On \( L \)-fuzzy partitioned automata

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Abstract

This paper is towards the study of theory of fuzzy automata with fuzzy partitions. Specifically, we study the concept of the \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy automaton. Further, we introduce the concept of a crisp-deterministic \( L \)-fuzzy automaton corresponding to the \( L \)-fuzzy partitioned automaton such that both accept the same \( L \)-fuzzy language. Finally, the notion of the fuzzified \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy partitioned automaton is introduced and a characterization of its \( L \)-fuzzy language is given.

Keywords: \( L \)-fuzzy automata; \( L \)-fuzzy languages; \( L \)-fuzzy partitions; \( L \)-fuzzy partitioned automata.

1 Introduction

Since the theory of fuzzy sets was introduced by Zadeh [43], fuzzy automata and languages have been studied as methods for bridging the gap between the precision of computer languages and vagueness. These studies were initiated by Santos [32], Wee [41], and Wee and Fu [42], and further developed by a number of researchers (cf., [18, 22, 25]). Fuzzy automata and languages with membership values in different lattice structures have attracted considerable attention from researchers in this area (cf., [1–3, 6, 10–18, 20, 21, 26–31, 33–35, 37, 38, 40]). Among these works, the work of Jin and his coworkers [14] is towards the algebraic study of fuzzy automata based on po-monoids; the work of Peeva is towards the study of minimizing the states of fuzzy automata and its application to study pattern recognition (cf., [26, 27]); the work of Kim, Kim and Cho [18] is towards the algebraic study of fuzzy automata theory; the work of Abolpour and Zahedi is towards the use of categorical concepts in the study of automata with membership values in different lattice structures (cf., [1–3]); the work of Horry and Zahedi [10] is towards the use fuzzy topologies for the study of a max-min general fuzzy automaton; the work of Das [6] is towards the fuzzy topological characterization of a fuzzy automaton; the work of Qiu is towards the algebraic and topological study of fuzzy automata theory based on residuated lattices (cf., [28–31]); the work of Li and Pedrycz [20] is towards the fuzzy automata based on lattice-ordered monoids; the work of Ćirić and his coworkers is towards the study of determinism in fuzzy automata theory (cf., [11–13]), and the work of Tiwari and his coworkers is towards the algebraic and topological study of fuzzy automata (cf., [33–35, 37, 38, 40]). In application point of view, fuzzy automata provide a useful surrounding for ambiguous computation and have shown their importance for solving meaningful problems in learning systems, pattern recognition and data base theory (cf., [4, 25, 27]).

In this paper specifically, we introduce and study

- the concept of the \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy automaton;
- the crisp-deterministic \( L \)-fuzzy automaton corresponding to the \( L \)-fuzzy partitioned automaton such that both accept same \( L \)-fuzzy language; and
- the notion of the fuzzified \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy partitioned automaton.

The content of this paper is arranged as follows. Section 2 contains preliminary information about the content of the paper. In Section 3, we introduce the concept of the \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy automaton. Further, we study the relationship among the \( L \)-fuzzy languages of the \( L \)-fuzzy partitioned automaton and \( L \)-fuzzy automaton. In Section 4, we introduce the crisp-deterministic
L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language. Finally, in section 5, the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton is introduced. Interestingly, we show that the L-fuzzy language of fuzzified L-fuzzy partitioned automaton can be obtained from the L-fuzzy language of the L-fuzzy partitioned automaton.

2 Preliminaries

In this section, we recall the concepts related to residuated lattices [5,39]; L-fuzzy relations [24,39]; L-fuzzy automata [7,23,36]; L-fuzzy languages [7,39], and L-fuzzy objects [23,24].

We begin with the following.

Definition 2.1. An algebra \( (L, \land, \lor, \circ, \rightarrow, 0, 1) \) is called complete residuated lattice if it satisfies the following conditions:

(i) \( (L, \leq, \land, \lor, 0, 1) \) is a complete lattice with the greatest element 1 and the least element 0;

(ii) \( (L, \circ, 1) \) is a commutative monoid; and

(iii) \( x \circ y \leq z \) iff \( x \leq y \rightarrow z \), for all \( x, y, z \in L \).

Throughout this paper, we assume \( L \) is a complete residuated lattice \( (L, \land, \lor, \circ, \rightarrow, 0, 1) \) and the L-fuzzy sets considered in this paper are in sense of [9], i.e., an L-fuzzy set \( A \) in a set \( X \) is a map \( A : X \rightarrow L \). For a nonempty set \( X \), \( \mathcal{F}(X) \) denotes the collection of all L-fuzzy sets in \( X \). Also, for \( x, y \in L \), \( x \rightarrow y = (x \rightarrow y) \land (y \rightarrow x) \) and \( \Lambda \) denotes an indexed set.

Definition 2.2. For L-fuzzy set \( A \) in a nonempty set \( X \), core of \( A \), denoted by \( \text{core}(A) \), is given as,

\[
\text{core}(A) = \{ x \in X : A(x) = 1 \}.
\]

Further, if \( \text{core}(A) \neq \emptyset \), then \( A \) is called normal L-fuzzy set.

