Fields with automorphism and valuation

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1 Introduction

The question has been of interest for decades: If a model-complete theory $T$ is augmented with axioms for an automorphism $\sigma$ of its models, does the resulting theory $T_\sigma$ have a model-companion?

It does have, when $T$ is the theory $\text{ACF}$ of algebraically closed fields. The model companion of $T_\sigma$ in this case is $\text{ACFA}$, studied by Macintyre [14] and Chatzidakis and Hrushovski [6] and others. However, $T_\sigma$ is not companionable when $T$ is $\text{ACFA}$ itself [13].

A more general result, established by Kikyo [12], is that if $T_\sigma$ is companionable, and $T$ is dependent, then $T$ must also be stable. In particular then, $T_\sigma$ cannot be companionable when $T$ is the theory $\text{ACVF}$ of algebraically closed valued fields in the signature of fields with a predicate for a valuation ring. Note that an automorphism $\sigma$ of a valued field induces an automorphism $\sigma_v$ of the the value group $\Gamma$ and an automorphism $\bar{\sigma}$ of the residue field. Then $T_\sigma$ is companionable when $T$ is the model companion of the theory of any of the following classes of valued fields:

1) valued $D$-fields [21],
2) isometric valued difference fields, where $\sigma_v(\gamma) = \gamma$ for all $\gamma$ in $\Gamma$ [3, 2],
3) contractive valued difference fields, where $\sigma_v(\gamma) > n\gamma$ for all positive $\gamma$ in $\Gamma$ and $n$ in $\omega$ [1],
4) multiplicative valued fields, where $\sigma_v(\gamma) = \rho\gamma$ for all $\gamma$ in $\Gamma$, for a certain constant $\rho$ [15].

Moreover,

5) $T_\sigma \cup T_v$ is companionable when $T$ is $\text{ACVF}$ and $T_v$ is a companionable theory of ordered abelian groups equipped with an automorphism [7] (in this case...
\((\Gamma, \sigma_v) \models T_v)\).

The corresponding model companion in each of the five cases satisfies an analogue of Hensel’s lemma for \(\sigma\)-polynomials (see [1, Definition 4.2]).

In this paper we consider the theory \(\text{FAV}\) of a valued field equipped with an automorphism of the field alone. There is no required interaction of valuation and the automorphism: the automorphism need not fix the valuation ring (setwise). The similar case of a differential field with an automorphism of the field alone was treated in [16]. Our main theorem, Theorem 11, is that \(\text{FAV}\) has a model companion, \(\text{FAV}^*\).

There is an obvious candidate for \(\text{FAV}^*\), since \(\text{FAV}\) is included in the union of two model-complete theories, namely \(\text{ACFA}\) and \(\text{ACVF}\). However, we show, as Theorem 12, that \(\text{ACFA} \cup \text{ACVF}\) is not model complete.

Our paper is organized as follows. In §2 we give axioms of \(\text{FAV}\). In §3 preliminaries about companionable theories are explained. Then in §4, Theorem 8 establishes a geometric axiomatization of \(\text{ACFA}\). Using this, in §5 we prove Theorems 11 and 12.

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2 Fields with an automorphism and a valuation

A signature sufficient for a first-order axiomatization of fields with an automorphism and a valuation is the signature \( \{+, -, \times, 0, 1\} \) of fields, augmented with

1) a singulary operator \( \sigma \) for the automorphism and
2) a singulary predicate \( \in \mathcal{O} \) for membership in the valuation ring.

We shall write the last two symbols after their arguments. The fields with an automorphism and a valuation are then axiomatized by the field axioms, along with axioms

\[
(x + y)^\sigma = x^\sigma + y^\sigma, \quad (x \cdot y)^\sigma = x^\sigma \cdot y^\sigma, \quad \exists y \ y^\sigma = x
\]

for a surjective endomorphism (which for a field is an automorphism), and axioms

\[
0 \in \mathcal{O},
\]
\[
x \in \mathcal{O} \land y \in \mathcal{O} \Rightarrow -x \in \mathcal{O} \land x + y \in \mathcal{O} \land x \cdot y \in \mathcal{O},
\]
\[
\exists y \ (x \notin \mathcal{O} \Rightarrow x \cdot y = 1 \land y \in \mathcal{O})
\]

for a valuation ring. It will be our habit, as here, to suppress outer universal quantifiers. For convenience, we introduce a singulary predicate \( \in \mathcal{M} \) for membership in the unique maximal ideal of the valuation ring. This means requiring

