Parametrized Complexity of Weak Odd Domination Problems

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Abstract

A weak odd dominated (WOD) set is a subset $B$ of vertices such that
$\exists C \subseteq V \setminus B, \forall v \in B, |N(v) \cap C| \equiv 1 \pmod{2}$. Given a graph $G$ of order $n$, $\kappa(G)$ denotes the size of the greatest WOD set, and $\kappa'(G)$ the size of the smallest non-WOD set. The maximum of $\kappa(G)$ and $n - \kappa'(G)$, denoted $\kappa_Q(G)$, has been originally introduced to characterize the threshold of graph-based quantum secret sharing schemes. The decision problems associated with $\kappa$, $\kappa'$ and $\kappa_Q$ have been proved to be NP-Complete.

In this paper, we prove that depending on the parametrization, the WOD problems are either FPT or equivalent to Oddset, and as a consequence, hard for $W[1]$. We also prove that the three WOD problems are hard for $W[1]$ even for bipartite graphs.

1 Introduction

Odd domination is a domination-type problem in which a set $C$ oddly dominates its odd neighbourhood $Odd(C) = \{u \in V, |N[u] \cap C| \equiv 1 \pmod{2}\}$.

The odd domination falls into the general framework of $(\sigma, \rho)$-domination [HKT99, Tel94]. The parametrized complexity of these problems have been studied, in particular in the parity cases [GKS09].

Weak odd domination is a variation of odd domination, which does not fall into the general framework of $(\sigma, \rho)$-domination, in which a Weak Odd Dominated (WOD) set is, given a graph $G = (V, E)$, a set $B \subseteq V$ such that there exists $C \subset V \setminus B$ with $B \subseteq Odd(C) := \{v \in V \setminus C, |N(v) \cap C| \equiv 1 \pmod{2}\}$, in other words, every vertex in $B$ has an odd number of neighbors in $C$. The Lemma 1 in [GJMP11] gives a good characterization of non-WOD sets: $B \subseteq V$ is not WOD if and only if $\exists C \subseteq B$ such that $|C| \equiv 1 \pmod{2}$ and $Odd(C) \subseteq B$. Since a subset of a WOD set is WOD and a superset of a non-WOD is non-WOD, we focus on the greatest WOD set and the smallest WOD set:

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Definition 1. Given a graph $G$, $\kappa(G) = \max_{B \text{ WOD}} |B| = \max_{C \subseteq V} |\text{Odd}(C)|$

Definition 2. Given a graph $G$, $\kappa'(G) = \min_{B \text{ not WOD}} |B| = \min_{C \subseteq V, |C| \equiv 1 \text{ (mod 2)}} |\text{Odd}(C)|$

These problems have been proved to be NP-complete even for bipartite graphs $[GJMP11]$. Several NP-Complete problems, like deciding whether a graph of order $n$ has a vertex cover of size at most $k$, have been proved to be fixed parameter tractable, i.e. they can be solved in time $f(k)n^O(1)$ for some computable function $f$. The parametrized complexity hierarchy $[DF99]: FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq XP$ is, roughly speaking, a way to distinguish the problems which are fixed parameter tractable (FPT) from those which are not.

In this paper, the parametrized complexity of weak odd domination problems is explored. The parametrized complexity of other variations of odd domination have been studied, in particular Oddset $[DFVW97]$ which is hard for $W[1]$, but up to now Oddset is not known to belong to any $W[i]$. We use Oddset as a reference problem to prove the hardness for $W[1]$ of weak odd domination problems.

The weak odd domination is strongly related to graph-based quantum secret sharing protocols, defined in $[MS08]$. These protocols are represented by graphs in which every vertex represents a player. It has been proved in $[GJMP11]$, that for a quantum secret sharing protocol based on a graph $G$ of order $n$, $\kappa_Q(G)$ defined as $\max(\kappa(G), n - \kappa'(G))$ is the minimal threshold such that any set of more than $\kappa_Q(G)$ players can recover the secret. Graphs with a small quantum threshold (i.e. $\kappa_Q(G) \leq 0.811n$ for a graph $G$ of order $n$) has been proved to exist using non constructive methods $[GJMP11]$. In fact, a random graph has a small $\kappa_Q$ high probability (see $[GJMP11]$ for details). Thus, deciding whether a graph has a small threshold is crucial for the generation of good graph-based quantum secret sharing protocols.

