Quantum Affine Algebra and Universal $R$-Matrix with Spectral Parameter, $II$

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Abstract:

This paper is an extended version of our previous short letter [1] and is attempted to give a detailed account for the results presented in that paper. Let $U_q(G^{(1)})$ be the quantized nontwisted affine Lie algebra and $U_q(G)$ be the corresponding quantum simple Lie algebra. Using the previously obtained universal $R$-matrix for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$, we determine the explicitly spectral-dependent universal $R$-matrix for $U_q(A_1)$ and $U_q(A_2)$. We apply these spectral-dependent universal $R$-matrix to some concrete representations. We then reproduce the well-known results for the fundamental representations and we are also able to derive for the first time the extremely explicit and compact formula of the spectral-dependent $R$-matrix for the adjoint representation of $U_q(A_2)$, the simplest nontrivial case when the tensor product of the representations is not multiplicity-free.
1 Introduction

This paper is an extended version of our previous short letter [1] where only the results have been announced and is partially attempted to account for the details for the results presented in that letter.

Quantum deformation of universal enveloping algebras, or for short, quantum algebra, is perhaps one of the most important discoveries in recent years in mathematics and theoretical physics [2][3]. The novelty in this theory is that it has a quasitriangular Hopf algebra structure. Namely, there exists a canonical element $R$, called the universal $R$-matrix, in the deformed algebra satisfying the spectral-independent quantum Yang-Baxter equation (QYBE) which plays a key role in CFT’s [5] and knot theory [4][6][7]. Integrable models [8][9][10], on the other hand, use spectral-dependent $R$-matrix which is the solution to the spectral-dependent QYBE.

Since Jimbo and Jones’ works [3][11], a central issue has been finding spectral parameter dependent $R$-matrix using the quantum group techniques (see, for example, [12]). The usual approach seems to be Jones’ ”Yang-Baxterization” procedure. That is, given some representation of braid group, one can in principal work out the spectral-dependent solution to QYBE by Yang-Baxterizing the former. This approach has been extensively applied to the case of so-called ”abelian Yang-Baxterization” where the tensor product of representations is multiplicity-free. In fact, as far as we know, all the previous research in literature has limited to this simple case. When the tensor product of representations is not multiplicity-free, Jones points out that ”non-abelian Yang-Baxterization” plays a role. Therefore, one may expect that one can not any more use the simple ansatz that spectral-dependent $R$-matrix takes the form of spectral-dependent scalar functions times spectral-independent projection operators, and thus makes it very difficult to solve the Jimbo-type equations [3].

We will present a new way of obtaining the spectral-dependent $R$-matrix for quantum simple Lie algebras. Our idea is, in some sense, to reverse the above process. More precisely, we start from the universal $R$-matrix of the quantum affine algebra $U_q(G^{(1)})$ and then apply it to finite-dimensional loop representations $V(z)$ of $U_q(G^{(1)})$ which is known to be isomorphic to the ones, $V \otimes \mathbb{C}(z, z^{-1})$ of the corresponding quantum simple Lie algebra $U_q(G)$. In this way, a spectral parameter appears automatically and we obtain the spectral parameter dependent solution to QYBE for the latter. Our approach has been partly initiated by Khoroshkin and Tolstoy’s work [13] who consider the simplest case of the fundamental representation of $U_q(A_1)$ and has classical analogue [14]. One of the advantages lying in our method is that the multiplicity-free and non-multiplicity-free cases can be treated in a unified fashion. As a matter of fact, we are able to get a spectral-dependent universal $R$-matrix for $U_q(A_1)$ and $U_q(A_2)$. Applying to some concrete representations, we are able to reproduce the well-known results for the fundamental representations and to obtain for the first time the extremaly explicit and compact formula for spectral-dependent $R$-matrix of $U_q(A_2)$ for the adjoint representation, the simplest nontrival
case when the tensor product of representations is not multiplicity-free.

The present paper is set in the following fashion. In section 1 and 2, we give some account for the fundamentals needed in this paper. In section 3, we give the universal $R$-matrix with the explicit spectral dependence for $U_q(A_1)$ and $U_q(A_2)$. In section 4, we apply the spectral-dependent universal $R$-matrix to some concrete representations and reproduce some well-known results. We also obtain an extremely explicit and compact formula for spectral-dependent $R$-matrix in the adjoint representation of $U_q(A_2)$. In section 5 we present some remarks. Finally, some extra details are put in Appendix A and B.

2 Quantum Affine Lie Algebras

We start with the definition of the nontwisted quantum affine Lie algebra $U_q(G^{(1)})$. Let $A^0 = (a_{ij})_{1 \leq i,j \leq r}$ be a symmetrizable Cartan matrix. Let $G$ stand for the finite-dimensional simple Lie algebra associated with the symmetrical Cartan matrix $A^0_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j)$, $i,j = 1,2,\ldots,r$, where $r$ is the rank of $G$. Let $A = (a_{ij})_{0 \leq i,j \leq r}$ be a symmetrizable, generalized Cartan matrix in the sense of Kac [15]. Let $G^{(1)}$ denote the nontwisted affine Lie algebra associated with the corresponding symmetric Cartan matrix $A_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j)$, $i,j = 0,1,\ldots,r$. Then the quantum algebra $U_q(G^{(1)})$ is defined to be a Hopf algebra with generators: $\{E_i, F_i, q^{h_i} \mid i = 0,1,\ldots,r\}$, $q^d$ and relations,

\[ q^{h} q^{h'} = q^{h+h'} \quad (h, h' = h_i \ (i = 0,1,\ldots,r), \ d) \]
\[ q^{h} E_i q^{-h} = q^{(h,\alpha_i)} E_i, \quad q^{h} F_i q^{-h} = q^{-(h,\alpha_i)} F_i \]
\[ [E_i, F_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \]
\[ (\text{ad}_q E_i)^{1-\alpha_{ij}} E_j = 0, \quad (\text{ad}_q^{-1} F_i)^{1-\alpha_{ij}} F_j = 0 \quad (i \neq j) \]

where

\[ (\text{ad}_q x_{\alpha}) x_{\beta} = [x_{\alpha}, x_{\beta}]_q = x_{\alpha} x_{\beta} - q^{(\alpha, \beta)} x_{\beta} x_{\alpha} \]

The algebra $U_q(G^{(1)})$ is a Hopf algebra with coproduct, counit and antipode similar to the case of $U_q(G)$: explicitly, the coproduct is defined by

\[ \Delta(q^{h}) = q^{h} \otimes q^{h}, \quad h = h_i, \ d \]
\[ \Delta(E_i) = q^{-h_i} \otimes E_i + E_i \otimes 1 \]
\[ \Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{h_i}, \quad i = 0,1,\ldots,r \]

Formulae for the counit and antipode may also be given, but are not required below.

Let $\Delta'$ be the opposite coproduct: $\Delta' = T \Delta$, $T(x \otimes y) = y \otimes x$, $\forall x,y \in U_q(G^{(1)})$. Then $\Delta$ and $\Delta'$ is related by the universal $R$-matrix $R$ in $U_q(G^{(1)}) \otimes U_q(G^{(1)})$ satisfying

\[ \Delta'(x) R = R \Delta(x), \quad x \in U_q(G^{(1)}) \]
\[ (\Delta \otimes \text{id}) R = R^{13} R^{23}, \quad (\text{id} \otimes \Delta) R = R^{13} R^{12} \]
We define an anti-involution $\theta$ on $U_q(G^{(1)})$ by

$$\theta(q^h) = q^{-h}, \quad \theta(E_i) = F_i, \quad \theta(F_i) = E_i, \quad \theta(q) = q^{-1}$$

(5)

which extend uniquely to an algebra anti-involution on all of $U_q(G^{(1)})$ so that $\theta(ab) = \theta(b)\theta(a), \ \forall a, b \in U_q(G^{(1)})$. Throughout the paper, we use the notations:

$$(n)_q = \frac{1 - q^n}{1 - q}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q_\alpha = q^{(\alpha,\alpha)}$$

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{(n)_q}, \quad (n)_q! = (n)_q(n-1)_q \ldots (1)_q$$

(6)

3 Universal $R$-Matrix for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$

This section is devoted to a brief review of the construction of the universal $R$-matrix for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$. We start with rank 2 case. Fix a normal ordering in the positive root system $\Delta_+$ of $A_1^{(1)}$:

$$\alpha, \alpha + \delta, ... , \alpha + n\delta, ... , \delta, 2\delta, ... , m\delta, ... , (\delta - \alpha) + l\delta, ... , \delta - \alpha$$

(7)

where $\alpha$ and $\delta - \alpha$ are simple roots; $\delta$ is the minimal positive imaginary root. Construct Cartan-Weyl generators $E_\gamma, \ F_\gamma = \theta(E_\gamma), \ \gamma \in \Delta_+$ of $U_q(A_1^{(1)})$ as follows: We define

$$\tilde{E}_\delta = [(\alpha,\alpha)]^{-1}_q [E_\alpha, E_{\delta - \alpha}]_q$$

$$E_{\alpha + n\delta} = (-1)^n (\text{ad} \tilde{E}_\delta)^n E_\alpha$$

$$E_{(\delta - \alpha) + n\delta} = (\text{ad} \tilde{E}_\delta)^n E_{\delta - \alpha}, ...$$

$$\tilde{E}_{n\delta} = [(\alpha,\alpha)]^{-1}_q [E_{\alpha + (n-1)\delta}, E_{\delta - \alpha}]_q$$

