On the growth speed of own values for the fourth order spectral problem with Radon–Nikodim derivatives

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Abstract. In the present paper, the growth rate of eigenvalues for the fourth-order spectral problem with nonsmooth solutions is obtained. It arises when we apply the Fourier method to the mathematical model, describing small free vibrations of a mechanical system consisting of pivotally connected rods. We assume that at the connection points there are springs that respond to rotation, while the system is in the external environment with localized features, leading to a loss of smoothness of the solution. The analysis of the problem is based on the pointwise approach proposed by Yu. V. Pokorny, and proved to be effective in studying linear boundary problems of the second and fourth orders with continuous solutions (an exact parallel with oscillation theory of ordinary differential equations is constructed).

1. Introduction
The method developed in [1] to estimate the growth rate of the eigenvalues of a classical boundary value problem with smooth solutions, was adapted in [2, 3] for two problems with nonsmooth solutions: one of different orders, and the other of the fourth order with a spectral parameter in the second derivative. In work [4], the growth rate of the eigenvalues was obtained, where a mathematical model implemented as a boundary value problem

\[
\begin{cases}
(p u''_{xx})'' - (ru'_{x})' + Q'_{x} u = \lambda M'_{x} u; \\
u(0) = u'(0) = u(\ell) = u'(\ell) = 0,
\end{cases}
\]

with derivatives in the measure was considered. This model describes small transverse vibrations of a rod placed in an external environment with localized features, leading to a loss of smoothness of the solution by the method of the separation of variables.

In the indicated papers, the pointwise method of Yu. V. Pokorny was used. Notice that a qualitative theory with nonsmooth solutions began to develop rapidly after the publication in 1999 of the work of Yu. V. Pokorny [5]. So there are monographs [6, 7], papers [8–11] in which thoroughly studied linear boundary value problems of the second order with derivatives in measures. The pointwise approach used in linear problems has been shown to be effective both in nonlinear problems [12, 13], in problems with discontinuous solutions [14–16], and in the fourth-order boundary value problems [17–19], as well as in the regional problems of hyperbolic
type [20, 21]. This efficiency is explained quite simply: when using derivatives in measures, the equation becomes pointwise defined, and makes it possible to use qualitative methods for analyzing solutions, in contrast to the theory of generalized functions. Indeed, when using the theory of Schwartz-Sobolev distributions, difficult problems are manifested. The first problem is that only weak solvability can be established, hence equations are not suitable for applications. The second problem which has not yet been solved arises when multiply the generalized function by the discontinuous one. The third problem is that equations in generalized functions are the equality of two functionals defined over the space of basic functions, and it is extremely difficult to apply methods of qualitative analysis to such equations.

In the present paper, we study the spectral problem

\[
\begin{align*}
\left\{ & \left( pu''_{2\mu}\right)'' \sigma - \left( ru'_{x}\right)_{\sigma} + Q'_{\sigma} u = \lambda M'_{\sigma} u; \\
& u(0) = u''_{2\mu}(0) = u(\ell) = u''_{2\mu}(\ell) = 0,
\end{align*}
\]

which also occurs when the variables are separated. The corresponding mathematical model describes small transverse vibrations of a chain of rods pivotally connected. We assume that at each such point there is a spring that responds to the rotation of the rods located to the left and to the right of the point of contacts. The system is placed in external environment with localized features. At the ends the physical system is pivotally fixed.

Notice that the function \(\mu(x)\) has jumps at the points of articulation, and in all the others points it is continuous. The positive on the segment \([0, \ell]\) function \(p(x)\) characterizes the material from which the rod is made; the function \(r(x)\) is the string tension force at the point \(x\); \(Q(x)\) determines the elastic response of the external environment, and \(M(x)\) is the mass of the segment \([0, x]\). The \(\sigma\) is measure generated by the function \(\sigma(x)\) contains all the features of the system: these are points at which there are localized features. By \(S(\sigma)\) we denote the set of discontinuity points of the function \(\sigma(x)\).

We consider a solution of problem (1) in the class \(E\) of absolutely continuous functions \(u(x)\) on \([0, \ell]\), whose first derivative \(u'_{\sigma}\) is absolutely continuous on \([0, \ell]\); the quasiderivative \(pu''_{2\mu}\) is absolutely continuous on \([0, \ell]\); \(pu''_{2\mu}\) is \(\sigma\)-absolutely continuous on \([0, \ell]\).

By a solution we mean a function belonging to the class \(E\) and satisfying the boundary conditions.

By the eigenvalue of a boundary value problem we mean any number \(\lambda\) for which problem (1) has a nontrivial solution. This function is called an eigenfunction corresponding to this eigenvalue.

