The Mellin transform of the square of Riemann’s zeta-function

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Abstract. Let \( Z_1(s) = \int_1^\infty |\zeta(\frac{1}{2} + ix)|^2 x^{-s} \, dx \) \((\sigma = \Re s > 1)\). A result concerning analytic continuation of \( Z_1(s) \) to \( \mathbb{C} \) is proved, and also a result relating the order of \( Z_1(\sigma + it) \) \((\frac{1}{2} \leq \sigma \leq 1, \, t \geq t_0)\) to the order of \( Z_1(\frac{1}{2} + it) \).

1. Introduction

Let \( Z_k(s) \), the (modified) Mellin transform of \(|\zeta(\frac{1}{2} + ix)|^{2k}\), denote the analytic continuation of the function defined initially by

\[
Z_k(s) := \int_1^\infty |\zeta(\frac{1}{2} + ix)|^{2k} x^{-s} \, dx \quad (k \in \mathbb{N}, \, \sigma = \Re s > c(k) \,(>1)).
\]

This function, when \( k = 2 \), was introduced by Y. Motohashi [15] (see also [16]), and its properties were further studied in [10] and [11]. The latter work also contains some results on the function \( Z_1(s) \), which is the principal object of the study in this paper. It was shown that \( Z_1(s) \) is regular for \( \sigma > -3/4 \), except for a double pole at \( s = 1 \). The principal part of the Laurent expansion of \( Z_1(s) \) at \( s = 1 \) is

\[
\frac{1}{(s-1)^2} + \frac{2\gamma - \log(2\pi)}{s-1},
\]

where \( \gamma = -\Gamma'(1) = 0.577215\ldots \) is Euler’s constant. M. Jutila [13] continued the study of \( Z_1(s) \) and proved that \( Z_1(s) \) continues meromorphically to \( \mathbb{C} \), having only a double pole at \( s = 1 \) and at most double poles for \( s = -1, -2, \ldots \).

The object of this note is to prove some new results on \( Z_1(s) \). First we make more precise Jutila’s result on the analytic continuation of \( Z_1(s) \). We have the following

THEOREM 1. The function \( Z_1(s) \) continues meromorphically to \( \mathbb{C} \), having only a double pole at \( s = 1 \), and at most simple poles at \( s = -1, -3, \ldots \). The principal part of its Laurent expansion at \( s = 1 \) is given by (1.2).

Our second aim is to prove an order result for \( Z_1(s) \). This is a result M. Jutila mentioned in [13], but the term that I initially claimed, namely \( t^{1-2\sigma+\varepsilon} \) is too optimistic. It appears that what can be proved is contained in

THEOREM 2. We have, for \( \frac{1}{2} \leq \sigma \leq 1, \, t \geq t_0 > 0 \),

\[
Z_1(\sigma + it) \ll \varepsilon t^{\frac{1}{2}-\sigma+\varepsilon} \max_{t-t^\varepsilon \leq v \leq t+t^\varepsilon} |Z_1(\frac{1}{2} + iv)| + (t^{\frac{9-16\varepsilon}{16\varepsilon}} + t^{-1}) \log t.
\]

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Corollary (see M. Jutila [13]). For \( \frac{1}{2} \leq \sigma \leq 1, \ t \geq t_0 \) we have

\[
Z_1(\sigma + it) \ll \varepsilon t^{\frac{1}{6}-\sigma+\varepsilon}.
\]

The bound \( Z_1(\frac{1}{2} + it) \ll \varepsilon t^{1/3+\varepsilon} \) was mentioned in [11], and its proof was elaborated by M. Jutila in [13]. If one inserts this bound in (1.3), then (1.4) follows immediately. We note that here and later \( \varepsilon \) denotes arbitrarily small constants, not necessarily the same ones at each occurrence. It can be also proved that, for any given \( \varepsilon > 0 \),

\[
\int_1^T |Z_k(\sigma + it)|^2 \, dt \gg \varepsilon T^{2-2\sigma-\varepsilon} \quad (k = 1, 2; \ \frac{1}{2} < \sigma < 1)
\]

The bound (1.5) for \( k = 2 \) appeared in my paper [10], and the proof of the bound when \( k = 1 \) is on similar lines, so that the details will not be given here. It is plausible that

\[
\max_{t-t^\varepsilon \leq v \leq t+t^\varepsilon} |Z_1(\frac{1}{2} + iv)| \gg 1
\]

holds, but at present I am unable to prove (1.6).

