THE PARABOLIC VERLINDE FORMULA: ITERATED RESIDUES AND WALL-CROSSINGS

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Abstract. We give a new proof for the parabolic Verlinde formula in all ranks based on a comparison of wall-crossings in Geometric Invariant Theory and certain iterated residue functionals. On the way, we develop a tautological variant of Hecke correspondences, calculate the Hilbert polynomials of the moduli spaces, and present a new, transparent, local approach to the $\rho$-shift problem of the theory.

1. Introduction

1.1. The Verlinde formula

The Verlinde formula is a strikingly beautiful statement in Enumerative Geometry motivated by quantum physics [Ver]. Our focus in this paper will be the more difficult, parabolic variant, which we briefly describe below.

Let $C$ be a smooth, complex projective curve of genus $g \geq 1$, and fix an auxiliary point $p \in C$. We will call a vector $c = (c_1 > c_2 > \cdots > c_r) \in \mathbb{R}^r$ satisfying $\sum c_i = 0$ and $c_1 - c_r < 1$ regular if no nontrivial subset of its coordinates sums to an integer. For such a $c \in \mathbb{R}^r$, there exists a smooth projective moduli space $P_0(c)$ ([Se, MS, B]), whose points are in one-to-one correspondence with the equivalence classes of pairs $(W, F)$, where $W \to C$ is a vector bundle of rank $r$ on $C$ with trivial determinant, $F$ is a full flag of the fiber $W_p$, and the pair satisfies a certain parabolic stability condition depending on $c$ (cf. §2.1). This condition roughly states that for a proper subbundle $W_1 \subset W$, the degree $\deg(W_1)$ is strictly smaller than the sum of a subset of the coordinates of $c$ depending on the position of $W_1$ with respect to $F$.

There is a natural way to associate to a positive integer $k$ and an integer vector $\lambda \in \mathbb{Z}^r$ satisfying $\lambda_1 + \cdots + \lambda_r = 0$ a line bundle $L(k; \lambda)$ on $P_0(c)$, in such a way that if $c = \lambda/k$, then $L(k; \lambda)$ is ample. The parabolic Verlinde formula is the following expression for the Euler characteristic of the ample line bundle $L(k; \lambda)$: assume $c = \lambda/k$ is regular; then

\begin{equation}
\chi(P_0(c), L(k; \lambda)) = N_{r,k} \cdot \sum \frac{(-i)^{\{j\}} \exp(2\pi i \hat{\lambda} \cdot x)}{\prod_{i<j} (2 \sin \pi (x_i - x_j))^{2g-1}}
\end{equation}

where $N_{r,k} = r(r(k + r)^{r-1})^{g-1}$, $\hat{\lambda} = \lambda + \frac{k}{r}(r-1, r-3, \ldots, 1-r)$, and the sum is taken over the finite set of those points in the interior of the parallellepiped

$\{x = (x_1, x_2, \ldots, x_r = 0) | 0 < x_i - x_{i+1} < 1 \text{ for } i = 1, \ldots, r-1\}$

which satisfy the conditions
• $(k + r)x \in \mathbb{Z}^r$
• $x_i - x_j \notin \mathbb{Z}$ for $1 \leq i < j \leq r$. 

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We note that this finite set is a set of lattice points in the interior of \((r - 1)!\) identical simplices (cf. the rhombus on Figure 1).

![Figure 1. The set of \(\lambda\)'s (left), and the finite set from (1) (right) for \(k = 6, r = 3\).](image)

**Remark 1.1.** Equality (1) remains valid in greater generality, for certain cases when \(\lambda/k\) is non-regular. This slightly more technical statement will be given in Theorems 4.7 and 4.8.

**Notation:** We denote the discrete sum in (1) depending on \(k, \lambda, r\) and \(g\) by \(\text{Ver}(k, \lambda)\). In what follows, the shift \(\frac{1}{2}(r - 1, r - 3, \ldots, 1 - r)\) will be denoted by \(\rho\), and thus we have \(\hat{\lambda} = \lambda + \rho\).

Equality (1), the parabolic Verlinde formula has attracted a lot of attention over the years, and there is a number of different proofs. There is a generalization of this formula associated to a simply connected compact Lie group, and the form presented here corresponds to the case of the group \(SU(r)\).

In this article, we give a novel proof of this result, which stands out with its technical simplicity. We believe the methods and ideas described in the paper will have other applications in Geometric Invariant Theory and the study of moduli spaces.

Below, we give a quick sketch of the strategy of the proof, treating the example of the case of rank 3 in §1.2-4. Next, we give a short guide to the contents of the paper in §1.5.

Our work has close relationship with several earlier approaches, and we describe these links in §1.6.

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1.2. The residue formula

The proof is based on 3 ideas. We will follow the arguments below for the case \(r = 3\). We fix thus an integer \(k > 1\) and an integer vector \(\lambda = (\lambda_1 > \lambda_2 > \lambda_3)\), such that \(\lambda_1 + \lambda_2 + \lambda_3 = 0\) and \(\lambda_1 - \lambda_r < k\).
We start with the study of the right hand side of \((1)\), which, for \(r = 3\), may be written in the following somewhat simplified form
\[
\text{Ver}(k, \lambda) = N_{3,k} \sum_{0 < n_2 < n_1 < k + 3} \frac{2i \sin 2\pi (\lambda_1 + 1)n_1 + \lambda_2 n_2}{8 \sin \pi \frac{n_1 - 3}{k + 3} \sin \pi \frac{n_2}{k + 3} \sin \pi \frac{n_1}{k + 3}} 2^{g-1},
\]
where \(n_1, n_2\) are integers. Using Theorem \([4.7]\) and Remark \([4.6]\) one can show that
\[
\text{Ver}(k, \lambda) = \begin{cases} 
p_+(k; \lambda), & \lambda_2 > 0, 
p_-(k; \lambda), & \lambda_2 < 0,
\end{cases}
\]
where \(p_+\) and \(p_-\) are two polynomials, given by the right hand sides of the expressions of Example \([4]\) on page \([17]\). We note two properties of \(p_{\pm}(k, \lambda)\):

A. The wall-crossing difference \(p_- - p_+\) has a relatively simple form (cf. Example \([5]\) with \(\lambda_1 + \lambda_3\) replaced by \(-\lambda_2)\):
\[
p_-(k; \lambda) - p_+(k; \lambda) = \text{Res} \frac{(-3(k + 3)^2)^g \cdot e^{(\lambda_1 + 1)y - \lambda_2 y}}{(1 - e^{(k+3)y})w_\Phi(x, y)2^{g-1}} dx dy,
\]
where \(w_\Phi(x, y) = 2\sinh(\frac{x}{2}) \cdot 2\sinh(\frac{y}{2}) \cdot 2\sinh(\frac{x+y}{2})\).

B. An easy calculation via substitution shows that for any permutation on 3 elements \(\sigma \in \Sigma_3\), our polynomials have the following symmetries:
\[
p_+(k; \sigma \cdot \lambda + \theta_1[k]) = (-1)^\sigma p_+(k; \lambda + \theta_1[k])
\]
and
\[
p_-(k; \sigma \cdot \lambda + \theta_0 \sigma[k]) = (-1)^\sigma p_-(k; \lambda + \theta_0 \sigma[k]),
\]
where
\[
\theta_0[k] = \frac{k}{3}(1, 1, -2) + (0, 1, -1) \quad \theta_1[k] = \frac{k}{3}(2, -1, -1) + (1, -1, 0).
\]

### 1.3. Wall-crossings in moduli spaces

Now consider the left hand side of \((1)\). It is easy to check that the set of isomorphism classes of parabolic bundles in \(P_0(c)\) remains unchanged as long as \(c_2\) does not change sign. Hence, effectively, we have two moduli spaces \(P_0(>0)\) and \(P_0(<0)\), corresponding to the two chambers separated by the red \((c_2 = 0)\) line in Figure \([1]\). Introduce the notation
\[
q_+(k; \lambda) = \chi(P_0(>), \mathcal{L}(k; \lambda)), \quad q_-(k; \lambda) = \chi(P_0(<), \mathcal{L}(k; \lambda))
\]
for the generalized Hilbert polynomials of these two spaces.

In \([5]\), we derive a simple formula \((47)\) for the wall-crossing difference in Geometric Invariant Theory. The formula has the form of a residue of an equivariant integral, taken with respect to the equivariant parameter. In our case, the space on which we integrate is the space of rank-3 parabolic bundles which split into a direct sum of a rank-2 and a rank-1 bundle. This equivariant integral may be evaluated using induction on the rank (cf. the detailed calculation in Example \([6]\) on page \([31]\), and the result is
\[
q_-(k; \lambda) - q_+(k; \lambda) = \text{Res} \frac{(-3(k + 3)^2)^g \cdot e^{\lambda_1 z + \lambda_2 u + z}}{w_\Phi(z, -u)2^{g-1}(1 - e^{(k+3)z})} dz du,
\]
where \(\Phi, \lambda_1, \lambda_2, \ldots\) are some constants.
where $u$ plays the role of the equivariant parameter, the generator of $H^*_c(\text{pt})$. This iterated residue coincides with the expression above after changing $(z,u)$ to $(x,-y)$, and thus we have

$$p_+ - p_- = q_+ - q_-.$$  \hspace{1cm} (5)

### 1.4. Hecke correspondences, Serre duality and the symmetry argument

Hecke correspondences between moduli spaces of bundles of different degrees were introduced by Narasimhan and Ramanan in [NR]. In §7 of our paper, we describe a "tautological" variant of this construction, which identifies the same space with several moduli spaces of parabolic bundles with different degrees and weights. Using this construction we can fiber our two moduli spaces, $P_0(>)$ and $P_0(<)$ over the moduli spaces of stable bundles (without parabolic structure) of degrees 1 and $-1$:

$$\text{Flag}_3 \to P_0(<) \to N_{-1}, \quad N_1 \leftarrow P_0(>) \leftarrow \text{Flag}_3^\prime,$$

where the fibers are full flags of 3-dimensional vector spaces. Serre duality applied to a $\text{Flag}_3$-bundle implies a $\Sigma_3$-antisymmetry of the Euler characteristics of line bundles on this space, and after careful identification of these bundles, we derive the same symmetry properties for the functions $q_\pm$ as we did for the polynomials $p_\pm$: $q_+(k;\lambda)$ satisfies (2), while $q_-(k;\lambda)$ satisfies (3).

The final argument is elegant: we can rearrange equation (5) describing the equality of wall-crossings as

$$p_+ - q_+ = p_- - q_-,$$

and we introduce the notation $\Theta(k;\lambda)$ for this polynomial. Then $\Theta$ satisfies both (2) and (3), and thus it is anti-invariant with respect to an affine Weyl group action in the plane for each fixed $k$. This implies that $\Theta(k;\lambda)$ vanishes and this completes the proof.

### 1.5. Contents of the paper

There are a number of complications which arise when $r > 3$. We will highlight these in this section, and also give a brief guide to the contents of the paper.

We start with a quick introduction into the theory of parabolic bundles in §2. Here we describe the line bundles we are considering, as well as the chamber structure of the space of parabolic weights induced by the stability condition. The combinatorics of the iterated residue formulas mentioned in §1.2 above is considerably more complicated in the higher rank case, and is best treated using the notion of diagonal bases of hyperplane arrangements introduced in [Sz1]; we review this construction in the special case of the $A_r$ root arrangement in §3.

Using this notion, in §4 we present a residue formula for the Verlinde sums on the right hand side of (1) obtained in [Sz2] (Theorems 4.4 and 4.7). It turns out that because of a standard $\rho$-shift type effect in the theory, this residue formula does not have a manifestly polynomial form on our chambers, and thus, we formulate our main result, Theorem 4.8 in two parts: in part I. we state the equality of the Euler characteristics of line bundles with a modified residue formula, which is manifestly polynomial on our chambers, and in part II. we state the equality of the modified formula with the original residue formula from [Sz2]. Part II. is proved in §10 while the the proof of part I. takes up the rest of the paper.
The parabolic Verlinde formula

At the end of §4, we present our wallcrossing formula for Verlinde sums in Proposition 4.16, which uses in an essential manner the yoga of diagonal bases (cf. property A. above for the case of \( r = 3 \)).

The geometric part of our work starts in §5 where we derive a simple general result, formula (47), for wallcrossings in GIT. We apply this result to parabolic moduli spaces in §6 and, using induction on the rank, obtain Theorem 6.11, the higher rank version of formula (4) above.

It is downhill from here: in §7 we describe the tautological Hecke correspondences we need in several places in the paper, and in §8 we derive the Weyl-symmetries of the polynomials \( q_{\pm} \), and finish the proof along the lines sketched above.

We are essentially done, but we hit a snag when checking the beginning of our induction on the rank: our argument does not work for \( r = 2 \). Roughly, the reason for this is that we need our simplex of parabolic weights to have at least 2 regular vertices, and for \( r = 2 \), we have only 1. The way out is to consider the moduli space with two punctures and then all the pieces fall in place. This argument is carried out in §9.

1.6. Historical remarks

There is a long list of proofs of the Verlinde formula, and we cannot do justice to all the approaches in this short introduction. We will thus focus on the historical lineage of our paper, and the works that are closest in spirit to what we do (cf. S for a more comprehensive overview).

The proofs of the Verlinde formula fall in two categories: proofs of the fusion rules and proofs that find some interpretation of the "Fourier transformed" discrete sum on the right hand side of (1); our work belongs to this second group. Another line of division concerns the model which one uses for the moduli spaces: via the Narasimhan-Seshadri correspondence, the moduli spaces of vector bundles may equally be presented as symplectic manifolds of certain types of flat connections on punctured Riemann surfaces, and this opens the way of using the methods of symplectic geometry. While these symplectic approaches lead to results equivalent to the ones coming out of the algebro-geometric setup, the fields of applications of the two approaches seem to be very different.

The idea of proving the Verlinde formula via wall crossings appeared in the seminal paper of Michael Thaddeus [Th2]. He used a geometric approach and managed to prove the Verlinde formula in rank 2 by crossing walls in the moduli of stable pairs. The master space construction, which plays a central role in our paper, first appeared in his works as well [Th1]. In a sense, our paper may be thought of as the completion of his program.

A paper closely related to our work is that of Jeffrey and Kirwan [JK], who also use the residue calculus introduced in [Sz1, Sz2]. This paper approaches the problem from a symplectic/cohomological point of view, and has a somewhat different angle form ours. The use of iterated residues is not quite as consistent in [JK] as in our work, and the parabolic case was not resolved from this point of view [J]. The geometric model used to represent the moduli spaces as quotients is rather complicated.

In a comprehensive paper [BL] covering the case of all compact groups, Bismut, Laborie used a differential-geometric approach to find the generating function
for the parabolic Verlinde formula. This work was the motivation for the residue formula in [Sz2], which is also used in the present paper.

In a remarkable series of papers of Alekseev, Meinrenken and Woodward [AMW], again, approaching the subject from the symplectic point of view, gave a direct proof of (1), using reduction in infinite dimensions. A general approach related to twisted K-theory was introduced by Meinrenken in [M]. We should also mention recent work by Loizides and Meinrenken in [LM], which employs the residue techniques of [Sz2].

Finally, we drew motivation from the paper of Teleman and Woodward [TW], where the Verlinde formula is put in the framework of localization in K-theory of stacks. This very impressive work is probably accessible to a small number of experts only. In the present article, we demonstrate, in particular, that the sophisticated tools employed in [TW], at least in this instance, may be replaced by a simple combinatorial device.

In summary, the virtues of this article are:

- A proof of the parabolic Verlinde formula which needs as background only the basics of GIT.
- The discrete sum, and the generating function giving the coefficients of the Hilbert polynomial are treated at the same time, and the \( \rho \)-shift is dealt with explicitly.
- A few technical innovations such that an efficient wall-crossing formula in GIT (Theorem 5.6) and the tautological Hecke correspondences keep the arguments simple, and the technical difficulties related to infinite dimensional quotients or singularities, in our approach, are absorbed by a combinatorial device: the theory of iterated residues.

2. Parabolic bundles

2.1. Definitions

Let \( C \) be a smooth complex projective curve of genus \( g \geq 2 \), and fix a point \( p \in C \).

- A parabolic bundle on \( C \) is a vector bundle \( W \) of rank \( r \) with a full flag \( F_0 \) in the fiber over \( p \):

\[
W_p = F_r \supseteq \ldots \supseteq F_1 \supseteq F_0 = 0
\]

and parabolic weights \( c = (c_1, \ldots, c_r) \) assigned to \( F_r, F_{r-1}, \ldots, F_1 \), satisfying the conditions

\[
c_1 > c_2 > \ldots > c_r \text{ and } c_1 - c_r < 1.
\]

- The parabolic degree and the parabolic slope of \( W \) are defined as

\[
\text{pardeg}(W) = \deg(W) - \sum_{i=1}^{r} c_i; \quad \text{slope}(W) = \frac{\text{pardeg}(W)}{\text{rank}(W)}.
\]

- A morphism \( f : W \to W' \) of parabolic bundles is a morphism of vector bundles satisfying \( f_p(F_i) \subseteq F'_j \) if \( c_{r-i+1} < c'_{r-j+1} \). In particular, an endomorphism of a parabolic bundle \( W \) is a vector bundle endomorphism preserving the flag \( F_0 \).

\[\text{For technical reasons, we have chosen a sign convention opposite to that in the majority of treatments in the literature.}\]
Denote by $\text{ParHom}(W, W')$ the sheaf of parabolic morphisms from $W$ to $W'$. Then there is a short exact sequence of sheaves

$$0 \to \text{ParHom}(W, W') \to \text{Hom}(W, W') \to T_p \to 0,$$

where $T_p$ is a torsion sheaf supported at $p$. The rank of $T_p$ is the number of pairs $(i, j)$, s.t. $c_i < c'_j$ (cf. [BH]).

If $W' \subset W$ is a subbundle of $W$, then both $W'$ and the quotient $W/W'$ inherit a parabolic structure from $W$ in a natural way (cf. [MS], definition 1.7).

A parabolic bundle $W$ is \textit{stable of weight} $c$, if any proper subbundle $W' \subset W$ satisfies $\text{parslope}(W') < \text{parslope}(W)$; and $W$ is \textit{semistable of weight} $c$, if the inequality is not strict.

\textbf{Remark 2.1.} Note that the parabolic stability condition depends on the parabolic weights only up to adding the same constant to all weights $c_i$.

\section*{2.2. Construction of the moduli spaces}

We start with a quick review of the construction of Mehta and Seshadri [MS] of the moduli space of stable parabolic bundles. It follows from Remark 2.1 that, without loss of generality, we can assume that the parabolic weights of a rank-$r$ degree-$d$ bundle belong to the simplex

$$\Delta_d = \left\{ (c_1, c_2, ..., c_r) \mid c_1 > c_2 > ... > c_r, c_1 - c_r < 1, \sum_{i} c_i = d \right\}.$$

\textbf{Definition 2.2.} We will call a vector $c = (c_1, ..., c_r) \in \mathbb{R}^r$ such that $\sum c_i \in \mathbb{Z}$ \textit{regular} if for any nontrivial subset $\Psi \subset \{1, 2, ..., r\}$, we have $\sum_{i \in \Psi} c_i \notin \mathbb{Z}$.

