BASIC MORSE-NOVIKOV COHOMOLOGY FOR FOLIATIONS

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Abstract. In this paper we find sufficient conditions for the vanishing of the Morse-Novikov cohomology on Riemannian foliations. We work out a Bochner technique for twisted cohomological complexes, obtaining corresponding vanishing results. Also, we generalize for our setting vanishing results from the case of closed Riemannian manifolds. Several examples are presented, along with applications in the context of l.c.s. and l.c.K. foliations.

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1. Introduction

We consider in what follows a smooth manifold $M$ endowed with a global 1− differential closed form $\theta$. The Morse-Novikov cohomology complex $(\Omega, d_\theta)$ (where $\Omega$ is the de Rham complex of the manifold $M$, while $d_\theta$ is the twisted derivative $d_\theta := d - \theta \wedge$) plays an important role when investigating aspects related to the geometry, topology and Morse theory of the underlying manifold $M$ (see e.g. [Pa]).

Cohomological complexes of this type were also introduced and studied by Lichnerowicz in the context of Poisson geometry [L] (in many papers this cohomology is also called Lichnerowicz cohomology).

Classical examples of Morse-Novikov cohomology are obtained on locally conformally symplectic manifolds and locally conformally Kähler manifolds [OV2, V3]. These manifolds admit local symplectic and Kähler structures which cannot be extended to the whole manifold. Instead, at the global level a closed 1− form is obtained (called the Lee form), and a Morse-Novikov cohomology naturally appears.

Twisted differential operators and Morse-Novikov cohomology can be canonically extended in the larger framework represented by Riemannian foliations (i.e. foliations with Riemannian structure which locally induces Riemannian submersions [R]). The transverse geometry of the foliations represents the extension of the geometry of Riemannian manifolds; the classical setting is obtained in the absolute case of manifolds foliated by points [Mo, T].

On Riemannian foliations defined on a closed manifold, such twisted basic cohomological objects are mostly used for the case when $\theta$ is related to the mean curvature form. For example, in [Do] the author studies the tenseness of Riemannian foliations (i.e. the existence of a metric with basic (projectable) mean curvature form). In [HR], twisted (modified) differentials are used as an alternative to the
classical approach in order to investigate the tautness of the foliation (i.e. the existence of a metric which turns all leaves into minimal submanifolds), and to perform basic harmonic analysis.

In this paper we stick with the general case of basic Morse-Novikov cohomology associated to a basic, closed 1-form, and present several instances in which the groups of this cohomology are trivial.

First of all, we generalize to our setting previous vanishing results known in the classical case when $\theta$ is parallel [LLMP] and, respectively, non-exact [GL]. Then we study the influence of the basic curvature on the basic Morse-Novikov cohomological groups. Beside the case when this curvature operator in non-negative, in the second part of the paper we work out a Bochner type technique specific for our particular framework.

Consequently, we obtain vanishing results which may hold even in the case when the basic curvature is not necessarily non-negative (in this case the basic de Rham cohomology groups may not be trivial).

Concerning the last result, two particular cases are relevant. These are the case of classical closed Riemannian manifolds and the case of the basic de Rham cohomology of Riemannian foliations. On the other side, the above vanishing results have their correspondents in the context of locally conformally symplectic and locally conformally Kähler foliations.

The paper is organized as follows. In the next section we present the main features of Riemannian foliations with basic curvature form, which represent our framework throughout this paper. In section 3 we introduce several technical tools we use in the rest of the paper. More precisely, the three subsections present the twisted Bott connection, the twisted basic curvature operator and a corresponding Weitzenböck formula. In section 4 we present the main results of the paper along with several examples. The consequences of these results for the setting of locally conformally symplectic and locally conformally Kähler foliations are briefly stated in the final section.

2. Preliminaries

2.1. Basic facts about foliations. We are going to consider in what follows a smooth, closed Riemannian manifold $(M, g, F)$ endowed with a foliation $\mathcal{F}$ such that the metric $g$ is bundle-like [H]; the dimension of $M$ will be denoted by $n$. We also denote by $T\mathcal{F}$ the leafwise distribution tangent to leaves, while $Q = T\mathcal{F}^\perp \simeq TM/T\mathcal{F}$ will be the transverse distribution. Let us assume $\dim T\mathcal{F} = p$, $\dim Q = q$, so $p + q = n$.

As a consequence, we get the following exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TM \to Q \to 0.$$ 

A corresponding exact sequence for the dual vector bundles also appears (see e. g. [H]). The canonical projection operators on the distributions $Q$ and $T\mathcal{F}$ will be denoted by $\pi_Q$ and $\pi_{T\mathcal{F}}$, respectively.

Throughout this paper we use local vector fields $\{e_i, f_a\}$ defined on a neighborhood of an arbitrary point $x \in M$, so that they determine an orthonormal basis at any point where they are defined, $\{e_i\}$ spanning the distribution $Q$ and $\{f_a\}$
spanning the distribution $T\mathcal{F}$. In what follows we use the classical ‘musical’ isomorphisms $\sharp$ and $\flat$ determined by the metric structure $g$. The coframe $\{e^i, f^a\}$ will be also employed, with $e^i := e^\flat_i$, $f^a := f^\flat_a$.

A standard linear connection employed for the study of the basic geometry of our Riemannian foliated manifold is the Bott connection $\nabla$ (see e.g. [T]); it is a metric and torsion-free connection. If we denote by $\nabla^M$ the Levi-Civita connection on $M$, then on the transverse distribution $Q$ we can define the connection $\nabla$ by the following relations
\[
\begin{align*}
\nabla_u w &= \pi_Q \left( [u, w] \right), \\
\nabla_v w &= \pi_Q \left( \nabla^M_v w \right),
\end{align*}
\]
for any smooth sections $u \in \Gamma(T\mathcal{F})$, $v, w \in \Gamma(Q)$.

The transverse divergence associated to the Bott connection is defined in the usual manner, as a trace operator:
\[
\text{div} \nabla := \sum_i g \left( \nabla_{e^i}, e^i \right).
\]

As in the case of Riemannian submersions, we investigate the geometric objects that can be locally projected on submanifolds transverse to the leaves. The restriction of the classical de Rham complex of differential forms $\Omega(M)$ to the complex of basic (projectable) differential forms generates the basic de Rham complex, defined as
\[
\Omega_b(F) := \{ \eta \in \Omega(M) \mid \iota_v \eta = 0, \mathcal{L}_v \eta = 0 \text{ for any } v \in \Gamma(T\mathcal{F}) \}.
\]
Here $\mathcal{L}$ is the Lie derivative along $v$, while $\iota$ stands for interior product. The basic de Rham derivative is defined also as a restriction of the classical derivative $d$, namely $d_b := d|_{\Omega_b(F)}$ (see e.g. [T]).

**Remark 2.1.** The basic de Rham complex is defined independently of the metric structure $g$, and in fact the groups of the basic de Rham cohomology are topological invariants [T]. Moreover, it is easy to see that basic forms are actually parallel along the leaves with respect to the connection $\nabla$.

A basic vector field is a vector field $v$ parallel along leaves with respect to $\nabla$. If the vector field is also transverse, i.e. is a section in the transverse distribution, $v \in \Gamma(Q)$, then $v^\flat \in \Omega_b(F)$. In the sequel, we use basic (projectable) vector fields $\{e_i\}$.

An interesting example of differential form which is not necessarily basic is represented by the mean curvature form (see e.g. [A, BEI]). It is denoted by $\kappa$ and it is defined as
\[
\kappa^\flat := \pi_Q \left( \sum_a \nabla^M_f f_a \right).
\]

According to [A], on any Riemannian foliation the mean curvature form can be decomposed in a unique way as the sum
\[
\kappa = \kappa_b + \kappa_o,
\]
where $\kappa_b \in \Omega_b(F)$ is the basic component of the mean curvature, $\kappa_o$ being its orthogonal complement with respect to the Fréchet $C^\infty$ topology. We note also that $\kappa_b$ is a closed 1-form [A].

