EXISTENCE OF LEAFWISE INTERSECTION POINTS IN THE UNRESTRICTED CASE

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Abstract. In this article, we study the question of existence of leafwise intersection points for contact manifolds which are not necessarily of restricted contact type.

1. Introduction

The study of existence of leafwise intersection points has become an important aspect of Hamiltonian dynamics. Leafwise intersection points interpolate between periodic orbits and Lagrangian intersection points. To be more precise, when a coisotropic submanifold in a $2n$ dimensional symplectic manifold of has codimension 0 resp. $n$, a leafwise intersection point coincides with a periodic orbit resp. a Lagrangian intersection point. In fact, the higher codimensional case has already been considered and explored in [Gü, Ka], yet in this paper we restrict our interests to the codimension one case.

Let $(\Sigma,\lambda)$ be a contact hypersurface in a symplectic manifold $(M,\omega)$, that is $d\lambda = \omega|_\Sigma$. The Hamiltonian vector field $X_F$ is defined implicitly by $i_{X_F}\omega = dF$ for a time-dependent Hamiltonian function $F \in C^\infty(S^1 \times M)$ where $dF$ is only the derivative with respect to $M$ and we call the time 1-map $\phi_F$ of its flow a Hamiltonian diffeomorphism. In addition, we denote by Ham($M,\omega$) the group of Hamiltonian diffeomorphisms defined on $(M,\omega)$ and Ham$_c$($M,\omega$) the group of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian functions. The symplectic structure $\omega$ determines the characteristic line bundle $L_{\Sigma} \subset T\Sigma$ over $\Sigma$:

$$L_{\Sigma} := \{ (x,\xi) \in T_x\Sigma \mid \omega_x(\xi,\zeta) = 0 \text{ for all } \zeta \in T_x\Sigma \}.$$ 

$\Sigma$ is foliated by the leaves of the characteristic line bundle and we denote by $L_x$ the leaf through $x \in \Sigma$ of the characteristic foliation. We note that these leaves are spanned by the Reeb vector field $R$ of $\lambda$ which is characterized by $\lambda(R) = 1$ and $i_Rd\lambda_{\Sigma} = 0$. Then a leafwise intersection point of $\phi \in \text{Ham}(M,\omega)$ is by definition a point $x \in \Sigma$ such that $\phi(x) \in L_x$.

In this paper, we consider $(\Sigma \times (-1,\infty), d((r + 1)\lambda))$ the symplectization of $(\Sigma,\lambda)$ of dimension $2n - 1$ where $r$ is the coordinate on $(-1,\infty)$.

**Question 1.1.** Given $\phi \in \text{Ham}_c(\Sigma \times (\vartheta_1,\vartheta_2), d((r + 1)\lambda))$ for $-1 < \vartheta_1 < 0 < \vartheta_2 < \infty$, does $\phi$ have a leafwise intersection point?

We give an affirmative answer to the question above for a class of Hofer small $\phi$ (see Section 2 for the definition of the Hofer norm $\| \cdot \|$) and symplectically fillable contact manifolds. Throughout this paper, we assume that $\Sigma \times (\vartheta_1,\vartheta_2)$ is symplectically embedded in a symplectic manifold $(M,\omega)$ of dimension $2n$.

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Definition 1.2. We denote by $\varphi(\Sigma, \lambda) > 0$ the minimal period of a Reeb orbit of $(\Sigma, \lambda)$ which is contractible in $M$. If there is no contractible Reeb orbit we set $\varphi(\Sigma, \lambda) = \infty$.

Definition 1.3. We call a symplectic manifold $(M, \omega)$ convex at infinity if $(M, \omega)$ is symplectomorphic to the positive part of the symplectization of a compact contact manifold at infinity. Furthermore, $(M, \omega)$ is called symplectically aspherical if one has the equality $\omega|_{\pi_2(M)} = 0$.

Theorem A. Let $(M, \omega)$ be closed (or convex at infinity) and symplectically aspherical and $(\Sigma, \lambda)$ be a contact hypersurface with $\Sigma \times (-\vartheta_1, \vartheta_2)$ being symplectically embedded in $M$. Moreover, we assume that $\phi = \phi_F$ for $F \in C^\infty(S^1 \times \Sigma \times (\vartheta_1, \vartheta_2))$ where $F$ is constant outside the region $\Sigma \times [\rho_1, \rho_2]$ for $\rho_1 \in (\vartheta_1, 0), \rho_2 \in (0, \vartheta_2)$ and has Hofer norm $\|F\| \leq \varphi(\Sigma, \lambda)$. Then $\phi_F \in \text{Ham}_c(M, \omega)$ has a leafwise intersection point.

Remark 1.4. A contact hypersurface $(\Sigma, \lambda)$ in a symplectic manifold $(M, \omega)$ is of restricted contact type if the contact 1-form $\lambda$ is defined on the whole symplectic manifold $M$ and $d\lambda = \omega$. The main difference of this paper with other results in [AF1], [Gi] and [Gi2] is that we drop the condition of restricted contact type. Thus, our ambient symplectic manifold need not be exact and therefore can be closed. Moreover we remove the condition needed in [AF1] that $\Sigma$ bounds a compact region in $M$ by means of the argument developed in [Ka].

On the other hand, for a special perturbation $F$, we are able to find a leafwise intersection point in a symplectization of $\Sigma$ even though the symplectization is not convex at infinity.

Definition 1.5. We denote the support of Hamiltonian vector field $X_F$ by

$$\text{Supp}X_F := \text{cl}\{(x, r) \in \Sigma \times (-1, \infty) | X_F(t, x, r) \neq 0 \text{ for some } t \in S^1\}. \quad (1.1)$$

Theorem B. Let $\phi \in \text{Ham}(\Sigma \times (-1, \infty), d((r + 1)\lambda))$ be of the form $\phi = \phi_F$ for some $F \in C^\infty(S^1 \times \Sigma \times (-1, \infty))$ where $X_F$ is generated by the Reeb vector field $R$ and the Liouville vector field $Y$ (defined before Proposition 2.1). If $\text{Supp}X_F$ is compact and $\|F\| \leq \varphi(\Sigma, \lambda)$, then $\phi$ has a leafwise intersection point.

Remark 1.6. If the Weinstein conjecture holds, we can show Theorem B in an easier way. It is reduced to find a self intersection point of $S^1$ in $S^1 \times (-1, \infty)$ where $S^1$ is diffeomorphic to the Reeb orbit since there exist at least one periodic Reeb orbit. Therefore Theorem B follows with the assumption $\|\phi\| \leq e(S^1)$ where $e(S^1)$ is the displacement energy of $S^1$ in $\mathbb{R}^2$.

1.1. Idea of the proofs. Leafwise intersection points arise as critical points of a perturbed Rabinowitz action functional. Therefore our proof is based on Rabinowitz Floer homology as in [AF1]. The difference to [AF1] is that we do not assume restricted contact type of $\Sigma$. In order to overcome this difficulty, we apply an auxiliary Rabinowitz action functional similar as in [CFP]. However, in our situation, we have to perturb the auxiliary Rabinowitz action functional. Using the auxiliary Rabinowitz action functional we show that the moduli space of gradient flow lines of original Rabinowitz action functional can be compactified. To prove Theorem B, we compare the difference of the two action functionals and examine the energy of holomorphic curves. And then we notice that for special $F$, gradient flow lines of Rabinowitz action functional remain in a tubular neighborhood, that means, gradient flow lines do not go to infinity and thus we do not need the condition of convex at infinity.

