Face Numbers of 4-Polytopes and 3-Spheres

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Abstract

Steinitz (1906) gave a remarkably simple and explicit description of the set of all $f$-vectors $f(P) = (f_0, f_1, f_2)$ of all 3-dimensional convex polytopes. His result also identifies the simple and the simplicial 3-dimensional polytopes as the only extreme cases. Moreover, it can be extended to strongly regular CW 2-spheres (topological objects), and further to Eulerian lattices of length 4 (combinatorial objects).

The analogous problems “one dimension higher,” about the $f$-vectors and flag-vectors of 4-dimensional convex polytopes and their generalizations, are by far not solved, yet. However, the known facts already show that the answers will be much more complicated than for Steinitz’ problem. In this lecture, we will summarize the current state of knowledge. We will put forward two crucial parameters of fatness and complexity: Fatness $F(P) := \frac{f_1 + f_2 - 20}{f_0 + f_3}$ is large if there are many more edges and 2-faces than there are vertices and facets, while complexity $C(P) := \frac{f_3 - 20}{f_0 + f_3 - 10}$ is large if every facet has many vertices, and every vertex is in many facets. Recent results suggest that these parameters might allow one to differentiate between the cones of $f$- or flag-vectors of

- connected Eulerian lattices of length 5 (combinatorial objects),
- strongly regular CW 3-spheres (topological objects),
- convex 4-polytopes (discrete geometric objects), and
- rational convex 4-polytopes (whose study involves arithmetic aspects).

Further progress will depend on the derivation of tighter $f$-vector inequalities for convex 4-polytopes. On the other hand, we will need new construction methods that produce interesting polytopes which are far from being simplicial or simple — for example, very “fat” or “complex” 4-polytopes. In this direction, I will report about constructions (from joint work with Michael Joswig, David Eppstein and Greg Kuperberg) that yield

- strongly regular CW 3-spheres of arbitrarily large fatness,
- convex 4-polytopes of fatness larger than 5.048, and
- rational convex 4-polytopes of fatness larger than $5 - \varepsilon$.

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1. Introduction

Our knowledge about the combinatorics and geometry of 4-dimensional convex polytopes is quite incomplete. This assessment may come as a surprise: After all,

- 3-dimensional polytopes have been objects of intensive study since antiquity,
- properties of convex polytopes are essential to the geometry of Euclidean spaces,
- the regular polytopes (in all dimensions) were classified by Schläfi in 1850-52 [20], exactly 150 years ago, and
- modern polytope theory has achieved truly impressive results, in particular since the publication of Grünbaum’s volume [9] in 1967, thirty-five years ago.

Moreover, we have a rather satisfactory picture of 3-dimensional convex polytopes by now, where the essential combinatorial and geometric properties of 3-dimensional polytopes were isolated a long time ago. Here we mention the three results that will be most relevant to our subsequent discussion:

1. Steinitz [24] characterized the $f$-vectors $(f_0, f_1, f_2) \in \mathbb{Z}^3$ of the 3-dimensional convex polytopes: They are the integer points in the 2-dimensional convex polyhedral cone that is defined by the three conditions

$$f_0 - f_1 + f_2 = 2, \quad f_2 - 4 \leq 2(f_0 - 4), \quad f_0 - 4 \leq 2(f_2 - 4).$$

In particular, the $f$-vectors of the 3-dimensional polytopes are given as all the integer points in a rational polyhedral cone. Furthermore, Steinitz’ result includes the characterization of the polytopes with extremal $f$-vectors: The first inequality is tight if and only if the polytope is simplicial, while the second one is tight if and only if the polytope is simple.

2. In 1922, Steinitz [25] published a characterization of the graphs of 3-polytopes: They are all the planar, three-connected graphs on at least 4 nodes.

In modern terms (as reviewed below) and after some additional arguments the Steinitz theorem may be phrased as follows: Every connected finite Eulerian lattice of length 4 is the face lattice of a rational convex 3-polytope.

