Projective tilings and full-rank perfect codes*

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Abstract

A tiling of a vector space $S$ is the pair $(U, V)$ of its subsets such that every vector in $S$ is uniquely represented as the sum of a vector from $U$ and a vector from $V$. A tiling is connected to a perfect codes if one of the sets, say $U$, is projective, i.e., the union of one-dimensional subspaces of $S$. A tiling $(U, V)$ is full-rank if the affine span of each of $U, V$ is $S$. For finite non-binary vector spaces of dimension at least 6 (at least 10), we construct full-rank tilings $(U, V)$ with projective $U$ (both $U$ and $V$, respectively). In particular, that construction gives a full-rank ternary 1-perfect code of length 13, solving a known problem. We also discuss the treatment of tilings with projective components as factorizations of projective spaces.

Keywords: perfect codes, tilings, group factorization, full-rank tilings, projective geometry

1 Introduction

We are interested in factorizations (also known as tilings) of an elementary abelian group $\mathbb{Z}_p^n$ into the direct sum of two subsets, called tiles, and the connection of such factorizations with 1-perfect codes in Hamming spaces. This connection has been extensively studies in the binary case, where each tiling $(U, V)$ of $\mathbb{Z}_2^n$ corresponds to a 1-perfect binary code of length $|U| - 1$. To establish a similar correspondence (see Proposition 1 below) for an arbitrary prime $p$, one should require an additional property of $U$ to be a $\mathbb{Z}$-set, in the sense of [16]: $\mathbb{Z}_p \cdot U = U$. We study factorizations $(U, V)$ of $\mathbb{Z}_p^n$ with $U$ or both $U$ and $V$ satisfying this property or, more generally, factorizations $(U, V)$ of $\mathbb{F}_q^n$ (where $\mathbb{F}_q$ is the finite field of a prime power order $q$) such that $\mathbb{F}_q \cdot U = U$.

A notable particular consequence of the results of this paper is proving the existence of a ternary 1-perfect code of length 13 and rank 13, which was the last open question in the story of the study of possible values of the rank of 1-perfect codes. The next paragraph contains a brief survey of results in this area.
One-perfect codes in $\mathbb{F}_q^n$ ($q$ prime power) exist if and only if $n = (q^m - 1)/(q-1)$, $m \geq 1$. The case $m = 1$ is trivial and usually not considered. In general, the rank of a perfect code can possess any value from $n - m$ (linear and affine codes) to $n$ (full-rank codes); the exceptions are 1-perfect binary length-3 and length-7 codes and ternary length-4 codes, which are all of rank $n - m$. The construction of 1-perfect linear codes (i.e., of minimum possible rank $n - m$) originates to the work of Hamming [11], while in the most general form, for every $q$ and $m$, it was suggested by Golay [10]. For $q = 2$, $m \leq 3$ [31] and $q = 3$, $m = 2$, 1-perfect codes are unique up to equivalence and affine, which means in our context that there are no codes of non-minimum rank for those parameters. The first known construction of non-linear (i.e., of rank larger than the minimum possible value) binary 1-perfect codes, for any $m \geq 4$, was proposed by Vasil'ev [30]; Schönheim [22] generalized Vasil'ev's construction to any $q \geq 3$ and $m \geq 3$. In [7], Etzion and Vardy constructed binary 1-perfect codes of all admissible ranks, $m \geq 4$. Fraser and Gordon showed in [9] that permuting the symbols of the alphabet in one position of all codewords keeps the parameters of the code but can change its rank if $q \geq 4$ (4 is the first value of $q$ for which there are non-affine permutations of $\mathbb{F}_q$). This allows to construct 1-perfect codes of any admissible rank if $q \geq 4$ and $m \geq 2$; a code constructed in this way is equivalent to a linear code if we naturally consider the alphabet permutations as equivalence operations. In [20], Phelps and Villanueva constructed 1-perfect codes of all admissible ranks for $q \geq 3$ and $m \geq 4$ [20] and for non-prime $q \geq 4$ and $m \geq 3$ [18]. After the results above, only the existence of ternary 1-perfect codes of length $\frac{2^3 - 1}{2 - 1} = 13$ and ranks 12 and 13 remained open. The rank 12 case was solved by Romanov [21]. The current study presents, among other results, a solution for the last open case.

**Remark 1.** The switching technique used in [30], [22], [7], [20], [18] results in codes nonequivalent to a linear code in general. However, it remains unanswered whether we can reduce the rank of a code constructed there, as well as in the current paper if $q \geq 4$, by equivalence operations (probably, not); this question requires further studying. The following fact might be important in this context: if a code $D$ is obtained from a linear code $C$ by switching and the number of switched codewords is relatively small, then the original linear code $C$ can be uniquely reconstructed from $D$ as the closest code that is equivalent to a linear code [1]. It is also notable that no 1-perfect codes that are inequivalent to linear codes are known if $q$ is prime and $m = 2$ (for $q \leq 7$, such codes do not exist [12]).

In this paper, after Section 2 with preliminaries, we obtain the following results.

Firstly, in Section 3 for any field $\mathbb{F}$ of order larger than 2, we construct full-rank tilings $(U, V)$ of $\mathbb{F}^n$ such that $\mathbb{F} \cdot U = U$ ($n \geq 6$ even) or both $\mathbb{F} \cdot U = U$ and $\mathbb{F} \cdot V = V$ hold ($n \geq 10$ even, which improves a construction for $n \geq 12$ in [27]).

Secondly, in Section 4 we construct full-rank 1-perfect codes over $\mathbb{F}_q$, $q \geq 3$, in particular, a full-rank ternary 1-perfect code of length 13. Extending that theoretical result with a computer-aided search, we find such codes with different dimension of the kernel (the set of all periods of the code; constructing codes with different pairs [rank, kernel dimension] is also a known direction in the study of perfect codes [8], [2], [19]).

