From Additional Symmetries to Linearization of Virasoro Symmetries

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Abstract

We construct the additional symmetries and derive the Adler–Shiota–van Moerbeke formula for the two-component BKP hierarchy. We also show that the Drinfeld–Sokolov hierarchies of type D, which are reduced from the two-component BKP hierarchy, possess symmetries written as the action of a series of linear Virasoro operators on the tau function. It results in that the Drinfeld–Sokolov hierarchies of type D coincide with Dubrovin and Zhang’s hierarchies associated to the Frobenius manifolds for Coxeter groups of type D, and that every solution of such a hierarchy together with the string equation is annihilated by certain combinations of the Virasoro operators and the time derivations of the hierarchy.

Key words: additional symmetry, Adler–Shiota–van Moerbeke formula, two-component BKP hierarchy, Drinfeld–Sokolov hierarchy, Virasoro symmetry

1 Introduction

One of the attractive topics on hierarchies of integrable systems is to study their additional symmetries. Such symmetries can be represented in Lax equations with operators depending explicitly on the time variables as developed by Orlov and Schulman [20], and also in the form of Sato’s Bäcklund transformations as the action of vertex operators on tau function of the hierarchy [5]. These two strands are connected by a formula of Adler, Shiota and van Moerbeke [11] in consideration of the Kadomtsev–Petviashvili (KP) and the two-dimensional Toda lattice hierarchies aiming at finding the constraints satisfied by matrix integrals [2]. The Adler–Shiota–van Moerbeke (ASvM) formula for the KP hierarchy was also proved, in a more straightforward way, by Dickey [6].

In this note, we will construct the additional symmetries and derive the ASvM formula for the two-component BKP hierarchy, which was first introduced as a bilinear
equation by Date, Jimbo, Kashiwara and Miwa [3] and represented to a Lax form with pseudo-differential operators in [19]. This hierarchy is found interesting on aspects such as representation of Lie algebras, algebraic geometry, topological field theory and infinite-dimensional Frobenius manifolds [14, 21, 22, 29], as well as for the reductions of it to subhierarchies [14, 19, 27]. A direct reduction of the two-component BKP hierarchy is to the original (one-component) BKP hierarchy [5]. Recall that the ASvM formula for the BKP hierarchy was first obtained by van de Leur [18] with an algebraic method, and later proved by Tu [24] following the approach of Dickey [6]. Our first aim is to extend such results for the BKP hierarchy to that for the two-component case, along the line of Dickey and Tu.

Another important reduction of the two-component BKP hierarchy is to the Drinfeld–Sokolov hierarchy of type $D_n$, that is, the hierarchy associated to untwisted affine algebra $D_n^{(1)}$ with the zeroth vertex $c_0$ of the Dynkin diagram marked [7, 19] (see Section 5 for the Lax representation). Based on this fact, we will derive Virasoro symmetries for the $D_n$-type Drinfeld–Sokolov hierarchy by using the additional symmetries of its universal hierarchy. More exactly, let $\tau$ be the tau function of the Drinfeld–Sokolov hierarchy of type $D_n$ reduced from that of the two-component BKP hierarchy, then the Virasoro symmetries for the former hierarchy can be written as

$$\frac{\partial \tau}{\partial s_j} = V_j \tau, \quad j \geq -1 \quad (1.1)$$

where the operators $V_j$ satisfy

$$[V_i, V_j] = (i - j)V_{i+j} + \delta_{i+j,0} \frac{n}{12}(i^3 - i). \quad (1.2)$$

Virasoro symmetries represented in the form (1.1) are said to be linearized.

In a sense inspired by the connection between Gromov–Witten invariants and integrable systems [17, 25], Dubrovin and Zhang [11] proposed a project to classify integrable hierarchies satisfying certain axioms. These axioms are

(A1) the hierarchy carries a bi-Hamiltonian structure, the dispersionless limit of which is associated to a semisimple Frobenius manifold [8];

(A2) the hierarchy has a tau function defined by the tau-symmetric Hamiltonian densities;

(A3) linearization of Virasoro symmetries, namely, these symmetries are represented by the action of certain linear operators $L_j (j \geq -1)$ on the tau function, where the operators $L_j$ generate a half Virasoro algebra.

For example, both the Korteweg–de Vries (KdV) and the extended Toda hierarchies satisfy these axioms, and their tau functions give generating functions for Gromov–Witten invariants with target spaces of point and $\mathbb{C}P^1$ [17, 12, 25]. Furthermore, Dubrovin and Zhang conjectured that the Drinfeld–Sokolov hierarchies of ADE type all satisfy (A1)–(A3). This conjecture has been confirmed for the $A_n$-type hierarchy (equivalent to the $n$th KdV or Gelfand–Dickey hierarchy), but is still open for hierarchies of the other two types. In the case of the Drinfeld–Sokolov hierarchy of type $D_n$,
we will show that the operators $V_j$ are equal to $L_j$ up to a rescaling of the time variables. Therefore, in combination with the results in [9, 19], we conclude the following result.

**Theorem 1.1** The Drinfeld–Sokolov hierarchy of type $D_n$ coincides with the full hierarchy constructed by Dubrovin and Zhang starting from the Frobenius manifold on the orbit space of the Coxeter group $D_n$. Moreover, for this hierarchy every analytic solution $\tau$ satisfying the string equation $V_{-1}\tau = \partial \tau/\partial t_1$ admits the following constraints

$$\tilde{L}_j\tau = 0, \quad \tilde{L}_j = V_j - \frac{\partial}{\partial t_{(2n-2)(j+1)+1}}, \quad j \geq -1. \quad (1.3)$$

Here $t_{(2n-2)(j+1)+1}$ are time variables of the hierarchy and $\tilde{L}_j$ also obey the communication relation as in $(1.2)$.

We remark that the $D_n$-type Drinfeld–Sokolov hierarchy is equivalent to several versions of bilinear equations, which are constructed by Date, Jimbo, Kashiwara and Miwa [1], by Kac and Wakimoto [16, 26] in the context of representation theory, as well as by Givental and Milanov [13] in singularity theory.

This paper is arranged as follows. In next section we recall the definition of the two-component BKP hierarchy. In Sections 3 and 4, we will write down the additional symmetries and show the ASvM formula for this hierarchy. By reducing the two-component BKP hierarchy to the Drinfeld–Sokolov hierarchy of type $D_n$, the Virasoro symmetries (1.1) for this subhierarchy will be derived in Section 5. In Section 6, we will complete the proof of theorem 1.1. Finally, Section 7 is devoted to some remarks.

## 2 The two-component BKP hierarchy

Let us recall the definition of the two-component BKP hierarchy.

Assume $\mathcal{A}$ is an algebra of smooth functions of a spatial coordinate $x$, and the derivation $D = d/dx$ induces a gradation on $\mathcal{A}$ as

$$\mathcal{A} = \prod_{i \geq 0} A_i, \quad A_i \cdot A_j \subset A_{i+j}, \quad D(A_i) \subset A_{i+1}.$$

Denote $D = \{ \sum_{i \in \mathbb{Z}} f_i D^i \mid f_i \in \mathcal{A} \}$ and consider its two subspaces

$$D^- = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j \geq \max\{0, i-m\}} a_{i,j} D^j \mid a_{i,j} \in \mathcal{A}_j, m \in \mathbb{Z} \right\}, \quad (2.1)$$

$$D^+ = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j \geq \max\{0, m-i\}} a_{i,j} D^j \mid a_{i,j} \in \mathcal{A}_j, m \in \mathbb{Z} \right\}. \quad (2.2)$$
These subspaces form algebras with multiplication defined by the following product of monomials

\[ fD^i \cdot gD^j = \sum_{r \geq 0} \binom{i}{r} fD^r(g)D^{i+j-r}, \quad f, g \in A. \]

