Effective Action Approach to Heavy Particle Contributions and Wilsonian Renormalization

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Abstract

We work in theories with both light and heavy particles. A method to obtain an effective low energy action with respect to the light particle is presented. Thanks to Wilsonian renormalization, we obtain effective actions with finite number of local operators describing possible non-decoupling effects of heavy particles. The validity of the method is first checked by explicit computation of a specific observable, the $\rho$ - parameter, in the framework of effective theory. Then we discuss a procedure to obtain the full effective action with respect to light particles in a typical example of the system of $(t, b)$ doublet, gauge bosons and Higgs doublet, regarding only $t$ quark as a heavy particle.
1. Introduction

The Standard Model of elementary particle interactions is very successful, except for failing to explain recently claimed neutrino oscillation [1]. The model is expected to be an effective low energy theory of some more fundamental theory ("new physics") with a larger symmetry, such as Minimal Supersymmetric Standard Model (MSSM), Grand Unified Theory (GUT), super-string, ... The fact that the Standard Model is doing well in lower energies suggests that new heavy particles, characteristic to each "new physics", (superpartners, lepto-quark,...) do not practically affect the low energy theory or their effects are rather limited. At least in the type of new physics models where new heavy particles are expected to be decoupled from low energy world as in the case of superpartners, the effective low energy action should be well described by what we obtain just eliminating such heavy particles from the original action, though bare couplings and field normalizations get some modification: decoupling theorem [2]. Even in the type of new physics models where new heavy particles do affect low energy world (non-decoupling), as in the case of a model with heavy quark of fourth generation if it ever exists, the non-decoupling radiative corrections due to heavy particles are expected to be described by limited number of independent effective operators. For instance we know that only $S, T$ and $U$ parameters are enough to parametrize the non-decoupling effects on gauge boson self energies.

Our purpose in this work is to propose a method to construct the effective low energy action with respect to light particles, describing radiative corrections due to heavy particles, in theories with both light and heavy particles. Though usually non-decoupling effects of heavy particles are independently analyzed for each observable, it will be nice if we can get the full effective low energy action in a systematic way, since such obtained effective action should contain all informations of the possible non-decoupling effects. By the end of this article, we would like to show by taking a prototype model with only $(t,b)$ doublet, gauge bosons and Higgs doublet that this project is realized: we construct full effective action concerning gauge bosons, Higgs doublet and light $b$ quark, which results from the path-integrals of fermionic fields.

After explaining the outline of our method invoking Wilsonian renormalization, we check the validity of our method by focusing on the operators in the effective action which are relevant for a specific observable, i.e. the $\rho$- parameter. According to our method we obtain the necessary operators in two prototype models and calculate the $\rho$- parameter by use of these operators, to confirm that they reproduce the known results. Finally we demonstrate that to obtain the full effective action in a compact closed form is possible, provided we focus on the non-decoupling effects ($\Delta \rho \equiv \rho - 1$ is one of them [3]).

Key ingredient of our method is to utilize "Wilsonian renormalization" [4]. (In the present argument, Wilsonian renormalization just means to integrate out the higher momentum (short distance) part of the fields or to reduce the U.V. cutoff of the theory, and R.G.
The usefulness of Wilsonian renormalization may be understood as follows. A natural guess to obtain the effective theory is to (path-)integrate the heavy particles out from the theory, as heavy particles never appear in the external lines of low energy processes. It, however, turns out that such manipulation results in the appearance of non-local operators with respect to light particles. To see the situation, suppose that light and heavy particles, denoted by $l$ and $h$, respectively, are interacting via $L_{\text{int}} = \frac{\lambda}{2} h l l$ (with $\lambda$ being a coupling constant). The path-integral is equivalent to calculating Fig.1 and will produce effective lagrangian in momentum space,

$$L_{\text{eff}} = \frac{\lambda^2}{8} l^2 \frac{1}{m_h^2 - k^2} l^2 = \frac{\lambda^2}{8} (\frac{1}{m_h^2} l^4 - \frac{1}{m_h^4} l^2 \Box l^2 + \ldots),$$

