REVERSE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY
AND PLANAR PROJECTION OF A CONE\

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Abstract. Solutions to the so-called “reverse” Cauchy-Bunyakovsky-Schwarz (CBS) inequality form a circular cone in a real inner product space. Projecting nappe onto a hyperplane can yield: the whole hyperplane, a point, a nappe or a half-hyperplane that lacks the whole boundary but a point. A formula for aperture of the projected one-sided cone is given. In other words, information about the angle between vectors is used to assess the angle between their projections.

Key words. Cone, Nappe, Aperture, Planar projection, Angle, Reverse Cauchy inequality

AMS subject classifications. 15A63, 46C05, 26D15, 51M05, 51M04

1. Introduction and article overview. The Cauchy-Bunyakovsky-Schwarz (CBS) inequality

\begin{equation}
    u \cdot v \leq \|u\| \|v\|
\end{equation}

holds for any vectors \( u \) and \( v \) in a real inner product space \( H \). Moreover, the two sides are equal only for linearly dependant vectors \( u = \lambda v \) with scalar \( \lambda > 0 \) or in the case zero vector is involved. Therefore, the opposite inequality

\begin{equation}
    u \cdot v \geq \|u\| \|v\|
\end{equation}

will hold only as equality, for linearly dependant \( u \) and \( v \). Then, for any subspace \( V < H \) and a projection \( P : H \rightarrow V \), \( P(u) \) and \( P(v) \) are also linearly dependant. Thus,

\begin{equation}
    \text{inequality (1.2) } \implies \quad P(u) \cdot P(v) \geq \|P(u)\| \|P(v)\|.
\end{equation}

Certain relaxations of (1.2) are obtained by scaling the right hand side. If the corresponding scalar is less than \(-1\) or greater than \(1\) than the inequality is trivial, so all interesting cases are embodied in the so-called “reverse” CBS inequality\(^2\) (see \[3\]):

\begin{equation}
    u \cdot v \geq \cos \varphi \ \|u\| \|v\|.
\end{equation}

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\(^2\)First proved by Cauchy for real coordinate space \( \mathbb{R}^n \) (see \[2\] page 373).
However, for arbitrary projection $P$ in contrast with (1.3) inequality (1.4) \( \implies P(x) \cdot P(y) \geq \cos \varphi \| P(x) \| \| P(y) \| \).

**Example 1.1.** (*that proves (1.5)*) Let $u = (1, 1), v = (-1, 1)$. The angle between $u$ and $v$ is $\varphi = \frac{\pi}{2}$ and thus in this case (1.4) holds. Projection $P_1$ onto the first coordinate axis now makes (1.5) obvious because

$$P_1(u) \cdot P_1(v) = -1 \ngeq 0 = \cos \varphi \| P_1(u) \| \| P_1(v) \| .$$

Provided a projection $P : H \to V$ in a real inner product space and provided (1.4), a natural problem arises: find the smallest angle $\varphi_1 \in [0, \pi]$ such that

(1.6) \[ P(u) \cdot P(v) \geq \cos \varphi_1 \| P(u) \| \| P(v) \| . \]

Section 2 resolves that if one of the vectors in (1.4) is fixed, for example $v$, then the smallest angle $\varphi_1$ in (1.6) can be given in terms of the angle $\varphi$. Formula (1.6) will then be valid for all vectors $u$ that satisfy (1.4). Trivial cases are mentioned in Remark 2.3 and the case of projecting onto a line in Remark 2.11. All other cases resolve based on the angle $\psi$ between $v$ and $V^\perp$ (see Definition 2.2): $\varphi > \psi$: then Proposition 2.4 gives that the smallest $\varphi_1 = \pi$, $\varphi = \psi$: if $\psi < \pi/2$ then Propositions 2.7 and 2.8 establish that there is no minimal $\varphi_1$, but that the appropriate infimum is $\pi/2$, while if $\psi = \pi/2$ then Remark 2.9 shows that minimal $\varphi_1 = \pi/2$, $\varphi < \psi$: then the smallest $\varphi_1$ is given by formula (2.6) of Theorem 2.10.

