Does Knowledge Transfer Always Help to Learn a Better Policy?

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Abstract

One of the key approaches to save samples when learning a policy for a reinforcement learning problem is to use knowledge from an approximate model such as its simulator. However, does knowledge transfer from approximate models always help to learn a better policy? Despite numerous empirical studies of transfer reinforcement learning, an answer to this question is still elusive. In this paper, we provide a strong negative result, showing that even the full knowledge of an approximate model may not help reduce the number of samples for learning an accurate policy of the true model. We construct an example of reinforcement learning models and show that the complexity with or without knowledge transfer has the same order.

On the bright side, effective knowledge transferring is still possible under additional assumptions. In particular, we demonstrate that knowing the (linear) bases of the true model significantly reduces the number of samples for learning an accurate policy.

Keywords — Transfer Reinforcement Learning, Sample Complexity, Lower Bound, TV-distance

1 Introduction

Reinforcement learning (RL) is the framework of learning to control an unknown system through trial and error. Recently, RL achieves phenomenal empirical successes, e.g., AlphaGo (Silver et al., 2016) defeated the best human player in Go, and OpenAI used RL to precisely and robustly control a robotic arm (Andrychowicz et al., 2017). The RL framework is general enough such that it can capture a broad spectrum of topics, including health care, traffic control, and experimental design.
However, successful applications of RL in these domains are still rare. The major obstacle that prevents RL being widely used is its high sample complexity: both the AlphaGo and OpenAI arm took nearly a thousand years of human-equivalent experiences to achieve good performances.

One way to reduce the number of training samples is to mimic how human beings learn – borrow knowledge from previous experiences. In robotics research, a robot may need to accomplish different tasks at different time. Instead of learning every task from scratch, a more ideal situation is that the robot can utilize the similarities between the underlying models of these tasks and adapt them to future new jobs quickly. Another example is that RL agents are often trained in simulators and then applied to real-world (Ng et al., 2006; Itsuki, 1995; Dosovitskiy et al., 2017). It is still desirable to have their performance improved after seeing samples collected from the real-world. One might hope that agents from simulators (approximate models) can adapt to the real world (true model) faster than knowing nothing. Both examples lead to a natural question:

*If models are similar, can we achieve fast adaptation through knowledge transferring?*

This paper focuses on answering the above question. Suppose the true unknown model is a Markov Decision Process (MDP) $M$ and the RL agent is provided with an approximate model $M_0$ with

$$\text{dist}(M_0, M) \leq \beta,$$

where $\text{dist}(. , .)$ is a statistic distance and $\beta$ is a small scalar. We would like to study the sample complexity of learning a policy $\pi$ for $M$ such that its error$^1$ is at most $\varepsilon$ with $\varepsilon \ll \beta$ (i.e., the high-precision regime). For a fixed $\varepsilon$, a common wisdom would suggest that better $M_0$ (e.g. smaller $\beta$) can help reduce the sample complexity.

The most natural choice of $\text{dist}(. , .)$ is the total-variation (TV) distance between the transition kernels of $M_0$ and $M$. It is well-known (see e.g. Puterman 2014) that an optimal policy for $M_0$ has an error at most $O_M(\beta)$ in $M$, where $O_M$ hides the constants determined by the model. In this paper we show, however, to obtain an $\varepsilon$-optimal policy (the formal definition will be given in Sec. 1.2), the number of samples is of the form,

$$\Omega_M(\varepsilon^{-2})$$

when $\varepsilon \ll \beta$. Note that the sample complexity is independent of $\beta$. In particular, the complexity does not improve as $\beta$ becomes smaller (as long as $\varepsilon \ll \beta$). It renders the knowledge from a TV-distance ball of $M$ useless when pursing a high-precision control without further structural information of the model. In order to show the lower bound, we leverage techniques for proving hardness in the bandit literature (e.g. Mannor and Tsitsiklis 2004) and reinforcement learning (e.g. Azar et al. 2013) to carefully show that the approximate model does not provide critical information that matters for high-precision control of the true model. Therefore, learning high-precision policy does not benefit from the approximate model.

To complement the lower bound, we further investigate the possible structural information of a model that provably helps knowledge transferring. We show that if the unknown model is in the convex hull of a set of $K$ known base models, we are able to obtain a high-precision control with a number of samples significantly fewer than that of learning from scratch. Specifically, the number of

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$^1$The error of a policy is the difference between the values of the policy and the optimal policy.
samples is proportional to
\[ O(\text{poly}(K)) \]
rather than the much larger \(|S|\), the number of states in the model.

## 1.1 Related Work

Reducing sample complexity is a core research goal in RL. Many related sub-branches of RL, e.g., multi-task RL (Brunskill and Li, 2013; Ammar et al., 2014; Calandriello et al., 2014), lifelong RL (Abel et al., 2018; Brunskill and Li, 2014), and meta-RL (Al-Shedivat et al., 2017), provide different schemes to utilize experiences from previous tasks. Please also see a survey paper (Taylor and Stone, 2009) for more related works. However, these results focus on different special cases of knowledge utilization rather than understanding the fundamental question of whether an approximate model is useful for policy learning and what guarantees we can have.

In the area of Sim-to-Real\(^2\), some works point out that an imperfect approximate model may degrade the performance of learning and efforts have been made to address this issue, e.g (Kober et al., 2013; Buckman et al., 2018; Kalweit and Boedecker, 2017; Kurutach et al., 2018). There has been active empirical research, but little in theory is known. A more related work is Jiang 2018, who shows that even if the approximate model differs from the real environment in a single state-action pair (but which one is unknown), such an approximate model could still be information-theoretically useless. This is another interesting direction to look at. However, the statistic distance from such a model to the true model can be arbitrarily large and hence the policy of the approximate model does not have a guarantee on the true model. The limitation of the benefit that an approximate model could bring can also be found in Jiang and Li 2015, where the authors build a policy value estimator and use the approximate model to reduce variance. However, they demonstrate that, if no extra knowledge is provided, only the part of variance arising from the randomness in policy can be eliminated rather than the stochasticity in state transitions.

In order to take more advantage of the previous experiences, additional structure information is needed. A number of structure settings have been studied in the literature. For instance, in Brunskill and Li 2013, all models are assumed to be drawn from a finite set of MDPs with identical state and action spaces, but different reward and/or transition probabilities; in Abel et al. 2018, one study case requires that all models share the same transition dynamics and only reward functions change with a hidden distribution; in Mann and Choe 2012, a special mapping between the approximate model and the true model is assumed such that the approximate model can provide a good action-value initialization for the true model; in Calandriello et al. 2014, all tasks can be accurately represented in a linear approximation space and the weight vectors are jointly sparse; in Modi et al. 2019, every model’s transition kernel and reward function lie in the linear span of \(K\) known base models. To complement our lower bound, we study an MDP model that shares a similar information structure as in Modi et al. 2019. In contrast to Modi et al. 2019, our model is of infinite-horizon and the loss function is also different. Although not the main focus of this paper, our proposed model and algorithm provide another effective approach that supports knowledge transferring.

It is worth mentioning that the structure information such as the existence of a lower-dimensional knowledge-sharing space is not exclusive to RL. One can also find their applications in supervised multi-task learning, e.g. Kumar and Daume III 2012; Ruvolo and Eaton 2013; Maurer et al. 2013.

