ANOSOV ACTIONS: CLASSIFICATION AND THE ZIMMER PROGRAM

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In memory of Robert Zimmer who started these investigations

Abstract. Consider a volume preserving Anosov $C^\infty$ action $\alpha$ on a compact manifold $X$ by semisimple Lie groups with all simple factors of real rank at least 2. More precisely we assume that some Cartan subgroup $A$ of $G$ (or equivalently $G$) contains a dense set of elements which act normally hyperbolically on $M$ with respect to the orbit foliation of $A$. We show that $\alpha$ is $C^\infty$-conjugate to an action by left translations of a bi-homogeneous space $M\backslash H/\Lambda$, where $M$ is a compact subgroup of a Lie group $H$ and $\Lambda$ is a uniform lattice in $H$.

Crucially to our arguments, we introduce the notion of leafwise homogeneous topological Anosov $\mathbb{R}^k$ actions for $k \geq 2$ and provide their $C^0$ classification, again by left translations actions of a homogeneous space. We then use accessibility properties, the invariance principle of Avila and Viana, cohomology properties of partially hyperbolic systems by Wilkinson and lifting to a suitable fibration to obtain the classification of Anosov $G$ actions from the classification of topological Anosov $\mathbb{R}^k$ actions.

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1. Introduction

This paper is a contribution to the Zimmer Program of studying actions of higher rank semisimple Lie groups $G$ and their lattices $\Gamma$ on compact manifolds. More precisely one assumes that $G$ has real rank at least 2, and that $\Gamma$ is an irreducible lattice in $G$. This program was inspired by Margulis’ superrigidity theorem which classifies finite dimensional representations of such $\Gamma$ [39]. A classification of such is impossible in general as one can construct such actions starting with any flow via the induction procedure. However, classification or at least a detailed structural understanding might be possible under suitable geometric or dynamical assumptions on the action or underlying manifold. Zimmer formulated this program in his ICM address in 1986, and in other papers from the early 1980s, e.g. [58, 59, 60]. Margulis further cemented this in his list of problems for the new century [38, Problem 11].

Much progress has been made in recent years [18, 17]. On the one hand, Brown, Fisher and Hurtado made major progress on one of the main conjectures of the Zimmer program, the Zimmer Conjecture, that lattices in higher rank semisimple Lie groups cannot act on compact manifolds of dimension at most $d(G)$ where $d(G)$ can be calculated explicitly in terms of $G$ and the structure of its roots [5, 6, 7]. For $G = SL(n, \mathbb{R})$, this dimension is simply the optimal $n - 1$. For other groups though their conclusions are far from the expected and much work is left to be done.
key aspect of the proof is that such actions preserve a Riemannian metric, and hence are rather tame from point of view of dynamics.

On the other end of the dynamical spectrum lie hyperbolic actions, especially actions with an Anosov element. By this one usually means that some element \( g \in G \) acts normally hyperbolically w.r.t. the orbit foliation of an associated subgroup of \( G \) (e.g., the centralizer of an \( \mathbb{R} \)-split Cartan subgroup in a semisimple Lie group). That is, \( g \) uniformly expands and contracts complementary subbundles transverse to the orbit foliation of the associated subgroup. For lattices \( \Gamma \) this is simply that some element \( \gamma \in \Gamma \) acts via an Anosov diffeomorphism. Examples of such actions are known to exist on certain nilmanifolds \( N/\Lambda \) and finite quotients of such where \( \Lambda \) is a lattice in a (simply connected) nilpotent group \( N \). Conversely, if the underlying manifold is a nilmanifold, Brown, Rodriguez Hertz and Wang prove the beautiful global rigidity result that all actions of higher rank cocompact lattices are \( C^\infty \)-conjugate to one by automorphisms [8]. For non-uniform \( \Gamma \), they need to assume additional conditions such as being able to lift the action to the universal cover of \( N/\Lambda \). In addition, a well-known 50 year old question of Anosov and Smale asks whether Anosov diffeomorphisms only exist on nilmanifolds and their finite quotients [47]. A positive answer to this question would immediately imply the classification of Anosov actions at least for irreducible higher rank uniform lattices. However, at this point this question remains wide open.

Understanding the structure of Anosov actions on arbitrary compact manifolds for connected higher rank semisimple Lie groups \( G \) is similarly intriguing, and is the main goal of this paper. Anosov elements of \( G \) actions are actually never Anosov diffeomorphisms. Indeed, they will commute with their centralizers, typically non-discrete, and thus cannot act hyperbolically in the orbit directions of their centralizer. Really, Anosov \( G \) actions are ones which give the maximal amount of hyperbolicity possible for a \( G \)-action.

The simplest examples arise from Anosov actions of uniform lattices \( \Gamma \subset G \) by automorphisms of tori and nilmanifolds by the suspension construction (see Example 4.1). Other examples include the action by left translations of \( SO(p, q) \) on \( SO(p, q + 1)/\Lambda \) for suitable \( p, q \). These examples were introduced in [21], and we give a detailed description in Example 4.9.

As we just discussed, an Anosov action of a uniform lattice \( \Gamma \subset G \) on a nilmanifold \( N/\Lambda \) by automorphisms gives rise to an Anosov action of \( G \) via the suspension construction. Thus Anosov lattice actions by automorphisms occur naturally within \( G \) actions. (We warn though that this does not resolve the classification of Anosov actions by lattices since it is not clear that a potentially non-algebraic Anosov action induces an Anosov \( G \)-action.)

In this paper, we investigate the case of volume preserving Anosov actions of a higher rank semisimple Lie group \( G \), indeed we classify them, up to smooth equivalence, assuming the existence of a Cartan subgroup of \( G \) with many Anosov elements, in Theorem 2.2. While there are now many examples, they are all left translations of a bi-homogeneous space \( M\setminus H/\Lambda \), where \( M \) is a compact subgroup of a Lie group \( H \), \( \Lambda \) is a uniform lattice in \( H \) and where \( G \) embeds into \( H \). We will call such actions \textit{bi-homogeneous}. Note that \( G \) will normalize \( M \). Furthermore, these actions preserve natural bi-invariant affine structures on these spaces, themselves of algebraic nature (invariant under left translations by \( H \)). As for terminology however, bi-homogeneous is more precise than affine, and we will use it henceforth.

Prior works on hyperbolic actions in the Zimmer program have been deeply interwoven with understanding hyperbolic actions of higher rank abelian groups. Indeed, the Cartan subgroup of any higher rank semisimple Lie group \( G \) (without compact factors) is some \( \mathbb{R}^k \), \( k \geq 2 \). If its elements are sufficiently hyperbolic, one can then hope to use special knowledge about them, in particular classification of sufficiently hyperbolic \( \mathbb{R}^k \)-actions.
This is precisely what we do. Indeed, in a second, independent part, we classify Anosov actions of \( \mathbb{R}^k \) which intertwine certain homogeneous structures on certain dynamical foliations, and show that such actions are bi-homogeneous. Crucial to our approach is a deep understanding of actions of higher rank Abelian groups with a dense set of Anosov elements. While a classification of such is outstanding in general, and likely extremely difficult, we manage to do this here in Theorem 2.6 assuming existence of certain measurable solutions to coboundary equations. In our application to actions of semisimple Lie groups, we invoke Zimmer’s superrigidity theorem for cocycles to get these measurable solutions. Theorem 2.6 actually provides a framework for classification of higher rank actions of abelian groups which hopefully will prove useful in other problems.

At the heart of our approach lies a classification of certain Anosov actions of higher rank abelian groups, and building additional invariant structures which lead to classification. Connecting actions of semisimple groups and their lattices with those of higher rank abelian subgroups goes all the way back to Hurder’s proof of local rigidity of the action of \( SL(n, \mathbb{Z}) \) on the \( n \)-torus \( \mathbb{T}^n \). It was used again by Katok, Lewis and Zimmer in various works [32, 33, 34], then by Katok and Spatzier in their work on local rigidity [31]. As we will discuss below, these ideas were further developed by Goetze and Spatzier in [21].

Our approach to proving Theorem 2.6 refines the techniques of the recent work by Spatzier and Vinhage in [48] which gives a classification/structure theorem of the so-called totally Cartan actions, i.e. Anosov actions with the special property that maximal nontrivial intersections of stable manifolds of distinct elements are one-dimensional and the set of Anosov elements in \( \mathbb{R}^k \) is dense. In that case, one can solve the relevant cohomology problem directly, by using the one-dimensionality of these intersections strongly (cf. [30, 48]).

Let us now describe the work in more detail. Ultimately it is based on Zimmer’s deep insight that a classification of actions of higher rank semisimple Lie groups and their lattices may be possible, at least if they preserve geometric structures or have strong dynamical properties. This overarching vision was certainly based on Zimmer’s superrigidity theorem for cocycles [61]. As already mentioned, we use it very fruitfully in our work here. Zimmer’s result was measurable. He himself already realized in the early 1990s that versions with higher regularity could prove important, and formulated and proved a topological superrigidity theorem to that effect, in unpublished notes [61]. Later, Feres and Labourie pursued similar ideas in [16], and used them to prove various rigidity statements.

In Zimmer’s approach to topological superrigidity, he assumed existence of a Hölder section of a suitable bundle (with a bundle action by \( G \)) invariant under a parabolic subgroup of \( G \). This fits well with Anosov dynamics as stable bundles will furnish such objects. Goetze and Spatzier developed these ideas in [19] and used them to classify Cartan actions of semisimple Lie groups of real rank at least 3 [21]. Under various technical assumptions, they used this to prove existence of Hölder metrics along suitable foliations conformally invariant under some Cartan subgroup of \( G \). Then they get homogeneous structures along these foliations, which allowed them to prove smoothness of foliations and metrics. To be clear, this approach required that the acting group has real rank at least 3 and the superrigidity representation from Zimmer’s cocycle rigidity theorem is multiplicity-free, rather strong conditions indeed.

We overcome all these restrictions and more in our current work. While we use a radically different approach, we incorporate some of the prior ideas. In particular, finding homogeneous structures along suitable foliations is key, for us and for a variety of other rigidity problems, such as proving measure rigidity and local rigidity of higher rank abelian actions [31].

In the setting of actions of higher rank semisimple Lie groups, we get these leafwise homogeneous structures from Zimmer’s cocycle superrigidity theorem. A priori, they are only measurable,
and our first goal is to show they are Hölder. To this end, we use the invariance principle from partially hyperbolic dynamics that guarantees Hölderness of invariant structures assuming suitable accessibility properties [36, 2, 1, 28]. Under additional assumptions this had been used by Kalinin and Sadovskaya as well as Damjanović and Xu in [28, 13] to prove rigidity results for higher rank \( \mathbb{Z}^k \)-actions assuming accessibility. In this paper, we actually prove that the relevant structures are accessible in the presence of the action of a semisimple group.

For the final stage, we apply our second main result that \( \mathbb{R}^k \) Anosov actions which preserve Hölder leafwise homogenous structures are smoothly conjugate to bi-homogeneous actions. We still face the problem of combining different conjugacies for different Cartan subgroups to get a conjugacy for all of the \( G \)-action. To resolve it we use work by Zeghib on centralizers of homogeneous flows [56]. Then the conjugacy will extend to the centralizer of any Weyl chamber wall which is good enough to control all of \( G \).

In the end, all depends on our classification of Anosov \( \mathbb{R}^k \)-actions with leafwise homogeneous structures. While one might hope for a model by global homogeneous structures, natural examples show that one can only get bi-homogeneous models. We introduce a new construction to resolve this problem, by building a suitable compact group extension of the \( \mathbb{R}^k \) action, naturally built from the leafwise invariant conformal structures using frames. To our knowledge, this is the first time global rigidity with bi-homogeneous models was achieved in either the Zimmer program or the classification of actions of higher rank abelian groups. Even in rank one, such results are extremely rare and require significant additional structure. The few examples in rank one include entropy or exponent rigidity results for geodesic or contact flows (e.g. [3] or [10]). Our new methods work without this additional geometric data to produce a compact extension on which the action is actually homogeneous.

Our second main theorem, Theorem 2.6, then classifies Anosov actions of \( M \times \mathbb{R}^k \) where \( M \) is a compact group which preserves the leafwise homogeneous structures. Such turn out to be homogeneous. This is inspired by Spatzier and Vinhage’s classification of Cartan actions [48] though is considerably more complicated. The main idea is to take the isometry groups of the leafwise conformal structures to build a transitive action of a free product of the Lie groups. Then we show that this free product action actually factors through an actual Lie group, yielding our desired global homogeneous structure.

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2. Results

2.1. Rigidity for semisimple group actions. Let \( G \) be a real semisimple Lie group, \( A \subseteq G \) a split Cartan subgroup, and \( C = C_G(A) \) be the centralizer of \( A \) in \( G \). The centralizer always decomposes as a direct product of a compact group with \( A \), \( C = M \cdot A \). Let \( G \curvearrowright X \) be a \( C^\infty \) action of \( G \) on a smooth manifold \( X \).
If $F$ is a foliation invariant under action of an element $a$, we say that $a$ is \textit{normally hyperbolic} with respect to $F$, if there exists a continuous $a$-invariant splitting $TX = E_a^s \oplus T F \oplus E_a^u$, where $E_a^s$ (resp. $E_a^u$) are uniformly contracted (resp. uniformly expanded) by the action of $a$.

An element $a \in A$ will be called \textit{Anosov} if it acts normally hyperbolically with respect to the homogeneous foliation given by $C_G(A)$. Actions $G \curvearrowright X$ containing at least one Anosov element are called Anosov. In what follows we will require more Anosov elements:

\textbf{Definition 2.1.} If $G$ is a real semisimple Lie group, and $A \subset G$ a split Cartan subgroup, a $C^\infty$ action $G \curvearrowright X$ is called \textit{totally Anosov} (with respect to $A$) if for a dense set of $a \in A$, $a$ is normally hyperbolic with respect to the homogeneous foliation given by $C_G(A)$.

There are certain $G$-actions build from certain algebraic data, which we call \textit{algebraic}. Generally algebraic actions may be defined via group automorphisms and group multiplication. Actions defined via group multiplications are \textit{homogeneous actions}. Here we define \textit{bi-homogeneous} actions as follows: Let $H$ be a Lie group, and $q : G \rightarrow H$ be an embedding of $G$ into $H$. Suppose that $K \subset H$ is a compact subgroup commuting with $q(G)$ and $\Lambda \subset H$ is a cocompact lattice. Then the bi-homogeneous $G$-action $(q, H, K, \Lambda)$ is an action on $X = K \backslash H/\Gamma$ defined by:

$$g \cdot (Kh\Gamma) := K(q(g)h)\Gamma$$

The fact that $K$ commutes with $q(G)$ ensures the action is well-defined, and the fact that $K$ is compact will ensure that any right-invariant metric on $H$ which is bi-invariant under $K$ is well-defined on the quotient, and Haar measure on $H$ will project to a well-defined measure on $X$.

We have the following classification theorem.

\textbf{Theorem 2.2.} Suppose that every simple factor of a real semisimple group $G$ has real rank at least $2$, and let $G \curvearrowright X$ be a $C^\infty$ totally Anosov action with respect to some split Cartan subgroup $A \subset G$ such that the restriction of the action to $A$ preserves an invariant volume. Then the action is smoothly conjugate to a bi-homogeneous $G$-action.

The totally Anosov condition appearing in Theorem 2.2 relies on a distinguished abelian subgroup $A$ in which to find hyperbolic elements. We also get a formulation which is independent of such a subgroup. To do so, we make two important definitions:

The first is that of a hyperbolic element of an action $G \curvearrowright M$. Recall that if $F : V \rightarrow V$ is a linear transformation, $V$ splits as a sum of \textit{generalized eigenspaces}. Each such space corresponds to the sum of the blocks in the Jordan normal form of $F$ for a fixed eigenvalue. If $G$ is a Lie group with Lie algebra $g$, and $g \in G$, consider the splitting $g = g^+_g \oplus g^-_g \oplus g^0_g$, where $g^+_g$ denotes the generalized eigenspaces of $\text{Ad}(g)$ whose eigenvalues have modulus greater than $1$, $g^-_g$ is the sum of the generalized eigenspaces whose eigenvalues modulus less than $1$, and $g^0_g$ is the generalized eigenspace for eigenvalues of modulus $1$. Note that $g^\pm_g$ and $g^0_g$ are subalgebras and have corresponding connected Lie subgroups.

\textbf{Definition 2.3.} Let $G \curvearrowright M$ be a locally free $C^r$ group action, $r \geq 1$. We say that $g \in G$ is \textit{hyperbolic} for $G \curvearrowright M$ if there is a splitting $TM = E^g_+ \oplus E^0_g \oplus E^u_g$, where $E^0_g = g^0_g$ and $E^+_g, E^-_g$ are subbundles of $TM$ which contract uniformly under forward and backwards iterates of $g$, respectively.

Notice that $g$ always exponentially contracts $g^-_g$ and $g^+_g$ under forward and backward iterations, respectively. That $g$ is hyperbolic asks that these bundles can be extended to bundles $TM$ with the same property. Let $\mathcal{H}$ be the set of hyperbolic elements of $G \curvearrowright M$. Note that $\mathcal{H}$ is invariant under conjugation in $G$. 

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The other definition required is the Jordan-Chevalley projection. If $G$ is a semisimple Lie group, and $g \in G$, there exists a decomposition of $g$ called the Jordan-Chevalley decomposition as $g = k a n$, where $k$ is ad-compact, $a$ is $\mathbb{R}$-semisimple, $n$ is ad-unipotent, and $k$, $a$ and $n$ pairwise commute. The Jordan projection of $g$ is defined by $J(g) = a \pmod{W}$, where $W$ is taken to mean modulo the action of the Weyl group. The Jordan-Chevalley projection takes values in a Weyl chamber of $G$, which can be chosen with respect to any $\mathbb{R}$-split Cartan subgroup. For a thorough treatment of this topic, see [23, Section 4.2].

**Corollary 2.4.** Suppose that every simple factor of a real semisimple group $G$ has real rank at least 2, and let $G \acts X$ be a $C^\infty$ action. Let $\mathcal{H}$ be the set of hyperbolic elements for $G \acts M$. Assume either

- that $J(\mathcal{H})$ has dense image, or
- that $\mathcal{H}$ intersects the set of $\mathbb{R}$-semisimple elements in a dense set.

Then the action is smoothly conjugate to a bi-homogeneous $G$-action.

**Remark 2.5.** In fact, each of the conditions of Corollary 2.4 are equivalent to the totally Cartan assumption of Theorem 2.2. See Section 7.5.

### 2.2. Rigidity for Anosov $C = \mathbb{R}^k \times M$ actions.

Let $M$ be a compact Lie group and consider a locally free, volume preserving $C^\infty$ action of $C = A \times M$ on a smooth compact manifold $X$, where $A = \mathbb{R}^k$. If $a \in A$ is normally hyperbolic with respect to the orbit foliation of the action we say that $a$ is Anosov. An action with a dense set of Anosov elements in $A$ is called totally Anosov.

Given a volume preserving $C^2$-action as above, the Oseledec theorem for actions [4, Theorem 2.4] implies the existence of finitely many linear functionals $\chi: \mathbb{R}^k \to \mathbb{R}$, called Lyapunov functionals, and an action invariant measurable splitting $TX = \oplus E^\chi$ of the tangent bundle on a full volume set, such that Lyapunov exponent of $a$ in the direction of $v \in E^\chi(a)$ is $\chi(a)$. The sum of Oseledec distributions $E^\chi$ corresponding to positively proportional Lyapunov functionals is called the coarse Lyapunov distribution. Denote by $\Delta$ the set of finitely many coarse Lyapunov functionals for $\rho$ with respect to the volume.

We call the following the fundamental assumptions for a $C^2$ totally Anosov $\mathbb{R}^k \times M$ action.

**FA-1** For every $\lambda \in \Delta$, ker $\lambda \times M$ has a dense orbit.

**FA-2** The Oseledec splitting is Hölder continuous, and for each Oseledec space $E^\chi$, there exists a Hölder norm $||| \cdot |||_\chi$ on the bundle $E^\chi$ such that for every $a \in \mathbb{R}^k \times M$ and $v \in E^\chi$, $||| a_* v |||_\chi = e^{\chi(a)} ||| v |||_\chi$.

**Theorem 2.6.** If $\mathbb{R}^k \times M \acts X$ is a volume preserving totally Anosov $C^\infty$-action on a $C^\infty$-manifold $X$ satisfying assumptions (FA-1) and (FA-2), then there exists a double-homogeneous space $K \backslash H / \Gamma$ such that $\rho$ is $C^\infty$ conjugate to a bi-homogeneous action on some finite cover of $K \backslash H / \Gamma$.

We remark that Theorem 2.6 has also a low regularity version for $C^2$, see Theorem 13.4. In fact, assumptions (FA-1) and (FA-2) imply certain structure along the leaves of coarse Lyapunov foliations which lead to existence of topological conjugacy even for only Hölder actions with such leafwise structures, see Section 10.

Next, we state a consequence of the theorem above for the actions with certain strong accessibility property. Such strong accessibility holds for left translation actions by Cartan subgroups of semisimple Lie groups. The next theorem is in fact the main ingredient in the proof of Theorem 2.2. But first we define the strong accessibility property.
For a totally Anosov action the coarse Lyapunov distributions \( E_\lambda = \bigoplus_{i=1}^{n_\lambda} E^i_\lambda \) are sums of positively proportional Oseledets distributions, and they are integrable to coarse Lyapunov foliations \( W^\lambda \). Any \( a \) which lies in \( \ker \lambda \) for exactly one \( \lambda \in \Delta \) is called generic singular. Let \( \Delta(\hat{\lambda}) \) denote the set of coarse Lyapunov functionals not proportional to \( \lambda \). A generic singular element \( a \in \ker \lambda \) is \( \Delta(\hat{\lambda}) \)-accessible if any two points in \( X \) can be connected by a broken path whose legs lie in leaves of foliations \( W^\nu \) where \( \nu \in \Delta(\hat{\lambda}) \).

If \( \mathbb{R}^k \times M \curvearrowright X \) is an Anosov action, we say that it is strongly accessible if for every \( \lambda \in \Delta \), there exists \( a \in \ker \lambda \) such that \( a \) is \( \Delta(\hat{\lambda}) \)-accessible.

If \( \mathbb{R}^k \times M \curvearrowright X \) is a \( C^r \) Anosov action, we say that it is Oseledets conformal if there is an \( \mathbb{R}^k \)-invariant measurable conformal structure on each Oseledets space. By this we mean that there exists a measurable family of metrics \( ||\cdot||_\chi \) on the corresponding Oseledets spaces \( E^x \) such that \( ||a_\ast v||_\chi = e^{\chi(a)} ||v||_\chi \).

**Theorem 2.7.** Let \( \mathbb{R}^k \times M \curvearrowright X \) be a volume preserving, Oseledets conformal, strongly accessible totally Anosov \( C^\infty \) action. Then there exists a double-homogeneous space \( K \backslash H / \Gamma \) such that the action is \( C^\infty \) conjugate to a bi-homogeneous action on some finite cover of \( K \backslash H / \Gamma \).

3. OUTLINE OF THE ARGUMENTS

We will now describe the arguments in our work in more detail.

In Section 4, we first describe examples that exhibit the various difficulties we encounter in our classification. In particular, they explain the necessity to consider bi-homogeneous actions as models, both in the semisimple and also the higher rank abelian cases. We also explain how Anosov actions by lattices give rise to Anosov actions by semisimple groups.

In Section 5, we introduce the needed background from smooth dynamics, especially on partially hyperbolic systems. Critical for our work will be the notion of accessibility by distributions \( D_i \) of the tangent bundle. Roughly this means that any two points can be reached from each other by broken paths with legs tangent to some \( D_i \). For a partially hyperbolic map the \( D_i \) typically are stable and unstable distributions of some partially hyperbolic diffeomorphism.

We recall a powerful tool, the Invariance Principle, originally introduced by Ledrappier [36]. It was further developed by Avila and Viana, and then also Santamaria [2, 1]. Kalinin and Sadovskaya developed a version [28] which allows for direct application to the partially hyperbolic setting with center-bunching and accessibility conditions. More precisely, the principle shows Hölderness of measurable conformal structures invariant under a partially hyperbolic system with suitable accessibility properties.

At the end of this section, we explain how to apply the Invariance Principle to \( \mathbb{R}^k \) Anosov actions. For this, we introduce Lyapunov functionals and joint Oseledets splitting of the \( \mathbb{R}^k \) action as well as a common refinement, the coarse Lyapunov spaces and foliations they integrate to. These are simply sums of Oseledets space for positively proportional Lyapunov functionals. We later apply the Invariance Principle to the action of an element of \( \mathbb{R}^k \) which belongs to the kernel of a Lyapunov functional, a so called Lyapunov hyperplane.

The Invariance Principle has proved very useful in the rigidity of group actions before, especially as used by Damjanovic and Xu [13]. They overcome one principal difficulty, the regularity of Oseledets spaces within coarse Lyapunov foliations and related structures. The Invariance Principle could be circumvented in the works of Kalinin and Spatzier [30], and Spatzier and Vinhage [48], where the coarse Lyapunov foliations are one-dimensional and metric properties follow much easier. In this current work, the Invariance Principle and accessibility feature prominently again due to
the multidimensionality of the coarse Lyapunov foliations. Naturally, we need to prove the needed accessibility properties using that we have an action of a semisimple Lie group.

Section 6 reviews background material from group theory. Most important are the topological free product constructions for topological groups. Crucial will be various criteria when a topological group is actually a Lie group, most importantly for us one by Gleason and Palais [20]. However, later on in the final proofs, we will also employ the no small subgroups property of Montgomery and Zippin [42] and its application to inverse limits of Lie groups.

Section 7 gives the arguments for Anosov actions of semisimple Lie groups (Theorem 2.2), assuming the result for abelian actions. To state our main result in minimal terms of the $G$-action, we first use a Howe-Moore type argument to get invariance of a volume form by $G$ from that of a suitable one-parameter subgroup of the split Cartan.

Next in Section 7 we use Zimmer’s measurable cocycle rigidity theorem to get measurable conformal structures along the coarse Lyapunov foliations invariant under a suitable Cartan subgroup $A$. To prove that these conformal structures are Hölder, we invoke the Invariance Principle. For this we need to prove that a suitable sub-collection of coarse Lyapunov foliations is still accessible (Theorem 7.6). To prove this, we employ the structure of the acting semisimple group, in particular special facts about how to write elements of the Weyl group by products of unipotent elements.

Now the split Cartan subgroup $A$ preserves a Hölder conformal structure on each coarse Lyapunov foliation, has the right accessibility properties and has a dense set of Anosov elements by assumption. Thus we are ready to use the classification of such Anosov actions, the second main result of this paper (Theorem 2.7). We can conclude that the restriction of the $G$-actions to Cartan subgroups are smoothly conjugate to bi-homogeneous actions. Applying work by Zeghib (see Appendix C) allows us to combine these conjugacies to get one for the whole $G$ action.

In the remainder of the paper, Sections 9 through 12 we classify certain Anosov $\mathbb{R}^k \times M$ actions, and prove Theorems 2.6 and 2.7.

In Section 9, we construct leafwise homogeneous structures which intertwine with the given $\mathbb{R}^k \times M$ action. The idea is simple: the Lyapunov hyperplanes act by isometries on the associated coarse Lyapunov foliations. Each such hyperplane has dense orbits, so by taking limits they acts transitively on coarse Lyapunov leaves to provide the homogeneous structures.

From this we would like to get a simply transitive action of a Lie group on the coarse Lyapunov leaves. There is a complication however. When returning to the initial point of a leaf, we may rotate by isometries. To resolve this problem we construct a compact extension of the given $\mathbb{R}^k \times M$-action to an action of $\mathbb{R}^k \times M \times K$ for some suitable compact group $K$, essentially by passing to a suitable orthonormal frame bundle. For the lifted action we get group actions parameterizing the coarse Lyapunov foliations which intertwine with the $\mathbb{R}^k$-action in Theorem 9.15, which we axiomatize in the subsequent section (see also Proposition 13.1).

Section 10 introduces the notion of leafwise homogeneous topological Anosov actions and their classification in Theorem 10.7. This is a vast generalization of the work by Spatzier and Vinhage [48]. The key idea for the proof is that we have a natural transitive action of an infinite dimensional topological group, a free product of the groups defining the homogeneous structures on the coarse Lyapunov leaves. We show that this transitive action actually factors through a finite dimensional Lie group.

We think of elements fixing a given point $p$ as a cycle and need to show that they are independent of $p$. When the cycles belong to coarse Lyapunov spaces coming from opposite Lyapunov functionals, this is done in Section 10.

The general case is done in Section 11, through a careful study of various special types of cycles, in particular geometric commutators which correspond to taking Lie brackets in a Lie algebra. One
main point to remember here is that we do not have the necessary regularity to take brackets of vector fields as our objects are only Hölder.

One key lemma is Lemma 11.9 which shows that geometric commutators satisfy a cocycle like property with a polynomial correction term where the latter is independent of the base point. This is crucial for proving that the cycles are constant in the base point.

In Section 12 we consider arbitrary paths, and show that modulo the cycles which form well-defined group relations, they can be put in a canonical presentation. In particular, we may associate to an arbitrary path an equivalent one from a finite-dimensional family of presentations. The techniques and results of this section are similar to that of [48], with extra complications due to multidimensionality of the coarse Lyapunov foliations in the proof of regularity of the conjugacy (Theorem 13.4).

4. Examples

Throughout this section, $G$ denotes a semisimple Lie group, and $\Gamma \subset G$ is a lattice.

**Example 4.1** (Suspensions). Many $G$-actions come from a standard procedure called suspension or induction. Let $\Gamma \subset G$ be a (cocompact) lattice, and $\Gamma \curvearrowright X_0$ be a $C^\infty$ action of $\Gamma$ on $X_0$. The corresponding suspension space is the set $X = (G \times X)/\sim$, where $\sim$ denotes the relation in which $(g_1, x_1) \sim (g_2, x_2)$ if and only if there exists $\gamma \in \Gamma$ such that $g_2 = g_1 \gamma^{-1}$ and $x_2 = \gamma \cdot x_1$.

Notice that $G$ acts naturally on $X$ by $g \cdot (g', x) = (gg', x)$, which preserves equivalence classes. Furthermore, $G/\Gamma$ is a factor of $X$ under the projection $\pi(g, x) = g\Gamma$, and the restriction of the $G$-action to $\Gamma$ preserves $\pi^{-1}(e)$. The action of $\Gamma$ on $\pi^{-1}(e)$ is clearly $C^\infty$ conjugated to the action of $\Gamma$ on $X_0$.

**Remark 4.2.** In general, it is not clear how to conclude that a suspension action constructed as in Example 4.1 is totally Anosov. Indeed, the difficulty lies in concluding that if the action of $\gamma$ on $X_0$ is Anosov, then the action of a 1-parameter subgroup passing through $\gamma$ on $X$ is normally hyperbolic with respect to its centralizer in $G$. Even for very regular 1-parameter subgroup, this relationship is complex and nontrivial.

4.1. Homogenous examples. We begin by describing an alternate construction to the suspension when the $\Gamma$-action is algebraic.

**Example 4.3** (Algebraic suspensions). When the $\Gamma$-action used to construct the suspended action in Example 4.1 is algebraic, another equivalent construction shows that the suspended action is totally Anosov. Indeed, suppose that $X_0$ is a nilmanifold $X_0 = N/\Lambda$, and that there is a representation $\rho : G \to \text{Aut}(N)$ without zero weights such that $\rho(\Gamma)$ preserves $\Lambda$. In the case when $N = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, this data corresponds to a homomorphism $\rho : G \to SL(d, \mathbb{R})$ such that $\rho(\Gamma) \subset SL(d, \mathbb{Z})$.

Let $H$ denote the semidirect product $H = G \rtimes_\rho N$, and $\hat{\Gamma}$ denote the group $\Gamma \rtimes_\rho \Lambda$. Then suspension space $X$ constructed in Example 4.1 is diffeomorphic to $H/\hat{\Gamma}$ and the $G$-action is conjugate the homogeneous (left-translation) action. The diffeomorphism can be constructed immediately by writing an element of $H$ as $(g, n)$ where $g \in G$ and $n \in N$. Furthermore, since the representation $\rho$ has no zero weights, the action of $G$ is totally Anosov.

**Remark 4.4.** The construction described in Example 4.3 seems general, but is actually quite restrictive. Indeed, the difficulty lies in finding the representation $\rho$. Such representations seem to be guaranteed to exist from the Margulis superrigidity theorem. However, given a representation $\rho_0 : \Gamma \to \text{Aut}(N)$ which preserves $\Lambda$, the extension of $\rho_0$ to $G$ is usually only guaranteed to exist up to compact noise. This can be resolved by considering the lattice $\Lambda$ not in $G$, but in the product of
$G$ with a compact group $K$. The construction can proceed with these additional structures, but describes the suspension of the action $\Gamma \actson X_0$ as a bi-homogeneous action, rather than a homogeneous action. See Example 4.8.

**Example 4.5** (Embedding $\mathbb{R}$-split orthogonal groups). All actions described previously come from suspending a $\Gamma$-action. Here, we describe another class of actions which do not come from actions of lattices. Such actions were first described in [21]. Consider an embedding of the group $SO(n, n)$ into $SO(n, n + 1)$. Notice that both groups have the same real rank, $n$, and hence that an $\mathbb{R}$-split Cartan of $SO(n, n)$ is automatically an $\mathbb{R}$-split Cartan subgroup of $SO(n, n + 1)$. Furthermore, each group is $\mathbb{R}$-split, so the centralizer of an $\mathbb{R}$-split Cartan subgroup is discrete in both groups. Therefore, the translation action of $SO(n, n)$ on a compact quotient of $SO(n, n + 1)$ will be a totally Cartan action.

**Remark 4.6.** The special feature of the groups appearing in Example 4.5 is that both groups are $\mathbb{R}$-split. This procedure can be adapted to produce bi-homogeneous actions when the smaller group is still $\mathbb{R}$-split, and the centralizer of the smaller group in the larger group is compact. This happens for the group $SO(n, n)$ sitting inside $SO(n, m)$, $m \geq n$. See Example 4.9 for a precise description of this phenomena.

