Spin with Inertia

Toru Kikuchi

RIKEN Center for Emergent Matter Science (CEMS),
2-1 Hirosawa, Wako, Saitama 351-0198, Japan

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Abstract

We consider introducing finite moments of inertia to spin. Such inertia generally arises in spin effective dynamics, that is, when we incorporate the effect of environmental degrees of freedom (typically, conducting fermions) interacting with the spins. When the inertia is finite, new spin precession mode emerges, which is intrinsic to the system and caused by the spin Berry curvature itself. We discuss the effect of this inertia on resonance, spin waves and domain wall dynamics. We also discuss the equivalence between the dynamics of spin and those of a spinning top, which becomes explicit when we introduce the inertia.

1 Introduction

Particles have three fundamental quantities: mass, charge and spin. Different from the other two, spin itself is a dynamical quantity, and its dynamics have been widely studied and applied in science and technology. In particular, recent rapid growth of spintronics provides a stage where deeper understandings of spin dynamics directly lead to practical applications.

As is well known, the dynamics of spin are governed by the spin Berry phase term, and its equation of motion includes only the first-order time derivative of spin. This is natural because spin is an angular momentum and its equation of motion takes the familiar form, \((d/dt)(\text{angular momentum}) = (\text{torque})\). Without any torque, the solution of spin is only a static one. This is in contrast with, for example, the case of a massive point particle, which has inertia and can move at non-zero speed as its free motion. In this sense, spin
does not have inertia. In other words, the equation of motion of spin does not contain its second-order time derivative.

This inertia-less property of spin is characteristic of the dynamics in its original form. For systems where spin interacts with other degrees of freedom (typically, conducting electrons), the dynamics of spin are changed to be effective ones, which include the effects of such environmental degrees of freedom. For example, in ferromagnets, conducting electrons affect the dynamics of localized spins (spins of atoms localized at lattice cites). The effects are incorporated into the equation of motion of the spins, often perturbatively in powers of the derivative of the spins. The lowest order in that derivative expansion gives spin-transfer torque \[2, 3, 4, 5, 6, 7, 8\] and spin-relaxation torque \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]. Since these torques contain at most the first-order time derivative of the spins, the spins still do not have inertia at this order. However, in the next order of the derivative expansion, there should appear a term which contains the second-order time derivative of the spins, and this term will play the role of inertia of spin.

In this paper, we discuss spin effective dynamics with finite inertia: their basic properties and relevant phenomena. We take ferromagnets as a physical example, but the discussion itself will be more general. First, we perform derivative expansion of spin effective action and see that there appears the inertia of spin (section 2). The dynamics of spin with finite inertia are shown to be equivalent to those of a symmetric spinning top: the inertia of spin corresponds to the moment of inertia of the spinning top (section 3). Therefore, spin with inertia undergoes free precession or nutation as its intrinsic precession mode, just as a spinning top does (section 4). This intrinsic precession mode is caused by the spin Berry curvature itself, in contrast with that usual spin precession is caused by applied or effective magnetic fields. From phenomenological viewpoints, prediction of the existence of this intrinsic precession mode is the main content of this paper (but the numerical value of the frequency of this precession for ferromagnets turns out to be too high at the level of the analysis in this paper). We discuss experimentally observable phenomena involving the inertia, such as resonance, spin waves and domain wall motion (section 5).

2 Spin effective action and inertia

It is simple to see how the inertia of spin arises in its effective dynamics. Consider a system where spin \(n\) and the other environmental degrees of freedom \(c\) are interacting with each other. The total action is given by \(S_s[n] + S_e[c, n]\) where \(S_s[n]\) is the action of spin and
$S_c[c, n]$ is that of $c$ with its interaction with spin $n$. When we are interested only in the dynamics of spin $n$, it is convenient to integrate out $c$ and treat the effective action of spin $n$. The contribution from $c$ is given by path integration as

$$\exp\left(\frac{i}{\hbar}\Delta S_{\text{eff}}[n]\right) \equiv \int\mathcal{D}c \exp\left(\frac{i}{\hbar}S_c[c, n]\right). \quad (2.1)$$