3. The famous Koebe–Andreev–Thurston circle-packing theorem [26] implies that every combinatorial type of 3-polytope has a realization with all edges tangent to the unit sphere $S^2$. Furthermore, the representation is unique up to Möbius transformations; thus, in particular, symmetric graphs/lattices have symmetric realizations. (See Bobenko & Springborn [6] for a powerful treatment of this result, and for references to its involved history.)

This is the situation in dimension 3. The picture in dimension 4 is not only quite incomplete; it is also clear by now that the results for the case of 4-dimensional polytopes will be much more involved. So, it will be a substantial challenge for 2006 (the centennial of Steinitz’ little 2 1/2-page paper [24]) to characterize the closures of the convex cones of $f$-vectors and flag-vectors for 4-polytopes. This paper sketches some obstacles on the way as well as some efforts that have been undertaken or that should and will be made towards this goal. The obstacles are closely linked to the three results listed above:

1. The geometry of the set of $f$-vectors of 4-polytopes is rather intricate. It does not consist of all the integer points in its convex hull, its convex hull is not closed, and the cone it spans may be not polyhedral.
2. Moreover, the hierarchy covered by the second Steinitz theorem — connected Eulerian lattices, strongly regular CW spheres, convex polytopes, rational convex polytopes — does not collapse in dimension 4: The set of combinatorial types becomes increasingly restricted in this sequence.

3. Furthermore, the non-existence of edge-tangent representations for many types of 4-polytopes is an obstruction to the “E-construction” (see Section 6) that has recently produced sequences of interesting examples.

Nevertheless, there is hope: The boundary complex of a 4-dimensional polytope is 3-dimensional — thus we are in essence concerned with problems of 3-dimensional combinatorial geometry. That is, 4-dimensional polytopes and their faces can be effectively constructed, handled, and visualized. The tools that we have available in this context include dimensional analogy, Schlegel diagrams (see [29, Lect. 5]), a connection to tilings that will be outlined in Section 7, as well as computational tools (use Polymake [8]).

2. \textit{f}-Vectors and flag-vectors

The combinatorial type of a convex $d$-dimensional polytope $P$ ($d$-polytope, for short) is given by its face lattice $L(P)$: This is a finite graded lattice of length $d + 1$ which is Eulerian, that is, every non-trivial interval contains the same number of elements of odd and of even rank (cf. Stanley [22, 23]). Furthermore, for $d > 1$ this lattice is connected, that is, the bipartite graph of atoms and coatoms (elements of rank 1 vs. elements of rank $d$, corresponding to vertices vs. facets) is connected.

The primary numerical data of a polytope, or much more generally of a graded lattice, are the numbers $f_i = f_i(P)$ of $i$-dimensional faces ($i$-faces, resp. lattice elements of rank $d + 1$). More generally, one considers the $2^d$ flag numbers $f_S = f_S(P)$ (for $S \subseteq \{0, 1, \ldots, d - 1\}$) that count the chains of faces with one $i$-face for each $i \in S$. These are collected to yield the \textit{f}-vector $f(P) := (f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^d$, and the \textit{flag-vector} flag$(P) := (f_S : S \subseteq \{0, 1, \ldots, d - 1\})$ of the polytope or lattice. In terms of flag numbers, the bipartite graph of atoms and coatoms has $f_0 + f_{d-1}$ vertices and $f_{0,d-1}$ edges.

For general polytopes, the components of the \textit{f}-vector satisfy only one non-trivial linear equation, the Euler-Poincaré relation $f_0 - f_1 \pm \cdots \pm (-1)^{d-1}f_{d-1} = 1 + (-1)^{d-1}$. The flag-vector (with $2^d$ components, including the \textit{f}-vector) is highly redundant, due to the linear “generalized Dehn-Sommerville relations” (Bayer & Billera [3]) that allow one to reduce the number of independent components to $F_d - 1$, one less than a Fibonacci number. In particular, for $d = 3$ there is no additional information in the flag-vector, by $f_{01} = f_{12} = 2$, $f_{02} = f_{012} = 4$. For 4-polytopes, the full flag-vector is determined by

$$\text{flag}(P) := (f_0, f_1, f_2, f_3; f_{03}).$$

(We do \textit{not} delete one of the $f_i$ via the Euler-Poincaré relation, in order to explicitly retain the symmetry for dual polytopes.) As an example, the flag-vector of the 4-simplex is given by flag$(\Delta_4) = (5, 10, 10, 5; 20)$. The set of all \textit{f}-resp. flag-vectors of 4-polytopes will be denoted $\text{f-Vectors}(\mathcal{P}_4)$ resp. $\text{Flag-Vectors}(\mathcal{P}_4)$. 