Elements of the algebras \( D^- \) and \( D^+ \) are called pseudo-differential operators of the first type and the second type respectively. Note that in [19] a pseudo-differential operator of the first type takes the form \( \sum_{i<\infty} f_i D^i \) with \( f_i \in A \), namely, it only contains finite powers of \( D \). Here we have extended the notion \( D^- \) slightly, in order to include the Orlov–Schulman operators that will be introduced in next section. For any operator \( A = \sum_{i \in \mathbb{Z}} f_i D^i \in D^\pm \), its nonnegative part, negative part, residue and adjoint operator are respectively

\[ A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad \text{res } A = f_{-1}, \quad A^* = \sum_{i \in \mathbb{Z}} (-D)^i \cdot f_i. \quad (2.3) \]

Let

\[ L = D + \sum_{i \geq 1} u_i D^{-i} \in D^-, \quad \hat{L} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i \in D^+ \quad (2.4) \]

such that \( L^* = -DLD^{-1} \) and \( \hat{L}^* = -D\hat{L}D^{-1} \). The two-component BKP hierarchy is defined by the following Lax equations:

\[ \frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad \frac{\partial \hat{L}}{\partial t_k} = [(L^k)_+, \hat{L}], \quad \frac{\partial L}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, L], \quad \frac{\partial \hat{L}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, \hat{L}] \quad (2.5) \]

with \( k \in \mathbb{Z}_{\text{odd}}^+ \). Note that \( \partial/\partial t_1 = \partial/\partial x \); henceforth we assume \( t_1 = x \).

One can write the operators \( L \) and \( \hat{L} \) in a dressing form as

\[ L = \Phi D\Phi^{-1}, \quad \hat{L} = \hat{\Phi} D^{-1}\hat{\Phi}^{-1}. \quad (2.6) \]

Here

\[ \Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \hat{\Phi} = 1 + \sum_{i \geq 1} b_i D^i \quad (2.7) \]

are pseudo-differential operators over certain graded algebra that contains \( A \), and they satisfy

\[ \Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1}. \quad (2.8) \]

Given \( L \) and \( \hat{L} \), the dressing operators \( \Phi \) and \( \hat{\Phi} \) are determined uniquely up to a multiplication to the right by operators of the form (2.7) and (2.8) with constant coefficients. The two-component BKP hierarchy (2.5) can be redefined as

\[ \frac{\partial \Phi}{\partial t_k} = -(L^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial t_k} = ((L^k)_+ - \delta_{k1} \hat{L}^{-1})\hat{\Phi}, \quad \frac{\partial \Phi}{\partial \hat{t}_k} = -(\hat{L}^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{t}_k} = (\hat{L}^k)_+ \hat{\Phi} \quad (2.9, 2.10) \]
with \( k \in \mathbb{Z}_{\text{odd}}^+ \).

Denote \( t = (t_1, t_3, t_5, \ldots) \) and \( \hat{t} = (\hat{t}_1, \hat{t}_3, \hat{t}_5, \ldots) \). Introduce two wave functions

\[
\begin{align*}
    w(z) &= w(t, \hat{t}; z) = \Phi e^{\xi(t; z)}, \\
    \hat{w}(z) &= \hat{w}(\hat{t}, t; z) = \hat{\Phi} e^{\xi(\hat{t}; z)},
\end{align*}
\]

where the function \( \xi \) is defined as

\[
\xi(t; z) = \sum_{k \in \mathbb{Z}_{\text{odd}}^+} t_k z^k.
\]

The hierarchy (2.5) carries infinitely many bi-Hamiltonian structures [27, 28]. The Hamiltonian densities, which are proportional to the residues of \( L^k \) and \( \hat{L}^k \), satisfy the tau-symmetric condition. Hence given a solution of the two-component BKP hierarchy, there locally exists a tau function \( \tau(t, \hat{t}) \) such that

\[
\begin{align*}
    L w(z) &= z w(z), \\
    \hat{L} \hat{w}(z) &= z^{-1} \hat{w}(z).
\end{align*}
\]

Theorem 2.1 ([19]) The hierarchy defined by (2.9) and (2.10) is equivalent to the following bilinear equation

\[
\begin{align*}
    - \text{res}_{z=\infty} z^{-1} \tau(t - 2[z^{-1}], \hat{t}) \tau(t', \hat{t} + 2[z]) e^{\xi(t; z)} dz &= \text{res}_{z=0} z^{-1} \tau(t, \hat{t}) \tau(t' + 2[z], \hat{t}' - 2[z]) e^{\xi(t; z)} dz \\
    &\quad + \text{res}_{z=0} z^{-1} \tau(t', \hat{t}' - 2[z]) e^{\xi(t; z)} dz. \\
\end{align*}
\]

The proof of this theorem is mainly based on the following lemma.

Lemma 2.2 (see, for example, [5]) For any pseudo-differential operators \( P \) and \( Q \) of the same type, it holds that

\[
\text{res}_z(P e^{zx} \cdot Q^* e^{-zx}) = \text{res}(PQ).
\]

Here and below the residue of a Laurent series is defined as \( \text{res}_z \sum_i f_i z^i = f_{-1} \).

In (2.5) if \( \hat{L} \) vanishes, then one obtains the BKP hierarchy, of which the bilinear equation is (2.14) with the right hand side replaced by 1, see [5]. The following fact is observed from the bilinear equation (2.14).

Proposition 2.3 When either \( \hat{t} \) or \( t \) is fixed, the two-component BKP hierarchy is reduced to the BKP hierarchy with time variables \( t \) or \( \hat{t} \) respectively.
3 Orlov–Schulman operators and additional symmetries

We are to construct additional symmetries for the two-component BKP hierarchy by using the Orlov–Schulman operators, the coefficients of which may depend explicitly on the time variables of the hierarchy.

With the dressing operators given in (2.6), we introduce
\[ M = \Phi \Gamma \Phi^{-1}, \quad \hat{M} = \hat{\Phi} \hat{\Gamma} \hat{\Phi}^{-1}, \]
where
\[ \Gamma = \sum_{k \in \mathbb{Z}^{odd}} k t_k D_k^{-1}, \quad \hat{\Gamma} = x + \sum_{k \in \mathbb{Z}^{odd}} k \hat{t}_k D_k^{-1}. \]

If we assume the degrees of \( t_k \) and \( \hat{t}_k \) are equal to \( k \), then \( M \) and \( \hat{M} \) are pseudo-differential operators of the first and the second types respectively.

It is easy to see the following

\[ \text{Lemma 3.1} \]
The operators \( M \) and \( \hat{M} \) satisfy
\[ [L, M] = 1, \quad [\hat{L}^{-1}, \hat{M}] = 1; \quad (3.1) \]
\[ (2) \quad M w(z) = \partial_z w(z), \quad \hat{M} \hat{w}(z) = \partial_z \hat{w}(z); \quad (3.2) \]
\[ (3) \quad \frac{\partial M}{\partial t_k} = [(L^k)_+, \hat{M}], \quad \frac{\partial \hat{M}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, M]. \quad (3.3) \]

Given any pair of integers \((m, l)\) with \( m \geq 0 \), let
\[ A_{ml} = M^m L^l - (-1)^l L^{-l-1} M^m L, \quad \hat{A}_{ml} = \hat{M}^m \hat{L}^l - (-1)^l \hat{L}^{-l-1} \hat{M}^m \hat{L}^{-1}. \quad (3.4) \]

It is easy to check
\[ A_{ml}^* = -DA_{ml} D^{-1}, \quad \hat{A}_{ml}^* = -D\hat{A}_{ml} D^{-1}. \quad (3.5) \]

The following equations are well defined
\[ \frac{\partial \Phi}{\partial s_{ml}} = -(A_{ml})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{s}_{ml}} = (A_{ml})_+ \hat{\Phi}, \quad (3.6) \]
\[ \frac{\partial \Phi}{\partial \hat{s}_{ml}} = -(\hat{A}_{ml})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial s_{ml}} = (\hat{A}_{ml})_+ \hat{\Phi}. \quad (3.7) \]

We assume that these flows commute with \( \partial / \partial x \).

