COMPLETE CLASSIFICATION OF $H$-TYPE ALGEBRAS: I

KENRO FURUTANI, IRINA MARKINA

Abstract. Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra endowed with non-degenerate scalar product $\langle \cdot, \cdot \rangle$ and let $\mathcal{N} = V \oplus Z$, where $Z$ is the centre of the Lie algebra and $V$ its orthogonal complement with respect to the scalar product. We study the classification of the Lie algebras for which the space $V$ arises as a representation space of a Clifford algebra $\Cl(R^{r,s})$ and the representation map $J: \Cl(R^{r,s}) \to \End(V)$ is related to the Lie algebra structure by $\langle Jv, w \rangle = \langle z, [v, w] \rangle$ for all $z \in \mathbb{R}^{r,s}$ and $v, w \in V$. The classification is based on the range of parameters $r$ and $s$ and is completed for the Clifford modules $V$, having minimal possible dimension, that are not necessary irreducible. We find the necessary condition for the existence of a Lie algebra isomorphism according to the range of integer parameters $0 \leq r, s < \infty$. We present the constructive proof for the isomorphism map for isomorphic Lie algebras and defined the class of non-isomorphic Lie algebras.

1. Introduction

We are studying one special type of 2-step nilpotent Lie algebras. In the work [32] Métivier introduced 2-step real nilpotent Lie algebras $\mathfrak{n} = V \oplus Z$ with the center $Z$ such that the adjoint map $\text{ad}_z: \mathfrak{n} \to Z$ is surjective for any $x \in V$, where $V$ is the prescribed complement to the center. Equivalently, the bracket defines a vector valued anti-symmetric form $[\cdot, \cdot]: V \times V \to Z$, such that anti-symmetric real valued bilinear form $B(x, y) = \omega([x, y])$ is non-degenerate on $V$ for all $\omega \in Z^*$, $\omega \neq 0$. Particularly, it immediately implies that the space $V$ is even dimensional and $[V, V] = Z$ since $Z = \text{ad}_z(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$ for $x \in V$. These Lie algebras were introduced in order to study the analytic hypoellipticity and were called Lie algebras satisfying hypothesis $\text{H}$. The Lie algebras were also studied in [15] Definition 1.3 under the name non-singular, in [30] [33] as Lie algebras of Métivier group or in [27] as fat algebras since they are source of fat distributions.

Let us observe that if a 2-step nilpotent Lie algebra $\mathfrak{n}$ carries a positive definite product $\langle \cdot, \cdot \rangle$ on it, and the map $J_z: V \to V$ is defined by

$$\langle J_z x, y \rangle = \langle z, [x, y] \rangle = \langle z, \text{ad}_x(y) \rangle,$$

then $J_z$ is a non-singular linear map for any non-zero $z \in Z$ if and only if the Lie algebra $\mathfrak{n}$ is non-singular. The presence of a positive definite product is not restrictive at all, because Eberlein showed in [14] that any 2-step nilpotent Lie algebra is isomorphic to a standard metric form $(\mathcal{N}, \langle \cdot, \cdot \rangle_N)$, where $\mathcal{N} = \mathbb{R}^n \oplus W$, with $W \subset \mathfrak{so}(n)$ and the positive definite product defined by $\langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_{\mathbb{R}^n} + \langle \cdot, \cdot \rangle_{\mathfrak{so}(n)}$. Thus any 2-step nilpotent Lie algebra can be considered as a metric Lie algebra with the scalar product defined as above.

We are interested in those non-singular 2-step nilpotent Lie algebras, for which the map $J: Z \to \End(V)$ is a representation of a Clifford algebra $\Cl(R^{r,s})$. The map $J$ and the Lie brackets are related by (1) by making use a sign indefinite non-degenerate scalar product. The corresponding Lie algebras, which we denote by $\mathcal{N}_{r,s}(V)$, received the name pseudo $H$-type.

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Lie algebras. For the Clifford algebras $\text{Cl}(\mathbb{R}^{r,0})$, generated by the Euclidean space $\mathbb{R}^r$, the $H$-type algebras $\mathcal{N}_{r,0}(V)$ were introduced by Kaplan [24] and attracted a lot of attention [6, 10, 13, 23, 26, 34, 35]. The Lie algebras $\mathcal{N}_{r,0}(V)$ is a typical example of a standard metric form. For the Clifford algebras $\text{Cl}(\mathbb{R}^{r,s})$, generated by the Euclidean space $\mathbb{R}^{r,s}$, the pseudo $H$-type Lie algebras $\mathcal{N}_{r,s}(V)$ were introduced in [11], and studied in [12, 16, 18, 20, 21].

We study the isomorphism properties between the Lie algebras $\mathcal{N}_{r,s}(V)$. We show that the Lie algebras $\mathcal{N}_{r,s}(V)$ can not be isomorphic to $\mathcal{N}_{u,t}(V)$ unless $r = t$ and $s = u$ or $r = u$ and $s = t$. The present paper is the first part of the complete classification, where we concentrate on the classification of the Lie algebras based on the Clifford modules of minimal possible dimensions (which are not necessarily irreducible), admitting a scalar product making the representation map $J_z$ skew symmetric. Then, by making use the Atiyah-Bott periodicity for underlying Clifford algebras we extend the study to an arbitrary dimension pseudo $H$-type algebras. We also show that the Lie algebras based on the non-equivalent irreducible Clifford modules are isomorphic. We stress, that the isomorphic relation between the Clifford algebras and the associated pseudo $H$-type Lie algebras is not functorial. In some cases the isomorphic Clifford algebras lead to isomorphic Lie algebras, in other cases not.

Apart from being motivated by itself interesting mathematical question of the classification of Lie algebras, we want to mention here possible applications in other areas of mathematics. It was shown [10, 16] that the pseudo $H$-type Lie algebras admit the integer structure constants that in its turn, according to the Mal’cev theorem [31], guarantees the existence of lattices on the corresponding Lie groups. The factorization of pseudo $H$-type Lie groups by lattices gives a vast of new examples of nilmanifolds, which type strongly depend on the classification of pseudo $H$-type algebras [7, 8, 12, 21]. Nilmanifolds are related to the Grushin type differential operators descending from elliptic and sub-elliptic type operators on the corresponding pseudo $H$-type Lie groups. This kind of nilmanifolds allows precise construction of the spectral zeta function for the Grusin operator, [3, 4] and gives new examples of iso-spectral but non-diffeomorphic nilmanifolds (5).

Recently it was noticed that Tanaka prolongations of some pseudo $H$-type Lie algebras coincide with Tanaka prolongations of simple Lie algebras, factorized by parabolic subalgebras. It shows a close relation between the classification of pseudo $H$-type algebras and the theory of simple Lie algebras. The fact that Clifford algebras are pretty much useful in the orthogonal design, signal processing, space-time block coding, or computer vision, is well known [9, 19, 23, 36]. The structure of pseudo $H$-type algebras allows a new construction of orthogonal designs and possible wireless communications, as was shown in [18].

The article is organized in the following way. After the introduction we give necessary definitions and notations in Section 2, including the notion of admissible module and relation between Clifford algebras and pseudo $H$-type Lie algebras. We also describe the scheme of classification, that includes 4 steps. In the rest of sections we realise 3 steps of the classification.

2. Clifford algebras, modules, and pseudo $H$-type Lie algebras

2.1. Clifford algebras and representations. We use the notation $\mathbb{R}^{r,s}$ for the space $\mathbb{R}^{r+s}$ equipped with the non-degenerate symmetric bilinear form

$$\langle x, y \rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}, \quad x, y \in \mathbb{R}^{r,s}.$$ 

An orthonormal basis we denote by $\{z_1, \ldots, z_{r+s}\}$. Thus

$$\langle z_i, z_j \rangle_{r,s} = \epsilon_i(r,s)\delta_{i,j}, \quad \epsilon_i(r,s) = \begin{cases} 1, & \text{if } i = 1, \ldots, r, \\ -1, & \text{if } i = r+1, \ldots, r+s, \end{cases}$$
where $\delta_{i,j}$ is the Kronecker symbol. By $\text{Cl}_{r,s}$ we denote the Clifford algebra generated by $\mathbb{R}^{r,s}$, that is, the quotient algebra of the tensor algebra

$$\mathcal{T}(\mathbb{R}^{r+s}) = \mathbb{R} \oplus (\mathbb{R}^{r+s}) \oplus \left( \frac{2}{\mathbb{R}} \mathbb{R}^{r+s} \right) \oplus \left( 3 \times \mathbb{R}^{r+s} \right) \oplus \cdots,$$

divided by the two-sided ideal $I_{r,s}$ generated by the elements of the form $x \otimes x + \langle x, x \rangle_{r,s}$, $x \in \mathbb{R}^{r+s}$. The explicit determination of the Clifford algebras is given in [1] and they are isomorphic to matrix algebras presented [29]. We mention in [2] useful isomorphisms of Clifford algebras, related to $8$-periodicity, established in [1] and $(4 - 4)$-periodicity, see [29]. To denote isomorphic objects we use the symbol “$\cong$”.

$$\begin{align*}
\text{Cl}_{r,s} \otimes \text{Cl}_{0,8} & \cong \text{Cl}_{r,s+8} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16), \\
\text{Cl}_{r,s} \otimes \text{Cl}_{8,0} & \cong \text{Cl}_{r+8,s} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16), \\
\text{Cl}_{r,s} \otimes \text{Cl}_{4,4} & \cong \text{Cl}_{r+4,s+4} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16).
\end{align*}$$

An algebra homomorphism $\hat{J} : \text{Cl}_{r,s} \to \text{End}(U)$ is called representation map and the vector space $U$ is said to be the representation space. The representation space $U$ becomes Clifford $\text{Cl}_{r,s}$-module, where the multiplication is defined by $\phi u = \hat{J}_\phi u$, $u \in U$, $\phi \in \text{Cl}_{r,s}$. It is enough to define a linear map $J : \mathbb{R}^{r,s} \to \text{End}(U)$, satisfying $J^2_z = -\langle z, z \rangle_{r,s} \text{Id}_U$ for an arbitrary $z \in \mathbb{R}^{r,s}$. Then $J$ can be uniquely extended to the representation $\hat{J}$ by the universal property, see, for instance [22, 28, 29].

2.2. Admissible modules. Let $U$ be a Clifford $\text{Cl}_{r,s}$-module. We call the module $U$ admissible, if there is a non-degenerate symmetric bilinear form $\langle \ldots \rangle_U$ on $U$ such that the representation map $J$ satisfies the following condition:

$$\langle J_z x, y \rangle_U + \langle x, J_z y \rangle_U = 0 \quad \text{for all} \quad z \in \mathbb{R}^{r,s}, \quad x, y \in U. \quad (3)$$

We say that the map $J_z$ is skew symmetric with respect to the bilinear symmetric form $\langle \ldots \rangle_U$ and write $U = (U, \langle \ldots \rangle_U)$ for an admissible module. If $(U, \langle \ldots \rangle_U)$ is an admissible module with a non-degenerate scalar product, then it decomposes into the orthogonal sum of minimal dimensional admissible modules [16], since the orthogonal complement to an admissible submodule is an admissible module.

If $U$ is a $\text{Cl}_{r,0}$-module, then there always exists a positive definite scalar product $\langle \ldots \rangle_U$ such that $U$ becomes an admissible module. Particularly, any irreducible module is an admissible with respect to some positive definite scalar product. It allowed to Kaplan to introduce $H$-type Lie algebras in [24].

If $s > 0$, and $(U, \langle \ldots \rangle_U)$ is an admissible $\text{Cl}_{r,s}$-module, then the scalar product space $(U, \langle \ldots \rangle_U)$ has to be a neutral space [11], that is an even dimensional space, where the bilinear symmetric form has equal number of positive and negative eigenvalues. In this case an irreducible module need not be admissible.

Recall that the Clifford algebras $\text{Cl}_{r,s}$ with $r - s \neq 3(\text{mod} \ 4)$ admit only one irreducible module up to equivalence. Some of irreducible modules $V$ can be supplied with a scalar product with the property (3) and becomes an admissible module. In other cases the direct sum $V \oplus V$ must be taken in order to define the scalar product, see [11]. In both cases we call the obtained admissible module minimal admissible module. Thus, for the Clifford algebras $\text{Cl}_{r,s}$ with $r - s \neq 3(\text{mod} \ 4)$ the minimal admissible module is either $(V, \langle \ldots \rangle_V)$ or $(V \oplus V, \langle \ldots \rangle_{V \oplus V})$, where $V$ is an irreducible module. We will denote a minimal admissible module of the Clifford algebra $\text{Cl}_{r,s}$ by $V^{r,s}$.

We clarify now the structure of minimal admissible modules for $\text{Cl}_{r,s}$ with $r - s = 3(\text{mod} \ 4)$. In this case, there are two non-equivalent irreducible modules. Let $\{ z_1, \ldots, z_{r+s} \}$ be an orthonormal basis of $\mathbb{R}^{r,s}$ and $\{ J_{z_1}, \ldots, J_{z_{r+s}} \}$ the corresponding representation maps. The product $\Omega^{r,s} = \prod_{j=1}^{r+s} J_{z_j}$ is called the volume form. In the case of $r - s = 3(\text{mod} \ 4)$, it belongs
to the center of the Clifford algebra $\text{Cl}_{r,s}$ and $(\Omega^{r,s})^2 = \text{Id}$. Two non-equivalent irreducible modules are distinguished by the action of $\Omega^{r,s}$. We denote by $V_+$ the irreducible module, where the volume form acts as the identity operator and by $V_-$ the non-equivalent irreducible $\text{Cl}_{r,s}$-module, where the volume form $\Omega^{r,s}$ acts as the minus identity operator. If non of irreducible modules is admissible, then the minimal admissible module is one of the following forms $V_+ \oplus V_+$ or $V_+ \oplus V_-$ or $V_+ \oplus V_-$. A choice of a possible form depends on the value of index $s$ and it is explained in Proposition 1. The summary of possible structures of minimal admissible modules for all the cases is given in Table 1.

| $r + s \neq 3(\text{mod } 4)$ | $r + s = 3(\text{mod } 4)$ |
|---|---|
| $V$ or $V \oplus V$ | $\begin{cases} s \text{ is even} & \text{if } r \text{ is even and if the irreducible module } V_+ \text{ is admissible, then } V_- \text{ is also admissible and vice versa}; \\ 2. & \text{if } s \text{ is even and if one of irreducible modules is not admissible, then the other one is neither admissible. The minimal admissible module takes one of the forms: } V^{r,s}_+ = V_+ \oplus V_+ \text{ or } V^{r,s}_- = V_- \oplus V_-; \end{cases}$ |
| $\begin{cases} s \text{ is odd} & \text{if } V_+ \text{ or } V_- \text{ is admissible, then the minimal admissible module is one of the forms: } V^{r,s}_+ = V_+ \oplus V_+ \text{ or } V^{r,s}_- = V_- \oplus V_-; \end{cases}$ |

**Table 1. Structure of possible minimal admissible modules $V^{r,s}$**

**Proposition 1.** Let $\text{Cl}_{r,s}$ be a Clifford algebra with $r - s = 3(\text{mod } 4)$. The following cases are possible.

1. If $s$ is odd, then an irreducible module can not be admissible. The minimal admissible module is unique, up to an isomorphism, and has the form $V^{r,s}_+ = V_+ \oplus V_-;
2. If s$ is even and if the irreducible module $V_+$ is admissible, then $V_-$ is also admissible and vice versa;
3. If $s$ is even and if one of irreducible modules is not admissible, then the other one is neither admissible. The minimal admissible module takes one of the forms: $V^{r,s}_+ = V_+ \oplus V_+$ or $V^{r,s}_- = V_- \oplus V_-.$

**Proof.** To show the first claim we assume that $(V_+, \langle \cdot, \cdot \rangle_{V_+})$ is admissible. Then

$$\langle x, x \rangle_{V_+} = \langle \Omega^{r,s}(x), \Omega^{r,s}(x) \rangle_{V_+} = \prod_{i=1}^{r+s} \langle z_i, z_i \rangle_{V_+} = (-1)^s \langle x, x \rangle_{V_+}$$

for any $x \in V_+$. This shows that all the vectors $x \in V_+$ are null vectors and the scalar product $\langle \cdot, \cdot \rangle_{V_+}$ is degenerate. Thus the irreducible module $V_+$ can not be supplied with non-degenerate bilinear symmetric form, such that the maps $J_z$ satisfies (3). Similar arguments are valid for $V_-$. Thus if $(V^{r,s}_+, \langle \cdot, \cdot \rangle_{V^{r,s}_+})$ is a minimal admissible module, then $V^{r,s}_+$ has to contain both of $V_+$. The second statement is obvious. Before starting to prove the last statement, we note that if $r - s = 3(\text{mod } 4)$, then $r + s = 2s + 3(\text{mod } 4)$ is always odd and $\frac{r+s-1}{2} = s + 1(\text{mod } 2)$ is also odd in the case when $s$ is even. We assume now that non of two non-equivalent irreducible modules is admissible and we consider a minimal admissible module $(V^{r,s}_+, \langle \cdot, \cdot \rangle_{V^{r,s}_+})$. Then the volume form is an isometry by (1) and a symmetric operator because of

$$\langle \Omega^{r,s}(x), y \rangle_{V^{r,s}_+} = (-1)^{r+s} \langle x, J_{z_{r+s}} \cdots J_{z_1} y \rangle_{V^{r,s}_+} = (-1)^{r+s} (-1)^{\frac{r+s-1}{2}} \langle x, \Omega^{r,s}(y) \rangle_{V^{r,s}_+}.$$}

Thus, if $V^{r,s}_+$ contains two eigenspaces $V_+$ and $V_-$ of $\Omega^{r,s}$, then $V_+$ and $V_-$ have to be orthogonal non-degenerate subspaces of $(V^{r,s}_+, \langle \cdot, \cdot \rangle_{V^{r,s}_+})$ and therefore admissible modules. This contradicts to the assumption that non of irreducible modules is admissible.

In Table 2 we give the dimensions of minimal admissible modules $V^{r,s}_+$, $r, s \leq 8$. By the black colour we denote the dimensions of minimal admissible modules, that are also irreducible Clifford modules. The red colour is used for the minimal admissible modules which are direct product of two irreducible Clifford modules. The notation $N_{x,2}$ means that there are two minimal admissible modules.

We need a couple of more properties of the admissible modules, see [16].
**Table 2. Dimensions of minimal admissible modules**

| s/r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| 8   | 16 | 32 | 64 | 64\_k | 128 | 128 | 128 | 128 | 256 |
| 7   | 16 | 32 | 64 | 64   | 128 | 128 | 128 | 128 | 256 |
| 6   | 16 | 16\_k | 32 | 32   | 64 | 64\_k | 128 | 128 | 256 |
| 5   | 16 | 16 | 16 | 16   | 32 | 64 | 128 | 128 | 256 |
| 4   | 8  | 8  | 8  | 8    | 16 | 32 | 64 | 64\_k | 128 |
| 3   | 8  | 8  | 8  | 8    | 16 | 32 | 64 | 64   | 128 |
| 2   | 4  | 4\_k | 8  | 8    | 16 | 16\_k | 32 | 32   | 64 |
| 1   | 2  | 4  | 8  | 8    | 16 | 16 | 16 | 16   | 32 |
| 0   | 1  | 2  | 4  | 4\_k | 8  | 8  | 8  | 8\_k | 16 |

**Lemma 1.** [16] Let \((V, \langle \cdot, \cdot \rangle_V)\) be an admissible module and \(J_1, \ldots, J_l\) symmetric or anti-symmetric linear operators on \(V\) such that

1. \(J_k^2 = \text{Id}, k = 1, \ldots, l;\)
2. \(J_k J_j = -J_j J_k\) for all \(k, j = 1, \ldots, l.\)

Then for any \(v \in V\) with \(\langle v, v \rangle_V = 1\) there is a vector \(\tilde{v}\) satisfying:

\[
\langle \tilde{v}, J_k \tilde{v} \rangle_V = 0, \quad \text{and} \quad \langle \tilde{v}, \tilde{v} \rangle_V = 1, \quad k = 1, \ldots, l.
\]

If \(P\) is a linear operator on \(V\) such that \(P^2 = \text{Id}, P J_k = J_k P, k = 1, \ldots, l,\) and \(v \in V\) with \(\langle v, v \rangle_V = 1\), satisfies \(P v = v,\) then the vector \(\tilde{v}\) is also eigenvector of \(P\): \(P \tilde{v} = \tilde{v}.

**Remark 1.** Let \((V, \langle \cdot, \cdot \rangle_V)\) be an admissible module of a Clifford algebra \(\text{Cl}_{r,s}\). Then it can be easily seen form the definition of an admissible module, that the same module with the scalar product of the opposite sign \((V, -\langle \cdot, \cdot \rangle_V)\) is also an admissible module.

**2.3. Pseudo \(H\)-type algebras.** We give the definition of pseudo \(H\)-type algebras that is convenient for us to work. The equivalent definitions and their relations to Clifford algebras can be found in [2, 10, 11, 13, 16, 20, 24, 25, 26].

**Definition 1.** Let \((U, \langle \cdot, \cdot \rangle_U)\) be an admissible module of a Clifford algebra \(\text{Cl}_{r,s}\) and a map \(J: \text{Cl}_{r,s} \to \text{End}(U)\) a representation. A 2-step nilpotent Lie algebra \(U \oplus \mathbb{R}^{r,s}\) with the center \(\mathbb{R}^{r,s}\) and the Lie bracket defined via the relation

\[
\langle J_x x, y \rangle_U = \langle z, [x,y] \rangle_{r,s}, \quad z \in \mathbb{R}^{r,s}, \ x, y \in U,
\]

is called a pseudo \(H\)-type Lie algebra and is denoted by \(\mathcal{N}_{r,s}(U)\). If \(U = V^{r,s}\) is minimal admissible, then we write \(\mathcal{N}_{r,s}\). In Section [5] we prove the uniqueness of the algebra \(\mathcal{N}_{r,s}\). One of the particular consequences of Definition 1 is \(\langle J_x x, J_{x'} x' \rangle_U = \langle z, z' \rangle_{r,s} \langle x, x' \rangle_{r,s}\). Thus for an orthonormal basis \(\{z_1, \ldots, z_{r+s}\}\) the maps \(J_{z_j}: U \to U\) are isometries for \(j = 1, \ldots, r\) and anti-isometries for \(j = r+1, \ldots, r+s\).

