ATTRACTORS AND THEIR STABILITY ON BOUSSINESQ TYPE EQUATIONS WITH GENTLE DISSIPATION

ZHILIAN YANG*
School of Mathematics and Statistics, Zhengzhou University
No.100, Science Road, Zhengzhou 450001, China

PENGYAN DING
Institute of Applied Physics and Computational Mathematics
Beijing 100088, China

XIAOBIN LIU
School of Mathematics and Statistics, Zhengzhou University
No.100, Science Road, Zhengzhou 450001, China

(Communicated by Alain Miranville)

Abstract. The paper investigates longtime dynamics of Boussinesq type equations with gentle dissipation: $u_{tt} + \Delta^2 u + (-\Delta)^\alpha u_t - \Delta f(u) = g(x)$, with $\alpha \in (0, 1)$. For general bounded domain $\Omega \subset \mathbb{R}^N (N \geq 1)$, we show that there exists a critical exponent $p_\alpha \equiv \frac{N+2(2\alpha-1)}{(N-2)\alpha}$ depending on the dissipative index $\alpha$ such that when the growth $p$ of the nonlinearity $f(u)$ is up to the range: $1 \leq p < p_\alpha$, (i) the weak solutions of the equations are of additionally global smoothness when $t > 0$; (ii) the related dynamical system possesses a global attractor $A_\alpha$ and an exponential attractor $A_{\alpha,exp}$ in natural energy space for each $\alpha \in (0, 1)$, respectively; (iii) the family of global attractors $\{A_\alpha\}$ is upper semicontinuous at each point $\alpha_0 \in (0, 1]$, i.e., for any neighborhood $U$ of $A_{\alpha_0}$, $A_\alpha \subset U$ when $|\alpha - \alpha_0| \ll 1$. These results extend those for structural damping case: $\alpha \in [1, 2)$ in [31, 32].

1. Introduction. In this paper, we are concerned with longtime dynamics of Boussinesq type equations with gentle dissipation:

\begin{align*}
&u_{tt} + \Delta^2 u + (-\Delta)^\alpha u_t - \Delta f(u) = g(x) \quad \text{in } \Omega \times \mathbb{R}^+, \\
&u|_{\partial\Omega} = u_t|_{\partial\Omega} = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),
\end{align*}

where $\alpha \in (0, 1)$ is said to be a dissipative index, $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with the smooth boundary $\partial\Omega$, and the nonlinearity $f(u)$ and external force $g$ will be specified later.

Primary attention for Eq. (1) was paid to the case where the dissipative term $(-\Delta)^\alpha u_t$ is either absent (cf. [2, 3, 8, 16, 23]) or $\alpha = 0$ (weak damping, cf. [10, 11]), $\alpha = 1$ (structural damping, cf. [27, 28, 29]) and $\alpha = 2$ (strong damping, cf. [30]).

2000 Mathematics Subject Classification. Primary: 35B40, 35B41, 35B65; Secondary: 37L30, 37L15.

Key words and phrases. Boussinesq type equations, gentle dissipation, well-posedness, global attractor, exponential attractor, upper semicontinuity of attractors.

The authors are supported by NNSF of China (No. 11671367).

* Corresponding author.
For the physical background of them, one can see above-mentioned references in detail.

The study on the hyperbolic equations with fractional damping arises from [4]. In 1982, based on the results of empirical studies: there are always dissipative mechanisms acting within the linear elastic systems causing the energy to decrease during any positive time interval, Chen and Russell [4] presented an abstract mathematical model exhibiting the empirically observed damping rates in elastic systems:

\[ \ddot{x} + B \dot{x} + Ax = 0, \]  

where \( A \) (the elastic operator) and \( B \) (the dissipation operator) are two positive, self-adjoint operators with domains \( D(A) \) and \( D(B) \) dense in the Hilbert space \( X \) satisfying \( p_1 A^α \leq B \leq p_2 A^α \) for some constants \( 0 < p_1 < p_2 < \infty \) and \( 0 < α \leq 1 \).

More precisely, \( B \dot{x} = A^α \dot{x} \) is said to be structural damping when \( 1/2 \leq α \leq 1 \), gentle dissipation when \( 0 < α < 1/2 \) (cf. [5, 6, 7]). It is expected that this type of models will permit realistic simulation of various elastic systems wherein damping cannot be ignored.

According to above classification, \( (-\Delta)^α u_t \) in Eq. (1) is gentle dissipation when \( α \in (0, 1) \), structural damping when \( α \in [1, 2) \) for \( A = \Delta^2 \) (with hinged boundary condition (2)) and \( B = (-\Delta)^α = A^{α/2} \) here, and \( α \) is said to be a dissipative index.

Obviously, Eq. (1) is hyperbolic and its solutions have no additional smoothness when \( t > 0 \) if \( α = 0 \). It is well-known that the dissipative term \( (-\Delta)^α u_t \) (\( α > 0 \)) has a regularizing effect for the solutions of Eq. (1), i.e., it makes them be of additionally partial smoothness when \( t > 0 \). Recently, Yang et al. [31, 32] showed that when \( α \in [1, 2) \) (structural damping case), Eq. (1) is like parabolic, the related solution semigroup has a global attractor and a “partially strong” exponential attractor in natural energy space when the growth exponent \( p \) of \( f(u) \) is up to the supercritical range: \( 1 ≤ p < \tilde{p}_α = \frac{N+2(2α-1)}{(N-2(2α-1))} \), where \( α^+ = max\{a, 0\} \).

Moreover, the family of global attractors \( \{A_α\} \) is upper semicontinuous at the point \( α_0 \in [1, 2) \) in the following sense: for any neighborhood \( U \) of \( A_{α_0}, A_α \subset U \) when \( 0 < α - α_0 \) is small enough. For the investigations on the existence of global and exponential attractors for the case of \( α = 1 \) in phase space \( H^2 \cap H^1_0 \times L^2 \), one can see [22].

It is well known that the dissipative role of gentle dissipation is stronger than that of weak damping but weaker than that of structural damping. In this case, is the Eq. (1) like parabolic (as in the case of structural damping) or like hyperbolic (as in the case of weak damping)? What about the existence of its global and exponential attractors? What about the upper semicontinuity of perturbed attractors \( A_α \)? These questions are unsolved.

The aim of the present paper is to solve these questions. For general bounded domain \( Ω \subset \mathbb{R}^N (N \geq 1) \), we show that there exists a critical exponent \( p_α = \frac{N+2(2α-1)}{(N-2)} \) depending on the dissipative index \( α \) such that when the growth \( p \) of the nonlinearity \( f(u) \) is up to the range: \( 1 ≤ p < p_α \),

(i) Eq. (1) is like parabolic, i.e., its solutions possess additionally global smoothness when \( t > 0 \) (see Theorem 3.1);

(ii) the related solution semigroup \( \{S^α(t)\}_{t \geq 0} \) possesses a global attractor and an exponential attractor in natural energy space for each \( α \in (0, 1) \) (see Theorems 4.2 and 4.5);

(iii) the family of global attractors \( \{A_α\} \) is upper semicontinuous at the point \( α_0 \in (0, 1) \) when \( 1 ≤ p < p_{α_0} \) (especially, \( p_{α_0} = p^* = \frac{N+2}{(N-2)} \) when \( α_0 = 1 \), i.e., for any neighborhood \( U \) of \( A_{α_0}, A_α \subset U \) when \( |α - α_0| \ll 1 \) (see Theorem 5.3).
Here, the exponent \( p_\alpha \) is said to be critical because the uniqueness of solutions of problem (1)-(2) fails (or one can not get it) when \( p \geq p_\alpha \). Obviously, \( p_\alpha = p^* \) when \( \alpha = 1 \), which coincides with the critical exponent in [31].

We mention that there have been some recent researches on the well-posedness and longtime dynamics of nonlinear evolution equations with fractional damping, for example, the investigations on that of quasilinear wave equation with structural damping (cf. [13, 14, 20, 36]) and that of the semilinear wave equation with structural damping or gentle dissipation (cf. [24, 25, 34, 35]). And there have been extensive studies on the stability of exponential attractors when the perturbations are some coefficients of the evolution equations (cf. [17, 18, 19] and references therein). But it is challenging to construct robust families of exponential attractors when the perturbation is the dissipative index, which will be considered in our future work.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we discuss the well-posedness and the like parabolic properties of problem (1)-(2). In Section 4, we establish the existence of global and exponential attractors in natural energy space and discuss some properties of the global attractor. In Section 5, we investigate the upper semicontinuity of the family of global attractors \( \{A_\alpha\} \).

2. Preliminaries. For brevity, we use the following abbreviations:

\[ L^p = L^p(\Omega), \quad H^k = H^k(\Omega), \quad H = L^2, \quad V_2 = H^2 \cap H^1_0, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \quad \| \cdot \| = \| \cdot \|_{L^2}, \]

with \( p \geq 1 \), \( H^k \) are the \( L^2 \)-based Sobolev spaces and \( H^k_0 \) are the completion of \( C_0^\infty(\Omega) \) in \( H^k \) for \( k > 0 \). The notation \((\cdot, \cdot)\) for the \( H \)-inner product will also be used for the notation of duality pairing between dual spaces, the sign \( H_1 \hookrightarrow H_2 \) denotes that the functional space \( H_1 \) continuously embeds into \( H_2 \) and \( H_1 \hookrightarrow H_2 \) denotes that \( H_1 \) compactly embeds into \( H_2 \), \( C(\cdot, \cdot) \) denotes positive constants depending on the quantities appearing in the parenthesis.

Obviously, \( V_2 \hookrightarrow H \hookrightarrow V_2' \) (the dual space of \( V_2 \)). Define the operator \( A : V_2 \to V_2' \),

\[ (Au, v) = (\Delta u, \Delta v) \quad \text{for any} \quad u, v \in V_2. \]

Then, \( A \) is self-adjoint in \( H \) and strictly positive on \( V_2 \), so we can define the power \( A^s \) of \( A \) \((s \in \mathbb{R})\), and the spaces \( V_s = D(A^{\frac{s}{2}}) \) are Hilbert spaces with the scalar products and the norms

\[ (u, v)_s = (A^{\frac{s}{2}} u, A^{\frac{s}{2}} v), \quad \|u\|_V = \|A^{\frac{s}{2}} u\|, \quad s \in \mathbb{R}, \]

and \( V_{s_1} \hookrightarrow V_{s_2} \) for \( s_1 > s_2 \).

Rewriting Eq. (1) at an abstract level and applying \( A^{-\frac{s}{2}} \) to both sides, we obtain

\[ A^{-\frac{s}{2}} u_{tt} + A^{\frac{s}{2}} u + A^{\frac{s-1}{2}} u_t + f(u) = A^{-\frac{s}{2}} g, \]

\[ u(0) = u_0, \quad u_t(0) = u_1. \]

We denote the phase spaces

\[ X_\alpha = \begin{cases} V_{2-\alpha} \times V_{-\alpha}, & \alpha \in (0, 1/2], \\ V_{\alpha+1} \times V_{1-\alpha}, & \alpha \in (1/2, 1), \end{cases} \quad X = V_1 \times V_{-1}, \quad Y_\alpha = V_{-\alpha} \times V_{-\alpha-2}, \]

with \( \alpha \in (0, 1) \), which are equipped with usual graph norms, for example, \( \|(u, v)\|_X^2 = \|u\|_{V_1}^2 + \|v\|_{V_{-1}}^2 \). Obviously, they are Hilbert spaces and

\[ X_\alpha \hookrightarrow X \hookrightarrow Y_\alpha. \]
Assumption (H) (i) \( f \in C^1(\mathbb{R}) \), and when \( N \geq 2 \),
\[
\mu_f \equiv \liminf_{|s| \to \infty} \frac{f(s)}{s} > -\sqrt{\lambda_1}, \quad f'(s) \leq C(1 + |s|^{p-1}), \tag{6}
\]
where \( \lambda_1 \) is the first eigenvalue of \( A \), \( 1 \leq p < +\infty \) if \( N = 2 \); \( 1 \leq p \leq p^* = \frac{N+2}{N-2} \) if \( N \geq 3 \).
(ii) \( g \in V_-3, (u_0, u_1) \in X \), with \( \|(u_0, u_1)\|_X \leq R \).