1.2. History and related results. The problem of existence of leafwise intersection points was addressed by Moser [Mo]. Moser obtained existence results for simply connected $M$ and $C^1$-small $\phi$. Banyaga [Ba] removed the assumption of simply connectedness. Hofer [Ho] and
Ekeland-Hofer [EH] replaced the assumption of $C^1$-smallness by boundedness of the Hofer norm below a certain symplectic capacity for restricted contact type in $\mathbb{R}^{2n}$. Ginzburg [Gi] extended the Ekeland-Hofer results to subcritical Stein manifolds. Dragnev [Dr] obtained the result on the leafwise intersection problem to closed contact type submanifold in $\mathbb{R}^{2n}$. Albers-Frauenfelder [AF1] proved the existence of leafwise intersection points for a restricted contact hypersurface whenever a Hamiltonian diffeomorphism satisfies the same Hofer smallness assumption as in this article. By a different approach Gürel [Gü] also proved existence of leafwise intersection points of the extension of $F$ for higher codimensional restricted contact type under the Hofer smallness assumption as in this article. By a different approach Gürel [Gü] also proved existence of leafwise intersection points of restricted contact type in $\mathbb{R}^{2n}$.

We introduce a cutoff function $\phi$ such that $\supp\phi \subset (\varrho_1, \varrho_2)$. Then we can extend $\hat{\phi}$ to $\phi$ globally. Let $G : \mathbb{R} \to \mathbb{R}$ and $supp \phi \subset (\varrho_1, \varrho_2)$ such that $\phi(r) = r + 1$ for $r \in [\varrho_1, \varrho_2]$ and $\phi'(r) \leq 1 + \kappa$ for all $r \in \mathbb{R}$ and for some $\kappa > 0$ satisfying

$$\frac{1 + \varrho_1}{\varrho_1 - \varrho_2} < 1 + \kappa. \quad (2.2)$$

Then we have a global one form

$$\beta(y) := \begin{cases} 
\varphi(r)\lambda(x) & y = (x, r) \in \Sigma \times (\varrho_1, \varrho_2), \\
0 & y \in M - (\Sigma \times (\varrho_1, \varrho_2)).
\end{cases} \quad (2.3)$$

Let $\phi_Y^t$ be the flow of the Liouville vector field. Fix $\delta_1 > 0$ so that $\phi_Y^t|\Sigma$ is defined for $|t| \leq \delta_1$. Then we can define a function $\hat{G}$ near $\Sigma$ by $\hat{G}(\phi_Y^t(x)) = t$ for $x \in \Sigma$. Let $U_\delta := \{x \in M | |\hat{G}(x)| < \delta\}$ for $0 < \delta < \delta_1$ and choose $\delta_0 < \delta_1$ satisfying $U_{\delta_0} \subset \Sigma \times [\varrho_1, \varrho_2]$. Then we can extend $\hat{G}$ to $G \in C^\infty(M)$ to be defined on the ambient manifold $M$.

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2. Preliminaries

Let $(\Sigma, \lambda)$ be a contact hypersurface in a closed (or convex at infinity) and symplectically aspherical symplectic manifold $(M, \omega)$. In addition, we consider a non-autonomous Hamiltonian function $F \in C^\infty(S^1 \times \Sigma \times (\vartheta_1, \vartheta_2))$ which is constant outside the region $\Sigma \times [\rho_1, \rho_2]$ for $-1 < \vartheta_1 < \rho_1 < 0 < \rho_2 < \vartheta_2 < \infty$. We extend $F$ locally constant to the whole symplectic manifold $M$. The existence of leafwise intersection points of the extension of $F$ guarantees existence of leafwise intersection points of $F$ because the Hamiltonian vector field of the extension vanishes on $\Sigma \times ((\vartheta_1, \rho_1) \cup (\rho_2, \vartheta_2))$. For simplicity we denote the extension of $F$ by $F$ again. Moreover we additionally assume that $\Sigma$ bounds a compact region in $M$. In other words, $M - \Sigma$ consists of two component. This additional assumption will be removed in Step 4 in the proof of Theorem A by using the argument developed in [Ka].

We introduce the Liouville vector field $Y$ for $\Sigma$, that is $i_Y \omega = \lambda$ on $\Sigma \times (\vartheta_1, \vartheta_2)$. Then $Y$ has the following well-known properties.

Proposition 2.1. [HZ] Let the vector field $Y$ be the Liouville vector field for $(\Sigma, \lambda)$. Then $Y$ satisfies

$$L_Y \omega = \omega \quad \text{and} \quad Y \pitchfork T\Sigma. \quad (2.1)$$

We introduce a cutoff function $\varphi$ to extend $\lambda$ globally. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $supp \varphi \subset (\varrho_1, \varrho_2)$ such that $\varphi(r) = r + 1$ for $r \in [\rho_1, \rho_2]$ and $\varphi'(r) \leq 1 + \kappa$ for all $r \in \mathbb{R}$ and for some $\kappa > 0$ satisfying

$$\frac{1 + \varrho_1}{\varrho_1 - \varrho_2} < 1 + \kappa. \quad (2.2)$$

Then we have a global one form

$$\beta(y) := \begin{cases} 
\varphi(r)\lambda(x) & y = (x, r) \in \Sigma \times (\varrho_1, \varrho_2), \\
0 & y \in M - (\Sigma \times (\varrho_1, \varrho_2)).
\end{cases} \quad (2.3)$$

Let $\phi_Y^t$ be the flow of the Liouville vector field. Fix $\delta_1 > 0$ so that $\phi_Y^t|\Sigma$ is defined for $|t| \leq \delta_1$. Then we can define a function $\hat{G}$ near $\Sigma$ by $\hat{G}(\phi_Y^t(x)) = t$ for $x \in \Sigma$. Let $U_\delta := \{x \in M | |\hat{G}(x)| < \delta\}$ for $0 < \delta < \delta_1$ and choose $\delta_0 < \delta_1$ satisfying $U_{\delta_0} \subset \Sigma \times [\rho_1, \rho_2]$. Then we can extend $\hat{G}$ to $G \in C^\infty(M)$ to be defined on the ambient manifold $M$.
We also consider an almost complex structure $J$ on $\Sigma \times \mathbb{R}$ with time support on $(0, \chi)$. The inequality is obvious.

Proof. To prove a complex structure which interchanges the Reeb vector field $\phi$, computation shows $\lambda$ vanishes but $\phi$ and $\phi'$ do not. Let $F$ be a smooth function $\partial_r = \partial_r \mathbb{R}$ and $\partial_r = -\partial_r$. According to the previous Lemma 2.4 we only need to consider a Hamiltonian function $H$ which is SFT-like; that is, it splits on $\Sigma \times (\vartheta_1, \vartheta_2)$ with respect to $T(\Sigma \times (\vartheta_1, \vartheta_2)) = \ker \lambda \oplus (\ker d\lambda \oplus \partial_r)$. As $J|_{\ker d\lambda \oplus \partial_r}$ is a complex structure which interchanges the Reeb vector field $R$ with $\partial_r$, strictly speaking, $JR = \partial_r$ and $J\partial_r = -R$.

**Proposition 2.2.** For every $u \in TM$ the following inequality holds

$$d\beta(u, Ju) \leq (1 + \kappa)\omega(u, Ju) \quad (2.5)$$

Proof. For $u \in T(\Sigma \times (\vartheta_1, \vartheta_2))$, we can write $u = u_1 + u_2$ with respect to the decomposition $T(\Sigma \times (\vartheta_1, \vartheta_2)) = \ker \lambda \oplus (\ker d\lambda \oplus \partial_r)$. On $\Sigma \times (\vartheta_1, \vartheta_2)$, recall that we have chosen $\varphi$ as $\varphi(r) \leq r + 1$ and $\varphi'(r) \leq 1 + \kappa$ so that

$$d\beta(u, Ju) \leq \varphi(r)d\lambda(u_1, Ju_1) + \varphi'(r)dr \wedge \lambda(u_2, Ju_2) \leq (r + 1)d\lambda(u_1, Ju_1) + (1 + \kappa)dr \wedge \lambda(u_2, Ju_2)$$

$$\leq (1 + \kappa)\omega(u, Ju).$$

The last inequality follows from $\omega = d((r + 1)\lambda) = (r + 1)d\lambda + dr \wedge \lambda$. Outside of $\Sigma \times (\vartheta_1, \vartheta_2)$, $d\beta$ vanishes but $\omega(\cdot, J\cdot)$ is positive definite. Therefore the proposition is proved.

