Equivalence of the descents statistic on some (4,4)-avoidance classes of permutations

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ABSTRACT

In this article, we compute the generating function of the joint distribution of the first letter and descents statistics on six avoidance classes of permutations corresponding to two patterns of length four thereby demonstrating their equivalence. This distribution is in turn shown to be equivalent to the distribution on a restricted class of inversion sequences for the statistics that record the last letter and number of distinct positive letters, affirming a recent conjecture of Lin and Kim. Members of each avoidance class of permutations and also of the class of inversion sequences are enumerated by the \( n \)th large Schröder number, and thus, one obtains a new bivariate refinement of these numbers as a consequence. We make use of auxiliary combinatorial statistics to establish a system of recurrences for the distribution in question in each case and define special generating functions (specific to the class) based on the system. In some cases, we utilize the conjecture itself in a creative way to aid in solving the functional equations satisfied by these associated generating functions and in others use the kernel method.

1. Introduction

A permutation \( \pi = \pi_1 \cdots \pi_n \in S_n \) is said to contain \( \rho = \rho_1 \cdots \rho_m \in S_m \) where \( m \leq n \) if some subsequence of \( \pi \) is order isomorphic to \( \rho \). That is, there exist indices \( i_1 < i_2 < \cdots < i_m \) such that \( \pi_{i_a} > \pi_{i_b} \) if and only if \( \rho_a > \rho_b \) for all \( a \) and \( b \). Otherwise, \( \pi \) is said to avoid \( \rho \). In this context, \( \rho \) is referred to as a pattern. We say that \( \pi \) avoids the set \( K \) of patterns if it avoids each pattern in \( K \) and denote by \( S_n(K) \) the subset of \( S_n \) whose members avoid \( K \). The study of pattern avoidance in permutations has been an object of considerable attention in recent decades and the notion has been extended to several other finite discrete structures (see, e.g. Kitaev\textsuperscript{[5]} and references contained therein).

An inversion within \( \pi = \pi_1 \cdots \pi_n \in S_n \) is an ordered pair \( (a, b) \) where \( a, b \in [n] = \{1, 2, \ldots, n\} \) with \( a < b \) and \( \pi_a > \pi_b \). The inversion sequence of \( \pi \) is defined by \( x = x_1 \cdots x_n \), where \( x_i \) for each \( i \in [n] \) records the number of entries to the right of the letter \( i \) and smaller than \( i \). For example, if \( \pi = 621543 \in S_6 \), then \( x = 010125 \). Note that the
characterizing property of such sequences $x$ is $0 \leq x_i \leq i - 1$ for all $i$. The systematic study of patterns in inversion sequences is a topic that has only recently been initiated in [3, 11].

Let $I_n$ denote the set of all inversion sequences of length $n$ and $I_n(\geq, -, >)$ the subset of $I_n$ consisting of those $e = e_1 \cdots e_n$ in which there exist no indices $i < j < k$ such that both $e_i \geq e_j$ and $e_i > e_k$ hold. See [9] or [13] for an explanation of this notation and other similar patterns. By an ascent (descent) within a sequence $s = s_1 s_2 \cdots s_n$, we mean an index $i \in [n - 1]$ such that $s_i < s_{i+1}$ ($s_i > s_{i+1}$, respectively). Let $\text{asc}(s)$ ($\text{desc}(s)$) denote the number of ascents (descents) of $s$. If $s$ is a sequence having non-negative integral entries, then let $\text{dist}(s)$ denote the number of distinct positive letters appearing in $s$.

Lin and Kim made the following conjecture which provides a connection between inversion sequences and pattern avoidance in permutations.

**Conjecture 1.1 (Lin and Kim [9, Conj. 3.3]):** Let $(\nu, \mu)$ be a pair of patterns of length four. Then

$$\sum_{\pi \in S_\nu(\nu, \mu)} q^{\text{asc}(\pi)} v^{\text{last}(\pi)} = \sum_{e \in I_n(\geq, -, >)} q^{\text{dist}(e)} v^{\text{last}(e) + 1}$$

for the following six pairs $(\nu, \mu)$:

$$(4231, 3241), (4231, 2431), (4231, 3421), (2431, 3241), (3421, 2431), (3421, 3241).$$

Here, we find it more convenient to deal with the reversals of the patterns in each pair and consider alternatively the joint distribution of the descents and first letter statistics on the left-hand side above. Also, we will represent an inversion sequence $e = e_1 \cdots e_n$ using positive instead of non-negative entries (which can be achieved by adding one to each entry) so that $1 \leq e_i \leq i$ for all $i$. Then $\text{dist}(e)$ would need to be replaced by $\text{dist}(e) - 1$ and $\text{last}(e) + 1$ by $\text{last}(e)$ on the right side of Conjecture 1.1.

The main objective of this article is then to prove the following result.

**Theorem 1.2:** We have

$$\sum_{\pi \in S_\nu(\sigma, \tau)} q^{\text{desc}(\pi)} v^{\text{first}(\pi)} = \sum_{e \in I_n(\geq, -, >)} q^{\text{dist}(e) - 1} v^{\text{last}(e)}, \quad n \geq 1, \quad (1)$$

if and only if $(\sigma, \tau)$ is one of the following pattern pairs:

$$(1243, 1324), (1243, 1342), (1243, 1423), (1324, 1342), (1324, 1423), (1342, 1423),$$

where the inversion sequence $e$ is expressed using positive integers.

Combining the results that are proven in the subsequent sections implies Theorem 1.2; note that in several instances, the specific result in question follows from a more general one. We remark that the ‘only if’ direction in Theorem 1.2 follows from numerical evidence (upon considering the case $n = 6$), which eliminates all other possible pairs $(\sigma, \tau)$ of patterns of length four.

Let $S_{n,k}$ be given by

$$S_{n,k} = S_{n,k-1} + 2S_{n-1,k} - S_{n-1,k-1}, \quad 1 \leq k \leq n - 2,$$

with $S_{n,n} = S_{n,n-1} = S_{n,n-2}$ for $n \geq 3$ and $S_{1,1} = S_{2,1} = S_{2,2} = 1$; see entry A341695 in the OEIS [17]. One can show that $\sum_{k=1}^{n} S_{n,k} = S_n$, the $n$th large Schröder number (see [17, ...]}
In [12], it was proven that \(|S_{n,i}(\sigma, \tau)| = S_{n,i}\) holds for nine pairs of patterns of length four (and no others), including the six listed in Theorem 1.2, where \(S_{n,i}(\sigma, \tau)\) denotes the subset of \(S_n(\sigma, \tau)\) whose members start with \(i\). Here, a more technical argument is required to deal with the joint distribution of the first letter and descents statistics on \(S_n(\sigma, \tau)\) for the various pairs \((\sigma, \tau)\) and does not reduce to prior arguments when \(q = 1\). To establish Theorem 1.2, we show in each case that both sides of equality (1) have the same (ordinary) generating function by a computational approach. We make use of various techniques, including the kernel method [4] in some instances and educated guessing (aided at times by the conjecture itself) in others, to solve a system of functional equations satisfied by the associated generating functions in each case.

In particular, it is shown for each \((\sigma, \tau)\) that the generating function over \(n \geq 1\) of both sides of (1) is given by

\[
\frac{vx}{1 - vqx} + \frac{vx(vqx - v - x)t(vx)}{2(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)(vqx - vx - 1)} + \frac{(x + (qx^2 + qx + 3x^2 - 2x - 1)v)vx}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} - \frac{(2q^2x^2 + 3q^2x - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(qx - 1)vqx)v^3x^2}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)},
\]

where

\[
t(x) = \sqrt{(1 - 2q)^2x^2 - 2x(1 + 2q) + 1}.
\]

Our calculations in each case leading to the above generating function expression are supported by mathematical programming\(^1\). Note that the coefficient of \(v^i q^j\) in this expression yields a new refinement of \(S_n\). For other extensions of the Schröder numbers, see, e.g. [2, 6, 14–16]. For other recent results on various restricted classes of inversion sequences, see [1, 7, 8, 10, 20, 21] and references contained therein.

The organization of this article is as follows. In the next section, we perform the enumeration on the right side of (1) using an additional parameter that we call the height of an inversion sequence. Indeed, we are able to find the generating function of the joint distribution of the dist, last letter and height statistics on \(I_n(\geq, -, >)\). In the third and fourth sections, we treat the pattern pairs \((\sigma, \tau) = (1324, 1423)\) and \((1342, 1423)\), respectively, where we proceed by solving a more refined version of the problem in the first case and a generalization of the problem in the latter case. In the final section, we treat the remaining four pattern pairs by considering in addition the second letter statistic, which is necessary for finding recurrences satisfied by the joint distribution. These recurrences may then be rewritten in terms of generating functions, leading to a system of functional equations in each case.
We leave as an open question the problem of finding direct bijective proofs of the statistical equivalences in Theorem 1.2.

2. Statistics on a restricted class of inversion sequences

In this section, we determine the joint distribution of the last letter and dist statistics on $U_n = I_n(\geq, -, >)$. To aid in doing so, we consider an additional parameter on $I_n$ as follows.

By the height of $e = e_1 \cdots e_n \in I_n$, which will be denoted by $\text{hght}(e)$, we mean the greatest $x$ such that $x \geq y$, where $y$ is the successor of $x$. That is, $\text{hght}(e) = \max\{e_i : e_i \geq e_{i+1} \text{ and } 1 \leq i \leq n-1\}$. Note that one may restrict attention to $e_i$ corresponding to leftmost occurrences of letters in $e$ when determining $\text{hght}(e)$.

For example, if $n = 12$ and $e = 123345747789 \in I_{12}$, then $\text{hght}(e) = 7$. Note that indeed $e \in U_n$; to check this quickly, one need only verify that there are no letters in $\{1, 2\}$ to the right of the second 3 and no letters less than or equal 6 to the right of the second 4, where we have underlined the adjacencies in $e$ which determine the current height (for example, the height of the partial sequence $123345 \in U_6$ would be 3, which then becomes 7 after the addition of the next two letters 7,4).

Let $U_n(i, j)$ denote the subset of $U_n$ consisting of those $e$ such that last($e$) = $i$ and $\text{hght}(e) = j$. Note that $U_n(i, j)$ can be non-empty only when $1 \leq i \leq n$ and $1 \leq j \leq n - 1$. Further, $e = 12 \cdots n$ belongs to no subset $U_n(i, j)$ since there is no $x$ such that $x \geq y$ in the description above and hence will be counted separately. Observe further that, by the definitions, we have that $U_n(n, n-1)$ is empty since the first $n-1$ would have to occur in the penultimate position, which implies the last letter must belong to $[n-1]$ in order for a height of $n-1$ to be achieved.

Given $n \geq 2$, $1 \leq i \leq n$ and $1 \leq j \leq n - 1$, let $u_n(i, j) = u_n(i, j; q)$ be given by

$$u_n(i, j) = \sum_{e \in U_n(i, j)} q^{\text{dist}(e)-1},$$

and put zero for $u_n(i, j)$ otherwise. For example, we have

$$U_5(2, 4) = \{11142, 11242, 11342, 12142, 12242, 12342\},$$

and thus, $u_5(2, 4) = 4q^2 + 2q^3$.

The array $u_n(i, j)$ is determined recursively as follows.

**Lemma 2.1:** If $n \geq 3$, then

$$u_n(i, j) = \delta_{j,n-1} \cdot q^{n-2} + u_{n-1}(j, i) + \sum_{k=1}^{i-1} (u_{n-2}(j-1, k) + u_{n-1}(j-1, k)), \quad 1 \leq i < j \leq n - 1, \quad (3)$$

$$u_n(i, j) = q \sum_{\ell=1}^{i-1} u_{n-1}(\ell, j), \quad 1 \leq j < i \leq n, \quad (4)$$
and
\[ u_n(i, i) = \sum_{k=1}^{i-1} u_{n-1}(i, k) + \sum_{\ell=1}^{i} u_{n-1}(\ell, i), \quad 1 \leq i \leq n - 2, \]

with \( u_3(2, 2) = q \) and \( u_n(n - 1, n - 1) = q^{n-2} + q \sum_{i=1}^{n-2} \sum_{j=1}^{n-3} u_{n-2}(i, j) \) for \( n \geq 4 \) and initial values \( u_2(1, 1) = 1, u_2(2, 1) = 0. \)

**Proof:** The initial values when \( n = 2 \) follow from the definitions, so assume \( n \geq 3 \). To show (3), first note that for \( e \in U_n(i, j) \) where \( i < j \), we claim that the final two letters of \( e \) are \( j, \ i \). To realize this, note that \( \text{hght}(w) = j \) for \( w \in U_n \) implies \( w \) is expressible as \( w = \alpha j \ell \beta \), where \( 1 \leq \ell \leq j \). \( \alpha \) contains no letters greater than \( j - 1 \) and \( \beta \) contains no letters in \([j - 1]\). Then \( e \) belonging to \( U_n(i, j) \) where \( i < j \) implies that the section of \( e \) corresponding to \( \beta \) is empty, which yields the claim. Let \( e = e' i, \) where we assume \( e' \neq 12 \cdots (n - 1). \) Note that \( e' \) has all its letters in \([j - 1]\) except for the terminal \( j \), for otherwise, \( xji \) would occur with \( x \geq j > i \) and \( j, i \) the final two letters of \( e \), which is impossible. Thus, \( \text{hght}(e') = k \) for some \( 1 \leq k \leq j - 1 \). The successor \( t \) of the leftmost \( k \) in \( e' \) must then belong to \([k]\), for otherwise \( \text{hght}(e') \) would be strictly greater than \( k \). Thus, we must have \( k \leq i \), for otherwise \( kti \) would form an occurrence of the forbidden pattern.

If \( k = i \), then one may delete the terminal \( i \) from \( e \) resulting in a member of \( U_{n-1}(j, i) \) and hence there are \( u_{n-1}(j, i) \) possibilities. So assume \( k < i \). In this case, we have \( e = \delta k \ell t \rho, \) where \( 1 \leq t \leq k, \rho \geq 0, \delta \) is \((k - 1)\)-ary (i.e. contains only letters in \([k - 1]\)) and \( \rho \) has no letters in \([k]\) and ends in \( j, i \). Further, we have \( \rho = \rho^* ji, \) where \( \rho^* \) if non-empty contains letters in \([k + 1, j - 1]\) and is strictly increasing (for otherwise \( \text{hght}(e') = k \) would be violated). Then \( i > k \) implies \( i \) can only occur once or twice in \( \rho \) (and hence in \( e \)). If \( i \) occurs once in \( \rho \) (at the end), then deleting \( i \) results in a member of \( U_{n-1}(j, k) \) after reduction of letters greater than \( i \). Considering all possible \( k \) then gives \( q \sum_{k=1}^{i-1} u_{n-1}(j, k) \), where the \( q \) accounts for the additional distinct letter \( i \). If \( i \) occurs twice in \( \rho \) (once in \( \rho^* \) and again at the end), then deleting both of the letters \( i \) is seen to give \( q \sum_{k=1}^{i-1} u_{n-2}(j, k) \) possibilities. Combining the previous cases on \( k \) then implies (3) if \( j < n - 1 \). On the other hand, if \( j = n - 1 \), then \( e' = 12 \cdots (n - 1) \) is also possible, and thus, there is an additional member of \( U_n(i, n - 1) \) of weight \( q^{n-2} \), which completes the proof of (3) in all cases.