\(^2\)It stands for simulator-to-real-environment
1.2 Preliminaries

Notation. We use small letters (e.g., $s, a, t, p(s'|s, a)$) for scalars, capital letters (e.g., $R, V, Q$) for vectors or functions and (e.g., $N, K, L$) for some specific scalars, capital boldface letters (e.g., $P, P^\pi, U$) for matrices, and calligraphic letters (e.g., $\mathcal{S}, \mathcal{A}$) for sets. The cardinality of a set is denoted by $|\cdot|$. We use $[K]$ to represent the set $\{1, 2, \ldots, K\}$. The simplex in $\mathbb{R}^k$ is denoted by $\Delta^k := \{(x_1, x_2, \ldots, x_k)^\top | \sum_{i=1}^k x_i = 1, x_i \geq 0\}$. We abbreviate Kullback-Leibler divergence to KL and use $O, \Omega$ and $\Theta$ to denote leading orders in upper, lower, and minimax lower bounds, respectively; and we use $\tilde{O}, \tilde{\Omega}$ and $\tilde{\Theta}$ to hide the polylog factors.

Markov Decision Process. In this paper, we focus on the discounted Markov Decision Process (MDP) with an infinite horizon, while the same analysis straightforwardly extends to other settings of $\Delta$. We also denote the $\varepsilon$-optimal value function $V^\pi_\varepsilon(s)$ as such that its value (on any state $s \in \mathcal{S}$) is maximized over all policies, i.e.,

$$\forall \pi, s : \quad V^\pi(s) \leq V^{\pi_\varepsilon}(s).$$

We also denote the optimal $\varepsilon$-optimal policy $\pi^\varepsilon$ as $V^\varepsilon$. In practice, the optimal value/policy is in general not attainable. Therefore, it makes sense to study sub-optimal policies. Denote an $\varepsilon$-optimal policy $\pi$ for $\mathcal{M}$ as such that

$$\forall s \in \mathcal{S} : \quad 0 \leq V^\pi(s) - V^{\pi_\varepsilon}(s) \leq \varepsilon.$$
\(P(\cdot | s_i, \pi(s_i))^\top\), then by definition, it holds that
\[
V^\pi = R^\pi + \gamma P^\pi V^\pi.
\] (1)

**TV-distance for MDPs** To measure the closeness of two MDPs, \(M^1 := (S, A, P^1, R^1, \gamma)\) and \(M^2 := (S, A, P^2, R^2, \gamma)\), we introduce the following metric \(d_{TV}(\cdot, \cdot)\):
\[
d_{TV}(M^1, M^2) = \max \left\{ \|P^1 - P^2\|_\infty, \|R^1 - R^2\|_\infty \right\}.
\]
Note that the distance is only valid between MDPs with the same state and action spaces, and the discounted factor. The name TV comes from that due to the property of TV-distance, an optimal policy of \(M^1\) helps policy learning in \(M^2\). However, when a higher precision is desired, i.e., \(\varepsilon \ll \beta\), we show that the number of samples only depends on \(\varepsilon\), rather than \(\beta\), unless additional assumptions of the model are made. Such a conclusion is particularly striking since the knowledge of the approximate model from the vicinity of the true model is almost “useless.”

### 2 Problem Formulation and Illustration

We formalize our setting and the considered problem of knowledge transferring as below.

**Problem 1.** Suppose the unknown true model is an MDP \(M\) and an agent is provided with the full information of a prior model \(M_0\) satisfying \(M \in B_{TV}(M_0, \beta)\), where \(\beta > 0\) is a known constant. How many samples does it take to learn an \(\varepsilon\)-optimal policy for \(M\), where \(\varepsilon \ll \beta\) is an accuracy parameter?

Due to the property of TV-distance, an optimal policy of \(M_0\) is already a \(\beta/(1 - \gamma)\)-optimal policy for \(M\). It is natural to hope that a smaller \(\beta\) leads to a smaller sample complexity, i.e., the knowledge from \(M_0\) helps policy learning in \(M\). However, when a higher precision is desired, i.e., \(\varepsilon \ll \beta\), we show that the number of samples only depends on \(\varepsilon\), rather than \(\beta\), unless additional assumptions of the model are made. Such a conclusion is particularly striking since the knowledge of the approximate model from the vicinity of the true model is almost “useless.”

#### 2.1 A Basic Case

We illustrate our point with a simple case. Define two MDPs \(M_0\) and \(M\) as shown in Figure 1. Both of them have 5 states \(\{x, y_1, y_2, z_1, z_2\}\), where \(x\) has two actions \(a_1\) and \(a_2\) and the rest are all single-action states. After taking \(a_i\), \(x\) will transition to \(y_i\) deterministically. In \(M_0\), \(y_1\) and \(y_2\) transition to themselves with probabilities 0.5 and 0.4, respectively and to \(z_1\) and \(z_2\) with probability 0.5 and 0.6, respectively. In \(M\), each \(y_i\) transitions to itself with probability \(p_i\) and to \(z_i\) with probability \(1 - p_i\). \(p_1\) and \(p_2\) are unknown. In both models, \(z_1\) and \(z_2\) are absorbing states. They also have the same reward function: \(R(s, a, s') = 1\) if \(s' \in \{y_1, y_2\}\) and \(R(s, a, s') = 0\) otherwise.
Without loss of generality, we take $\beta = 0.2$ (the reader can easily generate similar examples for other values of $\beta$ following the same principle). Since $\mathcal{M} \in B_{TV}(\mathcal{M}_0, \beta)$, we have that $|p_1 - 0.5| \leq 0.1$ and $|p_2 - 0.4| \leq 0.1$.

![Figure 1:](image)

**Figure 1:** The left MDP is $\mathcal{M}_0$ and the right MDP is $\mathcal{M}$, where $|p_1 - 0.5| \leq 0.1$ and $|p_2 - 0.4| \leq 0.1$.

In $\mathcal{M}$, if $p_1 > p_2$, the optimal policy should have $\pi^*(x) = a_1$ and $V^*(x) = \frac{\gamma}{1 - \gamma p_1}$. If a policy $\pi$ returns $a_2$ for $x$, then its state value at $x$ is $V^\pi(x) = \frac{\gamma}{1 - \gamma p_2}$. Thus $\pi$ is $|\frac{\gamma}{1 - \gamma p_1} - \frac{\gamma}{1 - \gamma p_2}|$-optimal. When $\varepsilon < |\frac{\gamma}{1 - \gamma p_1} - \frac{\gamma}{1 - \gamma p_2}|$, to produce an $\varepsilon$-optimal policy, an algorithm must find out $p_1 > p_2$ with high probability. For $p_2 > p_1$, vice versa. Therefore, the problem of learning an $\varepsilon$-optimal policy is equivalent to identifying the larger value from $\{p_1, p_2\}$. The knowledge we can use from $\mathcal{M}_0$ is that $p_1 \in [0.4, 0.6]$ and $p_2 \in [0.3, 0.5]$, which does not help reduce the sample complexity due to the overlap.

### 2.2 Empirical Verification

Besides the previous simple case, we also do a numerical demonstration on a sailing problem (Vanderbei, 1996). In Figure 2, we generate two MDPs $\mathcal{M}_0$ and $\mathcal{M}$ with $\mathcal{M} \in B_{TV}(\mathcal{M}_0, 0.3)$. We compare the performances of two algorithms: 1. direct Q-learning (Watkins and Dayan, 1992) with transition samples from $\mathcal{M}$ (blue line); 2. use the full knowledge of $\mathcal{M}_0$ to generate a nearly optimal Q-function for $\mathcal{M}_0$, then use that Q-function to initialize the proceeding Q-learning algorithm with transition samples from $\mathcal{M}$ (red line). Both algorithms use the same batch of transition samples from $\mathcal{M}$. Since $\mathcal{M}_0$ is close to $\mathcal{M}$, the warm-start Q-learning is much better than the learning-from-scratch counterpart in the initial stage. However, these two curves overlap when they become closer to the optimal value, indicating similar sample complexities for both algorithms when pursuing a high-precision Q-value estimation.