(1)

where $m_h$ is the mass of $h$. One may naively expect that taking a limit, $m_h \to \infty$, the $L_{\text{eff}}$ disappears anyway and the non-locality is not a problem. Unfortunately it is not the case, as $k^2$ can easily exceeds $m_h^2$ in a Feynman diagram where light particle $l$ participates in the internal loop (Fig.2). In other words, the above mentioned limit and the integral of loop momentum are not commutative (of course, right answer should be obtained when we first perform loop integral and take the limit afterwards). Real problem is that even in a theory where decoupling of heavy particle is anticipated, the decoupling is not manifest at all, because of the presence of such non-local operator. In the above example, unless $\lambda$ is correlated with $m_h$, $h$ particle should be decoupled from the low energy world according to the decoupling theorem, though $L_{\text{eff}}$ cannot be simply neglected even for the limit of large $m_h$.

The situation would have changed if loop integral from the U.V. cutoff $\Lambda$ to some scale $\mu$, which is sufficiently smaller than $m_h$ ($\mu \ll m_h$), had been performed concerning the light particle (Wilsonian renormalization), i.e. if the U.V. cutoff of the theory had been reduced from $\Lambda$ to $\mu$. After such Wilsonian renormalization, the limit $m_h \to \infty$ can be safely taken and the problem of non-locality will disappear. Thus the decoupling is expected to become manifest. This is why we wish to invoke Wilsonian renormalization.

In the literature there is a long list of works discussing effective action formalism to deal with radiative corrections (see for instance [1],[2],[3]). Some of them [2],[3] are clearly related with our present work. Their prescriptions, however, do not utilize Wilsonian renormalization.
and are more oriented to the structure of low energy theory. In our approach the effective
theory is more directly constructed from the original theory.

2. Low energy effective action and Wilsonian renormalization

To illustrate our method to obtain an effective action $S_{\text{eff}}$, we consider a simple model
with a pair of light and heavy scalar particles, $l$ and $h$, with masses $m_l$ and $m_h$, $m_l \ll m_h$.
The partition function is given by a path-integral,

$$Z = \int [Dh][Dl]e^{iS[h,l]},$$

(2)

where $S$ is the original action of the system. Our aim is to derive low energy ($E \ll m_h$)
effective action (lagrangian) which is valid at the energy range $0 \leq E \leq \mu$, where the fixed
mass scale $\mu$ is the physical U.V. cutoff of the effective low energy theory and taken to be
sufficiently smaller than $m_h$, i.e. $m_l \ll \mu \ll m_h$. For instance, $m_h \sim 10^{16} \text{GeV}$ and $\mu$ can be
taken to be, say, $\sim 10^{15} \text{GeV}$, and $m_l \leq 10^2 \text{GeV}$, if the original theory is GUT type theory.

For such purpose, we divide each field into lower and higher momentum parts. Namely
$h = h_\mathbf{<} + h_\mathbf{>} \text{ and } l = l_\mathbf{<} + l_\mathbf{>}$, with $h_\mathbf{<}$ and $h_\mathbf{>}$ denoting heavy fields with momentum $|k| < \mu$
and $|k| > \mu$ ($|k| \equiv \sqrt{|k^\mu k_\mu|}$, after Wick rotation), etc..

Our method based on the Wilsonian renormalization is to perform path-integrals with
respect to $h_\mathbf{<}$, $h_\mathbf{>}$ and $l_\mathbf{>}$, so that we can get an effective action for $l_\mathbf{<}$ alone:

$$Z = \int [Dh_\mathbf{<}][Dh_\mathbf{>}][Dl_\mathbf{<}][Dl_\mathbf{>}]e^{iS[h_\mathbf{<}+h_\mathbf{>,}l_\mathbf{<}+l_\mathbf{>}]} = \int [Dl_\mathbf{<}]e^{iS_{\text{eff}}[l_\mathbf{<}]}$$

(3)
i.e., the effective action is given by

$$e^{iS_{\text{eff}}[l_\mathbf{<}]} = \int [Dh_\mathbf{<}][Dh_\mathbf{>}][Dl_\mathbf{>}]e^{iS[h_\mathbf{<}+h_\mathbf{>,}l_\mathbf{<}+l_\mathbf{>}]}.$$  

(4)

At the right-hand side of the above equation the ordering of the path-integrals should be
such that the integrals of higher momentum parts, $h_\mathbf{>,} l_\mathbf{>}$, are done before the integral of $h_\mathbf{<}$,
since if we perform $h_\mathbf{>}$ and $h_\mathbf{<}$ integrals first the problem of non-locality arises again. As
the matter of fact, we will see that the path integral of $h_\mathbf{<}$ always gives negligible contribution
to $S_{\text{eff}}$ and the only Wilsonian renormalization, i.e. the integrations of $h_\mathbf{>}$ and $l_\mathbf{>}$, yields
approximately correct answers of the effective action.

