Discrete transformation for matrix 3-waves problem in three dimensional space

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Abstract

Discrete transformation for 3-waves problem is constructed in explicit form. Generalization of this system on the matrix case in three dimensional space together with corresponding discrete transformation is presented also.
1 Introduction

The problem of three waves in two dimensions arises in different form in many branches of the mathematical physics. Its application to problems of radiophysics and nonlinear optics reader can find in [1]. In connection with the inverse scattering method it was investigated in [2] and considered in details in numerous further papers.

The goal of the present paper is generalize this system on the space of three dimension with simultaneous exchanging the unknown scalar functions on the operator valued ones. The last generalization allows to include into consideration the quantum region with the Heisenberg operators as unknown functions of the problem.

It is necessary to mention that numerous different (by the form) discrete transformation were used up to now with respect to two components hierarchies of integrable systems, which are connected with so called Darboux-Toda, Lotke-Volterra and Heisenberg substitutions. In the present paper we come to the substitution which connect six independent functions, which corresponds to $A_2$ algebra but not to $A_1$ as it was in the case of the two components systems. This substitution may be considered as the integrable mapping, connected six initial function with the six final ones. Substitutions of the present paper do not coincide with recently introduced in [3] so-called Ultra-Toda mappings. And the last comment.

The method of this paper without any difficulties can be generalized on the case of n-th wave problem. In this case the number of independent variables of substitution will be $(n \times (n + 1))$ which coincides with the number of positive and negative roots of $A_n$ algebra.

The traditional way for obtaining the system of equations for 3-waves problem in $(1 + 1)$ dimensions is the $L – A$ pair formalism

$$[\partial_x - u, \partial_t - v] = 0$$

with

$$u = \begin{pmatrix}
    c_1 \lambda & (c_2 - c_1)P & (c_3 - c_1)Q \\
    (c_1 - c_2)B & c_2 \lambda & (c_3 - c_2)A \\
    (c_1 - c_3)D & (c_2 - c_3)E & -(c_1 + c_2)\lambda
\end{pmatrix},$$

$$v = \begin{pmatrix}
    d_1 \lambda & (d_2 - d_1)P & (d_3 - d_1)Q \\
    (d_1 - d_2)B & d_2 \lambda & (d_3 - d_2)A \\
    (d_1 - d_3)D & (d_2 - d_3)E & -(d_1 + d_2)\lambda
\end{pmatrix}$$
where $c, d$ four arbitrary numerical parameters ($\sum c_i = 0, \sum d_i = 0$) and $(A, B, D, E, P, Q)$ are unknown functions of the problem. The system of equations for them has the form

$$(d_2 - d_1)P_x - (c_2 - c_1)P_t + \nu E = 0, \quad (d_3 - d_1)Q_x - (c_3 - c_1)Q_t - \nu PA = 0,$$

$$(d_3 - d_2)A_x - (c_3 - c_2)A_t + \nu QB = 0, \quad (d_1 - d_2)B_x - (c_1 - c_2)B_t - \nu AD = 0,$$

$$(d_1 - d_3)D_x - (c_1 - c_3)D_t + \nu BE = 0, \quad (d_2 - d_3)E_x - (c_2 - c_3)E_t - \nu DP = 0,$$

where $\nu = 3(c_1 d_2 - c_2 d_1)$. Soliton like solution for the last system of equations may be easily found with the help of the technique of the paper [4]. From these results it can be found the first steps of the discrete transformation with respect to which the last system is invariant. In present paper the reader have to consider the form of the discrete transformation as the lucky guess.

2 Discrete transformation

We can assume (but this is a direct corollary of the results of [4]) that the three "new" functions $(Q, A, P)$, denoted by the bar symbols, connected with the old ones as following

$$\bar{Q} = \frac{1}{D}, \quad \bar{A} = -\frac{B}{D}, \quad \bar{P} = \frac{E}{D}$$

satisfy the system (1). Then from the first and second equations of the first column it is possible to determine $\bar{E}, \bar{B}$ functions with the result

$$\bar{E} = -\frac{1}{(c_1 - c_3)} \frac{E}{D} (D_x - (c_2 - c_1)BE) + \frac{1}{(c_2 - c_3)} (E_x + (c_2 - c_1)DP),$$

$$\bar{B} = \frac{1}{(c_1 - c_3)} \frac{B}{D} (D_x - (c_2 - c_1)BE) - \frac{1}{(c_1 - c_2)} (B_x - (c_3 - c_2)AD).$$