The equation in (1) is defined on the special extension \([0, \ell]_{\sigma}\) of the segment \([0, \ell]\), in which every point \(\xi \in S(\sigma)\) is replaced by a triple of eigenelements \(\{\xi^{-}; \xi; \xi^{+}\}\). The set \([0, \ell]_{\sigma}\) is constructed as follows. On \([0; \ell]\) \(S(\sigma)\) we introduce the metric \(\rho(x; y) = |\sigma(x) - \sigma(y)|\). If \(S(\sigma) \neq \emptyset\), then \([0, \ell], \rho\) is an incomplete metric space. The standard completion leads to \([0, \ell]_S\), in which each \(\xi \in S(\xi)\) is replaced with a pair of eigenelements \(\{\xi^{-}; \xi^{+}\}\). The union of \([0, \ell]_S\) and \(S(\sigma)\) gives us \([0, \ell]_{\sigma}\).

In the first term on the left side of equality (1), the third differentiation with respect to \(x\) means that the quasiderivative \(pu''_{2\mu}(x)\) is continuous at all points, in particular, at the points \(\xi \in S(\mu)\), i.e.,

\[
p(\xi - 0)u''_{2\mu}(\xi - 0) = p(\xi) \frac{\Delta u'_{2\mu}(\xi)}{\Delta \mu(\xi)} = p(\xi + 0)u''_{2\mu}(\xi + 0).
\]

Finally, we obtain that at the points \(\xi\) belonging to the set \(S(\sigma)\), four conditions are satisfied: the continuity condition for \(u(x)\), conditions (2), and

\[
\Delta(pu''_{2\mu} + ru'_{x})(\xi) + u(\xi)\Delta Q(\xi) = \Delta F(\xi),
\]
(here $\Delta v(\xi)$ is the jump of the function $v(x)$ at the point $\xi$, i.e. $\Delta v(\xi) = v(\xi + 0) - v(\xi - 0)$).

Everywhere below, we assume that the following conditions are satisfied:

(i) $x, \mu(x), p(x), r(x), Q(x)$ and $F(x)$ are $\sigma$-absolutely continuous on $[0, \ell]_S$;

(ii) $p(x) > 0$ for all $x \in [0, \ell], \min_{[0, \ell]} \mu(x) > 0$;

(iii) integral $\int_0^\ell \frac{d\mu(x)}{p(x)}$ is finite;

(iv) $r(x) \geq 0$ for any $x \in [0, \ell]$;

(v) $Q(x)$ does not decrease on $[0, \ell]$.

These conditions provide unique solvability of the boundary value problem

$$
\begin{align*}
\left( \begin{array}{c}
u''(x) + (ru_x)'_\sigma + Q_x' u = \lambda F'_\sigma; \\
u(0) = u''(0) = 0; \\
u(\ell) = u''(\ell) = 0;
\end{array} \right)
\end{align*}
$$

where $F(x)$ is $\sigma$-absolutely continuous on $[0, \ell]$ function.

As shown in [18] the problem

$$
\begin{align*}
\left( \begin{array}{c}
u''(x) + (ru_x)'_\sigma + uQ_x' = F'_\sigma; \\
u(0) = u''(0) = 0; \\
u(\ell) = u''(\ell) = 0,
\end{array} \right)
\end{align*}
$$

is non-degenerate (a homogeneous problem has only a trivial solution). Therefore it has a unique solution for any $\sigma$-absolutely continuous function $F(x)$.

By $K(x, s)$ we denote the minimum of the following functional

$$
\Phi(u) = \int_0^\ell \frac{p u'^2}{2} dx + \int_0^\ell \frac{r u'^2}{2} dx + \int_0^\ell \frac{u^2}{2} dQ - \int_0^\ell u d\theta(x - s),
$$

where $\theta(x - s)$ is the Heaviside function. The existence of $K(x, s)$ is proved in exactly the same way as in [18].

Then problem (1) is equivalent to the integral equation

$$
u(x) = \lambda \int_0^\ell K(x, s) u(s) M'_\sigma(s) d\sigma,
$$

or, recalling the replacement theorem [8] we have

$$
u(x) = \lambda \int_0^\ell K(x, s) u(s) dM(s).
$$

It is easy to see that the integral operator

$$
(Au)(x) = \int_0^\ell K(x, s) u(s) dM(s)
$$

acts in $C[0, \ell]$ and is completely continuous. Therefore, its spectrum consists of eigenvalues having finite multiplicity. Moreover, since the problem is symmetric, i.e., for all $u, v \in E$ the
equality \((Lu, v) = (u, Lv)\) is true, we obtain that the spectrum is real. Let us show that for each eigenvalue there are no attached elements and the algebraic multiplicity is equal to one.

Suppose the opposite: for some eigenvalue \(\lambda_k\) there is a chain of adjoint nonzero vectors. Denote by \(\psi(x)\) a solution of the problem

\[
\begin{cases}
(pu_{xx}'' - ru_x')\varphi_k = \lambda_k M'_\sigma u + \varphi_k, \\
u(0) = u_x''(0) = 0; \\
u(\ell) = u_x''(\ell) = 0,
\end{cases}
\]

where \(\varphi_k(x)\) is the eigenfunction corresponding eigenvalue \(\lambda_k\).

