Strong solutions of the double phase parabolic equations with variable growth

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Abstract

This paper addresses the questions of existence and uniqueness of strong solutions to the homogeneous Dirichlet problem for the double phase equation with operators of variable growth:

\[ u_t - \text{div} \left( |\nabla u|^{p(z)-2} \nabla u + a(z)|\nabla u|^{q(z)-2} \nabla u \right) = F(z,u) \quad \text{in} \quad Q_T = \Omega \times (0,T) \]

where \( \Omega \subset \mathbb{R}^N, N \geq 2 \), is a bounded domain with the smooth boundary \( \partial \Omega \), \( z = (x,t) \in Q_T \), \( a : \overline{Q}_T \mapsto \mathbb{R} \) is a given nonnegative coefficient, and the nonlinear source term has the form

\[ F(z,v) = f_0(z) + b(z)|v|^{\sigma(z)-2}v. \]

The variable exponents \( p, q, \sigma \) are given functions defined on \( \overline{Q}_T \), \( p, q \) are Lipschitz-continuous and

\[ \frac{2N}{N+2} < p^* \leq p(z) \leq q(z) < p(z) + \frac{r}{2} \quad \text{with} \quad 0 < r < r^* = \frac{4p^-}{2N + p^- (N+2)}, \quad p^- = \min_{Q_T} p(z). \]

The initial function \( u_0 \) belongs to a Musielak-Sobolev space associated with the flux. We find conditions on the functions \( f_0, a, b, \sigma \) sufficient for the existence of a unique strong solution with the following global regularity and integrability properties:

\[ u_t \in L^2(Q_T), \quad |\nabla u|^{p(z)+\delta} \in L^1(Q_T) \quad \text{for every} \quad 0 < \delta < r^*, \]
\[ |\nabla u|^{s(z)}, a(z)|\nabla u|^{q(z)} \in L^\infty(0,T;L^1(\Omega)) \quad \text{with} \quad s(z) = \max\{2, p(z)\}. \]

The same results are established for the equation with the regularized flux

\[ \left( \epsilon^2 + |\nabla u|^2 \right)^{\frac{p(z)-2}{2}} \nabla u + a(z) \left( \epsilon^2 + |\nabla u|^2 \right)^{\frac{q(z)-2}{2}} \nabla u, \quad \epsilon > 0. \]

Keywords: singular and degenerate parabolic equation, double phase problem, variable nonlinearity, strong solution, higher integrability of the gradient, Musielak-Orlicz spaces

2010 MSC: 35K65, 35K67, 35B65, 35K55, 35K99

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1The first author acknowledges the support of the Research Grant from Czech Science Foundation, project GJ19-14413Y for the second part of this work.

2The second author acknowledges the support of the Research Grant MTM2017-87162-P, Spain.

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1. Introduction

Let $Ω ⊂ \mathbb{R}^N$ be a smooth bounded domain, $N \geq 2$ and $0 < T < \infty$. We consider the following parabolic problem with the homogeneous Dirichlet boundary conditions:

\[
\begin{aligned}
&u_t - \text{div} (|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u) = F(z, u) \quad \text{in} \ Q_T, \\
u(0) = u_0(x) \quad \text{in} \ Ω,
\end{aligned}
\]

(1.1)

where $z = (x, t)$ denotes the point in the cylinder $Q_T = Ω \times (0, T]$ and $Γ_T = \partial Ω \times (0, T)$ is the lateral boundary of the cylinder. The nonlinear source has the form

\[F(z, v) = f_0(z) + b(z)|v|^σ(z)^{-2}v.\]

(1.2)

Here $a > 0$, $b$, $p$, $q$, $σ$ and $f_0$ are given functions of the variables $z ∈ Q_T$.

Equations of the type (1.1) are often termed “the double phase equations”. This name, introduced in [14, 15], reflects the fact that the flux function $(|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2})\nabla u$ includes two terms with different properties. If $p(z) ≤ q(z)$ a.e. in the problem domain and $a(z)$ is allowed to vanish on a set of nonzero measure in $Q_T$, then the growth of the flux is determined by $p(z)$ on the set where $a(z) = 0$, and by $q(z)$ wherever $a(z) > 0$.

1.1. Previous work

The study of the double phase problems started in the late 80th by the works of V. Zhikov [40, 41] where the models of strongly anisotropic materials were considered in the context of homogenization. Later on, the double phase functionals attracted attention of many researchers. On the one hand, the study of these functionals is a challenging mathematical problem. On the other hand, the double phase functionals appear in a variety of physical models. We refer here to [6, 39] for applications in the elasticity theory, [5] for transonic flows, [7] for quantum physics and [10] for reaction-diffusion systems.

Equations (1.1) with $p ≠ q$ are also referred to as the equations with the $(p, q)$-growth because of the gap between the coercivity and growth conditions: if $p ≤ q$ and $0 ≤ a(x) ≤ L$, then for every $ξ ∈ \mathbb{R}^N$

\[|ξ|^p ≤ (|ξ|^{p-2} + a(x)|ξ|^{q-2})|ξ|^q ≤ C(1 + |ξ|^q), \quad C = \text{const} > 0.
\]

These equations fall into the class of equations with nonstandard growth conditions which have been actively studied during the last decades in the cases of constant or variable exponents $p$ and $q$. We refer to the works [1, 11, 14, 15, 21, 23, 27, 28, 31, 32, 34, 38] and references therein for the results on the existence and regularity of solutions, including optimal regularity results [21].

Results on the existence of solutions to the evolution double phase equations can be found in papers [8, 30, 37]. These works deal with the Dirichlet problem for systems of parabolic equations of the form

\[u_t - \text{div} a(x, t, \nabla u) = 0,\]

(1.3)

where the flux $a(x, t, \nabla u)$ is assumed to satisfy the $(p, q)$-growth conditions and certain regularity assumptions. As a partial case, the class of equations (1.3) includes equation (1.1) with constant exponents $p ≤ q$ and a nonnegative bounded coefficient $a(x, t)$. It is shown in [8, Th.1.6] that if

\[2 ≤ p ≤ q < p + \frac{4}{N+2},\]

then problem (1.1) with $F ≡ 0$ has a very weak solution.
\[ u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\Omega)) \quad \text{with} \quad u_t \in L^{\frac{p}{p-1}}(0, T; W^{-1,\frac{p}{p-1}}(\Omega)), \]

provided that \( u_0 \in W_0^{1, r}(\Omega) \), \( r = \frac{p(\gamma-1)}{\gamma-1} \). Moreover, \( |\nabla u| \) is bounded on every strictly interior cylinder \( Q_T^r \subset Q_T \) separated away from the parabolic boundary of \( Q_T \). In [30] these results were extended to the case

\[
\frac{2N}{N+2} < p < 2, \quad p \leq q < p + \frac{4}{N+2}.
\]

Paper [37] deals with weak solutions of systems of equations of the type (1.3) with \((p, q)\) growth conditions. When applied to problem (1.1) with constant \( p, q, b \equiv 0 \) and \( a(\cdot, t) \in C^\alpha(\Omega) \) with some \( \alpha \in (0, 1) \) for a.e. \( t \in (0, T) \), the result of [37] guarantees the existence of a weak solution

\[
u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),
\]

provided that the exponents \( p \) and \( q \) obey the inequalities

\[
\frac{2N}{N+2} < p < q < p + \frac{\alpha \min\{2, p\} \left(1 + \frac{2N}{p} \right)}{N+2}.
\]

The proofs of the existence theorems in [8, 32, 37] rely on the property of local higher integrability of the gradient, \( |\nabla u|^{p+\delta} \in L^1(Q_T^p) \) for every sub-cylinder \( Q_T^p \subset Q_T \). The maximal possible value of \( \delta > 0 \) indicates the admissible gap between the exponents \( p \) and \( q \) and vary in dependence on the type of the solution.

Equation (1.1) with constant exponents \( p \) and \( q \) furnishes a prototype of the equations recently studied in papers [9, 13, 21, 32] in the context of weak or variational solutions. The proofs of existence also use the local higher integrability of the gradient, but for the existence of variational solutions a weaker assumption on the gap \( q - p \) is required.

Nonlinear parabolic equations of the form (1.3) with the flux \( a(x, t, \nabla u) \) controlled by a generalized \( N \)-function were studied in [12] in the context of Musielak-Orlicz spaces. The class of equations studied in [12] includes, as a partial case, equation (1.1) with \( b \equiv 0 \) and variable exponents \( p, q \). It is shown that problem (1.1) with \( b = 0 \) and bounded data \( u_0 \) and \( f_0 \) admits a unique solution \( u \in L^\infty(0, T; L^2(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega)) \) with \( \nabla u \in L_M(Q_T; \mathbb{R}^N) \), where \( L_M \) denotes the Musielak-Orlicz space defined by the flux \( a(x, t, \xi) \). We refer here to [12, 22], the survey article [11] and references therein for the issues of solvability of elliptic and parabolic equations in the Musielak-Orlicz spaces.

1.2. Description of results

In the present work, we prove the existence of strong solutions of problem (1.1). By the strong solution we mean a solution whose time derivative is not a distribution but an element of a Lebesgue space, and the flux has better integrability properties than the properties prompted by the energy equality (the rigorous formulation is given in Definition 5.1). We consider first the case \( b \equiv 0 \). We show that if the exponents \( p, q \) and the data \( a, u_0, f_0 \) are sufficiently smooth, and if the gap between the exponents satisfies the condition

\[
\frac{2N}{N+2} < p(z) \leq q(z) < p(z) + \frac{2}{(N+2) + \frac{2N}{p^-}} \quad \text{in} \; Q_T, \quad p^- = \inf_{Q_T} p(z), \tag{1.4}
\]

then problem (1.1) has a strong solution \( u \) with

\[
u_{t} \in L^2(Q_T), \quad \text{ess sup}_{(0, T)} \mathcal{F}(u(\cdot, t), t) < \infty, \quad \mathcal{F}(u(\cdot, t), t) \equiv \int_{\Omega} (|\nabla u|^{p(z)} + a(z)|\nabla u|^{q(z)}) \, dx. \tag{1.5}
\]

Moreover, the solution possesses the property of global higher integrability of the gradient:

\[
\int_{Q_T} |\nabla u|^{p(z)+r} \, dz \leq C \quad \text{for every} \; 0 < r < \frac{4}{(N+2) + \frac{2N}{p^-}} \tag{1.6}
\]
with a finite constant $C$ depending only on $F(u_0, 0)$, $N$, $r$, and the properties of $p(\cdot)$ and $q(\cdot)$. The same existence result is valid for problem (1.1) with the regularized nondegenerate flux
\[
\left(\epsilon^2 + |\nabla u|^2\right)^{\frac{\sigma(z)-2}{2}} + a(z)\left(\epsilon^2 + |\nabla u|^2\right)^{\frac{\sigma(z)-2}{2}} \nabla u, \quad \epsilon > 0.
\]

In the case $b \neq 0$, the existence of strong solutions of problem (1.1) is proven under the additional assumption
\[
2 \leq \inf_{Q_T} \sigma(z) \leq \sup_{Q_T} \sigma(z) < 1 + \frac{1}{2} \inf_{Q_T} p(z),
\]
which restricts the considerations to the degenerate case $p(z) > 2$ in $Q_T$. For the uniqueness of strong solutions we assume that either $b(z) \leq 0$, or $\sigma \equiv 2$ in $Q_T$.

Property (1.6) of global higher integrability of the gradient turns out to be crucial for the proof of existence of strong solutions. Unlike the traditional approach based on the use of Caccioppoli type inequalities and scaling, estimate (1.6) is proved by means of an interpolation inequality of the Gagliardo-Nirenberg type. This method enables one to derive global estimates, although at expense of a stronger restriction on the gap between $p(z)$ and $q(z)$.

Inequality (1.6) allows one to extend the results to the multiphase equations
\[
u_t - \text{div}\left(|\nabla u|^{p(z)-2}\nabla u + \sum_{i=1}^K a_i(z)|\nabla u|^{q_i(z)-2}\nabla u\right) = F(z, u),
\]
provided each of $q_i(\cdot)$ satisfies (1.4).

2. The function spaces

We begin with a brief description of the Lebesgue and Sobolev spaces with variable exponents. A detailed insight into the theory of these spaces and a review of the bibliography can be found in [17, 19, 28]. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial \Omega$. Define the set
\[
\mathcal{P}(\Omega) := \{\text{measurable functions on } \Omega \text{ with values in } (1, \infty)\}.
\]

2.1. Variable Lebesgue spaces

Throughout the rest of the paper we assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial \Omega \in C^2$. Given $r \in \mathcal{P}(\Omega)$, we introduce the modular
\[
A_r(f) = \int_{\Omega} |f(x)|^{r(x)} \, dx \quad (2.1)
\]
and the set
\[
L^{r(\cdot)}(\Omega) = \{f : \Omega \to \mathbb{R} \mid \text{measurable on } \Omega, \ A_r(f) < \infty\}.
\]
The set $L^{r(\cdot)}(\Omega)$ equipped with the Luxemburg norm
\[
\|f\|_{r(\cdot), \Omega} = \inf \left\{\lambda > 0 : A_r\left(\frac{f}{\lambda}\right) \leq 1\right\}
\]
becomes a Banach space. By convention, from now on we use the notation
\[
r^- := \text{ess min}_{x \in \Omega} r(x), \quad r^+ := \text{ess max}_{x \in \Omega} r(x).
\]
If $r \in \mathcal{P}(\Omega)$ and $1 < r^- \leq r(x) \leq r^+ < \infty$ in $\Omega$, then the following properties hold.

(i) $L^{r(\cdot)}(\Omega)$ is a reflexive and separable Banach space.
2.2 Variable Sobolev spaces

Let us denote by $$W^{r}(\Omega)$$ where $$r$$ is a nonnegative function such that

$$\log \left( \frac{1}{\Omega} \right) \in L^{r}(\Omega)$$

and for all $$u \in L^{r}(\Omega)$$

$$\|u\|_{L^{r}(\Omega)} = \|u\|_{r(\cdot), \Omega} + \|\nabla u\|_{r(\cdot), \Omega}.$$ 

If $$r \in C^{0}(\Omega)$$, the Poincaré inequality holds: for every $$u \in W^{1,r}(\Omega)$$

$$\|u\|_{r(\cdot), \Omega} \leq C\|\nabla u\|_{r(\cdot), \Omega}. \tag{2.6}$$

Inequality (2.6) means that the equivalent norm of $$W^{1,r}(\Omega)$$ is given by

$$\|u\|_{W^{1,r}(\Omega)} = \|\nabla u\|_{r(\cdot), \Omega}. \tag{2.7}$$

Let us denote by $$C_{0}(\Omega)$$ the subset of $$\mathcal{P}(\Omega)$$ composed of the functions continuous on $$\Omega$$ with the logarithmic modulus of continuity:

$$p \in C_{0}(\Omega) \iff |p(x) - p(y)| \leq \omega(|x - y|) \quad \forall x, y \in \Omega, |x - y| < \frac{1}{2},$$

where $$\omega$$ is a nonnegative function such that

$$\limsup_{s \to 0^{+}} \omega(s) \ln \frac{1}{s} = C, \quad C = \text{const.}$$

If $$r \in C_{0}(\Omega)$$, then the set $$C^\infty(\Omega)$$ of smooth functions with finite support is dense in $$W^{1,r}(\Omega)$$. This property allows one to use the equivalent definition of the space $$W^{1,r}(\Omega)$$:

$$W^{1,r}(\Omega) = \left\{ \text{the closure of } C^\infty(\Omega) \text{ with respect to the norm } \| \cdot \|_{W^{1,r}(\Omega)} \right\}.$$ 

Given a function $$u \in W^{1,r}(\Omega)$$ with $$r \in C_{0}(\Omega)$$, the smooth approximations of $$u$$ in $$W^{1,r}(\Omega)$$ can be obtained by means of the Friedrichs mollifiers.
The following analogue of the Sobolev embedding theorem holds. Given \( p \in C^0(\Omega) \), \( 1 < p^- \leq p^+ < \infty \), let us introduce the Sobolev conjugate exponent

\[
p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\
\text{any number from } [1, \infty) & \text{if } p(x) \geq N.
\end{cases}
\]

If \( q \in C^0(\Omega) \) and \( \inf_\Omega (p^*(x) - q(x)) > 0 \), then for every \( u \in W^{1,p}(\Omega) \)

\[
\|u\|_{q(.)\Omega} \leq C\|u\|_{W^{1,p}(\Omega)}, \quad C = C(p^+, q^+, |\Omega|, N),
\]

and the embedding \( W^{1,p}(\Omega) \subset L^q(\Omega) \) is compact.

