Stochastic Variational Principles for Dissipative Equations with Ad vected Quantities

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Abstract
This paper presents symmetry reduction for material stochastic Lagrangian systems with advected quantities whose configuration space is a Lie group. Such variational principles yield deterministic as well as stochastic constrained variational principles for dissipative equations of motion in spatial representation. The general theory is presented for the finite-dimensional situation. In infinite dimensions we obtain partial differential equations and stochastic partial differential equations. When the Lie group is, for example, a diffeomorphism group, the general result is not directly applicable but the setup and method suggest rigorous proofs valid in infinite dimensions which lead to similar results. We apply this technique to the compressible Navier–Stokes equation and to magnetohydrodynamics for charged viscous compressible fluids. A stochastic Kelvin–Noether theorem is presented. We derive, among others, the classical

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deterministic dissipative equations from purely variational and stochastic principles, without any appeal to thermodynamics.

1 Introduction

The goal of this paper is to develop a Lagrangian symmetry reduction process for a large class of stochastic systems with advected parameters. The general theory, which yields both deterministic and stochastic constrained variational principles and deterministic, as well as stochastic reduced equations of motion, is developed for finite-dimensional systems. The resulting abstract equations then serve as a template for the study of infinite-dimensional stochastic systems, for which the rigorous analysis has to be carried out separately. The examples of the compressible Navier–Stokes equations and dissipative compressible magnetohydrodynamics equations, as well as their randomly perturbed counterparts are treated in detail. We recover with our method the classical dissipative fluid and magnetohydrodynamic equations without any appeal to thermodynamical considerations, except for the form of the internal energy density.

The stochastic version of Euler–Poincaré reduction introduced in Arnaudon et al. (2014) is close in spirit to Feynman’s viewpoint and, particularly, to the approach initiated in the eighties by Zambrini [c.f. Zambrini (2015) and references therein as well as Yasue (1981) and Nakagomi et al. (1981)]. It uses, as we do here, a main tool: the notion of generalized (or mean-value) derivative, that removes the contribution of the martingale part of the stochastic Lagrangian paths. This derivative was introduced in stochastic dynamics by Nelson (1967). We also refer to Arnaudon and Cruzeiro (2012), Arnaudon et al. (2014), Cipriano and Cruzeiro (2007), Koide and Kodama (2012) and references therein for various extensions on infinite-dimensional spaces and applications of this derivative in stochastic Euler–Poincaré reduction.

The crucial idea is that the generalized derivative contains a contraction term induced by noise (stochastic force) which gives rise to a second-order operator (such as the Laplacian) in the velocity equation of the stochastic model in continuum mechanics. In our approach, the noise, modeled by Brownian motion, is intrinsic to the behavior of Lagrangian paths and describes dissipation. In the theory of stochastic differential equations, a “transfer principle” (see Malliavin 1997) provides a way to transfer results of ordinary differential equations and first-order vector fields to stochastic differential equations and their corresponding second-order generators. Similarly, our methodology allows to proceed from group invariant variational principles in Geometric Mechanics to stochastic variational principles.

Then, the stochastic reduction procedure leads to characterizations of various partial differential equations whose viscous term only appears in relation with the Laplacian, such as the incompressible Navier–Stokes or the viscous Camassa–Holm equations. This stochastic Euler–Poincaré reduction is formulated on the group of volume preserving diffeomorphisms and the Lagrangian variables correspond to semimartingales.

This fundamental approach is quite different from modeling random perturbations caused by uncertainty, which can be introduced at various levels (cf. Holm (2019) for a discussion of this subject). One possible way, dating back to work in Bismut (1981) or Lázaro-Camí and Ortega (2008), is introducing noise in the action functional.
Moreover, in Cruzeiro et al. (2018), Holm (2015, 2019), based on these random action functionals, several stochastic variational principles have been introduced by taking the differentiation with respect to Lie derivatives along stochastic Lagrangian paths, which can be applied to characterize some stochastic partial differential equations. In this paper, we have separated the stochastic variational principles for deterministic equations and for stochastic ones. For deterministic equations, only the intrinsic noise plays a role. For stochastic equations, additional stochastic terms are considered in the action. As explained below, we have introduced the velocity derivative, the martingale part, as well as the contraction matrix, for a given stochastic Lagrangian path, which covers more information for a stochastic Lagrangian path. Based on these terms, we give new stochastic variational principles (see, e.g., Theorem 3.5) which can be applied to characterize the stochastic partial differential equation with more general viscous terms, including the stochastic compressible Navier–Stokes equation.

The theory of reduction of variational principles of mechanical systems with advected parameters, leading to Euler–Poincaré equations coupled with advection equations, and hence associated to semidirect products, has been developed in Holm et al. (1998). For continuum mechanical models, this method is particularly useful to characterize several kinds of evolutionary partial differential equations arising in conservative compressible fluids, such as the compressible Euler and ideal MHD equations (see, e.g., Holm et al. 1998, Section 7). Therefore, a first natural question arises whether it is possible to find a stochastic Euler–Poincaré reduction method that would characterize equations with viscous terms in compressible fluids, such as the compressible Navier–Stokes equation or the viscous compressible MHD equation. The main difficulty is that the generalized derivative, alluded to above, is not capable by itself to generate these viscosity terms since they do not appear only in connection with the generators of the underlying stochastic Lagrangian paths as in the case of incompressible fluids. The second natural question, amplifying the first one, is whether one can formulate a stochastic reduction procedure that would lead to interesting stochastic partial differential equations, appropriate for applications to continuum mechanics.

It is well known (Holm et al. 1998) that the Euler–Poincaré formulation naturally leads to Kelvin circulation theorems. The classical Kelvin Circulation Theorem for barotropic ideal fluids states that the circulation of the velocity around a closed loop moving with the fluid is constant in time. This statement is intimately connected to Poisson geometric properties of Euler’s ideal fluid equations (it characterizes the symplectic leaves in the phase space of Euler’s equations; see Marsden and Weinstein (1983)) and has important applications, for example, in the Lyapunov stability analysis of stationary solutions (see, e.g., Arnold 1965, 1969; Holm et al. 1983, 2002). For more general fluids, this theorem fails; instead of the vanishing of the time derivative of the circulation around a closed loop moving with the fluid, there is an explicit right hand side, responsible for generating circulation, involving advected quantities and the potential energy of the material. These identities are also known under the same name. For a general abstract formulation and a large class of examples of such Kelvin Circulation Theorems, see Holm et al. (1998, 2002). In addition, these Kelvin Circulation identities are equivalent to reformulations of the equations of motion that turn out to be convenient for the qualitative study of the fluid. It is natural hence to seek for a counterpart of such Kelvin–Noether identities appearing in stochastic Euler–
Poincaré reduction. Generally speaking, at least three types of possible stochastic perturbations can be considered: only in the Lagrangian (as in Arnaudon et al. (2014) and in some theorems of this paper), only in the Lagrangian velocity of the material loop around which the integral is taken (as in Holm (2015) and Cotter et al. (2020c)), or both (as in Holm (2019) and in some theorems of this paper). The decision where the stochastic perturbations are introduced depends on the nature of the problem to be modeled. As will be shown in the main theorems of this paper, different types of stochastic perturbations lead to deterministic or stochastic ordinary and partial differential equations.

There is a lot of freedom in the choice of the stochastic parameters used in the perturbations, namely of the diffusion coefficients in the Lagrangian paths. A future direction of research is to find a procedure that determines these parameters from observable data. The first steps in this direction have been taken in Holm (2015), Cotter et al. (2019, 2020a, b).

Summary of the main results. The main purpose in the paper is to solve the questions mentioned above. We summarize the main achievements of the paper.

1. We derive Euler–Poincaré equations for stochastic processes defined on semidirect product Lie algebras and give the associated deterministic constrained variational principle. In other words, we develop the semidirect Euler–Poincaré reduction for a large class of stochastic systems (see Theorems 3.5 and 3.12 below).

2. We study random action functionals, by introducing an additional stochastic force and defining the critical point for these new action functionals (see (3.3) and (3.9) below). Various stochastic partial differential equations, such as stochastic (both compressible and incompressible) Navier–Stokes or Euler equations and stochastic viscous MHD equations, are deduced from our stochastic reduction procedure.

3. We introduce a contraction matrix for the stochastic Lagrangian paths (see Sect. 2.2 below), which gives rise to a contraction force term in the action functional, capable to access separately, via reduction, each viscosity term, introduced usually by physical considerations, in the continuum mechanical model. In particular, we deduce the compressible Navier–Stokes and the viscous compressible MHD equations (Sect. 5) only from our stochastic variational principle, without any appeal to thermodynamics.

4. We prove a stochastic version of the Kelvin–Noether Circulation Theorem for our stochastic reduction procedure (see Theorem 4.1 below). Compared with the result in Holm et al. (1998), our (stochastic) evolution equations also depend on some martingales and some viscosity terms, in addition to the usual advected quantities (see Sect. 4).

Some comments on the new ideas are introduced in this paper.

1. As discussed earlier, the generalized derivative only produces a trace term on the contraction part of the associated stochastic Lagrangian path. In order to obtain different viscosity terms in the models of continuum mechanics, we have to investigate in more detail the effect of the contraction induced by the martingale term. To do this, we introduce a contraction matrix, which carries much more
information, involving each entry in the matrix, and not just their sum (as is the 
case for the generalized derivative).

2. Partially inspired by Cruzeiro et al. (2018) and Holm (2015), we also consider 
random perturbations of the action functionals so that the corresponding critical 
points satisfy a stochastic differential equation (a stochastic partial differential 
equation in the infinite-dimensional case). Therefore, our action functionals have 
integrands that consist of three parts: the Lagrangian, a contraction force, and 
a stochastic force, which model the Lagrangian structure, the viscosity, and the 
stochastic (martingale) nature of the action.

**Deterministic background.** The dynamics of many conservative physical systems 
can be described geometrically taking advantage of the intrinsic symmetries in their 
material description. These symmetries induce Noether conserved quantities and allow 
for the elimination of unknowns, producing an equivalent system consisting of new 
equations of motion in spaces with less variables and a non-autonomous ordinary 
differential equation, called the reconstruction equation. This geometric procedure is 
known as reduction, a method that is ubiquitous in symplectic, Poisson, and Dirac 
geometry and has wide applications in theoretical physics, quantum and continuum 
mechanics, control theory, and various branches of engineering. For example, in con-
tinuum mechanics, the passage from the material (Lagrangian) to the spatial (Eulerian) 
or convective (body) description is a reduction procedure. Of course, depending on the 
problem, one of the three representations may be preferable. However, it is often the 
case that insight from the other two representations, although apparently more intri-
cate, leads to a deeper understanding of the physical phenomenon under consideration 
and is useful in the description of the dynamics.

A simple example in which the three descriptions are useful and serve different 
purposes is free rigid body dynamics (e.g., Marsden and Ratiu 1999, Section 15). If one 
is interested in the motion of the attitude matrix, the material picture is appropriate. The 
classical free rigid body dynamics result, obtained by applying Hamilton’s standard 
variational principle on the tangent bundle of the proper rotation group $SO(3)$, states 
that the attitude matrix describes a geodesic of a left invariant Riemannian metric on 
$SO(3)$, characterized by the mass distribution of the body. However, as shown already 
by Euler, the equations of motion simplify considerably in the convective (or body) 
picture because the total energy of the rotating body, which in this case is just kinetic 
ergy, is invariant under left translations. The convective description takes place on 
the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ and is given by the classical Euler equations for a free 
rigid body, after implementing the Lie algebra isomorphism of $\mathbb{R}^3$ with $\mathfrak{so}(3)$ given by 
the cross product operation. Finally, the spatial description comes into play, because 
the spatial angular momentum is conserved during the motion and is hence used in 
the description of the rigid body motion.

The present paper uses exclusively Lagrangian mechanics, where variational prin-
ciples play a fundamental role since they produce the equations of motion. In continuum 
mechanics, the variational principle used in the material description is the standard 
Hamilton principle producing curves in the configuration space of the problem; these 
curves are critical points of the action functional. However, in the spatial and con-
vective representations, if the configuration space of the problem is a Lie group, the
induced variational principle requires the use of constrained variations, a fundamental result of Poincaré (1901); the resulting equations of motion are called today the \textit{Euler–Poincaré equations} (Marsden and Scheurle 1993; Marsden and Ratiu 1999, Section 13.5; Cendra et al. 2003). These equations have been vastly extended to include the motion of advected quantities (Cendra et al. 1998; Holm et al. 1998, 2002) as well as affine (Gay-Balmaz and Ratiu 2008) and non-commutative versions thereof that naturally appear in models of complex materials with internal structure (Gay-Balmaz and Ratiu 2009, 2011a) and whose geometric description has led to the solution of a long-standing controversy in the nematodynamics of liquid crystals (Gay-Balmaz et al. 2012, 2013). Euler–Poincaré equations have also very important generalizations to problems whose configuration space is an arbitrary manifold and the Lagrangian is invariant under a Lie group action (Cendra et al. 2001) as well as its extension to higher-order Lagrangians (Gay-Balmaz et al. 2011, 2012a, b). Lagrange–Poincaré equations turn out to model the motion of spin systems (Gay-Balmaz et al. 2009), long molecules (Ellis et al. 2010; Gay-Balmaz et al. 2012), free boundary fluids and elastic bodies (Gay-Balmaz et al. 2012), as well as charged and Yang–Mills fluids (Gay-Balmaz and Ratiu 2011a). There are also Lagrange–Poincaré theorems for field theory (Castrillón López et al. 2001; Castrillón López and Ratiu 2003; Castrillón López et al. 2000; Gay-Balmaz and Ratiu 2010; Ellis et al. 2011) and non-holonomic systems (Cendra et al. 2001). Lagrange–Poincaré equations also have interesting applications to Riemannian cubics and splines (Noakes and Ratiu 2016), the representation of images (Bruveris et al. 2011), certain classes of textures in condensed matter (Gay-Balmaz et al. 2015), and some control (Gay-Balmaz and Ratiu 2011b) and optimization (Gay-Balmaz et al. 2013) problems.

Variational principles play an important role in the design of structure preserving numerical algorithms. One discretizes both spatially and temporally such that the symmetry structure of the problem is preserved. Integrators based on a discrete version of Hamilton’s principle are called variational integrators (Marsden and West 2001). The resulting equations of motion are the discrete Euler–Lagrange equations and the associated algorithm for classical conservative systems is both symplectic as well as momentum-preserving and manifests very good long time energy behavior; see Lew et al. (2004a, b) for additional information. There are versions of such variational integrators for certain forced (Kane et al. 2000), controlled (Ober-Blöbaum et al. 2011), constrained holonomic (Leyendecker et al. 2008, 2010), non-holonomic (Kobilarov et al. 2010), non-smooth (Fetecau et al. 2003; Demoures et al. 2016), multiscale (Leyendecker and Ober-Blöbaum 2011; Tao et al. 2010), and stochastic (Bou-Rabee and Owhadi 2008) systems. Numerical modeling of stochastic Lie transport in hydrodynamics was carried out in Cotter et al. (2019). In the presence of symmetry, these systems can be reduced. However, today a general theory of discrete reduction in all of these cases is still missing and is currently being developed. If the configuration space is a Lie group, the first discretization of symmetric Lagrangian systems appears in Moser and Veselov (1991), motivated by problems in complete integrability; for an in-depth analysis of such problems, see Suris (2003).

There are various stochastic analogues for the above mentioned systems, depending on what phenomenon is modeled. The main directions, starting with variational principles, are either motivated by Feynman’s path integral approach to quantum mechanics...
or by stochastic optimal control. We already discussed the first one. The latter has its origins in the foundational work of Bismut (Bismut 1981, 1982) in the late seventies and in recent developments by Lázaro–Camí and Ortega (Lázaro-Camí and Ortega 2008, 2009a, b, c). Non-holonomic systems have been studied in the same spirit (Hochgerner and Ratiu 2015). All these works studied mainly stochastic perturbations of Hamiltonian systems. A very recent approach on the Lagrangian side, in Euler–Poincaré form, has been developed in Cruzeiro et al. (2018) and Holm (2015), where both the position and the momentum of the system are (independently) randomly perturbed, as well as the Lagrangian. The approach presented in this paper has natural versions for the models mentioned above. This is currently being investigated.

Building a Stochastic Geometric Mechanics theory is an ongoing effort by the mathematical and physical community. Among many others, we refer to recent work of D. Holm and collaborators (see Cotter et al. 2020a, b, c) and to Huang and Zambrini (2022).

Plan of the paper. In Sect. 2, we recall some basic probability notions necessary for the rest of the paper and give the crucial definition of the contraction matrix and martingale part for group valued semimartingales. Section 3 contains the first main result of the paper, namely the stochastic semidirect product Euler–Poincaré reduction for finite-dimensional Lie groups, both in left and right-invariant versions. We give the deterministic variational principle and the reduced equations of motion as well as their random deformations. In Sect. 4 we derive a stochastic Kelvin–Noether theorem. Section 5 presents the second main result of the paper, the reduction from the material to the spatial representation in infinite dimensions, which applies to the compressible Navier–Stokes equation and to the stochastic compressible magnetohydrodynamics equations. The stochastic reduction process recovers the standard deterministic equations in Eulerian representation as well as their random deformations.

2 The Derivative for Semimartingales

In Arnaudon et al. (2014), we gave the notion of generalized derivative for semimartingales taking values on some topological groups. In this section, we decompose a $G$-valued semimartingale (when the dimension of $G$ is finite, see, e.g., Emery 1989) into its velocity part, martingale part, and contraction part (matrix), which is crucial for our stochastic reduction procedure.

2.1 Some Probability Notions

We review in this subsection some basic notions of stochastic analysis on Euclidean spaces. We recall the concepts omitting the proofs, which can be found, for example, in Ikeda and Watanabe (1981).

We denote $\mathbb{R}^+ := [0, \infty[$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose we are given a family $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ of sub-$\sigma$-algebras of $\mathcal{F}$ which is non-decreasing (namely, $\mathcal{P}_s \subset \mathcal{P}_t$ for $0 \leq s \leq t$) and right-continuous, i.e., $\cap_{\varepsilon > 0} \mathcal{P}_{t+\varepsilon} = \mathcal{P}_t$ for all $t \in \mathbb{R}^+$. We then say that the probability space is endowed with a non-decreasing filtration.
A stochastic process $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $(\mathcal{F}_t)$-adapted if $X(t, \cdot) : \Omega \to \mathbb{R}^+$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. Typically, a filtration describes the past history of a process: one starts with a process $X$ and defines $\mathcal{F}_t$ to be the sigma-algebra generated by all sets $X(s, \cdot)^{-1}(B)$, with $0 \leq s \leq t$ and $B$ a Borel subset in $\mathbb{R}$. Then the process $X$ is automatically $(\mathcal{F}_t)$-adapted.

A stochastic process $M : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is a (\mathbb{R}-valued) martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ if

(i) $\mathbb{E}|M_\omega(t)| < \infty$ for all $t \geq 0$;

(ii) $M_\omega(t)$ is $(\mathcal{F}_t)$-adapted;

(iii) $\mathbb{E}_s(M_\omega(t))) = M_\omega(s)$ a.s. for all $0 \leq s < t$.

In the above definition, $\mathbb{E}$ denotes the expectation of the random variable with respect to the probability measure $\mathbb{P}$; $\mathbb{E}_s(M_\omega(t))) := \mathbb{E}[M_\omega(t)|\mathcal{F}_s]$, for each $s \geq 0$, is the conditional expectation of the random variable $M_\omega(t), t > s$, relative to the $\sigma$-algebra $(\mathcal{F}_s)$, i.e., $\Omega \ni \omega \mapsto \mathbb{E}_s[M_\omega(t)] \in \mathbb{R}$ is a $\mathcal{F}_s$-measurable function satisfying

$$\mathbb{E}\left[\mathbb{E}_s[M_\omega(t)]\chi_A(\omega)\right] = \mathbb{E}[M_\omega(t)]\chi_A(\omega), \quad \forall A \in \mathcal{F}_s,$$

where $\chi_A$ is the characteristic function of the set $A$. Thus, condition (iii) is equivalent to $\mathbb{E}[(M_\omega(t) - M_\omega(s))\chi_A(\omega)] = 0$ for all $A \in \mathcal{F}_s$ and all $t, s \in \mathbb{R}$ satisfying $t > s \geq 0$.

In this paper we shall only consider processes defined on compact time intervals $[0, T]$ which have continuous sample paths (i.e., continuous with respect to the time variable $t$ for almost all $\omega \in \Omega$).

If a martingale $M_\omega(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is continuous for a.s. $\omega \in \Omega$ and $\mathbb{E}[M_\omega(t)^2] < \infty$ for all $t \geq 0$, we say that $M$ has a quadratic variation $\mathbb{E}[M_\omega(t) - M_\omega(s)]^2$, for each $0 \leq s < t \leq T$. A martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ is a continuous, non-decreasing process with $M_\omega(0) = 0$ for a.s. $\omega \in \Omega$. Such a process is unique and coincides with the following limit (convergence in probability),

$$\lim_{n \to \infty} \sum_{(t_i, t_{i+1}] \in \sigma_n} (M_\omega(t_{i+1}) - M_\omega(t_i))^2,$$

where $\sigma_n$ is a partition of the interval $(0, t]$ and the mesh converges to zero as $n \to \infty$. Actually, the definition of the quadratic variation requires only right-continuity of $M$.

Moreover, for two martingales $M$ and $N$, under the same assumptions and conventions as given above, one can also define their covariation

$$[M_\omega, N_\omega]_t := \lim_{n \to \infty} \sum_{(t_i, t_{i+1}] \in \sigma_n} (M_\omega(t_{i+1}) - M_\omega(t_i))(N_\omega(t_{i+1}) - N_\omega(t_i)),$$

which extends the notion of quadratic variation. Clearly,

$$2[M_\omega, N_\omega]_t = [M_\omega + N_\omega, M_\omega + N_\omega]_t - [M_\omega, M_\omega]_t - [N_\omega, N_\omega]_t.$$

More generally, one can consider local martingales. A stopping time is a random variable $\tau : \Omega \to \mathbb{R}^+$ such that for all $t \geq 0$, $\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$. Then, a
stochastic process $M$ is a \textit{local martingale} if there exists a sequence of stopping times \{${\tau}_n \mid n \geq 1$\}, such that $\lim_{n \to \infty} {\tau}_n(\omega) = \infty$ a.s., and $M^{{\tau}_n}(t) := M(\min(t, \tau_n(\omega)))$ is a square integrable martingale for all $n \geq 1$, where $t \land {\tau}_n(\omega) := \min(t, \tau_n(\omega))$. Thus, for a local martingale $M$, we define $[M^{{\tau}_n}, M^{{\tau}_n}]_t := [M^n, M^n]_t$ if $t \leq {\tau}_n(\omega)$.

A real-valued \textit{Brownian motion} is a martingale $W(t)$ with continuous sample paths, $t \in \mathbb{R}^+$, such that $W^2(t) - t$ is a martingale; or, equivalently, such that $[W^{{\tau}_n}, W^{{\tau}_n}]_t = t$ for a.s. $\omega \in \Omega$.

A stochastic process $X : \Omega \times [0, T] \to \mathbb{R}$ is a (local) \textit{semimartingale} with respect to the non-decreasing filtration $(\mathcal{F}_t)_{t \geq 0}$ if, for every $t \geq 0$, it can be decomposed into a sum

$$X_\omega(t) = X_\omega(0) + M_\omega(t) + A_\omega(t),$$

where $M$ is a local martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ such that $M_\omega(0) = 0$ and $A$ is a càdlàg $(\mathcal{F}_t)_{t \geq 0}$-adapted process of locally bounded variation with $A_\omega(0) = 0$ a.s. (càdlàg = “continue à droite, limite à gauche” means, by definition, that $A$ is right-continuous with left limits at each $t \geq 0$; however, we consider only processes that are continuous in the time variable $t$, which is a standing assumption throughout this paper).

For a (local) semimartingale we define $[X_\omega, X_\omega]_t := [M_\omega, M_\omega]_t$.

Martingales and, in particular, Brownian motion, are not (a.s.) differentiable in the time variable (unless they are constant); therefore, one cannot integrate with respect to martingales as one does with respect to functions of bounded variation. We recall the definition of the two most commonly used stochastic integrals, the Itô and the Stratonovich integrals.

If $X$ and $Y$ are real-valued semimartingales with continuous sample paths such that for some $T > 0$,

$$\mathbb{E} \left[ \int_0^T |X_\omega(t)|^2 \, dt + \int_0^T |Y_\omega(t)|^2 \, dt \right] < \infty,$$

the \textit{Itô stochastic integral} in the time interval $[0, t]$, $0 < t \leq T$, with respect to $Y$ is defined as the limit in probability (if the limit exists) of the sums

$$\int_0^t X_\omega(s) \, dY_\omega(s) = \lim_{n \to \infty} \sum_{(t_i, t_{i+1}] \in \sigma_n} X_\omega(t_i)(Y_\omega(t_{i+1}) - Y_\omega(t_i)).$$

where $\sigma_n$ is a partition of the interval $(0, t]$ with mesh converging to zero as $n \to \infty$.

The \textit{Stratonovich stochastic integral} is defined by

$$\int_0^t X_\omega(s) \, dY_\omega(s) = \lim_{n \to \infty} \sum_{(t_i, t_{i+1}] \in \sigma_n} \frac{(X_\omega(t_i) + X_\omega(t_{i+1}))}{2} (Y_\omega(t_{i+1}) - Y_\omega(t_i))$$

whenever such limit exists.
These integrals do not coincide, in general, even though $X$ is a process with continuous sample paths (due to the lack of differentiability of the paths of $Y$). The Itô and the Stratonovich integrals are related by

$$\int_0^t X_\omega(s) \delta Y_\omega(s) = \int_0^t X_\omega(s) dY_\omega(s) + \frac{1}{2} \int_0^t d\left[ X_\omega, Y_\omega \right]_s. \tag{2.1}$$

If $f \in C^2(\mathbb{R})$, Itô’s formula states that

$$f(X_\omega(t)) = f(X_\omega(0)) + \int_0^t f'(X_\omega(s)) dX_\omega(s) + \frac{1}{2} \int_0^t f''(X_\omega(s)) d\left[ X_\omega, X_\omega \right]_s \tag{2.2}$$

This formula, for Stratonovich integrals, reads,

$$f(X_\omega(t)) = f(X_\omega(0)) + \int_0^t f'(X_\omega(s)) \delta X_\omega(s)$$

One advantage of Stratonovich integrals is that they allow the use of the same rules as those of the standard deterministic differential calculus. On the other hand an Itô integral with respect to a martingale $M$ is again a martingale (under the integrability condition $\mathbb{E} \left[ \int_0^T |X_\omega(t)|^2 d\left[ M_\omega, M_\omega \right]_t \right] < \infty$), a very important property. For example, we have, as an immediate consequence, that $\mathbb{E}_s \left[ \int_s^t X_\omega(r) dM_\omega(r) \right] = 0$ for all $0 \leq s < t$. This property does not hold for Stratonovich integrals.

