Optical scalars in spherical spacetimes

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Abstract

Consider a spherically symmetric spacelike slice through a spherically symmetric spacetime. One can derive a universal bound for the optical scalars on any such slice. The only requirement is that the matter sources satisfy the dominant energy condition and that the slice be asymptotically flat and regular at the origin. This bound can be used to derive new conditions for the formation of apparent horizons. The bounds hold even when the matter has a distribution on a shell or blows up at the origin so as to give a conical singularity.

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Relativists, especially those who are numerically inclined [1], have long known that regular spacelike slices often wrap around singularities rather than approaching them. In this letter we derive a new and remarkable relation which gives a bound on the optical scalars and which shows how slices which are asymptotically flat may be prevented from coming close to singularities.

Consider a spacelike slice through spacetime. The geometry of this slice cannot be chosen at will; it must satisfy the constraint equations. In the spherically symmetric case these constraints can be written as equations for the optical scalars. These equations, combined with the requirement of regularity at the origin and at infinity, force the optical scalars to remain bounded over the entire slice. The optical scalars are four-dimensional objects which we expect to become unboundedly large as one approaches a singularity. Thus regular spacelike slices are excluded from regions near singularities. This bound on the optical scalars also has a more immediate use. Over the years, we have been interested in developing criteria to determine when and if apparent horizons form [2,3]. In spherically symmetric systems the existence of an apparent horizon implies the existence of a black hole [4,5]. These bounds on the optical scalars allow us sharpen significantly our condition for the formation of apparent horizons.

We define a spherically symmetric spacetime as one having the metric

$$ds^2 = -\alpha^2(r,t)dt^2 + a(r,t)dr^2 + b(r,t)r^2d\Omega^2$$

(1)

where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta < \pi$ are the standard angle variables such that $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

The initial data for the Einstein equations are prescribed by giving the spatial geometry at $t = 0$, i.e., by specifying the functions $a(r,0)$ and $b(r,0)$, and by giving the extrinsic curvature (again at $t = 0$)

$$K^r_r = \frac{\partial_t a}{2a\alpha}, \quad K^\theta_\theta = K^\phi_\phi = \frac{\partial_t R}{\alpha R}, \quad trK = \frac{\partial_t (\sqrt{ab})}{\sqrt{ab\alpha}}$$

(2)

where the areal (Schwarzschild) radius $R$ is defined as
\[ R = r \sqrt{b} . \]  \hfill (3)

It is useful to define the mean curvature of a centered sphere in the initial hypersurface by

\[ p = \frac{2 \partial_r R \sqrt{a}}{\sqrt{a}} . \]  \hfill (4)

In a general spacetime the behaviour of a pencil of light rays is described by specifying a number of functions which describe the expansion and shear of the rays. In a spherically symmetric spacetime, however, we need specify only two. These objects can be expressed in terms of the initial data on any spacelike slice so they are simultaneously three-dimensional and four-dimensional scalars. These optical scalars are the divergence of future directed light rays

\[ \theta = \frac{2}{R} \frac{d}{\alpha dt} R = p - K_r^r + tr K , \]  \hfill (5)

and the divergence of past directed light rays

\[ \theta' = \frac{-2}{R} \frac{d}{\alpha dt} R = p + K_r^r - tr K , \]  \hfill (6)

where \( \frac{d}{\alpha dt} in = \frac{\partial}{\alpha} - \frac{\partial}{\sqrt{a}} \) and \( \frac{d}{\alpha dt} out = \frac{\partial}{\alpha} + \frac{\partial}{\sqrt{a}} \) are the full derivatives in the direction orthogonal to the centered sphere of ingoing and outgoing photons respectively. In flat space-time both quantities are positive and equal to \( 2/R \), where \( R \) is the radius of a sphere; hence each of the products \( R\theta \) and \( R\theta' \) equals 2.