2. Proof of Theorem 1

Let

\[
L_k(s) := \int_0^\infty |\zeta(\frac{1}{2} + ix)|^{2k} e^{-sx} \, dx \quad (\sigma = \Re s > 0, \ k \in \mathbb{N})
\]

be the Laplace transform of \( |\zeta(\frac{1}{2} + ix)|^{2k} \). We are interested in using the expression for \( L_1(s) \) when \( s = 1/T, \ T \to \infty \). Such a formula has been known for a long time and is due to H. Kober [14] (see also a proof in [17, Chapter 9]). The functions \( L_k(s) \) when \( k = 1, 2 \) were studied by F.V. Atkinson [1], [2]. The function \( L_2(s) \) was considered by the author [8], [9], and the approach to mean value of Dirichlet series by Laplace transforms by M. Jutila [12]. Kober’s result on \( L_1(s) \) is that, for any integer \( N \geq 0 \),

\[
L_1(\sigma) = \frac{\gamma - \log(2\pi \sigma)}{2 \sin \frac{\sigma}{2}} + \sum_{n=0}^N d_n \sigma^n + O_N(\sigma^{N+1}) \quad (\sigma \to 0+)
\]

where \( d_n \) are suitable constants. Note that for \( \sigma = 1/T, \ T \to \infty \) we have

\[
\frac{\gamma - \log \left( \frac{2\pi}{T} \right)}{2 \sin \left( \frac{1}{2T} \right)} = \frac{\log \left( \frac{T}{2\pi} \right) + \gamma}{T - \frac{1}{24T^3} + \frac{1}{16 \cdot 3^3 T^5} - \cdots}
\]

\[
= \left( \log \left( \frac{T}{2\pi} \right) + \gamma \right) \sum_{n=0}^{\infty} c_n T^{1-2n}
\]
with the coefficients $c_n$ that may be explicitly evaluated. Therefore for any integer $N \geq 0$

\begin{equation}
L_1 \left( \frac{1}{T} \right) = \left( \log \left( \frac{T}{2\pi} \right) + \gamma \right) \sum_{n=0}^{N} a_n T^{1-2n} + \sum_{n=0}^{N} b_n T^{-2n} + O_N(T^{-1-2N} \log T)
\end{equation}

with the coefficients $a_n, b_n$ that may be explicitly evaluated. In particular, we have

\begin{equation}
a_0 = 1, \quad b_0 = \pi.
\end{equation}

It is clear from Kober’s formula (2.2) that $a_0 = 1$. To see that $b_0 = \pi$ one can use the work
of Hafner–Ivić [4] (this was mentioned by Conrey et al. [3]), which will be stated now in
detail, as it will be needed also for the proof of Theorem 2. Let, as usual, for $T \geq 0$,

\begin{align*}
E(T) &= \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^2 \, dt - T \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right) \quad (E(0) = 0)
\end{align*}

denote the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|$. Let further

\begin{equation}
G(T) := \int_{0}^{T} E(t) \, dt - \pi T, \quad G_1(T) := \int_{0}^{T} G(t) \, dt.
\end{equation}

