$L^p - L^q$ ESTIMATES OF BERGMAN PROJECTOR ON THE MINIMAL BALL

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Abstract. We study the $L^p - L^q$ boundedness of Bergman projector on the minimal ball. This improves an important result of [5] due to G. Mengotti and E. H. Youssfi.

1. Introduction

G. Mengotti and E. H. Youssfi studied in [5] the $L^p - L^q$ boundedness of Bergman projector on the minimal ball. Here we improve their result by giving the $L^p - L^q$ boundedness of Bergman projector.

The minimal ball $B_*$ in $\mathbb{C}^n$ is defined as follows.

$$B_* = \{ z \in \mathbb{C}^n : |z|^2 + |z \cdot z| < 1 \}$$

where $z \cdot w = \sum_{j=1}^{n} z_j w_j$ for $z$ and $w$ in $\mathbb{C}^n$. This is the unit ball of $\mathbb{C}^n$ with respect to the norm $N_r(z) := \sqrt{|z|^2 + |z \cdot z|}$. The norm $N := N_r/\sqrt{2}$ was introduced by Hahn and Pflug in [4], where it was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidean norm in $\mathbb{R}^n$. More precisely, if $N$ is any complex norm in $\mathbb{C}^n$ such that $N(x) = |x| = \sqrt{\sum_{j=1}^{n} x_j^2}$ for $x \in \mathbb{R}^n$ and $N(z) \leq |z|$ for $z \in \mathbb{C}^n$, then $N_r(z)/\sqrt{2} \leq N(z)$ for $z \in \mathbb{C}^n$. Moreover, this norm was shown to be of interest in the study of several problems related to proper holomorphic mappings and the Bergman kernel, see for example [2, 3, 5, 6]. The domain $B_*$ is the first bounded domain in $\mathbb{C}^n$ which is neither Reinhardt nor homogeneous, and for which we have an explicit formula for its Bergman kernel. The study of $L^p - L^q$ estimates of Bergman projector on smooth homogeneous is rather well understood on the unit ball and Siegel domains, see for example [10], [11], [7], etc.

The authors of [5] developed a method for $L^p - L^q$ boundedness of Bergman projector on the minimal ball. Their argue consists to study the boundedness on an auxiliary complex manifold $M$. Next, to transfer the results obtained on $M$ to $B_*$ via a proper holomorphic mapping. Our strategy combine the method of [5], [10] and a new ingredient.

The plane of our research is the following. We first study the boundedness of certain class of integral operators on $M$ by using the generalized Schur’s test (see [10]) and the Forelli-Rudin estimates (see [5]). As consequence we obtain the Bergman projector estimate in $M$. Second we transplant the results obtained on $M$ to Bergman projector on the minimal ball.

2. Preliminaries

We first define the auxiliary complex manifold $M$. Let $n \geq 2$ and consider the nonsingular cone

$$H := \{ z \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 0, \ z \neq 0 \}.$$ 

This is the orbit of the vector $(1, i, 0, \ldots, 0)$ under the $SO(n+1, \mathbb{C})$--action on $\mathbb{C}^{n+1}$. It is well-known that $H$ can be identified with the cotangent bundle of the unit sphere $S^n$ in the

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Induced by the Haar probability measure of \(O\) \((2.1)\) we have, from \([5, Lemma\ 2.1]\), that

**Theorem A.** Let

\[
\text{The complex manifold } M \text{ is defined by}
\]

\[
M = \{ z \in \mathbb{H} : |z| < 1 \}
\]

The orthogonal group \(O(n+1, \mathbb{R})\) acts transitively on the manifold

\[
\partial M = \{ z \in \mathbb{C}^{n+1} : z \cdot z = 0 \text{ and } |z| = 1 \}
\]

