Pseudo-Goldstone gaps and order-by-quantum-disorder in frustrated magnets

Jeffrey G. Rau, Paul A. McClarty, and Roderich Moessner
Max-Planck-Institut für Physik komplexer Systeme, 01187 Dresden, Germany

(Dated: May 4, 2018)

In systems with competing interactions, continuous degeneracies can appear which are accidental, in that they are not related to any symmetry of the Hamiltonian. Accordingly, the pseudo-Goldstone modes associated with these degeneracies are also unprotected. Indeed, through a process known as “order-by-quantum-disorder”, quantum zero-point fluctuations can lift the degeneracy and induce a gap for these modes. We show that this gap can be exactly computed at leading order in $1/S$ in spin-wave theory from the mean curvature of the classical and quantum zero-point energies – without the need to consider any spin-wave interactions. We confirm this equivalence through direct calculations of the spin-wave spectrum to $O(1/S^2)$ in a wide variety of theoretically and experimentally relevant quantum spin models. We prove this equivalence through the use of an exact sum rule that provides the required mixing of different orders of $1/S$. Finally, we discuss some implications for several leading order-by-quantum-disorder candidate materials, clarifying the expected pseudo-Goldstone gap sizes in Er$_2$Ti$_2$O$_7$ and Ca$_3$Fe$_2$Ge$_6$O$_{12}$.

Goldstone’s theorem [1] connects the spontaneous breaking of a continuous symmetry to the presence of gapless excitations – a foundational result with applications in almost every branch of physics. Alternatively, gapless excitations can be generated by accidental degeneracies which are not symmetry enforced [2]. Just as continuous symmetries implicate the presence of gapless Goldstone modes, accidental degeneracies imply the presence of pseudo-Goldstone modes that are nearly gapless when (inevitably) these degeneracies are weakly lifted. These modes have been invoked to explain the appearance of unexpectedly low-lying excitations many contexts, ranging from from quantum chromodynamics [3] to high-temperature superconductors [4, 5] and quantum magnets [6–8]; perhaps the most well known example is the mass of the pion, which arises due to explicitly broken chiral symmetry [9, 10].

In lieu of explicit symmetry breaking, accidental degeneracies can also be lifted by fluctuations. Broadly referred to as “order-by-disorder” [6–8], this phenomenon has proven useful in understanding a wide variety of ordering phenomena in frustrated spin systems [7, 11–14], where accidental degeneracies are natural. In these systems, one of the most common, and least avoidable, types of fluctuation induced ordering is “order-by-quantum-disorder” [7, 8], where an accidentally degenerate manifold in the classical limit, $S \to \infty$, is lifted by quantum corrections at $O(1/S)$. Within non-interacting spin-wave theory [15], these contributions can be viewed as the zero-point energy of the harmonic spin-waves selecting some subset of the classically degenerate manifold [7, 8].

Due to this order-by-quantum-disorder, the pseudo-Goldstone modes associated with this accidental degeneracy must acquire a gap. Since in non-interacting spin-wave theory these modes are gapless, to obtain a finite gap one must include the effects of spin-wave interactions. While conceptually simple, such calculations have only been carried out for a few limited cases, mostly for simple isotropic Heisenberg-like models [8, 14, 16–19]. In more complex models with strong exchange anisotropy, such as in the rare-earth pyrochlores [20] or in Kitaev magnets [21, 22], these calculations can be complicated by the presence of three-magnon interactions that can lead to spontaneous magnon decay [23], even for collinear magnetic ground states.

In this paper, we offer a significant simplification, showing that the curvatures of classical and quantum zero-point energy densities computed at $O(1/S)$, are already sufficient to determine the pseudo-Goldstone gap exactly at $O(1/S^2)$. Explicitly, if the classically degenerate manifold is parameterized by $\phi$ with conjugate direction $\theta$ (see Fig. 1), the pseudo-Goldstone gap, $\Delta$, is given by

$$\Delta = \frac{1}{S} \sqrt{\frac{\partial^2 \epsilon}{\partial \phi^2}_0 \left( \frac{\partial^2 \epsilon}{\partial \phi^2}_0 - \frac{\partial^2 \epsilon}{\partial \theta \partial \phi}_0 \right)^2},$$

where the semi-classical energy density, $\epsilon(\phi, \theta)$, of the classical ground state at $(\phi, \theta)$ includes the classical $[O(S^2)]$ and quantum zero-point $[O(S)]$ contributions. This equivalence is far from evident in perturbation theory, as it involves mixing of different orders in $1/S$. This formula eliminates much of the burden of computing the pseudo-Goldstone gap, requiring only quantities from standard non-interacting spin-wave theory – without the consideration of spin-wave interactions – a computation considerably more straightforward to undertake in practice.

In light of this, we revisit a variety of models that exhibit order-by-quantum-disorder, including square and cubic
We then outline the key aspects of the proof of this equivalence, relegating the fine, technical details to the Supplemental Material (SM) [39]. Finally, we discuss some further applications of these results to the leading order-by-quantum-disorder candidates Er₇Ti₂O₁₂ and Ca₃Fe₂Ge₃O₁₁.

Spin-wave theory: We first review the physics of pseudo-Goldstone modes as they appear in linear spin-wave theory. As for the usual Goldstone modes, these can be classified into two types [37, 38], which we denote as I and II, which correspond to having non-conserved and conserved order parameters, respectively. For a type I pseudo-Goldstone mode the linear spin-wave dispersion vanishes linearly \( \sim |k| \), while for the type II case it vanishes quadratically \( \sim |k|^2 \), as illustrated in Fig. 2.

More explicitly, we can define the linear spin-wave Hamiltonian [15]

\[
S \sum_{k} \sum_{\alpha \beta} \left[ A_{k}^{\alpha \beta} a_{k \alpha}^\dagger a_{k \beta} + \frac{1}{2} \left( B_{k}^{\alpha \beta} a_{k \alpha}^\dagger a_{k \beta}^\dagger + h.c. \right) \right],
\]

where \( a_{k \alpha} \) is the (Fourier-transformed) Holstein-Primakoff boson with wave-vector \( k \) on sublattice \( \sigma \) of the (magnetic) unit cell. The matrices \( A_k \) and \( B_k \) depend on the classical ordering pattern and the exchange model; see SM [39] for details. The linear spin-wave spectrum is determined by the eigenvalues of the Bogoliubov dispersion matrix [40]

\[
\sigma^3 M_k \equiv \begin{pmatrix} A_k & B_k \\ B_k & -A_k^\dagger \end{pmatrix},
\]

where \( \sigma^3 = \text{diag}(+1, -1) \) is a block Pauli matrix. A pseudo-Goldstone mode appears as a zero in the linear spin-wave spectrum. Without loss of generality, we assume that this zero mode lies at the zone center, with \( M_0 \) being positive semi-definite, and \( M_k \) positive definite elsewhere (this is always possible for commensurate magnetic orders).

When spin-wave interactions are included, the excitation energies are indicated by the poles of the (retarded) magnon Green’s function

\[
G^R(k, \omega) = \left[ (\omega + i0^+) \sigma^3 - S M_k - \Sigma^R(k, \omega) \right]^{-1},
\]

where \( \Sigma^R(k, \omega) \) is the (retarded) self-energy. We use a formalism where the free magnon Green’s function is defined as a matrix that includes both the sublattice indices and the normal and anomalous contributions [40]. The effects of spin-wave interactions encoded in the self-energy can be computed perturbatively [41] in the limit \( 1/S \to 0 \) [42, 43]. Given the permutation symmetries of the three- and four-magnon vertices it is further helpful to use Hugenholtz diagrams [41] that account for these redundancies automatically.

At leading order, there are three classes of diagram for \( \Sigma^R(k, \omega) \), as illustrated in Fig. 3 [44]. One involves a single four-magnon interaction, with the remaining two diagrams involving two three-magnon interactions [23]. The first two

![FIG. 2. Schematic form of the spectrum of (a) type I and (b) type II pseudo-Goldstone modes. The type I modes have a linear spectrum \( \propto |k| \) at \( O(S^0) \), before the inclusion of spin-wave interactions, while the type II modes initially have a quadratic spectrum \( \propto |k|^2 \). For the type I case the interaction induced gap \( \Delta \) scales as \( S^{1/2} \), while for the type II case it scales as \( S^0 \). Heisenberg-compass models [16], Heisenberg-Kitaev-Γ models [24, 25] on the honeycomb lattice and \( J_1-J_2 \) models on the square and triangular lattices [7, 13, 14, 26]. For each model, we compute the gap both explicitly in interacting spin-wave theory – often for the first time – and then again using the curvature formula [Eq. (1)], confirming that they are indeed identical.

Finally, we consider applications; while order-by-quantum-disorder has a long theoretical history [6–8, 16, 27], there are only a handful of serious potential experimental candidates [8, 17, 28–31]. Two of the best material examples are the cubic Heisenberg anti-ferromagnet \( \text{Ca}_3\text{Fe}_2\text{Ge}_3\text{O}_{12} \) [8, 28], and the pyrochlore XY anti-ferromagnet \( \text{Er}_7\text{Ti}_2\text{O}_{12} \) [30–32], where the leading energetic effects [33] are naively [34–36] expected to be small [30]. In this context, the pseudo-Goldstone gap provides a quantitative benchmark which may be used to distinguish order-by-quantum disorder from more conventional energetic selection. Given knowledge of the models for these materials [28, 30], our result provides a straightforward way to estimate the pseudo-Goldstone gap observed experimentally, cleanly demonstrating the utility of these results.

The paper is organized as follows, first we present a general formalism for the gapping of pseudo-Goldstone modes in interacting spin-wave theory. Two cases are identified, type I and type II pseudo-Goldstone modes analogous to those associated with exact continuous symmetries [37, 38]. Next, we motivate the curvature formula [Eq. (1)] for the pseudo-Goldstone gap and, by direct calculations of the spin-wave spectrum at \( O(1/S^2) \), show that it holds exactly in a number of theoretically and experimentally relevant models, as summarized in Table I. We then outline the key aspects of the proof of this equivalence, relegating the fine, technical details to the Supplemental Material (SM) [39]. Finally, we discuss some further applications of these results to the leading order-by-quantum-disorder candidates \( \text{Er}_7\text{Ti}_2\text{O}_{12} \) and \( \text{Ca}_3\text{Fe}_2\text{Ge}_3\text{O}_{12} \).
diagrams, the “droplet” and “tadpole”, are static and thus effectively renormalize $M_k$. The final “bubble” diagram is dynamic and has non-trivial frequency dependence; when present this can lead to spontaneous decay of the one-magnon excitations into the two-magnon continuum [23].

The determination of the poles of the magnon Green’s function then proceeds perturbatively in the self-energy, with respect to $M_k$. For the type I case, one finds the relevant low-energy subspace of $\sigma_i M_0$ is similar to a defective Jordan block [40]. The pseudo-Goldstone gap, $\Delta$, is then given at leading order by

$$
\Delta = S^{1/2} \sqrt{2\Re \left[ V_0^2 \Sigma^2(0,0) \sigma_i M_0 V_0 \right] + O(S^{-1/2})},
$$

where $V_0$ characterizes part of the zero mode subspace [39]. For the type II case, there are two linearly independent eigenvectors of $\sigma_i M_0$ with eigenvalue zero, $V_0$ and $W_0$. The pseudo-Goldstone gap is then given by

$$
\Delta = \sqrt{\left(V_0^2 \Sigma^2(0,0)V_0\right)^2 - \left|V_0^4 \Sigma^2(0,0)W_0\right|^2 + O(S^{-1})},
$$

at leading order. A detailed derivation for both cases is provided in the SM [39].

The two types of pseudo-Goldstone modes follow distinct scalings with the spin length, for a type I pseudo-Goldstone mode the gap scales as $\Delta \sim O(S^{1/2})$, while for a type II pseudo-Goldstone mode it scales as $\Delta \sim O(S^0)$. Away from the zone center, the spectrum takes the low-energy form $\sim \sqrt{\epsilon^2 |k|^2 + \Delta^2}$ for type I modes, while for type II modes it takes the form $\sim \epsilon^2 |k|^2 + \Delta$, as shown schematically in Fig. 2.

**Curvature formula and semi-classical dynamics:** We now motivate the curvature formula for the pseudo-Goldstone gap, $\Delta$ [Eqs. (1)], through a heuristic semi-classical argument. We first need to understand the classical dynamics of the soft mode. In this regard, it is useful to construct a local (right-handed) frame $(\hat{x}_a, \hat{y}_a, \hat{z}_a)$ where $\hat{z}_a$ is the ordering direction, $\hat{x}_a$ is the soft-mode direction and $\hat{y}_a = \hat{z}_a \times \hat{x}_a$ [49]. If we parametrize the soft mode by an angle $\theta$, and the (locally) orthogonal directions by an angle $\phi$, we can define the transverse part of the classical spin as

$$
S^\perp_a = S (\phi \hat{x}_a + \theta \hat{y}_a),
$$

where the full spin configuration is given by the normalized vector

$$
S_a \approx S \left(1 - \frac{|S_a|^2}{S^2}\right)^{1/2} \hat{z}_a + S^\perp_a,
$$

to quadratic order in $\theta$ and $\phi$. For simplicity, we have assumed here that the soft mode is uniform, with the relative weight of the rotations not varying between sublattices – this assumption is not essential, and can be lifted [39], but holds for all the examples we consider below.

In terms of the spins, these low-energy degrees of freedom can be written as

$$
\phi = \sum \hat{x}_a \cdot S_{ia}, \quad \theta = \sum \hat{y}_a \cdot S_{ia},
$$

we see that these variables have the Poisson bracket $\{\phi, \theta\} = NS$ and thus essentially behave like a position and its canonically conjugate momentum. For the type I case, $\phi$ is classically soft, with no restoring force, while is $\theta$ is not, while for the type II case both are classically soft.

If we treat these collective coordinates as classical dynamical variables, then quantum fluctuations can be included in an ad-hoc way by using the semi-classical spin-wave energy density $\epsilon(\theta, \phi)$ as an effective potential; explicitly, we define

$$
\epsilon(\phi, \theta) \equiv \frac{1}{2N} \sum_{k\alpha} \epsilon_{k\alpha}(\phi, \theta),
$$

where $\epsilon_{k\alpha}(\phi, \theta)$ are the spin-wave energies found expanding about a classical ground state with finite $\phi, \theta$. For the type II case, this zero-point energy is well-defined for arbitrary $\phi$ and $\theta$, with the classical energy independent of both variables, while for the type I case, the zero-point energy is ill-defined for $\theta \neq 0$, and thus we fix $\theta = 0$. The curvatures of the total semi-classical energy are then defined via

$$
E(\phi, \theta) = \frac{N}{2} \left[ \left(\frac{\partial^2 \epsilon}{\partial \phi^2}\right)_0 \phi^2 + \left(\frac{\partial^2 \epsilon}{\partial \theta^2}\right)_0 \theta^2 + 2 \left(\frac{\partial^2 \epsilon}{\partial \theta \partial \phi}\right)_0 \phi \theta \right] + \cdots.
$$

These curvatures directly determine the normal mode frequency of $\theta$ and $\phi$ via the classical equations of motion

$$
\frac{d\phi}{dt} = + \frac{1}{S N} \epsilon_{\theta\phi} \phi + \frac{\partial^2 \epsilon}{\partial \phi \partial \phi} \phi \theta, \quad \frac{d\theta}{dt} = - \frac{1}{S N} \epsilon_{\theta \phi} \phi + \frac{\partial^2 \epsilon}{\partial \phi \partial \phi} \phi \theta,
$$

giving the pseudo-Goldstone gap shown in Eq. (1). This was derived first in the purely classical context by Smit and Belgers [50]. For all the cases of interest the cross term vanishes, so we omit $\left(\frac{\partial^2 \epsilon}{\partial \phi \partial \phi}\right)_0$ in what follows. Multiple sets of pseudo-Goldstone modes can be handled in a similar fashion, reducing to multiple independent copies of either the type I or type II structures described above (absent fine-tuning).

