Zeros of the Derivatives of Faber Polynomials
Associated with a Universal Covering Map

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Abstract

For a compact set \( E \subset \mathbb{C} \) containing more than two points, we study asymptotic behavior of normalized zero counting measures \( \{ \mu_k \} \) of the derivatives of Faber polynomials associated with \( E \). For example if \( E \) has empty interior, we prove that \( \{ \mu_k \} \) converges in the weak-star topology to a measure whose support is the boundary of \( g(D) \), where \( g : \{ |z| > r \} \cup \{ \infty \} \to \mathbb{C} \setminus E \) is a universal covering map such that \( g(\infty) = \infty \) and \( D \) is the Dirichlet domain associated with \( g \) and centered at \( \infty \).

Our results are counterparts of those of Kuijlaars and Saff (1995) on zeros of Faber polynomials.

1 Introduction

Let \( g \) be a function which is holomorphic in a neighborhood of infinity such that its Laurent series is of the form

\[
g(w) = w + \sum_{k=0}^{\infty} b_k w^{-k}. \tag{1.1}\]

Faber polynomials \( F_k \) associated with \( g \) are defined by the generating function

\[
\frac{g'(w)}{g(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \tag{1.2}
\]

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and the normalized derivatives of Faber polynomials, $P_k := (F_{k+1})'/((k + 1)$ for $k = 0, 1, 2, \ldots$, satisfy the equation

$$\frac{1}{g(w) - z} = \sum_{k=0}^{\infty} \frac{P_k(z)}{w^{k+1}}.$$  \hspace{1cm} (1.3)

For every $k$, both $F_k$ and $P_k$ are monic polynomials of degree $k$. To study more about Faber polynomials and their derivatives, see for example [3], [2] and [10].

For each $k$, let $\nu_k$ be the normalized zero counting measure of $F_k$; i.e.,

$$\nu_k = (2\pi k)^{-1} \Delta(\log |F_k|),$$

where $\Delta$ represents for the generalized Laplacian. Similarly, we denote by $\mu_k := (2\pi k)^{-1} \Delta(\log |P_k|)$ the normalized zero counting measure of $P_k$. Kuijlaars and Saff studied in [4] the limit behavior of the measures $\{\nu_k\}$, and we concern the limit behavior of $\{\mu_k\}$. Especially we are interested in the case when $g$ is a universal covering map.

Suppose that $E \subset \mathbb{C}$ is a compact set containing more than two points such that $\Omega := \mathbb{C}\setminus E$ is connected (but not necessarily simply-connected). By the Uniformization theorem (cf. [1], Chap. 10), there exists a unique number $r = r(E) > 0$ and a unique normalized universal covering map $g : \Lambda(r) := \mathbb{C}\{|w| \leq r\} \rightarrow \Omega$ which has a Laurent expansion of the form (1.1) at infinity. In this case Faber polynomials $\{F_k\}$, and their normalized derivatives $\{P_k\}$, associated with $g$ are also called Faber polynomials, or the normalized derivatives of Faber polynomials, respectively, associated with $E$.

Since $E$ contains more than two points, the domain $\Omega = \mathbb{C}\setminus E$ carries the unique hyperbolic (Poincaré) metric with constant curvature ($\equiv -1$), and we define $\tilde{E}$ as the union of $\partial E$ and the points in $\Omega$ which have more than one shortest curve to $\infty$ with respect to this metric.

**Example 1.** If $\mathbb{C}\setminus E$ is simply-connected, that is, $E$ is connected, then $\tilde{E} = \partial E$.

**Example 2.** Suppose $E$ is a compact set consisting of three points. Then $\tilde{E}$ is either a topological tripod or a line segment joining points in $E$. See Proposition [13]. For example, if $E = \{1, \eta, \eta^2\}$ where $\eta = \exp(2\pi i/3)$ is a third root of unity, then $\tilde{E} = \{\eta^j t : j = 0, 1, 2$ and $0 \leq t \leq 1\}$ (Figure 1). If $E = \{-1, 0, 1\}$, then $\tilde{E} = \{z \in \mathbb{R} : -1 \leq z \leq 1\}$ (Figure 2).
There is an alternative way to describe the set $\tilde{E}$. To explain this, let $\Gamma$ be the Fuchsian group such that $\Lambda(r)/\Gamma \cong \Omega$ and $g \circ \tau(z) = g(z)$ for all $\tau \in \Gamma$ and $z \in \Lambda(r)$. We denote by $\Gamma_\infty$ the orbit of $\infty$ under $\Gamma$ and by $d(\cdot, \cdot)$ the hyperbolic distance between two points in $\Lambda(r)$. The set

$$\mathcal{D} := \{ w \in \Lambda(r) : d(\infty, w) < d(\zeta, w) \text{ for all } \zeta \in \Gamma_\infty, \ \zeta \neq \infty \}$$

(1.4) is called the Dirichlet domain associated with $\Gamma$ (or $g$) and centered at $\infty$. It is a fundamental region (cf. [6], Section I-4). Then it will be shown in Corollary 10 that $\tilde{E} = \partial g(\mathcal{D})$, the boundary of the image of $\mathcal{D}$ under $g$.

Let $\mathcal{U}$ be the collection of compact sets in $\mathbb{C}$ with connected complements such that either $E$ is not connected, or $E$ is connected but $\partial E$ contains a singularity other than an outward cusp.

Our main result is:

**Theorem 3.** (1) $\tilde{E}$ is connected.

(2) If $E^\circ = \emptyset$, where $E^\circ$ denotes the interior of $E$, then the sequence $\{ \mu_k \}$ converges in the weak-star topology to the probability measure

$$\mu(z) = \frac{1}{2\pi} \Delta \log \delta(z)$$

(1.5)

where

$$\delta(z) = \limsup_{k \to \infty} |P_k(z)|^{1/k},$$

(1.6)

and the support of $\mu$ is $\tilde{E}$. If, in addition, $\mathbb{C} \setminus E$ is simply-connected, then $\mu$ is the equilibrium distribution of the compact set $E$. 

