KAM THEORY

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I

GROUP ACTIONS AND THE KAM PROBLEM
This book originated from courses taught by the first author in 2011 at the Dijon University (France) and in 2013 at the University of Ouargla (Algeria) on the Herman invariant tori conjecture.

The first author thanks M. Bahayou and A. Zeglaoui for the invitation in Ouargla and R. Uribe and A. Dubouloz for the invitation in Dijon. Thanks also to the audience and in particular to N. A’Campo, P. Cartier, Z. Fernane, A. Kessi, F. Laudenbach, D. Smai and N. Yousfi.
To the memory of V.I. Arnold (1937-2010).
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CHAPTER 1

Darboux theorems and action-angle variables

We review classical mechanics in its symplectic formulation and sketch proofs of the basic results like the existence of action-angle coordinates. Some familiarity with the language of global analysis is assumed.

1. Hamiltonian vector fields

A real $\mathcal{C}^\infty$ manifold $M$ is called symplectic if it is endowed with a closed 2-form $\omega$ which induces an isomorphism between the tangent and cotangent bundles:

$$TM \rightarrow T^*M, \ X \mapsto i_X\omega.$$ 

Here $i_X$ denotes the interior product of a differential form with a vector field $X$, that is:

$$i_X\omega(Y) = \omega(X,Y).$$

This isomorphism associates to each differential 1-form a unique vector field, which we call symplectically associated to it. Given a function $H : M \rightarrow \mathbb{R}$, the vector field $X_H$ associated to the 1-form $dH$ is called the Hamiltonian vector field of $H$.

The space $\mathbb{R}^{2n}$, equipped with the 2-form

$$\omega := \sum_{i=1}^{n} dq_i \wedge dp_i,$$

gives a basic example of a symplectic manifold. The isomorphism between tangent and cotangent bundles is given by

$$\partial_{q_i} \mapsto dp_i, \ \partial_{p_i} \mapsto -dq_i.$$ 

In this way we recover the classical definition of the Hamiltonian vector field

$$X_H := \sum_{i=1}^{n} (\partial_{p_i}H \partial_{q_i} - \partial_{q_i}H \partial_{p_i}),$$

as it is symplectically associated to $dH$. Its integral curves are the solutions to Hamilton's canonical equations of motion:

$$\begin{align*}
\dot{q}_i &= \partial_{p_i}H \\
\dot{p}_i &= -\partial_{q_i}H.
\end{align*}$$
A smooth map \( \varphi : M \rightarrow M' \) between two symplectic manifolds \((M, \omega)\) and \((M', \omega')\) is called \textit{symplectic}, if it preserves the symplectic forms:

\[
\varphi^*(\omega') = \omega.
\]

By a \textit{symplectomorphism} we will mean a symplectic diffeomorphism. A symplectomorphism \( \varphi : M \rightarrow M \) maps the Hamiltonian vector field of \( H \circ \varphi \) to that of \( H \). Therefore the qualitative behaviour of a dynamical system only depends on the orbit of Hamiltonian functions under the group of symplectomorphism.

If one can integrate the Hamiltonian vector field \( X_H \) up to time \( t \), we obtain a \textit{flow}

\[
\varphi_t := \varphi_{X_H}^t : M \rightarrow M, \quad \varphi_0 = \text{Id}_M, \quad \varphi_t \circ \varphi_s = \varphi_{t+s},
\]

which is a one parameter family of symplectomorphism depending on \( t \).

To show this, recall the formula

\[
\frac{d}{dt}(\varphi_t^* \omega) = \varphi_t^* (\mathcal{L}_{X_H} \omega)
\]

where \( \mathcal{L} \) stands for the Lie derivative. Applying Cartan’s formula

\[
\mathcal{L}_X = di_X + i_X d
\]

we find:

\[
\mathcal{L}_{X_H} \omega = di_{X_H} \omega + i_{X_H} d\omega = dH + 0 = 0.
\]

Hence \( \varphi_t^* \omega \) is constant in \( t \), and as \( \varphi_0 = \text{Id}_M \) we find \( \varphi_t^* \omega = \omega \).

So any function \( H \) gives rise to a differential \( dH \) and therefore to a vector field which preserves the symplectic form, but the converse is not always true. Vector fields preserving the symplectic form will be called \textit{symplectic vector fields}. These are usually called \textit{Hamiltonian vector fields}, but we wish to distinguish them from the vector fields defined by Hamiltonian functions (which are usually called \textit{exact Hamiltonian vector fields}). So we use the name Hamiltonian vector fields for the vector fields defined by a Hamiltonian function.

\textbf{Proposition 1.1.} In a symplectic manifold the closed one-forms are \textit{symplectically associated to symplectic vector fields}.

\textbf{Proof.} A vector field \( X \) is symplectic if and only if

\[
\mathcal{L}_X \omega = 0.
\]

By Cartan’s formula

\[
\mathcal{L}_X \omega = di_X \omega + i_X d\omega = di_X \omega.
\]

Therefore \( i_X \omega \) is a closed 1-form if and only if \( X \) is symplectic. \( \square \)
We denote by $\Omega^1(M)$ the vector space of $C^\infty$ real valued 1-forms on $M$ and let

$$\Omega^1_{\text{closed}}(M) := \{ \eta \mid d\eta = 0 \} \supset \Omega^1_{\text{exact}}(M) := \{ dH \mid H \in C^\infty(M) \}$$

be the sub-spaces of closed and exact 1-forms. The exterior derivative is a map

$$C^\infty(M) \to \Omega^1_{\text{closed}}(M)$$

which has the space of locally constant functions $H^0(M)$ as kernel, and the de Rham cohomology space $H^1(M)$ as cokernel, so we obtain an exact sequence of vector spaces

$$0 \to \Omega^1_{\text{exact}}(M) \to \Omega^1_{\text{closed}}(M) \to H^1(M) \to 0.$$  

Looking at the symplectically associated spaces of vector fields, the above exact sequence is converted into the exact sequence

$$0 \to \text{Ham}(M) \to \text{Symp}(M) \to H^1(M) \to 0,$$

where $\text{Ham}(M)$ denotes the vector space of Hamiltonian vector fields and $\text{Symp}(M)$ the space of vector fields which preserve the symplectic form. So we see that the difference between these two types of vector fields has a simple cohomological interpretation.

**Example 1.2.** Consider the $n$-dimensional torus $\mathbb{R}/\mathbb{Z}$; its points are described by $n$ angular coordinates $\theta_1, \ldots, \theta_n$. These $\theta_i$ are multivalued functions, but their differentials $\alpha_i := d\theta_i$ are well-defined closed 1-forms on the torus. We consider on the space $M := (\mathbb{R}/\mathbb{Z})^n \times \mathbb{R}^n$ the symplectic form

$$\omega = \sum_{i=1}^n d\theta_i \wedge dp_i,$$

where the $p_i$ are coordinates on the $\mathbb{R}^n$-factor. The cohomology classes $[\alpha_i], i = 1, 2, \ldots, n$ define a basis of the de Rham cohomology space

$$H^1(M) = \oplus_{i=1}^n \mathbb{R} [\alpha_i] \approx \mathbb{R}^n.$$  

These forms are associated to the symplectic vector fields:

$$X_1 = \partial_{p_1}, \ldots, X_n = \partial_{p_n}.$$  

Therefore any symplectic vector field on $M$ is of the form

$$\sum_{i=1}^n a_i \partial_{p_i} + X_H,$$

where

$$H : M \to \mathbb{R}$$

is a $C^\infty$ function and $a_1, \ldots, a_n \in \mathbb{R}$. 

2. Cotangent spaces

The cotangent space $M = T^*L$ of a manifold $L$ naturally inherits a symplectic form. The points of this manifold are pairs $(q, p)$, where $p \in T^*_qL$ is a covector at a point $q \in L$. The derivative of the bundle projection

$$\pi : M \rightarrow L$$

is a map

$$d\pi : TM \rightarrow TL.$$  

The action one-form $\alpha \in \Omega^1(M)$ is defined by the formula

$$\alpha_{q,p}(\xi) = p(d\pi_{q,p}(\xi)), \quad \xi \in T_{q,p}M.$$  

Then the exact two form:

$$\omega := d\alpha$$

is non-degenerate and is called the canonical symplectic form on $T^*L$.

Let us analyse this construction for $L = \mathbb{R}$. In this case, the cotangent bundle can be identified with the projection

$$\mathbb{R}^2 \rightarrow \mathbb{R}, (q, p) \mapsto q.$$  

If $\xi = (\xi_1, \xi_2) \in T_{(q,p)}\mathbb{R}^2$ is a tangent vector, then

$$d\pi_{q,p}(\xi) = \xi_1$$

so the action form is defined by

$$\alpha_{q,p}(\xi) = p\xi_1.$$
This means that 
\[ \alpha = pdq \]
and therefore \( \omega = dp \wedge dq \). Similarly, for the space \( \mathbb{R}^{2n} \), viewed as \( T^*\mathbb{R}^n \), we recover the symplectic form defined previously, up to a sign and in Example 1.2 we were in fact dealing with the cotangent space to the \( n \)-torus \( (\mathbb{R}/\mathbb{Z})^n \).

Note that the canonical symplectic form vanishes on the zero-section of \( T^*L \). More generally, an \( n \)-dimensional submanifold \( L \) of a \( 2n \)-dimensional symplectic manifold \( (M, \omega) \) is called \textit{Lagrangian} if \( \omega|_L = 0 \). In such a case the class of \( \omega \) vanishes inside the de Rham cohomology group \( H^2(L) \). In any neighbourhood of \( L \) which retracts onto \( L \), the symplectic form is exact. A form \( \alpha \) such that 
\[ d\alpha = \omega \]
is called an \textit{action form}.

**Proposition 1.3.** Consider the symplectic space \( \mathbb{R}^{2n} \) with coordinates \( q, p \) and symplectic form
\[ \omega = \sum_{i=1}^{n} dq_i \wedge dp_i. \]
Let \( L \subset \mathbb{R}^{2n} \) be a Lagrangian manifold given as the graph of a map
\[ f = (f_1, \ldots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n \]
over the \( q \)-space. Then there exists a function
\[ S : \mathbb{R}^n \longrightarrow \mathbb{R} \]
such that
\[ f = \nabla S = (\partial_{q_1} S, \partial_{q_2} S, \ldots, \partial_{q_n} S). \]
Conversely, any such graph is Lagrangian.

**Proof.** The action form
\[ \sum_{i=1}^{n} p_i dq_i \]
is closed on \( L \), thus the Poincaré lemma implies that it is the differential of a function \( S \):
\[ dS = \sum_{i=1}^{n} p_i dq_i, \]
hence \( p_i = \partial_i S \).

A function \( S \) as in the proposition is called a \textit{generating function} of \( L \). It is unique up to an additive constant.
3. The Darboux theorem

Recall that two mappings define the same germ along a subset, if their restriction to a common neighbourhood of the subset agree. We denote by 

\[ f : (M, X) \rightarrow (N, Y), \quad f(X) = Y \]

the germ of \( f : M \rightarrow N \) along \( X \). The following fundamental result is called the Darboux theorem.

**Theorem 1.4.** The germ of a \( 2n \)-dimensional symplectic manifold \( (M, \omega) \) at an arbitrary point is isomorphic to the germ of \( (\mathbb{R}^{2n}, \sum dq_i \wedge dp_i) \) at the origin.

**Proof.** Take a local chart 

\[ \varphi : (M, p) \rightarrow (\mathbb{R}^{2n}, 0) \]

at a point \( p \in M \). We get two symplectic forms on a neighbourhood of the origin in \( \mathbb{R}^{2n} \):

\[ \omega_0 = (\varphi^{-1})^* \omega, \quad \omega_1 = \sum_{i=1}^{n} dq_i \wedge dp_i. \]

The value of these forms at the origin are anti-symmetric bilinear forms which are conjugate by a linear transformation. Thus, up to a linear change of coordinates, we may and will assume that they are equal. It follows that

\[ \omega_t = (1 - t)\omega_0 + t\omega_1, \quad t \in [0, 1] \]

defines a 1-parameter family of symplectic forms in a sufficiently small contractible neighbourhood of the origin.

We now search for a 1-parameter family \( \varphi_t \) of symplectomorphisms such that:

\[ \varphi_t^* \omega_t = \omega_0. \]

We differentiate this equation with respect to \( t \) and as the right hand side is \( t \)-independent we obtain:

\[ 0 = \frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* (L_{X_t} \omega_t + \dot{\omega}_t), \quad \dot{\omega}_t := \frac{d}{dt} \omega_t, \]

where \( X_t \) is the time-dependent Hamiltonian vector field associated to \( v_t \). Composing with the inverse of \( v_t^* \), we get the equation:

\[ L_{X_t} \omega_t = -\dot{\omega}_t \]

Cartan’s formula shows that:

\[ L_{X_t} \omega_t = d_{X_t} \omega_t + i_{X_t} d\omega_t = d_{X_t} \omega_t. \]

As the neighbourhood was assumed to be contractible, the closed form \( \omega_t \) is in fact exact by the Poincaré lemma, so we have

\[ \omega_t = d\alpha_t. \]
It is therefore sufficient to solve the equation
\[ i_{X_t} \omega_t = -\dot{\alpha}_t \]
for the vector field \( X_t \), which is possible, since the interior product with a symplectic form is an isomorphism. As the forms \( \omega_1 \) and \( \omega_2 \) are equal at the origin, the vector field \( X_t \) vanishes at 0. Thus, the time 1 flow of the vector field \( X_t \) exists in a small neighbourhood of the origin and yields the sought for symplectomorphism. This proves the theorem. \( \Box \)

Coordinates such as in the above theorem are called Darboux coordinates, sometimes canonical coordinates. The existence of Darboux coordinates shows that there are no local symplectic invariants.

**Corollary 1.5.** Let \( L \subset M \) be a Lagrangian submanifold of a symplectic manifold \((M, \omega)\). The germ of \( L \) at a point is symplectomorphic to the germ at the origin of the zero section in \( T^* \mathbb{R}^n \) with its canonical symplectic structure.

**Proof.** Choosing Darboux coordinates at the point considered, we can reduce to the case \( M = T^* \mathbb{R}^n \) equipped with its canonical symplectic form. The \( \pi/2 \)-rotations in the planes \((q_i, p_i) \mapsto (-p_i, q_i)\)
are symplectomorphisms and by applying these, we may always assume that \( L \) is the graph of a map. By Proposition 1.3, this map is itself the gradient of a function \( S : \mathbb{R}^n \rightarrow \mathbb{R} \).

The map
\[ (q, p) \mapsto (q, p - \nabla S) \]
is a symplectomorphism which maps \( L \) to the zero section. \( \Box \)

4. The classical Darboux-Weinstein theorem

The theorem of Darboux implies the fundamental fact that a Lagrangian manifold has no local symplectic invariants. In some situations there are global versions of this result.

Our proof of the Darboux theorem consisted of two parts: starting with two different symplectic forms in a neighbourhood of the origin in \( \mathbb{R}^{2n} \), we first chose linear coordinates so that both symplectic forms agree at the origin. Then knowing that the linear path between the symplectic forms remains inside the space of symplectic forms, we applied the path homotopy method.

**Theorem 1.6.** Let \( L \) be compact submanifold of a symplectic manifold \( M \) and \( \omega_t, t \in [0, 1] \) a one parameter family of symplectic forms on \( M \) with \( C^\infty \) dependence on \( t \). Assume that the restrictions of de Rham
classes \([\omega_t]\) to \(H^2(L, \mathbb{R})\) are independent of \(t\). Then there exists a neighbourhood \(T \subset M\) of \(L\) such that the \(t\)-dependent symplectic manifolds \((T, \omega_t)\) are all symplectomorphic.

**Proof.** By compactness of the interval \([0, 1]\) it suffices to prove the theorem for sufficiently small values of \(t\). Choose a tubular neighbourhood \(T\) of \(L\). As \(T\) retracts to \(L\), the assumptions imply that the class of \(\partial_t \omega_t\) vanishes in \(H^2(T)\). This means that \(\partial_t \omega_t\) is exact in \(T\), that is, we find a family of 1-forms \(\alpha_t\) such that

\[
\partial_t \omega_t = d\alpha_t.
\]

As before one finds a time dependent vector field \(X_t\) such that

\[
i_{X_t} \omega_t = -\alpha_t
\]

By compactness of \(L\), the vector field can be integrated for sufficiently small times and its time \(t\)-flow sends \(\omega_0\) to \(\omega_t\). This concludes the proof of the theorem. \(\square\)

From this theorem we will deduce the following celebrated result:

**Theorem 1.7.** Any compact Lagrangian submanifold \(L\) of a symplectic manifold \((M, \omega)\) admits a neighbourhood symplectomorphic to a neighbourhood of \(L\) in \(T^*L\) with its standard symplectic structure.

In the proof we will make use of an important idea of Gromov, namely the existence of an almost complex structure adapted to a given symplectic form. Recall that that a complex structure on a real vector space \(E\) is a linear map \(J \in GL(E)\) with the property that \(J^2 = -\text{Id}\). It is said to be adapted to a linear symplectic form \(\omega\) on \(E\) if

\[
\omega(-, J-)\]

is an Euclidean scalar product on \(E\).

**Lemma 1.8.** Let \((E, \omega)\) be a symplectic vector space, \(L \subset E\) a linear Lagrangian subspace. A complex structure \(J \in GL(E)\) is adapted if and only if the following three conditions are satisfied:

(i) The quadratic form \(L \to \mathbb{R}, x \mapsto \omega(x, Jx)\) is positive definite,
(ii) \(L\) and \(JL\) are complementary,
(iii) \(\omega(x, y) = \omega(Jx, Jy)\) for any \(x, y \in E\).

**Proof.** We first check that the conditions are necessary. The restriction of a positive definite quadratic form to a vector sub-space remains positive definite, so clearly we have (i). If \(x \in L \cap JL\), then \(Jx\) lies also in \(L \cap JL\). As \(L\) is Lagrangian, we deduce that

\[
\omega(x, Jx) = 0
\]
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and this implies that \( x = 0 \) since \( \omega(-, J-) \) is positive definite, so we have \((ii)\). To verify check \((iii)\), let \( z = Jy \), using that \( \omega(-, J-) \) and \( \omega \) are respectively symmetric and antisymmetric, we get that:

\[
\omega(x, y) = \omega(x, -Jz) = \omega(z, -Jx) = \omega(Jy, -Jx) = \omega(Jx, Jy).
\]

Let us now prove that the conditions are sufficient. By \((iii)\) the form \( \omega(-, J-) \) is symmetric and by \((ii)\) any \( x \in E \) can be written as

\[
x = x_1 + Jx_2, \quad x_1, x_2 \in L.
\]

Then

\[
Jx = Jx_1 - x_2,
\]

and so

\[
\omega(x, Jx) = \omega(x_1, Jx_1) + \omega(x_1, -x_2) + \omega(Jx_2, Jx_1) + \omega(Jx_2, -x_2).
\]

The two terms in the middle are zero since \( L \) is Lagrangian, whereas by \((i)\) the first and last terms are \( \geq 0 \) and vanish only if \( x_1 \) and \( x_2 \) are both zero. \(\square\)

**Example 1.9.** Take \( E = \mathbb{R}^2 \) with the standard symplectic structure

\[
\omega : ((x, y), (x', y')) \mapsto xy' - x'y.
\]

A matrix defines an almost complex structure if its eigenvalues are \( \pm i \). Such a matrix is of the form

\[
J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}
\]

with \( a^2 + bc = -1 \). Therefore the set of complex structures is a two-sheeted hyperboloid. One sheet corresponds to the complex structure for which \( \omega(-, J-) \) is positive definite and the other one for which it is negative definite.
Theorem 1.10. Let $(E, \omega)$ be a linear symplectic space. The subset $\mathcal{J}_\omega$ of $GL(E)$ consisting of complex structures adapted to $\omega$ is a non-empty contractible manifold.

