Thin instantons in $G_2$-manifolds and Seiberg-Witten invariants

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March 28, 2013

Abstract

For two nearby disjoint coassociative submanifolds $C$ and $C'$ in a $G_2$-manifold, we construct thin instantons with boundaries lying on $C$ and $C'$ from regular $J$-holomorphic curves in $C$. We explain their relationship with the Seiberg-Witten invariants for $C$.

Contents

1 Introduction 2

2 Instantons of Dimension 2 and 3 4

2.1 Review of Symplectic Geometry 4
2.2 Counting Instantons in (almost) $G_2$-manifolds 5
2.3 Relationships to Seiberg-Witten Invariants 8
2.4 Deformation of Instantons 13

3 Dirac Equation on Thin 3-manifolds 21

3.1 A Simplified Model 21
3.2 First Eigenvalue Estimates 24
3.3 Schauder Estimates for Linear Model 30

4 Proof of the Main Theorem 42

4.1 Geometry of $\Sigma \subset \mathcal{C} \subset M$ 44
4.2 Linearization of Instanton Equation and Comparison with the Operator $\mathcal{D}$ 50
4.3 Quadratic Estimates 50
4.4 Perturbation Argument 68
5 Appendix: The Exponential-like Map $\exp$

1 Introduction

Intersection theory of Lagrangian submanifolds is an essential part of symplectic geometry. By counting the number of holomorphic disks bounding intersecting Lagrangian submanifolds, Floer and others defined the celebrated Floer homology theory. It plays an important role in mirror symmetry for Calabi-Yau manifolds and string theory in physics. In M-theory, Calabi-Yau threefolds are replaced by seven dimensional $G_2$-manifolds $M$ (i.e. oriented Octonion manifolds [26]). The analogs of holomorphic disks (resp. Lagrangian submanifolds) are instantons or associative submanifolds (resp. coassociative submanifolds or branes) in $M$ [25]. In [17], the Fredholm theory for instantons with coassociative boundary conditions has been set up. However, existence of instantons is still a difficult problem. As a first step, we want to give a construction modeled on the work of Fukaya and Oh [10] in symplectic geometry. As it was shown in [10], if we choose two nearby Lagrangian submanifolds in such a way that one is the graph of a closed one form on the other then the holomorphic disks bounding two is closely related to gradient flow lines of the one form. Searching for the analog in $G_2$-geometry leads us to study the following problem.

Problem: Given two nearby coassociative submanifolds $C$ and $C'$ in a (almost) $G_2$-manifold $M$. Relate the number of instantons in $M$ bounding $C \cup C'$ to the Seiberg-Witten invariants of $C$.

The basic idea is as follows: When the coassociative submanifold $C'$ is sufficiently close to $C$, then it is the graph of a self-dual two form on $C$. This two form is essentially a symplectic form on $C$ away from the intersection $C \cap C'$. Instantons bounding $C \cup C'$ would become holomorphic curves on $C$ when $C'$ collapses onto $C$ modulo the possible bubblings. By the seminal work of Taubes [36] on the equivalence of Gromov-Witten and Seiberg-Witten invariants, we expect that the number counted with algebraic weights of such instantons is given by the Seiberg-Witten invariants of $C$.

Settling the above problem completely is very difficult at the current stage. We treat the special case when $C$ and $C'$ are disjoint, i.e. $C$ is a symplectic four manifold in this paper. The basic tool is the gluing
technique. But even in this simpler case, the set up is quite different from the Lagrangian Floer theory (c.f. Fukaya and Oh [10]). Our domains are three dimensional instead of two dimensional and we have to deform the submanifolds rather than deforming the maps as was done in Floer theory. This is because we do not have the luxury of applying the conformal geometry in dimension two to transform the problem of finding holomorphic curves into the one of finding holomorphic maps. Furthermore, the linear theory is more difficult in this case since we have to deal with a problem which lacks uniform ellipticity, as we will explain in Section 3.3. Since the $G_2$ form is cubic, the needed quadratic estimate of the 3-dimensional instanton equation appears unavailable in the $L^p$ setting (see end of Section 4.3). So we set up the problem in Schauder setting, and in linear theory we need to go further from $L^p$ estimates to Schauder estimates. This is different from [10].

As $C'$ should be sufficiently close to $C$, we assume that they arise in a one-parameter smooth family of coassociative submanifolds $C_t$ with small $t$. Contracting with the normal vector field $n := dC_t/dt|_{t=0}$ for the infinitesimal deformation with the $G_2$-form $\Omega$, we obtain a closed self-dual two form $\iota_n \Omega \in \Omega^2_\pm (C_0)$. Using the induced metric, from $n$ one can define an almost complex structure $J = J_n$ on $C_0$ away from the zero set of $\iota_n \Omega$ (see (2) for details).

**Theorem 1** Suppose that $(M, \Omega)$ is a $G_2$-manifold and $\{ C_t \}$ is a one-parameter smooth family of coassociative submanifolds in $M$. When $\iota_n \Omega \in \Omega^2_\pm (C_0)$ is nonvanishing, then

1. (Proposition 6) If $\{ A_t \}$ is any one-parameter family of associative submanifolds (i.e. instantons) in $M$ satisfying

   $$\partial A_t \subset C_t \cup C_0, \quad \lim_{t \to 0} A_t \cap C_0 = \Sigma_0 \text{ in the } C^1\text{-topology,}$$

   then $\Sigma_0$ is a $J_n$-holomorphic curve in $C_0$.

2. (Theorem 27) Conversely, every regular $J_n$-holomorphic curve $\Sigma_0$ (namely those for which the linearization of $\bar{\partial} J_n$ on $\Sigma_0$ is surjective) in $C_0$ is the limit of a family of associative submanifolds $A_t$'s as described above.

Notice that in the product situation, where

$$(M, \Omega) := (X \times S^1, \text{Re } \Omega_X + \omega_X \wedge d\theta)$$

3
(c.f. Section 2.2) with \((X, \Omega_X)\) being a Calabi-Yau threefold, then our theorem would follow from the work of [10].

The paper is organized as follows: In Section 2, we first recall some basics of Floer theory, next we describe their \(G_2\)-counterparts, then we explain the connection between instantons and Seiberg-Witten invariants, and last we study the deformation of instantons with the aim to generalize to almost instantons. In Section 3, we first study the linear differential operator \(D\) (defined in (19)) on a type of thin 3-manifolds, which is a linear approximation of the instanton equation, then we give the \(L^2\) and Schauder estimates of its inverse \(D^{-1}\). In Section 4, we first compare the linearized instanton equation on almost instantons with the operator \(D\) on linear models, then we use the implicit function theorem to perturb almost instantons to true instantons, thus proving our main theorem.

Acknowledgments: The first author expresses his gratitude to J.H. Lee, Y.G. Oh, C. Taubes, R. Thomas and A. Voronov for useful discussions. The second author thanks S.L. Kong, G. Liu, Y.J. Lee, Y.G. Shi, L. Yin for useful discussions. The third author thanks the nice research environment in math departments of The Chinese University of Hong Kong and University of Minnesota, and useful discussion with T.J. Li. We thank the anonymous referees for pointing out many inaccuracies in our earlier versions and providing suggestions for improvement.

The work of the first author described in this paper was partially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK401908 and CUHK403709). The work of the second author described in this paper was partially supported by grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK403709). The third author was partially supported by RGC grant from the Hong Kong Government.

## 2 Instantons of Dimension 2 and 3

### 2.1 Review of Symplectic Geometry

Given any symplectic manifold \((X, \omega)\) of dimension \(2n\), there exists a compatible metric \(g\) so that the equation

\[
\omega(u, v) = g(Ju, v)
\]
defines a Hermitian almost complex structure
\[ J : T_X \to T_X, \]
that is \( J^2 = -\text{id} \) and \( g(Ju, Jv) = g(u, v) \).

A **holomorphic curve**, or **instanton** of dimension 2, is a two dimensional submanifold \( \Sigma \) in \( X \) whose tangent bundle is preserved by \( J \). Equivalently \( \Sigma \) is calibrated by \( \omega \), i.e. \( \omega|_{\Sigma} = vol_{\Sigma} \). By **algebraic counting** the number of instantons in \( X \), one can define a highly non-trivial invariant for the symplectic structure on \( X \), called the Gromov-Witten invariant.

When the instanton \( \Sigma \) has nontrivial boundary, then the corresponding boundary value problem would require \( \partial \Sigma \) to lie on a **Lagrangian submanifold** \( L \) in \( X \), i.e. \( \dim L = n \) and \( \omega|_{L} = 0 \). Floer studied the intersection theory of Lagrangian submanifolds and defined the Floer homology group \( HF(L, L') \) under certain assumptions.

Suppose that \( X \) is a compact Calabi-Yau manifold, i.e. the holonomy group of the Levi-Civita connection is inside \( SU(n) \), equivalently \( J \) is an integrable complex structure on \( X \) and there exists a holomorphic volume form \( \Omega_X \in \Omega^{n,0}(X) \) on \( X \) satisfying
\[
(-1)^{\frac{n(n-1)}{2}} (i/2)^n \Omega_X \wedge \bar{\Omega}_X = \omega^n/n!. 
\]
Under the mirror symmetry transformation, \( HF(L, L') \) is expected to correspond to the Dolbeault cohomology group of coherent sheaves in the mirror Calabi-Yau manifold.

A Lagrangian submanifold \( L \) in \( X \) is called a **special Lagrangian submanifold** with phase zero (resp. \( \pi/2 \)) if \( \text{Im} \Omega_X|_L = 0 \) (resp. \( \text{Re} \Omega_X|_L = 0 \)). With suitable choice of orientation of \( L \), \( L \) is calibrated by \( \text{Re} \Omega_X|_L \) (resp. \( \text{Im} \Omega_X|_L \)), that is \( \text{Re} \Omega_X|_L \) is the volume form of \( L \). They play important roles in the Strominger-Yau-Zaslow mirror conjecture for Calabi-Yau manifolds \([35]\).

When \( X \) is a Calabi-Yau threefold, there are conjectures of Vafa and others (e.g. \([3][14]\)) that relates the (partially defined) open Gromov-Witten invariant of the number of instantons with Lagrangian boundary condition to the large \( N \) Chern-Simons invariants of knots in three manifolds.

**2.2 Counting Instantons in (almost) \( G_2 \)-manifolds**

Notice that a real linear homomorphism \( J : \mathbb{R}^m \to \mathbb{R}^m \) being a Hermitian complex structure on \( \mathbb{R}^m \) is equivalent to the following conditions:
for any vector $v \in \mathbb{R}^m$ we have

1. $Jv$ is perpendicular to $v$.
2. $|Jv| = |v|$.

We can generalize $J$ to the case involving more than one vector. We call a skew symmetric bilinear map

$$\times : \mathbb{R}^m \otimes \mathbb{R}^m \to \mathbb{R}^m$$

a (2-fold) **vector cross product** if it satisfies

(i) $(u \times v)$ is perpendicular to both $u$ and $v$.
(ii) $|u \times v| = \text{Area of parallelogram spanned by } u \text{ and } v = |u \wedge v|$.

The obvious example of this is the standard vector product on $\mathbb{R}^3$. By identifying $\mathbb{R}^3$ with $\text{Im } \mathbb{H}$, the imaginary part of the quaternion numbers, we have

$$u \times v = \text{Im } \overline{vu}.$$ 

The same formula defines a vector cross product on $\mathbb{R}^7 = \text{Im } \mathbb{O}$, the imaginary part of the octonion numbers. Brown and Gray [15] showed that these two are the only possible vector cross product structures on $\mathbb{R}^m$ up to automorphism of $\mathbb{R}^m$.

Suppose that $M$ is a seven dimensional Riemannian manifold with a vector cross product $\times$ on each of its tangent spaces. The analog of the symplectic form is a degree three differential form $\Omega$ on $M$ defined as follows:

$$\Omega (u,v,w) = g (u \times v, w).$$

**Definition 2** Suppose that $(M,g)$ is a Riemannian manifold of dimension seven with a vector cross product $\times$ on its tangent bundle. Then

1. $M$ is called an **almost $G_2$-manifold** if $d\Omega = 0$.
2. $M$ is called a **$G_2$-manifold** if $\nabla \Omega = 0$ with $\nabla$ being the Levi-Civita connection.

**Remark 3** $M$ is a $G_2$-manifold if and only if its holonomy group is inside the exceptional Lie group $G_2 = \text{Aut } (\mathbb{O})$. The geometry of $G_2$-manifolds can be interpreted as the symplectic geometry on its knot space (see e.g. [25], [29]).
A typical family of examples of $G_2$-manifolds can be obtained via the product manifold $M := X \times S^1$ with $(X, \omega_X)$ being a Calabi-Yau threefold with a holomorphic volume form $\Omega_X$, and the $G_2$-form is given by

$$\Omega = \text{Re} \Omega_X + \omega_X \wedge d\theta.$$ 

Next we define the analogs of holomorphic curves and Lagrangian submanifolds in the $G_2$ setting.

**Definition 4** Suppose that $A$ is a three dimensional submanifold of an almost $G_2$-manifold $M$. We call $A$ an **instanton** or **associative submanifold**, if $TA$ is preserved by the vector cross product $\times$.

Harvey and Lawson \cite{HarveyLawson} showed that $A \subset M$ is an instanton if and only if $A$ is calibrated by $\Omega$, i.e. $\Omega|_A = \text{vol}_A$. This is in turn equivalent to $\tau|_{TA} = 0$ in our Lemma \ref{lemma:instanton} for $\tau$ defined in \cite{Joyce} (i.e. Corollary 1.7 in Section IV.1. of \cite{HarveyLawson} and Corollary 14 in \cite{Lawson}.

In M-theory, associative submanifolds are also called $M2$-branes. In the case when $M = X \times S^1$ with $X$ a Calabi-Yau threefold, $\Sigma \times S^1$ (resp. $L \times \{p\}$) is an instanton in $M$ if and only if $\Sigma$ (resp. $L$) is a holomorphic curve (resp. special Lagrangian submanifold with zero phase) in $X$.

A natural interesting question is to count the number of instantons in $M$. In the special case of $M = X \times S^1$, these numbers are related to the conjectural invariants proposed by Joyce \cite{Joyce} by counting special Lagrangian submanifolds in Calabi-Yau threefolds, and the product of holomorphic curves with $S^1$. This problem has been discussed by many physicists. For example Harvey and Moore discussed in \cite{HarveyMoore} the mirror symmetry aspects of these invariants; Aganagic and Vafa in \cite{AganagicVafa} related these invariants to the open Gromov-Witten invariants for local Calabi-Yau threefolds; Beasley and Witten argued in \cite{BeasleyWitten} that when there is a moduli of instantons, then one should count them using the Euler characteristic of the moduli space. The compactness issues of the moduli of instantons is a very challenging problem because the bubbling-off phenomena of (3-dimensional) instantons has not been well understood. This makes it very difficult to define an honest invariant by counting instantons.

Analogous to the Floer homology for Lagrangian intersections, when an instanton $A$ has a nontrivial boundary, $\partial A \neq \phi$, one should require it to lie inside a **brane** or a **coassociative submanifold** to make it a well-posed elliptic problem (see \cite{Joyce} for Fredholmness and index computation), i.e. submanifolds in $M$ where the restriction of $\Omega$ is
zero and have the largest possible dimension. Branes are the analog of Lagrangian submanifolds in symplectic geometry.

**Definition 5** Suppose that $C$ is a submanifold of an almost $G_2$-manifold $M$. We call $C$ a **coassociative submanifold** if

$$\Omega\big|_C = 0 \text{ and } \dim C = 4.$$ 

For example when $M = X \times S^1$ with $X$ a Calabi-Yau threefold, $H \times S^1$ (resp. $C \times \{p\}$) is a coassociative submanifold in $M$ if and only if $H$ (resp. $C$) is a special Lagrangian submanifold with phase $\pi/2$ (resp. complex surface) in $X$. In [25] J.H. Lee and the first author showed that the isotropic knot space $\mathcal{K}_{S^1} X$ of $X$ admits a natural holomorphic symplectic structure. Moreover $\mathcal{K}_{S^1} H$ (resp. $\mathcal{K}_{S^1} C$) is a complex Lagrangian submanifold in $\mathcal{K}_{S^1} X$ with respect to the complex structure $J$ (resp. $K$).

Constructing special Lagrangian submanifolds with zero phase in $X$ with boundaries lying on $H$ (resp. $C$) corresponds to the Dirichlet (resp. Neumann) boundary value problem for minimizing volume among Lagrangian submanifolds as studied by Schoen, Wolfson ([33], [34]) and [5]. For a general $G_2$-manifold $M$, the natural boundary value for an instanton is a coassociative submanifold. Similar to the intersection theory of Lagrangian submanifolds in symplectic manifolds, we propose to study the following problem: Count the number of instantons in $G_2$-manifolds bounding two coassociative submanifolds.

The product of a coassociative submanifold with a two dimensional plane inside the eleven dimension spacetime $M \times \mathbb{R}^{3,1}$ is called a $D5$-brane in M-theory. Counting the number of M2-branes between two D5-branes has also been studied in the physics literatures.

In general this is a very difficult problem. For instance, counting $S^1$-invariant instantons in $M = X \times S^1$ is the open Gromov-Witten invariant. However when the two coassociative submanifolds $C$ and $C'$ are close to each other, we can relate the number of thin instantons between them to the number of $J$-holomorphic curves in $C$ (Theorem [1]), hence by Taubes’ work, to the Seiberg-Witten invariant of $C$.

### 2.3 Relationships to Seiberg-Witten Invariants

To determine the number of instantons between nearby coassociative submanifolds, we first recall the deformation theory of compact coassociative submanifolds $C$ inside any $G_2$-manifold $M$, as developed by
McLean [28]. Given any normal vector field \( n \in \Gamma (\mathcal{N}_{C/M}) \), the interior product \( \iota_n \Omega \) is naturally a self-dual two form on \( C \) because of \( \Omega|_C = 0 \). This gives a natural identification,

\[
\Gamma (\mathcal{N}_{C/M}) \overset{\sim}{\longrightarrow} \Lambda^2_+ (C)
\]

\( n \mapsto \eta_0 = \iota_n \Omega. \) (1)

Furthermore infinitesimal deformations of coassociative submanifolds are parameterized by self-dual harmonic two forms \( \eta_0 \in H^2_+ (C) \), and they are always unobstructed, i.e. any such forms with sufficiently small norm give actual deformations to nearby coassociative submanifolds (see section 4 of [28]). Notice that the zero set of \( \eta_0 \) is the intersection of \( C \) with a infinitesimally nearby coassociative submanifold \( C_t \), that is

\[
\{ \eta_0 = 0 \} = \lim_{t \to 0} (C \cap C_t),
\]

where \( C = C_0 \) and \( \eta_0 = dC_t/dt|_{t=0} \).

Since

\[
\eta_0 \wedge \eta_0 = \eta_0 \wedge \ast \eta_0 = |\eta_0|^2 \ast 1,
\]

\( \eta_0 \) defines a natural symplectic structure on \( C_{\text{reg}} := C \setminus \{ \eta_0 = 0 \} \). If we normalize \( \eta_0 \) to \( \eta \),

\[
\eta = \eta_0 / |\eta_0|,
\]

then the equation

\[
\eta (u, v) = g (Ju, v)
\]
defines a Hermitian almost complex structure \( J \) on \( C_{\text{reg}} \). The \( J \) is determined by \( \eta \), which in turn is determined by \( n \), so we denote it by \( J_n \). More explicitly, for \( u \in T C_{\text{reg}} \),

\[
J_n (u) = |n|^{-1} n \times u. \quad \text{(2)}
\]

The next proposition says that when two coassociative submanifolds \( C \) and \( C' \) come together, not necessarily disjoint, then the limit of instantons bounding them will be a \( J_n \)-holomorphic curve \( \Sigma \) in \( C_{\text{reg}} \) with boundary \( C \cap C' \).

**Proposition 6** Let \( M \) be a \( G_2 \)-manifold. Suppose that for some \( \varepsilon_0 > 0 \), there is a smooth map

\[
\psi : C \times [0, \varepsilon_0] \longrightarrow M
\]
such that for each $t \in [0, \varepsilon_0]$, $\psi_t(\cdot) := \psi(\cdot, t)$ is a smooth immersion of $C$ into $M$ as a coassociative submanifold $C_t := \psi(C \times \{t\})$. Suppose that $n = dC_t/dt|_{t=0} \in \Gamma (N_{C_t}/M)$ is nowhere vanishing, and

$$\phi_t : \Sigma \times [0, t] \to M$$

is a smooth family of instantons in $M$ such that for each $t \in (0, \varepsilon_0)$, $\phi_t$ is an associative immersion with boundary condition

$$\phi_t(\Sigma \times \{0\}) \subset C_0 := \psi(C \times \{0\}) \subset C_t := \psi(C \times \{t\}).$$

Suppose that the $C^1$-limit of $\phi_t(\Sigma \times \{0\})$ exists as $t \to 0$. Then $\Sigma_0 := \lim_{t \to 0} \phi_t(\Sigma \times \{0\})$ is a $J_{\Sigma}$-holomorphic curve in $C_0$.

**Proof.** Let us denote the boundary component of $A_t =$Image($\phi_t$) in $C_0$ by $\Sigma_t$, i.e. $\Sigma_t := \phi_t(\Sigma \times \{0\})$. Let $w_t$ be a unit normal vector field for $\Sigma_t$ in $A_t$. We claim that $w_t$ is perpendicular to $C_0$. To see this, note that $TA_t$ being preserved by the vector cross product implies that

$$w_t = u \times v$$

for some tangent vectors $u$ and $v$ in $\Sigma_t$; In fact for any unit vector $u \in T\Sigma_t$, $v := w_t \times u \in T\Sigma_t$ by associativity condition of $A_t$ and $v \perp w_t$, and

$$u \times v = -u \times (u \times w_t)$$

$$= \tau (u, u, w_t) + g(u, w_t) u + g(u, u) w_t$$

$$= 0 + 0 + w_t = w_t,$$

where we have used in the second row the definition of $\tau$ in (3) and $\tau$ is a (vector-valued) form. Therefore given any tangent vector $w$ along $C_0$, we have

$$g(w, v) = g(u \times v, w) = \Omega (u, v, w) = 0,$$

where the last equality follows from $C_0$ being coassociative and $\Sigma_t \subset C_0$. Reparameterize $\phi_t : \Sigma \times [0, \varepsilon] \to M$ when necessary, then for small $t$ we can assume that $\frac{d}{ds}\phi_t(z, s)|_{s=0}$ is parallel to $w_t(z)$ for any $z \in \Sigma$. Noting that

$$\phi_t(z, 0) \subset C_0 \text{ and } \phi_t(z, t) \subset C_t,$$

we see $\lim_{t \to 0} w_t(z)$ is parallel to

$$\left( \frac{dC_t}{dt} \right)|_{t=0} = n \in \Gamma (\Sigma_0, N_{C_0}/M).$$
Therefore along $\Sigma_0$
\[
\lim_{t \to 0} w_t(z) = |n(z)|^{-1} n(z).
\]
For any $u_t \in T\Sigma_t$, $w_t \times u_t \perp w_t$ in associative $A_t$ so
\[
w_t \times u_t \in T\Sigma_t.
\]
Since $\Sigma_t \to \Sigma_0$ in $C^1$ topology, for any $u \in T\Sigma_0$, $u$ can be realized as the limit of $u_t$. Therefore
\[
J_n(u) = |n|^{-1} n \times u = \lim_{t \to 0} (w_t \times u_t) = \lim_{t \to 0} T\Sigma_t = T\Sigma_0,
\]
i.e. $\Sigma_0$ is a $J_n$-holomorphic curve in $C_0$ with respect to the almost complex structure $J_n$ defined (2).

The reverse of the above proposition is also true (Theorem 27). The Lagrangian analog of it was proven by Fukaya and Oh in [10]. On the other hand, by the celebrated work of Taubes, we expect that the number (counted with algebraic weights) of such holomorphic curves in $C_0$ equals to the Seiberg-Witten invariant of $C_0$. We conjecture the following statement.

**Conjecture:** Suppose that $C$ and $C'$ are nearby coassociative submanifolds in a $G_2$-manifold $M$. Then the number of instantons counted with algebraic weights in $M$ with small volume and with boundary lying on $C \cup C'$ is given by the Seiberg-Witten invariants of $C$.

The main result of our paper is to solve a special case of the above conjecture, namely, we will concentrate on the case that $C$ and $C'$ are both **compact** and they do **NOT** intersect. The basic ideas are (i) the limit of such instantons is a $J$-holomorphic curve for almost complex structure $J$ compatible to the (degenerated) symplectic form $\eta$ on $C$ coming from its deformations as coassociative submanifolds and this process can be reversed; (ii) the number of $J$-holomorphic curves in the four manifold $C$ should be related to the Seiberg-Witten invariant of $C$ by the work of Taubes ([37], [38]). Note that one only gets one symplectic form $\eta$ (and hence one almost complex structure $J$) from a given coassociative deformation of $C$, though of course one can get more (from different coassociative deformations).

---

$^1$In the remainder of the paper, we will always assume $C$ and $C'$ are compact and they do **NOT** intersect.
Suppose that $\eta$ is a self-dual two form on $C$ with constant length $\sqrt{2}$, in particular it is a (non-degenerate) symplectic form, and $\Sigma$ is a smooth holomorphic curve in $C$, possibly disconnected. If $\Sigma$ is regular in the sense that the linearized Cauchy-Riemann operator $D_\Sigma \bar{\partial}_J$ has trivial cokernel [36], then Taubes showed that the perturbed Seiberg-Witten equations,

\[ F^+_a = q(\psi) - r\sqrt{-1}\eta, \]

\[ D_{A(a)}\psi = 0, \]

have solutions for all sufficiently large $r$. Here $a$ is a connection on the complex line bundle $E$ over $C$ whose first Chern class equals the Poincaré dual of $\Sigma$, $PD[\Sigma]$, $F_a$ is the curvature 2-form of $E$ and $F^+_a$ is the projection of $F_a$ to $\bigwedge^2_+ (C)$, $\psi$ is a section of the twisted spinor bundle $S_+ = E \oplus (K^{-1} \otimes E)$ and $D_{A(a)}$ is the twisted Dirac operator, and $q(\cdot)$ is a certain canonical quadratic map from $S_+$ to $i \cdot \bigwedge^2_+ (C)$. The number of such solutions (counted with algebraic weights) is the Seiberg-Witten invariant $SW_C(\Sigma)$ of $C$. Furthermore the converse is also true, namely the Seiberg-Witten invariant $SW_C(\Sigma)$ is equal to the Gromov-Witten invariants counting holomorphic curves $\Sigma$. Thus Taubes established an equivalence between Seiberg-Witten theory and Gromov-Witten theory for symplectic four manifolds. This result has far reaching applications in four dimensional symplectic geometry.

For a general four manifold $C$ with nonzero $b^+(C)$, using a generic metric, any self-dual two form $\eta$ on $C$ defines a degenerate symplectic form on $C$, i.e. $\eta$ is a symplectic form on the complement of $\{ \eta = 0 \}$, which is a finite union of circles (see [12][21]). Therefore, one might expect to have a relationship between the Seiberg-Witten invariants of $C$ and the number of holomorphic curves with boundaries $\{ \eta = 0 \}$ in $C$. Part of this Taubes’ program has been verified in [37], [38].

