Doi-Hopf Modules over Weak Hopf Algebras

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Abstract

The theory of Doi-Hopf modules is generalized to Weak Hopf Algebras.

1 Introduction

The category $\mathcal{CM}(H)_A$ of Doi-Hopf Modules over the bialgebra $H$ was introduced in [7] and independently in [10]. It is the category of the modules over the algebra $A$ which are also comodules over the coalgebra $C$ and satisfy certain compatibility condition involving $H$. The study of $\mathcal{CM}(H)_A$ turned out to be very useful: It was shown in [7, 4] that many categories investigated independently before – such as the module and comodule categories over bialgebras, the Hopf modules category [15], and the Yetter-Drinfeld category [16, 14] – are special cases of $\mathcal{CM}(H)_A$. Using this observation many results known for module categories over bialgebras or Hopf algebras were generalized to this more general setting [5, 6].

In this paper we generalize the definition of Doi-Hopf modules to the case when $H$ is a Weak Bialgebra (WBA). Our definitions are supported by the fact that many results of [10, 5, 6] remain valid in this case.

Weak Bialgebras (Weak Hopf Algebras – WHA’s –) are generalizations of bialgebras (Hopf algebras) see [1, 2] and [12] (latter one using somewhat different terminology). In contrast to another direction of generalization, the quasi-Hopf algebras and weak quasi-Hopf algebras, WBA’s are coassociative. Though their counit is not an algebra map, their structure is designed such a way that their (left or right) (co-) module category carries a monoidal structure [12, 5] (and some more in the WHA case [3]).

WHA’s have relevance for example in describing depth 2 reducible inclusions [13].

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As the bialgebra (Hopf algebra) also the WBA (WHA) is a self-dual structure: The dual space of a finite dimensional WBA (WHA) carries naturally a WBA (WHA) structure \[1, 2\].

The paper is organized as follows: we define and examine the structures such as the weak Doi-Hopf datum (generalizing the Doi-Hopf datum of [7]) the weak Doi-Hopf module (generalizing the Doi-Hopf module of [7]) the weak smash product (generalizing the analogous notion of [14]) and the weak Doi-Hopf integral (generalizing definitions of [1, 8]). We illustrate these notions on the same four examples generalizing some classical examples of [7, 4].

2 The Weak Doi-Hopf Datum

In this Section \(H\) is a Weak Bialgebra (WBA) in the sense of \[2\] over the field \(k\). Its unit element is denoted by \(\mathbb{1}\), the product of the elements \(g, h \in H\) by \(gh\), the coproduct of \(h \in H\) by \(\Delta(h) = h_{(1)} \otimes h_{(2)}\) and the counit is denoted by \(\varepsilon\).

**Definition 2.1** Let \(H\) be a WBA over the field \(k\). The \(k\)-algebra \(A\) is a left \(H\)-comodule algebra if there exists a left weak coaction \(\rho\) of \(H\) on \(A\) which is also an algebra map. I.e. a map \(\rho : A \to H \otimes A\) such that

\[
\begin{align*}
(id_H \otimes \rho) \circ \rho &= (\Delta \otimes id_A) \circ \rho \quad (2.1a) \\
(\mathbb{1} \otimes a)\rho(1_A) &= (\Pi_R \otimes id_A) \circ \rho(a) \quad (2.1b) \\
\rho(ab) &= \rho(a)\rho(b) \quad (2.1c)
\end{align*}
\]

for all \(a, b \in A\). We use the standard notation \(\rho(a) = a_{<1>} \otimes a_{<0>}\) and \((\Delta \otimes id_A) \circ \rho(a) = a_{<2>} \otimes a_{<1>} \otimes a_{<0>} = (id_H \otimes \rho) \circ \rho(a)\).

The left weak coaction \(\rho\) is non-degenerate if \((\varepsilon \otimes id_A) \circ \rho = id_A\) or, equivalently, \((\varepsilon \otimes id_A) \circ \rho(1_A) = 1_A\). For non-degenerate left weak coactions \(\rho\) (2.1b) has an equivalent form (compare with [\[13\]]) \((\Delta \otimes id_A) \circ \rho(1_A) = (\mathbb{1} \otimes \rho(1_A)) (\Delta(\mathbb{1}) \otimes 1_A)\).

Similarly, \(A\) is a right \(H\)-comodule algebra if there exists a right weak coaction \(\rho\) of \(H\) on \(A\) which is also an algebra map. I.e. a map \(\rho : A \to A \otimes H\) such that

\[
\begin{align*}
((\rho \otimes id_H) \circ \rho &= (id_A \otimes \Delta) \circ \rho \quad (2.2a) \\
\rho(1_A)(a \otimes \mathbb{1}) &= (id_A \otimes \Pi_L) \circ \rho(a) \quad (2.2b) \\
\rho(ab) &= \rho(a)\rho(b) \quad (2.2c)
\end{align*}
\]

for all \(a, b \in A\). We also denote \(\rho(a) = a_{<0>} \otimes a_{<1>}\).

