Niebrzydowski Algebras and Trivalent Spatial Graphs

Paige Graves* Sam Nelson † Sherilyn Tamagawa‡

Abstract

We introduce Niebrzydowski algebras, algebraic structures with a ternary operation and a partially defined multiplication, with axioms motivated by the Reidemeister moves for Y-oriented trivalent spatial graphs and handlebody-links. As part of this definition, we identify generating sets of Y-oriented Reidemeister moves. We give some examples to demonstrate that the counting invariant can distinguish some Y-oriented trivalent spatial graphs and handlebody-links.

Keywords: Biquasiles, tribrackets, spatial graphs, handlebody-links

2010 MSC: 57M27, 57M25

1 Introduction

In [12] an algebraic structure known as ternary quasigroup was used for coloring planar knot complements. In [9], a related algebraic structure known as biquasile was introduced with axioms derived from the Reidemeister moves for oriented knots and links with elements used to color vertices for dual graph diagrams corresponding to regions in the planar complement of a knot diagram. Biquasile coloring invariants were enhanced with Boltzmann weights in [2] which are related to cocycles in the cohomology theory for ternary quasigroups in [13]. Biquasile colorings and ternary quasigroups were used to distinguish orientable surface links in [7] and [12]. In [10] structures known as virtual tribrackets were defined for coloring planar complements of oriented virtual link diagrams.

Spatial graphs, i.e. knotted graphs in R^3, were considered in terms of diagrammatic moves in [6]. Handlebody-links and their Reidemeister moves were considered in [3], and Y-orientations of trivalent spatial graphs and handlebody-links were considered in [4]. Quandle and biquandle colorings of handlebody-links have been considered in [5, 8], including algebraic structures such as G-families of quandles and biquandles, partially multiplicative quandles and biquandles and qualgebras. In [11], the algebraic structure known as bikei (see [1]) was extended by adding a partially defined multiplication to define a coloring structure for unoriented twisted virtual handlebody-links called twisted virtual bikeigebras.

In this paper we extend tribrackets to define coloring structures for planar complements of Y-oriented trivalent spatial graphs and Y-oriented handlebody-links, obtaining objects we call Niebrzydowski Algebras. These are sets with a ternary operation (·, ·) called a Niebrzydowski tribracket and a partially defined multiplication operation satisfying axioms coming from the Reidemeister moves for oriented trivalent spatial graphs. A certain subset of these objects satisfy the additional condition required by the IH-move, making them suitable for coloring Y-oriented handlebody-links.

The paper is organized as follows. In Section 2 we review the Reidemeister moves for Y-oriented trivalent spatial graphs and handlebody-links, identifying our preferred generating set of oriented moves. In Section 3 we introduce our algebraic structures and coloring rules and show that the cardinality of the set of colorings of a diagram representing an oriented trivalent spatial graph or a
Y-oriented handlebody-link defines an integer-valued invariant. We compute some explicit examples of the new invariant. We conclude in Section 4 with some questions for future work.

2 Trivalent Spatial Graphs

A (non-oriented) trivalent spatial graph is a finite graph with all vertices degree 3 embedded in $\mathbb{R}^3$, where two embedded graphs are considered equivalent if there is an ambient isotopy $\mathbb{R}^3 \to \mathbb{R}^3$ taking one to the other.

From [6] we know that two (non-oriented) trivalent spatial graphs are equivalent iff their diagrams are related by a finite sequence of the Reidemeister moves

We orient the edges of a trivalent spatial graph, forbidding sources and sinks at the vertices. Such an orientation is known as a Y-orientation.

Then, by listing all the possible orientations on all edges of each of $R4$ and $R5$, we see that we have the generating list of moves below.

\begin{center}
\begin{tabular}{cccc}
\text{R4.1} & \text{R4.2} & \text{R4.3} & \text{R4.4} \\
\includegraphics[width=0.2\textwidth]{R4-01} & \includegraphics[width=0.2\textwidth]{R4-02} & \includegraphics[width=0.2\textwidth]{R4-03} & \includegraphics[width=0.2\textwidth]{R4-04} \\
\text{R4.5} & \text{R4.6} & \text{R4.7} & \text{R4.8} \\
\includegraphics[width=0.2\textwidth]{R4-05} & \includegraphics[width=0.2\textwidth]{R4-06} & \includegraphics[width=0.2\textwidth]{R4-07} & \includegraphics[width=0.2\textwidth]{R4-08} \\
\text{R4.9} & \text{R4.10} & \text{R4.11} & \text{R4.12} \\
\includegraphics[width=0.2\textwidth]{R4-09} & \includegraphics[width=0.2\textwidth]{R4-10} & \includegraphics[width=0.2\textwidth]{R4-11} & \includegraphics[width=0.2\textwidth]{R4-12} \\
\text{R5.1} & \text{R5.2} & \text{R5.3} & \text{R5.4} \\
\includegraphics[width=0.2\textwidth]{R5-01} & \includegraphics[width=0.2\textwidth]{R5-02} & \includegraphics[width=0.2\textwidth]{R5-03} & \includegraphics[width=0.2\textwidth]{R5-04} \\
\end{tabular}
\end{center}
Lemma 1. We can obtain a generating set of Reidemeister moves on trivalent spacial graphs by taking one move from each of the following sets 

