Some results from algebraic geometry over complete discretely valued fields

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Abstract

This paper is concerned with algebraic geometry over complete discretely valued fields $K$ of equicharacteristic zero. Several results are given including: the canonical projection $K^n \times K^m \to K^n$ and blow-ups of the $K$-points of smooth $K$-varieties are definably closed maps; a descent property; a version of the Łojasiewicz inequality for continuous rational functions; curve selection for semialgebraic sets and the theorem on extending continuous hereditarily rational functions, established for the real field in our joint paper with J. Kollár. Our approach applies quantifier elimination due to Pas. By the transfer principle of Ax–Kochen–Ershov, all these results carry over to the case of Henselian discretely valued fields. Using different arguments in our subsequent paper [13], we establish them over Henselian real valued fields of equicharacteristic zero.

1. Introduction. We deal with complete discretely valued fields $K$ of equicharacteristic zero (with valuation $v$, valuation ring $R$ and residue field $k$). The $K$-points $X(K)$ of any $K$-algebraic variety $X$ inherit from $K$ a topology, called the $K$-topology. We prove that, for any $L$-definable subset $D$ of $K^n$, the canonical projection $D \times R^m \to D$ is definably closed in the $K$-topology (Theorem 1), and next draw several conclusions, including the theorem that blow-ups of the $K$-points of smooth $K$-varieties are definably closed maps, a descent property for such blow-ups, a version of the
Lojasiewicz inequality (Theorem 2) for continuous rational functions, curve selection for semialgebraic sets and the theorem on extending continuous hereditarily rational functions (Theorem 3), established for the real field in our joint paper [10] with J. Kollár. Observe that, via the transfer principle of Ax–Kochen–Ershov (see e.g. [3, Chap. II]), Theorem 1 and thus all results of this paper carry over to the case of Henselian discretely valued fields. Using different arguments in our subsequent paper (in preparation), we establish them over Henselian real valued fields of equicharacteristic zero.

One of the basic tools we use in the proof of Theorem 1 is quantifier elimination due to Pas ([14], Theorem 4.1) in the language $\mathcal{L}$ of Denef–Pas (cf. [6, 14]) with three sorts: the valued field $K$-sort, the value group $\Gamma$-sort and the residue field $k$-sort. The language of the $K$-sort is the language of rings; that of the $\Gamma$-sort is any augmentation of the language of ordered abelian groups (and $\infty$); finally, that of the $k$-sort is any augmentation of the language of rings. Additionally, we add a map $v$ from the field sort to the value group (the valuation), and a map $\overline{ac}$ from the field sort to the residue field (angular component map) which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring $R$ of $K$.

While there are several quantifier elimination results for algebraically closed valued fields, the situation is more delicate for valued fields which are not algebraically closed. The former results — expressed in natural languages being 1-sorted (the valued field), 2-sorted (the valued field and value group, and with the valuation map $v$) as well as 3-sorted (the valued field, value group and residue field, and with the valuation map $v$ and residue map) — are essentially due to Robinson [16]. Their proofs rely on the fact that the extensions of a given valuation to a finite normal field extension are all conjugate under the Galois group.

It is easy to check that the subsets of $K^n$, $n \in \mathbb{N}$, definable in the above 2-sorted language with the valuation map $v$ are precisely the subsets of $K^n$ definable in the expansion of the field $K$ by adding the unary relation symbol construed as the valuation ring $R$. Those sets shall be called $v$-definable. We now state the main result.

**Theorem 1.** Let $D$ be an $\mathcal{L}$-definable subset of $K^n$. Then the canonical projection $\pi : D \times R^m \longrightarrow D$ is definably closed in the $K$-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$. 
The proof of this theorem will be given in Section 2. By the transfer principle, Theorem 1 and the four corollaries stated below are valid over Henselian discretely valued fields of equicharacteristic zero. When the ground field $K$ is locally compact, they hold by a routine topological argument.

**Corollary 1.** Let $K \mathbb{P}^m$ be the projective space of dimension $m$ over $K$ and $D$ an $\mathcal{L}$-definable subset of $K^n$. Then the canonical projection $\pi : D \times K \mathbb{P}^m \to D$ is definably closed.