In higher dimensions, the difference between the Stratonovich and the Itô integral in Itô’s formula is given in terms of the Hessian of $f$ (see Sect. 2.2). In fact, suppose that $X$ is an $\mathbb{R}^d$-valued semimartingale; then Itô’s formula in $d$-dimensions (see also (2.2)) states that, for every $f \in C^2(\mathbb{R}^d)$,

$$f(X_\omega(t)) = f(X_\omega(0)) + \sum_{i=1}^d \int_0^t \partial_i f(X_\omega(s)) dX^i_{\omega}(s)$$

$$+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial^2_{i,j} f(X_\omega(s)) d\left[ X^i_{\omega}, X^j_{\omega} \right]_s$$

$$= f(X_\omega(0)) + \sum_{i=1}^d \int_0^t \partial_i f(X_\omega(s)) \delta X^i_{\omega}(s) \tag{2.3}$$

The Itô and the Stratonovich integral forms above are usually written in a differential form using the time derivative notations $dX^i_{\omega}(t)$ and $\delta X^i_{\omega}(t)$, respectively.

For independent Brownian motions $W^i$, $i = 1, \ldots, k$, we have

$$d\left[ W^i_{\omega}, W^j_{\omega} \right]_t = \delta_{ij} dt \tag{2.4}$$
where $\delta_{ij}$ denotes the Kronecker delta symbol. As the covariation of semimartingales is determined by their martingale parts, the following identities hold (see, e.g., Ikeda and Watanabe 1981),

$$d\llbracket W^i_t, l \rrbracket_t = 0 \quad \forall i = 1, \ldots, d, \quad d\llbracket l, l \rrbracket_t = 0, \quad (2.5)$$

where $l(t) = t$ is the identity (deterministic) function.

### 2.2 The Generalized Derivative and Martingale Part for (Topological) Group Valued Semimartingales

Let $G$ denote a topological group, endowed with a Banach manifold structure (possibly infinite-dimensional) whose underlying topology is the given one, such that all left (or right) translations $L_g$ (resp. $R_g$) by arbitrary $g \in G$ are smooth maps, where $L_g h := gh$, $R_g h := hg$, for all $g, h \in G$. Given a vector $v \in T_e G$, we denote by $v^L$ (resp. $v^R$) the left (resp. right) invariant vector field whose value at the neutral element $e$ of $G$ is $v$, i.e., $v^L(e) := T_e L_g v$ (resp. $v^R(g) := T_e R_g v$), where $T_e L_g : T_e G \to T_g G$ is the tangent map (derivative) of $L_g$ (and similarly for $R_g$). The operation $[v_1, v_2] := [v^L_1, v^L_2]_1$ for any $v_1, v_2 \in T_e G$, defines a (left) Lie bracket on $T_e G$. In this paper, we denote by $\mathfrak{g}$ the Lie algebra of $G$, which is the set of left invariant vector fields on $G$. When working with right invariant vector fields, we shall still use, formally, the left Lie bracket defined above, i.e., we shall never work with right Lie algebras; the bracket defined by right invariant vector fields is equal to the negative of the left Lie bracket defined above. Denote, as usual, by $\text{ad}_v : T_e G \to T_e G$ its dual map (the coadjoint action of $T_e G$ on its dual $T^*_e G$).

Suppose that $\nabla$ is a left invariant linear connection on $G$, i.e., $\nabla v_1^L v_2^L = [v_1, v_2]_e$ is a left invariant vector field, for any $v_1, v_2 \in T_e G$. Then we define $\nabla v_1 v_2 := \nabla v_1^L v_2^L_2$ for all $v_1, v_2 \in T_e G$. If right translation is smooth, in all the definitions above, we can replace left translation by right translation in a similar way. We also assume that the left invariant connection $\nabla$ is torsion free, namely

$$\nabla v_1 v_2 - \nabla v_2 v_1 = [v_1, v_2], \quad \text{for all} \quad v_1, v_2 \in T_e G.$$

For a fixed $g_1 \in G$, let $L_{g_2} L_{g_1} : T_{g_2} G \to T_{g_1 g_2} G$ be the tangent map (or derivative) of $L_{g_1}$ at the point $g_2 \in G$.

Let $G$ be endowed with a left invariant linear torsion free connection $\nabla$. The corresponding Hessian $\text{Hess} f(g) : T_g G \times T_g G \to \mathbb{R}$ of $f \in C^2(G)$ at $g \in G$ is defined by

$$\text{Hess} f(g)(v_1, v_2) := \tilde{v}_1 \tilde{v}_2 f(g) - \nabla \tilde{v}_1 \tilde{v}_2 f(g), \quad v_1, v_2 \in T_g G, \quad (2.6)$$

where $\tilde{v}_i$, $i = 1, 2$, are arbitrary smooth vector fields on $G$ such that $\tilde{v}_i(g) = v_i$. Since the connection is torsion free, $\text{Hess} f(g)$ is a symmetric $\mathbb{R}$-bilinear form on each $T_g G$.  

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In addition, Hess $f = \nabla^2 f = \nabla df$ (see, e.g., Emery 1989) is the covariant derivative associated with $V$ of the one-form $df$, where $d$ denotes the exterior differential.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a non-decreasing filtration $(\mathcal{F}_t)_{t \geq 0}$, a semimartingale with values in $G$ (with respect to $(\mathcal{F}_t)_{t \geq 0}$) is a $\mathcal{F}_t$-adapted stochastic process $g : \Omega \times \mathbb{R}^+ \to G$ such that, for every function $f \in C^2(G)$, $f \circ g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a real-valued semimartingale (on $(\Omega, \mathcal{F}, \mathbb{P})$), as introduced in Sect. 2.1 (see, e.g., Emery (1989) for the case of finite-dimensional Lie groups).

A semimartingale with values in $G$ is a $\nabla$- (local) martingale if

$$t \mapsto f(g_\omega(t)) - f(g_\omega(0)) - \frac{1}{2} \int_0^t \text{Hess } f(g_\omega(s)) d[g_\omega, g_\omega]_s ds$$

is a real-valued (local) martingale for any $f \in C^2(G)$, where $[g_\omega, g_\omega]_t$ is the quadratic variation of $g_\omega$. If $G$ is a finite-dimensional Lie group, then we define the quadratic variation by

$$(g_\omega, g_\omega)_t := d\left[\int_0^t P_s^{-1} \delta g_\omega(s), \int_0^t P_s^{-1} \delta g_\omega(s)\right]_t,$$

where $P_t : T_{g_\omega(0)}G \to T_{g_\omega(t)}G$ is the (stochastic) parallel translation along the (stochastic) curve $t \mapsto g_\omega(t)$ associated with the connection $V$; see, e.g., Emery (1989) or Ikeda and Watanabe (1981). Moreover, for some infinite-dimensional groups $G$ (for example the diffeomorphism group on a torus), the quadratic variation is also well defined; we refer the reader to Arnaudon et al. (2014) and Cipriano and Cruzeiro (2007) for details (see also Sect. 5 of this paper).

For a $G$-valued semimartingale $g_\omega(\cdot)$, suppose there exist an integer $m > 0$ and $\mathcal{F}_t$-adapted processes $v : \Omega \times \mathbb{R}^+ \to T_e G, w^i : \Omega \times \mathbb{R}^+ \to T_e G, M^i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, $1 \leq i \leq m$, such that $M^i$ is a ($\mathbb{R}$-valued) martingale with continuous sample paths, and for every $f \in C^2(G)$,

$$f(g_\omega(t)) = f(g_\omega(0)) + \sum_{i=1}^{m} \int_0^t \left\langle df(g_\omega(s)), T_e L_{g_\omega(s)}w^i_\omega(s) \right\rangle dM^i_\omega(s)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{m} \int_0^t \text{Hess } f(g_\omega(s))(T_e L_{g_\omega(s)}w^i_\omega(s), T_e L_{g_\omega(s)}w^j_\omega(s)) d[M^i_\omega, M^j_\omega]_s$$
$$+ \frac{1}{2} \sum_{i,j=1}^{m} \int_0^t \left\langle df(g_\omega(s)), T_e L_{g_\omega(t)}(\nabla w^i_\omega(s) w^j_\omega(s)) \right\rangle d[M^i_\omega, M^j_\omega]_s$$
$$+ \frac{1}{2} \sum_{i=1}^{m} \int_0^t \left\langle df(g_\omega(s)), d[w^i_\omega, M^i_\omega]_s \right\rangle + \int_0^t \left\langle df(g_\omega(s)), T_e L_{g_\omega(s)}v_\omega(s) \right\rangle ds.
$$

(2.7)
For such a $G$-valued semimartingale $g_\omega$, having the form (2.7) above, the following holds,

$$
dg_\omega(t) = T_e L_{g_\omega(t)} \left( \sum_{i=1}^{m} w^i_\omega(t) \delta M^i_\omega(t) + v_\omega(t) \, dt \right). \tag{2.8}
$$

Here $\delta$ denotes the Stratonovich integral (of the tangent vectors in $G$).

Note that although for a given left invariant connection $\nabla$, the choice of \{$(w^i_\omega, M^i_\omega) \mid 1 \leq i \leq m$\} in (2.7) may not be unique, the decomposition into the martingale part (which is $\sum_{i=1}^{m} w^i_\omega(t) \, dM^i_\omega(t)$, or the second summand in (2.7)), and the drift part without contraction (which is $T_e L_{g_\omega(t)} v_\omega(s) \, dt$, or the last summand in (2.7)) is unique. The three summands with $\frac{1}{2}$ in front represent the contraction.

Then we define the velocity derivative of $g_\omega(\cdot)$ by

$$
\frac{\partial g_\omega(t)}{dt} := T_e L_{g_\omega(t)} v_\omega(t), \tag{2.9}
$$

and the stochastic differential with respect to the martingale part of $g_\omega(\cdot)$ by

$$
d^\Delta g_\omega(t) := \sum_{i=1}^{m} T_e L_{g_\omega(t)} \left( w^i_\omega(t) \, dM^i_\omega(t) \right), \tag{2.10}
$$

where $dM^i_\omega(t)$ denotes the Itô integral with respect to the martingale $M^i_\omega(t)$. Note that the two terms above do not depend on the choice of the left invariant connection $\nabla$.

In order to obtain the viscous terms in associated stochastic Euler–Poincaré equations, we need to make a more detailed analysis of the contraction part of the semimartingale (or stochastic Lagrangian path) $g_\omega(\cdot)$. For a given left invariant connection $\nabla$ on $G$ and some fixed choice \{$(w^i_\omega, M^i_\omega) \mid 1 \leq i \leq m$\}, where both $w^i_\omega$ and $M^i_\omega$ are $\mathcal{F}_t$-adapted processes and $M^i_\omega$ are real valued martingales with continuous sample paths, we define the contraction matrix \(\frac{D^\nabla (w^i_\omega, M^i_\omega)}{dt}\) as the following $T_{g_\omega(t)} G$-valued $m \times m$ matrix:

$$
\left( \frac{D^\nabla (w^i_\omega, M^i_\omega)}{dt} g_\omega(t) \right)_{ij} := T_e L_{g_\omega(t)} \left( \nabla w^i_\omega(t) w^j_\omega(t) \frac{d[M^i_\omega, M^j_\omega]}{dt} + \frac{d[w^i_\omega, M^j_\omega]}{dt} \delta_{ij} \right), \quad 1 \leq i, j \leq m. \tag{2.11}
$$

Therefore, we can split the differential of a $G$-valued semimartingale (see (2.7)) into the velocity part, the Hessian (second-order) term, the martingale part, and the contraction part (more accurately the contraction matrix). Intuitively, the velocity part could be seen as the direction where the particles flow, the martingale part represents their random fluctuations, while the contraction part describes the contraction effect from the noise.
The term \( \left( \frac{D V_i(\omega, \mu)}{dt} \right)_{ij} \) corresponds to the contraction between the noises in the vector fields \( \omega_i^j \) and \( \omega_k^l \). Thus, the contraction matrix describes explicitly the behavior of the noises interaction along different vector fields (directions) \( \{ \omega_i^j \} \).

Let

\[
\text{Sum} \left( \frac{D V_i(\omega, \mu)}{dt} \right) := \sum_{i,j=1}^m \left( \frac{D V_i(\omega, \mu)}{dt} \right)_{ij} \in T_{g(t)} G
\]

denote the sum of all entries of the matrix \( \frac{D V_i(\omega, \mu)}{dt} \); for each fixed \( t \) this is a \( T_{g(t)} G \)-valued random variable.

Then it is easy to verify that for a \( G \)-valued semimartingale of the form (2.8) and any \( f \in C^2(G) \), the process

\[
N^f_t := f(g_0(t)) - f(g_0(0)) - \frac{1}{2} \int_0^t \text{Hess} f(g_0(s)) \text{d}[g_0, g_0]_s
- \frac{1}{2} \int_0^t \left\langle \frac{\text{d} f(g_0(s))}{\text{d} s}, \text{Sum} \left( \frac{D V_i(\omega, \mu)}{dt} \right)_{ij} \right\rangle \text{d} s
- \int_0^t \left\langle \frac{\text{d} f(g_0(s))}{\text{d} s}, \frac{\partial g_0(s)}{\text{d} s} \right\rangle \text{d} s
\]

is a real-valued local martingale.

We remark that by (2.9)–(2.11), the terms \( \frac{\partial}{\text{d} t}, \frac{d \Lambda}{\text{d} t}, \frac{D V_i(\omega, \mu)}{dt} \) are well defined for semimartingales with values in a finite-dimensional Lie group as well as in some infinite-dimensional groups (the diffeomorphism group on a torus for example); see, e.g., Arnaudon et al. (2014) or Sect. 5 below.

In the stochastic Euler–Poincaré reduction introduced in Sect. 3, the martingale part and the contraction part generate, respectively, the martingale term and the viscosity term in the associated (stochastic) Euler–Poincaré equation.

Moreover, when \( G \) is a finite-dimensional Lie group, for a \( G \)-valued semimartingale \( g_\omega(\cdot) \) of the form (2.8), we have the following equality (see, e.g., Emery 1989)

\[
\frac{D V g_\omega(t)}{dt} := P_t \left( \lim_{\epsilon \to 0} \mathbb{E}_t \left[ \frac{\eta_\omega(t + \epsilon) - \eta_\omega(t)}{\epsilon} \right] \right) = \frac{1}{2} \text{Sum} \left( \frac{D V_i(\omega, \mu)}{dt} \right) + \frac{\partial g_\omega(t)}{dt},
\]

where \( P_t : T_e G \to T_{g_\omega(t)} G \) is the stochastic parallel translation associated to \( \nabla \), \( \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | P_t] \) denotes the conditional expectation, and

\[
\eta_\omega(t) = \int_0^t P_s^{-1} \delta g_\omega(s) \in T_e G.
\]
Therefore, according to the definition, if a $G$-valued semimartingale $g_\omega(t)$ satisfies 
\[
\frac{D^V}{dt} g_\omega(t) = 0,
\]
then $g_\omega(t)$ is a $\nabla$-martingale.

In fact, $\frac{D^\nabla}{dt}$ is the generalized derivative in Arnaudon et al. (2014), which is a generalization for group-valued semimartingales of those in Cipriano and Cruzeiro (2007), Nelson (1967), Yasue (1981), Nakagomi et al. (1981), Zambrini (2015); it contains a single term formed by the sum of all elements in the contraction matrix. The generalized derivative is sufficient to generate the viscosity terms (second-order differential terms) in some partial differential equations through the stochastic reduction procedure. This is the case, for example, for the incompressible Navier–Stokes equation; see, e.g., Arnaudon et al. (2014), Cipriano and Cruzeiro (2007), Nelson (1967), Yasue (1981), Nakagomi et al. (1981) and Zambrini (2015). However, for a large class of equations in fluid mechanics, the viscous terms do not depend only on such kind of contraction terms; see, e.g., the compressible Navier–Stokes equation or the viscous MHD equation in Sect. 5. This is one of our motivations to introduce the decomposition of $\frac{D^V}{dt}$ above.

The generalized derivative coincides with the drift of a diffusion processes. It was commonly used since the beginning of Stochastic Analysis but was first associated with a dynamical interpretation, as a mean velocity, in the context of Nelson’s Stochastic Mechanics (1967).

Given a $\mathbb{R}^m$-valued martingale $M_\omega(t) = (M_\omega^1(t), \ldots, M_\omega^m(t))$, $t \in [0, T]$, which has continuous sample paths, (non-random) vectors $H_i \in T_eG$, $1 \leq i \leq m$, and a $\mathcal{F}_t$-adapted, $T_eG$-valued semi-martingale $u : \Omega \times [0, T] \to T_eG$, consider the following Stratonovich SDE on $G$,
\[
\begin{align*}
    dg_\omega(t) &= T_eLg_\omega(t) \left( \sum_{i=1}^m H_i \delta M_\omega^i(t) + u_\omega(t)dt \right), \\
    g_\omega(0) &= e.
\end{align*}
\]  

(2.13)

As explained in Emery (1989), given the connection $\nabla$, the difference (contraction term) between the Itô and Stratonovich integrals has the following form
\[
\sum_{i=1}^m \left( T_eLg_\omega(t) H_i \delta M_\omega^i(t) - T_eLg_\omega(t) H_i dM_\omega^i(t) \right) \\
= \frac{1}{2} \sum_{i=1}^m d\|[T_eLg_\omega(t) H_i, M_\omega^i(t)]_t \|
= \frac{1}{2} \sum_{i,j=1}^m T_eLg_\omega(t) (\nabla H_i H_j) d\|[M_\omega^i, M_\omega^j]_t \|.
\]

Therefore, Eq. (2.13) is equivalent to
\[
\begin{align*}
    dg_\omega(t) &= T_eLg_\omega(t) \left( \sum_{i=1}^m H_i dM_\omega^i(t) + \frac{1}{2} \sum_{i,j=1}^m \nabla H_i H_j d\|[M_\omega^i, M_\omega^j]_t + u_\omega(t)dt \right), \\
    g_\omega(0) &= e.
\end{align*}
\]  

(2.14)
If \( G \) is a finite-dimensional Lie group, there exists a unique strong solution for (2.13) (c.f. Ikeda and Watanabe 1981; Emery 1989) and hence also for (2.14). When \( G \) is the diffeomorphism group on a torus and \( u \) is less regular, a weak solution to (2.13) still exists (Arnaudon et al. 2014; Cipriano and Cruzeiro 2007) under suitable conditions on \( H_i \).

Applying Itô’s formula to the solution \( g_\omega(t) \) of (2.13) (see Emery (1989) if \( G \) is finite-dimensional and Arnaudon et al. (2014, Section 4.2) or Sect. 5 below if \( G \) is the diffeomorphism group on a torus), for every \( f \in C^2(G) \) we have,

\[
f(g_\omega(t)) = f(g_\omega(0)) + \sum_{i=1}^m \int_0^t \langle df(g_\omega(s)), TeL_{g_\omega(s)}H_i \rangle dM^i_\omega(s) 
+ \frac{1}{2} \int_0^t \text{Hess} f(g_\omega(s))d\|g_\omega, g_\omega\|_s + \int_0^t \langle df(g_\omega(s)), TeL_{g_\omega(s)}u_\omega(s) \rangle ds 
+ \frac{1}{2} \sum_{i,j=1}^m \int_0^t \langle df(g_\omega(s)), TeL_{g_\omega(s)}\nabla H_i H_j \rangle d\|M^i_\omega, M^j_\omega\|_s
\]

Actually, this last equality, valid for each \( f \in C^2(G) \), is a characterization of the solution of the stochastic differential equation (2.13) (or (2.14)), in a weak sense.

Clearly, by the definitions (2.9) and (2.10), we have

\[
\frac{d}{dt}g_\omega(t) = TeL_{g_\omega(t)}u_\omega(t),
\]

\[
d^\Delta g_\omega(t) = \sum_{i=1}^m (TeL_{g_\omega(t)}H_i) dM^i_\omega(t),
\]

\[
\left( D^\nabla (H_i, M^j_\omega)_i \right)_{ij} = TeL_{g_\omega(t)}(\nabla H_i H_j) d\|M^i_\omega, M^j_\omega\|_t.
\] (2.15)

### 3 Stochastic Semidirect Product Euler–Poincaré Reduction

In this section, partially inspired by Arnaudon et al. (2014), Cruzeiro et al. (2018) and Holm (2015), we extend the deterministic semidirect product Euler–Poincaré reduction, formulated and developed in Holm et al. (1998), to the stochastic setting. By such a reduction, we obtain a large class of partial differential equations and stochastic partial differential equations with various viscosity terms; see Sect. 5 below.

#### 3.1 Left Invariant Version

Let \( U \) be a vector space and \( U^* \) its dual, also denote by \( \langle \cdot, \cdot \rangle_U : U^* \times U \rightarrow \mathbb{R} \) the (weak) duality pairing. Suppose that \( G \) is a group endowed with a manifold structure making it into a topological group whose left translation is smooth. As discussed in Sect. 2.2, the tangent space \( TeG \) to \( G \) at the identity element \( e \in G \) is (isomorphic to) a Lie algebra.
Assume that $G$ has a left representation on $U$; therefore, there are naturally induced left representations of the group $G$ and the Lie algebra $T_eG$ on $U$ and $U^*$. All actions will be denoted by concatenation. Let $\langle \cdot, \cdot \rangle_{T_eG} : T^*_eG \times T_eG \to \mathbb{R}$ be the (weak) duality pairing between $T^*_eG$ and $T_eG$. Define the operator $\diamond : U \times U^* \to T^*_eG$ by

$$
\langle a \circ \alpha, v \rangle_{T_eG} := - \langle v a, a \rangle_U = \langle \alpha, v a \rangle_U, \quad v \in T_eG, \quad a \in U, \quad \alpha \in U^*. \quad (3.1)
$$

In fact, $a \circ \alpha$ is the value at $(a, \alpha)$ of the momentum map $U \times U^* \to T^*_eG$ of the cotangent lifted action induced by the left representation of $G$ on $U$.

Let $\mathcal{M}_m := \{ A = (a_{ij})^m_{i, j=1} \mid a_{ij} \in T_eG \}$ be the vector space of all $m \times m$ $T_eG$-valued matrices. Define $\mathcal{M} := \cup_{m=1}^\infty \mathcal{M}_m$. The dual of $\mathcal{M}_m$ is $\mathcal{M}_m^* := \{ \xi = (\xi_{ij})^m_{i, j=1} \mid \xi_{ij} \in T^*_eG \}$, the vector space of all $m \times m$ $T^*_eG$-valued matrices, relative to the pairing

$$
\langle \xi, A \rangle_{\mathcal{M}_m} := \sum_{i,j=1}^m \langle \xi_{ij}, a_{ij} \rangle_{T_eG}. \quad (3.2)
$$

Define $\mathcal{M}^* := \cup_{m=1}^\infty \mathcal{M}_m^*$.

Let $\mathcal{S}(G)$ denote the collection of $G$-valued semimartingales with smooth coefficients defined on the time interval $[0, T]$. In order to define the contraction matrix for $g_{\omega} \in \mathcal{S}(G)$ having the form (2.8), we need to fix a set of pairs $\{(w^i_\omega, M^i_\omega) \mid 1 \leq i \leq m \}$ in the martingale part of (2.8) (the first term of the right hand side of (2.8), i.e., the Itô integral). The hypotheses on this set of pairs remain the same: $w^i_\omega$ and $M^i_\omega$ are $\mathcal{P}_1$-adapted processes and $M^i_\omega$ are real valued martingales with continuous sample paths, for all $i = 1, \ldots, m$. We denote by $(g_{\omega}, w^i_\omega, M^i_\omega)_{i=1}^m$ an element in $\mathcal{S}(G)$ with a fixed choice $\{(w^i_\omega, M^i_\omega) \mid 1 \leq i \leq m \}$ in (2.8). Let $\mathcal{S}(G)$ be the collection of all these triples.

Given a (left invariant) linear connection $\nabla$ on $G$, a point $a_0 \in U^*$, a random (Lagrangian) function $l : \Omega \times [0, T] \times T_eG \times U^* \to \mathbb{R}$ such that $l_{\omega}(t)$ is $\mathcal{P}_1$-adapted for each $t \in [0, T]$, a (viscosity force) function $p : \mathcal{M} \times \mathcal{M} \times T_eG \to \mathbb{R}$, a (stochastic force) function $q : [0, T] \times T_eG \times U^* \to T^*_eG$, vectors $V_i \in T_eG$ (which are non-random), $1 \leq i \leq k$, and an $\mathbb{R}^k$-valued martingale $N_{\omega}(t)$, we define a stochastic action functional $J_{\nabla, a_0, l, p, q, (V_i, N_{\omega}), i=1}^k$ on $\mathcal{S}(G) \times \mathcal{S}(G) \times \mathcal{S}(G)$ by

---

1 Looking forward, we mention that the specification of the dual of $\mathcal{M}_m$ is only for convenience. In fact, we do not need to be that precise because Eq. (3.12) in Theorem 3.5 does not depend on the expression of $\frac{\delta p}{\delta A_j}$, $j = 1, 2$, and hence does not depend on the choice of the dual of $\mathcal{M}_m$ either. The reason is that in (3.14), with any choice for $\frac{\delta p}{\delta A_j}$, $j = 1, 2$, in a dual of $\mathcal{M}_m$, the value of $\frac{\delta p}{\delta A_1}(A_1, A_2, u), B_{\omega}(t, v)$ is the same and equals $\lim_{\varepsilon \to 0} \frac{1}{2} \{ p(A_1 + \varepsilon B_{\omega}, (A_2, u) \}. The same holds for $\frac{\delta p}{\delta A_2}$. Thus, the value of $K(A_1, A_2, u)$ is independent of the choice of the dual.
\[ J_{\nabla } \alpha_0, l, p, q, (V_1, N_0^\omega)_{i=1}^k \left( \left( g_1^0, w_1^i, M_0^i \right)_{i=1}^m, \left( g_2^0, w_2^i, M_0^i \right)_{i=1}^m, g_3^0 \right) \\
= \int_0^T \left( t, T_{g_1^0(t)} L_{g_1(t)}^{-1} \frac{\mathcal{D} g_1^0(t)}{dt} \right) dt \\
+ \int_0^T p \left( T_{g_1^0(t)} L_{g_1(t)}^{-1} \frac{\mathcal{D} g_1^0(t)}{dt}, T_{g_2^0(t)} L_{g_2(t)}^{-1} \frac{\mathcal{D} g_2^0(t)}{dt} \right) dt \\
+ \int_0^T q \left( T_{g_1^0(t)} L_{g_1(t)}^{-1} \frac{\mathcal{D} g_1^0(t)}{dt}, T_{g_1^0(t)} L_{g_1(t)}^{-1} d^\Delta g_1^0(t) \right) \\
- \frac{1}{k} \sum_{i=1}^k \int_0^T q \left( T_{g_1^0(t)} L_{g_1(t)}^{-1} \frac{\mathcal{D} g_1^0(t)}{dt}, T_{g_1^0(t)} L_{g_1(t)}^{-1} d^\Delta g_1^0(t) \right), \quad (3.3) \]

where \( \left( g_1^0, w_1^i, M_0^i \right)_{i=1}^m \in \mathcal{F}(G), \left( g_2^0, w_2^i, M_0^i \right)_{i=1}^m \in \mathcal{F}(G), T_{g_1^0(t)} L_{g_1(t)}^{-1} d^\Delta g_1^0(t) \) corresponds to the Itô integral on the vector space \( T_{eG} \), and

\[ \alpha_0(t) := g_3^0(t)^{-1} \alpha_0. \quad (3.4) \]

Note that \( J_{\nabla } \alpha_0, l, p, q, (V_1, N_0^\omega)_{i=1}^k \left( \left( g_1^0, w_1^i, M_0^i \right)_{i=1}^m, \left( g_2^0, w_2^i, M_0^i \right)_{i=1}^m, g_3^0 \right) \) for fixed time \( T \) is a random variable.

