Resonant Excitation of Disk Oscillations in Two-Armed-Deformed Disks and Application to High-Frequency QPOs

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Abstract

In previous papers we showed that in one-armed deformed disks, p-mode and g-mode oscillations are resonantly excited by horizontal resonance, and applied it to high-frequency quasi-periodic oscillations (QPOs) observed in low-mass X-ray binaries. In that model, the observed time variation of kHz QPOs is regarded as being the result of a time-dependent precession of the deformation. In this paper we consider another possible cause of the time variation of kHz QPOs. That is, we demonstrate that in two-armed deformed disks, p-mode and g-mode oscillations are excited by a vertical resonance, not by a horizontal resonance (horizontal resonance dampens them). Furthermore, we show that in the case of a vertical resonance, the frequencies of excited disk oscillations can vary with time if the vertical disk structure changes with time. A brief application of these results to the time variation of observed kHz QPOs is discussed.

Key words: accretion, accretion disks — neutron stars — quasi-periodic oscillations — resonance — two-armed disk deformation — X-rays; stars

1. Introduction

Many authors now think that high-frequency quasi-periodic oscillations (QPOs) observed in low-mass X-ray binaries are some kind of disk oscillations in a strong gravitational field, and that it gives a promising way to estimate the mass and spin of the central sources. High-frequency QPOs both in black-hole candidates and in neutron-star sources would have the same dynamical origin (Abramowicz et al. 2003), although there are some differences in their frequency variations.

Based on closeness to 3:2 of frequencies of twin QPOs, the importance of resonant phenomena was emphasized by Abramowicz and Kluźniak (2001). Subsequently, from a different context, Kato (2004, 2008a, 2008c) and Ferreira and Ogilvie (2008) pointed out the importance of resonant processes in a deformed (warped or eccentric) disks as an excitation process of high-frequency QPOs. That is, a non-linear interaction between disk oscillations and the deformed part of disks resonantly excites or dampens disk oscillations. There are two types of resonance. One is a horizontal resonance (Lindblad resonance), and the other is a vertical resonance. Inertial-acoustic oscillations (p-mode) and the gravity oscillations (g-mode) are found to be excited by the horizontal resonance.

The resonant excitation of the p-mode and g-mode oscillations by the horizontal resonance seems to occur most efficiently at the radius where the condition

$$\kappa = \frac{\Omega}{2}$$

is satisfied (Kato 2004, 2008a, 2008c), when the one-armed deformation of the disks has no precession, where \(\kappa(r)\) is the epicyclic frequency, \(\Omega(r)\) is the angular velocity of disk rotation and \(r\) is the radius from the central source on the disk plane. In the case of the Schwarzschild metric, the resonance occurs at \(4r_{\text{g}}\), where \(r_{\text{g}}\) is the Schwarzschild radius, defined by \(r_{\text{g}} = 2GM/c^2\), \(M\) being the mass of the central source. The frequencies of the p-mode and the g-mode oscillations excited by this resonance are

$$\omega = (m\Omega \pm \kappa)_{\text{res}}.$$  (2)

where the subscript \(\text{res}\) denotes the values at the resonant radius derived from equation (1) and \(m (= 1, 2, ..., \ldots)\) are the wavenumber of the oscillations in the azimuthal direction. The frequencies specified by equation (2) have ratios of some rational numbers.

The frequencies of oscillations given by equation (2) can qualitatively describe the high-frequency QPOs in black-hole and neutron-star X-ray binaries. In the case of neutron-star X-ray binaries, however, there is additional and challenging observational evidence that the frequencies of kHz QPOs change with time. If we want to describe this time change of kHz QPOs within the framework of the above-mentioned model, we must introduce the assumption that the one-armed disk deformation has time-dependent precession in the disks of neutron-star X-ray binaries, although such precession is not required in black-hole accretion disks. If the frequency of precession of disk deformation is denoted by \(\omega_{\text{p}}\), the resonant condition where the horizontal resonance occurs is changed from equation (1) to

$$\kappa = \frac{1}{2}(\Omega - \omega_{\text{p}}).$$  (3)

The resonant radius, \(r_{\text{res}}\), changes with a time change of \(\omega_{\text{p}}\), and thus the frequencies of resonantly excited disk oscillations

\[\text{1}\] In the case of neutron stars, the central sources have surfaces and the inner part of disks may have some influence on the stellar magnetosphere. In such situations, the warp might have precession (e.g., Meheut & Tagger 2009).
2 In the horizontal resonance, the resonant radius is determined by the radial
rotation as
\[ \frac{\rho r^2}{\rho_0} = c_s^2(r) , \]
where \( \rho_0 \) is the density on the equatorial plane \( z = 0 \), and \( H(r) \) is the half-thickness of the disk, \( r \) being the distance from the rotating axis of the disks.

The half-thickness, \( H \), of disks is related to the vertical epicyclic frequency, \( \Omega_\perp(r) \), by
\[ \Omega_\perp^2 H^2 = \frac{\rho_0}{\rho_0} = c_s^2(r) , \]
where \( c_s \) is the isothermal acoustic speed. The vertical epicyclic frequency, \( \Omega_\perp(r) \), is equal to the angular velocity of the Keplerian rotation, \( \Omega_\perp(r) \), in the case of a non-rotating central object, and practically equal to the angular velocity of the disk rotation, \( \Omega(r) \). Hereafter, however, we do not use \( \Omega_\perp \) or \( \Omega \) instead of \( \Omega_\perp \), so that we can trace back the effects of \( \Omega_\perp \) on the final results.