(8)

where $[\tilde{E}_{n\delta}, \tilde{E}_{m\delta}] = 0$ for any $n, m > 0$. Then

(i) for any $n > 0$, there exists a unique element $E_{n\delta}$ satisfying $[E_{n\delta}, E_{m\delta}] = 0$ for any $n, m > 0$ and the relation

$$\tilde{E}_{n\delta} = \sum_{k_1p_1 + ... + k_mp_m = n} \frac{q^{(\alpha,\alpha)} - q^{-(\alpha,\alpha)}}{p_1! \ldots p_m!} \sum_{p_{1} \ldots p_{m}} (E_{k_1\delta})^{p_1} \ldots (E_{k_m\delta})^{p_m}$$

(9)

(ii) the vectors $E_\gamma$ and $F_\gamma = \theta(E_\gamma), \ \gamma \in \Delta_+$ are the Cartan-Weyl generators for $U_q(A_1^{(1)})$. One has

Theorem: The universal $R$-matrix for $U_q(A_1^{(1)})$ may be written as

$$R = \prod_{n \geq 0} \exp_{q_\alpha}((q - q^{-1})(E_{\alpha + n\delta} \otimes F_{\alpha + n\delta}))$$

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\[ \exp \left( \sum_{n \geq 0} n[q_n^{-1}(q_{\alpha} - q_{\alpha}^{-1})(E_{n\delta} \otimes F_{n\delta})] \right) \cdot \prod_{n \geq 0} \exp_{q_n} ((q - q^{-1})(E_{(\delta - \alpha) + n\delta} \otimes F_{(\delta - \alpha) + n\delta})) \cdot q_{\delta}^{4h_{\alpha} \otimes h_{\alpha} + c \otimes d + d \otimes c} \] (10)

where \( c = h_{\alpha} + h_{\delta - \alpha} \). The order in the product (10) coincides with the chosen normal order (\( \prod \)).

We now consider rank 3 case. Let \( A_{sym}^0 = (a_{ij}^{sym}) \), \( i, j = 1, 2 \) and \( \Delta^0_+ \) respectively be symmetrical Cartan matrix and positive root system of rank 2 finite-dimensional simple Lie algebras \( A_2 \). In what follows we use \( A_{sym}^0 \) in the form

\[ A_{sym}^0 = (a_{ij}^{sym}) = \begin{pmatrix} (\alpha, \alpha) & (\alpha, \beta) \\ (\beta, \alpha) & (\beta, \beta) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \] (11)

The simple roots are \( \alpha, \beta \) and \( \delta - \psi \) with \( \psi = \alpha + \beta \) be the highest root of \( A_2 \).

**Proposition:** For \( U_q(A_2^{(1)}) \), we fix the following order in \( \Delta_+ \) of \( A_2^{(1)} \),

\[ \alpha, \alpha + \delta, ..., \alpha + m_1\delta, ..., \alpha + \beta, \alpha + \beta + \delta, ..., \alpha + \beta + m_2\delta, ..., \beta, \beta + \delta, ..., \beta + m_3\delta, ..., 2\delta, ... \]

\[ k\delta, ..., (\delta - \beta) + l_1\delta, ..., \delta - \beta, ..., (\delta - \alpha) + l_2\delta, ..., \delta - \alpha, ..., (\delta - \alpha - \beta) + l_3\delta, ..., \delta - \alpha - \beta \] (12)

where \( m_i, k_i, l_i \geq 0 \), \( i = 1, 2, 3 \). We set

\[ E_{\alpha + \beta} = [E_\alpha, E_\beta]_q, \quad E_{\delta - \alpha} = [E_\beta, E_{\delta - \alpha - \beta}]_q \]

\[ E_{\delta - \beta} = [E_\alpha, E_{\delta - \alpha - \beta}]_q \] (13)

and use the formula for \( E_{\gamma + n\delta} \) and \( E_{(\delta - \gamma) + n\delta}, \gamma \in \Delta^0_+ \),

\[ \tilde{E}_{\delta}^{(i)} = ((\alpha_i, \alpha_i))^{-1}[E_{\alpha_i}, E_{\delta - \alpha_i}]_q, \quad \alpha_i = \alpha, \beta, \alpha + \beta \]

\[ E_{\alpha_i + n\delta} = (-1)^n (\text{ad} \tilde{E}_{\delta}^{(i)})^n E_{\alpha_i} \]

\[ E_{\delta - \alpha_i + n\delta} = (\text{ad} \tilde{E}_{\delta}^{(i)})^n E_{\delta - \alpha_i}, \quad ... \]

\[ \tilde{E}_{n\delta}^{(i)} = ((\alpha_i, \alpha_i))^{-1}[E_{\alpha_i + (n-1)\delta}, E_{\delta - \alpha_i}]_q \] (14)

where \( [\tilde{E}_{n\delta}^{(i)}, \tilde{E}_{m\delta}^{(j)}] = 0 \) for any \( n, m > 0 \). One has the following statment similar to the case of \( U_q(A_1^{(1)}) \):

(i) there exists a unique element \( E_{n\delta}^{(i)} \), \( n > 0 \) satisfying \( [E_{n\delta}^{(i)}, E_{m\delta}^{(j)}] = 0 \) for any \( n, m > 0 \) and the relation \( (\alpha_i = \alpha, \beta) \)

\[ \tilde{E}_{n\delta}^{(i)} = \sum_{k_1p_1 + ... + k_mp_m = n} \frac{(q^{(\alpha_i, \alpha_i)} - q^{-(\alpha_i, \alpha_i)})\sum_{p_{i} - 1}}{p_{1}! ... p_{m}!} (E_{k_{1}\delta})^{p_{1}} ... (E_{k_{m}\delta})^{p_{m}} \] (15)

\[ 0 < k_1 < ... < k_m \]
(ii) the vectors \( E_\gamma \) and \( F_\gamma = \theta(E_\gamma) \), \( \gamma \in \Delta_+ \) are the Cartan-Weyl generators for \( U_q(A_2^{(1)}) \).

One can show \cite{13,16} (see, in particular, \cite{16}) the following

**Theorem:** For \( U_q(A_2^{(1)}) \), the universal \( R \)-matrix takes the explicit form

\[
R = \prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1})(E_{\alpha + n\delta} \otimes F_{\alpha + n\delta}) \right)
\prod_{n \geq 0} \exp_{q_n + \beta} \left( (q - q^{-1})(E_{\alpha + \beta + n\delta} \otimes F_{\alpha + \beta + n\delta}) \right)
\prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1})(E_{\beta + n\delta} \otimes F_{\beta + n\delta}) \right)
\exp \left( \sum_{n \geq 0} \sum_{i,j=1}^{2} C_{ij}^n(q)(q - q^{-1})(E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)}) \right)
\prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1})(E_{\delta - \beta} + n\delta \otimes F_{\delta - \beta} + n\delta) \right)
\prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1})(E_{\delta - \alpha} + n\delta \otimes F_{\delta - \alpha} + n\delta) \right)
\prod_{n \geq 0} \exp_{q_n + \beta} \left( (q - q^{-1})(E_{\delta - \alpha - \beta} + n\delta \otimes F_{\delta - \alpha - \beta} + n\delta) \right)
q^{\sum_{i,j=1}^{2} (a_{sym})_{ij} h_i \otimes h_j + c \otimes d + d \otimes d}
\tag{16}
\]

where \( c = h_0 + h_\psi \), the order in the product of \cite{16} is defined by \cite{12} and the constants \( C_{ij}^n(q) \) are given by

\[
(C_{ij}^0(q)) = (C_{ji}^n(q)) = \frac{n}{[n]_q} \frac{[2]_q^2}{q^{2n} + 1 + q^{-2n}} \begin{pmatrix} q^n + q^{-n} \quad (-1)^n \\ (-1)^n \quad q^n + q^{-n} \end{pmatrix}
\tag{17}
\]

4 Universal \( R \)-Matrix with Spectral Parameter

In this section we come to our main concern. We will determine explicitly spectral-dependent universal \( R \)-matrix for \( U_q(A_1) \) and \( U_q(A_2) \) by using the universal \( R \)-matrix \cite{11} and \cite{16} for the corresponding \( U_q(A_1^{(1)}) \) and \( U_q(A_2^{(1)}) \), respectively.

We state the following

**Lemma:** For any \( z \in \mathbb{C}^\times \), there is a homomorphism of algebras \( \text{ev}_z : U_q(A_1^{(1)}) \to U_q(A_1) \) given by

\[
\text{ev}_z(E_\alpha) = E_\alpha, \quad \text{ev}_z(F_\alpha) = F_\alpha, \quad \text{ev}_z(h_\alpha) = h_\alpha, \quad \text{ev}_z(c) = 0
\]

\[
\text{ev}_z(E_\beta) = zF_\alpha, \quad \text{ev}_z(F_\beta) = z^{-1}E_\alpha, \quad \text{ev}_z(h_\beta) = -h_\alpha
\tag{18}
\]

**Proof:** See the appendix A.

**Proposition:** (Omitting "ev\(_z\)"")

\[
E_{\alpha + n\delta} = (-1)^n z^n q^{-nh_\alpha} E_\alpha
\]

\[
E_{\beta + n\delta} = (-1)^n z^n q^{-nh_\alpha} E_{\beta}
\]
Theorem: The universal $R$-matrix of $U_q(A_1)$ with the explicit dependence of spectral parameter, $R(x,y) \equiv (ev_x \otimes ev_y)R$, can be written as the form,

$$R(x,y) = \prod_{n \geq 0} \exp q_\alpha \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n q^{-n h_\alpha} (E_\alpha \otimes F_\alpha q^{n h_\alpha}) \right)$$

$$\cdot \exp \left( \sum_{n>0} n[v]_{q_\alpha} (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} (E_{n\delta} \otimes F_{n\delta}) \right)$$

$$\cdot \prod_{n \geq 0} \exp q_\alpha \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n (F_\alpha q^{-n h_\alpha} \otimes q^{n h_\alpha} E_\alpha) \right) \cdot q^{\frac{1}{2} h_\alpha \otimes h_\alpha}$$

We now consider the case of $U_q(A_2^{(1)})$. We state

Lemma: For and $z \in \mathbb{C}^\times$, there is a homomorphism of algebras $ev_z: U_q(A_2^{(1)}) \to U_q(A_2)$ given by

$$ev_z(E_\alpha) = E_\alpha, \quad ev_z(F_\alpha) = F_\alpha, \quad ev_z(h_\alpha) = h_\alpha$$

$$ev_z(E_\beta) = E_\beta, \quad ev_z(F_\beta) = F_\beta, \quad ev_z(h_\beta) = h_\beta$$

$$ev_z(E_{\delta-\alpha}) = z F_{\alpha+\beta} q^{(h_\beta-h_\alpha)/3}, \quad ev_z(F_{\delta-\alpha}) = z^{-1} q^{(h_\alpha-h_\beta)/3} E_{\alpha+\beta}$$

$$ev_z(h_{\delta-\alpha}) = -h_{\alpha+\beta}, \quad ev_z(c) = 0$$

Proof: Straightforward calculations + induction in $n$, using (18) and the defining relations (8).
Proof: See the Appendix A.

