Finite Commutative Rings with a Sum-Zero Character

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Abstract

Let $R$ be a finite commutative ring and denote the multiplicative group of complex numbers of absolute value 1 by $\mathbb{C}_1$. We call a group homomorphism $\chi : (R, +) \rightarrow \mathbb{C}_1$ a character of $R$ and say that $\chi$ is sum-zero, if $\sum_{r \in I} \chi(r) = 0$ for each nonzero ideal $I$ of $R$. In this article, we prove that $R$ has a sum-zero character if and only if the number of maximal ideals and the number of minimal ideals of $R$ are equal, if and only if $R$ is a quasi-Frobenius ring. Also as an application, we prove that a generalization of the MacWilliams identity holds for the m-spotty weight enumerators of linear codes over $R$, if and only if $R$ has a sum-zero character, if and only if for every linear code $C$ over $R$, the dual of the dual $C$ is $C$ itself.

Keywords and Phrases: Sum-zero character; MacWilliams identity.
Mathematical Subject Classification: 13M99, 11T71.

1 Introduction

Throughout this paper, all rings are finite, commutative and with identity and $R$ denotes a ring.

Let $G$ be a finite Abelian group and denote the multiplicative group of complex numbers of absolute value 1 by $\mathbb{C}_1$. A group homomorphism $\chi : G \rightarrow \mathbb{C}_1$ is called a character of $G$. By a character of a ring $R$, we mean a character of the additive group of $R$. A character $\chi$ of $G$ is said be trivial (or principal) if $\chi(g) = 1$ for each $g \in G$. 

Definition 1.1. Assume that $\chi$ is a character of $R$. We say that $\chi$ is sum-zero, when $\sum_{r \in I} \chi(r) = 0$ for each nonzero ideal $I$ of $R$.

To see why such characters are important we need to recall some concepts of the theory of error correcting and error detecting codes. Suppose that $F = R^n$ be the free $R$-module of rank $n$. A submodule $C$ of $F$ is called a linear code of length $n$ over $R$ and each element of $C$ is called a codeword in $C$. Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in F$. By $x \cdot y$ we mean $\sum_{i=1}^n x_i y_i$. Now $C^\perp = \{ x \in F | \forall c \in C, xc = 0 \}$ is a again linear code of length $n$ over $R$, called the dual of $C$. Also the number of nonzero coordinates of $c$, denoted by $w(c)$, is called the Hamming weight of $c$.

Let $N = nb$ and $c = (c_{11}, c_{12}, \cdots, c_{1b}, c_{21}, \cdots, c_{2b}, \cdots, c_{n1}, \cdots, c_{nb})$ be a code of length $N$. Then $c_i = (c_{i1}, c_{i2}, \cdots, c_{ib})$ is called the $i$'th byte of $c$. Fix a number $1 \leq t \leq b$. An error $e$ is called a spotty byte error, if $t$ or fewer bits within a $b$-byte are in error ([10]). The number of spotty byte errors of an error $e$ is denoted by $w_M(e)$ and is called the $m$-spotty weight (or $m$-spotty $t/b$-weight) of $e$, that is, $w_M(e) = \sum_{i=1}^n \left\lceil \frac{w(e_i)}{t} \right\rceil$, where $e_i$ is the $i$'th byte of $e$. Note that in the case $t = 1$, this the usual Hamming weight.

The concepts of spotty byte errors and $m$-spotty weight of codewords, introduced in [10] and [9] respectively, are used in detecting and correcting multiple errors in byte error control codes which play an important role in computer memory systems (see, for example, [4, 6]).

The $m$-spotty $t/b$-weight enumerator of a linear code $C$ is $A_C(z) = \sum_{c \in C} z^{w_M(c)}$. In the case that $t = 1$ and $R$ is a field, the MacWilliams identity which relates $A_C(z)$ and $A_{C^\perp}(z)$ ([7, Theorem 5.2.9]) is a well-known theorem which is very useful in studying the properties of a linear code $C$ (see [7, Section 5.2]). Thus many authors have tried to generalize this identity.