The initial data must satisfy the constraints. These constraints, expressed in terms of \( \theta \) and \( \theta' \), can be written as

\[
\begin{align*}
\partial_t (\theta R) &= -8\pi R \rho - j_r \sqrt{a} - \frac{1}{4R} \left( \theta^2 R^2 - 4 - 4\theta tr KR^2 + \theta R (\theta R - \theta' R) \right), \\
\partial_t (\theta' R) &= -8\pi R \rho + j_r \sqrt{a} - \frac{1}{4R} \left( \theta'^2 R^2 - 4 + 4\theta' tr KR^2 + \theta' R (\theta' R - \theta R) \right),
\end{align*}
\]  \hfill (7) \hfill (8)
where \( l \) is the proper distance from the center, i.e., \( dl = \sqrt{a}dr \). \( \rho \) and \( j_r \) are the energy density and the current density of the sources that generate the gravitational field. Note that \( j_r/\sqrt{a} \) equals \( j.n \) where \( n \) is the unit normal in the radial direction. We will assume that the sources satisfy the dominant energy condition, \( \rho \geq |j| \). If the origin is regular, local flatness forces both optical scalars to satisfy the conditions \( \lim_{R \to 0} \theta R = \lim_{R \to 0} \theta' R = 2 \).

Asymptotic flatness also gives \( \lim_{R \to \infty} \theta R = \lim_{R \to \infty} \theta' R = 2 \).

The primary result of this calculation is a proof that if \( \theta R, \theta' R \) are bounded at the origin and at infinity they are bounded on the entire hypersurface. Define \( B = 4 \sup_{0 \leq r \leq \infty} (|RtrK|) \).

We prove

**Lemma 1.** Given spherical initial data that are regular at the origin and at infinity with sources that satisfy the dominant energy condition, both optical scalars are bounded on the entire hypersurface

\[-2 - B \leq \theta R, \quad \theta' R \leq 2 + B. \tag{9}\]

**Proof:** Let us assume that \( \theta R \geq 2 + B \) and \( \theta R \geq \theta' R \). Consider the non-source part of eqn.(7), i.e., \( (\theta^2 R^2 - 4 - 4\theta trKR^2 + \theta R(\theta R - \theta' R)) \). Since \( \theta R \geq 2 + B \), the first three terms are nonnegative while \( \theta R \geq \theta' R \) means that the last term is nonnegative. Therefore eqn.(7) implies that \( \partial_l(\theta R) \leq 0 \). Also, if \( \theta' R \geq 2 + B \) and \( \theta' R \geq \theta R \), a similar analysis of eqn.(8) gives \( \partial_l(\theta' R) \leq 0 \). Therefore, if \( \max(\theta R, \theta' R) \geq 2 + B \) then \( \partial_l[\max(\theta R, \theta' R)] \leq 0 \). Since the maximum starts at 2, and the derivative at \( 2 + B \) is negative, it cannot rise above \( 2 + B \). Hence \( 2 + B \) is an upper bound.

The argument for the lower bound works in exactly the same way. Let us assume that \( \theta R \leq -2 - B \) and that \( \theta R \leq \theta' R \). Again eqn.(7) means that \( \partial_l(\theta R) \leq 0 \). Hence if \( \min(\theta R, \theta' R) \leq -2 - B \) then \( \partial_l[\min(\theta R, \theta' R)] \leq 0 \). This means that one or the other of \( (\theta R, \theta' R) \) is driven more and more negative. However, asymptotic flatness demands that both rise up to \( +2 \) at infinity. Contradiction!

Let us stress that while \( B \) is a three-scalar which depends on the particular spacelike slice, \( \theta, \theta' \) and \( R \) are four-dimensional scalars, properties of the spacetime, which are independent
of the choice of foliation. Thus eqn.(9) places restrictions on the kind of regular spacelike slice that may enter particular regions of spacetime.

There are only two allowed topologies for globally regular, asymptotically flat, spherically symmetric, spacelike three-manifolds. They can either have $R^3$ topology with a regular center and one asymptotic end or $R \times S^2$ topology with two asymptotic ends, as in the Schwarzschild geometry. Lemma 1 holds in both cases.

