Multiple recurrence and the structure of probability-preserving systems

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Preface

In 1975 Szemerédi proved the long-standing conjecture of Erdős and Turán that any subset of \( \mathbb{Z} \) having positive upper Banach density contains arbitrarily long arithmetic progressions. Szemerédi’s proof was entirely combinatorial, but two years later Furstenberg gave a quite different proof of Szemerédi’s Theorem by first showing its equivalence to an ergodic-theoretic assertion of multiple recurrence, and then bringing new machinery in ergodic theory to bear on proving that. His ergodic-theoretic approach subsequently yielded several other results in extremal combinatorics, as well as revealing a range of new phenomena according to which the structures of probability-preserving systems can be described and classified.

In this work I survey some recent advances in understanding these ergodic-theoretic structures. It contains proofs of the norm convergence of the ‘non-conventional’ ergodic averages that underly Furstenberg’s approach to variants of Szemerédi’s Theorem, and of two of the recurrence theorems of Furstenberg and Katznelson: the Multidimensional Multiple Recurrence Theorem, which implies a multidimensional generalization of Szemerédi’s Theorem; and a density version of the Hales-Jewett Theorem of Ramsey Theory.

* * *

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Chapter 1

Introduction

The concerns of this work stem from the following remarkable result of Szemerédi ([Sze75]), which confirmed an old conjecture of Erdős and Turán ([ET36]).

**Szemerédi’s Theorem.**  For any $\delta > 0$ and $k \geq 1$ there is some $N_0 \geq 1$ such that if $N \geq N_0$ then any $A \subseteq \{1, 2, 3, \ldots, N\}$ with $|A| \geq \delta N$ includes a nontrivial $k$-term arithmetic progression: $A \supseteq \{a, a + n, \ldots, a + (k - 1)n\}$ for some $a \in \{1, 2, \ldots, N\}$ and $n \geq 1$.

This provides a considerable strengthening of a much older result of van der Waerden ([Wae27]), according to which any colouring of $\mathbb{N}$ using a bounded number of colours witnesses arbitrarily long finite arithmetic progressions that are monochromatic. Since any colouring with at most $c$ colours must have at least one colour class of upper Banach density at least $1/c$, van der Waerden’s Theorem can be deduced by applying Szemerédi’s Theorem to the intersection of that class with sufficiently long discrete intervals in $\mathbb{N}$.

Shortly after the appearance of Szemerédi’s ingenious combinatorial proof, Furstenberg gave a new proof of the above theorem in [Fur77] using a superficially quite different approach, relying on a conversion to a problem about probability-preserving dynamical systems.

Such a system consists of a probability space $(X, \Sigma, \mu)$ together with an invertible, measurable, $\mu$-preserving transformation $T : X \to X$. Furstenberg proved that all such systems enjoy a property of ‘multiple recurrence’:

**Multiple Recurrence Theorem.**  Whenever $(X, \Sigma, \mu)$ and $T$ are as above,
if $k \geq 1$ and $A \in \Sigma$ has $\mu(A) > 0$ then

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}(A) \cap \ldots \cap T^{-(k-1)n}(A)) > 0.$$  

In particular, there is some $n \geq 1$ such that

$$\mu(A \cap T^{-n}(A) \cap \ldots \cap T^{-(k-1)n}(A)) > 0.$$  

It is worth noting that analogously to this ergodic-theoretic proof of Szemerédi’s Theorem, it is possible to deduce the colouring theorem of van der Waerden from a multiple recurrence result in topological dynamics. We will not be concerned with this story here, but it is reported in detail in Furstenberg’s book \cite{Fur81}.

Shortly after the above result appeared, Furstenberg and Katznelson realized that the same basic method could be modified to apply to collections of commuting measure-preserving transformations, and proved the following in \cite{FK78}.

**Theorem A** (Multidimensional Multiple Recurrence Theorem). If $(X, \Sigma, \mu)$ is a probability space, $T_1, T_2, \ldots, T_d$ are commuting measurable invertible $\mu$-preserving self-maps of $X$ and $A \in \Sigma$ has $\mu(A) > 0$, then

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_1^{-n}(A) \cap \ldots \cap T_d^{-n}(A)) > 0.$$  

Of course this result implies one-dimensional multiple recurrence by setting $d := k$ and $T_i := T_i^n$ for $i = 0, 1, \ldots, k - 1$. In addition, Furstenberg and Katznelson were able to convert Theorem A back into a multidimensional combinatorial result generalizing Szemerédi’s Theorem.

**Multidimensional Szemerédi Theorem.** For any $\delta > 0$ and $d \geq 1$ there is some $N_0 \geq 1$ such that if $N \geq N_0$ then any $A \subseteq \{1, 2, \ldots, N\}^d$ with $|A| \geq \delta N^d$ includes the vertex set of the outer face of a nontrivial upright simplex:

$$A \supseteq \{a + ne_1, a + ne_2, \ldots, a + ne_d\}$$
for some $\mathbf{a} \in \{1, 2, \ldots, N\}^d$ and $n \geq 1$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ are the usual basis vectors of $\mathbb{Z}^d$.

This ergodic-theoretic approach to results in additive combinatorics has since developed into a whole subdiscipline, sometimes termed ‘Ergodic Ramsey Theory’; see, for instance, Bergelson’s survey [Ber96]. In particular, Furstenberg and Katznelson used this approach to prove a number of further results concerning some form of ‘recurrence’, culminating in the following density version of the classical Hales-Jewett Theorem [HJ63] proved in [FK91]:

**Theorem B** (Density version of the Hales-Jewett Theorem). For any $\delta > 0$ and $k \geq 1$ there is some $N_0 \geq 1$ such that if $N \geq N_0$ then any $A \subseteq [k]^N$ with $|A| \geq \delta k^N$ includes a combinatorial line: a subset $L \subseteq [k]^N$ of the form

$$L = \{ w \in [k]^N : w|_{[N]\setminus J} = w_0, \ & w_j \text{ is the same element of } [k] \text{ for all } j \in J \},$$

for some fixed nonempty $J \subseteq [N]$ and $w_0 \in [k]^{|N|\setminus J}$.

In fact, this result implies most of the other main results in density Ramsey Theory, including Szemerédi’s Theorem and its multidimensional generalization. This implication holds exactly as in the older setting of colouring Ramsey Theorems, which is well-treated in the book [GRS90] of Graham, Rothschild and Spencer.

In addition to achieving some striking new combinatorial results, Ergodic Ramsey Theory has also motivated new ergodic-theoretic questions, and has witnessed an ongoing interplay between insights into these two aspects of the subject.

One basic question that was resolved only recently is whether the ‘multiple ergodic averages’ studied in Theorems A and B above actually converge (that is, whether ‘$\lim \inf$’ can be replaced with ‘$\lim$’). In the case of the original Multiple Recurrence Theorem, this was finally shown to be so by Host and Kra in [HK05], following the establishment of several special cases and related results over two decades in [CL84, CL88a, CL88b, Zha96, FW96, HK01] (see also Ziegler’s paper [Zie07] for another proof of the Host-Kra result). The more general setting of Theorem A was then settled by Tao in [Tao08].

**Theorem C** (Norm convergence of nonconventional averages). For any commuting tuple of invertible measurable $\mu$-preserving transformations $T_1, T_2,$
\[ T_d \circ (X, \Sigma, \mu) \text{ and any functions } f_1, f_2, \ldots, f_d \in L^\infty(\mu), \text{ the multiple ergodic averages} \]
\[
\frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^{d} f_i \circ T_i^n
\]
converge in \( L^2(\mu) \) as \( N \to \infty \).

While the sequence of works preceding the proof of convergence in the one-dimensional setting of the Multiple Recurrence Theorem develops a large body of ergodic-theoretic machinery for the analysis of these averages, Tao departs quite markedly from those approaches and effectively converts the problem of convergence into a quantitative assertion concerning averages of \([-1, 1]\)-valued functions on large finite grids \( \{1, 2, \ldots, N\}^d \).

A new proof of Tao’s Theorem was given using classical ergodic-theoretic machinery in [Aus09]. It turns out that this convergence can be proved relatively quickly using a version of the older approaches, with the one new twist that starting from a system of commuting transformations of interest \( T_1, T_2, \ldots, T_d \circ (X, \Sigma, \mu) \) one must first pass to a carefully-chosen extended system \( \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_d \circ (\tilde{X}, \tilde{\Sigma}, \tilde{\mu}) \) (that is, a new system for which the original one is isomorphic to the action of the \( \tilde{T}_i \)'s on some globally invariant \( \sigma \)-subalgebra of \( \tilde{\Sigma} \): in ergodic-theoretic terms, the original system is a ‘factor’ of the new one). If the extension is constructed correctly then the asymptotic behaviour of the multiple ergodic averages associated to it admits a simplification allowing them to be compared with a similar system of averages involving only \( k-1 \) transformations; from this point convergence in \( L^2 \) follows quickly by induction on \( k \). The need for this extension also offers some explanation for the advantage that Tao gains in his approach to Theorem C by converting to the finitary, combinatorial world: during the course of his proof he constructs new functions from the initial data of the problem in ways that cannot be used to construct measurable functions in the ergodic-theoretic setting, but suitable measurable functions are available using the larger \( \sigma \)-algebra of the extended system.

Theorem C proves the convergence of the scalar averages appearing in Theorem A because
\[
\frac{1}{N} \sum_{n=1}^{N} \mu(T_1^{-n}(A) \cap T_2^{-n}(A) \cap \cdots \cap T_d^{-n}(A)) = \int \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} (f_i \circ T_i^n) \, d\mu
\]
when \( f_1 = f_2 = \ldots = f_d = 1_A \). Note that another re-proof of Tao’s theorem involving non-standard analysis has been given by Towsner in [Tow09], and that a different construction of some extensions of probability-preserving systems that can be used as in the proof of [Aus09] has since been given by Host in [Hos09].

Having found the extended systems appearing in the new proof of Theorem C, it turns out that they also afford a somewhat simplified description of the limiting value of the scalar averages appearing in Theorem A. These limiting values can always be expressed in terms of a certain \((d + 1)\)-fold self-joining of the system \((\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_d)\) (which appears already in the works of Furstenberg and Katznelson), and one finds that for the extended system this self-joining takes a special form. Crucially, that special form is precisely the hypothesis required to apply another result of Tao: the infinitary analog of the hypergraph removal lemma from [Tao07]. This leads fairly quickly to a new proof of Theorem A (and hence also one-dimensional multiple recurrence and their combinatorial consequences), which appeared in [Ausb].

A similar story is now known in the setting of Theorem B. For their proof of that theorem, Furstenberg and Katznelson first provided a correspondence with a class of stochastic processes enjoying stationarity with respect to some semigroup of transformations. This is broadly similar to Furstenberg’s original correspondence between Szemerédi’s Theorem and the Multiple Recurrence Theorem, but differs considerably in its details. Having built this bridge to a class of stochastic processes, Furstenberg and Katznelson then used analogs of their earlier structural results from the setting of probability-preserving \(\mathbb{Z}^d\)-actions to prove the ‘recurrence’ result that is the translation of Theorem B. Here, too, it turns out that the strategy of seeking extended systems in which the behaviour of interest is simplified leads to a new proof of that recurrence result, and so overall to a considerably shortened proof of Theorem B, where again the punchline is an implementation of Tao’s infinitary hypergraph removal. This new proof of Theorem B appears in [Ausb]. It was discovered simultaneously with the work of the Polymath project [Pola], which provided the first finitary, effective proof of that theorem, and the proof of [Ausb] used a key construction discovered by the members of that project (again, suitably translated to apply to the stochastic processes).

More recently still, in pursuit of some convergence results for ‘polyno-
mial’ analogs of the functional averages of Theorem C, it was found that a very abstract, unified approach could be given to the construction of the different extensions underlying the above-mentioned proofs of Theorems C, A and B. This rests on the notion of a system that is ‘sated’ relative to another class of systems. In this dissertation, the new proofs of the above results are re-told using this unifying language, and some speculations offered concerning some further extensions of this machinery.

Outline of the following chapters

In the next chapter we recall some basic definitions and conventions from the study of measurable dynamical systems, and then introduce the chief technical innovation on which most of the remaining chapters will rest: a special property of certain dynamical systems called ‘satedness’. The main result of that chapter, Theorem 2.3.2 asserts that any probability-preserving dynamical system admits extensions that enjoy this ‘satedness’ (where precisely what this means is relative to a choice of another class of systems).

In Chapter 3 we use the existence of sated extensions to prove Theorem C. After the introduction of another important technical device, the ‘Furstenberg self-joining’, this follows by a quick induction once the strategy of passing to a sated extension has been decided.

Chapter 4 is dedicated to Theorem A. In this case the use of sated extensions gives a relatively easy reduction of the proof to a case in which the Furstenberg self-joining (which describes the limiting averages of interest) admits a rather detailed structural description; but the use of that description to deduce the desired positivity of these averages is still rather involved. This requires an implementation of (a very slight modification of) Tao’s ‘infinitary hypergraph removal lemma’, which we will recall for completeness.

In Chapter 5 we prove Theorem B. This proof follows very closely that of Theorem A, notwithstanding that the category of dynamical systems in which the proof takes place is very different. However, the unusual features of this new category will require that we quickly re-examine the existence of sated extensions proved in Chapter 2 to check that a slightly modified version of that result holds here. After recalling Furstenberg and Katznelson’s original reformulation of Theorem B in terms of a ‘recurrence’ property of certain ‘strongly stationary’ stochastic processes, we establish this new notion
of ‘coordinatewise-satedness’ and show that in this world it implies a similar structure for certain joint distributions to that obtained for the Furstenberg self-joining in Chapter 4. The proof of Theorem B is then completed by another appeal to infinitary hypergraph removal, essential identical to that in Chapter 4.

Finally, Chapter 6 contains some speculations around an important question left open by our work. In the case of $\mathbb{Z}^d$-actions treated by Chapters 3 and 4, one can discern in the background a very general ergodic-theoretic meta-question concerning the possible joinings among systems enjoying various additional invariances. This is formulated precisely in Section 4.1 but in that section it is answered only in a special case that suffices for the proof of Theorem A. A more general answer would be very interesting in its own right, as well as potentially offering new insights on other generalizations of nonconventional average convergence and multiple recurrence. In Chapter 6 we will formulate a conjecture that would answer this question much more completely.
Chapter 2

Setting the stage

A handful of key technical ideas in ergodic theory will drive all of the proofs in the later chapters of this work. After recalling some standard definitions and notation in the first section below, we introduce two such key ideas: that of a subclass of a class of dynamical systems that has the property of being ‘idempotent’, and the constructions that this assumption of idempotence enables; and then the possibility of a system being ‘sated’ relative to such an idempotent class, together with the result that all systems have extensions that are sated in this way.

These preliminary sections provide the necessary background for Chapters 3 and 4 (and also Chapter 6). Unfortunately, the slightly unusual class of stochastic processes that appears in Chapter 5 is a little less willing to be analysed using this standard framework: the key ideas of idempotence and satedness will be central there too, but only after being modified to suit that class. The modifications will be explained early in that chapter, together with those small changes that must accordingly be made to the proofs in Sections 2.2 and 2.3. In principle one could give a unified treatment of all of these settings, but only at the expense of working with quite abstractly-defined categories of dynamical system and operations on them, in which our basic intuitions for the notions recalled in Section 2.1 may become obscured. Although more unified, that route seems to pose too great a risk to the clarity of the other chapters, and so we shall only indicate it in passing during Chapter 5.
2.1 Probability-preserving systems

Throughout this paper \((X, \Sigma)\) will denote a measurable space. Since our main results pertain only to the joint distribution of countably many bounded real-valued functions on this space and their shifts under some measurable transformations, by passing to the image measure on a suitable product space we may always assume that \((X, \Sigma)\) is standard Borel, and this will prove convenient for some of our later constructions. In addition, \(\mu\) will always denote a probability measure on \(\Sigma\). We shall write \((X^S, \Sigma^\otimes S)\) for the usual product measurable structure indexed by a set \(S\), and \(\mu^\otimes S\) for the product measure and \(\mu^\Delta S\) for the diagonal measure on this structure respectively. Given a measurable map \(\phi: (X, \Sigma) \to (Y, \Phi)\) to another measurable space, we shall write \(\phi^\# \mu\) for the resulting pushforward probability measure on \((Y, \Phi)\).

Suppose now that \(\Gamma\) is a discrete semigroup, and consider the class of all probability-preserving actions \(T: \Gamma \curvearrowright (X, \Sigma, \mu)\) on standard Borel probability spaces; these will be referred to as \(\Gamma\)-systems, and will often be denoted by either the quadruple \((X, \Sigma, \mu, T)\) or simply by a boldface letter such as \(X\). If \(\Lambda \leq \Gamma\) is a subgroup we denote by \(T^{\mid \Lambda}\) the \(\Lambda\)-action on \((X, \Sigma, \mu)\) defined by \((T^{\mid \Lambda})^\gamma := T^\gamma\) for \(\gamma \in \Lambda\), and refer to this as the \(\Lambda\)-subaction, and if \(X = (X, \Sigma, \mu, T)\) is a \(\Gamma\)-system then we write similarly \(X^{\mid \Lambda}\) for the system \((X, \Sigma, \mu, T^{\mid \Lambda})\) and refer to it as a subaction system.

A \(\Gamma\)-system \((X, \Sigma, \mu, T)\) is trivial if \(\mu\) is supported on a single point. Since any two such systems are measure-theoretically isomorphic simply by identifying these single points, we will usually refer to ‘the’ trivial system.

We will make repeated use of a handful of standard constructions and properties of \(\Gamma\)-systems.

Factors and joinings

A factor of the \(\Gamma\)-system \((X, \Sigma, \mu, T)\) is a globally \(T\)-invariant \(\sigma\)-subalgebra \(\Phi \leq \Sigma\). Relatedly, a factor map from one \(\Gamma\)-system \(T: \Gamma \curvearrowright (X, \Sigma, \mu)\) to another \(S: \Gamma \curvearrowright (Y, \Phi, \nu)\) is a measurable map \(\pi: X \to Y\) such that \(\nu = \pi^\# \mu\) and \(S^\gamma \circ \pi = \pi \circ T^\gamma\) for all \(\gamma \in \Gamma\). This situation is often signified by writing \(\pi: (X, \Sigma, \mu, T) \to (Y, \Phi, \nu, S)\). Factor maps comprise the natural morphisms between systems for a fixed acting semigroup.

To any factor map \(\pi\) is associated the factor \(\{\pi^{-1}(A): A \in \Phi\} \leq \Sigma\).
Two factor maps $\pi$ and $\psi$ are equivalent if these $\sigma$-subalgebras of $\Sigma$ that they generate are equal up to $\mu$-negligible sets, in which case we shall write $\pi \simeq \psi$; this clearly defines an equivalence relation among factors.

It is a standard fact that in the category of standard Borel spaces equivalence classes of factors are in bijective correspondence with equivalence classes of globally invariant $\sigma$-subalgebras under the relation of equality modulo negligible sets. A treatment of these classical issues may be found, for example, in Chapter 2 of Glasner [Gla03]. Given a globally invariant $\sigma$-subalgebra in $X$, a choice of factor $\pi: X \to Y$ generating that $\sigma$-subalgebra will be referred to as coordinatizing the $\sigma$-subalgebra.

More generally, the factor map $\pi: (X, \Sigma, \mu, T) \to (Y, \Phi, \nu, S)$ contains $\psi: (X, \Sigma, \mu, T) \to (Z, \Psi, \theta, R)$ if $\pi^{-1}(\Phi) \supseteq \psi^{-1}(\Psi)$ up to $\mu$-negligible sets. Another standard feature of standard Borel spaces is that this inclusion is equivalent to the existence of a factorizing factor map $\phi: (Y, \Phi, \nu, S) \to (Z, \Psi, \theta, R)$ with $\psi = \phi \circ \pi$ $\mu$-a.s., and that a measurable analog of the Schroeder-Bernstein Theorem holds: $\pi \simeq \psi$ if and only if a single such $\phi$ may be chosen that is invertible away from some negligible subsets of the domain and target. If $\pi$ contains $\psi$ we shall write $\pi \succ \psi$ or $\psi \prec \pi$.

If $\pi: X \to Y$ and $\psi: X \to Z$ are any two factor maps as above (not necessarily ordered), then the $\sigma$-subalgebra $\pi^{-1}(\Phi) \vee \psi^{-1}(\Psi)$ is another factor of $X$. In general we will write $\pi \vee \psi$ for an arbitrary choice of factor map coordinatizing this factor, and similarly for larger collections of factor maps.

Dual to the idea of a factor is that of an extension: if $X$ is a $\Gamma$-system, then an extension $X$ is another $\Gamma$-system $\tilde{X}$ together with a factor map $\pi: \tilde{X} \to X$.

More general than the notion of a factor is that of a joining: if $X_1, X_2, \ldots, X_k$ are $\Gamma$-systems then a joining of them is another $\Gamma$-system $X$ together with factor maps $\pi_i: X \to X_i$ such that these $\pi_i$ together generate the whole $\sigma$-algebra of $X$. Since their introduction by Furstenberg in [Fur67], joinings have become one of the most important concepts in the ergodic theorist’s vocabulary, as is well-demonstrated in Glasner’s book [Gla03].
**Partially invariant factors**

Given a $\Gamma$-system $X = (X, \Sigma, \mu, T)$, the $\sigma$-algebra $\Sigma^T$ of sets $A \in \Sigma$ for which $\mu(A \Delta T^{\gamma}(A)) = 0$ for all $\gamma \in \Gamma$ is $T$-invariant, so defines a factor of $X$. More generally, if $\Gamma$ is a group and $\Lambda \unlhd \Gamma$ then we can consider the $\sigma$-algebra $\Sigma_T\upharpoonright \Lambda$ generated by all $T\upharpoonright \Lambda$-invariant sets: we refer to this as the $\Lambda$-**partially invariant factor**. Note that in this case the condition that $\Lambda$ be normal is needed for this to be a globally $T$-invariant factor. Similarly, if $S \subseteq \Gamma$ and $\Lambda$ is the normal subgroup generated by $S$, we will sometimes write $\Sigma^T\upharpoonright S$ for $\Sigma^T\upharpoonright \Lambda$.