**Remark 3.1**

We explain intuitively why we want the action functional \( J_{\nabla } \alpha_0, l, p, q, (V_1, N_0^\omega)_{i=1}^k \) to have the form (3.3). When the Lagrangian \( l(v) = \frac{1}{2} \langle v, v \rangle_{\mathbb{R}^d} \), \( v \in \mathbb{R}^d \), is the kinetic energy, for a stochastic Lagrangian path \( dg_\omega(t) = dM_\omega(t) + u_\omega(t) dt = d^\Delta g(t) + \frac{\mathcal{D} g_\omega(t)}{dt} dt \) with \( M_\omega(t) \) being a \( \mathbb{R}^d \)-valued martingale, we can formally write the kinetic energy as follows

\[ \frac{1}{2} \int_0^T \left( T_{g_\omega(t)} L_{g_\omega(t)}^{-1} \frac{\mathcal{D} g_\omega(t)}{dt}, T_{g_\omega(t)} L_{g_\omega(t)}^{-1} \frac{\mathcal{D} g_\omega(t)}{dt} \right) dt \]

\[ + \int_0^T \left( T_{g_\omega(t)} L_{g_\omega(t)}^{-1} \frac{\mathcal{D} g_\omega(t)}{dt}, T_{g_\omega(t)} L_{g_\omega(t)}^{-1} d^\Delta g_\omega(t) \right) \]

\[ + \frac{1}{2} \int_0^T \left( T_{g_\omega(t)} L_{g_\omega(t)}^{-1} d^\Delta g_\omega(t), T_{g_\omega(t)} L_{g_\omega(t)}^{-1} d^\Delta g_\omega(t) \right) \]

\[ := I_1 + I_2 + I_3. \]

Here \( I_1 = \int_0^T l \left( T_{g_\omega(t)} L_{g_\omega(t)}^{-1} \frac{\mathcal{D} g_\omega(t)}{dt} \right) dt \) represents the kinetic energy of the velocity: it is the action functional in the deterministic case, based on which a standard Euler–Poincaré equation is obtained via the reduction procedure. The summand \( I_2 \) contains a stochastic differential for the martingale part of \( dg_\omega(t) \) and we can interpret it as the Itô integral with respect to this martingale. Concerning \( I_3 \), since it is not well-defined (it is almost-everywhere infinite), we drop this term in the action functional.
Notice that we keep the notation $I$ for the deterministic Lagrangian from which we start and regard $I_2$ as a random perturbation of $I_1$.

Besides the kinetic energy, we could also add some extra terms of the form

$$\sum_{i=1}^{k} \int_{0}^{T} \left( T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} \frac{\mathcal{D} g_{\omega}(t)}{dt}, V_{i} dN_{\omega_{i}}^{i}(t) \right)$$

which represents the external stochastic fluctuation for the velocity.

Therefore, we define an action functional as follows

$$J(g_{\omega}()) = \int_{0}^{T} \frac{1}{2} \left( T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} \frac{\mathcal{D} g_{\omega}(t)}{dt}, T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} \frac{\mathcal{D} g_{\omega}(t)}{dt} \right) dt$$

$$+ \int_{0}^{T} \left( T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} \frac{\mathcal{D} g_{\omega}(t)}{dt}, T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} d\Delta g_{\omega}(t) \right)$$

$$- \sum_{i=1}^{k} \int_{0}^{T} \left( T_{g_{\omega}(t)} L_{g_{\omega}(t)}^{-1} \frac{\mathcal{D} g_{\omega}(t)}{dt}, V_{i} dN_{\omega_{i}}^{i}(t) \right),$$

which, when we add the viscous term (defined by a viscosity force $p$ and the contraction matrix for $g_{\omega}$), is a particular case of (3.3) for $q(v,a) = v, \forall \nu \in \mathbb{R}^{d}, a \in U^*$ (we use here the identification of $T_{e} G$ with $T_{e} G$).

From now on, we write $J_{\nu}(V_{i}, N_{\omega_{i}}^{i})_{i=1}^{k}$ for $J_{\nu_{0}, l, p, q}(V_{i}, N_{\omega_{i}}^{i})_{i=1}^{k}$ for simplicity. In order to characterize the critical points of the action functional and to derive the corresponding Euler–Poincaré equation, it is necessary to consider a variation for $\left( g_{\omega_{1}}^{1}, w_{\omega_{1}}^{1,i}, M_{\omega_{1}}^{1,i} \right)_{i=1}^{m_{1}} \in \mathcal{F}(G)$ and $\left( g_{\omega_{2}}^{2}, w_{\omega_{2}}^{2,i}, M_{\omega_{2}}^{2,i} \right)_{i=1}^{m_{2}} \in \mathcal{F}(G)$.

For every $\varepsilon \in [0, 1)$ and $\mathcal{P}_{l}$-adapted process $g : \Omega \times [0, T] \rightarrow T_{e} G$ satisfying $g_{\omega}(0) = g_{\omega}(T) = 0$ and $g_{\omega}(\cdot) \in C^{1}([0, 1]; T_{e} G)$ a.s., let $e_{\omega, \varepsilon, g}(\cdot) \in C^{1}([0, T]; G)$ be the unique solution of the (random) time-dependent ordinary differential equation on $G$

$$\left\{ \begin{array}{l}
\frac{d}{dt} e_{\omega, \varepsilon, g}(t) = \varepsilon T_{e} L_{e_{\omega, \varepsilon, g}(t)} g_{\omega}(t), \\
e_{\omega, \varepsilon, g}(0) = e,
\end{array} \right. \quad (3.5)$$

where $g_{\omega}(t)$ denotes the derivative with respect to the time variable $t$. Note that this system implies $e_{\omega, 0, g}(t) = e$ a.s. for all $t \in [0, T]$.

From now on, in this section, we assume that $G$ is a finite-dimensional Lie group endowed with a left invariant linear connection $\nabla$ and $U$ is a finite-dimensional left $G$-representation space.

We give the following lemma concerning the variations induced by $e_{\omega, \varepsilon, g}$ on a semimartingale $g_{\omega} \in \mathcal{F}(G)$.

**Lemma 3.2** Suppose $g_{\omega} \in \mathcal{F}(G)$ has the form (2.8) and let

$$g_{\omega, \varepsilon, g}(t) := g_{\omega}(t) e_{\omega, \varepsilon, g}(t), \quad t \in [0, T], \quad \varepsilon \in [0, 1). \quad (3.6)$$
Then we have

\[ \begin{split}
     dg_{\omega, \varepsilon, g}(t) &= T_eL_{g_{\omega, \varepsilon, g}(t)} \left( \sum_{i=1}^{m} \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}(t) \delta M^i_{\omega}(t) \\
     &+ \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} v_{\omega}(t) dt + \varepsilon g_{\omega}(t) dt \right). 
\end{split} \tag{3.7} \]

**Proof** By Itô’s formula and recalling that the Leibniz rule holds for Stratonovich integrals, using (2.8), we have

\[ \begin{split}
     dg_{\omega, \varepsilon, g}(t) &= T_eL_{g_{\omega, \varepsilon, g}(t)} \left( \sum_{i=1}^{m} \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}(t) \delta M^i_{\omega}(t) + \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} v_{\omega}(t) dt \\
     &+ T_eL_{g_{\omega, \varepsilon, g}(t)} \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} \dot{v}_{\omega}(t) dt \right) \\
     &= T_eL_{g_{\omega, \varepsilon, g}(t)} \left( \sum_{i=1}^{m} \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}(t) \delta M^i_{\omega}(t) + \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} v_{\omega}(t) dt + \varepsilon g_{\omega}(t) dt \right),
\end{split} \]

where the last equality is due to (3.5).

Based on (3.7), it is natural to consider \( (g_{\omega, \varepsilon, g}, \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}, M^i_{\omega})_{i=1}^{m} \) as a deformation for \( (g_{\omega}, \omega^i, M^i_{\omega})_{i=1}^{m} \) with \( g_{\omega} \in \mathcal{D}(G) \) having the expression (2.8). Meanwhile, using definitions (2.9)–(2.11), it is easy to verify that

\[ \begin{split}
     T_{g_{\omega, \varepsilon, g}(t)}L_{g_{\omega, \varepsilon, g}(t)}^{-1} \frac{\partial g_{\omega, \varepsilon, g}(t)}{dt} &= \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} v_{\omega}(t) + \varepsilon g_{\omega}(t) \\
     T_{g_{\omega, \varepsilon, g}(t)}L_{g_{\omega, \varepsilon, g}(t)}^{-1} d^\Delta g_{\omega, \varepsilon, g}(t) &= \sum_{i=1}^{m} \left( \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}(t) \right) \delta M^i_{\omega}(t) \\
     &= \left( T_{g_{\omega, \varepsilon, g}(t)}L_{g_{\omega, \varepsilon, g}(t)}^{-1} \frac{\nabla (\text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} w^i_{\omega}, M^j_{\omega})_{i=1}^{m}}{g_{\omega, \varepsilon, g}(t)} \right)_{ij} \\
     &= \left( \nabla \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)} w^i_{\omega}(t) \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} \frac{\delta M^i_{\omega}}{dt} + \frac{\delta M^i_{\omega}}{dt} \text{Ad}_{\varepsilon_{\omega, \varepsilon, g}(t)}^{-1} \frac{\delta M^j_{\omega}}{dt} \right)_{ij}. \tag{3.8} \]

**Remark 3.3** Although by now we assume that \( G \) is a finite-dimensional Lie group, by the arguments in Arnaudon et al. (2014, Section 4.2) we know that (3.8) still holds when \( G \) is the diffeomorphism group on torus, see, e.g., (5.18) below. Hence, Theorem 3.5 stated below still holds for the diffeomorphism group on the torus (see Sect. 5). \( \square \)
Now we define the critical point for the action functional based on the variations introduced above. We say that \( \tilde{\omega} \in \mathcal{G} \) is a critical point of \( J^{\mathcal{V},(V_i,N^i_{i=1})} \) if for every \( \mathcal{P}_T \)-adapted process \( g_{\omega} \) satisfying \( g_{\omega}(\cdot) \in C^1([0, T]; T_e G) \) and \( g_{\omega}(0) = g_{\omega}(T) = 0 \) a.s., we have

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} J^{\mathcal{V},(V_i,N^i_{i=1})} \left( \left( g^1_{\omega,\varepsilon,\varepsilon} \right)_{i=1}^3, \left( g^2_{\omega,\varepsilon,\varepsilon} \right)_{i=1}^3 \right) = 0
\]

(3.9)

where

\[
g^j_{\omega,\varepsilon,\varepsilon}(t) := g^j_{\omega}(t)e_{\omega,\varepsilon,\varepsilon}(t), \quad t \in [0, T], \quad j = 1, 2, 3, \quad \varepsilon \in [0, 1).
\]

We emphasize the particular form of these deformations in the Lie group: they correspond to developments along (random) directions \( g_{\omega}(t) \).

**Remark 3.4** As will be seen in the applications presented in Sect. 5, the reason why we choose three different semimartingales in the variational principle (3.9) is that the viscosity constants in different equations may be different. ♦

Fixing (non-random) \( \{H^j_{i=1} m_j \} \in T_e G, j = 1, 2, 3 \), as well as \( \mathbb{R}^{m_j} \)-valued martingales \( M^j_{\omega}(t) = (M^1_{\omega}(t), \ldots, M^{m_j}_{\omega}(t)), j = 1, 2, 3 \), we consider \( g^j_{\omega}, H^j_{i=1} m_j \) \( g^j_{\omega} \) in \( \mathcal{G} \), \( j = 1, 2, 3 \), where \( g^j_{\omega} \) are the solutions of the following SDEs on \( G \),

\[
\begin{align*}
\frac{d g^j_{\omega}(t)}{dt} &= T_e L g^j_{\omega}(t) \left( \sum_{i=1}^{m_j} H^j_{i=1} \delta M^j_{\omega}(t) + u^j_{\omega}(t) \right), \\
g^j_{\omega}(0) &= e,
\end{align*}
\]

(3.11)

and where \( u^j_{\omega} \) is a \( \mathcal{P}_T \)-adapted, \( T_e G \)-valued semimartingale. Note that \( u^j_{\omega}(\cdot) \) is not given a priori and is the same for \( j = 1, 2, 3 \); we shall see below that it is the solution of a certain (stochastic) equation when \( \left( (g^1_{\omega}, H^1_{i=1} m_j), (g^2_{\omega}, H^2_{i=1} m_j), (g^3_{\omega}, H^3_{i=1} m_j) \right) \) is a critical point for \( J^{\mathcal{V},(H^j_{i=1} M^j_{\omega})_{i=1}^3} \).

### 3.2 Stochastic Variational Principle for Stochastic Differential Equations

In the theorem below we use the functional derivative notation. Let \( V \) be (a possibly infinite-dimensional) vector space and \( V^* \) a space in weak duality \( \langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R} \) with \( V \); in finite dimensions, \( V^* \) is the usual dual vector space, but in infinite dimensions it rarely is the topological dual. If \( f : V \rightarrow \mathbb{R} \) is a smooth function, then the functional derivative \( \frac{\delta f}{\delta a} \in V^* \), if it exists, is defined by \( \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon b) - f(a)}{\varepsilon} = \frac{\delta f}{\delta a} \) for all \( a, b \in V \).

In this section, we assume that \( l, p, q \) in the action functional \( J^{\mathcal{V},(V_i,N^i_{i=1})} \) are smooth with respect to all variables, except, of course, \( \omega \in \Omega \). Thus, in the theorem...
below, \( u \in T_e G, \alpha \in U^*, A_1, A_2 \in \mathcal{M} \), so that \( \frac{\delta l_{\alpha}}{\delta u} \in T_e^* G, \frac{\delta l_{\alpha}}{\delta \alpha} \in U, \frac{\delta p}{\delta A_1}, \frac{\delta p}{\delta A_2} \in \mathcal{M}^* \), and \( \frac{\delta p}{\delta u} \in T_e^* G \) are the partial functional derivatives of \( l : \Omega \times [0, T] \times T_e G \times U^* \rightarrow \mathbb{R} \) and \( p : \mathcal{M} \times \mathcal{M} \times T_e G \rightarrow \mathbb{R} \). Recall that \( \mathcal{M}_m := \{(a_{ij})_{i,j=1}^m \mid a_{ij} \in T_e G\} \), \( \mathcal{M}^*_m := \{(\xi_{ij})_{i,j=1}^m \mid \xi_{ij} \in T_e^* G\} \) relative to the pairing (3.2), \( \mathcal{M} := \cup_{m=1}^\infty \mathcal{M}_m \), and \( \mathcal{M}^* := \cup_{m=1}^\infty \mathcal{M}^*_m \).

**Theorem 3.5** Let \( l : \Omega \times [0, T] \times T_e G \times U^* \rightarrow \mathbb{R}, \ p : \mathcal{M} \times \mathcal{M} \times T_e G \rightarrow \mathbb{R}, \ q : [0, T] \times T_e G \times U^* \rightarrow T_e^* G \) such that \( \frac{\delta l_{\alpha}}{\delta u} \) is non-random and \( l_{\alpha}(t) \) is \( \mathcal{P}_1 \)-adapted. Suppose that the semimartingales \( g_{\omega}(\cdot) \), \( j = 1, 2, 3 \), have the form (3.11).

(i) Then \( (g_{\omega}^1, H_1^1, M_{\omega}^{1,i})_{i=1}^m, (g_{\omega}^2, H_2^2, M_{\omega}^{2,i})_{i=1}^m, (g_{\omega}^3) \) is a critical point of \( J_{\nu},(H_1^1, M_{\omega}^{1,i})_{i=1}^m \) (given in (3.3)) if and only if the \( \mathcal{P}_1 \)-adapted process \( u_{\omega}(t) \) coupled with the \( \mathcal{P}_1 \)-adapted process \( \alpha_{\omega}(t) \) (which is defined by (3.4)) satisfies the following (stochastic) semidirect product Euler–Poincaré equation for stochastic particle paths:

\[
\begin{align*}
\frac{d}{dt} & \left( \frac{\delta l_{\alpha}}{\delta u}(t, u_{\omega}(t), \alpha_{\omega}(t)) + \frac{\delta p}{\delta u}(\tilde{H}_{\alpha_{\omega}}, 1(t), \tilde{H}_{\alpha_{\omega}, 2}(t), u_{\omega}(t)) \right) \\
= & \sum_{i=1}^m a_{i}^u q(u_{\omega}(t), \alpha_{\omega}(t))dM_{\omega}^{1,i}(t) + ad_{u_{\omega}}^\nu(t, u_{\omega}(t), \alpha_{\omega}(t)) \\
+ & \left( \frac{\delta l_{\alpha}}{\delta u}(t, u_{\omega}(t), \alpha_{\omega}(t)) \right) \circ \alpha_{\omega}(t)dt + ad_{u_{\omega}}^\nu(t, u_{\omega}(t), \alpha_{\omega}(t)) \\
+ & K_{\omega}(t, \tilde{H}_{\alpha_{\omega}}, 1(t), \tilde{H}_{\alpha_{\omega}, 2}(t), u_{\omega}(t))dt \\
+ & d\alpha_{\omega}(t) = - \sum_{i=1}^m H_1^i \alpha_{\omega}(t)dM_{\omega}^{1,i}(t) \\
+ & \frac{1}{2} \sum_{i,k=1}^m H_3^i \alpha_{\omega}(t)d[M_{\omega}^{1,i}, M_{\omega}^{1,k}] + u_{\omega}(t)\alpha_{\omega}(t)dt.
\end{align*}
\]

Here the operation \( \circ \) is given by formula (3.1), \( \tilde{H}_{\alpha_{\omega}, j}(t) \in \mathcal{M}_{m_j}, \ j = 1, 2, \) is the \( m_j \times m_j \) matrix whose entries (which are in \( T_e G \)) are given by

\[
(\tilde{H}_{\alpha_{\omega}, j}(t))_{ik} = (\nabla_{H_j^i} H_j^k) \frac{d[M_{\omega}^{1,i}, M_{\omega}^{1,k}]_t}{dt}, \quad 1 \leq i, k \leq m_j,
\]

the operator \( K_{\omega} : [0, T] \times \mathcal{M} \times \mathcal{M} \times T_e G \rightarrow T_e^* G \) is defined for every \( \omega \in \Omega \) by

\[
(K_{\omega}(t, A_1, A_2, u), v)_{T_e G} = - \sum_{j=1}^2 \left\{ \frac{\delta p}{\delta A_j} (A_1, A_2, u), B_{\omega,j}(t, v) \right\}_{\mathcal{M}_{m_j}}.
\]

\[\forall t \in [0, T],\]

where \( A_j \in \mathcal{M}_{m_j}, \ j = 1, 2, u, v \in T_e G, \) and \( B_{\omega,j}(t, v) \in \mathcal{M}_{m_j} \) is the \( m_j \times m_j \) matrix whose entries are

\[
(\tilde{B}_{\omega,j}(t, v))_{ik} := (\nabla_{H_j^i} (ad_v H_j^k) + \nabla_{ad_v H_j^k} H_j^i) \frac{d[M_{\omega}^{1,i}, M_{\omega}^{1,k}]_t}{dt}, \quad 1 \leq i, k \leq m_j,
\]

\[t \in [0, T].\]
(ii) The first equation in (3.12) is equivalent to the stochastic dissipative Euler–Poincaré variational principle

\[
\frac{d}{d\varepsilon}\Bigg|_{\varepsilon=0} \left( \int_0^T l_\omega(t, u_{\omega,\varepsilon}(t), \alpha_{\omega,\varepsilon}(t)) dt + \int_0^T p(\tilde{H}_{\omega,1,\varepsilon}(t), \tilde{H}_{\omega,2,\varepsilon}(t), u_{\omega,\varepsilon}(t)) dt \right)
+ \int_0^T \langle q(t, u_{\omega,\varepsilon}(t), \alpha_{\omega,\varepsilon}(t)), \beta_{\omega,\varepsilon}(t) \rangle \\
- \sum_{i=1}^{m_1} \left( \int_0^T \langle q(t, u_{\omega,\varepsilon}(t), \alpha_{\omega,\varepsilon}(t)), H^1_i \rangle dM^1_{\omega,i}(t) \right) = 0
\]

(3.16)

on $T_eG \times U^*$, for variations of the form

\[
\begin{aligned}
\left. \frac{du_{\omega,\varepsilon}(t)}{d\varepsilon} \right|_{\varepsilon=0} &= \tilde{v}_\omega(t) + \text{ad}_{u_{\omega}(t)} v_{\omega}(t), \\
\left. \frac{d\alpha_{\omega,\varepsilon}(t)}{d\varepsilon} \right|_{\varepsilon=0} &= -v_{\omega}(t) \alpha_{\omega}(t), \\
\left. \frac{d\tilde{H}_{\omega,j,\varepsilon}(t)}{d\varepsilon} \right|_{\varepsilon=0} &= -B_{\omega,j}(t, v_{\omega}(t)), \ j = 1, 2, \\
\left. \frac{d\beta_{\omega,\varepsilon}(t)}{d\varepsilon} \right|_{\varepsilon=0} &= -\sum_{i=1}^{m_1} \int_0^t \text{ad}_{v_{\omega}(s)} H^1_i dM^1_{\omega,i}(s), \\
u_{\omega,0}(t) &= u_{\omega}(t), \alpha_{\omega,0}(t) = \alpha_{\omega}(t), \\
\beta_{\omega,0}(t) &= \sum_{i=1}^{m_1} \int_0^t H^1_i dM^1_{\omega,i}(s), \tilde{H}_{\omega,j,0}(t) = \tilde{H}_{\omega,j}(t),
\end{aligned}
\]

(3.17)

where $v_{\omega}(t)$ is an $\mathcal{P}_t$-adapted process such that $v_{\omega} \in C^1([0, T]; T_eG)$ and $v_{\omega}(0) = 0, v_{\omega}(T) = 0$ a.s. (Note that this variational principle is constrained and stochastic.)

Proof (i) Step 1. We start by proving that $\alpha_{\omega}(t) = g_3^3(t)^{-1}\alpha_0$ satisfies the second equation in (3.12).

Since $d\left( (g^3(t))^{-1} g^3(t) \right) = 0$, we have

\[
d\left( g_3^3(t) \right)^{-1} = -T_e R_{(g_3^3(t))^{-1}} T_{g_3^3(t)} L_{(g_3^3(t))^{-1}} dg_3^3(t),
\]

so replacing $dg_3^3(t)$ by its expression in (3.11) we obtain,

\[
\begin{aligned}
\left. \frac{dg_3^3(t)}{dt} \right|_{t=0} &= \left. \frac{dg_3^3(t)}{dt} \right|_{t=0} - \sum_{i=1}^{m_3} H^3_i \delta M^3_{\omega,i}(t) - u_{\omega}(t) dt, \\
g_3^3(0)^{-1} &= e.
\end{aligned}
\]

(3.18)
We now derive the stochastic differential equation satisfied by $\alpha_\omega(t)$:

$$
\begin{align*}
d\alpha_\omega(t) &= d \left( \int g_\omega^3(t)^{-1} \alpha_0 \right) = \left[ -T g_\omega^3(t) L g_\omega^3(t)^{-1} d g_\omega^3(t) \right] g_\omega^3(t)^{-1} \alpha_0 \\
&= - \sum_{i=1}^{m_3} H_i^3 \left( g_\omega^3(t)^{-1} \alpha_0 \right) \delta M_i^3(t) - u_\omega(t) \left( g_\omega^3(t)^{-1} \alpha_0 \right) dt,
\end{align*}
$$

(3.19)

Since we assume $U^*$ to be a finite-dimensional vector space, the difference between the Stratonovich and Itô integrals (see (2.1)) yields

$$
\begin{align*}
\sum_{i=1}^{m_3} \left( H_i^3 \left( g_\omega^3(t)^{-1} \alpha_0 \right) \right) \delta M_i^3(t) \\
= \sum_{i=1}^{m_3} \left( H_i^3 \left( g_\omega^3(t)^{-1} \alpha_0 \right) \right) d M_i^3(t) + \frac{1}{2} d \left[ H_i^3 \left( g_\omega^3(t)^{-1} \alpha_0 \right), M_i^3 \right].
\end{align*}
$$

By the same procedure as in (3.19), the (local) martingale part of $H_i^3(g_\omega^3(\cdot)^{-1} \alpha_0)$ is equal to $- \sum_{k=1}^{m_3} \int_0^t H_k^3 H_i^3(g_\omega^3(t)^{-1} \alpha_0) d M_k^3(t)$. Therefore, by (2.4) and (2.5) we derive

$$
\sum_{i=1}^{m_3} d \left[ H_i^3 \left( g_\omega^3(t)^{-1} \alpha_0 \right), M_i^3 \right]_t = - \sum_{i,k=1}^{m_3} H_k^3 (H_i^3(g_\omega^3(t)^{-1} \alpha_0)) d \left[ M_i^3, M_k^3 \right]_t.
$$

Using (3.19) we have,

$$
\begin{align*}
d\alpha_\omega(t) &= - \sum_{i=1}^{m_3} H_i^3 \alpha_\omega(t) d M_i^3(t) \\
&\quad + \frac{1}{2} \sum_{i,k=1}^{m_3} H_k^3 \left( H_i^3 \alpha_\omega(t) \right) d \left[ M_i^3, M_k^3 \right]_t - u_\omega(t) \alpha_\omega(t) dt,
\end{align*}
$$

(3.20)

which is the second equation in (3.12).