**Corollary 2.** Let $\phi_i$, $i = 0, \ldots, m$, be regular functions on $K^n$, $D$ be an $\mathcal{L}$-definable subset of $K^n$ and $\sigma : Y \to KA^n$ the blow-up of the affine space $K \mathbb{A}^n$ with respect to the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction

$$\sigma : Y(K) \cap \sigma^{-1}(D) \to D$$

is a definably closed quotient map.

Indeed, $Y(K)$ can be regarded as a closed subvariety of $K^n \times K \mathbb{P}^m$ and $\sigma$ the canonical projection.

Since the problem is local with respect to the target space, the above corollary immediately generalizes to the case where the $K$-variety $Y$ is the blow-up of a smooth $K$-variety $X$.

**Corollary 3.** Let $X$ be a smooth $K$-variety, $\phi_i$, $i = 0, \ldots, m$, regular functions on $X$, $D$ be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \to X$ the blow-up of the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction

$$\sigma : Y(K) \cap \sigma^{-1}(D) \to D$$

is a definably closed quotient map.

**Corollary 4** (Descent Property). Under the assumptions of Corollary 3, every continuous $\mathcal{L}$-definable function $g : \sigma^{-1}(D) \to K$ that is constant on the fibers of the blow-up $\sigma$ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \to K$.

Section 3 is devoted to a certain version of the Lojasiewicz inequality for continuous rational functions on $\mathcal{L}$-definable open (in the $K$-topology) subsets of a smooth $K$-variety and to curve selection for semialgebraic sets.
Having at our disposal the descent property and the Łojasiewicz inequality, we are able to adapt the proof of Proposition 10 on extending continuous hereditarily rational functions, from our joint paper with J. Kollár [10], to the case of Henselian discretely valued fields (and, at the same time, \(p\)-adic fields and their finite extensions). In fact, the lack of these two results was the only obstacle. This will be outlined in Section 4. In this fashion, all results from paper [10] hold over such fields, because so do all other results except Proposition 10 over any field with the density property (DP) (thus, in particular, over all complete real valued fields of characteristic zero).

2. Proof of Theorem 1. We begin with quantifier elimination due to Pas for Henselian valued fields \(K\) with residue field \(k\) of characteristic zero. These properties are, of course, definable in the language \(\mathcal{L}\) of Denef–Pas. We denote \(K\)-sort variables by \(x, y, z, \ldots\), \(k\)-sort variables by \(\xi, \zeta, \eta, \ldots\), and \(\Gamma\)-sort variables by \(k, q, r, \ldots\).

In the case of non algebraically closed fields, passing to the three sorts with additional two maps: the valuation and the residue map, is not sufficient. Quantifier elimination due to Pas holds for Henselian valued fields of equicharacteristic zero in the 3-sorted language with the valuation and the angular component map. Not all valued fields have an angular component map, but it exists whenever the valued field has a cross section or the residue field is \(\aleph_1\)-saturated (cf. [15]). In general, unlike for \(p\)-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets.

For both \(p\)-adic fields (Denef [6]) and general Henselian valued fields (Pas [14]), quantifier elimination was established by means of cell decomposition and a certain preparation theorem (for polynomials in one variable with definable coefficients) combined with each other. In the latter case, however, cells are no longer finite in number, but are parametrized by residue field variables.

Finally, let us mention that quantifier elimination based on the sort \(RV := K^*/(1 + m)\) (where \(K^* := K \setminus \{0\}\) and \(m\) is the maximal ideal of the valuation ring \(R\)) was introduced by Besarab [1]. This new sort binds together the value group and residue field into one structure. In paper [8, Section 12], quantifier elimination for Henselian valued fields of equicharacteristic zero, based on this sort, was derived directly from that by Robinson for algebraically closed
valued fields. Yet another, more general result, including Henselian valued fields of mixed characteristic, was achieved by Cluckers–Loeser [5] for so-called b-minimal structures (from "ball minimal"); in the case of valued fields, however, countably many sorts $RV_n := K^*/(1 + n m)$, $n \in \mathbb{N}$, are needed.

**Quantifier Elimination.** Let $(K, \Gamma, k)$ be a structure for the 3-sorted language $\mathcal{L}$ of Denef–Pas. Assume that the valued field $K$ is Henselian and of equicharacteristic zero. Then $(K, \Gamma, k)$ admits elimination of $K$-quantifiers in the language $\mathcal{L}$.