Then Hafner–Ivić [4] proved (see also [6])

\begin{equation}
G(T) = S_1(T; N) - S_2(T; N) + O(T^{1/4})
\end{equation}

with

\begin{align*}
S_1(T; N) &= 2^{-3/2} \sum_{n \leq N} (-1)^n d(n)n^{-1/2} \left( \text{arsinh} \sqrt{\frac{\pi n}{2T}} \right)^{-2} \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{-1/4} \sin(f(T, n)), \\
S_2(T; N) &= \sum_{n \leq N'} d(n)n^{-1/2} \left( \log \frac{T}{2\pi n} \right)^{-2} \sin(g(T, n)), \\
f(T, n) &= 2T \text{arsinh} \left( \sqrt{\frac{\pi n}{2T}} \right) + \sqrt{2\pi nT + \pi^2 n^2 - \frac{\pi}{4}}, \\
g(T, n) &= T \log \left( \frac{T}{2\pi n} \right) - T + \frac{\pi}{4}, \quad \text{arsinh} x = \log(x + \sqrt{x^2 + 1}),
\end{align*}

$AT < N < A'T$ ($0 < A < A'$ constants), $N' = \frac{T}{2\pi} + \frac{N}{2} - \sqrt{\frac{N^2}{4} + \frac{NT}{2\pi}}$.

We use Taylor’s formula (see [7, Lemma 3.2]) to simplify (2.6). Then we obtain

\begin{equation}
G(T) = 2^{-1/4} \pi^{-3/4} T^{3/4} \sum_{n=1}^{\infty} (-1)^n d(n)n^{-5/4} \sin(\sqrt{8\pi nT} - \frac{\pi}{4}) + O(T^{2/3} \log T),
\end{equation}
so that

\[
G(T) = O(T^{3/4}), \quad G(T) = \Omega_\pm(T^{3/4}).
\]

We use (2.6) with \( T = t, N = T, T \leq t \leq 2T \) and apply the first derivative test ([5, Lemma 2.1]) to deduce that

\[
\int_T^{2T} G(t) \, dt \ll T^{5/4},
\]

since the term \( O(1) \) in [5, eq. (15.29)] in Atkinson’s formula is in fact \( O(1/T) \). Hence

\[
G_1(T) = \int_0^T G(t) \, dt \ll T^{5/4}.
\]

To prove that \( b_0 = \pi \) (cf. (2.4)) we note that, using the definition of \( E(T) \) and applying integration by parts,

\[
L_1 \left( \frac{1}{T} \right) = \int_0^\infty e^{-t/T} \left( \log \left( \frac{t}{2\pi} \right) + 2\gamma + E'(t) \right) \, dt
= T \int_0^\infty e^{-x}(\log x + \log \frac{T}{2\pi} + 2\gamma) \, dx + \frac{1}{T} \int_0^\infty E(t)e^{-t/T} \, dt
\]

\[
= T \left( \log \left( \frac{T}{2\pi} \right) + \gamma \right) + \frac{1}{T^2} \int_0^t \int_0^\infty E(u) \, du \cdot e^{-t/T} \, dt
\]

\[
= T \left( \log \left( \frac{T}{2\pi} \right) + \gamma \right) + \frac{1}{T^2} \int_0^\infty (\pi t + O(t^{3/4}))e^{-t/T} \, dt
\]

\[
= T \left( \log \left( \frac{T}{2\pi} \right) + \gamma \right) + \pi + O(T^{-1/4}),
\]

where we also used the \( O \)-bound of (2.7) and

\[
\int_0^\infty e^{-x} \log x \, dx = \Gamma'(1) = -\gamma.
\]

A comparison of (2.3) and (2.9) proves that \( b_0 = \pi \), as asserted by (2.4).

We return now to the proof of Theorem 1. Let

\[
\hat{L}_k(s) := \int_1^\infty |\zeta(\frac{1}{2} + iy)|^{2k}e^{-ys} \, dy \quad (k \in \mathbb{N}, \sigma = \Re s > 0).
\]