Additionally, in Section 5 we discuss the connection of projective tilings with factorizations of projective geometries.
For more reading about factorizations of abelian groups, see [28]; for connections of tilings of binary spaces with perfect codes, see [4, 8]; for the problem of existence of full-rank factorizations of abelian groups, see [6].

2 Preliminaries

Let $S$ be a vector space over a field $F$. A pair $(U, V)$ of subsets of $S$ is called a tiling (of $S$) if every element of $S$ is uniquely represented as the sum $u + v$, $u \in U$, $v \in V$. The components $U$ and $V$ of a tiling are called tiles. We are mostly aimed of constructing tilings and codes in the vector space $F_q^n$ of the $n$-tuples over the finite field $F_q$ of order $q$. However, some of the definitions and results have sense for infinite vector spaces as well (see, e.g., [14], [15] for examples of infinite perfect codes). If the reader is not interested in abstract infinite generalizations, it is recommended to think that $F$ is always finite.

The rank, $\text{rank}(U)$, of a subset $U$ of a vector space $S$ is the dimension of the affine span $\langle U \rangle$ of $U$ (we use double and single brackets to distinguish between the affine span $\langle \cdot \rangle$ and the linear span $\langle \cdot \rangle$). If $\langle U \rangle = S$, then $U$ is said to be full-rank. A tiling $(U, V)$ is called full-rank if both $U$ and $V$ are full-rank.

A period of a subset $U$ of $S$ is an element $\bar{v}$ of $S$ such that $U + \bar{v} = U$. The set of all periods of $U$ is denoted by $\text{per}(U)$. (If $F = F_q$ and $p$ is the prime divisor of $q$, then $\text{per}(U)$ is necessarily an $F_p$-subspace of $F_q^n$ and is called the $p$-kernel of $U$.) The maximal subspace in the set of all periods of $U$ is called its kernel, $\ker(U)$ (also known as the $q$-kernel if $F = F_q$).

A set $U$ is called aperiodic if $\text{per}(U) = \{0\}$, where $0$ the all-zero vector. A tiling $(U, V)$ is called aperiodic if both $U$ and $V$ are aperiodic.

A subset $U$ of $S$ is called projective if it is the union of 1-dimensional subspaces of $S$, i.e., if $F \cdot U = U$. A tiling $(U, V)$ is called semiprojective if $U$ is projective. A tiling $(U, V)$ is called projective if both $U$ and $V$ are projective.

In the definitions above, $S$ can be considered as an abstract vector space and the basis does not play any role. This is not the case in the following group of definitions, where we consider the vector space $F^n$ of $n$-tuples over $F$, considered as vectors written is some fixed basis.

The Hamming weight $\text{wt}(x)$ of a vector $x$ in $F^n$ is the number of its nonzero coordinates. The Hamming distance between two vectors $\bar{x}$, $\bar{y}$ in $F^n$ is the number of coordinates in which they differ, i.e., $\text{wt}(\bar{y} - \bar{x})$. The radius-$r$ (Hamming) ball $B_r(\bar{x})$ centered in $\bar{x} \in F^n$ is the set of vectors at distance at most $r$ from $\bar{x}$; we denote $B_r = B_r(0)$.

A subset $C$ of $F^n$ is called an $r$-perfect code if $(B_r, C)$ is a tiling.

Nontrivial ($0 < r < n$) perfect codes over finite fields $F_q$ are known to exist if and only if $q = 2$, $r = \frac{n-1}{2}$, odd $n$ (binary repetition codes), $q = 2$, $r = 3$, $n = 23$ (the binary Golay code), $q = 3$, $r = 2$, $n = 11$ (the ternary Golay code), and $r = 1$, $n = \frac{2^m-1}{q-1}$, $m = 2, 3, \ldots$, for any prime power $q$ [29, 32]. So, all parameters of perfect codes (over fields) are known; however, there are many nonequivalent 1-perfect codes [30, 22] and the class of all such codes is hardly possible to characterize in any constructive terms.

The following proposition connects tilings and perfect codes. Its first claim generalizes [4, Propositions 7.1], where it was shown for $F_2$ (when all tilings are projective),
and also is a special case of [3] Theorem 2.1, where more general objects, coverings, are considered.

**Proposition 1.** Let \((U, V)\) be a semiprojective tiling of a vector space \(S\) over a field \(\mathbb{F}\). Let \(U^*\) be a complete set of mutually non-colinear representatives of \(U\). We assume that \(U^*\) is finite and denote \(N = |U^*|\) (in particular, \(N \cdot (|\mathbb{F}| - 1) = |U| - 1\)). Let the matrix \(H\) be formed by the elements of \(U^*\) as columns. If

\[
C = \{c \in \mathbb{F}^N : Hc \in V\},
\]

then

(i) \(C\) is a 1-perfect code in \(\mathbb{F}^N\),

(ii) \(\text{rank}(C) = \text{rank}(V_U) + N - r\),

(iii) \(\dim(\ker(C)) = \dim(\ker(V_U)) + N - r\), \(|\text{per}(C)| = |\text{per}(V_U)| \cdot |\mathbb{F}|^{N-r}\),

where \(V_U = V \cap \langle U \rangle\) and \(r = \text{rank}(U)\) (if \(U\) is full-rank, then \(V_U = V\) and \(r = \dim(S)\)).

**Proof.** Denote \(\overline{U} = \langle U \rangle\). Since \((U, V)\) is a tiling, every element of \(\overline{U}\) is uniquely represented as the sum of some \(\bar{u}\) from \(U\) and \(\bar{v}\) from \(V\); moreover, trivially \(\bar{u} \in \overline{U}\) and hence \(\bar{v} \in \overline{U}\) and \(\bar{v} \in V_U\). By the definition, \((U, V_U)\) is a tiling of \(\overline{U}\).

