Global stability of a piecewise linear macroeconomic model with a continuum of equilibrium states and sticky expectation

Pavel Krejčí∗  Harbir Lamba†  Dmitrii Rachinskii‡

Abstract

We consider piecewise linear discrete time macroeconomic models, which possess a continuum of equilibrium states. These systems are obtained by replacing rational inflation expectations with a boundedly rational, and genuinely sticky, response of agents to changes in the actual inflation rate in a standard Dynamic Stochastic General Equilibrium model. Both for a low-dimensional variant of the model, with one representative agent, and the multi-agent model, we show that, when exogenous noise is absent from the system, the continuum of equilibrium states is the global attractor. Further, when a uniformly bounded noise is present, or the equilibrium states are destabilized by an imperfect Central Bank policy (or both), we estimate the size of the domain that attracts all the trajectories. The proofs are based on introducing a family of Lyapunov functions and, for the multi-agent model, deriving a formula for the inverse of the Prandtl-Ishlinskii operator acting in the space of discrete time inputs and outputs.

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1 Introduction

Notions of friction and stickiness are widely accepted to exist in organizations, economies, and financial systems. Drawing intuition from mathematical physics, one can propose that a system with (dry) friction should have a continuum of equilibrium states. For example, an object can achieve an equilibrium on a curved surface at any point where the slope does not exceed the dry friction coefficient because friction balances the gravity. Notably, various hard-to-explain empirical regularities found in micro, macro, and organizational economic data, such as path-dependence, permanence, hysteresis, boom blessings and recession curses can be accounted for by the presence of many meta-stable states with the associated long timescale dynamics, which frictions introduce into an economic system.

∗Institute of Mathematics of the Czech Academy of Sciences, Czech Republic
†Department of Mathematical Sciences, George Mason University, VA, USA
‡Department of Mathematical Sciences, The University of Texas at Dallas, TX, USA
In other words, frictional effects at the micro-level might aggregate to macro-level long-term memory effects.

Based on this premise, we propose to test how the introduction of internal frictions affects dynamics of well-established single equilibrium economic models. It is a fundamental question what forms of friction actually arise from the behavior of economic agents, and how these can be modeled. In this paper, we take a phenomenological approach to modeling frictions using play operators, which are common in physics and engineering applications. In the context of economics, they are associated with the notions of ‘threshold’ and ‘inaction band’. These operators have been described in several equivalent ways including piecewise linear (PWL) functions and variational inequalities. We proceed by positing that inflation and inflation expectation are related by the play operator (or a combination thereof in the multi-agent variant of the model) in a version of a Dynamic Stochastic General Equilibrium (DSGE) macroeconomic model. Our analysis focuses on one particular feature of the adapted model with frictions. Namely, we consider local and global stability of this discrete time PWL system, which naturally possesses a continuum of equilibrium states.

In the remainder of this section some economic background and the structure of the paper are briefly discussed.

**Economic background and motivation.** Seminal work by the likes of Walras and Jevons in the late 1800’s steered the field of economics away from the domain of philosophy, and laid foundations for it to develop into a mathematically tractable, exact science. Early pioneers of what became known as Neoclassical Economics saw parallels between economic systems and the equilibrating forces in nature, and thus borrowed heavily from mathematical physics to derive both intuition and methodologies [1]. Despite concerns about the simplifying assumptions needed for this tâtonnement-driven view (e.g. [2,3]), the single-equilibrium approach is still considered good enough in many settings, and has not been supplanted by any widely-accepted, complete system. Significant developments have come from introducing concepts like Rational Expectations and Sticky Models. The former is the assumption of aggregate consistency in dynamic models [4]. This view admits that agent’s expectations about future uncertainties may be wrong individually, but in aggregate are in agreement with the model itself. In other words, although the future is not fully predictable, agents’ expectations are assumed not to be systematically biased and collectively use all relevant information in forming expectations of economic variables. Meanwhile the later includes the widely-used sticky models of Calvo [5] and the sticky-information of Mankiw and Reis [6]. These models are concerned with similar observations about the way in which real life agents do not instantaneously move to the ‘correct’ price or opinion but rather do so at a fixed rate and can be represented mathematically by introducing a delay term into the relevant equations. In the absence of noise the same optimal equilibrium solution will be reached as if the stickiness were absent. Continua of possible equilibria can also occur in such models (see for example [7,8]) but only in certain special cases (such as a passive interest-rate policy [5,9]) and are considered an extreme form of indeterminacy.

However, a robust empirical feature of economic output for developed countries is that output and output growth are non-normally distributed, exhibiting fat tails and excess kurtosis. In many models such booms and busts are explained by the occurrence of large (unpredictable) exogenous shocks followed by tranquil periods when nothing leads to non-normality. Explaining such regularities within linear unique-equilibrium economic models
often involves adding *a posteriori* assumptions such as the existence of an eigenvalue with largest modulus close to, but inside, the unit circle (the Unit Root Hypothesis). A better model should generate non-normality of the output data from within the theory.

The critique of unique-equilibrium models has a long history which we shall not attempt to detail here. For example, many have eloquently pointed out fundamental issues with the assumed equilibrating processes and the ways in which the “aggregation problem” was being solved (e.g. [10–13]). This approach is taken by multi-agent models including models of agents’ behavior under imperfect information, cognitive limitations, or endogenously generated “animal spirits” [14]. Here, the themes of confusion, rational inattention, simplification, and bounded rationality take center stage (e.g. the work of Sims and Gabaix). Non-normality in these models can be associated with cascading effects. Most of these models are however purely numerical, and incorporate assumptions inspired by the social sciences (e.g. psychology and sociology) in order to add more realistic behavior to the modeling construct of individual actors (much recent work has focused on the question of whether individual actors can ever be sufficiently rational).

Our approach is complementary to many of the above-referenced views. In this paper, we argue that a lingering constraint of single-equilibrium in all these models is unnecessary. Hysteresis is a well-understood property that explains the “stickiness” observed in many physical systems (e.g. plasticity, magnetism), and which explains how many equilibria can arise and be stable. Hysteresis has only recently been explicitly considered in economic time series such as the unemployment rate [15–17]. The form of stickiness that we use is, to our knowledge, new in a economic setting and differs from, for example, the stickiness of the Calvo pricing model [5] where hypothetical agents are only allowed to adjust (to the correct price) at a fixed rate. The way in which we incorporate stickiness into the model will be justified and described more fully below but, briefly, our sticky variables can only be in one of two modes. They are either currently ‘stuck’ at some value or they are being ‘dragged’ along by some other (related) variable because the maximum allowable difference between them has been reached. Hence, our agents are truly stuck (not just delayed) until forced to adjust by the discrepancy with the actual inflation rate. If an equilibrium is reached it is chosen by the prior states of the system, and a continuum of equilibria is an intrinsic feature of the model.

The research into how expectations are formed is extensive but far from conclusive, see for example [18–22]. However the idea of threshold effects and a ‘harmless interval’ of inflation is not new in economics [23–27]. A person may concurrently be subject to many or all of the limitations (e.g. rational inattention and bounded rationality and confusion, etc.), but for our purposes it is enough to assume that individual actors behave according to the play operator we describe, and we can remain relatively agnostic as to which mechanism may be driving the general features of the play operator (band of inaction, thresholds). In the absence of any exogenous forcing it would be very easy to distinguish between Calvo-type stickiness and the stuck-then-dragged behavior we investigate here — indeed Calvo stickiness would most likely be observed since agents could tell far more easily over time that, for example, their wage demands were too low and they were losing purchasing power. However, given the uncertainty of reality and the very limited cognitive skills or interest in forecasting of most economic agents, that may no longer hold.

**Structure of the paper.** In the next section, we present a discrete time 3-dimensional PWL macroeconomic model with sticky inflation expectations modeled by a play operator. The model uses the notion of a representative agent. When the exogenous noise
terms are absent from the system, it has a line segment of equilibrium points. Section 3 contains main results. In particular, in the system without exogenous noise, a simple condition ensures that the line segment of equilibrium points is the global attractor (Section 3.1). In the presence of uniformly bounded noise, we obtain an estimate of the globally attracting domain, which is proportional to the supremum norm of the noise. Interestingly, this estimate is uniform with respect to the parameter that controls the amount of stickiness in the expectation of future inflation rate. We then consider further variants of the model. First, we add stickiness into the response of the Central Bank (Section 3.2). This can destabilize the equilibrium states (leading to periodic, quasiperiodic or more complex dynamics [28] with the associated border collision bifurcations, which are typical of piecewise smooth systems [29]) but the system still possesses a bounded globally attracting domain. Then we consider a multi-agent variant of the model (Section 3.3). This is an \((n + 2)\)-dimensional PWL system with \(2n\) switching surfaces. We show that the \(n\)-dimensional continuum of equilibrium states of this system is the global attractor.

