NEW CONSERVATION FORMS AND LIE ALGEBRAS OF ERMAKOV-PINNEY EQUATION

ÖZLEM ORHAN
Istanbul Technical University, Faculty of Science and Letters
Department of Mathematical Engineering, 34469 Maslak
Istanbul, Turkey

TEOMAN ÖZER*
Istanbul Technical University, Faculty of Civil Engineering
Division of Mechanics, 34469 Maslak
Istanbul, Turkey

Abstract. In this study, we investigate first integrals and exact solutions of the Ermakov-Pinney equation. Firstly, the Lagrangian for the equation is constructed and then the determining equations are obtained based on the Lagrangian approach. Noether symmetry classification is implemented, the first integrals, conservation laws are obtained and classified. This classification includes Noether symmetries and first integrals with respect to different choices of external potential function. Furthermore, the time independent integrals and analytical solutions are obtained by using the modified Prelle-Singer procedure as a different approach. Additionally, for the investigation of conservation laws of the equation, we present the mathematical connections between the λ-symmetries, Lie point symmetries and the modified Prelle-Singer procedure. Finally, new Lagrangian and Hamiltonian forms of the equation are determined.

1. Introduction. Lie group of transformations is one of the most powerful methods for investigation of differential equations [16, 19]. There are many applications of Lie groups to deal with the problems in mechanics, mathematics and physics in the literature. From the mathematical point of view, the Ermakov-Pinney equation [6, 8] is an important example of a nonlinear ordinary differential equation, which describes the temporal dynamics of an effective scale factor of the universe in simple cosmological models of the so-called Friedmann-Robertson-Walker type. In addition, the Ermakov-Pinney equation is also related to the so-called nonlinear Schrödinger equation describing the wave function of Bose-Einstein condensates at the mean-field level [23, 25].

The application of Noether theorem in the concept of theory of Lie groups presents that the Noether symmetry of the action of a physical system has a corresponding conservation law. The natural form of the Lagrangian is defined in Noether theory and an important property of the Lagrangian is that conservation laws can be constructed by using its Lagrangian. Noether symmetries can also be used in finding of the first integrals of the nonlinear problems. The earliest studies

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* Corresponding author: Teoman Özer.
on Noether symmetries are due to German mathematician Emmy Noether [16]. In order to apply the Noether theorem, it is a fact that the differential equations should have a Lagrangian. Noether theorem gives variational symmetries corresponding to conservation laws for the associated Euler-Lagrange equations.

In the literature, Edmund Pinney presents the equation [24]

\[ \ddot{x} + \omega^2(t)x = \frac{1}{x^3}, \tag{1}\]

in which, \( \omega^2(t) \) is the external potential and the over dot denotes differentiation with respect to the time independent variable \( t \). Then Berkovich applies the method of factorization to problems related to Ermakov systems [1]. Furthermore Nucci [17] examines Lie symmetries of equation (1).

In the literature, there exists a method, which is called Prelle-Singer for solving first-order ordinary differential equations [26]. The Prelle-Singer method is based on the fact that if the given system of first order ordinary differential equation has a solution in terms of elementary functions, then the method ensures that this solution can be found. Later, Duarte [5] reconstructs the technique developed by Prelle and Singer and applies it to second-order ordinary differential equation. Their approach is based on the conjecture that if an elementary solution exists for the given second-order differential equation then there exists at least one elementary first integral. Recently, the generalized theory to higher order ordinary differential equations is introduced in the study [4]. It is important to say that the Prelle-Singer method has a strong mathematical relation with the \( \lambda \)-symmetry approach in order to investigate first integrals of nonlinear near differential equations [15].

This study is organized as follows. In section 2, we present some fundamental definitions about Noether theorem and Prelle-Singer method. In section 3, we discuss the nonlinear Ermakov-Pinney equation and the corresponding determining equations. This section also includes different cases corresponding to different choices of external potential coefficient. Furthermore, Noether point symmetries and first integrals for each different case are presented. In the section 4, we apply generalized Prelle-Singer method to Ermakov-Pinney equation and obtain Lie symmetry, first integral, \( \lambda \)-symmetry, integrating factor and Lagrangian-Hamiltonian functions of the equation. The last section summarizes some important results in the study.