It is difficult to calculate $\Delta S_{\text{eff}}[n]$ exactly, so we should perform some perturbative analysis. We here use derivative expansion, where $\Delta S_{\text{eff}}[n]$ is expanded in powers of $\partial_\mu n$ ($\mu = t, x, y, z$). The general form is

$$\Delta S_{\text{eff}}[n] = \int \frac{d^3x}{a^3} dt \left[ S_c B \cdot n + S_c \dot{\phi}(\cos \theta - 1) - \frac{J_c S_c^2}{2} (\partial_\mu n)^2 + \frac{m_s}{2} n^2 \right] + \mathcal{O}(\partial_\mu n^3). \quad (2.2)$$

with $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. We divide the Lagrangian density entirely by lattice constant $a$ so that each coefficient represents a quantity per each cite. Here, we have assumed for simplicity that the system is isotropic except the applied magnetic field $B$. Otherwise, there would appear other terms such as $\sum_i j^i \partial_i \phi (\cos \theta - 1)$ (which represents spin-transfer torque in ferromagnets) and $\sum_i v^i \partial_i n \cdot \dot{n}$ where $j^i$ and $v^i$ represent anisotropy. Adding this $\Delta S_{\text{eff}}[n]$ to the original spin action of the form

$$S_s[n] = \int \frac{d^3x}{a^3} dt \left[ S_l B \cdot n + S_l \dot{\phi}(\cos \theta - 1) - \frac{J_l S_l^2}{2} (\partial_\mu n)^2 \right], \quad (2.3)$$

we obtain the total spin effective action $S_{\text{eff}}[n] = S_s[n] + \Delta S_{\text{eff}}[n]$: \( (2.4) \)

$$S_{\text{eff}}[n] = \int \frac{d^3x}{a^3} dt \left[ SB \cdot n + S \dot{\phi}(\cos \theta - 1) - \frac{JS^2}{2} (\partial_\mu n)^2 + \frac{m_s}{2} n^2 \right] + \mathcal{O}(\partial_\mu n^3).$$

The first term is the Zeeman coupling, the second is the spin Berry phase and the third is spin-spin interaction. The final term is the inertial term of spin with the inertia $m_s$, coming totally from $\Delta S_{\text{eff}}[n]$.

A typical example of $S_s[c, n]$ is the s-d model \[19\], where conducting electrons $c$ interact with localized spins $n$:

$$S_s[c, n] = \int d^3x dt \bar{c} \left( i\hbar \partial_t + \frac{\hbar^2 \partial^2}{2m} + \epsilon_F + g n \cdot \sigma \right) c. \quad (2.5)$$

Here, $m$ is the mass of the conducting electrons, $\epsilon_F$ is the Fermi energy, and $g$ is the s-d interaction. To obtain derivative expansion for $\Delta S_{\text{eff}}[n]$, we perform $SU(2)$ gauge transformation $c \rightarrow U(x, t)c$ with an $SU(2)$ matrix $U$ acting on spin space, so that the s-d coupling becomes trivial, $\bar{c}(n \cdot \sigma)c \rightarrow \bar{c}c$. Due to this gauge transformation, there appears so-called spin gauge field $A_\mu = -iU^\dagger \partial_\mu U$ in \(2.5\) through $\partial_\mu \rightarrow \partial_\mu + iA_\mu$. This $A_\mu$ contains the
first-order derivative $\partial_{\mu} n$. Therefore, expanding $\exp(iS_{e}[c, n]/\hbar)$ in powers of $A_{\mu}$, we can calculate $\Delta S_{\text{eff}}[n]$ perturbatively in powers of $\partial_{\mu} n$. The inertia $m_{s}$ is calculated as

$$m_{s} = \frac{g}{g^{2} + (\hbar/\tau)^{2}} \frac{\hbar S_{c}}{2} \quad \text{where} \quad S_{c} \equiv \frac{\hbar^{3} k_{F}^{3} + \hbar^{3} k_{F}^{-3}}{6\pi^{2}}$$

with $\hbar^{2} k_{F}^{2}/2m \equiv \epsilon_{F} \pm g$. We include spin-independent elastic impurity scattering and evaluate $m_{s}$ in the lowest power of $\hbar/\epsilon_{F} \tau$, with $\tau$ the lifetime of the conducting electrons. This $m_{s}$ can be rewritten as

$$m_{s} = (k_{F}a)^{3} \frac{\hbar^{2}}{8\pi^{2}} \frac{g^{2}}{g^{2} + (\hbar/\tau)^{2}} \epsilon_{F} f(g/\epsilon_{F}), \quad f(x) \equiv \frac{1}{3x} \left[ (1 + x)^{1/2} - (1 - x)^{1/2} \right].$$