Thus there is a unique \((n+1, \mathbb{R})\)–invariant probability measure \(\mu\) on \(M\). This measure is induced by the Haar probability measure of \(O(n+1, \mathbb{R})\) (see \([5]\)). For any \(C^\infty\) function \(f\) on \(\mathbb{H}\) we have, from \([5, Lemma\ 2.1]\), that

\[
\int_{\mathbb{H}} f(z)\alpha(z) \wedge \overline{\alpha(z)} = m_n \int_{0}^{\infty} t^{2n-3} \int_{\partial M} f(t\xi)d\mu(\xi)dt
\]

provided that the integrals make sense. Moreover

\[
m_n = 2(n-1) \int_{z \in M} \alpha(z) \wedge \overline{\alpha(z)}.
\]

For all \(0 < p < \infty\) we consider Lebesgue space \(L^p(M)\) on the measure space \((M, (1 - |z|^2)^s \alpha(z) \wedge \overline{\alpha(z)})\). The Bergman space \(A^2(M)\) is the subspace of \(L^2(M)\) consisting of holomorphic functions. \(A^2(M)\) is the closed subspace of the Hilbert space \(L^2(M)\). There exists a unique orthogonal projection \(L^2(M)\) onto \(A^2(M)\). That is the weighted Bergman projection. Its explicit expression is the following.

\[
P_{s,M}f(z) = \int_{M} K_{s,M}(z,w)f(w)(1 - |w|^2)^s \alpha(w) \wedge \overline{\alpha(w)}
\]

where the so called kernel Bergman \(K_{s,M}\) (see \([5, Theorem\ 3.2]\)) is given by

\[
K_{s,M}(z,w) = C(n - 1 + (n + 1 + 2s)z \cdot \overline{w}).
\]

Here \(C\) is a certain constant that depends on \(n\) and \(s\). In this paper we consider the class of operators defined as follow.

\[
S_{M}f(z) = (1 - |z|^2)^{b_1} \int_{M} \frac{f(w)}{(1 - z \cdot \overline{w})^c}(1 - |w|^2)^{b_2} \alpha(w) \wedge \overline{\alpha(w)}
\]

and

\[
T_{M}f(z) = (1 - |z|^2)^{b_1} \int_{M} \frac{f(w)}{|1 - z \cdot \overline{w}|^c}(1 - |w|^2)^{b_2} \alpha(w) \wedge \overline{\alpha(w)}
\]

where \(b_1, b_2\) and \(c\) are any real numbers.

### 3. Statement of the auxiliaries results

The following auxiliaries results will play a key role for proving the main result of the paper.

**Theorem A.** Let \(b_1, b_2\) and \(c\) be real numbers. Let \(1 < p \leq q < \infty\), \(\max(-1, -1 - qb_1) < r < \infty\) and \(-1 < s < \infty\). Then the following assertions are equivalent.

(i) The operator \(T_{M}\) is bounded from \(L^p(M)\) to \(L^q(M)\).

(ii) The operator \(S_{M}\) is bounded from \(L^p(M)\) to \(L^q(M)\).
(iii) The parameters satisfy
\[ \begin{cases} 
  s + 1 < p(b_2 + 1) \\
  c \leq b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{p} 
\end{cases} \]

**Theorem B.** Let $b_1$, $b_2$ and $c$ be real numbers. Let $1 = p \leq q < \infty$, $\max(-1, -1 - qb_1) < r < \infty$ and $-1 < s < \infty$. Then the following assertions are equivalent.

(i) The operator $T_M$ is bounded from $L^1(B^q)$ to $L^p(B^r)$.

(ii) The operator $S_M$ is bounded from $L^1(B^q)$ to $L^p(B^r)$.