Proposition 2.1. [19, 39] Let \( (L, \land, \lor, \circ, \rightarrow, 0, 1) \) be a complete residuated lattice. Then for all \( x, y, z, x_j, y_j \in L \) and \( j \in \Lambda \), the following properties hold:

(i) \( x \leftrightarrow y = 1 \Leftrightarrow x = y \);

(ii) \( x \leftrightarrow y \leq y \rightarrow x \);

(iii) \( x \leftrightarrow y = y \leftrightarrow x \);

(iv) \( y \leq z \Leftrightarrow (x \circ y) \leftrightarrow (x \circ z) \); and

(v) \( x \circ (\lor (y_j : j \in \Lambda)) = \lor (x \circ y_j : j \in \Lambda) \) and

\[
(\lor (x_j : j \in \Lambda)) \circ y = \lor (x_j \circ y : j \in \Lambda).
\]

Definition 2.3. An L-fuzzy relation on a nonempty set \( X \) is a map \( E : X \times X \rightarrow L \). The L-fuzzy relation \( E \) is called

(i) reflexive if \( E(x, x) = 1, \forall x \in X \);

(ii) symmetric if \( E(x, y) = E(y, x), \forall x, y \in X \); and

(iii) transitive if \( E(x, y) \circ E(y, z) \leq E(x, z), \forall x, y, z \in X \).

A reflexive, symmetric, and transitive L-fuzzy relation on \( X \) is called an L-fuzzy similarity relation on \( X \).

Now, we recall the following concepts related to the L-fuzzy automata.

Definition 2.4. An L-fuzzy automaton is a system \( M = (Q, (M, *, e), T, I, F) \), where \( Q \) is a nonempty set of states, \( (M, *, e) \) is a monoid inputs, \( T : Q \times M \rightarrow L^Q \) is the transition function such that \( \forall p, q \in Q \) and \( \forall m, n \in M \),

\[
T(p,e)(q) = \begin{cases} 
1 & \text{if } p = q \\
0 & \text{if } p \neq q,
\end{cases}
\]

and for all \( (p, m * n) \in Q \), \( T(p, m)(r) \circ T(r, n)(q) : r \in Q \), \( I \in \mathcal{F}(Q) \) is the initial L-fuzzy state and \( F \in \mathcal{F}(Q) \) is the final L-fuzzy state.

A state \( q \in Q \) is called initial (resp. final) state of \( M \) if \( I(q) > 0 \) (resp. \( F(q) > 0 \)). An L-fuzzy automaton whose set of states is finite is called finite L-fuzzy automaton.

Definition 2.5. An L-fuzzy automaton \( M = (Q, (M, *, e), T, I, F) \) is called

(i) complete if for all \( m \in M \) and \( p \in Q \) there exists \( q \) such that \( T(p, m)(q) > 0 \);

(ii) deterministic if there is a unique initial state \( q_0 \) with \( I(q_0) > 0 \) and for all \( m \in M \) and \( p, q, r \in Q \), \( T(p, m)(q) > 0 \) and \( T(p, m)(r) > 0 \), then \( q = r \).

If \( M = (Q, (M, *, e), T, I, F) \) is a complete deterministic L-fuzzy automaton such that for all \( m \in M \) and \( p, q \in Q, T(p, m)(q) \in \{0,1\} \) and for unique initial state \( q_0, I(q_0) = 1 \), then \( M \) is called crisp-deterministic L-fuzzy automaton. In this case, there exists a function \( \delta : Q \times M \rightarrow Q \) such that for all \( p \in Q \) and \( m \in M \), \( \delta(p, m) = q \) iff \( T(p, m)(q) = 1 \). Such crisp-deterministic L-fuzzy automaton is denoted by \( (Q, (M, *, e), \delta, q_0, F) \).

Definition 2.6. An L-fuzzy language \( f_M : M \rightarrow L \) is

(i) accepted by an L-fuzzy automaton \( M = (Q, (M, *, e), T, I, F) \) if \( f_M(m) = \lor \{I(r) \circ T(r, m)(q) : r, q \in Q \}, \forall m \in M \); and
(ii) accepted by a crisp-deterministic $L$-fuzzy automaton $M = (Q, (M, *, e), \delta, q_0, F)$ if $f_M(m) = F(\delta(q_0, m)), \forall m \in M$.

**Definition 2.7.** Let $X$ be a nonempty set. A system $A = \{A_\lambda : \lambda \in \Lambda\} \in$ normal $L$-fuzzy sets in $X$ is an $L$-fuzzy partition of $X$ if $\{\text{core}(A_\lambda) : \lambda \in \Lambda\}$ is a partition of $X$. A pair $(X, A)$ is called a space with an $L$-fuzzy partition.

Now, we recall the following concept of $L$-fuzzy objects in spaces with $L$-fuzzy partitions.