Actually the effective action $S_{\text{eff}}$ can be calculated in two ways. One way is to directly
perform path-integral, as we will do in constructing the full effective action. Another way is
to calculate Feynman diagrams perturbatively. In the latter case, we may assign a double
line for a field with higher momentum and a single line for a field with lower momentum and
just calculate diagrams where the internal lines are either double line of heavy or light field
or single line of heavy field and external lines are all single lines of light fields.

A potentially serious problem of our method is concerning local gauge invariance. In our
approach, local gauge invariance may be explicitly broken by the presence of $\mu$ and U.V.
cutoff $\Lambda$. Actually the breakdown of gauge invariance due to $\mu$ is not a real problem, as the final result of some observable should not depend on the choice of $\mu$. A real problem might be caused by the presence of $\Lambda$. We, however, note that what we are really interested in are non-decoupling radiative corrections of heavy particles, which are all described by gauge invariant irrelevant (with mass dimension $d > 4$) operators provided we include Higgs field as well as light particles to form operators (the effects described by marginal or relevant operators can be absorbed in the renormalization of bare couplings and fields). The coefficients of the irrelevant operators are automatically finite and will not suffer from the problem originated from the regularization by $\Lambda$. For example, $\Delta \rho$ is described by a $d = 6$ operator including Higgs, though superficially it is an observable related with $d = 2$ operators, i.e. gauge boson mass-squared.

3. Effective action and the $\rho$-parameter

To check the validity of our method and to get some insight into the procedure toward the full construction of the effective action, we now derive a part of the effective action which is relevant for a specific observable, $\Delta \rho = \rho - 1$ or T-parameter [9], which is well-known as a typical example of the non-decoupling effect of heavy particle. The resultant effective action is then used to calculate the $\Delta \rho$, to confirm that it recovers the known results.

We take two typical examples of new physics contribution which give non-decoupling and decoupling effects on $\Delta \rho$: (a) the contribution of $t$ quark in a SU(2) doublet $(t,b)$ in the Standard Model, (b) the contribution of a SU(2) doublet of superpartners of light quarks, $(\tilde{u},\tilde{d})$, in MSSM.

(a) Doublet $(t,b)$ in the Standard Model

In this case the roles of $h$ and $l$ particles are played by $h$: $t$ quark, $l$: $b$ quark and SU(2) gauge bosons $W^+_{\mu}, W^-_{\mu}, W^3_{\mu}$, respectively. All our discussions in this work are restricted to the 1-loop level and for a while the presence of Higgs doublet is ignored except the role of providing quark and gauge boson masses. We assume $m_b, M_W \ll \mu \ll m_t$, though $t$ quark is not actually so heavy. If necessary $(t,b)$ doublet might be understood as the one of fourth generation, for instance. We adopt the way of calculating Feynman diagrams in order to derive the effective action which stems from the radiative corrections of $t_>, b_>$ and $t<_<$ fields, the part relevant for $\Delta \rho$. The contributing diagrams up to 1-loop level are shown in Fig.3. For the intermediate double lines the loop integral is over the momentum range $|k| \geq \mu$, while for intermediate single line of $t$ quark the loop integral is over the range $|k| \leq \mu$. The diagrams in Fig.3 are calculated to provide an effective lagrangian,

$$L_{\text{eff}} = (C_1 + \frac{1}{2} C_2) W^+_{\mu} W^-_{\mu} + \frac{1}{2} (C_1 - \frac{1}{2} C_2) W^3_{\mu} W^3_{\mu}$$

