Nonperturbative evaluation of quantum particle production in parametric resonance enhanced by noise

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Numerical studies are reported to support the idea to explain the particle production in late inflationary era based on the parametric resonance with a noise effect. Two nonperturbative renormalization group formulations are used to numerically calculate the time evolution of particle numbers. Firstly, the dynamical renormalization group (DRG) method is applied to sum up the secular contributions. Secondly, we derive an exact evolution equation of the particle number which turned out to be possible surprisingly owing to the presence of the noise effect. Our numerical results show a drastic enhancement of the particle production in agreement with earlier works qualitatively. A comparison is made of numerical results based on two methods. Our work provides nonperturbative and quantitative methods to evaluate the particle production for the inflationary cosmology.

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1 Introduction

Recent observations on the cosmic microwave background offer us strong evidence for the inflation at the beginning of the universe. Explosive particle production at the end of the inflation era is a crucial aspect of inflationary cosmology in order to realize the starting point of the successful big bang universe. It is supposed that a coherent oscillation of a scalar field, called the inflaton field, at the end of inflation period transfers the energy to matter particles very rapidly. Explicitly, an oscillation of the inflaton field induces a parametric resonance instability for any bosonic modes coupled to the field and leads to the explosive particle production [1–4].

The particle production by the parametric resonance has been discussed in many papers. The most simplified model is a harmonic oscillator receiving an external periodic force: its classical equation of motion is nothing but the Mathieu equation. While parametric resonance is a typical non-equilibrium phenomena, perturbative approximation is known to break down due to presence of the secular contributions in general non-equilibrium quantum systems. Therefore, nonperturbative analyses are required in order to evaluate parametric resonance quantum mechanically.

Among nonperturbative methods, those based on the renormalization group offers us powerful schemes for various dynamical problems [5].

The dynamical renormalization group (DRG) is frequently applied to pursue a dynamical evolution and to sum up the secular contributions. This corresponds to the Gell-Mann Low renormalization group in quantum field theories based on the perturbative expansion [6, 7].

The exact renormalization group has also been a powerful tool for the non-perturbative analyses. Recently the functional renormalization group for the non-equilibrium field theories are proposed [8, 9]. There a cutoff is introduced to the time of the propagators and the cutoff dependence of the 1PI functionals are derived. The cutoff time dependence is, of course, compatible with the time evolution by the equation of motion.

We found that the particle production in the parametric resonance can be analyzed with the functional renormalization group approach by introducing the cutoff time to the external forces [10]. It is shown that the particle number can be traced exactly by using the functional renormalization group equations. In this paper, we apply the method of Ref. [10] to our model to be given in the next section.

There are the resonant regions and the oscillating regions in the particle number spectrum. This just corresponds to the unstable regions and the stable regions in the parameter space of the Mathieu equation. Brandenberger et. al. pointed out that the stable region of the Mathieu equation disappears in the presence of noise [11, 12]. Moreover, it has been
proven that the production rate of the particle number, i.e., the Floquet exponent, is always enhanced irrespectively of the particle momentum. However an explicit way to evaluate the mean value of the particle number was not given there.

In this paper, we treat the non-equilibrium dynamics of the particle production in the real scalar quantum field theory. We note that the noise effect can be replaced by a self-interaction term with an imaginary coupling constant after taking the average over the various sample of noise functions. We formulate the renormalization equations for this system. Then the dynamical renormalization shows explicitly the disappearance of stable regions as well as the enhancement of the particle production rate. It is found, surprisingly, that the time evolution of the particle number can be traced exactly owing to the closed time path structure. We also study this exact evolution equation numerically.

In Section 2 our model of a real scalar field is explained. There the noise effect is given as a self-interaction. In Section 3 we consider the DRG for the particle numbers. In Section 4 we derive the exact evolution equation for the particle number, which may be regarded as the exact renormalization group equations. Section 5 is devoted to the numerical analyses of these renormalization group equations. There it will be shown that the enhancement of the particle number by the noise effect certainly occurs as expected. The last section is for the conclusion and the discussions.

We added two appendices to discuss the noise effect on the Mathieu equation. Appendix A recapitulates an important result of [11]: a solution grows exponentially even in stable regions of parameters. We propose a noble simulation method in Appendix B to observe the drastic growth of the particle number in this classical setting.

2 Model for the parametric resonance with noise

2.1 Model for the parametric resonance

We consider the following model for the parametric resonance in the environment of noise; the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \chi \partial^{\mu} \chi - \frac{1}{2} m_{\chi}^2 \chi^2 - \frac{1}{2} \kappa \phi \chi^2 - \frac{1}{2} g q(t) \chi^2,$$

where $\chi$ denotes a real scalar field and $q(t)$ represents the external homogeneous noise. The field $\phi$ is the inflaton field and oscillates in the end of the inflation period. We simply treat this as a homogeneous classical background given as $\phi(t) = \Phi \cos m_{\phi} t$. We do not take account of expansion of the universe and use the stationary flat metric.

When both of the external force and the noise are homogeneous, the model is reduced to an assembly of quantum mechanical systems by decomposing the scalar field as, $\chi(\mathbf{x}, t) =$
\[ \sum_k \chi_k(t) e^{ikx}. \]

Then the Hamiltonian is given by

\[
H = \sum_k \frac{1}{2} |\dot{\chi}_k(t)|^2 + \frac{1}{2} \left( \omega_k^2 + \kappa \Phi \cos \theta \phi + gq(t) \right) |\chi_k(t)|^2,
\]

(2)

where \( \omega_k^2 = k^2 + m^2 \chi \). We also define \( m \phi = \omega_0, \kappa \Phi = 2 \epsilon \omega_0^2, \omega_k^2 = \omega_0^2 a_k \). Then the equation of motion is given by

\[
\ddot{\chi}_k + \omega_0^2 \left( a_k + 2 \epsilon \cos \omega_0 t + gq(t) \right) \chi_k = 0.
\]

(3)

Without the noise term this equation is known as the Mathieu equation. It is well-known that the resonant region and the stable region forms a band structure in the parameter space of \( (\epsilon, a_k) \). In the resonance band the particle number increases exponentially, while it just oscillates in the stable band. The resonance occurs when \( \omega_k^2 \approx (n/2)^2 \omega_0^2, (n = 1, 2, 3 \ldots) \) for a small \( \epsilon \). However it has been shown that in Ref. [11] that the particle number increases exponentially irrespectively of the particle momentum \( k \) in the presence of the noise, and therefore the stable band disappears.