To show (4), first note that for \( w \in U_n(i, j) \) where \( j < i \), we must have \( w = w' ji t \rho, \) where \( w' \) is \((j - 1)\)-ary, \( 1 \leq t \leq j, \rho \geq 0 \) and \( \rho \) is strictly increasing on \([j + 1, j + 2, \ldots] \). Note that \( i > j \) implies \( \rho \) is non-empty. Then deleting the terminal \( i \) of \( w \) yields a member of \( U_{n-1}(\ell, j) \) for some \( \ell \) since the leftmost \( j \) (and its successor) are unaffected. Note that \(|\rho| > 1 \) implies \( j < \ell \leq i - 1 \), whereas \(|\rho| = 1 \) implies \( 1 \leq \ell \leq j \) (in the latter case, \( \ell = j \) if and only if \( p > 0 \) or \( p = 0 \) with \( t = j \)). Furthermore, \( i > j \) implies the terminal \( i \) is the only letter of its kind. Thus, considering all possible \( \rho \) yields formula (4).

To show (5), let \( w \in U_n(i, i) \), where \( 1 \leq i \leq n - 2 \). First, suppose \( w = w' ii \), where \( w' \) is \((i - 1)\)-ary. Then deleting the final \( i \) from \( w \) results in a member of \( U_{n-1}(i, k) \) for some \( k < i \), which accounts for the first sum on the right side. Note that \( i \leq n - 2 \) implies \( w' i \neq 12 \cdots (n - 1) \) and hence belongs to some \( U_{n-1}(i, k) \). On the other hand, if the leftmost \( i \) of \( w \) occurs to the left of the penultimate position, then deleting the final \( i \) results in a member of \( U_{n-1}(\ell, i) \) for some \( \ell \in [i] \). Note that \( \ell < i \) in this last case if and only if the successor of the leftmost \( i \) is less than \( i \) and occurs in the penultimate position of
w. Considering all possible \( \ell \) yields the second sum and completes the proof of (5). If \( i = n - 1 \), then \( w \in \mathcal{U}_n(n - 1, n - 1) \) implies \( w = w'(n - 1)(n - 1) \), where \( w' \in \mathcal{U}_{n - 2} \) with no restrictions. Then there are \( q^{n-2} + q \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} u_{n-2}(i, j) \) possibilities in this case, with the extra factor of \( q \) multiplying the sum accounting for the terminal letters \( n-1 \), which completes the proof.

Using Lemma 2.1, one gets the following values of the array \( u_n(i, j) \) when \( n = 3 \) and \( n = 4 \):

\[
\begin{align*}
\quad u_3(1, 1) = 1, & \quad u_3(1, 2) = u_3(2, 1) = u_3(2, 2) = u_3(3, 1) = q, & \quad u_3(3, 2) = 0 \\
\quad u_4(1, 1) = 1 & \quad u_4(1, 2) = q & \quad u_4(1, 3) = q + q^2 \\
\quad u_4(2, 1) = q & \quad u_4(2, 2) = 3q & \quad u_4(2, 3) = 2q^2 \\
\quad u_4(3, 1) = q + q^2 & \quad u_4(3, 2) = 2q^2 & \quad u_4(3, 3) = q + q^2 \\
\quad u_4(4, 1) = q + 2q^2 & \quad u_4(4, 2) = 2q^2 & \quad u_4(4, 3) = 0,
\end{align*}
\]

which may be verified using the definitions.

Let \( \mathcal{U}_n(v, w) = \mathcal{U}_n(v, w; q) \) be given by

\[
\mathcal{U}_n(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} u_n(i, j)v^iw^j, \quad n \geq 2,
\]

with \( \mathcal{U}_1(v, w) = 0 \). We wish to find \( \mathcal{U}_n(v, w) + v^nq^{n-1} \) for all \( n \geq 1 \) and, in particular, \( \mathcal{U}_n(v, 1) + v^nq^{n-1} \), where the additional term accounts for \( 12 \cdots n \). Define

\[
\mathcal{U}(x, v, w; q) = \mathcal{U}(x, v, w) = \sum_{n \geq 2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} u_n(i, j)v^iw^jx^n,
\]

where the \( q \) argument is often suppressed. Then

\[
\mathcal{U}(x, v, 1) + \sum_{n \geq 1} v^nq^{n-1}x^n = \mathcal{U}(x, v, 1) + \frac{vx}{1 - vqx}
\]

yields the generating function for the joint distribution on \( \mathcal{U}_n \) for \( n \geq 1 \) of the last letter and dist statistics (marked by \( v \) and \( q \), respectively).

To find \( \mathcal{U}(x, v, w) \), we define the further generating functions

\[
\mathcal{U}^+(x, v, w) = \sum_{n \geq 3} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} u_n(i, j)v^iw^jx^n,
\]

\[
\mathcal{U}^-(x, v, w) = \sum_{n \geq 2} \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} u_n(i, j)v^iw^jx^n
\]

and

\[
\mathcal{U}^0(x, v) = \sum_{n \geq 2} \sum_{i=1}^{n-1} u_n(i, v)x^n.
\]
Clearly,

\[ U(x, v, w) = U^+(x, v, w) + U^-(x, v, w) + U^0(x, vw), \]  

by the definitions. Rewriting (3)–(5) in terms of generating functions yields

\[ U^+(x, v, w) = \frac{v w^2 q x^3}{(1 - w q x)(1 - v w q x)} + x U^-(x, w, v) \]
\[ + \frac{v w q x^2}{1 - v} (U^-(x, w, v) - U^-(x, vw, 1)) \]
\[ + \frac{v w q x}{1 - v} (U^-(x, w, v) - U^-(x, vw, 1)) - w q x U(wx, 1, v) + v w q x U(vwx, 1, 1)), \]

(7)

\[ U^-(x, v, w) = \frac{v q x}{1 - v} (U^-(x, v, w) - v U^-(vx, 1, w)) + U^0(x, vw) - v U^0(vx, w)) \]
\[ + \frac{v q x}{1 - v} (U^+(x, 1, vw) - v U^+(vx, 1, w)), \]

(8)

\[ U^0(x, v) = \frac{v x^2}{1 - v q x} + x U^+(x, 1, v) + x U^0(x, v) + x U^-(x, v, 1). \]

(9)

Note that one might guess the form of the generating functions \( U^+, U^- \) when \( w = 1 \) and \( U^0 \) based on Conjecture 1.1 and the solution in the case of avoiding \( \{1324, 1342\} \) (see Theorem 5.2), which is obtained using the kernel method without guessing.

Then to find the solution to the system of functional equations, we implement the following steps:

- First assume that each of the generating functions \( U^+(x, v, 1), U^-(x, v, 1) \) and \( U^0(x, v) \) may be expressed as follows:

\[ F = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 - (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)t(vx)}{1 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4}, \]

where \( t(x) \) is given by (2).

- By comparing the coefficients of \( x^n \) in \( U^+(x, v, 1) \) and \( F \) for \( n = 0, 1, \ldots, 14 \), and solving for the unknowns \( a_i, b_i, c_i \), we obtain an explicit expression for \( U^+(x, v, 1) \), and likewise for \( U^-(x, v, 1) \) and \( U^0(x, v) \) (Note that this does not mean these expressions are necessarily the correct ones for the respective generating functions, but by programming, it is seen that they are correct up to the coefficient of \( x^{20} \).)

- Assume for now that we have the correct formulas for the generating functions \( U^+(x, v, 1), U^-(x, v, 1) \) and \( U^0(x, v) \).

- By taking \( w = 1 \) in (7) and (8), together with the expressions found above for \( U^+(x, v, 1), U^-(x, v, 1) \) and \( U^0(x, v) \), one obtains a system of equations in \( U^+(x, 1, v) \) and \( U^-(x, 1, v) \), which leads to explicit formulas for \( U^+(x, 1, w) \) and \( U^-(x, 1, w) \) (again these formulas are true up to the coefficient of \( x^{20} \)).

- Thus, by (7) and (8) and the expressions for \( U^+(x, v, 1), U^+(x, 1, w), U^-(x, v, 1), U^-(x, 1, w) \) and \( U^0(x, v) \), we obtain \( U^+(x, v, w) \) and \( U^-(x, v, w) \) explicitly.
• By (6), we then have an expression for $U(x, v, w)$. (Note that the formulas found in this way for $U(x, y, w)$, $U^+(x, v, w)$, $U^-(x, v, w)$ and $U^0(x, v)$ are all true up to $x^{20}$.)
• Hence, it remains to check that the formulas for $U(x, y, w)$, $U^+(x, v, w)$, $U^-(x, v, w)$ and $U^0(x, v)$ obtained in the previous steps actually satisfy (6)–(9), which can be done with the aid of any mathematical programming.

Thus, one obtains the following solution to the foregoing system of functional equations (6)–(9), where $t(x)$ is given by (2).

**Theorem 2.2:** The generating function $U(x, v, w)$ is given by

$$U(x, v, w) = U^+(x, v, w) + U^-(x, v, w) + U^0(x, vw),$$

where

$$U^0(x, v) = \frac{v^2x^2(2v-1)vx^-(v-3)vx^2 - 3v + 1 \pm v(v + 1)x^2(t(vx))}{2(2vx^2 - vx + v + x - 1)(vq - vx - 1)},$$

$$U^+(x, v, w) = -\frac{vwx^2vq^2wq + v + x)t(vwx)}{2(2vwq - vw + v + x - 1)(vq - vx + v + x - 1)}$$

$$-v^2wq^2 - wq + vx^2 - q + v^2wq + vwx - vx - v + 1$$

$$\frac{vwq^2((2v^2 - vwx + vwx^2 - 2v^2wq + vwx - v^2wq^2)}{2(2vwq - vw + v + x - 1)(vq - vx + v + x - 1)}$$

$$-v^2q - vwx - vx - v + 1)$$

$$U^-(x, v, w) = \frac{v^2wq^2vq(vwx - vx + vw + w - 1)t(vwx)}{2(2vwq - vw + v + x - 1)(vq - vx + v + x - 1)}$$

$$-v^2wq^2 - vwx^2 - 3v - 2v + 2v^2v + 2v - 1)x^2 - v^2wq^2$$

$$+2(2v^2q^2 - vq^2 - vwx - vx - w + 1)(vq - 1)$$

$$\times(2v^2wq - vw + vwx - vx - w + 1)$$

$$+vq + vwx^2 - vx - w + 1)$$

$$+2v^2q(v - 1)v^2q^2 - 3v - 2v + 2v^2v + 2v - 1)x + v + 1)v^2wq^2$$

$$+2(2v^2q^2 - vq^2 - vwx - vx - w + 1)(vq - 1)$$

$$\times(vq - 1)(2vwq - vw + vwx - vx - w + 1)$$

$$+2(q - 1)(vq - 1)(vq - vx + v + 1)v^3wq^3q$$

$$+vwx - vx - w + 1)(vq - 1)(2vwq - vw + vw + x - 1)$$
As a corollary, we have

$$U(x, v, 1) = \frac{xv(vqx - v - x)t(xv)}{2(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)(vqx - vx - 1)} + \frac{(x + (q^2 + qx + 3x^2 - 2x - 1)v)vx}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} - \frac{(2q^2x^2 + 3q^2x - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(q^2 + qx)v^3x^2}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)},$$

and, in particular,

$$U(x, 1, 1) = \frac{1 - (1 + 3q)x + q(2q - 1)x^2 - (1 - q)x\sqrt{(1 - 2q)^2x^2 - 2x(1 + 2q) + 1}}{2q(1 + x - qx)(1 - qx)}.$$

**Remark:** Several statistics on 021-avoiding inversion sequences (another subset of $I_n$ enumerated by the Schröder numbers) were studied in [8], and in light of these results, it would be interesting to find an Eulerian statistic on $I_n(021)$ that is equidistributed with dist on $I_n(\geq, -, >)$. Such a statistic would have the same distribution as the descents statistic on the six restricted classes of permutations given in Theorem 1.2.

### 3. The case (1324, 1423)

To find a recursive structure leading to recurrences in this case, we make use of a generating tree approach (see, e.g. [18, 19]). Consider forming members of $R_n = S_n(1324, 1423)$ from members of $R_{n-1}$, expressed using the elements of $[2, n]$, by inserting 1 appropriately. By an active site of $\pi \in R_n$, written using letters in $[2, n + 1]$, we mean a position where 1 may be inserted without introducing 1324 or 1423. One often refers to $\rho \in R_{n-1}$ in which the element 1 is inserted in forming $\pi \in R_n$ as the precursor of $\pi$.

Let $\text{act}(\pi)$ denote the number of (active) sites of $\pi$. Then it is seen that $\text{act}(\pi) = n + 1$ if and only if $\pi$ avoids both 213 and 312. On the other hand, suppose that the $i$th letter from the right within $\pi$, say $x$, starts a 213 or 312 and is the rightmost such letter of $\pi$ to do so (i.e. $i$ is minimal). Then $\text{act}(\pi) = i$, where $3 \leq i \leq n$, with the sites of $\pi$ corresponding to the $i$ possible positions of $\pi$ to the right of $x$ in which to insert 1. Note that $\text{act}(\pi) = i$ where $3 \leq i \leq n$ implies $\pi$ may be decomposed as

$$\pi = \alpha x \beta \gamma,$$

where $\alpha$ is of length $n - i$, $\beta$ is increasing with $|\beta| \geq 2$ and contains $\max(\beta \cup \gamma)$, $x > \min(\beta)$ and $\gamma$ is decreasing and possibly empty.

By the final descending run (fdr) within a permutation $\pi$, we mean the maximal string of consecutive letters ending with the last letter that is decreasing. Let $\text{fdes}(\pi)$ denote the length minus one of the fdr of $\pi$, which is the same as the number of descents that involve two adjacent elements in this run. We will refer to these descents as the final descents of $\pi$. Each position separating two letters that correspond to a final descent of $\pi$ is seen to be an active site. Note that $0 \leq \text{fdes}(\pi) \leq n - 1$, with $\text{fdes}(\pi) = 0$ if the last two letters of $\pi$ form an ascent and $\text{fdes}(\pi) = n - 1$ if $\pi = n(n - 1) \cdots 1$. In the decomposition of $\pi$ above, the final descents correspond to the positions directly preceding the letters of $\gamma$. 
Define \( \mathcal{R}_n(i, j) \) as the subset of \( \mathcal{R}_n \) consisting of those \( \pi \) for which \( \text{act}(\pi) = i \) and \( \text{fdesc}(\pi) = j \). Note that \( 0 \leq j \leq i - 2 \), with \( j = i - 2 \) occurring only when \( \pi \) is the decreasing permutation. Let \( r_n(i, j) = r_n(i, j; p, q) \) be defined by

\[
r_n(i, j) = \sum_{\pi \in \mathcal{R}_n(i, j)} p^{\text{first}(\pi)} q^{\text{desc}(\pi)}.
\]

For example, we have

\[
\mathcal{R}_5(4, 0) = \{23145, 32145, 34125, 35124, 42135, 43125, 45123, 52134, 53124, 54123\},
\]

which implies \( r_5(4, 0) = p^2 q(1 + p)^2 + p^3 q^2(1 + 2p + 3p^2) \).

