Finite-g Strings

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. The research described in this dissertation was carried out in the Department of Applied Mathematics and Theoretical Physics, Cambridge University, between September 2004 and March 2008. Except where reference is made to the work of others, all the results are original and based on the following works of mine:

1. “On the Dynamics of Finite-Gap Solutions in Classical String Theory”
   N. Dorey and B. Vicedo
   \textit{JHEP 0607}, 014 (2006) hep-th/0601194

2. “A Symplectic Structure for String Theory on Integrable Backgrounds”
   N. Dorey and B. Vicedo
   \textit{JHEP 0703}, 045 (2007) hep-th/0606287

3. “Semiclassical Quantisation of Finite-Gap Strings”
   B. Vicedo
   \textit{JHEP 0806}, 086 (2008) arXiv:0803.1605 [hep-th]

These papers are referred to as [1], [2] and [3] respectively in the bibliography. The content of Part II is taken mostly from [2]. Part III is based on all three papers [1–3].
and Part IV is entirely based on [3]. None of the original works contained in this dissertation has been submitted by me for any other degree, diploma or similar qualification.

The following is a list of my other publications, referred to as [4] and [5] in the bibliography. The main purpose of these papers is not discussed in this thesis although certain minor results from them are used:

4. “Giant Magnons and Singular Curves”

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*JHEP 0712*, 078 (2007) hep-th/0703180

5. “Large winding sector of AdS/CFT”

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Abstract

In view of one day proving the AdS/CFT correspondence, a deeper understanding of string theory on certain curved backgrounds such as $AdS_5 \times S^5$ is required. In this dissertation we make a step in this direction by focusing on $\mathbb{R} \times S^3$.

It was discovered in recent years that string theory on $AdS_5 \times S^5$ admits a Lax formulation. However, the complete statement of integrability requires not only the existence of a Lax formulation, but also that the resulting integrals of motion are in pairwise involution. This idea is central to the first part of this thesis.

Exploiting this integrability we apply algebro-geometric methods to string theory on $\mathbb{R} \times S^3$ and obtain the general finite-gap solution. The construction is based on an invariant algebraic curve previously found in the $AdS_5 \times S^5$ case. However, encoding the dynamics of the solution requires specification of additional marked points. By restricting the symplectic structure of the string to this algebro-geometric data we derive the action-angle variables of the system.

We then perform a first-principle semiclassical quantisation of string theory on $\mathbb{R} \times S^3$ as a toy model for strings on $AdS_5 \times S^5$. The result is exactly what one expects from the dual gauge theory perspective, namely the underlying algebraic curve discretises in a natural way. We also derive a general formula for the fluctuation energies around the generic finite-gap solution. The ideas used can be generalised to $AdS_5 \times S^5$. 
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Chapter 0

Introduction/Review

0.1 The AdS/CFT conjecture

Over the past thirty years there has been a fascinating rivalry between string theory on the one hand and gauge theories on the other in an attempt to describe the physics of the strong interaction. Indeed, string theory was originally invented as a way of describing some of the observed peculiarities of the strong force between quarks, the quarks being thought of in this theory as bound together by strings. But this theory of the strong force never had much success and with the advent of gauge theories it was soon discarded and replaced by the far more successful QCD which describes the interaction between quarks in terms of gauge fields. Later though string theory resurged as a possible candidate for unifying all the forces of nature. In this modern interpretation of string theory the strong force is now described by encapsulating QCD as a low energy part of its dynamics. The gauge fields however are now derived secondary objects of the theory, the fundamental objects being the strings themselves.

There is however yet another use of string theory discovered by ’t Hooft [6]
who realised that perturbation expansions of $SU(N)$ gauge field theory in the large $N$ limit resemble string theory genus expansions (see [7] for a review). Loosely speaking, in the $N \to \infty$ limit (with the ’t Hooft coupling $\lambda \equiv g_{YM}^2 N$ held fixed, $g_{YM}$ denoting the gauge theory coupling), each Feynman diagram of the $SU(N)$ gauge theory can be attributed a topology and the Feynman diagram expansion breaks up into a sum over topologies. Schematically we have for example for the free energy

$$\mathcal{F} = N^2 \begin{array}{c} \includegraphics{fig1.png} \\ \end{array} + 1 \begin{array}{c} \includegraphics{fig2.png} \\ \end{array} + \frac{1}{N^2} \begin{array}{c} \includegraphics{fig3.png} \\ \end{array} + \ldots$$

$$= \sum_{g=0}^{\infty} N^{2-2g} \sum_{l=0}^{\infty} c_{g,l} \lambda^l,$$

where each picture in the equation represents the sum over Feynman diagrams of the given topology. This reorganised sum of Feynman diagrams resembles a string perturbation expansion over Riemann surfaces with $1/N$ playing the role of the string coupling $g_S$ and the ’t Hooft coupling $\lambda$ related to Planck’s constant on the world sheet. More generally the $N \to \infty$ limit of correlation functions of $n$ (single-trace) gauge invariant operators $\hat{O}_j$ is schematically given by

$$\left\langle \prod_{j=1}^{n} \hat{O}_j \right\rangle = N^{2-n} \begin{array}{c} \includegraphics{fig4.png} \\ \end{array} + N^{-n} \begin{array}{c} \includegraphics{fig5.png} \\ \end{array} + \frac{1}{N^{2+n}} \begin{array}{c} \includegraphics{fig6.png} \\ \end{array} + \ldots$$

$$= \sum_{g=0}^{\infty} N^{2-2g-n} \sum_{l=0}^{\infty} c_{g,l}^{(n)} \lambda^l,$$

which in the string theory analogy resembles a correlation function of $n$ vertex operator insertions on the world sheet. In particular, any given gauge invariant operator $\hat{O}_j(x)$ should correspond to a certain string theory state $|\hat{O}_j\rangle$. Of course the
Feynman diagrams in perturbative ($\lambda \ll 1$) gauge theory are not literally smooth Riemann surfaces but the Feynman propagators merely suggest simplicial decompositions of Riemann surfaces. One can nevertheless imagine how in the $\lambda \gg 1$ regime, which requires a nonperturbative formulation of the theory, the number of vertices in a typical diagram would become huge and the Feynman diagrams would more closely approximate smooth Riemann surfaces. This beautiful observation about the large $N$ limit of gauge theories is at the heart of the concept of string/gauge dualities. Indeed, although the above analogy is far from rigourous it strongly suggests that gauge theories are intimately related to string theories on certain backgrounds, in that some gauge theories may admit dual descriptions in terms of string theories.

The AdS/CFT correspondence due to Maldacena [8] is a conjectured realisation of such a duality for a supersymmetric cousin of QCD, namely it relates four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$ to type IIB superstring theory on $AdS_5 \times S^5$ (see [9] for a review). Concretely, at large ’t Hooft coupling $\lambda \equiv g_{YM}^2 N \gg 1$, $\mathcal{N} = 4$ SYM theory is believed to have a dual description in terms of type IIB superstring theory on $AdS_5 \times S^5$ with equal radii of curvature $R$ such that $R/\alpha'^{1/2} = \lambda^{4/3}$. The string coupling in the AdS/CFT correspondence is not simply $1/N$ as above, but instead is given by

$$g_S = 4\pi g_{YM}^2 = \frac{4\pi \lambda}{N}.$$ 

The extra factor of $\lambda$ however does not affect the interpretation of the gauge theory perturbation expansions as genus expansions.

An important part of the AdS/CFT correspondence is establishing a ‘dictionary’ for translating the language of one theory into the other. That is, given a gauge theory operator $\hat{O}(x)$, we need a way of determining its dual string theory state $|\mathcal{O}\rangle$ and vice versa. For this it is helpful to classify the states of both theories
according to the global symmetries present. Both theories share the global (bosonic) symmetry group $SO(4, 2) \times SO(6)$: in gauge theory $SO(4, 2)$ corresponds to the conformal symmetry group (in $3 + 1$ dimensions) and $SO(6)$ to the R-symmetry (acting for instance in the fundamental representation on the scalar fields $\{\phi_i\}_{i=1}^{6} \text{ of } \mathcal{N} = 4 \text{ SYM}$), whereas on the string theory side $SO(4, 2) \times SO(6)$ is the target space symmetry. States on either side thus fall into representations of this global symmetry labelled by the eigenvalues $(E = \Delta, S_1, S_2, J_1, J_2, J_3)$ of the six Casimirs, the first three being for $SO(4, 2)$ and the last three for $SO(6)$. For instance the complex combinations $Z = \phi_1 + i\phi_2$, $W = \phi_3 + i\phi_4$ and $Y = \phi_5 + i\phi_6$ of the $SO(6)$ scalars have R-charges $(J_1, J_2, J_3)$ equal to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

Note that one of the Casimirs of $SO(4, 2)$ plays a distinguished role. In string theory this is the energy eigenvalue $E$ of the Hamiltonian $\mathcal{H}_{\text{string}}$ which generates time translation in $AdS_5$. And according to the AdS/CFT conjecture, it should be identified with the eigenvalue $\Delta$ of the Dilation operator $\mathcal{D}$ of $\mathcal{N} = 4 \text{ SYM}$. Therefore if $|\mathcal{O}\rangle$ is a string energy eigenstate of energy $E$ and $\hat{\mathcal{O}}(x)$ its dual gauge invariant conformal operator with anomalous dimension $\Delta$, namely

$$\mathcal{H}_{\text{string}}|\mathcal{O}\rangle = E \left( \frac{R^2}{\alpha'}, g_S \right) |\mathcal{O}\rangle, \quad \mathcal{D}\hat{\mathcal{O}}(x) = \Delta \left( \lambda, \frac{1}{N} \right) \hat{\mathcal{O}}(x)$$

then the AdS/CFT conjecture states that

$$\Delta \left( \lambda, \frac{1}{N} \right) = E \left( \frac{R^2}{\alpha'}, g_S \right). \quad (0.1.1)$$

Checking (0.1.1) for arbitrary $N$ seems a hopeless task since determining the energy spectrum of the string to all orders in $g_S$ would be incredibly difficult. A more modest goal, at least initially, would be to check the correspondence in the ’t Hooft limit $N \to \infty$ where all diagrams on the gauge theory side become planar, and the string theory becomes free, i.e. the worldsheet is topologically a sphere.
Even with this simplification the duality is still of strong/weak coupling type and is therefore very hard to test since the weak coupling regions of both theories (in which perturbative methods apply) are non-overlapping. Specifically, a conformal operator in the strong coupling limit $\lambda \gg 1$ should admit an equivalent description in terms of a classical string ($1/\sqrt{\lambda} \ll 1$), i.e. a worldsheet soliton. Conversely, a string moving on a highly curved background $\sqrt{\lambda} = R^2/\alpha' \ll 1$ should have an equivalent description as a weakly coupled ($\lambda \ll 1$) gauge field. This makes the conjecture very hard to prove since we only have access to perturbative methods on both sides of the correspondence.

0.2 The large Spin/R-charge limit

Despite the strong/weak coupling obstruction, it was realised in the work of Berenstein, Maldacena and Nastase [10] that explicit tests of the correspondence could be made (beyond sectors protected by supersymmetry) if one took the further limit $J \to \infty$ where $J$ is a certain charge, say $J_1$. This observation was later generalised in a series of papers by Frolov and Tseytlin [11–13] to larger sectors of the correspondence by taking multiple charges to infinity.

To first get an intuitive understanding of the significance of these large charge limits we go back to the picture of the Feynman graphs turning into Riemann surfaces. Focusing on the $SO(6)$ scalar sector of $\mathcal{N} = 4$ SYM, consider single-trace conformal operators

$$\hat{O} = \text{tr}(\hat{\Phi}_{i_1} \ldots \hat{\Phi}_{i_n}) = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} \quad (n = 6)$$

where in the pictorial representation the black dots each represent a single operator $\hat{\Phi}_i \in \{\hat{Z}, \hat{W}, \hat{Y}\}$. They form a closed chain by virtue of the trace in $\hat{O}$. Now the
2-point correlation function of $\hat{O}$ can be written symbolically as

$$\langle \hat{O}(x)\hat{O}(y) \rangle = \sum_{\text{all possible Wick contractions}}$$

where the right hand side represents the sum of all possible Wick contractions, i.e. Feynman diagrams connecting the operators $\hat{O}$ at $x$ and $y$. As before the Feynman diagrams suggest a simplicial decomposition of a Riemann surface (with boundaries). This simplicial decomposition may be refined in two ways: either one increases the coupling $\lambda$ as before to increase the number of vertices in these Feynman diagrams, or one can also increase the number $n$ of constituent operators $\hat{\Phi}_i$ in $\hat{O}$.

For example a BMN operator $\text{tr}(\hat{Z}^{J_1-2}\hat{W}\hat{Y}+\ldots)$ is made up of a large number of reference fields $\hat{Z}$ and a small number of other “impurity” fields $\hat{W}$ and $\hat{Y}$. Its string theory dual, the BMN string, is almost point-like and has angular momentum $J_1 = J$ on $S^5$. More generally an operator may contain a large amount of impurities such as $\text{tr}(\hat{Z}^{J_1}\hat{W}^{J_2}\hat{Y}^{J_3}+\ldots)$ with $J = J_1 + J_2 + J_3$. Its string theory dual, the Frolov-Tseytlin string, is spatially extended and spins with the three different angular momenta $J_1$, $J_2$ and $J_3$ on $S^5$. As explained above one expects such ‘long’ ($J \to \infty$) single-trace conformal operators to have a stringy behaviour even at weak coupling $\lambda \ll 1$.

Concretely, suppose one can expand both sides of (0.1.1) in terms of $\lambda/J^2$ and $1/J$. On the string side this is achieved by doing a semiclassical expansion in $1/J \propto 1/\sqrt{\lambda} \propto \alpha'$ with $\lambda/J^2$ held fixed. On the gauge side one could first expand in $\lambda \ll 1$ and then further expand each coefficient in $1/J$. When such expansions for the semiclassical energy $E$ and the perturbative anomalous dimension $\Delta$ exist and take on the similar form

$$J \left[ 1 + \sum_{n=1}^{\infty} \left( c_n + \sum_{k=1}^{\infty} \frac{c_{nk}}{J^k} \right) \left( \frac{\lambda}{J^2} \right)^n \right],$$

(0.2.1)
then their respective coefficients, for $c_n$ say, could be compared directly, even though they have been obtained differently from both sides of the duality.

With this procedure for making quantitative tests of the correspondence in place, the immediate goal from both sides of the duality is clear. From the gauge theory perspective one faces the problem of diagonalising the dilatation operator $\mathcal{D}$ on long single-trace conformal operators perturbatively in $\lambda \ll 1$. Since it commutes with the Casimirs of $SO(4,2) \times SO(6)$ it does not mix operators of different weights. For instance, its action on the complete set of operators $\hat{O}_{\alpha}^{J_1, J_2}$ composed solely of the two scalars $\hat{Z}$ and $\hat{W}$ is given by

$$\mathcal{D} \hat{O}_{\alpha}^{J_1, J_2}(x) = \sum_{\beta} \mathcal{D}_{\alpha \beta} \hat{O}_{\beta}^{J_1, J_2}(x). \tag{0.2.2a}$$

The problem is therefore reduced to diagonalising the matrix $\mathcal{D}_{\alpha \beta}$. However, since we are interested in the limit $J_1, J_2 \to \infty$ this simple diagonalisation task quickly becomes intractable without recourse to numerical methods.

The task on the string theory side is to obtain the semiclassical energy spectrum of strings on $AdS_5 \times S^5$ to leading order in $1/\sqrt{\lambda} \ll 1$. This in turn requires complete knowledge of the classical string motions on such a background. Restricting attention to the $SU(2)$ sector corresponding to the operators $\hat{O}_{\alpha}^{J_1, J_2}$ discussed above, the problem is reduced to finding the general solution to the equations of motion for a string moving on $\mathbb{R} \times S^3$. However, the equations of motion for the fields $\{X_i\}_{i=1}^4$ describing the embedding of the string into $S^3$,

$$\partial_\alpha \partial^\alpha X_i + \left( \sum_j \partial_\alpha X_j \partial^\alpha X_j \right) X_i = 0, \tag{0.2.2b}$$

are second order nonlinear partial differential equations subject to the constraint $\sum_{i=1}^4 X_i^2 = 1$. Solving them exactly therefore seems quite intractable as well.
0.3 Classical/Quantum Integrability

Fortunately, something of a miracle happens in both cases. By computing the 1-loop planar dilatation operator on single-trace operators of all six scalar fields of $\mathcal{N} = 4$ SYM, Minahan and Zarembo [14] discovered it was proportional to the Hamiltonian of the $\mathfrak{so}(6)$ integrable spin chain with nearest-neighbour interactions. Subsequently the complete one-loop planar dilatation operator of $\mathcal{N} = 4$ SYM was computed by Beisert [15, 16] and identified with an $\mathfrak{su}(2, 2|4)$ super spin chain by Beisert and Staudacher in [17]. Integrability also seems to persist at higher loops [18, 19]. For the purpose of this thesis we shall focus on the $SU(2)$ sector at one-loop where the planar dilatation operator reduces to the famous Heisenberg $\text{XXX}_{\frac{1}{2}}$ spin chain Hamiltonian which is *quantum integrable*. Specifically we have

\[
\mathcal{D}^{\text{planar}}_{SU(2)} = J + \frac{\lambda}{16\pi^2} \sum_{j=1}^{J} (1 - \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}) + O(\lambda^2) = J + \frac{\lambda}{4\pi^2} \hat{H}_{\text{XXX}_{\frac{1}{2}}} + O(\lambda^2),
\]  

(0.3.1)

where $\vec{\sigma}_j = (\sigma^\alpha_j)^3_{\alpha=1}$ is the set of Pauli matrices acting on the $j^{\text{th}}$ site of the spin chain. The tree-level term in (0.3.1) is just the common engineering dimension $J = J_1 + J_2$ of the operators $\hat{O}^{J_1,J_2}$, which is also just the length of the spin chain.

The fact that the one-loop planar dilatation operator (0.3.1) is integrable implies that it can be diagonalised analytically for any length $J$. As usual, the definition of quantum integrability requires the existence of a maximal set of commuting operators which includes the Hamiltonian. The construction of such operators in the Heisenberg $\text{XXX}_{\frac{1}{2}}$ spin chain proceeds in the usual way (see [20–22] for a general discussion on quantum integrable systems) by defining the Lax operator $\hat{L}_{j,a}(u) = u \mathbf{1}_j \otimes 1_a + \frac{i}{2} \sum_\alpha \sigma^\alpha_j \otimes \sigma^\alpha_a$ where $u \in \mathbb{C}$ is called the *spectral parameter*. Here the subscript $j$ indicates that the matrix acts on the $j^{\text{th}}$ site of the spin chain.
chain and the subscript $a$ indicates that the matrix acts on an extra ‘auxiliary’ site.
The main object of interest is the monodromy matrix $\hat{T}_a(u) = \hat{L}_{J,a}(u) \ldots \hat{L}_1,a(u)$
(which acts on all $J$ sites as well as the auxiliary site). Writing out the action on
the auxiliary site in matrix form it reads
\[
\hat{T}_a(u) = \begin{pmatrix}
\hat{A}(u) & \hat{B}(u) \\
\hat{C}(u) & \hat{D}(u)
\end{pmatrix}.
\]
Its trace over the auxiliary site $\hat{T}(u) = \text{tr}_a \hat{T}_a(u) = \hat{A}(u) + \hat{D}(u)$, the transfer matrix, generates the desired family of commuting operators since one can show [21]
\[
[\hat{T}(u), \hat{T}(v)] = 0, \quad \forall u, v \in \mathbb{C}.
\]
In particular the Hamiltonian can be extracted as $\hat{H}_{XXX,\frac{1}{2}} = \frac{i}{2} \left. \frac{d}{du} \log \hat{T}(u) \right|_{u=\frac{i}{2}} - \frac{J}{2}$. The diagonalisation of $\hat{H}_{XXX,\frac{1}{2}}$ can therefore be achieved by simultaneously diagonalising the whole family of operators $\hat{T}(u)$. For this one defines a reference state $|\Omega\rangle$ on the spin chain by the condition $\hat{C}(u)|\Omega\rangle = 0$ and looks for eigenvectors of the form
\[
|u_1, \ldots, u_M\rangle = \hat{B}(u_1) \ldots \hat{B}(u_M)|\Omega\rangle.
\]
(0.3.2)
This is akin to the Fock space construction where the operator $\hat{B}(u)$ creates a magnon excitation on the spin chain with rapidity $u$. One can show that (0.3.2) is
an eigenstate of the transfer matrix $\hat{T}(u)$ if and only if the parameters $u_j$ satisfy the famous Bethe equations which in this sector read [21, 22]
\[
\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^J = \prod_{k=1 \ (k \neq j)}^M \frac{u_j - u_k + i}{u_j - u_k - i}.
\]
(0.3.3)
The solutions $u_j \in \mathbb{C}$ of these equations are called Bethe roots.
To study the limit $J \to \infty$ of (0.3.3) one starts by taking its logarithm,

$$J \log \frac{u_j + i}{u_j - i} = \prod_{k=1}^{M} \log \frac{u_j - u_k + i}{u_j - u_k + i} - 2\pi i n_j,$$  
(0.3.4)

where the mode numbers $n_j \in \mathbb{Z}$ specify the branch of the logarithm. A careful study of these equations determines the location of the Bethe roots in the limit $J \to \infty$. Since all Bethe roots are of order $u_j \sim J$ it is convenient to introduce the scaled spectral parameter $x$ by $u = Jx$. If the number of mode numbers is finite, say $\{n_I\}_{I=1}^{K}$, and the number of Bethe roots with the same mode number is of order $J$ then one finds that the Bethe roots of a given mode number $n_I$ all agglomerate into a vertical ‘cut’ $C_I$ in the complex plane, see Figure 1. To characterise the density of the Bethe roots along the various cuts one introduces a function $p(x)$ on the complex plane called the quasi-momentum which can then be shown to have a simple pole at $x = 0$ and the property that its value jumps by $2\pi n_I$ across $C_I$ (see [23] for details). Moreover, its integral around any cut $C_I$ gives exactly the proportion of Bethe root lying on $C_I$ called the filling fraction,

$$S_I = \frac{1}{2\pi i} \oint_{A_I} p(x) dx, \quad I = 1, \ldots, K$$  
(0.3.5)

where $A_I$ is a contour around the cut $C_I$. Now by construction, a distribution of Bethe roots like the one in Figure 1 characterises the $J \to \infty$ limit of a single-trace eigen-operator of the one-loop planar dilatation operator (0.3.1). Therefore by the
reasoning of section 0.2 we expect it to match the description of a classical string solution on $\mathbb{R} \times S^3$. To see this we now turn to the string theory side.

Recall that the task there involves finding exact solutions to a set of non-linear second order partial differential equations (0.2.2b) subject to a constraint, which in general is impossible. Fortunately, it was discovered by Bena, Polchinski and Roiban [24] that the equations of motion for a superstring on $AdS_5 \times S^5$ can be formulated as a flatness condition for a 1-parameter family of currents $J(x)$ depending on a complex parameter $x \in \mathbb{C}$. This is a necessary condition for the theory to be \textit{classically integrable}. In the $SU(2)$ sector the lightcone components of these currents $J(x)$ are

$$J_\pm(x) = \frac{j_\pm}{1 \mp x}, \quad \partial_+ J_-(x) - \partial_- J_+(x) + [J_+(x), J_-(x)] = 0.$$  

This connection is built out of $j = -g^{-1}dg \in \mathfrak{su}(2)$ where $g \in SU(2)$ depends on the fields $\{X_i\}_{i=1}^4$ and specifies the embedding of the string into $SU(2) \simeq S^3$. The flatness condition (0.3.6) is equivalent to the equations of motion (0.2.2b). As we will show in this thesis, when written in this form (0.3.6) the equations of motion can be solved exactly.

As we review in chapter 6, the zero-curvature representation (0.3.6) of the equations of motion directly leads to the construction of an algebraic curve $\hat{\Sigma}$ equipped with a meromorphic differential $dp$, starting from a given solution $X_i^{sol}$ to (0.2.2b). In other words (0.3.6) provides an assignment

$$X_i^{sol} \longrightarrow (\hat{\Sigma}, dp).$$

Moreover, the pair $(\hat{\Sigma}, dp)$ is independent of the worldsheet $(\sigma, \tau)$-coordinates and therefore encodes the integrals of motion of the solution $X_i^{sol}$. Thus all solutions to (0.2.2b) on the string theory side are classified by their respective algebraic curves. In
the $SU(2)$ sector these curves are all hyperelliptic and can be represented in terms of cuts in the complex plane. In chapter 9 we will give a proof of the usual assumption that these cuts are all vertical in the complex plane, see Figure 2 (note that the path taken by the cuts is arbitrary as long as they join up all the branch points in pairs). The remarkable similarity between Figures 1 and 2 was first discovered by Kazakov, Marshakov, Minahan and Zarembo in their seminal paper [23] (see [25, 26] for shorter reviews). The quasi-momentum on the gauge theory side is identified here with the Abelian integral $p(x) = \int^x dp$ since its value also jumps across cuts $C_I$ by $2\pi n_I, n_I \in \mathbb{Z}$. It also has simple poles but this time they are at $x = \pm 1$ rather than $x = 0$. This is because to compare with the gauge theory one needs to scale the spectral parameter on the string theory side by setting $\tilde{x} = \frac{\sqrt{\lambda}}{\sqrt{2\pi J}} x$ so that $p(\tilde{x})$ now has poles at $\tilde{x} = \pm \sqrt{T}$ where $T \equiv \frac{\lambda}{16\pi^2 J^2}$. In the limit $\frac{\lambda}{J^2} \to 0$ the string theory then exactly reproduces the one-loop gauge theory result [23].

As we discussed above, by virtue of quantum integrability the one-loop planar dilatation operator $\hat{H}_{\text{XXX} \frac{1}{2}}$ belongs to a whole family of commuting operators encoded in the transfer matrix $\hat{T}(u)$. Likewise, as we will see in chapter 5 on the string theory side the energy is the first member of a whole hierarchy of conserved Poisson commuting charges encoded in a classical analogue $\text{tr} \Omega(x)$ of the transfer matrix. Now by construction, a distribution of Bethe roots characterises an eigen-operator of $\hat{T}(u)$ and an algebraic curve $(\hat{\Sigma}, dp)$ characterises a classical string solution. Therefore the matching of the classical string theory algebraic curve with the thermodynamic limit $(J \to \infty)$ of the one-loop Bethe root distribution provides

![Figure 2: Cut representation of a genus two Riemann surface.](image-url)
a complete check in the $SU(2)$ sector of the equality between the coefficients $c_1$ in the expansion (0.2.1) for the spectrum of the quantum operator $\hat{T}(u)$ on the one hand and the range of the classical phase-space function $\text{tr} \Omega(x)$ on the other. The construction of the algebraic curve was later generalised to the $SO(6)$ sector [27], to the non-compact $SL(2, \mathbb{R})$ sector [28] and eventually to the full supersymmetric case [29]. This curve was then successfully compared in [30] against the full spectrum of $\mathcal{N} = 4$ SYM single-trace operators in the Frolov-Tseytlin limit.

To take the comparison to the next order in $\frac{1}{T} \propto T$ it was shown in [23] that a further change of spectral parameter was necessary on the gauge theory side. If one first renames the spectral parameter $x$ as $\tilde{z}$, so that equations such as (0.3.5) now read the same with the relabelling $x \to \tilde{z}$,

$$S_I = \frac{1}{2\pi i} \oint_{\mathcal{A}_I} p(\tilde{z})d\tilde{z}, \quad I = 1, \ldots, K \quad (0.3.8)$$

then the change of spectral parameter $\tilde{z} \mapsto \tilde{x}$ required to match the string theory results (expressed in terms $\tilde{x} = \sqrt{T}x$) is defined by the Zhukovsky map

$$\tilde{z} = \tilde{x} + \frac{T}{\tilde{x}}. \quad (0.3.9)$$

This can also be written as $z = x + \frac{1}{x}$ in terms of the unscaled variables $x = \frac{1}{\sqrt{T}}\tilde{x}$ and $z = \frac{1}{\sqrt{T}}\tilde{z}$. As we will show in chapter 8 the spectral parameter $z = x + \frac{1}{x}$ is in fact the natural choice on the string theory side since it brings the symplectic structure to the canonical Darboux form. Furthermore, the filling fractions are also naturally expressed in terms of it, as in (0.3.8). With this change of variables the two-loop gauge theory result was shown to exactly match the next order in $T \propto \frac{1}{T}$ of the classical string theory algebraic curve (see [23, p27] for details). This provides a test of the correspondence in the $SU(2)$ sector at the level of the coefficient $c_2$ in the expansion (0.2.1). Despite this perfect agreement at two-loop,
the next coefficient $c_3$ in the expansion (0.2.1) on both sides of the correspondence were found to disagree, which has become known as the ‘three-loop discrepancy’ [31]. This mismatch however is not in conflict with the AdS/CFT correspondence and can be attributed to an order-of-limits effect [32, 33]. Indeed, on the string theory side one takes the classical limit $1/J \to 0$ before expanding in $\lambda' \equiv \lambda/J^2$ whereas on the gauge theory side the perturbation expansion in $\lambda$ precedes the expansion in $1/J$. In other words, the procedures described in section 0.2 for testing the AdS/CFT correspondence rely on the assumption that the following diagram [32]

$$
\delta(\lambda, J) = E(\lambda, J) \xrightarrow{J \to \infty} E(\lambda') \\
\delta_n(J) \xrightarrow{J \to \infty} \delta_n \neq E_n
$$

is commutative. Yet, assuming the AdS/CFT correspondence holds, the mismatch $\delta_3 \neq E_3$ at three-loop clearly shows otherwise and with hindsight the agreement for the coefficients $c_1$ and $c_2$ seems quite fortuitous.

One way to circumvent this difficulty would be to directly quantise string theory on $AdS_5 \times S^5$. The main objective of the work presented in this thesis was to make a step towards obtaining the leading semiclassical corrections to the string spectrum and possibly gain some insight in view of one day performing an exact quantisation of string theory on $AdS_5 \times S^5$. The more modest task of obtaining the semiclassical string spectrum would provide the set of coefficients $c_{n1}$ in the expansion (0.2.1) from the string theory side. These could then be perturbatively tested against the corresponding coefficients obtained from the gauge theory side. In this short introduction we have mostly been concerned with the $SU(2)$ sector corresponding classically to bosonic strings moving in an $\mathbb{R} \times S^3$ submanifold of $AdS_5 \times S^5$. This restriction is legitimate because at the classical level it is a consistent truncation of the full superstring theory on $AdS_5 \times S^5$. At the quantum level however,
even if we semiclassically quantise a solution in the subspace \( \mathbb{R} \times S^3 \) we know that quantum fluctuations will leave this subspace and so quantum mechanically one ought to consider the full target-space \( AdS_5 \times S^5 \). Despite this, in this thesis we will continue focusing on the subspace \( \mathbb{R} \times S^3 \subseteq AdS_5 \times S^5 \) as a toy model. The reason for doing this is that the \( SU(2) \) subsector is the only one for which the complete set of solutions is explicitly know \([1, 2]\), which is a necessary prerequisite for performing a semiclassical study of any system.

### 0.4 Outline of the thesis

**Part I** The first two chapters of this thesis contain all the necessary background material on the theory of Riemann surfaces \([34–42]\) and semiclassical quantisation of finite-dimensional systems \([43–52]\) required for Parts \( \text{III} \) and \( \text{IV} \) respectively. Since the theory of Riemann surfaces plays such an important role in Part \( \text{III} \) for completeness we cover the relevant aspects of it in some detail in chapter \( \text{I} \).

**Part II** In chapter \( \text{III} \) we give a review of bosonic strings theory on \( \mathbb{R} \times S^3 \) from the Lagrangian point of view and express it in terms of the \( SU(2) \) principal chiral model subject to the Virasoro constraints. In chapter \( \text{I} \) we rephrase everything from the Hamiltonian perspective discussing the implementation of the Virasoro and static gauge constraints in the Dirac formalism. Finally, in chapter \( \text{V} \) we tackle the question of integrability of bosonic strings on \( \mathbb{R} \times S^3 \). We start by reviewing the construction of the Lax connection and monodromy matrix in section \( \text{V.1} \) and the extraction of the local conserved charges in section \( \text{V.2} \). Section \( \text{V.3} \) is based on \([2]\) in which we show that the integrals of motion previously obtained are also in involution. This is the complete statement of integrability of string theory on \( \mathbb{R} \times S^3 \). We then exploit this in section \( \text{V.4} \) to construct the integrable hierarchy of the string as in \([3]\).

**Part III** In this Part we put to full use the integrability unveiled in Part \( \text{II} \) to
construct the general solution to the equations of motion for a string on $\mathbb{R} \times S^3$ following [1–3] as well as [4] for the last section. Section 6 is a review of the construction of the KMMZ curve [23] encoding the integrals of motion of a finite-gap solution. We show in section 7 that the reconstruction of the solution requires additional data, namely a finite set of points on the KMMZ curve. This completes the set of so-called algebro-geometric data. We express the general finite-gap solution explicitly in terms of this data using Riemann $\theta$-functions on the curve. In section 8 we derive the restriction of the symplectic structure of the string to the algebro-geometric data. The resulting finite-dimensional symplectic structure is canonical if the spectral parameter used is given by the Zhukovsky map. We then perform a standard change of variables to action-angle variables, obtaining explicit expressions for these in terms of the algebro-geometric data. In section 9 we discuss the necessary constraints on the data to obtain physical finite-gap solutions. In particular we derive the reality conditions on the KMMZ curve, showing that all the branch points must lie off the real axis in the $SU(2)$ sector.

Part IV In chapter 10 we use the knowledge of classical solutions acquired in Part III to perform a semiclassical analysis of bosonic string theory on $\mathbb{R} \times S^3$ from first principles. We derive a general and simple formula for extracting the fluctuation energies from the KMMZ curve in terms of a well defined meromorphic differential on the curve, namely the quasi-energy. We use these fluctuation energies to show formally (without regularising) that their sum leads to the discretisation of the KMMZ curve in the sense that all the fillings get half-integer quantised, including those of the singular points which are classically empty. The calculation therefore serves as a toy model for understanding from the finite-gap perspective the origin of the discretisation of the algebraic curve when leading order semiclassical corrections are included.
Figure 3: Chapter dependence guideline.
Part I

Background
Chapter 1

Riemann surfaces

“Donuts. Is there anything they can’t do?”

Homer Simpson

This chapter is intended as a self contained review, based on [34–42], of those aspects from the theory of Riemann surfaces relevant to Part III of this thesis. The most important concepts and results required in the theory of finite-gap integration are found in section 1.5. Section 1.6 is a discussion of singular algebraic curves which are fundamental to chapters 6 and 10. Finally, section 1.7 discusses the relation of a curve to its Jacobian, an object of great importance in Parts III and IV.

1.1 Definition & Examples

Consider a real two-dimensional (connected) topological manifold $M$, that is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^2$, and let \( \{U_{\alpha}\}_{\alpha \in A} \) be an open cover of $M$, i.e. $\cup_{\alpha \in A} U_{\alpha} = M$. Then the fact that $M$ is locally homeo-
omorph to \( \mathbb{R}^2 \) means we can find homeomorphisms \( z_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^2 \) called \textbf{local charts} from each \( U_\alpha \) to open subsets \( V_\alpha \subset \mathbb{R}^2 \). We are interested in doing complex analysis on \( M \) and so we use the homeomorphisms \( z_\alpha \) to locally equip \( M \) with the analytic structure of \( V_\alpha \subset \mathbb{R}^2 \simeq \mathbb{C} \). For instance, a function \( f : U_\alpha \to \mathbb{C} \) will be called \textbf{holomorphic} if \( f \circ z_\alpha^{-1} : \mathbb{C} \to \mathbb{C} \) is a holomorphic map in the usual sense. But for this analytic structure to have any meaning globally on \( M \) we need a compatibility condition between charts on overlapping sets \( U_\alpha \cap U_\beta \neq \emptyset \) ensuring that \( f \circ z_\alpha^{-1} \) is holomorphic iff \( f \circ z_\beta^{-1} \) is, for any \( f : U_\alpha \cap U_\beta \to \mathbb{C} \). Thus we say that two charts \( (U_\alpha, z_\alpha) \) and \( (U_\beta, z_\beta) \) are \textbf{(holomorphically) compatible} if

\[
t_{\alpha\beta} = z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)
\]
called the \textbf{transition function}, is holomorphic as a function from \( \mathbb{C} \) to \( \mathbb{C} \), \( c.f. \) for a differentiable manifold \( t_{\alpha\beta} \) is required to be differentiable. If the charts \( \{(U_\alpha, z_\alpha)\}_{\alpha \in \mathcal{A}} \) are all compatible they are said to form a \textbf{complex atlas} \( \mathcal{A} \) and two complex atlases \( \mathcal{A}, \tilde{\mathcal{A}} \) are compatible if \( \mathcal{A} \cup \tilde{\mathcal{A}} \) is a complex atlas. Any atlas \( \mathcal{A} \) can be extended to a \textbf{maximal atlas} \( \overline{\mathcal{A}} \) consisting of all charts compatible with \( \mathcal{A} \). A maximal atlas is also called a \textbf{complex structure}.

\textbf{Definition 1.1.1.} A \textbf{Riemann surface} is a real two-dimensional (complex one-
dimensional) connected manifold $M$ equipped with a complex structure.

**Remark** One great advantage of working with a Riemann surface as opposed to simply dealing with the underlying two-dimensional differentiable manifold is that one can apply all the local concepts and powerful theorems of complex analysis using the local homeomorphisms with $\mathbb{C}$. However, just as with differentiable manifolds, these local homeomorphisms are not canonical because they depend on the choice of chart $z_\alpha$, and so the only objects one can consider on a Riemann surface are ones whose definitions are chart invariant.

**Examples**

The following are basic examples of Riemann surfaces that will be important later:

- Any connected open domain $U \subset \mathbb{C}$ equipped with a single chart $(U, id)$.
- The Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ (the one-point compactification of $\mathbb{C}$) equipped with two charts $(U_1, z_1), (U_2, z_2)$

  
  
  $$
  U_1 = \mathbb{C}, \quad z_1 = z, \quad \text{and} \quad U_2 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}, \quad z_2 = 1/z,
  $$

  

  with holomorphic transition functions $t_{12}, t_{21} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \quad z \mapsto 1/z$.

- Any non-singular algebraic curve $C \subset \mathbb{C}^2$ defined by the zero-locus

  $$
  C = \{(x, y) \in \mathbb{C}^2 | P(x, y) = 0\}
  $$

  of a polynomial $P$ in $x$ and $y$. The non-singular criteria means that $\partial P/\partial x$ and $\partial P/\partial y$ never both vanish on $C$. By the implicit function theorem the variable $y$ (resp. $x$) can be taken as a local chart near points where $\partial P/\partial x \neq 0$ (resp. $\partial P/\partial y \neq 0$) and $x(y)$ (resp. $y(x)$) is analytic so this defines a complex structure on $C$. 

Remark: In the neighbourhood of a singular point \((x, y) \in C\), the curve \(C\) looks like an intersection of several complex-lines and so there is no neighbourhood of \((x, y)\) locally homeomorphic to \(C\). When encountering singular algebraic curves we will therefore have to desingularise them by a process to be explained later.

### 1.2 Holomorphic maps

**Definition 1.2.1.** A continuous mapping 

\[
f : M \to N
\]

between Riemann surfaces is called **holomorphic** (or **analytic**) if for every local chart \((U, z)\) on \(M\) and every local chart \((V, w)\) on \(N\) with \(U \cap f^{-1}(V) \neq \emptyset\), the mapping

\[
w \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to w(V)
\]

is holomorphic as a map from \(\mathbb{C}\) to \(\mathbb{C}\).

**Remark:** This definition is independent of the choice of charts \(z\) and \(w\) by holomorphicity of the transition functions to another set of charts \(z'\) and \(w'\). Moreover, because holomorphicity is a local concept, all the usual local properties of holomorphic functions on \(\mathbb{C}\) will persist for holomorphic maps. For instance, any holomorphic map \(f\) is open, i.e. \(f\) sends open sets \(U \subset M\) to open sets \(f(U) \subset N\).

A holomorphic mapping into \(\mathbb{C}\) is called a **holomorphic function.** A holomorphic mapping into \(\mathbb{C} \cup \{\infty\}\) is called a **meromorphic function.** The ring of holomorphic functions on \(M\) is denoted by \(\mathcal{H}(M)\) and the field of meromorphic functions on \(M\) by \(\mathcal{K}(M)\).
Local behaviour

A holomorphic function \( f : M \to N \) is locally injective around all but finitely many points of \( M \). That is, there exists a finite collection of points \( P_1, \ldots, P_n \in M \) such that for all other points \( P \in M \setminus \{P_1, \ldots, P_n\} \) the restriction \( f|_U \) to a neighbourhood \( U \subset M \setminus \{P_1, \ldots, P_n\} \) of \( P \) is injective. The points \( P_1, \ldots, P_n \) around which \( f \) fails to be locally injective are called branch points. These statements are made precise by the following Lemma:

**Lemma 1.2.2.** Let \( f : M \to N \) be a holomorphic map and \( P \in M \). Then there exists local charts \((U, z), (V, w)\) near \( P \in U, f(P) \in V \) such that \( F \equiv w \circ f \circ z^{-1} \) is given by

\[
F(z) = z^k, \quad k \in \mathbb{N}.
\]

**Proof.** Choose local charts \( \tilde{z} \) on \( M \) vanishing at \( P \) and \( w \) on \( N \) vanishing at \( f(P) \). Now \( F \) is holomorphic with \( F(0) = 0 \) so we can write it as \( F(\tilde{z}) = \tilde{z}^k g(\tilde{z}) \) for some \( g \) holomorphic with \( g(0) \neq 0 \). Since \( g \) is non-vanishing on a disc around the origin it has a \( k \)th root and so \( g(\tilde{z}) = h(\tilde{z})^k \). Defining a new coordinate \( z = \tilde{z} h(\tilde{z}) \) the result follows. \( \square \)

Thus a holomorphic map locally looks like the map \( z \mapsto z^k \). Hence in a small neighbourhood \( U \ni P \) the number of solutions to the equation \( f(Q) = R \) when \( R \in N \) approaches \( f(P) \) is \( k \). We see that the number \( k \) appearing in Lemma 1.2.2 has an invariant geometrical meaning for the map \( f \) and cannot depend of the choice of chart used to represent \( f \). It is called the **valency** or the **ramification number** of \( f \) at \( P \in M \). The number \( b_f(P) = k - 1 \) is called the **branch number** of \( f \) at \( P \in M \).

**Definition 1.2.3.** A point \( P \in M \) for which \( b_f(P) > 0 \) is called a **branch point** of \( f \).
Lemma 1.2.4. The branch points of a holomorphic map \( f : M \to N \) are isolated.

Proof. Let \( P \in M \) be a branch point of \( f \). Then by Lemma 1.2.2 there exists a neighbourhood \( U \ni P \) and coordinate \( z \) with \( z(P) = 0 \) for which \( f \) takes the local form \( F(z) = z^k, k > 1 \). But the map \( z \mapsto z^k \) is locally injective for \( z \neq 0 \) so \( b_f(Q) = 0 \) for any \( Q \in U \setminus \{ P \} \).

Corollary 1.2.5. If \( M \) is compact, then \( f : M \to N \) has finitely many branch points.

Global behaviour

The local property that a holomorphic map is open (which follows from Lemma 1.2.2) implies a far reaching global property of holomorphic maps on compact Riemann surfaces:

Theorem 1.2.6. Let \( M \) be compact and \( f : M \to N \) a non-constant holomorphic map. Then \( f \) is surjective (\( f(M) = N \)) and \( N \) is compact.

Proof. Since \( f \) is not constant, \( f(M) \) is open (a holomorphic mapping is open). But \( M \) is compact so \( f(M) \) is compact (the continuous image of a compact set is compact) and hence closed (a compact subset of a Hausdorff space is closed). So \( f(M) \) is a non-empty open and closed subset of \( N \), and since \( N \) is connected we have \( f(M) = N \).

In fact one can be a lot more precise. Not only is any \( Q \in N \) attained by \( f : M \to N \), but every \( Q \in N \) is assumed the same number of times, counting multiplicities.

Theorem 1.2.7. Let \( f : M \to N \) be a non-constant holomorphic function with \( M, N \) compact. Then there exists \( m \in \mathbb{N} \) such that for any \( Q \in N \) the equation
1.2. HOLOMORPHIC MAPS

\( f(P) = Q \) has precisely \( m \) solutions (counting multiplicities), i.e.

\[ \sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m, \quad \forall Q \in N. \]

Proof. Let \( Q \in N \). By Theorem 1.2.6 the equation \( f(P) = Q \) has at least one solution. The number of solutions \( m(Q) \) is finite because otherwise they would accumulate in \( N \) and hence \( f \) would be the constant map \( f : M \to Q \) (since a non-zero holomorphic function has isolated zeroes). Now by Lemma 1.2.2 there exists neighbourhoods \( V_Q \) of \( Q \) and \( U_i \) of \( P_i \in f^{-1}(Q) \) with respect to which \( f \) is of the local form \( z \mapsto z^{b_f(P_i) + 1} \) in \( U_i \). Since \( z \mapsto z^k \) has \( k \) zeroes near \( z = 0 \) it follows that \( m(Q) = \sum_i (b_f(P_i) + 1) \) is constant in \( V_Q \). By compactness one can cover \( N \) by finitely many \( V_Q \) and so \( m(Q) \) remains constant over \( N \). \( \square \)

We say that \( f : M \to N \) is an \textit{m-sheeted ‘branched’ covering} of \( N \), referring to the fact that branch points are the multiple solutions of \( f(P) = Q \), see Figure 1.2

![Figure 1.2: Branched covering.](image)

**Definition 1.2.8.** The number \( m \) is called the \textbf{degree} of \( f \) and we write \( m = \deg f \).

Applying Theorem 1.2.7 with \( N = \mathbb{C} \cup \{\infty\} \) implies that a non-constant meromorphic function \( f : M \to \mathbb{C}P^1 \) on a compact Riemann surface \( M \) assumes every value in \( \mathbb{C}P^1 \) the same number of times. In particular, \( f \) has as many zeroes as poles, provided they are counted correctly with multiplicities.
Remark. A single non-constant meromorphic function $f : M \to \mathbb{C}P^1$ completely determines the complex structure of $M$. Indeed, using Lemma 1.2.2 and the charts of $\mathbb{C}P^1$, a local chart vanishing at $P_0 \in M$ is constructed as follows (with $n = b_f(P_0) + 1$)

$$z(P) = (f(P) - f(P_0))^\frac{1}{n} \text{ if } f(P_0) \neq \infty, \quad \text{or} \quad z(P) = f(P)^{-\frac{1}{n}} \text{ if } f(P_0) = \infty.$$  

1.3 Topology

In this section we temporarily forget about the complex structure of Riemann surfaces and describe their topologies as real two-dimensional topological manifolds. Accordingly, all the charts on a surface $M$ in this section are homeomorphisms into $\mathbb{R}^2$, that is $z_\alpha : U_\alpha \to \mathbb{R}^2$. As before we still assume the surface is connected and hence path connected.

**Definition 1.3.1.** A manifold $M$ is orientable if there exists an atlas $(U_\alpha, z_\alpha)$ such that the transition functions $t_{\alpha\beta} = z_\beta \circ z_\alpha^{-1}$ preserve orientation.

**Proposition 1.3.2.** Every Riemann surface is orientable.

**Proof.** Holomorphic functions preserve orientation since by the Cauchy-Riemann equations the Jacobian of such a transformation $(x, y) \mapsto (x', y')$ is positive,

$$dx' \wedge dy' = \left[ \frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \frac{\partial y}{\partial x'} \right] dx \wedge dy' = \left[ \left( \frac{\partial x}{\partial x'} \right)^2 + \left( \frac{\partial x}{\partial y'} \right)^2 \right] dx' \wedge dy'.$$

$\square$

The following theorem and corollary give a complete classification of the possible topologies for a Riemann surface. The proof of Theorem 1.3.3, which we omit,
usually relies on the fact that every compact surface is triangulable [34] and proceeds by cutting and gluing the triangulation to arrive at the final desired polygon form:

**Theorem 1.3.3.** [34] Every compact orientable surface $M$ is homeomorphic either to the sphere $S^2$ or to a polygon with $4g$ edges $(\tilde{a}_i, \tilde{a}_i', \tilde{b}_i, \tilde{b}_i', i = 1, \ldots, g)$ identified pairwise in such a way that the orientations of these edges with respect to $M$ are opposite ($\tilde{a}_i' = \tilde{a}_i^{-1}, \tilde{b}_i' = \tilde{b}_i^{-1}$) and with all vertices identified.

**Remark** The $4g$-gon described by Theorem 1.3.3 is a lift of $M$ to its universal covering space $\tilde{M}$. We shall denote it $M_{\text{cut}}$ since it can be obtained from $M$ by cutting along certain cycles. The identification process described in Theorem 1.3.3 corresponds to applying the covering map $\tilde{\pi}: \tilde{M} \to M$, in other words $\tilde{\pi}(M_{\text{cut}}) \approx M$. The simply connected domain $M_{\text{cut}}$ will come in handy later for defining branches of multi-valued functions on $M$ and so we give it a name:

**Definition 1.3.4.** The $4g$-gon $M_{\text{cut}}$ of Theorem 1.3.3 is called the **normal form** of $M$.

In its normal form representation, the topology of $M$ is not very transparent since the edges and vertices still need to be identified following the prescription in Theorem 1.3.3. The next corollary describes the closed surface resulting from these identifications.

**Corollary 1.3.5.** [34] Every compact orientable surface $M$ is homeomorphic to a sphere with $g$ handles, that is to $S^2$ when $g = 0$ or to the $g$-fold connected sum of torii $T^1 \# T^1 \# \ldots \# T^1$ when $g \geq 1$.

**Proof.** Using Theorem 1.3.3 we just have to show that the normal form is homeomor-
phic to a \( g \)-fold connected sum of torii (a \textit{\( g \)-fold torus}). We proceed by induction on \( g \). We start by cutting the \( 4g \)-gon into two polygons. The first has the 4 edges \( \tilde{a}_1, \tilde{b}_1, \tilde{a}_1^{-1}, \tilde{b}_1^{-1} \) and a new edge \( \tilde{c} \). The second has the \( 4(g-1) \) remaining edges and the edge \( \tilde{c}^{-1} \).

Next we make the identification of edges and vertices in each of these two polygons using the induction hypothesis. We end up on the one hand with a torus with a disc cut out, whose boundary is \( \tilde{c} \), and on the other hand a \((g-1)\)-fold torus with a disc cut out, whose boundary is \( \tilde{c}^{-1} \).

It is clear from the last figure that gluing the \( \tilde{c} \) cycles back together corresponds to taking the connected sum of the torus with the \((g-1)\)-fold torus, which results in a \( g \)-fold torus. \( \square \)

\textbf{Definition 1.3.6.} The topological invariant \( g \in \mathbb{N} \) is called the \textit{genus} of \( M \).

\textbf{Fundamental group}

A curve \( \gamma \) in \( M \) is a continuous map \( \gamma : [0, 1] \to M \). It \textit{starts} at \( \gamma(0) \) and \textit{ends} at \( \gamma(1) \). If the start and end points coincide \( \gamma(0) = \gamma(1) \) then there is a natural
multiplication between closed curves starting and ending at $P \in M$, namely
\[
\gamma_1 \cdot \gamma_2(t) = \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t - 1) & \frac{1}{2} \leq t \leq 1. 
\end{cases}
\] (1.3.1)

If we allow reparametrisations of curves ($\gamma \mapsto \gamma \circ s$ where $s : [0, 1] \to [0, 1]$ with $\dot{s} > 0$), the above product has an obvious identity, $\iota : [0, 1] \to P$ and every curve $t \mapsto \gamma(t)$ has as inverse the same curve traversed in the opposite direction, $t \mapsto \gamma(1 - t)$. The resulting group however is far too big and not very useful. One can reduce its size considerably by taking a quotient:

**Definition 1.3.7.** Two curves $\gamma_1$ and $\gamma_2$ in $M$ both starting at $P$ and ending at $Q$ are called homotopic if there exists a continuous map $\gamma : [0, 1] \times [0, 1] \to M$ such that
\[
\gamma(t, 0) = \gamma_1(t), \quad \gamma(t, 1) = \gamma_2(t), \\
\gamma(0, \lambda) = P, \quad \gamma(1, \lambda) = Q.
\]
The equivalence class of curves homotopic to a given curve $\gamma$ is called its homotopy class and denoted $[\gamma]_\pi$.

The definition now descends to the quotient and gives a well-defined product between homotopy classes of curves based at $P$ by $[\gamma_1]_\pi \cdot [\gamma_2]_\pi = [\gamma_1 \cdot \gamma_2]_\pi$. The identity corresponds to the equivalence class $[\iota]_\pi$ of curves homotopic to the point $P$. With this multiplication, the homotopy classes of curves based at $P$ thus form a group denoted $\pi_1(M, P)$. Since $M$ is always assumed to be connected, any two points $P, Q \in M$ are connected by a path and the groups $\pi_1(M, P)$ and $\pi_1(M, Q)$ are isomorphic, although the isomorphism is not canonical because it depends on the homotopy class of the path joining $P$ and $Q$.

**Definition 1.3.8.** The fundamental group $\pi_1(M)$ of $M$ is the group $\pi_1(M, P)$ with any choice of base point $P \in M$. 
Let us now identify the fundamental group of a compact orientable surface $M$ by making use of the normal form representation $M_{\text{cut}}$ of Theorem 1.3.3 which lives in the universal cover $\tilde{M}$. Let $P \in M$ be the common image of the vertices of the $4g$-gon under the covering map $\tilde{\pi} : \tilde{M} \to M$. Define the projections $a_i := \tilde{\pi}(\tilde{a}_i)$ and $b_i := \tilde{\pi}(\tilde{b}_i)$ for $i = 1, \ldots, g$ under $\tilde{\pi}$ of all the edges of the $4g$-gon. These are all closed loops in $M$ through $P$ so it is convenient to use the base point $P$ to determine the fundamental group.

**Theorem 1.3.9.** $\pi_1(M)$ is generated by $[a_i]_\pi, [b_i]_\pi, i = 1, \ldots, g$ subject to the single relation

$$\prod_{i=1}^{g} [a_i]_\pi \cdot [b_i]_\pi \cdot [a_i]^{-1}_\pi \cdot [b_i]^{-1}_\pi = 1. \quad (1.3.2)$$

**Proof.** The lift $\tilde{c}$ to $\tilde{M}$ of any closed loop $c$ through $P$ is a sum of paths in $M_{\text{cut}}$ starting and ending on vertices. Such paths can clearly be retracted to portions of the boundary $\partial M_{\text{cut}}$ which is spanned by the edges $\tilde{a}_i, \tilde{a}_i^{-1}, \tilde{b}_i, \tilde{b}_i^{-1}, i = 1, \ldots, g$. We deduce that the homotopy classes $[a_i]_\pi, [b_i]_\pi, i = 1, \ldots, g$ generate the fundamental group $\pi_1(M, P) \approx \pi_1(M)$. The existence of a non-trivial lift $\tilde{c}$ which is contractible to an arbitrary point in $M_{\text{cut}}$ gives rise to a relation amongst these generators. Since $M_{\text{cut}}$ is simply connected, the only such cycle is $\partial M_{\text{cut}} = \prod_{i=1}^{g} \tilde{a}_i \cdot \tilde{b}_i \cdot \tilde{a}_i^{-1} \cdot \tilde{b}_i^{-1}$ which leads to the relation $[\partial M_{\text{cut}}]_\pi = 1$, namely (1.3.2). \qed

**First homology group**

A triangulation of $M$ consists of oriented vertices, edges and faces called 0-, 1- and 2-simplices respectively. A **0-simplex** is a point $P$ with an orientation, so either $(P)$ or $-(P)$. A **1-simplex** is a segment with endpoints $P_1, P_2$ and one of two possible orientations, either $(P_1, P_2)$ or $(P_2, P_1) = -(P_1, P_2)$. A **2-simplex** is a triangle with vertices $P_1, P_2, P_3$ and one of two possible orientations, either
1.3. TOPOLOGY

$(P_1, P_2, P_3)$ or $(P_1, P_3, P_2) = -(P_1, P_2, P_3)$. Formal sums $\sum m_i t_i$ ($m_i \in \mathbb{Z}$) of $n$-simplices $t_i$ are called $n$-chains and form a free abelian group $C_n(M)$ under addition. The requirement that simplices be oriented ensures that $C_n(M)$ is indeed a group, where the negative $-t_i$ is the simplex $t_i$ taken with opposite orientation.

One can define a natural sequence of boundary operations $\partial_n$ (all denoted $\partial$ when there is no ambiguity)

$$0 \xrightarrow{\partial_1} C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0, \quad (1.3.3)$$

given explicitly on 0-, 1- and 2-simplices by

$$\partial(P) = 0, \quad \partial(P_1, P_2) = (P_1) - (P_2), \quad \partial(P_1, P_2, P_3) = (P_1, P_2) + (P_2, P_3) + (P_3, P_1), \quad (1.3.4)$$

and extended to 0-, 1- and 2-chains by linearity. We define the subgroups of boundaries and cycles as $B_n(M) = \text{im}(\partial_{n+1}) = \partial C_{n+1}(M)$, and $Z_n(M) = \ker(\partial_n) = \{c \in C_n(M) | \partial c = 0\}$ respectively. It follows that the homomorphism $\partial_{n+1} : C_{n+1}(M) \rightarrow B_n(M)$ is surjective with kernel $Z_{n+1}(M)$ so $B_n(M) \approx C_{n+1}(M)/Z_{n+1}(M)$. It is trivial to check using (1.3.4) that $\partial^2 = \partial_n \partial_{n+1} = 0$ so that $B_n(M) \subset Z_n(M)$. Since these groups are abelian, $B_n(M)$ is normal in $Z_n(M)$ and their quotient $H_n(M) = Z_n(M)/B_n(M)$ is a group, called the $n$th homology group. It measures the deviation from exactness at the $n$th site of the sequence (1.3.3).

Now given $P \in M$, by definition $\partial(P) = 0$ so $Z_0(M) = C_0(M)$. But since any two points $P, Q \in M$ are related by a boundary $(P) = (Q) + \partial(P, Q)$ this means that $H_0(M)$ is generated by a single point $(P)$ and hence $H_0(M) = (P) \cdot \mathbb{Z} \approx \mathbb{Z}$. Next suppose the 2-chain $c = \sum m_i t_i$ is without boundary, $\partial c = 0$. Then $m_i = m_j$ when two triangles $t_i, t_j$ in the sum have adjacent edges. Since $c$ must be connected it follows that all the $m_i$ are equal so $Z_2(M)$ is generated by $M = \sum t_i$. Since also
\( B_2(M) = \{0\} \) it follows that \( H_2(M) = Z_2(M) \approx \mathbb{Z} \). From now on we focus on the remaining homology group.

**Definition 1.3.10.** The **first homology group** of \( M \) is defined as \( H_1(M) = Z_1(M)/B_1(M) \).

**Remark** \( H_1(M) \) can be shown not to depend on the triangulation used for \( M \). Therefore from now on the word ‘curve’ will refer to both continuous maps \( \gamma : [0,1] \to M \) and to 1-chains, the word ‘closed curve’ refers to continuous maps with \( \gamma(0) = \gamma(1) \) as well as 1-cycles and we use the word ‘boundary’ to designate curves which are 1-dimensional boundaries of domains in \( M \).

**Definition 1.3.11.** Two closed curves \( \gamma_1 \) and \( \gamma_2 \) in \( M \) are said to be **homologous** if

\[
\gamma_1 - \gamma_2 \in B_1(M).
\]

The equivalence class of curves homologous to a given \( \gamma \) is called its **homology class** which is an element of \( H_1(M) \) denoted \([\gamma]\) .

We now have two different equivalence relations on closed curves of \( M \): homotopy equivalence (Definition 1.3.7) on the one hand and homology equivalence (Definition 1.3.11) on the other. An obvious question to ask is whether or not these are related. It is obvious that homotopic curves \( \gamma_1, \gamma_2 \) are homologous since the homotopy is a continuous map \( \gamma : [0,1] \times [0,1] \to M \) which defines a tubular cobordism on \( M \) joining \( \gamma_1 \) and \( \gamma_2 \). The converse is false however since the cobordism from \( \gamma_1 \) to \( \gamma_2 \) can be more general (Figure 1.3). Recall from their respective definitions that the fundamental group \( \pi_1(M) \) is non-abelian whereas the first homology group \( H_1(M) \) is abelian. As it turns out the first homology group \( H_1(M) \) is the abelianisation of the fundamental group \( \pi_1(M) \). Specifically, defining the **commutator subgroup** \([\pi_1(M), \pi_1(M)] = \langle a \cdot b \cdot a^{-1} \cdot b^{-1} | a, b \in \pi_1(M) \rangle \) we have,
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Figure 1.3: Example of homologous cycles $\gamma_1, \gamma_2$ that are not homotopic.

Theorem 1.3.12. $H_1(M) \approx \pi_1(M)/[\pi_1(M), \pi_1(M)]$.

Proof. Since two homotopic curves are homologous, the map $\varphi : \pi_1(M) \to H_1(M), \ [\gamma]_\pi \mapsto [\gamma]$ is well defined. It is clearly a homomorphism since $\varphi([\gamma_1]_\pi \cdot [\gamma_2]_\pi) = \varphi([\gamma_1 \cdot \gamma_2]_\pi) = [\gamma_1 \cdot \gamma_2] = [\gamma_1] + [\gamma_2]$. Moreover $\varphi$ is surjective. Its kernel consists of $[\gamma]_\pi$ such that $[\gamma] = 0$. That is, $\gamma^{-1}$ can be chosen (up to homology) as the single boundary of a surface of genus $n$ (arbitrary) with a small disc removed.

The boundary $\gamma^{-1} \cdot \prod_{i=1}^{n} a_i \cdot b_i \cdot a_i^{-1}b_i^{-1}$ of the normal form of this surface being homotopic to a point implies $[\gamma]_\pi = \prod_{i=1}^{n} [a_i]_\pi \cdot [b_i]_\pi \cdot [a_i]_\pi^{-1} [b_i]_\pi^{-1} \in [\pi_1(M), \pi_1(M)]$. □

With the knowledge of the fundamental group it is now easy to compute the first homology group by the process of ‘abelianising’, that is, forgetting about the order in which we multiply cycles. In fact, since the relation (1.3.2) on the generators of $\pi_1(M)$ becomes trivial in the abelian case, the constraint disappears for $H_1(M)$ and we have

Corollary 1.3.13. $H_1(M)$ is the free abelian group generated by $[a_i], [b_i], i = 1, \ldots, g$.

Definition 1.3.14. The rank of $H_n(M)$ is called the Betti number and is denoted $b_n$. The Euler characteristic of $M$ is defined as $\chi(M) = b_0 - b_1 + b_2$. 

Remark The Euler characteristic is a topological invariant of $M$. Since $H_0(M) \approx \mathbb{Z}$ and $H_2(M) \approx \mathbb{Z}$ we have that $b_0 = b_2 = 1$. Furthermore, Corollary 1.3.13 implies $b_1 = 2g$. This leads to a simple expression for $\chi(M)$ in terms of the other known topological invariant, the genus $g$.

**Corollary 1.3.15.** A compact Riemann surface $M$ of genus $g$ has $\chi(M) = 2 - 2g$.

**Coverings**

In Theorem 1.2.7 we saw that every non-constant holomorphic map $f : M \to N$ between compact Riemann surfaces $M, N$ was a branched covering of $N$. In the present section we will give a topological property of branched coverings relating the Euler characteristics (and hence the genus) of the two surfaces $M, N$.

There is a simple way of computing the Euler characteristic of a compact surface $M$ using a triangulation of $M$.

**Proposition 1.3.16.** If a triangulation of $M$ has $F$ faces, $E$ edges and $V$ vertices then

$$\chi(M) = F - E + V.$$ 

**Proof.** Let $c_n = \text{rank } C_n(M)$, the number of $n$-simplices, and $z_n = \text{rank } Z_n(M)$. Then $\text{rank } B_n(M) = c_{n+1} - z_{n+1}$ so that $b_n = \text{rank } H_n(M) = z_n - c_{n+1} + z_{n+1}$. Hence $\chi(M) = z_0 - c_1 + c_2$, but $Z_0(M) = C_0(M)$ implies $z_0 = c_0$. $\square$

Recall from section 1.2 that a covering $f : M \to N$ can have only finitely many branch points, which are points $P \in M$ with positive branching number $b_f(P) > 0$. We define the **total branching number** as

$$b = \sum_{P \in M} b_f(P).$$
Theorem 1.3.17 (Riemann-Hurwitz). Let $f : M \to N$ be a branched covering of degree $m$ between compact surfaces $M$ and $N$, then

$$\chi(M) = m\chi(N) - b.$$  \[ \tag{1.3.5} \]

Proof. Let $B = \{ P \in M | b_f(P) > 0 \}$ be the set of branch points of $f$. Since $B$ is finite (Corollary 1.2.5) we can choose a triangulation $T$ of $N$ which includes all the points of $f(B)$ as vertices. Assume $T$ has $F$ faces, $E$ edges and $V$ vertices. Then the lift $f^{-1}(T)$ of $T$ to $M$ is a triangulation of $M$ with $mF$ faces, $mE$ edges and $mV - b$ vertices. Proposition 1.3.16 implies $\chi(N) = F - E + V$, $\chi(M) = mF - mE + mV - b$ and the result follows. \[ \square \]

Remark If we call $g$ the genus of $M$ and $\gamma$ the genus of $N$, then using Corollary 1.3.15 the Riemann-Hurwitz formula can be rewritten as

$$g = m(\gamma - 1) + 1 + \frac{b}{2}. \tag{1.3.5}$$

1.4 Differential structure

In this section we will exploit the real-differentiability of the local charts $z_\alpha : U_\alpha \to \mathbb{C}$ and use them to introduce differential calculus on $M$.

When working over the reals it is best to specify a local chart using real coordinates. So given a local complex coordinate $z : U \to \mathbb{C}$, the real and imaginary parts $z = x + iy$ define corresponding real coordinates $(x, y) : U \to \mathbb{R}^2$. In this chart, a local basis for the tangent space $T_p(M)$ at a point $P \in U$ is given by

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \tag{1.4.1}$$
and the dual basis of the cotangent space $T^*_p(M)$ is $\{dx, dy\}$. The local expression of a real-valued 1-form $\omega_R$ is given in terms of two differentiable functions $f, g : U \to \mathbb{R}$ as

$$\omega_R = f(x, y)dx + g(x, y)dy.$$ (1.4.2a)

Under a change of chart the components of a 1-form transform in such a way that the expression (1.4.2a) for the 1-form in terms of its components remains valid in the new chart. On a two dimensional manifold one can also define 2-forms. A local basis for these is given by $dx \wedge dy$ and a real-valued 2-form $\lambda_R$ is specified by a single function $h : U \to \mathbb{R}$ as

$$\lambda_R = h(x, y)dx \wedge dy.$$ (1.4.2b)

The component transforms under a change of chart in such a way that (1.4.2b) remains true.

However, when we come to treat the Riemann surface $M$ as a complex manifold, it will be natural to consider complex-valued functions $f : M \to \mathbb{C}$ rather than just functions into $\mathbb{R}$. It is therefore more appropriate to consider the complexifications $T(M)^\mathbb{C} \equiv T(M) \otimes \mathbb{C}$ and $T^*(M)^\mathbb{C} \equiv T^*(M) \otimes \mathbb{C}$ of the tangent and cotangent bundles respectively. The vectors (1.4.1) still define a basis of $T_p(M)^\mathbb{C}$ over the complex numbers and $\{dx, dy\}$ still provides a basis for $T^*_p(M)^\mathbb{C}$ over $\mathbb{C}$. Complex-valued differentials can now be specified using complex-valued components such as $f, g, h : U \to \mathbb{C}$ in (1.4.2). An alternative basis for $T(M)^\mathbb{C}$ is

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right).$$ (1.4.3)

Likewise we define the new dual basis of complex-valued 1-forms by

$$dz = dx + idy, \quad d\bar{z} = dx - idy.$$
These two differentials are independent since \(dz \land d\bar{z} = -2idx \land dy \neq 0\). In this new basis a complex-valued 1-form \(\omega\) is locally expressed in terms of two real-differentiable functions \(u, v : U \rightarrow \mathbb{C}\) as

\[
\omega = u(z)dz + v(z)d\bar{z}.
\]  

(1.4.4a)

For instance, the components of the real-valued 1-form \(\omega_{\mathbb{R}}\) in (1.4.2a) with respect to this basis are \(u = \frac{i}{2}(f - ig)\) and \(v = \bar{u}\). Likewise, in the new basis a complex-valued 2-form \(\lambda\) can be locally expressed in terms of a single function \(w : U \rightarrow \mathbb{C}\) as

\[
\lambda = w(z)dz \land d\bar{z}.
\]  

(1.4.4b)

The component of the real-valued 2-form \(\lambda_{\mathbb{R}}\) in (1.4.2b) with respect to this basis is \(w = \frac{i}{2}h\). Let us denote the spaces of complex-valued functions, 1-forms and 2-forms by \(\Omega^0(M), \Omega^1(M)\) and \(\Omega^2(M)\) respectively.

**Remark** The notation in (1.4.4) is slightly misleading: although the components \(u, v\) are functions of the local complex parameter \(z\), one can still have \(\frac{\partial u}{\partial \bar{z}} \neq 0\) and \(\frac{\partial v}{\partial \bar{z}} \neq 0\). Indeed, the statement that \(\frac{\partial w}{\partial \bar{z}} = 0\) for a complex valued function \(w = f + ig\) is equivalent to the Cauchy-Riemann equations \(\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}\).

**Differentials and integration**

Given a function \(f \in \Omega^0(M)\), its **exterior derivative** is a 1-form defined locally as

\[
df \equiv f_x dx + f_y dy = f_z dz + f_{\bar{z}} d\bar{z}.
\]

This definition is chart independent and so indeed defines a 1-form. We can extend this notion of exterior derivative to 1-forms \(\omega \in \Omega^1(M)\) given locally in (1.4.4a) by
defining
\[ d\omega \equiv du \wedge dz + dv \wedge d\bar{z} = (v_z - u_{\bar{z}})dz \wedge d\bar{z}. \] (1.4.5)

The second equality follows from the definition of exterior differential on functions. Finally, since the top forms on \( M \) are 2-forms, their exterior derivative must be zero. It is obvious from these definitions that the exterior derivative satisfies the usual cohomology property
\[ d^2 = 0. \] (1.4.6)

A 1-form \( \omega \) is **closed** if \( d\omega = 0 \) and it is **exact** if \( \omega = df \) for some function \( f \). Denoting the set of closed 1-forms as \( Z^1(M) = \{ \omega \in \Omega^1(M) | d\omega = 0 \} \) and the set of exact 1-forms as \( B^1(M) = d\Omega^0(M) \), the above condition \( (1.4.6) \) means that \( B^1(M) \subset Z^1(M) \), and since these are both vector spaces, the vector space quotient \( H^1_{dR}(M) = Z^1(M)/B^1(M) \) is also a vector space, called the **first de-Rham cohomology group** of \( M \). In fact we have a sequence
\[ 0 \rightarrow \Omega^0(M) \rightarrow \Omega^1(M) \rightarrow \Omega^2(M) \rightarrow 0, \] (1.4.7)
and \( H^1_{dR}(M) \) is the obstruction to this sequence being exact at the middle site.

As usual one can define integration of \( n \)-forms over \( n \)-chains. Integration therefore provides a natural pairing between \( \Omega^n(M) \) and \( C_n(M) \),
\[ \Omega^n(M) \times C_n(M) \rightarrow \mathbb{C}, \quad (\omega, c) \mapsto \int_c \omega. \] (1.4.8)

A 0-form \( f \in \Omega^0(M) \) is just a function and a 0-chain \( c \in C_0(M) \) is a finite sum of points \( c = \sum_\alpha n_\alpha P_\alpha, \ n_\alpha \in \mathbb{Z}, \ P_\alpha \in M \). In this case integration is defined as the evaluation map,
\[ \int_c f = \sum_\alpha n_\alpha f(P_\alpha). \]
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The integral of a 1-form \( \omega \in \Omega^1(M) \) along a 1-chain \( \gamma \in C_1(M) \) given by \( \gamma : [0, 1] \to M \) is also defined in the obvious way using local coordinates. If the path \( \gamma \) lies entirely inside a single chart \( z : U \to \mathbb{C} \) with respect to which \( \omega \) has the local expression given in (1.4.4a) then we define

\[
\int_\gamma \omega = \int_0^1 \left[ (u(x, y) + v(x, y)) \frac{dx}{ds} + i(u(x, y) - v(x, y)) \frac{dy}{ds} \right] ds,
\]

which is independent of both the choice of local chart on \( U \) and the parameter along \( \gamma \). If \( \gamma \) cannot be covered by a single chart we define the integral \( \int_\gamma \omega \) piecewise. Finally, one defines the integration of a 2-form \( \lambda \in \Omega^2(M) \) given locally as in (1.4.2b) over a domain \( D \) in the usual way by proceeding patchwise, where if \( D \subset U \) is contained in a single chart \( z : U \to \mathbb{C} \)

\[
\int_D \lambda = \int_{z(D)} -2iw(x, y) dx \wedge dy.
\]

One of the most interesting properties of integration is that the boundary operator \( \partial \) defined on chains \( C_n(M) \) and the exterior differential \( d \) defined on forms \( \Omega^n(M) \) are adjoint of each other with respect to the pairing (1.4.8).

**Theorem 1.4.1** (Stokes). Let \( \omega \in \Omega^n(M) \) and \( \gamma \in C_{n+1}(M) \) then

\[
\int_\gamma d\omega = \int_{\partial \gamma} \omega.
\]

As an immediate consequence the pairings (1.4.8) descend to pairings between cohomology and homology groups. The most important of these is

\[
H^1_{d\text{R}}(M) \times H_1(M) \to \mathbb{C}, \quad (\omega, c) \mapsto \int_c \omega. \tag{1.4.9}
\]

Given a closed 1-form \( \omega \in Z^1(M) \) we define,
**Definition 1.4.2.** The integral $\int_c \omega$ over a closed path $c$ is called a **period** of $\omega$.

The following lemma asserts that a closed 1-form is uniquely specified, up to exact forms, by its periods.

**Lemma 1.4.3.** A closed 1-form $\omega$ is exact if and only if all its periods vanish.

**Proof.** The ‘only if’ direction is obvious. To prove the ‘if’ statement, assume $\omega$ is closed and $\int_c \omega = 0$ for all $c$ with $\partial c = 0$. Then $f(P) \equiv \int_{P_0} \omega$ is well defined since it is independent of the path chosen, and by the fundamental theorem of calculus $df = d\left(\int_{P} \omega\right) = \omega$. □

It follows that if we consider the homology group with complex coefficients $H_1(M, \mathbb{C})$ as a vector space over $\mathbb{C}$ then the pairing (1.4.9) is non-degenerate and we have the following duality

$$H^1_{\text{dR}}(M) = (H_1(M, \mathbb{C}))^*,$$

between vector spaces over $\mathbb{C}$. Thus in particular $H^1_{\text{dR}}(M)$ is $2g$-dimensional.

If $c$ is a closed path it follows from corollary [1.3.13](#) that it can be written as

$$c \sim \sum_{i=1}^g n_i a_i + \sum_{i=1}^g m_i b_i, \quad n_i, m_i \in \mathbb{Z},$$

modulo boundaries, indicated by the symbol $\sim$ for homology equivalence. But it follows that for any closed 1-form $\omega$ we have the equality

$$\int_c \omega = \sum_{i=1}^g n_i \int_{a_i} \omega + \sum_{i=1}^g m_i \int_{b_i} \omega. \quad (1.4.10)$$

Therefore the set of $2g$ periods $\int_{a_i} \omega$ and $\int_{b_i} \omega$ form a basis of periods for $\omega$. They are called respectively **a- and b-periods** of $\omega$. Specifying these uniquely determines a
cohomology class: indeed if two closed 1-forms $\omega_1, \omega_2$ have the same $a$- and $b$-periods then $\int_c \omega_1 = \int_c \omega_2$ for any closed curve by (1.4.10) and hence $\omega_1 - \omega_2$ is exact by lemma 1.4.3 so $\omega_1$ and $\omega_2$ define the same cohomology class.

**Riemann bilinear identities**

There is a natural anti-symmetric inner-product between 1-forms on $M$ defined by,

$$\Omega^1(M) \times \Omega^1(M) \to \mathbb{C}, \quad (\omega_1, \omega_2) \mapsto \int_M \omega_1 \wedge \omega_2.$$ 

If both forms $\omega_1, \omega_2$ are closed then this inner-product depends only on their cohomology classes since for example $\int_M df \wedge \omega_2 = \int_M d(f \omega_2) = \int_{\partial M} f \omega_2 = 0$ using $\partial M = \emptyset$. The following proposition expresses this inner-product in terms of the $a$- and $b$-periods of the two 1-forms. The important relations (1.4.11) are known as the Riemann bilinear identities.

**Proposition 1.4.4.** Let $\omega_1, \omega_2 \in \Omega^1(M)$ be two closed 1-forms on $M$, then

$$\int_M \omega_1 \wedge \omega_2 = \sum_{i=1}^g \left[ \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{b_i} \omega_1 \int_{a_i} \omega_2 \right]. \quad (1.4.11)$$

**Proof.** Consider the normal form $M_{cut}$ of $M$. Since $M_{cut}$ is star-shaped and $\omega_1$ is closed we can write $\omega_1 = df$ in $M_{cut}$ where $f(P) = \int_{P_0}^P \omega_1$ with $P_0 \in M_{cut}$. Now using also the fact that $d\omega_2 = 0$ (in the second last equality) we have

$$\int_M \omega_1 \wedge \omega_2 = \int_{M_{cut}} \omega_1 \wedge \omega_2 = \int_{M_{cut}} df \wedge \omega_2 = \int_{M_{cut}} d(f \omega_2) = \int_{\partial M_{cut}} f \omega_2.$$ 

But the boundary $\partial M_{cut}$ consists of all the edges $\{ \tilde{a}_i, \tilde{b}_j, \tilde{a}_i^{-1}, \tilde{b}_i^{-1} \}_{i=1}^g$ so the last term
on the right hand side can be written more explicitly as

$$
\sum_{i=1}^{g} \left[ \int_{\tilde{a}_i} f \omega_2 + \int_{\tilde{b}_i} f \omega_2 + \int_{\tilde{a}_{i-1}} f \omega_2 + \int_{\tilde{b}_{i-1}} f \omega_2 \right].
$$

The contribution from the cycles $\tilde{a}_i$ and $\tilde{a}_{i-1}$ can be written as

$$
\int_{\tilde{a}_i} \left( \int_{P_0}^{P_i} \omega_1 - \int_{P_0}^{P'_i} \omega_1 \right) \omega_2,
$$

where $P_i$ denotes the integration point along the cycle $\tilde{a}_i$ and $P'_i$ the integration point on the cycle $\tilde{a}_{i-1}$ which is identified with $P_i$ on $M$ (see Figure 1.4). But this is just equal to

$$
\int_{\tilde{a}_i} \left( \int_{P_0}^{P_i} \omega_1 \right) \omega_2 = \int_{\tilde{a}_i} \left( - \int_{b_i} \omega_1 \right) \omega_2,
$$

or equivalently $- \int_{b_i} \omega_1 \int_{\tilde{a}_i} \omega_2$. Likewise, the contribution from the cycles $\tilde{b}_i$ and $\tilde{b}_{i-1}$ is $\int_{\tilde{a}_i} \omega_1 \int_{\tilde{b}_i} \omega_2$. Putting everything together, equation (1.4.11) now follows.

\[\square\]

### 1.5 Analytic structure

At last we exploit the analyticity of the transition functions between charts of $M$. Of course, everything up no now still holds but as we will see, the simple analyticity
requirement will lead to a wealth of extra structure on $M$.

