A PROOF OF THE $\ell$-ADIC VERSION OF THE INTEGRAL IDENTITY CONJECTURE FOR POLYNOMIALS

LÊ QUY THUONG

Abstract. It is well known that the integral identity conjecture is of prime importance in Kontsevich-Soibelman’s theory of motivic Donaldson-Thomas invariants for non-commutative Calabi-Yau threfolds. In this article we consider its numerical version and make it a complete demonstration in the case where the potential is a polynomial and the ground field is algebraically closed. The fundamental tool is the Berkovich spaces whose crucial point is how to use the comparison theorem for nearby cycles as well as the K"unneth isomorphism for cohomology with compact support.

1. Introduction

Let us start by outlining due to [13] on the concept of motivic Donaldson-Thomas invariants that concern the integral identity conjecture. These invariants is introduced in [12] in the framework for Calabi-Yau threfolds and the motivic Hall algebra. The latter generates the derived Hall algebra of To"en [18].

Let $\mathcal{C}$ be an ind-constructible triangulated $A_\infty$-category over a field $\kappa$. By giving a constructible stability condition on $\mathcal{C}$ one considers a collection of full subcategories $\mathcal{C}_V \subset \mathcal{C}$, with $V$ strict sectors in $\mathbb{R}^2$. The stability condition depends on homomorphisms $\eta : K_0(\mathcal{C}) \to \Gamma$ and $Z : \Gamma \to \mathcal{C}$, where $\Gamma$ is a free abelian group endowed with a skew-symmetric integer-valued bilinear form $\langle , \rangle$. A choice of $V$ gives rise to a cone $C(V, Z)$ contained in $\Gamma \otimes \mathbb{R}$ to which one associates a complete motivic Hall algebra $\hat{H}(\mathcal{C}_V)$. Define $A^{\text{Hall}}_V$ invertible in $\hat{H}(\mathcal{C}_V)$ as characteristic functions of the stacks of objects of $\mathcal{C}_V$. The generic elements satisfy the Factorization Property

$$A^{\text{Hall}}_V = A^{\text{Hall}}_{V_1} \cdot A^{\text{Hall}}_{V_2}$$

with $V = V_1 \sqcup V_2$ and the decomposition taken clockwisely.

If the field $\kappa$ has characteristic zero, motivic quantum torus $\mathcal{R}_\mathcal{C}$ is defined to be an associative algebra generated by symbols $\hat{e}_\gamma$, for $\gamma$ in $\Gamma$, with the usual relations

$$\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = [A^1_\kappa]^\frac{1}{2}(\gamma_1 \cdot \gamma_2) \hat{e}_{\gamma_1 + \gamma_2}, \hat{e}_0 = 1,$$

where $[A^1_\kappa]$ is the square root of $[A^k_\kappa]$. The coefficient ring $C_0$ for the quantum torus $\mathcal{R}_\mathcal{C}$ can be any commutative ring, where the two most important candidates should be a certain localization of the Grothendieck ring of algebraic $\kappa$-varieties and its $\ell$-adic version.

By choosing in addition the so-called orientation data (its existence depends on another conjecture) and using Denef-Loeser’s theory of motivic Milnor fiber (e.g. the motivic Thom-Sebastiani theorem) of the potential of an object of the category
C, by [12] Sec. 6, there is a map \( \Phi_V : \hat{H}(C_V) \to R_{C_V} \) for each \( V \), which is nice enough in the sense that if it was a homomorphism the Factorization Property would be preserved. This is in fact obstructed because of the lack of an assertion of the integral identity. In the case where the above \( C_0 \) is a certain localization of the ring \( \mathcal{M}_0^\beta \), one faces to the full version of the integral identity conjecture. If well passed, \( A_{\text{mot}} := \Phi_V(A_{\text{mot}}^\text{Hall}) \) would be invariants in the category of non-commutative Calabi-Yau threefolds, namely motivic Donaldson-Thomas invariants. Also, if \( C_0 \) is a variant of the Grothendieck ring \( K_0(D_{\text{const,aux}}(\text{Spec}(\kappa), \mathbb{Q}_\ell)) \), one meets the \( \ell \)-adic version of the conjecture, and in this case, the corresponding invariants are numerical Donaldson-Thomas invariants.

In the context of non-archimedean complete discretely valued fields \( K \) of equal characteristic zero, with valuation ring \( R \) and residue field \( \kappa \), Kontsevich-Soibelman define in [12] the motivic Milnor fiber \( S_{f, \kappa} \) of a formal function \( f : X \to \text{Spf}(R) \) at a closed point \( \kappa \) of the reduction \( X_0 \). To do this, they use Denef-Loeser’s formula on the motivic nearby cycle of a regular function (cf. [7, 8]) as well as the fact that resolution of singularities of \((X, X_0)\) exists (see Temkin [16]). Let \( f_{11} \) be the forgetful morphism for \( U \) a subvariety of \( X_0 \).

**Conjecture 1.1** (Integral identity [12]). Let \( f \) be in \( \kappa[[x, y, z]] \) invariant by the \( \kappa^\times \)-action of weight \((1, -1, 0)\) with \( f(0, 0, 0) = 0 \). Denote by \( X \) the formal neighborhood of \( \mathcal{A}^d_{\kappa} \) in \( \mathcal{A}^d_{\kappa} \) whose structural morphism \( f \) is induced by \( f(x, y, z) \). Denote by \( S \) the formal neighborhood of \( f \) in \( \mathcal{A}^d_{\kappa} \) whose structural morphism \( f \) is induced by \( f(0, 0, z) \). Then, the identity \( \int_{x \in \mathcal{A}^d_{\kappa}} S_{f, \kappa} = [\mathcal{A}^d_{\kappa}] \otimes S_{f, \kappa} \) holds in \( \mathcal{M}_0^\beta \).

Notice that we proved in [14] the regular version for a composition with a polynomial in two variables and for a function of Steenbrink type. The purpose of the present article is to show that the \( \ell \)-adic version of the integral identity conjecture holds if the series \( f \) is a polynomial and the ground field \( \kappa \) is an algebraically closed field of characteristic zero. Let \( R \psi \) denote the nearby cycles functor. This functor was defined earlier in [3, 4] and it will be recalled here in Subsection 2.5.

**Theorem 1.2.** Let \( \kappa \) be an algebraically closed field. If \( f \) is in \( \kappa[[x, y, z]] \) invariant by the \( \kappa^\times \)-action of weight \((1, -1, 0)\) with \( f(0, 0, 0) = 0 \), there is a canonical quasi-isomorphism of complexes: \( R\Gamma(X, \mathcal{A}^d_{\kappa}, \mathcal{R} \psi_f \mathcal{Q} \otimes_{\mathcal{A}^d_{\kappa}}) \cong R\Gamma(X, \mathcal{A}^d_{\kappa}, \mathcal{Q}) \otimes (R \psi f_{\kappa} \mathcal{Q}) \).

As an approach, we follow Kontsevich-Soibelman’s idea in [12] Prop. 9 using Berkovich spaces. The fundamental tools are the comparison theorem for nearby cycles and the Künneth isomorphism for étale cohomology with compact support.

The result in this article is part of the author’s thesis. He thanks his advisor François Loeser for such an interesting subject as well as many valuable suggestions and much patience. He thanks Vladimir Berkovich and Antoine Ducros for their answers to questions on Berkovich spaces. Especially, Ducros read carefully the earlier drafts of the manuscript and pointed out a serious mistake, so that the author can introduce this complete version.

2. Preliminaries on the Berkovich spaces

2.1. Notation. Let \( K \) be a non-archimedean complete discretely valued field \( K \) of equal characteristics zero, with valuation ring \( R \), maximal ideal \( m \) and residue field \( \kappa = R/m \).
2.2. From special formal schemes to analytic spaces. Let us remark that the main result of this article will only concern formal $R$-schemes topologically of finite type. It is however better to recall some preliminaries on the Berkovich spaces in a larger category that consists of special formal $R$-schemes.

A topological $R$-algebra $A$ is said to be \textit{special} if $A$ is a Noetherian adic ring such that, if $J$ is an ideal of definition of $A$, the quotient rings $A/J^n, n \geq 1$, are finitely generated over $R$. By \cite{4}, a topological $R$-algebra $A$ is special if and only if it is topologically $R$-isomorphic to a quotient of the special $R$-algebra $R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]$. An adic $R$-algebra $A$ is \textit{topologically finitely generated over $R$} if it is topologically $R$-isomorphic to a quotient algebra of the algebra of restricted power series $R\{T_1, \ldots, T_n\}$. Evidently, any topologically finitely generated $R$-algebra is a special $R$-algebra.

A formal $R$-scheme $\mathfrak{X}$ is said to be \textit{special} if $\mathfrak{X}$ is a separated Noetherian adic formal scheme and if it is a finite union of affine formal schemes of the form $\text{Spf}(A)$ with $A$ a special $R$-algebras. A formal $R$-scheme $\mathfrak{X}$ is \textit{topologically of finite type} if it is a finite union of affine formal schemes of the form $\text{Spf}(A)$ with $A$ topologically finitely generated $R$-algebras. It is a fact that the category of separated topologically of finite type formal $R$-schemes is a full subcategory of the category of $R$-special formal schemes, and both admit fiber products.