Proof. We fix two transverse linear Lagrangian subspaces $L_1, L_2 \subset E$. As $J$ is adapted to $\omega$, the quadratic form
$$q(\eta) := \omega(\eta, J\eta), \quad \eta \in L_1$$
is positive definite and $L_1$ and $JL_1$ provide a direct sum decomposition of $E$:
$$E = L_1 \oplus JL_1.$$If conversely $L'$ is any linear Lagrangian subspace transverse to $L_1$ and
$$q : L_1 \rightarrow \mathbb{R}$$is a positive definite quadratic form, we can recover an adapted complex structure $J$ on $E$ as follows: the $q$-orthogonal space to $x \in L_1$ is a hyperplane $H \subset L_1$. Therefore there is a unique $y \in L'$ which vanishes on $H$ such that
$$\omega(x, y) = q(x)$$Now define a complex structure $J$ by mapping $x$ to $y$ and $y$ to $-x$.

As the Lagrangian subspace $L_2$ is transverse to $L_1$, all other transverse Lagrangian subspaces $L'$ are obtained as graphs of the derivative of a quadratic function
$$S : L_2 \rightarrow \mathbb{R}.$$Thus the space of all adapted complex structures on our symplectic vector space is in one-to-one correspondence with the contractible set of pairs of quadratic forms $(S, q)$ such that $q$ is positive definite. □

Given a manifold $M$, we have a principal bundle
$$GL(TM) \rightarrow M,$$whose fibre above $x \in M$ is the linear group $GL(T_x M)$. A section of this bundle
$$J : M \rightarrow GL(TM)$$is called an almost complex structure if for any $x \in M$, we have
$$J(x)^2 = -\text{Id}.$$If now $M$ is equipped with a symplectic form $\omega$, the almost complex structure $J$ is called adapted (or also tamed) if $\omega(-, J-)$ is a Riemannian metric. From the above lemma one deduces the existence of adapted almost complex structures on symplectic manifolds. Indeed we have a bundle
$$\mathcal{J}_\omega(M) \rightarrow M$$whose fibre above $x \in M$ is the set of $\omega_x$-adapted structures on $T_x M$. This space is contractible and therefore the bundle admits sections.
Lemma 1.11. Let $\omega_1$ and $\omega_2$ be two symplectic forms on a manifold $M$. Assume that there exists a complex structure $J$ adapted both to $\omega_1$ and $\omega_2$. Then all two-forms lying on the positive cone

$$\alpha_1 \omega_1 + \alpha_2 \omega_2, \; \alpha_i \geq 0$$

are symplectic.

Proof. Put

$$\omega = \alpha_1 \omega_1 + \alpha_2 \omega_2, \; \alpha_i \geq 0.$$ 

Assume that $\omega(\xi, -)$ vanishes then

$$\omega(\xi, J\xi) = \alpha_1 \omega_1(\xi, J\xi) + \alpha_2 \omega_2(\xi, J\xi) = 0$$

But the $\omega_i(-, J-)$'s are euclidean scalar product thus

$$\omega_1(\xi, J\xi) \geq 0, \; \omega_2(\xi, J\xi) \geq 0$$

therefore the sum cannot be zero unless both terms vanish. Consequently $\xi = 0$. This proves the lemma. \hfill \Box

We can now prove the Darboux-Weinstein theorem.

Proof of theorem. The abstract normal bundle $NL$ to $L$ is the quotient of the restriction of the tangent bundle of $M$ to $L$, $TM|_L$, by the tangent bundle $TL$ to $L$. By the tubular neighbourhood theorem, there is a diffeomorphism $\phi$ from a neighbourhood of $L$ in $M$ to a neighbourhood of $L$ in the normal bundle $NL$. The interior product of vectors $v$ based at a point $p$ of $L$ with the symplectic form induces an isomorphism of $NL$ with the cotangent bundle $T^*L$ of $L$:

$$\nu : N_p L \longrightarrow T^*_p L, \; v \mapsto i_v \omega.$$ 

Therefore we reduced the theorem to the case $(M, \omega) = (T^*L, \omega)$ where $\omega$ is a priori not the standard symplectic form $\omega_{std}$ on $T^*L$.

We now choose almost complex structures $J$ and $J_{std}$ on $M = T^*L$ adapted to the symplectic forms $\omega$ and $\omega_{std}$.

First observe that the tangent space $T_{q,0}M$ is a direct sum

$$T_{q,0}M = T_qL \oplus JT_qL.$$

The inner product with the symplectic form gives an identification of $JT_qL$ with $T_q^*L$.

Using these data, we will now define a map

$$f : M \longrightarrow M$$

in the following way. The inclusion of bundles over $L$

$$i : T^*L \longrightarrow TM|_L,$$

is given pointwise by the inclusion:

$$T_q^*L \hookrightarrow T_{q,0}M.$$
Its composition with the linear bundle map
\[ JJ_{\text{std}}^{-1} : TM_{|L} \rightarrow TM_{|L} \]
defines a map
\[ JJ_{\text{std}}^{-1} \circ i : M \rightarrow TM_{|L}. \]
The map \( f \) is obtained by composing this map with the exponential map
\[ \exp : TM_{|L} \rightarrow M, \]
defined pointwise by the exponential maps
\[ \exp_q : T_{(q,0)}M \rightarrow M \]
for the Riemannian metric \( \omega(\cdot, J_{\text{std}} \cdot) \).

\[ f := \exp \circ JJ_{\text{std}}^{-1} \circ i : M = T^*L \leftrightarrow TM_{|L} \xrightarrow{J_{\text{std}}^{-1}} TM_{|L} \xrightarrow{\exp} M. \]

Clearly \( f \) restricted to \( L \) is the identity.

The derivative of this map is easily computed at points of \( L \):
\[ df_{|TL} = \text{Id} \]
\[ df_{|J_{\text{std}}TL} = JJ_{\text{std}}^{-1} \]

We assert that the form \( f^*\omega \) is \( J_{\text{std}} \)-adapted in any sufficiently small neighbourhood of \( L \). Indeed, for any point \( q \in L \) and any \( \xi \in T_q L \), we have
\[
\begin{align*}
  f^*\omega_{(q,0)}(\xi, J_{\text{std}}\xi) &= \omega_{(q,0)}(df(x)\xi, df(x)J_{\text{std}}\xi) \\
  &= \omega_{(q,0)}(\xi, J_{\text{std}}^{-1}J_{\text{std}}\xi) \\
  &= \omega_{(q,0)}(\xi, J\xi) > 0
\end{align*}
\]
As the condition of being adapted is open, this proves the assertion. Lemma 1.11 shows that the path
\[ \omega_t := t\omega_{\text{std}} + (1 - t)f^*\omega \]
remains inside the space of symplectic forms. Therefore, using Theorem 1.6 with \( M = T^*L \), we get that \((T^*L, \omega_{\text{std}})\) and \((T^*L, f^*\omega)\) are symplectomorphic, in a sufficiently small neighbourhood of \( L \). This concludes the proof of the theorem. \( \square \)

5. The relative Darboux-Weinstein theorem

As is customary in algebraic geometry, it is useful to look for a relative version of the Darboux-Weinstein theorem, where one considers families of manifolds over a base \( S \) and which specialise to it in the case that \( S \) reduces to a point.

A fibration of \( C^\infty \)-manifolds
\[ \pi : M \rightarrow S \]
is called symplectic, if it carries a two-form which restricts to a symplectic form on all the fibres \( M_s = \pi^{-1}(s) \). In the spirit of Grothendieck, \( M \) is viewed as a symplectic manifold over the base \( S \). Similarly, a Lagrangian manifold \( L \) over \( S \) is a fibration \( \rho : L \hookrightarrow S \), sitting in a diagram

Thus the fibre \( L_s \) at \( s \) is a Lagrangian submanifold of \( M_s \). Recall that for such a relative manifold \( \rho : L \rightarrow S \) there is a natural surjective map of vector bundles on \( L \):
\[ TL \rightarrow \rho^*(TS), \ v \mapsto d\rho(v). \]
The kernel of this map is \( T_S L \), called the relative tangent space. It consists of those tangent vectors of \( L \) that are tangent to the fibres of \( \rho \). In particular, if \( \rho \) is a trivial fibration with fibre \( L_0 \) then \( T_S L \) is simply \( TL_0 \times S \).

The dual of the relative tangent bundle we denote by \( T^*_S L \): it sits in an exact sequence
\[ 0 \rightarrow \rho^*(T^*S) \rightarrow T^*L \rightarrow T^*_S L \rightarrow 0 \]
of vector bundles on \( L \). The restriction of \( T^*_S L \) to the fibre \( L_s \) is just the cotangent bundle of \( L_s \):
\[ T^*_S L|_{L_s} = T^*L_s \]
The following parametric version of the Darboux-Weinstein theorem expresses the fact that there are no local invariants for families of symplectic manifolds.

**Theorem 1.12.** Let \( L \rightarrow S \) be a proper Lagrangian submanifold of a symplectic manifold \( M \rightarrow S \) over some base \( S \). The germ of \( M \rightarrow S \) along \( L \rightarrow S \) is, locally on the base \( S \), symplectomorphic to a the germ of \( T^*_S L \) along the zero section.

**Proof.** To adapt the proof the Darboux-Weinstein theorem to this parametric case, we use the relative variant of differential forms. The *relative de Rham complex* is defined by

\[
\Omega^*_\pi := \Omega^*_M / (\pi^* \Omega^1_S \wedge \Omega^*_M),
\]

where \( \Omega^p_N \) denotes the space of \( C^\infty p \)-forms on a manifold \( N \). If we take local coordinates \( s_1, \ldots, s_n \) which trivialise the fibration, two differential forms on \( M \) are equal as elements of \( \Omega^*_\pi \) if they are equal modulo a form of the type

\[
\sum \alpha_i \wedge ds_i, \quad \alpha_i \in \Omega^*_M.
\]

The cohomology of this relative de Rham complex is, locally on the base, the cohomology of the fibre tensored with the \( C^\infty \)-functions on the base. In particular, the cohomology class of a symplectic form vanishes in a neighbourhood of a Lagrangian fibre. The deformation argument remains the same with the only difference that we now consider the forms \( \omega_t \) entering the equation:

\[
i_{X_t} \omega_t = -\dot{\alpha}_t
\]

as relative differential forms with vanishing de Rham class. \( \square \)

### 6. Liouville integrability

The classical language of *Poisson brackets* is a very useful way to express the main features of symplectic geometry.

For two smooth functions

\[
f, g : M \rightarrow \mathbb{R},
\]

the Poisson-bracket of \( f \) and \( g \) is defined by

\[
\{f, g\} = \omega(X_f, X_g) = -\{g, f\}.
\]

It is readily seen that

\[
\{f, g\} = \mathcal{L}_{X_f}(g), \quad [X_f, X_g] = X_{\{f, g\}}.
\]

The Poisson-bracket is a *bi-derivation*

\[
\{f, gh\} = \{f, g\} h + \{f, h\} g, \quad \{fg, h\} = f\{g, h\} + g\{f, h\}
\]
which satisfies the Jacobi identity:
\[
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0
\]

The Darboux coordinates provide a local model for the Poisson bracket:
\[
\{ f, g \} = \sum_{i=1}^{n} \partial_{p_i} f \partial_{q_i} g - \partial_{q_i} f \partial_{p_i} g.
\]

More generally, a Poisson structure on a manifold is such an anti-symmetric bi-derivation satisfying the Jacobi identity. The dynamics of a Hamiltonian \( H : M \to \mathbb{R} \) can be expressed concisely by the statement
\[
\dot{G} = \{ H, G \},
\]

which indeed reduce to Hamilton’s equations of motion for Darboux coordinates. From this we see that a quantity
\[
G : M \to \mathbb{R}
\]
is preserved by the flow of \( H \) if and only if it Poisson-commutes with \( H \):
\[
\{ H, G \} = 0.
\]

Such a quantity is called a first integral. Any function \( \phi \) of \( H \) is a first integral:
\[
\mathcal{L}_{X_H} \phi(H) = \{ H, \phi(H) \} = 0
\]
In some cases the converse is true as well. For instance, endow \( M = \mathbb{R}^2 \) with its canonical symplectic structure \( \omega = dq \wedge dp \) and take \( H = p \). Then any first integral is a function \( G \) with the property that
\[
\{ H, G \} = 0 \iff \partial_q G = 0.
\]

so depends only on the variable \( p \). A Hamiltonian \( H \) is called Liouville integrable or simply integrable if there exists Poisson commuting functions \( H = f_1, \ldots, f_n \) with Hamiltonian vector fields that are linearly independent
\[
df_1 \wedge df_2 \wedge \ldots \wedge df_n \neq 0
\]
on an open dense subset of \( M \). The map \( f = (f_1, f_2, \ldots, f_n) \) is called a moment mapping. The smooth fibres of this map are automatically Lagrangian. Indeed, the Hamiltonian vector fields \( X_i \) of the \( f_i \)'s generate the tangent bundle and
\[
\omega(X_i, X_j) = \{ f_i, f_j \} = 0.
\]

In particular if \( f \) is smooth and proper, then it defines what is called a Lagrangian fibration. The Hamiltonian flows then induce a cocompact \( \mathbb{R}^n \)-action, and therefore, by general properties of \( \mathbb{R}^n \)-actions, the moment mapping is a fibration whose fibres are tori.

A moment mapping defined over some \( n \)-dimensional base
\[
f = (f_1, \ldots, f_n) : X \to S
\]
gives rise to a Lagrangian subspace in a symplectic manifold in the following way: the projection on the second factor gives a symplectic manifold

\[ M := X \times S \longrightarrow S \]

over the base \( S \). We denote the graph of \( f \) by

\[ L_f \subset M = X \times S. \]

It is a Lagrangian submanifold of \( M \) over \( S \). By the relative Darboux-Weinstein theorem, locally on \( S \), a neighbourhood of \( L_f \) in \( M \) is symplectomorphic to a neighbourhood of the zero section its cotangent bundle.

## 7. Action-angle variables

If a moment mapping \( f \) is proper, then its fibres have a co-compact \( \mathbb{R}^n \)-action coming from the commuting flows of \( f_1, \ldots, f_n \). The Lagrangian manifold \( L_f \) is therefore a fibration by tori over its base \( S \). Action-angle variables provide Darboux coordinates adapted to this situation. The action variables \( I_1, I_2, \ldots, I_n \) are functions of the \( f_i \), so they are constant on the tori of the fibration and have the further property that the integral curves of the corresponding Hamiltonian fields \( X_i \) all have a fixed period. The angle variables \( \theta_1, \theta_2, \ldots, \theta_n \) are the corresponding ‘time’ variables, so have the property that

\[ \omega = d\theta_1 \wedge dI_1 + d\theta_2 \wedge dI_2 + \ldots + d\theta_n \wedge dI_n. \]

Before we explain this in detail, let us start with some examples.

**Example 1.13.** The restriction of the map

\[ \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n, \ (q, p) \mapsto \left( \frac{1}{2}(q_1^2 + p_1^2), \ldots, \frac{1}{2}(q_n^2 + p_n^2) \right) \]

to the preimage \( X \) of any open subset \( S \subset \mathbb{R}_{>0}^n \) produces a Lagrangian fibration

\[ f : X \longrightarrow S. \]

The Hamiltonian vector field of \( f_i = \frac{1}{2}(q_i^2 + p_i^2) \) is

\[ X_i = p_i \partial_{q_i} - q_i \partial_{p_i}, \ i = 1, \ldots, n. \]

These vector fields \( X_i \) span the tangent spaces to the fibres. We get action-angle coordinates by putting

\[ I_i = f_i, \quad \theta_i = \arctan \frac{p_i}{q_i} \]

or equivalently

\[ p_i := \sqrt{I_i} \cos \theta_i, \quad q_i := \sqrt{I_i} \sin \theta_i. \]
Note that the function $\theta_i$ is multi-valued, but its differential $d\theta_i$ is equal to the closed one-form

$$\alpha_i := \frac{-q_i}{p_i^2 + q_i^2} dp_i + \frac{p_i}{p_i^2 + q_i^2} dq_i$$

These closed one-forms $\alpha_i$ form a dual basis to commuting vector fields $X_i$:

$$\alpha_i(X_j) = \delta_{ij}$$

Example 1.14. The above example is archetypical for any integrable system, but untypical in the sense that the action variables were very simple. In general the determination of the action-angle variables lead to integrals defining transcendental functions. Consider for example the case $n = 1$ and:

$$f(q,p) = p^2 + \frac{1}{2} q^2 - \frac{1}{3} q^3.$$ 

Let $S$ be a pointed disc of radius $r < 1/6$ and $X$ be the preimage of $S$ under the polynomial map $f$. This defines a locally trivial $S^1$-fibration, that we denote in the same way:

$$f : X \rightarrow S.$$
In that case, the action is the function

\[ I = \int_{\gamma_\varepsilon} pdq \]

where \( \gamma_\varepsilon \) is the small loop inside the level set

\[ \{(q, p) \in X : |f(q, p)| = \varepsilon\} \approx S^1. \]

By Stokes’ formula, this is just the area of the disc bounded by \( \gamma_\varepsilon \) and is thus given as an elliptic integral of the second kind. If we regard \( I \) as a function of \( q, p \), then the angle \( \theta \) is the time of this vector field. It is given by the indefinite integral

\[ \theta = \int_{(q, p)} dp \wedge dq \frac{dI}{dI} \]

and is defined only modulo a period. This is a generic feature: computing action-angle coordinates involves abelian integrals and theta functions.

Let us now consider the general case. We consider a proper moment map

\[ f = (f_1, \ldots, f_n) : X \rightarrow S, \]

pick a reference point \( 0 \in S \) and let \( L_0 := f^{-1}(0) \) be the corresponding reference torus. By shrinking \( S \), we may assume that \( S \) is contractible and the fibration is trivial. The total space \( X \) retracts to the torus \( L_0 \) and because the symplectic form \( \omega \) vanishes on \( L_0 \), the form \( \omega \) is exact on \( X \), so we can find an action form \( \alpha \) for \( \omega \):

\[ \omega = d\alpha. \]
For each cycle \( \gamma(0) \in H_1(L_0, \mathbb{Z}) \), we obtain, by parallel transport to neighbouring tori, a family of cycles \( \gamma(s) \in H_1(L_s, \mathbb{Z}) \). This defines an action integral:

\[
I_\gamma : S \longrightarrow \mathbb{R}, \quad s \mapsto I_\gamma(s) := \int_{\gamma(s)} \alpha.
\]

If we pick a basis \( \gamma_1(s), \gamma_2(s), \ldots, \gamma_n(s) \) for \( H_1(L_s, \mathbb{Z}) \), we obtain the action functions, or action variables, \( I_j \) by composing the action integrals with the map \( f \):

\[
I_j(q, p) = I_{\gamma_j}(f(q, p)).
\]

Note that a change of the homology basis induces an \( GL(n, \mathbb{Z}) \)-action on the possible choices for the action variables.