**Step 2.** Now we prove the first equation in (3.12). Recall from (3.5) that, for every $P_\varepsilon$-adapted process $g_\omega$ satisfying $g_\omega(\cdot) \in C^1([0, 1]; T \varepsilon G)$ and $g_\omega(0) = g_\omega(T) = 0$ a.s., $e_{\omega, \varepsilon, \tilde{g}}(\cdot) \in C^1([0, T]; G)$ a.s. uniquely solves the following (random) ordinary differential equation on $G$

$$
\frac{d}{dt} e_{\omega, \varepsilon, \tilde{g}}(t) = \varepsilon T \tilde{g} L e_{\omega, \varepsilon, \tilde{g}}(t) \dot{\tilde{g}}(t), \quad e_{\omega, \varepsilon, \tilde{g}}(0) = e.
$$

By Arnaudon et al. (2014, Lemma 3.1), we have

$$
\begin{align*}
\frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} e_{\omega, \varepsilon, \tilde{g}}(t) &= g_\omega(t), \\
\frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} e_{\omega, \varepsilon, \tilde{g}}(t)^{-1} &= -g_\omega(t), \text{ a.s.}
\end{align*}
$$

(3.21)
Since this computation is important in the proof, for the sake of completeness, we recall it below. Denoting by \( \frac{D}{Dt} \) and \( \frac{D}{D\varepsilon} \) the covariant derivatives, induced by \( \nabla \) on \( G \), along curves parametrized by \( t \) and \( \varepsilon \), respectively. Since the torsion vanishes, Gauss’ Lemma yields

\[
\frac{D}{Dt} \frac{d}{d\varepsilon} e_{\omega,\varepsilon,g}(t) = \frac{D}{D\varepsilon} \frac{d}{dt} e_{\omega,\varepsilon,g}(t) = \frac{D}{D\varepsilon} \left( \varepsilon T_{\varepsilon} L_{e_{\omega,\varepsilon,g}(t)} \dot{g}_0(t) \right)
\]

\[
= T_{\varepsilon} L_{e_{\omega,\varepsilon,g}(t)} \dot{g}_0(t) + \varepsilon \frac{D}{D\varepsilon} \left( T_{\varepsilon} L_{e_{\omega,\varepsilon,g}(t)} \dot{g}_0(t) \right)
\]

(3.22)

Taking \( \varepsilon = 0 \) and since \( e_{\omega,0,g}(t) = e \) for all \( t \), we obtain \( \frac{D}{Dt} \frac{d}{d\varepsilon} e_{\omega,0,g}(t) = \dot{g}_0(t) \). Moreover \( t \mapsto \frac{d}{d\varepsilon} |_{\varepsilon=0} e_{\omega,\varepsilon,g}(t) \) is a curve in the vector space \( T_{e} G \) and hence \( \frac{d}{dt} \frac{d}{d\varepsilon} |_{\varepsilon=0} e_{\omega,\varepsilon,g}(t) = \dot{g}_0(t) \). The first equality in (3.21) is then a consequence of \( \dot{g}_0(0) = 0 \) and \( \frac{d}{d\varepsilon} |_{\varepsilon=0} e_{\omega,\varepsilon,g}(0) = 0 \). Finally, since

\[
\frac{d}{d\varepsilon} e_{\omega,\varepsilon,g}(t)^{-1} = -T_{\varepsilon} R^{-1}_{\omega,\varepsilon,g}(t) T_{e_{\omega,\varepsilon,g}(t)} L^{-1}_{e_{\omega,\varepsilon,g}(t)} \frac{d}{d\varepsilon} e_{\omega,\varepsilon,g}(t),
\]

the second equality in (3.21) follows from the first.

Note that due to (3.21) we have \( \frac{d}{d\varepsilon} |_{\varepsilon=0} \text{Ad}_{e_{\omega,\varepsilon,g}(t)^{-1}} v = -\text{ad}_{g_0(t)} v, \quad v \in T_e G \). Combining this with (3.8) we have

\[
\frac{d}{d\varepsilon} |_{\varepsilon=0} \left( T_{e_{\omega,\varepsilon,g}(t)} L_{e_{\omega,\varepsilon,g}(t)}^{-1} \frac{d}{dt} g_{\omega,\varepsilon,g}(t) \right) = \dot{g}_0(t) + \text{ad}_{g_0(t)} g_0(t)
\]

(3.23)

\[
\frac{d}{d\varepsilon} |_{\varepsilon=0} \left( T_{e_{\omega,\varepsilon,g}(t)} L_{e_{\omega,\varepsilon,g}(t)}^{-1} d^\Delta g_{\omega,\varepsilon,g}(t) \right) = -\sum_{i=1}^{m_1} \text{ad}_{g_0(t)} H_i^1 d \text{M}_i^1 \omega(t)
\]

(3.24)

\[
\frac{d}{d\varepsilon} |_{\varepsilon=0} \left( T_{e_{\omega,\varepsilon,g}(t)} L_{e_{\omega,\varepsilon,g}(t)}^{-1} D_{\nabla, (H_{\omega, j}^1, M_{\omega}^{j, i})_{i=1}^{m_j}} g_{\omega,\varepsilon,g}(t) \right)_{km} = -(\nabla_{\text{ad}_{g_0(t)}} H_{k}^j \omega(t) + \nabla_{H_{k}^j} (\text{ad}_{g_0(t)} H_{k}^j)) \frac{d}{dt} [M_{\omega}^{j, k}, M_{\omega}^{j, m}]_{t}
\]

(3.25)

where \( g_{\omega,\varepsilon,g}(t) = g_{\omega}(t)e_{\omega,\varepsilon,g}(t), \quad H_{\omega, j}^i(t) := \text{Ad}_{e_{\omega,\varepsilon,g}(t)} H_{j}^i, \quad B_{\omega, j}(t, \cdot) \) is defined by (3.15) and we have applied the property \( [H_{\omega, j}^i, M_{\omega}^{j, k}] = 0 \) since \( H_{\omega, j}^i(\cdot) \) is a process with bounded variation.
Since \( g^j_{\omega, e, g}(t) := g^j_{\omega}(t)e_{\omega, e, g}(t) \) and \( e_{\omega, 0, g}(t) = e \) for all \( t \in [0, T] \), we conclude \( g^j_{\omega, 0, g}(t) = g^j_{\omega}(t) \), for all \( t \in [0, T] \), \( j = 1, 2, 3 \). Therefore,

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} g^3_{\omega, e, g}(t)^{-1} \alpha_0 = -g^3_{\omega}(t)^{-1} \left( \frac{d}{d\epsilon} \bigg|_{\epsilon=0} g^3_{\omega, e, g}(t) \right) g^3_{\omega}(t)^{-1} \alpha_0 \\
= - \left( \frac{d}{d\epsilon} \bigg|_{\epsilon=0} e_{\omega, e, g}(t) \right) g^2_{\omega}(t)^{-1} \alpha_0 \overset{(3.21)}{=} -\gamma_{\omega}(t) g^2_{\omega}(t)^{-1} \alpha_0 \\
= -\gamma_{\omega}(t) \alpha_{\omega}(t).
\]

(3.26)

Based on (3.3), (3.23)–(3.26) and noting that \( d^1 g^1_{\omega, e, g}(t) \bigg|_{\epsilon=0} = \sum_{i=1}^{m_1} H^1_i dM^1,i_{\omega}(t) \), we have

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} J_{\nabla,(H^1_i,M^1_i)_{i=1}^{m_1}} \left( \left( g^1_{\omega, e, g}, H^1_{\omega, i} \right)_{i=1}^{m_1}, \left( g^2_{\omega, e, g}, H^2_{\omega, i} \right)_{i=1}^{m_2}, g^3_{\omega, e, g} \right) \\
= \int_0^T \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( T_{g^1_{\omega, e, g}(t)} L_{g^1_{\omega, e, g}(t)} - 1 \frac{d^1 g^1_{\omega, e, g}(t)}{dt} \right) \right) dt \\
+ \int_0^T \left( \frac{\delta p}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( T_{g^1_{\omega, e, g}(t)} L_{g^1_{\omega, e, g}(t)} - 1 \frac{d^1 g^1_{\omega, e, g}(t)}{dt} \right) \right) dt \\
+ \int_0^T \left( \frac{\delta l_{\omega}}{\delta \alpha} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( T_{g^1_{\omega, e, g}(t)} L_{g^1_{\omega, e, g}(t)} - 1 \frac{d^1 g^1_{\omega, e, g}(t)}{dt} \right) \right) dt \\
= \int_0^T \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), \tilde{g}_{\omega}(t) + ad_{u_{\omega}(t)} \tilde{g}_{\omega}(t) \right) dt \\
- \int_0^T \left( \frac{\delta g_{\omega}(t) \alpha_{\omega}(t), \delta l_{\omega}}{\delta \alpha} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) dt \\
+ \int_0^T \left( \frac{\delta p}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), \tilde{g}_{\omega}(t) + ad_{u_{\omega}(t)} \tilde{g}_{\omega}(t) \right) dt \\
- \sum_{j=1}^{m_1} \int_0^T \left( \frac{\delta p}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right), B_{\omega, j}(t, \tilde{g}_{\omega}(t)) \right) dt \\
- \sum_{i=1}^{m_1} \int_0^T \left( q \left( t, u_{\omega}(t), \alpha_{\omega}(t) \right), ad_{\gamma_{\omega}(t)} H^1_i dM^1,i_{\omega}(t) \right) dt \\
= \int_0^T \left( -d \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \\
+ ad_{u_{\omega}(t)} \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \right) dt \\
+ ad_{u_{\omega}(t)} \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \right) dt \\
+ ad_{u_{\omega}(t)} \left( \frac{\delta l_{\omega}}{\delta u} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \right) \right) dt
Remark 3.6 As we shall see in Sect. 5, the conclusion of Theorem 3.5 still holds when

\[ G \] is the diffeomorphism group of a torus and the action of \( G \) on \( U^* \) is the pull-back map.

If \( G \) is a finite-dimensional Lie group and \( U \) is a finite-dimensional vector space, then (3.12) is a actually an SDE. However, when \( G \) is the diffeomorphism group, as illustrated in Sect. 5, (3.12) is a system of SPDEs.

Remark 3.7 The variation (3.17) is a stochastic version of Lin’s constrained variational principle (see, e.g., Holm et al. 1998, Theorem 1.2). In fact, if we take \( p = q = 0 \) and \( H^1_i = 0 \), (3.17) is the deterministic constrained variational principle in Holm et al. (1998, Theorem 1.2).

Remark 3.8 For simplicity, in Theorem 3.5 we assume that the contraction force \( p \) and stochastic force \( q \) are independent of the advection space \( U^* \). In fact, following the same procedure in the proof of Theorem 3.5, we could also characterize the critical points of an action functional even if \( p \) and \( q \) depend on \( U^* \).
3.3 Stochastic Variational Principle for Ordinary Differential Equations

If we take $q = 0$, and take the expectation in (3.4), through the stochastic reduction procedure in Theorem 3.5, we obtain a system of ODEs for the drift of the underlying stochastic paths (not the SDE in (3.12)).

Let $l$, $p$ be the same terms in Theorem 3.5 such that $l : [0, T] \times T_e G \times U^* \to \mathbb{R}$ is non-random. We define the action $\tilde{J}^\nabla$ on $\mathcal{F}(G) \times \mathcal{F}(G) \times \mathcal{F}(G)$, a random variable, by

$$
\tilde{J}^\nabla \left( \left( g^1_{\omega}, w^1_{\omega}, M^1_{\omega} \right)_{i=1}^{m_1}, \left( g^2_{\omega}, w^2_{\omega}, M^2_{\omega} \right)_{i=1}^{m_2}, g^3_{\omega} \right) := \int_0^T \left( \frac{\partial g^1_{\omega}(t)}{\partial t}, \tilde{\alpha}(t) \right) dt + \int_0^T p \left( \frac{\partial g^1_{\omega}(t)}{\partial t} \right) \frac{\partial^2 \nabla_{\omega} \left( w^1_{\omega}, M^1_{\omega} \right)_{i=1}^{m_1} g^1_{\omega}(t)}{\partial t}, T g^2_{\omega}(t) L g^3_{\omega}(t) \right) dt,
$$

where $g^j_{\omega}$, $j = 1, 2, 3$ are $G$-valued semimartingales with form (2.8) and $\tilde{\alpha}(t) := \mathbb{E}[\alpha_{\omega}(t)] = \mathbb{E}[g^3_{\omega}(t)^{-1} \alpha_0] \in U^*$ is non-random. The action functional $\tilde{J}^\nabla$ can be viewed as a deterministic counterpart of (3.3), where $q = 0$, $\alpha_{\omega}(t)$ is replaced by $\tilde{\alpha}(t)$, and there is no external stochastic force term (stochastic integral term).

Suppose also that deformations are of the form (3.10) with $g_{\omega}$ non-random (we write $g$ for $g_{\omega}$ in this section); then we can also define the critical point of $\tilde{J}^\nabla$ in the same way as in (3.9). To further simplify notation, we drop the index $\omega$ on some of the variables in the statement of the theorem below. For example, we write $u$ for $u_{\omega}$ (since $u$ is non-random) and define $[M^{j,i}, M^{j,k}]_t := [M^1_{\omega}, M^2_{\omega}]_t$ in order to emphasize that the quadratic variation is non-random (if this is the case).

**Theorem 3.9 (Stochastic reduction with deterministic drift and deformations)** Let the semimartingales $g^j_{\omega}(\cdot)$, $j = 1, 2, 3$, have the form (3.11) with $u \in C^1([0, T]; T_e G)$ and suppose that $[M^{j,i}, M^{j,k}]_t$, $1 \leq j \leq 3$, $1 \leq i, k \leq m_j$, are non-random.

(i) Then $\left( \left( g^1_{\omega}, H^1_{\omega} \right)_{i=1}^{m_1}, \left( g^2_{\omega}, H^2_{\omega} \right)_{i=1}^{m_2}, g^3_{\omega} \right)$ is a critical point of $\tilde{J}^\nabla$ if and only if $u(t)$ coupled with $\tilde{\alpha}(t)$ satisfies the following (ordinary differential) equation

$$
d \left( \frac{\delta l}{\delta u} \left( t, u(t), \tilde{\alpha}(t) \right) + \frac{\delta p}{\delta u} \left( \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) \right) = \text{ad}_u^w \left( \frac{\delta l}{\delta u} \left( t, u(t), \tilde{\alpha}(t) \right) \right) dt + \text{ad}_u^w \left( \frac{\delta p}{\delta u} \left( \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) \right) dt + \left( \frac{\delta l}{\delta \omega} \left( t, u(t), \tilde{\alpha}(t) \right) \right) \tilde{\alpha}(t) dt + K \left( t, \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) dt,
$$

$$
d \tilde{\alpha}(t) = \frac{1}{2} \sum_{i,k=1}^{m_3} H^3_k \left( H^3_i \tilde{\alpha}(t) \right) d \left[ [M^{3,i}, M^{3,k}]_t \right] - u(t) \tilde{\alpha}(t) dt,
$$

(3.28)
where \( \tilde{H}_1(t) \in \mathcal{M}_{m_1}, \tilde{H}_2(t) \in \mathcal{M}_{m_2}, \), \( K \) are the same terms as in Theorem 3.5 (except that we omit the subscript \( \omega \) in order to emphasize that these terms are non-random here).

(ii) The first equation in (3.28) is equivalent to the following stochastic variational principle

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left( \int_0^T l(t, u_\varepsilon(t), \tilde{\alpha}_\varepsilon(t))dt + \int_0^T p(\tilde{H}_1,\varepsilon(t), \tilde{H}_2,\varepsilon(t), u_\varepsilon(t))dt \right) = 0
\]

(3.29)
on \text{T}_eG \times U^* for variations of the form

\[
\begin{aligned}
\frac{du_\varepsilon(t)}{d\varepsilon} \bigg|_{\varepsilon=0} &= \hat{v}(t) + \text{ad}_u(t)v(t), \\
\frac{d\tilde{\alpha}_\varepsilon(t)}{d\varepsilon} \bigg|_{\varepsilon=0} &= -v(t)\tilde{\alpha}(t), \\
\frac{d\tilde{H}_j,\varepsilon(t)}{d\varepsilon} \bigg|_{\varepsilon=0} &= -B_j(t,v(t)), \ j = 1, 2, \\
u_0(t) &= u(t), \ \tilde{\alpha}_0(t) = \tilde{\alpha}(t), \ \tilde{H}_j,0(t) = \tilde{H}_j(t).
\end{aligned}
\]

where \( v \in C^1([0, T]; \text{T}_eG) \) with \( v(0) = 0, v(T) = 0 \) is non-random and \( B_j(t,v) \) is defined by (3.15). Note that the variational principle (3.29) is constrained; it is stochastic, even though \( l \) and \( p \) are deterministic, because in the variational principle the action functional is taken for stochastic Lagrangian paths (in fact, the contraction matrix for stochastic Lagrangian paths is used).

Proof (i) Since \( H^3_j, u \) are non-random and the action of \( \text{T}_eG \) on \( U^* \) is linear, we have

\[
\mathbb{E} \left[ H^3_j \left( H^3_i \alpha_\omega(t) \right) \right] = H^3_j \left( H^3_i \left( \mathbb{E} \left[ \alpha_\omega(t) \right] \right) \right) = H^3_j \left( \tilde{H}^3_i \left( \tilde{\alpha}(t) \right) \right),
\]

\[
\mathbb{E} \left[ u(t) \alpha_\omega(t) \right] = u(t) \mathbb{E} \left[ \alpha_\omega(t) \right] = u(t)\tilde{\alpha}(t)
\]

Then taking the expectation on both side of (3.20), we arrive at the second equation of (3.28).

Note that \( e_{\omega,\varepsilon,g} \) is non-random since \( g_\omega \) is non-random; from (3.26) we obtain

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mathbb{E} \left[ g_{3,\omega,\varepsilon,g}^{-1}(t)^{-1} \alpha_0 \right] = \mathbb{E} \left[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} g_{3,\omega,\varepsilon,g}^{-1}(t)^{-1} \alpha_0 \right] = -\mathbb{E} \left[ g(t) g_{3,\omega}^{-1}(t)^{-1} \alpha_0 \right] = -g(t)\tilde{\alpha}(t).
\]

Based on this and following the same computation as in (3.27) (note that here \( q = 0 \)), we get the first equation of (3.28).

(ii) By the same steps of (3.17), we derive the variations used in (3.29).
Remark 3.10 As we will see in Sect. 5, for the case that $G$ is a diffeomorphism group, the system (3.28) is a PDE with viscosity term. ♦

Remark 3.11 If we do not assume that $[M^j_i, M^k_j]$, $1 \leq j \leq 2$, $1 \leq i, k \leq m_j$, is non-random, then equation (3.28) may be a random ordinary differential equation (which is not a stochastic differential equation as in the stochastic equation (3.12)). ♦

If the covariation terms $[M^j_i, M^k_j]$, are non-random, then the variational principle (3.29) yields an ODE, namely the first equation in (3.28), as we just proved in Theorem 3.9. If, however, $[M^j_i, M^k_j]$ are random, the variational principle (3.29) still yields the first equation in (3.28), which is now an ODE but with random coefficients, where the random term only comes from $[M^j_i, M^k_j]$, this is not a stochastic differential equation because it misses the stochastic force function $q$ which is responsible for the stochastic character of the first equation in (3.12).

3.4 Right Invariant Version

Due to relative sign changes in the equations of motion and the dissipative constrained variational principle, with a view to applications for the spatial representation in continuum mechanics, we give below the right invariant version of Theorem 3.5.

Suppose that $G$ acts on the right on a vector space $U$ (we will write the action of $g \in G$ on $u \in U$ by $ug$ and similarly for the induced infinitesimal $g$-representation).

Thus, let $g_\omega(t)$ be a $G$-valued semimartingale of the form (right invariant version of (2.8))

$$
\frac{dg_\omega(t)}{dt} = T_e R_{g_\omega(t)} \left( \sum_{i=1}^{m} w^i_\omega(t) \delta M^i_\omega(t) + v_\omega(t) dt \right),
$$

(3.30)

where $T_e R_{g_\omega(t)}$ denotes the differential of the right translation $R_{g_\omega(t)}$ at the point $e$. For a fixed right invariant connection $\nabla$ on $G$ and $(w^i, M^i_\omega)_{i=1}^{m}$, where $w^i_\omega$ and $M^i_\omega$ are $\mathcal{P}_t$-adapted processes and $M^i_\omega$ are real valued martingales with continuous sample paths for all $i = 1, \ldots, m$, we define

$$
\left( \frac{D^{\nabla}(w^i_\omega, M^i_\omega)_{i=1}^{m}}{dt} g_\omega(t) \right)_{ij} := T_e R_{g_\omega(t)} \left( \nabla w^i_\omega(t) w^j_\omega(t) \frac{d[M^i_\omega, M^j_\omega]}{dt} + \frac{d[w^i_\omega, M^j_\omega]}{dt} \delta_{ij} \right), \quad 1 \leq i, j \leq m.
$$

The terms $\frac{D}{dt} g_\omega(t)$ and $d^Ag_\omega(t)$ are defined similarly as in the left invariant case (see the defining formulas (2.15)).

With $l : \Omega \times [0, T] \times T_e G \times U^* \to \mathbb{R}$, $p : \mathcal{M} \times \mathcal{M} \times T_e G \to \mathbb{R}$, $q : [0, T] \times T_e G \times U^* \to T_e^* G$, $V_i \in T_e G$, $N^i_\omega$, $1 \leq i \leq k$, satisfying the same conditions as those in Sects. 3.1 and 3.2 (for the left invariant case), the action functional $J^{\nabla}(V_i, N^i_\omega)_{i=1}^{k}$ is defined for the right invariant case by

\[\text{Springer}\]
Consider (right invariant) critical points of $J_{g_j}$ with $g_j$ of (3.7), to obtain Theorem 3.12 and 3.13 below, so we omit the proof here.

Then we have (the right invariant version of (3.7))

$\frac{d}{dt} g_j(t) = \alpha(t)(t, T_{g_j(t)} R_{g_j(t)}^{-1} \frac{\mathcal{D} V_j}{g_j(t)} + \alpha(t)(t, T_{g_j(t)} R_{g_j(t)}^{-1} \frac{\mathcal{D} V_j}{g_j(t)}).

(3.31)$

and

$\alpha(t) := \alpha g_j(t)^{-1}.

(3.32)$

As for the left invariant case, for every (random) $\mathcal{P}_1$-adapted process $g_\omega(\cdot)$ such that $g_\omega \in C^1([0, T]; T_e G)$, $g_\omega(0) = g_\omega(T) = 0$ a.s., and $\varepsilon \in [0, 1)$, let $e_{\omega, \varepsilon, g}(\cdot)$ be the unique solution of (the right invariant version of (3.5))

$\frac{d}{dt} e_{\omega, \varepsilon, g}(t) = \varepsilon T_e R_{e_{\omega, \varepsilon, g}(t)} g_\omega(t),

e_{\omega, \varepsilon, g}(0) = e.

(3.33)$

Define the deformation of $g_j(\cdot)$ by (the right invariant version of (3.6))

$g_{j, \varepsilon, g}(t) := e_{\omega, \varepsilon, g}(t) g_j(t), j = 1, 2, 3.

(3.34)$

Then we have (the right invariant version of (3.7))

$\frac{d}{dt} g_{\omega, \varepsilon, g}(t) = T_e R_{g_{\omega, \varepsilon, g}(t)}

\left( \sum_{i=1}^{m} \text{Ad}_{\omega, \varepsilon, g}(t) w_i(t) \delta M_i(t) + \text{Ad}_{\omega, \varepsilon, g}(t) v(t) \right)

+ \varepsilon \frac{\mathcal{D} V_i}{g_i(t)}

(3.35)$

Thus, we take $g_{\omega, \varepsilon, g}, \text{Ad}_{\omega, \varepsilon, g}(t) w_i(t, M_j(t))_{i=1}^{m}$ as a deformation of $g_{\omega, \varepsilon, g}(t, M_j(t))_{i=1}^{m}$ with $g_{\omega, \varepsilon, g} \in \mathcal{F}(G)$ having the expression (3.30). With such deformations, we can consider (right invariant) critical points of $J_{V_i(N_{\omega})_{i=1}^{m}}$ as in (3.9).

In the procedure leading to Theorem 3.5 and its proof, we can interchange all left actions and left translation operators by their right counterparts and use (3.35) instead of (3.7), to obtain Theorem 3.12 and 3.13 below, so we omit the proof here.
Theorem 3.12 Assume that $G$ is a finite-dimensional Lie group endowed with a right invariant linear connection $\nabla$, $H_i^j \in T_e G$, $1 \leq i \leq m_j$, and $M_i^j = \{M_{\omega}(t)^{j, i}\}_{i=1}^{m_j}$, $j = 1, 2, 3$ is an $\mathbb{R}^{m_j}$-valued martingale. Suppose that the semimartingales $g_{\omega}^j(\cdot) \in \mathcal{S}(G)$, $j = 1, 2, 3$, have the following form,

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathrm{d}g_{\omega}^j(t) = T_e R_{g_{\omega}^j(t)} \left( \sum_{i=1}^{m_j} H_i^j \delta M_{\omega}(t)^{j, i} + u_{\omega}(t) \mathrm{d}t \right), \\
g_{\omega}^j(0) = e,
\end{array} \right. \tag{3.36}
\end{align*}
\]

where $u_{\omega}$ is a $\mathcal{P}_1$-adapted, $T_e G$-valued semimartingale.

(i) If $\frac{\mathrm{d}l_i}{\mathrm{d}u}$ is non-random and $l_1(t)$ is $\mathcal{P}_1$-adapted, then \(\left( g_{\omega}^1, H_i^1, M_i^1 \right)_{i=1}^{m_1}, \left( g_{\omega}^2, H_i^2, M_i^2 \right)_{i=1}^{m_2}, g_{\omega}^3 \) is a critical point of $J^V(H_i^1, M_i^1)_{i=1}^{m_1}$ defined by (3.31) if and only if the $\mathcal{P}_1$-adapted, $T_e G$-valued semimartingale $u_{\omega}(\cdot)$ coupled with the $\mathcal{P}_1$-adapted, $U^*\omega$-valued semimartingale $\alpha_{\omega}(\cdot)$ satisfies the following equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathrm{d}\left( \frac{\mathrm{d}l_i}{\mathrm{d}u}(t, u_{\omega}(t), \alpha_{\omega}(t)) + \frac{\partial}{\partial t} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \\
\quad = - \sum_{i=1}^{m_1} \mathrm{ad}^\ast_{H_i} q(t, u_{\omega}(t), \alpha_{\omega}(t)) \mathrm{d}M_{\omega}^{\ast, i}(t) - \mathrm{ad}^\ast_{u_{\omega}(t)} \left( \frac{\partial}{\partial t} \left( t, u_{\omega}(t), \alpha_{\omega}(t) \right) \right) \mathrm{d}t \\
\quad - \mathrm{ad}^\ast_{u_{\omega}(t)} \left( \frac{\partial}{\partial t} \left( \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t) \right) \right) \mathrm{d}t + \left( \frac{\partial}{\partial t} (t, u_{\omega}(t), \alpha_{\omega}(t)) \right) \circ \alpha_{\omega}(t) \mathrm{d}t \\
\quad - K_{\omega}(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)) \mathrm{d}t, \\
\mathrm{d}\alpha_{\omega}(t) = - \sum_{i=1}^{m_3} \alpha_{\omega}(t) H_i^3 \mathrm{d}M_{\omega}^{\ast, i}(t) + \frac{1}{2} \sum_{i, k=1}^{m_1} \alpha_{\omega}(t) H_i^3 \mathrm{d}[M_{\omega}^{\ast, i}, M_{\omega}^{\ast, k}] - \alpha_{\omega}(t) u_{\omega}(t) \mathrm{d}t,
\end{array} \right. \tag{3.37}
\end{align*}
\]

where $\tilde{H}_{\omega, 1}(t)$, $\tilde{H}_{\omega, 2}(t)$, and $K_{\omega}$ are defined by (3.13), and (3.14) respectively.