A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of special formal schemes is of \textit{locally finite type} if locally it is isomorphic to a morphism of the form $\text{Spf}(B) \to \text{Spf}(A)$ with $B$ topologically finitely generated over $A$. The morphism $\varphi$ is of \textit{finite type} if it is a quasicompact morphism of locally finite type.

Due to \cite{4}, there is a canonical functor $\mathfrak{X} \mapsto \mathfrak{X}_n$ from the category of special formal $R$-schemes to that of (Berkovich) $K$-analytic spaces. In the affine case, the interpretation of this functor is explicit. Namely, if

$$\mathfrak{X} = \text{Spf}\left(R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]\right),$$

one has

$$\mathfrak{X}_n = E^n(0; 1) \times D^m(0; 1).$$

Also, if $\mathfrak{X} = \text{Spf}(A)$, where $A$ is a quotient of $R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]$ by an ideal $I$, then $\mathfrak{X}_n$ is the closed $K$-analytic subspace of $X = E^n(0; 1) \times D^m(0; 1)$ defined by the subsheaf of ideals $I\mathcal{O}_X$.

Generally, $\mathfrak{X}_n$ is defined by gluing in an appropriate manner of analytic spaces corresponding to affine formal schemes which covers $\mathfrak{X}$ (see \cite{4}).

\textbf{Remark 2.1.} (i) The functor $\mathfrak{X} \mapsto \mathfrak{X}_n$ takes a formal scheme topologically of finite type to a paracompact analytic space, and this functor commutes with fiber products.
(ii) The functor $X \mapsto \mathfrak{X}_n$ takes a morphism of finite type $φ : \mathfrak{Y} \to X$ to a compact morphism of $K$-analytic spaces $φ_η : \mathfrak{Y}_η \to \mathfrak{X}_\eta$. If $φ$ is finite (resp. flat finite), so is $φ_η$.

2.3. The reduction map. For a special formal $R$-scheme $\mathfrak{X}$, we denote by $\mathfrak{X}_0$ the closed subscheme of $\mathfrak{X}$ defined by the largest ideal of definition of $\mathfrak{X}$. Note that $\mathfrak{X}_0$ is a reduced Noetherian scheme, that the correspondence $\mathfrak{X} \mapsto \mathfrak{X}_0$ is functorial, and that the natural closed immersion $\mathfrak{X}_0 \to \mathfrak{X}$ is a homeomorphism. Moreover, the reduction $\mathfrak{X}_0$ is also a separated $κ$-scheme of finite type.

We now recall the construction of the reduction map in the affine case, that is for $\mathfrak{X} = \text{Spf}(A)$ with $A$ being an adic special $R$-algebra. Notice that Berkovich did this work in [3, 4] for any special formal $R$-scheme. The construction of the reduction map $π : \mathfrak{X}_η \to \mathfrak{X}_0$ for $\mathfrak{X} = \text{Spf}(A)$ runs as follows. Remark that each point $x$ of $\mathfrak{X}_η$ defines a continuous character $χ_x : A \to \mathcal{H}(x)$. In its turn, $χ_x$ defines a character $\overline{χ}_x : \mathcal{A}_0 = A/J → \mathcal{H}(x)$, where $J$ is the largest ideal of definition of $A$. Then we assign $π(x)$ to the kernel of $\overline{χ}_x$, which is a prime ideal of $\mathcal{A}_0$. This definition guarantees the compatibility of the reduction map with open immersion in the following meaning. If $\mathfrak{Y}$ is an open formal scheme of $\mathfrak{X}$, then the reduction maps for $\mathfrak{X}$ and $\mathfrak{Y}$ are compatible and $\mathfrak{Y}_0 \cong π^{-1}(\mathfrak{Y}_0)$.

2.4. Étale cohomology of analytic spaces. The theory of étale cohomology for Berkovich spaces (also called non-archimedean analytic spaces) is sharply developed in the long article [2]. Note that the groups $H^*(Y, \mathbb{Z}_η)$ and $H^*(Y, \mathbb{Q}_η)$ in the sense of derived functors are irrelevant, i.e. roughly speaking, they do not satisfy some “nice” properties which a cohomology theory should have. Grothendieck however pointed out that the following groups are relevant

$$\text{proj lim } H^*(Y, \mathbb{Z}/\ell^n\mathbb{Z}) \quad \text{and} \quad (\text{proj lim } H^*(Y, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_η} \mathbb{Q}_η.$$  

Thus from now on, we shall only consider these groups and denote them by $H^*(Y, \mathbb{Z}_η)$ and $H^*(Y, \mathbb{Q}_η)$, respectively (cf. [9, 11]). The same also holds for cohomology with compact support (cf. [9, 11]). Namely,

$$H_c^*(Y, \mathbb{Z}_η) := (\text{proj lim } H^*_c(Y, \mathbb{Z}/\ell^n\mathbb{Z})),$$

$$H_c^*(Y, \mathbb{Q}_η) := (\text{proj lim } H^*_c(Y, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_η} \mathbb{Q}_η.$$  

Let $\widehat{K}^s$ be the completion of a separable closure of $K$. For a $K$-analytic space $X$, there is a canonical morphism $b : \overrightarrow{X} := X \otimes_K \widehat{K}^s \to X$. Now fix such an $X$ and consider all the subspaces of its. If $Y$ is an analytic subspace of the $X$, denote by $\overrightarrow{Y}$ or by $Y \otimes_K \widehat{K}^s$ the preimage of $Y$ in $\overrightarrow{X}$ under $b$. The following are two of properties of the functor $Y \mapsto H^*_c(\overrightarrow{Y}, \mathbb{Q}_η)$ according to [2] Prop. 5.2.6, Cor. 7.7.3).

**Proposition 2.2** (Berkovich [2]). Let $Y, Y'$ be locally closed analytic subspaces of a given $K$-analytic space $X$.

(i) If $U$ is an open subspace of $Y$, $V := Y \setminus U$, there is an exact sequence

$$\cdots \to H^m_c(\overrightarrow{V}, \mathbb{Q}_η) \to H^{m+1}_c(\overrightarrow{U}, \mathbb{Q}_η) \to H^{m+1}_c(\overrightarrow{Y}, \mathbb{Q}_η) \to H^{m+1}_c(\overrightarrow{V}, \mathbb{Q}_η) \to \cdots.$$  

(ii) There is a canonical Küneth isomorphism of complexes

$$RΓ_c(\overrightarrow{Y}, \mathbb{Q}_η) \otimes RΓ_c(\overrightarrow{Y'}, \mathbb{Q}_η) \cong RΓ_c(\overrightarrow{Y \times Y'}, \mathbb{Q}_η).$$
2.5. The nearby cycles functor. A morphism \( \varphi : \mathcal{Y} \to X \) of special formal \( R \)-schemes is called \( \acute{e}tale \) if for any ideal of definition \( \mathcal{J} \) of \( X \) the morphism of schemes \( (\mathcal{Y}, \mathcal{O}_X/\mathcal{J}\mathcal{O}_Y) \to (X, \mathcal{O}_X/\mathcal{J}) \) is \( \acute{e}tale \). The reduction \( X_0 \) being the closed subscheme of \( X \) defined by the largest ideal of definition of \( X \), thus if the morphism \( \varphi : \mathcal{Y} \to X \) is \( \acute{e}tale \), the induced morphism \( \varphi_0 : \mathcal{Y}_0 \to X_0 \) is \( \acute{e}tale \).

By [2], a morphism of \( K \)-analytic spaces \( \varphi : Y \to X \) is \( \acute{e}tale \) if for each point \( y \in Y \) there exist open neighborhoods \( V \) of \( y \) and \( U \) of \( \varphi(y) \) such that \( \varphi \) induces a finite \( \acute{e}tale \) morphism \( \varphi : V \to U \). By a finite \( \acute{e}tale \) morphism \( \varphi : V \to U \) one means that for each affinoid domain \( W = \mathcal{M}(A) \) in \( U \), the preimage \( \varphi^{-1}(W) = \mathcal{M}(B) \) is an affinoid domain and \( B \) is a finite \( \acute{e}tale \) \( A \)-algebra. A morphism of \( K \)-analytic spaces \( \varphi : Y \to X \) is called quasi-\( \acute{e}tale \) if for any point \( y \in Y \) there exist affinoid domains \( V_1, \ldots, V_n \subset Y \) such that \( V_1 \cup \cdots \cup V_n \) is a neighborhood of \( y \) and each \( V_i \) may be identified with an affinoid domain in a \( K \)-affinoid space \( \acute{e}tale \) over \( X \). By definition, \( \acute{e}tale \) morphisms are also quasi-\( \acute{e}tale \).

Lemma 2.3 (Berkovich [4], Prop. 2.1). Assume that \( \varphi : \mathcal{Y} \to X \) is an \( \acute{e}tale \) morphism of special formal \( R \)-schemes. Then the following hold:

(i) \( \varphi_0(\mathcal{Y}_n) = \pi^{-1}(\varphi_0(\mathcal{Y}_0)) \), consequently \( \varphi_0(\mathcal{Y}_n) \) is a closed analytic domain in \( X_n \).

