Friedberg-Lee symmetry and tri-bimaximal neutrino mixing
in the inverse seesaw mechanism

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Abstract

The inverse seesaw mechanism with three pairs of gauge-singlet neutrinos offers a natural interpretation of the tiny masses of three active neutrinos at the TeV scale. We combine this picture with the newly-proposed Friedberg-Lee (FL) symmetry in order to understand the observed pattern of neutrino mixing. We show that the FL symmetry requires only two pairs of the gauge-singlet neutrinos to be massive, implying that one active neutrino must be massless. We propose a phenomenological ansatz with broken FL symmetry and exact $\mu$-$\tau$ symmetry in the gauge-singlet neutrino sector and obtain the tri-bimaximal neutrino mixing pattern by means of the inverse seesaw relation. We demonstrate that non-unitary corrections to this result are possible to reach the percent level and a soft breaking of $\mu$-$\tau$ symmetry can give rise to CP violation in such a TeV-scale seesaw scenario.

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I. INTRODUCTION

The fact that three active neutrinos possess non-degenerate but tiny masses is a striking signature of new physics beyond the standard model [1]. Perhaps the most popular approach towards understanding the small neutrino mass scale (< 1 eV) is the seesaw mechanism [2], which contains three right-handed neutrinos and retains $SU(2)_L \times U(1)_Y$ gauge symmetry. Although this mechanism can naturally work at a superhigh-energy scale ($\sim 10^{14}$ GeV) to generate tiny Majorana neutrino masses, it loses direct testability on the experimental side and causes a hierarchy problem on the theoretical side [3]. A straightforward way out is to lower the seesaw scale down to the TeV scale, an energy frontier to be soon explored by the Large Hadron Collider (LHC). But such a TeV-scale seesaw scenario inevitably suffers from a terrible fine-tuning of cancellations between the Yukawa coupling texture and the heavy Majorana mass matrix [4]. To resolve this unnaturalness problem built in the canonical (type-I) seesaw mechanism at low energies, some new interest has recently been paid to a relatively old idea — the inverse seesaw mechanism [5].

The inverse seesaw mechanism, which can be regarded as the simplest multiple seesaw picture [6], is an extension of the canonical seesaw mechanism by introducing three additional gauge-singlet neutrinos together with one gauge-singlet scalar. Its typical result for the effective mass matrix of three active neutrinos is $M_\nu = M_D (M_R^T)^{-1} M_\mu (M_R)^{-1} M_D^T$ in the leading-order approximation, where the scales of three mass matrices may naturally satisfy $M_R \gg M_D \gg M_\mu$. The smallness of $M_\nu$ can be attributed to both the smallness of $M_\mu$ and the smallness of $M_D/M_R$ at the TeV scale (i.e., $M_R \sim \mathcal{O}(1)$ TeV). It is therefore possible to get a balance between theoretical naturalness and experimental testability of the inverse seesaw scheme. Nevertheless, the inverse seesaw mechanism itself is impossible to interpret the observed pattern of neutrino mixing, which is composed of two large angles ($\theta_{12} \approx 34^\circ$ and $\theta_{23} \approx 45^\circ$) and one small angle ($\theta_{13} < 10^\circ$) [7], because the flavor structures of $M_R$, $M_D$ and $M_\mu$ are entirely unspecified. The latter can be determined by imposing certain flavor symmetries, but such flavor symmetries usually need to be broken in order to give rise to the correct neutrino mass spectrum and neutrino mixing pattern. For example, a proper combination of the $A_4$ flavor symmetry and the inverse seesaw mechanism at the TeV scale [8] may successfully predict the tri-bimaximal neutrino mixing pattern [9] (with $\theta_{12} = \arctan(1/\sqrt{2}) \approx 35.3^\circ$, $\theta_{13} = 0^\circ$ and $\theta_{23} = 45^\circ$), which is definitely consistent with current experimental data on solar, atmospheric, reactor and accelerator neutrino oscillations.

The present work aims to combine the inverse seesaw mechanism with a newly-proposed flavor symmetry — the Friedberg-Lee (FL) symmetry [10], so as to fix the flavor textures of $M_R$, $M_D$ and $M_\mu$ and thus predict the mass spectrum and mixing pattern of three active neutrinos at the TeV scale. As the FL symmetry requires a fermion mass term to be invariant under a space-time-independent translation of the relevant fermion fields, it can more reasonably be applied to the gauge-singlet neutrino sector instead of the active neutrino sector. We show that the FL symmetry forces one pair of the gauge-singlet neutrinos to be massless, leading to a simplified but viable version of the inverse seesaw mechanism which has two pairs of massive gauge-singlet neutrinos and allows one active neutrino to be massless. With the help of $\mu$-$\tau$ permutation symmetry, we consider a very simple FL symmetry breaking ansatz in the gauge-singlet neutrino sector and obtain the tri-bimaximal neutrino mixing pattern by means of the inverse seesaw relation. We find that non-unitary
corrections to this special mixing pattern are possible to reach the percent level if \( M_D/M_R \sim \mathcal{O}(0.1) \) holds at the TeV scale. We also demonstrate that a soft breaking of \( \mu-\tau \) symmetry can easily accommodate CP violation in such an inverse seesaw scenario.

**II. THE INVERSE SEESAW MECHANISM WITH FL SYMMETRY**

Let us work in the basis where the flavor eigenstates of three charged leptons are identified with their mass eigenstates throughout this paper. Different from the canonical seesaw mechanism with three right-handed neutrinos \( N_R^i \) (for \( i = 1, 2, 3 \)), the inverse seesaw scheme contains three additional gauge-singlet neutrinos \( S_R^i \) (for \( i = 1, 2, 3 \)) together with one gauge-singlet scalar \( \Phi \). Allowing for lepton number violation to a certain extent, one can write out the following gauge-invariant neutrino mass terms in the inverse seesaw mechanism:

\[
- \mathcal{L}_\nu = \bar{\nu}_L Y_\nu \tilde{H} N_R + \bar{N}_R^c Y'_\nu S_R \Phi + \frac{1}{2} \bar{S}_R^c M_\mu S_R + \text{h.c. ,} \tag{1}
\]

where \( \ell_L \) and \( \tilde{H} \equiv i\sigma_2 H^* \) stand respectively for the \( SU(2)_L \) lepton and Higgs doublets, \( Y_\nu \) and \( Y'_\nu \) are the \( 3 \times 3 \) Yukawa coupling matrices, and \( M_\mu \) is a symmetric Majorana mass matrix. After spontaneous gauge symmetry breaking, Eq. (1) becomes

\[
- \mathcal{L}_\nu' = \bar{\nu}_L M_D N_R + \bar{N}_R^c M_R S_R + \frac{1}{2} \bar{S}_R^c M_\mu S_R + \text{h.c. ,} \tag{2}
\]

where \( M_D \equiv Y_\nu \langle H \rangle \) and \( M_R = Y'_\nu \langle \Phi \rangle \) are the \( 3 \times 3 \) mass matrices. Then we arrive at the overall \( 9 \times 9 \) neutrino mass matrix \( \mathcal{M} \) in the flavor basis defined by \( (\nu_L, N_R^c, S_R^c) \) and their charge-conjugate states:

\[
\mathcal{M} = \begin{pmatrix}
0 & M_D & 0 \\
M_D^T & 0 & M_R \\
0 & M_R^T & M_\mu
\end{pmatrix} . \tag{3}
\]