The classical Arnold-Liouville-Mineur theorem can be stated as follows:

**Theorem 1.15.** Let \( f : X \longrightarrow S \) be a proper integrable system on a symplectic manifold \((M, \omega)\). There exists functions

\[
\theta_j : X \longrightarrow S^1,
\]

defined locally near a fibre, called the angles, such that the action-angle functions induce a local symplectomorphism

\[
X \longrightarrow (S^1)^n \times \mathbb{R}^n,
\]

where \((S^1)^n \times \mathbb{R}^n = T^*(S^1)^n\) is equipped with the standard symplectic form.

**Proof.** The Darboux-Weinstein theorem implies that we may assume that \( M = (S^1)^n \times \mathbb{R}^n = \{(q, p)\} \) equipped with its standard symplectic structure and that the Lagrangian fibre of \( f \) over \( s_0 \in S \) is the zero section. Up to a linear change of coordinates we may also assume that

\[
f_i(q, p) = p_i + \ldots
\]

where the dots stand for higher order terms in the \( p_i \)'s depending on the \( q \)-variables.

By the implicit function theorem, the fibres of \( f \) are the graphs of maps

\[
g_s : L \longrightarrow \mathbb{R}^n
\]

with

\[
f(q, p) = 0 \iff p = g_s(q).
\]

Comparing the Taylor series of \( f \) and \( g_s \), we see that

\[
g_s = s + o(\|s\|).
\]
We choose the cycle $\gamma_j(s)$ which projects to the $j$-th coordinate circle in the torus. Then the actions are defined by

$$I_j(q, p) = \int_{\gamma_j(s)} \sum_{i=1}^{n} p_i dq_i,$$

$$= \int_{\gamma_j(s)} \sum_{i=1}^{n} (g_i)_s(q) dq_i,$$

$$= (g_j)_s(q = 0),$$

$$= s_j + o(\|s\|)$$

$$= p_j + o(\|p\|).$$

The associated angles $\theta_j$ are of the form

$$\theta_j = q_j + o(\|q\|)$$

Thus the map

$$(q, p) \mapsto (\theta, I)$$

is a diffeomorphism and therefore a symplectomorphism. \hfill \Box

8. Integrable systems and the Gauss-Manin connection

The above proof for the existence of action-angle variables relies on blind computations. One of the beautiful aspects of classical integrable systems is that the construction of action-angle coordinates can also be explained in terms symplectic geometry of the action integrals $I_\gamma$ that we defined above.

We consider a torus bundle

$$f : X \longrightarrow S$$

defined by a proper smooth moment mapping over a contractible open set $S \subset \mathbb{R}^n$. We denote by $X_i$ the Hamiltonian fields of its components and choose a Lagrangian section of the torus bundle

$$\sigma : S \longrightarrow X, \quad \sigma^* \omega = 0.$$ 

The Hamiltonian flows of the $X_i$’s induce an $\mathbb{R}^n$-action on the fibres of $f$, which is, as any such $\mathbb{R}^n$-action, transitive on the fibres. Thus translating the section $\sigma$ by the Hamiltonian vector fields define sections starting at arbitrary points on a fibre. Thus for each point $x \in X$, we get a vector space $H_x \subset T_x X$ complementary to the tangent space of the fibre, that is, we get an Ehresmann connection on $X$.

In particular, each vector field $v$ on the base $S$ is lifted in a unique way to a vector field $\tilde{v}$ on $X$. 
Lemma 1.16. One has

\[ \omega = \sum_{i=1}^{n} \alpha_i \wedge df_i \]

where \( \alpha_i := \omega(\tilde{v}_i, -) \) and \( \tilde{v}_i := \tilde{\partial}_{s_i} \).

Proof. The vector fields \( \tilde{v}_1, \ldots, \tilde{v}_n \) lifting \( \partial_{s_1}, \ldots, \partial_{s_n} \) satisfy

\[ \omega(X_i, \tilde{v}_j) = df_i(\tilde{v}_j) = ds_i(\partial_{s_j}) = \delta_{ij}. \]

Moreover as the section \( \sigma \) is Lagrangian, we have:

\[ \omega(\tilde{v}_i, \tilde{v}_j) = 0. \]

Since the \( X_i \) pairwise commute, we also have

\[ \omega(X_i, X_j) = 0. \]

This means exactly that the symplectic form is \( \sum_{i=1}^{n} df_i \wedge \alpha_i. \)

\[ \square \]

Corollary 1.17. The lifted vector field \( \tilde{v} \) is symplectic:

\[ \mathcal{L}_{\tilde{v}} \omega = 0. \]

Proof. Any lifted vector field is a linear combination of the vector fields \( \tilde{v}_i \) with coefficients that are pulled back from \( S \).

By taking the interior product with the symplectic form \( \omega \), a vector field \( v \) is lifted to a one-form \( \alpha_v \). As symplectic vector fields correspond to closed one-forms, we obtain a map

\[ \Theta(S) \rightarrow \Omega^1(X)_{\text{closed}}, \quad v \mapsto \alpha_v \]

For each \( s \in S \), this induces a map \( T_s S \rightarrow H^1(L_s, \mathbb{R}), v \mapsto [\alpha_v] \).

Lemma 1.18. For each \( s \in S \), the map

\[ T_s S \rightarrow H^1(L_s, \mathbb{R}), v \mapsto [\alpha_v] \]

is an isomorphism.

Proof. The forms \( \alpha_1, \ldots, \alpha_n \) symplectically associated to the vector fields \( \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \) lifting \( \partial_{s_1}, \ldots, \partial_{s_n} \) form at each point the dual basis to hamiltonian vector fields \( X_1, \ldots, X_n \) and are therefore linearly independent.

Lemma 1.19. The one-form \( \alpha_v \) lifting a vector field \( v \in \Theta_S \) satisfies

\[ \mathcal{L}_{\tilde{v}} \alpha = \alpha_v + d(i_{\tilde{v}} \alpha). \]
Proof. By Cartan’s formula
\[ L_\tilde{\nu}\alpha = i_\tilde{\nu}\omega + d(i_\tilde{\nu}\alpha) \]
which means that the Ehresmann connection induces the usual Gauss-Manin connection on relative differential forms.

As a corollary to our discussion, we get the

**Theorem 1.20.** If \( \gamma_1(s), \gamma_2(s), \ldots, \gamma_n(s) \) form a basis for the lattice \( H_1(L_s, \mathbb{Z}) \), then the associated map
\[ \Psi : S \to T \subset \mathbb{R}^n, s \mapsto (I_{\gamma_1}(s), I_{\gamma_2}(s), \ldots, I_{\gamma_n}(s)) \]
is a local diffeomorphism.

Proof. By De Rham duality, if the \( \gamma_i(s) \)'s form a basis then integration over the \( \gamma_i(s) \)'s are linearly independent linear forms. \[\square\]

### 9. Affine structures and integrable systems

Let us keep the notation of the previous section.

The base space of a proper smooth moment map carries an affine structure, that we shall now define. Integration over a cycle \( \gamma(s) \in H_1(L_s, \mathbb{Z}) \) defines a linear function
\[ \int_{\gamma(s)} : H^1(L_s, \mathbb{R}) \to \mathbb{R}. \]
Identifying \( H^1(L_s, \mathbb{R}) \) with \( T_sS \), its kernel defines a hyperplane \( \gamma^\perp(s) \) in \( T_sS \). In this way, we obtain a distribution \( \gamma^\perp \) associated to a cycle and the level sets of \( L_\gamma \) are tangent to this distribution.

If \( \gamma_1(s), \gamma_2(s), \ldots, \gamma_n(s) \) form a basis for the lattice \( H_1(L_s, \mathbb{Z}) \) then the level-sets of the functions \( I_{\gamma_i} \) are tangent to the distribution and define an affine structure in the base of the moment map.
If we now consider these integrals of functions depending on the variables \((q, p)\):

\[
I_j(q, p) = I_{\gamma_j}(f(q, p)).
\]

We get a new integrable system \(I = (I_1, \ldots, I_n)\) for which the affine structure is now linear:

\[
X \xrightarrow{f} \I \xrightarrow{\gamma} S \xrightarrow{\Phi} T
\]

meaning that \textit{all action integrals} \(I_\gamma\) become \textit{linear functions} in the standard coordinates \(t_1, t_2, \ldots, t_n\) of \(\mathbb{R}^n \supset T\).

Via that map \(\Psi\), the distribution \(\gamma^\perp\) becomes the parallel distribution of linear hyperplanes.

Our original Lagrangian section of \(X \xrightarrow{\Phi} S\) induces a section of \(X \xrightarrow{\Psi} T\) by composition with the inverse of \(\Psi\). In this way, we lift the vector fields \(\partial_t\) to closed one-forms \(\beta_1, \beta_2, \ldots, \beta_n\), using the new Ehresmann connection.
As $I_\gamma(t)$ is now linear in $t$, the periods of the $\beta_i$’s are all constant. Thus we define:

$$\lambda_i := \int_{\gamma(t)} \beta_i = \partial_t I_\gamma(t) \in \mathbb{R}.$$  

The indefinite integral gives a well-defined map modulo the periods:

$$\int^*_{\sigma(t)} : L_t \longrightarrow \oplus_{i=1}^n \mathbb{R}/\lambda_i \mathbb{Z}, \quad x \longrightarrow (\int_{\sigma(t)}^x \beta_1, \int_{\sigma(t)}^x \beta_2, \ldots, \int_{\sigma(t)}^x \beta_n)$$

As the $\beta_i$’s are linearly independent this map is a local diffeomorphism and by definition of the periods $\lambda_i$, it is also one-to-one. Hence the induced torus fibration $X \longrightarrow T$ is trivialised.

Such a trivialisation is, in general, not possible if the base $S$ of the torus fibration is not contractible. Such examples occurs when we remove singular fibres from a larger family $\overline{X} \longrightarrow \overline{S}$.

As a rule, the base $S$ of the fibration is then not contractible, and interesting global issues arise.

The cycles $\gamma(s)$ may undergo monodromy under parallel transport along a loop in the base $S$, leading to a non-trivial representation of the fundamental group:

$$\rho : \pi_1(S,0) \longrightarrow \text{Aut}(H_1(L_0,\mathbb{Z})).$$

The lattices $\Lambda_s = H_1(L_s,\mathbb{Z})$ then form a non-trivial lattice bundle $\Lambda$ over $S$, commonly referred to as a local system. Although this phenomenon obstructs the existence of action-angle variables, the above constructions can be done locally on $S$. This leads to a version of action angle coordinates using a certain tautological symplectic torus bundle $T^*S/\Lambda$ associated to a Lagrangian section of $f : X \longrightarrow S$ that we will describe now.

As the fibre $L_s$ is a torus with transitive $\mathbb{R}^n$-action, the choice of a point $x \in L_s$ determines a map

$$\mathbb{R}^n \longrightarrow L_s, \quad 0 \mapsto x$$

and as this map is the universal covering map, the kernel can be identified with $H_1(L,\mathbb{Z})$. In this way $L_s$ is, after choosing a point as origin, identified with $H_1(L_s,\mathbb{R})/H_1(L_s,\mathbb{Z})$.

When we dualise the isomorphism

$$T_sS \longrightarrow H^1(L_s,\mathbb{R}), \quad v \mapsto [\alpha_v]$$

we get an isomorphism

$$T_sS \longrightarrow H_1(L_s,\mathbb{R}) \longrightarrow T^*_sS$$

Thus we get a natural lattice

$$\Lambda_s \subset T^*_sS$$
in the cotangent space at $s$, isomorphic to first homology group $H_1(L_s, \mathbb{Z})$ of the fibre. When we let $s$ run in $S$, we get a natural symplectic torus bundle attached to the situation:

$$T^*S/\Lambda \rightarrow S.$$  

isomorphic to

$$X \rightarrow S.$$  

This discussion can be summarised as follows

**Theorem 1.21.** Let $f : X \rightarrow S$ be a proper smooth integrable system with a Lagrangian section

$$\sigma : S \rightarrow X.$$  

The identification of the fibre of $f$ at $s$ with $H_1(L_s, \mathbb{R})/H_1(L_s, \mathbb{Z})$ induces a symplectomorphism of torus bundles over $S$

$$\begin{align*}
T^*S/\Lambda & \rightarrow X \\
& \downarrow \\
S & \rightarrow S
\end{align*}$$  

where $\Lambda_s$ is identified with $H_1(L_s, \mathbb{Z})$ via the isomorphism $T^*_s S \rightarrow H_1(L_s, \mathbb{R})$ induced by the section $\sigma$.

### 10. Bibliographical notes

The technique used to prove the Darboux theorem is called *Moser’s path homotopy method*. It was introduced in:

J. Moser, *On the volume elements on a manifold*, Trans. of the Am. Math. Soc. **120**(2), 286-294, (1965).

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Gromov, M. *Pseudo holomorphic curves in symplectic manifolds*, Inventiones Mathematicae, **82**(2), 307-347, (1985).
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**H. Mineur**, *Sur les systèmes mécaniques admettant n intégrales premières uniformes et l’extension à ces systèmes de la méthode de quantification de Sommerfeld*, C. R. Acad. Sci., Paris 200, 1571 - 1573, (1935).

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**V.I. Arnold**, *A theorem of Liouville concerning integrable problems of dynamics*, Sibirsk. Math. Zh. 4, 471 - 474, (1963).

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**H. Duistermaat**, *On global action-angle variables*, Communications on Pure and Applies Mathematics, 33(6), 687-706, (1980).

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and for more general integrable systems in:

**S. Vu Ngoc**, *Quantum monodromy in integrable systems*, Communications in mathematical physics, 203(2), 465-479, (1999).

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**M. Garay and D. van Straten** *Classical and quantum integrability*. Mosc. Math. Journal 10, 519-545, (2010).

There are some misprints in the paper which we (hopefully!) corrected here.
Classical sources on classical mechanics in the language of global analysis are:

V.I. Arnold, *Mathematical methods of classical mechanics*, Graduate texts in Mathematics (Vol. 60). Springer Verlag.

R. Abraham and J. Marsden, *Foundations of Mechanics*, Benjamin/Cummings Publishing Company, (1978).

For more detail on symplectic geometry, we also refer to:

D. McDuff, D. Salamon *Introduction to Symplectic Topology*, Oxford mathematical monographs, Clarendon Press, 1998.
CHAPTER 2

The KAM problem

We now come to a central theme of classical KAM-theory, namely that of the persistence of quasi-periodic motions after perturbation of an integrable Hamiltonian system. It is convenient to work with an algebraic model and formulate the problem in terms of certain Poisson-algebras associated to the algebraic torus, like the algebra of analytic Fourier series. We discuss the problem first on the level of formal power series and then discuss the special features that arise at the analytic level and identify the first order analytic obstruction.

1. Quasi-periodic motions

Let us now look at the motion for an integrable Hamiltonian $H = f(I)$ in action angle coordinates:

$$(\theta_1, \ldots, \theta_n, I_1, \ldots, I_n), \{I_j, \theta_k\} = \delta_{jk}.$$ 

Hamiltons equations of motion are

$$\begin{cases}
\dot{I}_j = 0 \\
\dot{\theta}_j = \partial_{I_j} f(I).
\end{cases}$$

These equations with initial condition

$I(0) = c, \theta(0) = \beta$

can easily be integrated:

$$\begin{cases}
I_j = c_j \\
\theta_j = \omega_j t + \beta_j
\end{cases}$$

where

$$\omega_i := \partial_{I_j} f(c).$$
The trajectory lies on the torus $I = c$ and such a motion is called \textit{quasi-periodic}. A quasi-periodic motion is dense on its torus, if the \textit{frequency vector} of the motion
\[
\omega := (\omega_1, \omega_2, \ldots, \omega_n)
\]
has $\mathbb{Z}$-independent coordinates. Of course, the vector $\omega$ will depend on $c$.

The question from which KAM-theory originates is the following:

\textit{Given an integrable system in action-angle coordinates, do such quasi-periodic motions persist after turning on a perturbation of the original Hamiltonian?}

In order to formulate this problem more precisely, it is convenient to introduce first an algebra-geometric model of the above situation. Then we will consider the formal and the analytic versions.

Rather then working with the angular variables $\theta_j$ it is convenient to use variables
\[
q_j := e^{i\theta_j}.
\]
For real values of $\theta_j$, the variable $q_j$ runs over the unit circle in the complex plane. Adding an imaginary part to $\theta_j$ changes the radius of the circle traced by $q_j$. So we are led to consider the \textit{algebraic torus}
\[
X := (\mathbb{C}^*)^n = \{(q_1, \ldots, q_n) \in \mathbb{C}^n : q_i \neq 0, i = 1, 2, \ldots, n\}.
\]
Note that $X$ has an anti-holomorphic involution

$$q_i \mapsto \frac{1}{q_i}$$

and the usual torus $(S^1)^n$ traced by all the angular variables $\theta_j$ is the part of the algebraic torus $X$ that is fixed by this involution, so it may be considered as the ‘real part’ of $X$ for the real structure defined by this anti-homomorphic involution.

Note that

$$\{I_j, q_k\} = \{I_j, e^{i\theta_k}\} = ie^{i\theta_k}\delta_{jk}.$$

We also will write $p_j$ for the conjugate action variables $-iI_j$, and note that then

$$\{p_j, q_k\} = -i\{I_j, e^{i\theta_k}\} = e^{i\theta_k}\delta_{jk} = q_k\delta_{jk}.$$

2. Algebraic model

We started our discussion of symplectic geometry in the context of differential geometry, but now we wish to take a more algebraic viewpoint. In differential geometry, the basic notion is that of a manifold, but in algebra one considers rings of functions on the manifold under consideration. Usually there are several natural choices for the types of functions one might want to consider: $C^\infty$, analytic, polynomial, convergent or formal power series.

The Poisson bracket of functions on a symplectic manifold gives the rings of functions the additional structure of a Poisson algebra. We recall that a Poisson algebra over a (commutative) ring $R$ is an $R$-algebra with an additional anti-symmetric operation $\{-,-\}$ which satisfy the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

and which is a bi-derivation, that is

$$g \mapsto \{f, g\}$$

is a derivation for any fixed $f$:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

As the bracket is anti-symmetric, it is also a derivation on the first variable. For a given real symplectic manifold $M$, the $\mathbb{R}$-algebra $C^\infty(M)$ is naturally a Poisson $\mathbb{R}$-algebra.

The algebraic model of the family of tori is the Laurent polynomial ring

$$A := \mathbb{C}[q, q^{-1}, p] := \mathbb{C}[q_1, q_2, \ldots, q_1^{-1}, q_2^{-1}, \ldots, q_n^{-1}, p_1, p_2, \ldots, p_n]$$
with the Poisson-bracket
\[ \{p_j, q_k\} = q_k \delta_{jk}. \]

The Poisson-algebra \( A \) can be seen as the affine coordinate ring of the cotangent bundle \( T^* X \) of our torus \( X \). The canonical symplectic form on that space is
\[ \omega := \sum_{j=1}^{n} \frac{dq_j}{q_j} \wedge dp_j \]
and leads precisely to the above Poisson-bracket. The fibres of the map
\[ T^* X \rightarrow \mathbb{C}^n, (q, p) \mapsto p = (p_1, p_2, \ldots, p_n) \]
are copies of \( X \) and the flow of any Hamiltonian \( H = f(p) \in \mathbb{C}[p_1, p_2, \ldots, p_n] \) preserves these fibres.