2.2. One-Armed Disk Deformation

We assume that the disks described above are deformed from an axisymmetric state by some external or internal cause. The deformation is a warp, or an eccentric deformation in the equatorial plane. They are assumed, for simplicity, to be time-independent.

The Lagrangian displacement associated with the deformation, \( \xi^W(r) \), is denoted by
\[ \begin{align*}
\xi^W_r &= \exp(-i\phi) \hat{\xi}^W_r(r) \mathcal{H}_n(z/H), \\
\xi^W_\phi &= \exp(-i\phi) \hat{\xi}^W_\phi(r) \mathcal{H}_n(z/H), \\
\xi^W_z &= \exp(-i\phi) \hat{\xi}^W_z(r) \mathcal{H}_n(z/H),
\end{align*} \]
where \( \phi \) is the azimuthal direction of the cylindrical coordinates \( (r, \phi, z) \); \( \mathcal{H}_n \) is the Hermite polynomial of order \( n^W \) with argument \( z/H \). In the case of eccentric deformation in the equatorial plane, \( n^W \) is zero \( (n^W = 0) \), while it is unity \( (n^W = 1) \) in the case of warp. It is noted that the number of nodes of \( \hat{\xi}^W_z \) is smaller than those of \( \hat{\xi}^W_r \) and \( \hat{\xi}^W_\phi \) by one. This is true even for disk oscillations [see equation (7)].

2.3. Disk Oscillations

Disk oscillations that are considered here are assumed to have a moderately short radial wavelength, so that it is shorter than the characteristic length of the radial variation of the disk structure. Furthermore, the wave motions are assumed to occur isothermally.

If the above approximations are adopted, the \( r \)- and \( z \)-dependences of disk oscillations on non-deformed disks are approximately separated (Okazaki et al. 1987). We can then express the displacement vector, \( \xi(r, t) \), associated with an oscillation mode of \( (\omega, m, n) \) as
\[ \begin{align*}
\xi_r(r, t) &= \exp[i(\omega t - m\phi)] \hat{\xi}_r(r) \mathcal{H}_n(z/H), \\
\xi_\phi(r, t) &= \exp[i(\omega t - m\phi)] \hat{\xi}_\phi(r) \mathcal{H}_n(z/H), \\
\xi_z(r, t) &= \exp[i(\omega t - m\phi)] \hat{\xi}_z(r) \mathcal{H}_{n-1}(z/H),
\end{align*} \]
where \( \omega \) is the frequency of the oscillations, and \( \mathcal{H}_n \) is the Hermite polynomials, as mentioned before. The integer \( n = (0, 1, 2, \ldots) \) specifies the number of nodes of \( \hat{\xi}_r \) (and \( \hat{\xi}_\phi \)) in the vertical direction \( (z \text{-direction}) \).

Hereafter, for simplicity, we characterize the oscillations and the disk deformations by the set of \( (\omega, m, n) \). For example,
a warp belongs to a mode characterized by (0,1,1) and an eccentric deformation on the disk plane by (0,1,0).

Here, classification of the disk oscillation modes is briefly summarized (for details, see Kato 2001; Kato et al. 2008). The oscillations with \( n = 0 \) occur predominantly on the equatorial plane. They are inertial-acoustic oscillations, and hereafter called “p-mode” oscillations. In the cases where \( n \geq 1 \), we have two kinds of oscillations. In one of them, \((\omega - m\Omega)^2 < k^2\)
while in the other one we have \((\omega - m\Omega) > n\Omega^2\). We call the former “g-modes”, and the latter “vertical p-modes”, except for some special cases mentioned below. In the case of \( n = 1 \) with \( m = 1 \), the latter oscillations are almost incompressible and have low frequencies. They are specially called “c-mode” oscillations. A warp is included to this type of oscillation.

In the resonant excitation problem to be examined in this paper, we group the above various oscillation modes into two classes and treat the oscillation modes in each class as a pack, due to the similarity of the mathematical treatment. One class is p-mode and g-mode oscillations, and the other one is vertical p-mode and c-mode oscillations.

2.4. Coupling between Deformation and Oscillations

Nonlinear couplings between the deformation specified by equations (6) and the disk oscillations given by equations (7) induce disk oscillations that are characterized by \((\omega, m \pm 1, \tilde{n})\), where \( \tilde{n} = n \pm 1 \) when the disk deformation is a warp, while \( \tilde{n} = n \) when the deformation is an eccentric deformation on the equatorial plane. Arbitrary combinations of \( \pm \) are allowed. We call the oscillations resulting from the coupling “intermediate oscillations”.

The intermediate oscillations have resonant interactions with the disks at particular radii. One of the resonances occurs at radii where the intermediate oscillations with \((\omega, m \pm 1, \tilde{n})\) have Lindblad resonances, which are specified by

\[
(\omega - (m \pm 1)\Omega)^2 - \kappa^2 = 0, \tag{8}
\]

where \( \kappa \) is the (horizontal) epicyclic frequency. This resonance is hereafter called “horizontal resonance”. Another one occurs at radii where the frequency \( \omega \) of the intermediate oscillations of \((\omega, m \pm 1, \tilde{n})\) becomes equal to the eigen-frequency of the vertical oscillations of the disks. The radii are characterized by

\[
(\omega - (m \pm 1)\Omega)^2 - \tilde{n}\Omega^2 = 0. \tag{9}
\]

This resonance is hereafter called “vertical resonance”.