\vspace{3em}

Figure 1: $E = \{1, \eta, \eta^2\}$ \quad Figure 2: $E = \{-1, 0, 1\}$
(3) Suppose $E^\circ$ is connected and $E \in \mathcal{U}$. Then there is a subsequence of 
\{\mu_k\} that converges in the weak-star topology to the measure $\mu$ in (1.5),
and its support is $\tilde{E}$. If, in addition, $\overline{\mathbb{C}} \setminus E$ is simply-connected, then $\mu$
is the equilibrium distribution of the compact set $E$.

The polynomials \{${\mathcal{P}_k}$\} provide some useful tools in other branches of mathematics. For example, A. Atzmon, A. Eremenko and M. Sodin showed in [2] that if $a$ is an element in a complex Banach algebra with unit and $E \subset \mathbb{C}$ is a compact set with connected complement, then the spectrum of $a$ is included in $E$ if and only if
\[
\limsup_{k \to \infty} \|P_k(a)\|^{1/k} \leq r,
\]
where \{${P_k}$\} and $r = r(E)$ are as before. They also showed in the same paper that for a given analytic germ $f(z) = \sum_{k=0}^{\infty} f_k z^{-k-1}$ at infinity with values in a Banach space, the polynomials \{${P_k}$\} can be used to determine whether $f$ has analytic continuation to $\Omega = \overline{\mathbb{C}} \setminus E$ or not.

For a formal Laurent series of the form (1.1), it is known (cf. [10]) that the zeros of $P_k(z)$ are exactly the eigenvalues of the leading $k \times k$ principal submatrix of the infinite Toeplitz matrix
\[
\begin{bmatrix}
b_0 & b_1 & b_2 & b_3 & \cdots \\
1 & b_0 & b_1 & b_2 & \cdots \\
0 & 1 & b_0 & b_1 & \cdots \\
0 & 0 & 1 & b_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

2 Known facts

Suppose the Laurent expansion of $g$ is given by (1.1), and we assume that $\rho_0 := \limsup_{k \to \infty} |b_k|^{1/k} < \infty$ so that the series is convergent in $|w| > \rho_0$. We also define $\delta_0$ as the smallest nonnegative number such that $g$ has a meromorphic extension to $\Lambda(\delta_0) = \{|w| > \delta_0\} \cup \{\infty\}$. Now for every $z \in \mathbb{C}$, we write $g^{-1}(z) := \{w \in \Lambda(\rho_0) : g(w) = z\}$ and $\tilde{g}^{-1}(z) := \{w \in \Lambda(\delta_0) : g(w) = z\}$.

Note that the multiplicity of a point $w \in \Lambda(\delta_0)$ is $m$ if $g'(w) = \cdots = g^{(m-1)}(w) = 0$ and $g^{(m)}(w) \neq 0$. The following definition is due to J. L. Ullman ([5], [6]).

**Definition 4.** For every nonnegative integer $p$, $\tilde{C}_p$ (or $C_p$) is the set of all points $z \in \mathbb{C}$ such that the points of largest absolute value in $\tilde{g}^{-1}(z)$ (or $g^{-1}(z)$, respectively) have total multiplicity $p$. 

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From the definition, one can easily see that $\tilde{C}_0 = \mathbb{C} \setminus g(\Lambda(\delta_0))$ is compact and $\tilde{C}_1$ is an open set containing a neighborhood of infinity.

The next two statements are analogous to Theorem 1.3 and Theorem 1.4 of [4], where the theorems below are proved with $C_0, C_1, F_k$ and $\nu_k$ in place of $\tilde{C}_0, \tilde{C}_1, P_k$ and $\mu_k$, respectively.

**Theorem 5 (cf. [4], Theorem 1.3).** If the interior of $\tilde{C}_0$ is empty, then the sequence $\{\mu_k\}$ converges in the weak-star topology to the measure $\mu$ in (1.5) and the support of $\mu$ is equal to $\partial \tilde{C}_1$. If, in addition, $\overline{C} = \tilde{C}_0 \cup \tilde{C}_1$, then $\mu$ is the equilibrium distribution of the compact set $\tilde{C}_0$.

**Theorem 6 (cf. [4], Theorem 1.4).** If the interior of $\tilde{C}_0$ is connected, then there is a subsequence of $\{\mu_k\}$ that converges in the weak-star topology to the measure $\mu$ in (1.5), and the support of $\mu$ is $\partial \tilde{C}_1$. If, in addition, $\overline{C} = \tilde{C}_0 \cup \tilde{C}_1$, then $\mu$ is the equilibrium distribution of the compact set $\tilde{C}_0$.

The proofs of Theorems 5 and 6 are given in Appendix A. In fact, one may check that these theorems can be shown by the same arguments for Theorems 1.3 and 1.4 in [4], provided that Ullman’s results used in [4] are replaced by those listed below.

**Lemma 7 ([9], Lemma 7; cf. [4], Lemma 2.2).** Every $z_0 \in \tilde{C}_p$, $p \geq 2$, has a neighborhood $B(z_0, \epsilon) := \{z : |z - z_0| < \epsilon\}$ such that $\partial \tilde{C}_1 \cap B(z_0, \epsilon)$ consists of a finite number of analytic Jordan arcs each joining $z_0$ to a point on the circle $\partial B(z_0, \epsilon)$. Any two arcs intersect only at $z_0$. The remaining points of $B(z_0, \epsilon)$ are in $\tilde{C}_1$.