Now choose an integer $d \gg 0$ such that $H^1(W) = 0$ and $W$ is generated by global sections for any rank-$r$ degree-$d$ semistable parabolic bundle $W$ of parabolic degree 0. Put $x = r(1-g) + d$ and consider the

- Grothendieck quas scheme $\text{Quot}(X, r)$ ([G]) parametrizing quotients $\mathcal{O}^X \to W$, where $W$ is a coherent sheaf of degree $d$ and rank $r$.
- This space is endowed with a universal bundle $UQ$, and a generically free action of the group $G = \text{PSL}(r)$, which does not, however, lift to $UQ$.
- Let $\text{LFQuot} \subset \text{Quot}(X, r)$ be the open subscheme consisting of locally free quotients $W$, such that the induced map $H^0(\mathcal{O}_X) \to H^0(W)$ is an isomorphism.
- Denote by $XQ$ the total space of the flag bundle $\text{Flag}(UQ_p)$ on $\text{LFQuot} \times p$. This space is endowed with the flag of vector bundles $\text{Fl}_1 \subset \cdots \subset \text{Fl}_{r-1} \subset \text{Fl}_r = UQ_p$.
- Let $k \in \mathbb{Z}$ and $(\lambda_1, ..., \lambda_r) \in \mathbb{Z}^r$, such that $\sum_{i=1}^r \lambda_i = kd$, and consider the line bundle

$$L(k; \lambda) = \det(UQ_p)^{k(1-g)} \otimes \det(\pi_* UQ)^{-k} \otimes (\text{Fl}_r/\text{Fl}_{r-1})^{\lambda_1} \otimes \cdots \otimes (\text{Fl}_1)^{\lambda_r}$$

on $XQ$, which does carry a $G$-linearization (lift of the $G$-action from $XQ$).
- Finally, assume $c \in \Delta_d$ is regular (cf. Definition 2.2 above) and define $\tilde{P}_d(c)$, the moduli space of stable parabolic weight-$c$ vector bundles on $C$ as the GIT quotient $XQ //^c G$ of $XQ$ with respect to any linearization $L(k; \lambda)$, such that $\lambda/k = c$. 

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Theorem 2.3 ([Se]). Assume that \( c \in \Delta_d \) is a regular weight vector. Then the moduli space \( \tilde{P}_d(c) \) is a smooth projective variety of dimension \( r^2(g-1)+\binom{r}{2}+1 \), whose points are in one-to-one correspondence with the set of isomorphism classes of stable parabolic bundles of weight \( c \) (cf. [2.7]).

Remark 2.4. Via the determinant map, the moduli space \( \tilde{P}_d(c) \) fibers over the Jacobian of degree-\( d \) line bundles with isomorphic fibers, and in this paper, we will focus on the moduli space

\[
P_d(c) = \{ W \in \tilde{P}_d(c) | \det W \cong \mathcal{O}(dp) \}.
\]

which is smooth, projective and has dimension \( (r^2-1)(g-1)+\binom{r}{2} \).

Remark 2.5. Note that tensoring with the line bundle \( \mathcal{O}(mp) \) induces an isomorphism: \( \otimes \mathcal{O}(mp) : P_d(c) \to P_{d+rm}(c) \), so the moduli spaces \( P_d(c) \), essentially, depend only on \( d \) modulo \( r \).

2.3. The Picard group of \( P_d(c) \)

For a regular \( c \in \Delta_d \), there exist universal bundles \( U \) over \( P_d(c) \times C \) endowed with a flag \( F_1 \subset \cdots \subset F_{r-1} \subset F_r = U_p \), and satisfying the obvious tautological properties. In general, such universal bundles \( U \), and hence the flag line bundles \( F_{i+1}/F_i \) are unique only up to tensoring by the pull-back of a line bundle from \( P_d(c) \). Nevertheless, we have the following statement, which is easy to verify.

Lemma 2.6. For \( k \in \mathbb{Z} \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \), such that \( \sum_{i=1}^r \lambda_i = kd \), the line bundle

\[
L_d(k; \lambda) = \det(U_p)^{k(1-g)} \otimes \det(\pi_\nu U)^{-k} \otimes (F_r/F_{r-1})^{\lambda_1} \otimes \cdots \otimes (F_1)^{\lambda_r}
\]

on \( P_d(c) \) is independent of the choice of the universal bundle \( U \).

Remark 2.7. The line bundle \( L(k; \lambda) \) defined in [2.2] descends to the line bundle \( L_d(k; \lambda) \) on the GIT quotient \( P_d(c) \).

Notation: We will say that \( U \) is normalized if the line subbundle \( F_1 \subset U_p \) is trivial. The parameter \( k \) is often called the level.

Let \( \omega \in H^2(C) \) be the fundamental class of our curve \( C \), and \( e_1, \ldots, e_{2g} \) a basis of \( H^1(C) \), such that \( e_i e_{i+g} = \omega \) for \( 1 \leq i \leq g \), and all other intersection numbers \( e_i e_j \) equal 0. For a class \( \delta \in H^*(P \times C) \) of a product, we introduce the following notation for its K"unneth components (cf. [W]):

\[
\delta = \delta(0) \otimes 1 + \sum_{i} \delta_{(1)}(e_i) \otimes e_i + \delta_{(2)}(e_i) \otimes \omega \in \bigoplus_{i=0}^{2} H^{*+i}(P) \otimes H^{1}(C).
\]

Later, we will need the following formula, which can be proved by a straightforward calculation.

Lemma 2.8. \( 2c_1(\mathcal{L}_d(r; d_1, \ldots, d)) = c_2(\text{End}_0(U_d))(2) \), where \( \text{End}_0 \) stands for traceless endomorphisms.

2.4. Walls and chambers

The central question we address in this paper is how the moduli space of stable parabolic bundles depends on the choice of parabolic weights. Let \( W \) be a vector bundle of degree \( d \) with a fixed full flag \( F_r \) of the fiber \( W_p \), and let us try to determine the structure of the set of parabolic weights \( c \in \Delta_d \) for which \( W \) is stable.
Clearly, for this we need to study the set of parabolic weights $c = (c_1, c_2, \ldots, c_r)$ for which one can find a proper subbundle $W' \subset W$ such that

$$\text{parslope}(W') = \text{parslope}(W) = 0.$$  

A subbundle $W' \subset W$ determines a short exact sequence of parabolic bundles

$$0 \to W' \to W \to W'' \to 0$$

and the position of $W'$ with respect to $F_*$ gives rise to a nontrivial partition of the set $\{1, 2, \ldots, r\}$ into two sets, $\Pi'$ and $\Pi''$ (cf. [MS], definition 1.7); the parabolic weights of $W'$ and $W''$ are then $c' = (c_i)_{i \in \Pi'}$ and $c'' = (c_i)_{i \in \Pi''}$, correspondingly.

The slope condition (8) translates into a pair of equivalent equalities:

$$d' = \sum_{i \in \Pi''} c_i, \quad d'' = \sum_{i \in \Pi''} c_i,$$

where $d', d'' = d - d'$ are the degrees of $W'$ and $W''$, respectively. This means that the critical values of $c \in \Delta_d$ for which (8) is possible lie on the union of affine hyperplanes (or \textit{walls}) defined by the equations

$$\sum_{i \in \Pi'} c_i = l, \text{ where } l \in \mathbb{Z}, \text{ and } \Pi' \subset \{1, 2, \ldots, r\} \text{ nontrivial.}$$

As only finitely many of these walls intersect the simplex $\Delta_d$, their complement is a finite union of open polyhedral \textit{chambers}. It is easy to verify that as we vary $c$ inside one of these chambers, the stability condition, and thus the moduli space $P_d(c)$ does not change.

\textbf{Example 1.} Consider the case of rank-3 degree-0 stable parabolic bundles with parabolic weights $c = (c_1, c_2, c_3) \in \Delta_0$. The set $\Delta_0$ is an open triangle with vertices $(0, 0, 0), (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ (cf. Figure 2), and there exist only two essentially different stability conditions. The wall separating the two regimes is given by the condition $c_2 = 0$. We write $P_0(\rangle)$ for the moduli space $P_0(c_1, c_2, c_3)$ with $c_2 > 0$, and $P_0(\langle)$ for $P_0(c_1, c_2, c_3)$ with $c_2 < 0$.

![Figure 2. The space of admissible parabolic weights for rank $r = 3$.](image)

3. \textbf{Wall-crossing in the Verlinde formula}

A key component of our approach is the notion of \textit{diagonal basis} and the associated generalized Bernoulli polynomials introduced for general hyperplane arrangements in [Sz1]. Using this formalism, we will be able to formulate our main result, Theorem 4.8.
3.1. Notation

We begin by setting up some extra notation for the space of parabolic weights introduced in §2.1.

- Let $V = \mathbb{R}^r/\mathbb{R}(1, 1, \ldots, 1)$ be the $r-1$-dimensional vector space, obtained as the quotient of $\mathbb{R}^r$. The dual space $V^*$ is then naturally represented as
  \[ V^* = \{ a = (a_1, \ldots, a_r) \in \mathbb{R}^r \mid a_1 + \cdots + a_r = 0 \}. \]

Let $x_1, x_2, \ldots, x_r$ be the coordinates on $\mathbb{R}^r$; given $a \in V^*$, we will write $\langle a, x \rangle$ for the linear function $\sum_i a_i x_i$ on $V$. We will sometimes identify this linear function with the vector $a$ itself.

- The vector space $V^*$ is endowed with a lattice $\Lambda$ of full rank:
  \[ \Lambda = \{ \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 + \cdots + \lambda_r = 0 \}. \]

In particular, for $1 \leq i \neq j \leq r$, we can define the element $\alpha^{ij} = x_i - x_j$ in $\Lambda$.

- Our arrangement is the set of hyperplanes $\{x_i = x_j\} \subset V$, $1 \leq i < j \leq r$. It will be convenient for us to think about this set as the set of roots of the $\Lambda_{r-1}$ root system with the opposite roots identified:
  \[ \Phi = \{ \pm \alpha^{ij} \mid 1 \leq i < j \leq r \}. \]

Note that $V^*$ carries a natural action of the permutation group $\Sigma_r$, permuting the coordinates $x_j$, $j = 1, \ldots, r$, and this action restricts to an action on $\Phi$ as well.

- The basic object of the theory is an ordered linear basis $B$ of $V^*$ consisting of the elements of $\Phi$. Let us denote the set of these objects by $\mathcal{B}$:
  \[ \mathcal{B} = \left\{ B = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \Phi^{r-1} \mid B \text{ basis of } V^* \right\} \]

- For $B \in \mathcal{B}$, we will write $\text{Fl}(B)$ for the full flag
  \[ \left[ V^* = \langle \beta^{[1]} \rangle, \langle \beta^{[2]} \rangle, \ldots, \langle \beta^{[r-1]} \rangle_{\text{lin}}, \ldots, \langle \beta^{[r-1]} \rangle_{\text{lin}}, \langle \beta^{[r-1]} \rangle_{\text{lin}} \right], \]
  where $\langle \cdot \rangle_{\text{lin}}$ stands for linear span.

3.2. Diagonal bases

**Definition 3.1.**

- For $\tau \in \Sigma_{r-1}$ and $B \in \mathcal{B}$, we will write $B \cup \tau$ for the permuted sequence $(\beta^{[\tau(1)]}, \beta^{[\tau(2)]}, \ldots, \beta^{[\tau(r-1)]})$.

- For two elements $B, C \in \mathcal{B}$ we will write $B \vdash C$ if for any $\tau \in \Sigma_{r-1}$, we have $\text{Fl}(B \cup \tau) \neq \text{Fl}(C)$.

- A subset $D \subset \mathcal{B}$ of $(r - 1)!$ elements is called a diagonal basis if for any two different elements $B, C \in D$, we have $B \vdash C$.

**Remark 3.2.** This definition is motivated by a construction [Sz1], which associates to each diagonal basis $D$ a pair of dual bases of the middle homology and the cohomology of the complexified hyperplane arrangement on $V \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\Phi$. The dimension of these (co)homology spaces is $(r - 1)!$. 
3.3. Combinatorial interpretation

This notion has the following purely combinatorial form.

- We can think of $\Phi$ as the edges of the complete graph on $r$ vertices.
- Then the set $\mathcal{B}$ may be thought of as the set of spanning trees of this graph with edges enumerated from 1 to $r - 1$. We will introduce the notation $\mathcal{B} \rightarrow \text{Tree}(\mathcal{B})$ for this ordered tree.
- In this language, the flag $\text{Fl}(\mathcal{B})$ corresponds to a sequence of $r$ nested partitions of the vertices (starting with the total partition into 1-element sets and ending with the trivial partition) associated to $\text{Tree}(\mathcal{B})$, the $j$th partition being the one induced by the first $j - 1$ edges. For example, the ordered tree $\alpha_{1,4}, \alpha_{2,3}, \alpha_{1,2}$ induces the same sequence of partitions as $\alpha_{1,3}, \alpha_{2,4}, \alpha_{1,2}$ (see Figure 3.3).

A diagonal basis $\mathcal{D}$ is then a set of $(r - 1)!$ ordered trees such that the $(r - 1)!$ partition sequences obtained by reordering the edges of any one of the ordered trees are different from the remaining elements of $\mathcal{D}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{\( \mathcal{B} = (\alpha_{1,3}, \alpha_{1,2}, \alpha_{3,4}) \)}
\end{figure}

3.4. Examples

There are essentially 2 known constructions of diagonal bases [Sz1].

The Hamiltonian basis. For each permutation $\sigma \in \Sigma_r$, we can define

\begin{equation}
\sigma(\mathcal{B}) = (\alpha_{\sigma(r-1), \sigma(r)}, \alpha_{\sigma(r-2), \sigma(r-1)}, \ldots, \alpha_{\sigma(1), \sigma(2)}) \in \mathcal{B}.
\end{equation}

The set $\mathcal{H}_m = \{\sigma(\mathcal{B}) \mid \sigma \in \Sigma_r, \sigma(1) = m\}$ is then a diagonal basis. In the combinatorial description, this diagonal basis corresponds to the set of Hamiltonian paths starting at vertex $m$, and endowed with the reversed natural ordering of edges.

Example 2. Here are some examples of Hamiltonian bases:

- for $r = 3$: $\mathcal{H}_1 = \{\alpha_{2,3}, \alpha_{1,2}, \alpha_{3,2}, \alpha_{1,3}\}$,
- and for $r = 4$:

\begin{equation}
\mathcal{H}_1 = \{(\alpha_{3,4}, \alpha_{2,3}, \alpha_{1,2}), (\alpha_{2,4}, \alpha_{3,2}, \alpha_{1,3}), (\alpha_{4,3}, \alpha_{2,4}, \alpha_{1,2}), (\alpha_{3,2}, \alpha_{4,3}, \alpha_{1,4}), (\alpha_{4,2}, \alpha_{3,4}, \alpha_{1,3}), (\alpha_{2,3}, \alpha_{4,2}, \alpha_{1,4})\}
\end{equation}

II. The no-broken-circuit bases. Let $\nu : \{1, \ldots, r(r - 1)/2\} \rightarrow \Phi$ be a total ordering, which we will represent as an order relation $< \nu$ on $\Phi$. To this ordering,
one can associate the following, so called noncommutative no-broken-circuit diagonal basis \([\text{Sz1}]\):

\[
D[r] = \{ (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in B \mid \beta^{[1]} \preceq \ldots \preceq \beta^{[r-1]}, \text{ and } \\
\alpha^{ij} \preceq \beta^{[m]} \Rightarrow (\alpha^{ij}, \beta^{[m]}, \ldots, \beta^{[r-1]}) \text{ linearly independent} \}.
\]

**Example 3.** Let \(\alpha^{1,3} \prec \alpha^{1,4} \prec \alpha^{2,3} \prec \alpha^{2,4} \prec \alpha^{1,2} \prec \alpha^{3,4}\) be the ordering of the positive roots for rank \(r = 4\). Then

\[
D[v] = \{ (\alpha^{1,3}, \alpha^{1,2}, \alpha^{3,4}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,3}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,4}), \\
(\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,3}), (\alpha^{1,3}, \alpha^{2,3}, \alpha^{3,4}) \}
\]

is the corresponding no-broken-circuit diagonal basis.

**Remark 3.3.** The hyperplane arrangement induced by \(\Phi\) is invariant under the natural action of \(\Sigma_r\) on the vector space \(V\). It follows easily from the definition that if \(D\) is a diagonal basis and \(\sigma \in \Sigma_r\) is a permutation, then \(\sigma(D)\) is also a diagonal basis.

### 4. The residue formula and the main result

In this section, we recall the residue formula from [Sz1] for \(\text{Ver}_p(k, \lambda)\), the discrete Verlinde sum on the right hand side of (1). The key feature of this formula is that it exposes the piecewise polynomial nature of \(\text{Ver}_p(k, \lambda)\), which is key for our wall-crossing analysis. While the objects are relatively simple, the formalism is heavy with notation, so we begin by describing the 1-dimensional case.

#### 4.1. The residue formula in dimension 1

The story begins with the Fourier series

\[
(11) \quad \frac{1}{(2\pi i)^m} \sum_{n \in \mathbb{Z}} \frac{\exp(2\pi i a n)}{n^m}
\]

for \(m \geq 2\), which is a periodic, piecewise polynomial function given by the formula

\[
\text{Res}_{x=0} \frac{\exp((a)x)}{1 - \exp(x)} \frac{dx}{x^m},
\]

where \((a)\) is the fractional part of the real number \(a\). The polynomial functions thus obtained on the interval \([0, 1]\) are called **Bernoulli polynomials**. The polynomial on the interval containing the real number \(c \in \mathbb{R} \setminus \mathbb{Z}\) is given by

\[
\text{Res}_{x=0} \frac{\exp((a - [c])x)}{1 - \exp(x)} \frac{dx}{x^m},
\]

where \([c]\) is the integer part of \(c\).

Now we pass to a trigonometric version of this formula, calculating finite sums of values of rational trigonometric functions over rational points with denominators equal to an integer \(k\).

We replace thus the rational function \(x^{-m}\) by the (hyperbolic) trigonometric function \(f(x) = (2 \sinh(x/2))^{-2m}\), and introduce an integer parameter \(\lambda\) related to
a via $ka = \lambda$. We consider the sum of values of the function $f$ over a finite set of rational points in analogy with (11):

$$\sum_{n=1}^{k-1} \frac{\exp(2\pi i \lambda n/k)}{(2\sin(\pi n/k))^2}.$$ 

where $\lambda, k \in \mathbb{Z}$. This sum is again periodic in $\lambda \mod k$, and for $m \geq 2$ we can evaluate it via the residue theorem as

$$(-1)^m \text{Res}_{z=1} \frac{z^{k\{\lambda/k\}}}{(z^{1/2} - z^{-1/2})^{2m}} \cdot \frac{k}{z(1-z^k)} \cdot \exp (\frac{\lambda/k \cdot x}{1 - \exp(x)}) \cdot f(x/k) \, dx.$$ 

Again, this is a piecewise polynomial function in the pair $(k, \lambda)$, which is polynomial in the cones bounded by the lines $\lambda = qk$, $q \in \mathbb{Z}$.

Note that in these calculations, a key role is played by the Bernoulli operator:

$$f \mapsto \text{Ber}[f](a) = \frac{f(x) \exp(ax) \, dx}{1 - \exp(x)},$$

which transforms meromorphic functions in the variable $x$ into polynomials in $a$, and plays the role of a generalized Fourier operator.

4.2. The multidimensional case

Now we return to the setup of §3 with the vector space $V$ endowed with the hyperplane arrangement $\Phi$. We introduce the notation $F_\Phi$ for the space of meromorphic functions defined in a neighborhood of $0$ in $V \otimes_\mathbb{R} \mathbb{C}$ with poles on the union of hyperplanes

$$\bigcup_{1 \leq i < j \leq r} \{x| \langle \alpha^i, x \rangle = 0\}.$$ 

In particular, the function

$$w_\Phi = \prod_{i < j} (2\sinh(\pi(x_i - x_j)))$$

is an element of $F_\Phi$.

To write down our residue formula, we need a multidimensional generalization of the notions of integer and fractional parts. Given a basis $B = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B}$ of $V^*$, and an element $a \in V^*$, we define $[a]_B$ and $\{a\}_B$ to be the unique elements of $V^*$ satisfying

- $[a]_B = a - \{a\}_B \in \Lambda$, and
- $\{a\}_B \in \sum_{j=1}^{[r-1]} [0, 1)\beta^{[j]}$.