Note that the interesting nearest-neighbor-interaction pattern of \( \mathcal{M} \) is guaranteed by an implementation of the global \( U(1) \times Z_4 \) symmetry with a proper charge assignment [6]. Note also that the mass scales of three sub-matrices in \( \mathcal{M} \) may naturally have a hierarchy \( M_R \gg M_D \gg M_\mu \), because the second mass term in \( \mathcal{L}_\nu' \) is not subject to the \( SU(2)_L \) gauge symmetry breaking scale and the third mass term in \( \mathcal{L}_\nu' \) violates the lepton number conservation [11]. In the leading-order approximation, one obtains the inverse seesaw relation for the effective mass matrix of three active neutrinos:

\[
M_\nu = M_D (M_R^T)^{-1} M_\mu (M_R)^{-1} M_D^T . \tag{4}
\]

It becomes obvious that \( M_\nu \to 0 \) holds in the limit \( M_\mu \to 0 \). For instance, \( M_\nu \sim \mathcal{O}(0.1) \) eV can easily be achieved from \( M_D/M_R \sim \mathcal{O}(10^{-2}) \) and \( M_\mu \sim \mathcal{O}(1) \) keV.

Without loss of generality, let us work in a basis where \( Y'_\nu \) (or equivalently, \( M_R \)) is diagonal; i.e., \( Y'_\nu = \text{Diag}\{y'_1, y'_2, y'_3\} \) with \( y'_i \equiv M_i/\langle \Phi \rangle \) and \( M_i \) (for \( i = 1, 2, 3 \)) being real and positive. Now we impose the FL translation [10] on both \( N_R^c \) and \( S_R^c \) fields:

\[
N_R^i = N_R^i + \xi_i \Theta , \quad S_R^i = S_R^i + \xi_i \Theta , \tag{5}
\]
where $\xi_i$ (for $i = 1, 2, 3$) are in general complex numbers, and $\Theta$ is a space-time-independent Grassmann parameter (i.e., $\Theta$ is anti-commuting and thus $\Theta^2 = 0$ holds). Note that the gauge-singlet neutrinos $N_R^i$ and $S_R^i$ do not interact with the gauge bosons of the standard model. Note also that the kinetic terms of $N_R^i$ and $S_R^i$ change under the above translation, but the resulting action is invariant just because $\Theta$ is independent of space and time [12]. Hence the whole Lagrangian of an inverse seesaw model will be invariant under the FL translation, if and only if we require three neutrino mass terms in $L_\nu$ to be invariant under the FL translation. This requirement simply implies

$$(Y_\nu)_{ij}\xi_j = 0; \quad \xi_i(Y_\nu')_{ij} = 0; \quad (Y_\nu')_{ij}\xi_j = 0; \quad (M_\mu)_{ij}\xi_j = 0$$

(6)

for $i, j = 1, 2, 3$. In other words, each mass matrix must have a zero eigenvalue. Because $Y_\nu'$ is diagonal, we may simply choose

$$Y_\nu' = \begin{pmatrix} y_1' & 0 & 0 \\ 0 & y_2' & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(7)

such that $\xi_1 = \xi_2 = 0$ and $\xi_3 \neq 0$ must hold to satisfy the second and third conditions given in Eq. (6). This in turn implies that the elements in the third column of $Y_\nu$ and those in the third row and the third column of $M_\mu$ must be vanishing, so as to satisfy the first, fourth and fifth conditions shown in Eq. (6). The textures of $Y_\nu$ and $M_\mu$ are therefore given as

$$Y_\nu = \begin{pmatrix} y_{11} & y_{12} & 0 \\ y_{21} & y_{22} & 0 \\ y_{31} & y_{32} & 0 \end{pmatrix},$$

$$M_\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

(8)

In this case, the symmetric $9 \times 9$ neutrino mass matrix $M$ in Eq. (3) can be simplified to an effective $7 \times 7$ neutrino mass matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times & 0 & 0 \\ \times & \times & \times & 0 & 0 & \times & \times \\ \times & \times & \times & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times & 0 & \times \\ 0 & 0 & 0 & \times & \times & 0 & \times \end{pmatrix},$$

(9)

where “$\times$” symbolically denotes an arbitrary (non-zero) matrix element. Because $\text{Det}M = 0$ holds, one active neutrino must be massless in this minimal inverse seesaw scenario. The phenomenology of such a simple scenario has partly been explored in Ref. [13]. Here we show that the FL symmetry applied to the gauge-singlet neutrino sector naturally explains why a pair of gauge-singlet neutrinos can be decoupled from the usual inverse seesaw mechanism.