We have computed the pseudo-Goldstone gap for a wide variety of models [39] using both non-linear spin-wave theory as described in Eqs. (5, 6), and using the curvature formula, as given in Eq. (1), which involves only linear spin-wave theory. The results are presented in Table I, where one can see that the two methods agree exactly for all models considered. This includes two- and three-dimensional models, isotropic and anisotropic models, models with and without magnon decay as well as realistic models for two experimental order-by-quantum-disorder candidates. Some details for each model,
as well as examples of how to define $\phi, \theta$ and compute the curvatures are provided in the SM [39].

Proof of formula: The equivalence between these two approaches can be understood as follows. The essential ingredient is to notice that the Holstein-Primakoff expansion should not depend on the choice of initial classical ground state about which one expands, so long as it is sufficiently close to the true ground state of the model. In other words, the expansion must “self-correct”, with the expectation values of the magnons giving the appropriate true ground state spin directions, order-by-order in $1/S$. We show this explicitly though a direct solution at $O(S)$ for a model without any accidental degeneracies in the SM [39]; the required cancellation then implies that the magnon energy at zero wave-vector can be related to the curvature of the classical energy density, as in the formula of Smit and Belgers [50].

To understand the implication of this self-correction at higher order, we must proceed more indirectly, since an exact solution of the $O(S^3)$ Hamiltonian is not possible. This statement can be made more formal if we define the rotated Hamiltonian

$$\mathcal{H}(\phi, \theta) = U(\phi, \theta)\mathcal{H}U(\phi, \theta),$$

where the rotation $U(\phi, \theta)$ produces the soft configurations of Eq. (7) from the state defined by $\xi_0$. The self-correction condition is then the (trivial) fact that the ground state energy of $\mathcal{H}(\phi, \theta)$ is independent of $\phi$ and $\theta$. If one considers the implications of this statement on the derivatives of the ground state energy of $\mathcal{H}(\phi, \theta)$, then at second order one finds that this implies the sum rule [39, 51]

$$U_\mu \sigma_3 \left[ \int d\omega \omega A(0, \omega) \right] \sigma_3 U_\nu = \frac{1}{SN} \left( \frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_{\xi_0},$$

where $\mu, \nu = \Theta, \Phi, \lambda_\phi = \phi$ and $\lambda_\theta = \theta$ and we define $U_\theta = (V_0 - W_0)/(i \sqrt{2})$, $U_\phi = (V_0 + W_0)/\sqrt{2}$ which span the zero-mode subspace [39]. The magnon spectral function $A(k, \omega)$ is

| Model                          | Parameters | Type | $\Delta$ | $(\frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu})_{\xi_0}$ | $S^{-1} \sqrt{(\frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu})_{\xi_0}}$ | Exp.   |
|--------------------------------|------------|------|----------|--------------------------------------------------------------------------------|----------------------------------------------------------------------------------|--------|
| Heisenberg-compass | $|K|/|J| \ll 1$ | I    | $0.52S^{\frac{1}{2}}|K|^2/|J|^2$ | $2|K|^2S^2$ | $0.134K^2S/|J|$ | $0.52S^{\frac{1}{2}}|K|^2/|J|^2$ |
| Heisenberg-compass | $|K|/|J| = -0.5$ | I    | $0.17JS^{\frac{1}{2}}$ | $|J|^2S$ | $0.028SJS$ | $0.17JS^{\frac{1}{2}}$ |
| Heisenberg-compass | $|K|/|J| < 1$ | I    | $0.093K^2S^2/|J|$ | $0.093K^2S^2/|J|$ | $0.093K^2S^2/|J|$ | $0.093K^2S^2/|J|$ |
| Heisenberg-compass | $|K|/|J| = 0.5$ | I    | $0.030JS$ | $0.030JS$ | $0.030JS$ | $0.030JS$ |
| Heisenberg-compass | $|K|/|J| = -0.5$ | I    | $0.024JS$ | $0.024JS$ | $0.024JS$ | $0.024JS$ |
| Heisenberg-Kitaev | $|K| < |J|$ | I+I | $1.16JS^{\frac{1}{2}}$ | $10JS^2$ | $2.74JS$ | $1.66JS^{\frac{1}{2}}$ |
| Heisenberg-Kitaev | $|K| < |J|$ | I+I | $0.43JS^{\frac{1}{2}}$ | $5JS^2$ | $0.038JS$ | $0.43JS^{\frac{1}{2}}$ |
| Heisenberg-$\Gamma$ | $|J|/|J| < +0.5$ | I    | $0.081JS^{\frac{1}{2}}$ | $1.5JS^2$ | $0.0047JS$ | $0.081JS^{\frac{1}{2}}$ |
| Heisenberg-$\Gamma$ | $|J|/|J| = +1.0$ | I    | $0.355JS^{\frac{1}{2}}$ | $3JS^2$ | $0.042JS$ | $0.355JS^{\frac{1}{2}}$ |
| Heisenberg $|J_1/J_2| < 1$ | I+I | $1.44JS^{\frac{1}{2}}$ | $4(2J_1 - J_2)S^{\frac{1}{2}}$ | $0.260JS_2$ | $1.44JS^{\frac{1}{2}}$ |
| Heisenberg $|J_1/J_2| = 0.5$ | I+I | $1.63JS^{\frac{1}{2}}$ | $6JS^2$ | $0.066JS$ | $0.63JS^{\frac{1}{2}}$ |
| Heisenberg $|J_1/J_2| = 0.5$ | I+I | $1.63JS^{\frac{1}{2}}$ | $6JS^2$ | $0.066JS$ | $0.63JS^{\frac{1}{2}}$ |
| Heisenberg $|J_1/J_2| = 0.5$ | I+I | $0.53JS$ | $0.53JS$ | $0.53JS$ | $0.53JS$ |
| Heisenberg $|J_1/J_2| = 0.5$ | I+I | $0.45JS$ | $0.45JS$ | $0.45JS$ | $0.45JS$ |
| Heisenberg $|J_1/J_2| = 0.5$ | I+I | $0.58JS$ | $0.58JS$ | $0.58JS$ | $0.58JS$ |
| Material                  | Parameters | Type | $\Delta$ | $(\frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu})_{\xi_0}$ | $S^{-1} \sqrt{(\frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu})_{\xi_0}}$ | Exp.   |
| Er$_2$Ti$_2$O$_7$ [30–32] | Savary et al. [30] | I    | $31.1 \mu eV$ | $157.5 \mu eV$ | $1.536 \mu eV$ | $31.1 \mu eV$ | $43 \mu eV$ [35, 47], $53 \mu eV$ [48] |
| Ca$_3$Fe$_2$Ge$_5$O$_{12}$ [8, 28] | Brueckel et al. [28] | I+I | $131 \mu eV$ | $2 \mu eV$ | $53.75 \mu eV$ | $131 \mu eV$ | $136 \mu eV$ [28] |

TABLE I. Calculations showing the equality of the pseudo-Goldstone gap, $\Delta$ computed from non-linear spin-wave theory [Eqs. (5,6)] and then independently from the curvatures of the classical and quantum zero-point energies [Eq. (1)]. For each model, the lattice, the exchange regime, the type of pseudo-Goldstone mode and several choice of parameters are listed. When available, additional theoretical or experimental estimates of the pseudo-Goldstone gap are listed.
defined as
\[
A(k, \omega) \equiv \frac{1}{2i} \left[ G^R(k, \omega) - G^A(k, \omega) \right],
\]
where the \( G^R(k, \omega) \) is the advanced magnon Green’s function.

Using this sum rule [Eq. (14)] one can show that, at \( O(S^0) \), the left-hand side is directly related to the pseudo-Goldstone gap, while the right-hand side is straightforwardly related to the curvatures of the classical energy density and quantum zero-point energy density at \( O(S^2) \) and \( O(S) \), respectively. Details of this proof are provided in SM [39]. This argument does not directly extend to higher orders in \( 1/S \) or to computing the energies of finite energy modes to \( O(S^0) \). We note that in broad strokes this argument bears some resemblance to the Witten-Veneziano formula [52, 53] for the mass of the \( \eta' \)-meson. In addition, the sum rule [Eq. (14)] is related to Dashen’s formula [54] for the mass of pseudo-Goldstone bosons, such as the pion, when chiral symmetry is broken (see the SM [39] for further details).

Discussion: We now discuss some applications to two leading experimental candidates for order-by-quantum-disorder. The first of these is the compound \( \text{Ca}_3\text{Fe}_2\text{Ge}_2\text{O}_{12} \), which is a three-dimensional \( S = 5/2 \) version of one of the canonical order-by-disorder models, the \( J_1-J_2 \) model [7, 13]. This system has a pair of type I pseudo-Goldstone modes, as well as two true Goldstone modes. Due to the low-symmetry of the lattice, there are several independent (isotropic) couplings, which have been estimated by comparison of the predictions of linear spin-wave theory with the inelastic neutron scattering spectrum at zero field [28]. One finds that the gap predicted by non-linear spin-wave theory, \( \sim 131 \mu\text{eV} \), is in good agreement with the \( 136 \mu\text{eV} \) observed experimentally [55]. This demonstrates sharply the utility of the method presented here, with the straightforward curvature calculation lending credence to our more involved non-linear spin-wave result. We also note that the size of the predicted gap supports the picture that \( \text{Ca}_3\text{Fe}_2\text{Ge}_2\text{O}_{12} \) is truly an example of order-by-quantum-disorder, and perhaps energetic corrections, such as biquadratic interactions [30], are small.

Finally we turn to \( \text{Er}_2\text{Ti}_2\text{O}_7 \), one of the more ideal material platforms for finding order-by-quantum-disorder [30, 31]. This is a three-dimensional \( S = 1/2 \) XY antiferromagnet with a single type I pseudo-Goldstone mode. Using the exchange parameters of Ref. [30], we find that the gap, computed directly in non-linear spin-wave theory as well as via the curvature formula is \( \sim 31 \mu\text{eV} \) [56]. This theoretical value is still somewhat lower than the 43 \( \mu\text{eV} \) [35, 47] and 53 \( \mu\text{eV} \) [48] that have been reported experimentally in \( \text{Er}_2\text{Ti}_2\text{O}_7 \). This disagreement could be the result of several factors, such as spin-wave theory being non-quantitative [57] at \( S = 1/2 \), uncertainties in the exchange parameters [30] or the presence of energetic corrections [34, 36] in addition to the order-by-quantum-disorder contribution.

There are many other experimental systems where order-by-quantum-disorder may be lurking, and where the results presented here would be useful. In the same vein as \( \text{Er}_2\text{Ti}_2\text{O}_7 \), order-by-quantum-disorder may play a role in the pyrochlores \( \text{Yb}_2\text{Ti}_2\text{O}_7 \) and its cousin \( \text{Yb}_2\text{Ge}_2\text{O}_7 \) [58, 59], and perhaps even in the ytterbium based spinels [60–63]. Order-by-disorder has also played a key role in the understanding of models [24, 64, 65] of Kitaev materials [22] such as \( \text{Na}_3\text{IrO}_3 \), \( \alpha-\text{RuCl}_3 \) and \( (\alpha,\beta,\gamma)-\text{Li}_2\text{IrO}_3 \), and also in related strongly spin-orbit coupled compounds [19, 66, 67].

On the more theoretical front, one could ask whether the methods discussed could also resolve larger, more extensive degeneracies. For example, one could consider systems with sub-extensive line or surface degeneracies [18, 68–70]. More drastically, one could consider a case like the the kagomé anti-ferromagnet, where the classical ground state is macroscopically degenerate [71–73]. If one expands about states that are expected to be selected by \( 1/S \) corrections, one finds that the linear spectrum has a large number of zero modes [74]. It would be interesting to compare the semi-classical approach outlined here to the approaches followed in Ref. [75].

Acknowledgments: We thank M. Zhitomirsky for a helpful discussion and M. Gingras and R. Coldea for collaborations on related topics. This work was in part supported by Deutsche Forschungsgemeinschaft (DFG) under grant SFB 1143.

[1] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).
[2] S. Weinberg, Phys. Rev. Lett. 29, 1698 (1972).
[3] S. Weinberg, The quantum theory of fields, Vol. 2 (Cambridge university press, 1995).
[4] E. Demler, W. Hanke, and S.-C. Zhang, Rev. Mod. Phys. 76, 909 (2004).
[5] R. M. Fernandes and A. V. Chubukov, Reports on Progress in Physics 80, 014503 (2016).
[6] J. Villain, R. Bidaux, J.-P. Carton, and R. Conte, Journal de Physique 41, 1263 (1980).
[7] C. L. Henley, Phys. Rev. Lett. 62, 2056 (1989).
[8] E. Shender, Sov. Phys. JETP 56, 178 (1982).
[9] Y. Nambu, Phys. Rev. Lett. 4, 380 (1960).
[10] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
[11] P. Chandra, P. Coleman, and A. I. Larkin, Phys. Rev. Lett. 64, 88 (1990).
[12] R. Moessner and J. T. Chalker, Phys. Rev. B 58, 12049 (1998).
[13] P. Chandra and B. Doucot, Phys. Rev. B 38, 9335 (1988).
[14] A. V. Chubukov and T. Joicouer, Phys. Rev. B 46, 11137 (1992).
[15] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).
[16] E. Belorizky, R. Casalegno, and J. Niez, Physica Status Solidi (b) 102, 365 (1980).
[17] A. B. Harris, A. Aharony, O. Entin-Wohlman, I. Y. Korenblit, R. J. Birgeneau, and Y.-J. Kim, Phys. Rev. B 64, 024436 (2001).
[18] T. Yildirim, A. B. Harris, and E. F. Shender, Phys. Rev. B 58, 3144 (1998).
[19] A. A. Aczel, A. M. Cook, T. J. Williams, S. Calder, A. D. Christianson, G.-X. Cao, D. Mandrus, Y.-B. Kim, and A. Paramekanti, Phys. Rev. B 93, 214426 (2016).
M. E. Zhitomirsky, M. V. Gvozdikova, P. C. W. Holdsworth, and J. G. Rau, S. Petit, and M. J. P. Gingras, Phys. Rev. B 96, 077204 (2014).