Section 2 is dedicated to the greatest WOD set problem. We prove a lower bound on $\kappa$ and then show that this implies that WOD Set Of Size At Least $k$ is FPT. On the other hand, we prove that WOD Set Of Size At Least $n - k$ is hard for $W[1]$ even for bipartite graphs by reduction from Oddset.

Section 3 follows a similar plan, we start with an upper bound on $\kappa'$, then prove using this bound that Non-WOD Set Of Size At Most $n - k$ is FPT, and finally that Non-WOD Set Of Size At Most $k$ is hard for $W[1]$ by reduction from WOD Set Of Size At Least $n - k$. Furthermore, we prove that the bipartite case is still hard for $W[1]$ by reduction from the general case.

In section 4 we prove that Quantum Threshold Of Size At Least $k$ is FPT and that Quantum Threshold Of Size At Least $n - k$ is hard for $W[1]$ even for bipartite graphs by reduction from Non-WOD Set Of Size At Most $k$. Finally, we show that all those problems are equivalent by proving a reduction from Oddset to Quantum Threshold Of Size At Least $n - k$. 
2 Greatest WOD set problem

First we consider the greatest WOD set problem. A direct parametrization of the greatest WOD set problem is:

**WOD Set Of Size At Least k**

input: A graph $G = (V, E)$ of order $n$
parameter: An integer $k$
question: Is there any $C \subseteq V$ such that $|\text{Odd}(C)| \geq k$? Or equivalently, is $\kappa(G) \geq k$?

First we show a lower bound on the greatest WOD set in order to produce an FPT algorithm that solves the problem.

**Lemma 1.** For any graph $G$ of order $n$ with no isolated vertex, $\kappa(G) \geq \sqrt{n}/2$.

*Proof.* We consider a greedy algorithm which inputs a graph $G$ with no isolated vertex and constructs a WOD set as follows: let $A$ be a set of vertices initialised as the empty set. The algorithm iteratively searches in $\text{Even}(A) = V \setminus (A \cup \text{Odd}(A))$ a vertex that has strictly more neighbours in $\text{Even}(A)$ than in $\text{Odd}(A)$, and add it to $A$, and so on.

The size of $\text{Odd}(A)$ increases at each step of the algorithm, so $|A \cup \text{Odd}(A)| \leq 2|\text{Odd}(A)|$. Moreover, at the end of the execution, every vertex in $\text{Even}(A)$ has more neighbours in $\text{Odd}(A)$ than in $\text{Even}(A)$. As a consequence, at the end of the algorithm, $|\text{Even}(A)| \leq \Delta |A \cup \text{Odd}(A)| \leq 2|\text{Odd}(A)|\Delta$ where $\Delta$ is the degree of the graph.

Since $n = |A| + |\text{Odd}(A)| + |\text{Even}(A)| = |A \cup \text{Odd}(A)| + |\text{Even}(A)|$, $n \leq 2|\text{Odd}(A)| + 2|\text{Odd}(A)|\Delta$, so $|\text{Odd}(A)| \geq \frac{n}{2(1+\Delta)}$. There are two cases: either $
 \Delta \geq \frac{\sqrt{n}}{2}$ which implies, since $\kappa(G) \geq \Delta$ \[\text{GIMPI}\], that $\kappa(G) \geq \frac{\sqrt{n}}{2}$; or $
 \Delta < \frac{\sqrt{n}}{2}$ which implies that $n > 4$ (since $\Delta \geq 1$), and as a consequence $|\text{Odd}(A)| \geq \frac{n}{2+\sqrt{n}} \geq \frac{\sqrt{n}}{2}$. So, in any case, $\kappa(G) \geq \frac{\sqrt{n}}{2}$. 

**Theorem 1.** WOD Set Of Size At Least k is FPT.

*Proof.* Let $A$ be an algorithm which inputs a graph $G$ of order $n$ and a parameter $k$. $A$ computes first, $G'$ the subgraph of $G$ obtained by removing all the isolated vertices, then $n'$ the order of $G'$ and finally:

- outputs true, when $k \leq \frac{\sqrt{n'}}{2}$,
- computes $\kappa(G') \geq k$, when $k > \frac{\sqrt{n'}}{2}$,

$A$ is an FPT algorithm that computes WOD Set Of Size At Least k since:

- Removing all isolated vertices from $G$ is done in polynomial time in $n$, and does not change the value of $\kappa$ since isolated vertices have no neighbour.
- When $k \leq \frac{\sqrt{n'}}{2}$ by Lemma I $\kappa(G) \geq k$. 