Finally, the proofs based on constructing a family of Lyapunov functions are presented in Section 4. In order to apply the Lyapunov function to the multi-agent model, we adapt a technique from the theory of hysteresis operators [30,31]. Namely, an explicit formula for the inverse of the Prandtl-Ishlinskii operator acting in the space of discrete time inputs and outputs is derived and used. We conclude with a summary of the main results and some suggestions for future work.

### 2 The model

#### 2.1 DSGE modeling framework

The standard approach to the problem of aggregating expectations is to introduce a ‘Representative Agent’ whose expectations are fully-informed and rational and consistent with the model itself. Here, an aggregation of boundedly rational agents into a similar Representative is required. Our approach is similar in spirit to that of De Grauwe [14] but we use a different model of boundedly rational agents’ behavior.

We start from a dynamic stochastic general equilibrium (DSGE) macroeconomics model, which includes aggregate demand and aggregate supply equations

\[
\begin{align*}
  x_t &= b_1 p_t + (1 - b_1) x_{t-1} + b_2 y_t + \eta_t, \\
  y_t &= (1 - a_1) y_{t-1} - a_2 (r_t - p_t) + \epsilon_t,
\end{align*}
\]

augmented with the interest rate-setting Taylor rule

\[
r_t = c_1 x_t + c_2 y_t + \xi_t,
\]

where \(y_t\) is output gap (or unemployment rate, or another measure of economic activity such as gross domestic product), \(x_t\) is inflation rate, \(r_t\) is interest rate, \(p_t\) is the economic agents’ aggregate expectation of future inflation rate, \(\eta_t, \epsilon_t, \xi_t\) are exogenous noise terms, and \(t \in \mathbb{N}\). The parameters satisfy

\[
0 \leq a_1 < 1, \quad 0 < b_1 < 1, \quad a_2, b_2, c_1, c_2 > 0.
\]

Of specific interest is the case \(a_1 = 0\).
This model is close to the model used in [14] but simpler in that we do not include
the aggregate expectation of the output gap and the correlation between the subsequent
values of the interest rate. The inclusion of such factors does not affect our most significant
qualitative observations, but would complicate some aspects of the rigorous analysis that
we present.

The novelty of our modeling strategy is in how we define the relationship between the
aggregate expectation of inflation $p_t$ and the inflation rate $x_t$.

2.2 Sticky expectation of inflation

We start from the empirical evidence cited above that individual agents’ expectations are
often sticky and may lag behind the currently observable values before they start to move.
We also posit that this gap between future expectations and current reality cannot grow
too large. We then imbue our now boundedly rational Representative Agent with these
same properties. More precisely, we assume the following rules that define the variations
of the expectation of future inflation rate $p_t$ with the actual inflation rate $x_t$ at integer
times $t$:

(i) The value of the difference $|p_t - x_t|$ never exceeds a certain bound $\rho$;
(ii) As long as the above restriction is satisfied, the expectation does not change, i.e.
    $|x_t - p_{t-1}| \leq \rho$ implies $p_t = p_{t-1}$;
(iii) If the expectation has to change, it makes the minimal increment consistent with
    constraint (i).

Rule (ii) introduces stickiness in the dependence of $p_t$ on $x_t$, while (i) states that
the expected inflation rate cannot deviate from the actual rate more than prescribed
by a threshold value $\rho$. Hence $p_t$ follows $x_t$ reasonably closely but on the other hand is
conservative because it remains indifferent to variations of $x_t$ limited to a (moving) window
$p - \rho \leq x \leq p + \rho$. The last rule (iii) enforces continuity of the relationship between $p_t$
and $x_t$ and, in this sense, can be considered as a technical modeling assumption that is
mathematically convenient.

Rules (i)–(iii) are expressed by the formula

$$p_t = x_t + \Phi_\rho(p_{t-1} - x_{t-1}), \quad t \in \mathbb{N} \tag{3}$$

with the piecewise linear saturation function

$$\Phi_\rho(v) = \begin{cases} 
\rho & \text{if } v \geq \rho, \\
 v & \text{if } -\rho < v < \rho, \\
-\rho & \text{if } v \leq -\rho. 
\end{cases} \tag{4}$$

Equations (1) and (2), completed with formulas (3) and (4), form a closed 3-dimensional
PWL model for the evolution of the aggregated variables $x_t, y_t, p_t$. Another variant of this

\footnote{More complicated variants of DSGE models, employing many more variables to represent different
sectors of the economy, are widely used by central banks to help determine interest rate policy [32–35]. DYNARE
software platform supporting both DSGE models relying on the rational expectations
hypothesis and some models where agents have limited rationality or imperfect knowledge of the state of
the economy is available at [36].}
Some further terminology will be useful. Denote by $S$ the set of all real sequences $x = (x_0, x_1, \ldots)$. For a given parameter $\rho > 0$, the play operator $p_\rho : [-\rho, \rho] \times S \to S$ is defined as the mapping which with a given initial condition $s_0 \in [-\rho, \rho]$ and a sequence $x \in S$ associates the sequence

$$p = (p_0, p_1, \ldots) = p_\rho[s_0, x]_t \in S$$

according to the formula (3) with $p_0 = x_0 - s_0$. The parameter $\rho$ is called the threshold, see Fig. 1 (left). A dual mapping $s_\rho : [-\rho, \rho] \times S \to S$, which is defined by the relationship

$$s_t = \Phi_\rho(x_t - x_{t-1} + s_{t-1}), \quad t \in \mathbb{N}$$

for an arbitrary pair $(s_0, x) \in [-\rho, \rho] \times S$, is known as the stop operator

$$s = (s_0, s_1, \ldots) = s_\rho[s_0, x]_t \in S,$$

see Fig. 1 (right). By definition the play and stop operators sum up to the identity:

$$p_t + s_t = x_t$$

for the sequences (5), (7).

One can think of the play operator as having two modes, see Fig. 2 (left). A ‘stuck mode’ where it will not respond to small changes in the input $x_t$ and a ‘dragged mode’
where the absolute difference between the input \( x_t \) and output \( p_t \) are at the maximum allowable and changes to the input, in the correct direction, will drag the output along with it. Further, in the context of our model, the output of the stop operator \( s_t \) measures the difference between the inflation rate and the expectation of the future inflation rate, hence \( s_t \) remains within the bound \( |s_t| \leq \rho \) at all times. We shall refer to the variable \( s_t = x_t - p_t \) as the perception gap. Interestingly the explicit relationship (6) has been observed in actual economic data [38, 39]. Fig. 2 (right) gives an interpretation of the stop operator combing an ideal spring and a dry friction element as used in mechanics.

3 Main results

3.1 Autonomous system

We first consider system (1)–(3) without noise terms:

\[
\begin{align*}
x_t &= b_1 p_t + (1 - b_1)x_{t-1} + b_2 y_t, \\
y_t &= (1 - a_1)y_{t-1} - a_2(c_1x_t + c_2y_t - p_t), \\
p_t &= x_t + \Phi_{\rho}(p_{t-1} - x_t).
\end{align*}
\]

Equilibrium points of this system form a line segment

\[
\mathfrak{A} = \left\{ (x_*, y_*, p_*)(u) : x_*(u) = \kappa u, \ y_*(u) = \frac{b_1}{b_2} u, \ p_*(u) = (\kappa - 1)u, \ |u| \leq \rho \right\}
\]

with

\[
\kappa = \frac{a_2(b_2 + b_1 c_2) + a_1 b_1}{a_2 b_2 (1 - c_1)}.
\]

**Theorem 3.1.** If \( c_1 > 1 \), then the line segment \( \mathfrak{A} \) of equilibrium points is the global attractor for system (8). Further, any trajectory converges to an equilibrium point \((x_*, y_*, p_*) \in \mathfrak{A} \).

We note that system (8) is written in an implicit form. If \( c_1 > 1 \) as in Theorem 3.1, it is easy to rewrite this system as an explicit PWL map \((x_t, y_t, p_t) = f(x_{t-1}, y_{t-1}, p_{t-1})\).

