Undirected Connectivity of Sparse Yao Graphs

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ABSTRACT
Given a finite set \( S \) of points in the plane and a real value \( d > 0 \), the \( d \)-radius disk graph \( G^d \) contains all edges connecting pairs of points in \( S \) that are within distance \( d \) of each other. For a given graph \( G \) with vertex set \( S \), the Yao subgraph \( Y_k[G] \) with integer parameter \( k > 0 \) contains, for each point \( p \in S \), a shortest edge \( pq \in G \) (if any) in each of the \( k \) sectors defined by \( k \) equally-spaced rays with origin \( p \). Motivated by communication issues in mobile networks with directional antennas, we study the connectivity properties of \( Y_k[G^d] \), for small values of \( k \) and \( d \). In particular, we derive lower and upper bounds on the minimum radius \( d \) necessary and sufficient for the connectivity of \( Y_k[G^d] \). We also show that, for \( d < 5 - \frac{2}{\sqrt{3}} \sqrt{k} \), the graph \( Y_3[G^d] \) can be disconnected, but \( Y_3[G^{2/\sqrt{3}}] \) is always connected. Finally, we show that \( Y_2[G^d] \) can be disconnected, for any \( d \geq 1 \).

1. INTRODUCTION
Let \( S \) be a finite set of points in the plane and let \( G = (S, E) \) be an arbitrary (undirected) graph with node set \( S \). The directed Yao graph \( Y_k[G] \) with integer parameter \( k > 0 \) is a subgraph of \( G \) defined as follows. At each point \( p \in S \), \( k \) equally-spaced rays with origin \( p \) define \( k \) cones. In each cone, pick a shortest edge \( pq \) from \( G \), if any, and add the directed edge \( \overrightarrow{pq} \) to \( Y_k[G] \). Ties are broken arbitrarily. The undirected Yao graph \( Y_k[G] \) ignores the directions of edges, and includes an edge \( pq \) if and only if either \( \overrightarrow{pq} \) or \( \overrightarrow{qp} \) is in \( Y_k[G] \).

For a fixed real value \( d > 0 \), let \( G^d(S) \) denote the \( d \)-radius disk graph with node set \( S \), in which two nodes \( p, q \in S \) are adjacent if and only if \( |pq| \leq d \). Most often we will refer to \( G^d(S) \) simply as \( G^d \), unless the point set \( S \) that defines \( G^d \) is unclear from the context. Under this definition, \( G_1 \) is the unit disk graph (UDG), and \( G^\infty \) is the complete Euclidean graph, in which any two points are connected by an edge. In this paper we study the connectivity of the undirected Yao graph \( Y_k[G^d] \), for small values \( k \in \{2, 3, 4\} \) and \( d \geq 1 \). Underlying our study is the assumption that \( G_1 \) is connected. (For example, \( G_1 \) can be thought of as the graph connecting all pairs of points that are within distance no greater than the length of the bottleneck edge in a minimum spanning tree for \( S \), normalized to one.) In this context, we investigate the following problem:

Let \( S \) be an arbitrary set of points in the plane, and suppose that the unit radius graph \( G_1 \) defined on \( S \) is connected. What is the smallest real value \( d \geq 1 \) for which \( Y_k[G^d] \) is connected?

Throughout the paper, we will refer to the minimum value \( d \) that renders \( Y_k[G^d] \) connected as the connectivity radius of \( Y_k[G^d] \).

Our research is inspired by the use of wireless directional antennas in building communication networks. Unlike an omnidirectional antenna, which transmits energy in all directions, a directional antenna can concentrate its transmission energy within a narrow cone; the narrower the cone, the longer the transmission range, for a fixed transmission power level. Directional antennas are preferable over omnidirectional antennas, because they reduce interference and extend network lifetime, two criteria of utmost importance in wireless networks operating on scarce battery resources.

Directed Yao edges can be realized with narrow directional antennas (otherwise called laser-beam antennas, to imply a small cone angle, close to zero). One attractive property of Yao graphs is that they can be efficiently constructed locally, because each node can select its incident edges based on the information from nodes in its immediate neighborhood only. This enables each node to repair the communication structure quickly in the face of dynamic and kinetic changes, providing strong support for node mobility.