It is seen that the \( r_n(i, j) \) can assume non-zero values only for \( 3 \leq i \leq n + 1 \) and \( 0 \leq j \leq i - 2 \) if \( n \geq 2 \), with \( r_1(2, 0) = p \). When \( n = 2 \), we have \( r_2(3, 1) = p^2 q \) and \( r_2(3, 0) = p \). If \( n = 3 \), then \( r_3(4, 0) = p \), \( r_3(4, 1) = pq(1 + p) \), \( r_3(4, 2) = p^3 q^2 \) and \( r_3(3, 0) = p^2 q(1 + p) \).

Note that one may regard \( p \) and \( q \) as indeterminates or, alternatively, as fixed positive integers. In the former case, one would regard members of the set \( \mathcal{R}_n(i, j) \) as being weighted according to certain parameter values, whereas in the latter, one can enumerate ‘coloured’ members of \( \mathcal{R}_n(i, j) \) wherein a first letter of size \( \ell \) is assigned one of \( p^\ell \) colours and each descent is assigned one of \( q \) colours, independently of the others. These interpretations of the polynomial \( r_n(i, j) \) may be used interchangeably when the context is clear and analogous interpretations apply to other distribution polynomials encountered.

The \( r_n(i, j) \) are given recursively as follows.

**Lemma 3.1:** We have

\[
r_n(j + 3, j) = pqr_{n-1}(j + 2, j - 1) + pq \sum_{i=j+3}^{n} r_{n-1}(i, j)
\]

\[
+ p \sum_{k=j+1}^{n-2} \sum_{i=k+2}^{n} r_{n-1}(i, k), \quad 0 \leq j \leq n - 3,
\]

\[
(10)
\]

\[
r_n(i, j) = pqr_{n-1}(i - 1, j - 1) + pr_{n-1}(i - 1, j) + pq \sum_{\ell = i}^{n} r_{n-1}(\ell, j), \quad 4 \leq j + 4 \leq i \leq n,
\]

\[
(11)
\]

\[
r_n(j + 2, j) = \delta_{j,n-1} \cdot p^n q^{n-1}, \quad 0 \leq j \leq n - 1,
\]

\[
(12)
\]

and

\[
r_n(n + 1, j) = \sum_{\ell = 1}^{n-1} p^\ell q\left(\frac{n - \ell - 1}{n - j - 2}\right), \quad 0 \leq j \leq n - 2.
\]

\[
(13)
\]

**Proof:** Formula (12) follows from observing that \( \mathcal{R}_n(j + 2, j) \) is non-empty if and only if \( j = n - 1 \), in which case it consists of the single element \( n(n - 1) \cdots 1 \) of weight \( p^n q^{n-1} \).

For (13), observe first that \( \mathcal{R}_n(n + 1, j) \) consists of \( \pi \) of the form \( \pi = \alpha \beta \), where \( \alpha \) is increasing and \( \beta \) is decreasing with \( |\beta| = j \). Then \( 0 \leq j \leq n - 2 \) implies \( \alpha \) is non-empty and we enumerate members of \( \mathcal{R}_n(n + 1, j) \) in this case according to the first letter \( \ell \) where
1 \leq \ell \leq n - 1. Then \( \alpha \) must contain \( n-j-2 \) members of \([\ell + 1, n - 1]\) in increasing order, with the remaining elements of \([n - 1] - \{\ell\}\) comprising \( \beta \). Thus, there are \( \binom{n-j-2}{n-j-1} \) members of \( R_n(n+1,j) \) with first letter \( \ell \), each having weight \( p^\ell q^j \). Considering all possible \( \ell \) then gives (13).

To show (10), suppose \( \pi \in R_n(j+3,j) \), where \( 0 \leq j \leq n - 3 \). Then \( \pi \) may be expressed as \( \pi = axyz\beta \), where \( \alpha \) or \( \beta \) is possibly empty, \( y < x, z \), and \( \beta \) is decreasing of length \( j \) with \( \max(\beta) < x \) if \( j > 0 \). Note that then the \( j+3 \) sites of \( \pi \) correspond to the positions to the right of the letter \( x \) and that either \( y = 1 \) or \( 1 \) occurs at the end of \( \pi \). For if not, then \( y < x, z \) and \( \beta \) decreasing would imply \( 1 \) lies in \( \alpha \) ensuring an occurrence of 1324 or 1423, as witnessed by \( 1xyz \). If \( 1 \) occurs at the end of \( \pi \), then there are \( pr_{n-1}(j+2,j-1) \) possibilities since the \( 1 \) would increase all parameter values (including \( act \) and \( fdesc \)) by one.

So assume \( 1 \) does not occur at the end of \( \pi \) and we consider cases based on the relative sizes of \( 1 \)’s neighbours as follows. Let us say that a pattern \( \tau \) occurs as a subword in \( \pi \) if the letters corresponding to an occurrence of \( \tau \) are consecutive entries of \( \pi \). We consider cases based on whether the element \( 1 \) within \( \pi \in R_n(j+3,j) \) is the middle letter in an occurrence of a 213 or 312 subword. In order for \( 1 \) to be involved in a 213, then \( 1 \) must be inserted directly preceding the \( (j+1) \)-st letter from the right within a member of \( R_{n-1}(i,j) \) for some \( j+3 \leq i \leq n \). Note that this increases the first letter by one as well as the number of descents (as \( 1 \) is placed within an ascen in this case). Considering all possible \( i \) then gives the first sum on the right side of (10). For \( 1 \) to be involved in a 312 instead, the precursor \( \rho \) of \( \pi \) must satisfy \( fdesc(\rho) = k \) where \( j+1 \leq k \leq n - 2 \), with \( 1 \) being inserted into the \( (j+1) \)-st final descent of \( \rho \) from the last. Note that \( act(\rho) = i \) for any \( i \in \{k+2, n\} \) since there is no restriction as to the number of additional sites of \( \rho \). Then \( \rho \in R_{n-1}(i,k) \) where \( i \) and \( k \) are as specified, with the number of descents unchanged by the insertion of \( 1 \). Considering all possible \( i \) and \( k \) then gives the second summation formula in and finishes the proof of (10).

A similar argument may be given for (11). First note that there are \( pqr_{n-1}(i-1,j-1) \) possible \( \pi \) obtained by inserting 1 at the end of the precursor \( \rho \) and \( pr_{n-1}(i-1,j) \) possibilities if 1 is inserted in the leftmost site of \( \rho \) (in which case 1 would be part of a 312 subword). On the other hand, if \( 1 \) is to be part of a 213, then it must be inserted into \( \rho \in R_{n-1}(\ell,j) \) for some \( i \leq \ell \leq n \) directly following the \( (i-1) \)-st letter from the right. Note that \( j+3 < i \leq \ell \) implies that there is an ascen between the \( (i-1) \)-st and \( (i-2) \)-nd rightmost letters of \( \rho \). Hence, insertion of \( 1 \) into this site introduces an additional descent, implying that there are \( pq \sum_{i=1}^{\ell} r_{n-1}(\ell,j) \) possible \( \pi \) in which \( 1 \) is part of a 213. Combining this case with the previous implies (11) and completes the proof.

By Lemma 3.1, one obtains the following non-zero values for \( r_n(i,j) \) when \( n = 4 \):

\[
\begin{align*}
\ r_4(3,0) &= (2p^2 + p^3)q + (p^3 + 2p^4)q^2, & \ r_4(4,0) &= (p^2 + p^3 + p^4)q, & \ r_4(5,0) &= p, \\
\ r_4(4,1) &= (p^2 + 2p^3 + 2p^4)q^2, & \ r_4(5,1) &= (2p + p^2)q, & \ r_4(5,2) &= (p + p^2 + p^3)q^2, \\
\ r_4(5,3) &= p^4 q^3,
\end{align*}
\]

which may be verified directly using the definitions.
Define
\[ R(x, v, w; p, q) = R(x, v, w) = \sum_{n \geq 2} \sum_{i=3}^{n+1} \sum_{j=0}^{i-2} r_n(i, j) v^i w^j x^n. \]

We seek a formula for \( R(x, 1, 1; p, q) \) and wish to show
\[ px + R(x, 1, 1; p, q) = \frac{px}{1 - pqx} + U(x, p, 1; q). \]

To aid in finding \( R(x, v, w) \), define the auxiliary generating function
\[ R^+(x, v, w) = \sum_{n \geq 4} \sum_{i=4}^{n-4} \sum_{j=0}^i r_n(i, j) v^j w^j x^n, \]
along with \( C(x, v) = \sum_{n \geq 2} \sum_{i=0}^{n-1} r_n(i+2, i) v^i x^n, \)
\( D(x, v) = \sum_{n \geq 3} \sum_{i=0}^{n-3} r_n(i+3, i) v^i x^n, \)
and \( E(x, v) = \sum_{n \geq 2} \sum_{j=0}^{n-2} r_n(n+1, j) v^j x^n. \) Clearly,
\[ R(x, v, w) = R^+(x, v, w) + v^2 C(x, vw) + v^3 D(x, vw) + vE(vx, w). \]

We now rewrite the recurrence relations from Lemma 3.1 in terms of generating functions. By (13), we have
\[
E(x, v) = \sum_{n \geq 2} \sum_{j=0}^{n-1} \sum_{i=1}^{n-1} p^i q^j \left( \frac{n - 1 - i}{n - j - 2} \right) v^j x^n = \sum_{i \geq 1} \sum_{n \geq 0} \sum_{j=0}^{n-1} p^i q^j \left( j + 1 - i \right) v^j x^{n+i+1}
\]
\[
= \sum_{i \geq 1} \sum_{n \geq 0} \sum_{j=1}^{n+i} p^i q^j \left( j + i \right) v^j x^{n+i+1} + \sum_{i \geq 1} \sum_{n \geq 0} \sum_{j=0}^{n-i} p^i q^j \left( j + 1 - i \right) v^j x^{n+i+1}
\]
\[
= \sum_{i \geq 1} \sum_{j=0}^{n+i} p^i q^j v^j x^{i+j+2} + \sum_{i \geq 1} \sum_{n \geq 0} \sum_{j=0}^{n} p^i q^j \left( j + 1 - i \right) v^j x^{n+i+1}
\]
\[
= \sum_{i \geq 1} \frac{p^i q^i v^i x^{i+2}}{(1-x)(1-x-qvx)} + \sum_{i \geq 0} \sum_{n \geq 0} \frac{p^{i+1} q^i v^i x^{n+2+i}}{1-x}.
\]

which implies
\[ E(x, v) = \frac{px^2}{(1-x-qvx)(1-pqx)}. \]

By (12), we have
\[ C(x, v) = \frac{p^2 qvx^2}{1-pqvx}. \]
By (10), we have

\[
D(x, v) = \sum_{n \geq 3} \sum_{i=0}^{n-3} pqr_{n-1}(i + 2, i - 1)v^i x^n + \sum_{n \geq 3} \sum_{i=0}^{n-3} \sum_{j=i+3}^{n} pqr_{n-1}(j, i)v^i x^n \\
+ p \sum_{n \geq 3} \sum_{i=0}^{n-2} \sum_{a=k+2}^{n-1} \sum_{a=k+2}^{n} r_{n-1}(a, k)v^i x^n \\
= pqvx \sum_{n \geq 3} \sum_{i=0}^{n-3} r_n(i + 3, i)v^i x^n + px \sum_{n \geq 2} \sum_{i=0}^{n-2} \sum_{j=i+3}^{n} r_n(j, i)v^i x^n \\
+ px \sum_{n \geq 2} \sum_{i=0}^{n-2} \sum_{a=k+2}^{n} \sum_{a=k+2}^{n+1} r_n(a, k)v^i x^n \\
= pqvx D(x, v) + px R^+(x, 1, v) + px E(x, v) + px D(x, v) \\
+ px \sum_{n \geq 2} \sum_{k=1}^{n-2} \sum_{i=0}^{n} r_n(a, k)v^i x^n \\
= pqvx D(x, v) + px R^+(x, 1, v) + px E(x, v) + px D(x, v) \\
+ px \sum_{n \geq 2} \sum_{a=3}^{n+1} \sum_{k=1}^{a-2} r_n(a, k) \left( \frac{1 - v^k}{1 - v} \right) x^n, \\
\]

which implies

\[
D(x, v) = pqvx D(x, v) + px R^+(x, 1, v) + px E(x, v) + px D(x, v) \\
+ \frac{px}{1 - v} (R^+(x, 1, 1) - R^+(x, 1, v)) \\
+ \frac{px}{1 - v} (E(x, 1) - E(x, v) + D(x, 1) - D(x, v) + C(x, 1) - C(x, v)). \quad (16)
\]

By (11), we have

\[
R^+(x, v, w) = pqvw \sum_{n \geq 4} \sum_{i=4}^{n-4} \sum_{j=0}^{i-4} r_n(i, j)v^i w^j x^n + px \sum_{n \geq 3} \sum_{i=3}^{n-3} \sum_{j=0}^{i-3} r_n(i, j)v^i w^j x^n \\
+ px \sum_{n \geq 3} \sum_{i=4}^{n+1} \sum_{j=0}^{n+1} r_n(a, j)v^i w^j x^n
\]

\[
= pqvw x + px D(x, v) + px E(x, v) + px D(x, v) \\
+ \frac{px}{1 - v} (E(x, 1) - E(x, v) + D(x, 1) - D(x, v) + C(x, 1) - C(x, v)). \quad (16)
\]
which implies

\[ R^+(x, v, w) = pqvw x R^+(x, v, w) + px R^+(x, v, w) + px^4 D(x, vw) \]

\[ + \frac{px}{1 - v} \left( v^3 R^+(x, 1, vw) - R^+(x, v, w) \right) + \frac{px^2 x}{1 - v} (v^2 E(x, vw) - E(vx, w)). \]

(17)