3.2 Sticky Central Bank response

The Central Bank policy can presumably exhibit stickiness too. To explore this scenario in this section we replace the Taylor rule (2) with the relation

\[
r_t = p_\sigma[c_1 x + c_2 y]_t + \xi_t \tag{10}
\]

also involving a play operator with an initial condition \( r_0 \) such that \( |r_0 - (c_1 x_0 + c_2 y_0)| \leq \sigma \). The play operator \( p_\sigma \) with a threshold \( \sigma \geq 0 \) independent of \( \rho \) in (10) should express the fact that the central bank’s decisions do not immediately follow the instantaneous value of \( c_1 x_t + c_2 y_t \), but they are activated only if the difference between \( r_t \) and \( c_1 x_t + c_2 y_t \) risks to exceed a given value \( \sigma \). For \( \sigma = 0 \), \( p_\sigma \) is the identity mapping and (10) becomes (2).
It is worth noting that for \( \sigma > 0 \) and a sequence \( \{v_t\}, r_t = p_\sigma[v_t] \) is the sequence with minimal variation in the \( \sigma \)-neighborhood of \( \{v_t\} \), that is, the implication

\[
\hat{r}_0 = r_0, \quad \hat{r}_t - v_t \leq \sigma \quad \forall t \in \mathbb{N} \cup \{0\} \quad \Rightarrow \quad \sum_{t=1}^{T} |\hat{r}_t - \hat{r}_{t-1}| \geq \sum_{t=1}^{T} |r_t - r_{t-1}|
\]

holds for every sequence \( \{\hat{r}_t\} \) and any \( T \in \mathbb{N} \) as a special case of [40, Theorem 1.2 and Corollary 1.6]. The meaning of (10) can thus be interpreted as the optimal strategy allowing to stay in a \( \sigma \)-neighborhood of the input sequence under minimal “cost” incurred by variations of the interest rate.

Re-writing equation (43) equivalently as

\[
r_t = c_1 x_t + c_2 y_t + \Phi_\sigma(r_{t-1} - c_1 x_{t-1} - c_2 y_{t-1}) + \xi_t,
\]

we obtain a 4-dimensional PWL system (1), (3), (12). In the absence of noise, i.e. when \( \eta_t = \epsilon_t = \xi_t = 0 \) for all \( t \), this system has the following set of equilibrium points:

\[
\mathfrak{M} = \left\{ (x_*, y_*, p_*, r_*) : x_* = \kappa u + \frac{v}{c_1 - 1}, \quad y_* = \frac{b_1}{b_2} u, \quad p_* = (\kappa - 1) u + \frac{v}{c_1 - 1}, \right. \\

r_* = \left( c_1 \kappa + \frac{b_1 c_2}{b_2} \right) u + \frac{v}{c_1 - 1}, \quad |u| \leq \rho, \quad |v| \leq \sigma \right\}.
\]

The subset

\[
\mathfrak{A}_0 = \{ (x_*, y_*, p_*, r_*) \in \mathfrak{M} : v = 0 \}
\]

of \( \mathfrak{M} \) can be considered as a natural embedding of the equilibrium set (9) into the 4-dimensional phase space of system (1), (3), (12).

It should be pointed out right away that stickiness in the Taylor rule can have a destabilizing effect on the equilibrium states. To see this, consider for simplicity the case when \( \rho = 0 \), which implies \( p = x \) and \( s = 0 \), i.e. we remove stickiness in the inflation expectation. Note that the system is locally linear in a vicinity of every equilibrium point belonging to \( \mathfrak{M} \) and satisfying \( |v| < \sigma \). Further, a simple calculation shows that the determinant of the linearization at such equilibrium points equals \((1 - a_1)(1 - b_1)/(1 - b_1 - a_2 b_2)\). If \( 1 - b_1 > a_2 b_2 > a_1 (1 - b_1) \), then this determinant is greater than 1, hence these equilibria are unstable (in particular, they are unstable in the important case of \( a_1 = 0, 1 - b_1 > a_2 b_2 \)). Numerical examples of several attractors including periodic orbits of different periods, a quasiperiodic orbit or a union of two equilibrium points corresponding to \( v = \pm \sigma \) (end points of the line segment of equilibrium points) can be found in [6].

The goal of this section is to estimate how far trajectories can deviate from the equilibrium set \( \mathfrak{A}_0 \) due to stickiness in the Taylor rule and a uniformly bounded noise.

**Theorem 3.2.** Let us consider system (1), (3), (12) with uniformly bounded exogenous terms \( \eta_t, \epsilon_t, \xi_t \) and with \( c_1 > 1 \). There exist constants \( L_1, L_2 \), which depend on the parameters \( a_1, a_2, b_1, b_2, c_1, c_2 \) but are independent of the threshold \( \rho \) of the inflation expectation (see (3)), such that every trajectory satisfies

\[
\limsup_{t\to\infty} |x_t - x_*(s_t)|, \limsup_{t\to\infty} |y_t - y_*(s_t)| \leq L_1 \sigma + L_2 m,
\]

where \( (x_*, y_*, p_*)(\cdot) \) is defined in (9), \( s_t = x_t - p_t, \sigma \) is the threshold of the play operator in the central bank’s policy (see (10)) and

\[
m = \sup \max_{t} \{ |\epsilon_t|, |\eta_t|, |\xi_t| \} < \infty.
\]
The proof presented below suggests an explicit upper bound for the coefficients $L_1, L_2$ in (14). Due to relations $p = x - s$ and $r = x - s$, estimates (14) imply similar estimates for $p_t$ and $r_t$:

$$\limsup_{t \to \infty} |p_t - p_s(s_t)| \leq L_1 \sigma + L_2 m,$$

$$\limsup_{t \to \infty} |r_t - c_1 x_s(s_t) - c_2 y_s(s_t)| \leq (c_1 + c_2)(L_1 \sigma + L_2 m) + \sigma + m.$$ (16) (17)

According to Theorem 3.2, estimates (14)–(17) are uniform with respect to $\rho$. In particular, if $\sigma = 0$ and hence system (1), (3), (12) becomes equivalent to system (1)–(3), then every trajectory converges to a neighborhood

$$\mathfrak{A}(R) = \{(x, y, p) : \min_{(x, y, p) \in \mathfrak{A}} \max\{|x - x_s|, |y - y_s|, |p - p_s|\} \leq R\}$$

of the set (9) of equilibrium points, and the size of this neighborhood is proportional to the supremum norm of the noise terms, $R = L_2 m$, and is independent of the threshold $\rho$ of inflation expectations (cf. Theorem 3.1).

### 3.3 A multi-agent model

Model (1)–(3) can be easily extended to account for differing types of agent with different inflation rate expectation thresholds. To this end, we replace the simple relationship (5) between $p_t$ and $x_t$ (which is equivalent to (3)) with the equation

$$p_t = \sum_{i=1}^{n} \nu_i p_{\rho_i}[\beta_i, x]_t,$$ (18)

Here the play operator $p_{\rho_i}$ models the expectation of inflation by the $i$-th agent; $p_t$ is the aggregate expectation of inflation; $\nu_i > 0$ is a weight measuring the contribution of agent’s expectation of inflation to the aggregate quantity; $\rho_i$ is an individual threshold characterizing the behavior of the $i$-th agent; $\beta_i$ is the initial condition for the corresponding play operator; and we assume the ordering $0 < \rho_1 < \cdots < \rho_n$.

Operator (18) is known as a (discrete) Prandtl-Ishlinskii (PI) operator $[30, 37, 41, 42]$. In analysis of this operator, it is convenient to restrict the set of initial conditions of the play operators in (18). In what follows, we assume that $|\beta_1| < \rho_1$ and $|\beta_i - \beta_{i-1}| \leq \rho_i - \rho_{i-1}$ for all $2 \leq i \leq n$. Further, the coefficients $\nu_i$ are assumed to fulfill the condition

$$\nu_i > 0 \quad \text{for} \quad i = 1, \ldots, n, \quad \nu_0 := 1 - \sum_{i=1}^{n} \nu_i \geq 0.$$ (19)

It will be useful to define the quantity $s_t$ similarly as in (7), that is

$$s_t = x_t - p_t = \nu_0 x_t + \sum_{i=1}^{n} \nu_i s_{\rho_i}[\beta_i, x]_t,$$ (20)

---

2The continuous time counterpart of this operator is widely used as a friction model in mechanical applications $[42]$ as well as for modeling elastoplastic systems $[43]$, constitutive laws of smart materials $[44]$, and material fatigue $[45]$. 9
where $s_{\rho}[\beta, x]_t = x_t - p_{\rho}[\beta, x]_t$ for all $t \in \mathbb{N} \cup \{0\}$.