In Appendices, we discuss the solutions of the equation motion given by (3). The solution of the Mathieu equation is known to enjoy the characteristic form as

\[
\chi_k(t) = P_k(t) e^{\gamma_k t},
\]

(4)

where \( P_k(t) \) is a periodic function. The factor \( \gamma_k \) is called the Floquet exponent. When this exponent has a real positive part, then the resonance occurs. In Ref. [11], it was shown that real part of the exponent is always enhanced in the presence of the noise. In Appendix A, we briefly sketch their argument. The classically defined particle number is evaluated numerically using the equation of motion (3) in Appendix B.

We may evaluate the particle number by perturbative expansion around the first resonant peak, \( \omega = \omega_0/2 \). If we parametrize as

\[
a_k = \frac{1}{4} + \epsilon \tilde{a}_k,
\]

(5)

then the Hamiltonian may be decomposed to the free part and the interaction part as

\[
H = \sum_k \left\{ \frac{1}{2} |\dot{\chi}_k(t)|^2 + \frac{1}{2} \omega_k^2 |\chi_k(t)|^2 \right\} + H' + H_q,
\]

(6)

\[
H' = \sum_k \frac{1}{2} \epsilon \omega_0^2 (\tilde{a}_k + 2 \cos \omega_0 t) |\chi_k(t)|^2 \equiv \sum_k \frac{1}{2} \epsilon f(t) |\chi_k(t)|^2,
\]

(7)

\[
H_q = \sum_k \frac{1}{2} gq(t) |\chi_k(t)|^2.
\]

(8)
In order to evaluate the particle number we introduce the particle creation and annihilation operators.

$$\chi_k(t) = \frac{1}{\sqrt{2\omega}} (a_k(t) + a_k^\dagger(t))$$  \hspace{1cm} (9)

Then the particle number is given by

$$n_k(t) = \text{Tr} \left\{ \rho(0) a_k^\dagger(t) a_k(t) \right\} \equiv \langle a_k^\dagger(t) a_k(t) \rangle,$$  \hspace{1cm} (10)

where $\rho(0)$ denotes the initial density matrix at $t = 0$. We assume that the initial state is given by the vacuum state $|0\rangle$, i.e., $\rho(0) = |0\rangle \langle 0|$. Then

$$n_k(t) = \langle 0| a_k^\dagger(t) a_k(t) |0\rangle$$  \hspace{1cm} (11)

We also define the anomalous particle numbers as

$$m_k(t) = \langle 0| a_k(t) a_k(t) |0\rangle,$$  \hspace{1cm} (12)

$$m_k^*(t) = \langle 0| a_k^\dagger(t) a_k^\dagger(t) |0\rangle,$$  \hspace{1cm} (13)

for the later conveniences. Similarly the two point functions are defined by

$$G_k(t, t') = \langle T_c \chi_k(t) \chi_k(t') \rangle,$$  \hspace{1cm} (14)

where $T_c$ denotes the time ordering along the closed time path. It will be shown later that both the particle numbers and the anomalous particle numbers are derived from the two point functions.

2.2 Statistical average of the white noise

Our present interest is the statistical average of the particle numbers obtained under an assembly of noise function $q$. We assume here that the noise is given by the Gaussian white noise such that

$$\langle q(t)q(t') \rangle = \sigma^2 \delta(t - t').$$  \hspace{1cm} (15)

Such noise could be caused by quantum fluctuations of the inflaton field or of various other environmental fields coupled to $\chi$, or could be some thermal fluctuations. In this paper we do not specify the origin of the noise and simply treat the parameters $g$ and $\sigma$ as free parameters.

Then the two point function after taking the statistical average of the noise is given by

$$G_k(t, t') = \langle T_c \chi_k(t) \chi_k(t') \rangle = \int Dq e^{-\frac{1}{2\sigma^2} \int dtq(t)^2} \langle T_c \chi_k(t) \chi_k(t') \rangle_q,$$  \hspace{1cm} (16)

where $\langle \rangle_q$ denotes the correlation obtained under a fixed noise function $q(t)$. 
The path integral representation for the two point function is useful. By the Keldish-Schwinger formalism, the path integral is given on the closed time path. Explicitly the two point function is given by

\[ G_k(t, t') = \int D\chi^+ e^{i\int dt (L_0[\chi^+] - L_0[\chi^-])} \times \int Dq e^{-\frac{1}{2\sigma^2} \int dtq(t)^2} e^{-\frac{i}{2g} \int dtq(t) \sum_k (\chi_k^+(t)^2 - \chi_k^-(t)^2)} \times \int D\chi^- e^{i\int dt \{ L_0[\chi^+] - L_0[\chi^-] \}} \times \int D\chi^- e^{i\int dt \{ L_0[\chi^+] - L_0[\chi^-] \}} \times \int D\chi^- e^{i\int dt \{ L_0[\chi^+] - L_0[\chi^-] \}} \times \int D\chi^- e^{i\int dt \{ L_0[\chi^+] - L_0[\chi^-] \}} = \int D\chi^+ D\chi^- \chi_k^+(t) \chi_k^-(t') e^{iS_{\text{eff}}[\chi^+, \chi^-]}, \]

where \( L_0 \) denotes the Lagrangian density for \( q = 0 \). The “effective action” \( S_{\text{eff}} \) is given explicitly by

\[ S_{\text{eff}} = \int dt \left\{ L_0[\chi^+] - L_0[\chi^-] \right\} + \frac{i \sigma^2 g^2}{8} \sum_k \sum_{k'} \left( \chi_k^+(t)^2 - \chi_k^-(t)^2 \right) \left( \chi_{k'}^+(t)^2 - \chi_{k'}^-(t)^2 \right) \right\}. \]

It should be noticed that there appear self-interaction terms with an imaginary coupling constant. Moreover, it is remarkable that the (\(+\)) fields and the (\(-\)) fields interact each other through these terms. Actually, it has been shown that such interaction appears generally in the Wilsonian effective action for a non-equilibrium field theory [8][9]. After integrating the quantum fluctuations with high frequencies, such stochastic noise terms as well as the dissipative terms appear for the lower frequency modes. Thus it would be more natural to treat such self-interaction terms from the beginning.