**Definition 2.8.** Let $(L, \mathcal{L})$ be a space with an $L$-fuzzy set $\mathcal{L} = \{L_a : a \in L\}$ be an $L$-fuzzy partition of $L$ such that $\forall a, b \in L, L_a(b) = a \leftrightarrow b$. Then an $L$-fuzzy object in a space with an $L$-fuzzy partition $(X, A)$ is a map $\{(A, \sigma) : (X, A) \rightarrow (L, \mathcal{L})\}$ such that

(i) $A : X \rightarrow L$ is a map;

(ii) $\sigma : \Lambda \rightarrow L$ is a map; and

(iii) $\forall \lambda \in \Lambda$ and $\forall x \in X, A_\lambda(x) \subseteq L_{\sigma(\lambda)}(A(x)) = \sigma(\lambda) \leftrightarrow A(x)$.

$R(X, A)$ denotes the set of all $L$-fuzzy objects in $(X, A)$.

Now, we recall the following from [8, 24].

Let $(X, A)$ be a space with an $L$-fuzzy relation $\pi$ on a set $\Lambda$ is defined as:

$$\pi(\lambda_1, \lambda_2) = (\lor \{A_{\lambda_2}(x) : x \in \text{core}(A_{\lambda_2})\}) \lor (\lor \{A_{\lambda_2}(x) : x \in \text{core}(A_{\lambda_2})\}), \forall \lambda_1, \lambda_2 \in \Lambda.$$

It can easily verified that $\pi$ is reflexive and symmetric $L$-fuzzy relation.

Now, consider the smallest $L$-fuzzy relation $\rho_{X,A}$ on a set $\Lambda$ with conditions:

$$\rho_{X,A}(\lambda_1, \lambda_2) \cap \rho_{X,A}(\lambda_2, \lambda_3) \subseteq \rho_{X,A}(\lambda_1, \lambda_3)$$

$$\pi(\lambda_1, \lambda_2) \leq \rho_{X,A}(\lambda_1, \lambda_2), \forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda.$$

In that case, $\rho_{X,A}$ is an $L$-fuzzy similarity relation on $\Lambda$.

Next, on the basis of $\rho_{X,A}$, the $L$-fuzzy relation $\delta_{X,A}$ on a set $X$ is defined as:

$$\delta_{X,A}(x_1, x_2) = \rho_{X,A}(\lambda_1, \lambda_2), \forall \lambda_1, \lambda_2 \in \Lambda, \forall x_1 \in \text{core}(A_{\lambda_1}), \text{and } \forall x_2 \in \text{core}(A_{\lambda_2}).$$

It can easily verified that $\delta_{X,A}$ is an $L$-fuzzy similarity relation on $X$.

**Proposition 2.2.** [24] For $(X, A)$ is a space with an $L$-fuzzy partition and $\Lambda = \{A_\lambda : \lambda \in \Lambda\}$ is an $L$-fuzzy partition of $X$, if $A : X \rightarrow L$ be a map. Then the following statements are equivalent.

(i) There exists the unique map $\sigma : \Lambda \rightarrow L$ such that $(A, \sigma) \in R(X, A)$; and

(ii) For given $\lambda \in \Lambda$, $x \in \text{core}(A_\lambda)$, and $x' \in X$, $A_\lambda(x') \leq A(x) \leftrightarrow A(x')$ holds.

Let $(X, A)$ be a space with an $L$-fuzzy partition and $\Lambda = \{A_\lambda : \lambda \in \Lambda\}$ be an $L$-fuzzy partition of $X$. Then $R_1(X, A)$ is defined as,

$$R_1(X, A) = \{(A : X \rightarrow L : (A, \sigma) \text{ is an } L\text{-fuzzy object in } (X, A) \text{ for some map } \sigma : \Lambda \rightarrow L\}.$$

### 3 The $L$-fuzzy partitioned automata

In this section, we introduce and study the concept of an $L$-fuzzy partitioned automaton corresponding to a given $L$-fuzzy automaton. Further, we establish the relationship among the $L$-fuzzy languages of the introduced $L$-fuzzy partitioned automaton and $L$-fuzzy automaton.

We begin with the following definition of the $L$-fuzzy automaton with $L$-fuzzy partition from [23].

**Definition 3.1.** A system $((X, A), (M, *, e), d)$ is called an $L$-fuzzy automaton with $L$-fuzzy partition, if

(1) $X$ is the set of state with an $L$-fuzzy partition $A$, where $A = \{A_\lambda : \lambda \in \Lambda\}$ is an $L$-fuzzy partition of $X$.

(2) $(M, *, e)$ is a monoid inputs; and

(3) $d : X \times M \rightarrow R_1(X, A)$ is a map such that $\forall x, y \in X, \forall x' \in \text{core}(A_\lambda), \text{and } \forall m, n \in M,

\begin{align*}
(i) & \quad d(x, e)(y) = \delta_{X,A}(x, y); \\
(ii) & \quad d(x, m * n)(y) = \lor \{d(x, m)(z) \lor d(z, n)(y) : z \in X\}; \\
(iii) & \quad d(x, m)(y) \cap A_\lambda(x) \subseteq A(x', y); \text{and}
\end{align*}

(iv) From the condition (iii), for each $x, x' \in \text{core}(A_\lambda), d(x, m) = d(x', m)$.