### 4.2 Bi-homogeneous examples

The first example of a bi-homogeneous action is an $\mathbb{R}^k$-action which does not extend to a semisimple group action.

**Example 4.7** (Weyl chamber flows on non-split groups). Fix a cocompact lattice $\Gamma \subset SL(n, \mathbb{C})$. Then let $X$ denote the double quotient space $Diag_U \backslash SL(n, \mathbb{C})/\Gamma$, where

$$Diag_U = \left\{ \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) : \theta_1, \ldots, \theta_n \in \mathbb{R}, \sum \theta_i = 0 \right\} \cong \mathbb{T}^{n-1}$$

is the group of unitary diagonal matrices.

Then the left-translation action of $Diag_{\mathbb{R}} = \{ (e^{t_1}, \ldots, e^{t_n}) : t_1, \ldots, t_n \in \mathbb{R}, \sum t_i = 0 \}$ is a totally Cartan $\mathbb{R}^{n-1}$ action.

We remark on some features of this example: first, since $U$ is compact, the space $X$ has the structure of an orbifold, and if $Diag_\mathbb{U} \cap \Gamma = \{ e \}$, then it is a manifold. Since $Diag_\mathbb{R}$ commutes with $U$, the left-translation action of $Diag_\mathbb{R}$ is well-defined. However, the groups $U_{ij}$ which consist of matrices with 1’s on the diagonal, any complex number in the $(i, j)$th position, and 0’s elsewhere, do not commute with $Diag_\mathbb{U}$. Hence while the coarse Lyapunov foliations have a canonical metric and each leaf has a fixed Euclidean structure, there does not exist a group action of $\mathbb{C}$ parameterizing the leaves.

**Example 4.8** (Suspensions of actions with “compact noise”). Let $G = SL(3, \mathbb{R}) \times SU(3)$ and $\Gamma \subset G$ be the $\mathbb{Z}$-points of some $\mathbb{Q}$-rational embedding of $\rho : SL(3, \mathbb{R}) \times SU(3) \to SL(d, \mathbb{R})$, $\Gamma = \rho^{-1}(SL(d, \mathbb{Z}))$. Then $\Gamma$ is a lattice in $G$ and $\rho$ is a representation of $G$. Define the semidirect product $H = G \rtimes \mathbb{R}^d$ in the usual way:

$$(g_1, v_1) * (g_2, v_2) = (g_1 g_2, \rho(g_2)^{-1} v_1 + v_2)$$

Let $\Lambda = \Gamma \rtimes \mathbb{Z}^d$ be the semidirect product of the corresponding lattices. We consider the space $X = SU(3) \backslash H/\Lambda$, and the action of $SL(3, \mathbb{R})$ on $X$ by left translations. As in Example 4.7, the action is well-defined since $SU(3)$ commutes with $SL(3, \mathbb{R})$. Furthermore, while $G$ splits as a direct product of $SL(3, \mathbb{R})$ and $SU(3)$ and the representation $\rho$ may be restricted to $SL(3, \mathbb{R})$, this restriction does not give the corresponding action of $\Gamma$ on $\mathbb{R}^d$. Therefore, $X$ is not a $SL(3, \mathbb{R}) \times \mathbb{R}^d$ homogeneous space.
If the representation $\rho$ does not have zero weights then this action of $SL(3, \mathbb{R})$ is totally Anosov. Here, while the root spaces are parameterized by group actions, the weight spaces in $\mathbb{R}^d$ will not be parameterized by group actions of the corresponding weight spaces, exactly because the group $SU(3)$ rotates each weight space.

This example can be easily generalized to any $\mathbb{R}$-split Lie group $H$ defined over $\mathbb{Q}$, its compact real form $K$, and a $\mathbb{Q}$-representation $\rho: H \times K \rightarrow SL(d, \mathbb{R})$.

**Example 4.9** (Embedding split groups in non-split groups). Let $H = SO(2, n)^0$, $n \geq 3$. We write $\text{Lie}(H)$ as the set of matrices

$$
\begin{pmatrix}
  t_1 & u & 0 & a & r_1 & r_2 & \ldots & r_{n-2}
  \\
  \dot{u} & t_2 & -a & 0 & s_1 & s_2 & \ldots & s_{n-2}
  \\
  0 & \dot{a} & -t_1 & -\dot{u} & \hat{r}_1 & \hat{r}_2 & \ldots & \hat{r}_{n-2}
  \\
  -\dot{a} & 0 & -u & -t_2 & \hat{s}_1 & \hat{s}_2 & \ldots & \hat{s}_{n-2}
  \\
  -\hat{r}_1 & -\hat{s}_1 & -r_1 & -s_1 & 0 & \theta_{12} & \ldots & \theta_{1(n-2)}
  \\
  -\hat{r}_2 & -\hat{s}_2 & -r_2 & -s_2 & -\theta_{12} & 0 & \ldots & \theta_{2(n-2)}
  \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
  \\
  -\hat{r}_{n-2} & -\hat{s}_{n-2} & -r_{n-2} & -s_{n-2} & -\theta_{1(n-2)} & -\theta_{2(n-2)} & \ldots & 0
\end{pmatrix}
$$

Notice that $\mathfrak{so}(2, 2)$ sits inside $\text{Lie}(H)$ with this presentation canonically as the upper left $4 \times 4$-block, and $\mathfrak{so}(n-2)$ sits inside as the bottom right $(n-2) \times (n-2)$-block. Furthermore, $\mathfrak{so}(2, 2)$ commutes with $\mathfrak{so}(n-2)$. Therefore, if $G = \exp(\mathfrak{so}(2, 2)) \subset H$ and $M = \exp(\mathfrak{so}(n-2)) \subset H$, we may construct an action of $G$ on $X = M\backslash H/\Lambda$, where $\Lambda$ is some fixed cocompact lattice in $H$. One may notice similarities with the previous examples: the action is Anosov and is well-defined because $G$ commutes with $M$. Furthermore, while the roots of $G$ act on $X$, the roots of $H$ do not act on the double quotient space. For instance, the subalgebra $\mathfrak{u}$ spanned by the coordinates $r_1, \ldots, r_{n-2}$ is a coarse Lyapunov foliation, normalized by $M$ and $M$ preserves an invariant metric on it. However, the action of $M$ is not trivial, so while $\exp(\mathfrak{u})$ acts on the homogeneous space $H/\Lambda$, it will not act on $M\backslash H/\Lambda$.

This is also an example of an abelian totally Anosov action by considering the action of the split Cartan subgroup. It can be further generalized to $SO(m, m)$ acting on $SO(m, n)$ quite easily or $SU(m, m)$ acting on $SU(m, n)$, respectively. However, the action of the Cartan subgroup of $SU(m, m)$ is not an abelian totally Anosov action: one must additionally quotient by $\text{Diag}_U \subset SU(m, m)$ on the left.

**Example 4.10** (Combining phenomena). We combine ideas in the last two examples to show one last feature. Let $H$ and $\Lambda$ be as in Example 4.9. $\Lambda$ is often obtained by taking a $\mathbb{Q}$-algebraic representation $\rho: H \times SO(n+2) \rightarrow SL(d, \mathbb{R})$, and letting $\Gamma = \rho^{-1}(SL(d, \mathbb{Z}))$. This construction, called restriction of scalars, requires the group $SO(n+2)$ to be there. One may proceed as in Example 4.8 and construct and example of $H$ on $SO(n+2) \backslash (H \times SO(n+2)) \times \mathbb{R}^d / \Gamma \times \mathbb{Z}^d$. The action of $H$ is Anosov in the sense of semisimple group actions, however, the restriction to the Cartan subalgebra of $H$ is not Anosov. Instead, the action of the centralizer of the split Cartan, $\mathbb{R}^2 \times SO(n-2)$ is Anosov. If one instead considers the quotient by $SO(n-2) \times SO(n+2)$ on the left, then one obtains a totally Anosov action of $\mathbb{R}^2$.

5. **Preliminaries on abelian actions**

5.1. Partially hyperbolic diffeomorphisms and center bunching. A $C^1$ diffeomorphism $f$ on a compact smooth Riemannian manifold $X$ is partially hyperbolic if there is a nontrivial $Df$–invariant
splitting $E_f^s \oplus E_f^c \oplus E_f^u$ of the tangent bundle $TX$ and a Riemannian metric on $X$ such that there exists continuous positive functions $\nu < 1, \nu' < 1, \gamma, \gamma'$ for which the following inequalities hold for any $x \in M$, $v^s \in E_f^s$, $v^u \in E_f^u$, $v^c \in E_f^c$:

$$\|Df^k(v^s)\| < \nu(x) < \gamma(x) < \|Df^k(v^u)\| < \gamma'(x)^{-1} < \nu'(x)^{-1} < \|Df^k(v^c)\|.$$  

The subbundles $E_f^s, E_f^c, E_f^u$ are called stable, unstable and center, respectively.

The bundles $E_f^s$ and $E_f^u$ are integrable to Hölder foliations $W_f^s$ and $W_f^u$. If $f$ is smooth than leaves of $W_f^s$ and $W_f^u$ are smooth.

Homogeneous actions examples in section 4 contain many partially hyperbolic elements which all have common center distribution and on this center distribution the action of these elements is isometric. In general we may not have such nice center behavior. For general partially hyperbolic diffeomorphisms we need to understand the extent to which non-conformality of $E^c$ is dominated by contraction and expansion in the directions $E^s_f$ and $E^u_f$. Such domination is called center bunching. Diffeomorphism $f$ is center bunched if the functions $\nu, \nu', \gamma, \gamma'$ can be chosen to satisfy

$$\nu < \gamma \gamma' \text{ and } \nu' < \gamma \gamma'.$$

If $f$ is $C^{1+\beta}$ then we use stronger version of center bunching condition which can imply stronger results such as ergodicity and Hölder regularity of distributions $E_f^s \oplus E_f^c \oplus E_f^u$ and $E_f^s \oplus E_f^c$ and $E_f^u$ (see [28] or [9] for more details). A $C^{1+\beta}$ diffeomorphism $f$ is strong center bunched if there is $\theta \in (0, \delta)$ and functions $\kappa, \kappa'$ such that $\max\{\nu^\theta, \nu'^\theta\} < \gamma \gamma', \nu \gamma^{-1} < \kappa^\theta$, $\nu' \gamma^{-1} < \kappa'^\theta$, and for all $x \in M$, for all $v^s \in E_f^s(x)$ and $v^u \in E_f^u(x)$:

$$\kappa(x) < \|Df(v)\|, \quad \|Df(v)\| < \kappa'(x)^{-1}.$$

5.2. Accessibility. A finite collection of foliations $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_r\}$ with $C^1$ leaves defines an accessibility relation on $X$. We say $x, y \in X$ are in the same accessibility class if they can be joined by a piecewise $C^1$ path such that every leg of the path is contained in a single leaf of one of the foliations in the collection $\mathcal{F}$. If there is a single accessibility class in $X$ we say that collection $\mathcal{F}$ is accessible. Given a partially hyperbolic diffeomorphism $f$ we say that $f$ is accessible if $\{W_f^s, W_f^u\}$ is accessible.

5.3. Invariant structures for $\mathbb{R}^k$ actions. Let $\rho : \mathbb{R}^k \to \text{Diff}(X)$ be an action with an ergodic invariant measure $\mu$. (In this paper $\mu$ is going to be the volume.) The Oseledets theorem for cocycles over abelian actions ([4, Theorem 2.4]) applied to the derivative cocycle of $\rho$, implies the existence of finitely many linear functionals $\chi : \mathbb{R}^k \to \mathbb{R}$ (called Lyapunov functionals), and a $\rho$-invariant measurable splitting $\oplus E^\chi$ of $TX$ (called the Oseledets decomposition), on a full $\mu$-measure set, such that for $a \in \mathbb{R}^k$ and $v \in E^\chi(x)$:

$$\lim_{a \to \infty} \frac{\log \|D_x \rho(a)v\| - \chi(a)}{\|a\|} = 0.$$  

The hyperplanes $\ker \chi \subset \mathbb{R}^k$ are called Weyl chamber walls, and the connected components of $\mathbb{R}^k - \cup \ker \chi$ are the Weyl chambers for $\mathcal{A}$ (with respect to $\mu$).

Two nonzero Lyapunov functionals $\chi_i$ and $\chi_j$ are coarsely equivalent if they are positively proportional: there exists $c > 0$ such that $\chi_i = c \cdot \chi_j$. This is an equivalence relation on the set of Lyapunov functionals, and a coarse Lyapunov functional is an equivalence class under this relation.

Here we consider $\mathbb{R}^k \times M$ actions which are totally Anosov; that is: there is a dense set of $a \in \mathbb{R}^k$ which act normally hyperbolically with respect to the $\mathbb{R}^k \times M$-orbit foliation. In this setting it is known that the coarse Lyapunov distributions $E^\lambda$ are integrable to Hölder foliations $W^\lambda$ which are called coarse Lyapunov foliations. The reason is that leaves of $W^\lambda$ are obtained as intersections of
leaves of stable foliations for all elements $a$ of the action for which $\lambda(a) < 0$. For more details see [48, Sections 4.1.4.2].

The following is a property of elements which lie in the Weyl chamber walls, which is known to hold for totally Anosov actions. Recall that by $\Delta$ we denote the set of finitely many coarse Lyapunov functionals and that any $a$ which lies in $\ker \lambda$ for exactly one $\lambda \in \Delta$ is called generic singular element.

**Lemma 5.1.** [Lemma 4.1 in [13]] Let $\rho$ be a totally Anosov action. For any $\lambda \in \Delta$, if $a \in \ker \lambda$ then $a$ has sub-exponential growth along $W^\lambda$. If $a \notin \ker \lambda$ then $a$ either contracts or expands $W^\lambda$ exponentially fast.

In particular, in our setting for any $\lambda \in \Delta$ and any generic singular element $a$ in $\ker \lambda$, $a$ can be viewed as a partially hyperbolic diffeomorphism acting on $X$, with center distribution $TO \bigoplus \oplus_{\lambda(a) = 0} E_{\lambda}$. 

**Lemma 5.2.** For any $\lambda \in \Delta$, any generic singular $a \in \ker \lambda$ is center bunched partially hyperbolic diffeomorphism with respect to the center distribution $TO \bigoplus \oplus_{\lambda(a) = 0} E_{\lambda}$.

5.4. **Some regularity theorems.** The following commonly used result will play an important role in proving the regularity of the conjugacies in Section 13.1. We will rely on both the $C^\infty$ and $C^{1,\theta}$ versions in the corresponding settings.

**Theorem 5.3** (Journé, [25]). Let $F : X \to Y$ be a continuous function between $C^\infty$ manifolds, $r \in \mathbb{Z}_+ \cup \{\infty\}$, $\theta > 0$, and assume $X$ has complementary transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ with uniformly $C^{r,\theta}$-leaves. Then if the restriction of $F$ to the leaves of the foliations $\mathcal{F}_i$ is uniformly $C^{r,\theta}$, then $F$ is $C^{r,\theta}$.

A version of the following result was first proved by Calabi and Hartman. Concerns about its proof were resolved when a new proof was provided by M. Taylor, whose formulation we use.

**Theorem 5.4** (Taylor, [50] Theorem 2.1). Let $\mathcal{O}, \Omega$ be open subsets of $\mathbb{R}^n$ and carry metric tensors $g = (g_{jk})$ and $h = (h_{jk})$, respectively. Assume $r \in \mathbb{Z}_+ \cup \{0\}$, $\theta \in [0, 1)$, $r + \theta > 0$, and $g_{jk} \in C^{r,\theta}(\mathcal{O})$, $h_{jk} \in C^{r,\theta}(\Omega)$. Let $\varphi : \mathcal{O} \to \Omega$. The following are equivalent:

1. $\varphi$ is a distance-preserving homeomorphism,
2. $\varphi$ is bi-Lipschitz and $\varphi^* h(x) = g(x)$, for almost every $x \in \mathcal{O}$,
3. $\varphi$ is a $C^1$ diffeomorphism and $\varphi^* h = g$,
4. $\varphi$ is a diffeomorphism of class $C^{r+1,\theta}$ and $\varphi^* h = g$.

**Theorem 5.5** (Chernoff-Marsden, [11]). Let $r \geq 1$, $X$ be a $C^r$ manifold, $G$ be a Lie group, and $\alpha : G \curvearrowright X$ be a group action such that for every $g \in G$, $\alpha(g)$ is a $C^r$ diffeomorphism. Then $\alpha : G \times X \to X$ is a $C^r$ action.

6. **More preliminaries**

6.1. **Some Lie group theory facts.**

**Definition 6.1.** The Weyl group with respect to $A$ is defined by $N_G(A)/C_G(A)$, where The Weyl group with respect to $A$ is defined by $N_G(A)/C_G(A)$ where $N_G(A)$ is the normalizer of $A$. Denote by $w_\beta$ the reflection about the hyperplane perpendicular to the root $\beta$. 
6.2. Free products of topological groups. Let $U_1, \ldots, U_r$ be topological groups. The topological free product of the $U_i$, denoted $\mathcal{P} = U_1 \ast \cdots \ast U_r$, is a topological group whose underlying group structure is exactly the usual free product of groups. That is, elements of $\mathcal{P}$ are given by

$$u_1 \ast \cdots \ast u_N$$

where each $u_k \in U_{i_k}$ for some associated sequence of symbols $i_k \in \{1, \ldots, r\}$. We call the sequence $(i_1, \ldots, i_k)$ the combinatorial pattern of the word. Each term $u_k$ is also called a leg and each word is also called a path. This is because in the case of a free product of connected Lie groups, the word can be represented by a path beginning at $e$, moving to $u_N$, then to $u_{N-1} \ast u_N$, and so on through the truncations of the word. The multiplication is given by concatenation of words, and the only group relations are given by

(6.1) $u \ast v = uv$, if $u, v$ belong to the same $U_i$ and

(6.2) $e^{(i)} = e \in \mathcal{P}$.

Notice that the relations (6.1) and (6.2) give rise to canonical embeddings of each $U_i$ into $\mathcal{P}$. We therefore identify each $U_i$ with its embedded copy in $\mathcal{P}$. The usual free product is characterized by a universal property: given a group $H$ and any collection of homomorphisms $\varphi_i : U_i \to H$, there exists a unique homomorphism $\Phi : \mathcal{P} \to H$ such that $\Phi_{|U_i} = \varphi_i$. The group topology on $\mathcal{P}$ may be similarly defined by a universal property, as first proved by Graev [22]:

**Proposition 6.2.** There exists a unique topology $\tau$ on $\mathcal{P}$ (called the free product topology) such that

1. each inclusion $U_i \hookrightarrow \mathcal{P}$ is a homeomorphism onto its image, and
2. if $\varphi_i : U_i \to H$ are continuous group homomorphisms to a topological group $H$, then the unique extension $\Phi$ is continuous with respect to $\tau$.

In the case when each $U_i$ is a Lie group (or more generally, a CW-complex), Ordman found a more constructive description of the topology [44]. Indeed, the free product of Lie groups is covered by a disjoint union of combinatorial cells.

**Definition 6.3.** Let $\mathcal{P}$ be the free products of groups $U_\beta$, where the $\beta$ ranges over some indexing set $\Delta$. A combinatorial pattern in $\Delta$ is a finite sequence $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ such that $\beta_i \in \Delta$ for $i = 1, \ldots, n$. For each combinatorial pattern $\bar{\beta}$, there is an associated combinatorial cell $C_{\bar{\beta}} = U_{\beta_1} \times \cdots \times U_{\beta_n}$. If each $U_\beta$ is a topological group, $C_{\bar{\beta}}$ carries the product topology from the topologies on $U_{\beta_i}$. Notice that each $C_{\bar{\beta}}$ has a map $\pi_{\bar{\beta}} : C_{\bar{\beta}} \to \mathcal{P}$ given by $(u_1, \ldots, u_N) \mapsto u_1 \ast \cdots \ast u_N$. Furthermore, if $C = \bigsqcup_{\bar{\beta}} C_{\bar{\beta}}$, $C$ carries a canonical topology in which each $C_{\bar{\beta}}$ is a connected component. Finally, $\pi : C \to \mathcal{P}$ is defined by setting $\pi(x) = \pi_{\bar{\beta}}(x)$ when $x \in C_{\bar{\beta}}$ (note that $\pi$ is onto).

**Lemma 6.4 ([51] Proposition 4.2).** If each $U_i$ is a Lie group, $\tau$ is the quotient topology on $\mathcal{P}$ induced by $\pi$. In particular, $f : \mathcal{P} \to Z$ is a continuous function to a topological space $Z$ if and only if its pullback $f \circ \pi_{\bar{\beta}}$ to $C_{\bar{\beta}}$ is continuous for every combinatorial pattern $\bar{\beta}$.

**Corollary 6.5.** If each $U_i$ is a connected Lie group, $\mathcal{P}$ is path-connected and locally path-connected.
\[ u_1 \ast u_2 \ast \cdots \ast u_N \cdot x = u_1(u_2(\ldots(u_N(x))\ldots)) \]

This can be observed to be an action of \( P \) immediately, and continuity can be checked with either the universal property (considering each action \( U_\alpha \circ X \) is a homomorphism from \( U_\alpha \) to \( \text{Homeo}(X) \)) or directly using the criterion of Lemma 6.4.

Suppose now that all groups \( U_\alpha \) are nilpotent and simply connected. Given a word \( u_1 \ast \cdots \ast u_m \) (which we often call a path as discussed above), we may associate a path in \( X \) defined by:

\[ \gamma \left( \frac{s + k - 1}{m} \right) = u_{i_m-k+1}^t(x_{k-1}), \ s \in [0,1], \ k = 1, \ldots, m, \]

where \( u^t \) is the one parameter subgroup passing through \( u \), \( x_0 \) is a base point and \( x_k = u_{i_m-k+1}(x_{k-1}) \).

This gives more justification for calling each term \( u_k \) a leg. The points \( x_k \) are called the \textit{break points} or \textit{switches} of the path.

We assume that for each \( g \in \mathbb{R}^k \times M \), there is an associated family of automorphisms \( g_* : U_\alpha \rightarrow U_\alpha \) for every \( \alpha \in \Delta \), and that the map \( g \mapsto g_* \) is a homomorphism from \( \mathbb{R}^k \times M \) to \( \prod_{\alpha \in \Delta} \text{Aut}(U_\alpha) \). Suppose that \( u_1 \ast \cdots \ast u_m \in P \) is an element of combinatorial length \( m \) with combinatorial pattern \((\alpha_1, \ldots, \alpha_m)\). Then define \( g_*: P \rightarrow P \) by:

\[ u_1 \ast \cdots \ast u_m \mapsto (g_*u_1) \ast \cdots \ast (g_*u_m). \]

One may check that \( g_* \) is a well-defined automorphism of \( P \) using relations (6.1) and (6.2), and noting that its inverse is \((g^{-1})_*\).

\textbf{Definition 6.6.} Let \( \hat{P} = (\mathbb{R}^k \times M) \ltimes P \), with the semidirect product structure given by

\[ (g_1, \rho_1) \cdot (g_2, \rho_2) = (g_1g_2, (g_2^{-1}\rho_1) \ast \rho_2). \]

The following proposition follows almost immediately from the definitions. A proof of the case when each \( U_\alpha \cong \mathbb{R} \) can be found in [48, Proposition 4.7].

\textbf{Proposition 6.7.} Let \( C \) be a closed, normal subgroup of \( P \), and \( H = P/C \) be the corresponding topological group factor of \( P \). If \( g_*C = C \) for all \( g \in \mathbb{R}^k \times M \), then \( g_* \) descends to a continuous homomorphism \( g_* \) of \( H \). Furthermore, if \( H \) is a Lie group with Lie algebra \( \mathfrak{h} \), and \( \pi_\alpha : U_\alpha \rightarrow H \) denotes the composition of inclusion into \( P \) and projection onto \( H \), then

1. each generating group \( U_\alpha \subset P \), \( d\pi_\alpha(\text{Lie}(U_\alpha)) \) is an invariant subspace of \( dg_* \) for every \( g \in \mathbb{R}^k \times M \),
2. if \( \xi \in \text{Lie}(U_\alpha) \) and \( \eta \in \text{Lie}(U_\beta) \) are eigenvectors of \( dg_* \) with eigenvalues of modulus \( \lambda_1 \) and \( \lambda_2 \), respectively, then \( \text{Lie}(\pi_\alpha X, \pi_\beta Y) \in \text{Lie}(H) \) is an eigenspace of \( dg_* \) with eigenvalue of modulus \( \lambda_1\lambda_2 \), and
3. if \( Z = [Z_1, [Z_2, \ldots, [Z_n, Z_0] \ldots]] \), with \( Z_k = X \) or \( Y \) for every \( k \), then \( Z \) is an eigenvector of \( dg_* \) with eigenvalue of modulus \( \lambda_1^u\lambda_2^v \) for some \( u, v \in \mathbb{Z}_+ \).

\textbf{6.3. Lie Criteria.} In this subsection, we recall two deep results for Lie criteria of topological groups. The first criterion was obtained by Gleason and Palais [20, Corollary 7.4]:

\textbf{Theorem 6.8 (Gleason-Palais).} If \( G \) is a locally path-connected topological group which admits an injective continuous map from a neighborhood of \( e \in G \) into a finite-dimensional topological space, then \( G \) is a Lie group.

Theorem 6.8 has an immediate corollary for actions of the path group \( P \) defined in Section 6.2:
**Corollary 6.9.** If \( \eta : P \curvearrowright X \) is a group action on a topological space \( X \), and there exists \( x_0 \in X \) such that \( C := \text{Stab}_\eta(x_0) \subset \text{Stab}_\eta(x) \) for every \( x \in X \), then \( C \) is normal and the \( \eta \) action descends to \( P/C \). If there is an injective continuous map from \( P/C \) to a finite-dimensional space \( Y \), then \( P/C \) is a Lie group.

**Proof.** We first show that \( C \) is normal. Let \( \sigma \in C \) and \( \rho \in P \). Then \( \sigma \cdot (\rho \cdot x_0) = \rho \cdot x_0 \), since \( \sigma \) stabilizes every point of \( X \). Therefore, \( \rho^{-1} \sigma \cdot x_0 = x_0 \), and \( \rho^{-1} \sigma \in C \), and \( C \) is a closed normal subgroup. By Corollary 6.5, \( P \), and hence all of its factors, are locally path-connected. Therefore, by Theorem 6.8, \( P/C \) is a Lie group if it admits an injective continuous map to a finite-dimensional space. \( \square \)

The second, more well-known, criterion was obtained by Gleason and Yamabe [49, Proposition 6.0.11], and plays a crucial role in the general solution of Hilbert’s fifth problem:

**Theorem 6.10 (Gleason-Yamabe).** Let \( G \) be a locally compact group. Then there exists an open subgroup \( G' \subset G \) such that, for any open neighborhood \( U \) of the identity in \( G' \) there exists a compact normal subgroup \( K \subset U \subset G' \) such that \( G'/K \) is isomorphic to a Lie group. Furthermore, if \( G \) is connected, \( G' = G \).

Recall that a locally compact group \( G \) has the no small subgroups property if for \( G' \) as in Theorem 6.10, there exists a neighborhood \( U \subset G' \) such \( U \) does not contain any compact normal subgroup besides \( \{e\} \). Such a group \( G \) then is automatically a Lie group, by Theorem 6.10.

The following corollary is often used, but a citation is not available. We provide a proof for completeness:

**Corollary 6.11.** If \( G \) is a separable, Hausdorff, locally compact, locally path-connected group, then \( G \) is an inverse limit of Lie groups with compact kernels. That is, we may construct the following commutative diagram describing \( G \):

\[
\begin{array}{ccc}
\cdots & \xrightarrow{q_3} & G_3 \\
& \xleftarrow{q_2} & \xrightarrow{q_1} G_2 \\
 & \xrightarrow{q_0} & G_1 \\
& \xleftarrow{p_1} & G_0 \\
& G & \xrightarrow{p_2} \xrightarrow{p_3} \end{array}
\]

Here, each \( G_i \) is a Lie group, \( p_i : G_i \to G_{i-1} \) and \( q_i : G \to G_i \) are surjective homomorphisms satisfying \( q_n = p_{n+1} \circ q_{n+1}, \ker p_n \) is compact, and \( \bigcap_{n=0}^\infty \ker q_n = \{e\} \subset G \).

**Proof.** Notice that since \( G \) is connected, \( G^0 = G \), and \( G/G^0 \) is compact. Choose a sequence \( U_n \) of neighborhoods of \( e \) such that \( \bigcap_{n=0}^\infty U_n = \{e\} \) (the existence of the sequence follows from separability). By Theorem 6.10, there exists a compact normal subgroup \( K_n \subset U_n \) such that \( G_n := G/K_n \) is a Lie group. Let \( K_n = \bigcap_{i=1}^n \tilde{K}_i \).

We claim that \( G'_n := G/K_n \) is a Lie group. Indeed, since \( G_n \) is a Lie group, there is an open set \( V_n \subset G_n \) such that the only subgroup of \( G_n \) contained in \( V_n \) is \( \{e\} \). Consequently, the only subgroups contained in \( V'_n \subset G \), the preimage of \( V_n \) in \( G_n \), must be contained in \( \tilde{K}_n \). Let \( W'_n = \bigcap_{i=1}^n V'_i \subset G \), and notice that since each \( V'_i \) is saturated by \( \tilde{K}_i \), \( W'_n \) is saturated by \( \tilde{K}_n = \bigcap_{i=1}^n \tilde{K}_i \). Then \( W'_n \) is a \( \tilde{K}_n \)-saturated neighborhood of \( K_n \subset G \). If \( W \) denotes the image of \( W'_n \) in \( G_n \), then \( W \) is open in \( G_n \). This implies that if \( L \) is a subgroup of \( G'_n \) contained in \( W_n \), then its preimage in \( G \) is a subgroup contained in \( W'_n \). By construction, it must be contained in each \( V'_n \) and therefore, must lie inside \( \bigcap_{i=1}^n \tilde{K}_i = K_n \). Therefore, \( L = \{e\} \) and \( G'_n \) is a Lie group, since it is locally compact and has no small subgroups.

By construction, \( G \) is exactly the projective limit of the groups \( G'_n \). \( \square \)
7. Proof of the result for semisimple Lie group actions

From assumptions in Theorem 2.2 there exists a totally Anosov $A$-action of a split Cartan subgroup of $G$ such that the $A$-action preserves the volume. We will show that this action satisfies all the properties in Theorem 2.7. By applying Theorem 2.7 gives us that the $A$ action is $C^\infty$ conjugate to a bi-homogeneous action (up to finite cover).

7.1. The $G$ action in Theorem 2.2 preserves volume.

Lemma 7.1. Let $X$ be a manifold with a $C^1$-action of the subgroup $A \times U$ on $X$, where $A$ is the maximal split Cartan subgroup in $G$ and $U$ is a one parameter unipotent subgroup $U$ in $G$. Suppose $A$ preserves a volume form $\omega$, and that $A$ has a dense set of recurrent orbits. Then $A \times U$ preserves $\omega$.

Proof. Let $u_s$ be the one-parameter group giving $U$, and $a_t$ the one generating $A$. Then we have the commutation relation $u_s e^{at} = a_t u_s -$ possibly after reparametrization of $a_t$. Let $g^s(x) = \log \text{Jac} \cdot u_s$. Then $g : X \times \mathbb{R} \to \mathbb{R}$ is $C^1$, and satisfies the relation $g^{se^{at}} = g^s$.

Now consider the derivative $h := \frac{\partial}{\partial s} g^s$. Then

$$e^t \cdot (h \circ a_t) = h.$$  

Then $h = 0$ along any recurrent orbit. As the latter form a dense set in $X$, $h \equiv 0$ on $X$. Since $h = \frac{\partial}{\partial s} g^s$, the fundamental theorem of calculus implies that $g^s \equiv g^0 = 1$. Hence the $u_s$ preserve $\omega$. □

Proposition 7.2. Let $G$ be a semisimple Lie group of the noncompact type, and $\rho : G \to \text{Diff}(X)$ an action on a manifold $X$. Suppose that a regular one-parameter subgroup $A = \{a_t\}$ of the maximal split Cartan preserves a volume form $\omega$ on $X$ and has a dense set of recurrent orbits. Then $G$ preserves $\omega$.

Proof. It is well-known that $G$ as above is generated by finitely many unipotent root subgroups $U_i$ which form skew product $A \times U_i$. Since $A$ is regular, the skew products are non-trivial. Hence the $A \times U_i$ preserve $\omega$ by Lemma 7.1. Therefore $G$ preserves $\omega$. □

7.2. Measurable conformal structure. As in Theorem 2.2 we let $G$ be a real semisimple group such that every simple factor of $G$ has real rank at least 2. The following is a foundational theorem in the study of higher rank Lie group actions, and actions of their lattices. It will give that the action of a split Cartan subgroup is Oseledets conformal.

Theorem 7.3 (Zimmer [57] Theorem 5.2.5). Given a cocycle $\beta : G \times X \to GL(n, \mathbb{R})$ over a measure preserving $G$-action $\rho$ on $X$, there exists a measurable map $\psi : X \to GL(n, \mathbb{R})$ and a homomorphism $\pi : G \to GL(n, \mathbb{R})$ such that the cocycle $\beta$ is cohomologous to $\pi$ via $\psi$:

$$\beta(g, x) = \psi(\rho(g, x)) \pi(g) \psi(x)^{-1}.$$  

Lemma 7.4. Given the $G$-action $\rho$ on $X$ as in Theorem 2.2, the $A$ action $\rho|_A$ is Oseledets conformal.
Proof. Given the G-action $\rho$ on $X$ as in Theorem 2.2, Theorem 7.3 gives us a measurable frame in which the derivative cocycle $D_2\rho(g)$ of the action $\rho$ is constant. Then the Lyapunov spectrum and Oseledets spaces for the totally Anosov action $\rho|_A : A \times X \to X$ are (measurably) identified with those of the representation $\pi|_A$. This implies that the representation has a diagonal form $\text{diag}(e^{\lambda_1}O(d_1), \ldots, e^{\lambda_r}O(d_r))$ where $O(d_i)$s are orthogonal groups of dimension $d_i$. The invariant subspaces $V_i$ of the diagonal representation carry a metric which via $\psi$ defines a measurable metric $\| \cdot \|_i$ on each $E_{\lambda_i}$ such that for $v \in E_{\lambda_i}$ and $a \in A$ we have: $\|a_vv\|_i = e^{\lambda(a)}\|v\|_i$. \hfill $\square$

7.3. Accessibility.

Definition 7.5. Fix some $\Omega \subset \Delta$. If $x, y \in X$, we say $y$ is $\Omega$-accessible from $x$ if there exists $x = z_0, \ldots, z_n = y$ such that for every $i = 0, \ldots, n - 1$ there exists $\lambda_i \in \Omega$ such that $z_{i+1} \in W_{\lambda_i}(z_i)$. The $\Omega$-accessibility class of $x$ is the set of all points $\Omega$-accessible from $x$, and is denoted $\text{Acc}_\Omega(x)$. The action is called $\Omega$-accessible if $\text{Acc}_\Omega(x) = X$ for some (equivalently, for every) $x \in X$.