The right weak coaction \(\rho\) is non-degenerate if \((id_A \otimes \varepsilon) \circ \rho = id_A\) or, equivalently, if \((id_A \otimes \varepsilon) \circ \rho(1_A) = 1_A\). For non-degenerate right weak coactions \(\rho\) (2.2b) has an equivalent form \((id_A \otimes \Delta) \circ \rho(1_A) = (1_A \otimes (\Delta(\mathbb{1})) (\rho(1_A) \otimes \mathbb{1})\).

The dual notion to comodule algebra is the module coalgebra defined as follows: The \(k\)-coalgebra \(C\) is a right \(H\)-module coalgebra if there exists a right weak action of \(H\) on \(C\) which is also a coalgebra map. I.e. a map \(\cdot : C \times H \to C\) such that

\[
\begin{align*}
(c \cdot g) \cdot h &= c \cdot (gh) \quad (2.3a) \\
c \cdot \Pi_L(h) &= \varepsilon_C(c_{(1)} \cdot h)c_{(2)} \quad (2.3b) \\
\Delta_C(c \cdot h) &= \Delta_C(c) \cdot \Delta(h) \quad (2.3c)
\end{align*}
\]
for all \( c \in C, \ g, h \in H \).

The right weak action \( \cdot \) is non-degenerate if \( c \cdot \mathbb{1} = c \ \forall c \in C \) or, equivalently, if \( \varepsilon_C(c \cdot \mathbb{1}) = \varepsilon_C(c) \ \forall c \in C \). For non-degenerate right weak actions \( \cdot \) (2.3b) has the equivalent reformulation as \( \varepsilon_C(c \cdot h) = \varepsilon(c \cdot \Pi^R(h)) \).

Similarly, \( C \) is a left \( H \)-module coalgebra if there exists a left weak action of \( H \) on \( C \) which is also a coalgebra map. I.e. a map \( \cdot \) : \( H \times C \rightarrow C \) such that

\[
\begin{align*}
g \cdot (h \cdot c) &= (gh) \cdot c \\
\Pi^R(h) \cdot c &= c_{(1)} \varepsilon_C(h \cdot c_{(2)}) \\
\Delta_C(h \cdot c) &= \Delta(h) \cdot \Delta_C(c)
\end{align*}
\]

for all \( c \in C, \ g, h \in H \).

The left weak action \( \cdot \) is non-degenerate if \( \mathbb{1} \cdot c = c \ \forall c \in C \) or, equivalently, if \( \varepsilon_C(\mathbb{1} \cdot c) = \varepsilon_C(c) \ \forall c \in C \). For non-degenerate left weak actions \( \cdot \) (2.4b) has the equivalent reformulation \( \varepsilon_C(h \cdot c) = \varepsilon(\Pi^L(h) \cdot c) \).

Notice, that in contrast to the case when \( H \) is an ordinary bialgebra the unit preserving property of \( \rho \) and the counit preserving property of \( \cdot \) are not required and the form of condition (b) in each group is somewhat different from the usual one.

**Definition 2.2** A right Weak Doi-Hopf datum is a triple \((H, A, C)\), where \( H \) is a WBA over \( k \), \( A \) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra.

A left Weak Doi-Hopf datum is a triple \((H, A, C)\) where \( H \) is a WBA over \( k \), \( A \) a right \( H \)-comodule algebra and \( C \) a left \( H \)-module coalgebra.

A (left or right) Weak Doi-Hopf datum is non-degenerate if both the weak coaction of \( H \) on \( A \) and the weak action of \( H \) on \( C \) are non-degenerate.

**Examples:**

1 Let \( H \) be a WBA over \( k \), \( A := H \) as an algebra with the coaction \( \rho := \Delta \), \( C := H^L \) with the coalgebra structure

\[
\Delta_{H^L}(a^L) := \Pi_{(2)} a^L \otimes S(\Pi_{(1)}) \equiv \Pi_{(2)} \otimes a^L S(\Pi_{(1)})
\]

\[
\varepsilon_{H^L}(a^L) := \varepsilon(a^L)
\]

and the action \( a^L \cdot h := \Pi_{(2)} \varepsilon(a^L h \Pi_{(1)}) \) for all \( a^L \in H^L, h \in H \). Then \((H, A = H, C = H^L)\) is a non-degenerate right Weak Doi-Hopf datum.

2 Let \( H \) be a WBA over \( k \), \( A := H^L \) as the subalgebra of \( H \) with the coaction \( \rho := \Delta|_{H^L} \), \( C := H \) as a coalgebra with the action \( c \cdot h := ch \) for all \( c, h \in H \). Then \((H, A = H^L, C = H)\) is a non-degenerate right Weak Doi-Hopf datum.