For $0 < x < 1$, the function $f(x)$ is a decreasing function from $f(0) = 1$ to $f(1) \approx 0.94$. Therefore, the value of the inertia is $m_{s} \sim \hbar^{2}/\epsilon_{F}$ when assuming $k_{F}a \sim \pi$ and $\hbar/\tau \lesssim g$. As we see below, spin with finite inertia has a typical precession mode with frequency $\omega_{0} \sim S/m_{s}$. Using $m_{s}$ above and assuming $S \sim \hbar$, this frequency becomes $\hbar \omega_{0} \sim \epsilon_{F} \sim 1\text{eV}$, which is unfortunately too high in this model. We hope that $m_{s}$ becomes bigger somehow in more realistic analysis or in other physical setups, and go on to general behavior of spin with inertia below.

The equation of motion (EOM) derived from the effective action (2.4) is [we drop $(\partial_{i} n)^{2}$ term until section 5]

$$S \dot{n} = -SB \times n + m_{s} \ddot{n} \times n.$$  \hspace{1cm} (2.8)

We see that the inertial term produces acceleration-dependent torque. We can rewrite this EOM, by taking vector product with $n$, as

$$m_{s} \ddot{n} = S n \times \dot{n} + SB - (SB \cdot n + m_{s} \dot{n}^{2}) n.$$  \hspace{1cm} (2.9)

This is identical with the EOM of a charged particle on a sphere under monopole magnetic field, when we identify $n$ as the position of the particle (the last term on the right-hand-side is the constraint force). The inertia of spin just corresponds to the mass of the particle. This ‘spin/charged particle’ analogy is useful throughout this paper.

### 3 Spin and spinning top

A spinning top is often employed as a cartoonish analogue of spin. In this section, we show that this analogy is indeed an exact one, particularly when the inertia of spin is finite.
First, let us write down their Lagrangians. The Lagrangian of spin, eq[2.4], is [dropping $\partial_i n^2$ term]

$$L_{\text{spin}} = \frac{m_s}{2} (\dot{\theta}^2 + 2 \dot{\phi}^2 \sin^2 \theta) + S \dot{\phi} (\cos \theta - 1) + BS \cos \theta,$$

while the Lagrangian of a spinning top in terms of the Euler angles is

$$L_{\text{top}} = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - \mu gl \cos \theta,$$

where $I_1, I_3$ are the moments of inertia, $\mu$ is the mass of the top, $g$ is the gravitational acceleration constant, and $l$ is the distance between the center of mass and the fixed point of the top. Here we have taken a symmetric spinning top and set two moments of inertia equal, $I_1 = I_2$. We take both the external magnetic field $B$ and the gravity in $z$ direction. Note that $L_{\text{spin}}$ is a function of $(\theta, \phi)$ while $L_{\text{top}}$ is that of $(\theta, \phi, \psi)$. Let us see below that they have equivalent dynamics about $(\theta, \phi)$.

The equivalence can be directly seen at the level of their EOMs. The EOM of spin is

$$\theta : \ m_s (\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) + S \dot{\phi} \sin \theta + BS \sin \theta = 0,$$

$$\phi : \ \frac{d}{dt} \left[ m_s \dot{\phi} \sin^2 \theta + S (\cos \theta - 1) \right] = 0,$$

while the EOM of spinning top is

$$\theta : \ I_1 (\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) + I_3 (\dot{\psi} + \dot{\phi} \cos \theta) \dot{\phi} \sin \theta - \mu gl \sin \theta = 0,$$

$$\phi : \ \frac{d}{dt} \left[ I_1 \dot{\phi} \sin^2 \theta + I_3 (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta \right] = 0,$$

$$\psi : \ \frac{d}{dt} \left[ I_3 (\dot{\psi} + \dot{\phi} \cos \theta) \right] = 0.$$  

(3.3)