\[
x \in \mathcal{M} \iff \exists y \ (x = 0 \lor (x \cdot y = 1 \land y \notin \mathcal{O})),
\]

or equivalently

\[
x \notin \mathcal{M} \iff \exists y \ (x \cdot y = 1 \land y \in \mathcal{O}). \quad (1)
\]
Because both the new predicate and its negation can thus be given existential definitions, use of the predicate does not affect the existence of a model-companion of the theory being axiomatized [17, Lem. 1.1, p. 427]. Let us denote this theory by FAV; officially, its signature is

$$\{+, -, \times, 0, 1, \sigma, \in \mathcal{O}, \in \mathcal{M}\}.$$  

For later use, we note that the identity

$$x \notin \mathcal{O} \iff \exists y \ (x \cdot y = 1 \land y \in \mathcal{M}) \quad (2)$$

holds in FAV.

For a valuation as such, we can introduce a new sort having signature $$\{+, 0, \infty, >\},$$ so that the valuation is a surjective function val from the original sort to the new sort that satisfies also

$$\text{val}(x) + \text{val}(y) = \text{val}(x \cdot y),$$  
$$0 = \text{val}(1),$$  
$$\infty = \text{val}(0),$$  
$$\text{val}(x) > \text{val}(y) \iff \exists z \ (y \cdot z = 1 \land x \cdot z \in \mathcal{M}).$$  

These rules ensure that the new sort is an ordered additive abelian group—the value group—with an additional element $$\infty$$ that is greater than all others, and

$$\infty + x = \infty = x + \infty.$$  

Also, val restricts to a homomorphism from the multiplicative group of units of the field onto the value group, and the kernel of this homomorphism is $$\mathcal{O} \setminus \mathcal{M},$$ which is the group of units
of the valuation ring $\mathfrak{O}$. Moreover,

$$\text{val}(x) \geq 0 \iff x \in \mathfrak{O},$$
$$\text{val}(x) > 0 \iff x \in \mathfrak{M},$$
$$\text{val}(x) < 0 \iff x \notin \mathfrak{O}.$$ 

As with the maximal ideal $\mathfrak{M}$, so with the value group, its official status does not matter for our purposes. Officially we shall not use the value group, and so we may write a typical model of $\text{FAV}$ as $(K, \sigma, \mathfrak{O})$. However, the value group may be useful for thinking things through.

3 Model-companions and -completions

The (Robinson) diagram of a structure $\mathfrak{A}$ in a signature $\mathcal{I}$ is the theory $\text{diag}(\mathfrak{A})$, in the signature $\mathcal{I}(A)$ (where $A$ is the domain of $\mathfrak{A}$), of structures in which $\mathfrak{A}$ embeds. This means $\text{diag}(\mathfrak{A})$ is axiomatized by all of the quantifier-free sentences of $\mathcal{I}$ with parameters from (the underlying set $A$ of) $\mathfrak{A}$ that are true in $\mathfrak{A}$. Thus $\text{diag}(\mathfrak{A})$ is also axiomatized simply by the atomic and negated atomic sentences of $\mathcal{I}(A)$ that are true in $\mathfrak{A}$.

When it exists, a model-companion of a theory $T_0$ is a theory $T_1$ in the same signature such that

1) for each $i$, every model of $T_i$ embeds in a model of $T_{1-i}$, that is, $T_{1-i} \cup \text{diag}(\mathfrak{A})$ is consistent whenever $\mathfrak{A}$ is a model of $T_i$; and

2) $T_1$ is model-complete, that is, $T_1 \cup \text{diag}(\mathfrak{A})$ is complete whenever $\mathfrak{A} \models T_1$. 
The model-companion of a theory is unique when it exists. It was “introduced by Barwise, Eklof, Robinson, and Sabbagh in 1969” [5, p. 609], these four logicians being known collectively as Eli Bers [10, p. 410]. The model-companion generalizes an earlier notion of Robinson [20, §4.3, p. 109]: the theory $T_1$ is a model-completion of $T_0$ in case $T_0 \subseteq T_1$ and $T_1 \cup \text{diag}(\mathcal{A})$ is consistent and complete whenever $\mathcal{A} \models T_0$. If $T_0$ has the model-companion $T_1$, then $T_1$ is a model-completion of $T_0$ just in case $T_0$ has the amalgamation property, that is, two models having a common submodel have a common supermodel (this is an exercise in Hodges [10, §8.4, exer. 9, p. 390] attributed to Eli Bers [8, Lem. 2.1, p. 254]).