2. Preliminaries.

2.1. Noether theorem and first integrals. Suppose that \( t \) is the independent variable, \( x = (x^1, \ldots, x^m) \) is the dependent variable and the derivative of \( x^\alpha \) with respect to \( t \) are

\[ x^\alpha_t = x^\alpha_1 = D_t(x^\alpha), \quad x^\alpha_s = D^s_t(x^\alpha), \quad s \geq 2, \quad \alpha = 1, 2, \ldots, m, \tag{2}\]

where \( D_t \) is the total derivative operator [3, 18], with respect to \( t \), which is defined as

\[ D_t = \frac{\partial}{\partial t} + x^\alpha_t \frac{\partial}{\partial x^\alpha} + x^\alpha_{tt} \frac{\partial}{\partial x^t}. \tag{3}\]

Here, the vector space of all differential functions of all finite orders is represented by \( \mathcal{A} \) that is universal space. Also, operators apart from total derivative operator (3) are defined on space \( \mathcal{A} \).
Definition 2.1. The operator
\[ \frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} + \sum_{s \geq 1} (-D_t)^s \frac{\partial}{\partial x^\alpha^s}, \quad \alpha = 1, 2, \ldots, m, \] (4)
is called the Euler operator or Euler-Lagrange operator.

Definition 2.2. The generalized operator is given by
\[ X = \xi \frac{\partial}{\partial t} + \eta^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{s \geq 1} \xi^\alpha_s \frac{\partial}{\partial x^\alpha_s}, \] (5)
where
\[ \xi^\alpha_s = D_s^t W^\alpha + \xi^s x^\alpha_s, \quad s \geq 2, \quad \alpha = 1, 2, \ldots, m, \] (6)
and \( W^\alpha \) is the Lie characteristic function
\[ \dot{W}^\alpha = \eta^\alpha - \xi^\alpha_t, \quad \alpha = 1, 2, \ldots, m. \] (7)
Here we can rewrite the generalized operator (5) in terms of characteristic function as below
\[ X = \xi D_t + W^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{s \geq 1} D_t^s (W^\alpha) \frac{\partial}{\partial x^\alpha_s}, \] (8)
and the Noether operator associated with a generalized operator \( X \) can be defined
\[ N = \xi + W^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{s \geq 1} D_t^s (W^\alpha) \frac{\partial}{\partial x^\alpha_s}. \] (9)

Now let us consider a \( k \)-th order system of ordinary differential equation
\[ E^\alpha(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(k)}) = 0, \quad \alpha = 1, 2, \ldots, m. \] (10)

Definition 2.3. The first integral of the system \( I \in A \) (9) can be written in the following form
\[ D_t I = 0. \] (11)
Then the expression (10) is called local conservation law for system (9). Furthermore, \( D_t I = Q^* E^\alpha \) is called the characteristic form of conservation law (10) where the functions \( Q^* = (\dot{Q}^1, \ldots, \dot{Q}^m) \) are the associated characteristic of the conservation law (10).

Definition 2.4. \[ \text{[11]} \] Let \( L = L(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(m)}) \in A, \; \alpha \leq k \) and nonzero functions \( f^\alpha \in A \) be a Lagrangian and \( X \) be a Lie-Bäcklund operator of the form of (5). If there exists a vector \( B \in A, \; B \neq NL + C, \; C = \text{constant} \), we have the following relation
\[ X^{(\alpha)} L + LD_t (\xi) = W^\alpha \frac{\delta L}{\delta x^\alpha} + D_t (B), \] (12)
where \( W = (W^1, \ldots, W^m) \), \( B(t, x) \) is the gauge function, and \( W^\alpha \in A \) then \( X \) is called a Noether operator corresponding to \( L \) and, \( X^{(\alpha)} \) is the \( \alpha \)-th prolongation of the generalized operator (8).