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• When $k > \sqrt{n'}$, $n' < 4k^2$. Since WOD Set Of Size At Least $k$ is in NP, this problem can be computed in time less than $f(n')$ for some increasing function $f$, so the time complexity is upperbounded by $f(4k^2)$.

Notice that WOD Set Of Size At Least $k$ is FPT only because the size of the greatest WOD set is bounded by a function of $n$. Thus, this is natural to consider the same problem but with a different parameterisation:

**WOD Set Of Size At Least $n - k$**

**Input**: A graph $G$ of order $n$

**Parameter**: An integer $k$

**Question**: Is $\kappa(G) \geq n - k$?

In order to prove that WOD Set Of Size At Least $n - k$ is $W[1]$-hard, we prove that it is harder than Oddset by an FPT reduction, since Oddset is known to be $W[1]$-hard [DFVW97].

**Theorem 2.** WOD Set Of Size At Least $n - k$ is harder than Oddset by an FPT-reduction.

**Proof.** For an instance $(G, k)$ of Oddset let $(G', k')$ be an instance of WOD Set Of Size At Least $n' - k'$ such that $G' = (A \cup D \cup E \cup c, E_1 \cup E_2 \cup E_3)$ and $k' = k + 1$ where:

- $A = \{a_u, \forall u \in R\}$
- $D = \{d_u, \forall u \in B, \forall 1 \leq i \leq k + 2\}$
- $F = \{f_i, \forall 1 \leq i \leq k + 2\}$

- $E_1 = \{c_i, \forall 1 \leq i \leq k + 2\}$
- $E_2 = \{c_u, \forall u \in R\}$
- $E_3 = \{f_i, \forall 1 \leq i \leq k + 2\}$

- If $\kappa(G') \geq n' - k'$, there exists a set $C$ of vertices in $G'$ such that $|Odd(C)| \geq n' - k'$. Since $F$ is an independent set of size greater than $k'$, $F$ has to be dominated by $C$, so $c \in C$. For every $u \in B$, the subset $D_u = \{d_u, \forall 1 \leq i \leq k + 2\}$ of $D$ is an independent set of size greater than $k'$, thus $D_u \subseteq Odd(C \cap A)$. Since $|Odd(C \cap A)| \geq n - k$, $|C \cap A| \leq k$. As a consequence $R' := \{u \in R, a_u \in C\}$ is a subset of $R$ such that $B \subseteq Odd(R')$ in $G$, and which size is smaller than $k$, so $(G, k)$ is a positive instance of Oddset.

- If $(G, k)$ is a positive instance of Oddset, then $\exists R' \subseteq R$, such that $|R'| \leq k$ and $B = Odd(R')$. Let $A' = \{a_u, u \in R'\}$, $|A' \cup \{c\}| \leq k + 1 = k'$ and $F \cup A \cup D \subseteq Odd(A')$, so $Odd(A \cup \{c\})$ is WOD Set Of Size At Least $n' - k'$.

\hfill \Box
Corollary 1. WOD Set Of Size At Least $n - k$ is hard for $W[1]$.

Notice that the graph used for the proof of theorem 2 (see figure 1) is bipartite so WOD Set Of Size At Least $n - k$ is hard for $W[1]$ even for bipartite graphs.

3 Smallest non-WOD set problem

In this section we consider the smallest non-WOD set problem. We show that the parametrized complexity depends on the choice of the parameter, like in the greatest WOD set problem, Non-WOD Set Of Size At Most $n - k$ is FPT 3 whereas Non-WOD Set Of Size At Most $k$ is hard for $W[1]$ 2

**Non-WOD Set Of Size At Most $n - k$**

- **input:** A graph $G = (V, E)$ of order $n$
- **parameter:** An integer $k$
- **question:** Is there any $C \subseteq V$ such that $|C| \equiv 1 \pmod{2}$ and $|C \cup \text{Odd}(C)| \leq n - k$? Or equivalently is $\kappa'(G) \leq n - k$?

The proof that the above problem is FPT is an algorithm mainly based on an upperbound on the size of non-WOD sets, and on a refinement of Theorem 1 [GJMP11] which states that for any graph $G$ of order $n$, $\kappa'(G) + \kappa(G) \geq n$, we prove that when $G$ has at least a universal vertex, $\kappa'(G) + \kappa(G) = n$.