**Theorem 1** ([10, 14, 16]). We fix an orthonormal basis \(\{z_k\}_{k=1}^{r+s}\) for \(\mathbb{R}^{r,s}\) and assume that \((V^{r,s}, \langle \cdot, \cdot \rangle_{V, r,s})\) is a minimal admissible module of \(\text{Cl}_{r,s}\) of dimension \(2N\). Then there exists an orthonormal basis \(\{x_i\}_{i=1}^{2N}\) for \(V^{r,s}\) such that

1. \(\langle x_i, x_j \rangle_{V, r,s} = \epsilon_i(N,N) \delta_{i,j};\)
2. For each \(k\), the operator \(J_{x_k}\) maps \(x_i\) to some \(x_j\) or \(-x_j\) with \(j \neq i;\)
3. There is a vector \(v \in V^{r,s}\), \(\langle v,v \rangle_{V, r,s} \neq 0\), such that all the basis \(\{x_i\}\) is obtained from \(v\) by action of \(J_{x_j}\), \(j = 1, \ldots, r+s\) or their product.
We call the basis \{x_i, z_j\} for \(N_{r,s}\) satisfying the properties of Theorem 1 an integral basis. Let \((W, \langle \cdot, \cdot \rangle_W)\) be a vector space with a non-degenerate scalar product. We say that a vector \(w \in W\) is positive if \(\langle w, w \rangle_W > 0\), negative if \(\langle w, w \rangle_W < 0\), and a null-vector if \(\langle w, w \rangle_W = 0\). We formulate some consequences of Theorem 1.

**Corollary 1.** If there exists an index \(i \in \{1, \ldots, 2N\}\) such that \(J_{zk}x_i = \pm J_{zi}x_i\), then \(k = l\). Hence any basis vector \(x_i\) is mapped to \(x_j \) or \(-x_j\) by at most one \(J_{zk}\).

**Proof.** If \(k \leq r\) then \(J_{zk}\) preserves positive and negative elements. If \(k > r\), then \(J_{zk}\) interchange the positive and negative elements. Therefore, under the assumption of the corollary only the cases \(k, l \leq r\) or \(k, l > r\) are possible. Assume \(k \neq l\). Then, from one hand \(\pm x_i = J_{zk}J_{zi}x_i\), but from the other hand

\[(J_{zk}J_{zi})^2 = -J_{zk}^2J_{zi}^2 = -\langle z_k, z_k \rangle_{r,s} \langle z_i, z_i \rangle_{r,s} \text{Id} = -\text{Id},\]

which contradicts to the existence of the eigenvalue 1 or \(-1\) of the operator \(J_{zk}J_{zi}\). \(\square\)

**Corollary 2.** Let \(N_{r,s}\) be a pseudo H-type algebra and \{x_i, z_j\} an integral basis. Set \([x_i, x_j] = \sum c_{ij}^k z_k\), then for fixed \(i\) and \(j\) the coefficients \(c_{ij}^k\) vanish for all but one \(k\) and in the later case \(c_{ij}^k = \pm 1\).

**Proof.** The proof follows from \(\langle J_{zk}x_i, x_j \rangle_{V_{r,s}} = \langle z_l, [x_i, x_j] \rangle_{r,s} = \begin{cases} c_{ij}^k & \text{if } k \leq r \\ -c_{ij}^k & \text{if } k > r. \end{cases}\) \(\square\)

**Corollary 3.** Let \(N_{r,s}\) be a pseudo H-type algebra, \{x_i, z_j\} an integral basis, and \([x_i, x_j] = \pm z_k\). Then

1. if either \(1 \leq i, j \leq N\) or \(N < i, j \leq 2N\) then \(z_k\) is positive, that is \(k \leq r\),
2. if \(1 \leq i \leq N < j \leq 2N\) then \(z_k\) is negative, i.e., \(k > r\).

**Proof.** We prove only the first statement, since the second one can be shown similarly. If we assume, by contrary, that \(k > r\), then \(J_{zk}\) should be an anti-isometry and

\[0 = \langle J_{zk}x_i, x_j \rangle_{V_{r,s}} = \langle z_k, [x_i, x_j] \rangle_{r,s} = \pm 1,\]

which is a contradiction. \(\square\)

### 2.4. Scheme for 4 step classification.

**Step 1.** We study the isomorphic and non-isomorphic cases of Lie algebras \(N_{r,s}\) and \(N_{s,r}\), \(r, s \leq 8\), \(r \neq s\) of equal dimensions, see Section 3. We also construct an automorphism of \(N_{r,r}\), \(r = 1, 2, 4\) having a special property and show that there is no such an automorphism of \(N_{3,3}\), see Theorem 6 and Corollary 7. Then the periodicity property (2) will be applied to extend these results to higher dimensional Lie algebras, see Theorems 4 and 5.

**Step 2.** If \(\dim(V^{r,s}) = 2 \dim(V^{s,r})\), then the Lie algebras \(N_{r,s}\) and \(N_{s,r}\) are not isomorphic simply because they have different dimension. We call these algebras trivially non-isomorphic. In this case we prove the isomorphism or non-isomorphism of the Lie algebras \(N_{r,s}(V^{r,s})\) and \(N_{s,r}(V^{s,r} \oplus V^{s,r})\), see Section 4.

**Step 3.** Let \(V^{r,s} = V_+\) or \(V^{r,s} = V_-\), where \(V_+, V_-\) are non-equivalent irreducible modules. We show that the Lie algebras \(N_{r,s}(V_+)\) and \(N_{s,r}(V_-)\) are isomorphic. An analogous question is considered when \(V^{r,s} = V_+ \oplus V_+\) or \(V^{r,s} = V_- \oplus V_-\). The isomorphism of Lie algebras particularly shows the uniqueness of the pseudo H-type algebra corresponding to two minimal admissible modules, see Section 5.
Step 4. The last step is devoted to the classification of Lie algebras, constructed from the multiple sum of minimal admissible modules. The admissible modules can differ either by the choice of the scalar product on it or they can be defined by non-equivalent representations.

In this paper we present 3 steps, finishing the classification of the Lie algebras whose complement to the centre is a minimal admissible module. We summarise the classification of the Step 1 among the basic pairs in Table 3. Here “d” stands for “double”, meaning that

| Table 3. Classification result after the first step |
|---------------------------------|
| | 8 | ∼ | 7 | d | d | d | d | ̸∼ | 6 | d | ̸∼ | h | |
| | 5 | d | ̸∼ | h | |
| | 4 | ̸∼ | h | h | h | ⟳ | |
| | 3 | d | ̸∼ | ⟲ | d | d | d | ̸∼ | d | ̸∼ | ⟲ | |
| | 2 | ̸∼ | h | ⟲ | d | d | ̸∼ | ̸∼ | h | ̸∼ | |
| | 1 | ̸∼ | ⟲ | d | ̸∼ | d | ̸∼ | ̸∼ | h | ̸∼ | |
| | 0 | ̸∼ | ̸∼ | h | ̸∼ | h | h | ̸∼ | |
| s/r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

dim $V_{r,s} = 2 \dim V_{s,r}$, and “h” (half) means that $\dim V_{r,s} = \frac{1}{2} \dim V_{s,r}$. The corresponding pairs are trivially non-isomorphic due to the different dimension of minimal admissible modules. The symbol $\cong$ denotes the Lie algebra having isomorphic pair, $\not\cong$ shows that the pair is non-isomorphic, the symbol $\⟳$ denotes the Lie algebra admitting a special type of automorphisms, and $\not\⟳$ denotes the Lie algebra not having this type of automorphism.

2.5. Remarks on Step 4 and further development. In the forthcoming paper [17] we will deal with Step 4, where we plan to consider the multiple sum $U = \oplus_i V_i$ of several minimal admissible modules $V_i = (V_\cdot, \langle \cdot, \cdot \rangle_V)$. Here different minimal admissible modules $V_i$ can have a common vector space $V$ but allows the scalar products of opposite sign. The minimal admissible modules can differ also by the choice of the irreducible modules for their construction: $V = V_+$ or $V = V_-$. Finally the minimal admissible modules can be both based on non-equivalent Clifford modules and admit the scalar products of opposite signs. Different combinations can give non-isomorphic Lie algebras $N_{r,s}(\oplus_i V_i)$.

We also aim to study the automorphism groups of the algebras $N_{r,s}$. They are determined by solving the equations arising during the construction of the map $A: V_{r,s} \rightarrow V_{r,s}$. The present paper indicates that it is reduced to the exact sequence

$$\{0\} \rightarrow K \rightarrow Aut(N_{r,s}) \rightarrow O(r, s) \rightarrow \{0\},$$

that defines the map $\Phi \rightarrow C$, see [6] for the form of $\Phi$. The last map is distinguished by the properties of $C$. In some cases $C^r C = \text{Id}$, as, for instance, in the case $N_{3,3}$, meanwhile for $N_{r,r}$, $r = 1, 2, 4$ one has $C^r C = \pm \text{Id}$. The map $C$ determines the map $A$ and the freedom in the construction of the map $A$ gives the kernel $K$. It can be seen from the present paper that it could be $K = \pm \text{Id}$ or $K = SO(2)$. In the forthcoming papers we aim to describe all the cases not only for the Lie algebras based on the minimal admissible modules, but also for the admissible modules of the type $U = \oplus_i V_i$.

3. Step 1: Lie algebras of minimal dimensions

3.1. Necessary condition of existence of an isomorphism. Let $A: U \rightarrow \bar{U}$ be a linear map. We denote by $A^\tau$ the adjoint map with respect to the scalar products on $(U, \langle \cdot, \cdot \rangle_U)$
and $(\tilde{U}, \langle \cdot, \cdot \rangle_{\tilde{U}})$:
\[ \langle A(x), y \rangle_{\tilde{U}} = \langle x, A^T(y) \rangle_U, \quad x \in U, \quad y \in \tilde{U}. \]

**Theorem 2.** Let \{\(U, \langle \cdot, \cdot \rangle_U; J\)\} and \{\(\tilde{U}, \langle \cdot, \cdot \rangle_{\tilde{U}}; \tilde{J}\)\} be admissible modules and representation maps of the Clifford algebras \(\text{Cl}_{r,s}\) and \(\text{Cl}_{\tilde{r},\tilde{s}}\), respectively. Assume that \(\dim U = \dim \tilde{U}\), \(r + s = \tilde{r} + \tilde{s}\), and that there is a Lie algebra isomorphism
\[ \Phi: \mathcal{N}_{r,s}(U) \to \mathcal{N}_{\tilde{r},\tilde{s}}(\tilde{U}) \]

between the corresponding pseudo H-type algebras. Then, necessarily, one of the cases \((r, s) = (\tilde{r}, \tilde{s})\) or \((r, s) = (\tilde{s}, \tilde{r})\) holds. Moreover, \(\Phi\) has to be of the form

\[ \Phi = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : \ \oplus_1^{r,s} \to \oplus_1^{\tilde{r},\tilde{s}}, \]

where \(A: U \to \tilde{U}\) and \(C: \mathbb{R}^{r,s} \to \mathbb{R}^{\tilde{r},\tilde{s}}\) are linear bijective maps satisfying the relation
\[ A^T \tilde{J}_z A = J_{C^r(z)} \ 	ext{for any} \ z \in \mathbb{R}^{\tilde{r},\tilde{s}}. \]  

There is no condition on \(B: U \to \mathbb{R}^{\tilde{r},\tilde{s}}\) and we may set \(B = 0\). Multiplying \(A\) by a suitable constant, we may assume that \(|\det (AA^T)| = 1\) and \(CC^r = \pm \text{Id}\).

**Proof.** If a Lie algebra isomorphism \(\Phi: \mathcal{N}_{r,s}(U) \to \mathcal{N}_{\tilde{r},\tilde{s}}(\tilde{U})\) exists, then it must be of the form \(\Phi\), since it maps the center to the center. The relation \(\Phi\) follows from the definition of Lie brackets
\[ \langle A^T \tilde{J}_z A(x), y \rangle_U = \langle \tilde{J}_z A(x), A(y) \rangle_{\tilde{U}} = \langle x, [A(x), A(y)] \rangle_{\tilde{r},\tilde{s}} = \langle z, C([x, y]) \rangle_{\tilde{r},\tilde{s}} = \langle C^r(z), x, y \rangle_U, \]
for all \(x, y \in U\) and \(z \in \mathbb{R}^{\tilde{r},\tilde{s}}\) which shows \(\Phi\). Conversely, if \(\Phi\) holds, then from \(\Phi\) we obtain \([A(x), A(y)] = C([x, y])\) and therefore the map \(\Phi = A \oplus C\) is a Lie algebra isomorphism. Note that \(\Phi\) implies that \(J_{C^r(z)}\) is singular, if and only if \(J_z\) is singular.

Let \(z_+\) and \(z_-\) be a positive and a negative vector in \(\mathbb{R}^{r,s}\), respectively. We set \(a_t = (1 - t)z_+ + tz_-\), \(0 \leq t \leq 1\). Then
\[ \langle a_0, a_0 \rangle_{r,s} = \langle z_+, z_+ \rangle_{r,s} > 0 \quad \text{and} \quad \langle a_1, a_1 \rangle_{r,s} = \langle z_-, z_- \rangle_{r,s} < 0. \]

There is \(t_0 \in (0, 1)\) with \(\langle a_{t_0}, a_{t_0} \rangle_{r,s} = 0\) and therefore \(J_{a_{t_0}}\) is singular. On the other hand, if \(z_1\) and \(z_2\) are orthonormal and both positive (negative) vectors in \(\mathbb{R}^{r,s}\) and \(b_t = (1 - t)z_1 + tz_2\), \(0 \leq t \leq 1\), then \(\langle b_t, b_t \rangle_{r,s} = (1 - t)^2 + t^2 > 0\), for all \(t \in [0, 1]\). This implies that \(J_{b_t}\) is non-singular for all \(t \in [0, 1]\). Hence, the operator \(C^r\) either preserves or reverses the sign of elements in \(\mathbb{R}^{r,s}\). These observations imply that only the cases \((r, s) = (\tilde{r}, \tilde{s})\) or \((r, s) = (\tilde{s}, \tilde{r})\) are possible, if \(r \neq s\).

For the remaining part of the proof we assume that \(r \neq s\) and \(\Phi: \mathcal{N}_{r,s}(U) \to \mathcal{N}_{s,r}(\tilde{U})\) is a Lie algebra isomorphism. Then \((A^T \tilde{J}_z A)^2 = J_{C^r(z)}^{2r} = -\langle C^r(z), C^r(z) \rangle_{r,s} \text{Id}_U\) by \(\Phi\) and therefore
\[ \det ((A^T \tilde{J}_z A)^2) = \det (AA^T)^2 \langle z, z \rangle_{s,r}^{2N} = \langle C^r(z), C^r(z) \rangle_{r,s}^{2N}, \]
where \(2N = \dim U = \dim \tilde{U}\). Since the operator \(C^r: \mathbb{R}^{s,r} \to \mathbb{R}^{r,s}\) reverses the sign of vectors we obtain
\[ |\det (AA^T)|^{1/N} \cdot \langle z, z \rangle_{s,r} = -\langle C^r(z), C^r(z) \rangle_{r,s} = -\langle z, CC^r(z) \rangle_{s,r}. \]

Multiplying \(A\) by a suitable constant we assume that \(|\det AA^T| = 1\) and \(CC^r = -\text{Id}\). \(\Box\)
Corollary 4. Let \( r \neq s \) and \( \Phi : \mathcal{N}_{r,s}(U) \to \mathcal{N}_{r,s}(\bar{U}) \) is a Lie algebra isomorphism, written in form \((7)\). Then \( CC^r = \text{Id} \). If \( r = s \) both cases \( CC^r = \pm \text{Id} \) are possible, see Theorem 6 and Corollary 7.

Lemma 2. Let \( \{U, \langle \ldots \rangle_U; \} \) and \( \{\bar{U}, \langle \ldots \rangle_{\bar{U}}; \} \) be admissible modules and representation maps of the Clifford algebras \( Cl_{r,s} \) and \( Cl_{s,r} \), respectively. Assume that \( r \neq s \). If
\[
\Psi = \begin{pmatrix} A \quad 0 \\ 0 \quad C \end{pmatrix} : \mathcal{N}_{r,s}(U) \to \mathcal{N}_{r,s}(\bar{U}) \quad \text{and} \quad \Phi = \begin{pmatrix} A \quad 0 \\ 0 \quad C \end{pmatrix} : \mathcal{N}_{r,s}(U) \to \mathcal{N}_{s,r}(\bar{U}),
\]
are Lie algebra isomorphisms, then the maps defined by
\[
\Psi^r = \begin{pmatrix} A^r \quad 0 \\ 0 \quad C^r \end{pmatrix} : \mathcal{N}_{r,s}(\bar{U}) \to \mathcal{N}_{r,s}(U) \quad \text{and} \quad \Phi^r = \begin{pmatrix} A^r \quad 0 \\ 0 \quad C^r \end{pmatrix} : \mathcal{N}_{s,r}(\bar{U}) \to \mathcal{N}_{r,s}(U),
\]
respectively, are Lie algebra isomorphisms as well.

Proof. First we show that \( \Psi^r \) defines a Lie algebra automorphism. According to \((7)\) and Corollary 4 we have
\[
(A^r \bar{J}_z A)^2 = -\langle C^r(z), C^r(z) \rangle_{r,s} \text{Id} = -\langle z, z \rangle_{r,s} \text{Id}, \tag{9}
\]
which implies \( AA^r \bar{J}_z AA^r \bar{J}_z AA^r = -\langle z, z \rangle_{r,s} AA^r \). Multiplying by \((AA^r)^{-1}\) from the right hand side we obtain
\[
AA^r \bar{J}_z AA^r \bar{J}_z = -\langle z, z \rangle_{r,s} \text{Id} = \bar{J}_z^2 \quad \Longrightarrow \quad AA^r \bar{J}_z AA^r = \bar{J}_z.
\]
Replacing \( A^r \bar{J}_z A \) by \( J_{C^r(z)} \), we get \( AA^r \bar{J}_z AA^r = AJ_{C^r(z)}A^r = \bar{J}_z = J_{CC^r(z)} \). Hence the map \( \Psi^r \) is a Lie algebra automorphism.

Next we show that \( \Phi^r \) defines a Lie algebra isomorphism. From \( C^r C = CC^r = \text{Id} \) we obtain
\[
(A^r \bar{J}_z A)^2 = J_{C^r(z)}^2 = -\langle C^r(z), C^r(z) \rangle_{r,s} \text{Id} = \langle z, z \rangle_{s,r} \text{Id} = -\bar{J}_z^2
\]
instead of \((9)\). It leads to \( -\bar{J}_z = AA^r \bar{J}_z AA^r = AJ_{C^r(z)}A^r \) by the same argument as above. Replacing \( z \) by \( C(z) \), we obtain \( AJ_{A^r} = \bar{J}_{C(z)} \), which proves the assertion. \( \square \)

The structure of a Lie algebra isomorphism inherits somehow \( \mathbb{Z}_2 \)-grading of the underlying Clifford algebras as shows the following lemma.

Lemma 3. Let \( \{z_i\}_{i=1}^{r+s} \) be an orthonormal basis of \( \mathbb{R}^{r+s} \) and the maps \( A \) and \( C \) as in Lemma 2. Then the following relations hold

1. If \( p = 2m, \ m \in \mathbb{N} \), then
\[
A \prod_{j=1}^{p} J_{z_j} = (-1)^m \prod_{j=1}^{p} \bar{J}_{C(z_j)} A, \quad A^r \prod_{j=1}^{p} \bar{J}_{z_j} = (-1)^m \prod_{j=1}^{p} J_{C^r(z_j)} A^r. \tag{10}
\]
\[
A^r A \prod_{j=1}^{p} J_{z_j} = \prod_{j=1}^{p} J_{z_j} A^r, \quad AA^r \prod_{j=1}^{p} \bar{J}_{C(z_j)} = \prod_{j=1}^{p} \bar{J}_{C(z_j)} AA^r. \tag{11}
\]

2. If \( p = 2m + 1, \ m \in \mathbb{N} \cup \{0\} \), then
\[
A \prod_{j=1}^{p} J_{z_j} A^r = (-1)^m \prod_{j=1}^{p} \bar{J}_{C(z_j)} A, \quad A^r \prod_{j=1}^{p} \bar{J}_{z_j} A = (-1)^m \prod_{j=1}^{p} J_{C^r(z_j)}. \tag{12}
\]
\[
A^r A \prod_{j=1}^{p} J_{z_j} A^r = - \prod_{j=1}^{p} J_{z_j}, \quad AA^r \prod_{j=1}^{p} \bar{J}_{z_j} AA^r = - \prod_{j=1}^{p} \bar{J}_{z_j} \tag{13}
\]
Proof. We only show the second parts of equalities, since the first parts can be obtained from them by transpositions. We assume that $C^r C = -\text{Id}$ and apply the induction arguments.

If $m = 0$ ($p = 1$) then (12) is reduced to (7). Assume now that (12) holds for $p = 2m + 1$.

Choose $z^*$ from the orthonormal basis $\{z_i\}_{i=1}^{p}$ and calculate

$$A^T \prod_{j=1}^{p} \tilde{J}_{z_j} \tilde{J}_{z^*} = A^T \prod_{j=1}^{p} \tilde{J}_{z_j} A A^{-1} \tilde{J}_{z^*} = (-1)^m \prod_{j=1}^{p} J_{C^r(z_j)} A^{-1} \tilde{J}_{z^*} (A^T)^{-1} A^T$$

$$= (-1)^{m+1} \left( \prod_{j=1}^{p} J_{C^r(z_j)} \right) J_{C^r(z^*)} A^T.$$

Thus, we proved (10) for $p = 2m + 1$. In the last equality we argued as follows. Since $\langle z^*, z^* \rangle_{s,r} = -\langle z^*, z^* \rangle_{s,r}$ and $(C^r(z^*), C^r(z^*))_{r,s} = (-\langle z^*, z^* \rangle_{s,r})$, we obtain

$$A^{-1} \tilde{J}_{z^*} (A^T)^{-1} = \left( A^T \tilde{J}_{z^*} A \right)^{-1} = \left( \frac{1}{\langle z^*, z^* \rangle_{s,r}} J_{C^r(z^*)} \right)^{-1} = -J_{C^r(z^*)}.$$

Thus in the previous step we, particularly showed that (10) is true for $m = 1$ ($p = 2$). Assume now that (10) holds for $p = 2m$, $m = 0, 1, \ldots$, then

$$A^T \left( \prod_{j=1}^{p} \tilde{J}_{z_j} \right) \tilde{J}_{z^*} = (-1)^m \prod_{j=1}^{p} J_{C^r(z_j)} A^T \tilde{J}_{z^*} = (-1)^m \left( \prod_{j=1}^{p} J_{C^r(z_j)} \right) J_{C^r(z^*)}.$$

Thus the assertion (12) holds for $p = 2m + 1$.

