Spacetime completeness of charged black holes in the four-dimensional Einstein-Gauss-Bonnet theory

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Abstract

By investigating the spacetime completeness of the charged black hole in the four-dimensional Einstein-Gauss-Bonnet theory, we find a contradiction between the completeness of spacetime and the smoothness of limiting from charged to uncharged cases. This contradiction stems from the ranges of the Einstein-Gauss-Bonnet parameter, which shows that the completeness is not only coordinate dependent but also parameter dependent. To verify the coordinate-and-parameter-dependent completeness of spacetime, we analyze mainly second-order phase transitions and quasinormal modes in the eikonal limit for the charged Einstein-Gauss-Bonnet black hole in the four-dimensional spacetime, where the resulting effects might be observed in future experiments when compared with that of the Reissner-Nordström black hole.

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1 Introduction

Traditionally, Lovelock’s theorem has shown [1, 2, 3] that Einstein’s general relativity is the unique theory of gravity in the four-dimensional spacetime. However, a novel Einstein-Gauss-Bonnet (EGB) theory in four dimensions, the so-called 4D EGB gravity, has recently been claimed [4] by the way of starting with a higher dimensional EGB theory and then rescaling the EGB parameter and taking the limit of the spacetime dimension to four. Correspondingly, its static spherically symmetric solution, the uncharged 4D EGB black hole has also been given. Later, the charged solution was provided [5] within the framework of the 4D EGB theory. Due to the novelty of this theory of gravity, other relevant solutions were obtained, such as the Bardeen black hole [6], the noncommutative inspired black hole [7], and the Born-Infeld black hole [8], etc. Nonetheless, the self consistency of this theory was seriously argued, e.g., from the points of view of field equations [9] and instability of vacua [10], and so on.

Besides the two issues on the self consistency mentioned above, we find a contradiction between the completeness of spacetime and the smoothness of limiting from charged to uncharged cases of the 4D EGB black holes. The charged case can be divided into two types depending on the positivity or negativity of the EGB parameter, where the type with the positive parameter violates the completeness of spacetime but guarantees the smoothness of the vanishing parameter, while the other type with the negative parameter behaves contrarily, i.e., it guarantees the completeness of spacetime but violates the smoothness of the vanishing parameter. Since the possible incomplete regions of spacetime are always hidden by the event horizons, we cannot judge the rationality of the 4D EGB black holes by the completeness of spacetime alone. With the aid of the characteristic variables of thermodynamics and dynamics, we try to distinguish the charged 4D EGB black hole from the Reissner-Nordström black hole and analyze whether their differences of characteristic variables can be expected to be observed in future experiments. In this paper we focus on the second-order phase transition and the quasinormal mode in the eikonal limit of the charged 4D EGB black hole.

The second-order phase transition point of black holes was first found by Daives [11, 12], known as “the Davies point”. When a black hole evolves past the Davies point, the sign of its heat capacity changes, indicating that the black hole has undergone a phase transition. As the heat capacity diverges at the Davies point, the phase transition is second order. The Davies points exist in the Reissner-Nordström (RN) black hole and the Kerr black hole, but not in the Schwarzschild black hole whose heat capacity is always negative. We pointed out [13] that the Davies point appears in the uncharged 4D EGB black hole. Here we shall further investigate the behaviors of the Davies point for the charged 4D EGB black hole.

