ON A NEW INVERSE SPECTRAL PROBLEM

A.M. AKHTYAMOV

1. Introduction. Inverse problems for the linear ordinary differential operators containing parameter has been studied by many authors ([1–7,10]). Operators pencils also intensively studied ([8–9]). But this two directions has been developed independently. The differential operators pencil recovery uniqueness theorem will be proved in our article. Novelty of this result is not equation coefficient recovery, but boundary conditions coefficients recovery. We will show that all conditions of the theorem are essential. We will cite also a few interesting examples.

2. The main result.

Let us consider the two following boundary–value problems

\[(1) \quad l(y, \lambda) = y'' + b\lambda y' + c\lambda^2 y = 0,\]
\[(2) \quad U_1(y) = a_{11} y(0) + a_{12} y(1) + a_{13} y'(0) + a_{14} y'(1) = 0,\]
\[(3) \quad U_2(y) = a_{21} y(0) + a_{22} y(1) + a_{23} y'(0) + a_{24} y'(1) = 0,\]

\[(1) \quad \tilde{l}(y, \lambda) = y'' + b\lambda y' + c\lambda^2 y = 0,\]
\[(4) \quad \tilde{U}_1(y) = \tilde{a}_{11} y(0) + \tilde{a}_{12} y(1) + \tilde{a}_{13} y'(0) + \tilde{a}_{14} y'(1) = 0,\]
\[(5) \quad \tilde{U}_2(y) = \tilde{a}_{21} y(0) + \tilde{a}_{22} y(1) + \tilde{a}_{23} y'(0) + \tilde{a}_{24} y'(1) = 0,\]

Here
\( \lambda \) is spectral parameter,
coefficients \( b, c, a_{ij}, \tilde{a}_{ij} \) are complex numbers and its independent
on parameter \( \lambda \),
\[
\text{rank}(a_{ij})_{2 \times 4} = \text{rank}(\tilde{a}_{ij})_{2 \times 4} = 2,
\]
\( x \in [0, 1] \), \( y = y(x) \in C^2[0, 1] \).

**Theorem.** If all nonzero eigenvalues of the boundary–value problems
(1), (2), (3) and (1), (4), (5) coincide, their multiplicities coincide, and, in addition, the following conditions are realized:

1. \( b^2 - 4c \neq 0 \),
2. \( b \neq 0 \),
3. \( c \neq 0 \),

then spectral problems themselves coincide, that is, the linear formes \( \tilde{U}_1(y), \tilde{U}_2(y) \) are linear expressed with the help of the linear formes \( U_1(y), U_2(y) \).

**Proof.**

Let:
- \( g(\lambda) \) be an arbitrary entire function,
- \( l \) be an arbitrary integer number,
- \( y_1(x, \lambda) = e^{\omega_1 \lambda x}, \quad y_2(x, \lambda) = e^{\omega_2 \lambda x} \) be a fundamental solves system
of the equation (1),
- \( \omega_1, \omega_2 \) be roots of the characteristic equation \( \omega^2 + b\omega + c = 0 \)
(according to condition of the theorem \( \omega_1 \neq \omega_2 \))
- \( \Delta(\lambda) \) be characteristic equation (1), (4), (5).

Under the condition \( \lambda \neq 0 \) the characteristic determinant of the following boundary–value problem

1. \( l(y, \lambda) = 0 \),
2. \( \tilde{U}_1(y) = 0 \),
3. \( \chi^l e^{g(\lambda)}(\tilde{U}_2(y)) = 0 \),
is the following function

$$\overline{\Delta}_1(\lambda) \equiv \begin{vmatrix} \tilde{U}_1(y_1(x, \lambda)) & U_1(y_2(x, \lambda)) \\ \lambda' e^{g(\lambda)} \tilde{U}_2(y_1(x, \lambda)) & \lambda' e^{g(\lambda)} \tilde{U}_2(y_2(x, \lambda)) \end{vmatrix}$$

(7) $$\equiv \lambda' e^{g(\lambda)} \overline{\Delta}(\lambda).$$

It follows from (7) that nonzero eigenvalues of spectral problems (1), (4), (5) and (1), (4), (6) coincide. Then according to the condition of the theorem, the nonzero eigenvalues of the spectral problems (1), (2), (3) and (1), (4), (6) also coincide.