We immediately obtain the following

**Corollary.** The 3-sorted structure $(K, \Gamma, k)$ admits full elimination of quantifiers whenever the theories of the value group $\Gamma$ and the residue field $k$ admit quantifier elimination in the languages of their sorts.

In this section, we deal with complete discretely valued fields $K$ of equicharacteristic zero, which are, by virtue of Cohen’s structure theorem, the quotient fields $K = k((t))$ of formal power series rings $k[[t]]$ in one variable $t$ with coefficients from a field $k$ of characteristic zero. The valuation $v$ and the angular component $ac$ of a formal power series are the degree and the coefficient of its initial monomial, respectively.

The additive group $\mathbb{Z}$ is an example of ordered $\mathbb{Z}$-group, i.e. an ordered abelian group with a (unique) smallest positive element (denoted by 1) subject to the following additional axioms:

$$\forall k \ k > 0 \ \Rightarrow \ k \geq 1$$

and

$$\forall k \ \bigvee_{r=0}^{n-1} \exists q \ k = nq + r$$

for all integers $n > 1$. The language of the value group sort will be the Presburger language of ordered $\mathbb{Z}$-groups, i.e. the language of ordered groups $\{<, +, -, 0\}$ augmented by 1 and binary relation symbols $\equiv_n$ for congruence modulo $n$ subject to the axioms:

$$\forall k, r \ k \equiv_n r \iff \exists q \ k - r = nq$$
for all integers $n > 1$. This theory of ordered $\mathbb{Z}$-groups in the Presburger language has quantifier elimination and definable Skolem (choice) functions. The above two countable axiom schemas can be replaced by universal ones when we augment the language by adding the function symbols $\lfloor \frac{k}{n} \rfloor$ (of one variable $k$) for division with remainder, which fulfil the following postulates:

$$\lfloor \frac{k}{n} \rfloor = q \iff \bigvee_{r=0}^{n-1} k = nq + r$$

for all integers $n > 1$. The theory of ordered $\mathbb{Z}$-groups admits therefore both quantifier elimination and universal axioms in the Presburger language augmented by division with remainder. Thus every definable function is piecewise given by finitely many terms and, moreover, is piecewise linear (see e.g. [4]).

In the residue field sort, we can add new relation symbols for all definable sets and impose suitable postulates. This enables quantifier elimination for the residue field in the augmented language. In this fashion, we have full quantifier elimination in the 3-sorted structure $(K, \mathbb{Z}, k)$ with $K = k((t))$.

Now we can readily pass to the proof of Theorem 1 which, of course, reduces easily to the case $m = 1$. So let $B$ be an $\mathcal{L}$-definable closed (in the $K$-topology) subset of $D \times \mathbb{R}_y \subset K^n \times \mathbb{R}_y$. It suffices to prove that if $a$ lies in the closure of the projection $A := \pi(B)$, then there is a point $b \in B$ such that $\pi(b) = a$.

Without loss of generality, we may assume that $a = 0$. Put

$$\Lambda := \{(v(x_1), \ldots, v(x_n)) \in \mathbb{Z}^n : x = (x_1, \ldots, x_n) \in A\}.$$

The set $\Lambda$ contains points all coordinates of which are arbitrarily large, because the point $a = 0$ lies in the closure of $A$. Hence and by definable choice, $\Lambda$ contains a set $\Lambda_0$ of the form

$$\Lambda_0 = \{(k, \alpha_2(k), \ldots, \alpha_n(k)) \in \mathbb{N}^n : k \in \Delta\} \subset \Lambda,$$

where $\Delta \subset \mathbb{N}$ is an unbounded definable subset and $\alpha_2, \ldots, \alpha_n : \Delta \rightarrow \mathbb{N}$ are increasing unbounded functions given by a term (because the function in one variable given by a term is either increasing or decreasing). We are going to recursively construct a point $b = (0, w) \in B$ with $w \in R$ by performing the following algorithm.
Step 1. Let

\[ \Xi_1 := \{(v(x_1), \ldots, v(x_n), v(y)) \in \Lambda_0 \times \mathbb{N} : (x, y) \in B\} \]