Next, we recall the definition of Hofer norm.

**Definition 2.3.** Let $F \in C^\infty_c(S^1 \times M, \mathbb{R})$ be a compactly supported Hamiltonian function. We set

$$||F||_+ := \int_0^1 \max_{x \in M} F(t, x)dt \quad ||F||_- := -\int_0^1 \min_{x \in M} F(t, x)dt = ||F||_+ \quad (2.7)$$

and

$$||F|| = ||F||_+ + ||F||_-.$$  

For $\phi \in \text{Ham}_c(M, \omega)$ the Hofer norm is

$$||\phi|| = \text{inf}\{||F|| \mid \phi = \phi_F\}. \quad (2.8)$$

**Lemma 2.4.** For all $\phi \in \text{Ham}_c(M, \omega)$

$$||\phi|| = ||\phi|| := \text{inf}\{||F|| \mid \phi = \phi_F, \ F(t, \cdot) = 0 \ \forall t \in [\frac{1}{2}, 1]\}. \quad (2.9)$$

Proof. To prove $||\phi|| \geq ||\phi||$, pick a smooth monotone increasing map $r : [0, 1] \to [0, 1]$ with $r(0) = 0$ and $r(\frac{1}{2}) = 1$. For $F$ with $\phi_F = \phi$ we set $F^r(t, x) := r(t)F(r(t), x)$. Then a direct computation shows $\phi_{F^r} = \phi_F$, $||F^r|| = ||F||$, and $F^r(t, x) = 0$ for all $t \in [\frac{1}{2}, 1]$. The reverse inequality is obvious.

Thanks to the previous Lemma 2.4 we only need to consider a Hamiltonian function $F$ with time support on $(0, \frac{1}{2})$ to prove the main theorems.
3. Rabinowitz Action functionals

3.1. Critical points. We denote by $\mathcal{L} \subset C^\infty(S^1, M)$ the component of contractible loops in $M$. For Hamiltonian functions $H$ and $F$ defined so far, the perturbed Rabinowitz action functional $A^H_F(v, \eta) : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$A^H_F(v, \eta) = -\int_{D^2} \bar{v}^* \omega - \int_0^1 F(t, v(t))dt - \eta \int_0^1 H(t, v(t))dt$$

(3.1)

where $\bar{v} : D^2 \to M$ is a filling disk of $v$. The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling discs.

We also define the auxiliary Rabinowitz action functional

$$\hat{A}^H_F(v, \eta) := -\int_{D^2} \bar{v}^* d\beta - \int_0^1 F(t, v(t))dt - \eta \int_0^1 H(t, v(t))dt .$$

(3.2)

where $\beta$ has been defined in (2.3). Furthermore, we will use the difference of two action functionals:

$$A := \hat{A}^H_F - A^H_F = \int_{D^2} \bar{v}^*(\omega - d\beta).$$

(3.3)

Critical points $(v, \eta) \in \text{Crit}A^H_F$ satisfy

$$\begin{cases} 
\partial_t v = X_F(t, v) + \eta X_H(t, v) \\
\int_0^1 H(t, v)dt = 0
\end{cases}$$

(3.4)

Albers-Frauenfelder observed that a critical point of $A^H_F$ gives rise to a leafwise intersection point.

**Proposition 3.1.** [AF1] Let $(v, \eta) \in \text{Crit}A^H_F$. Then $x = v(0)$ satisfies $\phi_F(x) \in L_x$. Thus, $x$ is a leafwise intersection point.

**Proof.** Since $F(t, \cdot)$ vanishes for $t \in (\frac{1}{2}, 1)$, we compute for $t \geq \frac{1}{2}$,

$$\frac{d}{dt} G(v(t)) = dG(v(t))[\partial_t v]$$

$$= dG(v(t)) \left[ X_F(t, v) + \eta X_H(t, v) \right] = 0$$

(3.5)

Since $\int_0^1 H(t, v(t))dt = 0$ and $G(v(t))$ is constant for $t \geq 1/2$, $v(t) \in H^{-1}(0) = G^{-1}(0) = \Sigma$ for $t \in [\frac{1}{2}, 1]$. On the other hand, $H$ has the time support on $(\frac{1}{2}, 1)$, $v$ solves the equation $\partial_t v = X_F(t, v)$ on $[0, \frac{1}{2})$. Therefore $v(\frac{1}{2}) = \phi_F^{1/2}(v(0)) = \phi_F(v(0))$ since $F = 0$ for $t \geq \frac{1}{2}$.

For $t \in (\frac{1}{2}, 1)$, $\partial_t v = \eta X_H(t, v)$ implies $x = v(0) = v(1) \in L(v(1))$. Thus we conclude that $x \in L_{\phi_F(x)}$, this is equivalent to $\phi_F(x) \in L_x$. \qed

3.2. Gradient flow lines. From now on, we allow $s$-dependence on $F$ as follows:

$F_s(t, x) = F_-(t, x)$ for $s \leq -1$, for some $F_-$ and $F_s(t, x) = F_+(t, x)$ for $s \geq 1$, for some $F_+$. Moreover $F_s(t, \cdot) = 0$ for $t \in (\frac{1}{2}, 1)$. We also choose a family $J(s)$ of compatible almost complex structures on $M$ so that they still are SFT-like and $J(s) = J_-$ for $s \leq -1$, for some $J_-$ and $J(s) = J_+$ for $s \geq 1$, for some $J_+$. 


Lemma 3.4. Let $\nabla_m^H$ be a gradient flow line of $A^H_{F_s}$. Then
\[
E(w) = \int_{-\infty}^{\infty} m( - \nabla_m^H(w(s)), \partial_s w(s)) ds
\]
\[
= - \int_{-\infty}^{\infty} dA^H_{F_s}(w(s))((\partial_s w(s)) ds
\]
\[
= - \int_{-\infty}^{\infty} \frac{d}{ds} (A^H_{F_s}(w(s))) ds + \int_{-\infty}^{\infty} (\partial_s A^H_{F_s})(w(s)) ds
\]
\[
= A^H_{F_s}(w_-) - A^H_{F_s}(w_+) - \int_{-\infty}^{1} \partial_s F_s(t, v) dt ds
\]
\[
\leq A^H_{F_s}(w_-) - A^H_{F_s}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds .
\]

Proof. It follows from the gradient flow equation (3.8).

Note. The time supports of $H$ and $F_s$ are disjoint.

On the tangent space $T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}) \cong T_v \mathcal{L} \times \mathbb{R}$, we define metric $m$ as follows:
\[
m_{(v, \eta)}((\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)) := \int_{0}^{1} \omega_{v}(\dot{v}_1, J\dot{v}_2) dt + \dot{\eta}_1 \dot{\eta}_2 .
\]

We also define another bilinear form $\tilde{m}$ with $\beta$.
\[
\tilde{m}_{(v, \eta)}((\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)) := \int_{0}^{1} d\beta_{v}(\dot{v}_1, J\dot{v}_2) dt + \dot{\eta}_1 \dot{\eta}_2 .
\]

Definition 3.2. A map $w \in (v, \eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ which solves
\[
\partial_s w(s) + \nabla_m^H(w(s)) = 0.
\]
is called a gradient flow line of $A^H_{F_s}$.