If moreover $\Gamma$ is Abelian and $T_1$ and $T_2$ are two commuting actions of $\Gamma$ on $(X, \Sigma, \mu)$, then we can define a third action $T_1T_2^{-1}$ by setting $(T_1T_2^{-1})^\gamma := T_1^\gamma T_2^{-\gamma^{-1}}$. Given this we often write $\Sigma^T_{T_1T_2}$ in place of $\Sigma^T_{T_1^{-1}T_2}$, and similarly for a larger number of actions of the same group.

**Relative independence**

If $\Sigma_i \geq \Xi_i$ are factors of $(X, \Sigma, \mu, T)$ for each $i \leq d$, then the tuple of factors $(\Sigma_1, \Sigma_2, \ldots, \Sigma_d)$ is **relatively independent** over the tuple $(\Xi_1, \Xi_2, \ldots, \Xi_d)$ if whenever $f_i \in L^\infty(\mu)$ is $\Sigma_i$-measurable for each $i \leq d$ we have

$$\int_X \prod_{i \leq d} f_i \, d\mu = \int_X \prod_{i \leq d} \mathbb{E}_\mu(f_i \mid \Xi_i) \, d\mu.$$

The information that various joint distributions are relatively independent will repeatedly prove pivotal in the following. Sometimes for brevity we will write that ‘$\Sigma_1$ is relatively independent from $\Sigma_2$, $\Sigma_d$, $\ldots$, $\Sigma_d$ over $\Xi_1$’ if $(\Sigma_1, \Sigma_2, \ldots, \Sigma_d)$ is relatively independent over $(\Xi_1, \Xi_2, \ldots, \Xi_d)$.

In case $\Gamma$ is a group (not just a semigroup, so each $T^\gamma$ is invertible) we can construct examples of this situation as follows. Suppose that $Y = (Y, \Phi, \nu, S)$ is a $\Gamma$-system and

$$\pi_i : X_i = (X_i, \Sigma_i, \mu_i, T_i) \longrightarrow Y$$

are extensions of it for $i = 1, 2, \ldots, k$. Then the **relatively independent product** of the systems $X_i$ over their factor maps $\pi_i$ is the system

$$\prod_{\{\pi_1 = \ldots = \pi_k\}} X_i = \left(\prod_{\{\pi_1 = \ldots = \pi_k\}} X_i, \bigotimes_{\{\pi_1 = \ldots = \pi_k\}} \Sigma_i, \bigotimes_{\{\pi_1 = \ldots = \pi_k\}} \mu_i, T_1 \times \cdots \times T_k\right)$$
where
\[ \prod_{\{\pi_1 = \ldots = \pi_k\}} X_i := \{ (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k : \pi_1(x_1) = \ldots = \pi_k(x_k) \}, \]
\[ \bigotimes_{\{\pi_1 = \ldots = \pi_k\}} \Sigma_i \]
is the restriction of \( \Sigma_1 \otimes \cdots \otimes \Sigma_k \) to this subset of \( X_1 \times \cdots \times X_k \), and
\[ \bigotimes_{\{\pi_1 = \ldots = \pi_k\}} \mu_i := \int_Y \bigotimes_{i=1}^k \mu_{i,y} \nu(dy) \]
with \( y \mapsto \mu_{i,y} \) an arbitrary choice of disintegration of \( \mu_i \) over \( \pi_i \). A quick check shows that the factors generated by the coordinate projections \( \phi_j : \prod_{\{\pi_1 = \ldots = \pi_k\}} X_i \rightarrow X_j \) are relatively independent over the common further factor map
\[ \pi_1 \circ \phi_1 \simeq \ldots \simeq \pi_k \circ \phi_k : \prod_{\{\pi_1 = \ldots = \pi_k\}} X_i \rightarrow Y. \]

In case \( k = 2 \) we write the relatively independent product more simply as \( X_1 \times_{\{\pi_1 = \pi_2\}} X_2 \), and in addition if \( X_1 = X_2 = X \) and \( \pi_1 = \pi_2 = \pi \) then we will abbreviate this further to \( X \times_\pi X \), and similarly for the individual spaces and measures.

The need for the invertibility of \( T \) in this construction arises in checking that \( \bigotimes_{\{\pi_1 = \ldots = \pi_k\}} \mu_i \) is invariant under the product action. For example, if \( k = 2 \) then the invariance of \( \mu_i \) under \( T_i \) implies that for each \( \gamma \in \Gamma \) the disintegrations \( \mu_{i,y} \) satisfy
\[ \int_Y (T_{i \gamma})_#^\# \mu_{i,y} \nu(dy) = \int_Y \mu_{i,y} \nu(dy). \]

However, to argue from here to the invariance of \( \mu_1 \otimes_{\{\pi_1 = \pi_2\}} \mu_2 \) we must know in addition that for \( \nu \)-almost every \( y \in Y \) there is a unique point \( S_\gamma^{-1}(y) \in Y \) such that \( (T_{i \gamma})_#^\# \mu_{i,S_\gamma^{-1}(y)} \) is supported on the fibre over \( y \). Given this and the essential uniqueness of disintegrations, the above equation implies that \( (T_{i \gamma})_#^\# \mu_{i,y} = \mu_{i,S_\gamma(y)} \) for \( \nu \)-almost every \( y \), from which it also follows that
\[ (T_1 \times T_2)_#^\# (\mu_{1,y} \otimes \mu_{2,y}) = (\mu_{1,S_\gamma(y)} \otimes \mu_{2,S_\gamma(y)}) \]
\( \nu \)-almost surely, so that integrating again with respect to \( y \) gives the desired invariance of \( \mu_1 \otimes_{\{\pi_1 = \pi_2\}} \mu_2 \). However, this latter argument is valid only if we can obtain the above equality pointwise in \( y \), and this can fail if \( T_{i \gamma} \) is not invertible.
Inverse limits

An inverse sequence of \( \Gamma \)-systems is a family of \( \Gamma \)-systems \( (X_m, \Sigma_m, \mu_m, T_m) \) together with factor maps

\[
\psi^m_k : (X_m, \Sigma_m, \mu_m, T_m) \rightarrow (X_k, \Sigma_k, \mu_k, T_k) \quad \text{for all } m \geq k
\]
satisfying the compatibility property that \( \psi^k_\ell \circ \psi^m_k = \psi^m_\ell \) whenever \( m \geq k \geq \ell \). From such a family one can construct an inverse limit

\[
\lim_{m \leftarrow} \left( (X_m, \Sigma_m, \mu_m, T_m, (\psi^m_k)_{m \geq k}) \right) =: (X, \Sigma, \mu, T)
\]
together with a sequence of factor maps

\[
\psi_m : (X, \Sigma, \mu, T) \rightarrow (X_m, \Sigma_m, \mu_m, T_m)
\]
such that \( \psi^m_k \circ \psi_m = \psi^m_\ell \) whenever \( m \geq k \), and such that the lifted factors \( \psi_m^{-1}(\Sigma_m) \) together generate the whole of \( \Sigma \). Moreover, subject to these stipulations this inverse limit is unique up to isomorphisms that intertwine all the factor maps \( \psi_m \). This construction is described, for example, in Section 6.3 of Glasner [Gla03].

2.2 Idempotent classes

In much of the following we will be concerned with properties of one system that are defined relative to some other class of systems.

Definition 2.2.1 (Idempotent class). A subclass \( C \) of \( \Gamma \)-systems is idempotent if it contains the trivial system and is closed under measure-theoretic isomorphism, inverse limits and joinings.

Note that our ‘classes’ need not be sets in the sense of ZFC. In all subsequent constructions involving these classes it will be clear that we need only some set-indexed family of members, and so we will not generally pass comment on this set-theoretic distinction. Alternatively, we could circumvent this issue altogether by working only with probability-preserving systems modelled by some Borel transformations and invariant probability measure on, say, the Cantor space, since any standard Borel system admits such a
model up to measure-theoretic isomorphism (see, for instance, Theorem 2.15 in [Gla03]).

**Examples** Suppose that $\Gamma$ is a group and that $\Lambda \trianglelefteq \Gamma$. Then the class of all $\Gamma$-systems for which the subaction of $\Lambda$ is trivial is easily seen to be idempotent. This important example will usually be denoted by $Z_0^\Lambda$ in the following.

More generally, for $\Lambda$ as above and any $n \in \mathbb{N}$ we let $Z_n^\Lambda$ denote the class of systems on which the $\Lambda$-subaction is a distal tower of height at most $n$, in the sense of direct integrals of compact homogeneous space data introduced in [Ausc] to allow for the case of non-ergodic systems. Standard results on the possible joinings and inverse limits of isometric extensions show that this class is idempotent (see [Ausc, Ausd]). Those arguments also allow us to identify certain natural idempotent subclasses of $Z_n^\Lambda$, such as the class $Z_{\text{Ab},n}^\Lambda$ of those systems with $\Lambda$-subaction a distal tower of height at most $n$ and in which each isometric extension is Abelian.

**Lemma 2.2.2.** If $C$ is an idempotent class of $\Gamma$-systems then any $\Gamma$-system $X$ has an essentially unique maximal factor in the class $C$.

**Proof** It is clear that under the above assumption the family of factors

$$\{ \Xi \leq \Sigma : \Xi \text{ is generated by a factor map to a system in } C \}$$

is nonempty (it contains $\{ \emptyset, X \}$, which corresponds to the trivial system), upwards directed (because $C$ is closed under joinings) and closed under taking $\sigma$-algebra completions of increasing unions (because $C$ is closed under inverse limits). There is therefore a maximal $\sigma$-subalgebra in this family.

**Definition 2.2.3.** If $C$ is an idempotent class then $X$ is a $C$-system if $X \in C$, and for any $X$ we write $\zeta_X^C : X \rightarrow CX$ for an arbitrarily-chosen coordinatization of its maximal $C$-factor given by the above lemma.

It is clear that if $\pi : X \rightarrow Y$ then $\zeta_X^C \supseteq \zeta_Y^C \circ \pi$, and so there is an essentially unique factorizing map, which we denote by $C\pi$, that makes the following diagram commute:
In addition, we shall abbreviate $X \times \zeta_X X$ to $X \times C X$, and similarly for the individual spaces and measures defining this relatively independent product.

The above lemma and definition explain the choice of the term ‘idempotent’, which is motivated by a more categorial viewpoint of such subclasses: if we identify such a class $C$ with a full subcategory of the category of $\Gamma$-systems with factor maps as morphisms, then the assignments $X \mapsto CX$, $\pi \mapsto C\pi$ define an autofunctor of this category which is idempotent.

The name we give for our next definition is also motivated by this relationship with functors.

**Definition 2.2.4 (Order continuity).** A class of $\Gamma$-systems $C$ is order continuous if whenever $(X_m)_{m \geq 0}, (\psi_m^k)_{m \geq k \geq 0}$ is an inverse sequence of $\Gamma$-systems with inverse limit $X$, $(\psi_m^k)_{m \geq 0}$ we have

$$\zeta_X^C = \bigvee_{m \geq 0} \zeta_{Cm} \circ \psi_m^k :$$

that is, the maximal $C$-factor of the inverse limit is simply given by the (increasing) join of the maximal $C$-factors of the contributing systems.

**Example** Although all the idempotent classes that will matter to us later can be shown to be order continuous, it may be instructive to exhibit one that is not. In case $\Gamma$ is an Abelian group, let us say that a system $X$ has a finite-dimensional Kronecker factor if its Kronecker factor $\zeta_1^X : X \rightarrow \mathbb{Z}_1^X$ can be coordinatized as a direct integral (see Section 3 of [Ausc]) of rotations on some measurably-varying compact Abelian groups all of which can be isomorphically embedded into a fibre repository $T^D$ for some fixed $D \in \mathbb{N}$ (this includes the possibility that the Kronecker factor is finite or trivial). It is now easy to check that the class of $\mathbb{Z}$-systems comprising all those that are either themselves finite-dimensional Kronecker systems, or have a Kronecker factor that is *not* finite-dimensional (so we exclude just those systems that have a
finite-dimensional Kronecker factor but properly contain it), is idempotent but not order continuous, since any infinite-dimensional separable group rotation can be identified with an inverse limit of finite-dimensional group rotations. ⊳

**Definition 2.2.5** (Hereditariness). An idempotent class $C$ is **hereditary** if it is also closed under taking factors.

**Definition 2.2.6** (Join). If $C_1$, $C_2$ are idempotent classes, then the class $C_1 \lor C_2$ of all joinings of members of $C_1$ and $C_2$ is clearly also idempotent. We call $C_1 \lor C_2$ the **join** of $C_1$ and $C_2$.

**Lemma 2.2.7** (Join preserves order continuity). If $C_1$ and $C_2$ are both order continuous then so is $C_1 \lor C_2$.

**Proof** Let $(X_m)_{m \geq 0}, (\psi_m^m)_{m \geq k \geq 0}$ be an inverse sequence with inverse limit $X, (\psi_m^m)_{m \geq 0}$. Then $\zeta_{C_1 \lor C_2}^X$ is the maximal factor of $X$ that is a joining of a $C_1$-factor and a $C_2$-factor (so, in particular, it must be generated by its own $C_1$- and $C_2$-factors), and hence it is equivalent to $\zeta_{C_1}^X \lor \zeta_{C_2}^X$. Therefore any $f \in L^\infty(\mu)$ that is $\zeta_{C_1}^X \lor \zeta_{C_2}^X$-measurable can be approximated in $L^2(\mu)$ by some function of the finite-sum form $\sum_p g_{p,1} \cdot g_{p,2}$ with each $g_{p,i} \in L^\infty(\mu)$ being $C_i$-measurable, and now since each $C_i$ is order continuous we may further approximate each $g_{p,i}$ by some $h_{p,i} \circ \psi_m$ for a large integer $m$ and some $C_i$-measurable $h_{p,i} \in L^\infty(\mu_m)$. Combining these approximations completes the proof. ⊳

**Examples** Of course, we can form the joins of any of our earlier examples of idempotent classes: for example, given a group $\Gamma$ and subgroups $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \triangleleft \Gamma$ we can form $Z_{\Lambda_1}^0 \lor Z_{\Lambda_2}^0 \lor \cdots \lor Z_{\Lambda_n}^0$. This particular example and several others like it will appear frequently throughout the rest of this work. Clearly each class $Z_{\Lambda}^0$ is hereditary, but in general joins of several such classes are not; we will see this explicitly in the first example of the next section. ⊳

The following terminology will also prove useful.

**Definition 2.2.8** (Joining to an idempotent class; adjoining). If $X$ is a system and $C$ is an idempotent class then a **joining of $X$ to $C$** or a **$C$-adjoining of $X$** is a joining of $X$ and $Y$ for some $Y \in C$.
2.3 Sated systems

The remainder of this dissertation concerns the consequences of one basic idea: that by extending a probability-preserving system, it is sometimes possible to impose on it some additional structure that makes its behaviour more transparent. For our later applications, a notion of ‘additional structure’ that is both useful and obtainable is best summarized by demanding that the system does not admit a nontrivial joining to systems drawn from various other special classes. We will soon show that all systems admit extensions for which some version of this is true. This idea, although very abstract and very simple, will repeatedly prove surprisingly powerful.

**Definition 2.3.1** (Sated system). Given an idempotent class $C$, a system $X$ is $C$-sated if whenever $\pi : \tilde{X} = (\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T}) \rightarrow X$ is an extension, the factor maps $\pi$ and $\zeta^\tilde{X}$ on $\tilde{X}$ are relatively independent over $\zeta^X \circ \pi = C \pi \circ \zeta^\tilde{X}$ under $\tilde{\mu}$. Phrased more pictorially, the two systems in the middle row of the commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow \zeta^\tilde{X} & & \downarrow \zeta^X \\
\tilde{C}X & \xleftarrow{C\pi} & CX
\end{array}
\]

are relatively independent over their common factor copy of the system $CX$.

An inverse sequence is $C$-sated if it has a cofinal subsequence all of whose systems are $C$-sated.

**Remark** This definition has an important precedent in Furstenberg and Weiss’ notion of a ‘pair homomorphism’ between extensions elaborated in Section 8 of [FW96].

**Example** If $X = (U, \text{Borel, Haar, } R_\phi)$ with $U$ a compact metrizable Abelian group, $\phi : \mathbb{Z}^2 \rightarrow U$ a dense homomorphism and $R_\phi$ the corresponding action of $\mathbb{Z}^2$ by rotations (so $R^n(\cdot) := \cdot + \phi(n)$), then $\mathbb{Z}_0^n X$ is coordinatized by the quotient homomorphism

$$U \rightarrow U/\phi(\mathbb{Z}_0),$$
and so $X$ is a member of $Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$ if and only if these quotients together generate the whole of $U$, hence if and only if $\phi(Z_{e_{1}}) \cap \phi(Z_{e_{2}}) = \{0\}$.

On the other hand, any ergodic action $X$ of $\mathbb{Z}^{2}$ by compact group rotations can be extended to a member of $Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$. To see this we first note that ergodicity is equivalent to the denseness of $\phi(Z_{2})$, and so in particular that $\phi(Z_{e_{1}}) + \phi(Z_{e_{2}}) = U$. It follows that the ‘larger’ group rotation system

$$\tilde{X} = (\tilde{U}, \text{Borel, Haar, } R_{\tilde{\phi}}),$$

where $\tilde{U} := \phi(Z_{e_{1}}) \oplus \phi(Z_{e_{2}})$ and the homomorphism $\tilde{\phi} : \mathbb{Z}^{2} \longrightarrow U^{2}$ is defined by

$$\tilde{\phi}(e_{1}) := (\phi(e_{1}), 0) \quad \text{and} \quad \tilde{\phi}(e_{2}) := (0, \phi(e_{2})),$$

is an extension of $X$ through the factor map

$$\tilde{U} \longrightarrow U : (x, y) \mapsto x + y.$$

Now $\tilde{X}$ clearly satisfies the above condition for membership of $Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$, since the quotients by $\phi(Z_{e_{i}})$ for $i = 1, 2$ are respectively the second and first coordinate projections. It follows that every such $X$ admits a $(Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}})$-adjoining that generates the whole of $X$, and which is therefore not relatively independent over any proper factor of $X$, and hence that $X$ itself is $(Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}})$-sated if and only if it is already in the class $Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$. This reasoning also shows that the class $Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$ is not hereditary.

A little more generally, if $X$ is a totally weakly mixing extension of an ergodic action $Y$ of $\mathbb{Z}^{2}$ by compact group rotations, then routine arguments show that $X$ is $(Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}})$-sated if and only if this is true of $Y$ (since a totally weakly mixing extension is relatively disjoint from any $Z_{0}^{e_{1}}$-system, and given this the Furstenberg-Zimmer Inverse Theorem implies that the $e_{2}$-invariant factor of any $Z_{0}^{e_{1}}$-adjoining of $X$ is also relatively independent from $X$ over its factor map to $Y$; see, for instance, Chapters 9 and 10 of [Gla03]). Therefore such an $X$ is $(Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}})$-sated if and only if $Y \in Z_{0}^{e_{1}} \vee Z_{0}^{e_{2}}$. \hspace{1cm} <

The crucial technical fact that turns satedness into a useful tool is the ability to construct sated extensions of arbitrary systems. This can be seen as a natural abstraction from Propositions 4.6 of [Aus09] and 4.3 of [Ausb], and appears in its full strength as Theorem 3.11 in [Ausd].
**Theorem 2.3.2** (Idempotent classes admit multiply sated extensions). If \((C_i)_{i \in I}\) is a countable family of idempotent classes then any system \(X_0\) admits an extension \(\pi : X \longrightarrow X_0\) such that

- \(X\) is \(C_i\)-sated for every \(i \in I\);
- the factors \(\pi\) and \(\bigvee_{i \in I} \zeta_{C_i}^X\) generate the whole of \(X\).

We shall prove this result after a preliminary lemma.

**Lemma 2.3.3.** If \(C\) is an idempotent class then the inverse limit of any \(C\)-sated inverse sequence is \(C\)-sated.

**Proof** By passing to a subsequence if necessary, it suffices to suppose that \((X_m)_{m \geq 0}, (\psi_m)_{m \geq k \geq 0}\) is an inverse sequence of \(C\)-sated systems with inverse limit \(X_\infty\), \((\psi_m)_{m \geq 1}\), and let \(\pi : X \longrightarrow X_\infty\) be any further extension and \(f \in L^\infty(\mu_\infty)\). We will commit the abuse of identifying such a function with its lift to any given extension when the extension in question is obvious. With this in mind, we need to show that

\[
E(f \mid \zeta_{\tilde{X}}) = E(f \mid \zeta_X^{\infty}).
\]

However, by the \(C\)-satedness of each \(X_m\), we certainly have

\[
E(E(f \mid \psi_m) \mid \zeta_{\tilde{X}}) = E(f \mid \zeta_X^{m}),
\]

and now as \(m \longrightarrow \infty\) this equation converges in \(L^2(\mu)\) to

\[
E(f \mid \zeta_{\tilde{X}}) = E\left(f \mid \bigvee_{m \geq 1} (\zeta_{C}^{X_m} \circ \psi_m)\right).
\]

By monotonicity we have

\[
\zeta_{\tilde{X}} \succ \zeta_X^{\infty} \succ \bigvee_{m \geq 1} (\zeta_{C}^{X_m} \circ \psi_m),
\]

and so by sandwiching the desired equality of conditional expectations must also hold. \(\square\)
Proof of Theorem 2.3.2  We first prove this for $I$ a singleton, and then in the general case.

Step 1  Suppose that $I = \{i\}$ and $C_i = C$. This case will follow from a simple ‘energy increment’ argument.

Let $(f_r)_{r \geq 1}$ be a countable subset of the $L^\infty$-unit ball $\{ f \in L^\infty(\mu) : \| f \|_\infty \leq 1 \}$ that is dense in this ball for the $L^2$-norm, and let $(r_i)_{i \geq 1}$ be a member of $\mathbb{N}^\mathbb{N}$ in which every non-negative integer appears infinitely often.

