Two-dimensional integrable theories with defects and fibre bundles over the circle

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Abstract
A procedure is described to associate fibre bundles over the circle to two-dimensional theories with defects which have their field equations and defects described by a zero curvature condition.

1 Introduction
The lagrangian formulation of two-dimensional integrable theories with certain discontinuities called defects has been the subject of some works in the last years. The theories analyzed include the cases of the Liouville, sine-Gordon and abelian affine Toda theories [1]-[2], the nonlinear Schrödinger theory [3], the complex sine-Gordon theory [4] and supersymmetric extensions of the sine-Gordon theory [5]-[6].

In the section 2 we review the Liouville theory with defect. In the section 3 we consider general theories with defects admitting a zero curvature description for their field equations and defects. We show that these field theories define in a natural way fibre bundles over the circle in such a way that theories in the absence of the defect correspond to trivial fibre bundles. In the appendixes A and B two calculations of the section 2 are given.

2 The Liouville theory with defect
The Liouville theory is an abelian conformal Toda theory. Its field equation is given in terms of a zero curvature condition

\[ \partial \bar{A} - \bar{\partial} A + [A, \bar{A}] = 0, \] (2. 1)

where

\[ A = B\varepsilon^{-1}B^{-1}, \] (2. 2)

\[ \bar{A} = -\varepsilon^{+} - \bar{\partial}BB^{-1}, \] (2. 3)
where $z \equiv t + x$, $\bar{z} \equiv t - x$, $\partial \equiv \frac{\partial}{\partial z} = (1/2)(\partial_t + \partial_x)$, $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = (1/2)(\partial_t - \partial_x)$, $A \equiv A_z = (1/2)(A_t + A_x)$, $\bar{A} \equiv A_{\bar{z}} = (1/2)(A_t - A_x)$, $B \equiv \exp(\phi h)$, $\varepsilon^+ \equiv \mu E_\alpha$, $\varepsilon^- \equiv \mu E_{-\alpha}$, $\mu \in \mathbb{R}$ and $(h, E_\alpha, E_{-\alpha})$ are the Chevalley generators of the Lie algebra $A_1$:

\[
[h, E_{\pm \alpha}] = \pm 2E_{\pm \alpha} \quad \text{and} \\
[E_\alpha, E_{-\alpha}] = h. \tag{2.4}
\]

The field equation is

\[
\partial \bar{\partial} \phi = \mu^2 e^{-2\phi}. \tag{2.6}
\]

(A parametrization $B \equiv \exp(-\phi h)$, $\varepsilon^+ \equiv \mu E_\alpha$ and $\varepsilon^- \equiv -\mu E_{-\alpha}$ would result in the equation $\partial \bar{\partial} \phi = \mu^2 e^{2\phi}$.)

There is a standard procedure \cite{7} to obtain the lagrangian of the Toda theories using their relation to the Wess-Zumino-Witten theory. The lagrangian of the Liouville theory is

\[
L = -\frac{k}{2\pi} \partial \phi \bar{\partial} \phi + \frac{k\mu^2}{2\pi} e^{-2\phi}, \tag{2.7}
\]

where $k$ is constant. The theory defined over all the real line $\mathbb{R}$, that is, $x \in \mathbb{R}$, in the absence of the defect, is denominated the theory in the bulk. The lagrangian of the Liouville theory with defect is given by

\[
L = \theta(-x)L_1 + \theta(x)L_2 + \delta(x)L_D \tag{2.8}
\]

where

\[
L_p = -\frac{k}{2\pi} \partial \phi_p \bar{\partial} \phi_p + \frac{k\mu^2}{2\pi} e^{-2\phi_p}, \tag{2.9}
\]

\[
L_D = \frac{k}{8\pi} (\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) + B(\phi_1, \phi_2), \tag{2.10}
\]

$p \in \{1, 2\}$, $\delta$ is the Dirac delta function, $\theta$ is the step function, $(d\theta(x)/dx) = \delta(x)$, with the choice $\theta(0) = 0$, that is, $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x \leq 0$. Note that this choice implies that

\[
\int_0^a \delta(x)dx = \int_0^a \frac{d\theta(x)}{dx}dx = 1 = -\int_a^0 \delta(y)dy = \int_0^a \delta(-x)dx, \tag{2.11}
\]

\[
\int_{-a}^0 \delta(x)dx = \int_{-a}^0 \frac{d\theta(x)}{dx}dx = 0 = -\int_0^{-a} \delta(y)dy = \int_0^a \delta(-x)dx, \tag{2.12}
\]