(ii) The induced morphism \( \mathcal{Y}_n \to X_n \) of \( K \)-analytic spaces is quasi-\( \acute{e}tale \).

For a \( K \)-analytic space \( X \), let \( X_{\acute{e}t} \) denote the quasi-\( \acute{e}tale \) site of \( X \) as in [3]. The quasi-\( \acute{e}tale \) topology on \( X \) is the Grothendieck topology on the category of quasi-\( \acute{e}tale \) morphisms \( U \to X \) generated by the pretopology for which the set of coverings of \( U \to X \) is formed by the families \( \{f_i : U_i \to U\}_{i \in I} \) such that each point of \( U \) has a neighborhood of the form \( f_i(V_1) \cup \cdots \cup f_i(V_n) \) for some affinoid domains \( V_1 \subset U_1, \ldots, V_n \subset U_n \). There is a morphism of sites \( \mu : X_{\acute{e}t} \to X_{\acute{e}t} \). Denote by \( X_{\acute{e}t} \) the category of sheaves of sets on \( X_{\acute{e}t} \). The functor \( \mu^* : X_{\acute{e}t} \to X_{\acute{e}t} \) is a fully faithful functor (cf. [3]).

Let \( X \) be a special formal \( R \)-scheme. By [3], the correspondence \( \mathcal{Y} \to \mathcal{Y}_0 \) induces an equivalence between the category of formal schemes \( \acute{e}tale \) over \( X \) and the category of schemes \( \acute{e}tale \) over \( X_0 \). We fix the functor \( \mathcal{Y}_0 \to \mathcal{Y} \) which is inverse to the previous correspondence \( \mathcal{Y} \to \mathcal{Y}_0 \). The composition of the functor \( \mathcal{Y}_0 \to \mathcal{Y} \) with the functor \( \mathcal{Y} \to \mathcal{Y}_n \) induces a morphism of sites \( \nu : X_{\acute{e}t} \to X_{\acute{e}t} \). By [4], this construction also holds over a separable closure \( K^s \) of \( K \), therefore we shall also denote by \( \nu \) the corresponding morphism of sites \( X_{\acute{e}t} \to X_{\acute{e}t} \), where \( X_{\acute{e}t} := X \otimes_{K^s} K \).

Now consider the composition of the functors \( \mu^* : X_{\acute{e}t} \to X_{\acute{e}t} \) and \( \nu_* : X_{\acute{e}t} \to X_{\acute{e}t} \), namely \( \nu_* \mu^* : X_{\acute{e}t} \to X_{\acute{e}t} \). This resulting functor composing with the pullback (or inverse image) functor of the canonical morphism \( X_{\acute{e}t} \to X_0 \) yields a functor \( \psi : X_{\acute{e}t} \to X_{\acute{e}t} \), which is called the nearby cycles functor (see [3, 4]). It is a left exact functor, thus we can involve right derived functors \( R^i \psi : \mathcal{S}(X_{\acute{e}t}) \to \mathcal{S}(X_{\acute{e}t}) \) and \( R\psi : D^+(X_{\acute{e}t}) \to D^+(X_{\acute{e}t}) \), the latter is exact while the others are right exact functors. If necessary, we can write \( R\psi_{ij} \) and \( R\psi_{ij} \) labeling \( f \) the structural morphism of \( X \).

Lemma 2.4 (Berkovich [4], Cor. 2.3). Let \( \varphi : \mathcal{Y} \to X \) be an \( \acute{e}tale \) morphism of special formal \( R \)-schemes and \( F \) in \( \mathcal{S}(X_{\acute{e}t}) \). Then for any \( m \geq 0 \) we have \( (R^m \psi F)|_{\mathcal{Y}_{\acute{e}t}} \cong R^m \psi(F|_{\mathcal{Y}_{\acute{e}t}}) \).
2.6. The comparison theorem for nearby cycles. By [4], Thm. 3.1, the comparison theorem for nearby cycles functor working on a henselian ring $R$. Let $\mathcal{E}$ be a scheme locally of finite type over $R$ with the structural morphism $f$; and let $\mathcal{E}_0$ be the zero locus of $f$, which is a $\kappa$-scheme. Then $\mathcal{E}_0 = \hat{\mathcal{E}}_0$, where the scheme on the right is the reduction of the completion $\hat{\mathcal{E}}$ of the scheme $\mathcal{E}$. For a subscheme $\mathcal{Y} \subset \mathcal{E}_0$, let $\hat{\mathcal{E}}_{/\mathcal{Y}}$ denote the formal $\mathfrak{m}$-adic completion of $\hat{\mathcal{E}}$ along $\mathcal{Y}$. A result of [4] shows that there is a canonical isomorphism of $K$-analytic spaces $(\hat{\mathcal{E}}_{/\mathcal{Y}})_\eta \cong \pi^{-1}(\mathcal{Y})$, where $\pi$ is the reduction map $\hat{\mathcal{E}}_\eta \rightarrow \mathcal{E}_0$. For a sheaf $\mathcal{F} \in \mathcal{E}_{\text{et}}$, with $\mathcal{E}_\eta := \mathcal{E} \otimes_R K$, let $\hat{\mathcal{F}}_{/\mathcal{Y}}$ denote the pullback of $\mathcal{F}$ on $(\hat{\mathcal{E}}_{/\mathcal{Y}})_\eta$. The nearby cycles functor for $\mathcal{E}$, for $\hat{\mathcal{E}}$ and for $(\hat{\mathcal{E}}_{/\mathcal{Y}})_\eta$ will be denoted by the same symbol $\psi$. If $\mathcal{Y}$ is an (ordinary) $\kappa$-scheme, we define $\underline{\mathcal{Y}} := \mathcal{Y} \otimes_\kappa \kappa^a$.

Theorem 2.5 (Berkovich [4], Thm. 3.1). Let $\mathcal{F}$ be an étale abelian constructible sheaf on $\mathcal{E}_0$. For $i \geq 0$, there is a canonical isomorphism $(R^i\psi\mathcal{F})|_{\underline{\mathcal{Y}}} \cong R^i\psi(\hat{\mathcal{F}}_{/\mathcal{Y}})$.

The previous theorem is widely known as the Berkovich’s comparison theorem for nearby cycles, while the full version is in fact stated for both nearby cycles functor and vanishing cycles functor and it is motivated by a conjecture of Deligne. Part of the conjecture claims that the restrictions of the vanishing cycles sheaves of a scheme $\mathcal{E}$ of finite type over a henselian discrete valuation ring to the subscheme $\mathcal{Y} \subset \mathcal{E}_0$ depend only on the formal $\mathfrak{m}$-adic completion $\hat{\mathcal{E}}_{/\mathcal{Y}}$ of $\mathcal{E}$ along $\mathcal{Y}$, and that the automorphism group of $\hat{\mathcal{E}}_{/\mathcal{Y}}$ acts on them. By proving this comparison theorem, Berkovich [4] provided the positive answer to Deligne’s conjecture.

The following corollary runs over any complete discretely valued field.

Corollary 2.6 (Berkovich [4], Cor. 3.6). Let $\mathcal{S}$ be an $R$-scheme of locally finite type, $\mathfrak{X}$ a special formal $\hat{\mathcal{S}}$-scheme which is locally isomorphic to the formal $\mathfrak{m}$-adic completion of a $\mathfrak{S}$-scheme of finite type along a subscheme of its reduction, $\mathcal{F}$ an étale sheaf on $\mathfrak{X}_\eta$ locally in the étale topology of $\mathfrak{X}$ isomorphic to the pullback of a constructible sheaf on $\mathfrak{S}_\eta$. Then $R\psi(\mathcal{F})$ is constructible and, for any subscheme $\mathcal{Y} \subset \mathfrak{X}_0$, there is a canonical isomorphism of complexes

$$R\Gamma(\underline{\mathcal{Y}}, (R\psi\mathcal{F})|_{\underline{\mathcal{Y}}}) \cong R\Gamma(\pi^{-1}(\mathcal{Y}), \mathcal{F}).$$

If, in addition, the closure of $\mathcal{Y}$ in $\mathfrak{X}_0$ is proper, there is a canonical isomorphism

$$R\Gamma_c(\underline{\mathcal{Y}}, (R\psi\mathcal{F})|_{\underline{\mathcal{Y}}}) \cong R\Gamma_{\pi^{-1}(\mathcal{Y})}(\mathfrak{X}_\eta, \mathcal{F}).$$

3. The polynomial $f$ and comparisons

From this section, the condition that $\kappa$ is an algebraically closed field will be used because of applying Berkovich’s comparison theorem for nearby cycles. Also, $R$ and $K$ will stand for $\kappa[[t]]$ and $\kappa((t))$, respectively.