\{R4.1, R4.2, R4.3, R4.7, R4.8, R4.9\},  
\{R4.4, R4.5, R4.6, R4.10, R4.11, R4.12\}  
\{R5.1, R5.2, R5.3, R5.7, R5.8, R5.9\}  
\{R5.4, R5.5, R5.6, R5.10, R5.11, R5.12\}  
\{R5.13, R5.14, R5.15, R5.19, R5.20, R5.21\}  
\{R5.16, R5.17, R5.18, R5.22, R5.23, R5.24\} 

Together with any minimal generating set of oriented classical Reidemeister moves.

For a list of minimal generating sets of oriented classical Reidemeister moves, see [14].

Proof. Through the sequence of moves given by

\[
\begin{align*}
R5 & \quad R2 \\
\end{align*}
\]

and by assigning different admissible combinations of orientations to each of the edges, we see that, together with \(R2\), we have the sequences of implications 

\[R5.1 \Rightarrow R5.9 \Rightarrow R5.2 \Rightarrow R5.7 \Rightarrow R5.3 \Rightarrow R5.8 \Rightarrow R5.1\]
and

\[ R5.4 \Rightarrow R5.12 \Rightarrow R5.5 \Rightarrow R5.10 \Rightarrow R5.6 \Rightarrow R5.11 \Rightarrow R5.4 \]

so the sets

\[ \{ R5.1, R5.2, R5.3, R5.7, R5.8, R5.9 \} \]

and

\[ \{ R5.4, R5.5, R5.6, R5.10, R5.11, R5.12 \} \]

give sets of equivalent moves.

Similarly, using the sequence of moves and assigning different admissible orientations to each of the edges, we see that, together with \( R2 \), we have the sequences of implications

\[ R5.13 \Rightarrow R5.21 \Rightarrow R5.14 \Rightarrow R5.19 \Rightarrow R5.15 \Rightarrow R5.20 \Rightarrow R5.13 \]

and

\[ R5.16 \Rightarrow R5.24 \Rightarrow R5.17 \Rightarrow R5.22 \Rightarrow R5.18 \Rightarrow R5.23 \Rightarrow R5.16 \]

Thus the sets

\[ \{ R5.13, R5.14, R5.15, R5.19, R5.20, R5.21 \} \]

and

\[ \{ R5.16, R5.17, R5.18, R5.22, R5.23, R5.24 \} \]

give sets of equivalent moves.

Now, we see that

\[ R4.1 \Rightarrow R4.7 \]
\[ R4.4 \Rightarrow R4.10 \]
\[ R4.2 \Rightarrow R4.9 \]
\[ R4.5 \Rightarrow R4.12 \]
\[ R4.3 \Rightarrow R4.8 \]
\[ R4.6 \Rightarrow R4.11 \]

and

\[ R4.7 \Rightarrow R4.1 \]
\[ R4.10 \Rightarrow R4.4 \]
\[ R4.8 \Rightarrow R4.3 \]
\[ R4.11 \Rightarrow R4.6 \]
\[ R4.9 \Rightarrow R4.2 \]
\[ R4.12 \Rightarrow R4.5 \]
so the sets

\{R4.1, R4.7\}, \{R4.2, R4.9\}, \{R4.3, R4.8\}, \{R4.4, R4.10\}, \{R4.5, R4.12\}, and \{R4.6, R4.11\}

are sets of equivalent moves, if we assume \(R2\).