$\forall a > 0$. The functional $B(\phi_1, \phi_2)$ is called the border function and it depends on the fields $\phi_1$ and $\phi_2$ but it does not depend on any of their derivatives. The border function is going to be determinated by an argument related to the conservation of
a modified momentum assigned to the theory. We have that
\[
\delta L \Bigg/ \delta \phi_1 = -\frac{k \mu^2}{\pi} \theta(-x) e^{-2\phi_1} + \delta(x) \left( \frac{\delta B}{\delta \phi_1} - \frac{k}{8\pi} \partial_t \phi_2 \right), \tag{2.13}
\]
\[
\delta L \Bigg/ \delta \phi_2 = -\frac{k \mu^2}{\pi} \theta(x) e^{-2\phi_2} + \delta(x) \left( \frac{\delta B}{\delta \phi_2} + \frac{k}{8\pi} \partial_t \phi_1 \right), \tag{2.14}
\]
\[
\partial_\mu \frac{\delta L}{\delta (\partial_\mu \phi_1)} = -\frac{k}{4\pi} \theta(-x) (\partial_t^2 - \partial_x^2) \phi_1 - \frac{k}{8\pi} \delta(-x) \partial_x \phi_1 + \frac{k}{8\pi} \delta(x) \partial_t \phi_2, \tag{2.15}
\]
\[
\partial_\mu \frac{\delta L}{\delta (\partial_\mu \phi_2)} = -\frac{k}{4\pi} \theta(x) (\partial_t^2 - \partial_x^2) \phi_2 - \frac{k}{8\pi} \delta(x) \partial_t \phi_1 + \frac{k}{4\pi} \delta(x) \partial_x \phi_2. \tag{2.16}
\]
The Euler-Lagrange equations
\[
\frac{\delta L}{\delta \phi_p} = \partial_\mu \left( \frac{\delta L}{\delta (\partial_\mu \phi_p)} \right), \tag{2.17}
\]
p \in \{1, 2\}, are equivalent to:

a) If \(x > 0\):
\[
\partial \bar{\phi}_2 = \mu^2 e^{-2\phi_2}. \tag{2.18}
\]

b) If \(x < 0\):
\[
\partial \bar{\phi}_1 = \mu^2 e^{-2\phi_1}. \tag{2.19}
\]

c) If \(x = 0\):
\[
\partial_t \phi_2 - \partial_x \phi_1 = \frac{4\pi}{k} \frac{\delta B}{\delta \phi_1} \text{ and } \tag{2.20}
\]
\[
\partial_x \phi_2 - \partial_t \phi_1 = \frac{4\pi}{k} \frac{\delta B}{\delta \phi_2}. \tag{2.21}
\]
The equations (2.20) and (2.21) are called the defect conditions. Note that we do not have a condition \(\phi_1 = \phi_2\) at \(x = 0\).

The energy-momentum tensor is given by
\[
T^{\mu\nu} = \sum_{p=1}^{2} g^{\nu\gamma} \frac{\delta L}{\delta (\partial_\mu \phi_p)} \partial_\gamma \phi_p - g^{\mu\nu} L \tag{2.22}
\]
where \(g_{00} = g^{00} = 1, g_{11} = g^{11} = -1, g_{01} = g^{10} = g^{01} = g^{10} = 0\) and \((t, x) = (x^0, x^1)\). Thus
\[
P = \int_{-\infty}^{0} T^{01} dx + \int_{0}^{\infty} T^{01} dx = \frac{k}{4\pi} \int_{-\infty}^{0} \frac{\partial \phi_1}{\partial t} \frac{\partial \phi_1}{\partial x} dx + \frac{k}{4\pi} \int_{0}^{\infty} \frac{\partial \phi_2}{\partial t} \frac{\partial \phi_2}{\partial x} dx. \tag{2.23}
\]
The time derivative of \(P\) is
\[
\frac{dP}{dt} \equiv \frac{dP^{(1)}}{dt} + \frac{dP^{(2)}}{dt} = \frac{k}{4\pi} \int_{-\infty}^{0} \left[ (\partial_t \partial_x \phi_1) \partial_t \phi_1 + (\partial^2_x \phi_1) \partial_x \phi_1 \right] dx
\]
\[
+ \frac{k}{4\pi} \int_{0}^{\infty} \left[ (\partial_t \partial_x \phi_2) \partial_t \phi_2 + (\partial^2_x \phi_2) \partial_x \phi_2 \right] dx. \tag{2.24}
\]
Using (2.19), we have
\[
\frac{dP^{(1)}}{dt} = \frac{k}{4\pi} \int_{-\infty}^{0} \left[ (\partial_t \partial_x \phi_1) \partial_t \phi_1 + \partial_x \phi_1 \left( \frac{\partial^2 \phi_1}{2} + \frac{4\pi}{k} \frac{\delta V_1}{\delta \phi_1} \right) \right] dx, \tag{2.25}
\]
where
\[
V_p \equiv -\frac{k\mu^2}{2}\pi e^{-2\phi_p}, \tag{2.26}
\]
p \in \{1, 2\}. Then
\[
\frac{dP^{(1)}}{dt} = \frac{k}{4\pi} \int_{-\infty}^{0} \partial_x \left[ \frac{(\partial_t \phi_1)^2}{2} + \frac{(\partial_x \phi_1)^2}{2} + \frac{4\pi}{k} V_1 \right] dx. \tag{2.27}
\]
The other term in (2.24) can be treated in a similar way. Thus
\[
\frac{dP}{dt} = \frac{k}{4\pi} \left[ \frac{(\partial_t \phi_1)^2}{2} + \frac{(\partial_x \phi_1)^2}{2} + \frac{4\pi}{k} V_1 \right] \bigg|_{-\infty}^{0} + \frac{k}{4\pi} \left[ \frac{(\partial_t \phi_2)^2}{2} + \frac{(\partial_x \phi_2)^2}{2} + \frac{4\pi}{k} V_2 \right] \bigg|_{0}. \tag{2.28}
\]
The kind of terms at ±∞ are already present in theory defined in the bulk (in the absence of the defect) and we are going to ignore them. Considering only the contribution from \(x = 0\) and using the expressions for \(\partial_x \phi_1\) and \(\partial_x \phi_2\) from (2.20) and (2.21), we have
\[
\frac{dP}{dt} = \left[ -\partial_t \phi_2 \frac{\delta B}{\delta \phi_1} - \partial_t \phi_1 \frac{\delta B}{\delta \phi_2} + \frac{2\pi k}{2} \left( \frac{\delta B}{\delta \phi_1} \right)^2 - \frac{2\pi k}{2} \left( \frac{\delta B}{\delta \phi_2} \right)^2 + V_1 - V_2 \right] \bigg|_{x=0}. \tag{2.29}
\]
If
\[
\left[ \frac{2\pi k}{\delta \phi_1} \left( \frac{\delta B}{\delta \phi_1} \right)^2 - \frac{2\pi k}{\delta \phi_2} \left( \frac{\delta B}{\delta \phi_2} \right)^2 + V_1 - V_2 \right] = 0, \tag{2.30}
\]
\[
\frac{\delta B(\phi_1, \phi_2)}{\delta \phi_1} = \frac{\delta M(\phi_1, \phi_2)}{\delta \phi_1}, \tag{2.31}
\]
\[
\frac{\delta B(\phi_1, \phi_2)}{\delta \phi_2} = \frac{\delta M(\phi_1, \phi_2)}{\delta \phi_2}, \tag{2.32}
\]
for some functional \(M\), in such a way that
\[
\frac{\delta^2 B}{\delta \phi_1^2} = \frac{\delta^2 B}{\delta \phi_2^2}, \tag{2.33}
\]
then
\[
\left( \frac{dP}{dt} \right) \bigg|_t = - \left( \frac{\partial}{\partial t} [M(t, x)] \right) \bigg|_{(t, x=0)} = - \left( \frac{d}{dt} [M(t, x = 0)] \right) \bigg|_t. \tag{2.34}
\]
and
\[ \frac{d}{dt} [P + M(t, x = 0)] = 0. \] (2.35)