(ii) The first equation in (3.37) is equivalent to the stochastic dissipative Euler–Poincaré variational principle

\[
\begin{align*}
\frac{\mathrm{d}}{\mathrm{d}t} & \left( \int_0^T l_{\omega}(t, u_{\omega, \epsilon}(t), \alpha_{\omega, \epsilon}(t)) \mathrm{d}t + \int_0^T p(\tilde{H}_{\omega, 1, \epsilon}(t), \tilde{H}_{\omega, 2, \epsilon}(t), u_{\omega, \epsilon}(t)) \mathrm{d}t \\
& + \int_0^T \langle q(t, u_{\omega, \epsilon}(t), \alpha_{\omega, \epsilon}(t)), d\beta_{\omega, \epsilon}(t) \rangle \\
& - \sum_{i=1}^{m_1} \left( \int_0^T \langle q(t, u_{\omega, \epsilon}(t), \alpha_{\omega, \epsilon}(t)), H_i^1 \rangle \mathrm{d}M_{\omega}^{1, i}(t) \right) \right) = 0 \tag{3.38}
\end{align*}
\]
on $T_eG \times U^*$, for variations of the form

$$
\begin{align*}
\frac{du_{\omega, \epsilon}(t)}{d\epsilon}
|_{\epsilon=0} & = v_{\omega}(t) - ad_{u_{\omega}(t)}v_{\omega}(t), \\
\frac{d\alpha_{\omega, \epsilon}(t)}{d\epsilon}
|_{\epsilon=0} & = -\alpha_{\omega}(t)v_{\omega}(t), \\
\frac{d\tilde{H}_{\omega, j, \epsilon}(t)}{d\epsilon}
|_{\epsilon=0} & = B_{\omega, j}(t, v_{\omega}(t)), \ j = 1, 2,
\frac{d\beta_{\omega, \epsilon}(t)}{d\epsilon}
|_{\epsilon=0} & = -\sum_{i=1}^{m_1} \int_0^t \text{ad}_{u_{\omega}(s)}H_i^1 dM_{\omega, i}^1(s), \\
u_{\omega, 0}(t) & = u_{\omega}(t), \alpha_{\omega, 0}(t) = \alpha_{\omega}(t), \beta_{\omega, 0}(t) = \sum_{i=1}^{m_1} \int_0^t H_i^1 dM_{\omega, i}^1(s), \tilde{H}_{\omega, j, 0}(t) = \tilde{H}_{\omega, j}(t),
\end{align*}
$$

(3.39)

where $v_{\omega}(t)$ is an $\mathcal{P}_1$-adapted process such that $v \in C^1([0, T]; T_eG)$ and $v(0) = 0$, $v(T) = 0$ a.s.

The right invariant version for the deterministic action functional in Theorem 3.9 is the following:

$$
\bar{J}^\nabla\left((g_1^1, w_{\omega, 1}^{1, i}, M_{\omega, 1}^{1, i})_{i=1}^{m_1}, (g_2^2, w_{\omega, 2}^{2, i}, M_{\omega, 2}^{2, i})_{i=1}^{m_2}, g_{\omega}^3\right)
= \int_0^T l\left(t, T_{g_{\omega}^1}(t)R_{g_{\omega}^1(t)}\frac{Dg_{\omega}^1(t)}{dt}, \tilde{\alpha}(t)\right) dt
+ \int_0^T p\left(T_{g_{\omega}^1(t)}R_{g_{\omega}^1(t)}\frac{Dg_{\omega}^1(t)}{dt}, T_{g_{\omega}^2(t)}R_{g_{\omega}^2(t)}\frac{Dg_{\omega}^2(t)}{dt}, T_{g_{\omega}^3(t)}R_{g_{\omega}^3(t)}\frac{Dg_{\omega}^3(t)}{dt}\right) dt.
$$

Here $l$ is non-random and $\tilde{\alpha}(t) := \mathbb{E}[\alpha_{\omega}(t)] \in U^*$ and $\alpha_{\omega}(t) := \alpha_{0}g_{\omega}^3(t)^{-1}$.

**Theorem 3.13** Suppose that the semimartingales $g_{\omega}^j(\cdot), \ j = 1, 2, 3$, have the form (3.36) with $u \in C^1([0, T]; T_eG)$ and suppose that $[M_{\omega, i}^{j, i}, M_{\omega, i}^{j, k}]_t$, $1 \leq j \leq 3$, $1 \leq i, k \leq m_j$, are non-random (we write $u$, $[M_{\omega, i}^{j, 1}, M_{\omega, i}^{j, k}]_t$ for $u_{\omega}$ and $[M_{\omega, i}^{j, 1}, M_{\omega, i}^{j, k}]_t$ in this theorem to emphasize that they are non-random). Consider deformations of the form (3.34) with $g$ non-random (we write $g$ for $g_{\omega}$ in this theorem because it is non-random).

(i) Then $\left((g_1^1, h_1^1, M_{\omega, 1}^{1, i})_{i=1}^{m_1}, (g_2^2, h_2^2, M_{\omega, 2}^{2, i})_{i=1}^{m_2}, g_{\omega}^3\right)$ is a critical point of $\bar{J}^\nabla$ if and only if $u(t)$ coupled with $\tilde{\alpha}(t)$ satisfies the following system of ordinary differential
equations

\[
\begin{aligned}
&\frac{d}{dt} \left( \frac{\delta L}{\delta u} (t, u(t), \bar{a}(t)) + \frac{\delta p}{\delta u} \left( \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) \right) \\
&= -\text{ad}^*_{u(t)} \left( \frac{\delta L}{\delta u} (t, u(t), \bar{a}(t)) \right) \, dt - \text{ad}^*_{u(t)} \left( \frac{\delta p}{\delta u} \left( \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) \right) \, dt \\
&\quad + \left( \frac{\delta L}{\delta a} (t, u(t), \bar{a}(t)) \right) \circ \bar{a}(t) \, dt - K \left( t, \tilde{H}_1(t), \tilde{H}_2(t), u(t) \right) \, dt, \\
\end{aligned}
\]

(3.40)

where \( \tilde{H}_1(t), \tilde{H}_2(t) \), \( \circ, K \) are the same terms as in Theorem 3.9.

(ii) The first equation in (3.40) is equivalent to the following stochastic variational principle

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} \left( \int_0^T l(t, u_\varepsilon(t), \tilde{a}_\varepsilon(t)) \, dt + \int_0^T p(\tilde{H}_{1,\varepsilon}(t), \tilde{H}_{2,\varepsilon}(t), u_\varepsilon(t)) \, dt \right) = 0
\]

(3.41)

on \( T_eG \times U^* \) for variations of the form

\[
\begin{aligned}
\frac{du_\varepsilon(t)}{d\varepsilon} \bigg|_{\varepsilon = 0} &= \dot{v}(t) - \text{ad}_{u(t)}v(t), \\
\frac{d\tilde{a}_\varepsilon(t)}{d\varepsilon} \bigg|_{\varepsilon = 0} &= -v(t)\tilde{a}(t), \\
\frac{d\tilde{H}_{j,\varepsilon}(t)}{d\varepsilon} \bigg|_{\varepsilon = 0} &= B_j(t, v(t)), \quad j = 1, 2, \\
u_0(t) = u(t), \quad \tilde{a}_0(t) = \tilde{a}(t), \quad \tilde{H}_{j,0}(t) = \tilde{H}_j(t),
\end{aligned}
\]

where \( v \in C^1([0, T]; T_eG) \) with \( v(0) = 0, v(T) = 0 \) is non-random and \( B_j(t, v) \) is defined by (3.15). Note that the variational principle (3.41) is constrained; it is stochastic, even though \( l \) and \( p \) are deterministic, because in the variational principle the action functional is taken for stochastic Lagrangian paths (in fact, the contraction matrix for stochastic Lagrangian paths is used).

### 4 Stochastic Kelvin–Noether Theorem

In this section we study a (stochastic) version of the Kelvin–Noether Theorem which holds for solutions of stochastic Euler–Poincaré equations with advection terms (see (3.12)).

Let \( G, U^*, l, q \) be as in Sect. 3. Suppose \( \mathcal{C} \) is a manifold and \( G \) acts on the left on \( \mathcal{C} \). Let \( \mathcal{K} : \mathcal{C} \times U^* \to T_e^*G \) be an equivariant map, i.e.,

\[
\left( \mathcal{K} \left( g^{-1}c, g^{-1}\alpha \right), \mu \right) = \left( \mathcal{K}(c, \alpha), \text{Ad}_{g^{-1}}\mu \right),
\]

\[
c \in \mathcal{C}, \quad \alpha \in U^*, \quad g \in G, \quad \mu \in T_e^*G,
\]

(4.1)
where \( \langle \cdot, \cdot \rangle \) denotes the weak pairing between \( T_e^{**} G \) and \( T_e^* G \).

As explained before, we identify the Lie algebra \( \mathfrak{g} \) with the tangent space \( T_e G \) at the unit element. As usual (see, e.g., Holm et al. 1998, Section 4), we define the Kelvin–Noether quantity \( I : C \times T_e G \times U^* \to \mathbb{R} \) by

\[
I(c, u, \alpha) := \left\langle \mathcal{K}(c, \alpha), \frac{\delta l}{\delta u}(u, \alpha) \right\rangle, \quad c \in C, \ u \in T_e G, \ \alpha \in U^*. \quad (4.2)
\]

Now we are ready to state the Kelvin–Noether Theorem for the solutions of (3.12).

**Theorem 4.1** Suppose that the \( \mathcal{P}_t \)-adapted process \( u_\omega(t) \) satisfies the first equation in (3.12) with \( \frac{\delta l}{\delta u}(\tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t), u_\omega(t)) \equiv 0, \ \alpha_\omega(t) = g_\omega^3(t)^{-1}c_0, \) and \( \frac{\delta l}{\delta u} \) non-random. Let the semimartingales \( g_\omega^j(\cdot) \in \mathcal{S}(G), \ j = 1, 2, 3, \) be defined by (3.11) using this \( u_\omega(t) \). For a fixed \( c_0 \in C, \) set \( c_\omega(t) := g_\omega^3(t)^{-1}c_0, \ \omega(t) := I(c_\omega(t), u_\omega(t), \alpha_\omega(t)). \) We have

\[
dI_\omega(t) = \left\langle \mathcal{K}(c_\omega(t), \alpha_\omega(t)), \sum_{i=1}^{m_1} \text{ad}^*_{H_1} q_\omega(t) \text{d} M_{\omega,i}^1(t) - \sum_{i=1}^{m_3} \text{ad}^*_{H_1} q_\omega(t) \text{d} M_{\omega,i}^3(t) \right\rangle
\]

\[
+ \left( \frac{\delta l}{\delta \alpha}(t) \circ \alpha_\omega(t) + K_\omega \left( t, \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t), u_\omega(t) \right) \right.
\]

\[
+ \frac{1}{2} \sum_{i,k=1}^{m_3} \text{ad}^*_{H_1} \text{ad}^*_{H_2} \left( \frac{\delta l}{\delta u}(t) \right) \frac{d\left[ M_{\omega,i}^1, M_{\omega,i}^3 \right]}{dt}
\]

\[
- \sum_{i=1}^{m_1} \sum_{k=1}^{m_3} \text{ad}^*_{H_1} q_\omega(t) \frac{d\left[ M_{\omega,i}^1, M_{\omega,k}^3 \right]}{dt} \left[ dt \right],
\]

where \( \frac{\delta l}{\delta u}(t) := \frac{\delta l}{\delta \alpha}(t, u_\omega(t), \alpha_\omega(t)), \frac{\delta l}{\delta \alpha}(t) := \frac{\delta l}{\delta \alpha}(t, u_\omega(t), \alpha_\omega(t)), \) \( q_\omega(t) := q(t, u_\omega(t), \alpha_\omega(t)), \) and \( \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t) \in \mathcal{M}, \ K_\omega : [0, T] \times \mathcal{M} \times \mathcal{M} \times T_e G \to T_e^* G \)

are defined by (3.13) and (3.14), respectively.

In particular, if \( \frac{\delta l}{\delta u}(t) = g_\omega^3(t), \ m_1 = m_3 = m, \) and \( H_1^1 = H_1^3 := H_1, \ M_{\omega,i}^1(t) = M_{\omega,i}^3(t), \ 1 \leq i \leq m, \) we have

\[
dI_\omega(t) = \left\langle \mathcal{K}(c_\omega(t), \alpha_\omega(t)), \frac{\delta l}{\delta \alpha}(t) \circ \alpha_\omega(t) \right\rangle dt + K_\omega \left( t, \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t), u_\omega(t) \right) dt
\]

\[
- \frac{1}{2} \sum_{i,k=1}^{m} \text{ad}^*_{H_1} \text{ad}^*_{H_2} q_\omega(t) \frac{d\left[ M_{\omega,i}^1, M_{\omega,k}^3 \right]}{dt} \left[ dt \right].
\]

**Proof** Due to (4.1) we have

\[
I_\omega(t) = \left\langle \mathcal{K} \left( g_\omega^3(t)^{-1}c_0, g_\omega^3(t)^{-1}c_0 \right), \frac{\delta l}{\delta u}(t, u_\omega(t), \alpha_\omega(t)) \right\rangle
\]

\[
= \left\langle \mathcal{K}(c_0, \alpha_0), \text{Ad}^*_{g_\omega^3(t)^{-1}} \frac{\delta l}{\delta u}(t, u_\omega(t), \alpha_\omega(t)) \right\rangle.
\]
By Itô’s formula, for any $T_e^*G$-valued semimartingale $\beta_\omega(\cdot)$, we have

$$d\text{Ad}^*_{g_3}(t)^{-1}\beta_\omega(t)$$

$$= \text{Ad}^*_{g_3}(t)^{-1}( - \text{ad}^*_{g_3}(t)^{-1}g_3(t)\beta_\omega(t) + \delta\beta_\omega(t))$$

$$= \text{Ad}^*_{g_3}(t)^{-1}( - \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}\beta_\omega(t)dM_\omega^{3,i}(t) - \text{ad}^*_{u_\omega(t)}\beta_\omega(t)dt + \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t)$$

(4.6)

$$+ d\beta_\omega(t) + \frac{1}{2} \sum_{i,k=1}^{m_3} \text{ad}^*_{H^i_3}d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t)$$

Combining (4.6) with (3.11) and (3.12) yields

$$d\left(\text{Ad}^*_{\delta_3}(t)^{-1}\frac{\delta l_\omega}{\delta u}(t)\right)$$

$$= \text{Ad}^*_{\delta_3}(t)^{-1}( - \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}\frac{\delta l_\omega}{\delta u}(t)dM_\omega^{3,i}(t) - \text{ad}^*_{u_\omega(t)}\frac{\delta l_\omega}{\delta u}(t)dt + \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}q_\omega(t)dM_\omega^{1,i}(t)$$

$$+ \text{ad}^*_{u_\omega(t)}\frac{\delta l_\omega}{\delta u}(t)dt + \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}\beta_\omega(t)d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t$$

$$+ \sum_{i,k=1}^{m_3} \text{ad}^*_{H^i_3}d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t)$$

Here we also apply the assumption $\frac{\delta l_\omega}{\delta u}(t, \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t), u_\omega(t)) \equiv 0$. Putting this into (4.5) and applying (4.2), we arrive at

$$dI_\omega(t) = \mathcal{K}(c_\omega, a_\omega) \cdot d\left(\text{Ad}^*_{\delta_3}(t)^{-1}\frac{\delta l_\omega}{\delta u}(t)\right)$$

$$= \mathcal{K}(c_\omega, a_\omega) \cdot \text{Ad}^*_{\delta_3}(t)^{-1}( \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}q_\omega(t)dM_\omega^{1,i}(t) - \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}\frac{\delta l_\omega}{\delta u}(t)dM_\omega^{3,i}(t)$$

$$+ \frac{\delta l_\omega}{\delta a}(t) \cdot a_\omega(t)dt + \sum_{i=1}^{m_3} \text{ad}^*_{H^i_3}\beta_\omega(t)d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t$$

$$+ \sum_{i,k=1}^{m_3} \text{ad}^*_{H^i_3}d\left[M_\omega^{3,i}, M_\omega^{3,k}\right]_t)$$

which proves (4.3).

If $\frac{\delta l_\omega}{\delta u}(t) = q_\omega(t)$, $m_1 = m_3 = m$, and $H^i_3 = H^i_3$, $M_\omega^{1,i}(t) = M_\omega^{3,i}(t)$, $1 \leq i \leq m$, then $\text{ad}^*_{H^i_3}\frac{\delta l_\omega}{\delta u}(t) = \text{ad}^*_{H^i_3}q_\omega(t)$ which, combined with (4.3), yields (4.4). □
Remark 4.2 If $H_i^j = 0$, $1 \leq i \leq m_j$, $j = 1, 2, 3$, then (4.3) becomes

$$d I_\omega(t) = \left\{ H(\omega(t), \alpha_\omega(t)), \frac{\delta I_\omega}{\delta \alpha}(t) \diamond \alpha_\omega(t) dt \right\},$$

thus recovering Holm et al. (1998, Theorem 4.1) for the deterministic Euler–Poincaré equation with advection term.

Remark 4.3 As we will see in Sect. 5, the term $K_\omega \left( t, \tilde{H}_\omega, 1(t), \tilde{H}_\omega, 2(t), u_\omega(t) \right)$ usually corresponds to some viscosity terms of the system.

5 Application to PDEs and SPDEs in Fluid Mechanics

We begin by recalling, from Ebin and Marsden (1970) and Marsden et al. (1972), the necessary standard facts about the group of diffeomorphisms on a smooth compact boundaryless $n$-dimensional manifold $M$. Then, when we present the compressible Navier–Stokes and MHD equations in the periodic case, we shall take $M = \mathbb{T}^3$, the usual three-dimensional flat torus.

Let $M$ be a smooth compact boundaryless $n$-dimensional manifold. Define

$$G^s := \left\{ g : M \to M \text{ is a bijection} \mid g, g^{-1} \in H^s(M, M) \right\},$$

where $H^s(M, M)$ denotes the manifold of Sobolev maps of class $s > 1 + \frac{n}{2}$ from $M$ to itself. The condition $s > \frac{n}{2}$ suffices to ensure the manifold structure of $H^s(M, M)$; only for such regularity class does the notion of an $H^s$-map from $M$ to itself make intrinsic sense. If $s > 1 + \frac{n}{2}$ (the additional regularity is needed in order to ensure that all elements of $G^s$ are $C^1$ and hence the inverse function theorem is applicable), then $G^s$ is an open subset in $H^s(M, M)$, so it is a $C^\infty$ Hilbert manifold. Moreover, it is a group under composition of diffeomorphisms, right translation by any element is smooth, left translation and inversion are only continuous, and $G^s$ is a topological group (relative to the underlying manifold topology) (see Ebin and Marsden 1970; Marsden et al. 1972); thus, $G^s$ is not a Lie group. Since $G^s$ is an open subset of $H^s(M, M)$, the tangent space $T_e G^s$ to the identity $e : M \to M$ coincides with the Hilbert space $\mathfrak{X}^s(M)$ of $H^s$ vector fields on $M$. Denote by $\mathfrak{X}(M)$ the Lie algebra of $C^\infty$ vector fields on $M$. The failure of $G^s$ to be a Lie group is mirrored by the fact that $\mathfrak{X}^s(M)$ is not a Lie algebra: the usual Jacobi–Lie bracket of vector fields, i.e., $[X, Y][f] = X[Y[f]] - Y[X[f]]$ for any $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, where $X[f] := df(X)$ is the differential of $f$ in the direction $X$, loses a derivative for finite differentiability class of vector fields and thus $[\cdot, \cdot] : \mathfrak{X}^s(M) \times \mathfrak{X}^s(M) \to \mathfrak{X}^{s-1}(M)$ is not an operation on $\mathfrak{X}^s(M)$. In general, the tangent space $T_\eta G^s$ at an arbitrary $\eta \in G^s$ is $T_\eta G^s = \left\{ U : M \to TM \text{ of class } H^s \mid U(m) \in T_{\eta(m)} M \right\}$. If $s > 1 + \frac{n}{2}$ and $X \in \mathfrak{X}^s(M)$, then its global (since $M$ is compact) flow $\mathbb{R} \ni t \mapsto F_t \in G^s$ exists and is a $C^1$-curve in $G^s$ (see, e.g., Marsden et al. 1972, Theorem 2.4.2). The candidate of what should have been the Lie group exponential map is $\exp : T_e G^s =
$\mathcal{X}^s(M) \ni X \mapsto F_i \in G^s$, where $F_i$ is the flow of $X$; however, exp does not cover a neighborhood of the identity and it is not $C^1$. Therefore, all classical proofs in the theory of finite-dimensional Lie groups based on the exponential map, break down for $G^s$. From now on, we shall always assume $s > 1 + \frac{n}{2}$.

Since right translation is smooth, each $X \in \mathcal{X}(M)$ induces a $C^\infty$ right invariant vector field $X^R \in \mathfrak{X}(G^s)$ on $G^s$, defined by $X^R(\eta) := X \circ \eta$. With this notation, we have the identity $[X^R, Y^R](e) = [X, Y]$, for any $X, Y \in \mathcal{X}(M)$. This is the analogue of saying that $\mathcal{X}(M)$ is the “right Lie algebra” of $G^s$ (as opposed to the usual “left Lie algebra”).

Assume, in addition, that $M$ is connected, oriented, and Riemannian; denote by $\langle \cdot, \cdot \rangle$ the Riemannian metric. Let $\mu_g$ be the Riemannian volume form on $M$, whose expression in local coordinates $(x^1, \ldots, x^n)$ is $\mu_g = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$, where $g_{ij} := (\partial/\partial x^i, \partial/\partial x^j)$ for all $i, j = 1, \ldots n$. Let $K : T(TM) \to TM$ be the connector of the Levi–Civita connection associated to a Riemannian metric, breaks down because $\eta$ is weak; the proof would only show uniqueness. However, $\eta \in \mathcal{T}_G s$ is weak metric is not right invariant (because of the Jacobian appearing in the change of variables formula in the integral).

The usual proof for finite-dimensional Lie groups, showing the existence of a unique Levi–Civita connection associated to a Riemannian metric, breaks down because $\langle \cdot, \cdot \rangle$ is weak; the proof would only show uniqueness. However, $K^0 : T(TG^s) \to TG^s$ given by $K^0(\mathcal{Z}_{U_\eta}) := K \circ \mathcal{Z}_{U_\eta}$, where $\mathcal{Z}_{U_\eta} \in T_\eta(TG^s)$, is a connector for the vector bundle $\tau_{G^s} : TG^s \to G^s$ (since $\tau_{G^s} \circ K^0 = \tau_{G^s} \circ \tau_{TG^s}$ and the vector bundles $\tau_{TG^s} : (T(M))^s \to TG^s$ and $\ker K^0 \oplus \ker \tau_{TG^s} \to TG^s$ are isomorphic). Here, $\mathcal{Z}_{U_\eta} \in T_{U_\eta}(TG^s)$ means that $\mathcal{Z}_{U_\eta} : M \to (T(M)$ satisfies $\tau_{TM}^s(\mathcal{Z}_{U_\eta}(m)) \in T_{\eta(m)}M$. The covariant derivative $\nabla^0$ on $G^s$ is defined by $\nabla^0 X := K^0 \circ \tau_{Y \cdot X} \cdot X$, for any $\mathcal{X}, \mathcal{Y} \in \mathcal{X}(G^s)$. This is the Levi–Civita connection associated to the weak metric $\langle \cdot, \cdot \rangle$ since it is torsion free ($\nabla^0_{X} \mathcal{Y} - \nabla^0_{Y} X = [\mathcal{X}, \mathcal{Y}]$, for all $\mathcal{X}, \mathcal{Y} \in \mathcal{X}(G^s)$, where $[\mathcal{X}, \mathcal{Y}]$ is the Jacobi–Lie bracket of vector fields on $G^s$) and $\langle \cdot, \cdot \rangle$-compatible
We formulate the theory presented in Sect. 3 for the infinite-dimensional group $G^s$; see Ebin and Marsden (1970, Theorem 9.1, page 128). Uniqueness of such a connection follows from the weak non-degeneracy of the metric $\langle \cdot, \cdot \rangle$. For $\overline{\nabla}^0$ there is an explicit formula [see, e.g., Misiołek 1993, (3.1)]:

$$\left( \overline{\nabla}^0_\chi \gamma \right)(\eta) = \left[ \frac{\partial}{\partial t} \left( \gamma(\eta_t) \circ \eta_t^{-1} \right) \right]_{t=0} + \nabla_{\chi(\eta) \circ \eta^{-1}} \left( \gamma(\eta) \circ \eta^{-1} \right) \circ \eta,$$

where $\nabla$ denotes the Levi–Civita connection on $M$, $\chi, \gamma \in \mathfrak{X}(G^s)$ and hence $\chi(\eta), \gamma(\eta) \in T_\eta G^s$, and $t \mapsto \eta_t$ is a $C^1$ curve in $G^s$ such that $\eta_0 = \eta$ and $\frac{d}{dt} |_{t=0} \eta_t = \chi(\eta)$. In Ebin and Marsden (1970) it is shown that the geodesic spray of the hydrodynamic Riemannian metric is a smooth vector field on on the tangent bundle of the subgroup of $G^s$ formed by volume preserving $H^s$-diffeomorphisms on $M$.

In this paper we need a right-invariant linear connection on $G^s$. We get it using the right-invariant metric on $G^s$ obtained by right translating the $L^2$ weak inner product on $\mathfrak{X}^1(M)$ to every point of $G^s$. This metric is not the hydrodynamic metric discussed above; the two weak Riemannian metrics coincide only on $\mathfrak{X}^1(M)$. There is an explicit formula for the right-invariant covariant derivative $\overline{\nabla}^0$ associated to this right-invariant weak metric (a particular case of Gay–Balmaz (2008, pages 44 and 46)):

$$\left( \overline{\nabla}^0_\chi \gamma \right)(\eta) = \left[ \frac{\partial}{\partial t} \left( \gamma(\eta_t) \circ \eta_t^{-1} \right) \right]_{t=0} + \nabla_{\chi(\eta) \circ \eta^{-1}} \left( \gamma(\eta) \circ \eta^{-1} \right) + \nabla \chi(\eta) \circ \eta^{-1} + \gamma(\eta) \circ \eta^{-1} \frac{\partial}{\partial t} \gamma(\eta)(\eta) \circ \eta^{-1} \right) \circ \eta, \quad (5.1)$$

where $\nabla$ is the Levi–Civita connection on $M$, $\chi, \gamma \in \mathfrak{X}(G^s)$ and hence $\chi(\eta), \gamma(\eta) \in T_\eta G^s$, and $t \mapsto \eta_t$ is a $C^1$ curve in $G^s$ such that $\eta_0 = \eta$ and $\frac{d}{dt} |_{t=0} \eta_t = \chi(\eta)$. As opposed to the hydrodynamic metric, the spray of this weak right-invariant metric is not smooth.