3.1. Resetting the data. Let $f(x, y, z)$ be in $\kappa[x, y, z]$ such that $f(0, 0, 0) = 0$ and $f(\tau x, \tau^{-1} y, z) = f(x, y, z)$ for $\tau \in \kappa^\times$. Let us consider the following $R$-schemes with the structural morphisms

$$\mathcal{E} := \text{Spec}(R[x, y, z]/(f(x, y, z) - t)) \rightarrow \text{Spec}(R),$$

$$\mathcal{W} := \text{Spec}(R[z]/(f(0, 0, z) - t)) \rightarrow \text{Spec}(R).$$
given by \( t = f(x, y, z) \), \( t = f(0, 0, z) \), respectively. Note that \( \mathcal{H}^{d_1} \) is a closed subvariety of \( \kappa \)-variety \( \mathcal{E}_0 = f^{-1}(0) \). We have identities \( X = \hat{\mathcal{E}}/\mathcal{H}^{d_1} \) and \( \mathfrak{X} = \hat{\mathcal{W}}/\mathcal{O} \), where the formal schemes on the left hand sides were already defined in first section.

Consider the reduction maps \( \pi : X_0 \to X_0 \) and \( \pi_W : \mathfrak{X}_0 \to \mathfrak{X}_0 \).

3.2. Applying the comparison theorem. Let \( f \) be the homogenization of \( f \), i.e. \( f(x, y, z, \xi) \) is homogeneous in \( d + 1 \) variables with \( f(x, y, z, 1) = f(x, y, z) \) and \( \deg(f) = \deg(f) = n \). Note that the \( R \)-scheme

\[
\mathbf{E} := \text{Proj} \left( R[x, y, z, \xi]/(f(x, y, z, \xi) - t\xi^n) \right)
\]

is locally of finite type. Let us consider the \( t \)-adic completion \( \hat{\mathbf{E}} \), which is a formal \( R \)-scheme canonically glued from the following affine formal \( R \)-schemes

\[
\begin{align*}
\text{Spf} \left( R\left[ x, y, z, \xi \right]/(f(x, y, z, \xi) - t\xi^n) \right) & \quad i = 1, \ldots, d_1, \\
\text{Spf} \left( R\left[ x, y, z, \xi \right]/(f(x, y, z, \xi) - t\xi^n) \right) & \quad j = 1, \ldots, d_2, \\
\text{Spf} \left( R\left[ \frac{x}{z_i}, \frac{y}{z_i}, \frac{z}{z_i}, \xi \right]/(f(\frac{x}{z_i}, \frac{y}{z_i}, \frac{z}{z_i}, \xi) - t\xi^n) \right) & \quad l = 1, \ldots, d_3, \\
\text{Spf} \left( R\left[ \frac{x}{\xi}, \frac{y}{\xi}, \frac{z}{\xi}, \xi \right]/(f(\frac{x}{\xi}, \frac{y}{\xi}, \frac{z}{\xi}, \xi) - t\xi^n) \right) & \cong \hat{\mathbf{E}}.
\end{align*}
\]

The reduction \( \hat{\mathbf{E}}_0 = \mathbf{E}_0 \) is the hypersurface \( \{ f = 0 \} \) in the projective space \( \mathbb{P}^d_\kappa \), it admits the inclusions \( \mathcal{H}^{d_1}_\kappa \subset \mathbf{E}_0 \subset \mathbf{E}_0 \).

Let \( \mathcal{H}^{d_1}_\kappa \) be the closure of \( \mathcal{H}^{d_1}_\kappa \) in \( \mathbf{E}_0 \). By construction, the embedding of \( \hat{\mathbf{E}} \) in \( \hat{\mathbf{E}} \) is an open immersion of formal \( R \)-schemes (thus it is an \( \acute{e} \)tale morphism). By [10] Cor. 10.9.9, the formal \( R \)-scheme \( X = \hat{\mathcal{E}}/\mathcal{H}^{d_1} \) can be identified to the fiber product of \( \hat{\mathbf{E}} \to \hat{\mathbf{E}} \) and \( \mathbf{X} := \hat{\mathbf{E}}/\mathcal{H}^{d_1}_\kappa \to \hat{\mathbf{E}} \). Since \( \acute{e} \)tale morphisms are preserved under base change, the induced morphism \( \hat{X} \to \mathbf{X} \) is also \( \acute{e} \)tale (it is even an open immersion).

Denote by \( \hat{f} \) the structural morphism of \( \mathbf{X} \), which is induced by \( f \). We shall use the following notation

\[
\begin{align*}
\star & : \mathbf{X}_\mathbf{E} \to \mathbf{X}_\mathbf{E} \text{ is the embedding of analytic spaces}, \\
\star & : \mathbf{X}_0 \to \mathbf{X}_0, \ k : \mathbf{X}_0 \setminus \mathbf{X}_0 \to \mathbf{X}_0, \ u : \mathcal{H}^{d_1}_\kappa \to \mathbf{X}_0 \text{ and } v : \mathcal{H}^{d_1}_\kappa \to \mathbf{X}_0 \text{ are the embeddings of } \kappa \text{-schemes (note that } v = j \circ u \).
\end{align*}
\]

Let \( F \) denote the constant sheaf \( (\mathbb{Z}/t^n\mathbb{Z})_{\mathbf{X}_\mathbf{E}} \) in \( \mathbf{S}(\mathbf{X}_\mathbf{E}) \), \( n \geq 1 \). By Lemma 2.4 for any \( m \geq 0 \), we have \( j^* R^m \psi_f(i_* F) \cong R^m \psi_f F \), hence \( j^* R^m \psi_f(i_* F) \cong j^* R^m \psi_f F \).

In the latter isomorphism, the complex on the right hand side can be fitted in the exact triangle

\[
\to j_* R^m \psi_f F \to R^m \psi_f(i_* F) \to k_* k^* R^m \psi_f(i_* F) \to .
\]

The functor \( v^* \) being exact, we have the following exact triangle

\[
(3) \quad \to u^* R^m \psi_f F \to v^* R^m \psi_f(i_* F) \to v^* k_* k^* R^m \psi_f(i_* F) \to .
\]

Observe that the support of the functor \( v^* \) is \( \mathcal{H}^{d_1}_\kappa \), which is a subset of \( \mathbf{X}_0 \), while that of \( k_* k^* \) is \( \mathbf{X}_0 \setminus \mathbf{X}_0 \), and the two subsets \( \mathcal{H}^{d_1}_\kappa \) and \( \mathbf{X}_0 \setminus \mathbf{X}_0 \) are disjoint in \( \mathbf{X}_0 \). This means \( v^* k_* k^* R^m \psi_f(i_* F) \cong 0 \), and one deduces that \( R^m \psi_f F|_{\mathcal{H}^{d_1}_\kappa} \cong R^m \psi_f(i_* F)|_{\mathcal{H}^{d_1}_\kappa} \).
We claim that \( R_i \) in the form

\[
\Gamma_c(\mathcal{H}_n, R\psi F)|_{\mathcal{H}_n}
\]

are quasi-isomorphisms. By the assumptions of Corollary \([2, \text{Cor. 5.2.4}]\) and the closure of \( \mathcal{H}_n \) in \( X_0 \) is proper as \( X_0 \) is. Let \( \pi \) be the reduction map \( X \rightarrow X_0 \). Then one deduces from Corollary \([2, \text{Cor. 5.2.4}]\) that

\[
\Gamma_c(\mathcal{H}_n, R\psi F)|_{\mathcal{H}_n} \cong \Gamma(\pi^{-1}(\mathcal{H}_n), (X_\pi, i_! F)).
\]

3.3. Shrinking analytic domains. Let us consider \( \Gamma(\pi^{-1}(\mathcal{H}_n), (X_\pi, i_! F)) \) as in (5). We remark that the analytic space \( X_\pi \) is the glueing of \( A := X_\pi \) together with other analytic spaces which correspond to the formal schemes in \([2, \text{Cor. 5.2.4}]\), each of which is a closed analytic domain in \( X_\pi \) (Lemma \([2, \text{Cor. 5.2.4}]\)). Similarly, \( \pi^{-1}(\mathcal{H}_n) \) is the glueing of \( X := \pi^{-1}(\mathcal{H}_n) \) together with others in the same way. Define \( P := X_\pi \setminus A \) and \( T := X_\pi \setminus \pi^{-1}(\mathcal{H}_n) \).