Thus the modified momentum \( P + M(t, x = 0) \) is conserved provided we have the relations (2.30)-(2.33).

The energy is given by
\[
E = \int_{-\infty}^{0} T^{00} \, dx + \int_{0}^{\infty} T^{00} \, dx
= \frac{k}{8\pi} \int_{-\infty}^{0} \left[ \frac{8\pi}{k} V_1 - (\partial_t \phi_1)^2 - (\partial_x \phi_1)^2 \right] \, dx
+ \frac{k}{8\pi} \int_{0}^{\infty} \left[ \frac{8\pi}{k} V_2 - (\partial_t \phi_2)^2 - (\partial_x \phi_2)^2 \right] \, dx.
\] (2.36)

Using (2.18) and (2.19) the time derivative of \( E \) can be evaluated as
\[
\frac{dE}{dt} = -\frac{k}{4\pi} (\partial_t \phi_1 \partial_x \phi_1) \bigg|_{-\infty}^{0} - \frac{k}{4\pi} (\partial_t \phi_2 \partial_x \phi_2) \bigg|_{0}^{\infty}.
\] (2.37)

Ignoring the terms at \( \pm \infty \) and using the expressions for \( \partial_x \phi_1 \) and \( \partial_x \phi_2 \) from (2.20) and (2.21), we have:
\[
\left( \frac{dE}{dt} \right) \bigg|_{t} = \left( \frac{\partial}{\partial t} [B(t, x)] \right) \bigg|_{(t,x=0)} = \left( \frac{d}{dt} [B(t, x = 0)] \right) \bigg|_{t} \] (2.38)

and
\[
\frac{d}{dt} [E - B(t, x = 0)] = 0. \] (2.39)

Thus the modified energy \( E - B(t, x = 0) \) is conserved. Note that we do not need the expressions (2.30)-(2.33) to get (2.39). We introduce \( \phi^+ \equiv \phi_1 + \phi_2 \) and \( \phi^- \equiv \phi_1 - \phi_2 \). Then
\[
\frac{\delta}{\delta \phi^+} = \frac{1}{2} \left( \frac{\delta}{\delta \phi_1} + \frac{\delta}{\delta \phi_2} \right) \quad \text{and} \quad \frac{\delta}{\delta \phi^-} = \frac{1}{2} \left( \frac{\delta}{\delta \phi_1} - \frac{\delta}{\delta \phi_2} \right). \] (2.40)

If we write
\[
B(\phi_1, \phi_2) \equiv B^+(\phi^+) + B^- (\phi^-), \] (2.42)
we see that (2.31)-(2.32) is solved by
\[
M = B^+ - B^-.
\] (2.43)

Using (2.40)-(2.42), we see that the equation (2.30) is equivalent to
\[
2 \frac{\delta B^+ \delta B^-}{\delta \phi^+ \delta \phi^-} = - \left( \frac{k}{2\pi} \right)^2 \mu^2 e^{-\phi^+} \sinh(\phi^-). \] (2.44)
This equation is solved by
\[ B^+ = \frac{k}{2\pi} \mu \lambda e^{-\phi^+} \quad \text{and} \quad (2.45) \]
\[ B^- = \frac{1}{2\pi} \frac{k}{2} \mu \cosh \phi^- , \quad (2.46) \]
where \( \lambda \) is an arbitrary constant.