The observation of gravitational waves from a binary black hole merger was first reported by the LIGO Scientific and Virgo collaborations [14] in 2016. Therefore, the gravitational wave opens a new window for us to study the properties of black holes. Quasinormal modes provide [15, 16, 17, 18] a characteristic variable of dynamics to depict the ringdown phase of black hole mergers, where their real parts correspond to the oscillating frequency of gravitational waves, while their imaginary parts to the damping time. They depend only on the inherent properties of spacetimes, so the differences of spacetimes will be intuitively reflected by the quasinormal modes. Normally the quasinormal modes of a black hole can only be computed numerically, but in the eikonal limit they can be derived [19].
analytically, i.e. the light ring/quasinormal mode correspondence, where the angular velocity of unstable null geodesics determines the real part of the quasinormal modes and the Lyapunov index does the imaginary part. Although this correspondence is not valid [20] for all field perturbations, it holds for our test field. Besides the quasinormal modes of a black hole, another important observable is the shadow of a black hole [21], where its values observed at infinity are inversely proportional to [22] the real part of the quasinormal modes in the eikonal limit. Thus, it is quite helpful to study the gravitational waves and shadows by the quasinormal modes in the eikonal limit for the charged 4D EGB black hole.

This paper is organized as follows. In Sec. 2, we investigate the constraints imposed by the completeness of spacetime on a charged 4D EGB black hole, and then reveal the contradiction between the charged and uncharged cases. Next, we examine in Sec. 3 the behaviors of the Davies point when the EGB parameter takes values in different ranges. We continue our discussions in Sec. 4 on the quasinormal modes in the eikonal limit. Finally, we give our conclusions in Sec. 5 where some comments and further extensions are included.

2 Charged 4D EGB black holes

The shape function of the charged 4D EGB black hole takes [5] the form,

\[ f(r) = 1 + \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} \right)} \right], \tag{1} \]

where \( \alpha \) stands for the EGB parameter, \( M \) the mass, and \( q \) the charge. Under the limit \( \alpha \rightarrow 0 \), the charged 4D EGB black hole turns back to the RN black hole with the shape function as follows,

\[ f_{\text{RN}}(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2}. \tag{2} \]

2.1 Horizons and singularities

By solving \( f(r) = 0 \) in Eq. (1), one can obtain event horizons of the charged 4D EGB black hole,

\[ r_{\pm_H} = M \pm \sqrt{M^2 - q^2 - \alpha}. \tag{3} \]

Now we turn to the analyses of singularities for the charged 4D EGB black hole by combining the shape function and the event horizons. When \( r \rightarrow 0 \), the asymptotic behavior of the term under a square root in Eq. (1) is

\[ 1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} \right) \longrightarrow -\frac{4\alpha q^2}{r^4}, \tag{4} \]

which means that the metric will become complex-valued if \( \alpha > 0 \). We notice that the appearance of a complex metric is different from the usual singularities that are only coordinate dependent, e.g. the singular point is at the origin. Here such a singularity is also parameter dependent, associating with the
positivity or negativity of the EGB parameter. In order to maintain the completeness of spacetime, i.e. to make the metric real anywhere, we impose the constraint,

\[ 1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} \right) \geq 0, \quad r \in [0, +\infty). \] (5)

When \( r > \frac{q^2}{2M} \), meaning \( \frac{2M}{r^3} - \frac{q^2}{r^4} > 0 \), the above condition becomes

\[ \alpha \geq -\frac{r^4}{4(2Mr - q^2)} \equiv h(r). \] (6)

Because \( h(r) \) takes its maximum at \( r = r_0 = \frac{2q^2}{3M} \),

\[ h(r_0) = -\frac{4q^6}{27M^4}, \] (7)

Eq. (6) gives one part of the constraint to \( \alpha \),

\[ \alpha \geq -\frac{4q^6}{27M^4}. \] (8)

Moreover, when \( 0 < r < \frac{q^2}{2M} \), meaning \( \frac{2M}{r^3} - \frac{q^2}{r^4} < 0 \), Eq. (5) implies

\[ \alpha \leq h(r). \] (9)

Considering

\[ \frac{dh(r)}{dr} = \frac{r^3(2q^2 - 3Mr)}{2q^2 - 2Mr^2} > 0, \] (10)

and \( h(r) \rightarrow 0 \) when \( r \rightarrow 0 \), we can see that \( h(r) \) increases monotonically in the range of \( 0 < r < \frac{q^2}{2M} \) and its minimum is zero. So, Eq. (9) gives the other part of the constraint to \( \alpha \),

\[ \alpha \leq 0. \] (11)

Combining Eq. (8) with Eq. (11), we get the constraint to \( \alpha \),

\[ -\frac{4q^6}{27M^4} \leq \alpha \leq 0, \] (12)

which is just from the requirement to keep a real metric.