**Lemma 3.1.** We have a quasi-isomorphism of complexes as follows

\[
\Gamma(\pi^{-1}(\mathcal{H}_n), (X_\pi, i_! F)) \cong \Gamma_X(A, F).
\]

**Proof.** Let \( i_\alpha \) be the embedding of an \( \tilde{K}^\alpha \)-analytic space \( \alpha \) in \( X_\pi \), \( i_{\alpha, \beta} \) the embedding of \( \alpha \) in \( \beta \) (thus \( i_A = i \)), and \( B := A \setminus X \). Now both sides of (6) can be rewritten as follows

\[
\Gamma(\pi^{-1}(\mathcal{H}_n), (X_\pi, i_! F)) \cong \Gamma_\pi(X_\pi, i_! F),
\]

\[
\Gamma_X(A, F) \cong \Gamma_\pi(X_\pi, i_! F).
\]

The supports of \( i_\beta \) and \( i_A \) are disjoint, hence \( h \) is a quasi-isomorphism. Rewrite \( h \) in the form \( h : \Gamma_i F \rightarrow i_A i^*_B A F \rightarrow \Gamma_i F \rightarrow i_A i^*_B A F \). The identity \( i_B = i \circ i_{B, A} \) implies the following isomorphisms of complexes

\[
\Gamma_i F \rightarrow i_A i^*_B A F \cong \Gamma_i F \rightarrow i_A i^*_B A F \cong i_A \Gamma_i F \rightarrow i_A i^*_B A F.
\]

We claim that \( \tilde{R}_\pi i_* = R\tilde{f}_\pi \). Indeed, one deduces from \([2, \text{Cor. 5.2.4}]\) and \( \tilde{f}_\pi \circ i = \tilde{f}_\pi \) that \( \tilde{R}_\pi \circ R\tilde{f}_\pi = R\tilde{f}_\pi \). That \( i_* = i_\pi \) is as \( A \) is closed in \( X_\pi \) (cf. Lemma \([2, \text{Cor. 5.2.4}]\)), while \( i_\pi \) is exact since the stalk \( (i_\pi F)_y \) is equal to \( F_y \) if \( y \in A \), and zero otherwise, thus \( R i_* = i_* \). Finally, taking the exact functor \( \tilde{R}_\pi \) to the quasi-isomorphism \( h \) yields a quasi-isomorphism of complexes

\[
\tilde{R}_\pi \Gamma_i F \cong \Gamma_i F \rightarrow i_A i^*_B A F \rightarrow \tilde{R}_\pi \Gamma_i F \rightarrow i_A i^*_B A F.
\]

This proves the lemma.
3.4. Description of \( A, X \) and \( D \). We notice that from now on we shall abuse the notation \( x, y, z \), and others, i.e. we shall use them parallelly with two different senses. Just before \((x, y, z)\) stands for a system of coordinates in \( \mathbb{A}^d (d = d_1 + d_2 + d_3) \), in what follow it will also denote the corresponding system of coordinates on the analytification \( \mathbb{A}^d_{\text{an}} \). Similarly, if \( \tau \) is an element in the group scheme \( G_{m, \kappa} \), we also write \( \tau \) for the corresponding element in \( G_{m, \kappa} \).

**Lemma 3.2.** With \( f \) as in Theorem 1.2, the analytic space \( X = \mathfrak{X}_\pi \) is the inductive limit of the compact domains

\[
A_{\gamma, \epsilon} := \{ (x, y, z) \in \mathbb{A}^d_{\mathfrak{K}^\ast, \text{Ber}} : |x| \leq \gamma^{-1}, |y| \leq \gamma \epsilon, |z| \leq \epsilon, f(x, y, z) = t \}
\]

with \( \gamma, \epsilon \) running over the value group \( |(K^*)^\ast| \) of the absolute value on \( K^* \) such that \( \gamma, \epsilon \in (0, 1) \) and \( \gamma, \epsilon \to 1 \). In the same way, \( X = \pi^{-1}(\mathbb{A}^d_{\text{an}}) \) is the inductive limit of

\[
X_{\gamma, \epsilon} := \{ (x, y, z) \in \mathbb{A}^d_{\mathfrak{K}^\ast, \text{Ber}} : |x| < \gamma^{-1}, |y| \leq \gamma \epsilon, |z| \leq \epsilon, f(x, y, z) = t \}.
\]

**Proof.** For each \( \gamma \in |(K^*)^\ast| \), choose an element \( \tau \gamma \) in \( G_{m, \kappa} \) such that its corresponding element \( \tau \gamma \) in \( G_{m, \kappa} \) takes absolute value \( \gamma \). Since \( f(\tau \gamma, x, \tau \gamma^{-1} y, z) = f(x, y, z) \), the following special \( R \)-algebras are isomorphic

\[
R(\tau \gamma x, \tau \gamma^{-1} y, z)/(f(x, y, z) - t) \cong R(x, y, z)/(f(x, y, z) - t).
\]

Setting

\[
A_\gamma := \left( \left( \text{Spf} \frac{R(\tau \gamma x, \tau \gamma^{-1} y, z)}{f(x, y, z) - t} \right) / \mathbb{A}^d_{\mathfrak{K}^\ast} \right) \mathfrak{X}_\pi,
\]

it is clear that

\[
A_\gamma = \{ (x, y, z) \in \mathbb{A}^d_{\mathfrak{K}^\ast, \text{Ber}} : |\tau \gamma x| \leq 1, |\tau \gamma^{-1} y| < 1, |z| < 1, f(x, y, z) = t \}
= \{ (x, y, z) \in \mathbb{A}^d_{\mathfrak{K}^\ast, \text{Ber}} : |x| \leq \gamma^{-1}, |y| < \gamma, |z| < 1, f(x, y, z) = t \}
\]

and that all the spaces \( A_\gamma \)'s, with \( \gamma \in |(K^*)^\ast| \), are analytically isomorphic. The latter implies an analytic isomorphism between any pair \( (A_\gamma, A_\gamma') \) with \( \gamma, \gamma' \) in \( |(K^*)^\ast| \) and, thus one can establish an inductive system

\[
\{ \{ A_\gamma \}, \{ A_\gamma \to A_\gamma' \}_{\gamma \prec \gamma'} : \gamma, \gamma' \in |(K^*)^\ast| \cap (0, 1) \}.
\]

Then \( A \) is exactly the inductive limit of this system \( \{ A_\gamma \} \) when \( \gamma \to 1 \). On the other hand, the space \( \{ y : |y| < \gamma \} \) is covered by the compact domains \( \{ z : |z| \leq \gamma e \} \) and the space \( \{ z : |z| < 1 \} \) is covered by the compact domains \( \{ z : |z| \leq \epsilon \} \) with \( \epsilon \in |(K^*)^\ast| \) and \( 0 < \epsilon < 1 \). Therefore \( A \) can be viewed as the inductive limit of \( A_{\gamma, \epsilon} \)'s as above with \( \gamma, \epsilon \in |(K^*)^\ast| \cap (0, 1) \) and \( \gamma, \epsilon \to 1 \).

The inductive system of \( X_{\gamma, \epsilon} \)'s whose limit describes \( X \) is defined by \( X_{\gamma, \epsilon} := A_{\gamma, \epsilon} \cap X \), transition morphisms induce from those in the system of \( A_{\gamma, \epsilon} \)'s. \( \square \)

We also remark that \( D := \pi^{-1}(0) \) is an open and locally compact analytic space, it can be covered by the following compact domains

\[
D_\epsilon := \{ z \in \mathbb{A}^d_{\mathfrak{K}^\ast, \text{Ber}} : |z| \leq \epsilon, f(0, 0, z) = t \}, \epsilon \in |(K^*)^\ast| \cap (0, 1).
\]

**Corollary 3.3.** Keeping the assumption of Theorem 1.2 and fixing a \( \gamma \in |(K^*)^\ast| \cap (0, 1) \), we have
(7) of complexes
Using comparison theorem.

4.1. Using comparison theorem. By Corollary 3.4, there is a quasi-isomorphism of complexes

\[
\text{\(\mathcal{R}G_c(\mathcal{A}_d, R\psi_f F|_{\mathcal{A}_d^{d}}) \xrightarrow{\text{qis}} \mathcal{R}\Gamma_{X_\gamma}(A_\gamma, F), \) \(F^\gamma_{\gamma}\) the pullback of \(F \in S(A)\) via \(A_\gamma \cong A\).
\]

(ii) \(\mathcal{R}G_c(\mathcal{A}_d, R\psi_f F|_{\mathcal{A}_d^{d}}) \xrightarrow{\text{qis}} \mathcal{R}\Gamma(D, G|_D), \) for \(G \in S(\mathcal{A}_\gamma)\).

Proof. By the description of \(A\) and \(X\), there are isomorphisms of analytic spaces \(A_\gamma \cong A\) and \(X_\gamma \cong X\) for a fixed \(\gamma\) in \(|(K^*)^*| \cap (0, 1)\). These together with (i), (ii) and Lemma 3.1 imply (i). Also, (ii) follows from Corollary 2.7.

Corollary 3.4. Keeping the assumption of Theorem 1.2 and fixing a \(\gamma \in |(K^*)^*| \cap (0, 1)\), we have

(i) \(\mathcal{R}G_c(\mathcal{A}_d, R\psi_f F|_{\mathcal{A}_d^{d}}) \xrightarrow{\text{qis}} \mathcal{R}\Gamma_{X_\gamma}(A_\gamma, F), \)

(ii) \(\mathcal{R}G_c(\mathcal{A}_d, R\psi_f F|_{\mathcal{A}_d^{d}}) \xrightarrow{\text{qis}} \mathcal{R}\Gamma(D), \)

4. Proof of Theorem 1.2

4.1. Using comparison theorem. By Corollary 3.4, there is a quasi-isomorphism of complexes

\[
\text{\(\mathcal{R}G_c(\mathcal{A}_d, R\psi_f F|_{\mathcal{A}_d^{d}}) \xrightarrow{\text{qis}} \mathcal{R}\Gamma_{X_\gamma}(A_\gamma, F), \) \(A_\gamma \cong A, \) \(X_\gamma \cong X\) for a fixed \(\gamma\) in \(|(K^*)^*| \cap (0, 1)\). The space \(A_\gamma\) is the analytic subspace of \(K^*\)-analytic space which is a union of the following increasing sequence of compact domains

\[A_{\gamma, \varepsilon} := \left\{ (x, y, z) \in \mathcal{A}_d, |x| \leq \gamma^{-1}, |y| \leq \gamma \varepsilon, |z| \leq \varepsilon, f(x, y, z) = t \right\}, \]

for \(\varepsilon \in |(K^*)^*| \cap (0, 1)\). The space \(X_\gamma\) is covered by the corresponding increasing sequence

\[X_{\gamma, \varepsilon} := \left\{ (x, y, z) \in \mathcal{A}_d, |x| < \gamma^{-1}, |y| \leq \gamma \varepsilon, |z| \leq \varepsilon, f(x, y, z) = t \right\}. \]

Denote \(B_\gamma := A_\gamma \setminus X_\gamma\) and \(B_{\gamma, \varepsilon} := A_{\gamma, \varepsilon} \setminus X_{\gamma, \varepsilon}\).