This notion naturally induces a chamber structure on $V^*$: we will call $a \in V^*$ regular if $a$ is a point of continuity for the functions $a \mapsto [a]_B$, $\{a\}_B$ for all $B \in \mathcal{B}$, i.e. when $\{a\}_B \in \sum_{j=1}^{[r-1]} (0, 1)\beta^{[j]}$. Now, for regular $a$ and $b$ we define the equivalence relation

$$a \sim b \text{ when } [a]_B = [b]_B \ \forall B \in \mathcal{B}.$$ 

The equivalence classes for this relation form a $\Lambda$-periodic system of chambers in $V^*$. 

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Convention: We will think of a partition \( \Pi \) of \( \{1, 2, \ldots, r\} \) into two nonempty sets as an ordered partition \( \Pi = (\Pi', \Pi'') \) such that \( r \in \Pi'' \), and we will call these objects nontrivial partitions for short.

Lemma 4.1. The equivalence classes of the relation \( \sim \) are precisely the chambers in \( V^* \) created by the walls parameterized by a nontrivial partition \( \Pi = (\Pi', \Pi'') \) of the first \( r \) positive integers, and an integer \( l \):

\[
S_{\Pi, l} = \{ c \in V^* | \sum_{j \in \Pi'} c_j = l \}
\]

Remark 4.2. Note that the walls given in (14) are precisely the same as the ones given in (9) for the case \( d = 0 \), where they play the role of walls separating the chambers of parabolic weights \( c \) in which the parabolic moduli spaces \( P_0(c) \) are naturally the same. This "coincidence" is precisely what we need for our comparative wall-crossing strategy. There is a small terminological issue here: the "chambers" in §2.4 are the intersections of the equivalence classes of \( \sim \) defined above with the open simplex \( \Delta_0 \) where the parabolic weights live (cf. Figures 2 and 4). We will use the term "chamber" in both cases if this causes no confusion.

Each element \( B = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B} \) defines an iterated version of the Bernoulli operator (12) on the space of functions \( \mathcal{F}_\Phi \): interpreting the elements \( a, \beta^{[l]} \in V^* \) as linear functions on \( V \), we define

\[
(15) \quad \text{iBer}_B[f(x)](a) = \frac{1}{(2\pi i)^{r-1}} \int_{Z_B} \frac{f(x) \exp(a, x) \, d\langle \beta^{[1]}, x \rangle \wedge \cdots \wedge d\langle \beta^{[r-1]}, x \rangle}{(1 - \exp(\langle \beta^{[1]}, x \rangle)) \cdots (1 - \exp(\langle \beta^{[r-1]}, x \rangle))},
\]

where the naturally oriented cycle \( Z_B \) is defined by

\[
Z_B = \{ v \in V \otimes_{\mathbb{R}} \mathcal{C} : \langle \beta^{[j]}, x \rangle| = \varepsilon_j, j = \ldots, r - 1 \} \subset V \otimes_{\mathbb{R}} \mathcal{C} \setminus \{ w_\Phi(x) = 0 \},
\]

with real constants \( \varepsilon_j \) satisfying \( 0 \leq \varepsilon_{r-1} \leq \cdots \leq \varepsilon_1 \). Thus again, \( \text{iBer}_B \) is a linear operator associating to a function in \( \mathcal{F}_\Phi \) a polynomial on \( V^* \).

Remark 4.3. Let us make a small remark about the computational aspects of the operator \( \text{iBer}_B \). Denoting the coordinate \( \langle \beta^{[j]}, x \rangle \) by \( y_j, j = 1, \ldots, r - 1 \), and writing
f and a in these coordinates: \( f(x) = \hat{f}(y) \), \( \langle a, x \rangle = \langle \hat{a}_s, y \rangle \), we can rewrite (15) as

\[
\text{iBer}_B[f(x)](a) = \text{Res}_{y_1=0} \ldots \text{Res}_{y_{r-1}=0} \frac{\hat{f}(y) \exp{\langle \hat{a}_s, y \rangle} \ dy_1 \wedge \cdots \wedge dy_{r-1}}{(1-\exp(y_1)) \ldots (1-\exp(y_{r-1}))},
\]

where iterating the residues here means that we keep the variables with lower indices as unknown constants, and then use geometric series expansions of the type

\[
\frac{1}{1-\exp(y_1-y_2)} = \frac{y_1-y_2}{1-\exp(y_1-y_2)}, \quad \frac{1}{1-\exp(y_1-y_2)} = \frac{y_1-y_2}{1-\exp(y_1-y_2)} \sum_{n=0}^{\infty} \frac{y_2^n}{y_1^{n+1}}.
\]

4.3. Invariance of diagonal bases and the main results

Diagonal bases have the following key invariance property.

**Theorem 4.4** ([Sz1]). Let \( f \in \mathcal{F}_D \), and \( c \in V^* \) be regular; let \( D \) be a diagonal basis of \( \Phi \). Then the functional (cf. (15) above)

\[
f \mapsto \sum_{B \in D} \text{iBer}_B[f(x)](a - [c]_B)
\]

transforming a meromorphic function \( f \in \mathcal{F}_D \) into a polynomial in the variable \( a \in V^* \) is independent of the choice of the diagonal basis \( D \). In particular, for regular \( a \in V^* \), the functional

\[
f \mapsto \sum_{B \in D} \text{iBer}_B[f(x)]([a]_B)
\]

transforms \( f \) into a well-defined piecewise polynomial function on \( V^* \), which is polynomial in each chamber.

As this functional is invariantly defined, it is not surprising that it is equivariant with respect to the symmetries of our hyperplane arrangement. For \( \sigma \in \Sigma_r \), we define, as usual

\[
\sigma \cdot f(x) = f(\sigma^{-1}x).
\]

This convention is consistent with (10).

**Lemma 4.5.** Let \( f \in \mathcal{F}_D \), and \( \sigma \in \Sigma_r \), and pick any diagonal basis \( D \). Then

\[
\sum_{B \in D} \text{iBer}_B[f(x)](\sigma \cdot a - [\sigma \cdot c]_B) = \sum_{B \in D} \text{iBer}_B[\sigma^{-1} \cdot f(x)](a - [c]_B)
\]

**Proof.** Indeed, it is sufficient to note that

\[
\langle \sigma \cdot a - [\sigma \cdot c]_B, x \rangle = \langle \sigma \cdot (a - [c]_B), x \rangle = \langle a - [c]_B, \sigma^{-1}(x) \rangle,
\]

perform the linear substitution \( x = \sigma(y) \), and conclude that

\[
\sum_{B \in D} \text{iBer}_B[f(x)](\sigma \cdot a - [\sigma \cdot c]_B) = \sum_{B \in D} \text{iBer}_B[\sigma^{-1} \cdot f(x)](a - [c]_B).
\]

Now the statement follows from the fact that \( \sigma \in \Sigma \) takes a diagonal basis to another diagonal basis (cf. Remark 3.3). \( \Box \)
Remark 4.6. By picking the Hamiltonian diagonal basis $H = \{ \sigma \cdot B_0 \mid \sigma \in \text{Stab}(1, \Sigma_r) \}$, we can turn the argument in the proof above around, and obtain the following formula:

$$
\sum_{B \in H} \text{Ber}[f(x)](a - [c]_B) = \sum_{\sigma \in \text{Stab}(1, \Sigma_r)} \text{Ber}([\sigma \cdot f(x)](\sigma \cdot a - [\sigma \cdot c]_B) = \\
\Res_{y_1 = 0} \cdots \Res_{y_{r-1} = 0} \sum_{\sigma \in \text{Stab}(1, \Sigma_r)} \frac{\sigma \cdot f(y) \exp \langle \sigma \cdot a - [\sigma \cdot c]_B, y \rangle \, dy_1 \wedge \cdots \wedge dy_{r-1}}{(1 - \exp(y_1)) \cdots (1 - \exp(y_{r-1}))},
$$

where

$$
B_0 = (y_1 = x_{r-1} - x_r, \ldots, y_{r-2} = x_2 - x_3, y_{r-1} = x_1 - x_2) \in B.
$$

Now we are ready to write down the residue formula for the Verlinde sums proved in [Sz2, Theorem 4.2]. Recall that we denoted by $\text{Ver}(k, \lambda)$ the finite sum on the right hand side of (1).

Theorem 4.7. Let $g \geq 1$, $k \in \mathbb{Z}_{>0}$, $\lambda \in \Lambda$, and let $D$ be any diagonal basis of $\Phi$. Introducing the notation $\hat{k} = k + r$, and $\hat{\lambda} = \lambda + \rho$, we have

$$
\text{Ver}(k, \lambda) = \hat{N}_{r,k} \sum_{B \in D} i\text{Ber}_B \left( w_{\Phi}^{1-2g}(x/\hat{k}) \right) \left( \hat{\lambda}/\hat{k} - [\hat{c}]_B \right),
$$

where $\hat{N}_{r,k} = (-1)^{(2)(g-1)} N_{r,k}$ (cf. (1)) and $\hat{c} \in V^*$ is a regular point in a chamber that contains $\hat{\lambda}/\hat{k}$ in its closure.

Now, if we look at our main goal (1): proving the equality

$$
\text{Ver}(k, \lambda) = \chi(P_0(\lambda/k), L_0(k; \lambda)),
$$

then we discover a rather embarrassing mismatch. Both sides are piecewise polynomial functions, however, for $\lambda/k$ in its closure.

- according to the HRR theorem, $\chi(P(\lambda/k), L_0(k; \lambda))$ is polynomial on the cones over the the equivalence classes (cf. (13)) of $\lambda/k$, while
- according to (18), $\text{Ver}(k, \lambda)$ is polynomial on the cones over the equivalence classes of $\hat{\lambda}/\hat{k}$,

and these conic partitions of $\{(k, \lambda) \mid \lambda/k \in \Delta \}$ could clearly be different (cf. Figure 5 for a sketch of this problem).

![Figure 5](image_url)

**Figure 5.** $\lambda/k$ is in the orange chamber, while $\hat{\lambda}/\hat{k}$ is in the green chamber.
Thus for (1) to be true, some miracle needs to occur, and these miracles are well-known in the area of "quantization commutes with reduction" [MS, V, 52V]. We will return to this problem in §10, but for now, we will be satisfied to use (18) to write down a (conjectural for the moment) formula for \( \chi(P_0(\lambda/k), L_0(k; \lambda)) \), which is manifestly polynomial on the cones where \( \lambda/k \) is in a fixed equivalence class.

Let us fix a regular \( c \in \Delta \) marking a particular chamber in \( \Delta \). The two cones \( \{(k; \lambda)| \lambda/k \sim c\} \) and \( \{(k; \lambda)| \hat{\lambda}/\hat{k} \sim c\} \) intersect along an open cone (this cone is shaded in orange on Figure 5), and on this intersection, the expression

\[
\sum_{B \in \mathcal{D}} \text{iBer}_B \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \hat{\lambda}/\hat{k} - [\lambda/k]_B \right)
\]

coincides with the right hand side of (18). As (20) is manifestly polynomial on each cone where \( \lambda/k \) is in a particular chamber in \( \Delta \), this expression will be then our main candidate for \( \chi(P_0(\lambda/k), L_0(k; \lambda)) \).

Our plan is thus to split the proof of (19) into three parts: the first is equality (18), and the other two are given in our main theorem below. We formulated all our statements in a manner that allows us to treat the cases when \( \lambda/k \) or \( \hat{\lambda}/\hat{k} \) are on a boundary separating two of our chambers in \( \Delta \).

**Theorem 4.8.** Let \( \lambda \in \Lambda \) and \( k \in \mathbb{Z}^{>0} \) be such that \( \lambda/k \in \Delta \). Let \( \zeta \) and \( \hat{\zeta} \in \Delta \) be regular elements, specifying two chambers in \( \Delta \), which contain \( \lambda/k \) and \( \hat{\lambda}/\hat{k} \) in their closures, correspondingly. Then for any diagonal basis \( \mathcal{D} \), the following two equalities hold:

(I.) \[ \chi(P_0(\zeta), L(k; \lambda)) = \mathbb{N}_{r,k} \sum_{B \in \mathcal{D}} \text{iBer}_B \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \hat{\lambda}/\hat{k} - [\zeta]_B \right) , \]

and

(II.) \[ \sum_{B \in \mathcal{D}} \text{iBer}_B \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \hat{\lambda}/\hat{k} - [\zeta]_B \right) = \sum_{B \in \mathcal{D}} \text{iBer}_B \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \hat{\lambda}/\hat{k} - [\hat{\zeta}]_B \right) . \]

**Remark 4.9.** Part (I) of the theorem implies that if \( \lambda/k \in \Delta \) is not regular, then

\[ \chi(P_0(c^{\pm}), L(k; \lambda)) = \chi(P_0(c^{-}), L(k; \lambda)), \]

for regular \( c^{\pm} \in \Delta \) in two neighboring chambers that contain \( \lambda/k \) in their closure (cf. Proposition 10.1 and Remark 10.4).

Before we proceed, we formulate a mild generalization of part (I) of our theorem. As observed above, if we fix a generic \( c \in \Delta \), and vary \( (\lambda, k) \) in such a way that \( \lambda/k \sim c \), then both sides of the equality (I) are manifestly polynomial, and thus we can extend the validity of this equality as follows.

**Corollary 4.10.** Let \( c \in \Delta \) be a regular element, which thus specifies a chamber in \( \Delta \) and a parabolic moduli space \( P_0(c) \) as well. Then for a diagonal basis \( \mathcal{D} \), an arbitrary weight \( \lambda \in \Lambda \), and a positive integer \( k \), we have

\[
\chi(P_0(c), L(k; \lambda)) = \mathbb{N}_{r,k} \sum_{B \in \mathcal{D}} \text{iBer}_B [w_{\Phi}^{1-2g}(x/\hat{k})] (\hat{\lambda}/\hat{k} - [c]_B) .
\]
Example 4. Let us write down these formulas in case of \( r = 3 \) explicitly. Let \( D \) be the diagonal basis from Example 2 then using Remark 4.6 we obtain
\[
\chi(P_0(<), \mathcal{L}(k; \lambda)) = (-1)^{g-1}(3k + 3)^g \\
\frac{\text{Res}_{y=0} \text{Res}_{x=0} (1 - e^{x(k+3)})(1 - e^{y(k+3)} w_{\Phi}(x, y)^{2g-1} dx dy)}{e^{\lambda_1 x + (\lambda_1 + \lambda_2) y - e^{\lambda_1 x + (\lambda_1 + \lambda_3) y + x}}}
\]
and
\[
\chi(P_0(>), \mathcal{L}(k; \lambda)) = (-1)^{g-1}(3k + 3)^g \\
\frac{\text{Res}_{y=0} \text{Res}_{x=0} (1 - e^{x(k+3)})(1 - e^{y(k+3)} w_{\Phi}(x, y)^{2g-1} dx dy)}{e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x} - e^{\lambda_1 x + (\lambda_1 + \lambda_3) y + x + y(k+3)}}
\]
where \( w_{\Phi}(x, y) = 2\sinh\left(\frac{y}{2}\right)2\sinh\left(\frac{y}{2} - \sinh\left(\frac{x+y}{2}\right)\right). \)

4.4. The walls

Our first step is to identify the wall-crossing terms of the residue formula (21), which originate in the discontinuities of the function \( c \mapsto \{c\}_B \). These discontinuities occur on "walls": the affine hyperplanes (14). The following is straightforward:

Lemma 4.11. Let \( S_{\Pi, l} \) be the wall defined by (14), and \( B = (\beta[1], \ldots, \beta[r-1]) \in B \) an ordered basis of \( V^* \). Then, as a function of \( c \), the fractional part function \( \{c\}_B \) has a discontinuity on the wall \( S_{\Pi, l} \) exactly when Tree(\( B \)) (cf. page 11) is a union of a tree on \( \Pi' \), a tree on \( \Pi'' \) (the enumeration of the edges is irrelevant here) and a single edge (which we will call the link) connecting \( \Pi' \) and \( \Pi'' \).

Notation: We will denote the element of \( B \) corresponding to this edge by \( \beta_{\text{link}} \); this vector thus depends on \( B \) and the partition \( \Pi \).

Now choose two regular elements \( c^+, c^- \in V^* \) in two neighboring chambers separated by the wall \( S_{\Pi, l} \), in such a way that
\[
[c^+_{\Pi'}] = 1 \quad \text{and} \quad [c^-_{\Pi'}] = 1 - 1,
\]
where
\[
c_{\Pi'} \overset{\text{def}}{=} \sum_{l \in \Pi'} c_l,
\]
and, as usual, \([q]\) stands for the integer part of the real number \( q \). Now introduce the notation
\[
p_{\pm}(k; \lambda) = \tilde{N}_{r, k} \sum_{B \in D} i \text{Ber}[w_{\Phi}^{1-2g}(x/\hat{k})](\lambda/\hat{k} - [c^\pm]_B)
\]
for the two polynomial functions in \( (k, \lambda) \) corresponding to \( c^+ \) and \( c^- \), respectively. We define the wall-crossing term in our residue formula (21) as the difference between these two polynomials:
\[
p_+(k; \lambda) - p_-(k; \lambda).
\]
Using Lemma 4.1 and (22), we obtain the following simple residue formula for this difference.
Let $(\Pi, 1)$, $c^+$ and $c^-$ be as above, and let us fix a diagonal basis $\mathcal{D} \subset \mathcal{B}$. Denote by $\mathcal{D}|\Pi$ the subset of those elements of $\mathcal{D}$, which satisfy the condition described in Lemma 4.11. Then

$$p_+(k, \lambda) - p_-(k, \lambda) = \mathcal{N}_{r,k} \sum_{B \in \mathcal{D}|\Pi} \text{iBer} \left[ (1 - \exp(\beta_{\text{link}}(x))) w_{\Phi}^{1-g} (x/\bar{k}) \right] \left( \lambda/\bar{k} - [c^+]_B \right),$$

where $\beta_{\text{link}}$ is the "link" element of $\mathcal{B}$ (depending on $\Pi$ and $B$) defined after Lemma 4.11.

Remark 4.13. Note that the multiplication by $1 - \exp(\beta_{\text{link}}(x))$ in (23) has the effect of canceling one of the factors in the denominator in the definition (15) of the operation iBer.

Example 5. Calculating the difference of two polynomials from Example 4 we get the wall-crossing term for rank 3 case:

$$p_-(k; \lambda) - p_+(k; \lambda) = (-3(k + 3)^2)^g \text{Res}_{y = 0} \text{Res}_{x = 0} e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x} \left( 1 - e^{x(k + 3)} \right) w_{\Phi}(x, y)^{2g-1} dx dy.$$

4.5. Wall-crossing and diagonal bases

Now we pass to the study of the combinatorial object $\mathcal{D}|\Pi$ defined in Lemma 4.12. One thing we will discover is that even though each diagonal basis consists of $(r - 1)!$ elements and the right hand side of (23) does not depend on the choice of $\mathcal{D}$, the number of elements in $\mathcal{D}|\Pi$ might vary with $\mathcal{D}$.

First we look at the case of the Hamiltonian basis $\mathcal{H}_1$. Form now on, we will use the notation $|\Pi'| = r'$ and $|\Pi''| = r''$ for a nontrivial partition $\Pi = (\Pi', \Pi'')$, (recall the convention $r \in \Pi''$). The following statement is easy to verify.

Lemma 4.14. Let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition, such that $1 \in \Pi'$ (the other case is analogous). Then

$$\mathcal{H}_1|\Pi = \{ \sigma(B) | \sigma(1) = 1, \text{ and } \sigma(\Pi') \in \Pi' \}.$$  

In particular, $|\mathcal{H}_1|\Pi| = (r' - 1)! \cdot r''!$.

It turns out that for our geometric applications, instead of $\mathcal{H}_1$, we will need to choose a particular nbc-basis, where the ordering is chosen to be consistent with $\Pi$.

