalence of thermodynamics of imperfect attractive Fermi and repulsive Bose gases is restricted to
two-dimensions where the imperfect Bose gas does not suffer the Bose-Einstein condensation.

Acknowledgments

M.N. acknowledges the support from National Science Center, Poland via grant 2014/15/B/ST3/02212.

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