A PERFORMANCE EVALUATION OF TRAPEZOIDAL VARIANTS FOR NUMERICAL CUBATURE

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Abstract

In this work, double integration cubature schemes of Trapezoid type have been focused. Recently, some derivative-based Trapezoid-type schemes have been proposed in literature incorporating derivatives at means of the limits of integration. We carry out the exhaustive performance evaluation of the existing closed Newton-Cotes Trapezoidal (CNCT) double integral scheme with its derivative-based variants in recent literature. The derivative-free and derivative-based rules are discussed in basic forms with local error terms and composite forms with global error terms. The performance of the rules on some double integrals in the form of observed order of accuracy, computational costs and error drops demonstrates the encouraging performance of the derivative-based trapezoidal variants over the derivative-free scheme performing numerical experiments.

Keywords: Cubature, Double integrals, Derivative-based schemes, Order of accuracy, computational cost, errors, Trapezoid

I. Introduction

When we have a single integral for numerical computation, we use the word quadrature, but when we have double or higher integrals, we use the word cubature. Burden and Faires in [IV] discussed numerical computation of areas and volumes for irregular regions. In literature mostly quadrature rules for numerical computation of single integrals are available, but it is also important to evaluate volumes using double integrals is [XII].

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For the numerical integration of closed Newton-Cotes quadrature rules, a novel family of quadrature rules was presented in [XIX], which increases two orders of precision over the classical closed Newton-Cotes formulas. In [III] a new numerical integration algorithm was proposed after modification, in a combined hybrid way to reduce errors.

A new efficient midpoint derivative-based quadrature scheme of trapezoid-type for the Riemann-Stieltjes integrals was developed by Memon et al. in 2020 [XI], which efficiently modified [XX]. A comparison of the polynomial collocation method for the solution of integral equations with uniformly-spaced quadrature rules was performed by Shaikh [XVI] in 2019.

Using arithmetic, geometric and harmonic means derivative-based closed Newton-Cotes rules Ramachandran et al. in 2016 [XIV] compared the results with the existing closed Newton-Cotes quadrature (CNC) rules. A modified four-point quadrature rule for numerical integration by using 2nd order derivative instead of 4th order derivative was proposed by Shaikh et al. in 2016 [XVII], which modified efficiently the work of Zhao et al. [XIX].

For the approximation of integrals, unlimited past methods with their modifications are available in [V], [II] and [I] by Burg in 2012, Bailey and Borwein in 2011 and Babolian et al. in 2005, respectively. Some new families of open Newton-Cotes rules which include higher accuracy than that of the classical formulas and also involve derivatives were presented by Zafar et al in 2013 [XVIII]. Achievements which are related with this work are due to Dehghan and colleagues [VI], [VII], [VIII] in 2005-06, Jain in 2007 [IX], Pal in 2007 [XII], Sastry in 1997 [XV] and Petrovskaya in 2011 [XI].

In literature quadrature rules and their modifications for numerical approximations are available. The only existing closed Newton-Cotes cubature schemes for double integrals, as discussed in [XII] which are in basic form.

Here, the comparison of five newly developed derivative-based trapezoidal variants schemes has been discussed which involves partial derivatives with functional evaluations. Numerical experiments decide the performance of five proposed schemes in reducing errors, order of accuracy and computational cost.

II. General Formulation

The Double integrals in general form in two dimensions over rectangles is defined as

\[ V = \int_c^d \int_a^b f(x, y) dx \, dy \]  

(1)

Here volume of the surface defined by the integrand over the area element is \( V \), considering finite limits to evaluate double integral over the area element.

This work is presented for the comparison of the performance of five newly proposed derivative-based cubature schemes in basic and composite forms as compared to the existing CNCT scheme.

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III. Five Proposed Derivative-based Trapezoidal Numerical Cubature Schemes

The five proposed derivative-based trapezoidal numerical cubature schemes in the basic form are:

\[ PT(a; b, c; d) = \int_c^d \int_a^b f(x, y) \, dx \, dy = CNCT + \sum_{i=1}^{3} C_i \phi_i \]  

(2)

The coefficients depending on the limits are \( C_i = g_i(b - a, d - c) \) and the derivative terms are:

\[ \phi_1 = \sum_{x,y} f_{xx}(\mu_x, y), \phi_2 = \sum_{x,y} f_{yy}(x, \mu_y) \text{ and } \phi_3 = f_{xxy}(\mu_x, \mu_y). \]  

(3)

The five new derivative-based schemes are proposed using arithmetic, geometric, harmonic, heronian and centroidal means, respectively, AM, GM, HaM, HeM and CM namely AMT, GMT, HaMT, HeMT, CMT which are defined in (2). PT stands for the notation of proposed derivative-based cubature schemes of Trapezoidal-type. Table 1 shows the coefficients \( C_i \)'s in the basic forms, and the averages concerning \( x \) and \( y \).