To simplify our terminology, we will use the language of graphs and edges introduced in §3.3 and we will think of $\alpha^{ij} \in \Phi$ as an edge in the complete graph on $r$ vertices. To define the ordering $\nu$, we need to choose an edge between $\Pi'$, $\Pi''$; the choice is immaterial, but for simplicity we will set for $m^{\text{def}} = \max \{ i \in \Pi' \}$ and $r \in \Pi''$, and set $\beta_{\text{link}} = \alpha^{m,r}$ to be the smallest element according to $\nu$.

The $\nu$-ordered list of edges thus starts with $\beta_{\text{link}}$, and then continues with the remaining $r' \cdot r'' - 1$ edges connecting $\Pi'$ and $\Pi''$. Next we list the $r'(r' - 1)/2$ edges connecting vertices in $\Pi'$ in any order, and finally, we list the remaining edges, those connecting vertices in $\Pi''$.

Notation: We introduce the natural notation $\Phi'$ and $\Phi''$ for the $A_{r'}$ and $A_{r''}$ root systems corresponding to $\Pi'$ and $\Pi''$, and we denote by $\mathcal{D}[\nu]$, $\mathcal{D}'[\nu]$ and $\mathcal{D}''[\nu]$, the diagonal nbc-bases induced by the ordering $\nu$ on $\Phi$, $\Phi'$ and $\Phi''$, respectively.
Lemma 4.15. Given elements $B' \in D'[u]$ and $B'' \in D''[u]$, we can define an element of $D[u]$ as follows: we start with $\beta_{\text{link}}$, then append $B'$, and then continue with $B''$. This construction creates a one-to-one correspondence
\begin{equation}
D'[u] \times D''[u] \to D[u]/\pi;
\end{equation}
in particular, $|D[u]/\pi| = (r' - 1)! \cdot (r'' - 1)!$.

Finally, putting Lemmas 4.12 and 4.15 together, we arrive at the following elegant statement:

Proposition 4.16. Let $(\Pi, 1)$, $c^+$ and $c^-$ be as in Lemma 4.12 and let $D'$ and $D''$ be diagonal bases of $\Phi'$ and $\Phi''$ correspondingly. Then
\begin{equation}
p_+(k; \lambda) - p_-(k; \lambda) = (k + r)\tilde{N}_{r,k} \sum_{B' \in D'} \sum_{B'' \in D''} \text{Res}_{\beta_\text{link} = 0} \text{iBer}_{B'} \text{iBer}_{B''} \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \lambda/\hat{k} - [c^+]_B \right) d\beta_{\text{link}},
\end{equation}
where $\text{Res}_{\beta_\text{link} = 0} \text{iBer}_{B'} \text{iBer}_{B''} d\beta_{\text{link}}$ is simply $\text{iBer}_B$ (cf (15)) with $B$ obtained by appending $B'$, and then $B''$ to $\beta_{\text{link}}$, and with the factor $(1 - \exp(\langle \beta_{\text{link}}, x \rangle))$ removed from the denominator.

Remark 4.17. The expression
\begin{equation}
\text{Res}_{\beta_\text{link} = 0} \text{iBer}_{B'} \text{iBer}_{B''} \left[ w_{\Phi}^{1-2g}(x/\hat{k}) \right] \left( \lambda/\hat{k} - [c^+]_B \right) d\beta_{\text{link}}
\end{equation}
may equally be interpreted as follows. We write
\begin{equation}
\lambda \equiv \lambda/\hat{k} - [c^+]_B = m_{\text{link}} \beta_{\text{link}} + n' + n''
\end{equation}
according to the splitting of $B$, think of $w(x/\hat{k})$ as a function in $\mathcal{T}_{\Phi''}$ with some fixed values of the parameters from $B'$ and $\beta_{\text{link}}$, and then calculate
\begin{equation}
\text{iBer}_{B''} [w_{\Phi}^{1-2g}(x/\hat{k})](n'').
\end{equation}
The result will be a rational function $Q$ in the variables from $B'$ and $\beta_{\text{link}}$, and we proceed to calculate $\text{iBer}_{B'} [Q](n')$ to obtain a function $F$ in the variable $\beta_{\text{link}}$, and finally the answer is $\text{Res}_{\beta_{\text{link}} = 0} \exp(\langle \beta_{\text{link}}, x \rangle) F(\beta_{\text{link}}) d\beta_{\text{link}}$.

We observe that since the trees $\text{Tree}(B')$ and $\text{Tree}(B'')$ are disjoint, the order of the application of the operations $\text{iBer}_{B'}$ and $\text{iBer}_{B''}$ is immaterial.

5. Wall crossing in master space

Master spaces were introduced by Thaddeus in [Th1] in order to understand GIT quotients when varying linearizations. Following his footsteps, in this section, we describe a simple but very effective method to control the changes in the Euler characteristics of line bundles when crossing a wall in the space of linearizations.

5.1. Wall-crossing and holomorphic Euler characteristics

We begin by recalling the basic notions of Geometric Invariant Theory.

Let $X$ be a smooth projective variety over $\mathbb{C}$, and $G$ a reductive group acting on $X$. A linearization of this action is a line bundle $L$ on $X$ with a lifting of the $G$-action to a linear action on $L$. An ample linearization is $G$-effective, if $L^n$ has a nonzero $G$-invariant section for some $n > 0$; the space of such linearizations is called the $G$-effective ample cone; we denote this cone by $\text{Cone}_G(X)$.
For \( L \in \text{Cone}_G(X) \), we define the invariant-theoretic quotient \( X/\LL G \) as the \( \text{Proj} \) of the graded ring of invariant sections of the powers of \( L \):

\[
M_L = \text{Proj} \bigoplus_n H^0(X, L^n)^G.
\]

According to Mumford’s Geometric Invariant Theory [MF], there is a partition of \( X \) depending on \( L \):

\[
(26) \quad X = X^s[L] \cup X^{ss}[L] \cup X^{us}[L]
\]

into the set of stable, strictly semistable, and unstable points, such that there is a surjective map \( (X^s[L] \cup X^{ss}[L])/G \to M_L \), which is a bijection if \( X^{ss}[L] \) is empty, and the quotient \( X^s[L] \) is a smooth orbifold.

In [DH], Dolgachev and Hu studied the dependence of the GIT quotient \( M_L = X/\LL G \) on \( L \). They showed that \( \text{Cone}_G(X) \) is divided by hyperplanes, called walls, into finitely many convex chambers, such that when \( L \) varies within a chamber, the partition (26) and thus the GIT quotient \( M_L \) remains unchanged. Moreover, an ample effective linearization lies on a wall precisely when it possesses a strictly semistable point.

Now let us consider two neighboring chambers, with smooth GIT quotients \( M_+ \) and \( M_- \). We pick an arbitrary linearization \( \mathcal{L} \) of the \( G \)-action on \( X \), which descends to \( M_+ \) and \( M_- \). This last condition means that if \( S \subset G \) is the stabilizer of a generic point in \( X \), then \( S \) acts trivially on the fibers of \( \mathcal{L} \). We will call such linearizations \( \text{descending} \).

Thus, given such a descending linearization \( \mathcal{L} \) of the \( G \)-action on \( X \), we obtained two line bundles: one on \( M_+ \) and one on \( M_- \), which, by abuse of notation, we will denote by the same letter \( L \). Via taking Chern classes, this construction creates a correspondence between classes in \( H^2(M_+, \mathbb{Z}) \) and \( H^2(M_-, \mathbb{Z}) \), which we will assume to be an isomorphism of free \( \mathbb{Z} \)-modules. We will thus identify these lattices, and introduce the notation \( \Gamma \) for them:

\[
\Gamma = H^2(M_+, \mathbb{Z}) \simeq H^2(M_-, \mathbb{Z}).
\]

The walls mentioned above can be thought of as hyperplanes in \( \Gamma_R = \Gamma \otimes \mathbb{Z} R \).

Our goal in this section is to compare the holomorphic Euler characteristics \( \chi(M_+, \mathcal{L}) \) and \( \chi(M_-, \mathcal{L}) \), which are given by the Hirzebruch-Riemann-Roch theorem:

\[
\chi(M_{\pm}, \mathcal{L}) = \int_{M_{\pm}} \exp(c_1(\mathcal{L})) \text{Todd}(M_{\pm}).
\]

As this expression is manifestly polynomial in \( c_1(\mathcal{L}) \), we obtain thus two polynomials on \( \Gamma \), and our goal is to calculate their difference, the \( \text{wall-crossing term} \)

\[
(27) \quad \chi(M_+, \mathcal{L}) - \chi(M_-, \mathcal{L}).
\]

5.2. The master space construction

To simplify our setup, we will make some additional assumptions.

Assumptions 5.1.  

(1) The generic stabilizer of \( X \) is trivial.

(2) Let \( L_+ \) and \( L_- \) be two ample linearizations of the \( G \)-action on \( X \) from the adjacent chambers corresponding to the quotients \( M_+ \) and \( M_- \). Without loss of generality, we can assume that the linearization \( L_0 = L_+ \otimes L_- \) lies
on the single wall separating the two chambers, and that the interval connecting \( c_1(L_+) \) and \( c_1(L_-) \) in \( \Gamma_R = \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \) does not intersect any other walls.

(3) Let \( X^0 \) be the set of those semistable points \( x \in X^{ss}[L_0] \) which are not stable for \( L_\pm \):

\[
X^0 := X^{ss}[L_0] \setminus (X^s[L_+] \cup X^s[L_-])
\]

We assume that \( X^0 \) is smooth, and that for \( x \in X^0 \) the stabilizer subgroup \( G_x \subset G \) is isomorphic to \( \mathbb{C}^* \).

(4) Assume that there is a linearization \( \hat{L} \) of the \( G \)-action on \( X \) such that \( L_+ = L_- \otimes \hat{L}^n \) for some positive integer \( n \), and such that for each \( x \in X^0 \), the stabilizer subgroup \( G_x \) acts freely on \( L_x \backslash 0 \).

Now we introduce the master space construction of Thaddeus \([Th1] \). Consider the variety \( Y = \mathbb{P}(O \oplus \hat{L}) \), which is a \( \mathbb{P}^1 \)-bundle over \( X \) endowed with the additional \( \mathbb{C}^* \)-action \((1, t^{-1})\). As \( Y \) is a projectivization of a vector bundle on \( X \), it comes equipped with \( O(1) \), which is the standard \( G \times \mathbb{C}^* \)-equivariant line bundle. To simplify our notation, we will denote the same way the linearizations of the \( G \)-action on \( X \) and their pull-backs (with tautological \( G \)-action) to \( Y \).

The master space \( Z \) then is the GIT quotient of \( Y \) with respect to the linearization \( L_-(n) = L_- \otimes O(n) \):

\[
Z = Y \sslash_{L_-(n)} G,
\]

which inherits a \( \mathbb{C}^* \)-action from \( Y \). Some additional notation:

- We will denote this copy of \( \mathbb{C}^* \) by \( T \).
- the projection \( Y \to X \) by \( \pi \), and the quotient map \( Y^s \to Z \) by \( \psi \).
- Introduce the notation \( Y(0 : \cdot) \) and \( Y(\cdot : 0) \) for the two copies of \( X \) in \( Y \), corresponding to the two poles of the projective line; then \( Y \) is partitioned into 3 sets:

\[
Y = Y(0 : \cdot) \sqcup Y(\cdot : 0) \sqcup \hat{\mathbb{L}}^\circ,
\]

where \( \hat{\mathbb{L}}^\circ \) is the line bundle \( \hat{\mathbb{L}} \) with the zero-section removed. We will write \( \pi_0 \) for the restriction of \( \pi \) to \( \hat{\mathbb{L}}^\circ \). We can collect our maps on the following diagram.

\[
\hat{\mathbb{L}}^\circ \quad \xymatrix{ \quad \ar[r] & Y = \mathbb{P}(O \oplus \hat{L}) \supset Y^s \ar[r]^(0.5){\psi} & Z \ar[d]^{\pi} \quad \xymatrix{ \quad } \ar[r]_{\pi_0} & X}
\]

**Proposition 5.2.**

(1) There are embeddings

\[
t_- : M_- \to Z \quad \text{and} \quad t_+ : M_+ \to Z
\]

obtained as the quotients \( Y^s \cap Y(\cdot : 0)/G \) and \( Y^s \cap Y(0 : \cdot)/G \), correspondingly.

(2) The strictly semistable locus of \( Y \) with respect to the linearization \( L_-(n) \) is empty, and the GIT quotient \( Z = Y^s/G \) is smooth.

(3) There is an embedding \( t_0 : X^0/G \to Z \), obtained via \( \psi(\pi_0^{-1}(X^0)) \). We denote the image of \( t_0 \) by \( Z^0 \).

(4) The fixed point locus \( Z^T \) is the disjoint union of \( t_+(M_+) \), \( t_-(M_-) \), and \( Z^0 \).
Proof. (1)-(3) follow from [11, 4.2, 4.3]. To prove (4), first note that $Y(\cdot : 0)$ and $Y(0 : \cdot)$ are fixed by $T$, so we immediately obtain that $M_\pm \subset Z$ are fixed components. Also the $G$-action on $Y$ commutes with the $T$-action, so a point $\psi(y) \in \psi(\pi^{-1}(X))$ is fixed by $T$ if and only if the $T$-orbit $T \cdot y \subset \pi^{-1}(X)$ is contained in the $G$-orbit $G \cdot y \subset \pi^{-1}(X)$. Since $T \cdot y \subset \pi^{-1}(x)$ for some $x \in X$, we need $y \in \pi^{-1}(X^0)$. Moreover, for any $y \in \pi^{-1}(x) \subset \pi^{-1}(X^0)$, $T \cdot y = \pi^{-1}(x) = G_x \cdot y$, so a point $\psi(y) \in \psi(\pi^{-1}(X))$ is fixed by $T$ if and only if $\psi(y) \in \psi(\pi^{-1}(X^0)) = Z^0$. \hfill $\Box$

**Construction:** Given a $G$-equivariant vector bundle $E$ on $X$, we can construct a $T$-equivariant vector bundle $\zeta(E) \to Z$ on $Z$ by first pulling $E$ back from $X$ to $Y$, and endowing the resulting bundle $\pi^*E$ with the trivial action of $T$, and the action of $G$ pulled back from $X$. Then the identification of the normal bundles of the fixed point components of $T$ is fixed by $\pi^*E$ to $Z$. Then it is easy to verify the following.

**Lemma 5.3.** The restriction of the line bundle $\zeta(L)$ to $Z^0$ is trivial with $T$-weight $I$.

Before we formulate our wall-crossing formula, we need one more ingredient: the identification of the normal bundles of the fixed point components of $Z$.

**Lemma 5.4.**

1. The normal bundle on the component $M_+$ of $Z^T$ is $\zeta(L^{-1})|_{M_+}$, and the normal bundle of $M_-$ is $\zeta(L)|_{M_-}$.

2. The normal bundle $N_{Z^0}$ of $Z^0 = X^0/G \subset Z$ may be described as the descent of the normal bundle $N_{X^0}$ of $X^0 \subset X$. The weights of the action may be computed by fixing $x \in X^0$, identifying the stabilizer subgroup $G_x \subset G$ with $T$ via its action on the fiber $L_x$, and then considering the action of $G_x$ on $N_{X^0}$.

**Definition 5.5.** Given a $T$-vector bundle $V$ on a manifold on which $T$ acts trivially, the $T$-equivariant $K$-theoretical Euler class of $V^*$, which we denote by $E_t(V)$, may be described as follows: let $x_1, \ldots, x_n$ be the Chern roots of $V$, and $l_1, \ldots, l_n \in \mathbb{Z}$ be the corresponding $T$-weights. Then

$$E_t(V) = \prod_{j=1}^n (1 - t^{-l_j} \exp(-x_j)).$$

Now we are ready to write down our wall-crossing formula for (27). A key role will be played by the following notion: given a rational differential $1$-form on the Riemann sphere, let us denote taking the sum of residues at $0$ and at infinity by $\mu \mapsto \text{Res}_{t=0,\infty} \mu$:

$$\text{Res}_{t=0,\infty} \mu \overset{\text{def}}{=} \text{Res}_{t=0} \mu + \text{Res}_{t=\infty} \mu.$$  

**Theorem 5.6.** Let $L$ be a linearization of the $G$-action on $X$, and denote, as above, by $\zeta(L)$ the $T$-equivariant line bundle on $Z$ obtained by pull-back to $Y$ and descent to $Z$. If Assumptions [5.4] hold, then

$$\chi(M_+, L) - \chi(M_-, L) = \text{Res}_{t=0,\infty} \int_{Z^0} \frac{\text{ch}_t(\zeta(L)|_{Z^0})}{E_t(N_{Z^0})} \text{Todd}(Z^0) \frac{dt}{t},$$

where $N_{Z^0}$ is the $T$-equivariant bundle on $Z^0$ described in Lemma 5.4, $\text{ch}_t$ is the $T$-equivariant Chern character, and $E_t(N_{Z^0})$ is the $K$-theoretical Euler class of $N_{Z^0}$.  

$\square$
Proof of Theorem 5.6. The Atiyah-Bott fixed-point formula [AB1] applied to the line bundle $L$ on our master space $Z$ yields

$$\chi_t(Z, L) = \sum_{F \in Z^T} \int_F \frac{\text{ch}(\mathcal{L})}{E_t(N_F)} \text{Todd}(F),$$

where the sum is taken over the connected components of the fixed point locus $Z^T$.

In Proposition 5.2, we identified these components as $M_+, M_-$ and $Z^0$. According to Lemma 5.4, for $M_-$, the normal bundle is simply $\zeta(L)$, and thus the contribution of $M_-$ is equal to

$$\int_{M_-} \frac{\text{ch}(\mathcal{L}) \text{Todd}(M_-)}{1 - t \exp(-c_1(L))}.$$

A similar calculation gives the contribution of $M_+$ as

$$\int_{M_+} \frac{\text{ch}(\mathcal{L}) \text{Todd}(M_+)}{1 - t^{-1} \exp(c_1(L))}.$$

We observe that $\chi_t(Z, L)$ is a Laurent polynomial in $t$ since it is the alternating sum of $T$-characters of finite dimensional vector spaces. Thus, as a function of $t$, $\chi_t(Z, L)$ has poles only at $t = 0, \infty$, and by the Residue Theorem, we have

$$\text{Res}_{t=0,\infty} \frac{\chi_t(Z, L)}{t} = 0.$$

On the other hand, since

$$\text{Res}_{t=0,\infty} \frac{A}{1 - t^{-1}B} \frac{dt}{t} = -A \quad \text{and} \quad \text{Res}_{t=0,\infty} \frac{A}{1 - tB} \frac{dt}{t} = A,$$

we have

$$\text{Res}_{t=0,\infty} \int_{M_-} \frac{\text{ch}(\mathcal{L}) \text{Todd}(M_-)}{1 - t \exp(-c_1(L))} \frac{dt}{t} = \chi(M_-, L) \quad \text{and}$$

$$\text{Res}_{t=0,\infty} \int_{M_+} \frac{\text{ch}(\mathcal{L}) \text{Todd}(M_+)}{1 - t^{-1} \exp(c_1(L))} \frac{dt}{t} = -\chi(M_+, L).$$

Now, applying the functional $\text{Res}_{t=0,\infty}$ to the two sides of (29) multiplied by $dt/t$ gives us the desired result (47).

6. Wallcrossings in parabolic moduli spaces

In this section we apply Theorem 5.6 to wall crossings in the moduli space of parabolic bundles.