(iii) The parameters satisfy
\[ \begin{cases} 
  s < b_2 \\
  c = b_1 + b_2 - s + \frac{n+1+r}{q} \text{ or } s \leq b_2 \\
  c < b_1 + b_2 - s + \frac{n+1+r}{q} 
\end{cases} \]

4. **Statement of the main result**

To state the main result we need the following definitions. For any $s > 1$ we define the weighted measure $\nu_s$ on $B^*$ by $\nu_s(z) = (1 - N^2_s(z))^{-1} dv(z)$ where $v$ is the normalized Lebesgue measure on $B^*$. For all $0 < p < \infty$ we consider Lebesgue space $L^p(B^*)$ on the measure space $(B^*, |z| \cdot |z|^{-1} dv)$. The Bergman space $A^p(B^*)$ is the subspace of $L^p(B^*)$ consisting of holomorphic functions. It is well-known for $p = 2$ there exists a unique orthogonal projection from $L^2(B^*)$ onto $A^2(B^*)$. That is so called weighted Bergman projection and denoted $P_{s, B^*}$. Also, it is well-known that $P_{s, B^*}$ is an integral operator on $L^2(B^*)$. More precisely

\[ P_{s, B^*}(f) = \int_{B^*} K_{s, B^*}(z, w)f(w)dv_s(w) \]

and the so called Bergman kernel $K_{s, B^*}$ is explicitly given in [5, Theorem A] by

\[ K_{s, B^*}(z, w) = \frac{1}{(n^2 + n - s)v_s(B^*)} \frac{A(1 - z \cdot \bar{w}, z \cdot \bar{z})}{((1 - z \cdot \bar{w})^2 - z \cdot \bar{z})(1 - z \cdot \bar{w} - n + 1 + s)} \]

where

\[ A(X, Y) = \sum_{k=0}^{\infty} \binom{n + s + 1}{2k + 1} X^{n+s-2k-1}Y^k \]

\[ \times \left[ 2(n + s) - \frac{(n + 1 + 2s)(n + s - 2k)}{n + s + 1} (X^2 - Y) \right] \]

with

\[ \binom{n + s + 1}{2k + 1} = \frac{(n + s + 1)(n + s) \cdots (n + s - 2k + 1)}{(2k + 1)!} \]

The main result of the paper is the following.

**Theorem C.** Let $1 \leq p \leq q < \infty$, $-1 < \lambda, \hat{\lambda} < \infty$.

(i) For $1 < p \leq q < \infty$ the Bergman projector $P_{s, B^*}$ is bounded from $L^p(B^*)$ into $A^q_{\lambda}(B^*)$ if and only if

\[ \frac{\lambda + 1}{p} \leq \frac{\lambda + 1 + \hat{\lambda}}{q} \]

(ii) For $1 = p \leq q < \infty$ the Bergman projector $P_{s, B^*}$ is bounded from $L^1(B^*)$ into $A^q_{\lambda}(B^*)$ if and only if

\[ \frac{\lambda}{p} \geq n + \lambda \quad \text{or} \quad \frac{\lambda + 1 + \hat{\lambda}}{q} > n + 1 + \hat{\lambda} \]
To prove our results we need the following results.

**Lemma 4.1.** [5] Lemma 5.1] Let \( d \in \mathbb{N} \). For \( z \in \mathbb{M} \), \( c \in \mathbb{R} \), \( s > -1 \), define
\[
I_c(z) = \int_{\partial M} \frac{|z \bullet \xi|^{2d}}{|1 - z \bullet \xi|^{n+c}} d\mu(\xi)
\]
and
\[
J_{c,s}(z) = \int_{\partial M} \frac{|z \bullet w|^{2d}}{|1 - z \bullet w|^{n+c+s+t}} (1 - |w|^2)^s \alpha(w) \land \alpha(w)
\]
When \( c < 0 \), then \( I_c \) and \( J_{c,s} \) are bounded in \( \mathbb{M} \). When \( c > 0 \) then \( I_c(z) \simeq (1 - |z|^2)^{-c} \simeq J_{c,s}(z) \).

Finally, \( I_0(z) \simeq \log \frac{1}{1-|z|^2} \simeq J_{0,s}(z) \)

**Remark 4.2.** The symbol \( u(z) \simeq v(z) \) means that \( u(z)/v(z) \) has finite limit as \( |z| \) tends to 1.

The following results are the boundedness criterions for integral operators from \( L^p \) into \( L^q \) called generalize Schur’s test.