**Abelian differentials**

**Definition 1.5.1.** A differential $\omega \in \Omega^1(M)$ is called **holomorphic** (or Abelian of the first kind) if in any local chart $z : U \to \mathbb{C}$ it is given by a holomorphic function

$$\omega = f(z)dz, \quad f \in \mathcal{H}(U).$$

The differential $\bar{\omega}$ is called **anti-holomorphic**.

**Remark** This is well defined because in a different chart $z'$ we have $\omega = f(z(z'))(\partial_{z'}z)dz'$ and $f(z(z'))(\partial_{z'}z)$ is also holomorphic using the fact that $z' \mapsto z$ is.

The general complex-valued differential $\omega = udz + v d\bar{z}$ in (1.4.4a) is holomorphic if

$$v = 0, \quad u_{\bar{z}} = 0. \quad (1.5.1)$$

As we saw in the remark following (1.4.4a) the condition on $u$ is equivalent to the Cauchy-Riemann equations for $u$ and hence is equivalent to $u$ being holomorphic. Equation (1.5.1) together with (1.4.5) imply that every holomorphic differential is automatically closed

$$d\omega = 0.$$

We can therefore apply the Riemann bilinear identities (1.4.11) to holomorphic differentials. In particular,

**Lemma 1.5.2.** Let $\omega \not\equiv 0$ be a non-zero holomorphic differential on $M$, then

$$\Im \sum_{i=1}^{g} \int_{a_i} \omega \overline{\int_{b_i} \omega} < 0.$$
Proof. Let $\omega_1 = \omega$ and $\omega_2 = \bar{\omega}$ in (1.4.11) which in a local chart $U \subset M$ read $\omega = f(z)dz$ and $\bar{\omega} = \overline{f(z)}d\bar{z}$. Then $\omega \wedge \bar{\omega} = |f(z)|^2 dz \wedge d\bar{z} = -2i|f|^2 dx \wedge dy$, so that $i \int_U \omega \wedge \bar{\omega} > 0$ (this statement is coordinate independent by proposition 1.3.2). The integral over $M$ is defined patchwise so $i \int_M \omega \wedge \bar{\omega} > 0$. The result follows after rewriting the right hand side of (1.4.11) as $2i \sum_{i=1}^g \text{Im} \int_{a_i} \omega \int_{b_i} \bar{\omega}$.

Corollary 1.5.3. If $\omega \in \Omega^1(M)$ is holomorphic then

$$\int_{a_i} \omega = 0, \quad i = 1, \ldots, g \quad \Rightarrow \quad \omega \equiv 0.$$  

The set of all holomorphic differentials obviously forms a vector space over $\mathbb{C}$, which we denote $\mathcal{H}^1(M)$. Denote also the set of anti-holomorphic differentials as $\overline{\mathcal{H}}^1(M)$. Corollary 1.5.3 implies that $\dim \mathcal{H}^1(M) \leq g$ since if $\omega_1, \ldots, \omega_{g+1} \in \mathcal{H}^1(M)$ then some linear combination $\sum_{i=1}^{g+1} \alpha_i \omega_i$ must have vanishing $a$-periods and hence must itself vanish, $\sum_{i=1}^{g+1} \alpha_i \omega_i = 0$. In fact, as we will see later $\dim \mathcal{H}^1(M) = g$, and hence also $\dim \overline{\mathcal{H}}^1(M) = g$. But corollary 1.5.3 also implies that there are no non-zero exact holomorphic differentials on a compact Riemann surface $M$ without boundary\footnote{The assumption that $M$ is compact and without boundary is essential: if $D$ is the unit disc in $\mathbb{C}$ and $f$ is a function holomorphic in $D$ then $df$ is holomorphic and exact. Equally, if $f$ is entire in $\mathbb{C}$ then $df$ is holomorphic and exact in $\mathbb{C}$.}. So since $\mathcal{H}^1(M) \cap \overline{\mathcal{H}}^1(M) = \emptyset$, it follows that

$$H^1_{\text{dR}}(M) \cong \mathcal{H}^1(M) \oplus \overline{\mathcal{H}}^1(M).$$

Differentials of the form $\alpha = \omega_1 + \bar{\omega}_2$ with $\omega_1, \omega_2$ holomorphic are called **harmonic**. They can be expressed locally as $\alpha = dh$ where $h$ is a harmonic function.

Lemma 1.5.4. Let $\{a_i, b_i\}_{i=1}^g$ be a basis of $H_1(M)$. Then there exists a unique dual
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basis \{\omega_i\}_{i=1}^g of \mathcal{H}^1(M) which is normalised by the condition

\[ \int_{a_i} \omega_j = \delta_{ij}. \] (1.5.2)

Proof. Let \tilde{\omega}_1, \ldots, \tilde{\omega}_g be any basis of \mathcal{H}^1(M). By corollary 1.5.3 the \( g \times g \) matrix 
\[ A_{ij} = \int_{a_i} \tilde{\omega}_j \] is invertible (otherwise there exists \( \alpha_j \) s.t. \( \sum_j A_{ij} \alpha_j = 0 \) and thus \( \sum_j \alpha_j \tilde{\omega}_j = 0 \)). Then \( \omega_j \equiv \tilde{\omega}_k A_{kj}^{-1} \) is another basis of \( \mathcal{H}^1(M) \) with the desired property (1.5.2). \( \square \)

In order to get non-zero exact differentials we must therefore allow for singularities.

Definition 1.5.5. A differential \( \omega \in \Omega^1(M) \) is meromorphic if \( \omega \) is holomorphic in \( M \setminus \{P_1, \ldots, P_m\} \) and the behaviour around any \( P_i \in U \) in a local chart \( z : U \to \mathbb{C} \) (with \( z(P_i) = 0 \)) is given by a meromorphic function

\[ \omega = f(z)dz, \quad f(z) = \sum_{j=-N_i}^\infty f_j z^j, \quad N_i > 0, \quad f_{N_i} \neq 0. \] (1.5.3)

The set of all meromorphic differentials forms a vector space over \( \mathbb{C} \) which we denote \( \mathcal{K}^1(M) \). Note that the set \( S \) of poles of a meromorphic differential is discrete since meromorphic functions on \( \mathbb{C} \) have isolated poles. Moreover \( S \) is finite by compactness of \( M \), i.e. \( S = \{P_1, \ldots, P_m\} \).

Remark One could have defined a meromorphic differential \( \omega \) more concisely as one that has a local representation of the form

\[ \omega = f(z)dz, \quad f \in \mathcal{K}(M). \] (1.5.4)

However, since we did not allow differentials to take the value \( \infty \) in the previous section one must be careful. As a \( \mathbb{C} \)-valued differential, \( \omega \) in [1.5.4] is only defined on \( M' = M \setminus S \).
Definition 1.5.6. With the notation of (1.5.3) the order and residue of ω at \( P_i \in S \) are \( \text{ord}_{P_i} \omega = -N_i \) and \( \text{res}_{P_i} \omega = f_{-1} \) respectively. The singular part of ω at \( P_i \) is
\[
\sum_{j=-N_i}^{-1} f_j z^j, \quad \text{where} \quad z(P_i) = 0.
\]

Remark. The order is well defined as it has an invariant geometrical meaning (for much the same reason that the ramification number of a branch point was well defined, see the discussion after lemma 1.2.2), and the residue is chart independent because \( \text{res}_{P_i} \omega = \frac{1}{2\pi i} \int_{c_i} \omega \), where \( c_i \) is a counterclockwise cycle around \( P_i \). In general however the singular parts depend on the chart.

Proposition 1.5.7. Let ω be a meromorphic differential on a compact Riemann surface \( M \), then
\[
\sum_{P \in S} \text{res}_P \omega = 0.
\]

Proof. Consider the normal form \( M_{\text{cut}} \) of \( M \). Then
\[
\sum_{P \in S} \text{res}_P \omega = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{c_j} \omega = \frac{1}{2\pi i} \int_{\partial M_{\text{cut}}} \omega = 0,
\]
using holomorphicity of ω on \( M \setminus S \) in the second equality. The last equality follows from the fact that ω is single-valued on \( M \) so for instance \( \int_{a_i} \omega + \int_{a_{i-1}} \omega = 0 \).

Definition 1.5.8. An Abelian differential is of the first kind if it is holomorphic, of the second kind if it is meromorphic with vanishing residues and of the third kind otherwise.

Since an Abelian differential ω is closed on \( M \setminus S \), its primitive is locally well defined
\[
\Omega(P) = \int_{F_0}^P \omega. \tag{1.5.5}
\]
One can recover the Abelian differential from it by \( \omega = d\Omega \). It follows that \( \Omega(P) \) defines a meromorphic function on the whole of \( M \) only if ω is exact. More generally
the Abelian integral $\Omega(P)$ defined by (1.5.5) on $M$ will be multi-valued precisely when the cohomology class of $\omega$ is non-trivial which corresponds by lemma 1.4.3 to some of the periods of $\omega$ being non-zero. So consider a closed cycle $c$ on $M' = M \setminus S$. Because $M'$ has extra ‘punctures’ at the set $S$, a closed path on $M'$ is of the form

$$c \sim \sum_{i=1}^{g} n_i a_i + \sum_{i=1}^{g} m_i b_i + \sum_{j=1}^{m} k_j c_j, \quad n_i, m_i, k_j \in \mathbb{Z},$$

modulo boundaries, where $c_j$ is a cycle around $P_j$. In other words $\{a_i, b_i\}_{i=1}^{g}$ together with $\{c_j\}_{j=1}^{m}$ form a basis of $H_1(M')$. It follows that for the closed Abelian differential $d\Omega$ we have the equality

$$\int_{c} d\Omega = \sum_{i=1}^{g} n_i \int_{a_i} d\Omega + \sum_{i=1}^{g} m_i \int_{b_i} d\Omega + 2\pi i \sum_{j=1}^{m} k_j \text{res}_{P_j} d\Omega. \quad (1.5.6)$$

This equation is to be contrasted with the analogous formula (1.4.10) for the periods of regular differentials. Note however that the new term involving residues is only present when $d\Omega$ is of the third kind, and so in this case the multi-valuedness of the Abelian integral $\Omega$ is specified by the $a$- and $b$-periods of $d\Omega$ along with its residues.

Due to lemma 1.5.3 not all $a$-periods of an Abelian integral of the first can be zero. Now suppose $d\Omega$ is an Abelian differential of the second or third kind. In general its $a$-periods are non-trivial, say

$$A_i = \int_{a_i} d\Omega.$$

Consider subtracting from $d\Omega$ a combination of holomorphic differentials, by defining $d\hat{\Omega} = d\Omega - \sum_{j=1}^{g} \alpha_j \omega_j$. Clearly $d\hat{\Omega}$ has the same singular behaviour as $d\Omega$. However, the $a$-periods get shifted

$$\int_{a_i} d\hat{\Omega} = A_i - \alpha_i.$$
Therefore by choosing $\alpha_i = A_i$ one can set all the $a$-periods of $d\hat{\Omega}$ to zero.

**Definition 1.5.9.** We will say that an Abelian differential $d\Omega$ of the second or third kind is normalised if all its $a$-periods vanish, i.e. $\int_{a_i} d\Omega = 0, i = 1, \ldots, g$.

**Remark** By the discussion following equation (1.5.6), an Abelian differential $d\Omega$ of the third kind must be normalised with respect to a choice of $a$-cycles in the homology group $H_1(M')$ and not $H_1(M)$. Indeed, two $a$-cycles $a_i$ and $a'_i$ which are homologous in $H_1(M)$ are not necessarily homologous in $H_1(M')$ but $a'_i \sim a_i + \sum_{j=1}^{m} k_j c_j$ so that $\int_{a_i} d\Omega \neq \int_{a'_i} d\Omega$.

By the previous argument, any Abelian differential $d\Omega$ can be normalised by adjusting its holomorphic part. Moreover, the normalised differential is zero (i.e. $d\hat{\Omega} = 0$) if and only if $d\Omega$ was holomorphic. The following lemma shows that the normalised part $d\hat{\Omega}$ uniquely characterises the singular part of $d\Omega$.

**Lemma 1.5.10.** A normalised meromorphic differential $d\Omega$ is uniquely defined by the singular parts at each of its poles.

**Proof.** Suppose $d\Omega_1$ and $d\Omega_2$ are two normalised meromorphic differentials with the same set of poles and the same singular parts at these poles. Then $\omega = d\Omega_1 - d\Omega_2$ is holomorphic since the poles parts cancel out. But $\int_{a_i} \omega = 0$ since $d\Omega_1$ and $d\Omega_2$ are both normalised. It follows by lemma 1.5.3 that $\omega = 0$, namely $d\Omega_1 = d\Omega_2$. $\square$

**Examples**

We give two important examples of Abelian differentials denoted $\omega_P^{(n)}$ and $\omega_{PQ}$ of the second and third kinds respectively.

- Let $P \in M$ and $z$ a local coordinate around $P$ with $z(P) = 0$. Define a normalised Abelian differential of the second kind $\omega_P^{(n)}$ with singular parts at $P$ of the form

$$\omega_P^{(n)} = \frac{dz}{z^n}, \quad n \geq 2.$$
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Such a differential can be shown to exist and it is unique by lemma \[1.5.10\].
Note however that its definition depends on the local coordinate \(z\) at \(P\).

- Let \(P, Q \in M\). Introduce a normalised Abelian differential of the third kind \(\omega_{PQ}\) with singular parts at \(P\) and \(Q\) such that

\[
\ord_P \omega_{PQ} = \ord_Q \omega_{PQ} = -1
\]
\[
\res_P \omega_{PQ} = 1, \quad \res_Q \omega_{PQ} = -1.
\]

Such a differential can also be shown to exist and once again it is uniquely specified according to lemma \[1.5.10\]. This time however it does not depend on a choice of coordinates since it was defined in terms of invariants.

These differentials together with the \(g\) basis holomorphic differentials \(\omega_i\) form a complete basis of Abelian differentials on \(M\) in the sense that any Abelian differential \(d\Omega\) can be written as a finite linear combination of those

\[
d\Omega = \sum_{i=1}^g \alpha_i \omega_i + \sum_{P \in M} \sum_{n=2}^{N_P} \beta_{P,n} \omega_P^{(n)} + \sum_{Q,R \in M} \gamma_{Q,R} \omega_{QR},
\]  
(1.5.7)

where all but finitely many of the constants \(\alpha_i, \beta_{P,n}, \gamma_{Q,R} \in \mathbb{C}\) are zero. To arrive at (1.5.7) one first normalises \(d\Omega\) to obtain \(d\hat{\Omega}\) by subtraction of a (unique) linear combination of holomorphic differentials. One then reconstructs the finite singular part of \(d\hat{\Omega}\) from a linear combinations of the \(\omega_P^{(n)}, \omega_{QR}\) and invokes lemma \[1.5.10\]. The \(\gamma\) coefficients are note quite unique since for instance \(\omega_{PQ} + \omega_{QR} = \omega_{PR}\).

More Riemann bilinear identities

In section \[1.4\] we derived the Riemann bilinear identities \[1.4.11\] for closed differentials. Since holomorphic differentials are closed on \(M\) one can readily apply \[1.4.11\]
to them. In fact, for any $\omega_1, \omega_2 \in H^1(M)$ we have $\omega_1 \wedge \omega_2 = 0$ and so
\[
\sum_{i=1}^{g} \left[ \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{b_i} \omega_1 \int_{a_i} \omega_2 \right] = 0.
\] (1.5.8)

But now we must also allow for $\omega_1$ and $\omega_2$ to have singularities. In this case the Riemann bilinear identities receive extra contributions from the singularities.

**Proposition 1.5.11.** Let $d\Omega_1, d\Omega_2$ be Abelian differentials on $M$ where $d\Omega_1$ is not of the third kind, then
\[
\sum_{i=1}^{g} \left[ \int_{a_i} d\Omega_1 \int_{b_i} d\Omega_2 - \int_{b_i} d\Omega_1 \int_{a_i} d\Omega_2 \right] = 2\pi i \sum_{P \in M} \text{res}_P \Omega_1 d\Omega_2.
\] (1.5.9)

**Proof.** Consider once again the normal form $M_{\text{cut}}$ of $M$. Since $d\Omega_1$ is not of the third kind its Abelian integral $\Omega_1$ is single-valued in $M_{\text{cut}}$. Thus consider the meromorphic differential $\Omega_1 d\Omega_2$ on $M_{\text{cut}}$. Its integral around the boundary $\partial M_{\text{cut}}$ is
\[
\int_{\partial M_{\text{cut}}} \Omega_1 d\Omega_2 = \sum_{i=1}^{g} \left[ \int_{\tilde{a}_i} \Omega_1 d\Omega_2 + \int_{\tilde{b}_i} \Omega_1 d\Omega_2 + \int_{\tilde{a}_1^{-1}} \Omega_1 d\Omega_2 + \int_{\tilde{b}_1^{-1}} \Omega_1 d\Omega_2 \right],
\]
which by the exact same reasoning as in the proof of proposition [1.4.4] gives the left hand side of (1.5.9). On the other hand, $\Omega_1 d\Omega_2$ is holomorphic on $M_{\text{cut}} \setminus S$ where $S = \{P_1, \ldots, P_m\}$ is the finite set of singular points of $\Omega_1 d\Omega_2$. Therefore
\[
\int_{\partial M_{\text{cut}}} \Omega_1 d\Omega_2 = \sum_{j=1}^{m} \int_{c_j} \Omega_1 d\Omega_2,
\]
where $c_j$ is a small counterclockwise cycle around $P_j$. This last sum of integrals produces the right hand side of (1.5.9). \qed

**Corollary 1.5.12.**
\[
\int_{b_i} \omega_{PQ} = 2\pi i \int_{Q}^P \omega_i.
\] (1.5.10)
Proof. Apply proposition 1.5.11 to $d\Omega_1 = \omega_i$ and $d\Omega_2 = \omega_{PQ}$ and use

$$\sum_{P^i \in M} \text{res}_{P^i} \Omega_1 \omega_{PQ} = \Omega_1(P) - \Omega_1(Q) = \int_Q^P d\Omega_1. \quad \square$$

If the Abelian differentials $d\Omega_1, d\Omega_2$ are both of the third kind we cannot make use of proposition 1.5.11. Yet there is also a Riemann bilinear identity relating their periods. We will only need the case when $d\Omega_1 = \omega_{PQ}$ and $d\Omega_2 = \omega_{RS}$.

Proposition 1.5.13.

$$\int_S^R \omega_{PQ} = \int_Q^P \omega_{RS}. \quad (1.5.11)$$

Proof. Because $\omega_{PQ}$ has residues at the points $P, Q$ (assumed w.l.o.g. to lie in the interior of $M_{\text{cut}}$) we cannot write $\omega_{PQ} = df$ for some $f$ in $M_{\text{cut}}$. Yet if we introduce an extra ‘cut’ $[P, Q]$ between the points $P$ and $Q$ then $\omega_{PQ} = df$ is now exact on $M_{\text{cut}}' \equiv M_{\text{cut}} \setminus [P, Q]$. Consider the single-valued differential $f\omega_{RS}$ which is holomorphic on $M_{\text{cut}}' \setminus \{R, S\}$. As usual its integral around the boundary $\partial M_{\text{cut}}$ is

$$\int_{\partial M_{\text{cut}}} f\omega_{RS} = \sum_{i=1}^9 \left[ \int_{a_i} \omega_{PQ} \int_{b_i} \omega_{RS} - \int_{a_i} \omega_{RS} \int_{b_i} \omega_{PQ} \right] = 0, \quad (1.5.12)$$

where the last equality follows because $\omega_{PQ}$ and $\omega_{RS}$ are both normalised. On the other hand, since $f\omega_{RS}$ is holomorphic on $M_{\text{cut}}' \setminus \{R, S\}$ we have

$$\int_{\partial M_{\text{cut}}} f\omega_{RS} = 2\pi i \text{res}_R f\omega_{RS} + 2\pi i \text{res}_S f\omega_{RS} + \int_c f\omega_{RS}, \quad (1.5.13)$$
where \( c \) is the keyhole contour around the cut \([P, Q]\). Since the function \( f \) jumps by \( 2\pi i \text{res}_P \omega_{PQ} = 2\pi i \) across this cut the right hand side of (1.5.13) evaluates to

\[
2\pi i (f(R) - f(S)) + 2\pi i \int_{P}^{Q} \omega_{RS} = 2\pi i \left( \int_{S}^{R} \omega_{PQ} + \int_{P}^{Q} \omega_{RS} \right).
\]

Putting this result together with (1.5.12) yields (1.5.11). \qed

**Divisors**

**Definition 1.5.14.** A **divisor** on \( M \) is a formal finite sum of points

\[
D = \sum_{P \in M} m_P P, \quad m_P \in \mathbb{Z}.
\]

where \( m_P = 0 \) for all but finitely many points \( P \in M \).

We denote by \( \text{Div}(M) \) the **group of divisors** on \( M \), i.e. the free Abelian group generated by the points of \( M \). If \( D' = \sum_{P \in M} n_P P \) is another divisor the group operations are defined by,

\[
D + D' = \sum_{P \in M} (m_P + n_P)P, \quad -D = \sum_{P \in M} (-m_P)P,
\]

and the identity divisor is denoted by 0. This group is endowed with a natural homomorphism, called the **degree**

\[
\text{deg} : \text{Div}(M) \to \mathbb{Z}, \quad \sum_{P \in M} m_P P \mapsto \sum_{P \in M} m_P.
\]

There is an obvious partial ordering on the set of divisors defined by

\[
D \geq D' \iff m_P \geq n_P \forall P \in M.
\]
A divisor $D$ is said to be **positive** (or **integral** or **effective**) if $D \geq 0$.

A meromorphic function $f \in \mathcal{K}(M)$ on $M$ defines a divisor $(f)$ called a **principal divisor** as
\[
(f) = \sum_{P \in M} (\text{ord}_P f)P,
\] (1.5.14)
where $\text{ord}_P f$ is the order of $P$ if $f$ has a pole at $P$ or the multiplicity of $P$ if $f$ has a zero at $P$. Since $M$ is compact, theorem [1.2.7] implies that principal divisors have degree zero,
\[
\deg(f) = 0.
\] (1.5.15)

As it stands, the group $\text{Div}(M)$ does not have much structure and is rather huge. So consider the **linear equivalence** on the set of divisors defined as follows
\[
D \sim D' \quad \Leftrightarrow \quad \exists f \in \mathcal{K}(M) \text{ s.t. } (f) = D - D',
\]
and define the **divisor class group** $\text{Pic}(M)$ as the quotient $\text{Pic}(M) \equiv \text{Div}(M)/\sim$. In the same way that a function on $M$ defined a natural divisor by equation (1.5.14), a 1-form $\omega$ on $M$ also defines a divisor $(\omega)$ as
\[
(\omega) = \sum_{P \in M} (\text{ord}_P \omega)P.
\]
Note that the ratio of two meromorphic 1-forms $\omega_1, \omega_2 \in \mathcal{K}^1(M)$ is a meromorphic function $\omega_1/\omega_2 \in \mathcal{K}(M)$ with divisor $(\omega_1) - (\omega_2)$ and thus $(\omega_1) \sim (\omega_2)$. Therefore any meromorphic 1-form $\omega$ defines the same divisor class $K = (\omega) \in \text{Pic}(M)$ called the **canonical divisor** or **canonical class**. Equation (1.5.15) also implies that the degree of the canonical class is well defined since $\deg(\omega_1) = \deg(\omega_2)$.

Given a meromorphic function $f$, by definition its divisor of poles is equivalent to its divisor of zeroes. Conversely, given two **equivalent** divisors $D_0 = \sum_{i=1}^n P_i$
and \( D_\infty = \sum_{i=1}^n Q_i \) one can ask what meromorphic function \( f \) has the property that \( (f) = D_0 - D_\infty \). This question is answered by the following lemma. Equation \((1.5.16)\) will also be crucial later in discussions of section 1.7 in relation to the generalised Abel map and generalised Jacobians.

**Lemma 1.5.15.** Let \( f \) be meromorphic with divisor \( (f) = \sum_{i=1}^n (P_i - Q_i) \), then

\[
\frac{f(P)}{f(Q)} = \exp \sum_{i=1}^n \int_{Q_i}^{P_i} \omega_{PQ}, \quad (1.5.16)
\]

for any two points \( P, Q \in M \).

**Proof.** Using the Riemann bilinear identities \((1.5.11)\) the quantity in the exponent can be rewritten as \( \sum_{i=1}^n \int_Q^P \omega_{P_i Q_i} \). Since \( (f) = \sum_{i=1}^n (P_i - Q_i) \) the differential \( \frac{df}{f} \) has poles only at \( P_i \) with residue +1 and at \( Q_i \) with residue -1. But then

\[
\frac{df}{f} - \sum_{i=1}^n \omega_{P_i Q_i} = \sum_{j=1}^g c_j \omega_j, \quad (1.5.17)
\]

for some \( c_j \in \mathbb{C} \). Taking the \( a \)-periods of this equation leads to \( c_j = \int_{a_j} d \log f = 2\pi i m_j, m_j \in \mathbb{Z} \). On the other hand taking the integral from \( Q \) to \( P \) leads to

\[
\log \left( \frac{f(P)}{f(Q)} \right) = \sum_{i=1}^n \int_Q^P \omega_{P_i Q_i} + 2\pi i \sum_{j=1}^g m_j \int_Q^P \omega_j, \quad (1.5.18)
\]

which holds as an equality modulo \( 2\pi i \). However in the limit \( P_i \to Q_i \) we have \( f \to 1 \) and so the left hand side tends to zero modulo \( 2\pi i \). Likewise the first sum on the right hand side tends to zero in this limit because it can be written as \( \sum_{i=1}^n \int_{Q_i}^{P_i} \omega_{PQ} \). Since the very last term is discrete it must therefore always vanish modulo \( 2\pi i \), so we may set it to zero in \((1.5.18)\). Taking the exponential proves the lemma. \( \square \)

If we choose the function \( f \) to be normalised at \( Q \) say, so that \( f(Q) = 1 \), then
(1.5.16) gives a closed formula for the function \( f \) with \( (f) = \sum_{i=1}^{n} (P_i - Q_i) \), namely

\[
    f(P) = \exp \sum_{i=1}^{n} \int_{Q_i}^{P_i} \omega_{PQ}.
\]  

(1.5.19)

Of course, if the divisors \( D_0 = \sum_{i=1}^{n} P_i \) and \( D_{\infty} = \sum_{i=1}^{n} Q_i \) are not equivalent then (1.5.19) should not define a single valued function on the Riemann surface \( M \).

The Riemann-Roch theorem

Let \( D \) be an arbitrary divisor. We introduce the following vector space of meromorphic functions with prescribed zeroes and allowed poles,

\[
    L(D) = \{ f \in \mathcal{K}(M) \mid (f) \geq D \}.
\]

The content of this vector space is determined by the divisor \( D \) as follows: if a point \( P \in M \) figures in \( D \) with coefficient \( n > 0 \) then every \( f \in L(D) \) is forced to have a zero of order \( n \) at \( P \). If however \( Q \in M \) figures in \( D \) with coefficient \( m < 0 \) then any \( f \in L(D) \) is allowed to have at most a pole of order \(-m\) at \( Q \). In other words, if we split \( D = D_0 - D_{\infty} \) into two positive divisors \( D_0 = \sum_{j} n_j P_j \geq 0 \) and \( D_{\infty} = \sum_{k} m_k Q_k \geq 0 \) then a meromorphic function \( f \) is in \( L(D) \) provided it has zeroes of order at least \( n_j \) at \( P_j \) and poles of order at most \( m_k \) at \( Q_k \). We denote the dimension of this space as

\[
    r(D) = \dim L(D).
\]

Let us introduce a second vector space, containing meromorphic differentials with prescribed zeroes and allowed poles,

\[
    \Omega(D) = \{ \omega \in \mathcal{K}^1(M) \mid (\omega) \geq D \}.
\]
The description of this space is identical to \( L(D) \) but with the word ‘function’ replaced by the word ‘differential’. Its dimension we denote by

\[
i(D) = \dim \Omega(D).
\]

It is clear that \( r(D) \) and \( i(D) \) only depend on the divisor class of \( D \): if \( D_1 \sim D_2 \) then there exists \( h \in K(M) \) with \( (h) = D_1 - D_2 \) and multiplication by \( h \) defines vector space isomorphisms \( L(D_2) \to L(D_1) \) and \( \Omega(D_2) \to \Omega(D_1) \) and thus \( r(D_1) = r(D_2) \) and \( i(D_1) = i(D_2) \). Furthermore, these dimensions are related as follows

\[
i(D) = r(D - K). \tag{1.5.20}
\]

Indeed, if \( \omega_0 \) is any meromorphic differential its divisor is the canonical divisor \( (\omega_0) = K \) so that \( \omega \mapsto \omega/\omega_0 \) defines a vector space isomorphism \( \Omega(D) \to L(D - K) \).

We are now in a position to state one of the most important theorems on compact Riemann surfaces,

**Theorem 1.5.16 (Riemann-Roch).** Let \( M \) be a compact Riemann surface of genus \( g \) and \( D \) a divisor on \( M \). Then

\[
r(-D) = \deg D - g + 1 + i(D). \tag{1.5.21}
\]

**Corollary 1.5.17.** \( \dim \mathcal{H}^1(M) = g \).

**Proof.** Let \( D = 0 \) in (1.5.21). Since a meromorphic function \( f : M \to \mathbb{C}P^1 \) on a compact Riemann surface \( M \) is either constant or surjective by theorem 1.2.6 it follows that \( L(0) = \mathbb{C} \), i.e. \( r(0) = 1 \). But then \( i(0) = g \), so the space \( \Omega(0) = \mathcal{H}^1(M) \) of holomorphic differentials is \( g \) dimensional. \( \Box \)

**Corollary 1.5.18.** If \( \deg D < 0 \) then \( i(D) = -\deg D - 1 + g \).
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Proof. Again using theorem 1.2.6 we find that \( r(-D) = 0 \) since a meromorphic function \( f : M \to \mathbb{CP}^1 \) cannot have strictly more zeroes than poles. \( \square \)

**Corollary 1.5.19.** \( \deg K = 2g - 2 \).

**Proof.** Let \( D = K \) in (1.5.21). Using (1.5.20) we have that \( i(K) = r(0) = 1 \) and \( r(-K) = i(0) = g \) by corollary 1.5.17. \( \square \)

**Corollary 1.5.20.** Every compact Riemann surface \( M \) of genus zero is conformally equivalent to the Riemann sphere \( \mathbb{CP}^1 \).

**Proof.** Let \( P \in M \) then clearly \( r(-P) = 2 \) (since \( g = 0 \) and \( i(P) = 0 \) as \( \mathcal{H}^1(M) = \emptyset \)) so there exists a non-constant meromorphic function of degree one on \( M \) which is is a bijection by theorem 1.2.6. \( \square \)

It is obvious from theorem 1.2.6 that if \( D < 0 \) then \( -D \) is strictly positive and \( r(-D) = 0 \). Given a generic divisor \( D \geq 0 \) we would like to use the Riemann-Roch theorem to compute \( r(-D) \). According to (1.5.21) we need only determine \( i(D) \). Since we are assuming \( D \geq 0 \), this is the dimension of the space \( \Omega(D) \) of holomorphic differentials vanishing at \( D \). If \( \deg D \geq 2g - 1 \) then by corollary 1.5.19 there is no such differential and so \( i(D) = 0 \). Thus we have

\[
\begin{align*}
r(-D) &= 0, & \text{deg } D < 0 \\
&\geq 1 - g + \deg D, & 0 \leq \deg D < 2g - 1 \\
&= 1 - g + \deg D, & \deg D \geq 2g - 1
\end{align*}
\]

It remains to discuss positive divisors of the form \( D = P_1 + \cdots + P_n \) of degree \( \deg D = n \) in the range \( 0 \leq n < 2g - 1 \). Since the space of holomorphic differentials is of dimension \( g \) by corollary 1.5.17 the space \( \Omega(D) \) consists of the solutions \( c = \)
Now when $n \leq g$, the $n \times g$ matrix $M_{ji} = \omega_i(P_j)$ will typically be of rank $n$ except for very specific divisors $D$. Therefore generically in this case we will have $i(D) = g - n$ and hence $r(-D) = 1$. If however $n > g$, then the system (1.5.22) is over determined and generically has no solutions, except once again for very specific divisors $D$. So generically in this case we have $i(D) = 0$ and hence $r(-D) = n - g + 1$.

**Definition 1.5.21.** A positive divisor $D \geq 0$ is **special** if either $\deg D \leq g$, $r(-D) > 1$ or $\deg D \geq g$, $i(D) > 0$. It is **non-special** (or generic or in general position) if either $\deg D \leq g$, $r(-D) = 1$ or $\deg D \geq g$, $i(D) = 0$.

Of particular interest will be the case $\deg D = g$. Note also that if $D' = D + Q$ with $Q \in M$ then $D'$ is non-special whenever $D$ is non-special because $i(D) = 0 \Rightarrow i(D') = 0$ (from observing (1.5.22)).

**Moduli space at genus $g$**

Topologically speaking, the only invariant of a compact Riemann surface is its genus. That is, by theorem 1.3.5 any two Riemann surfaces $M, M'$ are homeomorphic if and only if they have the same genus. As it turns out, in two dimensions any two compact orientable surfaces $M, M'$ that are homeomorphic are also diffeomorphic. Thus at every genus $g \geq 0$ there is a unique differential structure up to diffeomorphisms. When it comes to analytic structures however things are very different. If we consider two Riemann surfaces $M, M'$ as equivalent when there is a biholomorphic mapping

$$f : M \to M',$$
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\( \text{i.e. a bijection with } f \text{ and } f^{-1} \text{ holomorphic, then it turns out that at every genus } g \geq 1 \text{ there is a continuous family of inequivalent Riemann surfaces. Denoting by } \mathcal{M}_g \text{ the moduli space of inequivalent Riemann surfaces at genus } g, \text{ the following important proposition is also a direct consequence of the Riemann-Roch theorem.} \)

\textbf{Proposition 1.5.22.}

\[
\dim \mathcal{M}_g = \begin{cases} 
0, & g = 0 \\
1, & g = 1 \\
3g - 3, & g \geq 2.
\end{cases}
\]

More generally it will be important to consider punctured Riemann surfaces. A \textbf{punctured} Riemann surface is simply a Riemann surface \( M \) marked at a finite set of ordered points \( \{P_\alpha \in M\}_{\alpha=1}^N \). We denote by \( \mathcal{M}_{g,N} \) the moduli space of punctured Riemann surfaces of genus \( g \) with \( N \) punctures.

\textbf{Corollary 1.5.23.} \textit{For any genus } \( g \) \text{ and number } \( N \) \text{ of punctures we have}

\[
\dim \mathcal{M}_{g,N} = 3g - 3 + N. \tag{1.5.23}
\]

\textit{Proof.} At genus zero there is only one Riemann sphere but its automorphism group is the Möbius group which has three complex parameters and hence allows one to fix three of the punctures to say 0,1 and \( \infty \). This leaves \( N - 3 \) free parameters.

At genus one there is a one parameter family of conformally inequivalent torii but one can fix a puncture to say 0 so \( \dim \mathcal{M}_{1,N} = 1 + (N - 1) = N. \)

Finally when \( g > 1 \) the automorphism group is finite so the dimension of the moduli space of Riemann surfaces of genus \( g \) with \( N \) punctures is simply \( 3g - 3 + N. \) In every case the formula \( 3g - 3 + N \) gives the correct count for \( \dim \mathcal{M}_{g,N}. \) \( \Box \)
1.6 Algebraic curves

Most examples of Riemann surfaces we will need are non-singular algebraic curves. These were already introduced in section 1.1 as the zero-locus of a polynomial $P$ in two complex variables $x, y$,

$$C = \{(x, y) \in \mathbb{C}^2 | P(x, y) = 0\}.$$  \hfill (1.6.1)

The non-singular condition is the requirement that at any point $(a, b) \in C$ the gradient of $P$ is non-vanishing, namely $dP(a, b) \neq 0$. Therefore in the immediate neighbourhood of any point $(a, b) \in C$ the curve (1.6.1) looks locally like

$$(x - a)\frac{\partial P}{\partial x}(a, b) + (y - b)\frac{\partial P}{\partial y}(a, b) = 0.$$  \hfill (1.6.2)

This is the equation for a line in $\mathbb{C}^2$, namely a copy of $\mathbb{C}$. In other words the non-singular condition means that $C$ is locally homeomorphic to $\mathbb{C}$ and an obvious local parameter is $x$ if $\partial P/\partial y \neq 0$ or $y$ if $\partial P/\partial x \neq 0$. In a neighbourhood where either local parameter works the transition functions $x(y)$ and $y(x)$ are holomorphic by the implicit function theorem. Therefore non-singular algebraic curves satisfy all the requirements of a Riemann surface.

Singularities

Oftentimes however an algebraic curve defined by (1.6.1) will be singular.

**Definition 1.6.1.** A point $(a, b) \in C$ is singular if $dP(a, b) = 0$.

In the neighbourhood of such a point the curve $C$ no longer looks like (1.6.2) since one has to look at subleading terms. The **multiplicity** of a singular point is
the smallest integer \( m \) such that

\[
\frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \neq 0,
\]

for some \( 0 \leq i, j \leq m \) such that \( i + j = m \). The curve \( C \) is then locally described by a homogeneous polynomial of degree \( m \) and (1.6.2) is replaced by

\[
\sum_{i+j=m} \frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \frac{(x-a)^i(y-b)^j}{i!j!} = 0. \tag{1.6.3}
\]

Since the left hand side polynomial is homogeneous in \((x-a)\) and \((y-b)\) of degree \( m \) it can be factored into a product of \( m \) linear polynomials and (1.6.3) is equivalent to a set of \( m \) linear equations \( \alpha_i(x-a) + \beta_i(y-b) = 0 \) where \( i = 1, \ldots, m \) and \((\alpha_i, \beta_i) \neq (0,0)\). Each of these linear equations defines a complex line in \( \mathbb{C}^2 \) which means that locally near a singular point the curve \( C \) looks like the intersection of several copies of \( \mathbb{C} \). The singular point is ordinary if the polynomial in (1.6.3) has no repeated factor. In this case the curve \( C \) looks locally like the intersection of \( m \) distinct lines.

**Definition 1.6.2.** A node is an ordinary singular point of multiplicity two.

By performing the birational change of variables \( X = \alpha_1(x-a) + \beta_1(y-b) \) and \( Y = \alpha_2(x-a) + \beta_2(y-b) \) a node can always be brought to the canonical form

\[
XY = 0. \tag{1.6.4}
\]

As depicted in the picture this consists of two copies of the complex line, namely \( X = 0 \) and \( Y = 0 \), intersecting at the common node \((X, Y) = (0,0)\). Performing the
birational change of coordinates $X = z - w$ and $Y = z + w$ leads to an equivalent representation of the node (1.6.4), namely $z^2 = w^2$. A singularity of the form $z^2 = w^3$ is called a cusp. More generally,

**Definition 1.6.3.** A singularity that can be brought to the local form

$$z^2 = w^m, \quad m \geq 4 \quad (1.6.5)$$

will be called a higher cusp if $m$ is odd and a higher node if $m$ is even.

Given a singular algebraic curve $C$, there are two standard ways of resolving singularities so as to obtain a Riemann surface which we now turn to. Afterwards we will describe the reverse procedures whereby one obtains singular curves from non-singular ones.

**Normalisation**

The first procedure for resolving singularities, known as normalisation (or desingularisation) consists of ‘blowing up’ each singular point into a finite set $S$ of points. The singular curve in this case is recovered by identifying each set of points $S$ to single points. In the case of the node (1.6.4) the singular point $(X, Y) = (0, 0)$ is doubled

$$X = 0, \quad Y = 0. \quad (1.6.6)$$

This results in two copies of the complex line which is obviously a Riemann surface. The line $X = 0$ admits $Y$ as a local parameter whereas $X$ is a local parameter on the line $Y = 0$. Concretely, normalisation can be achieved using a birational
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transformation\(^2\) as follows. Considering the node in the form \(z^2 = w^2\), we perform a birational transformation \((z, w) \mapsto (u, v)\) defined by \(z = uv\) and \(w = v\). This transformation has the desired feature that it is invertible except at the singular point itself \((z, w) = (0, 0)\). Indeed it transforms the node to \(u^2 = 1\) and so the singular point has been ‘blown up’ to a pair of points \((u, v) = (\pm 1, 0)\).

The normalisation of a singular point does not always result in the addition of points. Consider for example the cusp singularity \(z^2 = w^3\). It may be desingularised by the same birational transformation as we used for the node, resulting in the non-singular curve \(u^2 = v\). This time the singular point \((z, w) = (0, 0)\) gets mapped to the single point \((u, v) = (0, 0)\) which is a branch point of the map \((u, v) \mapsto v\). We conclude therefore that a cusp resolves into a branch point.

More generally, a higher node \(z^2 = w^{2r}\) may be desingularised by using the birational transformation \(z = uv^r\) and \(w = v\) which transforms it to \(u^2 = 1\). Thus as in the case of a node, the singular point has been ‘blown up’ to a pair of points \((u, v) = (\pm 1, 0)\). The case of a higher cusp \(z^2 = w^{2r+1}\) can also be desingularised by the same birational transformation yielding the non-singular curve \(u^2 = v\). So just as for the cusp, the singular point doesn’t get blown up but instead resolves into a single branch point. Since the birational transformations used to resolve singularities are always invertible away from the singular points in question we may resolve each of the finitely many singular points of an algebraic curve \(C\) by proceeding one at a time. This finite procedure results in a Riemann surface \(\hat{C}\) known as the normalisation of \(C\). Moreover, there is a continuous surjection

\[
\pi : \hat{C} \to C,
\]

which restricts to a biholomorphic map \(\pi : \hat{C} \setminus \pi^{-1}(S) \to C \setminus S\), where \(S\) is the finite

---

\(^2\) Another way to define the normalisation of a singular curve \(C\) is as the space of germs on \(C\).
set of singular points of $C$. In the present case $\pi^{-1}(S)$ is also finite and consists of at most twice as many points as $S$.

**Smoothing**

The other procedure for resolving singularities, known as smoothing (or deformation) consists of ‘perturbing’ the algebraic curve $C$ by a small parameter $t$. The original singular curve is recovered in the limit $t \to 0$. An example of a smoothing of the node (1.6.4) is

$$XY = t, \quad t \in \mathbb{C}.$$  \hspace{1cm} (1.6.7)

For $t \neq 0$ this curve is no longer singular and either $X$ or $Y$ may be used as local charts with the transition function $X = t/Y$ being holomorphic. The smoothed out node (1.6.7) is therefore a Riemann surface. To describe this surface locally it is sufficient to restrict the coordinates to within the unit disc $D = \{t \in \mathbb{C} \mid |t| < 1\}$. That is, given $t \in D$ we define

$$N_t = \{(X, Y) \in D^2 \mid XY = t\}. \hspace{1cm} (1.6.8)$$

Because $|Y| < 1$ it follows that $|X| > |t|$, and likewise $|Y| > |t|$ as a consequence of $|X| < 1$. Thus topologically the deformed node $N_t$ is the annulus (or cylinder) given by $|t| < |X| < 1$. To make use of both coordinates, it can also be modelled topologically as the two annuli $\sqrt{|t|} < |X| < 1$ and $\sqrt{|t|} < |Y| < 1$ glued together through the interior circle. In this picture the limit $t \to 0$ is more apparent and clearly gives two discs glued together at the origin, as in (1.6.4).
Defining the topological genus of a singular algebraic curve as the topological genus of its normalisation, smoothing a curve will increase its topological genus. For singular algebraic curves one can introduce an alternative genus to the topological genus, called the algebraic genus, defined as the genus of the smoothed out curve. It follows that the algebraic genus is invariant under the smoothing operation.

**Indentifying points**

The reverse process to normalisation consists in identifying certain points of a smooth algebraic curve $C$. Following [53] we define a modulus to be an effective divisor $m = \sum_{P \in C} n_P P, n_P \geq 0$. We refer to the finite set of points $P \in C$ for which $n_P > 0$ as the support $S$ of $m$. Then in the simplest case, a singular curve is obtained by collapsing the entire set $S$ to a single point $Q$ (more generally $S$ collapses to a smaller set $S'$). That is we define a singular curve as the set $C_m = (C \setminus S) \cup \{Q\}$. Notice that at the level of the curve no use was made of the multiplicities $n_P$ of each point $P$ in the modulus. These multiplicities enter in the definition of the allowed functions on the singular curve $C_m$. For instance [53, pp.61–62], if $m$ consists of two distinct points, namely $m = P_1 + P_2$ with $P_1 \neq P_2$ then it turns out that the resulting singular point $Q \in C_m$ is a node. In this case, a function on $C_m$ regular at $Q$ should arise from a function $f$ on $C$ which is regular at $P_1$ and $P_2$ but since these points are identified on $C_m$ we must also request that $f(P_1) = f(P_2)$ for $f$ to be single-valued on $C_m$. As another example, if $m = 2P$ then the curve $C_m$ is identical to $C$ since $S = \{P\}$ is a single point, however functions on $C_m$ are taken to be functions on $C$ with a vanishing first derivative at $P$. The singular point $Q \in C_m$ in this case turns out to be a cusp. In each case the original curve $C$ is the normalisation of the resulting singular curve $C_m$.

---

3The term ‘modulus’ makes sense with regards to corollary 1.5.23 since marking a point on a Riemann surface generically increases the dimension of the moduli space by one.
Recall that any two divisors $D, D' \in \text{Div}(C)$ are said to be equivalent $D \sim D'$ if there exists a meromorphic function $f$ on $C$ with divisor $(f) = D - D'$. On singular curves defined by a modulus $m$ as above we can also define an equivalence relation between divisors by defining a more stringent equivalence on $\text{Div}(C)$. First of all we say that a divisor $D \in \text{Div}(C)$ is prime to $S$ if it has no points in common with $S$. Two such divisors $D, D'$ are then said to be $m$-equivalent, written $D \sim_m D'$, if there exists a function $f$ on $C$ such that

$$
(f) = D - D', \quad \forall P \in S, \text{ ord}_P(f - 1) \geq n_P.
$$

The new second condition says that $f$ must take the value one at any $P \in S$ with multiplicity $n_P$. In particular $f$ takes the same value at all the points of $S$ which is required for $f$ to define a single-valued function on $C_m$. This new equivalence relation on $\text{Div}(C \setminus S)$ allows us to define the generalised divisor class group relative to $m$, denoted $\text{Pic}_m(C) \equiv \text{Div}(C \setminus S)/\sim_m$, of divisors prime to $S$ modulo $m$-equivalence. The main example we will need is that of a nodal curve (with a single node) for which $m = P_1 + P_2$ with $P_1 \neq P_2$. In this case (1.6.9) reads

$$
(f) = D - D', \quad f(P_1) = f(P_2) = 1.
$$

Degeneration

Recall that the smoothing procedure resulted in a 1-parameter family of Riemann surfaces $C_t$ for $t \neq 0$, with the original singular curve $C_0$ sitting at the limiting point $t = 0$. The reverse process of smoothing thus consists in pinching the family $C_t$ of Riemann surfaces by taking the limit $t \to 0$ to recover the singular curve $C_0$. One therefore has to construct a family $C_t$ of Riemann surfaces fibred over the unit disc $D = \{t \in \mathbb{C} \mid |t| < 1\}$ which is locally modelled on the smoothed node (1.6.8).
There are two different ways of obtaining a family $C_t$ of Riemann surface with a local neighbourhood modelled on the smoothed node $N_t$ (see [40, chapter III]):

- One can either take two distinct Riemann surfaces $M_1$ and $M_2$ punctured at $P_1$ and $P_2$ respectively with local coordinates $z_1$ and $z_2$ near these punctures and define $C_t = M_1 \sqcup N_t \sqcup M_2 / \mathcal{R}$. The quotient serves to specify the overlaps between the three surfaces $M_1$, $N_t$, and $M_2$ in the disjoint union. Specifically the relation $\mathcal{R}$ is defined as follows. A point near $P_1$ with local coordinate $z_1$ on $M_1$ is to be identified with the point of local coordinate $X = z_1$ on $N_t$. Similarly points of $M_2$ with local coordinate $z_2$ are identified with points of $N_t$ with local coordinate $Y = z_2$. Thus in the overlap we have by construction $z_2 = t/z_1$. The family $C_t$ then describes the **pinching of a cycle homologous to zero**.

- One can also take the two punctures $P_1$ and $P_2$ to be on the same Riemann surface $M$. In this case we define $C_t = M \sqcup N_t / \mathcal{R}'$. Once again the quotient specifies the overlap between the component surfaces $M$ and $N_t$ of the disjoint union. Here the relation $\mathcal{R}'$ is defined as follows. A point near $P_1$ with local coordinate $z_1$ on $M$ is to be identified with the point with local coordinate $X = z_1$ on $N_t$. Similarly points near $P_2$ with coordinate $z_2$ on $M$ are identified with points on $N_t$ with coordinate $Y = z_2$. Once more in the overlap we have $z_2 = t/z_1$. Here the family $C_t$ describes the **pinching of a non-zero homology cycle**.
We will be mostly concerned with the second possibility of pinching cycles homologous either to $a$- or $b$-cycles on $M$.

1.7 Jacobians

Consider the dual space $\mathcal{H}^1(M)^*$ of linear functionals $\mathcal{H}^1(M) \to \mathbb{C}$ on the space of holomorphic forms. By the integration mapping, every closed cycle $\gamma \in H_1(M,\mathbb{Z})$ defines an element of $\mathcal{H}^1(M)^*$ through its periods, namely

$$H_1(M,\mathbb{Z}) \to \mathcal{H}^1(M)^*, \quad \gamma \mapsto \left( \omega \mapsto \int_\gamma \omega \right).$$

An object of fundamental importance in the study of Riemann surfaces is the quotient of $\mathcal{H}^1(M)^*$ by the subgroup of periods $H_1(M,\mathbb{Z})$.

**Definition 1.7.1.** The Jacobian of $M$ is the quotient $J(M) = \mathcal{H}^1(M)^*/H_1(M,\mathbb{Z})$.

We can describe the Jacobian more explicitly using bases. So let $\{a_i, b_j\}$ be a canonical basis of $H_1(M)$ and $\omega_i$ the dual ‘normalised’ basis \((1.5.2)\) for the space of holomorphic differentials $\mathcal{H}^1(M)$. Since $\mathcal{H}^1(M)$ is a complex vector space of dimension $g$, its dual can be identified using the basis $\{\omega_j\}$ with the space $\mathbb{C}^g$ of complex column vectors. As for the period subgroup, the $a$-periods of the basis differentials $\omega_j$ being normalised by the condition $\int_{a_i} \omega_j = \delta_{ij}$ they define $g$ independent vec-
tors in \( \mathbb{C}^g \). The \( g \) remaining \( b \)-periods of the \( g \) basis holomorphic differentials are non-trivial and define an important \( g \times g \) matrix.

**Definition 1.7.2.** The period matrix \( \Pi \) is \( g \times g \) with components \( \Pi_{ij} = \int_{b_i} \omega_j \).

The period matrix has the following important properties,

**Lemma 1.7.3.** \( \Pi \) is symmetric and has positive definite imaginary part.

**Proof.** To show symmetry, apply the Riemann bilinear identities (1.5.8) to the normalised holomorphic differentials \( \omega_1 = \omega_i, \omega_2 = \omega_j \). To show positive definiteness of \( \text{Im} \, \Pi \), namely \( \sum_{i,j} c_i (\text{Im} \, \Pi_{ij}) c_j > 0 \), apply lemma 1.5.2 to \( \omega = \sum_{i=1}^g c_i \omega_i, c_j \in \mathbb{R} \). \( \square \)

In particular, since \( (\text{Im} \, \Pi) \) is positive definite it is invertible so that,

**Corollary 1.7.4.** The \( 2g \) columns of the full \( g \times 2g \) matrix of periods \( (1, \Pi) \) are linearly independent over \( \mathbb{R} \).

Hence the Jacobian is a **complex \( g \)-dimensional torus**, namely it is the quotient of \( \mathbb{C}^g \), viewed as a real vector space, by a real \( 2g \)-dimensional lattice\(^4\):

\[
J(M) = \mathbb{C}^g / \Lambda, \quad \Lambda \equiv 2\pi \mathbb{Z}^g \oplus 2\pi \Pi \mathbb{Z}^g.
\]  
(1.7.1)

Note that the Jacobian has an obvious Abelian group structure. Thus every Riemann surface \( M \) has associated with it a natural Abelian group \( J(M) \). Recall that we have already assigned an Abelian group to every Riemann surface \( M \), namely the divisor class group \( \text{Pic}(M) \), also called the **Picard group**. The Abel-Jacobi theorem states that the group \( \text{Pic}^0(M) \) of degree zero divisors modulo principal divisors and the Jacobian \( J(M) \) are isomorphic. The isomorphism is constructed using the Abel map which we now turn to.

\(^4\)The factors of \( 2\pi \) are conventions we adopt to simplify some of the notation later.
The Abel map

**Definition 1.7.5.** The Abel map $\mathcal{A} : M \to J(M)$ is defined relative to some base point $P_0 \in M$ by

$$P \mapsto \mathcal{A}(P) = 2\pi \int_{P_0}^{P} \omega \mod \Lambda,$$

where $\omega = (\omega_1, \ldots, \omega_g)^T$ is the vector of basis holomorphic forms.

**Remark** The integrals $\int_{P_0}^{P} \omega$ themselves are not well defined as they depend on the path $\gamma$ joining the base point $P_0$ to $P$. But if $\gamma'$ is another such path then $\gamma - \gamma'$ is closed so that the difference $2\pi \int_{\gamma} \omega - 2\pi \int_{\gamma'} \omega = 2\pi \int_{\gamma - \gamma'} \omega \in \Lambda$. For this reason equalities involving the Abel map should always be understood to be mod$\Lambda$ unless otherwise stated.

**Remark** The Abel map doesn’t depend on the choice of basis holomorphic forms since it can be written in a coordinate independent way as $\mathcal{A} : P \mapsto \left( \omega \mapsto 2\pi \int_{P_0}^{P} \omega \right)$.

The Abel map can be extended to the group of divisors $\text{Div}(M)$ by setting

$$\mathcal{A}\left( \sum_{P \in M} m_P P \right) = \sum_{P \in M} m_P \mathcal{A}(P),$$

which defines a group homomorphism $\mathcal{A} : \text{Div}(M) \to J(M)$. In particular, when acting on divisors of degree zero the Abel map $\mathcal{A} : \text{Div}^0(M) \to J(M)$ is easily seen not to depend on the base point $P_0$. Indeed, for $D = \sum_{\alpha=1}^{n} (P_\alpha - Q_\alpha)$ we have

$$\mathcal{A}\left( \sum_{\alpha=1}^{n} (P_\alpha - Q_\alpha) \right) = \sum_{\alpha=1}^{n} 2\pi \int_{P_0}^{P_\alpha} \omega - 2\pi \int_{P_0}^{Q_\alpha} \omega = \sum_{\alpha=1}^{n} 2\pi \int_{P_0}^{P_\alpha} \omega.$$  

It is a consequence of Abel’s theorem below that the Abel map on $\text{Div}^0(M)$ descends to a homomorphism

$$\mathcal{A} : \text{Pic}^0(M) \longrightarrow J(M)$$

between the groups $\text{Pic}^0(M)$ and $J(M)$. Moreover, this homomorphism is also
injective as a consequence of Abel’s theorem and surjective by Jacobi’s theorem. Thus the Abel map (1.7.3) provides an isomorphism between the degree zero Picard group Pic^0(M) on the one hand and the Jacobian J(M) on the other.

**Theorem 1.7.6** (Abel). A divisor \( D \in \text{Div}(M) \) is principal if and only if \( \deg D = 0 \) and \( \mathcal{A}(D) = 0 \).

**Proof.** The condition \( \deg D = 0 \) is obvious from (1.5.15). Let \( D = \sum_{i=1}^{n} (P_i - Q_i) \) and consider the function \( f(P) = \exp \sum_{i=1}^{n} \int_{Q_i}^{P} \omega_{P_iQ_i} \) in (1.5.19) which by lemma [1.5.13] has the right divisor \( (f) = \sum_{i=1}^{n} (P_i - Q_i) \). However this divisor is principal if and only if \( f \) is single-valued on \( M \). Since \( \omega_{P_iQ_i} \) is normalised with unit residues at its poles, this is the case if and only if \( \int_{b_i} \sum_{i=1}^{n} \omega_{P_iQ_i} \in 2\pi i \mathbb{Z} \). And by the Riemann bilinear identity (1.5.10) this is equivalent to \( \sum_{i=1}^{n} \int_{Q_i}^{P_i} \omega_j \in \mathbb{Z} \). \( \square \)

**Theorem 1.7.7** (Jacobi). Every point in \( J(M) \) is the image of an integral divisor of degree \( g \).

**Generalised Jacobians**

Consider the singular algebraic curve \( C_m \) described by a modulus \( m = P_1 + P_2, P_1 \neq P_2 \) on its normalisation \( C \). If the above construction of Jacobians for Riemann surfaces is to carry over to singular algebraic curves then the Abel map should be generalised. Indeed we would still like the Abel map to characterise divisors up to equivalence on \( C_m \). But we saw that divisors on \( C_m \) can be described as divisors on \( C \setminus S \) (where \( S \) was the support of \( m \)) subject to the stronger \( m \)-equivalence.

As we have seen, the nodal curve \( C_m \) can be resolved into two different Riemann surfaces: it can be desingularised to produce its normalisation \( C \) or it can be smoothed out to form a one-parameter family \( C_t \). In the first case the singular curve \( C_m \) is recovered by identifying \( P_1 \) with \( P_2 \) and in the second case by taking
$t \to 0$ to pinch off the extra handle. Both resolved curves being Riemann surfaces the above analysis applies to these, see Figure 1.8.

(a) Normalisation $C$  
(b) Singular $C_m$  
(c) Smoothing $C_t$

Figure 1.8: Normalisation and smoothing of the singular curve $C_m$.

In particular, each member of the family $C_t$, $t \neq 0$ can be assigned a Jacobian $J(C_t)$. We shall define the generalised Jacobian $J_m(C)$ associated with the singular curve $C_m$ as the limit of $J(C_t)$ as we take $t \to 0$. We now aim to give a more explicit description of $J_m(C)$ as a quotient much like equation (1.7.1) for the usual Jacobian. Recall that the construction of $C_t$ using two punctures on the same Riemann surface, as in Figure 1.7, lead to a Riemann surface with genus one higher since the smoothed out node gives it one extra handle. Let us define the canonical homology basis $\{a_I(t)\}_{i=0}^g$ of $C_t$ so that the extra $a_0(t)$-cycle goes around the smoothed out node with the extra $b_0(t)$-cycle intersecting $a_0(t)$ once, as illustrated in Figure 1.8 in the elliptic case $g = 1$. The singular limit $t \to 0$ corresponds then to pinching a particular $a_0$-cycle $\tilde{a}_0(t)$ to a point $Q$. We shall call $a_0(t)$ and $a_0'(t)$ the two cycles on either side of the shrinking cycle $\tilde{a}_0(t)$, as depicted in Figure 1.9.

Let $\{\omega_I(t)\}_{i=0}^g$ be the dual basis of holomorphic 1-forms canonically normalised as
It is clear from these relations that in the limit $t \to 0$ the 1-form $\omega_0(t)$ acquires poles at the points $P_1$ and $P_2$ on $C$ corresponding to the desingularisation of $Q$ with the following residues (For quantities taken at $t = 0$ we drop the argument for clarity and write for instance $a_I \equiv a_I(0), b_I \equiv b_I(0), \omega_I \equiv \omega_I(0), etc$)

$$\text{res}_{P_1} \omega_0 = \frac{1}{2\pi i} \int_{a_0} \omega_0 = \frac{1}{2\pi i}, \quad \text{res}_{P_2} \omega_0 = -\frac{1}{2\pi i} \int_{a_0} \omega_0 = -\frac{1}{2\pi i}.$$ 

Since $\omega_0$ has no further poles and $\int_{a_i} \omega_0 = 0$ for $i = 1, \ldots, g$ it uniquely determines the normalised Abelian differential of the third kind $\omega_0 = \frac{1}{2\pi i} \omega_{P_1P_2}$ on $C$. Moreover, the remaining differentials $\{\omega_i\}_{i=1}^g$ form a basis of holomorphic 1-forms on $C$ dual to the homology basis $\{a_i, b_i\}_{i=1}^g$ for $C$ by (1.7.4).

To identify the Jacobian $J(C_t)$ in the singular limit consider its period matrix $\Pi_{IJ}(t) = \int_{b_I(t)} \omega_J(t)$. Since the curve $b_0$ starts at $P_1$ and ends at $P_2$, the component $\Pi_{00}(t) = \int_{b_0(t)} \omega_0(t)$ will diverge in the limit $t \to 0$. All the other components $\Pi_{ij}(t)$ and $\Pi_{0j}(t)$ of the period matrix stay finite in this limit. It is clear now that the first column $\Pi_{I0}(t)$ tends to (an infinite multiple of) the unit vector $(1, 0, \ldots, 0)^T$. The lattice of periods $(1, \Pi(t))$ from corollary 1.7.4 thus becomes degenerate in the singular limit $t \to 0$ and is only spanned by $2g + 1$ linearly independent vectors

$$\begin{pmatrix} 1 & 0 & \Pi_{0j} \\ 0 & \delta_{ij} & \Pi_{ij} \end{pmatrix}. $$

Denoting by $\Lambda_m$ the lattice spanned by $2\pi$ multiples of these vectors, the generalised
Jacobian can therefore be defined as the quotient
\[ J_m(C) \equiv \mathbb{C}^{g+1}/\Lambda_m. \tag{1.7.5} \]

Because the lattice \( \Lambda_m \) is only spanned by \( 2g + 1 \) vectors linearly independent over \( \mathbb{R} \) and \( \mathbb{C}^g \) has dimension \( 2g + 2 \) over \( \mathbb{R} \), the lattice \( \Lambda_m \) is in some sense too small and as a result the quotient \( (1.7.5) \) is a non-compact algebraic group. Topologically it is simply \( J_m(C) \simeq J(C) \times \mathbb{C}^* \) with the \( \mathbb{C}^* \) factor being the origin of non-compactness.

**Definition 1.7.8.** The generalised Abel map \( \vec{A} : C \setminus S \to J_m(C) \) is defined relative to some base point \( P_0 \in C \) by
\[ P \mapsto \vec{A}(P) = 2\pi \int_{P_0}^P \vec{\omega} \mod \Lambda_m, \tag{1.7.6} \]
where \( \vec{\omega} = (\omega_0, \omega_1, \ldots, \omega_g)^T \) is the vector of basis holomorphic forms together with the Abelian differential of the third kind \( \omega_0 \).

**Remark** As for the usual Abel map, the integrals \( \int_{P_0}^P \vec{\omega} \) are not well defined since they depend on the path \( \gamma \) joining the base point \( P_0 \) to \( P \). But if \( \gamma' \) is another such path then it is straightforward to see that \( 2\pi \int_{\gamma} \vec{\omega} - 2\pi \int_{\gamma'} \vec{\omega} \in \Lambda_m \).

**Remark** Note that the points in the support \( S \) of the modulus \( \mathfrak{m} \) are avoided in the definition of the generalised Abel map since \( \omega_0 \) has poles there and so \( \vec{A}(P) \) would diverge there.

The generalised Abel map can also be extended to the group \( \text{Div}(C \setminus S) \) of divisors prime to \( S \) in the obvious way such that \( \vec{A} : \text{Div}(C \setminus S) \to J_m(C) \) is a group homomorphism. When acting on divisors of degree zero the Abel map \( \vec{A} : \text{Div}^0(C \setminus S) \to J_m(C) \) it does not depend on \( P_0 \). Moreover, by theorem 1.7.9 and theorem 1.7.10 below which are generalisations of Abel and Jacobi’s theorems,
this map on divisors of $C$ prime to $S$ descends to an isomorphism

$$\tilde{A} : \text{Pic}^0_m(C) \longrightarrow J_m(C)$$

between the **generalised Picard group** $\text{Pic}^0_m(C) = \text{Div}(C \setminus S)/\sim_m$ of degree zero divisors prime to $S$ modulo $m$-equivalence and the generalised Jacobian $J_m(C)$.

**Theorem 1.7.9** (generalised Abel). A divisor $D \in \text{Div}(C \setminus S)$ is of the form $D = (f)$ for some meromorphic function $f$ with $f(P_1) = f(P_2)$ if and only if $\deg D = 0$ and $\tilde{A}(D) = 0$.

**Proof.** By Abel’s theorem we have $D = (f)$ for some meromorphic function $f$ if and only if $\deg D = 0$ and $A(D) = 0$. Furthermore, it is immediate from lemma 1.5.15 that $f(P_1) = f(P_2)$ if and only if $A_0(D) = 0$. \hfill $\square$

We also have the following generalisation of Jacobi’s theorem [42].

**Theorem 1.7.10** (generalised Jacobi). Every point in $J_m(C)$ is the image of an integral divisor of degree $g + 1$.

**θ-functions**

**Definition 1.7.11.** The **Riemann θ-function** $\theta : \mathbb{C}^g \rightarrow \mathbb{C}$ is given by

$$z \mapsto \theta(z; \Pi) = \sum_{m \in \mathbb{Z}^g} \exp \{i\langle m, z \rangle + \pi i\langle \Pi m, m \rangle \}.$$  

where $\langle x, y \rangle = \sum_{i=1}^{g} x_i y_i$. When it is clear which period matrix we are using we shall omit it from the arguments and simply write $\theta(z) = \theta(z; \Pi)$.

It can be shown [36, pp.299–300] that the sum converges absolutely and uniformly on any compact subset of $\mathbb{C}^g$ and thus the Riemann θ-function is homolorphic
on the whole of \( \mathbb{C}^g \). Furthermore, it is obviously even \( \theta(-z) = \theta(z) \) and has the following important automorphy property under translation by lattice vectors \( 2\pi n + 2\pi \Pi m \in \Lambda \),

\[
\theta(z + 2\pi n + 2\pi \Pi m) = \exp \{-i\langle m, z \rangle - \pi i \langle \Pi m, m \rangle\} \theta(z). \tag{1.7.9}
\]

Note that although the Riemann \( \theta \)-function is defined on \( \mathbb{C}^g \), by the automorphy property its zeroes naturally live on the Jacobian \( J(M) \).

Combining the Riemann \( \theta \)-function with the Abel map \( \mathcal{A} \) we can define an interesting multi-valued function on \( M \). Let \( w \in \mathbb{C}^g \) be an arbitrary vector and consider the function \( P \mapsto \theta(\mathcal{A}(P) - w) \). Its zeroes are well defined on \( M \) and are characterised by the fundamental theorem of Riemann,

**Theorem 1.7.12** (Riemann). If \( P \mapsto \theta(\mathcal{A}(P) - w) \) does not vanish identically then it has exactly \( g \) zeroes \( P_1, \ldots, P_g \in M \) satisfying

\[
\mathcal{A}(P_1) + \cdots + \mathcal{A}(P_g) = w - \mathcal{K}, \tag{1.7.10}
\]

where \( \mathcal{K} \) is the vector of Riemann’s constants which depends only on \( M \) and the base point \( P_0 \) of the Abel map, given explicitly in components by

\[
\mathcal{K}_k = 2\pi \left[ \frac{1 + \Pi_{kk}}{2} - \sum_{j=1, j\neq k}^g \int_{a_j}^{a_j} \left( \int_{P_0}^{P_k} \omega_j \right) \omega_j \right]. \tag{1.7.11}
\]

Now let \( D > 0 \) be an integral divisor of degree \( \deg D = g \) and in view of equation (1.7.10) introduce the notation \( \zeta_D \equiv \mathcal{A}(D) + \mathcal{K} \). An important function that constitutes the building block for constructing functions on \( M \) with specified poles and zeroes is the following multi-valued function

\[
\psi_D : P \mapsto \theta(\mathcal{A}(P) - \zeta_D).
\]
The following theorem [36, p.313] asserts that a necessary and sufficient condition for $\psi_D$ to vanish identically is that the divisor $D$ be special.

**Theorem 1.7.13.** $\psi_D \not\equiv 0$ if and only if $i(D) = 0$.

Since the Riemann $\theta$-function is holomorphic the function $\psi_D$ has no poles, and by the automorphy property its zeroes are well defined on $M$. Therefore although $\psi_D$ is multi-valued its divisor $(\psi_D)$ is well defined on $M$ and we have

**Corollary 1.7.14.** If $D$ is non-special then $(\psi_D) = D$.

*Proof.* Since $D$ is non-special we have $i(D) = 0$ so that $\psi_D \not\equiv 0$ by theorem 1.7.13. But then Riemann’s theorem tells us that $\psi_D$ has exactly $g$ zeros $P_1, \ldots, P_g$ subject to the condition $\mathcal{A}(P_1) + \ldots + \mathcal{A}(P_g) = \zeta_D - \mathcal{K} = \mathcal{A}(D)$, namely

$$\mathcal{A}(P_1 + \ldots + P_g - D) = 0.$$ 

Now $\deg D = g$ implies $\deg (P_1 + \ldots + P_g - D) = 0$ and so by Abel’s theorem the divisor $P_1 + \ldots + P_g - D = (f)$ is principal, for some meromorphic function $f$. But since $i(D) = 0$ and $\deg D = g$, by the Riemann-Roch theorem $r(-D) = 1$ so that $f$ is constant and hence $P_1 + \ldots + P_g = D$. $\square$
Chapter 2

Semiclassical Approximations

In this chapter we review the necessary notions from semiclassical quantisation of finite-dimensional systems, based on [43–52], relevant for Part IV.

Consider a classical Hamiltonian system described by a $2n$ dimensional phase-space $(T^*X, \omega)$ with Hamiltonian $H : T^*X \to \mathbb{R}$. Classically we are interested in the trajectories of $H$, namely the integral curves of the vector field $X_H$ on $T^*X$ which solves Hamilton’s equation

$$\iota_{X_H} \omega = -dH. \quad (2.0.1)$$

The Hamiltonian is conserved along any trajectory since $X_H H = dH(X_H) = 0$. This constant value $E \in \mathbb{R}$ of $H$ defines the ‘energy’ of the trajectory which must therefore be constrained to the codimension one level set $\Sigma_E \equiv H^{-1}(E) \subset T^*X$.

Assume also that we have a desired quantisation of the system, that is we are given a self-adjoint operator $\hat{H} = H(x, -i\hbar \partial_x)$, for some choice of operator ordering, acting on $L^2(X)$. Quantum mechanically we are interested in the spectrum of this operator, namely the values of $E$ for which there exists a $\psi \in L^2(X)$ which solves
Schrödinger’s equation
\[ (\hat{H} - E)\psi = 0. \]  
(2.0.2)

The subject of semiclassical analysis is to understand how the two regimes are related in the limit $\hbar \to 0$. Therefore the immediate goal of semiclassical quantisation is to obtain the spectrum of $\hat{H}$ to leading order in $\hbar$ by solving the Schrödinger equation to that order,
\[ (\hat{H} - E)\psi = O(\hbar^2). \]  
(2.0.3)

The values $\{E_j^\hbar\}_{j=0}^\infty$ of $E$ for which this equation admits a solution for $\psi$ approximate the spectrum of $\hat{H}$ to order $O(\hbar)$.

One possible approach to obtain these values is to use what are known as trace formulae. The basic idea is to encode the spectrum in terms of a single function $n(E) \equiv \sum_{j=0}^\infty \delta(E - E_j^\hbar) = \text{tr} \delta(E - \hat{H})$ which one rewrites as
\[ n(E) = \text{Re} \frac{1}{\pi \hbar} \int_0^\infty dt \text{tr} e^{\frac{i}{\hbar}(E - \hat{H})t} = \text{Re} \frac{1}{\pi \hbar} \int_0^\infty dt e^{\frac{i}{\hbar}t} \int_{\text{p.o.} \gamma} e^{-\frac{i}{\hbar}\int_{\gamma} L} e^{-\frac{i}{\hbar}\int_{\gamma} L}, \]  
(2.0.4)

where the path integral is over closed paths $\gamma$ of period $t$ to account for the trace. In the semiclassical limit $\hbar \to 0$ we can evaluate the integral in the stationary phase approximation. If we assume that every periodic trajectory of the flow $X_H$ is isolated on the level set $\Sigma_E$ then the dominant contributions to the path integral will come from each isolated periodic orbit of the classical system. The result is known as the Gutzwiller trace formula. It ‘associates’ to each periodic orbit of the classical system a tower of semiclassical energy eigenvalues $\{E_j^\hbar\}$ of $\hat{H}$.

The connection between a periodic orbit $\gamma \in \Sigma_E$ and its associated spectrum $\{E_j^\hbar\}$ determined by the Gutzwiller trace formula is best understood in terms of the classical cylinder theorem [54, p576].

**Theorem 2.0.15** (Cylinder theorem). Let $\gamma \in H^{-1}(E)$ be a non-degenerate periodic
orbit of $X_H$. Then there exists $\epsilon > 0$ and $\Gamma : [E - \epsilon, E + \epsilon] \times S^1 \to T^*X$ such that for any $E_0 \in [E - \epsilon, E + \epsilon]$ the closed curve $\gamma_{E_0} = \Gamma(E_0, \cdot)$ is a periodic orbit of $X_H$ in $H^{-1}(E_0)$, see Figure 2.1.

Figure 2.1: Cylinder theorem: a periodic solution $\gamma_E$ on the energy level $H^{-1}(E)$ is contained in a one parameter family of periodic solutions of varying energy in the range $[E - \epsilon, E + \epsilon]$.

Now the Gutzwiller trace formula can also be written as a Bohr-Sommerfeld quantification condition (2.4.2) which essentially says that $\int_{\gamma_E} \lambda_{BS} \in \mathbb{Z}$ for a specific 1-form $\lambda_{BS}$ to be identified later. The Bohr-Sommerfeld condition therefore picks out a discrete set $\gamma_{E_j^h}$ of periodic orbits from the cylinder of theorem 2.0.15 whose energies $\{E_j^h\}$ are semiclassical approximations to eigenvalues of $\hat{H}$. This illustrates a very general feature of semiclassical analysis whereby analytic data of the quantum theory (here the spectrum of the operator $\hat{H}$) is related to geometric data of the classical theory (here the periodic orbits of the classical Hamiltonian $H$).

The assumption of non-degeneracy of the periodic orbits of the Hamiltonian flow $X_H$ on the energy surface $\Sigma_E$ is crucial in discussing the Gutzwiller trace formula: without it certain periodic orbits might no longer be isolated local minima of the action which complicates the stationary phase approximation. Yet this assumption easily breaks down, for instance when the system possesses just a single other first integral of motion, say $F$, since its flow $X_F$ then generates from $\gamma \subset \Sigma_E$.

---

1 $F$ is an integral of $X_H$ if $dF \neq 0$ almost everywhere and $X_H F = \{H, F\} = 0$. 

---
a continuous family of periodic orbits on the hypersurface $\Sigma_E$ itself. Indeed, if $\phi_t^H$ denotes the flow of $X_H$, so that $\gamma = \{\phi_t^H(p_0)\}_{t=0}^{T}$ is a periodic orbit through $p_0$, and $\phi_s^F$ the flow of $X_F$, then $\gamma_s = \{\phi_t^H \circ \phi_s^F(p_0)\}_{t=0}^{T}$ is a continuous family of periodic orbits containing $\gamma = \gamma_0$ (see Figure 2.2). Therefore the semiclassical approximation

![Figure 2.2: Degeneracy of periodic orbit on $\Sigma_E$ in presence of a symmetry.](image)

of the path integral in (2.0.4) will lead to a different semiclassical trace formula in the degenerate case.

Suppose the Hamiltonian system locally possesses a total of $p$ independent integrals of motion $F_1, \ldots, F_p$ where $1 < p \leq n$. Classically it is natural to consider all these integrals on the same footing as the Hamiltonian $H = H(F_1, \ldots, F_p)$. So rather than focusing on the Hamiltonian and its flow $X_H$ one should instead use the moment map

$$ F \equiv (F_1, \ldots, F_p) : T^*X \to \mathbb{R}^p $$

which generates a $p$-parameter flow $X_{F_i}$ through Hamilton’s equation

$$ \iota_{X_{F_i}} \omega = -dF_i, \quad i = 1, \ldots, p. \tag{2.0.5} $$

Each integral $F_j$ is conserved along these flows since $X_{F_i}F_j = dF_j(X_{F_i}) = 0$. In other words for any $f \in \mathbb{R}^p$ the level set $\Sigma_f \equiv F^{-1}(f) \subset T^*X$ is invariant under the flows $X_{F_i}$. The objects of classical interest here are the integral manifolds $\Lambda_f \subset \Sigma_f$ of the vector fields $X_{F_i}$. From now on we assume $f$ to be a regular value of $F$ meaning that $dF$ has maximal rank $p$. Then the integral manifold $\Lambda_f$ is a $p$-dimensional manifold
and assuming it is compact it must be a \( p \)-torus \( \mathbb{T}^p \). Furthermore, the level set \( \Sigma_f \) is of codimension \( p \) in \( T^*X \). The proper generalisation of the cylinder theorem 2.0.15 to Hamiltonian systems with integrals is [55, theorem 2.4 and lemma 2.6 pp.89-94]

**Theorem 2.0.16** (Generalised cylinder theorem). *With the above assumptions, let \( \Lambda_f \in \Sigma_f \) be an integral manifold of the \( X_{F_i} \). Then there exists a small neighbourhood \( \mathcal{U}_f \) of \( f \in \mathbb{R}^p \) and \( \Gamma : \mathcal{U}_f \times \mathbb{T}^p \to T^*X \) such that for any \( f_0 \in \mathcal{U}_f \) the \( p \)-torus \( \Lambda_{f_0} = \Gamma(f_0, \cdot) \) is an integral manifold of the \( X_{F_i} \) in \( \Sigma_{f_0} \).

