THE INVERSE PROBLEM OF OPTIMAL REGULATORS AND ITS APPLICATION

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ABSTRACT This paper presents a new solution to the inverse problem of linear optimal regulators to minimize a cost function and meet the requirements of relative stability in the presence of a constant but unknown disturbance. A state feedback matrix is developed using Lyapunov's second method. Moreover, the relationships between the state feedback matrix and the cost function are obtained, and a formula to solve the weighting matrices is suggested. The developed method is applied successfully to design the horizontal loops in the inertial navigation system.

1 Introduction

The inverse problem of optimal regulators is to find a suitable quadratic cost function for a linear time-invariable system with constant but unknown disturbance so that the optimal control law can meet the requirements of relative stability. In 1984, Juang and Lee presented a method of optimal pole assignment in a specified region. In 1986, Lee and Liaw presented a new method for finding the weighting matrices Q and R of inverse regulators without solution of the Ricatti equation. However, this method is not perfect for the description and prove of the theorems and lack of strictness. Moreover, the unit weighting matrix R may cause a non-negative weighting matrix Q (Lee and Liaw, 1986), which is inappropriate.

The following improvements of the results reported in the Ref. [3] are made in this paper: a state feedback matrix K that meets the relative stability constraint is derived using Lyapunov's second method so that the solution space is enlarged; some main theorems are amended. Meanwhile, this paper also proves that for any real matrix K, if R is an unit matrix, Q will be the non-negative. The method developed in this paper is finally used to design a single-axis horizontal loop of an inertial navigation system.

2 Inverse problem of regulators

Consider a completely controllable system

$$\dot{X}(t) = \hat{A}X(t) + \hat{B}\hat{u} + Fw$$ (1)

where $X \in R^n$, $\hat{u} \in R^m$, $w \in R^w$, $w$ is a constant vector with unknown disturbance, the matrix $\hat{A}$, $\hat{B}$ and $F$ have corresponding dimensions, $m \leq n$.

Assume the matrix $\hat{B}$ is of full rank, $\Re(F) \subset \Re(\hat{B})$.

Let $z(t) = \hat{u}(t) + \hat{w}$ in which $\hat{w} = Cw$, C is a coefficient matrix.

The cost function can be expressed as follows:

$$J = \int_{0}^{\infty} e^{2\omega t} [\hat{x}^T(t)\hat{Q}\hat{x}(t) + z^T(t)\hat{R}z(t) + \hat{u}^T(t)\hat{S}\hat{u}(t)]dt$$ (2)
where, \( \dot{Q} = Q^T \geq 0, \dot{R} = R^T > 0, \dot{S} = S^T > 0, \alpha \) is relative stability constraints, relative stability requires \( \alpha > 0 \).

The inverse problem of linear optimal regulators is to find the quadratic cost function (2) so that the control law \( \hat{u} \) determined by relative stability constraint is optimal.

Since \( \dot{Z}(t) = \hat{u}(t) \) \( \tag{3} \)

We can combine (3) and (1) into

\[
\begin{bmatrix}
\dot{X}(t) \\
\dot{Z}(t)
\end{bmatrix} =
\begin{bmatrix}
\dot{A} & \dot{B} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
X(t) \\
Z(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\hat{u}(t) \quad \tag{4}
\]

Define

\[
X(t) = [\dot{X}^T(t) \ Z(t)]^T
\]

\[
u(t) = \dot{u}(t) = \dot{z}(t)
\]

then Eq. (1) and Eq. (2) can be rewritten as:

\[
\dot{X}(t) = AX(t) + Bu(t) \quad \tag{5}
\]

and

\[
J = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt
\quad \tag{6}
\]

where

\[
A = \begin{bmatrix}
\dot{A} & \dot{B} \\
0 & 0
\end{bmatrix}, B = \begin{bmatrix} 0 \\
1
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
\dot{Q} & 0 \\
0 & \dot{R}
\end{bmatrix}, R = \dot{S}
\]

Thus, the inverse problem of optimal regulators turns to the following: given the plant dynamics (5) and relative steady constraints \( \alpha \), find a state feedback matrix \( K \) and weighting matrix \( (Q \geq 0 \text{ and } R > 0) \), so that \( u(t) = Kx(t) \) minimizes Eq. (6).

3 The form of the state feedback matrix

For a completely controllable \( n \) order continuous system, one criterion to evaluate its relative stability is that the real part and phase of all its closed-loop poles are within its limit. If \( S_r \) represents the desired work set, it is clear that \( S_r \) is the shaded parts in Fig. 1.