Remark 1. The first inequality in (6) implies that there exists a constant \( \theta < 0 \), \( \sqrt{\lambda_1} - \theta < 1 \) such that
\[
F(s) \geq -\frac{\theta}{2}s^2 - C, \quad f(s)s \geq -\theta s^2 - C,
\]
where and in the following \( F(s) \equiv \int_0^s f(\tau)d\tau \).

Lemma 2.1 (I). Let \( X \) be a Banach space, the set \( Z \subset C(\mathbb{R}^+; X) \), and \( \Phi : X \to \mathbb{R} \) be a continuous functional satisfying
\[
\sup_{t \in \mathbb{R}^+} \Phi(z(t)) \geq -\eta, \quad \Phi(z(0)) \leq K
\]
for some \( \eta, K \geq 0 \) and every \( z \in Z \). In addition, assume that for every \( z \in Z \) the function \( t \mapsto \Phi(z(t)) \) is continuously differentiable, and satisfies the differential inequality
\[
\frac{d}{dt} \Phi(z(t)) + \delta \|z(t)\|_X^2 \leq k
\]
for some \( \delta > 0 \) and \( k \geq 0 \) independent of \( z \). Then, for every \( \gamma > 0 \) there exists \( t_0 = \frac{\eta + K}{\gamma} > 0 \) such that
\[
\Phi(z(t)) \leq \sup_{\zeta \in X} \{ \Phi(\zeta) : \delta \|\zeta\|_X^2 \leq k + \gamma \}, \quad t \geq t_0.
\]

For simplicity, we restrict ourselves to the case \( N \geq 3 \) for the cases \( N = 1, 2 \) are easy. But all the results hold for the cases \( N = 1, 2 \).

3. Well-posedness.

Theorem 3.1. Let Assumption (H) be valid. Then problem (4)-(5) admits a weak solution \( u \equiv u^0 \), with \( (u, u_t) \in L^\infty(\mathbb{R}^+; X) \cap C_w(\mathbb{R}^+; X) \), and
\[
\|u(t)\|_{V_1}^2 + \|u_t(t)\|_{V_{-1}}^2 \leq \begin{cases} C_0(R) & \text{if } t > 0, \\ C_1 & \text{if } t \geq t_0 = t(R), \end{cases} \tag{7}
\]
\[
\int_0^t \|u_t(\tau)\|_{V_{-1}}^2 d\tau \leq C_0(R), \quad t > 0. \tag{8}
\]
In particular, when \( 1 \leq p < p_0 = \frac{N+2(2\alpha-1)}{(N-2)^2} \), (which means \( V_1 \hookrightarrow L^\infty(\frac{N(p\alpha-1)}{2\alpha}) \)), the solution further possesses the following properties:
(i) (Global regularity when \( t > 0 \)) For any \( a > 0, T > a \),
\[
(u_t, u_{tt}) \in \begin{cases} L^\infty(a, T; V_{-\alpha} \times V_{-\alpha-2}) \cap L^2(a, T; H \times V_{-2}), & \alpha \in (0, 1/2], \\ L^\infty(a, T; V_{1-\alpha} \times V_{\alpha-1}) \cap L^2(a, T; V_1 \times V_{-1}), & \alpha \in (1/2, 1), \end{cases}
\]
and

\[ \|u_t(t)\|_{V_\alpha}^2 + \|u_{tt}(t)\|_{V_{\alpha-2}}^2 + \int_t^{t+1} \left( \|u_t(\tau)\|^2 + \|u_{tt}(\tau)\|_{V_{\alpha-2}}^2 \right) d\tau \]

\[ \leq \left( 1 + \frac{1}{t^{1/\alpha}} \right) C_1(\alpha) C_0(R), \quad \alpha \in (0, 1), \quad t > 0, \]

where \( C_1(\alpha) = \left[ \frac{\alpha(1-\alpha)}{\alpha^3} \right]^{\frac{1-\alpha}{\alpha}} \); especially when \( \alpha \in (1/2, 1) \),

\[ \|u_t(t)\|_{V_{\alpha-1}}^2 + \|u_{tt}(t)\|_{V_{\alpha-2}}^2 + \int_t^{t+1} \left( \|u_t(\tau)\|_{V_{\alpha-2}}^2 + \|u_{tt}(\tau)\|_{V_{\alpha-2}}^2 \right) d\tau \]

\[ \leq \left( 1 + \frac{1}{t^{1/2}} \right) C_0(R), \quad t > 0. \]

Furthermore, if \( g \in V^- \), then for any \( a > 0 \),

\[ u \in \begin{cases} L^\infty(a, T; V_{2-\alpha}) \cap L^2(a, T; V_2), \quad \alpha \in (0, 1/2], \\ L^\infty(a, T; V_{\alpha+1}) \cap L^2(a, T; V_3), \quad \alpha \in (1/2, 1), \end{cases} \]

and

\[ \|u(t)\|_{V_{\alpha-1}}^2 + d \int_t^{t+1} \|u(\tau)\|_{V_2}^2 d\tau \]

\[ \leq \left( 1 + \frac{1}{t^{1/\alpha}} \right) C_1(\alpha) C(R, \|g\|_{V_{\alpha-2}}), \quad \alpha \in (0, 1), \quad t > 0, \]

where \( d = 1 \) if \( \alpha \in (0, 1/2] \); \( d = 0 \) if \( \alpha \in (1/2, 1) \); in particular when \( \alpha \in (1/2, 1) \),

\[ \|u(t)\|_{V_{\alpha+1}}^2 + \int_t^{t+1} \|u(\tau)\|_{V_{\alpha-1}}^2 d\tau \]

\[ \leq \left( 1 + \frac{1}{(a/2)^{2/\alpha}} \right) C(R, \|g\|_{V_{\alpha-1}}), \quad t > a/2. \]

(ii) The following energy identity

\[ E(\xi_u(t)) + \int_s^t \|u_t(\tau)\|_{V_{\alpha-1}}^2 d\tau = E(\xi_u(s)) \]

holds for every \( t > s \geq 0 \), where \( \xi_u = (u, u_t) \),

\[ E(u, v) = \frac{1}{2} (\|u\|_{V_2}^2 + \|v\|_{V_{\alpha-2}}^2) + \int_\Omega F(u) dx - (A^{-\frac{1}{2}} g, A^{-\frac{1}{2}} u), \quad (u, v) \in X. \]

(iii) (Strong continuity and stability)

\[ (u, u_t) \in C(\mathbb{R}^+; X), \]

and the solutions are Lipschitz stable in weaker space \( Y_\alpha = V_{-\alpha} \times V_{-\alpha-2} \), i.e.,

\[ \|(z, z_t)(t)\|_{Y_\alpha}^2 \leq C e^{kt} \|(z, z_t)(0)\|_{Y_\alpha}^2, \]

and quasi-stable in \( Y_\alpha \), i.e.,

\[ \|(z, z_t)(t)\|_{Y_\alpha}^2 \leq C \|(z, z_t)(0)\|_{Y_\alpha}^2 e^{-\kappa t} \]

\[ + C \int_0^t e^{-\kappa(t-\tau)} (\|z(\tau)\|_{V_{\alpha-1}}^2 + \|z_t(\tau)\|_{V_{\alpha-2}}^2) d\tau, \]

where \( k > 0 \) is a constant, \( \kappa \) denotes a small positive constant, \( z = u - v, u, v \) are two weak solutions of problem (4)-(5) corresponding to initial data \((u_0, u_1)\) and \((v_0, v_1)\), with \( \|u_0(u_1)\|_X + \|v_0(v_1)\|_X \leq R \), respectively.
Proof. We first give some a priori estimates to the solutions of problem (4)-(5). Using the multiplier \(u_t + \epsilon u\) in Eq. (4), we have
\[
\frac{d}{dt} H(\xi_u) + K(\xi_u) = 0, \tag{16}
\]
where \(\xi_u = (u, u_t)\),
\[
H(\xi_u) = E(\xi_u) + \epsilon \left( (A^{-\frac{1}{2}} u, A^{-\frac{1}{2}} u_t) + \frac{1}{2} \| u \|_{V_{\alpha-1}}^2 \right)
\geq \kappa \| u \|_{V_1}^2 + \| u_t \|_{V_{\alpha-1}}^2 - C \| g \|^2_{V_{\alpha-3}},
\]
\[
K(\xi_u) = \| u_t \|_{V_{2\alpha-1}}^2 - \epsilon \| u_t \|_{V_{\alpha-1}}^2 + \epsilon \left( \| u \|_{V_1}^2 + (f(u), u) - (A^{-\frac{1}{2}} g, A^{-\frac{1}{2}} u) \right)
\geq \frac{1}{2} \| u_t \|_{V_{\alpha-1}}^2 + \kappa \| u \|_{V_1}^2 + \| u_t \|_{V_{\alpha-1}}^2 - C \| g \|^2_{V_{\alpha-3}}
\]
for \(\epsilon > 0\) suitably small, where \(E(\xi_u)\) is as shown in (13) and we have used Remark 1 here. Hence,
\[
\frac{d}{dt} H(\xi_u) + \frac{1}{2} \| u_t \|_{V_{\alpha-1}}^2 + \kappa \| (u, u_t) \|_{X}^2 \leq C \| g \|^2_{V_{\alpha-3}}. \tag{17}
\]
By Assumption (H),
\[
H(\xi_u(0)) = C(\| u_0 \|_{V_1}^2 + \| u_t \|_{V_{\alpha-1}}^2 + \| u \|_{V_{\alpha-2}}^{p+1} + \| g \|_{V_{\alpha-3}}^2 + 1) \leq C_0(R)(\equiv K),
\]
where we have used the Sobolev embedding \(V_1 \hookrightarrow V_{\alpha-1}\) and \(\| u \|_{V_{\alpha-1}} \leq \max \{ \lambda_1^{-1/2}, 1 \} \| u \|_{V_1}\).

Applying respectively Lemma 2.1 (with \(\eta = k = C \| g \|^2_{V_{\alpha-3}}, \delta = \kappa, \gamma = 1, t_0 = \eta + K\)) and the Gronwall lemma to (17) gives estimate (7). Letting \(\epsilon = 0\) in (16) and integrating the resulting expression over \((0, t)\) give estimate (8).