T. Jolicoeur, E. Dagotto, E. Gagliano, and S. Bacci, Phys. Rev. B 42, 4800 (1990).

J. R. Tessman, Phys. Rev. 96, W13204 (2010).

J. G. Rau, E. K.-H. Lee, and H.-Y. Kee, Phys. Rev. Lett. 112, 077204 (2014).

J. Chaloupka, G. Jackeli, and G. Khaliullin, Phys. Rev. Lett. 109, 077204 (2012).

J. D. M. Champion, M. J. Harris, P. C. W. Holdsworth, A. S. M. Winter, A. A. Tsirlin, M. Daghofer, J. van den Brink, Y. J. Kim, A. Aharony, R. J. Birgeneau, F. C. Chou, O. Entin-Wohlman, R. W. Erwin, M. Greven, A. B. Harris, M. A. Kastner, I. Y. Korenblit, Y. S. Lee, and G. Shirane, Phys. Rev. B 83, 852 (1999).

L. Savary, K. A. Ross, B. D. Gaulin, J. P. C. Ruff, and L. Balents, Phys. Rev. Lett. 109, 167201 (2012).

M. E. Zhitomirsky, M. V. Gvozdikova, P. C. W. Holdsworth, and R. Moessner, Phys. Rev. Lett. 109, 077204 (2012).

J. D. M. Champion, M. J. Harris, P. C. W. Holdsworth, A. S. Wills, G. Balakrishnan, S. T. Bramwell, E. Cizmar, T. Fennell, J. S. Gardner, J. Lago, D. F. McMorrow, M. Orendäčová, A. Orendáčová, D. M. Paul, R. I. Smith, M. T. F. Telling, and A. Wildes, Phys. Rev. B 68, 020401 (2003).

These are thought to be biquadratic exchange [30] for Ca$_3$Fe$_2$Ge$_2$O$_{12}$ and multi-spin interactions [30, 34, 36] for Er$_3$Ti$_5$O$_{12}$.

P. A. McClarty, S. Curnoe, and M. J. P. Gingras, in Journal of Physics: Conference Series, Vol. 145 (IOP Publishing, 2009) p. 012032.

S. Petit, J. Robert, S. Guitteny, P. Bonville, C. Decorse, J. Olivlier, H. Mutka, M. J. P. Gingras, and I. Mirebeau, Phys. Rev. B 90, 064410 (2014).

J. G. Rau, S. Petit, and M. J. P. Gingras, Phys. Rev. B 93, 184408 (2016).

T. Brauner, Symmetry 2, 609 (2010).

H. Watanabe and H. Murayama, Phys. Rev. Lett. 108, 251602 (2012).

Supplemental material attached.

J.-P. Blaizot and G. Ripka, Quantum theory of finite systems, Vol. 3 (MIT press Cambridge, 1986).

J. W. Negele and H. Orland, Quantum many-particle systems (Westview, 1988).

F. J. Dyson, Phys. Rev. 102, 1217 (1956).

A. B. Harris, D. Kumar, B. I. Halperin, and P. C. Hohenberg, Phys. Rev. B 3, 961 (1971).

Practically, the sums involved in computing these diagrams are performed numerically on large (but finite) systems with periodic boundary conditions. In the presence of zero modes, we add a small magnon chemical potential $-\mu \sum \sigma_i \sigma_i'$ as a (physical) regulator. To recover the thermodynamic limit, one first takes the system size to be large ($L \to \infty$) before removing the regulator ($\mu \to 0$). An alternative regulator is to use of modified boundary conditions (e.g. anti-periodic) to remove the zero mode; these have the advantage of only having a single limit to evaluate ($L \to \infty$), but the disadvantage that they may break some symmetries of the original problem.

M. Gohlke, R. Verresen, R. Moessner, and F. Pollmann, Phys. Rev. Lett. 119, 157203 (2017).

R. R. P. Singh, W. Zheng, J. Oitmaa, O. P. Sukhov, and C. J. Hamer, Phys. Rev. Lett. 91, 017201 (2003).

E. LhoteJ, J. Robert, E. Ressochue, F. Damay, I. Mirebeau, J. Olivlier, H. Mutka, P. Dalmas de Réotier, A. Yauanc, C. Marin, C. Decorse, and S. Petit, Phys. Rev. B 95, 134426 (2017).

K. A. Ross, Y. Qiu, J. R. D. Copley, H. A. Dabkowska, and B. D. Gaulin, Phys. Rev. Lett. 112, 057201 (2014).

One may need to enlarge the unit cell to encode the soft-mode in this fashion.

J. Smit and H. G. Belgers, Philips Res. Rep. 10, 113 (1955).

V. Viswanath and G. Müller, The Recursion Method: Application to Many Body Dynamics, Vol. 23 (Springer Science & Business Media, 1994).

E. Witten, Nuclear Physics B 156, 269 (1979).

G. Veneziano, Nuclear Physics B 159, 213 (1979).

R. Dashen, Phys. Rev. 183, 1245 (1969).

Our result disagrees with previous estimates [8, 28] of this gap. We note that the methods used in Ref. [8] are non-standard, and the stated final numerical result was in error, and revised in subsequent work [28]. We find additional inconsistencies, with our result differing from that of Ref. [28]. If we restrict ourselves to $J_1 = J'_1$ (see SM [39]), as was done in Ref. [8], then our results differ from that stated in Ref. [28] by a factor of two. Including $J' > J_1$ further modifies our result from that of Ref. [28].

Our result differs from the estimate presented in Ref. [30] by a factor of $\sqrt{2}$. The simplicity of the curvature estimate presented here, relative to that carried out in Ref. [30], as well as the agreement with our explicit non-linear spin-wave calculation, leads us to trust our value of 31 $\mu$eV over the value 22 $\mu$eV reported in Ref. [30].

V. S. Maryasin and M. E. Zhitomirsky, Phys. Rev. B 90, 094412 (2014).

L. D. C. Jaubert, O. Benton, J. G. Rau, J. Oitmaa, R. R. P. Singh, N. Shannon, and M. J. P. Gingras, Phys. Rev. Lett. 115, 267208 (2015).

J. Robert, E. LhoteJ, G. Remenyi, S. Sahling, I. Mirebeau, C. Decorse, B. Canals, and S. Pettit, Phys. Rev. B 92, 064425 (2015).

G. C. Lau, R. S. Freitas, B. G. Ueland, P. Schiifer, and R. J. Cava, Phys. Rev. B 72, 054411 (2005).

T. Higo, K. Iritani, M. Halim, W. Higemoto, U. T. Ito, K. Kuga, K. Kimura, and S. Nakatsuji, Phys. Rev. B 95, 174443 (2017).

P. Dalmas de Réotier, C. Marin, A. Yaoaucn, C. Ritter, A. Maisuradze, B. Roesli, A. Bertin, P. J. Baker, and A. Amato, Phys. Rev. B 96, 134403 (2017).

J. G. Rau and M. J. Gingras, (2018), arXiv:1802.03024 [cond-mat.str-el].

S. Lee, E.-K.-H. Lee, A. Paramekanti, and Y. B. Kim, Phys. Rev. B 89, 014424 (2014).

I. Rousochatzakis, J. Reuther, R. Thomale, J. Robert, and I. Mirebeau, Phys. Rev. X 5, 011035 (2015).

J. Baillargeon and A. Avella, Phys. Rev. B 92, 184416 (2015).

F.-Y. Li, Y.-D. Li, Y. Yu, A. Paramekanti, and G. Chen, Phys. Rev. B 95, 085132 (2017).

U. Hizi and C. L. Henley, Journal of Physics: Condensed Matter 19, 145268 (2007).

D. Bergman, J. Alicea, E. Gull, S. Trebst, and L. Balents, Nature Physics 3, 487 (2007).

P. A. McClarty, P. Stasiak, and M. J. P. Gingras, Phys. Rev. B 89, 024425 (2014).
[71] J. T. Chalker, P. C. W. Holdsworth, and E. F. Shender, Phys. Rev. Lett. 68, 855 (1992).
[72] A. B. Harris, C. Kallin, and A. J. Berlinsky, Phys. Rev. B 45, 2899 (1992).
[73] D. A. Huse and A. D. Rutenberg, Phys. Rev. B 45, 7536 (1992).
[74] A. B. Harris, C. Kallin, and A. J. Berlinsky, Phys. Rev. B 45, 2899 (1992).
[75] A. Chubukov, Phys. Rev. Lett. 69, 832 (1992).
Supplemental material for “Order-by-quantum-disorder and the fate of pseudo-Goldstone modes”

Jeffrey G. Rau, Paul A. McClarty, and Roderich Moessner
Max-Planck-Institut für Physik komplexer Systeme, 01187 Dresden, Germany
(Dated: May 2, 2018)

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S1. LINEAR SPIN-WAVE THEORY

Consider a general translational invariant spin model

\[ H \equiv \frac{1}{2} \sum_{rr'} \sum_{\alpha\alpha'} S_{ra} J_{r-r',\alpha\alpha'} S_{r'\alpha'}, \]  

(S1)

where \( S_{ra} \) is a spin-\( S \) operator located in unit cell \( r \) with sublattice index \( \alpha \). The spin operators can be expressed in terms of Holstein-Primakoff bosons as

\[ S_{ra} = \sqrt{S} \left[ \left( 1 - \frac{n_{ra}}{2S} \right)^{1/2} a_{ra} \hat{e}_{a,-} + \left( 1 - \frac{n_{ra}}{2S} \right)^{1/2} a_{ra}^\dagger \hat{e}_{a,+} \right] + (S - n_{ra}) \hat{e}_{a,0}, \]  

(S2)

where \( n_{ra} \equiv a_{ra}^\dagger a_{ra} \). The vectors \( \hat{e}_{a,\pm}, \hat{e}_{a,0} \) define a local frame of reference; in a more conventional Cartesian basis one defines \( \hat{e}_{a,\pm} \equiv (\hat{x}_a \pm i\hat{y}_a) / \sqrt{2} \) and \( \hat{e}_{a,0} \equiv \hat{z}_a \). It is useful to write the exchange matrix in this frame

\[ J_{\alpha\alpha'}^{\mu\mu'} \equiv \hat{e}_{\alpha,\mu} J_{\delta,\alpha\alpha'} \hat{e}_{\alpha',\mu'}. \]  

(S3)

Expanding in powers of \( 1/S \) then yields a semi-classical expansion about the ordered state defined by \( \hat{e}_{a,0} \), typically chosen to be along the classical ordering direction. In this case at order \( O(S) \) in the Holstein-Primakoff operators one finds a quadratic form

\[ H = N S (S + 1) \epsilon_{cl} \left[ a_{k}\right]^\dagger \left( \begin{array}{cc} A_k & B_k \end{array} \right) \left( \begin{array}{c} A_k \\ B_k \end{array} \right) \left( \begin{array}{c} a_{k}^\dagger \\ a_{k} \end{array} \right) + O(S^{1/2}), \]  

(S4)

where \( N_s \) is the number of sublattices, \( N \) is the total number of sites and we have defined the Fourier transforms of the bosons as \( a_{ka} \equiv \sqrt{N}^{-1/2} \sum_r e^{-i k \cdot r} a_{ra} \) where \( N = N_c N_s \). The classical energy per site is defined as

\[ \epsilon_{cl} \equiv \frac{1}{2N_s} \sum_{\alpha\alpha'} \sum_{\delta} J_{\delta,\alpha\alpha'}^{00} \]  

(S5)

and the matrices \( A_k \) and \( B_k \) are given by

\[ A_k^{\alpha\alpha'} = J_{k,\alpha\alpha'}^{+-} - \delta_{\alpha\alpha'} \sum_{\mu} J_{0,\alpha\mu}^{00}, \]  

(S6a)

\[ B_k^{\alpha\alpha'} = J_{k,\alpha\alpha'}^{++}, \]  

(S6b)

where we have defined the Fourier transforms of the local exchange matrices as

\[ J_{k,\alpha\alpha'}^{\mu\mu'} = \sum_{\delta} J_{\delta,\alpha\alpha'}^{\mu\mu'} e^{i k \cdot \delta}. \]  

(S7)

The linear spin-wave Hamiltonian [Eq. (S4)] can be diagonalized by a Bogoliubov transformation. To do this one diagonalizes the modified matrix [1]

\[ \begin{pmatrix} A_k & B_k \\ -B_k^\dagger & -A_k^\dagger \end{pmatrix} \equiv \sigma_3 M_k \]  

(S8)
This yields pairs of eigenvectors $V_k\alpha$ and $W_{-k,\alpha} = \sigma_1 V^{*}_{-k\alpha}$ with eigenvalues $+\epsilon_{k\alpha}$ and $-\epsilon_{-k,\alpha}$. These vectors can be normalized such that

$$V^\dagger_k \sigma_3 V_k = +\delta_{\alpha\alpha'}, \quad W^\dagger_{-k\alpha} \sigma_3 W_{-k\alpha'} = -\delta_{\alpha\alpha'}, \quad W^\dagger_{-k\alpha} \sigma_3 V_{k\alpha'} = 0.$$ (S9)

One can then write the Hamiltonian in terms of diagonalized bosons, $\gamma_{k\alpha}$, as

$$H \equiv NS(S+1)\epsilon_{c1} + NS \epsilon_{qu} + S \sum_{k\alpha} \epsilon_{k\alpha} \gamma^\dagger_{k\alpha} \gamma_{k\alpha} + O(S^{1/2}),$$ (S10)

where we have identified the energy per site from quantum zero-point motion, $\epsilon_{qu}$, as

$$\epsilon_{qu} = \frac{1}{2N} \sum_{k\alpha} \epsilon_{k\alpha}.$$ (S11)

### A. Zero modes

The diagonalization of $\sigma_3 M_k$ can be carried out straightforwardly if $M_k$ is positive definite. Since we are interested in linear spin-wave problem with zero modes, we must treat the case where it is only positive-semi-definite in detail, following Blaizot and Ripka [1].