The discussion above shows that one cannot apply the theorems of Sect. 3 to the infinite-dimensional group $G^s$ directly. For infinite-dimensional problems, they serve only as a guideline and direct proofs are needed, which is what we do below. However, for each important formula, we shall point out the analogue in the finite-dimensional abstract setting of Sect. 3 which inspired the result, that still holds for the model presented here.

### 5.1 Stochastic Semidirect Product Euler–Poincaré Reduction for $G^s$

We formulate the theory presented in Sect. 3 for the infinite-dimensional group $G^s$. From now on we consider the case $M = \mathbb{T}^3$. We focus on the following type of SDEs on $G^s$,

$$\begin{align*}
\{ \text{d}g_\omega(t, \theta) = & \sum_{i=1}^m \mathcal{H}_i(g_\omega(t, \theta)) \delta M^i_\omega(t) + u_\omega(t, g_\omega(t, \theta)) \text{d}t \\
g_\omega(0, \theta) = & \theta, \quad \theta \in \mathbb{T}^3,
\end{align*}$$
where \( H_j \in \mathcal{X}^s(\mathbb{T}^3) \) is non-random, \( \{M^i_\omega(t)\}_{i=1}^m \) is a \( \mathbb{R}^m \)-valued martingale with continuous sample paths on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with respect to the filtration \( \mathcal{F}_t \), and \( u : \Omega \times [0, T] \to \mathcal{X}^s(\mathbb{T}^3) \) is such that \( u_\omega(t, \theta) \) is a (\( \mathcal{F}_t \)-adapted) semimartingale for every \( \theta \in \mathbb{T}^3 \).

In particular, here we take the constant vector fields \( H_1 = H_{1,v} = \sqrt{2}v(1, 0, 0), \ H_2 = H_{2,v} = \sqrt{2}v(0, 1, 0), \ H_3 = H_{3,v} = \sqrt{2}v(0, 0, 1) \) on \( \mathbb{T}^3 \), where \( v \geq 0 \) is a (viscosity) constant. This is understood in the trivialization \( T^3 = \mathbb{T}^3 \times \mathbb{R}^3 \), so \( H_1, H_2, H_3 : \mathbb{T}^3 \to \mathbb{R}^3 \) are constant maps. Let \( g^{v, M}_\omega(t, \theta) \) be the solution to the following SDE,

\[
\begin{aligned}
\mathsf{d}g^{v, M}_\omega(t, \theta) &= \sum_{i=1}^3 H_i \cdot \mathsf{d}M_i^\omega(t) + u_\omega(t, g^{v, M}_\omega(t, \theta)) \mathsf{d}t \\
g^{v, M}_\omega(0, \theta) &= \theta,
\end{aligned}
\tag{5.2}
\]

where \( M_\omega = \{M^1_\omega, M^2_\omega, M^3_\omega\} \) is a \( \mathbb{R}^3 \)-valued martingale with continuous sample paths.

By the standard theory of stochastic flows (see, e.g., Kunita (1990) and standard embedding theorems), if \( u_\omega \) is regular enough (with respect to the space variable), i.e., \( u_\omega \in C([0, T]; \mathcal{X}^{s'}(\mathbb{T}^3)) \) for some \( s' > s \) large enough, then \( g^{v, M}_\omega(t, \cdot) \in G^s \) for every \( t \in [0, T] \). From now on, for simplicity, we always assume \( u_\omega \) to be regular enough.

As in Holm et al. (1998, Section 6), let \( U^* \) be some linear space which can be a space of functions, densities, or differential forms on \( \mathbb{T}^3 \). The action of \( G^s \) on \( U^* \) is the pull-back map and the action of the “Lie algebra” \( T_eG^s \) on \( U^* \) is the Lie derivative.

If we take \( \alpha_0 = A_0(\theta) \cdot \mathsf{d}\theta := \sum_{i=1}^3 A_{0,i}(\theta) \mathsf{d}\theta_i \) to be a \( C^2 \)-one-form on \( \mathbb{T}^3 \), we derive the following result (see also an equivalent expression in Eyink (2009), equations (32)–(34) for the deterministic case). Formula (5.4) below is the analogue of the second equation in (3.37), derived here by hand for the infinite-dimensional group \( G^s \).

**Proposition 5.1** Let \( g^{v}_\omega(t) \) be given by (5.2) with \( M_\omega = W_\omega \), where \( W_\omega \) is a standard \( \mathbb{R}^3 \)-valued Brownian motion (i.e., \( W_\omega = (W^1_\omega, W^2_\omega, W^3_\omega) \) with \( W^i_\omega \), \( 1 \leq i \leq 3 \), independent \( \mathbb{R} \)-valued Brownian motions). Define

\[
\alpha_\omega(t, \theta) := \left( \alpha_0 g^{v}_\omega(t, \cdot)^{-1} \right)(\theta) = \left( \left( g^{v}_\omega(t, \cdot)^{-1} \right)^* \alpha_0 \right)(\theta)
\]

\[
:= A_\omega(t, \theta) \cdot \mathsf{d}\theta := \sum_{i=1}^3 A_{\omega,i}(t, \theta) \mathsf{d}\theta_i,
\]

where \( \left( g^{v}_\omega(t, \cdot)^{-1} \right)^* \) denotes the pull-back map by \( g^{v}_\omega(t, \cdot)^{-1} \), and

\[
\tilde{\alpha}(t, \theta) := \mathbb{E}[\alpha_\omega(t, \theta)] := \tilde{A}(t, \theta) \cdot \mathsf{d}\theta := \sum_{i=1}^3 \tilde{A}_i(t, \theta) \mathsf{d}\theta_i.
\]
Then $A_\omega$ satisfies the following SPDE,

$$
\begin{align*}
\frac{dA}{dt}_\omega, i(t, \theta) &= -\sum_{j=1}^{3} \sqrt{2} \nu \partial_j A_\omega, i(t, \theta, \omega) dW^j_\omega(t) \\
& \quad - \sum_{j=1}^{3} \left( u_\omega, j(t, \theta) \partial_j A_\omega, i(t, \theta) + A_\omega, j(t, \theta) \partial_i u_\omega, j(t, \theta) \right) dt \\
& \quad + \nu \Delta A_\omega, i(t, \theta) dt, \quad i = 1, 2, 3,
\end{align*}
$$

(5.3)

where we use the notation $u_\omega(t) := (u_\omega, 1(t), u_\omega, 2(t), u_\omega, 3(t))$ and $\partial_j$ and $\Delta$ stand for the partial derivative and the Laplacian with respect to the space variable $\theta$ of $A_\omega(t, \theta)$, respectively. Equation (5.3) can also be expressed as

$$
\frac{dA}{dt}_\omega(t, \theta) = -\sqrt{2} \nu \nabla A_\omega(t, \theta) \cdot dW_\omega(t) \\
- (u_\omega(t, \theta) \times \text{curl} A_\omega(t, \theta) - \nabla (u_\omega(t, \theta) \cdot A_\omega(t, \theta))) dt + \nu \Delta A_\omega(t) dt
$$

(5.4)

(the term $dA_\omega(t, \theta)$ above denotes the Itô differential of $A_\omega(t, \theta)$ with respect to the time variable).

Moreover, if $u_\omega$ is non-random (in this case we write $u$ for $u_\omega$), we have

$$
\begin{align*}
\partial_t \tilde{A}(t, \theta) &= -(u(t, \theta) \times \text{curl} \tilde{A}(t, \theta) - \nabla (u(t, \theta) \cdot \tilde{A}(t, \theta))) + \nu \Delta \tilde{A}(t, \theta), \\
\tilde{A}(0, \theta) &= A_0(\theta).
\end{align*}
$$

(5.5)

Proof We use the methods in Constantin and Iyer (2008, Lemma 4.1 and Proposition 4.2). It is not hard to see that, for the $C^2$ (note that we assume $u$ to be regular) spatial process $A_\omega, i(t, \theta)$, there exist adapted spatial processes $h_\omega, ij(t, \theta)$ and $z_\omega, i(t, \theta)$, $1 \leq i, j \leq 3$, such that,

$$
\begin{align*}
\frac{dA}{dt}_\omega, i(t, \theta) &= \sum_{j=1}^{3} h_\omega, ij(t, \theta) dW^j_\omega(t) + z_\omega, i(t, \theta) dt, \quad i = 1, 2, 3.
\end{align*}
$$

(5.6)

We compute below the expressions of $h_\omega, ij(t, \theta)$ and $z_\omega, i(t, \theta)$.

Notice that by the definition of $\alpha_\omega(t, \theta)$, $\left( g_\omega^\nu(t, \theta) \right)^* \alpha_\omega(t, \theta) = \alpha(0, \theta)$ is a constant with respect to the time variable, and

$$
\begin{align*}
\left( g_\omega^\nu(t, \theta) \right)^* \alpha_\omega(t, \theta) &= \sum_{j=1}^{3} \left( \sum_{i=1}^{3} A_\omega, i(t, g_\omega^\nu(t, \theta), \omega) V_\omega, ij(t, \theta) \right) d\theta_j,
\end{align*}
$$
where the process $V_{\omega,ij}(t, \theta) := \partial_j g_{\omega,i}^\nu(t, \theta)$ (here we use the notation $g_{\omega}^\nu(t) = (g_{\omega,1}^\nu(t), g_{\omega,2}^\nu(t), g_{\omega,3}^\nu(t))$). We get for each $1 \leq j \leq 3$,

$$d\left( \sum_{i=1}^{3} A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) V_{\omega,ij}(t, \theta) \right) = 0. \quad (5.7)$$

By (5.6) and the generalized Itô formula for spatial processes (see Kunita 1990, Theorem 3.3.1.), we have

$$dA_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) = \sum_{j=1}^{3} \left( h_{\omega,ij}(t, g_{\omega}^\nu(t, \theta)) + \sqrt{2} v \partial_j A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) \right) dW_j^\nu(t)$$

$$+ \left( z_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) + \nu A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) \right) dt + \sum_{j=1}^{3} \left( u_{\omega,j}(t, g_{\omega}^\nu(t, \theta)) \partial_j A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) + \sqrt{2} v \partial_j h_{\omega,ij}(t, g_{\omega}^\nu(t, \theta)) \right)^{\nu/Delta_{1}} A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) dt. \quad (5.8)$$

By the theory of the stochastic flows in Kunita (1990, Theorem 3.3.3.), from (5.2) we obtain

$$dV_{\omega,ij}(t, \theta) = \sum_{k=1}^{3} \delta_{k} u_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) V_{\omega,kj}(t, \theta) dt. \quad (5.9)$$

In particular, the martingale part of the above equality vanishes due to the fact that the diffusion coefficients $H_{\nu,\omega}$ in (5.2) are constant.

According to (5.8) and (5.9), for each $1 \leq j \leq 3$, the Itô differential with respect to the time variable equals

$$d\left( \sum_{i=1}^{3} A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) V_{\omega,ij}(t, \theta) \right)$$

$$= \sum_{i,k=1}^{3} \left( h_{\omega,ik}(t, g_{\omega}^\nu(t, \theta)) + \sqrt{2} v \partial_k A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) \right) V_{\omega,ij}(t, \theta) dW_k^\nu(t)$$

$$+ \sum_{i=1}^{3} \left( \sum_{k=1}^{3} \left( u_{\omega,k}(t, g_{\omega}^\nu(t, \theta)) \partial_k A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) + \sqrt{2} v \partial_k h_{\omega,ik}(t, g_{\omega}^\nu(t, \theta)) \right) \right)$$

$$+ z_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) + \nu A_{\omega,i}(t, g_{\omega}^\nu(t, \theta)) \right) V_{\omega,ij}(t, \theta) dt$$

$$+ \sum_{i,k=1}^{3} A_{\omega,k}(t, g_{\omega}^\nu(t, \theta)) \partial_i u_{\omega,k}(t, g_{\omega}^\nu(t, \theta)) V_{\omega,ij}(t, \theta) dt$$
Hence, from (5.7), we derive for each $1 \leq j, m \leq 3$,

$$
\sum_{i=1}^{3} \left( h_{\omega,i,m}(t, g_{\omega}^{v}(t, \theta)) + \sqrt{2}v \partial_{m} A_{\omega,i}(t, g_{\omega}^{v}(t, \theta)) \right) V_{\omega,ij}(t, \theta) = 0 \tag{5.10}
$$

and

$$
\sum_{i=1}^{3} \left( z_{\omega,i}(t, g_{\omega}^{v}(t, \theta)) + \nu \Delta A_{\omega,i}(t, g_{\omega}^{v}(t, \theta)) 
+ \sum_{k=1}^{3} \left( u_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \partial_{k} A_{\omega,i}(t, g_{\omega}^{v}(t, \theta)) + \sqrt{2}v \partial_{k} h_{\omega,ik}(t, g_{\omega}^{v}(t, \theta)) 
+ A_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \partial_{i} u_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \right) \right) V_{\omega,ij}(t, \theta) = 0.
\tag{5.11}
$$

As $\{V_{ij}(t, \theta, \omega)\}_{1 \leq i, j \leq 3}$ is a non-degenerate matrix-valued process (see, e.g., Kunita 1990), from (5.10) we deduce that, for each $1 \leq i, j \leq 3$,

$$
h_{\omega,i,j}(t, g_{\omega}^{v}(t, \theta)) = -\sqrt{2}v \partial_{j} A_{\omega,i}(t, g_{\omega}^{v}(t, \theta)).
\tag{5.12}
$$

Since $g_{\omega}^{v}(t, \theta)$ is invertible in $\theta$ we can derive the expression for $h_{\omega,i,j}$ not only as a function of $g_{\omega}^{v}(t, \theta)$ but also as a function (at the origin) of $(t, \theta)$. Indeed, noticing that, $\omega$-almost surely, $\theta \mapsto g_{\omega}(t, \theta)$ is a diffeomorphism for each fixed $t$, we get

$$
h_{\omega,i,j}(t, \theta) = -\sqrt{2}v \partial_{j} A_{\omega,i}(t, \theta), \quad \forall \theta \in \mathbb{T}^{3}, \tag{5.12}
$$

which is the expression for $h_{\omega,i,j}(t, \theta)$.

Since $\{V_{\omega,ij}(t, \theta, \omega)\}_{1 \leq i, j \leq 3}$ is non-degenerate, by (5.11), for each $1 \leq i \leq 3$,

$$
\sum_{k=1}^{3} \left( u_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \partial_{k} A_{\omega,i}(t, g_{\omega}^{v}(t, \theta)) 
+ \sqrt{2}v \partial_{k} h_{\omega,ik}(t, g_{\omega}^{v}(t, \theta)) 
+ A_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \partial_{i} u_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \right) 
+ A_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) \partial_{i} u_{\omega,k}(t, g_{\omega}^{v}(t, \theta)) = 0.
\tag{5.13}
$$

We use (5.12) in the above equation and the fact that $\theta \mapsto g_{\omega}(t, \theta)$ is a diffeomorphism for each fixed $t$, $\omega$-almost surely, and we obtain the expression for $z_{\omega,i}(t, \theta)$, namely,

$$
z_{\omega,i}(t, \theta) = \nu \Delta A_{\omega,i}(t, \theta) 
- \sum_{k=1}^{3} \left( u_{\omega,k}(t, \theta) \partial_{k} A_{\omega,i}(t, \theta) + A_{\omega,k}(t, \theta) \partial_{i} u_{\omega,k}(t, \theta) \right), \quad \forall \theta \in \mathbb{T}^{3}.
\tag{5.13}
$$
Combining (5.6), (5.12), and (5.13) proves (5.3). We can check that (5.3) is equivalent to (5.4) by direct computation.

If \( u_\omega(t, \cdot) = u(t, \cdot) \) is non-random, then it is easy to verify that

\[
\mathbb{E} [u(t, \theta) \times \text{curl} A_\omega(t, \theta)] = u(t, \theta) \times \mathbb{E} [\text{curl} A_\omega(t, \theta)]
= u(t, \theta) \times \text{curl} \mathbb{E} [A_\omega(t, \theta)] = u(t, \theta) \times \text{curl} \tilde{A}_\omega(t, \theta).
\]

\[
\mathbb{E} [\nabla (u_\omega(t, \theta) \cdot A_\omega(t, \theta))] = \nabla (u(t, \theta) \cdot \mathbb{E} [A_\omega(t, \theta)])
= \nabla \left( u(t, \theta) \cdot \tilde{A}(t, \theta) \right).
\]

So taking the expectation of the two sides of equation (5.4), (5.5) follows. \( \square \)

**Remark 5.2** In Proposition 5.1, \( U^* \) is taken to be a space of differential forms on \( \mathbb{T}^3 \). Note that the action of \( G^* \) on \( U^* \) is the pull-back map and the action of the “Lie algebra” \( T_c G^* \) on \( U^* \) is the Lie derivative. Then, for \( H_{1,v} = \sqrt{2}v(1, 0, 0), H_{2,v} = \sqrt{2}v(0, 1, 0), H_{3,v} = \sqrt{2}v(0, 0, 1) \), and \( \alpha_\omega(t, \theta) = A_\omega(t, \theta) \cdot d\theta \), we have

\[
\sum_{i=1}^{3} \alpha_\omega(t) H_{i,v} dW^i_\omega(t) = \sqrt{2}v (\nabla A_\omega(t, \theta) \cdot dW_\omega(t)) \cdot d\theta,
\]

\[
\sum_{i=1}^{3} \alpha_\omega(t) H_{i,v} H_{i,v} = \nu \Delta A_\omega(t, \theta) \cdot d\theta,
\]

\[
\alpha_\omega(t) u_\omega(t) = (u_\omega(t, \theta) \times \text{curl} A_\omega(t, \theta) - \nabla (u_\omega(t, \theta) \cdot A_\omega(t, \theta))) \cdot d\theta,
\]

which implies that (5.4) is just the second equation of (3.37).

In the same way, we can verify that (5.5) is the second equation of (3.40). \( \diamond \)

**Remark 5.3** By the same procedure as in the proof of Proposition 5.1, if \( \alpha_0 \) is replaced by another term, such as a function or a density, we can still prove the corresponding evolution equation for \( \alpha_\omega(t) := \alpha_0 g_\omega^v(t, \cdot)^{-1} = (g_\omega^v(t, \cdot)^{-1})^k \alpha_0 \).

For example, if \( \alpha_0 = \beta_0 : \mathbb{T}^3 \rightarrow \mathbb{R} \) is a \( C^2 \) function, then \( \alpha_\omega(t, \theta) \) satisfies the following stochastic transport equation,

\[
\begin{cases}
\text{d} \alpha_\omega(t, \theta) = -\sqrt{2}v (\nabla \alpha_\omega(t, \theta) \cdot dW_\omega(t) - u_\omega(t, \theta) \cdot \nabla \alpha_\omega(t, \theta)) dt + \nu \Delta \alpha_\omega(t, \theta) dt,
\alpha_\omega(0, \theta) = \beta_0(\theta).
\end{cases}
\]

This equation has been studied in Flandoli et al. (2010) which illustrates that the added stochastic force (noise) can turn an ill-posed ODE into a well-posed one.

If \( \alpha_0 = D_0(\theta) d^3\theta \) is a density (volume form), write \( \alpha_\omega(t, \theta) = D_\omega(t, \theta) d^3\theta \). Then \( D_\omega(t, \theta) \) satisfies the following equation

\[
\begin{cases}
\text{d} D_\omega(t, \theta) = -\sqrt{2}v (\nabla D_\omega(t, \theta) \cdot dW_\omega(t) - \text{div}(D_\omega u_\omega)(t, \theta)) dt + \nu \Delta D_\omega(t, \theta) dt,
D_\omega(0, \theta) = D_0(\theta).
\end{cases}
\]

(5.14)
Assume that \( u_\omega(\cdot) = u(\cdot) \) is non-random and \( \alpha_0 = D_0(\theta)d^3 \theta \) is a probability measure, let \( \hat{\alpha}(t) := E[\alpha_\omega(t)] := \tilde{D}(t, \theta)d^3 \theta \). Then \( \tilde{D}(t, \theta) \) satisfies the following forward Kolmogorov equation (or Fokker–Planck equation),

\[
\begin{align*}
\{ & d\tilde{D}(t, \theta) = -\text{div}(\tilde{D}u)(t, \theta)dt + \nu \Delta \tilde{D}(t, \theta)dt, \\
& \tilde{D}(0, \theta) = D_0(\theta). 
\end{align*}
\tag{5.15}
\]

Moreover, let \( \hat{g}_\omega^v(t, \theta) \) be the process satisfying

\[
d\hat{g}_\omega^v(t, \theta) = \sqrt{2\nu}dW_\omega(t) + u(t, \hat{g}_\omega^v(t, \theta))dt
\]
whose initial distribution is \( D_0(\theta)d^3 \theta \). Then for every \( t \in [0, T] \), the distribution of \( \hat{g}_\omega^v(t, \theta) \) is of the form \( \tilde{D}(t, \theta)d^3 \theta \), where \( \tilde{D}(t, \theta) \) satisfies (5.15). \( \diamond \)

**Remark 5.4** By carefully tracking the proof of Proposition 5.1, if we take \( M_\omega \) in (5.2) to be a general \( \mathbb{R}^3 \)-valued martingale, Eq. (5.4) becomes

\[
dA_\omega(t, \theta) = -\sqrt{2\nu}\nabla A_\omega(t, \theta) \cdot dM_\omega(t) - u_\omega(t, \theta) \times \text{curl}A_\omega(t, \theta)dt \\
+ \nabla(u_\omega(t, \theta) \cdot A_\omega(t, \theta))dt + \nu \sum_{i,j=1}^3 \partial_i \partial_j A_\omega(t) d[M^j_i, M^j_i]_t.
\] \( \diamond \)

For each \( \mathcal{F}_t \)-adapted process \( v \) such that \( v_\omega(\cdot, \cdot) \in C^1([0, 1]; \mathbb{R}^3(\mathbb{T}^3)) \) (for \( s \) large enough) with \( v_\omega(0, \theta) = v_\omega(T, \theta) = 0 \) a.s., the deformation (3.33) (for right invariant systems) in the formulation here is determined by the following stochastic flows \( e_{\omega, e, v}(t, \cdot) \in G^s \) (see, e.g., Arnaudon et al. (2014) and Cipriano and Cruzeiro (2007) for the deterministic counterpart)

\[
\begin{align*}
& \frac{de_{\omega, e, v}(t, \theta)}{dt} = \varepsilon \dot{v}_\omega(t, e_{\omega, e, v}(t, \theta)) \\
& e_{\omega, e, v}(0, \theta) = \theta.
\end{align*}
\tag{5.16}
\]

Setting \( g_{\omega, e, v}^v(t, \theta) := e_{\omega, e, v}(t, g_{\omega, e, v}^v(t, \theta)) \), where \( g_{\omega, e, v}^v \) is the solution to (5.2), and using such deformations, we can also define the critical point for an action functional in the same way as in (3.9), Sect. 3.

By the analysis in Arnaudon et al. (2014, Section 4.2) (especially (4.5)–(4.6) in Arnaudon et al. (2014)) for the (infinite-dimensional) group \( G^s \), we have

\[
dg_{\omega, e, v}^v(t) = T_e R_{g_{\omega, e, v}^v(t)} \left( \sum_{i=1}^3 H_{\omega, e, v}(t) \delta M_i \right) + Ad_{e_{\omega, e, v}(t)} u_\omega(t) + \varepsilon \dot{v}_\omega(t)dt.
\]
\tag{5.17}

where \( H_{\omega, i, e, v}(t) := Ad_{e_{\omega, e, v}(t)} H_{i, v} \). Based on the above equation for \( g_{\omega, e, v}^v \) and according to the definition of \( \frac{\partial g}{\partial t} \) and \( \frac{D}{\partial t} g_{\omega, e, v}^v \) (especially using the right-
invariant version of (2.15) and (3.8)), it is easy to verify that

\[
T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} \frac{d g_\omega^{v,M}(t,\theta)}{dt} = u_\omega(t,\theta),
\]

\[
\frac{d}{de}|_{e=0} \left( T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} \frac{d g_\omega^{v,M}(t,\theta)}{dt} \right) = \left( \text{ad}_{\omega_\theta}(t) u_\omega(t) \right)(\theta) = \left[ -v_\omega(t,\cdot), u_\omega(t,\cdot) \right](\theta),
\]

\[
T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} d g_\omega^{v,M}(t,\theta) = \sqrt{2} v d M_\omega(t),
\]

\[
\frac{d}{de}|_{e=0} \left( T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} d g_\omega^{v,M}(t,\theta) \right) = \left( \text{ad}_{\omega_\theta}(t) u_\omega(t) \right)(\theta) d M_\omega^I(t),
\]

\[
\frac{d}{de}|_{e=0} \left( T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} D_{g_\omega}^{v,M}(H_{i,v}^I, M_\omega^I)_{t=1} \right)_{ij} = \nabla_H^0 H_{j,v} \frac{d [M_\omega^I, M_\omega^J]}{dt} = 0,
\]

\[
\frac{d}{de}|_{e=0} \left( T_{g_\omega}^{v,M}(t,\theta) R_{g_\omega}^{v,M}(t,\theta)^{-1} D_{g_\omega}^{v,M}(H_{i,v}^I, M_\omega^I)_{t=1} \right)_{ij} = \left( \nabla_{\text{ad}_{\omega_\theta}(t)} H_{i,v} + \nabla_H^0 H_{i,v}(\text{ad}_{\omega_\theta}(t) H_{j,v}) \right) \frac{d [M_\omega^I, M_\omega^J]}{dt}
\]

\[
= 2 v \partial_i \partial_j v_\omega(t,\theta) \frac{d [M_\omega^I, M_\omega^J]}{dt},
\]

where \( \nabla^0 \) denotes the connection on \( \mathcal{X}(G^s) \) defined by (5.1) (in particular, we apply the property that \( \nabla^0 X Y(\theta) = \sum_{i,j=1}^3 X_i(\theta) \partial_i Y_j(\theta) \partial_j \) for every \( X = \sum_{i=1}^3 X_i(\theta) \partial_i \), \( Y = \sum_{i=1}^3 Y_i(\theta) \partial_i \in \mathcal{X}(\mathbb{T}^3) \) because the Christoffel symbols are zero, \( \mathbb{T}^3 \) being the flat torus).

Based on these formulas, the procedure on the variational principle in the proof of Theorem 3.12 and 3.13 also holds for the infinite-dimensional group \( G^s \) needed here. Hence, the first equation of (3.37) (also the first equation of (3.40)) remains true for \( G^s \).

Combining all the conclusions above, we deduce that Theorems 3.12 and 3.13 still hold for the infinite-dimensional group \( G^s \).

### 5.2 Compressible Navier–Stokes Equation

Suppose \( \nabla^0 \) is the right-invariant linear connection on \( \mathcal{X}(G^s) \) defined by (5.1). Let \( U^* \) denote the vector space of all densities on \( \mathbb{T}^3 \) and define \( \alpha_0 : = D_0(\theta) d^3 \theta \in U^* \). Let \( \mathcal{M}_m(G^s) \) be the collection of all \( m \times m \) matrices whose elements are in \( \mathcal{X}(\mathbb{T}^3) \) and define \( \mathcal{M}(G^s) : = \bigcup_{m=1}^\infty \mathcal{M}_m(G^s) \).
As in Holm et al. (1998), we take the dual space \((\mathcal{X}^s(\mathbb{T}^3))^*\) of \(\mathcal{X}^s(\mathbb{T}^3)\) to be the vector space \(\Omega^1(\mathbb{T}^3)\) of all differential one-forms on \(\mathbb{T}^3\) (here we fix the volume measure to be the Lebesgue measure on \(\mathbb{T}^3\)).

