QUANTIZATION OF DOUBLE COVERS OF NILPOTENT COADJOIN
T ORBITS I: NONCOMMUTATIVE MODELS

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Abstract. We construct by geometric methods a noncommutative model $E$ of the algebra of regular functions on the universal (2-fold) cover $\mathcal{M}$ of certain nilpotent coadjoint orbits $\mathcal{O}$ for a complex simple Lie algebra $\mathfrak{g}$. Here $\mathcal{O}$ is the dense orbit in the cotangent bundle of the generalized flag variety $X$ associated to a complexified Cartan decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ where $\mathfrak{p}^\pm$ are Jordan algebras by the TKK construction.

We obtain $E$ as the algebra of $\mathfrak{g}$-finite differential operators on a smooth Lagrangian subvariety in $\mathcal{M}$ where $\mathfrak{g}$ is given by differential operators $\pi_x^{\lambda_0}$ twisted according to a critical parameter $\lambda_0 = \frac{1}{2} \pm \frac{1}{4m}$. After Fourier transform, $E$ is a “quadratic” extension of the algebra $\mathcal{D}_0(X)$ of twisted differential operators for the (formal) $\lambda_0$th power of the canonical bundle.

Not only is $E$ a Dixmier algebra for $\mathcal{M}$, in the sense of the orbit method, but also $E$ has a lot of additional structure, including an anti-automorphism, a supertrace, and a non-degenerate supersymmetric bilinear pairing. We show that $E$ is the specialization at $t = 1$ of a graded (non-local) equivariant star product with parity.

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1. Introduction

Let $G$ be a simply-connected semisimple Lie group with maximal compact subgroup $U$. Let $\mathcal{O}$ be a coadjoint orbit of $G$ which is stable under dilations; so $\mathcal{O}$ is nilpotent, i.e., identifies with an adjoint orbit of nilpotent elements. Let $\mathcal{M}$ be a Galois cover of $\mathcal{O}$ with Galois group $\mathcal{S}$. Then $\mathcal{M}$ is a real symplectic manifold with KKS symplectic form $\omega$ and $\mathcal{M}$ is a Hamiltonian $G$-space. According to the orbit method and geometric quantization, quantization of $\mathcal{M}$ should give a unitary representation $E$ of $G \times \mathcal{S}$ whose $\mathcal{S}$-isotypic components are irreducible for $G$.

Now suppose $G$ is in fact a complex group. Then $\mathcal{M}$ has additional geometric structure: $\mathcal{M}$ is a complex symplectic manifold with complex structure $\mathbf{i}$ and holomorphic KKS form $\Omega$ where $\text{Re}\Omega = \omega$. Moreover the data $(\mathbf{i}, \Omega)$ extend to a $U$-invariant hyperkahler structure $(\mathbf{Kro})$. We now expect additional structure for the quantization. First $E$ should
be realized as a Hilbert space of $I$-holomorphic functions on $\mathcal{M}$. (Indeed, the hyperkahler data provides a “new” complex structure $J$ which polarizes $\omega$. Then $E$ should consist of $J$-holomorphic functions. Finally we can “rotate” $\mathcal{M}$ to move $J$ to $I$.) Second the Harish-Chandra module $\mathcal{E}$ of $U$-finite vectors in $E$ should be, as a representation of $G \times \mathcal{S}$, the algebra of $U$-finite $I$-holomorphic functions on $\mathcal{M}$; notice this is the algebra $R(\mathcal{M})$ of $I$-holomorphic functions which are regular in the sense of algebraic geometry.

Third, and most importantly, $\mathcal{E}$ should be a $(G \times \mathcal{S})$-equivariant noncommutative model of $R(\mathcal{M})$. This means that $G \times \mathcal{S}$ acts on $\mathcal{E}$ by algebra automorphisms and $\mathcal{E}$ has an invariant algebra filtration for which $\mathrm{gr} \, \mathcal{E}$ is equivariantly isomorphic, as a graded Poisson algebra, to $R(\mathcal{M})$. Here $R(\mathcal{M})$ has the Euler grading extending over $\frac{1}{2} \mathbb{N}$ (see [Moc], [B-K]). In particular, $\mathcal{E}^S$ is a noncommutative model of $R(\mathcal{O})$. Let $\mathfrak{g}$ be the Lie algebra of $G$. The Hamiltonian functions $\phi^x \in R(\mathcal{O})$, $x \in \mathfrak{g}$, (defined by the embedding of $\mathcal{O}$ into $\mathfrak{g}^*$) lift to elements $\psi^x \in \mathcal{E}^S$ such that $[\psi^x, \psi^y] = \psi^{[x,y]}$. Then $\mathcal{E}^S$ (or at least a subalgebra) is generated by the $\psi^x$ and is a primitive quotient $U(\mathfrak{g})/J$ of the universal enveloping algebra.

This additional structure suggests that a reasonable first objective of quantization is to construct an $\mathcal{S}$-equivariant noncommutative model $\mathcal{E}$ once we are given $\mathcal{M}$. Then the second objective is to establish unitarity using the model; see Remark 3.3.2 and [B2]. The case where the cover $\mathcal{M}$ is non-trivial, i.e., $\mathcal{M} \neq \mathcal{O}$, is particularly intriguing as then $\mathcal{E}$ must be strictly larger than $\mathcal{E}^S$ and so strictly larger than $U(\mathfrak{g})/J$.