It is worth remarking that the above conclusion does not depend on the chosen basis of $Y_\nu'$ (i.e., $Y_\nu'$ is diagonal), since a unitary transformation of either $N_R$ or $S_R$ does not change any
physical content of the inverse seesaw mechanism. To derive a phenomenologically-favored
neutrino mixing pattern from this interesting scheme, however, it is more convenient to
introduce a less stringent FL symmetry into the gauge-singlet neutrino sector. Here we
consider the case of $\xi_1 = \xi_2 = \xi_3 \equiv \xi$, which is equivalent to the original FL translation
(imposed on $\nu_L$ [10,14–16]). Then three mass terms in $L$ can keep invariant under the
translations $N_R^i \to N_R^i + \xi \Theta$ and $S_R^i \to S_R^i + \xi \Theta$ if they take the following forms:
$$\sum_{\alpha,i} N_{\alpha L}(M_D)_{\alpha i} N_R^i = \sum_{\alpha} A_{\alpha} N_{\alpha L}(N_R^3 - N_R^2) + \sum_{\alpha} B_{\alpha} N_{\alpha L}(N_R^2 - N_R^1) + \sum_{\alpha} C_{\alpha} N_{\alpha L}(N_R^1 - N_R^2),$$
$$\sum_{i,j} N_{R}^{ic}(M_R)_{ij} S_R^j = A(N_{R}^{3c} - N_{R}^{2c})(S_R^3 - S_R^2) + B(N_{R}^{2c} - N_{R}^{1c})(S_R^2 - S_R^1) + C(N_{R}^{1c} - N_{R}^{2c})(S_R^1 - S_R^2),$$
$$\sum_{i,j} S_{R}^{ic}(M_\mu)_{ij} S_R^j = a(S_{R}^{3c} - S_{R}^{2c})(S_R^3 - S_R^2) + b(S_{R}^{2c} - S_{R}^{1c})(S_R^2 - S_R^1) + c(S_{R}^{1c} - S_{R}^{2c})(S_R^1 - S_R^2),$$
where the Greek index runs over $(e, \mu, \tau)$ and the Latin index runs over $(1, 2, 3)$. The explicit
expressions of $M_D$, $M_R$, and $M_\mu$ turn out to be
$$M_D = \begin{pmatrix}
C_e - B_e & B_e - A_e & A_e - C_e \\
C_\mu - B_\mu & B_\mu - A_\mu & A_\mu - C_\mu \\
C_\tau - B_\tau & B_\tau - A_\tau & A_\tau - C_\tau
\end{pmatrix},$$
$$M_R = \begin{pmatrix}
B + C & -B & -C \\
-B & A + B & -A \\
-C & -A & A + C
\end{pmatrix},$$
$$M_\mu = \begin{pmatrix}
b + c & -b & -c \\
-b & a + b & -a \\
-c & -a & a + c
\end{pmatrix},$$
in which all the matrix elements are in general complex. It is easy to check that $\text{Det} M_D = \text{Det} M_R = \text{Det} M_\mu = 0$ holds, and thus each mass matrix has one zero eigenvalue. We see
that the above FL symmetry must be broken, at least for the second mass term of $L'$, such
that $\text{Det} M_R \neq 0$ holds to make the inverse seesaw formula in Eq. (4) applicable.

III. A FL SYMMETRY BREAKING ANSatz WITH $\mu$-$\tau$ SYMMETRY

There are certainly a variety of possibilities of breaking the FL symmetry. Here we follow
the spirit of Ref. [10] and consider a very simple symmetry breaking ansatz for the second
and third mass terms in Eq. (10):
$$\sum_{i,j} N_{R}^{ic}(M_R)_{ij} S_R^j \to \sum_{i,j} N_{R}^{ic}(M_R)_{ij} S_R^j + M_0 \sum_i N_{R}^{ic} S_R^i,$$
$$\sum_{i,j} S_{R}^{ic}(M_\mu)_{ij} S_R^j \to \sum_{i,j} S_{R}^{ic}(M_\mu)_{ij} S_R^j + \mu_0 \sum_i S_{R}^{ic} S_R^i,$$
(12)
where $M_0$ and $\mu_0$ are real and positive. To minimize the number of free parameters, we impose the $\mu$-$\tau$ permutation symmetry on these two mass terms (i.e., $B = C$ and $b = c$) and assume all of their parameters to be real. The resultant mass matrices read

$$M_R = M_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2B' & -B' & -B' \\ -B' & A' + B' & -A' \\ -B' & -A' & A' + B' \end{pmatrix},$$

$$M_\mu = \mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2b' & -b' & -b' \\ -b' & a' + b' & -a' \\ -b' & -a' & a' + b' \end{pmatrix},$$

in which $A' \equiv A/M_0$ (or $a' \equiv a/\mu_0$) and $B' \equiv B/M_0$ (or $b' \equiv b/\mu_0$) are defined to be two dimensionless parameters. The diagonalization of $M_R$ or $M_\mu$ is rather straightforward: $V_0^T M_R V_0 = \text{Diag}\{M_1, M_2, M_3\}$ and $V_0^T M_\mu V_0 = \text{Diag}\{\mu_1, \mu_2, \mu_3\}$, where $V_0$ is simply the tri-bimaximal mixing pattern [9]

$$V_0 = \begin{pmatrix} 2/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}. \tag{14}$$

Then we obtain three mass eigenvalues $M_1 = M_0(1 + 3B')$, $M_2 = M_0$, $M_3 = M_0(1 + 2A' + B')$; and similarly $\mu_1 = \mu_0(1 + 3b')$, $\mu_2 = \mu_0$, $\mu_3 = \mu_0(1 + 2a' + b')$. In terms of mass eigenvalues, $M_R$ and $M_\mu$ can be reexpressed as

$$M_R = M_1 X + M_2 Y + M_3 Z, \quad M_\mu = \mu_1 X + \mu_2 Y + \mu_3 Z, \tag{15}$$

where

$$X = \frac{1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix},$$

$$Y = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$Z = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{16}$$

Note that $X^2 = X$, $Y^2 = Y$, $Z^2 = Z$ and $XY = YX = XZ = ZX = YZ = ZY = 0$ hold. Note also that the inverse matrix of $M_R$ takes the same form as $M_R$ itself:

$$(M_R)^{-1} = M_1^{-1}X + M_2^{-1}Y + M_3^{-1}Z. \tag{17}$$

Now let us make a purely phenomenological assumption: $M_D = v1$ with $v \sim \langle H \rangle \approx 174$ GeV being the electroweak scale and $1$ being the identity matrix, just for the sake of simplicity. Then the inverse seesaw formula in Eq. (4) allows us to arrive at...
\[ M_\nu = \frac{v^2 \mu_1}{M_1^2} X + \frac{v^2 \mu_2}{M_2^2} Y + \frac{v^2 \mu_3}{M_3^2} Z. \] (18)

It is straightforward to show that this effective mass matrix can also be diagonalized by the unitary transformation \( V^T \bar{M}_\nu V_0 = \text{Diag}\{m_1, m_2, m_3\} \), where \( V_0 \) has been given in Eq. (14) and three neutrino masses \( m_i = v^2 \mu_i / M_i^2 \) (for \( i = 1, 2, 3 \)) directly reflect the salient feature of the inverse seesaw mechanism. In other words, the smallness of \( m_i \) is ascribed to both the smallness of \( \mu_i \) and that of \( v^2 / M_i^2 \). Current neutrino oscillation data only provide us with \( \Delta m_{21}^2 \equiv m_2^2 - m_1^2 \approx 7.7 \times 10^{-5} \text{ eV}^2 \) and \( \Delta m_{32}^2 \equiv m_3^2 - m_2^2 \approx \pm 2.4 \times 10^{-3} \text{ eV}^2 \) [7], and thus these two neutrino mass-squared differences can easily be reproduced from our result for \( m_i \) by adjusting six free parameters \( \mu_i \) and \( M_i \) (for \( i = 1, 2, 3 \)).