We will construct an inverse sequence $(X_m)_{m \geq 0}$, $(\psi^m_k)_{m \geq k \geq 0}$ starting from $X_0$ such that each $X_{m+1}$ is a $C$-adjoining of $X_m$. Suppose that for some $m_1 \geq 0$ we have already obtained $(X_m)_{m=0}^{m_1}$, $(\psi^m_k)_{m \geq k \geq 0}$ such that $\text{id}_{X_{m_1}} \simeq \zeta^{X_{m_1}} \cup \psi^{m_1}_0$. We consider two separate cases:

- If there is some further extension $\pi : \tilde{X} \to X_{m_1}$ such that
  \[
  \| E_{\tilde{\mu}}(f_{r_{m_1}} \circ \psi^{m_1}_0 \circ \pi | \tilde{X}_C) \|_2^2 > \| E_{\mu_{m_1}}(f_{r_{m_1}} \circ \psi^{m_1}_0 | \zeta^{X_{m_1}}) \|_2^2 + 2^{-m_1},
  \]
  then choose a particular $\pi : \tilde{X} \to X_{m_1}$ such that the increase
  \[
  \| E_{\tilde{\mu}}(f_{r_{m_1}} \circ \psi^{m_1}_0 \circ \pi | \tilde{X}_C) \|_2^2 - \| E_{\mu_{m_1}}(f_{r_{m_1}} \circ \psi^{m_1}_0 | \zeta^{X_{m_1}}) \|_2^2
  \]
  is at least half its supremal possible value over all extensions. By restricting to the possibly smaller subextension of $\tilde{X} \to X_{m_1}$ generated by $\pi$ and $\zeta^{\tilde{X}}$ we may assume that $\tilde{X}$ is itself a $C$-adjoining of $X_{m_1}$ and hence of $X_0$, and now we let $X_{m_1+1} := \tilde{X}$ and $\psi^{m_1+1} := \pi$ (the other connecting factor maps being determined by this one).

- If, on the other hand, for every further extension $\pi : \tilde{X} \to X_{m_1}$ we have
  \[
  \| E_{\tilde{\mu}}(f_{r_{m_1}} \circ \psi^{m_1}_0 \circ \pi | \tilde{X}_C) \|_2 \leq \| E_{\mu_{m_1}}(f_{r_{m_1}} \circ \psi^{m_1}_0 | \zeta^{X_{m_1}}) \|_2^2 + 2^{-m_1},
  \]
  then we simply set $X_{m_1+1} := X_{m_1}$ and $\psi^{m_1+1} := \text{id}_{X_{m_1}}$.

Finally, let $X_\infty$, $(\psi^m)_m \geq 0$ be the inverse limit of this sequence. We have
\[
\text{id}_{X_\infty} \simeq \bigvee_{m \geq 0} \psi^m \simeq \bigvee_{m \geq 0} (\zeta^X \cup \psi^m) \circ \psi^m \simeq \bigvee_{m \geq 0} (\zeta^X \circ \psi^m) \simeq \zeta^X \cup \psi_0,
\]
so \( X_\infty \) is still a \( C \)-adjoining of \( X_0 \). To show that it is \( C \)-sated, let \( \pi : \tilde{X} \to X_\infty \) be any further extension, and suppose that \( f \in L^\infty(\mu_\infty) \). We will complete the proof for Step 1 by showing that

\[
\mathbb{E}_{\tilde{\mu}}(f \circ \pi | \zeta_C) = \mathbb{E}_{\mu_\infty}(f | \zeta_C X_\infty) \circ \pi.
\]

Since \( X_\infty \) is a \( C \)-adjoining of \( X \), this \( f \) may be approximated arbitrarily well in \( L^2(\mu_\infty) \) by finite sums of the form \( \sum_p g_p \cdot h_p \) with \( g_p \) being bounded and \( \zeta_C X_\infty \)-measurable and \( h_p \) being bounded and \( \psi_0 \)-measurable, and now by density we may also restrict to using \( h_p \) that are each a scalar multiple of some \( f_{r,\psi_0} \). So by continuity and multilinearity it suffices to prove the above equality for just one such product \( g \cdot (f \circ \psi_0) \). Since \( g \) is \( \zeta_C X_\infty \)-measurable, this requirement now reduces to

\[
\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C) = \mathbb{E}_{\mu_\infty}(f \circ \psi_0 | \zeta_C X_\infty) \circ \pi.
\]

Since \( \zeta_C \tilde{X} \ni \zeta_C X_\infty \circ \pi \), this will follow if we only show that

\[
\|\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 = \|\mathbb{E}_{\mu_\infty}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2.
\]

Now, by the martingale convergence theorem we have

\[
\|\mathbb{E}_{\mu_m}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 \to \|\mathbb{E}_{\mu_\infty}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2
\]

as \( m \to \infty \). It follows that if

\[
\|\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 > \|\mathbb{E}_{\mu_\infty}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2
\]

then for some sufficiently large \( m \) we would have \( r_m = r \) (since each integer appears infinitely often as some \( r_m \)) but also

\[
\|\mathbb{E}_{\mu_m}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 = \|\mathbb{E}_{\mu_m}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2
\]

\[
\leq \|\mathbb{E}_{\mu_m}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2 - \|\mathbb{E}_{\mu_m}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2
\]

\[
< \frac{1}{2} \left( \|\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 - \|\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 \right)
\]

and

\[
\|\mathbb{E}_{\tilde{\mu}}(f \circ \psi_0 \circ \pi | \zeta_C)\|_2^2 > \|\mathbb{E}_{\mu_m}(f \circ \psi_0 | \zeta_C X_\infty)\|_2^2 + 2^{-m},
\]
so contradicting our choice of $X_{m+1} \rightarrow X_m$ in the first alternative in our construction above. This contradiction shows that we must actually have the equality of $L^2$-norms required.

**Step 2** The general case follows easily from Step 1 and a second inverse limit construction: choose a sequence $(i_m)_{m \geq 1} \in \mathcal{I}$ in which each member of $\mathcal{I}$ appears infinitely often, and form an inverse sequence $(X_m)_{m \geq 0}$, $(\psi_k^m)_{m \geq k \geq 0}$ starting from $X_0$ such that each $X_m$ is $C_{i_m}$-sated for $m \geq 1$. The inverse limit $X$ is now sated for every $C_i$, by Lemma 2.3.3. □

**Remark** Thierry de la Rue has shown me another proof of Theorem 2.3.2 in case $\Gamma$ is a group that follows very quickly from ideas contained in his paper [LRR03] with Lesigne and Rittaud, and which has now received a nice separate writeup in [Rue]. The key observation is that

*An idempotent class $C$ is hereditary if and only if every system is $C$-sated.*

This in turn follows from a striking result of Lemańczyk, Parreau and Thouvenot [LPT00] that if two systems $X$ and $Y$ are not disjoint then $X$ shares a nontrivial factor with the infinite Cartesian power $Y \times \infty$. Given now an idempotent class $C$ and a system $X$, let $C^*$ be the hereditary idempotent class of all factors of members of $C$, and let $Y$ be any $C$-system admitting a factor map $\pi : Y \rightarrow C^*X$ (such exists because by definition $C^*X$ is a factor of some $C$-system). Now forming $\tilde{X} := X \times \{\xi = \pi\} Y$ (so here is where we need $\Gamma$ to be a group), a quick check using the above fact shows that $C\tilde{X} = C^*\tilde{X}$, and that this is equivalent to the $C$-satedness of $\tilde{X}$. △
Chapter 3

The convergence of nonconventional averages

In this chapter Theorem C will be deduced from Theorem 2.3.2. This amounts to a rather simpler outing for many of the same ideas that will go into proving recurrence in the next chapter.

We first recall the Hilbert space version of a classical estimate due to van der Corput, which has long been a workhorse of Ergodic Ramsey Theory. After giving this its own section, the Furstenberg self-joining for a tuple of transformations is introduced, and then in the last section we show how the right instance of satedness implies that these enjoy some additional structure from which a proof of Theorem C follows quite quickly.

Notation

Before commencing with any of these proofs, we make a slight modification to the notation of the Introduction to be more in keeping with that of Chapter 2: rather than letting $T_1, T_2, \ldots, T_d$ denote a tuple of commuting individual transformations on $(X, \Sigma, \mu)$, we henceforth regard these as the subactions of the basis vectors $e_1, e_2, \ldots, e_d$ for a single $\mathbb{Z}^d$-action $T$. Theorem C is accordingly re-phrased as asserting that the averages

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T^{ne_1}) \cdot (f_2 \circ T^{ne_2}) \cdot \ldots \cdot (f_d \circ T^{ne_d})$$
converge in $L^2(\mu)$ for any $\mathbb{Z}^d$-system $(X, \Sigma, \mu, T)$. This slight increase in abstraction will prove worth tolerating when we come to various constructions of new actions from old during our later arguments, in which we will need to keep efficient track of how the action of one vector in $\mathbb{Z}^d$ may have been re-assigned to that of another. It follows that in the remainder of this work, a list such as ‘$T_1, T_2, \ldots, T_d$’ will denote a tuple of whole actions of some previously-decided group, rather than individual transformations.

3.1 The van der Corput estimate

This result and a related discussion can be found, for example, as Theorem 2.2 of Bergelson [Ber96].

**Proposition 3.1.1 (Van der Corput estimate).** Suppose that $(u_n)_{n \geq 1}$ is a bounded sequence in a Hilbert space $\mathcal{H}$. If the vector-valued averages

$$\frac{1}{N} \sum_{n=1}^{N} u_n$$

do not converge to 0 in norm as $N \to \infty$, then also the scalar-valued averages

$$\frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} \langle u_n, u_{n+m} \rangle$$

do not converge to 0 as $N \to \infty$ and then $M \to \infty$.

**Proof** For any fixed $H \geq 1$ we have

$$\frac{1}{N} \sum_{n=1}^{N} u_n \sim \frac{1}{N} \sum_{n=1}^{N} \frac{1}{H} \sum_{h=1}^{H} u_{n+h}$$

as $N \to \infty$, where the notation $w_N \sim v_N$ denotes that $w_N - v_N \to 0$ in $\mathcal{H}$. However, the squared norm of the right-hand double average may be estimated by

$$\left\| \frac{1}{N} \sum_{n=1}^{N} \frac{1}{H} \sum_{h=1}^{H} u_{n+h} \right\|^2 \leq \frac{1}{N} \sum_{n=1}^{N} \left\| \frac{1}{H} \sum_{h=1}^{H} u_{n+h} \right\|^2$$

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It follows that these averages must also not converge to 0 as $N \to \infty$ and then $H \to \infty$; but for large $H$ these can be expressed as averages of the averages
\[
\frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} \langle u_n, u_{n+m} \rangle
\]
for correspondingly large values of $M$, and so these also cannot converge to 0 as $N \to \infty$ and then $M \to \infty$, as required. □

3.2 The Furstenberg self-joining

Theorem C is proved by induction on $d$. In the first instance, this induction is enabled by a construction that is made possible once convergence is known for a smaller number of transformations, and which will also be central to the proof of Theorem A in the next chapter.

Thus, suppose now that for some $d \geq 1$ the convergence of Theorem C is known for all tuples of at most $d-1$ commuting transformations (so this assumption is vacuous if $d = 1$). Let $X = (X, \Sigma, \mu, T)$ be a $\mathbb{Z}^d$-system, and let $A_1, A_2, \ldots, A_d \in \Sigma$. By integrating and using the invariance of $\mu$ under $T^{e_1}$, our assumption applied to the transformations $T^{e_2-e_1}, \ldots, T^{e_d-e_1}$ implies that the scalar averages
\[
\frac{1}{N} \sum_{n=1}^{N} \mu(T^{-ne_1}(A_1) \cap T^{-ne_2}(A_2) \cap \cdots \cap T^{-ne_d}(A_d))
\]
\[
= \int_X 1_{A_1} \cdot \left( \frac{1}{N} \sum_{n=1}^{N} (1_{A_2} \circ T^{n(e_2-e_1)}) \cdots (1_{A_d} \circ T^{n(e_d-e_1)}) \right) d\mu
\]
converge as $N \to \infty$. Moreover, the limit takes the form $\mu^F(A_1 \times A_2 \times \cdots \times A_d)$ for some probability $\mu^F$ on $X^d$ that is invariant under the diagonal
The $\mathbb{Z}^d$-system $X^F := (X^d, \Sigma^\otimes d, \mu^F, T^\times d)$ is therefore a $d$-fold self-joining of $X$ through the $d$ coordinate projections $\pi_i : X^F \to X$. We refer to either $\mu^F$ or $X^F$ as the Furstenberg self-joining of $X$. Given functions $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$, by approximating each of them in $L^\infty$ using step functions we may extend the above definition of $\mu^F$ to the convergence

$$\frac{1}{N} \sum_{n=1}^{N} \int_X (f_1 \circ T^{ne_1}) \cdot (f_2 \circ T^{ne_2}) \cdot \cdots \cdot (f_d \circ T^{ne_d}) \, d\mu \to \int_{X^d} f_1 \otimes f_2 \otimes \cdots \otimes f_d \, d\mu^F$$

as $N \to \infty$.

In addition to its invariance under $T^\times d$, the definition of $\mu^F$ gives an additional invariance that will shortly prove crucial.

**Lemma 3.2.1.** Provided the limiting self-joining $\mu^F$ exists, it is also invariant under the transformation $T^{e_1} \times T^{e_2} \times \cdots \times T^{ed}$.

**Proof** For any $A_1, A_2, \ldots, A_d \in \Sigma$ we have

$$\mu^F((T^{e_1} \times T^{e_2} \times \cdots \times T^{ed})^{-1}(A_1 \times A_2 \times \cdots \times A_d))$$

$$= \lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu((T^{-ne_1} A_1) \cap \cdots \cap T^{-ne_d} A_d))$$

$$= \lim_{n \to \infty} \frac{1}{N} \sum_{n=2}^{N+1} \mu((T^{-ne_1} A_1) \cap \cdots \cap T^{-ne_d} A_d))$$

$$= \mu^F(A_1 \times A_2 \times \cdots \times A_d),$$

where the last equality follows because the discrete intervals $\{1, 2, \ldots, N\}$ and $\{2, 3, \ldots, N + 1\}$ asymptotically overlap in $1 - o(1)$ of their lengths. □

It will be important to know that Furstenberg self-joinings behave well under inverse limits. The following is another immediate consequence of the definition, and we omit the proof.
Lemma 3.2.2. If \( (X_m)_{m \geq 0}, (\psi^m_k)_{m \geq k \geq 0} \) is an inverse sequence with inverse limit \( X \), \( (\psi^m_k)_{m \geq k \geq 0} \) is an inverse sequence with inverse limit \( X \), then the Furstenberg self-joinings \( X^F \) form an inverse sequence under the factor maps \( (\psi^m)^{x_d} \) with inverse limit \( X^F \), \( (\psi^m)^{x_d} \).

\[ \square \]

3.3 The proof of convergence

The final observation needed before we prove Theorem C is that satedness implies a certain inverse result for the situation in which the functional averages

\[
S_N(f_1, f_2, \ldots, f_d) := \frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T^{ne_1}) \cdot (f_2 \circ T^{ne_2}) \cdot \ldots \cdot (f_d \circ T^{ne_d})
\]

do not converge to 0.

**Proposition 3.3.1.** Suppose that \( X \) is \( C \)-sated for the idempotent class

\[ C := \mathbb{Z}_{0}^{e_1} \vee \bigvee_{j=2}^{d} \mathbb{Z}_{0}^{e_1-e_j} \]

and that \( f_i \in L^\infty(\mu) \) for \( i = 1, 2, \ldots, d \). In addition, let \( \Phi := \Sigma^{T^{e_1}} \vee \bigvee_{j=2}^{d} \Sigma^{T^{e_1}-T^{e_j}} \), so this is a factor of \( X \). If

\[
S_N(f_1, f_2, \ldots, f_d) \not\rightarrow 0
\]

as \( N \rightarrow \infty \), then also \( \mathbb{E}(f_1 | \Phi) \neq 0 \).

**Remark** In the terminology of [FW96], which has since become standard in this area (and is roughly followed in [Aus09]), this asserts that for a \( C \)-sated system \( X \) the factor \( \Phi \) is **partially characteristic**.

**Proof** This rests on an appeal to the van der Corput estimate followed by a re-interpretation of what it tells us. Letting \( u_n := (f_1 \circ T^{ne_1}) \cdot (f_2 \circ T^{ne_2}) \cdot \ldots \cdot (f_d \circ T^{ne_d}) \), Proposition 3.1.1 and our assumption imply that the double
averages

\[
\frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} (u_n, u_{n+m})
\]

\[
= \frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} \int_X \left((f_1 \circ T^{m\epsilon_1}) \cdots (f_d \circ T^{m\epsilon_d}) \right)
\]

\[
\cdot \left((\overline{f_1} \circ T^{(n+m)\epsilon_1}) \cdots (\overline{f_d} \circ T^{(n+m)\epsilon_d}) \right) \, d\mu
\]

do not tend to 0 as \(N \to \infty\) and then \(M \to \infty\). However, simply by re-arranging the individual functions and recalling the definition of \(\mu^F\), the limit in \(N\) behaves as

\[
\frac{1}{M} \sum_{m=1}^{M} \frac{1}{N} \sum_{n=1}^{N} \int_X \left((f_1 \cdots f_d) \right)
\]

\[
\cdot \left((\overline{f_1} \cdots \overline{f_d} \circ (T^{\epsilon_1} \times \cdots \times T^{\epsilon_d})^m) \right) \, d\mu^F
\]

\[
= \frac{1}{M} \sum_{m=1}^{M} \int_{X^d} (f_1 \cdots f_d) \cdot \left((\overline{f_1} \cdots \overline{f_d} \circ \Sigma^{\otimes d} T^{\epsilon_1 \cdots \epsilon_d}) \right) \, d\mu^F.
\]

Now, since Lemma 3.2.1 gives that \(\mu^F\) is invariant under \(T^{\epsilon_1} \times T^{\epsilon_2} \times \cdots \times T^{\epsilon_d}\), the classical mean ergodic theorem allows us to take the limit in \(M\) to obtain

\[
\int_{X^d} (f_1 \cdots f_d) \cdot E_{\mu^F} \left((\overline{f_1} \cdots \overline{f_d} \circ (\Sigma^{\otimes d} T^{\epsilon_1 \cdots \epsilon_d}) \right) \, d\mu^F.
\]

Thus the van der Corput estimate tells us that this integral is non-zero. The proof is completed simply by re-phrasing this conclusion slightly. We have previously used \(\mu^F\) to define a \(\mathbb{Z}^d\)-system \(X^F\), but in light of Lemma 3.2.1 we may alternatively use it to define a \(\mathbb{Z}^d\)-system \(\tilde{X}\) by setting

\[
(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}) := (X^d, \Sigma^{\otimes d}, \mu^F),
\]

\[
\tilde{T}^{\epsilon_1} := T^{\epsilon_1} \times T^{\epsilon_2} \times \cdots \times T^{\epsilon_d}
\]
and
\[ \tilde{T}^{e_i} := (T \times d)^{e_i} \quad \text{for } i = 2, 3, \ldots, d \]
(thus, the basis direction \( e_1 \) is treated differently from the others). With this definition the first coordinate projection \( \pi_1 : X^d \to X \) still defines a factor map of \( \mathbb{Z}^d \)-systems \( \tilde{X} \to X \), because \( \tilde{T}^n \) does agree with \( T^n \) on the first coordinate in \( X^d \) for every \( n \). On the other hand, for \( i = 2, 3, \ldots, d \) the function \( f_i \circ \pi_i \in L^\infty(\mu^F) \) depends only on the \( i \)th coordinate in \( X^d \), and on this coordinate the transformations \( \tilde{T}^{e_1} \) and \( \tilde{T}^{e_i} \) agree, so that \( f_i \circ \pi_i \) is \( \tilde{T}^{e_i-} - \)invariant. Thus the nonvanishing
\[ \int_{X^d} (f_1 \otimes \cdots \otimes f_d) \cdot E_{\mu^F}(f_1 \otimes \cdots \otimes f_d \mid \tilde{T}^{e_1}) \, d\mu^F \neq 0 \]
asserts that the lifted function \( f_1 \circ \pi_1 \) has a nontrivial inner product with a function that is a pointwise product of \( \tilde{T}^{e_1} \)-measurable functions for \( i = 2, 3, \ldots, d \) and the function \( E_{\mu^F}(f_1 \otimes \cdots \otimes f_d \mid \tilde{T}^{e_1}) \), which is manifestly \( \tilde{T}^{e_1} \)-measurable. Therefore \( f_1 \circ \pi_1 \) has a nontrivial conditional expectation onto \( \tilde{T}^{e_1} \lor \bigvee_{j=2}^d \tilde{T}^{e_j} \), which is the \( \sigma \)-algebra generated by the factor map \( \tilde{X} \to C\tilde{X} \). On the other hand, by \( C \)-satedness \( f_1 \circ \pi_1 \) must be relatively independent from this \( \sigma \)-algebra over \( \Phi \), and so we also have \( E_{\mu}(f_1 \mid \Phi) \neq 0 \), as required. \( \square \)

**Proof of Theorem C** This proceeds by induction on \( d \). The case \( d = 1 \) is the classical mean ergodic theorem, so suppose now that \( d \geq 2 \), that we know the result for all tuples of at most \( d - 1 \) transformations and that we are given \( T : \mathbb{Z}^d \rtimes (X, \Sigma, \mu) \).

Let \( C \) be the class in Proposition 3.3.1. By Theorem 2.3.2 we may choose a \( C \)-sated extension \( \pi : \tilde{X} \to X \), and now since the corresponding inclusion \( L^\infty(\mu) \subseteq L^\infty(\mu^F) \) is an embedding of algebras that preserves the norms \( \| \cdot \|_2 \) it will suffice to prove convergence for the analogs of the averages \( S_N \) associated to \( \tilde{X} \). To lighten notation we henceforth assume that \( X \) itself is \( C \)-sated.