$$+ \frac{g^2}{2 m_t^2} \bar{b} W^-_{\mu} \gamma^\nu i(\gamma^\lambda \partial_\lambda) W^+_\nu \gamma^\sigma \frac{1}{2} \gamma_5 b. \quad (5)$$
The divergent coefficient $C_1$ is common for both charged and neutral gauge bosons and can be absorbed into redefinition of bare gauge boson masses in the original lagrangian. The coefficient $C_2$ gets finite contributions from the combination of Fig.3a and 3b and from Fig.3c:

$$C_2 = C_2^{(3a+3b)} + C_2^{(3c)},$$  \hspace{1cm} (6)

where

$$C_2^{(3a+3b)} \simeq \frac{3g^2}{64\pi^2}(m_t^2 - \mu^2),$$

$$C_2^{(3c)} \simeq \frac{-g^2 \mu^6}{64\pi^2 m_t^4}. \hspace{1cm} (7)$$

In the above expression $m_b$ has been ignored and among the $\mu$ dependent terms only the leading term of the expansion in $\mu^2/m_t^2 \equiv r \ll 1$ has been kept in each contribution. The remaining term in the second line of eq.(5) comes from the tree diagram Fig.3d. As emphasized in the introduction, thanks to the Wilsonian renormalization, taking only the leading term of the expansion in $\Box/m_t^2 \leq \mu^2/m_t^2 = r$ can be justified when we evaluate Fig.3d, thus making $L_{eff}$ a set of finite number of local operators.

Let us calculate $\Delta \rho$ by use of the derived effective lagrangian. In addition to the direct contributions from $C_2^{(3a+3b)}$ and $C_2^{(3c)}$, there are indirect contributions from the term in the second line of eq.(5) and from ordinary neutral current interaction of $b_\times$ field in the original lagrangian, through 1-loop diagrams shown in Fig.4a and Fig.4b, respectively:

$$\Delta \rho = \frac{1}{M_W^2} (C_2^{(3a+3b)} + C_2^{(3c)} + C_2^{(4a)} + C_2^{(4b)}), \hspace{1cm} (8)$$
with

\[ C_{(4a)}^2 \simeq \frac{-3g^2 m_t^4}{64\pi^2 m_t^2}, \]
\[ C_{(4b)}^2 \simeq \frac{3g^2 m_t^2}{64\pi^2 \mu^2}. \]  

(9)

\[ C_{(4a)}^2 \] and \[ C_{(4b)}^2 \] denote the contributions from Figs. 4a and 4b, respectively. In the summation of all contributions the leading \( \mu \) dependent term, \( \mu^2 \) term, cancel out between \( C_{(3a+3b)}^2 \) and \( C_{(4b)}^2 \), as we expected, and the well-known result \( \Delta \rho \) is recovered

\[ \Delta \rho \simeq \frac{3g^2 m_t^2}{64\pi^2 M_W^2}. \]  

(10)

Thus we have confirmed the validity of our method. It is interesting to note that the contributions of the \( t_< \) integration (single line of \( t \)) to the effective action yield contributions to the \( \Delta \rho \), denoted by \( C_{(3c)}^2 \) and \( C_{(4a)}^2 \), which are relatively suppressed by powers of \( r \). It should also be noticed that the main contribution to the \( \Delta \rho \) turns out to come from the \( C_{(3a+3b)}^2 \) alone, the result of the integration of higher momentum parts of \( t \) and \( b \) quarks.

Namely, the Wilsonian renormalization alone gives almost correct answer of the observable \( \Delta \rho \).

(b) Doublet \((\tilde{u}, \tilde{d})\) in MSSM

We briefly discuss the contribution of the pair of superpartners, \((\tilde{u}, \tilde{d})\), of light quarks \( u \) and \( d \) \((m_u, m_d \ll M_W)\) as a typical example of the case where the decoupling of new physics contribution is seen. In this case the roles of \( h \) and \( l \) particles are played by \( \tilde{u}, \tilde{d}, l: SU(2) \) gauge bosons \( W^+, W^-, W^3 \), respectively. Thus, following our method to obtain \( L_{\text{eff}} \), both higher and lower momentum parts of \((\tilde{u}, \tilde{d})\) should be integrated out and the sum of integrations just provides the final result of \( \Delta \rho \). So our purpose here is just to see the relative importance of \( C_{2(>)} \) and \( C_{2(<)} \), the contributions of single and double lines of \((\tilde{u}, \tilde{d})\), respectively. Ignoring left-right mixing of squarks, for simplicity, we get

\[ C_{2(>)} = \frac{3g^2}{64\pi^2 M_W^2} \frac{1}{M_W^2} \{ m_u^2 + m_d^2 - \frac{2m_u^2 m_d^2}{m_u^2 - m_d^2} \ln \frac{m_u^2}{m_d^2} - \frac{(m_u^2 - m_d^2)^2}{3m_u^4 m_d^4} \mu^6 \}, \]
\[ C_{2(<)} = \frac{g^2}{64\pi^2 M_W^2} \frac{1}{m_u^4 m_d^4} \mu^6, \]  