Using (14) and (15), we can write (16) and (17) as

\[
\begin{align*}
(1 - pqvx - px + \frac{px}{1 - v}) D(x, v) &= \left( px - \frac{px}{1 - v} \right) R^+(x, 1, v) + \frac{px}{1 - v} R^+(x, 1, 1) \\
&+ \frac{px}{1 - v} D(x, 1) + \frac{(qx - 1)(pqvx + px + px - p - 1)p^2 q x^3}{(pq - 1)(qx + x - 1)(pqvx - 1)(qvx - 1)^3}. \quad (18)
\end{align*}
\]

\[
\begin{align*}
\left(1 - pqvx - px + \frac{pqvx}{1 - v}\right) R^+(x, v, w) &= px^4 D(x, vw) + \frac{pqvx^4}{1 - v} R^+(x, 1, vw) \\
&- \frac{p^2 q^4 x^4}{(pqvx - 1)(qvx + x - 1)(qvx + vx - 1)}. \quad (19)
\end{align*}
\]

Replacing \( w \) with \( w/v \) in (19) then gives

\[
\begin{align*}
\left(1 - pqwx - px + \frac{pqvx}{1 - v}\right) R^+(x, v, w/v) &= px^4 D(x, w) + \frac{pqvx^4}{1 - v} R^+(x, 1, w) \\
&- \frac{p^2 q^4 x^4}{(pqwx - 1)(qwx + x - 1)(qwx + vx - 1)}. \quad (19)
\end{align*}
\]

Upon applying the kernel method and taking

\[
v = v_0(x, w) = \frac{1 - pqwx - px - \sqrt{(1 - pqwx - px + px)^2 - 4px(1 - pqwx)}}{2px},
\]


one gets

\[
R^+(x, 1, w) = \frac{px^3(v_0(x, w) - 1)}{(1 - pqwx)(1 - x - qxw)(1 - qxw - xv_0(x, w))} + \frac{v_0(x, w) - 1}{q} D(x, w).
\] (20)

Substituting (20) into (18) yields

\[
(q(1 - pqvx - pqx)(1 - v) + pqx - px(q(1 - v) - 1)(v_0(x, v) - 1)) D(x, v)
\]

\[
= px(v_0(x, 1) + q - 1)D(x, 1) + \frac{(q(1 - v) - 1)(v_0(x, v) - 1)p^2qx^4}{(1 - pqvx)(1 - x - qxv_0(x, v))}
\]

\[
+ \frac{(v_0(x, 1) - 1)p^2qx^4}{(1 - pqx)(1 - x - qx)(1 - qx - xv_0(x, 1))}
\]

\[
- \frac{(1 - qx)(1 - v)(pqvx + pqx + px - p - 1)p^2q^2x^3}{(1 - pqx)(1 - x - qx)(1 - pqvx)(1 - x - qxv_0)}.
\] (21)

Note that the solution of the equation

\[
q(1 - pqx - pqvx)(1 - v) + pqx - px(q(1 - v) - 1)(v_0(x, v) - 1) = 0
\]

is given by

\[
v = v_1(x) = \frac{1 + 2pqx - px - \sqrt{(2q - 1)^2p^2x^2 - 2(2q + 1)px + 1}}{4pqx}
\]

Thus, taking \(v = v_1(x)\) in (21) yields

\[
D(x, 1) = \frac{(v_0(x, v_1(x) - 1)p^3q^3}{(1 - pqvx_1(x))(1 - x - qxv_1(x))}
\]

\[
\times (1 - qxv_1(x) - xv_0(x, v_1(x)))(v_0(x, 1) - 1 + q)
\]

\[
- \frac{(v_0(x, 1) - 1)p^3q^3}{(1 - pqx)(1 - x - qx)(1 - qx - xv_0(x, 1))(v_0(x, 1) - 1 + q)}
\]

\[
- \frac{(1 - qx)(v_1(x) - 1)(pqvx_1(x) + pqx + px - p - 1)p^2q^2x^2}{(1 - pqx)(1 - x - qx)(1 - pqvx_1)(1 - x - qxv_1(x))(v_0(x, 1) - 1 + q)}.
\] (22)

Using (21) again, together with the expression for \(D(x, 1)\), we obtain explicitly the generating function \(D(x, v)\):

\[
K \cdot D(x, v) = p^3q^3x^2q^3v_0(x, v_0(x, v_1(x))(v_1(x) - v) - (p + 1)p^2q^2x^4v_0(x, v_1(x)
\]

\[
- (pq^2vx - pqvx - pqv - q + 1)p^2q^4v_0(x, v)
\]

\[
+ p^3q^2(qx - x - 1)x^4v_0(x, v_1(x))v_1(x)
\]

\[
- (pq^2v^2x - pqv - qv + q - 1)p^2q^4v_0(x, v_1(x))
\]

\[
- (p + 1)p^2q^2(qx - x - 1)x^3v_1(x)
\]

\[
- p^3q^2v(qx - x - 1)(pqvx - p - 1)x^3.
\] (23)
where

\[ K = (pqvx - 1)(qvx + xv_0(x, v) - 1)(pq^2(v^2 - 1)x + px(qv - q + 1)v_0(x, v)
\]
\[ - pqvx + 2pqx - px - qv + q)(pqxv_1(x) - 1)(qvx_1(x) + xv_0(x, v_1(x)) - 1). \]

Hence, by (20), we obtain a formula for \( R^+(x, 1, v) \) and then by (19), a formula for \( R^+(x, v, w) \):

\[
L \cdot R^+(x, v, w) = -p^3q^2v^4x^5(pq^2(v^2w^2 - vw - 1)x^2 + (pv^2 + 2pq - p)x^2 + (p - 1)qvw
\]
\[ + (pq - p + q)x - p)v_0(x, vw)v_0(x, v_1(x))v_1(x) - v^4q^2p^2x^5(p^2q^3v^3w^3x^2
\]
\[ - 2p^2q^2v^3w^3x - 2p^2q^2v^3w^2x^2 + 2pq^3v^2w^2x + p^2q^2vw + p^2qw
\]
\[ + pqvw - 2pqx + qvw - pq + px + p - q)v_0(x, vw)v_0(x, v_1(x))
\]
\[ + p^2q^2v^4x^4(p + 1)(pq^2v^2w^2x^2 - pq^2vw^2 - pq^2v^2x^2
\]
\[ + pqvw + pv^2x^2 + 2pq^2v - qvw + pq - px^2 - px + qx - p
\]
\[ \times v_0(x, vw)v_1(x) - v^6q^2p^2x^4(p^2q^2v^3w^3x - p^2q^3v^3w^3x
\]
\[ - p^2q^3v^3w^3x^2 - 2p^2q^3v^3w^2x^2 - 2p^2q^2v^2w^2x^2 + 2p^2q^3v^2w^2x
\]
\[ + p^2q^3v^3w^2x - p^2q^2v^2w^2x + 2pq^2v^2w^2x + p^2q^2v^2w^2x
\]
\[ + pq^2v^2w^2x^2 + 2pq^2v^2w^2x + p^2q^2v^2w^2x + p^2q^2v^2w^2x
\]
\[ + pq^2v^2 - p^2qvw - 2pqvw - pv^2 - pq^2x
\]
\[ - pqvw + 2pqx - qvw - px + q)v_0(x, vw) + v^5q^2p^3x^5
\]
\[ \times (pq^2v^2w^2x^2 - pq^2v^2w^2x^2 - pq^2v^2w^2x^2)
\]
\[ - pqvx^2 + 2pqx^2 + pv^2 - qvw + pqx + pvx - px^2 - px
\]
\[ + qx - p)v_0(x, v_1(x))v_1(x) + v^5q^2p^2x^5(p^2q^3v^3w^3x^2 + p^2q^2v^3w^2x^2
\]
\[ - 2p^2q^2v^2w^2x - 2pq^2v^2w^2x - p^2q^2v^2w^2x + p^2q^2v^2w^2x
\]
\[ - pq^2v^2w^2x + p^2q^2v^2w^2x + pq^2v^2w^2x + p^2q^2v^2w^2x
\]
\[ + pq^2v^2w^2x - pq^2v^2w^2x + pq^2v^2w^2x + 2pqvx^2 + pv^2 - qvw + px
\]
\[ + pvx - px^2 - px + qx - p)v_0(x, v_1(x)) + v^5q^2p^2x^4(p^2q^2v^3w^3x^3 - p^2q^3v^3w^3x
\]
\[ - p^2q^3v^3w^3x^2 + p^2q^3v^3w^3x^2 - p^2q^2v^3w^3x^3 - 2p^2q^3v^2w^2x
\]
\[ - p^2q^3v^2w^2x + 2p^2q^2v^2w^2x - p^3v^2w^2x + 2p^2q^2v^2w^2x
\]
\[ - p^2q^2v^2w^2x + p^2q^2v^2w^2x + p^2q^2v^2w^2x + pq^2v^2w^2x
\]
\[ - pq^2v^2w^2x + p^2q^2v^2w^2x + pq^2v^2w^2x + pq^2v^2w^2x - p^2qvw
\]
\[ + pq^2v^2w^2x - pq^2v^2w^2x - p^2qvw + pq^2v^2w^2x + 2pq^2v^2w^2x
\]
\[ + pq^2v^2w^2x + pq^2v^2w^2x - pq^2v^2w^2x - pq^2v^2w^2x
\]
\[ + 2pqx - qvw - px + q),\]
where
\[ L = (pqv^2wx - pqvw^2 + pqvx + pv^2x - pxv - v + 1)(qvwx + vx - v + 1)(pqvw - 1) \]
\[ \times (qvwx + xv_0(x, vw) - 1)(pq^2v^2w^2x + pqvw^2x_0(x, vw) - pq^2x - pqvw) \]
\[ - pqxv_0(x, vw) + 2pqx + pxv_0(x, vw) - px - qvw + q)(pqxv_1(x) - 1) \]
\[ \times (qxv_1(x) + xv_0(x, v_1(x)) - 1). \]

Therefore, we can state the following result.

**Theorem 3.2:** We have
\[ R(x, v, w) = R^+ (x, v, w) + v^2C(x, vw) + v^3D(x, vw) + vE(vx, w), \]
where \( R^+ (x, v, w) \) is as above and \( C(x, v) \), \( D(x, v) \) and \( E(x, v) \) are given in (15), (23) and (14), respectively.

As a corollary to this result, one can show (with the aid of programming) that
\[ vx + R(x, 1, 1; v, q) \]
\[ = \frac{vx}{1 - vqx} + \frac{vx(vqx - v - x)t(xv)}{2(vq^2x^2 - vq^2x^2 - qx + vx - v - x + 1)(vq^2 - vx - 1)} \]
\[ + \frac{(x + (qx^2 + qx + 3x^2 - 2x - 1)v)v^2 - vx}{2(vq^2 - vx - 1)(vq^2 - vx - 1)(vq^2x^2 - vq^2x^2 - qx + vx - v - x + 1)} \]
\[ - \frac{(2q^2x^2 + 3q^2x^2 - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(q-q)vq^2) vx}{2(vq^2 - vx - 1)(vq^2 - vx - 1)(vq^2x^2 - vq^2x^2 - qx + vx - v - x + 1)} \]
\[ = \frac{vx}{1 - vqx} + U(x, v, 1; q), \]
which establishes the desired equivalence of distributions in the case of (1324, 1423).

**4. The case (1342, 1423)**

To enumerate members of this class according to the number of descents, we consider a more general version rather than a refinement of the problem at hand. Given \( n \geq 1 \), \( 1 \leq m \leq n \) and \( 1 \leq i \leq n \), let \( \mathcal{D}_n(i, m) \) denote the subset of \( \mathcal{S}_{n,i}(1342, 1423) \) consisting of those members in which the letters \( n, n - 1, \ldots, n - m + 1 \) form a decreasing subsequence. Let \( \mathcal{D}_n(m) = \bigcup_{i=1}^n \mathcal{D}_n(i, m) \) for \( n \geq 1 \) and \( m \in [n] \). Define the distributions \( d_n(i, m) = d_n(i, m; q) \) and \( D_n(m) = D_n(m; q) \) by
\[ d_n(i, m) = \sum_{\pi \in \mathcal{D}_n(i, m)} q^{\text{desc}(\pi)} \]
and
\[ D_n(m) = \sum_{\pi \in \mathcal{D}_n(m)} q^{\text{desc}(\pi)}. \]
Note that \( D_n(m) = \sum_{i=1}^n d_n(i, m) \), by the definitions.
To aid in finding a recurrence for \(d_n(i, m)\), we consider the subset of \(D_n(m)\) comprising its indecomposable members. Let \(E_n(m)\) denote the subset of \(D_n(m)\) consisting of those \(\sigma\) which cannot be decomposed as \(\sigma = \sigma' \sigma''\) where \(\sigma'\) contains \([n - m + 1, n]\) and \(\sigma''\) is a permutation of \([a]\) for some \(a \geq 1\). Define \(e_n(m) = e_n(m; q)\) by

\[
e_n(m) = \sum_{\pi \in E_n(m)} q^{\text{desc}(\pi)}.
\]

Note that \(d_n(i, m)\) may assume non-zero values only when \(n \geq 1\) and \(i, m \in [n]\). Further, we have \(d_n(i, m) = 0\) if \(n - m + 1 \leq i \leq n - 1\), since elements of \([n - m + 1, n]\) must decrease, with \(d_n(n, n) = q^{n-1} = e_n(n)\) and \(d_n(i, n) = 0\) if \(i < n\). One may verify directly the following values of \(d_n(i, m)\) and \(e_n(m)\) when \(n = 3:\)

- \(d_3(1,1) = q + 1, d_3(1,2) = q, d_3(1,3) = 0;\)
- \(d_3(2,1) = 2q, d_3(2,2) = d_3(2,3) = 0;\)
- \(d_3(3,1) = d_3(3,2) = q + q^2, d_3(3,3) = q^2,\)

with \(e_3(1) = 1 + 2q, e_3(2) = 2q\) and \(e_3(3) = q^2\). It can be shown in general that \(d_n(1, n-1) = q^{n-2}\) and \(d_n(n, n-1) = (n-2)q^{n-2} + q^{n-1}\) for \(n \geq 2\), with \(e_n(n-1) = (n-1)q^{n-2}\).

The \(d_n(i, m)\) and \(e_n(m)\) satisfy the following system of intertwined recurrences.

**Lemma 4.1:** If \(n \geq 1\), then

\[
e_n(m) = D_n(m) - q \sum_{a=1}^{n-m} e_{n-a}(m)D_a(1), \quad 1 \leq m \leq n. \tag{24}
\]

If \(n \geq 2\), then

\[
d_n(m, n) = qD_{n-1}(m-1), \quad 2 \leq m \leq n, \tag{25}
\]

with \(d_n(n, 1) = qD_{n-1}(1)\). If \(n \geq 3\), then

\[
d_n(n-m, m) = qD_{n-2}(m-1) + q \sum_{j=1}^{n-m-1} d_{n-1}(j, m), \quad 2 \leq m \leq n-1, \tag{26}
\]

with \(d_n(n-1, 1) = qD_{n-2}(1) + q \sum_{j=1}^{n-2} d_{n-1}(j, 1)\). If \(3 \leq m + 2 \leq n\) and \(1 \leq i \leq n - m - 1\), then

\[
d_n(i, m) = d_{n-1}(i, m) + qD_{n-2}(n - i - 1) + q \sum_{j=1}^{i-1} d_{n-1}(j, m)

+ q \sum_{j=i+2}^{n-m} \sum_{a=0}^{i-1} e_{j-a-2}(j - i - 1)d_{n-j+a+1}(a + 1, m), \tag{27}
\]

with \(d_n(i, n) = \delta_{i,n} \cdot q^{n-1}\) for \(1 \leq i \leq n\), \(d_n(i, m) = 0\) for \(n - m + 1 \leq i \leq n - 1\) and \(d_2(1, 1) = 1, d_2(2, 1) = q\).