Equations (1), (2), (18) form an $(n+2)$-dimensional PWL system. Similarly, if the Taylor rule (2) is replaced with its sticky counterpart (12), we obtain a PWL system of dimension $n+3$. One can formulate natural analogs of Theorems 3.1, 3.2 for these systems. For example, let us consider the analog of Theorem 3.1 for the autonomous system

$$
\begin{align*}
x_t &= b_1 p_t + (1 - b_1)x_{t-1} + b_2 y_t, \\
y_t &= (1 - a_1)y_{t-1} - a_2(c_1 x_t + c_2 y_t - p_t)
\end{align*}
$$

(21)

coupled with formula (18) for the aggregate expectation of inflation.

**Theorem 3.3.** If $c_1 > 1$, then any trajectory of system (18), (21) converges to an equilibrium point of this system.

We note that equilibrium points of system (18), (21) form an $n$-dimensional parallelepiped in its phase space. The proof of Theorem 3.3 uses the constructions from the proof of Theorem 3.1 but additionally relies on an inversion formula for the Prandtl-Ishlinskii operator. This proof also shows a possible way to extend Theorem 3.2 to the multi-agent model (1), (12), (18) with sticky inflation expectation and exogenous noise.

### 4 Proofs

#### 4.1 Play and stop operators

For the reader’s convenience, we summarize here some well-known properties of the discrete time stop operator $s_t = s_{\rho}[s_0, x]_t$ and play operator $p_t = p_{\rho}[s_0, x]_t$ which are needed in the sequel.

**Lemma 4.1.** Let $\{x_t; t \in \mathbb{N} \cup \{0\}\}$ be a given sequence. Then $p_t, s_t$ satisfy (5), (7) if and only if $|s_t| \leq \rho$ for all $t \in \mathbb{N} \cup \{0\}$ and the variational inequality

$$
(p_t - p_{t-1}, s_t - z) \geq 0 \quad \forall t \in \mathbb{N} \cup \{0\}
$$

(22)

holds for every $z \in [-\rho, \rho]$.

**Proof.** Relations (5), (7) are equivalent to the series of implications

$$
p_t - p_{t-1} > 0 \Rightarrow p_t = x_t - \rho \Rightarrow s_t = \rho, \quad p_t - p_{t-1} < 0 \Rightarrow p_t = x_t + \rho \Rightarrow s_t = -\rho,
$$

which is in turn equivalent to (22) under the condition $|s_t| \leq \rho$ for all $t \in \mathbb{N}$. \(\blacksquare\)

For a generic sequence $\{z_t; t \in \mathbb{N} \cup \{0\}\}$, we introduce the notation

$$
\nabla_t z := z_t - z_{t-1} \quad \text{if} \quad t \geq 1; \quad \nabla_t^2 z := z_t - 2z_{t-1} + z_{t-2} \quad \text{if} \quad t \geq 2.
$$

(23)

We will systematically use the identity

$$
\nabla_t z z_t = \frac{1}{2} (z_t^2 - z_{t-1}^2) + \frac{1}{2} (\nabla_t z)^2.
$$

(24)

Choosing in (22) the value $z = s_{t-1}$, we obtain that $\nabla_t p \nabla_t s \geq 0$, hence

$$
\nabla_t p \nabla_t s \geq (\nabla_t s)^2.
$$

(25)
Furthermore, summing the inequalities
\[(p_t - p_{t-1})(s_t - s_{t-1}) \geq 0,\]
\[(p_{t-1} - p_{t-2})(s_{t-1} - s_t) \geq 0,\]
which follow from (22) by the choice \(z = s_{t-1}\) and \(z = s_t\), respectively, we obtain that \(\nabla_t^2 p \nabla_t s \geq 0\), hence
\[
\nabla_t^2 x \nabla_t s \geq \nabla_t^2 s \nabla_t s = \frac{1}{2}((\nabla_t s)^2 - (\nabla_{t-1} s)^2) + \frac{1}{2}(\nabla_t^2 s)^2, \tag{26}
\]
and similarly
\[
\nabla_t^2 x \nabla_t x = \frac{1}{2}((\nabla_t x)^2 - (\nabla_{t-1} x)^2) + \frac{1}{2}(\nabla_t^2 x)^2, \tag{27}
\]
which is a special case of identity (24) with \(z_t = \nabla_t s\) and \(z_t = \nabla_t x\).

**Lemma 4.2.** For a given sequence \(\{x_t; t \in \mathbb{N} \cup \{0\}\}\), put \(p_t = p_t[s_0, x_t]\), \(s_t = x_t - p_t\) with some given initial condition \(s_0 = [-\rho, \rho]\). Let \(q_t = x_t + \delta s_t = (1 + \delta)x_t - \delta p_t\) for some \(\delta > -1\). Then
\[
x_t = \frac{1}{1+\delta} \left( q_t + \delta p_{(1+\delta)}[1+\delta s_0, q_t] \right).	ag{28}
\]

**Proof.** We have \(q_t - p_t = (1 + \delta)s_t\), hence \(|q_t - p_t| \leq (1 + \delta)\rho\), and
\[
(p_t - p_{t-1}, q_t - p_t - (1 + \delta)\rho x) \geq 0 \quad \forall t \in \mathbb{N} \forall |x| \leq 1. \tag{29}
\]
By Lemma 4.1 this implies that \(p_t = p_{(1+\delta)}[(1 + \delta)s_0, q_t]\) and the assertion follows. \(\blacksquare\)

**Lemma 4.3.** For a given sequence \(\{x_t; t \in \mathbb{N} \cup \{0\}\}\), put \(p_t = p_t[s_0, x_t]\) with some given initial condition \(s_0 = [-\rho, \rho]\). Then for every \(t, j \in \mathbb{N}\) we have
\[
|p_{t+j} - p_t| \leq \max_{i=1,\ldots,j}\{|x_{t+i} - x_t|\}. \tag{30}
\]

**Proof.** We fix \(t \in \mathbb{N} \cup \{0\}, J \in \mathbb{N}\) and for \(j = 0, 1, \ldots, J\) set
\[
S_j = \max\{|p_{t+j} - p_t|^2, \max_{i=1,\ldots,j}\{|x_{t+i} - x_t|^2\}|.\]

The proof will be complete if we prove that the sequence \(\{S_j\}\) is nonincreasing for \(j = 0, 1, \ldots, J\). Indeed, then \(S_J \leq S_0\), which is precisely the desired statement.

Assume for contradiction that \(S_j > S_{j-1}\) for some \(j = 1, \ldots, J\). Then
\[
|p_{t+j} - p_t| > \max_{i=1,\ldots,j}\{|x_{t+i} - x_t|\}, \tag{31}
\]
\[
(p_{t+j} - p_t)^2 > (p_{t+j-1} - p_t)^2. \tag{32}
\]
Inequality (32) can be equivalently written as
\[
(p_{t+j} - p_{t+j-1})(p_{t+j} - p_t) > \frac{1}{2}(p_{t+j} - p_{t+j-1})^2 > 0. \tag{33}
\]
We now replace in (22) written for \(t + j\) instead of \(t\) the element \(z\) by \(s_t\) and obtain
\[
(p_{t+j} - p_{t+j-1})(s_{t+j} - s_t) \geq 0, \tag{34}
\]
hence, combining (33) with (34), we have
\[
(p_{t+j} - p_t)(s_{t+j} - s_t) \geq 0,
\]
which implies that
\[
(p_{t+j} - p_t)^2 \leq (p_{t+j} - p_t)(x_{t+j} - x_t)
\]
in contradiction with (31). This completes the proof of Lemma 4.3. \(\blacksquare\)
4.2 Long time asymptotics