Consider then the case where one has a single zero mode at $k = 0$; there is no loss of generality so long as all zero modes occur at commensurate wave-vectors. Now, the matrix $M_0$ is Hermitian, with $M_0 = M_0^\dagger$, and satisfies $\sigma_1 M_0 \sigma_1 = M_0^\dagger$. Since $M_0$ and $\sigma_0 M_0$ share zero modes, we classify the type of zero mode by the spectral properties of $M_0$. There are two possibilities: $M_0$ has a single zero eigenvector (type I) or $M_0$ has two linearly independent zero eigenvectors (type II). We define vectors [1] that span the zero mode subspace as $V_0$ and $W_0$ for both type I and type II cases. These can be chosen [1] to satisfy the normalization conditions [Eq. (S9)]

$$V^\dagger_0 \sigma_3 V_0 = +1, \quad W^\dagger_0 \sigma_3 W_0 = -1, \quad V^\dagger_0 \sigma_3 W_0 = 0,$$ (S12)

where $W_0 = \sigma_1 V_0^\dagger$. It useful to define the block forms of these matrices

$$V_0 \equiv \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, \quad W_0 \equiv \begin{pmatrix} Y_0^* \\ X_0^* \end{pmatrix}.$$ (S13)

Their relationship with the dispersion matrix depends on the type of the zero mode. One has

$$M_0 V_0 = M_0 W_0 = \begin{cases} (2\mu)^{-1} \sigma_3 (V_0 - W_0), & \text{type I} \\ 0 & \text{type II} \end{cases},$$ (S14)

where we have defined

$$\frac{1}{2\mu} \equiv V_0^\dagger M_0 V_0.$$ (S15)
The case of a type I zero mode corresponds to $\sigma_3 M_0$ being non-diagonalizable, with the zero mode subspace being a two by two Jordan block. This can be seen by projecting the dispersion into the subspace spanned by $V_0$ and $W_0$, where one finds

$$\begin{pmatrix} V_0^\dagger M_0 V_0 & V_0^\dagger M_0 W_0 \\ W_0^\dagger M_0 V_0 & W_0^\dagger M_0 W_0 \end{pmatrix} = \frac{1}{2\mu} \begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix} \quad \text{(type I)}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(type II)}. \quad (S16)$$

Classically, a type I zero mode can be understood as a mode that costs zero energy, but its canonically conjugate partner costs finite energy; the type II case corresponds to both having zero energy cost.

S2. NON-LINEAR SPIN-WAVE THEORY

Going to next order in $1/S$, one obtains a considerably more complex Hamiltonian. This can be written as a sum of the usual two-magnon terms, as well as three- and four-magnon interactions. We write

$$H = NS(S + 1)\epsilon_{c1} + S H_2 + S^{1/2} H_3 + S^0 H_4$$

where we define the individual pieces in symmetrized form as

$$H_2 = \frac{1}{2} \sum_{\alpha\beta} \sum_k \left[A_{k\alpha k}\hat{a}_{k\alpha}^\dagger \hat{a}_{k\beta}^\dagger + A_{-k\alpha k}^\dagger \hat{a}_{-k\alpha}^\dagger \hat{a}_{-k\beta} + \left(B_{k\alpha k}^\dagger \hat{a}_{k\alpha}^\dagger \hat{a}_{-k\beta} + \bar{B}_{k\alpha k}^\dagger \hat{a}_{-k\alpha}^\dagger \hat{a}_{k\beta}\right)\right],$$

(S18a)

$$H_3 = \frac{1}{2!} \sum_{\alpha\beta\mu\nu} \sum_{kk'} \left[T_{kk\alpha k\beta} \hat{a}_{k\alpha}^\dagger \hat{a}_{k\beta}^\dagger \hat{a}_{k+k'\mu}\hat{a}_{k+k'\nu}^\dagger + \bar{T}_{kk'\alpha k\beta}^\dagger \hat{a}_{k+k'\mu}^\dagger \hat{a}_{k+k'\nu}\hat{a}_{k\alpha}\hat{a}_{k\beta}\right],$$

(S18b)

$$H_4 = \frac{1}{N_c} \sum_{\alpha\beta\mu\nu q} \sum_{kk'q} \left[\frac{1}{(2!)^2} \sum_{kk'q} V_{kk'q}^{\alpha\beta\mu\nu} \hat{a}_{k+q,\alpha}^\dagger \hat{a}_{k',q,\beta} \hat{a}_{k+q,\mu} \hat{a}_{k',q,\nu} + \frac{1}{3!} \left(D_{kk'q}^{\alpha\beta\mu\nu} \hat{a}_{k\alpha}^\dagger \hat{a}_{k\beta}^\dagger \hat{a}_{q,\mu} \hat{a}_{q,\nu} \hat{a}_{k+k'+q,\nu} + \text{h.c.}\right)\right].$$

(S18c)

In terms of the local exchange matrices [Eq. (S3)] one can write

$$A_{k\alpha}^{\alpha\beta} = \mathcal{J}_{k,\alpha\beta}^{+-} - \delta_{\alpha\beta} \sum_{\mu} \mathcal{J}_{0,\alpha\mu}^{00},$$

(S19a)

$$B_{k\alpha}^{\alpha\beta} = \mathcal{J}_{k,\alpha\beta}^{++},$$

(S19b)

$$T_{kk'\alpha k\beta} = -\left[\delta_{\alpha\beta} \mathcal{J}_{k,\alpha\beta}^{00} + \delta_{\beta\alpha} \mathcal{J}_{k,\alpha\beta}^{00}\right],$$

(S19c)

$$V_{kk'q}^{\alpha\beta\mu\nu} = \delta_{\alpha\mu} \delta_{\beta\nu} \mathcal{J}_{k,\alpha\beta}^{00} + \delta_{\alpha\nu} \delta_{\beta\mu} \mathcal{J}_{k,\alpha\beta}^{00} - \left(\delta_{\mu\nu} \delta_{\beta\alpha} \mathcal{J}_{k+q,\alpha\beta}^{+-} + \delta_{\alpha\beta} \delta_{\nu\mu} \mathcal{J}_{k,\alpha\beta}^{+-}\right),$$

(S19d)

$$D_{kk'q}^{\alpha\beta\mu\nu} = -\frac{3}{4} \left(\delta_{\alpha\mu} \delta_{\beta\nu} \mathcal{J}_{k,\alpha\beta}^{++} + \delta_{\beta\mu} \delta_{\alpha\nu} \mathcal{J}_{k,\alpha\beta}^{++}\right),$$

(S19e)

where the four-magnon vertices have been left unsymmetrized for brevity.

When $1/S \to 0$, the interactions encoded in $H_3$ and $H_4$ can be included perturbatively since they are $O(S^{-1/2})$ and $O(S^{-1})$ with respect to the linear spin-wave parts. Our main interest is the (retarded) magnon Green’s function, including both normal and anomalous parts. We define the
real-frequency retarded Green’s functions as

\[
G^R_{\alpha\alpha'}(k, \omega) \equiv -i \int dt e^{i\omega t} \left\langle \Theta(t) \left[\langle a_{k\alpha}(t), a_{k\alpha'}^\dagger(0)\rangle\right] \right.,
\]

(S20a)

\[
G^R_{\alpha\alpha'}(k, \omega) \equiv -i \int dt e^{i\omega t} \left\langle \Theta(t) \left[\langle a_{k\alpha}(t), a_{-k\alpha'}(0)\rangle\right] \right.,
\]

(S20b)

\[
G^R_{\alpha\alpha'}(k, \omega) \equiv -i \int dt e^{i\omega t} \left\langle \Theta(t) \left[\langle a_{-k\alpha}^\dagger(t), a_{k\alpha'}^\dagger(0)\rangle\right] \right.,
\]

(S20c)

\[
G^R_{\alpha\alpha'}(k, \omega) \equiv -i \int dt e^{i\omega t} \left\langle \Theta(t) \left[\langle a_{-k\alpha}^\dagger(t), a_{-k\alpha'}(0)\rangle\right] \right.,
\]

(S20d)

where \(\langle \cdots \rangle\) is a ground state average. These can be organized into a matrix Green’s function

\[
G^R(k, \omega) \equiv \begin{pmatrix}
G^R_{\alpha\alpha'}(k, \omega) & G^R_{\alpha\alpha'}(k, \omega) \\
G^R_{\alpha\alpha'}(k, \omega) & G^R_{\alpha\alpha'}(k, \omega)
\end{pmatrix} \equiv \left[\sigma_3 (\omega + i0^+) - S M_k - \Sigma^R(k, \omega)\right]^{-1}.
\]

(S21)

The self-energy, \(\Sigma^R(k, \omega)\), is generated by interactions, first at \(O(S^0)\). As for the Green’s function, this contains both normal and anomalous parts. The advanced version of this Green’s function can be determined similarly, satisfying the relation \(G^A(k, \omega) = G^R(k, \omega)^\dagger\). The relevant diagrams in the magnon interactions are given in Fig. 3 of the main text.

The (sharp) magnon modes can then be determined from examination of the poles of the magnon Green’s function. This amounts to solving the non-linear eigenvalue equation

\[
\det \left(\omega + i0^+ - \sigma_3 \left[SM_k + \Sigma^R(k, \omega)\right]\right) = 0,
\]

(S22)

for the frequency \(\omega\). In the limit of \(1/S \to 0\), the eigenvalue problem can be solved by treating \(\Sigma^R(k, \omega)\) as a perturbation to the eigenvalues and eigenvectors of \(M_k\). Since the self-energy is in general non-Hermitian, there are quasi-normal modes that include both a real and imaginary part, with \(\omega \equiv E_{k\alpha} - i\Gamma_{k\alpha}\). Expanding about the energies, \(S\epsilon_{k\alpha}\), found at \(O(S)\) one finds the effective eigenvalue problem

\[
\sigma_3 \left[SM_k + \Sigma^R(k, S\epsilon_{k\alpha})\right] = \left(E_{k\alpha} - i\Gamma_{k\alpha}\right) V_{k\alpha},
\]

(S23)

for each mode found in linear spin-wave theory. If \(\epsilon_{k\alpha}\) is non-degenerate (generically) this can be via simple perturbation theory, giving

\[
E_{k\alpha} - i\Gamma_{k\alpha} = S\epsilon_{k\alpha} + V_{k\alpha}^\dagger \sigma_3 \Sigma^R(k, S\epsilon_{k\alpha}) V_{k\alpha} + O(S^{-1}).
\]

(S24)

Similarly \(V_{k\alpha} = V_{k\alpha} + O(S^{-1})\). The eigenvectors analogous to \(W_{-k\alpha}\), which we define as \(W_{-k,\alpha}\), can be determined in a similar fashion. We note that this perturbative assumption can fail under certain circumstances, even as \(1/S \to 0\); see Ref. [2] for details.

## A. Pseudo-Goldstone modes

The pseudo-Goldstone modes we are interested in have two features that distinguish them from the simple case presented above: (i) their energy at \(O(S)\) vanishes and (ii) the subspace associated
with these eigenvalues is two-dimensional, and so degenerate perturbation theory is generically needed. First, note that since the initial linear spin-wave energy is zero, we evaluate the self-energy at zero-frequency in this calculation. If one expands

\[ \Sigma^R(0, \omega) = \Sigma^R(0, 0) + \frac{\partial \Sigma^R(0, 0)}{\partial \omega} + \cdots, \]

one sees that the frequency derivative term carries an additional factor of \( O(S^{-1}) \) relative to the leading term and thus can be dropped. Second, we can ignore the distinction between the retarded and advanced self-energies, taking \( \Sigma^R(0, 0) \sim \Sigma^A(0, 0) \equiv \Sigma_0 \). Finding the pseudo-Goldstone energy thus amounts to simply diagonalizing the dispersion and self-energy (at zero wave-vector and frequency) projected into the zero-mode subspace.

We consider the two different types of pseudo-Goldstone mode. For the type II case, the projection of the dispersion part, \( M_0 \), vanishes and one thus needs to diagonalize only

\[ \sigma_3 \left( \begin{array}{cc} V_0^\dagger \Sigma_0 V_0 & V_0^\dagger \Sigma_0 W_0 \\ W_0^\dagger \Sigma_0 V_0 & W_0^\dagger \Sigma_0 W_0 \end{array} \right). \] (S26)

Symmetries of the self-energy tell us that \( \Sigma_0 = \Sigma_0^\dagger \) and thus diagonalizing this matrix yields two eigenvalues \( \pm \Delta \) with

\[ \Delta = \sqrt{(V_0^\dagger \Sigma_0 V_0)^2 - |W_0^\dagger \Sigma_0 W_0|^2} + O(S^{-1}). \] (S27)

We thus see that the type II pseudo-Goldstone gap scales as \( O(S^0) \). The type I pseudo-Goldstone mode is different since the projected dispersion matrix is not zero. One finds that

\[ \left( \begin{array}{cc} V_0^\dagger M_0 V_0 & V_0^\dagger M_0 W_0 \\ W_0^\dagger M_0 V_0 & W_0^\dagger M_0 W_0 \end{array} \right) = \frac{1}{2\mu} \left( \begin{array}{cc} +1 & +1 \\ +1 & +1 \end{array} \right). \] (S28)

One thus must diagonalize

\[ \frac{S}{2\mu} \left( \begin{array}{cc} +1 & +1 \\ -1 & -1 \end{array} \right) + \left( \begin{array}{cc} V_0^\dagger \Sigma_0 V_0 & V_0^\dagger \Sigma_0 W_0 \\ -W_0^\dagger \Sigma_0 V_0 & -W_0^\dagger \Sigma_0 W_0 \end{array} \right). \] (S29)

Using again that \( \Sigma_0 \) is Hermitian, as well as the definition of \( \mu \) [Eq. (S15)], one finds the pseudo-Goldstone gap

\[ \Delta = S^{1/2} \sqrt{2 \mathrm{Re} \left[ V_0^\dagger \Sigma_0 \sigma_3 M_0 V_0 \right]} + O(S^{-1/2}). \] (S30)

This scales as \( S^{1/2} \), distinct from that found in the type II case.

More practically, the evaluation of \( \Sigma_0 \) requires the evaluation of sums that involve the eigenvectors and eigenvalues of \( \sigma_3 M_k \). Due to the presence of the zero-modes, these are somewhat ill-defined. To circumvent this, we include a small, but finite, chemical potential for the magnons \( \sim \mu \sum_k a_k^\dagger a_k \), which plays the role of a physical pinning field. This resolves any singularities and renders the sums well-defined. The appropriate thermodynamic limit is then reached by converging the result in system size at fixed \( \mu \), then sending \( \mu \to 0 \). Note that the inclusion of \( \mu \), absent true Goldstone modes, directly implies that \( \Sigma^R(0, 0) = \Sigma^A(0, 0) \), since the two-magnon continuum then starts at \( 2\mu \). For most cases of interest the pseudo-Goldstone gap depends only weakly on the pinning field and thus the above procedure, while strictly required, is mostly unnecessary.
We now show how the gap computed from non-linear spin-wave theory can be directly related to the curvatures of the classical and quantum parts of the spin-wave energies computed in linear spin-wave theory. The strategy we take is based on the “self-correcting” nature of the spin-wave expansion: if one performs the Holstein-Primakoff expansion about a classical ground state sufficiently close to the true ground state, one should recover the true ground state properties, order by order in $1/S$. This then implies that there is some degree of independence with respect to the initial classical starting point.

We show this in a few stages: first we will give an explicit demonstration of this “self-correction” at the level of linear spin-wave theory. At higher order we proceed more implicitly, showing that the same physics is encoded in a sum rule for the spectral function. Using this sum rule will show that the curvature formula discussed in the main text holds. As a warm up we do this for gapped modes in linear spin-wave theory, before proceeding to the type I and type II pseudo-Goldstone modes of interest.

### A. Self-correcting linear spin-waves

Before delving into the main proof, we first show how this self-correcting behavior appears in practice. Consider the linear spin-wave theory discussed previously [Eq. (S4)], but expanded about a state which is not the classical ground state. In this case one finds additional terms in the Hamiltonian that are linear in the magnons.