A fundamental result in this field is due to Domínguez [Dq], and it shows that $\kappa_o$ can always be considered to be 0. More precisely, any Riemannian foliation defined
on a closed manifold can be turned into a foliation with basic mean curvature form by changing the bundle-like metric such that the transverse metric remains unchanged. As we plan to work with geometric objects related to the transverse metric structure, we can make the standard assumption of a basic mean curvature without actually restricting our framework (see e.g. [HR, T]).

And hence, from now on we shall assume that

\[ \kappa = \kappa_b \]

For any basic forms \( \alpha_1, \alpha_2 \in \Omega^p_b(\mathcal{F}) \), we extend the notation \( g(\alpha_1, \alpha_2) \) for the inner product canonically induced by the metric tensor \( g \) on \( \Omega^p_b(\mathcal{F}) \). Taking the integral on the closed manifold \( M \), we obtain the classical Hilbertian product

\[ \langle \alpha_1, \alpha_2 \rangle := \int_M g(\alpha_1, \alpha_2) \, d \mu_g, \]

where \( d \mu_g \) is the measure induced on \( M \) by \( g \).

The adjoint operator of \( d_b \) with respect to the above product, which is called the basic de Rham coderivative, may be also written as

\[ \delta_b := \sum_a -\iota_{f_a} \nabla f_a + \iota_{\kappa^z}. \]

2.2. Morse-Novikov cohomology with basic form. On a closed Riemannian manifold \( M \), using a closed differential 1-form \( \theta \), we can define the twisted de Rham derivative \( d_\theta : \Omega(M) \to \Omega(M) \),

\[ d_\theta := d - \theta \wedge, \]

where \( \Omega(M) \) is the de Rham complex defined on \( M \).

As \( \theta \) is closed, \( d_\theta^2 = 0 \), and the Morse-Novikov cohomological complex \( (\Omega(M), d_\theta) \) can be defined canonically. Note that if \( \theta \) is exact, \( \theta = df \), \( f \in C^\infty(M) \), then the mapping \( [\alpha] \to [e^{-f} \alpha] \) defines an isomorphism between the de Rham and Morse-Novikov cohomologies.

Also, the concept of Morse-Novikov cohomology can be easily extended to the basic Morse-Novikov cohomology on a Riemannian foliation. More precisely, assuming that \( \theta \) is a basic closed 1-form, then we can write the twisted basic de Rham derivative using the above defined basic de Rham differential operator \( d_b \) (see e.g. [IP])

\[ d_{b,\theta} := d_b - \theta \wedge, \]

and the basic Morse-Novikov complex \( (\Omega_b(\mathcal{F}), d_{b,\theta}) \) is constructed. The Morse-Novikov cohomology groups \( \{H^i_{b,\theta}(\mathcal{F})\}_{0 \leq i \leq q} \) (which can be regarded as twisted de Rham cohomology groups) are defined in the canonical manner, and a cohomological theory can be undertaken.

The particular case \( \theta = \frac{1}{2} \kappa \) is investigated in [HR], the authors obtaining vanishing results using curvature-type operators and an interesting interplay with the tautness properties of the foliation.

For general \( \theta \), in order to study these cohomological groups, we employ additional basic twisted differential operators compatible with \( d_{b,\theta} \).

We introduce these operators in the next section.
3. Basic twisted differential operators on Riemannian foliations

3.1. The twisted Bott connection. The main tool we use to describe and investigate the twisted cohomology groups $H_{b,\theta}^i(\mathcal{F})$ is a linear connection which we modify in a convenient way.

**Definition 3.1.** For a closed 1-form $\theta \in \Omega_b(\mathcal{F})$ and vector fields $v \in \Gamma(TM)$, $w \in \Gamma(Q)$ we define the twisted Bott connection $\nabla^\theta$

$$\nabla^\theta_v w := \nabla_v w - g(v, \theta^\sharp)w.$$

For the particular case when $\theta \equiv \frac{1}{2}\kappa$ we adopt the notation $\tilde{\nabla} := \nabla^\frac{1}{2}\kappa$. As this connection will play an important role in our further considerations, we choose to denote

$$\tilde{\nabla} := \nabla^\frac{1}{2}\kappa + \theta.$$

The connection in (1) is extended canonically on $\Omega^b_b(\mathcal{F})$, and for convenience will be denoted with $\tilde{\nabla}$ too.

**Remark 3.1.** As $\kappa^\sharp, \theta^\sharp \in \Gamma(Q)$, one can see that if $\alpha \in \Omega_b(\mathcal{F})$ and $v \in \Gamma(T\mathcal{F})$, then

$$\tilde{\nabla}_v^\theta \alpha = 0,$$

in other words, the basic forms remain parallel with respect to the new connection $\tilde{\nabla}^\theta$.

The interesting feature of the twisted Bott connection $\tilde{\nabla}^\theta$ is that it can be used to build up the twisted basic de Rham derivative. Let us denote $\tilde{\nabla}$

$$\tilde{d}_{b,\theta} := d_{b,\frac{1}{2}\kappa+\theta}.$$

An alternative way to construct this operator is the following:

$$\tilde{d}_{b,\theta} = \sum_i e_i \wedge \tilde{\nabla}^\theta_{e_i}.$$

We will also denote the cohomology groups associated to $\tilde{d}_{b,\theta}$ by $\tilde{H}_{b,\theta}^i(\mathcal{F})$, with $\tilde{H}_{b,\theta}^i(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^i(\mathcal{F})$.

For the particular case $\theta \equiv 0$, one reobtains the basic modified operator $\tilde{d}_b := \tilde{d}_{b,0}$, with $\tilde{d}_{b,\theta} = \tilde{d}_b - \theta \wedge$, and the corresponding cohomology complex $\tilde{H}_b^i(\mathcal{F}) := \tilde{H}_{b,0}^i(\mathcal{F})$, in accordance with [HR].

3.2. Computational properties of the twisted basic operators. In the following we investigate several computational properties of the above introduced twisted basic operators (again, for the case $\theta \equiv 0$ see also [HR][SVV]).

First of all we construct the adjoint operators $\tilde{\nabla}^\theta_{\ast v}$ and $\delta_{b,\theta} := \tilde{d}_{b,\theta}^{\ast}$. For the twisted Bott connection we obtain the following result.

**Lemma 3.1.** The (formal) adjoint operator associated to the differential operator $\tilde{\nabla}^\theta_v$ can be computed with the formula

$$\tilde{\nabla}^\theta_{\ast v} = -\tilde{\nabla}^\theta_v - \text{div}_v \tilde{\nabla} - 2g(v, \theta^\sharp).$$

Proof. We evaluate the expression

\[ S = \left\langle \tilde{\nabla}_v^\theta \alpha_1, \alpha_2 \right\rangle - \left\langle \alpha_1, \tilde{\nabla}_v^{\theta^*} \alpha_2 \right\rangle. \]

We obtain

\[ \left\langle \tilde{\nabla}_v^\theta \alpha_1, \alpha_2 \right\rangle = \left\langle \nabla_v \alpha_1, \alpha_2 \right\rangle - \frac{1}{2} \left\langle g(v, \kappa^\sharp) \alpha_1, \alpha_2 \right\rangle - \left\langle \alpha_1, \nabla_v g(\alpha_1, \alpha_2) \right\rangle \]

and

\[ \left\langle \alpha_1, \tilde{\nabla}_v^{\theta^*} \alpha_2 \right\rangle = - \left\langle \alpha_1, \nabla_v \alpha_2 \right\rangle + \frac{1}{2} \left\langle \alpha_1, g(v, \kappa^\sharp) \alpha_2 \right\rangle - \left\langle \alpha_1, \text{div} \nabla_v \alpha_2 \right\rangle. \]