Note that the neutrino mixing matrix \( V \) appearing in the charged-current interactions of three active neutrinos is not exactly \( V_0 \) used to diagonalize \( M_\nu \) in Eq. (18), just because of slight mixing between light and heavy neutrinos in the inverse seesaw mechanism. As shown in Appendix A, the charged-current interactions of three light Majorana neutrinos read

\[ -\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (e \mu \tau)_{L} \gamma^\mu V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L \begin{pmatrix} \nu^\prime_1 \\ \nu^\prime_2 \\ \nu^\prime_3 \end{pmatrix}_L + \text{h.c.}, \] (19)

in which \((e, \mu, \tau)\) and \((\nu_1, \nu_2, \nu_3)\) are the mass eigenstates of charged leptons and active neutrinos, respectively. The relationship between \( V \) and \( V_0 \) is given by \( V = (1 - \eta) V_0 \), where \( \eta \lesssim O(10^{-2}) \) signifies the slight deviation of \( V \) from \( V_0 \) and its approximate expression can be found in Eq. (A7). For the ansatz under consideration, we explicitly obtain

\[ \eta \approx \frac{1}{2} M_D (M_R)^{-2} M_D = \frac{1}{2} \left[ \frac{v^2}{M_1^2} X + \frac{v^2}{M_2^2} Y + \frac{v^2}{M_3^2} Z \right]. \] (20)

Current experimental constraints on the matrix elements of \( \eta \) are

\[ |\eta| < \begin{pmatrix} 5.5 \times 10^{-3} & 3.5 \times 10^{-5} & 8.0 \times 10^{-3} \\ 3.5 \times 10^{-5} & 5.0 \times 10^{-3} & 5.0 \times 10^{-3} \\ 8.0 \times 10^{-3} & 5.0 \times 10^{-3} & 5.0 \times 10^{-3} \end{pmatrix}, \] (21)

at the 90% confidence level [17]. Combining Eqs. (16), (20) and (21), we arrive at

\[ |\eta_{ee}| \approx \frac{1}{6} \left| \frac{2v^2}{M_1^2} + \frac{v^2}{M_2^2} \right| < 5.5 \times 10^{-3}, \]

\[ |\eta_{e\mu}| \approx \frac{1}{6} \left| \frac{v^2}{M_1^2} - \frac{v^2}{M_2^2} \right| < 3.5 \times 10^{-5}, \]

\[ |\eta_{\mu\mu}| \approx \frac{1}{12} \left| \frac{v^2}{M_1^2} + \frac{2v^2}{M_2^2} + \frac{3v^2}{M_3^2} \right| < 5.0 \times 10^{-3}, \]

\[ |\eta_{\mu\tau}| \approx \frac{1}{12} \left| \frac{v^2}{M_1^2} + \frac{2v^2}{M_2^2} - \frac{3v^2}{M_3^2} \right| < 5.0 \times 10^{-3}, \] (22)

together with \( |\eta_{e\tau}| = |\eta_{e\mu}| \) and \( |\eta_{\mu\tau}| = |\eta_{\mu\mu}| \) due to the \( \mu-\tau \) symmetry of Hermitian \( \eta \). If \( M_i \sim O(1) \text{ TeV} \), then \( M_1 \approx M_2 \) is expected from the stringent constraint on \( |\eta_{e\mu}| \). This
result accordingly implies $|\eta_{ee}| \approx v^2/M_1^2 < 1.1 \times 10^{-2}$, from which $M_1 > 1.6$ TeV can be extracted. It is in general difficult to get a limit on $M_3$. But if $M_1 \approx M_2 \approx 1.8$ TeV is taken, for example, then $|\eta_{\mu\mu}| < 5.0 \times 10^{-3}$ leads us to $M_3 > 1.7$ TeV.

With the help of Eqs. (14) and (20), it is straightforward to obtain an interesting expression of $V = (1 - \eta)V_0$ in this inverse seesaw scenario:

$$V = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 - \frac{v^2}{2M_1^2} & 0 & 0 \\ 0 & 1 - \frac{v^2}{2M_2^2} & 0 \\ 0 & 0 & 1 - \frac{v^2}{2M_3^2} \end{pmatrix}.$$  \hspace{1cm} (23)

This simple but instructive result clearly shows the deviation of $V$ from $V_0$. In particular, $V_{e3} = (V_0)_{e3} = 0$ holds; i.e., the non-unitary effect does not contribute to this smallest neutrino mixing matrix element.

We remark that the above FL symmetry breaking ansatz is viable and suggestive for building a realistic inverse seesaw model at the TeV scale. To generate non-vanishing $V_{e3}$ and CP violation, however, one has to invoke a different FL symmetry breaking pattern with one or more non-trivial $CP$-violating phases.

**IV. SOFT $\mu$-$\tau$ SYMMETRY BREAKING AND CP VIOLATION**

We proceed with the FL symmetry breaking ansatz in Eq. (12) but allow soft $\mu$-$\tau$ symmetry breaking for $M_R$ and $M_\mu$ (i.e., $C = B^*$ and $c = b^*$ with $B$ and $b$ being complex):

$$M_R = M_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2ReB' & -B' & -B'^* \\ -B' & A' + B' & -A' \\ -B'^* & -A' & A' + B'^* \end{pmatrix},$$

$$M_\mu = \mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2Reb' & -b' & -b'^* \\ -b' & a' + b' & -a' \\ -b'^* & -a' & a' + b'^* \end{pmatrix},$$  \hspace{1cm} (24)

where $M_0$ (or $\mu_0$) and $A'$ (or $a'$) are real. As shown in Appendix B, the inverse matrix of $M_R$ takes the same texture as $M_R$ itself:

$$(M_R)^{-1} = \frac{1}{M_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2ReB'' & -B'' & -B''^* \\ -B'' & A'' + B'' & -A'' \\ -B''^* & -A'' & A'' + B''^* \end{pmatrix},$$  \hspace{1cm} (25)

where

$$A'' = -\frac{A' + (2A'\text{Re}B' + |B'|^2)}{1 + 2(A' + 2\text{Re}B') + 3(2A'\text{Re}B' + |B'|^2)},$$

$$B'' = -\frac{B' + (2A'\text{Re}B' + |B'|^2)}{1 + 2(A' + 2\text{Re}B') + 3(2A'\text{Re}B' + |B'|^2)},$$  \hspace{1cm} (26)
which can easily be read off from Eq. (B3) in Appendix B. Furthermore, we show that
\((M_R^T)^{-1}M_{\mu}(M_R)^{-1}\) is also of the same texture:

\[
(M_R^T)^{-1}M_{\mu}(M_R)^{-1} = \frac{\mu_0}{M_0^2} \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2\text{Re}\hat{B} & -\hat{B} & -\hat{B}^* \\ -\hat{B} & \hat{A} + \hat{B} & -\hat{A} \\ -\hat{B}^* & -\hat{A} & \hat{A} + \hat{B}^* \end{pmatrix} \right],
\]