In addition, we need to consider the restrictions from horizons, i.e. the horizon radii should be real. Based on Eq. (3), we have the following constraint,

\[ M^2 - q^2 - \alpha \geq 0. \] (13)

For the case of \( q^2 \leq M^2 \), the above condition does not give extra restrictions to Eq. (12), while for the case of \( q^2 > M^2 \), the combination of Eq. (13) and Eq. (12) leads to a tighter form than Eq. (12),

\[ -\frac{4q^6}{27M^4} \leq \alpha \leq M^2 - q^2. \] (14)
Note that we shall verify this inequality soon later by checking $-\frac{4q^6}{27M^6} \leq M^2 - q^2$.

For the sake of concision in the following demonstrations, we introduce the rescaling of variables,

$$\frac{q}{M} \to Q, \quad \frac{\alpha}{M^2} \to a,$$

with which we rewrite the constraint conditions that consist of Eq. (12) and Eq. (14) in a dimensionless form,

$$-\frac{4}{27}Q^6 \leq a \leq 0, \quad \text{for} \quad Q^2 \leq 1,$$

$$-\frac{4}{27}Q^6 \leq a \leq 1 - Q^2, \quad \text{for} \quad Q^2 > 1.$$

Now we compensate the proof for the inequality Eq. (14) or its dimensionless form Eq. (16b), that is, to check $1 - Q^2 \geq -\frac{4}{27}Q^6$. To this end, we define a function,

$$g(Q) = 1 - Q^2 + \frac{4}{27}Q^6,$$

and then prove $g(Q) \geq 0$ for $Q^2 > 1$. We draw the graph of $g(Q)$ with respect to $Q$ in Fig. 1 from which we can finish our proof. We pay attention to the two zero points of $g(Q)$ located at $Q = \pm \sqrt{\frac{3}{2}}$, where $a$ has one unique value, $a = -\frac{1}{2}$. This means that the charged EGB black hole has only one horizon when the rescaled EGB parameter equals $-\frac{1}{2}$, i.e. the outer and inner horizons merge.

![Figure 1: The function $g(Q)$ is tangent to the horizontal axis $Q$ at the two points, $(\pm \sqrt{\frac{3}{2}}, 0)$.](image)

Besides the rescaled charge and EGB parameter, see Eq. (15), we rescale the radial coordinate $r$ by

$$\frac{r}{M} \to x,$$

and then rewrite the horizons Eq. (3) as

$$x_{H}^\pm = 1 \pm \sqrt{1 - Q^2 - a},$$
where it seems that \( a \) takes the range from minus infinity to zero. In fact, there are no solutions for \( f(r) = 0 \) if \( a < -\frac{1}{2} \). Let us make a detailed analysis. It is obvious that the algebraic equation, \( f(r) = 0 \), can be simplified to be

\[
\sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} \right)} = 1 + \frac{2\alpha}{r^2},
\]

where the both hand sides should not be negative. However, the right hand side may be negative when \( \alpha < 0 \), which gives rise to an extra restriction to the range of \( \alpha \). In order to find out such an extra condition, we substitute \( r = r_H^+ \) into the right hand side of Eq. (20) and thus obtain the extra constraint for \( \alpha \),

\[
1 + \frac{2\alpha}{\left( M + \sqrt{M^2 - q^2 - \alpha} \right)^2} \geq 0,
\]

or its dimensionless form,

\[
2a \geq -(1 + \sqrt{1 - Q^2 - a})^2.
\]

Now we can fix the physical region for the charged 4D EGB black hole by combining the constraints Eqs. (16a), (16b), and (22) and depict it in Fig. 2, where it is obvious that \( a < -\frac{1}{2} \) is beyond this parameter region.