The limited number of antennas per node (1 to 4 in practice), raises the fundamental question of connectivity of Yao graphs \( Y_k \), for small values of \( k \). If the communication graph induced by antennas operating in omnidirectional mode is connected, by how much must an antenna radius increase to guarantee that \( k \) laser-beam antennas at each node, pointing in the direction...
of the $Y_k$ edges, preserve connectivity? In this paper we focus our attention on small $k$ values (2, 3 and 4) corresponding to the number of antennas commonly used in practice.

1.1 Prior Results

Yao graphs have been extensively studied in the area of computational geometry, and have been used in constructing efficient wireless communication networks [5 8 7 4]. Applying the Yao structure on top of a dense communication graph, in order to obtain a sparser graph, is a very natural idea. Most existing results concern Yao graphs $Y_k[G^\infty]$ with $k \geq 6$. These graphs exhibit nice spanning properties, in the sense that the length of a shortest path between any two nodes $p, q \in Y_k[G^\infty]$ is only a constant times the Euclidean distance $|pq|$ separating $p$ and $q$ [2 1 3]. In the context of using laser-beam antennas to realize $Y_k$, however, these results could only be applied if 6 or more antennas were available at each node, which is a rather impractical requirement. Few results exist on Yao graphs $Y_k$, for small values of $k$ (below 6). It has been shown that $Y_2[G^\infty]$ and $Y_3[G^\infty]$ are not spanners [6], and that $Y_4[G^\infty]$ is a spanner [3]. However, as far as we know, no results exist on $Y_k[G^\infty]$, for any fixed radius $d \geq 1$.

1.2 Our Results

We develop lower and upper bounds on the connectivity radius of $Y_2$, $Y_3$ and $Y_4$, relative to the unit radius. (Recall that our assumption that the unit radius disk graph $G^1$ is connected.) We prove tight lower and upper bounds equal to $\sqrt{2} \approx 1.414$ on the connectivity radius $d$ of $Y_2[G^d]$. Surprisingly, we prove a smaller upper bound equal to $\frac{2}{\sqrt{3}} \approx 1.155$ on the connectivity radius $d$ of $Y_3[G^d]$. This is somewhat counterintuitive, as one would expect that fewer outgoing edges per node (3 in the case of $Y_3$ compared to 4 in the case of $Y_4$) would necessitate a higher connectivity radius, however our results show that this is not always the case. We also derive a lower bound of $5 - \frac{2}{\sqrt{3}} \approx 1.056$ on the connectivity radius $d$ of $Y_4[G^d]$, leaving a tiny interval $[1.056, 1.155]$ on which the connectivity of $Y_4$ remains uncertain. Finally, we show that $Y_3[G^d]$ can be disconnected, for any fixed value $d \geq 1$.

1.3 Definitions

Let $S$ be a fixed set of points in the plane. At each node $p \in S$, let $r_1, r_2, \ldots, r_k$ denote the $k$ rays originating at $p$, with $r_1$ horizontal along the $+$x axis (see Figure 1 for $k = 3$). Let $C_i(p)$ to denote the half-open cone delimited by $r_i$ and $r_{i+1}$, including $r_i$ but excluding $r_{i+1}$. (Here we use $r_{k+1}$ to mean $r_1$.) For any point $p \in S$, let $x(p)$ denote the $x$–coordinate of $p$ and $y(p)$ denote the $y$–coordinate of $p$. For any $p, q \in S$, let $|pq|$ denote the Euclidean distance between $p$ and $q$. For any point $p \in S$ and any real value $\delta > 0$, let $D(p, \delta)$ be the closed disk with center $p$ and radius $\delta$.

2. CONNECTIVITY OF $Y_4$

In this section we derive tight lower and upper bounds on the connectivity radius $d$ for $Y_4[G^d]$. Recall that our work relies on the assumption that $G^1$ is connected.

**Theorem 2.1.** There exist point sets $S$ with the property that $G^1(S)$ is connected, but $Y_3[G^d]$ is disconnected, for any $1 \leq d < \sqrt{2}$.