Definition 2.5. \( X \) is a Noether point symmetry corresponding to Lagrangian of the system of differential equations (9) if there exists a function \( B(t, x) \). In addition, \( X \) is a Noether point symmetry corresponding to a Lagrangian of the equation, then \( I \) is a first integral associated with \( X \), which is given by the expression \[ I = \xi L + (\eta - x' \xi) L_x' - B. \] (13)
2.2. Lagrangian and Hamiltonian description. Let’s consider the equation

\[ \ddot{x} = \frac{P(t,x,\dot{x})}{Q(t,x,\dot{x})}, \]  

and the associated differential form

\[ \frac{P}{Q} dt - d\dot{x}. \]  

By adding a differential form of \(S(t,x,\dot{x})(\dot{x}dt - dx)\) to the form (15) where \(S\) is an unknown function, it is possible to consider the equation of the form

\[ \left(\frac{P}{Q} + S\dot{x}\right)dt - (Sdx + d\dot{x}). \]  

The modified Prelle-Singer method is based on finding a function \(S\) such that the differential form (16) is proportional to the differential form

\[ dI = It dt + Ix dx + I\dot{x} d\dot{x}, \]  

for some function \(I(t,x,\dot{x})\). This means that there exists a function \(R\) such that

\[ dI = R\left(\frac{P}{Q} + S\dot{x}\right)dt - (Sdx + d\dot{x}). \]  

The existence of the functions \(S, I\) and \(R\) satisfying (17) implies that

\[ I_t = R\left(\frac{P}{Q} + S\dot{x}\right), \quad I_x = -RS, \quad I\dot{x} = -R. \]  

The compatibility conditions for systems (18) yield

\[ A(S) = -\phi_x + S\phi_x + S^2, \quad A(R) = -R(S + \phi_x), \quad R_x = R_x S + RS_x, \]  

where \(\phi = \frac{P}{Q}\) and \(A\) is the operator associated with equation (15).

Furthermore, assuming the existence of a Hamiltonian one can write

\[ I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x}) \]  

where \(L(x,\dot{x})\) is the Lagrangian and \(p\) is the canonically conjugate momentum, thus we have

\[ \frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}} \dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}} \dot{x}. \]  

From equation (21) we identify the conjugate momentum

\[ p = \int \frac{I_x}{\dot{x}} d\dot{x} + f(x), \]  

where \(f(x)\) is an arbitrary function of \(x\). We take \(f(x) = 0\) and substituting the first integral \(I\) into equation (22) and integrating it, we can obtain the expression for the canonical momentum \(p\).

3. Noether symmetries of Ermakov-Pinney equation. We now consider Noether symmetry classification of the Ermakov-Pinney equation \[12, 27\]

\[ \ddot{x} + \omega^2(t)x = \frac{1}{x^3}, \]  

in which, \(x\) is the spatially dependent field of interest and \(\omega^2(t)\) is the external potential and the over dot denotes differentiation with respect to the independent variable \(t\).

For the Ermakov-Pinney equation (23), we can write the Euler-Lagrange operator

\[ \frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - D_t \frac{\partial}{\partial x_t^\alpha} + D_t^2 \frac{\partial}{\partial x_{tt}^\alpha}, \]  

(24)
and the Lagrangian \( L \) for the Ermakov-Pinney equation (23) can be written as
\[
L = \frac{1}{2} \ddot{x}^2 - \frac{1}{2} \omega^2(t) x^2,
\]
and if we apply Euler-Lagrange operator (24) to Lagrangian (25), then we obtain
\[
\frac{\delta L}{\delta x} = -x\omega^2(t) - \ddot{x}.
\]
In addition, if we rewrite the Ermakov-Pinney equation in the following form
\[
-x\omega^2(t) - \ddot{x} = -\frac{1}{x^3},
\]
then, the equation (26) becomes
\[
\frac{\delta L}{\delta x} = -\frac{1}{x^3}.
\]

In relation (11), the Lagrangian (25) has at most first order derivatives and then we can take \( \alpha = 1 \) and write the following definition
\[
W^1 = (\eta - \xi \dot{x})(-\frac{1}{x^3}) = \frac{\xi \dot{x}}{x^3} - \frac{\eta}{x^3},
\]
and \( D_t(B) \) is defined in the form
\[
D_t(B) = B_t + \dot{x}B_x.
\]

By application of the first prolongation of the generalized operator (8) \( X \) to Lagrangian (25), we get
\[
X L = \xi (-\omega(t)\dot{\omega}(t)x^2) + \eta (-x\omega^2(t)) + \eta^1 \dot{x},
\]
where \( \eta^1 \) is defined in the form
\[
\eta^1 = \eta_t + (\eta_x - \xi_t)\dot{x} - \xi_x(\dot{x})^2.
\]

