LARGE SETS OF COMPLEX AND REAL EQUIANGULAR LINES

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ABSTRACT. Large sets of equiangular lines are constructed from sets of mutually unbiased bases, over both the complex and the real numbers.

1. Introduction

The angle between vectors $x_j$ and $x_k$ of unit norm in $\mathbb{C}^d$ is $\arccos|\langle x_j, x_k \rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product. A set of $m$ distinct lines in $\mathbb{C}^d$ through the origin, represented by vectors $x_1, \ldots, x_m$ of equal norm, is equiangular if for some real constant $a$ we have

$$|\langle x_j, x_k \rangle| = a \text{ for all } j \neq k.$$ 

The number of equiangular lines in $\mathbb{C}^d$ is at most $d^2$ [4], and when the vectors are further constrained to lie in $\mathbb{R}^d$ this number is at most $d(d+1)/2$ (attributed to Gerzon in [9]). It is an open question, in both the complex and real case, whether the upper bound can be attained for infinitely many $d$, although in both cases $\Theta(d^2)$ equiangular lines exist for all $d$. Specifically, König [8] constructed $d^2 - d + 1$ equiangular lines in $\mathbb{C}^d$ where $d - 1$ is a prime power, and de Caen [3] constructed $2(d+1)^2/9$ equiangular lines in $\mathbb{R}^d$ where $(d+1)/3$ is twice a power of 4. By extending vectors using zero entries as necessary, we can derive sets of $\Theta(d^2)$ equiangular lines from these direct constructions for all $d$.

Two orthogonal bases $\{x_1, \ldots, x_d\}, \{y_1, \ldots, y_d\}$ for $\mathbb{C}^d$ are unbiased if

$$|\langle x_j, y_k \rangle| = \frac{1}{\sqrt{d}} \text{ for all } j, k.$$ 

A set of orthogonal bases is a set of mutually unbiased bases (MUBs) if all pairs of distinct bases are unbiased.

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The number of MUBs in $\mathbb{C}^d$ is at most $d + 1$ [4, Table I], which can be attained when $d$ is a prime power by a variety of methods [5], [7], [10]. The number of MUBs in $\mathbb{R}^d$ is at most $d/2 + 1$ [4, Table I], which can be attained when $d$ is a power of 4 [1], [2].

The authors recently gave a direct construction of $d^2/4$ equiangular lines in $\mathbb{C}^d$, where $d/2$ is a prime power [6]. We show here how to generalize the underlying construction to give $\Theta(d^2)$ equiangular lines in $\mathbb{C}^d$ and $\mathbb{R}^d$ directly from sets of complex and real MUBs.

2. The Construction

We associate an ordered set of $m$ vectors in $\mathbb{C}^d$ with the $m \times d$ matrix formed from the vector entries, using the ordering of the set to determine the ordering of the vectors.

**Theorem 1.** Suppose that $B_1, B_2, \ldots, B_r$ form a set of $r$ MUBs in $\mathbb{C}^d$, each of whose vectors has all entries of unit magnitude, where $r \leq d$. Let $a_1, a_2, \ldots, a_t$ be constants in $\mathbb{C}$, where $t \geq 1$. Let $B_j(v)$ be the set of $d$ vectors formed by multiplying entry $j$ of each vector of $B_j$ by $v \in \mathbb{C}$, and let $L(v) = \bigcup_{j=1}^{r} B_j(v)$ (considered as an ordered set). Then all inner products between distinct vectors among the $rd$ vectors of

$$\left[ L(a_1) \ L(a_2) \ \ldots \ L(a_t) \ L \left( t + 1 - \sum_{j=1}^{t} a_j \right) \right]$$

in $\mathbb{C}^{(t+1)d}$ have magnitude $\sum_{j=1}^{t} |a_j - 1|^2 + \left| \sum_{j=1}^{t} (a_j - 1) \right|^2$ or $(t + 1)\sqrt{d}$.

**Proof.** Write $A = \{a_1, a_2, \ldots, a_t, t+1-\sum_{j=1}^{t} a_j\}$ for the set of arguments $v \in \mathbb{C}$ taken by $L(v)$ in the construction. We consider two cases, according to whether distinct vectors of $L(v)$ originate from the same basis or from distinct bases.

In the first case, consider the inner product of distinct vectors of $L(v)$ constructed from vectors from the same basis $B_j$. Since the original vectors are orthogonal, this inner product is $z(|v|^2 - 1)$ for some $z$ of unit magnitude that depends only on the original two vectors. Since each occurrence of $L(v)$ uses the same ordering, the inner product of the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ is therefore $z \sum_{v \in A} (|v|^2 - 1)$, which equals $z \left( \sum_{j=1}^{t} |a_j - 1|^2 + \left| \sum_{j=1}^{t} (a_j - 1) \right|^2 \right)$ after straightforward algebraic manipulation.