Lemma 1 has a number of interesting consequences. Let us assume, for a moment, that the trace of the extrinsic curvature vanishes, i.e., that the initial data define a maximal slice. This means that $B \equiv 0$ and Lemma 1 implies that $|\theta R|, |\theta' R| \leq 2$. A surface on which $\theta < 0$ is called a trapped surface; such surfaces play a key role in the singularity theorems of general relativity. Eqn.(7) can be used to derive

$$\partial_l (\theta R^2) = -8\pi R^2 (\rho - \frac{j_r}{\sqrt{a}}) + 1 + \frac{1}{4} \theta R (2\theta' R - \theta R). \quad (10)$$

Let $L(S)$ be the geodesic (proper) radius of a sphere $S$; $R(S)$ its areal radius; $M(S) = \int_{V(S)} \rho dV$ the total mass inside $S$; and $P(S) = \int_{V(S)} \frac{j_r}{\sqrt{a}} dV$ be the total radial momentum. Integrating (10) noting that $4\pi R^2 dl = 4\pi \sqrt{a} R^2 dr$ is the proper volume, we get

$$(\theta R^2)(S) = -2(M - P)(S) + L(S) + \frac{1}{4} \int_0^{L(S)} \theta R (2\theta' R - \theta R) dl. \quad (11)$$

We can see that $\frac{1}{4} \int_0^{L} dl \theta R (2\theta' R - \theta R) \leq \frac{1}{4} \int_0^{L} dl (\theta' R)^2 \leq L$, where the first inequality comes from the trivial estimate $2ab - a^2 \leq b^2$ and the second from Lemma 1. Therefore

$$(\theta R^2)(S) \leq -2(M - P)(S) + 2L(S) \quad (12)$$

for any surface $S$. In particular, if $M - P \geq L$ at any given sphere $S$ then $\theta(S)$ must be negative. Thus we have proven:

**Theorem 1.** Under conditions of Lemma 1, assuming $tr K \equiv 0$, if the difference between the total rest mass $M(S)$ and the radial momentum $P(S)$ exceeds the proper radius $L(S)$ of a sphere $S$, $M(S) - P(S) > L(S)$, then $S$ is trapped.
This theorem improves our earlier result [2], in which we got a similar result but with $L$ replaced by $\frac{7}{6}L$ and the weaker conclusion that there exists a trapped surface inside $S$. The difference is due to the fact that we now impose the somewhat stronger condition that $\rho - |j| \geq 0$, whereas in [2] we used $\rho + \frac{3}{32\pi}(K^r_r)^2 \geq 0$. Since the new conditions in Theorem 1 eliminate tachyons this is a real difference. The constant $\frac{7}{6}$ also appears in our criteria for the formation of cosmological black holes [3]; we believe that these can also be improved to 1.

The meaning of Theorem 1 is transparent. Radially ingoing matter $j_r \leq 0$ helps form apparent horizons. The presence of outgoing matter, i.e., when $P(S)$ becomes positive, has to be compensated for by a greater matter density. In the extremal case of radially outgoing photons, when $M(S) = P(S)$, apparent horizons cannot form. This follows from our Theorem 2 below.

Theorem 1 is sharp in the sense that there exists an initial value configuration when the inequality saturates. This is a 3-geometry created by a shell of moving matter; the explicit calculation will be done elsewhere. The case in which $P = 0$ was discussed in [2] and the corresponding criterion (with the same constant 1, as above) was shown to be the best possible.

It is interesting that we obtain an exact criterion with the constant 1; this suggests that Theorem 1 is part of a more complex true statement that can be formulated for general nonspherical spacetimes. It suggests also that $M(S)$ is a sensible measure of the energy of a gravitational system that might appear as a part of a quasilocal energy measure in nonspherical systems.

We also obtain a necessary condition for the formation of apparent horizons. In [3] we found a criterion based on asymptotic data outside a collapsing system. [2] states that $M(S) > \frac{L}{2}$ must be satisfied if $S$ is trapped in the case of moment of time symmetry data. The same holds true if the matter is moving under some stringent conditions on the sign of the momentum density [4]. Here we will derive a different (and not particularly interesting, although exact) estimate. The most important assumption we make is that $\theta'$ is everywhere
positive on the initial hypersurface. Just as $\theta \leq 0$ guarantees a singularity to the future, $\theta' \leq 0$ guarantees a singularity to the past. Therefore, data which arises from a regular past must have positive $\theta'$.