Theorem 7.6. Let $G \curvearrowright X$ be an action satisfying the assumptions of Theorem 2.2. Then for any coarse Lyapunov foliation $W^\lambda$, the action is $\Delta(\hat{\lambda})$-accessible.

Proof. We begin by showing the following

Lemma 7.7. The action by $A$ is $\Delta$-accessible.

Proof. For a partially hyperbolic system with integrable $W^{cs, cu, c}$, by transversality for any two points $x, y$ can be linked by finite $c, s, u$ paths. In our case, it is easy to see we only need to show that any two points $x, y$ in the same $A \times M$-leaf, $y$ is $\Delta$-accessible from $x$. Assume $y = ax$, where $a \in A \times M$, since G is semi-simple, there is a sequence of nilpotent elements $u_i \in G$ such that each $u_i$ is in the “root-subgroup” with respect to $A$ and $a = u_n \cdot u_{n-1} \cdots u_1$, let $z_i := u_i \cdots u_1 \cdot x$, then $z_{i+1} \in W_{\lambda_i}(z_i)$ for some root $\chi_i$ and $y$ is $\Delta$ accessible from $x$. \hfill $\square$

To show $\Delta(\hat{\lambda})$-accessibility, by the last lemma it is enough to prove for any two points $x, y$ in the same $W^\lambda$ leaf, $y$ is $\Delta(\hat{\lambda})$-accessible from $x$.

Recall that the Weyl group $N_G(A)/C_G(A)$ (where $N_G(A)$ is the normalizer of $A$) acts on the weights and roots. Denote by $w_\beta$ the reflection about the hyperplane perpendicular to the root $\beta$. Its well known that the Weyl group action on weight and roots is generated by $w_\beta$. Moreover, we have the following fact from the representation theory of semisimple Lie groups.

Lemma 7.8. For any weight $\lambda$, there exist root $\beta$, and weight $\gamma$ not proportional to $\lambda$ such that $w_\beta(\gamma) = \lambda$.

We return now to the proof of Theorem 7.6. We fix $\lambda$ and consider $\beta, \gamma$ which satisfy the previous lemma. The following lemma can be found for example in [15, Lemma 1.3].

Lemma 7.9. There is a representative of $w_\beta$ in $N_G(A)$ can be written as $u_\beta \cdot v_{-\beta} \cdot u_\beta$, where $u_\beta$ and $v_{-\beta}$ are in the unipotent subgroups corresponding to the root $\beta$ and $-\beta$ respectively.

By the previous lemma we show the following:

Lemma 7.10. For any representative $w$ of $w_\beta$, $w(W^\gamma) = W^\lambda$.

Proof. Replacing $w$ by $w^{-1}$, it is enough to show $wW^\gamma \subset W^\lambda$. By definition of coarse Lyapunov foliation, $W^\gamma = \cap_{a; \gamma(a) < 0} W^a$, $W^\lambda = \cap_{a; \lambda(a) < 0} W^a$. Therefore we only need to prove the following properties:

(1) $w(W^\gamma)$ is an $A$-invariant foliation.
(2) for any $a$ such that $\lambda(a) < 0$, $a$ uniformly contracts $w(W^\gamma)$.

Property (1) follows from the fact that for any $a \in A$,

$$aw(W^\gamma) = w(w^{-1}aw)(W^\gamma) = w(w_\beta(a))(W^\gamma) = w(W^\gamma).$$

To show (2) observe that by the last equation we know that $a$ contracts $w(W^\gamma)$ if and only if $w_\beta(a)$ contracts $W^\gamma$, i.e. $\gamma(w_\beta(a)) < 0$, notice that $w_\beta(\lambda) = \gamma$ and $w_\beta$ is an isometry, so

$$\gamma(w_\beta(a)) = w_\beta(\lambda)(w_\beta(a)) = \lambda(a)$$

which implies (2). \qed

By Lemmas 7.9 and 7.10, we know that any two points $x, y$ on the same $W^\lambda$ leaf can be linked by finitely many $W^\beta, W^{-\beta}, W^\gamma$ paths. Since $\pm \beta, \gamma \in \Delta(\hat{\lambda})$, we finish the proof. \qed

7.4. **Conclusion of the proof of Theorem 2.2.** Let $\rho : \mathcal{G} \to \text{Diff}^\infty(X)$ be the $G$-action as in Theorem 2.2. Then there is some split Cartan subgroup $A$ of $G$ such that $\rho(A)$ is a totally Anosov volume preserving action. Due to Section 7.1 the $G$ action $\rho$ preserves volume as well. Then Sections 7.2 and 7.3 imply that the action $\rho(A)$ satisfies the conditions of Theorem 2.7. Therefore there is a homogeneous space $K/H/\Gamma$ such that $\rho(A)$ is conjugate to a bi-homogeneous action on a finite cover of $K/H/\Gamma$. Thus, it is clear that we may lift the $A$-action to a homogeneous action on $H/\Gamma$. By applying the lifting lemma (see Theorem D.1), the whole $\rho(G)$ action lifts to a $G$-action on $H/\Gamma$ such that the maximal split Cartan subgroup $A$ of $G$ acts homogeneously, i.e. by left multiplication.

We wish to show that the $G$ action is homogeneous, not just its restriction to $A$. For this, we use Theorem C.1, noting that since every simple factor of $G$ has rank at least two, for every root $\chi$ of $G$, the action of $\ker \chi \subset A$ is ergodic. Since $A$ is an $\mathbb{R}$-split Cartan subgroup, the action of $\ker \chi$ is $\mathbb{R}$-semisimple. Hence Theorem C.1 applies, and since $U_\chi$, the root subgroup of $G$ corresponding to $\chi$, commutes with $\ker \chi$ and is $C^\infty$. So, by Theorem C.1, $U_\chi$ acts by affine maps.

Finally, observe that since the $U_\chi$ generate $G$ as $\chi$ varies over all roots of $G$, the action of $G$ is affine. Hence any conjugate of $A$ will also be homogeneous, since the conjugation of a homogeneous action by an affine transformation is also homogeneous. Since $G$ is semisimple, the conjugates of $A$ generate $G$ and the action of $G$ on $H/\Gamma$ is by translations.

7.5. **Proof of Corollary 2.4.** We must show that the assumptions of Theorem 2.2 are satisfied when either of the assumptions of Corollary 2.4 are satisfied. Assume that $\mathcal{J}(H)$ has dense image. We claim that if the Jordan decomposition of $g \in G$ is $g = k\alpha n$, and $g$ is a hyperbolic element for $G \propto X$, then $a$ is a hyperbolic element for $G \propto X$. This implies the result, since for a fixed Cartan subgroup of $G$, there is a dense subset which is conjugate to a hyperbolic element. Since an open dense subset of that Cartan subgroup satisfies that $Z_G(a) = g_0^\mathbb{R}$, it follows that there is a dense set of Anosov elements. Indeed, we may choose a Riemannian metric on $X$ which is invariant under $k$, since $k$ belongs to a compact subgroup. Furthermore, the condition that $g$ is hyperbolic for $G \propto X$ is exactly that it is normally hyperbolic with respect to the orbit foliation of $\exp(E_g^0)$. In particular, the bundles $E_g^s$ and $E_g^u$ are unique, and since $k, a$ and $n$ commute with $g$, each of them must preserve the subbundles as well. Hence, $\|Dg|_{E_g^s}(x)\| = \|D(an)|_{E_g^s}(x)\|$. Since $E_g^s$ is uniformly contracting under $g$, there exists $C > 0$ and $0 < \lambda < 1$ such that $\|Dg^k|_{E_g^s}(x)\| \leq C\lambda^k$. We now appeal to the following

**Lemma 7.11.** Let $n \in G$ be an $\text{ad}$-unipotent element. Then for every $\varepsilon > 0$, there exists some $C' > 0$ such that $C'e^{-k\varepsilon} \leq \|Dn^k(x)\| \leq C' e^{k\varepsilon}$. 

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Let us apply the lemma before proceeding with the proof. Notice that

$$\left\| D^{n^k} E_g (x) \right\| = \left\| D^{n^{-k}} \circ D^{g^k} E_g (x) \right\| \leq C' e^{k \varepsilon} \cdot C e^{k \lambda} = C C' e^{(\lambda+\varepsilon)k}.$$  

Since $\varepsilon$ is arbitrary, we may choose it to be $-\lambda/2$, so that $\lambda + \varepsilon$ is still negative. Therefore, $a$ contracts the bundle $E^u_g$ exponentially. A symmetric argument works for $E^u_g$.

**Proof of Lemma 7.11.** Fix $v \in TX$, and let

$$\chi(v) = \limsup_{k \to \infty} \frac{1}{k} \log \left\| D^{n^k}(v) \right\|$$

denote the (upper) Lyapunov exponent of $n$ on the vector $v$. Observe that

$$\chi(v) \leq \sup_{v \in TX} \left\| D^n(v) \right\| / \|v\| < \infty,$$

so $\chi$ is a bounded function on $TX$. Furthermore, since $n$ is ad-unipotent and belongs to a semisimple Lie group, if it is nontrivial, there exists a renormalizing element $b \in G$ such that $b^{-1} nb = n^2$ (this follows from the Jacobson-Morozov theorem). Then direct computation shows that

$$\chi(Db(v)) = \limsup_{k \to \infty} \frac{1}{k} \log \left\| D^{n^k}(Db(v)) \right\| = \limsup_{k \to \infty} \frac{1}{k} \log \left\| D(b^{-1} n^k b)(v) \right\|$$

$$= \left\| D(n^{2k})(v) \right\| = 2 \chi(v).$$

Even though this is taken along the subsequence of even iterates, notice that $D(n^{2k+1}) = Dn^{2k} \circ Dn = Dn^{2(k+1)} \circ Dn^{-1}$. Since $Dn$ is uniformly bounded, the lim sup along even terms and odd terms coincides. Since $\chi$ must remain bounded, we conclude that $\chi(v) = 0$ for all $v \in TX$. One inequality in the claim follows immediately. The other follows from an identical analysis using lim inf and the lower exponent.

Now we turn to the second assumption, that $\mathcal{H}$ intersects the set of $\mathbb{R}$-semisimple elements in a dense set. Let $\mathcal{S}$ denote the set of such elements. Fix a Cartan subgroup $A \subset G$. Any $\mathbb{R}$-semisimple element of $G$ is conjugate to an element of $A$, and the map $J|_{\mathcal{S}}$ is onto. In particular, $J(\mathcal{H})$ must be dense if $\mathcal{H} \cap \mathcal{S}$ is dense in $\mathcal{S}$.

8. **Proof of Theorem 2.7 from Theorem 2.6**

We show that assumptions of Theorem 2.7, imply the fundamental assumptions in Theorem 2.6.

8.1. **Assumptions of Theorem 2.7 imply (FA-1).** Now we show that the accessibility assumption in Theorem 2.7 implies condition (FA-1) in Theorem 2.6. We use the result from [9] that a partially hyperbolic, volume preserving center bunched diffeomorphism is ergodic. By the strong accessibility assumption in Theorem 2.7, the generic singular element is an accessible partially hyperbolic diffeomorphism. By Lemma 5.2, it is also center bunched. This implies ergodicity for the generic singular element in $\text{Ker}\lambda$, which implies directly (FA-1).
8.2. **Assumptions of Theorem 2.7 imply (FA-2).** In this section we show that the measurable invariant metric in the assumption on Oseledets conformality in Theorem 2.7, can be improved given the assumptions in Theorem 2.7.

We are going to use the following version of the result based on the invariance principle as it appears in [28], but similar results can also be found in the initial works on the invariance principle [2] and [1].

**Theorem 8.1.** [28] Let \( f \) be a \( C^{1+\theta} \) partially hyperbolic volume preserving, strongly center-bunched, accessible diffeomorphism of a closed manifold \( X \). Let \( \pi : E \to X \) be a Hölder vector bundle over \( X \), and let \( \phi \) be a fiber bunched Hölder linear cocycle over \( f \), \( \phi : E \to E \), with coinciding Lyapunov exponents with respect to volume. Then the following hold:

1. Any almost-everywhere defined \( \phi \)-invariant measurable sub bundle \( V \subset E \) coincides almost everywhere with a Hölder continuous one.
2. Any almost-everywhere defined \( \phi \)-invariant conformal structure coincides almost everywhere with a Hölder continuous one.

Now let \( \rho \) be a \( C^\infty \) Anosov action of \( C = \mathbb{R}^k \times M \) which satisfies the conditions of Theorem 2.7. Then by applying Theorem 8.1 to generic singular elements in \( \rho \) we obtain the following corollary:

**Lemma 8.2.** Every Oseledec subspace \( E^\lambda \) of \( \rho \) is Hölder continuous. Within \( E^\lambda \) there is an \( A \)-invariant Hölder continuous invariant conformal structure.

**Proof.** In our setting for any coarse Lyapunov foliation \( W^\lambda \), a generic singular element \( a \) in \( \ker \lambda \) is by assumption a partially hyperbolic diffeomorphism with center distribution \( TO \oplus E_\lambda \oplus E_{-\lambda} \), or \( TO \oplus E_\lambda \). In the rest of the argument we assume the former, since it is the more general case. Moreover, the generic singular \( a \) is accessible (by assumptions of Theorem 2.7) and by Lemma 5.2 it is center bunched partially hyperbolic. Since it is a singular element, it has zero Lyapunov exponents in the direction of the center distribution \( TO \oplus E_\lambda \oplus E_{-\lambda} \) and non-zero exponents in all other directions.

Consider the Hölder continuous linear cocycle \( Da|_{E_\lambda \oplus E_{-\lambda}} \) over \( a \), since \( a \in \ker \lambda \), \( Da|_{E_\lambda \oplus E_{-\lambda}} \) has zero Lyapunov exponents. By Oseledec theorem and by assumption on Oseledets conformality in Theorem 2.7, the Oseledec subspaces within \( E_\lambda \) and \( E_{-\lambda} \) are measurable \( A \)-invariant subbundles, and within each of them there is an \( A \)-invariant measurable conformal structure. Therefore by Theorem 8.1 all the measurable invariant objects we obtained coincide with Hölder continuous invariant ones almost everywhere. \( \square \)

We will use the following result from [54] to obtain Hölder regularity of the measurable metric in the assumption of the main Theorem 2.7.

**Theorem 8.3.** [54] Let \( f : X \to X \) be a \( C^{1+\theta} \) strongly center bunched conservative partially hyperbolic diffeomorphism. Assume that \( f \) is accessible. Let \( \phi : X \to \mathbb{R} \) be a Hölder continuous function, then any measurable solution of the cohomological equation

\[
\phi = \Phi \circ f - \Phi + c
\]

coincides with a Hölder continuous solution almost everywhere.

**Lemma 8.4.** The metric in the assumption of Theorem 2.7 is Hölder continuous (within each Oseledec subspace).

**Proof.** By Lemma 8.2 the Oseledec subspaces and measurable conformal structures in the assumption of Theorem 2.7 are actually Hölder. As a consequence, to show our lemma we have to solve
some cocycle equations. We fix a smooth background metric on $X$. It induces a Hölder continuous metric $g_\lambda$ within each Oseledec subspace $E^\lambda$, then $g_\lambda$ induces a Hölder continuous volume form $\nu_\lambda$ within $E^\lambda$. For any $a \in A$, if we let $q(x,a) := \text{Jac}_{\nu_\lambda}(Da|_{E^\lambda})$, then $q$ is Hölder continuous. Moreover we have

**Lemma 8.5.** There exists a Hölder continuous function $\phi : X \to \mathbb{R}$ such that for any $b \in A$

\[
\phi(x) \cdot \phi(bx)^{-1} = e^{\dim(E^\lambda)\lambda(b)}
\]

**Proof.** First we show our lemma for generic $a \in \ker \lambda$. By the assumption on Oseledets conformality of Theorem 2.7 we get the existence of a measurable function $\phi$ satisfying our lemma (consider the measurable volume form induced by the measurable metric). Then by Theorem 8.3, we can actually upgrade the regularity of $\phi$ to be Hölder, since $a$ is $C^2$, volume preserving, accessible and center-bunched.

Now we show that for any general $b \in A$, $b$ satisfies our lemma. We can define a Hölder continuous volume form on $E^\lambda$ by $\nu'_\lambda := \phi \cdot \nu_\lambda$. By our construction of $\phi$, $\nu'_\lambda$ is invariant under the action of $ta$ for $t \in \mathbb{R}$. By commutativity there exists a Jacobian function $\psi$ such that $b_*\nu'_\lambda = \psi \cdot \nu'_\lambda$ is also a $ta$–invariant volume form on $E^\lambda$. Since $ta$ acts transitively on $X$, $\psi$ is a constant function. Notice that $b$ acts on $E^\lambda$ with Lyapunov exponents equal to $\lambda(b)$. Therefore because $\psi$ is constant, we complete the proof of the lemma. \qed

Combining the Hölder continuous volume form $\nu'_\lambda = \phi \cdot \nu_\lambda$ and the Hölder continuous conformal structure on $E^\lambda$ we easily get the Hölder continuous metric as claimed in the Lemma 8.4. \qed

We summarize now the conclusions of this section: Let $\rho$ be an action satisfying assumptions of Theorem 2.7. Each coarse Lyapunov distribution $E_\lambda$ of $\rho$ decomposes into Oseledets spaces with exponents positively proportional to $\lambda$: $E_\lambda = \bigoplus_{i=1}^{n_\lambda} E_{ci\lambda}$ and the following hold:

1. There exists a Hölder continuous Oseledets decomposition.

\[ TX = \bigoplus_{\lambda \in \Delta} (\bigoplus_{i=1}^{n_\lambda} E_{ci\lambda}). \]

2. There exists a Hölder continuous Riemannian metrics (inner products). $\langle \cdot, \cdot \rangle_{c_i\lambda}$ on $E_{ci\lambda}$ such that for all $v \in E_{ci\lambda}$

\[ \|a_*v\| = e^{c_i\lambda(a)}\|v\|. \]

3. Let $a \in \ker \lambda$, and $y \in W^a_\lambda(x)$. Let $H_{x,y}^{s,a} : E_\lambda(x) \to E_\lambda(y)$ be the stable holonomy map for $a$. Then $H_{x,y}^{s,a}(E_{ci\lambda}) = E_{ci\lambda}$ and

\[ H_{x,y}^{s,a} : E_{ci\lambda} \to E_{ci\lambda} \]

is an isometry with respect to the inner products above constructed from $a_y^{-n}I_{a_x,a_y}a_y^n|_{E_\lambda(x)} \to H_{x,y}^{s,a}$ where for $x, y$ two nearby points we let $I_{xy} : E_\lambda(x) \to E_\lambda(y)$ be a linear identification which is Hölder close to the identity.

We note that claim (3) is an application of the invariance principle [2] or Theorem 8.1 above. Since $a$ has 0 Lyapunov exponents on $E_\lambda$, by Theorem 8.1, the stable holonomy $H_{x,y}^{s,a}$ preserves each invariant sub bundle almost everywhere. And by Hölder continuity of $H_{x,y}^{s,a}$ and $E_{ci\lambda}$, we know the stable holonomy preserves each $E_{ci\lambda}$ everywhere. By [28], the stable holonomy preserves the conformal structure within each $E_{ci\lambda}$ everywhere. To show (3) we only need to prove $H_{x,y}^{s,a}$ preserves the volume form induced by the Hölder continuous metric within each $E_{ci\lambda}$. But it is not hard to see that the Jacobian of $H_{x,y}^{s,a}|_{E_{ci\lambda}}$ is exactly the holonomy of the one-dimensional cocycle $\text{Jac}(Da|_{E_{ci\lambda}})$, then as a consequence, the stable holonomy preserves the invariant volume form within each $E_{ci\lambda}$, as a result, the stable holonomy is an isometry.

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Moreover, since the $M$-action in the assumptions of Theorem 2.7 commutes with the $\mathbb{R}^k$ action, we may assume without loss of generality that the $||\cdot||_{\alpha,i}$ are invariant under the $M$-action by averaging its pushforwards under $M$ with respect to the Haar measure on $M$. All this implies that (FA-2) holds for actions satisfying assumptions in Theorem 2.7.

9. Construction of a large group of isometries and construction of the fibration

In this section we describe the first steps in the proof of Theorem 2.6. We will use assumptions ((FA-1)) and ((FA-2)) first to construct a simply transitive subgroup of isometries of $W^\lambda(x)$.

After that, we construct a compact extension over $X$ which is a genuinely higher-rank leafwise homogeneous Anosov action. Roughly speaking the main feature of these actions is that they come together with groups of isometries acting transitively on the leaves of coarse Lyapunov foliations.

A key point here is the following: for each $x \in X$, we will be able to construct a simply transitive subgroup of isometries of $W^\lambda(x)$. All such groups will be isomorphic to one another, denoted by $N^\lambda$. However, this does not immediately give an action of a group on the entire manifold. Indeed, in Example 4.7, each coarse Lyapunov leaf is a copy of $C$, but there is no global action of $C$ on $X$ which parameterizes the leaves on $X$. This can be resolved by passing to some compact fiber bundle over $X$ (which is hinted at in Example 4.7). In case when $W^\lambda$ are 1-dimensional, as in [48] does not occur. In Section 9.2, we will carefully choose a compact fiber bundle extension $\hat{X}$ over $X$ so that the group $N^\lambda$ acts on it in a canonical way. Similar problem occurs for higher-rank semisimple Lie group actions such as Example 4.8.

9.1. Constructing a large group of isometries. Throughout this section we assume that $\mathbb{R}^k \times M \subset X$ is a $C^\infty$ totally Anosov action satisfying (FA-1) and (FA-2). In particular, for $v,w \in T_x W^\lambda = \oplus_{i=1}^{n_\lambda} E_i^\lambda(x)$ we have the metric

$$\langle v, w \rangle = \sum_{i=1}^{n_\lambda} \langle v_i, w_i \rangle_{E_i^\lambda},$$

where $v_i, w_i$ are the $E_i^\lambda$ components of $v$ and $w$ respectively. Denote by $\text{Isom}(W^\lambda(x))$ the group of isometries with respect to this metric.

**Definition 9.1.** Call an isometry $\phi : W^\lambda(x) \to W^\lambda(x)$ harnessed if $\phi_*$ preserves the Oseledets splitting. Let $\text{Isom}^\lambda_H(x) := \text{Isom}_H(W^\lambda(x))$ be the group of harnessed isometries.

Note that the group of harnessed isometries $\text{Isom}^\lambda_H(x)$ is a closed subgroup of $\text{Isom}(W^\lambda(x))$ because the limits in $\text{Isom}^\lambda(x)$ preserve Oseledets frame.

The main outcome of this section is the following proposition:

**Theorem 9.2.** Let $\mathbb{R}^k \times M \subset X$ be a $C^{(1,\theta)}$ (or $C^\infty$ resp.) totally Anosov action satisfying (FA-1) and (FA-2).

Fix a coarse Lyapunov foliation $W^\lambda$. Then there exists a nilpotent Lie group $N^\lambda$ such that for all $x \in X$ there exists a subgroup $N^\lambda_x \subset \text{Isom}_H(W^\lambda(x))$ with the following properties:

1. $N^\lambda_x$ isomorphic to $N^\lambda$,
2. for all $h \in N^\lambda_x$, $h_*$ preserves the Oseledets splitting $TW^\lambda = \oplus_{i=1}^{n_\lambda} E_i^\lambda$,
3. $N^\lambda_x$ acts transitively on $W^\lambda(x)$,
4. $a N^\lambda_x a^{-1} = N^\lambda_x$ for all $a \in \mathbb{R}^k \times M$.
5. $N^\lambda_x$ acts by $C^{(1,\theta)}$ (or $C^\infty$ resp.) diffeomorphisms.

We first easily show the following property of harnessed isometries:
Lemma 9.3. Isom\(_H^\lambda(ax) = a \circ Isom_H^\lambda(x) \circ a^{-1}.

Proof. Choose any harnessed isometry \( \phi \in Isom_H^\lambda(x) \) at \( x \). \( \phi \) is a harnessed isometry if and only if \( \phi \) at every point of \( W^\lambda(x) \) is an orthogonal matrix that preserves the Oseledets subspaces. Then note that \( a \circ \phi \circ a^{-1} \) satisfies the same properties. It clearly preserves the Oseledets splitting, and \( a \) undoes what \( a^{-1} \) does on each Oseledets space.

\( \square \)

We will use the following Lemma, the proof of which is essentially the same as the proof of Theorem 2.8 in [21], for completeness we sketch the proof here.

Lemma 9.4. Let \( x, y \in X \). If there exists a sequence \( a_k \in \ker \lambda \) such that \( \lim_{k \to \infty} a_k(x) \to y \). Then there exists a subsequence \( k_j \) and an isometry map \( a_0 : W^\lambda(x) \to W^\lambda(y) \) such that \( a_0 = \lim_{j \to \infty} a_{k_j}|_{W^\lambda(x)}, Da_0 = \lim_{j \to \infty} Da_{k_j}|_{W^\lambda(x)} \), where \( D \) denotes the derivative.

Proof. For the first claim, the proof is essentially the same as that of Proposition 2.9 in [21]. For completeness we could sketch the proof here. The idea is to use \( C^\infty \) metric to approximate original Hölder continuous metric to show that if \( a_{k_i}(x_i) \to y_i, i = 1, 2 \) then \( d_{W^\lambda}(x_1, x_2) \geq (1-\epsilon) d_{W^\lambda}(y_1, y_2) \) for any small \( \epsilon > 0 \). Notice that by classical diagonal argument, we can find a subsequence \( k_j \) such that \( a_{k_j} \) converges on a dense set \( \{x_i\} \subset W^\lambda(x) \). Then combining with the last inequality we could extend \( \lim a_{k_j} \) to a Lipchitz continuous map \( a_0 : W^\lambda(x) \to W^\lambda(y) \), with Lipchitz constant bounded by 1.

By similar approximation argument, we could show that \( a_0 \) is invertible and \( a_0^{-1} \) is also Lipchitz continuous with Lipchitz constant bounded by 1, therefore \( a_0 \) is an isometry on \( W^\lambda(x) \).

For the second claim, by the first claim and (FA-1), as the proof of Theorem 2.8 of [21] we could easily build a homogeneous space structure on every \( W^\lambda \) leaf such that for any point \( x \) the group \( Isom(W^\lambda(x)) \) acting transitively on \( W^\lambda(x) \). Since the metric is Hölder continuous, by Rauch-Taylor Theorem (Theorem 5.4) we know the action of \( Isom(W^\lambda(x)) \) is \( C^{1+} \). Therefore the \( C^\infty \) homogeneous space structures of \( W^\lambda \) leaves are \( C^{1+} \) equivalent to the original differentiable structure. Therefore by the corresponding proof in [21], we know that although the metric along \( W^\lambda \) is only Hölder continuous but locally the uniqueness of length minimizing geodesics holds. As a consequence, if we take the subsequence \( a_{k_j} \to a_0 \) in the first claim, since the derivative of an isometry is determined by how geodesics mapped, we get that \( Da_{k_j} \) also converges to \( Da_0 \). For more details see [21].

\( \square \)

Similarly we can show the transitivity of the \( Isom_H^\lambda(x) \) action for points with dense \( \ker \lambda \) orbit.

Lemma 9.5. If \( x \) has a dense ker \( \lambda \) orbit, then \( Isom_H^\lambda(x) \) acts transitively on \( W^\lambda(x) \).

Proof. If \( x \) has a dense ker \( \lambda \) orbit, then for any \( x' \in W^\lambda(x) \), we can find a sequence \( a_k \in ker \lambda \) such that \( \lim_{k \to \infty} a_k x = x' \). As a consequence of lemma 9.4 we can pick a subsequence of \( a_k \) such that this subsequence uniformly converges to a harnessed isometry \( a_0 \) on \( W^\lambda(x) \) such that \( a_0(x) = x' \). Since \( x' \) was arbitrary, this implies that \( Isom_H^\lambda(x) \) acts transitively on \( W^\lambda(x) \).

\( \square \)

The next lemma combines the previous two lemmas.

Lemma 9.6. Suppose \( x \in X \) has a dense ker \( \lambda \)–orbit, then for any \( y \in X \) the group \( Isom_H^\lambda(y) \) is isomorphic \( Isom_H^\lambda(x) \), and \( Isom_H^\lambda(y) \) acts transitively on \( W^\lambda(y) \).

Proof. Let \( y \in X \). If \( x \in X \) has a dense ker \( \lambda \)–orbit, there exists \( a_k \in ker \lambda \) such that \( \lim_{k \to \infty} a_k x = y \). By Lemma 9.4 there exists a subsequence \( k_j \) and an isometry map \( a_0 : W^\lambda(x) \to W^\lambda(y) \) such
that\ a_0 = \lim_{j \to \infty} a_{k_j} |_{W^\lambda(x)},\ Da_0 = \lim_{j \to \infty} Da_{k_j} |_{W^\lambda(x)}\). Since each \(a_{k_j}\) is a harnessed isometry from \(W^\lambda(x)\) to \(W^\lambda(a_{k_j} x)\), \(a_0\) is a harnessed isometry from \(W^\lambda(x)\) to \(W^\lambda(y)\) as well. And \(a_0\) induces an isomorphism between \(\text{Isom}_H^\lambda(x)\) and \(\text{Isom}_H^\lambda(y)\) by Lemma 9.3. This proves the first claim in the Lemma.

For the second claim in the lemma, pick any \(y' \in W^\lambda(y)\); as in the previous part of the proof we can construct a harnessed isometry \(a_0\) from \(W^\lambda(x)\) to \(W^\lambda(y)\) and \(a_0(x) = y\). Let \(a_0^{-1} \cdot y' := x'\). By Lemma 9.5, for any \(x' \in W^\lambda(x)\), there exists an \(a' \in \text{Isom}_H^\lambda(x)\) such that \(a' x = x'\), then \(a_0 \circ a' \circ a_0^{-1}\) is a harnessed isometry in \(\text{Isom}_H^\lambda(y)\) which maps \(y\) to \(y'\).

\[\square\]

The remaining part of the argument is to show that the group from the previous lemma has a large nilpotent subgroup \(N^\lambda\) which is the group we need in Proposition 9.2.

**Lemma 9.7.** At a periodic point \(p\), \(\text{Isom}_H^\lambda(p) = K_p^\lambda \ltimes N_p^\lambda\) where \(K_p^\lambda\) is a compact Lie group and \(N_p^\lambda\) is simply connected nilpotent Lie group, and \(N_p^\lambda\) acts simply transitively on \(W^\lambda(p)\).

Lemma 9.7 is a direct corollary of the following more general proposition.

**Lemma 9.8.** Let \((X, d)\) be a connected complete Riemannian manifold, \(G\) be a locally compact topological group acting isometrically and transitively on \(X\). If there exists a strictly contracting \(a \in \text{Diff}(X)\) such that a normalizes the \(G\)-action, then

1. there exists a unique nilpotent subgroup \(N\) normal in \(\text{Isom}(X)\) and acting simply and transitively on \(X\);
2. one can isometrically identify \(X\) with \(N\), the metric \(d\) on \(X\) is identified with the left-invariant metric on \(N\);
3. for any \(p_0 \in X\), \(\text{Isom}(X)\) is the semi direct product of \(N\) with the isotropic group \(K_{p_0} := \{g \in \text{Isom}(M), kp_0 = p_0\}\);
4. for any \(p_0 \in X\), \(G\) is the semi direct product of \(N\) with the isotropic group \(K'_{p_0} := \{g \in G, kp_0 = p_0\}\).

**Proof.** By Theorem 5.4 we know all elements of \(G\) are \(C^1\), hence by [11], \(G\) is a locally compact Lie group and the action by \(G\) is a \(C^1\) Lie group action. By completeness of \((X, d)\), there exists a unique \(a\) fixed point \(p \in X\). Since \(aG a^{-1} = G\), the conjugacy by \(a\) induces an Lie group automorphism of \(G\), hence a Lie algebra automorphism \(\Phi_a\) of \(\mathfrak{g} := \text{Lie}(G)\).

Denote by \(V_\mu\) the generalized eigenspace of \(\Phi_a\) for eigenvalue \(\mu \in \mathbb{C}\). Using the fact that \([V_{\mu_1}, V_{\mu_2}] \subset V_{\mu_1+\mu_2}\) we get that \(n := \oplus_{|\mu|<1} V_{\mu}\) forms a nilpotent Lie subalgebra of \(\mathfrak{g}\). We denote by \(N\) the connected nilpotent Lie group \(\exp(n)\), then \(aN a^{-1} = N\).

