Problem of Estimation of Fractional Derivative

for a Spectral Function of Gaussian Stationary Processes

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\section*{Abstract.}

We study the problem of nonparametric estimation of the fractional derivative of unknown spectral function of Gaussian stationary sequence (time series) and show that these problems is well posed with the classical speed of convergence when the order of derivative is less than 0.5.

We prove also the asymptotical unbiasedness and normality of offered estimates with optimal speed of convergence.

For the construction of the confidence region in some functional norm we establish the Central Limit Theorem in correspondent space of continuous functions for offered estimates.

\textit{Key words and phrases:} Fractional derivatives and integrals of a Riemann-Liouville type, weak convergence of distributions in Banach spaces, spectral function and spectral density, sample, estimate, confidence region, periodogram, multiple stochastic integral, majorizing measure method, asymptotical normality, bias, stationary Gaussian random process (time series), Ibragimov’s theorem, Fejer kernel and approximation, Central Limit Theorem in Banach space, Lebesgue-Riesz spaces, random variable, vector (r.v.), and random process (r.p.).

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\section{Notations. Statement of problem.}

"Fractional derivatives have been around for centuries but recently they have found new applications in physics, hydrology and finance", see [28].

Another applications: in the theory of Differential Equations are described in [29]; in statistics see in [1], [4], see also [14], [12], [32]; in the theory of integral equations etc. see in the classical monograph [45].
We consider here the problem of the nonparametric estimation of the fractional derivative for a spectral function of Gaussian stationary sequence.

We will prove that if the order of the fractional derivative $\alpha$ is positive and is strictly less than $1/2$, then this problem is well posed. In particular, the speed of convergence of offered (asymptotical) unbiased estimate is $1/\sqrt{n}$, as in the case of estimation of ordinary spectral function $F(\lambda)$; they are asymptotical normal still in uniform norm.

Our results improve ones in the books and articles [2], chapters 7-9; [4], [5], [16], chapter 5; [17], [31], chapter 5, section 5.13; [32] etc.

In particular, it is proved in the preprint [32], see also [4], that the problem of estimation of fractional derivative for function of distribution $F^{(\alpha)}(x)$, under condition $\alpha < 1/2$ is well posed with the optimal rate of convergence $1/\sqrt{n}$, when $n$ is the volume of the sample, and is announced an analogues fact about the fractional derivative of spectral function.

Let $\alpha = \text{const} \in (0, 1)$; and let also $g = g(x)$, $x \in R$ be certain measurable numerical function. We recall for reader convenience that the fractional derivative of a Riemann-Liouville type of order $\alpha$, $\alpha \in (0, 1)$:

\[
D_\alpha^x g(x) = \frac{d}{dx} \int_0^x \frac{g(t) \, dt}{(x-t)^\alpha}, \quad (1.1)
\]

see, e.g. the classical monograph of S.G.Samko, A.A.Kilbas and O.I.Marichev [45], pp. 33-38; see also [29].

Hereafter $\Gamma(\cdot)$ denotes the ordinary $\Gamma$ function.

We agree to take $D_\alpha^x g(x_0) = 0$, if at the point $x_0$ the expression $D_\alpha^x g(x_0)$ does not exists.

Notice that the operator of the fractional derivative is non-local, if $\alpha$ is not integer non-negative number.

Note also that for the considered further functions this fractional derivative there exists almost everywhere.

Recall also that the fractional integral $I_\alpha^x [\phi](x) = I_\alpha^x [\phi](x)$ of a Riemann-Liouville type of an order $\alpha$, $0 < \alpha < 1$ is defined as follows:

\[
I_\alpha^x [\phi](x) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^x \frac{\phi(t) \, dt}{(x-t)^{1-\alpha}}, \quad x, \, t > 0. \quad (1.2)
\]

It is known (theorem of Abel, see [45], chapter 2, section 2.1) that the operator $I_\alpha^x [\cdot]$ is inverse to the fractional derivative operator $D_\alpha^x [\cdot]$, at least in the class of absolutely continuous functions.

Evidently, (Kolmogorov’s theorem), the problem of distribution function estimation ($\alpha = 0$) is well posed in the uniform norm. V.D.Konakov in [19] proved in contradiction that the problem of the spectral density estimation, i.e. when $\alpha = 1$, is ill posed.

Roughly speaking, the result of V.D.Konakov may be reformulated as follows. Certain problem of statistical estimation is well posed if there exists an continuous
in natural distance estimate (more exactly, a sequence of such continuous estimates) such that the speed of convergence is equal (or less than) \(1/\sqrt{n}\). As a rule these estimations are asymptotically normal.

2 Main result: estimation of fractional derivatives of spectral function.

Let us consider in this section the classical problem of estimation of fractional derivatives for spectral function, or equally the problem of fractional integral for spectral density estimation.

Let \(\eta_k, k = 1, 2, \ldots, n\) be real valued, centered: \(E\eta(k) = 0\) Gaussian distributed stationary random sequence (time series) with (unknown) even covariation function \(r = r(m)\), spectral function \(F(\lambda), \lambda \in [0, 2\pi]\), \(F(0+) = F(0) = 0\), and with spectral density \(f(\lambda)\), (if there exists):

\[
\int_{[-\pi,\pi]} \cos(\lambda m) dF(\lambda) = \int_{[-\pi,\pi]} \cos(\lambda m) f(\lambda) d\lambda, \tag{2.1}
\]

so that

\[
F(\lambda) = \int_0^\lambda f(t) dt = I^{(1)}[f](\lambda); \quad F(0) = F(0+) = 0.
\]

We can and will suppose without loss of generality that the spectral density, i.e. the function \(f = f(\lambda)\) to be continued on the whole axis \(R^1\) as a \(2\pi\) periodical (continuous) function \(f(\lambda) = f(\lambda + 2\pi)\).

The periodogram of this sequence will be denoted by \(J_n(\lambda), 0 \leq \lambda \leq 2\pi\) :

\[
J_n(\lambda) := (2\pi n)^{-1} \left| \sum_{k=1}^{n} e^{ik\lambda} \eta(k) \right|^2; \quad i^2 = -1. \tag{2.2}
\]

We intend here to estimate the fractional derivative \(F^{(\alpha)}(\lambda)\) of the spectral function \(F(\lambda)\).

