Nonlocal interaction between ultracold atoms trapped in optical lattices can give rise to interesting quantum many-body phenomena. However, its realization usually demands unconventional techniques, for example, the artificial gauge fields or higher-orbit Feshbach resonances, and is not highly controllable. Here, we propose a valid and feasible scheme for realizing a tunable finite-range interaction for spinless fermions immersed into the bath of bosons. The strength of the effective interaction for the fermionic subsystem is artificially tunable by manipulating bosons, ranging from the repulsive to the attractive regime. In addition, the interaction distance is locked to the hopping of bosons, making the finite-range interaction perfectly clean for the fermionic subsystem. Specifically, we find that, by introducing an additional staggered hopping of bosons, the proposal is readily applied to search the Majorana corner modes in such a spinless system, without the implementation of complex artificial gauge fields, which is totally distinct from existing results reported in spinful systems. Therefore this scheme provides a potential platform for exploring the unconventional topological superfluids and other nontrivial phases induced by long-range interactions in ultracold atoms.

I. INTRODUCTION

Recent progress of ultracold atoms in optical lattices offers a remarkable platform for the simulations and discoveries of many-body phenomena [1–3], e.g., the superfluid–Mott insulator transition for a Bose-Hubbard model [4]. Extensive studies were made based on ultracold atoms that are locally interacted [5]. While in the presence of the nonlocal atomic interaction, it can lead to exotic quantum many-body behaviors, including the supersolids, density waves and topological phases [6–30], and hence deserves more investigations. Intuitively in cold atoms, the nonlocal interaction may be introduced via the higher-orbit Feshbach resonance, but usually faces technical shortcomings such as the heating effect or three-body loss [31–33]. Therefore, alternative ways for synthesizing effective nonlocal interaction are rather expected in current atomic physics.

A series of proposals were reported to generate effective nonlocal interaction based on the nontrivial interplays or configurations, including the Rashba spin-orbit coupling [34–39], optical lattices with additional designs [40–43], and higher-orbital band physics [44, 45]. However in these proposals, the resulting effective interaction cannot be generalized to arbitrary distance because of the limits of constructions. Another way to engineer effective long-range interaction is through the exchange of mediating particles such as the cavity photons [16, 46–50]. Successful examples include the highly magnetic atoms [15, 51–55], dipolar moments [56–59] and Rydberg atoms [60–63]. Unfortunately, such photon-mediated interactions are unavoidably accompanied by dissipation, moreover the short- and long-range interactions are simultaneously present and cannot be individually controlled. On the other hand, in the Bose-Fermi mixture [64–69], due to the high manipulations of bosons, the induced boson-assisted interaction is possible to be artificially controllable for the fermionic system.

In this paper, we follow this line and propose a scheme for realizing an effective finite-range interaction in systems composed of atomic mixtures. The main features of the proposal are as follows: (i) An alternative mecha-
nism for superfluids stemming from the repulsive interactions is introduced. In our proposal, the boson-boson and boson-fermion interactions are both repulsive \[64, 70, 71\], the engineered effective interaction for the fermionic sub-system is artificially controllable, and can range from the repulsive regime to the attractive one mainly by an introduced staggered bosonic potential. (ii) The engineered effective interaction is non-local, with a long-range interaction distance that is locked to the hopping distance of bosons. This makes it possible for obtaining various patterns of atomic interactions beyond the s-wave one, since the hopping is artificially controllable. Therefore it can provide a perfectly clean platform to control and investigate the quantum systems with long-range interactions, compared with those schemes based on the dipolar moments or Rydberg atoms \[56–63\]. (iii) In the presence of a staggered hopping for bosons, the effective interaction takes the staggered modulation and the extra degree of freedom contributes to supporting higher-order topological superfluids phase accompanied by Majorana corner modes even in such a spinless system without implementation of complex artificial gauge fields, which is not reported in existing results that focus on spinful systems.

