Bad semidefinite programs: they all look the same

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A pair of Semidefinite Programs (SDP)

\[ \begin{align*}
\sup_x & \quad c^T x \\
\inf_Y & \quad B \circ Y \\
\text{s.t.} & \quad \sum_{i=1}^m x_i A_i \preceq B \\
& \quad Y \succeq 0 \\
& \quad A_i \circ Y = c_i \ (i = 1, \ldots, m).
\end{align*} \]

Here

- \( A_i, B \) are symmetric matrices, \( c, x \in \mathbb{R}^m \).
- \( A \preceq B \) means that \( B - A \) is symmetric positive semidefinite (psd).
- \( A \circ B = \sum_{i,j} a_{ij} b_{ij} \).
Conic LPs and SDPs

- Common framework for LP and SDP: both $\mathbb{R}^n_+$ and psd matrices are closed convex cones.

- A set $C$ is a cone, if $x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C$.

- Linear objective, conic constraint both in LP and SDP, and many other interesting problems, notably SOCPs.
Why is SDP important: applications in

- 0–1 Integer programming
- Approx algorithms
- Chemical engineering
- Chemistry
- Coding theory
- Control theory
- Combinatorial opt
- Discrete geometry
- Eigenvalue optimization
- Facility planning
- Finance
- Geometric optimization
- Global optimization
- Graph visualization
- Inventory theory
- Machine learning
- Matrix analysis
- PDEs
- Probability theory
- Robust optimization
- Signal processing
- Statistics
- Structural optimization
SDP theory and applications

- **Nice duality theory**: see later
- **Applications**: see textbooks by
  - Boyd-Vandenberghe
  - Ben-Tal-Nemirovskii
- **Algebraic geometry**:
  - Nie-Sturmfels 2010
  - von Bothmer-Ranestad 2009
  - Gouveia, Parrilo, Thomas, 2010
  - Book by Blekherman, et al, 2013
- **Polynomial optimization**:
  - Lasserre 2000 –
  - Parrilo 2000 –
  - Nie 2000 –
  - Helton-Vinnikov 2003
SDP duality

The primal-dual pair of SDPs:

\[
\begin{align*}
\sup_x c^T x & \quad \text{inf}_Y B \bullet Y \\
\text{s.t. } \sum_{i=1}^m x_i A_i & \preceq B \\
& \quad Y \succeq 0 \\
& \quad A_i \bullet Y = c_i \quad (i = 1, \ldots, m).
\end{align*}
\]
SDP duality

The primal-dual pair of SDPs:

\[
\sup_x \ c^T x \quad \quad \quad \quad \quad \quad \quad \quad \inf_Y \ B \bullet Y \\
\text{s.t.} \quad \sum_{i=1}^m x_i A_i \leq B \quad \quad \quad \quad \quad \quad Y \succeq 0 \\
\quad \quad \quad \quad \quad \quad A_i \bullet Y = c_i (i = 1, \ldots, m).
\]

Easy: If \( x \) and \( Y \) are feasible, then \( c^T x \leq B \bullet Y \).
SDP duality

The primal-dual pair of SDPs:

\[
\begin{align*}
\sup_x c^T x & \quad \inf_Y B \bullet Y \\
\text{s.t.} \quad \sum_{i=1}^m x_i A_i & \preceq B & Y & \succeq 0 \\
A_i \bullet Y & = c_i (i = 1, \ldots, m).
\end{align*}
\]

Easy: If \( x \) and \( Y \) are feasible, then \( c^T x \leq B \bullet Y \).

Ideal situation: \( \exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \bullet \bar{Y} \).
SDP duality

The primal-dual pair of SDPs:

\[
\begin{align*}
\sup_x \quad & c^T x & \quad \inf_Y \quad & B \cdot Y \\
\text{s.t.} \quad & \sum_{i=1}^m x_i A_i \preceq B & \quad \text{Y} \succeq 0 \\
& A_i \cdot Y = c_i (i = 1, \ldots, m).
\end{align*}
\]

Easy: If \( x \) and \( Y \) are feasible, then \( c^T x \leq B \cdot Y \).

Ideal situation: \( \exists \bar{x}, \exists \bar{Y} : c^T \bar{x} = B \cdot \bar{Y} \).