We show first that \(\oplus_{|\mu|>1} V_\mu \subset \text{Lie}(K_p)\), where \(K_p\) is the stabilizer of \(p\) in \(G\). Assume, for a contradiction, that \(Z \in \oplus_{|\mu|>1} V_\mu\) has \(\exp(tZ) \cdot p \neq p\) for all sufficiently small \(t > 0\). Let \(\delta_t = d(\exp(tZ) \cdot p, p)\), and note that \(\delta_t > 0\) for all \(t > 0\), and \(\delta_t \to 0\) as \(t \to 0\). Since \(a\) is contracting, for sufficiently small \(\varepsilon > 0\) and some \(0 < \lambda < 1\),

\[(9.1)\quad a^{-1} \cdot (B(p, \varepsilon) \setminus B(p, \lambda \varepsilon)) \cap B(p, \lambda \varepsilon) = \emptyset.\]

By the intermediate value theorem, we may find, for every \(\varepsilon > 0\), some \(t\) such that \(\delta_t \in (\lambda \varepsilon, \varepsilon)\). Finally, without loss of generality, we may assume that \(\text{Ad}(a^{-1})\) preserves \(B(0, \delta) \subset \text{Lie}(G)\) for every \(\delta > 0\) by replacing \(a\) with a sufficiently large power.
Now, observe that \( a^{-1} \exp(tZ) \cdot p = \exp(t \text{Ad}(a^{-1}Z)) \cdot p \). Since \( Z \) is in the sum of positive generalized eigenspaces, \( \exp(t \text{Ad}(a^{-1}Z)) \cdot p \in B(p, \lambda \varepsilon) \). This is a contradiction to (9.1), so \( \oplus_{|\mu|>1} V_\mu \subset \text{Lie}(K_p) \).

Now let \( \mathfrak{k} := \oplus_{|\mu|=1} V_\mu \). Note that \( \mathfrak{k} \) is a subalgebra since \( \text{Ad}(a) \) is an automorphism of the Lie algebra, so whenever \( ||\text{Ad}(a)k|| \) and \( ||\text{Ad}(a)\varepsilon|| \) are bounded above and below by polynomials for positive and negative values of \( k \), so is \( ||\text{Ad}(a)^k, \text{Ad}(a)\varepsilon|| \). This property characterizes the generalized eigenspaces of modulus 1.

We claim that \( \exp(tZ) \cdot p = p \) for all \( Z \in \mathfrak{k} \). Let \( \mathfrak{k}_0 \) be the subalgebra in \( \mathfrak{k} \) such that for all \( Z \in \mathfrak{k}_0, t \in \mathbb{R}, \exp(tZ) \cdot p = p \). Assume now \( \mathfrak{k}_0 \neq \mathfrak{k} \). Choose \( \varepsilon \) small enough such that \( \exp \) is injective and close to identity at an \( \epsilon \) ball around 0 in \( \mathfrak{g} \). A useful fact is that for fixed \( \varepsilon' \ll \varepsilon \) small enough, for any \( Y \) in \( \mathfrak{k} \setminus \mathfrak{k}_0 \) such that \( ||Y|| < \varepsilon' \), \( d(\exp(Y) \cdot p, p) \) positive and has order \( O(||Y|| \cdot |\angle(Y, \mathfrak{k}_0)||) \) (if \( \mathfrak{k}_0 = \{0\} \), then without loss of generality we could assume the angle to be constant 1).

We pick now an arbitrary \( Z \in \mathfrak{k} \setminus \mathfrak{k}_0 \) and denote by \( Y_n := \Phi^n(Z) \), then \( ||Y_n|| \) has order at least \( ||Z|| \), up to a polynomial factor of \( n \). Take \( t_n = \epsilon_n^2 \). Then notice that both \( t_n \) and the angle \( \angle(Y_n, \mathfrak{k}_0) \) are bounded and either do not decay as \( n \to \infty \) or if they decay they do so at most polynomially fast. So we have \( d(\exp(t_nY_n)) \cdot p, p) \) has the order \( O(\epsilon^2 \cdot |\angle(Y_n, \mathfrak{k}_0)||) \), which cannot decay exponentially fast.

On the other hand:

\[
d(\exp(t_nY_n)) \cdot p = d(a^n \exp(t_nZ)a^{-n} \cdot p, p) = d(a^n \exp(t_nZ) \cdot p, p) \leq O(||a||^n d(\exp(t_nZ) \cdot p, p))
\]
decays exponentially fast due to our choice of \( t_n \). Then we get a contradiction. In summary, \( \mathfrak{k}_0 = \mathfrak{k} \).

Recall that \( G \) acting on \( X \) transitively, therefore for arbitrary small open neighborhood \( B \) of identity in \( G \), \( B \cdot p \) contains \( p \) as an interior point. Since \( \exp(\mathfrak{t}) \) fixes \( p \), we get \( (B \cap N \cdot p \) contains a open neighborhood of \( p \). Hence \( N \cdot p \subset a^{-n} \cdot (B \cap N) \cdot a^n(p) \), \( n \to \infty \) contains exponentially large neighborhood of \( p \), hence \( N \cdot p = X \), i.e. \( N \) acting transitively on \( X \).

In summary we showed that \( G \) hence \( \text{Isom}(X) \) contains a nilpotent Lie subgroup acting transitively on \( X \). Thus \( X \) is a homogeneous nilmanifold as defined in [55]. Then (1)-(3) of Lemma 9.8 follow from Theorem 2 in [55], and (4) is an easy corollary of (3).

**Proof of Lemma 9.7.** Let \( (\text{Isom}^G_H(p), W^\lambda(p)) \) of Lemma 9.7 be \( G, M \) respectively in the Lemma 9.8. To complete the proof of Lemma 9.7 we only need to show the existence of global contracting of \( W^\lambda(p) \) which is guaranteed by the existence of \( a \in C \) such that \( \lambda(a) < 0 \), and \( ap = p \). This is possible since the stabilizer of \( p \) contains \( Z^k \) as a cocompact subgroup. Hence the stabilizer of \( p \) in \( C = \mathbb{R}^k \times M \) is not contained in the kernel of any Lyapunov functional.

Lemma 9.7 together with Lemma 9.6 implies that the group \( (\text{Isom}^G_H(x)) \) for any \( x \) is isomorphic to \( K^\lambda \ltimes N^\lambda \). We show that the splitting is canonical in the sense that \( K^\lambda \) and \( N^\lambda \) are isomorphic to \( K^\lambda_x \) and \( N^\lambda_x \), respectively. As in Lemma 9.7 \( K^\lambda \) is compact and \( N^\lambda \) is simply connected nilpotent. Since \( \text{Isom}^G_H(x) \) acts transitively, so does the subgroup \( N^\lambda_x \). This completes the proof of Proposition 9.2.

**Lemma 9.9.** For any \( x \), \( \text{Lie}(N^\lambda) \) is canonically isomorphic to \( \oplus E^{c^\lambda}(x) \) as a vector space. For every \( \lambda \in \mathbb{R} \), the map scaling each \( E^{c^\lambda} \) by \( e^{\lambda r} \) is an automorphism of \( \text{Lie}(N^\lambda) \).

**Proof.** As the proof of Lemma 9.7 we know that we can identify \( N^\lambda \) harnessed isometrically with any \( W^\lambda(x) \). Therefore we can canonically identify (harnessed isometrically) \( \text{Lie}(N^\lambda) \) with the tangent space \( \oplus E^{c^\lambda}(x) \). And it induces a splitting of \( \text{Lie}(N^\lambda) \). As the proof of Lemma 9.6, this splitting is actually independent of the choice of \( x \). As the proof of Lemma 9.7 we could take a \( p \) with a
compact $C$-orbit and $ap = p$ such that $\lambda = \lambda(a) < 0$, then without loss of generality we can assume that $\text{Lie}(N^\lambda)$ is isomorphic to $\text{Lie}(N^\lambda_p) = \oplus E^{c_i_\lambda}_p$.

Picking suitable basis elements for the $E^{c_i_\lambda}$ to show the claim we only need to show that $[E^{c_i_\lambda}, E^{c_j_\lambda}] \subset E^{(c_i+c_j)_\lambda}$.

Let $X, Y$ are invariant vector fields tangent to $E^{c_i_\lambda}, E^{c_j_\lambda}$ respectively, with the non-vanished $[X, Y]$. Then we can find a subsequence $n_j \to \infty$ such that

$$\frac{1}{n_j} \log \|a^{n_j}_* [X, Y]\| = \frac{1}{n_j} \log \|[a^{n_j}_* X, a^{n_j}_* Y]\| = \frac{1}{n_j} \log e^{(c_1+C_j)_\lambda \|([k_1^{n_j}_* X, (k_2^{n_j}_*)_y Y]\| \to e^{(c_1+C_j)_\lambda \|X, Y]\|$$

for suitable $k_i \in SO(E^{c_i_\lambda})$ for which $k_i^{n_j} \to 1$, $i = 1, 2$. By (FA-2), the limit of $\frac{1}{n_j} \log \|a^{n_j}_* [X, Y]\|$ can be decided by its behavior along a subsequence $n_j$. This implies that $[E^{c_i_\lambda}, E^{c_j_\lambda}] \subset E^{(c_i+c_j)_\lambda}$.

Lemma 9.10. If $\mathbb{R}^k \times M \curvearrowright X$ is a $C^{1,\beta}$ totally Anosov actions satisfying (FA-1) and (FA-2), then for every $\lambda$, then each $g \in \text{Isom}(W^{\lambda}(x))$ (which acts transitively by Lemma 9.2) is a $C^{1,\beta}$ diffeomorphism.

Proof. From assumptions (FA-1) and (FA-2) we have that the regularity of the metric in the Osseledets spaces is Hölder with some Hölder exponent $\beta$. Theorem 5.4 implies that the isometries are $C^{1,\beta}$-transformations on each leaf.

If in addition we assume that the given action is $C^\infty$, we obtain two additional results:

Lemma 9.11. If the action is $C^\infty$, both the subspaces $E^{c_i_\lambda}$ and the Hölder metric $\langle \cdot, \cdot \rangle$ are $C^\infty$ along the coarse Lyapunov leaves of $W^\lambda$.

Proof. Recall that the action of the harnessed isometry group $\text{Isom}_H(W^{\lambda}(x))$ is obtained from taking limits of elements $a \in \ker \lambda$. In normal form coordinates (cf. Appendix A), these maps are given by sub resonance polynomials. Therefore, the limits of such maps are given by polynomials, and are hence $C^\infty$. Therefore, $\text{Isom}_H(W^{\lambda}(x))$ has a subgroup of $C^\infty$ isometries acting on $W^{\lambda}(x)$. Since $E^{c_i_\lambda}_x = h_s(E^{c_i_\lambda})$ for any harnessed isometry $h$ such that $h(x) = y$. Therefore each $E^{c_i_\lambda}$ can be viewed a $C^\infty$ homogeneous graph of a continuous mapping from $W^\lambda(x)$ to corresponding Grassmannian space over $W^\lambda(x)$, hence this graph is locally compact, therefore by the result in [45] we conclude that the splitting is $C^\infty$ along $W^\lambda$. Similarly, the a priori Hölder continuous metric $\langle \cdot, \cdot \rangle$ can be viewed as a $C^\infty$ homogeneous graph of a continuous mapping from $W^\lambda(x)$ to corresponding modular space of quadratic forms over $W^\lambda(x)$, then by the same proof we know it is actually $C^\infty$ along $W^\lambda(x)$.

As a direct corollary of last lemma and Rauch-Taylor Theorem 5.4, we have

Lemma 9.12. If $\mathbb{R}^k \times M \curvearrowright X$ is a $C^\infty$ totally Anosov action satisfying (FA-1) and (FA-2), then for every $\lambda$, each $g \in \text{Isom}(W^{\lambda}(x))$ is a $C^\infty$ diffeomorphism.

9.2. Construction of the fibration. For a given coarse Lyapunov exponent $\lambda$, let $N^\lambda$ denote the group constructed in Theorem 9.2, and $N^\lambda_x$ denote the simply transitive subgroup of $\text{Isom}_H(W^\lambda(x))$. Then let $\mathcal{X}^\lambda(x)$ denote the set of all vector fields on $W^\lambda(x)$ invariant under the action of $N^\lambda_x$. Recall that $W^\lambda$ have smooth leaves.

Lemma 9.13. $\mathcal{X}^\lambda(x) \cong \text{Lie}(N^\lambda(x))$ for every $x \in X$, and $\mathcal{X}^\lambda(x) = \mathcal{X}^\lambda(y)$ if $y \in W^\lambda(x)$.
Proof. This follows from a general fact: if a Lie group $H$ acts simply transitively on a space $Y$ by diffeomorphisms, the space of invariant vector fields under the action can be pulled back to the space of either right- or left-invariant vector fields on $H$, which is by definition Lie($H$). The last claim follows from the fact that $X_\lambda(x)$ depends only on the manifold $W^\lambda(x)$ and does not depend on $x$ itself (recall that the group of harnessed isometries acting simply transitively on a leaf is uniquely determined). 

We now define a fiber bundle $\tilde{X}$ over $X$, which will be a principal $K$-bundle for some compact group $K$. This bundle will have a canonical Hölder structure, but will not a priori have a well-defined smooth structure.

Lemma 9.14. Fix a point $x_0 \in X$ with a dense ker $\lambda \times M$ orbit. Fix any isomorphism $I_\lambda : \text{Lie}(N^\lambda) \to X_\lambda(x_0)$, and fix an orthonormal frame $\hat{\mathbf{f}}$ subordinate to the Oseledets splitting at $x_0$. Let $f_0(\lambda) = I_\lambda^{-1}(\hat{\mathbf{f}})$ now be a fixed frame, which determines a grading of $N^\lambda$ induced by pulling back the corresponding Oseledets subspaces: $\text{Lie}(N^\lambda) = \bigoplus_{i=1}^n E_i^\lambda$.

a) Define the $\lambda$-fiber at $x$, $F_x^\lambda$ to be the set of all Lie algebra homomorphisms $\varphi : \text{Lie}(N^\lambda) \to X_\lambda(x)$ such that $\varphi$ maps $f_0(\lambda)$ to an orthonormal frame, and the grading of $\text{Lie}(N^\lambda)$ defined by $f_0(\lambda)$ to the Oseledets decomposition at $x$.

b) Define $K^\lambda$ to be the collection of Lie algebra automorphisms of $\text{Lie}(N^\lambda)$ which preserve the grading of $\text{Lie}(N^\lambda)$, and act isometrically within each of the graded subspaces.

c) Define $\tilde{X}$ be the bundle whose fibers are $F_x = \prod_{\lambda \in \Delta} F_x^\lambda$ and let $K = \prod_{\lambda \in \Delta} K^\lambda$.

Then $F_x^\lambda$ is non-empty for every $x$, the $\lambda$-bundle $\tilde{X}^\lambda$ with fibers $F_x^\lambda$ is a principal $K^\lambda$-bundle, and $\tilde{X}$ is a principal $K$-bundle.

Proof. Note first that $F_x^\lambda$ is non-empty for every $x$. Clearly $F_{x_0}^\lambda$ is non-empty because it contains $\varphi = I_\lambda$ by construction. Then for $x$ in the the ker $\lambda$-orbit of $x_0$, $a_* \varphi$ is in $F^\lambda_x$. Taking limits gives that each $F_x^\lambda$ is non-empty.

Next, notice that if $k \in K^\lambda$ and $\varphi \in F^\lambda_x$, then $\varphi \circ k \in F^\lambda_x$. Furthermore, if $\varphi_1, \varphi_2 \in F^\lambda_x$, then $\varphi = \varphi_1 \circ \varphi_2^{-1}$ is a graded isometric automorphism of $\text{Lie}(N^\lambda)$. Therefore the $\lambda$-bundle $\tilde{X}^\lambda$ with fibers $F_x^\lambda$ is a principal $K^\lambda$-bundle, and consequently $\tilde{X}$ is a principal $K$-bundle.

Denote by $p = (x, \{\varphi_\lambda\})$ a point of $\tilde{X}$, where $x$ is the point of $X$ that $p$ projects to, and $\varphi_\lambda : \text{Lie}(N^\lambda) \to X_\lambda(x)$ is the homomorphism data obtained from the $F_x^\lambda$ fiber. Let $\Psi_a : \text{Lie}(N^\lambda) \to \text{Lie}(N^\lambda)$ denote the automorphism which expands the space $E_i^\lambda$ by $e^{ci\lambda(a)}$ (which is a graded automorphism by Lemma 9.9). Note that for $a \in M$, $\Psi_a$ is the identity.

Theorem 9.15. There exists a closed subgroup $\hat{K} \subset K$, and a subbundle $\hat{X} \subset \tilde{X}$ which is a principal $\hat{K}$-bundle such that:

1) The $\mathbb{R}^k \times M$-action lifts to $\tilde{X}$ by the formula:

\[
(9.3) \quad a \cdot (x, \{\varphi_\lambda\}) = (a \cdot x, \{a_* \varphi_\lambda \Psi_a^{-1}\}),
\]

it commutes with the $\hat{K}$-action, and $\mathbb{R}^k$ has a dense orbit in $\hat{X}$.

2) The foliations $W^\lambda(x)$ lift uniquely to foliations $\hat{W}^\lambda(x) \subset \hat{X}$ which contract exponentially for any $a \in \mathbb{R}^k$ with $\lambda(a) < 0$. The foliations $\hat{W}^\lambda(x)$ inherit a smooth structure from $W^\lambda$.

3) There are actions of $N_\lambda \subset \hat{X}$ with $C^{1,\theta}$ evaluation maps whose orbits are the leaves of the $\hat{W}^\lambda$-foliations.
(4) The action of $N_{\lambda}$ is normalized by $\hat{K}$ and $\mathbb{R}^k$. In particular, for every $a \in \mathbb{R}^k$ and $u \in N_{\lambda}$:

\begin{equation}
 au a^{-1} \cdot p = \Psi_a(u) \cdot p.
\end{equation}

(5) The action of $M$ on $X$ lifts to an action on $\hat{X}$ which is normalized by $\hat{K}$, and in fact commutes (after passing to a finite cover).

**Proof.** We break up the proof into several steps. Our first step is to build the action of $\mathbb{R}^k$ on $\hat{X}$. We then identify the subbundle $\hat{X}$ on which $\hat{K}$ has a dense orbit.

**Proof of (1).** The formula (9.3) for the lift of the $\mathbb{R}^k \times M$ action provided in the statement of the theorem can easily be verified as a lift. Notice that if $I : W^\lambda(a \cdot x) \to W^\lambda(a \cdot x)$ is an element of $N_{a \cdot x}$, then $a^{-1} \circ I \circ a : W^\lambda(x) \to W^\lambda(x)$ is an isometry as well. This is verified immediately by considering the derivative of the map. Each map making up the composition preserves the Oseledets splitting, and is conformal. The expansion and contraction rates in each subspace have perfect cancellation. Therefore, it is an element of $N_{X \cdot}$, and for $v \in X_{\lambda}(x)$ we have $I_* a_v a_v = a_v(a^{-1} I_* a_v a_v) = a_v v$, so $a_v v \in X_{\lambda}(a \cdot x)$. It is also clear that $a_* v$ is a Lie algebra homomorphism, from general differential geometry, brackets are taken to brackets. Therefore, $a_* v : \text{Lie}(N_{\lambda}) \to X_{\lambda}(a \cdot x)$ is a Lie algebra homomorphism, but scales the Oseledets spaces by exactly $e^{\lambda \cdot a \cdot v}$. Therefore, precomposing with the automorphism $\hat{\Psi}_{a^{-1}}$ yields an element of $F_{a \cdot x}$, as desired. Doing this for each $\lambda$ produces the formula (9.3).

That the lifted action of $\mathbb{R}^k$ commutes with the $K$-action on the fibers follows from the fact that $\hat{\Psi}_a$ commutes with every graded isometric automorphism of $\text{Lie}(N_{\lambda})$. Indeed, if $k = (k_{\lambda})_{\lambda \in \Delta} \in K$, then:

\begin{align*}
ak \cdot (x, \{\varphi_{\lambda}\}) &= a \cdot (x, \{\varphi_{\lambda} k_{\lambda}\}) \\
&= (a \cdot x, \{a \varphi_{\lambda} k_{\lambda} \Psi_{a^{-1}}\}) \\
&= (a \cdot x, \{a \varphi_{\lambda} \Psi_{a^{-1}} k_{\lambda}\}) \\
&= k \cdot (a \cdot x, \{a \varphi_{\lambda} \Psi_{a^{-1}}\}) \\
&= k a \cdot (x, \{\varphi_{\lambda}\}).
\end{align*}

**Proof of (2).** We claim that for any $a \in \Delta$ the foliation $W^\alpha(x)$ lifts uniquely to foliation $\hat{W}^\alpha(p)$ when $p = (x, \{\varphi_{\lambda}\})$ such that if $q \in W^\alpha(p)$, then $d(a^q p, a^q q) \to 0$ exponentially for any $a$ such that $\alpha(a) < 0$. If $u \in N_{\lambda}$, we may write $u = \exp(v)$ with $v \in \text{Lie}(N_{\lambda})$, since the exponential map is a diffeomorphism for nilpotent groups. If $p = (x, \{\varphi_{\lambda}\})$, let $y = \exp(\varphi_{\lambda}(v, x)$, where the exponential of a vector field is defined to be the time-one map of its induced flow. Then $y \in W^\lambda(x)$, so there exists a unique $q \in \hat{W}^\lambda(p)$ such that $q = (y, \{\varphi'_{\lambda}\})$ for some collection of homomorphisms $\varphi'_{\lambda}$.

We claim that $\varphi'_{\lambda} = \varphi_{\lambda}$. First, the equality makes sense: the $\lambda$-leaf of $x$ and $y$ coincide, and the maps $\varphi_{\lambda}$ and $\varphi'_{\lambda}$ are into vector fields on the corresponding leaves, see Lemma 9.13. By the formula (9.3) for the lifted action, the dynamics of $a \in \mathbb{R}^k$ such that $\lambda(a) < 0$, is determined by global action of $a$ on the leaves (which is contracting) and a fixed automorphism $\hat{\Psi}_a \in \text{Lie}(N_{\lambda})$. This implies that $(y, \{\varphi_{\lambda}\})$ is the point on the $\hat{W}^\lambda(p)$ leaf which is the lift of $y$. Then by the uniqueness of the lift we have $\varphi_{\lambda}$ and $\varphi'_{\lambda}$.

**Proof of (3).** Now we define the action of $N_{\lambda}$ on $\hat{X}$ through two conditions, the first is that for any $p = (x, \{\varphi_{\lambda}\})$, and any $g \in N_{\lambda}$, the point $g(p)$ must lie in $\hat{W}^\lambda(p)$. To fix the precise locus of $g(p)$, we only need to know its projection to $X$. A priori there is no canonical way to define $N_{\lambda}$ action on $W^\lambda(x)$, but the additional homomorphism data $\varphi_{\lambda}$ induced the canonical action of $N_{\lambda}$ on $W^\lambda(x)$ and we denote by $y \in W^\lambda(x)$ the action of $g \in N_{\lambda}$ of $x$. Hence $g(p)$ is well-defined, i.e.
the unique intersection of \( \tilde{W}^\lambda(p) \) with the fiber \( \tilde{K}(y) \) of \( y \). The actions have \( C^{1,\theta} \) evaluation maps since they are translations in an isometry group, and the isometry group action is \( C^{1,\theta} \) by Lemma 8.4 and Theorem 5.4.

**Proof of (4).** We show here the equation (9.4). If \( a \in \mathbb{R}^k, \ u = \exp(v) \in N^\lambda \) for \( v \in \text{Lie}(N^\lambda) \), we wish to compute \( aua^{-1} \cdot p \) for some \( p = (x, \{\varphi_\lambda\}) \in \mathcal{X} \). Since the \( \tilde{W}^\lambda \)-foliations are equivariant under \( a \) and invariant under \( u \), it suffices to verify (9.4) on projected point \( x \). Notice that

\[
aua^{-1} \cdot (x, \{\varphi_\lambda\}) = au \cdot (a^{-1} \cdot x, \{(a^{-1})_* \varphi_\lambda \Psi_a\}) = a \cdot (\exp((a^{-1})_* \varphi_\lambda \Psi_a(v))a^{-1} \cdot x, \{\varphi_\lambda\})
\]

for some collection \( \varphi_\lambda' \). Therefore, the base point of \( aua^{-1} \cdot p \) is given by \( \exp((a^{-1})_* \varphi_\lambda \Psi_a(v))a^{-1} \cdot x \).

The basepoint of \( aua^{-1} \cdot p \) is therefore given by \( a \cdot \exp((a^{-1})_\ast \varphi_\lambda \Psi_a(v))a^{-1} \cdot x \). Since \( \exp((a^{-1})_\ast \varphi_\lambda \Psi_a(v)) \) is the time-one map of the flow induced by \( \exp((a^{-1})_* \varphi_\lambda \Psi_a(v)) \) at \( a^{-1} \cdot x \), \( a \exp((a^{-1})_\ast \varphi_\lambda \Psi_a(v)) \) is the time-one map of the flow induced by \( a_\ast (a^{-1})_\ast \varphi_\lambda \Psi_a(v) = \varphi_\lambda \Psi_a(v) \) at \( x \). But this is exactly the basepoint of \( \Psi_a(u) \cdot p \), as claimed.

**Proof of (5).** As for the last claim on lifting the action of \( M \), let \( \varphi_\lambda \) be an isomorphism \( \text{Lie}(N^\lambda) \to \mathcal{X}_\lambda(x) \). Then so is \( m_\ast(\varphi_\lambda) \) for \( m \in M \). Since \( M \) commutes with the \( \mathbb{R}^k \) action, the eigenvalues of \( m_\ast(\varphi_\lambda) \) are the same as for \( \varphi_\lambda \). Note however that \( m \in M \) moves the base points on \( \mathcal{X} \).

Finally since \( M \) normalizes \( \tilde{K} \) and is a connected compact group, we can assume that \( M \) and \( \tilde{K} \) commute, possibly after passing to some finite cover.

In conclusion of the preparatory steps in the proof of Theorem 2.6, given an action \( \rho \) as in Theorem 2.6 we have constructed an extension of \( \rho \) as described in Theorem 9.15. This is the starting point for the rest of the proof of Theorem 2.7. In the next section we describe the class of actions to which the extension constructed in Theorem 9.15 belongs. We call them leafwise homogeneous topological Anosov actions (see Definition 10.4) and we show that, if genuinely higher rank, such actions are essentially homogeneous. This is our main technical result: Theorem 10.7 in the subsequent section. The rest of the paper is dedicated to its proof.

## 10. Leafwise homogeneous topological Anosov actions

Let \( M \) be a connected compact group. In this section, we define and develop properties of certain \( \mathbb{R}^k \times M \) actions on topological spaces which includes the important features of smooth Anosov actions as axioms (see Proposition 13.1). We begin by defining such actions:

Suppose \( X \) is a finite dimensional compact metric space. Let \( M \) be a compact connected Lie group, and \( \Delta \) a set of nonvanishing real linear functionals on \( \mathbb{R}^k \) up to positive scalar multiple (i.e. elements of \( (\mathbb{R}^k)^* / \sim \), where \( f \sim g \) if and only if \( f = \lambda g \) for some \( \lambda > 0 \)). For each \( \alpha \in \Delta \), we let \( [\alpha] \) denote the a subset of linear functionals belonging to the class \( \alpha \). We abusively let \( \alpha \) denote the “smallest” element of \( [\alpha] \). That is, we write:

\[
[\alpha] = \{\alpha = c_1 \alpha, c_2 \alpha, c_3 \alpha, \ldots, c_\ell \alpha\}, \text{ where } 1 = c_1 < c_2 < \cdots < c_\ell
\]

for some \( \ell = \ell(\alpha) \). Given such an \( [\alpha] \), an element \( a \in \mathbb{R}^k \), and a graded vector space \( V = \bigoplus_{i=1}^\ell E_i \), there exists a uniquely defined isomorphism \( a_\ast : V \to V \), by letting \( a_\ast|E_i \) be defined by scalar multiplication by \( e^{c_i \alpha(a)} \). We call each \( E_i \) an Oseledets space and \( a_\ast \) the graded homothety induced by \( a \).
Given a collection $a_1, \ldots, a_n \in \mathbb{R}^k$, let $\Delta^-(\{a_i\}) = \{ \chi \in \Delta : \chi(a_i) < 0 \text{ for every } i = 1, \ldots, n \}$ (we similarly define $\Delta^+$ as the set of weights with positive evaluations on every $\chi \in \Delta$). A subset of weights $\Phi \subset \Delta$ is called stable if there exists $a \in \mathbb{R}^k$ such that $\Phi \subset \Delta^-(a)$.

We recall the following definitions from [48, Section 5.7]. Let $a \in \mathbb{R}^k$ and let $\Phi$ be a stable subset of weights. We introduce a (not necessarily unique) order on the set $\Phi$. Choose $\mathbb{R}^2 \cong V \subset \mathbb{R}^k$ which contains $a$ and for which $\gamma_1|_V$ is proportional to $\gamma_2|_V$ if and only if $\gamma_1$ is proportional to $\gamma_2$ for all $\gamma_1, \gamma_2 \in \Phi$ (such choices of $V$ are open and dense). Fix some nonzero $\chi \in V^*$ such that $\chi(a) = 0$ ($\chi$ is not necessarily a weight). Then $\beta|_V \in V^* \cong \mathbb{R}^2$ for every $\beta \in \Phi$ and $\Phi|_V = \{ \beta|_V : \beta \in \Phi \}$ is contained completely on one side of the line spanned by $\chi$.

**Definition 10.1.** The ordering $\beta < \gamma$ if and only if $\angle(\chi, \beta|_V) < \angle(\chi, \gamma|_V)$ is called the circular ordering of $\Phi$ (induced by $\chi$ and $V$) and is a total order on $\Phi$. If the set $\Phi$ is understood, we let $|\alpha, \beta| = |\alpha, \beta|_{\Phi}$ denote the set of weights $\gamma \in \Phi$ such that $\alpha \leq \gamma \leq \beta$.

While each $\beta \in \Delta$ is only defined up to positive scalar multiple, this is still well-defined since the circular ordering on $\mathbb{R}^2$ is invariant under orientation-preserving linear maps.

**Definition 10.2.** If $\alpha, \beta \in \Delta$, let $\Sigma(\alpha, \beta) \subset \Delta$ (called the $\alpha, \beta$-cone) be the set of $\gamma \in \Delta$ such that $\gamma = \sigma \alpha + \tau \beta$ for some $\sigma, \tau > 0$. We may identify $\Sigma(\alpha, \beta)$ as a subset of the first quadrant of $\mathbb{R}^2$ by using the coordinates $(\sigma, \tau)$. The canonical circular ordering on $\Sigma(\alpha, \beta) \cup \{\alpha, \beta\}$ is the counterclockwise order in the first quadrant.

The Weyl chambers of $\Delta$ are the connected components of $\mathbb{R}^k \setminus \bigcup_{\alpha \in \Delta} \ker \alpha$. An element $a \in \mathbb{R}^k$ is called regular if $a$ belongs to a Weyl chamber. The following lemma first appeared in [48, Lemma 5.32], and the proof is identical (it is purely linear algebra).

**Lemma 10.3.** Let $\alpha, \beta \in \Delta$ be linearly independent and $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the Weyl chambers such that $\alpha$ and $\beta$ are both negative on every $\mathcal{W}_i$. For each such chamber, choose an arbitrary $a_j \in \mathcal{C}_j$. Then $\Sigma(\alpha, \beta) \cup \{\alpha, \beta\} = \Delta^-(\{a_i\})$.

As noted at the start of this section, we axiomatize the key structures obtained from the dynamics of an Anosov $\mathbb{R}^k \times M$-action. Most of the following properties were deduced in the previous section for smooth Anosov actions. The reason for formulating the definition in the topological setting is twofold: first, it allows us to identify which properties of smooth systems are important and reference those properties quickly. Second, it highlights the breadth of the geometric approach we employ. In particular, we will be able to obtain a purely topological rigidity result without the use of derivatives from these conditions.

**Definition 10.4.** We suppose we have a continuous, locally free action of $\mathbb{R}^k \times M$ on $X$ as described above, sets $\Delta$ and $[\alpha]$ as defined above, and the properties (TA-1)-(TA-7) as described below. We call such an action a leafwise homogeneous topological Anosov action of $\mathbb{R}^k \times M$.

(1) The $\mathbb{R}^k$ action has a dense orbit.
(2) For all $\alpha \in \Delta$, there is a nilpotent group $N_\alpha$ with actions $N_\alpha \lhd X$. These actions have locally bi-Lipschitz evaluation maps onto their images and are globally Hölder.
(3) For all $\alpha \in \Delta$ and $g \in \mathbb{R}^k \times M$, there exist semisimple automorphisms $g_* : N_\alpha \to N_\alpha$ such that for all $u \in N_\alpha$

$$gug^{-1} \cdot x = (g_* u) \cdot x.$$  

Assume also that the $\text{Lie}(N_\alpha)$ automorphism induced by $g_*$ is a graded homothety for $g \in \mathbb{R}^k$.

(Note: we will use the same notation $g_*$ for both the automorphism of the Lie group and Lie algebra. Since these operate on different spaces, the context will make clear which is intended.)
The set of $\mathbb{R}^k \times M$-closed orbits is dense.

If $a_1, \ldots, a_m \in \mathbb{R}^k$ is a list of regular elements, and $\{\chi_1, \ldots, \chi_r\}$ is a circular ordering of $\Delta^{-1}(\{a_i\})$, the evaluation map from $C(\chi_1, \ldots, \chi_r)$ is injective (recall the definition of $C(\chi_1, \ldots, \chi_r)$ from Definition 6.3). Its image is denoted $W^s_{(a_1, \ldots, a_m)}(x)$.

If $a_1, \ldots, a_m \in \mathbb{R}^k$ are regular elements, then $W^s_{(a_1, \ldots, a_m)}(x)$ as defined in (TA-5) is exactly the set of points $y \in X$ such that $d(a_i^t x, a_i^t y) \to 0$ for every $i = 1, \ldots, m$.

For any ordering $\beta = (\beta_1, \ldots, \beta_n)$ of $\Delta$ which lists every weight exactly once, there exist arbitrarily small open sets $U \subset C_{\beta} \times (\mathbb{R}^k \times M)$ containing the trivial path such that for every $x \in X$, the restriction of the evaluation map at $x$ from $U \to X$ is onto a neighborhood of $x$.

We say that the action is genuinely higher rank if it also satisfies:

For every $\alpha \in \Delta$, $(\ker \alpha \times M) \cdot x$ is dense for some $x \in X$.