**Proof:** The conditions above stated last may be verified using the definitions. To show (24), note that \(\pi \in D_n(m)\) is either indecomposable or of the form \(\pi = \pi' \pi''\), where \(\pi'\) contains \([n - m + 1, n]\) and \(\pi''\) is a permutation of \([a]\) for some \(1 \leq a \leq n - m\) with \(a\) maximal.
The range of $a$ implies that both $\pi'$ and $\pi''$ are non-empty and hence a descent occurs between the last letter of $\pi'$ and the first of $\pi''$. Considering all possible $a$ then yields (24). Formula (25) follows from removing $n$ from $\pi \in D_n(n, m)$, which results in a member of $D_{n-1}(m-1)$ if $m \geq 2$ or of $D_{n-1}(1)$ if $m = 1$. To show (26), first note that $\pi \in D_n(n - m, m)$ for $n \geq 3$ must have second letter $j < n - m$ or $j = n$, for $j \in [n - m + 1, n - 1]$ is disallowed as the elements of $[n - m + 1, n]$ are to decrease. If $j < n - m$, then removal of $n - m$ from $\pi$ results in a member of $D_{n-1}(j, m)$, whereas if $j = n$, then removal of both $n - m$ and $n$ yields a member of $D_{n-2}(m - 1)$ if $m \geq 2$ or of $D_{n-2}(1)$ if $m = 1$. Since a removed letter is part of a descent in either case, formula (26) follows from considering all possible values of $j$.

To show (27), suppose $\pi \in D_n(i, m)$ where $n \geq m + 2$ and $1 \leq i \leq n - m - 1$. If $\pi$ starts $i$, $j$ for some $j < i$, or starts $i(i + 1)$, then there are clearly $q \sum_{j=1}^{i-1} d_{n-1}(j, m)$ and $d_{n-1}(i, m)$ possibilities, respectively. If $\pi$ starts with $i$, $n$, then both letters may be deleted, wherein it is required that the elements of $[i + 1, n - 1]$ decrease. Since $i < n - m$, we have that $[n - m + 1, n - 1]$ is contained in $[i + 1, n - 1]$, and thus, there are $q D_{n-2}(n - i - 1)$ possibilities for the remaining letters of $\pi$, where the factor of $q$ accounts for the descent of $\pi$ arising due to $n$. This completes the proof of (27) in the case when $i = n - m - 1$, so assume $i < n - m - 1$. In this case, it is possible for the second letter to be greater than $i$, but equal to neither $i + 1$ nor $n$.

So suppose $\pi \in D_n(i, m)$ where $1 \leq i \leq n - m - 2$ has second letter $j \in [i + 2, n - m]$. Note first that the elements of $[i + 1, j - 1]$ must occur (in decreasing order) prior to any elements of $[j + 1, n]$, for otherwise a 1423 or 1342 would occur. Let $C$ denote the section of $\pi$ starting with $i + 1$ and ending with the predecessor of the leftmost element of $[j + 1, n]$. We claim that for some $c \in C$, it must be the case that $\pi$ can be decomposed as $\pi = \alpha c \beta$, where the section $\alpha c$ comprises all elements in an interval of the form $[d, j]$ for some $d \in [i]$.

Assuming for now the claim (which is shown below), we proceed to complete the proof of (27). Consider the leftmost $c$ for which $\pi$ may be decomposed as $\pi = \alpha c \beta$ such that $\alpha c$ is of the form $[a + 1, j]$ for some $0 \leq a \leq i - 1$. Then $\alpha c - [i, j]$ has length $j - a - 2$, with $[i + 1, j - 1]$ decreasing so that it is enumerated by $e_{j-a-2}(j - i - 1)$. Let $\beta^*$ denote the subsequence of $\pi$ consisting of the letters in $[i] \cup \beta$. Note that $\beta^*$ is a $(1342, 1423)$-avoiding permutation of the elements from the set $[a] \cup [j + 1, n] \cup [i]$ that starts with $i$. As there are no restrictions on $\beta^*$ imposed by $\alpha c$ in terms of avoidance (and conversely), we have that $\beta^*$ is enumerated by $d_{n-j+a+1}(a + 1, m)$ since its $m$ largest elements must form a decreasing subsequence. Note also that there is an additional descent previously unaccounted for that is attributed to the successor of $j$ in $\pi$, as it belongs to $[i - 1] \cup [j - 1]$. Thus, considering all possible $a$ and $j$ yields $q \sum_{j=i+2}^{n-m} \sum_{a=0}^{i-1} e_{j-a-2}(j - i - 1) d_{a-j+a+1}(a + 1, m)$ additional members of $D_n(i, m)$ and combining this with the previous cases yields (27).

We now prove the claim above concerning the decomposition of $\pi$. Note that if $\pi$ starts $ij(j - 1)(j - 2) \cdots (i + 1)$, then one may take $c = i + 1$ and $d = i$ in the decomposition, so assume that this is not the case. Let $\pi = \ell_1 \rho_1 \cdots \ell_r \rho_r$, where $\ell_1, \ldots, \ell_r$ denotes the complete set of left-right minima of $\pi$ and the $\rho_i$ are possibly empty. By a unit of $\pi$, we mean a section $\ell_i \rho_i$ for some $1 \leq i \leq r$. By the assumption that $\pi$ does not start with the string $ij(j - 1)(j - 2) \cdots (i + 1)$, it must be the case that $i + 1$ belongs to some unit having the first letter $u$ where $u \in [i - 1]$.

Let $S$ denote the set of letters in $[u + 1, i - 1]$ occurring to the right of $u$ within $\pi$. Let $S_1, S_2$ denote respectively the subsets of $S$ comprising those letters lying to the left of or to
the right of \( i + 1 \) within \( \pi \). If \( S_2 = \emptyset \), then all letters of \([u + 1, i - 1]\) occur to the left of \( i + 1 \) and one may take \( c = i + 1 \) and \( d = u \) in the decomposition above. So assume \( S_2 \neq \emptyset \) and note that \( \max(S_1) < \min(S_2) \) if \( S_1 \) is also non-empty, for otherwise, \( \pi \) would contain a 1342 in which the roles of 1 and 4 are played by \( u \) and \( i + 1 \). Let \( \ell \) denote the letter from the set \([u] \cup S_1\) occurring furthest to the right within \( \pi \). Then \( \ell \) directly precedes the letter \( p \) for some \( p \in [i + 1, j - 1] \), which implies that the string \( \ell \rho(p - 1) \cdots (i + 1) \) occurs. Note that the letters \( p, p - 1, \ldots, i + 2 \) in this case are all seen to be extraneous concerning the avoidance of 1342 and 1423.

Let \( \pi = \gamma \pi' \), where the first letter of the section \( \pi' \) is \( \ell \) and hence \( \pi' \) has length \( n - (j - p) - (i - u - r) \) where \( r = |S_2| \). Let \( \tau \) denote the \((1342, 1423)\)-avoiding permutation of \([q]\), where \( q = n + u + r - j + 1 \), obtained from \( \pi \) by taking the section \( \pi' \), deleting the extraneous letters \( p, p - 1, \ldots, i + 2 \) and standardizing the resulting sequence. Note that \( \ell \) in \( \pi \) corresponds to \( u \) in \( \tau \) since all elements of \([u, \ell - 1]\) must occur to the left of \( \ell \), whence \( \tau \in D_q(u, m) \) where \( 1 \leq u \leq q - m - 2 \). Further, the second letter of \( \tau \) (which corresponds to \( i + 1 \) in \( \pi \)) equals \( u + r + 1 \) and hence belongs to \([u + 2, q - m]\) as \( S_2 \neq \emptyset \).

We now proceed inductively on \( n \geq 4 \) to prove the claim, noting that it is trivial to verify when \( n = 4 \) (as only 1324 is possible). Since \( \tau \) is of length strictly less than \( n \) and satisfies the requirements concerning its first two letters, we have by the induction hypothesis that \( \tau \) may be decomposed as \( \tau = \alpha' \beta' \), where \( \alpha' \) is such that \( \alpha' \beta' \) comprises an interval of the form \([d', u + r + 1]\) for some \( d' \in [u] \). Then the letter of \( \pi \) that corresponds to \( \alpha' \) within \( \tau \) furnishes the desired \( c \), which completes the induction and proof.

We define the generating functions

\[
D(x, v, w) = \sum_{n \geq 1} \sum_{m=1}^{n} \sum_{i=1}^{n} d_n(i, m) v^i w^{m-1} x^n
\]

and

\[
E(x, v) = \sum_{n \geq 1} \sum_{m=1}^{n} e_n(m) v^{m-1} x^n.
\]

We split \( D(x, v, w) \) into three further generating functions as follows:

\[
D_1(x, v, w) = \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{i=1}^{n-m-1} d_n(i, m) v^i w^{m-1} x^n,
\]

\[
D_2(x, v, w) = \sum_{n \geq 2} \sum_{m=1}^{n-1} d_n(n - m, m) v^{n-m} w^{m-1} x^n
\]

and

\[
D_3(x, v, w) = \sum_{n \geq 1} \sum_{m=1}^{n} d_n(n, m) v^n w^{m-1} x^n.
\]

Note that

\[
D(x, v, w) = D_1(x, v, w) + D_2(x, v, w) + D_3(x, v, w).
\]
By (24), we have

\[
E(x, v) = \sum_{n \geq 1} \sum_{m = 1}^{n} e_n(m) v^{m-1} x^n
\]

\[
= \sum_{n \geq 1} \sum_{m = 1}^{n} D_n(m) v^{m-1} x^n - q \sum_{n \geq 1} \sum_{m = 1}^{n} \sum_{a=1}^{n-m} e_{n-a}(m) D_a(1) v^{m-1} x^n
\]

\[
= D(x, 1, v) - q \sum_{n \geq 2} \sum_{a=1}^{n-1} \sum_{m=1}^{n-a} e_{n-a}(m) D_a(1) v^{m-1} x^n
\]

\[
= D(x, 1, v) - qE(x, v)D(x, 1, 0).
\]

By (25), we have

\[
D_3(x, v, w) = \sum_{n \geq 1} d_n(n, 1) v^n x^n + \sum_{n \geq 2} \sum_{m=2}^{n} d_n(n, m) v^n w^{m-1} x^n
\]

\[
= qvxD(vx, 1, 0) + vx + qvwx \sum_{n \geq 1} \sum_{m=1}^{n} \sum_{i=1}^{n} d_n(i, m) v^n w^{m-1} x^n
\]

\[
= qvxD(vx, 1, 0) + vx + qvwxD(vx, 1, w).
\]

By (26), we have

\[
D_2(x, v, w) = \sum_{n \geq 2} d_n(n - 1, 1) v^{n-1} x^n + \sum_{n \geq 3} \sum_{m=2}^{n-1} d_n(n - m, m) v^{n-m} w^{m-1} x^n
\]

\[
= vx^2 + \sum_{n \geq 3} \left( q \sum_{i=1}^{n-2} d_{n-2}(i, 1) + q \sum_{i=1}^{n-2} d_{n-1}(i, 1) \right) v^{n-1} x^n
\]

\[
+ \sum_{n \geq 3} \sum_{m=2}^{n-1} \left( q \sum_{i=1}^{n-2} d_{n-2}(i, m-1) + q \sum_{j=1}^{n-1-m} d_{n-1}(j, m) \right) v^{n-m} w^{m-1} x^n
\]

\[
= vx^2 + qvx^2 \sum_{n \geq 1} \sum_{i=1}^{n} d_n(i, 1) v^n x^n + qx \sum_{n \geq 1} \sum_{i=1}^{n} d_n(i, 1) v^n x^n
\]

\[
- qx \sum_{n \geq 1} d_n(n, 1) v^n x^n
\]

\[
+ \sum_{n \geq 3} \sum_{m=2}^{n-1} \left( q \sum_{i=1}^{n-2} d_{n-2}(i, m-1) + q \sum_{j=1}^{n-1-m} d_{n-1}(j, m) \right) v^{n-m} w^{m-1} x^n
\]

\[
= (1 - q)vx^2 + qx\left( 1 + (1 - q)vx \right)D(vx, 1, 0)
\]

\[
+ \sum_{n \geq 3} \sum_{m=2}^{n-1} \left( q \sum_{i=1}^{n-2} d_{n-2}(i, m-1) + q \sum_{j=1}^{n-1-m} d_{n-1}(j, m) \right) v^{n-m} w^{m-1} x^n
\]
\[
= (1 - q)vx^2 + qx(1 + (1 - q)vx)D(vx, 1, 0) \\
+ qvw^2 \sum_{n \geq 1} \sum_{m=1}^{n} \sum_{i=1}^{n} d_n(i, m) v^{n-m} w^{m-1} x^n \\
+ q \sum_{n \geq 3} \sum_{m=2}^{n-2} \sum_{j=1}^{n-1} d_{n-1}(j, m) v^{n-m} w^{m-1} x^n \\
= (1 - q)vx^2 + qx(1 + (1 - q)vx)D(vx, 1, 0) + qwx^2 D(vx, 1, w/v) \\
+ q \sum_{n \geq 3} \sum_{m=2}^{n-2} \sum_{j=1}^{n-1} d_{n-1}(j, m) v^{n-m} w^{m-1} x^n \\
= (1 - q)vx^2 + qx(1 + (1 - q)vx)D(vx, 1, 0) + qwx^2 D(vx, 1, w/v) \\
+ qv \sum_{n \geq 2} \sum_{m=1}^{n} \sum_{j=1}^{n-1} d_n(j, m) v^{n-m} w^{m-1} x^n - qv \sum_{n \geq 2} \sum_{j=1}^{n-1} d_n(j, 1) v^{n-1} x^n \\
= (1 - q)vx^2 + qx(1 + (1 - q)vx)D(vx, 1, 0) + qwx^2 D(vx, 1, w/v) \\
+ qx \sum_{n \geq 2} \sum_{m=1}^{n} \sum_{j=1}^{n} d_n(j, m) v^{n-m} x^n - qx \sum_{n \geq 1} \sum_{j=1}^{n} d_n(j, 1) v^{n} x^n \\
+ qx \sum_{n \geq 1} d_n(n, 1) (vx)^n \\
= (1 - q)vx^2 + qx(1 + (1 - q)vx)D(vx, 1, 0) + qwx^2 D(vx, 1, w/v) \\
+ qx(D_1(vx, 1, w/v) + vD_2(x, v, w)) - qxD(vx, 1, 0) \\
+ qx(qvxD(vx, 1, 0) + vx) \\
= vx^2 + qvx^2 D(vx, 1, 0) + qwx^2 D(vx, 1, w/v) \\
+ qx(D_1(vx, 1, w/v) + vD_2(x, v, w)).
\]