We consider expanding about states that are related to the ground state by small rotations about the local $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{g}}_\alpha$ directions. We will assume these deformations describe the degrees of freedom involved in the low-lying, long-wavelength spin-waves of the problem. Explicitly, if $\hat{e}_{\alpha,0}$ and $\hat{e}_{\alpha,\pm}$ define the local frames for the classical ground state, then we define

\[
\hat{e}_{\alpha,0}(\zeta) \equiv \left(1 - |\zeta|^2\right)\hat{e}_{\alpha,0} - \hat{\zeta}\hat{e}_{\alpha,0} + O(\zeta^3),
\]

\[
\hat{e}_{\alpha,0}(\zeta) \equiv \left(1 - |\zeta|^2\right)\hat{e}_{\alpha,0} + \hat{\zeta}\hat{e}_{\alpha,0} + O(\zeta^3),
\]

\[
\hat{e}_{\alpha,-}(\zeta) \equiv \left(1 - |\zeta|^2\right)\hat{e}_{\alpha,-} - \hat{\zeta}\hat{e}_{\alpha,0} + O(\zeta^3),
\]

where the complex variable $\zeta \equiv (\phi + i\theta)/\sqrt{2}$ encodes both rotations and $|\zeta| \ll 1$.

Performing the Holstein-Primakoff expansion one then finds the Hamiltonian

\[
H(\zeta) = S(S + 1)N_{\text{el}}(\zeta) + S H_2(\zeta) + S^{3/2}H_1(\zeta) + O(S^{1/2}).
\]

The quadratic part we have encountered previously [Eq. (S4)], with

\[
H_2(\zeta) \equiv \frac{1}{2} \sum_k \left(|a_k^\dagger\right)^\dagger \begin{pmatrix} A_k(\zeta) & B_k(\zeta) \end{pmatrix} \begin{pmatrix} a_k^\dagger \\ a_{-k} \end{pmatrix}.
\]

\[
(S33)\]
The linear part appears as
\[
H_1(\zeta) = \sum_\alpha \left[ \lambda_\alpha(\zeta) a_{0,\alpha}^\dagger + \tilde{\lambda}_\alpha(\zeta) a_{0,\alpha} \right] = \left( \lambda(\zeta)^\dagger \lambda(\zeta)^\top \right) \left( \begin{array}{c} a_0 \\ a_0^\dagger \end{array} \right),
\]
where we have defined
\[
\begin{pmatrix} \lambda(\zeta) \\ \lambda(\zeta)^\dagger \end{pmatrix} = N^{1/2} M_0 \begin{pmatrix} \zeta \hat{u} \\ \bar{\zeta} \tilde{u} \end{pmatrix},
\]
where the vector \( \hat{u} \) is uniform over the sublattices, with \( \hat{u}_\alpha \equiv 1/\sqrt{N_s} \).

Our statement of self-correction is then that the ground state energy must be independent of \( \phi \) and \( \theta \) order by order in \( 1/S \). At first this appears somewhat contradictory: the constant classical part depends on \( \phi \) and \( \theta \) and is \( O(S^2) \); how can terms at \( O(S) \) and \( O(S^{3/2}) \) cancel it? We can see how this works out directly by shifting the bosons to remove the linear term. Define the shifted boson operators
\[
\begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix} = \begin{pmatrix} b_0 \\ b_0^\dagger \end{pmatrix} - S^{1/2} M(\zeta)^{-1} \begin{pmatrix} \lambda(\zeta) \\ \lambda(\zeta)^\dagger \end{pmatrix}.
\]

In terms of these shifted bosons, the Hamiltonian is then
\[
S(S+1)N\varepsilon_{cl}(\zeta) + \frac{S}{2} \sum_k \left( [b_k^\dagger]^\top b_{-k}^\top \right) \left( \begin{array}{cc} A_k(\zeta) & B_k(\zeta) \\ B_k^\dagger(\zeta) & A_{-k}^\dagger(\zeta) \end{array} \right) \left( \begin{array}{c} b_k \\ b_{-k}^\dagger \end{array} \right) - \frac{S^2}{2} \left( \lambda(\zeta)^\dagger \lambda(\zeta)^\top \right) M(\zeta)^{-1} \left( \begin{array}{c} \lambda(\zeta) \\ \lambda(\zeta)^\dagger \end{array} \right).
\]

Since the rotation is only evaluated to \( O(\zeta^2) \) we should also expand
\[
\varepsilon_{cl}(\zeta) = \varepsilon_{cl}(0) + \frac{1}{2} \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta^2} \right)_0 \theta^2 + \frac{1}{2} \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \phi^2} \right)_0 \phi^2 + 2 \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta \partial \phi} \right)_0 \theta \phi + O(\zeta^3).
\]

We thus see that the \( O(S^2) \) contribution to the energy, to quadratic order in \( \zeta \), is then given by
\[
S^2 N \left\{ \varepsilon_{cl}(0) + \frac{1}{2} \left[ \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta^2} \right)_0 \theta^2 + \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \phi^2} \right)_0 \phi^2 + 2 \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta \partial \phi} \right)_0 \theta \phi \right] - \frac{1}{2} \left( \bar{\zeta} \hat{u}^\top \zeta \tilde{u}^\top \right) M_0 \left( \begin{array}{c} \zeta \hat{u} \\ \bar{\zeta} \tilde{u} \end{array} \right) \right\}.
\]

To connect with the notation for the zero modes, as well as some notation we will introduce later, define
\[
U_\phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{u} \\ \tilde{u} \end{array} \right), \quad U_\theta = \frac{i}{\sqrt{2}} \left( \begin{array}{c} \hat{u} \\ -\tilde{u} \end{array} \right).
\]

One then sees that the shift induced part of the \( O(S^2) \) contributions are given as a matrix element of the vector \( \phi U_\phi + \theta U_\theta \). For the \( O(S^2) \) part of the energy to be independent of \( \phi \) and \( \theta \), one then must require
\[
\left( \frac{\partial^2 \varepsilon_{cl}}{\partial \phi^2} \right)_0 = U_\phi^\dagger M_0 U_\phi, \quad \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta^2} \right)_0 = U_\theta^\dagger M_0 U_\theta, \quad \left( \frac{\partial^2 \varepsilon_{cl}}{\partial \theta \partial \phi} \right)_0 = U_\phi^\dagger M_0 U_\theta.
\]
This can be related to the energy of the spin-wave mode in the subspace spanned by these two vectors. Note that
\[ U^\dagger_\phi \sigma_3 U_\phi = U^\dagger_\theta \sigma_3 U_\theta = 0, \quad U^\dagger_\phi \sigma_3 U_\theta = +i. \quad (S41) \]

Projecting the eigenvalue equation for the zone center spin-wave into this subspace then yields the two by two problem
\[
\det \left[ S \left( \begin{array}{cc}
U^\dagger_\phi M_0 U_\phi & U^\dagger_\phi M_0 U_\theta \\
U^\dagger_\theta M_0 U_\phi & U^\dagger_\theta M_0 U_\theta 
\end{array} \right) \right] - \omega \left( \begin{array}{cc}
0 & -i \\
+i & 0 
\end{array} \right) = 0. \quad (S42)
\]

There are two solutions, \( \pm \epsilon_0 \), where
\[
\epsilon_0 = S \sqrt{(U^\dagger_\phi M_0 U_\phi)(U^\dagger_\phi M_0 U_\theta) - |U^\dagger_\theta M_0 U_\phi|^2}, \quad (S43)
\]
where we used that \( U^\dagger_\theta M_0 U_\phi = U^\dagger_\phi M_0 U_\theta \). The energy of this mode is then directly related to the curvatures, with Eq. (S40) and Eq. (S43) implying that
\[
\epsilon_0 = S \sqrt{\left( \frac{\partial^2 \epsilon_{cl}}{\partial \theta^2} \right)_0 \left( \frac{\partial^2 \epsilon_{cl}}{\partial \phi^2} \right)_0 - \left( \frac{\partial^2 \epsilon_{cl}}{\partial \theta \partial \phi} \right)_0^2}. \quad (S44)
\]

Alternatively, this relation can be worked explicitly by expressing the curvatures and \( \epsilon_0 \) directly in terms of the relevant exchange matrices.

We have thus seen that enforcing the condition that the ground state energy is independent of the initial state at \( O(S^2) \), we have found a relation between a spin-wave energy computed at \( O(S) \) and curvatures of the classical energies, computed at \( O(S^2) \). Our goal in the next few sections is to establish the same kind of relation, but between a spin-wave energy at \( O(S^0) \) or \( O(S^{1/2}) \) and the curvatures of the classical and quantum energies at \( O(S^2) \) and \( O(S) \). Since we will not be able to solve the theory explicitly at \( O(S^0) \), as we did at \( O(S) \) above, our methods will have to be somewhat more indirect. However the essence of the argument remains the same.

### B. First moment sum rule and spectral functions

To attack this problem for interacting spin-waves, we will first define a rotated Hamiltonian that will serve as the starting point of the Holstein-Primakoff expansion. Since we want to leverage an exact sum rule, we will do this at the level of spin operators, rather than Holstein-Primakoff bosons (as was done in Sec. S3 A).

To motivate this, define the explicit zero-mode magnons [1]
\[
\gamma_0 = \sum_\alpha \left[ X_0^\alpha a_{0,\alpha} - Y_0^\alpha a_{0,\alpha}^\dagger \right] = V_0^\dagger \sigma_3 \begin{pmatrix} a_0^\dagger \\ a_0 \end{pmatrix} = - \left( \begin{pmatrix} a_0^\dagger \\ a_0 \end{pmatrix}^\dagger \right) \sigma_3 W_0, \quad (S45a)
\]
\[
\gamma_0^\dagger = \sum_\alpha \left[ X_0^\alpha a_{0,\alpha}^\dagger - Y_0^\alpha a_{0,\alpha} \right] = -W_0^\dagger \sigma_3 \begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix} = \left( \begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix}^\dagger \right) \sigma_3 V_0. \quad (S45b)
\]
By combining these, we can create a pair of Hermitian operators that can be defined directly in terms of the spins. We define a position-like operator, \( \Phi \), and momentum-like operator, \( \Theta \), as

\[
\Phi \equiv \sum_{\alpha} \left[ (\text{Re}X^\alpha_0 - \text{Re}Y^\alpha_0) S^x_{\alpha r} + (\text{Im}X^\alpha_0 + \text{Im}Y^\alpha_0) S^y_{\alpha r} \right] = \sqrt{\frac{S N}{2}} \left[ \gamma^\alpha_0 + \gamma_0 \right] + O(S^{-1/2}), \tag{S46a}
\]

\[
\Theta \equiv \sum_{\alpha} \left[ -(\text{Im}X^\alpha_0 - \text{Im}Y^\alpha_0) S^x_{\alpha r} + (\text{Re}X^\alpha_0 + \text{Re}Y^\alpha_0) S^y_{\alpha r} \right] = i \sqrt{\frac{S N}{2}} \left[ \gamma^\alpha_0 - \gamma_0 \right] + O(S^{-1/2}), \tag{S46b}
\]

which satisfy \([\Phi, \Theta] = iNS\) at leading order in \(1/S\) and map to combinations of the zero-mode magnons. These operators represent the general form of the soft directions discussed in the main text; given a spin model, the appropriate semi-classical variables can thus be obtained from the vectors that characterize the zero mode in linear spin-wave theory, via Eq. (S46).

Using these two operators, \( \Phi \) and \( \Theta \), we define the local spin rotation

\[
U(\phi, \theta) \equiv e^{-i\phi \Phi} e^{-i\theta \Theta} = e^{-i\theta \Theta} e^{-i\phi \Phi} e^{-i\theta \Phi} e^{-i\phi \Theta}.
\] \(\text{(S47)}\)

Note that since \([\Phi, \Theta] = iNS\) is a constant, the order of operations is unimportant at leading order in \(1/S\). This can be used to define the transformed Hamiltonian \( \mathcal{H}(\phi, \theta) \equiv U(\phi, \theta) \mathcal{H}(\phi, \theta) U(\phi, \theta) \) where \( \mathcal{H}(0, 0) = H \). The Hamiltonian can be expanded as

\[
\mathcal{H}(\phi, \theta) \sim e^{i\theta \Theta} \left( H + i\phi [\Phi, H] - \frac{\delta^2}{2} [\Phi, [\Phi, H]] + \cdots \right) e^{-i\Theta},
\]

\[
\sim H + i(\theta \Theta[H] + \phi [\Phi, H]) - \frac{1}{2} \left( \phi^2 [\Phi, [\Phi, H]] + 2\phi \Theta [\Phi, [\Theta, H]] + \Theta^2 [\Theta, [\Theta, H]] \right).
\]

We thus can write the derivatives

\[
\left( \frac{\partial^2 \mathcal{H}}{\partial \phi^2} \right)_0 = -[\Phi, [\Phi, H]], \quad \left( \frac{\partial^2 \mathcal{H}}{\partial \phi \partial \theta} \right)_0 = -[\Phi, [\Theta, H]],
\]

\[
\left( \frac{\partial^2 \mathcal{H}}{\partial \theta^2} \right)_0 = -[\Theta, [\Phi, H]], \quad \left( \frac{\partial^2 \mathcal{H}}{\partial \theta \partial \phi} \right)_0 = -[\Theta, [\Theta, H]]. \tag{S48}
\]

Now while the Hamiltonian is not invariant under this transformation, the ground state energy is trivially invariant under this operation.

If we thus define the ground state energy of \( \mathcal{H}(\phi, \theta) \) to be \( E(\phi, \theta) \), then one has \( E(\phi, \theta) = E(0, 0) \) identically, implying that all derivatives with respect to \( \phi \) and \( \theta \) vanish. The vanishing of these derivatives has implications for the correlation functions and expectation values of \( \Phi \) and \( \Theta \).

For example, the vanishing of the first derivatives implies the (trivial) statement that \( \langle [\Phi, H] \rangle = \langle [\Theta, H] \rangle = 0 \), where \( \langle \cdots \rangle \) is a ground state average with respect to \( H \). The implications from the second derivative are less trivial, and imply sum rules for the spectral function of these operators.