**Lemma 2.** For every graph $G$ of order $n$ with an universal vertex, $\kappa'(G) + \kappa(G) = n$.

**Proof.** Let $C$ be the greatest WOD set of $\overline{G}$, so there is a set $D$ such that $C = \text{Odd}(D) \Rightarrow |\text{Odd}(D)| = \kappa(\overline{G})$. There are two cases: if $|D| \equiv 1 \pmod{2}$, since $v \in \text{Odd}(D)$ in $\overline{G}$ if and only if $v \notin \text{Odd}(D)$ in $G$, so $\text{Even}(D)$ in $\overline{G}$ is $\text{Odd}(D)$ in $G$, thus $D \cup \text{Odd}(D)$ is a non-WOD set in $G$. And since $n = |D| + |\text{Odd}(D)| + |\text{Even}(D)|$, then $\kappa'(G) + \kappa(G) \leq n$. The Theorem 1 of [GJMP11] shows that $\kappa'(G) + \kappa(G) \geq n$ so $\kappa'(G) + \kappa(G) = n$. If $|D| \equiv 0$
(mod 2), then let v be the universal vertex in G, v is isolated in $\overline{G}$, so $D' = D \cup v$ is of an odd size and have $|\text{Odd}(D)| = \kappa(\overline{G})$, thus the first case is applicable. □

**Lemma 3.** For every graph G of order n with no universal vertex, $\kappa'(G) \leq n - \sqrt[4]{n}$.

**Proof.** According to Lemma 1 in [GJMP11], there is a non-WOD set of size at most $n - k$ if and only if there exists a set $D$ of vertices such that $|D \cup \text{Odd}(D)| \leq n - k$ and $|D| \equiv 1 \pmod{2}$. As a consequence, notice that $\kappa'(G) \leq n - \sqrt[4]{n}$ if there exists a subset $D$ such that $|D| \equiv 1 \pmod{2}$ and $|\text{Odd}(D)| \geq \sqrt[4]{n}$ in $\overline{G}$. Indeed, when $|D| \equiv 1 \pmod{2}$, $v \in \text{Odd}(D)$ in $\overline{G}$ if and only if $v \notin \text{Odd}(D)$ in G. In the rest of the proof, we prove the existence of such a set $D$ using the lemma [1]. Since G has no universal vertex, $\overline{G}$ has no isolated vertex, so $\kappa'(G) \geq \sqrt[4]{n}$. Thus there exists a non empty set $C$ such that $|\text{Odd}(C)| \geq \sqrt[4]{n}$ in $\overline{G}$. If $|C| \equiv 1 \pmod{2}$ then, according to the previous remark, $\kappa'(G) \leq n - \sqrt[4]{n}$. Otherwise, let $v$ be any vertex in $C$. If $|N(v)| \geq \sqrt[4]{n}$ in $\overline{G}$ let $D := v$, otherwise $|\text{Odd}(C \setminus v)| \geq \sqrt[4]{n} - |N(v)| \geq \sqrt[4]{n}$ in $\overline{G}$ and let $D := C \setminus \{v\}$. In both cases, $|D| \equiv 1 \pmod{2}$ and $|\text{Odd}(D)| \geq \sqrt[4]{n}$ in $\overline{G}$, so $\kappa'(G) \leq n - \sqrt[4]{n}$. □

**Theorem 3.** Non-WOD Set Of Size At Most $n - k$ is FPT.

**Proof.** Let A be an algorithm which inputs a graph $G$ of order n and a parameter $k$, and :

- outputs true if $G$ has no universal vertex and $k \leq \sqrt[4]{n}$.
- computes $\kappa'(G) \leq n - k$ if $G$ has no universal vertex and $k > \sqrt[4]{n}$.
- computes $\kappa(\overline{G}) \geq k$ if $G$ has at least one universal vertex.

A is an FPT algorithm that computes WOD Set Of Size At Least $k$ since:

- If $G$ has no universal vertex, then by Lemma 3 if $k \leq \sqrt[4]{n}$, $\kappa'(G) \leq n - k$. Moreover, if $k > \sqrt[4]{n}$ then $n < 16k^2$, so since Non-WOD Set Of Size At Most $n - k$ is in NP, this problem can be computed in time less than $f(n)$ for some increasing function $f$, so the time complexity is upperbounded by $f(16k^2)$.
- If $G$ has at least a universal vertex, then according to lemma [2] $\kappa'(G) \leq n - k$ if and only if $\kappa(\overline{G}) \geq k$.