(11)

and

\[ \Delta \rho = \frac{1}{M_W^2} (C_{2(>)} + C_{2(<)}) \]
\[ = \frac{3g^2}{64\pi^2 M_W^2} \frac{1}{M_W^2} \{ m_u^2 + m_d^2 - \frac{2m_u^2 m_d^2}{m_u^2 - m_d^2} \ln \frac{m_u^2}{m_d^2} \}, \]  

(12)

thus recovering the known one \([8]\). We note that \( m_u^2 = M_{\text{SUSY}}^2 + m_u^2 \) and \( m_d^2 = M_{\text{SUSY}}^2 + m_d^2 \) \((M_{\text{SUSY}}: \text{SUSY breaking scale})\), and under \( m_u, m_d \ll M_{\text{SUSY}} \) the \( \Delta \rho \) is suppressed by
1/M_{SUSY}^2 \text{ as } \Delta \rho \simeq \frac{g^2}{64\pi^2} \frac{(m_u^2 - m_d^2)^2}{M_W^2 M_{SUSY}^2}; \text{ decoupling occurs. Or we may just say } S_{\text{eff}} \simeq 0 \text{ (except for some renormalization effects to the original action), as expected. We learn that the main contribution comes from } C_{2(>)} \text{, again from the Wilsonian renormalization alone.}

4. Full effective action

So far we have retained only the operators in $S_{\text{eff}}$ which are relevant for a specific observable, i.e. $\Delta \rho$. We finally generalize the argument and present a systematic way to construct full effective action $S_{\text{eff}}$, obtained by the "integration" of heavy particles. In this article we take the Standard Model with one quark generation (t, b) as the typical case, and will construct full low energy effective action $S_{\text{eff}}$ which stems from the "integration" of $t_>$, $t_<$ and $b_>$ fields. The effective action $S_{\text{eff}}$ turns out to be given as a finite sum of operators with respect to SU(2) gauge bosons $W_+^\mu$, $W_-^\mu$, $W_3^\mu$, U(1) gauge boson $B_\mu$, Higgs doublet $\phi$ and $b_<$, provided only non-decoupling effects are kept. The non-decoupling effects will be extracted by taking a formal limit of $m_t$, $\mu \to \infty$, keeping the ratio $r = \frac{\mu^2}{m_t^2}$ a small constant ($r \ll 1$).

Here we take the way to directly perform path-integral of fermionic fields, as the original lagrangian is quadratic in the fermionic fields,

$$
L = (\bar{t} \ B \bar{b}) \{i\partial^\mu \gamma_\mu - \begin{pmatrix} m_t & 0 \\ 0 & m_b \end{pmatrix} \}
+ \left[ \left( g W_\mu^i \sigma^i \right) \frac{1}{2} + \frac{1}{6} g' B_\mu \right] \gamma^\mu L + g' \left( \begin{array}{c} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \end{array} \right) B_\mu \gamma^\mu R \right) \}
(13)

$$

where we have ignored the Higgs doublet $\phi$ except for the role of giving quarks their masses (it will be finally recovered in the discussion below), and $L$ and $R$ are chiral-projection operators. We perform path-integrals of $t_>$, $b_>$ and $t_<$ fields, though we will eventually see that the contribution of $t_<$ integration is less important.