Moreover, \(\mathbf{1}\) is equivalent to \(C = \{c \in \mathbb{F}^N : Hc \in V_U\}\) because \(Hc\) belongs to \(\overline{U}\) and cannot belong to \(V \setminus \overline{U}\).

To prove (i), we consider an arbitrary \(\bar{x}\) from \(\mathbb{F}^N\) and denote \(\bar{s} = H\bar{x}\). Since \(\bar{s} \in \overline{U}\) and \((U, V_U)\) is a tiling of \(\overline{U}\), we find that \(\bar{s}\) is uniquely represented in the form \(\bar{s} = \bar{u} + \bar{v}\), \(\bar{u} \in U\), \(\bar{v} \in V_U\). If \(\bar{u} = 0\), then \(\bar{u} = H0\); otherwise, by the definition of the matrix \(H\), \(\bar{u}\) is uniquely represented as a multiple of one of its columns, i.e., \(\bar{u} = H\bar{b}\) for some \(\bar{b}\) in \(B_1 \setminus \{0\}\). In both cases, \(\bar{u} = H\bar{b}\) for some unique \(\bar{b}\) in \(B_1\). Denoting \(\bar{c} = \bar{x} - \bar{b}\), we see that \(H\bar{c} = \bar{y} - \bar{u} = \bar{v} \in V_U\) and hence \(\bar{c} \in C\). So, \(\bar{x} = \bar{c} + \bar{b}\), where \(\bar{c} \in C\) and \(\bar{b} \in B_1\). We also see \(\bar{b}\) from \(B_1\) and hence \(\bar{c}\) from \(C\) are uniquely determined from \(\bar{x}\). Therefore, by the definitions, \((B_1, C)\) is a tiling and \(C\) is a 1-perfect code.

Next, \(H\) acts as a linear operator from \(\mathbb{F}^N\) onto \(\overline{U}\). Hence, it maps \(\langle C \rangle\) to \(\langle V_U \rangle\). Since the preimage (under \(H\)) of every vector in \(\overline{U}\) is an affine subspace of dimension \(N - r\), we find \(\dim(\langle C \rangle) = \dim(\langle V_U \rangle) + (N - r)\) and \(\text{rank}(C) = \text{rank}(V_U) + N - r\), which proves (ii).

(iii) is straightforward from the obvious fact that multiplication by \(H\) maps \(\ker(C)\) to \(\ker(V_U)\) and \(\text{per}(C)\) to \(\text{per}(V_U)\). \(\square\)

**Remark 2.** The theorem above shows the connection of 1-perfect codes and projective tilings. Tilings of a vector space over \(\mathbb{F} = \mathbb{F}_q\), where \(q\) is a power of a prime \(p\), are essentially factorizations of an elementary \(p\)-group. As shown in [24], some perfect codes can be constructed from factorizations of non-elementary abelian \(p\)-groups.
3 Constructing full-rank semiprojective and projective tilings

The construction of a tiling in the proofs of the following two theorems exploits the idea from Step 1 of the proof of [26, Theorem 2] (see also [5, Theorem 2.5], where that proof is given in a notation close to ours and for any $m \geq 3$), but we modified it to make $U$ (both $U$ and $V$, in Theorem 2) projective.

**Theorem 1.** If $F$ is a field of cardinality $|F|$ larger than 2 and $m$ is an integer, $m \geq 3$, then there is a full-rank aperiodic semiprojective tiling $(U, V)$ of $F^{2m}$ with $|U| = |V| = |F|^m$.

**Proof.** Let $(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_m)$ be the natural basis of $F^{2m}$. For convenience, we identify $\bar{x}_{m+i} = \bar{x}_i$ and $\bar{y}_{m+i} = \bar{y}_i$. We start with constructing a non-full-rank semiprojective tiling $(H, V)$. Let

$$H = \langle \bar{x}_1, \ldots, \bar{x}_m \rangle, \quad V = \langle \bar{y}_1 \rangle + \cdots + \langle \bar{y}_m \rangle,$$

It is not difficult to see that $(H, V)$ is a tiling (indeed, $V$ is obtained from $\langle \bar{y}_1, \ldots, \bar{y}_m \rangle$ by adding periods of $H$ to some elements). It is easy to see that $V$ is full-rank and aperiodic (see (iii) and (v) below), and it remains to modify $H$ with making it full-rank and aperiodic too (but keeping projective). The modification is based on the following fact.

(*) If $L$ is a subset of $F^{2m}$ with a period $\bar{x}_i$, then $L + V = (L + \gamma \bar{y}_i) + V$ for every $\gamma$ in $F$. To prove it, we assume $i = 1$ without loss of generality. Since $\bar{x}_1$ is a period of $L$, subtracting it from some elements of $V$ does not change the sum $L + V$. Hence,

$$L + V = L + V',$$

where $V' = \langle \bar{y}_1 \rangle + V_2 + \cdots + V_m$. Now we see that $\gamma \bar{y}_1$ is a period of $V'$, and so

$$L + V' = L + \gamma \bar{y}_1 + V'.$$

Again, $\bar{x}_1$ is a period of $L + \gamma \bar{y}_1$, and

$$L + \gamma \bar{y}_1 + V' = L + \gamma \bar{y}_1 + V,$$

Equalities (2)–(4) prove (*).