If vector $v$ is fixed, then based on the established intuition from Euclidean space, the set of all solutions to (1.4) forms a one-sided (directed) circular cone $\mathbf{K}(v, \varphi)$ (nappe) $u \in K(v, \varphi) \subseteq H$ with aperture $2\varphi$, apex null vector and oriented axis (nappe side) given by the vector $v$. Then (1.6) means that $P(u)$ belongs to the nappe $K_V(P(v), \varphi_1)$ in a subspace $V < H$. Can we then say that the projection of a cone onto a subspace is a cone in that subspace? We would still need to prove that $K_V(P(v), \varphi_1) \subseteq P(K(v, \varphi))$.

Section 3 explores the presented geometric viewpoint. Theorem 3.6 shows that the projection of a one-sided cone onto a subspace $V$ can be: a point, a one-sided cone (including the whole subspace $V$ and its half-space) or in a special case a half-subspace (nappe in $V$ with aperture $\pi$) that lacks the whole boundary but the apex. All the

and $(v_i)_{i=1,\ldots,n}$ in the real coordinate space $\mathbb{R}^n$. For example, Pólya-Szegő inequality (see [4]) estimates $\cos \varphi$ in (1.4) based on lower and upper bounds of coordinates $m_u \leq u_i \leq M_u$ and $m_v \leq y_i \leq M_v$, Cassels’ inequality ([5, page 330]) and its refinement by Andrca and Badea ([1]) provide a bound on $\cos \varphi$ in (1.4) based on the bounds of the ratio $m \leq u_i/v_i \leq M$, etc.

3 Equation (1.4) will be taken as the foundation for directed cone in Definition 3.1.
possible cases exist already in $\mathbb{R}^3$ and so the geometric intuition about the cones in Euclidean space may serve to better understand (1.4) in any real inner product space. Remark 3.7 notes that projections of a cone onto a hyperplane abide to the same rules of Theorem 3.6. Example 3.8 uses Theorem 3.6 to find a cone in $H$ that projects to a given cone in a subspace $V$. The article is closed with an interesting fact that can be interpreted this way: given any aperture less than $\pi$, when the dimension of space is large enough, a one-sided cone with such aperture can be fitted in an orthant (hyperoctant).

2. Notation and results. Assumption 2.1. (included throughout):

1. $H$ is an inner product space over $\mathbb{R}$, dot between the vectors is denoting their inner product, $\|x\| = \sqrt{x \cdot x}$ is denoting vector norm and $O$ is denoting zero vector,
2. $V < H$ is a vector subspace,
3. $P : H \rightarrow V$ is a projection onto $V$, i.e. linear map with range $V$ and $P^2 = P$,
4. $x = x_1 + x_2$ denotes a unique decomposition of $x \in H$ as the sum of two orthogonal vectors with $x_1 = P(x) \in V$ and $x_2 = x - P(x) \in V^\perp$.

Definition 2.2. $\psi_V(x)$ is the angle between a vector $x$ and a subspace $V^\perp$ defined by:

$$\psi_V(x) = \begin{cases} \arctan \frac{\|P(x)\|}{\|x - P(x)\|}, & \text{if } P(x) \neq x, \text{ i.e. } x_2 \neq O, \\ \frac{\pi}{2}, & \text{if } P(x) = x, \text{ i.e. } x_2 = O. \end{cases}$$

Remark 2.3. (Trivial cases) If $H = \{O\}$ then $V = H$, both sides of inequalities (1.4) and (1.6) reduce to zero no matter what scalar is in place of $\cos \varphi$ and $\cos \varphi_1$ respectfully. Similarly whenever $V = \{O\}$, (1.6) reduces to zero no matter what scalar is in place of $\cos \varphi_1$. Same happens when $v = O$. On the other hand, whenever $V = H$ (1.4) implies (1.6) for any $\varphi_1 \leq \varphi$. In case $\varphi = 0$ we have already seen in (1.3) that the smallest $\varphi_1 = 0$. These cases are hereafter excluded.

Assumption 2.4. (Nontrivial case) $\dim H \geq 2$ and $1 \leq \dim V < \dim H$.