From (4) and (5), similarly to the classical case, we get

\[ S = \int_M v(g(\alpha_1, \alpha_2)) \, d\mu_g + \int_M \text{div} v g(\alpha_1, \alpha_2) \, d\mu_g \]

\[ + \int_M g(\nabla^M_f v, f_a) g(\alpha_1, \alpha_2) \, d\mu_g \]

\[ = \int_M \left( v(g(\alpha_1, \alpha_2)) + \text{div} v g(\alpha_1, \alpha_2) \right) \, d\mu_g \]

\[ = \int_M \text{div} (g(\alpha_1, \alpha_2) v) = 0 \quad \text{by Green theorem, [PO].} \]

Then, \( \tilde{\nabla}_v^{\theta^*} \) is in fact the adjoint operator of \( \tilde{\nabla}_v^\theta \). \( \square \)

We now compute the adjoint operator \( \tilde{\delta}_{b, \theta} = \tilde{d}_{b, \theta}^* \). Consider first the operator

\[ \tilde{\delta}_b := \sum_i -\iota_{e_i} \nabla_{e_i} + \frac{1}{2} \iota_{\kappa^\sharp} \]

\[ = \delta_b + \frac{1}{2} \iota_{\kappa^\sharp} \]

which is known to be the adjoint of \( \tilde{d}_b \), [HR]. Then, the operator

\[ \tilde{\delta}_{b, \theta} := \tilde{\delta}_b - \iota_{\theta^\sharp} \]

\[ = \sum_i -\iota_{e_i} \tilde{\nabla}_{e_i}^{\theta^*} - 2\iota_{\theta^\sharp} \]

is the adjoint of \( \tilde{d}_{b, \theta} \).

Let \( v \) and \( w \) be transverse basic vector fields and let \( \alpha \) be a basic differential form. We define the Clifford product

\[ v \cdot \alpha := v^\flat \wedge \alpha - \iota_v \alpha. \]

We remark that \( v \cdot v \cdot \alpha = -\|v\|^2_{g} \alpha \), where \( \|v\|_g := \sqrt{g(v, v)}. \)
We define the corresponding Dirac-type operator \( \tilde{D}_{b, \theta} \) using (2) and (6)
\[
\tilde{D}_{b, \theta} := \tilde{d}_{b, \theta} + \tilde{\delta}_{b, \theta} = \sum_i (e_i \wedge \tilde{\nabla}_{e_i} \theta - \iota_{e_i} \tilde{\nabla}_{e_i} \theta) - 2t_{\theta}.
\]

As in the classical case, a Laplace-type operator related to \( \tilde{D}_{b, \theta} \) can be defined,
\[
\tilde{\Delta}_{b, \theta} := \tilde{d}_{b, \theta} \tilde{\delta}_{b, \theta} + \tilde{\delta}_{b, \theta} \tilde{d}_{b, \theta}.
\]

**Remark 3.2.** \( \tilde{\Delta}_{b, \theta} \) is a transverse elliptic operator defined on the Riemannian foliation, with the same symbol as \( \Delta_b \).

For \( \theta = 0 \), we obtain the twisted operator \( \tilde{\Delta}_b \) employed in [HR]; furthermore, for \( \theta = -\frac{1}{2} \kappa \) we actually obtain the basic Laplace operator \( \Delta_b \) (see e.g. [RP, T]).

\[(\Omega_b(F), \tilde{d}_{b, \theta})\] is a transverse elliptic complex and hence, similarly to [HR, Proposition 2.3] (for the classical case when the manifold is foliated by points see e.g. [Gi, V2]), the following Hodge type decomposition holds good:

**Theorem 3.1.** ([IP]) The basic cohomology \( \Omega_b(F) \) can be written as a direct sum
\[
\Omega_b(F) = \text{Im}(\tilde{d}_{b, \theta}) \oplus \text{Im}(\tilde{\delta}_{b, \theta}) \oplus \text{Ker}(\tilde{\Delta}_{b, \theta}).
\]

If we denote \( H^p(\tilde{\Delta}_{b, \theta}) := \text{Ker}(\tilde{\Delta}_{b, \theta}) |_{\Omega_b(F)} \), with \( 0 \leq p \leq n \), then
\[
H^p(\Delta_b) \simeq \tilde{H}^p_{b, \frac{1}{2} \kappa + \theta}(F).
\]

Now, concerning the twisted Bott connection and the Clifford product, we have

**Lemma 3.2.** The following Leibniz rule holds:
\[
\tilde{\nabla}^\theta_v (w \cdot \alpha) = \nabla_v w \cdot \alpha + w \cdot \tilde{\nabla}^\theta_v \alpha.
\]

**Proof.** We have
\[
\tilde{\nabla}^\theta_v (w \cdot \alpha) = \nabla_v (w \cdot \alpha) - g(v, \frac{1}{2} \kappa^2 + \theta^2) w \cdot \alpha
\]
\[
= \nabla_v w \cdot \alpha + w \cdot (\nabla_v \alpha - g(v, \frac{1}{2} \kappa^2 + \theta^2) \alpha),
\]
and the result follows from the very definition of \( \tilde{\nabla}^\theta_v \). \( \square \)

**Lemma 3.3.** The following relation holds:
\[
\iota_w \tilde{\nabla}^\theta_v = \tilde{\nabla}^\theta_v \iota_w - \iota_{\nabla_v w}.
\]

**Proof.** We can write
\[
\iota_w \tilde{\nabla}^\theta_v = \iota_w (\nabla_v - g(v, \frac{1}{2} \kappa^2 + \theta^2))
\]
\[
= \nabla_v \iota_w - \iota_{\nabla_v w} - g(v, \frac{1}{2} \kappa^2 + \theta^2) \iota_w
\]
\[
= \tilde{\nabla}^\theta_v \iota_w - \iota_{\nabla_v w}.
\] \( \square \)
The following two equations relating standard operators on Riemannian foliations are the natural extension of classical results from calculus on differentiable manifolds. The proofs are similar to the classical case.

**Lemma 3.4.** For any basic vector fields \( v, w \) and basic form \( \alpha \in \Omega^b(F) \), we have

\[
\iota_w (v \cdot \alpha) = \iota_w v^b \alpha - v \cdot \iota_w \alpha,
\]

\[
\mathcal{L}_v \alpha - \nabla_v \alpha = \sum_i e_i \wedge \iota \nabla_{e_i} v \alpha.
\]

3.3. The twisted basic curvature operator. In this subsection we show that the curvature operator associated to the twisted Bott connection coincides in fact with the basic curvature operator and depends only on the transverse metric.

Let \( \gamma \) be a closed basic 1-form and denote by \( R^\gamma_{\flat} v, w \) the basic curvature operator written using the connection \( \nabla^\gamma \) and the transverse basic vector fields \( v, w \). We have the following useful relation:

**Lemma 3.5.** The curvature operator \( R^\gamma_{\flat} v, w \) does not depend on \( \gamma \), namely

\[
R^\gamma_{\flat} v, w = R v, w.
\]

**Proof.** Starting with the definition of the curvature operator, we obtain for \( R^\gamma_{\flat} v, w \)

\[
R^\gamma_{\flat} v, w = \nabla^\gamma v \nabla^\gamma w - \nabla^\gamma w \nabla^\gamma v - [v, w]_{\gamma}.
\]

Furthermore, we compute:

\[
\nabla^\gamma v \nabla^\gamma w = (\nabla v - g(v, \gamma^\sharp))(\nabla w - g(w, \gamma^\sharp))
= \nabla v \nabla w - g(v, \gamma^\sharp) \nabla w - g(\nabla w, \gamma^\sharp)
- g(w, \nabla v \gamma^\sharp) - g(v, \gamma^\sharp) \nabla v + g(v, \gamma^\sharp) g(w, \gamma^\sharp).
\]

Similarly, we have:

\[
\nabla^\gamma w \nabla^\gamma v = \nabla w \nabla v - g(w, \gamma^\sharp) \nabla v - g(\nabla v, \gamma^\sharp)
- g(v, \nabla w \gamma^\sharp) - g(v, \gamma^\sharp) \nabla w + g(w, \gamma^\sharp) g(v, \gamma^\sharp),
\]

and

\[
\nabla^\gamma [v, w] = \nabla [v, w] - g([v, w], \gamma^\sharp).
\]

Now, as \( d\gamma = 0 \), we have:

\[
g(w, \nabla v \gamma^\sharp) = g(v, \nabla w \gamma^\sharp).
\]

On the other hand, the Levi-Civita connection is symmetric, and hence

\[
g(\nabla v w - \nabla w v - [v, w], \gamma^\sharp) = 0.
\]

The conclusion follows. \( \Box \)

**Remark 3.3.** As \( \tilde{R}^\theta = R^{2\kappa+\theta} \), we also have \( \tilde{R}^\theta = R \).
3.4. A twisted basic Weitzenböck formula. We present now the Weitzenböck-type formula for the Laplace operator $\Delta_{b,\theta}$, canonically constructed employing the derivative $\bar{d}_{b,\theta}$ on $\Omega_b(\mathcal{F})$.