Substituting \(\psi(x)\) into the equation from (6), multiplying the resulting identity by \(\varphi_k(x)\) and integrating over the measure \(\sigma\) along the segment \([0, \ell]\), we get the equality

\[
\int_0^\ell (p\psi_{xx}'' \varphi_k) d\sigma - \int_0^\ell (r\psi_x' \varphi_k) d\sigma + \int_0^\ell Q'_\sigma \varphi_k d\sigma = \lambda_k \int_0^\ell M'_\sigma \varphi_k d\sigma + \int_0^\ell \varphi_k^2 d\sigma. \tag{7}
\]

Let us integrate the first and second integral of equality (7) in parts (the first integral is four times, the second is two). We have

\[
\int_0^\ell (p\psi_{xx}'' \varphi_k) d\sigma = (p\psi_{xx}'' \varphi_k) \bigg|_0^\ell - p\psi_{xx}'' \psi_x' \bigg|_0^\ell + p\varphi_k'' \psi_x' \bigg|_0^\ell - (p\varphi_k'' \psi_x') \bigg|_0^\ell + \int_0^\ell (p\varphi_k'' \varphi_k) d\sigma = \int_0^\ell (p\varphi_k'' \varphi_k) d\sigma \tag{8}
\]

and

\[
\int_0^\ell (r\psi_x' \varphi_k) d\sigma = r\psi_x' \varphi_k \bigg|_0^\ell - r\varphi_k' \psi_x' \bigg|_0^\ell + \int_0^\ell (r\varphi_k') \psi d\sigma = \int_0^\ell (r\varphi_k') \psi d\sigma. \tag{9}
\]

All integrands are equal to zero, since \(\varphi_k(x)\) and \(\psi(x)\) satisfy the boundary conditions. Thus with respect to equalities (8) and (9), relation (7) takes the form:

\[
\int_0^\ell [(p\varphi_k'' \varphi_k) - (r\varphi_k') \psi_x' + Q'_\sigma \varphi_k] \psi d\sigma = \lambda_k \int_0^\ell M'_\sigma \varphi_k d\sigma + \int_0^\ell \varphi_k^2 d\sigma. \tag{10}
\]

But \(\varphi_k(x)\) is an eigenfunction corresponding to the eigenvalue \(\lambda_k\). Hence, (10) takes the form:

\[
\int_0^\ell \varphi_k^2 d\sigma = 0. \quad \text{Due to the continuity of } \varphi_k(x), \text{ we have the identity } \varphi_k(x) \equiv 0. \quad \text{The latter contradicts to the non-triviality of } \varphi_k(x).
\]

The geometric simplicity of each eigenvalue is proved in [22].

2. The main results

**Theorem.** Let conditions 1–5 be satisfied; \(\{\lambda_n\}\) are eigenvalues of problem (1). Then the series

\[
\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{2/5+\delta}}
\]
converges for any $\delta > 0$.

Proof. The eigenvalues of spectral problem (1) are defined as the zeros of the Fredholm operator, which in our case is defined as follows (see, for example, [1])

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n,$$

where

$$A_n = \int_0^\ell \cdots \int_0^\ell \left| \begin{array}{cccc} K(s_1, s_1) & K(s_1, s_2) & \cdots & K(s_1, s_n) \\ K(s_2, s_1) & K(s_2, s_2) & \cdots & K(s_2, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_n, s_1) & K(s_n, s_2) & \cdots & K(s_n, s_n) \end{array} \right| dM(s_1) \cdots dM(s_n).$$

In the same way as in [1], the convergence of the series is proved for all $\lambda$. However, it is impossible to apply the scheme used in [1] in our case, since $K(x, s)$ does not have a continuous derivative with respect to $x$.

However, the differences $K(s_{i+1}, s_j) - K(s_i, s_j)$ $(i = 1, 2, \ldots, n - 1, j = 1, 2, \ldots, n)$ for some $\kappa_{ij}$ enclosed between $\inf_{x, s \in [0, \ell]} K'_x(x, s)$ and $\sup_{x, s \in [0, \ell]} K'_x(x, s)$, can be written as follows

$$K(s_{i+1}, s_j) - K(s_i, s_j) = \kappa_{ij}(s_{i+1} - s_i).$$

Since $K(x, s)$ is a solution of the equation $Lu = \theta(x - s)$, the function $K'_x(x, s)$ is bounded on the whole square $[0, \ell] \times [0, \ell]$. Therefore, the quantities $\kappa_{i,j}$ are bounded in the aggregate by some constant $C$.