Recall that the problem of \(F(\cdot)\) estimation is well posed, theorem of I.A.Ibragimov, [17], in contradiction to the problem of spectral density \(f(\cdot) = F^{(1)}(\cdot)\) estimation.

We assume as before \(0 < \alpha < 1/2\), and denote \(\beta = 1 - \alpha; \quad \beta \in (1/2, 1)\).

Heuristic arguments. We have using the group properties of the fractional derivative-integral operators

\[
F^{(\alpha)} = D^\alpha[F] = D^\alpha I^1[f] = D^\alpha D^{-1}[f] = D^{\alpha-1}[f] = I^{1-\alpha}[f] = I^\beta[f] \approx I^\beta[J_n]. \tag{2.3}
\]
Thus, we can offer as an estimation of \( F^{(\alpha)} \) the following statistics

\[
F_{\alpha,n}(\lambda) := \frac{1}{\Gamma(\beta)} \int_0^\lambda \frac{J_n(t)}{(t-\lambda)^{1-\beta}} \, dt = \frac{1}{\Gamma(1-\alpha)} \int_0^\lambda \frac{J_n(t)}{(t-\lambda)^{\alpha}} \, dt.
\]  

(2.4)

Let us introduce a sequence as \( n = 1, 2, \ldots \) of the normed and mean zero random processes

\[
\zeta_n(\lambda) := \sqrt{n} \left( F_{\alpha,n}(\lambda) - \mathbf{E} F_{\alpha,n}(\lambda) \right)
\]

(2.5)

and also a Gaussian centered separable random process \( \zeta(\lambda) = \zeta_\infty(\lambda) \) with covariation function

\[
\Theta^{(f)}_\alpha(\lambda, \mu) = \Theta_\alpha(\lambda, \mu) \overset{\text{def}}{=} \text{Cov}(\zeta_\infty(\lambda), \zeta_\infty(\mu)) = \mathbf{E} \zeta(\lambda) \cdot \zeta(\mu) := \frac{4\pi}{\Gamma^2(1-\alpha)} \int_0^{\lambda \wedge \mu} \frac{f^2(\nu) \, d\nu}{(\lambda - \nu)^\alpha (\mu - \nu)^\alpha},
\]

(2.6)

so that

\[
\sigma^2_\alpha(\lambda) \overset{\text{def}}{=} \text{Var} \zeta_\infty(\lambda) = \Theta_\alpha(\lambda, \lambda) = \frac{4\pi}{\Gamma^2(1-\alpha)} \int_0^\lambda \frac{f^2(\nu) \, d\nu}{(\lambda - \nu)^{2\alpha}} = \frac{4\pi \Gamma(1-2\alpha)}{\Gamma^2(1-\alpha)} \cdot I^2[f^2](\lambda) < \infty,
\]

(2.7)

as long as the function \( f = f(\lambda) \) is presumed to be continuous and \( \alpha < 1/2 \).

**Theorem 2.1.** Suppose as before \( 0 < \alpha < 1/2 \) and that the spectral density \( f(\lambda) \) there exists and is continuous and strictly positive on the (closed) circle \([0, 2\pi]\):

\[
\exists C_1, C_2, \ 0 < C_1 < C_2 < \infty \ \Rightarrow \ C_1 \leq f(\lambda) \leq C_2,
\]

(2.8)

in particular

\[
f(0) = f(0+) = f(2\pi - 0) = f(2\pi).
\]

Our statement: the sequence of the distributions of continuous random processes \( \zeta_n(\cdot) \) converges weakly as \( n \to \infty \) in the space of continuous periodical functions \( C^*(0, 2\pi) \), i.e. in the Prokhorov-Skorokhod sense, to the distribution of the continuous random process \( \zeta_\infty(\cdot) \) : for arbitrary continuous bounded functional \( G : C^*(0, 2\pi) \to \mathbb{R} \)

\[
\lim_{n \to \infty} \mathbf{E}G(\zeta_n(\cdot)) = \mathbf{E}G(\zeta_\infty(\cdot)).
\]

(2.9)

**Proof.**

1. Note first of all that all the considered here r.p. \( \zeta_n(\lambda), \ z_\infty(\lambda) \) are continuous with probability one. Indeed, the covariation function \( \Theta_\alpha(\lambda, \mu) \) satisfies the Hölder’s condition with exponent \( \alpha - \epsilon \) for arbitrary value \( \epsilon \in (0, \alpha/2) \). For instance, the
function \( \lambda \to \sigma^2(\lambda) = \sigma^2_{\alpha}(\lambda) \) in (2.7) being represented as a fractional integral from the continuous function \( f^2(\cdot) \), obeys the module continuity of a form

\[
\omega(\sigma^2, h) \leq C(\alpha) \ h^{\alpha-1/p} \ |f^2|_p, \ p > 2/\alpha, \tag{2.10}
\]

\[
|g(\cdot)|_p := \left[ \int_0^{2\pi} |g(\lambda)|^p \ d\lambda \right]^{1/p},
\]

see [45], chapter 1, section 3.3., pp. 66-71.

The general case \( \lambda \neq \mu \) may be establish analogously using at the same statement in the book [45].

Hereafter as ordinary \( \omega(g(\cdot), h), \ h \in [0, 2\pi] \) denotes the module of continuity of (possible continuous and periodical) function \( g = g(\lambda) : \)

\[
\omega(g(\cdot), h) \overset{\text{def}}{=} \sup_{|\lambda - \mu| \leq h} |g(\lambda) - g(\mu)|. \tag{2.11}
\]

2. I.A.Ibragimov in proved in [17] in particular the Central Limit Theorem for the sequence of r.p.

\[
\tau_n(\lambda) := \sqrt{n} \ [F_n(\lambda) - EF_n(\lambda)]
\]

in the space \( C^*(0, 2\pi) \), where

\[
F_n(\lambda) = \int_0^\lambda J_n(t) \ dt
\]

is ordinary empirical spectral function.