II. EFFECTIVE FINITE-RANGE INTERACTION

We consider the fermionic atoms immersed into a quantum degenerate gas of bosonic atoms, resulting in a Bose-Fermi mixture as shown in Fig. 1. They are respectively confined in two-dimensional (2D) optical lattice potentials \(V_B(r) = V_B[\cos^2(k_Lx) + \cos^2(k_Ly)]\) and \(V_F(r) = V_F[\cos^2(k_Lx) + \cos^2(k_Ly)]\). Here \(k_L = \pi/a\) with the lattice constant \(a\). \(V_B\) and \(V_F\) are the potential depths and can be controlled individually. To capture a clear physics picture, we first consider a simple case of the model Hamiltonian written in the following form,

\[
H = H_B + H_F + H_{BF}.
\]

The first part describes the bosonic part,

\[
H_B = \int d\mathbf{r} \psi_b^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m_b} \nabla^2 + V_B(\mathbf{r}) - \mu_b + \Gamma(\mathbf{r})\right)\psi_b(\mathbf{r}) + g_B \int d\mathbf{r} \psi_b^\dagger(\mathbf{r})\psi_c^\dagger(\mathbf{r})\psi_b(\mathbf{r})\psi_b(\mathbf{r}).
\]

Here \(\psi_b\) is the bosonic operator. \(m_b\) is the mass of bosons. \(g_B\) is the bare boson-boson interaction strength in free space. The on-site potential of bosons is composed of the chemical potential \(\mu_b\) and a spatially modulated potential \(\Gamma(\mathbf{r}) = \Gamma \sin(k_Lx)\sin(k_Ly)\). The staggered potential will play a crucial role in a tunable effective interaction and we will discuss it later. The second part describes the fermionic subsystem,

\[
H_F = \int d\mathbf{r} \psi_c^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m_c} \nabla^2 + V_F(\mathbf{r}) - \mu_c\right)\psi_c(\mathbf{r}).
\]

Here \(\psi_c\) is the fermionic operator. \(m_c\) is the mass of fermions. \(\mu_c\) is the fermionic chemical potential. The last part describes the boson-fermion interaction,

\[
H_{BF} = g_{BF} \int d\mathbf{r} \psi_b^\dagger(\mathbf{r})\psi_c(\mathbf{r})\psi_b(\mathbf{r})\psi_c(\mathbf{r}),
\]

with the bare boson-fermion interaction strength \(g_{BF}\). We use the tight-binding approximation to study the system. The field operators \(\psi_b\) and \(\psi_c\) can be expanded in terms of Wannier wave functions \(W_B(\mathbf{r})\) and \(W_F(\mathbf{r})\), which are centered at lattice sites, describing the fields operators as \(\psi_b(\mathbf{r}) = \sum_j W_B(\mathbf{r} - \mathbf{r}_j)b_j\) and \(\psi_c(\mathbf{r}) = \sum_j W_F(\mathbf{r} - \mathbf{r}_j)c_j\). Here \(\mathbf{r}_j = ja\) denotes the coordinate of the \(j\)th site, and \(b_j\) and \(c_j\) denote the annihilation operators for bosons and fermions on the \(j\)th site respectively. Then the total lattice Hamiltonian takes the final form:

\[
H = -\sum_{i\neq j} t_{bc}b_i^\dagger b_j + \sum_j \left[\mu_b + (-1)^j \Gamma\right]n_{bj}^j + \sum_j U n_{bj}^j n_{bj}^j - \sum_{i\neq j} t_{bc}^\dagger c_j - \sum_j \mu_c n_{cj}^j + \sum_j V n_{cj}^j n_{cj}^j.
\]

Here \(n_{bj}^j = \lambda_j^\dagger\lambda_j\) (\(\lambda = b, c\)) denotes the density operator, and \(t_{bc}\) is the corresponding hopping magnitude. The on-site modulated potential \(\Gamma\) exhibits a checkerboard pattern. The boson-boson interaction strength \(U = g_B \int d\mathbf{r} |W_B(\mathbf{r})|^4\) and boson-fermion interaction strength \(V = g_{BF} \int d\mathbf{r} |W_B(\mathbf{r})W_F(\mathbf{r})|^2\). We assume the
strengths $U$ and $V$ are both positive, yielding the repulsive interaction.

It is known that Bose gases invoke a transition from a superfluid to a Mott insulator phase by means of a tunable boson-boson interaction [4, 72, 73]. Specifically in the strong interaction regime, the system reduces to a hard-core bosonic model that shares similar properties to a Fermi gas. Furthermore, bosons are tightly bounded in lattice sites, thus each site is singly occupied which resembles a Mott insulator behavior, as shown in Fig. 1. This opens the way for quantum simulation and engineering using ultracold atoms [64, 74], particularly in the studies on quantum magnetism [75, 76]. In this work, we follow this trick and apply it to generate an effective finite-range interaction. For simplicity, we first investigate the engineering of the nearest-neighbor (NN) one. It can invoke topological nontrivial properties that are absent in systems with only the on-site one.