The effective action $S_{\text{eff}}$ is given as

$$
S_{\text{eff}} = S_{\text{eff}(>)} + S_{\text{eff}(<)}
$$

(14)

where $S_{\text{eff}(>)}$ and $S_{\text{eff}(<)}$ denote the contributions due to the path-integrals of $t_>$, $b_>$ and $t_<$, respectively. First, $S_{\text{eff}(>)}$ is given as the $\text{Tr ln}$ of an operator in the quadratic form of $t_>$, $b_>$ and $t_<$ fields in the original lagrangian, which reads, after ignoring a constant term,

$$
S_{\text{eff}(>)} \simeq i \text{Tr}(>) \sum_{n=2}^4 \frac{1}{n} \left( \frac{1}{i \partial^\nu \gamma_\mu - m_t} \frac{0}{i \partial^\nu \gamma_\mu - m_b} \right) \left[ i \left( g W_\mu^i \sigma^i \right) \frac{1}{2} + \frac{1}{6} g' B_\mu \right] \gamma^\mu L 
+ g' \left( \begin{array}{c} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \end{array} \right) B_\mu \gamma^\mu R \right) \}
(15)

$$

where $\text{Tr}(>)$ means that in the fermion propagators, $S_F(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \text{ etc.}$, only higher momentum part $|k| \geq \mu$ should be integrated. Strictly speaking, this condition may
not be satisfied by all propagators in the internal lines, if the momenta of external gauge bosons are taken into account. But as long as the external momenta are low, \( |p^2| \leq \mu^2 \), the violation of the condition will occur only at a small shell of internal momentum \( k^\mu \), and we can just perform the integral of \( k^\mu \) at the range \( |k| \geq \mu \). It should be noticed that the summation of \( n \) ends up at \( n = 4 \) (gauge boson 4-point function). This is because the operators with \( n > 4 \) is suppressed by \( 1/m_t \) and/or \( 1/\mu \) and disappears at the formal limit, i.e. decoupled from the low energy theory. Further (Taylor-) expanding the gauge fields around a space-time point, we will get operators with derivatives. Let the number of the derivative be \( m \). Then actually only operators satisfying \( n + m \leq 4 \) survive under the formal limit as the candidate of possible non-decoupling contributions. So we end up with a finite number of local operators, as we expected. Among the remaining ones, some operators are accompanied by divergent coefficient functions and just contribute to the renormalizations of existing operators in the original lagrangian. What we are really interested in are the operators which do not exist in the original lagrangian whose coefficient functions should have all informations of the genuine non-decoupling contributions and are all finite. The problem of breakdown of local gauge invariance due to U.V. cutoff \( \Lambda \), therefore, does not appear in these coefficients. In the case of \( n = 2 \) (gauge boson 2-point functions), it is known that there exist three independent operators with finite coefficients which are related to the S, T and U parameters \([4]\). In the case of gauge boson 3-point functions \((n = 3)\) it turns out that there appears additional four independent operators to describe the non-decoupling effects \([10],[11]\). As for the case of \( n = 4 \), we have not worked out how many additional operators exist. For instance, \( \Delta \rho \) is essentially same as the T parameter, and is understood as one example of such finite coefficient functions of \( S_{eff} \). Next, the path-integral of \( t_\langle \rangle \) can be done by completing a square in the relevant part of the original lagrangian. The result reads as

\[
S_{eff(\langle \rangle)} \simeq i Tr_{\langle \rangle} \sum_{n=2}^{4} \frac{1}{n} \left( \frac{i}{i\partial^\mu \gamma_\mu - m_t} \right) i\left[ \left( \frac{g^2}{2} W_\mu^3 + \frac{g'}{6} B_\mu \gamma^\mu L + \frac{2g'}{3} B_\mu \gamma^\mu R \right) \right]^n \\
+ \int d^4x \left[ \frac{g^2}{2} \bar{b} W^\mu - \gamma^\mu i\gamma^\lambda \partial_\lambda W^\nu + \gamma^\mu \right] L b,
\]

where \( Tr_{\langle \rangle} \) means that in the t quark propagator only lower momentum part \( |k| \leq \mu \) should be integrated. These contributions are less important compared with \( S_{eff(\rangle \rangle)} \), as they are relatively suppressed by the powers of \( r = \mu^2/m_t^2 \), coming from the suppression of t quark propagator by \( 1/m_t \) and the limitation of phase space, \( |k| \leq \mu \). This situation can be explicitly confirmed in the case of \( \Delta \rho \), by comparing \( C_2^{(3a+3b)} \) with either \( C_2^{(3c)} \) or \( C_2^{(4a)} \), corresponding to the contribution shown in the first or the second lines of eq.(16). Thus combining \( S_{eff(\rangle \rangle)} \) with the remaining part of the original action with respect to light fields including b quark (with suitable renormalizations), we get whole theory in low-energies.