**Theorem 2.** Assume a regular maximal slice on which the sources satisfy the dominant energy condition. Let $S$ be the innermost trapped surface and let $(R\theta') > \epsilon > 0$ inside $S$. Then

$$M(S) - P(S) \geq \frac{\epsilon L}{2}.$$  

**Proof.** As before, we consider (11), which reads

$$\theta R^2 = -2(M - P) + L + \frac{1}{4} \int_0^L dl \theta R (2\theta' R - \theta R).$$  \hspace{1cm} (13)$$

Inside $S$, $R\theta$ is positive. We seek a lower bound on the last term on the right hand side of (13). Let $t = R\theta$, $u = R\theta'$; from Lemma 1 we know $|t|, |u| \leq 2$, so our task consists in estimating $2tu - t^2$ for $0 \leq t \leq 2, \epsilon \leq u \leq 2$. We know that $2tu - t^2 \geq F(t) = 2t\epsilon - t^2$. The only extremum of $F(t)$ is a maximum at $t = \epsilon$. The minimum must occur at the endpoints and it is easy to show that $2tu - t^2 \geq F(t) \geq 4\epsilon - 4$. Inserting this into (13) yields

$$\theta(S) R^2 \geq -2(M - P)(S) + L(S) + \frac{1}{4} \int_0^{L(S)} dl (4\epsilon - 4)$$

$$= -2(M - P)(S) + \epsilon L,$$  \hspace{1cm} (14)$$

that is, since $\theta(S) = 0$,

$$M(S) - P(S) \geq \frac{\epsilon L}{2}.$$  

Hence Theorem 2 is proven.

The inequality of Theorem 2 becomes an equality in the case of a spherical shell. The geometry inside the shell is flat and $\theta' R = 2$. The necessary condition that the shell be trapped is that $M - P > L$. In [2] we proved this in the special case when $P = 0$.

It is clear that the analysis performed here can include cases where the sources are distributions rather than classical functions; in particular, we have no difficulty with shells of matter. All we get on crossing the shell is a downward step in $\theta$ and $\theta'$. More interestingly, we can extend the analysis to include weak singularities at the origin.
Let us begin by considering a conical singularity \[8\]. Consider a metric of the form

\[ dS^2 = dr^2 + a^2 r^2 d\Omega^2. \]  

(15)

The scalar curvature of this metric is \((3) R = 2(1 - a^2)/r^2\). A moment of time symmetry data set is one for which \(j^i\) and \(K^{ij} \equiv 0\). For such data sets the constraints reduce to \((3) R = 16\pi \rho\). For the above metric we get \(\rho = (1 - a^2)/8\pi r^2\). The dominant energy condition reduces to the positivity of \(\rho\), which implies \(a^2 \leq 1\). For this metric we can also compute the mean curvature \(p\), which in this case equals both \(\theta\) and \(\theta'\), to get \(p = 2/r = 2a/R\). Hence we get \(|pR| \leq 2\). However, the argument of Lemma 1 only requires that \(\theta R, \theta'R\) be bounded at the origin. Therefore we have shown that Lemma 1 holds for moment of time symmetry data with a conical singularity at the origin. The conical singularity in question is determined by the deficit of the solid angle \(4\pi (1 - a^2)\). We will show that a similar result holds true for general nonmaximal data.

Let us consider initial data such that \(trK\) is finite while \(R\theta \to X\) and \(R\theta' \to Y\) as \(R \to 0\). Let us also assume that \(\partial_l (R\theta)\) and \(\partial_l (R\theta')\) are finite at \(R = 0\). There are terms on the right hand side of eqns.(7) and (8) which seem to diverge like \(1/R\). The source term will have the same sort of \(1/R\) divergence if \(8\pi R^2 \rho \to \alpha\) and \(8\pi R^2 j_r/\sqrt{a} \to \beta\), just as in the case of the conical singularity. The coefficient of this \(1/R\) term must vanish. This gives us a pair of equations, one from (7) and one from (8)

\[ \alpha - \beta + \frac{1}{2}X^2 - \frac{1}{4}XY - 1 = 0; \]  

(16)

\[ \alpha + \beta + \frac{1}{2}Y^2 - \frac{1}{4}XY - 1 = 0. \]  

(17)

By adding these equations we get

\[ 4\alpha = 4 - X^2 - Y^2 + XY; \]  

(18)

and by subtracting

\[ 4\beta = X^2 - Y^2. \]  