Furthermore, if we assume the \(R5\)-type moves and \(R1\), the sequence of moves

![Diagram](image)

gives us that

\[
R4.2 \Rightarrow R4.7 \quad R4.1 \Rightarrow R4.9 \quad R4.6 \Rightarrow R4.10 \quad R4.4 \Rightarrow R4.11
\]

Similarly, we have

![Diagram](image)

which tells us that, together with the classical Reidemeister moves and the \(R5\)-types moves,

\[
R4.3 \Rightarrow R4.7 \quad R4.1 \Rightarrow R4.8 \quad R4.5 \Rightarrow R4.10 \quad R4.4 \Rightarrow R4.12
\]

So, we have sets of equivalent moves

\[
\{R4.1, R4.2, R4.3, R4.7, R4.8, R4.9\}
\]

and

\[
\{R4.4, R4.5, R4.6, R4.10, R4.11, R4.12\}.
\]

Putting this all together, we have equivalence classes of moves given by

\[
\{R4.1, R4.2, R4.3, R4.7, R4.8, R4.9\},
\{R4.4, R4.5, R4.6, R4.10, R4.11, R4.12\}
\]

\[
\{R5.1, R5.2, R5.3, R5.7, R5.8, R5.9\}
\]

\[
\{R5.4, R5.5, R5.6, R5.10, R5.11, R5.12\}
\]

\[
\{R5.13, R5.14, R5.15, R5.19, R5.20, R5.21\}
\]

\[
\{R5.16, R5.17, R5.18, R5.22, R5.23, R5.24\}
\]

So, for a reduced Reidemeister moves set, we only need to choose one from each class, together with a minimal generating set of classical oriented Reidemeister moves.

By choosing a specific move from each equivalence class, we have the following corollary.

**Corollary 2.** The moves
together with a generating set of oriented classical Reidemeister moves form a generating set of oriented Reidemeister moves for trivalent spatial graphs.

3 Niebrzydowski Algebras

We begin with a modification of a definition from [12].

**Definition 1.** Let \( X \) be a set. A ternary operation \([\cdot, \cdot, \cdot] : X \times X \times X \to X\) on \( X \) is a Niebrzydowski tribracket or just a tribracket if it satisfies the conditions

(i) Any three of the four \( \{a, b, c, d\} \) in the equation \([a, b, c] = d\) determines the fourth,

(ii) For all \( a, b, c, d \in X \) we have

\[
[a, b, [b, c, d]] = [a, [a, b, c], [[a, b, c], c, d]]
\]

\[
[[a, b, c], c, d] = [[[a, b, [b, c, d]], [b, c, d]], d].
\]

**Example 1.** Any module over a ring \( R \) has a Niebrzydowski bracket structure defined by setting

\([a, b, c] = xa - xyb + yc\)

where \( x, y \in R^X \) are units in \( R \). This structure is known as an Alexander tribracket; see [10] for more.

**Example 2.** We can specify a Niebrzydowski algebra structure on a finite set \( X = \{1, 2, \ldots, n\} \) by listing the operation tables. Specifically, the Niebrzydowski bracket can be expressed as a 3-tensor, i.e. a vector whose entries are matrices, with the convention that to find \([a, b, c]\) we look in matrix \( a \), row \( b \) column \( c \). For example, in the tribracket structure on \( \{1, 2, 3\} \) specified by

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1
\end{bmatrix}
, 
\begin{bmatrix}
2 & 3 & 1 \\
1 & 2 & 3 \\
3 & 1 & 2
\end{bmatrix}
, 
\begin{bmatrix}
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 2 & 3
\end{bmatrix}
\]
we have $[1, 2, 3] = 2$ and $[2, 3, 1] = 3$.

**Definition 2.** Let $X$ be a set with a Niebrzydowski tribracket $[,]$. Then $X$ is a *Niebrzydowski algebra* if $X$ has a partially defined product $a, b \mapsto ab$ satisfying

(i) Any two of the three $\{a, b, c\}$ in $ab = c$ determines the third,

(ii) For all $a, b \in X$ we have

$$[a, ab, b] = ab \quad (iv.i)$$

and

(iii) For all $a, b, c \in X$ we have

$$a[a, b, c] = [a, b, bc] \quad (v.i)$$
$$[a, b, c]c = [ab, b, c] \quad (v.ii)$$
$$[a, b, c] = [a, [ab, [ab, b, c]]] \quad (v.iii)$$

(iv.i) $[a, b, c] = [a, ab, bc] \quad (v.iv)$.

**Example 3.** The tribracket in example 2 has eight partial products, given by the operation matrices

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 & - \\ -2 & 1 & - \\ 2 & - & 3 \end{bmatrix}, \begin{bmatrix} 1 & - & 2 \\ 3 & 2 & - \\ -1 & 3 & 2 \end{bmatrix}$$

where $a -$ indicates an undefined product.