We say that the action has SRB measures if it also satisfies:

For every pair $\alpha, \beta \in \Delta$, there exists a fully supported (not necessarily ergodic) measure $\mu$ which is invariant under $\ker \beta \times M$ and has absolutely continuous disintegrations along the $N_\alpha$-foliation.

Remark 10.5. Several properties listed here may follow from others. In particular, we believe that properties (TA-4), (TA-6) and (TA-9) can be deduced from the other conditions of the definition under a transitivity assumption. Since the proofs are straightforward when working with smooth systems, we add them as additional conditions here. In Section 13 we deduce that the extension constructed in Section 9 of a smooth Anosov action satisfying conditions (FA-1) and (FA-2) is leafwise homogeneous topological Anosov action with SRB measures. See also Section 4.6 and Remark 14.6 of [48].

Remark 10.6. While many examples of smooth Anosov $\mathbb{R}^k \times M$ satisfy assumptions (TA-1)-(TA-9), some do not. In particular, actions with nontrivial Jordan blocks will fail to satisfy (TA-3) and actions with rank one factors will fail to satisfy (TA-8). Among homogeneous actions, these are the actions which fail to satisfy the conditions and we believe them to be the only ways smooth Anosov $\mathbb{R}^k \times M$-actions fail to satisfy the conditions (although conjecturally they are all still homogeneous). In particular, $\mathbb{R}^k \times M$ actions which are restrictions of actions of semisimple groups are all leafwise homogeneous topological Anosov actions (see Proposition 13.1).

Note that the genuinely higher rank assumptions force that $k \geq 2$ unless we are discussing trivial actions (when $\Delta = \emptyset$ and the $\mathbb{R}^k \times M$ action is transitive). Our main (extremal) technical result of the next few sections follows. We require an additional assumption of integral Lyapunov coefficients, defined after Definition 11.2. This will always hold for $C^2$-actions by Lemma 13.3.

Theorem 10.7. Let $\alpha$ be a genuinely higher-rank leafwise homogeneous topological Anosov action with integral Lyapunov coefficients and SRB measures. Then there exists a Lie group $H$, an embedding of $\mathbb{R}^k \times M$ into $H$, a cocompact lattice $\Lambda \subset H$, and a homeomorphism $\phi : X \to H/\Lambda$ which conjugates the $\mathbb{R}^k \times M$-action to the natural actions by left translation on $H/\Lambda$ by $\mathbb{R}^k \times M$.

Throughout the remainder of Section 10, we assume that $\mathbb{R}^k \times M \curvearrowright X$ is a genuinely higher-rank leafwise homogeneous topological Anosov action unless otherwise stated.
10.1. Basic dynamical properties.

**Lemma 10.8.** Suppose that \( \mathbb{R}^k \times M \curvearrowright X \) is a leafwise homogeneous topological Anosov action. There exists \( \varepsilon_0 > 0 \) such that if \( d(ax, ay) < \varepsilon_0 \) for all \( a \in \mathbb{R}^k \), then \( y \in (\mathbb{R}^k \times M) \cdot x \).

*Proof.* Choose \( \varepsilon_0 \) as in so that if \( d(x, y) < \varepsilon_0 \), then \( y \) is in the image of the maps as in (TA-7). Then fix a regular \( a \in \mathbb{R}^k \) and write \( y = g a_1 a_2 x \), where \( a_1 \) is a product of the stable legs of \( a \) listed in a circular ordering, and \( a_2 \) is a product of the unstable legs of \( a \), listed in a circular ordering. Since \( d(a^t x, a^t y) < \varepsilon_0 \) for \( t \to \pm \infty \), by (TA-5), the endpoints and basepoints of \( a_1 \) and \( a_2 \) expand and contract in opposite behaviors. Therefore, unless \( a_1 \) and \( a_2 \) are both trivial, the points \( x \) and \( y \) will eventually separate by at least \( \varepsilon_0 \), a contradiction. \( \square \)

**Lemma 10.9.** For every \( x \in X \), \( W^\alpha(x) := N_\alpha x \) consists of the set of points \( y \in X \) such that \( d(a^t x, a^t y) \to 0 \) as \( t \to \infty \) for all \( a \) such that \( \alpha(a) < 0 \). Furthermore, \( W^\alpha(x) = \bigcap_{\alpha(a) < 0} W^s_a(x) \), and \( W^s_{(a_1, \ldots, a_m)}(x) = \bigcap W^s_{a_i}(x) \).

*Proof.* The proof is immediate from (TA-6). Indeed, notice that given any coarse Lyapunov exponent \( \alpha \) we may choose a collection \( \{a_1, \ldots, a_m\} \) such that \( \Delta^-(\{a_1, \ldots, a_m\}) = [\alpha] \). Indeed, given any collection which already contains \( [\alpha] \), if \( \Delta^-(\{a_1, a_2, \ldots, a_m\}) \cap [\chi] \neq \emptyset \) for some \( \chi \in \Delta \setminus [\alpha] \), we may remove a weight \( \chi \) by adding a such that \( \alpha(a) < 0 \) and \( \chi(a) > 0 \). Again applying (TA-6) to each \( W^s_{a_i}(x) \) independently, it is clear that their intersection must be \( W^\alpha(x) \). \( \square \)

We now describe a basic operation that is critical to our analysis of the way the group actions of \( N_\alpha \) interact with one another: the geometric commutator. This is in contrast to the infinitesimal commutator, which only exist when the space \( a \) has a smooth structure and the actions of \( N \) are known to be at least \( C^1 \). Even in the smooth setting, our foliations have smooth leaves but are only Hölder transversally, so we instead use a coarser version of the commutator which works even for leafwise homogeneous topological Anosov actions.

Recall that if \( \alpha \) and \( \beta \) are coarse Lyapunov exponents, \( \Sigma(\alpha, \beta) \) is the set of exponents which can be written as \( \sigma \alpha + \tau \beta \), where \( \sigma, \tau > 0 \) (Definition 10.2). Notice that while \( \alpha \) and \( \beta \) are only coarse exponents, the set \( \Sigma \) is still well-defined since we consider all positive linear combinations. Given a group \( G \) and elements \( g, h \in G \), we use the following convention for group commutators:

\[
[u, v] = v^{-1}u^{-1}vu.
\]

**Lemma 10.10.** Fix a leafwise homogeneous topological Anosov action. Let \( \alpha \) and \( \beta \) be non-proportional coarse Lyapunov exponents, \( u \in N_\alpha \) and \( v \in N_\beta \). Then for every \( x \), there exists a unique collection of elements \( w_i := \rho_{\gamma_i}^{\alpha, \beta}(u, v, x) \in N_{\gamma_i} \), where \( \gamma_i \) ranges over all coarse Lyapunov exponents in \( \Sigma(\alpha, \beta) \) listed so that \( (\alpha, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta) \) is the canonical circular ordering, satisfying:

\[
w_n * \cdots * w_2 * w_1 * [u, v] \cdot x = x.
\]

Furthermore, the functions \( \rho_{\gamma_i}^{\alpha, \beta} \) are continuous in all three variables, and satisfy the following equivariance property for any \( g \in \mathbb{R}^k \times M \):

\[
g_* \rho_{\gamma_i}^{\alpha, \beta}(u, v, x) = \rho_{\gamma_i}^{\alpha, \beta}(g_* u, g_* v, g x)
\]

*Proof.* Consider the points \( y = [u, v] \cdot x \), and notice that if \( a \in \mathbb{R}^k \) satisfies \( \alpha(a), \beta(a) < 0 \), then \( d(a^t x, a^t y) \to 0 \). In particular, if \( \{a_1, \ldots, a_m\} \) are elements such that \( \Delta^-(\{a_1, \ldots, a_m\}) = \{\alpha, \gamma_1, \ldots, \gamma_n, \beta\} \) (such a choice of \( \{a_i\} \) exists by Lemma 10.3), then \( y \in W^s_{(a_1, \ldots, a_m)}(x) \). Hence, by (TA-5) there exist elements \( u' \in N_\alpha \), \( v' \in N_\beta \), \( w_i \in N_{\gamma_i} \) such that
Lemma 10.13. Let \( \alpha, \beta \) be a collection of coarse Lyapunov exponents listed in a circular ordering and \( u_i, v_i \in N_{\gamma_i} \). If \( u_1 \cdots u_n \cdot x = v_1 \cdots v_n \cdot x \) for every \( x \in X \), then \( v_i = u_i \) for \( i = 1, \ldots, n \).

Proof. Suppose that \( u_1 \cdots u_r \cdot x = v_1 \cdots v_r \cdot x \) for every \( x \in X \), where \( u_i, v_i \in N_{\alpha_i} \). Then

\[
u_1 \cdots u_r \ast v_r^{-1} \ast \cdots \ast v_1^{-1}\]

stabilizes every point of \( X \). Picking some \( a \in \ker \alpha_r \) such that \( \alpha_i(a) < 0 \) for all \( i = 1, \ldots, r - 1 \) implies that

\[
(a_s u_1) \ast \cdots \ast (a_s u_{r-1}) \ast u_r \ast v_r^{-1} \ast (a_s v_{r-1})^{-1} \cdots (a_s v_1)^{-1}
\]

also stabilizes every point of \( X \). Letting \( \alpha_i(a) \to \infty, i < r \) implies that \( u_r v_r^{-1} \) stabilizes every point of \( X \). Since the action of \( N_{\alpha_1} \) is faithful, \( u_r = v_r \). Iterating this procedure by choosing \( a_i \in \ker \beta_i \) such that \( \alpha_j(a_i) < 0 \) for \( j = 1, \ldots, i - 1 \) inductively shows \( u_i = v_i \), \( i = 1, \ldots, r - 1 \).

\[\square\]

10.2. Groups generated by opposite weights. In this section, we study the interaction of the groups \( N_\alpha \) and \( N_{-\alpha} \), where \( \alpha \in \Delta \) is a weight such that \( -\alpha \in \Delta \). We will show that they fit into a Lie group action, and establish certain structural features.

Lemma 10.14. If \( y \in \overline{\ker \alpha \cdot x} \), then \( \overline{\ker \alpha \cdot y} \subseteq \overline{\ker \alpha \cdot x} \).

Proof. Suppose that \( z \in \overline{\ker \alpha \cdot y} \) and \( a \in \ker \alpha \) be such that \( d(ay, z) < \varepsilon \). Since \( a : X \to X \) is continuous, we may choose \( \delta > 0 \) such that if \( d(y', z') < \delta \), then \( d(ay', z') < \varepsilon \). Then choose \( b \in \ker \alpha \) such that \( d(bx, y) < \delta \). Then by construction, \( d(abx, z) < \varepsilon \) and \( z \in \overline{\ker \alpha \cdot x} \).

The following lemma gives a topological analogue of an ergodic decomposition for the action of \( \ker \alpha \). A topological ergodic decomposition does not always exist, but using the hyperbolic structures, we can obtain a partition into closed sets on which \( \ker \alpha \) is transitive.

Lemma 10.15. Fix \( \alpha \in \Delta \). There is an \( \mathbb{R}^k \)-invariant residual set of points \( x_0 \in X \) such that \((M \times \ker \alpha) \cdot x_0 \) is dense and either

1. \( \ker \alpha \cdot x_0 \) is dense, or
2. \( \ker \alpha \cdot x_0 \) is not dense.
(2) \( \mathcal{F}_\alpha(m) := m \cdot \ker \alpha \cdot x_0 \), \( m \in M \) are a family of closed sets that partition \( X \) and each atom is saturated by \( W^\beta \)-leaves for every \( \beta \in \Delta \) (including \( \beta = \alpha \)). The indexing of the partition \( \{ \mathcal{F}_\alpha(m) : m \in M \} \) by \( m \) depends on \( x_0 \), but the partition itself is independent of the choice of \( x_0 \) from the residual set.

In case (2), for every \( a \in \mathbb{R}^k \) and \( m \in M \), there exists \( m' \in M \) such that \( a \mathcal{F}_\alpha(m) = \mathcal{F}_\alpha(m') \).

Proof. Define \( A_\alpha = \ker \alpha \) and let \( W^{A_\alpha} \) be the accessibility class of a regular element \( a \in A_\alpha \). That is, \( W^{A_\alpha} \) is the set of points reached by finitely many legs in \( W^\beta \), \( \beta \neq \pm \alpha \). Since periodic points are dense by (TA-4), by Lemma 4.21 of [48], the set of points for which \( W^{A_\alpha}(x) \subset A_\alpha \cdot x \) is residual. Since \( W^{A_\alpha}(g \cdot x) = g W^{A_{\alpha}}(x) \) and \( A_\alpha \cdot g \cdot x = g A_\alpha \cdot x \), the same is true for every point along the \( \mathbb{R}^k \times M \)-orbit of such a point \( x \). We may therefore choose \( x_0 \) such that \( W^{A_\alpha}(x_0) \subset A_\alpha \cdot x_0 \) and \( (M \times A_\alpha) \cdot x_0 \) is dense in \( X \). Notice that since the \( M \)-action normalizes the \( N_\beta \) actions and \( M \times A_\alpha \) actions, the same is true for \( m \cdot x_0 \) for every \( m \in M \).

To see that the sets \( \mathcal{F}_\alpha(m) = m \cdot A_\alpha x_0 = A_\alpha m x_0 \) partition the space \( X \), observe first that they cover the space since \( (M \times A_\alpha) \cdot x \) is dense by (TA-8), and \( M \) is compact. We now show that they partition the space. Indeed, assume \( \mathcal{F}_\alpha(m_1) \cap \mathcal{F}_\alpha(m_2) \neq \emptyset \), so that there exists \( y \in A_\alpha m_1 x_0 \cap A_\alpha m_2 x_0 \). We will show that \( m_1 x_0 \in A_\alpha m_2 x_0 \), which implies that \( A_\alpha m_1 x_0 \subset A_\alpha m_2 x_0 \) by Lemma 10.13 (the opposite inclusion follows similarly).

By assumption, \( A_\alpha m_1 x_0 \) each contain their respective \( W^{A_\alpha}(m_i x_0) \). Since \( y \in A_\alpha m_1 x_0 \cap A_\alpha m_2 x_0 \), there exist points \( x_k \in A_\alpha m_1 x_0 \) and \( x' k \in A_\alpha m_2 x_0 \) such that \( x_k, x'_k \to y \). Choose \( k \) large enough so that we may find a connection from \( x_k \) to \( x'_k \) in the following way: \( x_k \) is connected to a point \( z_k \) by moving along a small piece of \( \mathbb{R}^k \times M \)-orbit, a small \( \alpha \)-leaf and then a small \( -\alpha \)-leaf. That is, \( z_k = v u g x_k \), where \( v \in N_\alpha, u \in N_\alpha \) and \( g \in \mathbb{R}^k \times M \) are all small group elements. Then \( z_k \) is connected to \( x'_k \) by the \( W^\beta \) leaves, \( \beta \neq \pm \alpha \). Such a connection is possible when \( x_k \) and \( x'_k \) are close by (TA-7).

Now, by construction, \( z_k \in W^{A_\alpha}(x'_k) \), so since \( W^{A_\alpha}(x'_k) \subset A_\alpha x'_k = A_\alpha m x_0 \) by the genericity assumption, we may find \( b_k \in A_\alpha \) such that \( b_k m x_0 \to z_k \). But if \( x_k = a_k m x_0 \), then \( z_k = v u g a_k m x_0 \), where \( u \in N_\alpha, v \in N_\alpha \) and \( g \in \mathbb{R}^k \times M \) become smaller with \( k \). Since \( a_k \in \ker \alpha, a_k^{-1} z_k = v u g m x_0 \). Therefore, given \( \varepsilon > 0 \), by choosing \( k \) sufficiently large, we guarantee that \( d(a_k^{-1} z_k, m x_0) < \varepsilon \). Fixing such a \( k \), we may find \( \delta > 0 \) such that if \( d(z, z_k) < \delta \), then \( d(a_k^{-1} z, m x_0) < \varepsilon \). Since \( b_k m x_0 \to z_k \), we may choose \( n \) so large to obtain that \( d(b_k m x_0, z_k) < \delta \). Then by construction, \( d(a_k^{-1} b_k m x_0, m x_0) < \varepsilon \). Therefore, since \( \varepsilon \) was arbitrarily small, and \( a_k^{-1} b_k \in A_\alpha \), \( m x_0 \in A_\alpha m x_0 \).

Finally, we claim that every coarse Lyapunov leaf is contained in an atom of \( \mathcal{F}_\alpha \). This is true by choice of \( x_0 \) for \( W^\beta \), \( \beta \neq \pm \alpha \), we now show it for \( W^{\pm \alpha} \). Indeed, notice that by definition, if \( a \in \mathbb{R}^k \), then \( a \cdot m \cdot A_\alpha x_0 = m \cdot A_\alpha (a \cdot x_0) \) and that if \( y \in W^{\alpha}(x_0) \), then \( d(a^n \cdot x_0, a^n \cdot y) \to 0 \) (similarly for \( W^{-\alpha}(x_0) \)). If the points were on different \( \mathcal{F}_\alpha \)-partition elements, they would remain a fixed positive distance from one another (since the atoms continue to differ by a fixed, non-identity \( m \in M \)), a contradiction.

Finally, notice that if we had chosen \( x_1 \) and \( x_0 \) from the residual set of points which contain \( W^{A_\alpha} \) to define the orbit closures, the proof would not change. In particular, the partition depends only on picking orbit closures which contain these sets, so the partition is into "maximal \( \ker(\alpha) \)-orbit closures." In particular, \( a \mathcal{F}_\alpha(m) \) must be another partition element, proving the last claim of the lemma.

\( ^1 \)By "maximal orbit closures" property it is obvious that for any \( a \in \mathbb{R}^k, a \mathcal{F}_\alpha(m) \) is contained in some partition element since \( a \mathcal{F}_\alpha(m) \) is an orbit closure of some point, then considering the \( -\alpha \) action we get the equality.
Fix a weight $\alpha \in \Delta$ such that $-\alpha \in \Delta$ as well.

**Lemma 10.15.** There exists a connected Lie group $G_\alpha$ and a continuous group action $G_\alpha \curvearrowright X$ such that

1. $\text{Lie}(G_\alpha) \cong \text{Lie}(N_\alpha) \oplus g_0 \oplus \text{Lie}(N_{-\alpha})$, where $g_0 \subset \mathbb{R}^k \oplus \text{Lie}(M)$,
2. the inclusion of $\text{Lie}(N_{\pm\alpha})$ in $\text{Lie}(G_\alpha)$ induces a local isomorphism from $N_{\pm\alpha}$ onto its image,
3. the action of $\exp_{G_\alpha}(\text{Lie}(N_{\pm\alpha}))$ coincides with the existing action of $N_{\pm\alpha}$,
4. the subgroups $\exp_{G_\alpha}(\text{Lie}(N_\alpha))$ and $\exp_{G_\alpha}(\text{Lie}(N_{-\alpha}))$ generate $G_\alpha$, and
5. the action of $\exp_{G_\alpha}(g_0)$ coincides with the existing action as a subgroup of $\mathbb{R}^k \times X$.

**Proof.** Consider the stabilizer of a point, $C_\alpha(x)$, under the action of the free product $N_\alpha * N_{-\alpha}$. Then $\ker \alpha$ acts trivially on every leg), $C_\alpha(x) = a_\ast C_\alpha(x) = C_\alpha(a \cdot x)$. Choose $x_0$ as in Lemma 10.14, and let $F_\alpha$ be the corresponding partition. Since the atoms of $F_\alpha$ are saturated by coarse Lyapunov leaves, $N_\alpha * N_{-\alpha}$-orbits are contained in a single $\ker \alpha$-orbit closure, so the cycles at any point contain those of $m \cdot x_0$ for some $m \in M$. Therefore, by Theorem 6.8 at each $m \in M$ the group $G_\alpha(m) = (N_\alpha * N_{-\alpha})/C_\alpha(m \cdot x_0)$ is Lie, and has a canonical Hölder action $G_\alpha(m) \curvearrowright F_\alpha(m)$ into which the group actions of $N_\alpha$ and $N_{-\alpha}$ canonically embed.

We wish to show that the group $G_\alpha(m)$ is independent of $m \in M$, and that there is a global group action $G_\alpha \curvearrowright X$. Recall that if $g \in \mathbb{R}^k \times M$, $g$ normalizes the $N_{\pm\alpha}$-actions. Let $g_\alpha$ denote the induced automorphism of $N_\alpha * N_{-\alpha}$. Any globally-defined map preserving the coarse Lyapunov foliations will take cycles to cycles, so $\mathcal{C}_\alpha(g \cdot n) = g_\alpha \mathcal{C}_\alpha(n)$. Since $g_\alpha$ is an automorphism of $N_\alpha * N_{-\alpha}$ taking $C_\alpha(m)$ to $C_\alpha(g \cdot m)$, $g_\alpha$ induces an isomorphism $g_\alpha : G_\alpha(m) \to G_\alpha(g \cdot m)$.

As a remark, note that this determines the isomorphism class of $G_\alpha(m)$, but that this is insufficient for our purposes. To obtain a group action on $X$, we need to know the cycles $C_\alpha(m)$ are constant, which is to say that the generating relations for the groups are the same. This is equivalent to showing that the maps $\beta_m : \text{Lie}(N_\alpha) \otimes \text{Lie}(N_{-\alpha}) \to \text{Lie}(G_\alpha)$ defined by $\beta_m(X,Y) = [X,Y]_{\text{Lie}(G_\alpha(m))}$ are independent of $m$ in some sense. It is possible that the groups $G_\alpha(m)$ have a graded family of isomorphisms, even though they do not induce the same actions. In fact, we will show that each $\text{Lie}(G_\alpha(m))$ sits inside a global vector space independent of $m$ and that the maps $\beta_m$ are constant there, proving constancy of relations and hence of $C_\alpha(m)$. This is made more precise below. So we now consider the Lie group structure of $G_\alpha(m)$. Notice that $N_{\pm\alpha}$ has a grading that corresponds to the Oseledets splitting, $\text{Lie}(N_\alpha) = E^u \oplus E^{u\alpha} \oplus \cdots \oplus E^{u_{-1}\alpha}$ and $\text{Lie}(N_{-\alpha}) = E^{-d_1\alpha} \oplus E^{-d_2\alpha} \oplus \cdots \oplus E^{-d_2\alpha}$. Suppose that $Z \in E^{c_0\alpha}$ and $Y \in E^{-d_\alpha}$. Then $\exp_{G_\alpha(m)}(t[Z,Y]_{G_\alpha(m)})$ is a Hölder curve in $X$, but a priori only a Hölder curve (recall that by (TA-2), the evaluation maps of the $N_{\pm\alpha}$ are assumed to be locally bi-Lipschitz, but the global actions of $N_{\pm\alpha}$, and hence $G_\alpha(m)$, are only Hölder). For each $a \in \mathbb{R}^k$, $a_\ast$ is an isomorphism between $G_\alpha(m)$ and $G_\alpha(m')$, where $m'$ is such that $aF_\alpha(m) = F_\alpha(m')$. Let us consider the asymptotic behavior $\exp_{G_\alpha(m)}(t[Z,Y]_{G_\alpha(m)})$. If $x \in F_\alpha(m)$ and $d_j > c_i$, then as $\alpha(a) \to \infty$:

$$d(ax, a\exp_{G_\alpha(m)}(t[Z,Y]_{G_\alpha(m)})x) = d(\exp_{G_\alpha(m')}((ta_\ast [Z,Y]_{G_\alpha(m)})ax, ax)$$

$$= d(\exp_{G_\alpha(m')}((te^{(c_i-d_j)\alpha(a)})[Z,Y]_{G_\alpha(m')}ax, ax)$$

$$\to 0.$$
From this we conclude that if \( c_i \neq d_j \), \([Z,Y]_{G_{\alpha}(m)} \in \text{Lie}(N_{\pm \alpha}) \). Assume that \( c_i > d_j \), so \([Z,Y]_{G_{\alpha}(m)} \in \text{Lie}(N_{\alpha}) \) (the opposite case is identical). Choose any \( b \in \mathbb{R}^k \) such that \( \alpha(b) > 0 \), and let \( m \in M \). Then \( bF_{\alpha}(m) = F_{\alpha}(gm) \) for some \( g \in M \), and \( bg^{-1} \) fixes \( F_{\alpha}(m) \). Choose any norm on \( \text{Lie}(G_{\alpha}(m)) \) such that the action of \( M \) is isometric. Then \((bg^{-1})_{\ast} \) is an automorphism of \( G_{\alpha}(m) \) and if \( Z' = g_{\ast}Z \) and \( Y' = g_{\ast}Y \). Then, using that the \( M \)-action is isometric:

\[
(bg^{-1})_{\ast}[Z,Y]_{G_{\alpha}(m)} = [(bg^{-1})_{\ast}Z,(bg^{-1})_{\ast}Y]_{G_{\alpha}(m)}, \quad \text{so}
\]

\[
\| (bg^{-1})_{\ast}[Z,Y]_{G_{\alpha}(m)} \| = \left| e^{(c_i-d_j)\alpha(b)} [Z',Y']_{G_{\alpha}(gm)} \right| \]

Moreover since \((bg^{-1})_{\ast} \) preserves the Oseledets splitting in \( \text{Lie}(N_{\pm \alpha}) \), by similar proof we have

\[
\| (bg^{-1})_{\ast}[Z,Y]_{G_{\alpha}(m)} \| = e^{(c_i-d_j)\alpha(b)} \left| [Z,Y]_{G_{\alpha}(m)} \right|. \]

Since \([Z,Y]_{G_{\alpha}(m)} \in \text{Lie}(N_{\alpha}) \), this implies that \([Z,Y]_{G_{\alpha}(m)} \) is in the direct sum of generalized eigenspaces of \((bg^{-1})_{\ast} \) with eigenvalues whose moduli are \( e^{(c_i-d_j)\alpha(b)} \) and hence that \([Z,Y]_{G_{\alpha}(m)} \in E^{(c_i-d_j)\alpha} \).

In the case when \( c_i = d_j \), then the flow generated by \([Z,Y] \) commutes with the \( \mathbb{R}^k \) action, and therefore its orbits are contained in the \( \mathbb{R}^k \times M \)-orbits by Lemma 10.8. Define \( \varphi(x) \) to be the element of \( A \cdot M \) such that \( \exp([Z,Y]) \cdot x = \varphi(x) \cdot x \). Then \( \varphi \) is constant on \( \text{ker} \) orbits, and hence \( \varphi(x) \) is constant on \( F_{\alpha}(m) \). Therefore, the flow generated by \([Z,Y] \) is a one-parameter subgroup of \( \mathbb{R}^k \times M \) on \( F_{\alpha}(m) \). Therefore, \( \text{Lie}(G_{\alpha}(m)) \) has a canonical vector space embedding into \( V = \text{Lie}(N_{\alpha}) \oplus \text{Lie}(\mathbb{R}^k \times M) \oplus \text{Lie}(N_{-\alpha}) \) (note that for now, \( V \) is only a vector space since there is no well-defined Lie bracket defined on all of \( V \)).

Finally, we wish to show that if \( Z \in \text{Lie}(N_{\alpha}) \) and \( Y \in \text{Lie}(N_{-\alpha}) \), then \([Z,Y]_{G_{\alpha}(m)} = [Z,Y]_{G_{\alpha}(m')} \) for any \( m, m' \in M \) (where we think of each \( \text{Lie}(G_{\alpha}(m)) \) as a vector subspace of \( V \)). Notice that by bilinearity of the Lie bracket, we may assume that \( Z \in E^{c_i\alpha} \) and \( Y \in E^{-d_j\alpha} \). We first prove the claim when \( F_{\alpha}(m') = aF_{\alpha}(m) \). With these assumptions, using the fact that \([Z,Y] \in E^{(c_i-d_j)\alpha(a)} \) if \( c_i \neq d_j \), or otherwise commutes with the \( \mathbb{R}^k \)-action, we see that

\[
[Z,Y]_{G_{\alpha}(m)} = (a^{-1})_{\ast}[a_{\ast}Z,a_{\ast}Y]_{G_{\alpha}(m')}
\]

\[= e^{(d_j-c_i)\alpha(a)} [e^{c_i\alpha(a)} Z, e^{-d_j\alpha(a)} Y]_{G_{\alpha}(m')} \]

\[= [Z,Y]_{G_{\alpha}(m')} . \]

Therefore, the group \( G_{\alpha}(m) \) is constant on \( \mathbb{R}^k \)-orbits, and hence \( C_{\alpha}(x_0) \), the space of cycles in \( N_{\alpha} \) and \( N_{-\alpha} \) is constant on a dense set by (TA.1). The sets \( C_{\alpha}(x) \) are semi-continuously varying in the following sense: if \( \lim_{n \to \infty} x_n = x \), and \( \sigma_n \in C_{\alpha}(x_n) \) is a sequence of cycles converging to a cycle \( \sigma \), then \( \sigma \in C(x) \). By density of the \( \mathbb{R}^k \) orbit and this semicontinuity, \( C_{\alpha}(x_0) \) is contained in \( C_{\alpha}(x) \) for every \( x \in X \). Therefore, there is a Lie group \( G_{\alpha} = (N_{\alpha} * N_{-\alpha})/C_{\alpha}(x_0) \) which acts on the total space \( X \), as described.

11. Polynomial forms of geometric commutators

**Theorem 11.1.** If \( \mathbb{R}^k \times M \curvearrowright X \) is a leafwise homogeneous topological Anosov action satisfying the assumptions of Theorem 10.7, then the functions \( \rho^{\alpha,\beta}(u,v,x) \) are independent of \( x \).
11.1. Reduction to Oseledets subspaces. If $\alpha$ is a coarse Lyapunov exponent, recall from the definition of a topological Anosov action that $\text{Lie}(N_\alpha)$ splits as a direct sum of Oseledets subspaces. That is, $\text{Lie}(N_\alpha) = E^{\geq\alpha} \oplus \cdots \oplus E^{\leq\alpha}$. Let $e(u) = \exp(u)$ for ease of notation, and log denote its inverse (which exists because $N_\alpha$ is nilpotent). Then by (TA-3), if $u \in E^{\geq\alpha}$, $a \in \mathbb{R}^k$,}

$$a_* e(u) = e(e^{\alpha a}(u)).$$

**Definition 11.2.** Given Oseledets spaces $E^{\geq\alpha}$ and $E^{\leq\beta}$, let $ho^{\alpha\beta} : E^{\geq\alpha} \times E^{\leq\beta} \times X \to \text{Lie}(N_\gamma)$ be defined by

$$\rho^{\alpha\beta}(u, v, x) = \pi_{m}(\log \rho^{\alpha\beta}(e(u), e(v), x)),$$

where $\pi_{m} : \text{Lie}(N_\gamma) \to E^{\geq\alpha} \times E^{\leq\beta}$ is the projection onto the corresponding Oseledets space induced by the Oseledets splitting.

We now turn to the last technical assumption which appears in Theorem 10.7. If $c^{\gamma}_{\alpha\beta} = \sigma c^{\alpha \gamma} + \tau c^{\beta \gamma}$ for some $\sigma, \tau > 0$, we call $\sigma$ and $\tau$ the Lyapunov coefficients of $c^{\gamma}_{\alpha\beta}$ with respect to $c^{\alpha \gamma}$ and $c^{\beta \gamma}$. We say that an action has integral Lyapunov coefficients if $\rho^{\alpha\beta} \equiv 0$ whenever both Lyapunov coefficients are less than 1.

**Lemma 11.3.** If for every pair of Oseledets subspaces $E^{\geq\alpha}, E^{\leq\beta}$ with $\alpha$ and $\beta$ nonproportional, the functions $\rho^{\alpha\beta}$ are independent of $x$, then the functions $\rho^{\alpha\beta}$ are independent of $x$ for every $\alpha, \beta \in \Delta$.

**Proof.** Let $u \in N_\alpha$ and $v \in N_\beta$. Then we may write $u = e(u_1)e(u_2)\cdots e(u_{\ell(\alpha)})$ and $v = e(v_1)e(v_2)\cdots e(v_{\ell(\beta)})$, where $u_i \in E^{\geq\alpha}$ and $v_j \in E^{\leq\beta}$. We wish to compute $[u, v]$ using commutator relations coming from commutators from the exponentials of Oseledets subspaces. Using their expression in terms of Oseledets subspaces, we may write

$$[u, v] = e(u_1)\cdots e(u_{\ell(\alpha)}) \cdot e(v_1)\cdots e(v_{\ell(\beta)}) \cdot e(-u_{\ell(\alpha)})\cdots e(-u_1) \cdot e(-v_{\ell(\beta)})\cdots e(-v_1).$$

We wish to push the $e(-u_{\ell(\alpha)})$ term past the $e(v_i)$-terms to cancel with $e(u_{\ell(\alpha)})$. Since we have constant commutator relations among the exponentials of Oseledets spaces, we may do so, but accumulate their commutators along the way. We may choose to put them on the left or right whenever we commute, we choose to put them on the left. That is, we write $e(v_i)e(-u_{\ell(\alpha)}) = ge(-u_{\ell(\alpha)})e(v_i)$, where $g$ is a geometric commutator of $e(-v_i)$ and $e(u_{\ell(\alpha)})$ determined by the $\rho$-function, which is constant. Recall that $g$ is a product of elements from the coarse Lyapunov groups strictly between $\alpha$ and $\beta$, which are constant by the assumption of the lemma.

The element $g$ itself may be decomposed as a product of Oseledets subspaces, which we may push past $e(v_1)\cdots e(v_{\ell-1})$ by the same method, accumulating new terms on the left in the process between the weights of $g$ and $\beta$. These terms are independent of $x$ since we only commute terms coming from exponentials of Oseledets spaces. Since $g$ will always take values in weights between $\alpha$ and $\beta$, there is a clear induction on $\# \Sigma(\alpha, \beta)$, which terminates since there are only finitely many such weights. We may express $[u, v]$ as

$$[u, v] = e(u_1)\cdots e(u_{\ell(\alpha)})ge(-u_{\ell(\alpha)})e(v_1)\cdots e(v_{\ell(\beta)}) \cdot e(-u_{\ell(\alpha)-1})\cdots e(-u_1) \cdot e(-v_{\ell(\beta)})\cdots e(-v_1),$$

where $g$ is a product of exponentials of Oseledets space from coarse Lyapunov exponents that lie strictly between $\alpha$ and $\beta$, which is still independent of $x$. Using the same procedure as above, we
may push all \( g \)-terms to the far left, and cancel the \( e(u_{\ell}(\alpha)) \)-term with its inverse to obtain the following expression, with \( g' \) independent of \( x \):

\[
[u, v] = g' e(u_1) \cdots e(u_{\ell}(\alpha) - 1) \cdot e(v_1) \cdots e(v_{\ell}(\beta)) \cdot e(-u_{\ell}(\alpha) - 1) \cdots e(-u_1) \cdot e(-v_{\ell}(\beta)) \cdots e(-v_1).
\]

We now repeat this process until we have canceled each \( e(u_i) \)-term which then further allows for the cancelling of all \( e(v_j) \)-terms, leaving only a product exponentials of Oseledets spaces from \( \Sigma(\alpha, \beta) \), which is independent of \( x \).