In the remaining part of this section, $\mathcal{M} = (Q, (M, *, e), T, I, F)$ is an $L$-fuzzy automaton and $(Q, Q)$ is a space with an $L$-fuzzy partition, where $Q = \{Q_\alpha : \alpha \in \Lambda\}$ is an $L$-fuzzy partition of $Q$.

Now, we introduce the concept of the $L$-fuzzy partitioned automata.

**Definition 3.2.** Let $\mathcal{M} = (Q, (M, *, e), T, I, F)$ be an $L$-fuzzy automaton. Then the $L$-fuzzy partitioned automaton corresponding to $\mathcal{M}$, denoted by $\mathcal{P}$, is the system $\mathcal{P} = ((Q, Q), (M, *, e), T, I, F)$, where

(i) $Q$ is the set of state with an $L$-fuzzy partition $Q$, where $Q = \{Q_\alpha : \alpha \in \Lambda\}$ is an $L$-fuzzy partition of $Q$;
(ii) \((M, *, e)\) is a monoid inputs; and

(iii) \(T_1(p, m)(q) = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, m)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\}, \forall p, q \in Q \text{ and } \forall m \in M\), where \(\delta_Q, q(q_1, q_2) = p_{Q, q}(q_1, q_2), \forall q_1, q_2 \in \Lambda, \forall q_1 \in \text{core}(Q_1), \text{ and } \forall q_2 \in \text{core}(Q_2)\).

**Proposition 3.1.** Let \(M = (Q, (M, *, e), T, I, F)\) be an L-fuzzy automaton. Then the L-fuzzy partitioned automaton \(P = ((Q, Q), (M, *, e), T, I, F)\) corresponding to \(M\) is an L-fuzzy automaton with L-fuzzy partition.

Proof. (i) Let \(p, s \in Q, s' \in \text{core}(Q), \text{ and } m \in M\). Then

\[
T_1(p, m)(s) \leftrightarrow T_1(p, m)(s') = \begin{cases} \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, m)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\} & \text{if } T_1(p, m) \in R_1(Q, Q) \\ \delta_Q, q(p, s) \leftrightarrow \delta_Q, q(p, s') & \text{if } T_1(p, m) \not\in R_1(Q, Q) \end{cases}
\]

Thus \(Q_0(s) \leq T_1(p, m)(s) \leftrightarrow T_1(p, m)(s')\), which implying that \(T_1(p, m) \in R_1(Q, Q)\).

(ii) Let \(p, q \in Q\). Then

\[
T_1(p, e)(q) = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, e)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\} = \delta_Q, q(p, q).
\]

Conversely,

\[
T_1(p, e)(q) = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, e)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\} = \delta_Q, q(p, q) \leq Q_0(s).
\]

Thus \(T_1(p, e)(q) = \delta_Q, q(p, q)\).

(iii) Let \(p, q \in Q\) and \(m, n \in M\). Then

\[
T_1(p, m \circ n)(q) = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, m \circ n)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\} = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, m)(r_2) \circ \delta_Q, q(r_2, q) : r_1, r_2 \in Q\} = \bigvee \{\delta_Q, q(p, r_1) \circ T(r_1, m)(r_2) \circ \delta_Q, q(r_2, q) : r, r_1, r_2 \in Q\}.
\]

Thus \(f_M \subseteq f_P\).

The following proposition is to be established the relationship between L-fuzzy languages of the introduced L-fuzzy partitioned automaton and L-fuzzy automaton.

**Proposition 3.2.** Let \(P = ((Q, Q), (M, *, e), T, I, F)\) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \(M = (Q, (M, *, e), T, I, F)\). Then \(f_M \subseteq f_P\).

Proof. Let \(m \in M\). Then

\[
f_P(m) = \bigvee \{I(r) \circ T_1(r, m)(q) \circ F(q) : r, q \in Q\} = \bigvee \{I(r) \circ \delta_Q, q(r_1) \circ T(r_1, m)(r_2) \circ \delta_Q, q(r_2, q) \circ F(q) : r, r_1, r_2, q \in Q\} \geq \bigvee \{I(r) \circ \delta_Q, q(r_1) \circ T(r_1, m)(q) \circ \delta_Q, q(q) \circ F(q) : r, q \in Q\} = \bigvee \{I(r) \circ T(r, m)(q) \circ F(r)\} = f_M(m).
\]

Thus \(f_M \subseteq f_P\).

**Proposition 3.3.** Let \(P = ((Q, Q), (M, *, e), T, I, F)\) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \(M = (Q, (M, *, e), T, I, F)\), where \(Q = \{Q_0 : \alpha \in \Lambda\}\) such that for all \(\alpha \in \Lambda\), there exists unique \(q \in Q\) with \(Q_0(q) = 1\) and 0, otherwise. Then \(f_P = f_M\).
Proof. Let \( m \in M \). Then
\[
f_P(m) = \bigvee \{ I(r) \circ T_1(r, m)(q) \circ F(q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ (\bigvee \delta_Q, Q(r, r_1) \circ T(r_1, m)(r_2) \\
\odot \delta_Q, Q(r_2, q) : r_1, r_2, q \in Q) \circ F(q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ \delta_Q, Q(r, r_1) \circ T(r_1, m)(q) \circ \delta_Q, Q(q, q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ T(r, m)(q) \circ F(q) \}
\]
\[
f_M(m).
\]
Thus \( f_P = f_M \).