**Theorem 4.3.** [10] Theorem 1] Let \( \nu_1 \) and \( \nu_2 \) be positive measures on the space \( \Omega \) and let \( K(z,w) \) be a non-negative measurable function on \( \Omega \times \Omega \). Let \( T \) be the integral operator with kernel \( K \), defined as follows.
\[
Tf(z) = \int_{\Omega} f(w)K(z,w)d\nu_1(w)
\]
Suppose \( 1 < p \leq q < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and suppose there exist \( \gamma \) and \( \delta \) such that \( \gamma + \delta = 1 \). If there exist positive functions \( h_1 \) and \( h_2 \) with positive constants \( C_1 \) and \( C_2 \),
\[
\int_{\Omega} h_1(w)^\gamma K(z,w)^{p\gamma} d\nu_1(w) \leq C_1 h_2(z)^{p\gamma}
\]
for almost all \( z \in \Omega \), and
\[
\int_{\Omega} h_2(z)^{\delta} K(z,w)^{q\delta} d\nu_2(z) \leq C_1 h_1(w)^{q\delta}
\]
for almost all \( w \in \Omega \), then \( T \) is bounded from \( L^p(\Omega,\nu_1) \) into \( L^q(\Omega,\nu_2) \) and the norm of the operator does not exceed \( C_1^{1/p} C_2^{1/q} \).

**Theorem 4.4.** [10] Theorem 2] Let \( \nu_1 \) and \( \nu_2 \) be positive measures on the space \( \Omega \) and let \( K(z,w) \) be a non-negative measurable function on \( \Omega \times \Omega \). Let \( T \) be the integral operator with kernel \( K \), defined as follows.
\[
Tf(z) = \int_{\Omega} f(w)K(z,w)d\nu_1(w)
\]
Suppose \( 1 = p \leq q < \infty \) and suppose there exist \( \gamma \) and \( \delta \) such that \( \gamma + \delta = 1 \). If there exist positive functions \( h_1 \) and \( h_2 \) with positive constants \( C_1 \) and \( C_2 \) such that
\[
\text{ess sup}_{w \in \Omega} h_1(w)K(z,w)^\gamma d\nu_1(w) \leq C_1 h_2(z)
\]
for almost all \( z \in \Omega \), and
\[
\int_{\Omega} h_2(z)^{\delta} K(z,w)^{q\delta} d\nu_2(z) \leq C_1 h_1(w)^{q\delta}
\]
for almost all \( w \in \Omega \), then \( T \) is bounded from \( L^1(\Omega,\nu_1) \) into \( L^q(\Omega,\nu_2) \) and the norm of the operator does not exceed \( C_1^{1/q} C_2^{1/q} \).
5. Sufficient conditions for $L^p - L^q$ estimates of $S_M$

In this section the main ingredient is the generalize Schur’s test. We are begining by the following lemma.

**Lemma 5.1.** Let $b_1$, $b_2$, and $c$ be real numbers. Let $1 < p \leq q < \infty$, $\max(-1,-1-qb_1) < r < \infty$ and $-1 < s < \infty$. If

$$
\begin{align*}
\begin{cases}
   s + 1 < p(b_2 + 1) \\
   c \leq b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{p}
\end{cases}
\end{align*}
$$

then $S_M$ is bounded from $L^p(\mathbb{M})$ to $L^q(\mathbb{M})$.