Suppose that $\eta$ is a nowhere vanishing self-dual harmonic two form on a coassociative submanifold $C$ in a $G_2$-manifold $M$. For any holomorphic curve $\Sigma$ in $C$, we want to construct an instanton in $M$ bounding $C$ and $C'$, where $C'$ is a small deformation of the coassociative submanifold $C$ along the normal direction given by $\eta$. Notice that $C$ and $C'$ do not intersect. We will construct such an instanton using a perturbation argument which requires a lower bound on the first eigenvalue for the appropriate elliptic operator. Recall that the deformation of an instanton is governed by a twisted Dirac operator. We will reinterpret it as a complexified version of the Cauchy-Riemann operator in Section 3.1.
2.4 Deformation of Instantons

To construct an instanton \( A \) in \( M \) from a holomorphic curve \( \Sigma \) in \( C \), we need to perturb an almost instanton \( A' \) to a honest one using a quantitative version of the implicit function theorem. Let us first recall the deformation theory of instantons \( A \) ([18] and [25]) in a Riemannian manifold \((M, g)\) with a parallel (or closed) \( r \)-fold vector cross product

\[
\times : \Lambda^r TM \to TM.
\]

In our situation, we have \( r = 2 \). By taking the wedge product with \( TM \) we obtain a homomorphism \( \tau \),

\[
\tau : \Lambda^{r+1} TM \to \Lambda^2 TM \cong \Lambda^2 T^* M,
\]

where the last isomorphism is induced from the Riemannian metric. As a matter of fact, the image of \( \tau \) lies inside the subbundle \( g_M \perp M \) which is the orthogonal complement of \( g_M \subset \mathfrak{so}(TM) \cong \Lambda^2 T^* M \), the bundle of infinitesimal isometries of \( TM \) preserving \( \times \). That is,

\[
\tau \in \Omega^{r+1} \left( M, g_M^\perp \right).
\]

**Lemma 7** ([18], [25]) An \( r + 1 \) dimensional submanifold \( A \subset M \) is an instanton, i.e. \( TA \) is preserved by \( \times \), if and only if

\[
\tau|_A = 0 \in \Omega^{r+1} \left( A, g_M^\perp \right).
\]

This lemma is important in describing deformations of an instanton. Mclean [28] used this to show that the normal bundle to an instanton \( A \) is a twisted spinor bundle over \( A \) and infinitesimal deformations of \( A \) are parameterized by twisted harmonic spinors.

In our present situation, \((M, g)\) is a \( G_2 \)-manifold. Using the cross product, we can identify \( g_M^\perp \subset \Lambda^2 T^* M \cong \Lambda^2 TM \) with the tangent bundle \( TM \), i.e. for \( u \wedge v \in \Lambda^2 TM \), we identify it with \( w \in TM \) that \( w = u \times v \). Then we can also characterize \( \tau \in \Omega^3 (M, TM) \) by the following formula,

\[
(*\Omega) (u, v, w, z) = g(\tau (u, v, w), z),
\]

More explicitly,

\[
\tau (u, v, w) = -u \times (v \times w) - g(u, v) w + g(u, w) v. \quad (3)
\]
Therefore \( A \subset M \) is an instanton if and only if \( *_A (\tau|_A) = 0 \in T_M|_A \).

**Example.** The \( G_2 \) manifold \( \mathbb{R}^7 \):

\[
\mathbb{R}^7 \simeq \text{Im} \mathbb{O} \simeq \text{Im} \mathbb{H} \oplus \mathbb{H} = \{(x_1 i + x_2 j + x_3 k, x_4 + x_5 j + x_6 k + x_7 k)\},
\]

the standard basis consists of \( e_i = \frac{\partial}{\partial x_i} \) \((i = 1, 2, \ldots, 7)\), the multiplication \( \times \) for \((a, b), (c, d) \in \text{Im} \mathbb{H} \oplus \mathbb{H} \simeq \text{Im} \mathbb{O} \) is

\[
(a, b) \times (c, d) = (ac - d^* b, da + bc^*)
\]

(Cayley–Dickson construction), where \( z^* \) denotes the conjugate of the quaternion \( z \). The \( G_2 \) form \( \Omega \) is

\[
\Omega = \omega^{123} - \omega^{167} - \omega^{527} - \omega^{563} - \omega^{154} - \omega^{264} - \omega^{374},
\]

the form \( \tau \) is the following ((5.4) in [28])

\[
\tau = (\omega^{256} - \omega^{247} + \omega^{346} - \omega^{357}) \frac{\partial}{\partial x_1} + (\omega^{156} - \omega^{147} - \omega^{345} + \omega^{367}) \frac{\partial}{\partial x_2} + (\omega^{245} - \omega^{267} - \omega^{146} - \omega^{157}) \frac{\partial}{\partial x_3} + (\omega^{567} - \omega^{127} + \omega^{136} - \omega^{235}) \frac{\partial}{\partial x_4} + \left(\omega^{126} - \omega^{467} + \omega^{137} + \omega^{234}\right) \frac{\partial}{\partial x_5} + \left(\omega^{457} - \omega^{125} - \omega^{134} + \omega^{237}\right) \frac{\partial}{\partial x_6} + \left(\omega^{124} - \omega^{456} - \omega^{135} - \omega^{236}\right) \frac{\partial}{\partial x_7},
\]

where \( \omega^{ijk} = dx_i \wedge dx_j \wedge dx_k \). \( \text{Im} \mathbb{H} \oplus \{0\} \) is associative (i.e. an instanton), and \( \{0\} \oplus \mathbb{H} \) is coassociative.

As a matter of fact, if \( A \) is already close to being an instanton, then we only need the normal components of \( *_A (\tau|_A) \) to vanish.

**Proposition 8** There is a positive constant \( \delta \) such that for any 3-plane \( A \) in \((\mathbb{R}^7, \Omega)\) with \(|\tau|_A| < \delta\), \( A \) is an instanton if and only if \( *_A (\tau|_A) \in T_A \).

**Proof.** McLean observed (From formula (5.6) in [28]) that if \( A_t \) is a family of linear subspaces in \( M \cong \mathbb{R}^7 \) with \( A_0 \) an instanton, then

\[
*_A \left( \frac{d\tau|_{A_t}}{dt} \right) \bigg|_{t=0} \in N_{A_0/M} \subset T_M|_{A_0}.
\]

We may assume that \( A \) is spanned by \( e_1, e_2 \) and \( \tilde{e}_3 = e_3 + \sum_{a=4}^{7} t_a e_a \) for some small \( t_a \)’s where \( e_i \)'s are a standard basis for \( \mathbb{R}^7 \), in particular \( e_1 \times
\( e_2 = e_3 \). This is because the natural action of \( G_2 \) on the Grassmannian \( Gr(2,7) \) is transitive. An easy computation (c.f. equation (5.4) in [28]) shows that the normal component of \(* (\tau|_A) \) in \( N_{A/M} \) is given by

\[
* (\tau|_A) \perp = -t_5 (e_4) \perp + t_4 (e_5) \perp + t_7 (e_6) \perp - t_6 (e_7) \perp,
\]

where \((\cdot) \perp\) denote the orthogonal projection onto \( N_{A/M} \). When \( t_\alpha \)'s are all zero, we have \((e_\alpha) \perp = e_\alpha\) for \( 4 \leq \alpha \leq 7 \). In particular, they are linearly independent when \( t_\alpha \)'s are small. In that case, \(* (\tau|_A) \perp = 0\) will actually imply that \( t_\alpha = 0 \) for all \( \alpha \), i.e. \( A \) is an instanton in \( M \). Hence the proposition.

This proposition will be needed later when we perturb an almost instanton to an honest one. We also need to identify the normal bundle \( N_{A/M} \) to an instanton \( A \) with a twisted spinor bundle over \( A \) as following [28]: We denote \( P \) to be the \( SO(4) \)-frame bundle of \( N_{A/M} \). Using the identification

\[
(a : SO(4) = (Sp(1) \times Sp(1)) / \pm (1,1) \to SO(\mathbb{H}),
\]

\[
(p, q) \cdot y = p y \bar{q}, \text{ with } p, q \in Sp(1) \text{ and } y \in \mathbb{H}.
\]

Mclean [28] showed that the normal bundle \( N_{A/M} \) can be identified as an associated bundle to \( P \) for the representation \( SO(4) \to SO(\mathbb{H}) \) given by \((p, q) \cdot y = p y \bar{q}\). The spinor bundle \( S \) of \( A \) is associated to \( P \) for the representation \( s : SO(4) \to SO(\mathbb{H}) \) given by \((p, q) \cdot y = y \bar{q}\). Let \( E \) be the associated bundle to \( P \) for the representation \( e : SO(4) \to SO(\mathbb{H}) \) given by \((p, q) \cdot y = p y\). Then because the 3 representations \( a, s \) and \( e \) have the relation

\[
a = s \circ e,
\]

we obtain

\[
N_{A/M} \cong S \otimes \mathbb{H} E.
\]

An alternative proof of \( N_{A/M} \cong S \otimes \mathbb{H} E \) using explicit frame identification is contained in the proof of next theorem.

We re-derive McLean’s theorem on deformation of associative submanifolds \( A \) in a \( G_2 \) manifold \( M \). The original proof (Theorem 5.2 in [28]) is not quite precise: the associative form \( \tau \in \Omega^3 (M, TM) \) is \textit{vector-valued} rather than a usual differential form, so the pull back operation and Cartan-formula need to be clarified. The key is to define suitable notion of pull back for vector-valued forms. Our calculation is flexible and can be extended to almost associative submanifolds in later sections. Other proofs were given in [2] and [16].
Theorem 9 (McLean) Under the correspondence of normal vector fields with twisted spinors, the Zariski tangent space to associative submanifolds at an associative submanifold $A$ is the space of harmonic twisted spinors on $A$, that is the kernel of the twisted Dirac operator.

Proof. For any section $V$ of $N_{A/M}$ with $C^0$ norm smaller than the injectivity radius $\delta_0$ of $M$, we define a nonlinear map

$$F : \Gamma (N_{A/M}) \to \Omega^3 (A, i^* TM),$$

where for the embedding $\exp V : A \to M$, $(\exp V)^*$ pulls back the differential form part of the tensor $\tau$, and $T_V : T_{\exp p (V)} M \to T_p M$ pulls back the vector part of the tensor $\tau$ by parallel transport along the geodesic $\exp p (V)$. There is an ambiguity of the form part and vector part of tensor $\tau$ up to a scalar function-valued matrix transform $\Theta$ and $\Theta^{-1}$ respectively, but by the linearity of $T_V$ and $(\exp V)^*$ on scalar function factors one can easily show the definition of $F$ is independent on such $\Theta$, so $F$ is well-defined. We make $F$ more explicit by using a good frame. At a point $p \in A$, we pick two orthonormal vectors $\{W_1, W_2\}$ in $T_p A$, then with respect to the induced connection $\nabla^A$ on $A$, we parallel transport $\{W_1, W_2, W_3 = W_1 \times W_2\}$ from $p$ to a neighborhood $B$ in $A$ along geodesic rays from $p$. From the construction we see

$$\nabla^A_{W_i} W_j (p) = 0, \text{ for } 1 \leq i, j \leq 3$$

Then at any $q \in B \subset A$ we have orthonormal basis

$$\{W_1 (q), W_2 (q), W_3 (q) = W_1 (q) \times W_2 (q)\}$$

spanning $T_q A$. (This uses that $W_1 (q) \times W_2 (q)$ and the parallely transported $W_3 (p)$ both orthogonal to $W_1 (q)$ and $W_2 (q)$ in 3 dimensional $A$). We further choose a smooth normal unit vector field $W_4$ on $B$ in $M$ as following: we choose a $W_4 \in N_{A/M} (p)$ then use the parallel transport of $N_{A/M}$ with respect to the induced connection $\nabla^\perp$ in $N_{A/M}$ defined as $\nabla^\perp = \perp \nabla$ from the metric on $M$, where $\perp : TM \to N_{A/M}$ is the natural projection. Thus on $B$,

$$W_4 \perp \{W_1, W_2, W_1 \times W_2\}.$$ 

Using Lemma A.15 in [13] (Cayley–Dickson construction) we can uniquely extend $\{W_\alpha (q)\}_{1,2,3,4}$ to basis $\{W_\alpha (q)\}_{\alpha = 1,2,\ldots,7}$ of $T_q M$, such that

$$W_{i+3} (q) = W_i (q) \times W_4 (q)$$

16
for \( i = 1, 2, 3 \), and the correspondence

\[
T_q M \ni W_\alpha \leftrightarrow e_\alpha \in \Im \mathcal{O}, \quad \alpha = 1, 2, \ldots, 7
\]  \hspace{1cm} (8)

preserves the inner product \( \cdot \) and cross product \( \times \), where \( \{ e_\alpha \}_{\alpha=1,2,\ldots,7} \)

is the standard basis of \( \mathbb{R}^7 \cong \Im \mathcal{O} \) defined in previous example. So locally \( N_{A/M} \) is trivialized as \( B \times \mathbb{H} \), and at any \( q \in B \), by the above basis \( W_\alpha \) we have algebra isomorphism

\[
T_q M = T_q A \oplus N_{A/M} (q) \cong \Im \mathbb{H} \oplus \mathbb{H} \cong \Im \mathcal{O}
\]

that smoothly depends on \( q \in B \). Hence \( N_{A/M} \) is a quaternion valued bundle over \( A \) isomorphic to \( \mathbb{S} \otimes \mathbb{H} E \). From our construction we also have

\[
\nabla_{W_i} W_k (p) = 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad k = 4, 5, 6, 7,
\]  \hspace{1cm} (9)

because \( \nabla_{W_i} W_4 (p) = 0 \) by construction of \( W_4 \), and for \( k = 5, 6, 7 \) say

\[
\nabla_{W_i} W_5 (p) = \perp (\nabla_{W_i} (W_1 \times W_4)) (p) = \perp (\nabla_{W_i} W_1 \times W_4 + W_1 \times \nabla_{W_i} W_4) (p)
\]

\[
= \nabla_{W_i} W_1 (p) \times W_4 + W_1 \times \nabla_{W_i} W_4 (p) = 0,
\]  \hspace{1cm} (10)

where the second row is because \( N_{A/M} (p) \times W_1 \subset T_p A \) (for \( \Im \mathcal{O} \cong \Im \mathbb{H} \oplus \mathbb{H}, \ (0, \mathbb{H}) \times (0, 1) \subset (\Im \mathbb{H}, 0) \) by (4)) and \( W_1 \times T_p A \subset T_p A \) by associative condition. We remark that the parallel transport in \( N_{A/M} \) w.r.t \( \nabla^\perp \) is an isometry, for if \( \nabla_T W = 0 \) for section \( W \) in \( N_{A/M} \) then

\[
\nabla_T \langle W, W \rangle = 2 \langle \nabla_T W, W \rangle = 2 \left\langle \nabla_T^\perp W, W \right\rangle = 0.
\]  \hspace{1cm} (11)

Next for each \( q \in B \), we parallel transport the frame \( \{ W_\alpha (q) \}_{\alpha=1,2,\ldots,7} \) along geodesical rays emanating from \( q \) in \( M \) in \( N_{A/M} (q) \) directions up to length \( \delta_0 \). This extends the frame to a tubular neighborhood of \( A \) in \( M \). Then \( \nabla_V W_\alpha (q) = 0 \) for any \( V \in \Gamma (N_{A/M}) \). If we write

\[
\tau = \omega^\alpha \otimes W_\alpha \quad (\alpha = 1, 2, \ldots, 7)
\]

following Einstein’s summation convention, then

\[
F (V) (q) = (\exp V)^* \omega^\alpha (q) \otimes T_V W_\alpha (\exp_q V).
\]
We have
\[
F'(0) V = \left. \frac{d}{dt} \right|_{t=0} F(tV) = \left. \frac{d}{dt} \right|_{t=0} \left[ (\exp tV)^* \omega^\alpha \otimes T_i V W_\alpha \right] = L_V \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha = d(i_V \omega^\alpha) \otimes W_\alpha + i_V d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha
\] (12)

Since \( \nabla \tau = 0 \) and \( \nabla_V W_\alpha (q) = 0 \) by the parallel property of \( \tau \) and \( W_\alpha \), we have
\[
0 = \nabla_V \tau(q) = \nabla_V \omega^\alpha \otimes W_\alpha(q) + \omega^\alpha \otimes \nabla_V W_\alpha(q) = \nabla_V \omega^\alpha(q) \otimes W_\alpha(q),
\]
and so \( \nabla_V \omega^\alpha(q) = 0 \). Since \( \nabla = d + A \) and in normal coordinates the connection 1-form \( A \) vanishes at \( q \) along the fiber direction of \( N_{A/M} \), we have \( i_V d\omega^\alpha(q) = 0 \). Therefore by (12), at \( q \) we have
\[
F'(0) V(q) = d(i_V \omega^\alpha) \otimes W_\alpha(q).
\] (13)

By our choice of \( W_\alpha \), the \( \tau = \omega^\alpha \otimes W_\alpha \) at \( q \) is the standard form \( \tau \), and by the parallel property of the cross product \( \times \) and \( \tau \), in the neighborhood of \( q \) in \( M \) the \( \tau \) is also of standard form as \( \tau \), in the sense that the coordinate vector \( \frac{\partial}{\partial \omega^\alpha} \) is replaced by the frame \( W_i \), and \( \omega^{ijk} = dx^i \wedge dx^j \wedge dx^k \) is replaced by \( W_i^* \wedge W_j^* \wedge W_k^* \) where \( W_i^* \) is the dual vector of \( W_i \). Namely
\[
\tau = (W_2^* \wedge W_5^* \wedge W_6^* - W_2^* \wedge W_3^* \wedge W_7^* + \cdots) \otimes W_1 + \text{similar terms}.
\]
This is because \( \nabla_V (\tau(W_i, W_j, W_k)) = 0 \) for \( V \in \Gamma(N_{A/M}) \), by \( \nabla \tau = 0 \) and that \( \{W_\alpha\}_{\alpha=1,2,\ldots,7} \) is parallel in \( N_{A/M} \) directions. Therefore from (13), similar to the way of deriving (5.6) in \cite{28}, and noting \( dW_i^*(p) = 0 \) \((i = 1, 2, \cdots, 7)\) from (7) and (9), we get
\[
F'(0) V(p) = d(i_V \omega^\alpha) \otimes W_\alpha(p) = DV(p) \otimes W_1^* \wedge W_2^* \wedge W_3^*(p) = DV(p) \otimes dvol_A(p),
\] (14)
where for \( V = V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7 \),
\[
DV(p) = -(V_{1}^{5} + V_{2}^{6} + V_{3}^{7}) W_4 + (V_{1}^{4} + V_{3}^{6} - V_{2}^{7}) W_5 + (V_{2}^{4} - V_{3}^{5} + V_{1}^{7}) W_6 + (V_{4}^{4} + V_{2}^{5} - V_{1}^{6}) W_7
\]
is the twisted Dirac operator (5.2) in [28] and also (17) below, with $V_i^k = dV^k(W_i)$, and $dvol_A = W_1^* \wedge W_2^* \wedge W_3^*$ is the induced volume form on $A \subset M$ since $\{W_\alpha\}_{\alpha=1,2,3}$ is the orthonormal basis of $TA$. Since $p \in A$ and the section $V$ are arbitrary, we have

$$F'(0) = D \otimes dvol_A. \quad (15)$$

Note that both sides of the above identity are independent on the choice of the frame $\{W_\alpha\}_{\alpha=1,2,3}$. If the normal vector field $V$ is induced from deformation of associative submanifolds $\{A_t\}_{0 \leq t \leq \varepsilon_0}$, i.e.,

$$A_t = \exp U(t) \cdot A \text{ for } U(t) \in \Gamma (N_{A/M}) \text{ and } U'(0) = V,$$

then differentiating $F(U(t)) = 0$ at $t = 0$ and using (15) we get $DV = 0$ on $A$, namely $V$ is a harmonic twisted spinor.

**Remark 10**

1. The vector field $V$ is only defined on $A \subset M$, but along the geodesic rays from $A$ in the fiber directions of $N_{A/M}$, one can extend $V$ to an open neighborhood of $A$ by parallel transport, therefore the Lie derivative $L_V \omega$ for any 3-form $\omega$ on $M$ makes sense in this neighborhood. However, the Lie derivative of $\omega$ restricted on $A$, namely $i_A^* (L_V \omega)$ for the inclusion $i_A : A \hookrightarrow M$, is actually independent of the extension of $V$, as mentioned in [28]. One can see this from the Cartan formula

$$L_V \omega = d(i_V \omega) + i_V d\omega$$

as follows: On $A$, the second term $i_V d\omega$ is independent on the extension of $V$. For the first term $d(i_V \omega)$, since we restrict it on $A \subset M$, one can directly check that this term only involves the derivatives of the section $V$ and $\omega$ in tangent directions of $A$, for any derivative in normal direction will contribute a covector not in $T^* A$ and make the corresponding summand in $i_A^* (L_V \omega)$ vanish. So $i_A^* (L_V \omega)$ is independent on the extension of $V$ and $\omega$ to $M$.

2. In the preceding proof, the normal vector field $V$ is used only to ensure that $\exp V : A \rightarrow M$ is an embedding and give $d(i_V \omega^\alpha) \otimes W_\alpha$ a twisted Dirac operator interpretation on the normal bundle $N_{A/M}$. Actually, to write down the linearization of $F$, one only needs the normal bundle of $A$ in $M$ in differential topology sense (namely at any $p \in A$, $N_{A/M}(p) + T_p A = T_p M$, but not necessarily $N_{A/M}(p) \perp T_p A$). If we denote the differential topological normal
bundle by $N_{A/M}^{top}$, then for section $V \in \Gamma \left( N_{A/M}^{top} \right)$ with small $C^0$ norm, $\exp V : A \to M$ is an embedding so we can define the linearization of $F$ as before.

3. The linearization formula

$$F'(0) V = d (i_V \omega^\alpha) \otimes W_\alpha + i_V d \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha \quad (16)$$

holds for any section $V \in \Gamma \left( N_{A/M}^{top} \right)$, and any tangent frame $\{W_a\}_{i=1,2,...,7}$ along any submanifold $A \subset M$ (not necessarily associative). The term $d (i_V \omega^\alpha) \otimes W_\alpha$ is the principal symbol term of the first order linear differential operator $F'(0)$ and behaves functorially under the diffeomorphism between two manifolds. To get $F'(0) V(p) = d (i_V \omega^\alpha) \otimes W_\alpha(p)$, one only needs to choose the frame $\{W_\alpha\}_{\alpha=1,...,7}$ that is parallel along the curves $\exp_p(tv)$ for all $p \in A$ and $v \in N_{A/M}^{top}(p)$.

4. On an almost instanton $A$, its normal bundle is not closed under $\times$ by $TA$ in general, so the linearized instanton equation $F'(0) V$ can NOT be interpreted as a twisted Dirac operator. For this reason, we will seldom use the twisted Dirac operator, but mainly (16) to do computations and estimates of $F'(0) V$, with the aid of good local frames $\{W_a\}_{i=1,2,...,7}$.

5. At a point $p$ in a general 3-manifold $A \subset M$ with volume form $dvol_A$, we can write the linearization

$$F'(0) V(p) = G \circ V(p) \otimes dvol_A(p) ,$$

where $G$ is a first order linear differential operator, whose principal symbol depends on the algebraic relation of the frame $\{W_a\}$ under the products $\times$ and $\cdot$, and the volume form on $A$. Besides the associative $\{W_a\}_{i=1,2,3}$ and coassociative $\{W_a\}_{i=4,5,6,7}$ type frame, there may be some other type of frames leading to a meaningful differential operator $G$.

By McLean’s theorem, the normal bundle to any instanton $A$ is a twisted spinor bundle $\mathbb{S}$ over $A$ corresponding to the representation $SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = pyq$. Let $\mathcal{D}$ be the twisted Dirac operator on $A$. We want to write it down explicitly in local coordinates.
near $p$. We let $\{W_\alpha\}_{\alpha=1}^7$ be a local orthonormal frame constructed as above. Suppose

$$V = V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7$$

is a normal vector field to $A$ and we write the covariant differentiation of $V$ as $\nabla(V) := V^i W_\alpha \otimes \omega^i$ with $\{\omega^i\}$ being the co-frame dual to $\{W_\alpha\}$, then the twisted Dirac operator is

$$D = W_1 \times \nabla_1 + W_2 \times \nabla_2 + W_3 \times \nabla_3,$$

(17)

where $\nabla_i := \nabla^W_{W_i}$ ($i = 1, 2, 3$). For instance when the $G_2$ manifold is $\text{Im} \mathcal{O} \simeq \text{Im} \mathbb{H} \oplus \mathbb{H}$ and the instanton is $\text{Im} \mathbb{H}$, then by viewing $V$ as a $\mathbb{H}$ valued function, $V = V^4 + iV^5 + jV^6 + kV^7$, the twisted Dirac operator is $D = i\nabla_1 + j\nabla_2 + k\nabla_3$. Expression (17) can be easily shown to be independent on the choice of orthonormal basis $\{W_\alpha\}_{i=1,2,3}$ of $TA$, thus $D$ is globally defined on $A$. From (17) and (9) we have

$$D V(p) = (W_1 \times \nabla_1 + W_2 \times \nabla_2 + W_3 \times \nabla_3) \,(V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7)
= - (V^5 + V_2^6 + V_3^7) \ W_4 + (V_1^4 + V_3^6 - V_2^7) \ W_5
+ (V_2^4 - V_3^5 + V_1^7) \ W_6 + (V_3^4 + V_2^5 - V_1^6) \ W_7.$$  

(18)

We remark that away from $p$ the expression of $D V$ may have more 0-th order terms in general.

## 3 Dirac Equation on Thin 3-manifolds

### 3.1 A Simplified Model

To motivate our analytical estimates for later sections, let us first consider a simplified model. Suppose that $A_\varepsilon = [0, \varepsilon] \times \Sigma$ is a three manifold and $(x_1, z)$ are coordinates on $A_\varepsilon$. On $A_\varepsilon$ we put a warped product metric $g_{A_\varepsilon,h} = h(z) \, dx_1^2 + g_\Sigma$, where $\Sigma$ is a Riemann surface with a background metric $g_\Sigma$ and $h(z) > 0$ is a $C^\infty$ function on $\Sigma$. Let $e_1$ be the unit tangent vector field on $A_\varepsilon$ normal to $\Sigma$, namely along the $x_1$-direction and $e_1 = h^{-1/2}(z) \frac{\partial}{\partial x_1}$. We introduce a first order linear differential operator $\mathcal{D}$ that

$$\mathcal{D} = e_1 \cdot \nabla_1 + \bar{\partial} = e_1 \cdot h^{-1/2}(z) \frac{\partial}{\partial x_1} + \bar{\partial},$$

(19)
where $\nabla_1 = \nabla_{e_1} = \hbar^{-1/2}(z) \frac{\partial}{\partial x_1}$, and $\bar{\partial} = (\bar{\partial}, \bar{\partial}'')$ is the Dirac operator on the Dolbeault complex $\Omega^0_\mathbb{C}(L)$ of Hermitian line bundle $L$ over the Riemann surface $\Sigma$ with a connection $\nabla$ (c.f. Proposition 3.67 of [6] or Proposition 1.4.25 of [31], by taking the Kahler manifold to be $\Sigma$ and the Hermitian line bundle to be $L$). Here $\bar{\partial}$ acts on the spinor bundle $S_\Sigma := S^+ \oplus S^-$ via the following identification

\[
\begin{align*}
\Omega^0_\mathbb{C}(L) \oplus \Omega^0_\mathbb{C}(L) & \xrightarrow{(\bar{\partial}, \bar{\partial}'')} \Omega^0_\mathbb{C}(L) \oplus \Omega^0_\mathbb{C}(L) \\
S^+ \oplus S^- & \xrightarrow{\bar{\partial}} S^- \oplus S^+
\end{align*}
\]

(20)

where on the left $S^+$ and $S^-$ are identified to complex line bundles $L$ and $L \otimes \Lambda^{0,1}(\Sigma)$ respectively, $\bar{\partial} : \Omega^0_\mathbb{C}(L) \to \Omega^0_\mathbb{C}(L)$ is the Dolbeault operator and $\bar{\partial}'' : \Omega^0_\mathbb{C}(L) \to \Omega^0_\mathbb{C}(L)$ is its formal adjoint operator. The Dolbeault operator $\bar{\partial}$ depends on the complex structures $j$ on $\Sigma$, $J$ on $L$, and the connection $\nabla$ of $L$.