**Proof.** We construct a point set $S$ that meets the conditions of the theorem. Note that $d < \sqrt{2}$ implies that $1 - \sqrt{d^2 - 1} > 0$, meaning that there exists a real value $\epsilon$ such that $0 < \epsilon < 1 - \sqrt{d^2 - 1}$, which is equivalent to

$$1 + (1 - \epsilon)^2 > d^2$$

Let $p$ and $q$ be the endpoints of a vertical segment of length 1, with $p$ below $q$. In Figure 2 the segment $pq$ is shown slightly slanted to the left, merely to reinforce our convention that $pq \in C_2(p)$ and $qp \in C_4(q)$. Shoot a horizontal ray from $p$ leftward, then slightly rotate it clockwise about $p$ by a tiny angle $\alpha$, so that the ray lies entirely in $C_3(p)$. Distribute points $a_1, a_2, \ldots, a_r$ in this order along this ray such that $|pa_1| = 1 - \epsilon$, and $|a_i a_{i+1}| = 1$, for each $i$. Let $b_i$ be the point symmetric to $a_i$ with respect to the midpoint of $pq$. Let

$$S = \{ p, q, a_i, b_i \mid 1 \leq i \leq r \}.$$
In the limit, as $\alpha$ approaches 0, the angle $\angle a_1pq$ approaches $\pi/2$ and $|a_1q| = \sqrt{1 + (1 - \epsilon)^2} > d$. This means that $a_1q$ is not an edge in $G^d$. Because $|a_ib_j| > |a_1q| > d$ for each $i, j \geq 1$, we have that no $a-$point is directly connected to a $b-$point in $G^d$. It follows that the graph $G^d$ is a path (depicted in Figure 2).

We now show that $pq \not\in Y_4[G^d]$, which along with the fact that $G^d$ is a path, yields that claim that $Y_4[G^d]$ is disconnected. First note that $p\overrightarrow{q}$ is not an edge in $Y_4[G^d]$ as depicted in Figure 2b. This is because $a_1$ is in the same cone $C_2(p)$ as $q$, and $|pa_1| = 1 - \epsilon < 1 = |pq|$. Similarly, $\overrightarrow{qp}$ is not an edge in $Y_4[G^d]$, because $b_1$ is in the same cone $C_2(q)$ as $p$, and $|qb_1| = 1 - \epsilon < 1 = |qp|$. We conclude that $Y_4[G^d]$ is disconnected. □

Upper Bound $d = \sqrt{2}$

We now show that $Y_4[G^d]$ is always connected for $d = \sqrt{2}$, matching the lower bound from Theorem 2.1

First we introduce a few definitions. For any pair of points $a, b$, let $d_\infty(a, b)$ denote the $L_\infty$ distance between $a$ and $b$, defined as

$$d_\infty(a, b) = \max\{|x(a) - x(b)|, |y(a) - y(b)|\}$$

Let $\mathcal{J}(a, b)$ be the square with corner $a$ whose boundary contains $b$, of side length $d_\infty(a, b)$ (see Figure 3b). The following inequalities follow immediately from the fact that $ab$ is a line segment inside $\mathcal{J}(a, b)$:

$$d_\infty(a, b) \leq |ab| \leq d_\infty(a, b)\sqrt{2}$$ (1)

**Theorem 2.2.** For any point set $S$ such that $G^1(S)$ is connected, $Y_4[G^{\sqrt{2}}]$ is also connected.

**Proof.** The proof is by contradiction. Assume to the contrary that $G^1$ is connected, but $Y_4[G^{\sqrt{2}}]$ is disconnected. Then $Y_4[G^{\sqrt{2}}]$ has at least two connected components, say $J_1$ and $J_2$. Since $G^1 \subseteq G^{\sqrt{2}}$ is connected, there is an edge $pq \in G^1$, with $p \in J_1$ and $q \in J_2$. To derive a contradiction, consider two points $a, b \in S$, with $a \in J_1$ and $b \in J_2$, that minimize $d_\infty(a, b)$. Then

$$d_\infty(a, b) \leq d_\infty(p, q) \quad \text{(by choice of } ab)$$

$$\leq |pq| \quad \text{(by 1)}$$

$$\leq 1 \quad \text{(because } pq \in G^1)$$

This along with the second inequality from 1 implies $|ab| \leq \sqrt{2}$, therefore $ab \in G^{\sqrt{2}}$. To simplify our analysis, rotate $S$ so that $b$ lies in the lower half of $C_1(a)$.