We say that a theory is inductive if every union of a chain of models is a model. Robinson’s name for such a theory was $\sigma$-persistent; but since we are already using the symbol $\sigma$ for a field automorphism, we prefer the simpler term for the kind of theory in question. By the Chang–Łoś–Suszko Theorem [10, 6.5.9, p. 297], A theory $T$ is inductive if and only if it is precisely the theory $T_{\forall\exists}$ axiomatized by the $\forall\exists$ (or $\forall_2$) consequences of $T$.

By a system we shall mean a (finite) conjunction of atomic and negated atomic formulas. For theories, having a model-companion or -completion means having an appropriate condition for when systems over a given model have solutions in a larger model. We recall first Robinson’s equivalent formulation of when inductive theories have model-completions; we review also the proof, for the sake of the variations that we shall state and use.

**Theorem 1** (Robinson [20, §5.5, p. 128]). *For an inductive theory $T$ to have a model-completion, a sufficient and necessary condition is that, for every system $\varphi(x, z)$ in the signa-
ture of $T$, there is a quantifier-free formula $\vartheta(x, y)$ in that signature such that, for all models $\mathcal{M}$ of $T$, for all tuples $a$ of parameters from $M$ having the same length as $x$, the following conditions are equivalent:

(i) $\varphi(a, z)$ is soluble in some model of $T \cup \text{diag}(\mathcal{M})$,

(ii) $\vartheta(a, y)$ is soluble in $\mathcal{M}$ itself.

When such $\vartheta$ do exist, then the model-completion of $T$ is the theory $T^*$ axiomatized by the sentences

$$\exists y \vartheta(x, y) \Rightarrow \exists z \varphi(x, z),$$

along with axioms for $T$ itself.

The sentence (4) is equivalent to the $\forall \exists$ sentence $\exists z (\vartheta(x, y) \Rightarrow \varphi(x, z))$, outer universal quantifiers being suppressed.

Proof of Robinson’s theorem. For the necessity of the given condition, suppose $T$ has the model-completion $T^*$. For every system $\varphi(x, z)$ in the signature of $T$, for every model $\mathcal{M}$ of $T$, for every tuple $a$ of parameters from $M$ such that (i) holds, since every model of $T$ embeds in a model of $T^*$, we can conclude from the completeness of $T^* \cup \text{diag}(\mathcal{M})$ that

$$T^* \cup \text{diag}(\mathcal{M}) \vdash \exists z \varphi(a, z).$$

By Compactness and the Lemma on Constants [10, 2.3.2, p. 43], there is a quantifier-free formula $\vartheta_{(\mathcal{M}, a)}(x, y)$ of the signature of $T$ such that

$$T^* \vdash \exists y \vartheta_{(\mathcal{M}, a)}(x, y) \Rightarrow \exists z \varphi(x, z).$$

By Compactness again, for the given system $\varphi$, there is a disjunction $\vartheta$ of finitely many of the formulas $\vartheta_{(\mathcal{M}, a)}$ such that for
every model $\mathcal{M}$ of $T$, for every tuple $a$ of parameters from $M$ having the length of $x$, if (i), then (ii). If conversely (ii), then $\varphi(a, z)$ is soluble in every model of $T^* \cup \text{diag}(\mathcal{M})$, by (5); but such a model is a model of $T \cup \text{diag}(\mathcal{M})$, and so (i) holds.

For the sufficiency of Robinson's condition, we first show that every model $\mathcal{M}$ of $T$ embeds in a model of the theory $T^*$ having the axioms (4) in addition to those of $T$. Here we shall use (ii) implies (i), but not the converse. For every system $\varphi(x, z)$ in the signature of $T$, for all $a$ and $b$ from $M$ such that $\mathcal{M} \models \vartheta(a, b)$, for some model $\mathcal{N}$ of $T \cup \text{diag}(\mathcal{M})$, the sentence

$$\exists z \varphi(a, z)$$

is true in $\mathcal{N}$. This sentence being existential and thus preserved in larger models, by Zorn's Lemma and inductivity of $T$, we can move the last of the three bold quantifiers to the front: in some model $\mathcal{M}'$ of $T \cup \text{diag}(\mathcal{M})$, for all systems $\varphi(x, z)$, for all $a$ and $b$ from $M$, the sentence

$$\exists z (\vartheta(a, b) \Rightarrow \varphi(a, z))$$

is true in $\mathcal{N}$. Now we can form the chain

$$\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}'' \subseteq \cdots,$$

whose limit is a model of $T^*$. Thus $T^* \cup \text{diag}(\mathcal{M})$ is consistent.