The operators which do not exist in the original lagrangian, the genuine effects of heavy particle, can be clearly extracted, once we introduce Higgs doublet field. All quantum
corrections should be written by local gauge invariant operators, if the VEV is replaced by
the original Higgs doublet, and the operators which do not exist in the original lagrangian
should be written as operators with higher mass dimension \((d > 4)\), i.e. by irrelevant
operators. So these operators have automatically finite coefficient functions and can be
clearly separated from marginal or relevant operators with \(d \leq 4\). One problem of this
project is that infinite number of higher dimensional operators may potentially participate
in describing the non-decoupling effects of t quark. This is because once we have a gauge
invariant operator we may trivially extend it by putting powers of \(\phi^\dagger \phi\) (\(\phi\: Higgs\ doublet\)).
These extended operators may be accompanied by additional inverse powers of \(v^2\), the VEV
of the Higgs field. After replacing \(\phi\) by its VEV, however, \(\phi^\dagger \phi/v^2\) just gives 1 and all of these
operators play the role to describe the non-decoupling effects on an equal footing. (In new
physics theories where decoupling of heavy particles are expected, such higher dimensional
operators are suppressed by the inverse powers of a new gauge invariant large mass scale,
say \(M\), as \(\phi^\dagger \phi/M^2\) and the problem does not exist). This problem may be evaded if we
utilize non-linear realization of the Higgs doublet, \(\phi = U(0, v)\) with \(U = \exp(iG^a \sigma^a/2v)\). For
simplicity we have ignored the physical Higgs field \(h_0\), but it will be recovered shortly. Now
powers of \((\phi^\dagger \phi)/v^2\) just gives 1 and the problem disappears. Thus our procedure is just to
replace the quark mass matrix as
\[
\begin{pmatrix}
m_t & 0 \\
0 & m_b
\end{pmatrix}
\rightarrow U \begin{pmatrix}
m_t & 0 \\
0 & m_b
\end{pmatrix} R + \begin{pmatrix}
m_t & 0 \\
0 & m_b
\end{pmatrix} U^\dagger L.
\] (17)

After a change of variables of path-integration utilizing a field dependent local SU(2)_L trans-
formation,
\[
\begin{pmatrix}
t \\
b
\end{pmatrix}_L \rightarrow \begin{pmatrix}
t' \\
b'
\end{pmatrix}_L = U^\dagger \begin{pmatrix}
t \\
b
\end{pmatrix}_L,
\] (18)
in order to eliminate \(U\) from the replaced quark mass matrix, the path-integration in terms of
the new variables yields the final result of the effective action,
\[
S_{eff} \simeq i Tr(>) \sum_{n=2}^{4} \frac{1}{n!} \left\{ \left( \frac{i}{\partial^\mu \gamma_\mu - m_t (1 + h_0 v)} \right) \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix} [U^\dagger (iD_\mu U) \gamma^\mu L
\right. \\
+ \frac{1}{6} g' B^\mu \gamma^\mu L + g' \left( \frac{3}{2} \begin{pmatrix} 0 & \gamma^\mu R \\
0 & -\frac{1}{3} \end{pmatrix} B^\mu \gamma^\mu R \right) \right\}^n
- ("relevant\ operators\),
\] (19)
where \(D_\mu\) is SU(2) covariant derivative, \(D_\mu = \partial_\mu - igW^a_\mu \sigma^a/2\), and the contribution from
\(t_<\) integration has been ignored. In this final result we have recovered the Higgs field \(h_0\)
by a simple replacement, \(m_q \rightarrow m_q (1 + h_0 v)\). Again we may retain only operators, whose
mass dimensions do not exceed 4. As the field \(U\) is dimensionless, the irrelevant operators
(relevant or marginal operators) in the linear realization of Higgs doublet correspond to
operators in which the sum of the numbers of \(U\) field, gauge bosons and derivative exceeds
4 (does not exceed 4). The subtraction of "relevant operators\ in the above expression
should be understood in such a sense. For instance, focusing on the operators in $S_{\text{eff}}$ which is quadratic in SU(2) gauge fields ($n = 2$) and without any derivative, we get two types of operators. One is $Tr[(D_{\mu}U)^{\dagger}D^{\mu}U]$ ($Tr$ should be understood to stand for the trace of matrix elements ). This is a ”relevant” operator and should be subtracted. Another one is $Tr[(U^{\dagger}iD_{\mu}U)\sigma_{3}(U^{\dagger}iD_{\mu}U)\sigma_{3}]$, which is clearly an ”irrelevant ” operator. We can easily check that setting $U = 1$ the operator gives a gauge boson mass-squared term corresponding to $\Delta \rho$, $W^{1\mu}W^{1\mu} + W^{2\mu}W^{2\mu} - W^{3\mu}W^{3\mu}$. The coefficient of this operator is proportional to