In the second case, consider vectors of $L(v)$ constructed from vectors from distinct bases $B_j, B_k$. Let these constructed vectors be

$$x = (x_1 \ x_2 \ \ldots \ vx_j \ \ldots \ x_d),$$
$$y = (y_1 \ y_2 \ \ldots \ vy_k \ \ldots \ y_d).$$
The inner product of $x$ and $y$ in $L(v)$ is
\[
x_1\overline{y_1} + \cdots + vx_j\overline{y_j} + \cdots + x_k\overline{y_k} + \cdots + x_d\overline{y_d} = \sum_{\ell=1}^{d} x_{\ell}\overline{y_{\ell}} + (v-1)x_j\overline{y_j} + (\overline{v}-1)x_k\overline{y_k}.
\]
Therefore the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product
\[
(t+1) \sum_{\ell=1}^{d} x_{\ell}\overline{y_{\ell}} + x_j\overline{y_j} \sum_{v\in A} (v-1) + x_k\overline{y_k} \sum_{v\in A} (\overline{v}-1) = (t+1) \sum_{\ell=1}^{d} x_{\ell}\overline{y_{\ell}},
\]
because $\sum_{v\in A} v = t + 1$. Now, all of the entries $x_{\ell}, y_{\ell}$ have unit magnitude by assumption, and so $|\sum_{\ell=1}^{d} x_{\ell}\overline{y_{\ell}}| = \sqrt{d}$ by the MUB property (1). Therefore the concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product of magnitude $(t+1)\sqrt{d}$. □

Remark. Lemma 6.2 of [6] describes the special case $t = 1$ and $r = d$ of Theorem 1, in which the MUBs are constrained to arise from a $(d, d, d, 1)$ relative difference set in an abelian group according to the construction method of [5]; the permutation $\pi$ given in [6, Lemma 6.2] can be dropped without loss of generality.

Corollary 2. Let $t$ be a positive integer and let $d$ be a prime power. There exist $d^2$ equiangular lines in $\mathbb{C}^{(t+1)d}$.

Proof. There exists a set of $d+1$ MUBs in $\mathbb{C}^d$ for which one of the bases is the standard basis [10]. After appropriate scaling, all entries of each of the vectors of the remaining $d$ bases therefore have unit magnitude, using [11]. So we may apply Theorem 1 with $r = d$.

There are infinitely many choices of $a_1, a_2, \ldots, a_t \in \mathbb{C}$ for which the two magnitudes in the conclusion of Theorem 1 are equal, one such choice being $a_j = 1 + d^{1/4}/\sqrt{t}$ for each $j$. □

Corollary 3. Let $t$ be a positive integer and let $d$ be a power of 4. There exist $d^2/2$ equiangular lines in $\mathbb{R}^{(t+1)d}$.

Proof. There exists a set of $d/2 + 1$ MUBs in $\mathbb{R}^d$ for which one of the bases is the standard basis [11], [2]. Apply Theorem 1 with $r = d/2$ and take, for example, $a_j = 1 + d^{1/4}/\sqrt{7}$ for each $j$ to obtain real equiangular lines. □

The proof of Theorem 1 shows that the magnitude of the inner product of distinct vectors is $\sum_{v\in A}(|v|^2 - 1)$ or $(t+1)\sqrt{d}$. In the construction of Corollaries 2 and 3 the constants $a_j$ are chosen so that these magnitudes are equal, and the inner product of each concatenated vector with itself is $\sum_{v\in A}(|v|^2 + d - 1)$. It follows that the common angle for the sets of equiangular lines constructed in Corollaries 2 and 3 is $\arccos(1/(1+\sqrt{d}))$ for all $t$, regardless of the choice of the constants $a_j$. 
Theorem 1 can be generalized as follows. Let $c_1, \ldots, c_t$ be real constants, and take the $rd$ vectors of
\[
\begin{bmatrix}
  c_1 L(a_1) & c_2 L(a_2) & \ldots & c_t L(a_t) & L\left(1 + \sum_{j=1}^{t} c_j^2 (1 - a_j)\right)
\end{bmatrix}
\]
in $\mathbb{C}^{(t+1)d}$. Then all inner products between distinct vectors have magnitude
\[
\sum_{j=1}^{t} c_j^2 |1 - a_j|^2 + \left|\sum_{j=1}^{t} c_j^2 (1 - a_j)\right|^2 \text{ or } (1 + \sum_{j=1}^{t} c_j^2) \sqrt{d}.
\]
If $a_1, a_2, \ldots, a_t$ and $c_1, c_2, \ldots, c_t$ are chosen so that these two magnitudes are equal, the common angle of the resulting set of equiangular lines is again $\arccos(1/(1 + \sqrt{d}))$.

Remark. The case $t = 1$ and $d = 4$ of Corollary 3 constructs 8 equiangular lines in $\mathbb{R}^8$ having the form $[L(a) \ L(2 - a)]$, where $a \in \{1 \pm \sqrt{2}\}$. We can extend this to a set $[L(a) \ L(2 - a) \ L(2 - a) \ L(a)]$ of 16 equiangular lines in $\mathbb{R}^8$, where $a \in \{1 \pm \sqrt{2}\}$; this extension does not seem to generalize easily to larger values of $d$.

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