2.5. Resonant Excitation of Oscillations

The intermediate oscillations interact nonlinearly with the disk deformation, after having the resonance mentioned above, to feedback to the original oscillations. This feedback process amplifies or dampens the original oscillations. Detailed analyses (Kato 2004, 2008a, 2008c) show that in both cases of horizontal and vertical resonances, the growth rate of oscillations, \(-\omega_0\) (\( \omega_0 \) being the imaginary part of frequency of oscillations), can be expressed in the form of

\[
-\omega_0 \propto \frac{\text{sign}[\omega - (m \pm 1)\Omega]_{\text{res}}}{E}, \tag{10}
\]

where \( E \) is the wave energy of the original oscillations \((\omega, m, n)\) in consideration, and \(\text{sign}[\omega - (m \pm 1)\Omega]_{\text{res}}\) is the signature of \(\omega - (m \pm 1)\Omega\) at the resonant radius, \(r_{\text{res}}\).4 The value of the proportional coefficient of equation (10) depends, of course, on the modes of oscillations and types of resonances. However, in some typical cases, the coefficient is always negative definite (Kato 2004, 2008a, 2008c). This means that the condition of excitation of disk oscillations is

\[
\frac{\text{sign}[\omega - (m \pm 1)\Omega]_{\text{res}}}{E} < 0. \tag{11}
\]

This condition allows us to have a simple physical interpretation. We first notice that for a resonance to occur efficiently, the resonant radius, \(r_{\text{res}}\), must be in the radial region where both the original and the intermediate oscillations dominate. Furthermore, we notice that in general a wave with \((\omega, m)\) has a negative energy if the wave is dominated inside the corotation radius given by \(\omega - m\Omega = 0\), while it is positive if the wave is outside the corotation radius. This consideration suggests that we can regard \(\text{sign}[\omega - (m \pm 1)\Omega]_{\text{res}}\) as the signature of the wave energy, \(E_{\text{int}}\), of the intermediate oscillation. Furthermore, the signature of the wave energy of the original oscillation is the same as the sign of \((\omega - m\Omega)_{\text{res}}\). Based on these considerations, we can write the amplification condition (11) as

\[
\frac{E_{\text{int}}}{E} < 0 \tag{12}
\]

or

\[
\frac{\text{sign}[\omega - (m \pm 1)\Omega]_{\text{res}}}{\text{sign}(\omega - m\Omega)_{\text{res}}} < 0. \tag{13}
\]

This condition can be interpreted in the following way. If an oscillation with positive energy \((E > 0)\) interacts resonantly, at a resonant radius, with an intermediate oscillation with negative energy \((E_{\text{int}} < 0)\), both oscillations are amplified by energy flowing from the intermediate oscillation to the original oscillation. The original oscillation is amplified, since it has \(E > 0\) and receives positive energy. The intermediate oscillation also grows by losing energy, since \(E_{\text{int}} < 0\). If the case of \(E < 0\) and \(E_{\text{int}} > 0\), the resonance also amplifies the oscillations. In this case, the direction of energy flow is opposite: it flows from the original oscillation to the intermediate oscillation at the resonant radius.

2.6. Resonant Radius and Type of Resonance that Excites Oscillations

It is noted that the resonant condition [equation (8) or (9)] alone does not uniquely determine the radius of the resonance. An additional restriction is necessary. We assume that the original oscillations are most strongly excited when the resonance occurs near the boundary of their propagation region. Near the boundary the oscillations have a long radial wavelength and their group velocity in the radial direction vanishes, i.e., they

\[\text{\footnotesize{4 It is noted that in some special cases of } n \geq 2 \text{ with } m \geq 2, \text{ the latter “vertical p-mode” oscillations can also have low frequencies (Silbergleit et al. 2001), although they are not incompressible.}}\]
stay there for a long time compared with in other places, and thus will grow there most strongly. Hence, when we consider p- and g-mode oscillations, we assume that an additional condition to be adopted to specify the resonant radius is

$$\left(\omega - m\Omega\right)^2 - \kappa^2 = 0,$$

(14)

since this represents the boundary of the propagation region of p- and g-mode oscillations. In the case where we consider vertical p-mode oscillations and c-mode oscillations, on the other hand, we assume that the resonance occurs at

$$\left(\omega - m\Omega\right)^2 - n\Omega_{\perp}^2 = 0,$$

(15)

for the same reason as stated above.

As mentioned in subsection 2.3, disk oscillations can be grouped into two classes in studying the present excitation problem. The first one is p- and g-mode oscillations. The second class is vertical p-mode and c-mode oscillations. Concerning the type of resonances, we also have two types, i.e., the horizontal and vertical resonances. Hence, we have four cases in combination of the set of oscillations and resonances. Among them, the excitation of disk oscillations occurs in the case where the oscillations are p- and g-modes and the resonance is horizontal (Kato 2004, 2008a, 2008c). Combining equations (8) and (14), we find that the resonance in this growing case occurs at the radius where the condition

$$\kappa = \frac{1}{2}\Omega$$

(16)

is satisfied. The radius where this condition is satisfied is $4r_g$, i.e., $r_{\text{res}} = 4r_g$, when the metric is the Schwarzschild one. If the metric is the Kerr, the radius becomes smaller than $4r_g$.

The frequencies of the disk oscillations that are excited there are [see equation (14)]

$$\omega = (m\Omega \pm \kappa)_{\text{res}},$$

(17)

Applications of these resonantly-excited oscillations to high-frequency QPOs have been made by Kato and Fukue (2006) and Kato (2008b).