□

Now we consider the other parametrisation of the problem, which is shown to be hard for $W[1]$ by reduction from WOD Set Of Size At Least $n - k$.

**Non-WOD Set Of Size At Most $k$**

input: A graph $G$

parameter: An integer $k$

question: Is $\kappa'(G) \leq k$?
**Theorem 4.** Non-WOD Set Of Size At Most k is harder than WOD Set Of Size At Least n – k by an FPT-reduction.

![Figure 2: Reduction from WOD Set Of Size At Least n – k to Non-WOD Set Of Size At Most k](image)

**Proof.** Given an instance \((G, k)\) of WOD Set Of Size At Least \(n – k\), let \((G', k')\) be an instance of Non-WOD Set Of Size At Most \(k'\) such that \(G' = (V \cup A \cup c, E \cup E_1 \cup E_2)\) and \(k' = k + 2\) where:

\[
A = \{a_i, \forall 1 \leq i \leq k \leq k + 3\} \\
E_1 = \{ua_i, \forall u \in V, \forall 1 \leq i \leq k \leq k + 3\}, \\
E_2 = \{a_i, c, \forall 1 \leq i \leq k \leq k + 3\}
\]

- If \((G, k)\) is a positive instance, there exists a subset \(C\) of vertices such that \(|Odd(C)| \geq n – k\). There are two cases: if \(|C| \equiv 0 (\text{mod} 2)\) then let \(C' = C \cup \{a\}\), where \(a\) is any vertex in \(A\). In this case \(A \subseteq Even(C')\), \(c \in Odd(C')\), and for any \(v \in Odd(C)\), \(v \notin Odd(C')\), so \(|C' \cap Odd(C')| \leq k + 2\). Otherwise, if \(|C| \equiv 1 (\text{mod} 2)\), let \(C' = C \cup \{a, c\}\), where \(a\) is any vertex in \(A\). \(c\) is added to keep \(A \subseteq Even(C')\), the rest of the solution does not change so \(|Odd(C') \cup C'| \leq k + 2\).

- If \((G', k')\) is a positive instance of Non-WOD Set Of Size At Most k, then there exists \(C'\) such that \(|C' \cap Odd(C')| \leq n – k\). So there are two cases: there is one or no vertex of \(C'\) in \(A\) since \(C'\) is of an odd size. If there is none then \(A\) belongs to \(Odd(C')\), since it is connected to all other vertices of \(G'\), so this case has no solutions since \(|A| > k'\). So there is one vertex of \(A\) in \(C'\), thus \(G\) is in \(Odd(C')\) like \(c\). Since \(|C' \cap Odd(C')| \leq k\) and \(a \in C'\) with \(a\) any vertex of \(A\), then \(|Odd(C' \cap V)| \geq n – k\) which means that \((G, k)\) is a positive instance of WOD Set Of Size At Least \(n – k\).

\(\square\)

**Corollary 2.** Non-WOD Set Of Size At Most k is hard for W[1].

The proof of the W[1]-hardness of Non-WOD Set Of Size At Most k does not respect the bipartition of the graph, however we prove that the problem is W[1] even for bipartite:
Figure 3: Reduction from NON-WOD Set Of Size At Most $k$ to NON-WOD Set Of Size At Most $k$, $G$ bipartite

**Theorem 5.** NON-WOD Set Of Size At Most $k$ is hard for $W[1]$ even for bipartite graphs.

**Proof.** Given an instance $(G, k)$ of NON-WOD Set Of Size At Most $k$ let $(G', k')$ be a bipartite instance with:

$G' = (A \cup B \cup D \cup F \cup H, E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5), k' = 2k$

$A = \{a_u, \forall u \in V\}$, $B_1 = \{b_{1,u}, \forall u \in V\}$, $B_2 = \{b_{2,u}, \forall u \in V\}$

$D = \{d_{i,u,j}, \forall i \in \{1, 2\}, \forall u \in V, \forall 1 \leq j \leq 2k + 1\}$

$F = \{f_{i,u,j,l}, \forall i \in \{1, 2\}, \forall u \in V, \forall 1 \leq j, l \leq 2k + 1\}$

$H = \{h_i, \forall 1 \leq i \leq 2k + 1\}$

$E_1 = \{a_{u}b_{i,v}, \forall i \in \{1, 2\}, \forall u \in V\}$, $E_2 = \{a_{u}b_{2,u}, \forall u \in V\}$