Proposition: (Omitting again "ev")

\[
E_{\alpha+n\delta} = (-1)^n z^n q^{-n\alpha} E_{\alpha} q^{-n(h_\alpha+2h_\beta)/3} \\
F_{\alpha+n\delta} = (-1)^n z^{-n} q^{n(h_\alpha+2h_\beta)/3} F_{\alpha} q^{n\alpha} \\
E_{\alpha+\beta+n\delta} = (-1)^n z^n q^{-n\alpha+\beta} E_{\alpha+\beta} q^{n(h_\beta-h_\alpha)/3} \\
F_{\alpha+\beta+n\delta} = (-1)^n z^{-n} q^{n(h_\alpha-h_\beta)/3} F_{\alpha+\beta} q^{n\alpha+\beta} \\
E_{\beta+n\delta} = (-1)^n [2] q^{-n} z^n q^n \left\{ \left( \text{ad}'_{q^{-1}E} \right)^n E_{\beta} \right\} q^{n(h_\beta-h_\alpha)/3} \\
F_{\beta+n\delta} = [2] q^{-n} z^n q^{n(h_\beta-h_\alpha)/3} \left( \text{ad}'_{q^{-1}F} \right)^n F_{\beta} \\
E_{(\delta-\beta)+n\delta} = [2] q^{-n} z^{n+1} q^{-n} \left\{ \left( \text{ad}'_q E \right)^n (\text{ad}'_{q^{-2}E_{\alpha}}) F_{\alpha+\beta} \right\} q^{n+1}(h_\alpha-h_\beta)/3 \\
F_{(\delta-\beta)+n\delta} = (-1)^n [2] q^{-n} z^{-n-1} q^{n+1}(h_\alpha-h_\beta)/3 (\text{ad}'_q F)^n (\text{ad}'_{q^2 E_{\alpha+\beta}}) F_{\alpha} \\
E_{(\delta-\alpha)+n\delta} = (-1)^n z^{n+1} q^{n+1}(h_\alpha+2h_\beta)/3 F_{\alpha} q^{-n\alpha} \\
F_{(\delta-\alpha)+n\delta} = (-1)^n z^{-n-1} q^{n\alpha} E_{\alpha} q^{n+1}(h_\alpha+2h_\beta)/3 \\
E_{(\delta-\alpha-\beta)+n\delta} = (-1)^n z^{n+1} q^{n+1}(h_\beta-h_\alpha)/3 F_{\alpha+\beta} q^{-n\alpha+\beta} \\
F_{(\delta-\alpha-\beta)+n\delta} = (-1)^n z^{-n-1} q^{n\alpha+\beta} E_{\alpha+\beta} q^{n+1}(h_\alpha-h_\beta)/3 \\
\tilde{E}_{n\delta}^{(\alpha)} = (-1)^{n-1} [2] q^{-1} z^n \left( E_{\alpha} F_{\alpha} - q^{-2n} F_{\alpha} E_{\alpha} \right) q^{n}(h_\alpha-q^{n}(h_\alpha+2h_\beta)/3 \\
\tilde{E}_{n\delta}^{(\beta)} = (-1)^n [2] q^{-1} z^{n-1} q^{(n+1)h_\alpha+2h_\beta}/3 (F_{\alpha} E_{\alpha} - q^{2n} E_{\alpha} F_{\alpha}) \\
\tilde{F}_{n\delta}^{(\beta)} = (-1)^n [2] q^{-n-2} \left\{ \left( \text{ad}'_{q^{-n+2}E} \right) \cdot \left( \text{ad}'_{q^{-1}E} \right)^{n-1} E_{\beta} \right\} q^{n(h_\beta-h_\alpha)/3} \\
\tilde{F}_{n\delta}^{(\beta)} = [2] q^{-n} q^{-1} q^{n(h_\alpha-h_\beta)/3} \left( \text{ad}'_{q^{-n+2}E} \right) \cdot \left( \text{ad}'_{q^{-1}F} \right)^{n-1} F_{\beta} \\
\text{where}
\]

\[
(\text{ad}'_Q A) \cdot B \equiv AB - QBA \\
E = (\text{ad}'_{q^{-1}E_{\beta}})(\text{ad}'_{q^{-2}E_{\alpha}}) F_{\alpha+\beta} \\
F = (\text{ad}'_q (\text{ad}'_{q^2 E_{\alpha+\beta}}) F_{\alpha}) F_{\beta} \\
E' = E_{\alpha+\beta} F_{\alpha} - q^2 F_{\alpha} E_{\alpha+\beta} \\
F' = E_{\alpha+\beta} F_{\alpha+\beta} - q^{-2} F_{\alpha} E_{\alpha+\beta} \\
(25)
\]

Proof: Straightforward computations + induction in \( n \), by using (23) and the defining relations of generators, eqs. (14) and (13).

We define the primed quantities, motivated by the form of (24),

\[
\tilde{E}_{n\delta}^{(\alpha)} \equiv z^n \tilde{E}_{n\delta}^{(\alpha)} , \quad \tilde{E}_{n\delta}^{(\beta)} \equiv z^n \tilde{E}_{n\delta}^{(\beta)} \\
\tilde{F}_{n\delta}^{(\alpha)} \equiv z^n \tilde{F}_{n\delta}^{(\alpha)} , \quad \tilde{F}_{n\delta}^{(\beta)} \equiv z^n \tilde{F}_{n\delta}^{(\beta)} \\
E_{n\delta}^{(\alpha)} \equiv z^n E_{n\delta}^{(\alpha)} , \quad F_{n\delta}^{(\alpha)} \equiv z^n F_{n\delta}^{(\alpha)}
\]
\[ E_{n\delta}^{(\beta)} = z^n E_{n\delta}^{(\beta)} , \quad F_{n\delta}^{(\beta)} = z^{-n} F_{n\delta}^{(\beta)} \]
\[ E_{\beta+n\delta}^{(\beta)} = z^n E_{\beta+n\delta}^{(\beta)} , \quad F_{\beta+n\delta}^{(\beta)} = z^{-n} F_{\beta+n\delta}^{(\beta)} \]
\[ E_{(\delta-\beta)+n\delta} = z^{n+1} E_{(\delta-\beta)+n\delta}^{(\beta)} , \quad F_{(\delta-\beta)+n\delta} = z^{-n-1} F_{(\delta-\beta)+n\delta}^{(\beta)} \]

Then the primed quantities do not depend on the parameter \( z \) and \( E_{\tilde{n}\delta}^{(i)} , \ F_{\tilde{n}\delta}^{(i)} , \ E_{\beta+n\delta}^{(i)} , \ F_{\beta+n\delta}^{(i)} , \ E_{(\delta-\beta)+n\delta}^{(i)} , \ F_{(\delta-\beta)+n\delta}^{(i)} \) can be easily read off by comparing (26) with (24); moreover, similar to the \( U_q(A_1^{(1)}) \) case, \( E_{n\delta}^{(i)} , \ F_{n\delta}^{(i)} \) are determined by the following equalities of formal series: \( (\alpha_i = \alpha , \ \beta) \)

\[ (q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{k=1}^{\infty} \tilde{E}_{k\delta}^{(i)} u^{-k} = \exp \left( (q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{l=1}^{\infty} \tilde{E}_{l\delta}^{(i)} u^{l} \right) - 1 \]
\[ -(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{k=1}^{\infty} \tilde{F}_{k\delta}^{(i)} u^{-k} = \exp \left( -(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{l=1}^{\infty} \tilde{F}_{l\delta}^{(i)} u^{l} \right) - 1 \]

which are the variants of (15). The above considerations and (16) then give rise to