The dual space to \( W = W^{1,q}(\Omega) \) is the set of linear bounded functionals over \( W \): \( W' = W^{-1,q'}(\Omega) \).

\( W' \) consists of the vectors \( G = (g_0, g_1, \ldots, g_N) \), \( g_i \in L^{q'}(\Omega) \), such that for every \( u \in W \)

\[
(G, u)_{W',W} = \int_\Omega \left( g_0 u + \sum_{i=1}^N g_i D_i u \right) dx.
\]

Since the equivalent norm of \( W \) is given by \( \|u\|_{W^{1,p}} \), for every \( G \in W' \) there exists a function \( F = (F_1, \ldots, F_N) \) such that \( F_i \in L^{q'}(\Omega) \) and for every \( u \in W \)

\[
(G, u)_{W',W} = \sum_{i=1}^N \int_\Omega F_i D_i u dx.
\]

### 2.3. Spaces of functions depending on \( x \) and \( t \)

For the study of parabolic problem \( \square \) we need the spaces of functions depending on \( z = (x,t) \in Q_T \). Given a function \( q \in C^{0,0}(\overline{Q_T}) \), we introduce the spaces

\[
V_q(t,\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u|^q(x,t) \in L^1(\Omega)\}, \quad t \in (0, T),
\]

\[
W_q(\Omega) = \{u : (0,T) \rightarrow V_q(.)\Omega \mid u \in L^2(Q_T), |\nabla u|^q \in L^1(Q_T)\}.
\]

The norm of \( W_q(.)\Omega \) is defined by

\[
\|u\|_{W_q(.)\Omega} = \|u\|_{L^2Q_T} + \|\nabla u\|_{L^qQ_T}.
\]

Since \( q \in C^{0,0}(\overline{Q_T}) \), the space \( W_q(.)\Omega \) is the closure of \( C_c^\infty(Q_T) \) with respect to this norm.

### 2.4. Musielak-Sobolev spaces

Let \( a_0 : \Omega \rightarrow [0, \infty) \) be a given function, \( a_0 \in C^{0,1} \big( \overline{\Omega} \big) \). Assume that the exponents \( p(x), q(x) \in C^{0,1} \big( \overline{\Omega} \big) \) take values in the intervals \( (p^-, p^+), (q^-, q^+) \), and \( p(x) \leq q(x) \) in \( \Omega \). Set

\[
r(x) = \max\{2, p(x)\}, \quad s(x) = \max\{2, q(x)\}
\]

and consider the function

\[
H(x,t) = t^{r(x)} + a_0(x)t^{s(x)}, \quad t \geq 0, \quad x \in \Omega.
\]

The set

\[
L^H(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable}, \rho_H(u) = \int_\Omega H(x, |u|) dx < \infty \right\}
\]

equipped with the Luxemburg norm

\[
\|u\|_H = \inf \left\{ \lambda > 0 : \rho_H \left( \frac{u}{\lambda} \right) \leq 1 \right\}
\]
becomes a Banach space. The space $L^H(\Omega)$ is separable and reflexive \[22\]. By $\mathcal{V}(\Omega)$ we denote the Musielak-Sobolev space

$$\mathcal{V}(\Omega) = \{ u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega) \}$$

with the norm

$$\|u\|_{\mathcal{V}} = \|u\|_{H} + \|\nabla u\|_{H}.$$ 

The space $\mathcal{V}_0(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm of $\mathcal{V}(\Omega)$.

### 2.5. Dense sets in $W_0^{1,p(\cdot)}(\Omega)$ and $\mathcal{V}_0(\Omega)$

Let $\{\phi_i\}$ and $\{\lambda_i\}$ be the eigenfunctions and the corresponding eigenvalues of the Dirichlet problem for the Laplacian:

$$(\nabla \phi_i, \nabla \psi)_{\Omega,2} = \lambda_i (\phi_i, \psi) \quad \forall \psi \in H_0^1(\Omega).$$

The relations

$$\left[ \begin{array}{c} f, g \end{array} \right]_k = \left( \begin{array}{c} (\Delta \phi_i, \Delta \psi)_{2,\Omega} \\ (\Delta \phi_i, \Delta \psi)_{H^1(\Omega)} \end{array} \right)$$

define an equivalent inner product on $H^k_D(\Omega)$: $\|f\|_{H^k_D(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i^k \|f_i\|^2_{L^2(\Omega)}$, where $f_i, g_i$ are the Fourier coefficients of $f, g$ in the basis $\{\phi_i\}$ of $L^2(\Omega)$. The corresponding equivalent norm of $H^k_D(\Omega)$ is defined by $\|f\|_{H^k_D(\Omega)} = \left[ f, f \right]_k$.

Let $f^{(m)} = \sum_{i=1}^m f_i \phi_i$ be the partial sums of the Fourier series of $f \in L^2(\Omega)$. The following assertion is well-known.

**Proposition 2.1.** Let $\partial \Omega$ be a $C^k$ domain. A function $f \in C^k(\Omega)$, $k \geq 1$, $k \geq 1$, can be represented by the Fourier series in the system $\{\phi_i\}$, convergent in the norm of $H^k(\Omega)$, if and only if $f \in H^k_D(\Omega)$. If $f \in H^k_D(\Omega)$, then the series $\sum_{i=1}^\infty \lambda_i^k f_i^2$ is convergent, and its sum is bounded by $C\|f\|_{H^k(\Omega)}^2$ with an independent of $f$ constant $C$, and $\|f^{(m)} - f\|_{H^k(\Omega)} \to 0$ as $m \to \infty$. If $k \geq \left\lceil \frac{N}{2} \right\rceil + 1$, then the Fourier series in the system $\{\phi_i\}$ of every function $f \in H^k_D(\Omega)$ converges to $f$ in $C^{k-\left\lceil \frac{N}{2} \right\rceil - 1}(\Omega)$.

**Proposition 2.2** [22, Th. 4.7, Proposition 4.10]. Let $\partial \Omega$ be a Lipschitz domain $C^k(\Omega)$. Then the set $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

Let us denote $P_m = \text{span}\{\phi_1, \ldots, \phi_m\}$, where $\phi_i$ are the solutions of problem \[2.10\].

**Lemma 2.1.** If $\partial \Omega$ is a $C^k$ domain, then

$$k \geq N \left( \frac{1}{2} + \frac{1}{N} - \frac{1}{p^+} \right), \quad p^+ = \max_{\Omega} p(x),$$

and $p \in C_{\text{log}}(\Omega)$, then $\bigcup_{m=1}^\infty P_m$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

**Proof.** Given $v \in W_0^{1,p(\cdot)}(\Omega)$ we have to show that for every $\epsilon > 0$ there is $m \in \mathbb{N}$ and $v_m \in P_m$ such that $\|v - v_m\|_{W_0^{1,p(\cdot)}(\Omega)} < \epsilon$. Fix some $\epsilon > 0$. By Proposition \[2.2\] there is $v_\epsilon \in C_c^\infty(\Omega) \subset H^k(\Omega)$ such that $\|v - v_\epsilon\|_{W_0^{1,p(\cdot)}(\Omega)} < \epsilon/2$. By Proposition \[2.1\], $v_\epsilon^{(m)}(x) = \sum_{i=1}^m v_\epsilon \phi_i(x) \in H^k(\Omega)$ and $v_\epsilon^{(m)} \to v_\epsilon$ in $H^k(\Omega)$,
3. Assumptions and main results

Let \( p, q : \Omega \mapsto \mathbb{R} \) be measurable functions satisfying the conditions

\[
\frac{2N}{N + 2} < p_− \leq p(z) \leq p_+ \quad \text{in } \Omega, \\
\frac{2N}{N + 2} < q_− \leq q(z) \leq q_+ \quad \text{in } \Omega, \\
p^+, q^+ = \text{const}.
\]

Moreover, let us assume that \( p, q \in W^{1,\infty}(\Omega) \) as functions of variables \( z = (x, t) \): there exist positive constants \( C^*, C^{**}, C_*, C_{**} \) such that

\[
\text{ess sup}_{\Omega} |\nabla p| \leq C^*, \quad \text{ess sup}_{\Omega} |p_t| \leq C^*, \\
\text{ess sup}_{\Omega} |\nabla q| \leq C_{**}, \quad \text{ess sup}_{\Omega} |q_t| \leq C^{**}.
\]

Therefore for every \( \delta > 0 \) there is \( m \in \mathbb{N} \) such that \( \|v_i - v_i^{(m)}\|_{H^k(\Omega)} < \delta \). Since \( k, N, q \) satisfy condition (2.11), the embeddings \( H^k_\mathcal{D}(\Omega) \subset W^{1,\varphi^+}_0(\Omega) \subset W^{1,\varphi^+}_0(\Omega) \) are continuous:

\[
\|w\|_{W^{1,\varphi^+}_0(\Omega)} \leq C \|w\|_{W^{1,\varphi^+}_0(\Omega)} \leq C' \|w\|_{H^k(\Omega)} \quad \forall w \in H^k_\mathcal{D}(\Omega)
\]

with independent of \( w \) constants \( C, C' \). Set \( C'\delta = \epsilon/2 \). Then

\[
\|v_\epsilon - v_\epsilon^{(m)}\|_{W^{1,\varphi^+}_0(\Omega)} \leq C'\epsilon \|v_\epsilon - v_\epsilon^{(m)}\|_{H^k(\Omega)} \leq C'\delta = \frac{\epsilon}{2}.
\]

It follows that

\[
\|v - v_\epsilon^{(m)}\|_{W^{1,\varphi^+}_0(\Omega)} \leq \|v - v_\epsilon\|_{W^{1,\varphi^+}_0(\Omega)} + \|v_\epsilon - v_\epsilon^{(m)}\|_{W^{1,\varphi^+}_0(\Omega)} < \frac{\epsilon}{2} + C'\delta = \epsilon.
\]

**Corollary 2.1.** If \( p \in C_\log(\overline{Q}_T) \) and condition (2.11) is fulfilled, then

\[
\left\{ v(x,t) : v = \sum_{i=1}^{\infty} v_i(t)\phi_i(x), \quad v_i(t) \in C^{0,1}[0,T] \right\} \text{ is dense in } W_p(\Omega)(Q_T).
\]

**Proposition 2.3.** Let \( \partial \Omega \in \text{Lip} \) and \( a_0, p, q \in C^{0,1}(\overline{\Omega}) \). If \( p(x) \leq q(x) \) in \( \Omega \) and

\[
s^+ \leq \frac{1}{s^+} \leq 1 + \frac{1}{N}, \quad s^+ = \max\{2, q(x)\}, \quad r^- = \min\{\max\{2, p(x)\}\},
\]

then \( C^s_\infty(\Omega) \cap V(\Omega) \) is dense in \( V_0(\Omega) \).

The assertion of Proposition 2.3 follows from [13, Th.3.1] or [26, Th.6.4.7]. A straightforward checking of all conditions listed in [26] is given in [16, Theorem 2.21].

**Lemma 2.2.** If \( a_0, p, q \in C^{0,1}(\overline{\Omega}) \) and \( \partial \Omega \in C^k \) with

\[
k \geq N \left( \frac{1}{2} + \frac{1}{N} - \frac{1}{s^+} \right), \quad (2.12)
\]

then \( \bigcup_{m=1}^{\infty} \mathcal{P}_m \) is dense in \( V_0(\Omega) \).

We omit the detailed proof which is an imitation of the proof of Lemma 2.1 by Proposition 2.3 \( C^s_\infty(\Omega) \) is dense in \( V_0(\Omega) \), and since \( C^s_\infty(\Omega) \subset H^k_\mathcal{D}(\Omega) \) every \( v_\epsilon \in H^k_\mathcal{D}(\Omega) \) can be approximated by \( v_\epsilon^{(m)} \in \mathcal{P}_m \).
The modulating coefficient $a(\cdot)$ is assumed to satisfy the following conditions:

$$a(z) \geq 0 \text{ in } \overline{Q}, \quad a \in C([0,T];W^{1,\infty}(\Omega)), \quad \text{ess sup}_{Q_T}|a_t| \leq C_a, \quad C_a = \text{const.} \quad (3.3)$$

We do not impose any condition on the null set of the function $z$ in conditions (3.2). It is possible that $p(z) < 2$ and $q(z) > 2$ at the same point $z \in Q_T$.

**Definition 3.1.** A function $u : Q_T \mapsto \mathbb{R}$ is called a strong solution of problem (1.1) if

1. $u \in W_{q(\cdot)}(Q_T), u_t \in L^2(Q_T), |\nabla u| \in L^\infty(0,T;L^{s(\cdot)}(\Omega))$ with $s(\cdot) = \max\{2,p(\cdot)\}$,
2. for every $\psi \in W_{q(\cdot)}(Q_T)$ with $\psi_t \in L^2(Q_T)$
   \[\int_{Q_T} u_t \psi \, dz + \int_{Q_T} (|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2}) \nabla u \cdot \nabla \psi \, dz = \int_{Q_T} F(z,u)\psi \, dz, \quad (3.4)\]
3. for every $\phi \in C_0^\infty(\Omega)$
   \[\int_{Q_T} (u(x,t) - u_0(x))\phi \, dx \to 0 \quad \text{as } t \to 0. \]

The main results are given in the following theorems.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded domain with the boundary $\partial \Omega \subset C^k, k \geq 2+\left\lfloor \frac{N}{2} \right\rfloor$. Assume that $p(\cdot), q(\cdot)$ satisfy conditions $\text{(3.1)}, \text{(3.2)}$, and there exists a constant

$$r \in (0,r^*), \quad r^* = \frac{4p^*}{p^*(N+2) + 2N},$$

such that

$$p(z) \leq q(z) \leq p(z) + \frac{r}{2} \text{ in } \overline{Q}. \quad (3.5)$$

If $a(\cdot)$ satisfies conditions $\text{(3.3)}$ and $b \equiv 0$, then for every $f_0 \in L^2(0,T;W^{1,2}_0(\Omega))$ and $u_0 \in W^{1,2}_0(\Omega)$ with

\[\int_{\Omega} \left( |\nabla u_0|^2 + |\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)} \right) \, dx = K < \infty \quad (3.6)\]

problem (1.1) has a unique strong solution $u$. This solution satisfies the estimate

\[\|u_t\|_{L^2(0,T;W^{2,2}_0(\Omega))} + \text{ess sup}_{(0,T)} \int_{\Omega} \left( |\nabla u|^{s(z)} + a(z)|\nabla u|^{q(z)} \right) \, dx + \int_{Q_T} |\nabla u|^{p(z) + r} \, dz \leq C \quad (3.7)\]

with the exponent $s(z) = \max\{2,p(z)\}$ and a constant $C$ which depends on $N, \partial \Omega, T, p^-, q^-, r, s(z), K$. The constants in conditions $\text{(3.2)}, \text{(3.3)}, \|f_0\|_{L^2(0,T;W^{2,2}_0(\Omega))}$ and $K$.