In particular, in [8], a MacWilliams type identity for the $m$-spotty weight enumerators of linear codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ is established, where $u^2 = 0$. This identity is then generalized to linear codes over the ring $\mathbb{F}_2[u]/(u^2)$ in [3]. In both of these papers, the authors first defined explicitly a character of the ring under consideration and proved, in our terms, that this character is sum-zero. These sum-zero characters play a key role in proving the main theorems of the articles.

Here, in Section 2, we prove that $R$ has a sum-zero character if and only if the number of maximal ideals and the number of minimal ideals of $R$ are equal, if and only if $R$ is a quasi-Frobenius ring, that is, an Artinian self-injective ring (see (2.5)). Moreover, in Section 3, we prove that $R$ satisfies a generalization of Macwilliams relation for the $m$-spotty weight enumerators of linear codes, if and only if $R$ has a sum-zero character,
if and only if \((C^\perp)^\perp = C\) for any linear code over \(R\) (see (3.5)).

## 2 A Characterization of Rings with a Sum-Zero Character

First, we present an example which shows that there are rings which do not have any sum-zero characters.

**Example 2.1.** Let \(R = \mathbb{Z}[x,y]/I\), where \(I = \langle x^2, y^2, xy \rangle\). Denote the images of \(x, y\) in \(R\) by \(x, y\), respectively and assume that \(\chi\) is a sum-zero character of \(R\). Note that for each \(r \in R\), we have \(\chi(r)^2 = \chi(2r) = \chi(0) = 1\), so \(\chi(r) = \pm 1\). If \(\chi(x) = 1\), then \(\sum_{r \in \langle x \rangle} \chi(r) = 2 \neq 0\), thus \(\chi(x) = -1\). By a similar argument \(\chi(y) = \chi(x + y) = -1\). Therefore, we get \(-1 = \chi(x + y) = \chi(x)\chi(y) = 1\). From this contradiction we conclude that there is no sum-zero character on \(R\).

We will use the following lemmas to characterize rings with a sum-zero character.

**Lemma 2.2** ([7, Theorem 5.2.1]). Let \(\chi\) be a character of a finite Abelian group \(G\). Then

\[
\sum_{g \in G} \chi(g) = \begin{cases} 
0 & \text{\(\chi\) is nontrivial} \\
|G| & \text{\(\chi\) is trivial}
\end{cases}
\]

**Lemma 2.3.** Let \(G\) be a finite Abelian group and \(0 \neq g \in G\). Then there is a character \(\chi\) of \(G\) with \(\chi(g) \neq 1\).

**Proof.** Note that \(G \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{q_i}\) for some \(k \in \mathbb{N}\) and prime powers \(q_i\). Assume that \(j\) is such that \(p_j(g) \neq 0\) where \(p_j\) is the canonical projection \(G \to \mathbb{Z}_{q_j}\). Set \(\omega\) to be the \(q_j\)’th primitive root of unity. Then it is easy to see that the character \(\chi\) defined by \(\chi(\bar{a}) = \omega^{\bar{a}}\) for \(\bar{a} \in \mathbb{Z}_{q_j}\) and \(\chi(x) = 1\) for every \(x \in \mathbb{Z}_{q_i}\) with \(i \neq j\), has the required property. \(\square\)

Now we can state and prove the main results of this section. Recall that the *socle* of \(R\) is the sum of minimal ideals of \(R\) (see [1, p. 118]).

**Theorem 2.4.** Assume that \(R\) has a unique maximal ideal \(\mathfrak{M}\). Then \(R\) has a sum-zero character if and only if \(R\) has a unique minimal ideal.