This section is devoted to the study of the asymptotic behavior of system (1), (10) as $t \to \infty$. Put $z_t = s_t [s_0, c_1, p + c_2 y_t] = c_1 x_t + c_2 y_t - r_t + \xi_t$. This allows us, with the notation (23), to rewrite (1), (10) in the form

$$ (1 - a_1) \nabla_t y + (a_1 + a_2 c_2) y_t + a_2 (c_1 - 1) x_t + a_2 s_t = a_2 (z_t - \xi_t) + \epsilon_t, $$

$$ (1 - b_1) \nabla_t x + b_1 s_t - b_2 y_t = \eta_t. $$

As a consequence of (35), we have

$$ (1 - b_2) \nabla_t^2 x + b_1 \nabla_t s - b_2 \nabla_t y = \nabla_t^2 \eta $$

with the notation (23). This enables us to eliminate $y_t$ from the system (35) and reformulate it as a second order equation

$$ \nabla_t^2 x + A \nabla_t x + B \nabla_t s + C x_t + D s_t = h_t, \quad t \geq 2, $$

with positive constants

$$ A = \frac{a_1 + a_2 c_2}{1 - a_1}, \quad B = \frac{b_1}{1 - b_1}, \quad C = \frac{a_2 b_2 (c_1 - 1)}{(1 - a_1)(1 - b_1)}, \quad D = \frac{b_1 (a_1 + a_2 c_2) + a_2 b_2}{(1 - a_1)(1 - b_1)}, $$

and with the right-hand side

$$ h_t = \frac{1}{(1 - a_1)(1 - b_1)} ((1 - a_1) \nabla_t \eta + b_2 (a_2 (z_t - \xi_t) + \epsilon_t) + (a_1 + a_2 c_2) \eta_t). $$

The sequence $\{h_t\}$ contains the term $z_t$ which is bounded above by $\sigma$, and the noise terms $\epsilon_t, \xi_t, \eta_t$, and $\nabla_t \eta$.

Equation (36) always has a solution $x_t$ at each time step $t$, since the right-hand side of (36) is for each fixed $t$ a bounded function of $x_t$ and the left hand side is an increasing piecewise linear function of $x_t$. In some cases, the solution may not be unique if the coefficient $D^* := a_2 b_2 / ((1 - a_1)(1 - b_1))$ in front of $x_t$ on the right-hand side is large. Our computations below show, however, that all solutions have the same asymptotic convergence towards a small neighborhood of a particular equilibrium point depending on the trajectory.

4.2.1 Auxiliary estimate 1

We put

$$ q_t := C x_t + D s_t, $$

and multiply the equation (36) by $\nabla_t q = C \nabla_t x + D \nabla_t s$. Putting

$$ V_t^1 := \frac{1}{2} \left( C (\nabla_t x)^2 + D (\nabla_t s)^2 + q_t^2 \right) $$

and using the relations (24)–(25) we obtain that

$$ V_t^1 - V_{t-1}^1 + C \left( \frac{1}{2} (\nabla_t^2 x)^2 + \frac{D}{2} (\nabla_t^2 s)^2 + AC (\nabla_t x)^2 + (BC + AD + BD) (\nabla_t s)^2 \right) + \frac{1}{2} (\nabla_t q)^2 \leq h_t \nabla_t q. $$

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4.2.2 Auxiliary estimate 2

We now rewrite (36) in the form
\[ \nabla^2 t x + \frac{A}{C} \nabla_t q + \left( B - \frac{AD}{C} \right) \nabla_t s + q_t = h_t \] (42)
with \( q_t \) given by (39), and multiply it by \( q_t \). We use (24) again and find constants \( E, F > 0 \) depending on \( A, B, C, D \) such that
\[ \nabla_t x q_t - \nabla_t q_t - \frac{1}{2} q_t^2 \leq \frac{1}{2} (\nabla_t x)^2 + A (\nabla_t x)^2 + h_t q_t. \] (43)

We now set
\[ V^0_t := \nabla_t x q_t + \frac{A}{2C} q_t^2, \] (44)
so that (43) has the form
\[ V^0_t - V^0_{t-1} + \frac{1}{2} q_t^2 \leq F \left( \frac{1}{2} (\nabla_t x)^2 + A (\nabla_t x)^2 \right) + h_t q_t. \] (45)

Finally, we choose \( \lambda > 0 \) such that \( \lambda F < C \), and \( \lambda V^0_t \geq - \frac{1}{2} V^0_t \) for all \( t \in \mathbb{N} \), and put \( W_t = V^0_t + \lambda V^0_t \). Then, due to (41) and (45), there exists a constant \( \mu > 0 \) such that
\[ W_t - W_{t-1} + \mu W_t \leq L |h_t| \sqrt{W_t} \] (46)
for all \( t \in \mathbb{N} \) as a consequence of (40), (41), and (45), with some constant \( L > 0 \).

4.2.3 Asymptotic behavior: Proof of Theorems 3.1–3.2

Let \( c_1 > 1 \). If \( h_t = 0 \), that is, no noise is present and the reaction of the central bank is instantaneous with \( \sigma = 0 \), then \( W_t \) is a Lyapunov function of the system which decays exponentially to 0. In particular, \( q_t \) defined by (39) converge exponentially to \( q_\infty = 0 \). Then, it follows from Lemmas 4.2 and 4.3 that \( x_t \) converge to some value \( x_\infty \), hence \( s_t = s_0[x_0, x]^t \) converge to some \( s_\infty \) such that
\[ C x_\infty + D s_\infty = 0, \] (47)
and \( p_t = x_t - s_t \to x_\infty - s_\infty \). Further, equations (8) imply that \( y_t \to y_\infty \) and the point \( (x_\infty, y_\infty, p_\infty) \) belongs to the set (9). This completes the proof of Theorem 3.1.

In the case of a general right-hand side \( h_t \), we have for every \( t > T > 0 \) as a consequence of (46) that
\[ W_t \leq (1 + \mu)^{-T} W_{T-T} + L \sum_{j=T-T+1}^{t} |h_j| \sqrt{W_j} (1 + \mu)^{j-1-t}. \] (48)

Assume that there exists \( T > 0 \) such that for all \( t_0 \in \mathbb{N} \) we have
\[ \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} |h_j| \leq \hat{\sigma} . \] (49)
Then it follows from (48) that
\[ W_t \leq (1 + \mu)^{-T} W_{t-T} + \frac{LT\hat{\sigma}}{1 + \mu} \max_{j=t-T+1, \ldots, t} \sqrt{W_j} \]  
(50)
for all \( t > T \). Assume first that there exists \( t > T \) such that \( W_t \geq W_j \) for all \( j = t-T, \ldots, t \). Then (50) yields that
\[ W_t \leq (1 + \mu)^{-T} W_t + \frac{LT\hat{\sigma}}{1 + \mu} \sqrt{W_t}, \]
hence,
\[ W_t \leq \left( \frac{LT\hat{\sigma}(1 + \mu)^{T-1}}{(1 + \mu)^T - 1} \right)^2. \]
This implies in particular that \( W_t \) is bounded and we put
\[ W^* := \limsup_{t \to \infty} W_t < \infty. \]

For an arbitrary \( \delta > 0 \) we find \( t_0 \) sufficiently large such that for all \( t > t_0 - T \) we have \( W_t \leq W^* + \delta \). Then for \( t > t_0 \) we obtain from (50) that
\[ W_t \leq (1 + \mu)^{-T} (W^* + \delta) + \frac{LT\hat{\sigma}}{1 + \mu} \sqrt{W^* + \delta}, \]
hence,
\[ W^* \leq (1 + \mu)^{-T} (W^* + \delta) + \frac{LT\hat{\sigma}}{1 + \mu} \sqrt{W^* + \delta}, \]
that is,
\[ W^* + \delta \leq \frac{LT\hat{\sigma}(1 + \mu)^{T-1}}{(1 + \mu)^T - 1} \sqrt{W^* + \delta} + \frac{\delta(1 + \mu)^T}{(1 + \mu)^T - 1}, \]
and we conclude that
\[ W^* \leq \left( \frac{LT\hat{\sigma}(1 + \mu)^{T-1}}{(1 + \mu)^T - 1} \right)^2 + \frac{\delta(1 + \mu)^T + 1}{(1 + \mu)^T - 1}. \]  
(51)
Since \( \delta > 0 \) is arbitrary, we obtain that
\[ \limsup_{t \to \infty} W_t \leq \left( \frac{LT\hat{\sigma}(1 + \mu)^{T-1}}{(1 + \mu)^T - 1} \right)^2. \]  
(52)

