Comment on “Uncertainty Relation for Photons”

Zhi-Yong Wang, Cai-Dong Xiong, and Qi Qiu

School of Optoelectronic Information, University of Electronic Science and Technology of China, Chengdu 610054, CHINA

In a recent interesting Letter [Phys. Rev. Lett. 108, 140401 (2012)] I. Bialynicki-Birula and his coauthor have derived the uncertainty relation for the photons in three dimensions. However, some of their arguments are problematical, and this impacts their conclusion.

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1. The authors of Ref. [1] have confused helicity with spin

Let \( \hat{A}^\mu(x) = (\hat{A}^0, \hat{A}) \) be a four-dimensional (4D) electromagnetic potential, in the Lorentz gauge condition, it can be expanded as (\( k \cdot x = \omega t - k \cdot r \)):

\[
\hat{A}^\mu(x) = \frac{\text{d}^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{s=0}^{3} \eta_\mu(k,s)[c(k,s)\exp(-ik \cdot x) + c^+(k,s)\exp(ik \cdot x)],
\]

where \( \eta_\mu(k,s) \ (s = 0,1,2,3 \ , \mu = 0,1,2,3) \) are four 4D polarization vectors, the four indices of \( s=0,1,2,3 \) describe four kinds of photons, respectively. One can choose

\[
\begin{align*}
\eta_\mu(k,0) &= (1,0,0,0), & \eta_\mu(k,1) &= (0,\varepsilon(k,1)) \\
\eta_\mu(k,2) &= (0,\varepsilon(k,2)), & \eta_\mu(k,3) &= (0,\varepsilon(k,3)),
\end{align*}
\]

where \( \varepsilon(k,i) \ (i = 1,2,3) \) are the 3D linear polarization vectors whose matrix forms are

\[
\varepsilon(k,1) = \frac{1}{|k|} \begin{pmatrix}
  k_2^2k_3 + k_2^2 |k| \\
  k_1^2 + k_2^2 \\
  k_1k_2(k_3 - |k|) \\
  -k_1
\end{pmatrix}, \quad \varepsilon(k,2) = \frac{1}{|k|} \begin{pmatrix}
  k_1k_2(k_3 - |k|) \\
  k_1^2 + k_2^2 \\
  k_1^2 |k| + k_2^2k_3 \\
  -k_1
\end{pmatrix}, \quad \varepsilon(k,3) = \frac{k}{|k|} \begin{pmatrix}
  1 \\
  k_1 \\
  k_2 \\
  k_3
\end{pmatrix}.
\]

One has \( \varepsilon(k,1) \times \varepsilon(k,2) = \varepsilon(k,3) = k/|k| \). Obviously, let \( k = (0,0,k_3) \) and \( k_3 = |k| \geq 0 \),
one has \( \varepsilon(k,1) = (1,0,0) \), \( \varepsilon(k,2) = (0,1,0) \), and \( \varepsilon(k,3) = (0,0,1) \), where \( \varepsilon(k,1) \) and \( \varepsilon(k,2) \) (perpendicular to \( k \)) are two transverse polarization vectors, while \( \varepsilon(k,3) \) (parallel to \( k \)) is the longitudinal polarization vector, in the 3D space they satisfy the orthonormality and completeness relations (\( T \) denotes the matrix transpose, \( I_{3\times3} \) denotes the \( 3 \times 3 \) unit matrix):

\[
e^T(k,i)\varepsilon(k,j) = \delta_{ij}, \quad \sum_i \varepsilon(k,i)e^T(k,i) = I_{3\times3}, \quad i,j = 1,2,3.
\]

(4)

The spinor representations of \( \varepsilon(k,i) \) (\( i = 1,2,3 \)) form the circular polarization vectors, i.e.,

\[
e'_1(k) = \varepsilon_1(k) = \frac{\varepsilon(k,1) + i\varepsilon(k,2)}{\sqrt{2}} = \frac{1}{\sqrt{2} |k|} \begin{pmatrix} k_1k_3 - ik_2 |k| \\ k_1 - ik_2 \\ k_2k_3 + ik_1 |k| \\ k_1 - ik_2 \\ -(k_1 + ik_2) \end{pmatrix}, \quad e_0(k) = \varepsilon(k,3),
\]