Proposition 2.5. Suppose Assumptions 2.1 and 2.4. Let $v \in H$, $v \neq O$ and $\varphi \in (\psi_V(v), \pi]$. Then there exists $u \in H$ such that

$$u \cdot v \geq \cos \varphi \|u\| \|v\| \quad \text{and} \quad P(u) \cdot P(v) = (-1) \|P(u)\| \|P(v)\|.$$
Suppose \( v_1 \neq O \) and \( v_2 \neq O \), then \( \pi/2 > \psi_V(v) > 0 \). Let \( u(t) = tv_1 + v_2 \) and

\[
(2.3) \quad f(t) = u(t) \cdot v - \cos \varphi \|u(t)\| \|v\| = \|v_1\|^2 t + \|v_2\|^2 - \cos \varphi \sqrt{t^2 \|v_1\|^2 + \|v_2\|^2} \sqrt{\|v_1\|^2 + \|v_2\|^2}.
\]

Evidently \( f \) is continuous. It will be useful to substitute tangent for cosine

\[
(2.4) \quad \cos \psi_V(v) = \frac{1}{\sqrt{1 + \tan^2 \psi_V(v)}} = \frac{1}{\sqrt{1 + \frac{\|v_1\|^2}{\|v_2\|^2}}} = \frac{\|v_2\|}{\sqrt{\|v_1\|^2 + \|v_2\|^2}}.
\]

After plugging \( t = 0 \) in (2.3) use \( \sqrt{\|v_1\|^2 + \|v_2\|^2} = \|v_2\| / \cos \psi_V(v) \) to establish

\[
f(0) = \|v_2\|^2 - \frac{\cos \varphi}{\cos \psi_V(v)} \|v_2\|^2 > 0.
\]

Then by continuity of \( f \) there exists some \( t_0 < 0 \) such that \( f(t_0) > 0 \). Now \( u = u(t_0) \) satisfies conclusion (2.2): first part because \( 0 < f(t_0) = u(t_0) \cdot v - \cos \varphi \|u(t_0)\| \|v\| \) and the second as \( u_1 \cdot v_1 = tv_1 \cdot v_1 = -\|t_0v_1\| \|v_1\| = -\|v_1\| \|v_1\| \).

The final case is when \( v_1 \neq O \) and \( v_2 = O \), then \( \pi \geq \varphi > \psi_V(v) = \pi/2 \). As \( \dim V < \dim H \) there exists \( z \in H \) such that \( z \perp v_1 \) and \( \|z\| = 1 \). Let \( u(t) = tv_1 + z \) and

\[
g(t) = u(t) \cdot v - \cos \varphi \|u(t)\| \|v\| = \|v_1\|^2 t - \cos \varphi \|v_1\| \sqrt{t^2 \|v_1\|^2} + 1.
\]

As \( \cos \varphi < 0 \) therefore \( g(0) = -\cos \varphi \|v_1\| > 0 \). Therefore by continuity of \( g \) there exists some \( t_0 < 0 \) such that \( g(t_0) > 0 \). Now \( u = u(t_0) \) satisfies conclusion (2.2): the first part because \( 0 < g(t_0) = u(t_0) \cdot v - \cos \varphi \|u(t_0)\| \|v\| \) and the second as \( u_1 \cdot v_1 = t_0v_1 \cdot v_1 = -\|t_0v_1\| \|v_1\| = -\|v_1\| \|v_1\| \).

Now we lay aside the case of \( \dim V = 1 \) until Remark 2.11. In the meantime we use the following set of presumptions.

**Assumption 2.6.** (Nontrivial case with \( \dim V \geq 2 \)) \( \dim H \geq 3 \) and \( 2 \leq \dim V < \dim H \).

**Proposition 2.7.** Suppose Assumptions 2.1 and 2.6 Let \( v \in H, v \neq O, \psi_V(v) > 0 \) and \( \varepsilon \in (0,1] \). Then there exists \( u \in H \) such that

\[
u \cdot v \geq \cos \psi_V(v) \|u\| \|v\| \quad \text{and} \quad P(u) \cdot P(v) = \varepsilon \|P(u)\| \|P(v)\|.
\]

**Proof.** We use notation \( v = v_1 + v_2 \) as in Assumption 2.14. \( \|v_1\| > 0 \) as \( \psi_V(v) > 0 \). Without loss of generality we assume that \( \|v_1\| = 1 \) because if the statement is true for vector \( v/\|v_1\| \) then it is also true for \( v \).
As \( \dim V \geq 2 \) there exist \( z \in V \) such that \( z \perp v_1 \) and \( \|z\| = \sqrt{1 - \epsilon^2} \). Let 
\[
    u(t) = \varepsilon v_1 + tv_2 + z,
\]
then \( P(u(t)) = \varepsilon v_1 + z \), \( \|P(u(t))\| = \sqrt{\epsilon^2 \|v_1\|^2 + \|z\|^2} = 1 \) and 
\[
    P(u(t)) \cdot v_1 = \varepsilon = \epsilon \|P(u(t))\| \|v_1\|.
\]
So we only need to find \( t \) such that 
\[
    u(t) \cdot v \geq \cos \psi_V(v) \|u(t)\| \|v\|.
\]
Its existence follows because the following real function (differentiable and strictly increasing with \( \lim_{t \to +\infty} f(t) = \varepsilon > 0 \)) assumes positive values:
\[
    f(t) = u(t) \cdot v - \cos \psi_V(v) \|u(t)\| \|v\| = \varepsilon + t \|v_2\|^2 - \|v_2\| \sqrt{1 + t^2 \|v_2\|^2}.
\]