The following lemma is clear.
Lemma 3.2 The flows (3.6) and (3.7) satisfy

\[
\frac{\partial L}{\partial \dot{s}_{ml}} = - (\dot{A}_{ml}), \quad \frac{\partial \hat{L}}{\partial \dot{s}_{ml}} = [\dot{A}_{ml}],
\]
\[
\frac{\partial M}{\partial \dot{s}_{ml}} = - (\dot{A}_{ml}), \quad \frac{\partial \hat{M}}{\partial \dot{s}_{ml}} = [\dot{A}_{ml}],
\]
\[
\frac{\partial w(z)}{\partial \dot{s}_{ml}} = - (\dot{A}_{ml})w(z), \quad \frac{\partial \hat{w}(z)}{\partial \dot{s}_{ml}} = (\dot{A}_{ml})\hat{w}(z),
\]

where \( \dot{s}_{ml} = s_{ml}, \dot{\hat{s}}_{ml} \) correspond to \( \dot{A}_{ml} = A_{ml}, \dot{\hat{A}}_{ml} \) respectively.

Proposition 3.3 The flows (3.6) and (3.7) commute with those in (2.9) and (2.10) that compose the two-component BKP hierarchy. Namely, for any \( \dot{s}_{ml} = s_{ml}, \dot{\hat{s}}_{ml} \) and \( \dot{t}_k = t_k, \dot{\hat{t}}_k \) one has

\[
\left[ \frac{\partial}{\partial \dot{s}_{ml}}, \frac{\partial}{\partial \dot{\hat{t}}_k} \right] = 0, \quad m \geq 0, \quad l \in \mathbb{Z}, \quad k \in \mathbb{Z}_{\text{odd}}.
\]

Proof The proposition is checked case by case with the help of Lemmas 3.1 and 3.2. For example,

\[
\left[ \frac{\partial}{\partial \dot{s}_{ml}}, \frac{\partial}{\partial \dot{t}_k} \right] \hat{\Psi} = \left[ (L^k) + \delta_{k1}L^{-1}, (\dot{A}_{ml}) + \delta_{k1}(\dot{\hat{A}}_{ml})^{-1} \right] \hat{\Psi} = 0.
\]

The proposition is proved. \( \square \)

Proposition 3.3 implies that the flows (3.6) and (3.7) give symmetries for the two-component BKP hierarchy. Such symmetries are called the additional symmetries.

Remark 3.4 The vector fields \( \partial/\partial s_{0,2i+1} \) (resp. \( \partial/\partial \hat{s}_{0,-2i+1} \)) cannot be identified to \( 2\partial/\partial t_{2i+1} \) (resp. \( 2\partial/\partial \hat{t}_{2i+1} \)). In fact, they act differently on either \( M \) or \( \hat{M} \).

Now we compute the commutation relation between the additional symmetries. Observe that the commutator \( X = [A_{ml}, A_{m'\ell'}] \) is a polynomial in \( M \) and \( L^{\pm 1} \), and it satisfies \( X^* = -DXD^{-1} \), hence there exist constants \( c^{qr}_{ml,m'\ell'} \) such that

\[
[A_{ml}, A_{m'\ell'}] = \sum_{q,r} c^{qr}_{ml,m'\ell'} A_{qr}.
\]

In the same way, one also has \( [\dot{A}_{ml}, \dot{A}_{m'\ell'}] = \sum_{q,r} c^{qr}_{ml,m'\ell'} \dot{A}_{qr} \). For example,

\[
c^{qr}_{0l,0\ell'} = 0, \quad c^{qr}_{1,2i+1;1,2j+1} = 4(i-j)\delta_{q1}\delta_{r,2(i+j)+1}.
\]
Proposition 3.5 Acting on the dressing operators $\Phi$ and $\hat{\Phi}$ (or the wave functions $w$ and $\hat{w}$), the vector fields of the additional symmetries (3.6) and (3.7) satisfy

\[
\begin{bmatrix}
\frac{\partial}{\partial s_{ml}}, \frac{\partial}{\partial s_{m'l'}}
\end{bmatrix} = -\sum_{q,r} c_{qr}^{m,l,m'l'} \frac{\partial}{\partial s_{qr}},
\]
(3.9)

\[
\begin{bmatrix}
\frac{\partial}{\partial \hat{s}_{ml}}, \frac{\partial}{\partial \hat{s}_{m'l'}}
\end{bmatrix} = \sum_{q,r} c_{qr}^{m,l,m'l'} \frac{\partial}{\partial \hat{s}_{qr}},
\]
(3.10)

\[
\begin{bmatrix}
\frac{\partial}{\partial \hat{s}_{ml}}, \frac{\partial}{\partial \hat{s}_{m'l'}}
\end{bmatrix} = 0
\]
(3.11)

It means that these vector fields generate a $w_{\infty}^B \times w_{\infty}^B$-algebra (cf., for example, [24]).

Proof The conclusion follows from a straightforward calculation. \qed

To prepare for the next section, we introduce two generating functions of operators as

\[
Y(\lambda, \mu) = -\sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l}(A_{m,m+l})_-, \quad (3.12)
\]
\[
\hat{Y}(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l}(\hat{A}_{m,m+l})_+ \quad (3.13)
\]

with parameters $\lambda$ and $\mu$.

Lemma 3.6 The action of the generators (3.12) and (3.13) on the wave functions \[2.11\] reads

\[
2Y(\lambda, \mu)w(z) = -w(-\lambda)\partial_x^{-1}(w_x(\mu)w(z) - w(\mu)w_x(z)) + w(\mu)\partial_x^{-1}(w_x(-\lambda)w(z) - w(-\lambda)w_x(z)), \quad (3.14)
\]
\[
2\hat{Y}(\lambda, \mu)\hat{w}(z) = \hat{w}(-\lambda)\partial_x^{-1}(\hat{w}_x(\mu)\hat{w}(z) - \hat{w}(\mu)\hat{w}_x(z)) - \hat{w}(\mu)\partial_x^{-1}(\hat{w}_x(-\lambda)\hat{w}(z) - \hat{w}(-\lambda)\hat{w}_x(z)) \quad (3.15)
\]

with $\hat{x} = \hat{t}_1$. Here the subscripts $x$ and $\hat{x}$ mean the derivatives with respect to them.

Proof Let us check the second equality first. The property (3.5) implies $(\hat{A}_{m,m+l})_+(1) = 0$, then using Lemmas 2.2 and 3.1 we have

\[
\begin{aligned}
&(\hat{A}_{m,m+l})_+\hat{w}(z) \\
&= \sum_{i \geq 1} \text{res}_z \left( \hat{A}_{m,m+l} D^{-i-1} \right) D^i \hat{w}(z) \\
&= \sum_{i \geq 1} \left( \text{res}_z \left( \hat{M}^m \hat{\Phi} D^{m+l} e^{x \xi + \xi(t; -\zeta)} \cdot (\hat{\Phi}^{-1} D^{-i-1})^* e^{-x \xi - \xi(t; -\zeta)} \right) e^{x \xi + \xi(t; -\zeta)} \cdot (\hat{\Phi}^{-1} D^{-i-1})^* e^{-x \xi - \xi(t; -\zeta)} \right) D^i \hat{w}(z)
\end{aligned}
\]
\[
\begin{align*}
&= \sum_{i \geq 1} \left( \text{res}_\zeta \left( \zeta^{m+1} \partial^m_\zeta \hat{w}(\zeta) \cdot (-D)^{-i} \zeta^{-1} \hat{w}(-\zeta) \right) \\
&\quad - (-1)^{m+i} \text{res}_\zeta \left( \partial^m_\zeta \zeta^{m+1-i} \hat{w}(\zeta) \cdot (-D)^{-i} \hat{w}(-\zeta) \right) \right) D^i \hat{w}(z) \\
&= - \text{res}_\zeta \left( \zeta^{m+1-i} \partial^m_\zeta \hat{w}(\zeta) \cdot D^{-1}(\hat{w}(-\zeta)\hat{w}_x(z)) \right) \\
&\quad + (-1)^{m+i} \text{res}_\zeta \left( \partial^m_\zeta \left( \zeta^{m+1-i} \hat{w}(\zeta) \right) \cdot D^{-1}(\hat{w}(-\zeta)\hat{w}_x(z)) \right).
\end{align*}
\]

