Resonances and spectral properties of detuned OPO pumped by fluctuating sources

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Twin beam fluctuations are analyzed for detuned and mismatched OPO configurations. Resonances and frequency responses to the quantum noise sources (quantum and pump amplitude/phase fluctuations) are examined as functions of cavity decay rates, excitation parameter and detuning. The dependence of self- and mutual correlations of beam amplitudes and phases on detuning, mismatch and damping parameters is discussed.

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I. INTRODUCTION

Continuous wave twin beams owe their popularity to the mutual quantum correlations. It has been shown theoretically (see f.i. [1,2]) and proved experimentally (see f.i. [3,4]) that this feature can be exploited for replacing classical fields in shot noise limited optical measurements. In particular, the spectrum difference of the two beams is shaped like a Lorentzian with a rather good noise suppression at zero frequency. On the other hand, Fabre et al. [5] have called the attention on the squeezing exhibited by each beam for pump intensity larger than four times the threshold value. They have also shown that additional squeezing can be imposed on each beam by using the photocurrent detected with the other one.

These properties are contrasted by unsuited damping coefficients of the OPO cavity, imperfect tuning of the three mode resonances, limited intensity and amplitude/phase fluctuations of the laser pump. First Lane et al. [12] drew the attention on the critical dependence of the noise suppression in the difference spectrum on the cavity mismatch, as experimentally confirmed by several authors (see f.i. [3,4]). Stable and narrow line lasers emit in general low intensity beams. Then, for observing single beam squeezing it is essential to lower the threshold by reducing the pump mode decay rate. This solution makes the OPO a triply resonant device featuring relaxation oscillations [15–17]. Due to the unavoidable OPO crystal losses the amplitude/phase fluctuations of the pump contribute to the difference spectral noise at low frequency, where quantum noise suppression is expected. The situation becomes worse and worse as the cavity is detuned from the resonant configuration.

These few remarks point to the several parameters which the features of an OPO depend on, namely, cavity damping coefficients, mismatch of these parameters, degree of excitation above threshold, crystal absorption, deviation from resonance condition (detuning), and pump amplitude/phase fluctuations. The last ones play a prominent role in the lower part of the frequency spectra of all relevant quantities. Many papers have been published discussing the role of some of these parameters. Here we aim to discuss analytically the twin beam fluctuations by encompassing systematically the many parameters mentioned above.

The representation of the three modes (signal/idler \(k = 1, 2\), pump \(k = 0\)) in the form \(a_k = r_k e^{-i\phi_k} (1 + \mu_k)\) lends itself to separate the steady state amplitude \(r_k\) from the fluctuating relative amplitude \(\mu_k\) and phase \(\phi_k\). In regard to the fluctuations, the OPO behaves as a forced linear system. Then, the Fourier transforms \(\hat{a}_k(\omega)\) of the combination \(\delta a_k = \mu_k - i\phi_k\) and the adjoint \(\hat{a}_k^*(\omega) \equiv \hat{a}_k(-\omega^*)\) form an algebraic system. The dependence of the \(\delta a_k(\omega)\) on the noise sources is embodied in the frequency responses \(K_{kl}^{0,\omega}(\omega)\) of the \(k\)-th mode to the \(l\)-th noise source. These responses are analytic functions of \(\omega\) with, all together, six complex poles (system resonances) in the half-plane \(\text{Im}(\omega) \geq 0\). Their location (generally complex or purely imaginary) influences the field correlations in the time domain and, hence, the relative frequency spectra. For example, when the poles have large real parts the system exhibits relaxation oscillations, responsible of typical peaks in photocurrent spectra. In short the \(K_{kl}^{0,\omega}(\omega)\) and their poles provide a link among the OPO, pump laser parameters and the statistical features of the twin beams.

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While several authors have shown that the unavoidable cavity mismatch reduces the mode correlation, less attention has been paid to the effects of detuning, apart from the analysis of Ref. \[1\] dedicated to OPO with pump bandwidths much larger than the signal/idler ones, i.e. in the adiabatic limit. For large detunings bistability, self-pulsing and chaos may appear \[13,19\], while for moderate deviations from cavity resonance, modifications of the spectra of the single beams and their differences are expected. In presence of detuning the decay constants become complex, \(\kappa_k = \gamma_k + i\varphi_k\), and the equations of motion of \(\mu_k\) do not separate from those of \(\phi_k\). In particular, the pump phase diffusion leaks into the single mode amplitude spectrum by enhancing the disturbing effects of the classical pump fluctuations. From the analytic point of view, the detuning has the effect of breaking the symmetry of the frequency responses, \(K_{kl}^{0,\pm}(\omega) \neq K_{kl}^{0,\pm}(-\omega)\), so that \(\mu_k(t)\) and \(\phi_k(t)\) do not commute with the respective quantities at different times.

The model, herein discussed, is an extension of the quantum analysis of Graham and Haken \[20,21\] and Fabre et al. \[17\]. In previous works the variance and covariance of the OPO beams have been analyzed by representing the fields either as operators \[15\] or classical variables. While Reid and Drummond \[23\] have taken in full account the signal/idler ones. The evolution of the mode amplitudes \(a_0\) (pump), \(a_1\) and \(a_2\) (signal/idler) of an OPO is described by a system of Langevin equations

\[
\frac{\partial}{\partial t} \begin{pmatrix} a_1 \\ a_2 \\ a_0 \end{pmatrix} = \begin{pmatrix} -\kappa_1' a_1 + 2\chi a_0 a_1^* + R_1' \\ -\kappa_2' a_2 + 2\chi a_0 a_2^* + R_2' \\ -\kappa_0' a_0 - 2\chi^* a_1 a_2 + \epsilon + R_0' \end{pmatrix}
\]

with \(R_k'(t)\) independent fluctuating delta-correlated Langevin forces and \(\epsilon = |\epsilon|e^{-i\phi_p}\) with the pump phase \(\phi_p(t)\) undergoing a slow diffusion, \(\langle (\phi_p(t) - \phi_p(0))^2 \rangle = 2\Delta\nu L\tau\). The coefficients \(\kappa_k'\) \((k = 0, 1, 2)\) stand for

\[
\kappa_k' = \gamma_k + \varsigma_k + i\varphi_k = \gamma_k + i\varphi_k = \kappa_k + \varsigma_k
\]

with \(\gamma_k\) and \(\varsigma_k\) the cavity mode damping rates, associated respectively with the output mirror and the other loss mechanisms (mainly due to the non-linear crystal), \(\varphi_k\) the detunings and \(2\chi\) the strength of the parametric interaction.

The operators \(R_k'\) are proportional to the vacuum fluctuation \(b_{in_k,1}\) entering the system through the output mirror and to the noise \(b_{in_k,2}\) associated with the crystal losses,

\[
R_k' = \sqrt{2\gamma_k}b_{in_k,1} + \sqrt{2\varsigma_k}b_{in_k,2} \equiv R_k + \sqrt{2\varsigma_k}b_{in_k,2}
\]

II. EQUATIONS OF MOTION

The evolution of the mode amplitudes \(a_0\) (pump), \(a_1\) and \(a_2\) (signal/idler) of an OPO is described by a system of Langevin equations

\[
\frac{\partial}{\partial t} \begin{pmatrix} a_1 \\ a_2 \\ a_0 \end{pmatrix} = \begin{pmatrix} -\kappa_1' a_1 + 2\chi a_0 a_1^* + R_1' \\ -\kappa_2' a_2 + 2\chi a_0 a_2^* + R_2' \\ -\kappa_0' a_0 - 2\chi^* a_1 a_2 + \epsilon + R_0' \end{pmatrix}
\]
Applying now the time reversal to the above Langevin equations \[(4,22)\] it can be easily shown that output, internal and input fields \((b_{\text{out}k,1}, a_k, b_{\text{in}k,1})\) are connected in presence of detuning by the standard boundary conditions \[(11)\]

\[
b_{\text{out}k,1} = \sqrt{2\gamma_k}a_k - b_{\text{in}k,1}
\]

Finally, the amplitude-phase representation

\[
a_k = r_ke^{-i\phi_k} (1 + \mu_k)
\]

will be adopted in the following, with \(r_k\) the steady state amplitude, \(\phi_k\) the phase, and \(\mu_k\) the relative amplitude fluctuation.

III. STEADY-STATE SOLUTION

Dropping the Langevin forces and the fluctuating amplitudes, the system \[(1)\] admits the steady state solution (see Eq. \[(5)\])

\[
\alpha_k = \langle a_k \rangle = r_0e^{-i\phi_0}
\]

\((\alpha_k^* = \langle a_k^* \rangle)\). Although \(\phi_k\) fluctuates it has been included in order to get average values of some phase combinations.