And in a self-consistent way determine from the second and first equations of the second column $\bar{D}$. We will not present here this not very simple expression, because in few lines below we will have observable expression for this value. Straightforward but tedious calculations show that the third equation of the first column is also satisfied ($\bar{D}_3 = -\bar{B}\bar{E}$).
For further consideration it is more suitable to introduce three dependent variables ($\xi + \eta + \sigma = 0$)

$\xi = (d_2-d_1)t+(c_2-c_1)x, \quad \eta = (d_3-d_2)t+(c_3-c_2)x, \quad \sigma = (d_1-d_3)t+(c_1-c_3)x.$

In each pairs of variables ($\xi, \eta$), ($\xi, \sigma$), ($\eta, \sigma$) the differentiation operators take the form

$$\begin{pmatrix}
\partial_1 & \equiv & \left(\frac{d_2-d_1}{\nu}x - \frac{(c_2-c_1)}{\nu}t\right) \\
\partial_2 & \equiv & \left(\frac{d_3-d_2}{\nu}x - \frac{(c_3-c_2)}{\nu}t\right) \\
\partial_3 & \equiv & \left(\frac{d_1-d_3}{\nu}x - \frac{(c_1-c_3)}{\nu}t\right)
\end{pmatrix} = 
\begin{pmatrix}
-\partial_\eta & \partial_\sigma & \partial_\sigma - \partial_\eta \\
\partial_\xi & \partial_\xi - \partial_\sigma & -\partial_\sigma \\
\partial_\eta - \partial_\sigma & -\partial_\xi & \partial_\eta
\end{pmatrix}$$

Really the explicit form the generators of differentiation via ($\xi, \eta, \sigma$) variables will be not essential. Now the system (1) looks much more attractive

$$P_1 = -QE, \quad A_2 = -BQ, \quad Q_3 = -PA$$

$$B_1 = -AD, \quad E_2 = -DP, \quad D_3 = -EB \quad (2)$$

In the last form the system is obviously invariant with respect to permutation of the indexes of differentiation with the simultaneous corresponding exchanging of unknown functions. The discrete transformation of the beginning of this section may be rewritten in more symmetrical form (we will denote it with the help of the symbol $T_3$)

$$\bar{Q} = \frac{1}{D}, \quad \bar{A} = -B, \quad \bar{P} = \frac{E}{D},$$

$$\bar{B} = D\left(\frac{B}{D}\right)_2, \quad \bar{E} = -D\left(\frac{E}{D}\right)_1, \quad \frac{D}{D} = DQ - (\ln D)_{1.2}$$

By the permutation indexes (1, 3) (together with corresponding exchanging of unknown functions) it is possible to obtain the $T_1$ discrete transformation with respect to which the system (2) is also invariant

$$\bar{P} = \frac{1}{B}, \quad \bar{Q} = \frac{A}{B}, \quad \bar{E} = \frac{D}{B},$$

$$\bar{D} = B\left(\frac{D}{B}\right)_2, \quad \bar{A} = -B\left(\frac{A}{B}\right)_3, \quad \frac{B}{B} = BP - (\ln B)_{2.3}$$
And at last the discrete transformation $T_2$ has the form

$$\bar{A} = \frac{1}{E}, \quad \bar{B} = \frac{D}{E}, \quad \bar{Q} = -\frac{P}{E},$$

$$\bar{D} = -E(\frac{D}{E})_1, \quad \bar{P} = E(\frac{P}{E})_3, \quad \bar{E} = EA - (\ln E)_{1,3}$$

In the form presented above substitutions $T_i$ may be considered as a mapping, connected six initial (unbar) functions with six final (bar) ones. From the other side each substitution may be considered as the infinite dimensional chain of equations. For instance the corresponding chain of equation in the case of $T_1$ substitution has the form

$$\frac{B^{n+1}}{B^n} - \frac{B^n}{B^{n-1}} = -(\ln B^n)_{2,3}, \quad D^{n+1} = B^n(\frac{D^n}{B^n})_2, \quad A^{n+1} = -B^n(\frac{A^n}{B^n})_3 \quad (3)$$

$$E^{n+1} = -\frac{D^n}{B^n}, \quad Q^{n+1} = \frac{A^n}{B^n}$$

In the first row we have the lattice like system connected 3 unknown functions ($B, D, A$) in each point of the lattice. The first chain for $B$ functions is exactly well known two dimensional Toda lattice.