\(p_1 : \Xi_1 \rightarrow \mathbb{N}\) be the canonical projection into the last coordinate and

\[ \beta_1(k) := \sup_{p_1^{-1}(k, \alpha_2(k), \ldots, \alpha_n(k)) \in \mathbb{N} \cup \{\infty\}, \ k \in \Lambda_0}. \]

If \(\limsup_{k \to \infty} \beta_1(k) = \infty\), there is a sequence \((x^{(\nu)}, y^{(\nu)}) \in B, \ \nu \in \mathbb{N}\), such that

\[ v(x_1^{(\nu)}), \ldots, v(x_n^{(\nu)}), v(y^{(\nu)}) \to \infty \]

when \(\nu \to \infty\). Since the set \(B\) is a closed subset of \(D \times R_y\), we get

\[ (x^{(\nu)}, y^{(\nu)}) \to 0 \in B \quad \text{when} \ \nu \to \infty, \]

and thus \(w = 0\) is the point we are looking for. Here the algorithm stops.

Otherwise

\[ \Lambda_1 = \{l_1\} \subset \Xi_1 \]

for some infinite definable subset \(\Lambda_1\) of \(\Lambda_0\) and \(l_1 \in \mathbb{N}\). The set

\[ \{(v(x_1), \ldots, v(x_n); \overline{ac}(y)) \in \Lambda_1 \times \mathbb{k} : (x, y) \in B, \ v(y) = l_1\} \]

is definable in the language \(\mathcal{L}\). By quantifier elimination, it is given by a quantifier-free formula with variables only from the value group \(\Gamma\)-sort and the residue field \(\mathbb{k}\)-sort. Therefore there is a finite partitioning of \(\Lambda_1\) into definable subsets over each of which the fibres of the above set are constant, because quantifier-free \(\mathcal{L}\)-definable subsets of the product \(\mathbb{Z}^n \times \mathbb{k}\) of the two sorts are finite unions of the Cartesian products of definable subsets in \(\mathbb{Z}^n\) and in \(\mathbb{k}\), respectively. One of those definable subsets, say \(\Lambda_1'\), must be infinite. Consequently, for some \(\xi_1 \in \mathbb{k}\), the set

\[ \Xi_2 := \{(v(x_1), \ldots, v(x_n), v(y - \xi_1 l_1^i)) \in \Lambda_1' \times \mathbb{N} : (x, y) \in B\} \]

contains points of the form \((k, l) \in \mathbb{N}^{n+1}\), where \(k \in \Lambda_1'\) and \(l > l_1\).

Step 2. Let \(p_2 : \Xi_2 \rightarrow \mathbb{N}\) be the canonical projection into the last coordinate and

\[ \beta_2(k) := \sup_{p_2^{-1}(k, \alpha_2(k), \ldots, \alpha_n(k)) \in \mathbb{N} \cup \{\infty\}, \ k \in \Lambda_1'}. \]
If \( \limsup_{k \to \infty} \beta_2(k) = \infty \), there is a sequence \((x^{(\nu)}, y^{(\nu)}) \in B, \nu \in \mathbb{N} \), such that
\[
v(x_1^{(\nu)}), \ldots, v(x_n^{(\nu)}), v(y^{(\nu)} - \xi_1 t^{l_1}) \to \infty \quad \text{when } \nu \to \infty.
\]
Since the set \( B \) is a closed subset of \( D \times R \), we get
\[
(x^{(\nu)}, y^{(\nu)}) \to (0, \xi_1 t^{l_1}) \in B \quad \text{when } \nu \to \infty,
\]
and thus \( w = \xi_1 t^{l_1} \) is the point we are looking for. Here the algorithm stops.

Otherwise \( \Lambda_2 \times \{l_2\} \subset \Xi_2 \) for some infinite definable subset \( \Lambda_2 \) of \( \Lambda'_1 \) and \( l_2 > l_1 \). Again, for some \( \xi_2 \in \mathbb{k} \), the set
\[
\Xi_3 := \{(v(x_1), \ldots, v(x_n), v(y - \xi_1 t^{l_1} - \xi_2 t^{l_2})) \in \Lambda'_2 \times \mathbb{N} : (x, y) \in B\}
\]
contains points of the form \((k, l) \in \mathbb{N}^{n+1}, \) where \( k \in \Lambda'_2, \Lambda'_2 \) is an infinite definable subset of \( \Lambda_2 \) and \( l > l_2 \).