The expansion of (11) by using the definition of the first prolongation of the Noether operator and relations (29)-(32) is written as below
\[
\eta \ddot{x} + \eta_x \dot{x}^2 - \xi \omega(t)\dot{\omega}(t)x^2 - \eta x\omega^2(t) - \xi_t \dot{x}^2 - \xi_x \dot{x}^3 + \frac{1}{2} \xi_t \dot{x}^2 - \frac{1}{2} \xi_t \dot{x}^2 \omega^2(t) + 
\]
\[
\frac{1}{2} \xi_x \dot{x} x^3 - \frac{1}{2} \xi_x \dot{x}^2 (t)x^2 + \frac{\eta}{x^3} - \frac{\dot{x}}{x^3} \omega^2(t)x - B_t - \dot{x}B_x = 0.
\]

The usual separation by powers of derivatives of \( x \) (33) reduces to the following determining equations
\[
-\frac{1}{2} \xi_x = 0, \quad \eta_x - \frac{1}{2} \xi_t = 0, \quad \eta_t - \frac{1}{2} \xi_x \omega^2(t) - \frac{\xi}{x^3} - B_x = 0,
\]
\[
\frac{\eta}{x^3} - \xi \omega(t)\dot{\omega}(t)x^2 - \eta x\omega^2(t) - \frac{1}{2} \xi_t \omega^2(t)x^2 - B_t = 0.
\]

To find the infinitesimals \( \xi \) and \( \eta \), the determining equations (34) should be solved together. First, from the solution of the equation (34) we have \( \xi = a(t) \) where \( a(t) \) is a function of \( t \). The solution of equation (34) is
\[
\eta = \frac{1}{2} \dot{a}(t)x + b(t),
\]
where \( b(t) \) is a function of \( t \). Thus, if we differentiate (34) with respect to \( t \) and \( \xi = a(t) \) with respect to \( x \) then we can eliminate the function \( B(t, x) \) from equations (34) and obtain the following equation

\[
\frac{3b(t)}{x^4} + \omega^2(t)b(t) + 2x\omega^2(t)\dot{a}(t) + 2xa(t)\omega(t)\dot{\omega}(t) + \ddot{b}(t) + \frac{1}{2}x\dddot{a}(t) = 0, \tag{36}
\]

which is a differential equation including unknown functions \( \omega(t), a(t) \) and \( b(t) \). Using these equations one can classify Noether symmetries and corresponding first integrals of the nonlinear Ermakov-Pinney equation (23) based on different forms of the external potential and differential relations for \( a(t) \) and \( b(t) \).

**Case 1.  \( \omega(t) = k(constant) \)**  In equation (36) if we consider \( \omega(t) = k \) then we obtain the following differential equation

\[
(k^2 + \frac{3}{x^4})b(t) + 2k^2x\dot{a}(t) + \ddot{b}(t) + \frac{1}{2}x\dddot{a}(t) = 0. \tag{37}
\]

In (36) it is clear that \( 2k^2\dot{a}(t) + \frac{1}{2}\dddot{a}(t) = 0, \) and the functions \( a(t) \) and \( b(t) \) are found

\[
a(t) = c_3 - \frac{c_2\cos(2kt)}{2k} + \frac{c_1\sin(2kt)}{2k} \quad \text{and} \quad b(t) = 0. \tag{38}
\]

From the solutions of \( a(t) \) and \( b(t) \) we obtain the following infinitesimal functions

\[
\xi = c_3 - \frac{c_2\cos(2kt)}{2k} + \frac{c_1\sin(2kt)}{2k}, \quad \eta = \frac{1}{2}x(c_1\cos(2kt) + c_2\sin(2kt)), \tag{39}
\]

and the corresponding Noether symmetries are

\[
X_1 = \frac{\sin(2kt)}{2k} \frac{\partial}{\partial t} + \frac{1}{2}x\cos(2kt) \frac{\partial}{\partial x}, \\
X_2 = \frac{1}{2}x \sin(2kt) \frac{\partial}{\partial x} - \frac{\cos(2kt)}{2k} \frac{\partial}{\partial t}, \\
X_3 = \frac{\partial}{\partial t}. \tag{40}
\]

By using relations (30) and (31) the function \( B(t, x) \)

\[
B(t, x) = \frac{1}{4kx^2}(2kc_3 + (2k^2x^4 - 1)c_2\cos(2kt) + (1 - 2k^2x^4)c_1\sin(2kt)), \tag{41}
\]

is determined, where \( c_i, i = 1, 2, 3 \) are constants. Thus, the first integrals (conserved forms) for the Ermakov-Pinney equation (23) can be determined by using expression (23) and by considering each group parameter \( c_i \).