3 Let \( H \) be a WBA over \( k \), \( A := H \) as an algebra with the coaction \( \rho := \Delta \), \( C := H \) as a coalgebra with the action \( c \cdot h := ch \) for all \( c, h \in H \). Then \((H, A = H, C = H)\) is a non-degenerate right Weak Doi-Hopf datum.

4 Let \( K \) be a WHA over \( k \), \( H := K^{op} \otimes K \) as a bialgebra. \((K^{op}) \) is the bialgebra with the same coalgebra structure as \( K \) and the opposite algebra structure.) \( A := K \) as an algebra with the coaction \( \rho(a) := (S^{-1}(a_{(3)}) \otimes a_{(1)}) \otimes a_{(2)} \) for all \( a \in K, C := K \)
as a coalgebra with the action $c \cdot (a \otimes b) = a b c$ for all $c \in K, (a \otimes b) \in H$. Then $(H = K^{op} \otimes K, A = K, C = K)$ is a non-degenerate right weak Doi-Hopf datum.

Let us call a (left or right) weak Doi-Hopf datum finite dimensional if all $H, A$ and $C$ are finite dimensional as $k$-spaces. There is a well defined notion of duality for finite dimensional weak Doi-Hopf data sending a left weak Doi-Hopf datum to a right one and vice versa:

Introduce the following notations: For any finite dimensional $k$-space $M$ let $\hat{M}$ denote the dual $k$-space. If $A$ is a finite dimensional algebra then by $\hat{A}$ we mean the dual space equipped with the dual coalgebra structure. Similarly, for a finite dimensional coalgebra $C$ denote the dual algebra by $\hat{C}$ and finally for a finite dimensional bialgebra $H$ denote the dual bialgebra by $\hat{H}$.

Proposition 2.3 For a (non-degenerate) right weak Doi-Hopf datum $(H, A, C)$ the triple $(\hat{H}, \hat{C}, \hat{A})$ is a (non-degenerate) left weak Doi-Hopf datum – called the dual of $(H, A, C)$ – with

$$\hat{\rho}(\hat{c}): = b_i \triangleright \hat{c} \otimes \beta_i$$
$$\phi \cdot \hat{a}: = (\phi \otimes \hat{a}) \circ \rho$$

where $\hat{c} \in \hat{C}$, $\{b_i\}$ is any basis in $H$ and $\{\beta_i\}$ is the dual basis in $\hat{H}$, $\phi \in \hat{H}$, $\hat{a} \in \hat{A}$ and $(h \triangleright \hat{c})(d) = \hat{c}(d \cdot h)$ for $\hat{c} \in \hat{C}, d \in C, h \in H$.

Similarly, for a (non-degenerate) left weak Doi-Hopf datum $(H, A, C)$ the triple $(\hat{H}, \hat{C}, \hat{A})$ is a (non-degenerate) right weak Doi-Hopf datum – called the dual of $(H, A, C)$ – with

$$\hat{\rho}(\hat{c}): = \beta_i \otimes \hat{c} \triangleleft b_i$$
$$\hat{a} \cdot \phi: = (\hat{a} \otimes \phi) \circ \rho$$

with the obvious notation. The above duality transformation is involutive.

Proof: The transformations (2.5) and (2.6) are obviously inverses of one other. One easily checks that (2.2a) for $(\hat{H}, \hat{C}, \hat{A})$ is equivalent to (2.3a) on $(H, A, C)$, (2.2b) to (2.3b), (2.2c) to (2.3c), (2.4a) to (2.1a), (2.4b) to (2.1b) and (2.4c) to (2.1c).

The non-degeneracy of the weak coaction $\rho$ of $\hat{H}$ on $C$ is equivalent to the non-degeneracy of the weak action of $H$ on $C$ while the non-degeneracy of the action of $\hat{H}$ on $\hat{A}$ is equivalent to the non-degeneracy of the weak coaction $\rho$ of $H$ on $A$ both in the left and right cases.

3 The Weak Doi-Hopf Module

Definition 3.1 The $k$-space $M$ is a right weak Doi-Hopf module over the right weak Doi-Hopf datum $(H, A, C)$ if it is a non-degenerate right $A$-module and a non-degenerate left $C$-comodule i.e. there exists an action $\cdot : M \times A \rightarrow M$ for which $m \cdot_A m = m \forall m \in M$ and a coaction $\rho_M : M \rightarrow C \otimes M$ for which $(\varepsilon_C \otimes \text{id}_M) \circ \rho_M = \rho_M$ such that the compatibility condition

$$\rho_M(m \cdot a) = m_{-1} \cdot a_{-1} \otimes m_{0} \cdot a_{0}$$

(3.1)
holds for $\rho_M(m) \equiv m_{<1>} \otimes m_{<0>}$. 