Let us consider \(f := f_{\gamma} : A_\gamma \cong A \to \mathcal{M}(\mathcal{K})\) and \(f_{\gamma, \varepsilon}\), the restriction of \(f\) to \(A_{\gamma, \varepsilon}\).

Lemma 4.1. For any \(m \geq 1\) and \(F \in S(A_\gamma)\), there is a canonical isomorphism of groups

\[H^m_{X_{\gamma, \varepsilon}} (A_\gamma, F) \cong \text{proj lim}_{\varepsilon \to 1} H^m_{X_{\gamma, \varepsilon}} (A_{\gamma, \varepsilon}, F). \]

Proof. The functors \(H^m_{X_{\gamma, \varepsilon}} (A_\gamma, -)\) are the derived functors of the global section functor \(H^m_{X_{\gamma, \varepsilon}} (A_{\gamma, -})\) defined by

\[H^m_{X_{\gamma, \varepsilon}} (A_\gamma, F) = \ker(F(A_\gamma) \to F(B_\gamma)), \]

the kernel of the restriction homomorphism \(F(A_\gamma) \to F(B_\gamma)\). Note that if \(J\) is an injective abelian sheaf then the pullback of \(J\) on \(B_\gamma\) is acyclic and the homomorphism \(J(A_\gamma) \to J(B_\gamma)\) is surjective. Take an injective resolution of \(F\), namely
By the universality of the projective limit, there are canonical morphisms

\[ 0 \rightarrow F \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots, \]

and consider the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow \ker(\alpha_0) \xrightarrow{d^0} \ker(\alpha_1) \xrightarrow{d^1} \ker(\alpha_2) \xrightarrow{d^2} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow J^0(A_\gamma) \xrightarrow{d^0_{A_\gamma}} J^1(A_\gamma) \xrightarrow{d^1_{A_\gamma}} J^2(A_\gamma) \xrightarrow{d^2_{A_\gamma}} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow J^0(B_\gamma) \xrightarrow{d^0_{B_\gamma}} J^1(B_\gamma) \xrightarrow{d^1_{B_\gamma}} J^2(B_\gamma) \xrightarrow{d^2_{B_\gamma}} \cdots
\end{array}
\]

Then we have

\[ H^m_{X,\gamma}(A_\gamma, F) = \ker(H^m(A_\gamma, F) \rightarrow H^m(B_\gamma, F)) \cong \ker(d^m)/\text{im}(d^{m-1}). \]

Analogously, we consider the surjections, say, \( \alpha_{m,\epsilon} : J^m(A_{\gamma,\epsilon}) \rightarrow J^m(B_{\gamma,\epsilon}). \)

Then we have \( H^m_{X,\gamma}(A_{\gamma,\epsilon}, F) \cong \ker(d^m)/\text{im}(d^{m-1}). \) Then we can use the arguments of \([\text{2}, \text{Lemma 6.3.12}]\) to complete the proof. Note that in this situation the following condition is satisfied: For any \( 0 < \epsilon < 1, \) for any \( \epsilon < \epsilon', \epsilon'' < 1, \) the image of \( H^{-1}_{X,\gamma,\epsilon'}(B_{\gamma,\epsilon'}, F) \) and that of \( H^{-1}_{X,\gamma,\epsilon''}(A_{\gamma,\epsilon''}, F) \) coincide in \( H^{-1}_{X,\gamma,\epsilon}(A_{\gamma,\epsilon}, F) \) under the restriction homomorphisms (see \([\text{5}, \text{Lemma 7.4}]\) for a similar argument). \( \Box \)

Here is an important corollary of \([\text{7}]\) and Lemma \([\text{4.1}]\).

**Corollary 4.2.** There is a canonical quasi-isomorphism of complexes

\[ R\Gamma_c(A^{d_1}_K, R\psi_j^*Q_{\ell}|_{A^{d_1}_K}) \xrightarrow{\text{qis}} \text{proj lim}_{\epsilon \rightarrow 1} R\Gamma_{X,\gamma,\epsilon}(A_{\gamma,\epsilon}, Q_{\ell}). \]

**Proof.** We deduce from \([\text{7}]\) and properties of the mapping cone functor that

\[ R\Gamma_c(A^{d_1}_K, R\psi_j^*Q_{\ell}|_{A^{d_1}_K}) \xrightarrow{\text{qis}} R\Gamma_{X,\gamma}(A_{\gamma}, Q_{\ell}) \]

\[ \cong Rf_*^*\text{Cone}(Q_{\ell} \rightarrow i_{B_{\gamma,\epsilon}}^*i_{A_{\gamma,\epsilon}}^*Q_{\ell}) \]

\[ \cong \text{Cone}(Rf_*^*Q_{\ell} \rightarrow Rf^*|_{B_{\gamma}})_*Q_{\ell}). \]

By the universality of the projective limit, there are canonical morphisms

\[ Rf_*^*Q_{\ell} \rightarrow \text{proj lim}_{\epsilon \rightarrow 1} Rf_*^*Q_{\ell}, \]

\[ R(f^*|_{B_{\gamma}})_*Q_{\ell} \rightarrow \text{proj lim}_{\epsilon \rightarrow 1} R(f^*|_{B_{\gamma,\epsilon}})_*Q_{\ell}. \]
Lemma 4.3. \((i)\)

and \(Y_{10}\)

Similarly, one can write

\[ \text{we write} \]

\[ \text{deduces from (8) that} \]

\[ \text{Lemma 4.1.} \]

This morphism of complexes in fact induces the cohomological isomorphisms in Lemma 4.1. \(\square\)

The second part of Corollary 3.4 asserts that

\[ (R\psi_{\tilde{f}}_3 Q_\ell)^0 \overset{\text{qis}}{\rightarrow} R\Gamma(D_\ell, Q_\ell). \]

The space \(D\) is open and locally compact, which is covered by the compact domains \(D_\ell = \{ z \in \mathbb{A}^r_{K^\circ, \text{Ber}} : |z| \leq \epsilon, f(0, 0, z) = t \} \), for \(\epsilon \in [(K^*)^* \cap (0, 1). \quad \text{By}[2] \text{Lem.} 6.3.12] \), there is a canonical isomorphism of cohomology groups

\[ H^m(D_\ell, Q_\ell) \overset{\text{c}}{\cong} \text{proj lim}_{\ell \rightarrow 1} H^m(D_\ell, Q_\ell) \]

for any \(m \geq 0\). Thus by the same arguments as in the proof of Corollary 4.2 one deduces from (8) that

\[ (R\psi_{\tilde{f}}_3 Q_\ell)^0 \overset{\text{qis}}{\rightarrow} \text{proj lim}_{\ell \rightarrow 1} R\Gamma(D_\ell, Q_\ell). \]

(Compare this with [5] Lem. 7.4.]

4.2. Using K"unneth isomorphism. We now use the K"unneth isomorphism for cohomology with compact support mentioned in Proposition 2.2 (iii). To begin, we write \(A_{\gamma, \epsilon}\) as a disjoint union of \(A_{\gamma, \epsilon} = A_{\gamma, \epsilon}^0 \sqcup A_{\gamma, \epsilon}^1\) of analytic spaces

\[ A_{\gamma, \epsilon}^0 := \{(x, y, z) \in A_{\gamma, \epsilon} : |x||y| = 0\}, \]

\[ A_{\gamma, \epsilon}^1 := \{(x, y, z) \in A_{\gamma, \epsilon} : |x||y| \neq 0\}. \]

Similarly, one can write \(X_{\gamma, \epsilon}\) as a disjoint union of analytic spaces

\[ X_{\gamma, \epsilon}^0 := \{(x, y, z) \in X_{\gamma, \epsilon} : |x||y| = 0\}, \]

\[ X_{\gamma, \epsilon}^1 := \{(x, y, z) \in X_{\gamma, \epsilon} : |x||y| \neq 0\}. \]