Then we have by absolute convergence, taking \( \sigma \) sufficiently large and making the change of variable \( xy = t \),

\[
\int_0^\infty \hat{L}_k(x)x^{s-1} \, dx = \int_0^\infty \left( \int_1^\infty |\zeta(\frac{1}{2} + iy)|^{2k}e^{-xy} \, dy \right) x^{s-1} \, dx
\]

\[
= \int_1^\infty |\zeta(\frac{1}{2} + iy)|^{2k} \left( \int_0^\infty x^{s-1}e^{-xy} \, dx \right) \, dy
\]

\[
= \int_1^\infty |\zeta(\frac{1}{2} + iy)|^{2k}y^{-s} \, dy \int_0^\infty e^{-t}t^{s-1} \, dt = Z_k(s)\Gamma(s).
\]
Further we have
\[ \int_0^\infty \bar{L}_1(x)x^{s-1} \, dx = \int_0^1 \bar{L}_1(x)x^{s-1} \, dx + \int_1^\infty \bar{L}_1(x)x^{s-1} \, dx \]
\[ = \int_1^\infty \bar{L}_1(1/x)x^{-1-s} \, dx + A(s) \quad (\sigma > 1), \]
where
\[ A(s) := \int_1^\infty \bar{L}_1(x)x^{s-1} \, dx \]
is an entire function. Since
\[ \bar{L}_1(1/x) = L_1(1/x) - \int_0^1 |\zeta(1/2 + iy)|^2 e^{-y/x} \, dy \quad (x \geq 1), \]
it follows from (2.11) with \( k = 1 \) by analytic continuation that, for \( \sigma > 1 \),
(2.12)
\[ Z_1(s)\Gamma(s) = \int_1^\infty L_1(1/x)x^{-1-s} \, dx - \int_1^\infty \left( \int_0^1 |\zeta(1/2 + iy)|^2 e^{-y/x} \, dy \right) x^{-1-s} \, dx + A(s) \]
\[ = I_1(s) - I_2(s) + A(s), \]
say. Clearly for any integer \( M \geq 1 \)
(2.13)
\[ I_2(s) = \int_1^\infty \int_0^1 |\zeta(1/2 + iy)|^2 \left( \sum_{m=0}^{M} \frac{(-1)^m}{m!} \left( \frac{y}{x} \right)^m + O_M(x^{-M-1}) \right) \, dy \, x^{-1-s} \, dx \]
\[ = \sum_{m=0}^{M} \frac{(-1)^m}{m!} h_m \cdot \frac{1}{m+s} + H_M(s), \]
say, where \( H_M(s) \) is a regular function of \( s \) for \( \sigma > -M - 1 \), and
\[ h_m := \int_0^1 |\zeta(1/2 + iy)|^2 y^m \, dy \]
is a constant. Inserting (2.3) in \( I_1(s) \) in (2.12) we have, for \( \sigma > 1 \),
(2.14)
\[ I_1(s) = \int_1^\infty \left( \log \frac{x}{2\pi} + \gamma \right) \sum_{n=0}^{N} a_n x^{-2n-s} \, dx + \int_1^\infty \sum_{n=0}^{N} b_n x^{-1-2n-s} \, dx + K_N(s) \]
\[ = \sum_{n=0}^{N} a_n \left( \frac{1}{(2n+s-1)^2} + \frac{\gamma - \log 2\pi}{2n+s-1} \right) + K_N(s), \]
say, where \( K_N(s) \) is regular for \( \sigma > -2N \). Taking \( M = 2N \) it follows from (2.12)–(2.14) that
(2.15)
\[ Z_1(s)\Gamma(s) = \sum_{n=0}^{N} a_n \left( \frac{1}{(2n+s-1)^2} + \frac{\gamma - \log 2\pi}{2n+s-1} \right) \]
\[ + \sum_{m=0}^{2N} \frac{(-1)^m}{m!} h_m \cdot \frac{1}{m+s} + R_N(s), \]
say, where $R_N(s)$ is a regular function of $s$ for $\sigma > -2N$. This holds initially for $\sigma > 1$, but by analytic continuation it holds for $\sigma > -2N$. Since $N$ is arbitrary and $\Gamma(s)$ has no zeros, it follows that (2.15) provides meromorphic continuation of $Z_1(s)$ to $\mathbb{C}$. Taking into account that $\Gamma(s)$ has simple poles at $s = -m$ ($m = 0, 1, 2, \ldots$) with residues $(-1)^m / m!$, and that near $s = 1$ we have the Taylor expansion