Then, for $n \geq 4$ we have

$$\left| \begin{array}{cccc} K(s_1, s_1) & K(s_1, s_2) & \cdots & K(s_1, s_n) \\ K(s_2, s_1) & K(s_2, s_2) & \cdots & K(s_2, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_n, s_1) & K(s_n, s_2) & \cdots & K(s_n, s_n) \end{array} \right| = (s_n - s_{n-1}) = \cdots =$$

$$\left| \begin{array}{cccc} K(s_1, s_1) & K(s_1, s_2) & \cdots & K(s_1, s_n) \\ K(s_2, s_1) & K(s_2, s_2) & \cdots & K(s_2, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_n, s_1) & K(s_n, s_2) & \cdots & K(s_n, s_n) \end{array} \right| = (s_2 - s_1)(s_3 - s_2) \cdots (s_n - s_{n-1}) =$$

$$\left| \begin{array}{cccc} K(s_1, s_1) & K(s_1, s_2) & \cdots & K(s_1, s_n) \\ K(s_2, s_1) & K(s_2, s_2) & \cdots & K(s_2, s_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_n, s_1) & K(s_n, s_2) & \cdots & K(s_n, s_n) \end{array} \right| = (s_2 - s_1)(s_3 - s_2) \cdots (s_n - s_{n-1}) \times$$

$$\times (\mu(s_3) - \mu(s_1))(\mu(s_4) - \mu(s_2)) \cdots (\mu(s_n) - \mu(s_{n-2}))).$$
where $\chi_{m,j} = \frac{k_{m+1,j} - k_{m,j}}{\mu(\sigma_{m+2}) - \mu(\sigma_{m})}$ ($m = 1, 2, \ldots, n - 2$, $j = 1, 2, \ldots, n$). It is easy to see that the quantities $\chi_{m,j}$ are bounded together.

Applying the Hadamard inequality and estimates

$$\left| (s_2 - s_1)(s_3 - s_2) \ldots (s_n - s_{n-1}) \right| \leq \left( \frac{\ell}{n-1} \right)^{n-1},$$

$$\left| (\mu(s_3) - \mu(s_1))(\mu(s_4) - \mu(s_2)) \ldots (\mu(s_n) - \mu(s_{n-2})) \right| \leq \left( \frac{\mu(\ell) - \mu(0)}{n-2} \right)^{n-2},$$

for $A_n$ (for $n \geq 4$) we have

$$|A_n| \leq C^n \cdot n^{\frac{\pi}{\ell}} \cdot (M(\ell) - M(0))^n \left( \frac{\ell}{n-1} \right)^{n-1} \cdot \left( \frac{\mu(\ell) - \mu(0)}{n-2} \right)^{n-2} =$$

$$= \frac{1}{\ell} (C(M(\ell) - M(0))\ell(\mu(\ell) - \mu(0)))^n \frac{n^{\frac{\pi}{\ell}}}{(n-1)^{n-1}(n-2)^{n-2}}.$$ 

Since for any fixed positive $\varepsilon$

$$\lim_{n \to \infty} \frac{n-1}{n^{\varepsilon n}} = \lim_{n \to \infty} \frac{n-2}{n^{\varepsilon n}} = 0,$$

we have for a sufficiently large $n$ (depending on $\varepsilon$) that inequality

$$|A_n| \leq \frac{1}{\ell} (C(M(\ell) - M(0)))^n n^{-\frac{\pi}{\ell} + \varepsilon n}$$

holds. The proved inequality, according to the general theory of entire functions [23], [24], shows that the growth order of the function $D(\lambda)$ is not higher than $\frac{2}{\delta} - \varepsilon$ for any $\varepsilon \in (0; \frac{1}{2})$.

Therefore, $D(\lambda)$ has an order of growth not higher than $2/5$, therefore, for an arbitrary $\delta > 0$ the series

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{2/5+\delta}}$$

converges. The theorem is proved.

Let us show that the proved theorem allows us to justify the possibility of applying of the Fourier method to a mathematical model describing small free transverse vibrations of the system described at the beginning of the work. The mathematical model of such system has the form

$$M'(x) \frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x} \left( p u'' + (ru')' \right)_x - uQ'(x),$$

$$u(0, t) = u''(0, t) = 0;$$

$$u(\ell, t) = u''(\ell) = 0;$$

$$u(x, 0) = \psi_0(x);$$

$$u'(x, 0) = \psi_1(x),$$

where $u(x, t)$ is deviation of the system from the equilibrium position of the point $x$ at time $t$.

The functions $\psi_0(x)$ and $\psi_1(x)$ are the initial deviation and initial velocity of the system, respectively.

By the solution (12) we mean a function $u(x, t)$ which turns the equation in (12) into an identity that holds almost everywhere with respect to measure $\sigma$, and satisfies the initial and boundary conditions.

We consider a solution of problem (12) a class of variables $u(x, t)$ that are absolutely continuous in the aggregate, the first multibrand derivative of $u_x(x, t)$ is $\mu$-absolutely continuous
on \([0, \ell]\) for every fixed \(t\); the quasiderivative \(p_\ell u''_\mu(x, t)\) is absolutely continuous on \([0, \ell]\) for every fixed \(t\); \((p_\ell u''_\mu)'_x\) is \(\sigma\)-absolutely continuous on \([0, \ell]\); \(u(x, t)\) have continuous derivatives of the second order in the variable \(t\) for fixed \(x \in [0; \ell]\).