**Proof.** To use generalize Schur’s test we first consider the following tools.

$$
\begin{align*}
h_1(z) &= (1 - |z|^2)^{-u}, \\
 h_2(w) &= (1 - |w|^2)^{-v}, \\
 dv_1(z) &= (1 - |z|^2)^s \alpha(z) \land \overline{\alpha(z)}, \\
 dv_2(w) &= (1 - |w|^2)^s \alpha(w) \land \overline{\alpha(w)}
\end{align*}
$$

and

$$
K(z, w) = \frac{(1 - |z|^2)^{b_1}(1 - |w|^2)^{b_2} - s}{|1 - z \bullet w|^{b_1+b_2 - \frac{n+1+r}{q} - \frac{n+1+s}{p}}.}
$$

Second, observe that if $c \leq b_1 + b_2 + \frac{n+1+r}{q} - \frac{n+1+s}{p}$ then

$$
|S_M f(z)| \leq |T_M f(z)| \leq 2^{b_1+b_2 + \frac{n+1+r}{q} - \frac{n+1+s}{p}} |T f(z)|.
$$

Thus the boundedness of $S_M$ arises from $T$ where

$$
T f(z) = \int_M f(w) K(z, w) dv_1(w)
$$

To do this we adopt the following notations.

$$
\begin{align*}
c &= b_1 + b_2 - s + \frac{n+1+s}{p'} + \frac{n+1+r}{q}
\end{align*}
$$

and

$$
\begin{align*}
\tau &= \frac{n+1+s}{p'} + \frac{n+1+r}{q}
\end{align*}
$$

Let us choose

$$
\begin{align*}
t &= \frac{\frac{n+1+s}{p'} + v - u}{\tau}
\end{align*}
$$

where $u$ and $v$ will be determined. It is easy to see that

$$
\begin{align*}
1 - t &= \frac{\frac{n+1+r}{q} + u - v}{\tau}
\end{align*}
$$

$$
\begin{align*}
\int_M h_1(w)^{p'} K(z, w)^{q'} dv_1(w) &= (1 - |z|^2)^{b_1} J_{c_1, s_1}(z)
\end{align*}
$$

where $c_1 = cp' + p'u - (b_2 - s)tp' - n - s - 1$ and $s_1 = cp' + p'u - (b_2 - s)tp' - n - s - 1$,

$$
\begin{align*}
\int_M h_2(z)^q K(z, w)^{(1-t)q} dv_2(z) &= (1 - |w|^2)^q J_{c_2, s_2}(w)
\end{align*}
$$

where $c_2 = cq(1-t) + qv - b_1q(1-t) - n - r - 1$ and $s_2 = cq(1-t) + qv - b_1q(1-t) - n - r - 1$.

It is clear that from (5.2) and (5.3) we get

$$
\begin{align*}
c - b_1 - b_2 + s &= \tau.
\end{align*}
$$
We achieve the lemma’s proof by invoking Theorem 4.3. Now we are going to prove (5.11). First, from Lemma 4.1 combined with (5.6) and (5.7) we obtain that.

\[ u \]

Otherwise we claim that there exist two real numbers \( -1 < s < 1 \). From easy calculus we have that.

\[ \text{Second, we choose } u \text{ and } v \text{ such that} \]

\[ \]

Finally, by combining (5.2), (5.3), (5.4), (5.5) and (5.14) we prove easily (5.11).

\[ \square \]

**Lemma 5.2.** Let \( b_1, b_2, \) and \( c \) be real numbers. Let \( 1 \leq p \leq q < \infty, \max(-1, -1 - qb_1) < r < \infty \) and \(-1 < s < \infty \). If

\[ \left\{ \begin{array}{l}
  s + 1 < p(b_2 + 1) \\
  c \leq b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{r} \\
 \end{array} \right. \]

then \( S_M \) is bounded from \( L^1(M) \) to \( L^q(M) \).

