Spatial-Homogeneity of Stable Solutions of Almost-Periodic Parabolic Equations with Concave Nonlinearity

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Abstract
We study the spatial-homogeneity of stable solutions of almost-periodic parabolic equations. It is shown that if the nonlinearity satisfies a concave or convex condition, then any linearly stable almost automorphic solution is spatially-homogeneous; and moreover, the frequency module of the solution is contained in that of the nonlinearity.

1 Introduction
We consider the semilinear parabolic equation with Neumann boundary condition

\begin{equation}
\begin{aligned}
&u_t = \Delta u + f(t, u, \nabla u), \quad t > 0, \quad x \in \Omega \\
&\frac{\partial u}{\partial n} |_{\partial \Omega} = 0, \quad t > 0
\end{aligned}
\end{equation}

where \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain and \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}; (t, u, p) \mapsto f(t, u, p) \) together with its first and second derivatives are almost periodic in \( t \) uniformly for \( (u, p) \) in any compact
subset of $\mathbb{R} \times \mathbb{R}^n$. Such equation is ubiquitous throughout the modeling of population dynamics and population ecology. The almost periodicity of the nonlinearity $f$ captures the growth rate influenced by external effects which are roughly but not exactly periodic, or environmental forcing which exhibits different, non-commensurate periods.

In cases where $f$ is independent of $t$ (i.e., the autonomous case) or $f$ is time-periodic with period $T > 0$ (i.e., the time $T$-periodic case), it has been known that stable equilibria or $T$-periodic solutions are not supposed to possess spatial variations on a convex domain. For instance, in terms of an autonomous equation on a convex domain $\Omega$ with $f$ being independent of $\nabla u$, Casten and Holland [2] and Matano [10] proved that any stable equilibrium is spatially-homogeneous (i.e., without any spatial structure). In other words, any spatially-inhomogeneous equilibrium on a convex domain must be unstable. Later, Hess [6] considered the time $T$-periodic equation and showed that all stable $T$-periodic solutions are spatially-homogeneous on a convex domain $\Omega$.

When the system (1.1) is driven by a time almost periodic forcing, there usually exist almost automorphic solutions rather than almost periodic ones. As a matter of fact, the appearance of almost automorphic dynamics is a fundamental phenomenon in almost periodically forced parabolic equations [16–20]. We also refer to [7–9,12–14,23] on the study of almost automorphic dynamics in different types of almost-periodic differential systems. Among many others, Shen and Yi [20] showed that any stable almost automorphic solution of (1.1) is spatially-homogeneous on a convex domain $\Omega$.

Besides the convexity of the domain, the convexity or concavity of the nonlinearity $f$ in (1.1) (i.e., the function $f(t, \cdot, \cdot) : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is convex or concave for all $t \in \mathbb{R}$) can be thought as an alternative condition which guarantees that any spatially-inhomogeneous equilibrium and time $T$-periodic solution are unstable in the autonomous case (Casten and Holland [2]) and the time $T$-periodic case (Hess [6]), respectively.

The present paper is mainly focusing on the almost periodically forced equation (1.1). We will show that, if $f(t, \cdot, \cdot) : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is convex or concave for all $t \in \mathbb{R}$, then any linearly stable almost automorphic solution $u(t, x)$ (see Definition 2.2) of (1.1) is spatially-homogeneous; and moreover, the frequency module of $u(t, x)$ is contained in that of $f$ (see Theorem 3.1).

Our result can be viewed as an effective supplement of the above-mentioned result in [20]; for the concavity or convexity of $f$, instead of for convex domains. It also generalizes to multi-frequency driven systems from that in the autonomous cases [2] and time-periodic cases [6].

The paper is organized as follows. In Section 2, we review the basic notations and concepts involving skew-product semiflows, linearly stable and almost periodic (automorphic) functions which will be useful in our discussions. In Section 3, we prove the spatial-homogeneity of linearly stable almost automorphic solutions to (1.1) under the assumption that the nonlinearity $f$ is concave or convex.
2 Notations and Preliminary Results

2.1 Skew-product Semiflows and Linearly Stable Solutions

Let \( Y \) be a compact metric space with metric \( d_Y \) and \( \mathbb{R} \) be the additive group of reals. A real flow \((Y, \mathbb{R})\) (or \((Y, \sigma)\)) is a continuous mapping \( \sigma : Y \times \mathbb{R} \to Y, (y, t) \mapsto y \cdot t \) satisfying: (i) \( \sigma(y, 0) = y \); (ii) \( \sigma(y, s), t) = \sigma(y, s + t) \) for all \( y \in Y \) and \( s, t \in \mathbb{R} \). A subset \( E \subset Y \) is invariant if \( \sigma(y, t) \in E \) for each \( y \in E \) and \( t \in \mathbb{R} \), and is called minimal or recurrent if it is compact and the only non-empty compact invariant subset of it is itself. By Zorn’s Lemma, every compact and \( \sigma \)-invariant set contains a minimal subset. Moreover, a subset \( E \) is minimal if and only if every trajectory is dense in \( E \).