**Theorem:** The universal \( R \)-matrix of \( U_q(A_2) \) with the explicit dependence of spectral parameter, \( R(x,y) \equiv (ev_x \otimes ev_y)R, \) takes the form,

\[ R(x,y) = \prod_{n \geq 0} \exp_{q_{\alpha}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n \left( q^{-n h_\alpha} E_{\alpha} q^{-n(h_\alpha + 2h_\beta)/3} \otimes q^{n(h_\alpha + 2\beta)/3} F_{\alpha} q^{n h_\alpha} \right) \right) \]
\[ \cdot \prod_{n \geq 0} \exp_{q_{\alpha + \beta}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n \left( q^{-n h_{\alpha + \beta}} E_{\alpha + \beta} q^{n(h_\beta - h_\alpha)/3} \otimes q^{n(h_\beta - h_\alpha)/3} F_{\alpha + \beta} q^{n h_{\alpha + \beta}} \right) \right) \]
\[ \cdot \prod_{n \geq 0} \exp_{q_{\beta}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^n \left( E_{\beta+n\delta}^{(\beta)} \otimes F_{\beta+n\delta}^{(\beta)} \right) \right) \]
\[ \cdot \exp \left( \sum_{n > 0} \sum_{i,j = 1}^{2} \mathcal{C}_{nij}(q)(q - q^{-1})(E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)}) \right) \]
\[ \cdot \prod_{n \geq 0} \exp_{q_{\alpha}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} \left( E_{(\delta-\beta)+n\delta}^{(\beta)} \otimes F_{(\delta-\beta)+n\delta}^{(\beta)} \right) \right) \]
\[ \cdot \prod_{n \geq 0} \exp_{q_{\alpha}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} \left( q^{-(n+1)(h_\alpha + 2h_\beta)/3} F_{\alpha} q^{-n h_\alpha} \otimes q^{n h_\alpha} E_{\alpha} q^{(n+1)(h_\alpha + 2h_\beta)/3} \right) \right) \]
\[ \cdot \prod_{n \geq 0} \exp_{q_{\alpha + \beta}} \left( (q - q^{-1}) \left( \frac{x}{y} \right)^{n+1} \left( q^{(n+1)(h_\beta - h_\alpha)/3} F_{\alpha + \beta} q^{-n h_{\alpha + \beta}} \right) \right) \]
\[ \cdot \sum_{i,j=1}^{2} \left( a_{ij}^{-1} \right) \left( \otimes q^{n h_{\alpha + \beta}} E_{\alpha + \beta} q^{(n+1)(h_\alpha - h_\beta)/3} \right) \]

\[ \text{(28)} \]

5 Applications

To illustrate the general theory developed in the previous section, we present a detailed study of the spectral-dependent \( R \)-matrix for some concrete and interesting representations.

First consider the \( U_q(A_1^{(1)}) \) case. Let \( V_l , \ l \in \mathbb{Z}_+ \) denote the \((l+1)\)-dimensional module of \( U_q(A_1) \) (spin \( l/2 \) representation) with basis \( \{ v_m^{(l)} | 0 \leq m \leq l \} \). We have
**Proposition:** For spin $l/2$ representation of $U_q(A_1)$, we have

\[
\begin{align*}
    h_\alpha v_m^{(l)} &= (l - 2m)v_m^{(l)} \\
    E_\alpha v_m^{(l)} &= [l - m + 1]_q v_{m-1}^{(l)} \\
    F_\alpha v_m^{(l)} &= [m + 1]_q v_{m+1}^{(l)} \\
    E_{n\delta} v_m^{(l)} &= [2]_q^{-1} \frac{(-1)^{n-1}}{n} q^{nm} \left( [n(l - m)]_q - q^{-n(l+2)[nm]} \right) v_m^{(l)} \\
    F_{n\delta} v_m^{(l)} &= [2]_q^{-1} \frac{(-1)^{n-1}}{n} q^{-nm} \left( [n(l - m)]_q - q^{n(l+2)[nm]} \right) v_m^{(l)}
\end{align*}
\]

where it is understood that $v_m^{(l)}$ is identically zero if $m > l$ or $m < 0$.

**Proof:** Straightforward computations + induction in $n$.

(i) for spin $1/2$ representation, we have from (29)

\[
\begin{align*}
    h_\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
    E_{n\delta} &= [2]_q^{-1} \frac{(-1)^{n-1}}{n} [n]_q \begin{pmatrix} 1 & 0 \\ 0 & -q^{-2n} \end{pmatrix}, \quad F_{n\delta} = [2]_q^{-1} \frac{(-1)^{n-1}}{n} [n]_q \begin{pmatrix} 1 & 0 \\ 0 & -q^{2n} \end{pmatrix}
\end{align*}
\]

Using (22), it follows from (22) that

\[
R_{1/2,1/2}(x, y) = f_q(x, y) \cdot \begin{pmatrix} 1 & 0 \\ q^{-1}(y-x) & y^{-1}q^{-2} \\ y^{-1}q^{-2} & y^{-1}q^{-2} \\ y^{-1}q^{-2} & y^{-1}q^{-2} \\ 1 \end{pmatrix}
\]

where

\[
f_q(x, y) = q^{1/2} \cdot \exp \left( \sum_{n>0} \frac{q^n - q^{-n}}{q^n + q^{-n}} (x/y)^n \right)
\]

and use has been made of the notation:

\[
(A \otimes B) = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1N}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1}B & A_{M2}B & \cdots & A_{MN}B \end{pmatrix}
\]

We thus reproduce the well-known result \([4]\), up to a scalar factor $f_q(x, y)$. In \([13]\), KT obtained (31) directly from (10).

(ii) for spin 1 representation, (22) give rise to

\[
\begin{align*}
    h_\alpha &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & [2]_q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & [2]_q & 0 \end{pmatrix}
\end{align*}
\]
\[ F'_{n\delta} = [2]_q^{-1}(\frac{(-1)^{n-1}}{n})[n]_q \begin{pmatrix} q^n + q^{-n} & 0 & 0 \\ 0 & -q'^n(q^{2n} - q^{-2n}) & 0 \\ 0 & 0 & -q^{2n}(q^n + q^{-n}) \end{pmatrix} \] (34)

We now apply (22) to \( V_{1/2} \otimes V_1 \) with \( V_1 \) being the spin-1 representation of \( U_q(A_1) \). Using (34), we obtain from (22),

\[ R_{1/2,1}(x,y) = \frac{q^2(y - q^{-1}x)}{y - qx} \cdot \left( e_{11} + e_{66} + \frac{q^{-2}(y - qx)}{y - q^{-3}x}(e_{33} + e_{44}) + \frac{yq^{-1}(1 - q^{-2})}{y - q^{-3}x}e_{24} + \frac{xq^{-1}(1 - q^{-2})}{y - q^{-3}x}e_{53} \right) \] (35)

where \( e_{ij} \) is the matrix satisfying \( (e_{ij})_{kl} = \delta_{ik}\delta_{jl} \) and \( e_{ij}e_{kl} = \delta_{jk}e_{il} \).

We now turn to the \( U_q(A_2^{(1)}) \) case. We state

**Proposition:** The explicit form of generators on the fundamental representation of \( U_q(A_2) \) is given by

\[
\begin{align*}
    h_\alpha &= \text{diag}(1, -1, 0) , \quad h_\beta = \text{diag}(0, 1, -1) \\
    E_\alpha &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad F_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
    E_{\alpha+\beta} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad F_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
    E'_{\beta+n\delta} &= q^{-2n-3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad F'_{\beta+n\delta} = q^{2n+3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
    E'_{(\delta-\beta)+n\delta} &= (-1)^n q^{-2n+3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
    F'_{(\delta-\beta)+n\delta} &= (-1)^n q^{2n+3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
    E'_{n\delta} &= [2]_q^{-1}(-1)^{n-1}[n]_q q^{-n/3} \text{diag} \left( 1, -q^{-2n}, 0 \right) \\
    F'_{n\delta} &= [2]_q^{-1}(-1)^{n-1}[n]_q q^{n/3} \text{diag} \left( 1, -q^{2n}, 0 \right) \\
    E'_{n\delta} &= [2]_q^{-1}(n)q^{-n-3/2} \text{diag} \left( 0, -q^{-2n}, 0 \right) \\
    F'_{n\delta} &= [2]_q^{-1}(n)q^{n+3/2} \text{diag} \left( 0, -q^{2n}, 0 \right)
\end{align*}
\] (36)
We apply (28) to \( V(3) \otimes V(3) \), where \( V(3) \) stands for the fundamental representation of \( U_q(A_2) \). Using (36) we get from (28). It follows from (28) that

\[
R_{(3),(3)}(x, y) = f_q(x, y) \cdot \left( e_{11} + e_{99} + \frac{q^{-1}(y-x)}{y-q^{-2}x}(e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \frac{y(1-q^{-2})}{y-q^{-2}x}(e_{24} + e_{37} + e_{68}) + \frac{x(1-q^{-2})}{y-q^{-2}x}(e_{42} + e_{73} + e_{86}) \right)
\]

(37)

where

\[
f_q(x, y) = q^{2/3} \cdot \exp \left( \sum_{n>0} \frac{q^{2n} - q^{-2n}}{q^{2n} + 1 + q^{-2n}} \frac{(x/y)^n}{n} \right)
\]

(38)

We thus reproduce the well-known result (3), up to a scalar factor \( f_q(x, y) \).