But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

This is bad, since we would like a certificate of optimality.
Pathology # 1: nonattainment in dual

Primal:

\[
\sup 2x_1 \quad \iff \quad \sup 2x_1
\]

\[
s.t. \quad x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad s.t. \quad \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0
\]
Pathology # 1: nonattainment in dual

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Only feasible \( x_1 \) is \( x_1 = 0 \).
Pathology # 1: nonattainment in dual

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\text{s.t. } x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{s.t. } \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0
\end{align*}
\]

Only feasible \(x_1\) is \(x_1 = 0\).

Dual: Dual variable is \(Y \succeq 0\).

\[
\begin{align*}
\inf y_{11} & \\
\text{s.t. } y_{11} & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \succeq 0
\end{align*}
\]
Pathology # 1: nonattainment in dual

Primal:

\[
\begin{align*}
\sup & \quad 2x_1 \\
\text{s.t.} & \quad x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\end{align*}
\]

\[\Leftrightarrow \]

\[
\begin{align*}
\sup & \quad 2x_1 \\
\text{s.t.} & \quad \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \\
\end{align*}
\]

Only feasible \( x_1 \) is \( x_1 = 0 \).

Dual: Dual variable is \( Y \succeq 0 \).

\[
\begin{align*}
\inf & \quad y_{11} \\
\text{s.t.} & \quad \begin{pmatrix} y_{11} & 1 \\ 1 & y_{22} \end{pmatrix} \succeq 0 \\
\end{align*}
\]

Here \( \inf = 0 \), but not attained: Any \( y_{11} > 0 \), \( y_{22} = 1/y_{11} \) is feasible, but \( y_{11} = 0 \) is not.
Pathology # 2: positive duality gap

Primal:

\[
\sup \ x_2 \\
\text{s.t. } x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Pathology # 2: positive duality gap

Primal:

\[ \sup x_2 \]

\[ s.t. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Only feasible \( x_2 \) is \( x_2 = 0 \).
Pathology # 2: positive duality gap

Primal:

\[
\sup x_2 \\
\text{s.t. } x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Only feasible \( x_2 \) is \( x_2 = 0 \).

Dual value is 1, and it is attained.
Terminology

Definition:

• The system \( P_{SD} = \{ x \mid \sum_{i=1}^{m} x_i A_i \leq B \} \) is well-behaved, if for all \( c \) such that

\[
\sup\{ c^T x \mid x \in P_{SD} \}
\]

is finite,

the dual program has the same value, and it attains.

• Badly behaved, otherwise.

• We would like to understand well/badly behaved systems.
Some literature

• Conic LPs may be badly behaved when $K$ is not polyhedral.

• **Borwein-Wolkowicz 1981** Facial reduction: theoretical construction of well behaved system

• **Ramana 1995** Extended dual for SDP

• **Ramana, Tunçel, Wolkowicz, 1997** Facial reduction implies correctness of extended dual

• **Klep, Schweighofer, 2013** Related duals based on algebraic geometry.

• **Waki, Muramatsu, 2013; Pataki 2013:** Simpler facial reduction algorithms.

• **P 2007** Closedness of linear image of a closed, convex cone
Motivation

The systems

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

are both badly behaved.
Motivation

The systems

\[
\begin{pmatrix}
\begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array}
\end{pmatrix}
\preceq
\begin{pmatrix}
\begin{array}{c}
1 \\
0 \\
\end{array}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\end{pmatrix}
+ \begin{pmatrix}
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{array}
\end{pmatrix}
\preceq
\begin{pmatrix}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

are both badly behaved.

Curious similarity:

- “Hanging off” diagonals;

- if we delete 2nd row and 2nd column in all matrices in the second system, and delete the first matrix, we get back the first system.
Why all bad SDPs look the same

• Semidefinite system:

\[
(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B
\]
Why all bad SDPs look the same

• Semidefinite system:
  \[(P_{SD}) \sum_{i=1}^{m} x_i A_i \preceq B\]

• W.l.o.g. the max (rank) slack is
  \[Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} .\]

Then \((P_{SD})\) badly behaved \(\iff\) \(\exists V\) a lin. combination of the \(A_i\) and \(B\) as
\[V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix} , \text{ where } V_{22} \succeq 0, \text{ } \mathbb{R}(V_{12}^T) \not\subseteq \mathbb{R}(V_{22}) .\]
Why all bad SDPs look the same