Taking account of the Sobolev embedding: \(V_1 \hookrightarrow L^{p+1}, L^{1+1/p} \hookrightarrow V_{\alpha-1}\), and making use of Eq. (4) and estimate (7), we obtain
\[
\| u_t \|_{V_{\alpha-2}} \leq C(\| u \|_{V_{\alpha-1}} + \| u \|_{V_1} + \| u_t \|_{V_1} + \| g \|_{V_{\alpha-3}} + 1) \leq C_0(R), \quad t > 0, \tag{18}
\]
where we have used the fact \(V_{\alpha-1} \hookrightarrow V_{2\alpha-3}\) and
\[
\| u \|_{V_{2\alpha-3}} \leq \max \{ \lambda_1^{-1/2}, 1 \} \| u \|_{V_{\alpha-1}}.
\]

When \(1 \leq p < p_\alpha\), formally differentiating Eq. (4) with respect to \(t\) gives that \(v = u_t\) solves
\[
A^{-\frac{1}{2}} v_{tt} + A^{\frac{1}{2}} v = A^{\frac{1}{2}} f_t + f'(u) v = 0. \tag{19}
\]
(i) (Global regularity when \(t > 0\)) When \(\alpha \in (0, 1)\), using the multiplier \(A^{-\frac{1}{2}} v_t + \epsilon A^{-\frac{1}{2}} v\) in (19) yields
\[
\frac{d}{dt} H_1(\xi_v) + (1 - \epsilon) \| v_t \|_{V_{\alpha-2}}^2 + \epsilon \| v \|_2^2 = -(f'(u)v, A^{-\frac{1}{2}} v_t + \epsilon A^{-\frac{1}{2}} v), \tag{20}
\]
where \(\xi_v = (v, v_t)\) and
\[
H_1(\xi_v) = \frac{1}{2} \left( \| v \|_{V_{\alpha-1}}^2 + \| v_t \|_{V_{\alpha-2}}^2 + \epsilon \| v \|_{V_{\alpha-2}}^2 \right) + \epsilon (A^{-\frac{1}{2}} v, A^{-\frac{1}{2}} v_t) \tag{21}
\]
for $\epsilon$ suitably small. By the Sobolev embedding $V_1 \hookrightarrow L^{\frac{N(p-1)}{2(p-\delta)}}$ for $1 \leq p < p_\alpha$, $0 < \delta \ll 1$ and the interpolation theorem,
\begin{align*}
|\langle f'(u)v, A^{-\frac{1}{2}}v \rangle + \epsilon A^{-\frac{1}{2}}v| &\leq C(1 + \|u\|^{p-2}v_1 + \|A^{-\frac{1}{2}}v_1\|_{\frac{N(p-1)}{2(p-\delta)}} + \epsilon\|A^{-\frac{1}{2}}v\|_{\frac{N(p-1)}{2(p-\delta)}}) \\
&\leq C(1 + \|u\|^{p-2}v_1 + \|v_1\|_{V^{2,0}_{-\delta}} + \epsilon\|v\|_{V^{2,0}_{-\delta}}) \\
&\leq \epsilon\left(\frac{1}{4}\|v\|^2 + \|v_1\|^2_{V^{2,0}_{-\delta}}\right) + C\|v\|^2_{V^{2,0}_{-\delta}} + \|v_1\|^2_{V^{2,0}_{-\delta}}
\end{align*}
where we have used the fact
\begin{equation}
\|v\|^2_{V^{2,0}_{-\delta}} \leq \left\{ \begin{array}{ll}
\max\{\lambda_1^{-(1+\delta)/4}, 1\}\|v\|^2_{V^{2,0}_{-\delta}}, & \text{if } \alpha \in (0, 1/2], \\
C\|v\|^{1-\delta+2\alpha}\|v\|^{2+\delta-2\alpha}_{V^{2,0}_{-\delta}}, & \text{if } \alpha \in (1/2, 1).
\end{array} \right.
\end{equation}
Inserting (22) into (20) and using (7), (18) and (21), we obtain
\begin{equation}
d \frac{dt}{H_1(\xi_t) + \kappa H_1(\xi_t) + \frac{1}{2}\|v_t\|^2_{V^{2,0}_{-\delta}} + \kappa\|v\|^2} \leq C_0(R).
\end{equation}
When $0 \leq t \leq 1$, multiplying (24) by $t^{1/\alpha}$ and using the interpolation theorem arrive at
\begin{align*}
\frac{d}{dt}&[t^{1/\alpha}H_1(\xi_t) + \kappa t^{1/\alpha}H_1(\xi_t) + \frac{1}{2}t^{1/\alpha}(\kappa\|v\|^2 + \|v_t\|^2_{V^{2,0}_{-\delta}})] \\
&\leq C_0(R) + \frac{1}{\alpha}Ct^{1-\alpha}(\|v\|^2_{V^{2,0}_{-\delta}} + \|v_t\|^2_{V^{2,0}_{-\delta}}) \\
&\leq \frac{1}{\alpha}Ct^{1-\alpha}(\|v\|^2_{V^{2,0}_{-\delta}} + \|v_t\|^2_{V^{2,0}_{-\delta}}) + C_0(R) \\
&\leq \frac{1}{4}t^{1/\alpha}(\kappa\|v\|^2 + \|v_t\|^2_{V^{2,0}_{-\delta}}) + C_1(\alpha)C_0(R),
\end{align*}
where we have used the interpolation
\begin{align*}
\frac{1}{\alpha}Ct^{1-\alpha}(\|v\|^2_{V^{2,0}_{-\delta}} + \|v_t\|^2_{V^{2,0}_{-\delta}}) &\leq \frac{1}{4}t^{1/\alpha}(\kappa\|v\|^2 + \|v_t\|^2_{V^{2,0}_{-\delta}}) + C_1(\alpha)C(\|v\|^2_{V^{2,0}_{-\delta}} + \|v_t\|^2_{V^{2,0}_{-\delta}}),
\end{align*}
with $C_1(\alpha) = \left[\frac{4(1-\alpha)}{\alpha}\right]^{1-\alpha}$. Applying the Gronwall lemma to (24) over $(1, t)$ and exploiting (25) yield
\begin{equation}
\|u_t(t)\|^2_{V^{2,0}_{-\delta}} + \|u_{tt}(t)\|^2_{V^{2,0}_{-\delta}} \leq C_1(\alpha)C_0(R), \quad t > 1.
\end{equation}
Integrating (24) over $(t, t+1)$ and making use of (25)-(26), we obtain
\begin{equation}
\int_{t}^{t+1} \left(\|u(t)\|^2 + \|u_{tt}(t)\|^2_{V^{2,0}_{-\delta}}\right)dt \leq \left(1 + \frac{1}{t^{1/\alpha}}\right)C_1(\alpha)C_0(R), \quad t > 0.
\end{equation}
The combination of (25)-(26) and (27) gives (9).
In particular, when \( \alpha \in (0, 1) \), taking account of the Sobolev embedding \( V_1 \hookrightarrow L^{2Np/(2N-1)} \), \( L^{2Np/(2N-1)} \hookrightarrow V_{-a} \) for \( 1 \leq p < p_\alpha \), we infer from Eq. (4) that
\[
\|u\|_{V_{-a}} \leq \|A^{-\frac{\alpha}{2}}u_t\|_{V_{-a}} + \|A^{\frac{2-\alpha}{2}}u_t\|_{V_{-a}} + \|f(u)\|_{V_{-a}} + \|A^{-\frac{\alpha}{2}}g\|_{V_{-a}}
\leq C(\|u_t\|_{V_{-a}} + \|u\|_{\frac{2Np}{2N-1}} + \|g\|_{V_{-a}} + 1)
\leq C(\|u_t\|_{V_{-a}} + \|u\|_{V_{-a}} + \|u\|_{V_{-a}}^p + \|g\|_{V_{-a}} + 1)
\leq \left(1 + \frac{1}{t^{1/\alpha}}\right)C_1(\alpha)C(R, g\|_{V_{-a}}), \quad t > 0.
\]
Hence estimate (11) holds.

Especially, when \( \alpha \in (0, 1/2) \), taking account of \( V_1 \hookrightarrow L^{2p_\alpha} \), we have (taking \( \alpha = 0 \) above)
\[
\int_t^{t+1} \|u(\tau)\|^2_{V_{-a}} d\tau \leq C \int_t^{t+1} (\|u_t(\tau)\|^2_{V_{-a}} + \|u(\tau)\|^2_{V_{-a}} + 1) d\tau
\leq \left(1 + \frac{1}{t^{1/\alpha}}\right)C_1(\alpha)C(R, g\|_{V_{-a}}), \quad t > 0.
\]

Hence estimate (11) holds.

When \( \alpha \in (1/2, 1) \), using the multiplier \( A^{-\frac{\alpha}{2}}v_t + \epsilon v \) in (19), we obtain
\[
\frac{d}{dt}H_2(\xi_\nu) + (1 - \epsilon)\|v_t\|^2_{V_{-a}} + \epsilon\|v\|^2_{V_1} = -(f'(u)v, A^{-\frac{\alpha}{2}}v_t + \epsilon v),
\]  
(29)
where
\[
H_2(\xi_\nu) = \frac{1}{2} \left(\|v_t\|^2_{V_{-a}} + \|v_t\|^2_{V_{-a-1}} + \epsilon\|v\|^2_{V_1} + \epsilon(A^{-\frac{\alpha}{2}}v, A^{-\frac{\alpha}{2}}v_t)
\sim\|v_t\|^2_{V_{-a}} + \|v_t\|^2_{V_{-a-1}}
\]
for \( \epsilon \) suitably small. By Assumption (H), the Sobolev embedding \( V_1 \hookrightarrow L^{N(N/2-1)}(1 \leq p < p_\alpha) \), and the interpolation theorem, we have
\[
|\langle f'(u)v, A^{-\frac{\alpha}{2}}v_t + \epsilon v \rangle|
\leq C(1 + \|v\|^2_{V_1} + \epsilon)\|v\|_{V_{2a-1}} + \epsilon\|v\|^2_{V_1} + \epsilon\|v\|^2_{V_{-a-1}}
\leq C(1 + \|v\|^2_{V_1} + \epsilon)\|v\|_{V_{2a-1}} + \epsilon\|v\|^2_{V_{-a-1}}
\leq \epsilon\left(\frac{1}{4}\|v\|^2_{V_1} + \|v\|^2_{V_{-a-1}}\right) + C(\|v\|^2_{V_{-a}} + \|v\|^2_{V_{-3}}).
\]  
(30)
Inserting (30) into (29) yields
\[
\frac{d}{dt}H_2(\xi_\nu) + \kappa H_2(\xi_\nu) + \frac{1}{2}\|v_t\|^2_{V_{-a}} + \frac{\epsilon}{2}\|v\|^2_{V_1} \leq C_0(R).
\]  
(31)
When \( 0 < t \leq 1 \), multiplying (31) by \( t^{2/\alpha} \) gives
\[
\frac{d}{dt}\left(t^{2/\alpha}H_2(\xi_\nu)\right) + \kappa t^{2/\alpha}H_2(\xi_\nu) + \frac{1}{2}t^{2/\alpha}\epsilon\|v\|^2_{V_1} + \|v_t\|^2_{V_{-a-1}}
\leq C_0(R) + \frac{2}{\alpha}Ct^{\frac{2-\alpha}{2}}(\|v\|^2_{V_{1-a}} + \|v_t\|^2_{V_{-a-1}})
\leq C_0(R) + \frac{2}{\alpha}Ct^{\frac{2-\alpha}{2}}(\|v\|^2_{V_1} + \|v_t\|^2_{V_{-a-1}})
\leq \frac{1}{4}t^{2/\alpha}\epsilon(\|v\|^2_{V_1} + \|v_t\|^2_{V_{-a-1}}) + C_0(R),
\]  
(32)
\[
\|u_t(t)\|^2_{V_{-a}} + \|u_t(t)\|^2_{V_{-a-1}} \leq \frac{1}{t^{2/\alpha}}C_0(R), \quad 0 < t \leq 1.
\]
where we have used the interpolation

\[ C \frac{2}{\alpha} \frac{\alpha^{\frac{2-n}{2}}}{\alpha} \left( \| v \|_{V_1}^{\frac{\alpha}{\alpha}} \| v \|_{V_{\infty}}^{\frac{\alpha}{\alpha}} + \| v_t \|_{V_{\infty}}^{\frac{\alpha}{\alpha}} \| v_t \|_{V_{\infty}}^{\frac{\alpha}{\alpha}} \right) \]

\[ \leq \frac{1}{4} t^{2/\alpha} (\kappa v v_{11}^2 + \| v_t \|_{V_{\infty}}^{\frac{\alpha}{\alpha}}) + C_2(\alpha) C \left( \| v \|_{V_{\infty}}^{\frac{\alpha}{\alpha}} + \| v_t \|_{V_{\infty}}^{\frac{\alpha}{\alpha}} \right), \]

and where \( C_2(\alpha) = \left( \frac{4(2-\alpha)}{\alpha} \right)^{\frac{2-n}{2}} \leq C \) for \( \alpha \in (1/2, 1) \). When \( t > a/\alpha \), applying the Gronwall lemma to (31) over \((1, t)\) and using (32), we get

\[ \| u(t) \|_{V_{1-\alpha}}^2 + \| u_t(t) \|_{V_{1-\alpha}}^2 \leq C_0(R), \quad t > 1. \]

Integrating (31) over \((t, t + 1)\) and taking advantage of (32)-(33), we obtain

\[ \int_t^{t+1} \left( \| u(t) \|_{V_{1-\alpha}}^2 + \| u_t(t) \|_{V_{1-\alpha}}^2 \right) dt \leq \left( 1 + \frac{1}{t^{2/\alpha}} \right) C_0(R), \quad t > 0. \] (34)

The combination of (32)-(33) and (34) gives (10).