(5)

where \( e'_1(k) \) denotes the complex conjugate of \( e_1(k) \) (while \( e'_1(k) \) denotes the hermitian conjugate of \( e_1(k) \), and so on). Using Eq. (5) one can prove the orthonormality and completeness relations as follows:

\[
e^\dagger_\lambda(k)e^\lambda_\lambda(k) = \delta_{\lambda\lambda'}, \quad \sum_{\lambda} e^\lambda_\lambda(k)e^\dagger_{\lambda'}(k) = I_{3\times3}, \quad \lambda, \lambda' = \pm 1,0.
\]

(6)

In 3D space, the spin matrices of the electromagnetic field are:

\[
\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(7)

They form the spin matrix vector \( \tau = (\tau_1, \tau_2, \tau_3) \). Let \( \tau \cdot k = \tau_1k_1 + \tau_2k_2 + \tau_3k_3 \), using Eqs. (5) and (7) one can prove that

\[
\frac{\tau \cdot k}{|k|} e_\lambda(k) = \lambda e_\lambda(k), \quad \lambda = \pm 1,0.
\]

(8)

Eq. (8) implies that \( e_0(k) \) (parallel to \( k \)) denotes the longitudinal polarization vector,
while $e_{\pm}(k)$ (perpendicular to $k$) correspond to the right- and left-hand circular polarization vectors, respectively, and $\lambda = \pm 1, 0$ represent the spin projections in the direction of $k$ (i.e., $\lambda = \pm 1, 0$ represent the helicities of photons).

For simplicity let us consider the electromagnetic field in vacuum. In the units of $\hbar = c = 1$, in terms of the electromagnetic field intensities $\hat{E}(r,t)$ and $\hat{B}(r,t)$ one can define the Riemann-Silberstein vectors $\hat{F}^{(\pm)}(r,t) = [\hat{E}(r,t) \pm i\hat{B}(r,t)]/\sqrt{2}$. Substituting Eq. (1) into $\hat{E} = -\nabla \hat{A}^0 - \partial \hat{A}^\mu / \partial t$ and $\hat{B} = \nabla \times \hat{A}$ one can prove that (note that $\omega = |k|$)

$$\hat{E} = \int \frac{d^4k}{\sqrt{2\omega(2\pi)^3}} \sum_{i=1}^3 \{ |k| \varepsilon(k,i) [ \hat{b}(k,i) \exp(-ik \cdot x) + \hat{b}^\dagger(k,i) \exp(ik \cdot x)] \}, \quad (9)$$

$$\hat{B} = \int \frac{d^4k}{\sqrt{2\omega(2\pi)^3}} \sum_{i=1}^3 \{ k \times \varepsilon(k,i) [ \hat{b}(k,i) \exp(-ik \cdot x) + \hat{b}^\dagger(k,i) \exp(ik \cdot x)] \}, \quad (10)$$

where

$$\hat{b}(k,1) = ic(k,1), \quad \hat{b}(k,2) = ic(k,2), \quad \hat{b}(k,3) = i[\hat{c}(k,3) - \hat{c}(k,0)]. \quad (11)$$

Using Eq. (3) one can obtain $k \times \varepsilon(k,1) = |k| \varepsilon(k,2)$, $k \times \varepsilon(k,2) = -|k| \varepsilon(k,1)$, and $k \times \varepsilon(k,3) = (0,0,0)$. Therefore, when the electromagnetic field is described by the 4D electromagnetic potential $\hat{A}^\mu(x)$, there involves four 4D polarization vectors $\eta^\mu(k,s)$ ($s = 0,1,2,3$) together describing four kinds of photons; while described by the electromagnetic field intensities $\hat{E}$ and $\hat{B}$, there only involves three 3D polarization vectors $\varepsilon(k,i)$ ($i = 1,2,3$), and Eq. (11) shows that the $i = 1,2$ solutions describe two kinds of transverse photons ($s=1, 2$), while the $i = 3$ photons correspond to the admixture of the longitudinal ($s=3$) and scalar ($s=0$) photons. According to QED, only those state vectors (say, $|\Phi\rangle$) are admitted for which the expectation value of the Lorentz gauge condition is satisfied: $\langle \Phi | \partial^\mu \hat{A}_\mu | \Phi \rangle = 0$, which implies that
\[
\{\phi | \hat{b}(k,3) | \phi \} = i \{\phi | [\hat{c}(k,3) - \hat{c}(k,0)] | \phi \} = 0. \tag{12}
\]