### 3 Lower Bound of Transfer Learning from a TV-distance Ball

In this section, we formally prove that an approximate model does not help when learning a high-precision policy. In particular, we show the following lower bound.
Figure 2: A toy test comparison between direct Q-learning and warm-start Q-learning with a nearly optimal Q-value of $M_0$ as initialization.

Theorem 1. (Main Result) Let $M$ be an unknown MDP. Suppose MDP $M_0$ is given and it satisfies $M \in B_{TV}(M_0, \beta)$. There exists $\varepsilon_0, \delta_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, the sample complexity of learning an $\varepsilon$-optimal policy for $M$ with probability at least $1 - \delta$ is
\[
\Omega \left( \frac{N}{(1 - \gamma)^3 \varepsilon^2} \log \left( \frac{1}{\delta} \right) \right).
\]

As shown in Azar et al. 2013 and Sidford et al. 2018; Agarwal et al. 2019, the sample complexity of directly learning an $\varepsilon$-optimal policy for an MDP with high probability under a generative model is
\[
\tilde{\Theta} \left( \frac{N}{(1 - \gamma)^3 \varepsilon^2} \log \left( \frac{N}{\delta} \right) \right).
\]

We conclude that for any $\beta > 0$, when $\varepsilon$ is small enough, the sample complexity of learning with prior knowledge is at least as hard as learning without prior knowledge, if we only know the true model lies in a small TV-distance ball of the approximate model. As any online algorithm can be applied in the generative model case, the lower bound automatically adapts to the online setting as well.

Before starting the proof, we give the following definition about the correctness of RL algorithms.

Definition 1. (($M_0, \beta, \varepsilon, \delta$)-correctness) Given $\beta > 0$ and a prior model $M_0$, we say that an RL algorithm $A$ is ($M_0, \beta, \varepsilon, \delta$)-correct if for every $M \in B_{TV}(M_0, \beta)$, $A$ can output an $\varepsilon$-optimal policy with probability at least $1 - \delta$.

Next, we construct a class of MDPs. We are going to select one model $M^0$ from the class as prior knowledge. Then we show that if an RL algorithm $A$ learns with samples significantly fewer than the lower bound, there would always exist an MDP $M \in B_{TV}(M_0, \beta)$ such that $A$ cannot be ($M_0, \beta, \varepsilon, \delta$)-correct. Hence, the lower bound complexity is established.

Construction of the Hard Case We define a family of MDPs $M$. These MDPs have the structure as depicted in Figure 3. The state space $S$ consists of three disjoint subsets $X$ (gray nodes), $Y_1$ (green nodes), and $Y_2$ (blue nodes). The set $X$ includes $K$ states $\{x_1, x_2, \ldots, x_K\}$ and each of them has $L$ available actions $\{a_1, a_2, \ldots, a_L\} =: A$. States in $Y_1$ and $Y_2$ are all of single-action. In
total, there are \( N := 3KL \) state-action pairs. For state \( x \in \mathcal{X} \), by taking action \( a \in \mathcal{A} \), it transitions to a state \( y_1(x, a) \in \mathcal{Y}_1 \) with probability 1. Note that such a mapping is one-to-one from \( \mathcal{X} \times \mathcal{A} \) to \( \mathcal{Y}_1 \). For state \( y_1(x, a) \in \mathcal{Y}_1 \), it transitions to itself with probability \( p_M(x, a) \in (1/2, 1) \) and to a corresponding state \( y_2(y_1) \in \mathcal{Y}_2 \) with probability \( 1 - p_M(x, a) \). \( p_M(x, a) \) can be different for different models. All states in \( \mathcal{Y}_2 \) are absorbing. The reward function is: \( R(s, a, s') = 1 \), if \( s' \in \mathcal{Y}_1 \); \( R(s, a, s') = 0 \), otherwise.

**Figure 3:** The class of MDPs considered in the proof of Theorem 1. Nodes represent states and arrows show transitions. \( \mathcal{X} \) consists of all grey nodes. \( \mathcal{Y}_1 \) comprises of all green nodes. Blue nodes form \( \mathcal{Y}_2 \).

\( \mathcal{M} \) is a generalization of a multi-armed bandit problem used in Mannor and Tsitsiklis 2004 to prove a lower bound on bandit learning. A similar example is also shown in Azar et al. 2013 to prove a lower bound on reinforcement learning without any prior knowledge. For an MDP \( M \in \mathcal{M} \), it is fully determined by the parameter set \( \{ p_M(x_k, a_l), k \in [K], l \in [L] \} \). And its Q-function has the values:

\[
Q_M(x, a) = \frac{1}{1 - \gamma p_M(x, a)}, \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A}.
\]

**Prior Model \( \mathcal{M}_0 \) and Hypotheses of \( \mathcal{M} \)** Now, we select a prior model \( \mathcal{M}_0 \in \mathcal{M} \) with

\[
p_{\mathcal{M}_0}(x, a) \equiv \frac{4\gamma - 1}{3\gamma} =: p_0, \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A},
\]

where \( \gamma \) is the discounted factor. We restrict \( \gamma \in (0.4, 1) \), then \( p_0 \in (1/2, 1) \). Given \( \beta > 0 \), let \( \mathcal{A} \) be an \( (\mathcal{M}_0, \beta, \varepsilon, \delta) \)-correct algorithm. Denote by \( \beta' := \min(\beta, 1 - p_0) \). We consider \( 1 + K(L - 1) \) possibilities of \( \mathcal{M} \):

\[
\mathcal{M}_1 : \begin{cases} p_{\mathcal{M}_1}(x_k, a_1) = p_0 + \alpha_1, & \forall k \in [K], \\ p_{\mathcal{M}_1}(x_k, a_l) = p_0, & \forall k \in [K], l \neq 1; \end{cases}
\]

for every \( k \in [K], l \neq 1 \), \( \mathcal{M}_{k, l} : \begin{cases} p_{\mathcal{M}_{k, l}}(x_k, a_1) = p_0 + \alpha_2, \\ p_{\mathcal{M}_{k, l}}(x_k, a_{l'}) = p_{\mathcal{M}_1}(x_{k'}, a_{l'}), & \forall (k', l') \neq (k, l), \end{cases} \)

where \( \alpha_1 \) is selected such that

\[
Q_{\mathcal{M}_1}(x_k, a_1) - Q_{\mathcal{M}_1}(x_k, a_l) = \frac{1}{1 - \gamma(p_0 + \alpha_1)} - \frac{1}{1 - \gamma p_0} = 2\varepsilon
\]
and \( \alpha_2 = \frac{4(1-\gamma p_0)^2 \varepsilon}{\gamma} \) such that

\[
Q_{M_k,l}(x_k, a_l) - Q_{M_k,l}(x_k, a_1) = \frac{1}{1 - \gamma(p_0 + \alpha_2)} - \frac{1}{1 - \gamma(p_0 + \alpha_1)} \geq 2\varepsilon.
\]

Note that the parameter set of \( M_1 \) differs from that of \( M_0 \) only on action \( a_1 \) and the parameter set of \( M_{k,l} \) differs from that of \( M_1 \) only on pair \((x_k, a_1)\). When \( \varepsilon \leq \frac{\beta^2}{8(1-\gamma p_0)^2} \), \( 0 < \alpha_1 < \alpha_2 \leq \beta/2 \). Thus, all models above lie in \( B_{TV}(M_0, \beta) \). We refer to them as hypotheses of \( M \). Every hypothesis gives a probability measure over the same sample space. We denote by \( E_1, P_1 \) and \( E_{k,l}, P_{k,l} \) the expectation and probability under hypothesis \( M_1 \) and \( M_{k,l} \), respectively. These probability measures capture both the randomness in the corresponding MDP and the randomization carried out by the algorithm \( \mathcal{A} \), for example its sampling strategy. It is worth mentioning that in Azar et al. 2013, the authors implicitly assume that the sampling numbers to different states are determined before the start of the algorithm and do not change during learning (this is due to their conditionally independence argument in Lemma 18). Such an assumption does not apply to adaptive sampling strategy. In our result, adaptive sampling is included.