The defect conditions, given by the equations (2.20)-(2.21), are equivalent to
\[ \partial (\phi_1 - \phi_2)|_{x=0} = 2\mu \lambda e^{-\phi_1}|_{x=0} \quad \text{and} \quad (2.47) \]
\[ \bar{\partial} (\phi_1 + \phi_2)|_{x=0} = \frac{\mu}{\lambda} \sinh (\phi^-)|_{x=0}. \quad (2.48) \]

Suppose that the equations (2.47) and (2.48) hold not only at \((t, x = 0)\) but at all \((t, x)\). Then taking the \( \bar{\partial} \) derivative of (2.47) and using (2.48), results
\[ \bar{\partial} \partial (\phi_1 - \phi_2) = \mu^2 (e^{-2\phi_1} - e^{-2\phi_2}). \quad (2.49) \]
Similarly, taking the \( \partial \) derivative of (2.48) and using (2.47), results
\[ \bar{\partial} \partial (\phi_1 + \phi_2) = \mu^2 (e^{-2\phi_1} + e^{-2\phi_2}). \quad (2.50) \]

We see that (2.49) and (2.50) imply that \((\phi_1 \text{ is a solution of the Liouville equation}) \iff (\phi_2 \text{ is a solution of the Liouville equation})\). That is, if the equations (2.47) and (2.48) were valid at all the points \((t, x)\) they would be Bäcklund transformations. As (2.47) and (2.48) are supposed to hold only at the point \((t, x = 0)\), they are said to be frozen Bäcklund transformations.

From (2.2) and (2.3) we can obtain \( A_t \) and \( A_x \). It is convenient to us to apply a gauge transformation to \((A_t, A_x)\) with a group element given by \( g = e^{\frac{\phi}{2}} \) to obtain \((A^g_t, A^g_x)\). To simplify our notation we are just going to write \((A_t, A_x)\) instead \((A^g_t, A^g_x)\):
\[ A_t = -\mu e^{-\phi} E_\alpha + \mu e^{-\phi} E_{-\alpha} + \frac{\partial_x \phi h}{2} \quad \text{and} \quad (2.51) \]
\[ A_x = \mu e^{-\phi} E_\alpha + \mu e^{-\phi} E_{-\alpha} + \frac{\partial_t \phi h}{2}. \quad (2.52) \]

As the curvature transforms as \( F^g_{tx} = g F_{tx} g^{-1} \), we see that \((F^g_{tx} = 0) \iff (F_{tx} = 0)\). That is, the zero curvature condition
\[ \partial_t A_x - \partial_x A_t + [A_t, A_x] = 0 \quad (2.53) \]
is equivalent to the Liouville equation (2.6), where \( A_t \) and \( A_x \) are given by (2.51) and (2.52).
Now we want to describe a procedure to obtain the field equations \((2.18) - (2.19)\) and the defect conditions \((2.20) - (2.21)\) as a zero curvature condition in the following sense: Define

\[
A_t^{(p)} \equiv -\mu e^{-\phi} E_\alpha + \mu e^{-\phi} E_{-\alpha} + \frac{\partial_x \phi_p h}{2}, \quad (2.54)
\]

\[
A_x^{(p)} \equiv \mu e^{-\phi} E_\alpha + \mu e^{-\phi} E_{-\alpha} + \frac{\partial_t \phi_p h}{2}, \quad (2.55)
\]

\[
D_1 \equiv \partial_x \phi_1 - \partial_t \phi_2 + \frac{4\pi \delta B}{k \delta \phi_1} \quad \text{and} \quad (2.56)
\]

\[
D_2 \equiv \partial_x \phi_2 - \partial_t \phi_1 - \frac{4\pi \delta B}{k \delta \phi_2}, \quad (2.57)
\]

\[p \in \{1, 2\}.\] Note that \(D_1 = 0\) and \(D_2 = 0\) are equivalent to \((2.20)\) and \((2.21)\).

Let \(a, b \in \mathbb{R}, a < 0 < b\). Define

a) \(\forall (t, x) \in \mathbb{R}^2\) such that \(x < b:\)

\[
\hat{A}_t^{(1)} \equiv [\theta(x - a) + \theta(a - x)] A_t^{(1)} - \frac{1}{2} \theta(x - a) D_1 h \quad \text{and} \quad (2.58)
\]

\[
\hat{A}_x^{(1)} \equiv \theta(a - x) A_x^{(1)} \quad (2.59)
\]

b) \(\forall (t, x) \in \mathbb{R}^2\) such that \(x > a:\)

\[
\hat{A}_t^{(2)} \equiv [\theta(x - b) + \theta(b - x)] A_t^{(2)} - \frac{1}{2} \theta(b - x) D_2 h \quad \text{and} \quad (2.60)
\]

\[
\hat{A}_x^{(2)} \equiv \theta(x - b) A_x^{(2)} \quad (2.61)
\]

Let

\[
\partial_t \hat{A}_x^{(1)} - \partial_x \hat{A}_t^{(1)} + [\hat{A}_t^{(1)}, \hat{A}_x^{(1)}] = 0 \quad \text{and} \quad (2.62)
\]

\[
\partial_t \hat{A}_x^{(2)} - \partial_x \hat{A}_t^{(2)} + [\hat{A}_t^{(2)}, \hat{A}_x^{(2)}] = 0. \quad (2.63)
\]

One can verify, as explained in the appendix A, that:

a) The equation \((2.62)\) implies for \(x < a\) in the equation \((2.53)\) calculated at \((A_t, A_x) = (A_t^{(1)}, A_x^{(1)})\) and this is equivalent to \((2.19)\).

b) The equation \((2.63)\) implies for \(x > b\) in the equation \((2.53)\) calculated at \((A_t, A_x) = (A_t^{(2)}, A_x^{(2)})\) and this is equivalent to \((2.18)\).

c) The equation \((2.62)\) implies for \(x = a\) in \(D_1 = 0\) and this is equivalent to \((2.20)\).

d) The equation \((2.63)\) implies for \(x = b\) in \(D_2 = 0\) and this is equivalent to \((2.21)\).

e) In the intersection \(a < x < b\), the equations \((2.62)\) and \((2.63)\) imply that

\[
\partial_x \phi_1 = \partial_x \phi_2 = 0. \quad (2.64)
\]
The idea is that we can take \( |a| \) and \( b \) so small as we want and in this case the conditions a) to d) would correspond to the field equations and defect conditions as expressed in (2.18) - (2.21).