Define the Lagrangian \(l : \Omega \times [0, T] \times \mathcal{X}^s(\mathbb{T}^3) \times U^* \to \mathbb{R}\) by

\[
l(w_\omega(t), \alpha_\omega(t)) := \int_{\mathbb{T}^3} \left( \frac{D_\omega(t, \theta)}{2} |w_\omega(t, \theta)|^2 - D_\omega(t, \theta)e(D_\omega(t, \theta)) \right) d^3\theta,
\]

where \(w_\omega(t) \in \mathcal{X}^s(\mathbb{T}^3)\), \(\alpha_\omega(t) := D_\omega(t, \theta)d^3\theta \in U^*\), \(\forall \omega \in \Omega, \forall t \in [0, T]\),

\(e(D_\omega)\) is the fluid’s specific internal energy, and the pressure \(p_\omega\) is given by \(p_\omega = \frac{D_\omega}{d^3\omega}\). We assume that both \(D_\omega(t)\) and \(p_\omega(t)\) are \(\mathcal{P}_1\)-adapted. See Holm et al. (1998, Section 7) for more details on such Lagrangians. Here we use a random version, since \(w_\omega(t)\) is the trace operator and

\[
\langle \omega(\cdot) \rangle := \int_{\mathbb{T}^3} \omega(\cdot) d^3\theta = \sum_{i,j=1}^m \int_{\mathbb{T}^3} P_i(u(\theta))(B)_{ij}(\theta)) d^3\theta,
\]

\(\forall A \in \mathcal{M}_n(\mathbb{G}^s), B \in \mathcal{M}_m(\mathbb{G}^s), \forall u \in \mathcal{X}^s(\mathbb{T}^3)\), so \(u(\theta), (B)_{ij}(\theta) \in \mathbb{R}^3 \forall \theta \in \mathbb{T}^3\)

\((5.21)\)

where \(\text{Tr} : \mathcal{M}(\mathbb{G}^s) \to \mathcal{X}^s(\mathbb{T}^3)\) is the trace operator and \(P_i : \mathbb{R}^3 \to \mathbb{R}\) is the projection operator defined by

\[
P_i(x_1, x_2, x_3) := \begin{cases} x_i, & \text{if } 1 \leq i \leq 3, \\ 0, & \text{if } i > 3. \end{cases}
\]

We take the stochastic force \(q : [0, T] \times \mathcal{X}^s(\mathbb{T}^3) \times U^* \to (\mathcal{X}^s(\mathbb{T}^3))^*\) to be \(q(t, u, \alpha) := \langle u, \cdot \rangle\).

With \(\nabla^0, l, \tilde{p}, q, \alpha_0\) given above, we define an action functional \(J^0 := \int \nabla^0(H_{\omega}, W_{\omega}^{1, i})^{m_1}_{i=1} + (g_{\omega}, W_{\omega}^{2, i})^{m_2}_{i=1} + g^{3}_{\omega})\) according to (3.31) as follows

\[
J^0 \left( \left( g^{1}_{\omega}, W^{1, i}_{\omega}, M^{1, i}_{\omega} \right)_{i=1}^{m_1}, \left( g^{2}_{\omega}, W^{2, i}_{\omega}, M^{2, i}_{\omega} \right)_{i=1}^{m_2}, g^{3}_{\omega} \right) := \int_{0}^{T} \int_{\mathbb{T}^3} \left( \frac{1}{2} |w_\omega(t, \theta)|^2 D_\omega(t, \theta) - D_\omega(t, \theta)e(D_\omega(t, \theta)) \right) d^3\theta dt
\]
\[
+ \int_0^T \bar{p} \left( \mathbf{D}^{\nu_0, (w^1_{\omega i} \cdot M^1_{\omega i})_{i=1}^{m_1}} s^1_{\omega}(t), \mathbf{D}^{\nu_0, (w^2_{\omega i} \cdot M^2_{\omega i})_{i=1}^{m_2}} s^2_{\omega}(t), \omega, \omega(t) \right) \, dt
+ \int_0^T \int_{\mathbb{T}^3} \langle w_{\omega i}(t, \theta), d\Xi(t, \theta) \rangle \, d^3 \theta \, dt
- \sum_{i=1}^3 \sqrt{2\nu} \int_0^T \int_{\mathbb{T}^3} w_{\omega i}(t, \theta) d^3 \theta \, dW_i(t),
\]
\[
\forall \left( \left( g^1_{\omega}, w^1_{\omega i}, M^1_{\omega i} \right)_{i=1}^{m_1}, \left( g^2_{\omega}, w^2_{\omega i}, M^2_{\omega i} \right)_{i=1}^{m_2}, g^3_{\omega} \right) \in \mathcal{T}(G^2) \times \mathcal{T}(G^3) \times \mathcal{T}(G^5),
\]

(5.22)

where
\[
w_{\omega}(t, \cdot) := T_{g^1_{\omega}(t)} R_{g^1_{\omega}(t)}^{-1} \left( \frac{\partial g^1_{\omega}(t)}{\partial t} \right) = (w_{\omega,1}(t, \cdot), w_{\omega,2}(t, \cdot), w_{\omega,3}(t, \cdot)),
\]
\[
d\Xi(t, \cdot) := T_{g^1_{\omega}(t)} R_{g^1_{\omega}(t)}^{-1} \left( d^3 g^1_{\omega}(t) \right),
\]
\[
D_{\omega}(t, \theta) d^3 \theta := \left( g^3_{\omega}(t, \cdot) \right)^* a_0,
\]

and \(W_{\omega}\) is a standard \(\mathbb{R}^3\)-valued Brownian motion.

Let \(g^\nu_{\omega}\) be the solution of (5.2) with \(\nu > 0\) and \(M_\omega = W_\omega\) the same \(\mathbb{R}^3\)-valued Brownian motion as in the definition of \(J^\nu\) above; we recall this equation in the first line of (5.23) below. In particular, \(g^0_{\omega}\) is a solution of (5.2) with the same \(u_{\omega}\) and \(\nu = 0\), which is an ODE for each fixed \(\omega \in \Omega\). Explicitly, suppose that \(g^\nu_{\omega}\) and \(\tilde{g}^\nu_{\omega}\) are the solutions of the following SDEs,

\[
\begin{cases}
    d g^\nu_{\omega}(t, \theta) = \sum_{i=1}^3 H_{1,i} d W_i(t) + u_{\omega}(t, g^\nu_{\omega}(t, \theta)) dt, & g^\nu_{\omega}(0, \theta) = \theta, \\
    d \tilde{g}^\nu_{\omega}(t) = \sum_{i=1}^3 H_{1,i} d W_i(t) + u_{\omega}(t, \tilde{g}^\nu_{\omega}(t)) dt, & \tilde{g}^\nu_{\omega}(0, \theta) = \theta,
\end{cases}
(5.23)
\]

where \(W_{\omega} = (W^1_{\omega}, W^2_{\omega}, W^3_{\omega})\) is a standard \(\mathbb{R}^3\)-valued Brownian motion, i.e., \(W^1_{\omega}, W^2_{\omega}, W^3_{\omega}\) are independent \(\mathbb{R}\)-valued Brownian motions, and the three components of \(\tilde{W}_{\omega}\) are equal, i.e., \(\tilde{W}^1_{\omega} = \tilde{W}^2_{\omega} = \tilde{W}^3_{\omega}\) are \(\mathbb{R}\)-valued Brownian motions. The constant maps \(H_{1,i} := \sqrt{2\nu}(1, 0, 0), H_{2,i} := \sqrt{2\nu}(0, 1, 0), H_{3,i} := \sqrt{2\nu}(0, 0, 1), H_{1,\mu} := \sqrt{2\mu}(1, 0, 0), H_{2,\mu} := \sqrt{2\mu}(0, 1, 0), H_{3,\mu} := \sqrt{2\mu}(0, 0, 1) : \mathbb{T}^3 \to \mathbb{R}^3, \nu, \mu \geq 0,\)
define vector fields on \(\mathbb{T}^3\) using the trivialization \(TT^3 = \mathbb{T}^3 \times \mathbb{R}^3\). Suppose \(u : \Omega \times [0, T] \to \mathcal{X}(\mathbb{T}^3)\) satisfies that \(u_{\omega}(t, \theta)\) is a \(\mathcal{P}_1\)-adapted semimartingale for every \(\theta \in \mathbb{T}^3\).

We can thus characterize the critical points \(\left( g^\nu_{\omega}, H_{1,i}, W^i_{\omega} \right)_{i=1}^3, \left( \tilde{g}^\nu_{\omega}, H_{1,i}, \tilde{W}^i_{\omega} \right)_{i=1}^3, g^0_{\omega}\) (when \(m_1 = m_2 = 3\)) of \(J^\nu\) as follows.

**Theorem 5.5** (SPDE case) \(\left( g^\nu_{\omega}, H_{1,i}, W^i_{\omega} \right)_{i=1}^3, \left( \tilde{g}^\nu_{\omega}, H_{1,i}, \tilde{W}^i_{\omega} \right)_{i=1}^3, g^0_{\omega}\) is a critical point of \(J^\nu\), using deformations (5.17) induced by (5.16), if and only if \((u_{\omega}, D_{\omega})\)
satisfies the following stochastic compressible Navier–Stokes equation,

\[
\begin{align*}
\mathrm{d}u_\omega(t) &= -u_\omega(t) \cdot \nabla u_\omega(t) \, \mathrm{d}t - \frac{1}{D_\omega(t)} \left( \sqrt{2} \nabla \nabla u_\omega(t) \cdot \mathrm{d}W_\omega(t) - v \Delta u_\omega(t) \, \mathrm{d}t \right) \\
\mathrm{d}D_\omega(t) &= -\mu \nabla \text{div} u_\omega(t) \, \mathrm{d}t + \nabla p_\omega(t) \, \mathrm{d}t,
\end{align*}
\]  
(5.24)

where \( D_\omega(t, \theta) \, \mathrm{d}^3 \theta := (g^{0}_{\omega}(t, \cdot)^{-1})^* (D_0(\theta) \, \mathrm{d}^3 \theta) \).

**Theorem 5.6 (PDE case)** Let

\[
\mathcal{J} \left( \left( g^{1}_{\omega}, w^{1,i}_{\omega}, M^{1,i}_{\omega} \right)_{i=1}^{m_1}, \left( g^{2}_{\omega}, w^{2,i}_{\omega}, M^{2,i}_{\omega} \right)_{i=1}^{m_2}, g^{3}_{\omega} \right) = \int_0^T \int_{\mathbb{R}^3} \left( \frac{1}{2} |w_\omega(t, \theta)|^2 \tilde{D}(t, \theta) - \tilde{D}(t, \theta) e \left( t, \tilde{D}(t, \theta) \right) \right) \, \mathrm{d}^3 \theta \, \mathrm{d}t
\]

\[
+ \int_0^T \tilde{p} \left( D \nabla_0 \left( (w^{1,i}_{\omega}, M^{1,i}_{\omega})_{i=1}^{m_1} \right), \left( D \nabla_0 \left( (w^{2,i}_{\omega}, M^{2,i}_{\omega})_{i=1}^{m_2} \right), g^{3}_{\omega} \right) \right) \, \mathrm{d}t,
\]

\( \forall \left( \left( g^{1}_{\omega}, w^{1,i}_{\omega}, M^{1,i}_{\omega} \right)_{i=1}^{m_1}, \left( g^{2}_{\omega}, w^{2,i}_{\omega}, M^{2,i}_{\omega} \right)_{i=1}^{m_2}, g^{3}_{\omega} \right) \in \mathcal{J}(G^t) \times \mathcal{J}(G^t) \times \mathcal{J}(G^t),
\]

where the specific internal energy \( e(t, \tilde{D}) \) is non-random, the non-random pressure is given by \( p = \tilde{p} \frac{\partial c}{\partial \tilde{D}} \), and

\[
w_{\omega}(t, \cdot) := T_{g^{1}_{\omega}(t)} R_{g^{1}_{\omega}(t)^{-1}} \left( \frac{\partial g^{1}_{\omega}(t)}{\partial t} \right), \quad \tilde{D}(t, \cdot) \, \mathrm{d}^3 \theta = \mathbb{E} \left[ \left( g^{3}_{\omega}(t, \cdot)^{-1} \right)^* \alpha_0 \right].
\]

Suppose that \( u_\omega = u \) in (5.2) and (5.23) is non-random and the deformations \( (5.16) \) are defined with \( v \) non-random. Then \( \left( g^{v}_{\omega}, H_{i,v}, W_{i,v} \right)_{i=1}^{3}, \left( g^{\mu}_{\omega}, H_{i,\mu}, \tilde{W}_{i,\mu} \right)_{i=1}^{3}, g^{0}_{\omega} \) is a critical point of \( \mathcal{J} \), using deformations \( (5.17) \) induced by \( (5.16) \), if and only if (the non-random variables) \( (u, \tilde{D}) \) satisfy the following (deterministic) classical Navier–Stokes equations for compressible fluid flow

\[
\begin{align*}
\frac{\partial u(t)}{\partial t} &= -(u(t) \cdot \nabla u(t)) \, \mathrm{d}t + \frac{1}{D(t)} \left( \nabla \Delta u(t) \, \mathrm{d}t + \mu \nabla \text{div} u(t) \, \mathrm{d}t - \nabla p(t) \, \mathrm{d}t \right), \\
\frac{\partial \tilde{D}(t)}{\partial t} &= -\text{div} \left( u(t) \tilde{D}(t) \right) \, \mathrm{d}t.
\end{align*}
\]  
(5.25)

**Proof of Theorem 5.5.** By (5.14), we know that \( D_\omega(t, \theta) \) satisfies the second equation of (5.24). As explained above, since Theorem 3.12 still holds for \( G^t \), it suffices to show that the first equation of (5.24) is just the first one in (3.37) for our model.
Relations (5.18)–(5.20), combined with the definition (5.23) of $\tilde{g}^\mu_{\omega}$, yield the identities

$$T_{\tilde{g}^\mu_{\omega}(t, \theta)} R_{\tilde{g}^\mu_{\omega}(t, \theta)^{-1}} \left( \frac{D V^0_i (H_{i, \mu}, \tilde{W}^i_{\omega})}{dt} \right)_{i j} = \nabla^0_{H_{j, \mu}} \frac{\tilde{W}^i_{\omega}}{dt} = 0,$$

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} T_{\tilde{g}^\mu_{\omega, \varepsilon}(t, \theta)} R_{\tilde{g}^\mu_{\omega, \varepsilon}(t, \theta)^{-1}} \left( \frac{D V^0_i (H_{i, \mu}, \tilde{W}^i_{\omega})}{dt} \right)_{i j} = \left( \nabla^0_{ad_{\omega}(t) H_{i, \mu}} H_{j, \mu} + \nabla^0_{H_{i, \mu}} (ad_{\omega}(t) H_{j, \mu}) \right) \frac{d[\tilde{W}^i_{\omega}, \tilde{W}^j_{\omega}]}{dt} = 2 \mu \partial_i \partial_j v_{\omega}(t, \theta).$$

(5.26)

For every $u, \tilde{u} \in \mathcal{X}^r (\mathbb{T}^3)$, $\alpha = D(\theta) d^3 \theta \in U^*$, we easily get the following formulas (recall that the stochastic force is given by $q(t, u, \alpha) := \langle u, \cdot \rangle$):

$$\frac{\delta l}{\delta u} (t, u, \alpha) = u D,$$

$$ad^*_H (u D) = (\text{div} u) u D + \nabla (D u) + \frac{D}{2} \nabla (|u|^2),$$

$$\sum_{i = 1}^3 ad^*_H q dW^i_{\omega}(t) = \sqrt{2} \nabla u \cdot dW_{\omega}(t),$$

$$\left( \frac{\delta l_{\omega}}{\delta \alpha} (t, u, \alpha) \right) \diamond \alpha = \frac{D}{2} \nabla (|u|^2) - \nabla p_{\omega}(t).$$

The last equality is obtained by repeating the argument in Holm et al. (1998, Section 7) (especially (7.4), (7.18), and (7.19)), even though $p_{\omega}(t)$ is random.

On the other hand, for every $A, A \in \mathcal{M}_n (G^3)$ and $B, B \in \mathcal{M}_m (G^3)$, we have,

$$\left( \frac{\delta \tilde{p}}{\delta A} (A, B, u), \tilde{A} \right) = \frac{1}{2} \int_{\mathbb{T}^3} u(\theta) \cdot \text{Tr} (\tilde{A}(\theta)) d^3 \theta,$$

$$\left( \frac{\delta \tilde{p}}{\delta B} (A, B, u), \tilde{B} \right) = \frac{1}{2} \sum_{i, j = 1}^m \int_{\mathbb{T}^3} P_i (u(\theta)) P_j ((\tilde{B})_{ij}(\theta)) d^3 \theta,$$

$$\left( \frac{\delta \tilde{p}}{\delta u} (A, B, u), \tilde{u} \right) = \frac{1}{2} \int_{\mathbb{T}^3} \tilde{u}(\theta) \cdot \text{Tr} (A(\theta)) d^3 \theta + \frac{1}{2} \sum_{i, j = 1}^m \int_{\mathbb{T}^3} P_i (\tilde{u}(\theta)) P_j ((B)_{ij}(\theta)) d^3 \theta.$$

(5.28)

Hence, using $M^1_{\omega} = W_{\omega}$, and (5.18), (5.26), (5.28), we get the formula for the operator $K$ defined by (3.14):

$$\langle K_{\omega} (t, A, B, u), \tilde{u} \rangle = - \sum_{i = 1}^3 \int_{\mathbb{T}^3} u(\theta) \cdot \left( \nabla^0_{[\tilde{u}, H_{i, v}], H_{i, v}} + \nabla^0_{H_{i, v}} [\tilde{u}, H_{i, v}] \right) (\theta) d^3 \theta$$

$$- \sum_{i, j = 1}^3 \int_{\mathbb{T}^3} u_i (\theta) P_j \left( \nabla^0_{[\tilde{u}, H_{i, \mu}], H_{j, \mu}} + \nabla^0_{H_{i, \mu}} [\tilde{u}, H_{j, \mu}] \right) (\theta) d^3 \theta.$$
that is,

\[ K_\omega (t, A, B, u) = -\nu \Delta u - \mu \nabla \text{div} u, \quad A, B \in \mathcal{M}_3(G^3), \quad u \in \mathcal{X}^s(T^3), \quad t \in [0, T], \]

\[ \frac{\delta \tilde{p}}{\delta u}(0, 0, u) = 0. \]  \hspace{1cm} (5.29)

Thus, combining the equalities above, the first equation in (3.37) becomes

\[ d(u_\omega(t)D_\omega(t)) = -\sqrt{2\nu} \nabla u_\omega(t) \cdot dW_\omega(t) - (\text{div} u_\omega(t)) u_\omega(t)D_\omega(t)dt \]

\[ -u_\omega(t) \cdot \nabla (u_\omega(t)D_\omega(t)) dt + \nu \Delta u_\omega(t)dt + \mu (\text{div} u_\omega(t))dt - \nabla p_\omega(t)dt. \]  \hspace{1cm} (5.30)

Using the second equation of (5.24), we get

\[ u_\omega(t)dD_\omega(t) = - (\text{div} u_\omega(t)) u_\omega(t)D_\omega(t)dt - (u_\omega(t) \cdot \nabla D_\omega(t)) u_\omega(t)dt \]  \hspace{1cm} (5.31)

and together with (5.30), we obtain the first equation of (5.24).

Proof of Theorem 5.6. This follows carrying out the same computations as in the previous proof and the one in Theorem 3.13.

Remark 5.7 We emphasize that the usual Navier–Stokes equations for compressible fluids (5.25) were deduced from our stochastic variational principle, without any appeal to thermodynamic considerations in order to get the dissipative terms; these terms appear entirely due to the type of stochastic processes we consider.

Remark 5.8 For the incompressible case, i.e., \( D_\omega(t, \theta) = 1 \), Eq. (5.24) becomes

\[ du_\omega(t) = -\sqrt{2\nu} \nabla u_\omega \cdot dW_\omega(t) - u_\omega(t) \cdot \nabla u_\omega dt + \nu \Delta u_\omega dt - \nabla p_\omega(t) dt, \]

which is a stochastic incompressible Navier–Stokes equation.

Remark 5.9 \( \tilde{p} = 0 \) in the definition of \( J' \) (formula (5.22)) and following the same steps as in Theorem 5.5, it is easy to verify that the associated critical point \( (u_\omega(t), D_\omega(t)) \) of \( J' \) satisfies the following stochastic compressible Euler equation,

\[ \begin{cases} 
du_\omega(t) = -u_\omega(t) \cdot \nabla u_\omega(t) dt - \frac{1}{D_\omega(t)^{\frac{1}{2}}} \left( \sqrt{2\nu} \nabla u_\omega(t) \cdot dW_\omega(t) + \nabla p_\omega(t)dt \right), \\
D_\omega(t) = -\text{div} (u_\omega(t)D_\omega(t)) dt.
\end{cases} \]  \hspace{1cm} (5.32)

Remark 5.10 We illustrate here how the contraction force \( \tilde{p} \), defined by (5.21), gives rise to the term modeling viscosity in the compressible Navier–Stokes equation (and MHD equation later). Other choices for the contraction force \( \tilde{p} \) yield different dissipative equations.
For example, let $\tilde{p} : \mathcal{M}(G^s) \times X^s(\mathbb{T}^3) \times U^* \to \mathbb{R}$ be defined by

$$\tilde{p} \left( A, u, D(\theta)d^3\theta \right) = \frac{1}{2} \int_{\mathbb{T}^3} \text{Tr}(A)(\theta) \cdot u(\theta) D(\theta)d^3\theta, \quad (5.32)$$

where $A \in \mathcal{M}(G^s), u \in X^s(\mathbb{T}^3), D(\theta)d^3\theta \in U^*$. Define the action functional $J^\nu$ in the same way as in Theorem 5.5 with $\tilde{p}$ replaced by the expression in (5.32).

As explained in Remark 3.8, although $\tilde{p}$ depends on $U^*$, we can repeat the procedure in Theorem 5.5 to show that

$$\left( g^x_{\omega}, H_i, \nu, W_i^{i} \right)_{i=1}^3 \left( g^x_{\omega}, H_i, \mu, \tilde{W}_i^{i} \right)_{i=1}^3 \left( g^x_{\omega} \right)$$

is a critical point of $J^\nu$ if and only if $(u_{\omega}(t), D_{\omega}(t))$ satisfies the following system of equations:

$$
\begin{aligned}
\frac{du_{\omega}(t)}{dt} &= -u_{\omega}(t) \cdot \nabla u_{\omega}(t) dt + 2\nu \langle \nabla u_{\omega}(t), \nabla \log D_{\omega}(t) \rangle dt + \nu \Delta u_{\omega}(t) dt \\
-\frac{\sqrt{2\nu \nabla u_{\omega}(t)}}{D_{\omega}(t)} \cdot dW_{\omega}(t) - \frac{\nabla D_{\omega}(t)}{D_{\omega}(t)} dt,
\end{aligned}
\quad
dD_{\omega}(t) = -\frac{\nabla D_{\omega}(t) \cdot dW_{\omega}(t)}{\Delta} - \text{div} \left( u_{\omega}(t) D_{\omega}(t) \right) dt + \nu \Delta D_{\omega}(t) dt.
\$$

The term $\langle \nabla u_{\omega}(t), \nabla \log D_{\omega}(t) \rangle$ in the equation above is crucial for energy dissipation. This term also appears in Brenner’s model; see, e.g., Brenner (2005a, b), Feireisl and Vasseur (2010).

\subsection*{5.3 Compressible MHD Equation}

Let $\alpha_0 := (b_0(\cdot), B_0(\cdot) \cdot dS, D_0(\theta)d^3\theta)$, where $b_0$ is a $C^2$ function on $\mathbb{T}^3, B_0(\theta) \cdot dS$ is an exact two-form on $\mathbb{T}^3$, i.e., there is some one-form $A_0(\theta) \cdot d\theta$ such that

$$B_0(\theta) \cdot dS = d\left( A_0(\theta) \cdot d\theta \right) = \sum_{1 \leq j < k \leq 3, i \neq j, i \neq k} (\text{curl}A_0(\theta))_j d\theta_j \wedge d\theta_k, \quad (5.33)$$

and $D_0(\theta)d^3\theta$ is a density on $\mathbb{T}^3$. We let $U^*$ denote the vector space of all such triples $(b(\cdot), B(\cdot) \cdot dS, D(\theta)d^3\theta)$.

As in Holm et al. (1998, equation (7.16)), let $I : \Omega \times \{0, T\} \times X^s(\mathbb{T}^3) \times U^* \to \mathbb{R}$ be defined by

$$I(w_\omega(t), \omega_\omega(t), B_\omega(t), D_\omega(t))$$

$$= \int_{\mathbb{T}^3} \left( \frac{D_\omega(t, \theta)}{2} |w_\omega(t, \theta)|^2 - D_\omega(t, \theta)e(D_\omega(t, \theta), \omega_\omega(t, \theta)) - \frac{1}{2} |B_\omega(t, \theta)|^2 \right) d^3\theta,$$

where for every $\omega \in \Omega, t \in [0, T]$, we have $w_\omega(t) \in X^s(\mathbb{T}^3), b_\omega(t) \in C^2(\mathbb{T}^3)$ is the entropy function, $B_\omega(\theta) \cdot dS$ is an exact differential two-form as in (5.33) representing the magnetic field in the fluid, $D_\omega(t)d^3\theta$ is a density on $\mathbb{T}^3$ representing the mass density of the fluid, and $e$ is the fluid’s specific internal energy (a $C^2$ function of two real variables).
The pressure $p$ and the temperature $T$ of the fluid are given in terms of a thermodynamic equation of state for the specific internal energy $e$, namely $de = -p\left(\frac{\partial}{\partial \rho}\right) + T(t) d\rho$, i.e., $p = D^2 \frac{de}{d\rho}$, $T = \frac{de}{d\rho}$. We assume that the stochastic processes $p_\omega(t) := p(D_\omega(t), b_\omega(t)) \in C^2(\mathbb{T}^3)$ and $T_\omega(t) := T(D_\omega(t), b_\omega(t)) \in C^2(\mathbb{T}^3)$ are $\mathcal{F}_t$-measurable for all $t$. Note that $e_\omega(t) := e(D_\omega(t), b_\omega(t))$ and $l_\omega(t) := l(w_\omega(t), b_\omega(t), B_\omega(t), D_\omega(t))$ are also stochastic processes. As explained in Holm et al. (1998, Section 7) it is assumed that $e_\omega^2 := \frac{\partial p_\omega}{\partial D_\omega} > 0$, where $e_\omega$ is the adiabatic sound speed.