Hence
\[
\hat{Y}(\lambda, \mu) \hat{w}(z) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} \left( - \text{res}_\zeta \left( \zeta^{m+1-l} \partial^m_\zeta \hat{w}(\zeta) \cdot D^{-1}(\hat{w}(-\zeta)\hat{w}_x(z)) \right) \\
+ (-1)^{m+l} \text{res}_\zeta \left( \partial^m_\zeta \left( \zeta^{m+1-l} \hat{w}(\zeta) \right) \cdot D^{-1}(\hat{w}(-\zeta)\hat{w}_x(z)) \right) \right)
\]
\[
= - \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \partial^m_\lambda \hat{w}(\lambda) \cdot D^{-1}(\hat{w}(-\lambda)\hat{w}_x(z)) \\
+ \sum_{l=-\infty}^{\infty} (-\lambda)^{-l} \text{res}_\zeta \left( \left( \zeta + \mu - \lambda \right)^{-l} \hat{w}(\zeta + \mu - \lambda) D^{-1}(\hat{w}(-\zeta)\hat{w}_x(z)) \right)
\]
\[
= - \hat{w}(\mu)D^{-1}(\hat{w}(-\lambda)\hat{w}_x(z)) + \hat{w}(-\lambda)D^{-1}(\hat{w}(\mu)\hat{w}_x(z)).
\]

It leads to
\[
2\hat{Y}(\lambda, \mu) \hat{w}(z) = \hat{w}(-\lambda)D^{-1}(\hat{w}(\mu)\hat{w}_x(z) - \hat{w}_x(\mu)\hat{w}(z)) \\
- \hat{w}(\mu)D^{-1}(\hat{w}(-\lambda)\hat{w}_x(z) - \hat{w}_x(-\lambda)\hat{w}(z)). \tag{3.16}
\]

To rewrite this equality as (3.15), we denote \( \rho = \text{res} \hat{L} \) and recall \( \partial_x \hat{w}(z) = -\hat{L}_- \hat{w}(z) = -D^{-1}(\rho \hat{w}(z)). \) Since
\[
\begin{align*}
\partial_x D^{-1}(\hat{w}(\mu)\hat{w}_x(z) - \hat{w}_x(\mu)\hat{w}(z)) \\
= D^{-1}(\hat{L}_- \hat{w}(\mu)\hat{w}_x(z) - \hat{L}_- \hat{w}_x(\mu)\hat{w}(z)) \\
= D^{-1}(\rho \hat{w}(\mu)\hat{w}_x(z) + \hat{w}(\mu)D^{-1}(\rho \hat{w}(z)) \\
= \hat{w}_x(\mu)\hat{w}(z) - \hat{w}(\mu)\hat{w}_x(z),
\end{align*}
\]
then (3.16) is recast to (3.15). The verification of the equality (3.14) is easier. The lemma is proved. \( \square \)

**Remark 3.7** Suppose \( \hat{t} \) is fixed, then \( w(z) \) is a wave function of the BKP hierarchy. In this case, the equality (3.14) is just equation (2.4) in [24], with \( Y(\lambda, \mu) = -\lambda Y_B(\lambda, \mu) \) where \( Y_B(\lambda, \mu) \) is the notation used there.

4 Vertex operators and the Adler–Shiota–van Moerbeke formula

Let us consider the additional symmetries acting on the tau function of the two-component BKP hierarchy.
Here \(\cdot\) stands for the normal-order product, which means that 
\[
\dot{\tau} = \eta_{p} \tau = \frac{\partial \tau}{\partial t}. 
\]

Clearly \([\dot{\tau}, \dot{t}] = 2k\delta_{k,-l}\) with \(\dot{p} = p\) or \(\dot{\tau}\). Introduce \(\dot{p}(z) = \sum_{k \in \mathbb{Z}^{\text{odd}}} \dot{p}_{k} z^{-k}/k\), and define vertex operators
\[
X(\lambda, \mu) = e^{p(\lambda)-p(\mu)} \; ; \quad \hat{X}(\lambda, \mu) = e^{\dot{p}(\lambda^{-1})-\dot{p}(\mu^{-1})} \; . 
\]
Here \(";\) stands for the normal-order product, which means that \(\dot{p}_{k \geq 0}\) must be placed to the right of \(\dot{p}_{k < 0}\).

If either \(\hat{t}\) or \(t\) is fixed, the two-component BKP hierarchy is reduced to the usual BKP hierarchy. Hence by using Sato’s Bäcklund transformation for the latter (see [5]), the vertex operators \(\hat{X}(\lambda, \mu)\) provide infinitesimal transformations on the space of tau functions of the two-component BKP hierarchy. Namely, given any solution \(t^{\text{rel}}(\lambda, \mu)\) of order \(O(2)\), the functions \(\dot{\tau}(z) + \epsilon\hat{X}(\lambda, \mu)\tau(t, \dot{t}) + O(\epsilon^{2})\) also satisfy (2.14) modulo terms of order \(O(\epsilon^{2})\) as \(\epsilon \to 0\).

**Theorem 4.1 (ASvM formula)** For the two-component BKP hierarchy the following equalities hold true
\[
\hat{X}(\lambda, \mu)\dot{\tau}(z) = 2\frac{\mu - \lambda}{\mu + \lambda} Y(\lambda, \mu)\dot{\tau}(z). 
\]

Here the actions of \(\hat{X}(\lambda, \mu)\) on the wave functions are generated by their actions on the tau function (recall (2.13)).

**Proof** The proof is similar to that for the case of the BKP hierarchy [21], so we only sketch the main steps.

First, the bilinear equation (2.14) yields the following Fay identity
\[
\begin{align*}
\frac{s_{0} - s_{1} s_{0} - s_{2} s_{0} - s_{3}}{s_{0} + s_{1} s_{0} + s_{2} s_{0} + s_{3}} \times & \tau(t + 2[s_{0}] + 2[s_{1}] + 2[s_{2}] + 2[s_{3}], \dot{t} + 2[\dot{s}_{0}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}]) \tau(t, \dot{t} + 2[\dot{s}_{0}]) \\
+ & \sum_{c.p.(s_{1}, s_{2}, s_{3})} \frac{s_{1} - s_{0} s_{1} + s_{2} s_{1} + s_{3}}{s_{1} + s_{0} s_{1} - s_{2} s_{1} - s_{3}} \times \tau(t + 2[s_{2}] + 2[s_{3}], \dot{t} + 2[\dot{s}_{1}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}]) \tau(t + 2[s_{0}] + 2[s_{1}], \dot{t} + 2[\dot{s}_{0}]) \\
= & \frac{\dot{s}_{0} - \dot{s}_{1} \dot{s}_{0} - \dot{s}_{2} \dot{s}_{0} - \dot{s}_{3}}{\dot{s}_{0} + \dot{s}_{1} \dot{s}_{0} + \dot{s}_{2} \dot{s}_{0} + \dot{s}_{3}} \times \tau(t + 2[\dot{s}_{1}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}], \dot{t} + 2[\dot{s}_{0}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}]) \tau(t + 2[s_{0}], \dot{t}) \\
+ & \sum_{c.p.(\dot{s}_{1}, \dot{s}_{2}, \dot{s}_{3})} \frac{\dot{s}_{1} - \dot{s}_{0} \dot{s}_{1} + \dot{s}_{2} \dot{s}_{1} + \dot{s}_{3}}{\dot{s}_{1} + \dot{s}_{0} \dot{s}_{1} - \dot{s}_{2} \dot{s}_{1} - \dot{s}_{3}} \times \tau(t + 2[\dot{s}_{1}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}], \dot{t} + 2[\dot{s}_{0}] + 2[\dot{s}_{2}] + 2[\dot{s}_{3}]) \tau(t + 2[s_{0}], \dot{t})
\end{align*}
\]
\[ \times \tau(t + 2[s_1] + 2[s_2] + 2[s_3], \dot{t} + 2[\dot{s}_2] + 2[\dot{s}_3])\tau(t + 2[s_0], \dot{t} + 2[\dot{s}_0] + 2[\dot{s}_1]). \]  

Here \(s_i\) and \(\dot{s}_i\) are parameters, and “c.p.” stands for “cyclic permutation”.