To be more precise, we introduce following I.A.Ibragimov also a centered separable Gaussian random process \( \tau(\lambda), \ \lambda \in [0, 2\pi] \) with covariation function

\[
\text{Cov}(\tau(\lambda), \tau(\mu)) = 4\pi \int_0^{\min(\lambda, \mu)} f^2(x) \ dx. \tag{2.13}
\]

Define a non-negative function \( \beta = \beta(\lambda), \ \lambda \in [0, 2\pi] \)

\[
\beta^2(\lambda) := 4\pi \int_0^\lambda f^2(x) \ dx, \tag{2.14}
\]

and let us introduce also the following continuous distance function

\[
d_\beta(\lambda, \mu) := |\beta(\lambda) - \beta(\mu)|. \tag{2.15}
\]

Then the r.p. \( \tau(\lambda) \) may be represented in the sense of distributional coincidence as follows:

\[
\tau(\lambda) \overset{d}{=} B(\beta(\lambda)),
\]

where \( B(t) \) is ordinary Brownian motion (Wiener’s process).

Obviously, the r.p. \( \tau(\cdot) \) is continuous a.e.

I.A.Ibragimov proved also that as \( n \to \infty \) in the space \( C^*[0, 2\pi] \)
\[ \text{Law}(\tau_n(\cdot)) = \text{Law}(\sqrt{n} [F_n(\cdot) - \mathbf{E}F_n(\cdot)]) \rightarrow \text{Law}(\tau(\cdot)). \quad (2.16) \]

See also [7], [10], [23].

Therefore, the finite-dimensional distributions of r.p. \( \zeta_n(\cdot) \) converges to ones for the r.p. \( \zeta(\cdot) \), which are of course also Gaussian.

Moreover,

\[ \sup_n \sup_{\lambda \in [0, 2\pi]} \mathbf{E}\tau_n^2(\lambda) \leq C_1 < \infty, \quad (2.17a) \]

\[ \sup_n \mathbf{E}[\tau_n(\lambda) - \tau_n(\mu)]^2 \leq C_2 \, d_\beta(\lambda, \mu), \quad (2.17b) \]

see [17].

3. Let us calculate the covariation function of the r.p. \( \tau = \tau(\lambda) \).

More detail, we propose

\[ \lim_{n \to \infty} n \cdot \text{Cov} \{F_{n, \lambda}(\cdot), F_{n, \mu}(\cdot)\} = \frac{4\pi \Gamma(1 - 2\alpha)}{\Gamma^2(1 - \alpha)} \cdot D^{\alpha}_{\lambda} \left[D^{\alpha}_{\mu} [f^2]\right] = \]

\[ \frac{4\pi}{\Gamma^2(1 - \alpha)} \cdot \int_0^{\lambda \wedge \mu} \frac{f^2(\nu) \, d\nu}{(\lambda - \nu)^\alpha (\mu - \nu)^\alpha} =: \Theta_{\alpha}(\lambda, \mu). \quad (2.18) \]

Note that the last integral is finite since the function \( f \) is bounded and \( \alpha < 1/2 \).

This assertion (2.18) follows immediately from the following proposition, see the fundamental monograph of T.W. Anderson [2], chapter 5, page 564-572, theorem 9.3.1: if \( w(\lambda, \nu) \) is non-negative integrable relative the second variable function, then

\[ \lim_{n \to \infty} \mathbf{E} \int_0^{2\pi} w(\lambda, \nu) J_n(\nu) \, d\nu = \lim_{n \to \infty} \int_0^{2\pi} w(\lambda, \nu) \mathbf{E} J_n(\nu) \, d\nu = \]

\[ \int_0^{2\pi} w(\lambda, \nu) f(\nu) \, d\nu, \]

\[ \lim_{n \to \infty} n \cdot \text{Cov} \left( \int_0^{2\pi} w(\lambda_1, \nu) J_n(\nu) \, d\nu, \int_0^{2\pi} w(\lambda_2, \nu) J_n(\nu) \, d\nu \right) = \]

\[ 4\pi \int_0^{2\pi} w(\lambda_1, \nu) \, w(\lambda_2, \nu) \, f^2(\lambda) \, d\lambda, \]

with remainder terms. We choose \( w(\lambda, \nu) = |\lambda - \nu|^{-\alpha} \); it is easy to verify that all the conditions of the mentioned result are satisfied.

Recall also that the considered stationary sequence \( \{\eta(k)\} \) is Gaussian, i.e. without cumulant function.

4. It remains only to establish the weak compactness of the distributions \( \zeta_n(\cdot) \) in the space of continuous functions \( C^* [0, 2\pi] \). Note first of all that

\[ C_3 |\lambda - \mu| \leq d_\beta(\lambda, \mu) \leq C_4 |\lambda - \mu|. \quad (2.19) \]
Further, the r.p. \( \tau_n(\lambda) \) can be represented as a two-dimensional mean zero stochastic integral over a Gaussian stochastic measure, or equally in the terminology of the article [21] "square Gaussian random vectors (variables)" (SGV) or more generally "square Gaussian random process" (SGP).

5. We will use some facts about these random vectors and processes, see [6], [21]. Introduce following V.V.Buldygin and Yu.V.Kozachenko the next function

\[
\phi(\lambda) := \ln \left( (1 - |\lambda|)^{-1/2} e^{-|\lambda|/2} \right), \quad |\lambda| < 1, \tag{2.20}
\]

and \( \phi(\lambda) = +\infty \) otherwise. V.V.Buldygin and Yu.V.Kozachenko in [6] have proved that

\[
\sup_{\eta \in \text{SGV}} E \exp \left( \lambda \eta / \sqrt{2 \text{Var} \eta} \right) \leq \exp \phi(\lambda), \tag{2.21}
\]

where in (2.21) the supremum is calculated over all the non-trivial random variable \( \eta \) from the set SGV, which may be defined on arbitrary probability space.

Since for the r.v. of the form \( \eta = \pm (\xi^2 - 1) \), where the r.v. \( \xi \) has a standard normal distribution, in the inequality (2.21) take place the equality, we conclude

\[
\sup_{\eta \in \text{SGV}} E \exp \left( \lambda \eta / \sqrt{2 \text{Var} \eta} \right) = \exp \phi(\lambda). \tag{2.21a}
\]

6. The relations (2.21) (and (2.21a)) may be transformed. The function \( \phi = \phi(\lambda) \) generated so-called Banach space \( B(\phi) \) as follows.