Using the multiplier \( Au \) in Eq. (4) arrives at

\[ \frac{1}{2} \frac{d}{dt} \| u \|_{V_{1-\alpha}}^2 + \| u \|_{V_{3}}^2 + (f'(u) \nabla u, \nabla (A \frac{1}{2} u)) \]

\[ = -(A^{-\frac{1}{2}} u_{tt} - A^{-\frac{1}{2}} g, Au) \leq \frac{1}{4} \| u \|_{V_{3}}^2 + C(\| u \|_{V_{1-\alpha}}^2 + \| g \|_{V_{1-\alpha}}). \]

Due to

\[ |(f'(u) \nabla u, \nabla (A \frac{1}{2} u)|) \leq C(1 + \| u \|_{V_{1-\alpha}}^{p-1} \| \nabla u \|_{V_{1-\alpha}}^{2-\alpha} \| \nabla (A \frac{1}{2} u) \|_{V_{1-\alpha}}^{2}) \]

\[ \leq C(1 + \| u \|_{V_{1-\alpha}}^{p-1}) \| u \|_{V_{3}} \| u \|_{V_{1-\alpha}} \| g \|_{V_{3}} \]

\[ \leq \frac{1}{4} \| u \|_{V_{3}}^2 + C(\| u \|_{V_{1-\alpha}}^2 + \| g \|_{V_{1-\alpha}}). \]

and by virtue of (7), we have

\[ \frac{d}{dt} \| u(t) \|_{V_{1-\alpha}}^2 + \| u(t) \|_{V_{3}}^2 \leq C(\| u \|_{V_{1-\alpha}}^2 + C(R, \| g \|_{V_{1-\alpha}}). \] (35)

For any \( a > 0 \), when \( a/2 < t \leq a/2 + 1 \), multiplying (35) by \((t - a/2)^{2-\alpha} \) and making use of the interpolation theorem and (34) yield

\[ \frac{d}{dt} \left( t - a/2 \right)^{2-\alpha} \| u(t) \|_{V_{1-\alpha}}^2 \right) + \left( t - a/2 \right)^{2-\alpha} \| u(t) \|_{V_{3}}^2 \]

\[ \leq C \| u \|_{V_{1-\alpha}}^2 + C(R, \| g \|_{V_{1-\alpha}}) + \frac{2}{2-\alpha} (t - a/2)^{2-\alpha} \| u \|_{V_{1-\alpha}}^2 \| u \|_{V_{3}} \]

\[ \leq \frac{1}{2} \| u \|_{V_{1-\alpha}}^2 + C(R, \| g \|_{V_{1-\alpha}}). \]

\[ \| u(t) \|_{V_{1-\alpha}}^2 \leq \frac{1}{(t - a/2)^{2-\alpha}} \left( 1 + \frac{1}{(a/2)^{2-\alpha}} \right) C(R, \| g \|_{V_{1-\alpha}}), \quad a/2 < t \leq a/2 + 1. \] (36)

When \( t > a/2 + 1 \), applying the Gronwall lemma to (35) over \((a/2 + 1, t)\) and exploiting (34) and (36), we obtain

\[ \| u(t) \|_{V_{1-\alpha}}^2 \]

\[ \leq C(\| u(a/2 + 1) \|_{V_{1-\alpha}}^2 + \int_{a/2+1}^{t} e^{-\left( t - \tau \right)} \left( C \| u \|_{V_{1-\alpha}}^2 + C(R, \| g \|_{V_{1-\alpha}}) \right) d\tau \] (37)

\[ \leq C(R, \| g \|_{V_{1-\alpha}}), \quad t > a/2 + 1. \]
Integrating (35) over \((t, t + 1)\) and making use of (36)-(37), we get
\[
\int_{t}^{t+1} \|u(\tau)\|_{V_2}^2 d\tau \leq \left(1 + \frac{1}{(a/2)^{2/\alpha}} + \frac{1}{(t-a/2)^{2/\alpha}}\right)C(R, \|g\|_{V_{-1}}), \quad t > a/2. \tag{38}
\]
The combination of (36)-(38) leads to (12).

Based on estimates (7)-(8) (which obviously hold for the Galerkin approximations) and the Galerkin method, one can easily prove that problem (4)-(5) possesses a weak solution \(u\), with \((u, u_t) \in L^\infty(\mathbb{R}^+; X) \cap C_w(\mathbb{R}^+; X)\), and by the lower semi-continuity of weak limit, estimates (7)-(12) hold for \(u\). We omit the process here.

(ii) (Energy identity) On account of (35), we have (13) holds for \(t \geq s > 0\).

Letting \(s \to 0^+\) in (13), we have that, for any \(a > 0\),
\[
\int_a^T (f(u), u_t) d\tau \leq \begin{cases} 
C \int_a^T [(1 + \|u\|_{L^p}^p)] |u_t| d\tau, & \alpha \in (0, 1/2), \\
C \int_a^T [(1 + \|u\|_{L^p}^{p-1}) |u_t| \|u_t\|_{L^\infty}^1] d\tau, & \alpha \in (1/2, 1), \\
lC_1(\alpha) C(a, T), & \alpha \in (0, 1),
\end{cases}
\]
that is, \(f(u)u_t \in L^1([a, T] \times \Omega)\), then we can use the multiplier \(u_t\) in Eq. (4) and energy identity (13) holds for \(t \geq s > 0\).

Letting \(s \to 0^+\) in (13), we have
\[
E_s = \lim_{s \to 0^+} E(\xi_u(s)) = E(\xi_u(t)) + \int_0^t \|u_t(\tau)\|_{V_{-1}}^2 d\tau.
\]
Repeating the same arguments as in [32] shows that \(E(\xi_u(0)) = E_s\), i.e., (13) holds for each \(t \geq s \geq 0\).

(iii) (Strong continuity and stability) By using energy identity (13) and repeating the same proof as in [32], one easily gets
\[
\lim_{t \to 0} \|u(t, u_t(t))\|_{V_1 \times V_{-1}}^2 = \|(u(0), u_t(0))\|_{V_1 \times V_{-1}}^2,
\]
that is, \(\|\xi_u(t)\|_{X} \in C[0, T]\). By the uniform convexity of the phase space \(X\) and \((u, u_t) \in C_u([0, T], X)\), we have \((u, u_t) \in C([0, T], X)\).

Let \(z = u - v\), where \(u, v\) are two weak solutions of problem (4)-(5) corresponding to initial data \((u_0, u_1)\) and \((v_0, v_1)\), respectively. Obviously, \(z\) solves
\[
A^{-\frac{1}{2}}z_t + A^{\frac{1}{2}}z + A^{-\frac{\alpha+1}{2}}z_t + f(u) - f(v) = 0, \tag{39}
\]
\[
z(0) = z_0 \equiv u_0 - v_0, \quad z_t(0) = z_1 \equiv u_1 - v_1.
\]
Using the multiplier \(A^{-\frac{\alpha+1}{2}}z_t + \epsilon A^{-\frac{1}{2}}z\) in Eq. (39), we obtain
\[
\frac{d}{dt} H_3(\xi_z) + (1 - \epsilon)\|z_t\|_{V_{-2}}^2 + \epsilon\|z\|_{V_{-2}}^2
\]
\[
= - (f(u) - f(v), A^{-\frac{\alpha+1}{2}}z_t + \epsilon A^{-\frac{1}{2}}z)
\]
\[
\leq C(1 + \|u\|_{L^p}^{p-1} \|u_t\|_{L^\infty}^1 + \|v\|_{L^p}^{p-1} \|v_t\|_{L^\infty}^1) \|z\| \left(\|A^{-\frac{1}{2}}z\|_{V_{-\frac{2}{\alpha+2}}}^2 + \|A^{-\frac{\alpha+1}{2}}z_t\|_{V_{-\frac{2\alpha}{\alpha+2}}}^2\right)
\]
\[
\leq C\|z\| \left(\|z\|_{V_{2a-2-\delta}} + \|z_t\|_{V_{-2-\delta}}\right)
\]
\[
\leq \frac{1}{4} \epsilon \|z\|_{V_{-2}}^2 + \|z_t\|_{V_{-2}}^2 + C(\|z\|_{V_{-2}}^2 + \|z_t\|_{V_{-2}}^2),
\]
where \(\xi_z = (z, z_t)\).
\[
H_3(\xi_z) = \|z\|_{V_{-\alpha}}^2 + \|z_t\|_{V_{-\alpha-2}}^2 + \epsilon [2(A^{-\frac{1}{2}}z, A^{-\frac{1}{2}}z_t) + \|z\|_{V_{-\alpha-2}}^2] \sim \|z\|_{V_{-\alpha}}^2 + \|z_t\|_{V_{-\alpha-2}}^2
\]
for $\varepsilon$ suitably small, and where we have used (23) (replacing $v$ by $z$ there). Therefore,

$$
\frac{d}{dt} H_3(\xi_t) + \kappa(H_3(\xi_t) + \|z\|^2 + \|z_t\|^2) \leq C(\|z\|^2_{Y_{\alpha-1}} + \|z_t\|^2_{Y_{\alpha-3}}), \quad (40)
$$

with $\kappa > 0$ suitably small. Applying the Gronwall lemma to (40) gives (15). The formula (14) follows from (15) for $Y_\alpha \hookrightarrow V_{-1} \times V_{-3}$. \hfill $\square$

**Remark 2.** (i) One sees from property (i) that when $t > 0$, the regularity of the weak solutions as $\alpha \in (1/2, 1)$ is stronger than that as $\alpha \in (0, 1/2]$.

(ii) A simple calculation shows that $\lim_{\alpha \to 0} C_1(\alpha) = \infty$, which means that the regularity of the weak solutions (when $t > 0$) fails for $\alpha = 0$ (weak damping case), and $C_2(\alpha)$ is bounded for $\alpha \in (1/2, 1)$.

(iii) In order to investigate the upper semicontinuity of the global attractor $A_\alpha$ on $\alpha$, it is necessary to clarify the dependency and independency of the control constants on $\alpha$ in Theorem 3.1, respectively.