From Lemma 7 we have an important corollary.

**Corollary 8 (cf. [4], Corollary 2.3).**

$$\partial \tilde{C}_1 = \partial \tilde{C}_0 \cup \bigcup_{p \geq 2} \tilde{C}_p.$$  

A point $\xi \in \mathbb{C}$ is called a limit point of the zeros of $\{P_k\}$ if there exist an increasing sequence $\{k_j\}$ and a zero $\xi_j$ of $P_{k_j}$ for each $j$ such that $\xi = \lim_{j \to \infty} \xi_j$.

**Theorem 9 ([9]; cf. [4], Theorem 1.2).** All limit points of the zeros of $\{P_k\}$ are in $\mathbb{C} \setminus \tilde{C}_1$. Every boundary point of $\tilde{C}_1$ is a limit point of the zeros of $\{P_k\}$.

**Proof.** The first statement is Lemma 3 of [9], and the second statement is given in the proof of Lemma 11 in the same paper. □
Lemma 10 ([9], Lemma 12). The boundary of the unbounded component of $\tilde{C}_1$ is a connected set.

Lemma 11 ([9], Lemma 8; cf. [4], Lemma 2.4). For every $\epsilon_0 > 0$ and $z_0 \in \tilde{C}_p$, $p \geq 2$, there are $\epsilon_1 > 0$ and $z_1 \in \tilde{C}_q$, $q \geq 2$, such that

\begin{align*}
B(z_1, \epsilon_1) &\subset B(z_0, \epsilon_0), \\
B(z_1, \epsilon_1) \cap \tilde{C}_0 = \emptyset, \\
B(z_1, \epsilon_1) \cap \tilde{C}_1 = D_1 \cup D_2,
\end{align*}

where $D_1$ and $D_2$ are disjoint non-empty domains. Moreover, there exist analytic functions $f_1$ and $f_2$ on $B(z_1, \epsilon_1)$ such that

\begin{align*}
|f_1(z)| > |f_2(z)|, & \quad z \in D_1, \\
|f_i(z)| = \delta(z), & \quad z \in D_i, \ i = 1, 2,
\end{align*}

where $\delta(z)$ is defined in (1.6).

Lemma 12 ([9]; cf. [4], Lemma 3.1). (a) For every $z \in \tilde{C}_0$,

$$\delta(z) = \delta_0,$$

where $\delta_0$ is defined in the first paragraph of this section.

(b) For every $z \notin \tilde{C}_0$,

$$\delta(z) = \max\{|w| : w \in \tilde{g}^{-1}(z)\}.$$

(c) For every $z \in \tilde{C}_1$,

$$\delta(z) = \lim_{k \to \infty} |P_k(z)|^{1/k};$$

i.e., the lim sup in (1.6) can be replaced by lim.

An immediate corollary of (2.3) and (2.4) is:

Corollary 13 (cf. [4], Lemma 3.1). $\delta(z)$ is a continuous function on $\mathbb{C}$.

3 Proof of Theorem

Suppose $E \subset \mathbb{C}$ is a compact set containing more than two points such that its complement $\Omega = \mathbb{C} \setminus E$ is connected, and let $g : \Lambda(r) \to \Omega$ be the uniformizing map with Laurent expansion (1.1) at infinity.
Lemma 14. For given \( z \in \Omega \), there is a one-to-one correspondence between shortest curves (with respect to the hyperbolic metric in \( \Omega \)) from \( z \) to \( \infty \) and points in \( \tilde{g}^{-1}(z) \) with largest absolute value.

Proof. Suppose \( \gamma : [a, b] \to \Omega \) is a shortest curve from \( z \) to infinity such that \( \gamma(a) = z \) and \( \gamma(b) = \infty \), and let \( \tilde{\gamma} : [a, b] \to \Lambda(r) \) be the lifting curve of \( \gamma \) such that \( \tilde{\gamma}(b) = \infty \). Because \( g \) is a local isometry between \( \Lambda(r) \) and \( \Omega \) with hyperbolic metrics on them, we know that \( \tilde{\gamma} \) is a shortest curve in \( \Lambda(r) \); i.e., the trace of \( \tilde{\gamma} \) is the ray
\[
\{ tw : 1 \leq t \leq \infty \}
\]for \( w = \tilde{\gamma}(a) \). Now if \( |w_0| > |\tilde{\gamma}(a)| \) for some \( w_0 \in \tilde{g}^{-1}(z) \), let \( \alpha \) be the ray \([3.1]\) with \( w = w_0 \). Then since \( |w_0| > |\tilde{\gamma}(a)| \), \( \alpha \) is shorter than \( \tilde{\gamma} \), hence \( g(\alpha) \) is shorter than \( g(\tilde{\gamma}) = \gamma \). Since \( g(\alpha) \) connects \( z \) to \( \infty \), this contradicts the choice of \( \gamma \), proving \( |\tilde{\gamma}(a)| = \max\{|w| : w \in \tilde{g}^{-1}(z)\} \). Conversely, if we have a point \( w_0 \in \tilde{g}^{-1}(z) \) such that \( |w_0| = \max\{|w| : w \in \tilde{g}^{-1}(z)\} \), then by the same argument above the image of the ray \([3.1]\) with \( w = w_0 \) is a shortest curve in \( \Omega \) connecting \( z = g(w_0) \) and \( \infty \). Therefore the map \( \gamma \mapsto \tilde{\gamma}(a) \) is bijective between the set of shortest curves from \( z \) to \( \infty \) and the set of points in \( \tilde{g}^{-1}(z) \) with largest absolute value. The lemma follows. \( \Box \)

Note that \( g'(w) \neq 0 \) for any \( w \in \Lambda(r) \) since \( g \) is a covering map. Therefore the following corollary is an immediate consequence of Lemma 14.