Note that the intersection of the domains in which \( (\hat{A}_t^{(1)}, \hat{A}_x^{(1)}) \) and \( (\hat{A}_t^{(2)}, \hat{A}_x^{(2)}) \) are defined is \((t, x) \in \mathbb{R}^2 \) such that \( a < x < b \). Suppose that in this intersection the two connections are related by a gauge transformation \([1]\). In this case, \(2.62 \iff 2.63\). Note that in the intersection \( a < x < b \) we have \( \hat{A}_x^{(1)} = \hat{A}_x^{(2)} = 0 \). Then the group element \( g \) that gives the gauge transformation must be \( x \)-independent. We need to find \( g \) such that

\[
\hat{A}_t^{(1)} = g\hat{A}_t^{(2)} g^{-1} - (\partial_t g) g^{-1}. \tag{2.65}
\]

We can verify, as explained in the appendix B, that

\[
g = e^{-\frac{\phi_h}{\lambda}} e^{2\lambda E_a} e^{\frac{\phi_h}{\lambda}} \tag{2.66}
\]

where \( \lambda \) is the constant introduced in the equations (2.45) - (2.46). Note that \( g \) is \( x \)-independent as a consequence of (2.64) but it can be \( t \)-dependent.

### 3 Two-dimensional integrable theories with defects and fibre bundles over the circle

In this section we consider general theories which have the following structure: their lagrangians have the form

\[
L = \theta(-x)L_1 + \theta(x)L_2 + \delta(x)L_D. \tag{3.1}
\]

The field equations and defect conditions corresponding to (3.1) can be expressed as a zero curvature condition associated to two sets of gauge potentials. That is, it is possible to define

a) \( \hat{A}_t^{(1)} \) and \( \hat{A}_x^{(1)} \), \( \forall(t, x) \in \mathbb{R}^2 \) such that \( x < b \);

b) \( \hat{A}_t^{(2)} \) and \( \hat{A}_x^{(2)} \), \( \forall(t, x) \in \mathbb{R}^2 \) such that \( x > a \); where \( a < 0 < b \), such that

\[
\begin{align*}
\partial_t \hat{A}_x^{(1)} - \partial_x \hat{A}_t^{(1)} + [\hat{A}_t^{(1)}, \hat{A}_x^{(1)}] &= 0 \quad \text{and} \\
\partial_t \hat{A}_x^{(2)} - \partial_x \hat{A}_t^{(2)} + [\hat{A}_t^{(2)}, \hat{A}_x^{(2)}] &= 0 \tag{3.2}
\end{align*}
\]

correspond to the field equations (for \( x < a \) and \( x > b \)) and defect conditions (for \( x = a \) and \( x = b \)) in an analogous way to that described in the section 2 in the case of the Liouville theory with defect. In the intersection, \( (t, x) \in \mathbb{R}^2 \) such that \( a < x < b \), \(3.2\) and \(3.3\) imply that the fields of the theory obey some additional set of equations, as \(2.64\) in the case of the Liouville theory with defect. We
suppose that in this intersection the two sets of gauge potentials are related by a
gauge transformation

\[ \hat{A}_t^{(1)} = g \hat{A}_t^{(2)} g^{-1} - (\partial_t g) g^{-1}, \quad (3.4) \]
\[ \hat{A}_x^{(1)} = g \hat{A}_x^{(2)} g^{-1} - (\partial_x g) g^{-1}, \quad (3.5) \]

where \( g \) is a group element.

That is the situation we are proposing to analyze. The key role is played by the

group element \( g \). We suppose that the defect is placed at \( x = 0 \), but this can be
generalized easily. As \( a \) and \( b \) are arbitrary, to simplify our discussion, we are going
to take \( |a| = b = r > 0 \).

Note that in the Liouville theory with defect we were in fact dealing with the
Minkowski \( \mathbb{M}k^2 \) space-see (2,22). That is, the differentiable manifold \( \mathbb{R}^2 \) with
the metric tensor \( g^{\text{metric}} = dt \otimes dt - dx \otimes dx \) (in global coordinates) and a Levi-
Civita connection. In this section we consider the differentiable manifold \( \mathbb{R}^2 \) with the
constant metric tensor given by the general expression \( g^{\text{metric}} = \epsilon_t dt \otimes dt + \epsilon_x dx \otimes dx \),
where \( \epsilon_t = \pm 1 \) and \( \epsilon_x = \pm 1 \) (in global coordinates) and a Levi-Civita connection.
The Euclidean space \( E^2 \) corresponds to \( \epsilon_t = 1 \) and \( \epsilon_x = 1 \).

Now we are going to review the definition of fibre bundles. Then we are going
to see how a fibre bundle can be constructed from a minimal information and apply
that process to our case.