We also note that in the sense of algebraic geometry, quasi-periodic motion can be defined “over any field \( K \)” by replacing the coefficients \( \mathbb{C} \) by \( K \), and then considering the Poisson-algebra
\[ A = K[q, q^{-1}, p] \]
with the Poisson-bracket as defined before.

3. Perturbation theory

To describe a perturbation of a Hamiltonian \( H \in A \) we “add a parameter \( t \) to the ring \( A \)” and consider the ring of Laurent polynomials
\[ B := K[q, q^{-1}, p, t]. \]
Here \( t \) is a parameter, which is supposed to be a central element, i.e. an element that Poisson-commutes with all elements \( f \in B \):
\[ \{t, f\} = 0. \]

By a deformation or perturbation of \( H \) we mean any element \( H' \in B \) which reduces to \( H \) when we put \( t = 0 \), which means that one can write it in the form
\[ H' = H + tQ, \quad Q \in B. \]
Note that we can consider \( A \) as a subring of \( B \), consisting of functions that do not depend on \( t \), but also as factor ring of \( B \): \( A = B/tB \).

One of the main ideas of perturbation theory is that by applying systematically appropriate automorphisms of the algebra \( B \), one may try to bring the perturbed Hamiltonian \( H + tQ \) to a simpler form or reduce it to a specific normal form.
If we consider two \( C^\infty \) symplectic manifolds \((M, \omega)\) and \((N, \omega')\), then any symplectic map \( \varphi : M \to N \) produces a map of Poisson-algebras
\[
\phi := \varphi^* : C^\infty(N) \to C^\infty(M), f \mapsto f \circ \varphi,
\]
which is a Poisson morphism. In general a Poisson morphism is defined as a map between two Poisson \( R \)-algebras \( A \) and \( B \)
\[
\phi : A \to B
\]
such that
\[
\phi(\lambda f) = \lambda \phi(f), \quad \phi(f + g) = \phi(f) + \phi(g),
\phi(fg) = \phi(f)\phi(g), \quad \phi(\{f,g\}) = \{\phi(f), \phi(g)\}
\]
for all \( \lambda \in R, f, g \in A \).

A Poisson automorphism is an invertible Poisson morphism
\[
\phi : A \to A,
\]
and can be seen as a generalisation of the notion of symplectomorphism.

A Casimir element is an element \( c \) that Poisson-commutes with all elements of \( A \):
\[
\{c, a\} = 0 \quad \text{for all} \quad a \in A.
\]
We call a Poisson automorphism \( \phi \) central, if \( \phi(c) = c \) for all Casimir elements.

The rings \( A \) and \( B \) introduced above however is too small for many of the constructions we wish to perform. We will encounter several Poisson algebras similar to \( A \) and \( B \), that consist of certain power series rather than polynomials. These power series can be formal or convergent in some of the variables. We will introduce them in the sequel, but will usually call them \( A \) and \( B \).

**Definition 2.1.** If \( A \) is a Poisson algebra over a field of characteristic 0 and \( S \in B := A[[t]] \), then the series
\[
\varphi := e^{t\{-, S\}} = \text{Id} + t\{-, S\} + \frac{t^2}{2!}\{-, S\}, S\} + \ldots
\]
is called the formal flow of \( S \).

**Proposition 2.2.** The formal flow
\[
\varphi : B \to B
\]
is a central Poisson-automorphism.
Proof. As the terms in the series of $\varphi$ consist of Poisson-brackets, it is clear that $\varphi$ is central. One has

$$\varphi(f) \cdot \varphi(g) = (f + t\{f, S\} + \ldots)(g + t\{g, S\} + \ldots)$$

$$= f \cdot g + t(\{\{f, S\}g\} + f\{g, S\}) + \ldots$$

$$= \{f, g\} + t\{fg, S\} + \ldots$$

$$= \varphi(f \cdot g),$$

where the derivation property of the Poisson-bracket is used. Furthermore

$$\{\varphi(f), \varphi(g)\} = \{f + t\{f, S\} + \ldots, g + t\{g, S\} + \ldots\}$$

$$= \{f, g\} + t(\{\{f, S\}, g\} + f\{g, S\}) + \ldots$$

$$= \{f, g\} + t\{\{f, g\}, S\} + \ldots$$

$$= \varphi(\{f, g\}),$$

where one uses the Jacobi-identity.

4. The formal model

In this section and in the next one, we denote by $K$ any field of characteristic zero and consider the rings

$$A := K[q, q^{-1}][[p]], \quad B := A[[t]] = K[q, q^{-1}][[p, t]]$$

of formal power series in $p = (p_1, \ldots, p_n)$ and a central element $t$ whose coefficients are Laurent polynomials in $q = (q_1, \ldots, q_n)$ with the Poisson bracket

$$\{p_j, q_k\} = q_k\delta_{jk}.$$

Note that $K[[p]] \subset A$ and thus we can and will consider $A$ as a $K[[p]]$-algebra. There is also a natural averaging map

$$av : A \rightarrow K[[p]]$$

of 'taking the average over the torus'. It is the $K[[p]]$-linear map that maps each monomial $q^I$, $I \neq 0$ to 0, whereas $av$ is the identity on $K[[p]] \subset A$.

The powers of the maximal ideal of $K[[t]]$ filter the ring $B$. We write

$$f = g + (t)^n$$

if $f - g$ is of order $n$, i.e., it is a power series with monomials of degree not smaller than $n$ in the $t$ variable. More generally, given an ideal we write $f = g + I$ if $f - g$ belongs to some ideal $I$. We also write $(p)$ for the ideal $(p_1, p_2, \ldots, p_n)$ generated by the $p_i$’s.

We will consider a special class of Hamiltonians. We have seen that a Hamiltonian written in action-angle variables $(q, p)$ depends only on the
action variables $p$. Therefore we start with a Hamiltonian $H \in K[[p]]$ without constant term:

$$H = \sum_{I \in \mathbb{N}^n} \omega_I p^I = \sum_{i=1}^n \omega_i p_i + (p)^2.$$  

We will call such a Hamiltonian integrable, as clearly

$$\{H, p_i\} = 0, \quad i = 1, 2, \ldots, n.$$  

The vector $\omega := (\omega_1, \omega_2, \ldots, \omega_n)$ is called the frequency vector of $H$ (at 0):

$$\omega = \left( \frac{\partial H}{\partial p_1}(0), \frac{\partial H}{\partial p_2}(0), \ldots, \frac{\partial H}{\partial p_n}(0) \right).$$  

If $I \in \mathbb{Z}^n$ is an integral vector, we write

$$(\omega, I) := \sum_{i=1}^n \omega_i I_i$$

for the euclidean scalar product of $\omega$ and $I$.

The Hamiltonian $H$ is called non-resonant, if the $\omega_1, \omega_2, \ldots, \omega_n$ are $\mathbb{Z}$-independent:

$$(\omega, I) \neq 0 \quad \text{for all} \quad I \in \mathbb{Z}^n \setminus \{0\}$$

Otherwise it is called resonant and in that case a non-zero vector $I \in \mathbb{Z}^n$ such that

$$(\omega, I) = 0$$

is called a resonance of $H$.

For instance, $H = p_1 + ap_2$ is resonant if and only if $a \in \mathbb{Q}$.

**Proposition 2.3.** For a non-resonant $H \in K[[p]]$ the map

$$\{H, -\} : A \to A$$

has $K[[p]]$ as kernel and the $K[[p]]$-module generated by the monomials $q^I, I \neq 0$ as image.

**Proof.** We can consider $A = K[q, q^{-1}][[p]]$ as a $K[[p]]$-algebra with the monomials $q^I, I \in \mathbb{Z}^n$ as $K$-basis. As $H \in K[[p]]$, the map $\{H, -\} : A \to A$ is $K[[p]]$-linear. Furthermore, due to the Poisson-commutation rule

$$\{p_k, q_j\} = q_j \delta_{kj},$$

the map $\{H, -\}$ is diagonal in the monomial basis and we can write

$$\{H, q^I\} = ((\omega, I) + (p))q^I,$$

where $(-, -)$ denotes the Euclidean scalar product. As by assumption

$$(\omega, I) \neq 0$$
and all elements of the form
\[ c + (p), \ c \neq 0, \ c \in K \]
are invertible in the local ring \( K[[p]] \), it follows that the eigenvalue of \( \{H, -\} \) associated to \( q^I \) is invertible for \( I \neq 0 \). As a consequence, the kernel of \( \{H, -\} \) is \( K[[p]] \) and furthermore the image contains all monomials \( q^I, \ I \neq 0 \). □

So in the non-resonant case, the image of the map \( \{H, -\} \) consists exactly of the series with average equal to zero. One can express the above proposition by saying that for a non-resonant \( H \in K[[p]] \) we have an exact sequence of the form
\[
0 \rightarrow K[[p]] \rightarrow A \xrightarrow{\{H, -\}} A_{av} \xrightarrow{\rightarrow} K[[p]] \rightarrow 0,
\]
which says that kernel and cokernel of \( \{H, -\} \) are both isomorphic to \( K[[p]] \).

5. Formal stability

The following proposition states that a formal deformation of an integrable Hamiltonian remains integrable in a formal neighborhood of a non-resonant torus.

**Proposition 2.4.** Let \( H + tQ \in B \) be a deformation of an integrable Hamiltonian
\[
H = \sum_{I \in \mathbb{N}^n} \omega_I p^I = \sum_{i=1}^{n} \omega_i p_i + (p)^2.
\]
If \( H \) is non-resonant, then there exists a central Poisson automorphism \( \varphi : B \rightarrow B \)
such that
\[
\varphi(H + tQ) \in K[[p, t]].
\]

**Proof.** Let us write \( (t^n) \subset B \) for the terms that are divisible by \( t^n \), so we can write

\[
H + tQ = H(p) + (t)
\]

We will construct inductively a sequence of central Poisson automorphisms \( \varphi_0, \varphi_1, \varphi_2, \ldots \) of \( B \) such that
\[
\varphi_n(H + tQ) = P_n + (t^{n+1})
\]
and where
\[
P_n \in K[[p, t]], \ P_n = P_{n-1} + (t^n)
\]
We take \( \varphi_0 = Id, \ P_0 = H \). Assume we have constructed the sequence up to \( \varphi_n \). Then we look to the next order in \( t \) and write:
\[
\varphi_n(H) = P_n(p, t) + t^{n+1}Q_{n+1}(q, q^{-1}, p) + (t^{n+2}).
\]
By transferring the part of $Q_{n+1}$ that does not contain $q$ to the $P_n$ part ("averaging"), we arrive at the form
\[ \varphi_n(H + tQ) = \tilde{P}_n(p, t) + t^{n+1} \tilde{Q}_{n+1}(q, q^{-1}, p) + (t^{n+2}), \]
where we can assume that $\tilde{Q}_{n+1}$ does not contain pure $p$-monomials. As $H$ is assumed to be non-resonant, we can find $S_{n+1} \in A = K[q, q^{-1}][[p]]$ such that
\[ \{H, S_{n+1}\} + \tilde{Q}_{n+1}(p, q, q^{-1}) \in K[[p]]. \]
As $K$ is assumed to be of characteristic zero, the map
\[ \psi_{n+1} = e^{t^{n+1} \{ -S_{n+1}\}} \]
is a well-defined Poisson automorphism of $B$. As
\[ \psi_{n+1}(f) = f + t^{n+1} \{ f, S_{n+1} \} + \frac{t^{2n+2}}{2!} \{ \{ f, S_{n+1} \}, S_{n+1} \} + \ldots \]
the automorphism
\[ \varphi_{n+1} := \psi_{n+1} \varphi_n \]
has the property that
\[ \varphi_{n+1}(H + tQ) = P_{n+1} + (t^{n+2}). \]
This proves the statement by induction on $n$. \[ \square \]

Given a mechanical system defined by an Hamiltonian of the form described by the proposition, the change of variables constructed above reduces the motion to a quasi-periodic one. In fact the proposition gives the classical method for performing computations in celestial mechanics. However, one should notice that this reduction to the normal form is done by means of formal power series and convergence issues are not considered. In practice, one performs this reduction only up to a certain order, and the solution one obtains by truncation of higher order terms can only be expected to be asymptotic to the real solution.

Example 2.5. Consider the Hamiltonian function $H = p \in \mathbb{C}[q, q^{-1}, p]$ and its deformation
\[ H' = p + tp^2 + tpq + tpq^{-1}. \]
Keeping in mind that $\{ p, q \} = q, \{ f(p), q \} = f'(p)q$ the equations of motions are:
\[ \dot{q} = \{ H', q \} = q + t(q^2 + 2pq + 1), \]
\[ \dot{p} = \{ H', p \} = t(pq^{-1} - pq) \]
We put
\[ S = pq - pq^{-1} \]
so that
\[ p^2 + \{ S, H \} = p^2 + pq + pq^{-1} \]
is the perturbation. The symplectomorphism \( \varphi = e^{t\{-S\}} \) eliminates the first order term in \( H' \), but creates terms of higher order in \( t \):
\[
\varphi(H') = p + tp^2 - 2t^2(p^2q^{-1} + p^2q + p) + \ldots
\]
Hence, neglecting the terms of order 2 in \( t \), we get the quasi periodic motion
\[
\begin{align*}
\dot{q} &= (1 + 2tp)q, \\
\dot{p} &= 0
\end{align*}
\]
which can easily be integrated
\[
\begin{align*}
q(\tau) &= a e^{\tau+2b\tau}, \\
p(\tau) &= b
\end{align*}
\]
where \( \tau \) denotes the time.

Another difficulty which appears in celestial mechanics is the occurrence of resonances. In the resonant case, there exist resonance vectors \( I \neq 0 \) with \( (\omega, I) = 0 \) and in that case the corresponding resonant monomial \( q^I \) is not in the image of the map
\[
\{H, -\} : A \rightarrow A.
\]
The above procedure then still gives a sequence of Poisson automorphisms, but it is stopped when we arrive at the first resonant monomial: if \( H \) is of the form
\[
H = P_n(p) + t^{n+1}Q_{n+1}(p, q, q^{-1}) + (t^{n+2})
\]
and \( Q_{n+1}(p, q, q^{-1}) \) contains a resonant monomial, then the equation
\[
\{H, S_{n+1}\} + Q_{n+1}(p, q, q^{-1}) \in K[[p]].
\]
cannot be solved, hence we see:

**Proposition 2.6.** Let
\[
H = \sum_{I \in \mathbb{N}^n} \omega_I p^I = \sum_{i=1}^{n} \omega_i p_i + (t^2)
\]
be such that the frequencies \( \omega_1, \ldots, \omega_n \) are \( \mathbb{Z} \)-dependent. There exists perturbations \( H + tQ \) of \( H \) which cannot be reduced to a function of the \( p_i \)'s (and \( t \)) by a Poisson automorphism.

The proposition is of course a very weak statement, but nevertheless, the above simple fact already shows a dichotomy between the different tori of an integrable system. For instance the Hamiltonian
\[
H = p_1 - 2p_2
\]
is resonant, \( I = (2, 1) \) is a resonance vector and we have the resonant monomial \( q_1^2q_2 \). With the perturbation \( p_1 - 2p_2 + tq_1^2q_2 \) we are in the
situation of Proposition 2.6: the resonant monomial $q_1^2 q_2$ cannot be suppressed by a formal symplectomorphism. But if we make a scaling

$$(p_1, p_2) \mapsto ((1 + \lambda) p_1, p_2)$$

with $\lambda$ irrational, we are in the situation of Proposition 2.4, and the monomial can be transformed away. So already at the formal level, the situation is subtle. This is one of the manifestations of formal KAM theory that we will explore more in detail in a later section.

The beautiful non-integrability theorem of Poincaré goes far beyond this elementary remark. It states that over $\mathbb{R}$ and $\mathbb{C}$, the only first integrals are, in general, the functions of the Hamiltonian itself.

This non-integrability theorem can be understood in terms of global analysis using transversality arguments: we fix a time $T$ and consider the space $\Omega(M)$ of $T$-periodic loops in $M$. These $T$-periodic orbits are exactly the critical points of the action functional

$$\Omega(M) \to \mathbb{R}, \gamma \mapsto \int_{\gamma} pdq.$$ 

Using Smale’s transversality theorem, one can show that for a general $H$, the critical points of this function are isolated, hence the periodic orbits with period $T$ are isolated. If there were an independent first integral $F$, the flow of $F$ would transform each $T$-periodic orbit into a one-parameter family of $T$-periodic orbits, contradiction the fact that they are isolated. Therefore the flow of $F$ should be stationary at all periodic orbits and Poincaré’s theorem reduces to showing that $F$ is then a function $H$.

6. Analytic model and holomorphic Fourier series

Our analysis of the formal case reveals the inherent algebraic structures involved. But the problem we really want to deal with, is the analytic and not the formal case. We will describe a particular Poisson-algebra

$$\mathbb{C}\{q, q^{-1}, p\}$$

that is of importance for the Kolmogorov invariant torus theorem. We start with some preparations.

Consider a $2\pi$-periodic function $f$ that is holomorphic on a neighbourhood of the real axis. It has an expansion into a (convergent) Fourier series

$$f(\theta) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}.$$
Due to the compactness of the interval $[0, 2\pi]$, the function $f$ is in fact holomorphic on a strip 

$$B_t := \{ \theta = x + iy \in \mathbb{C} : x \in \mathbb{R}, \, |y| < t \}.$$ 

There exist a well-known relation between the rate of vanishing of the coefficients $a_n$ and the width $t$ of the strip.

**Proposition 2.7.** The function $f$ extends holomorphically to $B_t$ if and only if 

$$|a_n| = O(e^{-|n|s})$$ 

for any $s < t$.

**Proof.** Let $s < t$ and pick any $s' \in ]s, t]$. Consider the Hilbert space $L^2([0, 2\pi] \times [-s', s'], \mathbb{C})$ with Hermitian scalar product 

$$(f, g) := \frac{1}{2\pi} \int_{[0,2\pi] \times [-s', s']} f \cdot \overline{g} \, dx \, dy.$$ 

The functions $e_n := e^{inx} = e^{inx-ny}$ form an orthogonal set with 

$$(e_n, e_n) = \frac{1}{2\pi} \int_{-s'}^{s'} \int_0^{2\pi} e^{inx-ny} e^{-inx-ny} \, dx \, dy = \frac{e^{2ns'} - e^{-2ns'}}{2n}.$$ 

From the expansion of $f$ as Fourier-series 

$$f = \sum_{n \in \mathbb{Z}} a_n e^{i\theta}$$ 

we get 

$$(f, e_n) = a_n (e_n, e_n).$$ 

Hence we obtain 

$$|a_n| (e_n, e_n) = |(f, e_n)| \leq \|f\| \|e_n\|$$ 

or 

$$|a_n| \leq \frac{\|f\|}{\|e_n\|} \leq C e^{-|n|s}$$ 

for an appropriate choice of $C$. Conversely, if the coefficients decrease exponentially, then the Fourier series is convergent inside some compact subset of the strip and thus defines a holomorphic function. \qed

Now if the Fourier series $f = \sum_{n \in \mathbb{Z}} a_n e^{i\theta}$ is analytic in a strip $B_t$, the associated series 

$$\sum_{n \in \mathbb{Z}} a_n q^n$$ 

is analytic in the annulus 

$$\{ q \in \mathbb{C} \mid e^{-t} < |q| < e^t \}$$ 

These annuli form a fundamental system of neighbourhoods of the circle 

$$S^1 := \{ q \in \mathbb{C} \mid |q| = 1 \}.$$
Definition 2.8. We put
\[ \mathbb{C}\{q,q^{-1}\} := \left\{ \sum_{n=-\infty}^{\infty} a_n q^n \mid \exists t > 0 \, \forall n : |a_n| \leq O(e^{-|n|t}) \right\} \]
and call it the ring of holomorphic Fourier series.