Corollary 15. For each \( z \in \Omega \), \( z \in \tilde{E} \) if and only if \( z \in \tilde{C}_p \) for some \( p \geq 2 \).

Let \( \mathcal{D} \) be the Dirichlet domain associated with \( g \) and centered at \( \infty \), as we introduced in Section 1.

Corollary 16. \( \tilde{E} = \partial g(\mathcal{D}) \).

Proof. Because \( \mathcal{D} \) is a fundamental region, \( g(\overline{\mathcal{D}}) = \Omega \) and \( g\big|\mathcal{D} \) is injective. Therefore \( \partial \mathcal{E} = \partial \mathcal{D} \subset \partial g(\mathcal{D}) \), \( E^c \cap \partial g(\mathcal{D}) = \emptyset \), and \( \partial g(\mathcal{D}) \cap \Omega = g(\partial \mathcal{D}) \). Hence to prove the corollary, it suffices to show that \( E \cap \Omega = g(\partial \mathcal{D}) \cap \Omega = g(\partial \mathcal{D}) \).

Let \( d(\cdot, \cdot) \) denote the hyperbolic distance between two points in \( \Lambda(r) \), and let \( \Gamma \) be the corresponding Fuchsian group such that \( \Lambda(r)/\Gamma \cong \Omega \). Then for each \( w_1, w_2 \in \Lambda(r) \) and \( \tau \in \Gamma \), \( d(w_1, w_2) = d(\tau(w_1), \tau(w_2)) \). Therefore, \( d(w, \infty) \leq d(w, \tau(\infty)) \) for all \( \tau \in \Gamma \) if any only if \( d(w, \infty) \leq d(\tau(w), \infty) \) for all \( \tau \in \Gamma \), or \( |w| \geq |\tau(w)| \) for all \( \tau \in \Gamma \). This means that \( w \in \overline{\mathcal{D}} \) if and only if \( |w| = \max\{|\xi| : \xi \in \tilde{g}^{-1}(g(w))\} \). Now the corollary follows from Lemma 14 because we have \( w \in \partial \mathcal{D} \) if and only if there exist \( w' \in \partial \mathcal{D}\setminus\{w\} \) and \( \tau \in \Gamma \) such that \( w = \tau(w') \), that is, \( g(w) = g(w') \) (\( \Box \), p. 37). \( \Box \)
Note that the proof of this corollary also shows that \( \Omega \setminus \tilde{E} = g(\mathcal{D}) \).

**Proof of Theorem 3.** Suppose \( E \notin \mathcal{U} \). Then \( \Omega \) is simply-connected, hence \( \tilde{E} = \partial E = \partial \Omega \) is connected. If \( E \in \mathcal{U} \), then as shown in [4] (p. 444), \( g \) cannot be extended to \( \Lambda(r_0) \) for any \( r_0 < r \); i.e., \( r = \delta_0 \), where \( \delta_0 \) is defined in the first paragraph of Section 2. Therefore we have \( E = \tilde{C}_0 \), hence \( \Omega \setminus \tilde{E} = \tilde{C}_1 \) by Corollary 15, i.e., \( \tilde{C}_1 = g(\mathcal{D}) \). Now Lemma 10 implies the statement (1) of Theorem 3 since \( \tilde{C}_1 = g(\mathcal{D}) \) is connected and \( \tilde{E} = \partial g(\mathcal{D}) \).

To prove (2) of Theorem 3, assume that \( E \cap \mathcal{U} = \emptyset \). If \( E \) is not connected, then we have \( \delta_0 = r \), or \( \tilde{C}_0 = E \) by Corollaries 8 and 12. Since \( \tilde{C}_0 = E \) has empty interior, Theorem 5 implies that \( \{\mu_k\} \) converges to \( \mu \) and its support is \( \partial \tilde{C}_1 = \tilde{E} \). If \( E \) is connected, we consider two cases: \( \delta_0 < r \) and \( \delta = r \). If \( \delta_0 < r \), then every point \( z \in E \) corresponds to at least 2 points on \( |w| = r \) counting multiplicity, hence \( \tilde{C}_0 = \emptyset \), \( \delta(z) \) is constant \((= r)\) on \( E \), and \( E = \bigcup_{j \geq 2} \tilde{C}_j = \partial \tilde{C}_1 \). If \( \delta_0 = r \), then \( \delta(z) \) is constant \((= r = \delta_0)\) on \( E = \tilde{C}_0 \) by Lemma 12(a) and \( E = \tilde{E} = \partial \tilde{C}_1 \) as before. In any case \( E = \tilde{E} = \partial \tilde{C}_1 \) and \( \delta(z) \) is constant on \( E \). Therefore by Theorem 5, the sequence \( \{\mu_k\} \) converges to \( \mu = (2\pi)^{-1} \Delta \log \delta \) and its support is \( E = \tilde{E} \). Furthermore, it is the equilibrium distribution of \( E \) since \( \delta \) is constant on \( E \). This proves (2) of Theorem 3.

The statement (3) of Theorem 3 follows from Theorem 6 by the same argument as above.