Let \( X, Y \) be metric spaces and \((Y, \sigma)\) be a compact flow (called the base flow). Let also \( \mathbb{R}^+ = \{ t \in \mathbb{R} : t \geq 0 \} \). A skew-product semiflow \( \Pi^t : X \times Y \to X \times Y \) is a semiflow of the following form

\[
\Pi^t(u, y) = (\varphi(t, u, y), y), \quad t \geq 0, (u, y) \in X \times Y,
\]

(2.1)
satisfying (i) \( \Pi^0 = \text{Id}_X \) and (ii) the co-cycle property: \( \varphi(t + s, u, y) = \varphi(s, \varphi(t, u, y), y) \cdot t \) for each \((u, y) \in X \times Y \) and \( s, t \in \mathbb{R}^+ \). A subset \( E \subset X \times Y \) is positively invariant if \( \Pi^t(E) \subset E \) for all \( t \in \mathbb{R}^+ \). The forward orbit of any \((u, y) \in X \times Y \) is defined by \( \mathcal{O}^+(u, y) = \{ \Pi^t(u, y) : t \geq 0 \} \), and the \( \omega \)-limit set of \((u, y) \) is defined by \( \omega(u, y) = \{ (\tilde{u}, \tilde{y}) \in X \times Y : \Pi^{t_n}(u, y) \to (\tilde{u}, \tilde{y})(n \to \infty) \text{ for some sequence } t_n \to \infty \} \).

A flow extension of a skew-product semiflow \( \Pi^t \) is a continuous skew-product flow \( \tilde{\Pi}^t \) such that \( \tilde{\Pi}^t(u, y) = \Pi^t(u, y) \) for each \((u, y) \in X \times Y \) and \( t \in \mathbb{R}^+ \). A compact positively invariant subset is said to admit a flow extension if the semiflow restricted to it does. Actually, a compact positively invariant set \( K \subset X \times Y \) admits a flow extension if every point in \( K \) admits a unique backward orbit which remains inside the set \( K \) (see [20, part II]). A compact positively invariant set \( K \subset X \times Y \) for \( \Pi^t \) is called minimal if it does not contain any other nonempty compact positively invariant set than itself.

Let \( X \) be a Banach space and the cocycle \( \varphi \) in (2.1) be \( C^1 \) for \( u \in X \), that is, \( \varphi \) is \( C^1 \) in \( u \), and the derivative \( \varphi_u \) is continuous in \( u \in X \), \( y \in Y \), \( t > 0 \); and moreover, for any \( v \in X \),

\[
\varphi_u(t, u, y)v \to v \quad \text{as} \quad t \to 0^+,
\]

uniformly for \((u, y) \) in compact subsets of \( X \times Y \). Let \( K \subset X \times Y \) be a compact, positively invariant set which admits a flow extension. Define \( \Phi(t, u, y) = \varphi_u(t, u, y) \) for \((u, y) \in K \), \( t \geq 0 \). Then the operator \( \Phi \) generates a linear skew-product semiflow \( \Psi \) on \((X \times K, \mathbb{R}^+)\) associated with (2.1) over \( K \) as follows:

\[
\Psi(t, v, (u, y)) = \Phi(t, u, y)v, \Pi^t(u, y)), \quad t \geq 0, (u, y) \in K, v \in X.
\]

(2.2)

For each \((u, y) \in K \), define the Lyapunov exponent \( \lambda(u, y) = \limsup_{t \to \infty} \frac{\ln ||\Phi(t, u, y)||}{t} \), where \( || \cdot || \) is the operator norm of \( \Phi(t, u, y) \). We call the number \( \lambda_K = \sup_{(u, y) \in K} \lambda(u, y) \) the upper Lyapunov exponent on \( K \).
**Definition 2.1.** \( K \) is said to be *linearly stable* if \( \lambda_K \leq 0 \).