We now show $T^* \cup \text{diag}(\mathcal{M})$ is complete by induction on the complexity of sentences. The theory is complete with respect to existential sentences, namely $\exists_1$ sentences, since (i) implies (ii). Indeed, suppose the sentence $\exists z \varphi(a, z)$ is true in some model of $T^* \cup \text{diag}(\mathcal{M})$, where $\varphi$ is quantifier-free in the signature of $T$, and $a$ is from $M$. Since $\varphi$ is a disjunction of systems, it is enough to assume $\varphi$ itself is a system. Since
\( T \subseteq T^* \), we have (i) and therefore (ii). Since the formula \( \vartheta \) here is quantifier-free, the sentence \( \vartheta(a, b) \) belongs to diag(\( M \)) for some \( b \) from \( M \). Since the sentence (4) is an axiom of \( T^* \), we conclude
\[
T^* \cup \text{diag}(M) \vdash \exists y \, \varphi(a, y).
\]
Thus \( T^* \cup \text{diag}(M) \) is complete with respect to \( \exists_1 \) sentences.

Suppose now that for some positive integer \( n \), for all models \( M \) of \( T \), the theory \( T^* \cup \text{diag}(M) \) is complete with respect to \( \exists_n \) sentences. For an arbitrary model \( M \) of \( T \), let \( \varphi(x, z) \) be a \( \forall_n \) formula, and let \( a \) be a tuple of parameters from \( M \) such that the \( \exists_{n+1} \) sentence \( \exists z \, \varphi(a, z) \) is true in some model \( N \) of \( T^* \cup \text{diag}(M) \). Then for some \( c \) from \( N \), the sentence \( \varphi(a, c) \) is true in \( N \). Since \( N \models T \), by inductive hypothesis we have
\[
T^* \cup \text{diag}(M) \vdash \exists z \, \varphi(a, z).
\]
By Compactness, there is a quantifier-free formula \( \psi(x, y) \) such that
\[
N \models \exists y \, \psi(a, y), \quad T^* \vdash \exists y \, \psi(x, y) \Rightarrow \exists z \, \varphi(x, z).
\]
Since again \( N \) is a model of \( T^* \cup \text{diag}(M) \), which is complete with respect to existential sentences, we can conclude
\[
T^* \cup \text{diag}(M) \vdash \exists z \, \varphi(a, z).
\]
Thus \( T^* \cup \text{diag}(M) \) is complete with respect to \( \exists_{n+1} \) sentences.

By induction, \( T^* \cup \text{diag}(M) \) is complete. \( \square \)

A model \( M \) of a theory \( T \) is **existentially closed** if \( T \cup \text{diag}(M) \) is complete with respect to existential formulas. For Theorem 1 then, the proof of completeness of \( T^* \cup \text{diag}(M) \) is a generalization of the proof of the following.
Porism 2 (Robinson’s Test [20, 4.2.1, p. 92]). A theory $T$ is model-complete, provided that all of its models are existentially closed.

For an inductive theory $T$ with a model $\mathcal{M}$, for every system $\varphi(x, z)$ in the signature of $T$, for all $a$ from $M$, for some model $\mathcal{N}$ of $T \cup \text{diag}(\mathcal{M})$, if $T \cup \text{diag}(\mathcal{M}) \cup \{\exists z \varphi(a, z)\}$ is consistent, then $\varphi(a, z)$ is solved in $\mathcal{N}$. By the method of the proof that every model of $T$ embeds in a model of $T^*$, we can again put the last bold quantifier in front and go on to obtain the following.

Porism 3. Every model of an inductive theory embeds in an existentially closed model.

The two porisms lead to the following result, now standard [5, 3.5.15, p. 198].

Theorem 4 (Eklof and Sabbagh [8, 7.10–3, pp. 286–8]). An inductive theory $T$ has a model companion $T^*$ if and only if the models of $T^*$ are precisely the existentially closed models of $T$.