We now consider a very interesting case: to extract the spectral dependent \( R \)-matrix in the adjoint representation of \( U_q(A_2) \). As one knows, this is simplest nontrivial example where the tensor product is not multiplicity-free. To this effect, we introduce the so-called Gelfand-Tsetlin basis vector \( |(m)\rangle \) given by

\[
|(m)\rangle = \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} \end{pmatrix} \right\rangle
\]

(39)

It can be shown that the action of generators on the GT basis vectors reads

\[
h_\alpha |(m)\rangle = (2m_{11} - m_{12} - m_{22})|(m)\rangle
\]

\[
h_\beta |(m)\rangle = (2m_{12} + 2m_{22} - m_{11} - m_{13} - m_{23} - m_{33})|(m)\rangle
\]

\[
F_\alpha \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} \end{pmatrix} \right\rangle = \left\{ [m_{11} - m_{22}]_q [m_{12} - m_{11} + 1]_q \right\}^{1/2} \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} - 1 \end{pmatrix} \right\rangle
\]

\[
F_\beta \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} \end{pmatrix} \right\rangle = \left\{ \frac{[m_{12} - m_{11}]_q [m_{13} - m_{12} + 1]_q [m_{23} - m_{12}]_q [m_{33} - m_{12} - 1]_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22}]_q} \right\}^{1/2}
\]

\[
\times \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} - 1 & m_{22} \\ m_{11} \end{pmatrix} \right\rangle
\]

\[
+ \left\{ \frac{[m_{22} - m_{11} - 1]_q [m_{13} - m_{22} + 2]_q [m_{23} - m_{22} + 1]_q [m_{33} - m_{22}]_q}{[m_{12} - m_{22} + 2]_q [m_{12} - m_{22}]_q} \right\}^{1/2}
\]

\[
\times \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} - 1 \\ m_{11} \end{pmatrix} \right\rangle
\]

(40)
The matrix elements of \( E_\alpha \) and \( E_\beta \) are given by the transpose of the ones of \( F_\alpha \) and \( F_\beta \), respectively. Now for the adjoint representation, we have the following 8 state vectors:

\[
\begin{align*}
\phi_1 &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 \\ 0 \end{pmatrix}, & \phi_2 &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 \\ 0 \end{pmatrix}, & \phi_3 &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 \\ 0 \end{pmatrix} \\
\phi_4 &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 \end{pmatrix}, & \phi_5 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix}, & \phi_6 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\phi_7 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 \end{pmatrix}, & \phi_8 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
\]

Therefore, one has

**Proposition:** The matrix form of generators in the adjoint representation of \( U_q(A_2^{(1)}) \) is given by

\[
\begin{align*}
h_\alpha &= \text{diag}(1, -1, 2, 0, 0, -2, 1, -1), & h_\beta &= \text{diag}(1, 2, -1, 0, 0, 1, -2, -1) \\
E_\alpha &= e_{12} + [2]_q^{-1} e_{34} + [2]_q^{-1} e_{46} + e_{78}, & F_\alpha &= e_{21} + [2]_q^{-1} e_{43} + [2]_q^{-1} e_{66} + e_{87} \\
E_\beta &= e_{13} + [2]_q^{-1} e_{24} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{25} + [2]_q^{-1/2} e_{47} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{57} + e_{68} \\
F_\beta &= e_{31} + [2]_q^{-1/2} e_{42} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{52} + [2]_q^{-1/2} e_{74} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{75} + e_{86} \\
E_{\alpha+\beta} &= -[2]_q^{-1/2} q^{-2} e_{14} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{15} - q^{-1} e_{26} + e_{37} + [2]_q^{-1/2} q e_{48} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{58} \\
F_{\alpha+\beta} &= -[2]_q^{-1/2} q^2 e_{41} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{51} - q e_{62} + e_{73} + [2]_q^{-1/2} q^{-1} e_{84} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} e_{85} \\
E'_{\beta+n\delta} &= q^n e_{13} + [2]_q^{-1/2} q^n e_{24} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{-3n} e_{25} + \\
&\quad + [2]_q^{-1/2} q^{-3n} e_{47} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^n e_{57} + q^{-3n} e_{68} \\
F'_{\beta+n\delta} &= q^{-n} e_{31} + [2]_q^{-1/2} q^{-n} e_{42} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{3n} e_{52} + \\
&\quad + [2]_q^{-1/2} q^{3n} e_{74} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{-n} e_{75} + q^{3n} e_{86} \\
E'_{(\delta-\beta)+n\delta} &= -q^{n+2} e_{31} - [2]_q^{-1/2} q^{n+3} e_{42} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{-3n-1} e_{52} \\
&\quad - [2]_q^{-1/2} q^{-3n-3} e_{74} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{n+1} e_{75} - q^{-3n-2} e_{86}
\end{align*}
\]
\[ F'_{(\delta - \beta) + n\delta} = -q^{-n-2} e_{13} - [2]^{-1/2} q^{-n-3} e_{24} - \left( \frac{[3]}{[2]} \right)^{1/2} q^{3n+1} e_{25} \]
\[ - [2]^{-1/2} q^{3n+3} e_{47} - \left( \frac{[3]}{[2]} \right)^{1/2} q^{-n-1} e_{57} - q^{3n+2} e_{68} \]

\[ E'_{n\delta}^{(\alpha)} = [2]^{-1} (-1)^{n-1} \frac{[n]}{n} \left\{ q^{-n} e_{11} - q^{-3n} e_{22} + (q^n + q^{-n}) e_{33} + q^{-n} (q^{2n} - q^{-2n}) e_{44} - q^{-2n} (q^n + q^{-n}) e_{66} q^n e_{77} - q^{-n} e_{88} \right\} \]

\[ F'_{n\delta}^{(\alpha)} = [2]^{-1} (-1)^{n-1} \frac{[n]}{n} \left\{ q^n e_{11} - q^{3n} e_{22} + (q^n + q^{-n}) e_{33} + q^n (q^{2n} - q^{-2n}) e_{44} - q^{2n} (q^n + q^{-n}) e_{66} q^n e_{77} - q^n e_{88} \right\} \]

\[ E'_{n\delta}^{(\beta)} = [2]^{-1} [2]^{-1} \frac{[n]}{n} \left\{ q^{2n} e_{11} + (q^{2n} + q^{-2n}) e_{22} - e_{33} - q^n (q^n - q^{-n}) e_{44} + q^{-n} (q^n - q^{-n}) (q^{2n} + 1) e_{55} + q^{-2n} e_{66} - q^{-2n} (q^{2n} + q^{-2n}) e_{77} - q^{-4n} e_{88} \right\} \]

\[ F'_{n\delta}^{(\beta)} = [2]^{-1} [2]^{-1} \frac{[n]}{n} \left\{ q^{-2n} e_{11} + (q^{2n} + q^{-2n}) e_{22} - e_{33} + q^n (q^n - q^{-n}) e_{44} - q^n (q^n - q^{-n}) (q^{2n} + 1) e_{55} + q^{2n} e_{66} - q^n (q^{2n} + q^{-2n}) e_{77} - q^{4n} e_{88} \right\} \]

\[ (42) \]

**Proof:** Straightforward calculations + induction in \( n \).

**Proposition:** We have the following properties for the generators in \( (42) \),

\[ (E_\alpha)^2 = [2] q e_{36}, \quad (E_\alpha)^3 = 0, \quad (F_\alpha)^2 = [2] q e_{63}, \quad (F_\alpha)^3 = 0 \]
\[ (E_{\alpha+\beta})^2 = -[2] q^{-1} e_{18}, \quad (E_{\alpha+\beta})^3 = 0, \quad (F_{\alpha+\beta})^2 = -[2] q e_{63}, \quad (F_{\alpha+\beta})^3 = 0 \]
\[ (E'_{\beta+n\delta})^2 = [2] q^{-2n} e_{27}, \quad (E'_{\beta+n\delta})^3 = 0, \quad (F'_{\beta+n\delta})^2 = [2] q^{2n} e_{72}, \quad (F'_{\beta+n\delta})^3 = 0 \]
\[ (E'_{(\delta-\beta)+n\delta})^2 = [2] q^{2n} e_{72}, \quad (E'_{(\delta-\beta)+n\delta})^3 = 0 \]
\[ (F'_{(\delta-\beta)+n\delta})^2 = [2] q^{2n} e_{27}, \quad (F'_{(\delta-\beta)+n\delta})^3 = 0 \]

\[ (43) \]

**Proof:** Easily checked.

We now apply \( (28) \) to \( V_{(8)} \otimes V_{(8)} \), where \( V_{(8)} \) is the adjoint representation of \( U_q(A_2) \). Inserting \( (42) \) into \( (28) \), we see that in the expansion of each \( q \)-exponential only three terms survive thanks to the celebrated properties of generators, eq. \( (43) \). Thus one is able to work out the infinite products in \( (28) \). The contributions from the imaginary root vectors in \( (28) \) can also be worked out term by term and written as a very compact form. The final result may be put in the explicit and compact form,

\[ R_{(8),(8)}(x, y) = \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \frac{\left( \frac{x}{y} \right)^n}{(y^2 - x^2)(y - q^2 x)} e_{36} \otimes e_{63} \right\} \]
\[ \cdot \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \frac{\left( \frac{x}{y} \right)^n}{(y^2 - x^2)(y - q^2 x)} e_{36} \otimes e_{63} \right\} \]
\begin{align*}
&+ [2] q^{-1} (q^{-1})^2 \frac{y^2 (y + q^4 x)}{(y^2 - x^2) (y - q^2 x)} e_{18} \otimes e_{81} \\
&\cdot \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^n \left( E'_{\beta + n\delta} \otimes F'_{\beta + n\delta} \right) + \right. \\
&+ [2] q^{-1} (q^{-1})^2 \frac{y^2 (y + q^4 x)}{(y^2 - x^2) (y - q^2 x)} e_{27} \otimes e_{72} \\
&\cdot \{ \text{imaginary root vectors contribution} \} \\
&\left. \cdot \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \left( E'_{(\delta - \beta) + n\delta} \otimes F'_{(\delta - \beta) + n\delta} \right) + \right. \\
&+ [2] q^{-1} (q^{-1})^2 \frac{x^2 (y + q^4 x)}{(y^2 - x^2) (y - q^2 x)} e_{72} \otimes e_{27} \\
&\cdot \left. \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \left( E'_{(\delta - \alpha) + n\delta} \otimes F'_{(\delta - \alpha) + n\delta} \right) + \right. \\
&+ [2] q^{-1} (q^{-1})^2 \frac{x^2 (y + q^4 x)}{(y^2 - x^2) (y - q^2 x)} e_{63} \otimes e_{36} \\
&\cdot \left. \left\{ 1 + (q - q^{-1}) \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^{n+1} \left( E'_{(\delta - \alpha - \beta) + n\delta} \otimes F'_{(\delta - \alpha - \beta) + n\delta} \right) + \right. \\
&+ [2] q^{-1} (q^{-1})^2 \frac{x^2 (y + q^4 x)}{(y^2 - x^2) (y - q^2 x)} e_{81} \otimes e_{18} \\
&\cdot \left\{ q^2 \sum_{i=1}^{8} (1 - \delta_{i4} - \delta_{i5}) (e_{i2} \otimes e_{i2}) + q (e_{11} \otimes e_{22} + e_{11} \otimes e_{33} + e_{22} \otimes e_{66} + \right.ight. \\
&+ e_{33} \otimes e_{77} + e_{66} \otimes e_{88} + e_{77} \otimes e_{88} + \{ \leftrightarrow \}) + q^{-1} (e_{11} \otimes e_{66} + e_{11} \otimes e_{77} + \right.ight.ight.ight. \\
&\left. \left. + e_{22} \otimes e_{33} + e_{22} \otimes e_{88} + e_{33} \otimes e_{88} + e_{66} \otimes e_{77} + \right. \right.ight.ight.ight. \\
&\left. \left. \{ \leftrightarrow \} \right) + q^{-2} (e_{11} \otimes e_{88} + e_{22} \otimes e_{77} + e_{33} \otimes e_{66} + \{ \leftrightarrow \}) \right\}
\end{align*}