A noncommutative model $\mathcal{E}$ is an example of a Dixmier algebra, that is an overring”, with certain finiteness properties, of a primitive quotient of $U(\mathfrak{g})$. Vogan ([Vog1],[Vog2]) has developed this approach to quantization of $\mathcal{M}$ (at least the first objective) as a natural extension of the orbit method in the Dixmier sense (a correspondence between primitive ideals and orbits). Joseph has developed Dixmier algebra theory from a more ring-theoretic perspective; see [Jos] and references therein, especially the paper [J-S] with Stafford. For results particularly related to our paper, see also [McG1],[McG2],[Moc],[Zah].

In this paper, we construct by geometric methods a noncommutative model $\mathcal{E}$ for the universal cover $\mathcal{M}$ in a family of cases where $\mathcal{M}$ has degree 2. This quantization has some very nice properties and demonstrates, in a rather non-trivial way, some basic paradigms of quantization. See in particular Corollary 4.0.3 and Theorem 4.0.4. Not only is $\mathcal{E}$ a Dixmier algebra for $\mathcal{M}$ but also $\mathcal{E}$ has a lot of additional structure, including an anti-automorphism, a supertrace, and a non-degenerate supersymmetric bilinear pairing.

The nilpotent orbits we work arise in the following way (see §2 and especially Table I). We assume $\mathfrak{g}$ is simple and has a symmetric subalgebra $\mathfrak{k}$ with one-dimensional center so that $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$. We further assume that there is a non-constant homogeneous primitive $\mathfrak{k}$-semi-invariant polynomial function $F$ on $\mathfrak{p}^-$; then $\mathfrak{p}^-$ is a Jordan algebra by the TKK construction and $F$ is its Jordan norm. We take $\mathcal{O}$ to be the orbit of a generic element of $\mathfrak{p}^-$ and then (on account of $F$) the universal cover $\mathcal{M}$ is 2-fold and so $\mathcal{S} = \mathbb{Z}_2$.

In fact $\mathcal{O}$ has a more geometric realization as the unique Zariski open dense $G$-orbit in the cotangent bundle $T^*X$ of a (generalized) flag variety $X$. Here $X$ is $G/Q^\pm$ where $Q^\pm$ is the parabolic subgroup with Lie algebra $\mathfrak{q}^\pm = \mathfrak{k} \oplus \mathfrak{p}^\pm$. See §2.3 and, for an example, §2.3. A nice fact is that all regular functions on $\mathcal{O}$ extend to $T^*X$.

Each algebra $R(\mathcal{O}) = R(T^*X)$ has a natural family of noncommutative models which we study in §3. These are the algebras $\mathcal{D}^\lambda(X) = \mathcal{D}(X, N^\lambda)$ of twisted differential operators, equipped with the order filtration, where $N$ is the canonical line bundle and $\lambda$ is any
complex number. (If \( \lambda \) is not integral then \( N^\lambda \) must be interpreted formally.) The functions \( \phi^x \) lift to twisted vector fields \( \eta^x_\lambda \in D^\lambda(X) \) which generate \( D^\lambda(X) \) so that \( D^\lambda(X) = \mathcal{U}(\mathfrak{g})/J^\lambda \). For general \( \lambda \), \( D^\lambda(X) \) is a simple ring (see Proposition 3.2.2) and so \( J^\lambda \) is a maximal 2-sided ideal.

With this as our starting point, the problem is to build a filtered overring of \( D^\lambda(X) \) which is then a noncommutative model of \( R(M) \). To do this we introduce a complex algebraic Lagrangian submanifold \( \tilde{Z} \) in \( M \). Here \( \tilde{Z} \) covers (the orbital variety) \( Z = (F \neq 0) = \mathcal{O} \cap p^- \) and \( \tilde{Z} \) admits the function \( w = \sqrt{F} \).

We embed \( D^\lambda(X) \) into the algebra \( D(\tilde{Z}) \) of differential operators on \( \tilde{Z} \) by the sequence

\[
D^\lambda(X) \hookrightarrow D(p^+) \xrightarrow{\pi} D(p^-) \hookrightarrow D(Z) \hookrightarrow D(\tilde{Z})
\]

Here the first map is restriction to the big cell identified with \( p^+ \), the second is the Fourier transform (following [Goi]), the third is restriction to (the Zariski open dense set) \( Z \), and the fourth map is defined by lifting differential operators. See §4.

For each value of \( \lambda \) we realize \( \mathfrak{g} \) inside \( D(p^-) \) by the operators \( \pi^x_\lambda = \mathcal{F}(-\eta^x_\lambda) \). Then \( \pi^x_\lambda \) is multiplication by \( x \) if \( x \in p^+ \), \( \pi^x_\lambda \) is a twisted vector field if \( x \in \mathfrak{t} \), or \( \pi^x_\lambda \) has order 2 if \( x \in p^- \). These \( \pi^x_\lambda \) are familiar in representation theory because they make \( S(p^+) \) into a lowest weight representation, in fact a generalized Verma module for \( \mathfrak{q}^- \). See §4.3. Our first main result (Theorem 5.1.1) is

**Theorem 1.0.1.** Let \( \mathcal{E}^\lambda \) be the \( \mathfrak{g} \)-finite part of \( D(\tilde{Z}) \) with respect to the operators \( [\pi^x_\lambda, \cdot] \). Then \( \mathcal{E}^\lambda \) has a natural \( \mathfrak{g} \)-stable algebra filtration and we have canonical inclusions \( R(O) \subseteq \text{gr} \mathcal{E}^\lambda \subseteq R(M) \).