**Proof.** First assume that \(\chi\) is a sum-zero character of \(R\). Suppose that \(\frac{R}{\mathfrak{M}} \cong \mathbb{F}_q\), where \(q = p^k\) for a prime number \(p\) and \(k \in \mathbb{N}\). Since \(\mathfrak{M}\) is the only maximal ideal of \(R\), we have \(\mathfrak{M}V = 0\) where \(V\) denotes the socle of \(R\). Therefore, \(V\) is an \(\frac{R}{\mathfrak{M}}\)-module, that is,
a \mathbb{F}_q\text{-vector space. Also ideals of } R \text{ contained in } V \text{ are exactly the } \mathbb{F}_q\text{-subspaces of } V. 

Assume that \dim_{\mathbb{F}_q} V = s.

Let \omega be a \(p\)'th primitive root of unity and \(\psi : (\omega) \to \mathbb{F}_p\) be the map with \(\psi(\omega^i) = i\). Since \(\chi(v)^p = \chi(pv) = \chi(0) = 1\) for each \(v \in V\) and because \(\chi|_V\) is nontrivial by (2.2), we have \(\chi(V) = \langle \omega \rangle\). Set \(U\) to be the kernel of the \(\mathbb{F}_p\)-linear transformation \(\phi = \psi \circ \chi|_V\). Since \(\phi\) is onto, we see that \(\dim_{\mathbb{F}_p} U = sk - 1\).

According to (2.2), \(\chi|_V\) is nontrivial for each one-dimensional \(\mathbb{F}_q\)-subspace \(V'\) of \(V\) (because \(V'\) is an ideal of \(R\)). So \(V' \not\subseteq U\) and \(\dim_{\mathbb{F}_p} V' \cap U \leq k - 1\). Thus \(|V' \setminus U| \geq p^k - p^{k-1}\). There are \(\frac{q^s - 1}{q - 1}\) different one-dimensional \(\mathbb{F}_q\)-subspaces of \(V\), each two of which have zero intersection. Thus we must have \(|V \setminus U| \geq \frac{q^s - 1}{q - 1}(p^k - p^{k-1})\). Hence

\[
p^{ks} - p^{ks-1} \geq \frac{q^s - 1}{q - 1}(p^k - p^{k-1}) \iff p^{k(s-1)} \geq \frac{q^s - 1}{q - 1} \iff q^{s-1} \geq \sum_{i=0}^{s-1} q^i \iff s = 1.
\]

Consequently, \(V\) is one-dimensional as a \(\mathbb{F}_q\)-vector space and hence \(R\) has exactly one minimal ideal.

Conversely, assume that \(R\) has exactly one minimal ideal \(I\). Let \(0 \neq r \in I\). By (2.3), there is a character \(\chi\) of \(R\) with \(\chi(r) \neq 1\). Since for each ideal \(J\) of \(R\), we have \(r \in I \subseteq J\), we conclude that \(\chi|_J\) is nontrivial. Hence \(\chi\) is sum-zero by (2.2).

\(\square\)

**Corollary 2.5.** A finite commutative ring \(R\) has a sum-zero character, if and only if the number of maximal ideals and the number of minimal ideals of \(R\) are the same.

**Proof.** Note that \(R\) is Artinian and hence by [2, Theorem 8.7] \(R \cong R_1 \times R_2 \times \cdots \times R_k\), where each \(R_i\) is local. Thus the ideals of \(R\) are of the form \(I_1 \times I_2 \times \cdots \times I_k\) where \(I_i\) is an ideal of \(R_i\). Therefore, the number of maximal ideals of \(R\) (that is, \(k\)) and the number of minimal ideals of \(R\) are the same, if and only if each \(R_i\) has a unique minimal ideal, if and only if each \(R_i\) has a sum-zero character \(\chi_i\) by (2.4). But if \(\chi\) is a sum-zero character of \(R\), then it is easy to see that \(\chi_i = \chi|_{I_i}\) is a sum-zero character of \(R_i\) and conversely, if each \(\chi_i\) is a sum-zero character of \(R_i\), then \(\chi((r_1, r_2, \ldots, r_k)) = \prod_{i=1}^{k} \chi_i(r_i)\) is a sum-zero character of \(R\).