$E_3 = \{b_{i,u}d_{i,u,j}, \forall i \in \{1, 2\}, \forall u \in V, \forall 1 \leq j \leq 2k + 1\}$

$E_4 = \{d_{i,u,j}f_{i,u,j,l}, \forall i \in \{1, 2\}, \forall u \in V, \forall 1 \leq j, l \leq 2k + 1\}$

$E_5 = \{f_{i,u,j,l}h_p, \forall i \in \{1, 2\}, \forall u \in V, \forall 1 \leq j, l, p \leq 2k + 1\}$

- If $(G, k)$ is a positive instance, then $\exists C \subseteq V$ such that $|C \cup \text{Odd}(C)| \leq k$. So let $C' = \{a_u, \forall u \in C\}$ be the one in $G'$. Since $A$ is connected to $B_1$ by the same neighbourhood relation as in $G$, then $|\text{Odd}(C') \cap B_1| = |\text{Odd}(C)|$. And since $A$ is connected to $B_2$ by the same neighbourhood relation as in $G$ plus a matching, then $|\text{Odd}(C') \cap B_2| = |\text{Odd}(C)\Delta C|$. So the solution is of size $|C| + |\text{Odd}(C)| + |\text{Odd}(C)\Delta C| = 2|C \cup \text{Odd}(C)| \leq 2k$.

- If $(G', k')$ is a positive instance, then $\exists C' \subseteq V'$ such that $|C' \cup \text{Odd}(C')| \leq 2k$. So then let notice that $H$ and each components of $D$ and $F$ are greater than $k$, that taking an even number of vertices of $C'$ in one of those sets don’t change the problem, and that taking an odd number is equivalent to taking one. Therefore, no vertex of $H$ or $F$ can be in $C'$ or
it will be in the odd, the same for $F$ with $H$, for $D$ with $F$ and $B$ with $D$. So $C'$ is a subset of $A$ and as seen before it corresponds to a solution of size $k$ in $G$.

\[\square\]

4 Quantum Threshold problem

In this section we consider the quantum threshold problem. The quantum threshold $\kappa_Q(G)$ of a graph $G$ of order $n$ is defined as $\kappa_Q(G) = \max(\kappa(G), n - \kappa'(G))$. Like the greatest WOD set problem, the problem associated with $\kappa_Q$ admits a parametrization which is FPT due to a lower bound on the size of the quantum threshold that induce an FPT algorithm.

**Quantum Threshold Of Size At Least $k$**

**input:** A graph $G$

**parameter:** An integer $k$

**question:** Is $\kappa_Q(G) \geq k$?

**Theorem 6.** quantum threshold of size at least $k$ is FPT.

**Proof.** In [JMP12a] this is proved that for any graph $G$ of order $n$, that $\kappa_Q(G) \geq \frac{n}{2}$, using quantum information methods. So let $A$ be an algorithm which inputs $G$ a graph of order $n$ and $k$ a parameter, then outputs $true$ if $k \leq \frac{n}{2}$ or computes $\kappa_Q(G) \geq k$ if $k > \frac{n}{2}$. $A$ is an FPT algorithm which computes quantum threshold of size at least $k$ since if $k \leq \frac{n}{2}$ then $\kappa_Q(G) \geq k$, and if $k \leq \frac{n}{2}$, then $n < 2k$ and since quantum threshold of size at least $k$ is NP-complete in the general case by Theorem 3 of [GJMP11], it can be computed in time bounded $f(n) \leq f(2k)$, $f$ a crushing function.

The second parametrisation of this problem is proved to be hard for $W[1]$ by a reduction from non-WOD set of size at most $k$.

**Quantum Threshold Of Size At Least $n - k$**

**input:** A graph $G$ of order $n$

**parameter:** An integer $k$

**question:** Is $\kappa_Q(G) \geq n - k$?

**Theorem 7.** quantum threshold of size at least $n - k$ is harder than non-WOD set of size at most $k$ by an FPT-reduction.

**Proof.** Given $(G, k)$ an instance of non-WOD set of size at most $k$ let $(G^{k+1}, k)$ be an instance of quantum threshold of size at least $n - k$ where $G^p$ is the graph obtained by copying $p$ times $G$. Since $\kappa_Q(G) = \max(\kappa(G), n - \kappa'(G))$ by Lemma 6 of [GJMP11] then there are two possibilities for $\kappa_Q(G^{k+1}) \geq (k + 1)n - k$:  

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\( \kappa(G^{k+1}) \geq (k+1)n - k \iff (k+1)\kappa(G) \geq (k+1)n - k \) since all copies of the graph are independent so the best solution is the sum of the best solution in each copy. Thus \( \kappa(G) \geq n - \frac{k}{k+1} \), but \( \frac{k}{k+1} < 1 \) so \( \kappa(G) \geq n \).