When $h(z)$ is a constant, from the spinor bundle $S_\Sigma = S^+ \oplus S^-$ over $\Sigma$ one can construct a spinor bundle $S$ over the odd dimensional manifold $A_\varepsilon := [0, \varepsilon] \times \Sigma$ by taking the Cartesian product of $S_\Sigma$ with $[0, \varepsilon]$ (see Chapter 22 in [8]). When there is no confusion, we also write $S = S^+ \oplus S^-$ where the $S^\pm$ are the Cartesian products of the $S^\pm$ of $S_\Sigma$ with $[0, \varepsilon]$. [8] also constructs a Dirac operator on the spinor bundle $S \to A_\varepsilon$ right from the Dirac operator $\bar{\partial}$ on the spinor bundle $S_\Sigma \to \Sigma$, and it is $D = e_1 \cdot \nabla_1 + \bar{\partial}$ as in (19). For general $h(z)$, our $D$ is a Dirac type operator but not necessarily a genuine Dirac operator.

Let’s recall the Clifford multiplication of $TA_\varepsilon$ on $S$. Since $S$ is the Cartesian product of $S_\Sigma$ with $[0, \varepsilon]$, it enough to define the Clifford multiplication $\cdot$ of $e_1$ on $S_\Sigma$. Let $\sigma$ be the volume element of the Clifford bundle $Cl(\Sigma)$, $\sigma^2 = 1$. Then $S^+$ and $S^-$ are the $\pm 1$ eigenbundles of $\sigma$ and $S_\Sigma = S^+ \oplus S^-$. Since the Clifford multiplication $\cdot$ of $e_1$ on $S_\Sigma$ satisfies $e_1^2 = -1$, the natural choice of the action $e_1 \cdot$ on $S_\Sigma$ is to let $S^+$ and $S^-$ be the $\pm i$ eigenbundles, namely for $(u, v) \in S^+ \oplus S^-$, $e_1 \cdot (u, v) = (iu, -iv)$. The connection of $S$ along $x_1$ direction is trivial.

Given $z_0 \in \Sigma$, in its neighborhood in $\Sigma$ we can choose a complex coordinate $z = x_2 + ix_3 \in \mathbb{C}$ with $\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ orthonormal at $z_0$. We locally trivialize the spinor bundle $S_\Sigma = S^+ \oplus S^- \to \Sigma$ as

\[
\mathbb{H} = \mathbb{C} + \mathbb{C}j = \mathbb{C}^2 \to \mathbb{C}.
\]
We may choose the trivialization such that the Dirac operator
\[
\left( \partial, \partial^* \right) \bigg|_{z_0} = (\partial_z, -\partial_z).
\]

We may write the section \( V \) of \( S \rightarrow A_\varepsilon \) as
\[
V = (u, v) := u + v j \in S^+ \oplus S^-.
\]
with
\[
u = V^4 + i V^5 \in S^+ \simeq \mathbb{C},
\]
\[
v = V^6 + i V^7 \in S^+ \simeq \mathbb{C}, \quad \text{and} \quad v j \in S^+ j = S^-.
\]

With these understood, we may write (19) as
\[
D V = \left[ \begin{array}{cc}
i & 0 \\
0 & -i
\end{array} \right] \left( h^{-1/2} (z) \frac{\partial}{\partial x_1} + \left[ \begin{array}{cc}
0 & -i \partial^* \\
i \partial & 0
\end{array} \right] \right) \left[ \begin{array}{c}
u \\
v
\end{array} \right]
\]
at \( z_0 \)
\[
= \left[ \begin{array}{cc}
i & 0 \\
0 & -i
\end{array} \right] \left( h^{-1/2} (z) \frac{\partial}{\partial x_1} + \left[ \begin{array}{cc}
0 & i \partial \\
i \partial & 0
\end{array} \right] \right) \left[ \begin{array}{c}
u \\
v
\end{array} \right]
\]
\[
= (\nabla_1 u + i \partial_z v) + (\nabla_1 v + i \overline{\partial}_z u) \cdot j \cdot i
\]
\[
= (i \nabla_1 u - \partial_z v) + (-i \nabla_1 v + \overline{\partial}_z u) \cdot j,
\] (22)

where at \( z_0 \) we have the identification
\[
\overline{\partial} = \overline{\partial}_z = \frac{1}{2} (\nabla_2 + i \nabla_3), \quad -\overline{\partial}^* = \partial_z = \frac{1}{2} (\nabla_2 - i \nabla_3)
\]
with \( \nabla_i = \nabla \frac{\partial}{\partial x_i} \) for \( i = 2, 3 \) and \( \nabla_1 = h^{-1/2} (z) \frac{\partial}{\partial x_1} \). We will also denote
\[-i \overline{\partial} = \partial^+ \quad \text{and} \quad i \overline{\partial} = \partial^-
\]
respectively. They satisfy \( \partial^+ = (\partial^-)^* \).

This implies that the equation \( DV = 0 \) is equivalent to the following equations, an analog of Cauchy-Riemann equation,
\[
\partial^- u + h^{-1/2} (z) \frac{\partial v}{\partial x_1} = 0,
\]
\[
\partial^+ v + h^{-1/2} (z) \frac{\partial u}{\partial x_1} = 0.
\] (23)

We put the boundary condition
\[
v|_{\partial A_\varepsilon} = 0.
\] (24)

(The more precise formulation will be given in (25)). It is similar to the totally real boundary condition for \( J \)-holomorphic maps. When
$h(z)$ is a constant, it is a special case of the theory of boundary value problems for Dirac operators developed in [8]. For general $h(z)$, that the boundary value problem is Fredholm follows from our elliptic estimates in Subsection 3.3. Also see [17] for relevant discussion.

Later in Section 4, we will apply the above $D$ to the case when $h(z) = |n(z)|^2$, where $n(z) = \frac{d}{dt}C_t|_{t=0}$ is the normal vector field on $C_0$ coming from the deformation of coassociative submanifolds $C_t$, and the Hermitian line bundle $L$ is the normal bundle $N_{\Sigma_0/C_0}$ of the $J_n$-holomorphic curve $\Sigma_0$ in a coassociative manifold $C_0$, whose connection $\nabla$ is the normal connection from the induced metric of $\Sigma_0 \subset C_0$.

### 3.2 First Eigenvalue Estimates

In this subsection we will establish a quantitative estimate of the eigenvalue of the linearized operator for the simplified model $A_\varepsilon = [0, \varepsilon] \times \Sigma$ with warped product metric $g_{A_\varepsilon}, h = h(z) \, dx_1^2 + g_\Sigma$, where $\Sigma$ is a compact Riemann surface, and $h(z)$ is a smooth function on $\Sigma$ with $\frac{1}{K} \leq h \leq K$ for some constant $K > 0$. The volume form of this metric on $A_\varepsilon$ is

$$dvol_{A_\varepsilon} = h^{\frac{1}{2}}(z) \, dvol_\Sigma dx_1.$$

We introduce the following function spaces for spinors $V = (u, v)$ over $A_\varepsilon$.

**Definition 11** Let $S$ be the spinor bundle over $(A_\varepsilon, g_{A_\varepsilon})$ and $V$ be a smooth section of $S$,

1. We define the norm

$$\|V\|_{L^{m,p}(A_\varepsilon, S)} := \left( \sum_{\alpha + \beta \leq m} \int_0^\varepsilon \int_\Sigma \left| (\nabla_{x_1})^\alpha (\nabla_\Sigma)^\beta V \right|^p h^{\frac{1}{2}}(z) \, dvol_\Sigma dx_1 \right)^{1/p}$$

and

$$\|V\|_{C^m(A_\varepsilon, S)} := \sum_{\alpha + \beta \leq m} \sup_{A_\varepsilon} \left| (\nabla_{x_1})^\alpha (\nabla_\Sigma)^\beta V \right|$$

where $\nabla_{x_1}$ and $\nabla_\Sigma$ are the covariant differentiation along $x_1$-direction and $\Sigma$-directions (i.e. two tangent directions on $\Sigma$) respectively with respect to the metric $g_{A_\varepsilon}$, and the $L^p$-norm is with respect to $g_{A_\varepsilon}$ too. By the standard Sobolev embedding theorem,
we have an $\varepsilon$ independent constant $C$ such that for any smooth section $V$,

$$
\|V\|_{L^p(A_\varepsilon, S)} \leq C \|V\|_{L^{m,p}(A_\varepsilon, S)}, \text{ for } p \leq \frac{3q}{3 - mq}
$$

$$
\|V\|_{C^m(A_\varepsilon, S)} \leq C \|V\|_{L^{l,p}(A_\varepsilon, S)}, \text{ for } p \geq \frac{3}{l - m}
$$

as long as $\varepsilon$ is bounded below and above by some universal constant. For the later purpose let us fix $\varepsilon \in [1/2, 3/2]$.

2. We define the function spaces

$$
L^{m,p}(A_\varepsilon, S) := \left\{ V = (u, v) \in \Gamma (A_\varepsilon, S) \, | \, \|V\|_{L^{m,p}(A_\varepsilon, S)} < +\infty \right\}
$$

and $L^{m,p}_-(A_\varepsilon, S)$ (resp. $L^{m,p}_+(A_\varepsilon, S)$) be the closure (with respect to the norm $\|\cdot\|_{L^{m,p}(A_\varepsilon)}$) of the subspace of smooth sections $V = (u, v) \in \Gamma (A_\varepsilon, S)$ such that $v \in C_0^\infty (A_\varepsilon \setminus \partial A_\varepsilon)$ (resp. $u \in C_0^\infty (A_\varepsilon \setminus \partial A_\varepsilon)$), where $C_0^\infty (A_\varepsilon \setminus \partial A_\varepsilon)$ denotes the space of smooth functions with compact support inside $A_\varepsilon \setminus \partial A_\varepsilon$. Let us also introduce the space

$$
C^m(A_\varepsilon, S) := \left\{ V = (u, v) \in \Gamma (A_\varepsilon, S) \, | \, \|V\|_{C^m(A_\varepsilon, S)} < +\infty \right\},
$$

$$
C^m_-(A_\varepsilon, S) := \left\{ V = (u, v) \in \Gamma (A_\varepsilon, S) \, | \, \|V\|_{C^m(A_\varepsilon, S)} < +\infty, v|_{\partial A_\varepsilon} = 0 \right\}.
$$

Let $D$ be the operator (19) in our linear model. We will impose boundary conditions for sections $V$ that $D$ acts on. It is known (c.f. [8] Theorem 21.5) that the Dirac operators

$$
D_\pm := D|_{L^{1,2}_\pm} : L^{1,2}_\pm (A_\varepsilon, S) \to L^2 (A_\varepsilon, S)
$$

(25)

give well-posed local elliptic boundary problems and their formal adjoint operators are $D_\pm^* = D_\mp$. This boundary condition restricted on smooth sections $V = (u, v)$ in $L^{1,2}_- (A_\varepsilon, S)$ means

$$
v|_{\partial A_\varepsilon} = 0, \text{ i.e. } v (0, \Sigma) = v (\varepsilon, \Sigma) = 0.
$$

(26)

The following theorem compares the first eigenvalue for Dirac operator $\bar{\partial}$ on the Riemann surface $\Sigma$ with the operator $D$ on product three manifold $A_\varepsilon$. The theorem refers to the $L^2$ metric which is defined as
follows: Let \( U, V \) be two sections in \( L^2 \left( \mathbb{H}, S \right) \). We let the inner product \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{H}, S)} \) be

\[
\langle U, V \rangle_{L^2(\mathbb{H}, S)} := \int_{[0, \varepsilon]} \int_{\Sigma} \langle U, V \rangle (x_1, z) h^\frac{1}{2} (z) \, dvol \Sigma \, dx_1,
\]

where in the integral the \( \langle \cdot, \cdot \rangle \) is the inner product in fibers of the spinor bundle \( S \). We let

\[
|V (x_1, z)|^2 := \langle V (x_1, z), V (x_1, z) \rangle
\]

\[
\|V\|^2_{L^2(\mathbb{H}, S)} := \langle V, V \rangle_{L^2(\mathbb{H}, S)}.
\]

**Theorem 12** Suppose \( \lambda_{\partial^-} \) is the first eigenvalue of the Laplacian \( \Delta \Sigma = \partial^+ \partial^- \) acting on the space \( L^{1,2}_- (\Sigma, S^+) \) and \( \lambda_{\partial^+} \) is the first eigenvalue of the Laplacian \( \Delta \Sigma = \partial^- \partial^+ \) acting on the space \( L^{1,2}_+ (\Sigma, S^-) \). Let

\[
\lambda_{\mathcal{D}_\pm} := \inf_{0 \neq V \in L^{1,2}_\pm (\mathbb{H}, S)} \frac{\|DV\|^2_{L^2(\mathbb{H}, S)}}{\|V\|^2_{L^2(\mathbb{H}, S)}}.
\]

(Notice the boundary conditions \( (25) \) and \( (26) \) for \( V \).) Then for the operator \( \mathcal{D}_\pm : L^{1,2}_\pm (\mathbb{H}, S) \to L^2 (\mathbb{H}, S) \), we have

\[
\lambda_{\mathcal{D}_\pm} \geq \min \left\{ \lambda_{\partial \pm}, \frac{2}{K \varepsilon^2} - K \left\| h^{-\frac{1}{2}} \right\|^2_{C^0 (\Sigma)} \right\},
\]

where \( K > 0 \) is some constant such that \( \frac{1}{K} \leq h(z) \leq K \) for all \( z \in \Sigma \).

**Proof.** In the following we will assume the volume form \( dvol_{\mathbb{H}} = dvol \Sigma \, dx_1 \) to simplify calculation. General cases can be reduced to this case by observing

\[
\frac{\|DV\|^2_{L^2(\mathbb{H}, S)}}{\|V\|^2_{L^2(\mathbb{H}, S)}} \geq \frac{\int_{\mathbb{H}} |DV|^2 K^{-1/2} dvol \Sigma \, dx_1}{\int_{\mathbb{H}} |V|^2 K^{1/2} dvol \Sigma \, dx_1} = \frac{1}{K} \frac{\int_{\mathbb{H}} |DV|^2 dvol \Sigma \, dx_1}{\int_{\mathbb{H}} |V|^2 dvol \Sigma \, dx_1}
\]

from the condition \( \frac{1}{K} \leq h(z) \leq K \). It is enough to consider the case \( \mathcal{D}_- : L^{1,2}_- (\mathbb{H}, S) \to L^2 (\mathbb{H}, S) \). The \( \mathcal{D}_+ \) case is similar. For any \( V = (u, v) \in L^{1,2}_- (\mathbb{H}) \), we have

\[
\langle DV, DV \rangle_{L^2(\mathbb{H}, S)} = \int_{\mathbb{H}} \left( h^{-1} (z) \left| \frac{\partial V}{\partial x_1} \right|^2 + 2 \left( h^{-\frac{1}{2}} (z) \frac{\partial V}{\partial x_1} \right) \left[ \begin{array}{cc} 0 & \partial^+ \\ \partial^- & 0 \end{array} \right] V \right) + |\partial^+ v|^2 + |\partial^- u|^2 \, dvol_{\mathbb{H}}
\]

26
Using the formula $\partial^- = (\partial^+)^*$, we have

$$
\begin{align*}
\int_{\mathcal{A}_\epsilon} \left\langle h^{-\frac{1}{2}} (z) \frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V \right\rangle d\nuol_{\mathcal{A}_\epsilon} \\
= \int_{\mathcal{A}_\epsilon} \left( \left\langle h^{-\frac{1}{2}} \frac{\partial u}{\partial x_1}, \partial^+ v \right\rangle + \left\langle h^{-\frac{1}{2}} \frac{\partial v}{\partial x_1}, \partial^- u \right\rangle \right) d\nuol_{\mathcal{A}_\epsilon}
\end{align*}
$$

(∵ $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ orthogonal decomposition)

$$
\begin{align*}
&= \int_{\mathcal{A}_\epsilon} \left( \left\langle \partial^- \left( h^{-\frac{1}{2}} \frac{\partial u}{\partial x_1} \right), v \right\rangle - \left\langle h^{-\frac{1}{2}} v, \partial^- \left( \frac{\partial u}{\partial x_1} \right) \right\rangle \right) d\nuol_{\mathcal{A}_\epsilon} \\
&\quad + \int_{\{\epsilon\} \times \mathcal{S}} \left\langle h^{-\frac{1}{2}} v, \partial^- u \right\rangle d\nuol_{\mathcal{S}} - \int_{\{0\} \times \mathcal{S}} \left\langle h^{-\frac{1}{2}} v, \partial^- u \right\rangle d\nuol_{\mathcal{S}} \\
&= \int_{\mathcal{A}_\epsilon} \left( \left\langle \partial^- \left( h^{-\frac{1}{2}} \right) \cdot \frac{\partial u}{\partial x_1}, v \right\rangle + \left\langle h^{-\frac{1}{2}} \partial^- \left( \frac{\partial u}{\partial x_1} \right), v \right\rangle \\
&\quad - \left\langle h^{-\frac{1}{2}} v, \partial^- \left( \frac{\partial u}{\partial x_1} \right) \right\rangle \right) d\nuol_{\mathcal{A}_\epsilon} \text{ (since $v$ vanishes on $\partial\mathcal{A}_\epsilon$)}
\end{align*}
$$

(· · the above 2nd and 3rd terms cancel)

$$\tag{28}$$

Therefore

$$
\begin{align*}
2 \left| \int_{\mathcal{A}_\epsilon} \left\langle h^{-\frac{1}{2}} (z) \frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V \right\rangle d\nuol_{\mathcal{A}_\epsilon} \right| \\
\leq 2 \int_{\mathcal{A}_\epsilon} \left| \left\langle \partial^- \left( h^{-\frac{1}{2}} \right) \cdot \frac{\partial u}{\partial x_1}, v \right\rangle \right| d\nuol_{\mathcal{A}_\epsilon} \\
\leq \left\| h^{-\frac{1}{2}} \right\|_{C^1(\Sigma)} \int_{\mathcal{A}_\epsilon} 2 \left| \frac{\partial u}{\partial x_1}, v \right| d\nuol_{\mathcal{A}_\epsilon} \\
\leq \int_{\mathcal{A}_\epsilon} \left( \frac{1}{K} \left| u_{x_1} \right|^2 + K \left\| h^{-\frac{1}{2}} \right\|^2_{C^1(\Sigma)} \left| v \right|^2 \right) d\nuol_{\mathcal{A}_\epsilon}.
\end{align*}
$$

\tag{29}

In order to estimate $\int_{\mathcal{A}_\epsilon} h^{-1} (z) |V_{x_1}|^2 d\nuol_{\mathcal{A}_\epsilon}$, we notice that, for any fixed point $p \in \Sigma$, $v|_{[0,\epsilon] \times \{p\}}$ can be treated as a $C$-valued function over the
interval [0, \varepsilon] with boundary value v(0, z) = 0. Hence

\[
\int_0^\varepsilon |v|^2 \, dx_1 = \int_0^\varepsilon \left( \int_0^{x_1} \frac{\partial v}{\partial x_1} (t) \, dt \right)^2 \, dx_1 \\
\leq \int_0^\varepsilon \left( \int_0^{x_1} ds \left( \int_0^{x_1} \left| \frac{\partial v}{\partial x_1} (t) \right|^2 \, dt \right) \right) \, dx_1 \\
= \int_0^\varepsilon x_1 \, dx_1 \int_0^\varepsilon \left| \frac{\partial v}{\partial x_1} (t) \right|^2 \, dt \\
\leq K \frac{\varepsilon^2}{2} \int_0^\varepsilon h^{-1}(z) \left| \frac{\partial v}{\partial x_1} (t) \right|^2 \, dt,
\]

where the last inequality is by \( \frac{1}{K} \leq h(z) \leq K \) for all \( z \in \Sigma \). Putting (29), (30) in \( \langle DV, DV \rangle_{L^2(L^2(S_\varepsilon, S))} \), we have

\[
\langle DV, DV \rangle_{L^2(L^2(S_\varepsilon, S))} \\
\geq \int_{L^2(S_\varepsilon, S)} \left( h^{-1}(z) |u_{x_1}|^2 + |\partial^{-} u|^2 + h^{-1}(z) |v_{x_1}|^2 + |\partial^{+} v|^2 \right) \\
- 2 \left( \left\langle h^{-\frac{1}{2}}(z) \frac{\partial V}{\partial x_1}, \left[ \begin{array}{cc} 0 & \partial^{+} \\ \partial^{-} & 0 \end{array} \right] V \right\rangle \right) \, dvol_{L^2(S_\varepsilon, S)} \\
\geq \int_{L^2(S_\varepsilon, S)} \left( \left( h^{-1}(z) - \frac{1}{K} \right) |u_{x_1}|^2 + |\partial^{-} u|^2 + h^{-1}(z) |v_{x_1}|^2 \\
- K \left\| h^{-\frac{1}{2}} \right\|_{C^1(S)}^2 |v|^2 \right) \, dvol_{L^2(S_\varepsilon, S)} \quad (by \ (29)) \\
\geq 0 + \lambda_{\partial^{-}} \int_{L^2(S_\varepsilon, S)} |u|^2 dvol_{L^2(S_\varepsilon, S)} + \left( \frac{2}{K \varepsilon^2} - K \left\| h^{-\frac{1}{2}} \right\|_{C^1(S)}^2 \right) \int_{L^2(S_\varepsilon, S)} |v|^2 dvol_{L^2(S_\varepsilon, S)} \quad (by \ definition \ of \ \lambda_{\partial^{-}} \ and \ (30))
\]

\[
\geq \min \left\{ \lambda_{\partial^{-}}, \frac{2}{K \varepsilon^2} - K \left\| h^{-\frac{1}{2}} \right\|_{C^1(S)}^2 \right\} \left( \int_{L^2(S_\varepsilon, S)} \left( |u|^2 + |v|^2 \right) \, dvol_{L^2(S_\varepsilon, S)} \right)
\]

\[
= \min \left\{ \lambda_{\partial^{-}}, \frac{2}{K \varepsilon^2} - K \left\| h^{-\frac{1}{2}} \right\|_{C^1(S)}^2 \right\} \left\| V \right\|_{L^2(L^2(S_\varepsilon, S))}^2.
\]

For general volume form \( dvol_{L^2(S_\varepsilon, S)} = h^{\frac{1}{2}}(z) \, dvol_{L^2(dx_1, S)} \), by (27) there is an extra factor \( \frac{1}{K} \) for the lower bound of the \( L^2 \) eigenvalue \( \lambda_{D_{\varepsilon}} \). Hence the result. ■

Suppose \( V = (u, v) \in C^\infty(A_{\varepsilon}, S) \cap L^{1,2}(A_{\varepsilon}, S) \) and \( W = (f, g) \in C^\infty(A_{\varepsilon}, S) \cap L^2(A_{\varepsilon}, S) \) and \( D \) is the operator [19]. Then with respect to
the induced volume form $dvol_{h\varepsilon} = h^{1/2}(z) d\Sigma dx_1$ on $(A_\varepsilon, g_{h\varepsilon}, h)$ we have

$$
\int_{h\varepsilon} \langle DV, W \rangle dvol_{h\varepsilon} = \int_0^\varepsilon \int_\Sigma \left[ i \left( h^{-1/2} u x_1 + \partial^+ v \right), f \right] - \left[ i \left( h^{-1/2} v x_1 + \partial^- u \right), g \right] h^{1/2} d\Sigma dx_1
$$

$$
= i \int_0^\varepsilon \int_\Sigma \left[ - \langle u, f x_1 \rangle + \left( h^{1/2} v, \partial^- f \right) - \left( h^{1/2} u, \partial^+ g \right) + \langle v, g x_1 \rangle \right] d\Sigma dx_1
$$

$$
+ i \int_\Sigma \left( \langle u, f \rangle |_\delta - \langle v, g \rangle |_\delta \right) d\Sigma
$$

$$
= \int_0^\varepsilon \int_\Sigma \left[ \langle u, f x_1 - h^{1/2} \partial^+ g \rangle + \langle v, g x_1 + h^{1/2} \partial^- f \rangle \right] d\Sigma dx_1
$$

$$
+ i \int_\Sigma \left( \langle u, f \rangle |_\delta - \langle v, g \rangle |_\delta \right) d\Sigma
$$

$$
= \int_{h\varepsilon} \langle DV, W \rangle dvol_{h\varepsilon} + i \int_\Sigma \left( \langle u, f \rangle |_\delta - \langle v, g \rangle |_\delta \right) d\Sigma
$$

(31)

since $h$ is independent of $x_1$. This is the Green’s formula.

When $V \in L^1_{\subset} (A_\varepsilon, S)$ and $W \in L^1_+ (A_\varepsilon, S)$, the above boundary terms are zero so we have

$$
\int_{h\varepsilon} \langle DV, W \rangle dvol_{h\varepsilon} = \int_{h\varepsilon} \langle V, DW \rangle dvol_{h\varepsilon}.
$$

This implies that $D$ is a self adjoint operator from $L^1_{\subset} (A_\varepsilon, S)$ to $L^1_+ (A_\varepsilon, S)$ in the sense of $S$. Since $L^1_{\subset} (A_\varepsilon, S)$ is dense in $L^2 (A_\varepsilon, S)$, this implies that $D : L^1_{\subset} (A_\varepsilon, S) \to L^2 (A_\varepsilon, S)$ is surjective if and only if $\ker D|_{L^1_{\subset} (A_\varepsilon, S)} = 0 \subset L^1_+ (A_\varepsilon, S)$. By Theorem 12 for small enough $\varepsilon$, we have

$$
\ker D|_{L^1_+ (A_\varepsilon, S)} = 0 \iff \ker \partial^- \partial^+ = 0 \iff \ker \partial^+ = 0,
$$

where the last “$\iff$” is because $(\partial^+)^\ast = \partial^-$. Hence we have obtained the following result.

**Theorem 13** Let $\lambda_{\partial^-}$ and $\lambda_{\partial^+}$ be the first eigenvalue for $\partial^+ \partial^-$ and $\partial^- \partial^+$ respectively. For $D : L^1_{\subset} (A_\varepsilon, S) \to L^2 (A_\varepsilon, S)$, if $\varepsilon$ is sufficiently
small, then we have

\[ \ker \partial^- = \{0\} \iff \lambda_{\partial^-} > 0 \Rightarrow \mathcal{D} \text{ injective}, \]
\[ \ker \partial^+ = \{0\} \iff \lambda_{\partial^+} > 0 \Rightarrow \mathcal{D} \text{ surjective}. \]

Especially if both \( \lambda_{\partial^-}, \lambda_{\partial^+} > 0 \), then \( \mathcal{D} \) is one-to-one and onto.