We say by definition that the centered random variable (r.v) \( \eta \) defined on some sufficiently rich probability space belongs to the space \( B(\phi) \), if there exists some non-negative constant \( \gamma \geq 0 \) such that

\[
\forall \lambda \in R \Rightarrow E \exp(\lambda \eta) \leq \exp(\phi(\gamma \lambda)). \tag{2.22}
\]

The minimal value \( \gamma \) satisfying (2.22) is called a \( B(\phi) \) norm of the variable \( \eta \), write

\[
\| \xi \|_{B(\phi)} = \inf \{ \gamma, \gamma > 0 : \forall \lambda \Rightarrow E \exp(\lambda \xi) \leq \exp(\phi(\gamma \lambda)) \}. \tag{2.23}
\]

The space \( B(\phi) \) with respect to the norm \( \| \cdot \|_{B(\phi)} \) and ordinary algebraic operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables from the so-called exponential Orliczs space \( (\Omega, F, P), N(\cdot) \) with \( N\) -- function

\[
N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) := \sup_{\lambda} (\lambda u - \phi(\lambda)).
\]

The detail investigation of alike spaces see in [20], [31], [24].

7. The Ibragimov’s results (2.17a) and (2.17b) may be rewritten as follows

\[
\sup_n \sup_{\lambda \in [0,2\pi]} \| \tau_n(\lambda) \|_{B(\phi)} \leq C_5 < \infty, \tag{2.24a}
\]

\[
\sup_n \| \tau_n(\lambda) - \tau_n(\mu) \|_{B(\phi)} \leq C_6 d^{1/2}_3(\lambda, \mu), \tag{2.24b}
\]
as long as \(\tau_n(\lambda)\) is a two-dimensional stochastic integral.

Recall, see [20], [31] that the norm \(|| \cdot ||_{B\phi}\) is equivalent on the (closed) subspace of mean zero random variables with the following norm

\[
|||\eta|||_{G\psi} \overset{\text{def}}{=} \sup_{p \geq 2} \left[ ||\eta||_{B\phi}^{p} \right]^{\frac{1}{p}},
\]

where \(\psi(p) = p\). This means that on the set of centered variables from the space \(B(\phi) S^0 := \{\eta : E\eta = 0\} \cap B(\phi)\)

\[
0 < \inf_{\eta \in S^0} \left[ \frac{||\eta||_{B\phi}^{p}}{|||\eta|||_{G\psi}} \right] \leq \sup_{\eta \in S^0} \left[ \frac{||\eta||_{B\phi}^{p}}{|||\eta|||_{G\psi}} \right] < \infty.
\]

Therefore

\[
\sup_n \sup_{\lambda \in [0,2\pi]} ||\tau_n(\lambda)||_{G\psi} \leq C_7 < \infty,
\]

\[
\sup_n ||\tau_n(\lambda) - \tau_n(\mu)||_{G\psi} \leq C_8 d_{\beta}^{1/2}(\lambda,\mu)
\]

for some positive finite constants \(C_7, C_8\).

8. Let now \(p\) be fixed number greatest than \(1/\alpha\); for example

\[
p := p_0 \overset{\text{def}}{=} \frac{8}{1 - 2\alpha}.
\]

We deduce on the basis of inequalities (2.27a) and (2.27b)

\[
\sup_n \sup_{\lambda \in [0,2\pi]} \left| \tau_n(\lambda) \right|_{B\phi} \leq C_9 p < \infty,
\]

\[
\sup_n \left| \tau_n(\lambda) - \tau_n(\mu) \right|_{B\phi} \leq C_{10} p |\lambda - \mu|^{1/2}.
\]

We intend to apply the so-called majorizing measures method, see e.g. [34], choosing as a capacity of the majorizing measure the ordinary Lebesgue measure. The direct application of the proposition 2.1 from [34] gives us in the considered case the estimation

\[
|\tau_n(\lambda) - \tau_n(\mu)| \leq C(p) X(n,p) |\lambda - \mu|^{1/2 - 2/p},
\]

or equally

\[
\omega(\tau_n(\cdot),h) \leq C(p) X(n,p) h^{1/2 - 2/p}, \quad 0 \leq h \leq 2\pi,
\]

where the sequence as \(n = 1,2,\ldots\) of non-negative r.v. \(X_{n,p}\) is such that

\[
\sup_n X_{n,p}^p = 1.
\]

Note that \(p_0 = 8/(1 - 2\alpha)\) and a fortiori \(1/2 - 2/p_0 > 0\).

9. We know that the r.p. \(\zeta_n(\cdot)\) is the fractional derivative from \(\tau_n(\cdot)\):
\[
\zeta_n(\lambda) = D^\alpha \tau_n(\lambda)
\]
and in addition \(\tau_n(0) = 0\). We apply the inequality for such a function
\[
\omega(D^\alpha f, h) \leq C(\alpha) \cdot \int_0^h \frac{\omega(f, t) \, dt}{t^{1+\alpha}},
\]
see [45], p. 250-253, theorem 3.16; and get
\[
\omega(\zeta_n, h) \leq C \cdot \int_0^h \frac{t^{1/2-2/p} \, dt}{t^{1+\alpha}} \leq C(\alpha) \cdot X_{n,p} h^{(1-2\alpha)/8}.
\]
Since \(\alpha \in (0,1/2)\), we conclude taking into account (2.31) that the sequence of r.p. \(\zeta_n(\cdot)\) satisfies the famous Prokhorov’s criterion [39] for weak compactness of the (Borelian) probability measures in the space of continuous functions.

This completes the proof of theorem 2.1.

**Theorem 2.2.** Let all the conditions of theorem 2.1. be satisfied. Suppose in addition
\[
\lim_{n \to \infty} \sqrt{n} \omega(f, 1/n) \, |\ln \omega(f, 1/n)| = 0.
\]

We propose that the sequence of the distributions of continuous random processes
\[
\theta_n(\lambda) \overset{\text{def}}{=} \sqrt{n} \left( F_{n,\alpha}(\lambda) - F(\lambda) \right)
\]
converges weakly as \(n \to \infty\) in the space of continuous periodical functions \(C^*(0,2\pi)\), i.e. in the Prokhorov-Skorokhod sense, to the distribution of at the same continuous random process \(\zeta_\infty(\cdot)\).