As in Sect. 5.2, we work with a contraction force $\tilde{p} : \mathcal{M}(G^5) \times \mathcal{M}(G^3) \times \mathcal{X}^5(\mathbb{T}^3) \rightarrow \mathbb{R}$, defined by (5.21), and a stochastic force $q : \mathcal{X}^5(\mathbb{T}^3) \times \mathcal{U}^5 \rightarrow (\mathcal{X}^5(\mathbb{T}^3))^*$, defined by

$$q(u, \alpha) := \langle u, \cdot \rangle, \ \forall u \in \mathcal{X}^5(\mathbb{T}^3).$$

With $\nabla^0, l, \tilde{p}, q, \alpha_0, (H_{i,v}, W_\omega^j)_{i=1}^3$ as in Sect. 5.2, we define the action functional $J^\nu_1 := J^\nu_1(H_{i,v}, W_\omega^j)_{i=1}^3$ according to (3.31), which in this particular case becomes

$$J^\nu_1 \left( \left( g_\omega^1, w_\omega^j, M_\omega^1 \right)_{i=1}^{m_1}, \left( g_\omega^2, w_\omega^j, M_\omega^2 \right)_{i=1}^{m_2}, g_\omega^3, g_\omega^4, g_\omega^5 \right) = \int_0^T l_\omega(t, w_\omega(t, \cdot), B_\omega(t, \cdot), b_\omega(t, \cdot), D_\omega(t, \cdot)) dt$$

$$+ \int_0^T \tilde{p} \left( \frac{\nabla^0(W_\omega^j, M_\omega^1)_{i=1}^{m_1} g^1_\omega(t)}{dt} + \frac{\nabla^0(W_\omega^j, M_\omega^2)_{i=1}^{m_2} g^2_\omega(t)}{dt}, w_\omega(t) \right) dt$$

$$+ \int_0^T \int_{\mathbb{T}^3} \left( w_\omega(t, \theta), d\Xi_\omega(t, \theta) \right) d^3 \theta$$

$$- \sum_{i=1}^3 \sqrt{2v} \int_0^T \int_{\mathbb{T}^3} w_\omega^i(t, \theta) d^3 \theta dW^i(t),$$

for all $\left( g_\omega^1, w_\omega^j, M_\omega^1 \right)_{i=1}^{m_1}, \left( g_\omega^2, w_\omega^j, M_\omega^2 \right)_{i=1}^{m_2}, g_\omega^3, g_\omega^4, g_\omega^5 \in \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5)$,

where

$$w_\omega(t, \cdot) := T_{g_\omega^1(t)} R_{g_\omega^1(t)}^{-1} \left( \frac{\partial g^1_\omega(t)}{dt} \right) = (w_{\omega,1}(t, \cdot), w_{\omega,2}(t, \cdot), w_{\omega,3}(t, \cdot))$$

$$d\Xi_\omega(t, \cdot) := T_{g_\omega^1(t)} R_{g_\omega^1(t)}^{-1} \left( d^\alpha g^1_\omega(t) \right)$$

$$D_\omega(t, \cdot) d^3 \theta := \left( g_\omega^3(t, \cdot) \right)^{-1} \left( D_0(\theta) d^3 \theta \right)$$

$$B_\omega(t, \theta) \cdot dS := \left( g_\omega^4(t, \cdot) \right)^{-1} \left( B_0(\theta) \cdot dS \right)$$

$$b_\omega(t, \theta) := \left( g_\omega^5(t, \cdot) \right)^{-1} b_0,$$
and $W_\omega(t)$ is a standard $\mathbb{R}^3$-valued Brownian motion.

Let $g^v_\omega$ be the solution of (5.2) with $v > 0$ and $M_\omega = W_\omega$ the same $\mathbb{R}^3$-valued Brownian motion as in the definition of $J^v_1$ above. Although in the definition of $J^v_1$, five semimartingales are needed, we can define its critical points in the same way as in (3.9) along deformations (5.16). Moreover, the critical points

$$\left(\left(g^v_\omega, H_{i,v}, W^i_\omega\right)_{i=1}^3, \left(g^\mu_\omega, H_{i,\mu}, \tilde{W}^i_\omega\right)_{i=1}^3, g^0_\omega, g^{v_1}_\omega, g^{v_2}_\omega\right)$$

of $J^v_1$ are characterized as follows.

**Theorem 5.11 (SPDE case)**

$$\left(\left(g^v_\omega, H_{i,v}, W^i_\omega\right)_{i=1}^3, \left(g^\mu_\omega, H_{i,\mu}, \tilde{W}^i_\omega\right)_{i=1}^3, g^0_\omega, g^{v_1}_\omega, g^{v_2}_\omega\right)$$

is a critical point of $J^v_1$ if and only if the following stochastic compressible MHD equations hold for $(u_\omega(t), b_\omega(t), B_\omega(t), D_\omega(t))$,

$$
\begin{align*}
\frac{du_\omega(t)}{dt} &= -u_\omega(t) \cdot \nabla u_\omega(t) dt - \frac{1}{D_\omega(t)} \left(\sqrt{2\nu} \nabla u_\omega(t) \cdot dW_\omega(t) - \nu \Delta u_\omega(t) dt - \mu \nabla \text{div} u_\omega(t) dt - B_\omega(t) \times \text{curl} B_\omega(t) dt + \nabla p_\omega(t) dt\right), \\
\frac{dB_\omega(t)}{dt} &= -\sqrt{2\nu} \nabla B_\omega(t) \cdot dW_\omega(t) + \text{curl}(u_\omega(t) \times B_\omega(t)) dt + v_1 \Delta B_\omega(t) dt, \\
\frac{db_\omega(t)}{dt} &= -\sqrt{2\nu} \nabla b_\omega(t) \cdot dW_\omega(t) - u_\omega(t) \cdot \nabla b_\omega(t) dt + v_2 \Delta b_\omega(t) dt,
\end{align*}
$$

(5.35)

where $g^v_\omega$ and $g^\mu_\omega$ are the solution of the SDE (5.2) and (5.23) respectively, $D_\omega(t, \theta) d^3\theta = \left(g^0_\omega(t, \cdot)^{-1}\right)^* (D_0(\theta) d^3\theta)$, $B_\omega(t, \theta) \cdot dS := (g^v_\omega(t, \cdot)^{-1})^* (B_0(\theta) \cdot dS)$, $b_\omega(t, \theta) := (g^{v_1}_\omega(t, \cdot)^{-1})^* b_0$.

**Theorem 5.12 (PDE case)**

$$
\begin{align*}
J^v_1 \left(\left(g^1_\omega, w^{1,i}_\omega, M^{1,i}_\omega\right)_{i=1}^{m_1}, \left(g^2_\omega, w^{2,i}_\omega, M^{2,i}_\omega\right)_{i=1}^{m_1}, g^3_\omega, g^4_\omega, g^5_\omega\right) \\
&= \int_0^T l \left(t, w_\omega(t, \cdot), \tilde{B}_\omega(t, \cdot), \tilde{b}_\omega(t, \cdot), \tilde{D}_\omega(t, \cdot)\right) dt \\
&\quad + \int_0^T \tilde{\mathcal{P}} \left(\frac{D^{v_1}_\omega(w^{1,i}_\omega, M^{1,i}_\omega)_{i=1}^{m_1} g^1_\omega(t)}{dt} - \frac{D^{v_2}_\omega(w^{2,i}_\omega, M^{2,i}_\omega)_{i=1}^{m_2} g^2_\omega(t)}{dt}, w_\omega(t)\right) dt, \\
&\forall \left(\left(g^1_\omega, w^{1,i}_\omega, M^{1,i}_\omega\right)_{i=1}^{m_1}, \left(g^2_\omega, w^{2,i}_\omega, M^{2,i}_\omega\right)_{i=1}^{m_2}, g^3_\omega, g^4_\omega, g^5_\omega\right) \in \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5) \times \mathcal{F}(G^5),
\end{align*}
$$

where $l$ is non-random (hence the pressure $p(t)$ and temperature $T(t)$ are non-random), $w_\omega(t, \cdot) := T_{g^1_\omega(t)} R_{g^1_\omega(t)^{-1}} \left(\phi_{g^1_\omega(t)}(\cdot)\right)$, $\tilde{B}(t, \theta) \cdot dS := \mathbb{E}\left[(g^{v_1}_\omega(t, \cdot)^{-1})^* (B_0(\theta) \cdot dS)\right]$, $\tilde{b}(t, \theta) := \mathbb{E}\left[(g^{v_2}_\omega(t, \cdot)^{-1})^* b_0\right]$, $\tilde{D}(t, \cdot) d^3\theta = \mathbb{E}\left[(g^0_\omega(t, \cdot)^{-1})^* (D_0(\theta) d^3\theta)\right]$.

Suppose $u_\omega = u$ is non-random in (5.2), (5.23) and that in the deformation $v$ in (5.16) is also non-random. Then $\left(g^v_\omega, H_{i,v}, W^i_\omega\right)_{i=1}^3, \left(g^\mu_\omega, H_{i,\mu}, \tilde{W}^i_\omega\right)_{i=1}^3, g^0_\omega, g^{v_1}_\omega, g^{v_2}_\omega$ is a critical point of $J^1_1$ if and only if $\left(u, \tilde{B}, \tilde{D}, \tilde{b}\right)$ satisfies the following
compressible MHD equations

\[
\begin{align*}
\text{d}u(t) &= -u(t) \cdot \nabla u(t) \, dt + \frac{1}{\text{D}(t)} \left( \nu \Delta u(t) + \mu \nabla \text{div}(u(t)) - \tilde{B}(t) \times \text{curl}\tilde{B}(t) \right) dt - \nabla p(t) \, dt, \\
\text{d}\tilde{D}(t) &= -\text{div}(u(t) \cdot \tilde{D}(t)) dt, \\
\text{d}\tilde{B}(t) &= \text{curl}(u(t) \times \tilde{B}(t)) dt + v_1 \Delta \tilde{B}(t) dt, \\
\text{d}\tilde{b}(t) &= -u(t) \cdot \nabla \tilde{b}(t) dt + v_2 \Delta \tilde{b}(t) dt,
\end{align*}
\tag{5.36}
\]

here, \( g^\nu_{\omega} \) and \( \tilde{g}^\mu_{\omega} \) are the solution of the SDE (5.2) and (5.23), respectively.

**Proof** (Theorem 5.11) Equation (5.14) implies that \( D_\omega(t, \theta) \) satisfies the second equation of (5.35). Since \( B_0(\theta) \cdot \text{d}S = \text{d}(A_0(\theta) \cdot \text{d}\theta) \) for some one-form \( A_0(\theta) \cdot \text{d}\theta \), it follows that

\[
B_\omega(t, \theta) \cdot \text{d}S = \left( g^{\nu_1}_{\omega}(t, \theta) \right)^{-1} \ast \left( B_0(\theta) \cdot \text{d}S \right)
= \left( g^{\nu_1}_{\omega}(t, \theta) \right)^{-1} \ast \left( B_0(\theta) \cdot \text{d}\theta \right)
= \text{d}((g^{\nu_1}_{\omega}(t, \theta)^{-1}) \ast (A_0(\theta) \cdot \text{d}\theta))(\theta)
= \text{d}(A_\omega(t, \theta) \cdot \text{d}\theta),
\]

where

\[
A_\omega(t, \theta) \cdot \text{d}\theta := \left( g^{\nu_1}_{\omega}(t, \theta)^{-1} \right) \ast \left( A_0(\theta) \cdot \text{d}\theta \right), \quad \text{curl}A_\omega(t) := B_\omega(t).
\]

By Proposition 5.1, Eq. (5.4) holds for \( A_\omega(t) \) with viscosity constant \( \nu = v_3 \), and hence \( B_\omega(t) = \text{curl}A_\omega(t) \) satisfies the third equation of (5.35). We also have \( \nabla \cdot B_\omega(t) = \nabla \cdot (\text{curl}A_\omega(t)) = 0 \).

In the same way, we verify that the fourth equation in (5.35) holds for \( b_\omega(t) \).

According to Theorem 3.12, (5.20), (5.29) (which implies that \( \frac{\delta \theta}{\delta t} (\tilde{H}_{\omega,1}, \tilde{H}_{\omega,2}, u_\omega(t)) \equiv 0 \) here), we conclude that \( \left( (g^0_{\omega}, H_i, v, W_i)_{i=1}^3, \left( \tilde{g}^\mu_{\omega}, H_i, \mu, \tilde{W}_i \right)_{i=1}^3, g^{\nu_1}_{\omega}, g^{\nu_2}_{\omega}, g^{\nu_3}_{\omega} \right) \) is a critical point of \( J_1 \) if and only if the following equation holds

\[
\text{d} \left( \frac{\delta l_{\omega}}{\delta u} \right)(t) = -\sum_{i=1}^3 \text{ad}^*_{H_{i,v}} q(t, u_\omega, b_\omega, B_\omega, D_\omega) dW_{i_\omega}(t) - \text{ad}^*_{u_\omega(t)} \frac{\delta l_{\omega}}{\delta u} dt + \frac{\delta l_{\omega}}{\delta B} \cdot b_\omega dt
+ \frac{\delta l_{\omega}}{\delta D} \cdot B_\omega dt + \frac{\delta l_{\omega}}{\delta D} \cdot D_\omega dt - K_\omega \left( t, \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t), u_\omega(t) \right) dt,
\tag{5.37}
\]

where \( K_\omega, \tilde{H}_{\omega,1}(t), \tilde{H}_{\omega,2}(t) \) are the same terms as in (3.37).

From the computation in Holm et al. (1998, Section 7), particularly (7.4), (7.18), and (7.19), we get,

\[
\frac{\delta l_{\omega}}{\delta B} \cdot b_\omega + \frac{\delta l_{\omega}}{\delta B} \cdot B_\omega + \frac{\delta l_{\omega}}{\delta D} \cdot D_\omega = \frac{D_\omega}{2} \nabla \left( |u_\omega|^2 \right) + B_\omega \times \text{curl}B_\omega - \nabla p_\omega.
\]

Combining all of the above with (5.18)–(5.20), (5.27)–(5.29), into (5.37) yields,

\[
\text{d} \left( u_\omega(t) D_\omega(t) \right) = -\sqrt{2v} \nabla u_\omega(t) \cdot \text{d}W_\omega(t) - (\text{div}u_\omega(t)) u_\omega(t) D_\omega(t) dt
\]
\[-u_\omega(t) \cdot \nabla (u_\omega(t) D_\omega(t)) \, dt + B_\omega(t) \times \text{curl} B_\omega(t) \, dt \]
\[+ \nu \Delta u_\omega(t) \, dt + \mu \nabla \text{div} u_\omega(t) \, dt - \nabla p_\omega(t) \, dt. \quad (5.38)\]

Putting the second equation of (5.35) into (5.38), we derive the first equation in (5.35).

\[\Box\]

**Proof of Theorem 5.12** The proof of (5.36) follows by repeating the same computations as above and the ones in the proof of Theorem 3.13.

\[\Box\]

**Remark 5.13** The reason for choosing processes \(g_{vi}\) with different constants \(v_i\) is that the viscosity constants in Eq. (5.35) are different.

\[\Diamond\]

**Remark 5.14** In particular, if we take \(D(t) = 1, b(t) = 1\) for every \(t \in [0, T]\) in (5.36), we obtain the following incompressible viscous MHD equations (see, e.g., Sermange and Temam 1983),

\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p + B \times \text{curl} B = \nu/Delta_1 \, u \\
\partial_t B = \text{curl}(u \times B) + \nu_1 \Delta B \\
\text{div} u = 0.
\end{cases}
\]

\[\Diamond\]

### 5.4 Stochastic Kelvin–Noether Theorem in Continuum Mechanics

We now apply the results on Sect. 4 in continuum mechanics. Following the formulation in Holm et al. (1998, Section 6), we take here \(G = G^s\) (so \(T_e G = X^s(T^3)\)), \(U\) a linear space whose formal dual \(U^*\) represents the advection terms (such as mass density, entropy, the magnetic field viewed as a differential two-form, etc.), \(\mathcal{C} = \{\gamma \in C([0, 1]; G^s) \mid \gamma(0) = \gamma(1)\}\) the set of all continuous \(G^s\)-valued loops.

As explained in Holm et al. (1998, Section 6), the dual \((X^s(T^3))^* = \Omega^s(T^3) \otimes \text{Den}(T^3)\), where \(\Omega^s(T^3)\) denotes the space of \(H^s\)-differential one-forms on \(T^3\), and \(\text{Den}(T^3)\) is the set of all densities on \(T^3\). Given a mass density \(\rho = \rho(\theta) d^3 \theta\), we define the circulation map \(\mathcal{K} : \mathcal{C} \times U^* \to (X^s(T^3))^*\) by

\[
\langle \mathcal{K}(\gamma, a), \alpha \rangle = \oint_{\gamma(\cdot)} \frac{\alpha}{\rho}, \quad \gamma \in \mathcal{C}, \quad a \in U^*, \quad \alpha \in (X^s(T^3))^*.
\]

Since \(\alpha \in (X^s(T^3))^* = \Omega^s(T^3) \otimes \text{Den}(T^3), \rho \in \text{Den}(T^3), \text{ and } \frac{\alpha}{\rho} \in \Omega^s(T^3),\) the circulation integral above is well-defined.

Let \(\mathcal{L}_u\) denote the Lie derivative in the direction \(u \in X^s(T^3)\). As shown in Holm et al. (1998, Page 37, formula (6.2)), we have

\[
ad^*_u V = \mathcal{L}_u V, \quad u \in X^s(T^3), \quad V \in (X^s(T^3))^*.
\]

Suppose that \(g_{\omega}^{vi}(\cdot)\) is the solution of (5.2) with \(M_\omega = W_\omega\) being a standard \(\mathbb{R}^3\)-valued Brownian motion and \(\tilde{g}_\omega^{vi}(\cdot)\) is the solution to (5.23). As illustrated above, Theorem 3.12 still holds for the infinite-dimensional group \(G^s\).

\[\square\]
So, for a given Lagrangian functional $l : \Omega \times [0, T] \times \mathcal{X}^s(\mathbb{T}^3) \times U^* \to \mathbb{R}$ such that $\frac{\delta l}{\delta u}$ is non-random, and supposing that $\tilde{p} : \mathcal{M} \times \mathcal{M} \times \mathcal{X}^s(\mathbb{T}^3) \to \mathbb{R}$, $q : [0, T] \times \mathcal{X}^s(\mathbb{T}^3) \times U^* \to \mathbb{R}$ are the same terms as in Sect. 5.2, we define the action functional $J^{\mathbb{V},(H_{i,v},W_{i})_{i=1}^{3}}$ by (5.22). Hence, by Theorem 3.12 and the computations in Sect. 5.1 above, we conclude that $(5.20)$ and $(5.28)$ imply $\tilde{H}_1(t) = \tilde{H}_2(t) \equiv 0$ and $\frac{\delta \tilde{p}}{\delta u} (0, 0, u_\omega(t)) \equiv 0$,

$$
d \frac{\delta I_\omega}{\delta u}(t) = -\sqrt{2} \sum_{i=1}^{3} \partial_t u_\omega(t) dW^i_\omega(t) - \text{ad}^*_{u_\omega(t)} \left( \frac{\delta I_\omega}{\delta u}(t) \right) dt + \frac{\delta I_\omega}{\delta \alpha}(t) \cdot a_\omega(t) dt + v \Delta u_\omega(t) dt + \mu \nabla \text{div} u_\omega(t) dt,
$$

$$
\text{d}a_\omega(t) = -\sum_{i=1}^{3} \mathcal{L}_{H_{i,v1}} a_\omega(t) dW^i_\omega(t) - \mathcal{L}_{u_\omega(t)} a_\omega(t) dt + \frac{1}{2} \sum_{i=1}^{3} \mathcal{L}_{H_{i,v1}} \mathcal{L}_{H_{i,v1}} a_\omega(t) dt,
$$

(5.39)

where $u_\omega(\cdot)$ denotes the drift in (5.2), $\frac{\delta I_\omega}{\delta u}(t) := \frac{\delta I_\omega}{\delta u}(t, u_\omega(t), a_\omega(t))$, and $\frac{\delta l}{\delta a}(t) := \frac{\delta l}{\delta a}(t, u_\omega(t), a_\omega(t))$. Here, we have also applied the property that $q(t, u, \alpha) = (u, \cdot)$.

Given $\gamma_0 \in \mathcal{C}$, the Kelvin–Noether quantity $I : \mathcal{C} \times \mathcal{X}^s(\mathbb{T}^3) \times U^* \to \mathbb{R}$ is defined by

$$
I_\omega(t) := \oint_{\gamma_0(t)} \frac{1}{\rho_\omega(t)} \frac{\delta l_\omega}{\delta u}(t),
$$

(5.40)

where $\gamma_0(t)(\cdot) := \gamma_0(g^v_\omega(t), \cdot)$, $g^v_\omega(t)$ is a solution of the first equation in (5.23), $(u_\omega(t), a_\omega(t))$ is a solution of equation (5.39), and $\rho_\omega(t) = \rho((g^v_\omega)^{-1}(t, \theta)) d^3 \theta$.

The stochastic Kelvin–Noether Theorem on $G^3$ takes the following form.

**Proposition 5.15** Let $I_\omega(t)$ be defined by (5.40) and suppose that $(u_\omega(t), a_\omega(t))$ satisfies (5.39) with $v = v_1$ and $\frac{\delta l}{\delta u} = q = (u_\omega, \cdot)$. Then we have

$$
\text{d}I_\omega(t) = \oint_{\gamma_0(t)} \frac{1}{\rho_\omega(t)} \left( \frac{\delta I_\omega}{\delta a}(t) \cdot a_\omega(t) + \mu \nabla \text{div} u_\omega(t) \right) dt.
$$

(5.41)

**Proof** By definition of $\gamma_0(t)(\cdot)$ and the change of variables formula, we obtain

$$
I_\omega(t) = \oint_{\gamma_0(t)} \frac{1}{\rho_\omega(t)} \frac{\delta I_\omega}{\delta u}(t) = \oint_{\gamma_0(t)} \frac{1}{\rho_\omega(t)} \left( g^v_\omega(t) \right)^* \left[ \frac{\delta I_\omega}{\delta u}(t) \right]
$$

(5.42)
By carefully tracking the proof of Proposition 5.1, we know that the following right invariant version of (4.6) still holds
\[
\frac{d}{dt} \delta_{\mathbf{u}}(t) = \left\{ \sum_{i=1}^{3} \text{ad}_{H_{i,v}}^{\mathbb{R}} \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dW^{i}_{\omega}(t) + \text{ad}_{\mathbf{u}}^{\mathbb{R}} \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dt + \frac{d}{dt} \delta_{\mathbf{u}}(t) \right) + \frac{1}{2} \sum_{i=1}^{3} \left( \text{ad}_{H_{i,v}}^{\mathbb{R}} \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dt + 2 \text{ad}_{H_{i,v}}^{\mathbb{R}} d\left[ W^{i}_{\omega}(t), \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right] \right) \right\} .
\]

Then, replacing here \( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \) by its expression given in (5.39) and using the identity
\[
\langle u_{\omega}(t), \cdot \rangle = g_{\omega}(t) = \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) ,
\]
we get
\[
\frac{d}{dt} \delta_{\mathbf{u}}(t) = \left\{ \sum_{i=1}^{3} \text{ad}_{H_{i,v}}^{\mathbb{R}} \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dW^{i}_{\omega}(t) + \text{ad}_{\mathbf{u}}^{\mathbb{R}} \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dt + \frac{d}{dt} \delta_{\mathbf{u}}(t) \right) + \frac{1}{2} \sum_{i=1}^{3} \left( \text{ad}_{H_{i,v}}^{\mathbb{R}} \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) dt + 2 \text{ad}_{H_{i,v}}^{\mathbb{R}} d\left[ W^{i}_{\omega}(t), \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right] \right) \right\} \cdot
\]

Combining this with (5.42) yields
\[
\frac{dI_{\omega}(t)}{\rho_{\omega}(t)} = \int_{\gamma_{\omega}(t)} \frac{1}{\rho_{\omega}(t)} \left( g_{\omega}(t) \right)^{\mathbb{R}} \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) dt
\]
\[
= \int_{\gamma_{\omega}(t)} \frac{1}{\rho_{\omega}(t)} \left( g_{\omega}(t) \right)^{\mathbb{R}} \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) dt + \mu \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \cdot \mathbf{u}_{\omega}(t) dt
\]
\[
= \int_{\gamma_{\omega}(t)} \frac{1}{\rho_{\omega}(t)} \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) \left( \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \right) dt + \mu \frac{\delta_{\mathbf{u}}}{\delta \mathbf{u}}(t) \cdot \mathbf{u}_{\omega}(t) dt
\]
which finishes the proof of (5.41).
\[
\square
\]

\textbf{Remark 5.16} It is worthwhile to note that the second summand in the integrand of (5.41) is the bulk viscosity term appearing in the classical Navier–Stokes equations for compressible fluid flow, except that here, the velocity is random.

\textbf{Remark 5.17} Consider the following stochastic incompressible MHD equation (i.e., \( D_{\omega}(t) \equiv 1, b_{\omega}(t) \equiv 1 \)),
\[
\begin{cases}
\frac{d\mathbf{u}_{\omega}(t)}{dt} + \mathbf{u}_{\omega}(t) \cdot \nabla\mathbf{u}_{\omega}(t) dt + \nabla p_{\omega}(t) dt + \mathbf{B}_{\omega}(t) \times \text{curl}\mathbf{B}_{\omega}(t) dt = v\Delta\mathbf{u}_{\omega}(t) dt + \sqrt{2v} \nabla \mathbf{u}_{\omega}(t) \cdot dW_{\omega}(t), \\
\frac{d\mathbf{B}_{\omega}(t)}{dt} = \text{curl}(\mathbf{u}_{\omega}(t) \times \mathbf{B}_{\omega}(t)) dt + v\Delta\mathbf{B}_{\omega}(t) dt + \sqrt{2v} \nabla \mathbf{B}_{\omega}(t) \cdot dW_{\omega}(t), \\
\text{div}\mathbf{u}_{\omega}(t) = 0.
\end{cases}
\]

Note that
\[
l \left( \mathbf{u}_{\omega}(t), \mathbf{B}_{\omega}(t) \right) = \int_{T^{3}} \left( \frac{1}{2} |\mathbf{u}_{\omega}(t, \theta)|^{2} - \frac{1}{2} |\mathbf{B}_{\omega}(t, \theta)|^{2} \right) d^{3}\theta.
\]
It is easy to verify that \( \frac{d\mathbf{l}}{du} = q = \langle u_\omega, \cdot \rangle \), so we can apply Theorem 4.1 to obtain
\[
\frac{d}{dt} I_\omega(t) = \oint_{\gamma_\omega(t)} \frac{1}{\rho_\omega(t)} (\mathbf{B}_\omega(t) \times \text{curl} \mathbf{B}_\omega(t) + v \Delta u_\omega(t) - v \Delta \mathbf{B}_\omega(t)) \, dt.
\]

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