Similarly, $M$ is a left weak Doi-Hopf module over the left Doi-Hopf datum $(H, A, C)$ if it is a non-degenerate left $A$-module (with $A$-action $\cdot$) and a non-degenerate right $C$-comodule (with $C$-coaction $\rho_M$) such that

$$\rho_M(a \cdot m) = a_{<0>} \cdot m_{<0>} \otimes a_{<1>} \cdot m_{<1>}.$$  \hfill (3.2)

The category $^C \mathcal{M}(H)_A$ has as objects the finite dimensional right weak Doi-Hopf modules $M$ over the right weak Doi-Hopf datum $(H, A, C)$ and arrows $T : M \to M'$ which intertwine both the $A$-actions and the $C$-coactions:

$$T(m \cdot a) = T(m) \cdot a \quad \rho_{M'} \circ T = (\text{id}_C \otimes T) \circ \rho_M$$ \hfill (3.3)

for all $m \in M, a \in A$.

Similarly, $^A \mathcal{M}(H)^C$ is the category of finite dimensional left weak Doi-Hopf modules over the left Doi-Hopf datum $(H, A, C)$.

Let us see what categories $^C \mathcal{M}(H)_A$ are in our earlier examples:

**Examples:**

1 $^C \mathcal{M}(H)_A$ is equivalent to $\mathcal{M}_{A=H}$, the category of right $H$-modules. The equivalence functor $F : ^C \mathcal{M}(H)_A \to \mathcal{M}_A$ is the forgetful functor.

2 $^C \mathcal{M}(H)_A$ is equivalent to $^C H \mathcal{M}$, the category of left $H$-comodules. The equivalence functor $\hat{F} : ^C \mathcal{M}(H)_A \to ^C \mathcal{M}$ is the forgetful functor.

3 $^C \mathcal{M}(H)_A$ is equivalent to $H \mathcal{M}_H$, the category of weak Hopf modules [1, 2] over $H$.

4 $^C \mathcal{M}(H)_A$ is equivalent to $^{YD(K^\text{cop})^{op}}$, the category of (some twisted version of) Yetter-Drinfeld modules over $H$. (For its definition see the Appendix).

**Proposition 3.2** Let $(H, A, C)$ be a finite dimensional right weak Doi-Hopf datum and $(\hat{H}, \hat{C}, \hat{A})$ its dual. Then the categories $^C \mathcal{M}(H)_A$ and $^\hat{C} \mathcal{M}(\hat{H})_{\hat{A}}$ are equivalent.

**Proof:** Let us define the functor $D : ^C \mathcal{M}(H)_A \to ^\hat{C} \mathcal{M}(\hat{H})_{\hat{A}}$

$$D(M) : = \hat{M} \text{ as a } k \text{-space} \quad \hat{c} \cdot \mu : = (\hat{c} \otimes \mu) \circ \rho_M \quad \hat{\rho}_{\hat{M}}(\mu) : = a_i \triangleright \mu \otimes \alpha^i$$

$$D(T) : = T^t$$ \hfill (3.4)

where $M$ is an object and $T$ an arrow in $^C \mathcal{M}(H)_A$. $^t$ means transposition of linear operators, $\hat{c} \in \hat{C}, \mu \in \hat{M}$, $(a \triangleright \mu)(m) = \mu(m \cdot a)$ for $a \in A, \mu \in \hat{M}, m \in M$, $\{a_i\}$ is a basis for $A$ and $\{\alpha^i\}$ is the dual basis for $\hat{A}$. One checks by direct calculation that $D$ defines an equivalence functor. \hfill [3]

**Proposition 3.3** Let $(H, A, C)$ be a non-degenerate right weak Doi-Hopf datum. Then the forgetful functor $F : ^C \mathcal{M}(H)_A \to \mathcal{M}_A$ has a left adjoint and $\hat{F} : ^C \mathcal{M}(H)_A \to ^C \mathcal{M}$ has a right adjoint.
Proof: Our proof is constructive. Define $G : \mathcal{M}_A \to C^\mathcal{M}(H)_A$ by

$$G(M): = C \cdot 1_{A_{-1}} < 1_A > \otimes M \cdot 1_{A_{0}} < 1_A > \quad \text{as a space}$$

$$(c \otimes m) \cdot a = c \cdot a_{-1} > \otimes m \cdot a_{0} < 1_A >$$

$$\rho_{G(M)}: = (\Delta_C \otimes \text{id}_M)|_{G(M)}$$

$$G(T): = (\text{id}_C \otimes T)$$ (3.5)

for $M$ an object and $T$ an arrow in $C^\mathcal{M}(H)_A$, $a \in A, (c \otimes m) \in G(M) \subset C \otimes M$.