Proposition 3.4. If the L-fuzzy partitioned automaton \( P = (Q, \mathcal{Q}, (M, \ast, e), T_1, I, F) \) corresponding to given L-fuzzy automaton \( M = (Q, (M, \ast, e), T, I, F) \) is crisp-deterministic L-fuzzy automaton, then \( Q = \{ Q_\alpha : \alpha \in \Lambda \} \) such that for all \( \alpha \in \Lambda \), there exists unique \( q \in Q \) with \( Q_\alpha(q) = 1 \) and 0, otherwise.

Proof. Follows from Proposition 3.1.

4 Determinization of L-fuzzy partitioned automata

In this section, we introduce the crisp-deterministic L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language.

Definition 4.1. Let \( P = (Q, \mathcal{Q}, (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( M = (Q, (M, \ast, e), T, I, F) \). Then the crisp-deterministic L-fuzzy automaton corresponding to \( P \), denoted by \( F \), is the system \( F = (F(Q), (M, \ast, e), T_F, I_F, F_F) \), where

(i) \( F(Q) = \{ \mu : \mu : Q \to L \} \) is the set of states;

(ii) \( (M, \ast, e) \) is a monoid inputs;

(iii) \( T_F : F(Q) \times M \to F(Q) \) is a transition function such that for all \( \forall A \in F(Q) \), \( q \in Q \), and \( m \in M \),
\[
T_F(A, m)(q) = \bigvee \{ A(r) \circ T(r, m)(q) : r \in Q \};
\]

(iv) \( I_F \in F(Q) \) is an initial state such that \( \forall Q \in Q \),
\[
I_F(q) = \bigvee \{ I(r) \circ \delta_Q, Q(r, q) : r \in Q \};
\]

(v) \( F_F : F(Q) \to L \) is a final L-fuzzy state such that for all \( A \in F(Q) \),
\[
F_F(A) = \bigvee \{ A(r) \circ \delta_Q, Q(r, q) \circ F(q) : r, q \in Q \}.
\]

Proposition 4.1. Let \( P = (Q, \mathcal{Q}, (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( M = (Q, (M, \ast, e), T, I, F) \) and \( \mathcal{F} = (\mathcal{F}(Q), (\mathcal{M}, \ast, e), T_F, I_F, F_F) \) be a crisp-deterministic L-fuzzy automaton corresponding to \( P \). Then \( f_F = f_P \).

Proof. Let \( m \in M \). Then
\[
f_F(m) = \bigvee \{ I(r) \circ T_F(r, m)(q) \circ F(q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ (\bigvee \delta_Q, Q(r, r_1) \circ T(r_1, m)(r_2) \\
\odot \delta_Q, Q(r_2, q) : r_1, r_2, q \in Q) \circ F(q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ \delta_Q, Q(r, r_1) \circ T(r_1, m)(q) \circ \delta_Q, Q(q, q) : r, q \in Q \}
\]
\[
= \bigvee \{ I(r) \circ T(r, m)(q) \circ F(q) \}
\]
\[
f_M(m).
\]
Thus \( f_F = f_P \).

5 The fuzzified L-fuzzy partitioned automata

In this section, we introduce and study the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton. Further, we study the L-fuzzy language of such fuzzified L-fuzzy partitioned automaton in terms of L-fuzzy language of the L-fuzzy partitioned automaton.

Definition 5.1. Let \( P = (Q, \mathcal{Q}, (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( M = (Q, (M, \ast, e), T, I, F) \). The fuzzified L-fuzzy partitioned automaton corresponding to \( P \), denoted by \( W \), is the system \( W = (Q, \mathcal{Q}(Q), (\mathcal{M}(M), \odot, 1_\mathcal{M}), T_2, I, F) \), where

(i) \( (\mathcal{M}(M), \odot, 1_\mathcal{M}) \) is a monoid inputs, where \( \mathcal{M}(M) = \{ A : A : M \to L \} \) and \( 1_\mathcal{M} \in \mathcal{M}(M) \) such that \( \forall x \in M \),
\[
1_\mathcal{M}(x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{if } x \neq e \end{cases},
\]

(ii) \( T_2 : (p, A)(q) = \bigvee \{ A(m) \circ T_1(p, m)(q) : m \in M \}, \forall p, q \in Q \) and \( A \in \mathcal{A}(M) \).