First recall the definition of the spectral function of a pair of operators \( O \) and \( O' \)

\[
A_{OO'}(\Omega) \equiv \frac{1}{2i} \left[ G_{OO'}^R(\Omega) - G_{OO'}^A(\Omega) \right], \tag{S49}
\]
where \( G^R_{00'}(\omega) \) and \( G^A_{00'}(\omega) \) are the associated retarded and advanced Green’s functions for these operators. The vanishing of the second derivatives then implies the sum rules
\[
\int d\Omega \, \Omega \, A_{0\Phi}(\Omega) = \left( \frac{\partial^2 \mathcal{H}}{\partial \phi \partial \bar{\phi}} \right)_0, \quad \int d\Omega \, \Omega \, A_{\Phi 0}(\Omega) = \left( \frac{\partial^2 \mathcal{H}}{\partial \phi \partial \bar{\phi}} \right)_0, \\
\int d\Omega \, \Omega \, A_{\Phi \Phi}(\Omega) = \left( \frac{\partial^2 \mathcal{H}}{\partial \theta \partial \bar{\theta}} \right)_0, \quad \int d\Omega \, \Omega \, A_{\Phi 0}(\Omega) = \left( \frac{\partial^2 \mathcal{H}}{\partial \theta \partial \bar{\theta}} \right)_0. \tag{S50}
\]
We refer to these collectively as the first moment sum rule. We stress that the expectation value, \langle \cdots \rangle, is with respect to the ground state of \( H \), not the ground state of \( \mathcal{H}(\theta, \phi) \). Alternatively, these relations are a special case of more general expressions for the moments of the spectral function [3, 4].

These spectral functions can be related to the spectral function of the magnon operators. From Eq. (S46), we express these as
\[
A_{\Phi}(\Omega) = \frac{SN}{2} \left[ A_{0,\gamma_0}(\Omega) + A_{\gamma_0,0}(\Omega) + A_{\gamma_0,\gamma_0}(\Omega) + A_{0,\gamma_0}(\Omega) \right] + O(S^{-1/2}), \tag{S51a}
\]
\[
A_{\Phi 0}(\Omega) = \frac{iSN}{2} \left[ A_{0,\gamma_0}(\Omega) - A_{\gamma_0,0}(\Omega) + A_{\gamma_0,\gamma_0}(\Omega) - A_{0,\gamma_0}(\Omega) \right] + O(S^{-1/2}), \tag{S51b}
\]
\[
A_{\Phi \Phi}(\Omega) = \frac{iSN}{2} \left[ A_{0,\gamma_0}(\Omega) - A_{\gamma_0,0}(\Omega) + A_{\gamma_0,\gamma_0}(\Omega) - A_{0,\gamma_0}(\Omega) \right] + O(S^{-1/2}), \tag{S51c}
\]
\[
A_{\Phi 0}(\Omega) = \frac{SN}{2} \left[ A_{0,\gamma_0}(\Omega) + A_{\gamma_0,0}(\Omega) - A_{\gamma_0,\gamma_0}(\Omega) - A_{0,\gamma_0}(\Omega) \right] + O(S^{-1/2}). \tag{S51d}
\]

It’s useful to relate this back to the Green’s functions we have define for the magnons. Using the retarded and advanced magnon Green’s functions [Eq. (S21)] we can define the magnon spectral function
\[
A(k, \Omega) = \frac{1}{2i} \left[ G^R(k, \Omega) - G^A(k, \Omega) \right]. \tag{S52}
\]
Then each of these pieces can be expressed in terms of the one-magnon spectral function via
\[
A_{0,\gamma_0}(\Omega) = +V^+_0 \sigma_3 A(0, \Omega) \sigma_3 V_0, \quad A_{\gamma_0,0}(\Omega) = -V^+_0 \sigma_3 A(0, \Omega) \sigma_3 W_0, \\
A_{\gamma_0,\gamma_0}(\Omega) = -W^+_0 \sigma_3 A(0, \Omega) \sigma_3 V_0, \quad A_{0,\gamma_0}(\Omega) = +W^+_0 \sigma_3 A(0, \Omega) \sigma_3 W_0. \tag{S53}
\]
This can be made cleaner via the definition of some rotated basis vectors for the zero-mode subspace; define
\[
U_\theta = \frac{V_0 + W_0}{\sqrt{2}}, \quad U_\Phi = \frac{V_0 - W_0}{i \sqrt{2}}. \tag{S54}
\]
From the normalization conditions on \( V_0 \) and \( W_0 \) [Eq. (S9)] one has that
\[
U^*_{\mu} \sigma_3 U_\Phi = U^*_{\mu} \sigma_3 U_\theta = 0, \quad U^*_{\mu} \sigma_3 U_\theta = +i. \tag{S55}
\]
The set of spectral functions can be expressed compactly if we define an index \( \mu, \nu = \Phi \) or \( \Theta \), we can write
\[
A_{\mu \nu}(\Omega) = SN U^*_{\mu} \left[ \sigma_3 A(0, \Omega) \sigma_3 \right] U_{\nu}. \tag{S56}
\]
Coming back to the first moment sum rule, we then have that
\[ U_\mu^\dagger \sigma_3 \left[ \int d\Omega \: \Omega \: A(0, \Omega) \right] \sigma_3 U_\nu = \frac{1}{S N} \left( \frac{\partial^2 H}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0, \tag{S57} \]
where \( \lambda_\phi = \phi \) and \( \lambda_\theta = \theta \). This is the key result that will allow us to establish the equivalence statement.

C. Warm up: No pseudo-Goldstone mode

As an example consider the non-interacting case where there is no accidental degeneracy and no zero mode. If one performs a spin-wave expansion of \( \mathcal{H}(\phi, \theta) \) then one will find an \( O(S^2) \) classical contribution to the energy given as
\[ \mathcal{H}(\phi, \theta) = S^2 N \epsilon_{c1}(\phi, \theta) + O(S), \tag{S58} \]
where \( \epsilon_{c1}(\phi, \theta) \) is the classical energy density in the rotated configuration. The right hand side of the sum rule [Eq. (S57)] then reads
\[ \frac{1}{S N} \left( \frac{\partial^2 H}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0 = S \left( \frac{\partial^2 \epsilon_{c1}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0 + O(S^0). \tag{S59} \]
For the left side of the sum rule, we consider the magnon spectral function at \( O(S) \). In terms of the appropriate eigenvectors and eigenvalues of \( \sigma_3 M_0 \) one has
\[ A(0, \omega) = \sum_\alpha \left[ \delta(\omega - \epsilon_{0,\alpha}) V_{0,\alpha} V_0^\dagger - \delta(\omega + \epsilon_{0,\alpha}) W_{0,\alpha} W_0^\dagger \right]. \tag{S60} \]
The first moment is then given by
\[ \int d\omega \: \omega \: A(0, \omega) = \sum_\alpha \epsilon_{0,\alpha} \left[ V_{0,\alpha} V_0^\dagger + W_{0,\alpha} W_0^\dagger \right] = \sigma_3 M_0 \sigma_3, \tag{S61} \]
where we have used the spectral decomposition of \( M_0 \) [1]. Putting these two together one finds that
\[ U_\mu^\dagger M_0 U_\nu = S \left( \frac{\partial^2 \epsilon_{c1}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0. \tag{S62} \]
We thus see that the energies of the spin-wave modes at \( O(S) \) in the subspace spanned by \( U_\theta, U_\phi \) can be determined from the curvatures of the classical energy, just as in Sec. S3 A.

D. Type II pseudo-Goldstone mode

We now consider a case with a type II pseudo-Goldstone mode, as this is a bit simpler than the type I case. Due to the presence of the accidental degeneracy, the spin-wave expansion is stable to \( O(S) \) for any choice of \( \theta \) and \( \phi \). Consider the rotated Hamiltonian
\[ \mathcal{H}(\phi, \theta) = \frac{1}{2} \sum_{r r'} \sum_{\alpha \alpha'} S_{r \alpha}^r \left[ R_\alpha^r(\phi, \theta) J_{r - r', \alpha' \alpha} R_{\alpha'}(\phi, \theta) \right] S_{r' \alpha'}, \tag{S63} \]
where \( R_\alpha(\phi, \theta) \) is the local rotation matrix associated with the operator \( U(\phi, \theta) \). We perform our Holstein-Primakoff expansion about the ground state of \( \phi = \theta = 0 \) problem which, given we have rotated exchanges, is equivalent to performing the expansion about the state with finite \( \phi \) and \( \theta \). Since the classical ground state is stable for arbitrary \( \phi \) and \( \theta \), we can write

\[
\mathcal{H}(\phi, \theta) = S(S + 1)N\epsilon_{\xi}(0, 0) + S N\epsilon_{\text{qu}}(\phi, \theta) + S \sum_{k\alpha} \epsilon_{k\alpha}(\phi, \theta)\gamma_{k\alpha}^\dagger \gamma_{k\alpha} + O(S^{1/2}),
\]

(S64)

where we have noted that \( \epsilon_{\xi}(\phi, \theta) = \epsilon_{\xi}(0, 0) \). The second derivative is then

\[
\frac{1}{SN} \left( \frac{\partial^2 \mathcal{H}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_{0} = \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_{0} + \frac{1}{N} \sum_{k\alpha} \left( \frac{\partial^2 \epsilon_{k\alpha}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_{0} \langle \gamma_{k\alpha}^\dagger \gamma_{k\alpha} \rangle + O(S^{-1}).
\]

(S65)

The magnon expectation value is with respect to the \( \phi = \theta = 0 \) ground state, and thus is zero in linear spin-wave theory. We can thus say that \( \langle \gamma_{k\alpha}^\dagger \gamma_{k\alpha} \rangle \sim O(S^{-1}) \), and since \( \epsilon_{k\alpha}(\phi, \theta) \sim O(S^0) \), these terms are unimportant. The right hand side of the sum rule is then simply given by the curvature of the quantum zero-point energy density

\[
\frac{1}{SN} \left( \frac{\partial^2 \mathcal{H}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_{0} = \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_{0} + O(S^{-1}).
\]

(S66)

Consider now the left-hand side of the sum rule. Since the right-hand side has a leading contribution of \( O(S^0) \), we must also identify such a contribution at this order from the first moment of the spectral function. As discussed in Sec. S2, at \( O(S^0) \) the spin-wave energies can be determined by perturbatively including the \( O(S^{1/2}) \) and \( O(S^0) \) interaction terms from the Holstein-Primakoff expansion. This produces a spectral function of the form

\[
A(0, \omega) = \sum_{\alpha} \left[ \mathcal{L}(\omega - E_{0,\alpha}, \Gamma_{0,\alpha}) V_{0,\alpha}^\dagger V_{0,\alpha}^\dagger - \mathcal{L}(\omega + E_{0,\alpha}, \Gamma_{0,\alpha}) W_{0,\alpha}^\dagger W_{0,\alpha}^\dagger \right],
\]

(S67)

where the notation for the quasi-normal modes is defined in Sec. S2 and \( \mathcal{L}(\omega, \gamma) \equiv \gamma/((\omega^2 + \gamma^2)/\pi) \) is a Lorentzian centered at zero-energy with width \( \gamma \). Most of the terms in this sum do not survive projection into the zero-mode subspace. Recall that the solution of Eq. (S23) implies that

\[
U_{\mu}^\dagger \sigma_3 V_{0,\alpha}^\dagger \sigma_3 U_{\nu}^\dagger \sigma_3 V_{0,\alpha}^\dagger \sigma_3 U_{\nu}^\dagger \sigma_3 \sim O(S^{-2}),
\]

(S68)

for eigenvectors that do not arise from the zero mode subspace that appears in linear spin-wave theory. Since the energies, \( E_{0,\alpha} \), outside the zero-mode subspace have a piece \( \sim O(S) \), this means their total contribution to the first moment is \( O(S^{-1}) \) and thus can be ignored.

We thus can restrict only to those eigenvectors that arise from the zero mode subspace, which we denote as \( V_0 \) and \( W_0 \). We thus can write

\[
U_{\mu}^\dagger \sigma_3 \left[ \int d\omega \omega A(0, \omega) \right] \sigma_3 U_{\nu} = U_{\mu}^\dagger \sigma_3 \left[ \Delta \left( V_0^\dagger V_0^\dagger + W_0^\dagger W_0^\dagger \right) \right] \sigma_3 U_{\nu} + O(S^{-1}),
\]

where \( \Delta \) is the pseudo-Goldstone gap induced by spin-wave interactions. Note that we have set the width to zero for these modes; this is exact perturbatively in the interactions for the lowest lying mode, embodied in the assumption \( \Sigma^R(0, 0) \sim \Sigma^A(0, 0) \) articulated in Sec. S2 A.
Since they are obtained via degenerate perturbation theory, the vectors $\mathbf{V}_0$ and $\mathbf{W}_0$ are related to the unperturbed eigenvectors, $\mathbf{V}_0$ and $\mathbf{W}_0$, by a change of basis. Because of this, one can show that the matrix elements that appear in the first moment are identical to those found upon projecting the sum of the effective dispersion matrix, $\mathbf{M}_0 + \Sigma_0$ (see Sec. S2 A). Explicitly one finds

$$
\begin{align*}
V_0^\dagger \left[ (S \mathbf{M}_0 + \Sigma_0) \mathbf{V}_0 \right] &= W_0^\dagger \left[ (S \mathbf{M}_0 + \Sigma_0) \mathbf{W}_0 \right] = V_0^\dagger \sigma_3 \left( \Delta \mathbf{V}_0 + \mathbf{W}_0 \mathbf{W}_0^\dagger \right) \sigma_3 V_0, \\
W_0^\dagger \left[ (S \mathbf{M}_0 + \Sigma_0) \mathbf{V}_0 \right] &= (V_0^\dagger \mathbf{V}_0 + \mathbf{W}_0 \mathbf{W}_0^\dagger) \sigma_3 V_0.
\end{align*}
$$

(S69a)

(S69b)

Using the sum rule [Eq. (S57)], and the definitions of $U_\mu$ [Eq. (S54)] we can then see that

$$
\begin{align*}
V_0^\dagger [S \mathbf{M}_0 + \Sigma_0] V_0 &= \frac{1}{2} \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \theta^2} \right)_0 + \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \phi^2} \right)_0, \\
W_0^\dagger [S \mathbf{M}_0 + \Sigma_0] V_0 &= \frac{1}{2} \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \theta^2} \right)_0 - \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \phi^2} \right)_0 - i \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \theta \partial \phi} \right)_0.
\end{align*}
$$

(S70a)

(S70b)

The pseudo-Goldstone gap for the type II case is then found using Eq. (S27), giving the final result

$$
\Delta = \sqrt{\left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \phi^2} \right)_0 - \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \theta^2} \right)_0^2} \sim O(S^0).
$$

(S71)

**E. Type I pseudo-Goldstone mode**

Finally, we consider the type I case. This is somewhat more involved given there is only a single soft-direction. We assume that the coordinates $\phi$ and $\theta$ are chosen such that $\theta = 0$, $\phi \neq 0$ represents the soft direction. Because of this, the spin-wave expansion is stable only for $\theta = 0$ and thus we have to be somewhat careful. Consider first the dependence on $\theta$ with $\phi = 0$

$$
\mathcal{H}(0, \theta) = S^2 N \left[ \epsilon_{\text{cl}}(0, 0) + \frac{1}{2} \theta^2 \left( \frac{\partial^2 \epsilon_{\text{cl}}}{\partial \theta^2} \right)_0 \right] + O(S).
$$

(S72)

We thus see that there is an $O(S^2)$ contribution to this second derivative, with

$$
\frac{1}{SN} \left( \frac{\partial^2 \mathcal{H}}{\partial \theta^2} \right)_0 = S \left( \frac{\partial^2 \epsilon_{\text{cl}}}{\partial \theta^2} \right)_0 + O(S^0).
$$

(S73)

Next consider the case where $\theta = 0$, but $\phi \neq 0$, where there is no classical contribution, but a well defined spin-wave expansion at $O(S)$. One writes

$$
\mathcal{H}(\phi, 0) = S(S + 1) N \epsilon_{\text{cl}}(0, 0) + S N \epsilon_{\text{qu}}(\phi, 0) + S \sum_{k \alpha} \epsilon_{k \alpha}(\phi, 0) \gamma_{k \alpha}^{\dagger} \gamma_{k \alpha} + O(S^{1/2}).
$$

(S74)

Following the discussion in Sec. S6 D, we can then see that

$$
\frac{1}{SN} \left( \frac{\partial^2 \mathcal{H}}{\partial \phi^2} \right)_0 = \left( \frac{\partial^2 \epsilon_{\text{qu}}}{\partial \phi^2} \right)_0 + O(S^{-1}).
$$

(S75)
Now all that remains is the cross-term. The classical part of this vanishes due to the choice of $\phi$.