It should be noted that, before, the differential equation in model (12) is specified pointwise in accordance with the concept of Yu. V. Pokorny, due to a local equation of coupling at a point. This allows, in contrast to the theory of generalized functions, to talk about a strong solution of the mathematical model being studied.

The application of the classical scheme of separation of variables, that is, the search of a solution in the form of a product, leads us to the spectral problem:

\[
\begin{align*}
LX & \equiv (-pX''_{xx})_{x\sigma} + (rX'_x)'_{\sigma} - Q'_\sigma X(x) = -\lambda X(x)M'_\sigma(x), \\
X(0) = X''_{xx}(0) & = 0, \\
X(\ell) = X''_{xx}(\ell) & = 0.
\end{align*}
\tag{13}
\]

Thus we have problem (1) studied above.

Let \(\{\lambda_n\}_{n=1}^\infty\) be the eigenvalues of spectral problem (13). Denote by \(\varphi_n(x)\) the eigenfunction corresponding to the eigenvalue \(\lambda_n\). Moreover, we can consider it normalized:

\[
\int_0^\ell \varphi_n^2(x)M'_\sigma(s)\,d\sigma = 1.
\]

Otherwise, divide \(\varphi_n(x)\) by \(\sqrt{\int_0^\ell \varphi_n^2(x)M'_\sigma(s)\,d\sigma}\). Applying the adapted classical scheme for our case, we can easily see that the series \(% \sum_{k=1}^\infty c_k^2 \lambda_k\), where

\[
c_k = \int_0^\ell f(x)\varphi_n(x)M'_\sigma(x)\,d\sigma,
\]

converges. It follows that the Fourier series \(\sum_{n=1}^\infty c_n\varphi_n(x)\) converges uniformly and absolutely on \([0; \ell]\).

Notice that \(\varphi_k(x)\) is orthogonal with a weight of \(M'_\sigma(x)\) (moreover, integration is carried out over least \(\sigma\)), and for all \(N\) the inequality

\[
\int_0^\ell M'_\sigma(x)f^2(x)d\sigma \geq \sum_{k=1}^N c_k^2
\]

holds. Thus we have the analogue of Bessel’s inequality

\[
\int_0^\ell M'_\sigma(x)f^2(x)d\sigma \geq \sum_{k=1}^\infty c_k^2.
\]

Since each eigenvalue has an algebraic multiplicity of 1, the system of eigenfunctions is complete.

**Theorem 2.** Let \(\varphi_k(x)\) be an amplitude function normalized corresponding to the natural frequency \(\lambda_k\). Then there is a constant \(C^*\) such that for all \(k\) and \(x\) the estimate

\[
\max \left\{ \max_{0 \leq x \leq \ell} |\varphi_k'(x)|, \max_{0 \leq x \leq \ell} |\varphi_k''(x)|, \sup_{0 \leq x \leq \ell} |p(\varphi_k''_{xx})_x| \right\} \leq C^* \lambda_k
\]

holds.

Since \(\varphi_k(x)\) is an amplitude function, we have the identity

\[
\varphi_k(x) \equiv \lambda_k \int_0^\ell K(x, s)\varphi_k(s)\,dM(s),
\]

Theorem 2. Let \(\varphi_k(x)\) be an amplitude function normalized corresponding to the natural frequency \(\lambda_k\). Then there is a constant \(C^*\) such that for all \(k\) and \(x\) the estimate

\[
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\]

holds.

Since \(\varphi_k(x)\) is an amplitude function, we have the identity

\[
\varphi_k(x) \equiv \lambda_k \int_0^\ell K(x, s)\varphi_k(s)\,dM(s),
\]
where $K(x, s)$ is the influence function; $LX = F'_{L}$, $X(0) = 0$, $X_{x_{0}}'(0) = 0$, $X(\ell) = 0$, $X_{x_{0}}''(\ell) = 0$; $F(x)$ an arbitrary $\sigma$-absolutely continuous function on $[0, \ell]$. From (16) we find

$$|\varphi''_{k}(x)| = \lambda_{k} \int_{0}^{\ell} K'_{x_{0}}(x, s)|\varphi''_{k}(s)| dM(s) \leq \sup_{x, s}|K''_{x_{0}}(x, s)| \cdot C_{1} \cdot \lambda_{k}(M(l) - M(0)).$$

The expressions $|\varphi''_{k,xx}|$ and $|(p\varphi''_{k,xx})_{x}|$ are evaluated in the same way.