![Figure 2](image_url)

Figure 2: The horizontal axis denotes the absolute value of the rescaled charge \( Q \) and the vertical axis the rescaled EGB parameter \( a \). The shadow determines the parameter region allowed by the set of constraints Eqs. (16a), (16b), and (22), in which the completeness of spacetime is guaranteed in the charged 4D EGB black hole.
2.2 Contradictions between the charged and uncharged 4D EGB black holes

The shape function of the uncharged 4D EGB black hole has the form,

\[ f_{\text{un}}(r) = 1 + \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 + \frac{8\alpha M}{r^3}} \right]. \]  (23)

In order to ensure the smoothness of limiting from the charged to uncharged cases, Eq. (1) should go back to Eq. (23) when \( q \to 0 \). Naively, it seems no problem to reach the aim. However, based on the physically allowed region of parameters shown in Fig. 2, we find that the EGB parameter \( \alpha \) or the rescaled EGB parameter \( a \) goes to zero simultaneously when \( q \to 0 \). This implies that the charged 4D EGB black hole with \( \alpha < 0 \) or \( a < 0 \) will become the Schwarzschild black hole instead of the uncharged 4D EGB black hole in the limit of \( q \to 0 \), which gives rise to a contradiction between the charged and uncharged cases. In summary, the smoothness of limiting from the charged to uncharged 4D EGB black holes is broken when the (rescaled) EGB parameter is negative.

Now we know that the completeness of spacetime and the smoothness of vanishing charge cannot be coexisting in the charged 4D EGB black hole. We have to keep only one of them. Obviously, we have two options to do so. One option is to maintain the smoothness of vanishing charge, which requires \( \alpha > 0 \) or \( a > 0 \). On the one hand, this choice ensures the self consistency of the 4D EGB theory when the charged case is transformed into the uncharged case. On the other hand, however, it results in the imperfection of spacetime, i.e. the appearance of singularities. But the imperfect region of spacetime is always surrounded by event horizons and can never be observed by the observer outside. As a result, such an imperfect region of spacetime is in fact invisible. Alternatively, the other option is to keep the completeness of spacetime, which requires \( \alpha < 0 \) or \( a < 0 \). In this way, we may regard the charged 4D EGB theory as a specific theory that goes naturally to the Schwarzschild black hole in the vanishing charge. We have no criteria to distinguish the two choices only by the analyses of metrics. Next, we shall study second-order phase transitions and quasinormal modes in the eikonal limit for the charged 4D EGB black hole in order to find out differences between the two options in characteristic variables of thermodynamics and dynamics.

3 Second-order phase transitions

In this section we will discuss the behaviour of the Davies points associated with the second-order phase transitions in the 4D-CEGB black hole.

Traditionally, the Davies point has been gotten by calculating \( 1/C_q = 0 \), where the heat capacity \( C_q \) is divergency, so we calculate the heat capacity with fixing charge and EGB parameter and write it as dimensionless form

\[ \frac{C_q}{M^2} = \frac{1}{M^2} \left( \frac{\partial M}{\partial T} \right)_q = \frac{1}{M^2} \left( \sqrt{1 - Q^2 - a} \right) \cdot \frac{4\pi(x^+_{H}^3 + 2x^+_H a)^2}{-x^+_{H}^4 + 5x^+_{H}^2 a + 2a^2 + Q^2(3x^+_{H}^2 + 2a)} \]  (24)

, then we can get the master equation of the Davies point

\[ -x^+_{H}^4 + 5x^+_{H}^2 a + 2a^2 + Q^2(3x^+_{H}^2 + 2a) = 0 \]  (25)
and the related curve for $a > 0$ is showed in Figure 3. The Davies point corresponds to the RN black hole when $a = 0$, and it can be seen that the charge mass ratio $\frac{q}{M}$ of the 4D-EGB black hole at the Davies point is always smaller than that of the RN black hole.