According to Floer’s interpretation, gradient flow equation (3.8) needs to be interpreted as $v : \mathbb{R} \times S^1 \to M$ and $\eta : \mathbb{R} \to \mathbb{R}$ solving
\[
\begin{align*}
\partial_s v + J_s(v)(\partial_s v - \eta X_H(t, v) - X_{F_s}(t, v)) &= 0 \quad \text{ } \\
\partial_s \eta - \int_{0}^{1} H(t, v) dt &= 0. \quad \text{(3.9)}
\end{align*}
\]

Definition 3.3. The energy of a map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ is defined as
\[
E(w) := \int_{-\infty}^{\infty} ||\partial_s w||^2_m ds .
\]

Lemma 3.4. Let $w$ be a gradient flow line of $\nabla_m^H$. Then
\[
E(w) \leq A^H_{F_s}(w_-) - A^H_{F_s}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds .
\]

where $w_{\pm} = \lim_{s \to \pm \infty} w(s)$ and the negative part of Hofer norm $|| \cdot ||_- \text{ has been defined in Definition 2.3.}$ Moreover, equality hold if $\partial_s F_s = 0$.
Proposition 3.5. If \((v, \eta) \in \mathcal{L} \times \mathbb{R}\) and \((\hat{v}, \hat{\eta}) \in T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}) = \Gamma(S^1, v^*TM) \times \mathbb{R}\) then the following assertion holds.

\[
d\hat{A}^H_F(v, \eta)(\hat{v}, \hat{\eta}) = \hat{m}\left(\nabla_m A^H_F(v, \eta), (\hat{v}, \hat{\eta})\right).
\]  

(3.13)

Proof. It holds

\[
dH = i_{X_H} \omega = i_{X_H} d\beta \quad \text{and} \quad dF = i_{X_F} \omega = i_{X_F} d\beta.
\]  

(3.14)

We know that

\[
i_{X_F} d\beta = i_{X_F} (\varphi(r) d\lambda + \varphi'(r) dr \wedge \lambda),
\]

\[
i_{X_F} \omega = i_{X_F} d((r+1) \lambda) = i_{X_F}((r+1) d\lambda + dr \wedge \lambda).
\]  

(3.15)

On the region \([\rho_1, \rho_2] \times \Sigma, \varphi(r) = r + 1\) implies \(\omega = d\beta\). Outside the region \([\rho_1, \rho_2] \times \Sigma, i_{X_F} \omega = 0 = i_{X_F} d\beta\) by assumption. The other equality that \(i_{X_H} \omega = i_{X_H} d\beta\) is analogous to the above since we have chosen \(\delta_0\) so that \(U_{\delta_0} \subset \Sigma \times [\rho_1, \rho_2]\).

Next, we note the formula of \(\nabla_m A^H_F\):

\[
\nabla_m A^H_F = \begin{pmatrix}
J(v)(\partial_t v - \eta X_H(t, v) - X_F(t, v)) \\
- \int_0^1 H(t, v) dt
\end{pmatrix}.
\]

Now it directly follows

\[
d\hat{A}^H_F(v, \eta)(\hat{v}, \hat{\eta}) = \int_0^1 d\beta(\partial_t v, \hat{v}) - \int_0^1 dF(t, v)(\hat{v})dt - \eta \int_0^1 dH(t, v)(\hat{v})dt - \hat{\eta} \int_0^1 H(t, v)dt
\]

\[
= \int_0^1 d\beta(\partial_t v, \hat{v}) - \omega(\eta X_H(t, v) + X_F(t, v), \hat{v}) dt - \hat{\eta} \int_0^1 H(t, v)dt
\]

\[
= \int_0^1 d\beta(\partial_t v - \eta X_H(t, v) - X_F(t, v), \hat{v}) dt - \hat{\eta} \int_0^1 H(t, v)dt
\]

\[
= \hat{m}\left(\nabla_m A^H_F(v, \eta), (\hat{v}, \hat{\eta})\right).
\]  

(3.16)

\[\square\]

Proposition 3.6. Let a gradient flow line \(w = (v, \eta)\) of \(A^H_{F,s}\) converge asymptotically to \(w_\pm := \lim_{s \to \pm\infty} w(s)\). Then the following inequality holds.

\[
A(w_-) - \kappa E(w) \leq A(w(s)) \leq A(w_+) + \kappa E(w).
\]  

(3.17)
Proof. Using Proposition 2.2 and Proposition 3.5,

\[
\frac{d}{ds} A(w) = \frac{d}{ds} \hat{A}^H_{F_s}(w) - \frac{d}{ds} H_{F_s}(w) \\
= \hat{A}^H_{F_s}(w)(\partial_s w) - dA^H_{F_s}(w)(\partial_s w) + \partial_s \hat{A}^H_{F_s}(w) - \partial_s A^H_{F_s}(w) \\
= m\left(\nabla_m A^H_F(v, \eta), \partial_s w\right) - m\left(\nabla_m A^H_F(v, \eta), \partial_s w\right) + \int_0^1 \partial_s F_s dt - \int_0^1 \partial_s F_s dt \\
= \int_0^1 (d\beta - \omega)(-\partial_s v, J\partial_s v) dt - \left(\int_0^1 H(t, v) dt\right)^2 + \left(\int_0^1 H(t, v) dt\right)^2 \\
= \int_0^1 (\omega - d\beta)(\partial_s v, J\partial_s v) dt \\
\geq - \int_0^1 \kappa \omega(\partial_s v, J\partial_s v) dt.
\]

Integrate both sides of (3.18) with respect to \(s\) from \(-\infty\) to \(s_0 \in \mathbb{R}\), then we get

\[
A(w(s_0)) - A(w_-) = \int_{-\infty}^{s_0} \frac{d}{ds} A(w(s)) ds \\
\geq -\kappa \int_{-\infty}^{s_0} \int_0^1 \omega(\partial_s v, J\partial_s v) dtds \\
\geq -\kappa E(w).
\]

(3.19)

On the other hand, integrate from \(s_0\) to \(+\infty\) and obtain

\[
A(w(s_0)) - A(w_+) = - \int_{s_0}^{\infty} \frac{d}{ds} A(w(s)) ds \\
\leq \kappa \int_{s_0}^{\infty} \int_0^1 \omega(\partial_s v, J\partial_s v) dtds \\
\leq \kappa E(w).
\]

(3.20)

Combine above two inequalities (3.19) and (3.20), then the proposition follows immediately.

\[\square\]

Proposition 3.7. \(\hat{A}^H_{F_s}\) has uniform bounds along gradient flow lines of \(A^H_{F_s}\) in terms of the asymptotic data, that is the action values of \(A^H_{F_s}\) and \(\hat{A}^H_{F_s}\) at \(w\pm\).

Proof. At first, let us show that \(A^H_{F_s}\) is uniformly bounded along a gradient flow line \(w(s)\).

\[
0 \leq - \int_{s_1}^{s_2} dA^H_{F_s}(w(s))(\partial_s w) ds \\
= A^H_{F_{s_1}}(w(s_1)) - A^H_{F_{s_2}}(w(s_2)) - \int_{s_1}^{s_2} \int_0^1 \partial_s F_s(t, v) dtds \\
\leq A^H_{F_{s_1}}(w(s_1)) - A^H_{F_{s_2}}(w(s_2)) + \int_{s_1}^{s_2} ||\partial_s F_s|| ds.
\]

(3.21)
From above inequality we obtain
\[ A_{F_{s_2}}^H(w(s_2)) \leq A_{F_{s_1}}^H(w(s_1)) + \int_{-\infty}^{\infty} ||\partial_s F_s|| - ds \]
(3.22)

\[ A_{F_{s_2}}^H(w(s_1)) \geq A_{F_{s_1}}^H(w(s_1)) - \int_{-\infty}^{\infty} ||\partial_s F_s|| - ds \]
Therefore for any \( s_0 \in \mathbb{R} \), it holds
\[ |A_{F_{s_1}}^H(w(s_0))| \leq \max\{A_{F_{s_1}}^H(w(s_1)), -A_{F_{s_1}}^H(w(s_1))\} + \int_{-\infty}^{\infty} ||\partial_s F_s|| - ds. \]  
(3.23)