Suppose that \( f_1, f_2, \ldots, f_{d+1} \in L^\infty(\mu) \). Letting \( \Phi := \Sigma T^{e_1} \lor \bigvee_{j=2}^d \Sigma T^{e_j} \), we see that the function \( f_1 - E(f_1 \mid \Phi) \) has zero conditional expectation onto \( \Phi \), and so by the multilinearity of \( S_N \) and Proposition 3.3.1 we have that
\[
S_N(f_1, f_2, \ldots, f_d) - S_N(E(f_1 \mid \Phi), f_2, \ldots, f_d)
= S_N(f_1 - E(f_1 \mid \Phi), f_2, \ldots, f_d) \to 0
\]
in $L^2(\mu)$ as $N \rightarrow \infty$. It therefore suffices to prove convergence with $f_1$ replaced by $E(f_1 \mid \Phi)$, or equivalently under the assumption that $f_1$ is $\Phi$-measurable.

However, this implies that $f_1$ may be approximated in $\| \cdot \|_2$ by finite sums of the form $\sum_p g_p \cdot h_{2,p} \cdot h_{3,p} \cdots h_{d,p}$ in which each $g_p$ is $T^{e_1}$-invariant and each $h_{j,p}$ is $T^{e_j-e_1}$-invariant. Since the operator

$$f_1 \mapsto S_N(f_1, f_2, \ldots, f_d)$$

is linear and uniformly continuous in $L^2(\mu)$ for fixed bounded $f_2, f_3, \ldots, f_d$, it therefore suffices to prove convergence in case $f_1$ is simply one such product, say $gh_2h_3\cdots h_d$. For this function, however, we can re-arrange our averages as

$$S_N(f_1, f_2, \ldots, f_d) = \frac{1}{N} \sum_{n=1}^N ((gh_2h_3\cdots h_d) \circ T^{ne_1}) \circ (f_2 \circ T^{ne_2}) \cdots (f_d \circ T^{ne_d})$$

$$= g \cdot \frac{1}{N} \sum_{n=1}^N ((f_2h_2) \circ T^{ne_2}) \cdots ((f_dh_d) \circ T^{ne_d}) = gS_N(1_X, f_2h_2, \ldots, f_dh_d),$$

since $g \circ T^{ne_1} = g$ and $h_j \circ T^{ne_1} = h_j \circ T^{ne_j}$ for each $j = 2, 3, \ldots, d$. Now the averages appearing on the right are uniformly bounded in $\| \cdot \|_\infty$ and involve only the $d - 1$ transformations $T^{e_2}, T^{e_3}, \ldots, T^{e_d}$, and so the inductive hypothesis gives their convergence in $\| \cdot \|_2$. Since $\|g\|_\infty < \infty$ this gives also the convergence of the left-hand averages in $\| \cdot \|_2$, as required. □

**Remark** In fact the above proof gives a slight strengthening of Theorem C, in that the convergence is uniform in the location of the interval of averaging: that is, the averages

$$\frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^d f_i \circ T^{ne_i}$$

converge in $L^2(\mu)$ for any sequence of increasingly long finite intervals $I_N \subset \mathbb{Z}$, and the limit does not depend on the choice of these intervals. This result is treated in full in [Aus09].
Chapter 4

Multiple recurrence for commuting transformations

In this chapter we deduce Theorem A from Theorem 2.3.2. Coupled with Furstenberg and Katznelson’s correspondence principle from [FK78], this gives a new proof of the Multidimensional Szemerédi Theorem, but we will not recount that correspondence here since it is already well-known from that paper and several subsequent accounts, such as those in the books [Fur81] of Furstenberg and [TV06] of Tao and Vu.

After introducing a more convenient reformulation of Theorem A below, we first introduce a very general meta-question that covers most of the ergodic-theory we need. We then show how it specializes to give quite detailed information on the Furstenberg self-joining corresponding to a tuple of commuting transformations. From this the proof of Theorem A follows by appealing to a version of Tao’s infinitary hypergraph removal lemma.

We will continue the practice begun in the previous chapter of writing a tuple of commuting transformations as $T^{e_1}, T^{e_2}, \ldots, T^{e_d}$ for some $\mathbb{Z}^d$-action $T$. The convergence result of the previous chapter implies that for any such $T^{e_1}, T^{e_2}, \ldots, T^{e_d}$ the Furstenberg self-joining $\mu^F$ of Section 3.2 exists. Knowing this, Theorem A about the limit infima of scalar averages is a consequence of the following more general result:

**Theorem 4.0.2.** If $T : \mathbb{Z}^d \curvearrowright (X, \Sigma, \mu), \mu^F$ denotes the Furstenberg self-joining of the transformations $T^{e_1}, T^{e_2}, \ldots, T^{e_d}$ and $A_1, A_2, \ldots, A_d \in \Sigma$
then
\[ \mu^F(A_1 \times A_2 \times \cdots \times A_d) = 0 \quad \Rightarrow \quad \mu(A_1 \cap A_2 \cap \cdots \cap A_d) = 0. \]

Indeed, in case \( A_i = A \) for each \( A \) this assertion is precisely the contrapositive of Theorem A. However, the formulation of Theorem 4.0.2 has the great advantage of allowing us to manipulate the sets \( A_i \) separately in setting up a proof by induction.

### 4.1 The question in the background

Having reformulated our goal in this chapter as Theorem 4.0.2, it becomes clear that it is really an assertion about the joint distribution of the coordinate projections \( \pi_i : X^d \to X, i = 1, 2, \ldots, d \) under \( \mu^F \).

By Lemma 3.2.1 \( \mu^F \) is an invariant measure for the action \( \vec{T} \) of the larger group \( \mathbb{Z}^{d+1} \) defined by setting
\[ \vec{T}^\mathbb{Z}d \{ 0 \} := T \times d \quad \text{and} \quad \vec{T}^{\mathbb{Z}e_i + 1} := T^{e_i} \times \cdots \times T^{e_d}. \]

Thus this defines a \( \mathbb{Z}^{d+1} \)-system \( \vec{X} \) in which the Furstenberg self-joining \( X^F \) corresponds to the subaction of \( \mathbb{Z}^d \oplus \{ 0 \} \). The key to our proof is the observation that the coordinate projections \( \pi_i \) now define factor maps of \( \vec{X} \) onto a collection of \( \mathbb{Z}^{d+1} \)-systems \( X_1, X_2, \ldots, X_d \) for each of which some one-dimensional subgroup of \( \mathbb{Z}^{d+1} \) acts trivially: specifically, this is so with \( X_i = (X_i, \Sigma_i, \mu_i, T_i) \) defined simply by ‘doubling up’ the \( \mathbb{Z}e_i \)-subaction of \( T \):
\[ (X_i, \Sigma_i, \mu_i) := (X, \Sigma, \mu), \quad T_i^\mathbb{Z}d \{ 0 \} := T \quad \text{and} \quad T_i^{\mathbb{Z}e_i + 1} := T^{e_i}. \]

It follows immediately from these specifications that \( \pi_i \circ \vec{T} = T_i \circ \pi_i \) and that \( X_i \in \mathbb{Z}^{\mathbb{Z}e_i - e_i} \).

Having made these observations, our principal results on \( \mu^F \) will fall within the pattern of the following:

**Meta-question:**

Given subgroups \( \Gamma_1, \Gamma_2, \ldots, \Gamma_r \leq \mathbb{Z}^D \) and \( \mathbb{Z}^d \)-systems \( (X_i, \Sigma_i, \mu_i, T_i) \) for \( i = 1, 2, \ldots, r \) such that \( T_i \upharpoonright \Gamma_i = \text{id} \), what do these partial invariances imply about the possible joinings of these \( \mathbb{Z}^D \)-systems?
The first stage in proving Theorem 4.0.2 will boil down to a handful of special cases of this question. In this section we show that a partial answer covering all of the cases we need can be given quite easily, subject to an algebraic constraint on the subgroups $\Gamma_i$ and an allowance to pass to extended systems.

First, it is instructive to understand the simple case $r = 2$:

**Lemma 4.1.1.** If the systems $X_i$ are $\Gamma_i$-partially invariant for $i = 1, 2$, then any joining of them is relatively independent over their factors $\Sigma_i^{T_i|\Gamma_1+\Gamma_2}$.

**Proof** Suppose $\pi_i : (Y, \Phi, \nu, S) \rightarrow (X_i, \Sigma_i, \mu_i, T_i)$ is a joining of the two systems and consider subsets $A_i \in \Sigma_i$. In addition let $(F_N)_{N \geq 1}$ be a Følner sequence of subsets of $\Gamma_1$. Then the invariance of $\nu$ and the Mean Ergodic Theorem give

$$
\nu(\pi_1^{-1}(A_1) \cap \pi_2^{-1}(A_2)) = \lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{\gamma \in F_N} \int_Y (1_{A_1} \circ \pi_1)(1_{A_2} \circ T_2^\gamma \circ \pi_2) \, d\nu = \lim_{N \rightarrow \infty} \int_Y (1_{A_1} \circ \pi_1)(\left(1 \sum_{\gamma \in F_N} 1_{A_2} \circ T_2^\gamma \circ \pi_2\right) \circ \pi_2) \, d\nu = \int_Y (1_{A_1} \circ \pi_1)(E_{\mu_2}(A_2 | \Sigma_2^T|\Gamma_1) \circ \pi_2) \, d\nu.
$$

Since $T_2^\Gamma = \text{id}$ the factor $\Sigma_2^T|\Gamma_1$ consists of sets that are invariant under the whole group $\Gamma_1 + \Gamma_2$, and hence agrees with $\Sigma_2^T|\Gamma_1+\Gamma_2$. Arguing similarly with the roles of $X_1$ and $X_2$ reversed, this shows that that above is equal to

$$
\int_Y (E_{\mu_1}(A_1 | \Sigma_1^T|\Gamma_1+\Gamma_2) \circ \pi_1)(E_{\mu_2}(A_2 | \Sigma_2^T|\Gamma_1+\Gamma_2) \circ \pi_2) \, d\nu,
$$

as required. □

For $r \geq 3$ we will not obtain an answer as complete as the above. However, a natural generalization is available for certain special tuples of subgroups, subject to the further provision that we may replace the originally-given systems $X_i$ with some extensions of them. The extensions, of course, will be said extensions, and for them the picture is given by the following.
Theorem 4.1.2. Suppose that

\[ Z^D \cong \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_r \oplus \Lambda \]

is a direct sum decomposition of \( Z^D \) into the subgroups \( \Gamma_i \) and some auxiliary subgroup \( \Lambda \), and that \( X_i \in Z^\Gamma_0 \) for \( i = 1, 2, \ldots, r \) are systems such that each \( X_i \) is \( C_i \)-sated for

\[
C_i := \bigvee_{j \leq r, j \neq i} Z^{\Gamma_i + \Gamma_j}_0.
\]

Then for any joining \( \pi_i : Y \to X_i, i = 1, 2, \ldots, r \), the factors \( \pi_i^{-1}(\Sigma_i) \) are relatively independent over their further factors

\[
\pi_i^{-1}\left( \bigvee_{j \leq r, j \neq i} \Sigma_i \mid (\Gamma_i + \Gamma_j) \right).
\]

Proof. This is a simple appeal to the definition of satedness. We will show that \( \pi_1^{-1}(\Sigma_1) \) is relatively independent from \( \bigvee_{j=2}^r \pi_j^{-1}(\Sigma_j) \) over \( \pi_1^{-1}\left( \bigvee_{j=2}^r \Sigma_i \mid (\Gamma_i + \Gamma_j) \right) \), the cases of the other factors being similar.

Let \( \Gamma := \Gamma_2 \oplus \cdots \oplus \Gamma_r \oplus \Lambda \leq Z^D \), so this complements \( \Gamma_1 \) in \( Z^D \), and let \( Y = (Y, \Phi, \nu, S) \). From \( S \) we may construct a new \( \nu \)-preserving \( Z^D \)-action \( S' \) by defining

\[
(S')^m + n := S^n \quad \text{for all } m \in \Gamma_1, n \in \Gamma.
\]

Let \( Y' := (Y', \Phi, \nu, S') \), so manifestly \( Y' \in Z^\Gamma_0 \). Similarly define the systems \( X'_i = (X_i, \Sigma_i, \mu_i, T'_i) \) for \( i = 2, 3, \ldots, r \), so these also have trivial \( \Gamma_1 \)-subactions and hence in fact lie in the classes \( Z^{\Gamma_1}_0 + \Gamma_1 \). Since \( T_i^m = \text{id}_{X_1} \) for all \( m \in \Gamma_1 \) by assumption, we see that \( \pi_1 \circ S' = T_1 \circ \pi_1 \), so \( \pi_1 \) still defines a factor map \( Y' \to X_1 \). On the other hand, we also have

\[
\pi_i \circ (S')^m + n + p = \pi_i \circ S^n + p = T_1^n + p \circ \pi_i = T^n + p \circ \pi_i = (T_i')^m + n + p \circ \pi_i
\]

whenever \( i = 2, 3, \ldots, r \) and \( m \in \Gamma_1, n \in \Gamma_1, \text{ and } p \in \bigoplus_{j \neq 1, i} \Gamma_j \oplus \Lambda \).

Therefore \( \pi_i \) is a factor map \( Y' \to X'_i \) for \( i = 2, 3, \ldots, r \), and so \( Y' \) is a joining of \( X_1 \) with members of the classes \( Z^{\Gamma_1 + \Gamma_i}_0 \) for \( i = 2, 3, \ldots, r \); that is, \( Y \) is a \( C_1 \)-adjoining of \( X_1 \). By the assumption of \( C_1 \)-satedness, it follows that this adjoining is relatively independent over the maximal \( C_1 \)-factor of \( X_1 \).
which equals $\bigvee_{j=2}^{r} \Sigma_{\gamma=1}^{\gamma_T(1+\Gamma_\gamma)}$, as required.

**Example** Without the assumption of satedness, more complicated phenomena can appear in the joint distribution of three partially-invariant systems. For example, let $(X, \Sigma, \mu, T)$ be the $\mathbb{Z}^3$-system on the two-torus $\mathbb{T}^2$ with its Borel $\sigma$-algebra and Haar measure defined by $T^{e_1} := R(\alpha,0)$, $T^{e_2} := R(0,\alpha)$ and $T^{e_3} := R(\alpha,\alpha)$, where $R_q$ denotes the rotation of $\mathbb{T}^2$ by an element $q \in \mathbb{T}^2$ and we choose $\alpha \in \mathbb{T}$ irrational. In this case we have natural coordinatizations of the partially invariant factors $\zeta^{T^{e_i}} : X \rightarrow \mathbb{T}$ given by

$$
\zeta^{T^{e_1}}_0(t_1, t_2) = t_2, \quad \zeta^{T^{e_2}}_0(t_1, t_2) = t_1 \quad \text{and} \quad \zeta^{T^{e_3}}_0(t_1, t_2) = t_1 - t_2.
$$

It follows that in this example any two of $\Sigma^{T^{e_1}}$, $\Sigma^{T^{e_2}}$ and $\Sigma^{T^{e_3}}$ are independent, but also that any two of them generate the whole system (and so overall independence fails).

In fact, it is possible to give a fairly complete answer to our meta-question in the case of any three $\mathbb{Z}$-subactions of some $\mathbb{Z}^D$-action, without the simplifying power of extending our systems. However, that answer in general requires the handling of extensions of non-ergodic systems by measurably-varying compact homogeneous space data: it is contained in Theorem 1.1 of [Ausc], in which such extensions are studied in suitable generality. The full formulation of that Theorem 1.1 is rather lengthy, and will not be repeated here; and it seems clear that matters will only become more convoluted for larger $r$.

Theorem 4.1.2 already suffices for the coming applications, but it is natural to ask about more general collections of subgroups $\Gamma_i \leq \mathbb{Z}^D$. In fact it is possible to do slightly better than Theorem 4.1.2 with just a little extra effort: the same conclusion holds given only that these subgroups are **linearly independent**, in the sense that for any $n_i \in \Gamma_i$ we have

$$n_1 + n_2 + \cdots + n_r = 0 \Rightarrow n_i = 0 \forall i \leq r.$$

Indeed, given this linear independence, one can let $\Delta := \Gamma_1 + \Gamma_2 + \cdots + \Gamma_r$, and now argue as in the above proof to deduce that the conclusion holds provided that $X_1$ is $C_1$-sated among all $\Delta$-systems. However, it is not quite obvious that this is the same as being $C_1$-sated among $\mathbb{Z}^D$-systems. This turns out to be
true, but it requires the key additional result that whenever $\Delta \leq \Lambda$ are discrete Abelian groups, $X$ is a $\Lambda$-system and $\alpha : Y \to X|_{\Delta}$ is an extension of the $\Delta$-subaction, there is an extension of $\Lambda$-systems $\beta : Z \to X$ that fits into a commutative diagram

$$
\begin{array}{c}
Z|\Delta \\
\downarrow \beta \\
X|\Delta \\
\downarrow \alpha \\
Y.
\end{array}
$$

The elementary but slightly messy proof of this can be found in Subsection 3.2 of [Ausd].

What happens when there are linearly dependences among the subgroups $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$? An answer to this question could have several applications to understanding multiple recurrence, but it is also clearly of broader interest in ergodic theory. At present the picture remains unclear, but a number of recent works have provided answers in several further special cases, and in moments of optimism it now seems possible that a quite general extension of Theorem 4.1.2 (using satedness relative to a much larger list of classes of system) may be available. A more precise conjecture in this vein will be formulated in Chapter 6.

**Remark** Before leaving this section, it is worth contrasting the feature seen above that linear independence is helpful with previous works in this area. In the early study of special cases of Theorems B or C it was generally found that the analysis of powers of a single transformation (or correspondingly of arithmetic progressions in $\mathbb{Z}$) revealed more usable structure and was thus more tractable than the general case. Of course, Furstenberg’s original Multiple Recurrence Theorem preceded Theorem B; and the conclusion of Theorem C was known in many such ‘one-dimensional’ cases long before the general case was treated (see [CL84, CL88a, CL88b, FW96, HK, Zie07], although we note that Conze and Lesigne did also treat a two-dimensional case of Theorem C, and that in [Zha96] Zhang extended this result to three dimensions subject to some additional assumptions).

The same phenomenon is apparent in the search for finitary, quantitative approaches to Szemerédi’s Theorem and its relatives. Indeed, a purely finitary proof of the Multidimensional Szemerédi Theorem appeared only recently in works of Rödl and Skokan [RS04], Nagle, Rödl and Schacht [NRS06].
and Gowers [Gow07], building on the development by those authors of sufficiently powerful hypergraph variants of Szemerédi’s Regularity Lemma in graph theory. Furthermore, the known bounds for how large $N_0$ must be taken in terms of $\delta$ and $k$ are far better for Szemerédi’s Theorem than for its multidimensional generalization, owing to the powerful methods developed by Gowers in [Gow98, Gow01], which extend Roth’s proof for $k = 3$ from [Rot53] and are much more efficient than the hypergraph regularity proofs. As yet these methods have resisted extension to the multidimensional setting, except in one two-dimensional case recently treated by Shkredov [Shk05]. This story is discussed in much greater depth in Chapters 10 and 11 of [TV06].

Running counter to this trend, the value of linear independence for the present work is a consequence of our strategy of passing to extensions of probability-preserving systems. Although such extensions can lose any a priori algebraic structure (such as being a $\mathbb{Z}^D$-action in which the transformations $T^{e_i}$ are actually all powers of one fixed transformation), the various instances of satedness that it allows us to assume will furnish enough power to drive all of our subsequent proofs. These instances of satedness will all be relative to joins of different classes of partially invariant systems, and, as illustrated by the above proof of Theorem 4.1.2, the usefulness of this kind of satedness will rely on the ability to construct new systems for which the corresponding subgroups behave in specified ways. With this in mind it is natural that having those subgroups linearly independent removes a potential obstacle from these arguments, and that answering our meta-question for sated systems will be more difficult when the subgroups exhibit some linear dependences. ◁

4.2 More on the Furstenberg self-joining

We now return to the study of the Furstenberg self-joining $\mu^F$ introduced in the previous chapter, with the goal of deriving a structure theorem for it as a consequence of Theorem 4.1.2 in case $X$ is sated with respect to enough difference classes. In order to formulate this structure theorem, we first settle on some more bespoke notation.

In the following we shall make repeated reference to certain factors assembled from the partially invariant factors of our $\mathbb{Z}^d$-action $T$, so we now give these factors their own names. They will be indexed by subsets of
This is immediate from the definition: if \( e \subseteq [d] \), or more generally by subfamilies of the collection \( \binom{[d]}{\geq 2} \) of all subsets of \([d]\) of size at least 2. On the whole, these indexing subfamilies will be up-sets in \( \binom{[d]}{\geq 2} \): \( I \subseteq \binom{[d]}{\geq 2} \) is an up-set if \( u \in I \) and \( [d] \supseteq v \supseteq u \) imply \( v \in I \). For example, given \( e \subseteq [d] \) we write \( \langle e \rangle := \{ u \in \binom{[d]}{\geq 2} : u \supseteq e \} \) (note the non-standard feature of our notation that \( e \in \langle e \rangle \) if and only if \( |e| \geq 2 \)): up-sets of this form are principal. We will abbreviate \( \langle \{ i \} \rangle \) to \( \langle i \rangle \). It will also be helpful to define the depth of a non-empty up-set \( I \) to be \( \min \{ |e| : e \in I \} \).

The corresponding factor for \( e = \{ i_1, i_2, \ldots, i_k \} \subseteq [d] \) with \( k \geq 2 \) is \( \Phi_e := \sum_{T^{e_{i_1}}=T^{e_{i_2}}=\ldots=T^{e_{i_k}}} \), so this is the partially invariant factor for the \((k - 1)\)-dimensional subgroup

\[
\mathbb{Z}(e_{i_1} - e_{i_2}) + \mathbb{Z}(e_{i_1} - e_{i_3}) + \cdots + \mathbb{Z}(e_{i_1} - e_{i_k}).
\]

More generally, given a family \( A \subseteq \binom{[d]}{\geq 2} \) we define \( \Phi_A := \bigvee_{e \in A} \Phi_e \).

From the ordering among the factors \( \Phi_e \) it is clear that \( \Phi_I = \Phi_A \) whenever \( A \subseteq \binom{[d]}{\geq 2} \) is a family that generates \( I \) as an up-set, and in particular that \( \Phi_e = \Phi_{\langle e \rangle} \) when \( |e| \geq 2 \).