The known facts about and partial description of the sets $f$-Vectors($\mathcal{P}_4$) and Flag-Vectors($\mathcal{P}_4$) have been reviewed in detail in Grünbaum [9, Sect. 10.4], Bayer [2], Bayer & Lee [4, Sect. 3.8], and Höppner & Ziegler [11]. Here we will be concerned only with the linear known conditions that are tight at flag($\Delta_4$), and concentrate of the case of $f$-vectors rather than flag-vectors.

3. Geometry/Topology/Combinatorics

It pays off to study the $f$- and flag-vector problems with respect to the following hierarchy of four models—a combinatorial, a topological, and two geometric ones (where the last one includes arithmetic aspects):

- **Eulerian lattices**: Let $\mathcal{L}_4$ be the class of all connected Eulerian lattices of length 5, as defined/reviewed above. (More restrictively, one could require that all intervals of length at least 3 must be connected.)

- **Cellular spheres**: Let $\mathcal{S}_4$ be the class strongly regular cellulations of the 3-sphere, that is, of all regular cell decompositions of $S^3$ for which any intersection of two cells is a face of both of them (which may be empty). These objects appear as “regular CW 3-spheres with the intersection property” as in Björner et al. [5, pp. 203, 223]; following Eppstein, Kuperberg & Ziegler [7] we call them “strongly regular spheres.” The intersection property is equivalent to the fact that the face poset of the cell complex is a lattice.

- **Convex polytopes**: $\mathcal{P}_4$ denotes the combinatorial types of convex 4-polytopes.

- **Rational convex polytopes**: $\mathcal{P}^Q_4$ will denote the combinatorial types of convex 4-polytopes that have a realization with rational (vertex) coordinates.

We have natural inclusions

$$\mathcal{P}^Q_4 \subset \mathcal{P}_4 \subset \mathcal{S}_4 \subset \mathcal{L}_4.$$  

The first inclusion is strict due to the existence of non-rational 4-polytopes (Richter-Gebert [18]), the second one due to the known examples of non-realizable triangulated 3-spheres, the third since any strongly regular cell decomposition (e.g., a triangulation) of a compact connected 3-manifold without boundary has a connected Eulerian face lattice of length 5.

For each of the four classes we define its cone of flag-vectors, that is, the closure of the cone with apex flag($\Delta_4$) that is spanned by all vectors of the form flag($P$) − flag($\Delta_4$): For each family of combinatorial types we denote by Flag-Vectors(·) the set of flag-vectors, and by $f$-Cone(·) resp. Flag-Cone(·) the corresponding closures of the cones of $f$-vectors resp. flag-vectors. So we get the inclusions

$$f\text{-Vectors}(\mathcal{P}^Q_4) \subset f\text{-Vectors}(\mathcal{P}_4) \subset f\text{-Vectors}(\mathcal{S}_4) \subset f\text{-Vectors}(\mathcal{L}_4),$$

and

$$f\text{-Cone}(\mathcal{P}^Q_4) \subset f\text{-Cone}(\mathcal{P}_4) \subset f\text{-Cone}(\mathcal{S}_4) \subset f\text{-Cone}(\mathcal{L}_4),$$

and similarly for flag-vectors — but can we separate them? Is any of these inclusions strict? At the moment, that does not appear to be clear, not even if we consider the sets of $f$- or flag-vectors themselves rather than just the closures of the cones they span!
4. Fatness and complexity