It is sufficient to show (11) for $p = 2$. We have by (10)

$$A^T A \prod_{j=1}^{p} J_{z_j} A^T A = (-1)^m A^T A \prod_{j=1}^{p} J_{C^r(z_j)} A^T \tilde{J}_{z^*} = (-1)^m \left( \prod_{j=1}^{p} J_{C^r(z_j)} \right) J_{C^r(z^*)}.$$

Identity (13) can be deduced from (12). We obtain

$$A^T A \prod_{j=1}^{p} J_{z_j} A^T A = (-1)^m A^T A \prod_{j=1}^{p} J_{C^r(z_j)} = (-1)^m \left( \prod_{j=1}^{p} J_{C^r(z_j)} \right) J_{C^r(z^*)}.$$

and since $p$ is odd the equality (13) follows. \qed

**Remark 2.** It is clear that the result of Lemma 3 does not depend on the permutation of the basis elements. We emphasise that the existence of a Lie algebra isomorphism $\Phi = A \oplus C: N_{r,s}(U) \rightarrow N_{s,r}(U)$ is equivalent to the requirement that relation (7) holds. Moreover, (7) implies all the equalities listed in Lemma 3.

### 3.2. Observations on general structure of a possible isomorphism

We set up the notations. We denote by $\{z_1, \ldots, z_{r+s}\}$ an orthonormal basis for $\mathbb{R}^{r+s}$, where $\langle z_i, z_j \rangle_{r,s} = \epsilon_i(r, s) \delta_{i,j}$. A linear map $P: V_r \rightarrow V_s$ such that $P^2 = \text{Id}$ is called an involution. The eigenspaces of an involution $P$ we denote by $E^k_P$, where $k \in \{1, -1\}$ according to the eigenvalue. In order to denote the intersection of eigenspaces of several mutually commuting involutions $P_j$, $j = 1, \ldots, N$, we use multi-index $I = (k_1, \ldots, k_N)$ and write $E^I = \cap_{j=1}^{N} E_{P_j}^{k_j}$.

The basis for $\mathbb{R}^{r,s}$ we denote by $\{w_s, \ldots, w_{r+1}, w_r, \ldots, w_1\}$ with first “$s$” elements being positive and the last “$r$” vectors being negative. Therefore, the representation maps $J: \text{Cl}_{s,r} \rightarrow \text{End}(V^{s,r})$ satisfies

$$\tilde{J}_{w_j} = -\text{Id}_{V^{s,r}}, \quad j = s + r, \ldots, r + 1, \quad \tilde{J}_{w_j}^2 = \text{Id}_{V^{s,r}}, \quad j = r, \ldots, 1.$$
In general, operators and other objects related to the Clifford algebra $\text{Cl}_{r,s}$ will be denoted by letters $P, E, R, \ldots$, meanwhile the operators, associated to the Clifford algebra $\text{Cl}_{s,r}$ will carry the tilde on the top: $\tilde{P}, \tilde{E}, \tilde{R}, \ldots$. At the end we formulate an immediate corollary of Lemma 3 that will be used frequently in the paper.

**Corollary 5.** Let $r \neq s$ and assume that there is a Lie algebra isomorphism $\Phi = A \oplus C: N^{r,s} \to N^{s,r}$ with $A: V^{r,s} \to V^{s,r}$, $C: \mathbb{R}^{r,s} \to \mathbb{R}^{s,r}$, where we set $C(z_j) = w_j$ and $C'(w_j) = -z_j$. Let $P_j$, $j = 1, \ldots, N$ be mutually commuting isometric involutions on $V^{r,s}$ obtained by product of some $J_{z_k}$. Let $\tilde{P}_j$ be mutually commuting isometric involutions on $V^{s,r}$ obtained from $P_j$ by changing $J_{z_k}$ to $\tilde{J}_{w_k}$ and such that $A\tilde{P}_j = \tilde{P}_j A$, $j = 1, \ldots, N$. We denote by $E^I$ and $\tilde{E}^I$ the common eigenspaces of $P_j$ and $\tilde{P}_j$, respectively. Then

1. the map $A$ can be written as $A = \oplus A_I$, where $A_I: E^I \to \tilde{E}^I$ for any choice of $I = (k_1, \ldots, k_N)$;
2. if $\prod_{j=1}^p J_{z_j}: E^I \to E^I$ for some $I$, then $\prod_{j=1}^p \tilde{J}_{w_j}: \tilde{E}^I \to \tilde{E}^I$, and

\[
A_I \prod_{j=1}^p J_{z_j} = \begin{cases} (-1)^m \prod_{j=1}^p \tilde{J}_{w_j}(A_I^{-1})^{-1}(x_I), & \text{if } p = 2m + 1, \\ (-1)^m \prod_{j=1}^p \tilde{J}_{w_j} A_I(x_I), & \text{if } p = 2m, \end{cases} \quad x_I \in E_I,
\]

\[
A_I^p \prod_{j=1}^p \tilde{J}_{w_j} = \begin{cases} (-1)^{m+1} \prod_{j=1}^p J_{z_j}(A_I^{-1})^{-1}(y_I), & \text{if } p = 2m + 1 \\ (-1)^m \prod_{j=1}^p J_{z_j} A_I^{-1}(y_I), & \text{if } p = 2m, \end{cases} \quad y_I \in \tilde{E}_I.
\]

Corollary 5 gives an idea of a possible construction of an isomorphism $\Phi = A \oplus C: N^{r,s} \to N^{s,r}$. Choosing the bases $\{z_j\}_{j=1}^{r+s}$ for $\mathbb{R}^{r,s}$ and $\{w_j\}_{j=1}^{s+r}$ for $\mathbb{R}^{s,r}$ we define the map $C: \mathbb{R}^{r,s} \to \mathbb{R}^{s,r}$, by setting $C(z_j) = w_j$ and $C'(w_j) = -z_j$. Further, if we find mutually commuting isometric involutions $P_j$ and $\tilde{P}_j$, $j = 1, \ldots, N$, acting on $V^{r,s}$ and $V^{s,r}$, respectively, we can reduce the construction of the map $A: V^{r,s} \to V^{s,r}$ to the construction of the maps $A_I: E^I \to \tilde{E}^I$. Finally, we set $A = \oplus A_I$. Theorem 3 states that, under some conditions, the construction of all maps $A_I$ can be obtained from the only one map $A_1: E^1 \to \tilde{E}^1$, where we denote $E^1 = \bigcap_{j=1}^N E_{P_j}$.

**Theorem 3.** We set $C(z_j) = w_j$ and $C'(w_j) = -z_j$ for orthonormal bases $\{z_j\}_{j=1}^{r+s}$ for $\mathbb{R}^{r,s}$ and $\{w_j\}_{j=1}^{s+r}$ for $\mathbb{R}^{s,r}$. Let $P_j$, $j = 1, \ldots, N$ be mutually commuting isometric involutions on $V^{r,s}$ obtained by product of some $J_{z_k}$ and $\tilde{P}_j$ be mutually commuting isometric involutions on $V^{s,r}$ obtained from $P_j$ by changing $J_{z_k}$ to $\tilde{J}_{w_k} = \tilde{J}_{C(z_k)}$. We denote by $E^I$ and $\tilde{E}^I$ the common eigenspaces of $P_j$ and $\tilde{P}_j$, respectively, and set $E^1 = \bigcap_{j=1}^N E_{P_j}$ and $\tilde{E}^1 = \bigcap_{j=1}^N \tilde{E}_{P_j}$. We assume also that

(a) there are maps $G_I: E^1 \to E^I$ for all multi-indices $I$, written in the form of product $G_I = \prod J_{z_i}$, and
(b) there exists a map $A_1: E^1 \to \tilde{E}^1$ such that

\[
A_1 \prod_{j=1}^p J_{z_j} = \begin{cases} (-1)^m \prod_{j=1}^p \tilde{J}_{C(z_j)}(A_1^{-1})^{-1}, & \text{if } p = 2m + 1, \\ (-1)^m \prod_{j=1}^p \tilde{J}_{C(z_j)} A_1, & \text{if } p = 2m, \end{cases} \quad (14)
\]
for any choice of the product $\prod_{j=1}^p J_{z_j}$ that leaves invariant the space $E^1$.

Then there is a map $A: V^{r,s} \rightarrow V^{s,r}$ such that $\Phi = A \oplus C: \mathcal{N}^{r,s} \rightarrow \mathcal{N}^{s,r}$ is the Lie algebra isomorphism.

Proof. We define the maps $A_I: E^I \rightarrow \tilde{E}^I$ by the following

$$A_I = \begin{cases} 
(-1)^m \tilde{G}_I (A^{-1}_1)^r G^{-1}_{1}, & \text{if } G_I = \prod_{j=1}^{p=2m+1} J_{z_j}, \quad \tilde{G}_I = \prod_{j=1}^{p=2m+1} \tilde{J}_{w_j}, \\
(-1)^m \tilde{G}_I A_I G^{-1}_I, & \text{if } G_I = \prod_{j=1}^{p=2m} J_{z_j}, \quad \tilde{G}_I = \prod_{j=1}^{p=2m} \tilde{J}_{w_j}.
\end{cases} \quad (15)$$

Here and further $\tilde{J}_{w_k} = \tilde{J}_{C(z_k)}$. For the convenience we also write the adjoint maps.

$$A_I^T = \begin{cases} 
(-1)^{m+1} G_I A_I^{-1} \tilde{G}_I^{-1}, & \text{if } G_I = \prod_{j=1}^{p=2m+1} J_{z_j}, \quad \tilde{G}_I = \prod_{j=1}^{p=2m+1} \tilde{J}_{w_j}, \\
(-1)^m G_I A_I \tilde{G}_I^{-1}, & \text{if } G_I = \prod_{j=1}^{p=2m} J_{z_j}, \quad \tilde{G}_I = \prod_{j=1}^{p=2m} \tilde{J}_{w_j}.
\end{cases} \quad (16)$$

Then we set $A = \oplus A_I$. We only need to check the condition $AJ_{z_I} A^T = \tilde{J}_{C(z_I)}$ for any $z_I$ from the orthonormal basis for $\mathbb{R}^{r,s}$.

Observe the following facts. The spaces $E^I$ are mutually orthogonal because if $P_j(x) = x$, and $P_j(y) = -y$ for some isometry $P_j$, then

$$\langle x, -y \rangle_{V^{r,s}} = \langle P_j(x), P_j(y) \rangle_{V^{r,s}} = \langle x, y \rangle_{V^{r,s}} \implies \langle x, y \rangle_{V^{r,s}} = 0. \quad (17)$$

Thus $V^{r,s} = \oplus E^I$, and $V^{s,r} = \oplus \tilde{E}^I$, where the direct sums are orthogonal. The maps $G_I$ and $\tilde{G}_I$ are invertible and

$$G^{-1}_I = (\prod_{j=1}^p J_{z_j})^{-1} = (-1)^p \prod_{j=1}^p \langle z_j, z_j \rangle_{V^{r,s}}^{-1} \prod_{k=0}^{p-1} J_{z_{p-k}}. \quad (18)$$

Lemma 3 implies that

$$(A^{-1}_I)^r \prod_{j=1}^p J_{z_j} A^{-1}_I = (-1)^{m+1} \prod_{j=1}^p \tilde{J}_{C(z_j)}; \quad A_1 \prod_{j=1}^p J_{z_j} A^{-1}_I = (-1)^m \prod_{j=1}^p \tilde{J}_{C(z_j)}, \quad (19)$$

if $p = 2m + 1$, $m = 0, 1, \ldots$, and

$$(A^{-1}_I)^r \prod_{j=1}^p J_{z_j} A^{-1}_I = (-1)^m \prod_{j=1}^p \tilde{J}_{C(z_j)}; \quad A_1 \prod_{j=1}^p J_{z_j} A^{-1}_I = (-1)^m \prod_{j=1}^p \tilde{J}_{C(z_j)}, \quad (20)$$

if $p = 2m$, $m = 1, \ldots$.

We choose $J_{z_0}$ and $y \in V^{s,r} = \oplus \tilde{E}^I$. Then we write $y = \oplus y_I$ with $y_I \in \tilde{E}^I$. Thus we distinguish the cases when the map $G_I$ is the product of odd or even number of representation maps $J_{z_I}$. Moreover, for the multi-index $I$ we find a multi-index $K$ such that $G^{-1}_K J_{z_0} G_I$ leaves invariant the space $E^1$. Since $G_K$ can also be product of even or odd number of $J_{z_k}$, we differ
the following cases:

\[
AJ_{z_{j_0}} A^T y_1 = A_K J_{z_{j_0}} A^T_I y_I = \\
\begin{cases}
(-1)^{k+m+1} \widetilde{G}_K (A^{-1}_I)^t G_{I}^{-1} J_{z_{j_0}} G_I A^{-1}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m+1} J_{z_i}, \ G_K = \prod_{l=1}^{2k+1} J_{z_l}, \\
(-1)^{k+m+1} \widetilde{G}_K A_I G_{K}^{-1} J_{z_{j_0}} G_I A^{-1}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m+1} J_{z_i}, \ G_K = \prod_{l=1}^{2k} J_{z_l}, \\
(-1)^{k+m} \widetilde{G}_K (A^{-1}_I)^t G_{I}^{-1} J_{z_{j_0}} G_I A^{-1}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m} J_{z_i}, \ G_K = \prod_{l=1}^{2k+1} J_{z_l}, \\
(-1)^{k+m} \widetilde{G}_K A_I G_{K}^{-1} J_{z_{j_0}} G_I A^{-1}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m} J_{z_i}, \ G_K = \prod_{l=1}^{2k} J_{z_l}, \\
\end{cases}
\]

by definitions \((15)\) and \((16)\) of \(A_I\) and \(A^T_I\). Now we observe that

\[
\prod_{l=1}^{q} \frac{1}{(z_l, z_l)} \prod_{n=0}^{q-1} \widetilde{J}_{C(z_{q-n})} = \prod_{l=1}^{q} \frac{1}{(C(z_l), C(z_l))} \prod_{n=0}^{q-1} \widetilde{J}_{C(z_{q-n})} = \prod_{n=0}^{q-1} \widetilde{J}_{C(z_{q-n})} = \widetilde{G}_K^{-1}.
\]

Counting the number of elements in the product \(G_{K}^{-1} J_{z_{j_0}} G_I\) and using \((18)\) for \(G_K\), we apply corresponding formulas from \((19)\) or \((20)\), and then use \((21)\). We obtain

\[
AJ_{z_{j_0}} A^T y_1 = \\
\begin{cases}
(-1)^{4k+2m+4} \widetilde{G}_K \widetilde{G}_K^{-1} \widetilde{J}_{C(z_{j_0})} \widetilde{G}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m+1} J_{z_i}, \ G_K = \prod_{l=1}^{2k+1} J_{z_l}, \\
(-1)^{4k+2m+2} \widetilde{G}_K \widetilde{G}_K^{-1} \widetilde{J}_{C(z_{j_0})} \widetilde{G}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m+1} J_{z_i}, \ G_K = \prod_{l=1}^{2k} J_{z_l}, \\
(-1)^{4k+2m+2} \widetilde{G}_K \widetilde{G}_K^{-1} \widetilde{J}_{C(z_{j_0})} \widetilde{G}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m} J_{z_i}, \ G_K = \prod_{l=1}^{2k+1} J_{z_l}, \\
(-1)^{4k+2m+2} \widetilde{G}_K \widetilde{G}_K^{-1} \widetilde{J}_{C(z_{j_0})} \widetilde{G}_I \widetilde{G}_I^{-1} y_I & \text{if } G_I = \prod_{i=1}^{2m} J_{z_i}, \ G_K = \prod_{l=1}^{2k} J_{z_l}, \\
\end{cases}
\]

Thus \(AJ_{z_{j_0}} A^T y_1 = \widetilde{J}_{C(z_{j_0})} y_I\) and we finish the proof. \(\square\)

3.3. Isomorphic Lie algebras. We start from the construction of the isomorphism of Lie algebras \(N_{r,s}\) and \(N_{s,r}\) in 8 basic cases. We also show the existence of Lie algebra automorphisms \(\Psi : N_{r,r} \to N_{r,r}, \ r = 1, 2, 4\) such that \(\Psi = A \oplus C, CC^r = -Id\). This allows to apply the periodicity arguments in Theorems \([7]\) and \([8]\) for the classification of higher dimensional Lie algebras.

In the forthcoming theorems in order to build a Lie algebra isomorphism we start from the construction of a convenient basis for the space \(E^1 = \cap_{j=1}^N E_{P_j}\), where \(P_j, j = 1, \ldots, N\) are some mutually commuting isometric involutions. To construct the basis we need to find a vector \(v \in E^1\) with \(<v,v>_{V_{r,s}} = 1\). Therefore, one has to be sure that the restriction of \(<\cdot,\cdot>_{V_{r,s}}\) to \(E^1\) is positive definite or neutral. The following lemmas describe sufficient conditions for that. In the case when \(E^1\) is one dimensional we change the sign of the scalar product on the module space if it needs, see Remark \([4]\).

**Lemma 4.** Let \((V, <\cdot,\cdot>_V)\) be a neutral scalar product space and \(P : V \to V\) an isometric involution. Then we have the following cases.
1) If a linear map $R: V \to V$ is an isometry such that $PR = -RP$, then each of
eigenspaces $E^1$ and $E^{-1}$ of $P$ is a neutral scalar product space with respect to the
restriction of the scalar product $\langle \cdot, \cdot \rangle_V$ on $E^1$ and $E^{-1}$.
2) If a linear map $R: V \to V$ is an anti-isometry such that $PR = -RP$, then the
restriction of $\langle \cdot, \cdot \rangle_V$ on each of $E^1$, $E^{-1}$ is non-degenerate neutral or sign definite,
3) If a linear map $R: V \to V$ is an anti-isometry such that $PR = RP$, then the restriction
of $\langle \cdot, \cdot \rangle_V$ on each of $E^1$, $E^{-1}$ is non-degenerate neutral.

Proof. To show the first statement we observe that the isometry $R$ acts as an isometry from $E^1$
to $E^{-1}$. Since the eigenspaces $E^1$ and $E^{-1}$ are orthogonal, see (17), the scalar product $\langle \cdot, \cdot \rangle_V$
restricted to each $E^1$, $E^{-1}$ is non-degenerate. If the scalar product restricted to $E^1$ would
be positive definite, then the scalar product restricted to $E^{-1}$ would be also positive definite,
since the map $R$ is an isometry which contradicts the assumption that space $(V, \langle \cdot, \cdot \rangle_V)$
is neutral. The same arguments show that the restriction to $E^1$ could not be negative definite.
So the scalar product restricted to $E^1$ and therefore to $E^{-1}$ should be neutral.

In order to prove the second statement, we note that since $R: E^1 \to E^{-1}$ is an anti-isometry,
the restriction of $\langle \cdot, \cdot \rangle_V$ to $E^1$ can be sign definite and the restriction of $\langle \cdot, \cdot \rangle_V$ to $E^{-1}$ will
have opposite sign due to neutral nature of $(V, \langle \cdot, \cdot \rangle_V)$.

In the third case since the eigenspaces $E^1$ and $E^{-1}$ are invariant under $R$ but contains posi-
tive and negative vectors, then each of them is decomposed into subspaces of equal dimension
and the restriction of $\langle \cdot, \cdot \rangle_V$ on these subspaces is sign definite but of opposite signs. Thus
$E^1$ and $E^{-1}$ are neutral spaces.

Lemma 5. Let $(V, \langle \cdot, \cdot \rangle_V)$ be a neutral scalar product space. Let $P_1, \ldots, P_N$
be isometric mutually commuting involutions defined on $(V, \langle \cdot, \cdot \rangle_V)$ and $R_1, \ldots, R_N, R_{N+1}$
linear anti-isometric operators on $V$ such that

\[
P_1 R_1 = -R_1 P_1, \quad P_2 R_2 = R_2 P_1, \quad \ldots \quad P_1 R_N = R_N P_1, \quad P_1 R_{N+1} = R_{N+1} P_1,
P_2 R_2 = -R_2 P_2, \quad \ldots \quad P_2 R_N = R_N P_2, \quad P_2 R_{N+1} = R_{N+1} P_2,
\]

\[
\vdots
\]

\[
P_N R_N = -R_N P_N, \quad P_N R_{N+1} = R_{N+1} P_N.
\]

Then each common eigenspace $E^1$ of $P_1, \ldots, P_N$ is a non-trivial and neutral scalar product
space.

Proof. Let us assume that $P_1$ and $R_1$, $R_2$ satisfies the conditions: $P_1 R_1 = -R_1 P_1$ and $P_1 R_2 =
R_2 P_1$. The non-degeneracy of the restriction can be shown as in Lemma [4] The presence of
the operator $R_1$ ensures that the restriction of $\langle \cdot, \cdot \rangle_V$ to $E^1_{P_1}$ or $E^{-1}_{P_1}$ is neutral or sign definite
and the spaces $E^1_{P_1}$ and $E^{-1}_{P_1}$ have equal dimension. Since $R_2$ preserves $E^1_{P_1}$ and it is an
anti-isometry, the space $E^1_{P_1}$ contains both positive and negative vectors forming subspaces
of equal dimension. The same arguments, applied to $E^{-1}_{P_1}$. Thus, spaces $E^1_{P_1}$ and $E^{-1}_{P_1}$ are,
actually, neutral spaces.

Now we repeat the arguments applying them to the neutral spaces $E^1_{P_1}$ and $E^{-1}_{P_1}$ and the
operators $P_2$ and $R_2$, $R_3$. After $N$ steps we finish the proof.