Nonzero eigenvalues of the problem (1), (2), (3) are the roots of the following entire function

$$\Delta(\lambda) \equiv \begin{vmatrix} U_1(y_1(x, \lambda)) & U_1(y_2(x, \lambda)) \\ U_2(y_1(x, \lambda)) & U_2(y_2(x, \lambda)) \end{vmatrix}$$

([6, p.27]).

It follows from Weierstrass theorem about an entire function representation by its roots that

(8) $$\overline{\Delta}_1(\lambda) \equiv \lambda^k e^{f(\lambda)} \Delta(\lambda),$$

where $f(\lambda)$ is a certain entire function, and $k$ is a certain integer number. We assume, that $g(\lambda) \equiv f(\lambda)$, $l = k$ at the equation (6).

Then it follows from (7) and (8) that

$$\Delta(\lambda) \equiv \overline{\Delta}(\lambda).$$

That is

$$\begin{vmatrix} U_1(y_1(x, \lambda)) & U_1(y_2(x, \lambda)) \\ U_2(y_1(x, \lambda)) & U_2(y_2(x, \lambda)) \end{vmatrix} = \begin{vmatrix} \tilde{U}_1(y_1(x, \lambda)) & \tilde{U}_1(y_2(x, \lambda)) \\ \tilde{U}_2(y_1(x, \lambda)) & \tilde{U}_2(y_2(x, \lambda)) \end{vmatrix}.$$ 

From here

$$a_{11}a_{22} - a_{21}a_{12} - \bar{a}_{11}\bar{a}_{22} + \bar{a}_{21}\bar{a}_{12}(y_1(0)y_2(1) - y_1(1)y_2(0))$$
\[(a_{12}a_{23} - a_{13}a_{21} - \tilde{a}_{11}\tilde{a}_{23} + \tilde{a}_{13}\tilde{a}_{21})(y_1(0)y_2'(0) - y'_1(0)y_2(0))\]
\[+ (a_{11}a_{24} - a_{21}a_{14} - \tilde{a}_{11}\tilde{a}_{24} + \tilde{a}_{21}\tilde{a}_{14})(y_1(0)y_2'(1) - y'_1(1)y_2(0))\]
\[+ (a_{23}a_{12} - a_{13}a_{22} - \tilde{a}_{23}\tilde{a}_{12} + \tilde{a}_{13}\tilde{a}_{22})(y_1(1)y_2'(0) - y'_1(0)y_2(1))\]
\[+ (a_{12}a_{24} - a_{22}a_{14} - \tilde{a}_{12}\tilde{a}_{24} + \tilde{a}_{22}\tilde{a}_{14})(y_1(1)y_2'(1) - y'_1(1)y_2(1))\]
\[+ (a_{13}a_{24} - a_{23}a_{14} - \tilde{a}_{13}\tilde{a}_{24} + \tilde{a}_{23}\tilde{a}_{14})(y_1(0)y_2'(0) - y'_1(0)y_2(0)) = 0.\]

If we substitute the functions
\[y_1(x, \lambda) = e^{\omega_1 \lambda x}, \quad y_2(x, \lambda) = e^{\omega_2 \lambda x}\]
for the preceding identity, we obtain
\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{11} & \tilde{a}_{12} \\
  \tilde{a}_{21} & \tilde{a}_{22}
\end{pmatrix}
(e^{\omega_2 \lambda} - e^{\omega_1 \lambda})
\]
\[+ \begin{pmatrix}
  a_{11} & a_{13} \\
  a_{21} & a_{23}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{11} & \tilde{a}_{13} \\
  \tilde{a}_{21} & \tilde{a}_{23}
\end{pmatrix}
(\omega_2 \lambda - \omega_1 \lambda)
\]
\[+ \begin{pmatrix}
  a_{11} & a_{14} \\
  a_{21} & a_{24}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{11} & \tilde{a}_{14} \\
  \tilde{a}_{21} & \tilde{a}_{24}
\end{pmatrix}
(\omega_2 \lambda e^{\omega_2 \lambda} - \omega_1 \lambda e^{\omega_1 \lambda})
\]
\[+ \begin{pmatrix}
  a_{12} & a_{13} \\
  a_{22} & a_{23}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{12} & \tilde{a}_{13} \\
  \tilde{a}_{22} & \tilde{a}_{23}
\end{pmatrix}
(\omega_2 \lambda e^{\omega_1 \lambda} - \omega_1 \lambda e^{\omega_2 \lambda})
\]
\[+ \begin{pmatrix}
  a_{12} & a_{14} \\
  a_{22} & a_{24}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{12} & \tilde{a}_{14} \\
  \tilde{a}_{22} & \tilde{a}_{24}
\end{pmatrix}
(\omega_2 \lambda e^{(\omega_1 + \omega_2)\lambda} - \omega_1 \lambda e^{(\omega_1 + \omega_2)\lambda})
\]
\[+ \begin{pmatrix}
  a_{13} & a_{14} \\
  a_{23} & a_{24}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{13} & \tilde{a}_{14} \\
  \tilde{a}_{23} & \tilde{a}_{24}
\end{pmatrix}
\omega_1 \omega_2 \lambda^2 (e^{\omega_2 \lambda} - e^{\omega_1 \lambda}) = 0.
\]