Once again the Bohr-Sommerfeld conditions for such a system should pick out a discrete set of \( p \)-torii whose levels \( f_i \) provide semiclassical approximations to eigenvalues of the operators \( \hat{F}_i \). In other words these levels are such that the joint system of Schrödinger equations admits a solution for \( \psi \in L^2(X) \),

\[
(\hat{F}_i - f_i)\psi = O(\hbar^2), \quad i = 1, \ldots, p.
\]

(2.0.6)

This is the analogue of (2.0.3) for a Hamiltonian system with symmetries. Note that for (2.0.6) to admit a solution at all requires that \( [\hat{F}_i, \hat{F}_j] = O(\hbar^3) \) which is the semiclassical analogue of \( \{ F_i, F_j \} = 0 \).

The extreme case \( p = n \) corresponds to an integrable system which possesses the maximum number \( n \) of independent Poisson commuting first integrals. For such a system we must have \( \Lambda_f = \Sigma_f \) since \( \Lambda_f \subset \Sigma_f \) and both manifolds have the same dimension \( n \). In the following we will focus on this case since all intermediate cases can be obtained from it as we will see in section 2.4 The path integral treatment of integrable systems would lead to a semi-classical trace formula known as the Berry-Tabor formula [46, 47].

Despite the geometrical appeal of the path integral approach to semiclassical quantisation it is hard to discuss the issues of operator ordering within this framework. Indeed, thinking in terms of phase-space path integrals, since every-
thing in the integrand itself is classical, any information about quantum ordering is neatly tucked away in the definition of the regularisation used in the phase-space path integral measure \([d\gamma]\). The standard choice of discretisation of the path integral measure involves the mid-point prescription which corresponds to the Weyl-ordering prescription in the operator formalism. In particular the quantum Hamiltonian is the Weyl-ordered classical Hamiltonian, i.e. \(\hat{H} = \text{Op}_h^W(H)\). In order to deal with operator ordering issues, it is therefore more convenient to work directly with operators and attempt to solve Schrödinger’s equation (2.0.3) or (2.0.6) order by order in \(\hbar\). This is also mathematically better defined than path integral methods, although both lead to the same Bohr-Sommerfeld conditions which are necessary and sufficient conditions on the energy \(E\) (respectively the levels \(f\)) for the existence of a solution to (2.0.3) (respectively (2.0.6)).

A convenient operator formalism for discussing semiclassical quantisation involves pseudo-differential operators (referred to as ΨDOs for short). In section 2.1 we give a very brief introduction to ΨDOs and their relevance for treating semiclassical quantisation. We use it to discuss the issue of operator ordering in an integrable system in section 2.2. In section 2.3 we will show how the Bohr-Sommerfeld quantisation conditions are modified by the presence of a subprincipal symbol which reflects a choice of ordering.

To get an intuitive idea of how operator ordering ambiguities arise even at the semiclassical level to affect the quantisation conditions, it is instructive to consider the simple example of the harmonic oscillator for which the leading order quantisation is exact. The classical harmonic oscillator Hamiltonian is \(H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\), and the action variable of the closed path of energy \(E\) is given by

\[
I = \frac{1}{2\pi} \int_{H=E} p\,dx = \frac{E}{\omega}.
\]
By promoting the variables $x, p$ to operators $\hat{x}, \hat{p}$ there is only one reasonable choice of ordering in the Hamiltonian, namely the Weyl-ordered Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The spectrum of such an operator is well known to be $E_n = (n + \frac{1}{2})\hbar\omega$, $n \in \mathbb{N}$ so that the spectrum of the Weyl-ordered action variable $\hat{I} = \frac{1}{\omega}\hat{H}$ is simply,

$$\text{Spec (} \hat{I} \text{)} \subset \left( \mathbb{Z} + \frac{1}{2} \right)\hbar,$$

where the index of $\frac{1}{2}$ by which the spectrum is shifted from $\hbar\mathbb{Z}$ is known as the Maslov index in the context of Bohr-Sommerfeld quantisation. Now since we are given at the outset only the classical Hamiltonian, we could always choose to quantise it with a more perverse choice of ordering. For instance, if we rewrite the classical Hamiltonian as $H = \omega a a^*$ where $a \equiv \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{ip}{2m} \right)$ and after promoting everything to operators request that in the quantum Hamiltonian the $\hat{a}$ sits to the right of the $\hat{a}^\dagger$ then we obtain the normal-ordered Hamiltonian $\hat{H} : = \omega\hbar\hat{a}^\dagger\hat{a}$, where $[\hat{a}, \hat{a}^\dagger] = 1$. The corresponding normal-ordered action operator is given by $\hat{I} : = \hbar\hat{a}^\dagger\hat{a}$ whose spectrum is easily seen to consist of integer multiples of $\hbar$, \n
$$\text{Spec (} \hat{I} \text{)} \subset \mathbb{Z}\hbar.$$

We observe that the Maslov index is precisely cancelled by the shift from Weyl-ordering to normal-ordering.

## 2.1 Pseudo-differential operators

The passage from a classical system on phase-space $T^*X$ to its quantum counterpart involves promoting the algebra of classical observable $C(T^*X)$ to a noncommutative algebra $\mathcal{A}$ of operators. Classically, the Poisson algebra of observables is uniquely specified by the choice of a symplectic structure $\omega = \sum_i d\xi_i \wedge dx_i$ and the Poisson
bracket of two observables \( f, g \in C(T^*X) \) is then defined by \( \{f, g\} = \omega(X_f, X_g) \), where \( X_H \) denotes the Hamiltonian vector field associated to any function \( H \in C(T^*X) \) satisfying \( i_{X_H} \omega = -dH \). To pass to quantum mechanics, the prescription of **canonical quantisation** is to promote the special functions \( x_i, \xi_i \in C(T^*X) \) to operators \( \hat{x}_i, \hat{\xi}_i \) and the symplectic structure \( \omega = \sum_i d\xi_i \wedge dx_i \) to the Weyl algebra \([\hat{x}_i, \hat{\xi}_j] = i\hbar \delta_{ij}\) which admits the unique representation \( \hat{x}_i = x_i, \hat{\xi}_i = -i\hbar \partial / \partial x_i \equiv -i\hbar \partial_i \) in terms of differential operators on \( L^2(X) \). The problem that remains after canonical quantisation is to associate with any other given observable \( f \in C(T^*X) \) (function of \( x_i, \xi_i \)) a (pseudo-)differential operator \( \hat{f} \) on \( L^2(X) \), and it is immediately obvious that this is by no means unique. Many different operators correspond to the same classical function: for instance, given any \( t \in \mathbb{R} \), the differential operator \( tx_1 \partial_1 + (1-t) \partial_1 \cdot x_1 \) is a possible candidate for the quantisation of the function \( x_1 \xi_1 \). In other words, it is not possible to specify the operator ordering in an operator \( \hat{f} \) starting from just single function \( f \in C(T^*X) \). However, with an infinite set of functions \( f_k \in C(T^*X) \) it turns out to be possible to associate a unique operator \( \hat{f} \) by canonical quantisation. Such a set defines a function of \( \hbar \) through the asymptotic expansion

\[
 f(x, \xi; \hbar) \sim \sum_{k \geq 0} f_k(x, \xi) \hbar^k. \tag{2.1.1}
\]

We refer to such a \( \hbar \)-dependent function \( f(\hbar) \in C(T^*X) \) as a **classical (Weyl) symbol**, which is technically required to satisfy certain estimates, such as all its partial derivatives being uniformly bounded by some order function [45].

Without going into details of the construction, we now state the map from symbols to **pseudo-differential operators**\(^2\) (\( \Psi \text{DO} \) for short). Given a symbol \( f(\hbar) \), we define the corresponding \( \Psi \text{DO} \) by specifying its action on \( u \in L^2(X) \) using

\(^2\)When the symbol \( f(x, \xi; \hbar) \) is a polynomial in \( x, \xi \) the associated operator is an ordinary partial differential operator. To include the more general case when \( f(x, \xi; \hbar) \) might not be a polynomial we talk about pseudo-differential operators.
the \textbf{Weyl quantisation} formula (see [45, chapter 2] for details)

\[
(\text{Op}_\hbar^W (f(h))u) (x) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot \xi} f \left( \frac{x + y}{2}, \xi; \hbar \right) u(y) dy d\xi.
\]

It is important to note here that the choice of Weyl quantisation in the definition of the \(\Psi DO\) from its symbol does not limit us to having only Weyl ordered \(\Psi DOs\). Indeed, the operator \(\text{Op}_\hbar^W (f(h))\) is Weyl ordered only when the corresponding Weyl symbol is \(\hbar\)-independent. So it is precisely the subleading terms in the asymptotic expansion (2.1.1) of the symbol \(f(x, \xi; \hbar)\) which account for the different possible choices of orderings in the definition of the \(\Psi DO\). For example, the Weyl ordered operator of the classical observable \(x_1 \xi_1\) is given simply by the Weyl symbol \(x_1 \xi_1\), namely

\[
\text{Op}_\hbar^W (x_1 \xi_1) = \frac{-i\hbar}{2} (x_1 \partial_1 + \partial_1 \cdot x_1),
\]

whereas the left ordered operator \(-i\hbar x_1 \partial_1\) which corresponds to the same classical observable \(x_1 \xi_1\) as \(\text{Op}_\hbar^W (x_1 \xi_1)\) is given by a Weyl symbol with a subleading term in \(\hbar\) since

\[
\text{Op}_\hbar^W \left( x_1 \xi_1 + \frac{i\hbar}{2} \right) = -i\hbar x_1 \partial_1.
\]

Naturally the right ordered operator \(-i\hbar \partial_1 \cdot x_1\) has Weyl symbol \(x_1 \xi_1 - \frac{i\hbar}{2}\). A general \(\Psi DO\) \(A\) always has a unique Weyl symbol, which is a \(\hbar\)-dependent function \(f(x, \xi; \hbar)\) denoted \(\sigma^W(A)\). The leading non-zero term in the asymptotic expansion (2.1.1) of this Weyl symbol is called the \textbf{principal symbol}, denoted \(\sigma^W_0(A)\), and the subleading term is called the \textbf{subprincipal symbol}, denoted \(\sigma^W_{\text{sub}}(A)\). For instance, if \(f_0(x, \xi) \neq 0\) then \(\sigma^W_0(A) = f_0(x, \xi)\) and \(\sigma^W_{\text{sub}}(A) = f_1(x, \xi)\hbar\).

An important object for the study of quantum integrability is the commutator \([A, B]\) of two operators \(A\) and \(B\). In the present context of \(\Psi DOs\) one can show that if \(A, B\) are \(\Psi DOs\) then their commutator \([A, B]\) is also a \(\Psi DO\) with principal
symbol
\[ \sigma_0^W([A, B]) = -i\hbar \{ \sigma_0^W(A), \sigma_0^W(B) \}, \] (2.1.2)

(so that \(-i\hbar\sigma_0^W\) is a Lie algebra homomorphism) and subprincipal symbol
\[ \sigma_{\text{sub}}^W([A, B]) = -i\hbar \{ \sigma_0^W(A), \sigma_{\text{sub}}^W(B) \} - i\hbar \{ \sigma_{\text{sub}}^W(A), \sigma_0^W(B) \}. \] (2.1.3)

### 2.2 Integrable systems

As explained in section 2.1, one can keep track of operator orderings in the language of pseudo-differential operators by retaining subleading terms beyond the principal symbol in the full Weyl symbol of an operator. In most applications of the theory of \(\Psi\)DOs the quantities of interest are specified as \(\Psi\)DOs at the outset so that their full Weyl symbol is known. In the present case however we start from a classical system specified by its phase-space \((T^*X, \omega)\) and the set of classical observables of interest are \(F_1, \ldots, F_n, H\). Quantising this classical system requires an operator ordering prescription for obtaining operators from the corresponding classical observables. At the semiclassical level this boils down to the specification of an extra function, the subprincipal symbol, for each classical observable. Specifically, given a classical observable \(f_0 \in C(T^*X)\), we construct
\[ \hat{f} = \text{Op}_\hbar^W(f_0 + f_1\hbar), \]

where the presence of the subprincipal symbol \(f_1 \in C(T^*X)\) reflects the operator ordering ambiguities already manifesting themselves at the semiclassical level. Every possible choice of a function \(f_1 \in C(T^*X)\) corresponds to a different prescription for the operator ordering in \(\hat{f}\) at order \(O(\hbar)\). The principal symbol \(f_0 = \sigma_0^W(\hat{f})\) is the corresponding classical observable.
Recall the definition of an integrable system, which roughly speaking is one which possesses the maximum possible number of independent integrals of motion.

**Definition 2.2.1.** A Hamiltonian system \((T^*X, H)\) is said to be **classically integrable** if there exists \(n = \dim X\) functions \(F_1, \ldots, F_n \in C(T^*X)\) such that

\[
\begin{align*}
(1') & \quad dF_1 \wedge \ldots \wedge dF_n \neq 0 \text{ almost everywhere}, \\
(2') & \quad \{F_i, F_j\} = 0, \quad \forall i, j = 1, \ldots, n, \\
(3') & \quad H = H(F_1, \ldots, F_n).
\end{align*}
\]

Conditions (2') and (3') together imply that the \(F_i\) are in fact integrals of motion, \(X_H F_i = 0\). In other words, \(T^*X\) admits a torus action with moment map

\[
F \equiv (F_1, \ldots, F_n) : T^*X \to \mathbb{R}^n.
\]

When dealing with an integrable system it is convenient to treat all the integrals of motion on the same footing as the Hamiltonian \(H\) itself. At regular values \(f\) of \(F\), the level sets \(F^{-1}(f)\) define \(n\)-torii (in the compact case) and foliate \(T^*X\)

\[
\mathbb{T}^n \hookrightarrow T^*X \xrightarrow{F} \mathbb{R}^n.
\]

This foliation allows one to define canonical action-angle coordinates with the action variables \(\{I_i\}_{i=1}^n\) parametrising the base \(\mathbb{R}^n\) and the conjugate angle variables \(\{\theta_i\}_{i=1}^n\), each taking values in \([0, 2\pi]\), parametrising the independent cycles of the torus \(\mathbb{T}^n\).

The condition (2') can be phrased as \(\omega(X_{F_i}, X_{F_j}) = 0\) which says that the pullback of the symplectic form \(\omega\) to a level set \(\Lambda_f\) vanishes. In other words, the Liouville form \(\alpha\) defined as a primitive of \(\omega = d\alpha\) is closed on \(\Lambda_f\).

**Definition 2.2.2.** A \(\Psi DO\) \(\hat{H}\) is **semiclassically integrable** if there exists \(n\) \(\Psi DOs\) \(\hat{F}_1, \ldots, \hat{F}_n\) with principal symbols \(F_i = \sigma^W_0(\hat{F}_i)\) such that
\( dF_1 \wedge \ldots \wedge dF_n \neq 0 \) almost everywhere,

\[ [\hat{F}_i, \hat{F}_j] = O(\hbar^3), \quad \forall i, j = 1, \ldots, n, \]

\[ \hat{H} = H(\hat{F}_1, \ldots, \hat{F}_n) + O(\hbar^2) \text{ for some function } H. \]

Notice that we only require commutativity modulo \( O(\hbar^3) \) in property (2); it guarantees in particular that the operator \( H(\hat{F}_1, \ldots, \hat{F}_n) \) in (3) is free of operator ordering ambiguities certainly up to \( O(\hbar^3) \), so that property (3) makes sense. Property (2) is to be contrasted with the definition of full quantum integrability which requires exact commutativity \( [\hat{F}_i, \hat{F}_j] = 0 \). Now since \( \sigma^W_0([\hat{F}_i, \hat{F}_j]) = -i\hbar\{F_i, F_j\} \) by (2.1.2) and \( \sigma^W_0(\hat{H}) = H(F_1, \ldots, F_n) \), it follows that the principal symbols \( F_i = \sigma^W_0(\hat{F}_i) \) satisfy all three properties \((1')-(3')\) above for a classically integrable system with Hamiltonian \( H = \sigma^W_0(\hat{H}) \). This means that any semiclassically integrable system exhibits at leading order the full geometric structure of the underlying classically integrable system given by its principal symbols. In particular, the level set \( \Lambda_f \equiv F^{-1}(f) \) of the moment map \( F = (F_1, \ldots, F_n) : T^*X \to \mathbb{R}^n \) is a Lagrangian \( n \)-torus and foliates phase-space \( T^*X \) as we let \( f \) vary.

But the notion of semiclassical integrability contains more information than that of its underlying classical integrable structure [51, 52]. Property (1) only contributes at leading order since it is a statement about the principal symbols \( F_i \) alone, whereas property (2) at \( O(\hbar^2) \) yields an equation for the subprincipal symbols \( F^s_i = \sigma^W_{\text{sub}}(\hat{F}_i) \) of the \( \hat{F}_i \) using (2.1.3),

\[
0 = \frac{i}{\hbar} \sigma^W_{\text{sub}}([\hat{F}_i, \hat{F}_j]) = \{F_i, F^s_j\} + \{F^s_i, F_j\}. \tag{2.2.1}
\]

It is possible to interpret these equations geometrically so as to supplement the geometrical structure of the underlying classical integrable system defined by principal symbols. For this we define the **subprincipal form** \( \kappa \) on \( \Lambda_f \) by specifying its action
on the basis vectors $X_{F_i}$ at any point of $\Lambda_f$ through [51]

$$\kappa(X_{F_i}) = -F_i^s, \quad i = 1, \ldots, n. \quad (2.2.2)$$

It then follows immediately from (2.2.1) that $\kappa$ is closed since

$$d\kappa(X_{F_i}, X_{F_j}) = X_{F_i}\kappa(X_{F_j}) - X_{F_j}\kappa(X_{F_i}) - \kappa([X_{F_i}, X_{F_j}]) = -X_{F_i}F_j^s + X_{F_j}F_i^s - \kappa([F_i, F_j]) = \{F_i, F_j^s\} + \{F_j, F_i^s\} - 0 = 0.$$

Hence the operator ordering in the $\hat{F}_i$ can be accounted for at the semiclassical level by specifying a closed 1-form $\kappa$ on the Liouville $n$-torus $\Lambda_f$. And in fact it is clear from (2.2.2) that every choice of a closed 1-form $\kappa \in \Omega^1(\Lambda_f)$ corresponds to a different choice of operator ordering in the definition of the $\hat{F}_i$.

To summarise, the classical and semiclassical integrability conditions can both be expressed as the closure of the Liouville form $\alpha$ and subprincipal form $\kappa$ respectively on the level set $\Lambda_f$,

$$\text{Classical : } \quad d\alpha = 0 \quad \text{on } \Lambda_f, \quad (2.2.3a)$$

$$\text{Semiclassical : } \quad d\kappa = 0 \quad \text{on } \Lambda_f. \quad (2.2.3b)$$

### 2.3 Bohr-Sommerfeld conditions

We are interested in the joint spectrum of the $\hat{F}_i$ up to $O(\hbar)$ which requires solving the eigenvalue problem to that order

$$(\hat{F}_i - f_i)\psi = O(\hbar^2). \quad (2.3.1)$$
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The Bohr-Sommerfeld conditions are conditions on the set \( \{ f_i \} \) for the existence of a solution to these coupled pseudo-differential equations. Their rigorous derivation is rather involved (see for instance [51,52]) so here we would just like to outline how the subprincipal symbol comes about in these conditions. To solve (2.3.1) locally one considers a local patch \( V \subset \Lambda_f \) on which \( \pi : T^*X \to X \) is a diffeomorphism and uses the WKB ansatz

\[
\psi_{WKB} = e^{i\phi - 1 + i\phi_0} \rho + O(\hbar)
\]  (2.3.2)

on \( U = \pi(V) \subset X \) where the nature of \( \rho \) will be specified shortly. If we denote by \( \iota_{d\phi - 1} : U \hookrightarrow T^*X \) the 1-form \( d\phi - 1 \) viewed as a map then equation (2.3.1) can be shown [43] at leading order in \( \hbar \) to imply to the so called eikonal equation

\[
\text{im } \iota_{d\phi - 1} = V \subset \Lambda_f.
\]  (2.3.3)

Therefore \( \iota_{d\phi - 1} : U \to V \) with \( \pi \circ \iota_{d\phi - 1} = \text{id}_U \) so that \( \iota_{d\phi - 1} = \pi|_V^{-1} \). By a property of the tautological 1-form \( \alpha [43, \text{lemma 3.23 p29}] \), namely \( d\phi - 1 = \iota^*_{d\phi - 1} \alpha \), we then have

\[
d\pi|_V^* \phi - 1 = \alpha.
\]

In other words, \( \pi|_V^* \phi - 1 \) is a local solution to the classical integrability condition (2.2.3a). If \( \rho \) is a half-density\(^3\) on \( U \subset X \) then the subleading order of (2.3.1) implies the so called transport equation which can be written invariantly as [44, theorem 11.11 p126]

\[
\left( -i\mathcal{L}_{X_{F_i}} + F_i^s \right) \left( \pi|_V^* e^{i\phi_0} \rho \right) = 0.
\]  (2.3.4)

\(^3\)Since the product of two half-densities is a density of weight one there is a natural inner-product on half densities \( \langle \rho_1, \rho_2 \rangle = \int_M \rho_1 \rho_2 \) which makes the completion into a Hilbert space.
Writing $a = \pi|_V^*\phi_0$, since $e^{ia}$ is a function we have $\mathcal{L}_{X_{\pi}} e^{ia} = i X_{\pi} d e^{ia}$. Now using (2.2.2) we can rewrite (2.3.1) as

$$[d\pi|_V^*\phi_0(X_{F_i}) - \kappa(X_{F_i})](\pi|_V^*\rho) = i\mathcal{L}_{X_{F_i}} (\pi|_V^*\rho).$$  (2.3.5)

Therefore provided the subprincipal symbols are real this equation implies on the one hand that $\pi|_V^*\rho$ is an invariant half-density on $\Lambda_f$, i.e. $\mathcal{L}_{X_{\pi}} \pi|_V^*\rho = 0$, and on the other hand that

$$d\pi|_V^*\phi_0 = \kappa.$$  

But this just says that $\pi|_V^*\phi_0$ is a local solution to the subleading integrability condition (2.2.3b). To summarise, in a neighbourhood $V \subset \Lambda_f$ where $\pi|_V$ is a diffeomorphism the eigenvalue equation (2.2.3a) is solved by (2.3.2) if $\phi_{-1}$ and $\phi_0$ are primitives of the Liouville form $\alpha$ and the subprincipal form $\kappa$ respectively.

However, one runs into problems at caustic points where $\pi$ is singular (see Figure 2.3). A way around this problem was proposed by Maslov, the idea being to obtain a solution of (2.3.1) which is localised and defined patchwise on $\Lambda_f$ (near caustics one uses the “momentum” projection $\pi_p$ of $T^*X$ onto a typical fibre of $T^*X$ instead of $\pi$). Since this wave-function is defined on the whole of $\Lambda_f$ and the level set $\Lambda_f$ is compact, the single-valuedness of this global solution requires its phase to be

Figure 2.3: Caustics of the Lagrangian submanifold $\Lambda_f$
an integer multiple of $2\pi$. The phase turns out to be that of the local WKB solutions \( \psi_{\text{WKB}} \) introduced above but with additional Maslov index corrections (coming from the caustics). The single-valuedness of this phase leads to the Bohr-Sommerfeld-Maslov quantisation conditions (see [51] for a nice review).

**Theorem 2.3.1** (Bohr-Sommerfeld-Maslov). The eigenvalue problem [2.3.1] has a solution if and only if

\[
\frac{1}{2\pi \hbar} \int_{\gamma_i} \alpha + \frac{1}{2\pi} \int_{\gamma_i} \kappa = N_i + \frac{\mu_{\gamma_i}}{4} + O(\hbar), \quad i = 1, \ldots, n
\]  

(2.3.6)

where \( \gamma_i \) is a basis of \( H_1(\Lambda_f, \mathbb{R}) \) with Maslov indices \( \mu_{\gamma_i} \in \mathbb{Z} \) and integers \( N_i \in \mathbb{Z} \).

Note in particular the presence of the subprincipal form \( \kappa \) which as we have argued is related to operator ordering ambiguities in going from a classically integrable system to its quantum (or just semiclassically integrable) integrable counterpart. It has the effect of shifting the spectrum of the action variables similar to what happened in the case of the harmonic oscillator when we changed quantisation from Weyl to normal ordering. In the cases where all the operators are chosen to be Weyl ordered, in particular the \( \hat{F}_i \), we have \( \kappa = 0 \) and (2.3.6) reduces to the EBK quantisation conditions. From now on we shall always assume that the cohomology class \([\kappa]\) \( \in \mathbb{H}^1(\Lambda_f) \) of the subprincipal form \( \kappa \) vanishes. The reason for this assumption is that the result is simpler to express in this case and moreover it will give results that agree with those of [56–58]. With this assumption, the Bohr-Sommerfeld-Maslov conditions simplify

\[
\frac{1}{2\pi \hbar} \int_{\gamma_i} \alpha = N_i + \frac{\mu_{\gamma_i}}{4} + O(\hbar), \quad i = 1, \ldots, n.
\]  

(2.3.7)

We stress that this assumption does not imply the choice of Weyl ordering since it only corresponds to setting the subprincipal symbol to zero, whereas Weyl ordering corresponds to setting all the lower order Weyl symbols to zero as well.
2.4 Bohr-Sommerfeld for degenerate torii

The derivation of the Bohr-Sommerfeld-Maslov conditions (2.3.6) or (2.3.7) essentially consisted in quantising a Lagrangian \( n \)-torus \( \Lambda_f \) by constructing a wavefunction localised around it. However, even though the level set \( \Lambda_f \equiv F^{-1}(f) \) is indeed a Lagrangian \( n \)-torus for almost every value of the integrals of motion \( f_1, \ldots, f_n \) in an integrable system, there exists interesting level sets \( F^{-1}(f) \) in phase-space where this is not the case. This happens at the (measure zero) set of critical values of the map \( F = (F_1, \ldots, F_n) \). Consider for instance the two-dimensional harmonic oscillator with different frequencies and total Hamiltonian

\[
H = \frac{p_1^2}{2} + \frac{1}{2}\omega_1^2 x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}\omega_2^2 x_2^2 = H_1 + H_2,
\]

(2.4.1)

whose integrals of motion are given by \( H_1, H_2 \). For non-zero values \( E_1, E_2 \neq 0 \) of \( H_1, H_2 \) the level sets \( H^{-1}(E_1, E_2) \) consists of two ellipses, in other words a Lagrangian 2-torus. However, if say \( E_2 = 0 \) the level set \( H^{-1}(E_1, 0) \) consists of just a single ellipse (Figure 2.4). The same thing is true when \( E_1 = 0 \) and at the point

\[
T_1 = \frac{2\pi}{\omega_1}, \quad T_2 = \frac{2\pi}{\omega_2}
\]

Figure 2.4: Periodic orbit with \( H_2 = 0 \) of energy \( H = H_1 = E \).

where \( E_1 = E_2 = 0 \) the level set consists of just a single point. One can draw a picture of the phase-space in the region where \( \mathcal{E} \equiv \{(E_1, E_2) : E_i \geq 0, i = 1, 2\} \) which is foliated by 2-tori in the interior of \( \mathcal{E} \) but with the fibres over the boundary \( \partial \mathcal{E} \setminus \{(0, 0)\} \) being ellipses and the fibre over the point \((0, 0)\) being just a single point,
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see Figure 2.5. Note that the set of critical values $\partial \mathcal{E}$ is of measure zero. However,

Figure 2.5: The phase-space of the two-dimensional harmonic oscillator.

if we are interested in the semiclassical spectrum of the two-dimensional harmonic oscillator in the region near $\partial \mathcal{E}$ then a modification of the Bohr-Sommerfeld-Maslov quantisation conditions (2.3.7) is required so that it applies to isotropic $p$-torii which are the level sets of a limited number $p < n$ of integrals of motion $F_1, \ldots, F_p$.

It was pointed out by Voros [48, 49] that the Bohr-Sommerfeld conditions (2.3.7) for the apparently more restrictive case of an integrable system may be used to obtain the Bohr-Sommerfeld conditions in all other intermediate cases, namely the partially integrable one (with $p < n$ integrals of motion) and even the non-degenerate case $p = 1$ (where $H$ is the only integral). If the system has $p$ independent observables $\mathbf{F} = (F_1, \ldots, F_p)$ in involution (with $H = H(\mathbf{F})$), then on each codimension $p$ level set $\Sigma_f = \mathbf{F}^{-1}(f)$ the system has a $p$-torus $\Lambda_f \subset \Sigma_f$ generated by the vector fields $X_{F_i}$. Each of these $p$-torii is surrounded by an $n$-torus of the linearised system to which the Bohr-Sommerfeld-Maslov conditions (2.3.7) may be applied. This results in a set of Bohr-Sommerfeld conditions for the cycles on the $p$-torus which include stability angles for the small fluctuations in the directions transverse to this $p$-torus. The derivation of these Bohr-Sommerfeld conditions from those in the integrable case (2.3.7) are a bit lengthy but the derivation in the more general
case 1 < p < n is conceptually the same as the p = 1 case. We will therefore outline the proof [48, 49] only in the latter case.

Let \( \gamma \subset \Sigma_E \) be a periodic orbit of energy \( E \). We henceforth assume that \( E \) is a regular value of \( H \) so that \( \Sigma_E \) is a smooth codimension one submanifold of \( T^*X \). Given a point \( p_0 \in \gamma \), we call a section of \( \gamma \) at \( p_0 \) a smooth codimension one surface \( S \subset \Sigma_E \) transverse to \( \gamma \) and intersecting it at \( p_0 \). We then define the

\[ P = d\psi_{p_0} : T_{p_0}S \to T_{p_0}S. \]

We say that the periodic orbit \( \gamma \) is non-degenerate if and only if 1 is not an eigenvalue of the Poincaré map. This is a way of saying that \( \gamma \) is isolated on \( \Sigma_E \) in the sense that there are no periodic orbits on \( \Sigma_E \) arbitrarily close to it. The cylinder
Theorem then applies to $\gamma$ which therefore belongs to a family $\gamma_E$. Furthermore, $\gamma$ is said to be **stable** when the eigenvalues of the Poincaré map come in complex conjugate pairs of the form $(e^{i\nu_{\alpha}}, e^{-i\nu_{\alpha}})$ with $\nu_{\alpha} \in \mathbb{R}$. The angles $\nu_{\alpha}$ are then called the **stability angles**. In particular, for a non-degenerate curve all the stability angles are non-zero.

**Theorem 2.4.1.** Let $\gamma \in H^{-1}(E)$ be a stable non-degenerate periodic orbit of $X_H$. Then

$$\int_{\gamma} \alpha = 2\pi \left( N + \frac{\mu_{\gamma}}{4} \right) + \sum_{\alpha=2}^{n} \left( n_{\alpha} + \frac{1}{2} \right) \nu_{\alpha} \right] \hbar + O(\hbar^2), \quad (2.4.2)$$

with $N \in \mathbb{Z}$, $n_{\alpha} \in \mathbb{N}$ and $n_{\alpha} \ll |N|$, is a sufficient condition on $E$ for the existence of a solution to the Schrödinger equation in (2.0.3).

**Proof.** [48, 49] Since $\gamma$ is stable the Poincaré map is merely a product of rotations by angles $\nu_{\alpha}$ in $n - 1$ disjoint planes $\mathbb{R}^2_{\alpha} \subset T_{p_0}S$. In other words, every point $p_0 \in \gamma$ of the stable isolated periodic orbit $\gamma$ is surrounded by an infinitesimal torus $S_{F_2}^1 \times \ldots \times S_{F_n}^1$, where $S_{F_{\alpha}}^1 = \{ x_{\alpha} \in \mathbb{R}^2_{\alpha} : ||x_{\alpha}||^2 = F_{\alpha} \} \subset \mathbb{R}^2_{\alpha}$, which is preserved by the Poincaré map to first approximation in $F_{\alpha} \ll 1$. By the cylinder theorem the periodic orbit $\gamma$ belongs to a continuous family $\gamma_E$ parametrised by the energy $E$, and so one could now apply the Bohr-Sommerfeld-Maslov quantisation conditions to the family of torii $\Lambda \equiv \gamma_E \times S_{F_2}^1 \times \ldots \times S_{F_n}^1$ just constructed (see Figure 2.7). They read

$$\int_{S_{F_{\alpha}}^1} \alpha = 2\pi \left( n_{\alpha} + \frac{1}{2} \right) \hbar + O(\hbar^2), \quad \alpha = 2, \ldots, n$$

$$\int_{\tilde{\gamma}} \alpha = 2\pi \left( N + \frac{\mu_{\gamma}}{4} \right) \hbar + O(\hbar^2),$$

where $\tilde{\gamma}$ is the closed path on $\Lambda$ consisting of a classical path going from $T_{p_0}S$ once around $\Lambda$ back to $T_{p_0}S$ and the set of arcs of angles $-\nu_{\alpha}$ on $T_{p_0}S$ to close off this classical path (see red curve in Figure 2.7).
2.4. BOHR-SOMMERFELD FOR DEGENERATE TORI

Consider the 2-dimensional surface $\Gamma$ bounded by the periodic orbit $\gamma$ and the closed curve $\tilde{\gamma}$, constructed in the obvious way: at any point $t \neq 0$ along the curve $\gamma(t)$, $\Gamma$ looks locally like $\{\gamma(t) + \tau y(t) | 0 < t < T, 0 \leq \tau \leq 1\}$ where $y(t)$ is the transversal vector to $\gamma$ joining the points $\gamma(t)$ and $\tilde{\gamma}(t)$. At $t = 0$ we complete the surface by adding the sections of the disc of angle $-\nu_\alpha$ on $T_{p_0}S$. Then by Stokes’s theorem we have

$$\left( \int_{\tilde{\gamma}} - \int_{\gamma} \right) \alpha = \int_{\partial \Gamma} \alpha = \int_{\Gamma} \omega.$$ 

On the part of $\Gamma$ corresponding to $t \neq 0$ we have $\omega|_{t=0} = 0$ since the tangent space to $\Gamma$ is spanned by $X_H$ and the transversal vector $y$ (since $y$ lies in the energy surface $\Sigma_E$). And since $\Gamma_{t=0}$ looks like sections of angle $-\nu_\alpha$ of the disc of radius $\sqrt{F_\alpha}$ it follows that

$$\left( \int_{\tilde{\gamma}} - \int_{\gamma} \right) \alpha = \int_{\Gamma_{t=0}} \omega = - \sum_{\alpha=2}^{n} \nu_\alpha F_\alpha.$$ 

On the other hand we have that

$$\int_{S^1_{F_\alpha}} \alpha = \int_{D^1_{F_\alpha}} \omega = 2\pi F_\alpha,$$

where $D^1_{F_\alpha}$ is the disc in $\mathbb{R}^2_\alpha$ bounded by the circle $S^1_{F_\alpha}$. The last equality follows by

\[\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.7.png}
\caption{The infinitesimal torus around a stable isolated periodic orbit $\gamma$ illustrated in the case $n = 2$ where there is only one stability angle $\nu_\alpha$ and $T_{p_0}S = \mathbb{R}^2_\alpha$.}
\end{figure}
a direct computation. Combining everything we obtain \( (2.4.2) \).

Since the periodic orbit \( \gamma \subset \Sigma_E \) in fact belongs to a continuous 1-parameter family \( \gamma_E \) of periodic orbits parametrised by the energy \( E \) according to the cylinder theorem \( 2.0.15 \) what the condition \( (2.4.2) \) does is pick out a discrete set of periodic orbits \( \gamma_{E_j} \), in a neighbourhood of the level set \( \Sigma_E \), whose energies \( E_j^\hbar \) approximate eigenvalues of \( \hat{H} \) to leading order in \( \hbar \) (see Figure 2.8).

![Figure 2.8: Bohr-Sommerfeld semi-classical spectrum: the discrete set of periodic orbits \( \gamma_{E_j} \) shown in blue have energies \( E_j^\hbar \) approximating eigenvalues of \( \hat{H} \) to \( O(\hbar^2) \).](image)

The more general case of a system which has a total of \( p \) independent observables \( F_1, \ldots, F_p \) in involution (with \( H = H(F_1, \ldots, F_p) \)), where \( p \) lies in the range \( 1 < p < n \) is a straightforward generalisation (see [48,49] for details). In this case there is a different Poincaré map for each basis cycle \( \gamma_k \in H_1(\Lambda_f), k = 1, \ldots, p \) on the \( p \)-torus \( \Lambda_f \). Each has its own stability angles \( \nu^k, \alpha = p+1, \ldots, n \) for oscillations in the transverse directions to the \( p \)-torus and \( \Lambda_f \) is **stable** if these are all real.

**Theorem 2.4.2.** Let \( \Lambda_f \subset \Sigma_f \) be a stable integral manifold of the \( X_{F_i} \). Then

\[
\int_{\gamma_k} \alpha = 2\pi \left( N_k + \frac{\mu_\gamma}{4} \right) + \sum_{\alpha=p+1}^{n} \left( n^k_\alpha + \frac{1}{2} \right) \nu^k_\alpha \hbar + O(\hbar^2),
\]

with \( N_k \in \mathbb{Z}, n^k_\alpha \in \mathbb{N} \) and \( n^k_\alpha \ll |N_k| \), are sufficient conditions on \( f_1, \ldots, f_p \) for the existence of a solution to the Schrödinger equations \( (2.0.6) \).
To illustrate the use of the modified Bohr-Sommerfeld conditions \((2.4.2)\) for an isolated orbit let us go back to the case of the two-dimensional harmonic oscillators \((2.4.1)\). This system is obviously integrable and the exact spectrum of \(H\) is

\[
E_{n_1,n_2} = \left( n_1 + \frac{1}{2} \right) \hbar \omega_1 + \left( n_2 + \frac{1}{2} \right) \hbar \omega_2.
\]

However, suppose for the sake of argument that we can only solve classically for the Hamiltonian \(H_1\) and wish to obtain the spectrum of \(H = H_1 + H_2\) by perturbation as describe above. Then consider a particular motion of the Hamiltonian \(H_1\) of total energy \(H_1 = E\), through the point \((p_1, x_1, p_2, x_2) = (p_0, 0, 0, 0)\) say, see Figure 2.4. This defines a 1-parameter family of periodic orbits parametrised by their energy \(H = H_1 = E\). It is clear that the \((p_2, x_2)\)-plane gives a Poincaré section of the orbit through the point \((p_0, 0, 0, 0)\) since all orbits of \(H_1\) have the same period \(T_1 = \frac{2\pi}{\omega_1}\).

The prescription for determining the stability angles of this orbit is to consider small perturbations around it within the same energy level \(H = E\). If the periods of the two harmonic oscillators are different, \(T_1 \neq T_2\), then after a length of time \(T_1\), the motion in the \((p_2, x_2)\)-plane does not close and there is a deficit angle of \(\nu = \omega_2 \cdot T_1\), see Figure 2.9. The tower of energy levels corresponding to the periodic motion in

![Figure 2.9: Perturbed trajectory of energy \(H = H_1 + H_2 = E\).](image)

Figure 2.9 is therefore given by the Bohr-Sommerfeld condition \((2.4.2)\) which reads

\[
I_1 = \left[ \left( n_1 + \frac{1}{2} \right) + \left( n_2 + \frac{1}{2} \right) \frac{\nu}{2\pi} \right] \hbar + O(\hbar^2)
\]
and hence $E_{n_1,n_2} = \omega_1 \cdot I_1 = (n_1 + \frac{1}{2}) \hbar \omega_1 + (n_2 + \frac{1}{2}) \hbar \omega_2 + O(\hbar^2)$ so that the Bohr-Sommerfeld condition is actually exact to first order in $\hbar$ on the harmonic oscillator.
Part II

Classical Integrability of String Theory on $\mathbb{R} \times S^3$
Chapter 3

Strings on $\mathbb{R} \times S^3$

In this chapter we start by presenting two equivalent ways of modelling bosonic strings moving on $\mathbb{R} \times S^3$. One can either view $S^3$ as embedded in $\mathbb{R}^4$ and describe the string by a $\sigma$-model action, or view $S^3$ as the group manifold $SU(2)$ and describe the string in terms of a principal chiral model action. We subsequently only study the latter in great detail. It has a number of gauge symmetries which are unphysical and thus it is desirable to fix these in order to be left only with the physical degrees of freedom. At the end of the day the gauge fixed string is described by a principal chiral model action with flat metric subject to an added constraint.

3.1 Action

$\sigma$-model on $\mathbb{R} \times S^3$

Consider a bosonic string moving on $\mathbb{R} \times S^3$, where the factor $\mathbb{R}$ corresponds to time. This string is described by the embedding of a two dimensional worldsheet $W$ into the target manifold $\mathbb{R} \times S^3$. If we think of $S^3$ as the unit sphere in $\mathbb{R}^4$ then
the configuration of such a string is specified by a field $X_0$ describing the embedding into $\mathbb{R}$ and fields $X_i, i = 1, \ldots, 4$ subject to the constraint $\sum_i X_i^2 = 1$ describing the embedding into $S^3$. To fix the metric conventions, we choose the signatures $(- +)$ on the worldsheet and $(- + + +)$ on $\mathbb{R} \times S^3$.

The action for such a string is given by

$$S = \frac{-\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[ \sqrt{-\gamma^{\alpha\beta}} \left( \sum_{i=1}^{4} \partial_{\alpha} X_i \partial_{\beta} X_i - \partial_{\alpha} X_0 \partial_{\beta} X_0 \right) + \Lambda \left( \sum_{j=1}^{4} X_j^2 - 1 \right) \right].$$

(3.1.1)

Here $\gamma^{\alpha\beta}$ is the worldsheet metric, $\gamma = \det(\gamma_{\alpha\beta})$ and $\Lambda$ is a Lagrange multiplier constraining the string to the unit sphere $S^3 \subset \mathbb{R}^4$. The equations of motion for the various fields are

$$X_i : \quad \partial_{\alpha} \partial^{\alpha} X_i - \Lambda X_i = 0, \quad (3.1.2a)$$

$$X_0 : \quad \partial_{\alpha} \partial^{\alpha} X_0 = 0, \quad (3.1.2b)$$

$$\gamma_{\alpha\beta} : \quad T^{\alpha\beta} = 0, \quad (3.1.2c)$$

$$\Lambda : \quad \sum_j X_j^2 = 1, \quad (3.1.2d)$$

where

$$T^{\alpha\beta} \equiv \frac{\partial L}{\partial \gamma_{\alpha\beta}} = -\frac{\sqrt{\lambda}}{4\pi} \sqrt{-\gamma} \left( G^{\alpha\beta} - \frac{1}{2} \gamma^{\alpha\beta} \gamma_{\rho\sigma} G^{\rho\sigma} \right).$$

(3.1.3)

is the energy-momentum tensor and $G_{\alpha\beta} \equiv \sum_{i=1}^{4} \partial_{\alpha} X_i \partial_{\beta} X_i - \partial_{\alpha} X_0 \partial_{\beta} X_0$ is the pullback of the target space metric to the worldsheet. Multiplying equation (3.1.2a) by $X_i$, summing over $i = 1, \ldots, 4$ and making use of (3.1.2d) yields the value of the Lagrange multiplier $\Lambda = -\sum_j \partial_{\alpha} X_j \partial^{\alpha} X_j$. Substituting this value of the Lagrange multiplier back into (3.1.2a) gives rise to a set of nonlinear differential equations for the fields $X_i$

$$\partial_{\alpha} \partial^{\alpha} X_i + \left( \sum_j \partial_{\alpha} X_j \partial^{\alpha} X_j \right) X_i = 0. \quad (3.1.4)$$
3.1. ACTION

The nonlinearity is a consequence of the curvature of the background $S^3$ on which the string is moving. Unlike the linear equations for a string moving through flat space, the equations of motion (3.1.4) are a lot harder to solve in full generality. Yet we will show in Chapter 5 that these equations are integrable which means that they can in principle be solved.

Although the non-linear equations (3.1.4) have been solved explicitly using algebro-geometric methods [59] we shall work instead with a different model for strings moving through $\mathbb{R} \times S^3$. We shall exploit the group structure of $S^3$ and rewrite the string action as a principal chiral model on $SU(2)$. This is mainly to follow the literature on AdS/CFT [60] in which superstring theory on $AdS_5 \times S^5$ is described by a coset superspace model with target space $\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$. Moreover, in terms of this description the algebro-geometric construction had already been initiated in [23, 28, 29] for various subsectors as well as for the full theory.

$SU(2)$ principal chiral model

Since the sphere $S^3$ is isomorphic to the group $SU(2)$, the motion of the bosonic string in the $S^3$ manifold can also be formulated in terms of a field $g$ taking values in $SU(2)$ by defining

$$
g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix}.
$$

(3.1.5)

We immediately observe that $\det g = \sum_j X_j^2$ so that the constraint for the string to lie on $S^3$ is solved when $g \in SU(2)$. Furthermore, rewriting the $S^3$ part of the action (3.1.1) in terms of this new field one finds

$$
\sum_{i=1}^4 \partial_\alpha X_i \partial_\beta X_i = -\frac{1}{2} \text{tr} \left( g^{-1} \partial_\alpha gg^{-1} \partial_\beta g \right).
$$
This is precisely the principal chiral model action for the $SU(2)$-valued field $g$. Defining the $su(2)$-valued worldsheet current $j_\alpha = -g^{-1}\partial_\alpha g$ we can rewrite the $\sigma$-model action (3.1.1) as the following principal chiral model action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{-\gamma} \gamma^{\alpha\beta} \left[ \frac{1}{2} \text{tr}(j_\alpha j_\beta) + \partial_\alpha X_0 \partial_\beta X_0 \right].$$

(3.1.6)

Introducing the form notation $j = -g^{-1}dg = j_0 d\tau + j_1 d\sigma$ one can rewrite (3.1.6) more compactly as

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \left[ \frac{1}{2} \text{tr}(j \wedge^* j) + dX_0 \wedge^* dX_0 \right].$$

(3.1.7)

The dependence on the worldsheet metric $\gamma_{\alpha\beta}$ is now hidden in the Hodge $\ast$ operation. The current $j$ is identically flat from its definition so the equations of motion now read

$$g : \quad d \ast j = 0, \quad dj - j \wedge j = 0,$$

(3.1.8a)

$$X_0 : \quad d \ast dX_0 = 0,$$

(3.1.8b)

$$\gamma_{\alpha\beta} : \quad T^{\alpha\beta} = 0,$$

(3.1.8c)

where the induced metric is

$$G_{\alpha\beta} = -\frac{1}{2} \text{tr}(j_\alpha j_\beta) - \partial_\alpha X_0 \partial_\beta X_0$$

(3.1.9)

when expressed in terms of the principal chiral model fields.

**Remark** The second equation in (3.1.8a) is the condition for the existence of a matrix $g \in SU(2)$ such that $j = -g^{-1}dg$. Indeed, $dg + gj = 0$ implies $dj - j \wedge j = 0$ and conversely, if $dj - j \wedge j = 0$ then $j$ is a flat connection so the path ordered exponential $g = \text{Pexp}(\int^x -j)$ is path independent and solves $dg + gj = 0$. Thus (3.1.8a) is equivalent to $d(*g^{-1}dg) = 0$ which in turn is
From now on we shall treat only this model of strings on \( \mathbb{R} \times S^3 \).

\section*{3.2 Symmetries}

\subsection*{Global}

The action (3.1.6) is invariant under constant shifts in the time \( X_0 \). The Noether current is \( \frac{\sqrt{\lambda}}{2\pi} \sqrt{-\gamma} \partial_{0} X_0 \), and the corresponding Noether charge is the space-time energy of the string

\[ \Delta = \frac{\sqrt{\lambda}}{2\pi} \int_{0}^{2\pi} d\sigma \sqrt{-\gamma} \partial_{0} X_0. \]

The action (3.1.6) also has a global \( SU(2)_L \times SU(2)_R \) symmetry \( g \mapsto U_L g U_R \), where \( U_L \) and \( U_R \) are constant matrices. The Noether current corresponding to \( SU(2)_R \) is the current \( j = -g^{-1} dg \) introduced above whereas the Noether current for the \( SU(2)_L \) symmetry is \( l = -dg g^{-1} = gj g^{-1} \). The corresponding Noether charges are

\begin{align*}
SU(2)_R : \quad Q_R & = \frac{\sqrt{\lambda}}{4\pi} \int_{\gamma} \ast j, \quad (3.2.1a) \\
SU(2)_L : \quad Q_L & = \frac{\sqrt{\lambda}}{4\pi} \int_{\gamma} \ast l, \quad (3.2.1b)
\end{align*}

where \( \gamma \) is any curve winding once around the worldsheet, expressing the conserva-
tion of these Noether charges, e.g.

\[ \int_{\gamma_2} * j - \int_{\gamma_1} * j = \int_{\partial D} * j = \int_{D} d * j = 0. \]

Notice that the \( SU(2)_R \) current \( j \) which appears in the action (3.1.7) is invariant under the action of \( SU(2)_L \). On the other hand the \( SU(2)_R \) symmetry acts non-trivially on the current

\[ j \mapsto U_R^{-1} j U_R. \]  

(3.2.2)

**Local**

The string action in either of the above forms (3.1.1) or (3.1.6) is invariant under general reparametrisations of the worldsheet

\[ (\sigma, \tau) \mapsto (\sigma', \tau'), \]  

(3.2.3) with the fields \( X_0 \) and \( X_i \) (or equivalently \( g \) in (3.1.5)) transforming as scalars and \( \gamma_{\alpha\beta} \) as the components of a \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \) tensor. That is, if \( \sigma^\alpha \mapsto \sigma^\alpha + \epsilon^\alpha(\sigma, \tau) \) denotes the infinitesimal version of (3.2.3) then

\[ \delta_\epsilon X_i = \epsilon^\alpha \partial_\alpha X_i, \quad \delta_\epsilon X_0 = \epsilon^\alpha \partial_\alpha X_0, \]

\[ \delta_\epsilon \gamma_{\alpha\beta} = -\nabla_\alpha \epsilon_\beta - \nabla_\beta \epsilon_\alpha, \]

where \( \nabla_\alpha \) is the covariant derivative for the metric \( \gamma_{\alpha\beta} \). Also, since \( \sqrt{-\gamma} \) is a scalar density it behaves as \( \delta_\epsilon \sqrt{-\gamma} = \partial_\alpha (\epsilon^\alpha \sqrt{-\gamma}) \) under infinitesimal diffeomorphisms.

Noting that the Lagrangian in either (3.1.1) or (3.1.7) is a scalar density, so that
\[ \delta \mathcal{L} = \partial_\alpha (\epsilon^\alpha \mathcal{L}), \]
leads to the following on-shell conserved current by Noether’s theorem
\[ \nabla_\alpha j^\alpha \simeq 0, \quad j^\alpha = 2\epsilon_\beta T^{\alpha\beta}, \] (3.2.4)
where \( \simeq \) indicates an on-shell equality. However, we are dealing with a gauge transformation, since \( \epsilon_\beta \) is an arbitrary function of \((\sigma, \tau)\), and so expanding the conservation equation (3.2.4) in derivatives of \( \epsilon_\beta \) leads to two equations, known as Noether identities,
\[ \epsilon_\beta : \quad \nabla_\alpha T^{\alpha\beta} \simeq 0, \] (3.2.5a)
\[ \nabla_\alpha \epsilon_\beta : \quad T^{\alpha\beta} \simeq 0. \] (3.2.5b)

Equation (3.2.5a) says that the energy-momentum tensor \( T^{\alpha\beta} \) is conserved on-shell as it should be since \( T^{\alpha\beta} \) is the Noether current for the global part of the diffeomorphism group with \( \epsilon^\alpha = \text{const} \). The corresponding Noether charges are the components of the worldsheet energy-momentum vector, generating \( \sigma^\alpha \mapsto \sigma^\alpha + \epsilon^\alpha \) and given by
\[ P_\alpha = \int_0^{2\pi} d\sigma 2T_0^\alpha. \] (3.2.6)
However, equation (3.2.5b) shows that in fact \( T^{\alpha\beta} \) itself vanishes on-shell. This we already knew from the equation of motion for \( \gamma_{\alpha\beta} \) but the statement that the conserved charges vanish on-shell is reminiscent of gauge theories: as we will see in the next chapter (3.2.5b) corresponds to a secondary constraint in the Hamiltonian formalism.

The string action is also invariant under Weyl transformations of the metric
\[ \gamma_{\alpha\beta} \mapsto e^\phi \gamma_{\alpha\beta}, \] (3.2.7)
where \( \phi(\sigma, \tau) \) is an arbitrary function on the worldsheet. This is a symmetry because
3.3 Gauge fixing

The reason for wanting to fix the gauge, *i.e.* the coordinates on the worldsheet, is that all the remaining degrees of freedom will be physical.

Conformal gauge

Since the worldsheet is topologically a sphere every metric on it is conformally equivalent. This is certainly true for Riemannian metrics (Euclidean signature) but can also be shown [61] in the case at hand of a pseudo-Riemannian metric (Lorentzian signature). In other words, it is possible to choose coordinates \((\sigma, \tau)\) on the worldsheet with respect to which the metric \(\gamma_{\alpha\beta}\) assumes the conformal form

\[
\gamma_{\alpha\beta} = e^{\phi} \eta_{\alpha\beta} = e^{\phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3.1}
\]

We shall henceforth always assume such a choice of coordinates, known as conformal gauge. The presence of the prefactor \(e^\phi\) is irrelevant, at least classically, because of Weyl invariance. In this gauge the worldsheet energy and momentum defined in (3.2.6) read

\[
P_0 = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \left[ \frac{1}{2} \text{tr}(j_0^2 + j_1^2) + (\partial_0 X_0)^2 + (\partial_1 X_0)^2 \right], \tag{3.3.2a}
\]

\[
P_1 = -\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left[ \frac{1}{2} \text{tr}(j_0 j_1) + \partial_0 X_0 \partial_1 X_0 \right]. \tag{3.3.2b}
\]
3.3. GAUGE FIXING

Note also for later that in conformal gauge, the global Noether charges $Q_R, Q_L$ defined in (3.2.1) read

$$Q_R = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma j_0, \quad Q_L = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma l_0. \quad (3.3.3)$$

The gauge choice (3.3.1) however does not completely fix the gauge. Indeed, any coordinate transformation $(\sigma, \tau) \mapsto (\sigma', \tau')$ that changes the metric $\gamma_{\alpha\beta}$ only up to an overall factor

$$\gamma_{\alpha\beta} \mapsto \Lambda \gamma_{\alpha\beta}, \quad (3.3.4)$$

will not affect the gauge choice (3.3.1) since the factor $e^{\phi}$ is arbitrary and can absorb $\Lambda$. Such a coordinate transformation is known as a conformal transformation. The group of conformal transformations is an infinite dimensional subgroup of the diffeomorphism group (3.2.3) which possesses the following infinite number of Noether currents

$$j^\alpha = 2\epsilon_\beta T^{\alpha\beta},$$

where $\epsilon_\alpha$ is a conformal Killing vector, i.e. it satisfies the infinitesimal form of (3.3.4) which is $\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha = \lambda \gamma_{\alpha\beta}$. In particular, for $\epsilon_\alpha = \text{const}$ we recover the global diffeomorphisms of equation (3.2.6) which are nothing but rigid translations in $\sigma$ and $\tau$. Thus even after imposing the condition (3.3.1) there remains a residual gauge invariance in the form of the conformal group, which we will have to fix by imposing further gauge conditions. But in order to determine these extra conditions we have to first understand what the general conformal transformation looks like, which requires solving (3.3.4).
Static gauge

A more suitable coordinate system for discussing conformal transformations of the metric (3.3.4) are light-cone coordinates: given a coordinate system \((\sigma, \tau)\) for which the metric takes the form (3.3.1), i.e. \(ds^2 = e^\phi(-d\tau^2 + d\sigma^2)\), we define \(\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)\). In these coordinates the metric becomes

\[ ds^2 = -4e^\phi d\sigma^+ d\sigma^- . \tag{3.3.5} \]

It follows easily that the only way for a transformation \(\sigma^\pm \mapsto \sigma'^\pm\) to be conformal is to have \(\frac{\partial \sigma'^+}{\partial \sigma^-} = \frac{\partial \sigma'^-}{\partial \sigma^+} = 0\), up to the trivial permutation \(\sigma'^+ \leftrightarrow \sigma'^-\). In other words,

\[ \sigma^+ \mapsto \sigma'^+ = f^+(\sigma^+), \quad \sigma^- \mapsto \sigma'^- = f^-(\sigma^-), \tag{3.3.6} \]

where \(f^\pm\) are arbitrary invertible functions. We will now fix this residual gauge symmetry by imposing a second condition on top of (3.3.1).

Note that the equation of motion (3.1.8b) for \(X_0\) is decoupled from the other fields and hence can be solved separately. Written in terms of light-cone coordinates it reads \(\partial_+ \partial_- X_0 = 0\) and has the general solution

\[ X^\text{sol}_0(\sigma, \tau) = X^+_0(\sigma^+) + X^-_0(\sigma^-). \]

One would now like to apply a residual gauge transformation \((\sigma, \tau) \mapsto (\sigma', \tau')\) given by (3.3.6) with \(\kappa f^\pm = X^\pm_0 (\kappa \neq 0)\) in order to bring \(X^\text{sol}_0\) to the simpler form

\[ X^\text{sol}_0(\sigma', \tau') = \kappa \sigma'^+ + \kappa \sigma'^- = \kappa \tau'. \tag{3.3.7} \]

The coefficient \(\kappa\) is not arbitrary but proportional the space-time energy \(\Delta\) of the string since \(\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \dot{X}^\text{sol}_0(\sigma, \tau) = \kappa \sqrt{\lambda}\) using the coordinate system \((\sigma', \tau').\)
The condition (3.3.7) is called the static gauge condition.

**Remark** Note that since the $f^\pm$ in (3.3.6) must be invertible functions, the transformation to (3.3.7) just described is possible only if the $X^\pm_0$ are themselves invertible [61]. We will assume from now on that this is the case for the solution $X^\text{sol}_0$.

In static gauge the worldsheet energy and momentum (3.3.2) simplify further to

$$P_0 = -\sqrt{\lambda} \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \frac{1}{2} \text{tr}(j_0^2 + j_1^2) - \frac{\sqrt{\lambda}\kappa^2}{2},$$

(3.3.8a)

$$P_1 = -\sqrt{\lambda} \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \text{tr}(j_0j_1).$$

(3.3.8b)

**Symplectic reduction**

By definition of the static gauge condition (3.3.7) the $\tau$ coordinate is now completely fixed. But this still leaves the possibility of performing conformal transformations (7.1.4) that fix $\tau$. It is easy to show that the only such transformations are rigid translations in $\sigma$

$$\tau' \mapsto \tilde{\tau}, \quad \sigma' \mapsto \tilde{\sigma} + b, \quad b \in \mathbb{R}.$$  

(3.3.9)

This is generated by the worldsheet momentum $P_1$. Thus, working in conformal static gauge, (3.3.1) together with (3.3.7), the original gauge invariance of the full string action is completely fixed except for the global transformation (3.3.9). We will therefore have to make sure that physical states are invariant under this global symmetry. This can be achieved by symplectic reduction onto the level set $P_1 = 0$. 
3.4 Virasoro constraints

It is important to note that even after fixing the metric using conformal gauge (3.3.1), the equations of motion for the metric $\gamma_{\alpha\beta}$ still carry nontrivial information and must therefore be retained. They become constraints on the other dynamical fields known as the Virasoro constraints.

When working in conformal gauge a lot of expressions simplify if we use light-cone coordinates in which the metric (3.3.5) is off-diagonal $\gamma_{\pm\pm} = 0, \gamma_{+-} = \gamma_{-+} = -2$. For instance, the tracelessness of the energy momentum tensor $\gamma_{\alpha\beta} T^{\alpha\beta} = 0$ implies in light-cone coordinates that $T^{+-} = T^{-+} = 0$. Moreover, from its definition (3.1.3) the remaining components of $T^{\alpha\beta}$ take on the simple form

$$T_{\pm\pm} = \frac{\sqrt{\lambda}}{4\pi} \left( \frac{1}{2} \text{tr} j_{\pm}^2 + (\partial_{\pm} X_0)^2 \right),$$

(3.4.1)

where $j_{\pm} = j_0 \pm j_1$ are the components of the current $j$ in light-cone coordinates.

In static gauge, since $X_0$ has been used to specify the worldsheet $\tau$ coordinate, only the current $j$ remains and the Virasoro constraints simplify to

$$\frac{1}{2} \text{tr} j_{\pm}^2 = -\kappa^2.$$

(3.4.2)

In fact, since the static gauge condition has fixed all the residual gauge invariance except for the rigid $\sigma$-translation of equation (3.3.9), and since the latter is generated by the worldsheet momentum $P_1$, it is convenient to postpone imposing the condition $P_1 = 0$, which is one of the Virasoro constraints (3.4.2). Thus we split the Virasoro constraints (3.4.2) into two parts. The first set of constraints read,

$$\frac{1}{2} \text{tr} j_{\pm}^2 = -\kappa_{\pm}^2.$$

(3.4.3)
where $\kappa_{\pm}$ are two independent constants. After imposing (3.4.3) the worldsheet energy and momentum (3.3.8) become $P_0 = \mathcal{E} - \sqrt{\lambda} \kappa^2 / 2$ and $P_1 = \mathcal{P}$ respectively, where

$$
\mathcal{E} = \frac{\sqrt{\lambda}}{4} (\kappa_+^2 + \kappa_-^2), \quad \mathcal{P} = \frac{\sqrt{\lambda}}{4} (\kappa_+^2 - \kappa_-^2)
$$

(3.4.4)

are the energy and momentum of the principal chiral field $j$. The remaining Virasoro constraint is the vanishing of the worldsheet momentum $\mathcal{P} = 0$ which corresponds to setting $\kappa_+ = \kappa_- = \kappa$. When imposing this last Virasoro constraint one must also identify string configurations related by rigid $\sigma$-translations, which amounts to performing the symplectic reduction of the previous subsection. We note finally that the vanishing of the worldsheet energy $P_0 = 0$ gives the string mass-shell condition, relating the energy of the principal chiral model $\mathcal{E}$ to the space time energy

$$
\mathcal{E} = \frac{\sqrt{\lambda}}{2} \kappa^2 = \frac{\Delta^2}{2 \sqrt{\lambda}}
$$

(3.4.5)
Chapter 4

Hamiltonian formalism

In the following chapter we will set up the Hamiltonian formalism for the action (3.1.6). The gauge invariance of the string is generated by some primary and secondary first-class constraints. The primary ones $p^{\alpha\beta} \approx 0$ are completely fixed by imposing conformal gauge in section 4.2 whereas the secondary ones $T^{\alpha\beta} \approx 0$ are almost entirely fixed using static gauge in section 4.4. The rigid $\sigma$-translation generated by the constraint $P_1 \approx 0$ remains and has to be fixed by a final symplectic reduction. The complete procedure for gauge fixing the string is summarised in the following diagram,

\[
\begin{align*}
  p^{\alpha\beta} &\approx 0, T^{\alpha\beta} \approx 0 \\
  \text{Conformal gauge} &\rightarrow \gamma_{\alpha\beta} \approx \eta_{\alpha\beta} \\
  T^{\alpha\beta} &\approx 0 \\
  \text{Static gauge} &\rightarrow X_0 - \frac{\pi}{\sqrt{\lambda}} \approx \pi^0 - \frac{p}{\lambda} \approx 0 \\
  P_1 &\approx 0 \\
  \text{Symplectic reduction} &\rightarrow \text{Physical d.o.f.}
\end{align*}
\]
As a result of fixing the gauge we must replace the Poisson bracket by a Dirac bracket which we introduce in section 4.4. We also explain how the reduced dynamics for the physical degrees of freedom arises from the time-dependence of the static gauge condition.

4.1 SU(2) principal chiral model

To set up the Hamiltonian formalism for the action (3.1.6) we start by identifying the canonical variables. For this we need to choose the variables we shall take as our canonical coordinates. Let us first choose a particular basis $t_a$ of the Lie algebra $su(2)$ with structure constants $f_{abc}$ and normalised such that

$$[t_a, t_b] = f_{ab}^c t_c, \quad \text{tr}(t_a t_b) \equiv k_{ab} = -\delta_{ab}, \quad a, b, c = 1, 2, 3.$$ 

Note that $\text{tr}(AB) = A_a B^a = -A_a B_a$ in terms of components $A = A^a t_a, B = B^a t_a$ with respect to this basis $t_a$. For concreteness we set $t_a = \frac{i}{\sqrt{2}} \sigma_a$ where $\sigma_a$ are the Pauli matrices.

Following [62], we take the components of the spatial part of the current $j$ as the first set of canonical variables $q^a(\sigma) = j_1^a(\sigma)$. We choose the other canonical coordinates to be the target-space time coordinate $q^0(\sigma) = X_0(\sigma)$ and the components of the worldsheet metric $\gamma_{\alpha\beta}$. The components $j_0^a(\sigma)$ are related to the time derivative of $q^a$ and hence are not independent coordinates. Indeed, by the flatness of the current $j$ we have $\partial_0 j_1 - \partial_1 j_0 = [j_0, j_1]$ so that

$$\dot{q}^a = \partial_\sigma j_0^a - [j_1, j_0]^a = \nabla_1 j_0^a, \quad (4.1.1)$$

where $\nabla_1$ is the covariant derivative for the connection $j_1 = j_1^a t_a$.
4.1. \textit{SU}(2) PRINCIPAL CHIRAL MODEL

We now determine the conjugate momenta. The absence of derivatives of the worldsheet metric in the action immediately implies that the conjugate momentum to $\gamma_{\alpha\beta}$ vanishes,

$$p^{\alpha\beta} = \frac{\delta S}{\delta \gamma_{\alpha\beta}} \equiv 0.$$  \hspace{1cm} (4.1.2)

This is a primary constraint on the Hamiltonian system which will be partly responsible for the gauge invariance. The conjugate momentum of the time coordinate $X_0$ is given by

$$\pi_0(\sigma) = \frac{\delta S}{\delta X_0(\sigma)} = \frac{\sqrt{\lambda}}{2\pi} \sqrt{-\gamma^{0\alpha}} \partial_\alpha X_0(\sigma).$$ \hspace{1cm} (4.1.3)

Finally, the computation of the conjugate momenta of the coordinates $q^a$ is a little bit more involved. One has

$$\pi_a(\sigma) = \frac{\delta S}{\delta \dot{q}^a(\sigma)} = \frac{\sqrt{\lambda}}{8\pi} \int d\sigma' d\tau' \sqrt{-\gamma^{0\alpha}} \frac{\delta j^h_0(\sigma') j_{ab}(\sigma')}{\delta \dot{q}^a(\sigma)},$$

and using equation (4.1.1) one can write $\frac{\delta j^h_0(\sigma')}{\delta \dot{q}^a(\sigma)} = \nabla_1^{-1} \left( \delta^h_0 \delta(\sigma - \sigma') \delta(\tau - \tau') \right)$. Then using this relation and integrating by parts we end up with

$$\pi_a(\sigma) = -\frac{\sqrt{\lambda}}{4\pi} \nabla_1^{-1} \left( \sqrt{-\gamma^{0\alpha}} j_{\alpha a}(\sigma) \right).$$ \hspace{1cm} (4.1.4)

In other words, $\nabla_1 \pi^a(\sigma) = -\frac{\sqrt{\lambda}}{4\pi} \sqrt{-\gamma^{0\alpha}} j^a_\alpha(\sigma)$ for $a = 1, 2, 3$.

Hamiltonian

We now have enough information to define the Hamiltonian corresponding to the action (3.1.6). Introducing capital letter indices $A = 0, 1, 2, 3$, it is given by

$$H_0 = \int d\sigma \left( \pi_A(\sigma) \partial_\sigma q^A(\sigma) \right).$$
where \( L = \frac{\sqrt{\lambda}}{4\pi} \sqrt{-\gamma^{\alpha\beta}} \left( \frac{1}{2} \text{tr}(j_\alpha j_\beta) + \partial_\alpha X_0 \partial_\beta X_0 \right) \) is the Lagrangian. After a little algebra it can be simplified to

\[
H_0 = \int d\sigma \left[ -\frac{\sqrt{-\gamma}}{\gamma_{11}} \left( \frac{2\pi}{\sqrt{\lambda}} \nabla_1 \pi^a \nabla_1 \pi_a + \frac{\sqrt{\lambda}}{8\pi} j_1^a j_1 a + \frac{\pi}{\sqrt{\lambda}} (\pi^0)^2 + \frac{\sqrt{\lambda}}{4\pi} (\partial_1 X_0)^2 \right) \right.
\]

\[
\left. + \frac{\gamma_{01}}{\gamma_{11}} (-\nabla_1 \pi_a j_1^a + \pi^0 \partial_1 X_0) \right].
\]

It is convenient to define a different parametrisation of the metric \( \gamma_{\alpha\beta} \) as

\[
\lambda^\pm = \frac{-\sqrt{-\gamma} \pm \gamma_{01}}{\gamma_{11}}, \quad \xi = \ln \gamma_{11}.
\] (4.1.5)

We see immediately that the first two parameters \( \lambda^\pm \) are invariant under Weyl transformations \( (3.2.7) \) whereas \( \xi \) transforms as \( \xi \mapsto \xi + \phi \). Just as the action \( (3.1.6) \) was Weyl invariant since it depended only on the Weyl invariant combination \( \sqrt{-\gamma^{\alpha\beta}} \), the Weyl invariance of the Hamiltonian is explicit from its sole dependence on the Weyl invariant variables \( \lambda^\pm \). Indeed, if we define the worldsheet energy and momentum densities as

\[
H_0 = \frac{2\pi}{\sqrt{\lambda}} \nabla_1 \pi^a \nabla_1 \pi_a - \frac{\sqrt{\lambda}}{8\pi} j_1^a j_1 a - \frac{\pi}{\sqrt{\lambda}} (\pi^0)^2 - \frac{\sqrt{\lambda}}{4\pi} (\partial_1 X_0)^2,
\] (4.1.6a)

\[
H_1 = -\nabla_1 \pi_a j_1^a + \pi^0 \partial_1 X_0,
\] (4.1.6b)

and define the combinations \( T_\pm = -H_0 \pm H_1 \) given explicitly by

\[
T_\pm = \frac{\sqrt{\lambda}}{4\pi} \left[ \frac{1}{2} \text{tr} \left( \frac{4\pi}{\sqrt{\lambda}} \nabla_1 \pi \mp j_1 \right) \right] + \left( \frac{2\pi}{\sqrt{\lambda}} \pi^0 \pm \partial_1 X_0 \right)^2,
\] (4.1.7)

then the Hamiltonian reads

\[
H_0 = \int d\sigma \left( \frac{\lambda^+}{2} T_+ + \frac{\lambda^-}{2} T_- \right).
\] (4.1.8)
Poisson brackets

The full set of canonical Poisson brackets between the generalised coordinates and their conjugate momenta are,

\[ \{ q^A(\sigma), q^B(\sigma') \} = \{ \pi_A(\sigma), \pi_B(\sigma') \} = 0 \] (4.1.9)

\[ \{ \pi_B(\sigma), q^A(\sigma') \} = \delta^A_B \delta(\sigma - \sigma'). \]

There are also Poisson brackets between the metric variables \( \lambda^{\pm} \) and their conjugate momenta (defined later in (4.1.13)) but we won’t be needing those at any stage.

We can derive from (4.1.9) the Poisson brackets between the variables \( \nabla_1 \pi_a \) and \( q^b \) that appear in the Hamiltonian, for example

\[ \{ \nabla_1 \pi^a(\sigma), q^b(\sigma') \} = \{ q^b(\sigma'), \partial_\sigma \pi^a(\sigma) \} - f^{abc} \{ q^b(\sigma'), q_d(\sigma) \pi_c(\sigma) \} \\
= \partial_\sigma \left( k^{ab} \delta(\sigma' - \sigma) \right) - f^{abc} q_d(\sigma) \delta^b_c \delta(\sigma' - \sigma) \\
= f^{abc} q_c(\sigma) \delta(\sigma - \sigma') + k^{ab} \delta'(\sigma - \sigma'). \] (4.1.10a)

Similarly we have

\[ \{ \nabla_1 \pi^a(\sigma), \nabla_1 \pi^b(\sigma') \} = f^{abc} \nabla_1 \pi_c(\sigma) \delta(\sigma - \sigma'). \] (4.1.10b)

As for the canonical variables \( \pi^0, X_0 \), since the coordinate \( X_0 \) only appears differentiated with respect to \( \sigma \), the following Poisson bracket is more useful

\[ \{ \pi^0(\sigma), \partial_1 X_0(\sigma') \} = -\delta'(\sigma - \sigma'). \] (4.1.11)
Using (4.1.10) and (4.1.11) one can derive the following algebra for the variables $T_{\pm}$,

\[
\{T_+(\sigma), T_+(\sigma')\} = \pm \frac{\sqrt{\lambda}}{8\pi} [T_+(\sigma) + T_+(\sigma')] \delta'(\sigma - \sigma'),
\]

\[
\{T_+(\sigma), T_-(\sigma')\} = 0.
\]  

(4.1.12)

**Constraints**

The next step in the Hamiltonian analysis is to determine the constraints. In terms of the new variables (4.1.5), the vanishing of the conjugate momentum of the metric $\gamma_{\alpha\beta}$ in (4.1.2) reads

\[
\pi_+^\lambda = \frac{\delta S}{\delta \lambda_+} \equiv 0, \quad \pi_\xi = \frac{\delta S}{\delta \xi} \equiv 0.
\]  

(4.1.13)

These are three primary constraints of the Hamiltonian system. According to the general theory of constrained Hamiltonian systems [63, 64], one must demand that these constraints be preserved in time under the Hamiltonian (4.1.8), which can lead to a further set of constraints. Indeed here we find

\[
\dot{\pi}_-^\lambda \approx 0 \quad \Rightarrow \quad T_\pm \approx 0,
\]  

(4.1.14)

whereas $\dot{\pi}_\xi \approx 0$ and $\dot{T}_\pm \approx 0$ (which follows from (4.1.12)) do not lead to any further constraints. One can do away with the canonical variables $\xi, \pi_\xi$ very easily: together they form a pair of second-class constraints since by definition $\{\xi, \pi_\xi\} = 1$ but since they do not appear in any of the physical variables (everything is Weyl invariant and $\pi_\xi \equiv 0$) they can simply be discarded (formally by defining an appropriate Dirac bracket).

The new constraints in (4.1.14) are called secondary constraints because they follow from the equations of motion as opposed to primary constraints which follow from the definitions of the conjugate momenta. However, equation (4.1.12) shows
that these constraints are first-class constraints since they form a closed algebra. In fact, the constraints \[4.1.14\] are simply the Virasoro constraints again. A simple way to see this is to go back to the primary constraints but in the form \[4.1.2\] and again determine the condition for their preservation in time,

\[
0 \approx \dot{p}^{\alpha \beta} = \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial \dot{\gamma}^{\alpha \beta}} \right) = \frac{\partial L}{\partial \gamma^{\alpha \beta}} \equiv T^{\alpha \beta}.
\]

In the second last equality we have made use of the Euler-Lagrange equations of motion and the fact that the Lagrangian \(L\) is independent of \(\partial_{\sigma} \gamma^{\alpha \beta}\), whereas the last equality is the definition of the energy-momentum tensor.

In the theory of constrained Hamiltonians [63, 64] one should always include the constraints in the Hamiltonian itself by the method of Lagrange multipliers. Thus one replaces the original Hamiltonian \(H_0\) in \(4.1.8\) with the total Hamiltonian

\[
H_T = H_0 + \int d\sigma \left[ \rho_+ T_+ + \rho_- T_- + \rho_+^\lambda \pi_+^\lambda + \rho_-^\lambda \pi_-^\lambda \right], \quad (4.1.15)
\]

\[
= \int d\sigma \left[ \left( \frac{\lambda^+}{2} + \rho_+ \right) T_+ + \left( \frac{\lambda^-}{2} + \rho_- \right) T_- + \rho_+^\lambda \pi_+^\lambda + \rho_-^\lambda \pi_-^\lambda \right]. \quad (4.1.16)
\]

The effect of the constraints in Hamilton’s equations corresponds to the ability to perform arbitrary gauge transformations on top of the true dynamical evolution of the system. Notice though that the original Hamiltonian \(H_0\) in \(4.1.8\) is itself a combination of the Virasoro constraints and hence vanishes on the constraint surface. This situation is typical of generally covariant theories.

### 4.2 Conformal gauge

We are now in a position to discuss conformal gauge fixing. Using the coordinate invariance generated by \(T_{\alpha \beta}\) we wish to fix \(\gamma_{\alpha \beta}\) to the flat metric \(\eta_{\alpha \beta} = \text{diag}(-1, 1)\).
This can be done in the Hamiltonian formalism by imposing the constraint \( c_{\alpha \beta} = \gamma_{\alpha \beta} - \eta_{\alpha \beta} \approx 0 \) by hand, which in terms of the metric variables (4.1.5) reads

\[
c_\pm = \lambda_\pm + 1 \approx 0.
\] (4.2.1)

This gauge fixing condition is second-class with respect to the constraints (4.1.13) since

\[
\{ \pi_{\pm}, c_\pm \} = \{ \pi_{\pm}, \lambda_{\pm} \} = 1.
\]

However, both constraints \( c_\pm \approx \pi^\lambda_{\pm} \approx 0 \) commute with the Virasoro constraints (4.1.7) since the latter doesn’t have any explicit dependence on the metric variables \( \lambda_{\pm} \) nor on their conjugate momenta \( \pi^\lambda_{\pm} \), as can be seen in (4.1.7). It thus follows that the matrix of Poisson brackets \( C_{ab} = \{ \phi_a, \phi_b \} \) between all the constraints \( \phi_a = (T_\pm, c_\pm, \pi^\lambda_{\pm}) \) takes the following schematic form

\[
C_{ab} = \{ \phi_a, \phi_b \} = \begin{pmatrix}
T & c & \pi^\lambda \\
* & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

the important point being that the second class constraints \( c_\pm \approx \pi^\lambda_{\pm} \approx 0 \) form an independent block of their own in the matrix of Poisson brackets \( C_{ab} \). In fact it follows that the inverse matrix has the same property

\[
C_{ab}^{-1} = \begin{pmatrix}
T & c & \pi^\lambda \\
* & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 \\
\end{pmatrix}
\]
so that the constraints \( c_\pm \approx \pi_\pm^A \approx 0 \) can be dealt with by defining a Dirac bracket

\[
\{ F, G \}^* = \{ F, G \} - \{ F, c_+ \} \{ \pi_+^A, G \} - \{ F, c_- \} \{ \pi_-^A, G \}.
\]

Clearly \( \{ F, G \}^* = \{ F, G \} \) whenever \( \{ F, c_\pm \} = \{ G, c_\pm \} = 0 \). But this is the case for arbitrary functions \( F, G \) of the canonical variables \( \lambda^\pm, \pi_A, q^A \), i.e. that do not depend on \( \pi_\pm^A \). After imposing conformal gauge we will retain the notation \( \{ \cdot, \cdot \} \) for the Dirac bracket instead of \( \{ \cdot, \cdot \}^* \) since the Poisson bracket won’t be needed any longer.