### 3.3 Schauder Estimates for Linear Model

In this section we will develop the necessary linear theory for the equation

\[ \mathcal{D} V = W \] on \( A_\varepsilon = \Sigma \times [0, \varepsilon] \)

with a warped product metric \( g_{h \varepsilon, h} := h(z) \, dx_1^2 + g_\Sigma \), where \( \mathcal{D} \) is the operator (19) in our linear model. The key issue is to estimate the operator norm of the inverse operator of \( \mathcal{D} \) with explicit dependence of \( \varepsilon \), as \( \varepsilon \) goes to zero. When \( \varepsilon \) is away from zero, say \( \varepsilon \in [1/2, 3/2] \), we have \( \varepsilon \)-free Schauder estimates. For \( \varepsilon \) small, we overcome the difficulty coming from \( \varepsilon \) by choosing an appropriate integer \( k \) so that \( k\varepsilon \in [1/2, 3/2] \) and we extend any solution \( V = (u, v) \) on \( A_\varepsilon \) to \( A_{k\varepsilon} \) in an \( L^p \) sense by suitable reflection. However much care will be needed to obtain the \( C^\alpha \)-estimate, because after the reflection of \( W \) across the boundary of \( A_\varepsilon \), it will no longer be continuous in general. This problem will be resolved in the case (ii) part of the proof of the following theorem.

Estimating the operator norm of \( \mathcal{D}^{-1} \) in the Schauder setting rather than in the \( L^p \) setting is crucial. Since our goal is to construct instantons \( A \) governed by the equation \( \tau|_A = 0 \), which involves the associative 3-form \( \tau \), for deformations of an approximate solution \( A \) by normal vector fields \( V \) in \( M \) that are in \( W^{1,p} \) class, the nonlinear equation will have cubic terms of \( \nabla V \) that are outside the \( L^p \) space. The cubic terms also cause difficulty for obtaining desired quadratic estimates for the implicit function theorem in the \( L^p \) setting (see Remark 25).

The Schauder estimates are harder to obtain than for the Cauchy-Riemann type equations, partly because in our equation (23), the derivative of \( v \) only controls the \( \partial_{x_1} \) and \( \bar{\partial}_z \) derivatives of \( u \), not the full derivatives.

We recall the definition of Hölder norms for functions \( f \) on a domain...
\[ \Omega \text{ in } \mathbb{R}^n: \]
\[
[f]_{\alpha;\Omega} := \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},
\]
\[
[f]_{C^0(\Omega)} = \|f\|_{C^0(\Omega)} + [f]_{\alpha;\Omega},
\]
\[
[f]_{C^{1,\alpha}(\Omega)} = \|f\|_{C^1(\Omega)} + [\nabla f]_{\alpha;\Omega}.
\]

Using trivialization of the bundle \( \mathcal{S} \) over \( A_\varepsilon \), the Hölder norms for sections \( V \) of the bundle \( \mathcal{S} \) are defined by patching the (finitely many) \( C^{1,\alpha} \)-coordinate charts on \( A_\varepsilon \).

**Theorem 14** Let \( D : L^{1,2}_-(A_\varepsilon, \mathcal{S}) \rightarrow L^2(A_\varepsilon, \mathcal{S}) \) be the operator \((23)\) defined on \( A_\varepsilon = \Sigma \times [0, \varepsilon] \) with warped product metric \( g_{h,h} := h(z) \, dx_1^2 + g_\Sigma \). Suppose that the first eigenvalues for \( \partial^- \partial^+ \) and \( \partial^+ \partial^- \) are bounded below by \( \lambda > 0 \). Then for any \( 0 < \alpha < 1 \) and \( p > 3 \) there is a positive constant \( C = C(\alpha, p, \lambda, h) \) independent of \( \varepsilon \) such that for any \( V \in C^{1,\alpha}_-(A_\varepsilon, \mathcal{S}) \) and \( W \in C^\alpha(A_\varepsilon, \mathcal{S}) \) satisfying
\[
DV = W
\]
we have
\[
C \|V\|_{C^{1,\alpha}_-(A_\varepsilon, \mathcal{S})} \leq \varepsilon^{-\left(\frac{2}{p} + 2\alpha\right)} \|W\|_{C^\alpha(A_\varepsilon, \mathcal{S})}.
\]
In other words, there exists a right inverse
\[
Q_\varepsilon : C^\alpha(A_\varepsilon, \mathcal{S}) \rightarrow C^{1,\alpha}_-(A_\varepsilon, \mathcal{S})
\]
of \( D : C^{1,\alpha}_-(A_\varepsilon, \mathcal{S}) \rightarrow C^\alpha(A_\varepsilon, \mathcal{S}) \) such that \( \|Q_\varepsilon\| \leq C \varepsilon^{-\left(\frac{2}{p} + 2\alpha\right)} \).

**Proof.** For the operator \((23)\), it is known (c.f. [8] Theorem 21.5) that the Dirac operators
\[
D_\pm := D|_{L^{1,2}_\pm} : L^{1,2}_\pm(A_\varepsilon, \mathcal{S}) \rightarrow L^2(A_\varepsilon, \mathcal{S})
\]
give well-posed local elliptic boundary problems. Using the orthogonal decomposition \( \mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \) we write sections \( V = (u, v) \in C^\infty(A_\varepsilon, \mathcal{S}) \) and \( W = (w_1, w_2) \in C^\infty(A_\varepsilon, \mathcal{S}) \), where \( u, w_1 \) are sections of \( \mathcal{S}^+ \) and \( v, w_2 \) are sections of \( \mathcal{S}^- \). So the equation \( DV = W \) may be explicitly written as (c.f. equation \((23)\))
\[
\begin{align*}
h^{-1/2}(z) \, w_{x_1} + \partial^+ v &= w_1 \\
h^{-1/2}(z) \, v_{x_1} + \partial^- u &= w_2
\end{align*}
\]
with \( v|_{\partial A_\varepsilon} = 0 \).
To make the exposition more transparent we will assume that \( h \equiv 1 \), and it is clear from the proof below that the argument works equally well for any \( h(z) \in C^\infty(\Sigma) \) such that \( \frac{1}{K} \leq h(z) \leq K \) for some constant \( K > 0 \). For any \( 0 < \varepsilon < 3/2 \) we take an integer \( k = k(\varepsilon) > 0 \) (which depends on \( \varepsilon \)) such that

\[
k(\varepsilon) \varepsilon \in [1/2, 3/2].
\]

For notation brevity, we will write \( k \) for \( k(\varepsilon) \) in the remainder of our paper. Then \( A_{k\varepsilon} = \Sigma \times [0, k\varepsilon] \) is a product region whose second component has length uniformly bounded below and above for any \( \varepsilon \). We divide the estimates into two cases: Case (i): Suppose that \( w_1 = 0 \), then we will have along the boundary \( \partial A_\varepsilon \), \( u_{x_1} = 0 \) since \( \nu = 0 \). Since \( \nu|_{\partial A_\varepsilon} = 0 \), we can extend \( \nu \) from \( A_\varepsilon \) to \( A_{k\varepsilon} \) by odd reflection along the walls \( \Sigma \times \{ j\varepsilon \} \) with \( 0 \leq j \leq k - 1 \), while still keeping \( \nu \) in \( C^1, \alpha(A_{k\varepsilon}) \).

Similarly we consider an even extension of \( u \) to \( A_{k\varepsilon} \), then \( u \) is still in \( C^1, \alpha(A_{k\varepsilon}) \) since \( u_{x_1}|_{\partial A_\varepsilon} = 0 \). The extension formula is

\[
v(x, z) = \begin{cases} -v((2j + 2)\varepsilon - x, z) & \text{for } x \in [(2j + 1)\varepsilon, (2j + 2)\varepsilon] \\ v(x - 2j\varepsilon, z) & \text{for } x \in [2j\varepsilon, (2j + 1)\varepsilon] \end{cases},
\]

\[
u(x, z) = \begin{cases} u((2j + 2)\varepsilon - x, z) & \text{for } x \in [(2j + 1)\varepsilon, (2j + 2)\varepsilon] \\ u(x - 2j\varepsilon, z) & \text{for } x \in [2j\varepsilon, (2j + 1)\varepsilon] \end{cases}.
\]

This will induce an even extension of \( w_2 \) so that the equation \( DV = W \) is satisfied in the \( C^\alpha \) sense on \( A_{k\varepsilon} \). The motivation of even and odd extension is to have an \( \varepsilon \)-independent \( L^p \) and Schauder estimate. By differentiating both sides of the equation \( v_{x_1} + \partial^- u = w_2 \) with respect to \( x_1 \), we obtain an equation which is equivalent to the Dirichlet problem of the second order elliptic equation

\[
v_{x_1x_1} - \partial^- \partial^+ v = \frac{\partial w_2}{\partial x_1} \text{ and } v|_{\partial A_{k\varepsilon}} = 0,
\]

\[
u_{x_1x_1} - \partial^+ \partial^- u = -\partial^+ w_2,
\]

noting that \( \partial^- \partial^+ \) is a positive operator. (Since \( u, v \) are only in \( C^{1, \alpha} \), they should be understood as weak solutions of the above equations).

Since the \( C^\alpha \)-norm is preserved under the above extension, Schauder and \( L^p \) interior estimate for the second order elliptic equation on the region

\[A_{k\varepsilon} \subset A_{2k\varepsilon} \cup A_{-2k\varepsilon}\]
would then imply that there are constants $C(\alpha)$ and $\tilde{C}(p)$ independent of $\varepsilon$ (because $k \varepsilon \in [1/2, 3/2]$) such that

$$
\|w_2\|_{C^\alpha(A_\varepsilon, S^-)} + \|V\|_{C^\alpha(A_\varepsilon, S)} = \|w_2\|_{C^\alpha(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S^-)} + \|V\|_{C^\alpha(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)} \\
\geq C(\alpha) \|V\|_{C^{1,\alpha}(A_\varepsilon, S)} \\
= C(\alpha) \|V\|_{C^{1,\alpha}(A_\varepsilon, S)},
$$

(33)

and

$$
\|w_2\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S^-)} + \|V\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)} \geq 4\tilde{C}(p) \|V\|_{L^1_{-p}(A_\varepsilon, S)}
$$

(c.f. [13] section 8.11 Theorem 8.32 and [30] Theorem B.3.2 respectively), so

$$
\|w_2\|_{L^p(A_\varepsilon, S^-)} + \|V\|_{L^p(A_\varepsilon, S)} \geq \tilde{C}(p) \|V\|_{L^1_{-p}(A_\varepsilon, S)}
$$

by periodicity of reflection. We also have

$$
\|V\|_{C^0(A_\varepsilon, S)} \leq \|V\|_{C_{-3/2}^{1,p}(A_\varepsilon, S)} \leq C(p, \lambda) \|W\|_{C^0(A_\varepsilon, S)},
$$

whose proof is identical to that of (33) in the following case (ii). Plugging this in (33) we have

$$
C(\alpha) \|V\|_{C^{1,\alpha}(A_\varepsilon, S)} \leq (C(p, \lambda) + 1) \|w_2\|_{C^\alpha(A_\varepsilon, S^-)}. 
$$

(34)

Case (ii): suppose that $w_2 = 0$ and $w_1 \in C^\alpha(A_\varepsilon, S^+)$. Since $v = 0$, this implies that if we consider the odd extension of $v$ and the even extension of $u$ to $A_{2k\varepsilon} \cup A_{-2k\varepsilon}$ as in the previous case, then they induce an odd extension of $w_1$ so that the equation

$$
\begin{cases}
    u_{x_1} + \partial^+ v = w_1 \\
    v_{x_1} + \partial^- u = 0
\end{cases}
$$

(35)

is satisfied in the weak sense on $A_{2k\varepsilon} \cup A_{-2k\varepsilon}$. Notice that $w_1$ does not vanish on $\partial A_\varepsilon$ in general, so after the odd extension, $w_1$ is no longer continuous but still we have $w_1 \in L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)$ for $\forall p$. Since $A_{k\varepsilon}$ is a proper subdomain in $A_{2k\varepsilon} \cup A_{-2k\varepsilon}$ and $k \varepsilon \in [1/2, 3/2]$, the interior $L^p$-estimate then implies that there is a constant $\tilde{C}(p)$ independent of $\varepsilon$ such that

$$
\|w_1\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S^-)} + \|V\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)} \geq 4\tilde{C}(p) \|V\|_{L^1_{-p}(A_\varepsilon, S)}.
$$
By the periodicity of $w_1$ and $V$, that is
\[ \|w_1\|_{L^p(\mathbb{R},\mathbb{R})} + \|V\|_{L^p(\mathbb{R},\mathbb{R})} \geq \tilde{C}(p) \|V\|_{L^{1,p}(\mathbb{R},\mathbb{R})}. \] (36)
Then we proceed to the $L^p$ estimates of $V$ purely in terms of $W$. It follows from (36) and Theorem 12 that for $\varepsilon < 3/2$ we have
\[ \|W\|_{L^2(A_{\varepsilon},\mathbb{R})} \geq C(\lambda) \|V\|_{L^2(A_{\varepsilon},\mathbb{R})} \text{ for } V \in C^\infty_c(\mathbb{R},\mathbb{R}) \]
with the constant $C(\lambda)$ dependent on $\lambda_{p+}$ but independent of $\varepsilon$. To go from $L^2$ to $L^p$, note the following interpolation inequality
\[ \|V\|_{L^p(A_{\varepsilon},\mathbb{R})} = \left( \int_{A_{\varepsilon}} |V|^p \right)^{1/p} \leq \|V\|_{C^0(A_{\varepsilon},\mathbb{R})} \cdot \left( \int_{A_{\varepsilon}} |V| \right)^{1/p} \]
\[ \leq \frac{p-1}{p} \delta_{\varepsilon}^{p-1} \|V\|_{C^0(A_{\varepsilon},\mathbb{R})} + \frac{1}{p} \|V\|_{L^1(A_{\varepsilon},\mathbb{R})} \]
(By Young’s Inequality)
\[ = \frac{p-1}{p} \delta_{\varepsilon}^{p-1} \|V\|_{C^0(A_{\varepsilon},\mathbb{R})} + \frac{1}{p} \|V\|_{L^1(A_{\varepsilon},\mathbb{R})} \]
\[ \leq C \left( \delta_{\varepsilon}^{p-1} \|V\|_{L^{1,p}(A_{\varepsilon},\mathbb{R})} + \delta_{\varepsilon}^{-p} \|V\|_{L^{2}(A_{\varepsilon},\mathbb{R})} \right) \]
(By Sobolev embedding),
where $C$ is independent of $\varepsilon$ in the last inequality because $k\varepsilon \in [1/2, 3/2]$. Putting this in (36) we have
\[ \left( \tilde{C}(p) - C \delta_{\varepsilon}^{p-1} \right) \|V\|_{L^{1,p}(A_{\varepsilon},\mathbb{R})} \leq \|W\|_{L^p(A_{\varepsilon},\mathbb{R})} + C \delta_{\varepsilon}^{-p} \|V\|_{L^2(A_{\varepsilon},\mathbb{R})} \]
\[ \leq \|W\|_{L^p(A_{\varepsilon},\mathbb{R})} + \frac{C \delta_{\varepsilon}^{-p}}{C(\lambda)} \|W\|_{L^2(A_{\varepsilon},\mathbb{R})} \]
\[ \leq \|W\|_{L^p(A_{\varepsilon},\mathbb{R})} \left( 1 + \frac{C \delta_{\varepsilon}^{-p}}{C(\lambda)} \right). \]
So
\[ \|V\|_{L^{1,p}(A_{\varepsilon},\mathbb{R})} \leq \left( \tilde{C}(p) - C \delta_{\varepsilon}^{p-1} \right)^{-1} \left( 1 + \frac{C \delta_{\varepsilon}^{-p}}{C(\lambda)} \right) \|W\|_{L^p(A_{\varepsilon},\mathbb{R})}. \] (37)
Fixing a small $\delta$ such that $\tilde{C}(p) - C \delta_{\varepsilon}^{p-1} > 0$ and let the constant
\[ \tilde{C}(p, \lambda) := \left( \tilde{C}(p) - C \delta_{\varepsilon}^{p-1} \right)^{-1} \left( 1 + \frac{C \delta_{\varepsilon}^{-p}}{C(\lambda)} \right). \]
which is independent of \( \varepsilon \). Then for \( p > 3 \), by Sobolev embedding
\( C^{1-3/p}_-(\Omega_\varepsilon, \mathbb{S}) \hookrightarrow L^1_{-p}(\Omega_\varepsilon, \mathbb{S}) \) and (37) we have
\[
\|V\|_{C^{1-3/p}_-(\Omega_\varepsilon, \mathbb{S})} = \|V\|_{C^{1-3/p}_-(\Omega_\varepsilon, \mathbb{S})} \\
\leq C \|V\|_{L^{1-p}(\Omega_\varepsilon, \mathbb{S})} - C \cdot \tilde{C} (p, \lambda) \|W\|_{L^p(\Omega_\varepsilon, \mathbb{S})} \\
\leq C (p, \lambda) \|W\|_{C^0(\Omega_\varepsilon, \mathbb{S})} = C (p, \lambda) \|W\|_{C^0(\Omega_\varepsilon, \mathbb{S})},
\]
where the constant \( C (p, \lambda) = C \cdot \tilde{C} (p, \lambda) \left( \frac{3}{2} \text{vol} (\Sigma) \right)^{1/p} \) is independent of \( \varepsilon \). We remark that the above argument also works in case (i), so (38) holds in all cases. In the following we give the Schauder estimate of \( V = (u, v) \). Differentiating the second equation in (35) with respect to \( x_1 \), we get the Dirichlet problem of the second order elliptic equation for (weak solutions) \( v \) and \( u \):
\[
v_{x_1x_1} - \partial^+ \partial^+ v = -\partial^- w_1 \quad \text{and} \quad v|_{\partial \Omega_\varepsilon} = 0, \\
u_{x_1x_1} - \partial^+ \partial^+ u = \partial_{x_1} w_1.
\]
Recall that in [13] chapter 4, the non-dimensional Schauder norm \( \|f\|_{C^{k, \alpha}(\overline{\Omega})} \) is defined as
\[
\|f\|_{C^{k, \alpha}(\overline{\Omega})} = \sum_{j=0}^k d^j \left| D^j f \right|_{0, \Omega} + d^{k+\alpha} \left| D^k f \right|_{\partial \Omega},
\]
where \( d \) is the diameter of \( \Omega \). For component \( v \), since \( v|_{\partial \Omega_\varepsilon} = 0 \), the standard Schauder estimate for second order elliptic equation on radius \( \varepsilon \) half balls \( B_\varepsilon \) with centers on \( \partial \Omega_\varepsilon \) (c.f. [13] Section 8.11 for regularity of weak solutions and Section 4.4 equation (4.43)) gives
\[
\varepsilon \|w_1\|_{C^0(B_\varepsilon, \mathbb{S}^+)} + \|v\|_{C^0_-(B_\varepsilon, \mathbb{S})} \geq C (\alpha) \|v\|_{C^{1, \alpha}_-(B_{\varepsilon/2}, \mathbb{S})},
\]
or equivalently, by the definition of \( \|f\|_{C^{k, \alpha}(\overline{\Omega})} \),
\[
\varepsilon \|w_1\|_{C^0_-(B_\varepsilon, \mathbb{S}^+)} + \|v\|_{C^0_-(B_\varepsilon, \mathbb{S})} \\
\geq C (\alpha) \left[ \|v\|_{C^0_-(B_{\varepsilon/2}, \mathbb{S})} + \varepsilon \|\nabla v\|_{C^0_-(B_{\varepsilon/2}, \mathbb{S})} + \varepsilon^{1+\alpha} [\nabla v]_{\alpha_+(B_{\varepsilon/2}, \mathbb{S})} \right],
\]
and
\[
\varepsilon \|w_1\|_{C^0_-(B_\varepsilon, \mathbb{S}^+)} + \|v\|_{C^0_-(B_\varepsilon, \mathbb{S})} \geq C (\alpha) \varepsilon^{1+\alpha} \|v\|_{C^{1, \alpha}_-(B_{\varepsilon/2}, \mathbb{S})},
\]
where the constant $C(\alpha)$ is independent of $\varepsilon$. Covering $A_\varepsilon$ by such radius-$\varepsilon$ half balls and radius-$\varepsilon/2$ full balls centered on meridian $\Sigma \times \{\varepsilon/2\}$ and take supremum on $A_\varepsilon$, we have

$$\varepsilon \|w_1\|_{C^0(A_\varepsilon, S^+)} + \|v\|_{C^0(A_\varepsilon, S)} \geq C(\alpha) \varepsilon^{1+\alpha} \|v\|_{C^{-\alpha}_1(A_\varepsilon, S)}. \quad (43)$$

By combining this with the fact that $v|_{\partial A_\varepsilon}=0$, and the definition of the Schauder norm, we have

$$\|v\|_{C^0(A_\varepsilon, S)} \leq \|v\|_{C^{1-3/p}_1(A_\varepsilon, S)} \varepsilon^{1-3/p} \leq C(p, \lambda) \|w_1\|_{C^0(A_\varepsilon, S)} \varepsilon^{1-3/p}. \quad (44)$$

Plug these into (43) we obtain

$$C(\alpha) \|v\|_{C^{1-\alpha}_1(A_\varepsilon, S)} \leq \varepsilon^{-\alpha} \|w_1\|_{C^0(A_\varepsilon, S^+)} + C(p, \lambda) \|w_1\|_{C^0(A_\varepsilon, S)} \varepsilon^{-(3/p+\alpha)}. \quad (45)$$

For the component $u$, its boundary value is nonzero so we can not directly apply the above inequalities as $v$. First we derive the $C^0$ estimate of $u$, using the assumption that $\partial^+ \partial^-$ has trivial kernel on $\Sigma$. Consider the section $\overline{u}$ of the bundle $S^+ \to \Sigma$ defined as

$$\overline{u}(z) = \int_0^\varepsilon u(x_1, z) \, dx_1.$$

From equation (35) we have

$$\partial^- \overline{u}(z) = -\int_0^\varepsilon h^{-1/2}(z) \partial_{x_1} u(x_1, z) \, dx_1 = h^{-1/2}(z) (v(0, z) - v(\varepsilon, z)) = 0.$$

But from the assumption that $\partial^+ \partial^-$ has trivial kernel on $\Sigma$ we get that

$$\overline{u}(z) \equiv 0.$$

Let $\text{Re}\overline{u}$ and $\text{Im}\overline{u}$ be the real and imaginary parts of the section $\overline{u}$ (notice that $S^+$ is a complex line bundle). Then for any fixed $z \in \Sigma$,

$$\int_0^\varepsilon \text{Re} u(x_1, z) \, dx_1 = 0 = \int_0^\varepsilon \text{Im} u(x_1, z) \, dx_1.$$

By mean value theorem for the $\mathbb{R}$-valued function $\text{Re} u(x_1, z)$, there exists $s \in [0, \varepsilon]$ depending on $z$, such $\text{Re} u(s, z) = 0$. Therefore for $x_1 \neq s$, by the definition of the Schauder norm we have

$$|\text{Re} u(x_1, z)| \leq \frac{|\text{Re} u(x_1, z) - \text{Re} u(s, z)|}{|x_1 - s|^{1-\frac{3}{p}}} \cdot |x_1 - s|^{1-\frac{3}{p}} \leq \|\text{Re} u\|_{C^{1-3/p}(A_\varepsilon, S)} \varepsilon^{1-\frac{3}{p}}.$$
Then we have

\[ \| \Re u \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \leq \| \Re u \|_{C^{1-3/p}(\mathbb{A}_\varepsilon, \mathbb{S})} \varepsilon^{1-\frac{3}{p}} \]
\[ \leq \| V \|_{C^{1-3/p}(\mathbb{A}_\varepsilon, \mathbb{S})} \varepsilon^{1-\frac{3}{p}} \]
\[ \leq C(p, \lambda) \| w_1 \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \varepsilon^{1-\frac{3}{p}} \]

by (38). Similarly

\[ \| \Im u \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S}^+)} \leq C(p, \lambda) \| w_1 \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \varepsilon^{1-\frac{3}{p}}. \]

Combining these we get

\[ \| u \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S}^+)} \leq C(p, \lambda) \| w_1 \|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \varepsilon^{1-\frac{3}{p}}. \quad (46) \]

Now we are ready to derive the Schauder estimate of \( u \). Using the equation

\[ \begin{cases} u_{x1} + \partial^+ v = w_1 & \text{with } v|_{\partial \varepsilon} = 0, \\ v_{x1} + \partial^- u = 0 \end{cases} \]

the Schauder estimate of \( u_{x1} \) and \( \partial^- u \) can be reduced to that of \( v \). For the full covariant derivative \( \nabla_\Sigma u \) on \( \Sigma \), we observe that

\[ [\nabla_\Sigma u]_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})} \leq [\nabla_\Sigma u]^z_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})} + [\nabla_\Sigma u]^{x_1}_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})} \]

where \([\cdot]^z_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})}\) and \([\cdot]^{x_1}_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})}\) as defined in (32) are the Schauder \( \alpha \)-components for the \( \Sigma \) and \([0, \varepsilon]\) directions respectively. For \([\nabla_\Sigma u]^z_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})}\), on each slice \( \Sigma_s := \Sigma \times \{s\} \), we use the elliptic estimate on compact closed surface \( \Sigma_s \) to control \( \nabla_\Sigma u \) by \( \partial^- u \),

\[ [\nabla_\Sigma u]^z_{\alpha(\Sigma_s, \mathbb{S})} \leq |u|_{C^{1,0}(\Sigma_s, \mathbb{S})} \leq C \left( |\partial^- u|_{C^0(\Sigma_s, \mathbb{S})} + |u|_{C^0(\Sigma_s, \mathbb{S})} \right), \]

and then take the sup for \( 0 \leq s \leq \varepsilon \) to get

\[ [\nabla_\Sigma u]^z_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})} \leq C \left( |\partial^- u|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} + |u|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \right) \]
\[ = C \left( |v_{x1}|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} + |u|_{C^0(\mathbb{A}_\varepsilon, \mathbb{S})} \right). \quad (47) \]

Since the estimate for \( v_{x1} \) is known in (45), the only term left to estimate is \([\nabla_\Sigma u]^{x_1}_{\alpha(\mathbb{A}_\varepsilon, \mathbb{S})}\). For this we reduce \( u \) to the zero boundary value case by introducing

\[ \tilde{u} = u - \rho \left( \frac{x_1}{\varepsilon} \right) u(\varepsilon, z) - \left( 1 - \rho \left( \frac{x_1}{\varepsilon} \right) \right) u(0, z), \]
where $\rho : [0, 1] \to [0, 1]$, $\rho (0) = 0$, $\rho (1) = 1$ is a smooth cut-off function such that $\| \rho \|_{C^2, [0, 1]} \leq C$. Then $\tilde{u}$ satisfies

$$\tilde{u}_{x_1 x_1} - \partial^+ \partial^- \tilde{u} = \partial_{x_1} \tilde{w}_1 + g \quad \text{and} \quad \tilde{u}|_{\partial \xi} = 0,$$

where

$$\tilde{w}_1 = w_1 + \rho \left( \frac{x_1}{\varepsilon} \right) \left[ (u_{x_1} (\varepsilon, z) - u_{x_1} (0, z)) + (w_1 (0, z) - w_1 (\varepsilon, z)) \right]$$

and

$$g = \frac{1}{\varepsilon^2} \rho'' \left( \frac{x_1}{\varepsilon} \right) \left[ u (0, z) - u (\varepsilon, z) \right] + \frac{1}{\varepsilon} \rho' \left( \frac{x_1}{\varepsilon} \right) \left[ (u_{x_1} (0, z) - u_{x_1} (\varepsilon, z)) + (w_1 (\varepsilon, z) - w_1 (0, z)) \right].$$