**Theorem 3.2.** Let in the conditions of Theorem 3.1 $b \neq 0$.

(i) Assume that $b, \sigma$ are measurable bounded functions defined on $Q_T$,

\[\|\nabla b\|_{\infty,Q_T} < \infty, \quad \|\nabla \sigma\|_{\infty,Q_T} < \infty, \quad 2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}, \quad \sigma^- = \text{ess inf}_{Q_T}(\sigma(z)), \quad \sigma^+ = \text{ess sup}_{Q_T}(\sigma(z)). \quad (3.8)\]

Then for every $f_0 \in L^2(0,T;W^{1,2}_0(\Omega))$ and $u_0 \in W^{1,2}_0(\Omega)$ satisfying condition $\text{(3.6)}$ problem (1.1) has at least one strong solution $u$. The solution $u$ satisfies estimate $\text{(3.7)}$ with the constant depending on the same quantities as in the case $b \equiv 0$ and on $\|\nabla b\|_{\infty,Q_T}, \|\nabla \sigma\|_{\infty,Q_T}, \sigma^\pm, \text{ess sup}_{Q_T}|b|$. 

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(ii) The strong solution is unique if $p(\cdot), q(\cdot)$ satisfy the conditions of Theorem 3.1 and either $\sigma \equiv 2$, or $b(z) \leq 0$ in $Q_T$.

An outline of the work. In Section 4 we collect several auxiliary assertions. We present estimates on the gradient trace on $\partial \Omega$ for the functions from variable Sobolev spaces and formulate the interpolation inequality which enables us to prove global higher integrability of the gradient. This property turns out to be the key element in the proof of the existence theorems for problem (1.1) and the regularized problem (5.1).

A solution of problem (1.1) is obtained as the limit of the family of solutions of the nondegenerate problems (5.1) with the regularized fluxes
\[
\left( (\epsilon^2 + |\nabla u|^2)^{\frac{p(z) - 2}{2}} + a(z)(\epsilon^2 + |\nabla u|^2)^{\frac{q(z) - 2}{2}} \right) \nabla u, \quad \epsilon > 0.
\]

For every $\epsilon \in (0, 1)$ problem (5.1) is solved with the method of Galerkin. In Section 5 we formulate the problems for the approximations.

Section 6 is devoted to derive a priori estimates on the approximate solutions and their derivatives. For the convenience of presentation, we separate the cases when $b \equiv 0$ and the source function is independent of the solution, and $b \not\equiv 0$. Since no restriction on the sign of $b$ is imposed, in the latter case derivation of the a priori estimates requires additional restrictions on the range of the exponent $p$. The a priori estimates of Section 6 involve higher-order derivatives of the approximate solutions. This is where we make use of the interpolation inequalities of Section 4 to obtain the global higher integrability of the gradient which, in turn, yields uniform boundedness of the $L^q(\cdot)_Q$-norms of the gradients of the approximate solutions.

Theorems 3.1 and 3.2 are proven in Section 7. We show first that for every $\epsilon > 0$ the constructed sequence of Galerkin’s approximations contains a subsequence which converges to a strong solution $u_\epsilon$ of the regularized problem (5.1). The proof relies on the compactness and monotonicity of the fluxes. Existence of a solution to problem (1.1) is established in a similar way. We show that the solutions of the regularized problem (5.1) converge (up to a subsequence) to a solution of the problem (1.1).

Notation: Throughout the rest of the text, the symbol $C$ will be used to denote the constants which can be calculated or estimated through the data but whose exact values is unimportant. The value of $C$ may vary from line to line even inside the same formula. Whenever it does not cause a confusion, we omit the arguments of the variable exponents of nonlinearity and the coefficients. We will use the shorthand notation
\[
|v_{xx}|^2 = \sum_{i,j=1}^{N} |v_{x_i x_j}|^2.
\]

4. Auxiliary propositions

Until the end of this section, the notation $p(\cdot), q(\cdot), a(\cdot)$ is used for functions not related to the exponents and coefficient in (1.1) and (5.1).

Lemma 4.1 (Lemma 1.32, [3]). Let $\partial \Omega \in \text{Lip}$ and $p \in C^0(\overline{Q_T})$. Assume that $u \in L^\infty(0,T; L^2(\Omega)) \cap W^{1,p(\cdot)}_0(Q_T)$ and
\[
\text{ess sup}_{(0,T)} \|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_{Q_T} |\nabla u|^{p(z)} dz = M < \infty.
\]

Then
\[
\|u\|_{p(\cdot),Q_T} \leq C, \quad C = C(M, p^\pm, N, \omega),
\]

where $\omega$ is the modulus of continuity of the exponent $p(\cdot)$.

The proof in [3] is given for the case $\Omega = B_R(x_0)$. To adapt it to the general case, it is sufficient to consider the zero continuation of $u$ to a circular cylinder containing $Q_T$. 

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Let us accept the notation
\[
\beta_\varepsilon(s) = \epsilon^2 + |s|^2, \\
\gamma_\varepsilon(z,s) = (\epsilon^2 + |s|^2)^{\frac{\mu\varepsilon - 2}{2}} + a(z)(\epsilon^2 + |s|^2)^{\frac{\mu\varepsilon - 2}{2}}, \quad s \in \mathbb{R}^N, \quad z \in Q_T, \quad \epsilon \in (0,1).
\]

(4.1)

With certain abuse of notation, we will denote by \(\gamma_\varepsilon(x,s)\) the same function but with the exponents \(p, q\) and the coefficient \(a\) depending on the variable \(x \in \Omega\).

**Lemma 4.2** (Lemma 4.1, [4]). Let \(\partial \Omega \in C^1\), \(u \in C^2(\overline{\Omega})\) and \(u = 0\) on \(\partial \Omega\). Assume that
\[
p : \Omega \mapsto [p^-, p^+], \quad p^\pm = \text{const},
\]
\[
2N + 2 < p^-,
\]
\[
p(\cdot) \in C^0(\overline{\Omega}), \quad \text{ess sup}_\Omega |\nabla p| = L,
\]
\[
\int_\Omega \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|u_{xx}|^2 dx < \infty, \quad \int_\Omega u^2 dx = M_0, \quad \int_\Omega |\nabla u|^{p(x)} dx = M_1.
\]

(4.2)

Then for every
\[
\frac{2}{N+2} =: r_* < r < r^* := \frac{4p^-}{p^-(N+2)+2N}
\]
and every \(\delta \in (0,1)\)
\[
\int_{Q_T} \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|\nabla u|^{2} dx \leq \delta \int_{Q_T} \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|u_{xx}|^2 dx + C \left(1 + \int_\Omega |\nabla u|^{p(x)} dx\right)
\]
with an independent of \(u\) constant \(C = C(\partial \Omega, \delta, p^\pm, N, r, M_0, M_1)\).

**Theorem 4.1** (Theorem 4.1, [4]). Let \(\partial \Omega \in C^1\), \(u \in C^0([0,T]; C^2(\overline{\Omega}))\) and \(u = 0\) on \(\partial \Omega \times [0,T]\). Assume that
\[
p : Q_T \mapsto [p^-, p^+], \quad p^\pm = \text{const},
\]
\[
2N + 2 < p^-,
\]
\[
\text{ess sup}_{Q_T} |\nabla p| = L,
\]
\[
\int_{Q_T} \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|u_{xx}|^2 dz < \infty, \quad \sup_{(0,T)} \|u(t)||_{2,\Omega}^2 = M_0, \quad \int_{Q_T} |\nabla u|^{p(z)} dz = M_1.
\]

(4.5)

Then for every
\[
\frac{2}{N+2} = r_* < r < r^* = \frac{4p^-}{p^-(N+2)+2N}
\]
and every \(\delta \in (0,1)\) the function \(u\) satisfies the inequality
\[
\int_{Q_T} \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|\nabla u|^{2} dz \leq \delta \int_{Q_T} \beta_\varepsilon^{\frac{\mu\varepsilon-2}{2}}(\nabla u)|u_{xx}|^2 dz + C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz\right)
\]
with an independent of \(u\) constant \(C = C(\Omega, \partial \Omega, T, \delta, p^\pm, \omega, r, M_0, M_1)\).

**Lemma 4.3.** Let \(\Omega \subset \mathbb{R}^N, N \geq 2\) be a bounded domain with the boundary \(\partial \Omega \in C^2\), and \(a \in W^{1,\infty}(\Omega)\) be a given nonnegative function. Assume that \(v \in W^{1,2}(\Omega) \cap W^{1,2}_0(\Omega)\) and denote
\[
K = \int_{\partial \Omega} a(x)(\epsilon^2 + |\nabla v|^2)^{\frac{\mu\varepsilon-2}{2}}(\Delta v (\nabla v \cdot n) - \nabla (\nabla v \cdot n) \cdot \nabla v) dS, \quad \text{(4.7)}
\]
where \(n\) stands for the exterior normal to \(\partial \Omega\). There exists a constant \(L = L(\partial \Omega)\) such that
\[
K \leq L \int_{\partial \Omega} a(x)(\epsilon^2 + |\nabla v|^2)^{\frac{\mu\varepsilon-2}{4}}|\nabla v|^2 dS.
\]

(4.7)
Lemma 4.4 follows from the well-known assertions, see, e.g., [29, Ch.1, Sec.1.5] for the case $a \equiv 1$, $N \geq 2$, or [2, Lemma A.1] for the case of an arbitrary dimension. Fix an arbitrary point $\xi \in \partial \Omega$ and introduce the local coordinate system $\{y\}$ with the origin $\xi$. The system is chosen so that $y_N$ coincides with the direction $n$. There is a neighborhood of $\xi$ where $\partial \Omega$ is represented in the form $y_N = \omega(y_1, \ldots, y_{N-1})$ with a twice differentiable function $\omega$. In the local coordinates

$$I_{\partial \Omega} \equiv \Delta v(\nabla v \cdot n) - \nabla(\nabla v \cdot n) \cdot \nabla v = \sum_{i=1}^{N-1} (D^2_{y_i y_i} w - D^2_{y_i y_{N-1}} w D_{y_i} w),$$

where $w(y) = v(x)$, and

$$I_{\partial \Omega}(\xi) = - (D_{y_N} w(0))^2 \sum_{i=1}^{N-1} D^2_{y_i y_i} \omega(0) = - (\nabla v(\xi) \cdot n)^2 \sum_{i=1}^{N-1} D^2_{y_i y_i} \omega(0).$$

Since $\omega$ is twice differentiable, then $|I_{\partial \Omega}(\xi)| \leq C|\nabla v(\xi)|^2$ with a constant $C$ depending only on $N$ and $\sup |D^2_{y_i y_j} \omega(y)|$. Estimate (4.7) follows because $\xi \in \partial \Omega$ is arbitrary.

**Lemma 4.4.** Let $\partial \Omega$ be a Lipschitz-continuous surface and $a(\cdot)$ be a nonnegative function on $\overline{\Omega}$. Assume that $a, q \in W^{1,\infty}(\Omega)$, with

$$\|\nabla q\|_{\infty, \Omega} \leq L < \infty, \quad \|\nabla a\|_{\infty, \Omega} \leq L_0 < \infty.$$

There exists a constant $\delta = \delta(\partial \Omega)$ such that for every $u \in W^{1, q}(\Omega)$

$$\delta \int_{\partial \Omega} a(x)(e^2 + |u|^2)^{q(x)-2}|u|^2 dS \leq C \int_{\Omega} \left(a(x)|u|^{q(x)-1} |\nabla u| + a(x)|u|^{q(x)} |\ln |u|| + |u|^{q(x)+1}\right) dx \quad (4.8)$$

with a constant $C = C(q^+, L, L_0, N, \Omega)$.

**Proof.** By [22, Lemma 1.5.1.9] there exists $\delta > 0$ and $\mu \in (C^\infty(\overline{\Omega}))^N$ such that $\mu \cdot n \geq \delta$ a.e. on $\partial \Omega$. By the Green formula

$$\delta \int_{\partial \Omega} a(x)|u|^{q(x)} dS \leq \int_{\Omega} a(x)|u|^{q(x)} (\mu \cdot n) dS = \int_{\Omega} \text{div}(a(x)|u|^{q(x)} \mu) dx$$

$$= \int_{\Omega} \left(a(x) (q(x)|u|^{q(x)-q} u(\nabla u \cdot \mu) + |u|^{q(x)} \ln |u| (\nabla q \cdot \mu) + |u|^{q(x)} \mu) \right) dx$$

$$\leq q^+ \max_{\overline{\Omega}} |\mu| \int_{\Omega} a(x)|u|^{q(x)-1} |\nabla u| dx + \|\nabla q\|_{\infty, \Omega} \max_{\overline{\Omega}} |\mu| \int_{\Omega} a(x)|u|^{q(x)} |\ln |u|| dx + \max_{\overline{\Omega}} |\mu||\nabla a||_{L^\infty(\Omega)} \int_{\Omega} |u|^{q(x)} dx$$

$$\leq C \int_{\Omega} \left(a(x)|u|^{q(x)-1} |\nabla u| + a(x)|u|^{q(x)} |\ln |u|| + |u|^{q(x)}\right) dx$$

with $C = C(N, q^+, L, L_0, \Omega)$. This inequality implies (4.8) because

$$a(x)(e^2 + |u|^2)^{q(x)-2}|u|^2 \leq a(x)(e^2 + |u|^2)^{\frac{q(x)}{2}} \leq C + a(x)|u|^{q(x)}$$

with an independent of $u$ constant $C$.

**Corollary 4.1.** Under the conditions of Lemma 4.4 for every $\lambda \in (0, 1)$ and $\epsilon \in (0, 1)$

$$\int_{\partial \Omega} a(x)(e^2 + |u|^2)^{\frac{q(x)}{2}-2}|u|^2 dS \leq \lambda \int_{\Omega} a(x)(e^2 + |u|^2)^{\frac{q(x)}{2}-2} |\nabla u|^2 dx + L_0 \int_{\Omega} |u|^{q(x)} dx + L \int_{\Omega} a(z)|u|^{q(x)} |\ln |u|| dx + K \quad (4.9)$$
with independent of $u$ constants $K, L, L_0$.

**Proof.** We transform the first term on the right-hand side of (4.8) using the Cauchy inequality:

$$a|u|^{q-1}|\nabla u| \leq (a(e^2 + |u|^2)^{\frac{q-2}{2}}|\nabla u|^2)^{\frac{q-2}{q}} (a(e^2 + |u|^2)^{\frac{q-2}{2}})^{\frac{1}{q}} \leq \lambda a(e^2 + |u|^2)^{\frac{q-2}{2}}|\nabla u|^2 + Ca(e^2 + |u|^2)^{\frac{q}{2}}.$$

**Theorem 4.2.** Let $\partial \Omega \in C^2$, $u \in C^2(\Omega)$ and $u = 0$ on $\partial \Omega$. Assume that $a(\cdot)$ satisfies the conditions of Lemma 4.1. $p(\cdot)$ satisfies the conditions of Lemma 4.2, and

$$q : \Omega \mapsto [q^-, q^+] \subset \left(\frac{2N}{N + 2}, \infty\right), \quad q \in W^{1, \infty}(\Omega), \quad \text{ess sup}_\Omega |\nabla q| = L.$$

If for a.e. $x \in \Omega$

$q(x) < p(x) + r$ with $\frac{2}{N + 2} < r < \frac{4p^+}{p^-(N + 2) + 2N}$

then for every $\lambda \in (0, 1)$

$$\int_{\partial \Omega} \gamma_c(x, \nabla u)|\nabla u|^2 dS \leq \lambda \int_{\Omega} \gamma_c(x, \nabla u)|u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx\right)$$

with a constant $C$ depending on $\lambda$ and the constants $p^\pm$, $N, L, L_0$, but independent of $u$.