One can thus impose the constraints and thereafter forget about the metric degrees of freedom \( \lambda_\pm, \pi_\pm^A \) (i.e. \( \gamma_{\alpha \beta}, p^{\alpha \beta} \)) altogether. Therefore even in the Hamiltonian framework it is legitimate to work in the conformal gauge right from the outset, and set the worldsheet metric to be flat in the Hamiltonian. This corresponds in the variables (4.1.5) to setting (4.2.1), that is \( \lambda^\pm \approx -1 \), and the preservation of this gauge condition in time requires that \( \dot{c}_\pm = \dot{\lambda}^\pm \approx 0 \) which implies \( \rho^A_\pm = 0 \). The total Hamiltonian (4.1.15) then becomes

\[
H_T = \int d\sigma (\rho_+ T_+ + \rho_- T_-),
\]

where we have shifted the definitions of \( \rho_\pm \) by \( -\frac{1}{2} \).

### 4.3 Current algebra

From now on we shall assume that the metric is flat. In this case the covariant derivative of the momenta variables \( \nabla_1 \pi_a(\sigma) \) are related to \( j^0_a \) alone, as equation
\[ \nabla_1 \pi^a(\sigma) = \frac{\sqrt{\lambda}}{4\pi} j_0^a(\sigma). \]  

(4.3.1)

In fact, owing to the fact that the momenta \( \pi^a \) never appear without a covariant derivative, it is possible to rewrite every expression in terms of the current components \( j_0^a, j_1^a \) alone rather than the canonically conjugate variables \( \pi^a, j_1^a \) and it will be convenient to do so. The Hamiltonian, given by (4.1.6), for example reads

\[ H_0 = -\frac{\sqrt{\lambda}}{8\pi} (j_0^a j_0^a - j_1^a j_1^a) - \frac{\pi}{\sqrt{\lambda}} (\pi^0)^2 - \frac{\sqrt{\lambda}}{4\pi} (\partial_1 X_0)^2, \]  

(4.3.2a)

\[ H_1 = -\frac{\sqrt{\lambda}}{4\pi} j_0^a j_1^a + \pi^0 \partial_1 X_0. \]  

(4.3.2b)

Equivalently, the Virasoro constraints (4.1.7) now read

\[ T_\pm = \frac{\sqrt{\lambda}}{4\pi} \left[ \frac{1}{2} \text{tr} j_\mp^2 + \left( \frac{2\pi}{\sqrt{\lambda}} \pi^0 \pm \partial_1 X_0 \right)^2 \right] \approx 0, \]  

(4.3.3)

where \( j_\pm = j_0 \pm j_1 \) are the components of the current \( j \) in light-cone coordinates. With the substitution \( \pi^0 = -\frac{\sqrt{\lambda}}{2\pi} X_0 \) we notice that the variables \( T_\pm \) are nothing but the light-cone components \( T_{\mp\mp} \) of the energy-momentum tensor (3.4.1).

Rewriting also the Poisson brackets (4.1.10) by eliminating the three conjugate momenta \( \pi_a \) in favour of the current components \( j_0^a \) we obtain,

\[ \{ j_1^a(\sigma), j_1^b(\sigma') \} = 0, \]  

(4.3.4a)

\[ \frac{\sqrt{\lambda}}{4\pi} \{ j_0^a(\sigma), j_1^b(\sigma') \} = f^{abc} j_{1c}(\sigma) \delta(\sigma - \sigma') + k^{ab} \delta'(\sigma - \sigma'), \]  

(4.3.4b)

\[ \frac{\sqrt{\lambda}}{4\pi} \{ j_0^a(\sigma), j_0^b(\sigma') \} = f^{abc} j_{0c}(\sigma) \delta(\sigma - \sigma'). \]  

(4.3.4c)

The key feature to note about these fundamental brackets is the presence of the
derivative of a delta function $\delta'(\sigma - \sigma')$ on the right hand side of \eqref{4.3.4b}. Because of this term the brackets \eqref{4.3.4} are usually described as \textbf{non-ultralocal}. As we will see in the next chapter, the non-ultra local term will be the main source of problems in proving integrability of string theory on $\mathbb{R} \times S^3$, giving rise to ambiguities which will have to be dealt with properly.

At this stage however there is no apparent difficulty in dealing with the brackets \eqref{4.3.4}. For example, one can use them to show that the $SU(2)_R$ symmetry is generated by the Noether charge $Q_R$ defined in \eqref{3.2.1}. Indeed, we find from the last two brackets \eqref{4.3.4b}, \eqref{4.3.4c} that the Noether charge $Q_R$ acts on the $SU(2)_R$ current $j$ as expected

$$\{\epsilon \cdot Q_R, j\} = [\epsilon, j] = \delta \epsilon j,$$  \tag{4.3.5}$$

where $\epsilon = \epsilon^a t_a \in \mathfrak{su}(2)$ is infinitesimal, $\epsilon \cdot Q_R = \text{tr}(\epsilon Q_R) = \epsilon_a Q_R^a$ and $Q_R$ is given in conformal gauge by \eqref{3.3.3}. Moreover, the brackets \eqref{4.3.4} correctly leads to the Hamiltonian version of the equations of motion \eqref{3.1.8a}, namely

$$\{P_0, j_0\} = \partial_1 j_1,$$  \tag{4.3.6a}$$
$$\{P_0, j_1\} = \partial_1 j_0 + [j_0, j_1],$$  \tag{4.3.6b}$$
$$\{P_1, j_\alpha\} = \partial_1 j_\alpha, \quad \alpha = 0, 1.$$  \tag{4.3.6c}$$

where $P_\alpha = \int d\sigma \mathcal{H}_\alpha$ is the worldsheet energy-momentum vector and $\mathcal{H}_\alpha$ are given in \eqref{4.3.2}. If we interpret $P_0$ as generating the $\tau$-flow on phase-space, \textit{i.e.} $\{P_0, j_\alpha\} = \partial_0 j_\alpha$, then equations \eqref{4.3.6a} and \eqref{4.3.6b} are equivalent to $\partial_0 j_0 = \partial_1 j_1$ and $\partial_0 j_1 - \partial_1 j_0 = [j_0, j_1]$ respectively, which are the equations of motion \eqref{3.1.8a} for $j$ in components.
4.4 Static gauge

As already discussed in section 3.3, the constraint (4.2.1) by itself isn’t sufficient to fix the gauge invariance since the group of conformal transformations that leave the metric $\eta_{\alpha\beta}$ invariant up to an overall factor remains as the residual gauge group. We therefore have to impose further gauge fixing conditions.

The static gauge condition was defined by the single equation (3.3.7) for the general solution $X_0^{\text{sol}}(\sigma, \tau)$ of the field $X_0$. However, at any given time $\tau$, a solution $X_0^{\text{sol}}$ not only determines the configuration of the field $X_0(\sigma, \tau) = X_0^{\text{sol}}(\sigma, \tau)$ but also its momentum through the defining formula (4.1.3) which in conformal gauge reads $\pi^0(\sigma, \tau) = -\frac{\sqrt{\lambda}}{2\pi} \partial_0 X_0^{\text{sol}}(\sigma, \tau)$. Therefore in the Hamiltonian formalism the static gauge condition really consists of two constraints,

$$X_0 + \frac{p_0}{\sqrt{\lambda}} \tau \approx 0, \quad \pi^0 - \frac{p_0}{2\pi} \approx 0.$$  \hfill (4.4.1)

As before, the constant of proportionality, which here we denote $p_0$ since it is the zero-mode of the momentum $\pi^0$, is constrained by the space time energy $\Delta$ of the string since

$$p_0 = \int_0^{2\pi} d\sigma \pi^0(\sigma, \tau) = -\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \dot{X}_0(\sigma, \tau) = -\Delta.$$

In section 4.2 we imposed conformal gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ which had the effect of fixing the gauge invariance generated by the primary constraints $p^{\alpha\beta}$. But there are also secondary constraints, the Virasoro constraints (4.3.3) which remain unfixed and generate a residual gauge invariance. This will be fixed by imposing static gauge. Even though the Virasoro constraints $T_{\pm}$ by themselves are first class by equation (4.1.12), the static gauge conditions fail to commute with these and among themselves (since $\{\pi^0(\sigma), X_0(\sigma')\} = \delta(\sigma - \sigma') \neq 0$), so that the full set of constraints
becomes second-class.

However, as discussed in section 3.3, the static gauge still doesn’t completely fix the residual gauge invariance since it leaves the possibility of performing a rigid $\sigma$-translation, which is generated by the worldsheet momentum. Thus we start by isolating this generator among the Virasoro constraints, which we do by decomposing both the Virasoro constraints (4.3.3) and static gauge conditions (4.4.1) into Fourier modes.

### Fourier modes

Introduce the modes $L_n, \tilde{L}_n$ of the current part of the $T_{\pm}$ in (4.3.3), namely $\frac{1}{2} \text{tr} j_{\pm}^2$, by

$$
L_n = \frac{\sqrt{\lambda}}{8\pi} \int_0^{2\pi} e^{in\sigma} \frac{1}{2} \text{tr} j_+^2(\sigma) d\sigma, \quad \tilde{L}_n = \frac{\sqrt{\lambda}}{8\pi} \int_0^{2\pi} e^{-in\sigma} \frac{1}{2} \text{tr} j_-^2(\sigma) d\sigma.
$$

These are easily seen to satisfy the following algebra

$$
\{L_m, L_n\} = i(m - n)L_{m+n}, \\
\{L_m, \tilde{L}_n\} = 0, \\
\{\tilde{L}_m, \tilde{L}_n\} = i(m - n)\tilde{L}_{m+n},
$$

which follows from the Virasoro algebra (4.1.12) for the $T_{\pm}$. Define also the modes $\alpha_n, \tilde{\alpha}_n$ of $X_0$ and $\pi^0$ as

$$
\alpha_n = \frac{\lambda^{\frac{1}{4}}}{\sqrt{2\pi}} \int_0^{2\pi} e^{-in\sigma} \frac{1}{2} \left( -\frac{2\pi}{\sqrt{\lambda}} \pi^0(\sigma) - \partial_\sigma X_0(\sigma) \right) d\sigma, \quad n \neq 0 \\
\tilde{\alpha}_n = \frac{\lambda^{\frac{1}{4}}}{\sqrt{2\pi}} \int_0^{2\pi} e^{in\sigma} \frac{1}{2} \left( -\frac{2\pi}{\sqrt{\lambda}} \pi^0(\sigma) + \partial_\sigma X_0(\sigma) \right) d\sigma, \quad n \neq 0 \\
x_0 = \frac{1}{2\pi} \int_0^{2\pi} X_0(\sigma) d\sigma, \quad p_0 = \int_0^{2\pi} \pi^0(\sigma) d\sigma.
$$
Their algebra easily follows from the defining bracket \( \{ \pi^0(\sigma), X_0(\sigma') \} = \delta(\sigma - \sigma') \), namely

\[
\{ \alpha_m, \alpha_n \} = im\delta_{m+n}, \quad \{ \alpha_m, \tilde{\alpha}_n \} = 0,
\]

\[
\{ \tilde{\alpha}_m, \alpha_n \} = im\delta_{m+n}, \quad \{ p_0, x_0 \} = 1.
\]

In terms of these modes, the Virasoro constraints (4.3.3) and static gauge fixing conditions (4.4.1) read

Virasoro: \( L_n \approx \tilde{L}_n \approx 0 \) \( (n \neq 0) \), \( L_0 \approx \tilde{L}_0 \approx -\frac{p_0^2}{4\sqrt{\lambda}} \).

Static gauge: \( \alpha_n \approx \tilde{\alpha}_n \approx 0 \) \( (n \neq 0) \), \( x_0 + \frac{p_0}{\sqrt{\lambda}} \tau \approx 0 \).

Yet these include the generator \( L_0 - \tilde{L}_0 \) of rigid translations \( \sigma \rightarrow \sigma + b \). Therefore setting aside this rigid transformation to deal with it later by symplectic reduction, the set of relevant constraints now read

Virasoro: \( L_n \approx \tilde{L}_n \approx 0 \) \( (n \neq 0) \), \( \gamma_0 \equiv (L_0 + \tilde{L}_0) + \frac{p_0^2}{2\sqrt{\lambda}} \approx 0 \), (4.4.5a)

Static gauge: \( \alpha_n \approx \tilde{\alpha}_n \approx 0 \) \( (n \neq 0) \), \( c_0 \equiv x_0 + \frac{p_0}{\sqrt{\lambda}} \tau \approx 0 \), (4.4.5b)

This separation of the constraint \( P_1 \approx 0 \) from the Virasoro constraints is just a rephrasing in Hamiltonian terms of equation (3.4.3) in section 3.3. Indeed, in the present language we have \( L_0 = -\frac{\sqrt{\lambda}}{4}\kappa^2_+, \tilde{L}_0 = -\frac{\sqrt{\lambda}}{4}\kappa^2_- \) and \( p_0 = -\sqrt{\lambda}\kappa \) so that

\[
P_0 = -L_0 - \tilde{L}_0 - \frac{p_0^2}{2\sqrt{\lambda}}, \quad P_1 = -L_0 + \tilde{L}_0,
\]

is equivalent to equation (3.4.4). The energy and momentum of the principal chiral field are \( \mathcal{E} = -L_0 - \tilde{L}_0 \) and \( \mathcal{P} = -L_0 + \tilde{L}_0 \) respectively. Now although we postpone imposing the Virasoro constraint \( \mathcal{P} \approx 0 \) (because there is no corresponding gauge
fixing condition in static gauge (4.4.1), the Virasoro constraint $P_0 \approx 0$ is imposed alongside the static gauge fixing conditions (4.4.1). As we saw in section 3.4 this condition has the effect of equating the principal chiral model energy with the space-time energy of the string,

$$\mathcal{E} \approx \frac{p_0^2}{2\sqrt{\lambda}} = \frac{\Delta^2}{2\sqrt{\lambda}} \quad (4.4.6)$$

**Dirac brackets**

The static gauge condition (4.4.5b) fixes all the modes of $X_0, \pi^0$ except for $p_0$ which leaves the degrees of freedom of the principal chiral fields $j$ and $p_0$. But the last Virasoro constraint in (4.4.5a) determines $p_0$ as a function of $j$ through the combination $L_0 + \tilde{L}_0$. We shall refer to the degrees of freedom remaining after imposing conformal static gauge and the Virasoro constraints (4.4.5) as the reduced phase-space.

**Definition 4.4.1.** *The reduced phase-space* $\mathcal{P}^\infty$ *is parameterised by the current* $j(\sigma)$ *subject to the constraints* $L_n \approx \tilde{L}_n \approx 0, n \neq 0$.

The physical degrees of freedom can now be described by a simple symplectic reduction of the reduced phase-space $\mathcal{P}^\infty$ onto the level set $P_1 \approx 0$:

$$\begin{array}{c}
P_1 \approx 0 \quad \iota \quad \mathcal{P}^\infty \\
\pi \downarrow \\
\text{Physical phase-space}
\end{array}$$

Since the constraints (4.4.5) defining $\mathcal{P}^\infty$ are second-class, fixing them requires introducing a Dirac bracket. The matrix of Poisson brackets $C'_{ab} = \{\chi_a, \chi_b\}$ between all the second-class constraints $\chi_a$ in (4.4.5) takes the following schematic form.
weakly (i.e. on the constraint surface $\chi_a \approx 0$)

$$C'_{ab} = \{\chi_a, \chi_b\} \approx \begin{pmatrix} 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} \gamma_0 \\ c_0 \\ L_n \\ \tilde{L}_n \\ \alpha_n \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(4.4.7)

with inverse $C'^{-1}_{ab}$ of exactly the same form. But when working in conformal static gauge it is enough to consider functions $F, G$ of $j$ which are independent of $X_0, \pi^0$ (and therefore commute with the constraints $\alpha_n, \tilde{\alpha}_n, c_0 = x_0 + p_0 \tau / \sqrt{\lambda}$). It follows from (4.4.7) that for such functions the Dirac bracket takes the form

$$\{F, G\}_{D.B.} = \{F, G\} - \sum_{n,m \neq 0} \{F, L_n\}\{L_n, L_m\}^{-1}\{L_m, G\} - \sum_{n,m \neq 0} \{F, \tilde{L}_n\}\{\tilde{L}_n, \tilde{L}_m\}^{-1}\{\tilde{L}_m, G\}. \quad (4.4.8)$$

Here $\{L_n, L_m\}^{-1}$ denotes the matrix inverse of $\{L_n, L_m\}$, likewise for $\{\tilde{L}_n, \tilde{L}_m\}^{-1}$. There are no terms involving $L_0 + \tilde{L}_0$ because the corresponding components in the inverse matrix $C'^{-1}_{ab}$ all vanish. If either of the two functions $F, G$ happen to be invariant under residual gauge transformations generated by $L_n, \tilde{L}_n, n \neq 0$ then their Dirac and Poisson brackets are equal

$$\{F, G\}_{D.B.} = \{F, G\}.$$

The expression (4.4.8) for the Dirac bracket can be simplified further. Using the Virasoro algebra (4.4.3) one finds the weak equalities $\{L_n, L_m\} \approx 2inL_0 \delta_{m+n}$
and \( \{ \tilde{L}_n, \tilde{L}_m \} \approx 2in\tilde{L}_0\delta_{m+n} \), the (matrix) inverses of which are

\[
\{L_n, L_m\}^{-1} \approx \frac{i}{2nL_0}\delta_{m+n}, \quad \{\tilde{L}_n, \tilde{L}_m\}^{-1} \approx \frac{i}{2n\tilde{L}_0}\delta_{m+n}.
\]

The Dirac bracket (4.4.8) then takes the simpler form

\[
\{F, G\}_{D.B.} \approx \{F, G\} - \frac{i}{2L_0} \sum_{n \neq 0} \frac{1}{n} \{F, L_n\}\{L_{-n}, G\} - \frac{i}{2\tilde{L}_0} \sum_{n \neq 0} \frac{1}{n} \{F, \tilde{L}_n\}\{\tilde{L}_{-n}, G\}.
\]  

This bracket provides a non-degenerate symplectic structure on the reduced phase-space \( P^\infty \). To close this chapter we determine the reduced dynamics on \( P^\infty \) with respect to this Dirac bracket (4.4.9).

### Reduced dynamics

In a generally covariant theory such as string theory, ‘time’ cannot be an observable since arbitrary time-reparametrisations are allowed. In other words time is pure-gauge and the only quantities one can talk about are constants of the motion. But instead of talking about gauge-invariant quantities we have chosen to isolate the physical degrees of freedom by explicitly breaking the time-reparametrisation invariance through the use of gauge fixing conditions. And because such gauge conditions single out a special time, it makes sense to talk about the reduced dynamics, with respect to this time, of the degrees of freedom parameterising the reduced phase-space \( P^\infty \).

Naively one would guess that the dynamics on \( P^\infty \) is generated simply by the total Hamiltonian \( H_T \) if we use the Dirac brackets. From a physical point of view this must obviously be wrong since otherwise the Hamiltonian being weakly zero \( H_T \approx 0 \)
would imply that every function $F$ with no explicit time dependence is actually time independent $\dot{F} \approx \{H_T, F\}_{\text{D.B.}} \approx 0$. The reason why $H_T$ gives the wrong dynamics on $\mathcal{P}^\infty$ is because the static gauge fixing conditions (4.4.1) are $\tau$-dependent and implementing such constraints in Dirac’s theory of constrained Hamiltonian systems turns out to be far from obvious. Indeed, using the usual equations of motion the $\tau$-dependent constraint $c_0 = x_0 + \frac{p_0}{\sqrt{\lambda}} \tau \approx 0$ is not preserved under time evolution because

$$\frac{dc_0}{d\tau} = \frac{\partial c_0}{\partial \tau} + \{H_T, c_0\}_{\text{D.B.}} \approx \frac{\partial c_0}{\partial \tau} = \frac{p_0}{\sqrt{\lambda}} \not\approx 0.$$  

A correction term needs to be added to the equations of motion in order to accommodate for the $\tau$-dependence of the constraint $c_0 \approx 0$. For an arbitrary functions $F$ with explicit time dependence the equations of motion now read [64, p110, ex. 4.8]

\begin{align*}
\frac{dF}{d\tau} &= \frac{\partial F}{\partial \tau} + \{H_T, F\}_{\text{D.B.}} - \frac{\partial c_0}{\partial \tau} \{\gamma_0, c_0\}^{-1}\{\gamma_0, F\}, \quad (4.4.10a) \\
&= \frac{\partial F}{\partial \tau} + \{H_T, F\}_{\text{D.B.}} - \{\gamma_0, F\}, \quad (4.4.10b)
\end{align*}

where $\gamma_0 = L_0 + \tilde{L}_0 + \frac{p_0^3}{2\sqrt{\lambda}} \approx 0$ is the only Virasoro constraint that has a non-zero Poisson bracket with $c_0 \approx 0$. It is immediate from (4.4.10a) that now we have $\frac{dc_0}{d\tau} \approx 0$ and all the other constraints are also preserved, since their Poisson bracket with $\gamma_0$ is weakly zero. Note that the correction term is just a gauge transformation whose role is to maintain the dynamics on the constraint surface $c_0 \approx 0$, much like the Dirac bracket ensures that time-independent second-class constraints are preserved in time.

An undesirable feature of (4.4.10) is that it isn’t written in terms of the Dirac bracket. However, for functions $F$ which only depend on the principal chiral fields $j$ one can show that $\{p_0, F\}_{\text{D.B.}} = -\frac{\sqrt{\lambda}}{p_0} \{L_0 + \tilde{L}_0, F\}$. Indeed, going back to the matrix

\footnote{Any complete gauge fixing in a generally covariant theory always requires imposing time-dependent gauge fixing conditions.}
of Poisson brackets (4.4.7) we have \( C'_{c_0 \gamma_0} = \{ c_0, \gamma_0 \} \) but \( C'_{c_0 \gamma_0}^{-1} = 1/\{ \gamma_0, c_0 \} \), thus

\[
\{ p_0, F \}_{\text{D.B.}} = \{ p_0, F \} - \{ p_0, c_0 \} \frac{1}{\{ \gamma_0, c_0 \} \{ \gamma_0, F \}}.
\]

Now \( \{ p_0, c_0 \} = 1 \), \( \{ \gamma_0, c_0 \} = \frac{p_0}{\sqrt{\lambda}} \) and \( \{ p_0, F \} = 0 \) by assumption on \( F \) so the result follows. Using this result the equation of motion (4.4.10b) can be rewritten for such functions of the physical variables as

\[
\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + \left\{ H_T + \frac{p_0^2}{2\sqrt{\lambda}}, F \right\}_{\text{D.B.}}.
\] (4.4.11)

Thus we observe that the equations of motion on the reduced phase-space \( \mathcal{P}^\infty \) are generated not by the total Hamiltonian \( H_T \) (which is weakly zero) but by a shifted Hamiltonian

\[
H^* \equiv H_T + \frac{p_0^2}{2\sqrt{\lambda}} \approx \frac{p_0^2}{2\sqrt{\lambda}}.
\] (4.4.12)

A careful generalisation of Dirac’s analysis of constrained Hamiltonian systems to include time-dependent constraints (hence allowing the use of time-dependent gauge fixing conditions) was given in [65, 66] and also leads to the same conclusion. There the presence of time-dependent constraints leads to a shift in the 1-form \( dH_T \mapsto dH_T + A \) so that the reduced dynamics \( \iota(v)\omega^* = dH_T + A \) can still be described by Hamilton’s equations in terms of the Dirac bracket provided \( A \) is locally exact.

A simple computation in the formalism of [65, 66] shows that \( A = -d(L_0 + \tilde{L}_0) \) and hence the total Hamiltonian gets shifted by the same amount (4.4.12) since \( -L_0 - \tilde{L}_0 \approx \frac{p_0^2}{2\sqrt{\lambda}} \) by the Virasoro constraints (4.4.5).

The equation of motion for the reduced dynamics (4.4.11) has an obvious interpretation. It says that the energy \( \mathcal{E} = -L_0 - \tilde{L}_0 \) of the principal chiral model generates worldsheet \( \tau \)-translations on \( \mathcal{P}^\infty \) (from now on we assume \( F \) has no explicit
dependence of $\tau$)
\[
\frac{dF}{d\tau} = \{\mathcal{E}, F\}_{\text{D.B.}}. \tag{4.4.13a}
\]

But using the zero-mode parts of the Virasoro constraints (4.4.6) and static gauge fixing conditions (4.4.5b) we see that the dynamics (4.4.13a) is equivalent to the global translation symmetry in the target time $X_0$.
\[
\frac{dF}{dx_0} = \{\Delta, F\}_{\text{D.B.}},
\]

which is generated by the space-time energy $\Delta = -p_0$ of the string. In conclusion, although the worldsheet coordinates have been fixed, we have done so using the $\tau$-dependent static gauge fixing conditions which relate the worldsheet time $\tau$ to the target time $X_0$. As a result, the global $X_0$-translation symmetry gives rise to non-trivial $\tau$-dynamics for the remaining degrees of freedom of the string. Since at this stage the vanishing of the worldsheet momentum $\mathcal{P} \approx 0$ hasn’t yet been imposed, the momentum $\mathcal{P}$ of the principal chiral model still generates worldsheet $\sigma$-translations as in (4.3.6c),
\[
\frac{dF}{d\sigma} = \{\mathcal{P}, F\}_{\text{D.B.}}. \tag{4.4.13b}
\]

Unlike (4.4.13a) however the $\sigma$-dynamics (4.4.13b) are not physical and must be removed at the end of the day by symplectic reduction to the level set $\mathcal{P} \approx 0$. 
Chapter 5

Integrability

“When makes the desert beautiful, said the little prince, is that somewhere it hides a well…”

Antoine de Saint-Exupéry, Le Petit Prince

5.1 Conserved charges

When working in conformal and static gauge, the only field that remains unfixed is the principal chiral field $j$. The equations of motion of the string reduce to $j$ being both conserved and flat (3.1.8a)

$$d \ast j = 0, \quad (5.1.1a)$$

$$dj - j \wedge j = 0. \quad (5.1.1b)$$

These are two first-order differential equations for the current $j$ which express abelian and non-abelian conservation laws for $\ast j$ and $j$ respectively as we now explain.

1 “What makes the desert beautiful, said the little prince, is that somewhere it hides a well…”
First of all, as we discussed in section 3.2, the current $j$ is actually the Noether current for the global $SU(2)_R$ symmetry whose conservation is equivalent to equation (5.1.1a). The corresponding Noether charge defined in (3.2.1a) is the integral $\int_\gamma *j$ around a closed loop $\gamma$ of non-trivial homotopy on the worldsheet. In geometrical terms its conservation is a consequence of Stokes’ theorem as already discussed in section 3.2.

\[
\int_{\gamma_2} *j - \int_{\gamma_1} *j = \int_{\partial D} *j = \int_D d * j = 0. \quad (5.1.2)
\]

Secondly, the current $j$ is flat by equation (5.1.1b). This was a consequence of its definition $j = -g^{-1} dg$. But this property also leads to a very nice conservation law. Indeed, consider the parallel transporter $\hat{\Psi}(\gamma)$ with $j$ as connection along a path $\gamma$ on the worldsheet,

\[
\hat{\Psi}(\gamma) = P \exp \int_\gamma j. \quad (5.1.3)
\]

**Theorem 5.1.1** (non-abelian Stokes’). *If $j$ is a lie-algebra valued 1-form and $D$ is a simply connected region then*

\[
P \exp \int_{\partial D} j = A \exp \int_D \hat{\Psi}(\gamma)^{-1}(dj - j \wedge j)\hat{\Psi}(\gamma),
\]

*where $A$ is some “surface ordering” and $\gamma$ is a path joining the base point of $\partial D$ to the integration point $x \in D$. 

Corollary 5.1.2. If $j$ is flat and $D$ is simply connected then

$$P \exp \int_{\partial D} j = 1.$$  

We deduce from corollary 5.1.2 that the parallel transporter $\hat{\Psi}(\gamma)$ defined by (5.1.3) only depends on the homotopy class of $\gamma$ with fixed endpoints $x, y$. Now consider the parallel transporter around a closed loop $\gamma_x$ based at $x$ and winding once around the worldsheet. Note that the base-point $x$ is important here since we are considering path-ordered exponentials. Corollary 5.1.2 implies that $\hat{\Psi}(\gamma_x)$ is independent of the path, provided it still starts and ends at $x$ after winding a single time around the worldsheet. This is not quite a conservation law in the sense of (5.1.2) since it only gives $\hat{\Psi}(\gamma_2) = \hat{\Psi}(\gamma_1)$ if the paths $\gamma_1$ and $\gamma_2$ are both bound at the same point $x$. We would like a relation between $\hat{\Psi}(\gamma_1)$ and $\hat{\Psi}(\gamma_2)$ for two general loops $\gamma_1, \gamma_2$ as in (5.1.2). But corollary 5.1.2 also provides such a relation when the base points $x$ and $y$ of $\gamma_1$ and $\gamma_2$ are different, namely

$$\hat{\Psi}(\gamma_2) = \hat{\Psi}(\gamma)\hat{\Psi}(\gamma_1)\hat{\Psi}(\gamma)^{-1},$$

where $\gamma$ is a path connecting the base points $x$ and $y$. This is a sort of “non-abelian” conservation law. In fact it implies more than one “abelian” conservation law because each eigenvalue of $\hat{\Psi}(\gamma_i)$ is separately conserved, which follows from

$$\det \left( \lambda I - \hat{\Psi}(\gamma_1) \right) = \det \left( \lambda I - \hat{\Psi}(\gamma_2) \right).$$  

(5.1.4)
Lax connection

Since flat currents lead to conserved quantities it makes sense to consider the combination

\[ J = \alpha j + \beta * j, \]  

(5.1.5)

and try to adjust the constants \( \alpha, \beta \) to render \( J \) flat. It is straightforward to show from (5.1.1) using the rules \( ** = +1 \) and \( a \wedge b = - * a \wedge b \) for any (lie-algebra valued) 1-forms \( a, b \) that

\[ dJ - J \wedge J = -(\alpha^2 - \alpha - \beta^2) j \wedge j. \]

Notice that the right hand side is proportional to the amount by which \( *j \) fails to be flat, namely \( *j \wedge *j = - j \wedge j \). We see that \( J \) is flat provided \( \alpha^2 - \alpha - \beta^2 = 0 \). This is a single constraint on the two parameters of (5.1.5) admitting two solutions \( \alpha = \frac{1}{1-x^2}, \beta = \pm \frac{x}{1-x^2} \), parameterised by a single variable \( x \). Both solutions are related by \( x \to -x \) so this construction provides a 1-parameter family of flat currents \( J(x) \).

**Definition 5.1.3.** The Lax connection is the 1-parameter family of 1-forms on the worldsheet

\[ J(x) = \frac{1}{1 - x^2} (j - x * j), \quad x \in \mathbb{C}. \]  

(5.1.6)

By construction \( dJ - J \wedge J = \alpha (dj - j \wedge j) + \beta d* j \) and so we have the following

**Lemma 5.1.4.** The Lax connection \( J(x) \) is flat if and only if \( j \) is on-shell, i.e.

\[ dJ(x) - J(x) \wedge J(x) = 0 \quad \Leftrightarrow \quad \begin{cases} 
  d \ast j = 0, \\
  d j - j \wedge j = 0.
\end{cases} \]

Note that the flatness condition (along with the whole formalism that will
5.1. CONSERVED CHARGES

follow from it) is invariant under gauge transformations

\[ J(x) \mapsto \tilde{g} J(x) \tilde{g}^{-1} + (d \tilde{g}) \tilde{g}^{-1}, \quad (5.1.7) \]

where the matrix \( \tilde{g}(x, \sigma, \tau) \) is an arbitrary function of the spectral parameter \( x \) and the worldsheet space and time coordinates \( \sigma, \tau \). In particular, for the purpose of discussing the integrals of motion the Lax connection (5.1.6) is by no means special. Indeed in section 5.2 we shall make use of the gauge freedom (5.1.7) to move to a more appropriate gauge for identifying the local conserved charges.

Monodromy

Owing to the flatness of the current \( J(x) \), it is now natural to consider parallel transporters on the worldsheet using \( J(x) \) as the connection,

\[
\hat{\Psi}(\gamma, x) = P \exp \int_{\gamma} J(x). \quad (5.1.8)
\]

As before, the object leading to “non-abelian” conservation laws is the transporter around a path of non-trivial homotopy.

Definition 5.1.5. The Monodromy matrix is the parallel transporter

\[
\Omega(x, \sigma, \tau) = P \exp \int_{[\gamma(\sigma, \tau)]} J(x),
\]
where $\gamma(\sigma, \tau)$ is a loop starting and ending at $(\sigma, \tau)$ that winds once around the worldsheet.

By the non-abelian Stokes’ theorem this definition only depends on the homotopy class $[\gamma(\sigma, \tau)]$ of the curve $\gamma(\sigma, \tau)$ with both end-points fixed at $(\sigma, \tau)$. In particular, since the path $\sigma \in [0, 2\pi]$ at fixed time $\tau$ is in this homotopy class, if we write the Lax connection in components as $J(x) = J_0(x)d\tau + J_1(x)d\sigma$ then we can write the monodromy matrix as

$$\Omega(x, \sigma, \tau) = P\exp \left[ \int_{\sigma}^{\sigma+2\pi} d\sigma' J_1(x, \sigma', \tau) \right]. \quad (5.1.9)$$

Furthermore, by using the non-ablelian Stokes’ theorem to change the base point $(\sigma, \tau)$, the monodromy matrix has the following immediate property

**Lemma 5.1.6.** The $(\sigma, \tau)$-evolution of $\Omega(x, \sigma, \tau)$ is isospectral, i.e.

$$\Omega(x, \sigma', \tau') = \hat{\Psi}(\gamma, x)\Omega(x, \sigma, \tau)\hat{\Psi}(\gamma, x)^{-1}, \quad (5.1.10)$$

where $\gamma$ is a path from $(\sigma, \tau)$ to $(\sigma', \tau')$.

From now on we may sometimes omit the explicit dependence on the base point $(\sigma, \tau)$ when it is clear and abbreviate $\Omega(x, \sigma, \tau)$ as $\Omega(x)$. 
5.2 LOCAL CONSERVED CHARGES

Integrals of motion

Once more the isospectral evolution of the $\Omega(x)$ in lemma 5.1.6 means that all its eigenvalues are conserved since the characteristic polynomial

$$\Gamma(x, \Lambda) = \det (\Lambda 1 - \Omega(x, \sigma, \tau)),$$

is independent of $(\sigma, \tau)$. However, the novelty when considering $J(x)$ as connection instead of $j$ is that the conserved eigenvalues are now functions of $x \in \mathbb{C}$. Taylor expanding each eigenvalue in $x$ therefore provides an infinite number of integrals of motion.

The infinitesimal version of (5.1.10), that is its leading order in $\delta \sigma = \sigma' - \sigma$ and $\delta \tau = \tau' - \tau$, gives a differential equation governing the $(\sigma, \tau)$-dependence of the monodromy matrix,

$$[d - J(x), \Omega(x)] = 0.$$  

(5.1.12)

It is evident that any power of $\Omega(x)$ also satisfies the same equation, or equivalently in components $\partial_\alpha \Omega(x)^n = [J_\alpha(x), \Omega(x)^n]$, $\alpha = 0, 1$. Taking the trace yields another way of characterising the conservation of the eigenvalues of $\Omega(x)$,

$$\partial_0 \text{tr } \Omega(x)^n = \partial_1 \text{tr } \Omega(x)^n = 0.$$  

(5.1.13)

5.2 Local conserved charges

Conserved charges can be of two different types: local or non-local. A conserved charge is **local** if it is the integral of a local density, otherwise it is **non-local**, such as when the density itself is an integral. Any charge arising from a continuous symmetry through the use of Noether’s theorem is always local since it is the integral $\int_\gamma *j$ of
a current \( *j \) which is a local expression of the fields and whose local conservation is expressed as \( d * j = 0 \). It is clear also that any closed 1-form which is a local expression of the fields would give rise to a local conserved charge. However, since the conserved charges \( \text{tr} \Omega(x)^n \) arose from a non-abelian conservation law \( dJ(x) - J(x) \wedge J(x) = 0 \), it is not obvious that any of them are local. Although non-local charges are very interesting we will not be concerned with them here and so we would like a way of extracting only local charges from the generator of charges \( \text{tr} \Omega(x)^n \).

If it were possible to diagonalise the Lax connection \( J(x) \) then the non-abelian conservation law for the current \( J(x) \) would reduce to \( dJ(x) = 0 \) and immediately provide infinitely many local charges. The following theorem [67, p66] shows that this “abelianisation” is possible in a neighbourhood of the singularities \( x = \pm 1 \) of the Lax connection.

**Theorem 5.2.1.** Around the points \( x = \pm 1 \) there exists regular local periodic gauge transformations

\[
J(x) \mapsto J'_{(\pm)}(x) = g_{(\pm)}(x)J(x)g_{(\pm)}(x)^{-1} + dg_{(\pm)}(x)g_{(\pm)}(x)^{-1}, \quad (5.2.1)
\]

such that \( J'_{(\pm)}(x) = \sum_{n=-1}^{\infty} J_n^{(\pm)}(x \mp 1)^n \) are diagonal. In particular, \( dJ'_{(\pm)}(x) = 0 \).

In the gauge of theorem 5.2.1 the conservation laws become abelian \( dJ'_{(\pm)}(x) = 0 \) and so one can immediately assert the existence of an infinite number of local charges

\[
Q_n^{(\pm)} = \int_{\gamma} J_n^{(\pm)}(x), \quad n = -1, 0, \ldots \quad (5.2.2)
\]

These are the coefficients of \( Q^{(\pm)}(x) = \int_{\gamma} J'_{(\pm)}(x) = \sum_{n=-1}^{\infty} (x \mp 1)^n \int_{\gamma} J_n^{(\pm)} \) which is
conserved by Stokes’ theorem,
\[
\int_{\Gamma_2} J_\ell'(x) - \int_{\Gamma_1} J_\ell'(x) = \int_{\partial D} J_\ell'(x) = \int_D J_\ell'(x) = 0.
\]

Moreover, because the connection is diagonal, the path ordering in the definition of the monodromy matrix is not necessary in this gauge. Therefore around \( x = \pm 1 \) the monodromy matrix transforms under the gauge transformation of theorem 5.2.1 to the following very simple diagonal form,
\[
\Omega(x) \rightarrow g_{(\pm)}(x)\Omega(x)g_{(\pm)}(x)^{-1} = \exp \left[ \sum_{n=-1}^{\infty} Q_n^{(\pm)}(x \mp 1)^n \right], \quad (5.2.3)
\]
where we have used the fact that \( g_{(\pm)}(x) \) are periodic in \( \sigma \). In particular, the local charges (5.2.2) can be extracted from \( \text{tr} \Omega(x)^n \) by expanding around \( x = \pm 1 \), as claimed. Since \( j_\pm \in \mathfrak{su}(2) \) implies \( \det \Omega(x) = 1 \), it follows that all the diagonal matrices \( Q_n^{(\pm)} \) are proportional to the third Pauli matrix \( \sigma_3 = \text{diag}(1, -1) \).

**Definition 5.2.2.** The local charges are given by
\[
Q_n^{(\pm)} = \frac{1}{2t} \text{tr} (Q_n^{(\pm)} \sigma_3), \quad n = -1, 0, \ldots
\]

Recall that apart from satisfying the equations of motion (5.1.1), the current \( j \) must also solve the Virasoro constraints (3.4.3)
\[
\frac{1}{2} \text{tr} j_\pm^2 = -\kappa_\pm^2.
\]

Up to now we have not yet implemented these in the Lax formalism. The next proposition makes first use of these constraints to compute the first local charges.
Q^{(\pm)}_{-1}. Note however that a complete treatment of the Virasoro constraints will have to wait until we switch over to the Hamiltonian formalism in the next section.

**Proposition 5.2.3.** The first charges are equal to $Q^{(\pm)}_{-1} = -\pi \kappa_{\pm}$. In particular,

$$g_{(\pm)}(x)\Omega(x)g_{(\pm)}(x)^{-1} = \exp \left[ -\frac{i\pi \kappa_{\pm}}{x \mp 1} \sigma_3 + O((x \mp 1)^0) \right] \quad \text{as } x \rightarrow \pm 1. \quad (5.2.4)$$

**Proof.** The asymptotics of the first component $J_1(x)$ of the Lax connection near $x = \pm 1$ are

$$J_1(x) = \frac{1}{2 \mp 1} \frac{j_{\pm}}{x \mp 1} + O((x \mp 1)^0), \quad \text{as } x \rightarrow \pm 1.$$  

But because the gauge parameters $g_{(\pm)}(x) = \sum_{n=0}^{\infty} g_{(\pm)}^n(x \mp 1)^n$ are regular it follows from (5.2.1) that $J'_\pm(x)$ has a simple pole at $x = \pm 1$ and the coefficients $g_{0,\pm}$ are the matrices diagonalising $j_{\pm}$. In other words,

$$J'_\pm(x) = -\frac{1}{2} \frac{j_{\pm}^{\text{diag}}}{x \mp 1} + O((x \mp 1)^0), \quad \text{as } x \rightarrow \pm 1.$$  

It remains to compute the eigenvalues of $j_{\pm}$. But since $j_{\pm} \in \mathfrak{su}(2)$ one has $\det j_{\pm} = -\frac{1}{2} \text{tr} j_{\pm}^2$. So the Virasoro constraint (3.4.3) may be rewritten as $\det j_{\pm} = \kappa_{\pm}^2$ and using $\text{tr} j_{\pm} = 0$ the eigenvalues of $j_{\pm}$ are therefore $i\kappa_{\pm}$ and $-i\kappa_{\pm}$.  

Recall from equation (3.4.4) that the energy and momentum of the principal chiral field $j$ are given by $E \pm P = \sqrt{\frac{\lambda}{2}} \kappa_{\pm}^2$ and therefore are directly related to the squares of the first charges $Q^{(\pm)}_{-1}$, namely

$$E \pm P = \frac{\sqrt{\lambda}}{2\pi^2} \left( Q^{(\pm)}_{-1} \right)^2. \quad (5.2.5)$$

We note for later that the light-cone components $J_{\pm}(x) = J_0(x) \pm J_1(x) = \frac{j_{\pm}}{1 \pm x}$ of the Lax connection are diagonalised by $g_{0,\pm}(x) = g_{(\pm)}(\pm 1)$ and thus take on the following
simple form

\[ J_{\pm}(x') = \frac{iK_{\pm}}{1 \mp x'} g_0^{(\pm)} - i\sigma_3 g_0^{(\pm)}. \]  

(5.2.6)

**Noether charges**

We have just seen that an infinite number of local charges can be extracted from the expansion of the monodromy matrix at the special points \( x = \pm 1 \), in particular the principal chiral field energy and momentum \[ (5.2.5) \]. It turns out that the Noether charges of the global \( SU(2)_R \times SU(2)_L \) symmetries can be easily extracted from asymptotics of the monodromy matrix at other points. As we now show, the Noether charge \( Q_R \) (resp. \( Q_L \)) of the \( SU(2)_R \) (resp. \( SU(2)_L \)) symmetry is the first non-trivial coefficient in the expansion of \( \Omega(x) \) at \( x = \infty \) (resp. \( x = 0 \)). The higher coefficients of the expansions at these points are all related to non-local charges \[ [68] \] and will therefore not interest us.

The asymptotic expansion of the connection \[ (5.1.6) \] at \( x = \infty \)

\[ J(x) = \frac{1}{x} \ast j + O \left( \frac{1}{x^2} \right), \]

leads to the following asymptotic expansion of the monodromy matrix at \( x = \infty \)

\[ \Omega(x) = P \exp \int_{[\gamma(\sigma,\tau)]} \left( \frac{1}{x} \ast j + O \left( \frac{1}{x^2} \right) \right) \]

\[ = 1 + \frac{1}{x} \frac{4\pi Q_R}{\sqrt{\lambda}} + O \left( \frac{1}{x^2} \right), \text{ as } x \to \infty. \]

(5.2.7a)

The asymptotics of the connection at \( x = 0 \) is \( J(x) = j - x \ast j + O (x^2) \), so that

\[ d - J(x) = d - j + x \ast j + O (x^2), \]

\[ = g^{-1} (d + x \ast l + O (x^2)) g, \]
where \( l = -dg g^{-1} = gjg^{-1} \). Now because the field \( g(\sigma, \tau) \) is periodic in \( \sigma \) it follows that the asymptotic expansion of the monodromy matrix near \( x = 0 \) is given by

\[
\begin{align*}
g\Omega(x)g^{-1} & = P\exp \left( \int_{[\gamma(\sigma, \tau)]} -x \ast l + O(x^2) \right), \\
& = 1 - x \frac{4\pi Q_L}{\sqrt{\lambda}} + O(x^2), \quad \text{as } x \to 0. \quad (5.2.7b)
\end{align*}
\]

Since the Noether charges \( Q_R \) and \( Q_L \) are conserved classically, we may fix them to lie in a particular direction of \( \text{su}(2) \) and take them for example to be proportional to the third Pauli matrix \( \sigma_3 \)

\[
Q_R = \frac{1}{2i} R\sigma_3, \quad Q_L = \frac{1}{2i} L\sigma_3, \quad R, L \in \mathbb{R}_+.
\]

where \( R \) and \( L \) are constants of the motion. By restricting the Noether charges in this way we focus on the subset of ‘highest weight’ solutions to the equations of motion. There is however no loss of generality in doing so since all other solutions can be obtained by applying a combination of \( SU(2)_R \) and \( SU(2)_L \) to such a ‘highest weight’ solution. With this restriction the asymptotic expansions \([5.2.7]\) reduce to

\[
\begin{align*}
\Omega(x) & = 1 - \frac{1}{2} \frac{2\pi R}{x} \sigma_3 + O \left( \frac{1}{x^2} \right), \quad \text{as } x \to \infty, \quad (5.2.8a) \\
g\Omega(x)g^{-1} & = 1 + x \frac{2\pi L}{\sqrt{\lambda}} \sigma_3 + O(x^2), \quad \text{as } x \to 0. \quad (5.2.8b)
\end{align*}
\]

### 5.3 Involution of conserved charges

In section \([5.1]\) we saw that given a solution \( j \) to the equations of motion \([5.1.1]\) one could construct a 1-parameter family of flat 1-forms on the worldsheet, which in turn lead to the existence of an infinite number of integrals. However, for the system in question to be integrable requires also that these integrals of motion be in
pairwise involution. To study this question we must now turn to the Hamiltonian framework, introduced in chapter 4.

After having studied solutions of the equations of motion it is straightforward to pass to a Hamiltonian analysis once we realise that the space of solutions of the equations of motion (5.1.1) is in one to one correspondence with phase-space. Indeed, any given solution \( j^{\text{sol}}(\sigma, \tau) \) of (5.1.1) determines a point in phase-space by restriction to a chosen time slice, say \( \tau = 0 \), that is \( j(\sigma) = (j^{\text{sol}}_0(\sigma, 0), j^{\text{sol}}_1(\sigma, 0)) \). Conversely, any point \( j(\sigma) \) in phase-space determines a unique solution \( j^{\text{sol}}(\sigma, \tau) \) whose initial condition at \( \tau = 0 \) is \( j^{\text{sol}}(\sigma, 0) = j(\sigma) \). However, as we saw in chapter 3 the current \( j \) must also satisfy the Virasoro constraints (3.4.3)

\[
\frac{1}{2} \text{tr} j^2 = -\kappa^2.
\]

If these constraints are satisfied by the initial conditions \( j^{\text{sol}}_\pm(\sigma, 0) \) then using the equations of motion we have \( \partial_\tau \frac{1}{2} \text{tr}(j^{\text{sol}})^2 = \pm \partial_\tau \frac{1}{2} \text{tr}(j^{\text{sol}})^2 \) so that the same constraints are also satisfied by full solution \( j^{\text{sol}}(\sigma, \tau) \). Therefore the space of solutions satisfying (3.4.3) is in one to one correspondence with the reduced phase-space \( \mathcal{P}^\infty \) introduced in chapter 4.

Rephrased in the Hamiltonian formalism, the content of section 5.1 is as follows. One can define a 1-parameter family of \( \mathfrak{su}(2) \)-valued functions on phase-space

\[
j(\sigma) \mapsto J_1(\sigma, x) = \frac{1}{1 - x^2} (j_1(\sigma) + x j_0(\sigma)), \quad x \in \mathbb{C},
\]

with the property, following from lemma 5.1.6 that its path ordered exponential (5.1.9)

\[
j(\sigma) \mapsto \Omega(x, \sigma) = \text{Pexp} \left[ \int_{\sigma}^{\sigma + 2\pi} d\sigma' J_1(x, \sigma') \right],
\]

has a simple isospectral evolution under Hamilton’s equations (4.4.13). And in
particular, equation (5.1.13) shows that the 1-parameter family of functions $j(\sigma) \mapsto \text{tr} \Omega(x)$ is invariant under the $\tau$- and $\sigma$-flows which in the Hamiltonian formalism are generated by $\mathcal{E}$ and $\mathcal{P}$ respectively. In other words, the upshot of section 5.1 rephrased in Hamiltonian terms should read

$$\{\mathcal{E}, \text{tr} \Omega(x)^n\}_{\text{D.B.}} = \{\mathcal{P}, \text{tr} \Omega(x)^n\}_{\text{D.B.}} = 0,$$

(5.3.3)

We will rederive this result within the Hamiltonian formalism by in fact proving a much stronger result.

What we are seeking to show using the Hamiltonian formalism is that the conserved charges obtained in the previous section are in pairwise involution. But this statement is equivalent to showing that

$$\{\text{tr} \Omega(x)^n, \text{tr} \Omega(x')^m\} = 0, \quad \forall n, m \in \mathbb{N}.$$

(5.3.4)

However, since we are working on the reduced phase-space all statements must be made with respect to the Dirac bracket instead of the Poisson bracket. So the ultimate goal of this section is to show that (5.3.4) also holds for Dirac brackets,

**Theorem 5.3.1.** The traces of powers of the monodromy matrix generate quantities in involution with respect to the Dirac bracket (4.4.9), i.e.

$$\{\text{tr} \Omega(x)^n, \text{tr} \Omega(x')^m\}_{\text{D.B.}} = 0, \quad \forall n, m \in \mathbb{N}.$$

(5.3.5)

This is the full statement of Liouville integrability of string theory on $\mathbb{R} \times S^3$ in conformal static gauge. From section 5.2 we know that $\mathcal{E}$ and $\mathcal{P}$ can be obtained from $\text{tr} \Omega(x)^n$ in the limit $x \to \pm 1$ and thus (5.3.3) is a trivial consequence of (5.3.5).

In the following we shall adopt tensor notation for all brackets. We define the
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Poisson bracket between two $2 \times 2$ matrices $A$ and $B$ as

$$\{A \otimes B\} = \int d\sigma \left( \frac{\delta A}{\delta \pi^a(\sigma)} \otimes \frac{\delta B}{\delta q^a(\sigma)} - \frac{\delta A}{\delta q^a(\sigma)} \otimes \frac{\delta B}{\delta \pi^a(\sigma)} \right), \quad (5.3.6)$$

where the operation $\otimes$ on the right hand side denotes the usual tensor product.

This notation conveniently encodes all the Poisson brackets between the various components of $A$ and $B$. For example, if $A, B \in \mathfrak{su}(2)$ in components are $A = A^a t_a$ and $B = B^a t_a$ then by definition (5.3.6) we have $\{A \otimes B\} = \{A^a, B^b\} t_a \otimes t_b$.

$\{J_1, J_1\}$ algebra

The monodromy matrix (5.3.2) being the path ordered exponential of the space component (5.3.1) of the Lax connection, we will need the Poisson bracket $\{J_1, J_1\}$ in order to construct the Poisson bracket of monodromy matrices.

The set of Poisson brackets $\{J_1^a(\sigma, x), J_1^b(\sigma', x')\}$ can be easily obtained from the fundamental brackets of currents $\{j_1^a(\sigma), j_1^b(\sigma')\}$ in (4.3.4). Introducing $\eta := \kappa_{ab} t_a \otimes t_b = t_a \otimes t^a$, called the Casimir tensor, they can be written as

$$\frac{\sqrt{\lambda}}{4\pi} \{J_1(\sigma, x) \otimes J_1(\sigma', x')\} = \left[ -\frac{\eta}{x - x'}, \frac{x^2}{1 - x^2} J_1(\sigma, x) \otimes 1 + \frac{x^2}{1 - x^2} 1 \otimes J_1(\sigma, x') \right] \delta(\sigma - \sigma')$$

$$+ \frac{x + x'}{(1 - x^2)(1 - x'^2)} \eta \delta(\sigma - \sigma'). \quad (5.3.7)$$

This bracket has the form of the fundamental Poisson bracket $\{J_1, J_1\}$ for a non-
ultralocal integrable system formulated by Maillet [69,70]

\[ \{ J_1(\sigma, x) \otimes J_1(\sigma', x') \} = [r(\sigma, x, x'), J_1(\sigma, x) \otimes 1 + 1 \otimes J_1(\sigma', x')] \delta(\sigma - \sigma') - [s(\sigma, x, x'), J_1(\sigma, x) \otimes 1 - 1 \otimes J_1(\sigma', x')] \delta(\sigma - \sigma') \]

\[ -(r(\sigma, x, x') + s(\sigma, x, x') - r(\sigma', x, x') + s(\sigma', x, x')) \delta'(\sigma - \sigma'), \] (5.3.8)

These brackets involve a pair of matrices \( r \) and \( s \). Notice that the \( r \) matrix can be removed from the \( \delta' \)-term using the identity \((r(\sigma') - r(\sigma)) \delta'(\sigma - \sigma') = r'(\sigma) \delta(\sigma - \sigma')\) valid for any function \( r \) (as can be seen by integrating the left hand side against a test function \( \psi(\sigma) \)). Thus the non-ultralocality of the bracket is accounted for by the matrix \( s \) alone. Indeed, the bracket (5.3.8) is a non-trivial generalisation of the standard ultralocal bracket which corresponds to setting \( s = \partial_x r = 0 \). In the present case the matrices \( r \) and \( s \) are constant (independent of \( \sigma \) and \( \tau \))

\[ s(x, x') = -\frac{2\pi}{\sqrt{\lambda} (1 - x^2)(1 - x'^2)} \eta, \] (5.3.9a)

\[ r(x, x') = -\frac{2\pi}{\sqrt{\lambda} (x - x')(1 - x^2)(1 - x'^2)} \eta. \] (5.3.9b)

The description of the principal chiral model in terms of Maillet’s \((r, s)\)-matrix formalism and the corresponding formulae \([5.3.9]\) for the \((r, s)\)-matrices were first obtained in [71].

\( \{ T, T \} \) and \( \{ T, J_1 \} \) algebras

The next step towards the algebra of monodromy matrices is the algebra of transition matrices. A transition matrix is defined relative to an interval \([\sigma_1, \sigma_2]\) as

\[ T(\sigma_1, \sigma_2, x) = P_{\text{exp}} \int_{\sigma_2}^{\sigma_1} d\sigma J_1(\sigma, x). \] (5.3.10)
The monodromy matrix is then simply a special transition matrix whose interval wraps the circle fully once, that is \( \Omega(x, \sigma) = T(\sigma + 2\pi, \sigma, x) \).

Now the transition matrix \((5.3.10)\) is the unique solution of either of the two following differential equations with boundary condition \( T(\sigma_2, \sigma_2, x) = 1 \),

\[
\frac{\partial T}{\partial \sigma_1}(\sigma_1, \sigma_2, x) = J_1(\sigma_1, x)T(\sigma_1, \sigma_2, x), \quad \frac{\partial T}{\partial \sigma_2}(\sigma_1, \sigma_2, x) = -T(\sigma_1, \sigma_2, x)J_1(\sigma_2, x). \tag{5.3.11}
\]

Considering the first of these, its variation is

\[
\frac{\partial \delta T}{\partial \sigma_1}(\sigma_1, \sigma_2, x) = \delta J_1(\sigma_1, x)T(\sigma_1, \sigma_2, x) + J_1(\sigma_1, x)\delta T(\sigma_1, \sigma_2, x)
\]

with initial condition \( \delta T(\sigma_1, \sigma_1, x) = 0 \), to which the unique solution is easily seen to be \([62]\)

\[
\delta T(\sigma_1, \sigma_2, x) = \int_{\sigma_2}^{\sigma_1} d\sigma T(\sigma_1, \sigma, x)\delta J_1(\sigma, x)T(\sigma, \sigma_2, x). \tag{5.3.12}
\]

But now using the definition of the Poisson bracket \((5.3.6)\) along with equation \((5.3.12)\) it is easy to relate the bracket of transition matrices \(\{T, T\}\) or the bracket \(\{T, J_1\}\) to the bracket of currents \(\{J_1, J_1\}\). Specifically we find

\[
\{T(\sigma_1, \sigma_2, x) \otimes T(\sigma_1', \sigma_2', x')\} = \int_{\sigma_2}^{\sigma_1} d\sigma \int_{\sigma_2'}^{\sigma_1'} d\sigma' \left( T(\sigma_1, \sigma, x) \otimes T(\sigma_1', \sigma', x') \right) \\
\times \left\{ J_1(\sigma, x) \otimes J_1(\sigma', x') \right\} \left( T(\sigma, \sigma_2, x) \otimes T(\sigma', \sigma_2', x') \right), \tag{5.3.13a}
\]

\[
\{T(\sigma_1, \sigma_2, x) \otimes J_1(\sigma_3, x')\} \\
= \int_{\sigma_2}^{\sigma_1} d\sigma \left( T(\sigma_1, \sigma, x) \otimes 1 \right) \left\{ J_1(\sigma, x) \otimes J_1(\sigma_3, x') \right\} \left( T(\sigma, \sigma_2, x) \otimes 1 \right). \tag{5.3.13b}
\]
Plugging the bracket (5.3.8) into these equations one finds after a bit of algebra [70]

\[
\{T(\sigma_1, \sigma_2, x) \otimes T(\sigma'_1, \sigma'_2, x')\} \\
= + \epsilon(\sigma'_1 - \sigma'_2)\chi(\sigma; \sigma'_1, \sigma'_2) \\
\times T(\sigma, \sigma, x) \otimes T(\sigma', \sigma, x') (r(\sigma, x, x') - s(\sigma, x, x')) T(\sigma, \sigma_2, x) \otimes T(\sigma, \sigma'_2, x')|_{\sigma = \sigma_2}^{\sigma = \sigma_1} \\
+ \epsilon(\sigma_1 - \sigma_2)\chi(\sigma; \sigma_1, \sigma_2) \\
\times T(\sigma, \sigma, x) \otimes T(\sigma', \sigma, x') (r(\sigma, x, x') + s(\sigma, x, x')) T(\sigma, \sigma_2, x) \otimes T(\sigma, \sigma'_2, x')|_{\sigma = \sigma'_1}^{\sigma = \sigma'_2},
\]

(5.3.14a)

where \(\epsilon(\sigma) = \text{sign}(\sigma)\) is the usual sign function and \(\chi(\sigma; \sigma_1, \sigma_2)\) is the characteristic function of the interval \((\sigma_1, \sigma_2)\), and

\[
\{T(\sigma_1, \sigma_2, x) \otimes J_1(\sigma_3, x')\} \\
= - 2(\delta(\sigma_3 - \sigma_1) - \delta(\sigma_3 - \sigma_2))(T(\sigma_1, \sigma_3, x) \otimes 1)s(\sigma, x, x')(T(\sigma_3, \sigma_2, x) \otimes 1) \\
+ \epsilon(\sigma_1 - \sigma_2)\chi(\sigma_3; \sigma_1, \sigma_2)(T(\sigma_1, \sigma_3, x) \otimes 1) \\
\times [(r + s)(\sigma, x, x'), J_1(\sigma_3, x) \otimes 1 + 1 \otimes J_1(\sigma_3, x')](T(\sigma_3, \sigma_2, x) \otimes 1).
\]

(5.3.14b)

**Maillet regularisation**

It follows from the algebra (5.3.14a) that the function,

\[
\Delta^{(1)}(\sigma_1, \sigma_2, \sigma'_1, \sigma'_2; x, x') = \{T(\sigma_1, \sigma_2, x) \otimes T(\sigma'_1, \sigma'_2, x')\}
\]

is well defined and continuous where \(\sigma_1, \sigma_2, \sigma'_1, \sigma'_2\) are all distinct, but it has discontinuities proportional to \(2s\) precisely across the hyperplanes corresponding to some of the \(\sigma_1, \sigma_2, \sigma'_1, \sigma'_2\) being equal. Defining the Poisson bracket \(\{T \otimes T\}\) for coinciding intervals \((\sigma_1 = \sigma'_1, \sigma_2 = \sigma'_2)\) or adjacent intervals \((\sigma'_1 = \sigma_2\) or \(\sigma_1 = \sigma'_2)\) requires defining
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the value of the discontinuous matrix-valued function $\Delta^{(1)}$ at its discontinuities.

Remark The discontinuities encountered here are all proportional to the matrix $s$ and hence are absent in the ultralocal case ($s = \partial_r r = 0$), as it should be. Heuristically, the reason for this difference can be understood from equation 5.3.13a which expresses the $\{ T, T \}$ bracket as a double integral of the $\{ J_1, J_1 \}$ bracket. In the ultralocal case where the bracket $\{ J_1, J_1 \}$ contains only $\delta$-singularities, the bracket $\{ T, T \}$ is thus a continuous function. However, in the non-ultralocal case where the bracket $\{ J_1, J_1 \}$ contains also $\delta'$-singularities, its double integral $\{ T, T \}$ will still be a distribution, and indeed it contains characteristic functions $\chi$ which are discontinuous.

It is shown in [69] that requiring antisymmetry of the Poisson bracket and the derivation rule to hold imposes the symmetric definition of $\Delta^{(1)}$ at its discontinuous points; for example at $\sigma_1 = \sigma'_1$ we must define

$$
\Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1, \sigma'_2; x, x') = \lim_{\epsilon \to 0^+} \frac{1}{2} \left( \Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1 + \epsilon, \sigma'_2; x, x') + \Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1 - \epsilon, \sigma'_2; x, x') \right),
$$

and likewise for all other possible coinciding endpoints. This definition is equivalent to assigning the value of $\frac{1}{2}$ to the characteristic function $\chi$ at its discontinuities. Having thus defined $\Delta^{(1)}$ at its discontinuities we now have a definition of the Poisson bracket $\{ T \otimes T \}$ for coinciding and adjacent intervals consistent with the antisymmetry of the Poisson bracket and the derivation rule. However this definition of the $\{ T \otimes T \}$ Poisson bracket does not satisfy the Jacobi identity as is shown in [69], so that in fact no strong definition of the bracket $\{ T \otimes T \}$ with coinciding or adjacent intervals can be given without violating the Jacobi identity [69]. It is nevertheless possible [69, 72] to give a weak\(^2\) definition of this bracket for coinciding or adjacent

\(^2\)The bracket is weak in the sense that any multiple Poisson bracket of $T$’s can be given a meaning which cannot be reduced to its similarly defined constituent Poisson brackets, i.e. the multiple Poisson bracket $\{ T \otimes \{ \ldots \{ T \otimes T \} \ldots \} \}$ with $n$ factors of $T$ must be separately defined for each $n$. 

intervals in a way that is consistent with the Jacobi identity as follows: consider the multiple Poisson bracket of \((n + 1)\) transition matrices

\[
\Delta^{(n)} \left( \sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}, x^{(1)}, \ldots, x^{(n+1)} \right) \\
= \left\{ T \left( \sigma_1^{(1)}, \sigma_2^{(1)}, x^{(1)} \right) \otimes \cdots \otimes T \left( \sigma_1^{(n)}, \sigma_2^{(n)}, x^{(n)} \right) \otimes T \left( \sigma_1^{(n+1)}, \sigma_2^{(n+1)}, x^{(n+1)} \right) \right\}
\]

which is unambiguously defined and continuous where \(\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}\) are all distinct, but again is discontinuous across the hyperplanes defined by some of the points \(\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}\) being equal. The values of \(\Delta^{(n)}\) at its discontinuities are defined by employing a point splitting regularisation followed by a total symmetrisation limit [69]. For example, we define its value at \(\sigma_1^{(i)} = \sigma_1, i = 1, \ldots, n + 1\) by

\[
\Delta^{(n)} \left( \sigma_1, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, x^{(1)}, \ldots, x^{(n+1)} \right) = \lim_{\epsilon \to 0^+} \frac{1}{(n + 1)!} \sum_{p \in S_{n+1}} \Delta^{(n)} \left( \sigma_1 + p(1)\epsilon, \sigma_2^{(1)}, \ldots, \sigma_1 + p(n + 1)\epsilon, \sigma_2^{(n+1)}, x^{(1)}, \ldots, x^{(n+1)} \right),
\]

and similarly one defines the value of \(\Delta^{(n)}\) at all other discontinuities. With the function \(\Delta^{(n)}\) being defined at its discontinuities we now have the definition of a weak bracket which reduces to the normal Poisson bracket on quantities for which the latter is continuous. It is shown in [69] that the Jacobi identity for transition matrices with coinciding or adjacent interval is now satisfied in terms of this weak bracket (\(\Delta^{(2)}\) being the relevant quantity in this case).

\{\Omega, \Omega\} algebra

Using this regularisation procedure we now derive an expression for the Poisson bracket between two monodromy matrices in the periodic case under consideration,
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The Poisson bracket of two generic transition matrices $T(\gamma, x)$ and $T(\gamma', x')$ on $S^1$, defined relative to two different paths $\gamma$ and $\gamma'$ on $S^1$, is given by

$$\{T(\gamma, x) \otimes T(\gamma', x')\} = \sum_{\tilde{\gamma}'} \{T(\tilde{\gamma}, x) \otimes T(\tilde{\gamma}', x')\},$$

where the sum is over lifts $\tilde{\gamma}'$ of $\gamma'$ to $\mathbb{R}$. An example of these lifted paths is shown in Figure 5.1. Let us now apply this formula to compute the Poisson bracket between two transition matrices $\Omega(x, \sigma)$ and $\Omega(x', \sigma)$ on $S^1$. The common interval $\gamma$ of both matrices stretches once around the full circle and so it follows that if we take $\tilde{\gamma} = [\sigma, \sigma + 2\pi]$ to be the lift of the interval of $\Omega(x, \sigma)$ then there are only three possibilities for the lift $\tilde{\gamma}'$ of the interval of $\Omega(x', \sigma)$ which give a non-zero

Figure 5.1: Example of a path lifting required in computing Poisson brackets of transition matrices on $S^1$ of the form $\{T(\gamma, x) \otimes T(\gamma', x')\}$.
contribution to the right hand side of (5.3.16), namely

\[ [\sigma - 2\pi, \sigma], \quad [\sigma, \sigma + 2\pi], \quad [\sigma + 2\pi, \sigma + 4\pi]. \quad (5.3.17) \]

Since the corresponding three brackets \( \{ T(\tilde{\gamma}, x) \otimes T(\tilde{\gamma}', x') \} \) on \( \mathbb{R} \) are over coinciding or adjacent intervals they need to be regularised by the procedure described above. Let us start by considering the coinciding interval bracket \( \{ T(\sigma + 2\pi, \sigma, x) \otimes T(\sigma + 2\pi, \sigma, x') \} \). There are 4 different possible point splittings of the endpoints, each giving the same contribution (using (5.3.14a))

\[ r(x, x') (\Omega(x, \sigma) \otimes \Omega(x', \sigma)) - (\Omega(x, \sigma) \otimes \Omega(x', \sigma)) r(x, x') \]

in the limit of coinciding points. On the other hand, the adjacent interval brackets (corresponding to the first and last choices for \( \tilde{\gamma}' \) in (5.3.17)) each have two possible point splittings and together they contribute, in the coinciding end-point limit,

\[ (\Omega(x, \sigma) \otimes 1) s(x, x') (1 \otimes \Omega(x', \sigma)) - (1 \otimes \Omega(x', \sigma)) s(x, x') (\Omega(x, \sigma) \otimes 1) \]

to the Poisson bracket of two monodromy matrices. The sum of the last two expressions gives the right hand side of (5.3.16) which yields the sought-after (weak) Poisson bracket between two monodromy matrices on \( S^1 \)

\[ \{ \Omega(x, \sigma) \otimes \Omega(x', \sigma) \} = [r(x, x'), \Omega(x, \sigma) \otimes \Omega(x', \sigma)] \]
\[ + (\Omega(x, \sigma) \otimes 1) s(x, x') (1 \otimes \Omega(x', \sigma)) \]
\[ - (1 \otimes \Omega(x', \sigma)) s(x, x') (\Omega(x, \sigma) \otimes 1). \quad (5.3.18) \]

As a specific check of (5.3.18) we show that the \( SU(2)_R \) symmetry is canonically realised on \( \Omega(x) \) via the weak Poisson bracket [72]. Recall from equation (5.2.7a) that the global Noether charge \( Q_R \) can be read off from the asymptotic
expansion of the monodromy matrix at \( x = \infty \). Then starting with equation \((5.3.18)\) multiplied by \( x(\epsilon \otimes 1) \) and taking the trace over the first tensor product space followed by the limit \( x \to \infty \) one deduces, using also the asymptotics \( r(x, x') \sim_{x \to \infty} \frac{2\pi}{\sqrt{\lambda}} \frac{1-2x'^2}{x(1-x'^2)} \eta \) and \( s(x, x') \sim_{x \to \infty} \frac{2\pi}{\sqrt{\lambda}} \frac{1}{x(1-x'^2)} \eta \), that

\[
\{ \epsilon \cdot Q_R, \Omega(x') \} = [\epsilon, \Omega(x')].
\]

In other words, the right Noether charge \( Q_R \) generates the correct transformation on \( \Omega(x) \), which we expect to be

\[
\Omega(x) \to U_R^{-1} \Omega(x) U_R,
\]

provided we use the weak bracket \((5.3.18)\).

**\{tr \Omega, tr \Omega\} algebra**

Now consider the bracket \( \{ \Omega(x, \sigma)^n \otimes \Omega(x', \sigma)^m \} \) for any \( n, m \in \mathbb{N} \), which can easily be reduced to \((5.3.18)\) as follows (omitting the \( \sigma \)-dependence)

\[
\{ \Omega(x)^n \otimes \Omega(x')^m \} = nm \left( \Omega(x)^{n-1} \otimes 1 \right) \{ \Omega(x) \otimes \Omega(x') \} \left( 1 \otimes \Omega(x)^{m-1} \right).
\]

Then using the standard notational shorthands \( \frac{1}{A} = A \otimes 1 \) and \( \frac{2}{A} = 1 \otimes A \), and taking the trace over both factors of the tensor product we find

\[
\{ tr \Omega(x)^n, tr \Omega(x')^m \} = nm tr_{12} \left( \frac{1}{\Omega(x)^{n-1}} \frac{2}{\Omega(x')}^{m-1} \left\{ \frac{1}{\Omega(x)}, \frac{2}{\Omega(x')} \right\} \right)
\]

\[
= nm tr_{12} \left[ r(x, x') + s(x, x'), \frac{1}{\Omega(x)^n} \frac{2}{\Omega(x')^m} \right],
\]
where in the second line we have used (5.3.18). In conclusion we have arrived at the desired Poisson bracket

$$\{\mathrm{tr} \Omega(x)^n, \mathrm{tr} \Omega(x')^m\} = 0.$$  (5.3.19)

Because this bracket is zero it can be understood as defining a bracket in the strong sense and without recourse to any regularisation. We deduce from this last relation that the invariants of the system encoded in the quantity $\mathrm{tr} \Omega(x)^n$ are in involution with respect to the Poisson bracket.

$$\{\mathrm{tr} \Omega, \mathrm{tr} \Omega\}_{\text{D.B.}}$$

As explained in chapter 3 we always choose to work in conformal static gauge in order to isolate the physical degrees of freedom of the string. This is done by imposing static gauge conditions to fix the gauge invariance generated by the Virasoro constraints. But within the Hamiltonian description of chapter 4 these constraints together form a set of second class constraints and so to consistently impose them one must replace Poisson brackets by the Dirac bracket (4.4.9). However, as formula (4.4.9) shows, this distinction between Poisson and Dirac is unnecessary when one of the arguments is invariant under conformal transformations generated by $L_n, \tilde{L}_n, n \neq 0$. We now show that the generator of conserved charges $\mathrm{tr} \Omega(x)^n$ is conformally invariant so that

$$\{\mathrm{tr} \Omega(x)^n, F\}_{\text{D.B.}} = \{\mathrm{tr} \Omega(x)^n, F\}$$

for an arbitrary function $F$ of the principal chiral model fields $j$. As a special case we deduce that the involution property (5.3.19) also holds with respect to the Dirac bracket.

To show the conformal invariance of $\mathrm{tr} \Omega(x)^n$ let us start with the Poisson
5.3. INVOLUTION OF CONSERVED CHARGES

Once again, Poisson brackets on $S^1$ are computed by working on the universal cover $\mathbb{R}$. So let $\sigma_1 = \sigma + 2\pi$, $\sigma_2 = \sigma$ and $\sigma_3 = \sigma'$ in (5.3.14b) to obtain the Poisson bracket $\{\Omega(\sigma, x) \otimes J_1(\sigma', x')\}$. This easily leads to the Poisson brackets $\{\Omega(\sigma, x) \otimes j_\pm(\sigma')\}$ after noting from the definition of $J_1(x)$ that $J_1(0) = \frac{1}{2}(j_+ - j_-)$ and $\lim_{x \to \infty}(-x)J_1(x) = \frac{1}{2}(j_+ + j_-)$, in particular

\[
\{\Omega(\sigma, x) \otimes j_\pm(\sigma')\} = (T(\sigma + 2\pi, \sigma', x) \otimes 1) \times 
\left( (\sigma' - \sigma - 2\pi) - \delta(\sigma' - \sigma) \right) \frac{4\pi}{\sqrt{\lambda}} \frac{1 \pm x}{1 - x^2} \eta + \chi(\sigma'; \sigma + 2\pi, \sigma) \times 
\left[ -\frac{2\pi}{\sqrt{\lambda}} \frac{2x}{1 - x^2} \eta, (x \pm 1)J_1(\sigma', x) \otimes 1 \pm 1 \otimes \frac{1}{2}(j_+(\sigma') - j_-(\sigma')) \right] 
\times (T(\sigma', \sigma, x) \otimes 1).
\]

Using the identity $\text{tr}_2(\eta 1 \otimes A) = A$ for any matrix $A \in \mathfrak{su}(2)$ one can show that after multiplying the above equation by $1 \otimes j_\pm(\sigma')$ and taking the trace $\text{tr}_2$ over the second tensor factor the commutator disappears and we are left with

\[
\left\{ \Omega(\sigma, x), \frac{1}{2} \text{tr}_2 j_\pm^2(\sigma') \right\} = \frac{4\pi}{\sqrt{\lambda}} (\delta(\sigma' - \sigma - 2\pi) - \delta(\sigma' - \sigma)) T(\sigma + 2\pi, \sigma', x) J_\pm(\sigma', x) T(\sigma', \sigma, x),
\]

where $J_\pm(\sigma', x) = j_\pm(\sigma')/(1 \mp x)$. Next we multiply both sides by $e^{\mp in\sigma'}$ and integrate over $\sigma'$ from 0 to $2\pi$. However, since we are on the universal cover $\mathbb{R}$ of $S^1$ we get two non-zero contributions, namely from the integrations over the two lifts $[0, 2\pi]$ and $[2\pi, 4\pi]$ (assuming $\sigma \in (0, 2\pi)$). From the definition (4.4.2) of the Virasoro generators we can write the result as follows

\[
\begin{align*}
\{\Omega(\sigma, x), L_n\} &= \frac{1}{2} e^{in\sigma} [J_+(\sigma, x), \Omega(\sigma, x)], \\
\{\Omega(\sigma, x), \tilde{L}_n\} &= \frac{1}{2} e^{-in\sigma} [J_-(\sigma, x), \Omega(\sigma, x)].
\end{align*}
\]

(5.3.20)
Note that in the above calculation it is because of the presence of the $s$-matrix, which arises from non-ultralocality of the Poisson brackets of the model, that we end up with the correct transformation property for $\Omega(x)$ under conformal transformations. Finally, since the right hand sides are commutators, taking the trace shows that $\text{tr} \Omega(x)^m$ is invariant under conformal transformations generated by $L_n, \tilde{L}_n$, namely

$$\{\text{tr} \Omega(x)^m, L_n\} = \{\text{tr} \Omega(x)^m, \tilde{L}_n\} = 0.$$ 

As we have already argued, this immediately implies the involution of the conserved charges with respect to the Dirac bracket

$$\{\text{tr} \Omega(x)^n, \text{tr} \Omega(x')^m\}_{\text{D.B.}} = 0.$$ 

This completes the proof of theorem 5.3.1.

\{\Omega, \Omega\}_{\text{D.B.}} \text{ algebra}

In fact we can prove a much stronger result that will be useful later in chapter 8. Combining the relations (5.3.20) for the conformal transformation of the monodromy matrix with the definition (4.4.9) of the Dirac bracket we can compute the Dirac algebra $\{\Omega(x), \Omega(x')\}_{\text{D.B.}}$. One finds, using the fact that the partial sums of $\sum_{n \neq 0} \frac{1}{n}$ vanish, that it is identical to the Poisson algebra, namely

**Proposition 5.3.2.** The Dirac bracket between two monodromy matrices on $S^1$ is given on the reduced phase-space $\mathcal{P}^\infty$ by

$$\{\Omega(x) \otimes \Omega(x')\}_{\text{D.B.}} \approx [r(x,x'), \Omega(x) \otimes \Omega(x')]$$

$$+ (\Omega(x) \otimes 1) s(x,x') (1 \otimes \Omega(x'))$$

$$- (1 \otimes \Omega(x')) s(x,x') (\Omega(x) \otimes 1).$$

(5.3.21)
5.4 The string hierarchy

An immediate consequence of theorem 5.3.1 is that the charges \( \text{tr} \Omega(x)^n \) are not only conserved under the \( \tau \)- and \( \sigma \)-flows generated by \( \mathcal{E} \) and \( \mathcal{P} \) but also under the flows generated by the infinite number of charges \( \text{tr} \Omega(x')^m \) themselves. In particular this is true for the flows generated by all the local charges \( Q_{n}^{(\pm)} \). It follows that if we treat any two local charges \( Q_{m}^{(\pm)} \) and \( Q_{n}^{(\pm)} \) as Hamiltonians instead of \( \mathcal{E} \) and \( \mathcal{P} \) then the corresponding equations of motion will be integrable since they also admit the infinite number of conserved charges \( \text{tr} \Omega(x)^n \). In the light of section 5.1 we therefore expect these equations of motion to admit a Lax representation in terms of some Lax connection with components \( J_{m,\pm}(x) \) and \( J_{n,\pm}(x) \). In the following section we show that this is indeed the case and derive the corresponding expressions for the Lax matrix \( J_{n,\pm}(x) \) associated with the local charge \( Q_{n}^{(\pm)} \).