In the sequel, we fix \( \varepsilon \in (0, \varepsilon_0) \) and \( \delta \in (0, \delta_0) \), where \( \varepsilon_0 \) and \( \delta_0 \) will be determined later. Let \( t^* = \frac{c_1}{(1-\gamma)^22\varepsilon^2 \log \left( \frac{1}{4\delta} \right)} \),

where \( c_1 > 0 \) is to be determined later. We also define \( T_{k,l} \) the number of samples that algorithm \( \mathcal{A} \) calls from the generative model with input state \( y_1(x_k, a_l) \) till \( \mathcal{A} \) stops (these sample calls are not necessarily consecutive). For every \( k \in [K], l \neq 1 \), we define the following three events:

\[
A_{k,l} = \{ T_{k,l} \leq 4t^* \},
\]

\[
B_{k,l} = \{ \mathcal{A} \text{ outputs a policy } \pi \text{ with } \pi(x_k) = a_1 \},
\]

\[
C_{k,l} = \left\{ S_{k,l} - p_0 T_{k,l} \leq \sqrt{2p_0(1-p_0)T_{k,l} \log(1/4\delta)} \right\},
\]

where \( S_{k,l} \) is the sum of rewards (non-discounted) by calling the generative model \( T_{k,l} \) times with input state \( y_1(x_k, a_l) \). For these events, we have the following lemmas.

**Lemma 1.** For any \( k \in [K], l \neq 1 \), if \( E_1[T_{k,l}] \leq t^* \), \( P_1(A_{k,l}) > 3/4 \).

**Proof.**

\[
t^* \geq E_1[T_{k,l}] > 4t^*P_1(T_{k,l} > 4t^*) = 4t^*(1 - P_0(T_1 \leq 4t^*)).
\]

Thus, \( P_1(A_{k,l}) > 3/4 \). \( \blacksquare \)

**Lemma 2.** For any \( k \in [K], l \neq 1 \), if \( \delta < 1/16 \), \( P_1(C_{k,l}) \geq 3/4 \).

**Proof.** When \( l \neq 1 \), under hypothesis \( M_1 \), \( p_{M_1}(x_k, a_l) = p_0 \). By definition, the instant rewards from state \( y_1(x_k, a_l) \) are i.i.d. Bernoulli-\( p_0 \) random variables. Denote by \( \epsilon := \sqrt{2p_0(1-p_0)T_{k,l} \log(1/4\delta)} \). By Chernoff-Hoeffding bound and \( p_0 > 1/2 \), we have that

\[
P_1 \left( S_{k,l} - p_0 T_{k,l} \leq \sqrt{2p_0(1-p_0)T_{k,l} \log(1/4\delta)} \right)
\]
\[ \geq 1 - \exp \left( -KL \left( p_0 + \frac{\epsilon}{T_{k,l}} \parallel p_0 \right) \cdot T_{k,l} \right) \geq 1 - \exp \left( -\frac{\epsilon^2}{2p_0(1-p_0)T_{k,l}} \right) = 1 - 4\delta. \]

Thus, when $\delta < 1/16$, $\mathbb{P}_1(C_{k,l}) \geq 3/4$. \hfill \blacksquare

Now, we set $\delta_0$ as $1/16$, then for $\delta \in (0, \delta_0)$, $\mathcal{A}$ should return a policy $\pi$ such that when $\mathcal{M} = \mathcal{M}_1$, $\pi(x_k) = a_1$ for every $k \in [K]$ with probability at least $1 - \delta$, i.e. $\mathbb{P}_1(B_{k,l}) \geq 1 - \delta \geq 1 - 1/4$, for all $k \in [K]$ and $l \neq 1$. Define the event $\mathcal{E}_{k,l} := A_{k,l} \cap B_{k,l} \cap C_{k,l}$. Combining the results above, we have that

\[ \mathbb{P}_1(\mathcal{E}_{k,l}) > 1 - 3/4 = 1/4, \quad \forall \, k \in [K], l \neq 1. \]

Next, we show that if the expectation of number of samples in $\mathcal{A}$ on any $y_1(x_k, a_l)$ is less than $t^*$, then $B_{k,l}$ occurs with probability greater than $\delta$ under the hypothesis $\mathcal{M}_{k,l}$.

**Lemma 3.** Let $\varepsilon_0 = \frac{\beta\gamma}{8(1-\gamma p_0)^2}$. For any $k \in [K], l \neq 1$, when $\varepsilon \in (0, \varepsilon_0)$, if $\mathbb{E}_1[T_{k,l}] < t^*$, then $\mathbb{P}_{k,l}(B_{k,l}) > \delta$.

**Proof.** Given $k \in [K]$ and $l \neq 1$, we denote by $W$ the length-$T_{k,l}$ random sequence of the instant rewards by calling the generative model $T_{k,l}$ times with the input state $y_1(x_k, a_l)$. As one can see, if $\mathcal{M} = \mathcal{M}_1$, this is an i.i.d. Bernoulli-$p_0$ sequence; if $\mathcal{M} = \mathcal{M}_{k,l}$, this is an i.i.d Bernoulli-$(p_0 + \alpha_2)$ sequence. We define the likelihood function $L_{k,l}$ by letting

\[ L_{k,l}(w) = \mathbb{P}_{k,l}(W = w) \]

for every possible realization $w$. This function can be used to define a random variable $L_{k,l}(W)$, where $W$ is the sample path of the random sequence. Following the previous notation, $S_{k,l}$ is the sum of rewards, i.e. the total number of getting 1s in $W$. Then we have the likelihood ratio $L_{k,l}(W)/L_1(W)$ as

\[ \frac{L_{k,l}(W)}{L_1(W)} = \frac{(p_0 + \alpha_2)^{S_{k,l}(1-p_0 - \alpha_2)T_{k,l}-S_{k,l}}}{(p_0)^{S_{k,l}(1-p_0)T_{k,l}-S_{k,l}}} = \left(1 + \frac{\alpha_2}{p_0}\right)^{S_{k,l}} \left(1 - \frac{\alpha_2}{1-p_0}\right)^{T_{k,l}-S_{k,l}} = \left(1 + \frac{\alpha_2}{p_0}\right)^{S_{k,l}} \left(1 - \frac{\alpha_2}{1-p_0}\right)^{S_{k,l}\frac{1-p_0}{p_0} \left(1 - \frac{\alpha_2}{1-p_0}\right)^{T_{k,l}-S_{k,l}/p_0}}. \]