\[
I_1 = \frac{1}{2}\cos(2kt)x\dot{x} + \frac{\sin(2kt)(k^2x^4 - x^2\dot{x}^2 - 1)}{4kx^2}, \\
I_2 = \frac{2k\sin(2kt)x^3\dot{x} + \cos(2kt)(1 - k^2x^4 + x^2\dot{x}^2)}{4kx^2}, \\
I_3 = -\frac{2k + 2k^3x^4 + 2kx^2\dot{x}^2}{4kx^2}. \tag{42}
\]

For this choice, the Lagrangian and the Hamiltonian functions are

\[
L = \frac{1}{2}x^2 - \frac{1}{2}k^2(t)x^2 \quad \text{and} \quad H = \frac{1}{2}(p^2 + k^2x^2), \tag{43}
\]

where the conjugate momentum is \( p = \dot{x} \). And the invariant solution corresponding to first integral \( I_3 \) is

\[
x(t) = -\frac{(-1)^{\frac{k}{8}}e^{-ik(t-c_1)}\sqrt{4 + e^{4ik(t-c_1)}}}{2\sqrt{k}}. \tag{44}
\]
Case 2. $\omega(t) = t^n$, $n \geq 1$ In equation (36) if we consider $\omega(t) = t^n$, then we obtain
\[ 2nt^{2n-1}xa(t) + (t^{2n} \frac{3}{n^4})b(t) + 2t^{2n}x\dot{a}(t) + \ddot{b}(t) + \frac{1}{2}x\ddot{u}(t) = 0, \] (45)
which gives $a(x) = 0$ and $b(x) = 0$. Then $\xi, \eta$, and the gauge function are determined as
\[ \xi = 0, \quad \eta = 0, \quad B(t, x) = c_1, \] (46)
where $c_1$ is a constant. The corresponding first integral is $I = c_1$. For this choice, the Lagrangian and Hamiltonian functions are
\[ L = \frac{1}{2}(\dot{x}^2 - t^{2n}x^2) \quad \text{and} \quad H = \frac{1}{2}(p^2 + t^{2n}x^2), \] (47)
where the conjugate momentum is $p = \dot{x}$.

Case 3. $\omega(t)$ is arbitrary From equation (36), we have $b(t) = 0$. And the equation (36) becomes,
\[ 2xa(t)\omega(t)\dot{\omega}(t) + 2x\omega(t)^2\dot{a}(t) + \frac{1}{2}x\ddot{a}(t) = 0. \] (48)
And if we solve the function $\omega(t)$ in terms of $a(t)$ one can find
\[ w(t) = \frac{\sqrt{4c_1 + \dot{a}(t)^2 - 2a(t)\ddot{a}(t)}}{2a(t)}, \] (49)
For this case it is clear that the infinitesimal functions are
\[ \xi = c_2, \quad \eta = 0, \quad B(t, x) = \frac{c_2}{2x^2} + c_3, \] (50)
where $c_1, c_2, c_3$ are constants and the generator is
\[ X = \frac{\partial}{\partial t}, \] (51)
and the first integral is
\[ I = -\frac{1}{2}(\frac{1}{x^2} + \dot{x}^2). \] (52)
For this choice, the Lagrangian is
\[ L = \frac{1}{2}x^2 - \frac{1}{2}\left(\frac{\sqrt{4c_1 + \dot{a}(t)^2 - 2a(t)\ddot{a}(t)}}{2a(t)}\right)^2x^2, \] (53)
the Hamiltonian function corresponding to Lagrangian is
\[ H = \frac{1}{2}\left(\frac{\sqrt{4c_1 + \dot{a}(t)^2 - 2a(t)\ddot{a}(t)}}{2a(t)}\right)(p^2 + x^2), \] (54)
where the conjugate momentum is $p = \dot{x}$.