We call the anti-isometric operators $R_1, \ldots, R_N, R_{N+1}$, described in Lemma [5] complementary
operators to the family $P_1, \ldots, P_N$. In some of situations the operator $R_{N+1}$ can be omitted, but we still call the system of operators $R_1, \ldots, R_N$ complementary.

We say that three operators $i, j, k: V^r,s \to V^r,s$ form a quaternion structure if they satisfy the
relation

\[
i^2 = j^2 = k^2 = -\text{Id}_{V^r,s}, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.
\]  

(22)

Theorem 4. The Lie algebras $N_{r,0}$ and $N_{0,r}$ are isomorphic for $r = 1, 2, 4, 8$. 

We conclude that the matrix for the map $A$ are used to define the orthonormal basis $\{v, J_z v, z\}$ for $N_{1,0}$, and $\{u, J_w u, w\}$ for $N_{0,1}$.

The isomorphism map $\Phi: N_{1,0} \rightarrow N_{0,1}$ is given by

$$v \mapsto u \quad J_z v \mapsto J_w u, \quad z \mapsto w,$$

and the non-vanishing commutators are $[v, J_z v] = z$, and $[u, J_w u] = w$. Here $A^T u = v$, $A^T J_w u = -J_z v$, $C^T(w) = -z$. We see from the commutation relations that the Lie algebras $N_{1,0}$ and $N_{0,1}$ are isomorphic to the Heisenberg algebra.

**Case $r = 2$.** The minimal admissible module $V^{1,0}$ is isometric to $\mathbb{R}^{4,0}$ and $V^{0,1}$ is isometric to $\mathbb{R}^{2,2}$. We choose $v \in V^{2,0}$ and $u \in V^{0,2}$, with $\langle v, v \rangle_{V^{1,0}} = \langle u, u \rangle_{V^{0,2}} = 1$ and construct the orthonormal bases:

$$\{x_1 = v, x_2 = J_z v, x_3 = J_{z^2} v, x_4 = J_{z_1} J_{z_2} v, z_1, z_2\} \quad \text{for} \quad N_{2,0},$$

$$\{y_1 = u, y_2 = J_w u, y_3 = J_{w^2} u, y_4 = J_{w_2} J_{w_1} u, w_1, w_2\} \quad \text{for} \quad N_{0,2}.$$

The isomorphism $\Phi$ is given by $x_j \mapsto y_j$, $j = 1, \ldots, 4$ and $z_k \mapsto w_k$, $k = 1, 2$ and then it is extended by linearity. The non-vanishing commutation relations on $N_{2,0}$ are

$$[x_1, x_2] = z_1, \quad [x_1, x_3] = z_2, \quad [x_2, x_4] = -z_2, \quad [x_3, x_4] = z_1,$$

and, correspondingly, for the basis of $N_{0,2}$.

**Case $r = 4$.** The minimal admissible module $V^{4,0}$ is isometric to $\mathbb{R}^{8,0}$. We choose an isometric involution $P = J_{z_1} J_{z_2} J_{z_3} J_{z_4}$ on $V^{4,0}$ and write $V^{4,0} = E_P^1 \oplus \bar{E}_P^1$. The operators $i = J_{z_1} J_{z_2}$, $j = J_{z_2} J_{z_3}$, $k = J_{z_2} J_{z_4}$, define a quaternion structure on $E_P^1$, commute with $P$ and therefore leave invariant the space $E_P^1$. Let $v \in V^{4,0}$ be such that $\langle v, v \rangle_{V^{4,0}} = 1$ and $P(v) = v$. Then

$$\{x_1 = v, x_2 = i(v), x_3 = j(v), x_4 = k(v)\} \quad \text{is an orthonormal basis for} \quad E_P^1.$$

The minimal admissible module $V^{0,4}$ is isometric to $\mathbb{R}^{14,4}$. Let $\bar{P} = J_{w_1} J_{w_2} J_{w_3} J_{w_4}$ and we write $V^{0,4} = \bar{E}_P^1 \oplus \bar{E}_P^{-1}$. The complementary operators are $\bar{P}_1 = \bar{J}_{w_3}$ and $\bar{P}_2 = \bar{J}_{w_1} \bar{J}_{w_2}$. Choose $u \in \bar{E}_P^1$ such that $\langle u, u \rangle_{V^{0,4}} = 1$. Then the operators $\bar{i} = \bar{J}_{w_1} \bar{J}_{w_2}$, $\bar{j} = \bar{J}_{w_1} \bar{J}_{w_3}$, $\bar{k} = -\bar{J}_{w_1} \bar{J}_{w_4}$ are used to define the orthonormal basis $\{y_1 = u, y_2 = \bar{i}(u), y_3 = \bar{j}(u), y_4 = \bar{k}(u)\}$ for $\bar{E}_P^1$.

Now we construct the map $\Phi = A \oplus C$ by setting $C(z_k) = w_k$, $C^T(w_k) = -z_k$ and $A = A_1 \oplus A_{-1}$ according to Corollary 5. To construct $A_1: E_P^1 \rightarrow \bar{E}_P^1$, we write $A_1(v) = (a_1 + a_2 \bar{i} + a_3 \bar{j} + a_4 \bar{k})u$. Moreover, $A_1$ has to satisfy the relations $A_1 \bar{i} = -\bar{i} A_1$, $A_1 \bar{j} = -\bar{j} A_1$, $A_1 \bar{k} = \bar{k} A_1$. Thus we obtain

$$A_1(x_2) = -\bar{i} A_1(v), \quad A_1(x_3) = -\bar{j} A_1(v), \quad A_1(x_4) = \bar{k} A_1(v).$$

We conclude that the matrix for the map $A_1$ is given by

$$A_1 = \begin{pmatrix}
  a_1 & a_2 & a_3 & -a_4 \\
  a_2 & -a_1 & -a_4 & -a_3 \\
  a_3 & a_4 & -a_1 & a_2 \\
  a_4 & -a_3 & a_2 & a_1
\end{pmatrix}.$$
The map $J_{z_1}: E^1_P \to E^{-1}_P$ is used to define $A_{-1}: E^1_P \to E^{-1}_P$ by $A_{-1} = \tilde{J}_{w_1} (A^{-1}_1)^* J_{z_1}^{-1}$. The proof of this case is finished by applying Theorem 3.

**Case** $r = 8$. Recall that the minimal admissible module $V^{8,0}$ is isometric to $\mathbb{R}^{16,0}$.

We fix the mutually commuting isometric involutions acting on $V^{8,0}$:

$$P_1 = J_{z_1} J_{z_2} J_{z_3} J_{z_4}, \quad P_2 = J_{z_1} J_{z_2} J_{z_5} J_{z_6}, \quad P_3 = J_{z_1} J_{z_2} J_{z_7} J_{z_8}, \quad P_4 = J_{z_1} J_{z_2} J_{z_5} J_{z_7}.$$

The common eigenspaces $E^l$ are one dimensional. We construct an orthonormal basis for $V^{8,0}$ starting from a vector $v \in E^1 = \cap_{j=1}^4 E^1_{P_j}$ such that $\langle v, v \rangle_{V^{8,0}} = 1$:

$$x_1 = v, \quad x_2 = J_{z_1} J_{z_2} v, \quad x_3 = J_{z_1} J_{z_3} v, \quad x_4 = J_{z_1} J_{z_4} v, \quad x_5 = J_{z_1} J_{z_5} v, \quad x_6 = J_{z_1} J_{z_6} v, \quad x_7 = J_{z_1} J_{z_7} v, \quad x_8 = J_{z_1} J_{z_8} v, \quad x_9 = J_{z_2} v, \quad x_{10} = J_{z_2} v, \quad x_{11} = J_{z_3} v, \quad x_{12} = J_{z_4} v, \quad x_{13} = J_{z_5} v, \quad x_{14} = J_{z_6} v, \quad x_{15} = J_{z_7} v, \quad x_{16} = J_{z_8} v. \quad (23)$$

Analogously, the isometric involutions $\tilde{P}_j$, obtained by changing $J_{z_j}$ to $\tilde{J}_{w_j}$ in $P_j$, $j = 1, 2, 3, 4$, are used to construct an orthonormal basis for $V^{8,0}$ by changing $x_k = \prod_{l} J_{z_{j_l}} v$ to $y_k = \prod_{l} \tilde{J}_{w_{j_l}} u$, where $u \in \tilde{E}^1$ with $\langle u, u \rangle_{V^{8,0}} = 1$. The complementary anti-isometric operators $\tilde{R}_1 = \tilde{J}_{w_1} \tilde{J}_{w_5}$, $\tilde{R}_2 = \tilde{J}_{w_6}$, and $\tilde{R}_3 = \tilde{J}_{w_7}$ guarantee that the space $\tilde{E} = \cap_{j=1}^3 \tilde{E}^1_{\tilde{P}_j}$ is two dimensional neutral. If the restriction of $\langle \ldots \rangle_{V^{8,0}}$ to the space $\tilde{E}^1_{\tilde{P}_4} \cap \tilde{E}$ is not positive definite, then we change the sign of the scalar product by Remark 1.

We claim that the map $\Phi = A \oplus C: \mathcal{N}^{8,0} \to \mathcal{N}^{0,8}$ such that

$$A(v) = u, \quad A(x_j) = -y_j, \quad j = 2, \ldots, 8, \quad A(x_j) = y_j, \quad j = 9, \ldots, 16,$$

and $C(z_k) = w_k$, $C^*(w_k) = -z_k$, $k = 1, \ldots, 8$ is the Lie algebra isomorphism. We show it by checking the commutators. First observe, that the structure of involutions implies that for any $1 < i < j \leq 8$ there is $1 < k \leq 8$ such that

$$J_{z_i} J_{z_j} = \pm J_{z_i} J_{z_k} \quad \text{and simultaneously} \quad \tilde{J}_{w_i} \tilde{J}_{w_j} = \pm \tilde{J}_{w_i} \tilde{J}_{w_k}. \quad (24)$$

The second observation is that $[x_i, x_j] = [y_i, y_j] = 0$ if either $1 \leq i, j \leq 8$ or $9 \leq i, j \leq 16$. Indeed, for instance, for any $1 \leq i \leq 8$ and $1 \leq j \leq 8$, we calculate

$$\langle [x_i, x_j], z_j \rangle_{V^{8,0}} = \langle [v, J_{z_1} J_{z_j} v], z_j \rangle_{V^{8,0}} = \langle J_{z_2} J_{z_j} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = \langle J_{z_2} J_{z_j} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = \langle J_{z_2} J_{z_j} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = \pm \langle J_{z_2} J_{z_k} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = \pm \langle J_{z_2} J_{z_k} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = 0,$$

for any $1 < k \leq 8$. Analogously, the rest of the cases is proved by using (24).

To show that $\Phi = A \oplus C$ is a Lie algebra isomorphism, we need to check $C[x_i, x_j] = [A(x_i), A(x_j)]$ for any choice of $i = 1, \ldots, 8$ and $j = 9, \ldots, 16$, since all other commutators vanish. We calculate for any $k = 1, \ldots, 8$, $i = 1$, and $j = 9, \ldots, 16$

$$\langle C[x_1, x_j], w_k \rangle_{V^{8,0}} = -\langle [x_1, x_j], z_k \rangle_{V^{8,0}} = -\langle J_{z_k} J_{z_i} J_{z_j} v, z_k \rangle_{V^{8,0}} = -\langle z_k, z_i \rangle_{V^{8,0}} \delta_{k,l},$$

$$\langle [A(x_1), A(x_j)], w_k \rangle_{V^{8,0}} = \langle [y_1, y_j], w_k \rangle_{V^{8,0}} = \langle \tilde{J}_{w_1} \tilde{J}_{w_j} u, \tilde{J}_{w_1} \tilde{J}_{w_j} u \rangle_{V^{8,0}} = \langle w_k, w_l \rangle_{V^{8,0}} \delta_{k,l}. $$

Since $-\langle z_k, z_i \rangle_{V^{8,0}} \delta_{k,l} = \langle w_k, w_l \rangle_{V^{8,0}} \delta_{k,l}$, we obtain that $C[x_1, x_j] = [A(x_1), A(x_j)]$. We continue and calculate for any $k = 1, \ldots, 8$, $i = 2, \ldots, 8$, and $j = 9, \ldots, 16$

$$\langle C[x_i, x_j], w_k \rangle_{V^{8,0}} = \langle [x_i, x_j], C^*(w_k) \rangle_{V^{8,0}} = -\langle [x_i, x_j], z_k \rangle_{V^{8,0}} = -\langle J_{z_2} J_{z_j} J_{z_m} v, J_{z_i} J_{z_j} v \rangle_{V^{8,0}} = -\tau \langle J_{z_2} J_{z_i} J_{z_m} v, J_{z_j} J_{z_i} v \rangle_{V^{8,0}} = -\tau \langle z_1, z_i \rangle_{V^{8,0}} \langle z_m, z_n \rangle_{V^{8,0}} \delta_{m,n}, \quad (25)$$

where $\tau = 2$, and so on.
where $\varepsilon = \pm 1$ and depends on number of permutations and sign in (24). Analogously
\[
\langle [A(x_i), A(x_j)], w_k \rangle_{0,8} = \langle [-y_i, y_j], w_k \rangle_{0,8} = -\varepsilon \langle J_{w_k} J_{w_1} J_{w_2} J_{w_3} u, J_{w_4} J_{w_5} u \rangle_{V,0,8} = -\varepsilon \langle w_1, w_1 \rangle_{0,8} \langle w_2, w_3 \rangle_{0,8} \delta_{m,n},
\]
where the value of $\varepsilon$ is the same as in (25), due to the same number of permutations and the equalities in (21).

Then the basis
\[
\{c, \tilde{J} \}_{0,8} = \{A(x_1), A(x_2)\}
\]
and
\[
\{b, \tilde{J} \}_{0,8} = \{w_1, w_2\}
\]
with the same
\[
\{c, \tilde{J} \}_{0,8} = \{A(x_1), A(x_2)\}
\]
and
\[
\{b, \tilde{J} \}_{0,8} = \{w_1, w_2\}
\]
are neutral 4-dimensional spaces invariant under the action of quaternion structure (26), which allows to find a convenient basis of $V_{1,1}$ with $\langle v, v \rangle_{V_{1,1}} = 1$. Then the basis $\{x_1 = v, x_2 = i(v), x_3 = j(v), x_4 = k(v)\}$ for $E^1$ is orthonormal by Lemma 1, where we set $\tilde{J} = j$. Analogous calculations we make for the Lie algebra $N_{2,5}$. The mutually
\[
\text{commuting isometric involutions and the anti-isometric complementary operators are}
\]
\[
\tilde{P}_1 = J_{w_1} J_{w_2} J_{w_3} J_{w_4}, \quad \tilde{P}_2 = J_{w_1} J_{w_3} J_{w_5} J_{w_7}, \quad \tilde{R}_1 = J_{w_1} J_{w_3} J_{w_5} J_{w_7} J_{w_9}, \quad \tilde{R}_2 = J_{w_5} J_{w_7} J_{w_9} J_{w_{11}}.
\]

Let us assume that there is an isomorphism $\Phi: N_{5,2} \rightarrow N_{2,5}$, $\Phi = A \oplus C$, $A: V_{5,2} \rightarrow V_{2,5}$, where we define $\tilde{C}(w) = \Phi(w) = z_j = w_j$, $j = 1, \ldots, 7$. Then according to Lemma 3 the map $A: V_{5,2} \rightarrow V_{2,5}$ has to satisfy the relations
\[
AP_j = \tilde{P}_j A, \quad Ai = -i A, \quad Aj = j A, \quad Ak = -k A.
\]
Thus, we apply Corollary 5 and set $A = \oplus A_j$. To construct $A_1 : E^1 \to \tilde{E}^1$ we write $A_1(v) = (a_1 + a_2 \mathbf{i} + a_3 \mathbf{j} + a_4 \mathbf{k}) u$, $a_j \in \mathbb{R}$. Since

$$A_1(x_2) = -\mathbf{i} A_1(v), \quad A_1(x_3) = \mathbf{j} A_1(v), \quad A_1(x_4) = -\mathbf{k} A_1(v),$$

we obtain the matrices for the map $A_1$ and $A_1^T$:

$$A_1 = \begin{pmatrix} a_1 & a_2 & -a_3 & a_4 \\ a_2 & -a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & -a_1 \end{pmatrix}, \quad A_1^T = \begin{pmatrix} a_1 & a_2 & -a_3 & -a_4 \\ a_2 & -a_1 & -a_4 & a_3 \\ a_3 & -a_4 & a_1 & -a_2 \\ -a_4 & -a_3 & a_2 & -a_1 \end{pmatrix}. \quad (27)$$

To find relations between $a_j$, we observe that $J_{z_5}$ preserves $E^1$ and $\widetilde{J}_{w_5}$ preserves $\tilde{E}^1$ and therefore they have to satisfy the relation $A_1^T \widetilde{J}_{w_5} A_1 = -J_{z_5}$. In order to calculate the matrices for $J_{z_5}$ and $\widetilde{J}_{w_5}$ we observe that the isometric involution $T = J_{z_1} J_{z_2} J_{z_3}$ commutes with $P_1$ and $P_2$ and $E^1 \cap \tilde{E}^1 = E^1$. Therefore $Tv = v$ and we obtain $J_{z_5} v = -\mathbf{i} v$. To find the matrix $\widetilde{J}_{w_5}$ we note that the isometric involution $\widetilde{T} = \widetilde{J}_{w_1} \widetilde{J}_{w_2} \widetilde{J}_{w_3} \widetilde{J}_{w_4}$ commutes with $\widetilde{P}_j$, $j = 1, 2$ and $E^1 = E^1 \oplus E^{-1}_{T^{-1}}$, where the eigenspaces $E^1_{T^{-1}}$ and $E^{-1}_{T^{-1}}$ of $\widetilde{T}$ are neutral. Thus, we can assume that $\widetilde{T}u = u$, which leads to $\widetilde{J}_{w_5} u = \widetilde{J}_u u$. Then

$$A_1^T \widetilde{J}_{w_5} A_1 = A_1^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} A_1 = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -J_{z_5}. \quad (28)$$

Thus, equation (28) leads to three relations

$$-2a_2 a_3 + 2a_1 a_4 = -1, \quad a_1 a_2 + a_3 a_4 = 0, \quad a_1^2 - a_2^2 + a_3^2 - a_4^2 = 0,$$

giving the solution

$$a_2 = a_3, \quad a_1 = -a_4, \quad a_1^2 + a_3^2 = \frac{1}{2}. \quad (29)$$

The operator $T = J_{z_1} J_{z_2} J_{z_3}$ leaves invariant subspace $E^1$ and therefore we have to check the equality $A_1 J_{z_1} J_{z_2} J_{z_3} A_1^T = -\widetilde{J}_{w_1} \widetilde{J}_{w_2} \widetilde{J}_{w_3}$. We calculate

$$A_1 J_{z_1} J_{z_2} J_{z_3} A_1^T = -\widetilde{J}_{w_1} \widetilde{J}_{w_2} A_1 J_{z_5} A_1^T = -\widetilde{J}_{w_1} \widetilde{J}_{w_2} \widetilde{J}_{w_3},$$

where we used the relations $A_1 \mathbf{i} = -\mathbf{i} A_1$ and $A_1 J_{z_5} A_1^T = \widetilde{J}_{w_5}$.

To construct the remaining parts of the map $A$, we use Theorem 3 and the maps

$$G_{1,-1} = J_{z_5} : E^1 \to E^{1,-1}, \quad G_{-1,1} = J_{z_3} : E^1 \to E^{-1,1}, \quad G_{-1,-1} = J_{z_1} : E^1 \to E^{-1,-1}.$$ 

Observe that the solution (29) shows that $A_1 v$ is a null vector.

**Case $(r, s) = (5, 1)$**: In this case we change the arguments and use the isomorphism between $N_{5,2}$ and $N_{2,5}$. Assume that the map $\Phi = A \oplus C : N_{5,2} \to N_{2,5}$, and $C(z_k) = w_k$ is a Lie algebra isomorphism. Recall that minimal admissible modules $V^{5,2}, V^{2,5}$ and $V^{5,1}, V^{1,5}$ are irreducible and isometric to $\mathbb{R}^{8,8}$. The natural inclusions $\mathbb{R}^{5,1} \subset \mathbb{R}^{5,2}$ and $\mathbb{R}^{1,5} \subset \mathbb{R}^{2,5}$ define the Clifford action of $\text{Cl}_{5,1}$ and $\text{Cl}_{1,5}$ on $V^{5,1}$ and $V^{1,5}$, respectively by restrictions of the Clifford action of $\text{Cl}_{5,2}$ and $\text{Cl}_{2,5}$.

Let $\pi_- : \mathbb{R}^{5,2} \to \mathbb{R}^{5,1}$ be the projection map defined by

$$z_1 \mapsto z_1, \quad \ldots, \quad z_6 \mapsto z_6, \quad z_7 \mapsto 0,$$

and let $\pi_+ : \mathbb{R}^{2,5} \to \mathbb{R}^{1,5}$ be the projection defined by

$$w_1 \mapsto w_1, \quad \ldots, \quad w_6 \mapsto w_6, \quad w_7 \mapsto 0.$$
Then the map

\[ \text{Id} \oplus \pi_- : \mathcal{N}_{5,2} = V^{5,2} \oplus \mathbb{R}^{5,2} \to \mathcal{N}_{5,1} = V^{5,1} \oplus \mathbb{R}^{5,1} \]

is a Lie algebra homomorphism with kernel \( K_- = \text{span}\{z_7\} \). Also \( \text{Id} \oplus \pi_+ \) is a Lie algebra homomorphism from \( \mathcal{N}_{2,5} \) to \( \mathcal{N}_{1,5} \) with kernel \( K_+ = \text{span}\{w_7\} \). Then the isomorphism \( \Phi \) induces an isomorphism \( \overline{\Phi} \) between \( \mathcal{N}_{5,1} \) and \( \mathcal{N}_{1,5} \) by

\[
\begin{align*}
\{0\} & \longrightarrow K_- \longrightarrow \mathcal{N}_{5,2} \xrightarrow{\text{Id} \oplus \pi_-} \mathcal{N}_{5,1} \longrightarrow \{0\} \\
\{0\} & \longrightarrow K_+ \longrightarrow \mathcal{N}_{2,5} \xrightarrow{\text{Id} \oplus \pi_+} \mathcal{N}_{1,5} \longrightarrow \{0\}.
\end{align*}
\]

Hence the Lie algebras \( \mathcal{N}_{5,1} \) and \( \mathcal{N}_{1,5} \) are isomorphic.