Our starting point is the \( \{ T, J_1 \} \) Poisson bracket (5.3.14b). Let \( \sigma_1 = \sigma + 2\pi, \sigma_2 = \sigma, \sigma_3 = \sigma' \) in (5.3.14b) and identify the monodromy matrix as \( \Omega(\sigma, x) = T(\sigma + 2\pi, \sigma, x) \) then

\[
\{ \Omega(\sigma, x) \otimes J_1(\sigma', x') \} \\
= (T(\sigma + 2\pi, \sigma', x) \otimes 1)[(r + s)(x, x'), J_1(\sigma', x) \otimes 1 + 1 \otimes J_1(\sigma', x')](T(\sigma', \sigma, x) \otimes 1) \\
- 2(\delta(\sigma' - \sigma - 2\pi) - \delta(\sigma' - \sigma))(T(\sigma + 2\pi, \sigma', x) \otimes 1)s(x, x')(T(\sigma', \sigma, x) \otimes 1).
\]

Taking the trace over the first factor of the tensor product we observe that the left hand side of this equation becomes independent of \( \sigma \). Likewise, the first term on the right hand side also becomes independent of \( \sigma \) using the translation invariance of the transition matrix \( T \) by \( 2\pi \) since we are working on the circle \( S^1 \), i.e. \( T(\sigma + 2\pi, \sigma', x) = T(\sigma, \sigma' - 2\pi, x) \). This shows that the last term must also be independent of \( \sigma \) after taking the trace over the first tensor factor and hence one can substitute its value.
at \( \sigma \neq \sigma' \) which is zero. We therefore end up with, after using (5.3.11)

\[
\{ \text{tr} \Omega(x), J_1(\sigma', x') \} = \partial_{\sigma'} J(\sigma', x, x') + [J(\sigma', x, x'), J_1(\sigma', x')],
\]

(5.4.1)

where

\[
J(\sigma', x, x') = \text{tr}_1 [\{ \Omega(\sigma', x) \otimes 1 \}(r + s)(x, x')].
\]

(5.4.2)

If we interpret the Poisson bracket \( \{ \text{tr} \Omega(x), J_1(\sigma', x') \} \) in (5.4.1) as the “time” derivative of \( J_1(\sigma', x') \) with respect to the “time” generated by the Hamiltonian \( \text{tr} \Omega(x) \) then (5.4.1) takes exactly the form of a zero-curvature equation. One can also obtain the equations of motion for the monodromy matrix with respect to the Hamiltonian \( \text{tr} \Omega(x) \). Starting from the Poisson algebra of the monodromies (5.3.18) and taking the trace over the first factor of the tensor product as above yields

\[
\{ \text{tr} \Omega(x), \Omega(\sigma', x') \} = [J(\sigma', x, x'), \Omega(\sigma', x')].
\]

(5.4.3)

Once again, if we interpret the Poisson bracket \( \{ \text{tr} \Omega(x), \Omega(x') \} \) as a “time” derivative, this last equation take the same form as the \((\sigma, \tau)\)-evolution equation of the monodromy matrix (5.1.12). So equations (5.4.1) and (5.4.3) both suggest that (5.4.2) is the Lax matrix corresponding to all the higher order flows generated by the Hamiltonians \( \text{tr} \Omega(x) \), just as \( J_0 \) and \( J_1 \) were the Lax matrices generating \( \tau \)- and \( \sigma \)-flows respectively. However, what we are really interested in are the Lax matrices corresponding to the local charges of section 5.2. And according to theorem 5.2.1 these are related to the coefficients of the Taylor expansion of (5.4.2) around \( x = \pm 1 \).
5.4. THE STRING HIERARCHY

Lax matrices

Using expressions (5.3.9) for the \((r, s)\)-matrices their sum which enters in (5.4.2) is given by

\[
(r + s)(x, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{2x^2}{(x-x')(1-x^2)} \eta.
\]

Now by definition, \(\eta = t_a \otimes t^a\) where the \(\text{su}(2)\) generator \(t^a\) is related to the Pauli matrices as \(t^a = \frac{i}{\sqrt{2}} \sigma_a\). Therefore the Lax matrix (5.4.2) can be written more explicitly as

\[
J(\sigma', x, x') = -\frac{\pi}{\sqrt{\lambda}} \frac{2x^2}{(x-x')(1-x^2)} \text{tr} [\Omega(\sigma', x) \sigma_a] \sigma_a.
\] (5.4.4)

Now it is straightforward to show that for any matrix \(A \in \text{SL}(2, \mathbb{C})\) the following is true

\[
V^{-1} \text{tr}[A \sigma_a] \sigma_a V = \sigma_3, \quad \text{where} \quad V^{-1} AV = \text{diag} (\lambda_+, \lambda_-),
\] (5.4.5)

i.e. \(V\) is the matrix of eigenvectors of \(A\) and \(\lambda_\pm\) are the eigenvalues. Since \(\Omega(\sigma', x)\) has unit determinant let us denote its eigenvalues by \(e^{\pm ip(x)}\). Let us also denote the corresponding matrix of eigenvectors as \(\Psi(\sigma', x)\) or simply \(\Psi(x)\), omitting the \(\sigma'\) dependence for clarity. In particular we know from equation (5.2.3) that in a neighbourhood of \(x = \pm 1\) we have

\[
\Psi(x) = g(\pm)(x)^{-1}, \quad p(x) = \sum_{n=-\infty}^\infty Q_n^{(\pm)}(x \mp 1)^n.
\] (5.4.6)

The identity (5.4.5) applied to the Lax matrix (5.4.4) corresponding to \(\text{tr} \Omega(x)\) yields

\[
\text{tr} \Omega(x) \iff J(x, x') = \frac{4\pi i \sin p(x)}{\sqrt{\lambda}(1-1/x^2)} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}.
\] (5.4.7)

**Remark** From now on we indicate the correspondence between a charge \(Q\) and its Lax matrix \(L(x')\) by the shorthand notation \(Q \leftrightarrow L(x')\). It is to be understood as meaning that \(Q\) and \(L(x')\) are related by an equation of the form \(\{Q, J_1(x')\} = \partial_{\sigma'} L(x') + [L(x'), J_1(x')]\). For instance (5.4.7) is to be read as (5.4.4).
But since \( \text{tr} \Omega(x) = 2 \cos p(x) \), it follows that the Lax matrix responsible for the flow of the Hamiltonian \( p(x) \) is

\[
p(x) \longleftrightarrow J(x, x') = -\frac{2\pi i}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'}.
\]  

(5.4.8)

Now by expanding this around \( x = \pm 1 \) and using (5.4.6) we can extract the Lax matrices associated with each local charge \( Q_{n-1}^{(\pm)} \), namely for \( n \geq 0 \)

\[
Q_{n-1}^{(\pm)} \longleftrightarrow \tilde{J}_{n,\pm}(x') = \text{res}_{x=\pm 1} (x \mp 1)^n J(x, x').
\]

(5.4.9)

Using the straightforward identity for any rational matrix \( M(x) \) with singularities at \( x = \pm 1 \)

\[
\text{res}_{x=\pm 1} \frac{M(x)}{x - x'} dx = -(M(x'))_{\pm 1},
\]

(5.4.10)

where \( (M(x'))_{\pm 1} \) denotes the pole part of \( M(x') \) at \( x' = \pm 1 \), one can recast the Lax matrix (5.4.9) in the much more useful form

\[
Q_{n-1}^{(\pm)} \longleftrightarrow \tilde{J}_{n,\pm}(x') = \left( \frac{2\pi i}{\sqrt{\lambda}} \frac{x'^2}{x'^2 - 1} \frac{g^{(\pm)}(x')^{-1}\sigma_3g^{(\pm)}(x')}{(x' \mp 1)^n} \right)_{\pm 1},
\]

(5.4.11)

where we have used the asymptotics (5.4.6) of \( \Psi(x') \) near \( x' = \pm 1 \). At the zeroth level \( n = 0 \) equation (5.4.11) reads

\[
Q_{-1}^{(\pm)} \longleftrightarrow \tilde{J}_{0,\pm}(x') = \pm \frac{\pi i}{\sqrt{\lambda}} \frac{g^{(\pm)}(\pm 1)^{-1}\sigma_3g^{(\pm)}(\pm 1)}{x' \mp 1}.
\]

This Lax matrix is almost equal to \( J_{\pm}(x') \) given in (5.2.6). So let us introduce an alternative basis \( J_{n,\pm} \) of Lax matrices whose zeroth level \( n = 0 \) will correspond exactly to the components of the Lax connection \( J_{\pm} \). It follows from (5.4.11) that
we have the following correspondence between integral of motion and Lax matrix
\[
\frac{\sqrt{\lambda}}{2\pi^2} Q_{-1}^{(\pm)} Q_{n-1}^{(\pm)} \quad \longleftrightarrow \quad J_{n,\pm} \equiv \frac{\sqrt{\lambda}}{2\pi^2} \left( Q_{-1}^{(\pm)} \tilde{J}_{n,\pm} + Q_{n-1}^{(\pm)} \tilde{J}_{0,\pm} \right). \tag{5.4.12}
\]

We see from (5.2.5) and (5.2.6) that the zeroth level \( n = 0 \) of this hierarchy is precisely the Lax connection \( J_{\pm} \) associated with \( \mathcal{E} \pm \mathcal{P} \), hence \( J_{0,\pm} = J_{\pm} \) as desired. So we define,

**Definition 5.4.1.** The **string hierarchy** is generated by the Hamiltonians
\[
H_{n,\pm} \equiv \frac{\sqrt{\lambda}}{2\pi^2} Q_{-1}^{(\pm)} Q_{n-1}^{(\pm)}. \tag{5.4.13}
\]

**Higher times**

At this point we can also define a hierarchy of times \( \tilde{t}_{n,\pm} \) parameterising the flows generated by the Hamiltonians \( Q_{n-1}^{(\pm)} \) of (5.4.9), namely we define
\[
\partial_{\tilde{t}_{n,\pm}} = \left\{ Q_{n-1}^{(\pm)}, \cdot \right\}_{D.B.}.
\]

However since it is preferable to work in terms of the alternative basis of Lax matrices \( J_{n,\pm}(x') \) which reduced to the Lax connection \( J_{\pm}(x') \) at the zeroth level, we define the corresponding higher times,

**Definition 5.4.2.** The **hierarchy of times** \( t_{n,\pm} \) of the hierarchy are defined by
\[
\partial_{t_{n,\pm}} = \left\{ H_{n,\pm}, \cdot \right\}_{D.B.}.
\]

When we will need to be explicit about the dependence of a function \( f \) on all the higher times we will write simply \( f(t) \) using the notation \( \{t\} \) for the complete set of times \( \{t_{0,\pm}, t_{1,\pm}, \ldots\} \). Let us also denote the multi-indices labelling the hierarchy,
such as \((n, +)\), using capital letters, \(e.g.\ N = (n, s)\) where \(n \in \mathbb{N}\) and \(s = \pm 1\).

**Zero-curvature**

Going back to equation (5.4.3), if we follow the prescription that lead us from (5.4.7) to (5.4.9), namely of dividing through by \(-2 \sin p(x)\) and taking the residue at \(x = \pm 1\) one easily shows,

\[
[\partial_{t_N} - \tilde{J}_N(x'), \Omega(x')] = 0. \tag{5.4.14}
\]

By linearity of the definition (5.4.12) of the Lax matrices \(J_N(x')\) in terms of the \(\tilde{J}_N(x')\) and using the fact that the local charges \(Q_n^{(\pm)}\) are constant with respect to the higher times \(\tilde{t}_M\) we deduce the following,

**Proposition 5.4.3.** The evolution of the monodromy matrix under the hierarchy of times (5.4.2) is governed by

\[
[\partial_{t_N} - J_N(x'), \Omega(x')] = 0, \tag{5.4.15a}
\]

which is exactly of the form (5.1.12).

*Proof.* \(\partial_{t_n, \pm} \Omega(x') = \frac{\sqrt{\lambda} \omega}{2\pi} \{Q_{n-1}^{(\pm)} \Omega(x'), \Omega(x')\}_{\text{D.B.}} + \frac{\sqrt{\lambda} \omega}{2\pi} \{Q_{n-1}^{(\pm)} \Omega(x'), \Omega(x')\}_{\text{D.B.}}\), which using (5.4.14) equals \(\frac{\sqrt{\lambda} \omega}{2\pi} \{Q_{n-1}^{(\pm)} \tilde{J}_{n, \pm}(x') + Q_{n-1}^{(\pm)} \tilde{J}_{0, \pm}(x'), \Omega(x')\} = [J_{n, \pm}(x'), \Omega(x')]. \)

Finally we derive the evolution equations for the Lax matrices (5.4.12) under the hierarchy of times in definition (5.4.2) and show that they take the zero-curvature form. We closely follow an argument given in [67, p51-52] for finite-dimensional systems which applies readily here.
Proposition 5.4.4. The Lax matrices \([5.4.12]\) satisfy the zero-curvature condition
\[
[\partial_{tN} - J_N(x'), \partial_{tM} - J_M(x')] = 0. \tag{5.4.15b}
\]

Proof. As for proposition \(5.4.3\) we first prove that the zero-curvature equation holds for the matrices \(\tilde{J}_M(x')\) and times \(\tilde{t}_M\), namely
\[
\partial_{\tilde{t}_N} \tilde{J}_N(x') - \partial_{\tilde{t}_N} \tilde{J}_M(x') = [\tilde{J}_M(x'), \tilde{J}_N(x')]. \tag{5.4.16}
\]
Equation \(5.4.15b\) will then follow by linearity and the constancy of the \(Q^{(\pm)}_{n-1}\).

Writing the monodromy matrix as \(\Omega(x') = \Psi(x') \text{diag}(e^{ip(x)}, e^{-ip(x)})\Psi(x')^{-1}\), equation \(5.4.14\) implies
\[
\left[ \Psi(x')^{-1} \left( \partial_{\tilde{t}_N} \Psi(x') \right) - \Psi(x')^{-1} \tilde{J}_N(x')\Psi(x'), \text{diag}(e^{ip(x)}, e^{-ip(x)}) \right] = 0. \tag{5.4.17}
\]

But any \(2 \times 2\) matrix commuting with a diagonal matrix must itself be diagonal, and therefore we may write
\[
\partial_{\tilde{t}_N} \Psi(x') = \tilde{J}_N(x')\Psi(x') + \Psi(x')D(x'), \tag{5.4.18}
\]
for some unknown diagonal \(2 \times 2\) matrix \(D(x')\). Let \(N = (n, s_n)\) and \(M = (m, s_m)\), then
\[
\partial_{\tilde{t}_M} \tilde{J}_N(x') = \left[ \tilde{J}_M(x'), \frac{2\pi i}{\sqrt{\lambda}} \frac{x'^2}{x'^2 - 1} \Psi(x')\sigma_3\Psi(x')^{-1} \left( x' - s_n \right)^n \right]_{s_n}, \tag{5.4.19}
\]
where we have made use of \(5.4.18\) and the subscript on the commutator means we take the pole part of the whole commutator at \(x' = s_n\). Let us start by assuming that \(s_n \neq s_m\), then \(\tilde{J}_M(x')\) is regular at \(x' = s_n\) and only the pole part at \(x' = s_n\) of
the second term in the commutator contributes which is just \( \tilde{J}_N(x') \), so

\[
\partial_{t_M} \tilde{J}_N(x') = [\tilde{J}_M(x'), \tilde{J}_N(x')]_{sn},
\]

and likewise we also have \( \partial_{t_N} \tilde{J}_M(x') = [\tilde{J}_N(x'), \tilde{J}_M(x')]_{sm} \). Since \([ \tilde{J}_M(x'), \tilde{J}_N(x') ]\) is rational with poles only at \( x' = \pm 1 \) and vanishes at \( x' = \infty \) it can be written as a sum over its pole parts, namely

\[
[\tilde{J}_M(x'), \tilde{J}_N(x')] = [\tilde{J}_M(x'), \tilde{J}_N(x')]_{+1} + [\tilde{J}_M(x'), \tilde{J}_N(x')]_{-1}.
\]

But because \( s_n \neq s_m \) we have \( \{s_m, s_n\} = \{\pm 1\} \) and the zero-curvature condition (5.4.16) follows. If instead we assume that \( s_n = s_m \), then we have

\[
\left[ \frac{2\pi i}{\sqrt{\lambda}} \frac{x'^2}{x'^2 - 1} \frac{\Psi(x')\sigma_3 \Psi(x')^{-1}}{(x' - s_n)^n}, \tilde{J}_M(x') - \frac{2\pi i}{\sqrt{\lambda}} \frac{x'^2}{x'^2 - 1} \frac{\Psi(x')\sigma_3 \Psi(x')^{-1}}{(x' - s_n)^m} \right]_{sn} = 0
\]

since both arguments in the commutator are regular at \( x' = s_n = s_m \). The zero-curvature equation (5.4.16) again readily follows from the above equation and (5.4.19).

\[\square\]

**Gauge redundancy**

The form of the zero-curvature equations (5.4.15b) is invariant under gauge transformations. If \( \tilde{g}(t) \) is an arbitrary matrix depending on all the higher times \( \{t\} \) then the new Lax connections defined by the transformation

\[
J_M(x') \mapsto \tilde{g} J_M(x') \tilde{g}^{-1} + (\partial_{t_M} \tilde{g}) \tilde{g}^{-1}
\]

also satisfy the zero-curvature equations (5.4.15b). The gauge transformation parameter \( \tilde{g} \) will always be taken to be independent of the spectral parameter \( x \). This
choice obviously preserves the analytic properties of the Lax matrices. However, starting from the form (5.4.12) of the Lax matrices which are all expressed as singular parts, the transformation (5.4.20) will generically add a term constant in \( x \). Therefore the Lax matrices (5.4.12) correspond to the gauge choice

\[ J_M(\infty) = 0. \] (5.4.21)

When solving the zero-curvature equation we will use this gauge choice to extract the Lax matrices in the form (5.4.12). In particular, extracting the current \( j \) from the Lax connection \( J(x) \) will require bringing the latter to the defining form (5.1.6) and this is achieved by imposing \( J(\infty) = 0. \)

Even after imposing the gauge choice (5.4.21) there remains a residual gauge transformation by constant matrices \( \tilde{g} \). Requiring also that gauge transformations preserve the reality conditions on the Lax matrices will lead to the further restriction \( \tilde{g} \in SU(2) \). But this residual symmetry is nothing but the global \( SU(2)_R \) symmetry (3.2.2) of the original equations of motion.

\(^3\)It is also possible to choose \( \tilde{g} \) to depend on \( x \) and still preserve the analytic properties of the Lax matrices. The corresponding gauge transformations (5.4.20) give rise to Bäcklund transformations which allow one to construct new solutions from old ones. For a review of such dressing methods see [73, 74] and [67, p74-79] as well as [75, 76] for an application in the context of strings on \( \mathbb{R} \times S^5 \).
Part III

Finite-Gap Integration of String Theory on $\mathbb{R} \times S^3$
Chapter 6

Some curves

“Drama is life with the dull bits cut out.”

Sir Alfred Joseph Hitchcock

One of the key ingredients of chapter 5 that eventually lead to complete integrability was the 1-parameter family of flat currents $J(x)$ on the worldsheet which crucially depended on an auxiliary complex parameter $x \in \mathbb{C}$, called the spectral parameter. Expanding the eigenvalues of $\Omega(x)$ in this variable produced an infinite number of integrals of motion. Now instead of expanding in the spectral parameter to extract individual integrals of motion, consider the characteristic polynomial \( 5.1.11 \) of the monodromy matrix $\Omega(x)$ which neatly encodes all the integrals of motion,

$$
\Gamma(x, \Lambda) \equiv \det(\Lambda \mathbf{1} - \Omega(x)).
$$

The presence of a spectral parameter makes the characteristic polynomial depend
on two complex variables which therefore defines a curve \( \Gamma \subset \mathbb{C}^2 \) via the equation

\[
\Gamma(x, \Lambda) = 0.
\]

It follows that to every solution \( j \) of the equations of motion one can assign an
invariant curve \( \Gamma \) which encodes all its integrals of motion. The major problem with
this curve however is that it is infinitely singular and non-algebraic so the object of
this chapter is to desingularise it and obtain a Riemann surface on which we can
perform complex analysis in the subsequent chapters.

### 6.1 The spectral curve

Since the evolution of the monodromy matrix \( \Omega(x) \) with respect to all the higher
times \( \{t\} \) is isospectral by (5.4.15a), its characteristic equation defines a complex
curve in \( \mathbb{C}^2 \) independent of all the higher times,

**Definition 6.1.1.** The spectral curve \( \Gamma \) is a curve in \( \mathbb{C}^2 \) defined by

\[
\Gamma : \Gamma(x, \Lambda) \equiv \det(\Lambda 1 - \Omega(x)) = 0.
\]

It is a 2-sheeted branched cover in the sense that \( \hat{\pi} : \Gamma \to \mathbb{C}, (x, \Lambda) \mapsto x \) is surjective
and two to one (almost everywhere). Indeed, since \( \Omega(x) \) is \( 2 \times 2 \) it has at most two
distinct eigenvalues \( \Lambda_\pm(x) \) with corresponding points \( \mathfrak{P}_\pm = (x, \Lambda_\pm) \in \Gamma \) in \( \hat{\pi}^{-1}(x) \).
Note that \( \Omega(x) \) having unit determinant means

\[
\Lambda_+(x)\Lambda_-(x) = 1.
\]

But at values of \( x \) for which these eigenvalues coincide, \( \hat{\pi}^{-1}(x) \) is a single point on
\( \Gamma \) which can be either a branch point or a singular point. Note also that \( \Gamma \) admits
6.1. **THE SPECTRAL CURVE**

A natural holomorphic involution

\[ \hat{\sigma} : \Gamma \to \Gamma, \quad (x, \Lambda) \mapsto (x, \Lambda^{-1}) \]  

(6.1.3)

with the property that \( \hat{\pi} \circ \hat{\sigma} = \hat{\pi} \) and it is clear from (6.1.2) that \( \hat{\sigma} \) interchanges the points \( \mathcal{P}_\pm \in \hat{\pi}^{-1}(x) \) for any \( x \in \mathbb{C} \). Moreover, the fixed points of \( \hat{\sigma} \) precisely correspond to the branch points and singular points of \( \Gamma \).

Although the spectral curve is a very natural curve to consider it is not algebraic. To see this recall that the evolution of the monodromy matrix can be expressed in terms of the differential equation (5.4.15a). We deduce from Poincaré’s theorem on the analytic dependence of solutions on the initial conditions and parameters\(^1\) that \( \Omega(x) \) is holomorphic in \( \mathbb{C} \setminus \{\pm 1\} \). From its asymptotics at infinity (5.2.7a), \( \Omega(x) \) is also holomorphic at \( x = \infty \). On the other hand, proposition 5.2.3 shows that in a neighbourhood of the points \( x = \pm 1 \) the eigenvalues \( \Lambda_\pm(x) \) have essential singularities from which it follows that (6.1.1) does not define an algebraic curve since \( \Gamma(x, \Lambda) = (\Lambda_+(x) - \Lambda)(\Lambda_-(x) - \Lambda) \) is not rational in \( x \in \mathbb{C} \).

To determine the values of \( x \in \mathbb{C} \) over which the cover \( \hat{\pi} : \Gamma \to \mathbb{C} \) branches we consider the discriminant of the polynomial \( \Gamma(x, \cdot) \),

\[ \Delta_\Gamma(x) = (\Lambda_+(x) - \Lambda_-(x))^2 \]  

(6.1.4)

and let \( \mathcal{Z}_\Gamma = \{ x \in \mathbb{C} | \Delta_\Gamma(x) = 0 \} \) be its set of zeroes. This corresponds to the set of \( x \in \mathbb{C} \) where the two eigenvalues coincide \( \Lambda_+(x) = \Lambda_-(x) \). In particular, \( \Omega(x) \) is diagonalisable for all \( x \in \mathbb{C} \setminus \mathcal{Z}_\Gamma \) and at any \( x_0 \in \mathcal{Z}_\Gamma \) we have \( \Lambda_+(x_0) = \pm 1 \) by (6.1.2). Now since \( \Delta_\Gamma(x) \) is meromorphic on \( \mathbb{C} \setminus \{\pm 1\} \) its zeroes must be isolated so that the set \( \mathcal{Z}_\Gamma \) is discrete. However, since \( \Delta_\Gamma(x) \) has essential singularities at

\(^1\)The solution \( x(t) \in \mathbb{C}^n \) to the differential equation \( \frac{dx}{dt} = F(t, x) \) depends holomorphically on the initial value \( x_0 \in \mathbb{C}^n \) and on any other parameter provided the vector function \( F \) itself depends holomorphically on these parameters.
Consider a point $x_0 \in \mathcal{Z}_\Gamma$. The order of the zero $x_0$ of $\Delta_\Gamma(x)$ determines the behaviour of the eigenvalues $\Lambda_{\pm}(x)$ near $x_0$ because from (6.1.2) and (6.1.4) we find

$$\Lambda_{\pm}(x) = \Lambda_{\pm}(x_0) \pm \sqrt{\Delta_\Gamma(x)} + O(x - x_0).$$

(6.1.5)

In particular, if $\Delta_\Gamma(x) = O(x - x_0)$ then $x_0$ corresponds to a branch point since

$$\Lambda_{\pm}(x) = \Lambda_{\pm}(x_0) \pm \alpha \sqrt{x - x_0},$$

(6.1.6)

and analytic continuation around $x_0$ locally interchanges the two eigenvalues.

**Remark** This means that the functions $\Lambda_{\pm}(x)$ are not globally well defined in the $x$-plane, and so expressions involving them should be handled with care. Nevertheless their sum $(\Lambda_{+}(x) + \Lambda_{-}(x))$ and product $\Lambda_{+}(x)\Lambda_{-}(x)$ are well defined analytic functions in $x \in \mathbb{C} \setminus \{\pm 1\}$ since $\Gamma(x, \Lambda) = (\Lambda_{+}(x) - \Lambda)(\Lambda_{-}(x) - \Lambda)$ is. For instance $\Delta_\Gamma(x) = (\Lambda_{+}(x) + \Lambda_{-}(x))^2 - 4\Lambda_{+}(x)\Lambda_{-}(x)$ is well defined for $x \in \mathbb{C} \setminus \{\pm 1\}.$

On the other hand,

**Proposition 6.1.2.** If $\Delta_\Gamma(x) = O((x - x_0)^n), \ n > 1$ then $\mathfrak{P}_0 = (x_0, \Lambda_{+}(x_0)) \in \Gamma$ is a singular point.

**Proof.** From the definition (6.1.1) we have $\Gamma(x, \Lambda) = (\Lambda_{+}(x) - \Lambda)(\Lambda_{-}(x) - \Lambda)$ so

$$\frac{\partial \Gamma(x_0, \Lambda_{+}(x_0))}{\partial x} = \left[ -\Lambda_{+}(x_0) \frac{d}{dx} (\Lambda_{+}(x) + \Lambda_{-}(x)) + \frac{d}{dx} (\Lambda_{+}(x)\Lambda_{-}(x)) \right] \bigg|_{x=x_0}$$

$$= -\frac{1}{4} \left[ (\Lambda_{+}(x) + \Lambda_{-}(x))^2 - 4\Lambda_{+}(x)\Lambda_{-}(x) \right] \bigg|_{x=x_0}$$

$$= -\frac{1}{4} \frac{d\Delta_\Gamma(x_0)}{dx} = 0,$$

where in the second line we have written $\Lambda_{+}(x_0)$ as $\frac{1}{2} (\Lambda_{+}(x_0) + \Lambda_{-}(x_0)),$ and in
6.1. **THE SPECTRAL CURVE**

the last line we used the definition of $\Delta_\Gamma(x)$. The last equality follows because by assumption $x_0$ is a multiple root of $\Delta_\Gamma(x)$. But since $\Lambda = \Lambda_+(x_0)$ is a double root of $\Gamma(x_0, \Lambda)$,

$$\frac{\partial \Gamma}{\partial \Lambda}(x_0, \Lambda_+(x_0)) = 0,$$

and so we conclude that $(x_0, \Lambda_+(x_0)) \in \Gamma$ is indeed a singular point. □

It follows that all points $\Psi \in \Gamma$ with $\hat{\pi}(\Psi) \in \mathcal{Z}_\Gamma$ are either branch points or singular points. By equation (6.1.5) and proposition 6.1.2 the singular points are locally of the form

$$(\Lambda - \Lambda_0)^2 = (x - x_0)^n, \quad n \geq 2$$

where $\Lambda_0 = \Lambda_+(x_0)$. These types of singularities were discussed in chapter [1]. When $n = 2$ this is a node and for $n = 3$ it is a cusp. Higher order singularities are either higher nodes or higher cusps depending on whether $n$ is even or odd respectively.

Since every point $\Psi \in \Gamma$ corresponds to an eigenvalue of $\Omega(\hat{\pi}(\Psi))$, let us denote by $\mathcal{E}_\Gamma(\Psi)$ the corresponding eigenspace with $\dim \mathcal{E}_\Gamma(\Psi) \leq 2$. The following proposition exhibits a fundamental difference between node-like and cusp-like singularities with regard to their respective eigenspaces.

**Proposition 6.1.3.** If $\Psi \in \Gamma$ is not a node-like singularity then $\dim \mathcal{E}_\Gamma(\Psi) = 1$.

*Proof.*** This is obvious for $\hat{\pi}(\Psi) \in \mathcal{C} \setminus \mathcal{Z}_\Gamma$. So let $x_0 \in \mathcal{Z}_\Gamma$ and $\Psi_0 = (x_0, \Lambda_+(x_0)) \in \Gamma$. Assume that $\Psi_0$ is either a branch point, a cusp or a higher cusp. All these cases fall into the same category for which $\Delta_\Gamma(x) = O(x - x_0)^{2r+1}, r \in \mathbb{N}$. Let us also denote the components of the monodromy matrix as

$$\Omega(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}.$$

Since zeroes of $\Delta_\Gamma(x)$ are isolated, in a small enough neighbourhood of $x_0$ the mon-
odromy matrix $\Omega(x)$ has two distinct eigenvectors which are easily shown to be

$$\psi_\pm(x) = \left( 1, \frac{D(x) - A(x)}{2B(x)} \pm \frac{\sqrt{A_\Gamma(x)}}{B(x)} \right)^T.$$

Now since we are assuming $\Delta_\Gamma(x) = O((x - x_0)^{2r+1}$ and because $B(x) = O((x - x_0)^n)$, $A(x) - D(x) = O((x - x_0)^m)$ for some non-negative integer $n$ and $m$, it follows that $\psi_\pm \to \begin{pmatrix} 1 \\ a \end{pmatrix}$ or $\psi_\pm \to \begin{pmatrix} o \\ 1 \end{pmatrix}$ and either way $\dim \mathcal{E}_\Gamma(\mathcal{P}_0) = 1$. \hfill \Box

Whereas proposition \ref{prop:6.1.3} establishes that most points of $\Gamma$ correspond to a single eigenvector of the monodromy matrix, it does not forbid nodes and higher nodes to have a two-dimensional eigenspace. But as we know from chapter \ref{chap1} these node-like singularities are blown up into a pair of regular points upon desingularisation. We therefore anticipate a crucial property of the normalisation of the spectral curve, namely that to each of its points corresponds a unique eigenvector.

However if we want the curve $\Gamma$ to have finite topological genus (i.e. finitely many branch points and cusp-like singularities) then it must have an infinite number of node-like singularities accumulating at $x = \pm 1$. With $\Gamma$ being so singular it is not obvious how to normalise it. In the next section we follow a standard approach for obtaining an algebraic curve $\Sigma$ from $\Gamma$ which can then be normalised in the usual way to obtain a Riemann surface $\hat{\Sigma}$.

### 6.2 The algebraic curve

Thus far we have constructed a 1-dimensional complex curve $\Gamma$ from any given solution. The problem however is that this curve is either of infinite genus or highly singular and in order to make use of the powerful tools of complex analysis we need instead a finite-genus Riemann surface. But for the normalisation of $\Gamma$ to have finite genus, the curve $\Gamma$ itself must also be finite-genus. The class of solutions giving rise
6.2. THE ALGEBRAIC CURVE

to such finite-genus spectral curves will be called finite-gap solutions\(^2\). The standard way to introduce these solutions is as follows,

**Definition 6.2.1.** A finite-gap solution is one that is independent of some given combination of the higher-times of the hierarchy, i.e. \( \sum c_N \partial t_N j = 0, c_N \in \mathbb{C} \).

Consider the zero-curvature equations from the hierarchy

\[
\partial_t N J_M(x) - \partial_t M J_N(x) + [J_M(x), J_N(x)] = 0.
\]

Taking the sum over \( N \) weighted by the coefficients \( c_N \) and using the finite-gap condition \( \sum c_N \partial t_N J_M(x) = 0 \) we obtain an equation of the form

\[
\partial_t M L(x) = [J_M(x), L(x)],
\]

where we have introduced the Lax matrix \( L(x) \equiv \sum c_N J_N(x) \). Equation (6.2.1) takes exactly the same form as the evolution equation (5.4.1 5a) for \( \Omega(x) \). It says that the evolution of the Lax matrix \( L(x) \) with respect to all the higher times is also isospectral which once more provides an invariant curve \( \Sigma \) in \( \mathbb{C}^2 \). However, because \( L(x) \) is rational in \( x \) with poles of finite order at \( x = \pm 1 \) the resulting curve \( \Sigma \) is now algebraic, as opposed to the spectral curve defined in terms of \( \Omega(x) \) with essential singularities at \( x = \pm 1 \).

**Definition 6.2.2.** The algebraic curve \( \Sigma \subset \mathbb{C}^2 \) is defined by

\[
\Sigma : \Sigma(x, y) \equiv \det(y1 - L(x)) = 0.
\]

Since this new curve is algebraic it may be normalised in the usual way to obtain

---

\(^2\)The notion of a ‘gap’ originates from the KdV equation for which these methods were first developed. There the branch cuts of the spectral curve all lie on the real axis and correspond to forbidden gaps in the spectrum of some operator. Unfortunately this terminology does not reflect the general situation for which a more suggestive term would be ‘finite-genus’ or perhaps ‘finite-g’.\n
a Riemann surface. We denote the **normalised algebraic curve** as \( \hat{\Sigma} \), equipped with the normalisation map

\[
\pi_\Sigma : \hat{\Sigma} \to \Sigma.
\]  

(6.2.3)

We now ask how this Riemann surface \( \hat{\Sigma} \) is related to the spectral curve \( \Gamma \).

Just as for the spectral curve \( \Gamma \) one can define the discriminant

\[
\Delta_\Sigma(x) = (y_+(x) - y_-(x))^2
\]

of the polynomial \( \Sigma(x, \cdot) \) as well as its set of zeroes \( Z_\Sigma \). Defining also the eigenspace \( \mathcal{E}_\Sigma(P) \) corresponding to a point \( P = (x, y) \in \Sigma \), propositions 6.1.2 and 6.1.3 readily apply to the algebraic curve \( \Sigma \) without modification. In particular, since the normalisation (6.2.3) blows up each node-like singularity of \( \Sigma \) to a pair of regular points on \( \hat{\Sigma} \), if we define the eigenspaces \( \mathcal{E}_\hat{\Sigma}(P) \) corresponding to points of \( \hat{\Sigma} \) in the obvious way then we have the following important result,

**Proposition 6.2.3.** \( \forall P \in \hat{\Sigma}, \dim \mathcal{E}_\hat{\Sigma}(P) = 1. \)

Going back to the evolution equation (5.4.15a) of the monodromy matrix under the higher times we see that in the case of a finite-gap solution, for which \( \sum_N c_N \partial_t \Omega(x) = 0 \), one has

\[
[L(x), \Omega(x)] = 0.
\]

It follows that if \( \psi(\mathcal{P}) \) where \( \mathcal{P} = (x, \Lambda) \in \Gamma \) is an eigenvector of \( \Omega(x) \) with eigenvalue \( \Lambda \) then

\[
(\Omega(x) - \Lambda \mathbf{1})(L(x)\psi(\mathcal{P})) = 0.
\]

So if \( \mathcal{P} = (x, \Lambda) \in \Gamma \) is not a node-like singularity then proposition 6.1.3 implies that \( L(x)\psi(\mathcal{P}) \) must be proportional to \( \psi(\mathcal{P}) \) so that

\[
L(x)\psi(\mathcal{P}) = y\psi(\mathcal{P}), \quad \hat{\pi}(\mathcal{P}) = x, \quad (6.2.4a)
\]
where \( y \) is one of the two eigenvalues of \( L(x) \). If \( P = (x, y) \in \Sigma \) also isn’t a node-like singularity of \( \Sigma \) then the analogue of proposition 6.1.3 for \( \Sigma \) implies that there exists a unique eigenvector \( \psi'(P) \) of \( L(x) \) with eigenvalue \( y \) such that

\[
L(x)\psi'(P) = y\psi'(P), \quad \dot{\pi}(P) = x, \tag{6.2.4b}
\]

and hence from equations (6.2.4) we have \( \psi'(P) = \psi(\mathfrak{g}) \). But then by continuity at the node-like singularities of \( \Gamma \) and \( \Sigma \) this equality must also hold at these points. We conclude that \( \Omega(x) \) and \( L(x) \) have the same eigenvectors for all \( x \in \mathbb{C} \) even though they do not have the same eigenvalues (because they define different curves). We shall denote the eigenvector by the same symbol \( \psi \) whether it lives on \( \Gamma \), \( \Sigma \) or \( \hat{\Sigma} \).

Now consider the eigenvector \( \psi(P) \) at any \( P \in \hat{\Sigma} \). By definition, the monodromy matrix is the parallel transporter around the worldsheet of the string and so writing the dependence on \( \sigma \) explicitly we have

\[
\psi(P, \sigma + 2\pi) = \Omega(x, \sigma)\psi(P, \sigma) = \Lambda\psi(P, \sigma). \tag{6.2.5}
\]

Hence \( \Lambda \) can be written as a quotient of two functions of \( P \in \hat{\Sigma} \) and so is well defined on \( \hat{\Sigma} \). Thus \( \hat{\Sigma} \) can be thought of as an (infinite) normalisation of \( \Gamma \) in the sense that there is a continuous surjection

\[
\pi_{\Gamma} : \hat{\Sigma} \to \Gamma, \quad P = (x, y) \mapsto \mathfrak{g} = (x, \Lambda(P)), \tag{6.2.6}
\]

whose restriction \( \pi_{\Gamma} : \hat{\Sigma} \setminus \pi^{-1}(S) \to \Gamma \setminus S \), with \( S \) denoting the set of singular points of \( \Gamma \), is a holomorphic bijection. In particular the spectral curve \( \Gamma \) has the same finite topological genus as \( \hat{\Sigma} \), i.e. \( g_\Gamma = g_\Sigma = g_{\hat{\Sigma}} \equiv g \).
The Riemann surface

We can be more explicit about the algebraic form of the normalisation $\hat{\Sigma}$ of the algebraic curve $\Sigma$. Since the matrix $L(x) = \sum N_c N_J(x)$ appearing in (6.2.2) is traceless (because all the $J_N$ are) it can be written out in components as

$$L(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix}$$

where each entry is a rational function of $x \in \mathbb{C}$. The defining equation (6.2.2) for $\Sigma$ then simplifies to

$$\Sigma : y^2 = -\det L(x) = a(x)^2 + b(x)c(x). \quad (6.2.7)$$

Multiplying this equation throughout by an appropriate perfect square $(Q(x))^2$, where $Q(x)$ is a polynomial, and redefining $y \mapsto Q(x)y$ it is possible to turn the right hand side of (6.2.7) into a polynomial, say $P(x)$. If this polynomial contains any repeated factors one may further divide throughout by another perfect square to eliminate them and so we may assume without loss of generality that $P(x)$ contains no repeated factors. The resulting non-singular curve $y^2 = P(x)$ is simply the normalisation $\hat{\Sigma}$ in algebraic form and the various redefinitions of $y$ to achieve this form are nothing but the birational transformations required to normalise $\Sigma$. Since $x = \infty$ is not a branch point of the spectral curve $^3$ it can’t be a branch point of $\hat{\Sigma}$ which means that the polynomial $P$ must be of even degree. And because the curve $\hat{\Sigma}$ has genus $g$ by definition, the degree of $P$ must be precisely $2g + 2$ (by the Riemann-Hurwitz formula (1.3.5)) since in the hyperelliptic case the total

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$^3$It follows from the asymptotics (5.2.8a) of the monodromy matrix at $x = \infty$ that the spectral curve generically takes the form $(\Lambda - 1)^2 = C^2/x^2$ near $x = \infty$ where $C \neq 0$ is a constant, hence $\Psi_\infty = (\infty, 1) \in \Gamma$ corresponds to a node. By the same token $\Psi_0 = (0, 1) \in \Gamma$ is shown not to be a branch point using the asymptotics (5.2.8b) of the monodromy matrix at $x = 0$. 

---
branching number \( b \) appearing in the formula is equal to the number of branch points. Therefore the normalised algebraic curve takes the following final form

\[
\hat{\Sigma} : \quad y^2 = \prod_{i=1}^{g+1} (x - u_i)(x - v_i),
\]

(6.2.8)

where at this stage the branch points \( \{u_i, v_i\}_{i=1}^{g+1} \) are arbitrary complex numbers. It is evident that one can always represent the curve (6.2.8) by introducing branch cuts in the complex plane, joining up the \( 2g+2 \) branch points in pairs (Figure 6.1).

Figure 6.1 shows that statements such as “the point \( P \in \hat{\Sigma} \) lies on the top sheet” are not invariant under changes of the representation of \( \hat{\Sigma} \) in terms of cuts.

The involution (6.1.3) of \( \Gamma \) induces a holomorphic involution of \( \hat{\Sigma} \), called the hyperelliptic involution,

\[
\hat{\sigma} : \quad \hat{\Sigma} \to \hat{\Sigma}, \quad (x, y) \mapsto (x, -y).
\]

It has the effect of interchanging the two sheets of (6.2.8) with the branch points \( \{u_i, v_i\}_{i=1}^{g+1} \) as fixed points.
6.3 Quasi-momentum

The normalisation \( \hat{\Sigma} \) of the spectral curve \( \Gamma \) being a Riemann surface, it is a much more desirable curve to work with than the spectral curve itself. Therefore when discussing finite-gap solutions we will always work with \( \hat{\Sigma} \) and forget about the spectral curve altogether. This is a legitimate step to take provided we have a way of recovering the spectral curve from the Riemann surface. For instance if we specify the function \( \Lambda(P) \) on the curve \( \hat{\Sigma} \) then the spectral curve is simply the image of \( \hat{\Sigma} \) under the normalisation map (6.2.6). Therefore the pair \( (\hat{\Sigma}, \Lambda) \) contains sufficient information to characterise the spectral curve. However, as we explain below, the function \( \Lambda(P) \) is not meromorphic since its ‘branches’ \( \Lambda_{\pm}(x) \) have essential singularities at \( x = \pm 1 \). The goal of this section is to replace \( \Lambda(P) \) by an Abelian differential \( dp \) on \( \hat{\Sigma} \).

Since \( \Lambda(P) \) is well defined on \( \hat{\Sigma} \) it can obviously be represented by two functions \( \Lambda_{\pm}(x) \) living on the top and bottom sheets respectively which ‘match up’ along the cuts. These are the same ‘branches’ of \( \Lambda(P) \) as in section 6.1, but the advantage of having introduced branch cuts is that these functions \( \Lambda_{\pm}(x) \) are now well defined on the cut planes and moreover they are distinct from one another (see equation (6.1.6) and the remark following it). We are now able to unambiguously specify the essential singularities of the function \( \Lambda \) by giving them for its branches \( \Lambda_{\pm} \). Because these essential singularities are located at \( x = \pm 1 \), we need to be specific about the position of the different cuts relative to the points \( x = \pm 1 \) since moving a cut over either of these points will swap the relative definitions of \( \Lambda_{\pm}(x) \) and \( \Lambda_{\mp}(x) \) at these points (see Figure 6.2). We therefore introduce an equivalence relation on representations of \( \hat{\Sigma} \) in terms of cuts, where two representations are equivalent if the cuts of one can be deformed within the punctured Riemann sphere \( \mathbb{C}P^1 \setminus \{\pm 1\} \) to the cuts of the other. It is straightforward to see that there are only two such
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Figure 6.2: Moving a single cut over the point \( x = -1 \) as in the figure interchanges the values of the branches \( \Lambda_+ (x) \leftrightarrow \Lambda_- (x) \) in the shaded region.

...equivalence classes and that they obviously both specify the same Riemann surface \( \hat{\Sigma} \). Given a representative of one equivalence class, one can obtain a representative of the other class by crossing say \( x = -1 \) with just a single cut.

Now with respect to a given equivalence class of cuts, the essential singularities of the function \( \Lambda \) on \( \hat{\Sigma} \) can be uniquely specified in terms of those of its branches \( \Lambda_\pm (x) \), which can be read off from (5.2.4). Among the two equivalence classes of cuts to choose from, we shall pick the one with respect to which the asymptotics of \( \Lambda_\pm (x) \) near \( x = \pm 1 \) take the following form

\[
\Lambda_\pm (x) = \exp \left[ \mp \frac{i \pi \kappa_\pm}{x \pm 1} + O ((x \pm 1)^0) \right], \quad \text{as } x \to \pm 1, \tag{6.3.1}
\]

To obtain the representation of \( \Lambda(P) \) with respect to the other equivalence class of cuts one simply flips the sign in the exponent at \( x = -1 \).

Remark Since the equations of motion are invariant under the interchange \( \sigma \leftrightarrow \tau \) of worldsheet space and time coordinates, applying such a transformation to a given solution will generate another solution. It will turn out that when applied to the present solution this transformation will change the definition of \( \Lambda(P) \) so that the asymptotics (6.3.1) will now be valid with respect to the other equivalence class of cuts. Thus the two different equivalence classes of cuts give two different ways of defining the function \( \Lambda \) on \( \hat{\Sigma} \) by (6.3.1). Both will turn out to give...
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solutions related by the discrete symmetry $\sigma \leftrightarrow \tau$.

Since the functions $\Lambda_{\pm}(x)$ are by definition the eigenvalues of the monodromy matrix $\Omega(x)$ it follows using also (6.1.2) that

$$\Lambda(P) + \frac{1}{\Lambda(P)} = \text{tr} \Omega(\hat{\pi}(P)).$$

And because $\Omega(x)$ is holomorphic in $\mathbb{C} \setminus \{\pm 1\}$ we conclude that the function $\Lambda$ can have no poles or zeroes in $\hat{\Sigma} \setminus \hat{\pi}^{-1}(\{\pm 1\})$.

Although the function $\Lambda$ is enough to recover the spectral curve $\Gamma$ from $\hat{\Sigma}$ as we have already discussed, its essential singularities are not a very desirable feature. It is best therefore to replace the function $\Lambda$ with a meromorphic differential $dp$ defined by

$$dp = -id\Lambda = -id\log \Lambda.$$  \hspace{1cm} (6.3.2)

Since $\Lambda$ has no poles or zeroes in $\hat{\Sigma} \setminus \hat{\pi}^{-1}(\{\pm 1\})$ it follows that the poles of $dp$ can only come from the points $P \in \hat{\Sigma}$ with $\hat{\pi}(P) = \pm 1$. In fact these poles are easily derived from the behaviour (6.3.1) of $\Lambda$ at these points and one finds

$$dp(x^\pm) = \mp d \left( \frac{\pi \kappa^\pm}{x - 1} \right) + O \left( (x - 1)^2 \right), \quad \text{as} \quad x \to +1,$$

$$dp(x^\pm) = \mp d \left( \frac{\pi \kappa^\pm}{x + 1} \right) + O \left( (x + 1)^2 \right), \quad \text{as} \quad x \to -1.$$  \hspace{1cm} (6.3.3)

Here we have introduced the following notation: if $x$ is not a branch point then $x^\pm \in \hat{\Sigma}$ denotes the pair of points in $\hat{\pi}^{-1}(x)$ ($x^+$ living on one of the cut planes, $x^-$ on the other), whereas if $x$ is a branch point then $x^+ = x^- = \hat{\pi}^{-1}(x)$ is a single point. For instance we can rewrite the branches of the function $\Lambda$ as $\Lambda_{\pm}(x) = \Lambda(x^\pm)$. Because $\Lambda$ is a well defined function on $\hat{\Sigma}$, it follows from the definition (6.3.2) of $dp$
that its integral around any closed loop is an integer multiple of $2\pi$, and in particular

$$\int_{a_i} dp = 2\pi m_i, \quad \int_{b_i} dp = 2\pi n_i, \quad m_i, n_i \in \mathbb{Z}.$$ 

At this point we must be more specific about the choice of homology basis on $\hat{\Sigma}$. We define the basis of $a$-cycles as loops encircling $g$ different cuts. The $b$-cycles are defined from the $g$ remaining independent cycles such that they have canonical intersections with the $a$-cycles, i.e. $a_i \cap b_j = \delta_{ij}$, $i, j = 1, \ldots, g$. The resulting basis of $a$- and $b$-cycles is called canonical. An example is shown in Figure 6.3.

![Figure 6.3: Canonical $a$- and $b$-cycles for a genus two curve.](image)

**Remark** Such a basis always exists (indeed the homology basis constructed in section 1.3 is canonical) but is by no means unique: if $\kappa = (a, b)^T$ and $\kappa' = (a', b')^T$ are homology bases then $\kappa' = X\kappa$ with $X \in SL(2, \mathbb{Z})$. The condition for $\kappa$ being canonical is $\kappa \circ \kappa^T = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that the new basis $\kappa'$ is also canonical provided $XJX^T = J$, that is $X \in Sp(2g, \mathbb{Z})$.

As usual we may choose to normalise this differential (and we denote the normalised differential by the same symbol) so that

$$\int_{a_i} dp = 0, \quad \int_{b_i} dp = 2\pi n_i, \quad n_i \in \mathbb{Z}. \quad (6.3.4)$$

According to lemma 1.5.10 it is then uniquely defined by its asymptotics at $x = \pm 1$ given in (6.3.3). As an immediate consequence it is easy to see that it has the
following behaviour under pullback by the hyperelliptic involution,

\[ \hat{\sigma}^* dp = -dp. \]

**Remark** For the same reason as with \( dp \), this property under pullback by the hyperelliptic involution also holds for any *normalised* Abelian differential of the second or third kind whose poles are invariant under \( \hat{\sigma} \) and whose singular parts at the poles \( x^+ \) and \( x^- = \hat{\sigma}x^+ \) are opposite. This is the case for the third kind Abelian differential \( \omega_{x^+x^-} \) and we have \( \hat{\sigma}^* \omega_{x^+x^-} = -\omega_{x^+x^-} \). In fact, it even holds for *normalised* Abelian differentials of the *first* kind. Indeed, the holomorphic basis differentials can be locally written as \( \omega_i = df_i \) for some holomorphic functions \( f_i \) so that \( \hat{\sigma}^* \omega_i = d(f_i \circ \hat{\sigma}) \). Since \( f_i \circ \hat{\sigma} \) are still holomorphic so are \( \hat{\sigma}^* \omega_i \). But by the choice of \( a \)-periods \( \hat{\sigma}a_i = -a_i \) and hence \( \int_{a_i} \hat{\sigma}^* \omega_j = \int_{\hat{\sigma}a_i} \omega_j = -\int_{a_i} \omega_j = -\delta_{ij} \). Lemma 1.5.4 implies \( \hat{\sigma}^* \omega_j = -\omega_j \).

The normalised differential \( dp \) and its Abelian integral will play a fundamental role in the sequel.

**Definition 6.3.1.** The **quasi-momentum** is the Abelian integral \( p(P) = \int^P dp \).

A consequence of normalising \( dp \) with respect to the chosen set of \( a \)-cycles is that the branches \( p_\pm(x) = p(x^\pm) \) of the quasi-momentum now define single-valued functions on the complex plane with cuts, even though the Abelian integral \( p(P) \) itself is multi-valued on the whole of \( \hat{\Sigma} \).

**Asymptotics**

The asymptotics of the differential \( dp \) near the points \( 0^\pm, \infty^\pm \) can be deduced from the asymptotics of the monodromy matrix \( \Omega(x) \) near \( x = 0, \infty \) for ‘highest weight’ solutions, namely equations (5.2.8) of chapter 5. They are directly expressed in
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terms of the Casimirs $R^2, L^2$ of $SU(2)_R \times SU(2)_L$ as follows

$$dp(x^\pm) = \mp d \left[ \frac{1}{x} \frac{2\pi R}{\sqrt{\lambda}} + O \left( \frac{1}{x^2} \right) \right], \quad \text{as } x \to \infty,$$

$$dp(x^\pm) = \pm d \left[ \frac{2\pi L}{\sqrt{\lambda}} + O \left( x^2 \right) \right], \quad \text{as } x \to 0. \quad (6.3.5)$$

**Remark** Just as when specifying the asymptotics (6.3.1) near $x = \pm 1$, here the positions of the different cuts relative to the points $x = \infty$ and $x = 0$ are important. Both asymptotics in (6.3.5) are valid with respect to one of two equivalence classes of cuts, two sets of cuts being equivalent if they can be deformed within $\mathbb{C}P^1 \setminus \{ \infty \}$ or $\mathbb{C}P^1 \setminus \{ 0 \}$ respectively.

We will always choose the base point for the quasi-momentum to be $\infty^+$, which fixes the additive constant such that $p(\infty^+) = 0$, namely

$$p(P) = \int_{\infty^+}^P dp.$$

Considering only points $P = x^+$ on the upper sheet and restricting also the integration path to lie on the upper sheet we find the following asymptotics for the quasi-momentum $p(x) \equiv \int_{\infty^+}^{x^+} dp$ itself,

$$p(x) = -\frac{1}{x} \frac{2\pi R}{\sqrt{\lambda}} + O \left( \frac{1}{x^2} \right), \quad \text{as } x \to \infty,$$

$$p(x) = 2\pi m + x \frac{2\pi L}{\sqrt{\lambda}} + O \left( x^2 \right), \quad \text{as } x \to 0. \quad (6.3.6)$$

where $m \in \mathbb{Z}$. 
The logarithmic derivative curve

Another way to obtain an algebraic curve from the spectral curve is to define a new matrix $L'(x)$ by \cite{27, 29}

$$
\Psi(x)^{-1}L'(x)\Psi(x) = -i \frac{\partial}{\partial x} \log \left( \Psi(x)^{-1}\Omega(x)\Psi(x) \right),
$$

where $\Psi(x)$ is the matrix of eigenvectors of the monodromy matrix $\Omega(x)$. This way the eigenvalues of $L'(x)$ are the logarithmic derivatives\footnote{The logarithm $\log f$ of a function is not well defined (it requires branch cuts) but its derivative $(\log f)'$ is well defined since the values of $\log f$ on different branches differ by constants.} of those of $\Omega(x)$, but the corresponding eigenvectors are unchanged. By the above discussion it is clear that the eigenvalues $\lambda_\pm(x) \equiv -i(\log \Lambda_\pm(x))'$ of $L'(x)$ are rational because they can be written as the quotient of two meromorphic differentials, namely $\lambda_\pm = \frac{dp}{dx}(x^\pm)$. The characteristic equation for $L'(x)$ thus defines another algebraic curve in $\mathbb{C}^2$.

**Definition 6.3.2.** The logarithmic derivative curve $\Sigma' \subset \mathbb{C}^2$ is defined by

$$
\Sigma': \Sigma'(x, \lambda) \equiv \det(\lambda 1 - L'(x)) = 0. \quad (6.3.7)
$$

This curve has the same normalisation $\hat{\Sigma}$ as the curves $\Gamma$ and $\Sigma$ with the obvious normalisation map

$$
\pi_{\Sigma'}: \hat{\Sigma} \to \Sigma', \quad P = (x, y) \mapsto (x, \lambda(P)), \quad \text{where} \quad \lambda(P) = \frac{dp}{dx}(P).
$$

To understand how the new curve $\Sigma'$ relates to the spectral curve $\Gamma$ we first relate the set of zeroes $Z_\Gamma$ of $\Delta_\Gamma(x)$ to the set of zeroes $Z_{\Sigma'}$ of the discriminant

$$
\Delta_{\Sigma'}(x) = (\lambda_+(x) - \lambda_-(x))^2
$$
of the curve $\Sigma'$. So consider a point $x^* \in \mathcal{Z}_\Gamma$, then $(\Lambda_+(x) - \Lambda_-(x))^2 = O ((x - x^*)^n)$ with $n \geq 1$ so that $\Lambda_+(x)/\Lambda_-(x) = 1 + O ((x - x^*)^{\frac{n}{2}})$. After taking the logarithmic derivative this leads to $(\lambda_+(x) - \lambda_-(x))^2 = O ((x - x^*)^{n-2})$, from which we read:

- $n = 1$: branch points of $\Gamma$ become square-root singularities of $\Sigma'$,
- $n = 2$: nodes of $\Gamma$ all disappear on $\Sigma'$,
- $n = 3$: cusps of $\Gamma$ become ordinary branch points of $\Sigma'$,
- $n \geq 4$: higher order singularities of $\Gamma$ persist on $\Sigma'$ with order $n - 2$.

Now because the curve $\Sigma'$ is algebraic, the discriminant $\Delta_{\Sigma'}(x)$ of the polynomial $\Sigma'(x, \cdot)$ is meromorphic on $\mathbb{CP}^1$ and so its set of zeros $\mathcal{Z}_{\Sigma'} \subset \mathbb{CP}^1$ is finite. This shows that the spectral curve $\Gamma$ has only a finite number of singular points of order $n > 2$, so that the singular points accumulating at $x = \pm 1$ must be nodes.

### 6.4 Moduli

At this point we have now replaced the spectral curve $\Gamma$ by a Riemann surface $\hat{\Sigma}$ equipped with an Abelian integral $\rho$ called the quasi-momentum. The purpose of this section is to count the number of independent moduli of the spectral curve and introduce a ‘good’ set of coordinates on the moduli space. This problem was solved in great generality by Krichever and Phong in [77, 78] where they devised a ‘universal’ and more systematic description of the moduli spaces of the spectral data for a large class of integrable systems. Specifically, the spectral data of those systems covered by [77] all consist of a Riemann surface $\hat{\Sigma}$ with $N$ punctures $(P_\alpha)_{\alpha=1}^N$ and two Abelian integrals $E$ and $Q$ with poles of orders at most $n = (n_\alpha)_{\alpha=1}^N$ and $m = (m_\alpha)_{\alpha=1}^N$ at the punctures. So the strategy of [77] is to consider the moduli space of all such Riemann surfaces (with the discrete parameters $g = \text{genus}(\hat{\Sigma}), N, n, m$
held fixed) called the **universal configuration space** \( \mathcal{M}_g(n, m) \) and introduce an explicit set of local coordinates on it. The moduli space for the spectral data of a specific integrable system then consists of a leaf in a foliation of \( \mathcal{M}_g(n, m) \) for some \( g, n, m \). Remarkably, or perhaps not so surprisingly, we will find that the moduli space for the spectral data at hand also admits such a description. We start by reviewing the construction of the universal configuration space \( \mathcal{M}_g(n, m) \) and the definition of a set of local coordinates \([77]\).

**The universal configuration space**

In the present subsection we closely follow the discussion in \([78]\). The first immediate goal is to determine the dimension of the universal configuration space \( \mathcal{M}_g(n, m) \). This is an easy consequence of the Riemann-Roch theorem.

**Lemma 6.4.1.** \( \dim \mathcal{M}_g(n, m) = 5g - 3 + 3N + \sum_{\alpha=1}^{N} (n_\alpha + m_\alpha) \).

**Proof.** By corollary \([1.5.18]\) of the Riemann-Roch theorem, the number of degrees of freedom of the Abelian differential \( dE \) with poles of order at most \( n_\alpha + 1 \) at \( P_\alpha, \alpha = 1, \ldots, N \) is \( \sum_{\alpha=1}^{N} (n_\alpha + 1) - 1 + g = N - 1 + g + \sum_{\alpha=1}^{N} n_\alpha \). The Abelian integral \( E(P) = \int_{P_0}^{P} dE \) has one extra degree of freedom corresponding to the choice of \( P_0 \) so \( E \) has a total of \( N + g + \sum_{\alpha=1}^{N} n_\alpha \) free parameters. Likewise the Abelian integral \( Q \) has \( N + g + \sum_{\alpha=1}^{N} m_\alpha \) degrees of freedom. Finally, by corollary \([1.5.23]\) the dimension of the moduli space of Riemann surfaces of genus \( g \) with \( N \) punctures is \( 3g - 3 + N \) for all \( g \geq 0 \). \( \square \)

The next goal is to determine a set of \( 5g - 3 + 3N + \sum_{\alpha=1}^{N} (n_\alpha + m_\alpha) \) functions on \( \mathcal{M}_g(n, m) \) with linearly independent differentials which would thus define a set of homolorphic coordinates on \( \mathcal{M}_g(n, m) \). Krichever and Phong introduced in \([77, 78]\) a convenient set of such functions with respect to which the moduli spaces of the
spectral data for many integrable systems locally correspond to level sets of some of these coordinates \((i.e.\text{ to leaves in } \mathcal{M}_g(n,m))\). We now review the construction of this coordinate system.

A fundamental ingredient for defining these coordinates is a certain meromorphic differential \(d\lambda\) which is central to the study of many integrable systems. It will also turn up naturally in chapter 8 and play a crucial role there when we come to study the symplectic structure of the string in the algebro-geometric context. Although the Abelian integrals \(E, Q\) are potentially multi-valued on \(\hat{\Sigma}\), they define single-valued branches on the normal form \(\hat{\Sigma}_{\text{cut}}\) (see definition 1.3.4) with extra cuts between the various punctures (for instance by joining \(P_1\) to \(P_\alpha\) for each \(\alpha = 2, \ldots, N\)). We make a choice of branch for the Abelian integral \(Q\) and define the 1-form

\[
d\lambda = QdE
\]

on \(\Sigma_{\text{cut}}\), which has a pole at each puncture \(P_\alpha\) of order \(n_\alpha + m_\alpha + 1\). This construction for defining \(d\lambda\) should be carried out in a continuous way locally on the universal configuration space \(\mathcal{M}_g(n,m)\). Next, in order to discuss the local behaviours of the various differentials \(d\lambda, dE, dQ\) and Abelian integrals \(Q, E\) at the punctures we also need to introduce a local set of charts \(w_\alpha\) near each puncture \(P_\alpha\). Such local charts are naturally provided by one of the Abelian integrals, say \(E\).

We are now in a position to define the set of local coordinates on \(\mathcal{M}_g(n,m)\) of [77, 78]. The first set of \(\sum_{\alpha=1}^{N}(n_\alpha + m_\alpha)\) coordinates are given by

\[
T_{\alpha,k} = \frac{1}{k} \text{res}_{P_\alpha}(w_\alpha^k d\lambda), \quad \alpha = 1, \ldots, N, \quad k = 1, \ldots, n_\alpha + m_\alpha.
\]

The next set of \(3N - 3\) coordinates are given by the residues of the differentials \(d\lambda,\)
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$dE$ and $dQ$ at the punctures \footnote{For the differentials $dE$ and $dQ$ which are well defined on $\hat{\Sigma}$ only $N - 1$ residues can be specified since the total sum of their residues must add up to zero by proposition 1.5.7.}

\[ R^\lambda_\alpha = \text{res}_{P_\alpha} d\lambda, \quad R^E_\alpha = \text{res}_{P_\alpha} dE, \quad R^Q_\alpha = \text{res}_{P_\alpha} dQ, \quad \alpha = 2, \ldots, N. \] (6.4.2b)

Finally the remaining $5g$ coordinates are given by periods of the differentials $d\lambda$, $dE$ and $dQ$, namely

\[ \tau_{a_i} = \int_{a_i} d\lambda, \quad \tau_{b_i} = \int_{b_i} d\lambda, \quad \tau_{a_i,E} = \int_{a_i} dE, \quad \tau_{b_i,E} = \int_{b_i} dE, \] (6.4.2c)

\[ \tau_{a_i,Q} = \int_{a_i} dQ, \quad \tau_{b_i,Q} = \int_{b_i} dQ, \] (6.4.2d)

\[ s_i = \int_{a_i} d\lambda, \quad i = 1, \ldots, g. \] (6.4.2e)

It is proved in [77] that these $5g - 3 + 3N + \sum_\alpha (n_\alpha + m_\alpha)$ functions (6.4.2) have linearly independent differentials and thus define a local holomorphic coordinate system for $M_g(n, m)$. Given such a coordinate system, one can consider the joint level set of all but the last $g$ coordinates (6.4.2e) and excluding also a certain number $l \leq N - 1$ of residues $R^\lambda_\alpha$. This defines a smooth foliation of $M_g(n, m)$ with the remaining $g + l$ coordinates defining a coordinate system $\{(s_i)_{i=1}^g, (R^\lambda_\alpha)_{\alpha=2}^{l+1}\}$ on each $(g + l)$-dimensional leaf.

The leaf

We now want to make use of the general framework reviewed in the previous subsection to count the independent moduli of the spectral data $\{\hat{\Sigma}, p\}$. Let us identify the quasi-momentum with the first Abelian integral, namely $E \equiv p$. The general setup requires choosing another Abelian integral $Q$. Our choice at this stage might seem rather ad hoc but it is guided by the results on the symplectic structure to be
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Indeed it will turn out that the moduli defined in this section are precisely the action variables of the string.

Since \( \hat{\Sigma} \) is hyperelliptic it also comes equipped with a holomorphic function \( x : \hat{\Sigma} \to \mathbb{CP}^1 \) of degree two which provides a coordinate chart in the neighbourhood of any point \( P \in \hat{\Sigma} \) that isn’t a branch point of the cover given by \( x \). The appropriate choice for the Abelian integral \( Q \) is the following meromorphic function on \( \hat{\Sigma} \)

\[
    z = x + \frac{1}{x}. \tag{6.4.3}
\]

This function clearly defines a double cover of the \( x \)-plane and thus has degree four on \( \hat{\Sigma} \). To make contact with the general construction we make the following identifications

\[
    E \equiv p, \quad Q \equiv z.
\]

By definition of the quasi-momentum (6.3.3) it has simple poles at the four points \( \{(+1)^\pm, (-1)^\pm\} \in \hat{\pi}^{-1}(\{\pm1\}) \) above \( x = \pm1 \). And by (6.4.3) we see that the function \( z \) has simple poles at the four points \( \{0^\pm, \infty^\pm\} \in \hat{\pi}^{-1}(\{0, \infty\}) \) above \( x = 0, \infty \). Therefore we have a total of \( N = 8 \) punctures. Because the Abelian integral \( Q = z \) is actually a function on \( \hat{\Sigma} \), here the 1-form (6.4.1) is a well defined and single-valued meromorphic differential on \( \hat{\Sigma} \),

\[
    d\lambda \equiv zdp.
\]

From the asymptotics of the quasi-momentum at \( x = \pm1 \) we can define local coordinates \( w_\pm \) near these points by setting \( E = 1/w_\pm \). Local coordinates around \( 0^\pm \) and \( \infty^\pm \) are provided by \( w_0 = x \) and \( w_\infty = 1/x \) respectively.

The residues of the differentials \( dz, dp, d\lambda \) and \( w_\alpha d\lambda \) can be easily computed
at all these punctures, for instance

\[ T_{\pm,1} = \text{res}(\pm1) \, w_{\pm}d\lambda = \text{res}(\pm1) \, w_{\pm} \frac{d}{d\lambda} \left( \frac{1}{w_{\pm}} \right) = -\text{res}(\pm1) \, \frac{dw_{\pm}}{w_{\pm}} = -z(\pm1) = \mp2. \]

For the residues at 0\( ^\pm \), \( \infty^\pm \) one must use the asymptotics (6.3.5) of the quasi-momentum at 0 and \( \infty \) respectively. All the residues are summarised in table 6.1 using the notation of the general construction. Furthermore, since the function \( z \)

| \( P_\alpha \) | \( (+1)^\pm \) | \( (-1)^\pm \) | \( 0^\pm \) | \( \infty^\pm \) |
|---|---|---|---|---|
| \( m \) | 1 | 1 | 0 | 0 |
| \( n \) | 0 | 0 | 1 | 1 |
| \( R_\alpha^E \) | 0 | 0 | 0 | 0 |
| \( R_\alpha^Q \) | 0 | 0 | 0 | 0 |
| \( R_\lambda \) | 0 | 0 | \( \pm \frac{2\pi L}{\sqrt{\lambda}} \) | \( \pm \frac{2\pi R}{\sqrt{\lambda}} \) |
| \( T_{\alpha,1} \) | -2 | 2 | 0 | 0 |

Table 6.1: Residues at the eight punctures.

is single-valued on \( \hat{\Sigma} \) all the periods of \( dz \) are zero whereas those of the normalised differential of the quasi-momentum \( dp \) are determined by (6.3.4) so we have

\[ \tau_{a_i, E} = \tau_{b_i, E} = 0, \quad \tau_{a_i, Q} = 0, \quad \tau_{b_i, Q} = 2\pi n_i. \quad (6.4.4) \]

The remaining \( g \) coordinates were defined in (6.4.2c). However, for conventional reasons we will scale these coordinates differently and set

\[ S_i = \frac{\sqrt{\lambda}}{8\pi^2 i} \int_{a_i} zdp, \quad i = 1, \ldots, g. \quad (6.4.5) \]

We see from table 6.1 that besides these \( g \) coordinates there are only two other tunable parameters in the general solution, namely the Casimirs of the global \( SU(2)_R \) and \( SU(2)_L \) symmetries which are expressible here in terms of residues on
the top sheet at infinity and zero respectively,

\[ R = -\frac{\sqrt{\lambda}}{2\pi} \text{res}_{\infty^+} z dp, \quad L = \frac{\sqrt{\lambda}}{2\pi} \text{res}_{0^+} z dp. \]

Recall however from section 3.2 that the current \( j \) is invariant under \( SU(2)_L \) but still transforms under \( SU(2)_R \) by (3.2.2). Since its components parametrise phase-space it follows that the action of the \( SU(2)_L \) symmetry on phase-space is trivial and does not play any part in the Hamiltonian formalism. We therefore fix the parameter \( L \) to define a leaf \( \mathcal{L} \) as the joint level set of all but the \( g+1 \) remaining parameters \( \{S_i\}_{i=1}^g \) and \( R \). Defining the following differential on \( \hat{\Sigma} \)

\[ \alpha = \frac{\sqrt{\lambda}}{4\pi} z dp, \quad (6.4.6) \]

the remaining \( g+1 \) coordinates parametrising the leaf are

\[ S_i = \frac{1}{2\pi i} \int_{a_I} \alpha, \quad i = 1, \ldots, g, \quad \frac{R}{2} = -\text{res}_{\infty^+} \alpha. \quad (6.4.7) \]

Equivalently, since the number of moduli precisely coincides with the number of cuts in the algebraic curve (6.2.8) one can parametrise \( \mathcal{L} \) by assigning a modulus to each cut. Specifically, for \( I = 1, \ldots, g+1 \) we define a cycle \( A_I \) to encircle the \( I^{th} \) cut \( C_I \) once counterclockwise on the top sheet. We can also define the dual cycles \( B_I \) as the contour going from \( \infty^+ \) to \( \infty^- \) through the \( I^{th} \) cut, see Figure 6.4.

![Figure 6.4: The cycle \( A_I \) and path \( B_I \) for the cut \( C_I \).](image-url)
Definition 6.4.2. The filling fraction for the $I^{th}$ cut is given by,

$$S_I = \frac{1}{2\pi i} \int_{A_I} \alpha = \frac{\sqrt{\lambda}}{8\pi^2 i} \int_{A_I} z dp. \quad (6.4.8)$$

The filling fractions are related to the variable $R$ and the parameter $L$ by

$$\sum_{I=1}^{g+1} S_I = -\text{res}_{\infty+} \alpha - \text{res}_{0+} \alpha = \frac{1}{2}(R - L). \quad (6.4.9)$$

The moduli space $\mathcal{L}$ is therefore a complex manifold with only orbifold singularities of dimension

$$\dim_{\mathbb{C}} \mathcal{L} = g + 1,$$

every point of which corresponds to an admissible curve $\hat{\Sigma}$ of genus $g$. 
Chapter 7

Algebro-geometric solutions

Given any (finite-gap) solution to the hierarchy of zero-curvature equations (5.4.15b) we have shown how to construct a Riemann surface \( \hat{\Sigma} \) equipped with an Abelian differential \( dp \), both of which are independent of the hierarchy of times. The goal of finite-gap integration (or algebro-geometric methods) \([67, 79–83]\) is to reconstruct the (finite-gap) solution itself after specifying further analytic data on \( \hat{\Sigma} \). A key part of the theory of Riemann surfaces which underlies this method of finite-gap integration is the construction of functions and differentials on a Riemann surface with prescribed singularities. The idea of finite-gap integration therefore is to identify a finite set of points on \( \hat{\Sigma} \) that will be the zeroes and poles of certain functions in terms of which the solution can be expressed. If this data is sufficient to uniquely determine these functions then it will also be enough to recover the solution.

In chapter 6 we have focused mostly on the integrals of motion of the solution,
namely the eigenvalues of the monodromy matrix, which we showed were encoded in the data \( \{ \hat{\Sigma}, dp \} \). To completely encode the monodromy matrix we are missing its dynamical part, which corresponds to its eigenvectors. However we have already argued in chapter 6 that these eigenvectors define a single-valued vector function \( \psi \) on \( \hat{\Sigma} \). To remove the arbitrary normalisation of \( \psi \) we introduce the normalised eigenvector denoted \( h \), with the suitable choices of normalisation conditions to be discussed later. As it turns out \( h(P, t) \) is in fact meromorphic in \( P \in \hat{\Sigma} \) with precisely \( g + 1 \) poles \( \hat{\gamma}_1(t), \ldots, \hat{\gamma}_{g+1}(t) \) (and hence also \( g + 1 \) zeroes) which explicitly depend on the hierarchy of times \( \{ t \} \). We can conveniently gather these points by defining the dynamical divisor

\[
\hat{\gamma}(t) \equiv \hat{\gamma}_1(t) + \cdots + \hat{\gamma}_{g+1}(t),
\]

which, as its name suggests, encodes the dynamics of the monodromy matrix. After making use of the gauge symmetry to set \( h_i(P_j) = \delta_{ij} \) where \( P_{1,2} = \infty^\pm \), it follows from the Riemann-Roch theorem that this data is enough to uniquely specify the components \( h_1 \) and \( h_2 \) of the normalised eigenvector \( h \) and hence also \( \Omega(x) \).

One would like to construct a similar set of functions that can be uniquely specified by some analytic data but in terms of which we can also write the solution. For this we exploit the hierarchy equations (5.4.15) which express the fact that the operators \( \partial_{t_M} - J_M(x) \) and \( \Omega(x) \) all commute among themselves and can thus be simultaneously diagonalised. Thus there exists an alternative normalisation of the eigenvector \( \psi(P, t) = \varphi(P, t)h(P, t) \) such that it solves the following linear system

\[
(\partial_{t_M} - J_M(x)) \psi(P, t) = 0, \quad \forall M.
\]

Unlike the normalised eigenvector \( h(P, t) \) above, the eigenvector \( \psi(P, t) \) is not meromorphic. Instead its components have essential singularities at the poles \( x = \pm 1 \).
of the Lax matrices and define what are called **Baker-Akhiezer functions**. If we are able to identify a set of analytic data which uniquely characterises this vector $\psi$ then the Lax connection could be recovered from it by the formula

$$J(x) = d\Psi(x)\Psi(x)^{-1},$$

(7.0.2)

where $\Psi(x) = (\psi(x^+), \psi(x^-))$ is the matrix constructed out of the pair of column eigenvectors at the points $x^\pm$ above $x \in \mathbb{C}$. Remarkably it turns out the only extra data needed to uniquely characterise the vector $\psi$ is the initial condition $\hat{\gamma}(0)$ of the dynamical divisor (7.0.1). All the dynamics can be recovered uniquely once the constant data $\{(\hat{\Sigma}, dp), \hat{\gamma}(0)\}$ has been specified. In particular, the time-dependence of the dynamical divisor (7.0.1) can be inferred from that of the vector $\psi$. The idea of finite-gap integration is illustrated in Figure 7.1: Every finite-gap solution

![Figure 7.1: Idea of finite-gap integration.](image)

is in one-to-one correspondence with a smooth Riemann surface $\hat{\Sigma}$ (of genus three in Figure 7.1) equipped with a set of marked points $\hat{\gamma}(0)$ (four of them in Figure 7.1). The pair $(\hat{\Sigma}, dp)$ encodes the integrals of motion of the solution whereas the dynamics is encoded in the marked points $\hat{\gamma}(t)$. Their exact motion on $\hat{\Sigma}$ is very complex, but what we find is that if we map $\hat{\Sigma}$ to its generalised Jacobian
is topologically a $g$-torus times a $\mathbb{C}^*$ factor) via the generalised Abel map then the motion becomes extremely simple, namely it linearises. The $(\sigma, \tau)$-motion of the string on the generalised Jacobian is like that of an infinitely rigid string wrapping one cycle of the torus and moving linearly in time along another direction.

### 7.1 The normalised eigenvector

In order to discuss the analytic properties of the eigenvector $\psi(P)$ at every $P \in \hat{\Sigma}$ we must first fix its normalisation. There are many ways one could normalise $\psi$ but to keep things simple we choose a linear normalisation condition and define the normalised eigenvector $h$ to satisfy

$$\alpha \cdot h(P) = 1,$$

(7.1.1)

where $\alpha \in \mathbb{C}^2$ is an arbitrary two component constant row vector. A common choice is $\alpha = (1, 0)$ which has the effect of setting the first component $h_1$ of $h$ equal to one. Although this might be the simplest possible normalisation condition it is not the most symmetric one. A more symmetric condition is $\alpha = (1, 1)$ which sets both components on an equal footing,

$$h_1(P) + h_2(P) = 1.$$

(7.1.2)

From now on we shall always stick to this normalisation for $h$.

**Lemma 7.1.1.** The components of $h$ are meromorphic functions on $\hat{\Sigma}$.

**Proof.** Let $\Delta(x, y)$ be the matrix of cofactors of $(L(x) - y1)$, which satisfy $(L(x) - y1)\Delta(x, y) = \Sigma(x, y)1$. Then for $(x, y) \in \Sigma$ we have $(L(x) - y1)\Delta(x, y) = 0$, so that every column of $\Delta(P)$ is proportional to the eigenvectors at $P$. Since $\Delta(P)$ is
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meromorphic (i.e. rational in $x$ and $y$) the result follows. □

**Definition 7.1.2.** A vector $v(P)$ is said to have a **pole** at $Q \in \hat{\Sigma}$ if at least one of its components $v_i(P)$ has a pole at $Q$.