• Semidefinite system:

\[(P_{SD}) \sum_{i=1}^{m} x_i A_i \preceq B\]

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• Ex: \(x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\)
Why all bad SDPs look the same

- Semidefinite system:
  \[(P_{SD}) \sum_{i=1}^{m} x_i A_i \preceq B\]

- W.l.o.g. the max (rank) slack is
  \[Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} .\]

Then \((P_{SD})\) badly behaved \(\Leftrightarrow \exists V\) a lin. combination of the \(A_i\) and \(B\) as

\[V = \begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{12}^T & V_{22} \end{pmatrix}, \quad \text{where } V_{22} \succeq 0, \quad \mathbb{R}(V_{12}^T) \subsetneq \mathbb{R}(V_{22}).\]

- Ex: \(x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\)
Why all bad SDPs look the same

• Semidefinite system:

\[
(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B
\]

• W.l.o.g. the max (rank) slack is

\[
Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then \((P_{SD})\) badly behaved \(\Leftrightarrow \exists V\) a lin. combination of the \(A_i\) and \(B\) as

\[
V = \begin{pmatrix} \overline{V_{11}} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0, \ R(V_{12}^T) \not\subseteq R(V_{22}).
\]

• Ex: \(x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\)
What is missing?

• Matrices $Z, V$ prove that $(P_{SD})$ is badly behaved.
• But: this is not yet a poly time, or easy to verify proof of bad behavior
Reformulations of

\[(P_{SD}) \sum_{i=1}^{m} x_i A_i \preceq B\]

are obtained by a sequence of:

- Rotate all matrices by \( T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} \), \( M \) orthogonal.
- \( B \leftarrow B + \sum_{i=1}^{m} \mu_i A_i \)
- \( A_i \leftarrow \sum_{j=1}^{m} \lambda_j A_j \) where \( \lambda_i \neq 0 \)

Reformulations preserve well/badly behaved status; preserve max slack; provide an equivalence relation
Theorem: \((P_{SD})\) is badly behaved \(\iff\) it has a reformulation:

\[
(P_{SD,\text{bad}}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]
Theorem: \((P_{SD})\) is badly behaved \iff it has a reformulation:

\[
(P_{SD,bad}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(\begin{pmatrix} G_i \\ H_i \end{pmatrix}\) lin. indep. 3) \(H_m \succeq 0\)
Theorem: \((P_{SD})\) is badly behaved \(\iff\) it has a reformulation:

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where

1) \(Z\) is max slack; 2) \(\begin{pmatrix} G_i \\ H_i \end{pmatrix}\) lin. indep. 3) \(H_m \succeq 0\)

Proof that \((P_{SD,bad})\) is badly behaved:
Theorem: \((P_{SD})\) is badly behaved \iff it has a reformulation:

\[
(P_{SD, \text{bad}}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(\begin{pmatrix} G_i \\ H_i \end{pmatrix}\) lin. indep. 3) \(H_m \succeq 0\)

Proof that \((P_{SD, \text{bad}})\) is badly behaved:

\(x\) feas. with slack \(S\) \Rightarrow \(\text{supp}(S) \subseteq \text{supp}(Z)\)

\(\Rightarrow x_{k+1} = \cdots = x_m = 0 \Rightarrow \sup -x_m = 0\)
Theorem: \((P_{SD})\) is badly behaved \iff it has a reformulation:

\[
(P_{SD,\text{bad}}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(\begin{pmatrix} G_i \\ H_i \end{pmatrix}\) lin. indep. 3) \(H_m \succeq 0\)

Proof that \((P_{SD,\text{bad}})\) is badly behaved:

\[
Y \succeq 0, Y \cdot Z = 0 \Rightarrow Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_{22} \end{pmatrix} \\
\Rightarrow Y \cdot \begin{pmatrix} F_m & G_m \\ G_m^T & H_m \end{pmatrix} \succeq 0 \\
\Rightarrow \text{no dual soln with value 0}
\]
Example: before reformulation