4. The extended Prelle-Singer method for Ermakov-Pinney equation. In this section we consider other types of the first integrals by using the Prelle-Singer procedure. This method provides not only the first integrals but also integrating factors. Moreover, one can define the Hamiltonian and Lagrangian forms of the differential equations by using this method. For this purpose we consider the first integrals and exact solutions of the Ermakov-Pinney equation by the approach related with the Prelle-Singer and $\lambda$-symmetry.
4.1. The time-independent first integrals of Ermakov-Pinney equation.
For the Ermakov-Pinney (23) one can write
\[ \phi = -\omega^2(t)x + \frac{1}{x^3}. \] (55)
If this equation has a first integral \( I(t, x, \dot{x}) = C \), with a constant \( C \), then the total differential for the first integral can be written
\[ dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0. \] (56)
Substituting equation (56) in the formula \( \phi dt - d\dot{x} = 0 \) and adding a null term \( S(t, x, \dot{x})\dot{x}dt - S(t, x, \dot{x})dx \), we obtain the following relation
\[ (\phi + S\dot{x})dt - Sdx - d\dot{x} = 0. \] (57)
Multiplying (56) by the factor \( R(t, x, \dot{x}) \) is named the integrating factor, hence we obtain
\[ dI = R(\phi + S\dot{x})dt - RSdx - R\dot{x} = 0. \] (58)
It is clear that equations (56) and (58) yield the following relations
\[ I_t = R(\phi + S\dot{x}), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \] (59)
Then using the compatibility conditions, namely \( I_{tx} = I_{xt}, I_{\dot{z}} = I_{\dot{z}t}, I_{\dot{z}\dot{x}} = I_{\dot{x}\dot{z}} \), (59) provide us the following system of coupled nonlinear differential equations in terms of \( S, R \) and \( \phi \)
\[ S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}} + S + S^2, \] (60)
\[ R_t + \dot{x}R_x + \phi R_{\dot{x}} = -\phi_x + S + S^2, \] (61)
\[ R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0, \] (62)
where the last equation (62) is called compatibility equation. In addition one can determine the first integral \( I \) by using \( R \) and \( S \) functions with the following relations
\[ I = r_1 - r_2 - \int [R + \frac{d}{dx}(r_1 - r_2)]d\dot{x}, \] (63)
where
\[ r_1 = \int R(\phi + \dot{x}s)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx. \] (64)
First of all, we consider the time-independent first integral case, that is \( I_t = 0 \). One can easily find \( S \) from the first equation in (59) like this
\[ S = \frac{-\phi}{\dot{x}} = \frac{x^3\omega^2 - 1}{x^3\dot{x}}, \] (65)
for \( \phi \) in (55). Substituting this form of \( S \) into equation (61) we get
\[ R_t + \dot{x}R_x + (-\omega^2 x + \frac{1}{x^3})R_x + R(\frac{x^3\omega^2 - 1}{x^3\dot{x}}) = 0. \] (66)
The equation (66) is a first order linear partial differential equation. To solve this equation we assume \( R \) of the form
\[ R = \frac{\dot{x}}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \] (67)
where \( A(x), B(x) \) and \( C(x) \) are functions of \( x \) and \( r \) is a constant. If we substitute (67) into the equation (66), then we obtain a set of equations in terms of \( \dot{x} \) and its powers. From the solutions of these equations, the function \( R \)
\[ R = \dot{x} \left( c_1 + c_3\dot{x}^2 + c_3(\frac{1}{x^2} + x^2\omega^2) \right)^{-r}, \] (68)
is found and if we substitute the functions \( R(68) \) and \( S(65) \) into the equations (60)-(62), it is possible to check that these equations are satisfied. Thus, one can determine the first integral of the Ermakov-Pinney equation from the relation (63)

\[
I = \frac{(c_3 + c_1 x^2 + c_3 x^2 \dot{x}^2 + c_3 x^4 \omega^2)(c_1 + c_3 (\frac{1}{x^2} + \dot{x}^2 + x^2 \omega^2))^{-r}}{2c_3(r - 1)x^2}.
\]

Furthermore, one can determine the corresponding conjugate momentum related with the first integral (69) for \( r = -2 \),

\[
p = \frac{1}{x^4} \left( \frac{2}{3} c_3^2 x^4 \dot{x}^5 + \frac{1}{5} c_3 x^2 \dot{x}^3 (c_3 + c_1 x^2 + c_3 x^4 \omega^2) + \dot{x} (c_3 + c_1 x^2 + c_3 x^4 \omega^2)^2 \right),
\]

then the corresponding Hamiltonian form related with the first integral (69),

\[
H = -\frac{(c_3 + c_1 x^2 + c_3 x^2 \dot{x}^2 + c_3 x^4 \omega^2)(c_1 + c_3 (\frac{1}{x^2} + \dot{x}^2 + x^2 \omega^2))^2}{3c_3 x^2}.
\]