(19)
Note that eqn.(18) implies that $\alpha \leq 1$. The weak energy condition gives $\alpha \geq |\beta|$. Let us assume that $\beta \geq 0$. Eqn.(19) now gives us $X^2 \geq Y^2$ and $Y = \pm \sqrt{X^2 - 4\beta}$. Substituted this into eqn.(18) to give

$$[3X^2 - 4(1 - \alpha + \beta)][X^2 - 4(1 - \alpha + \beta)] + 4X^2\beta^2 = 0. \tag{20}$$

The roots of this equation, if it has any, must lie in the range $4(1 - \alpha + \beta)/3 \leq X^2 \leq 4(1 - \alpha + \beta)$. Therefore we have shown that $2 \geq |X| \geq |Y|$. If we assume $\beta < 0$, we just reverse the roles of $X$ and $Y$. Hence we obtain

**Lemma 2.** Given $\rho \geq |j|$ and if all of $trK, \theta R, \theta' R, \partial_i \theta R, \partial_i \theta' R, (8\pi \int_0^R \rho \tilde{R}^2 d\tilde{R})/R$ are finite in the limit $R = 0$ then

$$2 \geq \lim_{R \to 0} |\theta R|, \lim_{R \to 0} |\theta' R|, \quad 1 \geq \frac{8\pi \int_0^R \rho \tilde{R}^2 d\tilde{R}}{R}. \tag{21}$$

From eqns.(4), (5) and (6) it is clear that

$$2\partial_i R = \frac{2\partial_i R}{\sqrt{\alpha}} = pR = \frac{\theta R + \theta' R}{2}. \tag{22}$$

This means that the spatial part of the metric (1) can be written, at least in a small neighbourhood of $R = 0$, as

$$\frac{16}{(R\theta + R\theta')^2} dR^2 + R^2 d\Omega^2. \tag{23}$$

The estimate derived in lemma 2 implies that, under the stated conditions, there can be at most a conical singularity at the origin, with solid angle deficit $4\pi(1 - \frac{(X+Y)^2}{16})$. Conical singularities have previously been investigated in 2+1 gravity \[9\]. In the 2+1 case the conical singularity can also be described by an angle deficit expressed in terms of the mean curvature: $2\pi(1 - pR)$. However, in the 2+1 case the geometry is locally flat but globally nontrivial and the deficit angle is related to a total mass \[9\]. In our case, the deficit angle is a local phenomenon caused by a mildly singular mass distribution at the origin, where $\rho$ diverges like $r^{-2}$.

Lemma 2 gives the desired bound $|\theta R|, |\theta' R| \leq 2$ at the origin so we get a generalized version of Lemma 1:
Lemma 1’. Assume an asymptotically flat nonmaximal slice, satisfying the dominant energy condition, such that \(4 \sup_{0 \leq R \leq \infty} |RtrK| = B\) is finite. Let the conditions of Lemma 2 be satisfied at the origin. Then

\[2 + B \geq |\theta R|, |\theta' R|.
\]

Theorems 1 and 2 hold under similar conditions.

As we have mentioned earlier, \(\theta R\) and \(\theta' R\) are defined for any point in a spherically symmetric spacetime geometry, independent of any foliation or choice of time. One consequence of Lemma 1 is that if a point exists in a spherical spacetime for which either \(|\theta R|\) or \(|\theta' R|\) is larger than 2 then we know that a regular, maximal, asymptotically flat slice cannot pass through this point.

Consider regular, asymptotically flat, spherically symmetric initial data which contain an apparent horizon. Let us now evolve the spacetime and look at the maximal Cauchy development of this data. We are guaranteed that a singularity will occur for a sufficiently large value of local proper time. It seems to us that there are only three realistic outcomes:

i) The singularity will be of the kind where \(R\theta \to -\infty\), as in the Schwarzschild singularity. Maximal slices (and any other slicing with a regular trace of the extrinsic curvature) do not cover the full Cauchy evolution. We get a “collapse of the lapse”. The foliation can continue for infinite time as seen by asymptotic observers but “freezes” near the interior.

ii) We get some sort of “bag of gold” forming, where \(R\) goes to zero at some finite value of the proper radius, \(L\), and part of the spacetime pinches off from the rest.

iii) A singularity appears with diverging mass density. This may be a shell-crossing singularity, a central conical singularity as we discussed above or some sort of ‘strong’ central singularity. We expect that the appearance (or otherwise) of these singularities would be determined by the matter equation of state.

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