From now on, we assume that $d = 0$, and we write $\Delta$ for the corresponding set of admissible parabolic weights $\Delta_0$. Recall from Section 2.2 that for regular $c \in \Delta$, the moduli space of stable parabolic bundles $P_0(c)$ is the GIT quotient $XQ_{/c} \sslash PSL(\chi)$, where $XQ$ is the subspace of the total space of the flag bundle over the Quot scheme. Let us fix a partition $\Pi = (\Pi', \Pi'')$ and an integer $l$, and introduce the notation $\Delta'_l$ and $\Delta''_{l'}$ for the simplices of parabolic weights of $\Pi'$ and $\Pi''$. Let $\phi \in \Sigma_r$ be the unique permutation which sends $\{1, \ldots, r'\}$ to $\Pi'$ preserving the order of first $r'$ and the last $r''$ elements. We choose $c^0 = (c^0_1, \ldots, c^0_{r'}) \in S_{\pi,l}$ and two regular
elements \( c^+, c^- \in \Delta \) in two neighboring chambers separated by the wall \( S_{\Pi_1} \), such that
\[
c^\pm = c^0 + \epsilon(\ldots, 0, 1, 0, \ldots, 0, -1)
\]
for some positive \( \epsilon \in \mathbb{Q} \), where 1 and \(-1\) are on the \( \phi(r') \)th and \( r' \)th places, respectively. Let
\[
c' = \sum_{i \in \Pi'} c^0_i x_i \in \Delta'_1 \quad \text{and} \quad c'' = \sum_{i \in \Pi''} c^0_i x_i \in \Delta''_{-1}.
\]
For \((k, \lambda) \in \mathbb{Z} \times \Lambda\), consider the polynomials
\[
q_{\pm}(k, \lambda) = \chi(P_0(c^\pm), \mathcal{L}_0(k; \lambda)).
\]
Our goal is to calculate the difference of these two polynomials.

**Notation:** To simplify our notation, from now on, we omit the index \( t \) from the symbols for equivariant characteristic classes.

### 6.1. The master space construction

We construct the master space \( Z \) from §5.2 using the following data:
- a smooth variety \( X = XQ \) (cf. §2.2);
- linearizations \( L^\pm = L(k; \lambda^\pm) \) of the \( G \)-action on \( X \) (cf. §2.2), such that \( \lambda^\pm / k = c^\pm \);
- the linearization \( \mathcal{L} = L(0; x_{\phi(r')} - x_r) \) of the \( G \)-action on \( X \).

The following statement is easy to verify.

**Lemma 6.1.** ([BH §3.2]) The subset \( X^0 \subset X \) is the set of points representing vector bundles \( W \) on \( C \), such that \( W \) splits as a direct sum \( W' \oplus W'' \), where \( W' \) and \( W'' \) are, respectively, \( c' \) and \( c'' \)-stable parabolic bundles. Therefore, we have the following description of the locus \( Z^0 \):
\[
Z^0 = \{ W = W' \oplus W'' | W' \in \tilde{P}_1(c'); W'' \in \tilde{P}_{-1}(c''); \det(W) \simeq 0 \}.
\]

**Remark 6.2.** Note that \( Z^0 \) is fibered over \( \text{Jac}^1 \) with fibre \( P_1(c') \times P_{-1}(c'') \) by the determinant map \( \tilde{P}_1(c') \to \text{Jac}^1 \) and
\[
H^*(Z^0, \mathbb{Q}) \simeq H^*(P_1(c') \times P_{-1}(c''), \mathbb{Q}) \otimes H^*(\text{Jac}^1, \mathbb{Q}).
\]

**Remark 6.3.** If the rank of the vector bundle \( W \in \tilde{P}_1(c) \) is 1, then \( c = 1 \) and \( \tilde{P}_1(1) \) is isomorphic to \( \text{Jac}^1 \), while \( P_1(1) \) is a point.

Now we need to verify the hypothesises of Theorem 5.6. First note that in our present construction \( X \) is not projective, however, it contains all semisimple points of the Quot scheme for all possible polarizations, and hence the missing points of the Quot scheme have no effect on any of our constructions.

Assumptions §5.1 (1)-(2) are trivially satisfied, so we study the action of the stabilizer \( G_x \subset \text{SL}_X \) of point \( x \in X \) on the fiber \( \mathcal{L}_x \).

- For a general point \( x \in X \) the stabilizer of \( x \) is the center \( Z_x \subset \text{SL}(X) \), which acts trivially on the fiber \( \mathcal{L}_x \).
- For \( x \in X^0 \), any element of the stabilizer of \( x \) induces an automorphism of the corresponding vector bundle \( W = W' \oplus W'' \), so \( G_x \simeq \mathbb{C}^* \times \mathbb{C}^* \subset \text{GL}_X \).

Then \((t_1, t_2) \in G_x \) is in \( \text{SL}_X \) if and only if \( t_1^x t_2^{x''} = 1 \), where \( x' = \chi(W') \) and \( x'' = \chi(W'') \). Note that \((t_1, t_2) \) acts on \( \mathcal{L}_x \) as \( t_1 t_2^{-1} \), and we need \( t_1 = t_2 \).
(hence $t^x = 1$) for this action to be trivial, so the stabilizer of any point in $\tilde{L}_x \setminus \emptyset$ is the center $Z_x \subset SL_x$.

Then the action of $G = \operatorname{PSL}(\chi)$ is free on $Y(0 : \cdot) \cup Y(\cdot : 0)$ and the action of $G_x \subset \operatorname{PSL}(\chi)$ on $\tilde{L}_x \setminus \emptyset$ induces an isomorphism $G_x \simeq \mathbb{C}^* \simeq T$.

Now by Theorem 6.9 the wall-crossing polynomial $q_-(k;\lambda) - q_+(k;\lambda)$ is equal to

\begin{equation}
\operatorname{Res}_{t=0,\infty} \int_{Z^0} \frac{\operatorname{ch}(\mathcal{L}_0(k;\lambda)|Z^0)}{E(N_{Z^0})} \operatorname{Todd}(Z^0) \frac{dt}{t}.
\end{equation}

Note that in our case, the $T$-action on $Z$ is free outside the fixed locus $Z^T$, so as a function in $t \in T$, the integral in (30) may have poles only at $t = 0, 1, \infty$. Then, using the Residue Theorem and substituting $t = e^u$, we conclude that (30) equals

\begin{equation}
- \operatorname{Res}_{u=0} \int_{Z^0} \frac{\operatorname{ch}(\mathcal{L}_0(k;\lambda)|Z^0)}{E(N_{Z^0})} \operatorname{Todd}(Z^0) \, du,
\end{equation}

and thus our goal is to calculate this integral.

Our first step is to identify the characteristic classes under the integral sign (cf. Proposition 6.9 for the result).

We start with the study of the restriction of the line bundle $\mathcal{L}_0(k;\lambda)$ to the fixed locus $Z^0 \subset Z$. Let $\beta$ be the Poincare bundle over $\operatorname{Jac} \times C$, such that $c_1(\beta)|_0 = 0$; define $\eta \in H^2(\operatorname{Jac})$ by $(\sum_i c_1(\beta)|_{(e_i)} \otimes e_i)^2 = -2\eta \otimes \omega$ (cf. §2.3), then (cf. [Z]) for any $m \in \mathbb{Z}$

\begin{equation}
\int_{\operatorname{Jac}} e^{\eta m} = m^g.
\end{equation}

Recall that for a parabolic weight $c = (c_1, \ldots, c_r) \in \Delta$ we have set $c_\Pi' = \sum_{i \in \Pi'} c_i$.

**Lemma 6.4.** Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda$, $k \in \mathbb{Z}^{>0}$ and let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition with $r \in \Pi''$. Let

\[ N' = \sum_{i \in \Pi'} \lambda_i x_i \quad \text{and} \quad N'' = \sum_{i \in \Pi''} \lambda_i x_i \]

and define $\delta$ by $(\lambda/k)|_{\Pi''} = 1 + \delta$. Then

\[
\operatorname{ch}(\mathcal{L}_0(k;\lambda)|Z^0) = e^{k\delta u} \exp \left( \frac{\eta k}{r'} + \frac{\eta k}{r''} \right).
\]

\[
\operatorname{ch}(\mathcal{L}_1(k;\lambda_1', \ldots, \lambda_r' - k\delta) \boxtimes \mathcal{L}_{-l}(k;\lambda_1'', \ldots, \lambda_r'') + k\delta)),
\]

where $\boxtimes$ denotes the external tensor product of line bundles on $P_1(c') \times P_{-1}(c'')$.

**Lemma 6.5.** Denote by $\tilde{U}'$ and $\tilde{U}''$ the universal bundles over $\tilde{P}_1(c') \times C$ and $\tilde{P}_{-1}(c'') \times C$ with the standard normalization (cf. §2.3), and denote by $\pi$ projections along $C$. Then the equivariant normal bundle to the fixed locus $Z^0 \subset Z$ is

\[ N_{Z^0} = R^1_\pi \operatorname{ParHom}(\tilde{U}', \tilde{U}'') \oplus R^1_\pi \operatorname{ParHom}(\tilde{U}'', \tilde{U}'), \]

where $T \simeq \mathbb{C}^*$-action has weights $(-1, 1)$. 
6.2. Calculation of the characteristic classes of $N_{Z^0}$

Before we calculate the equivariant K-theoretical Euler class of the conormal bundle $N^*_Z$, we need to introduce some notations. Recall that for $1 \leq i, j \leq r$, the differences $x_i - x_j \in V^*$ are linear functions on $V$, and the function $x_i - x_j$ corresponds to the linearization $L_0(0; x_i - x_j)$ on $X$, which descends to the line bundle $L_0(0; x_i - x_j)$ on the moduli space $P_0(c)$ (cf. §2.2). As in §5.2, we denote by $\zeta(L_0(0; x_i - x_j))$ the line bundle on $Z$ obtained by the pullback and then descent. Therefore, we obtain a correspondence between the linear functions $x_i - x_j$ and the T-equivariant line bundles on $Z$.

As $Z^0$ is a connected component of the fixed locus of the $T$-action on $Z$, and thus its equivariant cohomology factors: $H^*_T(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$. In particular, there are canonical embeddings $H^*(Z^0) \hookrightarrow H^*_T(Z^0)$ and $\mathbb{C}[u] \hookrightarrow H^*_T(Z^0)$.

Recall the definition of the permutation $\phi \in \Sigma_r$ given at the beginning of this chapter: $\phi$ takes the first $r'$ numbers to $\Pi'$, preserving the order of the first $r'$ and the last $r''$ elements. We introduce the symbols

$$z_i' - z_j' = c_1(\zeta(L_0(0; x_{\phi(i)} - x_{\phi(j)}))|_{Z^0}), \quad (1 \leq i, j \leq r')$$

$$z_i'' - z_j'' = c_1(\zeta(L_0(0; x_{\phi(r'+i)} - x_{\phi(r'+j)}))|_{Z^0}), \quad (1 \leq i, j \leq r'')$$

$$u = (z_i' - z_j') = c_1(\zeta(L_0(0; x_{\phi(r') - x_r}))|_{Z^0})$$

for the equivariant cohomology classes in $H^*_T(Z^0)$. The last equalities are consistent with Lemma 5.3.

**Remark 6.6.** Note that (cf. Remark 5.2)

$$z_i' - z_j' = c_1(\mathcal{F}_{r-i+1}^r \otimes (\mathcal{F}_{r-j+1}^r \otimes (\mathcal{F}_{r-r-i}^r)^*) \in H^2(P_1(c'))),$$

$$z_i'' - z_j'' = c_1(\mathcal{F}_{r-i+1}^r \otimes (\mathcal{F}_{r-j+1}^r \otimes (\mathcal{F}_{r-r-i}^r)^*) \in H^2(P_{-1}(c''))),$$

where $\mathcal{F}_i^r$ and $\mathcal{F}_i^r$ are the flag bundles (cf. §2.3) on $P_0(c')$ and $P_0(c'')$, correspondingly.

Taking into account these identifications, functions on $V$ give rise to equivariant cohomology classes on $Z^0$. To make the splitting $H^*_T(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$, explicit, however, we will write these classes in the form $f_u(z', z'')$, thinking of them as functions of the differences of the $z_i$'s and the differences of the $z_i''$s, depending on the parameter $u$. With this convention, we introduce

$$w_u(z', z'') = \prod_{i,j} \frac{2\sinh(z_i' - z_j')}{\phi(i) < \phi(r'+j)} \prod_{i,j} \frac{2\sinh(z_i'' - z_j'')}{\phi(r'+j) < \phi(i)},$$

$$\rho_u(z', z'') = \frac{1}{2} \sum_{i,j} (z_i' - z_j') + \frac{1}{2} \sum_{i,j} (z_i'' - z_j''),$$

where according to (33)

$$z_i' - z_j' = (z_i' - z_i'') + u - (z_j'' - z_i'') = c_1(\zeta(L_0(0; x_{\phi(i)} - x_{\phi(r'+j)}))|_{Z^0}) \in H^*_T(Z^0).$$

Now we are ready to write down our formula for the K-theoretical Euler class $E(N_{Z^0})$ (cf. definition 5.5 with $t = e^u$).
**Proposition 6.7.**

\[
E(N_{Z_0})^{-1} = (-1)^{tr + r' r'' (g - 1)} e^{-r l u} \exp \left( \frac{\eta r}{r'} + \frac{\eta r}{r''} \right) \ch_{\mathcal{L}_1}(r''; -l, ..., -l, -l + r l) \otimes \mathcal{L}_{-1}(r'; l, ..., l, l - r l).
\]

**Proof.** It follows from the short exact sequence (6) for parabolic morphisms that

\[
\ch(-\pi_0(\text{ParHom}(\tilde{U}'', \tilde{U}'))) = -\ch(\pi_0(\text{Hom}(\tilde{U}'', \tilde{U}'))) + \sum_{i,j} e^{z_i - z_j^r},
\]

and

\[
\ch(-\pi_0(\text{ParHom}(\tilde{U}', \tilde{U}''))) = -\ch(\pi_0(\text{Hom}(\tilde{U}', \tilde{U}''))) + \sum_{i,j} e^{z_i - z_j^r},
\]

so by Lemma 6.5,

\[
\ch(N_{Z_0}) = \ch(-\pi_0(\text{Hom}(\tilde{U}', \tilde{U}''))) \oplus -\pi_0(\text{Hom}(\tilde{U}', \tilde{U}''))
\]

(34)

\[
+ \sum_{i,j} e^{z_i - z_j^r} + \sum_{i,j} e^{z_i^r - z_j}.
\]

**Lemma 6.8.** We denote by \([f(x)]^W\) the multiplicative class of the vector bundle \(W\) given by the function \(f(x)\) in Chern roots of \(W\). Let \(S\) be a vector bundle on \(P \times C\) with T-weight 1, and \(\pi: P \times C \to P\) projection along the curve, then

\[
E(-\pi_0 S \oplus -\pi_0 S^*)^{-1} = (-1)^{rk(-\pi_0 S)} \frac{\exp(-\ch_2(S)(2))}{[(2\sinh(x/2))^{2g-2}]^{S_p}}.
\]

**Proof.** Note that

\[
E(-\pi_0 S)^{-1} = \left[ \frac{1}{1 - t^{-1} e^{-x}} \right]^{-\pi_0 S} = \left[ \frac{-te^{-x}}{1 - te^{-x}} \right]^{-\pi_0 S}
\]

and

\[
E(-\pi_0 S^*)^{-1} = \left[ \frac{1}{1 - te^{-x}} \right]^{-\pi_0 S^*} = \left[ \frac{1}{1 - te^{-x}} \right]^{-(\pi_0 S^*)^*}.
\]

Applying Serre duality and the Grothendieck-Riemann-Roch Theorem we obtain

\[
\ch(-\pi_0 S) + \ch((-\pi_0 S^*)^*) = \ch(-\pi_0 S) + \ch(\pi_0 (S \otimes K_C)) = \ch(-\pi_0 S) + \pi_0 (\ch(S \otimes K_C) \text{Todd}(C)) = \ch(-\pi_0 S) + \pi_0 (S + (2g - 2) \ch(S_p)) = (2g - 2) \ch(S_p),
\]

where \(K_C\) is the canonical sheaf on the curve \(C\), hence

\[
\left[ \frac{1}{1 - te^{-x}} \right]^{-\pi_0 S \oplus (-\pi_0 S^*)^*} = \left[ \frac{1}{(1 - te^{-x})^{2g-2}} \right]^{S_p} = \exp(-c_1(S_p)(g - 1)).
\]

Since

\[
[-te^{-x}]^{-\pi_0 S} = (-1)^{rk(-\pi_0 S)} \exp(c_1(-\pi_0 S))
\]

and by the Grothendieck-Riemann-Roch theorem

\[
\ch_1(-\pi_0 S) = \ch_1(S_p)(g - 1) - \ch_2(S)(2),
\]
we conclude that
\[-te^x]_{-n:S} = (-1)^{rk(-n:S)}\exp(c_1(S_p)(g-1))\exp(-ch_2(S)_{(2)}),\]
which finishes the proof of Lemma 6.8.

Note that the last two terms in (34) are the sums of Chern characters of line bundles, so they contribute the multiplicative factor
\[\exp(\rho_u^\chi(z',z''))/w_u^\chi(z',z'')\]
to the equivariant class $E(N_{Z_0})^{-1}$; and using Lemma 6.8 with $S = \text{Hom}(U', U'')$, we obtain that the inverse of the K-theoretical Euler class of the first term in (34) is
\[(-1)^{1+r' r'' (g-1)}w_u^\chi(z',z'')2^{-2g}\exp(-ch_2(\text{Hom}(U', U''))_{(2)}).
\]
Note that
\[-ch_2(\text{Hom}(U', U''))_{(2)} = \frac{1}{2}c_2(\text{End}_0(U' + U''))_{(2)} - \frac{1}{2}c_2(\text{End}_0(U'))_{(2)} - \frac{1}{2}c_2(\text{End}_0(U''))_{(2)} = c_1(L_0^1|_{Z_0} \otimes L_{-1}(-r'; -l_1, ..., -l) \boxtimes L_{-1}(-r''; l_1, ..., l))\].

The latter equality follows from Lemma 2.8. Finally, using Lemma 6.4 to calculate the Chern character of $L_0^1|_{Z_0}$, we obtain the formula for the class $E(N_{Z_0})^{-1}$, and the proof of the Lemma is complete.

### 6.3. The wall-crossing formula

Putting Lemma 6.4 and Proposition 6.7 together, we obtain the following.

**Proposition 6.9.** The wall-crossing term (37) is equal to
\[
\text{K Res}_{u=0} e^{(k\delta - rl)u} \int_{P_t(c') \times P_{-1}(c'')} \left[\left(w_u^\chi(z',z'')\right)^{1-2g}\exp(\rho_u^\chi(z',z''))\right] \cdot \left|\chi(L_t(k + r'; \lambda'_1 - l, ..., \lambda''_{r-1} - l, \lambda''_r - l - k\delta + rl))\boxtimes L_{-1}(k + r'; \lambda''_1 + l, ..., \lambda''_{r-1} + l, \lambda''_r + l + k\delta - rl))\right| \cdot \text{Todd}(P_t(c') \times P_{-1}(c'')) \cdot \text{Todd}(P_0(c'') \times P_0(c'')) \right| \cdot \text{du},
\]
where $\delta$ is a parameter depending on $\lambda$ and the wall $S_{11,1}$ (cf. Lemma 6.4) and $K$ is the constant $(-1)^{1+r' r'' (g-1)} \cdot (r(k+r))^{g} / (r'^r)^g$.