4.2. \( \lambda \)-symmetries determined from Lie point symmetries and the connection with Prelle-Singer method. The relationship between \( \lambda \)-symmetries, Lie point symmetries for the determination of the integrating factors and first integrals of second-order ordinary differential equation is very important from the mathematical point of view. If Lie point symmetries are known then one can easily construct \( \lambda \)-symmetries from them. Firstly, we suppose that the equation has Lie symmetries then we determine \( \lambda \)-symmetries and find corresponding first integrals by using relation between the null function \( S \) and \( \lambda \)-symmetries.

Now, we first follow an algorithm that gives \( \lambda \)-symmetries from Lie symmetries. For \( \dot{x} = \phi(t, x, \dot{x}) \), the vector field be in the form of

\[
A = \partial_t + \dot{x} \partial_x + \phi(t, x, \dot{x}) \partial_{\dot{x}}.
\]

In terms of \( A \), a first integral is any function in the form of \( I(t, x, \dot{x}) \) providing equality of \( A(I) = 0 \). Additionally, an integrating factor of equation is any function satisfying the following equation \( \mu [\dot{x} - \phi(t, x, \dot{x})] = D_t I \), where \( D_t \) is total derivative operator in the form of \( D_t = \partial_t + \dot{x} \partial_x + \ddot{x} \partial_{\ddot{x}} \). Thus \( \lambda \)-symmetries of second order differential equation can be obtained directly by using Lie symmetries of this same equation. Secondly, let \( v = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x} \) be a Lie point symmetry, where \( \xi(t, x) \) and \( \eta(t, x) \) are functions of their arguments. Then the characteristic function for \( v \) is \( Q = \eta - \xi \dot{x} \) and \( \lambda \)-symmetry is given by \( \lambda = \frac{\xi(Q)}{Q} \).

**Proposition 1.** Suppose that \( v = \partial_x \) is a \( \lambda \)-symmetry. If \( I \) is a first integral, then \( \mu = I_t \) is an integrating factor and \( -\mu \phi = I_t + \dot{x} I_{\dot{x}} \). The compatibility conditions for system (59) imply

\[
A(S) = -\phi_x + S \phi_{\dot{x}} + S^2, \quad A(R) = -R(S + \phi_{\dot{x}}), \quad R_x = R_{\dot{x}} S + R S_{\dot{x}}.
\]

The first equation in (74) yields that \( v = \partial_x \) is a \( \lambda \)-symmetry for \( \lambda = -S \). By writing the second and third equations of \( \lambda = -S \) (74) in terms of \( \lambda \) we obtain \( \mu = -R \). This reveals a role of \( \lambda = -S \) on the integration of equation, if we add \(-\lambda (\dot{x} dt - dx) \) to the differential form \( \phi dt - d\dot{x} \), then the resulting differential form
\[ \phi dt - d\dot{x} - \lambda (\dot{x} dt - dx) \] admits an integrating factor \( \mu \). Thus it can be said that the function \( S \) corresponds to Lie point symmetries of the equation, \( S = -\frac{\mu Q}{Q} \), where \( Q \) is the characteristic of some Lie point symmetry of the equation.

Now, we apply this algorithm to Ermakov-Pinney equation to obtain nontrivial \( \lambda \) and Lie symmetries.

**Proposition 2.** Using compatibility conditions (74), \( \lambda \)-symmetries corresponding to Lie point symmetries and the first integrals can be determined.