We construct $U$ by modifying $H$ in the following manner. For $i \in \{1, \ldots, m\}$ and $\gamma \in F \setminus \{0\}$, denote

$$H_{i,\gamma} = \langle \bar{x}_i \rangle + \gamma \bar{x}_{i+1}$$

and

$$U_{i,\gamma} = \langle \bar{x}_i \rangle + \gamma \bar{y}_i + \gamma \bar{x}_{i+1};$$

it is not difficult to observe that all these sets are mutually disjoint. Then, define

$$U = H \setminus \left( \bigcup_{i,\gamma} H_{i,\gamma} \right) \cup \left( \bigcup_{i,\gamma} U_{i,\gamma} \right)$$
(the unions are over \( i \in \{1, \ldots, m\} \) and \( \gamma \in F\backslash\{0\} \)). Now, we can claim that \((U, V)\) is a required full-rank aperiodic semiprojective tiling.

(i) \( U \) is projective because \( H, \{0\} \cup \bigcup_{\gamma \in F\backslash\{0\}} H_{i, \gamma} \), and \( \{0\} \cup \bigcup_{\gamma \in F\backslash\{0\}} U_{i, \gamma} \) are projective, \( i = 1, \ldots, m \).

(ii) \( U \) is full-rank. Indeed, it is easy to see that \( H \setminus \bigcup_{i, \gamma} H_{i, \gamma} \) still spans \( H \) (it includes the subset \((F\backslash\{0\})^m\), for example), while each \( U_{i, 1} \) adds \( y_i \) to the span, \( i = 1, \ldots, m \).

(iii) \( V \) is full-rank because \( \langle V \rangle = \langle \bar{x}_i, y_i \rangle, i = 1, \ldots, m \).

(iv) \( U \) is aperiodic. Indeed, consider the intersections of \( U \) with the cosets of the subspace \( H \). There are three types of such intersections: \( U \cap H; U \cap (H + \gamma y_i) = U_{i, \gamma}, i \in \{1, \ldots, m\}, \gamma \in F\backslash\{0\} \); and the empty intersections. Since \( U \cap H \) is the only intersection that is neither a line nor empty, we see that \( U \) has no periods out of \( H \). On the other hand, the two lines \( U_{1, 1} = U \cap (H + y_1) \) and \( U_{2, 1} = U \cap (H + y_2) \) has no common nonzero periods; hence, \( U \) has no nontrivial periods in \( H \).

(v) \( V \) is aperiodic. Indeed, the coset \( \bar{x}_1 + \ldots + \bar{x}_m + Y \) of \( Y = \langle y_1, \ldots, y_m \rangle \) contains only one element of \( V \); hence, there are no nontrivial periods of \( V \) in \( Y \). On the other hand, there are no other cosets of \( Y \) that intersect with \( V \) in exactly one element; this certifies that \( V \) has no periods out of \( Y \).

(vi) \((U, V)\) is a tiling. Indeed, \((H, V)\) is a tiling, and from (*) we see \( H_{i, \gamma} + V = U_{i, \gamma} + V \). Hence, replacing \( H_{i, \gamma} \) by \( U_{i, \gamma} \) does not change the tiling property.

It is quite natural to ask if the construction can be strengthened with making both tiles projective. The following variant of the construction do this for \( m \geq 5 \), while the existence of full-rank projective tilings of \( F_q^n \) for \( n < 10 \) remains an open challenging problem. Another construction of full-rank projective tilings of \( F_q^{2m} \) for \( m \geq 6 \) was suggested in [27].

**Theorem 2.** If \( F \) is a field of cardinality \(|F|\) larger than 2 and \( m \) is an integer, \( m \geq 5 \), then there is a full-rank aperiodic projective tiling \((U, V)\) of \( F^{2m} \) with \(|U| = |V| = |F|^m \)

**Proof.** The idea is the same as in the proof of Theorem 1 but now we cannot represent \( V \) as the sum \( V_1 + \ldots + V_m \) (indeed, if all \( V_i \) are projective, then such sum is a proper subspace of \( F^{2m} \) and \( V \) is not full-rank). Let \( B = \langle \bar{y}_1, \ldots, \bar{y}_m \rangle \), and we define

\[
V = \{ v(\bar{z}) : \bar{z} \in B \},
\]

where the function \( v(\cdot) \) is defined as

\[
v(0, \ldots, 0, z_1, \ldots, z_m) = ((z_2=0)z_1;0, (z_3=0)z_2;0, \ldots, (z_1=0)z_m;0, z_1, \ldots, z_m),
\]

where \( v?\lambda:\mu \) is a short notation for “if \( v \) then \( \lambda \) else \( \mu \). It is not difficult to see that for a 1-dimensional subspace \( L \) of \( B \), its image \( v(L) \) is also a 1-dimensional space (indeed, for each \( i \), the condition \( z_i = 0 \) is either constantly true or constantly false on all non-zero elements of \( L \), and so \( v(\cdot) \) is just a linear function on \( L \)). Hence, \( V \) is projective.

To define \( U \), we keep the notation \( H = \langle \bar{x}_1, \ldots, \bar{x}_m \rangle \) and

\[
U = H \setminus \left( \bigcup_{i, \gamma} H_{i, \gamma} \right) \cup \left( \bigcup_{i, \gamma} U_{i, \gamma} \right),
\]
but now

\[ H_{i,\gamma} = \langle \bar{x}_i, \bar{x}_{i+1} \rangle + \gamma \bar{x}_{i+2} \]

and \[ U_{i,\gamma} = \langle \bar{x}_i, \bar{x}_{i+1} \rangle + \gamma \bar{x}_{i+2} + \gamma \bar{y}_{i+1}. \]

For \( m \geq 5 \), all \( H_{i,\gamma} \) and \( U_{i,\gamma} \) are mutually disjoint (which is not true for \( m = 3, 4 \), and this is the only reason why we require \( m \geq 5 \)). The key observation to prove that \((U, V)\) is a tiling is now the following:

\((*)\) if \( L \) is a subset of \( \mathbb{F}^{2m} \) with periods \( \bar{x}_i \) and \( \bar{x}_{i+1} \), then \( L + V = (L + \gamma \bar{y}_{i+1}) + V \) for every \( \gamma \) in \( \mathbb{F} \). The proof of \((*)\) essentially repeats the similar for Theorem 1. We assume \( i = 1 \). Since \( \bar{x}_1 \) and \( \bar{x}_2 \) are periods of \( L \), varying the values in the first two coordinates of vectors of \( V \) does not change the sum \( L + V \). Hence,

\[ L + V = L + V', \quad (5) \]

where \( V' = \{v'(\bar{z}) : \bar{z} \in B\} \),

\[ v'(0, \ldots, 0, z_1, \ldots, z_m) = (0, 0, (z_1=0)?z_3:0, \ldots, (z_1=0)?z_m:0, z_1, z_2, z_3, \ldots, z_m) \].