In the strong interaction regime of $U$, the hopping magnitudes $t_b$ and $t_c$ are sufficiently less compared to $U$. In this way, we can suppose the bosonic bath is prepared into the Mott insulator phase, in which each site is filled with only one boson. Furthermore, we assume $t_b \gg t_c, \mu_c$, which yields the approximation that the fermionic subsystem does not affect the bosonic bath. In other words, the bosonic bath provides a background for the fermionic subsystem to generate effective interaction. At this time, we extract the nonperturbative Hamiltonian as $H_0 = \sum_{j}( -\mu_b + ( -1 )^j \Gamma )n_{j}^b + U n_{j}^b n_{j+1}^b + V n_{j}^b n_{j+1}^c$, and denote the Fock state of $H_0$ as $|\psi_j\rangle = |n_{j}^b n_{j+1}^b ; n_{j}^c n_{j+1}^c\rangle$. With respective to $H_0$, we regard the hopping of bosons as the perturbative term, $H_p = -\sum_{(i,j)} t_b n_{i}^b n_{j}^b$, here the summation $\sum_{(i,j)}$ takes over all NN sites. As discussed before, we supposed the single occupation of bosons in each site. We remark that if the hopping of fermions is isotropic, the number density of fermions is approximately uniform and constant. It indicates that the $V$ term in $H_0$ acts as an on-site energy shift of bosons. Then we can see that the subsystem described by the bosonic part of Hamiltonian (5) will reduce to the Bose-Hubbard model and can still form a Mott insulating state in the weak regime of $t_b/U [72, 73]$. Therefore, one can first prepare the bosonic subsystem into the Mott insulator phase with a single occupation in each site, and then adiabatically load in the staggered potential $\Gamma$ by employing the additional Stark shift or superlattice structure [77]. In this way, we can finally obtain the desired bosonic background with single occupation.

By taking the Pauli exclusion into considerations, the energy of $|\psi_j\rangle$ is split into 12 levels, which is shown in Fig. 2(a). When $U$ is large, the level transitions generated by $H_p$ are regarded as being fully far-detuned. The states $|11; n_{j}^c n_{j+1}^c\rangle$ with the initial occupation are macroscopically occupied, by contrast, the excited states $|02; n_{j}^c n_{j+1}^c\rangle$ and $|20; n_{j}^c n_{j+1}^c\rangle$ $(n_{j}^c, n_{j+1}^c = 0, 1)$ are extremely less occupied. The adiabatic elimination of the $|02; n_{j}^c n_{j+1}^c\rangle$ and $|20; n_{j}^c n_{j+1}^c\rangle$ manifolds gives rise to a second-order perturbation to the system. It is easily seen that $\mu_c$ does not affect the energy detuning of transitions in Fig. 2(a) because of the number conservation of the fermionic atoms, and $t_c$ does not contribute to the transitions from the initial states to the excited state manifolds. We then treat the bosonic bath as the reservoir of no further interest and concentrate on the fermionic subsystem. After performing detailed derivations in Appendix A, the effective Hamiltonian of the fermionic subsystem can be written as

$$H_{\text{eff}} = -\sum_{(i,j)} t_c c_i^c c_j - \sum_j \mu n_j^c + \sum_j U_{\text{eff}} n_j^c n_{j+1}^c.$$  (6)

Here the effective interaction strength is expressed as

$$U_{\text{eff}} = \sum_{\alpha_1, \alpha_2 = \pm} \frac{t_b^2}{U - \alpha_1 V/2 - \alpha_2 \Gamma} - \frac{4t_b^2 U}{(U^2 - \Gamma^2)},$$  (7)

and the chemical potential is given by $\mu = \mu_c + U_{\text{eff}} + 2t_b^2 U/(U^2 - \Gamma^2)$.

