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Monotone Dynamical Systems with Polyhedral Order Cones and Dense Periodic Points

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Abstract: Let $X \subset \mathbb{R}^n$ be a set whose interior is connected and dense in $X$, ordered by a closed convex cone $K \subset \mathbb{R}^n$ having nonempty interior. Let $T: X \approx X$ be an order-preserving homeomorphism. The following result is proved: Assume the set of periodic points of $T$ is dense in $X$, and $K$ is a polyhedron. Then $T$ is periodic.

Keywords: Dynamical systems; ordered spaces; convex cones; periodic orbits

1. Introduction

The following postulates and notation are used throughout:

- $K \subset \mathbb{R}^n$ (Euclidean $n$-space) is a solid order cone: a closed convex cone that has nonempty interior $\text{Int}(K)$ and contains no affine line.
- $\mathbb{R}^n$ has the (partial) order $\geq$ determined by $K$:
  \[ y \geq x \iff y - x \in K, \]
  referred to as the $K$-order.
- $X \subset \mathbb{R}^n$ is a nonempty set whose $\text{Int}(X)$ is connected and dense in $X$.
- $T: X \approx X$ is homeomorphism that is monotone for the $K$-order:
  \[ x \geq y \implies Tx \geq Ty. \]

A point $x \in X$ has period $k$ provided $k$ is a positive integer and $T^kx = x$. The set of such points is $\mathcal{P}_k = \mathcal{P}_k(T)$, and the set of periodic points is $\mathcal{P} = \mathcal{P}(T) = \bigcup_k \mathcal{P}_k$. $T$ is periodic if $X = \mathcal{P}_k$, and pointwise periodic if $X = \mathcal{P}$.

Our main concern is the following speculation:
Conjecture. If $\mathcal{P}$ is dense in $X$, then $T$ is periodic.

The assumptions on $X$ show that $T$ is periodic iff $T|\text{Int}(X)$ is periodic. Therefore we assume henceforth:

- $X$ is connected and open $\mathbb{R}^n$.

We prove the conjecture under the additional assumption that $K$ is a polyhedron, the intersection of finitely many closed affine halfspaces of $\mathbb{R}^n$:

**Theorem 1 (Main).** Assume $K$ is a polyhedron, $T : X \approx X$ is monotone for the $K$-order, and $\mathcal{P}$ is dense in $X$. Then $T$ is periodic.

For analytic maps there is an interesting contrapositive:

**Theorem 2.** Assume $K$ is a polyhedron and $T : X \approx X$ is monotone for the $K$-order. If $T$ is analytic but not periodic, $\mathcal{P}$ is nowhere dense.

**Proof.** As $X$ is open and connected but not contained in any of the closed sets $\mathcal{P}_k$, analyticity implies each $\mathcal{P}_k$ is nowhere dense. Since $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$, a well known theorem of Baire [1] implies $\mathcal{P}$ is nowhere dense. $\blacksquare$

The following result of D. Montgomery [4]$^*$ is crucial for the proof of the Main Theorem:

**Theorem 3 (Montgomery).** Every pointwise periodic homeomorphism of a connected manifold is periodic.

**Notation**

$i, j, k, l$ denote positive integers. Points of $\mathbb{R}^n$ are denoted by $a, b, p, q, u, v, w, x, y, z$.

$x \preceq y$ is a synonym for $y \succeq x$. If $x \preceq y$ and $x \neq y$ we write $x < y$ or $y > x$.

The relations $x \preceq y$ and $y \succeq x$ mean $y - x \in \text{Int}(K)$.

A set $S$ is totally ordered if $x, y \in S \implies x \preceq y$ or $x \succeq y$.

If $x \preceq y$, the order interval $[x, y]$ is $\{z : x \preceq z \preceq y\} = K_x \cap -K_y$.

The translation of $K$ by $x \in \mathbb{R}^n$ is $K_x : = \{w + x, w \in K\}$.

The image of a set or point $\xi$ under a map $H$ is denoted by $H\xi$ or $H(\xi)$. A set $S$ is positively invariant under $H$ if $HS \subset S$, invariant if $H\xi = \xi$, and periodically invariant if $H^k\xi = \xi$.

2. Proof of the Main Theorem

The following four topological consequences of the standing assumptions are valid even if $K$ is not polyhedral.

**Proposition 4.** Assume $p, q \in \mathcal{P}_k$ are such that

\[ p \preceq q, \quad p, q \in \mathcal{P}_k. \quad [p, q] \subset X. \]

Then $T^k((p, q]) = [p, q]$.

$^*$See also S. Kaul [3].
Proof. It suffices to take \( k = 1 \). Evidently \( T \mathcal{P} = \mathcal{P} \), and \( T[p, q] \subset [p, q] \) because \( T \) is monotone, whence \( \text{Int}([p, q]) \cap \mathcal{P} \) is positively invariant under \( T \). The conclusion follows because \( \text{Int}([p, q]) \cap \mathcal{P} \) is dense in \([p, q]\) and \( T \) is continuous.