Then, we will only take into account the transverse photons. Define
\[
\hat{a}_{\pm}(k) = \sqrt{\omega/2}[\hat{b}(k,1) \mp i \hat{b}(k,2)], \quad \hat{a}_0(k) = \sqrt{\omega/2}\hat{b}(k,3), \tag{13}
\]
one can prove that \( \hat{F}^{(\pm)}(r,t) = [\hat{E}(r,t) \pm i \hat{B}(r,t)]/\sqrt{2} \) are given by
\[
\hat{F}^{(1)}(r,t) = \int \frac{d^3k}{(2\pi)^{3/2}} \epsilon_i(k)[\hat{a}_i(k) \exp(-ik \cdot x) + \hat{a}^+_i(k) \exp(ik \cdot x)], \tag{14}
\]
\[
\hat{F}^{(-1)}(r,t) = \int \frac{d^3k}{(2\pi)^{3/2}} \epsilon_i(k)[\hat{a}_{-i}(k) \exp(-ik \cdot x) + \hat{a}^+_{-i}(k) \exp(ik \cdot x)], \tag{15}
\]
where Eq. (14) is equivalent to Eq. (9) of Ref. [1]. The annihilation and creation operators in Eq. (1) satisfy the commutation relations,
\[
[\hat{c}(k',s'), \hat{c}^+(k,s)] = -g_{ss'} \delta^{(3)}(k - k'), \quad s,s' = 0,1,2,3, \tag{16}
\]
where \( \delta^{(3)}(k' - k) = \delta(k'_1 - k_1) \delta(k'_2 - k_2) \delta(k'_3 - k_3) \), \( \ g_{ss'} = \text{diag}(1, -1, -1, -1) \). Using Eqs. (11), (13) and (16), one has
\[
[\hat{a}_\lambda(k), \hat{a}^+_\lambda'(k')] = \omega \delta_{\lambda \lambda'} \delta^{(3)}(k - k'), \quad \lambda, \lambda' = \pm 1, \tag{17}
\]
with the others vanishing. In particular, one has \( [\hat{a}_0(k), \hat{a}^+_0(k')] = 0 \).

Eq. (8) implies that the circular polarization vectors \( \epsilon_{\pm\lambda}(k) \) are the eigenvectors of photonic helicity operator (with the eigenvalues of \( \lambda = \pm 1 \), respectively), which implies that a photon with the polarization vector \( \epsilon_{\lambda}(k) \) has the spin projection of \( \lambda = \pm 1 \) onto the direction of the photon’s momentum, and then Eqs. (14) and (15) imply that \( \hat{F}^{(1)}(r,t) \) and \( \hat{F}^{(-1)}(r,t) \) describe the transverse photons with the helicities of \( \lambda = \pm 1 \), respectively.

On the other hand, taking \( \hat{F}^{(1)}(r,t) \) for example, if its positive-frequency part has the momentum of \( k \in (-\infty, +\infty) \), then its negative-frequency part has the momentum of \( -k \). As a result, if the state vector of \( \hat{a}^+_\lambda(k)|0\rangle \) has the spin of “up”, then the one of \( \hat{a}^+_\lambda(k)|0\rangle \)
has the spin of “down”, but both of them have positive helicity (because \( \hat{F}(r,t) \) has positive helicity). That is, both the annihilation and creation operators in Eq. (9) of Ref. [1] have positive helicity, while their spins are respectively “up” and “down”. Therefore, the authors of Ref. [1] have confused helicity with spin, and the statements before Eq. (9) in Ref. [1] are not appropriate.