We prove this in §5.2 by constructing a new filtration on \( D(\tilde{Z}) \) which extends the Fourier transform of the order filtration on \( D(p^+) \). We show its symbol calculus produces a symplectic open embedding of \( T^* \tilde{Z} \) into \( M \) and then we use the fact that \( R(M) \) is the \( \mathfrak{g} \)-finite part of its fraction field.

The condition now for \( \mathcal{E}^\lambda \) to be a noncommutative model of \( R(M) \) is that \( \text{gr} \mathcal{E}^\lambda = R(M) \). In our second main result (Theorem 5.1.1) we figure out which values of \( \lambda \) satisfy this. This involves the positive constant \( m \) attached to \( O \) by the property that the \( Q^+ \)-semi-invariant section in \( \Gamma(X, N^{-1}) \) has weight \( \chi^{2m} \) where \( \chi \) is the weight of \( w \).

**Theorem 1.0.2.** We have \( \text{gr} \mathcal{E}^\lambda = R(M) \) if and only if \( \lambda = \frac{1}{2} \pm \frac{1}{4m} \).

To prove this, we reduce (in Corollary 5.3.1) to showing that the function \( w \), regarded as a multiplication operator, is \( \mathfrak{g} \)-finite. Then in, §6.2-§6.4 we use Jordan algebra techniques to compute how \( w \) transforms under \( \mathcal{U}(p^-) \). We expect these Jordan techniques belie some deeper connection. Also it would be interesting to extend our result to describe the full set of rings \( \text{gr} \mathcal{E}^\lambda \) that appear as we vary \( \lambda \).

Let \( \lambda_0 = \frac{1}{2} - \frac{1}{4m} \) and \( \lambda_0' = \frac{1}{2} + \frac{1}{4m} \). If \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \), then \( \lambda_0 = \frac{1}{4} \), \( \lambda_0' = \frac{3}{4} \) and \( \mathcal{E}^{\lambda_0} \) is the Weyl algebra \( \mathbb{C}[[\frac{p}{\partial w}, w]] \). (See §4.3 Example 5.2.2 and Remark 6.1.3, Example 8.1.1 and Remark 8.4.3.) We are finding a new reason for the old result attaching \( \mathbb{C}[[\frac{p}{\partial w}, w]] \) to \( M \).

The two algebras \( \mathcal{E}^{\lambda_0} \) and \( \mathcal{E}^{\lambda_0'} \) are both anti-isomorphic and isomorphic (Corollary 5.4.1 and Proposition 5.5.1). It follows (Corollary 7.1.2) that \( J^{\lambda_0} \) coincides with \( J^{\lambda_0'} \) and is stable under the principal anti-automorphism \( \tau \) of \( \mathcal{U}(\mathfrak{g}) \). Thus our construction of \( \mathcal{E}^{\lambda_0} \) attaches to
\(\mathcal{M}\) just one 2-sided ideal of \(\mathcal{U}(g)\), namely the maximal ideal \(J^\lambda_0\). It is reasonable to attach to \(\mathcal{O}\) the maximal ideal \(J^{\frac{1}{2}}\). We may then think of the values \(\lambda_0\) and \(\lambda_0^*\) as representing some “quantum fluctuation” about \(\frac{1}{2}\), caused by passing from \(\mathcal{O}\) to \(\mathcal{M}\).

In \([\text{McG3}, \text{Tables 5-10}]\), McGovern attached, by a completely different method, a Dixmier algebra \(D\) to \(\mathcal{M}\) in each of our cases (except when \(g_\mathbb{R} = \mathfrak{so}(2, p)\) and \(p \) is odd). McGovern starts by manufacturing an infinitesimal character according to the recipes formulated in his Yale preprints and \([\text{McG3}]\). Then he invokes Moeglin’s construction in \([\text{Moe}]\) so that \(D\) is the \(g\)-finite part of the endomorphism ring of a certain (degenerate) simple Whittaker module. McGovern conjectures that \(\text{gr} D\) is isomorphic to \(R(\mathcal{M})\) and says that he can check this case-by-case when \(g\) is classical. By comparing infinitesimal characters (Corollary \([1.3.3]\) and Remark \([7.1.3]\)) and applying results of Moeglin on \(\text{gr} D\), we find independently that McGovern’s algebra \(D\) always coincides with our algebra \(\mathcal{E}^\lambda_0\). In particular, this proves McGovern’s conjecture in our one case \(g_\mathbb{R} = \mathfrak{e}_7(-25)\) where \(g\) is exceptional. It would be extremely valuable to find a geometric construction of the Whittaker module; we conjecture that this can done in the context of our construction of \(\mathcal{E}^\lambda_0\).

Now returning to Theorem \(1.0.1\), we have a built-in module for \(\mathcal{E}^\lambda_0\), namely the ring of regular functions on \(\tilde{Z}\). Consider the submodule \(\mathcal{H}\) generated by the constant function 1. We prove (Proposition \([7.2.1]\) and Corollary \([7.3.1]\))

**Corollary 1.0.3.** \(\mathcal{H}\) is a faithful simple module for \(\mathcal{E}^\lambda_0\). Moreover \(\mathcal{H}\) identifies with a subalgebra of \(R(\mathcal{M})\) which is maximal Poisson abelian. Then \(\mathcal{E}^\lambda_0\) is the \(g\)-finite part, with respect to the operators \([\pi^x, \cdot]\), of the algebra of differential operators on \(\mathcal{H}\).