$$
\begin{bmatrix}
54 & 46 & 50 & 4 \\
46 & -38 & 87 & -106 \\
50 & 87 & -60 & 296 \\
4 & -106 & 296 & -368
\end{bmatrix}
+x_1
\begin{bmatrix}
110 & 91 & 105 & -6 \\
91 & -72 & 171 & -210 \\
105 & 171 & -72 & 528 \\
-6 & -210 & 528 & -672
\end{bmatrix}
+x_2
\begin{bmatrix}
42 & 35 & 40 & 0 \\
35 & -28 & 67 & -82 \\
40 & 67 & -36 & 216 \\
0 & -82 & 216 & -272
\end{bmatrix}
+x_3
\begin{bmatrix}
36 & 30 & 35 & -2 \\
30 & -24 & 57 & -70 \\
35 & 57 & -24 & 176 \\
-2 & -70 & 176 & -224
\end{bmatrix}
+x_4
\begin{bmatrix}
389 & 323 & 370 & -12 \\
323 & -257 & 610 & -748 \\
370 & 610 & -288 & 1920 \\
-12 & -748 & 1920 & -2432
\end{bmatrix}
$$

Hard to tell if well or badly behaved
Example: after reformulation

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 2 & 1 \\
0 & 0 & 3 & -1 \\
2 & 3 & 0 & 2 \\
1 & -1 & 2 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 3 & -1 \\
0 & 0 & 2 & -1 \\
3 & 2 & 4 & 0 \\
-1 & -1 & 0 & 0
\end{pmatrix}
\leq \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

As before: \( x_3 = x_4 = 0 \Rightarrow \sup -x_4 = 0 \)

But: no dual solution with value 0
Theorem: \((P_{SD})\) is well behaved \iff it has a reformulation:

\[
(P_{SD,good}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]
Theorem: \((P_{SD})\) is well behaved \iff it has a reformulation:

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\]

where

1) \(Z\) is max slack; 2) \(H_i\) lin. indep. 3) \(H_i \cdot I = 0 \forall i\)
Corollaries:

- The question:
  Is \((P_{SD})\) well behaved?
  is in \(NP \cap coNP\) in real number model of computing.
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- The question:
  
  Is $(P_{SD})$ well behaved?
  
  is in $NP \cap coNP$ in real number model of computing.

- Certificate: reformulation, and proof that $Z$ is max rank slack.
Corollaries:

• The question:
  Is \((P_{SD})\) well behaved?
  is in \(NP \cap coNP\) in real number model of computing.

• Certificate: reformulation, and proof that \(Z\) is max rank slack.

• \((P_{SD})\) well behaved \(\Rightarrow\) for all \(c\) with a finite obj. value \(\exists\) optimal

\[
Y = \begin{pmatrix}
  r \\
  Y_{11} & 0 \\
  0 & Y_{22}
\end{pmatrix}
\]
Theorem cont’d: \((P_{SD})\) is well behaved \(\Leftrightarrow\) it has a reformulation:

\[
(P_{SD,\text{good}}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
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Theorem cont’d: \((P_{SD})\) is well behaved \iff it has a reformulation:

\[
(P_{SD,good}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(H_i\) lin. indep. 3) \(H_i \bullet I = 0 \ \forall i\)

- **Corollary:** we can generate all well behaved semidefinite systems: choose in sequence \(H_i, G_i, F_i\). Then do reformulation.
Theorem cont’d: \((P_{SD})\) is well behaved \iff it has a reformulation:

\[
(P_{SD,\,good}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(H_i\) lin. indep. 3) \(H_i \bullet I = 0 \ \forall i\)

- **Corollary**: we can generate all well behaved semidefinite systems: choose in sequence \(H_i, G_i, F_i\). Then do reformulation.

- **Corollary**: we can generate all linear maps under which the image of the psd cone is closed.
Theorem cont’d: \((P_{SD})\) is well behaved \iff \text{it has a reformulation:}

\[
(P_{SD, \text{good}}) \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,
\]

where

1) \(Z\) is max slack; 2) \(H_i\) lin. indep. 3) \(H_i \cdot I = 0 \ \forall i\)

• Corollary: we can generate all well behaved semidefinite systems: choose in sequence \(H_i, G_i, F_i\). Then do reformulation.

• Corollary: we can generate all linear maps under which the image of the psd cone is closed.