In the above derivation of $\tilde{w}_1$ and $g$ we have used the equation

$$\partial^+ \partial^- u = u_{x_1 x_1} - \partial_{x_1} w_1$$

from (39). By section 4.4 equation (4.46) we have on radius-$\varepsilon$ half ball $B_{\varepsilon}$,

$$\| \tilde{u} \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon \| \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon^2 | g |_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} \geq C (\alpha) \| \tilde{u} \|_{C^{1, \alpha} \left( B_{\varepsilon/2}, \mathbb{S} \right)}. $$

By the definition of the norm $\| f \|_{C^{\kappa, \alpha} \left( \Omega \right)}$ in (40), this is equivalent to

$$\| \tilde{u} \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon \| \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon^{1 + \alpha} \| \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon^2 | g |_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} \geq C (\alpha) \left[ \| \tilde{u} \|_{C^0 \left( B_{\varepsilon/2}, \mathbb{S} \right)} + \varepsilon \| \nabla \tilde{u} \|_{C^0 \left( B_{\varepsilon/2}, \mathbb{S} \right)} + \varepsilon^{1 + \alpha} \| \nabla \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon/2}, \mathbb{S} \right)} \right].$$

(48)

Therefore

$$\| \tilde{u} \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon \| \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon^{1 + \alpha} \| \nabla \tilde{w}_1 \|_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} + \varepsilon^2 | g |_{C^0 \left( B_{\varepsilon}, \mathbb{S} \right)} \geq C (\alpha) \varepsilon^{1 + \alpha} \| \nabla_{\Sigma} \tilde{u} \|_{C^0 \left( B_{\varepsilon/2}, \mathbb{S} \right)} \cdot$$

(49)

We are going to get rid of the tilde terms in the above inequality by the following rule: The terms with tilde differ from the original terms by $w_1$, $u$ and their derivatives. If the derivative is with respect to $x_1$ or $\partial^-$, then we reduce it to the previous estimates of $v$ using (35); If it is with respect to $\nabla_{\Sigma}$, the full derivative on $\Sigma$, then we use elliptic
estimate on each slice Riemann surface \( \{t\} \times \Sigma \). Notice that all \( \varepsilon \) powers are coupled with the order of derivatives so all terms are dimensionless, and after the enlargements the estimate of \( u \) is of correct \( \varepsilon \) power. The following is the precise calculation: From the definition of \( g, \tilde{w}_1 \) and \( \tilde{u} \) we can easily check that

\[
\begin{align*}
\| \varepsilon^2 g \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq C \left( \| u \|^\alpha_{(B_\varepsilon, \Sigma)} + \varepsilon \| u_{x_1} \|^\alpha_{(B_\varepsilon, \Sigma)} + \varepsilon \| w_1 \|^\alpha_{(B_\varepsilon, \Sigma)} \right) \\
\| \tilde{w}_1 \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq 3 \| w_1 \|^\alpha_{(B_\varepsilon, \Sigma)} + \| u_{x_1} \|^\alpha_{(B_\varepsilon, \Sigma)} \\
[\tilde{w}_1]_{\alpha; (B_\varepsilon, \Sigma)} &\leq C \left( [w_1]_{\alpha; (B_\varepsilon, \Sigma)} + \varepsilon \| u_{x_1} \|^\alpha_{(B_\varepsilon, \Sigma)} + \varepsilon \| w_1 \|^\alpha_{(B_\varepsilon, \Sigma)} \right) \\
[\tilde{w}_1]_{\alpha; (B_\varepsilon, \Sigma)}^2 &\leq C \left( [w_1]_{\alpha; (B_\varepsilon, \Sigma)} + [u_{x_1}]_{\alpha; (B_\varepsilon, \Sigma)}^2 \right) \\
\| \nabla \tilde{u} \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq 2 \| u \|^\alpha_{(B_\varepsilon, \Sigma)} \\
\| \nabla \Sigma \tilde{u} \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq \| \nabla \Sigma u \|^\alpha_{(B_\varepsilon, \Sigma)} , \\
\| \nabla_{x_1} \tilde{u} \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq C \left( \| \nabla_{x_1} u \|^\alpha_{(B_\varepsilon, \Sigma)} + \varepsilon \| u \|^\alpha_{(B_\varepsilon, \Sigma)} \right) \\
[\nabla \Sigma u]_{\alpha; (B_\varepsilon/2, \Sigma)}^2 &\leq C \left( [\nabla \Sigma \tilde{u}]_{\alpha; (B_\varepsilon/2, \Sigma)} + \varepsilon \| \nabla \Sigma u \|^\alpha_{(B_\varepsilon/2, \Sigma)} \right). \tag{50}
\end{align*}
\]

hence putting these back to \((49)\) and \((48)\), and noticing that from \((41)\) we can control \( \| u_{x_1} \|^\alpha_{(B_\varepsilon, \Sigma)} \) and \( \| \nabla \Sigma u \|^\alpha_{(B_\varepsilon, \Sigma)} \) that appeared in the above inequalities by the following:

\[
\begin{align*}
\varepsilon \| u_{x_1} \|^\alpha_{(B_\varepsilon, \Sigma)} &\leq C \left( \varepsilon \| \partial^+ v \|^\alpha_{(B_\varepsilon, \Sigma)} + \varepsilon \| w_1 \|^\alpha_{(B_\varepsilon, \Sigma)} \right) \quad \text{(by \((55)\))} \\
&\leq C \left( \| v \|^\alpha_{(A_\varepsilon, \Sigma)} + \varepsilon \| w_1 \|^\alpha_{(A_\varepsilon, \Sigma)} \right) + \varepsilon \| w_1 \|^\alpha_{(B_\varepsilon, \Sigma)} \quad \text{(by \((41)\))} \\
&\leq C \left( \| v \|^\alpha_{(A_\varepsilon, \Sigma)} + \varepsilon \| w_1 \|^\alpha_{(A_\varepsilon, \Sigma)} \right), \tag{51}
\end{align*}
\]

and

\[
\begin{align*}
\varepsilon \| \nabla \Sigma u \|^\alpha_{(\Sigma_s, \Sigma)} &\leq C \varepsilon \| \nabla \Sigma u \|^\alpha_{(\Sigma_s, \Sigma)} \\
&\leq C \varepsilon \left( \| \partial^- u \|^\alpha_{(\Sigma_s, \Sigma)} + \| u \|^\alpha_{(\Sigma_s, \Sigma)} \right) \quad \text{(by ellipticity on } \Sigma_s) \\
&\leq C \left( \varepsilon \| u_{x_1} \|^\alpha_{(\Sigma_s, \Sigma)} + \| u \|^\alpha_{(\Sigma_s, \Sigma)} \right) \quad \text{(by \((55)\))} \\
&\leq C \left( \varepsilon^{-\alpha} \| v \|^\alpha_{(A_\varepsilon, \Sigma)} + \varepsilon^{1-\alpha} \| w_1 \|^\alpha_{(A_\varepsilon, \Sigma)} + \varepsilon \| u \|^\alpha_{(\Sigma_s, \Sigma)} \right) \tag{52}
\end{align*}
\]
then from (49), (50) we get the estimate for \( \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})} \):

\[
\varepsilon^{1-\alpha} \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + \varepsilon \| u \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} + \varepsilon^{-\alpha} \| v \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\geq C\varepsilon \alpha \| \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})}.
\]

Combining (57) about \( \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})} \), we have the full control of \( \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})} \):

\[
\varepsilon^{1-\alpha} \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + \varepsilon \| u \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} + \varepsilon^{-\alpha} \| v \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\geq C\varepsilon \alpha \| \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})}.
\]

Combining with (51) and (52) about \( \| \nabla u \|_{C^0} \), we get

\[
\varepsilon \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + \varepsilon^{1+\alpha} \| u \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} + \varepsilon \| v \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\geq C\varepsilon \alpha \| \nabla\Sigma u \|^2_{C^\alpha(\mathcal{B}_{\varepsilon/2}, \mathcal{S})}.
\]

Covering \( \mathcal{A}_\varepsilon \) by radius-\( \varepsilon \) half balls \( \mathcal{B}_\varepsilon \) and radius-\( \varepsilon/2 \) full balls centered on meridian \( \Sigma \times \{ \varepsilon/2 \} \) and then taking supremum on \( \mathcal{A}_\varepsilon \), we derive

\[
\varepsilon \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + \varepsilon^{1+\alpha} \| u \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} + \varepsilon \| v \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\geq C\varepsilon \alpha \| \nabla\Sigma u \|^2_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S})}.
\]

Notice here \( v \) is involved on the left side to control \( u \), compared to similar inequality (13) about \( v \) which needs no \( u \). This is partly because in second order elliptic equations (39) of \( u \) and \( v \), the inhomogeneous term \( \partial_{x_1} w_1 \) is “more discontinuous” than \( -\partial^w w_1 \), for the (weak) derivative is taking in \( x_1 \) direction along which (the extended) \( w_1 \) is discontinuous. Plugging these into (44) again we obtain

\[
C\varepsilon \alpha \| u \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\leq \varepsilon^{2\alpha} \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + \varepsilon^{-\alpha} \| u \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} + \varepsilon^{-1+2\alpha} \| v \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \\
\leq \varepsilon^{2\alpha} \| w_1 \|_{C^\alpha(\mathcal{A}_\varepsilon, \mathcal{S}^+)} + C(p, \lambda) \| w_1 \|_{C^0(\mathcal{A}_\varepsilon, \mathcal{S})} \varepsilon^{-\frac{3+2\alpha}{2}}.
\]

The \( C^{1,\alpha} \) estimates (55) and (45) also hold when \( \mathcal{A}_\varepsilon \) is equipped with a warped product metric \( g_{k,\lambda} \) with \( h(z) \in C^\infty(\Sigma) \) such that \( \frac{1}{K} \leq h(z) \leq K \), up to the constant factor \( K > 0 \) on the right hand sides of (55) and (45), because they are derived by \( C^0 \) estimates (44), (46), and local Schauder estimates (12), (53) for \( u, v \) on half balls \( B_\varepsilon(p) \).
and $B_{\varepsilon/2}(p)$ for $p \in \partial A_\varepsilon = \{(0,z), (\varepsilon, z) | z \in \Sigma\}$, while when $\varepsilon \to 0$, the function $h(z)|_{B_\varepsilon(p)}$ converges to constant function $h(p)$ uniformly in $C^0$ norm in $B_\varepsilon(p)$, and correspondingly the second order equations of $u, v$ converge to those with constant coefficients as above. This is similar to “frozen coefficient method” in Schauder theory of second order elliptic equations. Finally we estimate the right inverse bound of $D$ in Schauder setting. By Theorem 13 the operator $D$ is surjective, so for any $W = (w_1, w_2)$ we can write it as the sum of $(w_1, 0)$ and $(0, w_2)$, whose preimages $V = D^{-1}(w_1, 0)$ and $D^{-1}(0, w_2)$ exist. Therefore we can reduce general cases to case (i) and (ii). Combining the $C^{1, \alpha}$ estimates (34) in case (i) and (55) and (45) for $u$ and $v$ in case (ii), we have

$$C(\alpha, p, \lambda) \|V\|_{C^{1, \alpha}(A_\varepsilon, S)} \leq \varepsilon^{-(\frac{2\alpha}{p} + 2\alpha)} \|W\|_{C^\alpha(A_\varepsilon, S)},$$

where $C(\alpha, p, \lambda) = C(\alpha)(1 + C(p, \lambda))^{-1}$ is independent on $\varepsilon$. Hence the result.

**Remark 15** (About the asymmetry of case (i) and case (ii)) In case (i), since $v|_{\partial A_\varepsilon} = 0$, after odd reflections of $v$ and even reflections of $u$ and $w_2$, the $V = (u, v)$ is still a $C^{1, \alpha}$ function on the domain $A_{k(\varepsilon)}$, so we can use the Schauder estimate of the domain $A_{k(\varepsilon)}$, which has uniform ellipticity; in case (ii), since $w_1|_{\partial A_\varepsilon} \neq 0$ in general, after odd reflections of $v$ and $w_1$, $w_1$ is no longer continuous on $A_{k(\varepsilon)}$, not to mention in $C^{1, \alpha}$. So we have to directly work on the thin domain $A_\varepsilon$ (which lacks uniform ellipticity as $\varepsilon \to 0$) for the Schauder estimates, with $\varepsilon$-dependent coefficients. The steps there are from $L^p$, $C^{1-\frac{2\alpha}{p}}, C^0$ to $C^{1, \alpha}$.

We remark that when $\lambda_{\partial^+} > 0$ but $\lambda_{\partial^-} = 0$ (i.e. $\ker \partial^- \neq \{0\}$), there still exists a right inverse $Q_\varepsilon : C^\alpha(A_\varepsilon, S) \to C^{1, \alpha}_-(A_\varepsilon, S)$ of $D$ with the operator norm bound $\|Q_\varepsilon\| \leq C\varepsilon^{-(\frac{2\alpha}{p} + 2\alpha)}$. To prove this, one needs to consider the restriction

$$D : L^{1, 2}_-(A_\varepsilon, S) \cap (\ker D) \to L^2(A_\varepsilon, S)$$

to construct the right inverse $Q_\varepsilon$, where “$\perp$” is the $L^2$ orthogonal comp-
plement, and replace the constants

\[
\lambda_D^- = \inf_{0 \neq V \in L^1_1(A_{\epsilon}, S)} \frac{\|DV\|^2_{L^2(A_{\epsilon}, S)}}{\|V\|^2_{L^2(A_{\epsilon}, S)}},
\]

\[
\lambda_{\partial^-} = \inf_{0 \neq V \in L^1_1(\Sigma, S^+)} \frac{\|\partial V\|^2_{L^2(\Sigma, S^+)}}{\|V\|^2_{L^2(\Sigma, S^+)}}
\]

in Theorem 12 by the constants

\[
\tilde{\lambda}_D^- = \inf_{0 \neq V \in L^1_1(A_{\epsilon}, S) \cap (\ker D)^{\perp}} \frac{\|DV\|^2_{L^2(A_{\epsilon}, S)}}{\|V\|^2_{L^2(A_{\epsilon}, S)}},
\]

\[
\tilde{\lambda}_{\partial^-} = \inf_{0 \neq V \in L^1_1(\Sigma, S^+) \cap (\ker \partial^-)^{\perp}} \frac{\|\partial V\|^2_{L^2(\Sigma, S^+)}}{\|V\|^2_{L^2(\Sigma, S^+)}}
\]

respectively to get the \(L^2\) estimate of \(V\), and in the \(C^0\) estimate of \(u\) in case (ii) of the above theorem use the fact that

\[u \in (\ker D)^{\perp} \Rightarrow \overline{u} \in (\ker \partial^-)^{\perp}\]

by integrating in the \(x_1\) direction. The details are left to readers.

From now on we fix \(p > 3\) sufficiently large and \(0 < \alpha < 1\) sufficiently small such that

\[0 < \frac{3}{p} + 3\alpha < \frac{1}{2}\]

This will be necessary for the implicit function theorem in the next section.

4 Proof of the Main Theorem

Let \(C_0 \subset M\) be a \textbf{compact} coassociative submanifold. Suppose that \(n\) is a normal vector field on \(C_0\) such that its corresponding self-dual two form,

\[\eta_0 = \iota_n \Omega \in \Lambda^2_+ (C_0)\]

is harmonic with respect to the induced metric. So \(\eta_0\) is actually a symplectic form on the complement of the zero set \(Z(\eta_0)\) of \(\eta_0\) in \(C_0\). Furthermore

\[J_n(u) := \frac{n}{|n|} \times u\]  \hspace{1cm} (56)
defines an almost complex structure $J_n$ on the $C_0 \setminus Z (\eta_0)$. Since deformations of coassociative submanifolds are unobstructed, we may assume that there is a one parameter family of coassociative submanifolds $\varphi : [0, \varepsilon] \times C_0 \longrightarrow M$ such that

$$\frac{\partial \varphi}{\partial t} \bigg|_{t=0} = n \in \Gamma (C_0, N_{C_0/M}).$$

For $\varepsilon$ small, let

$$\mathcal{C} := [0, \varepsilon] \times C_0, \quad C := \varphi (\mathcal{C}), \quad \text{and} \quad C_t := \varphi (\{t\} \times C_0).$$

then $\mathcal{C}$ is diffeomorphic to $\mathcal{C}$, and $\varphi (t, \cdot)$ is an embedding for $\forall t \in [0, \varepsilon]$. We remind the readers about the typefaces of our notations: both $\mathcal{C}$ and $\mathcal{C}$ are 5-dimensional, while each $C_t$ is 4-dimensional. Let

$$g_C := dt^2 \oplus g|_{C_0}$$

be the product metric on $\mathcal{C}$ and $\exp^C$ be the exponential map associated to the metric $g_C$.

In the remaining part of this article we assume that $\eta_0$ is nowhere vanishing on $C_0$, that implies that $(C_0, \eta_0)$ is a symplectic four manifold, and all coassociative submanifolds $C_t$'s are mutually disjoint. We are going to establish a correspondence between the regular $J_n$-holomorphic curves $\Sigma$ in $C_0$ and the instantons in $M$ with coassociative boundary conditions.

Given such a $\Sigma \subset C_0$, we denote

$$A_\varepsilon := [0, \varepsilon] \times \Sigma \quad \text{and} \quad A'_\varepsilon := \varphi (A_\varepsilon)$$

then $A'_\varepsilon$ is close to being associative in the sense that $|\tau|_{A'_\varepsilon} \leq K \varepsilon$ for some constant $K$ depending on the geometry of the family $\{C_t\}$ and $M$ for small $\varepsilon$. This is due to the smooth dependence of $\tau (\varphi (t, z))$ on $(t, z) \in [0, \varepsilon] \times \Sigma$, and that

$$\tau|_{T A'_\varepsilon|_{\varphi (0, \Sigma)}} = \tau|_{T A'_\varepsilon|_{\Sigma}} = 0,$$

since $\Sigma$ is a $J_n$-holomorphic curve, the tangent spaces $T \Sigma$ is closed under $J_n = \frac{n}{|n|} \times$, and $T A'_\varepsilon|_{\Sigma} = T \Sigma \oplus \text{span}\{n\}$.

We want to perturb $A'_\varepsilon$ to become an honest associative submanifold in $M$. In order to apply the implicit function theorem to obtain the desired perturbation for $A'_\varepsilon$, we need the estimates for the linearized
problem to behave well as \( \varepsilon \) approaches zero. Notice that for small \( \varepsilon \), the induced metric on \( A'_\varepsilon \) is close to be a warped product metric

\[
g_{h, \varepsilon} := h(z) \, dx_1^2 + g_{\Sigma}
\]
on \( A_\varepsilon \), where \( h(z) = |n(z)|^2 \) is the length squared of the normal vector \( n \). Namely

\[
(1 - K \varepsilon) \, g_{h, \varepsilon} \leq \varphi^* g_M \leq (1 + K \varepsilon) \, g_{h, \varepsilon}
\]

for some uniform constant \( K \). This is because for any vectors \( X, Y \in T_{t, z} A_\varepsilon \) and \( t \in [0, \varepsilon_0] \), the function \( G : [0, \varepsilon_0] \times TA_{\varepsilon_0} \times TA_{\varepsilon_0} \to \mathbb{R} \),

\[
G(t, X, Y) := g_M (d\varphi(t, z) X, d\varphi(t, z) Y)
\]
is smooth with respect to \((t, X, Y)\), bilinear in \((X, Y)\), and

\[
\begin{align*}
G(0, TA_{\varepsilon_0}|(0) \times \Sigma, TA_{\varepsilon_0}|(0) \times \Sigma) \\
= g_M \left( d\varphi|_{(0) \times \Sigma} (TA_{\varepsilon_0}|(0) \times \Sigma), d\varphi|_{(0) \times \Sigma} (TA_{\varepsilon_0}|(0) \times \Sigma) \right) \\
= g_{h_0, \varepsilon_0, h|(0) \times \Sigma}
\end{align*}
\]

where in the last identity we have used that \( d\varphi|_{(0) \times \Sigma} = id : T\Sigma \to T\Sigma \) and \( d\varphi|_{(0) \times \Sigma} : \frac{\partial}{\partial x_1} \to n \).

For the warped product metric on \( A_\varepsilon \) and the corresponding operator \( D \), the estimates for its inverse have been established in Theorem 14 in the previous section. The above discussion indicates that the linearization of the instanton equation on \( A'_\varepsilon \) may be compared with \( D \). We will first show they agree on \( \Sigma \) in the next subsection. Then the comparison on \( A'_\varepsilon \) is a small perturbation from their agreement on \( \Sigma \).

### 4.1 Geometry of \( \Sigma \subset C \subset M \)

We study the geometry of the \( J_n \)-holomorphic curves \( \Sigma \) in a \( G_2 \)-manifold \( M \). Let \( C = \bigcup_{0 \leq t \leq \varepsilon} C_t \) be the family of coassociative manifolds in the previous subsection and \( \Sigma \subset C = C_0 \). We remind the readers that the \( C \) is a 5-dimensional submanifold so its normal bundle in \( M \) has real rank 2, while \( C \) is a 4-dimensional submanifold. We recall the following results in Lemma 3.2 of [17].

**Proposition 16** 1. \( N_{\Sigma/C} \) and \( N_{C/M}|_{\Sigma} \) are complex line bundles with the almost complex structure \( J_n = \frac{\partial}{\partial n} \times \) on fibers.
2. $N_{C/M}|_{\Sigma} \simeq \wedge^0_1 (N_{\Sigma/C})$ as complex line bundles, where $L$ is the conjugate of a complex line bundle $L$, and $\wedge^0_1 (N_{\Sigma/C}) = N_{\Sigma/C} \otimes \wedge^0_1 (T^*\Sigma)$.

Note that the notations $\partial Y, X, \nu_X$ and $\mu_X$ in [17] correspond to our $\Sigma, C, N_{\Sigma/C}$ and $N_{C/M}|_{\Sigma}$ respectively.

In the following we will use the notation $N_{C/M}|_{\Sigma}$ rather than $\overline{N_{C/M}|_{\Sigma}}$, but we emphasize that the isomorphism $N_{C/M}|_{\Sigma} \simeq \wedge^0_1 (N_{\Sigma/C})$ is complex conjugate linear and should be regarded as an isomorphism between real vector bundles. Note that we do NOT complexify $N_{\Sigma/C}$, since it is already a complex line bundle, and more importantly, we want to use its complex structure $J_n = n \times$ to interplay with the $G_2$ geometry.

In the following Proposition we will show that $N_{\Sigma/C}$ and $N_{C/M}|_{\Sigma}$ are Hermitian line bundles, and $N := N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma} \to \Sigma$ is a Dirac bundle (in the sense of Definition 5.2 in [23]), i.e. $N$ is a left Clifford module over $\Sigma$ with respect to the $G_2$ multiplication $\times$, together with a Riemannian metric $\langle, \rangle$ and connection $\nabla^N$ on $N$ satisfying: (a) at each $p \in \Sigma$, for any $\sigma_1, \sigma_2 \in N_p$ and any unit vector $e \in T_p\Sigma$, $\langle e\sigma_1, e\sigma_2 \rangle = \langle e\sigma_1, e\sigma_2 \rangle$; (b) for any section $\varphi$ of $T\Sigma$ and section $\sigma$ of $N$, $\nabla^N (\varphi \times \sigma) = \left(\nabla^T\Sigma \varphi\right) \times \sigma + \varphi \times \nabla^N \sigma$.

**Proposition 17** Let $\nabla$ be the Levi-Civita connection of $(M, g)$. For any subbundle $L \to \Sigma_0$ of $TM|_{\Sigma_0}$, let the induced connection $\nabla^L$ of $L$ be

$$\nabla^L := \pi^L \circ \nabla,$$

where $\pi^L$ is the orthogonal projection of $TM|_{\Sigma}$ to subbundle $L$ according to the metric $g$. Then

1. For $L = N_{\Sigma/C}$, $N_{C/M}|_{\Sigma}$ or $T\Sigma$, the induced connection $\nabla^L$ is Hermitian, namely $\nabla^L J_n = 0$.

2. Let $N = N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$. Then with respect to the $G_2$ multiplication $\times$ as the Clifford multiplication and the induced connection $\nabla^N$, $N$ is a Dirac bundle.

**Proof.** For each $p \in \Sigma_0$, the subspace $T_p\Sigma_0 \oplus \text{span}\{n(p)\} \subset T_pM$ is associative since $\Sigma_0$ is $J_n$-holomorphic. By the same Cayley-Dickson construction in Theorem [9] we may choose orthonormal frame $\{W_\alpha\}_{\alpha=1,\ldots,7}$
in a neighborhood $B_\varepsilon(p) \subset \Sigma$ of $p$ satisfying standard $\cdot, \times$ relation as the basis of $\text{Im} \mathcal{O}$, such that

$$T\Sigma_0 = \text{span}\{W_2, W_3\}, \quad W_1 := W_2 \times W_3,$$

$$W_4 \in N_{\Sigma_0/C_0}, \quad W_{i+4} := W_i \times W_4 \text{ for } 1 \leq i \leq 3,$$

$$N_{\Sigma_0/C_0} = \text{span}\{W_4, W_5\}, \quad \text{and } N_{C/M}|_{\Sigma_0} = \text{span}\{W_6, W_7\}.$$  

For $i = 2, 3$, we have

$$\nabla_{W_i}^L J_n = \nabla_{W_i}^L (W_1 \times |L) = \pi^L (\nabla_{W_i}^L W_1) \times |L = \langle W_1, \nabla_{W_i}^L W_1 \rangle W_1 \times |L = 0,$$

where the third identity is because for $L = T\Sigma = \text{span}\{W_2, W_3\}$, $L = N_{\Sigma_0/C} = \text{span}\{W_4, W_5\}$ or $L = N_{C/M}|_{\Sigma_0} = \text{span}\{W_6, W_7\}$, under cross product $\times$, only the $W_1$ component of $\nabla_{W_i}^L W_1$ can preserve $L$, i.e.

$$W_1 \times L \subset L,$$  

and $(W_k \times L) \cap L = \{0\}$ for $2 \leq k \leq 7$

from octonion multiplication relation, and the last identity is because $\langle W_1, \nabla_{W_i}^L W_1 \rangle = \nabla_{W_i}^L (W_1, W_1) = 0$. Thus $\nabla_{W_i}^L J = 0$ for any $v \in T\Sigma_0$. Using the standard $G_2$ multiplication relation of the above “good” frame $\{W_\alpha\}_{\alpha=1,\ldots,7}$, it is easy to check that $N := N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$ is a Clifford module. To show $N$ is a Dirac bundle, we first note that for any unit vector $e \in T_p\Sigma$, $e \times$ is an isometry on $N$ by the property of $\times$; second, for any section $\varphi$ of $T\Sigma$ and section $\sigma$ of $N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$, with respect to the induced connection $\nabla^N$ on $N$ we have

$$\nabla^N (\varphi \times \sigma) = \pi^N \circ (\nabla^N \varphi \times \sigma + \varphi \times \nabla^N \sigma) = (\nabla^{T\Sigma} \varphi) \times \sigma + \varphi \times \nabla^N \sigma,$$

where in the second identity, the first term is because for distinct $i, j \in \{4, 5, 6, 7\}$, $W_i \times W_j \in \text{span}\{W_1, W_2, W_3\}$, and the second term is because $\text{span}\{W_1, W_2, W_3\}$ is closed under $\times$.  