(iv) Integrating (40) over $(0, T)$, taking account of $Y_\alpha \hookrightarrow V_{-1} \times V_{-3}$ and making use of estimate (14), we have

$$
\int_0^T \|z(t)\|^2_{H \times V_{-2}} d\tau \leq C(T)\|z(0)\|^2_{\tilde{V}_0} + C \int_0^T \|z(t)\|^2_{\tilde{V}_0} d\tau \leq C(T)\|z(0)\|^2_{\tilde{V}_0}. \tag{41}
$$

4. **Global and exponential attractors.** Let Assumption (H) be valid, with $1 \leq p < p_\alpha$ and $g \in V_{-1}$. Define the operator

$$
S_\alpha(t) : X \to X, \quad S_\alpha(t)(u_0, u_1) = \xi_\alpha(t) = (u_\alpha(t), u_\alpha^\ast(t)), \quad t \geq 0,
$$

where $u_\alpha$ is a weak solution of problem (4)-(5) (with gentle dissipation $A^{\frac{2}{2-q}}u_t$). By Theorem 3.1, $\{S_\alpha(t)\}_{t \geq 0}$ constitutes a semigroup on $X$ for each $\alpha \in (0, 1)$, which is continuous in weaker space $Y_\alpha$.

For simplicity, we denote $S_\alpha(t)$ by $S(t)$, $(u_\alpha, u_\alpha^\ast)$ by $(u, u_\ast)$ for each $\alpha \in (0, 1)$ in this section.

**Theorem 4.1.** Let Assumption (H) be valid, with $1 \leq p < p_\alpha$ and $g \in V_{-1}$. Then the solution semigroup $S(t) : X \to X$ is continuous.

**Proof.** When $t = 0$, the conclusion is obvious for $S(0) = I$. When $t > 0$, let $(u_0^\alpha, u_1^\alpha) \to (u_0, u_1)$ in $X$, and

$$
S(t)(u_0^\alpha, u_1^\alpha) = \xi_{u^\alpha}(t) = (u^\alpha(t), u_\alpha^\ast(t)), \quad S(t)(u_0, u_1) = \xi_u(t) = (u(t), u_\ast(t)).
$$

Noticing that $V_{\alpha+1} \hookrightarrow V_{-\alpha}$ for $\alpha \in (1/2, 1)$ (see (12)) and using the interpolation theorem, we get that for every $\alpha \in (0, 1)$,

$$
\|(u, u_\ast)\|_{V_1 \times V_{-1}} \leq C\|(u, u_\ast)\|_{V_{-\alpha} \times V_{-\alpha-2}} \|(u, u_\ast)\|_{V_{-\alpha} \times V_{-\alpha}} \leq C_1(\alpha)C(t, R)\|(u, u_\ast)\|_{V_{-\alpha} \times V_{-\alpha-2}} \tag{42}
$$

so

$$
\|\xi_{u^\alpha}(t) - \xi_u(t)\|_X \leq C_1(\alpha)C(t, R)\|\xi_{u^\alpha}(t) - \xi_u(t)\|_{\tilde{V}_0} \leq C_1(\alpha)C(t, R)\|\xi_{u^\alpha}(0) - \xi_u(0)\|_{\tilde{V}_0} \leq C_1(\alpha)C(t, R)\|\xi_{u^\alpha}(0) - \xi_u(0)\|_{X^0}.
$$

Therefore, $S(t)$ is continuous in $X$ for every $t > 0$. \hfill $\square$
Theorem 4.2. Let Assumption (H) be valid, with \(1 \leq p < p_\alpha\) and \(g \in V_{-1}\). Then the dynamical system \((S(t), X)\) possesses a global attractor \(\mathcal{A}_\alpha\), which is bounded in \(X_\alpha\), and
\[
\mathcal{A}_\alpha = \mathcal{M}^+(\mathcal{N}),
\]
where \(\mathcal{M}^+(\mathcal{N})\) is the unstable manifold emanating from the set \(\mathcal{N}\), and \(\mathcal{N}\) is the set of all the fixed points of \(S\), that is,
\[
\mathcal{N} = \{(u, 0) \in X| A^\frac{1}{2} u + f(u) = A^{-\frac{1}{2}} g\}.
\]
Moreover, \((S(t), X)\) is a gradient system, and every full trajectory \(\gamma = \{\xi_u(t)| t \in \mathbb{R}\}\) from \(\mathcal{A}_\alpha\) is of the following properties:

(i) \[
\lim_{t \to -\infty} \text{dist}_X \{(u(t), u_t(t))| \mathcal{N}\} = 0, \quad \lim_{t \to +\infty} \text{dist}_X \{(u(t), u_t(t))| \mathcal{N}\} = 0,
\]

(ii) \((u, u_t, u_{tt}) \in L^\infty(\mathbb{R}; X_\alpha \times V_v)\), and
\[
\sup_{\gamma \in \mathcal{A}} \sup_{t \in \mathbb{R}} (\|u(t)|X_\alpha + \|u_t(t)|V_v \leq C(\mathcal{A}_\alpha),
\]
where \(V_v = V_{-\alpha -2}\) if \(\alpha \in (0, 1/2]\); \(V_v = V_{-\alpha -1}\) if \(\alpha \in (1/2, 1)\).

Proof. Estimates (7) and (9)-(12) imply that the dynamical system \((S(t), X)\) has a bounded absorbing set \(B_0\) which is bounded in \(X_\alpha\). Without loss of generality we assume that \(B_0\) is forward invariant. Then the dynamical system \((S(t), X)\) is dissipative and the semigroup \(S(t)\) is uniformly compact for \(X_\alpha \hookrightarrow X\). Therefore, \((S(t), X)\) possesses a global attractor \(\mathcal{A}_\alpha = \omega(B_0)\), which is bounded in \(X_\alpha\) for \(\mathcal{A}_\alpha \subset B_0\).

We infer from (13) that the functional \(E(\xi_u)\) defined there is continuous on \(X\), and it is a strict Lyapunov function on \(X\). Therefore, the dynamical system \((S(t), X)\) is gradient, \(\mathcal{A}_\alpha = \mathcal{M}^+(\mathcal{N})\), and every full trajectory \(\gamma = \{\xi_u(t)| t \in \mathbb{R}\}\) from \(\mathcal{A}_\alpha\) is of property (i)-(ii) (cf. Theorem 2.28 in [12] and [33]).

Now, we investigate the exponential attractors. It follows from Theorem 4.2 that the dynamical system \((S(t), X)\) has a forward invariant absorbing set \(B_0\), which is bounded in \(X_\alpha\). Let
\[
\mathcal{B} = [B_0]|Y_\alpha,
\]
where \([ \cdot ]|Y_\alpha\) denotes the closure in \(Y_\alpha\). Obviously, \(\mathcal{B}\) is also a forward invariant absorbing set of \(S(t)\), which is closed in \(Y_\alpha\) and bounded in \(X_\alpha\). So the set \(\mathcal{B}\) equipped with \(Y_\alpha\)-norm forms a complete metric space, the operator \(S(t)\) is Lipschitz continuous on \(\mathcal{B}\) (w.r.t. \(Y_\alpha\)-topology), and \((S(t), \mathcal{B})\) (equipped with \(Y_\alpha\)-topology) constitutes a dissipative dynamical system.

We first give two lemmas which are indispensable for us to establish the existence of exponential attractors.

Lemma 4.3. [15] Let \(V : M \to M\) be a mapping defined on a bounded closed set \(M\) on a Banach space \(X\) with the norm \(\|\cdot\|\). Assume that there exist a Lipschitz mapping \(K\) from \(M\) into some Banach space \(\mathcal{Z}\), i.e.,
\[
\|Kv_1 - Kv_2\|_\mathcal{Z} \leq L_K\|v_1 - v_2\|, \quad \forall v_1, v_2 \in M,
\]
and a compact seminorm \(n_\mathcal{Z}(\cdot)\) on \(\mathcal{Z}\) such that
\[
\|Vv_1 - Vv_2\| \leq \eta\|v_1 - v_2\| + n_\mathcal{Z}(Kv_1 - Kv_2), \quad \forall v_1, v_2 \in M,
\]
where $0 < \eta < 1$ is a constant. Then for any $\theta \in (\eta, 1)$, the discrete dynamical system $(V^k, M)$ (M is equipped with the metric of $X$) has an exponential attractor $A_\theta$. Moreover,

$$\dim_f(A_\theta, X) \leq \left[ \ln \frac{1}{\theta} \right]^{-1} \ln m_Z \left( \frac{2L_K}{\theta - \eta} \right),$$

where $m_Z(R)$ is the maximal number of elements $z_i$ in the ball $\{ z \in Z ||z|| \leq R \}$ possessing the property $n_Z(z_i - z_j) > 1$ when $i \neq j$.

**Lemma 4.4.** Under the assumptions of Theorem 4.2, the dynamical system $\{ z(t) \}$ possesses the property $\nu_z$ in the ball $\{ z \in Z ||z|| \leq R \}$.

Proof. It follows from Eq. (39) that where $\xi$ is as shown in (39) and $\beta > \max\{ N/2 + 2, 4 \}$.

Proof. It follows from Eq. (39) that

$$\int_0^T ||z(t)||_{V_{\alpha\beta}}^2 dt \leq C(T)(\|z(0)\|_{Y_\alpha^0}^2,$$

(43) where $z$ is as shown in (39) and $\beta > \max\{ N/2 + 2, 4 \}$.

Proof. It follows from Eq. (39) that

$$\|z(t)||_{V_{\alpha\beta}}^2 \leq C(\|A^{1/2}z\|_{V_{\alpha+2}}^2 + \|A^{\alpha+1}z\|_{V_{\alpha\beta+2}}^2 + \|f(u) - f(v)||_{V_{\alpha\beta+2}}^2$$

$$\leq C(\|z\|^2 + \|z\|_{V_{\alpha\beta}}^2 + \|f(u) - f(v)||_{V_{\alpha\beta}}^2$$

(44) where we have used the Sobolev embeddings: $H \hookrightarrow V_{\alpha\beta}, V_{\alpha\beta} \hookrightarrow V_{\alpha\beta}, V_{\alpha\beta} \hookrightarrow L^\infty, L^1 \hookrightarrow V_{\alpha\beta}$ and the fact

$$\|f(u) - f(v)||_{L^1} \leq C(1 + \|u\|_{V_{\alpha\beta+2}}^{2\alpha+1} + \|v\|_{V_{\alpha\beta+1}}^{2\alpha+1})\|z\|^2$$

$$\leq \begin{cases} C(1 + \|u\|_{V_{\alpha+1}}^{2\alpha+1} + \|v\|_{V_{\alpha+1}}^{2\alpha+1})\|z\|^2, & \alpha \in (0, 1/2], \\ C(1 + \|u\|_{V_{\alpha+2}}^{2\alpha+1} + \|v\|_{V_{\alpha+2}}^{2\alpha+1})\|z\|^2, & \alpha \in (1/2, 1), \end{cases}$$

and where we have used the facts: $V_{\alpha\beta} \hookrightarrow L^{2\alpha+1}$ for $\alpha \in (0, 1/2]$ and $V_{\alpha\beta} \hookrightarrow L^{2\alpha+1}$ for $\alpha \in (1/2, 1)$. Integrating (44) over $(0, T)$ and making use of (41) yield (43).

**Theorem 4.5.** Under the assumptions of Theorem 4.2, the dynamical system $(\mathcal{S}(t), X)$ has an exponential attractor $\mathcal{A}_{\alpha\beta}^{exp}$.