Table 1: Coefficients and means used in the PT cubature schemes

| PT schemes | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( \mu_x \) | \( \mu_y \) |
|------------|-----------|-----------|-----------|---------|---------|
| AMT        | \(- \frac{(b-a)(d-c)^3}{24}\) | \(- \frac{(b-a)^3(d-c)}{24}\) | \(- \frac{(b-a)^3}{144}\) | \(a+b\) \(\frac{a+b}{2}\) | \(c+d\) \(\frac{c+d}{2}\) |
| GMT        | \(- \frac{(b-a)(d-c)^3}{24}\) | \(- \frac{(b-a)^3(d-c)}{24}\) | \(- \frac{(b-a)^3}{144}\) | \(\sqrt{ab}\) \(\sqrt{ab}\) | \(\sqrt{cd}\) \(\sqrt{cd}\) |
| HaMT       | \(- \frac{(b-a)(d-c)^3}{24}\) | \(- \frac{(b-a)^3(d-c)}{24}\) | \(- \frac{(b-a)^3}{144}\) | \(\frac{2ab}{a+b}\) \(\frac{2cd}{c+d}\) | \(\frac{2cd}{c+d}\) \(\frac{2cd}{c+d}\) |
| HeMT       | \(- \frac{(b-a)(d-c)^3}{24}\) | \(- \frac{(b-a)^3(d-c)}{24}\) | \(- \frac{(b-a)^3}{144}\) | \(\frac{a+\sqrt{ab}+b}{3}\) \(\frac{a+\sqrt{ab}+b}{3}\) | \(\frac{c+\sqrt{cd}+d}{3}\) \(\frac{c+\sqrt{cd}+d}{3}\) |
| CMT        | \(- \frac{(b-a)(d-c)^3}{24}\) | \(- \frac{(b-a)^3(d-c)}{24}\) | \(- \frac{(b-a)^3}{144}\) | \(\frac{2(a^2+ab+b^2)}{3(a+b)}\) \(\frac{2(c^2+cd+d^2)}{3(c+d)}\) | \(\frac{2(c^2+cd+d^2)}{3(c+d)}\) \(\frac{2(c^2+cd+d^2)}{3(c+d)}\) |

The local error terms in the proposed AMT, GMT, HaMT, HeMT and CMT schemes along with the precision are summarized in Table 2.
Table 2: Local error terms and degrees of precision of PT cubature rules

| PT schemes | Local Error terms                                                                 | Precision |
|------------|-----------------------------------------------------------------------------------|-----------|
| AMT        | \[-\frac{(b-a)^5}{480}(d-c) f_{xxxx}(\xi,\eta) - \frac{(b-a)(d-c)^5}{480} f_{yyyy}(\xi,\eta)\] | 3         |
| GMT        | \[-\frac{(b-a)^3}{24}(\sqrt{b} - \sqrt{a})^2 f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{24}(\sqrt{d} - \sqrt{c})^2 f_{yyy}(\xi,\eta)\] | 2         |
| HaMT       | \[-\frac{(b-a)^5}{24(a+b)} f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^5}{24(c+d)} f_{yyy}(\xi,\eta)\] | 2         |
| HeMT       | \[-\frac{(b-a)^3}{72}(\sqrt{b} - \sqrt{a})^2 f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^3}{72}(\sqrt{d} - \sqrt{c})^2 f_{yyy}(\xi,\eta)\] | 2         |
| CMT        | \[-\frac{(b-a)^5}{72(a+b)} f_{xxx}(\xi,\eta) - \frac{(b-a)(d-c)^5}{72(c+d)} f_{yyy}(\xi,\eta)\] | 2         |

**Definition 1.** The exact result and approximate result obtained by the proposed scheme, the highest positive integer where these results remains same, is defined to be the precision of the scheme.

**Definition 2.** If the degree of precision of a cubature scheme is M, then the leading local error term is the difference of exact and approximate evaluations of the integral for the (M+1)th order term in the Taylor’s series development of \(f(x, y)\) defined in the neighborhood of \((x_0, y_0)\).

The composite form of the proposed PT schemes in general, denoted as PT-Cn, may be defined as (21).

\[
PT - Cn = \int_a^b \int_c^d f(x, y) \, dx \, dy = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} PT(x_i; x_{i+1}, y_j; y_{j+1})
\]

In \(n^2\) elements, the averages are the means of each sub-square-element with \(b-a = nh\) and \(d-c = nk\) partitioning the original square element.

**IV. Numerical Experiments, Results and Discussion**

The four following tests of problems from the literature [XII] are selected, to check the performance of proposed AMT, GMT, HaMT, HeMT and CMT cubature schemes as compared to the existing scheme.