Going beyond the classical part requires the handling of terms linear in the magnons, as in Sec. S3 A. In this case the only piece of interest is the $O(S^{3/2})$ part, however, this is at most of order $O(\phi_0 \phi)$, and since $\langle a_0, \phi \rangle = O(S^{-1/2})$, this contribution is at most $O(S^0)$, with

$$\frac{1}{SN} \left( \frac{\partial^2 H}{\partial \theta \partial \phi} \right)_0 = O(S^0).$$

Due to the presence of the classical, $O(S^2)$, parts of the curvatures, this has no effect at leading order and thus can be ignored. Following the results of Sec. S3 D, we then can write

$$V_0^\dagger [S M_0 + \Sigma_0] V_0 = \frac{1}{2} \left[ S \left( \frac{\partial^2 \epsilon_{cl}}{\partial \theta^2} \right)_0 + \left( \frac{\partial^2 \epsilon_{qu}}{\partial \phi^2} \right)_0 \right],$$

$$W_0^\dagger [S M_0 + \Sigma_0] V_0 = \frac{1}{2} \left[ S \left( \frac{\partial^2 \epsilon_{cl}}{\partial \theta^2} \right)_0 - \left( \frac{\partial^2 \epsilon_{qu}}{\partial \phi^2} \right)_0 \right].$$

We thus have the final result, following Eq. (S30), that

$$\Delta = S^{1/2} \sqrt{\left( \frac{\partial^2 \epsilon_{cl}}{\partial \theta^2} \right)_0 \left( \frac{\partial^2 \epsilon_{qu}}{\partial \phi^2} \right)_0} + O(S^{-1/2}).$$

**S4. REMARKS ON SOME CONNECTIONS TO QUANTUM CHROMODYNAMICS**

The curvature formula presented in this paper bears some similarity to various formulas appearing in the literature on quantum chromodynamics (QCD) and chiral perturbation theory. Since, as it turns out, the similarities are more than merely superficial we briefly remark on them here.

Our work rests on the existence of accidental degeneracies in frustrated magnets which are not protected by symmetries of the full Hamiltonian. An analogous situation arises in the physics of the strong nuclear force [5]: when the bare $u, d, s$ quark masses are set to zero there is a set of global symmetries associated with axial charges – one for each of the eight generators of SU(3) – with 3 being the number of quark flavors. Although not exact symmetries of nature, it is fruitful to think along these lines because, in doing so, one realizes that the global symmetries are spontaneously broken leading to eight Goldstone modes. These can be associated with the lightest meson octet consisting of the $\pi, K$ and $\eta$ mesons – their masses arising from the explicit breaking of the axial vector symmetries. A formula due to Dashen [6]

$$m^2 \propto \langle [Q_5, [Q_5, H]] \rangle,$$

relates the meson mass to a double commutator of the Hamiltonian with generators of the approximate symmetries. The more widely known Gellmann-Oakes-Renner formula [7] follows from this relation. Dashen’s formula is analogous to the left-hand side of sum rule formula [Eq. (S57)], relating the first-moment of the spectral function to the gap of pseudo-Goldstone mode. In the context of QCD, the part of the Hamiltonian $\mathcal{H}$ that contributes to the mass comes from the explicit symmetry-breaking terms, such as the quark masses.
In addition to the physics of spontaneous and explicit breaking of SU(3) axial vector currents, there is a classically conserved U(1) axial charge which is broken by quantum fluctuations. Accordingly, there is also a ninth pseudoscalar meson $\eta'$ with a mass significantly larger than that of the pseudoscalar octet. Witten and Veneziano [8, 9] addressed the question of how to precisely frame the $\eta'$ mass as arising from the anomaly. Schematically, in the large-$N$ limit of SU($N$) Yang-Mills theory with massless quarks one finds [8]

$$m^2_{\eta'} \propto \left[ \frac{\partial^2 \epsilon_{YM}}{\partial \theta^2} \right]_0 \equiv \chi_{\infty} \propto \frac{1}{N},$$  \quad \text{(S80)}

where $m_{\eta'}$ is the mass of the $\eta'$-meson and $\chi_{\infty}$ is the “topological susceptibility”: the curvature of the energy of the pure gauge system with respect to the addition of the topological $\theta$-term that is associated with the U(1) anomaly [8]. In broad strokes this resembles the curvature formula: specifically, the $\eta'$ mass is analogous to the pseudo-Goldstone gap, and $\chi_{\infty}$ is analogous to the (mean) curvature of the classical and quantum zero-point energy densities and $N$ is analogous to $S$. The proof we give here for the pseudo-Goldstone gap is in a similar spirit to the argument presented by Witten: as in our case where a mixing of orders in perturbation theory ensures that the ground state energy is independent of the spin configuration, the small meson mass $m^2_{\eta'} \sim 1/N$, is needed to cancel a term at $O(N^2)$ to render the total energy independent of the $\theta$ angle [8].

We note in passing that order-by-disorder as discussed in condensed matter is similar to the notion of an anomaly in field theory in the sense that both involve the lifting of a classical symmetry by fluctuations. The resemblance is no deeper than this however, as while anomalies stem from the lack of invariance of the path integral measure, in the case of order-by-disorder accidental symmetries can usually be fine-tuned to become exact symmetries. Even so, the story of understanding the $\eta'$ mass in large-$N$ QCD from the axial U(1) anomaly does have some interesting parallels to our work.

### S5. ORDER-BY-QUANTUM DISORDER MODELS

We now discuss some details of the explicit calculations of the pseudo-Goldstone gap in a number of relevant quantum spin models. Where possible we will compare with other theoretical approaches or directly with experimental data. The key results have been presented in Table I of the main text.

#### A. Heisenberg-compass models

The simplest, and one of the earliest studied cases [10, 11] of order-by-quantum-disorder is a Heisenberg ferromagnet with a compass type anisotropy on square or cubic lattices. We write these models as

$$\sum_{ij\mu} \left[ JS_i \cdot S_{i+\mu} + KS_i^\mu S_{i+\mu}^\mu \right].$$  \quad \text{(S81)}
where $J < 0$ is ferromagnetic and $\mu$ is the bond direction. For $K = 0$ the classical ground state is a ferromagnet, $S_i = S \hat{n}$, with arbitrary overall direction $\hat{n}$. In the square lattice case, finite $K > 0$ selects $\hat{n} = \hat{z}$, with no residual degeneracy, while $K < 0$ fixes $\hat{n}$ to the $\hat{x}-\hat{y}$ plane with a $\text{U}(1)$ degeneracy. This is lifted by the quantum zero-point energy to select states aligned with the cubic axes, $\hat{n} = \hat{x}, \hat{y}$, with the associated pseudo-Goldstone mode being type I. A similar picture holds in the cubic case, except there is a residual type II $\text{O}(3)$ degeneracy for either sign of $K$, with the zero-point energy selecting the cubic axes [11]. As shown in Table I of the main text, the induced pseudo-Goldstone gap appears at next order in $1/S$. For small $|K|/|J|$ this yields a gap $\Delta \propto S^{1/2}|K|^{3/2}/|J|^{1/2}$ for the square lattice and $\Delta \propto K^2/|J|$ for the cubic lattice. Note that for both square and cubic models, the three-magnon interactions vanish, and thus there is no magnon decay.

We should note that the cubic-compass model was studied by Belorizky et al. [11], where order-by-quantum-disorder selection was computed, as well as the gap of the pseudo-Goldstone modes due to spin-wave interactions. In fact, a similar problem, order-by-quantum-disorder in cubic dipolar ferromagnets, was studied by Tessman [10], predating by several decades the seminal works of Villain et al. [12], Henley [13] and Shender [14]. The results of Belorizky et al. [11] are in quantitative agreement with ours where they overlap (see Table I of the main text and Eq. (29) of Ref. [11]). In their work, the relationship between the zero-point curvature and the gap was noted, and an argument was given for why this occurred. Their argument has some implicit assumptions that make it (a) incomplete and (b) not generalizable to the arbitrary order-by-quantum-disorder scenarios.

### B. Heisenberg-Kitaev-$\Gamma$ models

Another Heisenberg-compass model that has attracted recent attention is the Heisenberg-Kitaev [15] model and the related Heisenberg-Kitaev-$\Gamma$ model [16] defined on the honeycomb lattice. This reads

$$
\sum_{(i,j) \in \alpha(\beta)} \left[ JS_i \cdot S_j + KS_i^\alpha S_j^\alpha + \Gamma \left( S_i^\alpha S_j^\beta + S_i^\beta S_j^\alpha \right) \right],
$$

where the bonds have been divided into three types: $(x)yz$, $(x)yz$ and $(x)yz(z)$, as defined in Ref. [16].

First consider the Heisenberg-Kitaev limit, where $\Gamma = 0$. As a function of $J$ and $K$ there are four classical phases, a ferromagnet, anti-ferromagnet and the so-called zigzag and stripy phases [17]. All of these states have an accidental classical $\text{O}(3)$ degeneracy that is lifted by quantum fluctuations. In all cases the selection pins the moment direction to lie along the cubic spin axes [15, 17]. Since the stripy and zigzag phases can be mapped to the ferromagnet and antiferromagnet via a duality, and therefore do not need to be studied separately. The ferromagnetic case is similar to the cubic compass ferromagnet, showing a type II pseudo-Goldstone gap that scales as $\Delta \propto K^2/J$. The anti-ferromagnetic case is subtler: the Heisenberg limit has two type I Goldstone modes that are lifted by the introduction of the Kitaev exchange $K$. These two modes are lifted identically and
do not mix, with their gap scaling strongly with the Kitaev exchange, going as $\Delta \propto |K|S^{1/2}$. Both these cases do not have three-magnon interactions due to a residual $C_2$ spin rotation symmetry.

The phase diagram is more complex when $\Gamma \neq 0$, with many of the accidental degeneracies of the Heisenberg-Kitaev limit lifted [16]. We consider a region near the ferromagnetic limit where $J < 0, K = 0$ and $\Gamma > 0$. Here the spins are uniform, $S_i = S \hat{n}$, in the plane perpendicular to [111] with arbitrary orientation [16]. This accidental degeneracy is lifted by the quantum zero-point energy selecting directions equivalent to $\hat{n} = (y - x)/\sqrt{2}$. The pseudo-Goldstone mode is type I, with a gap that scales as $\Delta \sim S^{1/2}/\Gamma^2$. We note that, in contrast to the other cases discussed so far, this model includes three-magnon interaction terms and thus magnon decay.

C. $J_1$-$J_2$ models

Next, we consider one of the paradigmatic models of quantum-order-by-disorder, the $J_1$-$J_2$ anti-ferromagnet on the square lattice [13, 18]

$$J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle \langle ij \rangle \rangle} S_i \cdot S_j. \quad (S83)$$

For $J_1/J_2 = 0$, one sees that the two sublattices of the square lattices decouple and form two independent Néel anti-ferromagnets, with the relative orientation unconstrained. At the classical level this remains true for large $J_2$, until the transition to the usual Néel state at $J_1 = 2J_2$. There is thus an $O(3)$ degeneracy which is lifted by quantum fluctuations, in addition to the overall $O(3)$ due to spin rotation invariance. As shown by Henley [13], these fluctuations select a colinear state, with the two sublattices either aligned or anti-aligned.

Like the Heisenberg-Kitaev case, this has a pair of type-I pseudo-Goldstone modes, but, in addition, a pair of type I true Goldstone modes that remain gapless. As shown in Table I of the main text, the gap scales as $\Delta \propto J_1 S^{1/2}$ for $J_1 \ll J_2$, matching the appropriately defined mean curvature exactly. This case has also been studied in the $S = 1/2$ limit by series expansion techniques [19]. One finds that the relevant gap to triplet excitations (with $S = 1/2$) is in reasonable agreement with our $O(1/S^2)$ spin-wave results (see Table I of the main text).

The $J_1$-$J_2$ model on the triangular lattice has similar physics, though the ground state manifold is a bit more complex [20]. For small $J_2/J_1$ one finds the usual 120$^\circ$ structure, while for large $J_2/J_1$ classically one finds an incommensurate state [21]. At intermediate values, in the range $1/8 < J_2/J_1 < 1$ the ground state consists of any state in which the sum of the four spins on each rhombus of the lattice is zero [21]. This degeneracy is lifted by order-by-disorder, picking the maximally colinear state consisting of alternating ferromagnetic stripes [13, 20]. This has a pair of type II pseudo-Goldstone modes, in addition to a pair of type I true Goldstone modes that remain gapless. As shown in Table I of the main text, the gap scales as $\Delta \propto J_1 S^0$, and is largest near the boundaries, with a minimum of $\Delta \sim 0.44J_1$ near $J_2/J_1 \sim 0.42$. 

D. \textbf{Ca}_{3}\text{Fe}_{2}\text{Ge}_{3}\text{O}_{12}

We now consider an experimental candidate for order-by-quantum-disorder in a real materials, the garnet \text{Ca}_{3}\text{Fe}_{2}\text{Ge}_{3}\text{O}_{12} [14, 22–25]. The magnetic ions in \text{Ca}_{3}\text{Fe}_{2}\text{Ge}_{3}\text{O}_{12} are (spin-only) \text{Fe}^{3+} ions ($S = 5/2$), which form a pair of interpenetrating simple cubic lattices. Each site has only a single three-fold axis, thus there are two types of nearest-neighbor exchanges ($J_1$ and $J'_1$) and one second-neighbor exchange ($J_2$) [22]. We write this model schematically as

\begin{equation}
J_1 \sum_{\langle ij \rangle_{||}} S_i \cdot S_j + J'_1 \sum_{\langle ij \rangle_{\perp}} S_i \cdot S_j + J_2 \sum_{\langle \langle ij \rangle \rangle} S_i \cdot S_j,
\end{equation}

where the first neighbors $\langle ij \rangle$ have been partitioned into those along the local three-fold axis, denoted as $\langle ij \rangle_{||}$, and those perpendicular to it, denoted as $\langle ij \rangle_{\perp}$. These parameters have been estimated by comparison with the inelastic neutron scattering spectrum at zero field, where it was found that [22]

\begin{align*}
J_1 &= 0.078 \text{ meV}, \\
J'_1 &= 0.0265 \text{ meV}, \\
J_2 &= 0.053 \text{ meV},
\end{align*}

reproduces the experimental spectrum reasonably well. The classical ground states consist of two independent Néel antiferromagnets on the two cubic sublattices.