The fact that $G$ is a left adjoint of $F$ is justified by the existence of unit and counit natural homomorphisms $\rho : \text{id}_C \mathcal{M}(H)_A \to G \circ F$ and $\delta : F \circ G \to \text{id}_C \mathcal{M}_A$. Define them as

$$\rho_M : M \to G(M) \quad \rho_M(m): = m_{-1} > \otimes m_{0} < 1_A >$$

$$\delta_M : G(M) \to M \quad \delta_M: = (\varepsilon_C \otimes \text{id}_M)|_{G(M)}.$$ (3.6)

It is straightforward to show that $\rho_M \in (M, G(M))_{C^\mathcal{M}(H)_A}$, and $\rho$ is natural. The proof of $\delta_M \in (G(M), M)_{\mathcal{M}_A}$ lies on the following

Lemma 3.4 Let $(H, A, C)$ be a non-degenerate right weak Doi-Hopf datum. Then for any $c \in C$ and $a \in A$

$$(i) \quad \Delta_C(c \cdot a_{-1}>) \otimes 1_{A_{0}} < 1_A > = c_{(1)} \otimes c_{(2)} \cdot 1_{A_{-1}} > \otimes 1_{A_{0}} < 1_A >$$ (3.7)

$$(ii) \quad \Pi^L(a_{-1}>) \otimes a_{0} < 1_A > = \Pi^L(1_{A_{-1}} >) \otimes 1_{A_{0}} < 1_A > a.$$ (3.8)

Lemma 3.4 (ii) implies $\varepsilon_C(c \cdot a_{-1}>) a_{0} < 1_A > = \varepsilon_C(c \cdot 1_{A_{-1}} >) 1_{A_{0}} < 1_A > a$ and hence $\delta_M \in (G(M), M)_{\mathcal{M}_A}$. Naturality of $\delta$ is obvious.

One can proceed the same way in the case of $\hat{F}$ using now Lemma 3.4 (i). Define $\hat{G} : \mathcal{C}^\mathcal{M} \to C^\mathcal{M}(H)_A$ as

$$\hat{G}(M) := \{ \varepsilon_C(m_{-1} > \cdot a_{-1}>) m_{0} < 1_A > \otimes a_{0} < 1_A > | m \in M, a \in A \} \text{ as a space}$$

$$\rho_{\hat{G}(M)}(m \otimes a): = m_{-1} > \cdot a_{-1} > \otimes m_{0} < 1_A > \otimes a_{0} < 1_A >$$

$$\hat{G}(T) := T \otimes \text{id}_A$$ (3.9)

for $M$ an object and $T$ an arrow in $C^\mathcal{M}(H)_A$, $(m \otimes a) \in \hat{G}(M) \subset M \otimes A, b \in A$.

The unit and counit natural homomorphisms $\hat{\rho} : \text{id}_C \mathcal{M} \to \hat{F} \circ \hat{G}$ and $\hat{\delta} : \hat{G} \circ \hat{F} \to \text{id}_C \mathcal{M}(H)_A$ can be given by

$$\hat{\rho}_M : M \to \hat{G}(M) \quad \hat{\rho}_M(m): = \varepsilon_C(m_{-1} > \cdot 1_{A_{-1}} >) m_{0} < 1_A > \otimes 1_{A_{0}} < 1_A >$$

$$\hat{\delta}_M : \hat{G}(M) \to M \quad \hat{\delta}_M(m \otimes a): = m \cdot a$$ (3.10)
4 The Weak Smash Product

Definition 4.1 For the non-degenerate right weak Doi-Hopf datum $(H, A, C)$ define the weak smash product algebra $A \# \hat{C}$ as the $k$-space $1_{A<0>}A \otimes 1_{A<-1> \triangleright} \hat{C}$ equipped with the multiplication rule

$$(a \# \hat{c})(b \# \hat{d}) = (a_{<0>}b \# \hat{c}(a_{<-1>} \triangleright \hat{d})) \quad (4.1)$$

for $(a \# \hat{c}), (b \# \hat{d}) \in A \# \hat{C}$.

One checks that $(4.1)$ makes $A \# \hat{C}$ an associative algebra with unit element $1_{A<0>} \equiv 1_{A<-1> \triangleright} 1_{\hat{C}}$.

Let us see what algebras $A \# \hat{C}$ are in our earlier examples.

Examples:

1. $(A \equiv H) \#(\hat{C} \equiv \hat{H}^R)$ is isomorphic to $H$, the isomorphism being given by $\iota : A \# \hat{C} \to H$, $\iota = \text{id}_H \otimes \varepsilon_H$.

2. $(A \equiv H^L) \#(\hat{C} \equiv \hat{H})$ is isomorphic to $\hat{H}$, the isomorphism being given by $\iota : A \# \hat{C} \to \hat{H}$, $\iota : \varepsilon \otimes \text{id}_H$.

3. $(A \equiv H) \#(\hat{C} \equiv \hat{H})$ is isomorphic to the Weyl algebra or Heisenberg double $\hat{H} \rtimes H$, the isomorphism being given by $\iota : A \# \hat{C} \to \hat{H} \rtimes H$, $\iota(\mathbb{1}_2 a \# (\mathbb{1}_1 \rightarrow \phi)) = \phi a$. (In all of the examples $\rightarrow$ denotes the Sweedler’s arrow $\mathbb{1}_2$.)