By the definition of \( A \), we know
\[ |\widehat{A}_{F_{s_1}}^H(w(s))| \leq |A_{F_{s_1}}^H(w(s))| + |A(w(s))|, \]  
(3.24)

but both terms on the righthand side are uniformly bounded in terms of the asymptotic data, recall Lemma 3.4 and Proposition 3.6. Hence the proposition is proved. \( \square \)

**Theorem 3.8.** Let \( \mathcal{M} \) be a moduli space of gradient flow lines of \( A_{F_{s_1}}^H \) with uniform action bounds of \( A_{F_{s_1}}^H \) and \( \widehat{A}_{F_{s_1}}^H \) like \( 3.37 \). Then this moduli space is compact modulo breaking. More specifically, for a sequence \( \{w_n\}_{n \in \mathbb{N}} \) in \( \mathcal{M} \) and for every reparametrization sequence \( \sigma_n \in \mathbb{R} \) the sequence \( w_n(\cdot + \sigma_n) \) has a subsequence which converges in \( C_{\text{loc}}^{\infty}(\mathbb{R}, \mathcal{L} \times \mathbb{R}) \).

Moreover if \( w_n \) \( C_{\text{loc}} \)-converges to \( v \), we know \( E(v) \leq \lim \sup_{n \in \mathbb{N}} E(w_n) \) by the following calculation.
\[
E(v) = \int_{-\infty}^{\infty} ||\partial_s w||^2 ds = \lim_{T \to \infty} \int_{-T}^{T} ||\partial_s w||^2 ds \leq \lim_{T \to \infty} \lim_{n \in \mathbb{N}} E(w_n) = \lim_{n \in \mathbb{N}} E(w_n).
\]

This observation will be used later in \( 4.4 \).

**Proof.** If we establish the following facts, the proof of the theorem follows from standard arguments in Floer theory. For a sequence of elements \( \{w_n = (v_n, \eta_n)\}_{n \in \mathbb{N}} \) in \( \mathcal{M} \), we have

1. a uniform \( L^\infty \)-bound on \( v_n \),
2. a uniform \( L^\infty \)-bound on \( \eta_n \),
3. a uniform \( L^\infty \)-bound on the derivatives of \( v_n \).

(1) follows from the assumption “convex at infinity”. Once the \( L^\infty \)-bound on \( \eta_n \) is established, the \( L^\infty \)-bound on the derivatives of \( v_n \) follows from bubbling-off analysis together with the symplectic asphericity of \((M, \omega)\). Hence Theorem 3.11 finishes the proof. \( \square \)

**Lemma 3.9.** For \((v, \eta) \in \mathcal{L} \times \mathbb{R} \), there exist \( \epsilon > 0 \) and \( C > 0 \) such that
\[
||\nabla A_{F_{s_1}}^H(v, \eta)|| \leq \epsilon \quad \Rightarrow \quad |\eta| \leq C(|\widehat{A}_{F_{s_1}}^H(v, \eta)| + 1).
\]  
(3.25)

**Proof.** The proof of lemma proceeds in three steps.

**Step 1:** Assume that \( v(t) \) lies in \( U_\delta = \{x \in M \mid |G(x)| < \delta\} \) for all \( t \in (\frac{1}{2}, 1) \) where \( \delta = \min\{1, \delta_0/2\} \). Then there exists a constant \( C_1 > 0 \) such that
\[
|\eta| \leq C_1(|\widehat{A}_{F_{s_1}}^H(v, \eta)| + ||\nabla_m A_{F_{s_1}}^H(v, \eta)|| + 1).
\]  
(3.26)
We estimate

\[ |\tilde{A}_{F_s}^H(v, \eta)| = \left| \int_0^1 v^* \beta + \eta \int_0^1 H(t, v(t))dt + \int_0^1 F_s(t, v(t))dt \right| \]

\[ = |\eta \int_0^1 \beta(v)(X_H(t, v))dt + \int_0^1 \beta(v)(X_{F_s}(t, v))dt \]

\[ + \int_0^1 \beta(v)(\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v))dt + \eta \int_0^1 H(t, v(t))dt + \int_0^1 F_s(t, v(t))dt | \]

\[ \geq \left| \eta \int_0^1 \chi(t)\beta(v)(R(v))dt \right| - \left| \int_0^1 \beta(v)(\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v))dt \right| \]

\[ - \left| \eta \int_0^1 H(t, v(t))dt \right| - C_{\delta,F_s} \]

\[ \geq |\eta| - \delta|\eta| - C_{\delta}\|\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v)\|_{L^1} - C_{\delta,F} \]

\[ \geq (1 - \delta)|\eta| - C_{\delta}|\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v)\|_{L^2} - C_{\delta,F} \]

\[ \geq (1 - \delta)|\eta| - C_{\delta}|\nabla m A_{F_s}^H(v, \eta)|_m - C_{\delta,F} \]  

(3.27)

where \( C_{\delta} := |||v|_{L^\infty}, C_{\delta,F} := |||F|||_{L^\infty} + C_{\delta}|X_F|||_{L^\infty} \) and \( L^1, L^2 \)-norms on \( T, \mathcal{L} \) are taken with respect to the metric \( g(\cdot, \cdot) = \omega(\cdot, J^\cdot) \).

Thus we get

\[ |\eta| \leq \frac{1}{1 - \delta} \left( |\tilde{A}_{F_s}^H(v, \eta)| + C_{\delta}|\nabla m A_{F_s}^H(v, \eta)|_m + C_{\delta,F} \right). \]

(3.28)

This proves Step 1 with

\[ C_1 := \max \left\{ \frac{1}{1 - \delta}, \frac{C_{\delta}}{1 - \delta}, \frac{C_{\delta,F}}{1 - \delta} \right\}. \]

(3.29)

**Step 2:** There exists \( \epsilon > 0 \) with \( ||\nabla m A_{F_s}^H(v, \eta)||_m \geq \epsilon \) if there is \( t \in (\frac{1}{2}, 1) \) such that \( v(t) \notin U_\delta \).

If \( v(t) \in M - U_{\delta/2} \) for all \( t \in (\frac{1}{2}, 1) \) then easily we have

\[ ||\nabla m A_{F_s}^H(v, \eta)||_m \geq \left| \int_0^1 H(t, v(t))dt \right| \geq \frac{\delta}{2}. \]

(3.30)

Otherwise there exists \( t' \in (\frac{1}{2}, 1) \) such that \( v(t') \in U_{\delta/2} \). Thus we can find \( t_0, t_1 \in (\frac{1}{2}, 1) \) such that

\[ v(t_0) \in \partial U_{\delta/2}, \ v(t_1) \in \partial U_{\delta}, \ \text{and} \ \forall s \in [t_0, t_1], \ v(s) \in U_\delta - U_{\delta/2} \]  

(3.31)

or

\[ v(t_1) \in \partial U_{\delta}, \ v(t_0) \in \partial U_{\delta/2}, \ \text{and} \ \forall s \in [t_1, t_0], \ v(s) \in U_\delta - U_{\delta/2} \]

(3.32)
We treat only the first case; the later case is analogous. For $G := \max_{x \in U_s} ||\nabla G(x)||_g$, we have

\[
G||\nabla m A^H_{F_s}(v, \eta)||_m \geq G||\partial_t v - \eta X_H(v) - X_{F_s}(v)||_{L^2} \\
\geq G||\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v)||_L^1 \\
\geq \int_{t_0}^{t_1} ||\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v)||_g ||\nabla G(x)||_g dt \\
\geq \int_{t_0}^{t_1} \langle \nabla G(v(t)), \partial_t v - \eta X_H(t, v) - X_{F_s}(t, v) \rangle_g dt \\
= \int_{t_0}^{t_1} dG(v(t))(\partial_t v - \eta X_H(t, v) - X_{F_s}(t, v))dt \\
\geq |G(v(t_1)) - |G(v(t_0))| \\
= \delta. 
\]

Therefore Step 2 follows with $\epsilon = \min\{\frac{\delta}{2}, \frac{\delta}{20}\}$.