(44)

where "\{\leftrightarrow\}" denotes the interchange of the quantities in the space $X \otimes Y$; $E'_{\beta + n\delta}$, $F'_{\beta + n\delta}$, $E'_{(\delta - \beta) + n\delta}$, $F'_{(\delta - \beta) + n\delta}$ are given in (12) and

\begin{align*}
E'_{\alpha + n\delta} &= (-1)^n \left\{ q^{-2n} e_{12} + [2] q^{1/2} q^{-2n} e_{34} + [2] q^{1/2} e_{46} + e_{78} \right\} \\
F'_{\alpha + n\delta} &= (-1)^n \left\{ q^{2n} e_{21} + [2] q^{1/2} q^{2n} e_{43} + [2] q^{1/2} e_{64} + e_{87} \right\} \\
E'_{\alpha + \beta + n\delta} &= (-1)^n \left\{ -[2] q^{-1/2} q^{-2n - 2} e_{14} + \left( \frac{[3] q}{[2] q} \right)^{1/2} q^{-2n} e_{15} - q^{-1} e_{26} + \right. \\
&\left. + q^{-2n} e_{37} + [2] q^{-1/2} q e_{48} - \left( \frac{[3] q}{[2] q} \right)^{1/2} q^{-1} e_{58} \right\} \\
F'_{\alpha + \beta + n\delta} &= (-1)^n \left\{ -[2] q^{-1/2} q^{2n + 2} e_{41} + \left( \frac{[3] q}{[2] q} \right)^{1/2} q^{2n} e_{51} - q e_{62} + \right. \\
&\left. + q^{2n} e_{73} + [2] q^{-1/2} q^{-1} e_{84} - \left( \frac{[3] q}{[2] q} \right)^{1/2} q e_{85} \right\} \\
E'_{(\delta - \alpha) + n\delta} &= (-1)^n \left\{ q^{-2n-1} e_{21} + [2] q^{1/2} q^{-2n} e_{43} + [2] q^{1/2} e_{64} + q e_{87} \right\} \\
F'_{(\delta - \alpha) + n\delta} &= (-1)^n \left\{ q^{2n+1} e_{12} + [2] q^{1/2} q^{2n} e_{34} + [2] q^{1/2} e_{46} + q^{-1} e_{78} \right\}
\end{align*}
\[
E_{(\delta-\alpha-\beta)+n\delta} = (-1)^n \left\{ \begin{array}{l}
-|q|^{-1/2} q^{-2n+2} e_{41} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{-2n} e_{51} - q^2 e_{62} + \\
+ q^{-2n-1} e_{73} + |q|^{-1/2} q^{-1} e_{84} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q e_{85}
\end{array} \right. \\
F_{(\delta-\alpha-\beta)+n\delta} = (-1)^n \left\{ \begin{array}{l}
-|q|^{-1/2} q^{-2n-2} e_{14} + \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{2n} e_{15} - q^{-2} e_{26} + \\
+ q^{2n+1} e_{37} + |q|^{-1/2} q e_{48} - \left( \frac{[3]_q}{[2]_q} \right)^{1/2} q^{-1} e_{58}
\end{array} \right. \\
\right. \\
\right. \\
\{\text{imaginary root vectors contribution}\} = a'/a \sum_{i=1}^{8} (1 + (b/b' - 1)\delta_{i4} + (c/c' - 1)\delta_{i5}) (e_{ii} \otimes e_{ii}) + \\
+ a'(e_{11} \otimes e_{22} + e_{11} \otimes e_{33}) + aa'(e_{11} \otimes e_{44}) + a'c(e_{11} \otimes e_{55}) + a(e_{11} \otimes e_{66}) + \\
+ aa'c(e_{11} \otimes e_{77}) + ac(e_{11} \otimes e_{88}) + 1/a(e_{22} \otimes e_{11}) + 1/b'(e_{22} \otimes e_{33}) + \\
+ a'/b'(e_{22} \otimes e_{44}) + a'c(e_{22} \otimes e_{55}) + a'(e_{22} \otimes e_{66}) + aa'c/b'(e_{22} \otimes e_{77}) + \\
+ aa'c(e_{22} \otimes e_{88}) + 1/a(e_{33} \otimes e_{22}) + b(e_{33} \otimes e_{22}) + a'b(e_{33} \otimes e_{44}) + \\
+ ab(e_{33} \otimes e_{66}) + a'(e_{33} \otimes e_{77}) + a(e_{33} \otimes e_{88}) + 1/(aa')(e_{44} \otimes e_{11}) + \\
+ b/a(e_{44} \otimes e_{22}) + 1/(aa')(e_{44} \otimes e_{33}) + a'b(e_{44} \otimes e_{66}) + a'/b'(e_{44} \otimes e_{77}) + \\
+ aa'(e_{44} \otimes e_{88}) + 1/(aa')(e_{55} \otimes e_{11} + e_{55} \otimes e_{22}) + a'c(e_{55} \otimes e_{77} + e_{55} \otimes e_{88}) + \\
+ 1/a'(e_{66} \otimes e_{11}) + 1/a(e_{66} \otimes e_{22}) + 1/(aa')(e_{66} \otimes e_{33}) + 1/(aa')(e_{66} \otimes e_{44}) + \\
+ 1/b'(e_{66} \otimes e_{77}) + a'(e_{66} \otimes e_{88}) + 1/(aa')(e_{77} \otimes e_{11}) + b/(aa')(e_{77} \otimes e_{22}) + \\
+ 1/a(e_{77} \otimes e_{33}) + b/a(e_{77} \otimes e_{44}) + 1/(aa')(e_{88} \otimes e_{55}) + b(e_{88} \otimes e_{66}) + \\
+ a'(e_{88} \otimes e_{88}) + 1/(aa')(e_{88} \otimes e_{11}) + 1/(aa')(e_{88} \otimes e_{22}) + 1/a(e_{88} \otimes e_{33}) + \\
+ 1/(aa')(e_{88} \otimes e_{44}) + 1/(aa')(e_{88} \otimes e_{55}) + 1/a(e_{88} \otimes e_{66}) + e_{88} \otimes e_{77})
\right. \\
(45)
\right.
\]

in which we have defined

\[
\begin{align*}
 a &= \frac{y - q^2 x}{y - x}, & a' &= \frac{y - q^{-2} x}{y - x}, & b &= \frac{y - q^4 x}{y - q^2 x} \\
 b' &= \frac{y - q^{-4} x}{y - q^{-2} x}, & c &= \frac{y - q^6 x}{y - q^4 x}, & c' &= \frac{y - q^{-6} x}{y - q^{-4} x}
\end{align*}
\]

(46)

We see that (44) is an extremely explicit formula: the sums in (44) can be easily worked out.
We do this in the Appendix B.

## 6 Concluding Remarks

In this paper we have given a detailed account for the results presented in our previous short letter [1] where only the results have been announced.

We believe that along our line we may at least search for the solution to the following problems. Firstly, we may try to extend the above to other types of quantum affine algebras (twisted
or nontwisted). To this effect, we first have to answer the question how to quantize loop representations of the other types (type B, C, D, E and exotic) of groups. Secondly, we may wonder if there exist some kind of "universal" integrable lattice models which have our spectral-dependent \( R \)-matrix as their Boltzmann weights. Thirdly, we may consider the possibility of finding and computing eigenvalues of Casimir operators constructed from these spectral parameter dependent \( R \)-matrix which are expected to play some role in one dimensional open spin chains \[\text{[17][18]}\]. Finally, we believe our formula will be useful in quantizing the conformal affine Toda theories \[\text{[14]}\] and in the recently-developed \( q \)-deformed WZNW CFT’s \[\text{[19][20]}\]. These are problems now under consideration.