As a \(g\)-representation, \(\mathcal{H}\) is the direct sum of two lowest weight representations \(S(p^+\)) and \(wS(p^+)\). From this point of view, Corollary \(1.0.3\) is telling us how to locate \(\mathcal{E}^\lambda_0\) inside the (computable) algebra \(\text{End}_{g-fin}(\mathcal{H})\) by using the algebra structure on \(\mathcal{H}\).

Our noncommutative model \(\mathcal{E}^\lambda_0\) is naturally \(S\)-equivariant. This figures into the algebraic structure of \(\mathcal{E}^\lambda_0\). In particular the maximality of \(J^\lambda_0\) “induces upward” so that \(\mathcal{E}^\lambda_0\) is a simple ring (Corollaries \([1.3.1]\) and \([7.1.1]\)). See also Corollaries \([7.1.4]\) and \([7.7.1]\) for the decomposition of \(\mathcal{E}^\lambda_0\) as a \(\mathcal{U}(g)\)-bimodule.

In \([8.3]\) we focus on interpreting \(\mathcal{E}^\lambda_0\) as a quantization of \(R(\mathcal{M})\). We find in \([8.1]\) a natural quantization map \(q : R(\mathcal{M}) \to \mathcal{E}^\lambda_0\). Via \(q\), multiplication on \(\mathcal{E}^\lambda_0\) defines a new \((G \times S)\)-invariant map \(\circ\) on \(R(\mathcal{M})\). If \(\phi\) and \(\psi\) are homogeneous of degrees \(j\) and \(k\), then \(\phi \circ \psi = \sum_{p \in \mathbb{N}} C_p(\phi, \psi)\) where \(C_p(\phi, \psi)\) is homogeneous of degree \(j + k - p\). In fact, \(\circ\) deforms the Poisson algebra structure in the sense that \(C_0(\phi, \psi) = \phi \psi\) and \(C_1(\phi, \psi) - C_1(\psi, \phi) = \{\phi, \psi\}\).

Let \(T : R(\mathcal{M}) \to \mathbb{C}\) be the projection to the constant term defined by the Euler grading. We make \(R(\mathcal{M})\) into a supervector space where \(R^j(\mathcal{M})\) is even or odd according to whether \(j \in \mathbb{N}\) or \(j \in \mathbb{N} + \frac{1}{2}\). Our third main result (Theorem \([8.2.1]\), Corollaries \([8.3.1]\) and \([8.4.1]\)) is

**Theorem 1.0.4.** With respect to \(\circ\), \(R(\mathcal{M})\) is a noncommutative superalgebra with supertrace \(T\). The pairing \(Q(\phi, \psi) = T(\phi \circ \psi)\) on \(R(\mathcal{M})\) is \((G \times S)\)-invariant, supersymmetric, non-degenerate and orthogonal for the Euler grading.

Our product \(\circ\) on \(R(\mathcal{M})\) is the specialization at \(t = 1\) of a graded strongly \(g\)-invariant (non-local) star product. This has parity, i.e., \(C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi)\). If \(x \in g\) then \(\phi^x \circ \psi = \phi^x \psi + \frac{1}{2}\{\phi^x, \psi\} + \Lambda^x(\psi)\) where \(\Lambda^x\) is the \(Q\)-adjoint of \(\psi \mapsto \phi^x \psi\).
The parity condition, which is essential for star products, is not automatic but comes from an anti-automorphism $\beta$ of $\mathcal{E}_{\lambda_0}$ which extends the principal anti-automorphism $\tau$ of $\mathcal{U}(g)/J_{\lambda_0}$. We find that $\beta$ falls out of our comparison of $\mathcal{E}_{\lambda_0}$ with $\mathcal{E}_{\lambda_0}$ (Corollary 7.5.1).

Our star product is non-local in the sense that the operators $C_p(\cdot, \cdot)$ fail in general to be bi-differential. We know this because already the operators $\Lambda^x$ fail in general to be differential general (see [B2] and Remark 8.4.2).

We say $\circ$ is a Dixmier product because it makes $R(M)$ into a Dixmier algebra for $M$, equipped with extra structure. In [B2] we show, for a general nilpotent orbit cover $M$, how adding some axioms (for $\beta$, $\mathcal{T}$ and $Q$) to the usual definition of Dixmier algebra produces a Dixmier product on $R(M)$ and all the results in §8.1-§8.4. Thus these results persist even when $R(M)$ has non-trivial multiplicities.

In [B2] we show that our star product on $R(M)$ is “positive” in a sense which we define and consequently $R(M)$ becomes a unitary representation of $G$ (see Remark 8.3.2) made up of two irreducible components.

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2. The orbit cover $M$

2.1. Momentum construction of $\mathcal{O}$. Let $G$ be a connected and simply-connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G_\mathbb{R}$ be a real form of $G$. Let $K \subset G$ be the complexification of a maximal compact subgroup $K_\mathbb{R}$ of $G_\mathbb{R}$. Let $\mathfrak{g}_\mathbb{R}$, $\mathfrak{g}$, $\mathfrak{t}_\mathbb{R}$, $\mathfrak{t}$ be the Lie algebras of $G_\mathbb{R}$, $G$, $K_\mathbb{R}$, $K$. Let $g \mapsto \mathfrak{g}$, $x \mapsto \mathfrak{t}$, be the complex conjugation map. Then we have the Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ and its complexification $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$.