BTW, in terms of our circular polarization vector \( e_\lambda(k) \), one can express the normalized vector \( e(k) \) given by Eq. (10) in Ref. [1] as

\[
e(k) = \frac{k_1 - i k_2}{k_1^2 + k_2^2} e_1(k) = \exp(i \theta)e_1(k), \quad \theta = -\arctan(k_2/k_1),
\]

Then the normalized vector \( e(k) \) is also the eigenvector of Eq. (8) with the eigenvalue \( \lambda = 1 \), i.e., it is also the right-hand circular polarization vector. In fact, rotating \( e_{\pm_1}(k) \) round the wave number vector \( k \), one can obtain another right- and left-hand circular polarization vectors, respectively. The polarization vectors \( e_{\pm_1}(k) \) have the following properties (\( \lambda = \pm 1 \)):

\[
\begin{align*}
\hat{e}_\lambda^\dagger(-k)e_\lambda(k) &= 0, \quad \frac{\partial}{\partial k}\hat{e}_\lambda^\dagger(-k)e_\lambda(k) = (0, 0, 0), \quad \frac{\partial^2}{\partial k^2}\hat{e}_\lambda^\dagger(-k)e_\lambda(k) = 0, \\
\left[ \frac{\partial}{\partial k} e_\lambda^\dagger(k) \right] e_\lambda^\dagger(k) &= \frac{-i \lambda}{||k||}(k_2^2, 0, 0), \quad \frac{\partial}{\partial k} \left\{ \frac{\partial}{\partial k} e_\lambda^\dagger(k) e_\lambda^\dagger(k) \right\} = 0, \\
\left[ \frac{\partial^2}{\partial k^2} e_\lambda^\dagger(k) \right] e_\lambda^\dagger(k) &= \frac{-2}{||k||(||k||+k_3^2)}.
\end{align*}
\]

Note that \( \partial^2/\partial k^2 = (\partial/\partial k)\cdot(\partial/\partial k) = \partial^2/\partial k_1^2 + \partial^2/\partial k_2^2 + \partial^2/\partial k_3^2 \), and so on. The vector \( e_\lambda(k) \) is expressed in matrix forms, such that the scalar product of \( e_\lambda(k) \) with itself is \( e_\lambda^\dagger(k)e_\lambda(k) \), for example. However, the vector such as \( [\hat{\partial} e_\lambda^\dagger(k)/\hat{\partial} k] e_\lambda(k) \) is expressed in the usual form, because its vectorial property comes from \( \hat{\partial}/\hat{\partial} k \).

2. Eq. (15) (and then Eq. (27)) in Ref. [1] is wrong
To show this, similar to Ref. [1], we replace all field operators with the corresponding classical fields, and rewrite Eqs. (14) and (15) as \( (\hbar = c = 1) \)

\[
F^{(\lambda)}(r, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e_x(k)[a_x(k)\exp(-ik\cdot x) + a_{x}^\dagger(k)\exp(ik\cdot x)].
\] (23)

Note that our \( a_{\lambda}(k) \) are identical with \( f_{\lambda}(k) \) in Ref. [1], while our \( \hat{a}_{\lambda}(k) \) are identical with \( a_{\lambda}(k) \) in Ref. [1]. To calculate an energy moment, one should simultaneously take into account two kinds of transverse photons with the helicities of \( \lambda = \pm 1 \), which also lies in the fact that, the field quantities of \( E(r, t) \) and \( B(r, t) \) together are equivalent to the ones of \( F^{(1)}(r, t) \) and \( F^{(-1)}(r, t) \) together, rather than to \( F^{(1)}(r, t) \) only. Then, in spite of \( \left| F^{(1)} \right| = \left| F^{(-1)} \right| \), conceptually, the first and second moments of the classical energy density should be

\[
M_1 = \frac{1}{2} \int d^3r \left[ \left| F^{(1)} \right|^2 + \left| F^{(-1)} \right|^2 \right], M_2 = \frac{1}{2} \int d^3r r^2 \left[ \left| F^{(1)} \right|^2 + \left| F^{(-1)} \right|^2 \right].
\] (24)