Estimate (52) gives a uniform upper bound for the value of \( \limsup_{t \to \infty} W_t \) independent of the initial condition if the noise terms \( \epsilon_t, \xi_t, \eta_t \). In particular, if \( |h_t| \leq \hat{\sigma} \) for all \( t \in \mathbb{N} \), then formula (52) holds with \( T = 1 \), that is,
\[ \limsup_{t \to \infty} W_t \leq \left( \frac{L\hat{\sigma}}{\mu} \right)^2. \]  
(53)
Since by construction \( (Cx_t + Ds_t)^2 = q_t^2 \leq 2V_1^1 \leq 4W_1^1 \) and by definition \( x_*(s_t) = \kappa s_t = -s_tD/C \), it follows that
\[ \limsup_{t \to \infty} (x_t - x_*(s_t))^2 \leq \left( \frac{2L\hat{\sigma}}{\mu C} \right)^2. \]  
(54)
Further, the first equation in (1) implies that
\[ |p_{\ell_{\text{class}}} as a time discretization of a continuous time system and the increments of noise are of
3.2.

hence the second of the estimates (14) follows from the relations
\[ C(\nabla_t x)^2 \leq 2V_t^1 \leq 4W_t \]
and (53) combined with \( \hat{\sigma} = L'\sigma + L''m \) and (15). This completes the proof of Theorem
3.2.

A few remarks are in order. First, estimate (52) can be useful if system (1) is obtained
as a time discretization of a continuous time system and the increments of noise are of
class \( \ell_1 \). Second, counterparts of Theorems 3.1, 3.2 can be obtained by the same method
if \( p_t \) is allowed to depend also on \( y_t \), for example, if \( p_t = p_{\rho}[s_0, x + \delta y] \) with a small \( \delta > 0 \).

4.3 Stability of the multi-agent model

As mentioned earlier, the proof of Theorem 3.3 is parallel to the proof of Theorem 3.1 but
additionally relies on an inversion formula for the Prandtl-Ishlinskii operator. We start
by deriving the inverse operator.

4.3.1 Inversion of time discrete Prandtl-Ishlinskii operator

The main tool in our analysis is the following identity which, being inspired by the develop-
ments in [30, §34], is essentially due to M. Brokate, see [31, Proposition 2.2.16].

Lemma 4.4 (Brokate identity). Let \( \rho > 0, \sigma > 0, \beta \in [-\rho, \rho], \gamma \in [-\sigma, \sigma], \) and \( x \in S \) be
given. Put \( p = p_{\rho}[\beta, x], q = p_{\sigma}[\gamma, p] \). Then \( q = p_{\rho+\sigma}[\beta + \gamma, x] \). Moreover, the implication
\[ q_t - q_{t-1} \neq 0 \implies (p_t - p_{t-1})(q_t - q_{t-1}) > 0 \quad (55) \]
holds for every \( t \in \mathbb{N} \).

Proof. Put \( \varphi = p_{\rho+\sigma}[\beta + \gamma, x] \). We have by definition
\[ (p_t - p_{t-1})(x_t - p_t - \rho z) \geq 0, \quad (56) \]
\[ (q_t - q_{t-1})(p_t - q_t - \sigma z) \geq 0, \quad (57) \]
\[ (\varphi_t - \varphi_{t-1})(x_t - \varphi_t - (\rho + \sigma)z) \geq 0 \quad (58) \]
for all \( t \in \mathbb{N} \) and all \( z \in [-1, 1] \). In [37], we may choose \( \sigma z = p_{t-1} - q_{t-1} \) and obtain
\[ (q_t - q_{t-1})^2 \leq (q_t - q_{t-1})(p_t - p_{t-1}), \]
and the implication (55) follows. In particular, the inequality (56) remains valid if we
replace \( p_t - p_{t-1} \) with \( q_t - q_{t-1} \), that is,
\[ (q_t - q_{t-1})(x_t - p_t - \rho z) \geq 0 \quad (59) \]
for all \( t \in \mathbb{N} \) and all \( z \in [-1, 1] \). Adding (59) to (57) yields
\[ (q_t - q_{t-1})(x_t - q_t - (\rho + \sigma)z) \geq 0 \quad (60) \]
for all \( t \in \mathbb{N} \) and all \( z \in [-1, 1] \). We have indeed \( |x_t - q_t| \leq |x_t - p_t| + |p_t - q_t| \leq \rho + \sigma \), hence we may replace \((\rho + \sigma)z\) in (58) with \( x_t - q_t \), in (60) with \( x_t - \varphi_t \), and sum the two inequalities to obtain

\[
((q_t - \varphi_t) - (q_{t-1} - \varphi_{t-1}))(q_t - \varphi_t) \leq 0.
\] (61)

From (61) and (24) for \( z = q - \varphi \) it thus follows \((q_t - \varphi_t)^2 \leq (q_0 - \varphi_0)^2\) for all \( t \in \mathbb{N} \). We have \( \varphi_0 = x_0 - \beta - \gamma \), \( q_0 = p_0 - \gamma = x_0 - \beta - \gamma = \varphi_0 \), and this completes the proof. ■

We now recall the definition of a Prandtl-Ishlinskii operator as a linear combination of play operators. More specifically, let \( n \in \mathbb{N}, \nu_0, \ldots, \nu_n \in \mathbb{R}, 0 = \rho_0 < \rho_1 < \cdots < \rho_n \), and \( \beta_i \in [-\rho_i, \rho_i] \) be given numbers (in particular, \( \beta_0 = 0 \)). For \( x \in S \) and \( \beta = (\beta_1, \ldots, \beta_n) \) we put

\[
F[\beta, x] = \sum_{i=0}^{n} \nu_i p_{\rho_i}[\beta_i, x]
\] (62)

with the convention \( p_0[0, x] = x \). The mapping \( F : [-\rho_1, \rho_1] \times \cdots \times [-\rho_n, \rho_n] \times S \to S \) is called a (time and memory discrete) Prandtl-Ishlinskii operator.

**Hypothesis 4.5.** Let \( F \) be a Prandtl-Ishlinskii operator as in (62). The following conditions are assumed to hold:

(i) \( |\beta_i - \beta_{i-1}| \leq \rho_i - \rho_{i-1} \) for \( i = 1, \ldots, n \);

(ii) \( A_i := \sum_{j=0}^{i} \nu_j > 0 \) for \( i = 0, \ldots, n \).

We prove the following statement which shows that the inverse of a discrete Prandtl-Ishlinskii operator is again a Prandtl-Ishlinskii operator. In the continuous case, this result goes back to [46]. Finite collections of stops have been considered in [47]. A substantially more general situation in the space of regulated functions is considered in [48]. In fact, the explicit inversion formula presented below can also be deduced from [48, Corollary 3.3] which uses deeper results from the Kurzweil integration theory. In the discrete case, there exists an elementary proof that we present here. Note that Lemma 4.2 is a special case of Theorem 4.6 in the case \( n = 1 \).

**Theorem 4.6.** Let Hypothesis 4.5 hold. Define \( B_i := 1/A_i \) and

\[
\sigma_0 = \gamma_0 = 0, \quad \sigma_i - \sigma_{i-1} = A_{i-1}(\rho_i - \rho_{i-1}), \quad \gamma_i - \gamma_{i-1} = A_{i-1}(\beta_i - \beta_{i-1}),
\]

\[
\zeta_0 := B_0, \quad \zeta_i := B_i - B_{i-1}
\]

for \( i = 1, \ldots, n \). For an arbitrary \( x \in S \) put

\[
v = \sum_{i=0}^{n} \nu_i p_{\rho_i}[\beta_i, x].
\]

Then,

\[
x = \sum_{i=0}^{n} \zeta_i p_{\sigma_i}[\gamma_i, v].
\]

We start with an auxiliary identity.
Proposition 4.7. Let the hypotheses of Theorem 4.6 hold. For \( i = 1, \ldots, n \) put
\[
x^{(i)} = p_{\rho_i - \rho_{i-1}}[\beta_i - \beta_{i-1}, x^{(i-1)}], \quad v^{(i)} = p_{\sigma_i - \sigma_{i-1}}[\gamma_i - \gamma_{i-1}, v^{(i-1)}]
\]
with \( x^{(0)} = x, v^{(0)} = v \). Then for \( j = 1, \ldots, n \) we have
\[
v^{(j)} = A_j x^{(j)} + \sum_{i=j+1}^{n} \nu_i x^{(i)}.
\]