**Proposition 2.8.** Suppose Assumptions 2.1 and 2.6. Let \( v \in H, v \neq O \) such that \( \psi_V(v) > 0 \). Then \( \psi_V(v) < \pi/2 \) if and only if
\[
    \forall u \in H, (u \cdot v \geq \cos \psi_V(v) \|u\| \|v\| \text{ and } P(u) \neq O) \implies P(u) \cdot P(v) > 0.
\]

**Proof.** We use notation \( v = v_1 + v_2 \) and \( u = u_1 + u_2 \) as in Assumption 2.1.4. As \( \psi_V(v) > 0 \) so \( v_1 \neq O \).

Suppose \( \psi_V(v) < \pi/2 \), then \( v_2 \neq O \). Assume \( u_1 \neq O \), then \( \|u_2\| < \|u\| \) and \( \|v_2\| < \|v\| \).

(2.5) \[
    u_1 \cdot v_1 + u_2 \cdot v_2 = u \cdot v \geq \cos \psi_V(v) \|u\| \|v\| \tag{2.4}
\]

Therefore
\[
    \|u\| \|v_2\| - u_1 \cdot v_1 \leq u_2 \cdot v_2 \leq \|u_2\| \|v_2\| < \|u\| \|v_2\|
\]
and \( u_1 \cdot v_1 > 0 \) follows by subtraction of \( \|u\| \|v_2\| \) from both sides of previous inequality.

The above reasoning works also in case \( \dim V = 1 \).

Converse implication is proved by contraposition: \( \psi_V(v) \geq \pi/2 \) implies that there exists an \( O \neq u \in H \) such that \( u \cdot v \geq \cos \psi_V(v) \|u\| \|v\| \) and \( u_1 \cdot v_1 \leq 0 \). By (2.4) \( \psi_V(v) \geq \pi/2 \) is equivalent to \( \psi_V(v) = \pi/2, v_2 = O \) and \( \cos \psi_V(v) = 0 \). As \( \dim V \geq 2 \) there exists \( z \in V \) such that \( z \perp v_1 \) and \( \|z\| = 1 \). For \( u = z \) it is easily checked that 
\[
    u \cdot v = 0 \geq \cos \psi_V(v) \|u\| \|v\| \text{ and } u_1 \cdot v_1 = z \cdot v_1 = 0.
\]
This procedure is valid only when \( \dim V \geq 2 \).

**Remark 2.9.** (Case \( \psi_V(y) = \pi/2 \)) Suppose Assumptions 2.1 and 2.3. If \( \psi_V(v) = \pi/2 \) and \( \dim V \geq 2 \) then the second part of the proof of Proposition 2.8 shows that \( u \) can be found such that \( u \cdot v \geq 0 \), \( P(u) \neq O \) and \( P(u) \cdot P(v) = 0 \).

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\footnote{Formula references above and under (in)equality sign establish a link that can help to understand the relationship. This notation is used throughout the article.}
On the other hand, no matter what \( \dim V \), if \( \psi_V(v) = \pi/2 \), then \( v \in V \). Then, \( u \cdot v \geq 0 \) implies \( P(u) \cdot P(v) = u \cdot v \geq 0 \) and there can be no \( u \in H \) such that \( P(u) \cdot P(v) < 0 \).

**Theorem 2.10.** Suppose Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5. Let \( v \in H, v \neq O \), \( \varphi \in \langle 0, \psi(V) \rangle \) and

\[
\varphi_1 = \arccos \sqrt{\frac{\cos^2 \varphi - \cos^2 \psi_V(v)}{1 - \cos^2 \psi_V(v)}}.
\]

Then for arbitrary \( u \in H \):

\[
u \cdot v \geq \cos \varphi \|u\|\|v\| \implies P(u) \cdot P(v) \geq \cos \varphi_1 \|P(u)\|\|P(v)\|.
\]

Moreover, \( \varphi_1 \) in (2.6) is the smallest possible, as there exists \( u \in H \) such that \( u \cdot v = \cos \varphi \|u\|\|v\| \) and \( P(u) \cdot P(v) = \cos \varphi_1 \|P(u)\|\|P(v)\| \).