One cannot replace \( \mu_k \) in Theorem 3 by \( \nu_k = (2\pi k)^{-1} \Delta \log |F_k| \). To see this, let \( \eta = \exp(2\pi i/3) \), \( E = \{1, \eta, \eta^2\} \), and \( g : \Lambda(r) \to \Omega \) the universal covering map with Laurent expansion (1.1). As in Example 2,

\[
\tilde{E} = \{\eta^j t : j = 0, 1, 2 \text{ and } 0 \leq t \leq 1\}.
\]

Now we claim that there is a subsequence of \( \{\nu_k\} \) such that the support of its weak-star limit \( \nu \) is different from \( \tilde{E} \) (Figure 4).
Let \( \rho_0 \) be the maximum of absolute values of the finite poles of \( g \), and let \( \Omega_0 := g(\Lambda(\rho_0)) \). Then since \( \rho_0 > r = r(E) \) and

\[
\eta^{-1} g(\eta w) = g(w) \quad \text{for all } w \in \Lambda(r),
\]

one can easily see that \( C \setminus \Omega_0 = N_0 \cup N_1 \cup N_2 \), where \( N_0 \) is a closed neighborhood of 1 and \( N_j = \eta^j N_0, \ j = 1, 2 \). Therefore from Definition 4 and Corollary 2.3 of [4] (cf. Corollary 8),

\[
\partial C_1 = \partial C_0 \cup \bigcup_{p \geq 2} C_p = \bigcup_{j=1}^3 \partial N_j \cup (\tilde{E} \cap \Omega_0),
\]

which is the set shown in Figure 4.

By Lemma 3.1 of [4] (cf. Lemma 12), we have \( \limsup_{k \to \infty} |F_k(1)| = \rho_0 \). Thus there exists a subsequence \( \{F_{k_j}\} \) of \( \{F_k\} \) such that \( \lim_{j \to \infty} |F_{k_j}(1)| = \rho_0 \). Considering (3.2), we also have \( \lim_{j \to \infty} |F_{k_j}(\eta)| = \lim_{j \to \infty} |F_{k_j}(\eta^2)| = \rho_0 \). Now by Lemma 5.1 and the proof of Lemma 5.2 of [4] (cf. Lemmas 20 and 21), the sequence \( \{\nu_{k_j}\} \) converges in the weak-star topology to the measure

\[
\nu(z) = \frac{1}{2\pi} \Delta \log \left( \limsup_{k \to \infty} |F_k(z)|^{1/k} \right),
\]

and by Lemma 4.1 of [4] (cf. Lemma 19) the support of \( \nu \) is \( \partial C_1 \).
4 Compact sets consisting of three points

Let \( \Omega_{0,1} := \mathbb{C} \setminus \{0,1\} \) be a metric space equipped with the hyperbolic metric on it, and we use the notations \( H^+ := \{ \text{Im}(z) > 0 \} \), \( H^- := \{ \text{Im}(z) < 0 \} \), \( I_1 := \{ z \in \mathbb{R} : z < 0 \} \), \( I_2 := \{ z \in \mathbb{R} : 0 < z < 1 \} \), \( I_3 := \{ z \in \mathbb{R} : z > 1 \} \), and \( I := I_1 \cup I_2 \cup I_3 \). The following lemma is very trivial but plays a crucial role in our arguments later.

**Lemma 17.** Suppose \( z_1 \in H^+ \) and \( z_2 \in H^+ \cup I \). Then there is a unique shortest curve \( \gamma \) connecting \( z_1 \) and \( z_2 \). Moreover, the interior arc of \( \gamma \) does not intersect \( I \).

**Proof.** The proof is omitted here and left to the reader. In fact, one may prove it by considering the symmetric property of \( \Omega_{0,1} \).

Let \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) with the hyperbolic metric on it, and we assume that \( G : \mathbb{D} \to \Omega_{0,1} \) is a holomorphic universal covering map such that for a given point \( a \in I_1 \cup H^+ \), \( G(0) = a \). Let \( \Gamma \) be the modular group on \( \mathbb{D} \) such that \( \mathbb{D}/\Gamma \cong \Omega_{0,1} \) and \( G \circ \tau(z) = G(z) \) for all \( \tau \in \Gamma \), and we denote by \( \Gamma_0 \) the orbit of the origin under \( \Gamma \) and by \( \mathcal{D}_0 \) the Dirichlet domain with centered at the origin. Note that each component of \( G^{-1}(H^+) \) or \( G^{-1}(H^-) \) is a hyperbolic open triangle, and each side of such a triangle is a geodesic curve and a component of \( G^{-1}(I_j) \) for some \( j = 1, 2, 3 \). Now we consider the cases \( a \in I_1 \) and \( a \in H^+ \) separately.

If \( a \in I_1 \), there are two hyperbolic triangles \( \Delta^+ \) and \( \Delta^- \) in \( \mathbb{D} \) such that \( G(\Delta^+) = H^+ \), \( G(\Delta^-) = H^- \) and \( 0 \in \overline{\Delta^+} \cap \overline{\Delta^-} \). Suppose \( w \in \Delta^+ \). Then by Lemma 17 there exists a unique shortest curve \( \gamma : [t_0, t_1] \to H^+ \cup \{ a \} \) such that \( \gamma(t_0) = a \) and \( \gamma(t_1) = G(w) \). Let \( \tilde{\gamma} \) be the lifting curve such that \( \tilde{\gamma}(t_0) = 0 \). Because \( \gamma \) does not intersect \( I \), \( \tilde{\gamma} \) cannot intersect \( G^{-1}(I) \), hence \( \tilde{\gamma}(t_1) = w \). Now if \( G(w') = G(w) \) for some \( w' \neq w \), the shortest curve \( \alpha \) from \( w' \) to 0 intersects \( G^{-1}(I) \), thus \( G(\alpha) \) intersects \( I \). Therefore the length of \( G(\alpha) \) is strictly greater than the length of \( \gamma \), or the the length of \( \alpha \) is strictly greater than the length of \( \tilde{\gamma} \). This shows that \( w \in \mathcal{D}_0 \). Because a similar argument holds for any \( w \in \Delta^- \), we have

\[
\Delta^+ \cup \Delta^- \subset \mathcal{D}_0. \tag{4.1}
\]

Since \( \mathcal{D}_0 \) is a fundamental region, \( G \) is univalent in \( \mathcal{D}_0 \) and \( G(\overline{\mathcal{D}_0}) = \Omega_{0,1} \). Thus (4.1) in fact shows that

\[
\mathcal{D}_0 = \left( \overline{\Delta^+ \cup \Delta^-} \right)^\circ.
\]

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Note that in this case $G(\partial D_0) = I_2 \cup I_3$.