The Lie algebra isomorphism \( \overline{\Phi} : \mathcal{N}_{5,1} \to \mathcal{N}_{1,5} \) can be also induced by the isomorphism \( \Phi : \mathcal{N}_{6,1} \to \mathcal{N}_{1,6} \), which we will construct later in this theorem.

**Case** \((r, s) = (6, 2)\). The minimal admissible modules, that are also irreducible, of \( \text{Cl}_{6,2} \) and \( \text{Cl}_{2,6} \) are isometric to \( \mathbb{R}^{16,16} \). We fix mutually commuting isometric involutions

\[
P_1 = J_{z_1}J_{z_2}J_{z_3}J_{z_4}, \quad P_2 = J_{z_1}J_{z_2}J_{z_5}J_{z_6}, \quad P_3 = J_{z_1}J_{z_2}J_{z_7}J_{z_8},
\]

on \( V^{6,2} \) and the complementary anti-isometric operators \( R_1 = J_{z_3}J_{z_7}, \ R_2 = J_{z_5}J_{z_7}, \ R_3 = J_{z_7} \).

Define the quaternion structure \( i = J_{z_1}J_{z_2}, \ j = J_{z_1}J_{z_3}J_{z_5}J_{z_7}, \ k = J_{z_2}J_{z_3}J_{z_5}J_{z_7} \). Each of the space \( E^I = \bigoplus_{j=1}^3 E^I_{P_j} \) is isometric to \( \mathbb{R}^{2,2} \) according to Table 5. Denote \( E^I = \bigcap_{j=1}^3 E^I_{P_j} \) and fix a vector \( v \in E^1 \) such that \( \langle v, v \rangle_{V^{6,2}} = 1 \). Likewise we choose

\[
\begin{align*}
P_1 &= \overline{J}_{w_1} \overline{J}_{w_2} \overline{J}_{w_3} \overline{J}_{w_4}, \quad P_2 = \overline{J}_{w_1} \overline{J}_{w_2} \overline{J}_{w_5} \overline{J}_{w_6}, \quad P_3 = \overline{J}_{w_1} \overline{J}_{w_2} \overline{J}_{w_7} \overline{J}_{w_8} \quad \text{involutions,} \\
\overline{R}_1 &= \overline{J}_{w_1}, \quad \overline{R}_2 = \overline{J}_{w_5}, \quad \overline{R}_3 = \overline{J}_{w_1} \overline{J}_{w_3} \overline{J}_{w_5} \overline{J}_{w_7} \quad \text{complementary operators,} \\
\overline{i} &= \overline{J}_{w_1} \overline{J}_{w_2}, \quad \overline{j} = \overline{J}_{w_1} \overline{J}_{w_3} \overline{J}_{w_5} \overline{J}_{w_7}, \quad \overline{k} = -\overline{J}_{w_2} \overline{J}_{w_3} \overline{J}_{w_5} \overline{J}_{w_7} \quad \text{quaternion structure.}
\end{align*}
\]

The table of commutations is preserved if we change \( J_{z_k} \) to \( \overline{J}_{w_k} \).

Assume that there is a Lie algebra isomorphism

\[
\Phi = A \oplus C : \mathcal{N}_{6,2} \to \mathcal{N}_{2,6}, \quad C(z_k) = w_k, \quad C^T(w_k) = -z_k.
\]

Since \( AP_I = \overline{P}_I A \) we have \( A = \oplus A_I, \ A_I : E^I \to \overline{E}^I \), and

\[
A_I i = -\overline{I} A_I, \quad A_I j = \overline{J} A_I, \quad A_I k = -\overline{K} A_I \quad \text{for any} \ I = (k_1, k_2, k_3)
\]

by Corollary 5. The map \( A_I : E^1 \to \overline{E}^1 \) is defined by the relation \( A_I v = (a_1 + a_2 \overline{I} + a_3 \overline{J} + a_4 \overline{K}) u \) and (30) for \( I = (1, 1, 1) \). All other operators leaving the space \( E^1 \) invariant are linear combination of quaternion structure.
The maps $A_I$ for other multi-indices $I$ are constructed by Theorem 3 by making use of the following maps

\begin{align*}
G_{1,1,-1} &= J_{z_2}: E^1 \to E^{1,1,-1}, & G_{1,-1,1} &= J_{z_5}: E^1 \to E^{1,-1,1}, \\
G_{1,-1,-1} &= J_{z_2}J_{z_1}: E^1 \to E^{1,-1,1}, & G_{-1,1,1} &= J_{z_3}: E^1 \to E^{1,1,1}, \\
G_{-1,1,-1} &= J_{z_1}J_{z_5}: E^1 \to E^{-1,1,1}, & G_{-1,-1,1} &= J_{z_1}J_{z_7}: E^1 \to E^{-1,-1,1}, \\
G_{-1,-1,-1} &= J_{z_1}: E^1 \to E^{-1,-1,-1}.
\end{align*}

Case $(r, s) = (6, 1)$. The minimal admissible module $V^{6,1}$ is isometric to $\mathbb{R}^{8,8}$. The isometric involutions and the complementary anti-isometric operators are

$P_1 = J_{z_1}J_{z_2}J_{z_3}J_{z_4}, \quad P_2 = J_{z_1}J_{z_2}J_{z_5}J_{z_6}, \quad R_1 = J_{z_1}J_{z_7}, \quad R_2 = J_{z_5}J_{z_7}, \quad R_3 = J_{z_7}.$

The quaternion structure is $i = J_{z_1}J_{z_2}$, $j = J_{z_1}J_{z_3}J_{z_5}J_{z_7}$, and $k = J_{z_2}J_{z_3}J_{z_5}J_{z_7}$. Let $v \in E^1 = \bigoplus_{j=1}^2 E^1_R$ be such that $\langle v, \tilde{v} \rangle_{V^{6,1}} = 1$. Then $\{v, i(v), j(v), k(v)\}$ is an orthonormal basis for $E^1$. To show that the basis is orthogonal we argue as following. The operator $T = J_{z_2}J_{z_3}J_{z_5}$ is an isometry, commutes with $P_j$, $j = 1, 2$, and therefore it decomposes the space $E^1$ on two orthogonal subspaces: $E^1 = \text{span}\{v, k(v)\} \oplus \text{span}\{i(v), j(v)\}$, see (17). If it is necessary we change $v$ to $\tilde{v}$ by Lemma 1 where we set $\tilde{J} = k$.

Analogously, we fix the involutions and the anti-isometric complementary operators

\[
\begin{align*}
\tilde{P}_1 &= \tilde{J}_{w_1} \tilde{J}_{w_2} \tilde{J}_{w_3} \tilde{J}_{w_4}, & \tilde{P}_2 &= \tilde{J}_{w_1} \tilde{J}_{w_2} \tilde{J}_{w_5} \tilde{J}_{w_6}, & \tilde{R}_1 &= \tilde{J}_{w_1}, & \tilde{R}_2 &= \tilde{J}_{w_5}.
\end{align*}
\]

acting on $V^{1,6}$. Set the quaternion structure $\tilde{i} = \tilde{J}_{w_1}, \tilde{j} = \tilde{J}_{w_1} \tilde{J}_{w_3} \tilde{J}_{w_5} \tilde{J}_{w_7},$ and $\tilde{k} = -\tilde{J}_{w_2} \tilde{J}_{w_3} \tilde{J}_{w_5} \tilde{J}_{w_7}$. Choose a vector $u \in E^1$ such that $\langle u, u \rangle_{V^{1,6}} = 1$. By making use the quaternion structure we form an orthonormal basis on space $E^1$.

Is $\Phi = A \oplus C: \mathcal{N}_{6,1} \to \mathcal{N}_{1,6}$ an isomorphism, then it has to satisfy Lemma 3. We construct the map $A: V^{6,1} \to V^{1,6}, A = \oplus A_I$, by blocks $A_I: E^I \to E^I$. Put $A_1(v) = (a_1 + a_2 \tilde{i} + a_3 \tilde{j} + a_4 \tilde{k})u \neq 0$. Then by the action of quaternion structure

\[
A_1 i(v) = -\tilde{i} A_1(v), \quad A_1 j(v) = -\tilde{j} A_1(v), \quad A_1 k(v) = -\tilde{k} A_1(v),
\]

we find all the coefficients of $A_1$. The map $A_1$ must satisfies the condition $A_1^T \tilde{J}_{w_7} A_1 = -J_{z_7}$. Arguing as in the case of the construction of the isomorphism $\mathcal{N}_{5,2} \cong \mathcal{N}_{2,5}$, we find that $J_{z_7} v = -j(v)$ and $j_{w_7} u = -\tilde{i}(v)$ and the matrices $A$ and $A^T$ are given by (27). Thus we obtain the solution $a_2 = -a_3$, $a_1 = a_4$, $a_1^2 + a_3^2 = \frac{1}{2}$. We finish the proof by applying Theorem 3 and using the maps

\[
G_{1,-1} = J_{z_5}: E^1 \to E^{1,-1}, \quad G_{-1,1} = J_{z_3}: E^1 \to E^{1,1}, \quad G_{-1,-1} = J_{z_1}: E^1 \to E^{-1,-1}.
\]
Proof. Case $N_{1,1}$. The minimal admissible module $V^{1,1}$ is isometric to $\mathbb{R}^{2,2}$. We choose the basis $\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$ for $N_{1,1}$. Set $C(z_1) = z_2$, $C(z_2) = z_1$, and $C^{\tau}(z_1) = -z_2$, $C^{\tau}(z_2) = -z_1$. In order to satisfy Lemma 3, we define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A^{\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Case $N_{2,2}$. The minimal admissible module $V^{2,2}$ is isometric to $\mathbb{R}^{4,4}$. We fix the isometric involution $P = J_{z_1}J_{z_2}J_{z_3}J_{z_4}$ and choose the basis

$$\begin{cases} x_1 = v, & x_2 = J_{z_1}v, & x_3 = J_{z_2}v, & x_4 = J_{z_1}J_{z_2}v, \\ x_5 = J_{z_3}v, & x_6 = J_{z_4}v, & x_7 = J_{z_1}J_{z_3}v, & x_8 = J_{z_1}J_{z_4}v \end{cases} \quad \text{for} \quad V^{2,2},$$

where the vector $v$ is such that $Pv = v$ and $\langle v, v, v \rangle_{V^{2,2}} = 1$. We first listed the positive vectors of the basis and then negative vectors and therefore the matrix for the metric is the standard one: the diagonal matrix $I_{4,4}$ with first four diagonal entries 1 and the last four diagonal entries (-1). Satisfying the conditions of Lemma 3, we set

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Case $N_{4,4}$. The minimal admissible module $V^{4,4}$ is isometric to $\mathbb{R}^{8,8}$. We fix the isometric involutions

$$P_1 = J_{z_1}J_{z_2}J_{z_3}J_{z_4}, \quad P_2 = J_{z_1}J_{z_2}J_{z_5}J_{z_6}, \quad P_3 = J_{z_1}J_{z_2}J_{z_7}J_{z_8}, \quad P_4 = J_{z_1}J_{z_2}J_{z_5}J_{z_7}.$$ 

We choose $v$ such that $P_j(v) = v$, $j = 1, 2, 3, 4$, and $\langle v, v \rangle_{V^{4,4}} = 1$ and construct the basis for $V^{4,4}$, where we place first the positive vectors and then the negative ones.

$$\begin{cases} x_1 = v, & x_2 = J_{z_1}v, & x_3 = J_{z_2}v, & x_4 = J_{z_3}v, \\ x_5 = J_{z_4}v, & x_6 = J_{z_1}J_{z_2}v, & x_7 = J_{z_1}J_{z_3}v, & x_8 = J_{z_1}J_{z_4}v, \\ x_9 = J_{z_5}v, & x_{10} = J_{z_6}v, & x_{11} = J_{z_7}v, & x_{12} = J_{z_8}v, \\ x_{13} = J_{z_1}J_{z_5}v, & x_{14} = J_{z_1}J_{z_6}v, & x_{15} = J_{z_1}J_{z_7}v, & x_{16} = J_{z_1}J_{z_8}v. \end{cases} \quad \text{(31)}$$

Now we define $C$ as before by $C(z_j) = z_{9-j}$ and $C^{\tau}(z_j) = -z_{9-j}$, $j = 1, \ldots, 8$. The map $A$ is also similar to the previous cases. Namely, we set

$$A(v) = v, \quad A(J_{z_j}v) = J_{C(z_j)}v, \quad \text{and} \quad A^{\tau}(v) = v, \quad A^{\tau}(J_{z_j}v) = -J_{C^{\tau}(z_j)}v.$$ 

Then also

$$A(J_{z_j}J_{z_k}v) = \begin{cases} -J_{C(z_j)}J_{C(z_k)}v, & \text{if } J_{z_j}J_{z_k}v \text{ is positive}, \\ J_{C(z_j)}J_{C(z_k)}v, & \text{if } J_{z_j}J_{z_k}v \text{ is negative}, \end{cases}$$

and analogously for $A^{\tau}$. All conditions of Lemma 3 are satisfied. \qed

Theorem 7. If the Lie algebra $N_{r,s}$ is isomorphic to the Lie algebra $N_{s,r}$, then

1. the Lie algebras $N_{r,s+8k}$ and $N_{s+8k,r}$ are isomorphic;
2. the Lie algebras $N_{r+8k,s}$ and $N_{s,r+8k}$ are isomorphic;
3. the Lie algebras $N_{r+4k,s+4k}$ and $N_{s+4k,r+4k}$ are isomorphic.

for any $k = 1, 2, \ldots.$
Proof. Recall that if \((V^{r,s}, \dots)_{V^{r,s}}\) is a minimal admissible module, then the products
\[
V^{r,s} \otimes V^{0,8}, \quad V^{r,s} \otimes V^{8,0}, \quad V^{r,s} \otimes V^{4,4}
\]
are minimal admissible if \(V^{0,8}, V^{8,0}, \text{and } V^{4,4}\) are minimal admissible modules, see [16]. The scalar product on \(V^{r,s+8}\) is given by the product of bilinear symmetric forms on \(V^{r,s}\) and \(V^{0,8}\). Analogously, for the tensor products with \(V^{8,0}\) and \(V^{4,4}\). The representations are constructed as follows. Let \(\{\zeta_1, \ldots, \zeta_8\}\) be an orthonormal bases for \(\mathbb{R}^{0,8}, \mathbb{R}^{8,0}, \text{or } \mathbb{R}^{4,4}\), and \(\J_\zeta, \alpha = 1, \ldots, 8\) be the respective representations. Let \(J_{z_j}, j = 1, \ldots, r+s\) be representations of an orthonormal basis for \(\mathbb{R}^{r,s}\). We denote by \(\Omega^{0,8} = \prod_{\alpha=1}^8 \J_\zeta\) the volume form for \(\text{Cl}_{0,8}\) and analogously for others Clifford algebras. Set
\[
\J_{z_j} = J_{z_j} \otimes \Omega^{0,8} \quad \text{for } j = 1, \ldots, r+s,
\]
\[
\J_\alpha = \Id_{V^{r,s}} \otimes \J_\alpha \quad \text{for } \alpha = 1, \ldots, 8.
\]
Then the maps \(\J_{z_j}\) and \(\J_\alpha\) are representations of an orthonormal basis for \(\mathbb{R}^{r,s+8}\) as it was shown in [16]. If we substitute the volume form \(\Omega^{0,8}\) by \(\Omega^{8,0}\) or \(\Omega^{4,4}\), then we obtain the representations for the basis vectors of \(\mathbb{R}^{r,s+8}\) and \(\mathbb{R}^{r+4,s+4}\), respectively.

We start from the proof of the first case, since the rest can be proven similarly. Let \(\Phi = A \oplus C: N_{r,s} \rightarrow N_{s,r}\) and \(\Phi = \bar{A} \oplus \bar{C}: N_{0,8} \rightarrow N_{8,0}\) be the Lie algebra isomorphisms, with \(A: V^{r,s} \rightarrow V^{s,r}\) and \(\bar{A}: V^{0,8} \rightarrow V^{8,0}\). Let \(P_j, j = 1, \ldots, p\) and \(Q_k, k = 1, 2, 3, 4\), be mutually commuting isometric involutions on \(V^{r,s}\) and \(V^{0,8}\), respectively. Then \(\bar{P}_j = P_j \oplus \Id\) and \(\bar{Q}_k = \Id \oplus Q_k\) are mutually commuting isometric involutions on \(V^{r,s} \otimes V^{0,8}\). Let \(E^1 = \bigcap_{j=1}^p E^1_{P_j}, E^1 = \bigcap_{j=1}^4 E^1_{Q_j}\) and \(v \in E^1, \quad \langle v, v \rangle_{V^{r,s}} = 1, \quad u \in F^1, \quad \langle u, u \rangle_{V^{0,8}} = 1\).

We have \(\bar{P}_j(v \otimes u) = P_j(v) \otimes \Id(u) = v \otimes u\), and \(\bar{Q}_k(v \otimes u) = \Id(v) \otimes Q_k(u) = v \otimes u\). Therefore \(v \otimes u \in E^1 \otimes F^1\) and \(\langle v \otimes u, v \otimes u \rangle_{V^{r,s+8}} = 1\). The linear map \(\bar{A}: V^{r,s+8} \rightarrow V^{s+r+8}\) of a Lie algebra isomorphism \(\Phi = A \oplus C: N_{r,s+8} \rightarrow N_{s+r+8}\) should satisfy Lemma 3. Thus we could apply Corollary 3 and start the construction of \(A\) from the map \(\J_{1,1}: E^1 \otimes F^1 \rightarrow \tilde{E}^1 \otimes \tilde{F}^1\) and then extend it to an arbitrary \(\J_{1,1}: E^I \otimes F^J \rightarrow \tilde{E}^I \otimes \tilde{F}^J\). Observe that if \(\J_{z_j}\) leaves invariant the space \(E^1\), then the product \(\prod \J_{z_j}\) leaves the space \(E^1 \otimes F^1\) invariant and, analogously, if \(\J_\alpha\) leaves invariant the space \(F^1\), then \(\prod \J_\alpha\) leaves the space \(E^1 \otimes F^1\) invariant. As a consequence, we also obtain that the space \(E^1 \otimes F^1\) will be invariant under the action of \(\prod \J_{z_j}, \prod \J_\alpha\).

We denote by \(\{z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_8\}\) an orthonormal basis for \(\mathbb{R}^{r,s+8}\) with \(\{z_1, \ldots, z_r\}\) positive and \(\{z_{r+1}, \ldots, z_{r+s}, \zeta_1, \ldots, \zeta_8\}\) negative elements. Then, let \(\{w_{r+s}, \ldots, w_1, \omega_8, \ldots, \omega_1\}\) be an orthonormal basis for \(\mathbb{R}^{s+8,r}\) with \(\{w_{r+s}, \ldots, w_{r+1}, \omega_8, \ldots, \omega_1\}\) positive vectors and \(\{w_r, \ldots, w_1\}\) negative vectors.