3 Dynamical renormalization group

We first perform a perturbative expansion to evaluate the two point function. Suppose the density matrix is known at an arbitrary time \( t = \tau \). Namely we assume the particle numbers as well as the anomalous particle numbers are given at \( t = \tau \). Then the two point function for a free particles are found to be

\[ D_{\chi k}^>(t, t') = \langle \chi_k(t) \chi_k(t') \rangle = \frac{1}{2\omega} \left\{ e^{-i\omega(t-t')} (n_k(\tau) + 1) + e^{i\omega(t-t')} n_k(\tau) \right\} + e^{-i\omega(t+t')} \tilde{m}_k(\tau) + e^{i\omega(t+t')} \tilde{m}_k(\tau) \right\}. \]

Here we used the free particle evolution, \( a_k(t) = \exp[-i\omega(t - \tau)]a_k(\tau) \), and the rescaling of the anomalous particle numbers, \( \tilde{m}_k(\tau) = e^{2i\omega\tau} m_k(\tau) \).
In the first order of the perturbative expansion, the two point function is obtained as

\[ G_k^>(t, t') = D_k^>(t, t') - i\epsilon \int_\tau ds \ f(s) \ [D_k^>(t, s)D_k^>(t', s) - D_k^>(s, t)D_k^>(s, t')] \]

\[ -g^2\sigma^2 \int_\tau ds \ \{D_k^>(t, s)D_k^>(t', s) + D_k^>(s, t)D_k^>(s, t') \]

\[ -D_k^>(t, s)D_k^>(s, t') - D_k^>(s, t)D_k^>(t', s) \}D_k^>(s, s) \]

\[ = D_k^>(t, t') - \epsilon \int_\tau ds \ f(s) \ [D_k^>(t, s)\rho_k(t-s) + D_k^>(s, t)\rho_k(t'-s)] + g^2\sigma^2 \int_\tau ds \ \rho_k(t-s)\rho_k(t'-s)D_k^>(s, s). \] (21)

Here we introduced the spectral function for the free field as

\[ \rho_k(t-s) = i(D_k^>(t, s) - D_k^>(s, t)) \]

\[ = \frac{1}{\omega}\sin[\omega(t-s)]. \] (22)

Then we may perform the time integration by using the explicit form for the external force (7) and the two point function is found to be

\[ G_k^>(t, t') = D_k^>(t, t') + i\epsilon(t-\tau) \left\{ 2\cos\omega(t-t') \left( \tilde{m}_k(\tau) - \tilde{m}_k^*(\tau) \right) \right. \]

\[ + e^{i\omega(t+t')} \left( 2\tilde{a}_k\tilde{m}_k^*(\tau) + 2n_k(\tau) + 1 \right) \]

\[ - e^{-i\omega(t+t')} \left( 2\tilde{a}_k\tilde{m}_k(\tau) + 2n_k(\tau) + 1 \right) \}

\[ + g^2\sigma^2 \frac{1}{64\omega^3} \left( t-\tau \right) \left\{ 2\cos[\omega(t-t')] \left( 2n_k(\tau) + 1 \right) \right. \]

\[ - e^{i\omega(t+t')} \left( \tilde{m}_k^*(\tau) - e^{-i\omega(t+t')} \tilde{m}_k(\tau) \right) \}

\[ + \text{(non-secular terms)}. \] (23)

The contributions proportional to \( t-\tau \) are called the secular terms. Other non-secular terms are oscillating and small comparatively. Therefore, these are neglected in applying the DRG. It is seen that the perturbative expansion breaks down for \( \epsilon(t-\tau) > 1 \) due to the secular terms. The DRG is a simple method to sum up the secular contributions in all orders.

Now we derive the DRG equations for the particle numbers. The key point is that the two point function should be independent of choice of the “renormalization point” \( \tau. \)\(^1\) Explicitly

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\(^1\)This idea is corresponding with the Callan-Symanzik equation in the perturbation theory of the quantum field theories.
we impose

\[ \frac{d}{d\tau} G^>_k(t, t') = 0. \] (24)

Then we may find \( \tau \) dependence of \( n_k(\tau) \) as well as \( \tilde{m}_k(\tau) \) up to \( O(\epsilon^2, g^4) \) as

\[ \frac{dn_k}{d\tau} = i\omega_0 \epsilon (\tilde{m}_k - \tilde{m}_k^*) + g^2 \frac{\sigma^2}{4\omega_0^2} \left( n_k + \frac{1}{2} \right), \] (25)

\[ \frac{d\tilde{m}_k}{d\tau} = -i\omega_0 \epsilon (2\tilde{a}_k \tilde{m}_k + 2n_k + 1) - g^2 \frac{\sigma^2}{8\omega_0^2} \tilde{m}_k. \] (26)

In the absence of the noise term these equations are easily solved and the solution for
the particle number with the initial conditions, \( n_k(0) = m_k(0) = 0 \), is found to be

\[ n_k(t) = \frac{1}{1 - \tilde{a}_k^2} \sinh^2(\gamma_k t), \] (27)

where the Floquet exponent is given by

\[ \gamma_k = \epsilon \omega_0 \sqrt{1 - \tilde{a}_k^2}. \] (28)

This coincides exactly with the well-known results for the parametric resonance in the literature. Resonance occurs for the particle modes with \( |\tilde{a}_k| < 1 \), otherwise the particle number shows oscillation.