Now, varying the value of \( z_2 \), we see that \( \gamma \bar{y}_2 \) is a period of \( V' \), and so

\[ L + V' = L + \gamma \bar{y}_2 + V'. \quad (6) \]

Again, \( \bar{x}_1 \) and \( \bar{x}_2 \) are periods of \( L + \gamma \bar{y}_2 \), and

\[ L + \gamma \bar{y}_2 + V' = L + \gamma \bar{y}_2 + V. \quad (7) \]

Equalities \((5)-(7)\) prove \((*)\).

The rest of the proof of the theorem is similar to that of Theorem 1.

4 Full-rank perfect codes

From Theorem 1 and Proposition 1, we have the following.

**Corollary 1.** In \( \mathbb{F}_q^n \), \( n = \frac{q^m - 1}{q - 1} \), \( q \geq 3, m \geq 3 \), there is a full-rank 1-perfect code with the dimension \( n - 2m \) of the kernel.

As was noted in the introduction, the existence of full-rank 1-perfect codes in \( \mathbb{F}_3^{13} \) was an open problem, and by Corollary 1 we can construct such code with kernel dimension 7. Starting with that code and using the switching approach, full-rank 1-perfect codes in \( \mathbb{F}_3^{13} \) with kernel dimensions 6, 5, 4, and 3 were found by computer search (the two-coordinate switchings described in [23] was applied, while the traditional one-coordinate switching did not work for the given starting code in the meaning that the switched code was always equivalent to the original one).

**Theorem 3.** For each \( k \) from 3, 4, 5, 6, 7, there is a full-rank ternary 1-perfect code of length 13 with kernel dimension \( k \).

The examples of codes are available in [13]. As follows from [17], 7 is the maximum value of kernel dimension for full-rank 1-perfect codes in \( \mathbb{F}_3^{13} \), and so only the existence of codes of kernel dimension less than 3 remains unsolved for these parameters (both for full-rank codes and for codes of rank 12, see the rank–kernel table in [23]).
5 Factorizations in point-line geometries

This section does not contain any results apart from presenting a geometrical treatment of projective tilings in terms of factorizations of projective spaces. This interesting reformulation seems to be never observed before.

Let \( G \) be a projective space. We will say that two disjoint sets \( U, V \) of points of \( G \) form a factorization \((U, V)\) of \( G \) if for every point \( \bar{x} \) not in \( U \cup V \) there are unique points \( \bar{u} \) in \( U \) and \( \bar{v} \) in \( V \) such that \( \bar{x} \in \langle \bar{u}, \bar{v} \rangle \), where \( \langle \cdot \rangle \) is the minimum projective subspace that contains the points inside the brackets (in particular, \( \langle \bar{u}, \bar{v} \rangle \) is a line if \( \bar{u} \neq \bar{v} \)).

A set \( U \) of points is full-rank if \( \langle U \rangle = G \). A factorization \((U, V)\) is full-rank if both \( U \) and \( V \) are full-rank.

A point \( \bar{u} \) is called a period of a set \( U \) if \( U \) is the union of lines through \( \bar{u} \). A point set is aperiodic if it has no periods. A factorization \((U, V)\) is aperiodic if both \( U \) and \( V \) are aperiodic.

The following proposition connects factorizations of projective spaces with projective tilings of vector spaces. For a vector space \( S \), by \( \mathrm{PG}(S) \) we denote the projective space (of dimension \( \dim(S) - 1 \)) whose points are the 1-dimensional subspaces of \( S \) and lines corresponding to 2-dimensional subspaces of \( S \).

**Proposition 2.** Let \( U \) and \( V \) be two sets of points of a vector space \( S \). Assume that \( U \) and \( V \) are projective, i.e., \( U = \bigcup_{\bar{u} \in U} \bar{u} \) and \( V = \bigcup_{\bar{v} \in V} \bar{v} \) for some sets \( U \) and \( V \) of 1-dimensional subspaces of \( S \).

(i) \((U, V)\) is a tiling of \( S \) if and only if \((U, V)\) is a factorization of \( \mathrm{PG}(S) \).

(ii) \( U \) (similarly, \( V \)) is full-rank if and only if \( U \) is full-rank.

(iii) There is a period in \( U \) (similarly, in \( V \)) if and only if \( \dim(\ker(U)) \geq 1 \) (in the case of a prime-order field, it is sufficient to say that \( U \) has at least one nonzero period).

**Proof.** (i) We assume that \( U \cap V = \emptyset \) or, equivalently, \( U \cap V = \emptyset \), because otherwise neither \((U, V)\) is a tiling nor \((U, V)\) is a factorization. Under that assumption, \( \emptyset \) is uniquely represented as the sum of an element from \( U \) and an element from \( V \). Consider a nonzero vector \( \bar{y} \) and the 1-dimensional subspace \( \bar{y} \) containing it, \( \bar{y} = \langle \bar{y} \rangle \). It is easy to see the following:

- \( \bar{y} = \bar{u} + \bar{0} \), if and only if \( \bar{y} \in U \);
- \( \bar{y} = \bar{0} + \bar{v} \), if and only if \( \bar{y} \in V \);
- \( \bar{y} = \bar{u} + \bar{v} \), if and only if \( \bar{y} \in \langle \bar{u}, \bar{v} \rangle \setminus \{\langle \bar{u} \rangle, \langle \bar{v} \rangle\} \).