On the other hand, for any graph \( G \) of order \( n \) we have \( \kappa(G) < n \) so there is a contradiction.

\( (k+1)n - k' \geq (k+1)n - k \iff k' \geq (k+1)n - k \iff k' \geq k \) since all copies are independent the best solution is the best in one copy.

\[ \square \]

**Corollary 3.** Quantum Threshold Of Size At Least \( n - k \) is hard for \( W[1] \).

Notice that the reduction keeps the bipartition, so Quantum Threshold Of Size At Least \( n - k \) is \( W[1] \)-hard even for bipartite graphs.

Now we have proved that all the problems related to weak odd domination are hard for \( W[1] \), by successive FPT-reductions starting from Oddset. These problems are not only harder than Oddset but equivalent to Oddset. In order to prove that, we only need to prove that Oddset is harder than Quantum Threshold Of Size At Least \( n - k \).

**Theorem 8.** Oddset is harder than Quantum Threshold Of Size At Least \( n - k \) by an FPT-reduction.

**Proof.** Given an instance \((G,k)\) of Quantum Threshold Of Size At Least \( n - k \), let \((G',k')\) be an instance of Oddset with:

\[
G' = (A \cup \{d_1\} \cup \{d_2\} \cup \{c\}, E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \{d_1c\} \cup \{d_2c\})
\]

\[
A = \bigcup A_{i,j}, \forall i \in \{1,2\}, \forall j \in \{1,2,3,4,5\}
\]

\[
A_{i,j} = \{a_{i,j,u}, \forall u \in V\}, \quad E_1 = \{a_{1,2,u}a_{1,1,v}, \forall j \in \{4,5\}, \forall uv \in E\}
\]

\[
E_2 = \{d_{2,2,u}d_{2,j,v}, \forall j \in \{4,5\}, \forall uv \notin E\}
\]

\[
E_3 = \{a_{i,j,u}a_{i,1,u}, \forall i \in \{1,2\}, \forall j \in \{1,3\}, \forall l \in \{4,5\}, \forall u \in V\}
\]

\[
E_4 = \{d_{i,j,u}d_{i,1,u}, \forall i \in \{1,2\}, \forall j \in \{4,5\}, \forall uv \in E\}
\]

\[
E_5 = \{a_{1,2,u}a_{1,5,u}, \forall i \in \{2\}, \forall u \in V\}, \quad E' = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \{d_1c\} \cup \{d_2c\}
\]

And with \( B = S_{1,4} \cup S_{1,5} \cup \{c\}, A = S \setminus B \) and \( k = 2k + 1 \).

• If \((G,k)\) is a positive instance of Quantum Threshold Of Size At Least \( n - k \), then since \( \kappa_Q(G) = \max(\kappa(G), \kappa(G)) \) there are two cases: either \( \kappa(G) \) or \( \kappa(G) \) is greater than \( n - k \). If \( \kappa(G) \geq n - k \), then \( \exists C \) such that \( |\text{Odd}(C)| \geq n - k \). So let \( C' \subseteq R \) be equal to \( \{a_{1,2,u}, \forall u \in C\} \cup \{a_{1,1,u}, \forall u \in \text{Even}(C)\} \cup \{a_{1,5,u}, \forall u \in C \Delta \text{Even}(C)\} \cup \{d_2\} \) where \( \Delta \) is the symmetric difference. Since \( d_2 \in C' \), then \( c, A_{2,4} \) and \( A_{2,5} \) are in Odd\((C')\).