A coordinate bundle [3] \((E, \pi, M, F, G, \{U_i\}, \{\phi_i\})\) consists of the following elements:

a) A differentiable manifold \( E \) called the total space; a differentiable manifold \( M \) called the base space; a differentiable manifold \( F \) called the fibre.

b) A surjective map \( \pi : E \to M \) called the projection. The inverse image \( \pi^{-1}\{p\} \equiv F_p \) is called the fibre at \( p, \forall p \in M \).

c) A Lie group \( G \) called the structure group, which acts on \( F \) on the left. That is,
there is a differentiable map \( T : G \times F \to F \) such that \( T(g_1, T(g_2, f)) = T(g_1 g_2, f) \)
and \( T(e, f) = f, \forall g_1, g_2 \in G, \forall f \in F \), where \( e \) is the identity element of \( G \). (Note
that a Lie group is, by definition, a differentiable manifold.) We simplify the notation
writing \( T(g, f) \equiv g f \).

d) A set \( \{U_i\} \) which is an open covering of \( M (\bigcup_i U_i = M) \) and a set \( \{\phi_i\} \), where
each \( \phi_i \) is a diffeomorphism \( \phi_i : U_i \times F \to \pi^{-1}(U_i) \) such that \( \pi \phi_i(p, f) = p, \forall p \in U_i, \forall f \in F \). The map \( \phi_i \) is called the local trivialisation.

e) We define \( \phi_{i,p}(f) \equiv \phi_i(p, f), \forall p \in U_i, \forall f \in F \). Then the map \( \phi_{i,p} : F \to F_p \) is
a diffeomorphism. If \( U_i \cap U_j \neq \emptyset \), we require that \( t_{ij}(p) \equiv \phi_{i,p}^{-1} \phi_{j,p} : F \to F \) be an
element of \( G \). Then \( \phi_i \) and \( \phi_j \) are related by a \( C^\infty \) differentiable map
\( t_{ij} : U_i \cap U_j \to G \) as:

\[ \phi_j(p, f_j) = \phi_i(p, t_{ij}(p) f_j). \quad (3.6) \]

\( \{t_{ij}\} \) are denominated the transition functions.

Two coordinate bundles
\((E, \pi, M, F, G, \{U_i\}, \{\phi_i\})\) and \((E, \pi, M, F, G, \{V_i\}, \{\psi_i\})\) are said to be equivalent if
\((E, \pi, M, F, G, \{U_i\} \cup \{V_i\}, \{\phi_i\} \cup \{\psi_i\})\) is again a coordinate bundle. The fibre bundle is defined as an equivalence class of coordinate bundles and is usually denoted by \((E, \pi, M, F, G)\).

We require the consistency conditions:

a) \(\forall p \in U_i\)
\[ t_{ii}(p) = e. \] (3.7)

b) \(\forall p \in U_i \cap U_j\)
\[ t_{ij}(p) = [t_{ji}(p)]^{-1}. \] (3.8)

c) \(\forall p \in U_i \cap U_j \cap U_k\)
\[ t_{ij}(p)t_{jk}(p) = t_{ik}(p). \] (3.9)

If all the transition functions can be taken to be identity maps, the fibre bundle is called trivial. A trivial bundle is a product manifold \(M \times F\).

The minimal information to construct a fibre bundle \([8]\) is given by \(\{M, F, G, \{U_i\}, \{t_{ij}\}\}\), where \(M\) and \(F\) are differentiable manifolds, \(G\) is a Lie group acting on \(F\) by the left, \(\{U_i\}\) is an open covering of \(M\) and the set \(\{t_{ij}\}\) is such that each \(t_{ij} : U_i \cap U_j \to G\) is a \(C^\infty\) differentiable map satisfying (3.7)-(3.9).

We define
\[ E_0 = \bigcup_i U_i \times F \] (3.10)
and introduce an equivalence relation \(\sim\) between \((p, f) \in U_i \times F\) and \((q, f') \in U_j \times F\) if and only if \(p = q\) and \(f' = t_{ji}(p)f\).

The total space is defined by
\[ E \equiv E_0/\sim. \] (3.11)

Let us denote the equivalence class of \((p, f)\) by \([p, f]\). Then \([p, f] \in E\). The projection is given by
\[ \pi : [(p, f)] \to p \] (3.12)
and the local trivialisation is given by
\[ \phi_i : (p, f) \to [(p, f)]. \] (3.13)

It is possible to verify that this construction defines \([8]\) a coordinate bundle. Then the fibre bundle is the equivalence class containing this coordinate bundle.

Now let us apply this process to our case. We need to specify \(M, F, G, \{U_i\}, \{t_{ij}\}\). We take \(F\) as the Lie group itself. This corresponds to consider a principal fibre bundle. Of course, having a principal fibre bundle, one can construct its associated vector bundles \([8]\).