Now all that is left to do is to perform the integral, using an induction on the rank based on Corollary 4.10. We will begin with the case $l = 0$, as it is simpler. For $l = 0$, the integral from Proposition 6.9 has the form
\[
(35) \int_{P_t(c') \times P_{-1}(c'')} \left[w_u^\chi(z',z'')\exp(\rho_u^\chi(z',z''))\text{Todd}(P_t(c'))\text{Todd}(P_0(c'')) \cdot \chi(L_t(k + r'; \lambda'_1 - l, ..., \lambda''_{r-1} - l, \lambda''_r - k\delta))\boxtimes L_0(k + r'; \lambda''_1 + l, ..., \lambda''_{r-1} + l, \lambda''_r + k\delta)\right] \cdot \text{du},
\]
The inductive hypothesis (21) maybe cast in the following form

\[(36) \int \chi(L_0(k; \lambda)) \text{Todd}(P_0(c)) = \hat{N}_{r,k} \sum_{B \in \mathcal{D}} \text{iBer}[\exp(\lambda, x, \hat{k}) \cdot w(\lambda)_{\mathcal{B}}(x, \hat{k})^{1-2g}](\rho/\hat{k} - [c]_{\mathcal{B}}).\]

Now let us fix \(k\), and allow to vary \(\lambda\). We can extend this equality by linearity to arbitrary linear combinations of Chern characters of line bundles of the form
\[
\sum_i \chi(L_0(k; \lambda^i)) = \chi(L_0(k; 0)) \cdot \sum_i \chi(L_0(0; \lambda^i)).
\]

Since any polynomial on \(V\), up to a fixed degree may be represented as a linear combination of exponential functions of the form \(\exp(\lambda, x, \hat{k})\), formula (36) may be generalized in the following way.

**Lemma 6.10.** Let \(G(x)\) be a formal power series on \(V\), and denote by \(G(z)\) the characteristic class in \(H^*(P_0(c))\) obtained by the identification of functions on \(V\) and cohomology classes of \(P_0(c)\), described before the equation (33). Then we have

\[(37) \int \chi(L_0(k; 0)) G(z) \text{Todd}(P_0(c)) = \hat{N}_{r,k} \sum_{B \in \mathcal{D}} \text{iBer}[G(x, \hat{k}) \cdot w(\lambda)_{\mathcal{B}}(x, \hat{k})^{1-2g}](\rho/\hat{k} - [c]_{\mathcal{B}}).\]

Finally, let \(\mathcal{D}'\) and \(\mathcal{D}''\) be Hamiltonian bases (cf. §4.5). Since
\[
w_{\Phi'}(x, \hat{k}) w_{\Phi''}(x, \hat{k}) = w_{\Phi}(x, \hat{k}),
\]
where \(w_{\Phi'}, w_{\Phi''}\) and \(\rho', \rho''\) are naturally defined for the root systems \(\Phi'\) and \(\Phi''\) (cf. §4.5), the integral (35) is equal to

\[
\hat{N}_{r', k' + r''} \hat{N}_{r'', k' + r''} \sum_{B' \in \mathcal{D}'} \sum_{B'' \in \mathcal{D}''} \text{iBer} \text{iBer}[\exp(\lambda, x, \hat{k})^{1-2g} e^{\rho'(x, \hat{k})}]
\]

\[
((\lambda_{r'; 1}, ..., \lambda_{r'; r' - 1}, \lambda_{r'; r'} - k\delta)/\hat{k} - [c']_{\mathcal{B}'}, + (\lambda_{r' + 1}, ..., \lambda_{r'' + 1}, \lambda_{r'' + r''} + k\delta)/\hat{k} - [c'']_{\mathcal{B}''}).
\]

Identifying \(u\) (cf. (32)) with the "link" element of the diagonal basis \(D = (\alpha(\Phi(r'), \mathcal{D}' \check{\mathcal{D}}'))\) (cf. §4.5), and moving the factor \(e^{k\delta u}\) from Proposition 6.9 inside the argument of \text{iBer}, we obtain the proof of the following theorem for \(l = 0\).

**Theorem 6.11.** Let \(c^* \in \Delta\) be in the neighbouring chambers; then the wall-crossing term
\[
\chi(P_0(c^*), L_0(k; \lambda)) - \chi(P_0(c^-), L_0(k; \lambda))
\]
is equal to

\[(k + r) \hat{N}_{r,k} \sum_{B' \in \mathcal{D}'} \sum_{B'' \in \mathcal{D}''} \text{Res}_{\alpha(\Phi(r'), \mathcal{D}) = 0} \text{iBer} \text{iBer}[\exp(\lambda, x, \hat{k})^{1-2g}](\lambda/\hat{k} - [c^+]_{\mathcal{B}}) d\alpha(\Phi(r'), r),\]

where \(\mathcal{D}'\) and \(\mathcal{D}''\) are the diagonal bases of \(\Phi'\) and \(\Phi''\) (cf. §4.5) correspondingly.

**Remark 6.12.** Note that this wall-crossing term coincides with the one from Proposition 4.16.
Example 6. It follows from Example [1] that in case of rank 3, the permutation
\( \phi \in \Sigma_3 \) sends \((1,2,3)\) to \((1,3,2)\). Then \( u = c_1(F'_1 \otimes F''_1) \) and let \( z = z'_2 - z'_2 = c_1(F'_2 \otimes F''_2 \otimes F'''_2) \). Then the inverse of the K-theoretical Euler class of the conormal bundle is (cf. Proposition [6,7])

\[
\text{ch}(L) e^{\frac{y_1}{2}} e^{\frac{y_2}{2}} \left( 2\sinh \left( \frac{u}{2} \right) 2\sinh \left( \frac{z - u}{2} \right) \right)^{1-2g} ,
\]

where \( L = L_0(2;0,0) \) is a line bundle on the moduli space \( \mathcal{P}_0 \) of rank-2 degree-0 stable parabolic bundles. The Chern character of the restriction of the line bundle \( L_0(k;\lambda_1,\lambda_2,\lambda_3) \) to \( \Sigma \) is

\[
e^{-2y_u} \text{ch}(L^k_0) e^{\lambda_1 z + \lambda_2 u} .
\]

Hence the wall-crossing term

\[
\chi(P_0(<), L_0(k,\lambda)) - \chi(P_0(>), L_0(k,\lambda))
\]

is equal to

\[
- \left( \frac{3(k + 3)}{2} \right)^g \int_{u=0}^{e^{\lambda_2 u}} \left( \frac{2\sinh \left( \frac{u}{2} \right) 2\sinh \left( \frac{z - u}{2} \right) }{2\sinh \left( \frac{z-u}{2} \right) 2\sinh \left( \frac{z}{2} \right) } \right)^{2g-1-1} \text{Todd}(P_0) \, du.
\]

The integral is the Euler characteristis of a line bundle on a moduli space of degree-0 rank-2 stable parabolic bundles, so we can calculate it using the induction by rank. It is equal to

\[
(-1)^{g-1} (2(k+3))^g \int_{z=0} e^{(\lambda_1+1)z} (2\sinh \left( \frac{z-u}{2} \right) 2\sinh \left( \frac{z}{2} \right) )^{2g-1} (1-e^{(k+3)z}) \, dz ,
\]

so the wall-crossing term is

\[
(-3(k + 3)^2)^g \int_{u=0}^{e^{\lambda_1 z + \lambda_2 u + z}} \int_{z=0} w_q(z,u)^{2g-1} (1-e^{(k+3)z}) \, dz \, du ,
\]

where \( w_q(z,u) = 2\sinh \left( \frac{z-u}{2} \right) 2\sinh \left( \frac{z}{2} \right) \). Note that this is exactly the same polynomial as in Example [5] after changing \((z,u)\) to \((x,-y)\).

7. Tautological Hecke correspondences

If \( \lambda \neq 0 \), then we need one more step in our proof, which uses the Hecke correspondence to calculate the wall-crossing term [31].

7.1. The Hecke correspondence

Given a rank-\( r \) degree-\( d \) vector bundle \( W \) with a full flag \( 0 \subseteq F_1 \subseteq \ldots \subseteq F_r = W_p \) at \( p \), one can obtain a rank-\( r \) degree-\( d - 1 \) vector bundle \( W' \) with a full flag \( 0 \subseteq G_1 \subseteq \ldots \subseteq G_r = W'_p \) using the tautological Hecke correspondence construction as follows.

The evaluation map \( W \to W_p \) induces the short exact sequence of the associated sheaves of sections

\[
0 \to W' \xrightarrow{\alpha} W \to W_p/F_{r-1} \to 0
\]

on curve \( C \). Since \( W' \) is a kernel of \( \alpha \), it is a locally free sheaf, thus gives a rank-\( r \) vector bundle \( W' \) over \( C \) with \( \det(W') \simeq \det(W) \otimes \mathcal{O}(-p) \). The image of the associated morphism of vector bundles \( \alpha \) at the point \( p \) is \( F_{r-1} \subset W_p \), so
\( \alpha_p : W'_p \to W_p \) has a one-dimensional kernel \( G_1 \subseteq W'_p \). Moreover, compositions of \( \alpha_p \) with the quotient morphisms \( F_{r-1} \to F_{r-1}/F_1 \) induce a full flag of the corresponding kernels \( G_1 \subseteq \cdots \subseteq G_{r-1} \subseteq G_r = W'_p \) in \( W'_p \).

Denote this operator between the sets of isomorphism classes of degree-\( d \) and \( d - 1 \) vector bundles with a flag at \( p \) by

\[
\mathcal{H} : (W, F_p) \to (W', G_p).
\]

Similarly, for any \( m \geq 0 \), one can define the operator \( \mathcal{H}^m \) between the sets of isomorphism classes of degree-\( d \) and \( d - m \) vector bundles with a flag at the point \( p \) by iterating the above construction \( m \) times. Clearly, these maps are independent of the parabolic weights.

**Proposition 7.1.** Let \( c \in \Delta \) be a regular (cf. page[7]) point. Then the operator \( \mathcal{H} \) induces an isomorphism between the moduli spaces \( P_d(c_1, \ldots, c_r) \) and \( P_{d-1}(c_2, \ldots, c_r, c_1 - 1) \).

**Proof.** First, we need to show that if \( W \in P_d(c_1, \ldots, c_r) \) is a parabolic stable bundle with parabolic weights \( (c_1, \ldots, c_r) \), then \( W' \), its image under the Hecke operator \( \mathcal{H} \), is parabolic stable with respect to parabolic weights \( (c_2, \ldots, c_r, c_1 - 1) \). For this, consider the subbundle \( V' \subset W' \) and let \( \alpha(V') = V \subset W \) (cf. (38)) be its image. Since \( W \) is parabolic stable,

\[
\text{parslope}(V) < \text{parslope}(W) = \text{parslope}(W').
\]

We need to prove that \( \text{parslope}(V') < \text{parslope}(W') \). There are two possible cases:

- If \( \alpha \) maps \( V' \) to \( V \) isomorphically, then \( \text{deg}(V') = \text{deg}(V) \) and \( V_p \subset F_{r-1} \), hence \( \text{parslope}(V') = \text{parslope}(V) < \text{parslope}(W) \).
- Otherwise, \( \text{deg}(V') = \text{deg}(V) - 1 \), and \( V_p \) is not contained in \( F_{r-1} \), so one of the parabolic weights of \( V' \) is \( c_1 - 1 \). Then, as in the previous case, \( \text{parslope}(V') = \text{parslope}(V) \), and the result follows.

To show that the map \( \mathcal{H} \) is an isomorphism, note that \( \mathcal{H}^r \) maps

\[
P_d(c_1, c_2, \ldots, c_r) \to P_{d-1}(c_1 - 1, c_2 - 1, \ldots, c_r - 1).
\]

It is easy to check that given \( W \) and iterating the associated morphism of locally free sheaves of sections (38) \( r \) times, we obtain a subsheaf \( W' \subset W \) of sections of \( W \) which vanishes at the point \( p \). So the map (39) is just tensoring by \( O(-p) \), and hence it is an isomorphism. \( \Box \)

Now we can define an operator \( \mathcal{H}^m \) for any \( m \in \mathbb{Z} \), taking the inverse map if necessary. We will need the following statement, which follows from Proposition 7.1 and the construction of \( \mathcal{H}^m \).

**Corollary 7.2.** Let \( m \geq 0 \). Then under the isomorphism \( \mathcal{H}^m \) the line bundle \( \mathcal{L}_d(k; \lambda_1, \ldots, \lambda_r) \) corresponds to the line bundle \( \mathcal{L}_{d-m}(k; \lambda_{r-m+1}, \ldots, \lambda_r, \lambda_1 - k_r, \ldots, \lambda_{r-m} - k) \).

### 7.2. The effect of the Hecke correspondence on the integral

Recall that our goal is to calculate the wall-crossing term from Proposition 6.9. For simplicity, we assume that \( l \) is positive (the other case is analogous). We apply the Hecke operators \( \mathcal{H}^1 \) and \( \mathcal{H}^{-1} \) to the moduli spaces \( P_l(c') \) and \( P_{l-1}(c'') \) to obtain

\[
P_0' = P_0(c'_{l+1}, \ldots, c''_1, c'_1 - 1, \ldots, c'_1 - 1) \cong P_l(c')
\]

\[
P_0'' = P_0(c''_{r-m+1} + 1, \ldots, c''_r + 1, c'_1, \ldots, c''_{r-m} - 1) \cong P_{l-1}(c'').
\]
Recall (cf. page 10) that there is a natural action of the group $\Sigma_r$ on $V^*$, and hence (cf. page 27) on $H^2(P_1(c') \times P_{-1}(c''))$. Let $\tau' \in \Sigma_{r'}$ and $\tau'' \in \Sigma_{r''}$ be the cyclic permutations defined by

$$\tau' \cdot (c'_1 - 1, ..., c'_i - 1, c'_{i+1}, ..., c'_{r'}) = (c'_{i+1}, ..., c'_{r'}, c'_i - 1, ..., c'_1 - 1)$$

and

$$\tau'' \cdot (c''_1, ..., c''_{r''-1}, c''_{r''-1+1} + 1, ..., c''_1 + 1) = (c''_{r''-1+1} + 1, ..., c''_1 + 1, c''_1, ..., c''_{r''-1}).$$

And set $\tau = (\tau', \tau'') \in \Sigma_{r'} \times \Sigma_{r''} \subset \Sigma_r$. Note that

$$\tau' \cdot (-l + r'', -l + r', -l, ..., -l) = \tau' \cdot \rho' - \rho'$$

and

$$\tau'' \cdot (l, l, l - r'', ..., l - r') = \tau'' \cdot \rho'' - \rho'',$$

so applying the Hecke operator $H^2 \times H^{-1}$ to the wall-crossing term from Proposition 6.10 and using Corollary 7.2 we obtain that the wall-crossing term (31) is equal to

$$\int_{\mathcal{P}_0 \times \mathcal{P}_0''} \left( \tau \cdot w_{u_b}(z', z'')^{1-2g} e^{\lambda \cdot \rho(z', z'')} \right)$$

$$\cdot \left( \prod_{\mathcal{L}_0} (k + r''; \lambda'_1 - \hat{k}, ..., \lambda'_1 - \hat{k}, \lambda'_{r''-1} - \hat{k}, \lambda_{r''-1}, \lambda''_{r''-1}, \lambda''_{r''} - k \delta + rl) \right)$$

$$\cdot \left( \prod_{\mathcal{L}_0} (k + r'; \lambda''_1, ..., \lambda''_{r''-1}, \lambda''_{r''-1+1} + \hat{k}, ..., \lambda''_{r''} + \hat{k} + k \delta - rl) \right)$$

$$\cdot e^{\lambda' \cdot \rho'(z', z'') - \rho''(z', z'')} e^{\lambda'' \cdot \rho'(z', z'') - \rho''(z', z'')} \text{Todd}(\mathcal{P}_0) \text{Todd}(\mathcal{P}_0'') \ du.$$

As in §6.3 according to Lemma 6.10 we can calculate this integral using the induction on rank. Let $D'$ and $D''$ be two Hamiltonian diagonal bases. Then $\tau'(D')$ and $\tau''(D'')$ are also Hamiltonian diagonal bases (cf. Remark 3.3) and the integral in (40) is equal to

$$(-1)^{r'} \tilde{N}_{r', k+r''} \tilde{N}_{r''', k+r'} \sum_{B' \in \tau'(D')} \sum_{B'' \in \tau''(D'')}$$

$$\text{Iber} \text{Iber} \left[ \tau' \cdot (\lambda'_1 - \hat{k}, ..., \lambda'_1 - \hat{k}, \lambda'_{r''-1}, ..., \lambda'_{r''-1}, \lambda''_{r''} - k \delta + rl) / k - \right.$$  

$$\left[ \tau' \cdot (c'_1 - 1, ..., c'_1 - 1, c'_{1+1}, ..., c'_{r'}) \right] B' +$$

$$\left. \tau'' \cdot (\lambda''_1, ..., \lambda''_{r''-1}, \lambda''_{r''-1+1} + \hat{k}, ..., \lambda''_{r''} + \hat{k} + k \delta - rl) / \hat{k} - \right.$$  

$$\left[ \tau'' \cdot (c''_1, ..., c''_{r''-1}, c''_{r''-1+1} + 1, ..., c''_{r''} + 1) \right] B'' \right].$$

To arrive at Theorem 6.11 we need to make additional transformations of formula (41): first, we shift $\lambda'$ and $\lambda''$, and then we apply Lemma 4.5 to eliminate the cyclic permutation $\tau$.

Note that given an ordered basis $B \in B$ and an element $v \in V^*$ such that $\{v\}_B = 0$, for any weight $\lambda \in \Lambda$ and positive integer $k$ one have

$$\left( \lambda + \hat{k}v \right) / \hat{k} - [c + v]_B = \lambda / \hat{k} - [c]_B.$$
In particular, to perform the shift of $\lambda$ in (41), we use the following equality for any $B' \in D'$:

$$\begin{align*}
(43) \quad & (\lambda'_1 - \tilde{k}, ..., \lambda'_r - \tilde{k}, \lambda'_{r+1} - \tilde{k}, ..., \lambda'_{r'-1} - \tilde{k}, \lambda'_{r'} - k\delta + r1)/\tilde{k} - \\
& [(c'_1, ..., c'_{r'-1}, c'_{r'}) - (1, ..., 1, 0, ..., 0, -1)]_{B'} = \\
& (\lambda''_1, ..., \lambda''_{r'-1}, \lambda''_{r'} - k\delta + r1 - \tilde{1k})/\tilde{k} - [(c''_1, ..., c''_{r'-1}, c''_{r'}) - (1, ..., 1, 0, ..., 0, -1)]_{B''},
\end{align*}$$

which clearly remains true after changing $D'$ to $\tau'(D')$ and applying $\tau'$ to both sides of the equation. Similarly, shifting the last terms of (41) by $\tau''(0, ..., 0, -1, ..., -1, -1 + 1)$, we can rewrite (41) as

$$(-1)^{1r} \tilde{N}_{r', k + r''} \tilde{N}_{r, k + r'} \sum_{B' \in \tau'(D')} \sum_{B'' \in \tau''(D'')} \text{iBer}_B \text{iBer}_B' \sum_{B'' - e} \tau'' \cdot (\lambda''_1, ..., \lambda''_{r'-1}, \lambda''_{r'} + k\delta - r1 + \tilde{1k})/\tilde{k} - [\tau'' \cdot (c''_1, ..., c''_{r'-1}, c''_{r'}) - (1, ..., 1, 0, ..., 0, -1)]_{B''}.$$

Finally, identifying $u$ (cf. (53)) with the "link" element of the diagonal basis $\tau(D) = (\alpha^{\Phi}(r'), \tau(r) \tau'(D'))$ (cf. §4.5) and

- moving the factor $e^{(k\delta - r1)u}$ from (40) inside the argument of iBer$_B$, where $B = (\alpha^{\Phi}(r'), \tau(r) B' B'')$,
- applying (42) with $B = (\alpha^{\Phi}(r'), \tau(r) B' B'')$ and $v = 1\alpha^{\Phi}(r'), \tau(r)$,
- applying Lemma 4.5,
- and using the fact that

$$\tau^{-1} \cdot (w_{\Phi'}(x/\tilde{k})w_{\Phi''}(x/\tilde{k})) = (-1)^{1r}w_{\Phi'}(x/\tilde{k})w_{\Phi''}(x/\tilde{k}),$$

we obtain the formula of Theorem 5.11 for arbitrary $1 \in \mathbb{Z}$.