The signal/idler amplitudes \(\alpha_j (j = 1, 2)\) are different from zero for \(|\epsilon|\) above a threshold

\[
e^{th} = \frac{|\kappa_0'| \sqrt{|\kappa_1^*\kappa_2^*|}}{2\chi} = \frac{\gamma_0^* \sqrt{\gamma_1^* \gamma_2^*}}{2\chi \cos \psi} = |\kappa_0'| r_0
\]

only if the relative detunings \(\varphi_j\) (see Eq. \[(2)\]) are proportional to the damping factors \(\gamma_j' = \gamma_j + \kappa_j\) of the respective modes, i.e.

\[
\frac{\varphi_1}{\gamma_1^*} = \frac{\varphi_2}{\gamma_2^*} = \tan \psi, \quad \frac{\varphi_0}{\gamma_0^*} = \tan (\psi_0 - \psi)
\]

Most OPO cavities are stabilized on the pump mode, so that in the following we will put \(\psi = \psi_0\).

Amplitudes \(r_k\) and phases \(\phi_k\) of the three modes result interconnected

\[
|\kappa_1| r_1^2 = |\kappa_2| r_2^2 = (E - 1) |\kappa_0'| r_0^2
\]

\[
\langle \phi_1 + \phi_2 - \phi_0 \rangle = \psi
\]

with \(E\) related to the amplitude parameter \(E = |\epsilon/e^{th}|\) by

\[
E = \sqrt{E^2 - \sin^2 \psi + 1 - \cos \psi}
\]

In absence of detuning \(E\) reduces to \(E\). Notice that the relation between the frozen phases of the three beams (see f.i. Eq. (2.9) of Ref. [12]) depends on the detuning through the phase \(\psi\).

In turn \(r_0\) and \(\phi_0\) are respectively given by

\[
r_0 = \frac{\sqrt{|\kappa_1^* \kappa_2^*|}}{2\chi}
\]

\[
\langle \phi_p - \phi_0 \rangle = \psi - \psi_p
\]

with \(\phi_p\) the phase of the external pump and \(\psi_p = \arcsin (\sin \psi/E)\).

In the following the parameter

\[
C^2 = (E - 1) |\kappa_0'| r_0^2
\]

will be used.
The phase relations obtained for the steady state solutions are satisfied only approximately. In particular, the phases \( \phi_1 + \phi_2 - \phi_0 \) and \( \phi_0 - \phi_p \) (Eqs. (5-b) and (6-b)) undergo stable fluctuations, while \( \phi_1 - \phi_2 \) follows an undamped diffusion process. On the other hand, \( \phi \) is given by (12-a) with 

\[
\delta a_k = r_k e^{-i\phi_k} \delta \phi_k
\]

where \( \delta a_k = \mu_k - i\phi_k \) (Eq. (4)).

By exploiting the smallness of the quantities \( \phi_1 + \phi_2 - \phi_0 - \psi \) and \( \phi_0 - \phi_p + \psi - \psi_p \), the system can be linearized with respect to the fluctuating phases \( \delta \phi_k \) and amplitudes \( \mu_k \). Next, changing the phases \( \phi_k \) and \( \phi_p \) into \( \phi'_k = \phi_j - \frac{1}{2} \psi_p, \phi'_p = \phi_j + \psi - \psi_p \), and \( \phi'_p = \phi_p + \psi_p \), does not change the statistical properties of the system solutions. Thus, keeping the old symbols for the new quantities, the linearized system reads:

\[
\begin{align*}
\frac{\delta a_1}{\kappa_1^j} + \delta a_1 - \delta a_2 - \delta a_0 &= Z_1' \\
\frac{\delta a_2}{\kappa_2^j} + \delta a_2 - \delta a_0 - \delta a_1 &= Z_2' \\
e^{i\psi_0} \left( \frac{\delta a_0}{r_0^j} + \delta a_0 \right) + \delta a_1 + \delta a_2 &= Z_0' + Z_\psi + Z_\phi
\end{align*}
\]

with

\[
\begin{align*}
Z_j' &= e^{i\phi_j} R_j^j \quad (j = 1, 2) \\
Z_0' &= \frac{1}{E-1} e^{i\phi_0} R_0^j \frac{e^{i\psi_0} R_0^j}{r_0^j} \\
Z_\psi &= \frac{E}{E-1} e^{i\psi_p} \phi_p \\
Z_\phi &= -i \frac{E}{E-1} e^{i\psi_p} \phi_p
\end{align*}
\]

having denoted by \( \mu_c = \delta e/\epsilon \) and \( \phi_p \) the pump relative excess noise and phase fluctuation respectively.

The output field, having average intensity \( \bar{I}_j = 2\gamma_j r_j^2 = (2\gamma_j/\gamma'_j) \cos \psi C^2 \), is given by (see Eq. (4))

\[
b_{out, j} = \sqrt{2\gamma_j r_j^2} e^{-i\phi_j} \left( 1 + \mu_j - \frac{\kappa_j^j}{2\gamma_j} Z_j \right)
\]

where \( Z_j \) is given by \( (2a) \) with \( R_j^j \) replaced by \( R_j \).

Before concluding this section, it is worth looking at the approximations underlying the linear system (11). Products of phase-factors \( e^{i\phi} e^{-i\chi} \) can be represented by the generalized Campbell-Hausdorff formula (see f.i. \( [24] \) p. 35)

\[
e^{i\phi} e^{-i\chi} = \exp \left( i\phi - i\chi + \frac{1}{2} [\phi, \chi] + \frac{i}{12} [[\phi, \chi], \phi + \chi] \cdots \right)
\]

so that choosing \( \phi = \phi(t), \chi = \phi(t + dt) \) we have for the derivative of \( a_k(t) \)

\[
\dot{a}_k(t) = r_k e^{-i\phi_k} \left( \delta a_k(t) - \frac{1}{2} [\delta \phi_k(t), \phi_k(t)] + \frac{i}{6} \left[[\phi_k, \dot{\phi}_k], \phi_k\right] \cdots \right)
\]

which reduces to Eq. (10) at the first order in the fluctuating quantities.

V. FREQUENCY ANALYSIS

Now it is worth introducing the Fourier transforms
with $\hat{a}_k^\dagger (\omega) = \hat{a}_k^\dagger (-\omega^*)$ the adjoint of the Fourier component relative to the frequency $-\omega$. In the following we will consider generally complex frequencies $\omega$ and the symbol $\hat{\cdot}$, operating on the product of a c-function times an operator, will indicate the transformation $|f(\omega)|^2 = f^*(\omega)^2 |\hat{f}(\omega)|^2$.

In the frequency domain Eq. (11) reduces to the algebraic system:

$$
\begin{align*}
\Delta_1 \hat{a}_1 - \hat{a}_0 - \hat{a}_2^\dagger &= \hat{Z}_1' \\
\Delta_2 \hat{a}_2 - \hat{a}_0 - \hat{a}_1^\dagger &= \hat{Z}_2' \\
\Delta_0 \hat{a}_0 + \hat{a}_1 + \hat{a}_2 &= \hat{Z}_0' + \hat{Z}_c + \hat{Z}_\phi
\end{align*}
$$

with

$$
\Delta_j = 1 + i \frac{\omega}{\kappa_j}, \quad \Delta_0 = \left(1 + i \frac{\omega}{\kappa_0}\right) \frac{e^{i\psi_0}}{\mathcal{E} - 1}
$$

According to Eq. (12.c,d) $\hat{Z}_c = \frac{E_0}{\mathcal{E}} e^{i\psi_c} \hat{\mu}_c$ and $\hat{Z}_\phi = -i \frac{E_0}{\mathcal{E}} e^{i\psi_\phi} \hat{\phi}_p$ depend respectively on the pump amplitude excess noise $\hat{\mu}_c$ and on the phase $\hat{\phi}_p$.

Next, completing the system (15) with the respective $\hat{\cdot}$-transformed equations we obtain for the Fourier transforms $\hat{a}_k (\omega)$, and $\hat{a}_k^\dagger (\omega)$ a sixth-order algebraic system depending on the forcing terms $\hat{Z}_1'$, $\hat{Z}_c$, $\hat{Z}_\phi$, $\hat{Z}_2'$, $\hat{Z}_\phi^\dagger$. Being an OPO a stable system (see f.i. Ref. [12]), $\hat{a}_k (\omega)$, $\hat{a}_k^\dagger (\omega)$ may become singular only in the upper imaginary plane $\text{Im} (\omega) \geq 0$.