**Proof.** It is sufficient to justify that
\[
\lim_{n \to \infty} \sup_{\lambda \in [0,2\pi]} |E J_n(\lambda) - F(\lambda)| = 0.
\]

The expression for \(E J_n(\lambda)\) is given and investigated, e.g., in [2], chapter 8, sections 8.2 - 8.3:
\[
E J_n(\lambda) = \int_{-\pi}^\pi \Phi_n(\lambda - \nu) \, f(\nu) \, d\nu = [\Phi_n * f](\lambda),
\]
where \(\Phi_n(\cdot)\) is the well-known Fejer’s kernel
\[
\Phi_n(\lambda) = \frac{\sin^2(n\lambda/2)}{2\pi n \sin^2(\lambda/2)}.
\]
The error of the Fejer’s approximation \(\Phi_n * f - f\) in the uniform norm is investigated in many works: [9], [30], [40], pp. 339 - 341, [41] etc. For instance,
\[
\sup_{\lambda \in [0,2\pi]} \left| [\Phi_n * f](\lambda) - f(\lambda) \right| \leq C \cdot \omega(f, 1/n) \, |\ln \omega(f, 1/n)|,
\]
see [9], pp. 33 - 40. Therefore, the equality (2.35) there holds by virtue of condition (2.34).
Note that the condition (2.34) is satisfied if for example
\[
\omega(f, h) \leq C \, h^\beta, \quad 0 \leq h \leq 2\pi, \quad \beta = \text{const} > 1/2, \quad (2.37)
\]
see [9], pp. 33 - 40. Note only that the condition (2.37) is very weak.

T.W. Anderson in [2], chapter 8, section 8.3 imposed on the spectral density \( f(\lambda) \) a more strong restriction
\[
\sum_{m=0}^{\infty} \sigma(m) < \infty,
\]
where
\[
f(\lambda) = \sum_{m=0}^{\infty} \sigma(m) \cos(m\lambda).
\]

**Remark 2.1.** Emerging in the equality (2.7) the variable
\[
I^{2\alpha}[f^2](\lambda) = \frac{1}{\Gamma(2\alpha)} \int_0^{\lambda} \frac{f^2(\nu)d\nu}{(\lambda - \nu)^{1-2\alpha}}
\]
may be \( n^{-1/2} \) - consistent estimated as follows:
\[
I^{2\alpha}[f^2](\lambda) \approx \frac{1}{\Gamma(2\alpha)} \int_0^{\lambda} \frac{J^2_n(\nu)d\nu}{(\lambda - \nu)^{1-2\alpha}}.
\]

**Remark 2.2.** We have proved the asymptotical normality under certain conditions of the sequence of random processes
\[
\theta_n(\lambda) = \sqrt{n} \left\{ F_{n, \alpha}(\lambda) - F^{(\alpha)}(\lambda) \right\}
\]
as \( n \to \infty \) in the space \( C^*(0, 2\pi) \) of continuous functions. Therefore if the value \( n \) is "sufficiently great"
\[
P(\sqrt{n} \cdot \max_{\lambda} \left| \left\{ F_{n, \alpha}(\lambda) - F^{(\alpha)}(\lambda) \right\} \right| > u) \approx P(\max_\lambda |\zeta_\infty(\lambda)| > u), \quad u = \text{const} > 0.
\]

The asymptotical as \( u \to \infty \) behavior of the last probability is fundamental investigated in the monograph [38], see also [37]:
\[
P(\max_\lambda |\zeta_\infty(\lambda)| > u) \sim H(\alpha) \, u^{\kappa-1} \exp \left( -u^2/\sigma^2 \right),
\]
\[
H(\alpha), \quad \kappa = \text{const}, \quad \sigma^2 = \sigma^2(\alpha) = \max_{\lambda \in (0, 2\pi)} \Theta_\alpha(\lambda, \lambda).
\]

The last equalities may be used by construction of confidence region for \( F^{(\alpha)}(\cdot) \) in the uniform norm. Indeed, let \( 1 - \delta \) be the reliability of confidence region, for example, 0.95 or 0.99 etc. Let \( u_0 = u_0(\delta) \) be a maximal root of the equation
\[
H(\alpha) \, u_0^{\kappa-1} \exp \left( -u_0^2/\sigma^2 \right) = \delta,
\]
then with probability $\approx 1 - \delta$
\[
\sup_{\lambda \in (0, 2\pi)} \left| F_{\alpha,n}(\lambda) - F^{(\alpha)}(\lambda) \right| \leq \frac{u_0(\delta)}{\sqrt{n}}.
\]

3 CLT in Hölder spaces for spectral functions estimation.

Let $(X = \{x\}, d)$ be compact metric space relative some distance (or semi-distance) $d = d(x_1, x_2)$. The modified Hölder (Lipshitz) space $H^\alpha(d)$ consists by definition on all the numerical (real or complex) continuous relative the distance $d = d(x_1, x_2)$ functions $f : X \to R$ satisfying the addition condition
\[
\lim_{\delta \to 0^+} \frac{\omega(f, d, \delta)}{\delta} = 0. \quad (3.1)
\]
Here $\omega(f, \delta) = \omega(f, d, \delta)$ is as before uniform module of continuity of the (continuous) function $f$ relative the distance (metric) $d(\cdot, \cdot)$:
\[
\omega(f, d, \delta) = \omega(f, \delta) = \sup_{x_1, x_2 : d(x_1, x_2) \leq \delta} |f(x_1) - f(x_2)|. \quad (3.2)
\]

The norm of the space $H^\alpha(d)$ is defined as follows:
\[
||f||_{H^\alpha(d)} = \sup_{x \in X} |f(x)| + \sup_{d(x_1, x_2) > 0} \left\{ \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \right\}. \quad (3.3)
\]

The detail investigation of these spaces with applications in the theory of non-linear singular integral equations is undergoing in the first chapter of a monograph of Gusejnov A.I., Muchtarov Ch.Sh. [15]. We itemize some used facts about these spaces.

This modification of the classical Hölder (Lipshitz) space $H(d)$ in which the condition (3.1) do not be presumed and hence is not separable, is really separable Banach space, in particular is linear, normed and complete and in turn is a closed subspace of $H(d)$.

Note but the space $H^\alpha(d)$ may be trivial, i.e. may consists only on constant functions. Let for instance, $X$ be convex connected closed bounded domain in the space $R^m$, $m = 1, 2, \ldots$ and let $d(x_1, x_2) = |x_1 - x_2|$ be usual Euclidean distance. Then the space $H^\alpha(d)$ is trivial: $\dim H^\alpha(d) = 1$.

The space $H^\alpha(d^\beta)$, $\beta = \text{const} \in (0, 1)$ in this example in contradiction is not trivial.