We now return to the Furstenberg self-joining \( \mu^F \). For \( e = \{ i_1 < i_2 < \ldots < i_k \} \subseteq [d] \) we write \( \mu^F_e \) for the Furstenberg self-joining of the transformations \( T^{e_{i_1}}, T^{e_{i_2}}, \ldots, T^{e_{i_k}} \):

\[
\mu^F_e(A_1 \times \cdots \times A_k) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n e_{i_1}}(A_1) \cap \cdots \cap T^{-n e_{i_k}}(A_k)),
\]

so this clearly extends the definition of Section 3.2 in the sense that \( \mu^F_{[d]} = \mu^F \).

Of course, we know the existence of each \( \mu^F_e \) by the results of the previous chapter.

We next record some simple properties of the family of self-joinings \( \mu^F_e \) for \( e \subseteq [d] \). Given subsets \( e \subseteq e' \subseteq [d] \), in the following we write \( \pi_e \) for the coordinate projection \( X^{e'} \to X^e \), since the choice of \( e' \) will always be clear from the context.

**Lemma 4.2.1.** If \( e \subseteq e' \subseteq [d] \) then \( (\pi_e)_\# \mu^F_{e'} = \mu^F_e \).

**Proof** This is immediate from the definition: if \( e = \{ i_1 < i_2 < \ldots < i_k \} \subseteq e' = \{ j_1 < j_2 < \ldots < j_l \} \) and \( A_{i_j} \in \Sigma \) for each \( j \leq k \) then

\[
(\pi_e)_\# \mu^F_{e'}(A_{i_1} \times \cdots \times A_{i_k}) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n e_{j_1}}(B_{j_1}) \cap \cdots \cap T^{-n e_{j_l}}(B_{j_l})).
\]
where \( B_j := A_j \) if \( j \in e \) and \( B_j := X \) otherwise; but then this last average simplifies summand-by-summand directly to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n e_{i_1}}(A_1) \cap \cdots \cap T^{-n e_{i_k}}(A_k)) =: \mu_e^F(A_1 \times \cdots \times A_k),
\]
as required. \( \square \)

**Lemma 4.2.2.** For any \( e \subseteq [d] \) and \( A \in \Phi_e \) we have

\[
\mu_e^F(\pi_i^{-1}(A) \triangle \pi_j^{-1}(A)) = 0 \quad \forall i, j \in e:
\]
thus, the restriction \( \mu_e^F |_{\Phi^{e \otimes e}} \) is just the diagonal measure \( (\mu |_{\Phi_e})^{\Delta_e} \).

**Proof** If \( e = \{i_1 < i_2 < \ldots < i_k\} \) and \( A_j \in \Phi_e \) for each \( j \leq k \) then by definition we have

\[
\mu_e^F(A_1 \times A_2 \times \cdots \times A_k)
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n e_{i_1}}(A_1) \cap T^{-n e_{i_2}}(A_2) \cap \cdots \cap T^{-n e_{i_k}}(A_k))
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n e_{i_1}}(A_1 \cap A_2 \cap \cdots \cap A_k))
= \mu(A_1 \cap A_2 \cap \cdots \cap A_k),
\]
as required. \( \square \)

It follows from the last lemma that whenever \( e \subseteq e' \) the factors \( \pi_i^{-1}(\Phi_e) \leq \Sigma^{\otimes d} \) for \( i \in e \) are all equal up to \( \mu_{e'}^F \)-negligible sets. It will prove helpful later to have a dedicated notation for these factors.

**Definition 4.2.3 (Oblique copies).** For each \( e \subseteq [d] \) we refer to the common \( \mu_{[d]}^F \)-completion of the \( \sigma \)-subalgebra \( \pi_i^{-1}(\Phi_e) \), \( i \in e, \) as the **oblique copy** of \( \Phi_e \), and denote it by \( \Phi_e^F \). More generally we shall refer to factors formed by repeatedly applying \( \cap \) and \( \vee \) to such oblique copies as **oblique factors**.

We are now ready to derive the more nontrivial consequences we need from Theorem 4.1.2. These will appear in two separate propositions.
Proposition 4.2.4. For each pair $i \leq d$ let

$$C_i := \bigvee_{j \leq d, j \neq i} Z_0^{e_i - e_j}.$$ 

If $X$ is $C_i$-sated for each $i$ then the coordinate projections $\pi_i : X^d \to X$ are relatively independent under $\mu^F$ over the further factors

$$\pi_i^{-1} \left( \bigvee_{j \leq d, j \neq i} \sum_{T^e_i = T^e_j} \right) = \pi_i^{-1}(\Phi(i)).$$

Proof. This follows by applying Theorem 4.1.2 to the $\mathbb{Z}^{d+1}$-system $\overline{X}$ introduced at the beginning of the previous section. Indeed, as explained there the coordinate projections $\pi_i : \overline{X} \to X_i$ witness that $\overline{X}$ is a joining of the systems $X_i \in \mathbb{Z}_0^{(e_{d+1} - e_i)}$.

Let

$$D_i := \bigvee_{j \leq d, j \neq i} Z_0^{(e_i - e_{d+1}) + Z(e_j - e_{d+1})},$$

an idempotent class of $\mathbb{Z}^{d+1}$-systems. Now the assumption that $X$ is $C_i$-sated as a $\mathbb{Z}^d$-system implies that $X_i$ is $D_i$-sated as a $\mathbb{Z}^{d+1}$-system. Indeed, given any extension of $\mathbb{Z}^{d+1}$-systems $\pi : Y \to X_i$ the subaction $(D_i Y)^{(\mathbb{Z}^d \oplus \{0\})}$ is clearly a member of the class $C_i$, so the $C_i$-satedness of $X$ implies that $\pi$ is relatively independent from $\zeta_{D_i} : Y \to D_i Y$ over its further factor map $\zeta_X$, which agrees with $\zeta_{D_i}$ because the whole of $X_i$ is already $\mathbb{Z}(e_{d+1} - e_i)$-partially invariant.

Setting $\Gamma_i := \mathbb{Z}(e_i - e_{d+1})$ for $i = 1, 2, \ldots, d$ and $\Lambda := \mathbb{Z}e_{d+1}$, these subgroups define a direct-sum decomposition of $\mathbb{Z}^{d+1}$. Therefore Theorem 4.1.2 applies to tell us that the factors $\pi_i^{-1}(\Sigma)$ are relatively independent under $\mu^F$ over their further factors

$$\pi_i^{-1} \left( \bigvee_{j \leq d, j \neq i} \sum_{T^e_i = T^e_j} (Z(e_i - e_{d+1}) + Z(e_j - e_{d+1})) \right) = \pi_i^{-1}(\Phi(i)),$$

as required. \qed

For our second application of Theorem 4.1.2 we need a preparatory lemma.

Lemma 4.2.5. If $C \subseteq D$ are idempotent classes of $\Gamma$-systems for any discrete group $\Gamma$ and $X$ is $C$-sated, then $DX$ is also $C$-sated.
Proof If $X$ is $C$-sated and $\pi : Y \rightarrow DX$ is any extension, then the relatively independent product $\tilde{X} := X \times_{\{D\}^X=\pi} Y$ is an extension of $X$ through the first coordinate projection (it for the sake of using this relatively independent product that we need $\Gamma$ to be a group). Therefore by $C$-satedness the factor map $\zeta_\mathcal{E}^X$ is relatively independent from this coordinate projection over the further factor map $\zeta_\mathcal{E}^Y : X \rightarrow CX$ of the latter, and so the same must be true of $\zeta_\mathcal{E}^Y$. However, the factor map $\zeta_\mathcal{E}^Y$ is clearly contained in the factor map $\zeta_\mathcal{E}^X$ since $C \subseteq D$, and so it must actually equal $\zeta_\mathcal{E}^D \circ \zeta_\mathcal{E}^X : X \rightarrow C(DX)$. Hence $\pi$ is relatively independent from $\zeta_\mathcal{E}^X$ over its further factor map $\zeta_\mathcal{E}^DX$, as required.

\[ \square \]

Proposition 4.2.6. For each subset $e = \{i_1, i_2, \ldots, i_k\} \subseteq [d]$ let

\[ C_e := \bigvee_{j \in [d] \setminus e} Z_0(e_{i_1} - e_{i_2}) + \cdots + Z_0(e_{i_1} - e_{i_k}) + Z_0(e_{i_1} - e_{j}), \]

and suppose now that $X$ is $C_e$-sated for every $e$ (so this includes the assumption of the previous proposition when $e$ is a singleton). Then under $\mu^F$ the oblique factors have the property that $\Phi^F_I$ and $\Phi^F_I'$ are relatively independent over $\Phi^F_{I \cap I'}$ for any up-sets $I, I' \subseteq \{d\}_{\geq 2}$.

Proof Step 1 First observe that the result is trivial if $I \supseteq I'$, so now suppose that $I' = \langle e \rangle$ where $e$ is a maximal member of $\{d\}_{\geq 2} \setminus I$. Let $\{a_1, a_2, \ldots, a_m\}$ be the antichain of minimal elements of $I$, so that $\Phi^F_I = \bigvee_{k \leq m} \Phi^F_{a_k}$. The maximality assumption on $e$ implies that $e \cup \{j\}$ contains some $a_k$ for every $j \in [d] \setminus e$, and so $I \cap I'$ is precisely the up-set generated by these sets $e \cup \{j\}$ for $j \in [d] \setminus e$. We must therefore show that $\Phi^F_e$ is relatively independent from $\bigvee_{k \leq m} \Phi^F_{a_k}$ under $\mu^F$ over the common factor $\bigvee_{j \in [d] \setminus e} \Phi^F_{e \cup \{j\}}$.

Observe also that since $e \not\subseteq I$ we can find some $j_k \in a_k \setminus e$ for each $k \leq m$. Moreover, each $j \in [d] \setminus e$ must appear as some $j_k$ in this list, since it appears at least for some $k$ for which $a_k \subseteq e \cup \{j\}$.

Now Lemma 4.2.2 implies that $\Phi^F_{a_k}$ agrees with $\pi_{j_k}^{-1}(\Phi_{a_k})$ up to $\mu^F$-negligible sets. On the other hand, we clearly have $\pi_{j_k}^{-1}(\Phi_{a_k}) \leq \pi_{j_k}^{-1}(\Sigma)$, and so in fact it will suffice to show that $\Phi^F_e$ is relatively independent from $\bigvee_{j \in [d] \setminus e} \pi_{j_k}^{-1}(\Sigma)$ over $\bigvee_{j \in [d] \setminus e} \Phi^F_{e \cup \{j\}}$.
This alteration of the problem is important because it provides the linear independence needed to apply Theorem 4.1.2. Indeed, considering again the \( \mathbb{Z}^{d+1} \)-system \( \vec{X} \), in the present setting we see that the \( \sigma \)-subalgebras \( \Phi^F_e \) and \( \pi^{-1}_j(\Sigma) \) for \( j \in [d] \setminus e \)

constitute a collection of factors of \( \vec{X} \) that are partially invariant under the subgroups

\[
\Gamma_e := \mathbb{Z}(e_i - e_{d+1}) + \sum_{\ell \in \{i\}} \mathbb{Z}(e_i - e_\ell) \quad \text{and} \quad \Gamma_j := \mathbb{Z}(e_j - e_{d+1}) \quad \text{for} \ j \in [d] \setminus e
\]

respectively, where \( i \in e \) is arbitrary. On the one hand these subgroups can be inserted into a direct sum decomposition of \( \mathbb{Z}^{d+1} \), and on the other we may argue just as in the proof of Proposition 4.2.4 that the \( \mathbb{Z}^{d+1} \)-system defined by the factor \( \Phi^F_e \) is satated relative to the class \( \bigvee_{j \in [d] \setminus e} \mathbb{Z}_{\Gamma_e + \Gamma_j} \), using our satedness assumption on \( X \) and Lemma 4.2.5. The conclusion therefore follows from Theorem 4.1.2.

**Step 2** The general case can now be treated for fixed \( \mathcal{I} \) by induction on \( \mathcal{I} \). If \( \mathcal{I}' \subseteq \mathcal{I} \) then the result is clear, so now let \( e \) be a minimal member of \( \mathcal{I}' \setminus \mathcal{I} \) of maximal size, and let \( \mathcal{I}'' := \mathcal{I}' \setminus \{e\} \). It will suffice to prove that if \( F \in L^\infty(\mu^F) \) is \( \Phi^F_{\mathcal{I}''} \)-measurable then

\[
\mathbb{E}_{\mu^F}(F \mid \Phi^F_\mathcal{I}) = \mathbb{E}_{\mu^F}(F \mid \Phi^F_{\mathcal{I}' \setminus \mathcal{I}''}),
\]

and furthermore, by an approximation in \( \| \cdot \|_2 \) by finite sums of products, to do so only for \( F \) that are of the form \( F_1 \cdot F_2 \) with \( F_1 \) and \( F_2 \) being bounded and respectively \( \Phi^F_{\{e\}} \)- and \( \Phi^F_{\mathcal{I}''} \)-measurable. However, for such a product we can write

\[
\mathbb{E}_{\mu^F}(F \mid \Phi^F_\mathcal{I}) = \mathbb{E}_{\mu^F}(\mathbb{E}_{\mu^F}(F \mid \Phi^F_{\mathcal{I} \setminus \mathcal{I}''}) \mid \Phi^F_\mathcal{I}) = \mathbb{E}_{\mu^F}(\mathbb{E}_{\mu^F}(F_1 \mid \Phi^F_{\mathcal{I} \setminus \mathcal{I}''}) \cdot F_2 \mid \Phi^F_\mathcal{I}).
\]

By Step 1 we have

\[
\mathbb{E}_{\mu^F}(F_1 \mid \Phi^F_{\mathcal{I} \setminus \mathcal{I}''}) = \mathbb{E}_{\mu^F}(F_1 \mid \Phi^F_{(\mathcal{I} \cup \mathcal{I}'') \cap \{e\}}),
\]

and on the other hand \( (\mathcal{I} \cup \mathcal{I}'') \cap \{e\} \subseteq \mathcal{I}'' \) (because \( \mathcal{I}'' \) contains every subset of \( [d] \) that strictly includes \( e \), since \( \mathcal{I}' \) is an up-set), so \( (\mathcal{I} \cup \mathcal{I}'') \cap \{e\} = \mathcal{I}'' \cap \{e\} \) and therefore another appeal to Step 1 gives

\[
\mathbb{E}_{\mu^F}(F_1 \mid \Phi^F_{(\mathcal{I} \cup \mathcal{I}'') \cap \{e\}}) = \mathbb{E}_{\mu^F}(F_1 \mid \Phi^F_{\mathcal{I}''}).
\]
Therefore the above expression for $E_{\mu^F}(F_1 F_2 \mid \Phi^F_I)$ simplifies to

$$E_{\mu^F}(E_{\mu^F}(F_1 \mid \Phi^F_{I''}) \cdot F_2 \mid \Phi^F_I) = E_{\mu^F}(E_{\mu^F}(F_1 \cdot F_2 \mid \Phi^F_{I''}) \mid \Phi^F_I)$$

$$= E_{\mu^F}(E_{\mu^F}(F \mid \Phi^F_{I''}) \mid \Phi^F_I) = E_{\mu^F}(F \mid \Phi^F_{I''})$$

where the third equality follows by the inductive hypothesis applied to $I''$ and $I$.

4.3 Infinitary hypergraph removal and completion of the proof

Propositions 4.2.4 and 4.2.6 tell us a great deal about the structure of the probability measure $\mu^F$ for a system $X$ that is sated relative to all the necessary classes in terms of the partially-ordered family of factors

![Diagram showing the structure of the family of factors](image)

by showing that large collections of the $\sigma$-subalgebras appearing here are relatively independent over the collections of further $\sigma$-subalgebras that they have in common.
It is worth stressing at this point that we have not proved any such assertion for the joint distribution of all the original factors $\Phi_e \leq \Sigma$, but only for their oblique copies inside $\Sigma^{\otimes d}$. The problem of describing the joint distribution of the factors $\Phi_e$ themselves seems to be much harder, because it runs into precisely the difficulties with linear dependence discussed in Section 4.1. For example, if $e_1, e_2, e_3 \subseteq [d]$ are three subsets that are pairwise non-disjoint, then we have $\Phi_{e_i} = \Sigma^{T|r_{e_i}}$ for $\Gamma_{e_i} = \sum_{j,j' \in e_i} \mathbb{Z}(e_j - e_{j'})$, and these three subgroups are now clearly not linearly independent. In our analysis of the oblique factors $\Phi_e^F$ we carefully avoided a similar problem during Step 1 of the proof of Proposition 4.2.6, where we exploited the fact that $\Phi_e^F$ is contained modulo negligible sets in $\pi_j^{-1}(\Sigma)$ for any choice of $j \in e$, so that by making careful choices of the coordinates with which to express these oblique copies we were able to reduce the joint distribution of interest to the case covered by Theorem 4.1.2, involving only linearly independent subgroups. However, it seems clear that no similar trick will be available in the study of the factors $\Phi_e$.

Happily, however, we do not need any such more precise information to complete our proof of Theorem 4.0.2; in the remainder of this chapter we show how the structure proved above for $\mu^F$ suffices. This will proceed through a slight modification of Tao’s infinitary hypergraph removal lemma from [Tao07], which first appeared in the form given below in [Ausb].

**Proposition 4.3.1.** Suppose that $(X, \Sigma, \mu)$ is a standard Borel space and $\lambda$ is a $d$-fold coupling of $\mu$ on $(X^d, \Sigma^{\otimes d})$ with coordinate projection maps $\pi_i : X^d \rightarrow X_i$, and that $(\Psi_e)_e$ is a collection of $\sigma$-subalgebras of $\Sigma$ indexed by subsets $e \in \binom{[d]}{ \geq 2}$ with the following properties:

[i] if $e \subseteq e'$ then $\Psi_e \geq \Psi_{e'}$;

[ii] if $i, j \in e$ and $A \in \Psi_e$ then $\lambda(\pi_i^{-1}(A) \triangle \pi_j^{-1}(A)) = 0$, so that we may let $\Psi_e^\ell$ be the common $\lambda$-completion of the lifted $\sigma$-algebras $\pi_i^{-1}(\Psi_e)$ for $i \in e$;

[iii] if we define $\Psi_I^\ell := \bigvee_{e \in I} \Psi_e^\ell$ for each up-set $I \in \binom{[d]}{\geq 2}$, then the $\sigma$-subalgebras $\Psi_I^\ell$ and $\Psi'_I$, are relatively independent under $\lambda$ over $\Psi_{I \cap I'}^\ell$.

In addition, suppose that $\mathcal{I}_{i,j}$ for $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, k_i$ are collections of up-sets in $\binom{[d]}{\geq 2}$ such that $[d] \in \mathcal{I}_{i,j} \subseteq \langle i \rangle$ for each $i, j$, and that
the sets $A_{i,j} \in \Phi_{I_{i,j}}$ are such that

$$\lambda\left(\prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0.$$ 

Then we must also have

$$\mu\left(\bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} A_{i,j} \right) = 0.$$

**Proof of Theorem 4.0.2 from Proposition 4.3.1** Clearly the conclusion holds for a system $X$ if it holds for any extension of $X$, so by Theorem 2.3.2 we may assume that $X$ is $C_e$-sated for every $e \subseteq [d]$.

Now suppose that $A_1, A_2, \ldots, A_d \in \Sigma$ are such that $\mu^F(A_1 \times A_2 \times \cdots \times A_d) = 0$. Then by Proposition 4.2.4 we have

$$\mu^F(A_1 \times A_2 \times \cdots \times A_d) = \int_{X} \bigotimes_{i=1}^{d} E_{\mu}(1_{A_i} \mid \Phi_{i}) \, d\mu^F = 0.$$

The level set $B_i := \{E_{\mu}(1_A \mid \Phi_{i}) > 0\}$ (of course, this is unique only up to $\mu$-negligible sets) lies in $\Phi_{i}$, and the above vanishing requires that also $\mu^F(B_1 \times B_2 \times \cdots \times B_d) = 0$. Now setting $k_i = 1, I_{i,1} := \langle i \rangle$ and $A_{i,1} := B_i$ for each $i \leq d$, Lemma 4.2.2 and Proposition 4.2.6 imply that Proposition 4.3.1 applies to the partially invariant factors $\Phi_e$ and their oblique copies to give $\mu(B_1 \cap B_2 \cap \cdots \cap B_d) = 0$. On the other hand we must have $\mu(A \setminus B_i) = 0$ for each $i$, and so overall $\mu(A) \leq \mu(B_1 \cap B_2 \cap \cdots \cap B_d) + \sum_{i=1}^{d} \mu(A \setminus B_i) = 0$, as required.

The remainder of this chapter is given to the proof of Proposition 4.3.1. This proceeds by induction on a suitable ordering of the possible collections of up-sets $(I_{i,j})_{i,j}$, appealing to a handful of different possible cases at different steps of the induction. At the outermost level, this induction will be organized according to the depth of our up-sets.

The proof given below is taken essentially unchanged from [Ausb], where in turn the statement and proof were adopted with only slight modifications from [Tao07]. The reader may consult [Ausb] for an explanation of these modifications.
Definition 4.3.2. A family \((\mathcal{I}_{i,j})_{i,j}\) has the property \(P\) if it satisfies the conclusion of Proposition 4.3.1.

We separate the various components of the induction into separate lemmas.

Lemma 4.3.3 (Lifting using relative independence). Suppose that all up-sets in the collection \((\mathcal{I}_{i,j})_{i,j}\) have depth at least \(k\), that all those with depth exactly \(k\) are principal, and that there are \(\ell \geq 1\) of these. Then if property \(P\) holds for all similar collections having \(\ell - 1\) up-sets of depth \(k\), then it holds also for this collection.

Proof. Let \(\mathcal{I}_{i_1,j_1} = \langle e_1 \rangle, \mathcal{I}_{i_2,j_2} = \langle e_2 \rangle, \ldots, \mathcal{I}_{i_\ell,j_\ell} = \langle e_\ell \rangle\) be an enumeration of all the (principal) up-sets of depth \(k\) in our collection. We will treat two separate cases.