*Step 3* is carried out in the same way as the previous ones; and so on.

In this fashion, the algorithm either stops after a finite number of steps and then yields the desired point \( w \in R \) (actually, \( w \in \mathbb{k}[t] \)) such that \((0, w) \in B, \) or it does not stop and then yields a formal power series
\[
w := \xi_1 t^{l_1} + \xi_2 t^{l_2} + \xi_3 t^{l_3} + \ldots, \quad 0 \leq l_1 < l_2 < l_3 < \ldots
\]
such that for each \( \nu \in \mathbb{N} \) there exists an element \((x^{(\nu)}, y^{(\nu)}) \in B \) for which
\[
v(y^{(\nu)} - \xi_1 t^{l_1} - \xi_2 t^{l_2} - \ldots - \xi_\nu t^{l_\nu}) \geq l_\nu + 1 \geq \nu, \quad v(x_1^{(\nu)}), \ldots, v(x_1^{(\nu)}) \geq \nu.
\]
Hence \( v(y^{(\nu)} - w) \geq \nu, \) and thus the sequence \((x^{(\nu)}, y^{(\nu)}) \) tends to the point \( b := (0, w) \) when \( \nu \) tends to \( \infty \). Since the set \( B \) is a closed subset of \( D \times R \), the point \( b \) belongs to \( B \), which completes the proof. \( \square \)

3. **Łojasiewicz inequality and curve selection.** Consider a smooth \( K \)-variety \( X \) (it would be sufficient to assume that \( X \) is smooth at all points of \( X(K) \)). By a continuous rational function \( f \) on \( X(K) \) we mean a rational function \( f \) on \( X \) that extends continuously to \( X(K) \). In this section,
we assume that the ground field $K$ is a Henselian discretely valued field of equicharacteristic zero.

**Theorem 2** (Lojasiewicz Inequality.) Let $f, g$ be two continuous rational functions on $X(K)$ and $U$ be an $Ł$-definable open (in the $K$-topology) subset of $X(K)$. If
\[
\{x \in U : g(x) = 0\} \subset \{x \in U : f(x) = 0\},
\]
then there exist a positive integer $s$ and a continuous rational function $h$ on $U$ such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in U$.

**Proof.** Clearly, the functions $f, g$ are quotients of some regular functions on $X$:
\[
f = \frac{f_1}{f_2}, \quad g = \frac{g_1}{g_2}.
\]
We shall now apply a strong form of Hironaka’s transformation to a normal crossing (see e.g. [9, Chap. III] for references and relatively short proofs). Thus we can take a finite composite $\sigma : Y \longrightarrow X$ of blow-ups along smooth centers such that the pull-backs
\[
f_1^\sigma = f_1 \circ \sigma, \quad f_2^\sigma = f_2 \circ \sigma, \quad g_1^\sigma = g_1 \circ \sigma, \quad g_2^\sigma = g_2 \circ \sigma,
\]
are normal crossing divisors on $Y$ ordered with respect to divisibility relation. Since the quotients
\[
f^\sigma = \frac{f_1^\sigma}{f_2^\sigma} \quad \text{and} \quad g^\sigma = \frac{g_1^\sigma}{g_2^\sigma}
\]
are continuous on $Y(K)$, they are normal crossing divisors on $Y(K)$ too. Therefore, since
\[
\{y \in \sigma^{-1}(U) : g^\sigma(y) = 0\} \subset \{y \in \sigma^{-1}(U) : f^\sigma(y) = 0\},
\]
there is a positive integer $s$ and a normal crossing divisor $H$ on $Y(K) \cap \sigma^{-1}(U)$ vanishing on $\{y \in Y(K) \cap \sigma^{-1}(U) : f^\sigma(y) = 0\}$ such that
\[
(f^\sigma)^s(y) = H(y) \cdot g^\sigma(y) \quad \text{for all} \quad y \in Y(K) \cap \sigma^{-1}(U).
\]

Clearly, the regular function $H$ descends to the quotient $f^s/g$ over the set $\{x \in U : g(x) \neq 0\}$. On the other hand, $H$ vanishes on the fibres $\sigma^{-1}(x)$ if $x \in U$ and $g(x) = 0$. Hence and by the descent property (Corollary 4 to Theorem 1), $H$ descends to a continuous rational function $h$ on $U$. Then $f^s(x) = h(x) \cdot g(x)$ for all $x \in U$, as asserted. □
Remark. Such a version of the Lojasiewicz inequality holds over $p$-adic fields (and their finite extensions), because they are locally compact fields, and thus descent is available by a purely topological argument.