Second, take \(\dot{s}_0 = \dot{s}_3\) and \(\dot{s}_1 = -\dot{s}_2\), and then let all \(\dot{s}_i \to 0\). In this way \((4.3)\) is reduced to a Fay identity of the form for the BKP hierarchy. Differentiate it with respect to \(s_3\) and then let \(s_3 = 0\), consequently one obtains a differential Fay identity as follows

\[
\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right) (\tau(t + 2[s_1], \dot{t})\tau(t + 2[s_2], \dot{t}) - \tau(t + 2[s_1] + 2[\dot{s}_1] + 2[\dot{s}_2])\tau(t, \dot{t})) \\
= \left(\frac{1}{s_1} + \frac{1}{s_2}\right) (\partial_x \tau(t + 2[s_1], \dot{t}) \cdot \tau(t + 2[s_2], \dot{t}) - \tau(t + 2[s_1], \dot{t}) \cdot \partial_x \tau(t + 2[s_2], \dot{t})) \\
- \left(\frac{1}{s_1} - \frac{1}{s_2}\right) (\partial_x \tau(t + 2[s_1] + 2[\dot{s}_2], \dot{t}) \cdot \tau(t, \dot{t}) - \tau(t + 2[s_1] + 2[\dot{s}_1] + 2[\dot{s}_2]) \cdot \partial_x \tau(t, \dot{t})).
\]

(4.4)

Exchange \(s_i\) and \(\dot{s}_i\) and repeat the above procedure, then one gets another identity

\[
\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right) (\tau(t, \dot{t} + 2[\dot{s}_1])\tau(t, \dot{t} + 2[\dot{s}_2]) - \tau(t, \dot{t} + 2[\dot{s}_1] + 2[\dot{s}_2])\tau(t, \dot{t})) \\
= \left(\frac{1}{s_1} + \frac{1}{s_2}\right) (\partial_x \tau(t, \dot{t} + 2[\dot{s}_1]) \cdot \tau(t, \dot{t} + 2[\dot{s}_2]) - \tau(t, \dot{t} + 2[\dot{s}_1]) \cdot \partial_x \tau(t, \dot{t} + 2[\dot{s}_2])) \\
- \left(\frac{1}{s_1} - \frac{1}{s_2}\right) (\partial_x \tau(t, \dot{t} + 2[\dot{s}_1] + 2[\dot{s}_2]) \cdot \tau(t, \dot{t}) - \tau(t, \dot{t} + 2[\dot{s}_1] + 2[\dot{s}_2]) \cdot \partial_x \tau(t, \dot{t})).
\]

(4.5)

Observe that \((4.4)\) and \((4.5)\) were also derived in [23] as the first and the second differential Fay identities for the two-component BKP hierarchy.

Finally, by using the two differential Fay identities and Lemma 3.6, the equalities \((4.1)\) are verified after a straightforward calculation. Therefore the theorem is proved.

We want to represent the additional symmetries \((3.6)\) and \((3.7)\) with the tau function in \((2.12)\). To this end we expand the vertex operators in \((4.1)\) formally as (cf. [11, 12] and [31, 33])

\[
\hat{X}(\lambda, \mu) = \sum_{m=0}^{\infty} (\frac{\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} \hat{W}_l^{(m)},
\]

(4.6)

where

\[
\hat{W}_l^{(m)} = \text{res}_\lambda \left(\lambda^{m+l-1} \hat{\partial}_\mu^m |_{\mu=\lambda} \hat{X}(\lambda, \mu) \right).
\]

For convenience we assume \(\hat{p}_i = 0\) for even \(i\). It is easy to see

\[
W_l^{(0)} = \delta_{l,0} = \hat{W}_l^{(0)}, \quad W_l^{(1)} = \hat{p}_l, \quad \hat{W}_l^{(1)} = \hat{p}_{-l}, \\
W_l^{(2)} = \sum_{i+j=l} p_i p_j : -(l+1)p_i, \quad \hat{W}_l^{(2)} = \sum_{i+j=-l} \hat{p}_i \hat{p}_j : -(l+1)\hat{p}_{-l}.
\]
Introduce two operators $Z(\lambda, \mu)$ and $\dot{Z}(\lambda, \mu)$ as
\[
\dot{Z}(\lambda, \mu) = \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} (\dot{X}(\lambda, \mu) - 1).
\] (4.7)

They are expanded to the form
\[
\dot{Z}(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{m-l} \dot{Z}^{(m+1)}_l
\] (4.8)

with
\[
\dot{Z}^{(1)}_l = \dot{W}^{(1)}_l, \quad \dot{Z}^{(m+1)}_l = \frac{1}{m+1} \dot{W}^{(m+1)}_l + \frac{1}{2} \dot{W}^{(m)}_l \text{ for } m \geq 1.
\] (4.9)

Now we have a corollary of Theorem 4.1.

**Corollary 4.2** For the two-component BKP hierarchy, the additional symmetries (3.6) and (3.7) can be expressed via the tau function as
\[
\frac{\partial \tau}{\partial s_{m,m+l}} = \left( Z^{(m+1)}_l + \delta_{l0} c_m \right) \tau, \quad \frac{\partial \tau}{\partial s_{m,m+l}} = \left( \dot{Z}^{(m+1)}_l + \delta_{l0} \dot{c}_m \right) \tau
\] (4.10)

with certain constants $c_m$ and $\dot{c}_m$.

**Proof** Denote
\[
G(t; z) = \exp \left( - \sum_{k \in \mathbb{Z}^{2 \lambda}} \frac{2}{k} \frac{\partial}{\partial t_k} \right).
\]

By substituting the following equalities
\[
- (A_{m,m+l})_+ w(z) = \frac{\partial w(z)}{\partial s_{m,m+l}} = w(z)(G(t; z) - 1) \frac{\partial \tau / \partial s_{m,m+l}}{\tau},
\]
\[
\frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} X(\lambda, \mu) w(z) = w(z)(G(t; z) - 1) \frac{Z(\lambda, \mu) \tau}{\tau},
\]
\[
(\dot{A}_{m,m+l})_+ \dot{w}(z) = \frac{\partial \dot{w}(z)}{\partial s_{m,m+l}} = \dot{w}(z)(G(t; z) - 1) \frac{\partial \tau / \partial s_{m,m+l}}{\tau},
\]
\[
\frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} \dot{X}(\lambda, \mu) \dot{w}(z) = \dot{w}(z)(G(t; z) - 1) \frac{\dot{Z}(\lambda, \mu) \tau}{\tau}
\]
to the ASvM formula (4.2), one has
\[
\frac{\partial \tau}{\partial s_{m,m+l}} = (Z^{(m+1)}_l + c_{m,m+l}) \tau, \quad \frac{\partial \tau}{\partial s_{m,m+l}} = (\dot{Z}^{(m+1)}_l + \dot{c}_{m,m+l}) \tau
\]
with constants $c_{m,m+l}$ and $\dot{c}_{m,m+l}$. They are recast to the form (4.10) after a scaling transformation $\tau \mapsto \tau \exp \left( - \sum_{m \geq 0, l \neq m} (c_{ml} s_{ml} + \dot{c}_{ml} \dot{s}_{ml}) \right)$, which is allowed in the definition of the tau function (2.12). The corollary is proved. \(\square\)

The representation (4.10) shows that the vector fields $\partial / \partial s_{ml}$ and $\partial / \partial s_{ml}$, when acting on the tau function, generate a $W^{B}_{1+\infty} \times W^{B}_{1+\infty}$-algebra. Here $W^{B}_{1+\infty}$ is the central extension of $w^B_\infty$, see Proposition 5.5. It was observed by Adler, Shiota and van Moerbeke [1] the phenomenon that lifting the action of additional symmetries on wave functions to the action on tau functions implies a central extension of $w$-algebras.
5 Virasoro symmetries for Drinfeld–Sokolov hierarchies of type D