The effective interaction $U_{\text{eff}}$ has the following features: (i) In the simple case with $\Gamma = 0$, the interaction strength reduces to $U_{\text{eff}} = t_b^2 V^2/[U(U^2 - V^2/4)]$. This reveals that the sign of the effective interaction is solely determined by the boson-boson interaction $U$ when $U$ dominantly governs the physics. Particularly, the effective interaction is repulsive ($U_{\text{eff}} > 0$) since we assumed $U > 0$ in Fig. 2(b). (ii) In addition to the bare interaction strength, it provides an alternative way to tune $U_{\text{eff}}$ utilizing $\Gamma$. In Fig. 2(b) we show the dependence of $U_{\text{eff}}$ on $\Gamma$. We can see the sign of $U_{\text{eff}}$ is ambiguous and independent of $U$ when $\Gamma \neq 0$. From Eq. (7), the effective interaction can be demonstrated to be resonant at $\Gamma = U$ and $\Gamma = \pm U \pm V/2$. Across the resonance points, the sign of $U_{\text{eff}}$ is changed and a breakdown of the continuous controllability will be encountered between its attractive and repulsive interaction regimes. It shows that $\Gamma$ is the key to generating attractive effective interaction. (iii) The emergence of resonance points is because several occupied
states of the $|11; n_{j}^c n_{j+1}^c\rangle$ manifold degenerate to the excited states of the $|20; n_{j}^c n_{j+1}^c\rangle$ or $|02; n_{j}^c n_{j+1}^c\rangle$ manifolds. It tells the excited states are no longer isolated from the macroscopically occupied states, which can be revealed in Fig. 2(a). For this case, the results based on adiabatic elimination of excited states will be invalid near the resonance. As the consequence, in this work we only focus on the physics out of the resonance condition.

In the above discussions, we studied the simple case of the NN interaction. From Fig. 2 and the relevant formulas in Appendix A, we can see the interaction distance of fermions depends precisely on the hopping of bosons on arbitrary sites. It reveals that this proposal of the engineered effective interaction is readily extended to the longer-range ones, as long as the hopping of bosons is introduced between beyond-NN sites (e.g., finite-range hopping driven via Raman transitions [78, 79]). This can give rise to interesting competing phases of superfluids as well as charge density waves [16,80–82]. In this case, since additional optical fields are introduced, it is necessary to reduce the extra heating effect, such as preparing the laser fields far detuned during the Raman transitions. We remark that different from these works [16,46–50] based on the atom-cavity coupling, the effective long-range interaction is clean, because the interaction distance is locked to the hopping distance of bosons. This paves the way for realizing the interaction within a particular range for the fermionic subsystem.

For the sake of the anti-commutation relation of the $c$ operator, the effective interaction is known to support $p$-wave Cooper pairing of an odd parity if it is attractive. For the spinless Fermi gas, the attractive $p$-wave pairing can bring rich physics associated with interesting topological properties [83,84]. As aforementioned in the Introduction, the engineering of $p$-wave pairing usually faces practical difficulties. The proposal provides such a desirable routine to achieve the aim, because the proposal basically relies on the contact interaction that is attainable in current experimental techniques using the Bose-Fermi mixture. Furthermore, the proposal provides various methods for independently tuning the effective finite-range interaction $U_{\text{eff}}$ by $U$, $V$ (via the conventional Feshbach resonance technique) and $\Gamma$ (via the external optical field), both in terms of strength and spatial structure. Taking full advantage of our proposal, it indicates that the application in searching higher-order topological superfluids transitions is readily expected even in a spinless Fermi system, which motivates us forward and is studied next.

III. MAJORANA CORNER MODES

Higher-order topological superfluids can host Majorana zero modes (MZMs) whose dimensions are always lower than the traditional ones, providing that they have the same bulk dimensions. Particularly for the 2D case, the second-order topological superfluids support zero-dimensional (0D) Majorana zero modes localized at the corners instead of one-dimensional (1D) edges, known as the Majorana corner modes (MCMs). A variety of schemes based on solid-state systems [85–110] and ultracold atoms [111–113] were proposed for realizing the interesting MCMs, which mainly rely on complex lattice structures or artificial gauge fields, however spinless systems have not been involved. Since complex artificial gauge fields, e.g. spin-orbit coupling, cannot be applied in spinless systems, alternative approaches should be considered for the realization of higher-order topological superfluids.