$$i \int \frac{d^{4}k}{(2\pi)^{4}} \frac{(m_{t}^{2} - m_{b}^{2})^{2}k^{2}}{(k^{2} - m_{t}^{2})^{2}(k^{2} - m_{b}^{2})^{2}}$$

(20)

which exactly gives the formula of $\Delta \rho$. The presence of the factor $(m_{t}^{2} - m_{b}^{2})^{2}$ is what we expect from a symmetry argument \[12\]. $\Delta \rho$ or T-parameter is an observable which behaves as a 5-plet repr. of SU(2) and needs the $m_{t}^{2} - m_{b}^{2}$ factor, behaving as a triplet of SU(2), twice.

We finally ask a question, to what extent the result of Wilsonian renormalization alone is close to the exact result of $S_{\text{eff}}$. We have seen in this article that if we take a specific observable $\Delta \rho$, the $S_{\text{eff}(>)}$ alone gives almost correct answers in two examples of gauge models. The dominance of $S_{\text{eff}(>)}$, however, is not always true. In the case of $\Delta \rho$, the dominance of higher momentum part (double-line intermediate states) can be understood as a consequence of the fact that $\Delta \rho$ is concerned with gauge boson mass-squared, which of course has mass dimension 2. Therefore if we consider dimensionless observables, we expect to have the same order of logarithmic contributions both from $S_{\text{eff}(>)}$ and from the original action with respect to low momentum part of light particles. For instance the S-parameter \[9\] parametrizes a mixing between the field strengths of SU(2) and U(1) gauge bosons, and is dimensionless observable. In the case of (t, b) contribution, a part of S-parameter (proportional to B-L charge) behaves as $ln(m_{t}/m_{b})$. In our approach, $S_{\text{eff(>)}}$ and the low momentum part of b quark give contributions to the S-parameter behaving as $ln(m_{t}/\mu)$ and $ln(\mu/m_{b})$, respectively, which are comparable to each other.

5. Concluding remarks

In conclusion, we have proposed a systematic method to obtain effective low energy action for light particles in a system where both heavy and light particles coexist with a close relation, like the case of (t, b) system. Thanks to Wilsonian renormalization, we could get effective action with finite number of local operators describing possible non-decoupling effects of heavy particles. The validity of the method was first checked by the computation of the $\rho$ - parameter by use of the derived relevant part of the effective action. Then we constructed full effective action with respect to light particles, which contain all informations of the non-decoupling effects, by directly performing path-integrals of fermionic fields. A useful way to extract genuine contributions of heavy particles, described by irrelevant operators, was discussed to exist once we use non-linear realization of Higgs doublet, though we were
not interested in the Green functions of Higgs field themselves. As the irrelevant operators have finite coefficient functions, we did not suffer from the explicit breakdown of local gauge invariance due to the regularization by momentum cutoff. We have seen that only the path-integration of higher momentum parts of fields, i.e. Wilsonian renormalization alone, always gives practically correct answer for $S_{\text{eff}}$. In this work we did direct path-integration only for fermionic fields. If we also perform path-integrations of the higher momentum parts of gauge bosons and Higgs field, we will get an effective action including operators of fermionic fields only, which are out of scope of the present work. We hope that the procedure proposed in this paper is useful in deriving full low-energy effective action with finite number of local operators (healthy action !) starting from an original lagrangian, in any system with both heavy and light particles.

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