Proof. Define the operator

$$V^k = \mathcal{S}(kT) : \mathcal{B} \rightarrow \mathcal{B}, \quad k \in \mathbb{Z}^+.$$ We first show that the discrete system $(V^k, \mathcal{B})$ (equipped with $Y_{\alpha}$-topology) possesses an exponential attractor. We introduce the functional space

$$Z = \{ \xi_u = (u, u_t) \in L^2(0, T; Y_{\alpha}) | u, u_t \in L^2(0, T; V_{\alpha\beta}), \|\xi_u\|_Z < \infty \}$$

equipped with the norm

$$\|\xi_u\|_Z^2 = \int_0^T ((\|u(t)\|_{Y_{\alpha}}^2 + \|u(t)\|_{V_{\alpha\beta}}^2)dt.$$ Obviously, $Z$ is a Banach space. Define the mapping

$$K : \mathcal{B}(\subset Y_{\alpha}) \rightarrow Z, \quad K\xi_u = \xi_u(\cdot), \forall \xi_u \in \mathcal{B},$$

where $\xi_u(\cdot)$ means $\xi_u(t), t \in [0, T]$. By (14) and Lemma 4.4,

$$\|K\xi_u - K\xi_v\|_Z^2 = \int_0^T (||\xi_z(t)||_{Y_{\alpha}}^2 + ||z(t)||_{V_{\alpha\beta}}^2)dt \leq C(T)||\xi_z(0)||_{Y_{\alpha}}^2, \forall \xi_u, \xi_v \in Y_{\alpha},$$
where \( \xi_z(0) = \xi_u - \xi_v \). Estimate (15) means that
\[
\|V \xi_u - V \xi_v\|_{Y_alpha}^2 \leq n_T \|\xi_z(0)\|_{Y_alpha}^2 + n_T^2 (K \xi_u - K \xi_v),
\]
where \( n_T = C e^{-\kappa T} \) and
\[
n_Z(\xi_u) = C \|\xi_u(\cdot)\|_{L^2(0,T;V_{-1} \times V_{-3})}^2
\]
is a compact seminorm on \( Z \) for \( Z \leftrightarrow L^2(0,T;V_{-1} \times V_{-3}) \) (cf. [26]). Taking \( T : 0 < \eta_T < 1 \), we infer from Lemma 4.3 that the dynamical system \((V^k, B)\) has an exponential attractor \( \mathcal{A}_t \). Let
\[
\mathcal{A}_{exp}^\alpha = \bigcup_{0 \leq t \leq T} S(t) \mathcal{A}_t.
\]
By the standard argument (cf. [33]) one easily knows that \( \mathcal{A}_{exp}^\alpha \) is an exponential attractor of the dynamical system \((S(t), B)\) (equipped with \( Y_{alpha} \)-topology). And by virtue of additional regularity of the weak solutions of Eq. (4), the interpolation theorem and the technique used in [32], we easily recover the topology of the phase space \( X \) and show that \( \mathcal{A}_{exp}^\alpha \) is just an exponential attractor of the dynamical system \((S(t), X)\).

5. Upper semicontinuity of global attractors. In this section, we discuss the upper semicontinuity of the family of global attractors \( \{\mathcal{A}_\alpha\}_{\alpha \in (0,1]} \). In particular, when \( \alpha = 1, p < p^* \), the existence of global attractors for problem (4)-(5) has been established in [31].

Lemma 5.1 ([31]). Let Assumption (H) be valid, with \( 1 \leq p \leq p^* \) and \( g \in V_{-1} \). Then problem (4)-(5), with \( \alpha = 1 \), admits a unique weak solution \( u \in C(\mathbb{R}^+; X) \), estimates (7)-(8) hold (with \( \alpha = 1 \) there), and \( (u, u_t) \) depends continuously on initial data in \( X \). Moreover, if also \( f \in C^2(\mathbb{R}) \), and when \( N \geq 2 \),
\[
|f''(s)| \leq C(1 + |s|^{p^*-2}), \quad \text{with} \ 2 \leq p \leq p^*.
\]
Then the related dynamical system \((S^1(t), X)\) possesses an absorbing set \( B_1 \) and a global attractor \( \mathcal{A}_1 \), which are bounded in \( V_2 \times H \).

Lemma 5.2 ([9, 21]). Let \( X \) be a complete metric space, \( \{S^\lambda(t)\}_{\lambda \geq 0} \) be a continuous semigroup acting on \( X \) for each \( \lambda \in \Lambda \), and the dynamical system \((S^\lambda(t), X)\) possess a compact global attractor \( \mathcal{A}_\lambda \) for every \( \lambda \in \Lambda \). Assume that the following conditions hold:
(i) there exists a compact set \( D \subset X \) such that \( \mathcal{A}_\lambda \subset D \) for all \( \lambda \in \Lambda \);
(ii) if \( \lambda_k \rightarrow \lambda_0, x_k \in \mathcal{A}_{\lambda_k} \) and \( x_k \rightarrow x_0 \), then \( S^{\lambda_k}(t_0)x_k \rightarrow S^{\lambda_0}(t_0)x_0 \) for some \( t_0 > 0 \).
Then the family of attractors \( \mathcal{A}_\lambda \) is upper semicontinuous at the point \( \lambda_0 \), i.e.,
\[
\lim_{\lambda_k \rightarrow \lambda_0} \text{dist}_X \{\mathcal{A}_{\lambda_k}, \mathcal{A}_{\lambda_0}\} = 0.
\]

Theorem 5.3. Let Assumption (H) be valid, with \( 1 \leq p < p_{\alpha_0} = \frac{N+2(2\alpha_0-1)}{(N-2)p}, \alpha_0 \in (0,1) \), and \( g \in V_{-1} \). In particular, when \( \alpha_0 = 1 \), formula (46) holds. Then the family of global attractors \( \{\mathcal{A}_\alpha\} \) is upper semicontinuous at the point \( \alpha_0 \), i.e.,
\[
\lim_{\alpha \rightarrow \alpha_0} \text{dist}_X \{\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}\} = 0.
\]

Proof. Step 1: We first show that formula (47) holds for \( \alpha_0 \in (0,1) \). Without loss of generality we assume that \( \alpha \in [\gamma, \alpha_0 + \eta] \equiv \Lambda(\subset (0,1)) \) for \( \alpha \rightarrow \alpha_0 \), where \( \gamma \equiv \alpha_0 - \eta(>0) \), with \( \eta : 0 < \eta < \frac{1}{2} \min\{\alpha_0, 1 - \alpha_0\} \).
(i) Formally differentiating Eq. (4) with respect to $t$ and using the multiplier
\[ A^{-\frac{r+1}{2}}u_0^\alpha + \epsilon A^{-\frac{r}{2}}u_1^\alpha \quad (0 < \epsilon \ll 1) \] to the resulting expression, similar to the proof of (9) one easily gets that, for $1 \leq p < p_\gamma (\equiv \frac{N+2(2r-1)}{(N-2r-1)})$,
\[
\|u_0^\alpha (t)\|_{V_{-r}}^2 + \|u_1^\alpha (t)\|_{V_{-r-2}}^2 + \int_t^{t+1} (\|u_1^\alpha (\tau)\|^2 + \|u_0^\alpha (\tau)\|_{V_{-2}}^2) d\tau
\leq \left(1 + \frac{1}{t^{1/\gamma}}\right) C(R, \|g\|_{V_{-1}}), \quad t > 0,
\]
where $C(R, \|g\|_{V_{-1}})$ is a positive constant depending only on $\alpha_0$. Based on (48), repeating the proof of (11) (replacing $\alpha$ there by $\gamma$), one easily obtains
\[
\|u_\alpha^\alpha (t)\|_{V_{-r}}^2 \leq \left(1 + \frac{1}{t^{1/\gamma}}\right) C(\|g\|_{V_{-1}}).
\]

(ii) Let $\xi_u^\alpha (t) = (u^\alpha (t), u_0^\alpha (t)) = S^\alpha (t)(u_0^\alpha, u_1^\alpha)$ with $\alpha \in \Lambda$, $\xi_u^\alpha, u_0^\alpha \in A_\alpha$ with $\alpha \in \Lambda \setminus \{\alpha_0\}$, and $\lim_{\alpha \to \alpha_0} \|(u_0^\alpha, u_1^\alpha) - (u_0^{\alpha_0}, u_1^{\alpha_0})\|_X = 0$.

Then $(u_0^{\alpha_0}, u_1^{\alpha_0}) \in D$, estimates (48) and (49) (removing $1/t^{1/\gamma}$ there) hold for $u^\alpha$, with $\alpha \in \Lambda \setminus \{\alpha_0\}$, and the difference $z = u^\alpha - u^{\alpha_0}$ solves
\[
A^{-\frac{r}{2}} z_t + A^{\frac{r}{2}} z + A^{\frac{r-1}{2}} z_t + (A^{\frac{r-1}{2}} - A^{\frac{r-1}{4}}) u_0^{\alpha_0} + f(u^\alpha) - f(u^{\alpha_0}) = 0,
\]
\[ z(0) = u_0^\alpha - u_0^{\alpha_0}, \quad z_t(0) = u_1^\alpha - u_1^{\alpha_0}. \]

Using the multiplier $A^{-\frac{r+1}{4}} z_t + \epsilon A^{-\frac{r}{4}} z$ in Eq. (52), we have
\[
\frac{1}{2} \frac{d}{dt} H_4(\xi_z) + \|z_t\|_{V_{-r-2}}^2 - \epsilon \|z_t\|_{V_{-2}}^2 + \epsilon \|z\|^2 = - \left((A^{\frac{r-1}{2}} - A^{\frac{r-1}{4}}) u_0^{\alpha_0} + f(u^\alpha) - f(u^{\alpha_0}), A^{-\frac{r+1}{4}} z_t + \epsilon A^{-\frac{r}{4}} z\right),
\]
where $\xi_z = (z, z_t)$,
\[
H_4(\xi_z) = \|z\|_{V_{-r}}^2 + \|z_t\|_{V_{-r-2}}^2 + \epsilon [2(A^{-\frac{r}{2}} z, A^{-\frac{r}{4}} z_t) + \|z_t\|_{V_{-2}}^2] \leq \|z\|_{V_{-r}}^2 + \|z_t\|_{V_{-r-2}}^2.
\]
for $\epsilon > 0$ suitably small. When $1 \leq p < p_\gamma (< p_{\alpha_0})$,

$$
|f(u^n) - f(u^m)|, \quad A^\frac{\alpha - 2\gamma}{\alpha - 2}(z_t + cA^{-\frac{\gamma}{2}}z_t)
$$

\begin{align}
&\leq C(1 + \|u^n\|_{\|n - 1\|}^{\frac{1}{2}} + \|u^m\|_{\|n - 1\|}^{\frac{1}{2}})z_t(\|A^{-\frac{\gamma}{2}}z_t\|_{\frac{2N}{\alpha - 2}} + \|A^\frac{\alpha - 2\gamma}{\alpha - 2}z_t\|_{\frac{2N}{\alpha - 2}}) \\
&\leq C\|z\|_2(\|z\|_{V_{\gamma - 2}} + \|z_t\|_{V_{\gamma - 2}}) \\
&\leq \frac{1}{4}\epsilon(z^2 + \|z_t\|^2_{V_{\gamma - 2}}) + C(\|z\|^2_{V_{\gamma - 1}} + \|z_t\|^2_{V_{\gamma - 3}}),
\end{align}