**Proof.** We use notation \( v = v_1 + v_2 \) and \( u = u_1 + u_2 \) as in Assumption 2.4. Suppose \( v \) is fixed, \( 0 < \psi_V(v) \leq \pi/2 \) so \( v_1 \neq O \) and \( 0 \leq \varphi < \psi_V(v) \) so

\[
0 \leq \frac{\|v_2\|}{\|v_1\|} \quad \text{and} \quad \frac{\|v_2\|}{\|v_1\|} \geq \cos \psi_V(v) < \cos \varphi \leq 1.
\]

(2.7)

\[
\frac{\|v_2\|}{\|v_1\|} = \frac{\cos \psi_V(v)}{\sqrt{1 - \cos^2 \psi_V(v)}}
\]

When \( u_1 = O \) then (2.6) is trivial. The main part of the proof investigates

\[
\min \frac{u_1 \cdot v_1}{\|u_1\|\|v_1\|} = \min \frac{\cos \theta \|u\|\|v\| - u_2 \cdot v_2}{\|u_1\|\|v_1\|} = \min \cos \theta \sqrt{1 + \frac{\|u_2\|^2}{\|v_1\|^2} + 1 + \frac{\|v_2\|^2}{\|v_1\|^2}} - \frac{u_2}{\|u_1\|} \cdot \frac{v_2}{\|v_1\|}
\]

under the conditions that \( u_1 \neq O \) and \( u_1 \cdot v_1 + u_2 \cdot v_2 = u \cdot v = \cos \theta \|u\|\|v\| \geq \cos \varphi \|u\|\|v\| \) for some \( \theta \in [0, \varphi] \) that depends on \( u \) and \( v \). Under these conditions

\[
\sin \varphi \geq \sin \varphi \sqrt{1 + a^2 \sqrt{1 + b^2 - ab}}
\]

As \( \|u_2\|/\|u_1\| \geq 0 \) and \( v \) has been fixed from the start, together with condition (2.7), it is sufficient to examine real function \( a \mapsto f(a, b) \) for all \( a \geq 0 \) and a fixed \( b \) taking into account that \( \cos \varphi > b/\sqrt{1+b^2} \). Continuity, first and second derivative of \( g \)
show that $g$ is convex with the only argument of the minimum $a = b/\sqrt{\cos^2 \varphi (1+b^2)-b^2}$ and the minimum $\sqrt{\cos^2 \varphi (1+b^2)-b^2}$. Therefore

\[
(\bullet) \geq \sqrt{\cos^2 \varphi \left( 1 + \frac{\|v_2\|^2}{\|v_1\|^2} \right) - \frac{\|v_2\|^2}{\|v_1\|^2}} \sqrt{\cos^2 \varphi - \cos^2 \psi_V(v)} \frac{1}{1 - \cos^2 \psi_V(v)} > 0,
\]

so $u_1 \cdot y_1 \geq \sqrt{\cos^2 \frac{\varphi}{\cos^2 \varphi - \cos^2 \psi_V(v)} \|u_1\| \|v_1\|}$ whenever $\|u\| \neq 0$. The first part of the theorem has been proved without the premise $\dim V \geq 2$.

If $\dim V \geq 2$, $z \in V$ can be chosen such that $\|z\| = 1$ and $z \perp v_1$. It is straightforward to check that for $u = \cos \varphi_1 v_1 + \|v_1\| \sin \varphi_1 z + \frac{1}{\cos \varphi_1} v_2$:

\[
(2.10) \quad \|u_1\| = \sqrt{\cos^2 \varphi_1 \|v_1\|^2 + \|v_1\|^2 \sin^2 \varphi_1 \|z\|^2} = \|v_1\|
\]

\[
(2.11) \quad \|u\| = \sqrt{\|u_1\|^2 + \frac{\|v_2\|^2}{\cos^2 \varphi_1}} \|v_1\| \sqrt{1 + \frac{\|v_2\|^2}{\|v_1\|^2 \cos^2 \varphi_1}}
\]

\[
(2.12) \quad \|v\| = \|v_1\| \sqrt{1 + \frac{\|v_2\|^2}{\|v_1\|^2}} \left( \frac{\|v_2\|^2}{\|v_1\|^2} \right) \frac{\|v_1\|}{\sqrt{1 - \cos^2 \psi_V(v)}}
\]

\[
\frac{\|v_2\|^2}{\|v_1\|^2} \cos^2 \varphi = \frac{\|v_1\|^2}{\sqrt{1 - \cos^2 \psi_V(v)}} \cos \varphi \|u\| \|v\|
\]