Instead of linear combinations of face numbers or flag numbers (such as the toric h-vector, the cd-index etc. [4]), in the following we will rely on quotients of such. Thus we obtain homogeneous “density” parameters that characterize extremal polytopes. Such a quotient is the average vertex degree \( \frac{f_1}{f_0} = \frac{2f_1}{f_0} \). However, we will in addition normalize the quotients such that numerator and denominator vanish for the simplex; then every inequality of the form “our parameter \( \geq \) constant” translates into a linear inequality that holds with equality at the simplex. Thus instead of the average vertex degree we would use densities like \( \delta_0 := \frac{f_1 - 10}{f_0 - 5} \) or \( \delta_0 := \frac{f_1 + 2f_3 - 20}{f_0 + f_3 - 10} \). (They are not defined for the simplex.) For both of these densities equality in the valid inequality \( \delta_0 \geq 2 \) characterizes simple 4-polytopes.

We prefer such density parameters since they provide measures of complexity that are independent of the (combinatorial) “size” of the polytope. For example, they are essentially stable under various operations of “glueing” polytopes or “connected sums” of polytopes (as in [29, p. 274]). In terms of the closed cones of f- and flag-vectors, the density parameters measure “how close we are to the boundary” in terms of a “slope.” The following two parameters we call fatness and complexity:

\[
F(P) := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
5. The $f$-Vector cone for 4-polytopes

**Theorem** (cf. Bayer [2, Sect. 4], Eppstein et al. [7]).
In terms of the homogeneous coordinates $\varphi_0 := \frac{f_0 - 5}{f_1 + f_2 - 20}$ and $\varphi_3 := \frac{f_3 - 5}{f_1 + f_2 - 20}$, for $(f_1 + f_2 - 20, f_0 - 5, f_3 - 0)$-space, the five linear inequalities

(i) $\varphi_0 + 3\varphi_3 \leq 1$ (with equality for simplicial polytopes),
(ii) $3\varphi_0 + \varphi_3 \leq 1$ (with equality for simple polytopes),
(iii) $\varphi_0 \geq 0$ (close-to-equality if there are much fewer vertices than other faces),
(iv) $\varphi_3 \geq 0$ (close-to-equality if there are much fewer facets than other faces),
(v) $\varphi_0 + \varphi_3 \leq \frac{3}{5}$ (with equality if $F(P) = \frac{1}{\varphi_0 + \varphi_3} = \frac{3}{5}$, which forces $g_2^{\text{tor}}(P) = 0$)

define a 3-dimensional closed polyhedral cone with apex $\text{flag}(\Delta_4) = (0, 0, 0)$. It is a cone over a pentagon, which drawn to scale looks as follows:

The five linear inequalities are valid for $f$-Cone($P_4$), and they are tight and facet-defining for $f$-Cone($S_4$).

The interesting/crucial parts of this theorem are on the one hand the existence of arbitrarily fat objects, which was established for strongly cellular spheres in [7, Sect. 4] by the $M_g$-construction (see Section 6), and the last inequality (v), which follows for polytopes from $C(P) \geq 3$ [2], but whose validity for strongly regular spheres is an open problem.

Thus if $F(P) \geq \frac{3}{2}$ is valid for all strongly regular 3-spheres, then the five inequalities above give a complete linear description of $f$-Cone($S_4$). On the other hand, if fatness is not bounded for (rational) convex 4-polytopes, then the above system is a complete description of $f$-Cone($P_4$) resp. $f$-Cone($S_4$). But if one of the two ifs fails, then the two cones of $f$-vectors differ substantially!

One can attempt to give a similar description for the 4-dimensional cones Flag-Cone($P_4$) and Flag-Cone($S_4$). However, in this case the picture (compare Bayer [2, Sect. 2] and Höppner & Ziegler [11]) is much less complete, yet.

In both the $f$-vector and in the flag-vector case the simple polytopes and the simplicial polytopes appear as extreme cases, and they induce facets that meet only at the apex (the simplex). The $f$- and flag-vectors of simple/simplicial 4-polytopes and 3-spheres are well-known — a complete picture is given by the $g$-Theorem (McMullen [15]), which for 4-polytopes was first established by Barnette [1] and for 3-spheres by Walkup [27]).
6. Constructions

In order to prove completeness for linear descriptions of \( f \)- or flag-vector cones, one needs to have at one’s disposal enough examples or construction techniques for extremal polytopes that go beyond the usual classes of “simple and simplicial” polytopes (neighborly, stacked, random, etc.).