### 3. Resonant Excitation of Disk Oscillations in Two-Armed Disks

After the above preparation, we now proceed to the main purpose of this paper, i.e., to examine the case where the disks are deformed from an axisymmetric steady state into a state with a two-armed pattern. Different from the case of a one-armed pattern, a two-armed pattern cannot in general be stationary, i.e., it will be time-dependent, and wavy. What we need here is that a pattern has approximately a constant frequency for a time interval longer than the characteristic time by which disk oscillations grow. The origin of such a two-armed pattern is a problem to be discussed and clarified, but here we simply assume that such a pattern exists on the disks by some internal or external causes. For example, numerical three-dimensional magnetohydrodynamical (3D MHD) simulations of accretion disks (Machida & Matsumoto 2008) show that one-armed and two-armed patterns with slow rotation are produced in the innermost region of disks at a certain stage of disk evolution by magnetic-field stretching and reconnection processes.

The displacement vector, $\xi^T(r, t)$, associated with a two-armed deformation is now expressed as

$$\xi_t^T(r, t) = \exp[i(\omega^r T - 2\varphi)]\hat{\xi}_g^T(r)\hat{\mathcal{H}}_{n, \tau}(z/H),$$

$$\xi_\varphi^T(r, t) = \exp[i(\omega^r T - 2\varphi)]\hat{\xi}_g^T(r)\hat{\mathcal{H}}_{h, \tau}(z/H),$$

$$\xi_z^T(r, t) = \exp[i(\omega^r T - 2\varphi)]\hat{\xi}_g^T(r)\hat{\mathcal{H}}_{n, \tau-1}(z/H).$$

(18)

Here, $\omega^r$ is the angular velocity of the pattern, which is taken to be a free parameter. As the integer $n, T$, we are mainly interested in the cases of $n = 2$ and 3 (see section 6).

On such deformed disks, we impose disk oscillations. The displacement vector, $\xi(r, t)$, associated with the oscillations is described again by equation (7).

#### 3.1. Resonant Conditions

The nonlinear coupling between the disk deformation described by $(\omega^T, z)$ [see equation (18)] and the disk oscillations described by $(\omega, m, n)$ [see equation (7)] induces intermediate oscillations of $(\omega \pm \omega^T, m \pm 2, \hat{n})$, where $\hat{n} = n \pm n^T$. These intermediate oscillations have resonances with the disk rotation. The horizontal resonance occurs at [cf. equation (8)]

$$[\omega \pm \omega^T - (m \pm 2)\Omega]^2 - \kappa^2 = 0,$$

(19)

and the vertical resonance occurs at [cf. equation (9)]

$$[\omega \pm \omega^T - (m \pm 2)\Omega]^2 - \hat{n}\Omega_{\perp}^2 = 0.$$  

(20)

As mentioned in subsection 2.6, an additional condition is necessary to uniquely determine the radius of the resonance. We focus our attention again to the case where the resonance occurs at the radius where the group velocity of the original oscillations with $(\omega, m, n)$ has vanished to stay there for a long time. That is, when we consider p- and g-mode oscillations, we adopt [equation (14)]

$$\left(\omega - m\Omega\right)^2 - \kappa^2 = 0$$

(21)

as an additional condition. In the case where the excitation of the vertical p-mode and c-mode oscillations is examined, we adopt [equation (15)]

$$\left(\omega - m\Omega\right)^2 - n\Omega_{\perp}^2 = 0.$$  

(22)

There are four cases in combination of types of resonance and types of oscillation, i.e., two cases (horizontal or vertical resonance) for p- and g-mode oscillations, and two cases (horizontal or vertical resonance) for vertical p-mode and c-mode oscillations. Before examining these cases separately in subsection 3.3, we consider the excitation condition.

#### 3.2. Excitation Condition

In the case of one-armed deformation of disks, the condition of excitation of disk oscillations is given by equation (10). The mathematical procedures used to derive the condition are complicated (see Kato 2008a, 2008c), but they can be straightly extended to the case of two-armed deformation. The results show that the growth rate of oscillations, $-\omega_0$, can be expressed in the form of

$$-\omega_0 \propto \frac{\text{sign}[\omega \pm \omega_p - (m \pm 2)\Omega]_{\text{res}}}{E}.$$  

(23)
That is, compared with the case of the one-armed deformation, in the present case of two-armed deformation, \( \omega \) is changed to \( \omega \pm \omega_r \) and \( m \pm 1 \) is changed to \( m \pm 2 \).

The next problem is to examine the sign of the proportional coefficient on the right-hand side of equation (23). A straightforward generalization of the procedures of the one-armed deformation suggests that in some simplified cases (for example, the case where the non-linear coupling terms between the original oscillation and the deformation are constant in the resonant region\(^5\)), the proportional coefficient is negative definite, as in the case of one-armed deformation. When the proportional coefficient is negative definite, we again have a simple physical interpretation of equation (23) (see subsection 2.5). Considering them, we suppose that the amplification condition in the present two-armed case is given by

\[
\frac{\text{sign}(\omega \pm \omega_r - (m \pm 2)\Omega)_{\text{res}}}{\text{sign}(\omega - m\Omega)_{\text{res}}} < 0. \tag{24}
\]

In other words, the condition is also expressed as \( E^{\text{int}} / E < 0 \), and has a simple physical meaning.

3.3. Growing Cases and Their Resonant Radius

Based on the resonant conditions [relevant combinations of equation (19) or (20) to equation (21) or (22)] and the excitation condition [equation (24)], we now examine what types of oscillations (p- and g-mode oscillations or vertical p- and c-mode oscillations) are excited by what type of resonances (horizontal or vertical resonance), and where the radii are.