First suppose that two of the generating sets agree; by re-ordering if necessary we may assume that \(e_1 = e_2\). Clearly we can assume that there are no duplicates among the coordinate-collections \((\mathcal{I}_{i,j})_{i,j=1}^{k}\) for each \(i\) separately, so we must have \(i_1 \neq i_2\). However, if we now suppose that \(A_{i,j} \in \mathcal{I}_{i,j}\) for each \(i, j\) are such that

\[
\lambda \left( \prod_{i=1}^{d} (\bigcap_{j=1}^{k} A_{i,j}) \right) = 0,
\]

then by assumption [ii] the same equality holds if we simply replace \(A_{i_1,j_1} \in \langle e_1 \rangle\) with \(A'_{i_1,j_1} := A_{i_1,j_1} \cap A_{i_2,j_2}\) and \(A_{i_2,j_2} \) with \(A'_{i_2,j_2} := X\). Now this last set can simply be ignored to leave an instance of a \(\lambda\)-negligible product for the same collection of up-sets omitting \(\mathcal{I}_{i_2,j_2}\), and so property \(P\) of this reduced collection completes the proof.

On the other hand, if all the \(e_i\) are distinct, we shall simplify the last of the principal up-sets \(\mathcal{I}_{i_\ell,j_\ell}\) by exploiting the relative independence among the lifted \(\sigma\)-algebras \(\Psi_{i_\ell}^\dagger\). Assume for notational simplicity that \((i_\ell, j_\ell) = (1, 1)\); clearly this will not affect the proof. We will reduce to an instance of property \(P\) associated to the collection \((\mathcal{I}'_{i,j})\) defined by

\[
\mathcal{I}'_{i,j} := \begin{cases} 
\langle e_\ell \rangle \setminus \{e_\ell\} & \text{if } (i, j) = (1, 1) \\
\mathcal{I}_{i,j} & \text{else},
\end{cases}
\]

which has one fewer up-set of depth \(k\) and so falls under the inductive assumption.

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Indeed, by property [iii] under \( \lambda \) the set \( \pi_1^{-1}(A_{1,1}) \) is relatively independent from all the sets \( \pi_1^{-1}(A_{i,j}), \; (i, j) \neq (1, 1) \), over the \( \sigma \)-algebra \( \pi_1^{-1}(\Psi_{(e_i)\setminus\{e_1\}}) \), which is dense inside \( \Psi_{(e_i)\setminus\{e_1\}} \). Therefore

\[
0 = \lambda\left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = \int_X E_{\mu}(1_{A_{1,1}} | \Psi_{(e_1)\setminus\{e_1\}}) \circ \pi_1 \cdot \prod_{j=2}^{k_1} 1_{\pi_1^{-1}(A_{1,j})} \cdot \prod_{i=2}^{d} \prod_{j=1}^{k_i} 1_{\pi_1^{-1}(A_{i,j})} d\lambda.
\]

Setting \( A'_{1,1} := \{ E_{\mu}(1_{A_{1,1}} | \Psi_{(e_1)\setminus\{e_1\}}) > 0 \} \in \Psi_{(e_1)\setminus\{e_1\}} \) and \( A'_{i,j} := A_{i,j} \) for \( (i, j) \neq (1, 1) \), we have that \( \mu( A_{1,1} \setminus A'_{1,1} ) = 0 \) and it follows from the above equality that also \( \lambda\left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A'_{i,j} \right) \right) = 0 \), so an appeal to property P for the reduced collection of up-sets completes the proof.

\[ \square \]

**Lemma 4.3.4** (Lifting under finitary generation). Suppose that all up-sets in the collection \( (I_{i,j})_{i,j} \) have depth at least \( k \) and that among those of depth \( k \) there are \( \ell \geq 1 \) that are non-principal. Then if property P holds for all similar collections having at most \( \ell - 1 \) non-principal up-sets of depth \( k \), then it also holds for this collection.

**Proof** Let \( I_{i_1,j_1}, I_{i_2,j_2}, \ldots, I_{i_{\ell},j_{\ell}} \) be the non-principal up-sets of depth \( k \), and now in addition let \( e_1, e_2, \ldots, e_r \) be all the members of \( I_{i_{\ell},j_{\ell}} \) of size \( k \) (so, of course, \( r \leq \binom{k}{2} \)). Once again we will assume for simplicity that \( (i_{\ell}, j_{\ell}) = (1, 1) \). We break our work into two further steps.

**Step 1** First consider the case of a collection \( (A_{i,j})_{i,j} \) such that for the set \( A_{1,1} \), we can actually find finite subalgebras of sets \( B_s \in \Psi_{\{e_s\}} \) for \( s = 1, 2, \ldots, r \) such that \( A_{i_{\ell},j_{\ell}} \in B_1 \lor B_2 \lor \cdots \lor B_r \lor \Psi_{I_{i_{\ell},j_{\ell}} \cap (\bigcup_{s=1}^{\ell} B_s)} \) (so \( A_{1,1} \) lies in one of our non-principal up-sets of depth \( k \), but it fails to lie in an up-set of depth \( k + 1 \) only ‘up to’ finitely many additional generating sets). Choose \( M \geq \max_{s \leq r} |B_s| \), so that we can certainly express

\[
A_{1,1} = \bigcup_{m=1}^{M^{r}} (B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m)
\]
with \( B_{m,s} \in \mathcal{B}_s \) for each \( s \leq r \) and \( C_m \in \Psi_{\mathcal{I}_{1,1}}(\mathcal{I}_{2,k+1}) \). Inserting this expression into the equation

\[
\lambda\left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k} A_{i,j} \right) \right) = 0
\]

now gives that each of the \( M^r \) individual product sets

\[
\left( (B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m) \cap \bigcap_{j=2}^{k} A_{1,j} \right) \times \prod_{i=2}^{d} \left( \bigcap_{j=1}^{k} A_{i,j} \right)
\]

is \( \lambda \)-negligible.

Now consider the family of up-sets comprising the original \( \mathcal{I}_{i,j} \) if \( i = 2, 3, \ldots, d \) and the collection \( \langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_r \rangle, \mathcal{I}_{1,2}, \mathcal{I}_{1,3}, \ldots, \mathcal{I}_{1,k} \) corresponding to \( i = 1 \). We have broken the depth-\( k \) non-principal up-set \( \mathcal{I}_{1,1} \) into the higher-depth up-set \( \mathcal{I}_{1,1} \cap \left( |d| \geq k+1 \right) \) and the principal up-sets \( \langle e_s \rangle \), and so there are only \( \ell - 1 \) minimal-depth non-principal up-sets in this new family. It is clear that for each \( m \leq M^r \) the above product set is associated to this family of up-sets, and so an inductive appeal to property P for this family tells us that also

\[
\mu\left( (B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m) \cap \bigcap_{j=2}^{k} A_{1,j} \cap \bigcap_{i=2}^{d} \bigcap_{j=1}^{k} A_{i,j} \right) = 0
\]

for every \( m \leq M^r \). Since the union of these sets is just \( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k} A_{i,j} \), this gives the desired negligibility in this case.

**Step 2** Now we return to the general case, which will follow by a suitable limiting argument applied to the conclusion of Step 1. Since any \( \Psi_{e} \) is countably generated modulo \( \mu \), for each \( e \) with \(|e| = k \) we can find an increasing sequence of finite subalgebras \( \mathcal{B}_{e,1} \subseteq \mathcal{B}_{e,2} \subseteq \ldots \) that generates \( \Psi_{e} \) up to \( \mu \)-negligible sets. In terms of these define approximating sub-\( \sigma \)-algebras

\[
\Xi_{i,j}^{(n)} := \Psi_{\mathcal{I}_{i,j}}(\mathcal{I}_{2,k+1}) \cup \bigcup_{e \in \mathcal{I}_{i,j} \cap \left( |d| \right)} \mathcal{B}_{e,n},
\]

so for each \( \mathcal{I}_{i,j} \) these form an increasing family of \( \sigma \)-algebras that generates \( \Psi_{\mathcal{I}_{i,j}} \) up to \( \mu \)-negligible sets (indeed, if \( \mathcal{I}_{i,j} \) does not contain any sets of the
minimal depth \( k \) then we simply have \( \Xi^{(n)}_{i,j} = \Psi_{I_{i,j}} \) for all \( n \). Now property [iii] implies for each \( n \) that \( \Psi^\dagger_{I_{i,j}} \) and \( \bigvee_{(i,j) \neq (1,1)} \pi^{-1}_i(\Xi^{(n)}_{i,j}) \) are relatively independent over \( \pi^{-1}_1(\Xi^{(n)}_{1,1}) \).

Given now a family of sets \((A_{i,j})_{i,j}\) associated to \((I_{i,j})_{i,j}\), for each \((i,j)\) the conditional expectations \( E_\mu(1_{A_{i,j}} | \Xi^{(n)}_{i,j}) \) form an almost surely uniformly bounded martingale converging to \( 1_{A_{i,j}} \) in \( L^2(\mu) \). Letting

\[
B_{i,j}^{(n)} := \{ E_\mu(1_{A_{i,j}} | \Xi^{(n)}_{i,j}) > 1 - \delta \}
\]

for some small \( \delta > 0 \) (to be specified momentarily), it is clear that we also have \( \mu(A_{i,j} \triangle B_{i,j}^{(n)}) \to 0 \) as \( n \to \infty \). Let

\[
F := \prod_{i=1}^d \left( \bigcap_{j=1}^{k_i} B_{i,j}^{(n)} \right).
\]

We now compute using the above-mentioned relative independence that

\[
\lambda(F \setminus \pi^{-1}_i(A_{i,j})) = \int_{X^d} \left( \prod_{(i',j')} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) - 1_{A_{i,j}} \circ \pi_i \cdot \left( \prod_{(i',j')} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) d\lambda
\]

\[
= \int_{X^d} (1_{B_{i,j}^{(n)} \setminus A_{i,j}} \circ \pi_i) \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) d\lambda
\]

\[
= \int_{X^d} E_\mu(1_{B_{i,j}^{(n)} \setminus A_{i,j}} | \Xi^{(n)}_{i,j}) \circ \pi_i) \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) d\lambda
\]

for each pair \((i,j)\).

However, from the definition of \( B_{i,j}^{(n)} \) we must have

\[
E_\mu(1_{B_{i,j}^{(n)} \setminus A_{i,j}} | \Xi^{(n)}_{i,j}) \leq \delta 1_{B_{i,j}^{(n)}}
\]

almost surely, and therefore the above integral inequality implies that

\[
\lambda(F \setminus \pi^{-1}_i(A_{i,j})) \leq \delta \int_{X^d} (1_{B_{i,j}^{(n)} \circ \pi_i}) \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) d\lambda = \delta \lambda(F).
\]
From this we can estimate as follows:

\[ \lambda(F) \leq \lambda \left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) + \sum_{(i,j)} \lambda(F \setminus \pi_{i}^{-1}(A_{i,j})) \leq 0 + \left( \sum_{i=1}^{d} k_i \right) \delta \lambda(F), \]

and so provided we chose \( \delta < \left( \sum_{i=1}^{d} k_i \right)^{-1} \) we must in fact have \( \lambda(F) = 0 \).

We have now obtained sets \((B_{i,j}^{(n)})_{i,j}\) that are associated to the family \((\mathcal{I}_{i,j})_{i,j}\) and satisfy the property of lying in finitely-generated extensions of the relevant factors corresponding to the members of the \(\mathcal{I}_{i,j}\) of minimal size, and so we can apply the result of Step 1 to deduce that \(\mu\left( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} B_{i,j}^{(n)} \right) = 0\).

It follows that

\[ \mu\left( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} A_{i,j} \right) \leq \sum_{i,j} \mu(A_{i,j} \setminus B_{i,j}^{(n)}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \]

as required. \(\square\)

**Proof of Proposition [4.3.1]** We first take as our base case \(k_i = 1\) and \(\mathcal{I}_{i,1} = \{[d]\}\) for each \(i = 1, 2, \ldots, d\). In this case we know from property [ii] that for any \(A \in \Psi_{[d]}\) the pre-images \(\pi_{i}^{-1}(A)\) are all equal up to negligible sets, and so given \(A_1, A_2, \ldots, A_d \in \Psi_{[d]}\) we have \(0 = \lambda(A_1 \times A_2 \times \cdots \times A_d) = \mu(A_1 \cap A_2 \cap \cdots \cap A_d)\).

The remainder of the proof now just requires putting the preceding lemmas into order to form an induction with three layers: if our collection has any non-principal up-sets of minimal depth, then Lemma [4.3.4] allows us to reduce their number at the expense only of introducing new principal up-sets of the same depth; and having removed all the non-principal minimal-depth up-sets, Lemma [4.3.3] enables us to remove also the principal ones until we are left only with up-sets of increased minimal depth. This completes the proof. \(\square\)
Chapter 5

The Density Hales-Jewett Theorem

Much as for Szemerédi’s Theorem and its multidimensional generalization, the Ergodic Ramsey Theory approach to Theorem B begins by establishing its equivalence to a result about stochastic processes. We have deferred the introduction of the stochastic processes analog of Theorem B until now because it involves a less well-known family of processes than the tuples of commuting transformations that appear in Theorem A, and these new stochastic processes require a separate discussion. The proof from [FK91] of the correspondence between Theorem B and an assertion about these processes is also less well-known, and so we recall this in the first section below for completeness.

After formulating the stochastic processes result to which Theorem B is equivalent, we introduce an additional semigroup \( \Gamma \) of transformations on these processes and argue that we may reduce further to the case of processes whose distributions are invariant. This leaves us with a class of \( \Gamma \)-systems, on which we will bring a notion of satedness to bear. However, as promised at the beginning of Chapter 2, this first requires some modifications to that notion, effectively by imposing additional restrictions on the factor maps we allow in our theory of a kind not involved heretofore. With these modifications in place we will proceed to analogs of Propositions 4.2.4 and 4.2.6 and thence to the proof of Theorem B.
5.1 The correspondence with a class of stationary processes

Combinatorial notation

In addition to the finite spaces $[k]^N$ appearing in the statement of Theorem B, we will work with their union

$$[k]^* := \bigcup_{N \geq 1} [k]^N.$$ 

The spaces $[k]^N$ and $[k]^*$ are referred to as the $N$-dimensional and infinite-dimensional combinatorial spaces over the alphabet $[k]$ respectively. Most of this chapter will consider probabilities on product spaces indexed by $[k]^*$. 

If $A \subseteq [k]^N$ then we denote its density by $d(A) := \frac{|A|}{k^N}$; thus the assumption of Theorem B is that $N$ is sufficiently large in terms of $k$ and $d(A)$. 

Given two finite words $u, v \in [k]^*$ we denote their concatenation by either $uv$ or $u \oplus v$. For any finite $n$ we define an $n$-dimensional subspace of $[k]^*$ to be an injection $\phi : [k]^n \hookrightarrow [k]^*$ specified as follows: for some integers $0 = N_0 < N_1 < N_2 < \ldots < N_n$, nonempty subsets $I_1 \subseteq [N_1], I_2 \subseteq [N_2] \setminus [N_1], \ldots, I_n \subseteq [N_n] \setminus [N_{n-1}]$ and a word $w \in [k]^{N_n}$ we let $\phi(v_1 v_2 \cdots v_n)$ be the word in $[k]^*$ of length $N_n$ given by

$$\phi(v_1 v_2 \cdots v_n)_m := \begin{cases} w_m & \text{if } m \in [N_n] \setminus (I_1 \cup I_2 \cup \cdots \cup I_n) \\ v_i & \text{if } m \in I_i. \end{cases}$$

In these terms a combinatorial line is simply a 1-dimensional combinatorial subspace. 

Similarly, an infinite-dimensional subspace (or often just subspace) of $[k]^*$ is an injection $\phi : [k]^* \hookrightarrow [k]^*$ specified using some infinite sequence $0 = N_0 < N_1 < N_2 < \ldots$, nonempty subsets $I_{i+1} \subseteq [N_{i+1}] \setminus [N_i]$ and words $w_i \in [k]^{N_i}$, where for any $v \in [k]^n$ its image $\phi(v)$ has length $N_n$ and is given by the above formula with $w := w_n$. It is clear that the collection of all subspaces of $[k]^*$ forms a semigroup $\Gamma$ under composition.
Finally, let us define letter-replacement maps: give \( i \in [k] \) and \( e \subseteq [k] \), for each \( N \geq 1 \) we define \( r_{e,i}: [k]^N \rightarrow [k]^N \) by

\[
 r_{e,i}(w)_m := \begin{cases} 
    i & \text{if } w_m \in e \\
    w_m & \text{if } w_m \in [k] \setminus e
\end{cases}
\]

for \( m \leq N \), and let

\[
 r_{e,i} := \bigcup_{N \geq 1} r_{e,i}^N : [k]^* \rightarrow [k]^*
\]

(so clearly \( r_{e,i} \) actually takes values in the subset \( ([k] \setminus e) \cup \{i\})^* \subseteq [k]^* \).

**Reformulation in terms of stochastic processes**

The correspondence that Furstenberg and Katznelson establish for Theorem B is between dense subsets of the finite-dimensional combinatorial spaces \( [k]^N \) and stochastic processes indexed by the infinite-dimensional combinatorial space \( [k]^* \).

**Theorem 5.1.1** (Infinitary Density Hales-Jewett Theorem). For any \( \delta > 0 \), if \( \mu \) is a Borel probability measure on \( \{0, 1\}^{[k]^*} \) with the property that

\[
 \mu\{\mathbf{x} \in \{0, 1\}^{[k]^*} : x_w = 1 \} \geq \delta \quad \forall w \in [k]^* ,
\]

then there is a combinatorial line \( \phi: [k] \rightarrow [k]^* \) such that

\[
 \mu\{\mathbf{x} \in \{0, 1\}^{[k]^*} : x_{\phi(i)} = 1 \forall i \in [k]\} > 0.
\]

**Proof of Theorem B from Theorem 5.1.1** Clearly we may restrict our attention to \( k \geq 2 \). We will suppose that theorem B fails, and show that this would give rise to a counterexample to Theorem 5.1.1. We break this into two steps.

**Step 1** First observe that if \( N \geq L \geq 1 \) and \( A \subseteq [k]^N \) has \( d(A) > 1 - \frac{1}{k^L} \), then \( A \) necessarily contains a whole \( L \)-dimensional combinatorial subspace. Indeed, having density as high as this implies that each of the \( k^L \) subsets

\[
 A_u := \{ w \in [k]^{N-L} : u \oplus w \in A \} \quad \text{for } u \in [k]^L
\]
has density greater than $1 - \frac{1}{k^L}$, and so there must be some $w \in \bigcap_{u \in [k]^L} A_u$, implying that the subspace $[k]^L \hookrightarrow [k]^N : u \mapsto u \oplus w$ has image lying entirely in $A$.

In particular, letting $L = 1$, if we assume that Theorem B fails then we may let

$$\delta_0 := \sup \{ \delta > 0 : \text{Theorem B fails for subsets of density } \delta \}$$

and deduce that $0 < \delta_0 < 1$.

**Step 2** Now fix some integer $L \geq 1$ and let $A \subseteq [k]^N$ be a subset of density $d(A) = \delta > (1 + \frac{1}{2k^{L+1}})^{-1}\delta_0$ for some $N \geq L$ such that $A$ contains no combinatorial lines.

Let $N = L + M$ and decompose $[k]^N = [k]^L \oplus [k]^M$. For each $w \in [k]^L$ let

$$A_w := \{ v \in [k]^M : w \oplus v \in A \}.$$ 

Clearly

$$\frac{1}{k^L} \sum_{w \in [k]^L} d(A_w) = d(A) = \delta,$$

and on the other hand $d(A_w) < (1 + \frac{1}{2k^{L+1}})\delta$ for each $w$ once $N$ is sufficiently large, for otherwise $A_w$ would contain a combinatorial line by the definition of $\delta_0$. Therefore the above equation between densities and Chebyshev’s inequality require that in fact *every* $w \in [k]^L$ have $d(A_w) > \delta/2$.

Now defining the probability measure $\mu_L$ on $\{0, 1\}^{[k]^L}$ by

$$\mu_L\{(x_w)_{w \in [k]^L} : x_w = 1 \text{ for } w \in [k]^L \} := d(\{ v \in [k]^M : x_w = 1 \forall w \in [k]^L \})$$

for each $(x_w)_{w \in [k]^L} \in \{0, 1\}^{[k]^L}$, we see that for each $L$ we have produced a probability $\mu_L$ on $\{0, 1\}^{[k]^L}$ such that

$$\mu_L\{ x \in \{0, 1\}^{[k]^L} : x_w = 1 \} = d(A_w) \geq \delta/2 \geq \delta_0/4 \quad \forall w \in [k]^L$$

but

$$\mu_L\{ x \in \{0, 1\}^{[k]^L} : x_{\phi(i)} = 1 \forall i \in [k] \} = 0$$

for any combinatorial line $\phi : [k] \hookrightarrow [k]^L$. Finally defining $\mu := \bigotimes_{L \geq 1} \mu_L$, we obtain a measure that contradicts Theorem 5.1.1 with density $\delta_0/4$. □
Remark  The above proof is essentially taken from Proposition 2.1 of \cite{FK91}, where the reverse implication is also proved. 