We have now reduced \([u, v]\) to a product of exponentials of Oseledets spaces of \( \gamma \), where \( \gamma \in \Sigma(\alpha, \beta) \). To deduce that \( \rho^{\alpha, \beta}(u, v, x) \) is independent of \( x \), we write \( \Sigma(\alpha, \beta) \) in a circular ordering.

Then push all of the exponential terms from each the Oseledets subspace of the first coarse exponent to the left. Since we have constant relations among the exponentials of the Oseledets spaces, this is possible, accumulating their commutators on the right. The corresponding reduction yields an element, written in circular ordering, which is independent of \( x \). Uniqueness follows from Lemma 10.12.

\[ \Box \]

11.2. Setting up the inductions. We prove Theorem 11.1 using Lemma 11.3 by showing each \( \hat{\rho}^{\alpha, \beta}_{\alpha, \gamma} \) is constant. We use use three inductions. The outermost induction is on \( \#\Sigma(\alpha, \beta) \), we call this Induction I. In each step of the induction, we will show for Lyapunov exponents \( c_i^\alpha \alpha \) and \( c_j^\beta \beta \),

\[
(11.2) \quad \text{if } \hat{\rho}^{\alpha, \beta}_{\alpha, \gamma} \neq 0, \text{ then } c_m^\gamma = c_i^\alpha + \tau c_j^\beta \text{ for some } \tau, \sigma \in \mathbb{Z}_+, \text{ and}
\]

\[
(11.3) \quad \text{if } c_m^\gamma = c_i^\alpha + \tau c_j^\beta, \text{ then } \hat{\rho}^{\alpha, \beta}_{\alpha, \gamma}(u, v, x) \text{ is a polynomial which is } \sigma \text{-homogeneous in } u, \tau \text{-homogeneous in } v \text{ and independent of } x.
\]

We first state a key consequence of the induction. Let \( P_{[\alpha, \beta]} \) denote the group freely generated by the groups \( N_\gamma, \gamma \in \Sigma(\alpha, \beta) \cup \{ \alpha, \beta \} \).

**Lemma 11.4.** If (11.2) and (11.3) hold for all linearly independent \( \alpha, \beta \) such that \( \#\Sigma(\alpha, \beta) \leq n \), then for any such \( \alpha, \beta \),

- the action of \( P_{[\alpha, \beta]} \) factors through the action of a nilpotent Lie group \( N_{[\alpha, \beta]} \),
- \( \text{Lie}(N_{[\alpha, \beta]}) = \bigoplus_{\gamma \in \Sigma(\alpha, \beta) \cup \{ \alpha, \beta \}} \text{Lie}(N_\gamma) \), and
- the family of automorphisms \( \alpha^*_\gamma \) defined in (TA-3) descend to an automorphism of \( N_{[\alpha, \beta]} \).

**Proof.** Write \( \Sigma(\alpha, \beta) \cup \{ \alpha, \beta \} = \{ \alpha = \gamma_1, \gamma_2, \ldots, \gamma_r = \beta \} \) in the induced circular ordering. Let \( G \) denote the factor of the group \( P_{\alpha, \beta} \) modulo the commutator relations (10.3). By (11.3) and Lemma 11.3, they are independent of \( x \). We first claim that every \( \rho \in G \) can be written as

\[
(11.4) \quad u_1 * \cdots * u_r,
\]

where each \( u_i \in N_{\gamma_i} \) is unique. Indeed, any \( \rho \in G \) can be written as \( \rho = v_1 * \cdots * v_k \) (where each \( v_i \in N_{\beta_{\gamma_i}} \)).

We may begin by pushing all of the terms from the \( \beta_1 \) component to the left. We do this by looking at the first term to appear with \( \beta_1 \). Each time we pass it through, we may accumulate some \([u^{\beta_1}, v^{\beta_1}]\) which may be rewritten as \( \rho(u^{\beta_1}, v^{\beta_1}) \), having no \( \beta_1 \) terms, since we have quotiented by
the commutator relations (10.3). So we have shown that in \( G \), \( \rho \) is equal to \( u_1 \cdot \rho' \), where \( \rho' \) consists only of terms without \( \beta_1 \), and \( u_1 \in N_{\beta_1} \).

We now proceed inductively. We may in the same way push all \( \beta_2 \) terms to the left. Notice now that each time we pass through, the “commutator” \( \rho(u^{\beta_1}, v^{\beta_1}) \), \( j \geq 3 \) has no \( \beta_1 \) or \( \beta_2 \) terms. Iterating this process yields the desired presentation of \( \rho \).

Thus, every element of \( G \) has a unique presentation of the form (11.4), where the uniqueness follows from Lemma 10.12. The map which assigns an element \( \rho \) to such a presentation gives a an injective map from \( G \) to \( \prod N_{\beta_1} \) (but the map may not be a homomorphism of groups). By Lemma 6.4, it will be continuous once its lift to \( P_{\alpha,\beta} \) is continuous. In each combinatorial cell \( C_{\beta_1} \), the map is given by composition of the group multiplications in each \( N_{\beta_1} \) and the functions \( \rho^{\alpha,\beta}((.,.) \) evaluated on cell coordinates, which are continuous. Therefore, the lift is continuous, so the map from \( G \) is continuous.

Therefore, there is an injective continuous map from \( G \) to a finite-dimensional space, and \( G \) is a Lie group by Theorem 6.9. Fix \( a \) which contracts every \( \beta_i \). The fact that \( G \) is nilpotent follows from the fact that it has a contracting automorphism. \( \square \)

Each step of the outer induction on \( \# \Sigma(\alpha, \beta) \) will be proved using two further inductions. We introduce a partial order on \( \{(i, j) : 1 \leq i \leq \ell(\alpha), 1 \leq j \leq \ell(\beta)\} \) by saying that \((i_1,j_1) \leq (i_2,j_2)\) if and only if \( i_1 \leq i_2 \) and \( j_1 \leq j_2 \). The second induction will utilize this partial order: we will show (11.2) and (11.3) for a pair \( c_i^\alpha \) and \( c_j^\beta \) assuming that we have concluded it for all choices of \( c_m^\gamma \) and all \( c_i^\alpha, c_j^\beta \) such that \((i,j) \leq (i',j')\) and \((i,j) \neq (i',j')\). The base of this induction will then be \((\ell(\alpha), \ell(\beta))\), the unique maximal element. It is clear from the structure of the partial order that such an induction will exhaust all choices of \((i,j)\). We call this induction Induction II.

Given \( c_i^\alpha \) and \( c_j^\beta \), let

\[
[c_i^\alpha, c_j^\beta] = \left\{ c_m^\gamma : \rho_{c_m^\gamma} c_i^\alpha c_j^\beta \neq 0 \right\} \quad \text{and} \quad \Omega_l = \left\{ \sigma c_i^\alpha + \tau c_j^\beta : \sigma + \tau = l, \sigma, \tau > 0 \right\} \cap \{ c_m^\gamma : \gamma \in \Delta \}.
\]

Then there are finitely many values \( l_0 < l_1 < \cdots < l_m \) such that

\[
\Sigma(\alpha, \beta) = \bigcup_{p=0}^{m} \Omega_{l_p}.
\]

Given a subset \( S \) of Lyapunov exponents, and a weight \( c_i^\alpha \in \Delta \), let

\[
[c_i^\alpha, S] = \left\{ c_m^\gamma : \rho_{c_m^\gamma} c_i^\alpha c_j^\beta \neq 0 \text{ for some } c_j^\beta \in S \right\}.
\]

Let \( P_{(\alpha, \beta)} \), \( P_{(\alpha, \beta)} \), and \( P_{(\alpha, \beta)} \) denote the groups freely generated by the weights of \( \Sigma(\alpha, \beta) \), \( \Sigma(\alpha, \beta) \cup \{\alpha\} \) and \( \Sigma(\alpha, \beta) \cup \{\beta\} \), respectively. By the induction hypothesis, the action of each group factors through Lie groups which we denote by \( N_{(\alpha, \beta)} \), \( N_{(\alpha, \beta)} \) and \( N_{(\alpha, \beta)} \), respectively. Let

\[
F_1 = \bigoplus_{\gamma \in \Omega_1} E_{c_m^\gamma}, \quad \text{so that} \quad \text{Lie}(N_{(\alpha, \beta)}) = \bigoplus_{\alpha=1}^{m} F_{l_\alpha}.
\]

It now suffices to show Claims 11.2 and 11.3 for each weight \( \chi \in \Omega_{l_p} \). We will do this using a final induction on \( p \), starting from \( p = 0 \). We call this induction Induction III. We summarize each induction below, noting that the proof of (11.2) and (11.3) runs in a lexicographical ordering: for each step of Induction I, we do every step of Induction II, and for each of Induction II, we do every step of Induction III:

- **Induction I:** \( \# \Sigma(\alpha, \beta) \), base case \( \# \Sigma(\alpha, \beta) = 0 \).
• Induction II: Partial order on \((c_i^\alpha, c_j^\beta)\), base case maximal element, induction moves downward.

• Induction III: \(c_m^\gamma \in \Omega_l\), induction on \(l\), base case \(l = l_0\) smallest coefficients.

11.3. Proving the inductive steps. Fix \(l\), and given \(g \in N_{(\alpha, \beta)}\), write \(log \ g = \hat{g} + \tilde{g} + \check{g}\), where \(\hat{\check{g}}\) is the component of \(log \ g\) from the weight spaces of \(\Omega_{l'}\), \(l' < l\), \(\tilde{\check{g}}\) is the component from the weight spaces of \(\Omega_l\) and \(\hat{\tilde{g}}\) is the component from the weight spaces of \(\Omega_{l'}\), \(l' > l\).

By Induction I and Lemma 11.4, there are groups \(N_{(\alpha, \beta)}\), \(N_{(\alpha, \beta)}\) and \(N_{(\alpha, \beta)}\) generated by the weights \(\Sigma(\alpha, \beta)\), \(\{\alpha\} \cup \Sigma(\alpha, \beta)\) and \(\{\beta\} \cup \Sigma(\alpha, \beta)\). The following important lemma uses these group structures to describe the action of \(N_\alpha\) on \(N_{(\alpha, \beta)}\).

**Lemma 11.5.** If \(u \in E^{c_{\alpha}}\), then the conjugation action of \(exp(u)\) on \(N_{(\alpha, \beta)}\) preserves \(N_{(\alpha, \beta)}\). Furthermore, \(ad(u)\) a nilpotent automorphism such that \(ad(u)(F_i) \subset F_{i+1}\).

**Proof.** That \(ad(u)\) is unipotent follows from the fact that \(ad(u)(F_i) \subset F_{i+1}\), which we now show. Assume \(c_m^\gamma \in \Omega_l\). The Lie group \(N_{(\alpha, \beta)}\) also carries an automorphism \(a_\ast\) which expands \(E^{c_{\alpha}}\) by \(e^{c_\alpha(\alpha)}\) and \(E^{c_\alpha}\) by \(e^{c_\alpha(\alpha)}\). Therefore, \([E^{c_\alpha}, E^{c_\alpha}]\) consists of vectors which are expanded by \(e^{c_\alpha(\gamma)(\gamma)} e^{c_\alpha(\gamma)(\gamma)}\). Since \(a\) is arbitrary we conclude that \([E^{c_\alpha}, E^{c_\alpha}]\) \(\subset E^{c_\alpha} + E^{c_\alpha}\). Since \(c_m^\gamma \in \Omega_l\), the result follows.

A completely symmetric version holds for \(u \in E^{c_{\beta}}\). The following are immediate consequences of Lemma 11.5, and the Baker-Campbell-Hausdorff formula.

**Corollary 11.6.** If \(u \in E^{c_{\alpha}} \cup E^{c_{\beta}}\), and \(g = \exp(\hat{g} + \tilde{g} + \check{g}) \in N_{(\alpha, \beta)}\), let \(v = \exp(u) * g * \exp(-u)\). Then:

1. \(v \in N_{(\alpha, \beta)}\).
2. \(\tilde{v}\) is a polynomial of \(u\) and \(\hat{g}\).
3. \(\check{v}\) takes the following form:
   \[
   \check{v} = p(u, \hat{g}) + \tilde{g}
   \]
   for some polynomial \(p\) such that \(p(0, \cdot) = p(\cdot, 0) = 0\), and
4. \(\hat{v}\) is a polynomial in \(u\) and \(\log g\).

**Corollary 11.7.** If \(g_1, g_2 \in N_{(\alpha, \beta)}\) and we write \(g_3 = g_1 g_2\), then

\[
\hat{g}_3 = p_1(\hat{g}_1, \hat{g}_2) \quad \tilde{g}_3 = \tilde{g}_1 + \tilde{g}_2 + p_2(\hat{g}_1, \hat{g}_2) \quad \check{g}_3 = p_3(g_1, g_2),
\]
for some polynomials \(p_1, p_2\) and \(p_3\).

Notice that \(\rho^{\alpha, \beta}(u, v, x)\) is a formal product of elements from the groups \(N_\gamma, \gamma \in \Sigma(\alpha, \beta)\), written in a circular ordering. Therefore, it represents a unique element of \(N_{(\alpha, \beta)}\), which we abusively denote with the same notation. Fix \(c_m^\gamma \in \Omega_l\), and define a function \(r_{c_m^\gamma} c_{\alpha, \beta}(u, v, x)\) to be the \(E^{c_m^\gamma}\)-component of \(\log \rho^{\alpha, \beta}(u, e(v), x)\).

**Corollary 11.8.** \(r_{c_m^\gamma} c_{\alpha, \beta}(u, v, x) = \check{\rho}_{c_m^\gamma} c_{\alpha, \beta}(u, v, x) + p(u, v)\) for some polynomial \(p : E^{c_{\alpha}} \oplus E^{c_{\beta}} \to E^{c_m^\gamma}\) independent of \(x\).

**Proof.** The definitions of \(r_{c_m^\gamma} c_{\alpha, \beta}\) and \(\check{\rho}_{c_m^\gamma} c_{\alpha, \beta}\) are quite similar, the only difference being that \(\check{\rho}_{c_m^\gamma} c_{\alpha, \beta}\) uses the \(c_m^\gamma\)-component of \(\log \rho^{\alpha, \beta}\), while \(r_{c_m^\gamma} c_{\alpha, \beta}\) regards \(\rho^{\alpha, \beta}\) as an element of \(N_{(\alpha, \beta)}\), then takes the log and the \(c_m^\gamma\)-component. Therefore, we wish to compare the standard exponential
coordinate system on $N_{(\alpha, \beta)}$ and the coordinate system given by $(v_1, \ldots, v_n) \mapsto e(v_1) \ldots e(v_n)$, where $v_i \in N_{\gamma_i}$ and the $\gamma_i$ are listed in a circular ordering. By Corollary 11.7, the $c^\gamma_j\gamma$-components will differ only by polynomials that depend on the terms of the commutator coming from $\Omega_{l'}, l' < l$ (ie, the $i$-terms). Since by Induction III, such terms are polynomials, $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}$ and $\rho_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}$ differ by a polynomial in $u$ and $v$ which is independent of $x$. \hfill \square

Figure 1 gives an example of the structures above. We give a description of the features available for one step of the induction for this particular example. We assume that we are at the stage of the induction to analyze $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}$. Then $\Omega_0 = \{c_1^{\gamma_1}\gamma_1\}$, $\Omega_1 = \{c_2^{\gamma_2}\gamma_2\}$ and $\Omega_2 = \{c_2^{\gamma_1}, c_2^{\gamma_2}\gamma_2\}$, since they are the intersections of lines parallel to the one passing through $c_2^{\beta_j}$ and $c_2^\alpha$. At the first stage of the innermost induction, we would analyze only the function $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}$.

The crucial feature for the base step is that, by Lemma 11.5, no terms in the Oseledets space $c_1^{\gamma_1}\gamma_1$ can appear by commuting the $c^\beta_j\gamma_j$ with another weight in the figure. The second induction is necessary, due to the fact that $E^{c_2^\alpha}$ may only be a vector subspace of $\text{Lie}(N_a)$ and not be a subalgebra, and the algebraic properties of this subspace will be crucial in understanding the dependence of $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}(u,v,x)$ on $u$.

Luckily, some algebraic features remain. If $u_1, u_2 \in E^{c_2^\alpha}$, we may write $e(u_1 + u_2) = e(u_1)e(u_2) \cdot g$ for some $g \in N_a$. In fact, such a $g$ must lie in $\exp(E^{c_2^\alpha} \oplus E^{c_2^\alpha})$. By the second induction on the pairs $c_1^\alpha$ and $c_2^\beta_j$, we know that this additional term $g$ will have polynomial relations with $c_2^\beta_j$ and has polynomial relations with each $c^\gamma_j\gamma_m$ by the first induction on $\# \Sigma(\alpha, \beta)$. This allows the analysis to go through.

In the next step of the induction on the $\Omega_i$, $c_1^{\gamma_2}\gamma_2$ terms may appear when commuting the $c_1^{\gamma_1}\gamma_1$ terms with a multiple of $\alpha$ or $\beta$ (which will be needed in Lemma 11.9), but for this example, this is the only Lyapunov exponent strictly between $\alpha$ and $\beta$ with this property (by considering Figure 1 and Lemma 11.5). This can and will appear when analyzing how the function $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}(u,v,x)$ depends on $u$. Such terms will contribute polynomials by the induction hypothesis, leading to the final polynomial form.

We now return to the formal proof, assuming the induction hypotheses. Assume Claims 11.2 and 11.3 hold for $c^\gamma_j\gamma \in \Omega_{l'}, q < p$.

Fix $v \in E^{c_2^\beta_j}$ and let $\varphi(u, x) = \hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}(u,v,x)$. Notice that when $a \in \ker \beta$ and $c^\gamma_j\gamma = \sigma c_1^\alpha + \tau c_2^\beta$ (10.1), (TA-3) and the definition of $\hat{\rho}_{c^\gamma_j\gamma}^{c^\alpha, c^\beta_j}$ implies that

\begin{equation}
\varphi(u, x) = e^{-\sigma c^\gamma_j\gamma(a)} \varphi(c^\gamma_j\gamma(a) u, a \cdot x).
\end{equation}

We are now ready to establish the key lemma which gives a cocycle-like property to the function $\varphi$. While the proof requires checking some complicated details, the following lemma follows from two simple ideas: splitting a commutator into a sum of two commutators requires a conjugation and reordering, and with careful bookkeeping, the reordering and conjugation can be shown to contribute polynomial terms only. By the Baker-Campbell-Hausdorff formula, for homogeneous systems where the functions $\varphi$ are compositions of multiplication in a nilpotent Lie group, such polynomials will be nonvanishing unless $\sigma = \tau = 1$, and the cocycle equation without them will not hold. We assume that the induction hypotheses hold.

**Lemma 11.9.** $\varphi(u_1 + u_2, x) = \varphi(u_1, x) + \varphi(u_2, e(u_1)x) + p(u_1, u_2)$ for some polynomial $p : E^{c_1^\alpha} \times E^{c_1^\alpha} \rightarrow E^{c_2^\gamma}$ such that $p(0, \cdot) \equiv 0$ and $p(\cdot, 0) \equiv 0$. 43
Proof. We assume that \( \tilde{c}_m \in \Omega_{t} \). Recall that \( \varphi(u, x) = \tilde{\beta}_{\gamma}^{c_m^\alpha} (u, v, x) \) is the \( \gamma \)-component of the unique path \( \rho_{\gamma}^{c_m^\alpha} (u, v, x) \), written in circular ordering of the weights in \( \Sigma(\alpha, \beta) \), which connects \( [e(u), e(v)] \cdot x \) and \( x \). Given \( u_1, u_2 \in E^{c_m^\alpha} \), there exists \( q(u_1, u_2) \in \bigoplus_{\gamma > \alpha} E^{c_m^\alpha} \) such that \( e(u_1 + u_2) = e(q(u_1, u_2)) e(u_2) e(u_1) \). Since \( N_\alpha \) is nilpotent, \( q \) is a polynomial in \( u_1 \) and \( u_2 \). Notice that using only the free product relations, we get that:

\[
[e(u_1 + u_2), e(v)] = e(-v) * e(-u_1 - u_2) * e(v) * e(u_1 + u_2) = e(-v) * e(-u_1 - u_2) * e(v) * e(q(u_1, u_2)) * e(u_2) * e(u_1) = e(-v) * e(-u_1 - u_2) * e(v) * e(q(u_1, u_2)) * e(u_2) * (e(-v) * e(u_1) * e(v)) * e(u_1)
\]

The last equality is simply the second-to-last expression rewritten with color-coding. First, consider the red term. Since \( q \) takes values in \( \bigoplus_{\gamma > \alpha} E^{c_m^\alpha} \), we know the commutators of \( q \) with \( e(-v) \) are polynomial and independent of their basepoint by Induction II. Therefore we may rewrite the
red term as \( (e(-v) * e(-u_1) * e(-u_2) * e(\tau_0(u_1, u_2, v)) * (e(u_2) * e(u_1) * e(v))) \), where \( \tau_0 \in \text{Lie}(N_{(\alpha, \beta)}) \) is independent of \( x \), depending polynomially on \( u_1, u_2 \) and \( v \). Then by Induction I, we know how each of the conjugating terms act on the term \( \tau_0 \), which must be polynomially. Therefore, the entire first red term is independent of \( x \) and can be replaced by some \( e(\tau(u_1, u_2, v)) \) for some polynomial \( \tau \) taking values in \( \text{Lie}(N_{(\alpha, \beta)}) \).

Indeed, applying parts (2) and (3) of the corollary three times shows that the blue terms act on \( q \) of the group structure of \( N \) taking values in \( \tau \). Furthermore, since \( \alpha, \beta \) and is hence independent of \( y \), taking \( \tilde{g}(y) \) acts on \( y \). Therefore, the \( \Omega_{u_p} \) terms have exactly the prescribed form.

Lemma 11.10. If \( V \) and \( W \) are vector spaces and \( f : V \to W \) is a continuous function such that \( f(0) = 0 \) and

\[
(11.6) \quad f(v_1 + v_2) = f(v_1) + f(v_2) + p(v_1, v_2)
\]

for some polynomial \( p \), then \( f \) is a polynomial.

Proof. First, observe that since \( f(0 + v) = f(0) + f(v) + p(0, v) \), it follows that \( p(0, v) = 0 \). Therefore, \( p(v, w) \) has no constant terms and \( p \) is symmetric by (11.6). Furthermore, every term of \( p \) must have degree at least one in both \( v \) and \( w \) since it is symmetric. If \( q(v) = p(v, v) \), then every term of \( q \) has degree at least two. In particular, since each nonzero term of \( q \) is multiplied by at least \( 2^n \cdot 2^{-2n} \), \( \sum 2^n q(2^{-n}v) \) is summable and the sum converges to a polynomial of the same degree.

From (11.6), it follows that \( f(v) = 2f \left( \frac{1}{2}v \right) + q \left( \frac{1}{2}v \right) \). Inductively, it follows that \( f(v) = 2^n f(2^{-n}v) + \sum_{i=1}^{n} 2^{-i} q(2^{-i}v) \). Since the sum on the right hand side converges to a polynomial, it follows that \( \lim_{n \to \infty} 2^n f(2^{-n}v) \) converges to a vector uniformly bounded as \( v \) varies in a compact set. Therefore, the map \( D : v \mapsto \lim_{n \to \infty} 2^n f(2^{-n}v) \) is well-defined and satisfies

\[
D(v + w) = \lim_{n \to \infty} 2^n f \left( 2^{-n}(v + w) \right) = \lim_{n \to \infty} 2^n \left( f \left( 2^{-n}v \right) + f \left( 2^{-n}w \right) + p(2^{-n}v, 2^{-n}w) \right) = D(v) + D(w).
\]

Furthermore, since \( D(v) = f(v) - \sum_{i=1}^{\infty} 2^{i-1} q \left( 2^{-i}v \right) \), it follows that \( D \) is continuous and hence linear. Therefore, \( f(v) = D(v) + \sum_{i=1}^{\infty} 2^{i-1} q \left( 2^{-i}v \right) \), and \( f \) is a polynomial.
Recall that if we write \( c_i^m \gamma = \sigma c_i^\alpha \alpha + \tau c_i^\beta \beta \), we say that \( \sigma \) and \( \tau \) are the Lyapunov coefficients. Since we have assumed integral Lyapunov coefficients (Definition 11.2), we may assume that either \( \sigma \geq 1 \) or \( \tau \geq 1 \). We without loss of generality assume that \( \sigma \geq 1 \).

**Corollary 11.11.** The function \( \varphi(u, x) \) is a polynomial in \( u \), whose coefficients are functions of \( x \) which are constant along each the sets \( F_\beta(m) \) defined in Lemma 10.14.

**Proof.** By Lemma B.1, it suffices to show the following claim:

**Claim 11.12.1.** For every \( u, v \in E^c_\alpha \) such that \( ||u|| = 1 \), \( \varphi(tu + v, x) \) is a polynomial in \( t \) whose coefficients are functions of \( x \) which are constant along the sets \( F_\beta(m) \).

Claim 11.12.1 will follow from the following weaker claim:

**Claim 11.12.2.** For every \( u \in E^c_\alpha \) such that \( ||u|| = 1 \), \( \varphi(tu, x) \) is a polynomial in \( t \) whose coefficients are functions of \( x \) which are constant along the sets \( F_\beta(m) \).

Let us deduce Claim 11.12.1 from Claim 11.12.2. By Lemma 11.9,

\[
\varphi(tu + v, x) = \varphi(tu, x) + \varphi(v, e(tu) \cdot x) + p(tu, v).
\]

If we can show Claim 11.12.2, then it follows that \( \varphi(tu, x) \) is a polynomial in \( t \) whose coefficients are functions of \( x \) which are constant along the sets \( F_\beta(m) \). By Lemma 10.14 the atoms of \( F_\beta \) are saturated by the leaves of \( W^\alpha \). Since \( \varphi(st, y) \) is a polynomial in \( s \) whose coefficients are functions of \( y \) which are constant along atoms of \( F_\beta \), it follows that \( \varphi(v, e(tu) \cdot x) \) is independent of \( t \) and is constant as \( x \) moves within the sets \( F_\beta(m) \). We have shown that \( \varphi(tu + v, x) \) is the sum of a polynomial in \( t \) whose coefficients are functions of \( x \) which are constant along the sets \( F_\beta(m) \), a function of \( x \) which is constant along the sets \( F_\beta(m) \) and a polynomial in \( t \). Claim 11.12.1 follows.

So we aim to prove Claim 11.12.2. Fix \( u \in E^c_\alpha \) with \( ||u|| = 1 \), and for notational convenience, let \( g(t, x) = \varphi(tu, x) \). We claim that for every \( x \in X \) and Lebesgue almost every \( t_1 \in \mathbb{R} \), \( \frac{d}{dt} \bigg|_{t=t_1} g(t, x) \) exists. To prove the claim, it suffices to show that \( g \) is locally Lipschitz. By Lemma 11.9 and (11.5),

\[
||g(t, x) - g(t_1, x)|| = ||g(t - t_1, e(tu) \cdot x) + p(tu, (t - t_1)u)|| \\
\leq ||t - t_1||^\sigma ||g(1, a \cdot e(tu) \cdot x)|| + ||p(t_1u, (t - t_1)u)||
\]

for a suitable choice of \( a \in \ker \beta \). Recall that \( \sigma \geq 1 \) by the integrality assumption and the choice made directly before the statement of Corollary 11.11. Hence, since \( g(1, \cdot) \) is bounded, \( p \) is a polynomial and \( \sigma \geq 1 \), \( g(t, x) \) is Lipschitz in any compact neighborhood of \( t_1 \). Therefore, for almost every \( t_1 \in E^c_\alpha \), \( g \) is differentiable in \( t \) at \( t_1 \). Therefore, since \( g \) is differentiable in \( t \) at \( t = 0 \) on a dense set of each orbit of the one-parameter subgroup generated by \( u \), \( g \) is differentiable in \( t \) at \( t = 0 \) on a dense subset of \( X \).

By (11.5), the set of points for which \( g \) is differentiable in \( t \) is invariant under \( \mathbb{R}^k \). Therefore, if \( f(x) \) denotes the derivative of \( g \) in \( t \) at \( t = 0 \), \( f \) exists on a dense subset of \( X \). We claim that \( ||f(x)|| \leq B \) for some \( B \in \mathbb{R} \) whenever it exists. Indeed,

\[
|f(x)| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |g(\varepsilon, x)| = \lim_{\varepsilon \to 0} \varepsilon^\sigma |g(1, a_\varepsilon \cdot x)| \leq \sup_{y \in X} |g(1, y)|
\]

where \( a_\varepsilon \in \ker \beta \) is chosen appropriately (using (11.5)), since \( \sigma \geq 1 \). Notice that if \( a \in \ker \beta \), then again by (11.5) and the chain rule,
\[ f(a \cdot x) = e^{(\sigma-1)c_\alpha a(x)} f(x). \]

Therefore, either \( \sigma = 1 \) or \( f \equiv 0 \), since otherwise one may apply an element \( a \in \ker \beta \) with \( \alpha(a) \) arbitrarily large to contradict the boundedness of \( f \). Since \( \sigma = 1 \) or \( f \equiv 0 \), \( f \) is constant along \( \ker \beta \) orbits.

We claim that \( f \) is also constant along \( W^\alpha \) leaves whenever it exists. Assume it is not identically 0, otherwise the claim follows immediately. In this case \( \sigma = 1 \), and \( g \) must be a cocycle over the \( e(tu) \)-action by comparing Lemma 11.9 and (11.5). By (TA-9), there exists an SRB-like measure \( \mu \) which is invariant under the \( \ker \beta \)-action and has absolutely continuous disintegrations along \( N_\alpha \)-leaves. By standard Hopf argument, it follows that each ergodic component for the ergodic decomposition of \( \mu \) with respect to the \( \ker \beta \)-action is also absolutely continuous along \( N_\alpha \)-leaves.

Hence, for \( \mu \)-almost every point \( x \), \( f \) is constant at Lebesgue-almost every point of \( W^\alpha \) (since it is a \( \ker \beta \)-invariant function, and defined at Lebesgue almost every point of every \( W^\alpha \) leaf). Since the \( e(tu) \)-orbit foliation is a smooth subfoliation of \( N_\alpha \), it follows that it is absolutely continuous on almost every such orbit. Since \( f \) is the derivative of \( g \), and \( g \) is a cocycle over the \( e(tu) \)-flow, it follows that

\[ g(t, x) = \int_0^t f(e(ru) \cdot x) \, dr = v_x t \]

where \( v_x \in E^c_\alpha \gamma \) is the common value of \( f \) at almost every point of \( W^\alpha(x) \). Then \( g \) is linear on a dense set of \( W^\alpha \) leaves, and has derivative invariant under \( \ker \beta \). Since \( g \) is continuous, it follows that \( g(t, x) = v_x t \), where \( v_x \) is a derivative which depends only on the atom of \( F_\beta \). This concludes the case when \( \sigma = 1 \).

We return to the general case (either \( \sigma = 1 \) or \( f \equiv 0 \)). Let \( v_x \) denote the common value for \( f \) along \( W^\alpha(x) \) and \( \ker \beta \cdot x \). Fix \( t_1 \in \mathbb{R} \) such that \( f(e(t_1 u) \cdot x) \) exists (recall that the collection of such \( t_1 \) has full Lebesgue measure). Then by Lemma 11.9

\[
(11.7) \quad \frac{d}{dt} \bigg|_{t=t_1} g(t, x) = \frac{d}{dt} \bigg|_{t=0} \varphi((t + t_1)u, x) \\
= \frac{d}{dt} \bigg|_{t=0} \left( \varphi(t_1 u, x) + \varphi(tu, e(t_1 u) \cdot x) + p(t_1 u, tu) \right) = f(e(t_1 u) \cdot x) = v_x + q(t_1)
\]

where \( q \) is some polynomial independent of \( x \). Since we know the initial condition \( g(0, x) = 0 \), one may integrate \( v_x + q \) to get a polynomial form for \( g \) at each \( x \). Furthermore, since \( v_x \) does not vary as \( x \) moves along \( \ker \beta \), the coefficients of the polynomial are constant along the atoms of \( F_\beta \). This shows Claim 11.12.2, and finishes the proof. \( \square \)

**Proof of Claims 11.2 and 11.3.** Corollary 11.11 implies that for fixed \( v \), the function \( \rho_{c_\alpha, c_\beta}^j (u, v, x) \) is a polynomial in \( u \), whose coefficients are functions of the atoms of \( F_\beta \). It is not difficult to see that an analogous version of Lemma 11.9 holds when fixing \( u \) and varying \( v \). Indeed, if we define \( \psi(v, x) = \rho_{c_\alpha, c_\beta}^j (u, v, x) \), then a nearly identical proof shows that

\[
(11.8) \quad \psi(v_1 + v_2, x) = \psi(v_1, x) + \psi(v_2, e(v_1) * e(u) \cdot x) + p(u, v_1, v_2)
\]

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for some polynomial $p$. Furthermore, since we have already shown that $\hat{\rho}_{\alpha,c}^{i,j}(u,v,x)$ is a polynomial in $u$, it follows that the following equation for $\psi$ analogous to (11.5) for $\varphi$ holds for all $a \in \mathbb{R}^k$, not just $a \in \ker \alpha$:

$$\psi(v,x) = e^{-(c^\beta a)} \psi(e^{c^\beta a} v, a \cdot x).$$ (11.9)

Thus, $\psi$ is invariant under $\ker \beta$, and hence constant on each atom of $F_\beta$.