To carry out our study for the non-autonomous system (1.1), we embed it into a skew-product semiflow. Let \( f_\tau(t, u, p) = f(t + \tau, u, p) (\tau \in \mathbb{R}) \) be the time-translation of \( f \), then the function \( f \) generates a family \( \{ f_\tau | \tau \in \mathbb{R} \} \) in the space of continuous functions \( C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \) equipped with the compact open topology. Moreover, \( H(f) \) (the closure of \( \{ f_\tau | \tau \in \mathbb{R} \} \) in the compact open topology) called the hull of \( f \) is a compact metric space and every \( g \in H(f) \) has the same regularity as \( f \). Hence, the time-translation \( g \cdot t \equiv g_t (g \in H(f)) \) naturally defines a compact minimal flow on \( H(f) \) and equation (1.1) induces a family of equations associated to each \( g \in H(f) \),

\[
\begin{align*}
    u_t &= \Delta u + g(t, u, \nabla u), \quad t > 0, \quad x \in \Omega, \\
    \frac{\partial u}{\partial n} &= 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega.
\end{align*}
\]  

(2.3)

It follows from the standard theory of parabolic equations (see, e.g., [3]), for each \( u_0 \in C^1(\overline{\Omega}) \) satisfying \( \frac{\partial u_0}{\partial n} \) on \( \partial \Omega \), (2.3) admits a unique classical locally solution \( \varphi(t, \cdot; u_0, g) \) with \( \varphi(0, \cdot; u_0, g) = u_0 \).

Hereafter, we always assume that \( X \) is a fractional power space (see [5]) associated with the operator \( u \rightarrow -\Delta u, \mathcal{D} \rightarrow L^p(\Omega) \) such that \( X \hookrightarrow C^1(\overline{\Omega}) \) (\( X \) is compact embedded in \( C^1(\overline{\Omega}) \)), where \( \mathcal{D} = \{ u | u \in W^{2,p}(\Omega) \text{ and } \frac{\partial u}{\partial n}|_{\partial \Omega} = 0 \}, p > n \). For any \( u \in X \) and \( g \in H(f) \), (2.3) defines (locally) a unique solution \( \varphi(t, \cdot; u, g) \) in \( X \) is \( C^2 \) in \( u \) and is continuous in \( g \) and \( t \) within its (time) interval of existence. In the language of dynamic systems, there is a well defined (local) skew-product semiflow \( \Pi^t : X \times H(f) \rightarrow X \times H(f) \):

\[
\Pi^t(u, g) = (\varphi(t, \cdot; u, g), g \cdot t), \quad t > 0
\]  

(2.4)

associated with (2.3). By the standard a priori estimates for parabolic equations (see [3, 5]), if \( \varphi(t, \cdot; u, g)(u \in X) \) is bounded in \( X \) in the existence interval of the solution, then it is a globally defined classical solution. For any \( \delta > 0 \), \( \{ \varphi(t, \cdot; u, g) \} \) is relatively compact, hence the \( \omega \)-limit set \( \omega(u, g) \) is a nonempty connected compact subset of \( X \times H(f) \). Moreover, by [4, 5], \( \Pi^t \) restricted to \( \omega(u, g) \) is a (global) semiflow which admits a flow extension.

Let \( X^+ = \{ u \in X | u(x) \geq 0, x \in \Omega \} \). Denote by Int \( X^+ \) the interior of \( X^+ \). Clearly, \( \text{Int} X^+ \neq \emptyset \), since \( \{ u \in X | u(x) > 0 \text{ for } x \in \Omega, \frac{\partial u}{\partial n} < 0 \text{ for } x \in \partial \Omega \} \subset \text{Int} X^+ \). Thus, \( X^+ \) defines a strong ordering on \( X \) as follows:

\[
\begin{align*}
    u_1 &\leq u_2 \iff u_2 - u_1 \in X^+, \\
    u_1 &< u_2 \iff u_2 - u_1 \in X^+, \quad u_2 \neq u_1, \\
    u_1 &\ll u_2 \iff u_2 - u_1 \in \text{Int} X^+.
\end{align*}
\]

Immediately, we have the following lemma from [20, Lemma III. 5.1].

**Lemma 2.1.** The skew-product semiflow \( \Pi^t \) in (2.4) is strongly monotone, in the sense that: for any \((u, g) \in X \times H(f), v \in X \) with \( v > 0 \), one has \( \Phi(t, u, g)v \gg 0 \) for \( t > 0 \).
**Definition 2.2.** A bounded solution \( u(t, x) = \varphi(t, x; u_0, g) \) of (2.3)\((u_0 \in X) \) is **linearly stable** if it satisfies the following conditions:

(i) \( \omega(u_0, g) \) is linearly stable.

(ii) Let \( \Phi(t, s) \) \(( t \geq s \geq 0) \) be the solution operator of the following linearized equation along \( u(t, x): \)

\[
\begin{align*}
    v_t &= \Delta v + g_u(t, u, \nabla u)v + g_p(t, u, \nabla u)\nabla v \quad \text{in } \mathbb{R}^+ \times \Omega, \\
    \frac{\partial v}{\partial n} &= 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega.
\end{align*}
\]

(2.5)

Then \( \sup_{t \geq 0} \| \Phi(t, 0)v_0 \| < \infty \) for all \( v_0 \in X \).