Figure 3: This figure describes the behavior of the Davies points on the $a - Q$ plot when $a > 0$. The horizontal coordinate in the figure is the rescaled charge of the 4D-CEGB black hole, and the vertical coordinate is the rescaled EGB parameter. The blue line corresponds to the Davies point, the orange line represents the boundary of the physically allowed region, and the green line represents the evolution of the black hole.

Next, we will further investigate whether a second phase transition would happen during the evolutionary process of black hole. The evolutionary process of the black hole can be arbitrarily selected due to three variable parameters in the 4D-CEGB black hole. Therefore, we should limit the evolutionary process of the black hole. Considering that we fixed the charge and EGB parameters when calculating the heat capacity, we also fix these two parameters during the evolution of the black hole, while only changing mass of the black hole. And we’ll see later that this evolution is closely related to the evolution of temperature. Next we should determine the corresponding curve of the evolutionary process in Fig. 3. Since the dimensionless parameters $a$ and $Q$ satisfy the following relation,

\[
\frac{\alpha}{q^2} = \text{constant} = \frac{a}{Q^2}
\]  

(26)

on the $a - Q$ plot, the evolutionary process can be represented as curves

\[
a = cQ^2
\]  

(27)

where $c = \alpha/q^2$ are constants. Since $c$ ranges from 0 to infinity, the curve of evolution will always intersect the curve of the Davies point, which means that the black hole will always pass through the Davies points during its evolution.

Then, let’s turn to the situation $a < 0$. It can be seen that the charge mass ratio $\frac{q}{M}$ of the 4D-CEGB black hole at the Davies point is always larger than that of the RN black hole. In Fig 4, the shadow part
Figure 4: This graph describes the behavior of the Davies points on the $a - q$ plot when $a < 0$. The horizontal coordinate in the figure is the rescaled charge of the 4D-CEGB black hole, and the vertical coordinate is the rescaled EGB parameter. The blue line corresponds to the Davies point, and the shaded area is the physically allowed area, where the gray line and the red line are the upper and lower boundaries of the area respectively. The orange line represents the evolution of the black hole and intersects the blue and red lines at one point.

represents the physical region we discussed in Sec 2, and the blue line is related to the Davies point. The blue line and red line intersect in two points: $(0.956, -0.113)$ and $(\sqrt{\frac{3}{2}}, -\frac{1}{2})$. Here we also consider the process where the charge and EGB parameters are constants. The black hole is still evolving along curve $a = cQ^2$ just like in $a > 0$, but the $c$ is negative here. As can be seen from Fig 4, when the evolution curve passes through the lowest point, $c = -0.333$, which corresponds to the minimum value of $c$. Notice that the curve corresponding to the orange line in the figure is $a = -0.124Q^2$, so the black hole will be restricted in the physical region and cannot reach the Davies point when $-0.333 < c < -0.124$. It’s worth mentioning the special point $(\sqrt{\frac{3}{2}}, -\frac{1}{2})$. As we mentioned in Sec 2, the black hole solution in this point is unique and extreme. Now from Fig 4, we can see that this point is a second phase transition point and the black hole at this point can not evolve to other black holes in the evolutionary process. Meanwhile, other black holes can not evolve to the black hole at this point in the evolutionary process.

In summary, black holes can always reach the second phase transition point during the evolutionary process in $a > 0$ situation. However, if $a < 0$, black holes satisfying $-0.333 < \frac{a}{q^2} < -0.124$ will not reach the second phase transition point during the evolutionary process.