(54)

where we have used (23) (replacing $v$ and $\alpha$ there by $z$ and $\gamma$), and

$$
\left|\left((A^\frac{\alpha - 1}{\alpha} - A^\frac{\alpha - 1}{\alpha})u_t^\alpha, A^\frac{\alpha - 1}{\alpha}z_t + cA^{-\frac{\gamma}{2}}z_t\right)\right|
$$

\begin{align}
&\leq C\|(A^\frac{\alpha - 1}{\alpha} - A^\frac{\alpha - 1}{\alpha})u_t^\alpha\|_{V_{\gamma - 1}}(\|A^{-\frac{\gamma}{2}}z_t\|_{V_{\gamma + 1}} + \|A^{-\frac{\gamma}{2}}z_t\|_{V_{\gamma + 1}}) \\
&\leq \frac{1}{4}\epsilon(z^2 + \|z_t\|^2_{V_{\gamma - 2}}) + C\|(A^\frac{\alpha - 1}{\alpha} - A^\frac{\alpha - 1}{\alpha})u_t^\alpha\|^2_{V_{\gamma - 1}}.
\end{align}

(55)

Taking account of $Y_\gamma \rightarrow Y_{\gamma - 1} \times Y_{\gamma - 2}$, inserting (54)-(55) into (53), we get

$$
d\frac{d}{dt}H_4(\xi_t) \leq C H_4(\xi_t) + C\|(A^\frac{\alpha - 1}{\alpha} - A^\frac{\alpha - 1}{\alpha})u_t^\alpha\|^2_{V_{\gamma - 1}}.
$$

(56)

Noticing $X \rightarrow Y_\gamma$, applying the Gronwall lemma to (56) gives

$$
\|\xi_t(\tau)\|_{\gamma} \leq C e^{C_\tau}\|\xi_t(0)\|_{\gamma} + C \int_0^\tau e^{C(t - \tau)}\|((A^\frac{\alpha - 1}{\alpha} - A^\frac{\alpha - 1}{\alpha})u_t^\alpha(\tau))\|^2_{V_{\gamma - 1}} d\tau.
$$

(57)

By (7)-(8) and (18), $u_t^\alpha \in L^2(0, t; V_{\alpha - 1}) \cap C_w([0, t]; V_{\gamma - 1})$, then, for each $\tau \in [0, t]$,

$$
A^{-\frac{\gamma}{2}}u^\alpha_t(\tau) = \sum_{j=1}^{\infty} \lambda_j^\frac{\alpha - 1}{\alpha} c_j(\tau) w_j, \quad A^{-\frac{\gamma}{2}}u^\alpha_t(\tau) = \sum_{j=1}^{\infty} \lambda_j^\frac{\gamma}{4} c_j(\tau) w_j, \quad \beta \leq -\frac{1}{4},
$$

where $Aw_j = \lambda_j w_j, j = 1, 2, \ldots, \{w_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $H$, and for a.e. $\tau \in [0, t]$,

$$
\sum_{j=1}^{\infty} \left|\left(\lambda_j^\frac{\alpha - 1}{\alpha} - \lambda_j^\frac{2\alpha - \alpha - \alpha - 1}{\alpha} \right) c_j(\tau) \right|^2 = \left\|\left((A^\frac{\alpha - 1}{\alpha} - A^\frac{2\alpha - \alpha - \alpha - 1}{\alpha})u_t^\alpha(\tau)\right)^2_{V_{\gamma - 1}}\right\|_{V_{\gamma - 1}} < +\infty,
$$

(58)

$$
= \|(A^\frac{\alpha - 1}{\alpha} - A^\frac{2\alpha - \alpha - \alpha - 1}{\alpha})u_t^\alpha(\tau)\|^2_{V_{\gamma - 1}} \leq C\|u_t^\alpha(\tau)\|^2_{V_{\alpha - 1}} < +\infty,
$$

where we have used the fact: $\alpha - \gamma - 2 < \alpha_0 - 1$ for $\alpha \in \Lambda$. Then for any $\epsilon > 0$ there exists a constant $N = N(\alpha_0, \tau) > 0$ such that

$$
\sum_{j>N} \left|\left(\lambda_j^\frac{\alpha - 1}{\alpha} - \lambda_j^\frac{2\alpha - \alpha - \alpha - 1}{\alpha} \right) c_j(\tau) \right|^2 < \frac{\epsilon}{2},
$$

and there exists a $\eta_0 : 0 < \eta_0 < \eta$ such that

$$
\sum_{j=1}^{N} \left|\left(\lambda_j^\frac{\alpha - 1}{\alpha} - \lambda_j^\frac{2\alpha - \alpha - \alpha - 1}{\alpha} \right) c_j(\tau) \right|^2 < \frac{\epsilon}{2} \quad \text{as } 0 < |\alpha - \alpha_0| < \eta_0.
$$

By the arbitrariness of $\epsilon > 0$,

$$
\lim_{\alpha \rightarrow \alpha_0} \|(A^\frac{\alpha - 1}{\alpha} - A^\frac{2\alpha - \alpha - \alpha - 1}{\alpha})u_t^\alpha(\tau)\|^2_{V_{\gamma - 1}} = 0 \quad \text{a.e. } \tau \in [0, t].
$$

(59)

It follows from (58)-(59) and the Lebesgue dominated convergence theorem,

$$
\lim_{\alpha \rightarrow \alpha_0} \int_0^t e^{C(t - \tau)}\|((A^\frac{\alpha - 1}{\alpha} - A^\frac{2\alpha - \alpha - \alpha - 1}{\alpha})u_t^\alpha(\tau))\|^2_{V_{\gamma - 1}} d\tau = 0.
$$

(60)
The combination of (42) (replacing $\alpha$ there by $\gamma$), (57) and (60) yields
\[
\lim_{\alpha \to \alpha_0} \|(z, z_t)(t)\|_X \leq C_1(\alpha_0)C(t, R) \lim_{\alpha \to \alpha_0} \|(z, z_t)(t)\|_{\gamma X} = 0.
\]

Therefore, by Lemma 5.2 and the arbitrariness of $\eta$, the family of attractors $\{A_\alpha\}$ is upper semicontinuous at the point $\alpha_0$ for $1 \leq p < p_{\alpha_0}$.

Step 2: We show that formula (47) holds for $\alpha_0 = 1$. Without loss of generality we assume that $\alpha \in [\gamma_1, 1] \equiv \Lambda_1(\subset (0, 1])$ for $\alpha \to 1^-$, where $\gamma_1 = 1 - \eta(\eta > 0)$, with $\eta : 0 < \eta < \frac{1}{2}$.

Based on Theorem 3.1 and Lemma 5.1, similar to the proof of (10) (replacing the multiplier $A^{-\frac{1}{2}}u_{\alpha t} + \epsilon u_t$ there (see (29)) by $A^{-\frac{1}{2}}u_{\alpha t} + \epsilon u_t^\alpha$) and (12) (using the Sobolev embedding $V_{1+\alpha} \hookrightarrow V_{1+\gamma_1}$ there), we have that, for all $\alpha \in \Lambda_1$,
\[
\|u_\alpha(t)\|_{V_{1+\gamma_1}}^2 + \|u_\alpha(t)\|_{V_{1-\gamma_1-1}}^2 + \int_t^{t+1} \left(\|u_\alpha(t)\|_{V_1}^2 + \|u_\alpha(t)\|_{V_{-1}}^2\right) d\tau \\
\leq \left(1 + \frac{1}{t^{2/\gamma_1}}\right) C(R, \|g\|_{V_{-1}}), \quad t > 0,
\]
(61)

\[
\|u_\alpha(t)\|_{V_{1+\gamma_1}}^2 + \int_t^{t+1} \|u_\alpha(t)\|_{V_3}^2 d\tau \\
\leq \left(1 + \frac{1}{(a/2)^{2/\gamma_1}} + \frac{1}{(t-a/2)^{2/\gamma_1}}\right) C(R, \|g\|_{V_{-1}}), \quad t > a/2,
\]
(62)

where the constant $C(R, \|g\|_{V_{-1}})$ is independent of $\alpha \in \Lambda_1$. Therefore, the set $D$ defined in (50) (replacing $\Lambda$ there by $\Lambda_1$) is bounded in $V_{1+\gamma_1} \times V_{1-\gamma_1}$ and compact in $X$ for all $\alpha \in \Lambda_1$. And formula (51) holds (replacing $\Lambda$ there by $\Lambda_1$).

Let $\xi_{u\alpha}(t) = (u^\alpha(t), u_t^\alpha(t)) = S^\alpha(t)(u_0^\alpha, u_1^\alpha)$ with $\alpha \in \Lambda_1$,
\[
(u_0^\alpha, u_1^\alpha) \in A_\alpha \quad \text{with} \quad \alpha \in [\gamma_1, 1], \quad \text{and} \lim_{\alpha \to 1} \|(u_0^\alpha, u_1^\alpha) - (u_0^1, u_1^1)\|_X = 0.
\]

Then $(u_0^1, u_1^1) \in D$, estimates (61) and (62) (removing $1/t^{2/\gamma_1}$ and $1/(a/2)^{2/\gamma_1} + 1/(t-a/2)^{2/\gamma_1}$ there) hold for $u^\alpha$, with $\alpha \in [\gamma_1, 1]$, and the difference $z = u^\alpha - u^1$ solves
\[
A^{-\frac{1}{2}}z_{tt} + A^{\frac{1}{2}}z + A^{\frac{1}{2}}z_t + (A^{\frac{1}{2}} - I)u_{\alpha t}^\alpha + f(u^\alpha) - f(u^1) = 0,
\]
(63)

where $z(0) = u_0^\alpha - u_0^1$, $z_t(0) = u_1^\alpha - u_1^1$,

where the sign $I$ denotes the identity operator. Using the multiplier $A^{-\frac{1}{2}}z_t + \epsilon z$ in Eq. (63) yields
\[
\frac{d}{dt} H_5(\xi_z) + \|z_t\|_{V_{1-\gamma_1-1}}^2 - \epsilon \|z_t\|_{V_{-1}}^2 + \epsilon \|z\|_{V_3}^2 \\
= - \left(\left(A^{\frac{1}{2}} - I\right)u_{\alpha t}^\alpha + f(u^\alpha) - f(u^1), A^{-\frac{1}{2}}z_t + \epsilon z\right),
\]
(64)

where $\xi_z = (z, z_t)$.

\[
H_5(\xi_z) = \frac{1}{2}\left(\|z\|_{V_{1-\gamma_1}}^2 + \|z_t\|_{V_{1-\gamma_1-1}}^2\right) + \epsilon \left(\left(A^{-\frac{1}{2}}z, A^{-\frac{1}{2}}z_t\right) + \frac{1}{2}\|z\|_{V_{1-\gamma_1-1}}^2\right)
\sim \|z\|_{V_{1-\gamma_1}}^2 + \|z_t\|_{V_{1-\gamma_1-1}}^2
\]

then...
for \( \varepsilon > 0 \) suitably small. When \( 1 \leq p < p_{\gamma_1} (\leq p^*) \), similar to the proof of (30) (replacing \( \alpha \) by \( \gamma_1 \) and \( v \) by \( z \) there) we have

\[
\|(f(u^\alpha) - f(u^1), A^{-\frac{\gamma_1}{2}} z_t + \varepsilon z)\| \\
\leq C(1 + \|u^\alpha\|^{p-1}_{\frac{N}{N-\gamma_1}} + \|u^1\|^{p-1}_{\frac{N}{N-\gamma_1}})\|z\| \frac{2N}{N-2(\frac{\gamma_1}{2} + \frac{1}{2})} + \|z\| \frac{2N}{N-2(\frac{\gamma_1}{2} + \frac{1}{2})}
\]

\[
\leq C\|z\| v_1(\|z\|_{V_{1-\varepsilon}} + \|z\|_{V_{2\gamma_1-1-\varepsilon}})
\]

\[
\leq \frac{1}{4}(\varepsilon\|z\|^2_{V_1} + \|z\|^2_{V_{\gamma_1-1-\varepsilon}}) + C(\|z\|^2_{V_{1-\varepsilon}} + \|z_t\|^2_{V_{1-\varepsilon}}),
\]