We let the map \(\hat{C}: \mathbb{R}^{r,s+8} \rightarrow \mathbb{R}^{s+8,r}\) act on the basis by the following
\[
\hat{C}(z_j) = w_j, \quad \hat{C}(w_j) = -z_j, \quad \hat{C}(z_{r+1}) = \omega_1, \quad \hat{C}(\zeta_\alpha) = -\zeta_\alpha, \quad \alpha = 1, \ldots, 8.
\]
We define the map \(\hat{A}_{1,1}: E^1 \otimes F^1 \rightarrow \tilde{E}^1 \otimes \tilde{F}^1\) by its action on different type of products of \(\J_{z_j}\) and \(\J_\alpha\). Recall that \(\prod_{j=1}^p \J_{z_j} \prod_{\alpha=1}^q \J_\alpha = \prod_{j=1}^p J_{z_j} \otimes (\Omega^{0,8})^p \prod_{\alpha=1}^q J_\alpha\). Then we define
\[ \hat{A}_{1,1} \prod_{p} \mathcal{J}_j \prod_{a} \mathcal{J}_\alpha = \begin{cases} A_1 \prod_{j=1}^{p} J_{z_j} \otimes \mathcal{A}_1 (\Omega^{0,8})^p \prod_{a=1}^{q} J_{\zeta_a} = (-1)^{m+k} \prod_{j=1}^{p} \mathcal{J}_{C(z_j)}(A^*_1)^{-1} \otimes \Omega^{8,0} \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)}(\hat{A}^*_1)^{-1}, & \text{if } p = 2m + 1, \ q = 2k + 1, \\
A_1 \prod_{j=1}^{p} J_{z_j} \otimes \mathcal{A}_1 (\Omega^{0,8})^p \prod_{a=1}^{q} J_{\zeta_a} = (-1)^{m+k} \prod_{j=1}^{p} \mathcal{J}_{C(z_j)}(A^*_1)^{-1} \otimes \Omega^{8,0} \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} \hat{A}_1, & \text{if } p = 2m + 1, \ q = 2k, \\
A_1 \prod_{j=1}^{p} J_{z_j} \otimes \mathcal{A}_1 \prod_{a=1}^{q} J_{\zeta_a} = (-1)^{m+k} \prod_{j=1}^{p} \mathcal{J}_{C(z_j)} A_1 \otimes \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} \hat{A}_1, & \text{if } p = 2m, \ q = 2k. 
\end{cases} \tag{32} \]

We also can write the transposed map \( \hat{A}^*_1 \) by \( \hat{A}^*_1, \prod_{j=1}^{p} J_{z_j} \prod_{a} \mathcal{J}_\alpha = \)
\[ \begin{cases} A^*_1 \prod_{j=1}^{p} \mathcal{J}_{C(z_j)} \otimes \mathcal{A}_1 (\Omega^{8,0})^p \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} = (-1)^{m+k+1} \prod_{j=1}^{p} J_{z_j} A_1^{-1} \otimes \Omega^{8,0} \prod_{a=1}^{q} J_{\zeta_a} \hat{A}_1^{-1}, & \text{if } p = 2m + 1, \ q = 2k + 1, \\
A^*_1 \prod_{j=1}^{p} \mathcal{J}_{C(z_j)} \otimes \mathcal{A}_1 (\Omega^{8,0})^p \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} = (-1)^{m+k+1} \prod_{j=1}^{p} J_{z_j} A_1^{-1} \otimes \Omega^{8,0} \prod_{a=1}^{q} J_{\zeta_a} \hat{A}_1^{-1}, & \text{if } p = 2m + 1, \ q = 2k, \\
A^*_1 \prod_{j=1}^{p} \mathcal{J}_{C(z_j)} \otimes \mathcal{A}_1 \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} = (-1)^{m+k+1} \prod_{j=1}^{p} J_{z_j} A_1^{-1} \otimes \prod_{a=1}^{q} J_{\zeta_a} \hat{A}_1^{-1}, & \text{if } p = 2m, \ q = 2k + 1, \\
A^*_1 \prod_{j=1}^{p} \mathcal{J}_{C(z_j)} \otimes \mathcal{A}_1 \prod_{a=1}^{q} \hat{\mathcal{J}}_{C(\zeta_a)} = (-1)^{m+k} \prod_{j=1}^{p} J_{z_j} A_1^{-1} \otimes \prod_{a=1}^{q} J_{\zeta_a} \hat{A}_1^{-1}, & \text{if } p = 2m, \ q = 2k. 
\end{cases} \]

Then the maps \( G_{IJ} = G_I \otimes \mathcal{G}_J \colon E^1 \otimes F^1 \to E^I \otimes F^J \) will be used to define \( \hat{A}_{IJ} \colon E^I \otimes F^J \to \mathcal{E}^I \otimes \mathcal{F}^J \). Namely
\[ \hat{A}_{IJ} = \begin{cases} (-1)^{m} \mathcal{G}_{IJ}(\hat{A}^{-1}_{1,1})^{-1} G_{IJ}^{-1} & \text{if } p = 2m + 1, \\
(-1)^{m} \mathcal{G}_{IJ} \hat{A}_{1,1} G_{IJ}^{-1} & \text{if } p = 2m, \end{cases} \tag{33} \]
and
\[ \hat{A}^*_I_{IJ} = \begin{cases} (-1)^{m+1} G_{IJ} \hat{A}^{-1}_{1,1} \hat{G}_{IJ}^{-1} & \text{if } p = 2m + 1, \\
(-1)^{m} G_{IJ} \hat{A}^*_1 \hat{G}_{IJ}^{-1} & \text{if } p = 2m. \end{cases} \]

Thus, we obtain that the map \( \Phi = \hat{A} \otimes \mathcal{C} \), with \( \hat{A} = \oplus_{IJ} \hat{A}_{IJ} \), is a Lie algebra isomorphism from \( \mathcal{N}_{r,s+8} \) to \( \mathcal{N}_{s+8,r} \), according to Corollary 5. We recall that we can choose the following
map $\tilde{A}$: $\tilde{A}_1v = u$ and since the spaces $F^j$ are one dimensional, the corresponding maps $G_j: F^1 \to F^j$ are given by the basis $\{23\}$.

The third statement is proved analogously, where we change the map $\tilde{A}$: $V^{0,8} \to V^{8,0}$ to the map $\tilde{A}$: $V^{4,4} \to V^{4,4}$ constructed in Theorem 3. Then we use the definitions (32) and (33) to construct the isomorphism $\tilde{\Phi} = \tilde{A} \oplus \tilde{C}: \mathcal{N}_{r+4,s+4} \to \mathcal{N}_{s+4,r+4}$ by tensor product, where we change the volume forms $\Omega^{0,8}$ and $\Omega^{8,0}$ to $\Omega^{4,4}$. The maps $G_j: F^1 \to F^j$ are given by the basis $\{31\}$.

\begin{remark}
The reader can recognize in the construction of $\tilde{A}$ the $\mathbb{Z}^2$-graded tensor product. Indeed we write $A = A^0 \oplus A^1$ and $\tilde{A} = \tilde{A}^0 \oplus \tilde{A}^1$, where $A^0$ and $\tilde{A}^0$ act on the even product of generators $J_{z_j}$ and $A^1$ and $\tilde{A}^1$ act on the odd product of generators. Then formula (32) can be written as follows
\begin{equation}
\tilde{A}_1 = A_1 \oplus \tilde{A}_1 = (A_1 \oplus \tilde{A}_1)^0 \oplus (A_1 \oplus \tilde{A}_1)^1 = (A_0^0 \oplus A_1^0) \oplus (A_0^1 \oplus A_1^1) \oplus (A_0^1 \oplus A_1^0).
\end{equation}

This is not surprising, according to the $\mathbb{Z}_2$-graded structure of Clifford algebra and the isomorphism $\text{Cl}(\mathbb{R}^{r,s} \oplus \mathbb{R}^{p,q}) \cong \text{Cl}(\mathbb{R}^{r,s}) \oplus \text{Cl}(\mathbb{R}^{p,q})$, based on the $\mathbb{Z}^2$-graded tensor product $\otimes$.

\begin{theorem}
The following is true:
\begin{enumerate}
  \item the Lie algebras $\mathcal{N}_{r,r+8k}$ and $\mathcal{N}_{r+8k,r}$ are isomorphic for $r = 1, 2, 4$;
  \item the Lie algebras $\mathcal{N}_{r+4k,r+4k}$, $r = 1, 2, 4$ admit an automorphism $\Psi = A \oplus C$ with $CC^r = -\text{Id}$.
\end{enumerate}
\end{theorem}

\begin{proof}
The proof is literary the same as the proof of Theorem 7 where we need to change the volume forms $\Omega^{0,8}$ and $\Omega^{8,0}$ to $\Omega^{4,4}$.
\end{proof}

3.4. Non-isomorphic Lie algebras. We start from a small technical observation.

\begin{lemma}
Let $(V, \langle \ldots \rangle_V)$ be a neutral space and $T$ a linear map on $V$ with the properties: $T^2 = \text{Id}$ and the scalar product $(x, y) := \langle x, Ty \rangle_V$ is positive definite. Then there is no linear map $S$ on $V$ such that $S = S^r$, where $S^r$ is transposed with respect to $\langle \ldots \rangle_V$ and $STS = -T$.
\end{lemma}

\begin{proof}
Let us assume that a linear map $S: V \to V$ such that $S = S^r$ and $STS = -T$ exists. Then $\overset{\text{T}}{TS} = TS$, where $\overset{\text{T}}{S}$ is the transposition with respect to the positive definite scalar product $\langle \ldots \rangle_V$ and therefore
\begin{equation}
-T = STS = T(\overset{\text{T}}{S})TTS = T(\overset{\text{T}}{S})S \implies \overset{\text{T}}{SS} = -\text{Id},
\end{equation}
which is a contradiction.
\end{proof}

\begin{theorem}
The Lie algebras $\mathcal{N}_{r,s}$ and $\mathcal{N}_{s,r}$ for $(r, s) \in \{(3, 1), (3, 2), (3, 7), (3, 11)\}$ are not isomorphic.
\end{theorem}

\begin{proof}
CASE $(r, s) = (3, 1)$. The minimal admissible module $V^{3,1}$ is isometric to $\mathbb{R}^{4,4}$. We define the isometric involution $T = J_{z_1}J_{z_2}J_{z_3}$ and the orthonormal basis for $V^{3,1}$, starting from $v \in V^{3,1}$, $\langle v, v \rangle_{V^{3,1}} = 1$, and $Tv = v$:
\begin{align*}
x_1 &= v, & x_2 &= J_{z_1}v, & x_3 &= J_{z_1}v, & x_4 &= J_{z_3}v, \\
x_5 &= J_{z_4}v, & x_6 &= J_{z_4}J_{z_1}v, & x_7 &= J_{z_4}J_{z_2}v, & x_8 &= J_{z_4}J_{z_3}v,
\end{align*}
with $\langle x_k, x_k \rangle_{V^{3,1}} = -\langle x_{k+4}, x_{k+4} \rangle_{V^{3,1}} = 1$, $k = 1, \ldots, 4$. Moreover $T(x_i) = -x_i$, $i = 1, 2, 3, 4$ and $T(x_i) = x_i$, $i = 5, 6, 7, 8$. Assume that there is an isomorphism $\Phi: \mathcal{N}_{3,1} \to \mathcal{N}_{1,3}$, $\Phi = A \oplus C$ such that $A: V^{3,1} \to V^{1,3}$ and $C(z_j) = w_j$. Then the map $\Phi^r\Phi = A^rA \oplus -\text{Id}_{\mathbb{R}^3}$: $\mathcal{N}_{3,2} \to \mathcal{N}_{3,2}$ is an automorphism by Lemma 2. Denote $S = A^rA$ and obtain a contradiction as in Lemma 6 with $V = V^{3,1}$ and $T = J_{z_1}J_{z_2}J_{z_3}$.
\end{proof}
Case \((r, s) = (3, 2)\). We consider mutually commuting isometric involutions and the complementary anti-isometric operators

\[ P = J_{z_1}J_{z_2}J_{z_4}J_{z_5}, \quad T = J_{z_1}J_{z_2}J_{z_3}, \quad R_1 = J_{z_5}, \quad R_2 = J_{z_1}J_{z_4} \]

acting on \(V^{3,2}\). In Table 7 we show commutation relations of the involutions, complementary operators, and the representation maps \(J_{z_j}\). We conclude that the spaces \(E^1_P\) and \(E^{-1}_P\) are neutral. We pick up a vector \(v \in E^1_P\), \(\langle v, v \rangle_{V^{3,2}} = 1\) and construct an orthonormal basis for \(E^1_P\)

\[ x_1 = v, \quad x_2 = J_{z_1}J_{z_2}v, \quad x_3 = J_{z_1}J_{z_4}v, \quad x_4 = J_{z_2}J_{z_4}v \]

with \(\langle x_i, x_i \rangle_{V^{3,2}} = -\langle x_{i+2}, x_{i+2} \rangle_{V^{3,2}} = 1, \ i = 1, 2\). Table 7 also shows that

\[ Tx_1 = x_1, \quad Tx_2 = x_2, \quad Tx_3 = -x_3, \quad Tx_4 = -x_4. \]

Assuming now that there is an isomorphism \(\Phi: \mathcal{N}_{3,2} \to \mathcal{N}_{3,3}\), \(\Phi = A \oplus C\) such that \(A: V^{3,2} \to V^{2,3}\) and \(C(z_j) = w_j\), we obtain a contradiction by Lemma 6 with \(V = E^1_P\), \(S = A_1^1A_1\), and \(T = J_{z_1}J_{z_2}J_{z_3}\).

Case \((r, s) = (3, 7)\). We define the mutually commuting involutions

\[ P_1 = J_{z_1}J_{z_2}J_{z_5}J_{z_6}, \quad P_2 = J_{z_1}J_{z_2}J_{z_7}J_{z_8}, \quad P_3 = J_{z_1}J_{z_2}J_{z_9}J_{z_10}, \quad T = J_{z_1}J_{z_2}J_{z_3}, \]

and the complementary anti-isometric operators

\[ R_1 = J_{z_5}, \quad R_2 = J_{z_7}, \quad R_3 = J_{z_9}, \quad R_4 = J_{z_4}, \]

acting on \(V^{3,7}\).

Table 7. Commutation relations of operators on \(V^{3,2}\)

| \(J_{z_1}\) | \(J_{z_2}\) | \(J_{z_3}\) | \(J_{z_4}\) | \(J_{z_5}\) | \(J_{z_6}\) | \(J_{z_7}\) | \(J_{z_8}\) | \(J_{z_9}\) | \(R_1\) | \(R_2\) | \(R_3\) | \(R_4\) | \(Q\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(-1\) | \(-1\) | \(1\) | \(-1\) | \(-1\) | \(1\) | \(1\) | \(-1\) | \(-1\) | \(-1\) | \(1\) | \(1\) | \(1\) | \(-1\) |
| \(-1\) | \(-1\) | \(1\) | \(-1\) | \(-1\) | \(1\) | \(1\) | \(-1\) | \(-1\) | \(-1\) | \(-1\) | \(-1\) | \(-1\) | \(-1\) |

Table 8. Commutation relations of operators on \(V^{3,7}\)

Since the dimension of the minimal admissible module \(V^{3,7}\), which is also irreducible, is 64, the common eigenspace \(E^1 = \cap_{j=1}^3 E^1_{P_j}\) is 8-dimensional and neutral. We choose the following basis for \(E^1\), starting from \(v \in E^1\), \(\langle v, v \rangle_{V^{3,7}} = 1\) and making use the anti-isometric operator \(Q = J_{z_5}J_{z_7}J_{z_9}\)

\[ x_1 = v, \quad x_2 = J_{z_1}J_{z_2}v, \quad x_3 = J_{z_4}J_{z_1}Qv, \quad x_4 = J_{z_4}J_{z_2}Qv, \]

\[ x_5 = J_{z_3}v, \quad x_6 = J_{z_4}J_{z_3}v, \quad x_7 = J_{z_1}Qv, \quad x_8 = J_{z_2}Qv. \]

The basis is orthonormal by Lemma 6 and satisfies \(\langle x_j, x_j \rangle_{V^{3,7}} = -\langle x_{4+j}, x_{4+j} \rangle_{V^{3,7}} = 1\) and \(T(x_i) = x_i, \ i = 1, 2, 3, 4\) and \(T(x_i) = -x_i, \ i = 5, 6, 7, 8\).
due to the choice of the operators $R_4$ and $Q$. Thus we can apply Lemma 6 to the neutral space $V = E^1$ with operators $S = A^*_1 A_1$ and $T$. It finishes the proof.

**Case $(r, s) = (3, 11)$.** The minimal admissible modules $V^{11,3}$ and $V^{3,11}$ are isometric to $\mathbb{R}^{64,64}$. We choose a set of mutually commuting isometric involutions:

$$P_j = J_{z_1} J_{z_2} J_{z_3+2j} J_{z_4+2j}, \; j = 1, \ldots, 5, \quad T = J_{z_1} J_{z_2} J_{z_3},$$

acting on $V^{3,11}$. The complementary operators are $R_k = J_{z_3+2k}, \; k = 1, \ldots, 5$, and $R_6 = J_{z_4}$.

The space $E^1 = \bigcap_{j=1}^5 E^1_{P_j}$ is 4-dimensional neutral space. We choose the orthonormal basis

$$\{x_1 = v, \; x_2 = J_{z_1} J_{z_2} v, \; x_3 = J_{z_4} v, \; x_4 = J_{z_4} J_{z_1} J_{z_2} v\}$$

for $E^1$ with $v \in E^1$, $\langle v, v \rangle_{V^{3,11}} = 1$. It is easy to see that

$$T(x_j) = x_j, \quad T(x_{2+j}) = -x_{2+j}, \; j = 1, 2.$$

Thus, if we assume that there is an isomorphism $\Phi = A \oplus C : N_{3,11} \to N_{11,3}$, then the operator $S = A^*_1 A_1$ will act on $E^1$. Applying Lemma 6 to the neutral space $E^1$, operators $S$ and $T$, we obtain a contradiction. This finishes the proof. \hfill \Box

**Corollary 6.** There are no automorphism $\Phi = A \oplus C$ of $N_{3,3}$ with the condition $C^r C = -\text{Id}$.

**Proof.** If we assume that such an automorphism $\Psi$ exists, then it must induce an isomorphism between $N_{3,2}$ and $N_{2,3}$, which contradicts to Theorem 9. The constructive proof can be performed as follows. Let us assume the existence of an automorphism $\Psi = A \oplus C$ with $C^r C = -\text{Id}$. We fix mutually commuting isometric involutions and the complementary operators

$$P_1 = J_{z_1} J_{z_2} J_{z_4} J_{z_5}, \quad P_2 = J_{z_1} J_{z_3} J_{z_2} J_{z_6}, \quad T = J_{z_1} J_{z_2} J_{z_3}, \quad R_1 = J_{z_4}, \quad R_2 = J_{z_6}, \quad R_3 = J_{z_3} J_{z_5},$$

acting on $V^{3,3}$. Denote by $\{w_6, \ldots, w_1\}$ another orthonormal basis of $\mathbb{R}^{3,3}$, where $w_6, w_5, w_4$ are positive vectors and $w_3, w_2, w_1$ are negative. Put $C(z_i) = w_i$. The common eigenspace $E^1 = \bigcap_{j=1}^2 E^1_{P_j}$ is spanned by $\{x_1 = v, \; x_2 = J_{z_1} J_{z_2} v\}$, where $v = P_1(v) = P_2(v) = T(v)$ and $\langle v, v \rangle_{V^{3,3}} = 1$. Observe that $T(x_2) = -x_2$. Thus we obtain a contradiction as in Lemma 6 by setting $S = A^r A$ for the neutral space $E^1$. \hfill \Box

We can not apply directly the arguments of Theorem 7 to non-isomorphic pairs. Nevertheless, by a direct construction we obtain that the non-isomorphic properties are also respect the same periodicity.

**Theorem 10.** If $(r, s) \in \{(3, 1), \; (3, 2), \; (3, 7), \; (3, 11)\}$, then

1. the Lie algebra $N_{r+4k, s+4k}$ is not isomorphic to $N_{s+4k, r+4k}$ for any $k = 0, 1, 2, \ldots,$
2. the Lie algebra $N_{r,s+8k}$ is not isomorphic to $N_{s+8k, r}$ for any $k = 0, 1, 2, \ldots,$
3. the Lie algebra $N_{r+8k,s}$ is not isomorphic to $N_{s,r+8k}$ for any $k = 0, 1, 2, \ldots,$
Proof. Observe that if the Lie algebra $N_{r,s}$ has a system of $p$ mutually commuting isometric involutions, then the Lie algebra $N_{r+4k,s+4k}$ has $p + 4k$ mutually commuting isometric involutions. The dimensions of minimal admissible modules are related by $\dim(V^{r+4k,s+4k}) = 16 \dim(V^{r,s})$. Therefore, the dimension of the common eigenspace $E^1$, corresponding to eigenvalues $1$ of all the involutions, does not change and equal for $N_{r,s}$ and $N_{r+4k,s+4k}$ for any $k$.

The same argument valid for the Lie algebras $N_{r,s+8k}$ and $N_{s+8k,r}$. During the proof we show that for each value of $(r, s)$ in the statement of the theorem, we can apply Lemma 6 and deduce that $N_{r+4k,s+4k} \not\cong N_{s+4k,r+4k}$ and $N_{r,s+8k} \not\cong N_{s+8k,r}$ for any $k$.

We slightly change the notations. Denote by $\{z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s\}$ the orthonormal basis of $\mathbb{R}^{r,s}$ with $\langle z_k, z_k \rangle_{r,s} = 1$, $k = 1, \ldots, r$, and $\langle \zeta_j, \zeta_j \rangle_{r,s} = -1$, $j = 1, \ldots, s$.

CASE $(r, s) = (3, 1)$. The minimal admissible module $V^{3,1}$ has the isometric involution $T = J_{z_1}J_{z_2}J_{z_3}$. The minimal admissible module $V^{3+4k,1+4k}$ has the following mutually commuting isometric involutions

$$P_1 = J_{z_1}J_{z_2}J_{z_4}J_{z_5}, \quad P_2 = J_{z_1}J_{z_2}J_{z_6}J_{z_7}, \ldots, \quad P_{2k} = J_{z_1}J_{z_2}J_{z_{2k+4}}J_{z_{2k+5}},$$

$$P_{2k+1} = J_{z_1}J_{z_2}J_{z_4}J_{z_5}, \ldots, \quad P_{4k} = J_{z_1}J_{z_2}J_{z_{4k}}J_{z_{4k+1}}, \quad T = J_{z_1}J_{z_2}J_{z_3}.$$ 

The complementary operators are

$$R_l = J_{z_{2l+3}}J_{\zeta_1}, \quad l = 1, \ldots, 2k, \quad R_l = J_{\zeta_{2l-2k+1}}, \quad l = 2k + 1, \ldots, 4k, \quad R_{4k+1} = J_{\zeta_1}.$$ 

We choose the basis of $E^1 = \bigcap_{j=1}^{4k} E_{1,j}^{\bot}$, starting from $v \in E^1$, $\langle v, v \rangle_{V^{3+4k,1+4k}} = 1$. We also need an isometric operator $Q = \prod_{j=1}^{4k} R_j$. Thus we have

$$QP_j = -P_j Q, \quad QT = TQ, \quad J_{\zeta_1}P_j = P_j J_{\zeta_1}, \quad J_{\zeta_1}T = -T J_{\zeta_1}, \quad Q^2 = \text{Id}. \quad (35)$$

If it is necessary, we can apply Lemma 6 and find the following orthonormal basis

$$x_1 = v, \quad x_2 = J_{z_1}J_{z_2}v, \quad x_3 = J_{z_1}Qv, \quad x_4 = J_{z_2}Qv,$$

$$x_5 = J_{\zeta_1}v, \quad x_6 = J_{\zeta_1}J_{z_1}J_{z_2}v, \quad x_7 = J_{\zeta_1}J_{z_1}Qv, \quad x_8 = J_{\zeta_1}J_{z_2}Qv,$$ 

with $\langle x_k, x_k \rangle_{V^{3+4k,1+4k}} = -\langle x_{k+4}, x_{k+4} \rangle_{V^{3+4k,1+4k}} = 1$, $k = 1, \ldots, 4$ and

$$T(x_j) = x_j, \quad \text{and} \quad T(x_{4+j}) = -x_{4+j}, \quad j = 5, 6, 7, 8.$$ 

due to the choice of the corresponding operators. Thus assuming that there is a Lie algebra isomorphism $\Phi = A \oplus C : N_{3+4k,1+4k} \to N_{1+4k,3+4k}$, we define the map $S = A^T A : V^{3+4k,1+4k} \to V^{1+4k,3+4k}$ and obtain a contradiction by Lemma 6.