Because $\psi$ is a cyclic coordinate, its corresponding canonical momentum $M_3 \equiv I_3 (\dot{\psi} + \dot{\phi} \cos \theta)$ is conserved. Substituting this $M_3$ for $\psi$ in $\theta$- and $\phi$-equations in (3.4), we obtain the same equation as (3.3) with replacements

$$m_s \leftrightarrow I_1, \ S \leftrightarrow M_3, \ BS \leftrightarrow -\mu gl.$$  

(3.5)

We can see the correspondence explicitly through their Hamiltonians:

$$H_{\text{spin}} = \frac{p_\theta^2}{2m_s} + \frac{1}{2m_s} \frac{(M_\phi - S \cos \theta)^2}{\sin^2 \theta} - BS \cos \theta,$$

and

$$H_{\text{top}} = \frac{p_\theta^2}{2I_1} + \frac{1}{2I_1} \frac{(M_\phi - M_3 \cos \theta)^2}{\sin^2 \theta} + \mu gl \cos \theta + \frac{M_3^2}{2I_3}.$$  

(3.6)

(3.7)
where \( p_\theta \equiv \partial L/\partial \dot{\theta}, M_\phi \equiv \partial L/\partial \dot{\phi} \) are the canonical momenta of \( \theta \) and \( \phi \), respectively. These two Hamiltonians are completely the same with replacements (3.5). The last term in (3.7) does not contribute to the dynamics of \( \theta \) and \( \phi \), because in Hamiltonian formalism, the canonical momenta of cyclic coordinates are completely non-dynamical (constant) quantities.

The Lagrangians (3.1) and (3.2) have a simple relation. They are related by Legendre transformation about \( \psi \):

\[
L_{\text{spin}}(\theta, \dot{\theta}, \dot{\phi}; S) = L_{\text{top}}(\theta, \dot{\theta}, \dot{\phi}, \dot{\psi}) - S \dot{\psi} \bigg|_{\dot{\psi}=\psi(\theta,\phi,S)}
\]

with replacements (3.5). Here, as usual, we substitute for \( \dot{\psi} \) its corresponding canonical momentum \( S \) on the right-hand-side. The situation is quite similar to that of familiar centrifugal force problem. There, the original Lagrangian is given as \( L(r, \dot{r}, \dot{\phi}) = (m/2)(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) \), and we can obtain \( \phi \)-reduced Lagrangian by Legendre transformation about \( \phi \): \( L_{\text{red}}(r, \dot{r}; M) \equiv L - M\dot{\phi} = (m/2)\dot{r}^2 - M^2/2mr^2 - U(r) \). In this sense, spin dynamics is \( \psi \)-reduced dynamics of a spinning top. When we perform Legendre transformation also about \( \theta \) and \( \phi \), we are led to the same Hamiltonians (3.6) and (3.7).

As a rough understanding, the correspondence between spin and spinning top stems from intimate relation between spinning motion and magnetic field. This relation appears in various situations. For example, the EOM of a free particle in a coordinate frame rotating at angular velocity \( \Omega \) is \( m\ddot{x} = 2\Omega \times \dot{x} \) at the origin \( x = 0 \), as if it were subject to magnetic field \( B \approx \Omega \). Another example is Barnett effect and Richardson-Einstein-de Haas effect (rotation produces magnetization and vice versa). Due to this ‘spinning motion/magnetic field’ relation, the spinning motion of a spinning top corresponds to a monopole magnetic field, because the spinning angular velocity always points in the radial direction. This monopole is represented by the spin Berry phase term in the Lagrangian (3.1). While a spinning top has an explicit spinning degree of freedom \( \psi \), classical spin does not. Instead, its spinning motion is expressed by the presence of the monopole sitting at the center of the spin sphere.

Finally, let us comment on the usual behavior of spin with zero inertia, \( m_s = 0 \). This corresponds to setting \( I_1 (= I_2) = 0 \) under replacements (3.5). For an actual rigid body, any one of the principle moments of inertia is equal to or less than the sum of the other two, e.g., \( I_3 \leq I_1 + I_2 \). Thus, setting \( I_1 = I_2 = 0 \) leads to \( I_3 = 0 \). Therefore, usual spin with \( m_s = 0 \) corresponds to a ‘moment-of-inertia-less’ spinning top, with infinitely high spinning angular velocity \( \Omega_3 \equiv \dot{\psi} + \dot{\phi}\cos\theta \to \infty \) so that \( S = I_3\Omega_3 \) remains finite.
4 Intrinsic precession mode