We now pass to curve selection over fields which are not locally compact. While the real case goes back to papers \[2, 18\] (see also \[11, 12\]), the $p$-adic one was achieved in papers \[17, 7\]. By a semialgebraic subset of $K^n$ we mean a (finite) Boolean combination of sets of the form

$$\{x \in K^n : v(f(x)) \leq v(g(x))\},$$

where $f$ and $g$ are regular functions on $K^n$. We call a map $\varphi$ semialgebraic if its graph is a semialgebraic set.

**Curve Selection Lemma.** Let $A$ be a semialgebraic subset of $K^n$. If $0$ lies in the closure (in the $K$-topology) of $A \setminus \{a\}$, then there is a semialgebraic map $\varphi : R \to K^n$ given by restricted power series such that $\varphi(0) = a$ and $\varphi(R \setminus \{0\}) \subset A \setminus \{a\}$.

**Proof.** It is easy to check that every semialgebraic set is a finite union of sets of the form

$$\{x \in K^n : v(f_1(x)) <_1 v(g_1(x)) \} \cap \ldots \cap \{x \in K^n : v(f_r(x)) <_r v(g_r(x)) \},$$

where $f_1, \ldots, f_r, g_1, \ldots, g_r$ are regular functions and $<_1, \ldots, <_r$ stand for $\leq$ or $<$. We may assume, of course, that $A$ is a set of this form. As before, we take a finite composite $\sigma : Y \to KA^n$ of blow-ups along smooth centers such that the pull-backs of the coordinates $x_1, \ldots, x_n$ and

$$f_1^\sigma, \ldots, f_r^\sigma \quad \text{and} \quad g_1^\sigma, \ldots, g_r^\sigma$$

are normal crossing divisors ordered with respect to divisibility relation, unless one of those functions vanishes. Since the restriction $\sigma : Y(K) \to K^n$ is definably closed (Corollary 4 to Theorem 1), there is a point $b \in Y(K) \cap \sigma^{-1}(a)$ which lies in the closure of

$$B := Y(K) \cap \sigma^{-1}(A \setminus \{a\}).$$

Further,

$$Y(K) \cap \sigma^{-1}(A) = \{v(f_1^\sigma(y)) <_1 v(g_1^\sigma(y))\} \cap \ldots \cap \{v(f_r^\sigma(y)) <_r v(g_r^\sigma(y))\},$$
and thus $\sigma^{-1}(A)$ is in suitable local coordinates $y = (y_1, \ldots, y_n)$ near $b = 0$ a finite intersection of sets of the form

$$\{v(y^\alpha) \leq v(u(y))\}, \quad \{v(u(y)) \leq v(y^\alpha)\}, \quad \{v(y^\beta) < \infty\} \quad \text{or} \quad \{\infty = v(y^\gamma)\},$$

where $\alpha, \beta, \gamma \in \mathbb{N}^n$ and $u(y)$ is a regular function which vanishes nowhere.

The first case cannot occur because $b = 0$ lies in the closure of $B$; the second case holds in a neighbourhood of $b$; the third and fourth cases are equivalent to $y^\beta \neq 0$ and $y^\gamma = 0$, respectively. Consequently, since the pullbacks of the coordinates $x_1, \ldots, x_n$ are monomial divisors too, $B$ contains the set $(R \setminus \{0\}) \cdot c$ when $c \in B$ is a point sufficiently close to $b = 0$. Then the map

$$\varphi : R \longrightarrow K^n, \quad \varphi(z) = \sigma(z \cdot c)$$

has the desired properties. \hfill \Box

4. Extending continuous hereditarily rational functions. We first recall an elementary lemma from [10].

**Lemma 1.** If the ground field $K$ is not algebraically closed, then there are polynomials $G_r(x_1, \ldots, x_r)$ in any number of variables whose only zero on $K^r$ is $(0, \ldots, 0)$.