However it is explicitly seen that the noise term alters the nature of the solutions of the
equations (25) and (26). We may evaluate the Floquet exponent by assuming the asymptotic
form as \( n_k(t) \simeq A \exp(2\gamma_k t) \) and so on. Then the exponent is given by the largest eigenvalue
of the matrix

\[
\begin{pmatrix}
2\Delta & 0 & -2\omega_0 \epsilon \\
0 & -\Delta & 2\omega_0 \epsilon \tilde{a}_k \\
-2\omega_0 \epsilon & -2\omega_0 \epsilon \tilde{a}_k & -\Delta
\end{pmatrix},
\] (29)

where \( \Delta = g^2 \sigma^2 / 8\omega_0^2 \). For a small enough \( \Delta \) the exponent may be evaluated as

\[ \gamma_k = \epsilon \omega_0 \sqrt{1 - \tilde{a}_k^2} + \frac{\sigma^2 (1 + 2\tilde{a}_k^2)}{32\omega_0^2 (1 - \tilde{a}_k^2)} g^2 + O(\Delta^2). \] (30)

Thus it is found that the exponent is always enhanced: the second term is positive for
\( |\tilde{a}_k| < 1 \). Moreover even in the stable region of the Mathieu equation, \( |\tilde{a}_k| > 1 \), the Floquet exponent has real positive part and the resonance occurs. Thus we find that the stable region disappear at least near the first resonance peak. We will show the results by numerical
analysis of Eqs. (25) and (26) in section 5.

A comment is in order. The Floquet exponent for \( |\tilde{a}_k| > 1 \) is not the one given in Eq. (30). It is easy to confirm that the largest eigenvalue of the traceless matrix in Eq. (29) is positive for \( |\tilde{a}_k| > 1 \).
4 Exact evolution equations

The analysis by the DRG is based on the perturbative expansion, although the infinite series of the leading secular terms are summed up. Therefore the results are not always reliable for the broad resonance cases, \( \epsilon \geq O(1) \), or for the cases with large noise effects. Moreover, the DRG is not suitable for the purpose of studying the particle production for the wide range of the momentum modes simultaneously. As seen explicitly in the next section, the above DRG equation gives a good result for the momentum only around the first resonance peak.

It was found in Ref. [10] that the evolution of the particle numbers can be evaluated exactly by applying a variation of exact renormalization group equation in the absence of the self-interactions. The exact renormalization group equation is also consistent with time evolution by the equation of motion. In this section we apply this method to the present system with the noise effect and we find exact evolution equations.

Before going into this discussion, let us remind the basic equations in generic non-equilibrium field theories [13]. First we introduce the statistical function \( F_k(t, t') \) and the spectral function \( \rho_k(t, t') \) as

\[
G_k(t, t') = F_k(t, t') - \frac{i}{2} \rho_k(t, t') \text{sgn}(t - t').
\]

Therefore

\[
F_k(t, t') = \frac{1}{2} \left[ G^>(t, t') + G^<(t, t') \right] = \frac{1}{2} \langle \{ \chi_k(t), \chi_k(t') \} \rangle,
\]

\[
\rho_k = i \left[ G^>(t, t') - G^<(t, t') \right] = i \langle [\chi_k(t), \chi_k(t')] \rangle.
\]

It is noted that the relations \( \rho_k(t, t') |_{t=t'} = 0 \) and \( \partial_t \rho_k(t, t') |_{t=t'} = 1 \) hold.

The particle number \( n_k(t) \) may be evaluated from the statistical function as

\[
n_k(t) + \frac{1}{2} = \frac{1}{2\omega} \left( \partial_t \partial_{t'} + \omega^2 \right) \langle \chi_k(t)\chi_k(t') \rangle \bigg|_{t' \to t}
= \frac{1}{2\omega} \left( \partial_t \partial_{t'} + \omega^2 \right) F_k(t, t') \bigg|_{t' \to t}.
\]

Therefore we consider to evaluate the statistical functions. For a free particle system with a general density matrix \( \rho(t_0) \), the statistical function is explicitly given by

\[
F_k(t, t') = \frac{1}{2\omega} \left\{ 2\cos[\omega(t - t')] \left( n_k(t_0) + \frac{1}{2} \right) + e^{-i\omega(t+t')} \bar{m}_k(t_0) + e^{i\omega(t+t')} \bar{m}_k^*(t_0) \right\},
\]

\footnote{In the thermal equilibrium these functions are related to each other through the fluctuation-dissipation relation. However these functions are independent in generic non-equilibrium situations.}
while the spectral functions is given by Eq. (22).

Now we introduce a cutoff to the interaction time. Explicitly we define the two point function cutoff at \( \tau \), \( G_k^\tau (t, t') \), by

\[
G_k^\tau (t, t') = \int D\chi^+ D\chi^- \chi^+_k(t) \chi^-_k(t') \ e^{iS_{eff}[\chi^+, \chi^-]},
\]

where

\[
S_{eff} = \int dt \sum_k \left\{ \frac{1}{2} |\dot{\chi}_k(t)|^2 - \frac{1}{2} \omega^2 |\chi_k(t)|^2 \right\} - \int_0^\tau dt \left\{ \frac{1}{2} |\dot{\chi}_k(t)|^2 - \frac{1}{2} \omega^2 |\chi_k(t)|^2 \right\} \times
\]

\[-i \phi^2 \sigma^2 \sum_{k'} \sum_{k''} \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \right\}.
\]

Note that \( G_k^\tau (t, t') \) is independent of the cutoff time \( \tau \) when \( \tau > t, t' \). Therefore it is enough to consider the cases of \( 0 < \tau < t, t' \).