The minimal admissible module $V^{3,1+8k}$ has the following mutually commuting isometric involutions

$$P_1 = J_{z_1}J_{z_2}J_{z_4}J_{z_5}, \quad P_2 = J_{z_1}J_{z_2}J_{z_6}J_{z_7}, \ldots, \quad P_{4k} = J_{z_1}J_{z_2}J_{z_{4k}}J_{z_{4k+1}}, \quad T = J_{z_1}J_{z_2}J_{z_3}.$$ 

The complementary operators are $R_l = J_{z_{2l+3}}J_{\zeta_1}, \quad l = 1, \ldots, 4k$, and $R_{4k+1} = J_{\zeta_1}$. We also need the isometric operator $Q = \prod_{j=1}^{4k} R_j$. Choose the basis (36) and finish the proof by applying Lemma 6.

For the minimal admissible module $V^{3+8k,1}$ we choose the mutually commuting isometric involutions

$$P_1 = J_{z_1}J_{z_2}J_{z_4}J_{z_5}, \quad P_2 = J_{z_1}J_{z_2}J_{z_6}J_{z_7}, \ldots, \quad P_{4k} = J_{z_1}J_{z_2}J_{z_{4k}}J_{z_{4k+1}}, \quad T = J_{z_1}J_{z_2}J_{z_3}.$$ 

The complementary operators are $R_l = J_{z_{2l+3}}J_{\zeta_1}, \quad l = 1, \ldots, 4k$, and $R_{4k+1} = J_{\zeta_1}$ and $Q = \prod_{j=1}^{4k} R_j$. We choose the basis (36) and finish the proof by applying Lemma 6.
Case \((r, s) = (3, 7)\). This case is similar to the previous. Recall that for \(V^{3, 7}\) the mutually commuting involutions are

\[ P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, 2, 3, \quad T = J_{z_1}J_{z_2}J_{z_3}. \]

The complementary anti-isometric operators are

\[ R_l = J_{z_1+2l}, \quad l = 1, 2, 3, \quad R_4 = J_{z_1}, \quad Q = \prod_{l=1}^{3} J_{z_1+2l}. \]

For the minimal admissible module \(V^{3+4k, 7+4k}\) we choose the following involutions

\[ P_m = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad m = 1, \ldots, 2k, \quad P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, \ldots, 3 + 2k, \]

and \(T = J_{z_1}J_{z_2}J_{z_3}\). The complementary operators are

\[ R_p = J_{z_1+2p}^R, \quad p = 1, \ldots, 2k, \quad R_l = J_{z_1+2l}, \quad l = 1, \ldots, 3 + 2k, \quad R_{4+4k} = J_{z_1}, \]

and the isometry \(Q = \prod_{l=1}^{4k+4} R_j\). Since all the chosen operators satisfy (35), then we can take the basis (36) and finish the proof, applying Lemma 6.

For the minimal admissible module \(V^{3, 7+8k}\) we write

\[ P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, \ldots, 3 + 4k, \quad T = J_{z_1}J_{z_2}J_{z_3}, \]

\[ R_l = J_{z_1+2l}, \quad l = 1, \ldots, 3 + 4k, \quad R_{4+4k} = J_{z_1}, \quad Q = \prod_{l=1}^{4k+4} R_j. \]

For the minimal admissible module \(V^{3+8k, 7}\) we define

\[ P_m = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad m = 1, \ldots, 4k, \quad P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, 2, 3, \]

\[ T = J_{z_1}J_{z_2}J_{z_3}, \quad Q = \prod_{l=1}^{4k+4} R_j, \]

where

\[ R_l = J_{z_1+2l}^R, \quad l = 1, \ldots, 4k, \quad R_{4+4k} = J_{z_1}, \]

Case \((r, s) = (3, 11)\). We recall that the set of mutually commuting isometric involutions acting on \(V^{3, 11}\) is:

\[ P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, \ldots, 5, \quad T = J_{z_1}J_{z_2}J_{z_3}. \]

The complementary operators are \(R_j = J_{z_1+2j}, \quad j = 1, \ldots, 5\), and \(R_6 = J_{z_1}\). For the minimal admissible module \(V^{3+4k, 11+4k}\) we make the following modifications

\[ P_m = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad m = 1, \ldots, 2k, \quad P_j = J_{z_1}J_{z_2}J_{z_2}J_{z_3}, \quad j = 1, \ldots, 5 + 2k, \]

and \(T = J_{z_1}J_{z_2}J_{z_3}\). The complementary operators are

\[ R_p = J_{z_1+2p}^R, \quad p = 1, \ldots, 2k, \quad R_l = J_{z_1+2l}, \quad l = 1, \ldots, 5 + 2k, \quad R_{4+4k} = J_{z_1}, \]

The space \(E^1 = \bigcap_{j=1}^{5+4k} E^1_{p_j}\) is 4-dimensional neutral space. We choose the orthonormal basis:

\[ x_1 = v, \quad x_2 = J_{z_1}J_{z_2}v, \quad x_3 = J_{z_1}v, \quad x_4 = J_{z_1}J_{z_1}J_{z_2}v \]

with \(v \in E^1, \langle v, v \rangle_{V^{3+4k, 11+4k}} = 1\). Since \(T_1(x_j) = x_j, \ T_1(x_{2+j}) = -x_{2+j}, \ j = 1, 2\), we can finish the proof by applying Lemma 6.

It is clear what changing have to be done for the rest of the proof.
Then we finish the proof as in the previous case. We can apply now Lemma 6.

**Corollary 7.** There are no automorphisms of the algebra $\mathcal{N}_{3+4k,3+4k}$, $k = 0,1,\ldots$, of the form $\Psi = A \oplus C$ with $C^*C = -Id$.

**Proof.** If such an automorphism would exist, then it could induce an isomorphism between $\mathcal{N}_{3+4k,2+4k}$ and $\mathcal{N}_{2+4k,3+4k}$, which is a contradiction. The proof can be also obtained by method of Theorem 10 as in Corollary 6. \hfill \Box
4. Step 2: Trivially non-isomorphic Lie algebras

In this section we study the isomorphism between the Lie algebras $N_{r,s}$ and $N_{s,r}$, where one of the Lie algebras is constructed from minimal admissible Clifford module and another one is constructed by using the direct sum of two minimal admissible Clifford modules. We formulate one theorem, where we state all the cases that could be used for further applications of periodicity [2]. We continue to use the notation $V_{r,s}^r$ for minimal admissible modules and we write $U^{r,s}$ to denote a non-minimal admissible module. In the case when there are two minimal admissible modules, we write $V_{r,s}^r \cong V_+$ and $V_{r,s}^r \cong V_-$. We use the notation $N_{r,s}$ for the Lie algebra constructed by using the direct sum of two minimal admissible modules. Below in Theorem 11 we write $N_{r,s}^2 \cong N_{r,s}(V_{r,s}^r \oplus V_{r,s}^r)$ for the case of $r - s \neq 3(\text{mod } 4)$. In the cases $r - s = 3(\text{mod } 4)$ the Clifford algebra $C_{r,s}$ has two minimal admissible module and in this case we write $N_{r,s}^2 \cong N_{r,s}(V_{r,s}^r \oplus V_{r,s}^s)$.

**Theorem 11.** The following pairs of the Lie algebras are isomorphic

$$
N_{3,0}^2 \cong N_{0,3}, \quad N_{5,0}^2 \cong N_{0,5}, \quad N_{6,0}^2 \cong N_{0,6}, \quad N_{7,0}^2 \cong N_{0,7}, \\
N_{2,1} \cong N_{2,2}, \quad N_{4,1} \cong N_{1,4}, \quad N_{7,1} \cong N_{1,7}, \\
N_{4,2} \cong N_{4,4}, \quad N_{2,2} \cong N_{2,7}, \quad N_{6,3} \cong N_{3,6}.
$$

**Proof.** The scheme of the proof is the following: for each pair of Lie algebras $N_{r,s}^2$ and $N_{s,r}$ we find another pair $N_{l,m}$, $N_{m,l}$ of isomorphic algebras such that $V_{l,m}$ is isometric to $V_{s,r}$ and $U_{l,m}$. Then representations of the Clifford algebras $C_{l,m}$ and $C_{m,l}$ will induce actions on $V_{l,m}$ and $U_{l,m}$ that allow to induce the isomorphism $\Phi: N_{r,s} \rightarrow N_{r,s}^2$ from the existing isomorphism between $\Phi: N_{l,m} \rightarrow N_{m,l}$, as it is shown on the diagram:

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & K_- \\
\downarrow C & & \Phi \\
\{0\} & \longrightarrow & K_+ \\
\end{array}
\begin{array}{ccc}
N_{l,m} & \longrightarrow & N_{l,m} \\
I \oplus \pi_- & \longrightarrow & I \oplus \pi_+ \\
\Phi & \longrightarrow & \Phi \\
N_{s,r} & \longrightarrow & N_{s,r}^2 \\
\{0\} & \longrightarrow & \{0\}
\end{array}
$$

(37)

We consider case by case finding suitable isomorphic pairs of Lie algebras that will induce the isomorphism for the pairs listed in the statement of the theorem.

**Case** $N_{3,0}^2 \cong N_{0,3}$. Let $V_{4,0}$ be the minimal admissible module of the Clifford algebra $C_{4,0}$, then the natural inclusion $\mathbb{R}_{3,0}^2 \subset \mathbb{R}_{4,0}^3$ defines an admissible module $U_{3,0}^3$ of $C_{3,0}$. Then it must be $U_{3,0}^3 = V_{3,0}^+ \oplus V_{3,0}^-$, since the operator $J_{z_3} \Omega_{z_3}$ anti-commutes with the volume form $\Omega_{3,0} = J_{z_1} J_{z_2} J_{z_3}$. Thus $U_{3,0}^3$ includes both eigenspaces of $\Omega_{3,0}$ and $U_{3,0}^3$ is isometric to $V_{4,0}$.

The orthogonal projection $\pi_+: \mathbb{R}_{4,0} \rightarrow \mathbb{R}_{3,0}^3$ with kernel $K_+ = \text{span}\{z_1\}$ and the isometry map $I: V_{4,0} \rightarrow U_{3,0}^3$ define a surjective Lie algebra homomorphism $\rho = I \oplus \pi_+: N_{4,0} \rightarrow N_{3,0}^2$. Analogously, the orthogonal projection $\pi_-: \mathbb{R}_{0,4} \rightarrow \mathbb{R}_{0,3}^3$ with the kernel $K_- = \text{span}\{z_1\}$ and the isometry map $I: V_{0,4} \rightarrow V_{0,3}^3$ induce a surjective Lie algebra homomorphism $\rho = I \oplus \pi_-: N_{0,4} \rightarrow N_{0,3}$. Then the isomorphism $\Phi: N_{0,4} \rightarrow N_{4,0}$ induces an isomorphism $\Phi: N_{0,3} \rightarrow N_{3,0}^2$ by (37), since $\Phi(\zeta_1) = C(\zeta_1) = z_1$.

**Cases** $N_{5,0}^2 \cong N_{0,5}$, $N_{6,0}^2 \cong N_{0,6}$, $N_{7,0}^2 \cong N_{0,7}$. From now on we will only indicate the structure of $U_{r,s}^r$ and the isomorphic Lie algebras that induce the necessary isomorphism.

We have $U_{5,0}^5 = V_{5,0}^+ \oplus V_{5,0}^-$, $U_{6,0}^6 = V_{6,0}^+ \oplus V_{6,0}^-$, and $U_{7,0}^7 = V_{7,0}^+ \oplus V_{7,0}^-$. The isomorphisms $\Phi$ are induced from $\Phi: N_{8,0} \rightarrow N_{9,8}$.
Cases $N_{2,1} \cong N_{1,2}^2$, $N_{4,1} \cong N_{2,4}^2$, $N_{7,1}^2 \cong N_{1,7}$. Let $\Psi = A \oplus C : N_{2,2} \to N_{2,2}$, be a Lie algebra automorphism, such that
$$C(z_1) = z_4, \quad C(z_2) = z_3, \quad C(z_3) = z_2, \quad C(z_4) = z_1, \quad \text{and} \quad CC^T = -\text{Id}.$$
We have $U_1^2 = V_1^2 \oplus V_1^2$ and the isomorphism $\Phi : N_{2,1} \to N_{1,2}^2$ is induced by the automorphism $\Psi$.

We have $U_{1,4}^2 = V_{1,4}^2 \oplus V_{1,4}^2$, and the isomorphism $\Phi : N_{4,1} \to N_{2,4}^2$ is induced from $\Phi : N_{5,1} \to N_{1,5}$. Analogously, $U_{7,1}^1 = V_{7,1}^1 \oplus V_{7,1}^1$ and the isomorphism $\Phi : N_{7,1}^2 \to N_{1,7}$ is induced from $\Phi : N_{8,1} \to N_{1,8}$.

Cases $N_{4,2} \cong N_{2,4}^2$, $N_{2,7} \cong N_{2,7}$. We have $U_{2,4}^2 = V_{2,4}^2 \oplus V_{2,4}^2$, and the isomorphism $\Phi : N_{4,2} \to N_{2,4}^2$ is induced from $\Phi : N_{5,2} \to N_{2,5}$. One has $U_{7,2}^1 = V_{7,2}^1 \oplus V_{7,2}^1$, and the isomorphism $\Phi : N_{7,2} \to N_{2,7}$ is induced from $\Phi : N_{8,2} \to N_{2,8}$.

Cases $N_{4,3} \cong N_{3,4}^2$, $N_{5,3} \cong N_{2,5}^2$, $N_{6,3} \cong N_{2,6}^2$. We have $U_{3,4}^2 = V_{3,4}^2 \oplus V_{3,4}^2$, and the isomorphism $\Phi : N_{4,3} \to N_{3,4}^2$ is induced from the automorphism $\Psi$ of $N_{4,4}$.

One has the following modules $U_{3,k} = V_{3,k}^2 \oplus V_{3,k}^2$, and the isomorphism $\Phi : N_{k,3} \to N_{3,k}^2$ is induced from $\Phi : N_{k+1,4} \to N_{4,k+1}$ for $k = 5, 6$.

The last theorem is an application of the construction made in Theorem 7 to show the isomorphism of Lie algebras of high dimension.

**Theorem 12.** If the Lie algebra $N_{r,s}^2$ is isomorphic to the Lie algebra $N_{s,r}$, then

1. the Lie algebras $N_{r,s+8k}^2$ and $N_{s+8k,r}^2$ are isomorphic;
2. the Lie algebras $N_{r+8k,s}^2$ and $N_{s,r+8k}^2$ are isomorphic;
3. the Lie algebras $N_{r+4k,s+4k}^2$ and $N_{s+4k,r+4k}^2$ are isomorphic.

for any $k = 1, 2, \ldots$.

5. Step 3: uniqueness of minimal dimensional Lie algebras

Recall that we are classifying the resulting Lie algebras, constructed as $H$-type Lie algebras by making use of (probably) different admissible scalar products on the same representation space. In the present section we discuss the uniqueness of Lie algebras first with the same representation space but different scalar products and then the Lie algebras constructed from non-equivalent representation spaces.

**Proposition 2.** Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible module (not necessarily minimal) and denote by $V_1 = (V, -\langle \cdot, \cdot \rangle_V)$ an admissible module with the scalar product of the opposite sign, see Remark 7. Then the Lie algebras $N_{r,s}(V)$ and $N_{r,s}(V_1)$ are isomorphic under the map

$$N_{r,s}(V) = V \oplus \mathbb{R}^{r,s} \quad \mapsto \quad N_{r,s}(V_1) = V \oplus \mathbb{R}^{r,s}$$

$$(x, z) \quad \mapsto \quad (x, -z).$$

**Proposition 3.** Let $V$ be a representation space of a Clifford algebra $\text{Cl}_{r,s}$. We denote by $V_1 = (V, \langle \cdot, \cdot \rangle_V^{(1)})$ and $V_2 = (V, \langle \cdot, \cdot \rangle_V^{(2)})$ two minimal admissible modules with different scalar products. Then the Lie algebras $N_{r,s}(V_1)$ and $N_{r,s}(V_2)$ are isomorphic.

**Proof.** Any minimal admissible module is cyclic in the following sense. There is a vector, generating an orthonormal basis of the module by successive actions of the maps $J_z$, on this generating vector, see Theorem 7 item 3. We can find the generating vector in the space $E^1 = \cap E_{P_1}$, where $\{P_i\}$ is a set of the maximal number of mutually commuting isometric involutions. We choose two vectors $u, v \in E^1$ such that $\langle v, v \rangle_V^{(1)} = 1$, and $\langle u, u \rangle_V^{(2)} = 1$,.
where, if it is necessary, we can change the sign of the scalar products to be opposite according to Remark 1 and Proposition 2. Then these vectors generate the Clifford module, in the sense that $V$ is the span of all $\{J_{z_1} J_{z_2} \cdots J_{z_k} v\}$ and it is also the span of all $\{J_{z_1} J_{z_2} \cdots J_{z_k} u\}$. Moreover
\[
\langle J_{z_1} J_{z_2} \cdots J_{z_k} v, J_{z_1} J_{z_2} \cdots J_{z_{k'}} v \rangle_V^{(1)} = \langle J_{z_1} J_{z_2} \cdots J_{z_k} u, J_{z_1} J_{z_2} \cdots J_{z_{k'}} u \rangle_V^{(2)},
\]
for any choice of the basis vectors. Let us denote by $[\ , \ ]^{(k)}$, $k = 1, 2$ the brackets defined by scalar products $\langle \ , \ \rangle_V^{(k)}$. Then
\[
( z_\ell, [J_{z_1} \cdots J_{z_{k}} v, J_{z_1} \cdots J_{z_{k'}} v ]^{(1)}_{r,s} = \langle J_{z_2} J_{z_1} J_{z_1} \cdots J_{z_{k'}} v ]^{(1)}_{V},
\]
\[
\begin{align*}
\langle J_{z_2} J_{z_1} J_{z_1} \cdots J_{z_{k'}} u, J_{z_1} \cdots J_{z_{k'}} u \rangle^{(2)}_{V} &= \langle z_\ell, [J_{z_1} \cdots J_{z_{k}} u, J_{z_1} \cdots J_{z_{k'}} u ]^{(2)}_{V},
\end{align*}
\]
for any $l = 1, \ldots, r + s$. The map $\langle J_{z_1} J_{z_2} \cdots J_{z_{k}} v, z \rangle \mapsto (J_{z_1} J_{z_2} \cdots J_{z_{k}} u, z)$ is well defined and gives an isomorphism between the Lie algebras $N_{r,s}(V_1)$ and $N_{r,s}(V_2)$. \hfill $\square$

As it was mentioned in Section 2.2 some of the Clifford algebras $\text{Cl}_{r,s}$ have two minimal admissible modules, that correspond to two non-equivalent irreducible modules supplied with a neutral or a positive definite scalar product, making the representation maps skew-symmetric. We denote these modules by $V^{r,s}_+$ and $V^{r,s}_-$ and the corresponding Lie algebras by $N_{r,s}(V^{r,s}_+)$ and $N_{r,s}(V^{r,s}_-)$.\hfill $\square$

**Theorem 13.** If there are two minimal admissible $\text{Cl}_{r,s}$-modules $V^{r,s}_+$ and $V^{r,s}_-$, then the Lie algebras $N_{r,s}(V^{r,s}_+)$ and $N_{r,s}(V^{r,s}_-)$ are isomorphic.

The proof of Theorem 13 is contained in four lemmas. Lemma 7 is a reformulation of Lemma 3. Lemma 8 states general properties of Lie algebra isomorphism, Lemma 9 shows the isomorphism for lower dimensional cases and the last Lemma 10 is an application of the periodicity (2).\hfill $\square$

**Lemma 7.** Let $\{z_i\}_{i=1}^{r+s}$ be an orthonormal basis of $\mathbb{R}^{r,s}$ and $\Phi = A \oplus C : N_{r,s}(V^{r,s}_+) \to N_{r,s}(V^{r,s}_-)$ a Lie algebra isomorphism. Then the following relations hold
\[
\begin{align*}
A \prod_{j=1}^{p} J_{z_j} &= \begin{cases}
p \prod_{j=1}^{p} J_{C(z_j)} A, & \text{if } p = 2m, \\
p \prod_{j=1}^{p} J_{C(z_j)} (A^T)^{-1}, & \text{if } p = 2m + 1,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
A^T \prod_{j=1}^{p} J_{w_j} &= \begin{cases}
p \prod_{j=1}^{p} J_{C^T(w_j)} A^T & \text{if } p = 2m, \\
p \prod_{j=1}^{p} J_{C^T(w_j)} A^{-1}ER & \text{if } p = 2m + 1.
\end{cases}
\end{align*}
\]
for any $m \in \mathbb{N}$.

**Proof.** Recall, that a Lie algebra isomorphism $\Phi = A \oplus C : N_{r,s}(V^{r,s}_+) \to N_{r,s}(V^{r,s}_-)$ satisfies $C C^T = \text{Id}$ by Corollary 4. Then the proof follows literally the proof of Lemma 3 where one has to change the condition $CC^T = \text{Id}$ to $CC^T = \text{Id}$. \hfill $\square$

**Lemma 8.** Let $r - s = 3(\text{mod } 4)$ and the Clifford algebra $\text{Cl}_{r,s}$ admits two minimal admissible modules $V^{r,s}_\pm$. Assume that $\Phi = A \oplus C : N_{r,s}(V^{r,s}_+) \to N_{r,s}(V^{r,s}_-)$ is a Lie algebra isomorphism. Then $AA^T = - \det C \text{ Id}$ and $\det C = -1$ if $s = 0$.

**Proof.** Let $z_1, \ldots, z_{r+s}$ be orthonormal generators of the algebra $\text{Cl}_{r,s}$ and $V^{r,s}_\pm$ two non-equivalent minimal admissible modules of the algebra $\text{Cl}_{r,s}$ with the module actions, which
are denoted by \( J \) and \( \tilde{J} \), respectively. Both of admissible modules are irreducible and are distinguished by the actions of the volume forms \( \Omega^{r,s} = \prod_{j=1}^{r+s} J_{z_j} \) and \( \tilde{\Omega}^{r,s} = \prod_{j=1}^{r+s} \tilde{J}_{z_j} \), that is

\[
\Omega^{r,s} \equiv \text{Id} \text{ on } V^r_+, \text{ and } \tilde{\Omega}^{r,s} \equiv -\text{Id} \text{ on } V^r_-.
\]

Let us assume that there is an isomorphism \( \Phi = A \oplus C : N_{r,s}(V^r_+) \rightarrow N_{r,s}(V^r_-) \), where \( A : V^r_+ \rightarrow V^r_- \) and \( C : \mathbb{R}^r \rightarrow \mathbb{R}^r \) with \( CC^r = \text{Id} \) by Corollary 4. We set \( C(z_j) = \sum_{i=1}^{r+s} c_{ij} z_i \), \( j = 1, \ldots, r+s \). Then we have to satisfy the following

\[
AA^r = A\Omega^{r,s}A^r = \prod_{j=1}^{r+s} \tilde{J}_{C(z_j)} = \det C \prod_{j=1}^{r+s} \tilde{J}_{z_j} = \det C \tilde{\Omega}^{r,s} = -\det C \text{ Id} \quad \text{(38)}
\]

by Lemma 7 and the definition of the map \( C \).