We assume from now on that the real symmetric pair $(\mathfrak{g}_\mathbb{R}, \mathfrak{t}_\mathbb{R})$, or equivalently the complex symmetric pair $(\mathfrak{g}, \mathfrak{t})$, is Hermitian. This means that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{t}$ and there exists $x_0 \in \text{Cent} \mathfrak{t}_\mathbb{R}$ such that ad $x_0$ defines a complex structure on $\mathfrak{p}_\mathbb{R}$. Then we get the splitting $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^\pm$ are the $\pm i$-eigenspaces of ad $x_0$, and so we get

$$\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{t} \oplus \mathfrak{p}^- \quad (2.1)$$

Every Hermitian symmetric pair is a direct sum, in the obvious way, of Hermitian symmetric pairs $(\mathfrak{g}, \mathfrak{t})$ where $\mathfrak{g}$ is simple; such pairs are called irreducible.

Now $\mathfrak{p}^\pm$ are (complex conjugate) abelian Lie subalgebras of $\mathfrak{g}$. Let $U^\pm \subset G$ be the corresponding abelian subgroups. Then $Q^\pm = KU^\pm$ are parabolic subgroups of $G$ with Lie algebras $q^\pm = \mathfrak{t} \oplus \mathfrak{p}^\pm$. The coset spaces $G/Q^\pm$ are then (generalized) flag varieties of $G$. We put $X = G/Q^-$. We differentiate the $G$-action on $X$ to obtain an infinitesimal action

$$\mathfrak{g} \longrightarrow \mathfrak{vect}(X), \quad x \mapsto \eta^x \quad (2.2)$$

where the value of $\eta^x$ at a point $q$ is $\eta^x_q = \frac{d}{dt} |_{t=0} \exp (-tx) \cdot q$. Here $\mathfrak{vect}(X)$ denotes the Lie algebra of algebraic holomorphic vector fields on $X$ and by infinitesimal action we mean that (2.2) is a Lie algebra homomorphism.
The natural action of \( G \) on \( X \) induces a canonical action of \( G \) on the cotangent bundle \( T^*X \). This \( G \)-action is Hamiltonian with respect to the canonical symplectic form \( \Omega \) on \( T^*X \); throughout this paper symplectic means algebraic holomorphic symplectic. The \( G \)-equivariant moment map

\[
\mu : T^*X \rightarrow \mathfrak{g}^*
\]

is defined by \( \langle \mu(m), x \rangle = \mu^x(m) \) where \( \mu^x \) is the order one symbol of \( \eta^x \). In other words, the comorphism

\[
\mu^*: \mathfrak{g} \rightarrow R(T^*X), \quad x \mapsto \mu^x
\]

is a Lie algebra homomorphism where \( R(T^*X) \) is equipped with the Poisson bracket \( \{\cdot, \cdot\} \) defined by \( \Omega \). Thus \( \mu^* \) defines Hamiltonian \( \mathfrak{g} \)-symmetry on \( T^*X \). If we identify \( T^*X \) with the contracted product bundle \( G \times_{Q^-} (\mathfrak{g}/\mathfrak{q}^-)^* \) in the usual way, then \( \mu \) is the collapsing map. The following fact defines \( \mathcal{O} \) for us.

**Proposition 2.1.1.** The image of the moment map \( \mu \) is the closure of a single nilpotent coadjoint orbit \( \mathcal{O} \) in \( \mathfrak{g}^* \) so that

\[
\text{Cl}(\mathcal{O}) = \mu(T^*X) = G \cdot (\mathfrak{g}/\mathfrak{q}^-)^*
\]

Then \( \mathcal{O} \) is the Richardson orbit associated to \( Q^- \). The map \( \mu \) is generically 1-to-1 and \( \text{Cl}(\mathcal{O}) \) is normal. Consequently all regular functions on \( \mathcal{O} \) extend to \( \text{Cl}(\mathcal{O}) \) so that \( R(\mathcal{O}) = R(\text{Cl}(\mathcal{O})) \).

The moment map \( \mu \) is bijective over \( \mathcal{O} \) and \( \mu^{-1} \) defines a \( G \)-equivariant Zariski open embedding of complex algebraic manifolds

\[
\mathfrak{j} : \mathcal{O} \rightarrow T^*X
\]

Let \( \omega \) be the KKS symplectic form on \( \mathcal{O} \). Then \( \mathfrak{j} \) is symplectic; i.e., \( \mathfrak{j}^*\Omega = \omega \). The comorphism \( \mathfrak{j}^* : R(T^*X) \rightarrow R(\mathcal{O}) \) is an isomorphism of Poisson algebras.

To say the coadjoint orbit \( \mathcal{O} \) is nilpotent means that if we identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) using the complex Killing form \( (\cdot, \cdot)_\mathfrak{g} \) of \( \mathfrak{g} \), then the adjoint orbit corresponding to \( \mathcal{O} \) consists of nilpotent elements. This happens if and only if \( \mathcal{O} \) is stable under the dilation action of \( \mathbb{C}^* \) on \( \mathfrak{g}^* \). We often identify \( \mathcal{O} \) with its corresponding adjoint orbit.

Let \( \phi : \mathcal{O} \rightarrow \mathfrak{g}^* \) be the inclusion with comorphism

\[
\phi^* : \mathfrak{g} \rightarrow R(\mathcal{O}), \quad x \mapsto \phi^x
\]

Then \( \{\phi^x, \phi^y\} = \phi^{[x,y]} \) where the Poisson bracket on \( R(\mathcal{O}) \) is defined by the KKS form \( \omega \); this property determines \( \omega \) uniquely. So \( \phi^* \) is a Lie algebra homomorphism and thus defines Hamiltonian \( \mathfrak{g} \)-symmetry on \( \mathcal{O} \).