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Example 1.

\[ \int \int_{0}^{1} xe^{xy} \, dx \, dy \]

Example 2.

\[ \int \int_{1}^{3} \frac{1}{1+x+y} \, dx \, dy \]

Example 3.

\[ \int_{0}^{1} \frac{x}{(xy+1)^2} \, dy \, dx \]

Example 4.

\[ \int_{0}^{1} \frac{xy}{\sqrt{x^2 + y^2 + 1}} \, dy \, dx \]

By using MATLAB software, the correct decimal places’ approximations for Examples 1-4 which have been used to compute absolute errors are, respectively, 1.485339738238448, 0.454026674722594, 0.306852819440057 and 0.753421555101056.

The numerical difference between exact and approximate results are the absolute errors, which are used herein, by assigning elements \(n = 1, 2, 3, \ldots, 40\) so that the performance of the five proposed schemes can be observed.

In figures 1-4 the absolute errors for examples 1-4 are shown for elements \(n = 1, 2, 3, \ldots, 40\) of existing CNCT scheme and proposed AMT, GMT, HaMT, HeMT and CMT schemes. In four examples, all five proposed schemes reduced errors as compared to the existing scheme. From the proposed schemes, sometimes GMT, HaMT, HeMT and CMT leads in reducing errors, but mostly AMT is in top lead from the proposed five schemes, and it is applicable without any disadvantage in all double integrals, whereas GMT, HaMT, HeMT and CMT may not perform better in all problems. Even though, all proposed schemes are far better in reducing errors as compared to the existing scheme.

Table 3 shows the observed order of accuracy of CNCT and AMT, GMT, HaMT, HeMT, CMT schemes for example 1, here CNCT satisfies the theoretical order of accuracy which is 2, in 128 strips. AMT, GMT, HaMT obtained theoretical order of accuracies which are 4, 3, 3 in only 4 strips. HaMT, CMT obtained theoretical order of accuracies which are 3, 3 in only 2 strips.

Table 4 shows the observed order of accuracy of CNCT and AMT, GMT, HaMT, HeMT, CMT schemes for example 2, here CNCT satisfies the theoretical order of accuracy which is 2, in 32 strips. AMT, HaMT, CMT obtained theoretical order of accuracies which are 4, 3, 3 in only 2 strips. GMT obtained theoretical order of accuracies which are 3, 3 in only 2 strips. AMT, HaMT, CMT obtained theoretical order of accuracies which are 4, 3, 3 in only 2 strips. GMT obtained theoretical order of accuracies which are 3, 3 in only 2 strips.
accuracy which is 3, in 16 strips. HeMT obtained theoretical order of accuracy which is 3, in 4 strips.

Table 5 shows observed order of accuracy of CNCT and AMT, GMT, HaMT, HeMT, CMT schemes for example 3, here CNCT satisfies theoretical order of accuracy which is 2, in 32 strips. AMT, HeMT, CMT obtained theoretical order of accuracies which are 4,3,3 in only 2 strips. GMT obtained theoretical order of accuracy which is 3, in 16 strips. HaMT obtained theoretical order of accuracy which is 3, in 4 strips.

Table 6 shows the observed order of accuracy of CNCT and AMT, GMT, HaMT, HeMT, CMT schemes for example 4, here CNCT satisfies the theoretical order of accuracy which is 2, in 128 strips. AMT, HaMT, HeMT obtained theoretical order of accuracies which are 4,3,3 in only 2 strips. GMT obtained theoretical order of accuracy which is 3, in 16 strips. CMT obtained theoretical order of accuracy which is 3, in 4 strips.

Figures 5-6 describe the consumed computational cost of existing and proposed schemes, the figures clearly show the low computational cost of proposed schemes as compared to the existing scheme which consumes expensive computational cost in terms of several functional and partial derivatives evaluations.

V. Conclusion

Five numerical cubature schemes AMT, GMT, HaMT, HeMT and CMT have been proposed in the basic and composite form in this research work, which are efficient modifications of existing CNCT scheme from literature for the numerical computation of double integrals. The performance of proposed schemes to satisfy the theoretical order of accuracies is efficient than the existing scheme, moreover, low computational cost is consumed as well as absolute errors are reduced by the proposed schemes.

![Graph](image)

**Fig 1.** Absolute error distributions versus a number of elements of Trapezoidal variants for example 1.
Fig 2. Absolute error distributions versus a number of elements of Trapezoidal variants for example 2.

Fig 3. Absolute error distributions versus a number of elements of Trapezoidal variants for example 3.

Fig 4. Absolute error distributions versus a number of elements of Trapezoidal variants for example 4.