From the three assertions above, it is now obvious that \((U, V)\) satisfies the definition of a tiling if and only if \((U, V)\) satisfies the definition of a factorization.

(ii) is straightforward from the obvious \( \langle U \rangle = \mathrm{PG}(\langle U \rangle) \).

(iii) If all vectors from a 1-dimensional space \( \bar{x} = \langle \bar{x} \rangle \) are periods of \( U \), then for any \( \bar{y} \) in \( U \) it holds \( \langle \bar{y}, \bar{x} \rangle \subset U \), in particular, \( \langle \langle \bar{y} \rangle, \langle \bar{x} \rangle \rangle \subset U \). In this case, by the definition, \( \bar{x} \) is a period of \( U \).
Inversely, if \( \hat{x} = \langle\hat{x}\rangle \) is a period of \( \mathcal{U} \), then for any \( \hat{y} \) in \( \mathcal{U} \setminus \hat{x} \), the line \( \langle\hat{y}, \hat{x}\rangle \) is a subset of \( \mathcal{U} \), the subspace \( \langle\hat{y}, \hat{x}\rangle \) is a subset of \( \mathcal{U} \), and any vector from \( \hat{x} \) is a period of \( \mathcal{U} \).

The next two propositions are similar to known facts on factorizations of groups. They emphasize the fundamental value of full-rank and aperiodic factorizations.

**Proposition 3.** If \( (\mathcal{U}, \mathcal{V}) \) is a factorization of a projective space and \( \mathcal{U} \) is not full-rank, then \( (\mathcal{U}, \langle\mathcal{U}\rangle \cap \mathcal{V}) \) is a factorization of \( \langle\mathcal{U}\rangle \).

**Proof.** Let us consider a point \( \hat{y} \) in \( \langle\mathcal{U}\rangle \). Since \( (\mathcal{U}, \mathcal{V}) \) is a factorization of the whole projective space, exactly one from the following three cases takes place.

- \( \hat{y} \in \mathcal{U} \);
- \( \hat{y} \notin \mathcal{V} \); in this case, \( \hat{y} \in \langle\mathcal{U}\rangle \cap \mathcal{V} \);
- \( \hat{y} \in \{\langle\hat{u}, \hat{v}\rangle \setminus \{\hat{u}, \hat{v}\} \} \) for some unique \( \hat{u} \) from \( \mathcal{U} \) and \( \hat{v} \) from \( \mathcal{V} \); in this case, we have \( \hat{v} \in \langle\mathcal{U}\rangle \cap \mathcal{V} \).

By the definition, \( (\mathcal{U}, \langle\mathcal{U}\rangle \cap \mathcal{V}) \) is a factorization of \( \langle\mathcal{U}\rangle \).

For a projective space \( \mathcal{G} \) and a point \( \hat{x} \) in it, by \( \mathcal{G}/\hat{x} \) we denote the projective space (of the preceding dimension) whose points are the lines of \( \mathcal{G} \) through \( \hat{x} \) and lines are induced by the 2-dimensional subplanes of \( \mathcal{G} \) through \( \hat{x} \) (i.e., \( \mathcal{G}/\hat{x} \) is the “projection” of \( \mathcal{G} \) along the lines through \( \hat{x} \)). In particular, if \( \mathcal{G} = \text{PG}(\mathbb{F}_q^n) \), then the points of \( \mathcal{G}/\hat{x} = \text{PG}(\mathbb{F}_q^n)/\hat{x} \) are represented by 2-dimensional subspaces of \( \mathbb{F}_q^n \) that include the 1-dimensional subspace \( \hat{x} \), while the lines of \( \mathcal{G}/\hat{x} \) are represented by 3-dimensional subspaces of \( \mathbb{F}_q^n \) including \( \hat{x} \). For a set \( \mathcal{C} \) of points of \( \mathcal{G} \), by \( \mathcal{C}/\hat{x} \) we denote the following set of points of \( \mathcal{G}/\hat{x} \): \( \mathcal{C}/\hat{x} = \{\langle\hat{x}, \hat{y}\rangle : \hat{y} \in \mathcal{C}, \hat{y} \neq \hat{x}\} \).

**Proposition 4.** If \( \mathcal{G} \) is a projective geometry, \( (\mathcal{U}, \mathcal{V}) \) is a factorization of \( \mathcal{G} \), and \( \hat{x} \) is a period of \( \mathcal{U} \), then \( \langle\mathcal{U}/\hat{x}, \mathcal{V}/\hat{x}\rangle \) is a factorization of \( \mathcal{G}/\hat{x} \).

**Proof.** At first, we prove that

\[(*) \text{ every point of } \mathcal{G}/\hat{x} \text{ belongs to } \mathcal{U}/\hat{x}, \text{ or to } \mathcal{V}/\hat{x}, \text{ or to } \langle\hat{u}, \hat{v}\rangle \setminus \{\hat{u}, \hat{v}\} \text{ for some } \hat{u} \in \mathcal{U}/\hat{x} \text{ and } \hat{v} \in \mathcal{V}/\hat{x}.\]