The above discussion shows that $\mathbb{C}\{q,q^{-1}\}$ can be identified with algebra of germs of holomorphic functions along the unit circle.

This notion extends to $n$ variables $q_1, \ldots, q_n$ in an obvious way. We will be concerned with the Poisson algebras
\[ A := \mathbb{C}\{q,q^{-1},p\}, \quad B := \mathbb{C}\{q,q^{-1},p,t\} \]
with Poisson-bracket
\[ \{p_k, q_j\} = q_j \delta_{kj} \]
and $t$ a central element. In terms of the cotangent space $T^*X$ to the algebraic torus $X := (\mathbb{C}^*)^n$, we can say that the algebra $A$ consists of the germs of holomorphic functions on $T^*X$ along the central $n$-torus
\[ (S^1)^n = \{(q_1, q_2, \ldots, q_n) \in (\mathbb{C}^*)^n \mid |q_j| = 1\} \subset X. \]

7. First order analytic obstruction

In a previous section we saw that for a non-resonant $H \in K[[p]]$ the operator
\[ L = \{H,-\} : K[q,q^{-1}][[p]] \longrightarrow K[q,q^{-1}][[p]] \]
is diagonal in the monomial basis with kernel and cokernel isomorphic to $K[[p]]$ and series with vanishing average
\[ \left\{ \sum_{I \neq 0, J} a_I q^I p^J \right\} \]
as image. We showed that this fact implies that any perturbation of a non-resonant integrable Hamiltonian can formally be transformed to the integrable normal form.

As Poincaré already observed, an analogous statement does not hold at an analytic level. In fact, if we consider again an analytic integrable Hamiltonian $H \in \mathbb{C}\{p\}$, then the analogous operator
\[ L^{an} : \mathbb{C}\{q, q^{-1}, p\} \rightarrow \mathbb{C}\{q, q^{-1}, p\} \]
is of great complexity. Let us give an example and consider the Hamiltonian
\[ H = p_1 + \sqrt{2} p_2 + \frac{1}{2} p_2^2 \]
The frequency vector
\[ \omega := \left( \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2} \right) = (1, \sqrt{2} + p_2) \]
depends linearly on $p_2$ and we find:
\[ L^{an}(q_1^m q_2^{-n}) = (m - n(\sqrt{2} + p_2))q_1^m q_2^{-n} \]
and so the preimage under $L^{an}$
\[ \frac{1}{m - n(\sqrt{2} + p_2)} q_1^m q_2^{-n} \]
of the monomial $q_1^m q_2^{-n}$ has a pole at
\[ p_2^0 := \frac{m}{n} - \sqrt{2} \]
which is near the origin if $m/n$ is close to $\sqrt{2}$. By taking any (non-polynomial) convergent power series with such monomials:
\[ \sum_{m,n \geq 0} \alpha_{nm} q_1^m q_2^{-n} \]
we get an analytic series which is not in the image of $L^{an}$. So contrary to what happens in the formal case, the image of $L^{an}$ misses many non-trivial analytic functions and not only the the series with non-zero average, that is, series depending only on the $p$ variables. The image is therefore much smaller and it is non-trivial to see if a series belongs to it.

Let us return to the general case and consider a Hamiltonian
\[ H = \sum_{i=1}^{n} \omega_i p_i + (p)^2 \in \mathbb{C}\{p\}. \]
and a given perturbation

\[H + tQ, \quad S \in B\]

We try to find, as in the formal case, a Poisson-automorphism of the form

\[\varphi : B \rightarrow B, \quad \varphi(f) = e^{t\{f, S\}} = f + t\{f, S\} + (t^2)\]

that transforms \(H + tQ\) to an element of \(B\) that is independent of the \(q_i\). So we find

\[\varphi(H + tQ) = H + t\{H, S\} + tQ + (t^2),\]

that is, we need to find \(S \in B\) such that

\[\{H, S\} = -Q.\]

As we indicated above, this is very delicate condition.

However, this equation reduces drastically by restricting both sides to \(p = 0\): we get a new equation

\[\{H, S\}_{p=0} = -Q(q, p = 0)\]

where \(Q(q, p = 0) \in \mathbb{C}\{q, q^{-1}\}\). This equation reduces to

\[L_0(S) = \sum_{i=1}^{n} \omega_i q_i \partial_{p_i} S = -Q(q, p = 0) =: g\]

where

\[L_0 = \left\{ \sum_{i=1}^{n} \omega_i p_i, - \right\}\]

is the linear part of the Hamiltonian derivation.

We will consider the ring of holomorphic Fourier series as sitting inside the space of all formal Fourier series.

\[\mathbb{C}\{q, q^{-1}\} \subset \mathbb{C}[[q, q^{-1}]] := \{ \sum_{I \in \mathbb{Z}^n} a_I q^I \mid a_I \in \mathbb{C} \}.\]

In general, two series in \(\mathbb{C}[[q, q^{-1}]]\) can not be multiplied in the usual way, so it is not a ring, but only a vector space. But it will still be useful to to equip \(\mathbb{C}[[q, q^{-1}]]\) with the \textit{coefficient-wise product} \(\ast\):

\[\sum_{I} a_I q^I \ast \sum_{I} b_I q^I = \sum_{I} a_I b_I q^I\]

called the \textit{Hadamard} or \textit{convolution product}.

We see that the operator \(L_0\) is equal to taking the Hadamard product with the function

\[l = \sum_{I \in \mathbb{Z}^n \setminus \{0\}} (\omega, I) q^I.\]

\[L_0(S) = l \ast S = g\]
This equation is solved for $S$ as

$$S = h \star g$$

with

$$h := \sum_{I \in \mathbb{Z}^n \setminus \{0\}} (\omega, I)^{-1} q^I,$$

the Hadamard-inverse of $l$.

There is an obvious restriction: $(\omega, I)$ should not be equal to zero for any $I \neq 0$, otherwise $h$ is not defined. This is exactly the non-resonance condition. But even if this condition is satisfied, it might happen $h \star g$ is not an element of the ring $\mathbb{C}\{q, q^{-1}\}$ of holomorphic Fourier series.

We thus see that for homomorphic Fourier series there is an obstruction: $h \star f$ has to be holomorphic for any $f \in \mathbb{C}\{q, q^{-1}\}$.

This first order obstruction of analytic nature can be easily identified:

**Proposition 2.9.** The Hadamard product

$$h \star : \mathbb{C}[[q, q^{-1}]] \rightarrow \mathbb{C}[[q, q^{-1}]], \ f \mapsto h \star f$$

with the series

$$h = \sum_{I \in \mathbb{Z}^n} a_I q^I \in \mathbb{C}[[q, q^{-1}]]$$

maps the sub-algebra $\mathbb{C}\{q, q^{-1}\}$ to itself if and only for any $s > 0$ there exists a constant $C = C(s)$ such that the Fourier coefficients of $h$ satisfy the estimate:

$$|a_I| \leq C e^{||I||s}.$$

**Proof.** Assume that the coefficient $a_I$ of $h$ satisfy the above estimate and take any element

$$f(q) = \sum_I b_I q^I \in \mathbb{C}\{q, q^{-1}\}.$$

By definition of $\mathbb{C}\{q, q^{-1}\}$, there exists $t$ such that

$$|b_I| = O(e^{-||I||t}).$$

Take $s < t$, we get that

$$|a_I b_I| = O(e^{-||I||t-s)}).$$

Thus $h \star f \in \mathbb{C}\{q, q^{-1}\}$. Conversely if $h \star$ preserves the sub-algebra $\mathbb{C}\{q, q^{-1}\}$, then consider for any $s > 0$ the holomorphic series

$$f(q) = \sum_I e^{-||I||s} q^I \in \mathbb{C}\{q, q^{-1}\}.$$

We have $h \star f \in \mathbb{C}\{q, q^{-1}\}$, thus

$$|a_I e^{-||I||s}| = O(1).$$

Hence the coefficients of $h$ satisfy an estimate $|a_I| \leq C e^{||I||s}$. \qed
8. Bibliographical notes

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CHAPTER 3

The Kolmogorov invariant torus theorem

In this chapter we take a closer look at the non-resonance condition and introduce the so called Diophantine condition, and show that the set of Diophantine frequency vectors has full measure. Then we formulate Kolmogorov’s theorem on the existence of invariant tori and reinterpret the theorem in terms of infinite dimensional group actions.

1. Kolmogorov’s Diophantine condition

We have seen that in the context of perturbation theory we are given a frequency vector

$$\omega = (\omega_1, \omega_2, \ldots, \omega_n)$$

and we have to consider the Euclidean scalar product

$$\langle \omega, I \rangle$$

with lattice vectors $$I \in \mathbb{Z}^n$$. In the non-resonant case, the hyperplane $$\omega^\perp$$ orthogonal to $$\omega$$ intersects the lattice $$\mathbb{Z}^n$$ only at the origin. But of course, there will always be lattice points that are very close to the hyperplane. As a result, the series

$$h = \sum_I (\omega, I)^{-1}q^I$$

may have coefficients that grow very fast if $$|I|$$ becomes big: the problem of small denominators. We saw that the Hadamard product with a formal Fourier series

$$h = \sum a_Iq^I$$

will preserve the space of analytic Fourier series if the coefficients $$a_I$$ grow not faster than exponentially with $$|I|$$.

Rather than studying types of sub-exponential growth, we consider the simpler case of polynomial growth:

**Definition 3.1.** A vector $$\omega = (\omega_1, \ldots, \omega_n)$$ satisfies Kolmogorov’s Diophantine condition $$K(C, \nu)$$ if for all $$0 \neq I \in \mathbb{Z}^n$$

$$|(\omega, I)| \geq \frac{C}{\|I\|^{n-1+\nu}}.$$ 

Here $$C > 0$$ and $$\nu \in \mathbb{R}$$. (The shift by $$n-1$$ is conventional). We say that $$\omega$$ satisfies Kolmogorov’s condition if it satisfies $$K(C, \nu)$$ for some $$C$$ and $$\nu$$. 

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The condition \( K(C, \nu) \) means that the growth is bounded by a polynomial function:

\[
|\omega, I|^{-1} \leq \frac{\|I\|^{n-1+\nu}}{C}.
\]

In particular, Proposition 2.9 implies that if \( \omega \) satisfies Kolmogorov’s Diophantine condition, then the endomorphism

\[
h^* : \mathbb{C}[[q, q^{-1}]] \rightarrow \mathbb{C}[[q, q^{-1}]], \quad f \mapsto h^* f, \quad h := \sum_{I \neq 0} (\omega, I)^{-1} q^I
\]

preserves the sub-space \( \mathbb{C}\{q, q^{-1}\} \) of holomorphic Fourier series: the first analytic obstruction vanishes.

2. Diophantine approximation

The question how small \( |\omega, I| \) can be belongs to the field of Diophantine approximation. Let us first look at the case \( n = 2 \) and \( \omega = (1, \alpha), \ I = (p, -q) \). Then we have

\[
|(\omega, I)| = |q\alpha - p|,
\]

so this becomes small if the rational number \( \frac{p}{q} \) is a good approximation to \( \alpha \). If we subdivide the unit interval in \( N + 1 \) sub-intervals of length \( 1/(N + 1) \), then the fractional part of the \( N \) numbers

\[
\alpha, 2\alpha, 3\alpha, \ldots, N\alpha
\]

all fall in different intervals if \( 1/(N + 1) < \alpha \). As a consequence, at least one falls in the first or last interval, which means that one of these numbers differs by less than \( \leq 1/(N + 1) \) from an integer.\(^1\) So for all \( \alpha \) there exist infinitely many integers \( p, q \) so that

\[
|q\alpha - p| < \frac{1}{q}, \quad |\alpha - \frac{p}{q}| < \frac{1}{q^2}.
\]

In fact, such good rational approximations can be obtained from the continued fraction expansion of \( \alpha \). In higher dimension, one can show similarly that for any \( \omega \in \mathbb{R}^n \), there are always approximations such

\[
\forall C > 0, \exists I \in \mathbb{Z}^n, \ |(\omega, I)| \leq \frac{C}{\|I\|^{n-1}}.
\]

In the theory of Diophantine approximation, one usually says that the vector admits very good rational approximations if one can find an exponent bigger than \( n - 1 \). Kolmogorov’s Diophantine condition goes in the opposite direction.

\(^1\)This is Dirichlet’s pigeon hole principle.
It is easy to construct vectors which do not satisfy Kolmogorov’s Diophantine condition. Take:

\[ \alpha = \sum_{j \geq 0} 10^{-j!} . \]

The rational numbers:

\[ r_k = \sum_{j=0}^{k} 10^{-j!} \]

satisfy:

\[ |\alpha - r_k| \leq 2 \cdot 10^{-(k+1)!} . \]

The vector

\[ \omega = (1, \alpha) \]

admits very good rational approximations. To see it put:

\[ \beta_k = (\sum_{j=0}^{k} 10^{k!-j!}, 10^{k!}) \in \mathbb{Z}^2. \]

Then

\[ (\omega, \beta_k) = 10^{k!} \sum_{j \geq k+1} 10^{-j!} = O(10^{k!-(k+1)!}) \]

and

\[ \|\beta_k\| = O(10^{k!}) \]

therefore, for any \( \nu > 0 \), the sequence

\[ (\omega, \beta_k)\|\beta_k\|^{1+\nu} = O(10^{(\nu-k)k!}) \]

converges to zero.

Thus the vector \( \omega \) does not satisfy Kolmogorov’s Diophantine condition. On the other hand, a similar computation shows that for any \( I \in \mathbb{Z}^n \) with

\[ 10^{k!} \leq |I| < 10^{(k+1)!} \]

we have

\[ |(\omega, I)| \geq 10^{-kk!} . \]

Therefore

\[ |(\omega, I)^{-1}| \leq 10^{-kk!} \leq e^{10^{k!}} \leq e^{|I|}. \]

In particular, Proposition 2.9 implies that although \( \omega \) does not satisfies Kolmogorov’s Diophantine condition, the endomorphism

\[ h \ast : \mathbb{C}[[q, q^{-1}]] \rightarrow \mathbb{C}[[q, q^{-1}]], \ f \mapsto h \ast f, \ h := \sum_{I \neq 0} (\omega, I)^{-1} q^I \]

still preserves the space of holomorphic Fourier series.
3. Vectors satisfying the Kolmogorov Diophantine condition

Kolmogorov’s Diophantine condition is not a necessary condition for having an invariant torus theorem, but is sufficiently weak to allow the vectors satisfying it to form a set of positive measure. The condition also has also an abstract meaning in terms of operators as we shall see later. For the moment, we continue to investigate how many vectors satisfy the condition.

According to the well-known theorem of Liouville, for any algebraic non-rational number

\[ \alpha \in \mathbb{Q} \setminus \mathbb{Q} \]

of degree \( d \) then for some \( C > 0 \) one has

\[ |\alpha - \frac{p}{q}| \geq \frac{C}{q^d}, \]

so the vector \((1, \alpha)\) satisfies Kolmogorov’s Diophantine condition. Therefore the set of vectors which satisfy the condition obviously form a dense subset. But in fact one can prove that this set even has full measure.

Let us denote by

\[ \Omega(C, \nu) := \{ \omega \in \mathbb{R}^n : \forall 0 \neq I \in \mathbb{Z}^n, |(\omega, I)| \geq \frac{C}{\|I\|^{n-1+\nu}} \} \]

the set of vectors that satisfy Kolmogorov’s condition \( K(C, \nu) \).

**Proposition 3.2.** Assume that \( \nu > 0 \). Then for all \( R, C > 0 \) there is a constant \( k := k(\nu, R) \) such that

\[ \text{Vol}(B_R \setminus \Omega(C, \nu)) \leq k \cdot C \]

Here \( \text{Vol} \) is the Lebesgue measure and \( B_R \) the ball of radius \( R \).
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Proof. We consider a slightly more general situation. Let
\[ f : \mathbb{Z}^n \rightarrow \mathbb{R} \]
be a function that only depends on the Euclidean length \( \|I\| \) of \( I \in \mathbb{Z}^n \). Consider the set
\[ D(C) := \{ \omega \in B_R \mid \forall 0 \neq I \in \mathbb{Z}^n : |(\omega, I)| \geq Cf(I) \}. \]
The complement of this set in \( B_R \) can be written as
\[ D(C)^c = \bigcup_{0 \neq I \in \mathbb{Z}^n} B(C, I) \]
where \( B(C, I) \) is the set of \( \omega \in B_R \) that are 'bad' for \( C \) and \( I \):
\[ B(C, I) := \{ \omega \in B_R \mid |(\omega, I)| < Cf(I) \} \]
These \( \omega \)'s lie in a band of width
\[ 2Cf(I)/\|I\| \]
around the intersection of hyperplane \( I^\perp \) with the ball \( B_R \). So the volume of \( B(C, I) \) can be bounded by
\[ Vol(B(C, I)) \leq 2V_{n-1}(R)Cf(I)/\|I\|, \]
where \( V_{n-1}(R) \) is the volume of the \( n - 1 \)-dimensional radius \( R \)-ball. Now if the lattice sum
\[ \sum_{0 \neq I \in \mathbb{Z}^n} f(I)/\|I\| \]
exists and is finite, then the volume of \( \bigcup_{I \in \mathbb{Z}^n} B(C, I) \) is bounded by
\[ Vol \left( \bigcup_{0 \neq I \in \mathbb{Z}^n} B(C, I) \right) \leq kC, \quad k = 2V_{n-1}(R) \sum_{0 \neq I \in \mathbb{Z}^n} f(I)/\|I\|. \]
Now taking the intersection over \( C > 0 \) gives
\[ Vol(D^c) \leq \lim_{C \rightarrow 0} KC = 0. \]
Applying this to
\[ f(I) := 1/\|I\|^{n-1+\nu} \]
gives the result, as for any \( \nu > 0 \) one has
\[ \sum_I 1/\|I\|^{n+\nu} < \infty. \]

\[ \square \]

Corollary 3.3. For \( \nu > 0 \) the complement of the set
\[ \Omega(\nu) = \bigcup_{C>0} \Omega(C, \nu) \]
has measure zero.
Proof. That complement $\Omega(\nu)$ in the ball $B_R$ of radius $R$ is the set $D(R,\nu) := \bigcap_{C>0} B_R \setminus \Omega(C,\nu)$. According to the proposition, the volume of $B_R \setminus \Omega(C,\nu)$ goes to zero linearly with $C$, hence the measure of $D(R,\nu)$ is zero. By choosing a sequence of radii going to infinity, we see that the complement of $\Omega(\nu)$ is the union of countably many sets of measure zero, hence has itself measure zero. \qed

Similar considerations hold, of course, over the field of complex numbers. Note that for $\nu \leq 0$, the set $\Omega(\nu)$ is empty.