Robinson used Theorem 1 to prove that the theory $\text{DF}_0$ of fields of characteristic 0 with a derivation had a model-completion, $\text{DCF}_0$. But there are simpler practical approaches to obtaining model-completions; simpler still, if all we want are model-companions. First of all, in the proof of Theorem 1, we did not really need to extract the finite disjunction $\vartheta$ from all of the formulas $\vartheta_0(\mathcal{M}, a)$. Moreover, the proof that models of $T$ embed in models of $T^*$ did not require the formulas $\vartheta$ to be quantifier-free. Neither is this required for the observation that the models of $T^*$ are the existentially closed models of $T$. Thus we have the following.
**Porism 5.** For an inductive theory $T$ to have a model-
completion, a sufficient and necessary condition is that, for
every system $\varphi(x, z)$ in the signature of $T$, there is a set $\Theta$
of quantifier-free formulas $\vartheta(x, y)$ in that signature such that,
for all models $M$ of $T$, for all tuples $a$ of parameters from
$M$ having the same length as $x$, the following conditions are
equivalent:

(i) $\varphi(a, z)$ is soluble in some model of $T \cup \text{diag}(M)$,
(ii) $\vartheta(a, y)$ is soluble in $M$ itself for some $\vartheta$ in $\Theta$.

When such $\Theta$ do exist, then the model-
completion of $T$ is the
theory $T^*$ axiomatized by the sentences (4), where $\vartheta$ ranges
over $\Theta$, along with axioms for $T$ itself. If the formulas in the
sets $\Theta$ are not necessarily quantifier-free, the theory $T^*$ is still
the model-companion of $T$.

Simpler axiomatizations than Robinson’s for $\text{DCF}_0$ were
found by showing that the axioms need not explicitly concern
all systems [4, 19]. The general observation can be formulated
as follows.

**Porism 6.** Theorem 1 and its Porism 5 still hold, even if $\varphi$
is constrained to range over a collection of systems containing,
1) for each system $\psi(x, u)$ in the signature of $T$,
2) for each model $M$ of $T$,
3) for each tuple $a$ of parameters from $M$,
a system $\varphi(x, u, v)$ such that, if $\exists u \psi(a, u)$ is consistent with
$T \cup \text{diag}(M)$, then so is $\exists u \exists v \varphi(a, u, v)$, and

$$T \cup \text{diag}(M) \vdash \varphi(a, u, v) \Rightarrow \psi(a, u).$$  \hspace{1cm} (6)

We may refer to $\varphi(x, u, v)$ as a **refinement** of the system
$\psi(x, u)$. We shall apply Porism 6 when $T$ is $\text{FAV}$ or, as in the
next section, the theory of difference fields.
4 Difference fields

A **difference field** is a field equipped with an automorphism. As was observed in the introduction, the theory of difference fields in the signature \{+,-,\times,0,1,\sigma\} has a model-companion, called ACFA. Perhaps any axiomatization of ACFA can be made to serve present purposes; we shall derive the one that we shall use from Theorem 8, which is in the style of [16, Thm 3.1, p. 1337].

The following lemma will be the reason for condition (7) in Theorem 8. For notational economy, our set \( \omega \) of natural numbers is the set of von Neumann natural numbers, where

\[ n = \{0,\ldots,n-1\} = \{i: i < n\}. \]

**Fact 7.** Suppose \((K,\sigma)\) is a difference field, and \(I\) is a prime ideal of the polynomial ring \(K[X_j: j < n]\), and \(\tau\) is an embedding of \(m\) in \(n\). Write \(X_j + I\) as \(a_j\) whenever \(j < n\). For \(\sigma\) to extend to an automorphism of a field that includes \(K[a]\) so that \(a_i^\sigma = a_{\tau(i)}\) whenever \(i < m\), it is necessary and sufficient that

\[ f(a_i: i < m) = 0 \iff f(a_{\tau(i)}: i < m) = 0 \]

for all \(f\) in \(K[X_i: i < m]\).

**Theorem 8.** A difference-field \((K,\sigma)\) is existentially closed among all difference fields if and only if,

1) for all \(m\) and \(n\) in \(\omega\) such that \(m \leq n\),

2) for every injective function \(\tau\) from \(m\) into \(n\),

3) for every finite subset \(I_0\) of \(K[X_j: j < n]\), if \(I_0\) generates a prime ideal \((I_0)\) of \(K[X_j: j < n]\), and

\[ \{f(X_{\tau(i)}: i < m): f \in (I_0) \cap K[X_i: i < m]\} = (I_0) \cap K[X_{\tau(i)}: i < m], \quad (7) \]
then the system

\[
\bigwedge_{f \in I_0} f = 0 \land \bigwedge_{i < m} X_i^\sigma = X_{\tau(i)}
\]  

(8)

has a solution in \( K \) (the case \( m = 0 \) ensures that \( K \) is algebraically closed).