### 2.2 Almost Periodic and Almost Automorphic Functions

In this subsection, we always assume \( D \) is a non-empty subset of \( \mathbb{R}^m \).

**Definition 2.3.** A continuous function \( f : \mathbb{R} \times D \to \mathbb{R} \) is said to be **admissible** if for any compact subset \( K \subset D \), \( f \) is bounded and uniformly continuous on \( \mathbb{R} \times K \). \( f \) is \( C^r \) \(( r \geq 1) \) admissible if \( f \) is \( C^r \) in \( w \in D \) and Lipschitz in \( t \), and \( f \) as well as its partial derivatives to order \( r \) are admissible.

Let \( f \in C(\mathbb{R} \times D, \mathbb{R})(D \subset \mathbb{R}^m) \) be admissible. Then \( H(f) = \text{cl}\{f \cdot \tau : \tau \in \mathbb{R}\} \) (called the **hull of** \( f \) ) is compact and metrizable under the compact open topology (see [15, 20]), where \( f \cdot \tau(t, \cdot) = f(t + \tau, \cdot) \). Moreover, the time translation \( g \cdot t \) of \( g \in H(f) \) induces a natural flow on \( H(f) \) (cf. [15]).

**Definition 2.4.** A function \( f \in C(\mathbb{R}, \mathbb{R}) \) is **almost automorphic** if for every \( \{t'_k\} \subset \mathbb{R} \) there is a subsequence \( \{t_k\} \) and a function \( g : \mathbb{R} \to \mathbb{R} \) such that \( f(t + t_k) \to g(t) \) and \( g(t - t_k) \to f(t) \) pointwise. \( f \) is **almost periodic** if for any sequence \( \{t'_n\} \) there is a subsequence \( \{t_n\} \) such that \( \{f(t + t_n)\} \) converges uniformly. A function \( f \in C(\mathbb{R} \times D, \mathbb{R})(D \subset \mathbb{R}^m) \) is **uniformly almost periodic** (**automorphic**) in \( t \), if \( f \) is both admissible and almost periodic (**automorphic**) in \( t \in \mathbb{R} \).

**Remark 2.1.** If \( f \) is a uniformly almost automorphic function in \( t \), then \( H(f) \) is always **minimal**, and there is a residual set \( Y' \subset H(f) \), such that all \( g \in Y' \) is a uniformly almost automorphic function in \( t \). If \( f \) is a uniformly almost periodic function in \( t \), then \( H(f) \) is always **minimal**, and every \( g \in H(f) \) is uniformly almost periodic function (see, e.g. [20]).

Let \( f \in C(\mathbb{R} \times D, \mathbb{R}) \) be uniformly almost periodic (almost automorphic) and

\[
f(t, w) \sim \sum_{\lambda \in \mathbb{R}} a_\lambda(w)e^{i\lambda t} \quad (2.6)
\]

be a Fourier series of \( f \) (see [20, 22] for the definition and the existence of a Fourier series). Then \( S = \{a_\lambda(w) \neq 0\} \) is called the Fourier spectrum of \( f \) associated with Fourier series (2.6) and \( \mathcal{M} \) be the smallest additive subgroup of \( \mathbb{R} \) containing \( S(f) \) is called the frequency module of \( f \). Moreover, \( \mathcal{M}(f) \) is a countable subset of \( \mathbb{R} \) (see, e.g. [20]).
Lemma 2.2. Assume $f \in C(\mathbb{R} \times D, \mathbb{R})$ is a uniformly almost automorphic function, then for any uniformly almost automorphic function $g \in H(f)$, $\mathcal{M}(g) = \mathcal{M}(f)$.

Proof. See [20, Corollary I.3.7].

3 Spatial-homogeneity of Linearly Stable Solutions

In this section, we always assume that the function $(u, p) \mapsto f(t, u, p)$ in (1.1) is concave (resp. convex) for each $t \in \mathbb{R}$, that is, $f(t, \lambda u_1 + (1 - \lambda)u_2, \lambda p_1 + (1 - \lambda)p_2) \leq (\text{resp.} \geq) \lambda f(t, u_1, p_1) + (1 - \lambda)f(t, u_2, p_2)$ for any $\lambda \in [0, 1]$, $t \in \mathbb{R}$ and $(u_i, p_i) \in \mathbb{R} \times \mathbb{R}^n$, $i = 1, 2$. Clearly, $g(t, u, p)$ is also concave (resp. convex) for any $g \in H(f)$. We further assume that $f(t, \cdot, \cdot)$ is $C^2$ uniformly almost periodic. Our main result is the following theorem

Theorem 3.1. Assume that $f : (t, \cdot, \cdot) \mapsto f(t, \cdot, \cdot)$ is concave (or convex). Let $\varphi(t, \cdot, u_0, g) \in C^{1+\frac{1}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ ($\mu \in (0, 1]$) be a linearly stable almost automorphic (almost periodic) solution of (2.3), then $\varphi(t, \cdot, u_0, g)$ is spatially-homogeneous and is a solution of

$$u' = g(t, u, 0).$$  \tag{3.1}

Moreover, $\mathcal{M}(\varphi) \subset \mathcal{M}(f)$.