**Lemma 1** (Zhang, 1986)

For a time-invariable dynamic equation \( \dot{x} = Ax \), the necessary and sufficient conditions for its asymptotically stable zero solution is, for any given positive definite symmetric matrix \( N \), there exists only one positive definite symmetric matrix \( M \), so that

\[
A^TM + MA = -N
\]

**Lemma 2** (Zhang, 1986)

For a \( n \) order continuous controllable system \((A, B)\), if \( | \lambda_i | \frac{1 - \beta}{a\beta} (\frac{a}{1 - \beta}B + A) | < 1 \) then

\[
\lambda_i(A) \in S_r, i = 1, 2, \ldots, n,
\]

that is, \( R, \lambda_i(A) < \frac{1 - \beta}{a\beta}, | \lambda_i(A) | < \frac{a\beta}{1 - \beta} \).

**Lemma 3** (Zhang, 1986)

For any given positive definite symmetric \( n \times n \) matrix \( N \) and any positive \( a \) and \( \beta(0 \leq \beta < 1) \), there exists a solution of positive definite and symmetric matrix equation:

\[
\frac{1 - \beta}{a\beta} (A + \frac{a}{1 - \beta}I) + M = \frac{1 - \beta}{a\beta} (A + \frac{a}{1 - \beta}I) + \frac{1 - \beta}{a\beta} (A + \frac{a}{1 - \beta}I) = -N
\]

if and only if

\[
| \lambda_i (\frac{1 - \beta}{a\beta} (A + \frac{a}{1 - \beta}I)) | < 1, i = 1, 2, \ldots, n.
\]

**Theorem 1**

Assume that system (5) is controllable, and given matrix \( N = N^T > 0, M = M^T > 0 \), then there must exist a state feedback matrix \( K \) satisfying

\[
(BK)^TM + MBK = -\frac{a\beta}{1 - \beta} N - \left( A + \frac{a}{1 - \beta}I \right)^TM + M \left( A + \frac{a}{1 - \beta}I \right)
\]

where \( \lambda_i(A + BK) \in S_r, i = 1, 2, \ldots, n \).

**Proof**

If \( \lambda_i(A + BK) \in S_r \), from Lemma 2, there must exist equation as below:

![Fig. 1 Relative stability area (\( \alpha \) and \( \beta \) are relative stability constraints)](image-url)
Since $N = N^T > 0$ and $M = M^T > 0$, from Lemma 3, we have:

$$
\begin{align*}
\sum_{i=1}^{n} \left( \frac{1-\beta}{\alpha \beta} \left( A + BK + \frac{\alpha}{1-\beta} I \right) \right)^T M + \\
M \left[ \frac{1-\beta}{\alpha \beta} \left( A + BK + \frac{\alpha}{1-\beta} I \right) \right] = -N
\end{align*}
$$

By rearranging above equation, the Eq. (8) will be obtained.

4 Relationship between state feedback matrix and the cost function

Lemma 4 (Lee and Liaw, 1986)

For a completely controllable system (5), if rank $B = m$, $-KB$ is a symmetric and a semi-positive definite matrix, $A + BK$ is asymptotically stable, then there exists $R = R^T > 0$ and $K = -R^{-1}B^TP$, where $P = P^T > 0$ is a general solution of the following equation:

$$
P = K^TR(-RKB) + RK + Y \quad (10)
$$

where $Y = Y^T, B^TY = 0$. If matrix $KB$ is non-singular, then

$$
P = K^TR(-KB)^{-1}K + Y \quad (11)
$$

Theorem 2

Consider a completely controllable system (5), where $B$ is of full rank. If

(a) $\lambda_i(A + BK) \in S_+$, $i = 1, 2, \cdots, n$, 

(b) $-KB$ is a symmetric and a semi-positive definite matrix,

(c) $[K^TR(-RKB) + RK + Y](A + \frac{\alpha}{1-\beta}I) - (A + \frac{\alpha}{1-\beta}I)^T[K^TR(-RKB) + RK + Y]$ is symmetric and semi-positive definite, where $Y = Y^T, B^TY = 0$.

Then if $R$ is symmetric and positive definite, there must exist symmetric and non-negative matrices $P$ and $Q$ that satisfy:

$$
P = K^TR(-RKB) + RK + Y \quad (12)
$$

$$
Q = -P \left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T - \\
\left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T P + K^TRK \quad (13)
$$

where $a > 0, 0 < \beta < 1$.

Proof

From conditions (a), (b) and Lemma 4, there exists a symmetric and non-negative matrix $P$ that satisfies Eq. (12), since $(1 - \beta)/\alpha \beta > 0$, from condition (c) and Eq. (12), we get

$$
P \left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T - \\
\left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T P \geq 0
$$

From Lemma 3, when $a$ and $\beta$ are constrained, $P$ satisfies:

$$
P \left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T + \left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T P - PBR^{-1}B^TP + Q = 0 \quad (14)
$$

If $K = -R^{-1}B^TP$, instead of Eq. (14), we obtain Eq. (13). Because $R = R^T > 0$, there exists $K^TRK > 0$, and it is symmetric, so $Q$ is symmetric and non-negative matrix.