$$
\frac{1}{\Gamma(s)} = 1 + \gamma(s - 1) + \sum_{n=2}^{\infty} \frac{f_n}{n!} (s - 1)^n
$$

with $f_n = (1/\Gamma(s))^{(n)}|_{s=1}$, it follows that

$$
Z_1(s) = \frac{1}{(s-1)^2} + \frac{2\gamma - \log(2\pi)}{s-1}
+ \frac{1}{\Gamma(s)} \left\{ \sum_{n=1}^{N} a_n \left( \frac{1}{(2n+s-1)^2} + \frac{\gamma - \log 2\pi}{2n+s-1} \right) \right\}
+ \frac{1}{\Gamma(s)} \left( \sum_{m=0}^{2N} \frac{(-1)^m}{m!} h_m \cdot \frac{1}{m+s} \right) + U_N(s),
$$

(2.16)
say, where $U_N(s)$ is a regular function of $s$ for $\sigma > -2N$. Moreover, the function $1/(\Gamma(s)(2n+s-1))$ is regular for $n \in \mathbb{N}$ and $s \in \mathbb{C}$. Therefore (2.16) provides analytic continuation to $\mathbb{C}$, showing that besides $s = 1$ the only poles of $Z_1(s)$ can be simple poles at $s = 1 - 2n$ for $n \in \mathbb{N}$, as asserted by Theorem 1. With more care the residues at these poles could be explicitly evaluated.

### 3. Proof Theorem 2

We suppose that $X \gg t$ and start from

$$
Z_1(s) = \left( \int_1^X + \int_X^{\infty} \right) |\zeta(\frac{1}{2} + ix)|^2 x^{-s} \, dx \quad (\sigma = \Re s > 1).
$$

We use $|\zeta(\frac{1}{2} + ix)|^2 = \log(x/2\pi) + 2\gamma + (E(x) - \pi)'$ and integrate by parts to obtain

$$
Z_1(s) = \int_1^{t_{1-\varepsilon}} |\zeta(\frac{1}{2} + ix)|^2 x^{-s} \, dx + \int_{t_{1-\varepsilon}}^X |\zeta(\frac{1}{2} + ix)|^2 x^{-s} \, dx
+ \frac{X^{1-s}}{s-1} \left( \frac{1}{s-1} + \log X + 2\gamma - \log 2\pi \right)
- (E(X) - \pi)X^{-s} + s \int_{X}^{\infty} (E(x) - \pi)x^{-s-1} \, dx
= I_1 + I_2 + O(X^{-\sigma} + t^{-1}X^{1-\sigma} \log X) + I_3,
$$

(3.1)
provided \( E(X) = 0 \). But as proved in \([6]\), every interval \([T, T + C\sqrt{T}]\) with sufficiently large \( C > 0 \) contains a zero of \( E(t) \), hence we can assume that \( E(X) = 0 \) is fulfilled with suitable \( X \). Since \( E(x) \) is \( \asymp x^{1/4} \) in mean square (see e.g., \([5]\) or \([7]\)), it follows by analytic continuation that (3.1) is valid for \( \sigma > 1 \). By repeated integration by parts it is found that \( I_1 \ll t^{-1} \). The integral \( I_2 \) is split in \( O(\log T) \) subintegrals of the form

\[
J_K = \int_{K/2}^{5K/2} \varphi(x)|\zeta(\frac{1}{2} + ix)|^2 x^{-s} \, dx,
\]

where \( \varphi(x) \in C^\infty \) is a non-negative, smooth function supported in \([K/2, 5K/2]\) that is equal to unity in \([K, 2K]\). To bound \( J_K \) one can start from (1.1) with \( k = 1 \) and use the Mellin inversion formula (see \([10]\) for a detailed discussion concerning the analogous situation with the fourth moment of \( |\zeta(\frac{1}{2} + it)| \)), which yields

\[
|\zeta(\frac{1}{2} + ix)|^2 = \frac{1}{2\pi i} \int_{(1+\varepsilon)} \mathcal{Z}_1(s)x^{s-1} \, ds \quad (x > 1).
\]