We next consider the case $a \in H^+$. Let $\triangle_0$ be the triangle such that $0 \in \triangle_0$ and $G(\triangle_0) = H^+$. For each $j \in \{1, 2, 3\}$, we denote by $L_j$ the side of $\triangle_0$ such that $G(L_j) = I_j$, and let $\triangle_j$ be the hyperbolic triangle which is obtained by reflecting $\triangle_0$ with respect to $L_j$. Similarly, we denote by $\triangle_{j,k}$ the triangle obtained by reflecting $\triangle_j$ through the side over $I_k$, $k \neq j$. Finally, let $\zeta_{j,k}$ be the point in $\triangle_{j,k}$ such that $G(\zeta_{j,k}) = a$. See Figure 5.

![Figure 5: The case $a \in H^+$](image)

By Lemma 17 and the same argument as above, $\overline{\triangle}_0 \subset D_0$. Similarly, the closed triangle $\overline{\triangle}_{j,k}$ is contained in the Dirichlet domain with center at $\zeta_{j,k}$. Since $D_0$ is connected, we have

$$\overline{\triangle}_0 \subset D_0 \subset \bigcup_{j=0}^4 \triangle_j \cup \bigcup_{j=1}^3 L_j. \quad (4.2)$$

Let $A := \{\zeta_{j,k} : 1 \leq j, k \leq 3, j \neq k\}$. Then for all $w \in D_0$ and $\zeta \in \Gamma_0 \setminus (\{0\} \cup A)$, the shortest curve connecting these two points must pass through a point $w' \in \triangle_{j,k}$ for some $j$ and $k$, $j \neq k$. Therefore, denoting by $d_\mathbb{D}$ the hyperbolic distance in $\mathbb{D}$, we have

$$d_\mathbb{D}(w, \zeta) = d_\mathbb{D}(w, w') + d_\mathbb{D}(w', \zeta) > d_\mathbb{D}(w, w') + d_\mathbb{D}(w', \zeta_{j,k}) \geq d_\mathbb{D}(w, \zeta_{j,k}),$$
because $w' \in \triangle_{j,k}$ is contained in the Dirichlet domain with center at $\zeta_{j,k}$. This implies that

$$\mathcal{D}_0 = \bigcap_{\zeta \in A} \{ w : d_{\mathcal{D}}(w, 0) < d_{\mathcal{D}}(w, \zeta) \}. \quad (4.3)$$

We obtained $\triangle_{j,k}$ by reflecting $\triangle_0$ through a side over $I_j$ and then a side over $I_k$. The triangle $\triangle_0$ can be obtained from $\triangle_{k,j}$ by the same way. Therefore, there is $\tau \in \Gamma$ such that $\tau(0) = \zeta_{j,k}$ and $\tau(\zeta_{k,j}) = 0$. Then with the notation

$$\ell_{j,k} := \{ w : d_{\mathcal{D}}(w, 0) = d_{\mathcal{D}}(w, \zeta_{j,k}) \},$$

we have $\tau(\ell_{k,j}) = \ell_{j,k}$; i.e., $G(\ell_{j,k}) = G(\ell_{k,j})$. Also note that $\ell_{j,k}$ is a geodesic curve which separates the two sides of $\triangle_j$ lying over $I_j$ and $I_k$, because $\triangle_0$ and $\triangle_{j,k}$ are contained in the Dirichlet domains with centers at 0 and $\zeta_{j,k}$, respectively. Since each sides of $\triangle_j$ are also geodesic, we conclude that one end of $\ell_{j,k}$ approaches to the common vertex (at infinity) of $\triangle_0$ and $\triangle_{j,k}$, and $\ell_{j,k}$ intersects the side of $\triangle_j$ which is over $I_l$, $l \neq j, k$. In particular, this implies that $\ell_{j,k} \cap \ell_{j,k'} \neq \emptyset$ for $k \neq k'$.

Now one can easily see that $\mathcal{D}_0$ is a hexagon with exactly 3 vertices at infinity, which are in fact the vertices of $\triangle_0$. Moreover along $\partial \mathcal{D}_0$, finite and infinite vertices are placed alternatively and the $G$-images of the two sides sharing a common infinite vertex are same. Therefore, the three finite vertices are mapped to the same point, say $b$, and $G(\partial \mathcal{D}_0)$ is a tripod with center at $b$ such that each leg of it is a hyperbolic geodesic curve connecting $b$ to one of the points $0, 1, \infty$. Now we are ready to prove the following proposition.

**Proposition 18.** Suppose $E$ consists of three points. If the points in $E$ are on a straight line, $\tilde{E}$ is the line segment connecting points in $E$; i.e., $\tilde{E}$ is the convex hull of $E$. Otherwise $\tilde{E}$ is a tripod.