**Proof.** Indeed, take a polynomial

$$g(t) = t^d + a_1 t^{d-1} + \cdots + a_d \in K[t], \quad d > 1,$$

which has no roots. Then its homogenization

$$G_2(x_1, x_2) = x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d$$

is a polynomial in two variables we are looking for. Further, we can recursively define polynomials $G_r$ by putting

$$G_{r+1}(x_1, \ldots, x_r, x_{r+1}) := G_2(G_r(x_1, \ldots, x_r), x_{r+1}). \hfill \Box$$

We keep further the assumption that $K$ is a Henselian discretely valued field and, additionally, that it is not algebraically closed. We thus have at our disposal the descent property (Corollary 4 to Theorem 1) and the Łojasiewicz inequality (Theorem 2) (which hold also over locally compact
fields by a purely topological argument). We are therefore able, by repeating
mutatis mutandis its proof, to carry over Proposition 10 from [10] to the case
of such fields.

**Theorem 3** (Extending continuous hereditarily rational functions). Let
$X$ be a smooth $K$-variety and $W \subset Z \subset X$ closed subvarieties. Let $f$ be a
continuous hereditarily rational function on $Z(K)$ that is regular at all $K$-
points of $Z(K) \setminus W(K)$. Then $f$ extends to a continuous hereditarily rational
function $F$ on $X(K)$ that is regular at all $K$-points of $X(K) \setminus W(K)$.

**Sketch of the Proof.** We shall keep the notation from paper [10]. The main
modification of the proof in comparison with paper [10] is the definition of
the functions $G$ and $F_{2n}$ which improve the rational function $P/Q$. Now we
need the following corrections:

$$G := \frac{P \cdot Q^d}{Q_{G}(Q, H)}$$

and

$$F_{dn} := G \cdot \frac{Q_{dn}^{G}}{Q_{2n}(Q, H)} = \frac{P \cdot Q^d}{Q_{G}(Q, H)} \cdot \frac{Q_{dn}^{G}}{Q_{2n}(Q, H)}.$$**

where the positive integer $d$ and the polynomial $G_{2}$ are taken from Lemma 1
and its proof. It is clear that the restriction of $F_{dn}$ to $Z(K) \setminus W(K)$ equals $f_{2}$,
and thus Theorem 3 will be proven once we show that the rational function
$F_{dn}$ is continuous on $X(K)$ for $n \gg 1$.

We work on the variety $\pi : X_{1}(K) \rightarrow X(K)$ obtained by blowing up the
ideal $(PQ^{d-1}, G_{2}(Q, H))$. Equivalently, $X_{1}(K)$ is the Zariski closure of the
graph of $G$ in $X(K) \times K^{P1}$. Two open charts are considered:

- a Zariski open neighbourhood $U^{*}$ of the closure (in the $K$-topology) $Z^{*}$
of $\pi^{-1}(Z(K) \setminus W(K))$;
- $V^{*} := X_{1}(K) \setminus Z^{*}$, which is an open (in the $K$-topology) $L$-definable
subset of $X_{1}(K)$.

The subtlest analysis is on the latter chart, on which $F_{dn} \circ \pi$ can be written
in the form

$$F_{dn} \circ \pi = (P \circ \pi) \cdot \left( \frac{Q_{dn}^{G-1}}{H^{d}} \circ \pi \right) \cdot \left( \frac{Q^d}{G_{2}(Q, H)} \circ \pi \right) \cdot \left( \frac{H^{d}}{G_{2}(Q^{n}, H)} \circ \pi \right).$$

By the Łojasiewicz inequality (Theorem 2), the second factor extends to a
continuous rational function on $V^{*}$ for $n \gg 1$. 

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Via the descent property, it suffices to show that the rational function $F_{dn} \circ \pi$ (with $n \gg 1$) extends to a continuous function on $X_1(K)$ that vanishes on

$$E := \pi^{-1}(W(K)).$$

The proof of this fact goes now along the same line of reasoning as in paper [10], once we know that the factors

$$\frac{Q^{dn}}{G_2(Q^n, H)} \quad \frac{Q^d}{G_2(Q, H)} \quad \text{and} \quad \frac{H^d}{G_2(Q^n, H)}$$

are regular functions off $W(K)$ whose valuations are bounded from below. But this follows immediately from an auxiliary lemma.