**Proof.** Applying (4.9) to $|\nabla u|$ we obtain

$$\int_{\partial \Omega} a(x)(e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|\nabla u|^2 dS \leq \lambda \int_{\Omega} a(x)(e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|u_{xx}|^2 dx + L_0 \int_{\Omega} |\nabla u|^{q(x)} dx + L \int_{\Omega} |\nabla u|^{q(x)} |\nabla u|^2 dx + K,$$

with independent of $u$ constants $L, K, L_0$. Choose $0 < r_1 < r_2 < r^*$ so small that $q(x) + r_1 < p(x) + r_2$ and

$$|\nabla u|^{q(x)} |\nabla u|^2 \leq \begin{cases} |\nabla u|^{q(\cdot)+r_1}(|\nabla u|^{q(\cdot)-r_1} |\nabla u|) \leq C(r_1, q^+)|\nabla u|^{q(\cdot)+r_1} \quad &\text{if } |\nabla u| \geq 1, \\ |\nabla u|^{q^+} |\nabla u|^2 \leq C(q^-) \quad &\text{if } |\nabla u| \in (0, 1). \end{cases}$$

with a constant $C$ independent of $u$. Thus, there exists a constant $C$ such that

$$|\nabla u|^{q(\cdot)} |\nabla u|^2 \leq C(1 + |\nabla u|^{q(\cdot)+r_1}) \leq C(1 + |\nabla u|^{p(x)+r_2}) \in \Omega.$$

Using this inequality and then applying Lemma 4.2 we continue (4.11) as follows:

$$\int_{\partial \Omega} a(x)(e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|\nabla u|^2 dS \leq \lambda \int_{\Omega} a(x)(e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(\cdot)+r_2} dx\right)$$

$$\leq \lambda \int_{\Omega} a(x)(e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|u_{xx}|^2 dx + \lambda \int_{\Omega} (e^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}}|u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(\cdot)} dx\right)$$

$$= \lambda \int_{\Omega} \gamma_c(x, \nabla u)|u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(\cdot)} dx\right).$$

Adding to this inequality the inequality corresponding to $q = p$ and $a \equiv 1$, we arrive at (4.10).
Since \( \mathcal{P}_m \subset C^1(\Omega) \cap H^1_0(\Omega) \), the interpolation inequalities of this section remain true for every function \( w \in \mathcal{P}_m, m \in \mathbb{N} \).

5. Regularized problem

Given \( \epsilon > 0 \), let us consider the following family of regularized double phase parabolic equations:

\[
\begin{cases}
\partial_t u - \text{div}(\gamma_{\epsilon}(z, \nabla u)\nabla u) = F(z,u) & \text{in } Q_T, \\
u = 0 & \text{on } \Gamma_T, \\
u(0,.) = u_0 & \text{in } \Omega, \ \epsilon \in (0,1),
\end{cases}
\]  

(5.1)

where \( F(z,u) \) is defined in (4.1). By Lemma 2.2 the functions \( u_{\epsilon}^{(m)} \) are absolutely continuous and differentiable a.e. in \( (0,T_m) \), and the a priori estimates (6.26), (6.28) in the case \( \epsilon \neq 0 \), show that for every \( m \) the function \( u_{\epsilon}^{(m)}(x,T_m) \) belongs to \( \text{span}\{\phi_1, \ldots, \phi_m\} \) and satisfies the estimate

\[
\|\nabla u_{\epsilon}^{(m)}(\cdot,T_m)\|_{L^2(\Omega)}^2 + \mathcal{F}(u_{\epsilon}(\cdot,T_m),T_m) \leq C + \|f_0\|_{L^2(\Omega)}^2 + \|\nabla u_0^{(m)}\|_{L^2(\Omega)}^2 + \mathcal{F}(u_0^{(m)},0)
\]

with the function \( \mathcal{F} \) defined in (5.5) and a constant \( C \) independent of \( m \) and \( \epsilon \). This estimate allows one to continue each of \( u_{\epsilon}^{(m)} \) to the maximal existence interval \( (0,T) \).

5.1. Galerkin’s method

Let \( \epsilon > 0 \) be a fixed parameter. The sequence \( \{u_{\epsilon}^{(m)}\} \) of finite-dimensional Galerkin’s approximations for the solutions of the regularized problem (5.1) is sought in the form

\[
u_{\epsilon}^{(m)}(x,t) = \sum_{j=1}^{m} u_j^{(m)}(t)\phi_j(x)
\]  

(5.2)

where \( \phi_j \in W^{1,2}_0(\Omega) \) and \( \lambda_j > 0 \) are the eigenfunctions and the corresponding eigenvalues of problem (2.10).

The coefficients \( u_j^{(m)}(t) \) are characterized as the solutions of the Cauchy problem for the system of \( m \) ordinary differential equations

\[
\begin{cases}
(\gamma^{(m)})(t) = -\int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \nabla u_{\epsilon}^{(m)} \cdot \nabla \phi_j \, dx + \int_{\Omega} F(z, u_{\epsilon}^{(m)}) \phi_j \, dx, \\
u_j^{(m)}(0) = (u_0^{(m)}, \phi_j)_{2,\Omega}, \quad j = 1, 2, \ldots, m,
\end{cases}
\]  

(5.3)

where \( \gamma_{\epsilon} \) is defined in (4.1). By Lemma 2.2 the functions \( u_0^{(m)} \in \mathcal{P}_m \) can be chosen so that

\[
u_0^{(m)}(x) \in W^{1,2}_0(\Omega) \quad \text{if } \max_{x \in \Omega} q(x,0) \leq 2,
\]

\[
u_0^{(m)}(x) \in \mathcal{V}_0(\Omega) \quad \text{if } \max_{x \in \Omega} q(x,0) > 2.
\]  

(5.4)

By the Carathéodory existence theorem, for every finite \( m \) system (5.3) has a solution \( (u_1^{(m)}, u_2^{(m)}, \ldots, u_m^{(m)}) \) in the extended sense on an interval \( (0,T_m) \), the functions \( u_j^{(m)}(t) \) are absolutely continuous and differentiable a.e. in \( (0,T_m) \). The a priori estimates (6.15), (6.19) in the case \( b \equiv 0 \), and (6.26), (6.28) in the case \( b \neq 0 \), show that for every \( m \) the function \( u_{\epsilon}^{(m)}(x,T_m) \) belongs to \( \text{span}\{\phi_1, \ldots, \phi_m\} \) and satisfies the estimate

\[
\|\nabla u_{\epsilon}^{(m)}(\cdot,T_m)\|_{L^2(\Omega)}^2 + \mathcal{F}(u_{\epsilon}(\cdot,T_m),T_m) \leq C + \|f_0\|_{L^2(\Omega)}^2 + \|\nabla u_0^{(m)}\|_{L^2(\Omega)}^2 + \mathcal{F}(u_0^{(m)},0)
\]
6. A priori estimates

6.1. A priori estimates I: the case $b \equiv 0$

**Lemma 6.1.** Let $\Omega$ be a bounded domain with the boundary $\partial \Omega \in \text{Lip}$, $p(\cdot), q(\cdot)$ satisfy (3.3), $u_0 \in L^2(\Omega)$ and $f_0 \in L^2(Q_T)$. If $b \equiv 0$, then $u_{\epsilon}^{(m)}$ satisfies the estimates

$$\sup_{t \in (0,T)} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dz \leq C_1 \epsilon T (\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2)$$

and

$$\int_{Q_T} \left( |\nabla u_{\epsilon}^{(m)}|^{p(z)} + a(z) |\nabla u_{\epsilon}^{(m)}|^{q(z)} \right) \, dz \leq C_2 \int_{Q_T} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dz + C_3$$

where the constants $C_i$ are independent of $\epsilon$ and $m$.

**Proof.** By multiplying $j$th equation of (3.3) by $u_{j}^{(m)}(t)$ and then by summing up the results for $j = 1, 2, \ldots, m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 = \sum_{j=1}^{m} u_{j}^{(m)}(t)u_{j}^{(m)}(t) = -\sum_{j=1}^{m} u_{j}^{(m)}(t) \int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \nabla u_{\epsilon}^{(m)} \cdot \nabla \phi_j \, dx$$

$$+ \sum_{j=1}^{m} \int_{\Omega} f_0(x, t) \phi_j(x) u_{j}^{(m)}(t) \, dx$$

$$= -\int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dx + \int_{\Omega} f_0(x, t) u_{j}^{(m)} \, dx.$$

Using the Cauchy inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dx \leq \frac{1}{2} \|f_0(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2.$$  

Now, rewriting the last inequality in the equivalent form

$$\frac{1}{2} \frac{d}{dt} \left( e^{-t} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 \right) + e^{-t} \int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dx \leq \frac{e^{-t}}{2} \|f_0(\cdot, t)\|_{2,\Omega}^2$$

and integrating with respect to $t$, we arrive at the inequality

$$\sup_{t \in (0,T)} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 + \int_{Q_T} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 \, dx \, dt \leq C e^{T} (\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2)$$

where the constant $C$ is independent of $\epsilon$ and $m$. Since $a(z)$ is a nonnegative bounded function, the second assertion follows from (6.5) and the inequality

$$a(z)|\nabla u_{\epsilon}^{(m)}|^{q(z)} \leq a(z) \left( \epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2 \right)^{\frac{q(z)}{2}} \leq \begin{cases} 2a(z) \left( \epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2 \right)^{\frac{q(z)-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 \text{ if } |\nabla u_{\epsilon}^{(m)}| \geq \epsilon, \\ (2\epsilon^2)^{\frac{q(z)}{2}} a(z) \leq 2^{q(z)} a(z) \text{ otherwise.} \end{cases}$$

\[\square\]
Lemma 6.2. Let \( \Omega \) be a bounded domain with \( \partial \Omega \in C^k, k \geq 2 + \frac{N}{2} \). Assume that \( p(\cdot), q(\cdot) \) satisfies (3.1), (3.2), (3.3) and and \( a(\cdot) \) satisfies (3.4). If \( u_0 \in W_0^{1,2}(\Omega), f_0 \in L^2((0,T); W_0^{1,2}(\Omega)) \) and \( b \equiv 0 \), then for a.e. \( t \in (0,T) \) the following inequality holds:

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_\epsilon^{(m)}(\cdot,t) \|^2_{2,\Omega} + \int_0^t \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 \, dx \\
\leq C_1 \left( 1 + \int_0^t |\nabla u_\epsilon^{(m)}|^p(z) \, dx + \| \nabla u_\epsilon^{(m)}(\cdot,t) \|^2_{2,\Omega} + \| f_0(\cdot,t) \|^2_{W_0^{1,2}(\Omega)} \right)
\] (6.7)

with independent of \( m \) and \( \epsilon \) constants \( 0 < C_0 < \min\{p^-, 1\} \) and \( C_1 > 0 \).

Proof. Let us multiply each of equations in (3.1) by \( \gamma_ju_j^{(m)} \) and sum up the results for \( j = 1, 2, \ldots, m \):

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_j^{(m)}(\cdot,t) \|^2_{2,\Omega} = \sum_{j=1}^m \lambda_j u_j^{(m)} \partial_t u_j^{(m)}(t)
\]

\[
= \sum_{j=1}^m \lambda_j u_j^{(m)} \int_\Omega \text{div}(\gamma(z, \nabla u_\epsilon^{(m)}) \nabla u_j^{(m)}) \phi_j \, dx + \sum_{j=1}^m \lambda_j u_j^{(m)} \int_\Omega f_0(x,t) \phi_j \, dx
\]

\[
= - \int_\Omega \text{div}(\gamma(z, \nabla u_\epsilon^{(m)}) \nabla u_j^{(m)}) u_j^{(m)} \, dx - \int_\Omega f_0(x,t) u_j^{(m)} \, dx.
\] (6.8)

Since \( \partial \Omega \in C^k \) with \( k \geq 2 + \frac{N}{2} \), then \( u_\epsilon^{(m)}(\cdot,t) \in P_m \subset H^1_0(\Omega) \cap C^1(\overline{\Omega}) \). Therefore the first term on the right-hand of (6.8) can be transformed by means of the Green formula:

\[
- \int_\Omega \text{div} \left( \gamma(z, \nabla u_\epsilon^{(m)}) \nabla u_j^{(m)} \right) u_j^{(m)} \, dx
\]

\[
= - \int_\Omega \left( \sum_{k=1}^N \left( u_j^{(m)} \right)_{x_k} \right) \left( \sum_{i=1}^N \left( \gamma(z, \nabla u_\epsilon^{(m)}) (u_\epsilon^{(m)})_{x_i} \right) \right) \, dx
\]

\[
= - \int_{\partial \Omega} \Delta u_j^{(m)} \gamma(z, \nabla u_\epsilon^{(m)}) \nabla u_j^{(m)} \cdot n \, dS + \int_{\Omega} \sum_{k=1}^N \left( u_j^{(m)} \right)_{x_k} \gamma(z, \nabla u_\epsilon^{(m)}) (u_\epsilon^{(m)})_{x_i} \, dx
\]

\[
= - \int_{\partial \Omega} \gamma(z, \nabla u_\epsilon^{(m)}) \sum_{k=1}^N \left( u_j^{(m)} \right)_{x_k} (u_\epsilon^{(m)})_{x_k} \, dS
\]

\[
- \int_{\Omega} \sum_{k=1}^N \left( u_j^{(m)} \right)_{x_k} \gamma(z, \nabla u_\epsilon^{(m)}) (u_\epsilon^{(m)})_{x_k} \, dx
\]

\[
- \int_{\Omega} \gamma(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 \, dx + J_1 + J_2 + J_3 + J_4,
\]

where \( n = (n_1, \ldots, n_N) \) is the outer normal vector to \( \partial \Omega \),

\[
J_1 := \int_\Omega (2 - p(z))(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{(\epsilon^2 - 2)}{2}} \left( \sum_{k=1}^N \left( \nabla u_\epsilon^{(m)}, \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) \, dx
\]

\[
+ \int_\Omega (2 - q(z))a(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{(\epsilon^2 - 2)}{2}} \left( \sum_{k=1}^N \left( \nabla u_\epsilon^{(m)}, \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) \, dx,
\]

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The terms on the right-hand side of (6.9) are estimated in three steps. Indeed:

**Step 1:** estimate on $J = \int \gamma(z, \nabla u_{\epsilon}^{(m)}) \left( \Delta u_{\epsilon}^{(m)}(\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) - \nabla u_{\epsilon}^{(m)} \cdot \nabla(\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) \right) dS,$

$$J_a = - \int \sum_{i,k=1}^{N} a_{x_k}(u_{\epsilon}^{(m)})_{x_i} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{\alpha(z) - 2}{2}} (u_{\epsilon}^{(m)})_{x_k x_i} dS,$$

Substitution into (6.8) leads to the inequality

$$\frac{1}{2} \frac{d}{dt} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 + \int \gamma(z, \nabla u_{\epsilon}^{(m)}) (u_{\epsilon}^{(m)})_{xx}^2 dx = J_1 + J_2 + J_{\partial\Omega} + J_a - \int \nabla f_0 \cdot \nabla u_{\epsilon}^{(m)} dx \leq J_1 + J_2 + J_{\partial\Omega} + J_a + \frac{1}{2} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|f_0(\cdot, t)\|_{W_{0,\epsilon}^{1,2}(\Omega)}^2,$$

The terms on the right-hand side of (6.9) are estimated in three steps.