For the moment one has \( a_{\lambda}(k)a_{\lambda}^\dagger(k') = a_{\lambda'}^\dagger(k')a_{\lambda}(k) \) (\( \lambda, \lambda' = \pm 1 \)). For \( t = 0 \), one can prove that (see Appendix A)

\[
M_1 = -i \int d^3k [a_i(k)D_k a_i^\dagger(k) - a_{i}^\dagger(k)D_k a_i(k)],
\] (25)

\[
M_2 = \int d^3k \left\{ \frac{1}{|k|}[|a_i(k)|^2 + |a_{i}^\dagger(k)|^2] - [a_i(k)D_k a_i^\dagger(k) + a_{i}^\dagger(k)D_k a_i(k)] \right\},
\] (26)

where \( D_k = \partial/\partial k - iA \) and \( A = (-k_z, k_i, 0)/|k|(|k| + k_z) \). Our Eq. (26) is different from Eq. (15) in Ref. [1], such that in Ref. [1] the conclusion based on Eqs. (15) and (27) are questionable. Using the facts that \( M_1 \) is a real number and \( M_2 \) is nonegative definite, one can obtain some relations.

3. The definition Eq. (8) in Ref. [1] is not reasonable

The classical energy is
\[ M_0 = (1/2) \int \text{d}^3 r \left[ |F^{(0)}|^2 + |F^{(-)}|^2 \right] = \int \text{d}^3 k [d_1(k)^2 + d_{-1}(k)^2]. \]  

(27)

Let us denote

\[ \langle r^n \rangle = \frac{M_n}{M_0} = \frac{(1/2) \int \text{d}^3 r r^n \left[ |F^{(0)}|^2 + |F^{(-)}|^2 \right]}{(1/2) \int \text{d}^3 r \left[ |F^{(0)}|^2 + |F^{(-)}|^2 \right]}, \quad n = 1, 2. \]  

(28)

Obviously, \( \langle r^n \rangle = M_n/M_0 \) have the dimension of \([\text{length}]^n\). One can define

\[(\Delta r)^2 = \langle r^2 \rangle - \langle r \rangle^2. \]  

(29)

We do not think the definition Eq. (8) in Ref. [1] is reasonable. Instead, we think that \( \Delta r \) should be defined via Eq. (29).

To provide a heuristic insight, let us consider a single-mode field with the frequency \( \omega = |k| \) and let \( a_{-1}(k) = 0, |a_1(k)|^2 = 1 \). For the moment one has \( \partial a_1^*(k)/\partial k = 0 \), \( M_0 = 1, \quad \langle r \rangle = M_1 = -A, \quad \langle r^2 \rangle = M_2 = -L \), and then \( \Delta r = \sqrt{-L - |A|^2} = 1/|k| \).

4. Conclusion

The statements before Eq. (9) in Ref. [1] have confused the concept of helicity with that of spin; Eq. (15) in Ref. [1] should be replaced with our Eq. (26). As a result, in Ref. [1] the conclusions related to (15) and Eq. (27), etc., have not been proven. Moreover, conceptually, we do not think the definition Eq. (8) in Ref. [1] is reasonable. Instead, we think that \( \Delta r \) should be defined via Eq. (29).

*E-mail: zywang@uestc.edu.cn

[1] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. Lett. 108, 140401 (2012).