By (27), we have

\[
D_1(x, v, w) = \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{i=1}^{n-1-m} d_n(i, m) v^i w^{m-1} x^n \\
= \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{i=1}^{n-1-m} \left( d_{n-1}(i, m) + q \sum_{j=1}^{n-2} d_{n-2}(j, n - i - 1) \right) v^i w^{m-1} x^n \\
+ q \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{i=1}^{n-1-m} \sum_{j=1}^{n-1} d_{n-1}(j, m) v^i w^{m-1} x^n \\
+ \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{i=1}^{n-1-m} \left( q \sum_{j=i+2}^{n-m} e_{j-2-a}(j - 1 - i) d_{n-j+a+1}(a + 1, m) \right)
\]
\[ x(D_1(x, v, w) + D_2(x, v, w)) + \frac{qx^2}{1-w} \sum_{n \geq 1} \sum_{a=1}^n d_n(a, i) v^{n-i}(1 - w^i)x^n \]

\[ + \frac{qx}{1-v} \sum_{n \geq 3} \sum_{m=1}^{n-2} \sum_{j=1}^{n-1-m} d_n(j, m)(v^j - v^{n-m})w^{m-1}x^n \]

\[ + q^2 \sum_{m \geq 1} \sum_{n \geq 1} \sum_{j=1}^{n-1} \sum_{a=0}^{j-1} e_{j-a}(j + 1 - i)d_{n+m-j+a+1}(a + 1, m) \]

\[ \times v^j w^{m-1}x^{n+m} \]

\[ = x(D_1(x, v, w) + D_2(x, v, w)) + \frac{qx^2}{1-w} (D(vx, 1, 1/v) - wD(vx, 1, w/v)) \]

\[ + \frac{qx}{1-v} (vD_1(x, v, w) - D_1(vx, 1, w/v)) \]

\[ + q^2 \sum_{m \geq 1} \sum_{n \geq 1} \sum_{j=1}^{n-1} \sum_{a=0}^{j-1} e_{j-a}(j + 1 - i)d_{n+m-j+a+1}(a + 1, m) \]

\[ \times v^j w^{m-1}x^{n+m} \]

\[ = x(D_1(x, v, w) + D_2(x, v, w)) + \frac{qx^2}{1-w} (D(vx, 1, 1/v) - wD(vx, 1, w/v)) \]

\[ + \frac{qx}{1-v} (vD_1(x, v, w) - D_1(vx, 1, w/v)) \]

\[ + q^2 \sum_{m \geq 1} \sum_{n \geq 1} \sum_{j=1}^{n-1} \sum_{a=0}^{j-1} e_{j-a}(j + 1 - i)d_{n+m-j+a+1}(a + 1, m) \]

\[ \times v^j w^{m-1}x^{n+m} \]

Hence, there is the following system of functional equations.
Lemma 4.2: We have \( D(x, v, w) = D_1(x, v, w) + D_2(x, v, w) + D_3(x, v, w) \), where

\[
E(x, v) = \frac{D(x, 1, v)}{1 + qD(x, 1, 0)},
\]

\[
D_1(x, v, w) = x(D_1(x, v, w) + D_2(x, v, w)) + \frac{qv^2}{1 - w} (D(vx, 1, 1/v) - wD(vx, 1, w/v)) + \frac{q}{v} (vD_1(x, v, w) - D_1(vx, 1, w/v)) + \frac{q}{v} E(vx, 1/v)(D_1(x, v, w) + D_2(x, v, w)),
\]

\[
D_2(x, v, w) = vx^2 + qvx^2D(vx, 1, 0) + qwx^2D(vx, 1, w/v) + qx(D_1(vx, 1, w/v) + vD_2(x, v, w)),
\]

\[
D_3(x, v, w) = qvxD(vx, 1, 0) + vx + qvwxD(vx, 1, w).
\]

To solve these equations, we assume

\[
P_1 : D(x, v, 0) = vx + R(x, 1, 1; v, q),
\]

\[
P_2 : \frac{D_1(x, 1, w)}{D_2(x, 1, w)} = \frac{1 - x - t(x)}{2qx(-qx + x + 1)} - 1,
\]

where

\[
R(x, v) = \frac{vx(qvx - v - x)t(vx)}{2((q - 1)vx - 1)(q(q - 1)vx^2 - qx + (v - 1)(x - 1))},
\]

\[
-\frac{vx(2(q - 1)^2v^2x^3 - (2q^2 - 3q + 2)v^2x^2 + (3 - 2q^2)vx^2 + (3q - 2)vx + v^2x - v + x)}{2((q - 1)vx - 1)(q(q - 1)vx^2 - qx + (v - 1)(x - 1))}
\]

and \( t(x) \) is given by (2).

First, we will determine a potential solution to the system of equations in Lemma 4.2 with the aid of the assumptions \( P_1 \) and \( P_2 \). One may then verify that this potential solution indeed satisfies the equations in Lemma 4.2 (as well as \( P_1 \) and \( P_2 \)), and hence, it is the desired solution. Note that \( D(x, v, 0) \) is the actual generating function that is sought in this case since it enumerates all members of \( D_n(1) \).

By Lemma 4.2, we have

\[
D(x, 1, w) = D_1(x, 1, w) + D_2(x, 1, w) + D_3(x, 1, w),
\]

\[
D_2(x, 1, w) = x^2 + qx^2D(x, 1, 0) + qwx^2D(x, 1, w) + qx(D_1(x, 1, w) + D_2(x, 1, w)),
\]

\[
D_3(x, 1, w) = qxD(x, 1, 0) + x + qwxD(x, 1, w).
\]

Solving this system under \( P_1 \) and \( P_2 \), we get explicit formulas for \( D_j(x, 1, w) \) for \( j = 1, 2, 3 \) and \( D(x, 1, w) \). Hence, by Lemma 4.2, we have an explicit formula for \( E(x, v) \), which is given by

\[
E(x, v) = \frac{D(x, 1, v)}{1 + qD(x, 1, 0)}.
\]

Again by Lemma 4.2, we have

\[
D(x, v, w) = D_1(x, v, w) + D_2(x, v, w) + D_3(x, v, w),
\]
Theorem 4.3: We have $D(x, v, w) = D_1(x, v, w) + D_2(x, v, w) + D_3(x, v, w)$, where

$$D_1(x, v, w) = \frac{qv^2}{1-w} (D(vx, 1, 1/v) - wD(vx, 1, w/v)) - \frac{q}{1-v} D_1(vx, 1, w/v) + x (1 + \frac{q}{v} E(vx, 1/v)) D_2(x, v, w)\left(1 - x - \frac{q}{v} E(vx, 1/v) - \frac{qv^2}{1-v}\right),$$

$$D_2(x, v, w) = \frac{vx^2 + qvx^2 D(vx, 1, 0) + qw^2 D(vx, 1, w/v) + qx D_1(vx, 1, w/v)}{1-qvx},$$

$$D_3(x, v, w) = qvx D(vx, 1, 0) + vx + qvx D_1(vx, 1, w/v),$$

which leads to the following explicit formulas for the $D_j(x, v, w)$ and $D(x, v, w)$.

Theorem 4.3: We have $D(x, v, w) = D_1(x, v, w) + D_2(x, v, w) + D_3(x, v, w)$, where

$$D_1(x, v, w) = \frac{4vx^3(1 + q + q(1-q)vx)}{b_1 + b_2 t(vx)},$$

where

$$b_1 = (2q + 1)(q + 1)wv^2 - 2(w + 1)qvx - 2x + 2 - ((6q^3 - q^2 - 4q + 1)wv^2$$

$$- (6q^3 - 3q - 1)wv + (2 - 6q^2)x + 6q)vx$$

$$+ (2q - 1)(q - 1)(qx - 1)(2q - 1)wv^2 - 2v^2x^2,$$

$$b_2 = (2q - 1)(q + 1)wv^2 - 2(w + 1)qvx - 2x + 2 - (q - 1)(qx - 1)\times ((2q - 1)wv - 2)vx,$$

$$D_2(x, v, w) = \frac{4vx^2}{2 - (4qv + 2qw - 2v + w)x + v(w(2q - 1)^2x^2 + (2 + wx - 2qw)t(vx))},$$

$$D_3(x, v, w) = \frac{8vx}{6 - w + 2(v(2q + 2q - 1)x + v^2w(2q - 1)^2x^2}$$

$$+ (w + 2 - (2q - 1)vwx)t(vx).$$

Moreover,

$$E(x, v) = \frac{1 - x - 2qv + 2qx - t(x)}{2q(qv^2x - 2qv + vx - v + 2)}.$$  

Proof: By direct calculation (with the aid of mathematical programming), one may verify that each equation in Lemma 4.2 holds as well as $P_1$ and $P_2$.  

Remark: In light of the two different decompositions used in the proofs of Theorems 3.2 and 4.3 above, it would be interesting to find a bijection between $S_n(1324,1423)$ and $S_n(1342,1423)$ that preserves the descent and first letter statistics.

We close this section with some further discussion concerning our method of solution in this case. While we were unable to solve directly the system of functional equations in Lemma 4.2 for $D$ and the $D_i$, we thought it might be possible to find simpler auxiliary relations between the functions. Since $D(x, v, 0)$ is the generating function that is sought in the end in this case, the assumption $P_1$ is based on the conjecture itself with the desired formula already having been established in previous cases. As for the assumption $P_2$, it was convenient to have a simpler relation involving $D_1$ than that given in Lemma 4.2. After
some experimentation, we observed that the ratio \( \frac{D_1(x,1,w)}{D_2(x,1,w)} \) did not appear to depend on \( w \). Taking an ansatz that this ratio is rational in \( t(x) \), where \( t(x) \) is given by (2), we sought polynomials \( P(x, q) \), \( Q(x, q) \) and \( R(x, q) \) such that
\[
P(x, q) \frac{D_1(x,1,w)}{D_2(x,1,w)} = Q(x, q) + R(x, q) t(x).
\]

Assuming that the polynomials all have degree in \( x \) of at most \( d \) for some \( d > 0 \), we can determine their coefficients using the relation above and the first \( 3d + 3 \) terms of \( \frac{D_1(x,1,w)}{D_2(x,1,w)} \). One can subsequently increase the maximum allowed degree \( d \), observe if any of \( P, Q \) and \( R \) change and then cease the procedure once no changes have been observed after several increments. One thus obtains the statement of assumption \( P_2 \) as given. Then we can solve the simpler system for \( D(x, 1, w) \) and the \( D_t(x, 1, w) \), as detailed above, using the needed formula for \( D(x, v, 0) \) from \( P_1 \) and the relation from \( P_2 \) as the third equation involving the \( D_t(x, 1, w) \). This leads to formulas for \( D(x, v, w) \) and the \( D_t(x, v, w) \), which may subsequently be confirmed to satisfy the system in Lemma 4.2. By the uniqueness of power series solutions to this system and since each of the generating functions determined is analytic at \( x = 0 \), the solution that is found is the one sought. Note that uniqueness of the power series solutions follows from the fact that the equations in the system in Lemma 4.2 were themselves derived from a system of recurrences that uniquely determine the associated arrays (and would yield those same arrays recursively when computing the coefficients using these equations).

5. The remaining cases

In this section, we compute the joint distribution of the first letter and descents statistics in the remaining cases. To do so, we consider a common approach based on consideration of the second letter. Given \( n \geq 2 \) and \( i, j \in [n] \) with \( i \neq j \), let \( S_{n,i,j}(\sigma, \tau) \) denote the subset of \( S_n(\sigma, \tau) \) whose members start \( i, j \). Let \( a_n(i, j) = a_n(i, j; q) \) be defined by
\[
a_n(i, j) = \sum_{\pi \in S_{n,i,j}(\sigma, \tau)} q^{\text{desc}(\pi)}, \quad n \geq 2,
\]
for the particular pattern pair \((\sigma, \tau)\) under consideration. Given \( n \geq 2 \) and \( 1 \leq i \leq n \), let \( a_n(i) = \sum_{j \neq i} a_n(i, j) \), with \( a_1(1) = 1 \). Further, let \( a_n = \sum_{i=1}^{n} a_n(i) \) for \( n \geq 1 \). Note that \( a_n \) is itself a \( q \)-generalization of the Schröder numbers.

We now determine the distribution \( a_n(i, j) \) in the cases where
\[
(\sigma, \tau) = (1324, 1342), (1243, 1423), (1243, 1342), (1243, 1324).
\]
Note that for all the pattern pairs under consideration, we have
\[
a_n(i, j) = qa_{n-1}(j), \quad 1 \leq j < i \leq n.
\] (28)
So the task remains to write a recurrence for \( a_n(i, j) \) when \( j \geq i \). We observe the following obvious initial conditions for \( 1 \leq n \leq 3 \), which apply to all of the pattern pairs:
\[
a_1 = a_1(1) = 1, \quad a_2(1, 2) = 1, \quad a_2(2, 1) = q, \quad a_3(1, 2) = 1, \quad a_3(1, 3) = a_3(2, 1) = a_3(2, 3) = a_3(3, 1) = q \quad \text{and} \quad a_3(3, 2) = q^2.
\]
In all the cases, it is convenient to convert the recurrences to generating functions as follows. Define \( A(x, v, w) = A(x, v, w; q) \) by

\[
A(x, v, w) = \sum_{n \geq 2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_n(i, j) v^i w^j x^n,
\]

with

\[
A^+(x, v, w) = \sum_{n \geq 2} \sum_{i=1}^{n-1} \sum_{j=1}^{n} a_n(i, j) v^i w^j x^n \quad \text{and}
\]

\[
A^-(x, v, w) = \sum_{n \geq 2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} a_n(i, j) v^i w^j x^n.
\]

Note that \( A(x, v, w) = A^+(x, v, w) + A^-(x, v, w) \), by the definitions. Further auxiliary generating functions that are case dependent will be defined below.

### 5.1. The case \((1324, 1342)\)

The \( a_n(i, j) \) when \( i < j \) are given recursively for the pattern pair \((1324, 1342)\) as follows.

**Lemma 5.1:** We have

\[
a_n(i, n) = qa_{n-1}(i, n-1) + q \sum_{j=1}^{i} a_{n-2}(j), \quad 1 \leq i \leq n - 3, \quad (29)
\]

and

\[
a_n(n-2, n) = q^2a_{n-3} + q \sum_{j=1}^{n-3} a_{n-2}(j), \quad n \geq 4, \quad (30)
\]

with \( a_n(i, i+1) = a_{n-1}(i) \) for \( 1 \leq i \leq n - 1 \) and \( a_n(i, j) = 0 \) for \( 2 \leq i + 1 < j < n \).