Let’s discuss what will happen when a 4D-CEGB black hole evolves past the Davies point, and first review the definition of the Davies point

$$\frac{1}{C_q} = \left( \frac{\partial T}{\partial M} \right)_q = 0 \quad (28)$$
Considering the temperature during the evolution of black hole

\[ T = \int_i^l dT = \int_l \left( \frac{\partial T}{\partial M} \right)_q \, dM \quad (29) \]

where "\( l \)" means the evolutionary curve of the 4D-CEGB black hole, we can conclude that the Davies point corresponds to a saddle point of temperature change. And then we should determine whether that saddle point corresponds to a maximum or a minimum. First, we examine the relationship between the black hole heat capacity and the black hole mass when \( a > 0 \), as shown in Fig. 5, when the black hole mass increases and passes through the Davies point, the heat capacity changes from positive to negative, indicating that the Davies point corresponds to the maximum point of temperature and is the only saddle point. Then we look at the relationship between the heat capacity and the mass when \( a < 0 \), as showed in Figs. 5, 7, and there are two situations: when \(-0.124 < c < 0\), the Davies points exist in the process of evolution of the black hole, and when the mass of the black hole increases and passes through the Davies point, the heat capacity still goes from positive to negative, indicating that the Davies point corresponds to the maximum point of temperature and is the only saddle point. However, when \(-0.333 < c < -0.124\), the black hole will not pass through the Davies point during its evolution, and the temperature will increase monotonously with the mass of the black hole.

In summary, for the 4D-CEGB black hole of \( a > 0 \), there is always a saddle point at which the black hole temperature takes the maximum value, corresponding to the Davies point. When the black hole reaches this point, the temperature will change from increasing to decreasing. However, for the 4D-CEGB black hole of \( a < 0 \), there exists a special interval of \(-0.333 < c < -0.124\), in which the temperature of the black hole always increases monotonously with the mass of the black hole.

![Figure 5: This figure describes the relationship between the heat capacity of the black hole and the mass of the black hole when \( a > 0 \), where we take the charge \( q = 1 \) and the EGB parameter \( \alpha \) from 0.1 to 0.5. The vertical lines in the figure correspond to the Davies points.](image-url)
Figure 6: This figure describes the relationship between the heat capacity of the black hole and the mass of the black hole when $a < 0$, where we take the charge $q = 1$ and the EGB parameter $\alpha$ from -0.1 to -0.06. The vertical lines in the figure correspond to the Davies points.

Figure 7: This figure describes the relationship between the heat capacity of the black hole and the mass of the black hole when $a < 0$, where we take the charge $q = 1$ and the EGB parameter $\alpha$ from -0.16 to -0.124. At $-0.333 < \alpha < -0.124$, the black hole evolution will not pass through the Davies point.

4 Quasinormal modes in the eikonal limit

According to the light ring/quasi-normal modes correspondence[19], quasi-normal modes for the spherically symmetric black hole in the eikonal limit ($l \gg 1$) is of the following form

$$\omega = \Omega_c l - i(n + 1/2)|\lambda|$$

(30)
where the angular velocity $\Omega_c$ and the Lyapunov index $\lambda$ can be computed by

$$\Omega_c = \sqrt{f_c} = \frac{1}{r_c}, \quad \lambda = \sqrt{\frac{f_c(2f_c - r^2 f''_c)}{2r_c^2}},$$

and $f_c \equiv f(r_c), f''_c \equiv f''(r)|_{r=r_c}$, where the radius $r_c$ of photon sphere is determined by

$$2f(r) - r^2 f(r) = 0$$

We substitute the 4D-CEGB metric into the above equation, and get the relationship between $r_c$ and parameters of the 4D-CEGB black hole

$$r_c^4 - 9M^2 r_c^2 + 4M(3q^2 + \alpha)r_c - 4q^2(q^2 + \alpha) = 0$$

and relevant angular velocity $\Omega_c$ and the Lyapunov index $\lambda$ are

$$\Omega_c^2 = \frac{f_c}{r_c^2} = \frac{1}{2\alpha} \left[ 1 - \frac{1 + 4\alpha \left( \frac{2M}{r_c^2} - \frac{q^2}{r_c^2} \right)}{2} \right]$$