4. $(A \equiv K) \#(\hat{C} \equiv \hat{K})$ is isomorphic to the (twisted) Drinfel’d double $D(K_{op})_{op}$ (for its definition see the Appendix). The equivalence is given by $\iota : A \# \hat{C} \to D(K_{op})_{op}$, $\iota(\mathbb{1}_2 a \# (\mathbb{1}_1 \rightarrow \phi \leftarrow S^{-1}(\mathbb{1}_3))) = D(\phi)D(a)$.

Proposition 4.2 Let $(H, A, C)$ be a non-degenerate right weak Doi-Hopf datum such that $C$ is finite dimensional as a $k$-space. Then the categories $^C\mathcal{M}(H)_A$ and $\mathcal{M}_{A \# \hat{C}}$ are isomorphic.

Proof: We have the functor $P : ^C\mathcal{M}(H)_A \to \mathcal{M}_{A \# \hat{C}}$

$$P(M) = M \text{ as a } k \text{-space} \quad m \cdot (a \# \hat{c}) = \hat{c}(m_{<-1>})m_{<0>} \cdot a$$

$$P(T) = T \quad (4.2)$$

for $M$ an object and $T$ an arrow in $^C\mathcal{M}(H)_A$, $(a \# \hat{c}) \in A \# \hat{C}, m \in M$.

If $C$ is finite dimensional as a $k$-space then let $\{c_i\}$ be any basis for $C$ and $\{\gamma_i\}$ the dual basis for $\hat{C}$ and construct the inverse functor $P' : \mathcal{M}_{A \# \hat{C}} \to ^C\mathcal{M}(H)_A$ of $P$:

$$P'(M) = M \text{ as a } k \text{-space} \quad m \cdot a = m \cdot (1_{A<0>} \equiv a \# 1_{A<-1> \triangleright} 1_{\hat{C}})$$

$$\rho_M(m) = c_i \otimes m \cdot (1_{A<0>} \# 1_{A<-1> \triangleright} \gamma_i)$$

$$P'(T) = T \quad (4.3)$$

for $M$ an object and $T$ an arrow of $^C\mathcal{M}(H)_A$, $a \in A, m \in M$.
5 Integrals for Weak Doi-Hopf Data

Let \((H, A, C)\) be a non-degenerate right weak Doi-Hopf datum where \(H\) is a weak Hopf algebra with antipode \(S, F : \mathcal{C} \mathcal{M}(H)_A \to \mathcal{M}_A\) the forgetful functor, \(G\) its left adjoint as in Proposition 3.3. \(V\) be the \(k\)-space of the natural homomorphisms \(\nu : G \circ F \to \text{id}_{\mathcal{C} \mathcal{M}(H)_A}\), called the space of integrals for the weak Doi-Hopf datum \((H, A, C)\). We have a straightforward generalization of Theorem 2.3 of \([6]\):

**Theorem 5.1** The space \(V\) is isomorphic to the space \(V_4\):

\[
V_4 = \{ \gamma : C \to (C, A)_{\text{Lin}} \mid \forall c, d \in C \quad a \in A \quad \gamma(c)(d)a = a_{\leq 0} \gamma(c \cdot a_{\leq 1})(d \cdot a_{\leq -1}) \\
c_{(1)} \otimes \gamma(c_{(2)})(d) = d_{(2)} \cdot \gamma(c)(d_{(1)})_{\leq -1} \otimes \gamma(c)(d_{(1)})_{< 0}\}. 
\]

(5.1)

Furthermore the isomorphism \(f_4 : V \to V_4\) takes \(\nu \in V\) to a normalized element of \(V_4\) i.e. to an element \(\gamma \in V_4\) such that \(\gamma(c_{(1)})(c_{(2)}) = \varepsilon(c \cdot 1_{A_{\leq 1}})1_{A_{< 0}}\) if and only if \(\nu\) is a splitting of the unit natural homomorphism \(\rho : \text{id}_{\mathcal{C} \mathcal{M}(H)_A} \to G \circ F\).

The relevance of the existence of normalized elements in \(V_4\) is discussed in \([6]\).

Let us turn to the investigation of the space of integrals over the weak Doi-Hopf datum \((H, A, C)\) in our earlier examples. In doing so we make the additional assumption in the Examples 2 and 3 on \(H\) and in 4 on \(K\) to be a Frobenius WHA. Under this additional condition we identify the space of integrals for the weak Doi-Hopf datum \((H, A, C)\) with certain subspace of the smash product algebra \(A \# \hat{C}\).

Also the normalization condition is formulated as a relation in the algebra \(A \# \hat{C}\).

In all of the examples \(r\) be a non-degenerate right integral in \(H\) and \(\rho\) the dual right integral \((\hat{H})^\varepsilon\) in \(\hat{H}\).