The order-by-disorder physics in this material is a three-dimensional analogue of the $J_1$-$J_2$ square lattice model and has the same basic features: selection of colinear ordering and gapping of two type I pseudo-Goldstone modes. For the experimental parameters we find the gap is given by

\begin{equation}
\Delta = 83 \mu\text{eV} \cdot S^{1/2} + O(S^{-1/2}),
\end{equation}

giving $\sim 131 \mu\text{eV}$ for $S = 5/2$. This is in good agreement with the gap of 0.033THz = 136 \mu\text{eV} that has been observed experimentally [22]. There are some alternatives to an interpretation in terms of order-by-quantum-disorder – biquadratic exchange can directly lift the accidental degeneracy [26]. The agreement of the gap with experiment, coupled with the relatively large $S = 5/2$, suggests such additional interactions may not necessary. We note that the classical part of the curvatures can be worked out analytically, with \( \frac{\partial^2 \epsilon}{\partial \theta^2} \) must be computed numerically. This model includes only four-magnon interactions, with the three-magnon terms vanishing due to the isotropic interactions and colinear ground state.

E. \textbf{Er}_2\text{Ti}_2\text{O}_7

Finally, we consider the XY anti-ferromagnet \text{Er}_2\text{Ti}_2\text{O}_7 which is a leading candidate for experimental realization of order-by-quantum disorder. The appropriate pseudo-spin-1/2 model for the \text{Er}^{3+} ions take the form

\begin{equation}
\sum_{\langle ij \rangle} \left[ J_{zz} S_i^z S_j^z - J_{\pm} \left( S_i^+ S_j^- + S_i^- S_j^+ \right) + J_{z\pm} \left( \gamma_{ij} S_i^z S_j^+ + \text{h.c.} \right) + J_{z\pm} \left( \zeta_{ij} \left( S_i^z S_j^+ + S_i^+ S_j^z \right) + \text{h.c.} \right) \right].
\end{equation}
The exchange parameters have been determined by fitting the spin-wave spectrum in an applied magnetic field. Good agreement is found for the values \[ J_{zz} = -2.5 \cdot 10^{-2} \text{ meV}, \]
\[ J_{\pm} = +6.5 \cdot 10^{-2} \text{ meV}, \]
\[ J_{\pm\pm} = +4.2 \cdot 10^{-2} \text{ meV}, \]
\[ J_{z\pm} = -0.88 \cdot 10^{-2} \text{ meV}, \]
(S86)
where the phases \( \gamma_{ij}, \zeta_{ij} \) and lattice structure are given in detail in Ref. [26]. The classical ground states have the pseudo-spins uniformly in their local XY planes, \( S_i = S (\cos \phi \hat{x} + \sin \phi \hat{y}) \), with the overall angle, \( \phi \), arbitrary. Order-by-quantum-disorder selects the six so-called \( \psi_2 \) states, with \( \phi = n\pi/3 \) (\( n = 0, 1, \ldots, 5 \)) and yields a single type I pseudo-Goldstone mode, similar in spirit to the square compass ferromagnet discussed earlier. This has been argued to be one of the best cases for observing order-by-quantum-disorder as two-spin interactions of arbitrary range do not lift the classical degeneracy. For these exchange parameters one finds the gap to be
\[ \Delta = 44 \mu\text{eV} \cdot S^{1/2} + O(S^{-1/2}) \]
(S87)
giving \( \sim 31 \mu\text{eV} \) for \( S = 1/2 \). This number is comparable, though somewhat smaller, than the gaps that have been reported from experiment, which range from 43 \( \mu\text{eV} \) [27] to 53 \( \mu\text{eV} \) [28]. We note that the classical curvature \( \left( \frac{\partial^2 \epsilon}{\partial \theta^2} \right)_0 \) can be computed analytically, giving \( \left( \frac{\partial^2 \epsilon}{\partial \theta^2} \right)_0 = 6(2J_{\pm} + J_{z\pm})S^2 \), while \( \left( \frac{\partial^2 \epsilon}{\partial \phi^2} \right)_0 \) must be computed numerically. We note that this case includes both three- and four-magnon interactions due to the highly anisotropic exchange interactions.

### S6. CURVATURE CALCULATION EXAMPLES

In this Section we give a selection of representative cases on how to calculate the curvatures for the examples discuss in in the Table I of the main text. For the sake of brevity we adopt the shorthand
\[ \left( \frac{\partial^2 \epsilon}{\partial \theta^2} \right)_0 \equiv A_{\theta\theta}, \quad \left( \frac{\partial^2 \epsilon}{\partial \phi^2} \right)_0 \equiv A_{\phi\phi}, \quad \left( \frac{\partial^2 \epsilon}{\partial \theta \partial \phi} \right)_0 \equiv A_{\theta\phi}. \]
(S88)

#### A. Heisenberg-compass (cubic, ferromagnet)

This example is particularly simple, since there is a single sublattice and it is type II. Expanding about a state with ordering vector along the cubic \( \hat{z} \) axis it is clear that the relevant states are
\[ S_i^z = S [\phi \hat{x} + \theta \hat{y}]. \]
(S89)
We see then the curvatures correspond to simply rotating the ordering direction along the \( \hat{x} \) and \( \hat{y} \) directions. Due to the cubic symmetry this then further implies that \( A_{\theta\theta} = A_{\phi\phi} \equiv A \), with \( A_{\theta\phi} = 0 \), giving a gap equation \( \Delta = A/S \). To compute A we compute \( \epsilon_{qu}(\theta, 0) \) numerically for a series of small values of \( \theta \), fitting to a quadratic form to extract the curvature.
B. Heisenberg-Kitaev (honeycomb, Néel)

This example is type I+I, with a pair of identical pseudo-Goldstone modes and two sublattices (A and B). We focus on one of the pseudo-Goldstone modes, the other can be obtained similarly. One defines the frames

\[
\hat{x}_A = +\hat{x}, \quad \hat{y}_A = +\hat{y}, \quad \hat{z}_A = +\hat{z}, \quad \text{(S90a)}
\]

\[
\hat{x}_B = -\hat{x}, \quad \hat{y}_B = +\hat{y}, \quad \hat{z}_B = -\hat{z}. \quad \text{(S90b)}
\]

so that the uniform local \(\hat{x}_A, \hat{x}_B\) direction corresponds to a soft-mode. The second soft-mode can be obtained by switching the roles of \(\hat{x}\) and \(\hat{y}\). The spin configurations are then described by the transverse parts

\[
S^\perp_{iA} = S [+\phi \hat{x} + \theta \hat{y}], \quad \text{(S91a)}
\]

\[
S^\perp_{iB} = S [-\phi \hat{x} + \theta \hat{y}]. \quad \text{(S91b)}
\]

The classical, \(O(S^2)\) parts of the curvatures in these variables are simple, with

\[
A_{\theta\theta} = 2(3J + K)S^2, \quad \text{(S92)}
\]

and the \(A_{\phi\phi} \sim O(S)\), as given in Table I of the main text. To determine \(A_{\phi\phi}\) we computed \(\epsilon_{\text{eq}}(\phi, 0)\) numerically for the spin configurations above for a series of small values of \(\phi\). We then fit these values to a quadratic form to extract the curvature.

C. \(J-\Gamma\) model (honeycomb, ferromagnet)

This model orders ferromagnetically into the plane perpendicular to [111] and thus has a superficially more complicated structure. We define the vectors

\[
X = \frac{1}{\sqrt{6}} [\hat{x} + \hat{y} - 2\hat{z}], \quad \text{(S93a)}
\]

\[
Y = \frac{1}{\sqrt{2}} [\hat{y} - \hat{x}], \quad \text{(S93b)}
\]

\[
Z = \frac{1}{\sqrt{3}} [\hat{x} + \hat{y} + \hat{z}]. \quad \text{(S93c)}
\]

The selected ferromagnetic ground state for \(\Gamma > 0\) is along \(\hat{Y}\) (and its equivalents) so we parameterize

\[
\hat{x}_A = -\hat{X}, \quad \hat{y}_A = \hat{Z}, \quad \hat{z}_A = \hat{Y}, \quad \text{(S94a)}
\]

\[
\hat{x}_B = -\hat{X}, \quad \hat{y}_B = \hat{Z}, \quad \hat{z}_B = \hat{Y}. \quad \text{(S94b)}
\]

The relevant spin configurations are thus given by

\[
S^\perp_{iA} = S^\perp_{iB} = S [-\phi \hat{X} + \theta \hat{Z}]. \quad \text{(S95)}
\]
The classical contributions to the curvatures are thus simply given by

$$A_{\theta\theta} = 3\Gamma S^2,$$

(S96)

with $A_{\phi\phi} \sim O(S)$ generated only by quantum fluctuations, as given in Table I of the main text. To compute $A_{\phi\phi}$ we computed $\epsilon_{\text{qu}}(\phi, 0)$ numerically for the spin configurations above for a series of small values of $\phi$. We then fit these values to a quadratic form to extract the curvature.

### D. Heisenberg $J_1$-$J_2$ (square, stripe)

This example has two true type I Goldstone modes in addition to two type I pseudo-Goldstone modes. These modes are described by a unit cell with four sites, which we label as $A$ (at $0$), $B$ (at $\hat{x}$), $C$ (at $\hat{y}$) and $D$ (at $\hat{x} + \hat{y}$). The stripe, $(\pi, 0)$ order is then given by the vectors

$$\hat{z}_A = \hat{z}_D = +\hat{z}, \quad \hat{z}_B = \hat{z}_C = -\hat{z}.$$  

(S97)

The pseudo-Goldstone modes of interest are those that change the relative angle of the two sublattices. The appropriate local frames for one of these modes is then given by

$$\hat{x}_A = \hat{x}_B = +\hat{x}, \quad \hat{x}_C = \hat{x}_D = -\hat{x},$$

$$\hat{y}_A = \hat{y}_D = +\hat{y}, \quad \hat{y}_B = \hat{y}_C = -\hat{y}.$$  

(S98)

as in the Néel case, the second soft mode is identical in character, with the role of $\hat{x}$ and $\hat{y}$ axes switched. These define the soft spin configurations

$$S_{iA}^+ = S [ +\phi \hat{x} + \theta \hat{y}], \quad S_{iB}^+ = S [ +\phi \hat{x} - \theta \hat{y}],$$

$$S_{iC}^+ = S [ -\phi \hat{x} - \theta \hat{y}], \quad S_{iD}^+ = S [ -\phi \hat{x} + \theta \hat{y}].$$  

(S99)

This then yields the classical curvature for the $\theta$ direction as

$$A_{\theta\theta} = 4(2J_2 - J_1)S^2,$$  

(S100)

with the classical curvature $A_{\phi\phi} \sim O(S)$, as given in Table I of the main text. To compute $A_{\phi\phi}$ we computed $\epsilon_{\text{qu}}(\phi, 0)$ numerically for the spin configurations above for a series of small values of $\phi$. We then fit these values to a quadratic form to extract the curvature.

[1] J.-P. Blaizot and G. Ripka, *Quantum theory of finite systems*, Vol. 3 (MIT press Cambridge, 1986).
[2] M. E. Zhitomirsky and A. L. Chernyshev, Rev. Mod. Phys. **85**, 219 (2013).
[3] V. Viswanath and G. Müller, *The Recursion Method: Application to Many Body Dynamics*, Vol. 23 (Springer Science & Business Media, 1994).
[4] G. Rickayzen, *Green’s functions and condensed matter* (Courier Corporation, 2013).
[5] S. Weinberg, *The quantum theory of fields*, Vol. 2 (Cambridge university press, 1995).

[6] R. Dashen, Phys. Rev. 183, 1245 (1969).

[7] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[8] E. Witten, Nuclear Physics B 156, 269 (1979).

[9] G. Veneziano, Nuclear Physics B 159, 213 (1979).

[10] J. R. Tessman, Phys. Rev. 96, 1192 (1954).

[11] E. Belorizky, R. Casalegno, and J. Niez, physica status solidi (b) 102, 365 (1980).

[12] J. Villain, R. Bidaux, J.-P. Carton, and R. Conte, Journal de Physique 41, 1263 (1980).

[13] C. L. Henley, Phys. Rev. Lett. 62, 2056 (1989).

[14] E. Shender, Sov. Phys. JETP 56, 178 (1982).

[15] J. Chaloupka, G. Jackeli, and G. Khaliullin, Phys. Rev. Lett. 105, 027204 (2010).

[16] J. G. Rau, E. K.-H. Lee, and H.-Y. Kee, Phys. Rev. Lett. 112, 077204 (2014).

[17] J. Chaloupka, G. Jackeli, and G. Khaliullin, Phys. Rev. Lett. 110, 097204 (2013).

[18] P. Chandra and B. Doucot, Phys. Rev. B 38, 9335 (1988).

[19] R. R. P. Singh, W. Zheng, J. Oitmaa, O. P. Sushkov, and C. J. Hamer, Phys. Rev. Lett. 91, 017201 (2003).

[20] A. V. Chubukov and T. Jolicoeur, Phys. Rev. B 46, 11137 (1992).

[21] T. Jolicoeur, E. Dagotto, E. Gagliano, and S. Bacci, Phys. Rev. B 42, 4800 (1990).

[22] T. Brueckel, B. Dorner, A. Gukasov, V. Plakhty, W. Prandl, E. Shender, and O. Smirnow, Zeitschrift für Physik B Condensed Matter 72, 477 (1988).

[23] A. Gukasov, T. Brückel, B. Dorner, V. Plakhty, W. Prandl, E. Shender, and O. Smirnov, EPL (Europhysics Letters) 7, 83 (1988).

[24] T. Brückel, B. Dorner, A. Gukasov, V. Plakhty, W. Prandl, E. Shender, and O. Smirnov, Physica B: Condensed Matter 156, 308 (1989).

[25] T. Brückel, B. Dorner, A. Gukasov, and V. Plakhty, Physics Letters A 162, 357 (1992).

[26] L. Savary, K. A. Ross, B. D. Gaulin, J. P. C. Ruff, and L. Balents, Phys. Rev. Lett. 109, 167201 (2012).

[27] E. Lhotel, J. Robert, E. Ressouche, F. Damay, I. Mirebeau, J. Ollivier, H. Mutka, P. Dalmas de Réotier, A. Yaouanc, C. Marin, C. Decorse, and S. Petit, Phys. Rev. B 95, 134426 (2017).

[28] K. A. Ross, Y. Qiu, J. R. D. Copley, H. A. Dabkowska, and B. D. Gaulin, Phys. Rev. Lett. 112, 057201 (2014).