Note first that

$$\Delta_{b,\theta} = \bar{D}_{b,\theta}^2$$

$$= \sum_{i,j} (e_i \cdot \bar{\nabla}_{e_i}^\theta) (e_j \cdot \bar{\nabla}_{e_j}^\theta) - 2 \sum_i t_{\theta g} (e_i \cdot \bar{\nabla}_{e_i}^\theta) - 2 \sum_i (e_i \cdot \bar{\nabla}_{e_i}^\theta) t_{\theta g} + 4 t_{\theta g} t_{\theta g}. \tag{11}$$

As the last term vanishes, we compute below the other three terms.

For the first one, using Lemma 3.2 we get

$$\sum_{i,j} (e_i \cdot \bar{\nabla}_{e_i}^\theta) (e_j \cdot \bar{\nabla}_{e_j}^\theta) = \sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \bar{\nabla}_{e_j}^\theta + \sum_{i,j} e_i \cdot e_j \cdot \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_j}^\theta.$$

Let $\Gamma^k_{ij}$ be the Christoffel coefficients given by $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$, for the local orthonormal frame $\{e_i\}_{1 \leq i \leq q}$. As $e_i$ are basic, it turns out that $\Gamma^k_{ij}$ are basic functions. Then:

$$\sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \bar{\nabla}_{e_j}^\theta = \sum_{i,j,k} e_i \cdot -\Gamma^j_{ik} e_k \cdot \bar{\nabla}_{e_j}^\theta$$

$$= - \sum_{i,j,k} e_i \cdot e_k \cdot \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_j}^\theta,$$

and we obtain

$$\sum_{i,j} (e_i \cdot \bar{\nabla}_{e_i}^\theta) (e_j \cdot \bar{\nabla}_{e_j}^\theta) = \sum_i \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_i}^\theta - \sum_i \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_i}^\theta$$

$$+ \sum_{i<j} e_i \cdot e_j \cdot (\bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_j}^\theta - \bar{\nabla}_{e_j}^\theta \cdot \bar{\nabla}_{e_i}^\theta - \bar{\nabla}_{e_i}^\theta \cdot \nabla_{e_j} e_i - \bar{\nabla}_{e_j}^\theta \cdot \nabla_{e_i} e_j). \tag{12}$$

As

$$\sum_i \nabla_{e_i} e_i = - \sum_i g(\nabla_{e_i} e_j, e_i) e_j = - \sum_i \text{div} e_i e_i,$$

the ‘rough’ Laplace operator $(\bar{\nabla}^\theta)^2$ of the basic twisted connection satisfies the equation:

$$(\bar{\nabla}^\theta)^2 = \sum_i \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_i}^\theta - \sum_i \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_i}^\theta$$

$$= \sum_i -\text{div} e_i \bar{\nabla}_{e_i}^\theta - \sum_i \bar{\nabla}_{e_i}^\theta \cdot \bar{\nabla}_{e_i}^\theta$$

$$= \sum_i \bar{\nabla}_{e_i}^\theta \bar{\nabla}_{e_i}^\theta + 2 \sum_i g(\rho, \tau(\rho)) \bar{\nabla}_{e_i}^\theta$$

by Lemma 3.1.

Observe that, according to the definition of the twisted Bott connection, in (12) we do not have yet a basic curvature operator associated to $\bar{\nabla}^\theta$. However, using Remark 3.1 we obtain

$$\bar{\nabla}_{e_i}^\theta (\rho e_i) \alpha = 0,$$
for any $\alpha \in \Omega_b(F)$. Then, using also Remark 3.3 we denote
\[
\mathcal{R} := \sum_{i<j} e_i \cdot e_j \cdot (\mathring{\nabla}^\theta_{e_i e_j} \mathring{\nabla}^\theta_{e_i} - \mathring{\nabla}^\theta_{e_j e_i} \mathring{\nabla}^\theta_{e_j} - \mathring{\nabla}^\theta_{e_i e_j} - \mathring{\nabla}^\theta_{e_j e_i})
\]
\[
= \sum_{i<j} e_i \cdot e_j \cdot \mathring{\mathcal{R}}_{e_i e_j}^\theta
\]
\[
= \sum_{i<j} e_i \cdot e_j \cdot R_{e_i e_j}^\theta.
\]

With this, we finally obtain
\[
(iii) \quad \sum_{i,j} (e_i \cdot \mathring{\nabla}^\theta_{e_i}) (e_j \cdot \mathring{\nabla}^\theta_{e_j}) = \sum_i \mathring{\nabla}^\theta_{e_i} \mathring{\nabla}^\theta_{e_i} + 2 \sum_i g(e_i, \theta^j) \mathring{\nabla}^\theta_{e_i} + \mathcal{R}.
\]

As for the second term in (11), using Lemma 3.4 we get
\[
(ii) \quad \sum_i \frac{i}{\theta^i} g(e_i, \theta^j) \mathring{\nabla}^\theta_{e_i} = \sum_i \frac{i}{\theta^i} e^j \mathring{\nabla}^\theta_{e_i} - \sum_i e^i \cdot \frac{i}{\theta^i} \mathring{\nabla}^\theta_{e_i}
\]
\[
= \sum_i g(e_i, \theta^j) \mathring{\nabla}^\theta_{e_i} - \sum_i e_i \cdot \frac{i}{\theta^i} \mathring{\nabla}^\theta_{e_i}.
\]

For the third term we use Lemma 3.3
\[
(iv) \quad \sum_i (e_i \cdot \mathring{\nabla}^\theta_{e_i}) \frac{i}{\theta^i} = \sum_i e_i \cdot \frac{i}{\theta^i} \mathring{\nabla}^\theta_{e_i} + \sum_i e_i \cdot \frac{i}{\theta} \mathring{\nabla}^\theta_{e_i}.
\]

Plugging equations (13), (14), (15) in the relation (11), we end up with the corresponding version of Weitzenböck formula for the Laplace-type operator $\tilde{\Delta}_{b,\theta}$:
\[
\tilde{\Delta}_{b,\theta} = \sum_i \mathring{\nabla}^\theta_{e_i} \mathring{\nabla}^\theta_{e_i} + 2 \sum_i g(e_i, \theta^j) \mathring{\nabla}^\theta_{e_i} + \mathcal{R} - 2 \sum_i g(e_i, \theta^j) \mathring{\nabla}^\theta_{e_i}
\]
\[
+ 2 \sum_i e_i \cdot \frac{i}{\theta^i} \mathring{\nabla}^\theta_{e_i} - 2 \sum_i e_i \cdot \frac{i}{\theta^i} \mathring{\nabla}^\theta_{e_i} - 2 \sum_i e_i \cdot \frac{i}{\theta} \mathring{\nabla}^\theta_{e_i}
\]
\[
= \sum_i \mathring{\nabla}^\theta_{e_i} \mathring{\nabla}^\theta_{e_i} - 2 \sum_i e_i \cdot \frac{i}{\theta} \mathring{\nabla}^\theta_{e_i} + \mathcal{R}.
\]

**Remark 3.4.** In general, it is difficult to obtain a convenient basic Weitzenböck formula. First of all, if $\theta = 0$, then we obtain the classical form of such formula for $\tilde{\Delta}_b$ (for the particular case when the mean curvature $\kappa$ is also harmonic, see also [HR]). Secondly, as we noticed before, for the particular case when $\theta = -1/2 \cdot \kappa$, we obtain the classical basic Laplace operator, [RP]. Assuming that $\kappa$ is not only basic, but also parallel with respect to the Bott connection (with the corresponding topological consequences), we obtain again a standard form for (16).