**Examples:**

1 The space of Doi-Hopf integrals over \((H, A, C)\) is isomorphic to \(V_0 = \text{Center}H\).

Construct the isomorphism \(f : V_4 \to V_0\) as

\[
f(\gamma) = \gamma(\mathbb{I})(\mathbb{I}).
\]

(5.2)

The unique normalized element of \(V_0\) is the unit element \(\mathbb{I}\) of \(H\).

2 The space of the Doi-Hopf integrals is isomorphic to \(V_0 : = \langle \hat{H}^R \rangle \cap \hat{H}\), the commutant of the right subalgebra in \(\hat{H}\). Let us construct the isomorphism \(f : V_4 \to V_0\) as

\[
[f(\gamma)](h) = \varepsilon(\gamma(r)(h))
\]

(5.3)

for all \(h \in H\).

An element \(\xi \in V_0\) is normalized if

\[
\hat{S}^{-1}(\rho_{(2)})\xi\rho_{(1)} = \mathbb{I}
\]

holds in \(\hat{H}\).

The space \(V_0\) is not isomorphic to the space \(\mathcal{I}^L(\hat{H})\) of left integrals in \(\hat{H}\). It is its subspace \(\hat{H}^L\) which is isomorphic to \(\mathcal{I}^L(\hat{H})\) via the isomorphism \(g : \mathcal{I}^L(\hat{H}) \to \hat{H}^L\),
\( g(\lambda): = \hat{S}(\lambda \leftarrow r) \). It is but true that the existence of normalized elements in \( \mathcal{L}(\check{H}) \) and \( V_0 \) are equivalent.

3 The space of the Doi-Hopf integrals is isomorphic to \( V_0: = H' \cap (\check{H} \bowtie H) \), the commutant of \( H \) in the Weyl algebra. The isomorphism \( f : V_4 \to V_0 \) is given by

\[
 f(\gamma) : = \beta^i \gamma(r)(b_i)
\]

with the help of the basis \( \{ b_i \} \) of \( H \) and the dual basis \( \{ \beta^i \} \) of \( \check{H} \).

The element \( w \in V_0 \) is normalized if

\[
 \hat{S}^{-1}(\rho(2))w\rho(1) = 1_{\check{H} \bowtie H}
\]

holds in the Weyl algebra \( \check{H} \bowtie H \).

4 The space of the Doi-Hopf integrals is isomorphic to \( V_0: = \{ u \in \mathcal{D}(K^\text{op})^\text{op} | uD(b) = D(b(1))uD(S^{-1}(r)S^{-2}(b(2)) \to \rho) \} \). The isomorphism \( f : V_4 \to V_0 \) is given by

\[
 f(\gamma) : = D(\beta^i)D(\gamma(r)(b_i))
\]

with the help of the basis \( \{ b_i \} \) of \( H \) and the dual basis \( \{ \beta^i \} \) of \( \check{H} \).

\( u \in V_0 \) is normalized if

\[
 \hat{S}^{-1}(\rho(2))u\rho(1) = 1_{\mathcal{D}(K^\text{op})^\text{op}}
\]

holds in the double \( \mathcal{D}(K^\text{op})^\text{op} \).

6 Appendix: Yetter-Drinfel’d modules over WHA’s and Drinfel’d doubles

For the convenience of the reader we give here the generalization of the double construction due to Drinfel’d [9] and of the corresponding theory of Yetter-Drinfel’d modules [16, 14] to WHA’s.

**Definition 6.1** [3] Let \( H \) be a finite dimensional WHA over the field \( k \). Its Drinfel’d double \( \mathcal{D}(H) \) is the WHA defined below:

As a \( k \)-space \( \mathcal{D}(H) \) is an amalgamated tensor product \( H_L \equiv H_R \otimes H \equiv \check{H} \) with the amalgamation relations \( a^R \otimes \check{\mathbb{1}} \equiv \mathbb{1} \otimes (a^R \leftarrow \check{a}^R) \); \( (a^L \otimes \check{\mathbb{1}} \equiv \mathbb{1} \otimes (a^L \to \check{a}^L) \) for \( a^L \in H^L, a^R \in H^R \). Denote by \( \mathcal{D}(a)\mathcal{D}(\phi) \) the image of \( H \otimes \check{H} \ni a \otimes \phi \) under the amalgamation and \( \mathcal{D}(a) \equiv \mathcal{D}(a)\mathcal{D}(\mathbb{1}), \mathcal{D}(\phi) \equiv \mathcal{D}(\mathbb{1})\mathcal{D}(\phi) \).