\(\square\)

The ring \(R\) is called *quasi-Frobenius* (or *QF*) if \(R\) is Artinian and injective as an \(R\)-module (see [1, P. 333]). It follows easily from [1, Corollary 31.4], that a finite commutative ring \(R\) is QF if and only if the number of minimal ideals and number of maximal ideals of \(R\) are equal. Thus we get:
Corollary 2.6. A finite commutative ring has a sum-zero character, if and only if it is a QF-ring.

Also by [1, Theorem 30.7], for an Artinian ring (and in particular a finite ring) \( R \), being QF is equivalent to satisfying the double annihilator condition, that is, \( \text{Ann}(\text{Ann}(I)) = I \) for every ideal \( I \) of \( R \). The rings satisfying the double annihilator condition are also studied under the name co-m rings, see for example [5, p. 6].

3 A MacWilliams Type Relation for Rings with a Sum-Zero Character

In the sequel, we fix positive integers \( n, b \) and \( 1 \leq t \leq b \) and set \( N = nb \). Also as in the introduction, if \( c \) is a codeword of length \( N \), by \( c_i \) we mean the \( i \)'th byte of \( c \). Furthermore, \( C \) is assumed to be a linear code of length \( N \) over \( R \).

For a codeword \( c \in R^N \), let \( \alpha_s (1 \leq s \leq b) \) be the number of bytes with Hamming weight \( s \). Then \( w_M(c) = \sum_{s=0}^b \binom{s}{t} \alpha_s \). Assume that for each \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_b) \), \( A_\alpha \) denotes the number of codewords in \( C \) with the bytewise Hamming weight distribution \( \alpha \). Then

\[
  A_C(z) = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_b=n} A_\alpha \prod_{s=0}^b (z^\left \lceil \frac{s}{t} \right \rceil )^{\alpha_s},
\]

where \( A_C(z) \) is the m-spotty \( t/b \)-wight enumerator of \( C \).

Definition 3.1. Let \( r = |R| \). For any \( 0 \leq s \leq b \) set

\[
  g_s(z) = \sum_{i=0}^s \sum_{j=i}^b (-1)^i (r-1)^{j-i} \binom{b-s}{j-i} \binom{s}{i} z^\left \lceil \frac{j}{t} \right \rceil .
\]

Then we say that a ring \( R \) satisfies the generalized MacWilliams relation (or satisfies the GMR), if for every \( n, b, t \in \mathbb{N} \) with \( 1 \leq t \leq b \) and every linear code \( C \) of length \( N = nb \) over \( R \), we have

\[
  A_{C^\perp}(z) = \frac{1}{|C|} \sum_{\alpha_0+\alpha_1+\cdots+\alpha_b=n} A_\alpha \prod_{s=0}^b (g_s(z))^{\alpha_s},
\]

where \( A_\alpha \)'s are as above.

Note that in the case \( t = 1 \), \( g_s(z) = (1-z)^s(1+(r-1)z)^{b-s} \). Hence it can readily be checked that the equality of the above definition in the case \( t = 1 \) turns into

\[
  A_{C^\perp}(z) = \left( \frac{1+(r-1)z}{1+(r-1)z} \right)^N A_C \left( \frac{1-z}{1+(r-1)z} \right),
\]
which is the usual MacWilliams identity (see [7, Theorem 5.2.9]).

To characterize rings which satisfy the GMR, we need the following lemmas.

**Lemma 3.2.** Suppose that $C$ is a linear code of length $N$ over $R$ and $V$ is a $\mathbb{C}$-vector space. Let $\chi$ be a sum-zero character of $R$ and $f : R^N \to V$ be any map. If

$$\hat{f}(u) = \sum_{v \in R^N} \chi(u \cdot v) f(v)$$

for each $u \in R^N$, then

$$\sum_{v \in C^\perp} f(v) = \frac{1}{|C|} \sum_{u \in C} \hat{f}(u).$$

**Proof.** The proof is quite similar to [8, Lemma 2.8].

**Lemma 3.3.** Suppose that $\chi$ is a sum-zero character of $R$ and $y \in R$. Then

$$\sum_{x \in R} \chi(xy) = \begin{cases} 0, & y \neq 0 \\ |R|, & y = 0 \end{cases}.$$ 