**Proposition 5.** Assume \( a, b \in \mathcal{P}_k, a \ll b, \) and \([a, b] \subset X\). There is a compact arc \( J \subset \mathcal{P}_k \cap [a, b] \) that joins \( a \) to \( b \), and is totally ordered by \( \ll \).

*Proof.* An application of Zorn’s Lemma yields a maximal set \( J \subset [a, b] \cap \mathcal{P} \) such that: \( J \) is totally ordered by \( \ll \), \( a = \text{max} J \), \( b = \text{min} J \). Maximality implies \( J \) is compact and connected and \( a, b \in J \), so \( J \) is an arc (Wilder [7], Theorem I.11.23).

**Proposition 6.** Let \( M \subset X \) be a homeomorphically embedded topological manifold of dimension \( n - 1 \), with empty boundary.

(i) \( \mathcal{P} \) is dense in \( M \).

(ii) If \( M \) is periodically invariant, it has a neighborhood base \( \mathcal{B} \) of periodically invariant open sets.

*Proof.* (i) \( M \) locally separates \( X \), by Lefschetz duality [5] (or dimension theory [6]). Therefore we can choose a family \( \mathcal{V} \) of nonempty open sets in \( X \) that the family of sets \( \mathcal{V}_M := \{ V \cap M : V \in \mathcal{V} \} \) satisfies:

- \( \mathcal{V}_M \) is a neighborhood basis of \( M \),
- each set \( V \cap M \) separates \( V \).

By Proposition 5, for each \( V \in \mathcal{V} \) there is a compact arc \( J_V \subset \mathcal{P} \cap V \) whose endpoints \( a_V, b_V \) lie in different components of \( V \setminus M \). Since \( J_V \) is connected, it contains a point in \( V \cap M \cap \mathcal{P} \). This proves (i).

(ii) With notation as above, let \( B_V := [a_V, b_V] \setminus \partial [a_V, b_V] \). The desired neighborhood basis is \( \mathcal{B} := \{ B_V : V \in \mathcal{V} \} \).

From Propositions 4 and 6 we infer:

**Proposition 7.** Suppose \( p, q \in \mathcal{P}, \ p \ll q \) and \([p, q] \subset X\). Then \( \mathcal{P} \) is dense in \( \partial [p, q] \).

Let \( T(m) \) stand for the statement of Theorem 1 for the case \( n = m \). Then \( T(0) \) is trivial, and we use the following inductive hypothesis:

**Hypothesis (Induction).** \( n \geq 1 \) and \( T(n - 1) \) holds.

Let \( Q \subset \mathbb{R}^n \) be a compact \( n \)-dimensional polyhedron. Its boundary \( \partial Q \) is the union of finitely many convex compact \((n - 1)\)-cells, the *faces* of \( Q \). Each face \( F \) is the intersection of \( \partial [p, q] \) with a unique affine hyperplane \( E^{n-1} \). The corresponding *open face* \( F^o := F \setminus \partial F \) is an open \((n - 1)\)-cell in \( E^{n-1} \). Distinct open faces are disjoint, and their union is dense and open in \( \partial Q \).

**Proposition 8.** Assume \( p, q \in \mathcal{P}_k, \ p \ll q, \ [p, q] \subset X \). Then \( T|\partial [p, q] \) is periodic.

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This result is adapted from Hirsch & Smith [2], Theorems 5.11 & 5.15.
Proof. \([p,q]\) is a compact, convex \(n\)-dimensional polyhedron, invariant under \(T^k\) (Proposition 4). By Proposition 6 applied to \(M := \partial[p,q]\), there is a neighborhood base \(B\) for \(\partial[p,q]\) composed of periodically invariant open sets. Therefore if \(F^* \subset \partial[p,q]\) is an open face of \([p,q]\), the family of sets

\[
B_{F^*} := \{W \in B: W \subset F^*\}
\]

is a neighborhood base for \(F^*\), and each \(W \in B_{F^*}\) is a periodically invariant open set in which \(P\) is dense.

For every face \(F\) of \([p,q]\) the Induction Hypothesis shows that \(F^* \subset P\). Therefore Montgomery’s Theorem implies \(T|F^*\) is periodic, so \(T|F\) is periodic by continuity. Since \(\partial[p,q]\) is the union of the finitely many faces, it follows that \(T|\partial[p,q]\) is periodic.

To complete the inductive proof of the Main Theorem, it suffices by Montgomery’s theorem to prove that an arbitrary \(x \in X\) is periodic. As \(X\) is open in \(\mathbb{R}^n\) and \(P\) is dense in \(X\), there is an order interval \([a,b] \subset X\) such that

\[
a \ll x \ll b, \quad a, b \in P_k.
\]

By Proposition 5, \(a\) and \(b\) are the endpoints of a compact arc \(J \subset P_k \cap [a,b]\), totally ordered by \(\ll\). Define \(p, q \in J:\)

\[
p := \sup \{y \in J: y \leq x\}, \quad q := \inf \{y \in J: y \geq x\}.
\]

If \(p = q = x\) then \(x \in P_k\). Otherwise \(p \ll q\), implying \(x \in \partial[p,q]\), whence \(x \in P\) by Proposition 8.

Conflict of Interest

The author declares no conflicts of interest in this paper.

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