Further, if an another distance $r = r(x_1, x_2)$ on the source set $X$ is such that
\[
\forall x_1 \in X \Rightarrow \lim_{d(x_2) \to 0} \frac{d(x_1)}{r(x_1)} = 0, \quad (3.4)
\]
then the space $H^\alpha(d)$ is continuously embedded in the space $H^\alpha(r)$. 

\[11\]
We will write the equality (1.4) as follows: \( d \ll r \).

For instance, the distance \( r(x_1, x_2) \) may have a form

\[
r(x_1, x_2) = d^3(x_1, x_2), \quad \beta = \text{const} \in (0, 1).
\]

Of course, in the considered here problem \( X = [0, 2\pi] \), \( d(\lambda, \mu) = |\lambda - \mu|^\Delta, \quad \Delta = \text{const} \in (0, 1) \). We introduce hence the following Hölder’s spaces \( H^\alpha_\Delta \) over the circle \([0, 2\pi]\) consisting on all the periodical (continuous) functions with finite norm

\[
||f||_{H^\alpha_\Delta} \overset{def}{=} \sup_{\lambda \in [0,2\pi]} |f(\lambda)| + \sup_{|\lambda - \mu| > 0} \left\{ \frac{|f(\lambda) - f(\mu)|}{|\lambda - \mu|^\Delta} \right\},
\]

and such that

\[
\lim_{\delta \to 0^+} \left\{ \frac{\omega(f, h)}{h^\Delta} \right\} = 0.
\]

or equally

\[
\forall \mu \in [0,2\pi] \Rightarrow \lim_{\lambda \to \mu} \left\{ \frac{|f(\lambda) - f(\mu)|}{|\lambda - \mu|^\Delta} \right\} = 0.
\]

The classical CLT in Hölder’s spaces, i.e. CLT for the sums of independent random processes, with applications, is investigated in many works, see, e.g. [18], [35], [42], [43], [44].

Our aim in this section is investigation of the CLT for estimation of fractional derivative for spectral function.

**Theorem 3.1.** Let all the conditions of theorem 2.1 be satisfied. Let also \( \Delta \) be arbitrary number such that \( 0 \leq \Delta < 1/2 - \alpha \). The sequence of the distributions generated in Hölder the space \( H^\alpha_\Delta \) by the r.p. \( \zeta_n \) converges weakly as \( n \to \infty \) to the distribution in this space to at the same r.p. \( \zeta_\infty \).

**Proof.** To establish the weak compactness in these spaces, we return to the inequalities (2.32)-(2.33):

\[
\omega(\zeta_n, h) \leq C X_{n,p} \int_0^h \frac{t^{1/2 - 2/p}}{t^{1+\alpha}} \, dt \leq
\]

\[
C(\alpha, p) X_{n,p} h^{1/2 - \alpha - 2/p} \leq C(\alpha, \hat{p}) X_{n,p} h^{\Delta + \delta}, \quad \exists \delta = \text{const} > 0,
\]

if the value \( p = \hat{p} = \hat{p}(\alpha, \Delta, \delta) \) is sufficiently great. As before, \( \sup_n E X^p_{n,p} = 1 \).

We apply the Tchebychev’s inequality

\[
\sup_n P \left( \frac{\omega(\zeta_n, h)}{h^{\Delta + \delta}} > u \right) \leq \frac{C(\alpha, \hat{p})}{u^{\hat{p}}} \leq \epsilon
\]

for sufficiently greatest values \( u \). As long as the set of a (continuous) functions \( f : [0,2\pi] \to R \) such that

\[
\{ f : \omega(f, h) \leq u \cdot h^{\Delta + \delta} \}, \quad u = \text{const} < \infty
\]
is a shift - precompact set in the space $H^\alpha_\Delta$, see [15], chapter 1, we conclude that the main Prokhorov’s condition for weak compactness of probability measures [39] is satisfied.

The rest: convergence of finite-dimensional distributions, belonging the limit process $\zeta_\infty(\cdot)$ to the space $H^\alpha_\Delta$ is just.

**Corollary 3.1.** If we choose $\alpha = 0$, we get to the following extension of I.A.Ibragimov’s [17] result: for arbitrary value $\Delta$ from the interval $(0, 1/2)$ the sequence of distributions in the Banach space $H^\alpha_\Delta$ of the r.p. $\tau_n(\cdot)$ converges weakly as $n \to \infty$ to one for the r.p. $\tau(\cdot)$.

**Corollary 3.2.** If in addition to the conditions of theorem 3.1 the function $F = F(\lambda)$ satisfies the following restriction

$$\lim_{n \to \infty} \sqrt{n} \| F^{(\alpha)} \star \Phi_n - F^{(\alpha)} \| H^\alpha_\Delta = 0,$$

then the sequence of the distributions of Hölder continuous random processes

$$\theta_n(\lambda) \overset{\text{def}}{=} \sqrt{n} (F_{n,\alpha}(\lambda) - F^{(\alpha)}(\lambda))$$

converges weakly as $n \to \infty$ in the space $H^\alpha_\Delta$ in the Prokhorov-Skorokhod sense to the distribution of at the same centered Gaussian continuous random process $\zeta_\infty(\cdot)$.

The sufficient conditions for the equality (3.8) may be found in the articles [11], [22]. For instance, this equality is satisfied if

$$D^\alpha F \in H^\alpha_\Delta,$$

where as before $\Delta < \alpha - 1/2$, see [11], page 8, corollary 3.2.