**Cubical polytopes.** (all of whose proper faces are combinatorial cubes) form a natural class of polytopes. A very specific construction by Joswig & Ziegler [12] produced “neighborly cubical” polytopes as special projections of suitably deformed \( n \)-cubes to \( \mathbb{R}^4 \): These are rational cubical 4-polytopes \( C_n^4 \) with the graph of the \( n \)-cube (for \( n \geq 4 \), hence with flag-vectors

\[ \text{flag}(C_n^4) = (4, 2n, 3(n - 2), n - 2; 8(n - 2)) \cdot 2^{n-2}. \]

Thus we have rational 4-polytopes of fatness \( F(C_n^4) \) arbitrarily close to 5, and complexity \( C(C_n^4) \) arbitrarily close to 8. (One may also show that all cubical polytopes and spheres satisfy \( F(P) < 5 \) and \( C < 8 \).

**The E-construction.** Eppstein, Kuperberg & Ziegler [7] presented and analyzed a particular 4-dimensional construction that produces interesting example polytopes: Let \( P \subset \mathbb{R}^4 \) be a simple 4-polytope whose ridges (2-faces) are tangent to \( S^3 \); then its polar dual \( P^\Delta \) is a simplicial edge-tangent polytope. The **E-polytope** of \( P \), obtained as \( E(P) := \text{conv}(P \cup P^\Delta) \), is then 2-simple and 2-simplicial. It has fatness

\[ F(E(P)) = \frac{6f_0(P) - 10}{f_0(P) + f_3(P) - 5} < 6. \]

In [7], this construction was used to produce infinite families of 2-simple 2-simplicial polytopes — apparently the first of their kind. It was also used to construct 4-polytopes of fatness larger than 5.048 — currently this is the largest value that has been observed for convex polytopes. (All simple and simplicial polytopes have fatness smaller than 3.) We note, however, that the prerequisites for the E-construction are rather hard to satisfy. Obvious examples where they can be achieved arise from regular convex polytopes. On the other hand, it turned out that, for example, \( P^\Delta \) cannot be a stacked 4-polytope with more than 6 vertices! Moreover, for most examples the tangency-condition seems to force non-rational coordinates for \( P \), and hence for \( P^\Delta \). The analysis in [7] depends on a geometric analysis that puts the Klein model of hyperbolic geometry onto the interior of the 4-ball bounded by \( S^3 \).

Thus one may ask: Does the E-construction produce non-rational polytopes? Are there possibly flag-vectors of 2-simple 2-simplicial polytopes that cannot be realized by rational polytopes? While it seems quite reasonable that the E-polytope of the regular 120-cell, with flag-vector \( \text{flag}(E(P_{120})) = (720, 3600, 3600, 720; 5040) \), fatness \( F(E(P_{120})) > 5.02 \), and 720 biyramids over pentagons as facets, could be non-rational, current investigations (Paffenholz [16]) suggest that E-polytopes are less rigid than one would think at first glance, since in some cases the tangency conditions may be relaxed or dropped.
Fat 3-spheres: The $M_g$-construction. Based on a covering space argument, Eppstein, Kuperberg & Ziegler [7, Sect. 4] constructed a family of strongly regular CW 3-spheres whose fatness is not bounded at all. The construction starts with a perfect cellulation of the compact connected orientable 2-manifold $M_g$ of genus $g$ with $f$-vector $(1, 2g, 1)$ and “fatness” $\frac{f_0}{f_0 + f_2} = g$. Then one shows that there is a finite covering $\tilde{M}_g$ of $M_g$ whose induced cell decomposition (of the same “fatness” $g$) is strongly regular. Finally, from the standard embedding of $\tilde{M}_g \times I$ into $S^3$, where the interval $I$ is subdivided very finely and $\tilde{M}_g \times I$ gets the product decomposition, one obtains a cellulation of $S^3$ whose flag vector is dominated by the flag-vector of $\tilde{M}_g \times I$. This yields strongly regular cell decompositions of $S^3$ whose $f$-vector is approximately proportional to $(1, 2g, 1) \ast (1, 1) = (1, 2g + 1, 2g + 1, 1)$. Thus the resulting spheres have fatness arbitrarily close to $2g + 1$.