To search the potential higher-order topological phases, we introduce a spatial modulation to the hopping of bosons, which provides an extra degree of freedom in the spinless system. In particular, we assume the hopping magnitude of bosons is staggered along the $y$ direction, while it remains uniform along the $x$ direction. For simplicity, we introduce a dimensionless quantity $\eta$ to characterize the staggered structure. The perturbative term is then transformed as $H_p \rightarrow \sum_{j_x,j_y} -t_b (b_{j_x,j_y}^\dagger b_{j_x+1,j_y} + b_{j_x,j_y}^\dagger b_{j_x+1,j_y} - 1 +$
are nonzero, it outlines the superfluid phase. Under the staggered hopping, the fermionic system will be dimerized into an A-B sublattice structure. Since $U_{\text{eff}}$ directly depends on $t_b$ in Eq. (7), the effective NN interaction in Eq. (6) will simultaneously respond to the staggered pattern, and is rewritten as $\sum_{j} U_{\text{eff}} n_j^\alpha n_{j+1}^\alpha \rightarrow \sum_{j} U_{\text{eff}} n_j^\alpha n_{j+1}^\alpha + \eta n_j^a n_{j+1}^a n_{j+2}^a n_{j+1}^b + \eta n_j^b n_{j+1}^b n_{j+2}^a n_{j+1}^a$.

We exploit the mean-field Bogoliubov de Gennes (BdG) approach to study the superfluid phases of Fermi gases, which can capture the qualitative features and particularly the topological features of our interest. In the BdG approach, the order parameter of the superfluid phase is characterized by a nonzero Chern number $\nu=1$. Specifically, first-order topological superfluids phase, in addition to the trivial superfluid phase, of second-order topological superfluids phase, in addition to the trivial superfluid phase, in addition to the trivial superfluid phase. When $\nu=0$ for second-order topological superfluids phase, the BdG approach to study the superfluid phases of Fermi gases 

We demonstrate that the strength and distance of the staggered hopping can readily be realized using the existing techniques of ultracold atoms. Notice that in previous investigations we assumed that the hopping magnitude of the bosons needs to be larger than fermions. Intuitively, one can choose the bosonic atoms (e.g., $^7\text{Li}$ and the other fermionic atoms with the heavier mass (e.g., $^{171}\text{Yb}$) [116, 117]. We load the two species of atoms into optical lattices with the identical wavelength $\lambda_L = 1064\text{ nm}$ but of different trap depths, e.g., $V_R = 12\text{E}_R$ and $V_F = 5 \times \eta_\text{m} E_R \approx 0.2 E_R$. Here we take the recoil energy of the bosonic lattice $E_R = h^2/(2 m_b \lambda_L^2) \approx 25.1 \text{kHz}$ as the energy unit to simplify the discussion, and $\eta_\text{m} = 7/171$ denotes the mass ratio between the two atomic species. After the above setups, the NN hopping strengths can be determined as $t_b \approx 0.0123 E_R \approx 0.31 \text{kHz}$ and $t_c \approx 0.0658 \times \eta_\text{m} E_R \approx 0.22 t_b$. By preparing the staggered bosonic potential $\Gamma = 5.8 t_b \approx 0.071 E_R \approx 1.8 \text{kHz}$, the effective interaction strength $U_{\text{eff}} = -1.9 t_b \approx -8.7 t_c$, at which it is also known to support the superfluid phase in conventional Fermi gases [42]. The parameter setup used in Fig. 3 is therefore attainable. After obtaining the effective interaction, to realize the higher-order topological superfluids, we can synthesize the staggered hopping by applying a double-well structure to the optical lattice potential or using optical fields, which was also widely applied in searching nontrivial phases in ultracold atoms [118–123].

In summary, we proposed the scheme for engineering the finite-range interaction from the contact one using a fermionic optical lattice immersed into a bosonic bath. We demonstrate that the strength and distance of the effective fermionic interaction assisted with bosons are both highly tunable, which sharply differ from previous schemes based on Bose-Fermi mixtures [64–69]. Particularly by introducing staggered hopping of bosons, we find the fermionic subsystem undergoes a second-order topological transition and supports Majorana corner modes within experimental reach, yet to be reported in such a spinless system. Although further applications are required, especially with respect to the effects of long-range interactions in the extended Hubbard model which are still currently under active investigations [124–127], our results show a promising quantum simulation tool to investigate unconventional topological superfluids and other nontrivial phases induced by tunable finite-range interaction in ultracold atomic mixtures.