### 3 Some properties of the discrete transformations

All constructed above discrete transformations are invertible. This means that unbar unknown function may presented in terms of the bar ones. For instance $T_3^{-1}$ looks as

$$D = \frac{1}{Q}, \quad B = -\bar{A}/\bar{Q}, \quad E = \bar{P}/\bar{Q},$$

$$P = -\bar{Q}(\frac{\bar{P}}{\bar{Q}})_{2}, \quad A = \bar{Q}(\frac{\bar{A}}{\bar{Q}})_{1}, \quad \frac{Q}{\bar{Q}} = \bar{D}Q - (\ln \bar{Q})_{1,2}$$

It is not difficult to check by direct computation that discrete transformations $T_i$ are mutual commutative ($T_iT_j = T_jT_i$) on the solutions of the system (2).
We present below corresponding calculations to prove that $T_1T_2 = T_2T_1 = T_3$. Indeed result of the action of $T_1$ on some solution of the system (2) is the following

\[ P^1 = \frac{1}{B}, \quad Q^1 = \frac{A}{B}, \quad E^1 = -\frac{D}{B}; \]

\[ D^1 = B\left(\frac{D}{B}\right)_2, \quad A^1 = -B\left(\frac{A}{B}\right)_3, \quad B^1 = BP - (\ln B)_{2,3} \]

Action of the $T^2$ transformation on this solution leads to

\[ A^{21} = \frac{1}{E^1} = -\frac{B}{D}, \quad B^{21} = \frac{D^1}{E^1} = D\left(\frac{B}{D}\right), \]

\[ Q^{21} = -\frac{P^1}{E^1} = \frac{1}{D}, \quad D^{21} = -E^1\left(\frac{D^1}{E^1}\right)_1 = -\frac{D}{B}(B(\ln D)_2 - B_2)_1 = QD^2 - D(\ln D)_{12} \]

\[ P^{21} = E^1\left(\frac{P^1}{E^1}\right)_3 = \frac{E}{D}, \quad E^{21} = (E^1)^2 A^1 - E^1(\ln E^1)_{13} = -D\left(\frac{E}{D}\right)_{1} \]

The same calculation repeated in the back direction shows that $W^{1,2} = W^{2,1} = W^3$ - the result of application of the $T_3$ transformation to an initial solution $W$.

Thus from each given initial solution $W_0 \equiv (A, P, Q, E, B, D)$ of the system (2) it is possible to obtain the chain of solutions labeled by two natural numbers ($l_1, l_2$, or ($l_3$)) the number of application of the discrete transformations ($T_1, T_2, T_3$) to it (as it was shown above $T_1T_2 = T_2T_1 = T_3$).

The arising chain of equations with respect to ($D, B, E$) functions are exactly two dimensional Toda lattices. Their general solutions in the case of two fixed ends are well-known [5]. As reader will be seen soon this fact allows to construct the many soliton solutions of the 3-wave problem in the most straightforward way.

4 Resolving of discrete transformation chains

4.1 Two identities of Jacobi

We begin from the following obvious equalities for determinants of $n$-th order

\[ Det_n(T_n) \equiv D_n \begin{pmatrix} T^{n-1}_{n-1} & a \\ b & \tau \end{pmatrix} = D_{n-1}(T_{n-1})(\tau - bT^{n-1}_{n-1}a) \equiv D_{n-1}(T_{n-1})\tilde{\tau} \]
where $T_{n-1}$ is $(n - 1) \times (n - 1)$ matrix, $a, b$ are $(n - 1)$ dimensional column (row) vectors respectively and $\tau$ scalar.

By the same reason the following formula takes place

$$D_n \left( \frac{T_{n-1}}{b^1 \ a^1 \ \tau_{11} \ \tau_{12}} \right) = D_{n-2} (T_{n-2}) D_2 \left( \frac{T_{n-2}}{b^1 - b^2 \ T_{n-2}^{-1} \ a^2 \ \tau_{11} - b^2 \ T_{n-2}^{-1} \ a^2} \right)$$

where $a', b'$ are $(n - 2)$ dimensional columns (rows) vectors, $\tau_{i,j}$ components of 2-th dimensional matrix. It is obvious how relations of these types may be continued.