Observe that we can write \(X_{\gamma, \epsilon}^0\) as the product \(Y_{\gamma, \epsilon}^0 \times D_\epsilon\) with \(D_\epsilon\) as in Subsection 3.4 and \(Y_{\gamma, \epsilon}^0 := \{(x, y) \in \mathbb{A}^{d_1 + d_2}_{K^\circ, \text{Ber}} : |x||y| = 0, |x| < \gamma^{-1}, |y| \leq \gamma \epsilon\}. \) By the compactness of \(A_{\gamma, \epsilon}^0\), \(D_\epsilon\), and by the K"unneth isomorphism, we have

\[ R\Gamma_{X_{\gamma, \epsilon}^0}(A_{\gamma, \epsilon}^0, Q_\ell) \overset{\text{qis}}{\rightarrow} R\Gamma_c(Y_{\gamma, \epsilon}^0, Q_\ell) \otimes R\Gamma_c(D_\epsilon, Q_\ell) \]

\[ \text{(10) } \overset{\text{qis}}{\rightarrow} R\Gamma_c(Y_{\gamma, \epsilon}^0, Q_\ell) \otimes R\Gamma(D_\epsilon, Q_\ell). \]

Decompose \(Y_{\gamma, \epsilon}^0\) into a disjoint union of \(Y_{\gamma, \epsilon}^{0,1} := \{(x, 0) \in \mathbb{A}^{d_1 + d_2}_{K^\circ, \text{Ber}} : |x| < \gamma^{-1}\}

\[ \text{and } Y_{\gamma, \epsilon}^{0,2} := \{(0, y) \in \mathbb{A}^{d_1 + d_2}_{K^\circ, \text{Ber}} : 0 < |y| \leq \gamma \epsilon\}. \]

Lemma 4.3. \((i)\) \(R\Gamma_c(A_{\gamma, \epsilon}^0, Q_\ell) \overset{\text{qis}}{\rightarrow} R\Gamma_c(Y_{\gamma, \epsilon}^{0,1}, Q_\ell); \quad \text{(ii) } R\Gamma_c(Y_{\gamma, \epsilon}^{0,2}, Q_\ell) \overset{\text{qis}}{\rightarrow} 0; \quad \text{(iii) } R\Gamma_c(A_{\gamma, \epsilon}^0, Q_\ell) \overset{\text{qis}}{\rightarrow} R\Gamma_c(Y_{\gamma, \epsilon}^0, Q_\ell). \)
Proof: (i) For notational simplicity, let $F$ denote both constant sheaves $\mathbb{Z}/\ell^m\mathbb{Z}$ on $\mathbb{A}_{K^s}^{d_1}$ and on $\mathbb{A}_{K^s}^{d_1, an} = \mathbb{A}_{K^s, Ber}^{d_1}$. The comparison theorem for cohomology with compact support [2, Thm. 7.1.1] gives an isomorphism of groups

$$H^m_c(\mathbb{A}_{K^s}^{d_1}, F) \cong H^m_c(\mathbb{A}_{K^s}^{d_1, an}, F),$$

for any $m \geq 0$. Let $V = \mathbb{A}_{K^s}^{d_1, an} \setminus Y^{0,1}_{\gamma, \epsilon}$. By Proposition 5.2.6 (ii) of [2] (notice that Proposition 2.2 (ii) is the $\ell$-adic version of this result), we have an exact sequence

$$\cdots \to H^m_c(V, F) \to H^{m+1}_c(Y^{0,1}_{\gamma, \epsilon}, F) \to H^{m+1}_c(\mathbb{A}_{K^s}^{d_1, an}, F) \to H^{m+1}_c(V, F) \to \cdots .$$

We shall prove that $H^m_c(V, F) = 0$ for every $m$.

Let us choose an open covering $\{V_i\}_{i \in \mathbb{N}}$ of $V = \mathbb{A}_{K^s}^{d_1, an} \setminus Y^{0,1}_{\gamma, \epsilon}$ defined as follows:

$$V_i := \{ x \in \mathbb{A}_{K^s}^{d_1, an} : \gamma^{-1} \leq |x| < \gamma_i \},$$

where $\gamma^{-1} < \gamma_i < \gamma_j$ for every $i < j$. Choose an analogous open covering $\{V_{ijl}\}_{i,j \in \mathbb{N}}$ of $V_i \cap V_j$ for each pair $i, j$. Let $\alpha_i$ and $\alpha_{ijl}$ be the open embeddings $V_i \to V$ and $V_{ijl} \to V$, respectively. Then the following exact sequence

$$\bigoplus_{i,j,l} \alpha_{ijl}(F_{V_{ijl}}) \to \bigoplus_i \alpha_l(F_{V_i}) \to F_V \to 0$$

induces an exact sequence

$$\bigoplus_{i,j,l} H^m_c(V_{ijl}, F) \to \bigoplus_i H^m_c(V_i, F) \to H^m_c(V, F) \to 0.$$

The étale cohomology groups with compact support $H^m_c(V_{ijl}, F)$ and $H^m_c(V_i, F)$ clearly vanish for $m \geq 0$, thus $H^m_c(V, F) = 0$ for $m \geq 0$. By (12), one has $H^m_c(\mathbb{A}_{K^s}^{d_1, an}, F) \cong H^m_c(Y^{0,1}_{\gamma, \epsilon}, F)$ for $m \geq 0$, which together with (11) implies that $H^m_c(\mathbb{A}_{K^s}^{d_1}, F) \cong H^m_c(Y^{0,1}_{\gamma, \epsilon}, F)$ for $m \geq 0$. Now, since $\kappa$ is algebraically closed and $K^s$ is separably closed (for fields of characteristic zero the concepts “algebraically closed” and “separably closed” coincide), applying a result of SGA4 1/2 [3, Cor. 3.3], for $m \geq 0$, $H^m_c(\mathbb{A}_{K^s}^{d_1}, F) \cong H^m_c(\mathbb{A}_{K^s}^{d_1}, F)$. Therefore

$$H^m_c(\mathbb{A}_{K^s}^{d_1}, F) \cong H^m_c(Y^{0,1}_{\gamma, \epsilon}, F), \quad m \geq 0,$$

hence the $\ell$-adic version, namely, $H^m_c(\mathbb{A}_{K^s}^{d_1}, \mathbb{Q}_\ell) \cong H^m_c(Y^{0,1}_{\gamma, \epsilon}, \mathbb{Q}_\ell)$ for $m \geq 0$.

(ii) Let us denote by $F$ the constant sheaf $\mathbb{Z}/\ell^m\mathbb{Z}$, and consider the closed immersion $\mathcal{M}(\hat{K}^s) \to \mathcal{M}(\hat{K}^s\{\gamma^{-1}y\})$ of $\hat{K}^s$-analytic spaces. By [2, Cor. 4.3.2], there is an isomorphism of groups

$$H^m(\mathcal{M}(\hat{K}^s), F) \cong H^m(\mathcal{M}(\hat{K}^s\{\gamma^{-1}y\}), F)$$

for each $m \geq 0$. This leads an isomorphism of groups in the $\ell$-adic cohomology. Thus using the exact sequence in Proposition 2.2 (ii), we have $H_c(Y^{0,2}_{\gamma, \epsilon}, \mathbb{Q}_\ell) = 0$.

(iii) follows from (i) and (ii).
4.3. The final step of the proof. The aim of this subsection is to prove the following

\[(13) \quad R\Gamma_{X,\gamma} (\gamma, \epsilon, Q_\ell) \overset{\text{qis}}{\longrightarrow} R\Gamma_{X^0,\gamma} (\gamma, \epsilon, Q_\ell).
\]

Assume the quasi-isomorphism (13). Then there are quasi-isomorphisms of complexes, due to Corollary 4.2, (13), (10) and Lemma 4.3.

\[
R\Gamma_c (K^d_\gamma, R\psi|_{A^d_\gamma}) \overset{\text{qis}}{\longrightarrow} \lim_{\varepsilon \rightarrow 1} \left( R\Gamma_c (K^d_\gamma, Q_\ell) \otimes R\Gamma (D_\epsilon, Q_\ell) \right)
\]

This together with (9) implies Theorem 1.2.

To process a proof for (13), we write $R\Gamma_{X,\gamma} (\gamma, \epsilon, Q_\ell)$ and $R\Gamma_{X^0,\gamma} (\gamma, \epsilon, Q_\ell)$ in the following form:

\[
R\Gamma_{X,\gamma} (\gamma, \epsilon, Q_\ell) \overset{\text{qis}}{\longrightarrow} Rf^* \Gamma (\gamma, \epsilon, X, Q_\ell),
\]

\[
R\Gamma_{X^0,\gamma} (\gamma, \epsilon, Q_\ell) \overset{\text{qis}}{\longrightarrow} R(f^*|_{A^0_\gamma} \gamma, \epsilon, X, Q_\ell).
\]

where $A^0_\gamma := \{(x, y, z) \in A_\gamma : x = y = 0\}$ and $B^0_\gamma := B_\gamma \cap A^0_\gamma$. To abuse notation we shall use from now on $Q_{\ell x}$ in stead of $Q_{\ell x, \gamma, \epsilon}$.