**Step 3:** Proof of the lemma.

Step 2 yields that $v(t) \in U_\delta$ for all $t \in (\frac{1}{2}, 1)$. Thus we are able to apply Step 1 and it shows that $|\eta| \leq C(|\ddot{A}^H_{F_s}(v, \eta)| + 1)$ with $C = C_1 + \epsilon$. \hfill \Box

For a gradient flow line $w$ of $A^H_{F_s}$ and $\sigma \in \mathbb{R}$, we set

$$
\tau(w, \sigma) := \inf\{\tau \geq 0 \mid ||\nabla m A^H_{F_s}(w(\sigma + \tau))||_m \leq \epsilon\}. 
$$

**Lemma 3.10.** We have a bound on $\tau(w, \sigma)$ as follows:

$$
\tau(w, \sigma) \leq \frac{A^H_{F_s}(w_-) - A^H_{F_s}(w_+) + C_F}{\epsilon^2} 
$$

for $C_F := \int_{-\infty}^{\infty} ||\partial_t F_s||_- ds < \infty$.

**Proof.** Using Lemma 3.3, we compute

$$
\epsilon^2 \tau(w, \sigma) \leq \int_{\sigma}^{\sigma + \tau(w, \sigma)} ||\nabla m A^H_{F_s}(w)||_m^2 ds \\
\leq E(w) \\
\leq A^H_{F_s}(w_-) - A^H_{F_s}(w_+) + \int_{-\infty}^{\infty} ||\partial_t F_s||_- ds. 
$$

Dividing $\epsilon^2$ both sides, the lemma follows. \hfill \Box

**Theorem 3.11.** Assume that $w = (v, \eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ is a gradient flow line of $A^H_{F_s}$ for which there exist $a \leq b$ such that

$$
\ddot{A}^H_{F_s}(w(s)), A^H_{F_s}(w(s)) \in [a, b] \text{ for all } s \in \mathbb{R} 
$$

(3.37)
Then the $L^\infty$-norm of $\eta$ is uniformly bounded.

Proof. Using the Lemma 3.9 and Lemma 3.10,
\[
|\eta(\sigma)| \leq |\eta(\sigma + \tau(w, \sigma))| \int_\sigma^{\sigma + \tau(w, \sigma)} |\partial_s \eta(s)| \, ds
\leq C \left( |\hat{A}^H_{F_\omega}(v, \eta)| + 1 \right) + \tau(w, \sigma) \|H\|_{L^\infty}
\leq C \left( \max\{|a|, |b|\} + 1 \right) + \left( \frac{|b - a| + C_F}{\epsilon^2} \right) \|H\|_{L^\infty}.
\] (3.38)

As we have already mentioned, Theorem 3.11 completes the proof of Theorem 3.8.

4. Proof of theorem A

The proof of Theorem A proceeds in four steps. In first three steps we give a proof under the assumptions that $\|F\| < \varphi(\Sigma, \lambda)$ and $\Sigma$ splits $M$ into two components. Step 4 finally removes these additional assumptions.

Step 1. Theorem A holds when $\|F\| < \frac{\varphi(p,0)}{\varphi(p,0)} \varphi(\Sigma, \lambda)$.

Proof. For $0 \leq r$, we choose a smooth family of functions $\beta_r \in C^\infty(\mathbb{R}, [0, 1])$ satisfying
1. for $r \geq 1$: $\beta'_r(s) \cdot s \leq 0$ for all $s \in \mathbb{R}$, $\beta_r(s) = 1$ for $|s| \leq r - 1$, and $\beta_r(s) = 0$ for $|s| \geq r$,
2. for $r \leq 1$: $\beta_r(s) \leq r$ for all $s \in \mathbb{R}$ and $\text{supp} \beta_r \subset [-1, 1]$,
3. $\lim_{r \to \infty} \beta_r(s + r) =: \beta_{\infty}(s)$ exists, where the limit is taken with respect to the $C^\infty_{\text{loc}}$ topology.

We fix a point $p \in \Sigma$ and consider the moduli space
\[
\mathcal{M} := \left\{ (r, w) \in [0, \infty) \times C^\infty_{\text{loc}}(\mathbb{R}, \mathcal{L} \times \mathbb{R}) \mid w : \text{gradient flow line of } A^H_{\beta_r, F}, \quad w_-(p, 0), w_+ \in \Sigma \times \{0\} \right\}.
\] (4.1)

Claim: If there exists no leafwise intersection point, then $\mathcal{M}$ is compact. Moreover, its boundary consists of the point $(0, p, 0)$ only.

Proof of Claim. For $(r, w) \in \mathcal{M},$
\[
E(w) = - \int_{-\infty}^{\infty} dA^H_{\beta_r, F}(w(s))(\partial_s w) \, ds
\leq A^H_0(p, 0) - A^H_0(p, 0) + \int_{-\infty}^{\infty} \|\partial_s \beta_r(s) F\|_- \, ds
= \int_{-\infty}^{\infty} \|\beta'_r(s) F\|_- \, ds
= \int_{-\infty}^{0} \beta'_r(s) \|F\|_- \, ds - \int_{0}^{\infty} \beta'_r(s) \|F\|_+ \, ds
= \beta_r(0)(\|F\|_- + \|F\|_+)
\leq \|F\|.
\] (4.2)
Accordingly, we can also estimate
\[ |\eta| = |\mathcal{A}_0^H (\gamma, \eta)| \leq |\mathcal{A}_0^H (\gamma, \eta)| + |\mathcal{A}(\gamma, \eta)| \leq E(v) + \left| \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}(v(s)) ds \right| \leq \limsup_{n \to \infty} E(w_n) + \limsup_{n \to \infty} \left| \int_{-\infty}^{s_n} \frac{d}{ds} \mathcal{A}(w_n(s)) ds \right| \leq |F| + \limsup_{n \to \infty} \left( \sup_{s \in \mathbb{R}} \mathcal{A}(w_n(s)) - \mathcal{A}(p,0) \right) \leq |F| + \limsup_{n \to \infty} \left( \max \{ |\mathcal{A}(w_+)|, |\mathcal{A}(w_-)| \} + \kappa E(w) \right) \leq |F| + \kappa |F|.

If $|F| < \frac{1}{1+\varphi(\Sigma, \lambda)}$, then due to the previous estimation \[4.4\] we deduce the contradiction $|\eta| < \varphi(\Sigma, \lambda)$. Since we have chosen any $\kappa$ satisfying $\frac{1+\varphi_1}{\rho_1-\varphi_2} < 1 + \kappa$, taking the limit $\kappa \to \frac{1+\varphi_1}{\rho_1-\varphi_2} - 1$ we deduce a contradiction to the assumption $|F| < \frac{\varphi_1}{1+\varphi_2} \varphi(\Sigma, \lambda)$. This proves the Claim.