Now, solving the system (15) completed by the $\hat{\cdot}$-counterpart yields

$$
\hat{a}_k = \frac{A_k^1 C_k - B_k C_k^1}{A_k A_k^1 - B_k B_k^1}
$$

with the functions $A_k (\omega), B_k (\omega), C_k (\omega)$ defined in Appendix A.

Next, introducing the Fourier transformed quadratures of the output modes $\hat{\mu}_{\beta_k} = \frac{1}{2} (\hat{\beta}_k + \hat{\beta}_k^\dagger)$, and $\hat{\phi}_{\beta_k} = \frac{i}{2} (\hat{\beta}_k - \hat{\beta}_k^\dagger)$ we have (see Appendix A)

$$
\begin{align*}
\hat{\mu}_{\beta_j} &= \left( K_{jj}^0 - \frac{\kappa_j' \kappa_j}{2\kappa_j} \right) \otimes \hat{X}_j^0 + K_{jj}^0 \otimes \hat{X}_\sigma^0 + K_{jj}^0 \otimes \hat{X}_{e}^0 + K_{jj}^0 \otimes \hat{X}_{\phi}^0 + \left( \hat{X}_{0}^{\psi_0} + \frac{E}{\mathcal{E} - 1} \hat{X}_{e}^{\phi_\psi} - \frac{E}{\mathcal{E} - 1} \hat{X}_{\phi}^{\phi_\psi} \right) \\
\hat{\phi}_{\beta_j} &= i \left( K_{jj}^0 - \frac{\kappa_j' \kappa_j}{2\kappa_j} \right) \otimes \hat{X}_j^0 + i K_{jj}^0 \otimes \hat{X}_\sigma^0 + i K_{jj}^0 \otimes \hat{X}_{e}^0 + i K_{jj}^0 \otimes \hat{X}_{\phi}^0 + \left( \hat{X}_{0}^{\psi_0} + \frac{E}{\mathcal{E} - 1} \hat{X}_{e}^{\phi_\psi} - \frac{E}{\mathcal{E} - 1} \hat{X}_{\phi}^{\phi_\psi} \right)
\end{align*}
$$

where $K \otimes \hat{X}$ is a shorthand for $\frac{1}{2} \left( K (\omega) b(\omega) + K^* (-\omega^*) b^\dagger (\omega) \right)$.

The functions $K_{kl}^{0,\pm}$ represent the frequency responses to the quantum noise sources and the classical phase and amplitude fluctuations of the pump laser. They are analytic in $\omega$ with poles in the upper half plane $\text{Im} (\omega) \geq 0$ coincident with some zeros of $A_k A_k^1 - B_k B_k^1$, as shown in the following sections.

According to (17) the amplitudes and phases depend on the quadratures $X_0^\phi = \frac{1}{2} \left( e^{i\theta} Z_k + e^{-i\theta} Z_k^\dagger \right)$ of the $k$-th vacuum noise entering the cavity through the coupling mirror and that relative to the crystal loss $\sigma_k$, $X_\sigma^\theta$. The fluctuating amplitude $\mu_e$ and phase $\phi_p$ of the external pump are represented respectively by $X_e^\phi$ and $X_{\phi_p}^\theta$.

**VI. RESONANCES**

Important features of the OPO fluctuations can be investigated by extending the system (15) to complex values of $\omega$. This will be done by retaining the above definition of the operator $\hat{\cdot}$ transforming an analytic function of $\omega$, with poles $\Omega_e$, in a new one with poles $-\Omega_e^*$ and complex conjugate residues. The system of six unknown functions $\hat{a}_k (\omega)$, $\hat{a}_k^\dagger (\omega)$ is characterized by the sixth order characteristic polynomial $D (\omega) = D' (\omega) \hat{\omega}$ with a zero at the origin,
\[
D'(\tilde{\omega}) = \tilde{\omega}^5 + D^{(5)}\tilde{\omega}^4 + D^{(4)}\tilde{\omega}^3 + D^{(3)}\tilde{\omega}^2 + D^{(2)}\tilde{\omega} + D^{(1)}
\]

with \(\tilde{\omega}\) the frequency normalized to \(\gamma\). The coefficients \(D^{(j)}\) are reported in Appendix B (Eq. (37)).

For a detuned balanced cavity (\(\delta = |\tilde{\Omega}_1| = 0\)) \(D^{(1)}\) are independent of \(\tan \psi\) so that \(D'\) coincides with that of a resonant OPO having an effective excitation parameter \(E_{eff} = 1 + (\xi - 1) / \cos \psi\). In particular, \(D'\) factorizes into the product \(D_+ D_-\) of polynomials discussed in the following for the resonant case. This means that a detuned balanced OPO cannot be distinguished from a perfectly resonant one with an appropriate pump strength.

The six zeros \(\tilde{\Omega}_r\) \((r = 0, \ldots, 5)\) of \(D'(\tilde{\omega})\) represent the system resonances. In particular, the real parts of their values indicate the oscillation frequencies, whereas the imaginary ones stand for the corresponding damping parameters, i.e. the resonance widths. Since \(D(\tilde{\omega}) = D'(\tilde{\omega})\), each resonance \(\tilde{\Omega}_r\) having a finite real part is mirrored by \(-\tilde{\Omega}_r^\ast\). Apart from the one at the origin \((\tilde{\Omega}_0 = 0)\), the resonance \(\tilde{\Omega}_1 = i\omega_1\) is located on the positive imaginary axis, while the remaining four ones may be either purely imaginary or may have finite real parts. In fact, for excitation parameter varying in an interval depending on the cavity damping factors, one or both couples \(\tilde{\Omega}_r, -\tilde{\Omega}_r^\ast\) are replaced by two purely imaginary roots. In the case of non-vanishing real part \(\tilde{\Omega}_2 = -\tilde{\Omega}_3^\ast = \omega_2 + i\omega_2\) and \(\tilde{\Omega}_4 = -\tilde{\Omega}_5^\ast = \omega_4 + i\omega_4\): the system is characterized by two dampings \(w_2, w_4\), and frequencies \(\omega_2, \omega_4\). In particular, \(w_1 + 2w_2 + 2w_4\) is independent of the detuning, whereas for a balanced OPO \(w_1 = 2\).

The expressions of \(\tilde{\Omega}_r\) are rather lengthy and complicated, thus, for the sake of simplicity, we briefly discuss their major features, and dwell on two noticeable cases. Fig. 1 displays the dependence of the real and imaginary parts of \(\tilde{\Omega}_r\) on the pump parameter \(E\) for \(\psi = 0.3, \gamma_0 = 2\) and a 20\% mismatch \((\delta = 0.1)\). The resonance frequencies \(\tilde{\omega}_2\) and \(\tilde{\omega}_4\) start from the origin and grow approximately like \(\sqrt{E}\). The damping parameter \(\omega_1\) starts from zero and rapidly reaches the constant value of 2, whereas \(\omega_2 = 2\) is independent of \(E\), and \(\omega_4\) moves quickly away from 2 toward 1. This behavior is not very sensitive to the detuning angle.

At resonance \(D'\) factorizes into \(i\gamma^{(3)} D_+ D_-\), being \(D_-\), and \(D_+\) two polynomials of second and third order respectively,

\[
D_- (\tilde{\omega}) = \tilde{\omega}^2 - i\gamma^{(1)} \tilde{\omega} - 2\gamma^{(2)}, \quad D_+ (\tilde{\omega}) = \tilde{\omega}^3 - i\gamma^{(1)} \tilde{\omega}^2 - \gamma^{(2)} \tilde{\omega} + i\gamma^{(3)}
\]

where \(\gamma^{(1)} = 2 + \gamma_0', \gamma^{(2)} = 2E\gamma_0',\) and \(\gamma^{(3)} = 4(E - 1)\gamma_0' (1 - \delta^2)\).