If $ab \not\in Y_4[G^{\sqrt{2}}]$, then $ab$ connects $J_1$ and $J_2$, contradicting our assumption that $J_1$ and $J_2$ are disjoint connected components. So $ab \not\in Y_4[G^{\sqrt{2}}]$. However $ab \in G^{\sqrt{2}}$ and $b \in C_1(a)$, therefore there is $\overrightarrow{ac} \in Y_4[G^{\sqrt{2}}]$, with $c \in C_1(a)$ and $|ac| \leq |ab|$. If $c$ lies inside $\mathcal{J}(a, b)$, then $d_\infty(b, c) < d_\infty(a, b)$, because each of the horizontal and vertical distance between $b$ and $c$ is strictly smaller than the side length of $\mathcal{J}(a, b)$. This along with the fact that $bc$ connects $J_1$ and $J_2$, contradicts our choice of $ab$. So $c$ must lie outside of $\mathcal{J}(a, b)$ (but not outside of $D(a, |ab|)$, because $|ac| \leq |ab|$).

Let $e$ be the lower right corner of $\mathcal{J}(a, b)$, and let $f$ be intersection point between the boundary of $D(a, |ab|)$ and the horizontal ray through $a$ in the direction of $e$ (see Figure 3a). We will be using the fact that $|be| > |ef|$ (2)

(This follows from the fact that $\angle bfe = \angle aba > \angle bfe$, and the Law of Sines applied on $\triangle bef$.)

We now derive a contradiction to our choice of $ab$ as follows. If both $b$ and $e$ lie in the lower half of $C_1(a)$, as depicted in Figure 3a, then $|y(b) - y(c)| < d_\infty(a, b)$. Also $|x(b) - x(c)| < |ef|$, which by inequality 2 is no longer than $d_\infty(a, b)$. It follows that $d_\infty(b, c) < d_\infty(a, b)$, which along with the fact that $bc$ connects $J_1$ and $J_2$, contradicts our choice of $ab$. If $b$ and $e$ lie on opposite sides of the bisector of $C_1(a)$, as depicted in Figure 3b, then the vertical distance from $c$ to the top side of $\mathcal{J}(a, b)$ is smaller than $|ef|$, which in turn is smaller than $|be|$ (by inequality 2). It follows that $|y(c) - y(b)| < d_\infty(a, b)$. Also, because $c$ lies strictly to the right of $a$, we have that $|x(c) - x(b)| < d_\infty(a, b)$. These together show that $d_\infty(b, c) < d_\infty(a, b)$. This along with the fact that $bc$ connects $J_1$ and $J_2$, contradicts our choice of $ab$. □

Theorems 2.1 and 2.2 together establish matching lower and upper bounds (equal to $\sqrt{2}$) for the connectivity radius of $Y_4$.

3. CONNECTIVITY OF $Y_3$

The Yao graph $Y_3$ has three outgoing edges per node, compared to four outgoing edges in the case of $Y_4$. So one would expect that the radius necessary to maintain $Y_3$ connected would exceed the radius necessary to
maintain $Y_4$ connected. However, our results show that an antenna radius equal to $\frac{2}{\sqrt{3}} < \sqrt{2}$ suffices to maintain $Y_3$ connected. This is a surprising result, given that a radius of $\sqrt{2}$ is necessary and sufficient to maintain $Y_4$ connected, as established in the previous section.

**Theorem 3.1.** There exist point sets $S$ with the property that $G^1(S)$ is connected, but $Y_3[G^d]$ is disconnected, for any $1 \leq d < 5 - \frac{2}{3}\sqrt{35}$.

**Proof.** We construct a point set $S$ that satisfies the conditions of the theorem. Start with an isosceles trapezoid $pa_1b_1q$ of unit altitude and bases $pq$ and $a_1b_1$, with $|pq| = 1$ and $|a_1b_1| = 1 + \varepsilon$, for some small real value $0 < \varepsilon < 1$, to be determined later. Place a point $x$ on $pa_1$ at distance $|pa_1|/3$ from $p$, and a second point $y$ on $b_1q$ at distance $|b_1q|/3$ from $b_1$. Then simply reflect $px$ about the vertical line through $p$, and $qy$ about the vertical line through $q$. As we will later see, this places $px$ and $pq$ in the same cone of $p$ (after a $90^\circ$ counterclockwise rotation), so that $px$ and $pq$ compete in the edge selection process at $p$.