**Step 1:** estimate on $J_1.$ Since $a(z) \geq 0$ and $p(z) < q(z)$ in $Q_T,$ the term $J_1$ is merged in the left-hand side. Indeed:

$$J_1 = \int_{\{x \in \Omega, p(z) \geq 2\}} (2 - p(z)) \ldots + \int_{\{x \in \Omega, p(z) < 2\}} (2 - p(z)) \ldots + \int_{\{x \in \Omega, q(z) \geq 2\}} (2 - q(z)) \ldots + \int_{\{x \in \Omega, q(z) < 2\}} (2 - q(z)) \ldots \leq \int_{\{x \in \Omega, p(z) \geq 2\}} (2 - p(z))(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{\alpha(z) - 2}{2} - 1} \left( \sum_{k=1}^{N} \left( \nabla u_{\epsilon}^{(m)} \cdot \nabla (u_{\epsilon}^{(m)})_{x_k} \right)^2 \right) dx + \int_{\{x \in \Omega, q(z) < 2\}} (2 - q(z)) a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{\alpha(z) - 2}{2} - 1} \left( \sum_{k=1}^{N} \left( \nabla u_{\epsilon}^{(m)} \cdot \nabla (u_{\epsilon}^{(m)})_{x_k} \right)^2 \right) dx,$$

whence

$$|J_1| \leq \max \{0, 2 - p^-\} \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{\alpha(z) - 2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + \max \{0, 2 - q^-\} \int_{\Omega} a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{\alpha(z) - 2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx.$$
Step 2: estimate on \( J_2 \). By the Cauchy inequality, for every \( \delta_0 > 0 \)

\[
|J_2| \leq \frac{1}{2} ||q||_{\infty, \Omega} \int_{\Omega} \left( (\epsilon^2 + |\nabla u^m|)^{\frac{p(z)-2}{2}} \sum_{k,i=1}^N |(u^m_{i,k})_{x_k x_i}| \right) \times \left( (u^m_{i,k})_{x_k}|| \ln(\epsilon^2 + |\nabla u^m|)(\epsilon^2 + |\nabla u^m|)^{\frac{p(z)-2}{2}} \right) dx
\]

\[
+ \frac{1}{2} ||\nabla q||_{\infty, \Omega} \int_{\Omega} \left( (a(z))^{\frac{1}{2}} (\epsilon^2 + |\nabla u^m|)^{\frac{p(z)-2}{2}} \sum_{k,i=1}^N |(u^m_{i,k})_{x_k x_i}| \right) \times \left( (a(z))^{\frac{1}{2}} |(u^m_{i,k})_{x_k}|| \ln(\epsilon^2 + |\nabla u^m|)(\epsilon^2 + |\nabla u^m|)^{\frac{p(z)-2}{2}} \right) dx
\]

(6.10)

\[
\leq \delta_0 \int_{\Omega} \gamma_c(z, \nabla u^m) \sum_{k,i=1}^N |(u^m_{i,k})_{x_k x_i}|^2 dx
\]

\[
+ C_1 \int_{\Omega} \ln^2(\epsilon^2 + |\nabla u^m|)^2) \gamma_c(z, \nabla u^m)|\nabla u^m|^2 dx
\]

with a constant \( C_1 = C_1(C^*, C^{**}, N, \delta_0) \). Let us denote

\[
\mathcal{M} = C_1 \int_{\Omega} \ln^2(\epsilon^2 + |\nabla u^m|)^2) \gamma_c(z, \nabla u^m)|\nabla u^m|^2 dx.
\]

For \( \mu_1 \in (0, 1) \) and \( y > 0 \) the following inequality holds:

\[
y^\frac{p(z)}{2} \ln^2 y \leq \begin{cases} y^{\frac{p(z)}{2}} (y - \frac{1}{2} \ln^2 y) \leq C(\mu_1, p^+)(y^{\frac{p(z)}{2}}) & \text{if } y \geq 1, \\ y^\frac{p(z)}{2} \ln^2 y \leq C(p^+) & \text{if } y \in (0, 1). \end{cases}
\]

(6.11)

Let

\[
r_\ast = \frac{2}{N + 2} \text{ and } r^* = \frac{4p^-}{p^-(N + 2) + 2N}.
\]

(6.12)

Take the numbers \( r_1, r_2 \) such that

\[
r_1 \in (r_\ast, r^*), \quad r_2 \in (0, 1), \quad q(z) + r_2 \leq p(z) + r_1 < p(z) + r^*
\]

and estimate \( \mathcal{M} \) applying (6.11):

\[
\mathcal{M} \leq C \left( 1 + \int_{\Omega} (\epsilon^2 + |\nabla u^m|)^{\frac{p(z) + r_1 - 2}{2}} |\nabla u^m|^2 dx + \int_{\Omega} a(z)(\epsilon^2 + |\nabla u^m|)^{\frac{p(z) - r_1 - 2}{2}} |\nabla u^m|^2 dx \right)
\]

with a constant \( C = C(C_1, r_1, r_2) \). Let us transform the integrand of the second integral using the following inequality:

\[
(\epsilon^2 + |\nabla u^m|)^{\frac{p(z) + r_1 - 2}{2}} |\nabla u^m|^2 \leq (\epsilon^2 + |\nabla u^m|)^{\frac{p(z) + r_1}{2}} \leq 1 + (\epsilon^2 + |\nabla u^m|)^{\frac{p(z) + r_1}{2}} \leq 1 + \left\{ \begin{array}{ll} (2\epsilon^2)^{\frac{p(z) + r_1}{2}} & \text{if } |\nabla u^m| < \epsilon, \\ 2(\epsilon^2 + |\nabla u^m|^2)^{\frac{p(z) + r_1 - 2}{2}} |\nabla u^m|^2 & \text{if } |\nabla u^m| \geq \epsilon. \end{array} \right.
\]

(6.13)

Using (6.13) and the interpolation inequality of Lemma 4.12 we finally obtain

\[
\mathcal{M} \leq C \left( 1 + \int_{\Omega} (\epsilon^2 + |\nabla u^m|)^{\frac{p(z) + r_1 - 2}{2}} |\nabla u^m|^2 dx \right)
\]

\[
\leq \delta_1 \int_{\Omega} (\epsilon^2 + |\nabla u^m|)^{\frac{p(z) - 2}{2}} |(u^m_{i,k})_{x_k}|^2 dx + C \left( 1 + \int_{\Omega} |\nabla u^m| |\nabla u^m|^{dz} dx \right)
\]

(6.14)
with any \( \delta_1 \in (0,1) \) and \( C = C(\delta_1) \). Gathering (6.10) and (6.14), we finally obtain:

\[
|J_2| \leq \left( \delta_0 + \delta_1 \right) \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}|^2 \, dx + C \left( 1 + \int_{\Omega} |\nabla u^{(m)}|^{1/p(z)} \, dx \right)
\]

with a constant \( C \) depending on \( \delta_1 \) and \( \|a(\cdot,t)\|_{\infty,\Omega} \), but independent of \( \varepsilon \) and \( m \).

**Step 3:** estimates on \( J_\partial \) and \( J_{\partial \Omega} \). Let \( \rho \in (r_*, r^*) \) be such that \( 2q(z) - p(z) < p(z) + \rho < p(z) + r^* \).

Applying Young’s inequality and (6.13) we obtain the estimate

\[
|J_a| \leq \sum_{i,k=1}^{N} |a_{ik}| \| (u^{(m)})_{xx} \| \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}| \, dx
\]

\[
\leq \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}|^2 \, dx + C(\delta) \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} \, dx
\]

\[
\leq \tilde{\delta} \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}|^2 \, dx + C' \left( 1 + \int_{\Omega} |\nabla u^{(m)}|^{q} \, dx \right)
\]

where \( C' = C'(\|\nabla u\|_{\infty,\Omega}, N, q) \) is independent of \( \varepsilon \) and \( m \). By Lemma 4.2 we obtain

\[
|J_a| \leq \delta_2 \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}|^2 \, dx + C \left( 1 + \int_{\Omega} |\nabla u^{(m)}|^{p(z)} \, dx \right)
\]

for any \( \delta_2 \in (0,1) \) and a constant \( C \) independent of \( \varepsilon \) and \( m \).

To estimate \( J_{\partial \Omega} \) we use Lemma 4.3 and Theorem 4.2.

\[
|J_{\partial \Omega}| \leq \left| \int_{\partial \Omega} \gamma(r, \nabla u^{(m)})(\nabla u^{(m)}, \nabla (\nabla u^{(m)}), n) \, dS \right|
\]

\[
\leq C \int_{\partial \Omega} |\nabla u^{(m)}| |\nabla u^{(m)}| \, dS
\]

\[
\leq \delta_3 \int_{\Omega} \left( \varepsilon^2 + |\nabla u^{(m)}|^{2} \right)^{\frac{p(z)-2}{2}} |(u^{(m)})_{xx}|^2 \, dx + C \left( 1 + \int_{\Omega} |\nabla u^{(m)}|^{p(z)} \, dx \right)
\]

with an arbitrary \( \delta_3 \in (0,1) \) and \( C \) depending upon \( \delta_3, p, q, a, \partial \Omega \) and their differential properties, but not on \( \varepsilon \) and \( m \). To complete the proof and obtain (6.7), we gather the estimates of \( J_1, J_2, J_a, J_{\partial \Omega} \) and choose \( \delta_i \) so small that

\[
\min \{ 1, p^* - 1 \} = \sum_{i=0}^{3} \delta_i = \eta > 0.
\]

\[\]

**Lemma 6.3.** Under the conditions of Lemma 6.2,

\[
\sup_{(0,T)} \| \nabla u^{(m)}(\cdot, t) \|_{L^2(\Omega)}^2 + \int_{Q_T} \gamma(r, \nabla u^{(m)})(u^{(m)}_{xx})^2 \, dz \leq C' \left( 1 + \|u_0\|_{L^2(\Omega)}^2 + \|f_0\|_{L^2(0,T;W^{1,2}(\Omega))}^2 \right)
\]

and

\[
\int_{Q_T} |\nabla u^{(m)}|^{q(z)} \, dz + \int_{Q_T} |\nabla u^{(m)}|^{p(z)+r} \, dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^*}{p^* - (N + 2) + 2N}
\]

with constants \( C', C'', C''' \) independent of \( m \) and \( \varepsilon \).
Proof. Multiplying (6.7) by $e^{-2C_1 t}$ and simplifying, we obtain the following differential inequality:

\[
\frac{d}{dt} \left( e^{-2C_1 t} \| \nabla u^{(m)}_\varepsilon(t) \|^2_{L^2(\Omega)} \right) \leq C e^{-2C_1 t} \left( 1 + \int_{\Omega} |\nabla u^{(m)}_\varepsilon|^2 dx + \|f_0(\cdot, t)\|^2_{L^2_{0,T,W_{0,2}^{1,2}}(\Omega)} \right).
\]

Integrating it with respect to $t$ and taking into account (6.1) and (6.2) we arrive at the following estimate: for every $t \in [0, T]

\[
\| \nabla u^{(m)}_\varepsilon(t) \|^2_{L^2(\Omega)} \leq C e^{C(1 + \int_{\Omega} |\nabla u^{(m)}_\varepsilon|^2 dx + \|f_0(\cdot, t)\|^2_{L^2_{0,T,W_{0,2}^{1,2}}(\Omega)})}.
\]

Substitution of the above estimate into (6.1) gives

\[
\frac{1}{2} \int_{\Omega} \frac{d}{dt} \| \nabla u^{(m)}_\varepsilon(t) \|^2_{L^2(\Omega)} + C_0 \int_{\Omega} \beta(z, |\nabla u^{(m)}_\varepsilon|^2) dx \leq C_1 \left( 1 + \int_{\Omega} |\nabla u^{(m)}_\varepsilon|^2 dx + \|f_0(\cdot, t)\|^2_{L^2_{0,T,W_{0,2}^{1,2}}(\Omega)} \right).
\]

Integrating it with respect to $t$ and using (6.2) to estimate the integral of $|\nabla u^{(m)}_\varepsilon|^2$ on the right-hand side, we obtain

\[
\int_{Q_T} |\nabla u^{(m)}_\varepsilon|^2 d(z) \leq C e^{C(1 + \int_{\Omega} |\nabla u^{(m)}_\varepsilon|^2 dx + \|f_0(\cdot, t)\|^2_{L^2_{0,T,W_{0,2}^{1,2}}(\Omega)})}.
\]

To prove estimate (6.10), we make use of Theorem 4.1. Let us fix a number $r \in (r_*, r^*)$ with $r_*, r^*$ defined in (6.12). Split the cylinder $Q_T$ into the two parts $Q^+_T = Q_T \cap \{p(z) + r \geq 2\}$, $Q^-_T = Q_T \cap \{p(z) + r < 2\}$ and represent

\[
\int_{Q_T} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z) = \int_{Q^+_T} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z) + \int_{Q^-_T} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z) \equiv I_+ + I_-.
\]

Since

\[
I_+ \leq \int_{Q^+_T} (\epsilon^2 + |\nabla u^{(m)}_\varepsilon|^2) \frac{p(z)+r-2+2}{2} |\nabla u^{(m)}_\varepsilon|^2 d(z) \leq \int_{Q^+_T} (\epsilon^2 + |\nabla u^{(m)}_\varepsilon|^2) \frac{p(z)+r-2}{2} |\nabla u^{(m)}_\varepsilon|^2 d(z),
\]

the estimate on $I_+$ follows immediately from Theorem 4.1 and (6.15). To estimate $I_-$, we set $B_+ = Q^+_T \cap \{z : |\nabla u^{(m)}_\varepsilon| \geq \epsilon\}$, $B_- = Q^-_T \cap \{z : |\nabla u^{(m)}_\varepsilon| < \epsilon\}$. The estimate on $I_-$ follows from Theorem 4.1 and (6.15) because

\[
I_- = \int_{B_+} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z) = \int_{B_+} (\epsilon^2 + |\nabla u^{(m)}_\varepsilon|^2) \frac{p(z)+r-2}{2} |\nabla u^{(m)}_\varepsilon|^2 d(z) + \int_{B_-} \epsilon^{p(z)+r} d(z)
\]

\[
\leq \int_{B_+} (\epsilon^2 + |\nabla u^{(m)}_\varepsilon|^2) \frac{p(z)+r-2}{2} |\nabla u^{(m)}_\varepsilon|^2 d(z) + \epsilon^{p(z)+r} d(z)
\]

\[
\leq C \left( 1 + \int_{Q_T} (\epsilon^2 + |\nabla u^{(m)}_\varepsilon|^2) \frac{p(z)+r-2}{2} |\nabla u^{(m)}_\varepsilon|^2 d(z) \right).
\]

By combining the above estimates, using the Young inequality, and applying (6.15), (6.2) and Theorem 4.1 we obtain (6.16) with $r \in (r_*, r^*)$:

\[
\int_{Q_T} |\nabla u^{(m)}_\varepsilon|^2 d(z) + \int_{Q_T} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z) \leq 1 + \int_{Q_T} |\nabla u^{(m)}_\varepsilon|^{p(z)+r} d(z)
\]

\[
\leq C \left( 1 + \int_{Q_T} \beta(z, |\nabla u^{(m)}_\varepsilon|^2) d(z) \right).
\]

If $r \in (0, r_*)$, the required inequality follows from Young’s inequality. \qed
Proof. By multiplying (5.3) with \(m\) we rewrite (6.20) as

\[
C \text{ with an independent of } \epsilon \text{ and } m \text{ constant } C.
\]

**Remark 6.1.** Under the conditions of Lemma 6.3

\[
\int_{Q_T} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)}{p(z)-1}} \, dz \leq C, \quad \epsilon \in (0, 1),
\]

(6.17)

with an independent of \(\epsilon\) and \(m\) constant \(C\).