**Proof.** For the function \( \phi \) defined by (55) the operator \( A \)

\[
A = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} - \left( -\omega^2 x + \frac{1}{x^3} \right) \frac{\partial}{\partial x},
\]

is computed and from the relation \( \lambda = \frac{A(Q)}{Q} \), the \( \lambda \)-symmetry is found as below

\[
\lambda = \frac{(\omega^2 - \frac{1}{x^2})\xi + \dot{x} \eta}{\eta - \dot{x} \xi}.
\]

Using the relation between the functions \( S \) and \( \lambda \), that is \( S = -\lambda \), one can determine the null function \( S \) that satisfies equation (60), then if we substitute this form of \( S \) function into (60) we obtain over-determined systems of partial differential equations. From the solutions, one can determine easily infinitesimal functions \( \xi \) and \( \eta \) as \( \xi = 1 \) and \( \eta = 0 \) and the classical Lie point symmetry of Ermakov-Pinney equation as \( X = \frac{\partial}{\partial t} \). Thus, if we substitute the functions \( \xi \) and \( \eta \) into (76), we obtain

\[
S = -\lambda = -\frac{(\omega^2 - \frac{1}{x^2})\xi + \dot{x} \eta}{\eta - \dot{x} \xi}.
\]

To find a first integral \( w(t, x, \dot{x}) \) of \( v^{[\lambda, 1]} \), we consider a particular solution of the equation \( w_x + \lambda w_x = 0 \), where \( v^{[\lambda, 1]} \) is the first order \( \lambda \)-prolongation of the vector field \( v \). From the solution of this equation, the first integral is

\[
I = \frac{1 + x^2 \dot{x}^2 + x^4 \omega^2}{2x^2},
\]

and the corresponding integrating factor \( \mu = \dot{x} \). Since \( \mu = -R \) we have \( R = -\dot{x} \). In fact, it is possible to show that the functions \( S \) and \( R \) satisfy equations (60)-(62). Furthermore, one can determine the canonical conjugate momentum, the corresponding Lagrangian and the Hamiltonian functions related with the first integral (81) as follows

\[
p = \dot{x}, \quad L = -\frac{1 - x^2 \dot{x}^2 + x^4 \omega^2}{2x^2}, \quad H = \frac{1 + x^2 \dot{x}^2 + x^4 \omega^2}{2x^2}.
\]

Infinitesimals of Ermakov-Pinney equation are \( \xi = \sin(2\omega t) \) and \( \eta = \omega x \cos(2\omega t) \) and the corresponding classical Lie point symmetry of Ermakov-Pinney equation is \( X = \sin(2\omega t) \frac{\partial}{\partial t} + \omega x \cos(2\omega t) \frac{\partial}{\partial x} \). If we substitute the functions \( \xi \) and \( \eta \) into (76) we obtain \( \lambda \)-symmetry

\[
\lambda = \frac{2\dot{x} \omega \cos(2\omega t) + \frac{x^4 \omega^2 - 1}{x^4} \sin(2\omega t)}{2x \omega \cos(2\omega t) - \dot{x} \sin(2\omega t)}.
\]

In additional, for the infinitesimals \( \xi = \cos(2\omega t) \) and \( \eta = -\omega x \sin(2\omega t) \), Lie point symmetry of Ermakov-Pinney equation is \( X = \cos(2\omega t) \frac{\partial}{\partial t} - \omega x \sin(2\omega t) \frac{\partial}{\partial x} \).
and the $\lambda$-symmetry corresponding to this Lie symmetry
\[ \lambda = -\frac{(x\omega^2 - \frac{1}{x})\cos(2\omega t) - 2i\omega \sin(2\omega t)}{\dot{x} \cos(2\omega t) + 2i\omega \sin(2\omega t)}, \] (81)
is found. 

5. Conclusion. In this study we analyze conservation laws of nonlinear Ermakov-Pinney equation, which is second order nonlinear ordinary differential equation. We consider the application of the Lagrangian approach for the classification in this problem. For different external potential functions we obtain Noether point symmetry algebras. We here deal with the corresponding first integrals and some invariant solutions for each case. In addition, we determine the time-independent first integrals by using the modified Prelle-Singer approach and present the mathematical relations between these approaches in terms of the symmetry transformations namely, Lie point, and $\lambda$-symmetries. Moreover, we construct the appropriate Lagrangian and Hamiltonian forms from the time independent first integrals. The Prelle-Singer procedure is based on the fact that one can determine the explicit solutions satisfying all three determining equations (60)-(62). In our study, we consider specific ansatz forms to determine the null forms $S$, and integrating factor $R$. Using this procedure, $\lambda$ symmetries from Lie symmetries of Ermakov-Pinney equation are obtained. Finally, the corresponding first integrals, exact solutions and Hamiltonian forms are found from $\lambda$ symmetries of the equation.

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E-mail address: orhanozlem@itu.edu.tr  
E-mail address: tozer@itu.edu.tr