Definition 5.2. An L-fuzzy language \( f_w : \mathcal{M}(M) \to L \) is accepted by a fuzzified L-fuzzy partitioned automaton \( W = (Q, \mathcal{Q}(Q), (\mathcal{M}(M), \odot, 1_\mathcal{M}), T_2, I, F) \) if \( f_w(A) = \bigvee \{ I(r) \circ T_2(r, A)(q) \circ F(q) : r, q \in Q \}, \forall A \in \mathcal{M}(M) \).
Proposition 5.1. Let \( \mathcal{P} = ((Q, \mathcal{Q}), (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( \mathcal{M} = (Q, (M, \ast, e), T, I, F) \). Then the fuzzified L-fuzzy partitioned automaton \( \mathcal{W} = ((Q, \mathcal{Q}), (F(M), \circ, I), T_2, I, F) \) corresponding to \( \mathcal{P} \) is an L-fuzzy automaton with L-fuzzy partition.

Proof. (i) Let \( p, s, s' \in Q, s' \in \text{core}(Q_\alpha), \) and \( A \in F(M) \). Then

\[
T_2(p, A)(s) \leftrightarrow T_2(p, A)(s')
\]

\[
= \bigvee \{ A(m_1) \circ T_1(p, m_1)(s) : m_1 \in M \} \leftrightarrow \bigvee \{ A(m_2) \circ T_1(p, m_2)(s') : m_2 \in M \}
\]

\[
\geq \bigvee (A(m) \circ T_1(p, m)(s)) \leftrightarrow (A(m) \circ T_1(p, m)(s'))
\]

\[
= T_1(p, m)(s) \leftrightarrow T_1(p, m)(s')
\]

\[
= Q_\alpha(s).
\]

Thus \( Q_\alpha(s) \leq T_2(p, m)(s) \leftrightarrow T_2(p, m)(s') \), which implying that \( T_2(p, m) \in R_1(Q, \mathcal{Q}) \).

(ii) Let \( p, q \in Q \). Then

\[
T_2(p, 1_e)(q) = \bigvee \{ 1_e(m) \circ T_1(p, m)(q) : m \in M \}
\]

\[
= T_1(p)(q)
\]

\[
\delta_{Q_\alpha, p}(q).
\]

Thus \( T_2(p, 1_e)(q) = \delta_{Q_\alpha, p}(q) \).

(iii) Let \( p, q \in Q \) and \( A_1, A_2 \in F(M) \). Then

\[
T_2(p, A_1 \circ A_2)(q) = \bigvee \{ (A_1 \circ A_2) m \circ T_1(p, m)(q) : m \in M \}
\]

\[
= \bigvee \{ A_1(m_1) \circ A_2(m_2) \circ T_1(p, m)(q) : m = m_1 \ast m_2 \}
\]

\[
= \bigvee \{ A_1(m_1) \circ A_2(m_2) \circ T_1(p, m_1 \ast m_2)(q) : m_1, m_2 \in M \}
\]

\[
= \bigvee \{ A_1(m_1) \circ A_2(m_2) \circ T_1(p, m_1)(r) \circ T_1(r, m_1)(q) : r \in Q, m_1, m_2 \in M \}
\]

\[
= \bigvee \{ T_2(p, A_1)(r) \circ T_2(r, A_2)(q) : r \in Q \}.
\]

Thus \( T_2(p, A_1 \circ A_2)(q) = \bigvee \{ T_2(p, A_1)(r) \circ T_2(r, A_2)(q) : r \in Q \} \).

(iv) Let \( p, s, s' \in Q, s' \in \text{core}(Q_\alpha), \) and \( A \in F(M) \). Then

\[
T_2(p, A)(s) \circ Q_\alpha(p) = \bigvee \{ A(m) \circ T_1(p, m)(s) : m \in M \} \circ Q_\alpha(p)
\]

\[
= \bigvee \{ A(m) \circ T_1(p, m)(s) \circ Q_\alpha(p) : m \in M \}
\]

\[
\leq \bigvee \{ A(m) \circ T_1(s', m)(s) : m \in M \}
\]

\[
= T_2(s', A)(s).
\]

Thus \( T_2(p, A)(s) \circ Q_\alpha(p) \leq T_2(s', A)(s) \).

(v) From the condition (iv), \( T_2(s, A) = T_2(s', A), \) \( \forall s', s' \in \text{core}(Q_\alpha) \) and \( A \in F(M) \).

Hence \( W = ((Q, \mathcal{Q}), (F(M), \circ, I), T_2, I, F) \) is an L-fuzzy automaton with L-fuzzy partition.