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7 Appendix A

We consider here finite-dimensional loop representations of \( U_q(gl(n)^{(1)}) = U_q(A_{-1}^{(1)}) \) with the Chevalley generators \( \{E_i, F_i, q^{h_i}, 0 \leq i < n; q^d\} \) in which

\[
E_i \equiv E_{ii+1}, \quad F_i \equiv E_{i+1i}, \quad q^{h_i} \equiv E_{ii} - E_{i+1i+1}, \quad 1 \leq i < n, \quad q^{E_{nn}} \quad (47)
\]

are the usual Chevalley generators of \( U_q(gl(n)) \). We define

\[
E_{ij} = E_{ik}E_{kj} - q^{-1}E_{kj}E_{ik}, \quad i < k < j \\
E_{ij} = E_{ik}E_{kj} - qE_{kj}E_{ik}, \quad i > k > j \quad (48)
\]

and put

\[
E_\psi \equiv q^{E_{11} + E_{nn}}E_{1n}, \quad F_\psi \equiv E_{n1}q^{-E_{11} - E_{nn}}, \quad h_\psi \equiv E_{11} - E_{nn} \quad (49)
\]

then we have

**Proposition:** For any given \( z \in \mathbb{C}^\times \), there is a homomorphism of algebras \( \text{ev}_z: U_q(gl(n)^{(1)}) \to U_q(gl(n)) \), in terms of the Chevalley generators,

\[
\text{ev}_z(E_i) = E_i, \quad \text{ev}_z(F_i) = F_i, \quad \text{ev}_z(h_i) = h_i \\
\text{ev}_z(E_0) = zF_\psi, \quad \text{ev}_z(F_0) = z^{-1}E_\psi, \quad \text{ev}_z(h_0) = -h_\psi, \quad \text{ev}_z(c) = 0 \quad (50)
\]

**Proof:** To show they define a homomorphism \( U_q(gl(n)^{(1)}) \to U_q(gl(n)) \), one needs to check that the relations in \[\text{[4]}\] are satisfied. This is immediate except for the last two, which reduce to

\[
(ad_{q_{-1}}F_1)^{1+(\psi,\alpha_i)}F_0 = (ad_{q_{-1}}F_1)^{1+(\psi,\alpha_i)}E_\psi = 0 \quad (51)
\]
\[(\text{ad}_{q^{-1}}F_0)^{1+(\psi,\alpha_i)}F_i = (\text{ad}_{q^{-1}}E_\psi)^{1+(\psi,\alpha_i)}F_i = 0 \quad (52)\]

and two similar relations with interchange \(F_i \leftrightarrow E_i\), \(F_0 \leftrightarrow E_0\), \(q^{-1} \leftrightarrow q\). We now prove (52). First we consider the case: \(1 < i < i+1 < n\). In this case \((\psi,\alpha_i) = 0\), and the l.h.s. of (51) becomes

\[(\text{ad}_{q^{-1}}F_i)E_\psi = [F_i, E_\psi] = q^{E_{11} + E_{nn}}[E_{i+1}, E_{1n}] \quad (53)\]

which can be easily checked to be vanishing. We then consider the \(i = 1\) case. In this case the l.h.s. of (51) reads

\[(\text{ad}_{q^{-1}}F_1)^2E_\psi = (\text{ad}_{q^{-1}}E_{21})(\text{ad}_{q^{-1}}E_{21})E_\psi \quad (54)\]

One can easily show \((\text{ad}_{q^{-1}}E_{21})E_\psi = q^{E_{22} + E_{nn}}E_{2n}\). Inserting this into (54), one gets

\[= (\text{ad}_{q^{-1}}E_{21})q^{E_{22} + E_{nn}}E_{2n} = q^{E_{22} + E_{nn} - 1}[E_{21}, E_{2n}] = 0 \quad (55)\]

as required. Finally for \(i = n\), we see the l.h.s. of (51) reduces to

\[(\text{ad}_{q^{-1}}F_n)^2E_\psi = (\text{ad}_{q^{-1}}E_{n,n-1})(\text{ad}_{q^{-1}}E_{n,n-1})E_\psi \quad (56)\]

Some direct computations give

\[= q^{E_{11} + E_{nn} - 1}\left\{q^{-1}E_{n,n-1}[E_{n,n-1}, E_{1n}] - q[E_{n,n-1}, E_{1n}E_{n,n-1}]\right\} \quad (57)\]

which, using the directly checkable formula,

\[E_{n,n-1}, E_{1n}] = -q^{E_{n,n-1} - E_{nn}}E_{1n,n-1}, \quad (58)\]

is easily seen to be vanishing. We may similarly prove (52).

**Remark:** Since \(N \equiv \sum_{i=1}^{n} E_{ii}\) commutes with everything, therefore, if we set, instead of (19),

\[E_\psi \equiv q^{E_{11} + E_{nn} - \frac{2}{n} N}E_{1n}, \quad F_\psi \equiv E_{n1}q^{-E_{11} - E_{nn} + \frac{2}{n} N}E_{1n} \quad (59)\]

then the above proposition in this appendix still holds. It turns out that it is more convenient to use (53) as we did in the previous sections.

**8 Appendix B**

For completeness, in this appendix we work out the sums appearing in (14). We list the results below:

\[\text{the first sum} = \frac{y}{y - x}\left\{e_{12} \otimes e_{21} + [2]_{q}^{1/2}(e_{12} \otimes e_{33}) + [2]_{q}^{1/2}(e_{34} \otimes e_{21}) + [2]_{q}(e_{34} \otimes e_{43}) + [2]_{q}(e_{46} \otimes e_{64}) + [2]_{q}^{1/2}(e_{46} \otimes e_{87}) + [2]_{q}^{1/2}(e_{78} \otimes e_{64}) + e_{12} \otimes e_{87} + [2]_{q}(e_{34} \otimes e_{64}) + [2]_{q}^{1/2}(e_{34} \otimes e_{87})\right\} + \frac{y}{y - q^{-2}x}\left\{[2]_{q}^{1/2}(e_{12} \otimes e_{64}) + e_{12} \otimes e_{87} + [2]_{q}(e_{34} \otimes e_{64}) + [2]_{q}^{1/2}(e_{34} \otimes e_{87})\right\} + \]


\[
+ \frac{y}{y - q^2 x} \left\{ [2]_{q}^{1/2}(e_{14} \otimes e_{21}) + [2]_{q}(e_{46} \otimes e_{43}) + e_{78} \otimes e_{21} + [2]_{q}^{1/2}(e_{78} \otimes e_{43}) \right\}
\]

the second sum is
\[
\frac{y}{y - x} \left\{ 1/[2]_{q}(e_{14} \otimes e_{41}) - [3]_{q}^{1/2} / ([2]_{q})(e_{14} \otimes e_{51}) - 1/([2]_{q}^{1/2})(e_{14} \otimes e_{73}) - q^{2}[3]_{q}^{1/2} / [2]_{q}(e_{15} \otimes e_{41}) + [3]_{q} / [2]_{q}(e_{15} \otimes e_{51}) + ([3]_{q} / [2]_{q})^{1/2}(e_{15} \otimes e_{73}) + e_{26} \otimes e_{62} - 1/([2]_{q}^{1/2})(e_{26} \otimes e_{84}) + ([3]_{q} / [2]_{q})^{1/2}(e_{26} \otimes e_{85}) - q^{2}/[2]_{q}^{1/2}(e_{37} \otimes e_{41}) + ([3]_{q} / [2]_{q})^{1/2}(e_{37} \otimes e_{51}) + e_{37} \otimes e_{73} - q^{2}/[2]_{q}^{1/2}(e_{48} \otimes e_{62}) + e_{48} \otimes e_{84} - q^{2}[3]_{q}^{1/2} / [2]_{q}(e_{48} \otimes e_{85}) + ([3]_{q} / [2]_{q})^{1/2}(e_{58} \otimes e_{62}) - [3]_{q}^{1/2} / ([2]_{q}^{2})(e_{58} \otimes e_{84}) + [3]_{q} / [2]_{q}(e_{58} \otimes e_{85}) \right\}
\]

the third sum is
\[
\frac{y}{y - q^{-2} x} \left\{ 1/(q/[2]_{q})^{1/2}(e_{14} \otimes e_{62}) - 1/(q^{3}[2]_{q})(e_{14} \otimes e_{84}) + [3]_{q}^{1/2} / ([2]_{q})^{1/2}(e_{14} \otimes e_{85}) - q^{-1}([3]_{q} / [2]_{q})^{1/2}(e_{15} \otimes e_{62}) + [3]_{q}^{1/2} / ([2]_{q})^{1/2}(e_{15} \otimes e_{84}) - q([3]_{q} / [2]_{q})(e_{15} \otimes e_{85}) - q(e_{37} \otimes e_{62}) + 1/(q/[2]_{q})^{1/2}(e_{37} \otimes e_{84}) - q([3]_{q} / [2]_{q})^{1/2}(e_{37} \otimes e_{85}) \right\}
\]

the fourth sum is
\[
\frac{x}{y - x} \left\{ e_{13} \otimes e_{31} + 1/[2]_{q}^{1/2}(e_{13} \otimes e_{42}) + ([3]_{q} / [2]_{q})^{1/2}(e_{13} \otimes e_{75}) + 1/[2]_{q}^{1/2}(e_{24} \otimes e_{31}) + 1/[2]_{q}(e_{24} \otimes e_{42}) + [3]_{q}^{1/2} / [2]_{q}(e_{24} \otimes e_{75}) + [3]_{q} / [2]_{q}(e_{25} \otimes e_{52}) + [3]_{q}^{1/2} / [2]_{q}(e_{25} \otimes e_{74}) + ([3]_{q} / [2]_{q})^{1/2}(e_{25} \otimes e_{86}) + [3]_{q}^{1/2} / [2]_{q}(e_{47} \otimes e_{52}) + 1/[2]_{q}(e_{47} \otimes e_{74}) + e_{47} \otimes e_{86} + e_{57} \otimes e_{31} + [3]_{q}^{1/2} / [2]_{q}(e_{57} \otimes e_{42}) + [3]_{q}^{1/2} / [2]_{q}(e_{57} \otimes e_{75}) + [3]_{q} / [2]_{q}(e_{57} \otimes e_{85}) + 1/[2]_{q}^{1/2}(e_{68} \otimes e_{74}) + e_{68} \otimes e_{86}\right\}
\]