**Proposition 7.1.3.** The normalised eigenvector $h$ has $g + 1$ poles on $\hat{\Sigma}$.

**Proof.** Consider the function $W(x) = (\det H(x))^2$ where $H(x) = (h(x^+), h(x^-))$ is the matrix of the normalised eigenvectors at $x$. The function $W(x)$ is a well defined meromorphic function on the Riemann sphere since it depends symmetrically on the points $x^\pm$ above $x$. Counting multiplicities, it is obvious that

$$(\# \text{ poles of } W(x)) = 2 \times (\# \text{ poles of } h(P)).$$

Since the eigenvectors $h(x^\pm)$ corresponding to distinct eigenvalues $y_\pm$ of $L(x)$ are linearly independent it follows that $W(x)$ vanishes if and only if $x$ corresponds to a branch point, where two columns of $H(x)$ coincide ($\hat{\Sigma}$ is non-singular). Now suppose $x_0$ corresponds to a branch point $P \in \hat{\Sigma}$, and let $z$ be a local parameter on $\hat{\Sigma}$ around $P$. In this coordinate, the covering map $P \mapsto x$ takes the form $x - x_0 = O(z^2)$ near $z = 0$. Also, $\det H(x) = O(z)$ near $z = 0$ and so $W(x) = O(z^2) = O(x - x_0)$, and hence the multiplicity of the zero $x_0$ of $W(x)$ is equal to one, which is also the branching number of the corresponding branch point $P$. So summing over all branch points we have

$$(\# \text{ zeros of } W(x)) = (\text{total branching number of } P \mapsto x) = 2(N + g - 1),$$

where the last equality follows from the Riemann-Hurwitz relation [1.3.5]. But since $W(x)$ is a function meromorphic on the Riemann sphere, it has as many poles as zeroes (counting multiplicities) and the result follows. □
Recall that the present formalism derives from the hierarchy of zero-curvature equations (5.4.15b) which are invariant under gauge transformations (5.4.20). We now make use of this freedom to fix the normalised eigenvectors at infinity. Specifically, we apply the gauge transformation with parameter \( \tilde{g} = H(\infty)^{-1} \) where \( H(x) = (h(x^+), h(x^-)) \). Because eigenvectors of the monodromy matrix transform as \( h \mapsto \tilde{g} h \), we have in the new gauge

\[
h(\infty^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h(\infty^-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(7.1.3)

Notice that this gauge transformation preserves the normalisation of \( h \) because by the special choice \( \alpha = (1, 1) \) of normalisation in (7.1.1) we have \( \alpha H(x) = \alpha \).

**Remark** The gauge fixing condition (7.1.3) imposed so far also fixes part of the global \( SU(2)_R \) symmetry of the equations of motion (since the latter also acts on the eigenvectors as \( h \mapsto \tilde{g} h \)). Specifically, since in this gauge the eigenvectors of the monodromy matrix \( \Omega(x) \) are of the canonical form (7.1.3) it follows from the general form (5.2.7a) of \( \Omega(x) \) at \( x = \infty \) that the right Noether charge \( Q_R \) must be diagonal in this gauge. Thus the gauge fixing condition (7.1.3) restricts the \( SU(2)_R \) current \( j \) to the level set \( Q_R = \frac{1}{2} R \sigma_3 \), thereby breaking \( SU(2)_R \) to a diagonal \( U(1)_R \).

The residual gauge symmetry which preserves (7.1.3) consists of diagonal matrices \( \tilde{g}(t) = \text{diag}(d_1, d_2) \) whose action on the normalised eigenvector \( h \) is

\[
h \mapsto f(P)^{-1} \tilde{g} h,
\]

(7.1.4)

where \( f(P) = \alpha \cdot (\tilde{g} h(P)) = d_1 h_1(P) + d_2 h_2(P) \). The role of the function \( f(P) \) is to keep \( h \) normalised by (7.1.1). Since its poles are the same as those of \( h \) it has the effect of changing the divisor \( \hat{\gamma}(t) \) of poles of \( h \) to the equivalent divisor \( \hat{\gamma}'(t) \)

\(^1\Omega(\infty) \) is clearly invertible because \( \Omega(\infty) - 1 = 0 \). This in turn means that the eigenvectors \( h(\infty^+) \) and \( h(\infty^-) \) are linearly independent which implies \( H(\infty) \) is invertible.
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(\sim \hat{\gamma}(t)) of zeroes of \( f \). Let \([\hat{\gamma}(t)]\) denote the equivalence class of such divisors \( \hat{\gamma}(t) \).

Proposition 7.1.4. There is a 1−1 correspondence between residual gauges and representatives of the equivalence class \([\hat{\gamma}(t)]\).

Proof. A specific representative \( \hat{\gamma}'(t) = \hat{\gamma}_1'(t) + \ldots + \hat{\gamma}_{g+1}'(t) \) of the equivalence class \([\hat{\gamma}(t)]\) is uniquely specified by a single one of its points. Thus it suffices to show that for an arbitrary point \( \hat{\gamma}_1' \in \hat{\Sigma} \) there exists a unique \( \tilde{g} = \text{diag}(d_1, d_2) \) such that \( 0 = f(\hat{\gamma}_1') = d_1 h_1(\hat{\gamma}_1') + d_2 h_2(\hat{\gamma}_1') \). But since we can scale away \( d_1 \) in (7.1.4) this has the unique solution \( d_2 = -h_1(\hat{\gamma}_1')/h_2(\hat{\gamma}_1') \). \(\square\)

From now on we fix the residual gauge by choosing a representative \( \hat{\gamma}(t) \) from the equivalence class \([\hat{\gamma}(t)]\). It follows from proposition 7.1.3 and equation (7.1.3) that the components of the eigenvector \( h \) satisfy the following properties,

\[
(h_1) \geq -\hat{\gamma}(t) + \infty^-, \quad h_1(\infty^+) = 1, \quad (h_2) \geq -\hat{\gamma}(t) + \infty^+, \quad h_2(\infty^-) = 1. \tag{7.1.5}
\]

From now on we assume the divisor \( \hat{\gamma}(t) \) to be in general position. Let \( \gamma_-(t) + \infty^- \) and \( \gamma_+(t) + \infty^+ \) (deg \( \gamma_\pm(t) = g \)) be the divisors of zeroes of \( h_1 \) and \( h_2 \) respectively. We also assume \( \gamma_\pm(t) \) to be non-special, i.e. \( r(-\hat{\gamma}_+(t)) = 1 \), which implies that \( \hat{\gamma}(t) \) is non-special (by the remark following definition 1.5.21). The divisors \( \gamma_\pm(t) \) are defined uniquely by the following equivalence of divisors,

\[
\hat{\gamma}(t) \sim \gamma_\pm(t) + \infty^\pm. \tag{7.1.6}
\]

Proposition 7.1.5. Conditions (7.1.5) uniquely specify a meromorphic vector \( h \).

Proof. Suppose not and let \( h \) and \( h' \) be two vectors satisfying conditions (7.1.5). Consider the meromorphic function \( f_i(P) = h_i(P)/h_i'(P), i = 1, 2 \) of degree \( g \). Its
divisor or poles is contained in the divisor $\gamma'(t)$ or $\gamma'(t)$ of zeroes of $h'_1$ or $h'_2$ which by assumption is in general position. Thus $r(-\gamma'_\pm) = 1$ and $f_i$ must be constant by Riemann-Roch. But $f_1(\infty^+) = f_2(\infty^-) = 1$ so $f_i = 1$ and hence $h = h'$. □

**Remark** Suppose we chose to normalise $h$ by the condition $h_1(P) = 1$ instead of (7.1.2). This corresponds to multiplying the above eigenvector $h$ by $\frac{1}{h_1}$. The second component would then have divisor $\geq -\gamma - (t) - \infty^+ + \infty^+$ now with a pole forced at $\infty^-$. We now show that there exists a pair of functions $h_1, h_2$ which satisfy the conditions (7.1.5). To do this we construct explicit functions on $\hat{\Sigma}$ with the properties (7.1.5) in terms of Riemann $\theta$-functions. By proposition 7.1.5 these constructed functions must therefore be equal to the components of the normalised eigenvector. Proposition 7.1.6 below provides a reconstruction formula for reconstructing the normalised eigenvector $h$ from its analytic data, namely a divisor $\hat{\gamma}(t)$ on the Riemann surface $\hat{\Sigma}$. Let $w_1, w_{g+1}, w_{\infty}^\pm, w^\pm \in \mathbb{C}^g$ be defined as follows,

$$w_1 = \sum_{i=1}^g A(\hat{\gamma}_i(t)) + \mathcal{K}, \quad (7.1.7a)$$

$$w_{g+1} = \sum_{i=2}^{g+1} A(\hat{\gamma}_i(t)) + \mathcal{K}, \quad (7.1.7b)$$

$$w_{\infty}^\pm = A(\infty^\pm) + \sum_{i=2}^g A(\hat{\gamma}_i(t)) + \mathcal{K}, \quad (7.1.7c)$$

$$w^\pm = w_1 + w_{g+1} - w_{\infty}^\pm = A(\hat{\gamma}) - A(\infty^\pm) + \mathcal{K}. \quad (7.1.7d)$$

**Proposition 7.1.6.** The components $h_1, h_2$ of the normalised eigenvector $h$ are given by $h_1(P) = h_-(P)$ and $h_2(P) = h_+(P)$ where

$$h_\pm(P) = \frac{\theta(\mathcal{A}(\infty^\pm) - w_1) \theta(\mathcal{A}(\infty^\mp) - w_{g+1}) \theta(\mathcal{A}(P) - w_\infty^\pm) \theta(\mathcal{A}(P) - w^\pm)}{\theta(\mathcal{A}(\infty^\mp) - w_\infty^\pm) \theta(\mathcal{A}(\infty^\mp) - w^\pm) \theta(\mathcal{A}(P) - w_1) \theta(\mathcal{A}(P) - w_{g+1})}. \quad (7.1.6)$$

**Proof.** The first factor in this formula is merely a constant ensuring $h_\pm(\infty^\pm) = 1$. 

So we need to show that the second factor is a well defined function of \( P \) and has the right divisor. But as \( P \) is taken around a \( b \)-cycle the \( \theta \)-functions change by various factors (1.7.9) which cancel however by (7.1.7d). As a result \( h_\pm(P) \) are well defined.

The two \( \theta \)-functions in the denominator vanish at the points \( \hat{\gamma}_1(t), \ldots, \hat{\gamma}_g(t) \) and \( \hat{\gamma}_2(t), \ldots, \hat{\gamma}_{g+1}(t) \) respectively by corollary 1.7.14. Likewise the first \( \theta \)-function in the numerator vanishes at \( \infty^\pm \) and \( \hat{\gamma}_2(t), \ldots, \hat{\gamma}_g(t) \) but the latter \( g-1 \) zeroes cancel with the same zeroes in the denominator so that \( (h_\pm) \geq -\hat{\gamma}(t) + \infty^\pm \). \( \square \)

## 7.2 Baker-Akhiezer vector and linearisation

Equations (5.4.15) express the fact that the operators \( \partial_t - J_M(x) \) all commute among themselves as well as individually with the monodromy matrix \( \Omega(x) \). This means they can all be simultaneously diagonalised and there exists an eigenvector \( \psi(P,t) \) at every \( P \in \hat{\Sigma} \) with \( \hat{\pi}(P) = x \) which solves the following linear system

\[
\left( \partial_t - J_M(x,t) \right) \psi(P,t) = 0, \quad \forall M.
\]  

(7.2.1)

**Remark** Note that the vector equations (7.2.1) might not have global solution \( \psi \) for topological reasons: if the base space is not simply connected (which is the case here since the \( \sigma \) coordinate on the worldsheet is periodic), then even though the Lax connection \( J(x) \) is flat there are still closed paths with non-trivial holonomy, and hence a covariantly constant vector cannot exist globally on the base space. Thus if \( \psi(P) \) is a local solution on a neighbourhood \( U \) of the base space, then \( \Omega(x)\psi(P) = \Lambda(P)\psi(P) \) describes the same solution on \( U \).

Just as in the case of the normalised eigenvector \( h \) the aim is to identify the analytic properties of the vector \( \psi(P,t) \) which specify it uniquely. Since \( \psi(P,t) \) is
an eigenvector it can be written as a multiple of the normalised eigenvector,

$$\psi(P, t) = \varphi(P, t) h(P, t).$$  \hfill (7.2.2a)

Alternatively we can also write the solution to \((7.2.1)\) in the form

$$\psi(P, t) = \hat{\Psi}(x, t) h(P, 0),$$  \hfill (7.2.2b)

where \(\hat{\Psi}(x, t)\) is a formal solution to the matrix analogue of equation \((7.2.1)\), namely

$$\left(\partial_{t_M} - J_M(x, t)\right) \hat{\Psi}(x, t) = 0, \quad \forall M.$$  \hfill (7.2.3)

Indeed the vector \(\psi\) defined by \((7.2.2b)\) then trivially satisfies \((7.2.1)\). Furthermore, if we fix the initial condition to be \(\psi(P, 0) = h(P, 0)\) so that \(\hat{\Psi}(x, 0) = 1\), then by uniqueness of the solution to \((7.2.3)\) with initial condition \(\Omega(x, 0)\) it follows that \(\hat{\Psi}(x, t)\Omega(x, 0) = \Omega(x, t)\hat{\Psi}(x, t)\) and therefore \((7.2.2b)\) is indeed also an eigenvector of the monodromy matrix \(\Omega(x, t)\) with eigenvalue \(\Lambda(P)\). Having already identified the defining analytic properties of \(h\) we now use \((7.2.2)\) to determine those of \(\psi\).

The hierarchy of Lax matrices can be rewritten in the more transparent form

$$J_{n, \pm}(x) = \left(\hat{\Psi}(x) s_{n, \pm}(x)\sigma_3 \hat{\Psi}(x)^{-1}\right)_{\pm 1}. \hfill (7.2.4)$$

**Definition 7.2.1.** The singular parts \(s_{n, \pm}(x)\) are given by

$$s_{n, \pm}(x) = \left(\frac{i}{\pi} x^2 - 1 \left(\frac{Q_{n-1}^{(\pm)}}{(x \mp 1)^n} + \frac{Q_{-1}^{(\pm)}}{x \mp 1}\right)\right)_{\pm 1}. \hfill (7.2.5)$$

In the particular case of the zeroth level \(n = 0\) where the Lax matrix becomes
the Lax connection \( J_{0,\pm}(x) = J_{\pm}(x) \), the singular parts read

\[
s_{0,\pm}(x) = s_{\pm}(x) = \frac{iK_{\pm}}{1 \mp x}. \tag{7.2.6}
\]

**Lemma 7.2.2.** Let \( \psi(P,t) \) be the eigenvector which solves (7.2.1) with initial condition \( \psi(P,0) = h(P,0) \) then it is meromorphic on \( \hat{\Sigma} \setminus \{(\pm 1)^\pm\} \) with

\[
\begin{align*}
(\psi_1) & \geq -\hat{\gamma}(0) + \infty^-, \quad \psi_1(\infty^+) = 1, \\
(\psi_2) & \geq -\hat{\gamma}(0) + \infty^+, \quad \psi_2(\infty^-) = 1,
\end{align*} \tag{7.2.7a}
\]

and has the following asymptotic behaviour in a neighbourhood of \( (\pm 1)^\pm \in \hat{\Sigma} \),

\[
\begin{cases}
\psi_1(x^\pm, t) e^{\mp \sum_n s_{n,+(x)} t_{n,+}} = O(1), & \text{as } x \to +1, \\
\psi_1(x^\pm, t) e^{\mp \sum_n s_{n,-(x)} t_{n,-}} = O(1), & \text{as } x \to -1.
\end{cases} \tag{7.2.7b}
\]

**Proof.** Because \( J_M(x) \) only has poles at \( x = \pm 1 \) it follows by Poincaré’s theorem on holomorphic differential equations that \( \hat{\Psi}(x,t) \) is holomorphic outside \( x = \pm 1 \) since the initial condition \( \hat{\Psi}(x,0) = 1 \) is \( \hat{\Psi}(x,0) = 1 \) is holomorphic. It directly follows from (7.2.2b) that \( \psi(P,t) \) is meromorphic outside \( \hat{x}^{-1}(\pm 1) \) with poles at \( \hat{\gamma}(0) \). Moreover, using the gauge fixing condition \( J_M(\infty) = 0 \) we observe that \( \partial_{t_m,\pm} \hat{\Psi}(\infty, t) = 0 \) and hence \( \hat{\Psi}(\infty, t) = 1 \) by the choice of initial conditions. Equations (7.2.7a) now follow from (7.1.3) at \( t = 0 \).

Consider now the representation (7.2.2a) of \( \psi(P,t) \) which we can write as

\[
\Psi(x) = H(x) \Phi(x) \quad \text{where } \Psi(x) \text{ and } H(x) \text{ are the matrix of column eigenvectors } \psi \text{ and } h \text{ at } x \text{ respectively and } \Phi(x) = \text{diag}(\varphi(x^+), \varphi(x^-)).
\]

Since \( h \) is holomorphic in a neighbourhood of \( \hat{x}^{-1}(\pm 1) \) this means that \( H(x) \) is holomorphic near \( x = \pm 1 \). Rewriting (7.2.1) as a matrix equation \( \Psi^{-1}(x) \partial_{t_m,\pm} \Psi(x) = \Psi(x)^{-1} J_{m,\pm}(x) \Psi(x) \) we

\[\text{\footnote{The same conclusion does not hold for the vector } \psi(P,t) \text{ even though it satisfies the system (7.2.1). Indeed, we chose its initial condition to be } \psi(P,0) = h(P,0) \text{ which has poles at } \hat{\gamma}(0). \text{ Therefore we conclude that the components of } \psi(P,t) \text{ are holomorphic in } P \text{ away from both } \hat{x}^{-1}(\pm 1) \text{ and the points of } \hat{\gamma}(0).}\]
study it in a neighbourhood of $x = \pm 1$. It can be written as

$$
(\partial_{t_{m, \pm}} \Phi(x)) \Phi(x)^{-1} + H(x)^{-1} \partial_{t_{m, \pm}} H(x) = s_{m, \pm}(x)\sigma_3 + H(x)^{-1}V(x)H(x), \quad (7.2.8)
$$

where we have set $J_{m, \pm}(x) = H(x)s_{m, \pm}(x)\sigma_3 H(x)^{-1} + V(x)$ with $V(x)$ being the negative of the holomorphic part of $H(x)s_{m, \pm}(x)\sigma_3 H(x)^{-1}$ at $x = \pm 1$. The second term on the right hand side is clearly holomorphic at $x = \pm 1$. One can show that the second term on the left hand side also is. For this we need the evolution equation (7.2.15) of the normalised eigenvector that we will derive later in the proof of theorem 7.2.9. It reads in matrix form

$$
H(x)^{-1} \partial_{t_{m, \pm}} H(x) = H(x)^{-1}J_{m, \pm}(x)H(x) - \text{diag}(C(x^+), C(x^-)),
$$

where $C(P) = \alpha \cdot J_{m, \pm}(x)h(P)$. Therefore

$$
\text{diag}(C(x^+), C(x^-)) = s_{m, \pm}(x)\sigma_3 + \text{diag}(\alpha \cdot V(x)h(x^+), \alpha \cdot V(x)h(x^-)),
$$

where the second term is holomorphic at $x = \pm 1$. The first term is singular but cancels with the corresponding term in $H(x)^{-1}J_{m, \pm}(x)H(x) = s_{m, \pm}(x)\sigma_3 + H(x)^{-1}V(x)H(x)$. Hence the second terms in both the left and right hand sides of (7.2.8) are holomorphic at $x = \pm 1$ so that $\varphi(x^\pm)^{-1} \partial_{t_{m, \pm}} \varphi(x^\pm) = \pm s_{m, \pm}(x) + O(1)$ from which (7.2.7b) follows. □

Functions on a Riemann surface $\hat{\Sigma}$ satisfying properties like those in (7.2.7) are known as Baker-Akhiezer functions. They have essential singularities at certain punctures (7.2.7b) generalising the exponential map $z \mapsto \exp z$ which is holomorphic in $\mathbb{C}$ but has an essential singularity at $z = \infty$. Despite the fact that these functions are not meromorphic on $\hat{\Sigma}$ they still admit the notion of a degree since,

Lemma 7.2.3. The Baker-Akhiezer functions $\psi_i$ have an equal number of zeroes
and poles (counting multiplicities).

**Proof.** Consider the differential \( d \log \psi_i = d\psi_i/\psi_i \) on \( \hat{\Sigma} \). It is straightforward to show using property (7.2.7b) that \( d \log \psi_i \) is meromorphic in a neighbourhood of the punctures \( \hat{\pi}^{-1}(\pm 1) \). But since it is also meromorphic away from the punctures on \( \hat{\Sigma} \setminus \hat{\pi}^{-1}(\pm 1) \), \( d \log \psi_i \) defines a meromorphic differential on \( \hat{\Sigma} \). The lemma follows using \( \int_{\partial \hat{\Sigma}_{\text{cut}}} d \log \psi_i = 0 \) and the fact that \( d \log \psi_i \) has no residues at \( \hat{\pi}^{-1}(\pm 1) \). \( \square \)

Since we are assuming \( \hat{\gamma}(t) \) to be non-special the divisor \( \hat{\gamma}(0) \) of poles of \( \psi_i \) is also in general position which allows us to use the Riemann-Roch theorem to prove,

**Proposition 7.2.4.** Conditions (7.2.7) uniquely specify a Baker-Akhiezer vector \( \psi \).

**Proof.** Suppose there are two vectors \( \psi \) and \( \psi' \) satisfying conditions (7.2.7) and consider the function \( f_i(P) \equiv \psi_i(P)/\psi'_i(P) \), \( i = 1, 2 \). Since \( \psi_i \) and \( \psi'_i \) have the same essential singularities (7.2.7b) at \( \hat{\pi}^{-1}(\pm 1) \) they cancel in the definition of \( f_i \) which is therefore meromorphic. Its divisor of poles is contained in the divisor of zeroes of \( \psi'_i \) which is of degree \( g \) and by assumption is in general position. Thus \( f_i \) must be constants which are fixed to one by the conditions \( f_1(\infty^+) = f_2(\infty^-) = 1 \). \( \square \)

It remains to show that there exists a pair of function \( \psi_1, \psi_2 \) which satisfy all the conditions of (7.2.7). Once again existence is shown by explicit construction of such functions using the Riemann \( \theta \)-function as a building block. It follows from proposition 7.2.4 that the functions constructed below must be equal to the components of the Baker-Akhiezer vector \( \psi \) thus providing reconstruction formulae.

The main ingredient of these formulae is a certain normalised Abelian differential of the second kind \( dQ \). We let \( dQ \) have poles at the points \( \hat{\pi}^{-1}(\pm 1) \in \hat{\Sigma} \) with
singular parts defined in terms of (7.2.5) by

\[ dQ = -idS_\pm, \text{ as } x \to \pm 1, \]

where

\[
\begin{aligned}
S_+(x^\pm, t) &= \pm \sum_n s_{n,+}(x)t_{n,+}, \\
S_-(x^\pm, t) &= \pm \sum_n s_{n,-}(x)t_{n,-}.
\end{aligned}
\]

Its regular part is fixed uniquely by the normalisation condition \( \int_{a_i} dQ = 0 \). The \( b \)-periods define a vector in \( \mathbb{C}^g \). As in chapter 1 we denote \( \zeta_D = \mathcal{A}(D) + \mathcal{K} \).

**Proposition 7.2.5.** The components \( \psi_1, \psi_2 \) of the Baker-Akhiezer vector \( \psi \) are given by \( \psi_1(P) = \psi_+(P) \) and \( \psi_2(P) = \psi_-(P) \) where

\[
\psi_\pm(P) = h_\mp(P, 0) \frac{\theta \left( \mathcal{A}(P) + \int_b dQ - \zeta_{\gamma_\mp(0)} \right) \theta \left( \mathcal{A}(\infty^\pm) - \zeta_{\gamma_\mp(0)} \right)}{\theta \left( \mathcal{A}(P) - \zeta_{\gamma_\mp(0)} \right) \theta \left( \mathcal{A}(\infty^\pm) + \int_b dQ - \zeta_{\gamma_\mp(0)} \right)} \exp \left( i \int_{\infty^\pm}^P dQ \right).
\]

**Proof.** Since the \( \theta \)-functions are all holomorphic in a neighbourhood of \( x = \pm 1 \), it follows by definition of \( dQ \) that \( \psi_\pm \) have the right asymptotics (7.2.7b).

Among the four \( \theta \)-functions present only two of them depend on \( P \). The other two merely define overall constants ensuring \( \psi_\pm(\infty^\pm) = 1 \). So focusing on the \( P \) dependence we need to show that \( \psi_\pm(P) \) is a well defined function of \( P \), has the right divisor and the right asymptotics at \( x = \pm 1 \).

When \( P \) is taken around an \( a \)-cycle nothing changes because \( dQ \) is normalised and the \( \theta \)-functions are \( a \)-periodic. As \( P \) goes around the \( b_k \)-cycle the ratio of \( \theta \)-functions gets multiplied by \( \exp \left( -i \int_{b_k} dQ \right) \) which exactly cancels with the shift in the exponential of \( \psi_\pm(P) \), which is therefore well defined.

The \( \theta \)-function in the denominator vanishes at the \( g \) points of \( \gamma_\pm(0) \) which all cancel with the corresponding zeroes of \( h_\mp(P, 0) \) to leave \( (\psi_\pm) \geq -\hat{\gamma}(0) + \infty^\pm \). □

Recall from proposition 7.1.4 that the choice of a dynamical divisor \( \hat{\gamma}(t) \) for the normalised eigenvector \( h \) corresponded to a choice of residual gauge. However the Baker-Akhiezer only depends on the initial value \( \hat{\gamma}(0) \) of the divisor. Thus the choice
of an initial divisor \( \hat{\gamma}(0) \) in the construction of the Baker-Akhiezer vector should correspond to fixing only the constant part of the residual gauge. But the constant part of the residual gauge symmetry (7.1.4) corresponds precisely to the unfixed \( U(1)_R \) subgroup of the global \( SU(2)_R \) (in fact, before imposing reality conditions we are really dealing with a \( \mathbb{C}^* \) subgroup of \( SL(2, \mathbb{C})_R \), therefore

**Proposition 7.2.6.** The choice of an initial divisor in \( [\hat{\gamma}(0)] \) corresponds to a choice of initial value for the \( U(1)_R \) angle.

We can be a bit more specific about this connection between the divisor \( \hat{\gamma}(0) \) and the \( U(1)_R \) angle. Since the Baker-Akhiezer vector is defined as the solution to the linear system (7.2.1) with initial condition \( \psi(P, 0) = h(P, 0) \) it is easy to determine how it transforms under \( U(1)_R \). Indeed, the Lax matrices all transform by conjugation \( J_M(x) \rightarrow \tilde{g} J_M(x) \tilde{g}^{-1} \) where \( \tilde{g} = \text{diag}(W, W^{-1}) \in SL(2, \mathbb{C}) \). The initial condition being the normalised eigenvector it transforms as in (7.1.4), namely \( h(P, 0) \rightarrow f(P, 0)^{-1} \tilde{g} h(P, 0) \) where \( f(P, 0) = Wh_1(P, 0) + W^{-1}h_2(P, 0) \). It follows then that the Baker-Akhiezer vector transforms as \( \psi(P, t) \rightarrow f(P, 0)^{-1} \tilde{g} \psi(P, t) \) or equivalently in terms of the reconstructed components \( \psi_\pm \) of proposition 7.2.5

\[
\psi_\pm(P, t) \rightarrow f(P, 0)^{-1} W^{\pm 1} \psi_\pm(P, t),
\]

where we can write \( f(P, 0) = W \psi_1(P, 0) + W^{-1} \psi_2(P, 0) \). Proposition 7.2.7 below expresses exactly how the parameter \( W \) of a \( U(1)_R \) transformation depends on the two divisors \( \hat{\gamma}(0) \) and \( \hat{\gamma}'(0) \) related through this \( U(1)_R \) transform. We first need to define a normalised Abelian differential of the third kind \( \omega_\infty \) that will be essential in the description of the \( U(1)_R \) degree of freedom. It is defined by the residues \( \pm \frac{1}{2\pi i} \) at its simple poles \( \infty^\pm \in \hat{\Sigma} \). Using the notation of chapter 1 for the basis of normalised
Abelian differentials of the third kind $\omega_{PQ}$ it can also be written more explicitly as

$$\omega_\infty = \frac{1}{2\pi i} \omega_{\infty^+\infty^-}.$$  \hspace{1cm} (7.2.10)

**Proposition 7.2.7.** The $U(1)_R$ transformation $\tilde{g} = \text{diag}(W,W^{-1})$ which takes the initial divisor from $\hat{\gamma}(0)$ to $\hat{\gamma}'(0)$ is given explicitly by

$$W = \exp \frac{i}{2} \left( 2\pi i \sum_{j=1}^{g+1} \int_{\hat{\gamma}_j(0)}^{\hat{\gamma}'_j(0)} \omega_\infty \right).$$

*Proof.* Recall that the function $f(P,0)$ has poles at the initial divisor $\hat{\gamma}(0)$ and its zeroes define the ‘new’ initial divisor $\hat{\gamma}'(0)$. Furthermore it takes the values $f(\infty^\pm) = W^\pm$ at the points $\infty^\pm$. The result is now immediate by lemma 1.5.15. \hspace{1cm} \Box

**Linearisation**

Notice that the hierarchy of times enters linearly in the definition of the Baker-Akhiezer vector $\psi(P,t)$ through the essential singularity, which is a usual trait of finite-gap integration. All the time dependence of the Baker-Akhiezer vector, and hence of the solution, is encoded in the meromorphic differential $dQ$ which is linear in the hierarchy of times. In fact, we can define a differential associated to each time of the hierarchy by writing

$$dQ = \sum_n t_{n,+} d\Omega_{n,+} + \sum_n t_{n,-} d\Omega_{n,-} = \sum_N t_N d\Omega_N,$$

using the multi-index notation, where the normalised Abelian differentials of the second kind $d\Omega_{n,\pm}$ are defined uniquely by their respective behaviours at the points
$x = \pm 1$, namely

\[
d\Omega_{n,+}(x^\pm) = \mp ids_{n,+}(x) \quad \text{as } x \to +1,
\]
\[
d\Omega_{n,-}(x^\pm) = \mp ids_{n,-}(x) \quad \text{as } x \to -1.
\]

(7.2.12)

This correspondence between times of the hierarchy and Abelian differentials on $\hat{\Sigma}$

\[
t_{n,\pm} \mapsto d\Omega_{n,\pm}
\]

is a very general feature of finite-gap integration. In standard terminology one says that the differential couples to the time for obvious reasons from (7.2.11). As we saw in section 5.4 of chapter 5 every Hamiltonian corresponds to a Lax matrix which is responsible for generating the corresponding time in the Lax formalism. Here we see that every Hamiltonian also corresponds to a meromorphic differential on $\hat{\Sigma}$ responsible for generating the corresponding time in the finite-gap language. Notice the splitting between differentials singular at $x = +1$ and those singular at $x = -1$. These are related to left and right movers of the string. For instance, at the zeroth level $n = 0$ we have $\sigma^\pm \equiv \frac{1}{2}(\tau \pm \sigma) = t_{0,\pm}$ and $dq_\pm \equiv dq \pm dp = 2\pi d\Omega_{0,\pm}$, so

\[
t_{0,+}d\Omega_{0,+} + t_{0,-}d\Omega_{0,-} = \frac{1}{2\pi}(\sigma dp + \tau dq).
\]

The normalised Abelian differential $dp = \pi d\Omega_{0,+} - \pi d\Omega_{0,-}$ is nothing but the differential of the quasi-momentum defined by its asymptotics in (6.3.3). We see here that it couples to the worldsheet spatial coordinate $\sigma$ which justifies the nomenclature ‘quasi-momentum’ for its Abelian integral. The differential $dq = \pi d\Omega_{0,+} + \pi d\Omega_{0,-}$ on the other hand couples to the worldsheet time coordinate $\tau$ suggesting that,

**Definition 7.2.8.** The quasi-energy is the Abelian integral $q(P) = \int^P dq$.

Its differential $dq$ is the unique normalised Abelian differential of the second kind
defined by the following asymptotics,

\[ dq(x^\pm) = \mp d \left( \frac{\pi \kappa_+}{x-1} \right) + O \left( (x-1)^2 \right), \quad \text{as } x \to +1, \]
\[ dq(x^\pm) = \pm d \left( \frac{\pi \kappa_-}{x+1} \right) + O \left( (x+1)^2 \right), \quad \text{as } x \to -1. \]  

(7.2.13)

The linear time-dependence of the singular parts (i.e. of the exponents of the Baker-Akhiezer vector) has the profound consequence that the motion of the system can be mapped to a linear motion in an appropriate space, which is characteristic of all integrable systems. This is the statement of theorem 7.2.9 below. Before we can state the theorem we need to introduce some notation. It is evident from proposition 7.2.7 that the points \( \infty^\pm \) will play a particular role in characterising the \( U(1)_R \) degree of freedom. In particular the differential (7.2.10) plays an essential part. As in chapter 1 we therefore introduce a modulus

\[ m = \infty^+ + \infty^- \]

(which is an integral divisor) to encapsulate these special points at infinity. The generalised Jacobian \( J_m(\hat{\Sigma}) \) (sometimes also denoted \( J(\hat{\Sigma}, \infty^\pm) \)) relative to this modulus was defined in chapter 1 as well. It can be understood as the Jacobian associated to the singular algebraic curve obtained by identifying the points \( \infty^\pm \) on \( \hat{\Sigma} \). Besides the \( g \) canonical \( b \)-cycles we introduce a degenerate \( b_\infty \)-cycle starting at \( \infty^- \) and ending at \( \infty^+ \) and combine these into a \((g+1)\)-dimensional vector \( \vec{b} = (b_1, \ldots, b_g, b_\infty)^T \).

Following definition 1.7.8 of chapter 1 we also introduce the generalised Abel map \( \vec{A}(P) = 2\pi \int_{P_0}^P \vec{\omega} \) where here \( \vec{\omega} = (\omega_1, \ldots, \omega_g, \omega_\infty)^T \). Note that here we let the extra \( b \)-period \( b_\infty \) and the third kind Abelian differential \( \omega_\infty \) be the \((g+1)\)th component and not the 0th, just for notational convenience. Recall from chapter 1 that the generalised Jacobian is isomorphic via the generalised Abel map to the generalised
Picard group of degree zero divisors on $\hat{\Sigma} \setminus \{\infty^\pm\}$ modulo $m$-equivalence. Thus the divisor $\hat{\gamma}(t) - \hat{\gamma}(0)$ represents a point in $J_m(\hat{\Sigma})$ which by the following theorem has the amazing property that its motion is linear on $J_m(\hat{\Sigma})$. The quite lengthy proof is an adaptation of that in [67, pp.142–145] to include the $U(1)_R$ degree of freedom which as we have already know corresponds to a choice of divisor in the class $[\hat{\gamma}(0)]$.

**Theorem 7.2.9.** The motion of the dynamical divisor $\hat{\gamma}(t)$ on $\hat{\Sigma}$ is mapped by the generalised Abel map $\vec{A}$ to a linear motion on the generalised Jacobian $J_m(\hat{\Sigma})$,

$$\vec{A}(\hat{\gamma}(t)) = \vec{A}(\hat{\gamma}(0)) - \int_B dQ.$$  \hfill (7.2.14)

**Proof.** Consider the equation $\Omega(x)h(P) = \Lambda(P)h(P)$ for the normalised eigenvector. Differentiating this equation with respect to the higher time $t_N$ (with $N = (n, s)$ where $n \in \mathbb{N}$ and $s = \pm 1$) and using the evolution equation (3.4.15a) for the monodromy matrix we find

$$(\Omega(x) - \Lambda(P)) (\partial_{t_N} h(P) - J_N(x)h(P)) = 0.$$  

It follows then by uniqueness of the eigenvector at each point $P \in \hat{\Sigma}$ that

$$\partial_{t_N} h(P) = [J_N(x) - C(P)] h(P),$$  \hfill (7.2.15)

for some scalar function $C(P, t) \in \mathbb{C}$. Using the fact that the eigenvector $h(P)$ is normalised by the condition (7.1.1) we obtain an expression for this scalar, namely $C(P) = \alpha \cdot J_N(x)h(P)$. Next we introduce the following function depending on a small time difference $\delta t$,

$$\mathcal{N}(t, \delta t, P) = 1 + \delta t C(P, t) = 1 + \delta t \alpha \cdot J_N(x)h(P).$$  \hfill (7.2.16)

Working to first order in $\delta t$ one can then rewrite equation (7.2.15) in terms of this
function as follows,

\[ \mathcal{N}(t, \delta t, P) h(P, t + \delta t) = (1 + \delta t J_N(x)) h(P, t) + O(\delta t^2). \]

(7.2.17)

This relation allows us to read off the pole structure of \( \mathcal{N} \). Indeed, the right hand side of (7.2.17) has simple poles at \( \hat{\gamma}(t) \) from \( h(P, t + \delta t) \) and poles of order \( n+1 \) at \( s^+, s^- \) from \( J_N(x) = J_{n,s}(x) \). Since the left hand side must have the same poles this implies that the function \( \mathcal{N} \) must have simple poles at \( \hat{\gamma}(t) \) as well as poles of order \( n+1 \) at \( s^+, s^- \). Furthermore, to cancel off the undesired poles at \( \hat{\gamma}(t + \delta t) \) coming from \( h(P, t + \delta t) \) the function \( \mathcal{N} \) must also have simple zeroes at \( \hat{\gamma}(t + \delta t) \). We denote its remaining \( 2n + 2 \) zeroes as \( s^\pm_\alpha(t), \alpha = 1, \ldots, n + 1 \). By continuity, as \( \delta t \to 0 \) the zeroes \( s^\pm_\alpha(t) \) must converge to the poles \( s^\pm \) respectively since \( \mathcal{N}(t, 0, P) = 1 \). Moreover we also note from (7.2.16) that at both points \( \infty^\pm \) the function \( \mathcal{N} \) takes the value one (since \( J_N(\infty) = 0 \) and the components of \( h \) are regular at \( \infty^\pm \)). Thus

\[ (\mathcal{N}) = \hat{\gamma}(t + \delta t) - \hat{\gamma}(t) + \sum_{\alpha=1}^{n+1} s^+_\alpha(t) + \sum_{\alpha=1}^{n+1} s^-_\alpha(t) - (n + 1)s^+ - (n + 1)s^- , \]

\[ \mathcal{N}(t, \delta t, \infty^+) = \mathcal{N}(t, \delta t, \infty^-) = 1. \]

It now follows by the generalised Abel theorem [1.7.9] that

\[ \sum_{j=1}^{g+1} \int_{\hat{\gamma}_j(t)}^{\hat{\gamma}_j(t+\delta t)} \vec{\omega} = -\sum_{\alpha=1}^{n+1} \int_{s^+}^{s^+_\alpha(t)} \vec{\omega} - \sum_{\alpha=1}^{n+1} \int_{s^-}^{s^-_\alpha(t)} \vec{\omega}. \]

The left hand sides of the above equations multiplied by \( (\delta t)^{-1} \) tend to the time-derivative of the generalised Abel map of the divisor \( \hat{\gamma}(t) \) in the limit \( \delta t \to 0 \). To show (7.2.14) we therefore compute the right hand sides in this limit. We take \( \delta t \) sufficiently small so that all the zeroes \( s^\pm_\alpha \) are within a small neighbourhood \( U^\pm \) of the corresponding poles \( s^\pm \). Note that the differentials \( \vec{\omega} \) are all holomorphic in the
neighbourhoods $U^\pm$. Now letting $\vec{\sigma}^\pm(P) = \int_{s^\pm} \vec{\omega}$ be the local integral of $\vec{\omega}$ in $U^\pm$,

$$\frac{1}{\delta t} \sum_{\alpha=1}^{n+1} \int_{s^\pm} s^\pm_\alpha(t) \vec{\omega} = \frac{1}{\delta t} \sum_{\alpha=1}^{n+1} \vec{\sigma}^\pm(s^\pm_\alpha(t)) = \frac{1}{2\pi i} \int_{\partial U^\pm} \vec{\sigma}^\pm dC,$$

using the fact that the zeroes $s^\pm_\alpha(t)$ satisfy $C(s^\pm_\alpha, t) + (\delta t)^{-1} = 0$ and $\vec{\sigma}^\pm(s^\pm) = 0$ so that within the neighbourhood $U^\pm$ only the zeroes $s^\pm_\alpha(t)$ contribute. In the limit $\delta t \to 0$ all the zeroes tend to the single point $s^\pm$ so that

$$\lim_{\delta t \to 0} \frac{1}{\delta t} \sum_{\alpha=1}^{n+1} \int_{s^\pm} s^\pm_\alpha(t) \vec{\omega} = \text{res}_{s^\pm} \left( \vec{\sigma}^\pm dC \right) = -\text{res}_{s^\pm} \left( C\vec{\omega} \right).$$

These last residues can be computed explicitly using the definition of the function $C(P) = \alpha \cdot J_{n,s}(x) h(P)$ and the Lax matrices $J_{n,s} = (H(x)s_{n,s}(x)\sigma_3 H(x)^{-1})$. When computing the residue at $x = s$ one need not take the pole part in the expression for the Lax matrix. Thus we can write $J_{n,s} = H(x)s_{n,s}(x)\sigma_3 H(x)^{-1}$ and

$$\text{res}_{s^\pm} \left( C\vec{\omega} \right) = \text{res}_{s^\pm} \left( \alpha \cdot H(x)s_{n,s}(x)\sigma_3 H(x)^{-1} h(P) \vec{\omega} \right) = \pm \text{res}_{s^\pm} \left( s_{n,s}(x) \vec{\omega} \right).$$

Here we have also made use of the definition $H(x) = (h(x^+), h(x^-))$. This last expression can be rewritten in terms of the Abelian integral of the differentials $d\Omega_{n,s}$, namely $\pm \text{res}_{s^\pm} \left( s_{n,s}(x) \vec{\omega} \right) = i \text{res}_{s^\pm} \left( \Omega_{n,s}(P) \vec{\omega} \right)$. Finally we arrive at the following simple expressions for the time-derivatives of the generalised Abel maps $\vec{A}(\hat{\gamma}(t))$,

$$\frac{\partial}{\partial t_{n,s}} \vec{A}(\hat{\gamma}(t)) = 2\pi i (\text{res}_{s^+} + \text{res}_{s^-}) \Omega_{n,s}(P) \vec{\omega}. \quad (7.2.18)$$

Notice first of all that the left hand side is independent of the higher times $\{t\}$ and hence the dynamics of $\hat{\gamma}(t)$ is mapped to a linear flow under the generalised Abel map. Considering the first $g$ components of (7.2.18) and using the Riemann bilinear
identity (1.5.9) with \( d\Omega_1 = d\Omega_{n,s} \) and \( d\Omega_2 = \omega_i \) we find that

\[
2\pi i (\text{res}_s^+ + \text{res}_s^-)\Omega_{n,s}(P)\omega_i = -\int_{b_i} d\Omega_{n,s}.
\]

Finally for the \((g+1)^{\text{st}}\) component we use again the Riemann bilinear identity (1.5.9) but with \( d\Omega_1 = d\Omega_{n,s} \) and \( d\Omega_2 = \omega_\infty \) which reads

\[
2\pi i (\text{res}_s^+ + \text{res}_s^-)\Omega_{n,s}(P)\omega_\infty = -2\pi i (\text{res}_\infty^+ + \text{res}_\infty^-)\Omega_{n,s}(P)\omega_\infty.
\]

The left hand side is easily evaluated using the definition \( \omega_\infty = \frac{1}{2\pi i} \omega_\infty^+ + \omega_\infty^- \) to give

\[
-2\pi i (\text{res}_\infty^+ + \text{res}_\infty^-)\Omega_{n,s}(P)\omega_\infty = -\left(\Omega_{n,s}(\infty^+) - \Omega_{n,s}(\infty^-)\right) = -\int_{b_\infty} d\Omega_{n,s}
\]

and the theorem is proved. \(\square\)

### 7.3 The dual linear system

We now introduce the concept of the dual normalised eigenvector \( h^+ \) and the dual Baker-Akhiezer vector \( \psi^+ \). The purpose of these vectors is two-fold. First of all they will provide useful formulae for the inverses \( H(x)^{-1} \) and \( \Psi(x)^{-1} \) of the matrices \( H(x) = (h(x^+), h(x^-)) \) and \( \Psi(x) = (\psi(x^+), \psi(x^-)) \) which appear in most of the reconstruction formulae such as (7.0.2). Secondly the dual Baker-Akhiezer vector will be very useful in discussing reality conditions in chapter 9. Since the matrix \( g \) defined in (3.1.5) is \( SU(2) \)-valued \( g^\dagger = g^{-1} \) its inverse will be expressible in terms of the dual Baker-Akhiezer vector.

The dual vectors \( h^+ \) and \( \psi^+ \) are defined in essentially the same way as their usual counterparts \( h \) and \( \psi \) except that they are taken to be left eigenvectors of the
monodromy matrix $\Omega(x)$ as opposed to right eigenvectors. Specifically we have,

$$h^+(P,t) (\Omega(x,t) - \Lambda(P)1) = 0.$$ 

and $\psi^+(P,t) = \varphi^+(P,t)h^+(P,t)$. They are both row vectors and $h^+$ can be normalised by the condition $h^+ \cdot \alpha^T = 1$. The reason these dual eigenvectors provide formulae for $H(x)^{-1}$ and $\Psi(x)^{-1}$ respectively essentially boils down to,

**Lemma 7.3.1.** $\forall P \in \hat{\Sigma}, \quad h^+(P) \cdot h(\hat{\sigma}P) = 0$.

**Proof.** Recall that $\hat{\sigma}$ denotes the hyperelliptic involution. If $P$ corresponds to a point of the spectral curve for which $\Lambda(P) \neq \Lambda(\hat{\sigma}P)$ then

$$\Lambda(\hat{\sigma}P)h^+(P) \cdot h(\hat{\sigma}P) = h^+(P)\Omega(x)h(\hat{\sigma}P) = \Lambda(P)h^+(P) \cdot h(\hat{\sigma}P).$$

Thus the result holds at such points and remains true for all $P \in \hat{\Sigma}$ by continuity. □

**Remark** Note in particular that since a branch point $Q$ of $\hat{\Sigma}$ is a fixed point of the hyperelliptic involution $\hat{\sigma}$, namely $\hat{\sigma}Q = Q$, it follows that $h^+(Q) \cdot h(Q) = 0$.

If we define the meromorphic function $\eta(P) = h^+(P) \cdot h(P)$ then the row vector $H^+(P) = \eta(P)^{-1}h^+(P)$ satisfies the following orthogonality conditions with the normalised eigenvector $h$,

$$H^+(P) \cdot h(P) = 1, \quad H^+(P) \cdot h(\hat{\sigma}P) = 0. \quad (7.3.1)$$

Since by definition the matrix $H(x)$ is built out of the column vector $h(P)$ it follows from (7.3.1) that its inverse $H(x)^{-1}$ can be constructed using the row vector $H^+(P)$. Specifically we have proved the following

**Proposition 7.3.2.** $H(x)^{-1} = (H^+(x^+)^T, H^+(x^-)^T)^T$. 
We would like to obtain the analytic properties of the dual vectors $h^+(P)$ and $\psi^+(P)$ on the Riemann surface $\hat{\Sigma}$ so that they may also be reconstructed from a set of algebro-geometric data. Let us begin with the dual normalised eigenvector $h^+$.

The dual normalised eigenvector

We can extract the algebro-geometric data of the dual normalised eigenvector $h^+$ in a similar way to section 7.1 for the normalised eigenvector $h$. It is straightforward to see that lemma 7.1.1 and proposition 7.1.3 both remain true for $h^+$. We therefore define the dual dynamical divisor $\hat{\gamma}^+(t)$ to be the divisor of poles of $h^+(P,t)$ whose degree is again $\deg \hat{\gamma}^+(t) = g + 1$. Its equivalence class $[\hat{\gamma}^+(t)]$ is conveniently characterised by the following,

**Lemma 7.3.3.** Let $\Omega$ be a meromorphic differential with double poles at $\infty^\pm$ and zeroes at $\hat{\gamma}(t)$. Its $g + 1$ remaining zeroes are equivalent to the divisor $\hat{\gamma}^+(t)$ whose image under the generalised Abel map satisfies (where $B$ denotes the divisor of branch points of $\hat{\Sigma}$)

$$\tilde{A}(\hat{\gamma}(t)) + \tilde{A}(\hat{\gamma}^+(t)) = \tilde{A}(B).$$

(7.3.2)

**Proof.** Equation (7.3.2) easily follows from consideration of the function $\eta(P)$ which has poles at $\hat{\gamma}(t) + \hat{\gamma}^+(t)$, zeroes at the branch points of $\hat{\Sigma}$ and satisfies $\eta(\infty^\pm) = 1$.

Now consider the differential $\tilde{\Omega} = \eta(P)^{-1}dx$. It is easy to show that $dx$ has zeroes at the branch points and double poles at $\infty^\pm$. Thus $\tilde{\Omega}$ is of the form prescribed by the lemma with double poles at $\infty^\pm$ and zeroes at $\hat{\gamma}(t) + \hat{\gamma}^+(t)$.

Now let $\Omega$ be any other differential with double poles at $\infty^\pm$, zeroes at $\hat{\gamma}(t)$ and some other $g + 1$ zeroes at $\hat{\gamma}^+(t)$. Then $\Omega/\tilde{\Omega}$ is a meromorphic function with divisor $\hat{\gamma}^+(t) - \hat{\gamma}^+(t)$ which gives the required equivalence $\hat{\gamma}^+(t) \sim \hat{\gamma}^+(t)$. □

There is however one notable difference with the procedure of section 7.1 for
extracting the analytic data of the normalised eigenvector $h$. In that section we already exploited the gauge freedom of the zero-curvature equations which by now is completely fixed. Indeed in (7.1.3) we had used a gauge transformation to set the normalised eigenvectors at $x = \infty$ equal to the canonical basis, and then we used the residual gauge symmetry to pick a particular divisor $\hat{\gamma}(t)$ from the equivalence class $[\hat{\gamma}(t)]$. Thus when determining the analytic properties of the dual normalised eigenvector $h^+$ there is no longer any gauge freedom to exploit and we must stick to the gauge conditions used up to this point.

However, we know from lemma 7.3.1 that for instance $h^+(\infty^+)$ should be orthogonal to $h(\infty^-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and should be normalised by the condition $h^+ \cdot \alpha^T = 1$. From these conditions and the corresponding conditions on $h^+(\infty^-)$ we conclude
\begin{equation}
    h^+(\infty^+) = (1, 0), \quad h^+(\infty^-) = (0, 1).
\end{equation}

Therefore the gauge transformation that brought the normalised eigenvectors $h(\infty^\pm)$ to the canonical form (7.1.3) at the same time puts the dual normalised eigenvectors $h^+(\infty^\pm)$ in the desired form (7.3.3).

As in the case of the normalised eigenvector, equation (7.3.3) is invariant under residual gauge transformations which applied to dual vectors looks like,
\begin{equation}
    h^+ \mapsto f(P)^{-1}h^+ \tilde{g},
\end{equation}
where $\tilde{g} = \text{diag}(d_1, d_2)$ is diagonal and $f(P) = (h^+(P)\tilde{g}) \cdot \alpha^T = d_1 h_1^+(P) + d_2 h_2^+(P)$. It has the effect of swapping the pole divisor $\hat{\gamma}^+(t)$ of $h^+$ for an equivalent divisor $\hat{\gamma}'^+(t) \sim \hat{\gamma}^+(t)$. The difference now is that this residual gauge invariance (7.1.4) has already been used on the normalised eigenvector to pick its divisor of poles $\hat{\gamma}(t)$ from the equivalence class $[\hat{\gamma}(t)]$. Hence there is no freedom left to move around the dual dynamical divisor $\hat{\gamma}^+(t)$ in the equivalence class $[\hat{\gamma}^+(t)]$. Indeed changing $\hat{\gamma}^+(t)$ is
equivalent to multiplying the dual normalised eigenvector $h^+$ by a diagonal $\tilde{g}$ which will affect the orthogonality condition of lemma 7.3.1.

Nevertheless, let $\hat{\gamma}^+(t)$ be any divisor in the equivalence class $[\hat{\gamma}^+(t)]$. The corresponding dual normalised eigenvector $\hat{h}^+$ is likely to be expressed in the ‘wrong’ residual gauge and needs to be transformed by (7.3.4) so as to satisfy lemma 7.3.1.

**Proposition 7.3.4.** The rows $H^+(P)$ of the inverse matrix $H(x)^{-1}$ are given by

$$H^+_1(P) = \chi(P)\hat{h}^+_1(P), \quad H^+_2(P) = \frac{\chi(P)}{\chi(\infty^-)}\hat{h}^+_2(P), \quad (7.3.5)$$

where $\chi$ is the meromorphic function with zeroes at $\hat{\gamma}(t) + \hat{\gamma}^+(t)$, poles at the branch points and normalised by $\chi(\infty^+) = 1$.

**Proof.** Applying a residual gauge transformation (7.3.4) to $\tilde{h}^+$, the new normalised eigenvector $h^+ = f(P)^{-1}\tilde{h}^+\tilde{g}$ should satisfy the orthogonality condition (7.3.1). But this condition is equivalent to the statement of proposition 7.3.2 that $\eta(P)^{-1}h^+(P)$ constitutes the rows of the left inverse of the matrix $H(x)$. Since the left inverse is equal to the right inverse for finite dimensional matrices we also have

$$\sum_{P \in \hat{\pi}^{-1}(x)} \eta(P)^{-1}h^+_i(P)h_j(P) = \delta_{ij}. \quad (7.3.6)$$

Written in terms of the components of $\tilde{h}^+$ this condition reads

$$\sum_{P \in \hat{\pi}^{-1}(x)} \chi(P)d_i\tilde{h}^+_i(P)h_j(P) = \delta_{ij}, \quad (7.3.6)$$

where $\chi(P) = (\eta(P)f(P))^{-1}$. The parameters $d_i$ of the residual gauge transformation $\tilde{g} = \text{diag}(d_1, d_2)$ can now be deduced from (7.3.6) by taking the $x \to \infty$ limit. In particular since $\hat{h}^+$ also satisfies (7.3.3) one finds $d_1 = 1/\chi(\infty^+)$ and $d_2 = 1/\chi(\infty^-)$. 

Now since \((\eta) = B - \hat{\gamma}(t) - \hat{\gamma}'(t)\) and \((f) = \hat{\gamma}'(t) - \hat{\gamma}(t)\) where \(B\) is the divisor of branch points and \(\hat{\gamma}'(t)\) the pole divisor of \(h^+\), we deduce that
\[(\chi) = \hat{\gamma}(t) + \hat{\gamma}(t) - B.\]
Normalising \(\chi\) such that \(\chi(\infty^+) = 1\) we find \((7.3.5)\). □

**Remark** The factor of \(1/\chi(\infty^-)\) in the second component of \((7.3.5)\) corresponds to the residual gauge transformation required to turn \(\tilde{h}^+\) into the correct eigenvector \(h^+\) satisfying \((7.3.1)\).

Thus the upshot of proposition \((7.3.4)\) is that we may pick any divisor \(\hat{\gamma}(t)\) from the equivalence class \([\hat{\gamma}(t)]\) to be the dual dynamical divisor. The corresponding dual normalised eigenvector \(h^+\) then needs to be adjusted by a residual gauge transformation, determined by proposition \((7.3.4)\), before it can provide the rows of the inverse matrix \(H(x)^{-1}\).

After choosing a divisor \(\hat{\gamma}(t)\) from the equivalence class \([\hat{\gamma}(t)]\) it follows from the analogue of proposition \((7.1.3)\) for \(\tilde{h}^+\) and equation \((7.3.3)\) that the components of \(\tilde{h}^+\) satisfy the following properties,
\[
\begin{align*}
(\tilde{h}_1^+) &\geq -\hat{\gamma}(t) + \infty^- , \quad \tilde{h}_1^+(\infty^+) = 1 , \\
(\tilde{h}_2^+) &\geq -\hat{\gamma}(t) + \infty^+ , \quad \tilde{h}_2^+(\infty^-) = 1.
\end{align*}
\]

(7.3.7)

The remainder of the analysis of the dual eigenvector \(\tilde{h}^+\) is now identical to that of the eigenvector \(h^+\) but with \(\hat{\gamma}(t)\) replaced everywhere by \(\hat{\gamma}(t)\). In particular, proposition \((7.1.5)\) says that the conditions \((7.3.7)\) uniquely specify \(\tilde{h}^+\) and an analogous reconstruction formula as in proposition \((7.1.6)\) can be obtained for this vector.

Specifically, defining the vectors \(v_1, v_{g+1}, v_{\infty}^\pm, v^\pm \in \mathbb{C}^g\) as follows,
\[
\begin{align*}
v_1 & = \sum_{i=1}^g \mathcal{A}(\hat{\gamma}_i^+(t)) + \mathcal{K}, \quad v_{g+1} = \sum_{i=2}^{g+1} \mathcal{A}(\hat{\gamma}_i^+(t)) + \mathcal{K}, \\
v_{\infty}^\pm & = \mathcal{A}(\infty^\pm) + \sum_{i=2}^g \mathcal{A}(\hat{\gamma}_i^+(t)) + \mathcal{K}, \quad v^\pm = v_1 + v_{g+1} - v_{\infty}^\pm,
\end{align*}
\]

**Proposition 7.3.5.** The components \(\tilde{h}_1^+, \tilde{h}_2^+\) of the dual normalised eigenvector \(\tilde{h}^+\)
are given by $\tilde{h}_1^+(P) = k_-(P)$ and $\tilde{h}_2^+(P) = k_+(P)$ where

$$k_\pm(P) = \frac{\theta(\mathcal{A}(\infty^\mp) - v_1) \theta(\mathcal{A}(\infty^\mp) - v_{g+1})}{\theta(\mathcal{A}(\infty^\mp) - v_1^+) \theta(\mathcal{A}(\infty^\mp) - v_{g+1}^+)}. \theta(\mathcal{A}(P) - v_1^+) \theta(\mathcal{A}(P) - v_{g+1}^+).$$

The dual Baker-Akhiezer vector

We now wish to obtain a formula for the inverse of the matrix $\Psi(x)$ constructed from the Baker-Akhiezer vector $\psi(P,t)$. Since $\psi$ satisfies the linear system (7.2.1) it follows that $\Psi(x)$ satisfies the matrix analogue $(\partial_t M - J_M(x)) \Psi(x) = 0, \forall M$. The inverse matrix then solves the dual linear system

$$\partial_t M \Psi(x)^{-1} + \Psi(x)^{-1} J_M(x) = 0, \forall M. \quad (7.3.9)$$

Thus if $\Psi(x)^{-1}$ is to be built out of row vectors $\Psi^+(P)$ these should satisfy the analogue of this equation for row vectors, namely

$$\partial_t M \Psi^+(P) + \Psi^+(P) J_M(x) = 0, \forall M. \quad (7.3.10)$$

**Proposition 7.3.6.** Let $\Psi^+(P,t)$ be the row vector solution to (7.3.10) with initial condition $\Psi^+(P,0) = H^+(P,0)$ then its components can be written as

$$\Psi_1^+(P) = \chi_0(P) \tilde{\psi}_1^+(P), \quad \Psi_2^+(P) = \frac{\chi_0(P)}{\chi_0(\infty^-)} \tilde{\psi}_2^+(P), \quad (7.3.11)$$

where $\chi_0$ is the function $\chi$ taken at $t = 0$. Moreover, the functions $\tilde{\psi}^+_i$ are meromorphic on $\hat{\Sigma} \setminus \{(+1)^\pm\}$ with

$$\tilde{\psi}_1^+(\infty^-) = 1, \quad \tilde{\psi}_2^+(\infty^-) = 1, \quad (7.3.12a)$$
and have the following asymptotic behaviour in a neighbourhood of $(\pm 1) \in \hat{\Sigma}$,

\[
\begin{cases}
\tilde{\psi}^+_1(x^+, t)e^{\pm \sum_n s_{n,+}(x) t_{n,+}} = O(1), & \text{as } x \to +1, \\
\tilde{\psi}^+_1(x^+, t)e^{\pm \sum_n s_{n,-}(x) t_{n,-}} = O(1), & \text{as } x \to -1.
\end{cases}
\]  

(7.3.12b)

**Proof.** Let $\hat{\Psi}(x)$ be the formal matrix solution of the linear system (7.2.3) with initial condition $\hat{\Psi}(x, 0) = 1$. It follows that $\hat{\Psi}(x)^{-1}$ is a formal matrix solution to (7.3.9) with the same initial condition. We may then write the solution to (7.3.10) with the initial condition $\Psi^+_1(P, 0) = \Phi^+_1(P, 0)$ as $\hat{\Psi}^+(P, t) = H^+(P, 0)\hat{\Psi}(x, t)^{-1}$. Taking (7.3.11) as defining the functions $\tilde{\psi}^+_1$ and using (7.3.5) this can be rewritten

\[
\begin{pmatrix}
\tilde{h}^+_1(P, 0), \\
\chi_0(\infty^-)\tilde{h}^+_2(P, 0)
\end{pmatrix} = \begin{pmatrix}
\hat{\Psi}(x, t)^{-1}.
\end{pmatrix}
\]  

(7.3.13)

Now since $\hat{\Psi}(x, t)$ is holomorphic outside $x = \pm 1$ it follows from (7.3.13) that $\tilde{\psi}^+(P, t)$ is meromorphic outside $\hat{x}^{-1}(\pm 1)$ with the same pole divisor as $\hat{h}^+(P, 0)$, namely $\hat{x}^+(0)$. Moreover, since $\hat{\Psi}(\infty, t) = 1$ we have $\tilde{\psi}^+_1(\infty, t) = h^+_1(\infty, 0)$ and equations (7.3.12a) readily follow from (7.3.7) by setting $t = 0$.

Recall the matrix equation $\Psi(x) = H(x)\Phi(x)$ which was used in the proof of lemma (7.2.2) where $\Phi(x) = \text{diag}(\varphi(x^+), \varphi(x^-))$. We are now interested in the inverse matrices, namely $\Psi(x)^{-1} = \Phi(x)^{-1}H(x)^{-1}$. But this immediately shows that the singular parts of $\tilde{\psi}^+$, encoded in $\Phi(x)^{-1}$, are opposite to those of $\psi$, which were encoded in $\Phi(x)$, and (7.3.12b) follows.  

□

The conditions (7.3.12) are those of a Baker-Akhiezer vector (with respect to different data) and just as in proposition (7.2.4) they uniquely specify the vector $\tilde{\psi}^+$. This vector will be called the **dual Baker-Akhiezer vector**. One can also write down explicit formulae in terms of Riemann $\theta$-functions which satisfy (7.3.12), giving rise to reconstruction formulae for the components of $\tilde{\psi}^+$. Specifically, defining the
divisors $\delta_\pm(t)$ by the following equivalence

$$\hat{\gamma}^+(t) \sim \delta_\pm(t) + \infty^\pm,$$  \hspace{1cm} (7.3.14)

then the analogue of proposition 7.2.5 is obtained simply by making the replacements $h_\pm \to k_\pm$, $\gamma_\pm \to \delta_\pm$ and $dQ \to -dQ$. The result is the following,

**Proposition 7.3.7.** The components $\tilde{\psi}_1^+$, $\tilde{\psi}_2^+$ of the dual Baker-Akhiezer vector $\tilde{\psi}^+$ are given by $\tilde{\psi}_1^+(P) = \phi_+(P)$ and $\tilde{\psi}_2^+(P) = \phi_-(P)$ where

$$\phi_\pm(P) = k_\mp(P,0) \frac{\theta(\mathcal{A}(P) - \int_b dQ - \zeta_\delta(0)) \theta(\mathcal{A}(\infty^\pm) - \zeta_\delta(0))}{\theta(\mathcal{A}(P) - \zeta_\delta(0)) \theta(\mathcal{A}(\infty^\pm) - \int_b dQ - \zeta_\delta(0))} \exp\left(-i\int_{\infty^\pm}^P dQ\right).$$

### 7.4 Reconstruction formulae

**The $SL(2,\mathbb{C})_R$ current $j$**

The Lax connection $J(x)$ can be reconstructed from the formula

$$J(x) = d\Psi(x)\Psi(x)^{-1},$$  \hspace{1cm} (7.4.1)

where $\Psi(x) = (\psi(x^+), \psi(x^-))$ is the matrix of Baker-Akhiezer column vectors $\psi$ above $x$. However, in order to obtain expressions for the components $j_0, j_1$ of the current $j$ we must first show that the reconstructed Lax connection (7.4.1) takes the original form (5.1.6) for some current $j$. This is the content of theorem 7.3.1 below. The crux of the proof is a standard argument based on the uniqueness of the Baker-Akhiezer vector (see for instance [82, pp.93–94]).

**Remark** Note that even though the definition of $\Psi(x)$ depends on the order of the rows (so $\Psi(x)$ isn’t a properly defined function of $x$), the definitions (7.4.1) of $J(x)$ in terms of this matrix do not depend of the ordering of its columns and therefore the connection $J(x)$ obtained
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this way is a well defined function of the spectral parameter $x$.

**Remark** It was noted that a solution $\psi(P)$ to the auxiliary linear system (7.2.1) is determined locally only up to a power of the eigenvalue $\Lambda(P)$ of $\Omega(x)$ so that $\Psi(x)$ is also determined locally only up to right multiplication by a diagonal matrix $\text{diag}(\Lambda(x)^n, \Lambda(x)^{-n})$. But this constant right diagonal matrix cancels out in the definitions (7.4.1) of the Lax connection in terms of $\Psi(x)$ so that $J(x)$ is well defined globally on the base space.

**Theorem 7.4.1.** Given the Baker-Akhiezer vector $\psi$, the light-cone components $j_{\pm}$ of the $SL(2,\mathbb{C})_R$ current can be recovered by the formula

$$j_{\pm} = i\kappa_{\pm} \lim_{x \to \pm 1} (\Psi(x)\sigma_3\Psi(x)^{-1}).$$

(7.4.2)

**Proof.** Equation (7.2.7b) together with (7.2.6) gave the behaviour of the eigenvector $\psi$ near the essential singularities at $x = \pm 1$. Focusing on the zeroth level $n = 0$ of the hierarchy, namely the $(\sigma,\tau)$-dependence, we have

$$\psi(x^\pm) \sim_{x \to 1} O(1)e^{\pm i\kappa_{\pm}x^2}, \quad \psi(x^\pm) \sim_{x \to -1} O(1)e^{\pm i\kappa_{-}x_2}.$$

We may rewrite this behaviour in terms of the matrix $\Psi(x) = (\psi(x^+), \psi(x^-))$ near $x = \pm 1$ as follows

$$\Psi(x, \sigma, \tau) = \left( \begin{array}{c} \Psi_0(\sigma, \tau) + \sum_{s=1}^{\infty} \Psi_s(\sigma, \tau)(x-1)^s \end{array} \right)^{\frac{i\kappa_+}{1-x} \sigma_3} \text{ as } x \to 1,$$

$$\Psi(x, \sigma, \tau) = \left( \begin{array}{c} \Phi_0(\sigma, \tau) + \sum_{s=1}^{\infty} \Phi_s(\sigma, \tau)(x+1)^s \end{array} \right)^{\frac{i\kappa_-}{1+x} \sigma_3} \text{ as } x \to -1.$$

(7.4.3)

It is straightforward to derive from these expansions the asymptotics near $x = +1$

$$\left\{ \begin{array}{l}
(\partial_+ \Psi)^{-1} = \frac{i\kappa_+}{1-x} (\Psi_0 \sigma_3 \Psi_0^{-1}) + O(1) \\
(\partial_- \Psi)^{-1} = O(1)
\end{array} \right. \quad \text{as } x \to 1$$
and likewise near $x = -1$,

\[
\begin{cases}
(\partial_+ \Psi) \Psi^{-1} = O(1) \\
(\partial_- \Psi) \Psi^{-1} = \frac{i\kappa_-}{1 + x} (\Phi_0 \sigma_3 \Phi_0^{-1}) + O(1)
\end{cases}
\quad \text{as } x \to -1.
\]

However we also find from (7.2.7a) that $\Psi(x) = 1 + O\left(\frac{1}{x}\right)$ as $x \to \infty$ so that

\[
(\partial_\pm \Psi) \Psi^{-1} = O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty.
\]

Thus the above asymptotics at $x = \pm 1, \infty$ take the following form

\[
\begin{align*}
(\partial_+ \Psi) \Psi^{-1} &= J_+(x) + O(1), \quad (\partial_- \Psi) \Psi^{-1} = J_-(1) + O(1) \quad \text{as } x \to 1 \quad (7.4.4a) \\
(\partial_+ \Psi) \Psi^{-1} &= J_+(-1) + O(1), \quad (\partial_- \Psi) \Psi^{-1} = J_-(x) + O(1) \quad \text{as } x \to -1 \quad (7.4.4b) \\
(\partial_\pm \Psi) \Psi^{-1} &= J_\pm(\infty) + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty, \quad (7.4.4c)
\end{align*}
\]

where the matrices $J_\pm(x)$ here have been defined as

\[
J_+(x) = \frac{i\kappa_+}{1 - x} (\Psi_0 \sigma_3 \Psi_0^{-1}), \quad J_-(x) = \frac{i\kappa_-}{1 + x} (\Phi_0 \sigma_3 \Phi_0^{-1}).
\]  

(7.4.5)

To show that these are in fact the light-cone components of the Lax connection consider the following vector-valued functions

\[
f_\pm(P) = (\partial_\pm - J_\pm(x)) \psi(P) \quad (7.4.6a)
\]

\[
= \left[ (\partial_\pm \Psi(x)) \Psi(x)^{-1} - J_\pm(x) \right] \psi(P), \quad (7.4.6b)
\]

where $\hat{\pi}(P) = x$. From (7.4.6a) we see that on $\hat{\Sigma} \setminus \hat{\pi}^{-1}(\pm 1)$ the components of the vectors $f_\pm(P)$ have exactly the same \textit{constant} poles as $\psi(P)$ at $\hat{\gamma}(0)$ as well as the same \textit{constant} zeroes as the components of $\psi(P)$ at $\infty^\pm$ (see (7.2.7a) using the same gauge fixing condition $J_\pm(\infty) = 0$ as usual. Also from (7.4.6b) and using
the asymptotics at \( x = \pm 1 \) in (7.4.4a) and (7.4.4b), these vectors have essential singularities at \( x = \pm 1 \) of exactly the same form as those of the vector \( \psi \). Unlike the Baker-Akhiezer vector \( \psi(P) \) however, the vector \( f_\pm(P) \) may take on arbitrary \( \{t\}\)-dependent values at \( \infty^\pm \). Thus by the uniqueness of the Baker-Akhiezer vector we must have \( f_\pm(P) = D(t)\psi(P) \) where \( D(t) = \text{diag}(f_\pm(\infty^+), f_\pm(\infty^-)) \) is an undetermined diagonal matrix independent of \( P \in \hat{\Sigma} \). But the asymptotics at \( x = \infty \) in (7.4.4c) together with (7.4.6b) now show that in fact \( D(t) \) must be zero, so we conclude

\[
f_\pm(P) \equiv 0.
\]

Going back to the definition (7.4.6) of these vectors this implies that \( J_\pm(x) \) defined in (7.4.3) is exactly the reconstructed Lax connection (7.4.1), and hence the latter is indeed of the form (5.1.6).

One can easily check that the reconstructed currents (7.4.2) satisfy the first set of Virasoro constraints (3.4.3) since \( j_2^\pm = -\kappa_2^\pm \) so that

\[
\text{tr} j_2^\pm = -\kappa_2^\pm \text{tr} 1 = -2\kappa_2^\pm.
\]

Also, before having imposed any reality conditions on the algebro-geometric data the reconstructed current (7.4.2) takes values in \( \mathfrak{sl}(2,\mathbb{C}) \) since it is obviously invertible and traceless,

\[
\text{tr} j_\pm = i\kappa_\pm \text{tr} \sigma_3 = 0.
\]

**The \( SL(2,\mathbb{C}) \) embedding \( g \)**

Having shown that the Lax connection (7.4.1) reconstructed out of Baker-Akhiezer vectors takes precisely the form of a Lax connection constructed from a current \( j \) we were able to express the current \( j \) itself in terms of Baker-Akhiezer functions.
Now since the current $j$ is really of the form $j = -g^{-1}dg$ for some $g$, we would like to extract now a formula for the matrix $g$ in terms of Baker-Akhiezer functions. For this we can go back to equation (7.4.1) for $J(x)$ and rewrite it as

$$d\Psi(x)^{-1} + \Psi(x)^{-1}J(x) = 0.$$ 

And since we know $J(x)$ is of the form $J(x) = \frac{1}{1-x^2}(j - x \star j)$ we have $j = J(0)$ and setting $x = 0$ in the above equation we find

$$d\Psi(0)^{-1} + \Psi(0)^{-1}j = 0.$$ 

This is to be compared with the defining equation $dg + gj = 0$ for the matrix $g$. We see immediately from this comparison that the matrix $g$ can be reconstructed in terms of $\Psi(0)^{-1}$ whose rows we showed were dual Baker-Akhiezer vectors. Because $\det g = 1$ we would need to divide $\Psi(0)^{-1}$ by the square root of its determinant, but $d \det \Psi(0)^{-1} = \det \Psi(0)^{-1} \text{tr}(\Psi(0)d\Psi(0)^{-1}) = -\det \Psi(0)^{-1} \text{tr} j = 0$ so this is possible. However since we haven’t yet imposed reality conditions, at this stage we can only require that $g \in SL(2, \mathbb{C})$. In particular $g$ could be of the general form $g_{L}^{-1}\Psi(0)^{-1}g_{R}^{-1}$ where $g_{R}, g_{L} \in SL(2, \mathbb{C})$ are constant diagonal matrices. Such issues will only be resolved later in chapter 9 when we come to discuss reality conditions. We postpone the complete reconstruction of $g$ until then. At this point we have,

**Proposition 7.4.2.** Given the dual Baker-Akhiezer vector $\tilde{\psi}^{+}$, the matrix $g \in SL(2, \mathbb{C})_{R}$ can be recovered by the formula

$$g(t) = \sqrt{\det \Psi(0, t)} \cdot g_{L}^{-1}\Psi(0, t)^{-1}g_{R}^{-1},$$

where $g_{R}, g_{L} \in SL(2, \mathbb{C})$ are constant diagonal matrices.
Remark Recall that $\Psi(x)$ is determined locally only up to right multiplication by a diagonal matrix $\text{diag}(\Lambda(x^+)^{n+}, \Lambda(x^-)^{n-})$. However since $\Lambda(0^+) = 1$ and the reconstruction formula for $g(t)$ only depends on $\Psi(0)$ and it follows that this ambiguity is absent in $g(t)$. 
Chapter 8

Symplectic structure

The subject of the previous chapter was the reconstruction of the general finite-gap solution from the following piece of algebro-geometric data:

- A smooth algebraic curve $\hat{\Sigma}$ of genus $g$ equipped with a differential $dp$.
- A generic set of $g + 1$ points $\hat{\gamma}(0)$ on this curve.

At fixed genus $g$, different finite-gap solutions are obtained by varying the moduli of the pair $(\hat{\Sigma}, dp)$ and choosing different initial divisors $\hat{\gamma}(0)$ on this curve. As we saw in chapter 6, the correct interpretation of the moduli space of curves is as a $g + 1$ dimensional leaf $\mathcal{L}$ in the universal configuration space. Furthermore, since a non-special divisor of degree $g + 1$ uniquely determines a point in the generalised Jacobian $J_m(\hat{\Sigma})$ via the generalised Abel map, a more natural description for the initial divisor $\hat{\gamma}(0)$ is as the point $\mathcal{A}(\hat{\gamma}(0))$ in the generalised Jacobian $J_m(\hat{\Sigma})$. Then by theorem 7.2.9, the locus of the dynamical divisor $\hat{\gamma}(t)$ in $J_m(\hat{\Sigma})$ is a straight line through this point. The above algebro-geometric data at genus $g$ therefore corresponds to a point in the Jacobian bundle over $\mathcal{L}$ whose fibre over any point $\hat{\Sigma}$ of the base $\mathcal{L}$ is the generalised Jacobian $J_m(\hat{\Sigma})$. 

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This suggests an alternative way of picturing finite-gap solutions that will be useful later. Once we will have imposed reality conditions in chapter 9 the real slice of the generalised Jacobian will turn out to be a \((g + 1)\)-dimensional real torus and the base \(\mathcal{L}_\mathbb{R}\) will become \((g + 1)\)-dimensional over the reals. Therefore the dynamics of a finite-gap solution will correspond to linear motion on a \((g + 1)\)-torus which is very reminiscent of a finite-dimensional integrable system. In fact one can view the Jacobian bundle as the phase-space of a \((g + 1)\)-dimensional dynamical system. But if the algebro-geometric data is to be thought of as a finite-dimensional phase-space it must be equipped with a natural symplectic structure. Now the finite-gap construction provides a (reconstruction) map \(\mathcal{G}\) from the Jacobian bundle to the reduced phase-space \(\mathcal{P}^\infty\) which was introduced in chapter 4 as the space of solutions to the equations of motion satisfying the Virasoro and static gauge constraints (3.4.3), see Figure 8.1. But the space \(\mathcal{P}^\infty\) is equipped with a Dirac bracket from chapter 4 which can thus be pulled back to the algebro-geometric data using the map \(\mathcal{G}\). This way we obtain a ‘natural’ symplectic structure on the Jacobian bundle.

In practise the pullback will be achieved as follows. Recall from chapter 5 that the integrals of motion which parameterise the base \(\mathcal{L}\) can be obtained from the trace of the monodromy matrix \(\Omega(x)\). On the other hand, using a trick due to Sklyanin (see [84] for a review) we will show how to also extract the initial divisor \(\hat{\gamma}(0)\) from \(\Omega(x)\). Therefore the Dirac bracket \(\{\Omega(x), \Omega(x')\}\) between monodromies appropriately regularised à la Maillet (see chapter 5) can be used to read off the Dirac
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brackets of the algebro-geometric data, much like we obtained the Dirac brackets between integrals of motion already in chapter 5. As we will see below, our analysis for strings moving on $\mathbb{R} \times S^3$ can be thought of as a non-linear generalisation of the more familiar Hamiltonian analysis of strings in flat space. We will therefore begin by reviewing the standard discussion of the flat space case following [85, 86].

8.1 Strings in flat space

Consider a closed bosonic string moving on $(D + 1)$-dimensional Minkowski space with worldsheet fields $X^\mu(\sigma, \tau), \mu = 0, 1, \ldots, D$. In conformal gauge, the worldsheet equation of motion is the two-dimensional Laplace equation $\partial_+ \partial_- X^\mu = 0$. As the equation is linear, the general solution for closed string boundary conditions is given by the Fourier series,

$$X^\mu(\sigma, \tau) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in(\tau - \sigma)} + i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in(\tau + \sigma)}. \quad (8.1.1)$$

where the Fourier coefficients $\alpha^\mu_n$ and $\tilde{\alpha}^\mu_n$ correspond to classical oscillator coordinates for left- and right-moving modes respectively. For the purpose of drawing the analogy with finite-gap solutions it will be convenient to restrict attention to classical solutions with a finite number of oscillators turned on. Indeed we will see that these ‘finite-oscillator’ solutions are close analogs of the finite-gap solutions to string theory on $\mathbb{R} \times S^3$ and more generally on classically integrable backgrounds. Generic solutions can then be obtained as a limiting case.