**Theorem 3.** Let the following conditions be satisfied: 1) the functions $p(x), r(x), Q(x)$ and $M(x)$ are $\sigma$-absolutely continuous on $[0, \ell]$; 2) $M_{\sigma}'' \geq m_{0} > 0$; 3) $\psi_{i}(x)$ $(i = 1, 2)$ are absolutely continuous on $[0, \ell]$; $\psi''_{i}(x)$ are $\mu$-absolutely continuous on $[0, \ell]$; $p \psi''_{i}(x)$ are absolutely continuous on $[0, \ell]$; $p \psi''_{i}(x)$ are $\sigma$-absolutely continuous on $[0, \ell]$; 4) the functions $(L\psi_{0})(x)$ and $(L\psi_{1})(x)$ are continuous on $[0, \ell]$; 5) the function $(L\psi_{0})(x)$ and its derivative are absolutely continuous on $[0, \ell]$; 7) $\psi_{0}(0) = \psi''_{1}(0) = \psi(0) = (L\psi_{0})(0) = (L\psi_{0}')(0) = (L\psi_{0})(0) = (L\psi_{0})(0) = (L\psi_{0})(0) = (L\psi_{0})(0) = 0$. Then the function

$$u(x, t) = \sum_{k=1}^{\infty} \varphi_{k}(x) \left(A_{k} \cos \sqrt{\lambda_{k}} t + \frac{B_{k}}{\lambda_{k}} \sin \sqrt{\lambda_{k}} t \right),$$

where $\varphi_{k}(x)$ normalized amplitude function corresponding to the eigenvalue $\lambda_{k}$,

$$A_{k} = \int_{0}^{\ell} M'_{\sigma}(x) \varphi_{k}(x) \psi_{0}(x) d\sigma, \quad B_{k} = \int_{0}^{\ell} M'_{\sigma}(x) \varphi_{k}(x) \psi_{1}(x) d\sigma,$

is a solution of the mathematical model

$$\begin{cases}
M'_{\sigma}(x) \frac{\partial^{2} u}{\partial \sigma^{2}} = -\frac{\partial}{\partial \sigma} (pu''_{x_{0}}(x) + (ru')'_{x_{0}} - uQ'_{\sigma}), \\
u(0, t) = u''_{x_{0}}(0, t) = u(\ell, t) = u''_{x_{0}}(\ell, t) = 0, \\
u_{x}(x, 0) = \psi_{0}(x), \\
u_{x}(x, 0) = \psi_{1}(x).
\end{cases}$$

Moreover, series (17) can be differentiated by $t$ twice and by $x$ four times (first by $x$, then by $\mu$, $x$ and $\sigma$). Series (17) converges absolutely and even on the rectangle $[0, \ell] \times [0, T]$.

Let us estimate the coefficients of the Fourier series of the function $\psi_{0}(x)$. We have

$$A_{k} = \int_{0}^{\ell} \psi_{0}(x)M'_{\sigma}(x) \varphi_{k}(x) d\sigma = \int_{0}^{\ell} \psi_{0}(x) \frac{1}{\lambda_{k}} \left((p\varphi''_{k,xx})_{x_{0}} - (r\varphi'_{k,x})_{x_{0}} + Q'_{\sigma}\varphi_{k}\right) d\sigma =$$

$$= \int_{0}^{\ell} \psi_{0}(x) \frac{1}{\lambda_{k}} (p\varphi''_{k,xx})_{x_{0}} d\sigma - \int_{0}^{\ell} \psi_{0}(x) \frac{1}{\lambda_{k}} (r\varphi'_{k,x})_{x_{0}} d\sigma + \int_{0}^{\ell} \psi_{0}(x) \frac{1}{\lambda_{k}} Q'_{\sigma}\varphi_{k} d\sigma.$$

Let us integrate the first integral of the right-hand side of the last equality four times in parts, the second twice. With respect to the properties of the function $\varphi_{k}(x)$ we obtain

$$A_{k} = \frac{1}{\lambda_{k}} \left[ \psi_{0}(p\varphi''_{k,xx})_{x_{0}} \right]_{0}^{\ell} - \psi_{0}p\varphi''_{k,xx} \bigg|_{0}^{\ell} + \varphi'_{k} p\psi''_{k,xx} \bigg|_{0}^{\ell} - \varphi''_{k,xx} \varphi_{0} \bigg|_{0}^{\ell} + \int_{0}^{\ell} \varphi_{k}(x)(p\psi''_{k,xx})_{x_{0}} d\sigma -$$

$$-r\varphi'_{k,x} \varphi_{0} \bigg|_{0}^{\ell} + r\varphi'_{k,x} \varphi_{0} \bigg|_{0}^{\ell} - \int_{0}^{\ell} \varphi_{k}(x)(r\psi'_{k,x})_{x_{0}} d\sigma - \int_{0}^{\ell} \varphi_{k}(x)Q'_{\sigma}(x) \psi_{0}(x) d\sigma \right] =$$

$$= \frac{1}{\lambda_{k}} \int_{0}^{\ell} \varphi_{k}(x)L\psi_{0} d\sigma = \frac{1}{\lambda_{k}} \int_{0}^{\ell} \varphi_{k}(x)M'_{\sigma}(x) \left(\frac{L\psi_{0}}{M_{\sigma}}\right)(x) d\sigma.$$
The last equality means that the numbers \( \lambda_k A_k \) are the coefficients of the Fourier series of the function \( \left( \frac{L\psi_0}{M'_\sigma} \right)(x) \). Hence, the series \( \sum_{k=1}^{\infty} |\lambda_k^2 A_k^2| \) converges.