Then we may deduce the cutoff dependence of the two-point function as

\[
i \partial_\tau G_k^\tau (t, t') = \int D\chi^+ D\chi^- \chi^+_k(t) \chi^-_k(t') \ e^{iS_{eff}[\chi^+, \chi^-]}
\]

\[
\times \left\{ \frac{1}{2} \epsilon f(\tau) \sum_{k'} \left( |\chi^+_k(\tau)|^2 - |\chi^-_k(\tau)|^2 \right) \right\} \times
\]

\[-i \phi^2 \sigma^2 \sum_{k'} \sum_{k''} \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \left( \chi^+_k(t)^2 - \chi^-_k(t)^2 \right) \right\}
\]

\[
= \frac{1}{2} \epsilon f(\tau) \sum_{k'} \left\{ \langle \chi_k(t) \chi_k(t') \chi^2_k(\tau) \rangle - \langle \chi^2_k(\tau) \chi_k(t) \chi_k(t') \rangle \right\}
\]

\[-i \phi^2 \sigma^2 \sum_{k'} \sum_{k''} \left\{ \langle \chi_k(t) \chi_k(t') \chi^2_k(\tau) \rangle \chi^2_k(\tau) + \langle \chi^2_k(\tau) \chi_k(t) \chi_k(t') \rangle \right\}
\]

\[-\langle \chi^2_k(\tau) \chi_k(t) \chi_k(t') \rangle - \langle \chi^2_k(\tau) \chi_k(t) \chi_k(t') \rangle \right\}.
\]

Here we note that these correlation functions can be rewritten in terms of the commutators of the scalar fields, which is given by

\[
[\chi_k(t), \chi_{k'}(\tau)] = \left( D^\tau_k (t - \tau) - D^\tau_k (\tau - t) \right) \delta_{kk'}
\]

\[= -i \rho_{kk'} (t - \tau). \]

Eventually the cutoff dependence of the two-point function is found to be

\[
i \partial_\tau G_k^\tau (t, t') = -i \epsilon f(\tau) \left\{ G_k^\tau (t, \tau) \rho_k (t' - \tau) + G_k^\tau (\tau, t') \rho_k (t - \tau) \right\}
\]

\[+i \phi^2 \sigma^2 G_k^\tau (\tau, \tau) \rho_k (t - \tau) \rho_k (t' - \tau). \]
It should be noted that this evolution equation is exact.

If we decompose the cutoff two-point function as

\[ G_k(t, t')_\tau = F_k(t, t')_\tau - \frac{i}{2} \rho_k(t, t')_\tau, \]  

then it is immediately found that the evolution equations for these functions are given by

\[ \partial_\tau F_k(t, t')_\tau = -\frac{\epsilon f(\tau)}{\omega} \left[ F_k(t, \tau)_\tau \sin[\omega(t' - \tau)] + F_k(\tau, t')_\tau \sin[\omega(t - \tau)] \right] + g^2 \frac{\sigma^2}{\omega^2} F_k(\tau, \tau)_\tau \sin[\omega(t - \tau)] \sin[\omega(t' - \tau)], \]  

(42)

\[ \partial_\tau \rho_k(t, t')_\tau = -\frac{\epsilon f(\tau)}{\omega} \left[ \rho_k(t, \tau)_\tau \sin[\omega(t' - \tau)] + \rho_k(\tau, t')_\tau \sin[\omega(t - \tau)] \right]. \]  

(43)

We note

\[ \rho_k(t, 0)_\tau = 0 = 1 \]  

since the interactions are totally switched off for \( \tau = 0 \). Therefore, we find from Eq. (43)

\[ \partial_\tau \rho_k(t, t')_{\tau=0} = 0. \]  

(45)

This means that the spectral function is always given by the free form given by Eq. (22).\(^3\)

Evolution of the particle number is derived from the Eq. (42). Since the system is free for \( t, t' > \tau \), \( F_k(t, t')_\tau \) is represented in terms of \( n_k(\tau) \) and \( m_k(\tau) \). Therefore

\[ \partial_\tau F_k(t, t')_\tau = \frac{1}{2\omega} \left\{ 2\cos[\omega(t - t')] \dot{n}_k(\tau) + e^{-i\omega(t + t')} \dot{m}_k(\tau) + e^{i\omega(t + t')} \dot{m}_k^*(\tau) \right\}. \]  

(46)

Then, by noting that \( \dot{m}_k = e^{2i\omega\tau} m_k \), the evolution equations for the particle number and the anomalous particle number may be obtained from Eq. (42) exactly as\(^4\)

\[ \frac{dn_k}{d\tau} = \frac{i}{2\omega} \epsilon f(\tau) (m_k - m_k^*) + g^2 \frac{\sigma^2}{4\omega^2} (2n_k + 1 + m_k + m_k^*), \]  

(47)

\[ \frac{dm_k}{d\tau} = -2i\omega m_k - \frac{i}{2\omega} \epsilon f(\tau) (2n_k + 2m_k + 1) - g^2 \frac{\sigma^2}{4\omega^2} (2n_k + 1 + m_k + m_k^*). \]  

(48)

We may solve these equations by perturbative expansion. Then we can reproduce the DRG equations, after extracting the secular terms.

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\(^3\) If thermalization occurs, then the spectral function should evolve and approach to the form satisfying the fluctuation-dissipation relation. Therefore this system does not show thermalization. It seems to be caused by lack of self-interaction among different momentum modes.

\(^4\) As is discussed in Ref. [10], these equations may be regarded as the exact renormalization equations.
5 Numerical analyses

Here we study numerically how the noise affects the particle number spectra by using the exact evolution equations in (47) and (48) as well as the DRG equations in (25) and (26).

For a weak external force or for a small $\epsilon$, narrow resonance peaks appear in the momentum spectra of the particle number, while, for a strong external force, broad resonance peaks show up. The situations are called the narrow resonance and the broad resonance, respectively. In the following subsections, we separately consider these two situations. Numerical calculations are performed by choosing the parameters as $\omega_0 = 1$, $m_\chi = 0$ and $\sigma = 1$.

5.1 Noise effect to narrow resonance case

First, we consider a narrow resonance case with $\epsilon = 0.1$. Fig. 1 and Fig. 2 show the particle number spectra obtained with the exact evolution equations and the DRG equations, respectively. The particle numbers $n_k(t)$ are plotted as functions of $k = |k|$. The four lines in Fig. 1 show the particle number $n_k(t)$ at $t = 100$ under the noises with $g = 0.0, 0.1, 0.3, 0.5$ from the bottom to the top. Clearly, the particle numbers are strongly enhanced owing to the noise effect: the enhancement becomes more evident as the noise increases; for a large enough noise, even the peaks of resonances are not observed.