Hereafter, we only consider the case when $f$ is concave, because by a transformation from $u$ to $-u$, the convexity of nonlinearity $g$ can be changed into concavity.

Let $\varphi(t, \cdot, u_0, g) \in C^{1+\frac{1}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ be an almost automorphic solution of (2.3) with $u(0) = u_0$. Then, $\omega(u_0, g)$ is an almost automorphic minimal set; and hence, $\varphi(t, x; u_0, g)$ is well defined for all $t \in \mathbb{R}$. For brevity, we write $u(t, x) = \varphi(t, x; u_0, g)$ and define the following function $c : \mathbb{R} \to \mathbb{R}$ by

$$c(t) := \max_{x \in \overline{\Omega}} u(t, x), \quad t \in \mathbb{R}.$$  

Let $M(t) = \{ x \in \overline{\Omega} : u(t, x) = c(t) \}$. Then, similar as the arguments in [6, p.327], $c(t)$ is a Lipschitz continuous function and hence differentiable for a.e. $t \in \mathbb{R}$; define $\tilde{\mathbb{R}} = \{ t \in \mathbb{R} | c(t) \text{ is differentiable} \}$, then $\mathbb{R} \setminus \tilde{\mathbb{R}}$ is a set of zero measure and $c'(t)$ is continuous on $\tilde{\mathbb{R}}$; and moreover, $c'(t) = u_1(t, x)$ for any $t \in \tilde{\mathbb{R}}$ and $x \in M(t)$. Since $u \in C^{1+\frac{1}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ is an almost automorphic solution of (2.3), $c'(t) \in L^\infty(\mathbb{R})$. Moreover, we have the following

Lemma 3.2. $c(t)$ is an almost automorphic function.

Proof. Note that $u(t, x)$ is a uniformly almost automorphic function on $\mathbb{R} \times \overline{\Omega}$. Then, for any sequence $t_n \to \infty$, there are $v(t, x) \in H(u)$ (the hull of $u$) and a subsequence $\{ t_{n_k} \} \subset \{ t_n \}$, such that $u(t + t_{n_k}, x) \to v(t, x)$ and $v(t - t_{n_k}, x) \to u(t, x)$, uniformly for $(t, x) \in I \times \overline{\Omega}$, where $I$ is any compact set contained in $\mathbb{R}$. In other words, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\left\{ \begin{array}{l} u(t + t_{n_k}, x) - \epsilon < v(t, x) < u(t + t_{n_k}, x) + \epsilon \\ v(t - t_{n_k}, x) - \epsilon < u(t, x) < v(t - t_{n_k}, x) + \epsilon, \end{array} \right.$$  

Proof. See [20, Corollary I.3.7].
for any \( k > N \) and \((t, x) \in I \times \overline{\Omega}\). Therefore,

\[
\begin{aligned}
\max_{x \in \overline{\Omega}} u(t + t_{nk}, x) - \epsilon < \max_{x \in \overline{\Omega}} v(t, x) < \max_{x \in \overline{\Omega}} u(t + t_{nk}, x) + \epsilon \\
\max_{x \in \overline{\Omega}} v(t - t_{nk}, x) - \epsilon < \max_{x \in \overline{\Omega}} u(t, x) < \max_{x \in \overline{\Omega}} v(t - t_{nk}, x) + \epsilon,
\end{aligned}
\]

that is,

\[
|c(t + t_{nk}) - \max_{\overline{\Omega}} v(t, x)| < \epsilon \quad \text{and} \quad |c(t) - \max_{\overline{\Omega}} v(t - t_{nk}, x)| < \epsilon,
\]

for any \( k > N \) and \( t \in \mathbb{R} \). This implies that \( c(t) \) is an almost automorphic function. \( \square \)

Now, we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( w(t, x) = c(t) - u(t, x) \). Then, it is clear that \( w(t, x) \) is a uniformly almost automorphic function and \( w(t, x) \geq 0 \) on \( \mathbb{R} \times \overline{\Omega} \). Since \( u(t, x) \) is a solution of (2.3), denote \(-\Delta\) by \( A \), we have

\[
w_t + Aw = c'(t) - u_t + \Delta u = c'(t) - g(t, u, \nabla u) \tag{3.2}
\]

for all \( t \in \mathbb{R} \). Since \( g \) is concave,

\[
g(t, c, 0) \leq g(t, u, \nabla u) + \frac{\partial g}{\partial u}(t, u, \nabla u)w + \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, u, \nabla u)w_{x_i}. \tag{3.3}
\]

Let

\[
A(t) = A - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, u, \nabla u)\frac{\partial}{\partial x_i} - \frac{\partial g}{\partial u}(t, u, \nabla u).
\]

Together with (3.2)-(3.3), one has

\[
w_t + A(t)w \leq c'(t) - g(t, c, 0) := q(t)
\]

for all \( t \in \mathbb{R} \).