**Proof.** Let $g : \Lambda(r) \to \mathbb{D} \setminus E$ be a holomorphic universal map. We choose a linear transformation $T$ such that $T(E) = \{0, 1, \infty\}$ and $a := T(\infty) \in H^+ \cup I_1$. Since every linear transformation maps circles onto circles, we see that $a \in I_1$ if and only if $\infty$ is on the circle passing through the points in $E$; i.e., if and only if the points in $E$ are on a straight line. Now let $G(w) := T \circ g(r/w)$. Then $G$ is a holomorphic universal covering map from $\mathbb{D}$ to $\Omega_{0,1}$ such that $G(0) = a$. Now the proposition follows from Corollary 16 and the arguments preceding the proposition, since $T$ is a conformal map and the map $w \mapsto r/w$ sends the Dirichlet domain in $\mathbb{D}$ with center at the origin onto the Dirichlet domain in $\Lambda(r)$ with center at infinity. \hfill \qed
A Appendix: Proofs of Theorems 5 and 6

For a measure $\omega$ on $\mathbb{C}$, we denote its logarithmic potential by

$$P_\omega(z) := -\int_\mathbb{C} \log |z - \xi| d\omega(\xi)$$

and let $P(z) := \log \delta(z) = \limsup_{k \to \infty} k^{-1} \log |P_k(z)|$. Note that $\mu(z) = (2\pi)^{-1} \Delta P(z)$ by the definition of $P$ and (1.3).

**Lemma 19.** (a) $P$ is subharmonic on $\mathbb{C}$.

(b) $P$ is harmonic on $\tilde{C}_1 \cup (\tilde{C}_0)^c$, but not at points of $\partial \tilde{C}_1$.

(c) $\mu$ is a probability measure with support $\partial \tilde{C}_1$.

(d) $P_\mu(z) = -P(z)$ for all $z \in \mathbb{C}$.

**Proof.** Note that the limit superior of a sequence of subharmonic functions is subharmonic if it is upper semi-continuous. Since $P$ is upper semi-continuous by Corollary 13 and $k^{-1} \log |P_k|$ is subharmonic for all $k$, (a) follows.

By (2.3), $P$ is constant on $\tilde{C}_0$ hence harmonic on $(\tilde{C}_0)^c$. If $z \in \tilde{C}_1$, there exist a neighborhood $N$ of $z$ and an inverse branch of $g$, say $f$, defined on $N$ such that

$$|f(\xi)| = \max\{|\zeta| : \zeta \in g^{-1}(\xi)\}, \quad \text{for } \xi \in N.$$  

Therefore by (2.4), $P(\xi) = \log |f(\xi)|$ in $N$ hence harmonic since $|f(\xi)| > \delta_0 \geq 0$.

The function $P$ is not harmonic on $\partial \tilde{C}_0$ since Lemma 12 implies that $P(z) = \delta_0$ for all $z \in \tilde{C}_0$ but $P(z) > \delta_0$ if $z \notin \tilde{C}_0$. We next show that $P$ is not harmonic at a point $z \in \tilde{C}_p$, $p \geq 2$. If it is not the case, there exists a neighborhood $N$ of $z$ such that $P$ is harmonic on $N$. Then by Lemma 11 there exist a subdomain $N' \subset N$, two disjoint domains $D_1, D_2$ such that $D_1 \cup D_2 = N' \cap \tilde{C}_1$, and analytic functions $f_1$ and $f_2$ satisfying (2.1) and (2.2). This implies $P(\xi) - \log |f_2(\xi)|$ is a harmonic function which is positive on $D_1$ and zero on $D_2$, which is impossible. Therefore $P$ is not harmonic on $\bigcup_{p \geq 2} \tilde{C}_p$. Now (b) follows from Corollary 8.

Since $P$ is subharmonic by (a), $\mu = (2\pi)^{-1} \Delta P$ is a measure. Moreover by (b), the support of $\mu$ is $\partial \tilde{C}_1$. Therefore to show (c), it suffices to show that $\mu(\mathbb{C}) = 1$. Note that since $\mu$ has compact support and $P$ is harmonic off the set $\partial \tilde{C}_1$, the Riesz Decomposition Theorem (5, Theorem II.21) implies that

$$u(z) := P_\mu(z) + P(z)$$

(A.1)
is harmonic on any bounded domain $D$ containing $\partial \tilde{C}_1$. Because this is true for any arbitrary large domain $D$, $u$ is in fact harmonic on $\mathbb{C}$, hence constant.

Let $f$ be the inverse of $g$ defined on a neighborhood of $\infty$ such that $f(\infty) = \infty$. Then it is easy to see from (1.1) that $f(z) = z + O(1)$ as $z \to \infty$. Moreover by (2.3), $\delta(z) = |f(z)|$ for sufficiently large $z$. Thus $\mathcal{P}(z) = \log |f(z)| = \log |z| + o(1)$ as $z \to \infty$. But since $\mathcal{P}_\omega(z) = -\omega(\mathbb{C}) \log |z| + o(1)$ for any finite measure $\omega$ with compact support, we conclude from (A.1) that $\mu(\mathbb{C}) = 1$ and $u \equiv 0$, which shows (c) and (d) simultaneously. This completes the proof. \hfill \Box

Recall that $\mu_k(z) = (2\pi k)^{-1} \Delta(\log |P_k(z)|)$. Since $P_k$ is a monic polynomial of degree $k$, it can be shown that $\mathcal{P}_{\mu_k} = -k^{-1} \log |P_k|$ by the same argument for Lemma 19(d).

**Lemma 20.** Let $\omega$ be any weak-star limit of $\{\mu_k\}$. Then

$$
\mathcal{P}_\omega(z) = \mathcal{P}_{\mu}(z), \quad z \in \tilde{C}_1 \cup \partial \tilde{C}_1, \quad (A.2)
$$

$$
\mathcal{P}_\omega(z) \geq \mathcal{P}_{\mu}(z), \quad z \in \mathbb{C}. \quad (A.3)
$$

**Proof.** Suppose $\{\mu_k\}$ converges to $\omega$ in the weak-star topology. By (2.5) and Lemma 19(d),

$$
\lim_{j \to \infty} \mathcal{P}_{\mu_k}(z) = -\lim_{j \to \infty} k_j^{-1} \log |P_k(z)| = -\log \delta(z) = -\mathcal{P}(z) = \mathcal{P}_{\mu}(z)
$$

for all $z \in \tilde{C}_1$. Therefore the Lower Envelope Theorem ([5], Theorem 3.8) implies that $\mathcal{P}_{\mu}(z) = \mathcal{P}_\omega(z)$ for all $z \in \tilde{C}_1$ except on a set of logarithmic capacity zero. On the other hand, Theorem 7 implies that the support of $\mathcal{P}_\omega$ is contained in $\mathbb{C} \setminus \tilde{C}_1$, hence it is harmonic in $\tilde{C}_1$. Because $\mathcal{P}_{\mu} = -\mathcal{P}(z)$ is also harmonic in $\tilde{C}_1$ (Lemma 19(b)), we conclude that $\mathcal{P}_{\mu}(z) = \mathcal{P}_\omega(z)$ for all $z \in \tilde{C}_1$.