By our choice of $p_0, \alpha_2,$ and $\varepsilon$, it holds that $\alpha_2/(1-p_0) \in (0, 1/2]$ and $\alpha_2/p_0 \in (0, 1/2)$. With the fact that $\log(1-u) \geq -u - u^2$ for $u \in [0, 1/2]$ and $\exp(-u) \geq 1 - u$ for $u \in [0, 1]$, we have that

\[ \left(1 - \frac{\alpha_2}{1-p_0}\right)^{\frac{1-p_0}{p_0}} \geq \exp \left( \frac{1-p_0}{p_0} \left( -\frac{\alpha_2}{1-p_0} - \left( \frac{\alpha_2}{1-p_0}\right)^2 \right) \right) \]

\[ \geq \left(1 - \frac{\alpha_2}{p_0}\right) \left(1 - \frac{\alpha_2}{p_0(1-p_0)}\right). \]
Thus

\[
\frac{L_{k,l}(W)}{L_1(W)} \geq \left(1 - \frac{\alpha_2^2}{p_0^2}\right)^{S_{k,l}} \left(1 - \frac{\alpha_2^2}{p_0(1 - p_0)}\right)^{S_{k,l}} \left(1 - \frac{\alpha_2}{1 - p_0}\right)^{T_{k,l} - S_{k,l}/p_0}
\]

\[
\geq \left(1 - \frac{\alpha_2^2}{p_0^2}\right)^{T_{k,l}} \left(1 - \frac{\alpha_2^2}{p_0(1 - p_0)}\right)^{T_{k,l}} \left(1 - \frac{\alpha_2}{1 - p_0}\right)^{T_{k,l} - S_{k,l}/p_0}
\]
due to \(S_{k,l} \leq T_{k,l}\). Next, we proceed on the event \(E_{k,l}\). By definition, if \(E_{k,l}\) occurs, event \(A_{k,l}\) has occurred. Using \(\log(1 - u) \geq -2u\) for \(u \in [0, 1/2]\), it follows that

\[
\left(1 - \frac{\alpha_2^2}{p_0^2}\right)^{T_{k,l} - S_{k,l}/p_0} \geq \left(1 - \frac{\alpha_2^2}{p_0^2}\right)^{4t^*} \geq \exp\left(-8t^* \frac{\alpha_2^2}{p_0^2}\right) \geq (4\delta)^{7000c_1}.
\]

Using \(\log(1 - u) \geq -2u\) for \(u \in [0, 1/2]\), we have that

\[
\left(1 - \frac{\alpha_2^2}{p_0(1 - p_0)}\right)^{T_{k,l}} \geq \left(1 - \frac{\alpha_2^2}{p_0(1 - p_0)}\right)^{4t^*} \geq \exp\left(-8t^* \frac{\alpha_2^2}{p_0(1 - p_0)}\right) \geq (4\delta)^{7000c_1}.
\]

Further, we have that when \(E_{k,l}\) occurs, \(C_{k,l}\) also occurs. Therefore,

\[
\left(1 - \frac{\alpha_2}{1 - p_0}\right)^{T_{k,l} - S_{k,l}/p_0} \geq \left(1 - \frac{\alpha_2}{1 - p_0}\right)^{\sqrt{\frac{1 - p_0}{p_0}T_{k,l} \log(1/4\delta)}} \geq \left(1 - \frac{\alpha_2}{1 - p_0}\right)^{\sqrt{\frac{1 - p_0}{p_0}4t^* \log(1/4\delta)}}
\]

\[
\geq \exp\left(-\sqrt{16 \frac{\alpha_2^2}{p_0(1 - p_0)}t^* \log(1/4\delta)}\right)
\]

\[
\geq (4\delta)^{\sqrt{13000c_1}}.
\]

By taking \(c_1\) small enough, e.g. \(c_1 = 10^{-5}\), we have

\[
\frac{L_{k,l}(W)}{L_1(W)} \geq (4\delta).
\]

By a change of measure, we deduce that

\[
\mathbb{P}_{k,l}(B_{k,l}) \geq \mathbb{P}_{k,l}(E_{k,l}) = \mathbb{E}_{k,l}[1\mathbb{E}_{k,l}] = \mathbb{E}_1 \left[\frac{L_{k,l}(W)}{L_1(W)} 1_{E_{k,l}}\right] \geq 4\delta \ast 1/4 = \delta.
\]

If \(\mathcal{A}\) is \((\mathcal{M}^0, \beta, \epsilon, \delta)\)-correct, under hypothesis \(\mathcal{M}_{k,l}\), \(\mathcal{A}\) should produce a policy \(\pi\) such that \(\pi(x_k) = a_l\) with probability greater than \(1 - \delta\). Thus, we should have \(\mathbb{P}_{k,l}(B_{k,l}) < \delta\) for all \(k \in [K], l \neq 1\). From Lemma 3, it requires \(\mathbb{E}_1[T_{k,l}] > t^*\) for all \(k \in [K], l \neq 1\). In total, we need

\[
\Omega\left(N \frac{\exp(\gamma_\epsilon \log(1/\delta))}{(1 - \gamma_\epsilon)^2}\right)
\]
samples, which concludes our proof of Theorem 1.
4 A Case Study for Knowledge Transfer in Reinforcement Learning

In this section, we impose a new assumption on similarities among models such that transferring knowledge achieves fast adaptation. We consider a sequence of MDPs, where they have the same state and action spaces, and the discounted factor, but different transition dynamics and/or reward functions. At each time step $t$, we want to learn an $\varepsilon$-optimal policy for $M_t := (S, A, P_t, R_t, \gamma)$. The assumption we propose is a convex hull structure as stated below.

**Assumption 1.** Given a finite set of MDPs $M := \{M^1, M^2, \ldots, M^K\}$ where $M^k = (S, A, P^k, R^k, \gamma)$, we have $M_t \in \text{conv}(M)$\(^3\) for all $t > 0$. We have full knowledge of all MDPs in $M$ and access to a generative model of each $M_t$.

We define a set of matrices $V := \{V_{s,a} \in \mathbb{R}^{|S| \times K}, s \in S, a \in A\}$, where the $k$th column of $V_{s,a}$ is $P^k(\cdot|s,a)$. Since $M_t \in \text{conv}(M)$, there exists a vector $C_t \in \Delta^K$ such that

$$P_t(\cdot|s,a) = V_{s,a} C_t, \quad \forall (s,a) \in S \times A.$$  \hspace{1cm} (2)

We define a matrix $U \in \mathbb{R}^{|S|^2|A| \times K}$ by stacking all $V_{s,a}$ vertically, i.e.

$$U := \begin{bmatrix} V_{s_1,a_1} \\ V_{s_1,a_2} \\ \vdots \\ V_{s_{|S|},a_{|A|}} \end{bmatrix}.$$  

We make the following assumption about $U$.

**Assumption 2.** $U$ has full column rank.

Since $K$ is much smaller than $|S|^2|A|$, the assumption is easy to be satisfied in real applications. Then a direct result is:

**Lemma 4.** There exists a set $\{(s'_k, a'_k)\}_{k=1}^K$ such that the matrix formed by stacking all $V_{s'_k,a'_k}$ vertically has column rank $K$.

The proof is easy with basic linear algebra. Let $\{(s'_k, a'_k)\}_{k=1}^K$ be the set in Lemma 4. We define

$$U_{\text{trun}} \in \mathbb{R}^{K|S| \times K} := \frac{1}{K} \begin{bmatrix} V_{s'_1,a'_1} \\ V_{s'_2,a'_2} \\ \vdots \\ V_{s'_K,a'_K} \end{bmatrix}, \quad P_t \in \mathbb{R}^{K|S|} := \frac{1}{K} \begin{bmatrix} P_t(\cdot|s'_1,a'_1) \\ P_t(\cdot|s'_2,a'_2) \\ \vdots \\ P_t(\cdot|s'_K,a'_K) \end{bmatrix}. \quad (3)$$

Basically, we shrink the size of $U$ and normalize it to $U_{\text{trun}}$. Equation (2) reduces to

$$P_t = U_{\text{trun}} C_t.$$  \hspace{1cm} (4)

\(^3\ac M \in \text{conv}(M)\) if there exists a vector $C := [c_1, c_2, \ldots, c_K]^T \in \Delta^K$ such that for any $(s,a) \in S \times A$, $P(\cdot|s,a) = \sum_{k=1}^K c_k P^k(\cdot|s,a)$ and $R = \sum_{k=1}^K c_k R^k$. 