The result is the shaded polygon depicted in Figure 4. Simple calculations show that the vertical distance between $x$ and $y$ is $1/3$, and the horizontal distance between $x$ and $y$ is

$$1 - \frac{1}{3} \cdot \varepsilon - \frac{2}{3} \cdot \varepsilon = 1 - \frac{\varepsilon}{2}$$

It follows that $|xy|^2 = (\frac{1}{3})^2 + (1 - \frac{2}{3})^2 = 1 + \frac{9\varepsilon^2 - 36\varepsilon + 4}{36}$. We will later require that $|xy| > d$ and $|a_1b_1| > d$, so that neither $xy$ nor $a_1b_1$ is a candidate for $Y_3[G^d]$. These two inequalities reduce to

$$\left\{ \begin{array}{l} 9\varepsilon^2 - 36\varepsilon + 40 - 36d > 0 \\ 1 + \varepsilon > d \end{array} \right.$$  

Simple calculations yield the solution

$$1 \leq d < 5 - \frac{2}{3}\sqrt{35} \quad (3)$$

$$d - 1 < \varepsilon < 2 - \frac{2}{3}\sqrt{9d - 1}. \quad (4)$$

By the triangle inequality, $|xa_1| < 2/3 + \varepsilon/2$. It can be easily verified that the above constraints on $\varepsilon$ and $d$ yield $|xa_1| < 1$. Similarly, each of $px$, $qy$ and $qy$ has length less than $1$. Also note that

$$|xy| > |qy| > d,$$

since the horizontal distance between $x$ and $q$ is greater than the horizontal distance between $x$ and $y$, and the vertical distance is $1/3$ in both cases. Similarly, $|a_1y| > |xy| > d$.

We are now ready to construct $S$. Start by rotating the polygon $pxa_1b_1qy$ counterclockwise by $90^\circ$, so that it lies on its side, as in Figure 4. Shoot a horizontal ray rightward from $a_1$, then rotate it slightly clockwise so that it lies entirely in $C_3(a_1)$. Distribute points $a_2, a_3, \ldots, a_r$ at unit intervals along this ray. Let $b_i$ be the reflection of $a_i$ with respect to the horizontal through the midpoint of $pq$, for each $i > 1$. Our point set is

$$S = \{ p, q, x, y, a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r \}.$$  

The graph $G^1$ is a path (depicted in Figure 4) and is therefore connected. We now show that $Y_3[G^d]$ is disconnected.

By construction, the following inequalities hold: $|a_1b_1| > d$; $|xy| > |qy| > d$; $|a_1q| > |a_1y| > |xy| > d$; and $|a_iy| > |a_1b_1| > |a_1b_1| > d$, for any $i \geq 1$ and any $j \geq 0$ (because $\angle a_1b_1a_j$ is obtuse). By symmetry, similar arguments hold for the $b$—points as well. It follows that the graph $G^d$ is a path identical to $G^1$, therefore the removal of any edge from $G^d$ disconnects it.

Next we show that $pq \notin Y_3[G^d]$, which along with the observation above implies that $Y_3[G^d]$ is disconnected. By construction, $|px| < |pq|$. This along with the fact that both $x$ and $q$ lie in the same cone $C_1(p)$, implies that $p$ does not select $pq$ for inclusion in $Y_3[G^d]$. Similarly, $|qy| < |pq|$. This along with the fact that both $y$ and $p$ lie in the same cone $C_3(q)$, implies that $q$ does not select $qp$ for inclusion in $Y_3[G^d]$. These together show that $pq \notin Y_3[G^d]$, therefore $Y_3[G^d]$ is disconnected. □

**Upper Bound $d \leq 2/\sqrt{3}$.**

Next we derive an upper bound on the connectivity radius for $Y_3$. The approach adopted here is somewhat similar to the one employed in the proof of Theorem 2.2 but it uses a generalized distance function $d_R$ (in place of $d_\infty$), to measure the distance between two connected components. We define $d_R$ as follows. For any point $a \in S$ and any point $b \in C_i(a)$, let $R(a,b)$ denote the closed rhombus with corner $a$ and edges parallel to $r_i$ and $r_{i+1}$, whose boundary $\partial R(a,b)$ contains $b$ (see Figure 5). (Recall that $C_i(a)$ is the half-open cone with apex $a$ that includes $r_i$ and excludes $r_{i+1}$.) Define $d_R(a,b)$ to be the side length of $R(a,b)$. Clearly, $d_R(a,a) = 0$. Because our approach does not use the triangle inequality on $d_R$, we skip the proof that $d_R$ is a distance metric, and focus instead on the symmetry...
property of \( d_R \) (Property (i) of Lemma 3.2 below), and the relationship between \( d_R \) and the Euclidean distance.