We are able to regard the moduli space $\mathcal{M}$ as the zero set of a Fredholm section of a Banach bundle over a Banach manifold. Moreover, the Fredholm section is already transversal at the boundary point since the boundary is a constant solution and at a constant solution $A^H$ is Morse-Bott. Since $\mathcal{M}$ is compact by the previous claim, we can perturb a Fredholm section away from the boundary point to get a transverse Fredholm section whose zero set is a compact manifold with a single boundary point $(0,p,0)$. But such a manifold does not exist. This finishes the proof of Step 1 by contradiction.

\[ \square \]

**Step 2.** There exist a symplectic manifold $\hat{M}$ which is symplectically aspherical and convex at infinity and a symplectic embedding $\psi': \Sigma \times (\vartheta'_1, \vartheta_2) \to \hat{M}$ for any $-1 < \vartheta'_1 < \vartheta_2$.

\[ \text{1See the proof of Theorem A in [AF1] for the detail arguments.} \]
Proof. We have additionally assumed that a closed contact manifold \( \Sigma \) splits \( M \) into two components. We call a bounded component \( M_b \) with \( \partial M_b = \Sigma \). Then we get a new symplectic manifold \( (\hat{M}, \hat{\omega}) \) which is still symplectically aspherical and convex at infinity as follows:

\[
\hat{M} := M_b \cup_{\partial M_b} \Sigma \times [0, \infty)
\]

\[
\hat{\omega} = \begin{cases} 
\omega & \text{on } M_b, \\
\omega + (r + 1)\lambda & \text{on } \Sigma \times (\theta_1, \infty).
\end{cases}
\]

\( \hat{\omega} \) is well-defined since \( \omega \) equals to \( d((r + 1)\lambda) \) on \( \Sigma \times (\theta_1, 0] \subset M_1 \). Then we rescale the symplectic structure \( \hat{\omega} \). For arbitrary small \( 0 < \nu < 1 \), we have a rescaled symplectic manifold \( (\hat{M}, \nu \hat{\omega}) \). Then there is a symplectic embedding

\[
\psi_\nu : (\Sigma \times [-1 + \nu, \infty), d((r + 1)\lambda)) \longrightarrow (\Sigma \times [0, \infty), d((r + 1)\nu\lambda)) \subset (\hat{M}, \nu \hat{\omega})
\]

\[
(x, r) \mapsto (x, \frac{1}{\nu}(r - \nu + 1)).
\]

Therefore \( \Sigma \times (\theta'_1, \theta_2) \) can be embedded into \( (\hat{M}, \nu \hat{\omega}) \) via \( \psi' = \psi_\nu \) for \( 0 < \nu < 1 + \theta'_1 \), and it finishes the proof of Step 2. \( \Box \)

**Step 3.** Proof of Theorem A for the case that \( ||F|| < \varphi(\Sigma, \lambda) \) and \( \Sigma \) splits \( M \) into two components.

**Proof.** Let the Hofer norm of \( F \) be less than \( \varphi(\Sigma, \lambda) \). Thus we pick \( \theta'_1 > -1 \) satisfying

\[
||F|| < \frac{\rho_1 - \theta'_1}{1 + \rho_1} \varphi(\Sigma, \lambda).
\]

Then we can symplectically embed \( \Sigma \times (\theta'_1, \theta_2) \) to the symplectic manifold \( (\hat{M}, \nu \hat{\omega}) \) with \( 0 < \nu < 1 + \theta'_1 \) by Step 2. Thus Step 1 enable us to find a leafwise intersection point. \( \Box \)

**Step 4.** End of the proof of Theorem A.

**Proof.** In the proof of Step 4, our contact hypersurface \( \Sigma \) need not bound a compact region in \( M \). We consider a family of time-dependent Hamiltonian functions \( H_\nu \in C^\infty(S^1 \times M) \) for \( \nu \in \mathbb{N} \) where \( H_\nu(t, x) = \chi(t)G_\nu(x) \) such that

1. \( 0 < \epsilon_\nu < \min\{1, \delta_0/2\} \) converges to zero as \( \nu \) goes to infinity,
2. \( x \in \Sigma \),
3. \( G_\nu|_{M - U_{\delta_0}} = \text{constant} \),
4. \( G_\nu^{-1}(0) = \Sigma \times \{-\epsilon_\nu, \epsilon_\nu\} =: \Sigma_{-\epsilon_\nu} \cup \Sigma_{\epsilon_\nu} \).

We note that \( X_{G_\nu}|_{\Sigma_{\pm \epsilon_\nu}} = \pm R_{\pm \nu} \) where \( R_{\pm \nu} \) is the Reeb vector field on \( \Sigma_{\pm \epsilon_\nu} \), and we denote by \( \phi^t_{R_{\pm \nu}} \) the flow of the Reeb vector field \( R_{\pm \nu} \). Then according to Proposition 3.1, one of the followings holds: For \( (v_\nu, \eta_\nu) \in \text{Crit} A^{H_\nu}_F \),

\[
\phi^1_F(v_\nu(\frac{1}{2})) = v_\nu(0) = \phi^{\eta_\nu}_{R_{\pm \nu}}(v_\nu(\frac{1}{2})) \quad \text{or} \quad (4.10)
\]

\[
\phi^1_F(v_\nu(\frac{1}{2})) = v_\nu(0) = \phi^{\eta_\nu}_{R_{-\pm \nu}}(v_\nu(\frac{1}{2})). \quad (4.11)
\]
Given a perturbation $F$ with $||F|| < \varphi(\Sigma, \lambda)$, the following holds for a sufficiently large $\nu \in \mathbb{N}$.

$$||F|| < \min \{ \varphi(\Sigma_{-\epsilon}, (1 - \epsilon)\lambda), \varphi(\Sigma_{+\epsilon}, (1 + \epsilon)\lambda) \};$$

then Step 1, 2, and 3 guarantee the existence of critical points $(v_\nu, \eta_\nu)$ of $A_{F,\nu}^H$. For clarity, let $n_\nu$ be $-\eta_\nu$ resp. $\eta_\nu$ and $N_\nu$ be $R_{+\nu}$ resp. $R_{-\nu}$ if (4.10) resp. (4.11) holds. Thus we have

$$\phi^1_F(v_\nu(1/2)) = \phi^1_{N_\nu}(v_\nu(1/2)).$$

Then estimation (4.13) in Step 1 implies the following lemma.

**Lemma 4.1.** $n_\nu$ is uniformly bounded in terms of $\lambda$ and $F$.

**Proof.** We estimate like (4.13).

$$||F|| \geq |A_{F,\nu}^H(v_\nu, \eta_\nu)|$$

$$= \left| \int_0^1 v^*\lambda + \int_0^1 H_\nu(t, v_\nu(t))dt + \int_0^1 F(t, v_\nu(t))dt \right|$$

$$= \left| \int_0^1 \chi(t)\lambda(v_\nu) (\pm \eta_\nu R_{\pm\nu}(v_\nu) + X_F(t, v_\nu)) dt + \int_0^1 F(t, v_\nu(t))dt \right|$$

$$= \left| \frac{\pm \eta_\nu}{1 \pm \epsilon_\nu} + \int_0^1 \lambda(v_\nu) (X_F(t, v_\nu)) + \int_0^1 F(t, v_\nu(t))dt \right|.$$

Therefore we conclude

$$|n_\nu| = |\eta_\nu| \leq 2||F|| + 2||\lambda|_{[v_0/2]}||_{L^\infty} ||X_F||_{L^\infty} + 2||F||_{L^\infty}. \quad (4.14)$$

The two sequences of points $\{v_\nu(0)\}_{\nu \in \mathbb{N}}$ and $\{v_\nu(1/2)\}_{\nu \in \mathbb{N}}$ converge and we denote by

$$x_0 := \lim_{\nu \to \infty} v_\nu(0), \quad x_{1/2} := \lim_{\nu \to \infty} v_\nu(1/2). \quad (4.15)$$

Obviously $x_0$ and $x_{1/2}$ are points in $\Sigma$. Moreover we know that

$$x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi^1_F(v_\nu(1/2)) = \phi^1_F(\lim_{\nu \to \infty} v_\nu(1/2)) = \phi^1_F(x_{1/2}). \quad (4.16)$$

Furthermore, due to Lemma 4.1 we have a limit of $\{n_\nu\}_{\nu \in \mathbb{N}}$.

$$\lim_{\nu \to \infty} n_\nu =: n. \quad (4.17)$$

Thus we conclude that $x_0$ and $x_{1/2}$ lie on a same leaf:

$$x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi^1_{N_\nu}(v_\nu(1/2)) = \phi^1_{N}(x_{1/2}). \quad (4.18)$$

It directly follows

$$\phi^1_{N}(x_{1/2}) = \phi^1_F(x_{1/2}) \quad (4.19)$$

from equation (4.16) together with (4.17).