It is well-known that on a compact Kähler manifold with a Hermitian line bundle $L$, one can define a Dirac operator on the Dolbeault complex $\Omega^{0,*}_C(L)$ (c.f. Proposition 3.67 of [6] or Proposition 1.4.25 of [31]). Taking the Kähler manifold to be $\Sigma$ and the Hermitian line bundle to be $N_{\Sigma/C}$, we have the Dolbeault Dirac operator on $N_{\Sigma/C} \oplus \Lambda^0_C(N_{\Sigma/C})$. On the other hand, by the above Lemma, $N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$ is a Dirac bundle and has a canonically associated Dirac operator. The following Proposition compares the two Dirac operators.
Proposition 18 The Dolbeault Dirac operator on \( N_{\Sigma/C} \oplus \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) \) agrees with the Dirac operator on \( N_{\Sigma/C} \oplus N_{C/M|\Sigma} \), whose Clifford multiplication is the \( G_2 \) multiplication \( \times \) and connection is the induced connection from \( M \).

Proof. Given any \( p \in \Sigma \), we may further assume the “good” frame \( \{ W_\alpha \}_{\alpha=1,2,\ldots,7} \) on \( B_\varepsilon (p) \) in Proposition 17 satisfies
\[
\nabla^T_{W_i} W_j (p) = \nabla^{N_{\Sigma/C}}_{W_i} W_k (p) = \nabla^{N_{C/M|\Sigma}}_{W_i} W_{k+2} (p) = 0 \quad (57)
\]
for \( 2 \leq i,j \leq 3 \) and \( 4 \leq k \leq 5 \), where the \( \nabla^{T\Sigma_0}, \nabla^{N_{\Sigma/C}0} \) and \( \nabla^{N_{C/M|\Sigma}} \) are orthogonal projections of the Levi-Civita connection \( \nabla \) on \( M \) to \( T\Sigma_0, N_{\Sigma/C}0 \) and \( N_{C/M|\Sigma} \) respectively (with respect to the metric \( g \)).

To achieve (57), we can first require \( \nabla^{\Sigma_0} W_i W_j (p) = \nabla^{N_{\Sigma/C}0} W_i W_k (p) = \nabla^{N_{C/M|\Sigma}} W_i W_k + 2 \) for \( 2 \leq i,j \leq 3 \), and then use \( W_{i+4} := W_i \times W_4 \) for \( 1 \leq i \leq 3 \) to show all other covariant derivatives in (57) vanish at \( p \). Let \( Z = \frac{1}{\sqrt{2}} (W_2 - iW_3) \in T\Sigma_{C}^{1,0}, \quad \overline{Z} = \frac{1}{\sqrt{2}} (W_2 + iW_3) \in T\Sigma_{C}^{0,1} \) and \( \overline{Z}^* = \frac{W_2 - iW_3}{\sqrt{2}} \in T^*\Sigma_{C}^{0,1} \), then \( \langle \overline{Z}^*, Z \rangle = 1 \). We recall that the real bundle isomorphism \( f : \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) \simeq N_{C/M|\Sigma} \) from [17] (up to factor \( -\frac{1}{\sqrt{2}} \)) in local frame was given by
\[
\begin{align*}
\quad f : \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) = W_4 \otimes_C T^*\Sigma_{C}^{0,1} & \quad \rightarrow \quad W_4 \times T\Sigma_{C}^{1,0} = N_{C/M|\Sigma}, \\
W_4 \otimes_C \overline{Z}^* & \quad \mapsto \quad -\frac{1}{\sqrt{2}} W_4 \times Z,
\end{align*}
\]
which is complex conjugate linear and independent on the choice of \( W_4 \in N_{\Sigma/C} \). So we have the map
\[
\Phi := id \oplus f : N_{\Sigma/C} \oplus \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) \rightarrow N_{\Sigma/C} \oplus N_{C/M|\Sigma}
\]
whose action on the basis of (rank two real vector bundles) \( N_{\Sigma/C} \) and \( \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) \) is the following:
\[
\begin{align*}
\Phi : N_{\Sigma/C} & \xrightarrow{id} N_{\Sigma/C}, \quad \{ W_4, W_5 \} \rightarrow \{ W_4, W_5 \}, \\
\Phi : \Lambda^0_{\Sigma/C} (N_{\Sigma/C}) & \xrightarrow{f} N_{C/M|\Sigma}, \quad \{ W_4 \otimes \overline{Z}^*, W_5 \otimes \overline{Z}^* \} \rightarrow \{ W_6, W_7 \}.
\end{align*}
\]

(59)
This is because under $f$,

\[
W_4 \otimes \left( \frac{W_2^* - iW_3^*}{\sqrt{2}} \right) \rightarrow -\frac{1}{\sqrt{2}} \frac{W_4 \times W_2 - W_4 \times iW_3}{\sqrt{2}} = -\frac{W_6 - (W_1 \times W_4) \times W_3}{2} = W_6,
\]

\[
W_5 \otimes \left( \frac{W_2^* - iW_3^*}{\sqrt{2}} \right) = J_n W_4 \otimes \left( \frac{W_2^* - iW_3^*}{\sqrt{2}} \right)
\]

\[
\rightarrow -J_n (W_6) = -W_1 \times W_6 = W_7,
\]

where the second row is from the tensor property of $f$ and the last row is because $f : \bigwedge^{0,1}_{\mathbb{C}} (N_{\Sigma/C}) \rightarrow N_{C/M}|_{\Sigma}$ is complex conjugate linear. So for the sections

\[U = V^4 W_4 + V^5 W_5 + V^6 W_4 \otimes \bar{Z}^* + V^7 W_5 \otimes \bar{Z}^* \in \Gamma \left( N_{\Sigma/M} \oplus \bigwedge^{0,1}_{\mathbb{C}} (N_{\Sigma/C}) \right),\]

\[V = V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7 \in \Gamma \left( N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma} \right),\]

we have

\[\Phi (U) = V.\]

We compute the Dolbeault Dirac operator using the above frame. The Clifford multiplication $c$ on the Dolbeault complex $N_{\Sigma/C} \oplus \bigwedge^{0,1}_{\mathbb{C}} (N_{\Sigma/C})$ at $p$ is (see Section 1.4.3 in [31])

\[c \left( \frac{W_2 + iW_3}{\sqrt{2}} \right) : V^4 W_4 + V^5 W_5 \rightarrow \sqrt{2} \left( V^4 W_4 \otimes \bar{Z}^* + V^5 W_5 \otimes \bar{Z}^* \right),\]

\[c \left( \frac{W_2 - iW_3}{\sqrt{2}} \right) : V^6 W_4 \otimes \bar{Z}^* + V^7 W_5 \otimes \bar{Z}^* \rightarrow \sqrt{2} \left( -V^6 W_4 - V^7 W_5 \right),\]

where $e \left( \bar{Z}^* \right)$ is the wedge by $\bar{Z}^*$ and $i \left( \bar{Z} \right)$ is the contraction by $\bar{Z}$. Using (57) at $p$ and complex linearity, we have

\[
\nabla \left( \frac{W_2 + iW_3}{\sqrt{2}} \right) \left( V^4 W_4 + V^5 W_5 \right) (p)
\]

\[
= \frac{1}{\sqrt{2}} \left( V^4_2 W_4 + V^4_3 J_n W_4 + V^5_2 W_5 + V^5_3 J_n W_5 \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( (V^4_2 - V^5_3) W_4 + (V^5_2 + V^4_3) W_5 \right)
\]

and

\[
c \left( \frac{W_2 + iW_3}{\sqrt{2}} \right) \circ \nabla \left( \frac{W_2 + iW_3}{\sqrt{2}} \right) \left( V^4 W_4 + V^5 W_5 \right) (p)
\]

\[
= (V^4_2 - V^5_3) W_4 \otimes \bar{Z}^* + (V^5_2 + V^4_3) W_5 \otimes \bar{Z}^*.
\]
Similarly we have
\[
\nabla \left( \frac{w_2 - iw_3}{\sqrt{2}} \right) \left( V^6 W_4 \otimes \overline{Z^*} + V^7 W_5 \otimes \overline{Z^*} \right) (p)
\]
\[=
\frac{1}{\sqrt{2}} \left[ (V^6_2 + V^7_3) W_4 \otimes \overline{Z^*} + (-V^6_3 + V^7_2) W_5 \otimes \overline{Z^*} \right]
\]
and
\[
c \left( \frac{W_2 - iW_3}{\sqrt{2}} \right) \circ \nabla \left( \frac{w_2 - iw_3}{\sqrt{2}} \right) \left( V^6 W_4 \otimes \overline{Z^*} + V^7 W_5 \otimes \overline{Z^*} \right) (p)
\]
\[=
-(V^6_2 + V^7_3) W_4 - (-V^6_3 + V^7_2) W_5.
\]

The Dolbeault Dirac operator \( \overline{\partial} \) on \( N_{\Sigma/M} \oplus \wedge^{0,1}_{\Sigma} (N_{\Sigma/M}) \) is defined by
\[
\overline{\partial} = c \left( \frac{W_2 + iW_3}{\sqrt{2}} \right) \circ \nabla \left( \frac{w_2 + iw_3}{\sqrt{2}} \right) + c \left( \frac{W_2 - iW_3}{\sqrt{2}} \right) \circ \nabla \left( \frac{w_2 - iw_3}{\sqrt{2}} \right),
\]
so at \( p \) we have
\[
\overline{\partial} U (p) = -(V^6_2 + V^7_3) W_4 + (V^6_3 - V^7_2) W_5
\]
\[+
(V^4_2 - V^5_3) W_4 \otimes \overline{Z^*} + (V^5_2 + V^4_3) W_5 \otimes \overline{Z^*},
\]
and
\[
\Phi \left( \overline{\partial} U (p) \right) = -(V^6_2 + V^7_3) W_4 + (V^6_3 - V^7_2) W_5
\]
\[+
(V^4_2 - V^5_3) W_6 + (V^5_2 + V^4_3) W_7.
\]
On the other hand, the twisted Dirac operator on \( N := N_{\Sigma/C} \oplus N_{C/M} |_{\Sigma} \) is
\[
D := W_2 \times \nabla^N_{W_2} + W_3 \times \nabla^N_{W_3}.
\]
Using (57) at \( p \), we have
\[
DV (p) = (W_2 \times \nabla^N_{W_2} + W_3 \times \nabla^N_{W_3}) (V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7)
\]
\[=
(V^4_2 W_2 \times W_4 + V^5_3 W_3 \times W_4) + (V^5_2 W_2 \times W_5 + V^4_3 W_3 \times W_5)
\]
\[+
(V^6_2 W_2 \times W_6 + V^5_3 W_3 \times W_6) + (V^7_2 W_2 \times W_7 + V^5_3 W_3 \times W_7)
\]
\[=
-(V^6_2 + V^7_3) W_4 + (V^6_3 - V^7_2) W_5
\]
\[+
(V^4_2 - V^5_3) W_6 + (V^4_2 + V^5_3) W_7.
\]
Therefore
\[
\Phi \left( \overline{\partial} U (p) \right) = DV (p).
\]
Note that Dirac operators are independent on the choice of orthonormal basis at \( p \), and our \( p \in \Sigma \) is arbitrary, so \( \Phi (\overline{\partial}U) = DV \) on \( \Sigma \). The proof of the proposition is completed.

We now make connection to Subsection 3.1. Taking \( L = N_{\Sigma/C} \) in that subsection, then \( S^+ = N_{\Sigma/C} \) and \( S^- = \Lambda^{0,1}_C (N_{\Sigma/C}) \). Our assumption that \( \Sigma \) is regular implies \( \dim H^{0,1}_D (N_{\Sigma/C}) = 0 \), for \( H^{0,1}_D (N_{\Sigma/C}) \) corresponds to the cokernel of \( \overline{\partial} \). By the Dolbeault isomorphism, we have

\[
\dim H^1 (\Sigma, N_{\Sigma/C}) = \dim H^{0,1}_D (N_{\Sigma/C}) = 0.
\]

Since the dimension for the Seiberg-Witten moduli is 0, by the equivalence to Gromov-Witten moduli we have

\[
\dim H^0 (\Sigma, N_{\Sigma/C}) = \dim H^1 (\Sigma, N_{\Sigma/C}) = 0,
\]

for we only count \( J_n \)-holomorphic curves \( \Sigma \) of index 0. Hence for

\[
\overline{\partial} : \Omega^0 (N_{\Sigma/C}) \to \Omega^{0,1} (N_{\Sigma/C})
\]

\[
\overline{\partial}^* : \Omega^{0,1} (N_{\Sigma/C}) \to \Omega^0 (N_{\Sigma/C})
\]

in Subsection 3.1 ker \( \overline{\partial} \) and ker \( \overline{\partial}^* \) are trivial. Moreover by Theorem 13 the linear operator

\[
\mathcal{D} : L^{1,2}_{-} (A_\epsilon, \mathbb{S}) \to L^2 (A_\epsilon, \mathbb{S})
\]

is one-to-one and onto.

### 4.2 Linearization of Instanton Equation and Comparison with the Operator \( \mathcal{D} \)

To prove the main theorem (Theorem 27), we will construct a map

\[
F_\epsilon : C^{-m,\alpha} (N_{A_\epsilon/M}) \to C^{m-1,\alpha} (N_{A_\epsilon/M})
\]

such that the solution to the equation \( F_\epsilon (V) = 0 \) will give rise to an associative submanifold (instanton) with boundary lying on \( C_0 \cup C_\epsilon \). The spaces \( C^{m,\alpha} (N_{A_\epsilon/M}) \) and \( C^{-m,\alpha} (N_{A_\epsilon/M}) \) are defined by

\[
C^m (N_{A_\epsilon/M}) := \left\{ V \in \Gamma (N_{A_\epsilon/M}) \mid \|V\|_{C^m (N_{A_\epsilon/M})} < +\infty \right\},
\]

\[
C^{-m} (N_{A_\epsilon/M}) := \left\{ V \in \Gamma (N_{A_\epsilon/M}) \mid \|V\|_{C^{-m} (N_{A_\epsilon/M})} < +\infty, \text{ and } V|_{\varphi (\{0\} \times \Sigma)} \subset TC_0, V|_{\varphi (\{\epsilon\} \times \Sigma)} \subset TC_\epsilon. \right\}.
\]
We construct a three dimensional submanifold $A'_\varepsilon = \varphi (A_\varepsilon) \subset M$ by flowing $\Sigma$ along with $C_t$. For the spinor bundle $S \to A_\varepsilon$, we will construct an exponential-like map $\widetilde{\exp} : S \to M$, with the following properties of the differential $d\widetilde{\exp}|_{A'_\varepsilon}$ on $\{0\} \times \Sigma$:

1. On fiber directions of $S$, $d\widetilde{\exp}|_{\{0\} \times \Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge^0 \wedge^1 (N_{\Sigma/C}) \to N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$, where $f : \wedge^0 \wedge^1 (N_{\Sigma/C}) \to N_{C/M}|_{\Sigma}$ is the real vector bundle isomorphism in (58);

2. On base directions of $S$, $d\widetilde{\exp}|_{\{0\} \times \Sigma} = id : T\Sigma \to T\Sigma$ and $d\widetilde{\exp}|_{\{0\} \times \Sigma} : \frac{\partial}{\partial x_1} \to n(z)$;

3. On the boundary $\{0, \varepsilon\} \times \Sigma$ of $A_\varepsilon$, $\widetilde{\exp}|_{\{0\} \times \Sigma} (S^+ \oplus 0) \subset C_0$, and $\widetilde{\exp}|_{\{\varepsilon\} \times \Sigma} (S^+ \oplus 0) \subset C_\varepsilon$.

The construction of the map $\widetilde{\exp}$ is somewhat technical. To keep the main flow of our paper, we postpone it to the appendix.

To make the linear theory developed in the previous section applicable, we compare the linearization $DF_\varepsilon(0)$ with the operator $D$ in Section 3.1 by the following diagram to get $\varepsilon$-dependent bound of its right inverse:

$$
\begin{array}{ccc}
C^m_{-\alpha}(A_\varepsilon, S) & \xrightarrow{DF_\varepsilon(0)} & C^m_{-1,\alpha}(A_\varepsilon, S) \\
d\widetilde{\exp} \downarrow & & \downarrow d\widetilde{\exp} \\
C^m_{-\alpha}(N_{A'_\varepsilon}/M) & \xrightarrow{D} & C^m_{-1,\alpha}(N_{A'_\varepsilon}/M)
\end{array}
$$

Now we define the nonlinear map $F_\varepsilon$ with the important property that elements in $F_\varepsilon^{-1}(0)$ with small norm correspond to associative submanifolds in $M$ near $A'_\varepsilon$ for small $\varepsilon$. Given any $C_0 \cup C_\varepsilon$, we modify the metric $g$ near $C_0$ and $C_\varepsilon$ to make them totally geodesic. We denote this new smooth metric by $g_\varepsilon$, and we make $g_\varepsilon$ $C^1$-continuously depend on $\varepsilon$ in our construction. Let $\exp^{g_\varepsilon}$ be the exponential map of $g_\varepsilon$. Then $\exp^{g_\varepsilon}$ has the following properties:

1. For sections $V$ of $N_{A'_\varepsilon}/M$ with $C^0$ norm smaller than a fixed constant $\delta_0$ (depending on the uniform injectivity radius of the family of metrics $\{g_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$),

$$
\exp^{g_\varepsilon} V : A'_\varepsilon \to M
$$

is a smooth embedding.
2. For $V \in C_{m,\alpha}^m(NA_{\varepsilon}/M)$, let
\[
A_{\varepsilon}(V) := (\exp^{g_{\varepsilon}} V)(A'_{\varepsilon}).
\] (62)

Then $A_{\varepsilon}(V)$ is a submanifold of $M$ nearby $A'_{\varepsilon}$ satisfying the boundary condition
\[
\partial A_{\varepsilon}(V) \subset C_0 \cup C_{\varepsilon}.
\] (63)

3. For small $t \geq 0$, the family of embeddings $\exp^{g_{\varepsilon}}(tV) : A'_{\varepsilon} \to M$ satisfies
\[
\left. \frac{d}{dt} \right|_{t=0} \exp^{g_{\varepsilon}}(tV) = V.
\] (64)

Next we define $F_{\varepsilon} : C_{m,\alpha}^m(A'_{\varepsilon}, NA_{\varepsilon}/M) \to C_{m-1,\alpha}^m(A'_{\varepsilon}, NA_{\varepsilon}/M)$,
\[
F_{\varepsilon}(V) = *_{A'_{\varepsilon}} \circ \perp_{A'_{\varepsilon}} \circ (T_V \circ (\exp^{g_{\varepsilon}} V)^* \tau),
\] (65)

where

1. $(\exp^{g_{\varepsilon}} V)^*$ pulls back the differential form part of $\tau$,

2. $T_V : T_{\exp^{g_{\varepsilon}}(tV)}M \to T_pM$ pulls back the vector part of $\tau$ by the parallel transport with respect to $g$ along the path $\exp^{g_{\varepsilon}}(tV)$,

3. $\perp_{A'_{\varepsilon}} : TM|_{A'_{\varepsilon}} \to NA_{\varepsilon}/M$ is the orthogonal projection with respect to $g$,

4. $*_{A'_{\varepsilon}} : \Omega^3(A'_{\varepsilon}) \to \Omega^0(A'_{\varepsilon})$ is the quotient by the volume form $dvol_{A'_{\varepsilon}}$ induced from $g$.

We stress that $g_{\varepsilon}$ is only used to construct a map $\exp^{g_{\varepsilon}} : NA_{\varepsilon}/M \to M$ satisfying the coassociative boundary condition (63) and derivative condition (64). For our covariant derivatives, parallel transport, orthogonal projection and volume form, we still use the original metric $g$.

To better understand $F_{\varepsilon}$, we let
\[
P_{\varepsilon} := *_{A'_{\varepsilon}} \circ \perp_{A'_{\varepsilon}} : \Gamma(TM|_{A'_{\varepsilon}}) \otimes \Omega^3(A'_{\varepsilon}) \to \Gamma(NA_{\varepsilon}/M),
\]
and
\[
F^{g_{\varepsilon}}(V) := T_V \circ (\exp^{g_{\varepsilon}} V)^* \tau.
\] (66)

Then
\[
F_{\varepsilon}(V) = P_{\varepsilon} \circ F^{g_{\varepsilon}}(V).
\]
At any $p \in A'$, both $\perp_{A'}$ and $\ast_{A'}$ are linear operators between finite dimensional spaces, so $\|P_\varepsilon\| \leq C$, where the constant $C$ only depends on $\varphi$ and is uniform for all $0 < \varepsilon \leq \varepsilon_0$. We notice that $P_\varepsilon$ does not involve $V$ so the essential part of $F_\varepsilon (V)$ is $F^g_\varepsilon (V)$.

By Proposition 5 if $A'$ is sufficiently close to being associative, then there exists $\delta > 0$, such that for $\|V\|_{C_{-}^{1, \alpha} (A'_{\varepsilon}, N_{A'/M})} < \delta$,

$$F_\varepsilon (V) = 0 \iff F^g_\varepsilon (V) = 0 \iff A_\varepsilon (V) \text{ associative.}$$

We also note that $\partial A_\varepsilon (V) \subset C_0 \cup C_\varepsilon$ for $\varepsilon \in C_{-}^{1, \alpha} (A'_{\varepsilon}, N_{A'/M})$.

Before we compute $F^g_\varepsilon' (0) V$, we observe the following useful fact. The original exponential map $\exp^g : N_{A'/M} \to M$ does not satisfy the coassociative boundary condition (63), but satisfies the same derivative condition (64):

$$\frac{d}{dt} \bigg|_{t=0} \exp^g (tV) = V = \frac{d}{dt} \bigg|_{t=0} \exp^g_\varepsilon (tV) .$$

For smooth maps $f : A'_{\varepsilon} \to M$ and a smooth form $\tau$, the nonlinear map $\Gamma : f \to f^* \tau$ is differentiable with respect to $f$. Therefore

$$\frac{d}{dt} \bigg|_{t=0} (\exp^g (tV))^* \tau = \Gamma' (0) V = \frac{d}{dt} \bigg|_{t=0} (\exp^g_\varepsilon (tV))^* \tau. \quad (67)$$

Recall the $F (V)$ (6) defined in our proof of McLean’s theorem is

$$F (V) := T_V \circ (\exp^g V)^* \tau,$$

where the parallel transport $T_V$ is with respect to $g$ along the geodesic $\exp^g (tV)$. $F^g_\varepsilon (V)$ and $F (V)$ are different nonlinear maps, but (67) says that

$$F^g_\varepsilon' (0) V = F' (0) V. \quad (68)$$

We give an alternative proof of (68) in the following Lemma 19. The proof gives more information of $F^g_\varepsilon' (0) V$ when there is a good frame field for the vector-valued form $\tau$.

**Lemma 19** For any $V \in C_{-}^{1, \alpha} (N_{A'/M})$, $F^g_\varepsilon' (0) V = F' (0) V$.

**Proof.** For any $p \in A'_{\varepsilon}$, we choose a frame field $\{W_a\}_{1 \leq a \leq 7}$ in its neighborhood $B \subset A'_{\varepsilon}$, and then extend $V$ and $\{W_a\}_{1 \leq a \leq 7}$ to $M$ by the parallel transport with respect to $g$ along the curve $\exp^g_p (tV)$. We
write \( \tau = \omega^\alpha \otimes W_\alpha \) in the neighborhood of \( p \) in \( M \), following Einstein’s summation convention. Similar to (12), we compute

\[
F_{g^\varepsilon} (0) V = d (i_V \omega^\alpha) \otimes W_\alpha + i_V d \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha,
\]

where the covariant derivative \( \nabla \) is with respect to \( g \), and we have used that \( \frac{d}{dt}|_{t=0} \exp^{g^\varepsilon} (tV) = V \). By the parallel property of \( \tau \) and \( W_\alpha \) with respect to \( g \), we have

\[
i_V d \omega^\alpha \otimes W_\alpha = 0 = \omega^\alpha \otimes \nabla_V W_\alpha
\]

(also see Remark 10 item 3). The first term \( d (i_V \omega^\alpha) \big|_{A'_\varepsilon} \) only depends on the restriction of \( V \) and \( \omega^\alpha \) on \( A'_\varepsilon \). (Remark 10 item 1). Therefore

\[
F_{g^\varepsilon} (0) V = d (i_V \omega^\alpha) \otimes W_\alpha \big|_{A'_\varepsilon} = F' (0) V.
\]

\textbf{Remark 20} To apply the implicit function theorem to \( F_{\varepsilon} (V) \), in the remaining part of our paper we will only need the estimate of \( F'_{\varepsilon} (0) \), and the quadratic estimate (80) of \( F'_{\varepsilon} (V) \). Because of the above lemma, to compute \( F'_{\varepsilon} (0) = P_\varepsilon \circ F_{g^\varepsilon} (0) \), we can replace the metric \( g_\varepsilon \) by \( g \) in (66). This will simplify the exposition in many places. For our quadratic estimate (80), the proof uses no feature of the \( G_2 \) metric \( g \) and is valid for any Riemannian metrics, including \( g_\varepsilon \). The constant \( C \) in (80) is uniform for \( \{ g_\varepsilon \}_{0 \leq \varepsilon \leq \varepsilon_0} \), since this is a compact family of metrics \( C^1 \)-continuously depending on \( \varepsilon \). So although we used \( g_\varepsilon \) in the definition of \( F_{\varepsilon} (V) \) (65), from now on we will pretend \( g_\varepsilon \) is \( g \) in \( F_{\varepsilon} (V) \), and we will simply write \( \exp^g \) as \( \exp \).