Thus, when $\dim V \geq 2$, formula (2.6) gives the smallest possible $\varphi_1$. \[\blacksquare\]

**Remark 2.11.** (dim $V = 1$) Suppose Assumptions 2.1 and 2.4. Suppose $\dim V = 1$ and $\psi_V(v) > 0$. The argument used here is that $P(u)$ and $P(v)$ are scalars so that $P(u) P(v)$ can take only one out of two values: $|P(u)||P(v)|$ or $-|P(u)||P(v)|$. In case $\varphi > \psi_V(v)$ Proposition 2.9 gives the smallest $\varphi_1 = \pi$. In case $\varphi < \psi_V(v)$, the first part of Theorem 2.10 holds and so $\cos \varphi_1 > 0$, which by the featured argument implies that the smallest $\varphi_1 = 0$.

In case $\varphi = \psi_V(v) \leq \pi/2$: $P(u) P(v) > 0$ when $P(u) \neq 0$ (highlighted in the proof of Proposition 2.3) and in case $\varphi = \psi_V(v) = \pi/2$ by the second point of Remark 2.9 (4) implies $P(u) P(v) \geq 0$. Then, by the same featured argument the smallest $\varphi_1 = 0$. \[\blacksquare\]
3. **Application to projection of the cone on a subspace or a hyperplane.** The following definitions and result for cones in higher dimensions correspond well with natural observations about the three-dimensional Euclidean space.
**Definition 3.1.** A directed (one-sided) cone or nappe in the inner product space \( H \) over the field \( \mathbb{R} \) with apex \( a \in H \), axis direction given by \( v \in H \), \( v \neq O \) and half-aperture \( \varphi \in [0, \pi] \) is defined as

\[
K_H(a, v, \varphi) = \{ u \in H : (u - a) \cdot v \geq \cos \varphi \| u - a \| \| v \| \}.
\]

A directed cone with apex included but with the rest of the boundary excluded is:

\[
K'_H(a, v, \varphi) = \{ a \} \cup \{ u \in \Omega : (u - a) \cdot v > \cos \varphi \| u - a \| \| v \| \}.
\]

When the apex is \( O \), then notation is abbreviated: \( K_H(v, \varphi) = K_H(O, v, \varphi) \) and \( K'_H(v, \varphi) = K'_H(O, v, \varphi) \).

**Remark 3.2.** Dilation (Minkowski addition) is denoted as \( X \oplus Y = \{ x + y : x \in X \text{ and } y \in Y \} \). It is evident that \( K_H(a, v, \varphi) = K_H(v, \varphi) \oplus \{ a \} \) as:

\[
u \in K_H(a, v, \varphi) \iff u - a \in K_H(v, \varphi) \iff u \in K_H(v, \varphi) \oplus \{ a \}.
\]

**Remark 3.3.** By the CBS inequality for any \( u \) and \( u' = -u \): \( u' \cdot v \leq \| u' \| \| v \| \). Multiplying with \( -1 \) yields \( u \cdot v \geq (1) \| u \| \| v \| \). Therefore, \( K_H(v, \pi) = H \).

**Definition 3.4.** Given projection \( P \) from Assumption 2.1.3 the projection of the set \( \Omega \subseteq H \) onto the subspace \( V \) is \( P[\Omega] = \{ P(u) \in V : u \in \Omega \} \).

**Lemma 3.5.** Suppose Assumptions 2.1. Let \( u \in K_H(v, \varphi) \), \( V < H \), \( P(v) \neq O \), \( P(u) \neq O \) and \( P(u) \cdot P(v) = \cos \theta \| P(u) \| \| P(v) \| \). Then \( K_V(P(v), \theta) \subseteq P[K_H(v, \varphi)] \).