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Table-3 Observed order of accuracy of Trapezoidal variants for example 1

| m (Strips) | n (Intervals) | Observed order of accuracy | CNCT | AMT | GMT | HaMT | HeMT | CMT |
|------------|--------------|----------------------------|------|-----|-----|------|------|-----|
| 1          | 1            | NA                         | NA   | NA  | NA  | NA   | NA   | NA  |
| 2          | 2            | 2.263456                   | 4.195962 | 4.235355 | 3.902323 | 4.346329 | 3.611945 | NA  |
| 4          | 4            | 2.08709                    | 4.006048 | 3.901208 | exact    | 3.965818 | exact  | NA  |
| 8          | 8            | 2.023813                   | exact | exact | exact | exact | exact  | NA  |

Table-4 Observed order of accuracy of Trapezoidal variants for example 2

| M (Strips) | N (Intervals) | Observed order of accuracy | CNCT | AMT | GMT | HaMT | HeMT | CMT |
|------------|--------------|----------------------------|------|-----|-----|------|------|-----|
| 1          | 1            | NA                         | NA   | NA  | NA  | NA   | NA   | NA  |
| 2          | 2            | 2.007075                   | 3.807696 | 3.763696 | 3.810534 | 2.362379 | 3.642882 | NA  |
| 4          | 4            | 2.001881                   | exact | 3.921577 | exact    | 3.310766 | exact  | NA  |
| 8          | 8            | 2.00051                    | 3.977889 | exact | exact  | exact  | NA   | NA  |
| 16         | 16           | 2.000124                   | 3.994286 | NA   | NA   | NA   | NA   | NA  |
| 32         | 32           | 2.000053                   | exact | NA   | NA   | NA   | NA   | NA  |
| 64         | 64           | exact                      | NA   | NA   | NA   | NA   | NA   | NA  |
| 128        | 128          | 2.000066                   | NA   | NA   | NA   | NA   | NA   | NA  |

Table 5 Observed order of accuracy of Trapezoidal variants for example 3

| m (Strips) | n (Intervals) | Observed order of accuracy | CNCT | AMT | GMT | HaMT | HeMT | CMT |
|------------|--------------|----------------------------|------|-----|-----|------|------|-----|
| 1          | 1            | NA                         | NA   | NA  | NA  | NA   | NA   | NA  |
| 2          | 2            | 0.964222                   | 4.01648 | 5.662284 | 5.745421 | 3.618716 | 3.713892 | NA  |
| 4          | 4            | 1.484241                   | exact | 3.203444 | 2.992389 | exact | exact | NA  |
| 8          | 8            | 1.890426                   | exact | 3.703145 | exact | exact | exact | NA  |
| 16         | 16           | 1.973466                   | 3.774674 | NA   | NA   | NA   | NA   | NA  |
| 32         | 32           | 1.993437                   | exact | NA   | NA   | NA   | NA   | NA  |
| 64         | 64           | 1.998352                   | NA   | NA   | NA   | NA   | NA   | NA  |
| 128        | 128          | 1.999548                   | NA   | NA   | NA   | NA   | NA   | NA  |

Table 6 Observed order of accuracy of Trapezoidal variants for example 4

| m (Strips) | n (Intervals) | Observed order of accuracy | CNCT | AMT | GMT | HaMT | HeMT | CMT |
|------------|--------------|----------------------------|------|-----|-----|------|------|-----|
| 1          | 1            | NA                         | NA   | NA  | NA  | NA   | NA   | NA  |
| 2          | 2            | 2.045366                   | 4.154764 | 3.501015 | 3.625854 | 3.613894 | 0.362703 | NA  |
| 4          | 4            | 2.008972                   | 4.112882 | 3.674563 | exact    | exact  | 3.056241 | NA  |
| 8          | 8            | 2.00192                    | 4.023337 | 3.729474 | exact    | exact  | NA   | NA  |
| 16         | 16           | 2.000050                   | 4.005587 | 3.769024 | exact    | exact  | NA   | NA  |
| 32         | 32           | 2.000069                   | 4.001435 | exact | exact  | exact  | NA   | NA  |
| 64         | 64           | 2.000083                   | 4.000332 | NA   | NA   | NA   | NA   | NA  |
| 128        | 128          | 2.000022                   | 4.000033 | NA   | NA   | NA   | NA   | NA  |
Fig 5. Computational cost (in logarithm scale) to achieve an absolute error of at most $1E{-8}$ from Example 1.

Fig 6. Computational cost (in logarithm scale) to achieve an absolute error of at most $1E{-8}$ from Example 2.
Fig 7. Computational cost (in logarithm scale) to achieve an absolute error of at most 1E-08 from Example 3.

Fig 8. Computational cost (in logarithm scale) to achieve an absolute error of at most 1E-08 from Example 4.

Conflict of Interest:
There is no conflict of interest regarding this article.

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