(i) Horizontal resonance of p- or g-mode oscillations

In this case, as mentioned in subsection 3.1, the resonant radii are the places where both equations (19) and (21) are simultaneously satisfied. More explicitly, the horizontal resonances of p- and g-mode oscillations occur at radii where one of the following set of two equations are satisfied:

\[
\begin{align*}
(a) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \kappa \\
& \quad \omega - m\Omega = \kappa,
\end{align*}
\]

\[
\begin{align*}
(b) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \kappa \\
& \quad \omega - m\Omega = -\kappa,
\end{align*}
\]

\[
\begin{align*}
(c) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\kappa \\
& \quad \omega - m\Omega = \kappa,
\end{align*}
\]

\[
\begin{align*}
(d) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\kappa \\
& \quad \omega - m\Omega = -\kappa.
\end{align*}
\]

By inspection, we can see that the radii satisfying condition (a) or condition (d) are not interesting here, since the excitation condition (24) is not satisfied in these cases. If there are radii where condition (b) or (c) is satisfied, the oscillations are excited there, since the excitation condition (24) is satisfied there. From condition (b) or (c), we can see that such radii are \( \pm \omega^T \mp 2\Omega = 2\kappa \). Since \( \omega^T \) will be much smaller than \( \Omega \) in practical cases, we take as the resonant radius where oscillations are excited.

\[
\kappa = \Omega \pm \frac{\omega^T}{2}, \tag{26}
\]

where + is for the case of \( \omega^T < 0 \) and - is for \( \omega^T > 0 \), since \( \kappa < \Omega \). In the case where \( \omega^T \) is much smaller than \( \Omega \), this resonance occurs at an outer region of the disks, and the frequency ratios of excited oscillations are roughly 1:2:3\ldots. Such oscillations will be of interest, but are subjects outside the present issue.

(ii) Vertical resonance of p- or g-mode oscillations

In this case, the radii where resonance occurs efficiently are places where both equations (20) and (21) are simultaneously satisfied. These conditions can be written down in the following four cases:

\[
\begin{align*}
(a) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \tilde{n}^{1/2}\Omega \downarrow \\
& \quad \omega - m\Omega = \kappa,
\end{align*}
\]

\[
\begin{align*}
(b) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \tilde{n}^{1/2}\Omega \uparrow \\
& \quad \omega - m\Omega = -\kappa,
\end{align*}
\]

\[
\begin{align*}
(c) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\tilde{n}^{1/2}\Omega \downarrow \\
& \quad \omega - m\Omega = \kappa,
\end{align*}
\]

\[
\begin{align*}
(d) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\tilde{n}^{1/2}\Omega \uparrow \\
& \quad \omega - m\Omega = -\kappa.
\end{align*}
\]

As in the horizontal resonance, we see that cases (a) and (d) are uninteresting even if they have solutions, since the excitation condition (24) is not satisfied. Resonances resulting from case (b) or case (c), on the other hand, satisfy condition (24).

In cases of (b) and (c), the resonant radii are found to be

\[
\kappa = 2\Omega - \tilde{n}^{1/2}\Omega \mp \omega^T, \tag{28}
\]

where both signs of \( \pm \) are possible. The negative sign is for the case of (b) and positive one is for (c). Here, \( \omega^T \) has been assumed to be much smaller than \( \Omega \). This resonant condition is satisfied in the case of \( \tilde{n} = 2 \) and 3 at the inner region of relativistic disks.

If the resonant radius, \( r_{\text{res}} \), is determined by solving equation (28), the frequencies of resonantly excited oscillations are found to be

\[
\omega = (m\Omega \pm \kappa)_{\text{res}}. \tag{29}
\]

(iii) Horizontal resonance of vertical p-mode or c-mode oscillations

In this case, the set of equations to be used to determine resonant radii are [see equations (19) and (22)] the following:

\[
\begin{align*}
(a) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \kappa \\
& \quad \omega - m\Omega = n^{1/2}\Omega \downarrow,
\end{align*}
\]

\[
\begin{align*}
(b) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = \kappa \\
& \quad \omega - m\Omega = -n^{1/2}\Omega \downarrow,
\end{align*}
\]

\[
\begin{align*}
(c) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\kappa \\
& \quad \omega - m\Omega = n^{1/2}\Omega \downarrow,
\end{align*}
\]

\[
\begin{align*}
(d) : & \quad \omega \pm \omega^T - (m \pm 2)\Omega = -\kappa \\
& \quad \omega - m\Omega = -n^{1/2}\Omega \downarrow.
\end{align*}
\]

As in the previous studies of (i) and (ii), cases (a) and (d) are outside of our present interest, since the oscillations that satisfy the conditions do not satisfy the excitation condition (24). In the cases of (b) and (c), on the other hand, oscillations are
excited, and the radius is characterized by
\[ \kappa = 2\Omega - n^{1/2}\Omega_L \pm \omega^T. \] (31)
This expression for resonant radii is the same as relation (28), except that \( \bar{n} \) in relation (28) is now replaced by \( n \).

The frequencies of resonantly excited oscillations are
\[ \omega = (m\Omega \pm n^{1/2}\Omega_L)_{res}, \] (32)
where the resonant radius, \( r_{res} \), is now determined by equation (31).

Finally, it is noted that there is no resonance characterized by the set of equations (20) and (22), as long as \( n \) and \( \bar{n} \) are moderate integers.

4. Frequency Variation by Change of Vertical Disk Structure

Among resonantly excited oscillations discussed in the previous section, the p- and g-mode oscillations resulting from vertical resonance [i.e., the set described by equations (28) and (29)] are of interest in relation to the observed kHz QPOs, since their frequencies are in a reasonable frequency range, as discussed in previous papers.