(55)

and

\[
\left|\left((A^{-\frac{\gamma_1}{2}} - I)u^1_t, A^{-\frac{\gamma_1}{2}} z_t + \varepsilon z\right)\right| \\
\leq C\|((A^{-\frac{\gamma_1}{2}} - I)u^1_t\|_{V_{\gamma_1-1-\varepsilon}} + \|z\|_{V_{\gamma_1-1-\varepsilon}})
\]

(66)

\[
\leq \frac{1}{4}(\varepsilon\|z\|^2_{V_1} + \|z\|^2_{V_{\gamma_1-1-\varepsilon}}) + C\|((A^{-\frac{\gamma_1}{2}} - I)u^1_t\|_{V_{\gamma_1-1-\varepsilon}}^2.
\]

Taking account of \( V_{1-\gamma_1} \times V_{\gamma_1-1} \leftrightarrow V_{1-\varepsilon} \times V_{\varepsilon-3} \) and inserting (55)-(66) into (64), we get

\[
d\|H_5(\xi_\varepsilon) \|_{V_{1-\gamma_1} \times V_{\gamma_1-1}}^2 \\
\leq C e^{Ct}\|\xi_\varepsilon(0)\|_{X}^2 + C \int_0^t e^{C(t-\tau)}\|((A^{-\frac{\gamma_1}{2}} - I)u^1_t(\tau))\|_{V_{\gamma_1-1-\varepsilon}}^2 d\tau.
\]

(67)

By estimates (7)-(8) (with \( \alpha = 1 \) there), \( u^1_t \in L^2(0, t; H) \cap C_w([0, t]; V_{\gamma_1-1}) \), then for each \( \tau \in [0, t] \),

\[
A^{-\frac{\gamma_1}{4}} u^1_t(\tau) = \sum_{j=1}^{\infty} c_j(\tau) w_j, \quad A^{\beta} u^1_t(\tau) = \sum_{j=1}^{\infty} \lambda_j^{\beta+\frac{\gamma_1}{4}} c_j(\tau) w_j, \quad \beta \leq -\frac{1}{4},
\]

and for a.e. \( \tau \in [0, t] \),

\[
\sum_{j=1}^{\infty} \left|\left(\lambda_j^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - \lambda_j^{\frac{\gamma_1}{4} + \frac{\alpha}{2}}\right)c_j(\tau)\right|^2 = \left|\left(A^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - A^{\frac{\gamma_1}{4} + \frac{\alpha}{2}}\right)u^1_t(\tau)\right|^2
\]

(68)

\[
= \|(A^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - A^{\frac{\gamma_1}{4} + \frac{\alpha}{2}})u^1_t(\tau)\|_{V_{\gamma_1-1-\varepsilon}}^2 \leq C\|u^1_t(\tau)\|^2_{V_{\gamma_1-1-\varepsilon}} < +\infty.
\]

Therefore, for any \( \varepsilon > 0 \) there exists a constant \( M = M(\varepsilon) > 0 \) such that

\[
\sum_{j > M} \left|\left(\lambda_j^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - \lambda_j^{\frac{\gamma_1}{4} + \frac{\alpha}{2}}\right)c_j(\tau)\right|^2 < \varepsilon^2
\]

and there exists a \( \eta_0 : 0 < \eta < \eta_0 \) such that

\[
\sum_{j=1}^{M} \left|\left(\lambda_j^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - \lambda_j^{\frac{\gamma_1}{4} + \frac{\alpha}{2}}\right)c_j(\tau)\right|^2 < \varepsilon^2 \quad \text{as } 0 < 1 - \alpha < \eta_0.
\]

By the arbitrariness of \( \varepsilon > 0 \),

\[
\lim_{\alpha \to 1} \|(A^{\frac{\gamma_1}{4} - \frac{\alpha}{2}} - A^{\frac{\gamma_1}{4} + \frac{\alpha}{2}})u^1_t(\tau)\|_{V_{\gamma_1-1-\varepsilon}}^2 = 0 \quad \text{a.e. } \tau \in [0, t].
\]

(69)
By (68)-(69) and the Lebesgue dominated convergence theorem,
\[
\lim_{\alpha \to 1} \int_0^t e^{C(t-\tau)} \left\| (A_{\alpha}^{\gamma_1 \gamma_1 - 1} - I) u_I(t) \right\|_{V^{-\gamma_1 + 1}}^2 d\tau = 0. \tag{70}
\]

By virtue of the interpolation
\[
\left\| (u, u_t) \right\|_{V^{1/2} \times V^{-1}} \leq C \left\| (u, u_t) \right\|_{V^{1+\gamma_1} \times V^{-\gamma_1}}^{1/2} \left\| (u, u_t) \right\|_{V^{1-\gamma_1} \times V^{-\gamma_1 - 1}}^{1/2},
\]
estimates (67) and (70), we have
\[
\lim_{\alpha \to 1} \left\| (z, z_t) (t) \right\|_X \leq C(t, R) \lim_{\alpha \to 1} \left\| (z, z_t) (t) \right\|_{V^{1-\gamma_1} \times V^{-\gamma_1 - 1}}^{1/2} = 0.
\]
Therefore, by Lemma 5.2 and the arbitrariness of \(\eta\), the family of attractors \(\{A_\alpha\}\) is upper semicontinuous at the point \(\alpha_0 = 1\) for \(1 < p < p^*\).

**Acknowledgments.** The authors thank the referee for his valuable comments and suggestions. The research is supported by National Natural Science Foundation of China (Grant No. 11671367).

**REFERENCES**

[1] V. Belleri and V. Pata, Attractors for semilinear strongly damped wave equations on \(\mathbb{R}^3\), *Discrete Continuous Dynam. Systems - A*, 7 (2001), 719–735.

[2] J. L. Bona and R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, *Comm. Math. Phys.*, 118 (1988), 15–29.

[3] E. Cerpa and I. Rivas, On the controllability of the Boussinesq equation in low regularity, *J. Evol. Equ.,* (2018).

[4] G. Chen and D. L. Russell, A mathematical model for linear elastic systems with structural damping, *Quart. Appl. Math.*, 39 (1982), 433–454.

[5] S. P. Chen and R. Triggiani, Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems, *Lecture Notes in Math.*, 1354 (1988), Springer-Verlag, 234–256.

[6] S. P. Chen and R. Triggiani, Proof of extension of two conjectures on structural damping for elastic systems, *Pacific J. Math.*, 136 (1989), 15–55.

[7] S. P. Chen and R. Triggiani, Gevrey class semigroups arising from elastic systems with gentle dissipation: the case \(0 < \alpha < 1/2\), *Proceedings of AMS*, 110 (1990), 401–415.

[8] Y. Cho and T. Ozawa, On small amplitude solutions to the generalized Boussinesq equations, *Discrete Continuous Dynam. Systems - A*, 17 (2007), 691–711.

[9] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative System*, Typography, layout, ACTA, 2002.

[10] I. Chueshov and I. Lasiecka, Existence, uniqueness of weak solutions and global attractors for a class of nonlinear 2D Kirchhoff-Boussinesq models, *Discrete Continuous Dynam. Systems - A*, 15 (2006), 777–809.

[11] I. Chueshov and I. Lasiecka, On global attractor for 2D Kirchhoff-Boussinesq model with supercritical nonlinearity, *Communications in Partial Differential Equations*, 36 (2010), 67–99.

[12] I. Chueshov and I. Lasiecka, *Long-time behavior of second order evolution equations with nonlinear damping*, Memoirs of AMS, 195 (2008).

[13] I. Chueshov, Global attractors for a class of Kirchhoff wave models with a structural nonlinear damping, *J. Abstr. Differ. Equ. Appl.*, 1 (2010), 86–106.

[14] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, *J. Differential Equations*, 252 (2012), 1229–1262.

[15] I. Chueshov, *Dynamics of Quasi-Stable Dissipative Systems*, Springer, 2015.

[16] P. Deift, C. Tomei and E. Trubowitz, Inverse scattering and the Boussinesq equation, *Comm. Pure Appl. Math.*, 35 (1982), 567–628.

[17] M. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a singularly perturbed Cahn-Hilliard system, *Math. Nachr.*, 272 (2004), 11–31.
[18] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, Discrete Continuous Dynam. Systems - A, 10 (2004), 211–238.
[19] S. Gatti, A. Miranville, V. Pata and S. Zelik, Continuous families of exponential attractors for singularly perturbed equations with memory, Proc. R. Soc. Edinb., 140A (2010), 329–366.
[20] V. Kalantarov and S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, J. Differential Equations, 247 (2009), 1120–1155.
[21] L. V. Kapitanski and I. N. Kostin, Attractors of nonlinear evolution equations and their approximations, Leningrad Math. J., 2 (1991), 97–117.
[22] K. Li and S. H. Fu, Asymptotic behavior for the damped Boussinesq equation with critical nonlinearity, Appl. Math. Lett., 30 (2014), 44–50.
[23] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, J. Differential Equations, 106 (1993), 257–293.
[24] A. Savostianov, Strichartz estimates and smooth attractors for a sub-quintic wave equation with fractional damping in bounded domains, Adv. Differential Equations, 20 (2015), 495–530.
[25] A. Savostianov and S. Zelik, Smooth attractors for the quintic wave equations with fractional damping, Asymptotic Analysis, 87 (2014), 191–221.
[26] J. Simon, Compact sets in the space $L^p(0,T;B)$, Annali di Matematica Pura ed Applicata, 146 (1986), 65–96.
[27] V. Varlamov, On spatially periodic solutions of the damped Boussinesq equation, Differential Integral Equations, 10 (1997), 1197–1211.
[28] V. Varlamov, Eigenfunction expansion method and the long-time asymptotics for the damped Boussinesq equation, Discrete Continuous Dynam. Systems - A, 7 (2001), 675–702.
[29] V. Varlamov and A. Balogh, Forced nonlinear oscillations of elastic membranes, Nonlinear Anal. RWA., 7 (2006), 1005–1028.
[30] S. B. Wang and X. Su, Global existence and long-time behavior of the initial-boundary value problem for the dissipative Boussinesq equation, Nonlinear Anal. RWA, 31 (2016), 552–568.
[31] Z. J. Yang, Longtime dynamics of the damped Boussinesq equation, J. Math. Anal. Appl., 399 (2013), 180–190.
[32] Z. J. Yang and P. Y. Ding, Longtime dynamics of Boussinesq type equations with fractional damping, Nonlinear Analysis, 161 (2017), 108–130.
[33] Z. J. Yang, P. Y. Ding and L. Li, Longtime dynamics of the Kirchhoff equations with fractional damping and supercritical nonlinearity, J. Math. Anal. Appl., 442 (2016), 485–510.
[34] Z. J. Yang, Z. M. Liu and P. P. Niu, Exponential attractor for the wave equation with structural damping and supercritical exponent, Commun. Contemp. Math., 18 (2016), 155055.
[35] Z. J. Yang, Z. M. Liu and N. Feng, Longtime behavior of the semilinear wave equation with gentle dissipation, Discrete Continuous Dynam. Systems - A, 36 (2016), 6557–6580.
[36] Z. J. Yang and Z. M. Liu, Longtime dynamics of the quasi-linear wave equations with structural damping and supercritical nonlinearities, Nonlinearity, 30 (2017), 1120–1145.