### 4 Non-asymptotical approach.

We do not suppose in this section that $n \to \infty \ (n >> 1)$. More exactly, we intend to obtain here the upper and lower exponential estimate for the non-asymptotical probabilities for the following normed uniform deviations

$$W^\alpha_\alpha(u) \overset{\text{def}}{=} \sup_n P(\sqrt{n} \sup_{\lambda} |F_{n,\alpha}(\lambda) - E F_{n,\alpha}(\lambda)| > u))$$

and correspondingly

$$W_{\alpha}(u) \overset{\text{def}}{=} \sup_n P(\sqrt{n} \sup_{\lambda} |F_{n,\alpha}(\lambda) - F^{(\alpha)}(\lambda)| > u)), \ u \geq 1.$$

**Theorem 4.1.** Let all the conditions of theorem 2.1 be satisfied. Our statement: for some positive finite constants $C_1 = C_1(\alpha)$, $C_2 = C_2(\alpha)$, $C_1 \leq C_2$

$$\exp(-C_2(\alpha) u) \leq W^\alpha_\alpha(u) \leq \exp(-C_1(\alpha) u), \ u \geq 1.$$
Theorem 4.2. Let all the conditions of theorem 2.2 be satisfied. Our statement: for some positive finite constants \( C_3 = C_3(\alpha) \), \( C_4 = C_4(\alpha) \), \( C_3 \leq C_4 \)

\[
\exp(-C_4(\alpha)u) \leq W_\alpha(u) \leq \exp(-C_3(\alpha)u), \quad u \geq 1. \tag{4.3}
\]

Proof. Let us consider the random processes \( \zeta_n = \zeta_n(\lambda) \). We employ the inequality (2.33):

\[
\omega(\zeta_n, h) \leq C \int_0^h t^{1/2-2/p} dt,
\]

then

\[
\omega(\zeta_n, h) \leq C_5 X_{n,p} h^{1/2-2/p}, \quad p \geq 1/\alpha,
\]

where as before \( \sup_n \sup_p \mathbb{E}|X_{n,p}|^p = 1 \). On the other words,

\[
|\zeta_n(\lambda) - \zeta_n(\mu)| \leq C_5 X_{n,p} |\lambda - \mu|^{1/2-2/p}. \tag{4.4}
\]

Analogously

\[
|\zeta_n(\lambda)| \leq C_6 X_{n,p}. \tag{4.4a}
\]

As long as the random variables \( \{\zeta_n(\lambda)\}, \lambda \in [0, 2\pi] \) are also the two-dimensional stochastic integrals over Gaussian measure (Gaussian chaos), on the other words, belongs to the described above Banach space \( B(\phi) \). The so-called entropy condition \[36\] for the set \( [0, 2\pi] \) relative the distance \( |\lambda - \mu|^{1/2-2/p} \) for each the values \( p, p > 1/\alpha \) is satisfied, and we conclude using the main result of an article \[36\] that

\[
\sup_n \max_{\lambda \in [0,2\pi]} |\zeta_n(\lambda)| \in G\psi, \tag{4.5}
\]

where (recall) \( \psi(p) = p \).

The right-hand side of bilateral inequality (4.2) follows immediately from (4.5), see \[20\], \[31\]. The left-hand of (4.2) estimate is very simple:

\[
W_\alpha^o(u) \geq \mathbb{P}(|\zeta_1(\pi)| > u) \geq \exp(-C_2(\alpha)u).
\]

The second theorem 4.2 may be proved by means of theorem 2.2.

5 Concluding remarks.

A. Weight case.

Perhaps, it is interest to investigate the error in the uniform norm of the approximation of a form

\[
V(x) \cdot (W \cdot F)^{(\alpha)}(x) \approx V(x) \cdot (W \cdot F)_{n,\alpha}(x),
\]

or analogously
\[ V(x) \cdot (W \ast F)^{(\alpha)}(x) \approx V(x) \cdot (W \ast F)_{n,\alpha}(x), \]
or analogously
\[ V(x) \ast (W \ast F)^{(\alpha)}(x) \approx V(x) \ast (W \ast F)_{n,\alpha}(x), \]
where \( V(x), W(x) \) are two weight functions, for instance, \( V(x) = |x|^\gamma, W(x) = |x|^\Delta \), \( \gamma, \Delta = \text{const} \).

**B. Applications (possible) in statistics.**

The asymptotical tail behavior of the statistic \( \sup_\lambda |\theta_n(\lambda)| \) may be used perhaps in turn in statistics, for instance, for the verification of semi-parametrical hypotheses and detection of distortion times of signals etc.

**C. Non-centered sample.**

If the source stationary Gaussian random sequence \( \{\eta_k\}, k = 1, 2, \ldots \) is non-centered:

\[ \mathbb{E}\eta_k = a \neq 0, \; k = 1, 2, \ldots, n, \]
then we can replace as ordinary
\[ \eta_k := \eta_k^0 \stackrel{df}{=} \eta_k - n^{-1} \sum_{j=1}^{n} \eta_j = \eta_k - a_n, \]
where \( a_n = n^{-1} \sum_{j=1}^{n} \eta_j \) is consistent estimation for the value \( a \). Both the theorems 2.1 and 2.2 remains true under at the same conditions.

**D.** Perhaps, obtained above results may be extended on the multivariate time series by using of the results of the book [16], chapter 5, as well as on the non-Gaussian processes through the cumulant function and on the case of the ”continuous time”.

**References**

[1] R. J. Adler, R. E. Feldman and M. S. Taqqu (Eds.) *A Practical Guide to Heavy Tails: Statistical Techniques and Applications.* Birkhäuser, New York, 1998.

[2] Anderson T.W. *Statistics analysis of time series.* Willey, New York, (2011).

[3] Shuyang Bai, Mamikon S. Ginovyan, Murad S. Taqqu. *Functional Limit Theorems for Toeplitz Quadratic Functionals of Continuous time Gaussian Stationary Processes.* arXiv:1501.05574v1 [math.PR] 22 Jan 2015

[4] I. B. Bapna and Nisha Mathur. *Application of Fractional Calculus in Statistics.* Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 18, 849-856
[5] Andrea Borla and Costen Protopoescu. Nonparametric Estimation of the Fractional Derivative of a Function Distribution. Internet publication, PDF, (2014).

[6] Buldygin, V., Kozachenko, Yu. Metric Characterization of Random Variables and Random Processes. Am. Math. Soc., Providence, RI (2000). MR1743716

[7] Buldygin, V., Zayats, V. On the asymptotic normality of estimates of the correlation functions stationary Gaussian processes in spaces of continuous functions. Ukr. Math. J. 47(11), 14851497 (1995) (in Russian), MR1369560. doi:10.1007/BF01057918

[8] Buldygin V.V., Mushtary D.Ch., Ostrovsky E.I., Puchalskii A.W. New Trends in Probability Theory and Statistics. (1992), VSP (Utrecht, Tokyo, New York).

[9] J.K.Burkill. Lectures On Approximation By Polynomials. Tata Institute of Fundamental Research. Bombay, 1959.

[10] Rainer Dahlhaus. Asymptotic Normality of Spectral Estimates. Journal of Multivariate Analysis, 16, 412-431, (1985).