If we want to account for the observed frequency variation of kHz QPOs by the model described by equations (28) and (29), the frequency of two-armed pattern, \( \omega^T \), must change with time. The variation of \( \omega^T \) changes the resonant radius [see equation (28)], and thus the frequencies of oscillations [see equation (29)] are changed. This may be one of the possible causes of frequency changes of the observed kHz QPOs, but it is not clear whether a large variation of \( \omega^T \) required to explain the time variation of kHz QPOs is generally expected.

Here, one of the other possibilities of time variation of the resonant radius is considered. So far, we have assumed that the disk is vertically isothermal and oscillations also occur isothermally. We now relax this assumption, and consider the case where the pressure, \( p \), and density, \( \rho \), are distributed in the vertical direction with a polytropic relation, i.e., \( p = K\rho^{1+1/N} \), and the polytropic index \( N \) changes with time. It is noted that in polytropic disks the vertical integration of the vertical hydrostatic balance gives
\[ T_0(r,z) = T_{00}(r) \left( 1 - \frac{z^2}{H^2} \right). \] (33)
\[ \rho_0(r,z) = \rho_{00}(r) \left( 1 - \frac{z^2}{H^2} \right)^N, \] (34)
\[ p_0(r,z) = T_{00}(r) \left( 1 - \frac{z^2}{H^2} \right)^{1+N}, \] (35)
where subscript 0 represents the quantities in the equilibrium state, and 00 are those on the equatorial plane (e.g., Kato et al. 2008).

We consider adiabatic disk oscillations on such polytropic disks, assuming that the ratio of the specific heat, \( \gamma \), is related to \( N \) by \( \gamma = 1 + 1/N \). Compared with the case of isothermal disks, deriving the dispersion relation of oscillations in such polytropic disks is very complicated (e.g., Perez et al. 1997; Silbergleit et al. 2001). However, in the limiting case where the couplings between the horizontal and vertical motions in an oscillation mode are neglected, we can easily see that the eigen-frequency of local vertical oscillations with \( (\omega, m, n) \) is given by
\[ (\omega - m\Omega)^2 - \Psi_n\Omega^2 = 0, \] (36)
where for the fundamental \( (n = 1) \), the first overtone \( (n = 2) \) and the second overtone \( (n = 3) \), and we have (e.g., Kato 2005)
\[ \Psi_1 = 1, \]
\[ \Psi_2 = 2 + \frac{1}{N} = 1 + \gamma, \] (37)
\[ \Psi_3 = 3 + \frac{3}{N} = 3\gamma. \]
It is noted that in the case of isothermal disks, \( 1/N = 0 \) and \( \Psi_n = n \).

The above consideration suggests that the condition of vertical resonance, equation (20), is now changed to
\[ [\omega \pm \omega^T - (m \pm 2)\Omega^2] - \Psi_n\Omega^2 = 0. \] (38)
This relation shows that the resonant radius of the vertical resonance depends not only on \( \omega^T \), but also on \( \gamma \). In real accretion disks, a change of the mass accretion rate, for example, may bring about a change of the disk vertical structure. This change gives rise to a change of the radius of the vertical resonance, leading to a frequency change of resonantly excited disk oscillations.

The next subject to be investigated is how the excitation condition (24) is changed in the present case of polytropic disks. A detailed investigation of this is very complicated in any mathematical treatment, since normal-mode analyses of oscillations are troublesome in the case of polytropic disks. Let us first consider the separability of variables associated with oscillations. If the radial variation of oscillations is strong, a physical quantity associated with the oscillations, say \( V(r,z) \), is approximately separated as \( V_{r}(r)V(z) \). In the case of isothermal disks, \( V_{z}(z) \) is the Hermite polynomials [see equation (7)], while it is Gezenbauer polynomials in the case of polytropic disks (see Perez et al. 1997; Silbergleit et al. 2001). This makes the description of oscillations complicated.

Furthermore, when we want to describe non-linear couplings between oscillations, we must expand a product of Gezenbauer polynomials into a series of the Gezenbauer polynomials, and must use orthogonal relations among the polynomials to separate the oscillation modes. These are very complicated compared with in the case of Hermite polynomials. These facts make any detailed examination of excitation condition troublesome, compared with in the case of isothermal disks. Hence, a detailed derivation of the excitation condition in the case of polytropic disks is beyond the scope of this paper. Here, we must be satisfied with rough and physical considerations. That is, we assume that the excitation condition (24) still holds even in the case of polytropic disks, since it has a simple and reasonable physical meaning that will be free from a particular disk structure, as mentioned before.

Hereafter we focus our attention only on the case where p- and g-mode oscillations are excited by the vertical resonance. In this case the basic equations to be used to specify the resonant radius are the set of equations (38) and (29), and the
equation to be used to judge whether the oscillations are really excited is equation (24).