\(D_-\) has two roots \(\tilde{\Omega}_4, \tilde{\Omega}_5\) which are purely imaginary if \(\gamma^{(1,2)} > 8\gamma^{(2)}\), otherwise \(\tilde{\Omega}_4 = -\tilde{\Omega}_5^\ast\). In turn \(D_+\) has three purely imaginary roots \((\tilde{\Omega}_1 = i\omega_1, \tilde{\Omega}_2 = i\omega_2,\ and \tilde{\Omega}_3 = i\omega_3\) if

\[
\gamma^{(1,2)} > 3\gamma^{(2)}, \quad \gamma^{(1,2)} \gamma^{(2)} > 9\gamma^{(3)}, \quad (\gamma^{(1,2)} - 4\gamma^{(2)}) \gamma^{(2)} > 27\gamma^{(3)} + 2(2\gamma^{(1,3)} - 9\gamma^{(1)} \gamma^{(2)})
\]

According to these inequalities (derived by means of the Sturm sequence) the roots are all purely imaginary for \(1 < E < E_{max}\), with \(E_{max}\) depending on \(\delta\) and \(\tilde{\delta}\). In the example plotted in Fig. 2 for \(\delta = 0.05, E_{max}\) grows with \(\gamma_0'\) except for the presence of a peninsula in the interval \(\gamma_0' \approx 1.5 \pm 2.75\). Scaling the threshold with \(\gamma_0'\) (Eq. (18)), for \(\gamma_0' > 1\) the roots are purely imaginary for all realistic values of the excitation parameters. This explains why the relaxation oscillations (resonances with non-vanishing real parts) can be observed only for \(\gamma_0'\) not much greater than \(\gamma\).

While the poles of \(K^\frac{\tau}{jk}\) control the transient behavior of \(\langle \phi^2_j (\tau, 0) \rangle\), the relaxation oscillations correspond to those of \(K^\frac{\tau}{jk}\). In particular, for \(E\) rather large

\[
\tilde{\Omega}_1 \simeq i \left(1 - \delta^2\right), \quad \tilde{\Omega}_2 \simeq \frac{i}{2} \left(1 + \delta^2 + \gamma_0'\right) + \sqrt{2E\gamma_0'}
\]

Similar expressions were obtained in Ref. [10] on the basis of a rate equation model for the relaxation oscillations.

In Fig. 3 we have plotted the real and imaginary parts of \(\tilde{\Omega}_2, \tilde{\Omega}_3\) versus \(1 < E < 10\) for different decay rates \(\gamma_0' = 0.5, 2.5, 4.5,\) and \(6.5\) and \(\delta = 0\). These plots also display the dependence of the roots on the detuning angle \(\psi\) of a balanced system having an effective excitation parameter \(E_{eff}\) equal to \(E\).

In the adiabatic limit \((\gamma_0' \gg \gamma^\ast)\), see f.i. Ref. [11] \(D'(\tilde{\omega})\) reduces to a third order polynomial,

\[
D'(\tilde{\omega}) = \tilde{\omega}^3 + D^{(3)}\tilde{\omega}^2 + D^{(2)}\tilde{\omega} + D^{(1)}
\]

that is the OPO is characterized only by four poles, \(\tilde{\Omega}_0 = 0, \tilde{\Omega}_1 = i\omega_1, \tilde{\Omega}_2 = -\tilde{\Omega}_3^\ast = \tilde{\Omega}_2 + i\omega_2\) and the relaxation oscillations disappear.
Consequently, the noise correlations have in the frequency domain the following expressions:

\[ K_{jj}^0 (\omega) \approx \frac{1}{4} \left( 1 - \frac{\tan \psi}{E_{\text{eff}}/1} \right) \]

\[ K_{jj}^\pm (\omega) \approx -K_{jj}^\pm (\omega) \approx -\frac{1 - \delta^2}{4 \cos^2 \psi} \]

\[ K_{jj}^\mp (\omega) \approx \frac{(E_{\text{eff}} - 1) - (\tan^2 \psi - (E_{\text{eff}} - 1) + i E_{\text{eff}} \tan \psi) \hat{\delta}_j}{2E_{\text{eff}}} \]

where \((E_{\text{eff}} - 1) = (E - 1)/\cos \psi\) represents the effective distance from the threshold of a detuned device. Analogously the phase reduces for \(\omega \approx 0\) to

\[ \hat{\phi}_{jj} (\omega) \approx -\frac{1 - \delta^2}{4 \cos^2 \psi} \omega \left( \hat{X}_{j0} - \hat{X}_{jj}^0 \right) - \frac{(E_{\text{eff}} - 1) - (\tan^2 \psi - (E_{\text{eff}} - 1) + i E_{\text{eff}} \tan \psi) \hat{\delta}_j}{2E_{\text{eff}}} \otimes \hat{X}_{\phi p}^{+} + \psi_p \]

namely the detuning enhances, through the factors \(\cos^{-2} \psi\), the phase fluctuations associated to the quantum noise sources.

The mathematical difficulties associated with the singularity of \(K_{jj}^\pm, K_{jj}^\mp\) at the origin can be avoided by displacing slightly the zero \(\Omega_0 = 0 \rightarrow \Omega_0 = i \varepsilon \quad (\varepsilon > 0)\), and letting \(\varepsilon \rightarrow 0\) once performed the integrations.

The presence of poles with real parts larger than the imaginary ones is evidenced in the plots of \(|K_{jj}^0|^2, |K_{jj}^0|^2\) and \(|K_{jj}^0|^2\) reported in Fig. 4 for cavity parameters \(\tilde{\gamma}_0^0 = 1, \tilde{\delta} = 0.05, \text{and} \ E = 2, 3, 4\). They exhibit a behavior typical of triply resonant devices, namely peaked at frequencies increasing with the pump rate. While the peak height of the self-coupled response \(K_{jj}^0\) (Figs 4a) slightly decreases with \(E\), those of the cross-coupling \(K_{jj}^0\) (Figs 4b) and of the response \(K_{jj}^0\) (Figs 4c) to pump fluctuations keep, respectively, nearly constant and growing in height with \(E\). All these frequency responses are not very sensitive to the detuning.

VIII. SPECTRAL DENSITIES

For vacuum input fields, only antinormally ordered terms contribute to the variances, i.e.

\[ \langle \hat{b}_{in} (\omega) \hat{b}_{in}^\dagger (\omega') \rangle = \delta (\omega^* + \omega') \]

Consequently, the noise correlations have in the frequency domain the following expressions:

\[ \langle \hat{Z}_j^i (\omega) \hat{Z}_j^i (\omega') \rangle' = \frac{2 \gamma_j^i}{|K_j^i|^2} = \gamma_j^i \]

\[ \langle \hat{Z}_0^i (\omega) \hat{Z}_0^i (\omega') \rangle' = \frac{2 \gamma_0^i}{|K_0^i|^2} \frac{1}{E - 1} = \gamma_0^i \]

\[ \langle \hat{Z}_e (\omega) \hat{Z}_e (\omega') \rangle' = \langle \hat{Z}_e (\omega) \hat{Z}_e (\omega) \rangle' = \left( \frac{E}{E - 1} \right)^2 S_e (\omega) \]

\[ \langle \hat{Z}_\phi (\omega) \hat{Z}_\phi (\omega') \rangle' = \langle \hat{Z}_\phi (\omega) \hat{Z}_\phi (\omega) \rangle' = \left( \frac{E}{E - 1} \right)^2 S_\phi (\omega) \]

(22) all the other terms vanishing identically. \(S_e (\omega) = \langle |\mu_e (\omega)|^2 \rangle\) and \(S_\phi (\omega) = \Delta \nu_e^2 / \omega^2\) stand for the spectral densities of the pump amplitude and phase respectively. The apex (‘) indicates the omission on the right-side of the factor \(\delta (\omega^* + \omega') / C^2\).
A. Gaussian statistics

As a result of the superposition of uncorrelated spectral components, the single beam fluctuation amplitude $\delta a(t) = k\hat{a}$ is proportional through a coefficient $k$ to the annihilation operator $\hat{a}$ of an oscillator associated to a Gaussian density operator (see [25,26])

$$\rho = Ne^{-(n_\mu + n_\phi)K_0 + \frac{n_\mu - n_\phi}{2}(K_+ + K_-)} = Ne^{2nK_0 e^{uK_+} e^{-wK_-}}$$

(23)

with $2K_+ = \hat{a}^\dagger \hat{a}^2$, $2K_- = \hat{a}^2$, $4K_0 = \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger$ forming a realization of the SU(1,1) group, with the relevant algebraic structure displayed by the commutators $[K_-, K_+] = 2K_0$, $[K_0, K_\pm] = \pm K_\pm$. In the above equation $\rho$ has been expressed either as a single exponential depending on the coefficients $n_\mu$ and $n_\phi$ or as a disentangled product of exponentials depending on $u$, $v$ and $w$. The connection between these two representations is displayed by the relations

$$2u = \ln(-w/v)\ , \ v = \frac{\sinh\sqrt{n_\mu n_\phi} \sinh\left(\sqrt{n_\mu n_\phi} + \Theta\right)}{\sinh^2\Theta} , \ w = -\frac{\sinh\sqrt{n_\mu n_\phi}}{\sinh\left(\sqrt{n_\mu n_\phi} + \Theta\right)}$$

with $\cosh\Theta = (n_\mu + n_\phi)/(n_\mu - n_\phi)$. Here we limit ourselves to report the results of calculations based on the Wei-Norman technique [27] (see also [28]) which will be discussed elsewhere together with the calculation of the trace

$$Tr(e^{2uK_0 e^{vK_+} e^{-wK_-}}) = \frac{1}{\sqrt{(vw + 1) e^u - 2 + e^{-u}}}$$