**Lemma 3.2.** For any pair of points \( a, b \in S \) the following properties hold:

(i) \( d_R(a, b) = d_R(b, a) \).

(ii) \( |ab| \leq d_R(a, b) \).

(iii) \( |ab| \geq d_R(a, b) \sqrt{3} \).

**Proof.** To simplify our analysis, rotate \( S \) so that \( b \in C_1(a) \), as depicted in Figure 5. Consider the quadrilateral \( bcef \) from Figure 6, with sides \( ce \in \partial R(a, b) \) and \( bf \in \partial R(b, a) \); \( bc \) and \( ef \) are parallel, since they are both parallel to \( r_2 \); and \( \angle cef \) and \( \angle bfe \) are each 60°. These together show that \( bcef \) is an isosceles trapezoid, meaning that \( |ce| = |bf| \). Since \( |ce| = d_R(a, b) \) and \( |bf| = d_R(b, a) \), Property (i) holds.

Now note that \( R(a, b) \) is the union of two equilateral triangles of side length \( d_R(a, b) \), adjacent alongside the bisector of \( C_1(a) \). Also note that \( ab \) is a segment that connects \( a \) to the opposite side in one of these triangles - call it \( T \). It follows that \( ab \) is no longer than the side of \( T \) (thus yielding inequality (ii)), and no shorter than the height of \( T \) (thus yielding inequality (iii)). This completes the proof. \( \square \)

Following is an intermediate result that will help prove our main upper bound result stated in Theorem 3.4.

This intermediate result will simply rule out some configurations that will occur in the analysis of the main result. To follow the logical sequence of our analysis, the reader can skip ahead to Theorem 3.4 and refer back to Lemma 3.3 only when called upon from Theorem 3.4.

**Lemma 3.3.** Let \( a, b, c \in S \) be such that \( b, c \in C_i(a) \), for some \( i \in \{1, 2, 3\} \), and \( |ac| \leq |ab| \). Furthermore, assume that both \( b \) and \( c \) lie either in the half of \( C_i(a) \) adjacent to \( r_1 \) (excluding the bisector points), or in the half of \( C_i(a) \) adjacent to \( r_{i+1} \) (including the bisector points). Then \( d_R(b, c) < d_R(a, b) \).

**Proof.** To simplify our analysis, rotate \( S \) so that both \( b \) and \( c \) lie in the lower half of \( C_i(a) \) (adjacent to \( r_1 \)). Let \( \delta = d_R(a, b) \). By Lemma 3.2(ii), \( |ab| \leq \delta \). This along with \( |ac| \leq |ab| \) implies that \( c \in D(a, \delta) \). More precisely, \( c \) lies in a circular sector of angle 60°, formed by the intersection between \( D(a, \delta) \) and the lower half of \( C_i(a) \).

If \( c \in C_1(b) \), then \( R(a, b) \) and \( R(b, c) \) are similar (see Figure 7). This along with the fact that \( c \) lies in a same 60°-sector as \( b \) implies that \( d_R(b, c) < d_R(a, b) \). (The inequality is strict due to the fact that the lower half of the cone \( C_1(a) \) does not include the upper bounding ray.) If \( c \in C_3(b) \), then \( b \in C_1(c) \) (see Figure 8). This is Figure 5: Rhombus \( R(a, b) \) of side length \( d_R(a, b) \).

Figure 6: Lemma 3.2 (a) \( d_R(a, b) = d_R(b, a) \) (b) Relationship between \( d_R(a, b) \) and \( |ab| \).

Figure 7: Lemma 3.3 case \( c \in C_1(b) \): \( d_R(b, c) \leq d_R(a, b) \).