Proposition 5.2. Let \( \mathcal{P} = ((Q, \mathcal{Q}), (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( \mathcal{M} = (Q, (M, \ast, e), T, I, F), \)

\( \mathcal{W} = ((Q, \mathcal{Q}), (F(M), \circ, I), T_2, I, F) \) be the fuzzified L-fuzzy partitioned automaton corresponding to \( \mathcal{P} \), and \( W = A_1 \ast \ldots \ast A_n, \forall A_1, \ldots, A_n \in F(M) \). Then \( T_2(p, W)(q) = \bigvee \{ T_1(p, m_1 \ast \ldots \ast m_n)(q) : \circ A_1(m_1) \ast \ldots \ast A_n(m_n) \} \ast A_1(m_1), \ldots, A_n(m_n) \ast A_{n+1} \in F(M) \). Then

\[
T_2(p, W)(q) = \bigvee \{ T_2(p, A_1)(r) \circ T_2(r, A_{n+1})(q) : r \in Q \}
\]

\[
= \bigvee \{ \bigvee \{ T_1(p, m_1 \ast \ldots \ast m_n)(r) : \circ A_1(m_1) \ast \ldots \ast A_n(m_n) \} \circ A_{n+1}(r, m_{n+1})(q) : r \in Q, m_1, \ldots, m_{n+1} \in M \}
\]

\[
= \bigvee \{ T_1(p, m_1 \ast \ldots \ast m_n)(q) \ast A_1(m_1) \ast \ldots \ast A_n(m_n) \ast A_{n+1} \} : m_1, \ldots, m_{n+1} \in M \}
\]

The following is towards the L-fuzzy language of the introduced fuzzified L-fuzzy partitioned automaton in terms of the L-fuzzy language of the L-fuzzy partitioned automaton.

Proposition 5.3. Let \( \mathcal{P} = ((Q, \mathcal{Q}), (M, \ast, e), T_1, I, F) \) be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton \( \mathcal{M} = (Q, (M, \ast, e), T, I, F), \)

\( \mathcal{W} = ((Q, \mathcal{Q}), (F(M), \circ, I), T_2, I, F) \) be the fuzzified L-fuzzy partitioned automaton corresponding to \( \mathcal{P} \), and \( W = A_1 \ast \ldots \ast A_n, \forall A_1, \ldots, A_n \in F(M) \). Then \( f_W(W) = \bigvee \{ f_T(m_1 \ast \ldots \ast m_n) \circ A_1(m_1) \ast \ldots \ast A_n(m_n) : m_1, \ldots, m_n \in M \} \).
Thus partitioned automaton corresponding to a given \( L \)-fuzzy automaton have been used to introduce the fuzzified \( L \)-fuzzy language. Finally, the concept of the \( L \)-fuzzy partitioned automaton is introduced corresponding to the \( L \)-fuzzy automaton. Further, the crisp-deterministic \( L \)-fuzzy language of the partitioned automaton can be obtained from the \( L \)-fuzzy partitioned automaton. Interestingly, it is shown here obtained the relationship among the \( L \)-fuzzy languages of the \( L \)-fuzzy partitioned automaton and \( L \)-fuzzy automaton. Further, the crisp-deterministic \( L \)-fuzzy automaton is introduced corresponding to the \( L \)-fuzzy partitioned automaton such that both accept same \( L \)-fuzzy language. Finally, the concept of the \( L \)-fuzzy sets have been used to introduce the fuzzified \( L \)-fuzzy partitioned automaton corresponding to a given \( L \)-fuzzy partitioned automaton. Interestingly, it is shown here that the \( L \)-fuzzy language of fuzzified \( L \)-fuzzy partitioned automaton can be obtained from the \( L \)-fuzzy language of the \( L \)-fuzzy partitioned automaton.

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**References**

[1] K. Abolpour, M.M. Zahedi, Isomorphism between two BL-general fuzzy automata, Soft Computing 16 (2012) 729-736.

[2] K. Abolpour, M. M. Zahedi, BL-general fuzzy automata and accept behavior, Journal of Applied Mathematics and Computing 38 (2012) 103-118.

[3] K. Abolpour, M.M. Zahedi, General fuzzy automata based on complete residuated lattice-valued, Iranian Journal of Fuzzy Systems 14 (2017) 103-121.

[4] G. Bailador, G. Trivino, Pattern recognition using temporal fuzzy automata, Fuzzy Sets and Systems 61 (2009) 37-55.

[5] R. Bělohlávek, Fuzzy relational systems: foundations and principles, Springer Science and Business Media, New york, 2012.

[6] P. Das, A fuzzy topology associated with a fuzzy finite state machine, Fuzzy Sets and Systems, 105 (1999) 469-479.

[7] J. R. G. De Mendivil, J. R. Garitagoitia, Determinization of fuzzy automata via factorization of fuzzy states, Information Sciences 283 (2014) 165-179.

[8] L. Garmendia, R. González del Campo, V. López, J. Recasens, An algorithm to compute the transitive closure, a transitive approximation and a transitive opening of a fuzzy proximity, Mathware and soft computing 16 (2009) 175-191.

[9] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145-174.

[10] M. Horry, M.M. Zahedi, Some (fuzzy) topologies on general fuzzy automata, Iranian Journal of Fuzzy Systems 10 (2013) 73-89.

[11] J. Ignjatovi´ c, M. ´Ciri´ c, S. Bogdanovi´ c, Determinization of fuzzy automata with membership values in complete residuated lattices, Information Sciences 178 (2008) 164-180.

[12] J. Ignjatovi´ c, M. ´Ciri´ c, S. Bogdanovi´ c, T. Petkovi´ c, Myhill-Nerode type theory for fuzzy languages and automata, Fuzzy Sets and Systems 161 (2010) 1288-1324.