The free motion of a symmetric spinning top is precession of its symmetric axis. When external force such as the gravitational field is applied, this precession is moved as a whole in direction perpendicular to both the applied force and the radial direction of the center of the precession. The precession is called nutation especially when such external force is applied [20]. Note that this precession is intrinsic to the top itself and independent of the applied force. When the nutation radius is small, the frequency of this nutation is $M_3/I_1$ where $M_3$ is the angular momentum along the symmetric axis and $I_1$ is the moment of inertia along an arbitrary axis perpendicular to the symmetric axis. Since the dynamics of a spin and a spinning top are equivalent, the behavior of spin with finite inertia is completely the same. Spin has its intrinsic precession or nutation mode whose frequency is about $\omega_0 \sim S/m_s$ (see the replacements (3.5)).

We can understand this behavior of spin (and spinning top) also in the context of a charged particle, regarding the Lagrangian (3.1) as that of a massive charged point particle on a sphere subject to monopole magnetic field. The free precession of spin is just (a spherical version of) the cyclotron motion of the particle under the monopole field. The frequency of this intrinsic precession mode is $\omega_0 \sim S/m_s$. When external force such as magnetic field is applied to spin, the precession of spin is moved as a whole, which is just the $E \times B$ drift motion in the context of the particle. Note that in the EOM (2.9), the external magnetic field $B$ plays the role of an electric field for the charged particle. Thus, ‘the electric field $E$’ for this $E \times B$ drift is the external magnetic field applied to spin, while ‘the magnetic field $B$’ is the intrinsic monopole magnetic field of spin, i.e., the spin Berry curvature.

Therefore, the general motion of spin with inertia under applied magnetic field is a double precession: one is Larmor precession due to the applied magnetic field, while the other (nutation) is cyclotron motion due to the Berry curvature.

Let us see the $m_s \to 0$ limit. The free precession solution of the EOM (3.3) (with $B = 0$) is $\theta = \theta_0$ (const.), $\dot{\phi} = S/m_s \cos \theta_0$. Using (the initial value of) the speed of spin, $v \equiv |\dot{n}|$, which is related to the integration constant $\theta_0$ above through $v = S\dot{\phi} \sin \theta_0 = S^2 \tan \theta_0/m_s$, we can rewrite the solution in term of $v$ instead of $\theta_0$:

$$\theta = \arctan \left( \frac{m_s v}{S^2} \right), \quad \dot{\phi} = \sqrt{\left( \frac{v}{S} \right)^2 + \left( \frac{S}{m_s} \right)^2}. \quad (4.1)$$

Now when we take the $m_s \to 0$ limit with $v$ fixed, then $\theta \to 0$ and $\dot{\phi} \to \infty$. Thus, at this limit, the free precession is shrunked to a point, and the spin looks like it is stopping.
at that point. This is consistent with the behavior of usual spin with $m_s = 0$, where the free solution is only a static one. For the case where external magnetic field is applied, the nutation radius goes to zero as $m_s \to 0$, and what is left is the motion of the guiding center of $E \times B$ drift. We recognize this as the usual Larmor precession of spin due to external magnetic field.

We have seen in this section that the spin system has intrinsic precession mode, which is independent of externally applied field. The magnetic field causing this precession is the monopole magnetic field of spin, i.e., the spin Berry phase itself. Its frequency, $\omega_0 \sim S/m_s$, is infinite when the inertia of spin $m_s$ is zero, but goes down to finite value when $m_s$ becomes non-zero.

5 Phenomena

We briefly discuss experimentally observable phenomena involving the inertia of spin.