(27)

where

\[
\hat{A} = +A'' [(1 + A'' + B'') (1 + a' + b') + A'' a' + B'' b']
- B''\* [(1 + A'' + B'') b' + B'' (1 + 2\text{Re}b') - A'' b']
+ (1 + A'' + B'') [(1 + A'' + B'') a' + A'' (1 + a' + b'') - B'' b'] ,
\]

\[
\hat{B} = -A'' [(1 + 2\text{Re}B'') b'' + B''\* (1 + a' + b'') - B'' a']
+ B'' [(1 + 2\text{Re}B'') (1 + 2\text{Re}b') + B' b' + B''\* b'']
+ (1 + A'' + B'') [(1 + 2\text{Re}B'') b' + B'' (1 + a' + b') - B'' a'] ,
\]

(28)

which can directly be read off from Eq. (B6). Note that \(\hat{A}\) is real and \(\hat{B}\) is complex. Making
the same phenomenological assumption for \(M_D\) as in section III (i.e., \(M_D = v1\)), we simply
obtain the effective mass matrix of three active neutrinos from Eqs. (4) and (27) in this
inverse seesaw scenario:

\[
M_\nu = m_0 \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2\text{Re}\tilde{B} & -\tilde{B} & -\tilde{B}^* \\ -\tilde{B} & \tilde{A} + \tilde{B} & -\tilde{A} \\ -\tilde{B}^* & -\tilde{A} & \tilde{A} + \tilde{B}^* \end{pmatrix} \right]
\]

(29)

with \(m_0 = v^2\mu_0/M_0^2\). The small mass eigenvalues of \(M_\nu\) are therefore attributed to small \(\mu_0\)
as well as small \(v^2/M_0^2\) at the scale of \(M_0 \sim \mathcal{O}(1)\) TeV.

The diagonalization of \(M_\nu\) in Eq. (29) can be done by using the unitary transformation
\(U_0^\dagger M_\nu U_0^* = \text{Diag}\{m_1, m_2, m_3\}\), where

\[
U_0 = \begin{pmatrix} 2 & 1 & 0 \\ \sqrt{6} & \sqrt{3} & 1 \\ 1 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -i \sin \theta \\ 0 & 1 & 0 \\ -i \sin \theta & 0 & \cos \theta \end{pmatrix}
\]

(30)

and \(\tan 2\theta = \sqrt{3} \text{Im}\tilde{B}/(1 + \tilde{A} + 2\text{Re}\tilde{B})\) arises from the soft \(\mu-\tau\) symmetry breaking term of \(M_\nu\) (i.e., \(\text{Im}\tilde{B} \neq 0\)). Comparing this result with the standard parametrization of \(U_0\) in
terms of three mixing angles \(\theta_{12}, \theta_{13}, \theta_{23}\) and three CP-violating phases \(\delta, \rho, \sigma\) [1,18], we
immediately find

\[
\theta_{12} = \arcsin \left( \frac{1}{\sqrt{2} + \cos 2\theta} \right),
\]

\[
\theta_{13} = \arcsin \left( \frac{2}{\sqrt{6}} \sin \theta \right),
\]

(31)
together with $\theta_{23} = 45^\circ$, $\delta = 90^\circ$ and $\rho = \sigma = 0^\circ$. On the other hand, three mass eigenvalues of $M_\nu$ are given by

\begin{align*}
m_1 &= m_0 \left[ \sqrt{(1 + \hat{A} + 2 \text{Re} \hat{B})^2 + 3 (\text{Im} \hat{B})^2} - \hat{A} + \text{Re} \hat{B} \right], \\
m_2 &= m_0, \\
m_3 &= m_0 \left[ \sqrt{(1 + \hat{A} + 2 \text{Re} \hat{B})^2 + 3 (\text{Im} \hat{B})^2} + \hat{A} - \text{Re} \hat{B} \right].
\end{align*}

(32)

By adjusting four real parameters in Eq. (32), we can easily fit two observed neutrino mass-squared differences $\Delta m^2_{21}$ and $\Delta m^2_{32}$. For example, $m_0 \approx 8.8 \times 10^{-3}$ eV, $\hat{A} \approx 2.3$, $\text{Re} \hat{B} \approx -0.3$ and $\text{Im} \hat{B} \approx 0.3$ lead to a hierarchical neutrino mass spectrum: $m_1 \approx 0.15 m_0 \approx 1.3 \times 10^{-3}$ eV, $m_2 = m_0 \approx 8.8 \times 10^{-3}$ eV and $m_3 \approx 5.4 m_0 \approx 4.8 \times 10^{-2}$ eV, consistent with current neutrino oscillation data [7]. In this illustrative case, we can also obtain $\theta \approx 6^\circ$, which in turn predicts $\theta_{12} \approx 39^\circ$ and $\theta_{13} \approx 5^\circ$.

It is worth reiterating that the imaginary parts of $B'$, $b'$ and $\hat{B}$ represent the $\mu$-$\tau$ symmetry breaking effects of $M_R$, $M_\mu$ and $M_\nu$, respectively, in this phenomenological ansatz. They have nothing to do with the $\mu$-$\tau$ symmetry breaking effect in the charged-lepton sector, because the latter is characterized by the mass ratio $m_\tau/m_\mu \approx 17$ in the chosen basis where the charge-lepton mass matrix is diagonal and real.

Now we calculate the non-unitary corrections to $U_0$ in order to figure out the neutrino mixing matrix $V = (1 - \eta) U_0$, which is certainly more non-trivial in this CP-violating inverse seesaw scenario. By using the formula of $\eta$ given in Eq. (A7) and the expression of $(M_R)^{-1}$ shown in Eq. (25), we obtain

\begin{align*}
\eta_{ee} &\approx \frac{v^2}{2 M^2} \left[ (1 + 2 \text{Re} B'')^2 + 2 |B''|^2 \right], \\
\eta_{e\mu} &\approx \frac{v^2}{2 M^2} \left[ A'' B''* - (1 + 2 \text{Re} B'') B''* - (1 + A'' + B''*) B'' \right], \\
\eta_{\mu\mu} &\approx \frac{v^2}{2 M^2} \left[ A''^2 + |B''|^2 + |1 + A'' + B''|^2 \right], \\
\eta_{\mu\tau} &\approx \frac{v^2}{2 M^2} \left[ B'^2 - 2 A'' (1 + A'' + B'') \right],
\end{align*}