$$\lambda^2 = \frac{f_c(2f_c - r^2 f''_c)}{2r_c^2} = \frac{f(r_c)}{r_c^8 A^{3/2}} \left( -8q^4 - 4q^2(2r_c^2 + 4q^2) - 36M^2 r_c^4 + 32q^2 M r_c^6 \right)$$

where

$$A(r) = 1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} \right)$$

We can rewrite equations above in dimensionless form

$$\frac{x_c^4 - 9x_c^2 + 4(3Q^2 + a)x_c - 4Q^2(Q^2 + a)}{x_c^2} = 0$$

$$\Omega_c^2 M^2 = \frac{f(x_c)}{x_c^8 A^{3/2}} \left[ 1 - \frac{1 + 4a \left( \frac{2}{x^3} - \frac{Q^2}{x^4} \right)}{2} \right]$$

$$\lambda^2 M^2 = \frac{f(x_c)}{x_c^8 A^{3/2}} \left[ 1 - \frac{1 + 4a \left( \frac{2}{x^3} - \frac{Q^2}{x^4} \right)}{2} \right]$$

where $f$ and $A$ are

$$f(x) = 1 + \frac{x^2}{2a} \left[ 1 - \frac{1 + 4a \left( \frac{2}{x^3} - \frac{Q^2}{x^4} \right)}{2} \right]$$

$$A(x) = 1 + 4a \left( \frac{2}{x^3} - \frac{Q^2}{x^4} \right)$$

We show $\Omega_c M - \lambda M$ plots of $a > 0$ and $a < 0$ in Figs [8][9], respectively.

When $a > 0$, we can see from Fig. [8] that, with the increase of $a$, the range of the photon sphere angular velocity of the 4D-CEGB black hole gradually decreases, but is always included in the range of the RN black hole. Note that in the quasi-normal mode, the photon sphere angular velocity corresponds
Figure 8: This figure shows the relationship between the angular velocity of the black hole photon sphere and Lyapunov index when $a > 0$. In the figure, the horizontal coordinate is the dimensionless Lyapunov index, and the vertical coordinate is the dimensionless angular velocity. The red line in the figure represent the RN black hole and the other lines represent different EGB parameters.

Figure 9: This figure shows the relationship between the angular velocity of the black hole photon sphere and Lyapunov index when $a > 0$. In the figure, the horizontal coordinate is the dimensionless Lyapunov index, and the vertical coordinate is the dimensionless angular velocity. The black line in the figure represent the RN black hole and the other lines represent different EGB parameters.

to the oscillation frequency of quasi-normal mode, so we can’t distinguish the 4D-CEGB black hole from the RN black hole just by the oscillation frequency of quasi-normal mode when $a > 0$. Fortunately, the Lyapunov index of the 4D-CEGB black hole can be separated from the range of the RN black hole, and the separation becomes larger with the increase of $a$. The range of $\lambda M$ in the RN black hole is $[0.177, 0.196]$, and the minimum value of $\lambda M$ in 4D-CEBG black holes is 0.141, which is located in $a = 1, Q = 0$. It is noted that the imaginary part of quasi-normal mode is related to the damping time of
the gravitational wave,
$$\tau = \frac{1}{|\Omega_{c}|} \tag{41}$$
where the decay time represents the time taken for the amplitude to decay to $e^{-1}$, thus it can be seen that the damping time of the 4D-EGB black hole will be longer than the maximum damping time of the RN black hole in the case of choosing the appropriate parameter $a$, and we can calculate their maximum relative deviation as
$$\frac{\Delta \tau}{\tau} = \frac{0.141 - 0.177}{0.177} \approx 25.5\%. \tag{42}$$