Since the leaves of $W^\alpha$ and $W^\beta$ are contained in the atoms of $F_\beta$ by Lemma 10.14, the dependence of $\psi$ on $x$ does not affect (11.8). Therefore, on each atom of $F_\beta$, the map $\psi$ satisfies the assumption of Lemma 11.10 and on each atom of $F_\beta$, the function $\hat{\rho}_{\alpha,c}^{i,j}(u,v,x)$ is a polynomial in $u$ and $v$, with coefficients depending only on the atom.

We must show that the polynomials are independent of $x$. Notice that $\hat{\rho}_{\alpha,c}^{i,j}(u,v,x)$ is a family of polynomials whose coefficients depend on $x$, so (10.1) implies that the polynomial is $\sigma$-homogeneous in $u$ and $\tau$-homogeneous in $v$, since otherwise the coefficients would grow to $\infty$ by applying contracting or expanding elements of $\mathbb{R}^k$ (so, in particular, $\sigma, \tau \in \mathbb{Z}$ and (11.2) holds). Therefore, the polynomial is unchanged as $x$ moves along its $\mathbb{R}^k$-orbit by (10.1). Since there is a dense $\mathbb{R}^k$-orbit and $\hat{\rho}_{\alpha,c}^{i,j}$ is continuous in all variables, it follows that it is independent of $x$. Hence (11.3) holds. \qed

12. Partial homogeneity implies homogeneity

The goal of this section is to produce homogeneous structures related to the partial homogeneous structures coming from the $\mathbb{R}^k$- and $N_\alpha$-actions. The arguments and approach expand those of [48, Section 14] in several ways. While the overall scheme is similar, several new obstacles appear due to the presence of the compact group $M$ as well the coarse Lyapunov foliations being parameterized by general nilpotent Lie groups (rather than copies of $\mathbb{R}$). In particular, we build group actions to describe homogeneous spaces and use Corollary 6.11 to produce Lie structures on them.

We carefully piece together such homogeneous structures to build one on $X$. In particular, we use specific, computable relations between the flows of negatively proportional weights provided by the classification in Lemma 10.15. We will see that such relations yield a canonical presentation of paths, but only for a dense set of paths containing an open neighborhood of the identity in $\hat{P}_{\{\alpha,-\alpha\}}$.

Our approach is to find canonical presentations for words in $\hat{P}$ using only commutator relations and symplectic relations which we assume are constant and well-defined. By fixing a regular element, we will be able to use such relations to rearrange the terms in an open set of words to write them using only stable legs, then only unstable legs, then the action (Proposition 12.9). This will imply that the quotient group of $\hat{P}$ by the commutator relations and symplectic relations is locally compact, which allows us to use the structure of locally compact groups (Corollary 6.11). The core of the approach is Lemma 12.8, which gives the ability to commute stable and unstable paths.

In the end we will have shown that if an ideal factor has constant $\rho$-functions, it is conjugate to a homogeneous action. In particular, this holds for any maximal factor, which sets up the induction on factoring out by a chain of ideals in Section 11.

**Definition 12.1.** We say that a leafwise homogeneous topological Anosov action has *constant pairwise cycle structure* if

(CPCS-1) for each pair of nonproportional $\alpha, \beta \in \Delta$, $\gamma \in \Sigma(\alpha, \beta)$ and fixed $u \in N_\alpha$ and $v \in N_\beta$, $\hat{\rho}_{\alpha,\beta}(u,v,y)$ (see Definition 10.11) is independent of $y \in X$, and...
(CPCS-2) for each \( \alpha \in \Delta \) such that \(-\alpha \in \Delta \), the action of \( P_{\{\alpha, -\alpha\}} \) factors through a Lie group action on \( X \).

In applications, the conditions above are deduced from the genuinely higher-rank assumptions (TA-8) (indeed, they are consequences of Theorem 11.1 and Lemma 10.15, respectively). However, we do not need the genuinely higher-rank assumptions to conclude the main goal of this section:

**Theorem 12.2.** If \( \mathbb{R}^k \times M \acts X \) is a leafwise homogeneous, topological Anosov action with constant pairwise cycle structure, then the action is topologically conjugate to a translation action on a homogeneous space \( G/\Gamma \), with \( \Gamma \) discrete.

12.1. **Stable-unstable-neutral presentations.** We let \( \rho_{\alpha, \beta}(u, v) \) denote the common value of \( \rho_{\gamma}(u, v, x) \) for \( \alpha, \beta \in \Delta, \gamma \in \Sigma(\alpha, \beta) \). This is guaranteed to be independent of \( x \) since the action has constant pairwise cycle structure. Let \( \mathcal{P} = \mathcal{P}_\Delta \) be the group freely generated by the groups \( N_\alpha \). Let \( \mathcal{C}' \) be the smallest closed normal subgroup containing all cycles of the form

- \([u, v] + \rho_{\alpha, \beta}(u, v) \) for \( u \in N_\alpha \) and \( v \in N_\beta \) as described in Definition 10.11 and
- any element of \( \mathcal{P}_{\{\alpha, -\alpha\}} \) which factors through the identity of the Lie group action provided by (CPCS-2) (ie, \( \ker(\mathcal{P}_{\{\alpha, -\alpha\}}) \to G_\alpha \)), where \( G_\alpha \) is as in Lemma 10.15).

Since such cycles are cycles at every point by assumption, \( \mathcal{C}' \subset \mathcal{C}(x) := \text{Stab}_\mathcal{P}(x) \) and \( \mathcal{C}' \) is normal. Consider the quotient group \( \mathcal{G} = \mathcal{P}/\mathcal{C}'. \)

Fix a regular element \( a_0 \in \mathbb{R}^k \). The goal of this subsection is to show that any \( \rho \in \mathcal{G} \) can be reduced (via the relations in \( \mathcal{C}' \)) to some \( \rho_+ * \rho_- * \rho_0 \), with \( \rho_+ \) having only terms from \( N_\chi \) with \( \chi \in \Delta^+(a_0) \), \( \rho_- \) having only terms from \( N_{-\chi} \) with \( \chi \in \Delta^-(-a_0) \), and \( \rho_0 \) being a product of elements of \( \mathbb{R}^k \times M \) generated by symplectic pairs (see Lemma 10.15). Rather, we will show this for the group obtained by taking the semidirect product of the \( \mathbb{R}^k \times M \) with \( \mathcal{G} \) see Proposition 12.9. We begin by identifying well-behaved subgroups of \( \mathcal{G} \). Given a subset \( \Xi \subset \Delta \), let \( \mathcal{G}_{\Xi} \) denote the subgroup of \( \mathcal{G} \) generated by the subgroups \( N_\chi \), as \( \chi \) ranges over \( \Xi \). We say that \( \Xi \) is stable if \( \Xi \subset \Delta^-(a) \) for some \( a \in \mathbb{R}^k \). We say that it is closed if for any \( \alpha, \beta \in \Xi, \Sigma(\alpha, \beta) \subset \Xi \).

**Lemma 12.3.** If \( \Xi \) is stable, closed collection of weights, then \( \mathcal{G}_{\Xi} \) is a nilpotent Lie group.

**Proof.** The lemma follows as in the proof of Lemma 11.4. Indeed, the proof works verbatim after restricting the weights \( \chi \in \Xi \) to a generic \( \mathbb{R}^2 \subset \mathbb{R}^k \). Then one uses the circular ordering to find unique presentations of \( \mathcal{G}_{\Xi} \). See also [48], Section 5.2 and Lemma 17.5.

Let \( \chi \in \Delta \) be a weight such that \(-\chi \in \Delta \), and \( \beta \in \Delta \) be any linearly independent weight. Let \( \Xi = \{t\beta + s\chi : t \geq 0, s \in \mathbb{R} \} \cap \Delta \), and \( \Xi' = \{t\beta + s\chi : t > 0, s \in \mathbb{R} \} \cap \Delta = \Xi \setminus \{\chi, -\chi\} \).

**Proposition 12.4.** If \( \Xi \) is as above and \( \rho \in \mathcal{G}_{\Xi} \) is any element, then \( \rho = \rho_\chi * \rho_{\Xi} \), where \( \rho_\chi \in \mathcal{G}_{\{\chi, -\chi\}} \), and \( \rho_{\Xi} \in \mathcal{G}_{\Xi'} \). Furthermore, such a decomposition is unique.

**Proof.** The proof technique is the same as that of Lemma 11.4. Using constancy of commutator relations, we may push any elements of \( N_{\pm \chi} \) to the left, accumulating elements of \( \mathcal{G}_{\Xi'} \) as the commutator on the right.

To see uniqueness, suppose that \( \rho_\chi * \rho_{\Xi} = \rho'_\chi * \rho'_{\Xi} \). Then \( (\rho'_\chi)^{-1} * \rho_\chi = \rho'_{\Xi} * \rho_{\Xi}^{-1} \). But \( \mathcal{G}_{\Xi'} \) is a subgroup of \( \mathcal{G}_{\Xi} \) and it is clear that \( \mathcal{G}_{\Xi} \cap \mathcal{G}_{\{\chi, -\chi\}} = \{e\} \). Therefore, \( \rho'_\chi = \rho_\chi \) and \( \rho_{\Xi} = \rho'_{\Xi} \), and the decomposition is unique.

**Corollary 12.5.** If \( \Xi \) is as above, \( \mathcal{G}_{\Xi} \) is a Lie group. Furthermore, \( \mathcal{G}_{\Xi} \) has the semidirect product structure \( \mathcal{G}_{\{\chi, -\chi\}} \ltimes \mathcal{G}_{\Xi} \), with \( \mathcal{G}_{\Xi} \) a nilpotent group.
Proof. Notice that in the proof of Proposition 12.4, we get a unique expression by moving the elements of $G_{\{\chi,-\chi\}}$ to the left, and doing so accumulates only the $G_{3}$ terms. Therefore, the decomposition gives $G_{3}$ the structure of a semidirect product of $G_{\{\chi,-\chi\}}$ and $G_{\{\rho\}}$. These groups are Lie by (CPCS-2) and Lemma 12.3, respectively. The action of $G_{\{\chi,-\chi\}}$ on $G_{\{\rho\}}$ is continuous since the action of its generating subgroups corresponding to $\chi$ and $-\chi$ are given by commutators, which are continuous. Therefore, $G_{3}$ is the semidirect product of the Lie group $G_{\{\chi,-\chi\}}$ with the Lie group $G_{\{\rho\}}$, with a continuous representation, and is hence a Lie group.

The crucial tool in showing that $\mathcal{P}/\mathcal{C}'$ is Lie is to show that it is locally Euclidean. To that end, the crucial result is Lemma 12.8. Fix a regular element $a_{0} \in \mathbb{R}^{k}$, then define $G_{+} = G_{\Delta+(a_{0})}$ and $G_{-} = G_{\Delta-(a_{0})}$. Note that $G_{\pm}$ are nilpotent Lie groups by Lemma 12.3. Let $D_{\chi}$ be the subgroup of $G_{\{\chi,-\chi\}}$ which coincides with the $\mathbb{R}^{k} \times M$-action (see Lemma 10.15). Let $D \subset G$ be the group freely generated by all such $D_{\chi}$. Note that the action of each element of $D$ coincides with the action of some element of the $\mathbb{R}^{k} \times M$ action by construction. However, the action of $D$ is not obviously faithful, and may fail to be, as is the case for the Weyl chamber flow on $SL(3,\mathbb{R})$ where there are 3 symplectic pairs of weights, each generating one-parameter subgroups of $\text{Diag} \cong \mathbb{R}^{2}$.

Lemma 12.6. Suppose the $\mathbb{R}^{k}$ orbit of $x_{0}$ is dense. Then if $d \in D$ is a cycle at $x_{0}$, then $d$ is a cycle everywhere.

Proof. The action of the generators of $D$ coincides with the $\mathbb{R}^{k} \times M$ action, therefore the action of $D$ commutes with the $\mathbb{R}^{k}$ action. Then if $d$ is a cycle at $x_{0}$, $d$ is a cycle at any point in $\mathbb{R}^{k} \cdot x_{0}$, hence everywhere as the $\mathbb{R}^{k}$ orbit of $x_{0}$ is dense.

Denote by $\mathcal{C}_{D}$ the group of cycles in $D$ at a point $x_{0}$ with a dense $\mathbb{R}^{k}$-orbit (such a point exists by (TA-1)), and set $G = G/\mathcal{C}_{D}$ and $D = D/\mathcal{C}_{D}$. Notice that $G$ and $D$ are topological groups since $\mathcal{C}_{D}$ is a closed normal subgroup by Lemma 12.6. Furthermore, let $G_{\pm}$ denote the projections of the groups $G_{\pm}$ to $G$. The following lemma is immediate from the fact that the action of $D$ coincides with that of the $\mathbb{R}^{k} \times M$ action and Corollary 12.5:

Lemma 12.7. Let $\rho^{\pm} \in G_{\pm}$ (resp. $G_{\pm}(\bar{x})$) and $\rho^{0} \in D$. Then

$$\rho^{0} \rho^{\pm}(\rho^{0})^{-1} \in G_{\pm} \text{ (resp. } G_{\pm}(\bar{x})) .$$

The following is the basic commutation argument. It is an adaptation of Lemma 14.11 of [48], with changes to account for the nilpotent groups being multidimensional.

Lemma 12.8. For an open set of elements $\rho^{+} \in G_{+}$, $\rho^{-} \in G_{-}$ containing $\{e\} \times \{e\}$ there exist $(\rho^{+})' \in G_{+}$, $(\rho^{-})' \in G_{-}$ and $\rho^{0} \in D$ such that

$$\rho^{+} \ast \rho^{-} = (\rho^{-})' \ast (\rho^{+})' \ast \rho^{0} .$$

Furthermore, $(\rho^{+})'$, $(\rho^{-})'$ and $\rho^{0}$ depend continuously on $\rho^{+}$ and $\rho^{-}$.

Proof. Order the weights of $\Delta^{+}(a_{0})$ and $\Delta^{-}(a_{0})$ using a fixed circular ordering as $\Delta^{+}(a_{0}) = \{\alpha_{1}, \ldots, \alpha_{n}\}$ and $\Delta^{-}(a_{0}) = \{\beta_{1}, \ldots, \beta_{m}\}$. Since $\Delta^{+}(a_{0})$ is a stable subset, $G_{+}$ is a nilpotent group by Lemma 12.3. Therefore, we may write $\rho_{+} = u_{n} \ast \cdots \ast u_{1}$ for some $u_{i} \in N_{\alpha_{i}}$. We will inductively show that we may write the product $\rho^{+} \ast \rho^{-}$ as $u_{n} \ast \cdots \ast u_{k} \ast (\rho^{-})' \ast v_{k-1} \ast \cdots \ast v_{1} \ast \rho^{0}$ for some $v_{i} \in N_{\alpha_{i}}$, $(\rho^{-})' \in G_{-}$ and $\rho^{0} \in D$ (all of which depend on $k$, the index of the induction). Our given expression is the base case $k = 1$.

Suppose we have this for $k$. If $-\alpha_{k} \in \Delta$, then it must be in $\Delta_{-}$. Let $l(k)$ denote the index for which $\beta_{l(k)} = -\alpha_{k}$ if $-\alpha_{k}$ is a weight. Otherwise, since there is no weight negatively proportional to $\alpha_{k}$, we set $\beta_{l(k)} = -\alpha_{k}$ with $l(k)$ a half integer so that $-\alpha_{k}$ appears between $\beta_{l(k)-1/2}$ and
\( \beta_{l(k)+1/2} \). Then decompose \( \Delta \) into six (possibly empty) subsets: \( \{\alpha_k\}, \{-\alpha_k\}, \Delta_1 = \{\alpha_l : l < k\}, \Delta_2 = \{\alpha_l : l > k\}, \Delta_3 = \{\beta_l : l < l(k)\} \) and \( \Delta_4 = \{\beta_l : l > l(k)\} \). See Figure 2.

\[
\begin{align*}
\Delta_1 & \quad \Delta_2 \\
\Delta_3 & \quad \Delta_4
\end{align*}
\]

\( \{\chi : \chi(a_0) = 0\} \)

We let \( G_{\Delta_i} \) denote the subgroup of \( G \) generated by the \( \Delta_i \). Notice that \( \Delta_- = \Delta_3 \cup \{-\alpha_k\} \cup \Delta_4 \) (with \( \{-\alpha_k\} \) omitted if there is no weight of this form) is stable, so again, since \( G_- \) is nilpotent, \( (\rho^-)' \) may be expressed uniquely as \( q_3 \ast w \ast q_4 \) with \( q_3 \in G_{\Delta_3} \) and \( q_4 \in G_{\Delta_4} \) and \( w \in N_{-\alpha_k} \) (if \( -\alpha_k \) is not a weight, we omit this term). Now, \( \{\alpha_k\} \cup \Delta_2 \cup \Delta_3 \) is a stable set whose associated group is nilpotent. So \( u_k \ast q_3 = q_2 \ast (q_3)' \ast v_k \) for some \( q_2 \in G_{\Delta_2}, (q_3)' \in G_{\Delta_3} \) and \( v_k \in N_{\alpha_k} \). Notice that by iterating some \( a \in \mathbb{R}^k \) for which \( \alpha_k(a) = 0 \), and \( \beta(a) < 0 \) for all \( \beta \in \Delta_2 \cup \Delta_3 \), we actually know that \( v_k = u_k \). Thus, we have put our expression in the form:

\[
\begin{align*}
\sum \ast \cdots \ast u_k \ast (\rho^-)' \ast v_k \ast \cdots \ast v_1 \ast \rho^0 &= \sum \ast \cdots \ast u_k \ast (q_3 \ast w \ast q_4) \ast v_{k-1} \ast \cdots \ast v_1 \ast \rho^0 \\
&= \sum \ast \cdots \ast (u_k \ast q_3) \ast w \ast q_4 \ast v_{k-1} \ast \cdots \ast v_1 \ast \rho^0 \\
&= \sum \ast \cdots \ast u_{k+1} \ast (q_2 \ast (q_3)' \ast v_k) \ast w \ast q_4 \ast v_{k-1} \ast \cdots \ast v_1 \ast \rho^0
\end{align*}
\]

Now, there are two cases: \( -\alpha_k \not\in \Delta \) in which case \( v_k = e \). If \( -\alpha_k \in \Delta \), then as long as \( v_k \) and \( w \) are both sufficiently small, \( v_k \ast w = w' \ast v_k' \ast g \) for some \( w' \in N_{-\alpha_k}, v'_k \in N_{\alpha_k} \) and \( g \in D \) (since by Lemma 10.15, the corresponding subalgebras form a splitting of \( G_\alpha \)). Furthermore, notice that \( q_2 \) and the terms appearing before \( q_2 \) all belong to \( \Delta_2 \), so we may combine them to reduce the expression to:

\[
\sum \ast \cdots \ast v_k \ast (q_3)' \ast w' \ast v_k' \ast g \ast q_4 \ast v_{k-1} \ast \cdots \ast v_1 \ast \rho^0
\]

for some collection of \( u_k' \in N_{\alpha_k} \), and \( g \in D \). But by Lemma 12.7, \( g \) may be pushed to the right preserving the form of the expression and being absorbed into \( \rho^0 \). We abusively do not change these terms and drop \( g \) from the expression.

Now, we do the final commutation by commuting \( v_k' \) and \( q_4 \). Notice that \( \{\alpha_k\} \cup \Delta_1 \cup \Delta_4 \) is a stable subset. Therefore, we may write \( v_k' \ast q_4 \) as \( (q_4)' \ast v_k'' \ast q_1 \) with \( q_1 \in G_{\Delta_1}, (q_4)' \in G_{\Delta_4} \) and

\[
\begin{align*}
\Delta_1 & \quad \Delta_2 \\
\Delta_3 & \quad \Delta_4
\end{align*}
\]
Given a word \( \alpha \) corresponding to sequences of weights \( \{\chi_i\}_{i=1}^{n} \) and letting \( C_{\hat{\chi}} = \{u_1 \cdots u_n : u_i \in N_{\chi_i} \} \cong N_{\chi_1} \times \cdots \times N_{\chi_n} \). Then a neighborhood of the identity is a union of neighborhoods in each cell \( C_{\hat{\chi}} \) containing 0.

Let \( \pi : \hat{\mathcal{P}} \to \hat{G} \) denote the canonical projection, and note that \( \ker \pi \) is exactly the group generated by (conjugates of) commutator cycles, nontrivial symplectic cycles, cycles in \( \mathcal{C}_D \), and identifications of elements of \( \mathbb{R}^k \times M \) with their generated versions in \( \mathcal{P} \) (ie, elements of the form \( f(d)\rho_d^{-1} \)). Furthermore, notice that putting \( \Delta^+(a) \) and \( \Delta^-(a) \) in a circular ordering makes \( G_+ \times G_- \times (\mathbb{R}^k \times M) \) a combinatorial cell in \( \hat{\mathcal{P}} \).

**Proposition 12.9.** There exists an open neighborhood \( U \) of \( e \in \hat{\mathcal{P}} \) and a continuous, open map \( \Phi : U \to G_+ \times G_- \times (\mathbb{R}^k \times M) \) such that \( \pi \circ \Phi = \pi \).

**Proof.** We describe the map \( \Phi \), whose domain will become clear from the definition. Let \( \Delta^+(a_0) = \{\alpha_1, \ldots, \alpha_n\} \) and \( \Delta^-(a_0) = \{\beta_1, \ldots, \beta_m\} \) be the weights as described in the proof of Lemma 12.8. Given a word \( \rho = u_1 \cdots u_n, u_i \in N_{\chi_i}, \chi_i \in \Delta \) for every \( i \), we begin by taking all occurrences of \( \alpha_n \) in \( \rho \) and pushing them to the left, starting with the leftmost term. When we commute it past another \( \alpha_i \), we accumulate only other \( \alpha_j, i + 1 \leq j \leq n - 1 \), in \( \rho^{\alpha_i \alpha_n} \), which we may canonically present in increasing order on the right of the commutation. A similar statement holds for the commutation of \( \alpha_n \) with \( \beta_i \). We iterate this procedure as in the proof of Lemma 12.8 to obtain the desired presentation. Since the commutation operations involved are determined by the combinatorial type, the resulting presentation is continuous from the cell \( C_{\hat{\chi}} \). Furthermore, if one of the terms happens to be \( e \), the procedure yields the same result whether it is considered there or not. Thus, it is a well-defined continuous map from a neighborhood of the identity in \( \hat{\mathcal{P}} \) to \( G_+ \times G_- \times (\mathbb{R}^k \times M) \). It is continuous from \( \hat{\mathcal{P}} \) because it is continuous from each \( C_{\hat{\chi}} \).

Notice that in the application of Lemma 12.8, we require that all terms are sufficiently small. Thus, in each combinatorial pattern, since the algorithm is guaranteed to have a finite number of steps and swaps appearing, and each term appearing will depend continuously on the initial values of the terms, we know that for each \( \hat{\chi} \), some neighborhood of 0 will be in the domain of \( \Phi \), by the neighborhood structure described above.

Notice that the reduction of a word \( u \) to a word of the form \( u_+ \cdots u_- \cdots a \in G_+ \times G_- \times (\mathbb{R}^k \times M) \) uses only relations defining \( \hat{G} \). Therefore, if after the reductions, the same form is obtained, the original words must represent the same element of \( \hat{G} \). That is, \( \pi \circ \Phi = \pi \). \( \square \)

**Corollary 12.10.** The group \( \hat{G} \) is a Lie group.

**Proof.** Choose \( U \) is as in Lemma 12.9, and let \( K \subset \Phi(U) \) be a compact neighborhood of \( (e,e,e) \in G_+ \times G_- \times (\mathbb{R}^k \times M) \). Note that such a neighborhood exists since \( G_\pm \) and \( \mathbb{R}^k \times M \) are all Lie
groups using Lemma 12.3. Then $\Phi^{-1}(K)$ is a neighborhood of $e \in \hat{\mathcal{P}}$, and $\pi(\Phi^{-1}(K)) = \pi(K)$ is a neighborhood of $e \in \hat{G}$. Therefore, $\hat{G}$ is locally compact. Furthermore, since $\hat{G}$ is the factor of the locally path-connected group $\mathcal{P}$, $\hat{G}$ is locally path-connected. Hence $\hat{G}$ is a projective limit of Lie groups by Corollary 6.11. Hence $\hat{G}$ has an associated sequence $G_n$ of connected Lie groups, factor maps $q_n : \hat{G} \to G_n$, and projections $p_n : G_n \to G_{n-1}$ such that $\ker p_n$ is compact, $q_n = p_{n+1} \circ q_{n+1}$, and $\bigcap_{n=1}^{\infty} \ker q_n = \{e\}$ (see the diagram in Corollary 6.11).

We first claim that there exists $N, d \in \mathbb{N}$ such that $\dim(G_n) = d$ for all $n \geq N$. Indeed, note that since each $p_n$ is surjective, $\dim(G_n) \geq \dim(G_{n-1})$, so it suffices to show that $\dim(G_n) \leq d$ for all $n \in \mathbb{N}$ and some $d \in \mathbb{N}$. We have that $q_n(G_+) = q_n(G_-)$ and $q_n(\mathbb{R}^k \times M)$, and $q_n$ are Lie subgroups of $G_n$, and since $q_n \circ \pi \circ \Phi : G_+ \times G_- \times (\mathbb{R}^k \times M) \to G_n$ is an open map, we conclude that the map $(g_1, g_2, a) \mapsto g_1 \cdot g_2 \cdot a$ from $q_n(G_+) \times q_n(G_-) \times (\mathbb{R}^k \times M)$ is an open map. It follows that $\dim(G_n) \leq \dim(q_n(G_+)) + \dim(q_n(G_-)) + \dim(q_n(\mathbb{R}^k \times M)) \leq \dim(G_+) + \dim(G_-) + \dim(\mathbb{R}^k \times M)$, which is independent of $n$.

Since $\dim(G_n) = \dim(G_{n+1})$ for all $n \geq N$, $p_n$ is a local isomorphism for all $n \geq N + 1$. It follows that the algebras $\text{Lie}(G_n)$ are all isomorphic, and there exists a unique simply connected group $\hat{G}$ such that $\text{Lie}(\hat{G}) \cong \text{Lie}(G_n)$ for sufficiently large $n$. We may therefore construct local isomorphisms $f_n : \hat{G} \to G_n$ inductively by defining $f_{n+1}$ to be the unique Lie group homomorphism with derivative $(dp_n)^{-1} \circ df_n$. We therefore obtain the following commutative diagram:

By the universal property of inverse limits, there exists a unique homomorphism $F : \hat{G} \to \hat{G}$ such that $q_n \circ F = f_n$. We claim that the image of $F$ is exactly the path component of $\hat{G}$. Indeed, if $\gamma : [0, 1] \to \hat{G}$ is any path such that $\gamma(0) = e$, then $\gamma_n = q_n \circ \gamma$ is a path in $G_n$, and $p_n \circ \gamma_n = \gamma_{n-1}$. Since $f_n$ is a local isomorphism, there exists a unique $\tilde{\gamma}_n : [0, 1] \to \hat{G}$ such that $f_n \circ \tilde{\gamma}_n = \gamma_n$. Since $p_n \circ f_n = f_{n-1}$, the maps $\tilde{\gamma}_n$ all coincide, let $\tilde{\gamma} : [0, 1] \to \hat{G}$ denote the corresponding lift. Then by construction, $F \circ \tilde{\gamma} = \gamma$, and the endpoint of $\gamma$ can be reached in the image of $F$.

Finally, since $\hat{G}$ is path connected, the path identity component is exactly $\hat{G}$, so $F$ is onto. Therefore, $\hat{G}$ is the image of a Lie group, and therefore Lie.

\[ \quad \]

12.2. Description of the homogeneous spaces. By Corollary 12.10, $X$ is the homogeneous space of a Lie group $\hat{G}$, which is generated by subgroups which are images of $N_\alpha$, $\alpha \in \Delta$ and $\mathbb{R}^k \times M$. Furthermore, the group $\hat{G}$ is a factor of the group $\mathcal{P} = (\mathbb{R}^k \times M) \times \mathcal{P}$, where $\mathcal{P}$ is the free product of the groups $N_\alpha$, and let $\mathcal{C}$ denote the kernel of $\mathcal{P} \to \hat{G}$. Therefore, $\mathcal{C}$ is the normal closure of the group generated by commutator relations $\rho^{\alpha, \beta}(u, v, x)$ (which do not depend $x$), symplectic relations, and identifications of the diagonal elements of $G_\alpha$ with the $\mathbb{R}^k$-action.

Let $S(x) = \text{Stab}_\mathcal{C}(x)^0$, and notice that $S_1$ is the closed subgroup of a Lie group and therefore Lie.

Lemma 12.11. $\dim(S(x))$ is independent of $x \in X$. 53
Proof. Since $\hat{G}$ acts transitively on $X$ by (TA-7), it follows that $X$ is the homogeneous space of the Lie group $\hat{G}$. The result is now immediate since if $g \in \hat{G}$, $S(x)$ and $S(g \cdot x)$ are subgroups conjugated by $g$. □

Let $a_0 \in \mathbb{R}^k$ be a generic element (i.e., an element such that $\alpha(a_0) \neq 0$ for all $\alpha \in \Delta$ and $c_i \alpha(a_0) \neq c_j \beta(a_0)$ for all functionals $c_i \alpha \in [\alpha], c_j \beta \in [\beta]$.

Lemma 12.12. If $x$ has a dense $\mathbb{R}^k$-orbit and is $a_0^{\pm 1}$-recurrent, and $S(x) \neq \{e\}$, then $\text{Lie}(S(x))$ contains an element of an Oseledets subspace of $\text{Lie}(N_a)$ for some $\alpha \in \Delta$.

Proof. Suppose that $S(x) \neq \{e\}$, and consider $\text{Lie}(S(x)) \subset \text{Lie}(G)$. Notice that $x \mapsto \text{Lie}(S(x))$ is semi-continuous in the following sense: if $x_n \to x$ and $\mathbb{R}v_n \subset \text{Lie}(S(x_n))$, with $||v_n|| = 1$ and $v_n \to v$, then $\mathbb{R}v \subset \text{Lie}(S(x))$.

Now, simply notice that if $S(x) \neq \{e\}$ and $a_0^n \cdot x \to x$, then $S(x)$ contains its fastest Oseledets space (either forward or backward), or is contained in $(\mathbb{R}^k \times M) \times S$. $S(x)$ is transverse to $\mathbb{R}^k \times M$ since the action of $\mathbb{R}^k \times M$ is locally free by the definition of Lyapunov homogeneous topological Anosov action. Therefore, $S(x)$ contains an Oseledets space if it is nontrivial. □

Corollary 12.13. $S(x) = \{e\}$ for all $x \in X$.

Proof. If $S(x) \neq \{e\}$ for some point, it is nontrivial at every point by Lemma 12.11. Since $a_0^{\pm 1}$-recurrence and dense $\mathbb{R}^k$-orbit are both residual properties, we may find some $x_1$ such that $x_1$ has a dense $\mathbb{R}^k$-orbit and for which $\text{Lie}(S(x_1))$ contains an element of an Oseledets space by Lemma 12.12. But since $a_1 S(x_1) = S(a \cdot x_1)$ and Oseledets spaces are invariant under $a_1$, it follows that $\text{Lie}(S(x))$ contains that element for all $x$. This contradicts (TA-2), since locally bi-Lipschitz implies locally free. □

Proof of Theorem 12.2. We have just shown that $S(x) = \{e\}$, $\text{Stab}_G(x)$ is discrete. The group $\hat{G}$ is Lie by Corollary 12.10 and contains the $\mathbb{R}^k \times M$-action as a subgroup. The result follows. □

12.3. Proof of Theorem 10.7. By Theorem 11.1 and Lemma 10.15, any action satisfying the assumptions of Theorem 10.7 has pairwise constant cycle structure (recall Definition 12.1). Therefore, Theorem 10.7 is a consequence of Theorem 12.2.

13. Smooth Anosov Actions

In this section, we verify that smooth abelian actions of $\mathbb{R}^k \times M$ satsifying assumptions (FA-1) and (FA-2) must be leafwise-homogeneous topological Anosov actions. Much of the work establishing this was done in Section 9 (culminating in Theorem 9.15). The main things left to check are the integral Lyapunov coefficients property (recall the paragraph following Definition 11.2), as well as verifying that the conjugating map is smooth.

Proposition 13.1. If $\mathbb{R}^k \times M \acts X$ is a $C^\infty$ Anosov action satisfying conditions (FA-1) and (FA-2) of the introduction, then the compact extension of the action provided by Theorem 9.15 is a genuinely higher-rank leafwise homogeneous topological Anosov action with integral Lyapunov coefficients and SRB measures.

Proof. Properties (TA-1) and (TA-8) follow immediately from (FA-1). The group actions of Property (TA-2) come from Theorem 9.15(3), and the normalization property (TA-3) follows from (4) and (5) of that Theorem.

Property (TA-4) follows from the Anosov closing lemma for actions of $\mathbb{R}^k \times M$. The proof of Lemma 4.15 in [48] is written for $\mathbb{R}^k$-actions, but works verbatim for $\mathbb{R}^k \times M$. Then, as in Lemma
Recall that a common stable foliation is a foliation $W$ whose leaves are given by $\bigcap_{k=1}^n W_{\alpha_k}^s(x)$ for some collection $a_1, \ldots, a_k$ of Anosov elements of $\mathbb{R}^k$. Common stable foliations are H"older foliations with smooth leaves (see [48, Lemma 4.5]).

**Lemma 13.2.** Consider a common stable foliation $W$, which is the sum of coarse Lyapunov distributions $E_{\alpha_1}, \ldots, E_{\alpha_n}$. Let $\{\beta_1, \ldots, \beta_N\}$ be an arbitrary listing of Lyapunov functionals such that $\bigcup_{i=1}^n [\alpha_i] = \{\beta_1, \ldots, \beta_N\}$. Then the map

$$\phi_x : \bigoplus_{i=1}^N E_{\beta_i}^\circ \to X$$

defined by

$$\phi_x(u_1, \ldots, u_N) = \exp(u_1)\exp(u_2)\cdots\exp(u_N) \cdot x$$

is surjective from a neighborhood of 0 in $\bigoplus_{i=1}^N E_{\beta_i}^\circ$ onto a neighborhood of $x$ in $W(x)$.