(33)

together with $\eta_{e\tau} = \eta_{e\mu}^*$ and $\eta_{\tau\tau} = \eta_{\mu\mu}$ for Hermitian $\eta$. Here we have more free parameters to adjust, such that the values of $|\eta_{\alpha\beta}|$ (for $\alpha, \beta = e, \mu, \tau$) can fit their experimental upper bounds given in Eq. (21). Taking account of $|\eta_{e\mu}| < 3.5 \times 10^{-5}$, we may simply choose $\eta_{e\mu} \approx 0$ in the calculation of $V$. In addition, the smallness of both $\theta$ in $U_0$ and that of $|\eta_{ee}|$, $|\eta_{\mu\mu}|$ and $|\eta_{\mu\tau}|$ allow us to obtain an approximate expression of $V$ as follows:

\begin{align*}
V \approx \begin{pmatrix}
\frac{2}{\sqrt{6}} (1 - \eta_{ee}) & \frac{1}{\sqrt{3}} (1 - \eta_{ee}) & -i \frac{2\theta}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}} (1 - \eta_{+}) - i \frac{\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} (1 - \eta_{+}) & \frac{1}{\sqrt{2}} (1 - \eta_{-}) + i \frac{\theta}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}} (1 - \eta_{+}^*) + i \frac{\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} (1 - \eta_{+}^*) & -\frac{1}{\sqrt{2}} (1 - \eta_{-}^*) + i \frac{\theta}{\sqrt{6}}
\end{pmatrix},
\end{align*}

(34)
where $\eta_\pm \equiv \eta_{\mu\mu} \pm \eta_{\mu\tau}$ is defined, $\cos \theta \approx 1$ and $\sin \theta \approx \theta$ are taken, and the terms of $O(\theta \eta_{\alpha\beta})$ are omitted (for $\alpha\beta = ee, \mu\mu, \mu\tau$). It becomes quite obvious that the non-unitary corrections to $U_0$ contain a new CP-violating phase $\text{arg}(\eta_{\mu\tau})$, which is possible to give rise to an observable effect of CP violation in $\nu_\mu \to \nu_\tau$ and $\bar{\nu}_\mu \to \bar{\nu}_\tau$ oscillations.

To see the above point more clearly, let us calculate the following Jarlskog invariants [19] of leptonic CP violation:

$$J_{\mu\tau}^{13} \equiv \text{Im} \left( V_{\mu 1} V_{\tau 3} V^*_{\mu 3} V^*_{\tau 1} \right) \approx \frac{1}{3} \left( \text{Im} \eta_{\mu\tau} - \frac{\theta}{\sqrt{3}} \right),$$

$$J_{\mu\tau}^{23} \equiv \text{Im} \left( V_{\mu 2} V_{\tau 3} V^*_{\mu 3} V^*_{\tau 2} \right) \approx \frac{1}{3} \left( 2\text{Im} \eta_{\mu\tau} + \frac{\theta}{\sqrt{3}} \right).$$

(35)

It is the sum $J_{\mu\tau}^{13} + J_{\mu\tau}^{23} \approx \text{Im} \eta_{\mu\tau}$ that appears in the probabilities of $\nu_\mu \to \nu_\tau$ and $\bar{\nu}_\mu \to \bar{\nu}_\tau$ oscillations [20], and thus we have

$$P(\nu_\mu \to \nu_\tau) \approx \sin^2 \frac{\Delta_{32}}{2} + 2\text{Im} \eta_{\mu\tau} \sin \Delta_{32},$$

$$P(\bar{\nu}_\mu \to \bar{\nu}_\tau) \approx \sin^2 \frac{\Delta_{32}}{2} - 2\text{Im} \eta_{\mu\tau} \sin \Delta_{32},$$

(36)

where $\Delta_{32} \equiv \Delta m_{32}^2 L / (2E)$ with $E$ being the neutrino beam energy and $L$ being the baseline length. In obtaining Eq. (36), we have neglected those small non-unitary but CP-conserving effects, including the “zero-distance” effect [17]. Considering $|\text{Im} \eta_{\mu\tau}| \leq |\eta_{\mu\tau}| < 5.0 \times 10^{-3}$ as given in Eq. (21), we find that it is possible to have non-unitary CP violation at the percent level in a medium-baseline experiment of $\nu_\mu \to \nu_\tau$ and $\bar{\nu}_\mu \to \bar{\nu}_\tau$ oscillations (see Ref. [21] for more detailed and model-independent discussions).

V. SUMMARY

The origin of small masses of three active neutrinos, together with their unexpectedly large mixing angles, is a big puzzle in particle physics. Although a lot of attempts have been made in the past decade to solve this flavor problem, new ideas are eagerly wanted in the upcoming LHC era to achieve a balance between theoretical naturalness and experimental testability of the mechanisms of neutrino mass generation and flavor mixing. In the present work we have combined the inverse seesaw mechanism with the FL symmetry to fix the flavor textures of neutrino mass matrices at the TeV scale, such that both the neutrino mass spectrum and the neutrino mixing pattern can be calculated. To be explicit, we have applied the FL symmetry to the gauge-singlet neutrino sector and shown that it forces one pair of the gauge-singlet neutrinos to be massless, leading to a simplified but viable version of the inverse seesaw mechanism which has two pairs of massive gauge-singlet neutrinos and allows one active neutrino to be massless. Taking account of the exact $\mu-\tau$ permutation symmetry, we have proposed a very simple FL symmetry breaking ansatz in the gauge-singlet neutrino sector and then obtained the tri-bimaximal neutrino mixing pattern by means of the inverse seesaw relation. We find that non-unitary corrections to this interesting neutrino mixing matrix are possible to reach the percent level at the TeV scale. We have also demonstrated
that a soft breaking of $\mu$-$\tau$ symmetry can easily accommodate leptonic CP violation in such an inverse seesaw scenario.

We remark that it is technically more natural to make the inverse seesaw mechanism work at the TeV scale, although it contains much more degrees of freedom than the canonical (type-I) seesaw mechanism. We also remark that the FL symmetry discussed in this work, like some other flavor symmetries discussed in the literature, may serve as a phenomenological organizing principle for studying the flavor textures of neutrino mass matrices. How to break this symmetry is certainly an open question and involves a lot of arbitrariness, but our ansatz shows that there exist one or more simple symmetry breaking patterns which allow us to predict (or at least to understand) small neutrino masses and large neutrino mixing angles. As for the TeV-scale inverse seesaw picture under consideration, it is even possible to have observable effects, induced by the non-unitarity of the effective mixing matrix of three active neutrinos, in a delicate medium- or long-baseline neutrino oscillation experiment.

It is finally worth emphasizing that testing the unitarity of the light Majorana neutrino mixing matrix in neutrino oscillations and searching for the signatures of heavy Majorana (or pseudo-Dirac) neutrinos at the LHC can be complementary to each other, both qualitatively and quantitatively, in order to deeply understand the intrinsic properties of Majorana particles [22]. In spite of many challenges on the road ahead, we optimistically expect that some experimental breakthrough in this aspect will pave the way towards the true theory of neutrino mass generation, flavor mixing and CP violation.