8. **Affine Weyl symmetry and the proof of part I of Theorem 4.8**

In this section, we prove certain symmetry properties of our Hilbert polynomials on the left hand side of (1), and we finish the proof of part I of Theorem 4.8. We start with the basic instance of symmetry of Hilbert polynomials: relative Serre duality.

8.1. **Serre duality**

**Proposition 8.1.** Let $\mathcal{E} \to X$ be a rank 2 vector bundle over a smooth variety $X$, $\pi : Y = \mathbb{P}(\mathcal{E}) \to X$ its projectivization and $\omega_{X/Y}$ the relative cotangent line bundle. Then

$$\chi(Y, \pi^*(\mathcal{L} \otimes \omega_{X/Y}^m)) = -\chi(Y, \pi^*(\mathcal{L} \otimes \omega_{X/Y}^{m+1})$$

for any line bundle $\mathcal{L} \in \text{Pic}(X)$.

**Proof.** It follows from the short exact sequence

$$0 \to \Omega_{X/Y} \otimes \mathcal{O}_X \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_X \to 0,$$

that

$$\omega_{X/Y} = \wedge^2(\pi^*\mathcal{E}(-1)) = \pi^*(\wedge^2\mathcal{E} \otimes \mathcal{O}(-2)).$$
By Serre duality for families of curves [H] Ch III, §7-8 we have then
\[
\chi(Y, \pi^*(L \otimes (\mathcal{L}^2 E)^m) \otimes \mathcal{O}(-2m)) = - \chi(Y, \pi^*(L \otimes (\mathcal{L}^2 E)^m) \otimes \pi^*(\mathcal{L}^2 E)^{-2m+1} \otimes \mathcal{O}(2m - 2)).
\]

□

Now we can iterate this statement to the case of flag bundles.

**Proposition 8.2.** Let \( \pi : Y = \text{Flag}(E) \to X \) be a rank \( r \) flag bundle over \( X \). Let \( L \) be a line bundle on \( X \), and \( F_1, F_2, \ldots, F_r/F_{r-1} \) the standard flag line bundles on \( Y \). For \( k \in \mathbb{Z} \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda \) denote by
\[
L(k; \lambda) = (\pi^* L)^k \otimes (F_r/F_{r-1})^{\lambda_1} \otimes (F_{r-1}/F_{r-2})^{\lambda_2} \otimes \cdots \otimes F_1^{\lambda_r}.
\]
Consider the polynomial
\[
q(k; \lambda_1, \lambda_2, \ldots, \lambda_r) = \chi(Y, L(k; \lambda_1, \lambda_2, \ldots, \lambda_r))
\]
in \( (k, \lambda) \in \mathbb{Z} \times \Lambda \) and extend this definition to \( \mathbb{R} \times \mathbb{V}^* \). Then \( q(k; \lambda - \rho) \) is anti-invariant under the permutations of \( \lambda_1, \lambda_2, \ldots, \lambda_r \).

8.2. The Weyl anti-symmetry of the functions \( q_1 \) and \( q_{-1} \)

Armed with this statement, we are ready to take on the symmetries of the Hilbert polynomial of our parabolic moduli spaces. We note that the two sets \( \Delta_{\pm 1} \) of weights for degree \pm 1 stable parabolic bundles are simplices with one of their vertices at \( (\frac{1}{r}, \ldots, \frac{1}{r}) \) and \( (\frac{-1}{r}, \ldots, \frac{-1}{r}) \), correspondingly (cf. §2.2).

Denote by \( N_{\pm 1} \) the moduli spaces of rank \( r \) degree \pm 1 stable vector bundles and by \( \text{UN} \) any universal bundle over \( N_{\pm 1} \times \mathbb{C} \) (cf. e.g. [AB2]).

**Lemma 8.3.** Let \( \mathbf{c} = (c_1, \ldots, c_r) \) be a parabolic weight from the chamber in \( \Delta_1 \), which has as one of its vertices the (regular) point \( (\frac{1}{r}, \ldots, \frac{1}{r}) \). Then the moduli space \( P_1(\mathbf{c}) \) of rank-\( r \) degree-1 stable parabolic bundles is isomorphic to the flag bundle \( \text{Flag}(\text{UN}_0) \) over \( N_1 \). An analogous statement holds in the case of degree \(-1 \) and the point \( (\frac{-1}{r}, \ldots, \frac{-1}{r}) \in \Delta_{-1} \).

**Proof:** A simple calculation shows that the point \( (c_1, \ldots, c_r) \in \Delta_1 \), such that all \( c_1 > 0 \), lies inside the chamber in \( \Delta_1 \) with the vertex \( (\frac{1}{r}, \ldots, \frac{1}{r}) \). Hence it is enough to prove the first statement for the moduli space \( P_1(c_1, \ldots, c_r) \) with positive parabolic weights.

Moreover, it is sufficient to show that if \( (W, F_+) \) is a parabolic stable vector bundle which represents a point in \( P_1(c_1, \ldots, c_r) \), then \( W \) is stable as an ordinary bundle. Assume that \( W \) admits a proper subbundle \( W' \) with \( \text{slope}(W') \geq \text{slope}(W) = \frac{1}{r} \), then \( \text{deg}(W') \geq 1 \). Since all parabolic weights of \( W \) are positive, this implies that \( \text{parslope}(W') > 0 = \text{parslope}(W) \), and therefore \( W \) is parabolic unstable. The proof for degree \(-1 \) bundles is analogous. □

Denote the moduli spaces described above by \( P_1(\mathbf{c}) \) and \( P_{-1}(\mathbf{c}) \), correspondingly, and their images under the Hecke isomorphisms \( \mathcal{H} \) and \( \mathcal{H}^{-1} \) by \( P_0(\mathbf{c}) \) and \( P_0(\mathbf{c}) \).

The following statement is straightforward (cf. Lemma 2.8).

**Lemma 8.4.** The line bundles \( L_1(r; 1, \ldots, 1) \) and \( L_{-1}(r; -1, \ldots, -1) \) on \( P_1(\mathbf{c}) \) and \( P_{-1}(\mathbf{c}) \) defined in Lemma 2.8 may be obtained as pullbacks of the ample generators of the Picard groups \( \text{Pic}(N_{\pm 1}) \).
Example 7. In case of rank-3 parabolic bundles the moduli space \( P_1(c_1, c_2, c_3) \) with \( 2c_3 > c_1 + c_2 - 1 \) is a flag bundle over \( N_1 \) and it is isomorphic to the moduli space \( P_0(\rangle) \) from Example \( \text{[1]} \) while the moduli space \( P_{-1}(c_1, c_2, c_3) \) with \( 2c_1 < c_2 + c_3 + 1 \) is a flag bundle over \( N_{-1} \) and it is isomorphic to \( P_0(\langle) \).

Now we establish the Weyl anti-symmetry of the polynomials

\[
q_{-1}(k; \lambda_1, ..., \lambda_r) = \chi(P_0(\langle), L_0(k; \lambda_1, ..., \lambda_r))
\]

and

\[
q_1(k; \lambda_1, ..., \lambda_r) = \chi(P_0(\rangle), L_0(k; \lambda_1, ..., \lambda_r))
\]

defined on \( \mathbb{R} \times \Lambda \), as in Proposition \( \text{[8.2]} \). Let \( \tau \in \Sigma_r \) be the cyclic permutation, such that \( \tau \cdot (c_1, ..., c_r) = (c_2, ..., c_r, c_1) \), and consider two points in \( V^* \):

\[
\theta_1[k] = \frac{k+r}{r} \cdot (1, 1, ..., 1) - (k+r)x_\tau - \rho = \tau \cdot \left( \frac{k}{r} \cdot -k, \frac{k}{r}, ..., \frac{k}{r} \right) - \tau \cdot (\rho) = \left( \frac{k}{r} - \frac{r-1}{2} + 1, \frac{k}{r} - \frac{r-1}{2} + 2, ..., \frac{k}{r} - \frac{r-1}{2} + r - 1, -k + \frac{k}{r} - \frac{r-1}{2} \right)
\]

and

\[
\theta_{-1}[k] = -\frac{k+r}{r} \cdot (1, 1, ..., 1) + (k+r)x_1 - \rho = \tau^{-1} \cdot (-\frac{k}{r}, ..., -\frac{k}{r}, -\frac{k}{r} + k) - \tau^{-1} \cdot (\rho) = \left( -\frac{k}{r} + \frac{r-1}{2}, -\frac{k}{r} - \frac{r-1}{2}, -\frac{k}{r} - \frac{r-1}{2} + 1, ..., -\frac{k}{r} - \frac{r-1}{2} + r - 2 \right).
\]

Proposition 8.5. The polynomials \( q_1(k; \lambda + \theta_1[k]) \) and \( q_{-1}(k; \lambda + \theta_{-1}[k]) \) are anti-invariant under the action of the Weyl group by permutations of \( \lambda_1, ..., \lambda_r \).

Proof. Recall that the moduli space \( P_0(\rangle) \) is isomorphic to the flag bundle \( P_1(\rangle) \) over \( N_1 \) under the Hecke isomorphism \( H^{-1} \). Then using Corollary 7.2, Proposition 8.2 and Lemma 8.4 for any permutation \( \sigma \in \Sigma_r \) we obtain

\[
q_1(k; \sigma \cdot \lambda + \theta_1[k]) \overset{\text{def}}{=} \chi(P_0(\langle), L_0(k; \sigma \cdot \lambda + \theta_1[k])) \overset{7.2}{=} \chi(P_1(\rangle), L_1(k; \tau^{-1} \cdot \sigma \cdot \lambda + (\frac{k}{r}, ..., \frac{k}{r}) - \rho)) \overset{8.2, 8.4}{=} (-1)^{\sigma} \chi(P_1(\rangle), L_1(k; \tau^{-1} \cdot \lambda + (\frac{k}{r}, ..., \frac{k}{r}) - \rho)) \overset{7.2}{=} (-1)^{\sigma} \chi(P_0(\langle), L_0(k; \lambda + \theta_1[k])) \overset{\text{def}}{=} (-1)^{\sigma} q_1(k; \lambda + \theta_1[k]).
\]

The proof for \( q_{-1} \) is similar. \( \square \)

The two group actions in Proposition 8.5 may be combined in the following manner. For \( k \geq 0 \), we define an action of the **affine Weyl group** \( \Sigma \times \Lambda \) on \( \Lambda \times \mathbb{Z} \), which acts trivially on the second factor, the level, and the action at level \( k \) is given by setting

\[
\sigma \cdot \lambda = \sigma \cdot (\lambda + \rho) - \rho \quad \text{and} \quad \gamma \cdot \lambda = \lambda + (k + r)\gamma \quad \text{for} \ \sigma \in \Sigma, \ \gamma \in \Lambda.
\]

We denote the resulting group of affine-linear transformations of \( V^* \) by \( \hat{\Sigma}[k] \), and note that the action is defined in such a way that

\[
\sigma \cdot \lambda + \rho = \sigma \cdot (\lambda + \rho) \quad \text{and} \quad (\gamma \cdot \lambda + \rho)/\hat{k} = \gamma + (\lambda + \rho)/\hat{k}
\]
It is easy to verify that the stabilizer subgroup

$$\Sigma^+_r \overset{\text{def}}{=} \text{Stab}(\theta_1[k], \hat{\Sigma}[k]) \subset \hat{\Sigma}[k]$$

is generated by the transpositions \(s_{i,i+1}, 1 \leq i \leq r - 2\) and the reflection \(\alpha^{r-1,r} \circ s_{r-1,r}\);

similarly,

$$\Sigma^-_r \overset{\text{def}}{=} \text{Stab}(\theta_1[k], \hat{\Sigma}[k]) \subset \hat{\Sigma}[k]$$

is generated by \(s_{i,i+1}, 2 \leq i \leq r - 1\) and the reflection \(\alpha^{1,2} \circ s_{1,2}\).

Then Proposition 8.5 may be recast in the following form: the polynomial \(q_0(k; \lambda)\) is anti-invariant with respect to the copy \(\Sigma^+_r\) of the symmetric group \(\Sigma_r\), while \(q_{-1}(k; \lambda)\) is anti-invariant with respect to the copy \(\Sigma^-_r\) of the symmetric group \(\Sigma_r\).

The following statement is straightforward:

**Lemma 8.6.** Both subgroups \(\Sigma^+_r\) are isomorphic to \(\Sigma_r\) and for \(r > 2\), the two subgroups generate the affine Weyl group \(\hat{\Sigma}[k]\).

### 8.3. The Weyl anti-symmetry of the polynomials \(p_1\) and \(p_{-1}\)

Following (21), we define the two polynomials

$$p_{\pm 1}(k; \lambda) = \sum_{B \in D} \text{iBer}[w_0^{1-2g}(\chi/\hat{k})](\lambda/\hat{k} - [\theta_{\pm 1}]_B),$$

where \(\theta_1 = \frac{1}{s} \cdot (1, 1, \ldots, 1) - x_r\), and \(\theta_{-1} = -\frac{1}{s} \cdot (1, 1, \ldots, 1) + x_1\).

**Proposition 8.7.** The polynomial \(p_1(k; \lambda)\) is anti-invariant with respect to \(\Sigma^+_r\), and \(p_{-1}(k; \lambda)\) is anti-invariant with respect to \(\Sigma^-_r\).

**Proof.** We recall that the points \(\theta_{\pm 1}[k]\) are the fixed points of the actions of \(\Sigma^\pm_r\), and clearly \(\lim_{k \to x} \theta_{\pm 1}[k]/k = \theta_{\pm 1}\). This means that we can fix a small open ball \(D \subset V^*\) centered at \(\theta_1\) such that

$$\lambda/k \in D \implies \forall \sigma \in \Sigma^+: (\sigma \cdot \lambda + r)/\hat{k} \sim \theta_1.$$  

Then for \(\lambda/k \in D\) we have

$$p_1(k; \lambda) = \sum_{B \in D} \text{iBer}[w_0^{1-2g}(\chi/\hat{k})](\lambda/\hat{k})_B.$$

Now, let us consider a generator of \(\Sigma^+_r\) of the type \(\sigma = s_{i,i+1}, 1 \leq i \leq r - 2\). Using (44), and Lemma 4.5, and the fact that \(\sigma \cdot w_0 = -w_0\) we obtain

$$p_1(k; \sigma \cdot \lambda) = \sum_{B \in D} \text{iBer}[w_0^{1-2g}(\chi/\hat{k})](\sigma \cdot (\lambda/\hat{k})_B) = \sum_{B \in D} \text{iBer}[(-w_0)^{1-2g}(\chi/\hat{k})](\lambda/\hat{k})_B = -p_1(k; \lambda).$$

The case of the last generator \(\alpha^{r-1,r} \circ s_{r-1,r}\) is similar, but after the substitution, we need to use the equality \(\{\alpha^{r-1,r} + \hat{\lambda}/\hat{k}\}_B = \{\hat{\lambda}/\hat{k}\}_B\) to obtain \(p_1(k; \hat{\lambda}/\hat{k})_B = -p_1(k; \lambda)\).  

8.4. Proof of part I. of Theorem 4.8

Recall that in Lemma 4.1 we introduced a chamber structure on $\Delta \subset V^*$ created by the walls $S_{\Pi,l}$, where $\Pi = (\Pi', \Pi'')$ is a nontrivial partition, and $l \in \mathbb{Z}$. Before we proceed, we introduce some extra notation. Denote by

$$\tilde{\Delta} = \{(k; a) | a/k \in \Delta \} \subset \mathbb{R}^{>0} \times V^*$$

the cone over $\Delta \subset V^*$, and let

$$\tilde{\Delta}^{\text{reg}} = \{(k; a) | a/k \in \Delta \text{ is regular} \} \subset \tilde{\Delta}$$

be the set of its regular points. Denote by $\tilde{S}_{\Pi,l} \subset \tilde{\Delta}$ the cone over the wall $S_{\Pi,l} \subset \Delta$; then $\tilde{\Delta}^{\text{reg}}$ is the complement of the union of walls $\tilde{S}_{\Pi,l}$ in $\tilde{\Delta}$. Finally, denote by $\tilde{\Delta}^{\text{reg}}_\Lambda$ the intersection of the lattice $\mathbb{Z}^{>0} \times \Lambda$ with $\tilde{\Delta}^{\text{reg}}$.

By substituting $\zeta = \lambda/k$, we can consider the left-hand side and the right-hand side of formula I. of Theorem 4.8 as functions in $(k, \lambda) \in \tilde{\Delta}^{\text{reg}}$. We denote by $q(k; \lambda)$ and $p(k; \lambda)$ the left-hand side and the right-hand side, correspondingly.

We showed that $q(k; \lambda)$ and $p(k; \lambda)$ are polynomials on the cone over each chamber in $\Delta$ (cf. Theorem 4.4, 4.8). We proved that the wall-crossing terms, i.e., the differences between polynomials on neighbouring chambers, for $q(k; \lambda)$ (cf. Theorem 6.11) and for $p(k; \lambda)$ (cf. Proposition 4.10) coincide, hence there exists a polynomial $\Theta(k; \lambda)$ on $\mathbb{Z}^{>0} \times \Lambda$, such that the restriction of $\Theta(k; \lambda)$ to $\tilde{\Delta}^{\text{reg}}_\Lambda$ is equal to the difference $p(k; \lambda) - q(k; \lambda)$.

Now for $r > 2$, we can conclude that

$$\Theta(k; \lambda) = p_1(k; \lambda) - q_1(k; \lambda) = p_{-1}(k; \lambda) - q_{-1}(k; \lambda),$$

where $p_{\pm 1}(k; \lambda)$ and $q_{\pm 1}(k; \lambda)$ are the restrictions of $p(k; \lambda)$ and $q(k; \lambda)$ to two specific chambers defined in 8.3 and 8.2. Then, according to Propositions 8.5 and 8.7, the polynomial $\Theta(k; \lambda)$ is anti-invariant with respect to the action of the subgroups $\Sigma^+_{\lambda}$, and hence by Lemma 8.6, it is anti-invariant under the action of the entire affine Weyl group $\tilde{\Sigma}[k]$. It is easy to see that any such polynomial function has to vanish, and thus $p(k; \lambda) = q(k; \lambda)$, and this completes the proof of part I. of Theorem 4.8 for the case when $\lambda/k \in \Delta$ is regular.

As in Corollary 4.10, we can extend $p(k; \lambda)$ from the interior of each chamber to its boundary by polynomiality. Clearly, to prove part I. of Theorem 4.8 for the cases when $\lambda/k$ is not regular, it is sufficient to show, that these extensions from the chambers containing $\lambda/k$ in their closure give the same value on $(k; \lambda)$. It follows from Remark 10.4 that this is the case, and this completes the proof of part I. of Theorem 4.8 (cf. Remark 4.9).

9. Rank 2, two points

Unfortunately, the argument above does not work for $r = 2$, because, in this case, $\theta_1[k] = \theta_{-1}[k]$, the groups $\Sigma^+_{\lambda}$ and $\Sigma^+_{\lambda}$ coincide, and thus they do not generate the entire affine Weyl group. The way out is to pass to the 2-punctured case.

9.1. Wall-crossing

We will thus fix two points: $p, s \in \mathbb{C}$, and study the moduli space of rank-2, stable parabolic bundles $W$ with fixed determinant isomorphic to $O(pd)$, with parabolic structure given by a line $F_1 \subset W_p$ with weight $(c, -c)$, and a line $G_1 \subset W_s$ with weight $(a, -a)$.
Now we need to repeat the analysis of our work so far in this somewhat simpler case; some details thus will be omitted.

Set $d = 0$; then the space of admissible weights (cf. Figure 5) is a square

\[ \square = \{(c, a) \mid 1 > 2c > 0, 1 > 2a > 0, \} \]

which has two adjacent chambers defined by the conditions

\[ c > a \quad \text{and} \quad c < a. \]

Denote the corresponding moduli spaces by $P_0(c > a)$ and $P_0(c < a)$.