4. Kolmogorov’s invariant torus theorem: Statement

We now come to the formulation of the first important result in KAM theory, namely the theorem of Kolmogorov concerning the stability of invariant tori under certain conditions. The proof of the theorem will be given in chapter 11, after we have all technical machinery in place.

**Theorem 3.4.** Consider the Poisson algebras $A = \mathbb{C}\{q,q^{-1},p\}$ and $B = \mathbb{C}\{q,q^{-1},p,t\}$. Let $I = (p_1,p_2,\ldots,p_n) \subset A$ the ideal generated by the $p_i$’s. Consider an element $H \in A$ of the form

$$H = \sum_{i=1}^{n} \omega_i p_i + \sum_{i=1}^{n} \alpha_{ij} p_i p_j + (p)^3$$

If

(D) the vector $(\omega_i)$ satisfies the Kolmogorov Diophantine condition.

(N) the matrix $(\alpha_{ij})$ is invertible.

then the pair $(H,I)$ is homotopically stable.

Here by homotopic stability we mean the following: for any deformation $H + tQ \in \mathbb{C}\{q,q^{-1},p,t\}$ of $H$, there exists a central Poisson automorphism $\varphi$ of $\mathbb{C}\{q,q^{-1},p,t\}$, series $c(t) \in t\mathbb{C}\{t\}$ and $a_{ij} \in tB$ such that:

$$\varphi(H + tQ) = H + c(t) + \sum a_{ij} p_i p_j.$$ 

Condition (N) is called Kolmogorov’s non-degeneracy condition.

As each hamiltonian of the form at the right hand side has $p_1 = p_2 = \ldots = p_n = 0$ as invariant torus, we see that under the assumptions of the theorem, any hamiltonian $H + tQ$ admits a family of invariant Lagrangian manifolds parametrised by $t$, for $t$ small enough. As we shall see, the theorem also holds in the real analytic context, because
all our constructions commute with the conjugation $q_i \mapsto 1/\overline{q}_i$, the real part of the invariant manifolds are tori.

5. The one dimensional case

Although the case $n = 1$ of the Kolmogorov theorem is rather trivial, it already shows the origin of the non-degeneracy condition and its relation to symplectic but non-hamiltonian vector fields.

Let us take a closer look at the Hamiltonian

$$H(q, p) = \omega p + \alpha p^2.$$  

The conditions of the theorem become:

(D) $\omega \neq 0$,

(N) $\alpha \neq 0$.

The theorem produces for any perturbation $H + tQ$ a Poisson morphism $\varphi$ and an element $c(t) \in \mathbb{C}\{t\}$ such that

$$\varphi(H + tQ) = c(t) + \omega p + (p)^2.$$  

As the Hamiltonian flow preserves the level sets of $H$, we know that, in a neighbourhood of the zero section, the motion will take place along the curves

$$H = \text{constant}.$$  

These curves are diffeomorphic to circles, that is one-dimensional tori. Kolmogorov’s theorem tells us that for each $t$, we may select a circle on which the period of the motion is precisely $\omega$, as is the case for $H_0$. If we omit the non-degeneracy condition, then this is obviously wrong: take for instance

$$H = (\omega + t)p.$$  

For fixed value of $t$, the motion along the circles $p = \text{constant}$ has frequency $\omega + t$, so it cannot be equal to $\omega$ unless $t = 0$.

Let us consider the specific deformation

$$H' = \omega p + \alpha p^2 + tp$$  

of $H$ and let us compute the function $c$ and the Poisson automorphism $\varphi$ in this case.

First if we consider a Poisson automorphism induced by a Hamiltonian vector field

$$\varphi = e^{t\{-S\}}$$  

then the effect is

$$\varphi(H') = \omega p + \alpha p^2 + tp + t\{\omega p + \alpha p^2, S\} + (t^2).$$
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As \( p \) is not in the image of
\[
S \mapsto \{ \omega p + \alpha p^2, S \}
\]
we must use a non-hamiltonian vector field to get rid of the deformation:
\[
v = d(t)\partial_p.
\]

We have
\[
e^t v H' = \omega p + \alpha p^2 + tp + d(t)(\omega + 2\alpha p) + (t^2).
\]
Therefore we take
\[
d(t) = -\frac{t}{2\alpha}.
\]
The associated Poisson automorphism \( \varphi \) maps \( H \) to
\[
\varphi(H') = \omega p + \alpha p^2 - \frac{\omega}{\alpha} t + \frac{1}{4\alpha} t^2.
\]

In general, we will have an infinite series, but here the process stops at the first step.

The circle
\[
C_t := \{ p + \frac{1}{2\alpha} t = 0 \}
\]
is mapped to
\[
C_0 := \{ p = 0 \}
\]
The frequency of \( H' \) along \( C_t \) and \( H \) along \( C_0 \) are equal:
\[
\partial_p H'|_{C_t} = \omega + p + \frac{1}{2\alpha} t = \omega = \partial_p(H)|_{C_0}.
\]

This trivial example shows how the non-degeneracy condition works: it enables us to choose among the invariant circles the one for which the frequency is precisely \( \omega \). If \( \alpha \) is equal to 0, such a choice is no longer possible.

6. Kolmogorov’s theorem and group actions

We will now give a slight reformulation of Kolmogorov’s invariant torus theorem as a statement about a certain group action in the infinite dimensional vector space
\[
E := \mathbb{C}\{q, q^{-1}, p, t\}
\]
of analytic Fourier series depending on a parameter \( t \). Clearly, the group of central Poisson automorphisms acts on \( E \), but we will consider the subgroup \( G \) of elements \( \varphi \) which are tangent to the identity, i.e., whose restriction to \( t = 0 \) is the identity on \( \mathbb{C}\{q, q^{-1}, p\} \). When we fix an element \( H \in E \), the group \( G \) acts on the affine space
\[
H + M \subset E,
\]
of perturbations of \( H \), where \( M := tE \subset E \) consist of all the elements \( tQ \) we can add to \( H \).
We will take $H$ to be of the special form
\[ H = \sum_{i=1}^{n} \omega_i p_i + \sum_{i=1}^{n} \alpha_{ij} p_i p_j, \]
where the frequency $\omega = (\omega_i)$ and $(\alpha_{ij})$ are fixed. If we add to $H$ a special perturbation from the linear space
\[ F = t(\mathbb{C}\{t\} \oplus I^2) \subset M, \]
we obtain a Hamiltonian of the form
\[ H + tc(t) + \sum_{ij} a_{ij} p_i p_j \]
The elements of this affine space $H + F \subset H + M$ have clearly the special property that their flow preserves the subset $p_1 = p_2 = \ldots = p_n = 0$ that defines the torus.

Kolmogorov’s theorem can be formulated as follows:

**Theorem 3.5.** If $(\omega_i)$ is Diophantine and $(\alpha_{ij})$ is invertible, then any element of $H + M$ lies in the $G$-orbit of an element in $F$. In other words, the map
\[ G \times F \longrightarrow H + M, \quad (\varphi, \alpha) \mapsto \varphi(H + \alpha) \]
is surjective.

It means that any perturbation of $H$ can be transformed into a normal form belonging to $F$. As all elements of the normal form possess an invariant torus, any perturbation of $H$ admits an invariant torus, obtained from the torus $p_1 = p_2 = \ldots = 0$ in the normal form via the
automorphism $\varphi \in G$.

The Kolmogorov theorem gives an answer to a very general question in a particular case: given a group action on a space $M$, how can we ensure that a subset $F \subset M$ is a local transversal to the action?

When such a property holds we say that the corresponding element in $F$ is a normal form and $H + F$ is called a transversal to the action. Such a choice for $F$ is usually not unique and correspondingly there are different normal forms. The appropriate choice of $F$ is determined by utilitarian considerations.

In the next chapters of this book we will develop this point of view and define a general iteration scheme to bring elements to normal form. Furthermore, we will formulate a general statement of convergence that implies the above theorem as a special case.

7. Bibliographical notes

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Arnold pointed out that the set of vectors which satisfy Kolmogorov’s Diophantine condition and which lie on a submanifold might have zero measure. There are many results which prevent such pathologies, see for instance:

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CHAPTER 4

The generalised KAM problem

The original theorem of Kolmogorov pertains to the very special problem of perturbations of quasi-period motion on tori in a Hamiltonian system given in action-angle variables. It is of considerable interest to look for a more intrinsic and coordinate independent understanding of the theorem. This is indeed possible and a first step consists of replacing tori to more general Lagrangian subvarieties or coisotropic subvarieties that are invariant under the flow of a Hamiltonian $H$. This can be done in a general context of Poisson-algebras.

1. Invariant ideals

In this section we start with the standard symplectic space $M = K^{2n}$, with canonical coordinates $q, p$ and we assume that the field $K$ is algebraically closed of characteristic 0, e.g. $K = \mathbb{C}$. The ring of polynomial functions $A := K[q, p]$ is a Poisson-algebra with the standard Poisson-bracket

$$\{F, G\} = \sum_{i=1}^{n} \partial_{p_i} F \partial_{q_i} G - \partial_{q_i} F \partial_{p_i} G.$$

To a collection of polynomials

$$f_1, \ldots, f_k \in K[q, p]$$

one associates the affine variety:

$$V := \{(p, q) \in M : f_1(q, p) = \cdots = f_k(q, p) = 0\} \subset M.$$ 

This variety in fact only depends on the ideal $I = (f_1, \ldots, f_k) \subset A$ generated by the polynomials $f_i$, hence the notation $V = V(I)$. More generally, given a ring $A$ and an ideal $I$ there is an associated subscheme $V(I) \subset \text{Spec}(A)$.

Recall that the radical $\sqrt{I}$ of an ideal $I$ consists of those elements of the ring for which a power belongs to the ideal $I$. Hilbert’s Nullstellensatz is the statement that $\sqrt{I}$ is equal to the ideal of all polynomials that vanish on $V(I)$. A radical ideal is an ideal such that $\sqrt{I} = I$. So radical ideals $I$ can alternatively be characterised by the the property that any polynomial which vanishes on $V(I)$ belongs to $I$. 

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If $G$ is a first integral of $H$, then the level sets of $G$ are preserved by the Hamiltonian flow of $H$. One idea of Kolmogorov was to search for lower dimension invariant manifolds.

**Proposition 4.1.** Let $I \subset A$ be a radical ideal. The following assertions are equivalent:

i) $\{H, I\} \subset I$

ii) $V(I)$ is invariant under the formal flow of $H$.

If this applies, we say that $I$ is $H$-invariant.

**Proof.** We denote by $\varphi$ the formal flow of the Hamiltonian $H$, which was defined as the automorphism of $B := A[[t]]$ defined by the series $e^{t\{-H\}}$:

$$\varphi(f) = f + t\{f, H\} + \frac{t^2}{2!}\{\{f, H\}, H\} + \ldots$$

The ideal $I \subset A$ generates an ideal $IB \subset B$ consisting of series

$$h_0 + h_1t + h_2t^2 + \ldots$$

where the $h_i \in I$. The invariance of $V(I)$ under $\varphi$ means, by definition:

$$\varphi(\sqrt{IB}) \subset \sqrt{IB}$$

But if $I$ is radical, $IB$ is radical too, $\sqrt{IB} = IB$. To show this, it is sufficient to show that if $f^2 \in IB$ then in fact $f \in IB$. When we write $f$ as a series

$$f = f_0 + f_1t + f_2t^2 + \ldots, \quad f_i \in A$$

then $f^2 \in IB$ leads to equations

$$f_0^2 \in I, \quad 2f_0f_1 \in I, \quad 2f_0f_2 + f_1^2 \in I, \ldots$$

If $I$ is radical, the first equation gives $f_0 \in I$, the second gives no information on $f_1$, but from the third we deduce $f_1^2 \in I$, so $f_1 \in I$. Continuing this way, we get $f_i \in I$ for all $i = 0, 1, 2, \ldots$, that is $f \in IB$, hence $IB$ is radical.

It follows that $V(I)$ is $\varphi$-invariant if and only if

$$f \in I \implies \varphi(f) \in IB$$

By looking at the coefficient of $t$ at the right hand side we see that $\varphi$-invariance of $V(I)$ implies that $\{I, H\} \subset I$. On the other hand, if $\{I, H\} \subset I$, then for $f \in I$, all terms

$$f, \quad \{f, H\}, \quad \{\{f, H\}, H\}, \quad \ldots$$

of the series for $\varphi(f)$ belong to $I$, so $\varphi(f) \in IB$. Hence the conditions i) and ii) are equivalent. \qed
The $K$-algebra
\[ K[q, p]/I \]
can be interpreted as the ring of polynomial functions on the variety $V(I)$ defined by the ideal $I$. It is usually called the affine coordinate ring of $V(I)$.

The invariance condition $\{H, I\} \subset I$ implies that the map
\[ K[q, p]/I \rightarrow K[q, p]/I, \quad f \mapsto \{H, f\} \]
is well-defined. It is a derivation of the coordinate ring $K[q, p]/I$ and represents the restriction of the Hamiltonian field $X_H$ to $V(I)$. We use the same name for this induced map and consider $\{H, -\} \in \text{Der}(K[q, p]/I)$.

Recall that an ideal $I \subset K[q, p]$ is called involutive (or co-isotropic) if $\{I, I\} \subset I$.

If a radical ideal $I$ is involutive and $V(I)$ is non-empty, then $\dim V(I) \geq n$, and if $V(I)$ is of pure dimension $n$, then $V(I)$ is a Lagrangian subvariety, possibly with singularities.

**Proposition 4.2.** For any involutive ideal $I \subset K[q, p]$ there is a well-defined map
\[ I/I^2 \rightarrow \text{Der}(K[q, p]/I), \quad H \mapsto \{H, -\}. \]

**Proof.** As $\{I, I\} \subset I$, the ideal $I$ is $H$-invariant for any $H \in I$, so for each $H \in I$ we have a well-defined derivation
\[ \{H, -\} : K[q, p]/I \rightarrow K[q, p]/I \]
We obtain a map
\[ I \rightarrow \text{Der}(K[q, p]/I), \quad H \mapsto \{H, -\} \]
Let $(f_1, \ldots, f_n)$ be generators of $I$ and $H \in I$. Put
\[ H' = H + \sum_{ij} a_{ij} f_if_j \]
We have
\[ \{H + \sum_{ij} a_{ij} f_if_j, g\} = \{H, g\} + \sum_{ij} a_{ij} f_if_j \{f_j, g\} + \sum_{ij} a_{ij} f_j \{f_i, g\} \mod I \]
So the polynomials $H$ and $H'$ define the same derivation of the coordinate ring $K[q, p]/I$. This shows that the above map factors over $I/I^2$ and we get an induced map
\[ I/I^2 \rightarrow \text{Der}(K[q, p]/I), \quad H \mapsto \{H, -\}. \]
\[ \square \]
In algebraic geometry, the module $I/I^2$ is called the conormal module because when $V(I)$ is smooth, it is dual to the space of sections of the normal bundle. Geometrically, the above proposition states that the conormal bundle to an involutive manifold maps naturally to its tangent space. As we shall see, this elementary algebraic statement turns out to be quite fundamental in KAM theory, as it shows that for the study of the flow of $H$ along $V(I)$ the important object is not the Hamiltonian $H$ itself, but rather its class modulo $I^2$.

There is a second point: the Hamiltonian flow depends on the differential of the Hamiltonian and not on the Hamiltonian itself. Adding a constant does not change the dynamics, and therefore the important object is the class of the Hamiltonian modulo the subspace $I^2 \oplus K$.

Also, it is clear that the above algebraic set-up can be applied in a more general context: if $I$ is an involutive ideal in an arbitrary Poisson-algebra $A$, we obtain in precisely the same way a well defined map

$$I/I^2 \rightarrow \text{Der}(A/I), \quad H \mapsto \{H, -\}$$

In the general situation the role of the constants is taken over by the Casimir elements, i.e. the elements Poisson-commuting with everything:

$$H^0(A) := \{ f \in A \mid \{ f, g \} = 0 \text{ for all } g \in A \}.$$

**Definition 4.3.** We say that two Hamiltonians $H, H' \in A$ are $I^2$-equivalent, if

$$H' = H \mod H^0(A) + I^2$$

that is, if we can write

$$H' = H + c + i$$

with $c \in H^0(A)$ and $i \in I^2$.

If $H$ and $H'$ are $I^2$-equivalent, then the derivations $\{H, -\}$ and $\{H', -\}$ of $A/I$ coincide, so $H$ and $H'$ define exactly the same dynamics on the variety defined by $I$.

Let us now consider some elementary examples.

**Example 4.4.** Let $n = 1$ and consider the ideal $I$ generated by $p \in K[p, q]$. The Hamiltonians $H = p$ and $H' = p + qp^2 + 1$ are $I^2$-equivalent. Both Hamiltonian fields are equal along the line

$$V(I) = \{(q, p) \in K^2 : p = 0\}$$

although the second one is much more complicated over the whole space:
Example 4.5. Consider again the ring $K[q, q^{-1}, p]$ with the Poisson structure

$$\{p_j, q_k\} = q_k \delta_{jk}$$

The ideal $I$ generated by the $p_i$ is involutive and the factor ring

$$K[q, q^{-1}, p]/I \approx K[q, q^{-1}]$$

is the coordinate ring of our algebraic torus $V(I)$. Consider a linear Hamiltonian

$$H_0 = \sum_{i=1}^{n} \omega_i p_i.$$ 

Any Hamiltonian $H$ of the form

$$H = H_0 \mod (K \oplus I^2)$$

defines the same quasi-periodic motion on the torus $V(I)$ as $H_0$.

Via the isomorphism

$$K[q, q^{-1}, p]/I \approx K[q, q^{-1}]$$
the derivation

$$\{H_0, -\} : K[q, q^{-1}, p]/I \longrightarrow K[q, q^{-1}, p]/I$$

is identified with the map

$$\{H_0, -\} : K[q, q^{-1}] \longrightarrow K[q, q^{-1}], q^I \mapsto (\omega, I)q^I.$$
We see that the kernel reduces to $K$ if and only if the frequencies $\omega_i$’s are $\mathbb{Z}$-independent.

From a formal viewpoint, the situation is therefore similar to that of the dynamics along a non resonant Hamiltonian. However in our previous approach, the qualitative behaviour of the operator varies at a resonant torus. But when we restrict the dynamics to a specific torus we can avoid this problem.

2. Poisson cohomology and Poisson vector fields

We already encountered the space of Casimir elements:

$$ H^0(A) := \{ f \in A \mid \{ f, g \} = 0 \ \text{for all} \ g \in A \}. $$

As the notation already indicates, there exist also higher cohomology groups $H^i(A)$ that generalise the de Rham groups of a symplectic manifold. In particular, we will see there exists an exact sequence analogous to the exact sequence

$$ 0 \rightarrow \text{Ham} (M) \rightarrow \text{Symp} (M) \rightarrow H^1(M) \rightarrow 0 $$

of a symplectic manifold to the case of a general Poisson algebra.