Given an integer \( n \geq 3 \), assume the two-component BKP hierarchy (2.5) is constrained by
\[
L^{2n-2} = \hat{L}^2 = \mathcal{L},
\]
where
\[
\mathcal{L} = D^{2n-2} + \frac{1}{2} \sum_{i=1}^{n-1} D^{-1} (v_i D^{2i-1} + D^{2i-1} v_i) + D^{-1} \rho D^{-1} \rho.
\]
(5.1)
Then the equations in (2.5) are reduced to
\[
\frac{\partial \mathcal{L}}{\partial t_k} = [(L^k)_+, \mathcal{L}], \quad \frac{\partial \mathcal{L}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, \mathcal{L}], \quad k \in \mathbb{Z}_{\text{odd}}.
\]
(5.2)
These equations compose the Drinfeld–Sokolov hierarchy of type \( D_n \), which is associated to the affine algebra \( D_n^{(1)} \) and the zeroth vertex of its Dynkin diagram \([7, 19]\).

Note that the equations (5.2) can also be written as (2.9) and (2.10) via the dressing operators.

Generally, the additional symmetries (3.6) and (3.7) for the two-component BKP hierarchy do not admit the constraint \( L^{2n-2} = \hat{L}^2 \), hence they cannot be reduced to symmetries for the subhierarchy (5.2) directly. However, a reduction works for certain linear combinations of these symmetries.

For \( j \geq -1 \), let
\[
B_j = \frac{1}{4n-4} A_{1,(2n-2)j+1} + \frac{1}{4} \hat{A}_{1,-2j+1}.
\]
(5.3)
Namely,
\[
B_j = \frac{1}{4n-4} (ML^{(2n-2)j+1} + L^{(2n-2)j} ML) + \frac{1}{4} (\hat{M} \hat{L}^{2j-1} + \hat{L}^{2j} \hat{M}^{-1}).
\]
Note that the product between \( B_j \) (which may not lie in \( \mathcal{D}^- \cup \mathcal{D}^+ \), see Section 2) and the bounded operator \( \mathcal{L} \in \mathcal{D}^- \cap \hat{\mathcal{D}}^+ \) makes sense \([19]\). In fact, we have
\[
[B_j, \mathcal{L}] = -L^{(2n-2)(j+1)} + \hat{L}^{2(j+1)} = 0.
\]
(5.4)
**Remark 5.1** The last equality in (5.4) is not true whenever \( j < -1 \), for the reason that the operator \( \mathcal{L} \) has different inverses in \( \mathcal{D}^- \) and \( \mathcal{D}^+ \).

The equality (5.4) together with the property \( B_j^* = -DB_jD^{-1} \) implies that the following evolutionary equations are well defined
\[
\frac{\partial \mathcal{L}}{\partial s_j} = [(B_j)_-, \mathcal{L}] = [(B_j)_+, \mathcal{L}], \quad j \geq -1.
\]
(5.5)
Since the operator \( \mathcal{L} \) can be written as \( \mathcal{L} = \Phi D^{2n-2} \Phi^{-1} = \hat{\Phi} D^{-2} \hat{\Phi}^{-1} \) with dressing operators of the form (2.7) and (2.8), then the equations (5.5) can be redefined by either of the following
\[
\frac{\partial \Phi}{\partial s_j} = -(B_j)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial s_j} = (B_j)_+ \hat{\Phi}.
\]
(5.6)
Observe that \( \partial / \partial s_j \) is just the reduction of the combination
\[
\frac{1}{4n-4} \frac{\partial}{\partial s_{1,(2n-2)j+1}} + \frac{1}{4} \frac{\partial}{\partial \hat{s}_{1,-2j+1}}
\]
of the additional symmetries (3.6) and (3.7) for the two-component BKP hierarchy. This observation leads to the following consequence.

**Proposition 5.2** For the Drinfeld–Sokolov hierarchy of type \( D_n \) defined by (5.2), the following statements hold true.

1. The flows (5.5) give symmetries of the hierarchy, namely,
   \[
   \left[ \frac{\partial}{\partial s_j}, \frac{\partial}{\partial t_k} \right] = 0, \quad \left[ \frac{\partial}{\partial s_j}, \frac{\partial}{\partial \hat{t}_k} \right] = 0, \quad j \geq -1, \quad k \in \mathbb{Z}_{\text{odd}}^+.
   \]

2. Let \( w(z) \) and \( \hat{w}(z) \) be the wave functions defined as in (2.11), then
   \[
   \frac{\partial w(z)}{\partial s_j} = -(B_j)_{-} w(z), \quad \frac{\partial \hat{w}(z)}{\partial s_j} = (B_j)_{+} \hat{w}(z), \quad j \geq -1. \tag{5.7}
   \]

3. With a tau function \( \tau \) reduced from that of the two-component BKP hierarchy, the above symmetries can be written as
   \[
   \frac{\partial \tau}{\partial s_j} = V_j \tau, \quad j \geq -1. \tag{5.8}
   \]

Here the operators \( V_j \) read
\[
V_j = \frac{1}{8n-8} \sum_i : p_i p_{(2n-2)j-i} : + \frac{1}{8} \sum_i : \hat{p}_i \hat{p}_{2j-i} : + \delta_{j0} \frac{n}{24} \left( 1 + \frac{1}{2n-2} \right), \tag{5.9}
\]
and they obey the following commutation relation
\[
[V_i, V_j] = (i - j) V_{i+j} + \delta_{i+j,0} \frac{n}{12} (i^3 - i). \tag{5.10}
\]

**Proof** The first two assertions are clear. For the third item, according to Corollary 4.2 we see that the equalities (5.8) hold for
\[
V_j = \frac{1}{4n-4} Z_{(2n-2)j}^{(2)} + \frac{1}{4} \hat{Z}_{-2j}^{(2)} + \delta_{j0} \cdot \text{const}.
\]
These operators are expanded to (5.9) with the constant chosen appropriately (the chosen constant will be interpreted by the spectrum of a Frobenius manifold, see the following section). The commutation relation (5.10) is checked by a straightforward calculation, which is valid even for all \( i, j \in \mathbb{Z} \), see, for example, [15]. The proposition is proved. \( \square \)
6 Hierarchy associated to Frobenius manifold

The hierarchy (5.2) carries a bi-Hamiltonian structure given by the following compatible Poisson brackets (see Proposition 8.3 of [7], as well as [9, 27]):

\[ \{F, G\}_1(\mathcal{L}') = 2 \int \text{res} X((DY_+\mathcal{L}_-') - (\mathcal{L}_-Y_+D) + (\mathcal{L}_-Y_-D) - (DY_-\mathcal{L}_+)') \, dx, \]  
\[ (6.1) \]

\[ \{F, G\}_2(\mathcal{L}') = 2 \int \text{res} ((\mathcal{L}_+Y_+\mathcal{L}_-') - X\mathcal{L}_-(Y_-\mathcal{L}_') + \mathcal{L}_-Y_-\mathcal{L}_-') \, dx, \]
\[ (6.2) \]

in which \( \mathcal{L}' = D\mathcal{L} \), and the arbitrary local functionals \( F \) and \( G \) have gradients \( X = \delta F / \delta \mathcal{L}' \) and \( Y = \delta G / \delta \mathcal{L}' \) respectively. More precisely, one rescales the time variables as

\[ T^{\alpha,p} = \begin{cases} 
\frac{(2n - 2)\Gamma(p + 1 + \frac{2\alpha - 1}{2n - 2})}{\Gamma(\frac{2\alpha - 1}{2n - 2})} t_{(2n-2)p+2\alpha-1}, & \alpha = 1, \ldots, n-1; \\
\frac{2\Gamma(p + 1 + \frac{1}{2})}{\Gamma(\frac{1}{2})} \hat{t}_{2p+1}, & \alpha = n
\end{cases} \]