Figure 8: Lemma 3.3 \( d_R(b, c) \leq d_R(a, b) \) (a) \( c \in C_3(b) \) (b) \( c \in C_2(b) \).
case is similar to the previous one: $R(a, b)$ and $R(c, b)$ are similar, and $d_R(b, c) = d_R(c, b) < d_R(b, a)$. Finally, if $c \in C_2(b)$, then $c \in R(b, a)$ (see Figure 8), and $R(b, c) \subset R(b, a)$. It follows that $d_R(b, c) < d_R(b, a) = d_R(a, b)$.

**Theorem 3.4.** For any point set $S$ such that $G_1(S)$ is connected, $Y_2[G^d]$ is also connected, for $d = \frac{2}{\sqrt{3}}$.

**Proof.** The proof is by contradiction. Assume to the contrary that $G_1$ is connected, but $Y_2[G^d]$ is disconnected. Then $Y_3[G^d]$ has at least two connected components, say $J_1$ and $J_2$. Because $G_1 \subseteq G^d$ is connected, there is an edge $pq \in G^1$, with $p \in J_1$ and $q \in J_2$. We now show that $d_R(c, e) < d_R(a, b)$. Let $\delta_1$ be the length of the projection of $ce$ on the ray $r_2$ in the (horizontal) direction of $r_1$. Similarly, let $\delta_2$ be the length of the projection of $ce$ on $r_1$ in the direction of $r_2$. (See Figure 10.) Then $d_R(c, e) = \max\{\delta_1, \delta_2\}$. We prove that $d_R(c, e) < d_R(a, b)$ by showing that each of $\delta_1$ and $\delta_2$ is smaller than $d_R(a, b)$.

Figure 9: Theorem 3.4: case when $a$ and $d$ lie on a same side of the bisector of $C_2(b)$.

To simplify our analysis, rotate $S$ so that $b \in C_1(a)$. Because $J_1$ and $J_2$ are not connected in $Y_3[G^d]$, and because $a \in J_1$ and $b \in J_2$, we have that $ab \notin Y_3[G^d]$. However $ab \in G^d$ and $b \in C_1(a)$, therefore there is $\overline{ac} \in Y_3[G^d]$, with $c \in C_1(a)$ and $|ac| \leq |ab|$. If both $b$ and $c$ lie in the same half of $C_1(a)$ (bounded by one ray and the bisector of $C_1(a)$), then by Lemma 3.3 we have that $d_R(b, c) < d_R(a, b)$. This along with the fact that $bc$ connects $J_1$ and $J_2$, contradicts our choice of $ab$. Then $b$ and $c$ must lie on either side of the bisector of $C_1(a)$, as depicted in Figure 9.

Assume without loss of generality that $b$ lies in the lower half (excluding the bisector) of $C_1(a)$, and $c$ lies in the upper half (including the bisector) of $C_1(a)$. Next we focus on $C_2(b)$. Because $a \in C_2(b)$, $ba \in G^d$, and $ba \notin Y_3[G^d]$, there must exist $\overline{be} \in Y_3[G^d]$, with $e \in C_2(b)$ and $|be| \leq |ab|$. As before, if $e$ and $a$ lie in the same half of $C_2(b)$ (bounded by one ray and the bisector of $C_2(b)$), then by Lemma 3.3 we have that $d_R(e, a) < d_R(b, a) = d_R(a, b)$. This along with the fact that $ae$ connects $J_1$ and $J_2$ contradicts our choice of $ab$. It follows that $a$ and $e$ lie on either side of the bisector of $C_2(b)$, as depicted in Figure 10.

Figure 10: Theorem 3.4: case when $a$ and $d$ lie on opposite sides of the bisector of $C_2(b)$.

First note that $c$ must lie outside of $R(b, a)$. Otherwise, if $c$ were to lie inside $R(b, a)$, then $R(b, c) \subset R(b, a)$ (see Figure 11(a)). This would immediately imply that $d_R(b, c) < d_R(b, a) = d_R(a, b)$, which along with the fact that $bc$ connects $J_1$ and $J_2$, would contradict our choice of $ab$. So $c$ lies inside $D(a, |ab|)$ (because $|ac| \leq |ab|$), but outside of $R(b, a)$. Similar arguments show that $e$ lies inside $D(b, |ab|)$, but outside of $R(a, b)$. Let $pq$ be the top left side of $R(b, a)$ (marked with a thick line in Figure 10). By the observations above, $c$ and $e$ cannot lie below $p$ or above $q$. This implies that the horizontal projection of $ce$ on the ray $r_2$ is strictly shorter than the horizontal projection of $pq$ on $r_2$: $\delta_1 < d_R(a, b)$.