**Appendix A Proof of Eq. (26)**

Let us calculate the second moment of the classical energy density
\[ M_2 = \frac{1}{2} \int d^3rr^2 \left( |F^{(1)}|^2 + |F^{(-1)}|^2 \right) = \int d^3rr^2 |F^{(1)}|^2, \]  
\quad (a1)

where \(( k \cdot x = \omega t - k \cdot r )\)

\[ F^{(1)}(r, t) = \int \frac{d^3k}{(2\pi)^3/2} \epsilon_1(k) [a_1(k) \exp(-ik \cdot x) + a_1^*(k) \exp(ik \cdot x)]. \]  
\quad (a2)

Substituting (a2) into (a1), one has (note that the vectors \( e_i(k) \) is expressed in matrix form, such that the scalar product of \( e_i(k) \) with itself is \( e_i(k) e_i(k) \)):

\[ M_2 = \int d^3rr^2 \left[ \int \frac{d^3k'}{(2\pi)^3/2} \epsilon_1(k') [a_1^*(k') \exp(ik' \cdot x) + a_{-1}(k') \exp(-ik' \cdot x)] \right] \]

\[ = \int d^3k \int d^3k' \int r^2 \frac{d^3r}{(2\pi)^3} \]

\[ \{ e_i^*(k') a_1^*(k') e_i(k) a_1(k) \exp(ik' \cdot x) \exp(-ik \cdot x) \]
\[ + e_i^*(k') a_{-1}(k') e_i(k) a_{-1}^*(k) \exp(-ik' \cdot x) \exp(ik \cdot x) \]
\[ + e_i^*(k') a_1^*(k') e_i(k) a_{-1}(k) \exp(ik' \cdot x) \exp(-ik \cdot x) \]
\[ + e_i^*(k') a_{-1}(k') e_i(k) a_1(k) \exp(-ik' \cdot x) \exp(-ik \cdot x) \}

\[ = -\int d^3k \int d^3k' \int \frac{d^3r}{(2\pi)^3} \]

\[ \{ e_i^*(k') a_1^*(k') e_i(k) a_1(k) \frac{\partial^2}{\partial k'^2} \exp(ik' \cdot x) \exp(-ik \cdot x) \]
\[ + e_i^*(k') a_{-1}(k') e_i(k) a_{-1}^*(k) \frac{\partial^2}{\partial k'^2} \exp(-ik' \cdot x) \exp(ik \cdot x) \]
\[ + e_i^*(k') a_1^*(k') e_i(k) a_{-1}(k) \frac{\partial^2}{\partial k'^2} \exp(ik' \cdot x) \exp(-ik \cdot x) \]
\[ + e_i^*(k') a_{-1}(k') e_i(k) a_1(k) \frac{\partial^2}{\partial k'^2} \exp(-ik' \cdot x) \exp(-ik \cdot x) \} \quad . \]  
\quad (a3)

Using

\[ \int \frac{d^3r}{(2\pi)^3} \exp[\pm i(k' - k) \cdot r] = \delta^{(3)}(k' - k), \int \frac{d^3r}{(2\pi)^3} \exp[\pm i(k' + k) \cdot r] = \delta^{(3)}(k' + k), \]  
\quad (a4)

one has
Using \( \frac{\partial^2}{\partial k'^2} = \frac{\partial^2}{\partial k_1'^2} + \frac{\partial^2}{\partial k_2'^2} + \frac{\partial^2}{\partial k_3'^2} \), \( \delta^{(3)} (k' - k) = \delta (k'_1 - k_1) \delta (k'_2 - k_2) \delta (k'_3 - k_3) \), and

\[
f(x')[\frac{\partial^n}{\partial x^n} \delta (x' - x)] = (-1)^n \delta (x' - x) [\frac{\partial^n}{\partial x^n} f(x')], \quad (a6)
\]