The four lines in Fig. 1 show the particle number $n_k(t)$ at $t = 100$ under the noises with $g = 0, 0.1, 0.3, 0.5$ from the bottom to the top. Clearly, the particle numbers are strongly enhanced owing to the noise effect: the enhancement becomes more evident as the noise increases; for a large enough noise, even the peaks of resonances are not observed.
The line for \( g = 0 \), i.e., without a noise, shows the resonant peak around \( k = 1/2 \). The next peak is expected to appear around \( k = 1 \) and we observe a minor structure on the curve. We find two oscillating regions from \( k = 0 \) to \( k \sim 0.4 \) and \( k \sim 0.6 \) to \( k \sim 1.0 \), where \( n_k(t = 100) \) oscillates as a function of \( k \). In the presence of a noise, the particle number increases drastically and the oscillating regions diminish. This result agrees with the theorem given in Ref. [11], which implies that the generalized Floquet exponent in the presence of a noise is strictly positive and the particle production rate is enhanced irrespectively of the particle mode \( k \). A brief description of the theorem is given in the Appendix A.

![n_k vs k graph](image)

**Fig. 2** The particle numbers \( n_k \) at \( t = 100 \) obtained by the DRG equations with \( \epsilon = 0.1 \). The four lines are respectively for \( g = 0.0, 0.1, 0.3, 0.5 \) from the bottom to the top.

Fig. 2 shows the particle numbers evaluated by the DRG equations with the same parameter \( \epsilon = 0.1 \). The particle number increases in the presence of a noise and this happens even for oscillating regions. However, the DRG equation fails to capture the drastic enhancement as observed in Fig. 1.

In the DRG equations, we used a perturbative expansion at the center of the momentum mode \( k = 0.5 \). Therefore, in the absence of a noise, we find the same first peak both in Fig. 1 and Fig. 2 and, around this peak or \( k \sim 1/2 \), the DRG equations reproduced \( n_k \) similar to the exact evolution equations. The second resonant peak in Fig. 1 is absent in Fig. 2. This is due to the perturbative expansion mentioned above: in order to have the second peak, we need to make an expansion around \( k = 1 \). From a comparison of two figures around \( k = 1/2 \),
we also notice that the particle numbers are underestimated in the DRG equations in the presence of noises.

Next, we consider the time evolution of particle numbers for particular momentum modes both in the resonance band and the non-resonance band.

\[ n(t) \]

Fig. 3 (a) \( n_k(t) \) with \( k = 0.5 \) in the narrow resonance with \( \epsilon = 0.1 \). The results with the exact evolution equations (the blue lines) and the DRG equations (the red lines) for \( g = 0.0, 0.3 \).

Fig. 3 shows \( n_k(t) \) for the mode with \( k = 0.5 \) in the resonance band. The results from the exact evolutions and the DRG equations are plotted with the blue and red lines, respectively, for \( g = 0, 0.3 \), with and without a noise. In the absence of a noise, both equations produce almost the same result. However, for both set of equations, a noise gives different results for the particle production. The enhancement is underestimated in the DRG equations.

The results for the mode in the oscillating region with \( k = 0.8 \) are shown in Fig. 4. We observe the expected behaviors: \( n_k(t) \) oscillates in the absence of noise \((g = 0)\), while it resonates and increases in the presence of noise \((g = 0.3)\). For this mode, the DRG equations overestimate the particle production.

5.2 Noise effect to broad resonance case

Taking the parameter \( \epsilon = 0.8 \), we numerically studied the broad resonance case: the same set of simulations are performed to produce figures corresponding to Fig. 1 to Fig. 4, by using both the exact evolution equations and the DRG equations. The latter equations however,
Fig. 4 (b) $n_k(t)$ with $k = 0.8$ in the narrow resonance with $\epsilon = 0.1$. The results with the exact evolution equations (the blue lines) and the DRG equations (the red lines) for $g = 0, 0.3$. The results of two methods are indistinguishable for $g = 0$.

are derived with a perturbative expansion and for a large $\epsilon$ their results are not expected to be so reliable.

The generic features of the noise effect observed for the narrow resonance case is evident for the broad resonance case as well. The particle production rate rises irrespectively to the particle momentum $k$ and the effect wipes out the resonance structure present without a noise. One can observe this effect in Fig. 5 that plots the particle numbers $n_k(t)$ at $t = 100$ for the values of $g = 0, 0.1, 0.3, 0.5$.

6 Summary and further questions

In this article we discussed a scalar field theory with the non-perturbative methods in order to consider the expected explosive particle production at the end of the inflation epoch. The scalar field is coupled to an external force, the oscillating inflaton field, and exposed to a Gaussian white noise.

In Ref. [11] it has been shown that the particle production is enhanced by the noise effect. However a concrete method to evaluate the production rate has been missing. In this paper, we derived the evolution equations for the particle numbers in the non-equilibrium quantum field theory and reported numerical results.
The correlation functions for non-equilibrium quantum fields are represented by path integrals of the Schwinger-Keldish formalism. After taking average over the Gaussian noise, the noise effect is incorporated as self-interaction terms with a pure imaginary coupling constant. Therefore we discussed the particle numbers or the two point functions with such interaction terms and examined the parametric resonance phenomena.

Firstly, we obtained the DRG equations for the particle numbers based on the non-equilibrium perturbative expansion at the leading order. Then it is explicitly shown that the production rate is enhanced in the presence of the noise, as long as the expansion parameters are small enough.

Secondly, we derived the evolution equations for the particle numbers by introducing a cutoff to the interaction time and by evaluating the response under a shift of the cutoff time. The evolution equations may be regarded as the non-equilibrium functional renormalization group equations. Surprisingly, the evolution equations for the particle numbers turned out to be exact: owing to the special form of the noise interaction terms, there was no need to use any approximation to write down the evolutions equations.