\textbf{Proposition 21} For any section \( V_1 \) of \( S \) and section \( V_2 := d \exp \cdot V_1 \) of \( N_{A'_\varepsilon}/M \), we have

\[
\| F' (0) V_2 - (d \exp) \circ D V_1 \otimes d \text{vol}_{A'_\varepsilon} \|_{C^\alpha (A'_\varepsilon, N_{A'_\varepsilon}/M)} \leq C \varepsilon^{1-\alpha} \| V_1 \|_{C^{1,\alpha} (A_\varepsilon, S)},
\]

and

\[
\| F'_{\varepsilon} (0) V_2 - (d \exp) \circ D V_1 \|_{C^\alpha (A'_\varepsilon, N_{A'_\varepsilon}/M)} \leq C \varepsilon^{1-\alpha} \| V_1 \|_{C^{1,\alpha} (A_\varepsilon, S)},
\]

where \( D \) is the operator on \( S \) in linear model, and the constant \( C \) is uniform for all \( \varepsilon \).
Proof. For each $p = \varphi (0, z)$ in $\Sigma_0 := \varphi \{(0) \times \Sigma\} \subset A'_\varepsilon$, we choose “good” frame $\{W_\alpha\}_{\alpha=1,2,\ldots,7}$ on $B_\varepsilon (z) \subset \Sigma$ as before. Then we extend the frame to $U_\varepsilon (p) := \varphi ([0,\varepsilon] \times B_\varepsilon (z))$ such that

$$W_\alpha (\varphi (t, z)) = T_\gamma \cdot W_\alpha (\varphi (0, z)) \quad \text{for} \quad \alpha = 1, 2, \ldots, 7,$$

where $T_\gamma$ is the parallel transport along the path $\gamma := \varphi ([0, t] \times \{z\})$ by the Levi-Civita connection on $M$. We further extend $\{W_\alpha\}_{\alpha=1,2,\ldots,7}$ to a tubular neighborhood of $U_\varepsilon (p) \subset M$ by parallel transport in fiber directions of $N_{A'_\varepsilon}/M$. Both $F' (0) V_2$ and $DV_1$ are globally defined on $A'_\varepsilon$ and $A_\varepsilon$ respectively. We are going to compare them in $U_\varepsilon (p)$ using the “good” frame. We write $\tau = \omega^\alpha \otimes W_\alpha$ in the neighborhood of $p$. By the parallel property of $\{W_\alpha\}_{\alpha=1,2,\ldots,7}$, the 3-forms $\omega^\alpha$ are similar to the standard ones in $\text{Im} \mathcal{O}$, in the sense that $d\tau_i$ is replaced by $(W_i)^\ast$ for $i = 1, 2, \ldots, 7$. We still have the formula (12) at $q$:

$$F' (0) V_2 (q) = d (i_{V_2} \omega^\alpha) \otimes W_\alpha (q) + i_{V_2} d\omega^\alpha \otimes W_\alpha (q) + \omega^\alpha \otimes \nabla_{V_2} W_\alpha (q).$$

The first term $d (i_{V_2} \omega^\alpha) \otimes W_\alpha (q)$ is the principal symbol part of the differential operator $F' (0)$. We claim that

$$d (i_{V_2} \omega^\alpha) \otimes W_\alpha = (d\text{exp} \cdot DV_1) \otimes d\text{vol}_{A'_\varepsilon} + E (q) V_1$$

where $E (q)$ is a smooth tensor on $U_\varepsilon (p)$ with $\|E\|_{C^1(U_\varepsilon (p))} \leq C_3 \varepsilon$. To see this, we first compute $d (i_{V_2} \omega^\alpha) \otimes W_\alpha$. For section $V_2$ of $N_{A'_\varepsilon}/M$ we write

$$V_2 (q) = \Sigma_{\alpha=1}^7 \phi^\alpha (q) W_\alpha (q).$$

Similar to our proof of McLean’s theorem, we have

$$d (i_{V_2} \omega^\alpha) \otimes W_\alpha (q) |_{T_q A'_\varepsilon} = DV_2 (q) d\text{vol}_{A'_\varepsilon} + E_1 (q) V_2$$

where

$$DV_2 (q) = - (\phi_1^5 + \phi_2^6 + \phi_3^7) W_4 + (\phi_1^4 + \phi_3^6 - \phi_2^5) W_5 + (\phi_2^4 - \phi_3^5 + \phi_1^6) W_6 + (\phi_3^4 + \phi_2^5 - \phi_1^6) W_7,$$

and

$$\phi_i^k (q) := d\phi^k (q) (W_i)$$

The reason is the following: From our construction of $W_\alpha$, $\nabla W_i W_j (p) = 0$ for $1 \leq i, j \leq 7$, and $\omega^\alpha$ are similar to the standard ones in $\text{Im} \mathcal{O}$. Thus when $q = p$,

$$d (i_{V_2} \omega^\alpha) \otimes W_\alpha (p) |_{T_p A'_\varepsilon} = DV_2 (p) d\text{vol}_{A'_\varepsilon}.$$
If \( q \) is \( \varepsilon \)-close to \( p \), then in (71) the error term \( E_1(q) \) is of order \( \varepsilon \) in \( C^1 \) norm, because \( \nabla W_i W_j(q) = o(\varepsilon) \) in \( C^1 \) and \( \text{span}\{W_\alpha(q)\}_{\alpha=1,2,3} \) has \( \varepsilon \)-order deviation from \( T_q A'_{\varepsilon} \) in \( C^1 \). We want to show

\[
DV_2(q) = d\tilde{\exp} \circ \left[ \left( h^{-1/2}(z) e_1 \cdot \frac{\partial}{\partial x_1} + \bar{\partial} \right) V_1 \right] + E_2(q) V_2, \tag{73}
\]

where the error term \( E_2(q) \) is of order \( \varepsilon \) in \( C^1 \) norm, and the \( \bar{\partial} \) is the Dolbeault Dirac operator on \( N_{\Sigma/C} \oplus \wedge_{\mathbb{C}}^{0,1} (N_{\Sigma/C}) \). Similar to our argument for \( E_1(q) \), it is enough to show

\[
DV_2(p) = d\tilde{\exp}_{[0] \times \Sigma} \circ \left[ \left( h^{-1/2}(z) e_1 \cdot \frac{\partial}{\partial x_1} + \bar{\partial} \right) V_1(0,z) \right].
\]

We observe that

\[
DV_2(p) = DV_2(p) - \phi_5^i W_4 + \phi_4^i W_5 + \phi_7^i W_6 - \phi_6^i W_7
\]

from (72) and (61), where \( D \) is the twisted Dirac operator (60) of \( N_{\Sigma/C} \oplus N_{C/M}\text{|}\Sigma \) over \( \Sigma \) in previous subsection. By Proposition 18, we have

\[
d\tilde{\exp}_{[0] \times \Sigma} \cdot \bar{\partial} V_1 = \Phi (\bar{\partial} V_1) = DV_2.
\]

So it is enough to prove, at \( p = \varphi(0,z) \), that

\[
d\tilde{\exp}_{[0] \times \Sigma} \left[ h^{-1/2}(z) e_1 \cdot \frac{\partial}{\partial x_1} V_1(0,z) \right] = \left( -\phi_5^i W_4 + \phi_4^i W_5 + \phi_7^i W_6 - \phi_6^i W_7 \right)(p).
\]

From

\[
V_2(q) = \sum_{\alpha=4}^7 \phi^\alpha(q) W_\alpha(q) = \sum_{\alpha=4}^7 \phi^\alpha(q) \cdot T_\gamma W_\alpha(\varphi(0,z))
\]

we get

\[
V_1(t,z) = \sum_{\alpha=4}^7 \phi^\alpha(\varphi(t,z)) \cdot (d\tilde{\exp})^{-1} T_\gamma W_\alpha(\varphi(0,z)) = \sum_{\alpha=4}^7 \phi^\alpha(\varphi(t,z)) \cdot \Phi^{-1} W_\alpha(\varphi(0,z)) + E_3(q) V_2,
\]

where \( \{ \Phi^{-1} W_\alpha(\varphi(0,z)) \}_{\alpha=4}^7 \) is regarded as a frame of \( S \to A_{\varepsilon} \) that is invariant along \( t \) direction, and \( E_3(q) \) is of order \( \varepsilon \) in \( C^1 \) norm. The second identity of \( V_1(t,z) \) is because the maps \( d\tilde{\exp} \) and \( T_\gamma \), at \( t = 0 \), are maps \( \Phi \) and the identity map respectively on \( \text{span}\{W_\alpha\}_{4 \leq \alpha \leq 7} \), and for
We have the error term $E_3(q)$ of desired order by smoothness of $\varphi, \phi^\alpha$ and $W_\alpha$. Let

\[ \psi^\alpha(x_1, z) := \phi^\alpha(\varphi(x_1, z)), \text{ and } \psi_1^\alpha(x_1, z) = \frac{\partial}{\partial x_1} \psi^\alpha(x_1, z). \]

We have

\[ e_1 \cdot \frac{\partial}{\partial x_1} \left[ \sum_{\alpha=4}^7 \psi^\alpha(x_1, z) \cdot \Phi^{-1} W_\alpha(\varphi(0, z)) \right] = \Phi^{-1} \left[ W_1 \times \sum_{\alpha=4}^7 \psi_1^\alpha(x_1, z) \cdot W_\alpha(\varphi(0, z)) \right] = \Phi^{-1} \left[ -\psi_1^5 W_4 + \psi_1^4 W_5 + \psi_1^7 W_6 - \psi_1^6 W_7 \right]. \]  

(75)

By the chain rule, for $4 \leq \alpha \leq 7$,

\[ \psi_1^\alpha(0, z) = \nabla_{W_1} \phi^\alpha(\varphi(0, z)) \cdot h(z) \frac{\partial}{\partial x_1} = h(z) \frac{\partial}{\partial x_1} \phi^\alpha(\varphi(0, z)) \]  

(76)

where the factor $h(z)$ is from $\langle W_1, \frac{d}{dt} \varphi(t, z) \big|_{t=0} \rangle = h(z)$. Comparing (74) and (75), and noticing (76), we have

\[ d\exp_{\{0\} \times \Sigma} \circ DV_1 = DV_2. \]

Therefore for $q$ that is $\varepsilon$-close to $p$, claim (70) is proved. So we get

\[ \| d(i_{V_2} \omega^\alpha) \otimes W_\alpha - (d\exp \circ DV_1) \otimes dvol_{A'_\varepsilon} \|_{C^0(U, (p))} = \| E(q) V_2 \|_{C^0(U, (p))} \leq C_1 \varepsilon^{1-\alpha} \| V_2 \|_{C^{1,\alpha}(U, (p))}, \]

(77)

where the constant $C_1$ is uniform for all $\varepsilon$. The remaining term

\[ BV_2 := i_{V_2} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla V_2 W_\alpha \]

is a 0-th order linear operator on $V_2$, where $B = B(q)$ is a smooth tensor on $A'_\varepsilon$. Since $\nabla \tau = 0$, same as in our proof in Theorem 9 we have

\[ i_W d\omega^\alpha(p) = \nabla W_\alpha(p) = 0 \text{ for all } W \in \mathcal{N}_{A'_\varepsilon} \]

so we have $B(p) \big|_{\mathcal{N}_{A'_\varepsilon}} = 0$. Since $\text{dist}(p, q) \leq C_0 \varepsilon$ for some uniform constant $C_0$, and $\mathcal{N}_{A'_\varepsilon}$ is a smooth fiber bundle over $A'_\varepsilon$, we conclude that the operator norm

\[ \| B(q) \|_{\mathcal{N}_{A'_\varepsilon}} \leq C_2 \varepsilon \]

57
for some uniform constant $C_2$. From this it is easy to prove
\[
\|BV_2\|_{C^\alpha(U_\varepsilon(p))} \leq C_3 \varepsilon^{1-\alpha} \|V_2\|_{C^{1,\alpha}(A'_\varepsilon, N_{A'_\varepsilon}/M)}.
\] (78)

The constant $C_3$ is uniform for all $\varepsilon$, by the compactness of $\Sigma$ and finite covering of $A'_\varepsilon$ by $U_\varepsilon(p)$. Putting (78) and (77) in (69), we have
\[
\|F'(0) V_2 - (d\exp \cdot D V_1) \otimes dvol_{A'_\varepsilon}\|_{C^\alpha(U_\varepsilon(p))} \\
\leq C_4 \varepsilon^{1-\alpha} \|V_2\|_{C^{1,\alpha}(A'_\varepsilon, N_{A'_\varepsilon}/M)} \\
\leq C_5 \varepsilon^{1-\alpha} \|V_1\|_{C^{1,\alpha}(A'_\varepsilon, \mathbb{S})}
\]
where the constants $C_4$ and $C_5$ are uniform for all $\varepsilon$. Using finitely many $U_\varepsilon(p)$ covering $A'_\varepsilon$ and then taking supremum, we have
\[
\|F'(0) V_2 - (d\exp \cdot D V_1) \otimes dvol_{A'_\varepsilon}\|_{C^\alpha(A'_\varepsilon)} \leq C \varepsilon^{1-\alpha} \|V_1\|_{C^{1,\alpha}(A'_\varepsilon, \mathbb{S})}.
\] (79)

Applying $P_\varepsilon$ on the left hand side of the above inequality, and noticing that $(d\exp)_{(0) \times \Sigma} \circ DV_1 = DV_2$ is a section of $N_{A'_\varepsilon}/M$, we have
\[
P_\varepsilon \circ ((d\exp) \circ DV_1 (t, z) \otimes dvol_{A'_\varepsilon}) = (d\exp) \circ DV_1 (t, z) + E_3(q)V_1,
\]
where $E_3(q)$ is of order $\varepsilon$ in $C^1$ norm, because at $q = \varphi(t, z)$ the vector $(d\exp) \circ DV_1 (t, z)$ may have component of order $\varepsilon$ in $C^1$ norm orthogonal to $N_{A'_\varepsilon}/M(q)$. So we get
\[
\|F'(0) V_2 - (d\exp) \circ DV_1\|_{C^\alpha(A'_\varepsilon, N_{A'_\varepsilon}/M)} \\
\leq \|P_\varepsilon \circ (F'(0) V_2 - (d\exp) \circ DV_1 \otimes dvol_{A'_\varepsilon})\|_{C^\alpha(A'_\varepsilon, N_{A'_\varepsilon}/M)} + \|E_3(q)V_1\|_{C^\alpha(A'_\varepsilon)} \\
\leq C \varepsilon^{1-\alpha} \|V_1\|_{C^{1,\alpha}(A'_\varepsilon, \mathbb{S})}
\]
where $C$ is a uniform constant independent on $\varepsilon$.  

**Proposition 22** There exists a right inverse $\widetilde{Q}_{\varepsilon}^{true}$ of $F'_\varepsilon(0)$, such that
\[
\|\widetilde{Q}_{\varepsilon}^{true}\| \leq C \varepsilon^{-\left(\frac{d}{p} + 2\alpha\right)},
\]
where the constant $C$ is uniform for all $0 < \varepsilon \leq \varepsilon_0$.

**Proof.** From Theorem 14 for operator $D$ on spinor bundle $\mathbb{S}$ over $A_\varepsilon$, there is a right inverse $Q_\varepsilon$ of $D$ such that $\|Q_\varepsilon\| \leq C \varepsilon^{-\left(\frac{d}{p} + 2\alpha\right)}$. Let
\[
\widetilde{D} = d\exp \circ D \circ (d\exp)^{-1},
\]
\[
\widetilde{Q}_{\varepsilon} = d\exp \circ Q_\varepsilon \circ (d\exp)^{-1},
\]
58
then
\[ \| \widetilde{Q}_\varepsilon \| \leq \| \text{dexp} \| \cdot C\varepsilon^{-\left(\frac{\varepsilon}{\varepsilon} + 2\alpha\right)} \cdot \| (\text{dexp})^{-1} \| \leq C\varepsilon^{-\left(\frac{\varepsilon}{\varepsilon} + 2\alpha\right)}, \]
and
\[ F'_\varepsilon (0) \widetilde{Q}_\varepsilon - id = \left( F'_\varepsilon (0) - \widetilde{D} + \widetilde{D} \right) \widetilde{Q}_\varepsilon - id \]
\[ = \left( F'_\varepsilon (0) - \widetilde{D} \right) \widetilde{Q}_\varepsilon + \widetilde{D} \widetilde{Q}_\varepsilon - id \]
\[ = \left( F'_\varepsilon (0) - \widetilde{D} \right) \widetilde{Q}_\varepsilon, \]
where the last identity is because \( \widetilde{D} \widetilde{Q}_\varepsilon = \text{dexp} \circ D \widetilde{Q}_\varepsilon \circ (\text{dexp})^{-1} = id. \)

From the previous proposition \( \left\| \left( F'_\varepsilon (0) - \widetilde{D} \right) \right\| \leq C\varepsilon^{1-\alpha}. \)
Therefore
\[ \left\| F'_\varepsilon (0) \widetilde{Q}_\varepsilon - id \right\| \leq \left\| \left( F'_\varepsilon (0) - \widetilde{D} \right) \right\| \left\| \widetilde{Q}_\varepsilon \right\| \]
\[ \leq C\varepsilon^{1-\alpha} \cdot C\varepsilon^{-\left(\frac{\varepsilon}{\varepsilon} + 2\alpha\right)} < \frac{1}{2} \]
when \( \varepsilon \) is sufficiently small, by our assumption that \( 1 - \left(\frac{\varepsilon}{\varepsilon} + 3\alpha\right) > 0. \)
So \( \widetilde{Q}_\varepsilon \) is an approximate right inverse of \( F'_\varepsilon (0). \) The true right inverse \( \widetilde{Q}_\text{true} \) of \( F'_\varepsilon (0) \) is \( \widetilde{Q}_\text{true} = \widetilde{Q}_\varepsilon \left( F'_\varepsilon (0) \widetilde{Q}_\varepsilon \right)^{-1} \) and
\[ \left\| \widetilde{Q}_\text{true} \right\| \leq \left\| \widetilde{Q}_\varepsilon \right\| \left\| \left( F'_\varepsilon (0) \widetilde{Q}_\varepsilon \right)^{-1} \right\| \leq C\varepsilon^{-\left(\frac{\varepsilon}{\varepsilon} + 2\alpha\right)} \cdot 2 \leq C\varepsilon^{-\left(\frac{\varepsilon}{\varepsilon} + 2\alpha\right)}. \]

\section*{4.3 Quadratic Estimates}

\textbf{Proposition 23} There exists \( \delta_0 > 0 \) such that for all sections \( V_0, V \) of \( N_{A'/M} \) that \( \| V_0 \|_{C^{1,\alpha}} (A'_e, N_{A'/M}) < \delta_0, \) we have
\[ \left\| F'_\varepsilon (V_0) V - F'_\varepsilon (0) V \right\|_{C^{\alpha}} (A'_e, N_{A'/M}) \]
\[ \leq C \| V_0 \|_{C^{1,\alpha}} (A'_e, N_{A'/M}) \| V \|_{C^{1,\alpha}} (A'_e, N_{A'/M}), \]
where the constant \( C \) is independent on \( \varepsilon. \)
Proof. Because $F_{\varepsilon}(V) = P_{\varepsilon} \circ F(V)$ and $P_{\varepsilon}$ is a bounded linear operator independent on $V$, it is enough to prove the quadratic estimate of $F(V)$, namely

$$\| F'(V_0)V - F'(0)V \|_{C^{\alpha}(A_{\varepsilon}',N_{A_{\varepsilon}'}/M)} \leq C \| V_0 \|_{C^{1,\alpha}(A_{\varepsilon}',N_{A_{\varepsilon}'}/M)} \| V \|_{C^{1,\alpha}(A_{\varepsilon}',N_{A_{\varepsilon}'}/M)}.$$

For any $p = \varphi(t,z)$ on $A_{\varepsilon}'$, we choose a normal frame field $\{W_\alpha\}_{\alpha = 1,2,...,7}$ in its neighborhood $B_{\delta_0}(p)$ where $\tau = \omega^\alpha \otimes W_\alpha$, with $\delta_0$ = the injectivity radius of $M$. For the point $q := \exp_p V_0$ and if $|V_0| < \delta_0$, then we can assume this neighborhood covers $q$. Let the submanifold

$$A_{\varepsilon}(V) =: (\exp V)(A_{\varepsilon}'),$$

which is similar to (62), except that the section $V$ here is in $N_{A_{\varepsilon}'}/M$ instead of $S$, and the $\exp : N_{A_{\varepsilon}'}/M \to M$ is the exponential map on $M$. For the family of embeddings of submanifolds

$$\psi_t := \exp(V_0 + tV) : A_{\varepsilon} \to A_{\varepsilon}(V_0 + tV) \subset M,$$

we have

$$\psi_0(A_{\varepsilon}) = A_{\varepsilon}(V_0),$$

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t(q) = (d\exp_p V_0)V(p) := V_1(q)$$

for any $p \in A_{\varepsilon}' = A_{\varepsilon}(0)$. Arguing as in Theorem 9 from $\nabla \tau = 0$ we have $\nabla W_\alpha(q) = d\omega^\alpha(q) = 0$. We have

$$F'(V_0)V(p)$$

$$= \left. \frac{d}{dt} \right|_{t=0} F(V_0 + tV)(p)$$

$$= \left. \frac{d}{dt} \right|_{t=0} [\exp(V_0 + tV)^* \omega^\alpha(p) \otimes T_{V_0 + tV} W_\alpha(\exp_p(V_0 + tV))]$$

$$= \left. \frac{d}{dt} \right|_{t=0} (\exp(V_0 + tV)^* \omega^\alpha(p) \otimes W_\alpha(p)$$

$$+ (\exp V_0)^* \omega^\alpha(p) \otimes \left. \frac{d}{dt} \right|_{t=0} T_{V_0 + tV} W_\alpha(\exp_p(V_0 + tV)), (81)$$

60
where in the last equality we have used \( T_{V_0 + tV} W_\alpha \left( \exp_p (V_0 + tV) \right) = W_\alpha (p) \) by the parallel property of \( W_\alpha \). For the first term of (81), by the property of the exponential map, namely \( d \exp_q (0) = id_{T_q M} \), we have

\[
\exp_p (V_0 + tV) = \exp_q (tv_1 + t |V| \beta (V_0, V, t)) \circ \exp_p V_0
\]

where \( V_1 = \left( d \exp_{A'_\xi} V_0 \right) V \) is a vector field on \( A\xi (V_0) \), \( \beta (V_0, V, t) \) and \( \beta' (V_0, V, t) \) are uniformly bounded, and \( \lim_{t \to 0} \beta (V_0, V, t) = 0 \). Therefore

\[
\frac{d}{dt} \bigg|_{t=0} \exp (V_0 + tV)^* \omega^\alpha (p)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \left( \exp_{A'_\xi} V_0 \right)^* \exp_{A\xi (V_0)} (tv_1 + t |V| \beta (V_0, V, t))^* \omega^\alpha
\]

\[
= \left( \exp_p V_0 \right)^* L_{V_1} \omega^\alpha
\]

\[
= \left( \exp_p V_0 \right)^* \left( (d (i_{V_1} \omega^\alpha) + i_{V_1} d \omega^\alpha) \right)
\]

where in the third equality we have used that \( \lim_{t \to 0} \beta (V_0, V, t) = 0 \).

For the second term of (81), by Gauss-Bonnet formula, which compares the parallel transports along different paths by curvature integration, we have

\[
T_{V_0 + tV} W_\alpha \left( \exp_p (V_0 + tV) \right)
\]

\[
= T_{V_0} \circ T_{V_1 + t|V|\beta (V_0, V, t)} W_\alpha \left( \exp_q (tv_1 + t |V| \beta (V_0, V, t)) \right)
\]

\[
+ \int_{\Delta(V_0, tv)} R(\sigma) d\sigma W_\alpha (p),
\]

where \( R(\sigma) \) is the Ricci curvature tensor, \( \Delta (V_0, tv) \) is the 2-dimensional geodesic triangle with the vertices \( p, \exp_p V_0 \), and \( \exp_p (V_0 + tV) \), and \( \sigma \) is the area element. Therefore

\[
\frac{d}{dt} \bigg|_{t=0} \left( T_{V_0 + tV} W_\alpha \left( \exp_p (V_0 + tV) \right) \right)
\]

\[
= T_{V_0} \circ \frac{d}{dt} \bigg|_{t=0} \left( T_{V_1} W_\alpha \left( \exp_q tv \right) \right) + \frac{d}{dt} \bigg|_{t=0} \int_{\Delta(V_0, tv)} R(\sigma) d\sigma W_\alpha (p),
\]

where in the first term we have dropped the higher order term \( t |V| \beta (V_0, V, t) \) since it is irrelevant for derivative at 0. If we denote

\[
B(V_0, V) := \frac{d}{dt} \bigg|_{t=0} \int_{\Delta(V_0, tv)} R(\sigma) d\sigma,
\]

\[61\]
by the area formula of $\Delta (V_0,tV)$ it can be shown that

$$|B (V_0, V) W_\alpha (p)| \leq C_5 |V_0| |V| |W_\alpha| (p),$$

where $C_5$ is a constant only depending on $(M,g)$. Plugging these in \((81)\), we have

$$F' (V_0) \mid_{A_\epsilon (0)} V (p) = (\exp V_0)^* L_{V_1} \omega^\alpha \otimes W_\alpha (p) + (\exp V_0)^* \omega^\alpha \otimes T_{V_0} \nabla V W_\alpha (q) + B (V_0, V) W_\alpha (p)$$

$$F' (V_0) \mid_{A_\epsilon (0)} V (p) = ((\exp V_0)^* \otimes T_{V_0}) \circ [(L_{V_1} \omega^\alpha \otimes W_\alpha (q) + \omega^\alpha (q) \otimes \nabla V W_\alpha (q))$$

$$+ B (V_0, V) W_\alpha (p)$$

$$= ((\exp V_0)^* \otimes T_{V_0}) \circ [(d (i_{V_1} \omega^\alpha) + i_{V_1} d \omega^\alpha) \otimes W_\alpha (q) + \omega^\alpha (q) \otimes \nabla V W_\alpha (q)]$$

$$+ B (V_0, V) W_\alpha (p).$$

Here we have used that $T_{V_0} W_\alpha (q) = W_\alpha (p)$. The \((82)\) can be rewritten as

$$F' (V_0) \mid_{A_\epsilon (0)} V (p) = ((\exp V_0)^* \otimes T_{V_0}) \circ [F' (0) \mid_{A_\epsilon (V_0)} V (q)]$$

$$+ B (V_0, V) W_\alpha (p),$$

which means the derivative $F' (V_0)$ on $A_\epsilon (0)$ can be expressed by the derivative $F' (0)$ on $A_\epsilon (V_0)$ via the transform $(\exp V_0)^* \otimes T_{V_0}$, up to the curvature term $B (V_0, V) W_\alpha (p)$. If $V_0 = 0$, then $q = p$ and $V_1 = V$, together with

$$i_V d \omega^\alpha (p) = \nabla V W_\alpha (p) = B (0, V) = 0,$$

the formula \((82)\) is simplified as

$$F' (0) \mid_{A_\epsilon} V (p) = d (i_{V} \omega^\alpha) \otimes W_\alpha (p).$$

Therefore

$$F'' (V_0) V (p) - F' (0) V (p)$$

$$= [(\exp V_0)^* d (i_{V_1} \omega^\alpha) - d (i_{V} \omega^\alpha)] \mid_{A_\epsilon (0)} \otimes W_\alpha (p) +$$

$$((\exp V_0)^* \otimes T_{V_0}) \circ [i_{V_1} d \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla V W_\alpha] (p) + B (V_0, V) W_\alpha (p)$$

\((83)\)

\((84)\)

For the second row \((84)\), it is a 0-th order linear differential operator on $V \in C^{1,\alpha} (\Gamma (N_{A_\epsilon} / M))$, where

$$V_1 = (d \exp V_0) \mid_{A_\epsilon (0)} V = E (V_0) V.$$
Proof. Consider the 3 linear maps from $\delta$ where the constants have

Combining the two rows we get

For the first row (83),

$$
\| (\exp V_0)^* \otimes T_{V_0}) \circ \left[ i_{E(V_0)\cdot} d\omega_\alpha \otimes W_\alpha + \omega_\alpha \otimes \nabla_{E(V_0)\cdot} W_\alpha \right] \\
+ B (V_0, \cdot) W_\alpha,
$$

$H$ is a bounded operator since the terms $\exp V_0, (\exp V_0)^*, T_{V_0}, E (V_0)$ and $B (V_0, \cdot)$ as elements in $C^\alpha$ are differentiable for $V_0 \in C^1, \alpha \left( \Gamma \left( N_{A_\epsilon' \epsilon'}/M \right) \right)$ with bounded Frechet derivatives. The bound of the derivatives is uniform on $\varepsilon$ so $\| H \|$ is uniformly bounded. Since $H (V_0) = 0$ when $V_0 = 0$, we have

$$
\| H (V_0, V) \|_{C^\alpha} \leq C_8 \| V_0 \|_{C^1, \alpha(A'_\epsilon)} \| V \|_{C^1, \alpha(A'_\epsilon)}.
$$

For the first row [S3], $[(\exp V_0)^* d (i_{V_1} \omega_\alpha) - d (i_{V'} \omega_\alpha)]$ is a first order linear differential operator on $V \left( V_1 = (d \exp V_0) \mid_{A_\epsilon (0)} V \right)$, and only involves the covariant derivatives of $V$. In the next lemma we will show

$$
\| (\exp V_0)^* d (i_{V_1} \omega_\alpha) - d (i_{V'} \omega_\alpha) \|_{C^0 (A'_\epsilon)} \leq C_7 \| V_0 \|_{C^1, \alpha (A'_\epsilon)} \| V \|_{C^1, \alpha (A'_\epsilon)}.
$$

Combining the two rows we get

$$
\| F' (V_0) V - F' (0) V \|_{C^0 (A'_\epsilon, N_{A_\epsilon' \epsilon'}/M)} \\
\leq C \| V_0 \|_{C^1, \alpha (A'_\epsilon, N_{A_\epsilon' \epsilon'}/M)} \| V \|_{C^1, \alpha (A'_\epsilon, N_{A_\epsilon' \epsilon'}/M)}.
$$

Lemma 24 For any $V_0, V \in \Gamma \left( N_{A'_\epsilon \epsilon'}/M \right)$ with $\| V_0 \|_{C^1, \alpha(A'_\epsilon)} \leq \delta_0$, we have

$$
\| (\exp V_0)^* d (i_{V_1} \omega_\alpha) - d (i_{V'} \omega_\alpha) \|_{C^0 (A'_\epsilon)} \leq C_7 \| V_0 \|_{C^1, \alpha (A'_\epsilon)} \| V \|_{C^1, \alpha (A'_\epsilon)},
$$

where the constants $\delta_0$ and $C_7$ are independent on $\varepsilon$.