Proof. We refine an arbitrary system of difference equations and inequations as follows. Over a difference field \((K, \sigma)\), suppose a system has a solution \((a_i : i < k)\) from some larger model. Whenever \( i < j < k \), we may assume that \( a_i \neq a_j \) and that the system has the inequation \( X_i \neq X_j \) as one of its conjuncts. We obtain a refinement having the form (8) as follows.

1. In non-constant terms, repeatedly make the replacements of \((t + u)^\sigma,\ (-t)^\sigma,\ (t \cdot u)^\sigma\) with \(t^\sigma + u^\sigma,\ -t^\sigma,\ t^\sigma \cdot u^\sigma\) respectively, until \( \sigma \) is applied only to variables and constant terms.

2. For every atomic or negated atomic formula \( \varphi \) of the system that is not of the form \( X^\sigma = Y \), but in which \( X^\sigma \) appears as an argument, replace that argument with a new variable \( Y \), and introduce the new equation \( X^\sigma = Y \).

3. If for some \( i \) less than \( k \), there is not already an equation of the form \( X_i^\sigma = Y \), then introduce such an equation, \( Y \) being a new variable.

4. Replace any polynomial inequation \( f \neq g \) with

\[
(f - g) \cdot X = 1,
\]

where \( X \) is a new variable.
After indexing the new variables appropriately, we have that
1) for some \( m \) and \( n \) in \( \omega \) such that \( m \leq n \),
2) for some function \( \tau \) from \( m \) into \( n \),
3) for some finite subset \( I_0 \) of \( K[X_j : j < n] \),
our system has the form of (8). (It may be that some of the
hidden parameters are part of compound terms that involve
\( \sigma \); but such terms can just be understood as standing for the
appropriate elements of \( K \).) If \( \tau \) is not injective, then the
system must have equations \( X_i^\sigma = X_\ell \) and \( X_j^\sigma = X_\ell \), where
\( k \leq i < j < m \); but these equations imply \( X_i = X_j \), and so
we can replace \( X_j \) throughout with \( X_i \). Thus we may assume
\( \tau \) is injective. In case the ideal generated by \( I_0 \) is not prime,
still, for some \((a_i : k \leq i < n)\) in the larger difference field,
the new system has the solution \((a_i : i < n)\), and we can then
add enough equations \( f(X_j : j < n) = 0 \) that are satisfied by
\((a_i : i < n)\) so that \( I_0 \) becomes a set of generators of a prime
ideal \( \mathfrak{P} \), and \((a_i : i < n)\) is a generic point over \( K \) of the zero-set
of \( \mathfrak{P} \). In this case (7) is satisfied. For every solution \((b_i : i < n)\)
of the latest system, \((b_i : i < k)\) solves the original system.
Thus if \((K, \sigma)\) meets the given conditions, it is existentially
closed as a difference field.

Conversely, under the given conditions, every system of the
form (8) is indeed consistent with \((K, \sigma)\), by Fact 7: if \( a \) is a
generic zero of \( I_0 \), we can extend \( \sigma \) to an isomorphism from
\( K(a_i : i < m) \) to \( K(a_\tau(i) : i < m) \), and then to an automor-
phism of a field including \( K(a) \). In this way, \( a \) solves (8), so
this system must have a solution in \( K \), if \((K, \sigma)\) is existentially
closed as a difference field. \( \Box \)

If we did not already know that ACFA existed, the foregoing
theorem would prove it by Porism 6, since the conditions that
(8) must satisfy are first-order. This is so, because of the
existence of appropriate bounds on degrees of polynomials, as established in [22]. In particular, for all $n$ and $r$ in $\omega$, there are bounds $s$ and $t$ in $\omega$ such that, for all fields $K$, for all $m$ in $\omega$, for every ideal $I$ of $K[X_j : j < n]$ generated by a set $\{f_i : i < m\}$, each $f_i$ having degree $r$ or less,

1) the primeness of the ideal can be established by showing

$$gh \in I \& g \notin I \implies h \in I$$

for all polynomials $g$ and $h$ in $K[X_j : j < n]$ having degree $s$ or less, and

2) membership in $I$ by polynomials like $gh$ having degree $2s$ or less is established by polynomials of degree $t$ or less, in the sense that, if indeed $gh \in I$, then $gh = \sum_{i < m} g_i \cdot f_i$ for some $g_i$ having degree $t$ or less.