with \( p = 0, 1, 2, \ldots \). In particular, \( T^{1,0} = t_1 = x \). Then the hierarchy (5.2) is recast to the following bi-Hamiltonian form

\[ \frac{\partial F}{\partial T^{\alpha,p}} = \{F, H_{\alpha,p}\}_1 = \left( p + \frac{1}{2} + \mu_\alpha \right)^{-1} \{F, H_{\alpha,p-1}\}_2, \]
\[ (6.4) \]

where the densities of the Hamiltonians \( H_{\alpha,p-1} \) read

\[ h_{\alpha,p-1} = \begin{cases} 
\frac{\Gamma(\frac{2\alpha - 1}{2n - 2})}{(4n - 4)\Gamma(p + 1 + \frac{2\alpha - 1}{2n - 2})} \text{res} L^{(2n-2)p+2\alpha-1}, & \alpha = 1, \ldots, n-1; \\
\frac{\Gamma(\frac{1}{2})}{4\Gamma(p + 1 + \frac{1}{2})} \text{res} \hat{L}^{2p+1}, & \alpha = n,
\end{cases} \]

and the constants \( \mu_\alpha \) are

\[ \mu_\alpha = \begin{cases} 
\frac{2\alpha - n}{2n - 2}, & \alpha = 1, \ldots, n-1; \\
0, & \alpha = n.
\end{cases} \]

(6.5)

Introduce a family of differential polynomials

\[ \Omega_{\alpha,p;\beta,q} = \partial_x^{-1} \frac{\partial h_{\alpha,p-1}}{\partial T^{\beta,q}}, \quad \alpha, \beta \in \{1, 2, \ldots, n\}; \quad p, q \geq 0. \]
They satisfy $\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}$, which means that the densities $h_{\alpha,p}$ are tau-symmetric [11]. Hence a tau function $\tau$ of the integrable hierarchy (6.4) is defined by
\[
\frac{\partial^2 \log \tau}{\partial T^{\alpha,p} \partial T^{\beta,q}} = \Omega_{\alpha,p;\beta,q}. \tag{6.6}
\]

It is straightforward to see the following

**Lemma 6.1** For the hierarchy (5.2), the tau function given in (6.6) coincides with the one (2.12) reduced from that of the two-component BKP hierarchy.

The dispersionless limit of the Hamiltonian structures (6.1) and (6.2) are associated to a Frobenius manifold on the orbit space of the Coxeter group $D_n$ [8, 9, 29]. Let $\mathcal{M}$ denote this Frobenius manifold.

We lay out some necessary datas of $\mathcal{M}$. Take the symbol of the pseudo-differential operator $L$ in (5.1) as
\[
l(z) = z^{2n-2} + v_{n-1}z^{2n-4} + \cdots + v_{2}z^{2} + v_{1} + \rho^{2}z^{-2}.
\]

Let
\[
w^\alpha = \begin{cases} 
-\frac{2n-2}{2n-2\alpha - 1} \text{res}_{z=\infty} l(z) \frac{2n-2\alpha-1}{2n-2} dz, & \alpha = 1, 2, \ldots, n-1; \\
\frac{2}{\rho}, & \alpha = n, 
\end{cases} \tag{6.7}
\]
then $w^1, \ldots, w^n$ give a system of flat coordinates on $\mathcal{M}$. The potential $F(w^1, \ldots, w^n)$ of this Frobenius manifold is determined by
\[
\frac{\partial^3 F}{\partial w^\alpha \partial w^\beta \partial w^\gamma} = \frac{1}{2} \text{res}_{l'(z)=0} \frac{\partial w^\alpha l(z) \cdot \partial w^\beta l(z) \cdot \partial w^\gamma l(z)}{l'(z)} dz.
\]

On $\mathcal{M}$ the unity vector field is $e = \partial / \partial w^1$, the invariant metric and the multiplication are given by
\[
\langle \frac{\partial}{\partial w^\alpha}, \frac{\partial}{\partial w^\beta} \rangle = \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \frac{\partial^3 F}{\partial w^\alpha \partial w^\beta \partial w^\gamma}, \quad \frac{\partial}{\partial w^\alpha} \cdot \frac{\partial}{\partial w^\beta} = c^\gamma_{\alpha\beta} \frac{\partial}{\partial w^\gamma}, \quad c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} \frac{\partial^3 F}{\partial w^\epsilon \partial w^\alpha \partial w^\beta}.
\]

Here $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$ and summations over repeated Greek indices are assumed. The Euler vector field is
\[
E = \sum_{\alpha=1}^{n-1} \frac{n-\alpha}{n-1} w^\alpha \frac{\partial}{\partial w^\alpha} + \frac{n}{2n-2} w^n \frac{\partial}{\partial w^n},
\]
which satisfies $\text{Lie}_E F = (3 - d) F$ with $d = (n-2)/(n-1)$.

It is easy to see that the dispersionless limit of $h_{\alpha,-1}$ equals to $\eta_{\alpha\beta} w^\beta$. Hence one has the following lemma by virtue of the bi-Hamiltonian recursion relation.

---

1According to the notation of [11], one still needs to do a replacement $T^{\alpha,p} \mapsto \epsilon T^{\alpha,p}$ with a small parameter $\epsilon$, meanwhile the residue of a pseudo-differential operator becomes the coefficient of $(\epsilon D)^{-1}$. By dispersionless limit we mean the limit as $\epsilon \to 0$. 

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Lemma 6.2 (see, for example, [29]) The principal hierarchy [11] for the Frobenius manifold $M$ coincides with the dispersionless limit of the hierarchy (6.4).

By now we have explained that the hierarchy (6.4) satisfies the axioms (A1) and (A2) reviewed in Section 1. From the principal hierarchy to the full hierarchy, one still needs to consider the Virasoro symmetries.

Recall that the spectrum of a Frobenius manifold is a quadruple

$$(V, <, >, \mu, R),$$

where $V$ is some linear space with a flat metric $<, >$ and certain linear transformations $\mu$ and $R$. With this data, a series of operators $\{L_j \mid j \in \mathbb{Z}\}$ were constructed in [10, 11], which satisfy the Virasoro commutation relation

$$[L_i, L_j] = (i - j)L_{i+j} + \delta_{i+j,0} n \frac{n}{12} (i^3 - i).$$

Lemma 6.3 For all $j \geq -1$, the Virasoro operators $L_j$ associated to the Frobenius manifold $M$ coincide with $V_j$ in (5.9) derived from vertex operators. Hence the symmetries (5.8) are written as

$$\frac{\partial \tau}{\partial s_j} = L_j \tau, \quad j \geq -1.$$  

Proof To compute the operators $L_j$ for the Frobenius manifold $M$, we substitute into the formulae (2.29)-(2.32) of [10] with the following data

- the metric $<, >$ given by $\eta = (\eta_{\alpha\beta})$ with

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial w^\alpha \partial w^\beta} = \frac{1}{4n-4} \delta_{\alpha+\beta,n} + \frac{1}{4} \delta_{\alpha,n} \delta_{\beta,n};$$

- $\mu = \frac{2 - d}{2} 1 - \nabla E = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ given in (6.5), which satisfies

$$\eta \mu + \mu \eta = 0;$$

- $R = 0$ for the reason that none of the differences $\mu_\alpha - \mu_\beta$ is a nonzero integer.