5.1 Resonance

To test the existence of the inertia of spin experimentally, the most direct way is to see the resonance peak at the intrinsic frequency $\omega_0 \sim S/m_s$. First, let us see the resonance for free precession. Including the Gilbert damping and assuming $\mathbf{n} = (n_x, n_y, 1)$ with $n_x, n_y \ll 1$, the EOM is

$$m_s \ddot{\mathbf{n}} = S \mathbf{n} \times \dot{\mathbf{n}} + S \mathbf{B} - \alpha S \mathbf{n}.$$  \hspace{1cm} (5.1)

We apply AC magnetic field on $x$-$y$ plane, $\mathbf{B} = (B_x, B_y, B_z) = (B \cos \omega t, B \sin \omega t, 0)$. After enough time, the motion of spin converges to a stationary state where it precesses at frequency $\omega$. The solution of this state is given by $n_x + i n_y = A \exp \left( \frac{\alpha \omega_0}{\omega - \omega_0} \right)$ with $\omega_0 = \frac{S}{m_s}$. \hspace{1cm} (5.2)

The energy dissipation rate $I(\omega)$ is proportional to $\dot{n}^2$, which has resonance peak at the frequency $\omega_0 = S/m_s$.

Next, we consider resonance mode for Larmor precession accompanied by nutation. This time, we do not assume $\mathbf{n} = (n_x, n_y, 1)$ with $n_x, n_y \ll 1$. The EOM is eq. (5.1) with the Gilbert damping included. We apply two AC magnetic fields on $x$-$y$ plane and one DC field in $z$-direction: $\mathbf{B} = (B_x, B_y, B_z) = (b_1 \cos \omega_1 t + b_2 \cos \omega_2 t, b_1 \sin \omega_1 t + b_2 \sin \omega_2 t, B)$. We set $\omega_1$ be near $-B$ and $\omega_2$ near $S/m_s$. The motion of spin converges to Larmor precession of frequency...
Figure 1: Numerical result for the energy dissipation rate $I(\omega_1, \omega_2)$ as a function of two frequencies $\omega_1, \omega_2$ of applied AC magnetic fields: We apply two AC magnetic fields on $x$-$y$ plane and one DC field in $z$-direction, $B = (B_x, B_y, B_z) = (b_1 \cos \omega_1 t + b_2 \cos \omega_2 t, b_1 \sin \omega_1 t + b_2 \sin \omega_2 t, B)$. Then, $I(\omega_1, \omega_2)$ has two resonance peaks, one for Larmor precession due to the DC field, at $\omega_1 \sim -B$, and the other for accompanied nutation, at $\omega_2 \sim S/m_s$. The figure shows the numerical result at $B = 1 \text{GHz}$, $S/m_s = 10^3 \text{GHz}$, $b_1 = b_2 = 10^{-2} \text{GHz}$, $\alpha = 0.01$. The horizontal axes are in units of GHz and the vertical one is $I(\omega_1, \omega_2)$ divided by its maximum value.

$\omega_1$ with accompanying nutation of frequency $\omega_2$. The energy dissipation rate $I(\omega_1, \omega_2)$, which is proportional to $\dot{n}^2$, has two resonance peaks, at $\omega_1 \sim -B$ and at $\omega_2 \sim S/m_s$, respectively. Fig. 1 shows the numerical result of $I(\omega_1, \omega_2)$.

5.2 Spin Waves

We have so far dropped $(\partial_i n)^2$-term in the action (2.4). Let us include this and consider the resulting spin wave solutions. For simplicity, we set both the Gilbert damping and external magnetic field as zero, $\alpha = 0$ and $B = 0$. The EOM is

$$m_s \ddot{n} = S n \times \dot{n} + JS^2 \partial^2 \dot{n}. \quad (5.3)$$

When we assume $n = (n_x, n_y, 1)$ with $n_x, n_y \ll 1$, we obtain plane wave solutions $n_x + i n_y = Ae^{-i(\omega t - kx)}$ (with $A$ an arbitrary small constant) with two frequency modes

$$\omega = \frac{1}{2} S \left( -1 \pm \sqrt{1 + 4Jm_s k^2} \right) \sim \begin{cases} JSk^2 + O(k^4) \\ -(\frac{S}{m_s} + JSk^2) + O(k^4) \end{cases} \quad (5.4)$$
One is the usual spin wave mode in clockwise rotation seen from positive $z$-direction, and the other is essentially the free precession in counterclockwise rotation, which is mediated as spin wave by the interaction $J$ with neighboring spins. We take only the lowest order in powers of $k$ because we perform the derivative expansion from the beginning (2.4).