• Proof: \(\{(A_i \cdot Y)_{i=1}^{m} \mid Y \succeq 0\}\) is closed \iff \(\sum_{i=1}^{m} x_i A_i \preceq 0\) is well behaved.
Broader framework: Well- and badly behaved conic LPs

- Conic linear system, with $K$ closed, convex cone:

  \[(P) \quad Ax \leq_K b \quad (\iff b - Ax \in K)\]
Broader framework: Well- and badly behaved conic LPs

• Conic linear system, with $K$ closed, convex cone:

$$(P) \quad Ax \leq_K b \quad (\Leftrightarrow b - Ax \in K)$$

• Conic LP:

$$(P_c) \quad \sup \{ \langle c, x \rangle \mid x \text{ feasible in } (P) \}$$

• Ex: LP, SDP, SOCP, …
Well- and badly behaved conic LPs

• Conic linear system, with $K$ closed, convex cone:

$$\begin{align*}
(P) \quad & Ax \leq_K b \\
& (\Leftrightarrow b - Ax \in K)
\end{align*}$$

• Conic LP:

$$\begin{align*}
(P_c) \quad & \sup \{ \langle c, x \rangle \mid x \text{ feasible in (P)} \}
\end{align*}$$

• Ex: LP, SDP, SOCP, . . .

• $(P)$ is well-behaved, if

$$\sup(P_c) = \min(D_c) \forall c$$

where $(D_c)$ is dual program. Badly behaved if not well-behaved.
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• Known:
  
  $K$ polyhedral $\Rightarrow$ $(P)$ is well-behaved.
  $(P)$ Slater, i.e., $\exists x : b - Ax \in \text{ri } K \Rightarrow (P)$ is well-behaved.
Given $K$ closed, convex cone

- dual cone: $K^* = \{ y \mid \langle x, y \rangle \geq 0 \forall x \in K \}$

- $K$ is nice if $K^* + F\perp$ is closed for all $F$ faces of $K$. 
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- set of feasible directions at $z \in K$ – maybe not closed:
  \[
  \text{dir}(z, K) = \{ y \mid \exists \epsilon > 0 \text{ s.t. } z + \epsilon y \in K \}
  \]
A geometric result

• Conic system:

\[(P) \quad Ax \leq_K b\]

• Let \( z \in \text{ri}((R(A) + b) \cap K) \) (maximum slack)
A geometric result

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• Then: \((P)\) well behaved \iff

• \(R(A, b) \cap (\text{cl dir}(z, K) \setminus \text{dir}(z, K)) = \emptyset\).
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  - \( \text{R}(A, b) \cap (\text{cl dir}(z, K) \setminus \text{dir}(z, K)) = \emptyset. \)
- ⇒ is true, even if \( K \) is not nice
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- Then: \((P)\) well behaved \iff
  \[
  R(A, b) \cap (\text{cl \ dir}(z, K) \setminus \text{dir}(z, K)) = \emptyset.
  \]
- \(\implies\) is true, even if \(K\) is not nice
- \(K\) polyhedral, or \((P)\) Slater \((z \in \text{ri}K) \implies \text{dir}(z, K)\) closed.
The difference set

- So, the set
  
  \[ \text{cl} \ \text{dir}(z, K) \setminus \text{dir}(z, K) \]

  helps us understand conic duality.
The difference set

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$$\text{cl \ dir}(z, K) \setminus \text{dir}(z, K)$$

helps us understand conic duality.

• Example:

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$  

Then

$$V \in \text{cl \ dir}(Z, PSD) \setminus \text{dir}(Z, PSD)$$

iff

$$V = \begin{pmatrix} \overrightarrow{r} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{22} \succeq 0, \ R(V_{12}) \subsetneq R(V_{22}).$$
The difference set

• So, the set
  \[
  \text{cl } \text{dir}(z, K) \setminus \text{dir}(z, K)
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  \end{pmatrix}, \text{ where } V_{22} \succeq 0, \text{ R}(V_{12}^T) \nsubseteq \text{ R}(V_{22}).
  \]

• We recover characterization of badly behaved semidefinite systems.
Conclusion

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- Combinatorial type characterizations.
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• Corollaries:
  – Block-diagonality of all dual multipliers
  – Generating all well behaved systems
  – Generating all linear maps under which the image of the psd cone is closed.
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  – **Block-diagonality** of all dual multipliers
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  – **Generating** all linear maps under which the image of the psd cone is closed.

• More generally: conditions for well and badly behaved nature of a conic linear system

• **Exact characterization** when $K$ is nice.

• Latest version of paper is on Optimization Online.
Thank you!