Hence we have

$$L_{-1} = \sum_{p \geq 1} T^{\alpha,p} \frac{\partial}{\partial T^{\alpha,p-1}} + \frac{1}{2} \eta_{\alpha\beta} T^{\alpha,0} T^{\beta,0},$$

$$L_0 = \sum_{p \geq 0} \left( p + \frac{1}{2} + \mu_\alpha \right) T^{\alpha,p} \frac{\partial}{\partial T^{\alpha,p}} + \frac{1}{4} \text{tr} \left( \frac{1}{4} - \mu^2 \right),$$

$$L_1 = \sum_{p \geq 0} \left( p + \frac{1}{2} + \mu_\alpha \right) \left( p + \frac{3}{2} + \mu_\alpha \right) T^{\alpha,p} \frac{\partial}{\partial T^{\alpha,p+1}}.$$
\[ L_2 = \sum_{p \geq 0} \left( p + \frac{1}{2} + \mu_\alpha \right) \left( p + \frac{3}{2} + \mu_\alpha \right) \left( p + \frac{5}{2} + \mu_\alpha \right) T^{\alpha, p} \frac{\partial}{\partial T^{\alpha, p+2}} + \eta^{\alpha \beta} \left( \frac{1}{2} - \mu_\alpha \right) \left( \frac{1}{2} - \mu_\beta \right) \left( \frac{3}{2} - \mu_\beta \right) \frac{\partial}{\partial T^{\alpha, 1}} \frac{\partial}{\partial T^{\beta, 0}}. \]

(6.13)

By using (6.3) and after a straightforward calculation, we obtain \( L_j = V_j \) for \( j = -1, 0, 1, 2 \) where \( V_j \) are given in (5.9). In particular, the constant term in (6.12) is equal to the one in \( V_0 \). By virtue of the Virasoro commutation relations (5.10) and (6.9), the lemma is proved.

Getting Lemmas 6.2 and 6.3 together, we arrive at the following conclusion.

**Theorem 6.4** According to Dubrovin and Zhang’s construction, the hierarchy associated to the semisimple Frobenius manifold \( M \) and satisfying the axioms (A1)–(A3) coincides with the hierarchy (6.4), that is, the Drinfeld–Sokolov hierarchy of type \( D_n \).

**Remark 6.5** Observe that the definition of the tau function in (6.6) and the linearization of the Virasoro symmetries (6.10) are invariant with respect to the following scaling transformation

\[ F \mapsto \lambda F, \quad \frac{\partial}{\partial T^{\alpha, p}} \mapsto \sqrt{\lambda} \frac{\partial}{\partial \tilde{T}^{\alpha, p}}, \quad T^{\alpha, p} \mapsto \frac{1}{\sqrt{\lambda}} \tilde{T}^{\alpha, p}, \]

where \( F \) is the potential of the Frobenius manifold \( M \) and \( \lambda \) is an arbitrary nonzero parameter. Without considering the Virasoro symmetries, one can rescale \( F \) and \( T^{\alpha, p} \) separately and define different tau functions for the same hierarchy. That is why a tau function \( \tilde{\tau} = \tau^2 \) for the hierarchy (5.2) was obtained in [19]. In this sense the linearization of Virasoro symmetries helps to select a unique tau function.

We write the Virasoro operators \( L_j \) more precisely as \( L_j(\partial/\partial T; T) \) with \( T = \{ T^{\alpha, p} \} \). For \( j \geq -1 \), introduce

\[ \tilde{L}_j = L_j \left( \frac{\partial}{\partial \tilde{T}}; \tilde{T} \right), \quad \tilde{T} = \{ \tilde{T}^{\alpha, p} = T^{\alpha, p} - \delta^{\alpha}_1 \delta^p_1 \}. \]

(6.15)

Clearly the operators \( \tilde{L}_j \) also satisfy the Virasoro commutation relation as in (6.9). As an application of the property of linearization of Virasoro symmetries, we have the following theorem.

**Theorem 6.6** Suppose an analytic solution \( \tau \) of the hierarchy (6.4) satisfies the string equation

\[ L_{-1} \tau = \frac{\partial \tau}{\partial x}, \]

(6.16)

then it admits the following Virasoro constraints

\[ \tilde{L}_j \tau = 0, \quad j \geq -1. \]

(6.17)
Proof. Equation (6.16) is just $\partial \tau / \partial s = \partial \tau / \partial x$, which is represented with the wave functions as
\[ \frac{\partial w(z)}{\partial s} = (D - z)w(z) = -Lw(z), \quad \frac{\partial \hat{w}(z)}{\partial s} = D\hat{w}(z). \]

These equations together with (5.7) imply $B_{-1} = L$.

Recall for $j \geq -1$,
\[ B_j = \frac{1}{2n - 2}(ML^{(2n - 2)j + 1} + (n - 1)jL^{(2n - 2)j}) + \frac{1}{2}(\hat{M}\hat{L}^{2j - 1} - j\hat{L}^{2j}). \]

It is easy to see
\[ B_j = B_{-1}L^{j + 1} = L^{(2n - 2)(j + 1) + 1}. \]

Hence
\[ \frac{\partial \Phi}{\partial s_j} = -(B_j)\Phi = -(L^{(2n - 2)(j + 1) + 1})\Phi = \frac{\partial \Phi}{\partial t^{(2n - 2)(j + 1) + 1}}. \quad (6.18) \]

From (2.11) and (2.13) it follows that $\text{res } \Phi = -2\partial_x \log \tau$. Taking the residue of (6.18), we have
\[ \partial_x \left( \frac{1}{\tau} \left( \frac{\partial \tau}{\partial s_j} - \frac{\partial \tau}{\partial t^{(2n - 2)(j + 1) + 1}} \right) \right) = 0. \]

We make use of (6.10) and (6.3) to write the left hand side of the above equation as $\partial_x (\hat{L}_j \tau / \tau)$, thus
\[ \hat{L}_j \tau = f_j \tau, \quad j \geq -1, \quad (6.19) \]
where $f_j$ are functions of $\{T^\alpha,p \mid (\alpha, p) \neq (1, 0)\}$. In particular, the string equation (6.16) means $f_{-1} = 0$; we need to show $f_j = 0$ for all $j \geq 0$.

Now we apply the Virasoro commutation relation for the operators $\hat{L}_j$. Denote $\Delta_1 = \sum_{p \geq 1} \hat{T}^\alpha.p \cdot \partial / \partial T^\alpha.p^{-1}$, then
\[ 0 = \hat{L}_{-1} \tau = [\hat{L}_0, \hat{L}_{-1}] \tau = -f_0 \hat{L}_{-1} \tau - \Delta_1(f_0) \cdot \tau = -\Delta_1(f_0) \cdot \tau. \]

It means that $f_0$ is a constant. Similarly, considering
\[ [\hat{L}_1, \hat{L}_{-1}] \tau = 2\hat{L}_0 \tau = 2f_0 \tau, \quad (6.20) \]
\[ [\hat{L}_1, \hat{L}_0] \tau = \hat{L}_1 \tau = f_1 \tau, \quad (6.21) \]
we have
\[ -\Delta_1(f_1) = 2f_0, \quad -\Delta_0(f_1) = f_1, \]
where $\Delta_0 = \sum_{p \geq 0} (p + \frac{1}{2} + \mu_\alpha) \hat{T}^\alpha.p \cdot \partial / \partial T^\alpha.p$. Thus
\[ f_0 = 0, \quad f_1 = 0. \]

The case $j = 2$ is similar, and the cases $j \geq 3$ follow from a simple induction. The theorem is proved. \qed

Theorem [1.1] follows from the combination of Theorems [6.4] and [6.6].
7 Concluding remarks

We have obtained the linearized Virasoro symmetries for the Drinfeld–Sokolov hierarchy of type $D_n$ starting from the additional symmetries of its universal hierarchy, i.e., the two-component BKP hierarchy. Therefore we complete the proof that the Drinfeld–Sokolov hierarchy of type $D_n$ satisfies the axioms (A1)–(A3) in Section 1 and show that its tau function admits the Virasoro constraints. A tau function of the KdV or the extended Toda hierarchies that satisfies the Virasoro constraints plays an important role in the intersection theory on moduli spaces \cite{17, 12, 25}. Besides these hierarchies, now the Drinfeld–Sokolov hierarchy of type $D_n$ is involved in Dubrovin and Zhang’s construction \cite{11}. For the topological solution of this hierarchy, however, it is unknown whether there is a geometric illustration in moduli spaces or not. We plan to study it in the future. We also hope that our method, which is distinct from that used for the extended Toda hierarchy in \cite{12}, would be applied to study Virasoro constraints for other integrable hierarchies.

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