\( A_{1,4} \subset \text{Odd}(C') \) since \( A_{1,4} \) is connected to \( A_{1,2} \) with the same neighbourhood as in \( G \), so \( \{a_{1,4,u}, \forall u \in \text{Odd}(C)\} \subset \text{Odd}(C') \), and connected to \( A_{1,1} \) by a matching, so \( \{a_{1,4,u}, \forall u \in \text{Even}(C)\} \subset \text{Odd}(C') \). \( A_{1,5} \subset \text{Odd}(C') \) since \( A_{1,5} \) is connected to \( A_{1,2} \) with the same neighbourhood as in \( G \) plus a matching, so \( \{a_{1,4,u}, \forall u \in \text{Odd}(C)\} \Delta C \subset \text{Odd}(C') \), and connected to \( A_{1,1} \) by a matching, so \( \{a_{1,4,u}, \forall u \in \text{Even}(C)\} \Delta C \subset \text{Odd}(C') \).
Figure 4: Reduction from Quantum Threshold Of Size At Least $n - k$ to Oddset

So $B \subseteq \text{Odd}(C')$, and $|C'| = |C| + |\text{Even}(C)| + |\text{Even}(C)\Delta C| + 1 = 2|\text{Even}(C) \cup C| + 1 \leq 2k + 1$, thus $(G', k')$ is a positive instance of Oddset. If $\kappa(G) \geq n - k$ then this is similar permits to conclude.

- If $(G', k')$ is a positive instance of Oddset, then $\exists C' \subseteq R$ such that $B \subseteq \text{Odd}(C')$. $c$ has to be dominated either by $d_1$ or $d_2$. If $d_2 \in C'$, then $A_{2,4}$ and $A_{2,5}$ are in $\text{Odd}(C')$. Since $A_{1,4}$ is connected to $A_{1,1}$ by a matching at least one vertex of $A_{1,2}$ is in $C'$, so let $C = \{u \mid a_{1,2,u} \in C'\}$ be a set of vertices in $V$. By the same argument as above, $|C'| = 2|\text{Even}(C) \cup C| + 1$ so $(G, k)$ is a positive instance of Quantum Threshold Of Size At Least $n - k$. If $d_1 \in C'$, this is similar but in $\overline{G}$.

\[ \square \]

**Corollary 4.** All the following problems: WOD Set Of Size At Least $n - k$, Non-WOD Set Of Size At Most $k$ and Quantum Threshold Of Size At Least $n - k$ even for bipartite graph are equivalent to Oddset in parametrized complexity.

## 5 Conclusion

In this paper we have explored the parametrized complexity of weak odd domination problems: greatest WOD set, smallest non-WOD set and quantum threshold, and their respective related quantities $\kappa$, $\kappa'$ and $\kappa_Q$.

We have proved bounds on these quantities. These bounds imply that for any of those problems, there is a parametrization which is FPT. However, if one considers a more natural parametrization of these problems, it comes that
all the variants of weak odd domination problems are hard for \( W[1] \) even for bipartite graph. More precisely we show that they are equivalent to ODDSET.

A similar problem, the minimal degree up to local complementation that is close to finding \( \kappa' \), but without the constraint on the parity of the dominating set, has been proved to be NP-complete \cite{JMP12b} by using information theory, so it would be interesting to study its parametrized complexity.

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**References**

\[DF99\] R.G. Downey and M.R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.

\[DFVW97\] R.G. Downey, M.R. Fellows, A. Vardy, and G. Whittle. The parameterized complexity of some fundamental problems in coding theory. *CDMTCS Research Report Series*, 052, 1997.

\[GJMP11\] S. Gravier, J. Javelle, M. Mhalla, and S. Perdrix. On weak odd domination and graph-based quantum secret sharing. *arXiv:1112.2495v2*, 2011.

\[GKS09\] P.A. Golovach, J. Kratochvil, and O. Suchy. Parameterized complexity of generalized domination problems. *Discrete Applied Mathematics*, 160(6):780–792, 2009.

\[HKT99\] M. Halldórsson, J. Kratochvil, and J. A. Telle. Mod-2 independence and domination in graphs. *Graph-Theoretic Concepts In Computer Science*, 1665:101–109, 1999.

\[JMP12a\] J. Javelle, M. Mhalla, and S. Perdrix. New protocols and lower bound for quantum secret sharing with graph states. *arXiv:1109.1487v1, to appear in TQC 2012 proceedings*, 2012.

\[JMP12b\] J. Javelle, M. Mhalla, and S. Perdrix. On the minimum degree up to local complementation: Bounds and complexity. *arXiv:1204.4564v1, to appear in WG 2012 proceedings*, 2012.

\[MS08\] D. Markham and B. C. Sanders. Graph states for quantum secret sharing. *Physical Review A*, 78(4), 2008.

\[Tel94\] J. A. Telle. Complexity of domination-type problems in graphs. *Nordic Journal of Computing*, 1(1):157–171, 1994.