Indeed, consider a point of \( \mathcal{G}/\hat{x} \) neither in \( \mathcal{U}/\hat{x} \) nor in \( \mathcal{V}/\hat{x} \). That point, by the definition, is a line through \( \hat{x} \) in \( \mathcal{G} \), say \( \langle\hat{x}, \hat{y}\rangle \). Since \( (\mathcal{U}, \mathcal{V}) \) is a factorization of \( \mathcal{G} \) and \( \hat{y} \notin \mathcal{U} \cup \mathcal{V} \), there are \( \hat{u} \) in \( \mathcal{U} \) and \( \hat{v} \) in \( \mathcal{V} \) such that \( \hat{y} \in \{\hat{u}, \hat{v}\} \). If \( \hat{x} \in \{\hat{u}, \hat{v}\} \), then at least one of \( \hat{u}, \hat{v} \) is different from \( \hat{x} \) and hence \( \langle\hat{x}, \hat{y}\rangle = \langle\hat{u}, \hat{v}\rangle \) belongs to \( \mathcal{U}/\hat{x} \) or \( \mathcal{V}/\hat{x} \). Otherwise, \( \langle\hat{x}, \hat{y}\rangle \) belongs to the line between the points \( \langle\hat{x}, \hat{u}\rangle \) and \( \langle\hat{x}, \hat{v}\rangle \) in \( \mathcal{G}/\hat{x} \). In any case, \( (*) \) holds.

It remains to show that

\[(**) \text{ every point of } \mathcal{G}/\hat{x} \text{ belongs to only one set in } (*).\]
\(U/\dot{x}\) and \(V/\dot{x}\) are disjoint because \(U \cap V = \emptyset\) and \(\dot{x}\) is a period of \(U\).

Now consider the line \(\langle \dot{u}, \dot{v} \rangle\) for some \(\dot{u}\) in \(U/\dot{x}\) and \(\dot{v}\) in \(V/\dot{x}\). The line \(\dot{v}\) of \(G\) contains a point \(\dot{v}\) in \(V\). Moreover, all points of the line \(\dot{u}\) of \(G\) belong to \(U\), because \(\dot{x}\) is a period of \(U\). In the projective plain \(\langle \ddot{u}, \ddot{v} \rangle\), a 2-dimensional subspace of \(G\), every point \(\ddot{z}\), \(\ddot{z} \notin \ddot{u} \cup \ddot{v}\), belongs to the line \(\langle \dot{v}, \dot{u} \rangle\) between \(\dot{v}\) and some point \(\dot{u}\) in \(\ddot{u}\). Since \((U, V)\) is a factorization, such point \(\ddot{z}\) cannot belong to \(U\), or \(V\), or any line between \(U\) and \(V\) different from \(\langle \dot{v}, \dot{u} \rangle\). That means that \(\langle \dot{x}, \ddot{z} \rangle\) does not belong to \(U/\dot{x}\), or \(V/\dot{x}\), or any line between \(U/\dot{x}\) and \(V/\dot{x}\) different from \(\langle \dot{u}, \dot{v} \rangle\), which completes the proof of (**).

By (*) and (**), \((U/\dot{x}, V/\dot{x})\) satisfies the definition of a factorization.

**Problem 1.** Given a projective space, characterize all sizes of tiles of full-rank factorizations. In particular, which geometries have no full-rank factorizations (in the theory of group factorizations, that property is called the Rédei property [25, Ch. 9])? The same question, for aperiodic full-rank factorizations.

Similar factorizations can be considered for other geometries, not only projective spaces.

**Problem 2.** Are there nontrivial \((|U| > 1, |V| > 1)\) examples of factorizations \((U, V)\) of the affine spaces, in the sense above (i.e., when the space is partitioned into the sets \(U, V\), and \(\langle a, b \rangle \{a, b\}, a \in U, b \in V\)?)

For example, in the affine space \(\mathbb{F}_7^3\), if \(|U| = 5\) and \(|V| = 13\), then \(|U| + |V| + \sum_{a \in U, b \in V} |\langle a, b \rangle \{a, b\}| = 5 + 13 + 5 \cdot 13 \cdot 5 = 343\), which is exactly \(|\mathbb{F}_7^3|\). However, factorizations with such parameters do not exist (exhaustive search).

Also, factorizations into more than two sets can be considered in a similar manner. For example, for three sets, such factorization is a partition of the space into the sets \(U, V, W, \langle a, b \rangle \{a, b\}, \langle a, c \rangle \{a, c\}, \langle b, c \rangle \{b, c\}, \text{ and } \langle a, b, c \rangle \langle a, b \rangle \langle a, c \rangle \langle b, c \rangle, a \in U, b \in V, c \in W\).

**Data Availability Statement**

The dataset generated during the current study (examples of full-rank ternary \((13, 3^{10}, 3)_3\) perfect codes) is available in the IEEE DataPort repository [13].

**References**

[1] S. V. Avgustinovich and E. V. Gorkunov. Maximum intersection of linear codes and codes equivalent to linear. *J. Appl. Ind. Math.*, 13(4):600–605, 2019. DOI: 10.1134/S1990478919040021 translated from Diskretn. Anal. Issled. Oper. 26:4, 5–15 (2019).

[2] S. V. Avgustinovich, F. I. Solov’eva, and O. Heden. On ranks and kernels problem of perfect codes. *Probl. Inf. Transm.*, 39(4):341–345, 2003. DOI: 10.1023/B:PRIT.0000011272.10614.8c translated from *Probl. Peredachi Inf.*, 39(4): 30-34, 2003.
[3] A. Blokhuis and C. W. H. Lam. More coverings by rook domains. *J. Comb. Theory, Ser. A*, 36(2):240–244, March 1984. DOI: 10.1016/0097-3165(84)90010-4

[4] G. Cohen, S. Litsyn, A. Vardy, and G. Zémor. Tilings of binary spaces. *SIAM J. Discrete Math.*, 9(3):393–412, 1996. DOI: 10.1137/S0895480195280137

[5] S. den Breeijen. *Tilings Of Additive Groups*. Master’s thesis, Radboud University Nijmegen, 2018. https://www.math.ru.nl/~bosma/Students/SterredenBreeijenMSc.pdf