**Proof.** \( P(v) = O \) is excluded from lemma as case \( K(O, \theta) \) is not included in Definition 3.1. We use notation \( v = v_1 + v_2 \) and \( u = u_1 + u_2 \) as in Assumption 2.1.4. Note that \( v_1 \neq O \), \( u_1 \neq O \) and that \( u \in K_H(v, \varphi) \) corresponds to \( u \cdot v \geq \cos \varphi \| u \| \| v \| \).

We prove that for each \( w \in V \) such that \( w \cdot v_1 \geq \cos \theta \| w \| \| v_1 \| \) there exists \( z \in V^\perp \) such that \( u' = w + z \) satisfies \( u' \cdot v \geq \cos \varphi \| u' \| \| v \| \).

Take \( z = u^2 (\| w \| / \| u_1 \|) \), then \( \| u' \| = \| u \| \| u \| / \| u_1 \| \| u \| \) and

\[
u' \cdot v = w \cdot v_1 + \| w \| / \| u_1 \| u_2 \cdot v_1 \geq \cos \theta \| w \| \| v_1 \| + \| w \| / \| u_1 \| u_2 \cdot v_1 = \cos \theta \| u_1 \| \| v_1 \| + u_2 \cdot v_2 = \| w \| / \| u_1 \| (u_1 \cdot v_1 + u_2 \cdot v_2) = \| w \| / \| u_1 \| u \cdot v \geq \| w \| / \| u_1 \| \cos \varphi \| u \| \| v \| = \cos \varphi \| u' \| \| v \|. \]
Theorem 3.6. Suppose Assumptions (2.1) let \( v \in H, v \neq O \) and \( \varphi_1 \) as in (2.0). Based on assumptions on \( \varphi \) and \( \psi_V(v) \) in the first two columns, the projection of a nappe onto the subspace \( V \) is given in the last column of the following table:

| \( \varphi \in [0, \pi] \) | \( \psi_V(v) \in [0, \pi/2] \) | \( P[K_H(a,v,\varphi)] \) |
|--------------------------|---------------------------|-----------------------------|
| \( \varphi = 0 \)       | \( \psi_V(v) = 0 \)       | \( \{P(a)\} \)             |
| \( \varphi = \psi_V(v) \) | \( \psi_V(v) \in (0, \pi/2) \) | \( K_V^\varphi(P(a), P(v), \pi/2) \) |
|                          | \( \psi_V(v) = \pi/2 \)   | \( K_V(P(a), P(v), \pi/2) \) |
| \( \varphi < \psi_V(v) \) | \( \psi_V(v) > 0 \)       | \( K_V(P(a), P(v), \varphi_1) \) |
| \( \varphi > \psi_V(v) \) |                           | \( V \)                     |

Proof. Without the loss of generality it is enough to prove the theorem only for apex \( a = O \). The general case follows from Remark 3.2 as

\[
P[K_H(v, y, \varphi)] = P[K_H(y, \varphi) \oplus \{v\}] = P[K_H(y, \varphi)] \oplus \{P(v)\}.
\]

The case \( \text{dim} V = 0 \) is trivial as projection collapses everything to \( O \). When \( \text{dim} V = 1 \) there are just 4 different “cones” in \( V \) for \( \text{dim} V = 1 \): \( \{0\}, V, K_V(-1,0) \) and \( K_V(1,0) \) where “1” and “-1” correspond to the only two unit vectors in \( V \). Note also that in this case \( K_V^\varphi(P(v), \pi/2) = K_V(P(v), \pi/2) = K_V(P(v), 0) \). Remark 2.11 closes this case. The subsequent cases deal with \( \text{dim} V \geq 2 \).

Case 1. When \( \varphi = 0 \) note that \( \cos \varphi = 1 \) and (1.3) is in fact opposite CBS inequality (1.2). Thus \( K_H(v, 0) \) is directed line and so by (1.3) \( P[K_H(v, 0)] = K_V(P(v), 0) \) is directed line in direction \( P(v) \) in case \( P(v) \neq O \). Note also that formula (2.0) produces \( \varphi_1 = 0 \) when \( \varphi = 0 \). In the special case when \( P(v) = O \) (\( \psi_V(v) = 0 \)) then \( P(tv) = O \) for any \( t \in \mathbb{R} \) and therefore \( P[K_H(v, 0)] = \{O\} \).