The moments of $K_0$ and $K_\pm$ can be easily obtained by differentiating the operator $e^{2uK_0 e^{vK_+} e^{-wK_-}}$ with respect to $u$, $v$ and $w$. In particular,

$$\langle[\phi, \mu]\rangle = \frac{i}{2}k^2 , \ \langle\mu^2\rangle = \frac{k^2}{2}\left(\frac{1}{n_\mu} + \frac{1}{2}\right) , \ \langle\phi^2\rangle = \frac{k^2}{2}\left(\frac{1}{n_\phi} + \frac{1}{2}\right)$$

It is straightforward to extend the above representation to the twin pair $\delta a_1(t) = k\hat{a}$, $\delta a_2(t) = k\hat{b}$ for a balanced OPO,

$$\rho = N\exp\left(C_0^+ K_0^+ + C_0^- K_0^- + C^+ (K_+^+ + K_-^+)) + C^- (K_+^- + K_-^-)\right)$$

with $4K_{\pm}^\pm = \{\hat{a}^\dagger/\hat{a}\}^2 + \{\hat{b}^\dagger/\hat{b}\}^2$, $2K_0^\pm = K_{0a} + K_{0b}$, $2K_\pm = \{\hat{a}\hat{b}/\hat{a}^\dagger\hat{b}\}$ and $4K_0^\pm = \hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger$ forming an enveloping algebra of SU(1,1) characterized by the algebraic structure $2\left(K_0^+, K_{\pm}^\pm/\tau\right) = \pm K_{\pm}^\pm/\tau$, $2\left[K_{\pm}^\pm, K_{\mp}^\pm\right] = K_{\pm}^\pm - K_{\mp}^\pm$ all the other commutators vanishing identically. The disentangling of this operator will be discussed elsewhere.

Using the squeezing operator $S(\theta) = \exp\left[\frac{\theta}{2} (\hat{a}^2 - \hat{a}^2)\right]$ with $\tanh(2\theta) = 1/\cosh\Theta$, introduced by Gardiner and Zoller [29] for transforming a Gaussian density matrix into a thermal oscillator, we obtain in the present case $\rho = 2\sinh\frac{n_\mu n_\phi}{2} \exp(-2\sqrt{n_\mu n_\phi} K_0)$. Accordingly the twin beam fluctuations can be represented as a pair of squeezed and coupled thermal oscillators.

Finally, from the Gaussian statistics it descends for the phase factors $e^{i\phi_j(t)}$ correlation

$$\langle e^{i\phi_j(t)} e^{-i\phi_j(0)} \rangle = e^{\langle(\phi_j(t) - \phi_j(0))\phi_j(0)\rangle}$$

(24)

B. Correlations and frequency spectra

The various correlations can be represented as combinations of the functions

$$K_{\phi_j\phi_l}(\tau) = \sum_k \zeta_k \sum_{r=1}^5 e^{i\Omega_{r,j}^k} K_{jk,r}^+ K_{ik}^+(\Omega_{r}^+)$$

$$K_{\phi_j\phi_l}(\tau) = \sum_k \zeta_k \sum_{r=1}^5 e^{i\Omega_{r,j}^k} K_{jk,r}^0 K_{ik}^0(\Omega_{r}^+)$$

$$K_{\mu_j\mu_l}(\tau) = \sum_k \zeta_k \sum_{r=1}^5 e^{i\Omega_{r,j}^k} K_{jk,r}^0 K_{ik}^0(\Omega_{r}^+)$$

8
\[ K_{\mu_j \phi_1}(\tau) = \sum_k \zeta_k^l \sum_{r=1}^5 e^{i\Omega_r \tau} K_{jk,r}^0 K_{1k}^{\ast}(\Omega_r^r) \]
\[ K_{\phi_j \mu_1}(\tau) = \sum_k \zeta_k^l \sum_{r=1}^5 e^{i\Omega_r \tau} K_{jk,r}^\pi K_{1k}^0(\Omega_r^r) \]
(25)

with \( l = 0, 1, 2 \), the sums being extended to the 5 roots \( \Omega_r \) of the characteristic polynomial \( D' \) (i.e. except for \( \Omega_0 \)), and \( K_{jk,r}^0 \) standing for the residues of \( K_{jk,r}^0(\omega) \) at \( \Omega_r \).

We have for the correlations
\[ \langle (\phi_j(\tau) - \phi_j(0))(\phi_j(0)) \rangle = \frac{i}{4C^2} (K_{\phi_j \phi_j}(\tau) - K_{\phi_j \phi_j}(0)) - \Delta \nu_j \tau \]
\[ \langle \mu_j(\tau) \mu_j(0) \rangle = \frac{i}{4C^2} K_{\mu_j \mu_j}(\tau) \]
\[ \langle \mu_j(\tau) \phi_j(0) \rangle - \langle \phi_j(\tau) \mu_j(0) \rangle = \frac{1}{4C^2} (K_{\mu_j \phi_j}(\tau) + K_{\phi_j \mu_j}(\tau)) \]
(26)

and the commutators
\[ \langle [\phi_j(0), \phi_j(\tau)] \rangle = \frac{i}{2C^2} \text{Re} \{ K_{\phi_j \phi_j}(\tau) \} \]
\[ \langle [\mu_j(0), \mu_j(\tau)] \rangle = -\frac{i}{2C^2} \text{Re} \{ K_{\mu_j \mu_j}(\tau) \} \]
\[ \langle [\phi_j(0), \mu_j(\tau)] \rangle = \frac{1}{2C^2} \text{Im} \{ K_{\phi_j \mu_j}(\tau) \} \]
\[ \langle [\phi_j(0), \phi_j(\tau)] \rangle = \frac{i}{2C^2} \text{Re} \{ K_{\phi_j \mu_j}(\tau) \} \]
(27)

with \( \tau > 0 \). While (27a,b,c) vanish at resonance, the last expression does not. For a detuned OPO \( \mu_j(t) \) and \( \phi_j(t) \) do not commute with the respective quantities at different times while \( \mu_j(t) \) does not anticommute with \( \phi_j(t') \). By similar reasoning it can be also shown \( \langle \phi_j(0) \phi_j(0) \rangle \neq 0 \), that is the instantaneous frequency is correlated with the phase. In Fig. 5 the phase commutator has been plotted versus \( \tau \) for an OPO with \( \gamma_0 = 4, \delta = 0.2 \) and \( \psi = 0.1 \) and \( E = 1.5, 4 \) (Eq. 8). These plots exhibit respectively oscillatory and damped behaviors since for \( E = 4 \) the roots have non vanishing real parts, while in the other case they are almost purely imaginary.

The variance of the phase delay is given by
\[ \langle (\phi_j(\tau) - \phi_j(0))^2 \rangle = \frac{1}{2C^2} \text{Im} \{ K_{\phi_j \phi_j}(\tau) - K_{\phi_j \phi_j}(0) \} + 2\Delta \nu_j \tau \]
with
\[ \Delta \nu_j = F_\phi \Delta \nu_L + \frac{1}{8C^2} \left( 1 - \frac{\delta^2}{\cos^2 \psi} \right)^2 \]
\[ F_\phi = \left| \frac{(E_{eff} - 1) (\tan^2 \psi - (E_{eff} - 1) + iE_{eff} \tan \psi) \frac{\hat{\delta}_j E \cos \psi}{E_{eff} - 1}}{2E_{eff}} \right|^2 \]