**Proof.** As in proof of the Lemma 5.1 we have that. \( \tau = \frac{n+1+r}{q}, t = \frac{v-u}{r}, u < (b_2 - s)t \) and \(-tb_1 < v \). From easy calculus we have

\[ \max\left(\frac{1 - |z|^2}{2}, \frac{1 - |w|^2}{2}\right) \leq |1 - z \cdot \bar{w}| \]

This yields the following.

\[ \sup_{w \in M} h_1(w)K(z, w)^t \leq 2^{tb_1+tb_2-s+v-v}(1 - |z|^2)^{-v} \sup_{w \in M} |1 - z \cdot \bar{w}|^{tb_1+tb_2-s+v-v-tc} \]

\[ \leq 4^{tb_1+tb_2-s+v-v-tc/2}(1 - |z|^2)^{-v} \]

Otherwise, using the same method in Lemma 5.1 it is obvious to prove that.

\[ \int_M h_2(z)^q K(z, w)^{(1-t)q} \, d\nu(z) \leq C_2 h_1(w)^q \]

Finally, the lemma arises from Theorem 4.4.

\[ \square \]
Lemma 5.3. Let $b_1$, $b_2$, and $c$ be real numbers. Let $1 = p \leq q < \infty$, $\max(-1, -1 - q b_1)< r < \infty$ and $-1 < s < \infty$. If
\[
(5.16) \quad \left\{ \begin{array}{l}
 s = b_2 \\
 c \leq b_1 + \frac{n+1+r}{q}
\end{array} \right.
\]
then $S_m$ is bounded from $L^1_s(\mathbb{M})$ to $L^q_s(\mathbb{M})$.

Proof. Here, we consider $h_1(z) = 1$, $h_2(z) = (1 - |z|^2)^{-r}$, $K(z,w) = \frac{(1-|z|^2)^{b_1}}{|1-z\cdot w|^r}$ and $t = \frac{v}{c-b_1}$ where $c > 0$ and $\frac{1}{q'} + b_1 (1-t) < v < c-b_1$. Then
\[
\sup_{w \in \mathbb{M}} h_1(w)K(z,w)^t = \sup_{w \in \mathbb{M}} \frac{(1 - |z|^2)^{b_1}}{|1 - z \cdot w|^{v + t b_1}} \leq 2^t (1 - |z|^2)^{-v}
\]
Otherwise, from Lemma 4.1 we get that
\[
\int_{\mathbb{M}} h_2(z)^q K(z,w)^{(1-t)q} dv_2(z) = J_{c_3,s_3}(w) \leq C_2
\]
where $c_3 = (1-t)qc - b_1 (1-t)q + qv - n - r - 1 < 0$ and $s_3 = (1-t)b_1q - qv + r > -1$. □

6. Necessary conditions for $L^p - L^q$ estimates of $T_M$

Lemma 6.1. Let $b_1$, $b_2$, and $c$ be real numbers. Let $1 \leq p \leq q < \infty$, $\max(-1, -1 - q b_1)< r < \infty$ and $-1 < s < \infty$. If $T_M$ is bounded from $L^p_s(\mathbb{M})$ to $L^q_s(\mathbb{M})$ then
\[
(6.1) \quad \left\{ \begin{array}{l}
 s + 1 \leq p (b_2 + 1) \\
 c \leq b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{p}
\end{array} \right.
\]
and the strict inequality holds for $1 < p \leq q < \infty$.