5.3 Domain Wall

When easy-axis anisotropy $(KS^2/2) \cos^2 \theta$ is included to the action (2.4), the system has a static domain wall solution (which we take perpendicular to $z$-axis), $\cos \theta = \tanh[(z - X)/\lambda], \phi = \phi_0$ where $X$ and $\phi_0$ are integration constants, and $\lambda = \sqrt{J/K}$. Promoting $X$ and $\phi_0$ to dynamical variables and substituting the domain wall solution into the action (2.4), we obtain the Lagrangian of $X$ and $\phi_0$:

$$L[X, \phi_0] = \frac{M_w}{2} (\dot{X}^2 + \lambda^2 \dot{\phi}_0^2) + \frac{SN_w}{2\lambda} (\dot{X}\phi_0 - X \dot{\phi}_0)$$ (5.5)

with $M_w = m_s N_w / \lambda^2$ and $N_w = 2\lambda A/\alpha^3$ where $A$ is the cross sectional area of the domain wall. When we regard $(X, \lambda\phi_0)$ as the position of a particle on a cylinder, this Lagrangian (5.5) is that of a charged particle with magnetic field $SN_w/(2\lambda^2)$ perpendicular to the cylinder.

Therefore, introducing the inertia of spin has essentially the same effect on the dynamics of a domain wall as on those of spin itself. Originally, when the inertia is zero, the collective coordinate dynamics of a domain wall are equivalent to those of a massless charged particle on a cylinder $(X, \lambda\phi_0)$ subject to a perpendicular magnetic field. The inertia of spin introduces the mass into this particle. Then the domain wall obtains the intrinsic free precession mode, where $X$ and $\phi_0$ oscillates at frequency $\omega_0 \sim SN_w/\lambda^2 M_w = S/m_s$. This frequency corresponds to that of spin free precession discussed in section 4. This result is natural because the oscillation of the domain wall is caused by the free precession of each spin. When this domain wall is subject to externally applied magnetic field or potential (such as hard-axis anisotropy), the particle on a cylinder $(X, \lambda\phi_0)$ moves in the direction perpendicular to the applied force on the particle, accompanied by free precession (cyclotron motion) discussed above. For the particle, this is $E \times B$ drift motion, where ‘$E$’ is the external force and ‘$B$’ is the intrinsic magnetic field due to the spin Berry phase.

The discussion above is valid only when the inertia $m_s$ is so large that the intrinsic frequency $\omega_0 \sim S/m_s$ is less than the frequency of spin waves: $S/m_s < KS^2/\hbar \sim 100\text{GHz}$. When $S/m_s$ exceeds this frequency scale, the dynamics of the domain wall can no longer be described only by the collective coordinates $X$ and $\phi_0$ as (5.3), and spin wave excitation
should also be included into the dynamics. It is difficult to study analytically the motion of a domain wall with spin wave excitation (numerical simulation is performed in, e.g. [21], for \( m_s = 0 \) case). We expect that the intrinsic precession mode will be sustained when we apply properly adjusted AC field.

6 Summary and discussion

We introduced the notion of the inertia of spin, which arises in the derivative expansion of spin effective action. We observed that the dynamics of spin with finite inertia are equivalent to those of a symmetric spinning top, and found that the spin with finite inertia has its intrinsic precession mode, which is caused by the spin Berry curvature itself. We briefly discussed observable phenomena related to the inertia.

It is clear that the content of this paper can be generalized to systems other than spin. What we need are Berry curvature and derivative expansion of effective action: even when the dynamics of a system are originally governed by only the Berry phase and the system is inertia-less, there will arise the inertial term in the derivative expansion of the effective action. Many systems are then expected to have their intrinsic precession modes due to their Berry curvatures, which are shrinked when the inertia is zero but are resolved when it is finite.

As immediate future direction, it will be interesting to study the temporal-spatial derivative term, \( \sum_i v^j \dot{\mathbf{n}} \cdot \partial_i \mathbf{n} \), in effective action of spin \( \mathbf{n} \). Here, \( v^j \) is related to spin density-spin current correlation functions, and is generally non-zero when system has anisotropy.

Quantization of spin with finite inertia will also be interesting, where the total angular momentum is no longer the spin itself, \( S n \), but is the modified one, \( m_s n \times \dot{\mathbf{n}} + S n \). Since the dynamics are equivalent to those of a massive charged particle on a sphere with monopole background, the quantization is essentially the same as that for Landau level on a sphere [22] [23], where the quantized system is described by monopole harmonics [24].

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