This lemma follows from [46, Lemma 3.2], which uses ideas from [35], particularly Corollary 4.5. We do not offer a complete proof, but summarize the strategy: note that if the actions of $N_\alpha$ were smooth, this would be true from the inverse function theorem. One may then use smooth approximations of the one-parameter subgroups and show that in the limit the original action is recovered, and the onto property persists by topological degree arguments.

**Lemma 13.3.** If $\mathbb{R}^k \times M \to X$ is a $C^2$ Anosov action satisfying (FA-1) and (FA-2) of the introduction, then it has integral Lyapunov coefficients.
Let $W$ be the foliation whose leaves are tangent to $\bigoplus_{k=1}^{n} E^{\chi_k}$, which is a common stable manifold. Notice that choosing $a \in \ker \beta$, and perturbing by a very small amount will yield an element $a'$ close to $a$ for which \( \left( \bigoplus_{\chi \in A} E^\chi \right) \ominus \left( \bigoplus_{\chi \in B} E^\chi \right) \) is a dominated splitting of $TW$. Therefore, \( \bigoplus_{c^p\chi \in A} E^{c^p\chi} \) is tangent to a foliation $W^A$.

Assuming we have fixed $c^p_1 \alpha$ and $c^p_2 \beta$, if $\chi \in \Sigma(\alpha, \beta)$, let $\hat{N}_\chi$ be the nilpotent group tangent to the subalgebra

$$\hat{n}_\chi = \bigoplus_{c^p_\chi \in A} E^{c^p_\chi}.$$

This is a (possibly trivial) subalgebra since each $a \in \mathbb{R}^k$ acts by an automorphism of $N_\chi$, so by standard Lie theory, \( [E^{c^p_\chi} \chi, E^{c^q_\chi} \chi] \subset E^{(c^p_\chi + c^q_\chi)} \chi \). Notice that each leaf $W^A$ has local $C^0$ surjections given by the maps from $\hat{N}_\chi \times \hat{N}_\chi \times \ldots \hat{N}_\chi \to W^A(x)$ by

(13.1)

$$\left( u_1, \ldots, u_n \right) \mapsto u_1 \cdot u_2 \cdots u_n \cdot x.$$

The proof of this claim uses Lemma 13.2. See also Lemma 5.11 of [48]. Essentially it follows from transversality of the distributions, the difficulty being that they are only Hölder. Indeed, these show they are global coordinates. In particular, if both $c^p_\chi \chi_k$ and $c^q_\chi \chi_\ell \in A$, \( [c^p_\chi \chi_k, c^q_\chi \chi_\ell] \subset A \).

Similarly, if both $c^p_\chi \chi_k$ and $c^q_\chi \chi_\ell$ belong to $B'$, then they both belong to $C = \{ u \alpha + v \beta : v \geq 1 \} \cap \Delta$. By identical arguments, choosing a perturbation of $b \in \ker \alpha$, gives \( [B', B'] \subset [C, C] \subset C \subset A \cup B' \).

We now consider the case when $c^p_\chi \chi_k \in B'$ and $c^q_\chi \chi_\ell \in A$ with $|k - \ell| > 1$ (the case when $c^p_\chi \chi_k \in A$ and $c^q_\chi \chi_\ell \in B'$ follows from a symmetric argument). Let $x \in X$, $u \in E^{c^p_\chi \chi_k} \subset \text{Lie}(N_{\chi_k})$ and $v \in E^{c^q_\chi \chi_\ell} \subset \text{Lie}(N_{\chi_\ell})$. We construct points related to a geometric commutator in the following way

$$y = e(u) \cdot x \quad x' = e(v) \cdot x$$

$$y' = e(v) \cdot y \quad w = e(u) \cdot x'. $$

Notice that $y' = [e(-v), e(-u)] \cdot w$ (see Figure 3).

Let $\Sigma_{k\ell} := \{ \chi_{k+1}, \ldots, \chi_{\ell-1} \}$ be the set of coarse exponents strictly between $\chi_k$ and $\chi_\ell$. Each coarse exponent of $\Sigma_{k\ell}$ splits into Lyapunov exponents, let $\Omega_{k\ell}$ denote the set of Lyapunov exponents proportional to a coarse exponent of $\Sigma_{k\ell}$. Now write $\Omega_{k\ell}$ as

$$\Omega_{k\ell} = \{ \gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_m, \epsilon_1, \ldots, \epsilon_m \},$$

where $\{ \gamma_\bullet \}$, $\{ \delta_\bullet \}$ and $\{ \epsilon_\bullet \}$ are the exponents of $A \cap \Omega_{k\ell}$, $(B \setminus B') \cap \Omega_{k\ell}$ and $B' \cap \Omega_{k\ell}$, respectively, with each subset listed in an order to be clarified later.

We assume $y$ and $w$ are sufficiently close, to be determined later (if we show it for sufficiently small $u, v$, we may use the dynamics of $R^k \cap X$ and (9.4) to conclude it for arbitrary $u, v$).

Notice that the distribution $\bigoplus_{s=k+1}^{\ell-1} E^{\chi_s}$ is uniquely integrable to a foliation $W^{k\ell}$ with $C^2$ leaves since it is the intersection of stable manifolds for the action. Since $y' = [e(-v), e(-u)] \cdot w$, $y' \in Y^{k\ell}(w)$ by Lemma 10.10. Therefore, by Lemma 13.2 applied to the splitting $TY^{k\ell}$ into the bundles $E^s$, $E^b$ and $E^c$, there exists a path moving from $w$ to $y'$ which first moves along curves of the form $\exp(w)$, where $w \in E^{s}$, to arrive at a point $p$. Then similarly along exponential images of $w \in E^{b}$, to arrive at a point $q$ from $p$. Finally, we move along the exponential images of vectors in $E^c$ to connect $q$ to $y'$. In this way, $p$ is obtained from $w$ after moving along curves tangent to $B'$,
and \( q \) is obtained from \( p \) by moving along curves tangent to \( B \setminus B' \). Then \( q \) is also connected to \( y' \) via curves tangent to \( A \). See Figure 3.

**Figure 3.** A geometric commutator

Recall that \( |\alpha, \beta| \) denotes the set of coarse exponents between \( \alpha \) and \( \beta \), including \( \alpha \) and \( \beta \), and let \( W^{[\alpha, \beta]} \) denote the foliation whose leaves are tangent to the sum of these distributions (since it is a common stable manifold, it is a Hölder foliation with \( C^2 \) leaves). Choose any pair of \( C^2 \) discs \( D_1 \ni x, y, D_2 \ni x', q \) of dimension \( \sum_{\omega \in B} \text{dim}(E^\omega) \) transverse to \( W^A \) inside of \( W^{[\alpha, \beta]}(x) \). This is possible since \( x \) and \( y \) are connected via \( c^\chi_k \chi_k \in B' \) and \( x' \) and \( q \) are connected via only curves tangent to coarse Lyapunov foliations corresponding to exponents in \( B \). Therefore, \( x' \) and \( q \) are the images of \( x \) and \( y \) under the \( W^A \) holonomy from \( D_1 \) to \( D_2 \). By [37, Section 8.3, Lemma 8.3.1], there exist bi-Lipschitz coordinates for which the leaves of the foliation \( W^A \) are parallel Euclidean hyperspaces. In particular, the holonomy along \( W^A \) is uniformly Lipschitz independent of the choice of \( D_1 \) and \( D_2 \), given sufficiently good transversality conditions, so \( d(x, y)/d(x', q) \) is bounded above and below by a constant.

We claim that \( p = q \) (ie, that no weights of \( (B \setminus B') \cap \Omega_{kl} \) appear). Roughly, the reason is that such weights contract too slowly. Indeed, pick an element \( a' \in \ker \alpha \) such that \( \beta(a') = -1 \). We may perturb \( a' \) to an element \( a \) which is regular and such that \( \alpha(a) < 0 \), and such that if \( \delta \in B \setminus B' \), \( \chi_k(a) < \delta(a) < 0 \). This is possible because if \( \delta = \sigma \alpha + \tau \beta \in B \setminus B' \), then \( \tau < 1 \), so \( \delta(a') = -\tau > -1 = \beta(a') \geq c^\chi_k \chi_k(a') \), since when \( c^\chi_k \chi_k \in B' \), the \( \beta \) coefficient is at least 1. This is clearly an open condition for each \( \delta \), so we may choose \( a \) as indicated. We may also assume, by rescaling \( a \) as necessary, that \( \beta(a) = -1 \).

Since \( y = \exp(u)x \), we can estimate distance between iterates of \( x \) and \( y \) using the intertwining property (TA-3). Recall that since \( c^\chi_k \chi_k \in B' \), \( c^\chi_k \chi_k(a) < \beta(a) = -1 \). Therefore \( d_{W^A}(a^t \cdot x, a^t \cdot y) = e^{tc^\chi_k \chi_k(a)}d_{W^A}(x, y) < e^{-t}d_{W^A}(x, y) \) using the Hölder metric along the leaves of \( W^A \). Now, suppose \( p \neq q \). Recall that \( p \) and \( q \) are connected by legs in \( B \setminus B' = \{ \delta_1, \ldots, \delta_m \} \), so that there exist \( p =
\[ x_0, x_1, \ldots, x_m = q \text{ such that } x_s x_{s-1} \text{ are connected by a short curve everywhere tangent to } E^{\delta_s}. \]
Since the distributions \( E^{\delta_s} \) are transverse, if \( p \neq q \), there exists some \( s \) for which \( x_s \neq x_{s-1} \). Without loss of generality, we assume that \( \{\delta_s\} \) are ordered such that \( 0 > \delta_1(a) > \delta_2(a) > \cdots > \delta_m(a) > \chi_i(a). \)
Then let \( s_0 \) be the minimal \( s \) for which \( x_s \neq x_{s-1} \), and \( c_1 = \delta_{s_0}(a), c_2 = \delta_{s_0+1}(a). \) Notice that \( 0 > c_1 > c_2 > -1 \).
By minimality, we get \( x_{s_0-1} = p. \)

Let \( d \) denote the Riemannian distance on the manifold. Since for any \( \chi \in \Delta \), the distance along each \( W^x \) leaf is locally Lipschitz equivalent to the distance on the manifold, there exists \( L > 0 \) such that for all \( \chi \in \Delta \) and sufficiently close points \( z \in W^X(z') \), we have \( L^{-1}d_{W^X}(z, z') \leq d(z, z') \leq Ld_{W^X}(z, z') \).

Then after applying the triangle inequality, for sufficiently large \( t \), we get:
\[
d(a^t \cdot x', a^t \cdot q) \geq L^{-1}d_{W^X}(z, z') \leq d(z, z') \leq Ld_{W^X}(z, z').
\]

since by construction, apply the triangle inequality to all legs connecting \( a^t \cdot x_{s_0} \) and \( a^t \cdot q \) and \( a^t \cdot x' \) and \( a^t \cdot p \), which contract faster than \( e^{-ct} \) and \( e^{-t} \), respectively, since \( c_1 > c_2 > -1 \).
We may construct new disks \( D_1, D_2 \) with sufficient transversality conditions to \( W^A \) connecting \( a^t \cdot x \) and \( a^t \cdot y \), and \( a^t \cdot x' \) and \( a^t \cdot q \) (note that we may not simply iterate the disks \( D_1, D_2 \) forward, since \( W^A \) is not the fast foliation for \( a \), and the transversality may degenerate). Therefore, since each of the foliations along weights of \( B \) are uniformly transverse to those of \( A \), and the holonomies are Lipschitz with a uniform Lipschitz constant on sufficiently transverse discs, we arrive at a contradiction to the fact that \( d(a^t \cdot x, a^t \cdot y) < C^0 e^{-t} \), so \( p = q. \)

Therefore, the connection between \( w \) and \( y' \) only involves weights of \( A \cup B' \). By the induction hypothesis, the commutator of two weights in \( (A \cup B') \cap \Omega_{k\ell} \) produces only weights in \( (A \cup B') \cap \Omega_{k\ell} \).
Hence, for any \( x, \)
\[
e(v) \ast e(u) \cdot x = p_1 \ast e(u) \ast e(v) \cdot x = e(u) \ast e(v) \ast p_2 \cdot x
\]

for some paths \( p_1 \) and \( p_2 \) which only involve \( e(w) \) for some \( w \) in the weight spaces \( A \cup B' \), but may depend on \( x, u \) and \( v \). Using such relations, for any word in the weights \( A \cup B' \), we may put it in a desired circular ordering without weights in \( B \setminus B' \). This proves the inductive step, and hence the lemma, since \( B \setminus B' = \{\sigma \alpha + \tau \beta : \sigma, \tau < 1\} \cap \Delta. \)

\[ 13.1. \textbf{Regularity of conjugacies and proof of Theorem 2.6.} \text{In this section, we work in two regularities: } C^2 \text{ and } C^\infty. \]

\textbf{Theorem 13.4.} If \( \mathbb{R}^k \times M \curvearrowright X \) is a totally Anosov \( C^r \)-action on a \( C^\infty \)-manifold \( X \) for \( r = \infty \) or \( 2 \) satisfying assumptions (FA-1) and (FA-2), then there exists a (bi)homogeneous space \( K \setminus H/\Gamma \) and a \( C^r \)-diffeomorphism \( h : X \to K \setminus H/\Gamma \) such that, setting \( \tilde{a} = h \circ a \circ h^{-1} \) for any \( a \in \mathbb{R}^k \times M \), defines a bi-homogeneous action, where \( r' = \infty \) or \( (1, \theta) \) for any \( \theta \in (0, 1) \), respectively. Moreover, any \( C^0 \) conjugacy of the \( \mathbb{R}^k \times M \)-action is \( C^{r'}. \)

\textit{Proof.} By Proposition 13.1 and Theorem 10.7, it follows that there is a \( C^0 \)-conjugacy between the canonical lift \( \tilde{X} \) produced in Theorem 9.15 and a translation action on a homogeneous space \( H/\Gamma \). The \( C^0 \) conjugacy between \( \tilde{X} \) and \( H/\Gamma \) induces a conjugacy \( h \) between \( X \) and \( K \setminus H/\Gamma \). So it suffices to show that any \( C^0 \) conjugacy between \( X \) and \( K \setminus H/\Gamma \) is \( C^{r'}. \)

The coarse Lyapunov manifolds are parameterized by nilpotent group actions, which are subgroups of the full group \( H \) in the homogeneous model (and hence act smoothly on the space \( H/\Gamma \)).
They are conjugated to the actions $N_\alpha \ltimes X$ produced in Theorem 9.15, which act by $C^{r'}$ diffeomorphisms on their orbits by Lemma 9.12 or 9.10.

By standard Lie theoretic arguments (see, e.g., [42], Section 5.1), since each element is a diffeomorphism, the group action is $C^1$ in both cases. In the $C^\infty$ setting, it also follows immediately that the group action is $C^\infty$, and therefore provides $C^\infty$-coordinates. Therefore, the conjugacy is $C^\infty$ restricted to each leaf of a coarse Lyapunov foliations.

In the $C^2$-case, we still have that $h$ is $C^1$ along leaves of $W^\alpha$. Notice that if $\|\cdot\|_\alpha$ is the Hölder metric on $T W^\alpha$ constructed using (FA-2) in Section 9.1, then it is invariant under the isometries in Proposition 9.2. Then $h_*\|\cdot\|_\alpha$ is a norm which is invariant under right translation on each leaf of $W^\alpha$ in $H/\Gamma$ (notice that while right translation is not defined on all of $H/\Gamma$, it is well-defined on each $W^\alpha$-leaf). In particular, $h_*\|\cdot\|_\alpha$ is $C^\infty$ on each $W^\alpha$-leaf of $H/\Gamma$. Therefore, $h : W^\alpha(x) \to W^\alpha(h(x))$ is an isometry between a Hölder metric on the leaf in $\bar{X}$ and $C^\infty$ metric on the leaf in $H/\Gamma$, and is therefore $C^{r'}$ by Theorem 5.4.

The conjugacy $h$ is clearly $C^{r'}$ along the $\mathbb{R}^k \times K \times M$-orbits. The smoothness of $h$ will now follow from iterated applications of Theorem 5.3. Fix a regular element $a$. We show that $h$ is $C^{r'}$ along the foliations $W^s_a, W^u_a$ and $\mathbb{R}^k \times M \times K$. Then by Theorem 5.3, since $\mathbb{R}^k \times M \times K$ and $W^s_a$ foliate the center-stable manifold as transverse subfoliations, it will follow that $h$ is $C^{r'}$ along the center-stable foliation. Then again, since the conjugacy is $C^{r'}$ along the center-stable foliation and the unstable foliation, another application Theorem 5.3 shows that $h$ is $C^{r'}$.

So we must show that $h$ is $C^{r'}$ along the foliations $W^s_a$ and $W^u_a$. We show it for $W^s_a$; the proof for $W^u_a$ follows by considering $-a$. List the coarse exponents $\Delta^\alpha(a) = \{\alpha_1, \ldots, \alpha_n\}$ in circular ordering, so that there are foliations $\mathcal{W}_i$ such that $TW_i = \bigoplus_{j=1}^i T W^\alpha_j$. These foliations are obtained as intersections of stable manifolds from different Weyl chambers. We claim that $h$ is $C^{r'}$ along $\mathcal{W}_i$ by induction on $i$.

We have already established the base case of $i = 1$. Assume that $h$ is $C^{r'}$ along $\mathcal{W}_i$. Then by construction, $\mathcal{W}_{i+1}$ is a foliation whose leaves have two transverse subfoliations: $\mathcal{W}_i$ and $W^\alpha_{i+1}$. Since we know that $h$ is $C^{r'}$ along each by induction and since $W^\alpha_{i+1}$ is a coarse Lyapunov foliation, it follows that $h$ is $C^{r'}$ along $\mathcal{W}_{i+1}$ (again by Theorem 5.3). This proves the inductive step. Since $W^s_a = \mathcal{W}_n$, it follows that $h$ is $C^{r'}$ along $W^s_a$ and therefore, $h$ is $C^{r'}$. □

**APPENDIX A. NORMAL FORMS FOR CONTRACTING FOLIATIONS**

We recall some aspects of normal forms theory, following [27, 26] which contains optimal results in the uniformly contracting setting. If $f : X \to X$ is a $C^\infty$ diffeomorphism of a Riemannian manifold with norm $\|\cdot\|$ preserving a continuous foliation $\mathcal{W}$ with $C^\infty$ leaves, and $\chi = (\chi_1, \ldots, \chi_\ell)$ is an $\ell$-tuple of negative numbers such that $\chi_1 < \cdots < \chi_\ell < 0$, we say that $f$ has $(\chi, \varepsilon)$-spectrum on $\mathcal{W}$ if there is a splitting $T \mathcal{W} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$ into invariant subbundles such that for every $v \in \mathcal{E}_i$,

$$e^{\chi_i - \varepsilon} \leq \|df(v)\| / \|v\| \leq e^{\chi_i + \varepsilon}.$$ 

By Remark 4.2 of [26], this is sufficient to obtain the usual narrow band condition on the Mather spectrum if $\varepsilon$ is sufficiently small. Write a vector $v \in T_x \mathcal{W}$ in coordinates as $v = (v_1, \ldots, v_\ell)$, where $v_i \in \mathcal{E}_i$. A polynomial $q : \mathcal{E}_x \to \mathcal{E}_y$ is said to be $(s_1, \ldots, s_\ell)$-homogeneous if

$$q(\lambda_1 v_1, \ldots, \lambda_\ell v_\ell) = \lambda_1^{s_1} \cdots \lambda_\ell^{s_\ell} q(v_1, \ldots, v_\ell).$$
Then with a fixed \( \ell \)-tuple \((\chi_1, \ldots, \chi_\ell)\), say that a polynomial \( p : E_x \to E_y \) is of subresonance type (with respect to \( \chi \)) if

\[
p(v_1, \ldots, v_\ell) = (p_1(v_1, \ldots, v_\ell), \ldots, p_\ell(v_1, \ldots, v_\ell)),
\]

and for \( i = 1, \ldots, \ell, \) \( p_i \) is a sum of \((s_1, \ldots, s_\ell)\)-homogeneous polynomials such that \( \chi_i \leq \sum s_j \chi_j \).

It is of resonance type (with respect to \( \chi \)) if each \( p_i \) is a sum of \((s_1, \ldots, s_\ell)\)-homogeneous polynomials such that \( \sum s_j \chi_j = \chi_i \).

The following is a consequence of Theorem 4.6 of [26].

**Theorem A.1.** Let \( f : X \to X \) be a \( C^\infty \) diffeomorphism of a \( C^\infty \) manifold \( X \) preserving a continuous foliation \( W \) with \( C^\infty \) leaves. If \( \chi = (\chi_1, \ldots, \chi_\ell) \) is an \( \ell \)-tuple as above, then there exists a constant \( \varepsilon = \varepsilon(\chi) > 0 \) with the following property: if there exists a smooth Riemannian metric for which \( df|_{TW} \) has \((\chi, \varepsilon)\)-spectrum, then there exists a family of \( C^\infty \) diffeomorphisms \( H_x : T_x W \to W(x) \) such that

- (NF-1) for every \( x \in X \), \( H_{f(x)}^{-1} \circ f \circ H_x \) is a subresonance polynomial,
- (NF-2) for every \( x \in X \), \( dH_x = \text{Id} \),
- (NF-3) if \( G_x \) is any other such family, then \( G_x = H_x \circ p_x \), for some family of subresonance polynomials \( p_x \),
- (NF-4) if \( y \in W(x) \), then \( H_y = H_x \circ q_{x,y} \) for a composition of a translation with some subresonance polynomial \( q_{x,y} \),
- (NF-5) if \( g : X \to X \) is a \( C^\infty \) diffeomorphism which commutes with \( f \), then \( H_{g(x)}^{-1} \circ g \circ H_x \) is a subresonance polynomial.

**Appendix B. Polynomial functions along transverse foliations**

The following was shown by Margulis ([40, Lemma 4] or [41, Lemma 17]), who proved it for rational functions in the measurable setting assuming Lebesgue almost everywhere properties (cf. also [57, Theorem 3.4.4] and its proof.) We provide a proof for polynomials in the continuous case which is much simpler and more straightforward.

**Lemma B.1.** Let \( V \) be a vector space, and \( f : V \to \mathbb{R} \) be a continuous function such that \( t \mapsto f(v + tw) \) is a polynomial in \( t \) for every \( v, w \in V \). Then \( f \) is a polynomial.

**Proof.** We prove that for any collection of linearly independent elements \( v_1, \ldots, v_n \), the map

\[
(t_1, \ldots, t_n) \mapsto f(w + \sum t_i v_i)
\]

is a polynomial in \((t_1, \ldots, t_n)\) by induction on \( n \). Notice that the base case of \( n = 1 \) is the assumption of the lemma, and the case of \( n = \dim(V) \) proves the conclusion.

The degree of the polynomials must be uniformly bounded by some constant \( N \) since the map \( f \) is continuous. Assume we have shown it for subspaces of dimension \( n - 1 \), and consider a collection \( v_1, \ldots, v_n \). Then by assumption, fixing \( t_2, \ldots, t_n \) and letting \( t_1 \) vary yields a continuous family of polynomials in \( t_1 \), so

\[
(B.1) \quad f(t_1 v_1 + \cdots + t_n v_n) = a_0(t_2, \ldots, t_n) + a_1(t_2, \ldots, t_n)t_1 + \cdots + a_N(t_2, \ldots, t_n)t_1^N
\]

for some collection of continuous functions \( a_i : \mathbb{R}^{n-1} \to \mathbb{R} \). By induction, the functions

\[
p_k(t_2, \ldots, t_n) := f(kv_1 + t_2 v_2 + \cdots + t_n v_n)
\]

for every \( k \in \mathbb{R} \). Then for every \( k \),

\[
\sum_{r=0}^{N} \sum_{\ell=0}^{r} a_{r}(t_{\ell+2}, \ldots, t_{\ell+n}) k^\ell t_1^r
\]

is continuous in \((t_1, t_2, \ldots, t_n)\), hence is a polynomial in \((t_1, t_2, \ldots, t_n)\).
are polynomials. Applying (B.1) to the definition of the polynomials \( p_k \) yields the following system of equations:

\[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
2 & 4 & 8 & \ldots & 2^N \\
3 & 9 & 27 & \ldots & 3^N \\
\vdots \\
N + 1 & (N + 1)^2 & (N + 1)^3 & \ldots & (N + 1)^{N+1}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\vdots \\
a_N
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
p_{N+1}
\end{pmatrix}
\]

The matrix appearing is a Vandermonde matrix which is invertible, so each function \( a_i \) is actually a linear combination of the polynomials \( p_1, \ldots, p_N \). Therefore, (B.1) shows that the function is a polynomial.

\( \square \)

### Appendix C. Centralizers of ergodic homogeneous actions

**Theorem C.1.** Let \( G \) be a simply connected Lie group, \( \phi_t : G/\Gamma \to G/\Gamma \) be an ergodic homogeneous flow generated by an \( \mathbb{R} \)-semisimple element. Let \( Z_{\text{Lip}} \) denote the set of Lipschitz transformations commuting with \( \phi_t \). Then \( Z_{\text{Lip}} = Z_{\text{Aff}} \), the group of affine transformations commuting with \( \phi_t \).

The principle tool in proving Theorem C.1 is a result of Zeghib, which we summarize here. If \( X \) is a \( C^\infty \) manifold, a subset \( N \subset X \) is called rectifiable if it is the Lipschitz image of a bounded subset of \( \mathbb{R}^n \) for some \( n \). If \( N \) is a rectifiable subset, it has a well-defined Hausdorff measure, which we denote by \( \mu_N \).

**Theorem C.2** (Théorème A, [56]). Let \( G \) be a simply connected Lie group, \( \phi_t : G/\Gamma \to G/\Gamma \) be an ergodic homogeneous flow, generated by an \( \mathbb{R} \)-semisimple element. If \( N \subset X \) is a rectifiable, \( \phi_t \) invariant set, then \( \mu_N \) is \( \phi_t \)-invariant, and the ergodic components of \( \mu_N \) are the Haar measures on closed \( H \)-orbits in \( G/\Gamma \), where \( H \) is some fixed closed subgroup \( H \subset G \).

**Remark C.3.** In fact, Zeghib claims more, by working with bi-homogeneous flows, but we will only use the homogeneous flow version here.

**Proof of Theorem C.1.** Let \( f \in Z_{\text{Lip}} \), and consider the graph

\[
N = \{(x, f(x)) : x \in G/\Gamma \} \subset (G \times G)/(\Gamma \times \Gamma).
\]

By construction, since \( f \) commutes with \( \phi_t \), \( N \) is \( \phi_t \times \phi_t \)-invariant. Furthermore, since \( f \) is Lipschitz, \( N \) is rectifiable. Therefore, by Theorem C.2, \( \mu_N \), the Hausdorff measure on \( N \), is \( \phi_t \)-invariant. \( \phi_t \) is ergodic on \( G/\Gamma \), \( \phi_t \times \phi_t \) must be ergodic on \( N \) as well.

Therefore, there exists a unique subgroup \( H \subset G \times G \) such that \( N = (g, h) \cdot H/(\Gamma \times \Gamma) \) for some \( (g, h) \in G \times G \). Since \( (e\Gamma, f(e\Gamma)) \in N \), we may without loss of generality assume \( g = e \).

Finally, consider the Lipschitz transformation \( \pi : N \to G/\Gamma \) defined by \( \pi(x, y) = x \). Since \( N \) is the graph of a Lipschitz transformation, \( \pi \) is a Lipschitz homeomorphism. This further implies that \( \pi \) is a diffeomorphism, since \( N \) is a coset of \( H \). Hence, \( \dim(H) = \dim(G) \), and for each \( X \in \text{Lie}(G) \), there exists a unique \( \bar{F}(X) \in \text{Lie}(G) \) such that \( (X, \bar{F}(X)) \in \text{Lie}(H) \). It follows immediately that \( \bar{F} \) is a Lie algebra homomorphism since \( H \) is a subgroup. Let \( F : G \to G \) denote the lift of \( \bar{F} \) to \( G \).

It is immediate that \( H = \{(g, F(g)) : g \in G \} \). Therefore, from the definition of \( N \), we get that \( f(g\Gamma) = hF(g)\Gamma \), and \( f \) is affine, as claimed. \( \square \)
Appendix D. Lifting G-actions

Theorem D.1. Let r = ∞ or 2 and G be a real semisimple group such that every simple factor has real rank at least 2. Let G ∞ X be a C action which is totally Anosov with respect to some split Cartan subgroup A, and assume that

- π : Y → X is a C fiber bundle over X,
- the action of CG(A) lifts to a C action on Y,
- for each Weyl chamber C of A ⊂ G there exists some distinguished a ∈ C such that the lifted action of a on Y is partially hyperbolic with respect to E_y = T(π⁻¹(π(y))) ⊕ T(CG(A) · y), the sum of the tangent bundle to the fibers of π and orbits of CG(A).

Then there exists some continuous action of G, on Y which is a lift of the G-action in the sense that p : G → G is the canonical projection, π(g · x) = p(g) · π(x).

Furthermore, if there exists a continuous metric on Y for which d_{Tπ⁻¹(π(y))} is isometric for every a ∈ A and y ∈ Y, the G-action is C.

The proof of this theorem follows the scheme introduced by the first author and Katok in [14], extended by Zhenqi Wang in [52, 53] and made fully general by the third author in [51]. We summarize some important definitions before proceeding with the proof.

Let ∆_G denote the set of roots of G such that χ/2 is not a root, and for each χ ∈ ∆_G, let U_χ denote corresponding coarse Lyapunov subgroup. As described in Section 6.2, we consider the free product P of the groups U_χ, where χ ranges over ∆_G, and note that since the groups U_χ generate G, there exists a projection π : P → G such that the kernel C is exactly the expressions in P which yield contractible cycles on G.

Since G acts on X, any relations among the U_χ on G hold on X as well. Importantly, the commutator relations hold: if χ_1, χ_2 ∈ ∆_G are linearly independent, u ∈ U_χ_1 and v ∈ U_χ_2, then [u, v] ∈ U_χ s.t. χ = sχ_1 + tχ_2, χ_1, χ_2 ∈ Z_+ of particular interest:

Lemma D.2 (Lemma 4.7, [51]). Let P denote the free product of the coarse Lyapunov subgroups of G, and C_S ⊂ P denote the normal closure (in P) of the group generated by the relations v⁻¹ * u⁻¹ * v * u * ρ⁻¹ ∈ P, where ρ is any presentation of [u, v] in the group U_χ. Then C_S is a co-abelian subgroup of C, and every continuous action of C/C_S on a finite dimensional space is trivial.

Proof of Theorem D.1. Let a ∈ A be one of the distinguished elements of the Weyl chambers which lift to parially hyperbolic maps. Consider W_a,X and W_a,Y, the stable and unstable foliations of a as it acts on X and Y, respectively. Since the fibers of π are contained in the center foliation, it follows that π|W_a,X(y) is a C diffeomorphism. For every χ, we may therefore build an action U_χ ⊂ Y by choosing some a ∈ ℝ^k such that χ(a) < 0 and letting u · v denote the unique element of W_a,Y(y) which projects to u · x ∈ W_a,X(π(y)). Note that the action does not depend on the choice of a or Weyl chamber which a belongs to. By the universal property of (topological) free products, we may construct an action of P, the free product of the groups U_χ, on Y as well.

Fix any pair of linearly independent roots χ_1, χ_2 ∈ ∆_G. Then, by linear independence, there exists a ∈ ℝ^k such that χ_1(a), χ_2(a) < 0. If u ∈ U_χ_1 and v ∈ U_χ_2, then by Lemma D.2, there exists a w written as a product of elements from the groups U_χ, χ = sχ_1 + tχ_2, s, t ∈ Z_+ such that
\[ u, v \] w^{-1} = e, as elements in \( G \). Since the relations hold on \( G \), they act trivially on \( X \). Furthermore, since the orbits of \( U_{\chi_1}, U_{\chi_2} \) and every group appearing in the presentation of \( w \) are contained in \( W_{a,X} \), the relation holds when the groups act on \( Y \) as well.

Therefore, the restriction of the \( \mathcal{P} \)-action on \( Y \) to \( \mathcal{C} \) is trivial. Hence there exists a well-defined continuous action of \( \mathcal{C}/\mathcal{C}_{\mathcal{S}} \) on \( Y \). This action must be trivial by Lemma D.2. Finally, since the \( \mathcal{C} \) action is trivial, the action of \( \mathcal{P} \) induces an action of \( \tilde{G} = \mathcal{P}/\mathcal{C} \) on \( Y \), which by construction must cover the action on \( X \).

To prove the regularity of the action, notice that the lift of the actions of the groups \( U_{\chi} \) to \( Y \) is exactly determined by stable holonomies. Since, restricted to \( T\pi^{-1}(\pi(y)) \), the action is isometric, the stable holonomies will have arbitrarily good pinching properties. By now-standard arguments in partial hyperbolicity (see, eg, [43, Theorem 6.1]), it follows that the holonomies, and hence action of each of the groups \( U_{\chi} \) on \( Y \) is \( C^{r} \). Finally, since the \( U_{\chi} \) generate \( \tilde{G} \), every element of \( \tilde{G} \) is a \( C^{r} \) transformation of \( Y \). By Theorem 5.5, the action \( \tilde{G} \curvearrowright Y \) is a \( C^{r} \) group action. \( \square \)

**Remark D.3.** The smoothness of the lift can be obtained from a weaker assumption, namely center bunching. To obtain a \( C^{\infty} \) lift, one requires a more restrictive form of center bunching than that laid out in Section 5.1, which allows for no exponents on the fiber whatsoever (i.e., something imitating unipotent behavior). Since our application is to a compact group extension, and the action of a semisimple Lie group will always be conformal, we use this simpler version of the statement.

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