Similarly, we get that \( \lambda_k B_k \) are the coefficients of the Fourier series of the function \( (L\psi_1 M'_\sigma)(x) \) which is continuous on \([0, \ell]\). From the analogue of Bessel’s inequality (14) we have

\[
\sum_{k=1}^{\infty} (\lambda_k B_k)^2 \leq \int_0^\ell M'_\sigma(x) \left( \frac{L\psi_1(x)}{M'_\sigma(x)} \right) \, d\sigma.
\]

Thus the series \( \sum_{k=1}^{\infty} (\lambda_k B_k)^2 \) converges.

The series obtained by formal differentiation (17) have the form

\[
\frac{\partial u}{\partial x} = \sum_{k=1}^{\infty} \varphi_{kx}'(x) \left( A_k \cos \sqrt{\lambda_k} \, t + \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} \, t \right),
\]

\[
\frac{\partial^2 u}{\partial x \partial \mu} = \sum_{k=1}^{\infty} p \varphi_{kx\mu}'(x) \left( A_k \cos \sqrt{\lambda_k} \, t + \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} \, t \right),
\]

\[
\frac{\partial}{\partial x} \left( p \frac{\partial^2 u}{\partial x \partial \mu} \right) = \sum_{k=1}^{\infty} (p \varphi_{kx\mu}''(x))' \left( A_k \cos \sqrt{\lambda_k} \, t + \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} \, t \right),
\]

\[
\frac{\partial^2}{\partial x \partial \sigma} \left( p \frac{\partial^2 u}{\partial x \partial \mu} \right) = \sum_{k=1}^{\infty} (p \varphi_{kx\mu}''(x))_{x\sigma} \left( A_k \cos \sqrt{\lambda_k} \, t + \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} \, t \right) =
\]

\[
= \sum_{k=1}^{\infty} (r \varphi_{kx\sigma}')_x - \varphi_k Q'_\sigma + \lambda_k M'_\sigma \varphi_k,
\]

\[
\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \varphi_k(x) \left( -A_k \sqrt{\lambda_k} \sin \sqrt{\lambda_k} \, t + B_k \cos \sqrt{\lambda_k} \, t \right),
\]

\[
\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{\infty} \varphi_k(x) \left( -A_k \lambda_k \cos \sqrt{\lambda_k} \, t - B_k \sqrt{\lambda_k} \sin \sqrt{\lambda_k} \, t \right).
\]

Notice that of which is evaluated by the numerical series \( K \sum_{k=1}^{\infty} (\lambda_k |A_k| + \sqrt{\lambda_k} |B_k|) \), where \( K \) is constant. It remains to show that the series

\[
\sum_{k=1}^{\infty} (\lambda_k |A_k| + \sqrt{\lambda_k} |B_k|)
\]

converges. The convergence of the series was shown in Theorem 1, where \( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2/5+\varepsilon}} \) for any \( \delta > 0 \), which means that for \( \delta = 3/5 \). Then for any \( \varepsilon > 0 \), there exists \( N \) such that for any positive integers \( n \geq N \) and \( m \) the inequality \( \sum_{k=n}^{n+m} \frac{1}{\lambda_k} < \varepsilon \) holds. For these \( n \) and \( m \) we successively find (applying Cauchy’s inequality)

\[
\sum_{k=n}^{n+m} (\lambda_k |A_k| + \sqrt{\lambda_k} |B_k|) = \sum_{k=n}^{n+m} |A_k| \lambda_k^{3/2} \frac{1}{\sqrt{\lambda_k}} + \sum_{k=n}^{n+m} |B_k| \lambda_k \frac{1}{\sqrt{\lambda_k}} \leq
\]

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\[ \leq \sqrt{\sum_{k=n}^{n+m} A_k^2 \lambda_k^3} + \sqrt{\sum_{k=n}^{n+m} B_k^2 \lambda_k} \leq (\sqrt{\hat{A}} + \sqrt{\hat{B}}) \varepsilon, \]

where \( \hat{A} \) and \( \hat{B} \) denote the sums of the series \( \sum_{k=1}^{\infty} \lambda_k^3 A_k^2 \) and \( \sum_{k=1}^{\infty} (\lambda_k B_k)^2 \) respectively. Thus, the series \( \sum_{k=1}^{\infty} (\lambda_k |A_k| + \sqrt{\lambda_k} |B_k|) \) converges on the basis of Cauchy. The theorem is proved.