By the hypotheses of the Theorem, we have
\[\omega_1 + \omega_2 \neq 0, \quad \omega_1 - \omega_2 \neq 0, \quad \omega_1 \neq 0, \quad \omega_2 \neq 0.\]

Therefore functions
\[e^{\omega_2 \lambda} - e^{\omega_1 \lambda}, \quad \omega_2 \lambda - \omega_1 \lambda,
\]
\[\omega_2 \lambda e^{\omega_2 \lambda} - \omega_1 \lambda e^{\omega_1 \lambda}, \quad (\omega_2 \lambda e^{\omega_1 \lambda} - \omega_1 \lambda e^{\omega_2 \lambda}),
\]
\[\omega_2 \lambda e^{(\omega_1 + \omega_2)\lambda} - \omega_1 \lambda e^{(\omega_1 + \omega_2)\lambda}, \quad \omega_1 \omega_2 \lambda^2 (e^{\omega_2 \lambda} - e^{\omega_1 \lambda})
\]
are linear independent functions with respect to argument $\lambda$. (It is easily verified by the definition of functions linear independence.) Consequently

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix}$,  
$\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{23} \end{vmatrix}$,

$\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} = \begin{vmatrix} \tilde{a}_{12} & \tilde{a}_{14} \\ \tilde{a}_{22} & \tilde{a}_{24} \end{vmatrix}$,

$\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} = \begin{vmatrix} \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{23} & \tilde{a}_{24} \end{vmatrix}$.

It follows from preceding equations that all third order minors of the matrix

$$A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{23} & a_{24} & a_{23} & a_{24} \\
  \tilde{a}_{13} & \tilde{a}_{14} & \tilde{a}_{13} & \tilde{a}_{14} \\
  \tilde{a}_{23} & \tilde{a}_{24} & \tilde{a}_{23} & \tilde{a}_{24}
\end{pmatrix}$$

is equal to zero. That is why rank of the matrix $A$ is equal to two. The last means that the forms $\tilde{U}_1(y), \tilde{U}_2(y)$ are linear expressed by forms $U_1(y), U_2(y)$. As was to be proved.

**Remark.** One need two, three or more spectrums of the especially choosed problems for the differential operators recovery uniqueness. ([2–6]). But according to our theorem one needs only one spectrum for the operators pencil recovery uniqueness. It shows that "pencil case" differs from "nonpencil case".

3. **On essence of each condition of the Theorem.**

We will show now that all conditions of the Theorem are essential. We cite three examples, showing the importance of each condition of the theorem.

**Example 1** (First condition of the Theorem is missed: $b^2 - 4c = 0$, $b \neq 0$, $c \neq 0$).
The spectral problems

\begin{align*}
y'' - 2\lambda y' + \lambda^2 y &= 0, & y'' - 2\lambda y' + \lambda^2 y &= 0, \\
y(0) &= 0, & y(0) &= 0, \\
y'(1) &= 0, & y(1) + 2y'(1) &= 0
\end{align*}

have the same set of the eigenvalues. This set consists of only one eigenvalue $\lambda = -1$. However forms of boundary conditions of the first spectral problem are not linear expressed by the boundary conditions forms of the second spectral problem.

**Example 2** (Second condition of the Theorem is missed: $b^2 - 4c \neq 0$, $b = 0$, $c \neq 0$).

The spectral problems

\begin{align*}
y'' - \lambda^2 y &= 0, & y'' - \lambda^2 y &= 0, \\
y(0) + 2y'(0) &= 0, & y(0) &= 0, \\
y(1) &= 0, & y(1) - 2y'(1) &= 0
\end{align*}

have the same set of the eigenvalues, which is the same as roots of characteristic determinant $(1 + 2\lambda)e^{-\lambda} + (-1 + 2\lambda)e^\lambda$. However forms of boundary conditions of the first spectral problem are not linear expressed by the boundary conditions forms of the second spectral problem.