In a similar fashion we have for the phase sum variance
\[ \langle (\phi_j(\tau) + \phi_j(0))^2 \rangle = \frac{1}{4C^2} \text{Im} \{ K_{\phi_j \phi_j}(\tau) - K_{\phi_j \phi_j}(0) + K_{\phi_j \phi_j}(0) - K_{\phi_j \phi_j}(0) \} + 2\Delta \nu_j \tau \]
so that (see 23) in view of Eq. 24
\[ \langle e^{i\phi_j(\tau)} e^{-i\phi_j(0)} \rangle = e^{\Delta \nu_j \tau} \left( 1 + \frac{i}{4C^2} (K_{\phi_j \phi_j}(\tau) - K_{\phi_j \phi_j}(0)) \right) \]
\[ \langle e^{-i\phi_j(\tau)} e^{-i\phi_j(0)} \rangle = e^{\Delta \nu_j \tau} \left( 1 + \frac{i}{4C^2} (K_{\phi_j \phi_j}(\tau) - K_{\phi_j \phi_j}(0)) \right) \]
while the second-order correlations for the mode amplitudes $a_j, a_j^\dagger$ read

$$
\langle a_j^\dagger (\tau) a_j (0) \rangle = r_j^2 e^{-\Delta \nu_j \tau} \left[ 1 + \frac{i}{4C^2} \left( K_{\mu_j \mu_j} (\tau) + K_{\phi_j \phi_j} (\tau) - K_{\phi_j \phi_j} (0) - K_{\mu_j \phi_j} (\tau) - K_{\mu_j \phi_j} (0) \right) \right]
$$

$$
\langle a_j (\tau) a_j^\dagger (0) \rangle = r_j r_j' e^{-\Delta \nu_j \tau} \left[ 1 + \frac{i}{4C^2} \left( K_{\mu_j \mu_j} (\tau) + K_{\phi_j \phi_j} (\tau) - K_{\phi_j \phi_j} (0) - K_{\mu_j \phi_j} (\tau) - K_{\mu_j \phi_j} (0) \right) \right]
$$

Further, the intensity variances of the single beams are represented by

$$
\langle : a_j^\dagger (\tau) a_j (0) a_j^\dagger (0) a_j (0) : \rangle = r_j^4 (2 \mu_j (\tau) \mu_j (0) + \mu_j (0) \mu_j (\tau)) + i \langle \langle \phi_j (0) , \mu_j (\tau) \rangle \rangle
$$

where $: :$ standing for time-normal ordering. Analogously, for the respective spectra, normalized with respect to $r_j^4/2C^2$,

$$
:S_j := S_j + S_{[\mu \phi]_j}
$$

with $S_j$ separating in three terms

$$
S_j = S_{\mu_j} + \left( \frac{E}{\mathcal{E} - 1} \right)^2 S_{\epsilon_j} + \left( \frac{E}{\mathcal{E} - 1} \right)^2 S_{\phi_j}
$$

representing in the order the quantum and the pump amplitude and phase contributions

$$
S_{\mu_j} (\omega) = \sigma_{\mu_j}^2 (\omega) + \sigma_{\mu_j}^2 (-\omega)
$$

$$
S_{\epsilon_j} (\omega) = \sigma_{\epsilon_j}^2 (\omega) + \sigma_{\epsilon_j}^2 (-\omega)
$$

$$
S_{\phi_j} (\omega) = \sigma_{\phi_j}^2 (\omega) + \sigma_{\phi_j}^2 (-\omega)
$$

In turn, the normalized spectrum of the commutator [$\mu_j, \phi_j]$ splits in the components

$$
S_{[\mu, \phi]_j} (\omega) = \sigma_{[\mu, \phi]_j}^2 (\omega) + \sigma_{[\mu, \phi]_j}^2 (-\omega)
$$

with

$$
\sigma_{[\mu, \phi]_j}^2 (\omega) = \text{Re} \left\{ \sum_{k, r} \zeta_k \frac{K_{jk}^0}{\omega - \Omega_r} K_{jkr}^* (\Omega_r^*) \right\}
$$

Similar expressions hold for the output beams with $\bar{J}_j = 2\gamma_j r_j^2$ in place of $r_j^2$ and $\sigma_{\bar{J}_j}^2$ (Eq. (31) a) replaced by

$$
\sigma_{\bar{J}_j}^2 = \sum_{k \neq j} |K_{jk}^0|^2 \zeta_k + \left| K_{\bar{J}_j j}^0 - \frac{\zeta_j}{2\gamma_j} \right|^2 \gamma_j + \left| K_{j \bar{J}_j}^0 \right|^2 \gamma_j
$$

In Fig. 6, we have plotted $S_j$ (Eq. (31)), the spectrum measured by analyzing the photocurrent with a spectrum analyzer, for different detunings ($\psi = 0, 0.25, 0.5$) and pump parameter $E = 3$ by omitting the laser excess noise and phase fluctuations. For both $\delta_0 = 4$ (top) and 0.5 (bottom), in the case of zero and small detuning, $S_j$ starts from below the SNL, while, for $\psi = 0.5$, it starts just above the SNL. In all cases $S_j$ rapidly increases by reaching a peak in
correspondence of the relaxation oscillations frequency, and then decays monotonically toward the SNL. The height of the peak increases notably either with the detuning or by reducing the pump mode bandwidth.

Figure 7 describes the effects on the single beam spectrum of the pump excess noise at different detuning angles. \(\left(\frac{E}{I}\right)^2 S_{ej}(\omega)\) has been plotted for different detunings \(\psi = 0, 0.25, 0.5, \tilde{\delta} = 0.05,\) pump parameter \(E = 3,\) and crystal losses different for the signal/idler and the pump. For both cavity losses \(\gamma'_0 = 4\) (top) and 0.5 (bottom), \(\left(\frac{E}{I}\right)^2 S_{ej}\) increases with \(\psi\) and \(\tilde{\delta}\) by reaching a peak in correspondence of the relaxation oscillation. The behavior observed for \(0 < \gamma'_0 < 4\) leads us to conclude that for mode dampings of the same order, the detuning confines the observable squeezing to a small fraction of the twin–beam bandwidth. Moreover, it increases the amplitude of the excess laser noise transferred to the single beam spectrum, thus preventing the reduction below the shot noise level.

The difference spectrum \(S_d\) is obtained from \(S_j\) by replacing in (31), (32) and (33) \(K_{jk}^{0,\tilde{\delta}} = \left(I_1 K_{1k}^{\tilde{\delta}} - I_2 K_{2k}^{\tilde{\delta}}\right) / (I_1 + I_2)\) and analogously for the residues, \(\bar{\kappa}'\) and \(\bar{\gamma}'_0.\) In particular, the contributions of pump amplitude and phase fluctuations are represented by Eqs. (20)–(22) respectively, with \(\sigma_{\kappa}^2\) and \(\sigma_{\bar{\gamma}}^2\) replaced by \(\sigma_{\kappa'}^2\) and \(\sigma_{\bar{\gamma}'}^2,\) which are in turn defined by equations similar to (21)–(22) with \(K_{jk}^{0,\bar{\gamma}} = \left(I_1 K_{1k}^{\bar{\gamma}} - I_2 K_{2k}^{\bar{\gamma}}\right) / (I_1 + I_2).\) Being the spectral contributions of the laser excess noise and phase diffusion concentrated in the low frequency region, it is worth noting that (see Eq. (20–a))

\[
\Delta K_0^0(0) = \left(1 - i \tan \psi \right) \frac{\tilde{\delta}^{N}}{1 - \tilde{\delta}^2} \cos \psi
\]

so that

\[
\sigma_{\kappa'}^2(0) = \left(\frac{\tilde{\delta}^{N}}{1 - \tilde{\delta}^2}\right)^2 \left|\cos \psi \cos \psi_p + \frac{\sin \psi}{E_{eff}} \sin \psi_p\right|^2
\]

\[
\sigma_{\bar{\gamma}'}^2(0) = \left(\frac{\tilde{\delta}^{N}}{1 - \tilde{\delta}^2}\right)^2 \left|\cos \psi \sin \psi_p - \frac{\sin \psi}{E_{eff}} \cos \psi_p\right|^2
\]

These expressions evidence the deleterious effects of crystal losses and detuning also in the difference spectra. While for a resonant device only the excess noise influences the spectrum, in a detuned device also the phase diffusion contributes to the noise in proximity of the origin.

The plots of Fig. 8 show the difference spectrum \(S_d\) for an unbalanced resonant OPO \((\tilde{\delta} = 0.05, \psi = 0)\) for \(\gamma'_0 = 4\) (top) and 0.5 (bottom). The peak observed in the single beam spectrum for \(\gamma'_0 = 0.5\) (Figs. 6 and 7–bottom) is present also in the difference spectrum for the same damping ratio whereas for \(\gamma'_0 = 4\) (top) the difference spectrum keeps unchanged its sub–shot–noise character.