**Corollary 6.1.** Let condition (5.5) be fulfilled. Under the conditions of Lemma 6.3

\[
\|u_{\epsilon}^{(m)}\|_{q(\cdot), Q_T} \leq C \quad \text{with a constant } C \text{ independent of } m \text{ and } \epsilon.
\]

**Proof.** Condition (5.5) entails the inequality

\[
\frac{q(z)(p(z) - 1)}{q(z) - 1} \leq q(z) \leq p(z) + r.
\]

By Young’s inequality, the assertion follows then from (6.17):

\[
\int_{Q_T} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-1}{p(z)}} \, dz \leq C \left( 1 + \int_{Q_T} |\nabla u_{\epsilon}^{(m)}|^{p(z)+r} \, dz \right) \leq C.
\]

\[\square\]

**Lemma 6.4.** Assume that in the conditions of Lemma 6.3 \(u_0 \in W^{1,2}_0(\Omega) \cap V_0(\Omega)\). Then

\[
\|u_{\epsilon}^{(m)}\|^2_{2, Q_T} + \sup_{(0,T)} \int_{\Omega} \left( (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|)^{\frac{p(z)}{p(z)-1}} + a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|)^{\frac{2(p(z)-1)}{2}} \right) \, dx
\]

\[
\leq C \left( 1 + \int_{\Omega} \left( |\nabla u_0|^p + a(x,0)|\nabla u_0|^q \right) \, dx \right) + \|u_0\|^2_{2, Q_T}
\]

(6.19)

with an independent of \(m\) and \(\epsilon\) constant \(C\), which depends on the constants in conditions (3.2).

**Proof.** By multiplying (5.3) with \((u_{\epsilon}^{(m)})_t\) and summing over \(j = 1, 2, \ldots, m\) we obtain the equality

\[
\int_{\Omega} (u_{\epsilon}^{(m)})_t^2 \, dx + \int_{\Omega} \gamma(z, \nabla u_{\epsilon}^{(m)}) \cdot \nabla(u_{\epsilon}^{(m)})_t \, dx = \int_{\Omega} f_0(u_{\epsilon}^{(m)})_t \, dx.
\]

(6.20)

Using the identity

\[
a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)-2}{q(z)}} \nabla u_{\epsilon}^{(m)} \cdot \nabla(u_{\epsilon}^{(m)})_t = \frac{d}{dt} \left( \frac{a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)}{q(z)}}}{q(z)} \right)
\]

\[
+ \frac{a(z)q_t(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{2(p(z)-1)}{q(z)}}}{q^2(z)} \left( 1 - \frac{q(z)}{2} \ln((\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)) \right) - \frac{a_t(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)}{q(z)}}}{q(z)}
\]

we rewrite (6.20) as

\[
\|u_{\epsilon}^{(m)}(t, .)\|^2_{2, \Omega} + \frac{d}{dt} \int_{\Omega} \left( (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)}{p(z)-1}} + a(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{2(p(z)-1)}{2}} \right) \, dx
\]

\[
= \int_{\Omega} f_0(u_{\epsilon}^{(m)})_t \, dx - \int_{\Omega} p_t(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)}{p(z)-1}} \left( 1 - \frac{p(z)}{2} \ln(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2) \right) \, dx
\]

\[
- \int_{\Omega} a(z)q_t(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{2(p(z)-1)}{q(z)}} \left( 1 - \frac{q(z)}{2} \ln((\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)) \right) \, dx
\]

\[
+ \int_{\Omega} a_t(z)(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)}{q(z)}} \, dx
\]

(6.21)

\[\equiv \int_{\Omega} f_0(u_{\epsilon}^{(m)})_t \, dx + J_1 + J_2 + J_3.
\]
The first term on the right-hand side of (6.21) is estimated by the Cauchy inequality:

$$\left| \int_\Omega f_0(u^{(m)}_t) \, dx \right| \leq \frac{1}{2} \| (u^{(m)}_t) (\cdot, t) \|^2_{L^2(\Omega)} + \frac{1}{2} \| f_0 (\cdot, t) \|^2_{L^2(\Omega)}. \quad (6.22)$$

To estimate a priori estimates II: the case when the equation contains the nonlinear source.

We proceed to derive a priori estimates in the case when the equation contains the nonlinear source.

1) Let us multiply the $j$th equation of (5.3) by $u_j^{(m)}$ and sum up. In the result we arrive at equality (6.3) with the right-hand side containing the additional term

$$I_0 \equiv \int_\Omega b(z) |u^{(m)}_r| \sigma(z) \, dx.$$ 

Let $2(\sigma^+ - 1) < p^-$. Using the inequalities of Young and Poincaré we find that for every $t \in (0, T)$

$$|I_0| \leq B \left( 1 + \int_\Omega |u^{(m)}_r|^{2(\sigma^+ - 1)} \, dx + \int_\Omega |u^{(m)}_{r, t}|^2 \, dx \right)$$

$$\leq C_\delta + \delta \int_\Omega |u^{(m)}_r|^{p^-} \, dx + \int_\Omega |u^{(m)}_{r, t}|^2 \, dx$$

$$\leq C_\delta + \tilde{C} \delta \int_\Omega |\nabla u^{(m)}_r|^{p^-} \, dx + C \int_\Omega |u^{(m)}_{r, t}|^2 \, dx$$

$$\leq C_\delta + \tilde{C} \delta \int_\Omega |\nabla u^{(m)}_r|^{p} \, dx + C \int_\Omega |u^{(m)}_{r, t}|^2 \, dx$$

where $\delta \in (0, 1)$ is an arbitrary constant and $\tilde{C}$ is the constant from inequality (2.6) with $r = p^-$. We plug this estimate into (6.4) and use (6.6) with $a \equiv 1$ and $q$ substituted by $p$. Choosing $\delta$ sufficiently small, we transform (6.4) to the form

$$\frac{1}{2} \frac{d}{dt} \| u^{(m)}_r (\cdot, t) \|^2_{L^2(\Omega)} + (1 - C\delta) \int_\Omega |\nabla u^{(m)}_r| |\nabla u^{(m)}_{r, t}|^2 \, dx \leq C' \left( 1 + \| f_0 (\cdot, t) \|^2_{L^2(\Omega)} + \| u^{(m)}_{r, t} (\cdot, t) \|^2_{L^2(\Omega)} \right).$$

Integrating this inequality in $t$ we obtain the following counterpart of Lemma (6.4).
Lemma 6.5. Assume that $a(\cdot)$, $p(\cdot)$, $q(\cdot)$, $u_0$, $f_0$ satisfy the conditions of Lemma [6.4]. If $\sigma, b$ are measurable and bounded functions in $Q_T$ and $1 < \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}$, then

$$
\sup_{t \in (0,T]} \|u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_c(z, \nabla u^{(m)}_t)|\nabla u^{(m)}_t|^2 \, dz \leq C_1 e^{T}(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2) + C_0 \tag{6.23}
$$

and

$$
\int_{Q_T} \left( |\nabla u^{(m)}_t|^p(z) + a(z)|\nabla u^{(m)}_t|^p(z) \right) \, dx \, dt \leq C_2 \int_{Q_T} \gamma_c(z, \nabla u^{(m)}_t)|\nabla u^{(m)}_t|^2 \, dz + C_3 \tag{6.24}
$$

with independent of $\epsilon$ and $m$ constants $C_i$.

2) Estimate on $\|\nabla u^{(m)}_t(t)\|_{2,\Omega}$. We follow the proof of Lemma [6.2] multiplying each of equations in [5.3] by $\lambda u^{(m)}_t$ and summing the results we arrive at equality [6.8] with the additional term in the right-hand side. The new term can be transformed by means of integration by parts in $\Omega$:

$$
T_1 = \int_{\Omega} b(z)|u^{(m)}_t|^\sigma(z) - 2u^{(m)}_t|\Delta u^{(m)}_t| \, dx
\leq \int_{\Omega} (\sigma(z) - 1)|b(z)||u^{(m)}_t|^{\sigma(z) - 2}\,|\nabla u^{(m)}_t|^2 \, dx
+ \int_{\Omega} |u^{(m)}_t|^{\sigma(z) - 1}|\nabla b||\nabla u^{(m)}_t| \, dx + \int_{\Omega} |b(z)||u^{(m)}_t|^{\sigma(z) - 1}\ln ||u^{(m)}_t||\,|\nabla u^{(m)}_t||\nabla \sigma| \, dx
\equiv K_1 + K_2 + K_3.
$$

To estimate $K_3$ we assume that the functions $|b|$ and $|\nabla \sigma|$ are bounded a.e. in $Q_T$ and then apply the Cauchy inequality, [6.11], and the Poincaré inequality: if $2(\sigma^+ - 1) < p^-$, there exists a constant $\mu > 0$ such that $2(\sigma^+ - 1) + \mu \leq p^-$

$$
K_3 \leq C \left( 1 + \|u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u^{(m)}_t|^{2(\sigma(z) - 1 + \mu)} \, dx \right)
\leq C \left( 1 + \|\nabla u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u^{(m)}_t|^{2(\sigma^+ - 1 + \mu)} \, dx \right)
\leq C' \left( 1 + \|\nabla u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u^{(m)}_t|^{2(\sigma^+ - 1)} \, dx \right)
\leq C'' \left( 1 + \|\nabla u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u^{(m)}_t|^p \, dx \right).
$$

$K_2$ is estimated likewise: if $|\nabla b|$ is bounded a.e. in $Q_T$ and $2(\sigma^+ - 1) < p^-$, then

$$
K_2 \leq C \left( 1 + \|\nabla u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u^{(m)}_t|^{2(\sigma(z) - 1)} \, dx \right) \leq C' \left( 1 + \|\nabla u^{(m)}_t(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u^{(m)}_t|^p \, dx \right).
$$

To estimate $K_1$ we assume that $\sigma^- \geq 2$ and notice that the restriction on $p^-$ and $\sigma^+$ imposed to estimate $K_2$ and $K_3$ yields

$$
4 \leq 2\sigma^- \leq 2\sigma^+ < 2 + p^- \Rightarrow p^- > 2 \Rightarrow \sigma^+ < 1 + \frac{p^-}{2} < p^-.
$$

Using this observation and the Young inequality we estimate $K_1$ as follows:

$$
K_1 \leq C \left( \int_{\Omega} |\nabla u^{(m)}_t|^p \, dx + \int_{\Omega} |u^{(m)}_t|^p \frac{p(z)^2}{p(z)^2 - 2} \, dx \right)
\leq C \left( 1 + \int_{\Omega} |\nabla u^{(m)}_t|^p \, dx + \int_{\Omega} |u^{(m)}_t|^p \frac{p(z)^2 - 2}{p^2 - 2} \, dx \right)
\leq C' \left( 1 + \int_{\Omega} |\nabla u^{(m)}_t|^p \, dx + \int_{\Omega} |u^{(m)}_t|^p \, dx \right).
$$
Following the proof of Lemma 6.3 and taking into account the estimates on $\mathcal{K}$, we arrive at the inequality
\[
\sup_{(0,T)} \|\nabla u^{(m)}_\epsilon(\cdot, t)\|_{L^2(\Omega)}^2 + \int_{Q_T} \gamma(z, \nabla u^{(m)}_\epsilon)(u^{(m)}_\epsilon)_{xx}^2 \, dz \\
\leq Ce^{C'T} \left( 1 + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|f_0\|_{L^2(0,T;W^{1,2}_0(\Omega))}^2 \right) + C'' e^{C'T} \left( \int_{Q_T} |\nabla u^{(m)}_\epsilon|^p(\cdot, t) \, dz + \int_{Q_T} |u^{(m)}_\epsilon|^q(\cdot, t) \, dz \right)
\] (6.25)
with new constants $C$, $C'$, $C''$ which do not depend on $\epsilon$ and $m$. The last term on the right-hand side of this inequality is estimated by virtue of Lemma 4.1 and estimates (6.22), (6.24).

**Lemma 6.6.** Let the conditions of Lemma 6.6 be fulfilled. Then
\[
\sup_{(0,T)} \|\nabla u^{(m)}_\epsilon(\cdot, t)\|_{L^2(\Omega)}^2 + \int_{Q_T} \gamma(z, \nabla u^{(m)}_\epsilon)(u^{(m)}_\epsilon)_{xx}^2 \, dz \\
\leq Ce^{C'T} \left( \tilde{C} + \|u_0\|_{W^{1,2}_0(\Omega)}^2 + \|f_0\|_{L^2(0,T;W^{1,2}_0(\Omega))}^2 \right)
\] (6.26)
with an independent of $\epsilon$ and $m$ constants $C$, $C'$, and a constant $\tilde{C}$ depending only on $T$, and the quantities on the right-hand sides of (6.23), (6.24).

3) Estimate on $\|u^{(m)}_\epsilon\|_{2,Q_T}$. We follow the proof of Lemma 6.4. Multiplying (6.3) by $(u^{(m)}_\epsilon)_{tt}$ and summing the results we obtain equality (6.21) with the additional term on the right-hand side:
\[
\mathcal{M}_0 = \int_{\Omega} b(z)|u^{(m)}_\epsilon|^\sigma(z)u^{(m)}_\epsilon u^{(m)}_\epsilon_{t} \, dx.
\]

By Young’s inequality
\[
\mathcal{M}_0 \leq C \int_{\Omega} |u^{(m)}_\epsilon|^{2(\sigma(z) - 1)} \, dx + \frac{1}{2} \int_{\Omega} (u^{(m)}_\epsilon)_{t}^2 \, dx.
\]
Combining this inequality with (6.19) and taking into account the inequality $2(\sigma(z) - 1) < p(z)$ following from the inequality $2\sigma^+ - 1 < p^-$, we obtain
\[
\frac{1}{2} \|u^{(m)}_\epsilon\|_{L^2,\Omega}^2 + \sup_{(0,T)} \int_{\Omega} \left( \epsilon^2 + |\nabla u^{(m)}_\epsilon|^2 \right)^{\frac{\alpha(q-1)}{2}} + a(z)(\epsilon^2 + |\nabla u^{(m)}_\epsilon|^2)^{\frac{\alpha(q+1)}{2}} \, dx \\
\leq C \left( 1 + \int_{\Omega} \|u_0\|_{p(z,0)}^p + a(x,0)\|\nabla u_0\|_{q(z,0)}^q \right) \, dx + \|f_0\|_{L^2,\Omega}^2
\] (6.27)
\[
+ C' \left( 1 + \int_{Q_T} |u^{(m)}_\epsilon|^p(\cdot, t) \, dz \right).
\]
The last integral on the right-hand side is estimated by virtue of Lemma 4.1 and the estimates of Lemma 6.5.

**Lemma 6.7.** Let the conditions of Lemma 6.6 be fulfilled. Then
\[
\frac{1}{2} \|u^{(m)}_\epsilon\|_{L^2,\Omega}^2 + \sup_{(0,T)} \int_{\Omega} \left( \epsilon^2 + |\nabla u^{(m)}_\epsilon|^2 \right)^{\frac{\alpha(q-1)}{2}} + a(z)(\epsilon^2 + |\nabla u^{(m)}_\epsilon|^2)^{\frac{\alpha(q+1)}{2}} \, dx \\
\leq C \left( 1 + \int_{\Omega} \|u_0\|_{p(z,0)}^p + a(x,0)\|\nabla u_0\|_{q(z,0)}^q \right) \, dx + \|f_0\|_{L^2,\Omega}^2 + C'
\] (6.28)
with constants $C$, $C'$ independent of $\epsilon$ and $m$. 24
7. Existence and uniqueness of strong solution

In this section, we prove that the regularized problem \([5.1]\) and the degenerate problem \([1.1]\) have strong solutions and derive conditions of uniqueness of these solutions.