By generalizing the procedures in (ii) in subsection 3.3, we can easily see that the resonant radii, $r_{\text{res}}$, in the cases where oscillations are really excited are obtained by solving

$$\kappa = 2\Omega - \Psi^{1/2}/\Omega_L + \omega^T,$$

and the frequencies of the oscillations, $\omega$, are given by

$$\omega = (m\Omega \pm \kappa)_{\text{res}}.$$

5. Numerical Results

As is shown in equation (40), the frequencies of oscillations excited at the resonant radius are a discrete set, characterized by $m$ and $\pm$. As the typical frequencies, we take here $(\Omega - \kappa)_{\text{res}}$ and $(2\Omega - \kappa)_{\text{res}}$ as in the previous papers. The former oscillation has $m = 1$, while the latter has $m = 2$. We take the picture that the observed QPOs come from high-energy photons that are Comptonized in a corona surrounding a geometrically thin disk where the oscillations are generated. If this picture is adopted, the oscillations with $m = 1$ are observed in the twofold frequency (Kato & Fukue 2006). Based on this situation, we consider that $2(\Omega - \kappa)_{\text{res}}$ and $(2\Omega - \kappa)_{\text{res}}$ correspond to the typical twin frequencies of the observed kHz QPOs, and denote them as

$$2\omega_{LL} = 2(\Omega - \kappa)_{\text{res}}, \quad \omega_L = (2\Omega - \kappa)_{\text{res}}.$$  

The figures of frequencies $2\omega_{LL}$ and $\omega_L$ depend on the resonant radius, $r_{\text{res}}$, but the relation between $2\omega_{LL}$ and $\omega_L$ is free from detailed models determining $r_{\text{res}}$. That is, the relation depends only on the mass of the central object, $M$, and the spin parameter, $a_*$, representing the metric. The variations of such parameters as $\gamma$ and $\omega^T$ determine only the allowed range of variation on the $\omega_L-2\omega_{LL}$ curve. The $\omega_L-2\omega_{LL}$ relation is shown in figure 1 for some sets of $M$ and $a_*$. On this figure, the diagram showing the observed relation between the upper and lower frequencies of the twin kHz QPOs of some typical sources has been superposed, assuming that the lower and upper kHz QPOs correspond, respectively, to $2\omega_{LL}$ and $\omega_L$. In the region of $2\omega_{LL} > 600$ Hz, the $\omega_L-2\omega_{LL}$ relation seems to well describe observations if $M = 2.4 M_\odot$ and $a_* = 0$ are adopted.

Next, we examine which part of the $\omega_L-2\omega_{LL}$ curve is allowed in the case where $\gamma$ and $\omega^T$ vary in reasonable ranges. As preparation for this study, we first examine how the resonant radius varies upon changes of $\gamma$ and $\omega^T$. We first consider the case of $\omega^T = 0$ and $\tilde{a} = 2$. From equation (39), we see that in this case the resonant radius is described by

$$\kappa = 2\Omega - (1 + \gamma)^{1/2}/\Omega_L.$$  

Figure 2 shows how the resonant radius described by equation (42) changes as a function of $\gamma$ for two cases of $a_* = 0$ and $a_* = 0.3$. The change of the resonant radius by a change of $\omega^T$ is shown in figure 3 by adopting a positive sign in equation (39).

![Diagram showing the relation between $\omega_L$ and $2\omega_{LL}$ for some values of mass, $M$, and spin parameter, $a_*$. The set of $M$ (in units of $M_\odot$) and $a_*$ adopted to draw the curves are, from the uppermost to the lowermost curves, $(2.0, 0.2), (2.0, 0.0), (2.4, 0.2)$, and $(2.4, 0.0)$. Diagram showing the frequency correlation of the twin kHz QPOs of some neutron-star X-ray sources (taken from Abramowicz 2005) are superposed by assuming that the upper and lower kHz QPOs correspond, respectively, to $\omega_L$ and $2\omega_{LL}$. (Color Online).](image1)

![Diagram showing the relation between $r_{\text{res}}$ and $\gamma$.](image2)

![Diagram showing the relation between $r_{\text{res}}$ and $\omega^T$.](image3)
in figure 4 a straight curve of (1)
HBO is roughly (1
relation with the frequencies of kHz QPOs, i.e., the frequency of
lower kHz QPOs, the above conclusion seems to be robust,
relevant to specify the resonantly excited oscillations. If this
phenomena. That is, in addition to the condition of vertical
field anchored to the central source may be strong enough to
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More importantly, we should remember that our analyses
are based on an idea that QPOs are propagating transient
phenomena. That is, in addition to the condition of vertical
resonance, equation (38), we imposed a condition that the
oscillations that are excited most strongly are those whose
radial group velocity just vanishes at the resonant radius,
i.e., equation (21). Combining these two relations, we have
obtained the resonant radius, equation (39). The imposed condi-
tion (21), however, is not always clear as to whether it is most
relevant to specify the resonantly excited oscillations. If this
condition is relaxed, the resonant radius given by equation (39)
models of the resonance. This is true even in the case of one-
armed deformed-disks, as long as 2ωL and ωT are assumed
to correspond to the observed twin kHz QPOs.

There are a few possibilities to evade the above conclusion,
since some assumptions and simplifications are involved in the
present model. First, we have assumed that the angular velocity
of disk rotation, Ω, is Keplerian. In the central part of the
disks, this simplification might be violated, since a magnetic
field anchored to the central source may be strong enough to
modify the disk rotation.

6. Discussion

A comparison of the present disk oscillation model of QPOs
with the observed kHz QPOs suggests that the masses of the
central sources are around 2.4 M⊙, if they have no spin (see
figure 1). If they have spin, a larger mass is required to
describe observations by our model. This mass required by
our model seems to be rather large compared with that usually
supposed as the neutron-star mass, although it is not excluded
theoretically.