Proof. Suppose $1 < q < \infty$. Then the dual space $L^q_s(\mathbb{M})^*$ of $L^q_s(\mathbb{M})$ can be indentified with $L^q_s(\mathbb{M})$ under the integral paring
\[
<f,g> = \int_{\mathbb{M}} f(z)g(z) dv_2(z), \quad f \in L^q_s(\mathbb{M}), \quad g \in L^q_s(\mathbb{M})
\]
Moreover, by easy computation we have
\[
T_M^* g(z) = (1 - |z|^2)^{b_2-s} \int_{\mathbb{M}} \frac{(1 - |w|^2)^{b_1}}{|1 - z \cdot w|^r} g(w) dv_2(w)
\]
Let be $N$ a real number such that
\[
(6.2) \quad N > \max(-\frac{1+r}{q'}, -1 - r - b_1).
\]
Then from (2.1) we have
\[
(6.3) \quad \int_{\mathbb{M}} |f_N(z)|^q dv_2(z) = \frac{\omega(\partial \mathbb{M}) \Gamma(r + q'N + 1) \Gamma(n-1)}{2 \Gamma(r + q'N + n)}
\]
\[
(6.4) \quad T_M^* f_N(z) = C_N (1 - |z|^2)^{b_2-s} = C_N f_{b_2-s}(z)
\]
where
\[
f_N(z) = (1 - |z|^2)^N
\]
and
\[ C_N = \frac{(n-1)!\omega(\partial\Omega)\Gamma(b_1 + r + N + 1)}{(n-1)(n-2)2\Gamma(c)} \sum_{k=0}^{\infty} \frac{\Gamma(k+c)\Gamma(n-1+k/2)}{\Gamma(b_1 + r + N + n + k/2)} \]
Because \( T_Mf_N \) belongs to \( L^p_{\Omega}(\Omega) \) then from (6.4) and (6.5) we have \( s + p/(b_2 - s) > -1 \). This leads to \( s + 1 < p(b_2 + 1) \). Now, if we suppose \( 1 < p < \infty \) then \( T_Mf_N \) belongs to \( L^\infty(\Omega) \). This gives \( b_2 - s \geq 0 \). Thus \( s + 1 \leq 1(b_2 + 1) \). The case \( p = q = 1 \) is easy to prove. This completes the proof of the lemma. \( \square \)

**Lemma 6.2.** Let \( b_1, b_2, \) and \( c \) be real numbers. Let \( 1 < p \leq q < \infty \), \( \max(-1,-1-qb_1) < r < \infty \) and \(-1 < s < \infty \). Suppose \( T_M \) bounded from \( L^p(\Omega) \) to \( L^q(\Omega) \). Consider the following tree cases.

(i) \( 1 < p \leq q < \infty \) and \( s + 1 < p(b_2 + 1) \);
(ii) \( 1 = p \leq q < \infty \) and \( s < b_2 \);
(iii) \( 1 = p \leq q < \infty \) and \( s = b_2 \).
If (i) and (ii) hold then
\[ (6.5) \quad c \leq b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{p} \]
if (iii) hold then
\[ (6.6) \quad c < b_1 + b_2 - s + \frac{n+1+r}{q} + \frac{n+1+s}{p} \]

**Proof.** For any \( \xi \in \Omega \) we denote
\[ f_\xi(z) = \frac{(1 - |\xi|^2)^{n+1+b_2}|n-1+(n+1+2b_2)z\cdot\xi|}{(1-z\cdot\xi)^{n+1+b_2}} \]
Then (i) leads to
\[ (6.7) \quad \int_\Omega |f_\xi(z)|^p d\nu(z) \leq 2(n+1+b_2)(1 - |\xi|^2)^{p(n+1+b_2)-(n-1-s)}f_{p(n+1+b_2)-n-1-s}(\xi) \]
Otherwise, because \( g(\xi) = \frac{1}{(1-\xi\cdot\xi)^q} \) belongs to \( A^p_{\Omega}(\Omega) \) we have that
\[ T_M(f_\xi)(z) = \frac{(1 - |\xi|^2)^{b_1}(1 - |\xi|^2)^{n+1+b_2-(n+1+s)/p}T_M(g)(\xi)}{(1-|\xi|^2)^{b_1}(1 - |\xi|^2)^{n+1+b_2-(n+1+s)/p}} \]
From the boundedness of \( T_M \) we have that
\[ (6.8) \quad (1 - |\xi|^2)^{q(n+1+b_2)-(n+1+s)/p}f_{c_4,s_4}(\xi) \leq C \]
where \( C > 0, c_4 = qc - qb_1 - n - 1 - r - q(n+1+b_2) + (n+1+s)q/p \) and \( s_4 = qb_1 + r \). So, from Lemma 5.12 we have \( c_4 \leq 0 \). This gives (6.5). By the same way the case (ii) leads to (6.5). The case (iii) leads to \( (n+1+s)q/p - q(n+1+b_2) = s - b_2 = 0 \). So, from (6.8) combined with Lemma 5.12 we obtain easily (6.6). \( \square \)