Proof of Proposition 4.7. The definition of the play states that
\[
(x_t^{(i)} - x_{t-1}^{(i)})(x_t^{(i-1)} - x_t^{(i)}) - (\rho_i - \rho_{i-1})z \geq 0, \quad (63)
\]
\[
(v_t^{(i)} - v_{t-1}^{(i)})(v_t^{(i-1)} - v_t^{(i)}) - (\sigma_i - \sigma_{i-1})z \geq 0, \quad (64)
\]
for all \( i = 1, \ldots, n, t \in \mathbb{N}, \) and \( |z| \leq 1 \). For \( j = 1, \ldots, n \) put
\[
(\varphi_0^{(j)}, \varphi_1^{(j)}, \ldots) = \varphi^{(j)} := A_j x^{(j)} + \sum_{i=j+1}^{n} \nu_i x^{(i)}.
\]
Then we have
\[
\varphi^{(0)} = v^{(0)}, \quad \varphi_t^{(i)} = \varphi_t^{(i-1)} = A_{i-1}(x_t^{(i)} - x_t^{(i-1)}).
\]
Hence, we may choose in (63)-(64) suitable values of \( z \) in order to obtain
\[
(x_t^{(i)} - x_{t-1}^{(i)})(\varphi_t^{(i-1)} - \varphi_t^{(i)} - v_t^{(i-1)} + v_t^{(i)}) \geq 0, \quad (66)
\]
\[
(v_t^{(i)} - v_{t-1}^{(i)})(v_t^{(i-1)} - v_t^{(i)} - \varphi_t^{(i-1)} + \varphi_t^{(i)}) \geq 0, \quad (67)
\]
for \( i = 1, \ldots, n \) and \( t \in \mathbb{N} \). We now use the implication (55) to conclude that for all \( n \geq j \geq i \geq 1 \) we have
\[
(x_t^{(j)} - x_{t-1}^{(j)})(\varphi_t^{(i-1)} - \varphi_t^{(i)} - v_t^{(i-1)} + v_t^{(i)}) \geq 0, \quad (68)
\]
\[
(v_t^{(j)} - v_{t-1}^{(j)})(v_t^{(i-1)} - v_t^{(i)} - \varphi_t^{(i-1)} + \varphi_t^{(i)}) \geq 0.
\]
Summing the above inequalities over \( i = 1, \ldots, j \) yields (note that \( \varphi^{(0)} = v^{(0)} \) by (65))
\[
(x_t^{(j)} - x_{t-1}^{(j)})(\varphi_t^{(j)} - \varphi_t^{(j-1)}) \geq 0, \quad (v_t^{(j)} - v_{t-1}^{(j)})(\varphi_t^{(j)} - \varphi_t^{(j-1)}) \geq 0
\]
for \( j = 1, \ldots, n, t \in \mathbb{N} \). In particular, for \( j = n \), we have \( \varphi^{(n)} = A_n x^{(n)} \). Multiplying the first inequality of (68) by \( A_n \) and adding the second inequality yields
\[
(\varphi_t^{(n)} - v_t^{(n)} - \varphi_{t-1}^{(n)} + v_{t-1}^{(n)})(\varphi_t^{(n)} - v_t^{(n)}) \leq 0
\]
for all \( t \in \mathbb{N} \). Hence, by (24),
\[
(\varphi_t^{(n)} - v_t^{(n)})^2 \leq (\varphi_0^{(n)} - v_0^{(n)})^2. \quad (69)
\]
Note that by (53), \( \varphi_0^{(i)} - \varphi_0^{(i-1)} = A_{i-1}(x_0^{(i)} - x_0^{(i-1)}) = A_{i-1}(\beta_i - \beta_{i-1}) = \gamma_i - \gamma_{i-1} = v_0^{(i)} - v_0^{(i-1)} \), hence
\[
\varphi_0^{(i)} = v_0^{(i)} \quad \text{for } i = 0, \ldots, n. \quad (70)
\]
and from (69) it follows that
\[ \varphi^{(n)} = v^{(n)}. \]  
(71)

We now continue by backward induction and assume that \( \varphi^{(j)} = v^{(j)} \) for some \( 2 \leq j \leq n \). From (68) written with \( j - 1 \) instead of \( j \) we obtain, using the induction hypothesis, that
\[ -(x^{(j-1)}_t - x^{(j-1)}_{t-1})(v^{(j)}_t - \varphi^{(j)}_t - v^{(j-1)}_t + \varphi^{(j-1)}_t) \geq 0, \]
(72)
\[ -(v^{(j)}_t - v^{(j-1)}_t)(\varphi^{(j)}_t - v^{(j)}_t - \varphi^{(j-1)}_t + v^{(j-1)}_t) \geq 0. \]
(73)

We now add (73) to (67), multiply the sum of (72) with (66) by \( A \), and sum the two resulting inequalities to conclude that
\[ ((v^{(j)}_t - \varphi^{(j)}_t - v^{(j-1)}_t + \varphi^{(j-1)}_t) - (v^{(j)}_t - \varphi^{(j)}_t - v^{(j-1)}_t + \varphi^{(j-1)}_t))(v^{(j)}_t - \varphi^{(j)}_t - v^{(j-1)}_t + \varphi^{(j-1)}_t) \leq 0. \]

This is an inequality of type (24) which, together with the induction assumption \( \varphi^{(j)} = v^{(j)} \) yields
\[ (v^{(j-1)}_t - \varphi^{(j-1)}_t)^2 \leq (v^{(j-1)}_t - \varphi^{(j-1)}_t)^2. \]
(74)

Referring to (65) and (70) completes the proof. 
\[ \blacksquare \]

**Proof of Theorem 4.6** Let \( x^{(i)}, v^{(i)} \) be as in Proposition 4.7. By Lemma 4.4 we have \( v^{(i)} = p_{\sigma_i}[\gamma, v] \) for all \( i = 0, 1, \ldots, n \). We now use Proposition 4.7 and the summation by parts formula, which yields
\[ \sum_{j=0}^{n} \zeta_j v^{(i)} = \sum_{j=0}^{n} \zeta_j \left( A_j x^{(j)} + \sum_{i=j+1}^{n} \nu_i x^{(i)} \right) = x^{(0)} + \sum_{i=1}^{n} x^{(i)}(A_i \zeta_i + \nu_i B_{i-1}) = x, \]
and the proof is complete. 
\[ \blacksquare \]

### 4.3.2 Proof of Theorem 3.3

We can now use Theorem 4.6 to adapt the proof of Theorem 3.1 to the multi-agent case. The multi-agent system (18), (21) can still be rewritten as the equation
\[ \nabla_t^2 x + A \nabla_t x + B \nabla_t s + C x_t + D s_t = 0 \]
(75)
with positive constants \( A, B, C, D \) given by (37) and \( s_t \) as in (20). From the inequalities (25)–(27) we easily obtain their counterparts
\[ \nabla_t x \nabla_t s = \nu_0(\nabla_t x)^2 + \sum_{i=1}^{n} \nu_i \nabla_i x \nabla_t s^{(i)} \geq \nu_0(\nabla_t x)^2 + \sum_{i=1}^{n} \nu_i(\nabla_t s^{(i)})^2, \]
(76)
\[ \nabla_t^2 x \nabla_t s = \nu_0 \nabla_t^2 x \nabla_t x + \sum_{i=1}^{n} \nu_i \nabla_i^2 x \nabla_t s^{(i)} \geq \nu_0 \nabla_t^2 x \nabla_t x + \sum_{i=1}^{n} \nu_i \nabla_i^2 s^{(i)} \nabla_t s^{(i)} \]
\[ \geq \frac{\nu_0}{2}((\nabla_t^2 x)^2 - (\nabla_t x)^2)^2 + \sum_{i=1}^{n} \frac{\nu_i}{2}((\nabla_t s^{(i)})^2 - (\nabla_t s^{(i)})^2 + (\nabla_t^2 s^{(i)})^2), \]
(77)
where we denote \( s^{(i)} := s_{\rho_i}[\beta, x]. \)
As in Section 4.2, we define \( q_t \) by formula (39) and multiply equation (75) by \( \nabla_t q \). It follows from (76)–(77) that

\[
\nabla_t^2 x \nabla_t q = (C+\nu_0 D)\nabla_t^2 x \nabla_t x + D \sum_{i=1}^n \nu_i \nabla_t^2 x \nabla_t s^{(i)} \geq (C+\nu_0 D)\nabla_t^2 x \nabla_t x + D \sum_{i=1}^n \nu_i \nabla_t^2 s^{(i)} \nabla_t s^{(i)}
\]

\[
\geq \frac{C+\nu_0 D}{2} ((\nabla_t x)^2 - (\nabla_{t-1} x)^2 + (\nabla_t^2 x)^2) + \sum_{i=1}^n \frac{\nu_i D}{2} ((\nabla_t s^{(i)})^2 - (\nabla_{t-1} s^{(i)})^2 + (\nabla_t^2 s^{(i)})^2)
\]

and

\[
(A \nabla_t x + B \nabla_t s) \nabla_t q \geq (AC+\nu_0 (BC+AD))(\nabla_t x)^2 + BD(\nabla_t s)^2 + (BC+AD) \sum_{i=1}^n \nu_i (\nabla_t s^{(i)})^2.
\]