\section{5.2 Strongly stationary processes}

After introducing Theorem \ref{Thm:Stab}, Furstenberg and Katznelson make a further reduction to a special subclass of measures.

\begin{definition}[Semigroup action of combinatorial subspaces] If $\phi : [k]^N \mapsto [k]^*$ is a combinatorial subspace then for any product space $K^{[k]^*}$ we define the corresponding map $T_\phi : K^{[k]^*} \rightarrow K^{[k]^*}$ by
\[
(T_\phi(x))(w) := x_{\phi(w)} \quad \text{for } w \in [k]^N \text{ and } x = (x_u)_{u \in [k]^*} \in K^{[k]^*},
\]
and similarly define $T_\phi : K^{[k]^*} \rightarrow K^{[k]^*}$ in case $\phi : [k]^* \mapsto [k]^*$.
\end{definition}

\begin{definition}[Strongly stationary laws] A probability measure $\mu$ on the product $(K^{[k]^*}, \Psi^{[k]^*})$ for some standard Borel space $(K, \Psi)$ is strongly stationary if $T_\phi \# \mu = \mu$ for all subspaces $\phi \in \Gamma$. In this case the transformations $T_\phi$ give to $(K^{[k]^*}, \Psi^{[k]^*}, \mu)$ the structure of a probability-preserving $\Gamma$-system.
\end{definition}

\begin{lemma} If Theorem \ref{Thm:Stab} holds for all strongly stationary measures for any $\delta > 0$ then it holds for all measures satisfying the conditions of that theorem for any $\delta > 0$.
\end{lemma}

\begin{proof} This argument is again lifted directly from \cite{FK91}, and we only sketch the details. Given a measure $\mu$ satisfying the conditions of Theorem \ref{Thm:Stab} for some $\delta > 0$, by applying the Carlson-Simpson Theorem \cite{Car88} to arbitrarily fine finite open coverings of the finite-dimensional spaces of probability distributions on $\{0,1\}^{[k]^n}$ for increasingly large $n$, we obtain a subspace $\psi : [k]^* \mapsto [k]^*$ and an infinite word $w = w_1 w_2 \cdots \in [k]^\mathbb{N}$ such that the restricted laws
\[
T_{\psi(w_1 w_2 \cdots w_m \oplus \cdot \cdot \cdot)} \# \mu
\]
converge to a strongly stationary law as \( m \to \infty \), and since all one-dimensional marginals of the input law gave probability at least \( \delta \) to \( \{1\} \), the same is true of the limit measure. Finally, the subset of probability measures

\[
\{ \nu \in \Pr\{0, 1\}^{[k]^*} : \nu\{x \in \{0, 1\}^{[k]^*} : x_{\phi(i)} = 1 \ \forall i \leq k \} > 0 \}
\]

is finite-dimensional and open for any given line \( \phi : [k] \hookrightarrow [k]^* \), so if the limit measure is in this set the so is some image of the original measure. \( \square \)

An immediate consequence of the strong stationarity of a measure \( \mu \) is that for any two \( N \)-dimensional subspaces \( \phi, \psi : [k]^N \hookrightarrow [k]^* \) we have \( T_{\phi\#}\mu = T_{\psi\#}\mu \). In case \( N = 0 \) we refer to this common image measure as the point marginal \( \mu \) and denote it by \( \mu_{\text{pt}} \), and similarly in case \( N = 1 \) it is the line marginal of \( \mu \) and is denoted by \( \mu_{\text{line}} \). In these terms it is possible to give another, more convenient reformulation of Theorem 5.1.1.

**Theorem 5.2.4.** If \((K, \Psi)\) is a standard Borel space and \( \mu \) is a strongly stationary law on \((K^{[k]^*}, \Psi [k]^*)\) then for any \( A_1, A_2, \ldots, A_k \in \Psi \) we have

\[
\mu_{\text{line}}(A_1 \times A_2 \times \cdots \times A_k) = 0 \quad \Rightarrow \quad \mu_{\text{pt}}(A_1 \cap A_2 \cap \cdots \cap A_k) = 0.
\]

The resemblance to Theorem 4.0.2 is far from accidental!

The proof of Theorem 5.2.4 will involve a version of satedness for our systems of interest; however, here a slight subtlety creeps in. In the following we will need to work with only those \( \Gamma \)-systems that are of the form \((K^{[k]^*}, \Psi [k]^*, \mu, T)\) for some strongly stationary measure \( \mu \) (of course, the huge semigroup \( \Gamma \) could also have invariant measures for all sorts of other Borel actions, not of this form). On the other hand, the conclusion of Theorem 5.2.4 is not about the joint distribution of several copies of whole \( \Gamma \)-systems under some self-joining. Rather, it is about the joint distribution of some copies of just the ‘one-dimensional’ point marginal \((K, \Psi, \mu_{\text{pt}})\) under the line marginal: this is only a tiny fragment of the whole system \((K^{[k]^*}, \Psi [k]^*, \mu, T)\).

The way we can keep track of the structure of point and line marginals between different such systems is by restricting the kinds of factor map we allow.

**Definition 5.2.5.** Let \( A \) be the class of \( \Gamma \)-systems given by strongly stationary measures on product spaces indexed by \([k]^*\), as above.
A coordinatewise factor (or cw-factor) of \( X = (K^*[k], \Psi \otimes [k]^*, \mu, T) \in A \) is a \( \sigma \)-subalgebra of the form \( \Phi \otimes [k]^* \leq \Psi \otimes [k]^* \) for some \( \Phi \leq \Psi \). Slightly abusively, we will sometimes refer instead to the single-coordinate \( \sigma \)-subalgebra \( \Phi \) as a cw-factor. Likewise, a cw-factor map is a map of the form

\[
f^* : (K^*[k], \Psi \otimes [k]^*, \mu, T) \longrightarrow (L^*[k], \Xi \otimes [k]^*, \nu, T) : (x_w) \mapsto (f(x_w))_w
\]

for some Borel map \( f : (K, \Psi) \longrightarrow (L, \Xi) \), and \( f^* \) is a cw-isomorphism if \( f \) is measurably invertible away from some \( \mu^{pt} \) and \( \nu^{pt} \)-negligible sets (this is clearly equivalent to its being an isomorphism in the usual sense).

With \( f^* \) as above we shall sometimes refer to \( f \) as its corresponding single-coordinate map.

It is now easy to see that the class \( A \) is closed under joinings and inverse limits, provided that we interpret a joining of two systems \( (K^*[k], \Psi \otimes [k]^*, \mu, T) \) and \( (L^*[k], \Xi \otimes [k]^*, \nu, T) \) as a strongly stationary measure on \( (K \times L)^*[k]^* \) and that we restrict our attention to inverse sequences whose connecting maps are all cw-factor maps. We will henceforth refer to a subclass \( C \subseteq A \) as cw-idempotent if it is closed under cw-isomorphisms, joinings and inverse limits involving cw-factor maps, and now observe that all of the definitions and lemmas of Section 2.2 have direct analogs for cw-idempotent classes obtained simply by insisting that all morphisms be given by cw-factor maps. In particular, if \( C \) is a cw-idempotent class and \( X = (K^*[k], \Psi \otimes [k]^*, \mu, T) \in A \) then the maximal cw-C-factor of \( X \) is given by \( \Phi \otimes [k]^* \) where \( \Phi \) is the maximal \( \sigma \)-algebra in the family

\[
\{ \Xi \leq \Psi : \Xi \text{ is generated by some Borel map } f : (K, \Psi) \longrightarrow (K_1, \Psi_1) \\
\text{such that } (K_1^*[k]^*, \Psi_1 \otimes [k]^*, f^*_\# \mu, T) \in C \}.
\]

We will write a cw-factor map coordinatizing this maximal C-factor as \( \zeta_C \) for some map \( \zeta_C : K \longrightarrow CK \) of single-coordinate spaces.

Given these observations we can make our analog of Definition 2.3.1.

**Definition 5.2.6 (CW-sated systems).** For a cw-idempotent class \( C \subseteq A \), a system \( X \in A \) is cw-C-sated if for any cw-extension

\[
\pi^* : \tilde{X} = (\tilde{K}^*[k], \tilde{\Psi} \otimes [k]^*, \tilde{\mu}, T) \longrightarrow X
\]

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the single-coordinate maps $\pi : \tilde{K} \to K$ and $\tilde{\zeta}_C : \tilde{K} \to CK$ are relatively independent under $\tilde{\mu}^{pt}$ over $\zeta_C \circ \pi : \tilde{K} \to K \to CK$, where $\tilde{\zeta}_C^*$ and $\zeta_C^*$ coordinatize the maximal $C$-factors of $\tilde{X}$ and $X$ respectively.

**Theorem 5.2.7.** If $(C_i)_{i \in I}$ is a countable family of cw-idempotent classes then any system $X_0 \in A$ admits a cw-extension $\pi : X \to X_0$ that is cw-$C_i$-sated for every $i \in I$.

**Proof outline** This proceeds in exact analogy with the proof of Theorem 2.3.2. First, applying the argument for Lemma 2.3.3 to a bounded measurable function $f$ on the single-coordinate space of an inverse limit shows that an inverse limit of cw-$C_i$-sated systems through cw-factor maps is cw-$C_i$-sated. Next, given a system $X = (K[k]^*, \Psi \otimes [k]^*, \mu, T) \in A$, we show how to produce a cw-sated extension for a single cw-idempotent class $C$: first enumerate an $L^2$-dense sequence $(f_r)_{r \geq 1}$ in the unit ball of $L^\infty(\mu^{pt})$; then apply the same exhaustion argument as in Step 1 of Theorem 2.3.2 to produce an inverse sequence of cw-extensions

$$
\ldots \xrightarrow{(\zeta_{n+1}^*)} X_{n+1} = (K_{n+1}[k]^*, \Psi_{n+1} \otimes [n]^*, \mu_{n+1}, T) \xrightarrow{(\zeta_{n}^*)} X_n = (K_n[k]^*, \Psi_n \otimes [n]^*, \mu_n, T) \xrightarrow{(\zeta_{n-1}^*)} \ldots \to X
$$

such that for each $r$ it happens cofinally often that this extension is within a factor of 2 of achieving the optimal increase in the $L^2$-norm of the conditional expectation $E_{\mu^{pt}}(f_r \circ \psi_0^n | \zeta_C^{(n)})$ (where $\zeta_C^{(n)}$ is the single-coordinate map coordinatizing $CX_n$); and finally take the inverse limit of this sequence. Just as in the proof of Theorem 2.3.2 if this inverse limit were not cw-$C_i$-sated then this would lead to a contradiction with our assumption on the increase of $\|E_{\mu^{pt}}(f_r \circ \psi_0^n | \zeta_C^{(n)})\|_2$ for some finite $n$.

Finally the proof is completed by arguing that given a countable collection of cw-idempotent classes $C_i$, we can produce one long inverse sequence of extensions in which for each $i$ there is a cofinal subsequence of cw-$C_i$-sated systems, so that the inverse limit is cw-$C_i$-sated for every $i$.  \qed
This completes the modifications we need for our approach to Theorem B. Note that detailed proofs of the above results written in the setting of strongly stationary laws are given in [Ausa].

Remark  In principle one could give a complete unification of Chapter 2 with the above modifications to it by phrasing all of these results in terms of a general (not necessarily full) subcategory \( \text{Cat} \) of the category \( \Gamma\text{-Sys} \) of all \( \Gamma \)-systems, and adopting a flexible meaning for the term ‘relatively independent’. In this work we have preferred to draw a more informal parallel between our two settings of interest, but it may be instructive to deduce from the proofs of Chapter 2 what basic properties we really need for the existence of sated extensions and the various lemmas that support it. Although we leave the proof to the reader, it turns out that \( \text{Cat} \) must admit two basic constructions:

- it must have inverse limits;
- it must have generated factors: that is, if

\[
\begin{array}{c}
X \\
\downarrow \\
Y & \rightarrow & Z \\
\end{array}
\]

is a diagram in \( \text{Cat} \), then there is an essentially unique minimal system \( W \) that may be inserted into this diagram as

\[
\begin{array}{c}
X \\
\downarrow \\
Y & \rightarrow & W & \rightarrow & Z \\
\end{array}
\]

Note, interestingly, that it does not seem to be essential that any diagram such as

\[
\begin{array}{c}
X \\
\downarrow \\
Z \\
\end{array}
\]

\[
\begin{array}{c}
Y \\
\downarrow \\
X \\
\end{array}
\]
have a common extension of $X$ and $Y$ that can be inserted above it (of course, working in the whole of $\Gamma$-$\text{Sys}$ when $\Gamma$ is a group such a common extension is provided by the relatively independent product).

While these assumptions on $\text{Cat}$ are relatively innocuous, more drastic steps are needed if we are to accommodate the instances of relative independence appearing in both Theorem 2.3.2 and Theorem 5.2.7. The former of these asserts the relative independence of two whole factors of some extended system, whereas the latter concerns only the relative independent of functions of a single fixed coordinate within each of those factors (that is, relative independence under $\mu^{\text{pt}}$ rather than $\mu$). In order to treat these together, one could for example augment the category $\text{Cat}$ by attaching to each system some distinguished subalgebra of bounded measurable functions (the whole of $L^\infty$ in the first case, and the subalgebra of functions of $x_w$ for some distinguished $w \in [k]^*$ in the second), and then re-defining conditional expectation as an operator acting only between these subalgebras for different systems and satisfying the usual conditions of idempotence and agreement of integrals against functions in the target subalgebra.

Altogether these very abstract considerations seem more demanding than worthwhile, and I know of few other situations in which a non-standard example of an abstract category of systems having these properties has been useful in ergodic theory. One related area which could fit into this mould is the study of partial exchangeability in probability theory, for which we refer the reader to Kallenberg’s book [Kal02], the survey papers [Aus08, Ald] and the references given there.

5.3 Another appeal to the infinitary hypergraph removal lemma

The cw-idempotent classes for which we will apply Theorem 5.2.7 are as follows.

**Definition 5.3.1** (Partially insensitive processes). Given a subset $e \subseteq [k]$, a process $(K^{[k]^*}, \Psi^{[k]^*}, \mu, T) \in A$ is $e$-insensitive if its line marginal satisfies

$$x_i = x_j \quad \text{for } \mu^{\text{line}}\text{-a.e. } (x_1, x_2, \ldots, x_k) \in K^k \text{ for all } i, j \in e.$$

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We write $A_e \subseteq A$ for the subclass of all $e$-insensitive processes.

The persistence of $e$-insensitivity under inverse limits and joinings is immediate, and so we have:

**Lemma 5.3.2.** The class $A_e$ is cw-idempotent for each $e \subseteq [d]$. □

In parallel with the developments of Section 4.2, given an arbitrary process

$$X = (K^{[k]}^*, \Psi^{[k]}^*, \mu, T) \in A,$$

for each $e \subseteq [d]$ we let $\Phi_e$ denote the $e$-insensitive $\sigma$-subalgebra of $\Psi$, consisting of those $A \in \Psi$ such that $\mu^{\text{line}}(\pi^{-1}_i(A) \triangle \pi^{-1}_j(A)) = 0$ for all $i, j \in e$, where $\pi_i : K^k \to K$ is the coordinate projection. Letting $\zeta_e : (K, \Psi) \to (K_e, \Psi_e)$ be some map of standard Borel spaces such that $\Phi_e$ agrees with $\{\zeta_e^{-1}(E) : E \in \Psi_e\}$ modulo $\mu$-negligible sets, it follows that

$$\zeta_e^* : K^{[k]}^* \to K_e^{[k]}^* : (x_w)_w \mapsto (\zeta_e(x_w))_w$$

is a cw-factor map that coordinatizes $X \to A_e X$.

Directly from the definition of $\Phi_e$ we observe that if $i, j \in e$ then $\pi_i^{-1}(\Phi_e)$ and $\pi_j^{-1}(\Phi_e)$ differ only by $\mu^{\text{line}}$-negligible sets, and we denote their common $\mu^{\text{line}}$-completion by $\Phi_e^\dagger$. If now $I \subseteq \binom{[k]}{\geq 2}$ is an up-set, then similarly to the setup of Section 4.2 we define $\Phi_I := \bigvee_{e \in I} \Phi_e$ and $\Phi_I^\dagger := \bigvee_{e \in I} \Phi_e^\dagger$.

In terms of these definitions, the consequences of cw-satedness that we need are now essentially parallel to Propositions 4.2.4 and 4.2.6.

**Proposition 5.3.3.** For each $i \leq k$ let

$$C_i := \bigvee_{j \leq k, j \neq i} A_{\{i,j\}}.$$

If a system $X$ with strongly stationary measure $\mu$ is cw-$C_i$-sated for each $i$ then the $\sigma$-algebras $\pi_i^{-1}(\Psi) \leq \Psi^{\otimes k}$ are relatively independent under $\mu^{\text{line}}$ over the further factors

$$\pi_i^{-1}\left(\bigvee_{j \leq k, j \neq i} \Phi_{\{i,j\}}\right).$$
Proof. Clearly it will suffice to prove that \( \pi_1^{-1}(\Psi) \) is relatively independent from \( \pi_2^{-1}(\Psi) \lor \cdots \lor \pi_d^{-1}(\Psi) \) under \( \mu^{\text{line}} \) over

\[
\Xi := \bigvee_{j=2}^{k} \Phi_{\{1,j\}},
\]

since the cases of the other coordinates under \( \mu^{\text{line}} \) then follow by symmetry.

We prove this by contradiction, so suppose that \( f_1, f_2, \ldots, f_d \in L^\infty(\mu^{\text{pt}}) \) are such that

\[
\int_{K^k} f_1 \otimes f_2 \otimes \cdots \otimes f_k \, d\mu^{\text{line}} \neq \int_{K^k} \mathcal{E}(f_1 \mid \Xi) \otimes f_2 \otimes \cdots \otimes f_k \, d\mu^{\text{line}}.
\]

We will deduce from this a contradiction with the cw-satedness of \( \mu \). By replacing \( f_1 \) with \( f_1 - \mathcal{E}(f_1 \mid \Xi) \) it suffices to assume that \( \mathcal{E}(f_1 \mid \Xi) = 0 \) but that the left-hand integral above does not vanish.

For each \( j = 2, 3, \ldots, k \) recall the letter-replacement map \( r_{\{1,j\},j} : [k]^* \to [k]^* \) defined in Section 5.1. In view of the strong stationarity of \( \mu \), we may transport the above non-vanishing integral to any combinatorial line in \( [k]^* \): in particular, picking some \( w \in [k]^* \) for which \( w^{-1}\{j\} \neq \emptyset \) for every \( j \), the points \( \{w, r_{\{1,2\},2}(w), r_{\{1,3\},3}(w), \ldots, r_{\{1,k\},k}(w)\} \) form such a line, and so we have

\[
\int_{X^{[k]^*}} f_1(x_w) \cdot f_2(x_{r_{\{1,2\},2}(w)}) \cdot \cdots \cdot f_k(x_{r_{\{1,k\},k}(w)}) \, \mu(dx) = \kappa \neq 0.
\]

Now define the probability measure \( \lambda \) on \( (K \times K^{\{2,3,\ldots,k\}})^{[k]^*} \) to be the joint law under \( \mu \) of

\[
(x_w) \mapsto (x_w, x_{r_{\{1,2\},2}(w)}, x_{r_{\{1,3\},3}(w)}, \ldots, x_{r_{\{1,k\},k}(w)})_w.
\]

We see that all of its coordinate projections onto individual copies of \( K \) are still just \( \mu^{\text{pt}} \), the cw-factor map

\[
\phi^*_1 : (x_w, y_2, y_3, \ldots, y_k)_w \mapsto (x_w)_w
\]

has \( \phi^*_1 \# \lambda = \mu \), and the cw-factor map

\[
\phi^*_j : (x_w, y_2, y_3, \ldots, y_k)_w \mapsto (y_{j,w})_w
\]
for \( j = 2, 3, \ldots, k \) is \( \lambda \)-almost surely \( \{1, j\} \)-insensitive. Therefore through the cw-factor map \( \phi_1^* \) the law \( \lambda \) defines an extension of \( \mu \) as a measure space.

This new measure \( \lambda \) may not be strongly stationary, so may not define an extension of members of \( A \). However, we can now repeat the trick of Lemma 5.2.3 By the Carlson-Simpson Theorem there are a subspace \( \psi : [k]^* \rightarrow [k]^* \) and an infinite word \( w \in [k]^\mathbb{N} \) such that the pulled-back measures

\[
T_{\psi(u_1 u_2 \cdots u_n \oplus \cdot \cdot \cdot)} \lambda
\]

converge in the coupling topology on \( (K \times K^{(2, 3, \ldots, k)})[k]^* \) (recall that for couplings of fixed marginal measures this is compact; see Theorem 6.2 in [Gla03]) to a strongly stationary measure \( \tilde{\mu} \). Since \( \mu \) was already strongly stationary, we must still have \( \phi_1^* \# \tilde{\mu} = \mu \), and by the definition of the coupling topology as the weakest for which integration of fixed product functions is continuous it follows that we must still have, firstly, that

\[
\int_{(K \times K^{(2, 3, \ldots, k)})[k]^*} (f \circ \pi_u \circ \phi_1^*) \cdot \prod_{j \in [k]\setminus e} (h_j \circ \pi_u \circ \phi_j^*) \, d\tilde{\mu} = \kappa \neq 0
\]

for each \( u \in [k]^* \) (where now we may omit the assumption that \( u \) contains every letter at least once, by strong stationarity), and secondly that the cw-factors generated by the maps \( \phi_j^* \) are \( \{1, j\} \)-insensitive under \( \tilde{\mu} \), since this is equivalent to the assertion that for any \( A \in \Psi \) and line \( \ell : [k] \rightarrow [k]^* \) we have

\[
\int_{(K \times K^{(2, 3, \ldots, k)})[k]^*} 1_A(\phi_j(\ell(1))) \cdot 1_{K\setminus A}(\phi_j(\ell(j))) \, \tilde{\mu}(d\mathbf{z}) = 0
\]

and this is clearly a closed condition in the coupling topology.

It follows that this strongly stationary measure \( \tilde{\mu} \) gives a genuine cw-extension \( \phi_1^* : \tilde{X} \rightarrow X \) such that the lift of \( f_1 \circ \pi_1 \) as a function of any one coordinate must have a nontrivial inner product with some pointwise product of \( \{1, j\} \)-insensitive functions under \( \tilde{\mu} \) over \( j = 2, 3, \ldots, k \). Hence this lift has nonzero conditional expectation onto a \( \sigma \)-subalgebra of \( \Psi \otimes \Psi^{\otimes \{2, 3, \ldots, k\}} \) coordinatizing a cw-factor in the class \( C_1 \), but recalling our assumption that \( E(f_1 | \Xi) = 0 \), this provides the desired contradiction with cw-\( C_1 \)-satedness. \( \square \)
Proposition 5.3.4. For each $e \subseteq [k]$ let

$$C_e := \bigvee_{j \in [d] \setminus e} A_{e \cup \{j\}}.$$  

If $X$ is cw-$C_e$-sated for every $e$ then for any up-sets $\mathcal{I}, \mathcal{I}' \subseteq \binom{[k]}{\geq 2}$ the $\sigma$-subalgebras $\Phi^\downarrow_\mathcal{I}$ and $\Phi^\downarrow_{\mathcal{I}' \setminus \mathcal{I}}$ are relatively independent under $\mu^\line$ over $\Phi^\downarrow_{\mathcal{I} \setminus \mathcal{I}'}$.