[11] Borislav R. Draganov. Simultaneous approximation of functions by Fejer-type operators in a generalized Holder norm. Internet publication, 2014.

[12] Farida Enikeeva. Adaptive minimax estimation of a fractional derivative. Statistics Probability Letters, 76, (2006), 1441-1448.

[13] Ginovyan, M. S. On estimating the value of a linear functional of the spectral density of a Gaussian stationary process. Theory Probab. Appl., 33, (4), (1988), 722-726.

[14] Golubev, G.K., Enikeeva, F. (2001.) On the minimax estimation problem of a fractional derivative. Theory Probab. Appl. 46, 619-635.

[15] Guseinov A.I., Muchtarov Ch.Sh. Introduction to the theory of non-linear singular integral equations. Moskow, Nauka, (1980), (in Russian).

[16] E.J.Hannan. Multiple time series. The Australian National University, Canberra. John Willey and Sons Inc. New York, London, Sydney, Toronto, 1970.

[17] Ibragimov I.A. On Estimation of the Spectral Function of a Stationary Gaussian Process. Theory Probab. Appl., 8, 1963, (4), 366-401.

[18] Klicnarova Jana. Central limit theorem for Hölder processes on $R^m$ cube. Comment.Math.Univ.Carolin. 48, 1, (2007), 83-91.

[19] Konakov V.D. Non-Parametric Estimation of Density Functions. Theory Probab. Appl., 17, 2, (1973), pp. 361-365.
[20] Kozachenko Yu. V., Ostrovsky E.I. (1985). The Banach Spaces of random Variables of subgaussian type. Theory of Probab. and Math. Stat., (in Russian). Kiev, KSU, 32, 43-57.

[21] Yuriy Kozachenko, Viktor Troshki. A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process. Modern Stochastics: Theory and Applications, 1, (2014), 139-149, DOI: 10.15559/15-VMSTA17.

[22] R. A. Lasuriya. On the approximation of functions defined on the real axis by Fejer-type operators in the generalized Hölder metric. Mat. Zametki, 81 (2007), no. 4, 547 - 552 (in Russian); translation in Math. Notes, 81, (2007), no. 3-4, 483 - 488.

[23] Levit B. Ya. and Samarov A. M. A remark on estimation of spectral function. Problems Inform. Transmission, 14, (2), (1978), 61-66.

[24] E. Liflyand, E. Ostrovsky and L. Sirota. Structural properties of Bilateral Grand Lebesque Spaces. Turk. Journal of Math., 34, (2010), 207-219. TUBITAK, doi:10.3906/mat-0812-8

[25] Mark Meerschaert, Jeff Mortensen, and Hans-Peter Scheffler. Vector Grünwald formula for fractional derivatives. Internet electronic publication, 2014.

[26] K. Miller and B. Ross. (1993) An Introduction to Fractional Calculus and Fractional Differential Equations. Wiley, New York.

[27] Toshihiko Nishishiraho. Quantitative theorems on linear approximation processes of convolution operators in Banach spaces. Tohoku Math. J. (2), Volume 33, Number 1, (1981), 109-126.

[28] Mark Meerschaert, Jeff Mortensen, and Hans-Peter Scheffler. Vector Grünwald formula for fractional derivatives. Internet electronic publication, 2014.

[29] K. Miller and B. Ross. (1993) An Introduction to Fractional Calculus and Fractional Differential Equations. Wiley, New York.

[30] Toshihiko Nishishiraho. Quantitative theorems on linear approximation processes of convolution operators in Banach spaces. Tohoku Math. J. (2), Volume 33, Number 1, (1981), 109-126.

[31] Ostrovsky E.I. Exponential estimates for the random fields and its applications. (1999), Moskow-Obninsk, OINPE, (in Russian).

[32] E. Ostrovsky and L.Sirota. Well Posedness of the Problem of Estimation Fractional Derivative for a Distribution Function. arXiv:1412.6829v1 [math.ST] 21 Dec 2014
[33] E. Ostrovsky and L. Sirota. *Sharp Estimates for Module of Continuity of Fractional Integrals and Derivatives.* arXiv:1502.06170v1 [math.FA] 22 Feb 2015

[34] E. Ostrovsky and L. Sirota. *Simplification of the majorizing measure method, with development.* arXiv:1302.3202v1 [math.PR] 13 Feb 2013

[35] E. Ostrovsky and L. Sirota. *Central Limit Theorem in Hölder spaces in the terms of majorizing measures.* arXiv:1409.6054v1 [math.PR] 21 Sep 2014

[36] Ostrovsky E.I. *Estimations of distribution of maximum random field.* Teoriya veroiyatnostei i ee primeneniya. - 1997. - B.42. - N. 2. - pp. 482-494, (in Russian).

[37] V. I. Piterbarg, V. R. Fatalov. *The Laplace method for probability measures in Banach spaces.* Uspekhi Mat. Nauk, 1995, Volume 50, Issue 6 (306), 57-150.

[38] V.I.Piterbarg. *Asymptotical methods in the theory of Gaussian processes and fields.* American Mathematical Society, 1996.

[39] Prokhorov Yu.V. *Convergence of Random Processes and Limit Theorems of Probability Theory.* Probab. Theory Appl., (1956), V. 1, 177-238.

[40] Ronald A. DeVore, George G. Lorentz. *Constructive Approximation.* Springer-Verlag, (2009). Grundlehren der mathematischen Wissenschaften, 303; Berlin-Heidelberg-New York.

[41] R. A. DeVore. *The Approximation of Continuous Functions by Positive Linear Operators.* Lecture Notes in Mathematics, No. 293, Springer-Verlag, Berlin-Heidelberg-New York.

[42] Ratchkauskas A, Suquet Ch. *Central limit theorems in Hölder topologies for Banach space valued random fields.* Teor. Veroyatnost. i Primenen., 2004, Volume 49, Issue 1, Pages 109-125, (in Russian).

[43] A. Ratchkauskas, Ch. Suquet. *Necessary and sufficient condition for the Hölderian functional central limit theorem.* J. Theoret. Probab. 17 (2004) 221-243.

[44] Ratchkauskas A., V. Zemlys V. *Functional central limit theorem for a double-indexed summation process,* Liet. Mat. Rink., 45, (2005), 401-412.

[45] S. G. Samko, A. A. Kilbas and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers, Yverdon, 1993.