These elements of $X$ are interpreted as colors for regions in the planar complement of a $Y$-oriented trivalent spatial graph diagram according to the rule

The axioms come from our preferred generating set of $Y$-oriented Reidemeister moves. From the R4 moves, we have

\[ a \quad \Rightarrow \quad ab \]
\[ R4.10 \quad \Rightarrow \quad ab \]
and from the R5 moves we have

![Diagram](image)

**Remark 1.** The various oriented IH moves impose the idempotency-style condition that

\[ a(ab) = ab = (ab)b \]

for all \( a, b \in X \). We illustrate with two of the possible \( Y \)-oriented IH-moves; the other cases are...
The left and right invertibility of the partial product then require that \( ab = a = b \), so this condition implies that the multiplication is only defined for equal operands, i.e., if \( a \neq b \) then \( ab \) is not defined, and that we must have \( aa = a \). Thus, for handlebody-links the rule is that all three region colors around a vertex must be equal. We will call such a Niebrzydowski algebra an idempotent Niebrzydowski algebra.

Then by construction, we have

**Theorem 3.** Let \( X \) be a Niebrzydowski algebra and \( \Gamma \) a \( Y \)-oriented trivalent spatial graph. Then the number \( \Phi^X_Z(\Gamma) \) of \( X \)-colorings of any diagram of \( \Gamma \) is invariant under \( Y \)-oriented Reidemeister moves. If \( X \) is idempotent, then \( \Phi^X_Z(\Gamma) \) is an invariant of \( Y \)-oriented handlebody-links.

**Example 4.** The \( Y \)-oriented unknotted theta graph below has nine colorings by the Niebrzydowski algebra as depicted.

\[
X = \begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{bmatrix}, \begin{bmatrix}
2 & 3 & 1 \\
1 & 2 & 3 \\
3 & 1 & 2 \\
\end{bmatrix}, \begin{bmatrix}
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 2 & 3 \\
\end{bmatrix}, \begin{bmatrix}
1 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
\end{bmatrix}
\]
Hence we have counting invariant value $\Phi_X^Z(\Gamma) = 9$. This is distinct from the $Y$-oriented handcuff graph, which has only three $X$-colorings.

![Diagram](image)

We note that while these two graphs represent the same handlebody-knot, $X$ is not an idempotent Niebrzydowski algebra and thus can distinguish different trivalent spatial graphs which represent the same handlebody-knot. If we instead use the partial product given by the matrix

$$
\begin{bmatrix}
1 & - & - \\
- & 2 & - \\
- & - & 3
\end{bmatrix}
$$

then the resulting Niebrzydowski algebra is idempotent and both graphs have only the three constant colorings, reflecting the fact that they represent the same handlebody-knot.

**Example 5.** Idempotent Niebrzydowski algebra counting invariants can detect when handlebody-links have different genera. For example, the diagram on the left

![Diagram](image)

represents a handlebody-link of two genus 1 components (i.e., a classical link) while the one on the right represents a handlebody-link of a genus 2 component with a genus 1 component. These are distinguished by the counting invariant with respect to the idempotent Niebrzydowski algebra

$$X = \begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1
\end{bmatrix}, \begin{bmatrix}
2 & 3 & 1 \\
1 & 2 & 3 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 2 & 3
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
$$

with counting invariant values 27 and 3 respectively: checking the operation tensor of $X$ we verify that $[a, b, c] = d$ iff $[a, d, c] = b$, so the regions labeled $a, b, c$ can be freely chosen and we have $3^3 = 27$ $X$-colorings of the Hopf link, while for the handlebody-link on the right the coloring conditions $[a, a, b] = c$ and $[a, c, b] = a$ are satisfied only for $a = b = c$, yielding only three colorings.
Example 6. Let $X$ be the Niebrzydowski algebra given by

$$X = \begin{bmatrix} 
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 
\end{bmatrix}, \begin{bmatrix} 
2 & 3 & 1 \\
1 & 2 & 3 \\
3 & 1 & 2 
\end{bmatrix}, \begin{bmatrix} 
3 & 1 & 2 \\
2 & 3 & 1 \\
1 & 2 & 3 
\end{bmatrix}, \begin{bmatrix} 
- & 3 & - \\
- & - & 1 \\
2 & - & - 
\end{bmatrix}$$

and consider the two trivalent spatial graphs below.