**Proof.** If $y = 0$, then $\chi(xy) = 1$ for each $x \in R$ and the result follows. Suppose $y \neq 0$ and let $I$ be the annihilator ideal of $y$ in $R$. Note that $x_1 y = x_2 y \Leftrightarrow x_1 - x_2 \in I$. Thus

$$\sum_{x \in R} \chi(xy) = \sum_{y' \in Ry} \sum_{x y = y'} \chi(y') = |I| \sum_{y' \in Ry} \chi(y') = 0,$$

since $\chi$ is sum-zero.

For a codeword $v$ of length $b$, by $\text{supp}(v)$ we mean the set of indices $i$ such that $v_i \neq 0$.

**Lemma 3.4.** Let $u$ be a codeword of length $b$ over $R$ with $w(u) = s$. If $\chi$ is a sum-zero character of $R$ and $r = |R|$, then

$$\sum_{v \in E} \chi(u \cdot v) = (-1)^i(r - 1)^{j-i}\left(\begin{array}{c} b-s \\ j-i \end{array}\right),$$

where $E$ is the set of all $v \in R^b$ such that $\text{supp}(v) \cap \text{supp}(u) = \{a_1, a_2, \ldots, a_i\}$ and $w(v) = j$.

**Proof.** For each $v \in E$, we have $\chi(u \cdot v) = \chi\left(\sum_{k=1}^i u_{a_k} v_{a_k}\right) = \prod_{k=1}^i \chi(u_{a_k} v_{a_k})$. Noting that we have $(b-s)\left(\begin{array}{c} b-s \\ j-i \end{array}\right)$ ways to choose the $j-i$ nonzero coordinates of $v$ which are
not in \text{supp}(u), we see that

\[
\sum_{v \in E} \chi(u \cdot v) = \left( b - s \right) (r - 1)^{j - i} \sum_{0 \neq v_{a_1} \in R} \sum_{0 \neq v_{a_2} \in R} \cdots \sum_{0 \neq v_{a_i} \in R} \prod_{k=1}^{i} \chi(u_{a_k} v_{a_k})
= \left( b - s \right) (r - 1)^{j - i} \prod_{k=1}^{i} \chi(u_{a_k} v_{a_k})
= \left( b - s \right) (r - 1)^{j - i} \prod_{k=1}^{i} \left( \sum_{x \in R} \chi(x u_{a_k}) - \chi(0) \right)
= \left( b - s \right) (r - 1)^{j - i}(-1)^i,
\]

where the last equality holds by (3.3).

\[\square\]

**Theorem 3.5.** For a finite commutative ring \( R \) with \(|R| = r\) the following are equivalent.

(i) \( R \) has a sum-zero character \( \chi \);

(ii) \( R \) is QF;

(iii) The number of minimal ideals of \( R \) is equal to the number of maximal ideals of \( R \);

(iv) \( R \) satisfies the GMR;

(v) \(|C^\perp| = \frac{r^N}{|C|} \), for any linear code \( C \) of length \( N \) over \( R \);

(vi) \((C^\perp)^\perp = C\) for any linear code over \( R \).

**Proof.** (i)–(iii) are equivalent by (2.5) and (2.6).

(i) \(\Rightarrow\) (iv): Let \( C \) be a linear code of length \( N \) over \( R \) and \( A_\alpha \) be as in (3.1). Apply lemma (3.2) with \( f : R^N \to \mathbb{C}[z] \) defined by \( f(v) = z^{w_M(v)} \). Thus by (3.2), we get

\[
A_{C^\perp}(z) = \sum_{v \in C^\perp} f(v) = \frac{1}{|C|} \sum_{u \in C} \hat{f}(u).
\]