7.1. Regularized problem

**Theorem 7.1.** Let \(u_0, f, p, q, a, \partial \Omega\) satisfy the conditions of Theorem \([3.1]\) Then for every \(\epsilon \in (0, 1)\) problem \([5.1]\) has a unique solution \(u_\epsilon\) which satisfies the estimates

\[
\begin{align*}
\|u_\epsilon\|_{W^1(0, T; \Omega)} & \leq C_0, \\
\text{ess sup}_{(0, T)} \|u_\epsilon(\cdot, t)\|^2_{L^2(\Omega)} + \|u_\epsilon(t)\|^2_{L^2(\Omega)} + \text{ess sup}_{(0, T)} \|\nabla u_\epsilon(\cdot, t)\|^2_{L^2(\Omega)} & \leq C_0, \\
& + \text{ess sup}_{(0, T)} \int_{\Omega} \left( \epsilon^2 + |\nabla u_\epsilon^m(\cdot)|^2 \frac{m^2 (2^{q_0} - 3)}{2} + a(z)(\epsilon^2 + |\nabla u_\epsilon^m(\cdot)|^2 \frac{m^2 (2^{q_0} - 3)}{2}) \right) dx \leq C_0 \\
\end{align*}
\]

with a constant \(C_0\) depending on the data but not on \(\epsilon\). Moreover, \(u_\epsilon\) possesses the property of global higher integrability of the gradient: for every \(\delta \in (0, r^*)\), \(r^* = \frac{4p}{p - (N + 2) + 2N}\),

there exists a constant \(C = C\left(\partial \Omega, N, p^*, \delta, \|u_0\|_{W^{1, 2}_0(\Omega)}, \|f\|_{L^2(0, T; W^{1, 2}_0(\Omega))}\right)\) such that

\[
\int_{Q_T} |\nabla u_\epsilon|^p(z) + \delta dz \leq C.
\]

**Proof.** Let \(\epsilon \in (0, 1)\) be a fixed parameter. Under the assumptions of Theorem \([3.1]\) there exists a sequence of Galerkin approximations \(u_\epsilon^m\) defined by formulas \([5.2]\) which satisfies estimates \([5.1], [5.2], [6.15], [6.16], [6.18]\) and \([6.19]\). These uniform in \(m\) and \(\epsilon\) estimates enable one to extract a subsequence \(u_\epsilon^m\) (for which we keep the same name), and functions \(u_\epsilon, \eta_\epsilon, \chi_\epsilon\) such that

\[
\begin{align*}
&u_\epsilon^m \rightharpoonup u_\epsilon \text{ weakly in } L^\infty(0, T; L^2(\Omega)), \quad (u_\epsilon^m)_t \rightharpoonup (u_\epsilon)_t \text{ in } L^2(Q_T), \\
&\nabla u_\epsilon^m \rightharpoonup \nabla u_\epsilon \text{ in } (L^{p^*}(Q_T))^N, \quad \nabla u_\epsilon^m \rightharpoonup \nabla u_\epsilon \text{ in } (L^{q^*}(Q_T))^N, \\
&(\epsilon^2 + |\nabla u_\epsilon^m|^2)^{\frac{p-2}{2}} \nabla u_\epsilon^m \rightharpoonup \eta_\epsilon \text{ in } (L^{q^*}(Q_T))^N, \\
&(\epsilon^2 + |\nabla u_\epsilon^m|^2)^{\frac{p-2}{2}} \nabla u_\epsilon^m \rightharpoonup \chi_\epsilon \text{ in } (L^{q^*}(Q_T))^N.
\end{align*}
\]

In the third line we make use of the uniform estimate

\[
\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^m|^2)^{\frac{p-2}{2}} \nabla u_\epsilon^m dz \leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon^m|^p(z) + \frac{dz}{\epsilon^2} \right) \leq C,
\]

which follows from \([5.5]\) and \([6.10]\). The functions \(u_\epsilon^m\) and \((u_\epsilon^m)_t\) are uniformly bounded in \(L^\infty(0, T; W^{1,p^*}_0(\Omega))\) and \(L^2(0, T; L^2(\Omega))\) respectively, and \(W^{1,q^*}(\Omega) \subseteq W^{1,q-}(\Omega) \subseteq L^2(\Omega)\). By \([3.2]\) Sec.8, Corollary 4] the sequence \(\{u_\epsilon^m\}\) is relatively compact in \(C(0, T; L^2(\Omega))\), i.e., there exists a subsequence \(\{u_{\epsilon_k}^m\}\), which we assume coinciding with \(\{u_\epsilon^m\}\), such that \(u_\epsilon^m \rightarrow u_\epsilon\) in \(C([0, T]; L^2(\Omega))\) and a.e. in \(Q_T\). Let us define

\[
P_m = \left\{ \phi : \phi = \sum_{i=1}^m \psi_i(t)\phi_i(x), \psi_i \text{ are absolutely continuous in } [0, T] \right\}.
\]
Fix some \( m \in \mathbb{N} \). By the method of construction \( u^{(m)}_\varepsilon \in \mathcal{P}_m \). Since \( \mathcal{P}_k \subset \mathcal{P}_m \) for \( k < m \), then for every \( \xi_k \in \mathcal{P}_k \) with \( k \leq m \)

\[
\int_{Q_T} u^{(m)}_{\varepsilon t} \xi_k \, dz + \int_{Q_T} \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon) \nabla u^{(m)}_\varepsilon \cdot \nabla \xi_k \, dz = \int_{Q_T} f_0 \xi_k \, dz. \tag{7.4}
\]

Let \( \xi \in \mathcal{W}^{q(\varepsilon)}(Q_T) \). The space \( C^\infty([0, T]; C_0^\infty(\Omega)) \) is dense in \( \mathcal{W}^{q(\varepsilon)}(Q_T) \), therefore there exists a sequence \( \{\xi_k\} \) such that \( \xi_k \in \mathcal{P}_k \) and \( \xi_k \to \xi \in \mathcal{W}^{q(\varepsilon)}(Q_T) \). If \( U_m \to U \) in \( L^{q(\varepsilon)}(Q_T) \), then for every \( V \in L^{q(\varepsilon)}(Q_T) \) we have

\[
a(z)V \in L^{q(\varepsilon)}(Q_T) \quad \text{and} \quad \int_{Q_T} a U_m V \, dz \to \int_{Q_T} a UV \, dz.
\]

Using this fact we pass to the limit as \( m \to \infty \) in (7.4) with a fixed \( k \), and then letting \( k \to \infty \), we conclude that

\[
\int_{Q_T} u^{(m)}_{\varepsilon t} \xi \, dz + \int_{Q_T} \eta_\varepsilon \cdot \nabla \xi \, dz + \int_{Q_T} a(z) \, \chi_\varepsilon \cdot \nabla \xi \, dz = \int_{Q_T} f_0 \xi \, dz \tag{7.5}
\]

for all \( \xi \in \mathcal{W}^{q(\varepsilon)}(Q_T) \). To identify the limit vectors \( \eta_\varepsilon \) and \( \chi_\varepsilon \) we use the classical argument based on monotonicity. The flux function \( \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon) \nabla u^{(m)}_\varepsilon \) is monotone:

\[
(\gamma_\varepsilon(z, \xi) - \gamma_\varepsilon(z, \zeta)) \xi - \zeta \geq 0 \quad \text{for all } \xi, \zeta \in \mathbb{R}^N, \ z \in Q_T, \ \varepsilon > 0, \tag{7.6}
\]

see, e.g., [4, Lemma 6.1] for the proof. By virtue of (7.6), for every \( \psi \in \mathcal{P}_m \)

\[
\gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)|\nabla u^{(m)}_\varepsilon|^2 = \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)(\nabla u^{(m)}_\varepsilon - \nabla \psi) + \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)\nabla u^{(m)}_\varepsilon \cdot \nabla \psi \\
= (\gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)\nabla u^{(m)}_\varepsilon - \gamma_\varepsilon(z, \nabla \psi)\nabla \psi) \cdot (\nabla u^{(m)}_\varepsilon - \nabla \psi) \\
+ \gamma_\varepsilon(z, \nabla \psi)\nabla \psi \cdot (\nabla u^{(m)}_\varepsilon - \nabla \psi) + \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)\nabla u^{(m)}_\varepsilon \cdot \nabla \psi \geq \gamma_\varepsilon(z, \nabla \psi)\nabla \psi \cdot (\nabla u^{(m)}_\varepsilon - \nabla \psi) + \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)\nabla u^{(m)}_\varepsilon \cdot \nabla \psi. \tag{7.7}
\]

By taking \( \xi_k = u^{(m)}_\varepsilon \) in (7.4) we obtain: for every \( \psi \in \mathcal{P}_k \) with \( k \leq m \)

\[
0 = \int_{Q_T} (u^{(m)}_\varepsilon)_t u^{(m)}_\varepsilon \, dz + \int_{Q_T} \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)|\nabla u^{(m)}_\varepsilon|^2 \, dz - \int_{Q_T} f_0 u^{(m)}_\varepsilon \, dz \\
\geq \int_{Q_T} (u^{(m)}_\varepsilon)_t u^{(m)}_\varepsilon \, dz + \int_{Q_T} \gamma_\varepsilon(z, \nabla \psi)\nabla \psi \cdot (\nabla u^{(m)}_\varepsilon - \nabla \psi) \\
+ \int_{Q_T} \gamma_\varepsilon(z, \nabla u^{(m)}_\varepsilon)\nabla u^{(m)}_\varepsilon \cdot \nabla \psi \, dz - \int_{Q_T} f_0 u^{(m)}_\varepsilon \, dz.
\]

Notice that \( (u^{(m)}_\varepsilon, (u^{(m)}_\varepsilon)_t)_{2,Q_T} \to (u_t, u_x)_{2,Q_T} \), as \( m \to \infty \) as the product of weakly and strongly convergent sequences. This fact together with (7.6) means that each term of the last inequality has a limit as \( m \to \infty \). Letting \( m \to \infty \) and using (7.6), we find that for every \( \psi \in \mathcal{P}_k \)

\[
0 \geq \int_{Q_T} u_t u_{\varepsilon t} \, dz + \int_{Q_T} \gamma_\varepsilon(z, \nabla \psi)\nabla \psi \cdot (u_t - \psi) \, dz + \int_{Q_T} (\eta_\varepsilon + a(z)\chi_\varepsilon) \cdot \nabla \psi \, dz - \int_{Q_T} f_0 u_t \, dz \\
= \int_{Q_T} \left((\varepsilon^2 + |\nabla \psi|^2)^{(a(\varepsilon^2 + \varepsilon^2)) - 2} \nabla \psi - \eta_\varepsilon \right) \cdot \nabla (u_t - \psi) \, dz \\
+ \int_{Q_T} a(z) \left((\varepsilon^2 + |\nabla \psi|^2)^{(a(\varepsilon^2 + \varepsilon^2)) - 2} \nabla \psi - \chi_\varepsilon \right) \cdot \nabla (u_t - \psi) \, dz.
\]

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By the density of \( \bigcup_{k=1}^{\infty} \mathcal{P}_k \) in \( \mathcal{W}_{q(J)}(Q_T) \), the last inequality also holds for every \( \psi \in \mathcal{W}_{q(J)}(Q_T) \). Take \( \psi = u_\epsilon + \lambda \xi \) with a constant \( \lambda > 0 \) and an arbitrary \( \xi \in \mathcal{W}_{q(J)}(Q_T) \). Then
\[
\lambda \left[ \int_{Q_T} \left( \varepsilon^2 + |\nabla (u_\epsilon + \lambda \xi)|^2 \right)^{\frac{q-2}{2}} \nabla (u_\epsilon + \lambda \xi) \cdot \nabla \xi \, dz + \int_{Q_T} a(z) \left( \varepsilon^2 + |\nabla (u_\epsilon + \lambda \xi)|^2 \right)^{\frac{q-2}{2}} \nabla (u_\epsilon + \lambda \xi) - \chi \cdot \nabla \xi \, dz \right] \leq 0. \tag{7.8}
\]
Simplifying and letting \( \lambda \to 0 \) we find that
\[
\int_{Q_T} (\gamma_z(z, \nabla u_\epsilon) \nabla u_\epsilon - (\eta_z + a(z) \chi \epsilon)) \cdot \nabla \xi \, dz \leq 0 \quad \forall \xi \in \mathcal{W}_{q(J)}(Q_T),
\]
which is possible only if
\[
\int_{Q_T} (\gamma_z(z, \nabla u_\epsilon) \nabla u_\epsilon - (\eta_z + a(z) \chi \epsilon)) \cdot \nabla \xi \, dz = 0 \quad \forall \xi \in \mathcal{W}_{q(J)}(Q_T),
\]
The initial condition for \( u_\epsilon \) is fulfilled by continuity because \( u_\epsilon \in C([0, T]; L^2(\Omega)) \).

Uniqueness of the weak solution is an immediate byproduct of monotonicity. Let \( u, v \) be two solutions of problem \( \{5.1\} \). Take an arbitrary \( \tau \in (0, T] \), choosing \( u - v \) for the test function in equalities \( \{3.4\} \) for \( u \) and \( v \) in the cylinder \( Q_\tau = \Omega \times (0, \tau) \), subtracting the results and applying \( \{7.8\} \) we arrive at the inequality
\[
\frac{1}{2} \|u - v\|_{2, \Omega}(\tau) = \int_{Q_\tau} (u - v)(u - v) \, dz \leq 0.
\]
It follows that \( u(x, \tau) = v(x, \tau) \) a.e. in \( \Omega \) for every \( \tau \in [0, T] \).

Estimates \( \{7.1\} \) follow from the uniform in \( m \) estimates on the functions \( u_{\epsilon(m)} \) and their derivatives, the properties of weak convergence \( \{7.3\} \) and lower semicontinuity of the modular. Inequality \( \{6.17\} \) yields that for every \( \delta \in (0, r^*) \) the sequence \( \{\nabla u_{\epsilon(m)}\} \) contains a subsequence which converges to \( \nabla u_\epsilon \) weakly in \( (L^{p(m)} + \delta(Q_T))^N \), whence \( \{7.2\} \).

**Theorem 7.2.** Let in the conditions of Theorem \( \{7.1\} \) \( b \neq 0 \).

(i) Assume that \( b, \sigma \) are measurable and bounded functions in \( Q_T \)
\[
\|\nabla b\|_{\infty, Q_T} < \infty, \quad \|\nabla \sigma\|_{\infty, Q_T} < \infty, \quad 2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p}{2}.
\]
Then for every \( \epsilon \in (0, 1) \) problem \( \{5.1\} \) has at least one strong solution \( u \), which satisfies estimates \( \{7.3\}, \{7.2\} \).

(ii) The solution is unique if either \( \sigma \equiv 2 \), or \( b(z) \leq 0 \) in \( Q_T \) and \( \sigma^- \geq 1 \).

**Proof.** The proof is an imitation of the proof of Theorem \( \{7.1\} \). The estimates of Lemmas \( \{6.5\}, \{6.6\}, \{6.7\} \) allow one to extract a subsequence \( \{u_{\epsilon(m)}\} \) with the convergence properties \( \{7.3\} \). Let \( u_\epsilon \) be the pointwise limit of the sequence \( \{u_{\epsilon(m)}\} \). We have to show that for every \( \phi \in L^2(Q_T) \)
\[
\int_{Q_T} |u_{\epsilon(m)}|^{\sigma(z)-2} u_{\epsilon(m)} \phi \, dz \to \int_{Q_T} |u_\epsilon|^{\sigma(z)-2} u_\epsilon \phi \, dz.
\]
The sequence \( v_{m_k} = |u_{\epsilon(m_k)}|^{\sigma(z)-2} u_{\epsilon(m_k)} \) converges a.e. in \( Q_T \) to \( |u_\epsilon|^{\sigma(z)-2} u_\epsilon \) and is uniformly bounded in \( L^2(Q_T) \) because
\[
\int_{Q_T} v_{m_k}^2 \, dz = \int_{Q_T} |u_{(m_k)}^{(\varepsilon)}|^2 \, dz \leq C \left( 1 + \int_{Q_T} |u_{(m_k)}^{(\varepsilon)}|^p \, dz \right)
\]

It follows that there is \( v \in L^2(Q_T) \) such that \( v_{m_k} \rightharpoonup v \) in \( L^2(Q_T) \) and by virtue of pointwise convergence it is necessary that \( v = |u_{(\varepsilon)}|^{(\sigma(z)-2)} u \) a.e. in \( Q_T \).