The spatial graph $K_1$ on the left has three $X$-colorings

for a counting invariant value of $\Phi^X_{K_1}(K_1) = 3$. For the spatial graph $K_2$ on the right, we can observe that the partial multiplication structure of $X$ means that the only possible colorings around the
vertices are as depicted with \((a, b, c) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\).

However, for such a coloring to be valid, we would need
\[
[a, [a, c, b], b] = a
\]
and we verify that none of the three possibilities works:
\[
\begin{align*}
1, [1, 3, 2], 2 &= 1, 3, 2 = 3 \neq 1 \\
2, [2, 1, 3], 3 &= 2, 1, 3 = 1 \neq 2 \\
3, [3, 2, 1], 1 &= [3, 2, 1] = 2 \neq 3.
\end{align*}
\]
Hence there are no \(X\)-colorings of \(K_2\) and the counting invariant distinguishes these spatial graphs.

**Example 7.** Using our *Python* code, we find that the trivalent spatial graphs

are distinguished by their numbers of colorings by the Niebrzydowski algebra \(X\) specified by

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
2 & 3 & 4 & 1 \\
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2
\end{bmatrix}, \quad
\begin{bmatrix}
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1 \\
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3
\end{bmatrix}, \quad
\begin{bmatrix}
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1 \\
1 & 2 & 3 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 3 & 2 & 3 \\
2 & 1 & 3 & 4 \\
4 & 3 & 1 & 2 \\
3 & 2 & 1 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 2 & 1 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1 \\
4 & 1 & 2 & 3
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 3 & 3 & 2 \\
2 & 2 & 3 & 4 \\
4 & 3 & 1 & 2 \\
3 & 2 & 1 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 2 & 1 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1 \\
4 & 1 & 2 & 3
\end{bmatrix}
\]

with eight and four colorings respectively for the spatial graphs on the left and on the right.
4 Questions

We end with a few questions for future research.

We can’t help but notice that the fully-defined multiplication table for the tribracket in example 4 is a quandle table, namely the dihedral quandle of three elements. Is this a coincidence?

Because of the idempotency requirement, handlebody-link colorings tend to be fairly monochromatic. How many crossings in a handlebody knot (single component) are needed before we get a non-monochromatic coloring by an idempotent Niebrzydowski algebra?

As always, what enhancements of this counting invariant are possible? What are the connections with qualgebra colorings and qualgebra homology?

References

[1] S. Aksoy and S. Nelson. Bikei, involutory biracks and unoriented link invariants. *J. Knot Theory Ramifications*, 21(6):1250045, 13, 2012.

[2] W. Choi, D. Needell, and S. Nelson. Biquasile Boltzmann enhancements. arXiv:1704.02555, 2017.

[3] A. Ishii. Moves and invariants for knotted handlebodies. *Algebr. Geom. Topol.*, 8(3):1403–1418, 2008.

[4] A. Ishii. The Markov theorems for spatial graphs and handlebody-knots with Y-orientations. *Internat. J. Math.*, 26(14):1550116, 23, 2015.

[5] A. Ishii and S. Nelson. Partially multiplicative biquandles and handlebody-knots. In *Knots, links, spatial graphs, and algebraic invariants*, volume 689 of *Contemp. Math.*, pages 159–176. Amer. Math. Soc., Providence, RI, 2017.

[6] L. H. Kauffman. Invariants of graphs in three-space. *Trans. Amer. Math. Soc.*, 311(2):697–710, 1989.

[7] J. Kim and S. Nelson. Biquasile colorings of oriented surface-links. *Topology and its Applications*, 236:64 – 76, 2018.

[8] V. Lebed. Qualgebras and knotted 3-valent graphs. *Fund. Math.*, 230(2):167–204, 2015.

[9] D. Needell and S. Nelson. Biquasiles and dual graph diagrams. *J. Knot Theory Ramifications*, 26(8):1750048, 18, 2017.

[10] S. Nelson and S. Pico. Virtual tribrackets. arXiv, 2018.

[11] S. Nelson and Y. Zhao. Twisted virtual bikeigebra and twisted virtual handlebody-knots. arXiv:1711.04362, 2017.

[12] M. Niebrzydowski. On some ternary operations in knot theory. *Fund. Math.*, 225(1):259–276, 2014.

[13] M. Niebrzydowski. Homology of ternary algebras yielding invariants of knots and knotted surfaces. arXiv:1706.04307, 2017.

[14] M. Polyak. Minimal generating sets of Reidemeister moves. *Quantum Topol.*, 1(4):399–411, 2010.