In the next section we work out the Bochner technique for the basic Morse-Novikov cohomology, deriving corresponding conditions which imply the vanishing of the cohomology groups.

4. **Vanishing results for the basic Morse-Novikov cohomology**

In this section we investigate several situations when the groups of basic Morse-Novikov cohomology vanish.

We start by extending to our context a result from [LLMP], where the authors prove that if the closed form $\theta$ is also parallel, then the Morse-Novikov cohomology becomes trivial.
We shall need the following commutation formulae:

**Lemma 4.1.** The following equations are satisfied:

\[ [\tilde{\nabla}_\theta, \tilde{\partial}_b] = 0, \quad [\tilde{\nabla}_\theta, \tilde{\delta}_b] = 0. \]

**Proof.** We have

\[
\tilde{\nabla}_{\theta^t} = \sum_i e^i \wedge \nabla_{e_i} + \sum_i g(\theta^t, e_i) \nabla_{e_i} - \sum_i e^i \wedge \theta^t \nabla_{e_i}
\]

(17)

\[ = \sum_i e^i \wedge \nabla_{e_i} \theta^t + \theta^t \sum_i e^i \wedge \nabla_{e_i}
\]

\[ = \tilde{\partial}_b \theta^t + \theta^t \tilde{\partial}_b,
\]

where we use the fact that \( \theta \) is parallel.

We achieve the result in two steps. First of all, using (17) we show that \( \tilde{\partial}_b, \tilde{\delta}_b \) commute with \( \tilde{\nabla}_{\theta^t} \).

(18)

\[
\tilde{\nabla}_{\theta^t} \tilde{\partial}_b = \tilde{\partial}_b \tilde{\nabla}_{\theta^t} - \tilde{\nabla}_{\theta^t} \tilde{\partial}_b = \tilde{\nabla}_{\theta^t} \tilde{\nabla}_{\theta^t}.
\]

Taking adjoint operators, we obtain:

\[
\tilde{\nabla}_{\theta^t} \tilde{\delta}_b = \tilde{\delta}_b \tilde{\nabla}_{\theta^t}.
\]

Then, for the twisted operators, this gives:

(19)

\[
\tilde{\nabla}_{\theta^t} \tilde{\partial}_{b, \theta} = \tilde{\partial}_{b, \theta} \tilde{\nabla}_{\theta^t} - \tilde{\nabla}_{\theta^t} \tilde{\partial}_{b, \theta} = \tilde{\partial}_{b, \theta} \tilde{\nabla}_{\theta^t}.
\]

Again taking adjoint operators, we obtain

(20)

\[
\tilde{\nabla}_{\theta^t} \tilde{\delta}_{b, \theta} = \tilde{\delta}_{b, \theta} \tilde{\nabla}_{\theta^t}.
\]

\[ \square \]

The relation (17) allows us to establish the link between the operators \( \tilde{\partial}_{b, \theta}, \theta^t \) and the twisted connection \( \tilde{\nabla}_{\theta^t} \), namely

(21)

\[
\tilde{\nabla}_{\theta^t} - \text{Id} = (\tilde{\partial}_{b, \theta} - \theta^t \wedge) \tilde{\partial}_{b, \theta} + \theta^t \tilde{\partial}_{b, \theta} - \theta^t \wedge
\]

Now we can prove:

**Theorem 4.1.** Let \((M, \mathcal{F}, g)\) be a Riemannian foliation with closed manifold \(M\) and basic mean curvature \(\kappa\). If the basic, nontrivial 1-form \(\theta\) is parallel with respect to the Bott connection \(\nabla\), then

\[
\tilde{H}^i_{b, \theta}(\mathcal{F}) = H^i_{b, \frac{\partial}{2} + \theta}(\mathcal{F}) = 0
\]

for \(0 \leq i \leq q\), where \(H^i_{b, \frac{\partial}{2} + \theta}\) are the basic Morse-Novikov cohomology groups.
Proof: Let $\alpha \in \mathcal{H}^p(\tilde{\Delta}, \theta)$, where, as above, $\mathcal{H}^p(\tilde{\Delta}, \theta) = \text{Ker} \tilde{\Delta}_b, \theta|_{\Omega^{b}(F)}$. Then $\tilde{d}_b, \theta \alpha = 0$, $\tilde{\delta}_b, \theta = 0$, $\alpha$ being a harmonic form associated to $\tilde{\Delta}_b, \theta$. From Lemma 3.1, as $\theta$ is parallel it turns out that $\tilde{\nabla}_\theta^* = -\tilde{\nabla}_\theta$. We get
\[
\langle \tilde{\nabla}_\theta^* \alpha, \alpha \rangle = \langle \alpha, -\tilde{\nabla}_\theta^* \alpha \rangle,
\]
so
\[
(22) \quad \langle \tilde{\nabla}_\theta^* \alpha, \alpha \rangle = 0.
\]
Now, equation (21) implies:
\[
\tilde{\nabla}_\theta^* \alpha - \alpha = \tilde{d}_b, \theta (\iota_{\theta^*} \alpha),
\]
and $[\tilde{\nabla}_\theta^* \alpha] \equiv [\alpha]$ i.e. $\tilde{\nabla}_\theta^* \alpha$ and $\alpha$ lie in the same cohomology class of $H^p_{b, \frac{1}{2} \kappa + \theta}$. Using the commutation relations (19) and (20), we prove that $\tilde{\nabla}_\theta^* \alpha$ is also a harmonic form. Indeed,
\[
\tilde{d}_b, \theta \tilde{\nabla}_\theta^* \alpha = \tilde{\nabla}_\theta \tilde{d}_b, \theta \alpha = 0,
\]
\[
\tilde{\delta}_b, \theta \tilde{\nabla}_\theta^* \alpha = \tilde{\nabla}_\theta \tilde{\delta}_b, \theta \alpha = 0,
\]
and as a consequence we must have $\tilde{\nabla}_\theta^* \alpha = \alpha$. From (22) it follows that $\alpha = 0$, and hence $\mathcal{H}^p(\tilde{\Delta}, \theta) \equiv 0$, which proves the result. □

Example 4.1. We construct a Riemannian foliation endowed with a parallel basic 1-form and apply the above result.