Proof As for Proposition 4.2.6 we start with the case in which $\mathcal{I}' = \langle e \rangle$ for $e$ a member of $\binom{[d]}{\geq 2}$ of maximal size, and again just as for that proposition it suffices to show that $\Phi^\downarrow_e$ is relatively independent from $\bigvee_{j \in [k] \setminus e} \pi_j^{-1}(\Psi)$ over $\bigvee_{j \in [k] \setminus e} \Phi^\downarrow_{e \cup \{j\}}$ under $\mu^\line$.

Again this is best proved by deriving a contradiction with cw-satedness. Pick some $i \in e$, so $\Phi^\downarrow_e$ agrees with $\pi_i^{-1}(\Phi_e)$ up to negligible sets, let

$$\Xi := \bigvee_{j \in [k] \setminus e} \Phi_e \cup \{j\},$$

and suppose we have some $f \in L^\infty(\mu^\pt)$ that is $\Phi_e$-measurable and such that $E(f \mid \Xi) = 0$, and also $h_j \in L^\infty(\mu^\pt)$ for each $j \in [k] \setminus e$ such that

$$\int_{K^k} (f \circ \pi_i) \cdot \prod_{j \in [k] \setminus e} (h_j \circ \pi_j) \, d\mu^\line = \kappa \neq 0.$$

Arguing as for the preceding proposition, this nonvanishing can be transported to any combinatorial line in $[k]^*$, including to a line such as $\{r_{e,1}(w), r_{e,2}(w), r_{e,3}(w), \ldots, r_{e,k}(w)\}$ for any $w$ that contains every letter at least once. This gives

$$\int_{K^{|k|}} f(x_{r_{e,i}(w)}) \cdot \prod_{j \in [k] \setminus e} h_j(x_{r_{e,j}(w)}) \, d\mu = \kappa$$

for any such $w$, but since $f$ is $e$-insensitive we may replace the first factor in this integrand simply by $f(x_w)$.

It follows that if we define the probability measure $\lambda$ on $(K \times K^{[k] \setminus e})^{|k|}^*$ to be the joint law under $\mu$ of

$$\begin{aligned}
(x_w)_w &\mapsto (x_w, (x_{r_{e,j}(w)})_{j \in [k] \setminus e})_w
\end{aligned}$$

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then all of its coordinate projections onto individual copies of $K$ are still just $\mu_{pt}$, the cw-factor map

$$\phi^*: (x_w, (y_{j,w})_{j \in [k] \setminus e})_w \mapsto (x_w)_w$$

has $\phi^* \# \lambda = \mu$ and the cw-factor maps

$$\phi_j^*: (x_w, (y_{j,w})_{j \in [k] \setminus e})_w \mapsto (y_{j,w})_w$$

are $\lambda$-almost surely $(e \cup \{j\})$-insensitive. Therefore through $\phi^*$ the measure $\lambda$ is an extension of the measure $\mu$, and the above inequality gives a non-zero inner product for the lift of $f \circ \pi_u$ through $\phi^*$ with some product over $j \in [k] \setminus e$ of $(e \cup \{j\})$-insensitive functions under $\lambda$, which we can express as

$$\int_{K^{[k]^*}} (f \circ \pi_u \circ \phi^*) \cdot \prod_{j \in [k] \setminus e} (h_j \circ \pi_u \circ \phi_j^*) \, d\lambda = \kappa$$

for any $u \in [k]^*$ that contains each letter at least once.

To complete the proof, we may argue exactly as for Proposition 5.3.3 that within the not-necessarily-strongly-stationary law $\lambda$ we can find infinite-dimensional subspaces $\psi$ for which the corresponding image measures under $T_{\psi}$ converge in the coupling topology to a strongly stationary extension $\tilde{\mu}$ of $\mu$, and such that this extension preserves the feature that the lift of $f \circ \pi_u$ has a nontrivial inner product with a pointwise product of $(e \cup \{j\})$-insensitive functions under $\tilde{\mu}_{pt}$ for any word $u$. By our assumption that $E(f | \Xi) = 0$ this gives a contradiction with cw-$C_e$-satedness, as required.

The general case can now follow by induction on $I'$ for each fixed $I$ exactly as for Proposition 4.2.6. □

**Proof of Theorem 5.2.4** An initial application of Theorem 5.2.7 allows us to assume that $X$ is cw-sated for all the classes involved in Propositions 5.3.3 and 5.3.4.

Next, exactly as for the proof of Theorem 4.0.2 applying Proposition 5.3.3 shows that it suffices to prove Theorem 5.2.4 in case the sets $A_i$ lie in the $\sigma$-subalgebras $\Phi_{(i)} = \bigvee_{j \in [d] \setminus \{i\}} \Phi_{(i,j)} \leq \Psi_i$.

Finally, it follows from the definitions and Proposition 5.3.4 that the probability space $(K, \Psi, \mu_{pt})$, its self-coupling $\mu_{line}$ and the $\sigma$-subalgebras $\Phi_e$ and
their lifts $\Phi^1_{\bar{e}}$ for $e \subseteq [d]$ satisfy all the conditions of the ‘infinitary removal result’ Proposition 4.3.1, so another appeal to that proposition completes the proof. □

**Postscript to the above proof**

After the appearance of Furstenberg and Katznelson’s original, technically rather demanding proof of Theorem B in [FK91], considerable efforts were made to provide firstly a simpler proof, and more importantly one that could be made effective to deduce some quantitative bound on the necessary dependence of $N_0$ and $\delta$ and $k$.

Both of these goals were recently achieved by a large online collaboration, instigated by Tim Gowers and involving several other mathematicians, called Polymath1. Importantly, their new proof does give a dependence of $N_0$ on $\delta$ and $k$ similar to the dependence obtained for the Multidimensional Szemerédi Theorem by using the hypergraph regularity and removal lemmas. All these developments can be found online ([Polb]) and in the preprint [Polc].

Importantly, the infinitary proof of Theorem B that we have reported above relies on an observation that was originally taken from their work. I will not attempt an exact translation here since the lexicons of these two approaches are very different, but the outcome for stochastic processes is essentially the observation that an initially-given system $X \in A$ can be combined in a strongly stationary joining with some $\{1,j\}$-insensitive systems as in our proof of Proposition 5.3.3, which then gives some information on the structure of the original process $X$ (in our case by an appeal to cw-satedness).
Chapter 6

Coda: a general structural conjecture

It seems inadequate to finish this dissertation without discussing at least some of the issues obviously left open by the preceding chapters. Perhaps most interesting for ergodic theory is the meta-question introduced in Section 4.1 and in this last chapter I offer a few further speculations on what additional answers to it we might hope for.

Our first clue in this direction is offered by the works [HK05] of Host and Kra and [Zie07] of Ziegler, establishing the special case of Theorem C corresponding to different powers of a single ergodic transformation: that is, the result that if $T : \mathbb{Z} \curvearrowright (X, \Sigma, \mu)$ is ergodic and $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$ then the averages

$$S_N(f_1, f_2, \ldots, f_d) := \frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T^n) \cdot (f_2 \circ T^{2n}) \cdot \cdots \cdot (f_d \circ T^{dn})$$

converge in $L^2(\mu)$ as $N \to \infty$. Importantly, those two works both rest on a quite detailed result about ‘characteristic factors’ for these averages:

**Theorem 6.0.5** (Host-Kra Theorem). If $X = (X, \Sigma, \mu, T)$ is as above then there is a factor $\Phi \leq \Sigma$ that is characteristic for the averages $S_N$ in the sense that

$$S_N(f_1, f_2, \ldots, f_d) \sim S_N(\mathbb{E}(f_1 | \Phi), \mathbb{E}(f_2 | \Phi), \ldots, \mathbb{E}(f_d | \Phi))$$
in $L^2(\mu)$ for any $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$ as $N \to \infty$, and which can be generated by a factor map to a $(d-1)$-step pro-nilsystem: that is, it can be generated by some increasing sequence of factor maps

$$\pi_n : (X, \Sigma, \mu, T) \to (G_n/\Gamma_n, \text{Borel}, m_{G_n/\Gamma_n}, R_{g_n})$$

to systems that are given by rotations by elements $g_n$ on compact $(d-1)$-step nilmanifolds $G_n/\Gamma_n$.

**Remark** This notion of a characteristic factor is just a slight modification to that of a partially characteristic factor that we met in Proposition 3.3.1. In fact, Ziegler proves in [Zie07] that there is a unique minimal factor with the properties given by the above theorem, and in Leibman’s later treatment of these two proofs in [Lei05] it is shown that the pro-nilsystem characteristic factors constructed by Host and Kra are precisely these minimal factors.

This very surprising theorem asserts that for a completely arbitrary ergodic $\mathbb{Z}$-system $X$, its nonconventional averages $S_N$ are entirely controlled by some highly-structured factor of $X$, which can be expressed in terms of the very concrete data of rotations on nilmanifolds. In this informal discussion we will assume familiarity with the definition and basic properties of such ‘nilsystems’ here; they are treated thoroughly in [HK05] and the references given there.

Host and Kra and Ziegler’s proofs of the one-dimensional case of Theorem C proceed via two different approaches to Theorem 6.0.5. They are both rather longer than the proof in our Chapter 3, but using Theorem 6.0.5 they give a much more precise picture of the limit. On the other hand, the strategy used in our Chapter 3 simply cannot be specialized to the one-dimensional setting: it is essential for our approach that the result be formulated for the linearly independent directions $e_1, e_2, \ldots, e_d \in \mathbb{Z}^d$. This is because even if we are initially given a $\mathbb{Z}$-system $(X, \Sigma, \mu, T)$, we must re-interpret it as a $\mathbb{Z}^d$-system in order to pass to an extension that is sated in the way required by Proposition 3.3.1. To do this we define a new $\mathbb{Z}^d$-action $T'$ on $X$ by $(T')^{e_i} := T^i$, but once we ascend to our sated extension this special structure of a collection of powers of a single transformation will be lost, and so we can no longer focus on the special, one-dimensional case of convergence. In a sense, this quiet assumption of linear independence was a precursor to
the discussion of Section 4.1, we need the linear independence of the subgroups \( \mathbb{Z}e_i \leq \mathbb{Z}^d \) in order that a corresponding notion of satedness has useful consequences.

However, these two very different approaches to different cases of Theorem C do suggest a reconciliation of the issue raised at the end of Section 4.1: what becomes of our meta-question on the possibly joinings of \( \mathbb{Z}^D \)-systems \( X_i \in \mathbb{Z}^{\Gamma_i} \) if the subgroups \( \Gamma_i \) are not linearly independent? The centrepiece of this final chapter is a conjectural answer to this question. If true, it would offer the first step in a complete ‘interpolation’ between the structural result 6.0.5 of Host and Kra and our much softer result 4.1.2.

In order to formulate our conjecture, we first need some more notation. The notion of an isometric extension of ergodic probability-preserving systems and the fact that any such can be coordinatized as a skew-product extension over the base system by some compact homogeneous space are very classical; see, for instance, Glasner’s book [Gla03]. Here we will also assume familiarity with a natural but less common generalization of this theory to the case in which the base system is not necessarily ergodic, in which the fibres of our extension must be allowed to vary in a suitable ‘measurable’ way over the ergodic components of the base system. This theory is set up generally in [Ausc], where the lengthy but routine work of re-establishing all the well-known theorems from the ergodic case is carried out in full, and we will also adopt the basic notations of that paper.

**Definition 6.0.6** (Direct integral of pro-nilsystems). *If \( \Gamma \) is a discrete Abelian group then a \( \Gamma \)-system \( X = (X, \Sigma, \mu, T) \) is a direct integral of \( k \)-step pro-nilsystems if it admits a tower of factors*

\[
X = X_k \longrightarrow X_{k-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0
\]

*in which the action of \( \Gamma \) on \( X_0 \) is trivial, each extension \( X_i \longrightarrow X_{i-1} \) for \( i \geq 1 \) can be coordinatized as a relatively ergodic extension by measurably-varying compact metrizable Abelian group data*

\[
X_i \cong X_{i-1} \ltimes (A_i, m, \sigma_i)
\]

\[
X_{i-1}, \text{ canonical}
\]
(so the measurable group data $A_i \cdot \bullet$ really varies only over the base system $X_0$) and for each ergodic component $\mu_s$ of $\mu$ the resulting $k$-step Abelian distal ergodic $\Gamma$-system

$$(X, \Sigma, \mu_s, T) \cong (A_{1,s} \times A_{2,s} \times \cdots \times A_{k,s}, \text{Borel, Haar, } \sigma_1 \ltimes \sigma_2 \ltimes \cdots \ltimes \sigma_k)$$

is measure-theoretically isomorphic to an inverse limit of actions of $\Gamma$ by commuting rotations on $k$-step nilmanifolds.

**Remark** In fact it seems likely that the above class of systems can be set up in several different ways, which will presumably turn out to be equivalent. I have chosen the above definition here because I suspect it will ultimately prove relatively convenient for establishing the necessary properties of these systems, but an alternative has already appeared in the literature in the paper [CFH] of Chu, Frantzikinakis and Host.

The following lemma is now routine, given the ergodic case which is classical (it follows from the nilmanifold case of Ratner’s Theorem: see, for instance, [Lei07, Lei10]).

**Definition 6.0.7.** If $\Lambda \leq \Gamma$ is an inclusion of discrete Abelian groups, then the class $Z^\Lambda_{\text{nil},k}$ of those $\Gamma$-systems whose $\Lambda$-subactions are direct integral of $k$-step pro-nil systems is an idempotent class of $\Gamma$-systems. We refer to it as the class of $\Lambda$-partially $k$-step pro-nil systems.

We are now ready to offer our conjectural strengthening of Theorem 4.1.2 to the case of linearly dependent subgroups $\Gamma_i$.

**Conjecture 6.0.8** (General Structural Conjecture). Suppose that $\Gamma_i \leq \mathbb{Z}^d$ for $i = 1, 2, \ldots, r$ are subgroups among which there are no pairwise inclusions and $n_1, n_2, \ldots, n_r \geq 0$ are integers. Then depending on these data there are finite families of pairs

$$(\Lambda_{i,1}, m_{i,1}), (\Lambda_{i,2}, m_{i,2}), \ldots, (\Lambda_{i,k_i}, m_{i,k_i}) \quad \text{for } i = 1, 2, \ldots, r$$

such that each $m_{i,j} \geq 0$ is an integer and $\Lambda_{i,j} \leq \mathbb{Z}^d$ is a subgroup properly containing $\Gamma_i$ for each $i, j$, and for which the following holds.
If $X_i \in Z_{\text{nil},n_i}^\Gamma$ for each $i = 1, 2, \ldots, r$ and each $X_i$ is sated with respect to all possible joins of classes of the form $Z_{\text{nil},n}^\Gamma$ for $\Gamma \leq \mathbb{Z}^D$ and $n \geq 0$, then for any joining $\pi_i : Y \rightarrow X_i$, $i = 1, 2, \ldots, r$, the factors $\pi_i^{-1}(\Sigma_i)$ are relatively independent over their further factors

$$\pi_i^{-1}\left(\bigvee_{j=1}^{k_i} \Phi_{i,j}\right)$$

where $\Phi_{i,j}$ is the factor of $X_i$ generated by the factor map to $(Z_0^\Gamma \cap Z_{\text{nil},m_{i,j}}^\Lambda_{i,j})X_i$.

**Remark** We invoke the ‘no-inclusions’ condition on the subgroups $\Gamma_i$ in order to avoid degenerate cases. Without it, we might for example be asking for the collection of all possible joinings between two systems $X_i \in Z_{\text{nil},n_i}^\Gamma$ for $i = 1, 2$ with $\Gamma_1 \geq \Gamma_2$, and in this case Lemma 4.1.1 tells us something about the less constrained system $X_2$, but on the side of the more constrained system $X_1$ the joining may clearly be completely arbitrary.

In particular, the case in which $X_i$ has trivial $\Gamma_i$-subaction corresponds to $n_i = 0$, and in this case the above conjecture asserts that given enough satedness, the factors $\pi_i^{-1}(\Sigma_i)$ of the joining system are relatively independent over some further factors, each of which is assembled as a join of systems from the classes $Z_0^\Gamma \cap Z_{\text{nil},m_{i,j}}^{\Lambda_{i,j}}$. In particular, while each of these ingredients may not be partially invariant under any subgroup of $\mathbb{Z}^D$ strictly larger than $\Gamma_i$, for each them we do know something quite concrete (in terms of pro-nilsystems) about the subaction of some properly larger subgroup $\Lambda_{i,j} \geq \Gamma_i$.

Of course, the above conjecture does not strictly cover Theorem 4.1.2 since that gives much more precise information on the pairs $(\Lambda_{i,j}, m_{i,j})$ in case the $\Gamma_i$ are linearly independent: to wit, the $\Lambda_{i,j}$ are the sums $\Gamma_i + \Gamma_\ell$ for $\ell \neq i$, and $m_{i,j} = 0$. While a final understanding of Conjecture 6.0.8 would presumably also give a recipe for producing these pairs in the general case (and so would recover the exact details of our known special cases), the slightly incomplete formulation of Conjecture 6.0.8 seems ample for our present discussion, and as I write this any sensible guess as to its completion appears beyond reach.

Indeed, by itself Conjecture 6.0.8 seems very optimistic, so it is worth mentioning some special cases of it beyond Theorem 4.1.2 for which we have some supplementary evidence.
Firstly, if \( D = 2 \), each \( n_i = 0 \) and the \( \Gamma_i \) are pairwise linearly-independent one-dimensional subgroups \( \mathbb{Z} \mathbf{v}_i \leq \mathbb{Z}^2 \), then we can take a sensible guess at a more precise version of the above conjecture: that any joining of systems \( X_i \in \mathbb{Z}^k_{\Gamma_i} \) should be relatively independent over the maximal \((r - 1)\)-step pronilsystem factors \( X_i \rightarrow \mathbb{Z}_{\text{nil}, r} X_i \). Indeed, this would simply correspond to the Host-Kra Theorem in the case of the \( \mathbb{Z}^2 \)-system 

\[
\mathbf{X} := (X^k, \Sigma_{\otimes k}, \mu^F, \mathcal{T})
\]

with \( \mathcal{T}^{e_1} := T \times T \times \cdots \times T \) and \( \mathcal{T}^{e_2} := T \times T^2 \times \cdots \times T^k \), where now the subgroups are \( \Gamma_i = \mathbb{Z}(e_2 - ie_1) \) and the coordinate projections \( \pi_i : X^k \rightarrow X \) define factor maps to suitable \( \Gamma_i \)-partially-invariant \( \mathbb{Z}^2 \)-systems \( X_i \), constructed from \( X \) as in the proof of Proposition \[4.2.4\]. In fact, I strongly suspect that the methods of either [HK05] or [Zie07] could be adapted directly to proving this more general result on the possible joinings of such partially-invariant systems. Other, similar results on possible joinings of partially-invariant systems that do not require any extensions but would correspond to further special cases of Conjecture \[6.0.8\] have appeared in Frantzikinakis and Kra [FK05] (where nonconventional averages such as in our Theorem C are studied, but subject to some additional hypotheses on the individual ergodicity of several one-dimensional subactions), in Chu [Chu09] and in Chu, Frantzikinakis and Host [CFH]. In each of these cases, the joining in question has been either the Furstenberg self-joining of some tuple of commuting transformations, or the related Host-Kra self-joining (originally defined in [HK05] for the case of powers of a single transformation, and since adapted to the multi-directional case in [Hos09, Chu09, CFH]). However, in each of these cases it seems likely that the methods employed could be adapted to proving a corresponding instance of Conjecture \[6.0.8\].

Another special case of Conjecture \[6.0.8\] is the first beyond Theorem \[4.1.2\] that \textit{does} require an ascent to sated extensions, appears in [Ausd, Ause]. Indeed, the principal structural result of [Ause] can be phrased as asserting that if \( p_1, p_2 \) and \( p_3 \in \mathbb{Z}^2 \) are three directions which together with the origin \( 0 \in \mathbb{Z}^2 \) lie in general position, then for a sufficiently sated system \( X \) the Furstenberg self-joining \( \mu^F \) of the quadruple of transformations \( \text{id}, T^{p_1}, T^{p_2}, T^{p_3} \) is such that the coordinate projections \( \pi_i : X^{\{0,1,2,3\}} \rightarrow X \) are relatively independent over their further factors

\[
\pi_0^{-1}\left( \Sigma^{T^{p_1} = T^{p_2}} \vee \Sigma^{T^{p_1} = T^{p_3}} \vee \Sigma^{T^{p_2} = T^{p_3}} \vee \Sigma^\text{nil,2} \right)
\]
and

$$\pi_i^{-1}(\Sigma^{T_{p_i}} \vee \Sigma^{T_{p_i}=T_{p_j}} \vee \Sigma^{T_{p_i}=T_{p_k}} \vee \Sigma^{T_{m_{i,l}}}) \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.$$  

Arguing again as for Proposition 4.2.4, this would follow from a special case of Conjecture 6.0.8 (again with some more precise information on the pairs $(\Lambda_{i,j}, m_{i,j})$) when $D = 3$, $r = 4$ and $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ are four one-dimensional subgroups of $\mathbb{Z}^3$ any three of which are linearly independent.

At present no proof (or disproof) of Conjecture 6.0.8 seems to be at hand. Nevertheless, the various cases mentioned above do give me hope for it, and I strongly suspect that any result as powerful as this would constitute a major addition to our toolkit for approaching questions of multiple recurrence. For example, I would expect it to shed considerable new light on the Bergelson-Leibman conjecture on the convergence of ‘polynomial’ nonconventional averages [BL02]. For a recent discussion of these latter question see [Ausd, Ause], where the proof of an instance of this latter conjecture was the original motivation for the result on joint distributions mentioned above.

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