Now using results above let us transform the following expression

$$D_n \left( \frac{T_{n-1}}{b^1 \ a^1 \ \tau_{11}} \right) D_n \left( \frac{T_{n-1}}{b^2 \ a^2 \ \tau_{22}} \right)$$

$$D_{n-1} \left( \frac{T_{n-1}}{b^2 \ a^2 \ \tau_{21} \ \tau_{22}} \right) = D_{n-1} D_{n+1} \left( \frac{T_{n-1}}{b^1 \ a^1 \ \tau_{11} \ \tau_{12}} \right)$$

We will treated the last equality as the first Jacobi identity. By the same technique it is not difficult to show that the following equality takes place

$$D_n \left( \frac{T_{n-1}}{b^1 \ a^1 \ \tau} \right) D_{n+1} \left( \frac{T_{n-1}}{b^2 \ a^2 \ \tau} \right) - D_n \left( \frac{T_{n-1}}{b^2 \ a^2 \ \tau} \right) D_{n+1} \left( \frac{T_{n-1}}{b^1 \ a^1 \ \tau} \right) =$$

$$D_n \left( \frac{T_{n-1}}{a^1 \ \nu} \right) D_{n+1} \left( \frac{T_{n-1}}{a^2 \ \rho} \right) - D_n \left( \frac{T_{n-1}}{a^2 \ \rho} \right) D_{n+1} \left( \frac{T_{n-1}}{a^1 \ \nu} \right)$$

This equality we will use many times in what follows and will call second Jacobi identity. These identities can be generalized on the case of arbitrary semi-simple group. Reader can find these results in [3].

4.2 Concrete calculations

Let us take initial solution in the form

$$Q = A = P = 0, \quad B \equiv B(2), \quad E \equiv E(1). \quad D_3 = -BE$$

(4)
Application to this solution each of inverse transformations $T_i^{-1}$ is mean less via arising zeroes in denominators. The chain of equations under such boundary condition we will call as the chain with the fixed end from the left *from one side).

The result of application to such initial solution $l_3$ times $T_3$ transformation looks as (for the checking of this fact only two Jacobi identities of the previous subsection are necessary)

$$Q^{(l_3)} = (-1)^{l_3-1} \frac{\Delta_{l_3-1}}{\Delta_{l_3}}, \quad D^{(l_3)} = (-1)^{l_3} \frac{\Delta_{l_3}}{\Delta_{l_3}}, \quad \Delta_0 = 1$$

$$A^{(l_3)} = (-1)^{l_3} \frac{\Delta^B_{l_3}}{\Delta_{l_3}}, \quad P^{(l_3)} = \frac{\Delta^E_{l_3}}{\Delta_{l_3}}, \quad \Delta^B_0 = \Delta^E_0 = 0 \quad (5)$$

$$B^{(l_3)} = \frac{\Delta^B_{l_3+1}}{\Delta_{l_3}}, \quad E^{(l_3)} = (-1)^{l_3} \frac{\Delta^E_{l_3+1}}{\Delta_{l_3}}, \quad \Delta_{-1} = 0.$$

where $\Delta_n$ are minors of the n-th order of infinite dimensional matrix

$$\Delta = \begin{pmatrix}
D & D_2 & D_{22} & \ldots \\
D_1 & D_{12} & D_{122} & \ldots \\
D_{11} & D_{112} & D_{1122} & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix} \quad (6)$$

and $\Delta^E_{l_3}, \Delta^B_{l_3}$ are the minors of $l_3$ order in the matrices of which the last column (or row) is exchanged on the derivatives of the corresponding order on argument 1 of $E$ function (on argument 2 of the $B$ function in the second case).

In what follows the following notations will be used. $W^{l_3,l_1}, (W^{l_3,l_2})$ - the result of application discrete transformation $T^{l_3}T^{l_1} (T^{l_3}T^{l_2})$ to the corresponding component of the 3- wave field. $\Delta^{l_3,l_1} (\Delta^{l_3,l_2})$ - determinant of $l_3 + l_1 (l_3 + l_2)$ orders, with the following structure of its determinant matrix.

The first $l_3$ rows (columns) of it coincides with matrix of (6) and last $l_1, l_2$ rows (columns) constructed from the derivatives of $B, (E)$ functions with respect arguments 2, (1).