V. ACKNOWLEDGEMENTS

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To explicitly express the Hamiltonians in Eq. (A8), we introduce the following denotation,
\[ H_{n_1 n_2; m_1 m_2}^{n_1' n_2'} = \mathcal{P}_{n_1 n_2}^{m_1 m_2} H \mathcal{P}_{n_1' n_2'}^{m_1' m_2'}. \] (A9)
\[ H_{PP}^{(00)} = \sum_{(m_1 m_2) \neq (n_1 n_2)} \mathcal{P}_{n_1 n_2}^{m_1 m_2} (\mathcal{P}_{11}^{00} - H_{m_1 m_2; m_1 m_2}^{n_1 n_2})^{-1} H_{n_1 n_2; m_1 m_2}^{n_1' n_2'}. \] (A10)

Other nonzero off-diagonal terms can be obtained via the relations
\[ H_{n_1 n_2; m_1 m_2}^{n_1' n_2'} = (H_{m_1' m_2'; m_1' m_2'}^{n_1 n_2})^{\dagger} \] and
\[ H_{n_1 n_2; m_1 m_2}^{n_1' n_2'} = (m_1 \leftrightarrow m_2)^{\dagger} = (m_1' \leftrightarrow m_2')^{\dagger} = (n_1 \leftrightarrow n_2)^{\dagger}. \] Since there are four states for the \( \hat{P} \) subspace, we analyze them one by one. For \( |11; 00\rangle \), we get
\[ H_{PP}^{(00)} = \sum_{(m_1 m_2) \neq (n_1 n_2)} \mathcal{P}_{n_1 n_2}^{m_1 m_2} (\mathcal{P}_{11}^{00} - H_{m_1 m_2; m_1 m_2}^{n_1 n_2})^{-1} H_{n_1 n_2; m_1 m_2}^{n_1' n_2'}. \] (A11)

Likewise for \( |11; 11\rangle \), we get
\[ H_{PP}^{(11)} = -\left( \frac{t_2^2}{U - \Gamma} + \frac{t_2^2}{U + \Gamma} \right) |11; 11\rangle \langle 11; 11| + \frac{t_2^2}{U - \Gamma} |n_{j+1}^f n_{j+1}^f\rangle. \] (A12)

For \( |11; 01\rangle \), we get
\[ H_{PP}^{(10)} = -\left( \frac{2t_2^2}{2U - V - 2\Gamma} + \frac{2t_2^2}{2U + V + 2\Gamma} \right) |11; 01\rangle \langle 11; 01| \]
\[ = -\left( \frac{t_2^2}{U - V/2 - \Gamma} + \frac{t_2^2}{U + V/2 + \Gamma} \right) (1 - n_{j+1}^c) n_{j+1}^c. \] (A13)

For \( |10; 10\rangle \), we get
\[ H_{PP}^{(10)} = -\left( \frac{2t_2^2}{2U - V + 2\Gamma} + \frac{2t_2^2}{2U + V - 2\Gamma} \right) |10; 10\rangle \langle 10; 10| \]
\[ = -\left( \frac{t_2^2}{U - V/2 + \Gamma} + \frac{t_2^2}{U + V/2 - \Gamma} \right) n_{j+1}^c (1 - n_{j+1}^c). \] (A14)

After combining the above results, the final form of the effective Hamiltonian for the fermionic subsystem is expressed as
\[ H_{eff} = -\sum_{i \neq j} t_c c_i^\dagger c_j - \sum_j \mu n_j^c + \sum_{k < l} U_{eff} n_k^c n_l^c. \] (A15)
interaction (intermediate virtual state) are extremely less occupied, which can be adiabatically eliminated. The on-site boson-fermion bosonic hopping distance, i.e., site \( k \) here the distance of effective interaction is locked to the state. The effective interaction strength and the renormalized chemical potential are given as

\[
\mu_{\text{eff}} = \mu + U_{\text{eff}} + \frac{2t_b^2U}{U^2 - \Gamma^2}.
\] (A17)

When we consider the hopping of atoms in nearest-neighbor sites, the effective Hamiltonian (A15) is reduced to that expressed in Eq. (6) of the main text.