Let the set \(S^1(r)\) be defined by
\[ S^1(r) \equiv \{(t, x) \in \mathbb{R}^2 | t^2 + x^2 = r^2\}, \] (3.14)
where \( r \in \mathbb{R}, r > 0 \). We want to make \( S^1(r) \) a topological space. The open ball of radius \( R (R \in \mathbb{R}, R > 0) \) and center \((t_0, x_0) \in \mathbb{R}^2\) is defined as

\[
B^R(t_0, x_0) \equiv \{(t, x) \in \mathbb{R}^2| (t - t_0)^2 + (x - x_0)^2 < R^2\}.
\] (3. 15)

The usual topology of \( \mathbb{R}^2 \) is the one such that \( V \subseteq \mathbb{R}^2 \) is open if and only if, \( \forall p \in V, \exists B^R(p) \subseteq V \) for some radius \( R > 0 \). Note that, in particular, \( B^R(p) \) and \( \mathbb{R}^2 - \{p\} \) are open sets of \( \mathbb{R}^2, \forall p \in \mathbb{R}^2, \forall R > 0 \). We give to \( S^1(r) \) the topology induced by \( \mathbb{R}^2 \). That is, \( U \subseteq S^1(r) \) is open if and only if \( U = S^1(r) \cap V \), where \( V \) is some open set of \( \mathbb{R}^2 \). Note that, in particular, \( S^1(r) - \{p\} \) is an open set of \( S^1(r), \forall p \in S^1(r) \), because

\[
S^1(r) - \{p\} = S^1(r) \cap \{\mathbb{R}^2 - \{p\}\}.
\] (3. 16)

It is possible to give to the topological space \( S^1(r) \) a differentiable structure in such a way it becomes a differentiable manifold [8]. We take \( M \) as the differentiable manifold \( S^1(r) \).

We need to specify an open covering \( \{U_i\} \) of \( S^1(r) \). We define

\[
U_1 \equiv S^1(r) - \{(t, x) = (0, -r)\} \quad \text{and} \quad (3. 17)
\]

\[
U_2 \equiv S^1(r) - \{(t, x) = (0, r)\}.
\] (3. 18)

Note that \( (t, x) = (0, -r) \in S^1(r) \) and \( (t, x) = (0, r) \in S^1(r) \). Then \( S^1(r) = U_1 \cup U_2 \).

Note that

\[
U_1 \cap U_2 = A \cup B
\] (3. 19)

where

\[
A \equiv \{(t, x) \in S^1(r)| t > 0\}, \quad (3. 20)
\]

\[
B \equiv \{(t, x) \in S^1(r)| t < 0\} \quad \text{and} \quad (3. 21)
\]

\[
A \cap B = \emptyset.
\] (3. 22)

We need to specify \( \{t_{ij}\} \). As our open covering has only two open sets, we need to specify \( C^\infty \) differentiable maps \( t_{12} : U_1 \cup U_2 \to G, t_{21} : U_1 \cap U_2 \to G \) and we do not need the equation (3. 9). We take

\[
t_{21}(p) \equiv [t_{12}(p)]^{-1},
\] (3. 23)

in order to satisfy (3. 8).

As we said before, we are taking \( |a| = b = r > 0 \). Let

\[
I = \{(t, x) \in \mathbb{R}^2| -r < x < r\}.
\] (3. 24)

Then the group element \( g \) in the equations (3. 4) and (3. 5) defines a map \( g : I \to G \). Note that

\[
U_1 \cap U_2 = A \cup B \subset I.
\] (3. 25)
Now consider the restriction of $g$ to $A \cup B$. If $g : (A \cup B) \to G$ is a $C^\infty$ differentiable map, we can define $t_{12} : U_1 \cap U_2 \to G$ by
\[
t_{12}(p) \equiv g(p),
\]
$\forall p \in U_1 \cap U_2 = A \cup B$.

By last, suppose we have a theory without defect such that its field equations are expressed as a zero curvature condition in terms of the gauge potentials $A_t$ and $A_x$. We can define

a) $\forall (t, x) \in \mathbb{R}^2$ such that $x < b$
\[
\hat{A}_t^{(1)}(t, x) = A_t(t, x) \quad \text{and} \quad \hat{A}_x^{(1)}(t, x) = A_x(t, x).
\]

b) $\forall (t, x) \in \mathbb{R}^2$ such that $x > a$
\[
\hat{A}_t^{(2)}(t, x) = A_t(t, x) \quad \text{and} \quad \hat{A}_x^{(2)}(t, x) = A_x(t, x).
\]

Where $a < 0 < b$. Then (3.2) and (3.3) correspond to the field equations without defect and in the intersection $\{(t, x) \in \mathbb{R}^2 | a < x < b\}$ the gauge potentials are related by a gauge transformation (3.4) and (3.5) where $g$ is the identity element of $G$. It follows from (3.26) that all the transition functions are the identity element. Thus we have a trivial fibre bundle.

4 Appendix A

Using (2.58) and (2.59) we see that, $\forall (t, x) \in \mathbb{R}^2$ such that $x < b$, (2.62) is equivalent to:
\[
\theta(a - x)\partial_t A_x^{(1)} - [\delta(x - a) - \delta(a - x)]A_t^{(1)} - [\theta(x - a) + \theta(a - x)]\partial_x A_t^{(1)}
\]
\[
+ \frac{1}{2} \theta(x - a)\partial_x D_1 h + \frac{1}{2} \delta(x - a)D_1 h - \frac{1}{2} \theta(x - a)\theta(a - x)D_1 [h, A_x^{(1)}]
\]
\[
+ [\theta(x - a) + \theta(a - x)]\theta(a - x)[A_t^{(1)}, A_x^{(1)}] = 0. \quad (4.1)
\]