Since \( c'(t) \in L^\infty(\mathbb{R}) \), one has \( q \in L^\infty(\mathbb{R}) \). We now divide our proof into the following two cases: (i) \( q(t) \leq 0 \) for a.e. \( t \in \mathbb{R} \); (ii) \( q(t) > 0 \) on a set of positive measure.

Case (i). \( q(t) \leq 0 \) for a.e. \( t \in \mathbb{R} \). Let \( h \in L^\infty(\mathbb{R}, L^p(\Omega)) \) be defined from (3.2) by

\[
w_t + A(t)w =: h(t) \tag{3.4}
\]

and \( \Phi(t, s) \) be the fundamental solution associated with (3.4) (see Definition 2.2). If \( h \in C(\mathbb{R}, L^p(\Omega)) \), one can use the method of variation of constant to obtain

\[
w(t) = \Phi(t, 0)w(0) + \int_0^t \Phi(t, \tau)h(\tau)d\tau \tag{3.5}
\]
in \(L^p(\Omega)\) (see, e.g. [21, Theorem 5.2.2]). For general \(h \in L^\infty(\mathbb{R}, L^p(\Omega))\) in (3.4), similarly as the argument in [6, p.328-329], by using the strong continuity of \(\Phi(t, s)\) in \(s\) and [21, p.125, (5.33)], the following equation:

\[
\frac{\partial}{\partial s} (\Phi(t, s)w(s)) = \Phi(t, s)w_t(s) + \Phi(t, s)A(s)w(s) = \Phi(t, s)h(s)
\]  

(3.6)
can be established for any \(t \in \mathbb{R}\). Furthermore, \(\Phi(t, s)w(s)\) is in fact a Lipschitz continuous function of \(s\) from \(\mathbb{R}\) to \(L^p(\Omega)\) (hence, \(\Phi(t, s)w(s)\) is an absolutely continuous function of \(s\) in \(L^p(\Omega)\)). By using [1, Corollary A] and integrating \(s\) in (3.6) from 0 to \(t\), one can obtain (3.5).

Since \(h(\tau) \leq 0\) for a.e. \(\tau \in \mathbb{R}\), by strong positivity of \(\Phi\), one has \(\Phi(t, \tau)h(\tau) \leq 0\) for a.e. \(\tau \in [0, t]\) \((t > 0)\); and hence

\[
\int_0^t \Phi(t, \tau)h(\tau)d\tau \leq 0, \quad \forall t > 0.
\]

Therefore,

\[
w(t) \leq \Phi(t, 0)w(0).
\]  

(3.7)

Suppose that \(u(t, x)\) is not spatially-homogeneous. Then, \(w(0) > 0\) in \(C(\overline{\Omega})\) (i.e. \(w(0, x) > 0\) for all \(x \in \overline{\Omega}\), and \(w(0, \cdot) \neq 0\)). Noticing that the skew-product semiflow \(\Pi^f\) on \(X \times H(f)\) is strongly monotone (see Lemma 2.1), \(\omega(u_0, g)\) admits a continuous separation (see [20, Theorem II.4.4] or [11, Sec 3.5]) as follows: There exists continuous invariant splitting \(X = X_1(v, g) \oplus X_2(v, g)\) \((\forall (v, g) \in \omega(u_0, g))\) with \(X_1(v, g) = \text{span}\{\phi(v, g)\}, \phi(v, g) \in \text{Int}X^+\) and \(X_2(v, g) \cap X^+ = \{0\}\) such that

\[
\Phi(t, v, g)X_1(v, g) = X_1(\Pi^f(t, v, g)), \quad \Phi(t, v, g)X_2(v, g) \subset X_2(\Pi^f(t, v, g)).
\]  

Moreover, there are \(K, \gamma > 0\) satisfying

\[
\|\Phi(t, v, g)\|_{X_2(v, g)} \leq Ke^{-\gamma t}\|\Phi(t, v, g)\|_{X_1(v, g)}
\]

(3.9)
for any \(t \geq 0\) and \((v, g) \in \omega(u_0, g)\). Write \(w(0) = av_1 + v_2\) with \(v_1 \in X_1(u_0, g), \|v_1\| = 1\) and \(v_2 \in X_2(u_0, g)\). Since \(u(t, x)\) is linearly stable, \(\sup_{t \geq 0}\|\Phi(t, 0)v_1\|\) is bounded by Definition 2.2.