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Let $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ be the largest and smallest eigenvalues of $U_{\text{trun}}^T U_{\text{trun}}$, respectively. Note that $\lambda_{\text{min}} > 0$ due to the full column rank property of $U_{\text{trun}}$. We give an algorithm to learn an $\varepsilon$-optimal policy for $\mathcal{M}_t$. The algorithm is presented in Algorithm 1. For every $t > 0$, we first take samples to construct $P_t$'s empirical estimation $\hat{P}_t$. Then we find a vector $\hat{C}_t \in \Delta^K$ such that $U_{\text{trun}} \hat{C}_t \approx \hat{P}_t$. Next, we use $\hat{C}_t$ to form a model $\hat{\mathcal{M}}_t \in \text{conv}(\mathcal{M})$ as an approximation to the true $\mathcal{M}_t$. Finally, the algorithm returns a policy which is $\varepsilon/2$-optimal policy for $\mathcal{M}_t$. We will show in the proceeding that the output policy is $\varepsilon$-optimal for $\mathcal{M}_t$.

Algorithm 1 Transfer RL with a Convex Model Space

1: Input: $\mathcal{M} := \{\mathcal{M}^1, \mathcal{M}^2, \ldots, \mathcal{M}^K\}, \{(s'_k, a'_k)^T\}_{k=1}^K$, a generative model $GM, \varepsilon > 0, \delta > 0$, a zero vector $\widehat{P}_t \in \mathbb{R}^{K|\mathcal{S}|}$, $L = \left\lceil \frac{432K \lambda_{\text{max}}}{\varepsilon^2 (1-\gamma)^2 \lambda_{\text{min}}} \log \left( \frac{1}{4\delta} \right) \right\rceil$.

2: while $t > 0$ do

3: $i \leftarrow 1, \widehat{P}_t \leftarrow 0$;

4: while $i \leq L$ do

5: Sample $j \in [K]$ uniformly;

6: Sample $(s, r) \leftarrow GM(s'_j, a'_j)$;

7: Increment the $n$th coordinate in $\widehat{P}_t$ by 1, where $n = (j-1)|\mathcal{S}| + s$;

8: $i \leftarrow i + 1$;

9: end while

10: Normalize $\widehat{P}_t$: $\widehat{P}_t \leftarrow \frac{1}{L} \widehat{P}_t$;

11: Calculate $\widehat{C}_t := \text{Proj}_{\Delta^K}(U_{\text{trun}}^T U_{\text{trun}})^{-1} U_{\text{trun}} \widehat{P}_t$ ($U_{\text{trun}}$ as defined in Equation (3));

12: Formulate $\tilde{\mathcal{M}}_t := (\mathcal{S}, A, \sum_{k=1}^K c_t^k P^k, \sum_{k=1}^K c_t^k R^k, \gamma)$, where $[c_t^1, c_t^2, \ldots, c_t^K]^\top = \widehat{C}_t$;

13: Run any planning algorithm and get an $\varepsilon/2$-optimal policy $\pi_t$ for $\tilde{\mathcal{M}}_t$.

14: Output: $\pi_t$.

15: end while

Our proof consists of two steps: 1. we show in Lemma 5 that if the convex coefficients of two MDPs $\mathcal{M}, \tilde{\mathcal{M}} \in \text{conv}(\mathcal{M})$ are close under $\| \cdot \|_2$, then an $\varepsilon$-optimal policy $\pi$ for $\mathcal{M}$ is also nearly optimal for $\tilde{\mathcal{M}}$; 2. we show in Lemma 6 that the convex coefficients of $\tilde{\mathcal{M}}_t$ and $\mathcal{M}_t$ are close under $\| \cdot \|_2$.

Lemma 5. Suppose for MDPs $\mathcal{M} := (\mathcal{S}, A, P, R, \gamma)$ and $\mathcal{M} := (\mathcal{S}, A, \tilde{P}, \tilde{R}, \gamma)$, there exists two vectors $C := [c_1, c_2, \ldots, c_K] \in \Delta^K$ and $D := [d_1, d_2, \ldots, d_K] \in \Delta^K$ such that

$$P = \sum_{k=1}^K c_k P^k, \quad R = \sum_{k=1}^K c_k R^k;$$

$$\tilde{P} = \sum_{k=1}^K d_k P^k, \quad \tilde{R} = \sum_{k=1}^K d_k R^k.$$
If $\|C - D\|_2 \leq \alpha$, then an $\varepsilon'$-optimal policy for $\widehat{M}$ is \( (\varepsilon' + 6\alpha \sqrt{R}/(1 - \gamma)^2) \)-optimal for $M$.

Proof. Denote by $V^*$ and $\hat{V}^*$ the optimal value vectors for $M$ and $\widehat{M}$, respectively. Given an $\varepsilon'$-optimal policy $\pi$ for $\widehat{M}$, we denote by $V^\pi$ and $\hat{V}^\pi$ the value vectors following $\pi$ in $M$ and $\widehat{M}$, respectively. Then, by triangle inequality, we first have that

$$
\|V^\pi - V^*\|_\infty \leq \|V^\pi - \hat{V}^\pi\|_\infty + \|\hat{V}^\pi - V^\pi\|_\infty + \|V^\pi - V^*\|_\infty
$$

To bound $\|V^\pi - \hat{V}^\pi\|_\infty$, we notice that from Equation (1),

$$
V^\pi = (I - \gamma P^\pi)^{-1} R^\pi, \quad \hat{V}^\pi = (I - \gamma \hat{P}^\pi)^{-1} \hat{R}^\pi.
$$

By Gershgorin Circle Theorem (Gershgorin, 1931), the absolute values of all eigenvalues of $\gamma P^\pi$ are strictly smaller than 1. So $(I - \gamma P^\pi)^{-1} = I + \sum_{n=1}^\infty (\gamma P^\pi)^n$. Based on this, it holds that

$$
\|V^\pi - \hat{V}^\pi\|_\infty = \left\| (I - \gamma P^\pi)^{-1} R^\pi - (I - \gamma \hat{P}^\pi)^{-1} \hat{R}^\pi \right\|_\infty
$$

$$
= \left\| R^\pi - \hat{R}^\pi + \sum_{n=1}^\infty \gamma^n \left[ (P^\pi)^n R^\pi - (\hat{P}^\pi)^n \hat{R}^\pi \right] \right\|_\infty
$$

$$
\leq \|R^\pi - \hat{R}^\pi\|_\infty + \sum_{n=1}^\infty \gamma^n \left[ \| (P^\pi)^n - (\hat{P}^\pi)^n \| \right] + \| (\hat{P}^\pi)^n (R^\pi - \hat{R}^\pi) \|_\infty
$$

$$
\leq \|R^\pi - \hat{R}^\pi\|_\infty + \sum_{n=1}^\infty \gamma^n \left[ \left\| (P^\pi - \hat{P}^\pi) \left( \sum_{i=0}^{n-1} (P^\pi)^i (\hat{P}^\pi)^{n-1-i} \right) R^\pi \right\|_\infty + \|R^\pi - \hat{R}^\pi\|_\infty \right]
$$