In the last section we presented the various numerical results obtained with the DRG and evolution equations for the particle numbers. We examined both the narrow resonance case and the broad resonance case. It is clearly seen that the band structure in the momentum spectra of the particle numbers disappears due to the noise effect. The total production rate

Fig. 5 The particle numbers with $\epsilon = 0.8$ corresponding to a broad resonance band: $n_k$ at $t = 100$ obtained by the exact evolution equations for $g = 0, 0.1, 0.3, 0.5$. 
of particles is also found to be enhanced. The DRG equations are expected to be reliable only when both the noise and the external force are small. Our numerical results are consistent with this.

In Ref. [14], the authors extended their analysis to the cases of spatially inhomogeneous noise and claimed that the particle production rate is enhanced further compared to the homogeneous noise case. It would be interesting to extend our present analyses for the case of inhomogeneous noise. Furthermore, it is important to evaluate the noise effect and the particle production rate in realistic inflation scenarios. We leave these questions for future studies.

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A Mathieu equation with noise and the Floquet exponent

In this Appendix we consider a simplified differential equation given by

\[ \ddot{\chi} + (\omega^2 + p(t) + q(t))\chi = 0, \quad (A1) \]

where \( p(t) \) is a periodic function with a period \( T \) and \( q(t) \) denotes a noise function. Explicitly, we use \( p(t) = 2\epsilon \cos t \) and, therefore \( T = 2\pi \). We can choose the two independent solutions \( \chi_1(t) \) and \( \chi_2(t) \) so that \( \chi_1(0) = 1, \dot{\chi}_1(0) = 0, \chi_2(0) = 0, \dot{\chi}_2(0) = 1 \). Then the matrix

\[ \Phi_q(t, 0) = \begin{pmatrix} \chi_1(t) & \chi_2(t) \\ \dot{\chi}_1(t) & \dot{\chi}_2(t) \end{pmatrix} \quad (A2) \]

gives the fundamental solution (or transfer) matrix of this equation.

The fundamental solution satisfies the equation

\[ \dot{\Phi}_q = M(q(t), t)\Phi_q, \quad (A3) \]

with the initial conditions \( \Phi_q(0, 0) = I \). Here \( M(q(t), t) \) is the matrix

\[ M(q(t), t) = \begin{pmatrix} 0 & 1 \\ -[\omega^2 + p(t) + q(t)] & 0 \end{pmatrix}. \quad (A4) \]

It is noted that the determinant of \( \Phi_q(t) \) is the Wronskian of the differential equation (A1), which is a constant. Therefore \( \det\Phi_q(t) = 1 \), i.e. \( \Phi_q(t) \) is a matrix of SL(2, \( \mathbb{R} \)).
In the absence of noise the fundamental solution is given in the form

$$\Phi_0(t, 0) = P_0(t)e^{Ct},$$

(A5)

where $P_0(t)$ is a periodic matrix function with period $T$ and $C$ is a constant matrix. This is well-known as the Floquet theorem or the Bloch theorem. The largest real part of the eigenvalues of $C$, $\mu(0)$, is called the Floquet exponent. The matrix $\exp CT$ belongs to $\text{SL}(2, \mathbb{R})$, since $\Phi_0(T, 0) = \exp CT$. Therefore we find $\mu(0) \geq 0$. Resonance occurs when $\mu(0) > 0$, while $\mu(0) = 0$ in the stable region.

It has been shown in Ref. [11] that the Floquet exponent is always enhanced by the noise $q$. The authors defined the generalized Floquet exponent $\mu(q)$ by

$$\mu(q) = \lim_{N \to \infty} \frac{1}{NT} \log \| \prod_{j=1}^{N} \Phi_q(jT, (j-1)T) \|,$$

where $\| \|$ denotes the matrix norm. In order to see the noise effect to the Floquet exponent they introduced the reduced transfer matrix $\Psi_q(t, 0)$ by

$$\Phi_q(t, 0) = \Phi_0(t, 0)\Psi_q(t, 0) = P_0(t)e^{Ct}\Psi_q(t, 0).$$

(A7)

It is noted that $\Psi_q(t, 0)$ is also a $\text{SL}(2, \mathbb{R})$ matrix.

On the other hand, the Furstenberg’s theorem tells us that independent identically distributed random matrices $\Psi_j$ of $\text{SL}(2, \mathbb{R})$ enjoy

$$\lim_{N \to \infty} \frac{1}{NT} \log \| \prod_{j=1}^{N} \Psi_j \| \equiv \lambda > 0.$$

(A8)

Then we may apply the Furstenberg’s theorem to the reduced transfer matrix decomposed as

$$\Psi_q(NT, 0) = \prod_{j=1}^{N} \Psi_q(jT, (j-1)T),$$

(A9)

as long as the transfer matrices $\Psi_q(jT, (j-1)T)$ are given by independent and identically distributed random matrices. Thus they showed

$$\mu(q) = \mu(0) + \lambda > \mu(0).$$

(A10)

This result implies that the particle production rate is enhanced in the presence of noise irrespectively of the particle mode.
B  Numerical analysis of the noise effect in the Mathieu equation

Here we would like to confirm the implication of Eq. (A10) by numerically calculating the particle number

\[ n(t) = \frac{1}{2\omega}(\chi^2 + \omega^2\chi^2) \]

from the classical equation (A1). \( n(t) \) in Eq. (B1) may be regarded as the classical energy.

The technically important point turns out to be how we introduce a noise function. Naively we may divide the time region into intervals and approximate the noise function as a sequence of independent constants assigned to intervals [11]. However it is found that the Floquet exponent is not always enhanced in the resonant band. Rather the exponent is reduced at the resonant peak in the presence of noise. This seems to be contradicting with the above proof of the universal enhancement of the Floquet exponent.

Here we propose a different realization of the noise function. We represent the noise function as a superposition given by

\[ \nu(t) = \frac{a_0^{(j)}}{2} + \sum_{n=1}^{N} (a_n^{(j)} \cos nt + b_n^{(j)} \sin nt) \]

for each interval of \((j-1)T < t < jT\). The coefficients are given by random variables and are independent for different intervals;

\[ \langle a_n^{(j)} a_m^{(j')} \rangle = \delta_{n,m} \delta_{j,j'} \].

Then the white noise property, \( \langle \nu(t)\nu(t') \rangle = \delta(t-t') \) is maintained.