**IX. CONCLUSIONS**

The steady-state characteristics of the beams generated by an OPO depend on the decay constants of the cavity modes, the non linear crystal losses, the excitation parameter \(E,\) and the deviation from resonance (detuning). The last condition has been represented by an angle \(\psi.\) While for resonant devices the beam amplitudes are proportional to \(\sqrt{E - 1},\) in presence of detuning \(E\) has been replaced by an effective excitation parameter \(E.\) Similarly, the phase difference between the driving field and the pump mode is a function of the detuning angle \(\psi.\) The effects of these changes propagate to the fluctuating parts of the twin beams amplitudes.

Once linearized, the fluctuations of the twin beams are characterized by resonance frequencies which control the frequency responses \(K_{jk}^{0,\tilde{\delta}}\) to the different noise sources (quantum noise of signal/idler/pump and phase-amplitude fluctuations of the pump laser). These functions have in general six poles \(\Omega_p\) in the half plane \(\text{Im}(\omega) \geq 0.\) One, \(\Omega_0 = 0,\) always lies at the origin, a second one, \(\Omega_1 = i\omega_1,\) is located on the positive imaginary axis, while the remaining four ones may be purely imaginary or complex; when complex \((\Omega_2 = -\Omega_3, \Omega_4 = -\Omega_5)\) they are characterized by two damping constants \(\omega_2, \omega_4\) and frequencies \(\omega_2, \omega_4.\)

At resonance the \(K_{kl}^{0,\tilde{\delta}}\) have only three poles, different from the those of \(K_{kl}^{0,\bar{\gamma}}.\) Their locations in the complex plane depend on the excitation parameter \(E,\) the cavity damping coefficient \(\gamma'_0\) at the pump frequency, and the mismatch parameter \(\tilde{\delta}.\) For given \(\tilde{\delta}\) the couple of parameters \((E, \gamma'_0)\) corresponding to purely imaginary resonances forms, on the
the $E-\gamma^0_E$ plane, a connected region delimited by the straight line $E = 1$ and a curve representing the maximum excitation $E_{\text{max}}$ for which the poles are all imaginary. For $\gamma^0_E$ comprised in a particular interval the poles are imaginary for $1 < E < E_1$ and $E_2 < E < E_{\text{max}}$. Outside this region the two poles are complex conjugate and the spectrum exhibits a relaxation oscillation peak [16,17].

The amplitude quadrature responses $K^0_{jk}$ to the various noise terms are regular for $\omega \rightarrow 0$, contrariwise to the phase quadratures, $K^\pm_{jj}$ and $K^\pm_{jk}$, which present a pole. The difficulties of this pole have been bypassed by displacing it from the origin by a small quantity $\varepsilon$, and letting $\varepsilon \rightarrow 0$ at the end of the calculations. So doing the complex fluctuation $\mu-i\phi$ of each mode emerges as proportional to the annihilation operator $\tilde{\epsilon}^0$ at a squeezed thermal oscillator ($\mu-i\phi = k\tilde{\epsilon}$).

By exploiting the Gaussian statistics of the fluctuations second and fourth order correlations have been expressed by suitable combinations of the frequency responses and their residues. So doing, it has been filled the gap between the condition of each mode emerges as proportional to the annihilation operator $\tilde{\epsilon}^0$ of a squeezed thermal oscillator ($\mu-i\phi = k\tilde{\epsilon}$).

X. APPENDIX A

$A_k (\omega)$ and $B_k (\omega)$ depend on the functions $\Delta_k (\omega)$

$$
\begin{align*}
A_j &= \Delta_j + \frac{\Delta_j'}{\Delta_j'\Delta_0 + 1} - \frac{\Delta_0^\dagger}{\Delta_j'\Delta_0 + 1}, \\
B_j &= \frac{1}{\Delta_j'\Delta_0 + 1} + \frac{\Delta_j'}{\Delta_j'\Delta_0 + 1}, \\
A_0 &= \Delta_0 + \frac{\Delta_1^\dagger}{\Delta_2\Delta_1^\dagger - 1} - \frac{\Delta_1}{\Delta_1\Delta_2^\dagger - 1}, \\
B_0 &= \frac{1}{\Delta_2\Delta_1^\dagger - 1} + \frac{1}{\Delta_1\Delta_2^\dagger - 1}
\end{align*}
$$

while the $C_k (\omega)$ are linear combinations of the noise Fourier transforms $\tilde{Z}_k'$ (see Eqs. (17))

$$
C_j = \frac{\Delta^\dagger_j \tilde{Z}_0' - \tilde{Z}_j'}{\Delta_j'\Delta_0 + 1} + \frac{\Delta_j^\dagger \tilde{Z}_j'^\dagger}{\Delta_j'\Delta_0 + 1}, \\
C_0 = \tilde{Z}_0' - \frac{\Delta_1^\dagger \tilde{Z}_2' + \tilde{Z}_1'^\dagger}{\Delta_2\Delta_1^\dagger - 1} - \frac{\Delta_1 \tilde{Z}_1' + \tilde{Z}_2'^\dagger}{\Delta_1\Delta_2^\dagger - 1}
$$

For a detuned system ($\psi \neq 0$) $A_k$ differs from the adjoint $A_k^\dagger$, while $B_k$ coincides with $B_k^\dagger$.

Next, introducing the functions

$$
F_{k\pm} = \frac{A_k \pm B_k}{A_k A_k^\dagger - B_k^2}
$$

the frequency responses $K^0_{kl}$ defined by Eqs. (17) are given by

$$
\begin{align*}
K^0_{jj} &= F_j^\pm, \\
K^0_{j0} &= F_j^\pm \Delta_j' \Delta_0 + 1, \\
K^0_{0j} &= \frac{F_0 - F_0^\dagger \Delta_j'}{\Delta_j' \Delta_0 + 1}, \\
K^0_{00} &= \frac{F_0 - F_0^\dagger \Delta_j'}{\Delta_j' \Delta_0 + 1}, \\
\end{align*}
$$
and

\[ K_{jj} = F_j^+ \, , \quad K_{jj} = -\frac{F_j + F_j^+}{\Delta j' \Delta_0 + 1} \, , \quad K_{j j'} = \frac{F_j^+ \Delta j' - F_j}{\Delta j' \Delta_0 + 1} \]

Expressing \( F_k \) and \( \Delta_k \) as functions of \( \ddot{\omega} \) yields

\[
K^0_{jj} = -i \left( 1 + \tilde{\delta}_j \right) \frac{e^{i\psi} \ddot{\omega}^4 + K_{jj}^{(5)} \ddot{\omega}^3 + K_{jj}^{(4)} \ddot{\omega}^2 + K_{jj}^{(3)} \ddot{\omega} + K_{jj}^{(2)} \ddot{\omega}}{D'} \]
\[
K^0_{jj'} = -i \frac{1 - \ddot{\delta}^2 \ddot{\omega}^3 + K_{jj'}^{(4)} \ddot{\omega}^2 + K_{jj'}^{(3)} \ddot{\omega} + K_{jj'}^{(2)} \ddot{\omega} + K_{jj'}^{(1)} \ddot{\omega}}{D'} \]
\[
\begin{aligned}
K^0_{j j'} &= \tilde{\gamma}_0' (E_{\text{eff}} - 1) \left( 1 + \tilde{\delta}_j \right) \frac{\ddot{\omega}^3 + K_{j j'}^{(4)} \ddot{\omega}^2 + K_{j j'}^{(3)} \ddot{\omega} + K_{j j'}^{(2)} \ddot{\omega} + K_{j j'}^{(1)} \ddot{\omega}}{D'} \end{aligned}
\]

(35)

and

\[
\begin{aligned}
K^\pi_{jj} &= i \left( 1 + \tilde{\delta}_j \right) \frac{e^{i\psi} \ddot{\omega}^5 + K_{jj}^{(5)} \ddot{\omega}^4 + K_{jj}^{(4)} \ddot{\omega}^3 + K_{jj}^{(3)} \ddot{\omega}^2 + K_{jj}^{(2)} \ddot{\omega} + K_{jj}^{(1)} \ddot{\omega}}{D'} \frac{1}{\ddot{\omega}} \\
K^\pi_{jj'} &= -i \frac{1 - \ddot{\delta}^2 \ddot{\omega}^4 + K_{jj'}^{(4)} \ddot{\omega}^3 + K_{jj'}^{(3)} \ddot{\omega}^2 + K_{jj'}^{(2)} \ddot{\omega} + K_{jj'}^{(1)} \ddot{\omega}}{D'} \frac{1}{\ddot{\omega}} \\
K^\pi_{j j'} &= \tilde{\gamma}_0' (E_{\text{eff}} - 1) \left( 1 + \tilde{\delta}_j \right) \frac{\ddot{\omega}^4 + K_{j j'}^{(4)} \ddot{\omega}^3 + K_{j j'}^{(3)} \ddot{\omega} + K_{j j'}^{(2)} \ddot{\omega} + K_{j j'}^{(1)} \ddot{\omega}}{D'} \frac{1}{\ddot{\omega}}
\end{aligned}
\]

(36)

with \( D' (\ddot{\omega}) \) defined in (13).