Putting

\[
\tilde{V}_t^1 := \frac{1}{2} \left( (C+\nu_0 D)(\nabla_t x)^2 + D \sum_{i=1}^n \nu_i (\nabla_t s^{(i)})^2 + q_t^2 \right),
\]

we obtain (cf. (41) with \( h_t = 0 \))

\[
\tilde{V}_t^1 - \tilde{V}_{t-1}^1 + \frac{C+\nu_0 D}{2} (\nabla_t^2 x)^2 + \frac{1}{2} q_t^2 \geq (\nabla_t x - \nabla_{t-1} x) \nabla_t q + (AC+\nu_0 (BC+AD))(\nabla_t x)^2
\]

\[
+ BD(\nabla_t s)^2 + (BC+AD) \sum_{i=1}^n \nu_i (\nabla_t s^{(i)})^2 + \frac{1}{2} (\nabla_t q)^2 \leq 0.
\]

We continue as in Section 4.2 and multiply (75) by \( q_t \). Similarly to (43), we obtain

\[
\nabla_t x q_t - \nabla_{t-1} x q_{t-1} + \frac{A}{2C} (q_t^2 - q_{t-1}^2) + \frac{1}{2} q_t^2 \leq (\nabla_t x - \nabla_{t-1} x) \nabla_t q + E(\nabla_t s)^2 - \frac{A}{2C} (\nabla_t q)^2
\]

\[
\leq F \left( \frac{1}{2} (\nabla_t^2 x)^2 + (\nabla_t x)^2 + \sum_{i=1}^n \nu_i (\nabla_t s^{(i)})^2 \right)
\]

with some constants \( E, F > 0 \) depending only on \( A, B, C, D \). We set

\[
\tilde{V}_t^0 := \nabla_t x q_t + \frac{A}{2C} q_t^2
\]

and rewrite (78) in the form

\[
\tilde{V}_t^0 - \tilde{V}_{t-1}^0 + \frac{1}{2} q_t^2 \leq F \left( \frac{1}{2} (\nabla_t^2 x)^2 + (\nabla_t x)^2 + \sum_{i=1}^n \nu_i (\nabla_t s^{(i)})^2 \right),
\]

which is parallel to (44)–(45). We now define an auxiliary energy functional

\[
\tilde{W}_t = \tilde{V}_t^1 + \lambda \tilde{V}_t^0
\]

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with $\lambda > 0$ such that $\tilde{V}_t^0 \geq -\frac{1}{2} \tilde{V}_t^1$ and $\lambda < \min\{C + \nu_0 D, AC + \nu_0 (BC + AD), BC + AD\}$. We then find $\mu > 0$ and $L > 0$ such that for all $t \in \mathbb{N}$ we have the inequality

$$\tilde{W}_t - \tilde{W}_{t-1} + \mu \tilde{W}_t \leq 0.$$ 

Therefore, the decay of $\tilde{W}_t$ is exponential according to the formula

$$\tilde{W}_t \leq \left(\frac{1}{1 + \mu}\right)^t \tilde{W}_0,$$

hence also

$$\lim_{t \to \infty} q_t = 0. \quad (79)$$

Using (20), we can rewrite formula (39) as

$$q_t = (C + \nu_0 D)x_t + D \sum_{i=1}^{n} \nu_i s_{\rho_i}[\beta_i, x]_t = (C + D)x_t - D \sum_{i=1}^{n} \nu_i p_{\rho_i}[\beta_i, x]_t.$$ 

Hence, $q_t$ is given by a Prandtl-Ishlinskii operator of the form

$$q_t = \tilde{\nu}_0 x_t + \sum_{i=1}^{n} \tilde{\nu}_i p_{\rho_i}[\beta_i, x]_t$$

with $\tilde{\nu}_0 = C + D$, $\tilde{\nu}_i = -\nu_i D$ for $i = 1, \ldots, n$. The hypotheses of Theorem 4.6 are satisfied for

$$\tilde{A}_i := \sum_{j=0}^{i} \tilde{\nu}_j \geq C + D \left(1 - \sum_{j=0}^{i} \nu_j\right) \geq C > 0.$$ 

Consequently, by Theorem 4.6 we have

$$x_t = \tilde{\zeta}_0 q_t + \sum_{i=1}^{n} \tilde{\zeta}_i p_{\sigma_i}[\gamma_i, q]_t$$

with suitable constants $\tilde{\zeta}_i, \sigma_i, \gamma_i$. From (79) and from Lemma 4.3 we conclude that $x_t$ and $s_t$ are Cauchy sequences that converge to some limits $x_\infty$, $s_\infty$, respectively, which completes the proof of Theorem 3.3.

5 Conclusions

We have replaced rational expectations about future inflation with a form of boundedly rational aggregated ‘sticky’ expectation modeled by the play operator in a simple standard macroeconomic model. This single (and conceptually quite elementary) change transforms a unique equilibrium linear system to a PWL system with an entire continuum of equilibrium states. The PWL model with $n$ agents has $2^n$ switching surfaces and an $n$-dimensional continuum of equilibria. By constructing a Lyapunov function and developing a technique for inverting the Prandtl-Ishlinskii operator, we have shown that, when there is no exogenous noise, the continuum of equilibria is the global attractor of the system. The size of the basin of attraction of a particular equilibrium varies, generally becoming smaller towards the boundary of the set of equilibrium states.
If the presence of stickiness/frictions in economics does indeed induce a myriad of coexisting (metastable) equilibria then phenomena that are not possible (or require a posteriori model adjustments) in unique equilibrium models become not just feasible but inevitable. Perhaps the most obvious of these permanence, also known as remanence, where a system does not revert to its previous state after an exogenous shock is applied and then removed. It is of course a central concern of macroeconomics whether or not economies affected by, say, significant negative shocks can be expected to have permanently reduced productivity levels. For the models studied in this paper, after sufficiently small shocks (whether exogenous or applied by policy makers) the system will indeed revert to the same equilibrium but larger shocks will move the system from the basin of attraction of one equilibrium to the basin of attraction of a different one (at the same model parameters). The path to this new equilibrium may be long with a highly unpredictable endpoint. Furthermore, in the latter case the system will not exhibit a tendency to return to its pre-shocked state — the model displays true permanence. And the model parameters alone cannot determine which equilibrium a system is currently in without knowing important information about the prior states of the system — true path dependence. Hence, the model accounts for several hard-to-explain empirical regularities observed in economic data. This feature of the model is significant not just because it corresponds closely to actual economic events but it may have implications for forecasting and policy prescriptions too.

Our model of expectation formation is thus both mathematically tractable and has some basis in both observed data (see also [38, 39]) and models of bounded rationality. As such it provides a potentially useful, analytically tractable, alternative to staggered/delayed models — and one with additional complexity and explanatory power. Our choice of inflation expectations as the candidate for an initial investigation was influenced by the work of De Grauwe [14] on a different type of boundedly rational expectation formation process in a simple DSGE model. However, play operators are also a viable candidate for modeling other sticky economic variables at both the micro- and macro-economic levels. To demonstrate this, we used a play operator to represent sticky responses by the Central Bank. Although it has not been relevant to this paper play and stop operators, when combined appropriately [48] can have a remarkably simple aggregated response, even when connected via a network. This allows for (almost)-analytic solutions even when cascades and rapid transitions between states are occurring and will be the subject of future work. The same form of stickiness described above with the associated play operators have already been used to develop non-equilibrium asset-pricing models [49].

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