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References

[1] Lovitt W V *Linear integral equations* Moscow, 1957, 267 p
[2] Shabrov S A and Golovko N I About the velocity of increase of eigenvalue of a different order spectral problem with derivative on the measure 2015 *Proceedings of Voronezh State University. Series: Physics. Mathematics* 3, pp. 186–195
[3] Shabrov S A, Bugakova N I, Ilina O M and Shabrov M V On the rate of growth of the eigenvalues of one spectral problem with derivatives of the measure and a spectral parameter in the second derivative 2018 *Proceedings of Voronezh State University. Series: Physics. Mathematics* 3, pp. 203–207
[4] Shabrov S A, Bugakova N I and Shaina E A About the velocity of increase of eigenvalue of the fourth order spectral problem with derivative on the measure 2018 *Proceedings of Voronezh State University. Series: Physics. Mathematics* 4, pp. 207–215
[5] Pokorny Yu V The Stieltjes integral and measure derivatives in ordinary differential equations 1999 *Reports of the Russian Academy of Sciences* 364, 2 p. 167–169
[6] Pokorny Yu V, Penkin O M, Pryadiev V L et. al. *Differential equations on geometrical graphs* Moscow: Fizmatlit, 2004, 272 p
[7] Pokorny Yu V, Bakhtina J I, Zvereva M B and Shabrov S A *Stieltjes Oscillation Method in Spectral Problems* Moscow: Fizmatlit, 2009, 192 p
[8] Pokorny Yu V, Zvereva M B and Shabrov S A Oscillation theory of the Sturm-Liouville problem for impulsive problems 2008 *Russian Mathematical Surveys* 63, 1, pp. 111–154
[9] Pokornyi Yu V and Shabrov S A Toward a Sturm-Liouville theory for an equation with generalized coefficients 2004 *Journal of Mathematical Sciences* 119, 6, pp. 769–787
[10] Pokorny Yu V, Zvereva M B, Ishchenko A S and Shabrov S A Irregular Extension of oscillation theory of spectral problem Sturm-Liouville 2007 *Mathematical Notes* 82, 4, pp. 578–582
[11] Pokorny Yu V, Zvereva M B and Shabrov S A On Extension of the Sturm-Liouville Oscillation Theory to Problems with Pulse Parameters 2008 *Ukrainian Mathematical Journal* 60, 1, pp. 108–113
[12] Davydova M B and Shabrov S A On the number of solutions of a nonlinear boundary value problem with a Stieltjes integral 2011 *Proceedings of the University of Saratov. New series. Series: Mathematics. Mechanics. Computer* 11, 4, pp. 13–17
[13] Davydova M B and Shabrov S A Nonlinear comparison theorems for differential equations second-order derivatives of the Radon-Nikodym 2013 *Proceedings of Voronezh State University. Series: Physics. Mathematics* 1, pp. 155–160
[14] Pokornyi Yu V, Zvereva M B, Shabrov S A and Davydova M B Stieltjes differential pulsed problems with discontinuous solutions 2009, *Reports of Academy of Sciences* 428, 5, pp. 595–597
[15] Zvereva M B *Differential equations with discontinuous solutions: qualitative theory* Saarbruecken, 2012, 112 p
[16] Pokorny Yu V, Zvereva M B and Shabrov S A About problem of Sturm–Liouville for discontinuous strings 2004 *Proceedings of the universities. North Caucasus region. Mathematics and mechanics of continuous media. Special Issue. Rostov-on-Don* pp. 186–190
[17] Shabrov S A Mathematical model for small deformations of the rod system with internal features 2013 *Proceedings of Voronezh State University. Series: Physics. Mathematics* 1, pp. 232–250
[18] Baev A D, Shabrov S A, Golovanova F V and Meach Mon *The Function Of The Differential Impact Model Fourth Order* 2014 *Herald of the Voronezh Institute of Russian Ministry for Emergency Situations* 3 (12), pp. 65–73
[19] Shabrov S A A necessary condition for a minimum of a quadratic functional with Stieltjes integral 2012 Proceedings of the University of Saratov. New series. Series: Mathematics. Mechanic. Computer 12, 1, pp. 52–55

[20] Baev A D, Shabrov S A and Meach Mon Uniqueness Of The Solution Mathematical Model Of Forced String Oscillation Singularities 2014 Proceedings of Voronezh State University. Series: Physics. Mathematics 1, pp. 50–55

[21] Baev A D, Shabrov S A, Golovaneva F V and Meach Mon About Unique Classical Solution Mathematical Model Of Forced Vibrations Rod System With Singularities 2014 Proceedings of Voronezh State University. Series: Physics. Mathematics 2, pp. 74–80

[22] Shabrov S A Mathematical modeling and qualitative methods for the analysis of boundary value problems with derivatives as 2017 The dissertation for the degree of Doctor of Physics and Mathematics Voronezh, 412 p

[23] Titchmarsh E Function Theory Moscow, 1980, 464 p

[24] Levin B Ya Distribution of zeros of entire functions Moscow, 1956, 632 p