When $a < 0$, we can see from Fig. 9 that, with the increase of the absolute value of $a$, the photon sphere angular velocity and the Lyapunov index of 4D-CEGB black hole can both be out of the range of the RN black hole. And the point with maximum deviation is located in $a = -0.5$, $Q = \sqrt{3/2}$, where $\lambda M = 0.202$ and $\Omega_{c}M = 0.291$. The range of $\lambda M$ and $\Omega_{c}M$ in the RN black hole are $[0.177, 0.196]$ and $[0.192, 0.250]$, respectively. So, damping time of the 4D-CEGB black hole can be shorter than the minimum damping time of the RN black hole, and the maximum relative deviation between them is
$$\frac{\Delta \tau_{2}}{\tau_{2}} = \frac{0.202 - 0.196}{0.196} \approx -3.0\%. \tag{43}$$
where the negative sign means damping time of the 4D-CEGB black hole is smaller than the minimum damping time of the RN black hole. And oscillation frequency of the 4D-CEGB black hole can be higher than the maximum oscillation frequency of the RN black hole, and the maximum relative deviation between them is
$$\frac{\Delta \omega_{R2}}{\omega_{R2}} = \frac{0.291 - 0.250}{0.250} = 16.4\% \tag{44}$$
The shadow radius $R_{sh}$ observed at infinity of the black hole is closely related to angular velocity of the photon sphere
$$R_{sh} = \frac{r_{c}}{\sqrt{f(r_{c})}} = \Omega_{c}^{-1} \tag{45}$$
So, when $a < 0$, the shadow radius of the 4D-CEGB black hole can be smaller than the minimum shadow radius of the RN black hole, and the maximum relative deviation between them is
$$\frac{\Delta R_{sh}}{R_{sh}} = \frac{0.291 - 0.250}{0.250} = -14.1\% \tag{46}$$
where the negative sign means the radius of shadow in the 4D-CEGB black hole is smaller than the minimum shadow radius of the RN black hole. So in the case of $a < 0$, we will hopefully observe a shadow that is smaller than the smallest shadow of the RN black hole, which is not going to happen in the case of $a > 0$. 

5 Discussions and conclusions

In this paper, we reveal a contradiction between the completeness of spacetime and the smoothness of limiting from the charged to uncharged 4D EGB black holes. Its significance lies in the finding that the incomplete regions of the 4D EGB black holes depend on the ranges of the EGB parameter, which expands the concept of spacetime completeness from only the coordinate dependence to the coordinate-and-parameter dependence. In order to verify such a coordinate-and-parameter dependence, we try to distinguish the charged 4D EGB black hole from the Reissner-Nordström black hole of Einstein’s general relativity by investigating the behaviors of the Davies point and calculating the quasinormal modes in the eikonal limit. We summarize the following differences between the characteristic variables of thermodynamics and dynamics in the charged 4D EGB black hole and the same variables in the Reissner-Nordström black hole that might be observed in future experiments.

• When the EGB parameter is positive, the charged 4D EGB black hole will pass through one Davies point in the evolution with respect to mass, which indicates that there exists one unique saddle point with the maximum temperature. However, if the EGB parameter is negative and satisfies the constraint, $-0.333 < \frac{\alpha}{q^2} < -0.124$, the black hole will not meet a Davies point in the evolution and its temperature will not have an extremum, either. Compared with the second-order phase transition point of the RN black hole, the charge mass ratio $\frac{q}{M}$ at the Davies point is smaller than that of the RN black hole for the former case, while it is larger than that of the RN black hole for the latter case.

• When the EGB parameter is positive, the oscillating frequency of quasinormal modes of the charged 4D EGB black hole in the eikonal limit will not exceed the range of oscillating frequencies of the RN black hole, but the damping time can be longer than the maximum damping time of the RN black hole. When the EGB parameter is negative, the oscillating frequency can be higher than the maximum oscillating frequency of the RN black hole and the damping time shorter than the minimum damping time of the RN black hole.

• When the EGB parameter is positive, the charged 4D EGB black hole cannot be distinguished from the RN black hole. Nonetheless, the shadow radius of the former will probably be smaller than the minimum shadow radius of the latter when the EGB parameter is negative.

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