**Appendix B: Analytical descriptions for the generated Majorana corner modes**

1. Majorana representation

According to the generated higher-order topological superfluids and Majorana corner modes described in the main text, we start from the corresponding real-space Hamiltonian as the following form,

\[
H = \sum_{j} \left[ -t_x \sum_{\sigma=A,B} c_{\sigma,j}^\dagger c_{\sigma,j} - t_y (c_{B,j}^\dagger c_{A,j} + c_{A,j}^\dagger c_{B,j}^\dagger) \right.
\]
\[
+ \sum_{\sigma} \left( -\frac{\mu}{2} c_{\sigma,j}^\dagger c_{\sigma,j} + \Delta_x c_{\sigma,j}^\dagger c_{\sigma,j+\hat{x}}^\dagger \right)
\]
\[
+ \left( \Delta_y c_{B,j}^\dagger c_{A,j}^\dagger + \Delta_y^2 c_{A,j+\hat{x}}^\dagger c_{B,j}^\dagger \right) + \text{H.c.} \right].
\] (B1)

Here \( t_{x(y)} \) is the hopping strength of fermions along the \( x(y) \) direction, respectively. We neglect the \( y \)-directional hopping and represent the fermionic operators in terms of the Majorana operators \( \gamma_{\sigma,j}^{(1)} \) and \( \gamma_{\sigma,j}^{(2)} \), which satisfy the following relations:

\[
\gamma_{\sigma,j} = (\gamma_{\sigma,j}^{(1)} + i\gamma_{\sigma,j}^{(2)})/2, \quad \gamma_{\sigma,j}^\dagger = (\gamma_{\sigma,j}^{(1)} - i\gamma_{\sigma,j}^{(2)})/2. \] (B2)

The effective Hamiltonian is transformed as

\[
H = \frac{i}{2} \sum_{j} \left\{ \sum_{\sigma} [ (\Delta_x + t_c) \gamma_{\sigma,j}^{(1)}] \gamma_{\sigma,j+\hat{x}}^\dagger + (\Delta_x - t_c) \gamma_{\sigma,j}^{(1)} \gamma_{\sigma,j+\hat{x}}^\dagger \right.
\]
\[
- \mu \gamma_{\sigma,j}^{(1)} \gamma_{\sigma,j}^{(2)} + \Delta_y \gamma_{A,j}^{(1)} \gamma_{B,j}^{(1)} - \gamma_{A,j}^{(2)} \gamma_{B,j}^{(2)}
\]
\[
+ \Delta_y \gamma_{B,j}^{(1)} \gamma_{A,j}^{(1)} - \gamma_{B,j}^{(2)} \gamma_{A,j+\hat{x}}^\dagger \right\} .
\] (B3)

whose form reduces to the Benalcazar-Bernevig-Hughes (BBH) model [115] when \( \Delta_x = t_c \). It reveals that the second-order topological phase exists when \( |\mu| < 2t_x \) and \( \Delta y_1 < \Delta y_2 \) (i.e., \( \eta < 1 \)), which is consistent with the numeric results in Fig. 3(a) of the main text.
fermionic subsystem. The lattice model is dimerized into the edge theory \[86\]. We take superfluids and Majorana corner modes, here we perform for use in the edge theory.

\[\eta,\ldots\}\]

III and IV in Fig. 5. On the edge I, by expressing \(k\eta,\xi\) we decompose Hamiltonian (B4) as

\[
H_0(x) = \sqrt{|(2t_x + \mu)/t_x - (\Delta_x^2/t_x^2)|}, \quad \lambda_1 = \Delta_x/2t_x, \quad \lambda_2 = \Delta_x/2t_x, \quad \chi^1_\alpha \text{ denotes the eigenvector of } \tau_x \text{ that satisfies } \tau_x\chi^1_\alpha = \chi^1_\alpha.
\]

By expanding \(H_p\) in terms of \(\phi^I_\alpha(x)\), we obtain its matrix form with the elements expressed as

\[
H^I_{\alpha\beta}(k_y) = \int_0^{\pi} dx \phi^*_{\alpha}(y)H_p(-i\partial_x, k_y)\phi^I_\beta(y).
\]

Then the final form of the effective Hamiltonian is

\[
H^I(k_y) = \Delta_y k_y \sigma_x + (\Delta_y - \Delta_y^2)\sigma_y.
\]

Likewise, the low-energy effective Hamiltonians for edges II, III and IV can be obtained as:

\[
H^{II}(k_y) = 2\Delta_x k_x \sigma_y + (-2t_x - \mu)\tau_y,
\]

\[
H^{III}(k_y) = -\Delta_y k_y \sigma_x - (\Delta_y - \Delta_y^2)\sigma_y,
\]

\[
H^{IV}(k_y) = -2\Delta_x k_x \sigma_y + (2t_x + \mu)\tau_y.
\]