![Figure 6](image_url)

**Figure 6.** The space of admissible weights in the case of rank $r = 2$, two points.

Again, we have universal bundles over $P_0(c > a) \times C$ and $P_0(c < a) \times C$, which we will denote by the same symbol $U$; this bundle is endowed with two flags, $\mathcal{F}_1 \subset \mathcal{F}_2 = U_p$ and $\mathcal{G}_1 \subset \mathcal{G}_2 = U_s$. For $\mu, \lambda \in \mathbb{Z}$, we introduce the line bundle

\[ \mathcal{L}(k; \lambda, \mu) = \det(U_p)^{k(1-g)} \otimes \det(\pi_\varepsilon(U))^{-k} \]

\[ \otimes (\mathcal{F}_2/\mathcal{F}_1)^{\lambda} \otimes (\mathcal{F}_1)^{-\lambda} \otimes (\mathcal{G}_2/\mathcal{G}_1)^{\mu} \otimes (\mathcal{G}_1)^{-\mu}. \]

We repeat the construction of the master space from Section 5.1, choosing a point $(c^0, c^0)$ on the wall and two points

\[ (c, a)^\pm = (c^0, c^0) \pm \epsilon(1, 0) \in \square, \quad \epsilon \in \mathbb{Q}_{>0} \]

from the adjacent chambers. We can identify the fixed point set $Z^0$ as follows.

**Lemma 9.1.** The locus $Z^0$ defined in Proposition 5.2 is

\[ Z^0 \simeq \text{Jac}^0 \simeq \{V = L \oplus L^{-1} \mid L_p \subset \mathcal{F}_1, L_s^{-1} \subset \mathcal{G}_1\}. \]

As in §5.1, denote by $\mathcal{E}$ the universal bundle over $\text{Jac}^0 \times C$ normalized in such a way that $c_1(\mathcal{E}) = 0$ (cf. 7). Define

\[ \eta \in H^2(\text{Jac}) \quad \text{by} \quad \left( \sum_i c_1(\mathcal{E}_i) \otimes \mathcal{E}_1 \right)^2 = -2\eta \otimes \omega; \]

we have then $\int_{\text{Jac}} \text{e}^{m \eta} = m^g$ for $m \in \mathbb{Z}$.

Let $\pi : \text{Jac}^0 \times C \to \text{Jac}^0$ be the projection and $N_{Z^0}$ be the equivariant normal bundle to $Z^0$ in $Z$. Then, as in Lemma 6.3, Proposition 6.7 and Lemma 6.4 we obtain the identifications:

- $N_{Z^0} = \mathbb{R}_+ \pi_* (\text{ParHom}(\mathcal{E}, \mathcal{E}^{-1})) \oplus \mathbb{R}_+ \pi_* (\text{ParHom}(\mathcal{E}^{-1}, \mathcal{E}))$, where $\pi \simeq \mathbb{C}^*$-action has weights $(-1,1)$;
- $E(N_{Z^0})^{-1} = (-1)^g (2 \sin \left( \frac{\pi}{2} \right))^{-2g} \exp(4\eta)$;
- $\text{ch}_T (\mathcal{L}(k; \lambda, \mu)|_{Z^0}) = \exp(2k\eta) \exp(u(\lambda - \mu))$. 

Now we define the polynomials:

\[ h_>(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c > a), \mathcal{L}(k; \lambda, \mu)), \quad h_<(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c < a), \mathcal{L}(k; \lambda, \mu)). \]

and, applying Theorem 5.6, we obtain the following expression for their difference.

**Lemma 9.2.** The wall crossing term equals

\[ h_>(k; \lambda, \mu) - h_<(k; \lambda, \mu) = (-1)^g(2k + 4)^g \text{Res}_{u=0} \frac{\exp(u(\lambda - \mu))}{(2\sinh(\frac{u}{2}))^{2g}} \, du. \]

9.2 Symmetry

Denote by \( P_{-1}(c > a) \) the image of the moduli space \( P_0(c > a) \) under the Hecke isomorphism \( \mathcal{H} \) (cf. §7) at the point \( p \) and by \( P_{-1}(c < a) \) the image of the moduli space \( P_0(c < a) \) under the Hecke isomorphism \( \mathcal{H} \) at the point \( s \).

We have the following analogue of Lemma 8.3.

**Lemma 9.3.** Denote by \( N_{-1} \) the moduli space of rank-2 degree-\((-1)\) stable bundles on \( \mathbb{C} \) and by \( UN \) any universal bundle over \( N_{-1} \times \mathbb{C} \). Then the moduli spaces \( P_{-1}(c > a) \) and \( P_{-1}(c < a) \) are isomorphic to the bundle \( \mathbb{P}(UN_p) \times \mathbb{P}(UN_s) \) over \( N_{-1} \).

Denote by \( \mathcal{T}[p] \) and \( \mathcal{T}[s] \) the vertical tangent lines of \( \mathbb{P}(UN_p) \) and \( \mathbb{P}(UN_s) \), respectively, and by \( \mathcal{L}_{-1} \) the pullback of the ample generator of the Picard group of \( N_{-1} \) to \( \mathbb{P}(UN_p) \times \mathbb{P}(UN_s) \) (cf. Lemma 8.4). Then a simple calculation shows the following.

**Lemma 9.4.** Under the Hecke isomorphism \( \mathcal{H} \) at \( p \), the line bundle \( \mathcal{L}(2k; \lambda, \mu) \) on \( P_0(c > a) \) corresponds to the line bundle \( \mathcal{L}_{-1}^k \otimes \mathcal{T}[p]^{\lambda-k} \otimes \mathcal{T}[s]^{\mu} \) on \( P_{-1}(c > a) \).

Under the Hecke isomorphism \( \mathcal{H} \) at the point \( s \), \( \mathcal{L}(2k; \lambda, \mu) \) on \( P_0(c < a) \) corresponds to the line bundle \( \mathcal{L}_{-1}^k \otimes \mathcal{T}[p]^{\lambda} \otimes \mathcal{T}[s]^{\mu-k} \) on \( P_{-1}(c < a) \).

As in §8.2, applying Serre duality for families of curves (cf. Proposition 8.2) to the line bundles on the two \( \mathbb{P}^1 \times \mathbb{P}^1 \) bundles over \( N_{-1} \), we obtain that the polynomials \( h_>(k; \lambda, \mu) \) and \( h_<(k; \lambda, \mu) \) are anti-invariant under the action of the Weyl group \( \Sigma_2 \times \Sigma_2 \) with the center at \( \theta_1[k] = (\frac{k+1}{2}, \frac{-1}{2}) \) and \( \theta_2[k] = (\frac{-1}{2}, \frac{k+1}{2}) \), correspondingly (cf. Figure 7). In other words, we obtain the following 4 identities.

**Lemma 9.5.**

\[
\begin{align*}
&h_>(k; \lambda, \mu) = -h_>(k; \lambda, -\mu - 1) = -h_>(k; -\lambda + k + 1, \mu); \\
&h_<(k; \lambda, \mu) = -h_<(k; -\lambda - 1, \mu) = -h_<(k; \lambda, -\mu + k + 1). 
\end{align*}
\]

**Figure 7.** \( k = 4, r = 2, \) two points.
Now, define the polynomials
\[ \tilde{h}_>(k; \lambda, \mu) = (-1)^{g-1}(2k + 4)^g \text{Res}_{u=0} \frac{\exp(u(\lambda + \mu + 1)) - \exp(u(\lambda - \mu))}{(2\sinh(u))^2 g(1 - e^{u(k+2)})} du, \]
\[ \tilde{h}_<(k; \lambda, \mu) = (-1)^{g-1}(2k + 4)^g \text{Res}_{u=0} \frac{\exp(u(\lambda + \mu + 1)) - \exp(u(\lambda - \mu + k + 2))}{(2\sinh(u))^2 g(1 - e^{u(k+2)})} du, \]
and from here we can follow the logic of the proof of part I of Theorem 4.8.

**Proposition 9.6.** The polynomials introduced above, in fact, coincide:
\[ h_>(k; \lambda, \mu) = \tilde{h}_>(k; \lambda, \mu) \quad \text{and} \quad h_<(k; \lambda, \mu) = \tilde{h}_<(k; \lambda, \mu). \]

**Proof.** It is a simple exercise to show that \( \tilde{h}_>(k; \lambda, \mu) \) and \( \tilde{h}_<(k; \lambda, \mu) \) satisfy the identities appearing in Lemmas 9.2 and 9.5, and hence the polynomial
\[ \Theta(k; \lambda, \mu) = h_>(k; \lambda, \mu) - \tilde{h}_>(k; \lambda, \mu) = h_<(k; \lambda, \mu) - \tilde{h}_<(k; \lambda, \mu) \]
satisfies all four \( \Sigma_2 \)-symmetries listed in Lemma 9.5. These groups together generate a double action of the affine Weyl group \( \Sigma \) in \( \lambda \) and \( \mu \) separately, and this implies the vanishing of \( \Theta \).

As \( P_0(c > a) \) is a \( \mathbb{P}^1 \)-bundle over the moduli space of rank-2 degree-0 stable parabolic bundles \( P_0(c, -c) \), substituting \( \mu = 0 \) in \( \tilde{h}_> \), we obtain the Verlinde formula for rank 2.

**Corollary 9.7.**
\[ \chi(P_0(c, -c), L_0(k; \lambda)) = (-1)^{g-1}(2k + 4)^g \text{Res}_{u=0} \frac{\exp(u(\lambda + \frac{1}{2}))}{(2\sinh(u))^2 \left( g-1(1 - e^{u(k+2)}) \right)} du. \]

**10. The combinatorics of the \([Q, R] = 0\).**

In this section, we give a proof of the second part of Theorem 4.8. Let \( \lambda/k \in \Delta \), and fix a regular element \( \zeta \in \Delta \) in a chamber containing \( \lambda/k \) in its closure, and another regular element \( \zeta \in \Delta \) containing \( \lambda/k \) in its closure. Our goal is to prove the the equality \( p_c(k; \lambda) = p_c(k; \lambda) \), where we define
\[ p_c(k; \lambda) = N_{r,k} \sum_{B \in \mathcal{D}} i\text{Ber}[w^1_{\Phi}(x/k)] \langle \lambda/k - [c]_B \rangle \]
for a regular \( c \in \Delta \) and diagonal basis \( \mathcal{D} \). This is a subtle statement, which is a combinatorial-geometric projection of the idea of quantization commutes with reduction (or \([Q, R] = 0\) for short, cf. [MS, SzV]).

If \( \lambda/k \sim \lambda/k \), i.e. when \( \lambda/k \) and \( \lambda/k \) are regular elements in the same chamber in \( \Delta \), then \( p_c(k; \lambda) = p_c(k; \lambda) \) is a tautology. We assume thus that this is not the case, and denote by \( \delta(k, \lambda) \) the set of walls separating \( \zeta \) and \( \zeta \), or containing either \( \lambda/k \) or \( \lambda/k \) or both. Equivalently, the wall \( S_{1,1} \) belongs to \( \delta(k, \lambda) \) if
\[ (\lambda/k)_{1,1} \geq 1 \geq (\lambda/k)_{1,1}, \quad \text{or} \quad (\lambda/k)_{1,1} \leq 1 \leq (\lambda/k)_{1,1}, \]
where \( c_{1,1} = \sum_{i \in \mathcal{I}} c_i \) for an element \( c = (c_1, ..., c_T) \in V^* \). Clearly, there is a path in \( \Delta \) connecting \( \zeta \) and \( \zeta \), which intersects only walls from \( \delta(k, \lambda) \) in a generic points. Then to prove the equality \( p_c(k; \lambda) = p_c(k; \lambda) \), it is enough to show the following, at first sight somewhat surprising fact.
Proposition 10.1. Assume $g \geq 1$, $\lambda/k \in \Delta$, $S_{\Pi,1} \in S(k,\lambda)$ and let $c^{\pm} \in \Delta$ be two points in two neighboring chambers separated by the wall $S_{\Pi,1}$. Then

$$p_{c^+}(k;\lambda) = p_{c^-}(k;\lambda). \quad (47)$$

Proof. The difference of the two sides of (47) is expressed as a residue in (30). The integral in (30) is a rational expression in the variable $t$, and our plan is to show by degree count in $t$ and $t^{-1}$ that its residues at zero and at $\infty$ vanish. We define the degree of the quotient of two polynomials $R = P/Q$ of the variable $t$ as $\deg_{\tau}(R) = \deg_{\tau}(P) - \deg_{\tau}(Q)$, and we set $\deg_{\tau-1}(R) = \deg_{\tau}(R(t^{-1}))$. Then, clearly,

$$\deg_{\tau}(R) < 0 \implies \text{Res} \frac{R}{t} = 0 \quad \text{and} \quad \deg_{\tau-1}(R) < 0 \implies \text{Res} \frac{R}{t} = 0. \quad (48)$$

A convenient expression for (30) will be (40), where we change variables via $t = e^u$. In what follows, we will always tacitly assume this substitution, and we will write, for example, $\deg_{\tau \pm 1}(1/(e^u - e^{-u})) = -1$. Thus, we shall use a formula of the form $\text{Res}_{t=0} f(t) = \sum_{j>0} \frac{f^{(j)}(0)}{j!}$, and to show that this is zero, it is sufficient to show that $\deg_{\tau}(f) < 0$ and $\deg_{\tau-1}(f) < 0$.

Now we observe that the variable $u$ occurs only in the first line of (40), and thus, calculating the degrees in $t$ and $t^{-1}$ separately, we obtain the following formula:

$$\deg_{\tau \pm 1}(f) = \pm(k\delta - rt) + (1 - 2g)\deg_{\tau \pm 1}(\tau \cdot w_u^\lambda) + \deg_{\tau \pm 1}(\exp(\tau \cdot \rho_u^\lambda)). \quad (49)$$

Recall that here $\delta$ represents the distance of $\lambda/k$ from the wall $S_{\Pi,1}$, while $w_u^\lambda$ and $\rho_u^\lambda$ represent the parts of the Weyl denominator and the $\rho$-shift corresponding to roots connecting $\Pi'$ and $\Pi''$, respectively.

We begin the study of this expression with some simple remarks. We recall that the permutation $\tau$ preserves the partition $\Pi = (\Pi',\Pi'')$, and thus we have

$$\deg_{\tau \pm 1}(\tau \cdot w_u^\lambda) = \deg_{\tau \pm 1}(w_u^\lambda) = \frac{r' r''}{2}. \quad (50)$$

Using, in addition, that $\rho_u^\lambda$ is linear in $u$, we obtain

$$\deg_{\tau}(\exp(\tau \cdot \rho_u^\lambda)) = -\deg_{\tau-1}(\exp(\tau \cdot \rho_u^\lambda)) = \deg_{\tau}(\exp(\rho_u^\lambda)). \quad (51)$$

Combining these equalities, and assuming $g \geq 1$, we arrive at the following conclusion.

Lemma 10.2. The inequality

$$| (k\delta - rt) + \deg_{\tau}(\exp(\rho_u^\lambda)) | < \frac{r' r''}{2} \quad (52)$$

implies the vanishing of the wall-crossing term: equality (47).

Before we proceed, we introduce some notation. Denote by

$$\text{Inv}(\Pi) = \{(i,j) | \Pi' \ni i > j \in \Pi''\}$$

the set of “inverted” pairs of elements of the partition $\Pi$. The number of these pairs $|\text{Inv}(\Pi)|$ coincides with the standard notion of length of the shuffle permutation $\phi \in \Sigma_r$ introduced in §5.

Each pair $(i,j)$ which is not inverted contributes $+u/2$ to $\rho_u^\lambda$, while each inverted pair contributes $-u/2$, and thus we have

$$\deg_{\tau}(\exp(\rho_u^\lambda)) = \frac{r' r''}{2} - |\text{Inv}(\Pi)|. \quad (53)$$
Also, recall the notation $c_{\Pi'} = \sum_{i \in \Pi'} c_i$ for an element $c = (c_1, \ldots, c_r) \in V^*$; in particular, we have $(\lambda/k)_{\Pi'} = l + \delta$ and
\[
\rho_{\Pi'} = \sum_{i \in \Pi'} \frac{r + 1}{2} - i.
\]
The following is a simple exercise, whose proof will be omitted:
\begin{equation}
(51) \quad \deg_l(\exp(\rho^*_{\Pi'})) = \rho_{\Pi'}.
\end{equation}
Now we come to a key point of our argument.

**Lemma 10.3.** If the intersection of the wall $S_{\Pi,1}$ with $\Delta$ is non-empty, then
\begin{equation}
(52) \quad -\frac{r'r''}{2} < lr - \rho_{\Pi'} < \frac{r'r''}{2}.
\end{equation}

**Proof.** Pick a point $c = (c_1, \ldots, c_r)$ in the intersection $S_{\Pi,1} \cap \Delta$, and recall that for any $1 \leq i < j \leq r$, we have $0 < c_i - c_j < 1$, and
\[
\sum_{i \in \Pi'} c_i = -\sum_{j \in \Pi''} c_j = l.
\]
Then
\[
-|\text{Inv}(\Pi)| < \sum_{(i,j) \in \text{Inv}(\Pi)} (c_i - c_j) \leq \sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j),
\]
and, similarly,
\[
\sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j) < r'r'' - |\text{Inv}(\Pi)|.
\]
Now, since
\[
\sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j) = r'' \sum_{i \in \Pi'} c_i - r' \sum_{j \in \Pi''} c_j = lr,
\]
we can conclude
\[
-|\text{Inv}(\Pi)| < lr < r'r'' - |\text{Inv}(\Pi)|.
\]
In view of (50) and (51), these inequalities are equivalent to (52), and this completes the proof. \hfill \Box

Now we are ready to prove (47). The condition $S_{\Pi,1} \in S(k, \lambda)$, i.e. that $S_{\Pi,1}$ separates $\lambda/k$ and $\hat{\lambda}/k$ or contains $\lambda/k$ or $\hat{\lambda}/k$, may occur in two ways.\par
- $(\lambda/k)_{\Pi'} \geq 1 \geq (\hat{\lambda}/k)_{\Pi'}$, which is equivalent to the two inequalities: $\delta \geq 0$ and $lk + lr \geq \lambda_{\Pi'} + \rho_{\Pi'}$. After canceling $lk$ and reordering the terms, we can rewrite these as
\begin{equation}
(53) \quad 0 \geq k\delta - lr + \rho_{\Pi'} \geq \rho_{\Pi'} - lr.
\end{equation}
Using Lemma 10.3 then we can conclude that
\[
0 \geq k\delta - lr + \rho_{\Pi'} > -\frac{r'r''}{2},
\]
which, in view of the equality (51), implies the necessary estimate (49).
The second case is similar: $\frac{\lambda}{k} \leq l \leq (\lambda, \hat{k})_{II}^r$ is equivalent to $\delta \leq 0$ and $l \kappa + lr \leq \lambda_{II}^r + \rho_{II}^r$. This leads to

\begin{equation}
0 \leq k\delta - lr + \rho_{II}^r \leq \rho_{II}^r - lr,
\end{equation}

which, in turn, implies

\begin{equation}
0 \leq k\delta - lr + \rho_{II}^r < t' r'' \over 2,
\end{equation}

and hence (49).

This completes the proof of Proposition 10.1: indeed, a simple calculation shows that if $\lambda/k \in \Delta$ then $\hat{\lambda}/\hat{k} \in \Delta$, so the conditions of Lemma 10.3 hold. We have just shown that this implies (49), and according to Lemma 10.2 we can conclude the vanishing of the wall-crossing term (47).

Remark 10.4. Note that if $\lambda/k \in \Delta$ is non-regular, then it belongs to some wall from the set $S(k, \lambda)$. Hence proposition 10.1 implies that the right-hand side of formula (I.) of Theorem 4.8 is a well-defined function on the cone over $\Delta$: $\{(k, \lambda) \in \mathbb{Z}^{>0} \times \Lambda | \lambda/k \in \Delta\}$.

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