Case 2. Proposition 2.5 shows that when \( \psi_V(v) < \varphi \leq \pi \), then there is \( u \in K_H(v, \varphi) \) such that \( P(u) \cdot P(v) = \cos \varphi \|P(u)\| \|P(v)\| \). Therefore by Lemma 3.3 \( K_V(P(v), \pi) \subseteq P[K_H(v, \varphi)] \). Remark 3.3 yields \( V \subseteq P[K_H(v, \varphi)] \).

Case 3. When \( \varphi < \psi_V(v) \) note that \( P(v) \neq O \). The first part of Theorem 2.10 states that: \( u \in K_H(v, \varphi) \) implies \( P(u) \in K_V(P(v), \varphi_1) \), i.e. \( P[K_H(v, \varphi)] \subseteq K_V(P(v), \varphi_1) \). The second part of Theorem 2.10 establishes existence of \( u \in K_H(v, \varphi) \) such that \( P(u) \cdot P(v) = \cos \varphi_1 \|P(u)\| \|P(v)\| \). By Lemma 3.3 \( K_V(P(v), \varphi_1) \subseteq P[K_H(v, \varphi)] \).
Case 4. When \( \varphi = \psi_V(v) \in \langle 0, \pi/2 \rangle \), then by Proposition 2.7 and Lemma 3.5, \( K_V(P(v), \pi/2 - \varepsilon) \subseteq P[K_H(v, \varphi)] \). Furthermore, Proposition 2.8 gives that \( K_V(-P(v), \pi/2) \cap P[K_H(v, \varphi)] = \{O\} \), so \( P[K_H(v, \varphi)] = K_V'(P(v), \pi/2) \).

When \( \varphi = \psi_V(v) = \pi/2 \), then Remark 2.9 shows that \( u \in H \) can be found such that \( u \cdot v \geq \cos \psi_V(v) \|u\| \|v\| \) and \( P(u) \cdot P(v) = 0 \), but never \( P(u) \cdot P(v) < 0 \). By Lemma 3.6, \( P[K_H(v, \pi/2)] = K_V(P(v), \pi/2) \). Note also that formula (2.6) produces \( \varphi_1 = \pi/2 \) when \( \varphi = \cos \psi_V(v) \neq 1 \).

**Remark 3.7.** Suppose Assumptions 2.1 and let \( \Pi \subset H \) be a plane parallel to \( V < H \). Let \( d \) be the distance vector from \( V \) to \( \Pi \) such that \( d \perp V \) and \( \Pi = V \oplus \{d\} \). Projecting any \( u \) onto \( \Pi \) yields \( P_{\Pi}(u) = P(u) + d \) and projecting cone \( K_H(a, v, \varphi) \) on \( \Pi \) yields \( P[K_H(a, v, \varphi)] \oplus \{d\} \subseteq \Pi \) where \( P[K_H(a, v, \varphi)] \) is given in Theorem 3.6.

**Example 3.8.** Suppose Assumptions 2.1 and \( v \in H \) such that \( P(v) \neq O \). Which is the widest half aperture \( \varphi \) of a directed cone with apex \( a \in H \), axis and direction given by \( v \) such that the half aperture of a projected cone is at most \( \varphi_1 < \pi/2 \)?

Solving formula (2.6) for \( \varphi \) and using (2.1) for \( \psi_V(v) \) yields

\[
\varphi = \arccos \sqrt{\cos^2 \psi_V(v) + \cos^2 \varphi_1 - \cos^2 \psi_V(v) \cos^2 \varphi_1}.
\]

Theorem 3.6 establishes \( P[K_H(a, v, \varphi)] = K_V(P(a), P(v), \varphi_1) \) and any larger \( \varphi \) would yield aperture of projected cone larger then \( \varphi_1 \).

**Fact 3.9.** The widest half aperture of a one-sided cone that can fit inside an orthant of \( \mathbb{R}^n \) is \( \varphi = \arccos \sqrt{\frac{n-2}{n}} \).

**Proof.** All projections onto coordinate 2D planes of such a directed cone need to fit into a quadrant: directed cone with half aperture \( \varphi_1 = \pi/4 \) around directed axis \( P(y) = (1, 1) \). Therefore, the widest aperture of directed cone in question need to be around axis \( y = (1, 1, \ldots, 1) \). By formula (2.1) \( \psi = \arctan \sqrt{\frac{2}{n}} \) and \( \cos^2 \psi = \frac{n-2}{n} \).

Formula (2.6) yields \( \varphi = \arccos \sqrt{\frac{n-2}{n}} \) and Theorem 3.6 establishes the fact.

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