Since (8.1.1) is the general solution to the field equations, the fields $X^\mu(\sigma) = X^\mu(\sigma, 0)$ and $P^\mu(\sigma) = \dot{X}^\mu(\sigma, 0)$ restricted to a $\tau$-slice (taken at $\tau = 0$ without loss of generality) give a convenient parametrisation of the phase-space of the string (since we have not yet fixed the gauge this is not the physical phase-space). Written in
terms of the oscillator coordinates we find,

\[
X^\mu(\sigma) = x^\mu + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{in\sigma} + i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma},
\]

\[
P^\mu(\sigma) = p^\mu + \sum_{n \neq 0} \alpha_n^\mu e^{in\sigma} + \sum_{n \neq 0} \tilde{\alpha}_n^\mu e^{-in\sigma}.
\] (8.1.2)

Conversely the oscillator coefficients \(\alpha_n^\mu, \tilde{\alpha}_n^\mu\) as well as the centre of mass position and momenta \(x^\mu, p^\mu\) can be extracted from a generic phase-space configuration \(X^\mu(\sigma), P^\mu(\sigma)\) by the following relations

\[
\begin{align*}
\alpha_m^\mu &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ima} \frac{1}{2} (P^\mu(\sigma) - \partial_\sigma X^\mu(\sigma)) d\sigma, \quad m \neq 0 \\
\tilde{\alpha}_m^\mu &= \frac{1}{2\pi} \int_0^{2\pi} e^{ima} \frac{1}{2} (P^\mu(\sigma) + \partial_\sigma X^\mu(\sigma)) d\sigma, \quad m \neq 0 \\
x^\mu &= \frac{1}{2\pi} \int_0^{2\pi} X^\mu(\sigma) d\sigma, \quad p^\mu = \frac{1}{2\pi} \int_0^{2\pi} P^\mu(\sigma) d\sigma.
\end{align*}
\] (8.1.3)

Equations (8.1.3) are the inverse of the equations (8.1.2) and the transformation

\[
\{X^\mu(\sigma), P^\mu(\sigma)\} \mapsto \{x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu\}
\] (8.1.4)

is simply a change of variable on phase-space. The Poisson brackets which follow from the string action take the form,

\[
\{X^\mu(\sigma), X^\nu(\sigma')\} = \{P^\mu(\sigma), P^\nu(\sigma')\} = 0, \quad \{P^\mu(\sigma), X^\nu(\sigma')\} = \eta^{\mu\nu} \delta(\sigma - \sigma'),
\] (8.1.5)

and it is straightforward to rewrite these brackets in the new coordinate system as,

\[
\begin{align*}
\{\alpha_m^\mu, \alpha_n^\nu\} &= im\delta_{m+n} \eta^{\mu\nu}, \quad \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0, \\
\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} &= im\delta_{m+n} \eta^{\mu\nu}, \quad \{p^\mu, x^\nu\} = \eta^{\mu\nu}.
\end{align*}
\] (8.1.6)
So far we have discussed the full solution space of the equations of motion. The next step is to restrict to physical configurations of the string by fixing the residual gauge symmetry and imposing the Virasoro constraints. The former is achieved by defining light-cone coordinate \( X^\pm = X^0 \pm X^D \) and imposing the light-cone gauge condition \( X^- + \tau p^+ = 0, \quad P^+ = \frac{e^+}{2\pi} \) which fixes all the oscillator modes \( \alpha^+_n, \tilde{\alpha}^+_n \) of \( X^+ \) to zero and \( x^+ = -\tau p^+ \). With this choice, it is possible to solve the Virasoro constraint \( (P^- \pm X'^-) = (P^i \pm X'^i)^2/2p^+ \) explicitly to eliminate \( p^- \) and all the oscillator modes of \( X^- \) as well. The remaining independent degrees of freedom are

\[
\{x^i, p^i, x^-, p^+, \alpha^+_m, \tilde{\alpha}^+_n\} \quad (8.1.7)
\]

where the index \( i = 1, 2, \ldots, D - 1 \) runs over the transverse spacetime dimensions.

To find the Poisson brackets of the physical degrees of freedom one must follow the standard Dirac procedure for constrained systems. In the present case this is described in detail in [86]. The Virasoro constraint and the light cone gauge fixing condition together correspond to a system of second class constraints on phase-space. Fortunately, the resulting Dirac bracket for the transverse degrees of freedom is the same as their naive Poisson bracket, namely

\[
\{\alpha^+_m, \alpha^+_n\}_{\text{D.B.}} = im\delta_{m+n}\delta^{ij}, \quad \{\alpha^+_m, \tilde{\alpha}^+_n\}_{\text{D.B.}} = 0,
\]

\[
\{\tilde{\alpha}^+_m, \tilde{\alpha}^+_n\}_{\text{D.B.}} = im\delta_{m+n}\delta^{ij}, \quad \{p^i, x^j\}_{\text{D.B.}} = \delta^{ij}.
\]

(8.1.8)

These brackets are the starting point for canonical quantisation of the string which proceeds by the usual recipe of promoting brackets to commutators.

Classical string theory in flat space is trivially integrable as the corresponding equations of motion are linear. For comparison with the non-linear case, it will be convenient to exhibit integrability explicitly by constructing the corresponding action-angle variables. While keeping the centre of mass variables \( \{x^i, p^i\} \) we intro-
duce a new set of variables \( \{ \theta_n^j, S_n^j, \tilde{\theta}_n^j, \tilde{S}_n^j \}_{j=1}^{D-1} \) for the oscillator degrees of freedom by setting

\[
\alpha_n^j = \sqrt{n S_n^j} e^{i \theta_n^j}, \quad \tilde{\alpha}_n^j = \sqrt{n \tilde{S}_n^j} e^{i \tilde{\theta}_n^j}.
\] (8.1.9)

The variables \( S_n^j \) and \( \tilde{S}_n^j \) correspond to the classical analogs of the occupation numbers for the transverse oscillators. They can be shown to commute with the light-cone Hamiltonian governing the dynamics on reduced phase-space and therefore correspond to conserved charges. One may also check the involution condition

\[
\{ S_n^i, S_m^j \}_{\text{D.B.}} = \{ S_n^i, \tilde{S}_m^j \}_{\text{D.B.}} = \{ \tilde{S}_n^i, S_m^j \}_{\text{D.B.}} = 0.
\] (8.1.10)

Together with the momentum variables \( p^j \) these are the action variables of the flat space string. Furthermore, the angular variables \( \theta_n^j \) and \( \tilde{\theta}_n^j \) each have period \( 2\pi \) and are canonically conjugate to the corresponding action variables \( S_n^j \) since their non-vanishing Dirac brackets are

\[
\{ S_n^i, \theta_m^j \}_{\text{D.B.}} = \{ \tilde{S}_n^i, \tilde{\theta}_m^j \}_{\text{D.B.}} = \delta_{nm} \delta^{ij}.
\] (8.1.11)

Likewise from (8.1.8) the variables \( x^j \) are canonically conjugate to the \( p^j \). It follows immediately from Hamilton’s equations that the angle variables \( \{ x^j, \theta_n^j, \tilde{\theta}_n^j \} \) evolve linearly in time while, as above, the conjugate action variables \( \{ p^j, S_n^j, \tilde{S}_n^j \} \) remain constant, thus

\[
\begin{align*}
x^j(\tau) &= x^j(0) + p^j \tau, & p^j = \text{const.} \\
\theta_n^j(\tau) &= \theta_n^j(0) - n \tau, & S_n^j = \text{const.} \\
\tilde{\theta}_n^j(\tau) &= \tilde{\theta}_n^j(0) - n \tau, & \tilde{S}_n^j = \text{const.}
\end{align*}
\] (8.1.12)

We can summarise these results in a language more suitable for drawing the analogy with the non-linear case of strings on \( \mathbb{R} \times S^3 \). Using the change of variables
we can rewrite the general phase-space configuration (8.1.2) as

\[ X^j(\sigma) = x_j^i + i \sum_{n \neq 0} \frac{1}{n} \sqrt{n} S_n^j e^{i\theta_n^j + in\sigma}, \]
\[ P^j(\sigma) = p_j^i + \sum_{n \neq 0} \sqrt{n} S_n^j e^{i\theta_n^j + in\sigma}. \]

(8.1.13)

Recall that we are only considering ‘finite-oscillator’ solutions with a finite number of oscillator modes turned on. Therefore the sums in (8.1.13) are finite and the functions \( X^j, P^j \) depend only on a finite number of parameters. These are the angle variables \( \{x_j^i, \theta_n^j, \tilde{\theta}_n^j\} \) on the one hand and the action variables \( \{p_j^i, S_n^j, \tilde{S}_n^j\} \) on the other. Thus the pair \( X^j, P^j \) in (8.1.13) can be thought of as a map \( G : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^\infty \) from a finite-dimensional phase-space \( \mathcal{P}^{(k)} \) comprised of these non-vanishing modes of the string to the actual physical phase-space \( \mathcal{P}^\infty \) of the string. Moreover, the linear \( \tau \)-evolution (8.1.12) through the finite-dimensional phase-space gets mapped by (8.1.13) to the physical motion in \( \mathcal{P}^\infty \). In particular the function \( X^j \) alone maps the linear motion (8.1.12) to the general solution (8.1.1) in configuration space. We can say that a \( k \)-oscillator phase-space solution \( (X^j, P^j) : \mathbb{R}_\tau \rightarrow \mathcal{P}^\infty \) factors through \( \mathcal{P}^{(k)} \) parameterised by \( \{x_j^i, \theta_n^j, \tilde{\theta}_n^j, p_j^i, S_n^j, \tilde{S}_n^j\} \) since it decomposes as

\[ (X^j, P^j) : \mathbb{R}_\tau \xrightarrow{\tilde{\theta}} \mathcal{P}^{(k)} \xrightarrow{G} \mathcal{P}^\infty, \]

(8.1.14)

where the first map \( \tilde{\theta} \) is linear and the second is given by (8.1.13). This picture for strings in flat space is therefore very similar to the one presented at the start of this chapter for strings on \( \mathbb{R} \times S^3 \) where we had a finite-gap solution mapping the Jacobian bundle to the reduced phase-space \( \mathcal{P}^\infty \), as illustrated in Figure 8.1.

Equation (8.1.13) can be thought of as the flat space analogue of the reconstruction formula for the current \( j(\sigma) \) (theorem 7.4.1) in that it expresses the general phase-space configuration \( (X^i(\sigma), P^i(\sigma)) \) in terms of the finite data \( \{x^j, \theta_n^j, \tilde{\theta}_n^j, p^i, S_n^j, \tilde{S}_n^j\} \).
8.2 The geometric map

As we argued at the start of this chapter, the complete set of algebro-geometric data 
{\( (\hat{\Sigma}, dp), \hat{\gamma}(0) \)} for an arbitrary finite-gap solution \( j \) can be succinctly described as 
a point in the Jacobian bundle \( \mathcal{M}_{\mathbb{C}}^{(2g+2)} \) over \( \mathcal{L} \),

\[
J_m(\hat{\Sigma}) \to \mathcal{M}_{\mathbb{C}}^{(2g+2)} \to \mathcal{L},
\]

(8.2.1)

whose fibre over every point of the base, specified by a pair \( (\hat{\Sigma}, dp) \), is the generalised 
Jacobian \( J_m(\hat{\Sigma}) \) relative the usual modulus \( m = \infty^+ + \infty^- \). The finite-gap construction 
of chapter 7 (in particular theorem 7.4.1) thus defines an injective map, called 
the geometric map in the terminology of [78], from the algebro-geometric data 
\( \mathcal{M}_{\mathbb{C}}^{(2g+2)} \) into the space \( S_{\mathbb{C}}^\infty \) of complexified solutions \( j \in \mathfrak{sl}(2, \mathbb{C}) \) to the equations 
of motion of a string moving on \( \mathbb{R} \times S^3 \) which also satisfy the Virasoro and static 
gauge conditions (3.4.3),

\[
\mathcal{G} : \mathcal{M}_{\mathbb{C}}^{(2g+2)} \hookrightarrow S_{\mathbb{C}}^\infty.
\]

(8.2.2)

At the start of section 5.3 we described the reduced phase-space \( \mathcal{P}_{\infty} \) as the restriction 
of the space of solutions \( S_{\mathbb{C}}^\infty \) satisfying (3.4.3) to a chosen time slice. That is to say, 
setting all the higher times in the reconstruction map (8.2.2) to zero except for 
the worldsheet \( \sigma \)-coordinate provides an embedding of the Jacobian bundle into 
(complexified) reduced phase-space \( \mathcal{P}_{\mathbb{C}}^\infty \), namely

\[
\mathcal{G} : \mathcal{M}_{\mathbb{C}}^{(2g+2)} \hookrightarrow \mathcal{P}_{\mathbb{C}}^\infty.
\]

(8.2.3)

However, by virtue of theorem 7.2.9 the dependence on all the higher times can 
be recovered simply by composing the phase-space map (8.2.3) with a linear map 
determined by (7.2.14),

\[
\tilde{\theta} : \mathbb{R}^N \to \mathcal{M}_{\mathbb{C}}^{(2g+2)},
\]
which given a set of \( N \) higher times \( \{ t_i \}_{i=1}^N \) translates the Abel map \( \vec{A}(\hat{\gamma}) \) of a divisor \( \hat{\gamma} \) to the point \( \vec{A}(\hat{\gamma}) + \int_{\hat{\gamma}} dQ \) while staying on the same fibre of \( \mathcal{M}_C^{(2g+2)} \). So much like equation (8.1.14) in the flat space case, here the solution factors through the Jacobian bundle,

\[
j : \mathbb{R}^N \overset{\theta}{\to} \mathcal{M}_C^{(2g+2)} \overset{\theta}{\to} \mathcal{P}^\infty.
\]

(8.2.4)

The domain \( \mathbb{R}^N \) could be restricted to just \( \mathbb{R}_\tau \) if we are only interested in \( \tau \)-evolution.

Let \( \hat{\omega}_\infty \) denote the symplectic form on the reduced phase-space \( \mathcal{P}^\infty \) corresponding to the Dirac bracket (4.4.9) introduced in chapter [1]. The goal of the remainder of this chapter will be to compute the pullback of \( \hat{\omega}_\infty \) to the Jacobian bundle \( J_m(\hat{\Sigma}) \) by the geometric map (8.2.3). The result is the following,

**Theorem 8.2.1.** The pullback of the Dirac bracket on the reduced phase-space \( \mathcal{P}^\infty \) by the geometric map \( \mathcal{G} \) takes the simple form,

\[
\hat{\omega}_{2g+2} \equiv \mathcal{G}^* \hat{\omega}_\infty = \sum_{I=1}^{g+1} \delta S_I \wedge \delta \varphi_I,
\]

(8.2.5)

where \( S_I \) are the filling fractions (6.4.8). In particular we see that they precisely correspond to the action variables of the string. The conjugate angle variables \( \varphi_I \) are defined in terms of the divisor \( \hat{\gamma}(t) \) by

\[
\varphi_i = A_i(\hat{\gamma}(t)) - A_{g+1}(\hat{\gamma}(t)), \quad i = 1, \ldots, g
\]

\[
\varphi_{g+1} = -A_{g+1}(\hat{\gamma}(t)).
\]

(8.2.6)

To prove this theorem we will show how to express the algebro-geometric data in terms of the monodromy matrix \( \Omega(x) \), analogously to (8.1.3), and use this to read off their Dirac brackets from \( \{ \Omega(x), \Omega(x') \}_{\text{D.B.}} \). We already know from chapter [5] how to read off the integrals of motion from \( \Omega(x) \) and we have obtained their Dirac bracket (5.3.5), which is the analogue of (8.1.10) in flat space.
Extracting data

The divisor $\hat{\gamma}(t)$ of poles of $h(P,t)$ can be extracted from $\Omega(x)$ using a method due to Sklyanin \cite{84} as follows. We perform a similarity transformation on the monodromy matrix $\Omega(x)$ by $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{C})$ and define $\tilde{\Omega}(x) = L\Omega(x)L^{-1}$ with components

$$\tilde{\Omega}(x) = \begin{pmatrix} \tilde{A}(x) & \tilde{B}(x) \\ \tilde{C}(x) & \tilde{D}(x) \end{pmatrix}.$$ (8.2.7)

**Lemma 8.2.2.** Let $\hat{\gamma}_i \in \hat{\Sigma}$, $i = 1, \ldots, g + 1$ be the points of the divisor $\hat{\gamma} \equiv \hat{\gamma}(t)$ of poles of the normalised eigenvector $h$. Then the coordinates of the corresponding points on the spectral curve $\Psi_i = (x(\hat{\gamma}_i), \Lambda(\hat{\gamma}_i)) \in \Gamma$ satisfy

$$\tilde{B}(x(\hat{\gamma}_i)) = 0, \quad \Lambda(\hat{\gamma}_i) = \tilde{D}(x(\hat{\gamma}_i)) = \tilde{A}(x(\hat{\gamma}_i))^{-1}. \quad (8.2.8)$$

**Proof.** The normalised eigenvector $h(P)$ satisfies the system of equations

$$\Omega(x)h(P) = \Lambda(P)h(P), \quad \alpha \cdot h(P) = 1,$$

where as before $\alpha = (1,1)$. Since the components of $h(P)$ have poles at $\hat{\gamma}_i$ we introduce the residue vectors $h_i = \text{res}_{\hat{\gamma}_i} h(P)$ which satisfy the Sklyanin system

$$\Omega(x(\hat{\gamma}_i))h_i = \Lambda(\hat{\gamma}_i)h_i, \quad \alpha \cdot h_i = 0. \quad (8.2.9)$$

After the similarity transformation $\tilde{h}_i = Lh_i$, $\tilde{\Omega}(x(\hat{\gamma}_i)) = L\Omega(x(\hat{\gamma}_i))L^{-1}$ the system of equations \cite{82a} can be rewritten as

$$\tilde{\Omega}(x(\hat{\gamma}_i))\tilde{h}_i = \Lambda(\hat{\gamma}_i)\tilde{h}_i, \quad \left(\tilde{h}_i\right)_1 = 0.$$ \cite{8.2.7}

When expressed in terms of components \cite{8.2.7} this immediately implies \cite{8.2.8}. □
Notice that the variables \( \{ \Lambda(\hat{\gamma}_i) \}_{i=1}^{g+1} \) form a set of \( g+1 \) integrals of motion of the genus \( g \) finite-gap solution. For generic divisors \( \hat{\gamma} = \hat{\gamma}_1 + \ldots + \hat{\gamma}_{g+1} \) these variables are independent and therefore parameterise the base \( \mathcal{L} \) of the bundle \( \mathcal{M}_C^{(2g+2)} \). Since the knowledge of the remaining coordinates \( \{ x(\hat{\gamma}_i) \}_{i=1}^{g+1} \) completely specifies the divisor \( \hat{\gamma} \) they naturally provide coordinates along the fibres \( J_m(\hat{\Sigma}) \). The full set of coordinates \( \{ x(\hat{\gamma}_i) \}, \Lambda(\hat{\gamma}_i) \}_{i=1}^{g+1} \) of the points \( \mathfrak{P}_i \in \Gamma \) can be thus be thought of as the complete algebro-geometric data for genus \( g \) finite-gap solutions. Equations (8.2.8) will be our way of extracting the algebro-geometric data of a generic \( g \)-gap string. This is the non-linear analogue of extracting the Fourier coefficients of a finite-oscillator solution in the flat space case (8.1.3).

Because the matrix from which one reads off the divisor \( \hat{\gamma} \) isn’t exactly the monodromy matrix \( \Omega(x) \) but the similar matrix \( \tilde{\Omega}(x) = L \Omega(x) L^{-1} \), we will need the Dirac bracket \( \{ \tilde{\Omega}(x), \tilde{\Omega}(x') \}_{D.B.} \) between these new matrices.

**Lemma 8.2.3.** The Dirac algebra between monodromy matrices (5.3.21) is invariant under \( SL(2, \mathbb{C}) \) similarity transformations. In particular,

\[
\{ \tilde{\Omega}(x), \tilde{\Omega}(x') \}_{D.B.} \approx [r(x, x'), \tilde{\Omega}(x) \otimes \tilde{\Omega}(x')] \\
+ (\tilde{\Omega}(x) \otimes 1)s(x, x')(1 \otimes \tilde{\Omega}(x')) \\
- (1 \otimes \tilde{\Omega}(x'))s(x, x')(\tilde{\Omega}(x) \otimes 1). \tag{8.2.10}
\]

**Proof.** Let \( L \in SL(2, \mathbb{C}) \). Conjugating the Dirac bracket (5.3.21) by \( L \otimes L \) has the effect of replacing \( \Omega(x) \) by \( \tilde{\Omega}(x) = L \Omega(x) L^{-1} \) but also \( r(x, x') \) and \( s(x, x') \) by \( \tilde{r}(x, x') = (L \otimes L)r(x, x')(L^{-1} \otimes L^{-1}) \) and \( \tilde{s}(x, x') = (L \otimes L)s(x, x')(L^{-1} \otimes L^{-1}) \) respectively. However \( r(x, x') \) and \( s(x, x') \) are both multiples of \( \eta = t_a \otimes t^a \) which is invariant under \( SL(2, \mathbb{C}) \), i.e. \( (L \otimes L)\eta(L^{-1} \otimes L^{-1}) = \eta \), since infinitesimally for any \( \alpha \in \mathfrak{sl}(2, \mathbb{C}) \) we have \( [1 \otimes \alpha + \alpha \otimes 1, \eta] = 0 \). Therefore \( \tilde{r}(x, x') = r(x, x') \) and \( \tilde{s}(x, x') = s(x, x') \) and (8.2.10) follows. \( \square \)
8.3 Dirac brackets of algebro-geometric data

By lemma [8.2.2] the relevant components of \(\tilde{\Omega}(x)\) for extracting the algebro-geometric data are \(\tilde{A}(x)\) and \(\tilde{B}(x)\). Their Dirac brackets can then be deduced from the Dirac algebra between monodromies (8.2.10).

Lemma 8.3.1. Let \(\hat{r}(x, x')\) and \(\hat{s}(x, x')\) be defined as \(r(x, x')\) and \(s(x, x')\) respectively but without the factors of \(\eta\), i.e. \(r(x, x') = \hat{r}(x, x')\eta\) and \(s(x, x') = \hat{s}(x, x')\eta\). Then

\[
\begin{align*}
\{\tilde{A}(x), \tilde{A}(x')\}_{D.B.} &= (\tilde{B}(x)\tilde{C}(x') - \tilde{B}(x')\tilde{C}(x)) \hat{s}(x, x'), \\
\{\tilde{A}(x), \tilde{B}(x')\}_{D.B.} &= (\tilde{A}(x)\tilde{B}(x') + \tilde{A}(x')\tilde{B}(x)) \hat{r}(x, x') \\
&\quad + (\tilde{A}(x)\tilde{B}(x') + \tilde{D}(x')\tilde{B}(x)) \hat{s}(x, x'), \\
\{\tilde{B}(x), \tilde{B}(x')\}_{D.B.} &= 0.
\end{align*}
\]

(8.3.1)

Proof. Let us express the right hand side of (8.3.10) in terms of the components (8.2.7) of \(\hat{\Omega}(x)\). This requires the following ingredients

\[
\eta = \frac{1}{2} \begin{pmatrix} \sigma_3 & \sigma_1 - i\sigma_2 \\ \sigma_1 + i\sigma_2 & -\sigma_3 \end{pmatrix}, \quad \hat{\Omega}(x) \otimes \hat{\Omega}(x') = \begin{pmatrix} \tilde{A}(x)\tilde{\Omega}(x') & \tilde{B}(x)\tilde{\Omega}(x') \\ \tilde{C}(x)\tilde{\Omega}(x') & \tilde{D}(x)\tilde{\Omega}(x') \end{pmatrix},
\]

\[
1 \otimes \tilde{\Omega}(x') = \begin{pmatrix} \tilde{\Omega}(x') & 0 \\ 0 & \tilde{\Omega}(x') \end{pmatrix}, \quad \tilde{\Omega}(x) \otimes 1 = \begin{pmatrix} \tilde{A}(x)1 & \tilde{B}(x)1 \\ \tilde{C}(x)1 & \tilde{D}(x)1 \end{pmatrix}.
\]

Using these one can easily compute the following quantities

\[
\begin{align*}
\eta\tilde{\Omega}(x) \otimes \tilde{\Omega}(x') &= \frac{1}{2} \begin{pmatrix} \tilde{A}(x)\tilde{\Omega}(x')\sigma_3 + \tilde{C}(x)(\sigma_1 - i\sigma_2)\tilde{\Omega}(x') & \tilde{B}(x)\sigma_3\tilde{\Omega}(x') + \tilde{D}(x)(\sigma_1 - i\sigma_2)\tilde{\Omega}(x') \\
\tilde{A}(x)(\sigma_1 + i\sigma_2)\tilde{\Omega}(x') - \tilde{C}(x)\sigma_3\tilde{\Omega}(x') & \tilde{B}(x)(\sigma_1 + i\sigma_2)\tilde{\Omega}(x') - \tilde{D}(x)\sigma_3\tilde{\Omega}(x') \end{pmatrix}, \\
\tilde{\Omega}(x) \otimes \eta\tilde{\Omega}(x') &= \frac{1}{2} \begin{pmatrix} \tilde{A}(x)\tilde{\Omega}(x')\sigma_3 + \tilde{B}(x)\tilde{\Omega}(x')(\sigma_1 + i\sigma_2) & \tilde{A}(x)\tilde{\Omega}(x')(\sigma_1 - i\sigma_2) - \tilde{B}(x)\tilde{\Omega}(x')\sigma_3 \\ \tilde{C}(x)\tilde{\Omega}(x')\sigma_3 + \tilde{D}(x)\tilde{\Omega}(x')(\sigma_1 + i\sigma_2) & \tilde{C}(x)\tilde{\Omega}(x')(\sigma_1 - i\sigma_2) - \tilde{D}(x)\tilde{\Omega}(x')\sigma_3 \end{pmatrix}.
\end{align*}
\]
The Dirac brackets \((8.3.1)\) of various components of \(\tilde{\Omega}(8.2.10)\) using the above. In particular for the Dirac brackets we take the limit \(x \to x^T a\). Taking the limit we will consider the implications of the three relations \((8.3.1)\) in turn. First

**Proposition 8.3.2.** The Dirac brackets of the algebro-geometric data are

\[
\{ \tilde{\Omega}(x) \otimes 1 \eta 1 \otimes \tilde{\Omega}(x') = \frac{1}{2} \left( \begin{array}{ccc}
\tilde{A}(x)\sigma_3\tilde{A}(x') + \tilde{B}(x)(\sigma_1 + i\sigma_2)\tilde{B}(x') & \tilde{A}(x)(\sigma_1 - i\sigma_2)\tilde{A}(x') - \tilde{B}(x)\sigma_3\tilde{A}(x') \\
\tilde{A}(x)\sigma_3\tilde{A}(x') + \tilde{B}(x)(\sigma_1 + i\sigma_2)\tilde{B}(x') & \tilde{A}(x)(\sigma_1 - i\sigma_2)\tilde{A}(x') - \tilde{B}(x)\sigma_3\tilde{A}(x') \\
\tilde{C}(x)\sigma_3\tilde{C}(x') + \tilde{D}(x)(\sigma_1 + i\sigma_2)\tilde{D}(x') & \tilde{C}(x)(\sigma_1 - i\sigma_2)\tilde{C}(x') - \tilde{D}(x)\sigma_3\tilde{C}(x') \\
\tilde{C}(x)\sigma_3\tilde{C}(x') + \tilde{D}(x)(\sigma_1 + i\sigma_2)\tilde{D}(x') & \tilde{C}(x)(\sigma_1 - i\sigma_2)\tilde{C}(x') - \tilde{D}(x)\sigma_3\tilde{C}(x') \\
\end{array} \right),
\]

The Dirac brackets \((8.3.1)\) of various components of \(\tilde{\Omega}(x)\) can now be read off from \((8.2.10)\) using the above. In particular for the Dirac brackets

\[
\left\{ \tilde{A}(x), \tilde{A}(x') \right\}_{D.B.}, \quad \left\{ \tilde{A}(x), \tilde{B}(x') \right\}_{D.B.} \quad \text{and} \quad \left\{ \tilde{B}(x), \tilde{B}(x') \right\}_{D.B.}
\]

we take respectively the components \((11, 11), (11, 12)\) and \((12, 12)\) of the tensor product relation \((8.2.10)\). \(\square\)

Next we show that the relations \((8.3.1)\) imply non-trivial Dirac brackets between the complex variables \(\{x(\hat{\gamma}_i), \Lambda(\hat{\gamma}_i)\}_{i=1}^{g+1}\) comprising the algebro-geometric data.

**Proposition 8.3.2.** The Dirac brackets of the algebro-geometric data are

\[
\{ \Lambda(\hat{\gamma}_j), \Lambda(\hat{\gamma}_k) \}_{D.B.} = 0, \quad \text{(8.3.2a)}
\]

\[
\frac{\sqrt{\lambda}}{4\pi} \{ \Lambda(\hat{\gamma}_j), x(\hat{\gamma}_k) \}_{D.B.} = \Lambda(\hat{\gamma}_j) \frac{x(\hat{\gamma}_j)^2}{1 - x(\hat{\gamma}_j)^2} \delta_{jk} \quad \text{(8.3.2b)}
\]

\[
\{ x(\hat{\gamma}_j), x(\hat{\gamma}_k) \}_{D.B.} = 0. \quad \text{(8.3.2c)}
\]

**Proof.** We will consider the implications of the three relations \((8.3.1)\) in turn. First we take the limit \(x' \to x_{\hat{\gamma}_k} \equiv x(\hat{\gamma}_k)\) of \((8.3.1a)\). Using \((8.2.8)\) this gives

\[
\{ \tilde{A}(x), \Lambda(\hat{\gamma}_k)^{-1} \}_{D.B.} = \tilde{B}(x)\tilde{C}(x_{\hat{\gamma}_k})\tilde{s}(x, x_{\hat{\gamma}_k}).
\]

Taking the limit \(x \to x_{\hat{\gamma}_j}\) yields \(\{ \Lambda(\hat{\gamma}_j)^{-1}, \Lambda(\hat{\gamma}_k)^{-1} \}_{D.B.} = 0\), or equivalently \((8.3.2a)\).

We now turn to the Poisson bracket \((8.3.1b)\). Taking the limit \(x \to x_{\hat{\gamma}_j}\) first
gets rid of the terms proportional to \( \tilde{B}(x) \) (using \( \tilde{B}(x_{\gamma_j}) = 0 \)) and leaves

\[
\{ \tilde{A}(x_{\gamma_j}), \tilde{B}(x') \}_{\text{D.B.}} = \tilde{A}(x_{\gamma_j}) \tilde{B}(x') \left( \hat{r}(x_{\gamma_j}, x') + \hat{s}(x_{\gamma_j}, x') \right).
\]

Now using (8.2.8) we can write \( \tilde{B}(x') = (x' - x_{\gamma_k}) \tilde{B}_k(x') \) with \( \tilde{B}_k(x_{\gamma_k}) \neq 0 \), so that

\[
(x' - x_{\gamma_k}) \{ \tilde{A}(x_{\gamma_j}), \tilde{B}_k(x') \}_{\text{D.B.}} - \{ \tilde{A}(x_{\gamma_j}), x_{\gamma_k} \}_{\text{D.B.}} \tilde{B}_k(x') = \tilde{A}(x_{\gamma_j})(x' - x_{\gamma_k}) \tilde{B}_k(x') \left( \hat{r}(x_{\gamma_j}, x') + \hat{s}(x_{\gamma_j}, x') \right),
\]

where

\[
\hat{r}(x_{\gamma_j}, x') + \hat{s}(x_{\gamma_j}, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x_{\gamma_j}^2 + x'^2 - 2x_{\gamma_j}^2 x'^2}{(x_{\gamma_j} - x')(1 - x_{\gamma_j}^2)(1 - x'^2)} - \frac{2\pi}{\sqrt{\lambda}} \frac{x_{\gamma_j} + x'}{1 - x_{\gamma_j}^2}(1 - x'^2).
\]

Taking the limit \( x' \to x_{\gamma_k} \) with \( k \neq j \) kills everything but the second term on the left hand side, leaving \( \{ \tilde{A}(x_{\gamma_j}), x_{\gamma_k} \}_{\text{D.B.}} = 0 \), \( k \neq j \). Now setting \( k = j \) and taking the limit \( x' \to x_{\gamma_j} \) kills the \( \hat{s} \) term leaving

\[
-\{ \tilde{A}(x_{\gamma_j}), x_{\gamma_j} \}_{\text{D.B.}} = \frac{4\pi}{\sqrt{\lambda}} \tilde{A}(x_{\gamma_j}) \frac{x_{\gamma_j}^2}{1-x_{\gamma_j}^2},
\]

which is equivalent to (8.3.2b) by (8.2.8).

Finally, writing again \( \tilde{B}(x) = (x - x_{\gamma_j}) \tilde{B}_j(x) \), equation (8.3.1c) immediately leads to \( \{ x_{\gamma_j}, \tilde{B}(x') \}_{\text{D.B.}} = 0 \) which in turn implies (8.3.2c). \( \square \)

The algebro-geometric data needed to reconstruct a \( g \)-gap solution is a point on the Jacobian bundle \( \mathcal{M}^{(2g+2)}_\mathbb{C} \) specified by \( 2g+2 \) complex coordinates \( \{ x_{\gamma_i}, \Lambda(\gamma_i) \}_{i=1}^{g+1} \). Proposition 8.3.2 gives the complete set of Dirac brackets for these variables. To write these brackets in canonical form we perform the change of spectral parameter,

\[
z = x + \frac{1}{x}.
\]

We have already introduced this function in equation (6.4.3) of chapter 6 to discuss the moduli of the spectral curve. However, there the change of variable \( x \mapsto z \) was
unjustified. Here we see from (8.3.3b) below that the new spectral parameter $z$ is much better suited for discussions of the symplectic structure. Recalling also the definition (6.3.2) of the quasi-momentum we can rewrite the brackets (8.3.2) as

$$\{p(\hat{\gamma}_j),p(\hat{\gamma}_k)\}_{D.B.} = 0,$$

$$\frac{\sqrt{\lambda}}{4\pi i} \{p(\hat{\gamma}_j),z(\hat{\gamma}_k)\}_{D.B.} = \delta_{jk},$$

$$\{z(\hat{\gamma}_j),z(\hat{\gamma}_k)\}_{D.B.} = 0.$$

**Corollary 8.3.3.** The pullback of the Dirac bracket on the reduced phase-space $\mathcal{P}^\infty$ by the geometric map $\mathcal{G}$ is

$$\hat{\omega}_{2g+2} = \mathcal{G}^*\hat{\omega}_\infty = -\frac{\sqrt{\lambda}}{4\pi i} \sum_{i=1}^{g+1} \delta p(\hat{\gamma}_i) \wedge \delta z(\hat{\gamma}_i).$$

**Remark** Recall that the Abel map defines a local isomorphism from the group of divisors of degree $g + 1$ in the neighbourhood of a non-special divisor to the generalised Jacobian $J_m(\hat{\Sigma})$. But a divisor $\hat{\gamma} = \hat{\gamma}_1 + \ldots + \hat{\gamma}_{g+1}$ of degree $g + 1$ is nothing but an unordered set of $g + 1$ points $\{\hat{\gamma}_i\}_{i=1}^{g+1}$ on $\hat{\Sigma}$. Therefore the $(g + 1)$-symmetric product of the curve $\hat{\Sigma}$ can be locally identified via the Abel map with the generalised Jacobian $J_m(\hat{\Sigma})$. In particular, any symmetric expression in $\{\hat{\gamma}_i\}_{i=1}^{g+1}$, such as (8.3.4), naturally lives on $J_m(\hat{\Sigma})$.

**Action-angle variables**

The change of coordinates to action-angle variables is fairly standard (see for instance [77, 78] and [87, p.16]). We shall construct the complete set of action-angle variables starting from the algebro-geometric symplectic form (8.3.4) on $\mathcal{M}_C^{(2g+2)}$. 

It is useful at first to consider the universal curve bundle \( \mathcal{N} \) over the leaf \( \mathcal{L} \)
\[
\hat{\Sigma} \to \mathcal{N} \to \mathcal{L},
\]
whose fibre over every point of the base \( \mathcal{L} \) is the corresponding curve \( \hat{\Sigma} \). Now recall from chapter 6 that the \( \{S_i\}_{i=1}^{g} \) and \( R \) defined in (6.4.7) form a set of coordinates on the base \( \mathcal{L} \). They can be expressed in terms of the \( a_i \)-periods and residue at \( \infty^+ \) or \( \infty^- \) of the differential\footnote{Note that \( \tilde{\alpha} + i\alpha \equiv d\beta \) is locally exact where \( \beta = \frac{i\sqrt{\lambda}}{4\pi i}pz \) and \( \alpha \) was defined in (6.4.6).} \( \tilde{\alpha} \equiv -\frac{\sqrt{\lambda}}{4\pi i}pdz \) respectively,
\[
S_i = \frac{1}{2\pi} \int_{a_i} \tilde{\alpha}, \quad i = 1, \ldots, g, \quad \frac{R}{2} = \mp \frac{1}{2\pi} \oint_{\infty^\pm} \tilde{\alpha},
\]
where the contour integrals around the points \( \infty^\pm \in \hat{\Sigma} \) are taken counterclockwise. Note also that \( z \) can be taken as a local coordinate along the fibres of \( \mathcal{N} \). Denoting then by \( \delta \) the exterior derivative on the total space \( \mathcal{N} \), the differentials \( \delta z, \delta R \) and \( \{\delta S_i\}_{i=1}^{g} \) form a basis of differentials at every point of \( \mathcal{N} \). In this basis, the total exterior derivative of any function (or 1-form) \( f \) on \( \mathcal{N} \) can be separated as
\[
\delta f = \delta z \wedge \partial_z f + \frac{1}{2}\delta R \wedge \partial_R f + \sum_{i=1}^{g} \delta S_i \wedge \partial_{S_i} f \equiv df + \delta^\mathcal{L} f,
\]
where \( \delta^\mathcal{L} \) denotes the exterior derivative along the leaf \( \mathcal{L} \). In particular \( \delta z = dz \), \( \delta R = \delta^\mathcal{L} R \) and \( \delta S_i = \delta^\mathcal{L} S_i \). The differential \( \tilde{\alpha} \) on \( \hat{\Sigma} \), as in fact any differential on \( \hat{\Sigma} \), can be extended to a differential on \( \mathcal{N} \) by setting it to zero along \( \delta R \) and \( \{\delta S_i\}_{i=1}^{g} \).

Consider now its exterior derivative \( \delta \tilde{\alpha} \) on \( \mathcal{N} \)
\[
\delta \tilde{\alpha} = -\frac{\sqrt{\lambda}}{4\pi i} \delta p \wedge dz = \sum_{i=1}^{g} \delta S_i \wedge \partial_{S_i} \tilde{\alpha} + \frac{1}{2}\delta R \wedge \partial_R \tilde{\alpha}.
\]
Although \( \tilde{\alpha} \) is not single valued on \( \hat{\Sigma} \), the ambiguities in its definition are constant
8.3. DIRAC BRACKETS OF ALGEBRO-GEOMETRIC DATA

along the leaf $\mathcal{L}$. Indeed by equation (6.4.4) one can add to the differential $\tilde{\alpha}$ any integer multiple of $-\frac{\sqrt{5}}{2\pi}dz$, but this latter differential depends neither on $R$ nor on \( \{S_i\}_{i=1}^g \). It follows that $\partial_{S_i}\tilde{\alpha}$ and $\partial_{\overline{R}}\tilde{\alpha}$ are well defined. Furthermore, since none of the residues in table 6.1 depend on $S_i$, all the residues of $\tilde{\alpha} = d\beta - i\alpha$ are independent of $S_i$ and it follows that $\partial_{S_i}\tilde{\alpha}$ is holomorphic, i.e. $\partial_{S_i}\tilde{\alpha} = \sum_{j=1}^g c_{ij}\omega_j$. Using (8.3.5) we obtain $c_{ij} = 2\pi\delta_{ij}$. In contrast, we notice from table 6.1 that the residues of $\tilde{\alpha}$ at $\infty^\pm$ are proportional to $R$ so that $\partial_{\overline{R}}\tilde{\alpha}$ must have simple poles at these points. Specifically, using (8.3.5) its residues are found to be $\text{res}_{\infty^\pm} \partial_{\overline{R}}\tilde{\alpha} = \pm i$. In conclusion

$$\partial_{S_i}\tilde{\alpha} = 2\pi\omega_i, \quad i = 1, \ldots, g, \quad \partial_{\overline{R}}\tilde{\alpha} = -2\pi\omega_\infty,$$

where $\omega_\infty$ was defined in (7.2.10). Therefore (8.3.6) simplifies to

$$\delta\tilde{\alpha} = \sum_{i=1}^g \delta S_i \wedge 2\pi\omega_i - \frac{1}{2} \delta R \wedge 2\pi\omega_\infty.$$

The differential $\delta\tilde{\alpha}$ on $\mathcal{N}$ can be used to write down the symplectic form (8.3.4) as the following expression symmetric in the points $\hat{\gamma}_j \in \hat{\Sigma}$,

$$\hat{\omega}_{2g+2} = \sum_{j=1}^{g+1} \delta\tilde{\alpha}(\hat{\gamma}_j) = \sum_{i=1}^g \delta S_i \wedge \left( 2\pi \sum_{j=1}^{g+1} \omega_i(\hat{\gamma}_j) \right) - \frac{1}{2} \delta R \wedge \left( 2\pi \sum_{j=1}^{g+1} \omega_\infty(\hat{\gamma}_j) \right).$$

Recall that such a symmetric expression in the $\hat{\gamma}_j \in \hat{\Sigma}$ naturally lives on the generalised Jacobian $J_m(\hat{\Sigma})$ which can be locally (in the neighbourhood of a non-special divisor) identified with the $(g + 1)^{st}$ symmetric power of the curve $\hat{\Sigma}$ via the generalised Abel map $A : \hat{\Sigma}^{g+1}/S_{g+1} \rightarrow J_m(\hat{\Sigma})$ given explicitly by

$$\phi_i = A_i(\hat{\gamma}) = 2\pi \sum_{j=1}^{g+1} \int_{\hat{\gamma}_j} \omega_i, \quad i = 1, \ldots, g + 1,$$
where $\hat{\gamma} = \hat{\gamma}_1 + \ldots + \hat{\gamma}_{g+1}$, $\omega_{g+1} = \omega_{\infty}$ and $\phi_i$ are coordinates on $J_m(\hat{\Sigma})$. Now we can introduce an exterior derivative $\delta$ on the Jacobian bundle $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ by defining for any function (or 1-form) $f$,

$$\delta f = \sum_{i=1}^{g+1} \delta z_i \wedge \partial z_i f + \frac{1}{2} \delta R \wedge \partial R f + \sum_{i=1}^{g} \delta S_i \wedge \partial S_i f \equiv \sum_{i=1}^{g+1} d_i f + \delta^C f,$$

where $z_i$ is the local coordinate on the $i^{th}$ factor $\hat{\Sigma}$ of $\hat{\Sigma}^{g+1}/S_{g+1}$. It follows that

$$\delta \phi_i = \sum_{k=1}^{g+1} d_k \left( 2\pi \sum_{j=1}^{g+1} \int_{\hat{\gamma}_j} \omega_i \right) = 2\pi \sum_{k=1}^{g+1} \omega_k(\hat{\gamma}_k).$$

We can now write $\hat{\omega}_{2g+2}$ explicitly as a symplectic form on $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$, namely

$$\hat{\omega}_{2g+2} = \sum_{i=1}^{g} \delta S_i \wedge \delta \phi_i - \frac{1}{2} \delta R \wedge \delta \phi_{g+1}.$$ 

This can be further rewritten as

$$\hat{\omega}_{2g+2} = \sum_{i=1}^{g} \delta S_i \wedge \delta (\phi_i - \phi_{g+1}) + \delta \left( \frac{R - L}{2} - \sum_{i=1}^{g} S_i \right) \wedge \delta (-\phi_{g+1}), \quad (8.3.7)$$

where we have used the fact that $\delta L = 0$ since $L$ is fixed along the leaf $\mathcal{L}$ by definition. Recalling the definition of the $g + 1$ filling fractions $\{S_i\}_{i=1}^{g+1}$ in (6.4.8) and introducing the angle variables $\{\varphi_i\}_{i=1}^{g+1}$ as in (8.2.6), equation (8.3.7) becomes equivalent to (8.2.5) which completes the proof of theorem 8.2.1.

### 8.4 Quasi-actions

Remember that the Lax matrix in (5.4.8) is responsible for the flow of the Hamiltonian $p(x)$. Thus going back to the corresponding Hamilton equation written in Lax
form we have
\[
2\pi i \left\{ -\frac{\sqrt{\lambda}}{8\pi^2 i} \left( 1 - \frac{1}{x^2} \right) p(x), J_1(x') \right\}_{\text{D.B.}} = \left[ \partial_\sigma - J_1(x'), \frac{\Psi(x)\hat{\sigma}_3\Psi(x)^{-1}}{x - x'} \right]. \tag{8.4.1}
\]

Integrating this equation in \(x\) over the different \(a\)-cycles, and recalling the definition (6.4.7) of the first \(g\) action variables \(S_i = -\frac{\sqrt{\lambda}}{8\pi^2 i} \int_{a_i} \left( 1 - \frac{1}{x^2} \right) p(x)dx\) we find
\[
\{S_i, J_1(x')\}_{\text{D.B.}} = \left[ \partial_\sigma - J_1(x'), \frac{1}{4\pi} \int_{a_i} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'} dx \right]. \tag{8.4.2a}
\]

Similarly, integrating around the point \(x = \infty\) and recalling the definition (6.4.7) of the global \(SU(2)_R\) charge \(\frac{R}{2} = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\infty} \left( 1 - \frac{1}{x^2} \right) p(x)dx\) we find
\[
\frac{1}{2}\{R, J_1(x')\}_{\text{D.B.}} = \left[ \partial_\sigma - J_1(x'), -\frac{1}{4\pi} \oint_{\infty} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'} dx \right]. \tag{8.4.2b}
\]

Equations (8.4.2) simply say that the Hamiltonian flow of the action variables \(S_i\) and \(R\) are generated by the following respective Lax matrices
\[
S_i \quad \longleftrightarrow \quad A_i(x') = \frac{1}{4\pi} \int_{a_i} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'} dx, \tag{8.4.3}
\]
\[
\frac{R}{2} \quad \longleftrightarrow \quad -\frac{1}{4\pi} \oint_{\infty} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'} dx.
\]

Because any integral of motion can be expressed in terms of the action variables, one ought to be able to use equation (8.4.3) to derive the Lax matrix for any other integral of motion. We start with the following lemma, for which we introduce the cohomology group \(H^1(\hat{\Sigma}, \infty^{\pm})\) of the singular curve obtained from \(\hat{\Sigma}\) by identifying the points \(\infty^{\pm}\).

**Lemma 8.4.1.** The variation of the string hierarchy Hamiltonian \(H_{n,\pm}\) along the
leaf $\mathcal{L}$ depends only on the cohomology class of $d\Omega_{n,\pm}$ in $H^1(\hat{\Sigma}, \infty^\pm)$,
\[
\delta^\mathcal{L} H_{n,\pm} = \sum_{i=1}^{g} \delta^\mathcal{L} S_i \int_{b_i} d\Omega_{n,\pm} - \frac{1}{2} \delta^\mathcal{L} R \int_{\infty^-}^{\infty^+} d\Omega_{n,\pm}. \tag{8.4.4a}
\]

Using the filling fractions to parametrise $\mathcal{L}$ this can equivalently be written as
\[
\delta^\mathcal{L} H_{n,\pm} = \sum_{I=1}^{g+1} \delta^\mathcal{L} S_I \int_{B_I} d\Omega_{n,\pm}, \tag{8.4.4b}
\]
where $B_I$ is the contour going from $\infty^+$ to $\infty^-$ through the $I^{th}$ cut.

*Proof.* Using the Riemann bilinear identity (1.5.9) with $d\Omega_1 = d\Omega_{n,\pm}$ and $d\Omega_2 = \delta^\mathcal{L} \tilde{\alpha} = -\frac{\sqrt{\lambda}}{4\pi i} \delta^\mathcal{L} (pdz)$ we find
\[
- \sum_{i=1}^{g} \int_{b_i} d\Omega_{n,\pm} \int_{a_i} \delta^\mathcal{L} \tilde{\alpha} = 2\pi i \sum_{x=-1,1,\infty} (\text{res}_{x^+} + \text{res}_{x^-}) \Omega_{\pm} \delta^\mathcal{L} \tilde{\alpha}. \tag{8.4.5}
\]

Since $\Omega_{\pm}$ is regular at infinity but $\tilde{\alpha}$ has a simple pole at $\infty^\pm$ with opposite residues, the contribution from $x = \infty$ to (8.4.5) is easily evaluated to be
\[
2\pi i (\text{res}_{\infty^+} + \text{res}_{\infty^-}) \Omega_{\pm} \delta^\mathcal{L} \tilde{\alpha} = 2\pi i (\Omega_{\pm}(\infty^+) - \Omega_{\pm}(\infty^-)) \text{res}_{\infty^+} + \text{res}_{\infty^-} \delta^\mathcal{L} \tilde{\alpha} = \int_{\infty^-}^{\infty^+} d\Omega_{\pm} \int_{\infty^+}^{\infty^-} \delta^\mathcal{L} \tilde{\alpha}. \tag{8.4.6}
\]

Equation (8.4.6) now simplifies using the definitions (8.3.5) of the action variables $S_i$ and $R$ to
\[
\sum_{i=1}^{g} \delta^\mathcal{L} S_i \int_{b_i} d\Omega_{\pm} - \frac{1}{2} \delta^\mathcal{L} R \int_{\infty^-}^{\infty^+} d\Omega_{\pm} = -i \sum_{x=-1,1} (\text{res}_{x^+} + \text{res}_{x^-}) \Omega_{\pm} \delta^\mathcal{L} \tilde{\alpha}. \tag{8.4.6}
\]

The left hand side can be evaluated using the asymptotics (7.2.12) of the differentials $d\Omega_{n,\pm}$ at $x = \pm 1$, the explicit form (7.2.5) of the singular parts $s_{n,\pm}(x)$, the
expansion (5.4.6) of the quasi-momentum as well as the definition (5.4.13) of the string hierarchy Hamiltonians \( H_{n,\pm} \). The final result is equation (8.4.4a). Rewriting this equation as

\[
\delta L_{H_{n,\pm}} = g \sum_{i=1}^{g} \left( \int_{b_i} d\Omega_{n,\pm} - \int_{-\infty}^{\infty} d\Omega_{n,\pm} \right) \delta \mathcal{L} S_i - \left( \int_{-\infty}^{\infty} d\Omega_{n,\pm} \right) \delta \mathcal{L} \left( \frac{1}{2} R - \sum_{i=1}^{g} S_i \right),
\]

and using (6.4.9) along with the fact that \( L \) is constant along the leaf \( \mathcal{L} \) by definition gives equation (8.4.4b).

As a special case of lemma 8.4.1 consider the 0th level of the string hierarchy. We have \( H_{0,\pm} = \mathcal{E} \pm \mathcal{P} \) and \( d\Omega_{0,\pm} = dq_\pm/2\pi \) so that

\[\delta \mathcal{L} (\mathcal{E} \pm \mathcal{P}) = g \sum_{i=1}^{g} \delta \mathcal{L} S_i \int_{b_i} dq_\pm/2\pi \left( \mathcal{E} \pm \mathcal{P} \right), \quad (8.4.7a)\]

\[= \sum_{i=1}^{g+1} \delta \mathcal{L} S_i \int_{b_i} dq_\pm/2\pi. \quad (8.4.7b)\]

It follows immediately from lemma 8.4.1 that

\[\{H_{n,\pm}, \cdot\}_{\text{D.B.}} = \sum_{i=1}^{g} \left( \int_{b_i} d\Omega_{n,\pm} \right) \{S_i, \cdot\}_{\text{D.B.}} - \left( \int_{-\infty}^{\infty} d\Omega_{n,\pm} \right) \frac{1}{2} \{R, \cdot\}_{\text{D.B.}}. \]

Making use of the Lax matrix for the action variables (8.4.3) and the fact that the

\[\{S_i, \cdot\}_{\text{D.B.}} = \{R, \cdot\}_{\text{D.B.}}. \]

\[\mathcal{E} \pm \mathcal{P} \]
differentials $d\Omega_{n,\pm}$ are normalised we can write the Lax matrix for $H_{n,\pm}$ as

$$H_{n,\pm} \longleftrightarrow \frac{1}{4\pi} \sum_{i=1}^{9} \left[ \int_{a_i} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'} dx \int_{b_i} d\Omega_{n,\pm} - \int_{b_i} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'} dx \int_{a_i} d\Omega_{n,\pm} \right] + \frac{1}{4\pi} \oint_{\infty} \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'} dx \int_{\infty}^{\infty} d\Omega_{n,\pm}.$$ 

Written in this form we can apply a Riemann bilinear identity. Specifically we note that the Riemann bilinear identity (8.4.5) and the equation following it in the proof of lemma 8.4.1 are valid for any differential $\delta^x \tilde{\alpha}$ which has simple poles at $\infty^\pm$ of opposite residues there. But this is true of $\Psi(x)\sigma_3\Psi(x)^{-1}$ which has simple poles only at $x'^\pm, \infty^\pm$ since the poles of $\Psi(x)\sigma_3\Psi(x)^{-1}$ at the branch points cancel with the zeroes of $dx$. Furthermore, the residues at $\infty^\pm$ are opposite because viewing $\Psi(x) = (\psi(P), \psi(\tilde{\sigma}P))$ as a function $\Psi(P)$ on $\tilde{\Sigma}$ we have $\Psi(\tilde{\sigma}P) = \Psi(P)\sigma_1$ and hence $\Psi(\tilde{\sigma}P)\sigma_3\Psi(\tilde{\sigma}P)^{-1} = -\Psi(P)\sigma_3\Psi(P)^{-1}$. Therefore

$$H_{n,\pm} \longleftrightarrow -i \left( \text{res}_{x=1} + \text{res}_{x=-1} \right) \frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'} \Omega_{n,\pm}(x^+) dx, \quad (8.4.8)$$

where an overall factor of two came from the fact that we get equivalent contributions from both sheets, namely at $x^\pm = (+1)^\pm$ and $x^\pm = (-1)^\pm$. Note also importantly that there is no contribution from the apparent pole at $x = x'$ because this is not actually a pole of the Lax equation itself. This can be seen from (8.4.1) which is perfectly regular as $x$ approaches $x'$ since $[\partial_{t_{n,\pm}} - J_{n,\pm}(x'), \Psi(x')\sigma_3\Psi(x')^{-1}] = 0$ from (5.4.18) and the trivial fact that diagonal matrices commute. As already remarked in section 5.4, an equation such as (8.4.8) relating an integral of motion to a Lax matrix should really always be understood as a relation between two ingredients of a Lax equation. To evaluate the residues in (8.4.8) we use the identity (5.4.10) and
the asymptotics of the differentials $d\Omega_{n,\pm}$ at $x = \pm 1$. One finds

$$H_{n,\pm} \longleftrightarrow (\Psi(x')i\sigma_3\Psi(x')^{-1}\Omega_{n,\pm}(x'^+))_{\pm 1}$$

which is exactly the expression (7.2.4) for the hierarchy of Lax matrices, c.f. (5.4.12).

It is important to note that it was the multi-valuedness of the Abelian integral $\Omega_{n,\pm}(P) = \int P d\Omega_{n,\pm}$ (or equivalently the fact that $d\Omega_{n,\pm}$ had some non-trivial periods) which resulted in a non-zero answer for the corresponding Lax matrix. Indeed, the Lax matrix obtained by this argument clearly depends only on the cohomology class $[d\Omega_{n,\pm}] \in H^1(\hat{\Sigma}, \mathbb{C})$ of the Abelian differential $d\Omega_{n,\pm}$ one starts off with on the singular algebraic curve $\hat{\Sigma}/\{\infty^{\pm}\}$. One can see this explicitly from the equation preceding (8.4.8) or otherwise from (8.4.8) itself: suppose $d\Omega_{n,\pm}, d\Omega'_{n,\pm}$ are two representatives of the same cohomology class, then $d\Omega_{n,\pm} - d\Omega'_{n,\pm} = df$ is exact with $f(\infty^+) = f(\infty^-)$ and the corresponding difference of the expressions in (8.4.8) is

$$-\frac{i}{2} \sum_{P \in \{\pm 1\}^{\pm}} \text{res}_P \frac{\Psi(P)s_3\Psi(P)^{-1}}{x(P) - x'} f(P)dx,$$

where $\Psi(P) = (\psi(P), \psi(\hat{\sigma}P))$. But this is the sum over the residues of a well defined meromorphic differential on $\hat{\Sigma}/\{\infty^{\pm}\}$ (since $f(P)$ is single-valued and the residues at $\infty^{\pm}$ cancel against each other since $f(\infty^+) = f(\infty^-)$) and so is zero.

One could use the same trick as above to compute more explicitly the Lax matrices for the action variables (8.4.3). To apply the previous reasoning we write

$$\delta S_I = \sum_{J=1}^{g+1} \delta_{IJ} \delta S_J.$$

For the same argument to follow through we must introduce second kind Abelian
differentials \( dq^{(J)} \) with specific periods

\[
\int_{A_I} dq^{(J)} = 0, \quad \int_{B_I} dq^{(J)} = \delta_{IJ}.
\]

(8.4.9)

Such differentials exist: consider \( g + 1 \) independent differentials \( d\Omega_J \) from the hierarchy. Then \( A_{IJ} = \int_{B_I} d\Omega_J \) is invertible and \( dq^{(J)} = A_{IJ}^{-1} d\Omega_K \) have the desired property. Yet since the conditions (8.4.9) on the differentials \( dq^{(J)} \) uniquely specify their cohomology class in \( H^1(\hat{\Sigma}, \infty^\pm) \), by the preceding remark they are also sufficient to uniquely fix the resulting Lax matrix

\[
S_I \quad \longleftrightarrow \quad (\Psi(x') i\sigma_3 \Psi(x')^{-1} q^{(I)}(x'^+))_{+1} + (\Psi(x') i\sigma_3 \Psi(x')^{-1} q^{(I)}(x'^+))_{-1}.
\]

By the procedure of section 7.2 (see in particular the proof of lemma 7.2.2) these Lax matrices yield unique normalised Abelian differentials which satisfy (8.4.9), which we still denote \( dq^{(J)} \) by abuse of notation. Since the operations of constructing a Lax matrix from a given integral of motion and that of constructing an Abelian differential from a given Lax matrix are both linear, it follows that the equation for \( H_{n,\pm} \) in (8.4.4b) translates into an equation in terms of differential forms on \( \hat{\Sigma}/\{\infty^\pm\} \), namely

\[
d\Omega_{n,\pm} = \sum_{I=1}^{g+1} \left( \int_{B_I} d\Omega_{n,\pm} \right) dq^{(I)}.
\]

(8.4.10)

In particular at the 0th level \( n = 0 \) this equation provides an important formula for the differential of the quasi-energy that we will need in chapter 10,

\[
dq = \sum_{I=1}^{g+1} \left( \int_{B_I} dq \right) dq^{(I)}.
\]

(8.4.11)
Chapter 9

Real closed strings

“Reality continues to ruin my life.”
Calvin, Calvin and Hobbes

The method of finite-gap integration described in chapters 6 and 7 heavily relied on complex analysis and the theory of Riemann surfaces. This was to make use of powerful theorems such as the Riemann-Roch theorem to reconstruct solutions. All solutions obtained by this method are built out of a combination of meromorphic and Baker-Akhiezer functions \( \hat{\Sigma} \to \mathbb{C} \) from a Riemann surface \( \hat{\Sigma} \) into \( \mathbb{C} \). In particular, the phase-space coordinate of the string \( j \) reconstructed in theorem 7.4.1 is \( \mathfrak{sl}(2, \mathbb{C}) \)-valued and the corresponding embedding \( g \) of proposition 7.4.2 is \( SL(2, \mathbb{C}) \)-valued. However, the differential equations we set out to solve were all equations for physical strings whose embedding into the target space \( \mathbb{R} \times S^3 \) is described by an \( SU(2) \)-valued map \( g(\sigma, \tau) \in SU(2) \). Furthermore, the closed string boundary conditions require these embeddings to be \( 2\pi \)-periodic in \( \sigma \). It is therefore important to identify the subset of solutions among all those constructed by the finite-gap method which are both real (i.e. \( SU(2) \)-valued) and periodic in \( \sigma \).
The way to obtain real periodic solutions will simply be to restrict the allowed
algebro-geometric data. In the language of chapter 8 a genus \( g \) finite-gap solution
is a geometric map (8.2.2) from the \( 2g + 2 \) complex dimensional Jacobian bundle
\( \mathcal{M}_c^{(2g+2)} \) into the space of complexified solutions \( S_c^\infty \). The restriction to real algebro-
geometric data giving rise to real solutions through the geometric map (8.2.2) can
be identified with a sub-bundle \( \mathcal{M}_r^{(2g+2)} \) of \( \mathcal{M}_c^{(2g+2)} \). As we will see, the real slice of
the generalised Jacobian \( J_m(\hat{\Sigma}) \) is simply a real \((g + 1)\)-torus \( \mathbb{T}^{g+1} \) (with \( g + 1 \) factors of \( S^1 \)) and the real part \( L_R \) of the leaf \( L \) is parametrised by real
values of the filling fractions,

\[
\mathbb{T}^{g+1} \rightarrow \mathcal{M}_R^{(2g+2)} \rightarrow L_R.
\]

The restriction \( G'_R = G'|_{\mathcal{M}_R^{(2g+2)}} \) of the geometric map to the real bundle \( \mathcal{M}_R^{(2g+2)} \) is
an injective map from real algebro-geometric data to the space of real solutions \( S_R^\infty \),

\[
\mathcal{M}_c^{(2g+2)} \xrightarrow{G'} S_c^\infty \\
\mathcal{M}_R^{(2g+2)} \xrightarrow{G'_R} S_R^\infty
\]

By further restricting the real geometric map \( G'_R \) to a sub-bundle of the real
algebro-geometric data \( \mathcal{M}_R^{(2g+2)} \) corresponding to data satisfying certain periodicity
conditions, its image will consist of real periodic solutions.

9.1 Real curves

To identify the restrictions imposed by the reality conditions on the various curves
we go back to their respective definitions in chapter 6.
The spectral curve

The spectral curve is defined by equation (6.1.1) in terms of the monodromy matrix which was defined as the path-ordered exponential of the Lax connection,

\[ \Omega(x, \sigma, \tau) = P \exp \int_{\sigma}^{\sigma + 2\pi} J(x), \quad J(x) = \frac{1}{1 - x^2} (j - x^* j). \]

Now the requirement that the current \( j \in \mathfrak{su}(2) \) is equivalent to \( j^\dagger = -j \) which implies reality conditions on \( J(x) \) and \( \Omega(x) \) in turn. Specifically, for \( J(x) \) we have

\[ J(x)^\dagger = \frac{1}{1 - \bar{x}^2} (j^\dagger - \bar{x}^* j^\dagger) = -\frac{1}{1 - \bar{x}^2} (j - \bar{x}^* j) = -J(\bar{x}). \]

This implies the following reality conditions on \( \Omega(x) \)

\[ \Omega(x)^\dagger = \left( P \exp \int_{\sigma}^{\sigma + 2\pi} J(x) \right)^\dagger = P \exp \int_{\sigma}^{\sigma + 2\pi} J(x)^\dagger = P \exp \int_{\sigma}^{\sigma + 2\pi} -J(\bar{x}) = \left( P \exp \int_{\sigma}^{\sigma + 2\pi} J(\bar{x}) \right)^{-1} = \Omega(\bar{x})^{-1}. \quad (9.1.1) \]

In particular, for real values of \( x \in \mathbb{R} \) we have \( J(x) \in \mathfrak{su}(2) \) and \( \Omega(x) \in SU(2) \).

**Definition 9.1.1.** A curve \( C \) in \( \mathbb{C}^2 \) is real if it admits an anti-holomorphic involution \( \hat{\tau} : C \to C \). That is, \( \hat{\tau}^2 = 1 \) and for any function \( f \) on \( C \) holomorphic in a neighbourhood \( U \subset C \) the function \( f \circ \hat{\tau} \) is anti-holomorphic in \( \hat{\tau}(U) \).

A simple example of a real curve is the complex plane \( \mathbb{C} \) itself, which obviously admits complex conjugation \( \hat{\tau} : x \to \bar{x} \) as an anti-holomorphic involution. In particular a real curve is still ‘complex’ in the sense that it locally looks like \( \mathbb{C} \).

**Lemma 9.1.2.** The spectral curve \( \Gamma \) is real with anti-holomorphic involution

\[ \hat{\tau} : \quad \Gamma \to \Gamma, \quad (x, \Lambda) \mapsto (\bar{x}, \bar{\Lambda}^{-1}). \quad (9.1.2) \]
\textit{Proof.} Let \((x, \Lambda) \in \Gamma\) then by definition \(\det(\Lambda \mathbf{1} - \Omega(x)) = 0\). Taking the complex conjugate and using (9.1.1) yields \(\det(\bar{\Lambda} \mathbf{1} - \Omega(\bar{x})^{-1}) = 0\). Then provided \(\Lambda \neq 0\) and since \(\det \Omega(x) = 1 \neq 0\) we have \(\det(\Omega(\bar{x}) - \bar{\Lambda}^{-1} \mathbf{1}) = 0\) which means that \((\bar{x}, \bar{\Lambda}^{-1}) \in \Gamma\). Therefore the map \((x, \Lambda) \mapsto (\bar{x}, \bar{\Lambda}^{-1})\) sends the curve \(\Gamma\) to itself. Moreover it clearly squares to one and is anti-holomorphic. \(\square\)

This anti-holomorphic involution can be combined with the holomorphic involution \(\hat{\sigma}\) defined in (6.1.3). Together they generate a \(\mathbb{Z}_2 \times \mathbb{Z}_2\) group of involutions on \(\Gamma\) such that \(\hat{\sigma} \hat{\tau} = \hat{\tau} \hat{\sigma}\). Recall that each point \(x_0 \in \mathcal{Z}_\Gamma\) corresponding to degenerate eigenvalues of the monodromy matrix was a fixed point of \(\hat{\sigma}\). A slightly weaker statement is true for \(\hat{\tau}\).

\textbf{Lemma 9.1.3.} The discriminant \(\Delta_\Gamma(x)\) satisfies \(\overline{\Delta_\Gamma(x)} = \Delta_\Gamma(\bar{x})\). In particular, its set of zeroes \(\mathcal{Z}_\Gamma\) is invariant under \(\hat{\tau}\).

\textit{Proof.} Consider \(\Delta_\Gamma(\bar{x}) = (\Lambda_+ - \Lambda_-)^2\). By lemma 9.1.2 the eigenvalues \(\Lambda_{\pm}(\bar{x})\) above \(\bar{x}\) can equally be written \(\Lambda_{\pm}(x)^{-1}\). Thus \(\Delta_\Gamma(\bar{x}) = (\Lambda_+^{-1}(x) - \Lambda_-^{-1}(x))^2\) which can be rewritten as \(\overline{\Delta_\Gamma(x)}\) using (6.1.2). Hence \(x_0 \in \mathcal{Z}_\Gamma \Leftrightarrow \bar{x}_0 \in \mathcal{Z}_\Gamma\). \(\square\)

Recall from chapter 6 that the points \(x_0 \in \mathcal{Z}_\Gamma\) fall into one of two categories:

1. \(\Delta_\Gamma(x) = O(x - x_0)^{2r+1}\), i.e. \(x_0\) is a branch point or cusp-like singularity.

2. \(\Delta_\Gamma(x) = O(x - x_0)^{2r}\), i.e. \(x_0\) is a node-like singularity.

It follows from lemma 9.1.3 that the order of the zero is also preserved under the action of \(\hat{\tau}\). Thus branch points are mapped to branch points, cusps to cusps, and so on. The next lemma shows that branch points and cusp-like singularities must all lie off the real axis.

\textbf{Lemma 9.1.4.} If \(x_0 \in \mathcal{Z}_\Gamma \cap \mathbb{R}\) then \(x_0\) corresponds to a node-like singularity.
Proof. Let \( \mathfrak{P}_0 = (x_0, \Lambda_0) \in \Gamma \) with \( x_0 \in \mathcal{Z}_\Gamma \). We assume \( \hat{\tau} \mathfrak{P}_0 = \mathfrak{P}_0 \) (\( \Leftrightarrow x_0 \in \mathbb{R} \)) and show \( \mathfrak{P}_0 \) is node-like. Since \( x_0 \in \mathbb{R} \), we can write \( \Omega(x_0) \in SU(2) \) as

\[
\Omega(x_0) = \left( \begin{array}{cc}
A(x_0) & B(x_0) \\
-B(x_0) & A(x_0)
\end{array} \right).
\]

But then \( |A(x_0) - \Lambda_0|^2 + |B(x_0)|^2 = 0 \) since \( \Lambda_0 = \pm 1 \in \mathbb{R} \). Hence \( A(x_0) = \Lambda_0 \) and \( B(x_0) = 0 \) which implies that \( \Omega(x_0) \) is diagonal. Therefore \( \dim \mathcal{E}_\Gamma(\mathfrak{P}_0) = 2 \) and \( \mathfrak{P}_0 \) must be node-like by proposition 6.1.3. \( \Box \)

Recall that the involution \( \hat{\sigma} \) had the effect of interchanging the two sheets of the spectral curve \( \Gamma \). We wish to similarly describe the effect of \( \hat{\tau} \) on the individual sheets. Since the branch points and cusp-like singularities all come in complex conjugate pairs by lemma 9.1.4 we choose the cuts in the complex \( x \)-plane to be vertical, connecting a branch point with its reflection through the real axis (see figure 9.1). With this choice the set of cuts is invariant under \( \hat{\tau} \) which allows us to describe its effect as follows,

**Lemma 9.1.5.** The involution \( \hat{\tau} \) maps both sheets to themselves by \( x \mapsto \bar{x} \).

**Proof.** Consider the points \( (x, \Lambda_{\pm}(x)) \in \Gamma \) above \( x \in \mathbb{C} \) on the upper and lower sheet of the spectral curve. When \( x \in \mathbb{R} \) we have \( \Omega(x) \in SU(2) \) so that \( |\Lambda_{\pm}(x)| = 1 \) and hence \( \bar{\Lambda_{\pm}(x)}^{-1} = \Lambda_{\pm}(x) \). Now let \( x \in \mathbb{C} \). Then by equation (9.1.1) the eigenvalues \( \{\Lambda_{\pm}(\bar{x})\} \) of \( \Omega(\bar{x}) \) can equally be written \( \{\bar{\Lambda_{\pm}(x)}^{-1}\} \). Therefore by continuity, the equality \( \bar{\Lambda_{\pm}(x)}^{-1} = \Lambda_{\pm}(\bar{x}) \) which holds for \( x \in \mathbb{R} \) must also hold for all \( x \) in the cut plane. It follows that

\[
\hat{\tau}(x, \Lambda_{\pm}(x)) = (\bar{x}, \bar{\Lambda_{\pm}(x)}^{-1}) = (\bar{x}, \Lambda_{\pm}(\bar{x})).
\]

In other words, the point above \( x \) on the upper (respectively lower) sheet is mapped
The algebraic curve

The algebraic curve is defined by equation (6.2.2) in terms of a chosen combination of Lax matrices $L(x) = \sum_N c_N J_N(x)$ where $J_N(x) = (\Psi(x)s_N(x)\sigma_3\Psi^{-1}(x))_{\pm 1}$ and $s_N(x)$ are the singular parts defined in (7.2.5). Since $\Psi(x)$ is the matrix of eigenvectors of $\Omega(x)$ it satisfies $\Omega(x)\Psi(x) = \Psi(x)\text{diag}(\Lambda_+(x), \Lambda_-(x))$. Taking the hermitian conjugate followed by the inverse yields $\Omega(\bar{x})(\Psi(x)^\dagger)^{-1} = (\Psi(x)^\dagger)^{-1}\text{diag}(\Lambda_+(\bar{x}), \Lambda_-(\bar{x}))$ from which it follows that $\Psi(\bar{x}) = (\Psi(x)^\dagger)^{-1}D$ for some diagonal matrix $D$. It follows that $(\Psi(x)^{-1})^\dagger = \Psi(\bar{x})D^{-1}$ and $\Psi(x)^\dagger = D\Psi(\bar{x})^{-1}$ which combined with the fact that $s_N(x) = -s_N(\bar{x})$ and the assumption that $c_N \in \mathbb{R}$ gives

$$L(x)^\dagger = -L(\bar{x}). \quad (9.1.3)$$

The anti-holomorphic involution (9.1.2) of the spectral curve $\Gamma$ induces an anti-holomorphic involution (also denoted $\hat{\tau}$) on the algebraic curve $\Sigma$ which is easily obtained using (9.1.3).

**Lemma 9.1.6.** The algebraic curve $\Sigma$ is real with anti-holomorphic involution

$$\hat{\tau} : \Sigma \to \Sigma, \quad (x, y) \mapsto (\bar{x}, -\bar{y}). \quad (9.1.4)$$

**Proof.** Let $(x, y) \in \Sigma$ then by definition $\det(y1 - L(x)) = 0$. Taking the complex conjugate and using (9.1.3) yields $\det(\bar{y}1 + L(\bar{x})) = 0$ so that $(\bar{x}, -\bar{y}) \in \Sigma$. \qed

**Remark** It is straightforward to check that the statements of the lemmas 9.1.3, 9.1.4 and 9.1.5 equally apply to the algebraic curve $\Sigma$ with the involution (9.1.4). This is to be expected since $\Sigma$ is a (partial) normalisation of $\Gamma$. The proofs of the lemmas for $\Sigma$ are essentially the same.
as those for $\Gamma$ so we do not repeat them. We simply note that when $x \in \mathbb{R}$ the reality condition (9.1.3) says that $L(x) \in \mathfrak{su}(2)$ which can be used to prove the analogues of lemmas 9.1.4 and 9.1.5.

### The Riemann surface

The Riemann surface $\hat{\Sigma}$ was defined by equation (6.2.8). Because it is merely the normalisation of $\Sigma$, the involution $\hat{\tau} : \Sigma \to \Sigma$ naturally induces an anti-holomorphic involution on the Riemann surface $\hat{\Sigma}$ defined by exactly the same formula,

$$\hat{\tau} : \hat{\Sigma} \to \hat{\Sigma}, \quad (x, y) \mapsto (\bar{x}, -\bar{y}).$$

Since the full set of branch points $\{u_I, v_I\}_{I=1}^{g+1}$ of the Riemann surface $\hat{\Sigma}$ must be invariant under $\hat{\tau}$ by lemma 9.1.3 and none of them can be real by lemma 9.1.4, the only possibility is that they form complex conjugate pairs. We can therefore set $v_I = \bar{u}_I$ in (6.2.8) so that the general real Riemann surface takes the form,

$$\hat{\Sigma} : y^2 = \prod_{i=1}^{g+1} (x - u_i)(x - \bar{u}_i). \quad (9.1.5)$$

In order to specify the quasi-momentum which is normalised with respect to the $a$-cycles we must choose a canonical basis of $a$- and $b$-cycles. As in section 6.3 we will choose the $a$-cycles to encircle $g$ of the cuts. As for the canonically conjugate $b$-cycles, in the case of a real curve (9.1.5) it is convenient to choose them as shown in figure 9.1. The homology classes of these basis cycles are easily shown to have the following properties under the action of the anti-holomorphic involution $\hat{\tau}$

$$\hat{\tau}a_i \sim -a_i, \quad \hat{\tau}b_i \sim b_i + a_i + \sum_{j=1}^{g+1} a_j, \quad (9.1.6)$$

where $\sim$ denotes homology equivalence so that these expressions are to be un-
understood modulo cycles homologous to zero. In particular the \(a\)-cycles are pure imaginary. The reality conditions (9.1.6) on the basis homology cycles then in-

![Figure 9.1: Canonical choice of a- and b-cycles for a real curve.](image)

duce reality conditions on meromorphic differentials. For instance the vector \(\vec{\omega} = (\omega_1, \ldots, \omega_g, \omega_\infty)^T\), made up of the holomorphic differentials \(\{\omega_i\}_{i=1}^g\) and the third kind Abelian differential \(\omega_\infty\) defined in (7.2.10), has the following transformation under pullback by \(\hat{\tau}\). These are to be compared with the transformation property under pullback by \(\hat{\sigma}\) which reads \(\hat{\sigma}^*\vec{\omega} = -\vec{\omega}\).

**Lemma 9.1.7.** \(\hat{\tau}^*\vec{\omega} = -\vec{\omega}\).

**Proof.** The differentials \(\hat{\tau}^*\omega_i\) are holomorphic. Indeed, the holomorphic differentials \(\omega_i\) can locally be written as \(df_i\) for some holomorphic \(f_i\). Then locally we have \(\hat{\tau}^*\omega_i = d(f_i \circ \hat{\tau})\), where \(f_i \circ \hat{\tau}\) are holomorphic. Furthermore,

\[
\int_{a_i} \hat{\tau}^*\omega_j = \int_{a_i} \hat{\tau}^*\omega_j = \int_{\hat{\tau}a_i} \omega_j = -\int_{a_i} \omega_j = -\delta_{ij}.
\]

Therefore by lemma 1.5.4 we have \(-\hat{\tau}^*\omega_i = \omega_i\). As for the third kind differential \(\omega_\infty\), since its residues are pure imaginary it is clear that \(-\hat{\tau}^*\omega_\infty\) (which is also an Abelian differential of the third kind) has the same residues at the poles \(\infty^\pm\) and is
still normalised because

\[ \int_{a_i} \tau^* \omega_\infty = \int_{a_i} \hat{\tau}^* \omega_\infty = \int_{\tau a_i} \omega_\infty = -\int a_i \omega_\infty = 0. \]

It then follows by lemma 1.5.10 that \( -\hat{\tau}^* \omega_\infty = \omega_\infty \).

\[ \square \]

**Corollary 9.1.8.** Let \( D \geq 0 \) be an integral divisor. Then \( \bar{A}(\hat{\tau} D) = -\bar{A}(D) \).

**Proof.** Let \( D = P_1 + \ldots + P_n \) be an integral divisor of degree \( n \). Then

\[ \bar{A}(\hat{\tau} D) = 2\pi \sum_{i=1}^{n} \int_{\infty^+} P_i \omega = 2\pi \sum_{i=1}^{n} \int_{\infty^+} \hat{\tau}^* \omega = -2\pi \sum_{i=1}^{n} \int_{\infty^+} \bar{\omega} = -\bar{\bar{A}(D)}, \]

where the second equality holds on \( J_m(\hat{\Sigma}) \) since the path \( [\hat{\tau} P_i, \infty^+] \) is only equal to the path \( \hat{\tau} [P_i, \infty^+] \) modulo \( a- \) and \( b- \) cycles.

\[ \square \]

The reality condition on the homology basis also induces reality conditions on the differential \( d\bar{Q} \). Since this differential generates the linear flow on the generalised Jacobian \( J_m(\hat{\Sigma}) \) by theorem 7.2.9, the reality conditions on \( d\bar{Q} \) immediately allow us to conclude that the motion of the dynamical divisor \( \hat{\gamma}(t) \) on \( J_m(\hat{\Sigma}) \) is restricted to a real slice in the sense of the following lemma. In the next section we will describe the real slice of the Jacobian in more detail by discussing the reality conditions on the dynamical divisor itself.