**Example 3** (Third condition of the Theorem is missed: $b^2 - 4c \neq 0$, $b \neq 0$, $c = 0$).

The both spectral problems

\begin{align*}
y'' - \lambda^2 y &= 0, & y'' - \lambda^2 y &= 0, \\
y(0) + 2y'(0) &= 0, & y(0) &= 0, \\
y(1) &= 0, & y(1) - 2y'(1) &= 0
\end{align*}

both have not eigenvalues. Therefore the sets of the eigenvalues of the problems are coincided. However forms of boundary conditions of the first spectral problem are not linear expressed by the boundary conditions forms of the second spectral problem.
4. About some generalisations of the Theorem.

If coefficients of the boundary conditions depend on parameter $\lambda$, then conclusion of the Theorem generally speaking is not true. We cite an example, confirming this preposition.

**Example 4** (Coefficients of boundary conditions depend on parameter $\lambda$).

The spectral problems
\[
\begin{align*}
  y'' - 3\lambda y' + \lambda^2 y &= 0, &  y'' - 3\lambda y' + \lambda^2 y &= 0, \\
  \lambda y(0) + y'(0) &= 0, &  \lambda y(0) + 2y'(0) &= 0, \\
  4\lambda y(1) + y'(1) &= 0, &  2\lambda y(1) + y'(1) &= 0
\end{align*}
\]
have the same eigenvalues, which is the same as the roots of the characteristic determinant $\lambda^2(12e^{2\lambda} - 15e^\lambda)$. However forms of boundary conditions of the first spectral problem are not linear expressed by the boundary conditions forms of the second spectral problem. If coefficients of boundary conditions depend on parameter $\lambda$, then they can not be univalent recovered by the spectrum. It means that a generalisation of the Theorem on this way is not possible.

One can not univalent recover both coefficients of a equation and coefficients of all boundary conditions. We cite a corresponding example.

**Example 5** (Coefficients of equations are not coincided).

The spectral problems
\[
\begin{align*}
  y'' + 2\lambda y' + \lambda^2 y &= 0, &  y'' + 4\lambda y' + 5\lambda^2 y &= 0, \\
  y(0) &= 0, &  y(0) &= 0, \\
  y(\pi) &= 0, &  y(\pi) &= 0
\end{align*}
\]
have the same eigenvalues $\{\lambda_n\} = \pi n, \ n \in \mathbb{Z}$. Boundary conditions of the first spectral problem coincide with the boundary conditions of the second spectral problem. But coefficients of the both equations are different. Thus one can not univalent recover both the coefficients of the equation and the coefficients of the boundary conditions.
References

1. Borg G., Eine Umkehrung der Sturm–Liouvillschen Eigenwertaufgabe, Acta Math. 78, no.1 (1946), 1–96.
2. Denisov A.M., An introduction to the theory of the inverse problems [in Russian], Izd. MGU, Moscow, 1994.
3. Levinson N., The inverse Sturm–Liouville problem, Math. Tidsskr. Ser.B 13 (1949), 25–30.
4. Levitan B.M., The inverse Sturm–Liouville problems and applications [in Russian], Tr. Mosk. Matem. O-va 15 (1966), Moscow, 70–144.
5. Naymark M. A., Linear differential operators, Nauka, Moscow, 1968.
6. Sadovnichiy V.A., The inverse problem solve unique for the second order equation with the irreducible boundary conditions, the regularization sums of the part of the eigenvalues. Factorization of the characteristic determinant. [in Russian], Dokl. Akad. Nauk SSSR 206, no.2 (1972), 293–296.
7. Shkalikov A.A., Boundary value problems for ordinary differential equations with eigenparameter in the boundary condition. [in Russian], Tr. Sem. im. I.G. Petrovskogo 9 (1983), Izd. MGU, Moscow, 190–229.
8. Shkalikov A.A., Boundary value problems for ordinary differential equations with eigenparameter in the boundary condition. [in Russian], Tr. Sem. im. I.G. Petrovskogo 9 (1983), Izd. MGU, Moscow, 190–229.
9. Hinton D.V., An expansion theorem, Comm. Pure Appl. Math. 26 (1973), 715–729.
10. Hochstadt H., The inverse Sturm–Liouville problem, Comm. Pure Appl. Math. 26 (1973), 715–729.