As long as we assume that 2(Ω – κ)res(= 2ωL) and
(2ω – κ)res(= ωT) correspond, respectively, to the upper and
lower kHz QPOs, the above conclusion seems to be robust,
since the 2ωL–ωT relation does not depend on any detailed
parameters are, respectively, M = 2.4 M⊙ and a* = 0. The results are
shown in figure 4, where ωL and ωT are shown as functions of
2ωL. In the Z-sources, the frequency of the horizontal
branch oscillation (HBO) is known to change with a correlation
with the frequencies of kHz QPOs, i.e., the frequency of
HBO is roughly (1/15) of the lower kHz QPO frequency. So,
in figure 4 a straight curve of (1/15) × 2ωL has been added in
order to compare it with the [ωT–2ωL] relation obtained here.

The comparison of the line of (1/15) × 2ωL with the
[ωT–2ωL] curve in figure 4 suggests that if changes of
ωT and γ are not independent, but correlated, the present
model may describe the observed frequency correlation
between kHz QPOs and the horizontal branch QPOs (see the
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\[ \kappa = 2\Omega - (1 + \gamma)^{1/2} \Omega_L + \omega^T. \] (43)

In figure 3, γ = 4/3 has been adopted, and two cases of a* = 0
and a* = 0.3 are shown. The rres–ωT relation in the case of
κ = 2Ω – (1 + γ)1/2 – ωT is obtained by just changing the
sign of ωT in figure 3.

We next examine how much 2ωL and ωT vary, when ωT
and γ are changed within reasonable ranges. Two cases of
γ = 1 and γ = 4/3 are considered, and for each case of γ,
ωT is changed from 0 to –80 Hz (retrograde precession of
deformation), with M = 2.4 M⊙ and a* = 0. The results are
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is modified. For example, the resonant radius may differ for each oscillation mode, and the $2\omega_{\perp,1} - \omega_r$ relation is changed. Furthermore, we should notice that if the kHz QPOs are trapped oscillations, their frequencies are determined by a trapping condition, rather than by vanishing of the group velocity. Then, a different approach from the present paper is necessary to consider frequencies of QPOs (see Ferreira & Ogilvie 2008; F. Oktariani et al. 2009, in preparation).

In this paper we have considered the case of $\tilde{n} = 2$, i.e., the vertical component of the displacement vector associated with the intermediate oscillations has two nodes in the vertical direction. In the case of $\tilde{n} = 0$, there is no vertical resonance, and in the case of $\tilde{n} = 1$, the frequency of the vertical oscillation and that of the vertical resonance are always $\Omega_{\perp,2}$, independent of the vertical structure of the disks. Hence, $\tilde{n} = 2$ is the possible smallest value of $\tilde{n}$ where the vertical disk structure can affect the frequency of the resonant oscillations. Intermediate oscillations with $\tilde{n} = 2$ are realized when the set of $(n, n^T)$ is (0.2), (1.1), (1.3), or (2.0), · · · in the case of vertically isothermal disks. We assume that the situation does not change much even in the case of polytropic disks. Then, if we remember that we are now treating a deformation with $n \neq 0$, we can expect low-frequency deformation of disks for the following reason.

Silbergleit et al. (2001) examined global adiabatic disk oscillations in polytropic disks, corresponding to general considerations of the process deriving relations (38). By separating approximately a disturbance associated with disk oscillation, say $V(r, z)$, into a separated form, say $V_r(r)V_z(z)$, they solved the resulting wave equation by the WKB methods. They show that in the lowest order of approximations, the frequency of vertical oscillation, say $\omega$, is given by

$$\omega = \begin{cases} -(y + 1)^{1/2}\Omega_{\perp} + 2\Omega, & (n = 2), \\ -(3y)^{1/2}\Omega_{\perp} + 2\Omega, & (n = 3). \end{cases}$$

(45)

Here, the above results are applied to disk deformation, and thus $\omega$ and $\gamma$ in equation (45) are regarded, respectively, as $\omega^T$ and $n^T$. Equation (45) then suggests that a low-frequency disk deformation is expected when $n^T = 2$ and $n^T = 3$ for relevant figures of $\gamma$. For example, in the case of $n^T = 3$, a steady deformation of the disks is possible for $\gamma = 4/3$, and $\omega^T$ is negative (retrograde precession) for $\gamma > 4/3$, $|\omega^T|$ increasing with an increase of $\gamma$. This $\gamma$-dependence of $|\omega^T|$ is qualitatively the same as the $\omega^T - \gamma$ relation required to describe the observational correlation between kHz QPOs and HBOs. For example, let us consider the case where $\omega^T$ is related to $\gamma$ as $\omega^T = -100(y - 0.5)$ (retrograde precession), and $\gamma$ changes in the range of $\gamma = 2/3$ to $\gamma = 4/3$. The $2\omega_{\perp,1} - \omega_r$ and $|\omega^T| - \omega_{\perp,1}$ relations in this case are shown in figure 5. In this figure, the observed QPOs data for typical sources are superposed, assuming that $\omega_{\perp,1}$ corresponds to the upper kHz QPOs.

Finally, we should emphasize that in our QPO model, non-linear couplings and resonances that are considered are between the disk deformation and oscillations. Concerning the oscillations themselves, however, our resonant model is linear; non-linear and resonant processes among oscillations are not considered. If we want to describe the fact that the observed amplitudes of neutron-star twin QPOs change sign as the observed frequency ratio of the QPOs passes through the value 3 : 2 (Török 2009), non-linear resonant processes between twin QPOs should be considered, as Horák et al. (2009) did, and succeeded to describe it. The non-linear resonant processes among oscillations, however, are not considered in our resonant model, since in our model they are not main processes for determining the oscillations excited and their frequencies.

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6 We do not insist that $\omega^T$ and $\gamma$ should be correlated in this way. This is just an example.