### 2.2. The graded Poisson algebra \( R(\mathcal{O}) \)

We will say a (complex) commutative algebra \( \mathcal{A} \) is graded if \( \mathcal{A} \) is equipped with a vector space grading \( \mathcal{A} = \bigoplus_{p \in \mathbb{N}} \mathcal{A}^p \) such that \( \mathcal{A}^p \mathcal{A}^q \subseteq \mathcal{A}^{p+q} \). Here \( \mathbb{N} = \{0, 1, 2, \ldots\} \). In practice, \( \mathcal{A} \) will be a finitely generated algebra with \( \mathcal{A}^0 = \mathbb{C} \).

**Definition 2.2.1.** A graded Poisson algebra is a Poisson algebra \( \mathcal{A} \) together with an algebra grading \( \mathcal{A} = \bigoplus_{p \in \mathbb{N}} \mathcal{A}^p \) such that \( \{\mathcal{A}^p, \mathcal{A}^q\} \subseteq \mathcal{A}^{p+q-1} \). If also \( \mathcal{A}^p = 0 \) when \( p \notin \mathbb{N} \), then we say \( \mathcal{A} \) is an \( \mathbb{N} \)-graded Poisson algebra.
We have three examples on hand of \( \mathbb{N} \)-graded Poisson algebras. (i) The symmetric algebra \( S(\mathfrak{g}) \) with grading defined by polynomial degree and Poisson bracket defined by the Lie bracket on \( \mathfrak{g} \). (ii) \( R(T^*X) \) with Poisson bracket defined by \( \Omega \) and grading \( R(T^*X) = \bigoplus_{p \in \mathbb{N}} R^p(T^*X) \) where \( R^p(T^*X) \) is the subspace of functions which are homogeneous of degree \( p \) on the fibers of \( T^*X \) over \( X \). (iii) \( R(\mathcal{O}) \) with Poisson bracket defined by \( \omega \) and Euler grading \( R(\mathcal{O}) = \bigoplus_{p \in \mathbb{N}} R^p(\mathcal{O}) \) where \( R^p(\mathcal{O}) \) is the subspace of homogeneous degree \( p \) functions.

The natural extensions \( \mu^*: S(\mathfrak{g}) \to R(T^*X), \ P \mapsto \mu^P \), and \( \phi^*: S(\mathfrak{g}) \to R(\mathcal{O}), \ P \mapsto \phi^P \), are graded Poisson algebra homomorphisms. Plainly \( \mu \mathfrak{j} = \phi \) and so Proposition \( 2.1.1 \) gives

**Corollary 2.2.2.** The two maps \( \mu^* \) and \( \phi^* \) are surjective with the same kernel \( I \subset S(\mathfrak{g}) \). Then \( I \) is the ideal of functions vanishing on \( \mathcal{O} \).

### 2.3. The tube condition.

We define \( h \in \mathfrak{k} \) by \( h = -ix_0 \) so that \( \mathfrak{p}^\pm \) is the \( \pm 1 \)-eigenspace of \( \text{ad} \ h \). A Hermitian symmetric pair \((\mathfrak{g}, \mathfrak{k})\) is said to be of tube type if there exists \( e \in \mathfrak{p}^+ \) such that \( \mathfrak{s} = \mathbb{C}e + \mathbb{C}h + \mathbb{C}\overline{e} \) is a Lie subalgebra of \( \mathfrak{g} \) isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \) with bracket relations: \([h, e] = e, [h, \overline{e}] = -\overline{e}, [e, \overline{e}] = 2h\). In this case, \( e \) lies in \( \mathcal{O} \) and so \( \mathcal{O} = G \cdot e = G \cdot \overline{e} \). Clearly \((\mathfrak{g}, \mathfrak{k})\) is of tube type if and only if every one of its irreducible components is of tube type.

**Lemma 2.3.1.** Assume that \( \mathfrak{g} \) is simple. Then the fundamental group of \( \mathcal{O} \) is \( \mathbb{Z}_2 \) if \((\mathfrak{g}, \mathfrak{k})\) is of tube type or is trivial otherwise.

In Table 1, we write down the familiar list of irreducible Hermitian symmetric pairs \((\mathfrak{g}, \mathfrak{k})\) of tube type along with the real form \( \mathfrak{g}_\mathbb{R} \), the rank \( r \) of \((\mathfrak{g}, \mathfrak{k})\), and \( n = \dim \mathfrak{p}^\pm \). Note that \( \dim \mathcal{O} = 2 \dim X = 2n \). We specify the orbit \( \mathcal{O} \). In every case except one, the subalgebra \( \mathfrak{k} \subset \mathfrak{g} \) is unique up to conjugacy and hence we get a single orbit \( \mathcal{O} \). The exception is the case where \( \mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(4r) \), as then there are two choices for \( \mathfrak{k} \) and these give rise to two distinct orbits which are exchanged by outer automorphism. For the classical cases, we give the partition indexing \( \mathcal{O} \) (see e.g. [C-M]) and in the exceptional case we give the dimension of \( \mathcal{O} \) (this is enough since there is only one nilpotent orbit of that dimension). In Table 1, \( r \geq 1 \) and \( p \geq 2 \).