**Theorem 4.4.** With the previous notation and hypotheses, there is a canonical quasi-isomorphism of complexes

\[
Rf^*_\gamma \Gamma (Q_\ell \to i_{B^0_\gamma \times A^0_\gamma}, Q_\ell) \overset{\text{qis}}{\longrightarrow} R(f^*|_{A^0_\gamma} \gamma, \epsilon, X, Q_\ell).
\]

**Proof.** The space $A^1_\gamma := \{(x, y, z) \in A_\gamma : x, y, z \neq 0\}$ together with $A^0_\gamma$ composing a disjoint union of $A^1_\gamma$, there exists a canonical exact triangle

\[
Rf^*_\gamma \Gamma (Q_\ell \to i_{B^1_\gamma \times A^1_\gamma}, Q_\ell) \overset{\text{qis}}{\longrightarrow} R(f^*|_{A^0_\gamma} \gamma, \epsilon, X, Q_\ell)
\]

where $\gamma, \epsilon$ and $B^1_\gamma := B_\gamma \cap A^1_\gamma$. We are going to verify the following

\[
Rf^*_\gamma \Gamma (Q_\ell \to i_{B^1_\gamma \times A^1_\gamma}, Q_\ell) \overset{\text{qis}}{\longrightarrow} 0.
\]

Let us consider the action of $G^m_{\gamma, \epsilon}$ on $A^1_\gamma$ given by $\gamma, \epsilon$ for $(x, y, z) \in A^1_\gamma$. This $G^m_{\gamma, \epsilon}$-action is free, since $\gamma, \epsilon$ is of the form

\[
G^m_{\gamma, \epsilon} \cdot (x, y, z) \cap A^1_\gamma = \{(\tau x, \tau - 1 y, z) : \gamma, \epsilon \}
\]

for $(x, y, z) \in A^1_\gamma$. Also, an orbit of the action on $B^1_\gamma$ is of the form

\[
G^m_{\gamma, \epsilon} \cdot (x, y, z) \cap B^1_\gamma = \{(\tau x, \tau - 1 y, z) : \gamma, \epsilon \}
\]

for $(x, y, z) \in B^1_\gamma$. Furthermore, the $G^m_{\gamma, \epsilon}$-action has the following

**Property (\ast).** Every orbit on $A^1_\gamma$ intersects with $A^1_\gamma$ in a closed annulus $C$ and with $B^1_\gamma$ in a thin annulus contained in $C$.
Let $\mathcal{P}$ be the space of orbits of $\mathbb{G}^n_{m,K^\circ}$-action on $\mathbb{A}^d_{K^\circ,Ber}$. By Lemma 4.3 $\mathcal{P}$ admits an obvious structure of a $\hat{K}^\circ$-analytic space. The property (*) deduces that the restriction maps of the natural projection onto $\mathcal{P}$ on $A_{l,\varepsilon}^1$ and on $B_{l,\varepsilon}^1$, say, $a : A_{l,\varepsilon}^1 \to \mathcal{P}$ and $b : B_{l,\varepsilon}^1 \to \mathcal{P}$, are surjective. We remark that $\mathcal{P}^{\varepsilon}$ and $\mathcal{P}^{\varepsilon,1}$ factor through $a$ and $b$, respectively. Since one has a spectral sequence (the Leray spectral sequence, see Berkovich [2, Thm. 5.2.2])

$$H^n_c(\mathcal{P}, R^ma_!(\mathbb{Q}_R \to i_{B_{l,\varepsilon}^1,A_{l,\varepsilon}^1,*}\mathbb{Q}_R)) \Rightarrow R^{n+m}H^{1,*}\text{Cone}(\mathbb{Q}_R \to i_{B_{l,\varepsilon}^1,A_{l,\varepsilon}^1,*}\mathbb{Q}_R),$$

it suffices to verify that $Ra_!\text{Cone}(\mathbb{Q}_R \to i_{B_{l,\varepsilon}^1,A_{l,\varepsilon}^1,*}\mathbb{Q}_R)$ is quasi-isomorphic to 0. Let us consider the following exact triangle of complexes on $\mathcal{P}$:

$$\to Ra_!\mathbb{Q}_R \to Rb_!\mathbb{Q}_R \to Ra_!\text{Cone}(\mathbb{Q}_R \to i_{B_{l,\varepsilon}^1,A_{l,\varepsilon}^1,*}\mathbb{Q}_R)[+1].$$

Applying the Berkovich’s weak base change theorem [2 Thm. 5.3.1], we have

$$(R^ma_!(\mathbb{Q}_R)_\lambda \cong H^n_c(a^{-1}(\lambda), \mathbb{Q}_R), \quad (R^mb_!(\mathbb{Q}_R)_\lambda \cong H^n_c(b^{-1}(\lambda), \mathbb{Q}_R))$$

for $\lambda \in \mathcal{P}$ and $m \geq 0$. The embedding of the thin annulus $b^{-1}(\lambda)$ into the closed annulus $a^{-1}(\lambda)$ inducing an isomorphism on étale cohomology (here since $a^{-1}(\lambda)$ and $b^{-1}(\lambda)$ are compact, their étale cohomology and étale cohomology with compact support are the same), we obtain $(R^ma_!(\mathbb{Q}_R)_\lambda \cong (R^mb_!(\mathbb{Q}_R)_\lambda$. In other words, for $\lambda \in \mathcal{P}$ and $m \geq 0$,

$$R^ma_!(\mathbb{Q}_R \to i_{B_{l,\varepsilon}^1,A_{l,\varepsilon}^1,*}\mathbb{Q}_R)_\lambda \cong 0.$$  

This prove (15), which together with (14) implies the theorem. 

**Lemma 4.5.** There is a natural structure of an analytic space on the quotient

$$\mathcal{P} = (\mathbb{A}^{d_1+d_2}_{K^\circ,Ber} \setminus \{0\}) \times \mathbb{A}^{d_2}_{K^\circ,Ber}/\mathbb{G}^n_{m,K^\circ}.$$  

**Proof.** We endow $\mathcal{P}$ with the quotient topology, then obviously it is a compact Hausdorff space. The construction of analytic structure on $\mathcal{P}$ is analogous to that of the projective analytic spaces $\mathbb{P}^d_{K^\circ,Ber}$ where the natural $\mathbb{G}^n_{m,K^\circ}$-action on $\mathbb{A}^{d}_{K^\circ,Ber}$ is replaced by the $\mathbb{G}^n_{m,K^\circ}$-action given by $\tau \cdot (x, y, z) = (\tau x, \tau^{-1} y, z)$, which is also free. See [17] for the construction in detail of $\mathbb{P}^d_{K^\circ,Ber}$. 

**References**

[1] V. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monograph vol. 33, Amer. Math. Soc., Providence, RI, 1990.

[2] V. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. Inst. Hautes Étud. Sci. 78 (1993), 5-171.

[3] V. Berkovich, *Vanishing cycles for formal schemes*, Invent. Math. 115(3) (1994), 539-571.

[4] V. Berkovich, *Vanishing cycles for formal schemes II*, Invent. Math. 125(2) (1996), 367-390.

[5] V. Berkovich, *Vanishing cycles for non-archimedean analytic spaces*, J. Amer. Math. Soc. 9(4) (1996), 1187-1209.

[6] P. Deligne et al., *Cohomologie étale*, Lectures Notes in Math. 569 (1977), Berlin-Heidelberg-New York, Springer.

[7] J. Denef, F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), 201-232.

[8] J. Denef, F. Loeser, *Geometry on arc spaces of algebraic varieties*, Progr. Math. 201 (2001), 327-348.

[9] A. Ducros, *Étale cohomology of schemes and analytic spaces*, Note de cours de l'école d'été de juillet 2010 du projet ANR Espaces de Berkovich.
[10] A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique: I. Le langage des Schémas*, Publ. Math. Inst. Hautes Étud. Sci. 4 (1960), 5-228.

[11] E. Hrushovski, F. Loeser, *Monodromy and the Lefschetz fixed point formula*, arXiv:1111.1954[math.AG], [math.LO] 8 Nov 2011.

[12] M. Kontsevich, Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv: 0811.2435v1[math.AG] 16 Nov 2008.

[13] M. Kontsevich, Y. Soibelman, *Motivic Donaldson-Thomas invariants: summary of results*, arXiv: 0910.4315v2 [math.AG] 7 Feb 2010.

[14] Lê Quy Thuong, *On a conjecture of Kontsevich and Soibelman*, Algebra & Number Theory 6 (2012), no. 2, 389-404.

[15] J. Nicaise, J. Sebag, *Motivic Serre invariants, ramification, and the analytic Milnor fiber*, Invent. Math. 168 (1) (2007), 133173.

[16] M. Temkin, *Desingularization of quasi-excellent schemes in characteristic zero*, Adv. Math. 219 (2008), no. 2, 488–522.

[17] M. Temkin, *Introduction to Berkovich analytic spaces*, preprint , 52 pages.

[18] B. Toën, *Derived Hall algebras*, Duke Math. J. 135 (3) (2006), 587-615.

Institut de Mathématique de Jussieu, UMR 7586 CNRS, 4 place Jussieu, 75005 Paris, France (current)

E-mail address: leqthuong@math.jussieu.fr

Department of Mathematics, Vietnam National University, 334 Nguyen Trai Street, Hanoi, Vietnam

E-mail address: thuonglq@vnu.edu.vn