In the case \( s = 0 \) the scalar product on \( V^r_0 \) is positive definite and the matrix \( AA^r \) is positive. Therefore,

\[
AA^r = -\det C \text{ Id} \implies \det C = -1, \implies AA^r = \text{Id}.
\]

We present general ideas for the construction of a possible map \( A : V^r_+ \rightarrow V^r_- \) in this case. Observe that

\[
J_{z_i} x = \begin{cases} (-1)^i \prod_{j \neq i} J_{z_j} x, & \text{if } i = 1, \ldots, r, \\ (-1)^{i-1} \prod_{j \neq i} J_{z_j} x, & \text{if } i = r+1, \ldots, r+s, \end{cases} \quad \text{for any } x \in V^r_+.
\]

Since the map \( A \) has to commute with the product of any even number of representations, we obtain for \( i \in \{1, \ldots, r\} \)

\[
AJ_{z_i}A^r = (-1)^i \prod_{j \neq i} \tilde{J}_{C(z_i)} A A^r = \tilde{J}_{C(z_i)} \det C \tilde{\Omega}^{r,s}(-\det C) \text{ Id} = (\det C)^2 \tilde{J}_{C(z_i)} = \tilde{J}_{C(z_i)}
\]

and analogously for \( i \in \{r+1, \ldots, r+s\} \). Since \( A^{-1} = -\det C A^r \), in order to satisfy Lemma 7 we must define the map \( A : V^r_+ \rightarrow V^r_- \) by

\[
AJ_{z_i} = -\det C \tilde{J}_{C(z_i)} A, \quad AJ_{z_j} = \tilde{J}_{C(z_j)} A, \ldots \quad \text{(40)}
\]

We also obtain that

\[
\langle A(x), A(x) \rangle_{V^r_-} = \langle x, A^r A(x) \rangle_{V^r_+} = -\det C \langle x, x \rangle_{V^r_+}
\]

for any \( x \in V^r_+ \). Thus we see that if \( \det C = 1 \), then the map \( A \) became anti-isometry and in the case \( \det C = -1 \) the map \( A \) is an isometry. In the following lemma we give the precise construction of the isomorphism for the basic cases.

**Lemma 9.** The Lie algebras \( N_{r,s}(V^r_+) \) and \( N_{r,s}(V^r_-) \) are isomorphic for the set of indices \((r, s) \in \{(3, 0), (7, 0), (1, 2), (3, 4), (5, 2), (1, 6)\}\).  

**Proof.** Case \((r, s) = (3, 0)\). Making use of the notations of Lemma 8 we are interested in the construction of an isomorphism \( \Phi = A \oplus C : N_{3,0}(V^3_+) \rightarrow N_{3,0}(V^3_-) \), where \( A : V^3_+ \rightarrow V^3_- \) and \( C : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) with \( CC^r = \text{Id} \), \( \det C = -1 \). The spaces \( V^3_+ \) and \( V^3_- \) are spanned by orthonormal bases

\[
x_1 = v, \quad x_2 = J_{z_1} v, \quad x_3 = J_{z_2} v, \quad x_4 = J_{z_1} J_{z_2} v, \quad \text{ and }
\]

\[
y_1 = u, \quad y_2 = J_{w_1} u, \quad y_3 = J_{w_2} u, \quad y_4 = J_{w_1} J_{w_2} u,
\]

and

\[
\sum_{i=1}^{r+s} c_{ij} z_i = w_j, \quad j = 1, \ldots, r + s.
\]
respectively, for some \( \langle v, v \rangle_{V^+_{1,2}} = \langle u, u \rangle_{V^-_{1,2}} = 0 \). In order to construct the map \( A \) we assume
\[
A(v) = \sum_{i=1}^{4} a_i y_i, \quad A^2 = \langle A(v), A(v) \rangle_{V^+_{1,2}} = \langle v, A^T A(v) \rangle_{V^-_{1,2}} = 0.
\]
Then
\[
A(x_2) = AJ_{z_1} v = \tilde{J}_{w_1} A(v) = -a_2 y_1 + a_1 y_2 - a_4 y_3 + a_3 y_4, \\
A(x_3) = AJ_{z_2} v = \tilde{J}_{w_2} A(v) = -a_3 y_1 + a_4 y_2 + a_1 y_3 - a_2 y_4, \\
A(x_4) = AJ_{z_1} J_{z_2} = J_{w_1} J_{w_2} A(v) = -a_4 y_1 - a_3 y_2 + a_2 y_3 + a_1 y_4.
\]
Thus, for any orthogonal transformation \( C \in O(3) \), with \( \det C = -1 \) we can find a map \( A: V^+_{1,2} \to V^-_{1,2} \) such that \( \Phi = A \oplus C: N_1(V^+_{1,2}) \to N_1(V^-_{1,2}) \) is an isomorphism.

**Case** \( (r, s) = (1, 2) \). Notice that in this case \( V^+_{1,2} = V_+ \oplus V_+ \) and \( V^-_{1,2} = V_- \oplus V_- \), where \( V_+ \) are non-equivalent irreducible modules. We choose basis \( \tilde{1} \tilde{1} \tilde{1} \) and look for the map \( A: V^+_{1,2} \to V^-_{1,2} \) satisfying
\[
A(v) = \sum_{i=1}^{2} a_i y_i, \quad A^2 = \langle A(v), A(v) \rangle_{V^+_{1,2}} = \langle v, A^T A(v) \rangle_{V^-_{1,2}} = - \det C.
\]
We denote \( \det C = \epsilon \). Making calculations similar to \( \tilde{1} \tilde{1} \tilde{1} \), we find
\[
A = \begin{pmatrix}
a_1 & \epsilon a_2 & -\epsilon a_3 & -\epsilon a_4 \\
\epsilon a_2 & -\epsilon a_3 & \epsilon a_4 & -\epsilon a_1 \\
-\epsilon a_3 & \epsilon a_1 & -\epsilon a_4 & \epsilon a_2 \\
-\epsilon a_4 & \epsilon a_2 & \epsilon a_1 & -\epsilon a_3
\end{pmatrix}, \quad A^T = \begin{pmatrix}
a_1 & -\epsilon a_2 & \epsilon a_3 & \epsilon a_4 \\
\epsilon a_2 & -\epsilon a_3 & -\epsilon a_4 & \epsilon a_1 \\
-\epsilon a_3 & \epsilon a_4 & -\epsilon a_1 & \epsilon a_2 \\
-\epsilon a_4 & \epsilon a_1 & -\epsilon a_2 & -\epsilon a_3
\end{pmatrix}
\]
By direct calculations we obtain \( A^T A = - \det C \Id \). We conclude that for any transformation \( C \in O(1, 2) \) we can find a map \( A: V^+_{1,2} \to V^-_{1,2} \) such that \( \Phi = A \oplus C: N_1(V^+_{1,2}) \to N_1(V^-_{1,2}) \) is an isomorphism.

**Case** \( (r, s) = (7, 0) \). We choose the involutions
\[
P_1 = J_{z_1} J_{z_2} J_{z_3} J_{z_4}, \quad P_2 = J_{z_1} J_{z_2} J_{z_5} J_{z_6}, \quad P_3 = J_{z_1} J_{z_2} J_{z_5} J_{z_7},
\]
acting on the module \( V^+_{7,0} \). For the module \( V^-_{7,0} \) we fix the involutions \( \tilde{P}_j \), \( i = 1, 2, 3 \) changing the basis vectors \( z_j \) by \( w_j = C(z_j) \), \( j = 1, \ldots, 7 \). We take vectors \( v \in V^+_{7,0} \) and \( u \in V^-_{7,0} \) such that
\[
P_j(v) = v, \quad \tilde{P}_j(u) = u \quad \text{for} \quad j = 1, 2, 3, \quad \text{and} \quad \langle v, v \rangle_{V^+_{7,0}} = \langle u, u \rangle_{V^-_{7,0}} = 1.
\]
The 8 common eigenspaces of \( P_j \) are one dimensional and are spanned by the vectors \( v \) and \( J_{z_j} v \). Analogously, one dimensional eigenspaces of \( \tilde{P}_j \) are spanned by \( u \) and \( J_{w_j} u \). We set
\[
A(v) = \lambda u, \quad A J_{z_j} v = \lambda \tilde{J}_{w_j} u, \quad j = 1, \ldots, 7, \quad \lambda = \pm 1.
\]
Thus, for each \( C \in O(7) \) with \( \det C = -1 \) and \( \lambda = \pm 1 \) there is a Lie algebra isomorphism between \( N_{7,0}(V^+_{7,0}) \) and \( N_{7,0}(V^-_{7,0}) \).

**Case** \( (r, s) = (3, 4) \). By using the notations as in the previous case, we choose the mutually commuting isometric involutions
\[
P_1 = J_{z_1} J_{z_2} J_{z_4} J_{z_5}, \quad P_2 = J_{z_1} J_{z_2} J_{z_6} J_{z_7}, \quad P_3 = J_{z_1} J_{z_2} J_{z_5} J_{z_7},
\]
acting on \( V_{+}^{3,4} \) and a vector \( v \in V_{+}^{3,4} \) such that \( P_i(v) = v, i = 1, 2, 3, \langle v, v \rangle_{V_{+}^{3,4}} = 1 \). The 8 one dimensional common eigenspaces of involutions \( P_j, j = 1, 2, 3 \) are spanned by the vectors \( v \) and \( J_z v \) \((i = 1, \ldots, 7)\), respectively. For the module \( V_{-}^{3,4} \) we also choose the involutions \( \bar{P}_j, j = 1, 2, 3 \) with the same combinations of the generators and take a positive unit vector \( u \in V_{-}^{3,4} \) (if necessary, by changing the sign of the scalar product, see Remark \( \Box \)) such that \( \bar{P}_i(u) = u \) for \( i = 1, 2, 3 \).

The one dimensional common eigenspaces of the involutions \( \bar{P}_i \) are spanned by \( u \) and \( \bar{J}_{w_i}, i = 1, \ldots, 7 \). We may set \( A(v) = \lambda u \). Then we have \( A^r(u) = \lambda v \), according to the choice \( \langle u, u \rangle_{V_{+}^{3,4}} = 1 \). So that

\[
- \det C \cdot A^r(v) = - \det C \cdot \lambda^2 \cdot v = v \quad \Longrightarrow \quad \det C = -1,
\]

since \( \lambda^2 = 1 \).

Thus, there is a Lie algebra isomorphism \( A \oplus C \) between \( N_{3,4}(V_{+}^{3,4}) \) and \( N_{3,4}(V_{-}^{3,4}) \) with \( C \in O(3,4) \) and we can always assume that \( \det C = -1 \).

**Case** \((r, s) \in \{(5,2), (1,6)\} \). First we present arguments of existence of an isomorphism and then we give the constructive proof. In Theorem \( \Box \) it was shown that \( N_{5,2} \) is isomorphic to \( N_{2,5} \), where we implicitly assumed that \( \Omega^{5,2}(v) = -v \), which was used in the condition \( J_{z_5} v = -i(v) \). Thus we actually showed the isomorphism \( N_{5,2}(V_{+}^{5,2}) \cong N_{2,5}. \) The assumption \( \Omega^{5,2}(v) = v \) leads to \( J_{z_5} v = i(v) \), and thus \( N_{5,2}(V_{+}^{5,2}) \cong N_{2,5}. \) We conclude \( N_{5,2}(V_{-}^{5,2}) \cong N_{2,5}(V_{+}^{5,2}) \). The same arguments shows \( N_{1,6}(V_{-}^{1,6}) \cong N_{1,6}(V_{+}^{1,6}) \).

We propose the constructive proof now. Let \((r, s) = (5, 2)\). The mutually commuting isometric involutions and the complementary operators acting on \( V_{+}^{5,2} \)

\[
P_1 = J_{z_1} J_{z_2} J_{z_3} \bar{J}_{z_4}, \quad P_2 = J_{z_1} J_{z_2} J_{z_6} \bar{J}_{z_7}, \quad R_1 = J_{z_4} J_{z_6}, \quad R_2 = J_{z_5} J_{z_6}
\]

show that common eigenspaces are 4-dimensional and neutral. It is enough to construct the isomorphism by defining map \( A_1 : E^1 \to \bar{E}^1 \), since it can be extended to the map \( A : V_{+}^{5,2} \to V_{-}^{5,2} \) in a similar way as in Theorem \( \Box \). By making use of the quaternion structure

\[
i = J_{z_1} J_{z_2}, \quad j = J_{z_1} J_{z_3} J_{z_5} J_{z_7}, \quad k = J_{z_2} J_{z_3} J_{z_5} J_{z_7},
\]

we fix the orthonormal basis for \( E^1 \) as follows

\[
x_1 = v, \quad x_2 = i(v), \quad x_3 = j(v), \quad x_4 = k(v),
\]

where \( v \in E_1 = \bigcap_{i=1}^{2} \bar{E}_{P_i}^1 \) and \( \langle v, v \rangle_{V_{+}^{5,2}} = 1 \). The existence of such a vector \( v \) and the orthogonality of the basis is justified by Lemma \( \Box \). Analogously, we choose the involutions \( \bar{P}_i, i = 1, 2 \), and quaternion structure, acting on \( V_{-}^{5,2} \) by changing \( J_{z_5} \) to \( \bar{J}_{C(z_5)} \). It allows to get the orthonormal basis

\[
y_1 = u, \quad y_2 = \bar{i}(u), \quad y_3 = \bar{j}(u), \quad y_4 = \bar{k}(u),
\]

for some \( u \in \bar{E}_1 = \bigcap_{i=1}^{2} \bar{E}_{\bar{P}_i}^1 \) with \( \langle u, u \rangle_{V_{-}^{5,2}} = 1 \). We are looking for the map \( A_1 : V_{+}^{5,2} \to V_{-}^{5,2} \) satisfying \( A_1(v) = \sum_{i=1}^{4} a_i y_i \) and

\[
a_1^2 + a_2^2 - a_3^2 - a_4^2 = \langle A_1(v), A_1(v) \rangle_{V_{-}^{5,2}} = \langle v, A_1^* A_1(v) \rangle_{V_{+}^{5,2}} = - \det C.
\]
We use the property $A_1 \mathbf{i} = \hat{\mathbf{i}} A_1$ (and the same for $j, k$) according to (40), and calculate the matrix for $A_1$ and $A_1^T A_1$:

$$A_1 = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}, \quad A_1^T A_1 = \begin{pmatrix} a_1 & a_2 & -a_3 & -a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}$$

and

$$A_1^T A_1 = \begin{pmatrix} a_1^2 + a_2^2 - a_3^2 - a_4^2 & 0 & 2a_1 a_3 - 2a_2 a_4 & 2a_1 a_4 - 2a_2 a_3 \\ 0 & a_1^2 + a_2^2 - a_3^2 - a_4^2 & 2a_1 a_3 + 2a_2 a_4 & 2a_1 a_4 + 2a_2 a_3 \\ 2a_1 a_3 - 2a_2 a_4 & 2a_1 a_3 + 2a_2 a_4 & a_1^2 + a_2^2 - a_3^2 - a_4^2 & 2a_1 a_4 - 2a_2 a_3 \\ 2a_1 a_4 - 2a_2 a_3 & 2a_1 a_4 + 2a_2 a_3 & 2a_1 a_3 - 2a_2 a_4 & a_1^2 + a_2^2 - a_3^2 - a_4^2 \end{pmatrix}$$

In order to satisfy the condition $A_1^T A_1 = -\det C \text{ Id}$ we need to solve the equations

$$-2a_1 a_3 + 2a_2 a_4 = 0, \quad -2a_2 a_3 - 2a_1 a_4 = 0.$$

Thus, if $\det C = -1$, then we have to choose $a_3 = a_4 = 0$ and if $\det C = 1$, then $a_1 = a_2 = 0$. We conclude that for each $C \in O(5, 2)$ there is a Lie algebra isomorphism between $N_{5,2}(V^0_{\pm})$ and $N_{5,2}(V^2_{\pm})$.

Let now $(r, s) = (1, 6)$. The arguments are essentially the same as in the previous case. The mutually commuting isometric involutions acting on $V_{1,6}^1$ are $P_1 = J_{22} J_{23}, J_{24} J_{25}, P_2 = J_{22} J_{23} J_{26} J_{27}$. The quaternion structure is $i = J_{22} J_{23}, j = J_{21} J_{23} J_{25} J_{27}, k = J_{21} J_{22} J_{25} J_{27}$. We fix orthonormal bases for $E^1$ and $\overline{E}^1$ as in the previous case and construct the map $A_1$ as in (45). We come to the same conclusion that for each $C \in O(1, 6)$ there is a Lie algebra isomorphism between $N_{1,6}(V_{1,6}^1)$ and $N_{1,6}(V_{1,6}^1)$.

**Corollary 8.** If $\Phi = A \oplus C: N_{r,s}(V_{+}^{r,s}) \rightarrow N_{r,s}(V_{-}^{r,s})$ is a Lie algebra isomorphism, then we always assume that $\det C = -1$.

**Proof.** The cases of the indices $(r, s) = \{ (3, 0), (7, 0), (1, 6) \}$ contains in the proof of Lemma 9. For the rest of the cases in Lemma 9 if $\det C = -1$, then we compose the isomorphism $\Phi_1 = A \oplus C$ with the isomorphism $\Phi_\pm = \text{Id} \oplus - \text{Id}$ that gives the isomorphism between $N_{r,s}(V_1)$ with $V_1 = (V_{\pm}^{r,s}, \{ \cdot, \}_{V_{\pm}^{r,s}})$ and the Lie algebra $N_{r,s}(V_2)$ with $V_2 = (V_{\pm}^{r,s}, - \langle \cdot, \cdot \rangle_{V_{\pm}^{r,s}})$ by Proposition 8. The composition map $\Phi = \Phi_1 \circ \Phi_\pm = A \oplus (-C)$ will have the properties $\det C = -1$ since $r + s$ is odd and $AA^T = \text{Id}$ due to the change of the sign of the scalar product. \hfill \square

**Lemma 10.** If the Lie algebra $N_{r,s}(V_{+}^{r,s})$ is isomorphic to the Lie algebra $N_{r,s}(V_{-}^{r,s})$, then

1. the Lie algebras $N_{r,s+8k}(V_{+}^{r,s+8k})$ and $N_{r,s+8k}(V_{-}^{r,s+8k})$ are isomorphic;
2. the Lie algebras $N_{r+8k,s}(V_{+}^{r+8k,s})$ and $N_{r+8k,s}(V_{-}^{r+8k,s})$ are isomorphic;
3. the Lie algebras $N_{r+4k,s+4k}(V_{+}^{r+4k,s+4k})$ and $N_{r+4k,s+4k}(V_{-}^{r+4k,s+4k})$ are isomorphic.

for any $k = 1, 2, \ldots$.

**Proof.** The proof is similar to the proof of Theorem 8 and we only show the first statement, since the others can be obtained analogously. If $\Phi_1 = A \oplus C: N_{0,8} \rightarrow N_{0,8}$, with $CC^T = \text{Id}$ and $\Phi = A \oplus C: N_{r,s}(V_{+}^{r,s}) \rightarrow N_{r,s}(V_{+}^{r,s})$ with $CC^T = \text{Id}$, then the map $\hat{\Phi} = \hat{A} \oplus \hat{C}$ with $\hat{C} = C \oplus C$ and $\hat{A} = A \oplus A$ given by (43) is a Lie algebra isomorphism between $N_{r,s+8k}(V_{+}^{r,s+8k})$ and $N_{r,s+8k}(V_{-}^{r,s+8k})$.

First of all we observe that an automorphism $\tilde{\Phi} = \tilde{A} \oplus \tilde{C}: N_{r,s} \rightarrow N_{0,8}$, with $\tilde{C}C^T = \text{Id}$ always exists, where we can simply set $\tilde{C} = \text{Id}$ and $\tilde{A}: V_{0,8} \rightarrow V_{0,8}$ can be any map $A \in \text{GL}(8)$ satisfying $AJ_{2j}A^T = J_{2j}$, where $\{ z_1, \ldots, z_8 \}$ is an orthonormal basis for $\mathbb{R}^{0,8}$. 


We only need to define the map $\hat{A}: V_+^{s+8k} \to V_-^{s+8k}$. By making use of the notations of Theorem 7, we set

$$\hat{A} \prod_{j=1}^{p} \partial_j \prod_{\alpha=1}^{q} \partial_\alpha =$$

$$= \begin{cases}
A \prod_{j=1}^{p} J_{z_j} \otimes \tilde{A}(\Omega^{0,0}) \prod_{\alpha=1}^{q} J_{\zeta_\alpha} = \prod_{j=1}^{p} \tilde{J}_{C(z_j)}(A^\tau)^{-1} \otimes \Omega_{\alpha=1}^{q} \tilde{J}_{C(\zeta_\alpha)}(\tilde{A}^\tau)^{-1}, & \text{if } p = 2m + 1, \quad q = 2k + 1, \\
A \prod_{j=1}^{p} J_{z_j} \otimes \tilde{A} \prod_{\alpha=1}^{q} J_{\zeta_\alpha} = \prod_{j=1}^{p} \tilde{J}_{C(z_j)}(A^\tau)^{-1} \otimes \Omega_{\alpha=1}^{q} \tilde{J}_{C(\zeta_\alpha)} \tilde{A}, & \text{if } p = 2m + 1, \quad q = 2k,
\end{cases}$$

(46)

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K. Furutani: Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo, 2641 Yamazaki, Noda, Chiba (278-8510), Japan
E-mail address: furutani_kenro@ma.noda.tus.ac.jp

I. Markina: Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway
E-mail address: irina.markina@uib.no