[6] M. Dinitz. Full rank tilings of finite abelian groups. *SIAM J. Discrete Math.*, 20(1):160–170, 2006. DOI: 10.1137/S0895480104445794

[7] T. Etzion and A. Vardy. Perfect binary codes: Constructions, properties and enumeration. *IEEE Trans. Inf. Theory*, 40(3):754–763, 1994. DOI: 10.1109/18.335887

[8] T. Etzion and A. Vardy. On perfect codes and tilings: Problems and solutions. *SIAM J. Discrete Math.*, 11(2):205–223, May 1998. DOI: 10.1137/S0895480196309171

[9] O. H. Fraser and B. Gordon. Solution to a problem of A. D. Sands. *Glasgow Math. J.*, 20(2):115–117, 1979. DOI: 10.1017/S0017089500003803

[10] M. J. E. Golay. Notes on digital coding. *Proc. IRE*, 37(6):657, 1949. DOI: 10.1109/JRPROC.1949.233620.

[11] R. W. Hamming. Error detecting and error correcting codes. *Bell Syst. Tech. J.*, 29(2):147–160, 1950. DOI: 10.1002/j.1538-7305.1950.tb00463.x

[12] J. I. Kokkala, D. S. Krotov, and P. R. J. Östergård. On the classification of MDS codes. *IEEE Trans. Inf. Theory*, 61(12):6485–6492, Dec. 2015. DOI: 10.1109/TIT.2015.2488659

[13] D. Krotov. Perfect and related codes (dataset). IEEE Dataport, 2022. DOI: 10.21227/w856-4b70

[14] S. A. Malyugin. Systematic and nonsystematic perfect codes of infinite length over finite fields. *Sib. Elektron. Mat. Izv.*, 16:1732–1751, 2019. DOI: 10.33048/semi.2019.16.122 in Russian.

[15] S. A. Malyugin. Linear perfect codes of infinite length over infinite fields. *Sib. Elektron. Mat. Izv.*, 17:1165–1182, 2020. DOI: 10.33048/semi.2020.17.088

[16] C. Okuda. *The Factorization of Abelian Groups*. PhD thesis, The Pennsylvania State University, 1975.

[17] P. J. R. Östergård and S. Szabó. Elementary p-groups with the Rédei property. *International Journal of Algebra and Computation*, 17(1):171–178, 2007. DOI: 10.1142/S0218196707001429.
[18] K. T. Phelps, J. Rif` a, and M. Villanueva. Kernels and $p$-kernels of $p$-ary 1-perfect codes. *Des. Codes Cryptography*, 37(2):243–261, Nov. 2005. DOI: 10.1007/s10623-004-3989-x.

[19] K. T. Phelps and M. Villanueva. On perfect codes: Rank and kernel. *Des. Codes Cryptography*, 27(3):183–194, 2002. DOI: 10.1023/A:1019936019517.

[20] K. T. Phelps and M. Villanueva. Ranks of $q$-ary 1-perfect codes. *Des. Codes Cryptography*, 27(1-2):139–144, 2002. DOI: 10.1023/A:1016510804974.

[21] A. M. Romanov. On non-full-rank perfect codes over finite fields. *Des. Codes Cryptography*, 87(5):995–1003, May 2019. DOI: 10.1007/s10623-018-0506-1.

[22] J. Schönheim. On linear and nonlinear single-error-correcting $q$-ary perfect codes. *Inf. Control*, 12(1):23–26, 1968. DOI: 10.1016/S0019-9958(68)90167-8.

[23] M. Shi and D. S. Krotov. An enumeration of 1-perfect ternary codes. *Discrete Math.*, 346(7):113437(1–16), July 2023. DOI: 10.1016/j.disc.2023.113437.

[24] M. Shi, R. Wu, and D. S. Krotov. On $\mathbb{Z}_p\mathbb{Z}_{p^k}$-additive codes and their duality. *IEEE Trans. Inf. Theory*, 65(6):3841–3847, June 2019. DOI: 10.1109/TIT.2018.2883759.

[25] S. Szabó. *Topics in Factorization of Abelian Groups*, volume 29 of *Texts and Readings in Mathematics*. Hindustan Book Agency, Gurgaon, 2004. DOI: 10.1007/978-93-86279-22-4.

[26] S. Szabó. Constructions related to the Rédei property of groups. *J. Lond. Math. Soc., II. Ser.*, 73(3):701–715, June 2006. DOI: 10.1112/jlms/s1-73.3.701.

[27] S. Szabó. Full-rank factorings of elementary $p$-groups by $\mathbb{Z}$-subsets. *Indagationes Mathematicae*, 24(4):988–995, Nov. 2013. DOI: 10.1016/j.indag.2012.11.002.

[28] S. Szabó and A. D. Sands. *Factoring Groups into Subsets*, volume 257 of *Lecture Notes in Pure and Applied Mathematics*. Chapman and Hall/CRC, Boca Raton, FL, 2009. DOI: 10.1201/9781420090475.

[29] A. Tietäväinen. On the nonexistence of perfect codes over finite fields. *SIAM J. Appl. Math.*, 24(1):88–96, 1973. DOI: 10.1137/0124010.

[30] J. L. Wasiljew. Über dicht gepackte nichtgruppen-codes. In *Probleme der Kybernetik*, volume 8, pages 375–378. Akademie-Verlag, 1965. Translated from Problemy Kibernetiki 8: 337-339, 1962.

[31] S. K. Zaremba. Covering problems concerning Abelian groups. 27(2):242–246, Apr. 1952. DOI: 10.1112/jlms/s1-27.2.242.

[32] V. Zinoviev and V. Leontiev. The nonexistence of perfect codes over Galois fields. *Probl. Control Inf. Theory*, 2(2):123–132, 16–24[Engl. transl.], 1973.