At resonance the \( K^0_{jj'} \) reduce to

\[
\begin{aligned}
K^0_{j j'} &= -i \left( 1 + \tilde{\delta}_j \right) \frac{\ddot{\omega}^2 - i \left( 1 + \tilde{\gamma}_0' - \tilde{\delta}_j \right) \ddot{\omega} - \tilde{\gamma}_0' E \left( 1 - \tilde{\delta}_j \right)}{D_+} \\
K^0_{j j'} &= \frac{\ddot{\omega} + i \tilde{\gamma}_0' (E - 2)}{D_+} \\
K^0_{j j'} &= \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}} \\
K^0_{j j'} &= \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}} \\
K^0_{j j'} &= \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}}
\end{aligned}
\]

(36)

and

\[
\begin{aligned}
K^\pi_{j j'} &= -i \left( 1 + \tilde{\delta}_j \right) \frac{\ddot{\omega}^2 - i \left( 1 + \tilde{\gamma}_0' - \tilde{\delta}_j \right) \ddot{\omega} - \tilde{\gamma}_0' E \left( 1 - \tilde{\delta}_j \right)}{D_-} \frac{1}{\ddot{\omega}} \\
K^\pi_{j j'} &= \left( 1 - \tilde{\delta}^2 \right) \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}} \\
K^\pi_{j j'} &= \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}} \\
K^\pi_{j j'} &= \frac{\ddot{\omega} - i \tilde{\gamma}_0' E}{D_-} \frac{1}{\ddot{\omega}}
\end{aligned}
\]

In the adiabatic case \( (\tilde{\gamma}_0' \gg 1) \)

\[
\begin{aligned}
K^0_{j j'} &= -i \left( 1 + \tilde{\delta}_j \right) \frac{\ddot{\omega} - i E \left( 1 - \tilde{\delta}_j \right)}{\ddot{\omega}^2 - 2i E \ddot{\omega} - 4 (E - 1) \left( 1 - \tilde{\delta}_j \right) } \\
K^0_{j j'} &= -i \frac{(E - 2)}{\ddot{\omega} - 2i E \ddot{\omega} - 4 (E - 1) \left( 1 - \tilde{\delta}_j \right) } \\
K^0_{j j'} &= -i \frac{(E - 1)}{\ddot{\omega} - 2i E \ddot{\omega} - 4 (E - 1) \left( 1 - \tilde{\delta}_j \right) }
\end{aligned}
\]
and

\[ K_{j0} = -i \left( 1 + \delta_j \right) \frac{\tilde{\omega} - iE \left( 1 - \tilde{\delta}_j \right)}{\tilde{\omega} - i2E} \frac{1}{\tilde{\omega}} \]

\[ K_{jj} = E \left( 1 - \tilde{\delta}^2 \right) \frac{1}{\tilde{\omega} - i2E} \frac{1}{\tilde{\omega}} \]

\[ K_{j0} = -i \left( E - 1 \right) \left( 1 + \delta_j \right) \frac{1}{\tilde{\omega} - i2E} \]

XI. APPENDIX B

The coefficients \( D^{(j)} \) of the characteristic polynomial are given by

\[ D^{(5)} = -i2 \left( \tilde{\gamma}_0 + 2 \right) \]

\[ D^{(4)} = - \left( \tilde{\gamma}_0 + 2 \right)^2 - 4\tilde{\gamma}_0 - 4\tilde{\delta}^2 \tan^2 \psi - 4 \left( E_{eff} - 1 \right) \tilde{\gamma}_0 \]

\[ D^{(3)} = i4\tilde{\gamma}_0 \left[ E_{eff}\tilde{\gamma}_0 + 2 + 3 \left( E_{eff} - 1 \right) + \left( 1 + E_{eff} \right) \tilde{\delta}^2 \right] \]

\[ D^{(2)} = 4\tilde{\gamma}_0 \left[ \left( \left( E_{eff} - 1 \right)^2 + 3 \left( E_{eff} - 1 \right) + 1 \right) \tilde{\gamma}_0 + 2 \left( E_{eff} - 1 \right) - \left( 2 + \tilde{\gamma}_0 \right) \left( E_{eff} - 1 \right) - \tilde{\gamma}_0 \tan^2 \psi \right] \tilde{\delta}^2 \]

\[ D^{(1)} = -i8\tilde{\gamma}_0 E_{eff} \left( E_{eff} - 1 \right) \left( 1 - \tilde{\delta}^2 \right) \]

(37)

with \( \tilde{\gamma}_0 \) and \( \tilde{\delta} \) respectively the normalized pump mode damping and mismatch factor

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Figure captions

1. Zeros $\Omega_1, \Omega_2, \Omega_4$ of $D'(\omega)$ (see Eq. (3)), versus pump parameter $E$ for $\psi = 0.3$ and cavity parameters $\gamma'_0 = 2$, $\delta = 0.1$. $w_1 = \text{Im} \left[ \Omega_1 \right]$ (dashed), $\tilde{\omega}_2 = \text{Re} \left[ \Omega_2 \right]$ and $w_2 = \text{Im} \left[ \bar{\Omega}_2 \right]$ (dotted), and $\tilde{\omega}_4 = \text{Re} \left[ \bar{\Omega}_4 \right]$ and $w_4 = \text{Im} \left[ \bar{\Omega}_4 \right]$ (continuous). Notice that $\text{Re} \left[ \bar{\Omega}_{2,4} \right]$ vanishes for $E$ less a threshold value, in agreement with the discussion illustrated in Fig. 2.

2. Typical shape for $\delta = 0.05$ of the curve separating the points of the plane $\gamma'_0 - E$, for which the three resonance frequencies of the amplitude quadratures $\mu$ are purely imaginary (below), from those with two complex conjugate roots (above). Relaxation oscillations occur only for points lying above the curve.

3. Frequency $\tilde{\omega}_2 = \text{Re} \left[ \bar{\Omega}_2 \right]$ (continuous) and damping coefficient $w_2 = \text{Im} \left[ \bar{\Omega}_{2,3} \right]$ (dotted), of the relaxation oscillations at resonance ($\psi = 0$), versus the pump parameter $E$ (see Eq. (19)) for cavity decay rates $\gamma'_0 = 0.5, 2.5, 4.5, 6.5$ and $\delta = 0$.

4. In-phase absolute squared transfer functions $|K^0_{11} (\tilde{\omega})|^2, |K^0_{12} (\tilde{\omega})|^2$ and $|K^0_{10} (\tilde{\omega})|^2$ at resonance ($\psi = 0$) vs normalized frequency for $\gamma'_0 = 1, \delta = 0.05$, and pump parameters $E = 2, 3, 4$.

5. Average phase commutator $\langle [\phi_j (0), \phi_j (\tau)] \rangle$ versus $\tau$ for $\gamma'_0 = 4, E = 1.5, 4$ and $\psi = 0.1$.

6. Single mode spectra for pump cavity damping $\gamma'_0 = 4$ (top), $0.5$ (bottom), $\delta = 0.05$, pump parameter $E = 3$, detuning phases $\psi = 0$ (solid), $0.25$ (dashed), $0.5$ (dot–dashed), and crystal losses $\kappa_1 / \gamma = 0.3, \kappa_0 = \kappa_1 / 3$. The spectra have been normalized to the shot–noise–level.

7. Effects of the detuning on the transfer of pump excess noise to beam fluctuations. Plots parameters are as follow: $\gamma'_0 = 4$ (top), $0.5$ (bottom), $\delta = 0.05$, pump parameter $E = 3$, detuning phases $\psi = 0$ (solid), $0.25$ (dashed), $0.5$ (dot–dashed), and crystal losses $\kappa_1 / \gamma = 0.3, \kappa_0 = \kappa_1 / 3$.

8. Difference spectral density $S_d$ for cavity damping $\gamma'_0 = 4$ (top), $0.5$ (bottom), $\delta = 0.05$, pump parameter $E = 3$, zero detunings ($\psi = 0$), and crystal losses different for the pump and signal/idler $\kappa_{1,2} / \gamma = 0.3, \kappa_0 = \kappa_1 / 3$. The spectra have been normalized to the shot–noise–level.