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APPENDIX A: WEAK CHARGED-CURRENT INTERACTIONS

The standard charged-current interactions between three active neutrinos and three charged leptons are given by

\[ -\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (e^{\mu} \tau)_L \gamma^\mu \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_L W^- + \text{h.c.} \]  \hspace{1cm} (A1)

in the basis of their flavor eigenstates. Without loss of generality, we choose the basis in which the flavor eigenstates of three charged leptons are identified with their mass eigenstates. We proceed to diagonalize the 9 × 9 neutrino mass matrix \( \mathcal{M} \) in Eq. (3) so as to reexpress \( \mathcal{L}_{cc} \) in terms of the mass eigenstates of both charged leptons and neutrinos. Because \( \mathcal{M} \) is symmetric, it can be diagonalized by the following unitary transformation:

\[
\left( \begin{array}{ccc} V_{3 \times 3} & R_{6 \times 3} \\ S_{3 \times 6} & U_{6 \times 6} \end{array} \right) \mathcal{M} \left( \begin{array}{ccc} V_{3 \times 3} & R_{6 \times 3} \\ S_{3 \times 6} & U_{6 \times 6} \end{array} \right)^\dagger = \left( \begin{array}{cc} (\tilde{M}_\nu)_{3 \times 3} & 0 \\ 0 & (\tilde{M}_N)_{6 \times 6} \end{array} \right),
\]  \hspace{1cm} (A2)

in which \( \tilde{M}_\nu \) and \( \tilde{M}_N \) denote the diagonal mass matrices of light and heavy neutrinos, respectively, in the inverse seesaw mechanism. After this diagonalization, the neutrino flavor eigenstates \( \nu_\alpha \) (for \( \alpha = e, \mu, \tau \)) can be expressed in terms of the neutrino mass eigenstates \( \nu_i, N_i \) and \( S_i \) (for \( i = 1, 2, 3 \)) as

\[ \nu_\alpha L = \nu_i L + R \left( \begin{array}{c} N_i \\ S_i \end{array} \right) \]  \hspace{1cm} (A3)

Substituting this equation into Eq. (A1), we immediately arrive at

\[ -\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (e^{\mu} \tau)_L \gamma^\mu \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L + R \left( \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \right)_L W^- + \text{h.c.} \]  \hspace{1cm} (A4)

in the basis of mass eigenstates. As \( V \) and \( R \) belong to the same unitary transformation done in Eq. (A2), they satisfy the normalization condition \( VV^\dagger + RR^\dagger = 1 \) and the exact seesaw relation \( V\tilde{M}_\nu V^T + R\tilde{M}_N R^T = 0 \). Hence \( V \) itself must be non-unitary, and its deviation from unitarity is just measured by \( R \), which actually determines the collider signatures of heavy Majorana neutrinos at the LHC [6]. A global analysis of current experimental data shows that the strength of unitarity violation of \( V \) can at most reach the percent level [17].

Let us denote the slight deviation of \( V \) from a unitary matrix \( V_0 \) as follows: \( V = (1-\eta)V_0 \) with \( \eta \approx RR^\dagger/2 \) being a Hermitian matrix. Then one may simply follow Ref. [23] to derive the approximate inverse seesaw formula given in Eq. (4). Considering \( M_R \gg M_D \gg M_\mu \) and neglecting small non-unitary effects (i.e., \( V \approx V_0 \)), we obtain

\[ M_\nu \equiv V_0 \tilde{M}_\nu V_0^T \approx -(M_D \ 0) \left( \begin{array}{c} 0 \\ M_R \end{array} \right) \left( \begin{array}{c} M_R \\ M_\mu \end{array} \right)^{-1} \left( \begin{array}{c} M_D^T \\ 0 \end{array} \right) = M_D (M_R^T)^{-1} M_\mu (M_R)^{-1} M_D^T. \]  \hspace{1cm} (A5)
In this excellent approximation, $V_0$ is actually defined to be the unitary transformation used to diagonalize the effective $3 \times 3$ mass matrix of three light Majorana neutrinos (i.e., $M_\nu$). Because of

$$R \approx \left( M_D \ 0 \right) \left( \begin{array}{c|c} 0 & M_R \\ \hline M_R^T & M_\mu \end{array} \right)^{-1} U ,$$

where $U$ is approximately unitary to the same degree of accuracy which assures Eq. (A5) to hold, we finally arrive at

$$\eta \approx \frac{1}{2} R R^\dagger \approx \frac{1}{2} M_D (M_R^* M_R^T)^{-1} M_D^\dagger \quad \text{(A7)}$$

by taking account of $M_R \gg M_D \gg M_\mu$. This result allows us to estimate the non-unitary effects in $V$, once the textures of $M_D$ and $M_R$ are specified in an inverse seesaw model.

Note again that we have taken the natural hierarchy $M_R \gg M_D \gg M_\mu$ in deriving $M_\nu$ in Eq. (A5) and calculating $\eta$ in Eq. (A7). In fact, it is mathematically unnecessary to assume $M_R \gg M_\mu$ because the validity of Eq. (A5) is independent of the relative magnitude of $M_R$ and $M_\mu$. In other words, Eq. (A5) is also valid if $M_\mu \gg M_D$ holds, no matter whether $M_R$ is much larger or smaller than $M_\mu$. But we reiterate that $M_R \gg M_D \gg M_\mu$ is a physical condition of the inverse seesaw mechanism, in which the smallness of $M_\nu$ is naturally attributed to both the smallness of $M_\mu$ and the smallness of $M_D/M_R$.

**APPENDIX B: ALGEBRAIC PROPERTIES OF THE FL TEXTURE**

We refer to the following form of a mass matrix as the Friedberg-Lee (FL) texture:

$$F = f \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] + \left( \begin{array}{ccc} y + z & -y & -z \\ -y & x + y & -x \\ -z & -x & x + z \end{array} \right) \quad \text{(B1)}$$

where $f$, $x$, $y$ and $z$ are in general complex parameters. Given $\text{Det} F \neq 0$, it is easy to show that the inverse matrix of $F$ takes the same FL texture:

$$F^{-1} = \frac{1}{f} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] + \left( \begin{array}{ccc} y' + z' & -y' & -z' \\ -y' & x' + y' & -x' \\ -z' & -x' & x' + z' \end{array} \right) \quad \text{(B2)}$$

where

$$x' = -\frac{x + (xy + yz + zx)}{1 + 2(x + y + z) + 3(xy + yz + zx)} ,$$

$$y' = -\frac{y + (xy + yz + zx)}{1 + 2(x + y + z) + 3(xy + yz + zx)} ,$$

$$z' = -\frac{z + (xy + yz + zx)}{1 + 2(x + y + z) + 3(xy + yz + zx)} . \quad \text{(B3)}$$