**Lemma 2.** Let $g$ be the polynomial from the proof of Lemma 1. Then the set of values

$$v \left( \frac{t^d}{g(t)} \right) \in \mathbb{Z}, \quad t \in K,$$

is bounded from below.

In order to prove this lemma, observe that

$$v \left( \frac{t^d}{g(t)} \right) = v \left( 1 + \frac{a_1}{t} + \cdots + \frac{a_d}{t^d} \right)$$

for $t \in K$, $t \neq 0$. Hence the values under study are zero when $iv(t) < v(a_i)$ for all $i = 1, \ldots, d$. Therefore, we are reduced to analysing the case where

$$v(t) \geq k := \min \left\{ \frac{v(a_i)}{i} : i = 1, \ldots, d \right\}.$$ 

Denote by $\Gamma \simeq \mathbb{Z}$ the valuation group of $v$. We must thus show that the set of values $v(g(t)) \in \Gamma$ when $v(t) \geq k$ is bounded from above.

Take elements $a, b \in R$, $a, b \neq 0$, such that $aa_i \in R$ for all $i = 1, \ldots, d$, and $bt \in R$ whenever $v(t) \geq k$. Then

$$(ab)^d g(t) = (abt)^d + aba_1(abt)^{d-1} + (ab)^2a_2(abt)^{d-2} + \cdots + (ab)^d a_d =: h(abt),$$

where $h$ is a monic polynomial with coefficients from $R$ which has no roots in $K$. Clearly, it is sufficient to show that the set of values $v(h(t)) \in \Gamma$ when $t \in R$ is bounded from above.
Consider a splitting field $\tilde{K} = K(u_1, \ldots, u_d)$ of the polynomial $h$, where $u_1, \ldots, u_d$ are the roots of $h$. Let $\tilde{v}$ be a (unique) extension to $\tilde{K}$ of the valuation $v$, $\tilde{R}$ be its valuation ring and $\tilde{\Gamma}, \Gamma \subset \tilde{\Gamma} \simeq \mathbb{Z}$, its valuation group (see e.g. [19] Chap. VI, § 11] for valuations of algebraic field extensions). Then

$$u_1, \ldots, u_d \in \tilde{R} \setminus R \quad \text{and} \quad h(t) = \prod_{i=1}^{d} (t - u_i).$$

Since $R$ is a closed subring of $\tilde{R}$, which follows from the lemma below, there exists an $l \in \tilde{\Gamma}$ such that $\tilde{v}(t - u_i) \leq l$ for all $i = 1, \ldots, d$ and $t \in R$. Hence $v(h(t)) \leq dl$ for all $t \in R$, and thus the lemma follows. \hfill $\Box$

**Lemma 3.** The field $K$ is a closed subspace of its algebraic closure $\overline{K}$.

Indeed, Denote by $\text{cl} (E, F)$ the closure of a subset $E$ in $F$, and let $\tilde{K}$ and $L$ be the completions of $K$ and $\overline{K}$, respectively. We have

$$\text{cl} (K, \overline{K}) = \text{cl} (K, L) \cap \overline{K} = \tilde{K} \cap \overline{K}.$$ 

But, by the transfer principle of Ax-Kochen–Ershov, $K$ is an elementary substructure of $\tilde{K}$ and, a fortiori, is algebraically closed in $\tilde{K}$. Hence

$$\text{cl} (K, \overline{K}) = \tilde{K} \cap \overline{K} = K,$$

as asserted. \hfill $\Box$

In this fashion, we have outlined how to adapt the proof of Proposition 10 from paper [10] to the case of Henselian discretely valued fields. Note that all results of paper [10] except the theorem on extending continuous hereditarily rational functions hold over any fields with the (DP) property and thus, in particular, over all Henselian real valued fields of characteristic zero. An affirmative answer to the extension problem for the case of Henselian real valued fields of equicharacteristic zero will be given in our subsequent paper (in preparation).
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