**Proof:** Let \( \mathcal{A}_{n,i,j} = S_{n,i,j}(1324, 1342) \) and \( \mathcal{A}_{n,i} = S_{n,i}(1324, 1342) \). Note that the formula for \( a_n(i, i+1) \) when \( i < n \) and for \( a_n(i, j) \) when \( i + 1 < j < n \) may be verified using the definitions. To show (29), suppose \( \pi \in \mathcal{A}_{n,i,n} \), where \( 1 \leq i \leq n - 3 \), and let \( k \) denote the third letter of \( \pi \). Then we must have \( k = n-1 \), \( k = i+1 \) or \( k < i \), for \( i < k < n-1 \) would ensure an occurrence of 1324 or 1342. If \( k = n-1 \), then we may remove \( n \) resulting in a member of \( \mathcal{A}_{n-1,i,n-1} \), which accounts for the first term on the right side of (29). Note that the factor of \( q \) accounts for the descent occurring between \( n \) and \( n-1 \). If \( k < i \) or \( k = i+1 \), then both \( i \) and \( n \) may be deleted resulting in a member of \( \cup_{j=1}^{i} \mathcal{A}_{n-2,j} \). This is accounted for by \( q \sum_{j=1}^{i} a_{n-2}(j) \), the factor of \( q \) arising due to the descent occurring between the second and third letters within the enumerated permutations. Combining this case with the previous yields (29). To show (30), consider separately the cases when \( k = n-1 \) or \( k \leq n - 3 \). If \( k = n-1 \), then the first three letters \( n-2, n, n-1 \) may be deleted, resulting in \( \sigma \in S_{n-3}(1324, 1342) \) with no restrictions. This yields the \( q^2a_{n-3} \) term on the right side,
as there is a descent between \(n\) and \(n-1\) and also between \(n-1\) and the first letter of \(\sigma\) (note \(n \geq 4\) implies \(\sigma\) is non-empty). If \(k \leq n - 3\), then the first two letters may be deleted, which accounts for \(q \sum_{k=1}^{n-3} a_{n-3}(k)\) and completes the proof of (30).

By Lemma 5.1, one has for example when \(n = 4\) the following array:

\[
\begin{align*}
a_4(1, 2) &= 1 + q \quad a_4(1, 3) = 0 \quad a_4(1, 4) = q + q^2 \\
a_4(2, 1) &= q + q^2 \quad a_4(2, 3) = 2q \quad a_4(2, 4) = q + q^2 \\
a_4(3, 1) &= q + q^2 \quad a_4(3, 2) = 2q^2 \quad a_4(3, 4) = q + q^2 \\
a_4(4, 1) &= q + q^2 \quad a_4(4, 2) = 2q^2 \quad a_4(4, 3) = q^2 + q^3,
\end{align*}
\]

which may be verified using the definitions.

We now express the recurrences in Lemma 5.1 in terms of \(A(x, v, w)\) and \(A^\pm(x, v, w)\) as given above. Further, in this case, it is convenient to define the additional generating functions

\[
C(x, v) = \sum_{n \geq 2} \sum_{i=1}^{n-1} a_n(i, i + 1)v^i x^n, \quad D(x, v) = \sum_{n \geq 2} \sum_{i=1}^{n-1} a_n(i, n)v^i x^n.
\]

Then Lemma 5.1, together with (28), leads to the following system of functional equations:

\[
\begin{align*}
A^-(x, v, w) &= v^2 w qx^2 + \frac{v qx}{1 - v} (A(x, vw, 1) - v A(vx, w, 1)), \\
C(x, v) &= vx^2 + x A(x, v, 1), \\
D(x, v) &= qx D(x, v) - vq^2 x^3 A(vx, 1, 1) - v^2 q^2 x^4 + \frac{q x^2}{1 - v} (A(x, v, 1) - A(vx, 1, 1)) \\
&\quad + q x^2 A(vx, 1, 1) + v q x^2 (A(vx, 1, 1) + vx) + vx^2, \\
A^+(x, v, w) &= w C(x, vw) + D(wx, v) - v w^2 q x^2 A(vwx, 1, 1) - v w^2 x^2 - v w^2 w x^3.
\end{align*}
\]

Eliminating \(C(x, v)\) and \(D(x, v)\) from the preceding system, and using the fact \(A(x, v, w) = A^+(x, v, w) + A^-(x, v, w)\), implies that \(A(x, v, w)\) satisfies

\[
A(x, v, w) = \left( wx + \frac{v qx}{1 - v} \right) A(x, vw, 1) - \frac{v^2 q x}{1 - v} A(vx, w, 1)
+ \frac{v w^2 q x^2}{(1 - v)(wqx - 1)} A(vwx, 1, 1)
- \frac{w^2 q x^2}{(1 - v)(wqx - 1)} A(wx, v, 1) + \frac{(v w q x^2 - q x - w)v w x^2}{w q x - 1}. \tag{31}
\]

Taking \(w = 1\) in (31), and replacing \(x\) by \(x/v\), we obtain

\[
\begin{align*}
\left( 1 - \frac{x}{v} - \frac{q x}{1 - v} + \frac{q x^2}{v(1 - v)(qx - v)} \right) A(x/v, v, 1)
= \left( \frac{q x^2}{v(1 - v)(qx - v)} \right) A(x, 1, 1).
\tag{32}
\end{align*}
\]
Taking $v = v_0 = \frac{1 + x + t(x)}{2}$ in (32), where $t(x)$ is given by (2), then cancels out the left-hand side of (32) and leads to

$$A(x, 1, 1) = \frac{-1 + (1 + 2q)x + 2q(1 - q)x^2 + t(x)}{2q(qx - x - 1)}.$$  

Thus, by (32), we obtain an explicit formula for $A(x, v, 1)$. Substituting the expressions for $A(x, v, 1)$ and $A(x, 1, 1)$ into (31) then yields the following result.

Theorem 5.2: The generating function $A(x, v, w)$ which enumerates members of $S_n(1324, 1342)$ for $n \geq 2$ according to the joint distribution of the first and second letter statistics and the number of descents is given by (31), where

$$A(x, v, 1) = \frac{vx(vqx - v - x)t(vx)}{2(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)(vqx - vx - 1)} + \frac{(x + (q^2 + q + 3x^2 - 2x - 1)v)vx}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} - \frac{(2q^2x^2 + 3q^2x - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(q - 1)vq)x^3}{2(vqx - vx - 1)(vqx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} + \frac{v^2qx^2}{1 - vqx}.$$  

Taking $w = 1$ in Theorem 5.2, and comparing results with $U(x, v, 1)$ above, yields the following equality, which implies the desired equivalence of distributions.

Theorem 5.3: We have

$$A(x, v, 1) + vx = U(x, v, 1) + \frac{vx}{1 - vqx}.$$  

5.2. The case $(1243, 1423)$

In this and the next two subsections, it is demonstrated that $A(x, v, 1)$ is given by the same formula as in Theorem 5.2 above for each pattern pair under consideration. Note however that the distribution of the second letter statistic will be seen to be distinct in each case. For $(1243, 1423)$ (and the subsequent two cases), we extend the recurrence for $a_n(i, j)$ when $i < j$ found in [12] by introducing a further variable $q$ which tracks the number of descents.

This yields the following recurrence for $a_n(i, j)$ when $i < j$.

Lemma 5.4: We have

$$a_n(i, i + 1) = a_{n-1}(i, i + 1) + \sum_{a=1}^{i-1} \sum_{c=0}^{i-a-1} \sum_{b=a+1}^{i-c} q^{c+1} \times \binom{i - a - 1}{c} a_{n-c-2}(a, b), \quad 1 \leq i \leq n - 2, \quad (33)$$
with \( a_n(n - 1, n) = qa_{n-2} \),

\[
a_n(i, i + 2) = qa_{n-1}(i, i + 1) + a_{n-1}(i, i + 2)
+ \sum_{a=1}^{i-1} \sum_{c=0}^{i-a-1} \sum_{b=a+1}^{i-c+1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, b), \quad 1 \leq i \leq n - 3,
\]

(34)

with \( a_n(n - 2, n) = qa_{n-2} \), and

\[
a_n(i, j) = qa_{n-1}(i, j - 1) + \sum_{a=1}^{i-1} \sum_{c=0}^{i-a-1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, j - c - 2)
+ (1 - \delta_{j,n}) \cdot \left( a_{n-1}(i, j) + \sum_{a=1}^{i-1} \sum_{c=0}^{i-a-1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, j - c - 1) \right)
\]

(35)

for \( 4 \leq i + 3 \leq j \leq n \).

Define in this case

\[
C(x, v) = \sum_{n \geq 2} \sum_{i=1}^{n-1} a_n(i, i + 1) v^i x^n, \quad D(x, v) = \sum_{n \geq 3} \sum_{i=1}^{n-2} a_n(i, i + 2) v^i x^n.
\]

Then Lemma 5.4, together with (28), yields the following system:

\[
A^-(x, v, w) = qv^2 wx^2 + \frac{qv}{1-v} A(x, vw, 1) - \frac{qv^2 x}{1-v} A(vx, w, 1),
\]

(36)

\[
C(x, vw) = qvw x^2 A(vwx, 1, 1) + vw x^2 + qv^2 w^2 x^3 + xC(x, vw)
+ \frac{qx^2}{qvw + vw - 1} \left( vw A^+ \left( \frac{vw}{1 - qvw}, 1 - qvw, 1 \right) \right)
- (1 - qvw)A^+ \left( x, 1 - qvw, \frac{vw}{1 - qvw} \right),
\]

(37)

\[
D(x, vw) = qx(1-x)C(x, vw) + xD(x, vw)
+ \frac{qx^2}{vw(qvw + vw - 1)} \left( v^2 w^2 A^+ \left( \frac{vw}{1 - qvw}, 1 - qvw, 1 \right) \right)
- (1 - qvw)^2 A^+ \left( x, 1 - qvw, \frac{vw}{1 - qvw} \right),
\]

(38)

\[
A^+(x, v, w) = w(1-qwx)(1-x)C(x, vw) + w^2 (1-x)D(x, vw) + (1 + qw)x A^+(x, v, w)
+ \frac{qw^2 x^2}{qvw + v - 1} \left( (1 - qvw)A^+ \left( x, 1 - qvw, \frac{vw}{1 - qvw} \right) \right)
- vA^+(x, v, w)
\]
\begin{align*}
&+ \frac{qw^2}{v(qvw + v - 1)} \left( (1 - qvw)^2 A^+ (x, 1 - qvw, \frac{vw}{1 - qvw}) \right) \\
&- v^2 A^+ (x, v, w) \right) .
\end{align*}

Note that by (37) and (38), one can obtain a formula for \( C(x, vw) \) and \( D(x, vw) \) in terms of \( A \) and \( A^+ \).

By programming, one may verify that the solution of the preceding system of functional equations is given by the following result.

**Theorem 5.5:** Let \( r(x) = \sqrt{1 - x}\sqrt{4(q - 1)v^2w^2x^2 - 4qvw - x + 1} \). Then the generating function \( A(x, v, w) \) is given by \( A^+ (x, v, w) + A^- (x, v, w) \), where

**A^+ (x, v, w)**

\[
A^+ (x, v, w) = \frac{qw^2(1 - w^2)(1 - x)r(x)t(vwx)}{4((q - 1)x + 1)((q - 1)vwx - 1)(q(q - 1)v^2w^2x^2 - qwx + (v - 1)(x - 1))} \\
\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)) \\
+ \frac{qw^2(w^2 - 1)((q^2 - q + 1)vwx^2 - vwx - 2q - x + 1)r(x)}{4((q - 1)x + 1)((q - 1)vwx - 1)(q(q - 1)v^2w^2x^2 - qwx + (v - 1)(x - 1))} \\
\times ((q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)) \\
+ \frac{(2q(q - 1)v^2w^2x^2 - 2qvw + (2^2 - 1)(x - 1)\alpha x)}{2q - 1} \\
\alpha \beta \\
A^-(x, v, w)
\]

\[
A^- (x, v, w) = \frac{(x(qvx + vx - v - 1) - x(q(q - 1)v^2w^2x^2 +q^2x - v^2 + vx - 1)w}{2((q - 1)vwx - 1)(q(q - 1)v^2w^2x^2 - qvx + (vx - 1)(w - 1))} \\
\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)) \\
+ \frac{(2vw - 3v^2 + 2w - 2 + ((v + 1)(2q + 1) - (2q^2 + v^2 + 2q + 2v + 1)w + v(v + 1)(q + 2)w^2 + v^2(2q - 1)w^3)x)q^2wx^2}{2((q - 1)vwx - 1)(q(q - 1)v^2w^2x^2 - qvx + (vx - 1)(w - 1))} \\
\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)) \\
+ \frac{((q + 1)(2q + 1) + 2q(v + 1)(q - 2)w + ((2q - 2q^2 - 1)v^2 + q(q + 2)v - 2q^2 + 2q - 1)w^2 + q^2(v + 1)(2q - 1)w^3)qv^2wx^2}{2((q - 1)vwx - 1)(q(q - 1)v^2w^2x^2 - qvx + (vx - 1)(w - 1))} \\
\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)) \\
\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)).
\]
\[
\begin{align*}
&\frac{(4q^3 - q^2 - 2q + 1 - (2q^2 - 2q + 1)(v + 1)w + q^2v(2q - 1)w^2}{-(q - 1)(2q^2 - 2q + 1)qvwxqv^4w^2x^3} \\
&\times (q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)),
\end{align*}
\]

with

\[
\alpha = 4v^2w^2 - 6v^3w^2 - 2v^2w + vw^2 + 2v + w - 2
\]
\begin{align*}
&+ (2 - 2v - (2q^2 - 6q - 4v + 2v + 1)w + v(6qv^2 + 8v^2 - 5q - 4v - 2)w^2 \\
&+ v^2(4qv^2 - 4qv - 2v^2 - 6q + 4v - 1)w^3)x + (2(v - 1)(q - 1) \\
&+ (8q(1 - q)v - 6qv^2 - 2v^2 + 3q + 1)w - v(2(2q + 1)(q - 1) - 10q^2 \\
&- (q - 1)(4v - 1))w^2 + 2q^2(5q - 4)w^3)vwx^2 + (4q^2v + 2(2q - 1))v^2 \\
&+ (4v + 1)(1 - 2q) - 2v(5q^2 - 3q - 1)w - 2v^2(2q - 1)(q - 1)w^2)qv^2w^3x^3 \\
&+ 2q^2v^4w^5(2q - 1)(q - 1)x^4
\end{align*}

and

\[
\beta = 8v((q - 1)x + 1)((q - 1)vwx - 1)(qvwx - vw - 1)(q(q - 1)vwx^2 - qx \\
+ (v - 1)(x - 1))(q(q - 1)vwx^2 - qx + (vw - 1)(x - 1)).
\]

Moreover, \(C(x, v)\) and \(D(x, v)\) can be found by Equations (37) and (38), respectively.