On the other hand, we consider a perturbation $F$ with $||F|| = \varphi(\Sigma, \lambda)$. Set $F_\mu := \mu \cdot F$ for $\mu \in [0, 1)$ then $||F_\mu|| < \varphi(\Sigma, \lambda)$. By previous Step 1, 2, and 3 so far, we know the existence of a critical point of $A_{F_\mu}^H$, namely $(v_{\nu,\mu}, \eta_{\nu,\mu}) \in \text{Crit} A_{F_\mu}^H$. Using the same calculation in the proof of Lemma 4.1 we note that $\eta_{\nu,\mu}$ is uniformly bounded. Thus due to the Arzela-Ascoli’s
Corollary 5.2. Therefore the assertion (i) follows from the exactly same argument in Proposition 3.5.

Proof. Proofs of (i) and (ii) are almost the same as Proposition 3.5 and Proposition 2.2 so that using the computation in Proposition 2.2. We have assumed that \( \phi \) and \( \partial_s \phi \) are sufficiently smooth and \( \phi \) is a solution of the Hamiltonian system (2.2) for some Hamiltonian function \( H \). Then we obtain a global one form \( \beta = \varphi(r) \lambda \). Furthermore, we define action functionals \( A_{F_3}^H, \tilde{A}_{F_3}^H \) and \( A \) again with the new \( \beta \) as before.

**Proposition 5.1.** If \((v, \eta) \in L \times R \) and \((\hat{v}, \hat{\eta}) \in T_{(v, \eta)}(L \times R) \) then the following two assertions hold.

(i): \( d\tilde{A}_{F_3}^H(v, \eta)(\hat{v}, \hat{\eta}) = \tilde{m}\left( \nabla_m A_{F_3}^H(v, \eta), (\hat{v}, \hat{\eta}) \right) \),

(ii): \( (m - \tilde{m})(\hat{v}, \hat{\eta}) \geq 0 \).

**Proof.** Proofs of (i) and (ii) are almost the same as Proposition 3.3 and Proposition 2.2 respectively. We have chosen \( \varphi \) so that \( \varphi'(r) = 1 \) on \( \Sigma \times [\varrho^-, \varrho^+] \) and we have \( \varphi(r) < r + 1 \), \( \varphi'(r) \leq 1 \) for all \( r \in \mathbb{R} \) since \( \varrho^- > -1 \). Thus the assertion (ii) follows from

\[
\frac{d\beta(u, Ju)}{Ju} \leq \omega(u, Ju)
\]

using the computation in Proposition 2.2. We have assumed that \( X_F \) is spanned by \( R \) and \( Y \) so that

\[
i_{X_F} d\beta = i_{X_F} (dr \wedge \lambda) = i_{X_F} \omega.
\]

Therefore the assertion (i) follows from the exactly same argument in Proposition 3.3.

**Corollary 5.2.** The action value of the functional \( A = \tilde{A}_{F_3}^H - A_{F_3}^H \) is nondecreasing along a gradient flow line of \( A_{F_3}^H \).

**Proof.** Using Proposition 5.1 we estimate with a gradient flow line \( w(s) \) of \( A_{F_3}^H \).

\[
\frac{d}{ds} A(w(s)) = \frac{d}{ds} \left( \tilde{A}_{F_3}^H(w(s)) \right) - \frac{d}{ds} \left( A_{F_3}^H(w(s)) \right)
\]

\[
= d\tilde{A}_{F_3}^H(w)(\partial_s w) + (\partial_s \tilde{A}_{F_3}^H)(w) - dA_{F_3}^H(w)(\partial_s w) - (\partial_s A_{F_3}^H)(w)
\]

\[
= m(\partial_s w, \partial_s w) - \tilde{m}(\partial_s w, \partial_s w) + \int_0^1 \partial_s F_3(t, v) dt - \int_0^1 \partial_s F_3(t, v) dt
\]

\[
\geq 0.
\]
Corollary 5.3. \( A(w(s)) \) is identically zero for all \((r, w) \in M\), the moduli space defined in the proof of Theorem A.

Proof. We note that \( A(w_+\) = \( A(w_-\) = 0 since \( w_\pm = w(\pm\infty) \) are constants. Therefore the proof immediately follows from the previous corollary. \(\square\)

Proposition 5.4. Assume that \((r, w) = (r, v, \eta)\) is an element in \( M \). Then \( v \in C^\infty(\mathbb{R}, \mathcal{L}) \) remains in \( \Sigma \times [\varrho^- , \varrho^+] \).

Proof. Let us investigate the case that \( v(s, t) \) goes out of the region \( \Sigma \times [\varrho^- - \epsilon, \varrho^+ + \epsilon] \). Assume that \( v(s, t) \) does not lie in \( \Sigma \times [\varrho^- - \epsilon, \varrho^+ + \epsilon] \) for \( s_- < s < s_+ \). It means that there exists a nonempty open subset \( U \subset Z := (s_-, s_+) \times S^1 \) such that \( v(s, t) \in \Sigma \times ((-1, \varrho^- + \epsilon) \cup (\varrho^+ + \epsilon, \infty)) \) for \( (s, t) \in U \).

Using the previous corollary, we calculate
\[
0 = \int_{s_-}^{s_+} \frac{d}{ds} A(w(s)) = \int_{s_-}^{s_+} \int_0^1 (\omega - d\beta)(\partial_s v, J(v)\partial_s v)dt ds.
\]
(5.5)

The last equality holds since \( d\beta \) vanishes on \( \Sigma \times ((-1, \varrho^- - \epsilon) \cup (\varrho^+ + \epsilon, \infty)) \). However, \( (\omega - d\beta)(\partial_s v, J(v)\partial_s v) \) is bigger or equal to zero by the assertion (ii) in Proposition 5.1 and \( \int_U \omega(\partial_s v, J(v)\partial_s v)dtds > 0 \). Thus this case cannot occur and accordingly every gradient flow line of \( A_{HF}^H \) satisfying \( w(-\infty) = (p, 0) \) and \( w(\infty) \in \Sigma \) lies in \( \Sigma \times [\varrho^- - \epsilon, \varrho^+ + \epsilon] \). Taking the limit \( \epsilon \to 0 \), this finishes the proof of the proposition. \(\square\)

Remark 5.5. In the case that \( v(s, t) \) for \( s_- < s < s_+ \) goes entirely out the region \( \Sigma \times [\varrho^- , \varrho^+] \), we can show the above proposition more easily by using the energy argument. Since \( d\beta \) and \( H \) vanish outside of \( \Sigma \times [\varrho^- , \varrho^+] \), we calculate
\[
E(w)^{s_+}_{s_-} = \int_{s_-}^{s_+} ||\partial_s w||^2 ds = \int_{s_-}^{s_+} \frac{d}{ds} A(w(s)) \quad \text{(5.6)}
\]

It yields that \( w \) is constant when \( v \) is in the outside of the region, but such \( w \) never exist.

Proof of Theorem B. The previous proposition enable us to overcome the following problems, namely the \( L^\infty \)-bound on \( v \) and the \( L^\infty \)-bound on the derivatives of \( v \) although the symplectization of \( \Sigma \) is not convex at infinity. The \( L^\infty \)-bound on \( \eta \) is almost the same as what we showed and therefore Theorem 5.3 follows. Even easier, since \( \omega \) is exact on the symplectization, the bound follows from [AF1]. Hence Theorem A guarantees the existence of a leafwise intersection point. \(\square\)
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