**Lemma 9.1.9.** \( \bar{\tau}^* d\bar{Q} = d\bar{Q} \). In particular \( \int_b d\bar{Q} \in \mathbb{R}^{g+1} \).

**Proof.** \( \bar{\tau}^* d\bar{Q} \) is a second kind Abelian differential with the same singular parts as \( d\bar{Q} \), so the result follows by lemma 1.5.10. Moreover, \( \int_b d\bar{Q} = \int_b \hat{\tau}^* d\bar{Q} = \int_{\hat{\tau} b} d\bar{Q} = \int_{b} d\bar{Q} \) using the fact that \( d\bar{Q} \) is normalised together with the reality conditions (9.1.6) on the \( b- \) cycles and the fact that \( \hat{\tau} b_\infty \) is \( b_\infty \) plus a linear combination of \( a- \) cycles. \[ \square \]
We can also show that the reality conditions restrict the base $\mathcal{L}$ of the Jacobian bundle $\mathcal{M}_C^{(2g+2)}$ to a sub-leaf $\mathcal{L}_R$ as advertised at the start of this chapter. The next lemma shows that $\mathcal{L}_R \subset \mathcal{L}$ is parameterised by real filling fractions.

**Lemma 9.1.10.** The filling fractions are real, namely $s_I \in \mathbb{R}$, $I = 1, \ldots, g + 1$.

**Proof.** The differential $\alpha = \sqrt{\frac{1}{4\pi}} z dp$ is real by lemma 9.1.9 that is one can show $\bar{\tau^*} \alpha = \alpha$. Hence

$$S_I = -\frac{1}{2\pi i} \int_{A_I} \bar{\alpha} = -\frac{1}{2\pi i} \int_{A_I} \hat{\tau}^* \alpha = -\frac{1}{2\pi i} \int_{\hat{\tau}A_I} \alpha = \frac{1}{2\pi i} \int_{A_I} \alpha = S_I,$$

using the fact that $\hat{\tau}A_I = -A_I$, $I = 1, \ldots, g + 1$. ⊓⊔

### 9.2 Real divisor

Let the dual dynamical divisor $\hat{\gamma}^+(t)$ be the poles of the dual normalised eigenvector $h^+$ satisfying,

$$h^+(P) (\Omega(x) - \Lambda(P) \mathbf{1}) = 0, \quad h^+(P) \cdot \alpha^T = 1. \quad (9.2.1)$$

It is related in a very simple way to the dynamical divisor $\hat{\gamma}(t)$ which was defined as the poles of the ordinary normalised eigenvector $h$ satisfying,

$$(\Omega(x) - \Lambda(P) \mathbf{1}) h(P) = 0, \quad \alpha \cdot h(P) = 1. \quad (9.2.2)$$

**Lemma 9.2.1.** The reality condition on the dynamical divisor is $\hat{\gamma}^+(t) = \hat{\tau} \hat{\gamma}(t)$.

**Proof.** Taking the dual normalised eigenvector equation (9.2.1) at the point $\hat{\tau}P$ it
can be rewritten as

\[ h^+(\hat{\tau}P) \Lambda(\hat{\tau}P)^{-1} - h^+(\hat{\tau}P) \Omega(\bar{x})^{-1} = 0, \quad h^+(\hat{\tau}P) \cdot \alpha^T = 1. \] (9.2.3)

Using the reality conditions on the monodromy matrix we have \( \Omega(\bar{x})^\dagger \cdot \Omega(x) = 1. \)
Then \( h(\hat{\tau}(P))^\dagger \cdot h(P) = h(\hat{\tau}(P))^\dagger \Omega(\bar{x})^\dagger \cdot \Omega(x) h(P) = (\Omega(\bar{x}) h(\hat{\tau}(P)))^\dagger \cdot \Omega(x) h(P) = \Lambda(\hat{\tau}P)^\dagger \Lambda(P) h(\hat{\tau}(P))^\dagger \cdot h(P) \) so that \( \Lambda(\hat{\tau}P)^{-1} = \Lambda(P). \) Taking the hermitian conjugate of (9.2.3) it can now be rewritten as

\[ (\Omega(x) - \Lambda(P)1) (h^+(\hat{\tau}P))^\dagger = 0, \quad \alpha \cdot (h^+(\hat{\tau}P))^\dagger = 1. \] (9.2.4)

Now by proposition 6.1.3 there is a unique solution to (9.2.2) and hence (9.2.4) implies that \( h^+(P) = h(\hat{\tau}P)^\dagger. \) These vectors have poles at \( \hat{\gamma}^+(t) \) and \( \hat{\tau}\hat{\gamma}(t) \) respectively, so the result follows. \( \square \)

**Remark** Recall that in chapter 7 we could characterise the equivalence class \([\hat{\gamma}^+(t)]\) of the dual dynamical divisor using lemma 7.3.3 although quite implicitly. The divisor \( \hat{\gamma}^+(t) \) then had to be chosen arbitrarily from the class \([\hat{\gamma}^+(t)]\) and the reconstruction of the inverse matrix in proposition 7.3.4 required a residual gauge transformation because the corresponding normalised eigenvector \( h^+(P) \) was expressed in the ‘wrong’ residual gauge. Here lemma 9.2.1 says that the dual dynamical divisor \( \hat{\gamma}^+(t) \) corresponding to the poles of the dual normalised eigenvector \( h^+(P) \) in the ‘correct’ residual gauge can be immediately obtained from the dynamical divisor \( \hat{\gamma}(t) \) by complex conjugation. This avoids the worry of having to correct the residual gauge in reconstructing the inverse matrix as was done in proposition 7.3.4 since one can simply use proposition 7.3.2 if the correct dual dynamical divisor is known.

**Corollary 9.2.2.** Real divisors satisfy 
\[ 2\text{Im } \bar{A}(\hat{\gamma}(t)) = \bar{A}(B). \]

**Proof.** Combining equation (7.3.2) of lemma 7.3.3 with lemma 9.2.1 and corollary 9.1.8 we have \( \bar{A}(B) = \bar{A}(\hat{\gamma}(t)) + \bar{A}(\hat{\tau}\hat{\gamma}(t)) = \bar{A}(\hat{\gamma}(t)) - \bar{A}(\hat{\gamma}(t)) = 2\text{Im } \bar{A}(\hat{\gamma}(t)). \) \( \square \)
CHAPTER 9. REAL CLOSED STRINGS

Although the generalised Jacobian $J_m(\hat{\Sigma})$ is a non-compact Abelian group it turns out that its real slice, \textit{i.e.} the generalised Abel map of real divisors in lemma 9.2.1, is a real $(g+1)$-dimensional torus.

**Corollary 9.2.3.** The real slice of $J_m(\hat{\Sigma})$ is a $(g+1)$-torus $\mathbb{T}^{g+1}$ given explicitly by

$$\{ \vec{X} \in J_m(\hat{\Sigma}) \mid 2\text{Im } \vec{X} = \vec{A}(B) \} \subset J_m(\hat{\Sigma}).$$

(9.2.5)

It is a translation of the real torus $\mathbb{R}^{g+1}/2\pi \mathbb{Z}^{g+1} \subset J_m(\hat{\Sigma})$ by the vector $\vec{X}_0 \equiv \frac{1}{2} \vec{A}(B)$.

\textit{Proof.} Recall that the generalised Jacobian $J_m(\hat{\Sigma})$ is the quotient of $\mathbb{C}^{g+1}$ by the lattice $\Lambda_m$ spanned by $2\pi$ multiples of the $2g+1$ linearly independent vectors (over $\mathbb{R}$)

$$\begin{pmatrix}
\delta_{ij} & 0 & \Pi_{ij} \\
0 & 1 & \Pi_j
\end{pmatrix},$$

(9.2.6)

where $\Pi_j = \int_{b_j} \omega_\infty$ and $\Pi_{ij} = \int_{b_j} \omega_i$. Since $\text{Im } \Pi_{ij}$ is positive definite by lemma 1.7.3 the last $g$ column vectors in (9.2.6) have a non-zero imaginary part. Now any vector $\vec{X}$ belonging to (9.2.5) can be written as $\vec{X} = \vec{X}_0 + \vec{V}$ where $\text{Im } \vec{V} = 0$. Hence $\vec{V}$ must be a linear combination of the first $g+1$ real columns in (9.2.6) which span the real torus $\mathbb{R}^{g+1}/2\pi \mathbb{Z}^{g+1} \subset J_m(\hat{\Sigma})$. The real slice (9.2.5) is a translation by $\vec{X}_0$. \qed

\section*{9.3 Periodicity}

For closed strings the embedding field $g(\sigma, \tau)$ is periodic under $\sigma \rightarrow \sigma + 2\pi$. And just as for the reality conditions on $g(\sigma, \tau)$, this periodicity condition imposes restrictions on the allowed algebro-geometric data.

Because the configuration of a finite-gap string is specified by the position of the point $\vec{A}(\hat{\gamma}(t)) \in J_m(\hat{\Sigma})$ on the generalised Jacobian, a necessary condition for
the string to be closed is that the motion of this point be \( \sigma \)-periodic on \( J_m(\hat{\Sigma}) \). Yet we know from theorem 7.2.9 that the motion of this point is linear on \( J_m(\hat{\Sigma}) \) in all the higher times \( \{t\} \), and in particular in \( \sigma \) and \( \tau \). For a generic complex solution, since the generalised Jacobian is non-compact, the linear motion could very well never come back to itself. However, by corollary 9.2.3 the real slice of the Jacobian is a real \((g+1)\)-torus. Since this is compact, in the real case the linear motion of theorem 7.2.9 must wrap densely on the real slice of \( J_m(\hat{\Sigma}) \). Therefore all real finite-gap strings are quasi-periodic in all higher times \( \{t\} \). This is not too surprising since they describe a string moving on \( S^3 \) which is itself compact. Exact periodicity in any of the higher times \( t_N \) with period \( T_N \) is guaranteed if the vector \( T_N \int \vec{b} \, d\Omega_N \), which is real by lemma 9.1.9 happens to coincide with a lattice vector, namely

\[
T_N \int \vec{b} \, d\Omega_N \in 2\pi \mathbb{Z}^{g+1}.
\]

In particular, a finite-gap string is **closed** \( (i.e. \) invariant under \( \sigma \to \sigma + 2\pi \) if

\[
\int \vec{b} \, dp \in 2\pi \mathbb{Z}^{g+1}.
\]  
(9.3.1)

Most of these conditions are automatically satisfied. Indeed all the \( b \)-periods of \( dp \) are integer multiples of \( 2\pi \) by equation (6.3.4) which was a consequence of the single-valuedness of \( \Lambda(P) \) as a function on \( \hat{\Sigma} \). The only non-trivial condition in (9.3.1) is

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dp \in \mathbb{Z}.
\]  
(9.3.2)

In fact the linear motion on the generalised Jacobian only completely encodes the dependence of the current \( j = -g^{-1} dg \) on the higher times, but it is not sufficient to encode the time dependence of the embedding \( g \). To obtain the complete set of periodicity conditions on the algebro-geometric data one should instead consider the
embedding itself
\[ g(\sigma) = U_L \cdot \text{Pexp} \int_{\sigma} j, \]
where we have only explicitly written the dependence on \( \sigma \). Comparing this expression to the same expression translated by \( \sigma \rightarrow \sigma + 2\pi \) whose inverse is given by \( g^{-1}(\sigma + 2\pi) = \left( \text{Pexp} \int_{\sigma}^{\sigma+2\pi} j \right) \cdot U_L^{-1} \) we find
\[ g^{-1}(\sigma + 2\pi)g(\sigma) = \text{Pexp} \int_{\sigma}^{\sigma+2\pi} j = \Omega(0, \sigma). \]
Periodicity in \( \sigma \) of the embedding field \( g \) is therefore guaranteed provided \( \Omega(0) = 1 \).
Since the eigenvalues of the monodromy matrix at the origin are \( \Lambda(0^\pm) \) this condition can equally be written as conditions on the periods of \( dp \), namely
\[ \frac{1}{2\pi} \int_{\infty^+}^{0^\pm} dp \in \mathbb{Z}. \quad (9.3.3) \]
These conditions imply the earlier conditions (9.3.2) as it should be. Indeed, using the property \( \hat{\sigma}^*dp = -dp \) of the differential of the quasi-momentum we find
\[ \int_{\infty^\pm}^{0^\pm} dp = -\int_{\infty^\pm}^{0^\mp} \hat{\sigma}^*dp = -\int_{\infty^\mp}^{0^\pm} dp, \quad (9.3.4) \]
where the contour \( [0^\pm, \infty^\mp] \) is simply the image of the contour \( [0^\mp, \infty^\pm] \) under the holomorphic involution \( \hat{\sigma} \). But now breaking up the integral \( \int_{\infty^+}^{0^+} dp \) as follows
\[ \int_{\infty^+}^{0^+} dp = \int_{\infty^+}^{\infty^-} dp + \int_{\infty^-}^{0^-} dp + \int_{0^-}^{0^+} dp, \]
and doing the same for the integral \( \int_{\infty^+}^{0^-} dp \) it follows using (9.3.4) that
\[ \int_{\infty^+}^{0^\pm} dp = -\frac{1}{2} \left( \int_{\infty^+}^{\infty^-} dp \pm \int_{0^-}^{0^+} dp \right). \]
Thus we can write \( \int_{\infty^-}^{\infty} dp = -\int_{\infty^+}^{0^+} dp - \int_{0^-}^{\infty} dp \) and (9.3.3) implies (9.3.2) as claimed.

Finally let us use (9.3.4) to rewrite the full set of periodicity conditions (9.3.3) slightly differently as follows,
\[
\frac{1}{2\pi} \int_{\infty^\pm}^{0^+} dp \in \mathbb{Z}.
\]

(9.3.5)

### 9.4 Real closed finite-g strings

In the previous sections we have obtained necessary conditions on the algebro-geometric data for the finite-gap strings to be both real and closed. In this section we show that these conditions are also sufficient. That is, with algebro-geometric data satisfying the reality conditions and the periodicity conditions, the reconstructed current of theorem 7.4.1 is both \( su(2) \)-valued and \( 2\pi \) periodic in \( \sigma \). Moreover the reconstructed embedding of theorem 7.4.2 is \( SU(2) \)-valued and \( 2\pi \) periodic in \( \sigma \).

#### The \( SU(2)_R \) current \( j \)

Let \( \psi^+(P) \) be the dual Baker-Akhiezer vector defined by (7.3.12) with respect to the ‘correct’ dual dynamical divisor given in lemma 9.2.1 by \( \hat{\gamma}^+(0) = \hat{\tau}^\dagger \hat{\gamma}(0) \).

**Lemma 9.4.1.** \( \psi^+(P) = \psi(\hat{\tau}P)^\dagger \).

**Proof.** Consider the functions \( f_i(P) = \psi_i^+(P)/\psi_i(\hat{\tau}P) \). These are meromorphic functions with at most \( g \) poles (in general position) and hence are constant by the Riemann-Roch theorem. But by the normalisation conditions in (7.3.12a) and (7.2.7a) we have \( f_1(\infty^+) = f_2(\infty^-) = 1 \) so that \( f_i(P) \equiv 1 \). \( \square \)

It is instructive to give a second proof of lemma 9.4.1 but using the explicit reconstruction formulae of the two vectors \( \psi \) and \( \psi^+ \) in propositions 7.2.5 and 7.3.7.
Proof of lemma 9.4.1 (using reconstruction formulae). It follows from lemma 9.1.7 and the reality condition 9.1.6 on the b-cycles that the period matrix satisfies the following reality condition
\[ \bar{\Pi} = -\Pi - \Pi_0, \] (9.4.1)
where \((\Pi_0)_{ij} = \sum_{k \neq i} \delta_{kj} + 2\delta_{ij}\) has 1’s in all off-diagonal entries and 2’s along the diagonal. Using this, it follows from its definition 1.7.11 that the vector of Riemann’s constants is pure imaginary \(\bar{\mathcal{K}} = -\mathcal{K}\). In particular we have
\[ \bar{\zeta}_{\gamma \pm}(0) = \mathcal{A}(\gamma \pm(0)) + \mathcal{K} = -\mathcal{A}(\hat{\tau} \gamma \pm(0)) - \mathcal{K} = -\mathcal{A}(\delta \pm(0)) - \mathcal{K} = -\zeta_{\delta \pm}(0), \]
where the divisors \(\gamma \pm(t)\) and \(\delta \pm(t)\) of degree \(g\) were defined in 7.1.6 and 7.3.14 respectively. It also follows from (9.4.1) that the \(\theta\)-function defined in 1.7.8 satisfies the reality condition \(\bar{\theta}(z) = \theta(-\bar{z})\). This comes down to the following identity for the matrix \(\Pi_0\),
\[ \exp \{ \pi i (\Pi_0 \mathbf{m}, \mathbf{m}) \} = \exp \left\{ \pi i \sum_{i,j=1}^{g} (\Pi_0)_{ij} m_j m_i \right\} = \exp \left\{ \pi i \sum_{i=1}^{g} \left( \sum_{k \neq i} m_k + 2m_i \right) m_i \right\} = \exp \left\{ 2\pi i \sum_{i=1}^{g} \left( m_i^2 + \sum_{k > i} m_k m_i \right) \right\} = 1. \]
Finally, using the above and corollary 9.1.8 it is easily shown directly from the reconstruction formulae for the normalised eigenvector \(\mathbf{h}\) and the dual normalised eigenvector \(\mathbf{h}^+\) in propositions 7.1.6 and 7.3.5 that
\[ \bar{h}_{\pm}(\hat{\tau} P) = k_{\pm}(P). \]
Moreover, starting from the formulae in proposition 7.2.5 for the components of the Baker-Akhiezer vector, we can compute their conjugates evaluated at \(\hat{\tau} P\) and obtain
the formulae in proposition 7.3.7 for the dual Baker-Akhiezer vector. Specifically,

\[
\psi_{\pm}(\hat{\tau}P,0) = \frac{\theta\left( -\mathcal{A}(\hat{\tau}P) - \int_b \frac{dQ}{\mathcal{B}} + \zeta_{\gamma(0)} \right) \theta\left( -\mathcal{A}(\infty^\pm) + \zeta_{\gamma(0)} \right)}{\theta\left( -\mathcal{A}(\hat{\tau}P) + \zeta_{\gamma(0)} \right) \theta\left( -\mathcal{A}(\infty^\pm) - \int_b \frac{dQ}{\mathcal{B}} + \zeta_{\gamma(0)} \right)} \times \exp\left( -i \int_{\hat{\tau}P}^{\infty^\pm} \frac{dQ}{\tau} \right),
\]

which is the expression for \( \phi_{\pm}(P) \) in proposition 7.3.7 after using lemma 9.1.9.

**Corollary 9.4.2.** The inverse matrix of \( \Psi(x) = (\psi(x^+), \psi(x^-)) \) can be written as

\[
\Psi(x)^{-1} = \text{diag}(\chi_0(x^+), \chi_0(x^-)) \Psi(\bar{x})^\dagger,
\]

where \( (\chi_0) = \hat{\gamma}(0) + \hat{\tau}\hat{\gamma}(0) - B \) and \( \chi_0(\infty^\pm) = 1 \).

**Proof.** By proposition 7.3.6 the inverse matrix of \( \Psi(x) \) can be written as

\[
\Psi(x)^{-1} = \text{diag}(\chi_0(x^+), \chi_0(x^-)) \left( \psi^+(x^+)^T, \psi^+(x^-)^T \right)^T,
\]

where \( \chi_0(P) = \eta(P,0)^{-1} = (h^+(P,0) \cdot h(P,0))^{-1} \) has zeroes at \( \hat{\gamma}(0) \) and \( \hat{\tau}\hat{\gamma}(0) \), poles at the divisor of branch points \( B \) and is normalised at infinity, i.e.

\[
(\chi_0) = \hat{\gamma}(0) + \hat{\tau}\hat{\gamma}(0) - B, \quad \chi_0(\infty^\pm) = 1.
\]

Using lemma 9.4.1 we can now rewrite the matrix of dual Baker-Akhiezer vectors in terms of Baker-Akhiezer vectors.

**Theorem 9.4.3.** When using real and periodic algebro-geometric data, the recon-
structured current $j$ of theorem 7.4.1 is $su(2)$-valued and $\sigma$-periodic, i.e.

$$j_\pm \in su(2), \quad j_\pm(\sigma + 2\pi) = j_\pm(\sigma).$$

**Proof.** Using corollary 9.4.2 the reconstructed current (7.4.2) can be written as

$$j_+ = i\kappa_+ \Psi_0 \sigma_3 \text{diag}(\chi_0((+1)^+), \chi_0((+1)^-)) \Psi_0^\dagger,$$

$$j_- = i\kappa_- \Phi_0 \sigma_3 \text{diag}(\chi_0((-1)^+), \chi_0((-1)^-)) \Phi_0^\dagger,$$

where $\Psi_0, \Phi_0$ were defined in (7.4.3) as the leading terms in the expansion of $\Psi(x)$ at $x = \pm 1$. The defining properties of $\chi_0$ stated in corollary 9.4.2 also imply that $\chi_0(\hat{P}) = \chi_0(P)$ and hence $\chi_0((\pm 1)^+), \chi_0((\pm 1)^-) \in \mathbb{R}$. It is then immediate from (9.4.4) that $j_\pm^\dagger = -j_\pm$.

The $\sigma$-periodicity can be shown using the explicit reconstruction formulae for the Baker-Akhiezer and dual Baker-Akhiezer vectors in propositions 7.2.5 and 7.3.7. The arguments of the $\theta$-functions of both these vectors depends on $\sigma$ only through the combination $\frac{\sigma}{2\pi} \int_0^\infty dp \in \sigma \mathbb{Z}$. Then by the automorphy property (1.7.9) it follows that the $\theta$-function parts of the expressions in propositions 7.2.5 and 7.3.7 are invariant under $\sigma \rightarrow \sigma + 2\pi$. Now focusing on the exponential parts we can write

$$\Psi(x) = \text{diag} \left( 1, e^{i \int_{-\infty}^{\infty}^+ dQ} \right) \Theta_+(x) \text{diag} \left( e^{i \int_{-\infty}^{\infty}^+ dQ}, e^{i \int_{-\infty}^{\infty}^- dQ} \right),$$

$$\Psi(x)^{-1} = \text{diag} \left( e^{-i \int_{-\infty}^{\infty}^+ dQ}, e^{-i \int_{-\infty}^{\infty}^- dQ} \right) \Theta_-(x) \text{diag} \left( 1, e^{-i \int_{-\infty}^{\infty}^+ dQ} \right),$$

where $\Theta_\pm(x)$ contains the $\theta$-function part of these formulae. It now follows from the reconstruction formula (7.4.2) for $j_\pm$ that the current also depends on $\sigma$ through $\exp \left( \frac{i\sigma}{2\pi} \int_{-\infty}^{\infty}^+ dp \right) = \exp (i\pi n)$ for $n \in \mathbb{Z}$, which is also invariant under $\sigma \rightarrow \sigma + 2\pi$. Therefore the full reconstruction formula for $j_\pm$ is periodic in $\sigma$ of period $2\pi$. □
The $SU(2)$ embedding $g$

**Lemma 9.4.4.** Let $P = \hat{\tau}P \in \hat{\Sigma}$ be a fixed point of $\hat{\tau}$ then

$$
\psi_1(\hat{\sigma}P) = -A(P)\overline{\psi_2(P)}, \quad \psi_2(\hat{\sigma}P) = A(P)\overline{\psi_1(P)}
$$

where $A(P) = \chi_0(P)\det(\psi(P), \psi(\hat{\sigma}P))$.

**Proof.** By lemma 9.4.1 and equation (9.4.2) the rows of $\Psi(x)^{-1}$ take the form $\chi_0(P)\psi(P)\dagger$ and $\chi_0(\hat{\sigma}P)\psi(\hat{\sigma}P)\dagger$. It follows that

$$
\begin{pmatrix}
\overline{\psi_1(P)} & \overline{\psi_2(P)} \\
\overline{\psi_1(\hat{\sigma}P)} & \overline{\psi_2(\hat{\sigma}P)}
\end{pmatrix}
\begin{pmatrix}
\psi_1(P) \\
\psi_2(P)
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\chi_0(P)} \\
0
\end{pmatrix}.
$$

(9.4.5)

Multiplying by the inverse of the matrix on the left hand side we obtain

$$
\begin{pmatrix}
\psi_1(P) \\
\psi_2(P)
\end{pmatrix}
= -\frac{1}{D(P)\chi_0(P)}
\begin{pmatrix}
\overline{\psi_2(\hat{\sigma}P)} \\
-\overline{\psi_1(\hat{\sigma}P)}
\end{pmatrix},
$$

where $D(P) = \det(\psi(P), \psi(\hat{\sigma}P))$. Defining $A(P) = \chi_0(P)D(P)$ its conjugate is $A(P) = \chi_0(P)\overline{D(P)}$ since $\chi_0(P) \in \mathbb{R}$ for $\hat{\pi}(P) \in \mathbb{R}$ and the result follows.

**Theorem 9.4.5.** After a residual diagonal $SL(2,\mathbb{C})_L$ transformation $g_L$ the $SU(2)$ embedding can be recovered by the formula,

$$
g = \chi_0(0^+)\dagger
\begin{pmatrix}
\overline{\psi_1(0^+)} & \overline{\psi_2(0^+)} \\
-\overline{\psi_2(0^+)} & \overline{\psi_1(0^+)}
\end{pmatrix}
\in SU(2).
$$

**Proof.** By lemma 9.4.4 we can write $\Psi(0)$ as

$$
\Psi(0) = 
\begin{pmatrix}
\psi_1(0^+) & -\overline{\psi_2(0^+)} \\
\psi_2(0^+) & \overline{\psi_1(0^+)}
\end{pmatrix}
\text{diag}(1, \chi_0(0^+)\det \Psi(0)).
$$
This can equivalently be written as

\[
\frac{1}{\sqrt{\text{det} \Psi(0)}} \cdot \Psi(0) = \chi_0(0^+)^{\frac{1}{2}} \begin{pmatrix}
\psi_1(0^+) & -\psi_2(0^+)
\psi_2(0^+) & \psi_1(0^+)
\end{pmatrix} \text{diag}(S, S^{-1}),
\tag{9.4.6}
\]

where \( S = \sqrt{\chi_0(0^+)^{-1} \text{det} \Psi(0)} \). The diagonal matrix \( g_L = \text{diag}(S, S^{-1}) \) on the right hand side is nothing but an \( SL(2, C)_L \) residual transformation. Now the first component of (9.4.5) reads

\[
|\psi_1(0^+)|^2 + |\psi_2(0^+)|^2 = \frac{1}{\chi_0(0^+)}
\]

from which it follows that \( \chi_0(0^+) \) > 0 and

\[
\chi_0(0^+) \frac{1}{2} \begin{pmatrix}
\psi_1(0^+) & -\psi_2(0^+)
\psi_2(0^+) & \psi_1(0^+)
\end{pmatrix} \in SU(2).
\]

Removing the residual gauge transformation in (9.4.6) we have by proposition 7.4.2,

\[
g^{-1} = \frac{1}{\sqrt{\text{det} \Psi(0)}} \cdot \Psi(0) g_L^{-1} = \chi_0(0^+) \frac{1}{2} \begin{pmatrix}
\psi_1(0^+) & -\psi_2(0^+)
\psi_2(0^+) & \psi_1(0^+)
\end{pmatrix}.
\]

Inverting this proves the theorem. \( \square \)

Recall that the embedding matrix \( g \) encoded the fields \( X_i, i = 1, \ldots, 4 \) describing the embedding into \( S^3 \subset \mathbb{R}^4 \) through equation (3.1.5). Defining the complex fields \( Z_1 = X_1 + iX_2 \) and \( Z_2 = X_3 + iX_4 \) we have

**Corollary 9.4.6.** The embedding fields \( X_i, i = 1, \ldots, 4 \) are recovered in terms of the dual Baker-Akhiezer vector evaluated at \( 0^+ \) by

\[
Z_i = C \psi_i^+(0^+), \quad i = 1, 2
\tag{9.4.7}
\]

where \( C = \chi_0(0^+) \in \mathbb{R}_+ \) is a normalisation ensuring that \( |Z_1|^2 + |Z_2|^2 = 1 \).

**Proposition 9.4.7.** The reconstruction formulae (9.4.7) are \( 2\pi \) periodic in \( \sigma \).

**Proof.** The arguments of the \( \theta \)-functions in the reconstruction formulae for the dual
Baker-Akhiezer vector in propositions 7.3.7 depends on $\sigma$ only through the combination $\frac{\sigma}{2\pi} \int_b dp \in \sigma \mathbb{Z}$. Then by the automorphy property (1.7.9) the $\theta$-function part is invariant under $\sigma \to \sigma + 2\pi$. As for the exponentials, the $\sigma$-dependent parts are $\exp \left( -\frac{ia}{2\pi} \int_{\infty \pm}^{0 \pm} dp \right)$ which are clearly invariant under $\sigma \to \sigma + 2\pi$ by (9.3.5). \qed
Part IV

Applications
Chapter 10

Semiclassical strings on $\mathbb{R} \times S^3$

The method of semiclassical quantisation in field theory has been extensively developed by many authors in the 70’s using different approaches [46, 47, 88–93] (see also the books [94, 95] for a more or less complete survey and list of references). The aim of all these methods is to give a quantum mechanical meaning to extended classical solutions of the field equations which already classically exhibit particle like properties. The role played by such non-trivial classical solutions in the leading order quantisation of any field theory is evident from the path integral which is dominated by classical solutions in the $\hbar \to 0$ limit. It follows then that the applicability of semiclassical methods crucially relies on an explicit knowledge of classical solutions. Having studied the general finite-gap string in Part III we can now proceed with semiclassically quantising the string on $\mathbb{R} \times S^3$.

An important part in any approach to semiclassical quantisation is the treatment of the zero-modes (see [93] for a clear exposition of the problem and [88, 89, 94, 95] for various resolutions). Roughly speaking, if $\phi_{cl}$ is a solution to the field equations derived from an action $S[\phi]$ then a zero-mode of $\phi_{cl}$ is a (possibly hidden) symmetry of the equations of motion $S'[\phi] = 0$ which isn’t a symmetry of $\phi_{cl}$ itself.
If a classical solution has zero-modes then a naive semiclassical quantisation of the solution will fail. Indeed, suppose that $\phi_{\text{cl}}$ is not invariant under an infinitesimal symmetry $v$ of the equations of motion, then it follows immediately that $(v\phi_{\text{cl}}) \neq 0$ is in the kernel of the operator $S''[\phi_{\text{cl}}]$ which is therefore not invertible and so the propagator of the theory in the background $\phi_{\text{cl}}$ cannot be defined. The standard way around this difficulty is to treat the zero-mode directions separately using the method of **collective coordinates**. In short, collective coordinates parametrise the zero-mode directions, namely the flat directions in field space, along which the wave function will tend to spread out in the form of a plane wave. As a result the quantum counterpart of the solution $\phi_{\text{cl}}$ will acquire dynamics along these collective coordinates. Generally one has to perform a change of variables in field space to include the collective coordinates among the set of field variables and this can often only be done implicitly. A nice feature of the finite-gap construction is that it naturally lends itself to the separation of zero-modes.

To see why that is, recall from theorem 8.2.1 that the action variables $\{S_I\}_{I=1}^{g+1}$ act non-trivially on the angle variables $\{\varphi_I\}_{I=1}^{g+1}$, which parametrise the divisor $\hat{\gamma}(t)$ according to (8.2.6). Thus although each action variable generates an infinitesimal symmetry $v = \partial/\partial \varphi_I$ of the string equations of motion, the finite-gap string itself is not invariant under this symmetry. Therefore any $g$-gap string always has $g+1$ zero-modes for which the divisor $\hat{\gamma}(t)$ fills the role of collective coordinates. Alternatively, as we saw in chapter 8 the non-special divisor $\hat{\gamma}(t)$ can equally be described as a point $\tilde{A}(\hat{\gamma}(t))$ on the generalised Jacobian. Therefore any set of coordinates on the generalised Jacobian can be used as collective coordinates.

This leads to a very nice picture of finite-gap strings which ties in with the discussion of semiclassical quantisation of finite-dimensional systems in chapter 2. Indeed, the upshot of chapters 8 and 9 was that a finite-gap string could be thought
of as an embedding \( \mathcal{G}_\mathbb{R} : \mathcal{M}_{\mathbb{R}}^{(2g+2)} \hookrightarrow \mathcal{P}\mathbb{R}^\infty \) of a finite-dimensional integrable system

\[
\mathbb{T}^{g+1} \rightarrow \mathcal{M}_{\mathbb{R}}^{(2g+2)} \rightarrow \mathcal{L}_{\mathbb{R}}
\]

into the infinite dimensional reduced phase-space \( \mathcal{P}\mathbb{R}^\infty \) of the string. Or put another way, a finite-gap string describes a \((g+1)\)-parameter family of \((g+1)\)-torii in \( \mathcal{P}\mathbb{R}^\infty \) parameterised by the filling fractions \( \{ \mathcal{S}_I \}_{I=1}^{g+1} \). These torii are \textit{isotropic} since the pullback \((8.2.5)\) of the symplectic form \( \hat{\omega}_\infty \) to them is identically zero. Moreover, being finite-dimensional they are necessarily \textit{degenerate} isotropic torii of \( \mathcal{P}\mathbb{R}^\infty \). This is the necessary set-up to apply the Bohr-Sommerfeld conditions \((2.4.3)\) for the quantisation of a \( p \)-torus in an \( n \)-dimensional phase-space, where here the total phase-space is infinite dimensional so that \( n = \infty \) and \( p = g + 1 \).

In section \( 10.1 \) we start by recalling the method of semiclassical quantisation à la Dashen, Hasslacher and Neveu [88–90] when applied to the specific example of the breather solution in Sine-Gordon theory. We reformulate everything in a language that we hope will facilitate the conceptual understanding of the method in the finite-gap setting. In section \( 10.2 \) we will explicitly compute the stability angles of perturbations around a given finite-gap solution which appear in the Bohr-Sommerfeld conditions.

### 10.1 Analogy with Sine-Gordon breathers

Consider the example of the boosted Sine-Gordon breather solution [89,94,95]

\[
\phi_{r,v}(x,t) = \frac{4m}{\sqrt{\Lambda}} \tan^{-1} \left\{ \frac{\sqrt{\left( \frac{2m}{2\pi} \right)^2 - 1 \cdot \sin \left[ \left( \frac{2\pi}{v} \right) \cdot \frac{t - \frac{v}{2}x}{\sqrt{1 - v^2}} \right]} \cosh \left[ \sqrt{\left( \frac{2m}{2\pi} \right)^2 - 1 \cdot \left( \frac{2\pi}{v} \right)^2 \cdot \frac{t - \frac{v}{2}x}{\sqrt{1 - v^2}}} \right]} \right\}. \tag{10.1.1}
\]
This is really a two parameter family of solutions parametrised by their proper period $\tau$ and their velocity $v$, or equivalently by their energy $E$ and momentum $p$. To compute the (possibly continuous) spectrum of the corresponding quantum states it is always simpler at first to put the system in a very large but finite box of length $L$ by identifying $x \sim x + L$ so as to make the spectrum discrete, and then take the infinite volume limit $L \to \infty$ at the end. In this closed-loop world the breather solution (10.1.1) is periodic in $t$ of period $T$ provided $\tau$ and $v$ satisfy

$$T = l \frac{\tau}{\sqrt{1 - v^2}} = m \frac{L}{v}, \quad l, m \in \mathbb{N}.\]

If we were quantising the kink, we could move to its rest frame in which it is static and study small fluctuations in terms of eigenfrequencies. However, the breather is a little more complicated since it is time dependent in its rest frame, and because time dependent solutions are not point-like in field space, we need a way to characterise perturbations of the orbit as a whole. This was described in chapter 2 where we defined the Poincaré map. The idea was to consider the perturbation of a specific point on the orbit, evolve that perturbation under the equations of motion for roughly the period of the underlying solution, and compare the final perturbation with the original one. If the perturbation is stable then it will have merely rotated in which case the angle of rotation is called the stability angle. If instead the perturbation is unstable it will have grown exponentially in magnitude, which corresponds to the case of a complex stability angle. Finally, if the perturbation comes back exactly to itself, this means it describes a nearby periodic solution, and in general zero stability angles correspond to symmetries. In the case of the Sine-Gordon breather we therefore need to look for generic nearby solutions $\phi(x, t) = \phi_{\tau,v}(x, t) + \delta \phi$. This perturbed solution won’t be periodic in general, yet
because the linearised equation

$$\Box \delta \phi = (\cos \phi_{\tau,v}) \delta \phi$$  \hspace{1cm} (10.1.2)

is invariant under time translation by $T$ we can always write its solution as a superposition of eigenfunctions of time translation $\delta \phi(x, t + T) = e^{-i\nu \delta \phi(x, t)}$, where $\nu$ are their stability angles (another way to say this is that the time translation operator $\hat{T} : t \mapsto t + T$ commutes with the linearised operator $\hat{L} = \Box - \cos \phi_{\tau,v}$ and hence both operators can be simultaneously diagonalised. In particular the kernel of $\hat{L}$ is spanned by eigenfunctions of $\hat{T}$). Notice that the Sine-Gordon equation is invariant under arbitrary space and time translations, but the breather solution $\phi_{\tau,v}$ is not. As a result, $\partial \phi_{\tau,v}/\partial x$ and $\partial \phi_{\tau,v}/\partial t$ are both zero-modes, i.e. perturbations with zero stability angles. This is a special case of a much more general result,

**Lemma 10.1.1.** If a classical solution is not invariant under a symmetry of the action then it has a zero-mode.

*Proof.* Consider a periodic solution $\phi_{\text{cl}}$ of a field equation derived from an action $S[\phi]$, i.e. $S'[\phi_{\text{cl}}] = 0$, where $'$ denotes $\delta/\delta \phi$. If $v$ is an infinitesimal symmetry of the equations of motion, i.e. $v(S'[\phi]) = S''[\phi](v\phi)$, and suppose that $\phi_{\text{cl}}$ is not invariant under the symmetry then it follows immediately that $(v\phi_{\text{cl}}) \neq 0$ is in the kernel of the operator $S''[\phi_{\text{cl}}]$. Clearly it is a zero-mode since $v\phi_{\text{cl}}(t + T) = v\phi_{\text{cl}}(t)$. \hfill $\Box$

The task of finding nearby solutions to the breather is greatly facilitated by the fact that the Sine-Gordon equation is integrable, since we can use the Bäcklund transform to get new solutions from known solutions. In particular we can perturb our breather by adding a little breather of small amplitude on top of it (Figure 10.1). Studying double breather solutions in the limit where the small breather has vanishingly small amplitude corresponds to a linearised study of the Sine-Gordon
equation around the breather solution. So integrability gives us a convenient way of writing down explicit solutions to the linearised equation \((10.1.2)\) from which the stability angles of the breather may be read off.

Identifying the space of classical solutions with phase-space, for each \(\tau, v\) (or equivalently \(E, p\)) the breather solution \((10.1.1)\) is just a specific point in phase space. However, the existence of two zero-modes \(\frac{\partial \phi_{\tau,v}}{\partial x}\) and \(\frac{\partial \phi_{\tau,v}}{\partial t}\) for the breather solution indicates that it really belongs to a two parameter family of solutions with the same integrals of motion \(E, p\). These are the space and time translated breather solutions

\[ \phi_{\tau,v}(x + x_0, t + t_0). \]  

Since all the other stability angles of the breather are real, when we include first order quantum corrections the wavefunction will want to localise around not one breather, but around the whole two parameter family \((10.1.3)\) of breathers by spreading along the flat directions, namely the \(x_0\) and \(t_0\) directions. Along these directions the wavefunction will therefore be a plane wave, but since the \(t_0\)-direction is closed by periodicity of the breather solution the plane wave along it must have an integer number of peaks and troughs. In other words the change of phase of the wavefunction around this closed direction will have to be an integer multiple \(n\) of \(2\pi\). Along all the other non-zero stability angle directions the wavefunction will decay rapidly and, intuitively, for states with higher excitation number \(n_i\) it will extend further in these directions. The correct quantisation conditions encoding the semiclassical energy spectrum of the wavefunction localised around the family of breather solutions was
first derived by Dashen, Hasslacher and Neveu [88] and can be expressed as follows. If we define the ‘action’ of the breather solution as

\[ W(E) = \int_0^T dt \int dx \pi_{\tau,v}(x,t) \partial_0 \phi_{\tau,v}(x,t), \]  

then the DHN quantisation conditions read

\[ \frac{W(E)}{\hbar} = 2\pi n + \sum_{\nu_i > 0} \left( n_i + \frac{1}{2} \right) \nu_i + O(\hbar). \]  

Although the derivation of this formula is very complicated, it intuitively makes a lot of sense. In general the phase of the wavefunction in the semiclassical approximation is an action of the form \((10.1.4a)\) so the first term on the right hand side of \((10.1.4b)\) can be seen to come from the single-valuedness of the wavefunction along the compact \(t_0\)-direction whereas the correction from the sum over stability angles is related to the small fluctuations transverse to the \(t_0\) and \(x_0\) directions.

For the purpose of drawing the analogy between Sine-Gordon breathers and finite-gap strings it will be convenient to think of the conditions \((10.1.4)\) in more geometric terms in phase-space as follows. Since the breather in \((10.1.3)\) with \(x_0 = 0\) is periodic, it can be thought of as a closed orbit on the level set \(\Sigma_{E,p}\) of fixed \(E,p\). The direction along the orbit, parametrised by \(t_0\), corresponds to the zero-mode \(\partial \phi_{\tau,v}/\partial t\) of the breather. But since it has another zero-mode, namely \(\partial \phi_{\tau,v}/\partial x\), this orbit really belongs to a continuous family of periodic orbits, parametrised by \(x_0\), all contained in \(\Sigma_{E,p}\). However, because we are working in a periodically identified finite box, this two parameter \((x_0,t_0)\) family of breathers is in fact a torus \(\mathbb{T}_{E,p}^2\) lying within \(\Sigma_{E,p}\). And since all the other stability angles of the breather are non-zero, this means that \(\mathbb{T}_{E,p}^2\) is isolated on the level set \(\Sigma_{E,p}\) in the sense that it does not belong to a larger continuous family of periodic orbits within \(\Sigma_{E,p}\). Yet if we leave the level set \(\Sigma_{E,p}\), one can show that in a neighbourhood of \(\Sigma_{E,p}\) the torus
\( T^2_{E,p} \) persists, namely it belongs to a two parameter family of torii parametrised by \( E, p \). This was the content of the generalised cylinder theorem \[2.0.16\] in chapter 2. Looking back at the most general breather solution \[(10.1.3)\] it contains four independent parameters: the two parameters \( x_0, t_0 \) are parameters along the torus \( T^2_{E,p} \) whereas \( E, p \) parameterise the family of torii of the generalised cylinder theorem \[2.0.16\]. Now the effect of the quantisation condition \[(10.1.4)\] is to pick out a discrete set of breathers from this generalised cylinder of breathers \[(10.1.1)\], the energy and momentum of which approximate to order \( O(\hbar) \) the semiclassical energy spectrum of the quantum states localised around the breather solution. For instance, when applied to the Sine-Gordon breather the quantisation conditions \[(10.1.4)\] yield the following semiclassical spectrum \[89\]

\[
E_{k,n} = (p_k^2 + M_n^2)^{1/2}, \quad p_k = \frac{2\pi k}{L},
\]

where \( M_n = \frac{16m}{\gamma'} \sin \frac{n\gamma}{16} \) and \( \gamma' = \frac{\lambda}{m^2} \left( 1 - \frac{\lambda}{8\pi m^2} \right)^{-1} \), and in the infinite volume limit \( L \to \infty \) the momentum becomes continuous as expected.

The analogy with the finite-gap construction is as follows. Just as the generic breather \[(10.1.3)\] defined a four-parameter family of solutions, a finite-gap string defines a whole \((2g + 2)\)-parameter family of solutions parametrised by the algebro-geometric data. It can be written schematically as

\[
g = g\left( \sum_N t_N \bar{U}_N(\mathcal{S}) + \bar{D} \bigg| \mathcal{S} \right),
\]

where \( t_N \) are a set of \( g + 1 \) independent times (defined in section \[5.4\]), \( \bar{U}_N(\mathcal{S}) \) is some function of the filling fractions which play the role of the parameters \((\tau, v)\) or \((E, p)\) here. The vector \( \bar{D} \in \mathbb{C}^{g+1} \) is related to the initial divisor \( \hat{\gamma}(0) \) and is the exact analogue of the initial coordinates of the breather \((x_0, t_0)\). As already explained at the start of this chapter the \( g + 1 \) components of this vector correspond to \( g + 1 \)
zero-modes of the $g$-gap string, analogously to the breather case.

10.2 Perturbations of finite-gap strings

In view of applying a semiclassical quantisation formula like the one in (10.1.4) we must first determine all the stability angles of a given finite-gap string. So just as in the case of the Sine-Gordon breather, we would like to study perturbations of finite-gap strings obtained in chapter 7. Once again integrability will play a prominent role in solving the linearised equations. In fact, finding solutions to the linearised problem is very simple now that we have already fully exploited integrability to construct the most general finite-gap string. A perturbation of a given finite-gap string will simply be another ‘nearby’ finite-gap string. Recall from chapter 6 that the algebraic curve is hyperelliptic and can be represented by a set of $g + 1$ vertical cuts in the complex plane. How can one describe perturbations of the $g$-gap string corresponding to this curve? Playing the same game as for the Sine-Gordon breather where we used integrability to add another little breather on it, here we can just take a solution corresponding to a curve of genus one higher, but make the extra filling fraction very small, which corresponds to making the cut very small, see Figure 10.2. There

![Figure 10.2: Perturbation of a finite-gap solution.](image)

is an obvious analogy here between breathers in Sine-Gordon and cuts in bosonic strings on $\mathbb{R} \times S^3$ as one can think of a finite-gap string as a multi-breather solution consisting of finitely many breathers. Cuts with small filling fractions are analogous
to breathers of small amplitude as both describe perturbations. If we define the 
$a_i$-cycle ($i = 1, \ldots, g$) as in chapter 6 to encircle the $i$th cut counterclockwise (on
the upper sheet) then a perturbation of this kind clearly corresponds to pinching
an $a$-cycle of the algebraic curve. So we want to take the difference between the
solution before pinching an $a$-cycle and the solution after pinching the $a$-cycle; this
will give us a perturbation of the latter and we can then analyse its periodicity
properties to extract the corresponding stability angles. Notice however that any
given perturbation of a finite-gap string will have one stability angle defined for each
cycle on the generalised Jacobian, or equivalently for each macroscopic cut.

So given a $g$-gap solution $Z_i$ with underlying algebraic curve $\hat{\Sigma}$ of genus $g$ and
filling fractions $\{S_I\}_{I=1}^{g+1}$, we will obtain its stability angles by considering nearby
$(g + 1)$-gap solutions $Z_i + \delta Z_i$ with algebraic curves $\hat{\Sigma}^\epsilon$ of genus $g + 1$ with the same
macroscopic filling fractions $\{S_I\}_{I=1}^{g+1}$ and an extra small filling fraction $S_0 = O(\epsilon)$. The limit $\epsilon \to 0$ then corresponds to pinching the extra handle to zero size, so that
the limit curve $\hat{\Sigma}^0$ desingularises to the original curve $\hat{\Sigma}$, see Figure 10.3. The reason

for wanting the macroscopic filling fractions $\{S_I\}_{I=1}^{g+1}$ to be fixed is that we need to compute the perturbation of a finite-gap string within the level set determined by
these filling fractions (see chapter 2 as well as section 10.1).

Now since we are concerned with real finite-gap solutions, constructed from
real algebraic curves $\hat{\Sigma}$ (see chapter 9), the degeneration process in Figure 10.3
describing the perturbation should respect this reality condition. This forces us to consider degenerations through the pinching of imaginary cycles, namely the
10.2. PERTURBATIONS OF FINITE-GAP STRINGS

The process of pinching $a$-cycles on Riemann surfaces was discussed in relation to generalised Jacobians in section 1.7 of chapter I.

As we showed in chapter I, the dependence of the general finite-gap solution on the hierarchy of times $\{t\}$ is entirely encoded in the normalised Abelian differential of the second kind $dQ = \sum_N t_N d\Omega_N$ defined in (7.2.11) which enters the reconstruction formula as follows

$$Z_i = C_i \frac{\theta(2\pi \int_{P_i}^0 \omega - \int_b dQ - D; \Pi)}{\theta(\int_b dQ + D; \Pi)} \exp \left(-i \int_{P_i}^0 dQ\right),$$

(10.2.1)

where $P_1 = \infty^+$ and $P_2 = \infty^-$. In this expression we have hidden all the time independent part into the overall constants $C_i$ for clarity. A nearby solution $Z_i + \delta Z_i$ is constructed with the same formulae but using data on $\hat{\Sigma}^\epsilon$ to be specified below,

$$Z_i + \delta Z_i = C_i^\epsilon \frac{\theta(2\pi \int_{P_i}^0 \omega^\epsilon - \int_b dQ^\epsilon - \bar{D}; \bar{\Pi}^\epsilon)}{\theta(\int_b dQ^\epsilon + \bar{D}; \bar{\Pi}^\epsilon)} \exp \left(-i \int_{P_i}^0 dQ^\epsilon\right).$$

(10.2.2)

The ingredients of this deformed solution are as follows. First of all, since the underlying curve $\hat{\Sigma}^\epsilon$ has genus $g + 1$, the arguments of the $\theta$-functions for this curve are $(g + 1)$-component vectors, namely $\bar{D} = (D_0, D)^T \in \mathbb{C}^{g+1}$, $\bar{b}^\epsilon = (b_0^\epsilon, b^\epsilon)^T \in H^1(\hat{\Sigma}^\epsilon)$ are the $b$-periods of $\hat{\Sigma}^\epsilon$ and $\bar{\omega}^\epsilon = (\omega_0^\epsilon, \omega^\epsilon)^T$ its holomorphic differentials. In the singular limit $\epsilon \to 0$ one has $b^\epsilon \to b$ and $\omega^\epsilon \to \omega$ which are the $b$-cycles and the $g$ holomorphic differentials on $\hat{\Sigma}$ respectively. The extra $b$-cycle $b_0^\epsilon$ becomes a degenerate cycle on the curve $\hat{\Sigma}$, see Figure 10.3. As we showed in section 1.7.1 of chapter I, in the limit $\epsilon \to 0$ the extra holomorphic differential $\omega_0^\epsilon$ on $\hat{\Sigma}^\epsilon$ acquires a simple pole at the singular point and so becomes a normalised Abelian differential of the third kind. The Abelian differential $dQ^\epsilon$ on $\hat{\Sigma}^\epsilon$ is defined by the same singular parts (7.2.5) as $dQ$ at $x = \pm 1$ but could potentially acquire an extra simple pole at the singular point. However, because $dQ^\epsilon$ is normalised on $\hat{\Sigma}^\epsilon$, its residue there
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would vanish in the $\epsilon \to 0$ limit, so that in fact $d\mathcal{Q}^\epsilon \to d\mathcal{Q}$. One can also show that $C^\epsilon_i \to C_i$.

The important object in (10.2.2) when considering the singular limit $\epsilon \to 0$ is the period matrix $\tilde{\Pi}^\epsilon$ which admits the natural block form

$$
\tilde{\Pi}^\epsilon = \int_{\tilde{b}^\epsilon} \tilde{\omega}^\epsilon = \begin{pmatrix}
\Pi^\epsilon_{00} & \Pi^\epsilon
\Pi^\epsilon_0 & \Pi^\epsilon_T
\end{pmatrix}.
$$

(10.2.3)

The singular limits of each block follow from the above considerations of $\tilde{b}^\epsilon, \tilde{\omega}^\epsilon$ in the limit (see section 1.7 of chapter 1 for details). In particular, $\Pi^\epsilon \to \Pi$ as $\epsilon \to 0$ which is simply the period matrix of $\hat{\Sigma}$. The vectors $\Pi^\epsilon_0$ also stay finite in the limit. The top left component $\Pi^\epsilon_{00}$ on the other hand diverges in this limit, leading to a simplification of the Riemann $\theta$-function $\theta(\cdot; \tilde{\Pi}^\epsilon)$ as $\epsilon \to 0$ which becomes expressible in terms of the Riemann $\theta$-function $\theta(\cdot; \Pi)$ of $\hat{\Sigma}$. The result is expressed in the following lemma [40, 96],

**Lemma 10.2.1.** The behaviour of the Riemann $\theta$-function $\theta(\vec{z}; \tilde{\Pi}^\epsilon)$ associated with $\hat{\Sigma}^\epsilon$, where $\vec{z} = (z_0, z)^T \in \mathbb{C}^{g+1}$, has the following expansion in the limit $\epsilon \to 0$

$$
\theta(\vec{z}; \tilde{\Pi}^\epsilon) = \theta(\vec{z}; \Pi^\epsilon) + \left[ \theta(\vec{z} + \Pi^\epsilon_0; \Pi^\epsilon) e^{iz_0} + \theta(\vec{z} - \Pi^\epsilon_0; \Pi^\epsilon) e^{-iz_0} \right] e^{\pi i \Pi_{00}} + O(e^{2\pi i \Pi_{00}}).
$$

*Proof.* Using the fact that the imaginary part $\text{Im} \tilde{\Pi}^\epsilon$ of the period matrix $\tilde{\Pi}^\epsilon$ is positive definite we have $\text{Im} \Pi^\epsilon_{00} = \text{Im} \langle \tilde{\Pi}^\epsilon e^{(0)}, e^{(0)} \rangle > 0$, where $e^{(0)} = (1, 0, \ldots, 0)^T$. It follows that the quantity $e^{\pi i \Pi_{00}}$ tends to zero in the limit $\epsilon \to 0$. The result then follows from a straightforward expansion of $\theta(\vec{z}; \tilde{\Pi}^\epsilon)$ in terms of $e^{\pi i \Pi_{00}}$. \hfill $\square$

Now taking into account all the above limits and dropping all terms of order $O(\epsilon^2)$, a direct but tedious computation using lemma [10.2.1] shows that the difference
δZ_i between expressions (10.2.2) and (10.2.1) contains three types of contribution

\[
\delta Z_i = \left( \{ \text{periodic} \} + \{ \text{periodic} \} \times e^{i \int_{b_0} dQ} + \{ \text{periodic} \} \times e^{-i \int_{b_0} dQ} \right) \times e^{\pi i \Pi_0},
\]

(10.2.4)

\[
\delta Z_0^0 = \delta Z_0^+ + \delta Z_0^-, \]

where "\{periodic\}" denotes functions periodic in all the angle variables \(\varphi_I\) of the underlying finite-gap solution (10.2.1). The behaviour of each of the three perturbations in (10.2.4) under a shift \(\varphi_I \rightarrow \varphi_I + 2\pi\) of the \(I\)th angle variable is then

\[
\delta Z^0_i(\varphi_I + 2\pi) = \delta Z^0_i(\varphi_I),
\]

\[
\delta Z^\pm_i(\varphi_I + 2\pi) = e^{\pm 2\pi i \int_{b_0} dq^{(I)}} \delta Z^\pm_i(\varphi_I). \tag{10.2.5}
\]

The original perturbation \(\delta Z_i\) defined by opening up a small handle is therefore composed of three separate perturbations \(\delta Z^0_i, \delta Z^+_i\) and \(\delta Z^-_i\), each corresponding to different stability angles of the underlying solution (10.2.1). These stability angles can be read off directly from (10.2.5),

\[
\nu^{(I)}_0 = 0, \quad \nu^{(I)}_\pm = \pm 2\pi \int_{b_0} dq^{(I)}, \quad I = 1, \ldots, g + 1. \tag{10.2.6}
\]

The zero stability angles \(\nu^{(I)}_0\) are related to the \(\varphi_I\)-translation invariance of the equations of motion which is explicitly broken by the finite-gap string (10.2.1). These zero stability angles can be obtained much more directly by considering two neighbouring finite-gap strings with the same underlying curve \(\hat{\Sigma}\), but slightly different initial divisors \(\hat{\gamma}(0)\) and \(\hat{\gamma}^\epsilon(0)\) near each other on \(\hat{\Sigma}\). Since there are \(g + 1\) degrees of freedom in choosing the perturbed divisor \(\hat{\gamma}^\epsilon(0)\), for each angle \(\varphi_I, I = 1, \ldots, g\) this gives \(g + 1\) zero-modes, as one expects from the \(\varphi_J\)-translation invariance of the
equations of motion which the finite-gap string explicitly breaks,

\[ \nu_{0,J}^{(I)} = 0, \quad J = 1, \ldots, g + 1. \]

(10.2.7)

Now stability angles are only defined modulo \(2\pi\). But recall from section 9.3 of chapter 9 that for the solution to be periodic under \(\sigma \rightarrow \sigma + 2\pi\) required that the quasi-momentum \(dp\) satisfied the condition (9.3.2). Here we are interested in using the \(2\pi\) periodicity of the underlying solution (10.2.1) in the angle variables. This statement is equivalent to the quasi-actions satisfying

\[ 2\pi \int_{\infty^{-}}^{\infty^{+}} dq^{(I)} \in 2\pi \mathbb{Z}, \quad I = 1, \ldots, g + 1. \]

Therefore we can redefine the stability angles \(\nu_{\pm}^{(I)}\) as

\[ \nu_{\pm}^{(I)} = \pm 2\pi \left( \int_{b_{0}} dq^{(I)} + \int_{\infty^{-}}^{\infty^{+}} dq^{(I)} \right) = \pm 2\pi \int_{\mathcal{B}_{0}} dq^{(I)}, \]

(10.2.8)

where the contour \(\mathcal{B}_{0}\) runs from \(\infty^{+}\) on the top sheet to \(\infty^{-}\) on the bottom sheet, by going through the 0th cut, see Figure 10.4. In the singular limit \(\epsilon \rightarrow 0\) the 0th cut shrinks to a point, say \(P_{0}\) and so (10.2.8) yields

\[ \nu_{\pm}^{(I)} = \pm 2\pi \left( \int_{\infty^{+}}^{P_{0}} dq^{(I)} + \int_{\infty^{-}}^{\infty^{+}} dq^{(I)} \right) = \pm 2\pi \left( \int_{\infty^{+}}^{P_{0}} dq^{(I)} - \int_{\infty^{-}}^{\infty^{+}} dq^{(I)} \right) \]

\[ = \pm 2\pi \left( \int_{\infty^{+}}^{P_{0}} dq^{(I)} - \int_{P_{0}}^{\infty^{+}} dq^{(I)} \right) = \pm 2\pi \left( \int_{P_{0}}^{\infty^{+}} dq^{(I)} + \int_{\infty^{+}}^{P_{0}} dq^{(I)} \right) \]

\[ = \pm 4\pi q^{(I)}(P_{0}), \]

(10.2.9)

where \(q^{(I)}(P) \equiv \int_{\infty^{+}}^{P} dq^{(I)}\) with the integral running along the top sheet (the precise choice of contour then doesn’t matter since \(dq^{(I)}\) is normalised). By performing a
similar calculation to the one in (10.2.9) but on \( \int_{B_0} dp = 2\pi n_0, n_0 \in \mathbb{Z} \) which comes from \( 2\pi \) periodicity in \( \sigma \), one derives also an equation for the location of the singular point \( P_0 \), namely

\[
p(P_0) = n_0 \pi.
\]

(10.2.10)

The above analysis shows that to this singular point \( P_0 \) there corresponds two stability angles for each of the \( g + 1 \) cuts determined by the \( B_0 \)-period of corresponding quasi-action \( dq^{(I)} \) or

\[
v^{(I)}_{\pm} = \pm 4\pi q^{(I)}(P_0).
\]

(10.2.11)

Figure 10.4: The canonical cycles before (a) and after (b) shrinking of the 0th cut. Note that it doesn’t matter where this cut lies with respect to the other cuts, but for the sake of clarity of the figure we chose it to be the furthest to the left.

10.3 Semi-classical energy spectrum

Applying the Bohr-Sommerfeld conditions (2.4.3) of chapter 2 to the filling fractions, which were shown in chapter 8 to be the action variables of the string, we find

\[
\frac{S_I}{\hbar} = N_I + \frac{1}{2} + \sum_{\alpha = g+2}^{\infty} \left( n_\alpha + \frac{1}{2} \right) \frac{\nu^{(I)}_\alpha}{2\pi} + O(\hbar),
\]

(10.3.1)
where the sum is over positive stability angles and in the string theory context we have $\hbar = \frac{1}{\sqrt{\lambda}}$. In (10.3.1) we used the fact that the Maslov index for the $A_I$-cycle ($I = 1, \ldots, g + 1$) in the generalised Jacobian $J(\hat{\Sigma}, \infty^\pm)$ is simply $\mu_I = 2$.

Obtaining the energy spectrum from (10.3.1) is relatively straightforward assuming the system is semiclassically integrable, which guarantees that the action operators satisfy $[\hat{S}_i, \hat{S}_j] = O(\hbar^3)$. In the semiclassical regime the Hamiltonian is then defined by the same classical function of the actions $E_{cl}[S_1, \ldots, S_{g+1}]$ but evaluated on the action operators, that is

$$\hat{\mathcal{H}}_{\text{string}} = E_{cl}[\hat{S}_1, \ldots, \hat{S}_{g+1}] + O(\hbar^2).$$

It follows that the energy spectrum is simply the classical energy $E_{cl}$ evaluated on the eigenvalues of the action variables (10.3.1) namely

$$E = E_{cl}\left[N_1 h + \frac{\hbar}{2} + \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \frac{\nu_\alpha^{(1)}}{2\pi} h, \ldots, N_{g+1} h + \frac{\hbar}{2} + \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \frac{\nu_\alpha^{(g+1)}}{2\pi} h\right] + O(\hbar^2).$$

We now Taylor expand this using the fact that $N_I \gg n_\alpha$ and $\hbar \ll 1$ to obtain

$$E = E_{cl}\left[\left(N_1 + \frac{1}{2}\right) h, \ldots, \left(N_{g+1} + \frac{1}{2}\right) h\right] + \sum_{I=1}^{g+1} \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \frac{\partial E_{cl} \nu_\alpha^{(I)}}{\partial S_I} 2\pi h.$$

Using (8.4.7b) and (10.2.8) to express $\partial E_{cl}/\partial S_I$ and $\nu_\alpha^{(I)}$ respectively as $B$-periods,

$$E = E_{cl}\left[\left(N_1 + \frac{1}{2}\right) h, \ldots, \left(N_{g+1} + \frac{1}{2}\right) h\right] + \sum_{I=1}^{g+1} \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \int_{B_I} \frac{dq}{2\pi} \int_{B_\alpha} dq^{(I)} h,$$

where $B_\alpha$ is the contour running from $\infty^+$ to the singular point labelled $\alpha$ on the top
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sheet, and back on the bottom sheet to $\infty^−$. The sum over $I$ can now be performed using equation (8.4.11) which yields

$$E = E_{cl} \left[ \left( N_1 + \frac{1}{2} \right) \hbar, \ldots, \left( N_{g+1} + \frac{1}{2} \right) \hbar \right] + \sum_{\alpha=g+2}^{\infty} \left( n_{\alpha} + \frac{1}{2} \right) \int_{B_\alpha} \frac{dq}{2\pi \hbar}. \quad (10.3.2)$$

This is the main result of this section. It expresses the semiclassical energy spectrum corresponding to a finite-gap solution as the sum of two terms. The order $O(1)$ term is the classical energy of a finite-gap string evaluated on half-integer quantised filling fractions and the order $O(1/\sqrt{\lambda})$ term is an infinite sum over fluctuation energies $\delta E_\alpha$ for each singular point $P_\alpha$ of the spectral curve.

Equation (10.3.2) provides a closed form expression for the fluctuation energy $\delta E_\alpha$ of any singular point $P_\alpha$. The required ingredient is the differential of the quasi-energy $dq$ which is the Abelian differential on $\hat{\Sigma}$ uniquely defined by its asymptotics at $x = \pm 1$. The (pinched) contour $B_\alpha$ runs from $\infty^+$ to $P_\alpha$ on the top sheet, then back from $\hat{\sigma}P_\alpha$ to $\infty^-$ on the bottom sheet. Thus the integral can be evaluated more explicitly using the same argument as in (10.2.9). Combining this with the result of equation (10.2.10) we have proved,

**Theorem 10.3.1.** Let $P_\alpha \in \Gamma$ be any singular point of the spectral curve $\Gamma$. Then the value of the quasi-momentum at $P_\alpha$ is an integer multiple $n_\alpha \in \mathbb{Z}$ of $\pi$. Moreover, the value of the quasi-energy at $P_\alpha$ gives the fluctuation energy $\delta E_\alpha$ of $P_\alpha$, namely

$$p(P_\alpha) = n_\alpha \pi, \quad \delta E_\alpha = \frac{q(P_\alpha)}{\pi \sqrt{\lambda}}. \quad (10.3.3)$$

**Remark** These fluctuation energies can also be obtained more directly by computing the stability angles of a finite-gap string periodic in the worldsheet $\tau$-coordinate. For this one repeats the calculation of section 10.2 on a finite-gap solution $Z_i$ with $Z_i(\tau + T) = Z_i(\tau)$. Its perturbations $Z_i + \delta Z_i$ defined by opening up a cut on the underlying curve $\hat{\Sigma}$ of $Z_i$ are in general not periodic
in \( \tau \). We obtain the stability angles \( \nu_\alpha = \frac{T_q(P_\alpha)}{\pi} \) so that the fluctuation energies are given by \( \delta \mathcal{E}_\alpha = \frac{\nu_\alpha}{T_1} \sqrt{\lambda} \) which reproduces (10.3.3).

Note that the infinite sum \( \sum_{\alpha=g+2}^{\infty} (n_\alpha + \frac{1}{2}) \delta \mathcal{E}_\alpha \) in (10.3.2) is only formal and requires regularisation. Still, we can formally rewrite the main result (10.3.2) in a way that makes the quantisation of all the fillings apparent, including the fillings of the singular points. If we formally think of the function \( \mathcal{E}_{cl} \) as depending on the infinite set of filling fractions \( \{S_I\}_{I=1}^{g+1}, \{S_\alpha\}_{\alpha=g+2}^{\infty} \) (all but finitely many of which are turned off for the classical finite-gap solutions) then we can interpret the \( B_\alpha \)-period of \( dq/2\pi \) as \( \partial \mathcal{E}_{cl}/\partial S_\alpha \) using a formal analogue of (8.4.7b) for an infinite-gap solution. One can then resum the resulting Taylor expansion to obtain the following formal expression for the semiclassical energy spectrum

\[
\mathcal{E} = \mathcal{E}_{cl} \left[ \left( N_1 + \frac{1}{2} \right) \hbar, \ldots, \left( N_{g+1} + \frac{1}{2} \right) \hbar, \left( n_{g+2} + \frac{1}{2} \right) \hbar, \ldots \right]. \tag{10.3.4}
\]

We stress that this is only a formal derivation as rigorously one would have to regularise the divergent infinite sum over stability angles at the intermediate steps as well as subtract off the energy of the vacuum (i.e. the zero cut finite-gap solution). But formally at least the result of the above derivation is the following:

- The semiclassical energy spectrum is obtained by evaluating the classical energy function of an infinite-gap solution on filling fractions quantised to half-integer multiples of \( \hbar \).

- The infinite number of singular points of the spectral curve \( \det (\Omega(x) - y1) = 0 \) which accumulate at \( x = \pm 1 \) must be filled with half a unit of \( \hbar \) in their ground state with an additional integer multiple of \( \hbar \) for excitations.

**Remark** The energy \( \mathcal{E}_{cl} \) we have been using is not the space-time energy \( \Delta \) of the classical solution but rather the worldsheet energy. They are related by the simple formula (3.4.5).
Comparison with alternative approach

In [56] an alternative method was proposed for extracting the semiclassical energy spacing around any given classical solution from the algebraic curve $\hat{\Sigma}$ itself, without making use of the divisor $\hat{\gamma}(t)$ on $\hat{\Sigma}$ as we have done. The heart of the method resides in the assumption that the filling fractions $S_I$ become quantised in integer units at least in a semiclassical approximation. This was interpreted in the language of the gauge theory side by attributing to a single Bethe root one unit of filling fraction. In the semiclassical quantisation of a solution each cut of its algebraic curve thus turns into a large clump of Bethe roots with the filling fraction counting the number of such roots. The idea of [56] for obtaining the semiclassical energy spacings is then to compare the energies of two neighbouring classical solutions differing only by a single Bethe root. If the underlying solution is characterised by the quasi-momentum $p(x)$ and has $K = g + 1$ cuts $C_j$ with mode numbers $n_j \in \mathbb{Z}, j = 1, \ldots, K$,

$$p(x + i0) + p(x - i0) = 2\pi n_j, \quad x \in C_j, j = 1, \ldots, K, \quad (10.3.5)$$

then its perturbation is characterised by a perturbed quasi-momentum $p(x) + \delta p(x)$ with still the same $K$ cuts but also with an extra isolated Bethe root at $x_{K+1}$ with mode number $n_{K+1} \in \mathbb{Z}$,

$$p(x + i0) + \delta p(x + i0) + p(x - i0) + \delta p(x - i0) = 2\pi n_j, \quad x \in C_j, j = 1, \ldots, K, \quad (10.3.6a)$$

$$p(x_{K+1}) + \delta p(x_{K+1}) + p(x_{K+1}) + \delta p(x_{K+1}) = 2\pi n_{K+1}. \quad (10.3.6b)$$

By using (10.3.5) we may simplify (10.3.6a) to

$$\delta p(x + i0) + \delta p(x - i0) = 0, \quad x \in C_j, j = 1, \ldots, K. \quad (10.3.7a)$$
and since $\delta p(x)$ is small, to lowest order equation (10.3.6b) yields

$$p(x_{K+1}) = \pi n_{K+1},$$

(10.3.7b)

Equations (10.3.7) are the starting point in [56] for obtaining the semiclassical energy spacings by reading them off from $\delta p(x)$.

Let us now show that the semiclassical energy spacings obtained by this method agrees with the fluctuation energies of theorem (10.3.1). We know from (8.4.7b) that the variation of the energy $E$ of a classical solution as we vary the moduli $S_I$ is

$$\delta E = \sum_{i=1}^{g+1} \left( \int_{B_i} \frac{dq}{2\pi} \right) \delta S_i.$$

It follows that adding a single Bethe root (which would correspond to setting $\delta S_J = \hbar$ for some $J$) should increase the energy of the solution by

$$\delta E = \int_{B_J} \frac{dq}{2\pi} \hbar.$$

(10.3.8)

This is exactly the formula (10.3.3) for the fluctuation energies derived in this chapter. Moreover, equation (10.3.7b) is exactly the same formula as in (10.3.3) for the value of the quasi-momentum at a singular point. Thus theorem (10.3.1) predicts the same energy spacing (10.3.8) as we would expect if Bethe roots carried $\hbar = \frac{1}{\sqrt{\lambda}}$ units of filling fraction. Theorem (10.3.1) however was proved without any input from the gauge theory side and was derived by a purely string theoretic calculation.
Part V

Conclusions & Outlook
Integrability of string theory on $AdS_5 \times S^5$

It is now a very well established fact that the Metsaev-Tseytlin action [60] for type IIB superstrings on $AdS_5 \times S^5$ admits a Lax connection [24]. This connection gives rise through the usual construction of the monodromy matrix to a wealth of integrals of motion. However the existence of a Lax connection is only half the conditions required for Liouville integrability. Indeed, as we have stressed in chapter 5 it is also necessary that the integrals of motion be in pairwise involution with respect to the Poisson structure.

**Non-ultralocality.** The main obstacle in proving the involution property was the non-ultralocal nature of the Poisson brackets of the current (4.3.4). The problematic $\delta'$-term gives rise in the algebra of monodromy matrices to ambiguous $\chi$-terms containing the value of the characteristic functions $\chi(\sigma; \sigma_1, \sigma_2)$ at the endpoints $\sigma = \sigma_1, \sigma_2$. Yet no value can be given such that the anti-symmetry property and the derivation rule are satisfied without violating the Jacobi identity for the Poisson bracket of monodromies.

**Maillet regularisation.** A way around this problem proposed by Maillet [69,71,72] is to define a *weak* bracket by ‘temporarily’ giving independent definitions for each multiply nested Poisson bracket of monodromies. Using this weak bracket consistent with all the fundamental properties of the Poisson bracket one then follows the usual arguments to show that $\{\text{tr } \Omega(x), \text{tr } \Omega(x')\} = 0$. But since this final bracket is equal to zero, the Jacobi identity involving it obviously hold. This final bracket thus holds in the usual *strong* sense.

**String theory.** In chapter 5 we applied Maillet’s procedure to string theory on $\mathbb{R} \times S^3$. In particular we showed that the integrals of motion are in pairwise involution with respect to the Dirac bracket associated with Virasoro constraints and static gauge fixing conditions, thus proving the complete statement of integrability.
for strings on $\mathbb{R} \times S^3$. These arguments were later generalised to the case of bosonic strings on $AdS_5 \times S^5$ in a series of papers by Kluson [97–100] (see also [101, 102]).

Finite-gap strings on $\mathbb{R} \times S^3$

The fact that superstring theory on $AdS_5 \times S^5$ possesses an infinite number of integrals of motion has been thoroughly exploited in the literature (initiated by [23] in the $SU(2)$ sector and eventually in the general case by [29]) to completely classify the full set of classical solutions on $AdS_5 \times S^5$. More precisely, every finite-gap solution was assigned a finite-genus algebraic curve whose moduli encodes the integrals of motion. However, the algebraic curve is not enough to uniquely characterise a specific solution. The identification of the extra data and the reconstruction of the corresponding solution was the subject of Part III.

Finite-gap integration. The existence of a flat Lax connection $J(x)$ is the starting point in the theory of finite-gap integration [67, 79–82]. The key idea behind this method is that analytic functions are uniquely specified by only a finite amount of data, such as their poles and zeroes. In chapter 6 we constructed the KMMZ curve $\hat{\Sigma}$, equipped with a meromorphic differential $dp$, which provides an arena for doing complex analysis. We also showed that the eigenvectors of the monodromy matrix define a vector function $\psi(P)$ on $\hat{\Sigma}$. After normalising it we can determine its analytic properties.

The divisor. Choosing $\psi(P)$ to solve the equation $(d - J(x))\psi(P) = 0$ we find it is uniquely specified by $g + 1$ poles, its value $\left(\begin{array}{c} 1 \\ 0 \end{array}\right)$ and $\left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ at $\infty^\pm \in \hat{\Sigma}$ and essential singularities at $x = \pm 1$. The remarkable fact is that its divisor of poles $\hat{\gamma}(0)$ is static. Since the Lax connection $J(x)$ can be recovered from $\psi(P)$ which in turn can be reconstructed from its analytic data, we were able to reconstruct the
current $j$. As a quick check the general solution was shown in [4] to reduce in the elliptic case ($g = 1$) to the so called helical solutions of [103] obtained by the method of Pohlmeyer reduction. It would be very nice to extend this construction to larger sectors and in particular to the full case of bosonic strings on $AdS_5 \times S^5$.

**Induced symplectic structure.** Since a finite-gap solution is parametrised by the algebro-geometric data consisting of the KMMZ curve and the divisor $\check{\gamma}(0)$, it can be thought of as a map $\{ (\check{\Sigma}, dp), \check{\gamma}(0) \} \mapsto j$. In chapter 8 we obtained the pullback of the bracket (4.3.4) of currents $j$ to the algebro-geometric data by making use of the Maillet regularised bracket of monodromy matrices obtained in chapter 5. The remarkable result is that the induced bracket assumes the canonical Darboux form (8.3.3) when expressed in terms of two special Abelian integrals on $\check{\Sigma}$: the quasi-momentum $p$ and the Zhukovsky transform of the spectra parameter $x$,

$$z = x + \frac{1}{x}.$$ 

It would be very interesting to check whether this is still true for finite-gap strings on $AdS_5 \times S^5$. In view of ultimately quantising the string directly, the fact that the symplectic structure is canonical with respect to the spectral parameter $z$ strongly suggest the right variables for an exact quantisation.

**Reality conditions.** Since the method of finite-gap integration is so firmly grounded in complex analysis, the general solution it produces satisfies the complexification of the equations we set out to solve. In chapter 9 we obtained the necessary restrictions on the algebro-geometric data $\{ (\check{\Sigma}, dp), \check{\gamma}(0) \}$ for the reconstructed solution to describe a closed string on $\mathbb{R} \times S^3$. In particular the condition on the KMMZ curve is that its branch points come in complex conjugate pairs. It would be interesting to derive the analogous fact in the non-compact $AdS$ sectors where the dual gauge theory predicts that the branch points should all be real [28].
Semiclassical strings on $\mathbb{R} \times S^3$

In chapter 10 we performed a first principle semiclassical quantisation on the general finite-gap solution constructed in Part III. The main result of this analysis is the formula (10.3.3) for the fluctuation energies around a generic finite-gap solution. It was shown to agree with the implicit method of Gromov and Vieira [56] for extracting fluctuation energies from the spectral curve and on which the subsequent papers [57,58] relied. Our result (10.3.2) for the semiclassical spectrum is only formal since one would need to regularise the infinite sum over fluctuation energies as well as subtract from it the vacuum energy given by a zero-gap solution (i.e. the BMN string). In any case, such a regularisation would only be interesting in the full case of strings on $AdS_5 \times S^5$ where the fluctuations transverse to the subsector $\mathbb{R} \times S^3$ are included along with the fermions. More formally still, we showed that the energy spectrum can be obtained by evaluating the classical energy of an infinite-gap string (10.3.4) with all its infinite filling fractions quantised to half-integer multiples of $\hbar$,

$$E = E_{\text{cl}} \left[ \left( N_1 + \frac{1}{2} \right) \frac{1}{\sqrt{\lambda}} \right].$$

This result is to be interpreted as a limit of expressions where a finite but arbitrary number of first entries are of order $O(1)$ corresponding to the tree level order and the remaining infinite number of entries encode the 1-loop corrections of order $O(1/\sqrt{\lambda})$.

Finally, in view of ultimately obtaining an exact quantisation of string theory on $AdS_5 \times S^5$ we have argued that operator ordering issues will be of crucial importance since they already appear in the semiclassical analysis. By assuming for simplicity that the cohomology class of the subprincipal form vanished, our results for the fluctuation energies for the $SU(2)$ sector agreed with [56–58]. This rules out
many operator orderings for an exact quantisation and provides further hints as to how one might go about quantising string theory on $AdS_5 \times S^5$. 
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