$$
= \frac{\|R^\pi - \hat{R}^\pi\|_\infty}{1 - \gamma} + \sum_{n=1}^\infty \gamma^n \left[ \left\| (P^\pi - \hat{P}^\pi) R_n \right\| \right] \quad (5)
$$

where $R_n := \sum_{i=0}^{n-1} (P^\pi)^i (\hat{P}^\pi)^{n-1-i} R^\pi \in [0, n]|^{|S|}$. The third line is by triangle inequality and the fourth line is due to that $(P^\pi)^n$ is a transition matrix. For the first term in Equation (5), we have that

$$
\|R^\pi - \hat{R}^\pi\|_\infty = \max_{s \in S} \left| P(\cdot|s, \pi(s))^\top R(s, \pi(s), \cdot) - \hat{P}(\cdot|s, \pi(s))^\top \hat{R}(s, \pi(s), \cdot) \right|
$$

$$
\leq \max_{s \in S} \left| P(\cdot|s, \pi(s))^\top R(s, \pi(s), \cdot) - \hat{P}(\cdot|s, \pi(s))^\top R(s, \pi(s), \cdot) \right|
$$

$$
+ \max_{s \in S} \left| \hat{P}(\cdot|s, \pi(s))^\top \hat{R}(s, \pi(s), \cdot) - \hat{P}(\cdot|s, \pi(s))^\top \hat{R}(s, \pi(s), \cdot) \right|
$$

$$
= \max_{s \in S} \left| \sum_{k=1}^K (c_k - d_k) P^k(\cdot|s, \pi(s))^\top R(s, \pi(s), \cdot) \right| + \max_{s \in S} \left| \sum_{k=1}^K (c_k - d_k) \hat{P}(\cdot|s, \pi(s))^\top \hat{R}^k(s, \pi(s), \cdot) \right|
$$

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\[ \leq 2\|C - D\|_1 \leq 2\sqrt{K}\|C - D\|_2, \]  

(6)

where the second line is by triangle inequality and the last line is due to \( R^k(s, \pi(s), \cdot) \in [0, 1]|S| \). For the second term in Equation (5), observe that

\[
\| (\mathbf{P}^\pi - \hat{\mathbf{P}}^\pi) R_n \|_\infty = \max_{s \in S} \left| \sum_{k=1}^{K} (c_k - d_k) P^k(\cdot|s, \pi(s))^\top R_n \right|
\]

\[
\leq \| C - D \|_1 \| R_n \|_\infty \leq n\sqrt{K}\|C - D\|_2.
\]

(7)

Combining (6) and (7), we have the following result:

\[
(5) \leq 2\alpha\sqrt{K}/(1 - \gamma) + \alpha\sqrt{K}\gamma/(1 - \gamma)^2 \leq 3\alpha\sqrt{K}/(1 - \gamma)^2.
\]

To bound \( \| \hat{V}^* - V^* \|_\infty \), let \( \pi^* \) and \( \hat{\pi}^* \) be optimal policies for \( \mathcal{M} \) and \( \hat{\mathcal{M}} \), respectively. Then it holds that

\[ \hat{V}^* - V^* \leq \| \hat{\pi}^* - \hat{V}^\pi \|_\infty, \quad V^* - \hat{V}^* \leq \| V^\pi - \hat{V}^\pi \|_\infty. \]

Following the same steps as before, we have \( \| \hat{\pi}^* - \hat{V}^\pi \|_\infty \leq 3\alpha\sqrt{K}/(1 - \gamma)^2 \) and \( \| V^\pi - \hat{V}^\pi \|_\infty \leq 3\alpha\sqrt{K}/(1 - \gamma)^2 \). Therefore, \( \| \hat{V}^* - V^* \|_\infty \leq 3\alpha\sqrt{K}/(1 - \gamma)^2 \). The desired result is obtained.

Lemma 6. Let \( C_t \) be the solution of Equation (4). Then in Algorithm 1, for each \( t > 0 \), \( \| \hat{C}_t - C_t \|_2 \leq (1 - \gamma)^2\epsilon/(12\sqrt{K}) \) with probability at least \( 1 - \delta \).

Proof. As defined in Equation (3), \( \| P_t \|_1 = 1 \). Therefore, \( P_t \) can be taken as a distribution. The samples we take in Algorithm 1 are indeed all i.i.d. samples following \( P_t \). Therefore, \( \hat{P}_t \) is an empirical approximation of \( P_t \) with \( L \) samples. By Bernstein inequality for matrices (Tropp et al., 2015, Theorem 6.1.1), we have that

\[
P\left( \| \hat{P}_t - P_t \|_2 \leq \epsilon \right) \geq 1 - (1 + K|S|) \exp \left( \frac{-L^2\epsilon^2}{2L + 2L/3} \right) \geq 1 - (1 + K|S|) \exp \left( \frac{-L^2\epsilon^2}{3} \right).
\]

Taking \( \epsilon = \frac{\lambda_{\min}(1 - \gamma)^2}{12\sqrt{\lambda_{\max}\sqrt{K}}} \), when \( L = \left\lceil \frac{432K\lambda_{\max}}{\epsilon^2(1 - \gamma)^2\lambda_{\min}} \log\left( \frac{1 + K|S|}{\delta} \right) \right\rceil \), we have that

\[
P\left( \| \hat{P}_t - P_t \|_2 \leq \frac{\lambda_{\min}(1 - \gamma)^2}{12\sqrt{\lambda_{\max}\sqrt{K}}} \right) \geq 1 - \delta.
\]

Based on this, it holds that with probability at least \( 1 - \delta \),

\[
\| \hat{C}_t - C_t \|_2 = \| \text{Proj}_{\Delta^K} \left( (U_{\text{trun}}^\top U_{\text{trun}})^{-1}U_{\text{trun}}^\top \hat{P}_t \right) - (U_{\text{trun}}^\top U_{\text{trun}})^{-1}U_{\text{trun}}^\top P_t \|_2
\]

\[
\leq \| (U_{\text{trun}}^\top U_{\text{trun}})^{-1}U_{\text{trun}}^\top \hat{P}_t - (U_{\text{trun}}^\top U_{\text{trun}})^{-1}U_{\text{trun}}^\top P_t \|_2
\]

\[
\leq \frac{\sqrt{\lambda_{\max}}}{\lambda_{\min}} \| \hat{P}_t - P_t \|_2,
\]

where the second line is due to that \( \Delta^K \) is a convex set and \( C_t \in \Delta^K \). Therefore, \( \| \hat{C}_t - C_t \|_2 \leq \frac{\epsilon(1 - \gamma)^2}{12\sqrt{K}} \) with probability at least \( 1 - \delta \).
Combining Lemma 5 and 6, and the fact that $\pi_t$ is $\varepsilon/2$-optimal for $\tilde{M}_t$, we have the following result.

**Proposition 1.** Let $\varepsilon > 0$ and $\delta > 0$. Under Assumption 1 and 2, for any $t > 0$, with probability at least $1 - \delta$, Algorithm 1 returns an $\varepsilon$-optimal policy for $M_t$ with samples

$$O\left(\frac{K}{\varepsilon^2(1-\gamma)^2} \log\left(\frac{1 + K|S|}{\delta}\right)\right).$$

The sample complexity in Proposition 1 is irrelevant with $|S|$ which makes it significantly smaller than the lower bound $\Omega\left(\frac{N}{\varepsilon^2(1-\gamma)^2} \log(1/\delta)\right)$. Therefore, fast adaptation is achieved.