Let us denote by $\Theta_A$ the space of derivations of $A$ and consider the exterior algebra

$$ \bigwedge \Theta_A. $$

We extend the map

$$ A \rightarrow \Theta_A, \ f \mapsto \{ -, f \} $$

to the exterior algebra so that it becomes a differential graded algebra. The resulting map gives a complex

$$ C^\bullet (A) : 0 \rightarrow A \rightarrow \Theta_A \rightarrow \bigwedge^2 \Theta_A \rightarrow \cdots $$

called the Poisson complex. If we consider the bi-derivation $\{ -, - \}$ as an element $\pi \in \Lambda^2 \Theta_A$, then the differential of the complex is obtained by taking the Lie bracket with $\pi$. We denote by $H^\bullet (A)$ the cohomology of the Poisson complex. In the symplectic case, the Poisson complex is isomorphic to the de Rham complex. From this definition we see that indeed $H^0(A)$ is the space of Casimir elements.

A Poisson-derivation is a derivation $\theta$ that ‘preserves’ the Poisson-bracket:

$$ \theta \{ f, g \} = \{ \theta(f), g \} + \{ f, \theta(g) \} $$

and similarly, the space $H^1(A)$ can be interpreted as the space of such Poisson derivations, modulo those that are Hamiltonian. It fits inside the exact sequence:

$$ 0 \rightarrow \text{Ham} (A) \rightarrow \text{Poiss}(A) \rightarrow H^1(A) \rightarrow 0 $$
which provides a generalisation of the de Rham type sequence for the symplectic case.

Example 4.6. Consider the two-dimensional $K$-algebra
\[ A := K[q, q^{-1}][[p]] \]
of formal power series in $p$ whose coefficients are Laurent polynomials in $q$, with the Poisson bracket
\[ \{p, q\} = q. \]
A direct computation shows that:
\[ H^0(A) = K, \]
\[ H^1(A) = K\partial_p \]

Example 4.7. Consider the $2n$-dimensional $K$-algebra
\[ A := K[q, q^{-1}][[p]] \]
of formal power series in $p = (p_1, \ldots, p_n)$ whose coefficients are Laurent polynomials in $q = (q_1, \ldots, q_n)$, with the Poisson bracket
\[ \{p_j, q_j\} = q_j \]
and all other variables Poisson-commuting. The Poisson complex is isomorphic to the $n$-fold wedge product of the one considered above. According to Künneth formula:
\[ H^\bullet(A) = \bigwedge^\bullet \bigoplus_{i=1}^n K[\partial_{p_i}]. \]
So analogous to de Rham theory, we recover the cohomology of the $n$-torus.

3. Pairs and homotopic stability

We now describe a general framework in which one can formulate the problem posed by Kolmogorov. Of course, one can not hope for very general answers, and the answers one can give will very much depend on the particulars. Nevertheless, we hope that this general setup provides a useful conceptual framework.

Definition 4.8. A) pair $(H, I)$ in a Poisson-algebra $A$ consists of an element $H \in A$ and an $H$-invariant involutive ideal $I \subset A$:
\[ \{H, I\} \subset I, \quad \{I, I\} \subset I. \]
B) The normal space to a pair $(H, I)$, denoted $N(H, I)$, is defined by
\[ N(H, I) = A/\left(\{H, A\} + I^2 + H^0(A)\right). \]
Note that the subspace that is divided out is just a linear subspace and not an ideal.
Example 4.9. The ideal \( I = (p_1, p_2, \ldots, p_n) \) is involutive in the Poisson-algebra \( A := K[q, q^{-1}][[p]] \), which models the formal completion of the zero section of the cotangent bundle to the algebraic torus. For any \( H \in K[[p]] \) we obtain a pair \((H, I)\) in \( A \). We have seen that for a non-resonant Hamiltonian
\[
H = \sum_{i=1}^{n} \omega_i p_i
\]
the space \( K[[p]] \) complements the image of the operator \( \{ H, - \} \). Therefore the normal space to \((H, I)\) is the \( n \)-dimensional vector space
\[
N(H, I) = K[p_1] \oplus K[p_2] \oplus \cdots \oplus K[p_n]
\]
where we denote by \([-\cdot]\) the class of \(-\cdot\) in the factor space.

Example 4.10. Consider now the case of the symplectic two-dimensional algebra \( A := K[[q, p]] \) and consider the Hamiltonian
\[
H = pq
\]
and the invariant ideal \( I = (H) \). The Hamiltonian derivation
\[
\{ H, - \} : A \rightarrow A
\]
maps \( p^i q^j \) to \((i - j)p^i q^j \) therefore \( K[[pq]] \) complements the image of the operator. Thus the normal space to \((H, I)\) is the 1-dimensional vector space
\[
N(H, I) = K[pq]
\]
We want to describe the behaviour of pairs under deformation. So we consider Poisson-algebras \( B \) with a non-zero divisor \( t \) as central element. Hence one has an exact sequence
\[
0 \rightarrow B \overset{t}{\rightarrow} B \rightarrow A \rightarrow 0
\]
where \( A \) is the factor ring \( B/tB \). From any pair \((H', I')\) in \( B \) one obtains a pair \((H, I)\) in \( A \) by reduction modulo \( t \), i.e. by setting \( t = 0 \). We will only consider pairs \((H', I')\) with the flatness property:
\[
tf \in I' \implies f \in I'
\]
One then has a corresponding exact sequence
\[
0 \rightarrow B/I' \overset{t}{\rightarrow} B/I' \rightarrow A/I \rightarrow 0
\]
One wants to understand properties of the map

Pairs in \( B \rightarrow \) Pairs in \( A \), \((H', I') \mapsto (H, I)\)

The persistence property of invariant tori in KAM theory leads to the following general notion.
4. THE FORMAL KOLMOGOROV THEOREM

**Definition 4.11. (Persistence property)**

We say that a pair \((H, I)\) in \(A\) is persistent, if for any deformation \(H'\) of \(H\) there exists at least one pair \((H', I')\) in \(B\).

Of course, it is of great interest to find conditions that imply the persistence of a given pair.

In the classical situation the parameter \(t\) in \(A\) provides a trivial modification of the algebra \(A\), like \(B = A[[t]]\). In such cases we have the additional property that the canonical projection \(B \rightarrow A\) has a section \(A \hookrightarrow B\) and we can consider \(A\) as a subring of \(B\). In this situation one can formulate the following stronger property.

**Definition 4.12. (Homotopic stability property)**

We say that a pair \((H, I)\) is homotopically stable, if for any deformation \(H'\) of \(H\), there exists a central Poisson morphism \(\varphi : B \rightarrow B\), \(\varphi(t) = t, \varphi|_{A} = Id_{A}\) such that \(\varphi(H')\) is \(I^2\)-equivalent to \(H\), i.e. there exist \(c \in H^0(B)\), \(i \in I^2\) such that

\[\varphi(H') = H + c + i\]

**Proposition 4.13.** If the pair \((H, I)\) is homotopically stable, then it is persistent.

**Proof.** Take the ideal \(IB\). It has the same generators as \(I \subset A\), but now we look at what they generate in \(B\). Then take \(I' := \varphi^{-1}(IB)\). Clearly \(\{I', I'\} \subset I'\) and \(\{H', I'\} \subset I'\), so \((H', I')\) is a pair in \(B\) that lifts \((H, I)\).

4. The formal Kolmogorov theorem

The normal space \(N(H, I)\) of a pair can also be understood in terms of first order deformations, where we only look at the first order in \(t\) and ignore all higher order terms. Algebraically, this means that we work in the ring \(B = A[t]/(t^2)\). Note that given element \(Q \in A\) which maps to zero in \(N(H, I)\), then there exists \(h \in A\) and \(i \in I^2\) such that:

\[\{H, h\} = Q + i.\]

Then

\[\varphi = Id - t\{-, h\}\]

is an Poisson automorphism of the Poisson algebra

\[B = A[t]/(t^2)\]
that maps the first order deformation

\[ H' = H + tQ \]

to an element \( I^2 \)-equivalent to \( H \).

This remark can be lifted to obtain the following **generalised formal Kolmogorov theorem**.

**Theorem 4.14.** Let \( (H,I) \) be a pair in \( A \) and let \( B = A[[t]] \). If the map

\[ H^1(A) \rightarrow N(H,I), \quad v \mapsto [v(H)] \]

is surjective, then \( (H,I) \) is homotopically stable.

**Proof.** We will construct a Poisson automorphism \( \varphi \) of \( B = A[[t]] \) that brings a perturbation \( H + tQ \), \( Q \in B \) to the normal-form

\[ \varphi(H + tQ) = H + c + i \]

where \( c \in H^0(B) = H^0(A)[[t]] \), \( i \in I^2B \). Surjectivity of the above map means that all elements \( Q \) of \( A \) can be represented in the form

\[ Q = v(H) + c + i \]

where \( v \in \text{Poiss}(A) \), \( c \in H^0(A) \) and \( i \in I^2 \subset A \). We work modulo \((t^k)\) and use induction on \( k \) and construct a sequence of Poisson-automorphisms \( \varphi_0 = \text{Id}, \varphi_1, \varphi_2, \ldots \) with

\[ \varphi_{k+1} = \varphi_k + (t^k). \]

Assume that we found such a sequence of Poisson morphisms such that

\[ \varphi_k(H + tQ) = H + c_k + i_k + (t^k), \quad c_k \in H^0(B), i_k \in I^2 \]

If we look one order in \( t \) further we have

\[ \varphi_k(H + tQ) = H + c_k + i_k + a_k t^k + (t^{k+1}) \]

By assumption, we may find \( v_{k+1} \in \text{Poiss}(A) \) and \( \lambda \).

\[ v_{k+1}(H) = \]

Define

\[ a_{k+1} = a_k + t^k \alpha_k \in I^2 + H^0(A) \]

we get that

\[ e^{-t^k v_{k+1}} \varphi_k(H_0) = H_0 + a_{k+1} \text{ mod } (t^{k+1}). \]

This proves the theorem. □

**Example 4.15.** Let us consider the case \( n = 1 \), \( A = K[[q,p]] \) with

\[ H = pq \]

and let \( I = (H) \) be the ideal generated by \( H \). The coordinate lines are invariant under the Hamiltonian flow. We know that

\[ N(H, I) = K[pq] \]
We have

\[ H^0(A) = K, \quad H^1(A) = 0. \]

The map

\[ H^1(A) \rightarrow N(H, I), \quad v \mapsto [v(H)] \]

maps a zero-dimensional space to a one-dimensional space, so it cannot be surjective. There is indeed a non-trivial deformation of the pair \((H, I)\) namely

\[ H' = (1 + t)H, \quad I' = pq. \]

The difference with the torus case is that here there is a single special Lagrangian variety consisting of two lines. Along this variety there is a well defined frequency equal to \((1 + t)\).

**Example 4.16.** We make the following variation of the previous example: we let

\[ A = K[[\lambda, q, p]], \quad H = (1 + \lambda)pq, I = pq \]

where \(\lambda\) is a central element and let \(I = (H)\) be the ideal generated by \(H\). We have

\[ N(H, I) = K[[\lambda]][pq] \]

but now

\[ H^0(A) = K[[\lambda]], \quad H^1(A) = K[[\lambda]][\partial_\lambda] \]

The map

\[ H^1(A) \rightarrow N(H, I), \quad v \mapsto [v(H)] \]

is now an isomorphism. The pair \((H, I)\) is now homotopically stable.

Like in the example, the map involved in the theorem is in general a morphism of finite type modules over the ring of Casimir operators. Let us now work out the torus case in details.

**Proposition 4.17.** Let \(I\) be the ideal generated by \(p_1, \ldots, p_n\) in \(A := K[q, q^{-1}][[p]]\). Let \(H = \sum_{i=1}^n \omega_i p_i + \sum_{i,j} \alpha_{ij} p_i p_j + \cdots \in K[[p]]\) be a power series in the \(p\) variables such that

(R) the vector \(\omega = (\omega_1, \ldots, \omega_n)\) is non resonant,

(N) the \(n \times n\) matrix \((\alpha_{ij})\) is invertible

then these conditions are respectively equivalent to:

(R) \(N(H, I)\) is an \(n\)-dimensional vector space generated by the classes of \(p_1, \ldots, p_n\),

(N) the natural map \(H^1(A) \rightarrow N(H, I)\) is surjective.

**Proof.** The first assertion is due to the fact that the absence of resonances implies that \(\{H, -\}\) is diagonal in the monomial basis and
that its image consists of functions with zero mean value. For the second assertion note that
\[ H^0(A) = K[1] \]
\[ H^1(A) = \bigoplus_{i=1}^n K[\partial p_i] \]
and that the map
\[ K^n \rightarrow N(H, I), \ (c_1, \ldots, c_n) \mapsto \left[ \sum_{i=1}^n c_i \partial p_i H_0 \right] \]
is an isomorphism if and only if the matrix \((a_{ij})\) is invertible. This proves the theorem.
\[ \square \]

In particular the pair \((H, I)\) of the proposition is homotopically stable and we see that the formal version of Kolmogorov’s theorem is a special case of our general theorem.

We will see that in the analytic case, that is, Kolmogorov’s invariant torus theorem, persistence is also shown be showing homotopic stability.

5. Bibliographical notes

The Poisson complex was introduced in:
A. Lichnerowicz. *Les variétés de Poisson et leurs algèbres de Lie associées*. J. Differential Geom. 12 (1977), no. 2, 253–300.
CHAPTER 5

The Lie iteration

In the previous chapter, we stated Kolmogorov’s invariant torus theorem and its generalisations in terms of infinite dimensional group actions. In this sense, it is a particular case of a general theorem on group actions. It is fortunate that all basic geometrical ideas already appear in the finite dimensional situation, namely for Lie group actions and in fact even for linear group actions. So the natural action of a matrix Lie group $G \subset GL(V)$ on a finite dimensional vector space $V$ is a natural starting point for our investigation.

1. Local transversals to a group action

When a Lie group $G$ acts smoothly on a manifold $M$, we are given a map

$$\sigma : G \times M \longrightarrow M, \ (g, x) \mapsto g \cdot x.$$ 

If $a \in M$ is a point, we obtain the so-called orbit map

$$\sigma_a : G \longrightarrow M, \ g \mapsto g \cdot a,$$

whose image is the $G$-orbit through $a$. The derivative this map at the identity element $e \in G$ gives a linear map

$$\rho := d\sigma_a : \mathfrak{g} \longrightarrow T_a M, \ \xi \mapsto \xi(a) := d\sigma_a(\xi)$$

where $\mathfrak{g} = T_e G$ denotes the Lie-algebra of $G$. We call this map the infinitesimal action (at $a$). The image

$$\mathfrak{g} \cdot a := d\sigma_a(\mathfrak{g})$$

of this map is the tangent space at $a$ to the orbit through $a$ and might be called the $\mathfrak{g}$-orbit.
One may ask the following general question:

*If \( F \subset M \) is a submanifold such that \( T_a F \) is a transversal to the \( g \)-orbit of \( a \), is it true that \( F \) is a transversal to the \( G \)-action?*

*In other words: does \( T_a F + g \cdot a = T_a M \) implies that the map \( G \times F \to M, \ (g, b) \mapsto g \cdot b \) is locally surjective around \( a \)?*

It follows from the implicit function theorem that the answer is positive. In the next chapter we will describe a general iteration schema that provides a constructive solution. KAM theory is concerned with the analogous situation in infinite dimensions where no implicit function theorem is available, but the same iteration scheme still can be used. For example, as we will see, Kolmogorov’s invariant torus theorem can be proven using this iteration, if interpreted in an appropriate sense. For that we will introduce the notion of *Kolmogorov spaces*, which will be developed in the next chapter. The resulting general theory then has many applications.
We start with some basic examples of Lie group actions.

**Example 5.1.** Consider the action of the Lie group \( G = \mathbb{R} \) on \( \mathbb{R}^2 \) by translation along the vector \( u = (1, 0) \):

\[
\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad (t,x,y) \mapsto (x+t,y).
\]

The orbits are horizontal lines. Any curve which does not have an horizontal tangent at a given point is a transversal to the group action at that point.

**Example 5.2.** Consider the action of the Lie group \( G = \mathbb{R} \) acting on \( \mathbb{R}^2 \) by translation along a parabola:

\[
\mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (t,x,y) \mapsto (x+t,y-2xt-t^2).
\]

The orbit through \((x_0, y_0)\) is the parabola with equation

\[
y + x^2 = y_0 + x_0^2
\]

Any curve which is not tangent at a point to the corresponding parabola is a transversal at the point.

The infinitesimal action at \( a = (x_0, y_0) \) is the map

\[
\mathbb{R} \rightarrow \mathbb{R}^2, \quad t \mapsto (1, -2x_0)t.
\]

The \( g \)-orbit is the tangent to the parabola at \((x_0, y_0)\), so any curve which is not tangent at a point to the corresponding parabola is a transversal at the point. In particular, if \( x_0 = 0 \) the \( g \)-orbit is an horizontal line. The situation is summarised by the first picture of this chapter.
Example 5.3. Consider the natural action of the Lie group $G = \text{Gl}(n, \mathbb{R})$ on $\mathbb{R}^n$:

$$\text{GL}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \ (A, x) \mapsto Ax.$$  

The action is homogeneous at any point $x \neq 0$ and the orbit is $\mathbb{R}^n \setminus \{0\}$. Therefore there are two orbits: the origin and its complement.

Example 5.4. Consider the natural action of the Lie group $G = \text{SO}(n, \mathbb{R})$ on $\mathbb{R}^n$:

$$\text{SO}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \ (A, x) \mapsto Ax.$$  

The orbit of $a \neq 0$ is the $n - 1$ dimensional sphere $S^{n-1}$ of radius $\|a\|$ centred at the origin. Therefore the action is no longer homogeneous. A transversal to the orbit at $a$ is given by any manifold transversal to the sphere at $a$, for instance the straight line $\mathbb{R}a$.

The Lie algebra $g$ consists of antisymmetric matrices and its orbit at $a$ is the tangent space to our sphere at $a$. 

\[ F = \mathbb{R}a \]
\[ g \cdot a = T_a S^{n-1} \]
\[ G \cdot (a + \alpha) \]
2. Adjoint orbits for the linear group

If $G$ is a Lie group, it acts on itself via conjugation

$$G \times G \longrightarrow G, \ (g,h) \mapsto ghg^{-1}$$

As this action preserves the unit element $e \in G$, there is an induced linear action of $G$ on the tangent space $\mathfrak{g} = T_eG$

$$G \times \mathfrak{g} \longrightarrow \mathfrak{g}.$$ This is commonly called the adjoint action of a group on its Lie-algebra.