Because ACF admits full elimination of quantifiers, ACFA is the model-completion of the theory of difference fields that are algebraically closed as fields (compare to the last sentence in Porism 5).

5 A model-completion

We now consider the class of models $(K, \sigma, \mathcal{D})$ of $\text{FAV}$ such that

$$\exists x \, x \notin \mathcal{D}$$

and,

1) for all $m$ and $n$ in $\omega$ such that $m \leq n$,

2) for every injective function $\tau$ from $m$ into $n$,

3) for every finite subset $I_0$ of $\mathcal{D}[X_j : j < n]$,

4) for all subsets $\lambda$ of $n$ and $\kappa$ of $\lambda$,

if
a) $I_0$ generates a prime ideal $(I_0)$ of $K[X_j: j < n]$ such that the condition (7) in Theorem 8 holds, and

b) when $S$ is the ring $\mathfrak{O}[I_0 \cup \{X_\ell: \ell \in \lambda\}]$, the ideal of $S$ generated by the set $\mathfrak{M} \cup I_0 \cup \{X_k: k \in \kappa\}$ is proper, that is,

$$\left(\mathfrak{M} \cup I_0 \cup \{X_k: k \in \kappa\}\right)S \not\subseteq S,$$

then $K$ contains a common solution to the system (8) in Theorem 8 and the system

$$\bigwedge_{\ell \in \lambda} X_\ell \in \mathfrak{O} \land \bigwedge_{k \in \kappa} X_k \in \mathfrak{M}. \quad (10)$$

The case $m = 0 = \lambda$ ensures that $K$ is algebraically closed.

As the existentially closed difference-fields, characterized by Theorem 8, are just the models of a certain theory $\text{ACFA}$, so the models of $\text{FAV}$ just described are the models of a certain theory, which we shall call $\text{FAV}^*$. In particular, the condition (9) is first-order. Indeed, this condition means there are no $g_f$ and $h_k$ in $S$ such that

$$\sum_{f \in I_0} g_f \cdot f + \sum_{k \in \kappa} h_k \cdot X_k \equiv 1 \pmod{\mathfrak{M}}. \quad (11)$$

The ring $S$ is, for some subset $I_1$ of $I_0$, isomorphic to the quotient of the polynomial ring $\mathfrak{O}[[Y_f: f \in I_1] \cup \{X_\ell: \ell \in \lambda\}]$ by an element of bounded degree. We can also work over the residue field $\mathfrak{O}/\mathfrak{M}$, instead of $\mathfrak{O}$. Thus, by [22], for we can bound the degrees of the $g_f$ and $h_k$ that would make (11) true.

**Lemma 9.** Every model of $\text{FAV}$ embeds in a model of $\text{FAV}^*$.

**Proof.** Let $(K, \sigma, \mathfrak{O})$ be a model of $\text{FAV}$ such that,
1) for some $m$ and $n$ in $\omega$, where $m \leq n$,
2) for some injective function $\tau$ from $m$ into $n$,
3) for some finite subset $I_0$ of $\mathfrak{O}[X_j : j < n]$,
4) for some sets $\lambda$ and $\kappa$, where $\kappa \subseteq \lambda \subseteq n$,

we have that

a) $I_0$ generates a prime ideal $(I_0)$ of $K[X_j : j < n]$ such that the condition (7) in Theorem 8 holds, and

b) when $S$ is the ring $\mathfrak{O}[I_0 \cup \{X_\ell : \ell \in \lambda\}]$, then (9) holds.