Proof. Consider the 3 linear maps from $T_p M$ to $T_q M$, where $q = \exp_p V_0$:

$P_{A_\epsilon} : T_p M \rightarrow T_q M$, parallel transport along the geodesic $\exp_p t V_0$ for $0 \leq t \leq 1$,

d (\exp V_0) : T_p M \rightarrow T_q M$, tangent map of the diffeomorphism $\exp V_0 : A'_\epsilon \rightarrow A_\epsilon (V_0)$,

d (\exp_p) (V_0) : T_q (T_p M) \rightarrow T_q M$, tangent map of the exponential map $\exp_p : T_p M \rightarrow M$. 

63
where in the last one we identify $T_{V_0}(T_p M) \simeq T_p M$. Note that for diffeomorphism $d \left( \exp V_0 \right)$ we need a vector field $V_0$ on $A_\varepsilon$ but for $d \left( \exp_p \right) (V_0)$ we only need $V_0 \in T_p M$, so they are essentially different maps. But the 3 maps are very close to each other when $V_0$ is small. More precisely, for any $V_0, V \in \Gamma \left( N_{A'_\varepsilon/M} \right)$ with $\| V_0 \|_{C^{1,\alpha}(A'_\varepsilon)} \leq \delta_0$, we have comparison of the maps as the following:

\[
\begin{align*}
\quad d \left( \exp V_0 \right) V &= \text{Pal}_{V_0} V + h_1 (V_0) V, \\
\quad d \left( \exp_p \right) (V_0) V &= d \left( \exp V_0 \right) V + h_2 (V_0) V,
\end{align*}
\]

where for each $i = 1, 2$, the error term

\[ h_i (\cdot) : C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon/M} \right) \right) \to C^\alpha \left( \Gamma \left( \text{Hom} \left( TM|_{A'_\varepsilon}, TM|_{A_\varepsilon(V_0)} \right) \right) \right) \]

is differentiable with respect to the variable $V_0$, and

\[ h_i (V_0) = 0 \text{ for } V_0 = 0. \]

This is because for any fixed $p$, $\text{Pal}_{V_0(p)}$ and $d \left( \exp_p \right) (V_0)$ smoothly depend on $V_0$, and for the compact family $p \in A'_\varepsilon$, $\text{Pal}_{V_0}$ and $d \left( \exp_p \right) (V_0)$ inherits the smooth dependence on $V_0 \in C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon/M} \right) \right)$. For $d \left( \exp V_0 \right)$, since

\[ d \left( \exp V_0 \right) (x) = F_1 (x, V_0) \, dx + F_2 (x, V_0) \, dV_0 (x) \]

where $F_1 (x, y) = \frac{\partial}{\partial x} \exp_x y$ and $F_2 (x, y) = \frac{\partial}{\partial y} \exp_x y$ are bounded on $A_\varepsilon$, we see $d \left( \exp V_0 \right) \in C^\alpha$ smoothly depends on $V_0 \in C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon/M} \right) \right)$ with bounded Frechet derivative. This especially implies that

\[ \| h_i (V_0) V \|_{C^\alpha} \leq C_8 \| V_0 \|_{C^{1,\alpha}(A'_\varepsilon)} \| V \|_{C^{1,\alpha}(A'_\varepsilon)}, \quad (85) \]

for $i = 1, 2$. We have

\[
\begin{align*}
V_1 &= d \left( \exp V_0 \right) V + h_2 (V_0) V, \\
\quad d \left( \exp V_0 \right) W_\alpha (p) &= W_\alpha (q) + h_1 (V_0) W_\alpha (p), \\
\quad (\exp V_0)^* \omega^\alpha (q) &= \omega^\alpha (p) + h_3 (V_0) \omega^\alpha (q)
\end{align*}
\]

using the parallel property of $W_\alpha$ and $\tau$ (hence $\omega^\alpha$). (Here $h_3 (V_0)$ depends on $V_0 \in C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon/M} \right) \right)$ smoothly by similar reason of $h_1$ and $h_2$. Since $\omega^\alpha$ is a 3-form, $h_3 (V_0)$ can have terms of cubic power.

64
order of \( V_0 \). Therefore
\[
(\exp V_0)^* d (i V_1 \omega^\alpha) - d (i V \omega^\alpha) \\
= d \left[ (\exp V_0)^* (i d (\exp V_0 V + h_2 (V_0) V \omega^\alpha) - i V \omega^\alpha) \right] \\
= d \left[ (\exp V_0)^* i d (\exp V_0 V) \omega^\alpha + (\exp V_0)^* i h_2 (V_0) V \omega^\alpha - i V \omega^\alpha \right] \\
= d \left[ i V (\exp V_0)^* \omega^\alpha + (\exp V_0)^* i h_2 (V_0) V \omega^\alpha - i V \omega^\alpha \right] \\
= d \left[ i V (\omega^\alpha + h_3 (V_0) \omega^\alpha) + (\exp V_0)^* i h_2 (V_0) V \omega^\alpha - i V \omega^\alpha \right] \\
= d \left[ i V (h_3 (V_0) \omega^\alpha (q)) + (\exp V_0)^* i h_2 (V_0) V \omega^\alpha (q) \right].
\]

Using the property (S5) of \( h_i \) (\( i = 1, 2, 3 \)), we observe that each term in the above last identity in \( C^\alpha \) norm smoothly depends on \( V_0 \in C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon} / M \right) \right) \) and linearly on \( V \in C^{1,\alpha} \left( \Gamma \left( N_{A'_\varepsilon} / M \right) \right) \). Also when \( V_0 = 0 \), we have \((\exp V_0)^* d (i V_1 \omega^\alpha) - d (i V \omega^\alpha) = 0\). Therefore
\[
\|(\exp V_0)^* d (i V_1 \omega^\alpha) - d (i V \omega^\alpha)\|_{C^\alpha (A'_\varepsilon)} \leq C \|V_0\|_{C^{1,\alpha} (A'_\varepsilon)} \|V\|_{C^{1,\alpha} (A'_\varepsilon)}.
\]

**Remark 25** We also have some point estimates tied to the feature that \( \tau \) is a 3-form. It is well known that for any smooth embeddings \( \varphi : A_\varepsilon \to M \) and smooth sections \( V_0 \) of \( TM|_{A'_\varepsilon} \) with \( |V_0| \) smaller than the injectivity radius of \( M \),
\[
|d \exp_\varphi V_0| \leq C_6 (|d \varphi| + |\nabla V_0|),
\]
where \( C_6 \) is a constant only depending on \((M, g)\). Hence for the map \( \exp V_0 : TM|_{A'_\varepsilon} \to M \) and any section \( V \) of \( TM|_{A'_\varepsilon} \), at \( p \in A'_\varepsilon \) we have
\[
|(d \exp V_0)(p) V| \leq C_6 (|d \varphi| + |\nabla V_0|) |V(p)|.
\]
So for the first term (S3), we have
\[
|[(\exp V_0)^* d (i V_1 \omega^\alpha) - d (i V \omega^\alpha)] \otimes W_\alpha (p)| \\
\leq C_7 (|d \varphi| + |\nabla V_0|)^3 (|\nabla V| |V_0| + |V| |\nabla V_0| + |V||V_0|) (p),
\]
where the power 3 is because \( \omega^\alpha \) is a 3-form. For the next row (S4), it is a 0-th order linear differential operator on \( V \), where \( V_1 = (d \exp V_0)|_{A_\varepsilon(0)} V \). By the \( C^2 \) smoothness of \( \tau = \omega^\alpha \otimes W_\alpha \) and
\[
i_W d \omega^\alpha (p) = \nabla_W W_\alpha (p) = 0
\]
65
for any $W \in \Gamma(N_{\mathcal{A}'/M})$, at $q = \exp_p V_0$ we have

$$|i_{V_1} d\omega^\alpha(q)| \leq C_4 |V_0| |V|, \quad |\nabla_{V_1} W_\alpha(q)| \leq C_4 |V_0| |V|.$$  

Hence

$$|((\exp V_0)^* \otimes T_{V_0}) \circ [i_{V_1} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{V_1} W_\alpha](p) + B(V_0, V) W_\alpha(p)| \leq C_6^3 (|d\varphi| + |\nabla V_0|)^3 \cdot 2C_4 |V_0| |V| (p) + C_5 |V_0| |V| (p).$$

where the power 3 is because $\omega^\alpha$ is a 3-form. Combining all, we have

$$|F'(V_0) V - F'(0) V| (p) \leq \left[C_7 (|d\varphi| + |\nabla V_0|)^3 (|V_0| |V| + |\nabla V_0| |V| + |V_0| |\nabla V|) + C_5 |V_0| |V| \right] (p).$$

Because of the cubic power terms, it is difficult to get similar quadratic estimate in the $L^p$ setting, namely

$$\|F'(V_0) V - F'(0) V\|_{L^p(A'_{\mathcal{E}'},N_{\mathcal{A}'/M})} \leq C \|V_0\|_{W^{1,p}(A'_{\mathcal{E}'},N_{\mathcal{A}'/M})} \|V\|_{W^{1,p}(A'_{\mathcal{E}'},N_{\mathcal{A}'/M})}.$$  

This is one of the key reasons that we choose the Schauder setting for implicit function theorem, where the right inverse bound of $F'(0)$ is much harder to obtain than in the $L^p$ setting. In contrast, the Cauchy-Riemann operator of $J$-holomorphic curves is more linear in the $L^p$ setting: it has the quadratic estimate (see Proposition 3.5.3 in [30])

$$\|F'(V_0) V - F'(0) V\|_{L^p(\Sigma)} \leq C \|V_0\|_{W^{1,p}(\Sigma)} \|V\|_{W^{1,p}(\Sigma)}.$$  

4.4 Perturbation Argument

To find the zeros of $F_\varepsilon$, we are going to apply the following quantitative version of the implicit function theorem (c.f. Theorem 15.6 [9] or Proposition A3.4 in [30]).

**Theorem 26** Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces and $F : B_r(0) \subset X \to Y$ a $C^1$-map, such that

1. $(DF(0))^{-1}$ is a bounded linear operator with $\|(DF(0))^{-1} F(0)\| \leq A$ and $\|(DF(0))^{-1}\| \leq B$;
2. \(|DF(x) - DF(0)| \leq \kappa |x|_X\) for all \(x \in B_r(0)\);

3. \(2\kappa AB < 1\) and \(2A < r\).

Then \(F\) has a unique zero in \(B_{2A}(0)\).

To apply the above theorem, we define the map

\[
\tilde{F}_\varepsilon := \varepsilon^{-\left(\frac{1}{p}+2\alpha\right)}F_\varepsilon : C_{1-}^{1,\alpha}(\epsilon^*_\varepsilon, N_{A'/M}) \rightarrow C^\alpha(\epsilon^*_\varepsilon, N_{A'/M}).
\]

Then Proposition 22 implies that

\[
\left\| \left(DF_{\varepsilon}(0)\right)^{-1}\right\| \leq \varepsilon^{\frac{3}{p}+2\alpha}C_{\varepsilon}^{-\left(\frac{1}{p}+2\alpha\right)} = C.
\]

By Proposition 23, for \(\|V_0\|_{C_{1-}^{1,\alpha}(\epsilon^*_\varepsilon, N_{A'/M})} \leq \delta_0\) and any \(V \in C_{1-}^{1,\alpha}(\epsilon^*_\varepsilon, N_{A'/M})\),

\[
\left\| \left(DF_{\varepsilon}(V_0) - DF_{\varepsilon}(0)\right) V \right\|_{C^\alpha(\epsilon^*_\varepsilon, N_{A'/M})} \leq C_{\varepsilon}^{-\left(\frac{3}{p}+2\alpha\right)}\|V_0\|_{C_{1-}^{1,\alpha}(\epsilon^*_\varepsilon, N_{A'/M})}\|V\|_{C_{1-}^{1,\alpha}(\epsilon^*_\varepsilon, N_{A'/M})}.
\]

For \(F(V) = T_V \circ (\exp V)^*\tau\), fixing a volume form on \(\epsilon^*_\varepsilon\) we can regard \(F(V)\) as a section of \(TM|_{\epsilon^*_\varepsilon}\). We have

\[
\|F(0)\|_{C^\alpha(\epsilon^*_\varepsilon, TM)} = \|\varphi^*\tau\|_{C^\alpha(\epsilon^*_\varepsilon, TM)} = C\left\| (\varphi^*\tau)(0, z) + \int_0^{x_1} \frac{\partial}{\partial x_1} (\varphi^*\tau)(s, z) \, ds \right\|_{C^\alpha(\epsilon^*_\varepsilon, TM)}
\]

\[
= C\left\| \int_0^{x_1} \frac{\partial}{\partial x_1} \varphi^*\tau(s, z) \, ds \right\|_{C^\alpha(\epsilon^*_\varepsilon, TM)} \leq C\varepsilon^{1-\alpha}, \quad (86)
\]

where \((\varphi^*\tau)(0, z) = 0\) is because \(\varphi(0 \times \Sigma) = J_{\alpha-}\) holomorphic and then \(\tau|_{TM|_{\varphi(0 \times \Sigma)}} = 0\), and the last inequality is because for

\[
H(x_1, z) := \int_0^{x_1} \frac{\partial}{\partial x_1} \varphi^*\tau(s, z) \, ds
\]

where \(f(s, z) := \frac{\partial}{\partial x_1} \varphi^*\tau(s, z)\) is smooth on \([0, \varepsilon] \times \Sigma\), we have

\[
\|H\|_{C^0(\epsilon^*_\varepsilon)} \leq \int_0^\varepsilon \|f\|_{C^0(\epsilon^*_\varepsilon)} \, ds = \|f\|_{C^0(\epsilon^*_\varepsilon)} \varepsilon,
\]

\[
[H]_{\alpha; \epsilon^*_\varepsilon} \leq \int_0^\varepsilon [f]_{\alpha; \epsilon^*_\varepsilon} \, ds = [f]_{\alpha; \epsilon^*_\varepsilon} \varepsilon,
\]

\[
[H]_{\alpha; \epsilon^*_\varepsilon} \leq C[H]_{1; \epsilon^*_\varepsilon} \varepsilon^{1-\alpha} \leq \|f\|_{C^0(\epsilon^*_\varepsilon)} \varepsilon^{1-\alpha}.
\]
(Here \([f]_\alpha^{z}, [f]_\alpha^{x_1}\) are the Schauder components of \(f\) in the \(z\) and \(x_1\) directions of \(A_\varepsilon\)). Note that (86) implies that the tangent space of \(A_\varepsilon'\) is \(\varepsilon^{1-\alpha}\)-close to be associative.

By our construction

\[
F_\varepsilon (V) = \ast_{a_\varepsilon} \circ \perp_{a_\varepsilon'} \circ F (V),
\]

where \(\ast_{a_\varepsilon} \circ \perp_{a_\varepsilon'}\) is a bounded linear operator and \(T_V\) is a parallel transport, we have

\[
\|F_\varepsilon (0)\|_{C^{\alpha}(a_\varepsilon, N_{A_\varepsilon'}/M)} \leq C\varepsilon^{-(\frac{3}{p}+2\alpha)} \|F (0)\|_{C^{\alpha}(a_\varepsilon, N_{A_\varepsilon'}/M)} \leq C\varepsilon^{-(\frac{3}{p}+2\alpha)}\varepsilon^{1-\alpha} = C\varepsilon^{1-(\frac{2}{p}+3\alpha)}.
\]

Then we have

\[
\|D\tilde{F}_\varepsilon (0)^{-1}\| \|\tilde{F}_\varepsilon (0)\|_{C^{\alpha}(a_\varepsilon, S)} \varepsilon^{-(\frac{3}{p}+2\alpha)} \leq C\varepsilon^{1-\frac{6}{p}-5\alpha}.
\]

Note we have chosen that \(0 < \frac{3}{p} + 3\alpha < \frac{1}{2}\) (say \(\alpha = 1/12\) and \(p > 12\)) in the beginning, so \(1 - \frac{6}{p} - 5\alpha > 0\) then

\[
\|D\tilde{F}_\varepsilon (0)^{-1}\| \|\tilde{F}_\varepsilon (0)\|_{C^{\alpha}(a_\varepsilon, S)} \varepsilon^{-(\frac{3}{p}+2\alpha)} \leq C\varepsilon^{1-(\frac{2}{p}+3\alpha)} \varepsilon^{-(\frac{3}{p}+2\alpha)} = C\varepsilon^{1-\frac{6}{p}-5\alpha} \to 0
\]

as \(\varepsilon \to 0\). Theorem 26 then implies that there is a unique \(V_\varepsilon \in \Gamma (N_{A_\varepsilon'}/M)\) with

\[
\|V_\varepsilon\|_{C^{1,\alpha}(\beta_\varepsilon', N_{A_\varepsilon'}/M)} \leq 2C\varepsilon^{1-(\frac{2}{p}+3\alpha)}
\]

that solves \(\tilde{F}_\varepsilon (V_\varepsilon) = 0\), i.e. \(F_\varepsilon (V_\varepsilon) = 0\). Note that (86) and (87) together imply that the tangent space of \(A_\varepsilon (V_\varepsilon)\) is \(\varepsilon^{1-(\frac{2}{p}+3\alpha)}\)-close to be associative. By Proposition 8 for almost associative submanifolds \(A_\varepsilon (V_\varepsilon)\),

\[
F_\varepsilon (V_\varepsilon) = 0 \iff F (V_\varepsilon) = 0,
\]

while the latter means that \(A_\varepsilon (V_\varepsilon)\) is associative.

Finally, we obtain our main result:
Theorem 27  Suppose that $M$ is a $G_2$-manifold and $C_t$ is a one parameter family of coassociative submanifolds in $M$. Suppose that the self-dual two form $\eta = dC_t/dt|_{t=0} \in \Omega^2_+(C)$ is nonvanishing, then it defines an almost complex structure $J$ on $C_0$.

For any regular $J$-holomorphic curve $\Sigma$ in $C_0$, there is an instanton $A_\varepsilon$ in $M$ which is diffeomorphic to $[0, \varepsilon] \times \Sigma$ and $\partial A_\varepsilon \subset C_0 \cup C_\varepsilon$, for all sufficiently small positive $\varepsilon$.

Finally, we expect that any instanton $A$ in $M$ bounding $C_0 \cup C_t$ and with small volume must arise in the above manner. Namely we need to prove a $\varepsilon$-regularity result for instantons.

A few remarks are in order: First, counting such small instantons is basically a problem in four manifold theory because of Bryant’s result [7] which says that the zero section $C$ in $\Lambda^2_+ (C)$ is always a coassociative submanifold for an incomplete $G_2$-metric on its neighborhood provided that the bundle $\Lambda^2_+ (C)$ is topologically trivial. Second, when $\eta$ has zeros, the above Theorem 27 should still hold true. However using the present approach to prove it would require a good understanding of the Seiberg-Witten theory on any four manifold with a degenerated symplectic form as in Taubes program. But at least for regular $J_\eta$-holomorphic curves $\Sigma$ in $C_0$ that is away from the zero locus of $\eta$, the instantons $A_\varepsilon'$ in Theorem 27 still exist nearby $\Sigma \subset M$, for our gluing analysis only involves the local geometry of $\Sigma$ in $M$. Third, if we do not restrict to instantons of small volume, then we have to take into account the compactification of the moduli of instantons which has not been established yet, e.g. bubbling phenomenon as for pseudo-holomorphic curves, and gluing of instantons of big and small volumes similar to [32] in Floer trajectory case. Nevertheless, one would expect that if the volume of $A_\varepsilon$’s are small, then bubbling cannot occur, thus they would converge to a $J_\eta$-holomorphic curve in $C_0$.

5  Appendix: The Exponential-like Map $\mathcal{E}\exp$

We construct the exponential-like map $\mathcal{E}\exp : \mathbb{S} \to M$ satisfying the properties 1~3 in Subsection 4.2.

For any section $V = (u, v)$ of the spinor bundle $\mathbb{S} \to A_\varepsilon$, since the bundle $\mathbb{S} \to A_\varepsilon$ is a Cartesian product of the spinor bundle $\mathbb{S}_\Sigma \to \Sigma$ with the interval $[0, \varepsilon]$, we may alternatively regard $V (t, z)$ as a one parameter family of sections of the bundle $\mathbb{S}_\Sigma \to \Sigma$ for the parameter
t ∈ [0, ε]. Our goal is to obtain the deformation of \( A'_{\varepsilon} \subset M \) from \( V \). The idea is to deform \( A_{\varepsilon} \subset C := [0, \varepsilon] \times C \) using only the \( u \) component, then map the deformed \( A_{\varepsilon} \) to \( M \) by \( \varphi \), and at last deform it in \( M \) by the \( v \) component. The coassociative boundary condition is preserved under the map \( \tilde{\exp} \), since we separate the \( u \) and \( v \) deformations before and after the map \( \varphi \) respectively. The precise description is in order.

For each fixed \( t \in [0, \varepsilon] \), \( u(t, z) \) and \( v(t, z) \) are sections of the spinor bundle \( S_{\Sigma} \rightarrow \Sigma \). Using the real vector bundle isomorphism

\[
(id, f) : N_{\Sigma/C} \oplus \Lambda_{C}^{0,1}(N_{\Sigma/C}) \simeq N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma},
\]

\( u(t, z) \) and \( f(v(t, z)) \) are sections of \( N_{\Sigma/C} \) and \( N_{C/M}|_{\Sigma} \) respectively. For any fixed \( z \in \Sigma \), using the \( u \) component, we have a one parameter deformation \( \exp_{C}^{\Sigma} u(t, z) \) in \( C \) for \( 0 \leq t \leq \varepsilon \), where \( \exp_{C}^{\Sigma} \) is the exponential map associated to the induced metric of \( C \) in \( M \). Alternatively, we can view the deformation in the product space \( C := [0, \varepsilon] \times C \), such that the line \( [0, \varepsilon] \times \{ z \} \subset C \) is deformed to the curve \( \{ (t, \exp_{C}^{\Sigma} u(t, z)) \mid 0 \leq t \leq \varepsilon \} \subset C \). Let the curve \( \gamma_{u} \) in \( M \) be

\[
\gamma_{u}(t, z) := \varphi(t, \exp_{C}^{\Sigma} u(t, z)), \quad \text{for } 0 \leq t \leq \varepsilon \text{ and } z \in \Sigma.
\]

It is clear that

\[
\gamma_{u}(t, z) \subset C_{t} := \varphi(t, C)
\]

since \( \exp_{C}^{\Sigma} u(t, z) \subset C \).

For the \( v \) component, for each fixed \( t \in [0, \varepsilon] \), we have \( f(v(t, z)) \in N_{C/M}|_{\Sigma}(z) \). Let

\[
T_{\gamma_{u}(t, z)} : N_{C/M}|_{\gamma_{u}(0, z)} \longrightarrow N_{C/M}|_{\gamma_{u}(t, z)}
\]

be the parallel transport along the curve \( \gamma_{u}(s, z) \) for \( 0 \leq s \leq t \) with respect to the connection of \( N_{C/M} \) induced from the metric \( g \) on \( M \). We define the exponential-like map \( \tilde{\exp} \) as follows:

\[
\tilde{\exp} : S = S^{+} \oplus S^{-} \longrightarrow M, \quad V(t, z) = (u(t, z), v(t, z)) \quad \exp_{\gamma_{u}(t, z)}^{M} \left( T_{\gamma_{u}(t, z)} f(v(t, z)) \right)
\]

where \( \exp_{\gamma_{u}(x)}^{M} \) is the exponential map in \( M \). It follows from our construction that for any \( V \in C_{\Sigma}^{m}(A_{\varepsilon}, S) \) and \( x \in \partial A_{\varepsilon} = \{0, \varepsilon\} \times \Sigma \), \( \tilde{\exp} V \) satisfies the boundary condition \( \exp_{\partial A_{\varepsilon}} V \subset C_{0} \cup C_{\varepsilon} \), because

\[
(\tilde{\exp} V)(x) = \exp_{\gamma_{u}(x)}^{M}(0) = \gamma_{u}(x) \subset \varphi(\{0, \varepsilon\} \times C) = C_{0} \cup C_{\varepsilon}.
\]
By the definition of $\exp$, it is easy to see $\exp|_{A_\epsilon} = \varphi$, because $A_\epsilon$ is the zero section of $S$. Hence on base directions of $S \to A_\epsilon$, $d\exp|_{A_\epsilon} = d\varphi|_{A_\epsilon}$, especially
\[
d\exp|_{\{0\} \times \Sigma} = id : T\Sigma \to T\Sigma \text{ and } d\exp|_{\{0\} \times \Sigma} : \frac{\partial}{\partial x_1} \to n (z) .
\]
On fiber directions, that $d\exp|_{\{0\} \times \Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge_C^0 (N_{\Sigma/C}) \to N_{\Sigma/C} \oplus N_{C/M}|_{\Sigma}$ follows from $d\exp^M (0) = id$.

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