Consider a Hopf manifold constructed in the following manner (see e.g. [DO, V1]). On the complex manifold $\mathbb{C}^n \setminus \{0\}$ one considers the metric $g := |z|^2 \cdot g_0$, where $g_0$ is the canonical Euclidean metric, with $z := (z^1, \ldots, z^n)$. Let $\Delta$ be the cyclic group generated by the transformation $z \mapsto e^{2\pi i z}$. The quotient $\mathbb{C}H := (\mathbb{C}^n \setminus \{0\})/\Delta$, is a complex Hopf manifold. Moreover, the above metric is invariant with respect to the transformation, and a quotient metric, still denoted by $g$, is induced on $\mathbb{C}H$. If $S^1(1/\pi)$ is the circle of radius $1/\pi$, then the mapping $f : \mathbb{C}^n \setminus \{0\} \rightarrow S^1(1/\pi) \times S^{2n-1}$ defined as
\[
f(z) := \left( \frac{1}{\pi} e^{i \pi \ln |z|}, \frac{z}{|z|} \right)
\]
is invariant with respect to the above group action, and induces an isometry between $\mathbb{C}H$ and $S^1(1/\pi) \times S^{2n-1}$ [DO]. If $J$ is the complex structure, then the Kähler form $\omega(\cdot, \cdot) := g(\cdot, J \cdot)$ has the expression
\[
\omega = -i \frac{1}{2|z|^2} \sum_j dz^j \wedge d\bar{z}^j.
\]
One easily sees that $\omega$ satisfies the equation (see Section 5)
\[
d\omega = \theta \wedge \omega, \quad \text{with} \quad \theta = -\frac{1}{|z|^2} \sum_j (z^j d\bar{z}^j + \bar{z}^j dz^j).
\]
This $\theta$ is called Lee form and, in this case, it is parallel with respect to the Levi-Civita connection of $g$. Its metric dual, called the Lee field:
\[
B := \theta^* = - \sum_j \left( z^j \frac{\partial}{\partial \bar{z}^j} + \bar{z}^j \frac{\partial}{\partial z^j} \right)
\]
is also parallel, and consequently it is Killing, so we have (see e.g. [DO])
\[ L_B g = 0, \ L_B \theta = 0. \]

If \( \varphi \) is the flow generated by \( B \) and if \( T \in \mathbb{R} \) is fixed, then \( \varphi_T \) is an isometry of \( \mathbb{C}H \) which leaves \( g \) and \( \theta \) invariant. Then, the direct product space \( \hat{M} := \mathbb{C}H \times \mathbb{R} \) is endowed with a direct product Riemannian structure (on \( \mathbb{R} \) we just consider the canonical metric). Moreover, \( \hat{M} \) is foliated by the real lines, and \( \theta \) is in fact a basic 1-form which is parallel with respect to the Bott connection. We can now suspend (see e.g. [MQ]) the action of \( \varphi_T \) on \( \mathbb{C}H \) by considering the equivalence relation \( (y, x) \sim (\varphi_T(y), x + 1) \) on \( \hat{M} \) and taking the quotient
\[ M := \hat{M} / \sim. \]

We end up with a Riemannian foliation \((M, F)\) on which the basic 1-form \( \theta \) remains parallel. Applying Theorem 4.1, the basic Morse-Novikov cohomology groups \( \tilde{H}^i_{b,\theta}(F) \) for \( 0 \leq i \leq q \) are trivial for the above suspension of the Hopf manifold.

In what follows we investigate the weaker case when \( \theta \) is closed, but non-exact and non-parallel. We adapt the arguments from [GL] (see also [DO]) to our framework. More precisely, we prove the following statement:

**Theorem 4.2.** Let \((M, F, g)\) be a transversally oriented Riemannian foliation with the underlying manifold \( M \) closed and connected, and suppose the mean curvature \( \kappa \) basic. If the basic 1-form \( \theta \) is closed but not exact, then the top dimension basic Morse-Novikov cohomology group \( \tilde{H}^q_{b,\theta}(F) \) vanishes,

\[ \tilde{H}^q_{b,\theta}(F) = \tilde{H}^q_{b,\frac{1}{2}\kappa + \theta}(F) = 0. \]

**Proof.** Assume that \( \alpha \in \Omega^q_b(F) \), \( \alpha = f \cdot \text{vol}^Q \), where \( \text{vol}^Q \) is the (globally defined) transverse volume form. As \( \alpha \) and \( \text{vol}^Q \) are basic (leafwise invariant) differential forms, then \( f \) will be a basic (i.e. constant along leaves) smooth function. Locally we may write:

\[ \alpha = f e^1 \wedge \cdots \wedge e^q, \]

with respect to the basic orthonormal co-frame \( \{e^i\}_{1 \leq i \leq q} \). Clearly
\[ \tilde{d}_{b,\theta + \frac{1}{2}\kappa} (\alpha) = 0 \]
and let us assume that also
\[ \tilde{\delta}_{b,\theta + \frac{1}{2}\kappa} (\alpha) = 0. \]

Then we have
\[ \tilde{\delta}_{b,\theta + \frac{1}{2}\kappa} (\alpha) = - \sum_i e_i \wedge \nabla e_i \alpha - \iota_{\theta + \frac{1}{2}\kappa} \alpha \]
\[ = - \sum_i e_i \wedge \nabla e_i \alpha - \iota_{\theta} \alpha. \]

We consider the local descriptions \( \theta = \sum_i \theta_i e^i \) and also (23):
\[ \tilde{\delta}_{b,\theta + \frac{1}{2}\kappa} (\alpha) = - \sum_i \iota_{e_i} \nabla e_i \left( f e^1 \wedge \cdots \wedge e^q \right) - \iota_{\theta, e_i} \left( f e^1 \wedge \cdots \wedge e^q \right). \]

From here, as \( \text{vol}^Q \) is parallel with respect to \( \nabla \), we find
\[ e_i (f) + f \theta_i = 0, \]
for \(1 \leq i \leq q\). As \(f\) is basic, we obtain
\[
d_b f + f \theta = \sum_i e^i \wedge \nabla_{e^i} f + \sum_i f \theta_i e_i = 0.
\]
Assuming that the function \(f\) is nowhere vanishing, eventually changing the sign, we can write
\[
(24) \quad \theta = d \left(-\ln f\right),
\]
which is a contradiction with the initial assumption. Then the zero set of \(f\) cannot be empty.

Consider a finite open cover \(\{U_i\}_{i \in I}\) of \(M\), such that \((U_i, \varphi_i)\) are contractible foliated local maps. Then we can find some positive basic functions \(\psi_i\) satisfying the property
\[
(25) \quad \theta|_{U_i} = d (\ln \psi_i).
\]
In fact we can construct such function in a canonical way on a local transversal \(T\) such that the above relation is fulfilled for the local projection of the basic (projectable) 1-form \(\theta\) on \(T\), then we can take the pull-back of the function on \(U_i\), obtaining a local basic function. Then \((24)\) and \((25)\) imply \(f = c_i \psi_i\) on \(U_i\), with \(c_i \in \mathbb{R}\). If \(f = 0\) at some point then necessarily \(c_i = 0\), and \(f\) vanishes on a whole open neighborhood. So the zero set is open. Clearly the zero set of \(f\) is also closed, and hence it coincides with the connected manifold \(M\), and \(\alpha = 0\). Consequently there is no basic harmonic form of degree \(q\) with respect to \(\tilde{\Delta}_{b,\theta}\). Now the Hodge decomposition theorem yields \(\tilde{H}^q_{b,\theta}(\mathcal{F}) = H^q_{b,\frac{\alpha}{b}+\theta}(\mathcal{F}) = 0\).

\[
(\text{Remark 4.1.}) \quad \text{We point out that a theory of geometric objects which would be in the same time leafwise invariant and compactly supported is not easy to undertake, so the extension of the above result in the non-compact case is not trivial.}
\]

\[
(\text{Example 4.2.}) \quad \text{We now present an application for the above theorem. A classical Riemannian flow can be constructed starting with a matrix} \ A \in \text{SL}(2, \mathbb{Z}), \ \text{with} \ \text{Tr} A > 2. \ \text{If} \ \{\lambda_i\}_{1 \leq i \leq 2} \ \text{are the eigenvalues of} \ A, \ \text{it is easy to see that} \ \lambda_i \neq 1, \lambda_i > 0.
\]

Let \(\{v_i\}_{1 \leq i \leq 2}\) be the corresponding orthonormal eigenvectors. We denote by \(H\) the space \(\mathbb{R}^3\) regarded as
\[
(26) \quad \mathbb{R}^3 \equiv \mathbb{R} \times \mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R}v_1 \times \mathbb{R}v_2
\]
We endow \(H\) with a Lie group structure using the multiplication
\[
p \cdot p' := (t + t', \lambda'_1 \alpha' + \alpha, \lambda'_2 \beta' + \beta),
\]
for any \(p = (t, \alpha, \beta), \ p' = (t', \alpha', \beta') \in H\), with respect to the identification \((26)\) of \(\mathbb{R}^3\). Starting with the orthonormal basis \(\{e := (1, 0, 0), v_1, v_2\}\), we canonically construct three left invariant vector fields on \(H\) such that at any point \(p\) we have
\[
e_p = (1, 0, 0), \quad v_{1,p} = \lambda'_1 (0, 1, 0), \quad v_{2,p} = \lambda'_2 (0, 0, 1),
\]
which, in turn, generate the warped metric

\[
g = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_1^{-2t} & 0 \\
0 & 0 & \lambda_2^{-2t}
\end{pmatrix}.
\]

The manner we choose the matrix \( A \) insures that the standard subgroup \( \Gamma := \mathbb{Z} \times \mathbb{Z}_2 \) of \( \mathbb{R}^3 \) remains a discrete and co-compact subgroup of \( H \). Consequently, we obtain the quotient Lie group \( T^3_A := \Gamma \backslash H \). It also inherits the above left invariant geometric objects, which will be denoted in the same way, for convenience. The flow \( \varphi_2 \) generated by \( v_2 \) induces on our manifold a foliation structure \( F \), which can be proved to be a \( GA \)-foliated manifold, where \( GA \) is the the orientation preserving affine group \( \mathbb{Ca} \).