Now let us consider another mass matrix of the FL texture:
\[ D = d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \beta + \gamma & -\beta & -\gamma \\ -\beta & \alpha + \beta & -\alpha \\ -\gamma & -\alpha & \alpha + \gamma \end{pmatrix} , \]  

(B4)

where \( d, \alpha, \beta \) and \( \gamma \) are in general complex parameters. A lengthy but straightforward calculation shows that the seesaw-like mass matrix \( E \equiv DF^{-1}D^T \) takes the same texture as \( D \) and \( F \) do:

\[ E = \frac{d^2}{f} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \tilde{\beta} + \tilde{\gamma} & -\tilde{\beta} & -\tilde{\gamma} \\ -\tilde{\beta} & \tilde{\alpha} + \tilde{\alpha} & -\tilde{\alpha} \\ -\tilde{\gamma} & -\tilde{\alpha} & \tilde{\alpha} + \tilde{\gamma} \end{pmatrix} , \]  

(B5)

where

\[ \tilde{\alpha} = +\alpha [(1 + \alpha + \beta) (1 + x' + y') + \alpha x' + \beta y'] - \gamma [(1 + \alpha + \beta) y' + \beta (1 + y' + z') - \alpha z'] \\
+ (1 + \alpha + \gamma) [(1 + \alpha + \beta) - \gamma [(1 + \beta + \gamma) z' + \gamma (1 + x' + z') - \beta x'] + \beta [(1 + \beta + \gamma) (1 + y' + z') + \beta y' + \gamma z'] \\
+ (1 + \alpha + \beta) [(1 + \beta + \gamma) y' + \beta (1 + x' + y') - \gamma x'] , \]

\[ \tilde{\beta} = -\alpha [(1 + \beta + \gamma) z' + \gamma (1 + x' + z') - \beta x'] + \beta [(1 + \beta + \gamma) (1 + y' + z') + \beta y' + \gamma z'] \\
+ (1 + \alpha + \beta) [(1 + \beta + \gamma) y' + \beta (1 + x' + y') - \gamma x'] , \]

\[ \tilde{\gamma} = -\alpha [(1 + \beta + \gamma) y' + \beta (1 + x' + y') - \gamma x'] + \gamma [(1 + \beta + \gamma) (1 + y' + z') + \beta y' + \gamma z'] \\
+ (1 + \alpha + \gamma) [(1 + \beta + \gamma) z' + \gamma (1 + x' + z') - \beta x'] . \]  

(B6)

This interesting seesaw-invariant property is a salient feature of the FL texture.
REFERENCES

[1] Particle Data Group, C. Amsler et al., Phys. Lett. B 667, 1 (2008).
[2] H. Fritzsch, M. Gell-Mann, and P. Minkowski, Phys. Lett. B 59, 256 (1975); T.P. Cheng, Phys. Rev. D 14, 1367 (1976); P. Minkowski, Phys. Lett. B 67, 421 (1977); T. Yanagida, in Proceedings of the Workshop on Unified Theory and the Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, edited by P. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979), p. 315; S.L. Glashow, in Quarks and Leptons, edited by M. Lévy et al. (Plenum, New York, 1980), p. 707; R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).
[3] For a recent review, see: Z.Z. Xing, arXiv:0905.3903; and references therein.
[4] J. Kersten and A.Yu. Smirnov, Phys. Rev. D 76, 073005 (2007).
[5] D. Wyler and L. Wolfenstein, Nucl. Phys. B 218, 205 (1983); R.N. Mohapatra and J.W.F. Valle, Phys. Rev. D 34, 1642 (1986); E. Ma, Phys. Lett. B 191, 287 (1987).
[6] Z.Z. Xing and S. Zhou, Phys. Lett. B 679, 249 (2009).
[7] See, e.g., G.L. Fogli et al., Phys. Rev. D 78, 033010 (2008).
[8] M. Hirsch, S. Morisi, and J.W.F. Valle, arXiv:0905.3056.
[9] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. B 530, 167 (2002); Z.Z. Xing, Phys. Lett. B 533, 85 (2002); P.F. Harrison and W.G. Scott, Phys. Lett. B 535, 163 (2002); X.G. He and A. Zee, Phys. Lett. B 560, 87 (2003).
[10] R. Friedberg and T.D. Lee, High Energy Phys. Nucl. Phys. 30, 591 (2006); Annals Phys. 323, 1087 (2008); Annals Phys. 323, 1677 (2008); T.D. Lee, Nucl. Phys. A 805, 54 (2008).
[11] G. ’t Hooft, in Proceedings of 1979 Cargèse Institute on Recent Developments in Gauge Theories, edited by G. ’t Hooft et al. (Plenum Press, New York, 1980), p. 135.
[12] C. Jarlskog, Phys. Rev. D 77, 073002 (2008); arXiv:0806.2206.
[13] M. Malinsky, T. Ohlsson, Z.Z. Xing, and H. Zhang, Phys. Lett. B 679, 242 (2009).
[14] Z.Z. Xing, H. Zhang, and S. Zhou, Phys. Lett. B 641, 189 (2006).
[15] S. Luo and Z.Z. Xing, Phys. Lett. B 646, 242 (2007); Z.Z. Xing, Int. J. Mod. Phys. E 16, 1361 (2007); W. Chao, S. Luo, and Z.Z. Xing, Phys. Lett. B 659, 281 (2008); C.S. Huang, T.J. Li, W. Liao, and S.H. Zhu, Phys. Rev. D 78, 013005 (2008); T. Araki and R. Takahashi, arXiv:0811.0905; T. Araki and C.Q. Geng, arXiv:0906.1903.
[16] S. Luo, Z.Z. Xing, and X. Li, Phys. Rev. D 78, 117301 (2008).
[17] S. Antusch et al., JHEP 0610, 084 (2006); A. Abada et al., JHEP 0712, 061 (2007).
[18] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 517, 363 (2001).
[19] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); D.D. Wu, Phys. Rev. D 33, 860 (1986).
[20] Z.Z. Xing, Phys. Lett. B 660, 515 (2008).
[21] E. Fernandez-Martinez, M.B. Gavela, J. López-Pavón, and O. Yasuda, Phys. Lett. B 649, 427 (2007); Z.Z. Xing, Phys. Lett. B 660, 515 (2008); S. Luo, Phys. Rev. D 78, 016006 (2008); S. Goswami and T. Ota, Phys. Rev. D 78, 033012 (2008); G. Altarelli and D. Meloni, Nucl. Phys. B 809, 158 (2009); S. Antusch, M. Blennow, E. Fernandez-Martinez, and J. López-Pavón, Phys. Rev. D 80, 033002 (2009).
[22] For a recent review, see: Z.Z. Xing, Int. J. Mod. Phys. A 23, 4255 (2008).
[23] Z.Z. Xing, arXiv:0902.2469; Phys. Lett. B 679, 255 (2009).