[13] J. Ignjatovi´ c, M. ˇCiric, V. Simoovi´ c, Fuzzy relation equations and subsystems of fuzzy transition systems, Knowledge-Based Systems 38 (2013) 48-61.

[14] J.H. Jin, Q.G. Li, Y.M. Li, Algebraic properties of \( L \)-fuzzy finite automata, Information Sciences 234 (2013) 182-202.

[15] Y.B. Jun, Intuitionistic fuzzy finite state machines, Journal of Applied Mathematics and Computing 17 (2005) 109-120.

[16] Y.B. Jun, Intuitionistic fuzzy finite switchboard state machines, Journal of Applied Mathematics and Computing 20 (2006) 315-325.
[17] Y.B. Jun, Quotient structures of intuitionistic fuzzy finite state machines, Information Sciences 177 (2007) 4977-4986.

[18] Y.H. Kim, J.G. Kim, S.J. Cho, Products of $T$-generalized state machines and $T$-generalized transformation semigroups, Fuzzy Sets and Systems 93 (1998) 87-97.

[19] J. M. Ko, Y. Chan, Algebraic and topological structures on factorization of fuzzy sets, Journal of Intelligent and Fuzzy Systems 30 (2016) 1709-1718.

[20] Y. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids, Fuzzy Sets and Systems 156 (2005) 68-92.

[21] W. Lihua, D. Qiu, Automata theory based on complete residuated lattice-valued logic: Reduction and minimization, Fuzzy Sets and Systems 161 (2010) 1635-1656.

[22] D.S. Malik, J.N. Mordeson, M.K. Sen, Submachines of fuzzy finite state machine, Journal of Fuzzy Mathematics 2 (1994) 781-792.

[23] J. Močkór, Categories of fuzzy type automata in monads, IEEE International Conference (2017) 1-6.

[24] J. Močkór, M. Holčapek, Fuzzy objects in spaces with fuzzy partitions, Soft Computing 21 (2017) 7269-7284.

[25] J. N. Mordeson, D.S. Malik, Fuzzy Automata and Languages: Theory and Applications, Chapman and Hall/CRC, London/Boca Raton, 2002.

[26] K. Peeva, Behaviour, reduction and minimization of finite $L$-automata, Fuzzy Sets and Systems 28 (1988) 171-181.

[27] K. Peeva, Fuzzy acceptors for syntactic pattern recognition, International Journal of Approximate Reasoning 5 (1991) 291-306.

[28] D. Qiu, Automata theory based on complete residuated lattice-valued logic(I), Science in China Series F: Information Sciences 44 (2001) 419-429.

[29] D. Qiu, Automata theory based on complete residuated lattice-valued logic(II), Science in China Series F: Information Sciences 45 (2002) 442-452.

[30] D. Qiu, Automata theory based on quantum logic: some characterizations, Information and Computation 190 (2004) 179-195.

[31] D. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets and Systems 141 (2004), 391-414.

[32] E. S. Santos, Maximin automata, Information and Control 12 (1968) 367-377.

[33] A.K. Srivastava, S.P. Tiwari, A topology for fuzzy automata, Lecture Notes in Artificial Intelligence 2275 (2002) 485-490.

[34] S.P. Tiwari, V. Gautam, B. Davvaz, On minimal realization for a fuzzy language and Brzozowski’s algorithm, Journal of Intelligent and Fuzzy Systems 29 (2015) 1949-1956.

[35] S.P. Tiwari, S. Sharan, Fuzzy automata based on lattice-ordered monoids with algebraic and topological aspects, Fuzzy Information and Engineering 4 (2012) 155-164.

[36] S. P. Tiwari, A. K. Singh, S. Sharan, Fuzzy automata based on lattice-ordered monoid and associated topology, Journal of Uncertain Systems 6 (2012) 51-55.

[37] S.P. Tiwari, A. K. Singh, S. Sharan, V.K. Yadav, Bifuzzy core of fuzzy automata, Iranian Journal of Fuzzy Systems 12 (2015) 63-73.

[38] S.P. Tiwari, A.K. Srivastava, On a decomposition of fuzzy automata, Fuzzy Sets and Systems 151 (2005) 503-511.

[39] S. P. Tiwari, V. K. Yadav, P. Pal, B. K. Sharma, Minimal fuzzy realization for fuzzy behaviour: A bicategory-theoretic approach, Journal of Multiple-valued Logic and Soft computing 31 (2018) 105-121.

[40] S. P. Tiwari, V. K. Yadav, A. K. Singh, Construction of a minimal realization and monoid for a fuzzy language: a categorical approach, Journal of Applied Mathematics and Computing 47 (2015) 401-416.

[41] W.G. Wee, On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph. D. Thesis, Purdue University, 1967.

[42] W.G. Wee, K.S. Fu, A formulation of fuzzy automata and its application as a model of learning systems, IEEE Transaction on Systems Science and Cybernetics 5 (1969) 215-223.

[43] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.