5 Conclusion

In this paper, we show that transfer learning from a TV-distance neighbourhood cannot reduce the sample complexity of learning a high precision policy of an MDP, unless extra structural information is provided. We study the case where the new unknown model can be represented as a convex combination of a finite set of base models. In this setting, transfer learning achieves significantly lower sample complexity compared with learning from scratch.

References

Abel, D., Jinnai, Y., Guo, S. Y., Konidaris, G., and Littman, M. (2018). Policy and value transfer in lifelong reinforcement learning. In *International Conference on Machine Learning*, pages 20–29.

Agarwal, A., Kakade, S., and Yang, L. F. (2019). On the optimality of sparse model-based planning for markov decision processes. *arXiv preprint arXiv:1906.03804*.

Al-Shedivat, M., Bansal, T., Burda, Y., Sutskever, I., Mordatch, I., and Abbeel, P. (2017). Continuous adaptation via meta-learning in nonstationary and competitive environments. *arXiv preprint arXiv:1710.03641*.

Ammar, H. B., Eaton, E., Ruvolo, P., and Taylor, M. (2014). Online multi-task learning for policy gradient methods. In *International Conference on Machine Learning*, pages 1206–1214.

Andrychowicz, M., Wolski, F., Ray, A., Schneider, J., Fong, R., Welinder, P., McGrew, B., Tobin, J., PieterAbbeel, O., and Zaremba, W. (2017). Hindsight experience replay. In Guyon, I., Luxburg, U. V., Bengio, S., Wallach, H., Fergus, R., Vishwanathan, S., and Garnett, R., editors, *Advances in Neural Information Processing Systems 30*, pages 5048–5058. Curran Associates, Inc.

Azar, M. G., Munos, R., and Kappen, H. J. (2013). Minimax pac bounds on the sample complexity of reinforcement learning with a generative model. *Machine learning*, 91(3):325–349.

Brunskill, E. and Li, L. (2013). Sample complexity of multi-task reinforcement learning. In *Proceedings of the Twenty-Ninth Conference on Uncertainty in Artificial Intelligence*, pages 122–131. AUAI Press.
Brunskill, E. and Li, L. (2014). Pac-inspired option discovery in lifelong reinforcement learning. In *International conference on machine learning*, pages 316–324.

Buckman, J., Hafner, D., Tucker, G., Brevdo, E., and Lee, H. (2018). Sample-efficient reinforcement learning with stochastic ensemble value expansion. In *Advances in Neural Information Processing Systems*, pages 8224–8234.

Calandriello, D., Lazaric, A., and Restelli, M. (2014). Sparse multi-task reinforcement learning. In *Advances in Neural Information Processing Systems*, pages 819–827.

Denil, M., Agrawal, P., Kulkarni, T. D., Erez, T., Battaglia, P., and de Freitas, N. (2016). Learning to perform physics experiments via deep reinforcement learning. *arXiv preprint arXiv:1611.01843*.

Dosovitskiy, A., Ros, G., Codevilla, F., Lopez, A., and Koltun, V. (2017). Carla: An open urban driving simulator. In *Conference on Robot Learning*, pages 1–16.

Esteva, A., Robicquet, A., Ramsundar, B., Kuleshov, V., DePristo, M., Chou, K., Cui, C., Corrado, G., Thrun, S., and Dean, J. (2019). A guide to deep learning in healthcare. *Nature medicine*, 25(1):24–29.

Gershgorin, S. A. (1931). Über die abgrenzung der eigenwerte einer matrix. *Izv. Akad. Nauk, S. S. R. 7*, (6):749–754.

Itsuki, N. (1995). Soccer server: a simulator for robocup. In *JSAI AI-Symposium 95: Special Session on RoboCup*. Citeseer.

Jiang, N. (2018). Pac reinforcement learning with an imperfect model. In *Thirty-Second AAAI Conference on Artificial Intelligence*.

Jiang, N. and Li, L. (2015). Doubly robust off-policy value evaluation for reinforcement learning. *arXiv preprint arXiv:1511.03722*.

Kalweit, G. and Boedecker, J. (2017). Uncertainty-driven imagination for continuous deep reinforcement learning. In *Conference on Robot Learning*, pages 195–206.

Kober, J., Bagnell, J. A., and Peters, J. (2013). Reinforcement learning in robotics: A survey. *The International Journal of Robotics Research*, 32(11):1238–1274.

Kumar, A. and Daume III, H. (2012). Learning task grouping and overlap in multi-task learning. *arXiv preprint arXiv:1206.6417*.

Kurutach, T., Clavera, I., Duan, Y., Tamar, A., and Abbeel, P. (2018). Model-ensemble trust-region policy optimization. *arXiv preprint arXiv:1802.10592*.

Mann, T. A. and Choe, Y. (2012). Directed exploration in reinforcement learning with transferred knowledge. In *Ewrl*, pages 59–76.

Mannor, S. and Tsitsiklis, J. N. (2004). The sample complexity of exploration in the multi-armed bandit problem. *Journal of Machine Learning Research*, 5(Jun):623–648.
Maurer, A., Pontil, M., and Romera-Paredes, B. (2013). Sparse coding for multitask and transfer learning. In International conference on machine learning, pages 343–351.

Modi, A., Jiang, N., Tewari, A., and Singh, S. (2019). Sample complexity of reinforcement learning using linearly combined model ensembles. arXiv preprint arXiv:1910.10597.

Ng, A. Y., Coates, A., Diel, M., Ganapathi, V., Schulte, J., Tse, B., Berger, E., and Liang, E. (2006). Autonomous inverted helicopter flight via reinforcement learning. In Experimental robotics IX, pages 363–372. Springer.

Puterman, M. L. (2014). Markov Decision Processes.: Discrete Stochastic Dynamic Programming. John Wiley & Sons.

Ruvolo, P. and Eaton, E. (2013). Ella: An efficient lifelong learning algorithm. In International Conference on Machine Learning, pages 507–515.

Si, J. and Wang, Y.-T. (2001). Online learning control by association and reinforcement. IEEE Transactions on Neural networks, 12(2):264–276.

Sidford, A., Wang, M., Wu, X., Yang, L. F., and Ye, Y. (2018). Near-optimal time and sample complexities for solving discounted markov decision process with a generative model. arXiv preprint arXiv:1806.01492.

Silver, D., Huang, A., Maddison, C. J., Guez, A., Sifre, L., Van Den Driessche, G., Schrittwieser, J., Antonoglou, I., Panneershelvam, V., Lanctot, M., et al. (2016). Mastering the game of go with deep neural networks and tree search. nature, 529(7587):484.

Sutton, R. S., Barto, A. G., and Williams, R. J. (1992). Reinforcement learning is direct adaptive optimal control. IEEE Control Systems Magazine, 12(2):19–22.

Taylor, M. E. and Stone, P. (2009). Transfer learning for reinforcement learning domains: A survey. Journal of Machine Learning Research, 10(Jul):1633–1685.

Tropp, J. A. et al. (2015). An introduction to matrix concentration inequalities. Foundations and Trends® in Machine Learning, 8(1-2):1–230.

Vanderbei, R. (1996). Sailing strategies, an application involving stochastics, optimization, and statistics.

Watkins, C. J. and Dayan, P. (1992). Q-learning. Machine learning, 8(3-4):279–292.

Wiering, M. (2000). Multi-agent reinforcement learning for traffic light control. In Machine Learning: Proceedings of the Seventeenth International Conference (ICML’2000), pages 1151–1158.