**Table 1. The Orbits \( \mathcal{O} \)**

| \( \mathfrak{g} \)   | \( \mathfrak{k} \)       | \( r \) | \( n \) | \( \mathfrak{g}_\mathbb{R} \) | \( \mathcal{O} \) |
|------------------|------------------|------|------|------------------|------------------|
| \( \mathfrak{sp}(2r, \mathbb{C}) \) | \( \mathfrak{gl}(r, \mathbb{C}) \) | \( r \) | \( \frac{1}{2}r(r + 1) \) | \( \mathfrak{sp}(r, \mathbb{R}) \) | \( (2^r) \) |
| \( \mathfrak{sl}(2r, \mathbb{C}) \) | \( \mathfrak{sl}(\mathfrak{gl}(r, \mathbb{C}) \oplus \mathfrak{gl}(r, \mathbb{C})) \) | \( r \) | \( r^2 \) | \( \mathfrak{su}(r, r) \) | \( (2^r) \) |
| \( \mathfrak{so}(4r, \mathbb{C}) \) | \( \mathfrak{gl}(2r, \mathbb{C}) \) | \( r \) | \( (2r - 1) \) | \( \mathfrak{so}^*(4r) \) | \( (2^{2r})_{1, 11} \) |
| \( \mathfrak{e}_7 \) | \( \mathfrak{e}_6 \oplus \mathbb{C} \) | \( 3 \) | \( 27 \) | \( \mathfrak{e}_7(-25) \) | \( 54 \) |
| \( \mathfrak{so}(2 + p, \mathbb{C}) \) | \( \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(p, \mathbb{C}) \) | \( 2 \) | \( p \) | \( \mathfrak{so}(2, p) \) | \( (3, 1^{p-1}) \) |

The geometric interpretation of the tube condition is that the Hermitian symmetric space \( G_\mathbb{R}/K_\mathbb{R} \) is of tube type. The Jordan theoretic interpretation is that in TKK theory,
\(p^\pm\) is not just a Jordan triple system but also a Jordan algebra; cf. \(\S 3.3\). The invariant theoretic interpretation is given by the next lemma in terms of the algebra \(S(p^+)_{K-\text{semi}}\) of \(K\)-semi-invariants.

**Lemma 2.3.2.** Assume \(g\) is simple. Then \(S(p^+)_{K-\text{semi}} \neq \mathbb{C}\) if and only if \((g, \mathfrak{k})\) is of tube type. In the tube case, \(S(p^+)_{K-\text{semi}} = \mathbb{C}[F]\) is a polynomial ring in one homogeneous generator \(F\). Then \(F\) has degree \(r\) and \(F\) is unique up to scaling. The weight of \(F\) is \(\chi^2\) where \(\chi\) is a generator of the character group of \(K\).

This lemma defines \(\chi\) and then we extend \(\chi\) to a character of \(Q^\pm\) which is trivial on the unipotent radical.

**2.4. The universal cover \(\mathcal{M}\).** From now on we assume that \((g, \mathfrak{k})\) is an irreducible Hermitian symmetric pair of tube type. Then, by Lemma 2.3.1, \(O\) admits a universal 2-fold covering \(\mathcal{M}\). We can give a nice geometric construction of \(\mathcal{M}\) using Lemma 2.3.2 and the homogeneous function \(\phi^F \in R^*(O)\) defined by the function \(F\) introduced in Lemma 2.3.2.

**Proposition 2.4.1.** The function \(\phi^F\) is not a square in the field \(L = \mathbb{C}(O)\) of rational functions on \(O\). Let \(\tilde{L}\) be the field extension of \(L\) defined by adjoining
\[
\zeta = \sqrt{\phi^F}
\] (2.8)

The \(G\)-representation on \(L\) extends uniquely to \(\tilde{L}\) and \(\zeta\) is \(Q^+\)-semi-invariant of weight \(\chi\).

The normalization map \(\mathcal{M} \rightarrow O\) is a \(G\)-equivariant 2-fold covering \(\kappa : \mathcal{M} \rightarrow O\). \(R(\mathcal{M})\) is the algebra of \(G\)-finite functions in \(\tilde{L}\). Each irreducible \(G\)-representation occurring in \(R(\mathcal{M})\) has multiplicity one and is self-dual.

The function \(\zeta\) lies in \(R(\mathcal{M})\) and is the highest weight vector of a finite-dimensional irreducible \(G\)-representation \(V \subset R(\mathcal{M})\); \(V\) is given in Table 2. The algebra \(\mathbb{Z}_2\)-grading defined by the action of the Galois group \(S = \mathbb{Z}_2\) is
\[
R(\mathcal{M}) = R(O) \oplus R(O)V
\] (2.9)

**Explanation of Table 2.** In the first row, \(\wedge_0^r(\mathbb{C}^{2r})\) is the kernel of the map \(\wedge^r(\mathbb{C}^{2r}) \rightarrow \wedge^{r+2}(\mathbb{C}^{2r})\) defined by taking the wedge product with the symplectic form. In the third row, we obtain the the two half-spin representations, corresponding to the two orbits listed in Table 1. The other entries are clear.

We have several easy consequences of Proposition 2.4.1.

**Corollary 2.4.2.** The orbit \(G \cdot (e, \zeta)\) inside \(O \times V\) is a \(G\)-equivariant model for \(\mathcal{M}\) where the composition \(G \cdot (e, \zeta) \dashrightarrow O \times V \rightarrow O\) gives the 2-fold covering onto \(O\).