Many triangulated 3-spheres. Applied to the fat 3-spheres produced by the $M_g$-construction, the E-construction yields 3-spheres with substantially more non-simplicial facets than their number of vertices. Thus one obtains (Pfeifle [17]) that on a large number of vertices there are far more triangulated 3-spheres than there are types of simplicial 4-polytopes, thus resolving a problem of Kalai [14].

7. Tilings

There are close connections between $d$-polytopes (“polyhedral tilings of $S^{d-1}$”) and normal polyhedral tilings of $\mathbb{R}^{d-1}$. In particular, from 4-polytopes one may construct 3-dimensional tilings, for example by starting with a regular tiling of $\mathbb{R}^3$ by congruent tetrahedra and then replacing the tiles by Schlegel diagrams based on a simplex facet. (The converse direction, from tilings of $\mathbb{R}^3$ to 4-polytopes, is non-trivial: It hinges on non-trivial liftability restrictions; see Rybnikov [19].)

Normal tilings are face-to-face tilings of $\mathbb{R}^{d-1}$ by convex polytopes for which the inradii and circumradii of tiles are bounded from below resp. from above — see Grünbaum & Shephard [10, Sect. 3.2]. Of course, all components of an $f$-vector for tilings would be infinite, but one can try to define ratios, e.g. try to find the “average” number of vertices per tile.

The Euler formula for tilings. Thus, for $\rho > 0$, let $f_i(\rho)$ be the number of all faces of the tiling that intersect the interior of the ball $B^d(\rho)$ of radius $\rho$ around the origin. This yields a regular decomposition of an open $d$-ball into convex cells; via one-point-compactification (e.g. generated by stereographic projection) by one additional vertex we obtain a regular cell-decomposition of a $d$-sphere; thus we obtain [28] that for all $\rho > 0$

$$f_0(\rho) - f_1(\rho) + \cdots + (-1)^d f_{d-1}(\rho) = (-1)^{d-1}.$$  

In particular, this implies that if the limits $\phi_i := \lim_{\rho \to \infty} \frac{f_i(\rho)}{\sum_j f_j(\rho)}$ exist, then they satisfy $0 \leq \phi_i \leq \frac{1}{2}$. Furthermore, the existence of the limits $\phi_i$ is automatic for tilings that are invariant under a full-dimensional lattice of translations, such as the “tilings by Schlegel diagrams” suggested above. In this case the limits $\phi_i$ are
strictly positive, and they satisfy the Euler formula for tilings,
\[ \phi_0 - \phi_1 + \phi_2 \pm \cdots + (-1)^{d-1}\phi_{d-1} = 0. \]

For such tilings, we can also define flag-numbers \( \phi_S(\rho) \). Then the limit \( \rho \to \infty \) of any quotient such as \( F(\rho) := \frac{f_1(\rho) + f_2(\rho)}{f_0(\rho) + f_3(\rho)} \) exists, it is positive and finite, and it coincides with
\[ F(\mathcal{T}) := \frac{\phi_1 + \phi_2}{\phi_0 + \phi_3}. \]

One can construct “tilings by Schlegel diagrams” with a high (average) number of vertices per tile, or with a high number of tiles at each vertex. But are both achievable simultaneously? Equivalently, is there is a uniform upper bound on the fatness \( F(T) \) of normal tilings of \( \mathbb{R}^3 \)? This is not known, yet, but if fatness is bounded for normal 3-tilings, then it is bounded for 4-polytopes as well.

**Fat tilings.** Remarkably, there are tilings that have considerably larger fatness than the fattest polytopes we know. In particular, a modified E-construction applied to suitable Schlegel 3-diagrams [29, Lect. 5] of \( C_n \times C_n \) and embedding into a cubic tiling one obtains normal lattice-transitive tilings of fatness arbitrarily close to 6.

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