Similarly, using (2.60) and (2.61) we see that, $\forall (t, x) \in \mathbb{R}^2$ such that $x > a$, (2.63) is equivalent to:
\[
\theta(x - b)\partial_t A_x^{(2)} - [\delta(x - b) - \delta(b - x)]A_t^{(2)} - [\theta(x - b) + \theta(b - x)]\partial_x A_t^{(2)}
\]
\[
+ \frac{1}{2} \theta(b - x)\partial_x D_2 h - \frac{1}{2} \delta(b - x)D_2 h - \frac{1}{2} \theta(x - b)\theta(b - x)D_2 [h, A_x^{(2)}]
\]
\[
+ [\theta(x - b) + \theta(b - x)]\theta(x - b)[A_t^{(2)}, A_x^{(2)}] = 0. \quad (4.2)
\]
a) Let \( x < a \). Then, from (4.1),
\[
\partial_t \hat{A}^{(1)}_x - \partial_x \hat{A}^{(1)}_x + [\hat{A}^{(1)}_x, \hat{A}^{(1)}_x] = 0.
\] (4.3)

b) Let \( x > b \). Then, from (4.2),
\[
\partial_t \hat{A}^{(2)}_x - \partial_x \hat{A}^{(2)}_x + [\hat{A}^{(2)}_x, \hat{A}^{(2)}_x] = 0.
\] (4.4)

c) Let \( x = a \). Then, from (4.1),
\[
D_1 = 0.
\] (4.5)
d) Let \( x = b \). Then, from (4.2),
\[
D_2 = 0.
\] (4.6)
e) Let \( a < x < b \). Then, from (4.1) and (4.2),
\[
-\partial_x A^{(1)}_x + \frac{1}{2} (\partial_x D_1) h = 0 \quad \text{and} \quad 4.7
\]
\[
-\partial_x A^{(2)}_x + \frac{1}{2} (\partial_x D_2) h = 0. \tag{4.8}
\]

Using (2.42), (2.45), (2.46) and (2.54)-(2.57), we see that (4.7) and (4.8) are equivalent to
\[
-\mu \partial_x \phi_1 e^{-\phi_1} E_\alpha + \mu \partial_x \phi_1 e^{-\phi_1} E_{-\alpha} \\
+ \left[ -\frac{\partial_x \partial_\phi_2}{2} - \mu \lambda \partial_x (e^{-\phi_1 - \phi_2}) + \frac{\mu}{2\lambda} \partial_x [\sinh(\phi_1 - \phi_2)] \right] h = 0 \quad \text{and} \quad 4.9
\]
\[
-\mu \partial_x \phi_2 e^{-\phi_2} E_\alpha + \mu \partial_x \phi_2 e^{-\phi_2} E_{-\alpha} \\
+ \left[ -\frac{\partial_x \partial_\phi_1}{2} + \mu \lambda \partial_x (e^{-\phi_1 - \phi_2}) + \frac{\mu}{2\lambda} \partial_x [\sinh(\phi_1 - \phi_2)] \right] h = 0. \tag{4.10}
\]

From (4.9) and (4.10), we see that
\[
\partial_x \phi_1 = \partial_x \phi_2 = 0. \tag{4.11}
\]

5 Appendix B

We want to find the group element \( g \) in such a way to satisfy the equation (2.65). In order to get an expression which has no derivatives we write
\[
g = e^{-\text{grad}_h \phi_1} g_c e^{\text{grad}_h \phi_1}, \tag{5.1}
\]
where $g_c$ is a group element which is $t$-independent and $x$-independent. After some manipulation, we find that (2.65) is equivalent to

$$-(\mu e^{\phi_1-\phi_2})g_c E_\alpha g_c^{-1} - (\mu e^{-\phi_1-\phi_2})E_{-\alpha} + (\mu e^{-\phi_1-\phi_2})g_c E_{-\alpha} g_c^{-1}$$

$$+ (\mu e^{-\phi_1+\phi_2})E_\alpha + \left(\frac{2\pi}{k}\right) \left(\frac{\delta B}{\delta \phi_2}\right) g_c h g_c^{-1} + \left(\frac{2\pi}{k}\right) \left(\frac{\delta B}{\delta \phi_1}\right) h = 0.$$  (5.2)

We try a Gauss decomposition to $g_c$:

$$g_c = e^{\lambda_1 E_\alpha} e^{\lambda_2 h} e^{\lambda_3 E_{-\alpha}},$$  (5.3)

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are constants. Then, we find that

$$g_c E_{-\alpha} g_c^{-1} = e^{-2\lambda_2}(E_{-\alpha} + \lambda_1 h - \lambda_1^2 E_\alpha),$$  (5.4)

$$g_c h g_c^{-1} = h - 2\lambda_1 E_\alpha + 2\lambda_3 e^{-2\lambda_2}(E_{-\alpha} + \lambda_1 h - \lambda_1^2 E_\alpha)$$

and

$$g_c E_{-\alpha} g_c^{-1} = e^{2\lambda_2} E_\alpha - \lambda_3 (h - 2\lambda_1 E_\alpha - \lambda_3^2 e^{-2\lambda_2}(E_{-\alpha} + \lambda_1 h - \lambda_1^2 E_\alpha)).$$  (5.5)

Now, as (5.2) has terms proportional to $E_\alpha$, $E_{-\alpha}$ and $h$, we have three independent equations. By (2.42), (2.45) and (2.46), we see that each one of these equations has terms that are proportional to $e^{\phi_1-\phi_2}$, $e^{-\phi_1+\phi_2}$ and $e^{-\phi_1-\phi_2}$. Thus, we have nine equations for the three variables $\lambda_1$, $\lambda_2$ and $\lambda_3$. They can be solved and the result is

$$\lambda_1 = 2\lambda$$

and

$$\lambda_2 = \lambda_3 = 0,$$  (5.7)

where $\lambda$ is the constant introduced in the equations (2.45) and (2.46).

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