The result of additional application of $l_1$ times $T_1$ transformation to the solution (3) looks as

$$P^{(l_3,l_1)} = \frac{\Delta_{l_3,l_1-1}}{\Delta_{l_3,l_1}}, \quad B^{(l_3,l_1)} = \frac{\Delta_{l_3,l_1}}{\Delta_{l_3,l_1}}, \quad \Delta_0 = 1, \quad \Delta^{l_3-1} \equiv \Delta^E_{l_3}$$
\[
Q^{(l_3,l_1)} = (-1)^{l_3+l_1-1} \frac{\Delta l_3-1,l_1}{\Delta l_3,l_1}, \quad D^{(l_3,l_1)} = (-1)^{l_3+l_1} \frac{\Delta l_3+1,l_1}{\Delta l_3,l_1}, \quad (7)
\]
\[
E^{(l_3,l_1)} = (-1)^{l_3+l_1} \frac{\Delta l_3+1,l_1-1}{\Delta l_3,l_1}, \quad A^{(l_3,l_1)} = (-1)^{l_3+l_1} \frac{\Delta l_3-1,l_1+1}{\Delta l_3,l_1},
\]

We do not present the explicit form for components \(W^{(l_3,l_2)}\), which can be obtained without any difficulties from (7) by corresponding exchanging of the arguments and unknown functions.

5 Many-soliton solution of the scalar 3-waves problem

The system (2) allows the following reducing (under additional assumption that all operators of differentiation are the real ones \(\partial_\alpha = \partial_\alpha^*\))
\[
P = B^*, \quad A = E^*, \quad Q = D^*
\]

In this case the system (2) is reduced to three equations
\[
B_1 = -DE^*, \quad E_2 = -DB^*, \quad D_3 = -BE
\]
for three complex valued unknown functions \((E, B, D)\).

Now we would like to demonstrate how the multi-soliton solutions of the system (3) may be obtained with the help of the technique of discrete transformation in the most straightforward way.

With this aim let us consider the action of the direct and inverse \(T_i, T_i^{-1}\) transformations on the reduced solution of the system (3). The trick consists in the fact that discrete transformation does not conserve the condition of the reality (3) and starting from the solution of the reduced system we come back to solution of irreductible one and in some cases vice versa. We will denote the three dimensional vector \((Q, P, A)\) by the single symbol \(\vec{Q}\) and the by symbol \(\vec{D}\) three dimensional vector \((D, B, E)\). Then the result of actions of direct and inverse transformations on solution satisfying the condition of reality \(\vec{Q} = \vec{D}^*\) is the following
\[
T_i^n(\vec{D}, \vec{D}^*) = (t_i)^n(\vec{q}, \vec{d}), \quad T_i^{-n}(\vec{D}, \vec{D}^*) = (\vec{d}^*, \vec{q}^*)
\]

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where \( t_i \) are point like symmetries of the system (2)

\[
\begin{align*}
t_3(Q, P, A, D, B, E) &= (Q, -P, -A, D, -B, -E), \\
t_2(Q, P, A, D, B, E) &= (-Q, -P, A, -D, -B, E), \\
t_3(Q, P, A, D, B, E) &= (-Q, P, -A, -D, B, -E)
\end{align*}
\]

It is obvious that \( t_i^2 = 1 \). Thus if we apply \( 2n \) times discrete transformation to initial bad (nonreduced) solution \((0, \vec{D})\) and as a result obtain \(( t^n \vec{D}^*, 0)\) then in the middle of the chain we will have solution satisfying the condition of reality, which coincides with \( n \) soliton solution of the reduced system (3).

The solution of the chain with the boundary conditions \( \vec{Q} = 0 \) on the left end of the chain and \( \vec{D} = 0 \) on the right side we will call as the chain with fixed ends. Really condition \( \vec{D} = 0 \) is the system of equation from which initial functions \( D, B, E \) (see (3)) may be defined as the solutions of ordinary differential equations (see Appendix II).