To facilitate the discussion, we assume that \(t_x, \Delta_x, \Delta_y\), and \(\Delta_y^2\) are all positive hereafter. For edge I, the Hamiltonian \(H^I\) has the Dirac mass \(\Delta_y^2\). While for edge II, the Dirac mass is \(2t_x + \mu\) if accounting for the unified anticlockwise direction. It reveals that when \(\mu > -2t_x\) and \(\Delta_y < \Delta_y^2\), the Dirac mass changes its sign at the intersection of edges I and II, leading to the emergence of Majorana corner modes, which are analogous to the Jackiw-Rebbi zero modes \[129, 130\]. By repeating the same treatment at other gapless points like \(k = (\pi, 0)\), we finally conclude that Majorana corner modes appear as long as \(|\mu| < 2t_x\), which is consistent with Fig. 3(a) of the main text.

### 2. Edge theory for Majorana corner modes

To understand the emergent higher-order topological superfluids and Majorana corner modes, here we perform the edge theory \[86\]. We take \(t_y = 0\) first which can provide a simple picture. The continuum Hamiltonian at the low-energy limit can be obtained by expanding the wavevector \(k\) to the second order around the gapless point of \(k = (0, 0)\),

\[
H(k) = (t_x k_x^2 - 2t_x - \mu)\tau_z + 2\Delta_x k_x \tau_y + (\Delta_y - \Delta_y^2 + \Delta_y^2 k_y^2)\tau_x \sigma_y + \Delta_y k_y \tau_z \sigma_x.
\]

We label the four edges of the square lattice as I, II, III and IV in Fig. 5. On the edge I, by expressing \(k_x\) as \(-i\partial_x\), we decompose Hamiltonian (B4) as \(H = H_0 + H_p\), where

\[
H_0(-i\partial_x, k_y) = (-t_x \partial_x^2 - 2t_x - \mu)\tau_z - 2i\Delta_x \partial_x \tau_y,
\]

\[
H_p(-i\partial_x, k_y) = \Delta_y k_y \tau_x \sigma_y + (\Delta_y - \Delta_y^2)\tau_z \sigma_y.
\]

Here the insignificant \(k_y^2\) term was neglected due to the low-energy limit. By solving the eigenvalue equation \(H^I_0\phi^I_\alpha(x) = E\phi^I_\alpha(x)\) with \(E = 0\) under the boundary condition \(\phi^I_\alpha(0) = \phi^I_\alpha(+\infty) = 0\), we obtain the solution in the following form:

\[
\phi^I_\alpha(x) = N_x \sin(\lambda_2 x) \exp(-\lambda_2 x) \exp(ik_y y)\chi^I_\alpha.
\]

Here \(N_x = \sqrt{|(2t_x + \mu)/t_x - (\Delta_x^2/t_x^2)|}\), \(\lambda_1 = \Delta_x/2t_x\), \(\lambda_2 = \Delta_x/2t_x\), \(\chi^I_\alpha\) is the eigenvector of \(\tau_z\) that satisfies \(\tau_z\chi^I_\alpha = \chi^I_\alpha\).

### Appendix C: Edge-corner correspondence of second-order topological superfluids

The topological transitions process the different bulk-edge correspondences from the trivial to topological superfluids. In Fig. 6(a-e), we can see the gap of bulk bands closes and reopens across the first-order topological superfluids phase boundaries, associated with changed Chern numbers. By contrast in Fig. 6(c-e), the bulk bands keep open and \(C = 0\) when the system transits from the trivial phase to second-order topological superfluids phase. At this stage, the edge modes play the role of the bulk and its gap exhibits the closing and reopening, known as the edge-corner correspondence \[131\]. This is the manifest feature of the higher-order topological phase transition.
FIG. 6. Quasi-particle spectrum with cylindrical geometry for different $(\mu, \eta) = (-3t_c, 0.8)$ in (a), $(-3t_c, 0.68)$ in (b), $(-3t_c, 0.6)$ in (c), $(-2t_c, 0.6)$ in (d), and $(-t_c, 0.6)$ in (e). The blue lines mark the edge or corner states in each topological phase. (a), (c), (e) represent the first-order topological superfluids (FOTSF) phase, trivial superfluids (Trivial SF) phase and second-order topological superfluids (SOTSF) phase, respectively. We set the open boundary condition with $L = 200$ in $x$ for the left column and $y$ for the right column. All the figures are plotted under the self-consistent calculations of superfluids with $U_{\text{eff}} = -8.7t_c$.

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