Corollary 13 implies that $\mathcal{P}_{\mu}(z) = -\log \delta(z)$ is continuous if $\delta(z) > 0$. If $\delta(z) = 0$ (this happens only when $z \in \tilde{C}_0$ and $\delta_0 = 0$), $\mathcal{P}_{\mu}(z) = \infty$. Therefore we have

$$
\mathcal{P}_\omega(z) \leq \mathcal{P}_{\mu}(z), \quad \text{for all } z \in \partial \tilde{C}_1, \quad (A.4)
$$

because $\mathcal{P}_\omega$ is lower semi-continuous and $\mathcal{P}_\omega = \mathcal{P}_{\mu}$ on $\tilde{C}_1$.

If $z \in \tilde{C}_1$ and $z$ approaches to $\partial \tilde{C}_0$, we have $\mathcal{P}_{\omega}(z) = \mathcal{P}_{\mu}(z) \to -\log \delta_0$. Therefore, the minimum principle implies that $\mathcal{P}_{\omega}(z) \geq -\log \delta_0 = \mathcal{P}_{\mu}(z)$ for all $z \in \tilde{C}_0$. Note that combining this result with (A.4), we also have $\mathcal{P}_{\omega}(z) = \mathcal{P}_{\mu}(z)$ for all $z \in \partial \tilde{C}_0$. 14
Now it remains to show that $P_\omega(z) \geq P_\mu(z)$ for all $z \in \tilde{C}_p$, $p \geq 2$. But for sufficiently small $\epsilon$, Lemma 4 implies that the circle $\{ \xi : |z - \xi| = \epsilon \}$ is contained in $\tilde{C}_1$ except finitely many points. Since $P_\omega$ is superharmonic and $P_\omega(\xi) = P_\mu(\xi)$ for $\xi \in \tilde{C}_1$,

$$P_\omega(z) \geq \frac{1}{2\pi \epsilon} \int_{|z - \xi| = \epsilon} P_\omega(\xi)|d\xi| = \frac{1}{2\pi \epsilon} \int_{|z - \xi| = \epsilon} P_\mu(\xi)|d\xi|.$$ 

By letting $\epsilon \to 0$, we get $P_\omega(z) \geq P_\mu(z)$ since $P_\mu(\xi)$ is continuous at $z \in \tilde{C}_p$, $p \geq 2$. This completes the proof.

**Lemma 21.** Suppose $(\tilde{C}_0)^\circ \neq \emptyset$ and $U$ is a component of $(\tilde{C}_0)^\circ$. Then there exists a subsequence of $\{\mu_k\}$ that converges in the weak-star topology to a measure $\omega$ such that $P_\omega(z) = P_\mu(z)$ for all $z \in \overline{U}$.

**Proof.** Pick a point $z_0 \in U$. By (1.6) there is a subsequence $\{\mu_{kj}\}$ such that

$$\lim_{j \to \infty} P_{\mu_{kj}}(z_0) = -\lim_{j \to \infty} k_j^{-1}\log |P_{k_j}(z_0)| = -\log \delta_0.$$

Now let $\omega$ be a weak-star limit of a subsequence of $\{\mu_{kj}\}$. Then by the Principle of Descent ([5], Theorem 1.3), we have $P_\omega(z_0) \leq -\log \delta_0$. Since (A.2) says that $P_\omega(z) = -\log \delta_0$ for all $z \in \partial U$, the minimum principle implies that $P_\omega = -\log \delta_0 = P_\mu$ in $U$.

**Proofs of Theorems 5 and 6.** Suppose $(\tilde{C}_0)^\circ$ is empty and let $\omega$ be a weak-star limit of any convergent subsequence of $\{\mu_k\}$. Then by (A.2), we have $P_\omega(z) = P_\mu(z)$ for all $z \in \mathbb{C}$. Therefore $\omega = -(2\pi)^{-1}\Delta P_\omega = -(2\pi)^{-1}\Delta P_\mu = \mu$ and the first statement of Theorem 5 follows from Lemma 19. If, in addition, $\mathbb{C} = \tilde{C}_0 \cup \tilde{C}_1$, then $\mu$ is the equilibrium distribution of $\tilde{C}_0$ since the support of $\mu$ is $\partial \tilde{C}_1 = \partial \tilde{C}_0$ and $P_\mu$ is constant on $\tilde{C}_0$ ([7], Theorem III.15). This completes the proof of Theorem 5.

Now suppose $(\tilde{C}_0)^\circ$ is connected. Then by Lemma 21 there exists a subsequence $\{\mu_{kj}\}$ of $\{\mu_k\}$ converging to a measure $\omega$ such that $P_\omega = P_\mu$ in $\tilde{C}_0$. Since $P_\omega(z) = P_\mu(z)$ for all $z \in \mathbb{C}\setminus \tilde{C}_0$ by (A.2), this shows that $P_\omega = P_\mu$ in $\mathbb{C}$ hence $\omega = \mu$. Now Theorem 6 follows from Lemma 19.

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