In particular, taking \(w = 1\) in the preceding theorem gives

\[
A(x, v, 1) = \frac{vx(vqx - v - x)t(vx)}{2(vq^2x^2 - vqx^2 - qx + vx - x + 1)(vqx - vx - 1)}
\]
\begin{align*}
&+ \frac{x + (q^2x + qx + 3x^2 - 2x - 1)vv}{(vq - vx - 1)(vq - 1)(vq^2x^2 - vq^2x^2 - qx + vx - v - x + 1)} \\
&- \frac{(2q^2x^2 + 3q^2x - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(q - 1)vq)vx^2}{2(vq - vx - 1)(vq - 1)(vq^2x^2 - vq^2x^2 - qx + vx - v - x + 1)} \\
&+ \frac{v^2q^2x^2}{1 - vq},
\end{align*}

which agrees with the comparable generating function formula found above in the case (1324, 1342).

**Remark:** In determining a formula for \(A^+(x, v, w)\), we assume as an ansatz that there exist polynomials \(P, Q, R, S, T \in \mathbb{Z}[x, v, w, q]\) such that \(PA^+ = Q + Rt(vwx) + Sr(x) + Tr(x)t(vwx)\), where \(r(x)\) is as defined above and \(t(x)\) is given by (2). Proceeding in a manner comparable to the proof of Theorem 2.2 above, one can determine \(P, Q, R, S, T\) from the first several terms of \(A^+(x, v, w)\). Further, \(A^-(x, v, w)\) can be found by using (36) and the formula for \(A(x, v, 1)\), which is already known from prior cases and conjectured to hold in this case as well. Once \(A^+(x, v, w)\) and \(A^-(x, v, w)\) are determined, one may obtain \(C(x, v)\)
and $D(x, v)$ by taking $w = 1$ in (37) and (38). Then one may verify that the formulas found in this way for $A^+(x, v, w), A^-(x, v, w), C(x, v)$ and $D(x, v)$ satisfy the system (36)–(39) and indeed represent the desired solution. A similar idea applies in the next two subsections, where one only needs to find $P, Q$ and $R$ as no term involving $r(x)$ will be required.

### 5.3. The case $(1243, 1342)$

We have in this case the following recurrence for $a_n(i, j)$ when $i < j$.

**Lemma 5.6:** If $1 \leq i < j \leq n - 1$, then

\[
    a_n(i, j) = q \sum_{k=i+1}^{j-1} a_{n-1}(i, k) + \sum_{a=1}^{i-1} \sum_{c=0}^{j-c-2} \sum_{b=a+1}^{j-c+1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, b)
\]

\[
    + \delta_{i+j} \left( a_{n-1}(i, i+1) + \sum_{a=1}^{i-1} \sum_{c=0}^{j-a-1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, i-c) \right),
\]

with $a_n(i, n) = \sum_{j=1}^{i-1} a_{n-1}(i, j) + q \sum_{j=i+1}^{n-1} a_{n-1}(i, j)$ for $1 \leq i \leq n - 1$.

Define $C(x, v) = \sum_{n \geq 3} \sum_{i=1}^{n-1} a_n(i, i+1)x^n$. Then Lemma 5.6, together with (28), yields the following system:

\[
    A^-(x, v, w) = qx^2w^2 + \frac{vwx}{1-v}A(x, vw, 1) - \frac{v^2xq}{1-v}A(vx, w, 1),
\]

\[
    A^+(x, v, w) = vw^2x^2 + wxA^-(wx, v, 1) + qwxA^+(wx, v, 1)
\]

\[
    + \frac{qwx}{1-w}(A^+(x, v, w) - A^+(wx, v, 1))
\]

\[
    + \frac{q^2w^2x^2}{(qwx + v - 1)(qwx + vw - 1)} \left( A^+ \left( \frac{vx}{1 - qvx}, 1 - qxv, 1 \right) - A^+(x, 1 - qxv, \frac{vx}{1 - qvx}) \right)
\]

\[
    + \frac{qvwx^2}{(qwx + v - 1)(1-w)}(A^+(wx, v, 1) - A^+(x, v, w))
\]

\[
    + wc(x, vw) - qv^2w^2x^3 - qvw^2x^2A(vwx, 1, 1)
\]

\[
    - \frac{qvwx^2}{qwx + vw - 1} \left( A^+ \left( \frac{vx}{1 - qvx}, 1 - qxv, 1 \right) - A^+(x, 1 - qxv, \frac{vx}{1 - qvx}) \right)
\]

\[
    + \frac{qvx^2}{qvx + v - 1} \left( A^+ \left( \frac{vx}{1 - qvx}, 1 - qxv, 1 \right) - A^+(x, 1 - qxv, \frac{v}{1 - qvx}) \right)
\]

\[
    C(x, v) = qx^2x^3 + qvx^2A(vx, 1, 1)
\]
Theorem 5.7: The generating function $A(x, v, w)$ is given by $A^+(x, v, w) + A^-(x, v, w)$, where

\[
A^+(x, v, w) = \frac{(4vw - 6v^2w + v - 1 + (1 + (-2qw + 3q - 4w - 5)v + (6 - 4qw - 2q + 10w)v^2 + 2w(2q - 1)v^3)wx)vwx^2}{2((q - 1)vwx^2 - (2q - 1)vwx + (2vw - 1)(x - 1))(q(q - 1)vwx^2 - qwx + (v - 1)(wx - 1))((q - 1)vwx - 1)}
\]
\[
+ \frac{(4 + (6q^2w - 2q^2 + 7q - 4w - 9)v - (2w + 1)(2q - 1)v^2 - (4q^3vw - 6q^2vw + 2qvw + 3q^2 - 2qv + q + v - 3)vwx)v^2w^4x^4}{2((q - 1)vwx^2 - (2q - 1)vwx + (2vw - 1)(x - 1))(q(q - 1)vwx^2 - qwx + (v - 1)(wx - 1))((q - 1)vwx - 1)}
\]
\[
+ \frac{q(2q - 1)(q - 1)v^4w^6x^6}{2((q - 1)vwx^2 - (2q - 1)vwx + (2vw - 1)(x - 1))(q(q - 1)vwx^2 - qwx + (v - 1)(wx - 1))((q - 1)vwx - 1)}
\]

\[
A^-(x, v, w) = \frac{(vw^2 - (qv^2w^2 + qvw^2 - v^2w + v - w + 1)x + (q^2vw^2 - vw + q - w + 1)v^2x^2}{2(q(q - 1)vwx^2 - qx + (vw - 1)(x - 1))((q - 1)vwx - 1)(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))}
\]
\[
+ \frac{(2vw - 3vw^2 + 2w - 2 + ((v + 1)(2q + 1) - ((2q + 1)v^2 + 2(q + v) + 1)w + v(v + 1)(q + 2)w^2 + v^2(2q - 1)w^3)x)v^2wx^2}{2(q(q - 1)vwx^2 - qx + (vw - 1)(x - 1))((q - 1)vwx - 1)}
\]
\[
+ \frac{((1 + q)(2q + 1) + 2(q,v + 1)(q - 2)w - (2q^2v^2 - q^2v - 2q^2 + 2q^2 - 2qv + v^2 - 2q + 1)w^2 + qw(v + 1)(2q - 1)w^3)v^3wx^4}{2(q(q - 1)vwx^2 - qx + (vw - 1)(x - 1))((q - 1)vwx - 1)}
\]
\[
+ \frac{(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))}{2(q(q - 1)vwx^2 - qx + (vw - 1)(x - 1))((q - 1)vwx - 1)}
\]

Note that by (41) we have a formula for $C(x, v)$ in terms of $A$ and $A^+$. By programming, one can show that the preceding system has the following solution.
There is the following recurrence for $a_n(i,j)$ when $i < j$.

**Lemma 5.8:** We have

$$a_n(i, i + 1) = a_{n-1}(i, i + 1) + \sum_{a=1}^{i-1} \sum_{b=a+1}^{i} \sum_{c=0}^{i-b} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, b),$$

$$1 \leq i \leq n - 2,$$  \hspace{1cm} (42)

and

$$a_n(i, j) = a_{n-1}(i, j) + \sum_{a=1}^{i-1} \sum_{c=0}^{i-a-1} q^{c+1} \binom{i-a-1}{c} a_{n-c-2}(a, j-c-1),$$

$$3 \leq i + 2 \leq j \leq n - 1,$$  \hspace{1cm} (43)

with $a_n(i, n) = \sum_{j=1}^{i-1} a_{n-1}(i, j) + q \sum_{j=i+1}^{n-1} a_{n-1}(i, j)$ for $1 \leq i \leq n - 1$.

Define $C(x, v) = \sum_{n \geq 1} \sum_{i=1}^{n-1} a_n(i, i + 1)vx$. Then Lemma 5.8, together with (28), yields the following system:

$$A^-(x, v, w) = qv^2wx^2 + \frac{qv}{1-v} A(x, vw, 1) - \frac{qv^2}{1-v} A(vx, w, 1),$$
\[ C(x, v) = qv^2x^3 + qvx^2A(vx, 1, 1) + xC(x, v) + vx^3 \]
\[ + \frac{qvx^2}{qv + v - 1} \left( vA^+ \left( x, \frac{vx}{1 - qvx}, 1 - qvx, 1 \right) \right) \]
\[ - (1 - qvx)A^+ \left( x, 1 - qvx, \frac{v}{1 - qvx} \right) \]

\[ A^+(x, v, w) = wxA^-(wx, v, 1) + qwxA^+(wx, v, 1) + vw^2x^2 - qv^2w^3x^3 \]
\[ - vw^2x^3 + wC(x, vw) \]
\[ - qvw^2x^2A(vwx, 1, 1) + xA^+(x, v, w) - wxC(x, vw) \]
\[ + \frac{qwx^2}{qvwx + v - 1} \left( (1 - qvwx)A^+ \left( x, 1 - qvwx, \frac{vw}{1 - qvwx} \right) \right) \]
\[ - vA^+(x, v, w) \]

(44)

Note that by (44) we have a formula for \( C(x, v) \) in terms of \( A \) and \( A^+ \).

By programming, one can show that the preceding system has the following solution.

**Theorem 5.9:** The generating function \( A(x, v, w) \) is given by \( A^+(x, v, w) + A^-(x, v, w) \), where

\[ A^+(x, v, w) \]
\[ = \frac{(2v^2w + v - 1 + w(v(q - v)(2w + 1) - 2v + 1)x - vw^2(2q(q - 1)vw + 2q - v - 1)x^2 + qv^2v^2(q - 1)x^3)vw^2x^2t(vwx)}{2((q - 1)vwx - 1)((q - 1)vwx^2 - (2q + 1)vwx + (2w - 1)(x - 1))} \]
\[ \times (q(v - 1)vw^2x^2 - qwx + (wx - 1)(v - 1)) \]
\[ + \frac{(4vw - 6v^2w + v - 1 + 2(3 - q)v^2 + 3qv - 5v + 1 + 2(2q^2 - 2qv - v^2 - q + 5v - 2)vwx)vw^2x^2}{2((q - 1)vwx - 1)((q - 1)vwx^2 - (2q + 1)vwx + (2w - 1)(x - 1))} \]
\[ \times (q(v - 1)vw^2x^2 - qwx + (wx - 1)(v - 1)) \]
\[ + \frac{(q(7 - 2q)v + (1 - 2q)v^2 - 9v + 4 + 2(3q^2 - 2qv + v - 2)vwx)vwx^2w^4x^4}{2((q - 1)vwx - 1)((q - 1)vwx^2 - (2q + 1)vwx + (2w - 1)(x - 1))} \]
\[ \times (q(v - 1)vw^2x^2 - qwx + (wx - 1)(v - 1)) \]
\[ + \frac{(2q(3q - 2q^2 - 1)vw - 3q^2 + 2qv - q - v + 3 + (2q - 1)(q - 1)vwx)v^3wx^5}{2((q - 1)vwx - 1)((q - 1)vwx^2 - (2q + 1)vwx + (2w - 1)(x - 1))} \]
\[ \times (q(v - 1)vw^2x^2 - qwx + (wx - 1)(v - 1)) \]

\[ A^-(x, v, w) \]
\[ (v^2 - (q^2v^2 + qvw^2 - v^2w + v - w + 1)x + v(q^2vw^2 - vw + q - w + 1)x^2 - q^2w(q - 1)x^3)qv^2wx^2t(vwx) \]
\[ = \frac{2(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))(q - 1)vwx - 1) \times (q(q - 1)vw^2 - qx + (vw - 1)(x - 1))}{2(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))(q - 1)vwx - 1) \times (q(q - 1)vw^2 - qx + (vw - 1)(x - 1))} \]
\[ (2vw - 3vw^2 + 2w - 2 + ((v + 1)(1 + 2q) - (2qvw^2 + v^2 + 2q + 2v + 1)w + v(v + 1)(2 + q)w^2 + v^2(2q - 1)w^2)x)qv^2wx^2 \]
\[ + \frac{((1 + q)(1 + 2q) + 2q(v + 1)(q - 2)w + (q^2v + 2q(1 - q)v^2 - 2q^2 + 2qv - v^2 + 2q - 1)w^2 + qv(v + 1)(2q - 1)w^3)q^3wx^4}{2(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))(q - 1)vwx - 1) \times (q(q - 1)vw^2 - qx + (vw - 1)(x - 1))} + \frac{(4q^3 - q^2 - 2q + 1 - (2q^2 - 2q + 1)(v + 1)w + q^2v(2q - 1)w^2 - (q - 1) \times (2q^2 - 2q + 1)qvwx)q^4w^2x^5}{2(q(q - 1)v^2wx^2 - qvx + (vx - 1)(w - 1))(q - 1)vwx - 1) \times (q(q - 1)vw^2 - qx + (vw - 1)(x - 1))} \]

Moreover, \( C(x, v) \) can be found by Equation (44).

Hence,
\[ A(x, v, 1) = \frac{vx(vqx - v - x)t(vx)}{2(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)(vx - vx - 1)} \]
\[ + \frac{(x + (qx^2 + qx + 3x^2 - 2x - 1)v)vx}{2(vq^2x - vx - 1)(vx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} \]
\[ - \frac{(2q^2x^2 + 3qx^2 - qx^2 - qx - 3q + 2x - 1 + (1 - 2q)(q - 1)vq^2x^3)}{2(vq^2x - vx - 1)(vx - 1)(vq^2x^2 - vqx^2 - qx + vx - v - x + 1)} \]
\[ + \frac{v^2qx^2}{1 - vqx}, \]

as in the prior cases.

**Note**

1. The Maple files containing the relevant calculations and programming can be found at [http://math.haifa.ac.il/toufik/enum2005.html](http://math.haifa.ac.il/toufik/enum2005.html).

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