But

\[
\hat{f}(u) = \sum_{v \in R^N} \chi(u \cdot v) z^{w_M(v)} = \sum_{v_1 \in R^b} \sum_{v_2 \in R^b} \cdots \sum_{v_n \in R^b} \left( \prod_{k=1}^{n} \chi(u_k \cdot v_k) \right) \left( \prod_{k=1}^{n} z^{w(v_k)} \right)
= \prod_{k=1}^{n} \sum_{v_k \in R^b} \chi(u_k \cdot v_k) z^{w(v_k)}.
\]

Let \( s_k = w(u_k) \). According to (3.4), \( \sum_{v_k} \chi(u_k \cdot v_k) = (-1)^i (r - 1)^{j - i} \binom{b - s_k}{j - i} \), where the summation runs through all \( v_k \in R^b \) with \( w(v_k) = j \) and \(|\text{supp}(v_k) \cap \text{supp}(u_k)| = i \).
Thus if we set \( \binom{1}{k} = 0 \) for \( k < 0 \) or \( k > l \), then

\[
\sum_{v_k \in R^b} \chi(u_k \cdot v_k) = \sum_{i=0}^{s_k} (-1)^i (r-1)^{j-1} \binom{b-s_k}{j-i} \binom{s_k}{i}
\]

and hence \( \sum_{v_k \in R^b} \chi(u_k \cdot v_k) z^{\frac{w(v_k)}{w(c)}} = g_s(z) \).

Therefore, \( \hat{f}(u) = \prod_{k=1}^{n} g_{s_k}(z) = \prod_{s=0}^{b} (g_s(z))^\alpha_s \), where \( \alpha_s \) is the number of bytes of \( u \) with hamming weight \( s \). Now by (3.2),

\[
A_{C^\perp}(z) = \frac{1}{|C|} \sum_{u \in C} \hat{f}(u) = \frac{1}{|C|} \sum_{\alpha_0+\alpha_1+\ldots+\alpha_b=n} A_{\alpha} \prod_{s=0}^{b} (g_s(z))^\alpha_s.
\]

(iv) \( \Rightarrow \) (v): Apply (iv) with \( n = t = 1 \) and \( b = N \) to the code \( C \) to get

\[
A_{C^\perp}(z) = \frac{1}{|C|} \sum_{s=0}^{b} A'_{s} g_s(z),
\]

where \( A'_{s} \) denotes the number of codewords in \( C \) with \( w(c) = s \). As in the remarks after (3.1), in this case, \( g_s(z) = (1-z)^s(1+(r-1)z)^{b-s} \). Thus \( g_s(1) = 0 \) unless \( s = 0 \) and \( g_0(1) = r^b \). Also the only codeword with zero Hamming weight is 0, that is, \( A'_{0} = 1 \). Now the result follows from the fact that \( A_{C^\perp}(1) = |C^\perp| \), by the definition of weight enumerator of a code.

(v) \( \Rightarrow \) (vi): It is clear that always \( C \subseteq (C^\perp)^\perp \). But by (v), we have \( |C| = |(C^\perp)^\perp| \) and (vi) follows.

(vi) \( \Rightarrow \) (ii): Since every ideal \( I \) of \( R \) is a linear code of length 1 over \( R \), by (vi) we get \( \text{Ann}(\text{Ann}(I)) = I \) for each such \( I \). But this means that \( R \) is QF by [1, Theorem 30.7].

Example 3.6. Let \( R \) be the ring in Example (2.1) and \( \mathfrak{M} = \langle x, y \rangle \) be the maximal ideal of \( R \). Then \( |\mathfrak{M}| = 4 \) and \( \mathfrak{M}^\perp = \text{Ann}(\mathfrak{M}) = \mathfrak{M} \). Thus \( |\mathfrak{M}||\mathfrak{M}^\perp| \neq 8 = |R| \) and hence \( R \) does not satisfy the equivalent conditions of the previous theorem. Also it is easy to check that \( (\langle x \rangle^\perp)^\perp = \mathfrak{M} \neq \langle x \rangle \).

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