Figure 11: Theorem 3.4: (a) $c$ inside $R(b, a)$ (b) $d_R(c, e) < d_R(a, b)$.
in the direction of \( r_2 \): \( \delta_2 < d_R(a, b) \).

We have established that \( d_R(c, e) < d_R(a, b) \) (the rhombus \( R(c, e) \) is depicted in Figure (11b)). This along with the fact that \( ce \) connects \( J_1 \) and \( J_2 \), contradicts our choice of \( ab \). We conclude that \( G^d \) is connected.

Observe that our results leave a tiny gap between the lower bound of \( 5 - \frac{2}{\sqrt{3}} \approx 1.569 \) and the upper bound of \( \frac{\sqrt{3}}{2} \approx 1.555 \) on the connectivity radius \( d \) for \( Y_2[G^d] \). Nevertheless, both bounds beat the tight bound \( d = \sqrt{2} \approx 1.414 \) for the connectivity radius of \( Y_2[G^d] \).

4. CONNECTIVITY OF \( Y_2 \)

The point set \( S \) depicted in Figure 2 can be extended to show that \( Y_2[G^d] \) can be disconnected, for arbitrarily large \( d \). To see this, fix a real value \( d \geq 1 \), and distribute enough points \( a_i \) at unit interval along the leftward ray from \( p \), such that the leftmost point \( a_r \) is far enough from \( q \) – in particular, we require that it satisfies the inequality \( |a_r, q| > d \). Similarly, we require that the rightmost point \( b_s \) satisfies \( |b_s, p| > d \) (which follows immediately by symmetry). (Note that in this case \( d = \Omega(|S|) \)). Recall that the leftward ray from \( p \) is almost horizontal, so \( q \) and all the \( b \)-points lie above \( a_r \).

We now show that \( Y_2[G^d] \) is disconnected. First note that \( a_1 \) is the point closest to \( p \) in \( C_1(p) \), and that \( C_2(p) \) is empty. Therefore, the only edge \( Y_2[G^d] \) incident to \( p \) is \( pa_1 \). Also note that, for any \( i < r \), \( a_{i+1} \) is the point closest to \( a_i \) in \( C_1(a_i) \), and \( a_{i-1} \) is the point closest to \( a_i \) in \( C_2(a_i) \) (here we use \( a_0 \) to refer to the point \( p \)). Finally, \( q \) is the point closest to \( a_r \) in \( C_1(a_r) \). However, because \( |a_r, q| > d \), \( a_r, q \) is not in \( G^d \) and therefore \( a_r, q \) is not in \( Y_2[G^d] \). The arguments are symmetric for \( q \) and the \( b \)-points in \( S \). This shows that there is no edge in \( Y_2[G^d] \) connecting a point in \( \{ q, b_i \mid 1 \leq i \leq r \} \) to a point in \( \{ p, a_i \mid 1 \leq i \leq r \} \). We conclude that \( Y_2[G^d] \) is disconnected for connectivity radius values \( d = \Omega(|S|) \).

5. CONCLUSION

In this paper we establish matching lower and upper bounds on the connectivity radius for \( Y_4 \), and a tight interval on the connectivity radius for \( Y_3 \). Reducing the gap between the lower and upper ends of this interval remains open. These results show that a small increase in the radius of a directional antenna, (compared to the unit radius of an omnidirectional antenna,) renders an efficient communication graph for mobile wireless networks, provided that each node orients its \( k \in \{3, 4\} \) antennas in the direction of the \( Y_i \) edges. (Nodes are assumed to send messages in directional mode, and receive messages in omnidirectional mode). One key advantage of these graphs is that they can be quickly constructed locally, providing strong support for node mobility. We also establish that the connectivity radius for \( Y_2 \) may be arbitrarily large, which indicates that \( Y_2 \) is not a suitable communication graph for wireless networks that use narrow (laser-beam) directional antennas.

6. REFERENCES

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