Case (ia): \(\|\Phi(t, 0)v_1\|\) is bounded away from zero. In this case, there exist \(M \geq m > 0\) such that \(m \leq \inf_{t \geq 0}\|\Phi(t, 0)v_1\| \leq \sup_{t \geq 0}\|\Phi(t, 0)v_1\| \leq M\). Let \(\Gamma = \{\tau|\Phi(t, 0)v_1 \to \tau\} \subset X\). Since \(\sup_{t \geq 0}\|\Phi(t, 0)v_1\| \leq M\), by the regularity of \(\Phi(t, 0)\), one has \(\Gamma \neq \emptyset\). We further claim that \(\Gamma \subset \text{Int}X^+\) and \(\Gamma\) is a closed subset of \(X\). In fact, for any \(\tau \in \Gamma\), one can find a sequence \(\tau_n \to \infty\), such that \(\Phi(\tau_n, 0)v_1 \to \tau\). By virtue of (3.8), \(\Phi(\tau_n, 0)v_1 \in X_1(\Pi^{\tau_n}(u_0, g))\). Without loss of generality, one may assume that \(\Pi^{\tau_n}(u_0, g) \to (\tau, g) \in \omega(u_0, g)\).

This implies that \(\tau \in X_1(\tau, g) \subset \text{Int}X^+\). Note also that \(\|v\| \geq m > 0\). Then \(\tau \in \text{Int}X^+\).

Next, we prove that \(\Gamma\) is closed in \(X\). It suffices to prove that: if the sequence \(v_n \in \Gamma\) converges to some \(v^* \in X\), then \(v^* \in \Gamma\). Indeed, for any positive integer \(k \in \mathbb{N}\), there is \(n_k > 0\) such that \(\|v_n - v^*\| < \frac{1}{2k}\) for any \(n \geq n_k\), particularly, \(\|v_{n_k} - v^*\| < \frac{1}{2k}\). Noticing that \(v_{n_k} \in \Gamma\), there exists \(t_{n_k} \in \mathbb{R}^+\) such that \(\|\Phi(t_{n_k}, 0)v_1 - v_{n_k}\| < \frac{1}{2k}\); and hence, \(\|\Phi(t_{n_k}, 0)v_1 - v^*\| < \frac{1}{k}\). Without loss
of generality, one may assume $t_{n_k} \to \infty$ as $k \to \infty$, by letting $k \to \infty$, one has $\Phi(t_{n_k}, 0)v_1 \to v^*$ as $t_{n_k} \to \infty$, which means $v^* \in \Gamma$. Thus we have proved the claim.

Recall that $\omega(u_0, g)$ is an almost automorphic minimal set, there is a sequence $t_n \to \infty$ such that $\Pi^n(u_0, g) \to (u_0, g)$. By choosing a subsequence, still denoted by $t_n$, one has that $\Phi(t_n, 0)v_1 \to v^* \in X_1(u_0, g) \cap \text{Int}X^+$; in other words, there is a positive constant $a^*$ such that $v^* = a^*v_1$. Moreover, $\Phi(t, 0)a^*v_1 \in \Gamma$ for any fixed $t \in \mathbb{R}^+$. Therefore, $\Phi(t_n, 0)a^*v_1 \in \Gamma$. Observing that $\Phi(t, 0)$ is a linear operator and $\Gamma$ is a closed set, $\Phi(t_n, 0)v_1 \to (a^*)^2v_1 \in \Gamma$. Similarly, by repeating this argument, we have $(a^*)^nv_1 \in \Gamma$ for any $n \in \mathbb{N}$. Furthermore, by virtue of the boundedness of $\Gamma$, $a^* \leq 1$. If $0 < a^* < 1$, then it is not hard to see $0 \in \Gamma$, a contradiction to $\Gamma \subset \text{Int}X^+$. Therefore, $a^* = 1$. Note that $\sup_{t \geq 0} \|\Phi(t, 0)v_1\| \leq M$, by (3.9), $\|\Phi(t, 0)v_2\| \to 0$ as $t \to \infty$. By letting $t = t_n$ and $n \to \infty$ in (3.7), one has

$$w(0) \leq av_1.$$  

Therefore, $v_2 \leq 0$. Observing that $X_2(u_0, g) \cap X^+ = \{0\}$, $v_2 = 0$. Hence, $w(0) = av_1$ with $a \geq 0$. If $a > 0$, then $w(0) = av_1 \in \text{Int}X^+$, a contradiction to that $w(0) \notin \text{Int}X^+$. Thus, $a = 0$ and $u(t, x)$ is spatially-homogeneous.