In Fig. B1, evolutions of the particle number \( n(t) \) are shown in a logarithmic plot in the narrow resonance case. Explicitly the parameters of Eq. (A1) are set as \( \omega^2 = 1/4 \) and \( \epsilon = 0.1 \), which correspond to the first resonance peak. The noise function is set to be \( q(t) = 3\nu(t) \), where \( \nu(t) \) is given by Eq. (B2) with \( N = 5 \). In Fig. B1 the red line shows the particle number in the absence of the noise, and other blue lines show those obtained with various noise functions.

The particle numbers are found to grow exponentially. The Floquet exponents can be read off from the gradients of these lines. The generalized Floquet exponent \( \mu(q) \) is given by the average of the gradients of blue lines. It is clearly seen that the Floquet exponents are enhanced in the presence of noise. It is noted that only the lowest frequency modes of the noise function are effective, though we performed these calculations for \( N = 5 \). So it is thought that enhancement occurs because such a mode acts as a fluctuation to the amplitude of external force or \( \epsilon \).
Fig. B1  \(n(t)\) obtained by solving Eq. (A1) with \(\epsilon = 0.1\) and \(\omega^2 = 1/4\) (the first resonance peak). The noise function \(q(t) = 3\nu(t)\), where \(\nu(t)\) is given by Eq. (B2) with \(N = 5\). The red line shows \(n(t)\) in the absence of a noise, while the blue lines are \(n(t)\) obtained with various noise functions.

Fig. B2  \(n(t)\) obtained by solving Eq. (A1) with \(\epsilon = 0.1\) and \(\omega^2 = 1/4 + 1/10\) (in a slightly stable region off the first resonance peak). The noise function \(q(t) = \nu(t)\), where \(\nu(t)\) is given by Eq. (B2) with \(N = 5\). The red line shows \(n(t)\) in the absence of a noise, while the blue lines are \(n(t)\) obtained with various noise functions.
We also examined the case in a stable region of the Mathieu equation, by choosing the parameters as \( \epsilon = 0.1 \) and \( \omega^2 = 1/4 + 1/10 \). While the particle number obtained from a solution of the Mathieu equation is shown by the red line in Fig. B2, it oscillates between (0, 1). Then the Floquet exponent is given by a pure imaginary number.

The blue lines in Fig. B2 show the particle numbers obtained in the presence of the noise function \( q(t) = \nu(t) \), where \( \nu(t) \) is given by Eq. (B2) with \( N = 5 \). It is seen that the particle numbers grow exponentially.

Our numerical results show that the particle number defined in Eq. (B1) increases exponentially in the presence of a noise, irrespectively whether the parameters are stable or unstable regions for the Mathieu equation. Our result is consistent with the analytical result Eq. (A10) explained in Appendix A.

References

[1] L. Kofman, A. D. Linde and A. A. Starobinsky, “Reheating after inflation,” Phys. Rev. Lett. 73 (1994) 3195 doi:10.1103/PhysRevLett.73.3195 [hep-th/9405187].

[2] L. Kofman, A. D. Linde and A. A. Starobinsky, “Towards the theory of reheating after inflation,” Phys. Rev. D 56 (1997) 3258 doi:10.1103/PhysRevD.56.3258 [hep-ph/9704452].

[3] J. H. Traschen and R. H. Brandenberger, “Particle Production During Out-of-equilibrium Phase Transitions,” Phys. Rev. D 42 (1990) 2491. doi:10.1103/PhysRevD.42.2491

[4] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, “Universe reheating after inflation,” Phys. Rev. D 51 (1995) 5438 doi:10.1103/PhysRevD.51.5438 [hep-ph/9407247].

[5] K.G. Wilson and J.B. Kogut, Phys. Rept. 12, 75 (1974); F.J. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973); J. Polchinski, Nucl. Phys. B231, 269 (1984); C. Wetterich, Phys. Lett. B 301, 90 (1993); for a review, see for example, J. Berges, N. Tetradis and C. Wetterich, Phys. Rept. 363, 223 (2002).

[6] D. Boyanovsky, H. J. de Vega, R. Holman and M. Simionato, “Dynamical renormalization group resummation of finite temperature infrared divergences,” Phys. Rev. D 60 (1999) 065003 doi:10.1103/PhysRevD.60.065003 [hep-ph/9809346].

[7] D. Boyanovsky, H. J. de Vega and S. Y. Wang, “Dynamical renormalization group approach to quantum kinetics in scalar and gauge theories,” Phys. Rev. D 61 (2000) 065006 doi:10.1103/PhysRevD.61.065006 [hep-ph/9909369].

[8] T. Gasenzer and J. M. Pawlowski, “Towards far-from-equilibrium quantum field dynamics: A functional renormalisation-group approach,” Phys. Lett. B 670 (2008) 135. doi:10.1016/j.physletb.2008.10.049

[9] T. Gasenzer, S. Kessler and J. M. Pawlowski, “Far-from-equilibrium quantum many-body dynamics,” Eur. Phys. J. C 70 (2010) 423 doi:10.1140/epjc/s10052-010-1430-3 [arXiv:1003.4163 [cond-mat.quant-gas]].

[10] A. Higashiyama, A. Murakami, E. Nakamura, H. Terao, unpublished.

[11] V. Zanchin, A. Maia, Jr., W. Craig and R. H. Brandenberger, “Reheating in the presence of noise,” Phys. Rev. D 57 (1998) 4651 doi:10.1103/PhysRevD.57.4651 [hep-ph/9709273].

[12] R. H. Brandenberger, “Partial Differential Equations with Random Noise in Inflationary Cosmology,” arXiv:1407.4775 [math-ph].

[13] J. Rammer, “Quantum field theory of non-equilibrium states,” Cambridge, UK: Univ. Pr. (2007) 536 p

[14] V. Zanchin, A. Maia, Jr., W. Craig and R. H. Brandenberger, “Reheating in the presence of inhomogeneous noise,” Phys. Rev. D 60 (1999) 023505 doi:10.1103/PhysRevD.60.023505 [hep-ph/9901207].