one has

\[
M_2 = - \int d^3k \int d^3k' \left\{ \frac{\partial^2}{\partial k'^2} e_i^*(k') a_i (k') \exp(i \omega t') e_i (k) a_i (k) \exp(-i \omega t) \delta^{(3)} (k' - k) + \frac{\partial^2}{\partial k'^2} e_i^*(k') a_i (k') \exp(-i \omega t') e_i (k) a_i (k) \exp(-i \omega t) \delta^{(3)} (k' - k) \\
+ \frac{\partial^2}{\partial k'^2} e_i (k') a_i^* (k') \exp(i \omega t') e_i (k) a_i^* (k) \exp(i \omega t) \delta^{(3)} (k' + k) + \frac{\partial^2}{\partial k'^2} e_i (k') a_i^* (k') \exp(-i \omega t') e_i (k) a_i^* (k) \exp(-i \omega t) \delta^{(3)} (k' + k) \right\} = - \int d^3k \left\{ \frac{\partial^2}{\partial k'^2} e_i^*(k) a_i (k) \exp(i \omega t) e_i (k) a_i (k) \exp(-i \omega t) + \frac{\partial^2}{\partial k'^2} e_i^*(k) a_i (k) \exp(-i \omega t) e_i (k) a_i (k) \exp(i \omega t) \\
+ \frac{\partial^2}{\partial k'^2} e_i (k) a_i^* (k) \exp(i \omega t) e_i (k) a_i^* (k) \exp(i \omega t) + \frac{\partial^2}{\partial k'^2} e_i (k) a_i^* (k) \exp(-i \omega t) e_i (k) a_i^* (k) \exp(-i \omega t) \right\}. \quad (a7)
\]

where
\[ B_1 = \frac{\partial^2}{\partial k^2} e_i^t(k) a_i^t(k) \exp(i\omega t) e_i(k) a_i(k) \exp(-i\omega t), \quad (a8) \]
\[ B_2 = \frac{\partial^2}{\partial k^2} e_i^t(k) a_{-1}(k) \exp(-i\omega t) e_i(k) a_{-1}(k) \exp(i\omega t), \quad (a9) \]
\[ B_3 = \frac{\partial^2}{\partial k^2} e_i^t(-k) a_i^t(-k) \exp(i\omega t) e_i(k) a_i(k) \exp(i\omega t), \quad (a10) \]
\[ B_4 = \frac{\partial^2}{\partial k^2} e_i^t(-k) a_{-1}(k) \exp(-i\omega t) e_i(k) a_{-1}(k) \exp(-i\omega t). \quad (a11) \]

One can prove that
\[ e_i^t(-k)e_i(k) = 0, \quad \left[ \frac{\partial}{\partial k} e_i^t(-k) \right] e_i(k) = (0, 0, 0), \quad \left[ \frac{\partial^2}{\partial k^2} e_i^t(-k) \right] e_i(k) = 0, \quad (a12) \]
and then one has \( B_3 = B_4 = 0 \), it follows that
\[
M_2 = -\int \! \! d^3 k \\ \left\{ \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_i^t(k) \exp(i\omega t) e_i(k) a_i(k) \exp(-i\omega t) \right] + \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_{-1}(k) \exp(-i\omega t) e_i(k) a_{-1}(k) \exp(i\omega t) \right] \right\} \quad (a13)
\]
Here the vectors \( e_i(k) \) is expressed in matrix form, such that the scalar product of \( e_i(k) \) with itself is \( e_i^t(k)e_i(k) \). However, the vector such as \( [\frac{\partial}{\partial k} e_i^t(k) / \frac{\partial}{\partial k}] e_i(k) \) is expressed in the usual form, because its vectorial property comes from \( \frac{\partial}{\partial k} \). Let \( t = 0 \), (a13) becomes
\[
M_2 = -\int \! \! d^3 k \\ \left\{ \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_i^t(k) \exp(i\omega t) e_i(k) a_i(k) + \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_{-1}(k) \exp(-i\omega t) e_i(k) a_{-1}(k) \right] \right\} \quad (a14)
\]
where
\[
\frac{\partial^2}{\partial k^2} e_i^t(k) a_i^t(k) = \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_i^t(k) + 2\left[ \frac{\partial}{\partial k} e_i^t(k) \right] \left[ \frac{\partial}{\partial k} a_i^t(k) \right] + e_i^t(k) \frac{\partial^2}{\partial k^2} a_i^t(k), \quad (a15) \right]
\]
\[
\frac{\partial^2}{\partial k^2} e_i^t(k) a_{-1}(k) = \left[ \frac{\partial^2}{\partial k^2} e_i^t(k) a_{-1}(k) + 2\left[ \frac{\partial}{\partial k} e_i^t(k) \right] \left[ \frac{\partial}{\partial k} a_{-1}(k) \right] + e_i^t(k) \frac{\partial^2}{\partial k^2} a_{-1}(k). \quad (a16) \right]
\]
Substituting (a15) and (a16) into (a14), and define