7. **Proof of the Theorem A**

**Proof.** The assertion (i) implies (iii) follows from Lemma 6.1. It is obvious that (ii) implies (i). The assertion (iii) implies (ii) follows from Lemma 5.1. This completes the proof of the theorem. \( \square \)
8. PROOF OF THE THEOREM B

Proof. The assertion (i) implies (iii) follows from (ii) of Lemma 6.1 and (iii) of Lemma 6.2. It is obvious that (ii) implies (i). The assertion (iii) implies (ii) follows from Lemma 5.1 and Lemma 5.3. This achieves the proof of the theorem. □

Before proving Theorem C we recall the key tool which will be used. Let \( f : B^* \rightarrow C \) be a measurable function. We define a function \( I_Mf \) on \( M \) by

\[
(I_Mf)(z) = \frac{z_{n+1} f \circ F(z)}{(2(n+1))^{1/p}} = \frac{z_{n+1} f(z_1, \ldots, z_n)}{(2(n+1))^{1/p}}
\]

Lemma 8.1. [5, Lemma 4.1] For each \( p \geq 1 \) and \( \lambda > -1 \) the linear operator \( I_M \) is an isometry from \( L^p_{\lambda}(B^*) \) into \( L^p_{\lambda}(M) \). Moreover, we have \( P_{\lambda,M}I_M = I_MI_{\lambda,B^*} \) on \( L^p_{\lambda}(B^*) \).

Proposition 8.2. Let \( 1 \leq p \leq q < \infty \), \( -1 < \lambda, \bar{\lambda} < \infty \).

(i) For \( 1 < p \leq q < \infty \) the Bergman projector \( P_{s,M} \) is bounded from \( L^p_{\lambda}(M) \) onto \( A^q_{\bar{\lambda}}(M) \) if and only if

\[
\left\{ \begin{array}{l}
\lambda + 1 < p(s + 1) \\
\frac{s + 1 + \lambda}{p} - \frac{s + 1 + \bar{\lambda}}{q} > n + 1 + \lambda
\end{array} \right.
\]

(ii) For \( 1 = p \leq q < \infty \) the Bergman projector \( P_{s,M} \) is bounded from \( L^1_{\lambda}(M) \) onto \( A^q_{\bar{\lambda}}(M) \) if and only if

\[
\left\{ \begin{array}{l}
\lambda < s \\
\frac{s + 1 + \lambda}{q} \geq n + 1 + \lambda
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l}
\lambda \leq s \\
\frac{s + 1 + \lambda}{q} > n + 1 + \lambda
\end{array} \right.
\]

Proof. Let us choose in Theorem A and B \( c = n + 1 + \lambda, b_2 = s \) and \( b_1 = 0 \). Then it follows from Theorem A that \( P_{s,M} \) is bounded from \( L^p_{\lambda}(M) \) onto \( A^q_{\bar{\lambda}}(M) \) iff (iii) of Theorem A holds. Otherwise, from Theorem B it follows that \( P_{s,M} \) is bounded from \( L^1_{\lambda}(M) \) onto \( A^q_{\bar{\lambda}}(M) \) iff (iii) of Theorem B holds. This achieves the proof of the proposition. □

Remark 8.3. The assertion (i) of Proposition 8.2 improves Theorem 5.2 of [5]. Indeed, it suffices to take \( \lambda = \bar{\lambda} \) and \( p = q \geq 1 \).

9. PROOF OF THE THEOREM C

Proof. The equivalence of (i) and (iii) of Theorem A follows from Lemma 8.1. Also, the equivalence of (ii) and (iii) of Theorem B follows from Lemma 8.1. This completes the proof of the theorem. □

Remark 9.1. The assertion (i) of Theorem C improves an important result due to G. Mengotti and E. H. Youssfi [5]. Indeed, for \( \lambda = \bar{\lambda} \) and \( p = q \geq 1 \) we obtain the Theorem B of [5].

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