### 6 Matrix three waves problem in the space of three dimensions and its discrete transformation

In all calculations above we have never used (except of concrete resolving of discrete transformation chains) the condition that operators of differentiation are connected by the condition

\[ \partial_1 + \partial_2 + \partial_3 = 0 \]

as it follows from the definition of this operators. So we can consider the system (2) where all three operators are independent from each other and correspond to differentiation with respect to one of coordinates of three dimensional space. The second generalization consists in possibility to consider the unknown function in (3) as the operator valued ones. Of course in this case the order of the multiplications are essential and exactly coincides with fixed by the formula (3).
The discrete transformation in this case looks as

\[ \bar{Q} = D^{-1}, \quad \bar{A} = -BD^{-1}, \quad \bar{P} = D^{-1}E, \]
\[ \bar{B} = -D(BD^{-1})_2, \quad \bar{E} = -D(D^{-1}E)_1, \quad D^{-1}\bar{D} = QD - (D^{-1}D_2)_1 \]

By the same technique for \( T_1 \) we have

\[ \bar{P} = B^{-1}, \quad \bar{Q} = B^{-1}A, \quad \bar{E} = -DB^{-1}, \]
\[ \bar{D} = (DB^{-1})_2B, \quad \bar{A} = -B(B^{-1}A)_3, \quad \bar{BB}^{-1} = BP - (B_3B^{-1})_2 \]

And at last the discrete transformation \( T_2 \) looks as

\[ \bar{A} = E^{-1}, \quad \bar{B} = E^{-1}D, \quad \bar{Q} = -PE^{-1}, \]
\[ D = -E(E^{-1}D)_1, \quad P = (PE^{-1})_3E, \quad E^{-1}E = AE - (E^{-1}E_3)_1 \]

As in the scalar case the discrete transformations in the case under consideration are mutually commutative. The arising chains of equations for \((E, B, D)\) operator a valued functions (the matrices of the finite dimensions for instance) coincides with the investigated before matrix Toda chain. Explicit solutions for this chains of equations with the fixed ends reader can find in \([6]\). Uniting these results it is possible to construct multi soliton solutions of the matrix 3-wave problem in three dimensions similar to way proposed in \([7]\) for construction of multi soliton solutions for matrix Devay-Stewartson equation.

7 Outlook

The concrete results of the present paper are concentrated in explicit formulae for discrete transformations for 3-wave problem of the section two and their generalization on the matrix case (section 6).

But a no less important is the understanding how the method of the discrete transformation may be generalized on the case of multicomponent systems, connected with the semi simple algebras of the higher ranks \( r \). From results of the present paper it is clear that in the case of arbitrary semi simple algebra there are \( r \) independent basis mutually commutative discrete transformations. In what connection are this commutative objects with the
main ingredients of the representation theory of the group is very interesting
and intrigued question for further investigation.

And the last comment. The chain with two fixed ends can not be con-
sidered as the basis for some finite dimensional representation of the group
of the discrete transformation, if it is at all possible to apply term group for
it in this case. On the function at the end point of chain it is impossible
to act by direct transformation at the right side and inverse on the left end.
What is discrete transformation from the group theoretical point of view in
this case? We at this time have no answer on this question.

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9 Appendix I

In this appendix we would like to show, how it is possible to construct two-
dimensional integrable systems connected with $A_2$ algebra.

Let us consider the following $3 \times 3$ polynomial matrix

$$P(\lambda) = \begin{pmatrix}
\tilde{P}^{11}_{n_1 + 1} & P^{12}_{n_2} & P^{13}_{n_3} \\
\tilde{P}^{21}_{n_1} & P^{22}_{n_2 + 1} & P^{23}_{n_3} \\
\tilde{P}^{31}_{n_1} & P^{32}_{n_2} & P^{33}_{n_3 + 1}
\end{pmatrix}$$

(10)

where $P^{ij}_k$ are the polynomials of the degree $k$ (with respect to parameter
$\lambda$ and sign $\tilde{P}$ means that coefficient function on the highest degree of the
corresponding polynomial equal to unity.

Let us define coefficient function of all polynomials are defined from the
condition that between the columns of the matrix

$$\tilde{P} = P(\lambda) \exp(\tau_1 h_1 + \tau_2 h_2)$$

takes place the linear dependence in $(n_1 + n_2 + n_3 + 3)$ points of the $\lambda$ plane,
h_1, h_2 Cartan elements of $A_2$ algebra and $\tau_i = \phi_i(t, \lambda) + f_i(x, \lambda)$. $\phi_i, f_i$ are
arbitrary rational functions with respect to argument $\lambda$. 