We already know, as in the proof of Theorem 8, that the system (8) has a solution $a$ in a difference field $(L, \tilde{\sigma})$ of which $(K, \sigma)$ is a substructure; and we may require $a$ to be a generic solution of the field-theoretic part

$$\bigwedge_{f \in I_0} f = 0$$

of the system. We now show that $L$ has a valuation ring $\tilde{\mathfrak{O}}$ such that

$$K \cap \tilde{\mathfrak{O}} = \mathfrak{O}$$

and $a$ solves (10), that is,

$$\bigwedge_{\ell \in \lambda} a_\ell \in \tilde{\mathfrak{O}} \land \bigwedge_{k \in \kappa} a_k \in \tilde{\mathfrak{M}}. \quad (12)$$

We can do this by refining the proof of Chevalley’s theorem on extending valuations (for which see [9, Thm 3.1.1, p. 57]), or simply by using a refinement [23, Thm 5, p. 12] of the theorem itself. By this refinement, the sub-ring $S = \mathfrak{O}[I_0 \cup \{X_\ell : \ell \in \lambda\}]$ of $K(X_j : j < n)$ has a prime ideal $\mathfrak{P}$ that includes the proper ideal generated by $\mathfrak{M} \cup I_0 \cup \{X_k : k \in \kappa\}$; therefore some valuation ring $\mathfrak{O}^*$ of $K(X_j : j < n)$ with maximal ideal $\mathfrak{M}^*$ satisfies

$$S \subseteq \mathfrak{O}^*, \quad \mathfrak{P} = \mathfrak{M}^* \cap S.$$
In particular,

\[ \{X_\ell: \ell \in \lambda\} \subseteq \mathcal{O}^*, \ I_0 \cup \{X_k: k \in \kappa\} \subseteq \mathcal{M}^*, \ \mathcal{M}^* \cap \mathcal{O} = \mathcal{M}. \]

Now we can understand \( \mathcal{O}^*/(I_0)\mathcal{O}^* \) as a valuation ring of \( K(a) \). By Chevalley’s Theorem, we can extend this valuation ring to a valuation ring \( \tilde{\mathcal{O}} \) of \( L \). In this case (12) holds. \( \square \)

**Lemma 10.** \( \text{FAV}^* \) is model complete.

**Proof.** We proceed as in the proof of Theorem 8. Over a model of \( \text{FAV} \), supposing a system of atomic and negated atomic formulas has a solution \( (a_i: i < k) \) from some larger model, we transform the system into the conjunction of a system of the form (8) and a system of the form (10). We proceed as before, but now, since formulas \( f \in \mathcal{O} \) and \( f \in \mathcal{M} \) and their negations may appear, we can eliminate negations by applying (1) and (2), and we can replace \( f \in \mathcal{O} \) and \( f \in \mathcal{M} \) themselves with \( X \in \mathcal{O} \wedge f = X \) and \( X \in \mathcal{M} \wedge f = X \) respectively, where \( X \) is a new variable. \( \square \)

**Theorem 11.** \( \text{FAV}^* \) is the model companion of \( \text{FAV} \) and the model completion of the theory of models of \( \text{FAV} \) whose fields are algebraically closed.

**Proof.** The first part is the content of Lemmas 9 and 10. When the underlying field is required to be algebraically closed, then, by quantifier-elimination in the theory of such fields, the conditions that the systems (8) and (10) are to meet are given by a quantifier-free formula. \( \square \)

**Theorem 12.** The theory of models of \( \text{ACFA} \) that also have valuations is not the model companion of \( \text{FAV} \).
Proof. We show that there is a model \((K, \sigma, \mathcal{O})\) of FAV that is not a model of FAV\(^*\), although the reduct \((K, \sigma)\) is a model of ACFA.

It is known [11] that every nonprincipal ultraproduct of the algebraic closures of the fields of prime order, each equipped with its Frobenius automorphism, is a model of ACFA. Now let

\[(K, \sigma, \mathcal{O}) = \prod_p (\mathbb{F}_p(T)^{\text{alg}}, x \mapsto x^p, \mathcal{O}_T) / \mathcal{U}\]

for some nonprincipal ultrafilter \(\mathcal{U}\) on the set of primes and some valuation ring \(\mathcal{O}_T\) of each \(\mathbb{F}_p(T)\). For example, \(\mathcal{O}_T\) might be the \(T\)-adic valuation ring, consisting of those elements of \(\mathbb{F}_p(T)\) that, considered as functions of \(T\), are well-defined at 0. The structure is as desired since by (3) it satisfies

\[\forall x \ (\text{val}(x) > 0 \Rightarrow \text{val}(x^\sigma) > 0),\]

that is, \(\forall x \ (x \in \mathbb{M} \Rightarrow x^\sigma \in \mathbb{M})\), while in every model of FAV\(^*\) the system

\[x \in \mathbb{M} \land x^\sigma \notin \mathbb{M}\]

is soluble. \(\Box\)

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