Case (ib): $\inf_{t \geq 0} \|\Phi(t, 0)v_1\| = 0$. There is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $\|\Phi(t_n, 0)v_1\| < \frac{1}{n}$. When the sequence $\{t_n\}$ is bounded, there exist $t^* \in \mathbb{R}^+$ and a subsequence $t_{n_k} \to t^*$ as $k \to \infty$. Due to $\Phi(t, 0)v_1$ is continuous with respect to $t$, $\Phi(t^*, 0)v_1 = 0$, which contradicts to the strong positivity of $\Phi(t, 0)$. Thus, $\{t_n\}$ is unbounded. For simplicity, we assume $t_n \to \infty$ as $n \to \infty$. Again by (3.7), we have

$$0 \leq w(t_n) \leq a\Phi(t_n, 0)v_1 + \Phi(t_n, 0)v_2.$$  

(3.10)

For such $t_n$, by choosing a subsequence if necessary, one may assume that $\Pi^{t_n}(u_0, g) \to (u^*, g^*) \in \omega(u_0, g)$ and $c(t_n) \to c^*$. Let $t_n \to \infty$ in (3.10), one has $0 \leq w^* \leq 0$ where $w^* = c^* - u^*$. So, $w^*_0 = 0$, that is, $u^*(x) \equiv c^*$ on $\overline{\Omega}$ is spatially-homogeneous. By the minimality of $\omega(u_0, g)$, every point in $\omega(u_0, g)$ is spatially-homogeneous, thus, $u_0(x) = c(0)$ on $\overline{\Omega}$, a contradiction.

Thus, we have proved that $u(t, x)$ is spatially-homogeneous when $q(t) \leq 0$ a.e. in $\mathbb{R}$.

Case (ii). There is a positive measure subset $E$ in $\mathbb{R}$ such that $q(t) > 0$ for all $t \in E$. In the following, we will show that this case cannot occur. Actually, this can be proved by the same arguments in [6, p.329-330]. For the sake of completeness, we give a detailed proof below.

Suppose that there exists such subset $E \subset \mathbb{R}$. Then one can find some $t_0 \in \mathbb{R}$ such that $q(t_0) > 0$. Recall that $c'(t)$ is continuous on $\mathbb{R}$, there are nontrivial interval $[t_1, t_2] \subset \mathbb{R}$ and $\epsilon_0 > 0$ satisfying $q(t) \geq \epsilon_0$ for a.e. $t \in [t_1, t_2]$. By the concavity of $g(t, \cdot, \cdot, \cdot)$, we have

$$g(t, u, \nabla u) \leq g(t, c, 0) - \frac{\partial g}{\partial u}(t, c, 0)(c - u) - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, c, 0)(c - u)x_i.$$  

(3.11)

Let

$$\overline{A}(t) = A - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, c, 0) \frac{\partial}{\partial x_i} - \frac{\partial g}{\partial u}(t, c, 0)$$
and
\[ \overline{h}(t) = \frac{d}{dt}(c - u)(t) + \overline{A}(t)(c - u)(t). \]

Combing with (3.2) and (3.11), one can obtain \( \overline{h}(t) \geq q(t) \geq \epsilon_0 \) for a.e. \( t \) in \([t_1, t_2]\). On the other hand, similarly as in (3.5), we have
\[ (c - u)(t_2) = \Phi(t_2, t_1)(c - u)(t_1) + \int_{t_1}^{t_2} \Phi(t_2, s)\overline{h}(s)ds, \]
where \( \Phi(\cdot, \cdot) \) is the fundamental solution of \( u_t = \overline{A}(t)u \). Note that
\[ \int_{t_1}^{t_2} \Phi(t_2, s)\overline{h}(s)ds \geq \epsilon_0 \int_{t_1}^{t_2} \Phi(t_2, s)1ds \gg 0 \quad \text{in } C(\overline{\Omega}), \]
where \( 1 \) is the unit constant-function. Together with \( \Phi(t_2, t_1)(c - u)(t_1) \geq 0 \), it follows that \( (c - u)(t_2) \gg 0 \) in \( C(\overline{\Omega}) \), a contradiction to the definition of \( c \). So, Case (ii) cannot happen.

Therefore, we have proved that \( u(t, x) \equiv \varphi(t) \) is a spatially-homogeneous solution of (2.3); and moreover, it is an almost automorphic solution of (3.1). Finally, it follows from Lemma 2.2 and [20, Theorem III.3.4(c)] that \( \mathcal{M}(\varphi) \subset \mathcal{M}(g) = \mathcal{M}(f) \). Thus, we have completed the proof. \( \square \)

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