The last condition is equivalent to the following system of linear equations for defining the coefficient function (we present it here for elements of the first row)

\[ \tilde{P}_{n_1+1}^{11}(\lambda_s) + c_s \exp(\tau_2^s - 2\tau_1^s)P_{n_2}^{12}(\lambda_s) + d_s \exp-(\tau_2^s + \tau_1^s)P_{n_3}^{13}(\lambda_s) = 0 \quad (11) \]

\[ s = 1, 2, ..., (n_1 + n_2 + n_3 + 3), \tau_i^s \equiv \tau(\lambda_s) \]

\((n_1 + n_2 + n_3 + 3)\) exactly the number of coefficient function of polynomials of the first row. Thus (11) is the linear system of equation for their determination.

Let us now determinant \(\text{Det}(P(\lambda) = \text{Det}(\tilde{P}(\lambda))\). From (11) it follows that it is the polynomial of \((n_1 + n_2 + n_3 + 3)\) degree with unity coefficient before the highest term and from condition (11) that it has zeroes in \((n_1 + n_2 + n_3 + 3)\) points \(\lambda_s\) of the \(\lambda\) plane. Thus

\[ \text{Det}(P(\lambda)) = \text{Det}(\tilde{P}(\lambda)) = \Pi_{k=1}^{(n_1+n_2+n_3+3)}(\lambda - \lambda_k) \quad (12) \]

Now let us calculate the matrix \(\hat{P}\vec{P}^{-1}\), where \(\hat{f}\) means the differentiation with respect to one of two independent arguments of the problem \(x, t\). From the definition of the inverse matrix it follows that matrix elements of this matrix are the following ones

\[ (\hat{P}\vec{P}^{-1})_{\alpha,\beta} = \frac{\text{Det}(P_\beta \rightarrow \hat{P}_\alpha + P_\alpha(\hat{\tau}_{i+1} - \hat{\tau}_i))}{\Pi_{k=1}^{(n_1+n_2+n_3+3)}(\lambda - \lambda_k)}, \quad \tau_0 = \tau_3 = 0 \quad (13) \]

This symbolical form means that the determinant matrix of numerator arises after exchanging of the \(\beta\) row of the \(P\) matrix on the \(\alpha\) row of the matrix \(\hat{P}\) \(\exp - (\tau_1 h_1 + \tau_2 h_2)\).

It is not difficult to understand that matrix \(\hat{P}\vec{P}^{-1}\) possess all the same singularities as functions \(\tau\) by themselves.

Now let us illustrate situation on the example of three wave interaction, choosing \(\tau_1 = \lambda(c_1 t + c_2 x)\), \(\tau_2 = \lambda(d_1 t + d_2 x)\). Let us calculate in this case for instance \((\hat{P}_t \vec{P}^{-1})_{11}\). In connection with (13) we numerator determinant have

\[ \text{Det} \left( \begin{array}{cccc}
\hat{P}_{n_1+1} & \hat{P}_{n_1+1}c_1 \lambda & \hat{P}_{n_2} + P_{n_2} (c_2 - c_1) \lambda & \hat{P}_{n_3} - P_{n_3} c_2 \lambda \\
\text{P}_{n_1} & \text{P}_{n_1} & \text{P}_{n_2+1} & \text{P}_{n_3+1} \\
\text{P}_{n_1} & \text{P}_{n_1} & \text{P}_{n_2} & \text{P}_{n_3+1} \\
\text{P}_{n_1} & \text{P}_{n_1} & \text{P}_{n_2} & \text{P}_{n_3+1} \\
\end{array} \right) \]
It is obvious that between the columns of the matrix $\bar{P}$ the linear dependence takes place with the same coefficients and so numerator determinant has zeroes in the same points as determinant in enumerator. Computation the degrees of numerator shows that it is polynomial of the $(n_1 + n_2 + n_3 + 4)$ order and so considered matrix element is the linear function of the $\lambda$ parameter. From (12) it follows that it equal exactly $c_1 \lambda$. The same not conversion calculations show that matrix $\bar{P}_t \bar{P}^{-1}$ coincides with the $u$ matrix from the introduction after identification

$$P = (P^{12})^{n_2}_{n_2}, \quad Q = (P^{13})^{n_3}_{n_3}, \quad B = (P^{21})^{n_1}_{n_1},$$

$$A = (P^{23})^{n_3}_{n_3}, \quad D = (P^{31})^{n_1}_{n_1}, \quad E = (P^{32})^{n_2}_{n_2},$$

where values above are coefficients at the highest degree terms of the corresponding polynomial. These terms are known from the solution of the linear system (11) and so we have explicit solution of the system (1).