Let us take a closer look at it for the group $G = GL(n, \mathbb{R})$. Its Lie-algebra $\mathfrak{g}$ can be identified with the space $M(n, \mathbb{R})$ of $n \times n$ matrices and the adjoint action is given by conjugation

$$GL(n, \mathbb{R}) \times M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \ (P, A) \mapsto PAP^{-1}$$

To determine the corresponding infinitesimal action, we write down a one-parameter family of matrices near the identity, with a given $B \in M(n, \mathbb{R})$ as tangent vector:

$$P = \text{Id} + tB + o(t)$$

The inverse then is

$$P^{-1} = \text{Id} - tB + o(t)$$

and therefore

$$PAP^{-1} = A + t[B, A] + o(t).$$

Therefore the infinitesimal action at $A \in M(n, \mathbb{R})$ is the map

$$M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \ B \mapsto [B, A].$$

In order to describe a transversal to the $\mathfrak{g}$-orbit, it is convenient to use the Euclidean scalar product $\langle -, - \rangle$ on $\mathfrak{g} = M(n, \mathbb{R})$ given by the formula

$$\langle A, B \rangle := \text{Tr}(AB^T).$$

Here $-^T$ denotes the transposed matrix.

**Lemma 5.5.** The commutant

$$C(A) := \{ B \in M(n, \mathbb{R}) : [B, A] = 0 \}$$

of $A$ is mapped by $B \mapsto B^T$ to the orthogonal space of the $\mathfrak{g}$-orbit:

$$C(A)^T = (\mathfrak{g} \cdot A)^\perp.$$ 

**Proof.** The $\mathfrak{g}$-orbit of $A$ consist of all elements of the form $[A, X]$, where $X \in M(n, \mathbb{R})$. Now note the identity

$$\langle [A, X], B^T \rangle = \text{Tr}(AXB) - \text{Tr}(XAB) = \text{Tr}(BAX) - \text{Tr}(ABX) = \langle [B, A], X^T \rangle$$

So we see that $B^T$ belongs to the orthogonal space of the $\mathfrak{g}$-orbit of $A$ precisely when $B$ commutes with $A$. □
We compute a local transversal at a matrix $A$ in two extreme cases:

1) $A$ is diagonal with distinct eigenvalues
In this case, the characteristic polynomial has distinct roots. By transversality, the matrix $A$ admits a neighbourhood in which all matrices share the same property and are therefore all diagonalisable. In terms of group actions: the space of diagonal matrices is a local transversal to the adjoint action at $A$, which indeed is the transpose of the commutant. As an example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

then the space of matrices of the form

$$D_\lambda = \begin{pmatrix} 1 + \lambda_1 & 0 \\ 0 & 2 + \lambda_2 \end{pmatrix}$$

is a local transversal. In general it means that $A$ has a neighbourhood $U$ such that for all $B \in U$ there exists $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in GL(n, \mathbb{R})$ with the property that $PBP^{-1} = D_\lambda$.

2) $A$ is maximally nilpotent Jordan block. So we let

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

If a matrix $B$ commutes with $A$, $[B, A] = 0$, then $B$ preserves the filtration defined by the kernels of the powers of $A$:

$$\{0\} \subset \text{Ker} \ A \subset \text{Ker} \ A^2 \subset \ldots \text{Ker} \ A^{n-1} \subset \text{Ker} \ A^n = \mathbb{R}^n.$$  

and thus $B$ is seen to be of the form:

$$B = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{n-2} & \lambda_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_1 \end{pmatrix} = \sum_{i=1}^{n} \lambda_i A^{i-1}$$

The transpose of this space is a transversal to the orbit:

$$F = \left\{ \sum_{i=1}^{n} \lambda_i (A^{i-1})^T : \lambda_1, \ldots, \lambda_n \in \mathbb{C} \right\}.$$  

For instance, we start with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
The commutant consists of matrices of the form
\[
\begin{pmatrix}
\lambda_1 & \lambda_2 \\
0 & \lambda_1
\end{pmatrix}
\]
and the orthogonal complement to the orbit consists of matrices of the form
\[
\begin{pmatrix}
\lambda_1 & 0 \\
\lambda_2 & \lambda_1
\end{pmatrix}
\]
This means that there is a neighbourhood of \( A \) such that any matrix can be taken back to the normal form
\[
\begin{pmatrix}
\lambda_1 & 1 \\
\lambda_2 & \lambda_1
\end{pmatrix}
\]

3. The Lie iteration in the homogeneous case

The classical Heron iteration for finding surds and Newton’s more general root-finding iteration are quadratically convergent. The basic technical tool we will introduce now is an iteration scheme similar (but different) to the Newton iteration in the context of a Lie group action. To distinguish it from the Newton iteration, we call it the Lie iteration. Its efficiency relies on the following two facts:

1. The Lie group and the Lie algebra agree at first order.
2. The tangent space to the Lie group is isomorphic to the Lie algebra.

As we shall see, (1) implies that our iteration is quadratic, like for the Newton method and (2) implies that we only need to linearise the map at the identity, unlike the Newton method which requires global construction of inverses.

Let us first assume that \( V \) is infinitesimally \( G \)-homogeneous at a point \( a \in V \). This means that the infinitesimal action of the Lie algebra at \( a \)
\[
\rho := d\sigma_a : g \rightarrow V, \xi \mapsto \xi(a)
\]
is surjective. In this case the implicit function theorem tells us that \( V \) is locally \( G \)-homogeneous: for any \( b \in V \) small enough, there exists \( g \in G \) such that
\[
g(a) = a + b.
\]
The following iteration produces a sequence of elements in \( G \) which converges rapidly to \( g \). It is determined by the choice of a linear map
\[
j : V \rightarrow g
\]
that is a right inverse to the infinitesimal action, i.e
\[
\rho \circ j = \text{Id}.
\]
and the local isomorphism determined by the exponential map
\[
\exp : g \rightarrow G, \quad \xi \mapsto e^\xi.
\]
We start our iteration with:

\[ b_0 := b \]

and define

\[ \xi_0 := j(b_0). \]

The element

\[ e^{\xi_0}a \]

is close to \( a + b \) or equivalently

\[ e^{-\xi_0}(a + b) \]

is close to \( a \). The error of this approximation is

\[ b_1 = e^{-\xi_0}(a + b) - a \]

We set \( \xi_1 := j(b_1) \) and repeat the process. In this way we get sequences \((b_n), (\xi_n)\) which define the Lie iteration scheme in the homogeneous case:

\[
\begin{align*}
\xi_n &= j(b_n) \\
b_{n+1} &= e^{-\xi_n}(a + b_n) - a
\end{align*}
\]

Note that

\[ a + b_{n+1} = e^{-\xi_n}(a + b_n) = e^{-\xi_n}e^{-\xi_{n-1}}(a + b_{n-1}) = \cdots = \prod_{i \geq 0} e^{-\xi_i}(a + b_0). \]

and

\[ \xi_n(a) = d\rho_a(\xi_n) = d\rho_a(j(b_n)) = b_n \]

Orbit of the one parameter subgroup generated by \( \xi_1 \)

\[ e^{t\xi_1}a \]

\[ e^{t\xi_0}a \]

\[ b_2 = e^{\xi_1}a = \xi_2(a) \]

\[ b_1 = e^{\xi_0}a = \xi_1(a) \]

Orbit of the one parameter subgroup generated by \( \xi_0 \)
4. Quadratic convergence of the Lie iteration

We want to control the rate of convergence of this iteration. To do this we use the operator norm:

\[ \|\xi\| := \sup_{x \in \mathbb{R}^n} \|\xi(x)\|/\|x\|. \]

and observe that

\[ \log \prod_{i \geq 0} e^{\xi_i} \leq \sum_{i \geq 0} \|\xi_i\|. \]

In particular:

**Lemma 5.6.** Let \((\xi_i)\) be a sequence of operators of a Banach algebra. The infinite product \(\prod_{i \geq 0} e^{\xi_i}\) is converges in the operator norm provided that the sequence \((\|\xi_i\|)\) is summable.

The generalisation of this lemma to a more general situation will play a key role in KAM theory. We want to apply the lemma, so we must evaluate the rate of convergence of the sequence \((\xi_n)\). As

\[ b_n = \xi_n(a), \]
\[ b_{n+1} = e^{-\xi_n(a + b_n)} - a, \]

we obtain:

\[ b_{n+1} = e^{-\xi_n(a + \xi_n(a))} - a. \]

or equivalently:

\[ b_{n+1} = (e^{-\xi_n}(\text{Id} + \xi_n) - \text{Id})(a). \]

Thus the sequence \((\xi_n)\) is obtained by iteration of the function:

\[ F = j \circ f, \quad f(x) = e^{-x}(1 + x) - 1. \]

Note that \(f\) has a critical point at the origin:

\[ f(x) = (1 - x)(1 + x) - 1 + o(x^2) = -x^2 + o(x^2). \]

In particular, by Taylor’s formula, there exists a neighbourhood of the origin and a constant \(C > 0\) such that:

\[ \|f(\xi)\| \leq C\|\xi\|^2. \]

We are in the classical situation of a quadratic iteration: If

\[ \|f(x)\| \leq C\|x\|^2 \]

then

\[ \|f(f(x))\| \leq C\|f(x)\|^2 \leq C \cdot C^2 \|x\|^4 \]
\[ \|f(f(f(x)))\| \leq C\|f(f(x))\|^2 \leq C \cdot C^2 \cdot C^4 \|x\|^8 \]

so that in general

\[ \|f^{(n)}(x)\| \leq C^{2^n-1}\|x\|^{2^n} = \frac{1}{C}(C\|x\|)^{2^n} \]
and where \( f^{(n)} \) denotes the \( n \)-th iterate of \( f \).

So we see:

**Theorem 5.7.** If \( \|f(x)\| \leq C\|x\|^2 \) and \( \|x_0\| < 1/C \), then the sequence of iterates

\[
x_n := f^{(n)}(x)
\]

converges rapidly to zero:

\[
\|x_n\| \leq \frac{1}{C}\rho^{2^n}, \quad \rho := C\|x_0\| < 1.
\]

**Theorem 5.8.** Let \( V \) be a finite dimensional vector space, \( G \subset GL(V) \) a group acting linearly on \( V \) and \( \mathfrak{g} \) its Lie-algebra, \( a \in V \) a point. Assume that the map

\[
\rho = d\sigma_a : \mathfrak{g} \longrightarrow V, \xi \mapsto \xi(a)
\]

admits an inverse \( j \).

Then, for any \( b \in V \) small enough, the Lie-iteration produces a rapidly convergent sequence \((\xi_n)\) of elements in \( \mathfrak{g} \) such that

\[
a = \prod_{i \geq 0} e^{-\xi_i}(a + b).
\]

The classical Heron-Newton iteration is also quadratic, but it requires the computation of a inverse to the differential at every new step. In the context of a group action, we need only a single inverse \( j \) to the infinitesimal action. Also note that the quadraticity of the iteration scheme stems from the fact that the Lie-group \( G \) and the Lie-algebra \( \mathfrak{g} \) ‘agree up to first order’. But it is also important to be aware of the fact that the Lie iteration does neither require the group \( G \), nor the fact that \( \mathfrak{g} \) is its full Lie-algebra. This is especially important for the applications of the iteration in infinite dimensional situations, as it is not necessary to construct the complete group as an infinite dimensional manifold with tangent space \( \mathfrak{g} \).

**5. The Lie iteration in the general case**

We adapt our previous iteration to non-transitive actions. So let \( G \subset GL(V) \) be a Lie group acting naturally on the vector space \( V \). Let \( F \) be an \( \mathfrak{g} \)-transversal at some point \( a \in V \). By the implicit function theorem, it is also a \( G \)-transversal.

Given \( b \in V \), we want to find \( g \in G \) and \( \alpha \in F \) such that

\[
g(a + \alpha) = a + b
\]

To do so, we extend the infinitesimal action

\[
\rho : \mathfrak{g} \longrightarrow V, \quad \xi \mapsto \xi(a)
\]
to a surjective map

\[ \rho_a : F \times g \rightarrow V, \quad (\alpha, \xi) \mapsto \xi(a) + \alpha \]

Let

\[ j_a : V \rightarrow F \times g \]

be a right inverse to \( \rho_a \). We also put

\[ j : (a + F) \times V \rightarrow F \times g, \quad (a + \alpha, b) \mapsto j_{a+\alpha}(b) \]

Our problem is now the following: given \( b \in V \), we want to find \( g \in G \) and \( \alpha \in F \) such that

\[ g(a + \alpha) = a + b \]

To do so, we will modify the original iteration as follows. First we set \( b_0 := b \), \( a_0 := a \) and then consider \( j(a_0, b_0) = (\alpha_0, \xi_0) \). Then \( \alpha_0 \in F \) and \( \xi_0 \in g \) are elements such that

\[ b_0 = \xi_0(a_0) + \alpha_0 \]

Now the element \( e^{\xi_0}(a_0 + \alpha_0) \) will be close to \( a_0 + b_0 \) and we put

\[ a_1 := a_0 + \alpha_0, \]
\[ b_1 := e^{-\xi_0}(a_0 + b_0) - a_1, \]

We now can iterate this procedure and obtain sequences \( a_n, b_n, \alpha_n, \xi_n \) defining the Lie iteration in the general case:

\[
\begin{align*}
  a_{n+1} &:= a_n + \alpha_n, \\
  b_{n+1} &:= e^{-\xi_n}(a_n + b_n) - a_{n+1}, \\
  (\alpha_{n+1}, \xi_{n+1}) &:= j(a_{n+1}, b_{n+1})
\end{align*}
\]
The last equation expresses the fact that at each step we have the decomposition
\[ b_{n+1} = \xi_{n+1}(a_{n+1}) + \alpha_{n+1}. \]

Note that, just as in the iteration for the homogeneous case, one has
\[ a_{n+1} + b_{n+1} = e^{-\xi_n}(a_n + b_n) = e^{-\xi_n}e^{-\xi_{n-1}}(a_{n-1} + b_{n-1}) = \cdots = \prod_{i=0}^{n} e^{-\xi_i}(a + b) \]

Let us now rewrite this as the iteration of a mapping. If we substitute the relation
\[ b_{n+1} = \xi_n(a_n) + \alpha_n. \]
in
\[ b_{n+1} = e^{-\xi_n}(a_n + b_n) - a_{n+1}, \]
we obtain:
\[ b_{n+1} = e^{-\xi_n}(a_n + \xi_n(a_n) + \alpha_n) - a_n - \alpha_n = (e^{-\xi_n}(a_n + \xi_n(a_n)) - a_n) + (e^{-\xi_n} \alpha_n) - \alpha_n. \]
which can be written as
\[ b_{n+1} = A(\xi_n, a_n) + B(\xi_n, \alpha_n) \]
where
\[ A(\xi, a) = (e^{-\xi}(\text{Id} + \xi) - \text{Id})(a), \quad B(\xi, \alpha) = (e^{-\xi} - \text{Id})(\alpha). \]
Note that, for fixed \( a \), both \( A \) and \( B \) have a quadratic singularity at \( \xi = 0, \alpha = 0 \). From this we find
\[ (\alpha_{n+1}, \xi_{n+1}) = j(a_n + \alpha_n, A(\xi_n, a_n) + B(\xi_n, \alpha_n)) \]
The right hand side still involves \( a_n \), so we write:
\[ (a_{n+1}, (\alpha_{n+1}, \xi_{n+1})) = (a_n + \alpha_n, j(a_n + \alpha_n, A(\xi_n, a_n) + B(\xi_n, \alpha_n))) \]
The formula is a bit involved, but the only important point is that it is of the form
\[ (x_{n+1}, y_{n+1}) = (x_n + Ly_n, f(x_n, y_n)) \]
where \( L \) is a linear map and \( f(x, -) \) is quadratic. In our situation, we have \( x_n = a_n, y_n = (\alpha_n, \xi_n), Ly_n = \alpha_n \), and, as we observed:
\[ y \mapsto f(x, y) = j(x + Ly, A(x, y) + B(y)) \]
is quadratic.

We may adapt the fixed point theorem to this situation:

**Theorem 5.9.** Let \( E, F \) be Banach space and let
\[ F : B_E \times B_F \rightarrow E \times F, (x, y) \mapsto (x + Ly, f(x, y)) \]
be such that there exists a constant \( C \) with
\[ \|f(x, y)\| \leq C\|y\|^2 \]
for any \( x \in B_E \). Let \( y_0 \in B_F \) be such that

\[
C\|y_0\| \leq 1 \quad \text{and} \quad \frac{\|L\|}{C} \sum_{n \geq 0} (C\|y_0\|)^{2^n} < 1
\]

Then the sequence \((x_n, y_n) = F^n(x, y), \ x_0 = 0\) converges to a limit \((l, 0)\) and

\[
\|y_n\| \leq \rho^{2^n}, \quad \|x_n\| \leq \frac{\|L\|}{C} \sum_{i=1}^{n-1} \rho^{2^i}
\]

with \( \rho = C\|y_0\| \).

**Proof.** The proof is a small variation to the non-parametric case. As

\[
\|f(x, y)\| \leq C\|y\|^2
\]

we get that:

\[
\|y_n\| = \|f^{(n)}(x_0, y_0)\| \leq C^{2^n-1}\|y_0\|^{2^n} = \frac{1}{C}(C\|y_0\|)^{2^n}
\]

and

\[
\|x_n\| \leq \|L\| \|y_{n-1}\| + \|x_{n-1}\|
\]

\[
\leq \|L\| \|y_{n-1}\| + \|L\| \|y_{n-2}\| + \|x_{n-2}\|
\]

\[
\leq \ldots
\]

\[
\leq \|L\| \sum_{i=1}^{n-1} \|y_i\|.
\]

The main point of difference between the homogeneous and the parametric iteration is that in the first case we need only a single right inverse to the infinitesimal action at \( a \). In the parametric situation one needs such a right inverse at the complete sequence of points \( a = a_0, a_1, \ldots \) lying in the transversal slice. In practice this transversal consists of simple modification of our initial problem and is well under control.

**6. Bibliographical Notes**

In his lectures on singularity theory, Arnold used Lie groups actions as a preliminary step to the study of hypersurface singularities and their deformations. The study of \( GL(n, \mathbb{R}) \) orbits was studied in:

**Arnold V.I., On matrices depending on parameters**, Russian Mathematical Surveys 26(2), 29-43, (1971).
5. THE LIE ITERATION

Inspired by Kolmogorov’s invariant tori paper, the Lie iteration was formulated by Moser in the absolute case in:

J. Moser, *A rapidly convergent iteration method and non-linear partial differential equations II*, Ann. Scuola Norm Sup. Pisa - Classe di Scienze Sér. 3, **20**(3), 499-535, (1966).

see also:

J. Féjoz, J. and M. Garay., *Un théorème sur les actions de groupes de dimension infinie*, Comptes Rendus à l’Académie des Sciences, 348, 427-430, (2010).

M. Garay, *An Abstract KAM Theorem*, Moscow Math. Journal, Volume 14, Number 4, p.745-772, (2014).

Moser does not treat normal forms as an application of his implicit function theorems (now called Nash-Moser theorems). On the contrary, Moser noticed that the requirement of these theorems are in general not fulfilled. When possible application of implicit function theorems in the presence of small denominators remains a non trivial problem. For instance, following an idea of Herman this has been done by Bost for the case of the Kolmogorov theorem in:

J.-B. Bost, *Tores invariants des systèmes dynamiques hamiltoniens*, Sém. Bourbaki 639, Astérisque, 133-134, p. 113–157 (1986).
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*J. Moser* by Konrad Jacobs, Erlangen (1969).