\[ A = i \left[ \frac{\partial}{\partial k} e_i^+(k) e_i(k) \right] = \frac{1}{|k|(|k| + k_3)} (-k_2, k_1, 0), \quad L = \left[ \frac{\partial^2}{\partial k^2} e_i^+(k) e_i(k) \right] = \frac{-2}{|k|(|k| + k_3)}, \]  

one has (note that \( e_i^+(k) e_i(k) = 1 \))

\[ M_2 = - \int d^3k \{ \mathcal{L}[a_i^+(k)a_i(k) + a_{i-1}(k)a_{i-1}^+(k)] 
- 2i A \cdot \left[ \frac{\partial}{\partial k} a_i^+(k) \right] a_i(k) - 2i A \cdot \left[ \frac{\partial}{\partial k} a_{i-1}(k) \right] a_{i-1}^+(k) 
+ \left[ \frac{\partial^2}{\partial k^2} a_i^+(k) \right] a_i(k) + \left[ \frac{\partial^2}{\partial k^2} a_{i-1}(k) \right] a_{i-1}^+(k) \} \]  

(a18)

Let \( D_k = \partial / \partial k - i A \), using \( \frac{\partial}{\partial k} A = \partial A_1 / \partial k_1 + \partial A_2 / \partial k_2 + \partial A_3 / \partial k_3 = 0 \), one has

\[ a_i D_k^+ a_i^* = a_i \left( \frac{\partial}{\partial k} - i A \right) \left( \frac{\partial}{\partial k} - i A \right) a_i^* = a_i \left( \frac{\partial^2}{\partial k^2} a_i^* \right) - 2i A \cdot \left( \frac{\partial}{\partial k} a_i^* \right) a_i - |A|^2 |a_i|^2, \]  

\[ a_{i-1}^* D_k^+ a_{i-1} = a_{i-1}^* \left( \frac{\partial}{\partial k} - i A \right) \left( \frac{\partial}{\partial k} - i A \right) a_{i-1} = a_{i-1}^* \left( \frac{\partial^2}{\partial k^2} a_{i-1} \right) - 2i A \cdot \left( \frac{\partial}{\partial k} a_{i-1} \right) a_{i-1}^* - |A|^2 |a_{i-1}|^2. \]  

(a19)

(a20)

Using (a19) and (a20), and consider that \( L + |A|^2 = -1 / |k|^2 \) one can obtain Eq. (26), i.e.,

\[ M_2 = \frac{1}{2} \int d^3r r^2 \left( |F^{(1)}|^2 + |F^{(-1)}|^2 \right) = \int d^3r \left| F^{(1)} \right|^2 \]  

\[ = \int d^3k \left[ \frac{1}{|k|^2} [a_i(k)]^2 + [a_{i-1}(k)]^2 \right] - [a_i(k) D_k^+ a_i^+(k) + a_{i-1}(k) D_k^+ a_{i-1}^+(k)] \]  

(a21)

BTW, likewise, one can prove that

\[ M_1 = \frac{1}{2} \int d^3r \left[ |F^{(1)}|^2 + |F^{(-1)}|^2 \right] = \int d^3r \left| F^{(1)} \right|^2 \]  

\[ = -i \int d^3k [a_i(k) D_k^+ a_i^+(k) - a_{i-1}(k) D_k^+ a_{i-1}^+(k)] \]  

\[ M_0 = \frac{1}{2} \int d^3r \left[ |F^{(1)}|^2 + |F^{(-1)}|^2 \right] = \int d^3r \left| F^{(1)} \right|^2 \]  

\[ = \int d^3k [a_i(k)]^2 + [a_{i-1}(k)]^2 \]  

(a22)

(a23)