Here we replace the line of integration by the contour \( \mathcal{L} \), consisting of the same straight line from which the segment \([2 + \varepsilon - i, 1 + \varepsilon + i]\) is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole \( s = 1 \) of the integrand. By the residue theorem we have

\[
|\zeta(\frac{1}{2} + ix)|^2 = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_1(s)x^{s-1} \, ds + \log \left( \frac{x}{2\pi} \right) + 2\gamma \quad (x > 1).
\]

This gives

\[
J_K = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_1(w) \int_{K/2}^{5K/2} \varphi(x)x^{w-s-1} \, dx \, dw + O(t^{-1}).
\]

Repeated integration by parts in the \( x \)-integral shows that only the portion \(|3m - 3m s| \leq t^\varepsilon \) gives a non-negligible contribution. Hence replacing \( \mathcal{L} \) by the line \( \Re w = \frac{1}{2} \) we obtain

\[
J_K \ll \varepsilon \quad \max_{t-t^\varepsilon \leq v \leq t+t^\varepsilon} |\mathcal{Z}_1(\frac{1}{2} + iv)| + t^{-1}.
\]

To estimate \( I_3 \) in (3.1) we integrate by parts and use (2.7) and (2.8). We obtain

\[
\int_X^\infty (E(x) - \pi)x^{-s-1} \, dx = -G(X)X^{-s-1} + (s + 1) \int_X^\infty G(x)x^{-s-2} \, dx
\]

\[
= O(X^{-1/4 - \sigma}) - (s + 1)G_1(X)X^{-s-2} + (s + 1)(s + 2) \int_X^\infty G_1(x)x^{-s-3} \, dx
\]

\[
\ll X^{-1/4 - \sigma} + t^2 X^{-3/4 - \sigma}.
\]
Thus it follows from (3.1) and (3.2) that we have, for \( \frac{1}{2} \leq \sigma \leq 1, \ t \geq t_0, \)

\[
Z_1(\sigma + it) \ll_{\varepsilon} t^{\frac{1}{2} - \sigma + \varepsilon} \max_{t - t_0 \leq u \leq t + t_0} |Z_1(\frac{1}{2} + iv)| + (X^{1-\sigma}t^{-1} + t^{-1}) \log t
\]

+ \( X^{-\sigma} + tX^{-1/4-\sigma} + t^3X^{-3/4-\sigma}. \) \tag{3.3}

In (3.3) we choose \( X \) to satisfy

\[
X^{1-\sigma}t^{-1} = t^3X^{-3/4-\sigma}, \quad \text{i.e.,} \quad X = t^{16/7}.
\]

Then \( tX^{-1/4-\sigma} + X^{-\sigma} \ll t^3X^{-3/4-\sigma}, \) and (1.3) follows from (3.3).

In conclusion, one may ask what should be the true order of magnitude of \( Z_1(\sigma + it) \). This is a difficult question, but it seems reasonable to expect that

\[
Z_1(\sigma + it) \ll_{\varepsilon} t^{\frac{1}{2} - \sigma + 2\mu(\frac{1}{2}) + \varepsilon} \quad (\frac{1}{2} \leq \sigma \leq 1, \ t \geq t_0 > 0)
\] \tag{3.4}

holds, where

\[
\mu(\sigma) := \limsup_{t \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}
\]

is the Lindelöf exponent of \( \zeta(s) \). One could try to obtain (3.4) from (1.3) by refining the estimation of \( Z_1(s) \) in [11]. This procedure leads to an exponential sum with the divisor function which is estimated as \( \ll_{\varepsilon} t^{2\mu(\frac{1}{2}) + \varepsilon} \) (this, of course, is non-trivial and needs elaboration).

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