Corollary

If $R$ is a unit matrix in Theorem 2, there must exist symmetric and non-negative matrices $P$ and $Q$ that satisfy:

$$
P = K^TR(-RKB) + K + Y \quad (15)
$$

$$
Q = -P \left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T - \\
\left[ \frac{1-\beta}{\alpha \beta} \left( A + \frac{\alpha}{1-\beta} I \right) \right]^T P + K^TRK \quad (16)
$$

5 Procedures of designing inverse optimal regulator

(a) Give a symmetric and positive definite matrix $N$. Generally $N$ is a unit matrix, and

(b) Determine the range of $K$ that ensure $M > 0$ from Eq. (8);

(c) Examine if selected matrix $K$ satisfies conditions (b) and (c) in theorem 2, if not, go back to step(b), and select another matrix $K$ until $K$ satisfies the conditions;

(d) Set $R = I$, calculate $P$ and $Q$ by Eqs. (15) and (16);

(e) Set such a diagonal matrix $Q$ that $K$ becomes of the optimal feedback matrix;

(f) From $R = \dot{S}$ and $Q = \text{diag}(\dot{Q}/\dot{R})$, the cost function (2) which the original system required, is obtained.
6 An example

Our objective is to apply the method developed to the design of a single-axis horizontal loop in an inertial navigation system. It is required to design an optimal controller with which the relative stability of the system satisfies \( a = 1.3 \) and \( \beta = 0.5 \), and the navigation outputs are not affected by the constant gyro drift in the stability.

It is assumed that the derivation equation of the single-axis horizontal loop can be expressed as follows:

\[
\begin{align*}
d_{y} &= 2\theta \\
\theta &= -\delta v_{y} + \dot{u} + \epsilon_{x}
\end{align*}
\]

(17a)

(17b)

where \( \delta v_{y} \) is a Y-axis velocity error; \( \theta \) is the attitude angle of the platform; \( \dot{u} \) is the voltage which is loaded to the gyro moment; \( \epsilon_{x} \) is the gyro drift which is a constant but unknown disturbance.

Assume \( x(t) = [\delta v_{y} \ \theta \ \dot{u}]^{T}, u = \dot{u} \), there exists an extended state equation:

\[
\dot{x}(t) = 
\begin{bmatrix}
0 & 2 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} x(t) + 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u(t)
\]

(18)

Let the performance index be:

\[
J = \int_{0}^{T} e^{2\alpha t} (x^{T}Qx + u^{T}Ruy) dt
\]

(19)

From \( B^{T}Y = 0 \), we have:

\[
Y = \begin{bmatrix}
Y_{1} & Y_{2} & 0 \\
Y_{2} & Y_{3} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( Y_{1}, Y_{2} \) and \( Y_{3} \) are unknown. Set \( N = I \), then there exists the feedback gain matrix:

\[
K = [-3.2 \ 21.59 \ 8.3]
\]

Moreover, we have:

\[
M = 
\begin{bmatrix}
10,685.59 & 18,013.08 & 3,053.15 \\
18,013.08 & 31,603.09 & 5,474.52 \\
3,053.15 & 5,474.52 & 960.56
\end{bmatrix}
\]

where \( M \) matrix is a symmetric and positive definite. Set \( R = I \), then \( Y_{2} = 31.13, Y_{3} = 58.01 \), therefore, we have:

\[
P = 
\begin{bmatrix}
Y_{1} + 1.23 & 39.45 & 3.20 \\
39.45 & 114.17 & 21.59 \\
3.20 & 21.59 & 8.30
\end{bmatrix}
\]

To ensure \( P \geq 0, Q \geq 0 \), \( Y_{1} \) must be within an interval of \( 18.74 \leq Y_{1} \leq 166.88 \), then Eq. (2) is designed as the weighting matrix:

\[
R = 2.45
\]

\[
Q = 
\begin{bmatrix}
-4Y_{1} + 66.08 & -1.54Y_{1} - 2.69 & 0 \\
-1.54Y_{1} - 2.69 & -112.05 & 0 \\
0 & 0 & 2.45
\end{bmatrix}
\]

It is easy to test that the feature values of original system (17) are \( \pm \sqrt{2}j \), whereas the feature values of the optimal close-loop system are \( -2.9 \) and \( -2.7 \pm 0.8j \). Furthermore, we can also see that the system (17) is not affected by gyro drift \( \epsilon_{x} \) in the stability and required area.

In this paper a state feedback matrix \( K \) that meets the relative stability constraint \( s \) in the presence of a constant but unknown disturbance is developed using Lyapunov's second method. Moreover, the relationships between the state feedback matrix \( K \) and the cost function are obtained, and a formula to solve the weighting matrices is suggested. The single-axis horizontal loops in inertial navigation system have been used to illustrate the application of the method developed.

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