We compute the Lie brackets

\[
\begin{aligned}
[e, v_2] &= \ln \lambda_1 v_2, \\
[e, v_3] &= \ln \lambda_2 v_3, \\
[v_2, v_3] &= 0,
\end{aligned}
\]

and we use the classical Koszul formula for the metric \( g \) on \( T^3_A \) to obtain

\[
\begin{aligned}
g(\nabla_{v_2} v_2, e) &= \ln \lambda_1, \\
g(\nabla_{v_2} v_2, v_1) &= 0.
\end{aligned}
\]

Then \( \kappa^2 = \ln \lambda_2 e \). As \( \kappa^2 = dt \), we finally find

\[
\kappa = \ln \lambda_2 dt.
\]

It is now easy to see that on the compact quotient manifold \( T^3_A \) the basic form \( dt \) is closed but not exact. Furthermore, \( H^1_b(F) \equiv \mathbb{R} \mathbb{Ca} \), and the basic differential 1-forms \( \theta \) which are closed but not exact are precisely

\[
\theta = c dt + df, \quad \text{with } c \in \mathbb{R} \setminus \{0\} \text{ and } f \in \Omega^0_b(F).
\]

Applying Theorem 4.2 for the above considered Riemannian flow, we obtain the vanishing of the top dimension group of the basic Morse-Novikov cohomology complex \( (\Omega_b(F), d_b, \theta) \), i.e.

\[
\tilde{H}^2_{b, \theta}(F) = 0.
\]

This result can be regarded as a generalization of \( \mathbb{Ca} \) III, Proposition 2]. Indeed, for the particular choice \( \theta = -1/2 \cdot \ln \lambda_2 dt \), we obtain the vanishing of the basic cohomology group \( H^2_b(F) \), with the corresponding tautness consequences. For the particular case \( \theta = 0 \) see also \( \mathbb{HR} \).

In the following, we investigate the vanishing of the basic Morse-Novikov cohomology under the assumption of certain conditions related to the curvature-type operators. Note that in the classical case represented by closed Riemannian manifolds, if the de Rham cohomology vanishes, then all closed differential 1-forms are in fact exact. The Morse-Novikov cohomology complex, being isomorphic to de Rham complex (as we noticed in the previous section), vanishes too. If the curvature operator is non-negative and positive at some point, applying a well known result of Gallot and Meyer \( \mathbb{CM} \) on closed Riemannian manifolds, we obtain that the Morse-Novikov cohomology complex is trivial in this case.
In order to extend the above result in our framework, the only needed ingredient is the corresponding version of the result of Gallot and Meyer. This was achieved in [He, MRT], where the authors used arguments related to functional analysis and operator theory. We notice that, in accordance with Remark 3.4, for the particular case when $\kappa$ is basic and parallel, this fact can be derived in the classical fashion.

Consequently, we easily obtain the following result:

**Proposition 4.1.** If the Riemannian foliation $(M, \mathcal{F}, g)$ has non-negative, and positive at some point, basic curvature operator, then any closed basic 1-form $\theta$ is exact, and consequently

$$H^i_{b,\theta}(\mathcal{F}) = 0, \ 0 < i < q.$$

**Example 4.3.** As an application, we consider the case of a suspension foliation used by Connes [Co] (see also [Mo, Appendix E]). More precisely, let $S$ be a compact orientable surface of genus 2, with universal cover $\tilde{S}$, and define $\tilde{M} := SO(3, \mathbb{R}) \times \tilde{S}$. Let $h : \pi_1(S) \to SO(3, \mathbb{R})$ be a group homomorphism. Define a smooth diagonal action of $\pi_1(S)$ on $\tilde{M}$ by setting

$$R_{[\gamma]}(y, \hat{x}) = (h([\gamma]^{-1})(y), \hat{x} \circ [\gamma])$$

for each $[\gamma] \in \pi_1(S)$.

The quotient manifold $M := \tilde{M}/R$ is then a $SO(3, \mathbb{R})$-foliation. If, moreover, $h$ is injective, then the leaves are actually diffeomorphic to $\mathbb{R}^2$.

Endow $\tilde{M}$ with a direct product Riemannian metric, which is also invariant with respect to the above action $R$. Then the foliated manifold $M$ inherits a bundle-like metric. Concerning the transverse part, if the image of $h$ is represented by canonical mappings produced by taking left multiplications on $SO(3, \mathbb{R})$, then a left invariant metric $g$ can be defined in a standard way on our Lie group. For instance, in the Lie algebra $\mathfrak{o}(3, \mathbb{R})$ we choose $e_1, e_2$ and $e_3$ as

$$e_1 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the metric is constructed such that $\{e_i\}_{1 \leq i \leq 3}$ becomes an orthonormal basis.

As the group is compact, the metric is in fact bi-invariant, and the following formula can be used to compute the sectional curvature $k$ (see e.g. [Mi]).

$$k(e_i, e_j) = \frac{1}{4}g([e_i, e_j], [e_i, e_j]), \ 1 \leq i, j \leq 3.$$

As

$$[e_1, e_2] = e_3, \ [e_2, e_3] = e_1 \ [e_3, e_1] = e_2,$$

we obtain

$$k(e_1, e_2) = k(e_2, e_3) = k(e_3, e_1) = \frac{1}{4},$$

$SO(3, \mathbb{R})$ being in fact a distinguished compact Lie group admitting metrics with strictly positive curvature [Mi].

Proposition 4.1 now implies that the Morse-Novikov cohomology groups are trivial for the Connes foliation.

On the other hand, in [HR] Corollary 6.8, for the particular case when $\theta \equiv 0$, the authors obtain a vanishing result for the groups of basic Morse-Novikov cohomology which holds even in the case when the basic curvature operator is not necessarily
non-negative. Note that in this case the basic de Rham complex may not be trivial. Starting from this remark, we obtain the following vanishing result for a closed basic 1-form $\theta$.