\[
\frac{y}{y - q^{2} x} \left\{ ([3]_{q} / [2]_{q})^{1/2}(e_{13} \otimes e_{52}) + 1/[2]_{q}^{1/2}(e_{13} \otimes e_{74}) + e_{13} \otimes e_{86} + [3]_{q}^{1/2} / [2]_{q}(e_{24} \otimes e_{52}) + 1/[2]_{q}(e_{24} \otimes e_{74}) + 1/[2]_{q}^{1/2}(e_{24} \otimes e_{86}) + [3]_{q} / [2]_{q}(e_{57} \otimes e_{52}) + [3]_{q}^{1/2} / [2]_{q}(e_{57} \otimes e_{74}) + ([3]_{q} / [2]_{q})^{1/2}(e_{57} \otimes e_{86}) + \right\}
\]

\[
\frac{y}{y - q^{-4} x} \left\{ ([3]_{q} / [2]_{q})^{1/2}(e_{25} \otimes e_{31}) + [3]_{q}^{1/2} / [2]_{q}(e_{25} \otimes e_{42}) + [3]_{q} / [2]_{q}(e_{25} \otimes e_{75}) + 1/[2]_{q}^{1/2}(e_{47} \otimes e_{31}) + 1/[2]_{q}(e_{47} \otimes e_{42}) + [3]_{q}^{1/2} / [2]_{q}(e_{47} \otimes e_{75}) + e_{68} \otimes e_{31} + 1/[2]_{q}^{1/2}(e_{68} \otimes e_{42}) + ([3]_{q} / [2]_{q})^{1/2}(e_{68} \otimes e_{75}) \right\}
\]

the fourth sum is
\[
\frac{x}{y - x} \left\{ e_{31} \otimes e_{13} + 1/(q/[2]_{q})^{1/2}(e_{31} \otimes e_{24}) + q([3]_{q} / [2]_{q})^{1/2}(e_{31} \otimes e_{57}) + q/[2]_{q}^{1/2}(e_{42} \otimes e_{13}) + 1/[2]_{q}(e_{42} \otimes e_{24}) + q^{2}[3]_{q}^{1/2} / [2]_{q}(e_{42} \otimes e_{57}) + [3]_{q} / [2]_{q}(e_{52} \otimes e_{25}) + q^{2}[3]_{q}^{1/2} / [2]_{q}(e_{52} \otimes e_{47}) + q([3]_{q} / [2]_{q})^{1/2}(e_{52} \otimes e_{68}) + [3]_{q}^{1/2} / ([2]_{q}^{2})(e_{74} \otimes e_{25}) + 1/[2]_{q}(e_{74} \otimes e_{47}) + 1/[2]_{q}^{1/2}(e_{74} \otimes e_{68}) + q^{-1}([3]_{q} / [2]_{q})^{1/2}(e_{75} \otimes e_{13}) + [3]_{q}^{1/2} / ([2]_{q}^{2})(e_{75} \otimes e_{24}) + [3]_{q} / [2]_{q}(e_{75} \otimes e_{57}) + + q^{-1}([3]_{q} / [2]_{q})^{1/2}(e_{86} \otimes e_{25}) + q/[2]_{q}^{1/2}(e_{86} \otimes e_{47}) +
\]
\[ + e_{86} \otimes e_{68} \} + \frac{x}{y - q^4 x} \{ q^3([3]_q/[2]_q)^{1/2} (e_{31} \otimes e_{25}) + q^5/[2]_q^{1/2} (e_{31} \otimes e_{47}) + \\
+ q^4(e_{31} \otimes e_{68}) + q^4[3]_q^{1/2}/[2]_q(e_{42} \otimes e_{25}) + q^6/[2]_q(e_{42} \otimes e_{47}) + \\
+ q^5/[2]_q^{1/2} (e_{42} \otimes e_{68}) + q^2[3]_q/[2]_q(e_{75} \otimes e_{25}) + q^4[3]_q^{1/2}/[2]_q(e_{75} \otimes e_{47}) + \\
+ q^3([3]_q/[2]_q)^{1/2} (e_{75} \otimes e_{68}) \} + \frac{x}{y - q^3 x} \{ q^{-3}([3]_q/[2]_q)^{1/2} (e_{52} \otimes e_{13}) + \\
+ [3]_q^{1/2}/(q^4[2]_q) (e_{52} \otimes e_{24}) + [3]_q/(q^2[3]_q) (e_{52} \otimes e_{57}) + \\
+ 1/(q^5[2]_q^{1/2})(e_{74} \otimes e_{13}) + 1/(q^6[2]_q)(e_{74} \otimes e_{24}) + [3]_q^{1/2}/(q^4[2]_q)(e_{74} \otimes e_{57}) + \\
+ q^{-4}(e_{86} \otimes e_{13}) + 1/(q^5[2]_q^{1/2})(e_{86} \otimes e_{24}) + q^{-3}([3]_q/[2]_q)^{1/2} (e_{86} \otimes e_{57}) \}
\]

the fifth sum = \[ \frac{x}{y - x} \{ e_{21} \otimes e_{12} + [2]_q^{1/2}/q(e_{21} \otimes e_{34}) + q[2]_q^{1/2}(e_{43} \otimes e_{12}) + [2]_q(e_{43} \otimes e_{34}) + \\
[2]_q(e_{64} \otimes e_{46}) + [2]_q^{1/2}/q(e_{64} \otimes e_{78}) + q[2]_q^{1/2}(e_{87} \otimes e_{46}) + e_{87} \otimes e_{78} \}
\]

\[ + \frac{x}{y - q^{-2} x} \{ [2]_q^{1/2}/q(e_{21} \otimes e_{46}) + 1/q^2(e_{21} \otimes e_{78}) + [2]_q(e_{43} \otimes e_{46}) + [2]_q^{1/2}/q(e_{43} \otimes e_{78}) \}
\]

\[ + \frac{x}{y - q^{-2} x} \{ q[2]_q^{1/2} (e_{64} \otimes e_{12}) + [2]_q(e_{64} \otimes e_{34}) + q^2(e_{87} \otimes e_{12}) + q[2]_q^{1/2}(e_{87} \otimes e_{34}) \}
\]

the sixth sum = \[ \frac{x}{y - x} \{ 1/[2]_q(e_{41} \otimes e_{14}) - q^2[3]_q^{1/2}/[2]_q(e_{41} \otimes e_{15}) - q^3/[2]_q^{1/2}(e_{41} \otimes e_{37}) - \\
[3]_q^{1/2}/(q^2[2]_q) (e_{51} \otimes e_{14}) + [3]_q/[2]_q(e_{51} \otimes e_{15}) + q([3]_q/[2]_q)^{1/2} (e_{51} \otimes e_{37}) + \\
e_{62} \otimes e_{26} - q^3/[2]_q^{1/2} (e_{62} \otimes e_{48}) + q([3]_q/[2]_q)^{1/2} (e_{62} \otimes e_{58}) - 1/(q^3[2]_q^{1/2})(e_{73} \otimes e_{14}) + \\
q^{-1}([3]_q/[2]_q)^{1/2}(e_{73} \otimes e_{15}) + e_{73} \otimes e_{37} - 1/(q^3[2]_q^{1/2})(e_{84} \otimes e_{26}) + \\
1/[2]_q(e_{84} \otimes e_{48}) - [3]_q^{1/2}/(q^2[2]_q) (e_{84} \otimes e_{58}) + q^{-1}([3]_q/[2]_q)^{1/2} (e_{85} \otimes e_{26}) + \\
- q^2[3]_q^{1/2}/[2]_q(e_{85} \otimes e_{48}) + [3]_q/[2]_q(e_{85} \otimes e_{58}) \} + \frac{x}{y - q^{-2} x} \{ 1/[2]_q^{1/2}(e_{41} \otimes e_{26}) + \\
- q^3/[2]_q^{1/2}(e_{41} \otimes e_{48}) + q[3]_q^{1/2}/[2]_q(e_{41} \otimes e_{58}) - q^2([3]_q/[2]_q)^{1/2}(e_{51} \otimes e_{26}) + \\
+ q[3]_q^{1/2}/[2]_q(e_{51} \otimes e_{48}) - q^{-1}([3]_q/[2]_q)(e_{51} \otimes e_{58}) - 1/q^3(e_{73} \otimes e_{26}) + \\
1/[2]_q^{1/2}(e_{73} \otimes e_{48}) - q^2([3]_q/[2]_q)^{1/2}(e_{73} \otimes e_{58}) \} + \frac{x}{y - q^{-2} x} \{ 1/[2]_q^{1/2}(e_{62} \otimes e_{14}) + \\
- q^2([3]_q/[2]_q)^{1/2}(e_{62} \otimes e_{37}) - q^3(e_{62} \otimes e_{37}) - 1/(q^3[2]_q)(e_{84} \otimes e_{14}) + \\
+ [3]_q^{1/2}/(q[2]_q)(e_{84} \otimes e_{15}) + 1/[2]_q^{1/2}(e_{84} \otimes e_{37}) + [3]_q^{1/2}/(q[2]_q)(e_{85} \otimes e_{14}) + \\
- q([3]_q/[2]_q)(e_{85} \otimes e_{15}) - q^2([3]_q/[2]_q)^{1/2}(e_{85} \otimes e_{37}) \}
\]
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