Assume that \( u_1, u_2 \in W_{q(\cdot)}(Q_T) \) are two strong solutions of problem (5.1). The function \( u_1 - u_2 \) is an admissible test-function in the integral identities (7.4) for \( u_1 \). Combining these identities and using (7.6) we arrive at the inequality

\[
\frac{1}{2} \| u_1 - u_2 \|^2_{2, \Omega(t)} \leq \frac{1}{2} \| u_1 - u_2 \|^2_{2, \Omega(t)}(t) + \int_0^t \left( \gamma(z, \nabla u_1) \nabla u_1 - \gamma(z, \nabla u_2) \nabla u_2 \right) \cdot (u_1 - u_2) \, dz
\]

If \( \sigma \equiv 2 \), this inequality takes the form

\[
\frac{1}{2} \| u_1 - u_2 \|^2_{2, \Omega(t)}(t) \leq B \int_0^t \| u_1 - u_2 \|^2_{2, \Omega(t)}(\tau) \, d\tau, \quad t \in (0, T), \quad B = \text{ess sup}_{Q_T} b(z),
\]

whence \( \| u_1 - u_2 \|^2_{2, \Omega(t)}(0) = 0 \) in \( (0, T) \) by Grönwall’s inequality. Let \( b(z) \leq 0 \) in \( Q_T \). For \( \sigma(z) \geq 1 \) the function \( |u_{(\varepsilon)}|^{(\sigma(z)-2)} \) is monotone increasing as a function of \( s \), therefore \( \{ |u_1|^{(\sigma(z)-2)} u_1 - |u_2|^{(\sigma(z)-2)} u_2 \} (u_1 - u_2) \geq 0 \) a.e. in \( Q_T \) and

\[
\frac{1}{2} \| u_1 - u_2 \|^2_{2, \Omega(t)}(t) \leq 0 \quad \text{in} \quad (0, T).
\]

7.2. Degenerate problem. Proof of Theorems 3.1, 3.2

Let \( \{ u_{\varepsilon} \} \) be the family of strong solutions of the regularized problems (5.1) satisfying estimates (7.1). These uniform in \( \varepsilon \) estimates enable one to extract a sequence \( \{ u_{\varepsilon_k} \} \) and find functions \( u \in W_{q(\cdot)}(Q_T) \), \( \eta, \chi \in (L^{q(\cdot)}(Q_T))^N \) with the following properties:

\[
\begin{align*}
    & u_{\varepsilon_k} \rightharpoonup u \quad \ast\text{-weakly in} \quad L^\infty(0, T; L^2(\Omega)), \quad u_{\varepsilon_k} \rightharpoonup u \quad \text{in} \quad L^2(Q_T), \\
    & \nabla u_{\varepsilon_k} \rightharpoonup \nabla u \quad \text{in} \quad (L^{q(\cdot)}(Q_T))^N, \\
    & (\varepsilon_k^2 + |\nabla u_{\varepsilon_k}|^2)^{(\sigma(z)-2)/2} \nabla u_{\varepsilon_k} \rightharpoonup \eta \quad \text{in} \quad (L^{q(\cdot)}(Q_T))^N, \\
    & (\varepsilon_k^2 + |\nabla u_{\varepsilon_k}|^2)^{(\sigma(z)-2)/2} \nabla u_{\varepsilon_k} \rightharpoonup \chi \quad \text{in} \quad (L^{q(\cdot)}(Q_T))^N.
\end{align*}
\]

In the third line we make use of the uniform estimate

\[
\int_{Q_T} (\varepsilon^2 + |\nabla u_{\varepsilon_k}|^2)^{(\sigma(z)-2)/2} \, dz \leq C \left( 1 + \int_{Q_T} |\nabla u_{\varepsilon_k}|^{p(\cdot)+r} \, dz \right) \leq C,
\]

which follows from (5.3) and (7.2). Moreover, \( u \in C([0, T]; L^2(\Omega)) \). Each of \( u_{\varepsilon_k} \) satisfies the identity

\[
\int_{Q_T} u_{\varepsilon_k} \xi \, dz + \int_{Q_T} \gamma_{\varepsilon_k}(z, \nabla u_{\varepsilon_k}) \nabla u_{\varepsilon_k} \cdot \nabla \xi \, dz = \int_{Q_T} f_0 \xi \, dz, \quad \forall \xi \in W_{q(\cdot)}(Q_T), \tag{7.9}
\]

which yields

\[
\int_{Q_T} u_\xi \, dz + \int_{Q_T} (\eta + a(z) \chi) \cdot \nabla \xi \, dz = \int_{Q_T} f_0 \xi \, dz, \quad \forall \xi \in W_{q(\cdot)}(Q_T). \tag{7.10}
\]
To identify \( \eta \) and \( \chi \) we use the monotonicity argument. Take \( \xi = u_{e_k} \) in (7.9):

\[
\int_{Q_T} u_{e_k} u_{e_k} \, dz + \int_{Q_T} \gamma_{e_k}(z, \nabla u_{e_k}) \nabla u_{e_k} : \nabla u_{e_k} \, dz = \int_{Q_T} f_0 u_{e_k} \, dz.
\]  

(7.11)

According to (7.10), for every \( \phi \in W_{q(\cdot)}(Q_T) \)

\[
\int_{Q_T} \gamma_{e_k}(z, \nabla u_{e_k}) \nabla u_{e_k} : \nabla u_{e_k} \, dz \geq \int_{Q_T} \gamma_{e_k}(z, \nabla \phi) - (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi : \nabla (u_{e_k} - \phi) \, dz \\
+ \int_{Q_T} \gamma_{e_k}(z, \nabla u_{e_k}) \nabla u_{e_k} : \nabla \phi \, dz + \int_{Q_T} (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi : \nabla (u_{e_k} - \phi) \, dz \\
\equiv J_{1,k} + J_{2,k} + J_{3,k},
\]

where

\[
J_{2,k} \to \int_{Q_T} (\eta + a(z)\chi) : \nabla \phi \, dz,
\]

\[
J_{3,k} \to \int_{Q_T} (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi : \nabla (u - \phi) \, dz \quad \text{as} \quad k \to \infty.
\]

Since \( \left| \gamma_{e_k}(z, \nabla \phi) \nabla \phi - (|\nabla \phi|^{p-2} + a \frac{1}{\epsilon} (z)|\nabla \phi|^{q-2}) \nabla \phi \right| \to 0 \) a.e. in \( Q_T \) as \( k \to \infty \), and because the integrand of \( J_{1,k} \) has the majorant

\[
\left| (\epsilon^2_k + |\nabla \phi|^2)^{\frac{p^2}{2}} - |\nabla \phi|^{p-2} \nabla \phi \right|^p + \left| a \frac{z}{\epsilon^2_k} (z) ((\epsilon^2_k + |\nabla \phi|^2)^{\frac{p^2}{2}} - |\nabla \phi|^{q-2} \nabla \phi) \right|^q \leq C \left( (1 + |\nabla \phi|^2)^{\frac{p}{2}} + a(z)(1 + |\nabla \phi|^2)^{\frac{q}{2}} \right) \]

\[
\leq C \left( 1 + |\nabla \phi|^{p(z)} + a(z)|\nabla \phi|^{q(z)} \right),
\]

then \( J_{1,k} \to 0 \) by the dominated convergence theorem. Combining (7.10) with (7.11) and letting \( k \to \infty \) we find that for every \( \phi \in W_{q(\cdot)}(Q_T) \)

\[
\int_{Q_T} \left( (|\nabla \phi|^{p(z)-2} + a(z)|\nabla \phi|^{q(z)-2}) \nabla \phi - (\eta + a(z)\chi) \right) : \nabla (u - \phi) \, dz \geq 0.
\]

Choosing \( \phi = u + \lambda \zeta \) with \( \lambda > 0 \) and \( \zeta \in W_{q(\cdot)}(Q_T) \), simplifying, and then letting \( \lambda \to 0^+ \), we obtain the inequality

\[
\int_{Q_T} \left( (|\nabla u|^{p(z)-2} \nabla u + a(z)|\nabla u|^{q(z)-2} \nabla u) - (\eta + a(z)\chi) \right) : \nabla \zeta \, dz \geq 0 \quad \forall \zeta \in W_{q(\cdot)}(Q_T).
\]

Since the sign of \( \zeta \) is arbitrary, the previous relation is the equality. It follows that in (7.10) \( \eta + a(z)\chi \) can be substituted by \( |\nabla u|^{p(z)-2} \nabla u + a(z)|\nabla u|^{q(z)-2} \nabla u \). Since \( u \in C([0,T]; L^2(\Omega)) \), the initial condition is fulfilled by continuity. Estimates (7.11) follow from the uniform in \( \epsilon \) estimates of Theorem 3.2 and the lower semicontinuity of the modular exactly as in the proof of Theorem 3.1. Uniqueness of a strong solution is an immediate consequence of the monotonicity. Theorem 3.4 is proven.

To prove Theorem 3.2 we only have to check that \( |u_{e_k}|^{\sigma(z)-2} u_{e_k} \to |u|^{\sigma(z)-2} u \) in \( L^2(Q_T) \) (up to a subsequence). This is done as in the case of the regularized problem.

**Remark 7.1.** Under the assumption of the Theorem 3.1 or Theorem 3.2 and, in addition \( f_0 \in L^1((0,T]; L^\infty(\Omega)) \) and \( u_0 \in L^\infty(\Omega) \), the strong solution of the problem (1.11) is bounded and satisfies the estimate

\[
\|u(\cdot,t)\|_{\infty,\Omega} \leq e^{C_1 t} \|u_0\|_{\infty,\Omega} + e^{C_1 t} \int_0^t e^{-C_1 \tau} \|f_0(\cdot,\tau)\|_{\infty,\Omega} \, d\tau
\]

(7.12)

where \( C_1 = 0 \) if \( b(z) \leq 0 \) in \( Q_T \), or \( C_1 = \|b\|_{\infty, Q_T} \) if \( \sigma \equiv 2 \) (see [3, Ch.4,Sec.4.3,Th.4.3]).
References

[1] C. O. Alves and V. D. Rădulescu, *The Lane-Emden equation with variable double-phase and multiple regime*, Proc. Amer. Math. Soc., 148 (2020), pp. 2937–2952.

[2] S. Antontsev, I. Kuznetsov, and S. Shimarev, *Global higher regularity of solutions to singular p(x,t)-parabolic equations*, J. Math. Anal. Appl., 466 (2018), pp. 238–263.

[3] S. Antontsev and S. Shimarev, *Evolution PDEs with nonstandard growth conditions*, vol. 4 of Atlantis Studies in Differential Equations, Atlantis Press, Paris, 2015. Existence, uniqueness, localization, blow-up.

[4] R. Arora and S. Shimarev, *Strong solutions of evolution equations with p(x,t)-Laplacian: existence, global higher integrability of the gradients and second-order regularity*, J. Math. Anal. Appl., 493 (2021), pp. 124506, 31.

[5] A. Bahrouni, V. D. Rădulescu, and D. D. Repovš, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity, 32 (2019), p. 2481.

[6] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal., 63 (1976), pp. 337–403.

[7] V. Benci, P. d'Avenia, D. Fortunato, and L. Pisani, *Solutions in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal., 154 (2000), pp. 297–324.

[8] V. Bögelein, F. Duzaar, and P. Marcellini, *Parabolic equations with p,q-growth*, J. Math. Pures Appl. (9), 100 (2013), pp. 535–563.

[9] V. Bögelein, F. Duzaar, and P. Marcellini, *Parabolic systems with p,q-growth: a variational approach*, Archive for Rational Mechanics and Analysis, 210 (2013), pp. 219–267.

[10] L. Cherfils and Y. Il'Yasov, *On the stationary solutions of generalized reaction diffusion equations with p-q-laplacian*, Commun. Pure Appl. Anal., 4 (2005), p. 9.

[11] I. Chlebicka, *A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces*, Nonlinear Anal., 175 (2018), pp. 1–27.

[12] I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein, *Parabolic equation in time and space dependent anisotropic Musielak-Orlicz spaces in absence of Lavrentiev's phenomenon*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 36 (2019), pp. 1431–1465.

[13] ———, *Parabolic equation in time and space dependent anisotropic Musielak-Orlicz spaces in absence of Lavrentiev’s phenomenon*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 36 (2019), pp. 1431–1465.

[14] M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal., 218 (2015), pp. 219–273.

[15] ———, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal., 215 (2015), pp. 443–496.

[16] A. Crespo-Blanco, L. Gasiński, P. Harjulehto, and P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, arXiv:2103.08928v1, (2021).

[17] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.

[18] C. De Filippis, *Gradient bounds for solutions to irregular parabolic equations with (p,q)-growth*, arXiv preprint arXiv:2004.01452 (2020).

[19] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, vol. 2017 of Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[20] L. Diening, P. Nágele, and M. Růžička, *Monotone operator theory for unsteady problems in variable exponent spaces*, Complex Var. Elliptic Equ., 57 (2012), pp. 1209–1231.

[21] L. Esposito, F. Leonetti, and G. Mingione, *Sharp regularity for functional problems with (p,q) growth*, J. Differential Equations, 204 (2004), pp. 5–55.

[22] X. Fan, *Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method*, J. Math. Anal. Appl., 364 (2010), pp. 593–604.

[23] L. Gasiński and P. Winkert, *Existence and uniqueness results for double phase problems with convection term*, J. Differential Equations, 268 (2020), pp. 4183–4193.

[24] F. Giannetti, A. P. di Napoli, and C. Scheven, *On higher differentiability of solutions of parabolic systems with discontinuous coefficients and (p,q)-growth*, Proc. Roy. Soc. Edinburgh Sect. A, 150 (2020), pp. 419–451.

[25] P. Gmeineder, *Elliptic problems in non-smooth domains*, vol. 69 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR775683]. With a foreword by Susanne C. Brenner.

[26] P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, vol. 2236 of Lecture Notes in Mathematics, Springer, Cham, 2019.

[27] P. Hästö and J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, arXiv e-prints, (2019), p. arXiv:1902.00261.

[28] O. Kováčik and J. Rákosník, *On spaces L^p(x) and W k,p(x)*, Czechoslovak Math. J., 41(116) (1991), pp. 592–618.

[29] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1969. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2.

[30] W. Liu and G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations, 265 (2018), pp. 4311–4334.

[31] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations, 90 (1991), pp. 1–30.
[32] A variational approach to parabolic equations under general and p, q-growth conditions, Nonlinear Anal., 194 (2020), p. 111456.

[33] J. Ok, Regularity for double phase problems under additional integrability assumptions, Nonlinear Anal., 194 (2020), p. 111408.

[34] V. D. Rădulescu, Isotropic and anisotropic double-phase problems: old and new, Opuscula Math., 39 (2019), pp. 259–279.

[35] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[36] T. Singer, Parabolic equations with p, q-growth: the subquadratic case, Q. J. Math., 66 (2015), pp. 707–742.

[37] Existence of weak solutions of parabolic systems with p, q-growth, Manuscripta Math., 151 (2016), pp. 87–112.

[38] Q. Zhang and V. D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, J. Math. Pures Appl. (9), 118 (2018), pp. 159–203.

[39] V. Zhikov, On variational problems and nonlinear elliptic equations with nonstandard growth conditions, J. Math. Sci., 173 (2011), pp. 463–570.

[40] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR Izv., 29 (1987), p. 33.

[41] V. V. Zhikov, On Lavrentiev’s phenomenon, Russian J. Math. Phys., 3 (1995), pp. 249–269.