The algebra structure is defined by

\[
 \begin{align*}
 \mathcal{D}(a)\mathcal{D}(b) & = \mathcal{D}(ab) \\
 \mathcal{D}(\phi)\mathcal{D}(\psi) & = \mathcal{D}(\phi\psi) \\
 \mathcal{D}(\phi)\mathcal{D}(a) & = \mathcal{D}(a(2))\mathcal{D}(\phi(2))(\phi(1)|a(3))\langle \phi(3)|S^{-1}(a(1))\rangle.
\end{align*}
\]

One checks that (6.1) is compatible with the amalgamation relations and makes \( \mathcal{D}(H) \) an associative algebra with unit \( \mathcal{D}(\mathbb{1}) \equiv \mathcal{D}(\check{\mathbb{1}}) \).
The colagebra structure is given by
\[
\Delta_D(D(a)D(\phi)) = D(a(1))D(\phi(2)) \otimes D(a(2))D(\phi(1))
\]
\[
\varepsilon_D(D(a)D(\phi)) = \varepsilon(a(\phi \twoheadrightarrow \1)) \equiv \hat{\varepsilon}(\hat{1} \leftarrow a)\phi.
\] (6.2)

One checks that (6.2) makes $\mathcal{D}(H)$ a WBA. Finally the antipode is
\[
S_D(D(a)D(\phi)) = D(S^\ast(\phi))D(S(a))
\] (6.3)
making $\mathcal{D}(H)$ a WHA.

**Definition 6.2** Let $H$ be a WBA over the field $k$. The $k$-space $M$ is a right Yetter-Drinfel’d module over $H$ if it is a non-degenerate right $H$-module and a non-degenerate left $H$-comodule s.t.
\[
m_{< -1 >}a_{(1)} \otimes m_{< 0 >} \cdot a_{(2)} = a_{(2)}(m \cdot a_{(1)})_{< -1 >} \otimes (m \cdot a_{(1)})_{< 0 >}
\]
\[
m_{< -1 >}1_{(1)} \otimes m_{< 0 >} \cdot 1_{(2)} = m_{< -1 >} \otimes m_{< 0 >}
\] (6.4)
for all $m \in M, a \in A$.

Notice that if $H$ is also a WHA then (6.4) can be replaced by the single relation
\[
(m \cdot a)_{< -1 >} \otimes (m \cdot a)_{< 0 >} = S^{-1}(a_{(3)})m_{< -1 >}a_{(1)} \otimes m_{< 0 >} \cdot a_{(2)}.
\] (6.5)

By the category $\mathcal{YD}(H)$ we mean the category with objects the finite dimensional right Yetter-Drinfel’d modules over $H$ and arrows $T : M \rightarrow M'$ intertwining both the $H$-module and the $H$-comodule structures of $M$ and $M'$.

If $H$ is a finite dimensional WHA then by our Proposition 4.2 and Example 4. the category $\mathcal{YD}(H)$ is equivalent to the category of the right modules over the WHA $\mathcal{D}(H)$ hence carries (among others) a monoidal structure $\boxtimes$. It is not so obvious however that it is true for any WBA $H$:

**Proposition 6.3** Let $H$ be a WBA over the field $k$. Then the category $\mathcal{YD}(H)$ has a monoidal structure.

**Proof:** Our proof is constructive. For two objects $M, N$ and arrows $T, S$ of $\mathcal{YD}(H)$ let
\[
M \times N := M \cdot 1_{(1)} \otimes N \cdot 1_{(2)} \text{ as a } k \text{-space}
\]
\[
(m \otimes n) \cdot a := m \cdot a_{(1)} \otimes n \cdot a_{(2)}
\]
\[
\rho_{M \times N}(m \otimes n) := n_{< -1 >}m_{< -1 >}1_{(1)} \otimes m_{< 0 >} \cdot 1_{(2)} \otimes n_{< 0 >} \cdot 1_{(3)}
\]
\[
T \times S := (T \otimes S) \circ \Delta(1)
\] (6.6)
with $m \otimes n \in M \times N, a \in A$. The monoidal unit is
\[
H_L \text{ as a } k \text{-space}
\]
\[
a^L \cdot h := 1_{(2)}\varepsilon(a^L \cdot h \cdot 1_{(1)})
\]
\[
\rho_{H^L} := \Delta|_{H^L}
\] (6.7)
for \(a^L \in H^L, h \in H\).

The reader may check using some WBA calculus that all \(M \times N\) and \(H^L\) are Yetter-Drinfel'd modules over \(H\) if \(M\) and \(N\) are.

In order to prove that \(H^L\) is a monoidal unit for the category \(\mathcal{YD}(H)\) one has to construct the invertible intertwiners \(u^L_M \in (M, H^L \times M)_{\mathcal{YD}(H)}\), \(u^R_M \in (M, M \times H^L)_{\mathcal{YD}(H)}\) satisfying the triangle identities \([1]\) and being natural in \(M\). They are as follows:

\[
\begin{align*}
    u^L_M(m) &= 1_{(2)} \otimes m \cdot \Pi^L(1_{(1)}) \\
    u^R_M(m) &= m \cdot 1_{(1)} \otimes 1_{(2)}, \hspace{1cm} (6.8)
\end{align*}
\]

for all \(m \in M\) and all objects \(M\) of \(\mathcal{YD}(H)\).

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