10 Appendix II

In this appendix we would like to consider the simple example of soliton solution of 3-wave problem. We specially consider this simplest example in details to give the reader possibility to feel self-consistent of the whole construction of the present paper.

Let in notations of the 5-th section $l_3 = 2, l_1 = 0$. Condition that vector $\bar{D}^2 = 0$ is equivalent to the following system of equations

$$\Delta_3 = \Delta_3^B = \Delta_3^E = 0 \quad (14)$$

The first of this equations leads uniquely to explicit form of initial $D$ function

$$D = \phi_1(1)f_1(2) + \phi_2(1)f_2(2), \quad \phi_1 = \phi', \quad f_2 = \dot{f} \quad (15)$$

Using the initial conditions (4) equation $\Delta_3^E = 0$ may be rewritten consequently

$$B\Delta_3^E = -Det \begin{pmatrix} D & D_2 & D_1 \\ D_1 & D_{12} & D_{11} \\ D_{11} & D_{112} & D_{111} \end{pmatrix} =$$
\[(f_1 f_2 - \dot{f}_2 f_1) \text{Det} \begin{pmatrix} \phi_1 & \phi_2 & \phi_1' f_1 + \phi_2' f_2 \\ \phi_1' & \phi_2' & \phi_1'' f_1 + \phi_2'' f_2 \\ \phi_1'' & \phi_2'' & \phi_1''' f_1 + \phi_2''' f_2 \end{pmatrix} \]

Keeping in mind that \(\phi, f\) are the functions of the different arguments we conclude the last equations is equivalent to equality to zero of the two determinants of third order. The last condition in its turn can be rewritten as the system of equations

\[
\phi_1' = p \phi_1 + q \phi_2, \quad \phi_2' = s \phi_1 + t \phi_2 \\
\phi_1'' = p \phi_1' + q \phi_2', \quad \phi_2'' = s \phi_1' + t \phi_2' \\
\phi_1''' = p \phi_1'' + q \phi_2'', \quad \phi_2''' = s \phi_1'' + t \phi_2''
\]

(16)

From (16) it follows immediately that \((\phi_2 \neq c \phi_1) \quad p' = q' = s' = t' = 0\) and functions \(\phi_{1,2}\) are the solutions of the first row of (16)—the linear system of equation with the constant coefficients. Solution of this system is obvious

\[
\phi_1 = c_1 \exp \lambda_{11} + c_2 \exp \lambda_{21}, \quad \phi_2 = c_3 \exp \lambda_{11} + c_4 \exp \lambda_{21}.
\]

From the equation \(E \Delta_3^B\) by the same way we obtain

\[
f_1 = d_1 \exp \mu_{12} + d_2 \exp \mu_{22}, \quad f_2 = d_3 \exp \mu_{12} + d_4 \exp \mu_{22},
\]

where \(c, d, \lambda, \mu\) arbitrary numerical parameters.

The initial conditions

\[-D_3 = D_1 + D_2 = BE \equiv (b_1 \exp \mu_{12} + b_2 \exp \mu_{22})(e_1 \exp \lambda_{11} + e_2 \exp \lambda_{21})\]

allow using (15) allow determine parameters \(b, e\) and find one relation connected parameters \(c, d, \lambda, \mu\). Now let us calculate vector \(Q^{2,0}\) using explicit expressions for \(D, B, E\) functions. The last two ones we present in the following form \(E = p \phi_1 + q \phi_2, B = r f_1 + s f_2\).

\[
Q^{2,0} = -\frac{D}{D_2} = -\frac{\phi_1 f_1 + \phi_2 f_2}{D \left( \begin{array}{cc} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{array} \right) D \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)}
\]

\[
P^{2,0} = \frac{D \left( \begin{array}{cc} f_1 & p \\ f_2 & q \end{array} \right)}{D \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)}, \quad A^{2,0} = \frac{D \left( \begin{array}{cc} \phi_1 & \phi_2 \\ r & s \end{array} \right)}{D \left( \begin{array}{cc} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{array} \right)}
\]
Conditions of reality leads to other restriction on parameters involved. It is clear that two possibilities in the choice of parameters $\lambda$ and $\mu$ are possible $\lambda_2 = -\lambda_1^*$, $\lambda_1 = -\lambda_1^*$ and $\lambda_2 = -\lambda_2^*$. And the same limitations on parameters $\mu_2$. We do not present here explicit form for the other restrictions. This is pure algebraic manipulations.

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