**Theorem 4.3.** Let $(M, \mathcal{F}, g)$ be a Riemannian foliation with $M$ closed and the mean curvature $\kappa$ basic. Assume that $\theta$ is a closed 1-basic form and define the bi-linear map

$$
\beta_{\theta} : \Omega^1(\mathcal{F}) \times \Omega^1(\mathcal{F}) \to C^\infty(M), \text{ with } \beta_{\theta}(\cdot, \cdot) := \mathcal{L}_{\theta^*}g(\cdot, \cdot) + g(R\cdot, \cdot).
$$

If $\beta$ is non-negatively defined, and $\beta_x$ is positively defined at some point $x \in M$, then the basic Morse-Novikov cohomology groups vanish

$$
\tilde{H}^i_{b, \theta}(\mathcal{F}) = 0, \ 0 < i < q.
$$

**Proof.** We start from (16) and let $\alpha \in \Omega^i(\mathcal{F}), \ 0 < i < q$. Taking integrals on the closed manifold $M$, we obtain

$$
\langle \tilde{\Delta}_{b, \theta} \alpha, \alpha \rangle = \sum_{i} \left\| \tilde{\nabla}^\theta \alpha \right\|^2 - 2 \int_M g(\sum_i e_i \cdot \iota_{\nabla_{x_i} \theta^*} \alpha, \alpha) d\mu_g + \int_M g(R\alpha, \alpha) d\mu_g.
$$

where, for arbitrary $\alpha \in \Omega^i(\mathcal{F})$, we define $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. Then, in order to apply the Bochner technique, we take a closer look at the middle term. As the scalar product of forms of different degrees vanishes, Lemma 3.4 implies:

$$
g(\sum_i e_i \cdot \iota_{\nabla_{x_i} \theta^*} \alpha, \alpha) = g(\sum_i e_i \wedge \iota_{\nabla_{x_i} \theta^*} \alpha, \alpha)
$$

$$
= g(\mathcal{L}_{\theta^*} \alpha, \alpha) - g(\nabla_{\theta^*} \alpha, \alpha)
$$

$$
= -\frac{1}{2} \mathcal{L}_{\theta^*} g(\alpha, \alpha).
$$

This yields the formula

$$
\langle \tilde{\Delta}_{b, \theta} \alpha, \alpha \rangle = \sum_i \left\| \tilde{\nabla}^\theta \alpha \right\|^2 + \int_M (\mathcal{L}_{\theta^*} g(\alpha, \alpha) + g(R\alpha, \alpha)) d\mu_g.
$$

Arguing now as in the classical case, and considering the isomorphism $H^i(\tilde{\Delta}_{b, \theta}) \simeq H^i_{b, \frac{1}{2} \kappa + \theta}(\mathcal{F})$, we obtain the result. \[ \square \]

We outline the particular case of Theorem 4.3 obtained for $\theta = -\frac{1}{2} \cdot \kappa$. Then, the basic Morse-Novikov cohomology complex is again just the classical basic de Rham complex, and we get vanishing conditions suitable for Riemannian foliations with non-positive transverse curvature which are different from the previous results (see e.g. [He, MRT]).

**Corollary 4.1.** If the bi-linear map $\beta_{\theta}^-$ defined in (27) is non-negatively defined and positively defined at some point $x \in M$, then the basic cohomology groups are trivial,

$$
H^i_b(\mathcal{F}) = 0, \ 0 < i < q.
$$

Furthermore, as $H^1_b(\mathcal{F}) \equiv 0$, the foliation is taut. [A].
Finally, another particular case is represented by closed Riemannian manifolds. Again, this classical framework is obtained for the limiting case when the leaves are points. The mean curvature vanishes, the basic geometric objects become the classical ones, and we obtain the Bochner technique adapted for Morse-Novikov cohomology.

**Corollary 4.2.** Let us assume that \((M,g)\) is a closed Riemannian manifold of dimension \(n\) and \(\theta\) is a closed 1-differential form. If the bi-linear map

\[
\beta_\theta : \Omega(M) \times \Omega(M) \to C^\infty(M), \quad \text{with} \quad \beta_\theta(\cdot,\cdot) := L_\theta g(\cdot,\cdot) + g(\mathcal{R},\cdot),
\]

is non-negatively defined, and positively defined at some point \(x \in M\), then the groups of the Morse-Novikov cohomology vanish

\[
H^i_\theta(M) = 0, \quad \text{for} \quad 0 < i < n.
\]

5. **Applications to l.c.s. and l.c.K. foliations**

In this final section we apply the results obtained in the rest of the paper to the particular case represented by l.c.s. and l.c.K. foliations.

A *locally conformally symplectic* manifold (l.c.s.) is a differentiable manifold \(M\) of dimension \(2n\) endowed with a 2−differentiable form \(\omega\) which is locally conformal with a symplectic (i.e. closed and non-degenerate 2−differentiable form) [V3]:

\[
d(e^Uf \cdot \omega|U) = 0. \quad \omega \text{ will be also called *locally conformally symplectic structure*}.
\]

Following [DO, V1], we make the convention that the case when this procedure can be performed globally is not viewed as a particular case of l.c.s., but as an opposite case.

A condition equivalent to the definition is the existence of a global closed 1−form \(\theta\) (called *Lee form*) such that

\[
d\theta \omega := d\omega - \theta \wedge \omega = 0.
\]

Furthermore, assume the manifold to be complex and endowed with a metric \(g\) compatible with the complex structure \(J\). Then, if \(\omega\) is determined by \(J\) and \(g\) (i.e. \(\omega(\cdot,\cdot) := g(\cdot,J\cdot)\)), then the manifold is said to be *locally conformally Kähler*; in this latter case a local Kähler metric being obtained by a conformal change of the initial metric. If the Lee form \(\theta\) is parallel with respect to the Levi-Civita connection, then the manifold \(M\) is a *Vaisman manifold* (previously called *generalized Hopf manifold* [DO, V2]).

Regarding (28), we see that these types of differentiable manifolds have a natural Morse-Novikov cohomological complex (called also *adapted cohomology*), which encodes many interesting properties of the underlying manifolds (see [OV2, V2]).

The above defined geometric structures can be extended to the context of Riemannian foliations, the transverse geometry of the foliations corresponding to the geometry of the manifolds. We thus obtain *l.c.s. foliations, Kähler* and *l.c.K. foliations* and, respectively, *Vaisman foliations* (see [BD, IP]). We emphasize the interplay between the above defined spaces: for instance, classical Vaisman manifolds are examples of Kähler foliations of dimension 2 [BD].

The simplest examples of l.c.s. and l.c.K. foliations are represented by a smooth Riemannian submersion \(f : M \to N\), the base manifold \(N\) being a l.c.s. (l.c.K., respectively) manifold [BD]. In turn, the Proposition 4.1 offers a condition for the non-existence of such transverse structures.
Corollary 5.1. Let $(M, \mathcal{F})$ be a foliated manifold of codimension $2q$, with $M$ compact. Assume there is a bundle-like metric $g$ on $M$ such that the basic curvature operator is non-negatively defined and positively defined at some point. Then the foliation does not admit a transverse locally symplectic structure (and consequently there is no transverse l.c.K. structure with respect to any bundle-like metric defined on $(M, \mathcal{F})$).

The proof is straightforward applying our previous convention.

Now, assume that the basic curvature operator allows the existence of a transverse l.c.s. structure $\omega$ with the transverse Lee form $\theta$. Then, Theorem 4.3 provides vanishing conditions for the basic adapted cohomology.

Corollary 5.2. If the bi-linear form $\beta_\theta$ defined in (27) is non-negatively defined and positively defined at some point, then the groups of the adapted basic cohomology vanish

$$\tilde{H}^i_{b, \theta} = 0 \text{ for } 0 < i < 2q.$$

Note that the top dimension cohomology group cannot be addressed with the above result. In turn, this can be done using Theorem 4.2.

Corollary 5.3. If the basic Lee form $\theta$ is not exact, then $\tilde{H}^{2q}_{b, \theta} = 0$.

Remark 5.1. The above result stands as a generalization of [DO, Theorem 2.9] for the case when the manifold $M$ is compact.

In the final part of the paper we deal with Vaisman foliations. First of all we remark that the suspension foliation constructed in Example 4.1, endowed with a parallel Lee form is consequently a Vaisman foliation. On the other side, using Theorem 4.1 we see that the groups of the basic adapted cohomology are trivial.

Corollary 5.4. For any Vaisman foliation:

$$\tilde{H}^i_{b, \theta} = 0 \text{ for } 0 < i < 2q.$$

Remark 5.2. The above result is also an extension of [OV2], where the triviality of the adapted cohomology is derived directly from the structure theorem of compact Vaisman manifold [OV1]. For the general case of Riemannian foliations a similar attempt is not a trivial extension, as structural aspects of Riemannian foliations should be also considered [Mo].

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