**Remark 2.4.3.** In just one case, namely \(g = \mathfrak{sl}(2, \mathbb{C})\), the map \(\mathcal{M} \rightarrow O \times V \rightarrow V\) is an embedding. Here \(V = \mathbb{C}^2\). This accounts for why the \(\mathfrak{sl}(2, \mathbb{C})\) is so easy to write down; see \(\S 1\) and \(\S 2.5\).
Table 2. The Representation $V$

| $\mathfrak{g}_R$   | $V$                      |
|-------------------|--------------------------|
| $\mathfrak{sp}(r, \mathbb{R})$ | $\wedge_0^r(\mathbb{C}^{2r})$ |
| $\mathfrak{su}(r, r)$     | $\wedge^r(\mathbb{C}^{2r})$    |
| $\mathfrak{so}^*(4r)$    | $\mathbb{C}^{2^{2r-1}} = \pm \frac{1}{2}$-spin |
| $\mathfrak{e}_7(-25)$    | $\mathbb{C}^{56}$       |
| $\mathfrak{so}(2, -25)$  | $\mathbb{C}^{2+p}$       |

**Corollary 2.4.4.** The KKS form $\omega$ lifts to a $G$-invariant symplectic form $\tilde{\omega} = \kappa^* \omega$ on $\mathcal{M}$ which then defines a Poisson bracket on $R(\mathcal{M})$ and $\mathbb{C}(\mathcal{M})$. The $G$-representation on $R(\mathcal{M})$ corresponds to the $\mathfrak{g}$-representation

$$\Phi : \mathfrak{g} \to \text{End } R(\mathcal{M}), \quad \Phi^x = \{\phi^x, \cdot\} \quad (2.10)$$

As a Poisson algebra, $R(\mathcal{M})$ is generated by $R(\mathcal{O})$ and $\zeta$.

**Corollary 2.4.5.** Let $\mathcal{G} \subset R(\mathcal{M})$ be the set of functions which Poisson commute with $\phi^x$ for all $x \in \mathfrak{p}^+$. Then $\mathcal{G}$ is a maximal Poisson abelian subalgebra of $R(\mathcal{M})$. We have $\mathcal{G} = \{\phi^T + \phi^{T'} \zeta | T, T' \in S(\mathfrak{p}^+)\}$.

**Remark 2.4.6.** The $K$-types in $\mathcal{G}$ are in natural bijection with the $G$-types in $R(\mathcal{M})$.

The square of the $\mathbb{C}^*$-action on $\mathcal{O}$ lifts to the $\mathbb{C}^*$-action on $\mathcal{M}$ defined by $s(u, v) = (s^2u, s^r v)$ in the model of Corollary 2.4.2. Notice that $-1$ interchanges points in the cover $\kappa : \mathcal{M} \to \mathcal{O}$ if $r$ is odd, or acts trivially if $r$ is even. Let $R^j(\mathcal{M}) \subset R(\mathcal{M})$ be the space of homogeneous degree $2j$ functions where $j \in \frac{1}{2}\mathbb{Z}$. Then $R^p(\mathcal{O}) \subseteq R^p(\mathcal{M})$ for $p \in \mathbb{N}$.

**Corollary 2.4.7.** $R(\mathcal{M})$ is a graded Poisson algebra with respect to the Euler grading $R(\mathcal{M}) = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} R^j(\mathcal{M})$. We have $V \subseteq R^{\frac{1}{2}}(\mathcal{M})$.

**2.5. Example:** $\mathfrak{g}_R = \mathfrak{su}(r, r)$. Here $\mathfrak{g} = \mathfrak{sl}(2r, \mathbb{C})$ and complex conjugation on $\mathfrak{g}$ is the map $(A, B) \mapsto (-A^*, -B^*)$ where $A, B, C, D$ are complex $r \times r$ matrices. We can choose $\mathfrak{k}_\mathbb{R}$ so that

$$\mathfrak{k} = \{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A, D \in \mathfrak{gl}(r, \mathbb{C}), \text{Tr}(A + D) = 0\}$$

We can pick $x_0$ (which is unique up to sign) so that

$$\mathfrak{p}^+ = \{\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} | B \in \mathfrak{gl}(r, \mathbb{C})\}, \quad \mathfrak{p}^- = \{\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} | C \in \mathfrak{gl}(r, \mathbb{C})\}$$

Then $X$ identifies with the Grassmannian $Gr(r, 2r)$ of $r$-dimensional vector subspaces $L$ in $\mathbb{C}^{2r}$. A point in $T^*X$ corresponds to a pair $(p, L)$ where $p$ is a linear transformation $\mathbb{C}^{2r} / L \to L$. The moment map $\mu : T^*X \to \mathfrak{g}^*$ is given by $\mu(p, L) = \text{Tr}(xy_{p,L})$ where $y_{p,L}$ is the composite map $\mathbb{C}^{2r} \to \mathbb{C}^{2r} / L \overset{\mu}{\to} L \hookrightarrow \mathbb{C}^{2r}$. Thus $\mathcal{O}$ identifies with the nilpotent orbit $\{x \in \mathfrak{sl}(2r, \mathbb{C}) | x^2 = 0, \text{rank } x = r\}$. 

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We have $2h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where $I$ is the $r \times r$ identity matrix. We can choose $e = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ and then $\tau = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The polynomial $F \in S^r(\mathfrak{p}^+)$ is the determinant and so $\chi(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \text{Det} A$ and $\phi F(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \text{Det} C$.

3. Noncommutative models of $R(O)$

3.1. Noncommutative models. We will say a noncommutative algebra $\mathcal{B}$ is filtered if $\mathcal{B}$ is equipped with an increasing filtration $\mathcal{B} = \bigcup_{j \in \frac{1}{2}\mathbb{N}} \mathcal{B}_j$ such that $\mathcal{B}_j \mathcal{B}_k \subseteq \mathcal{B}_{j+k}$. The associated graded algebra is $\text{gr} \mathcal{B} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \text{gr}_j \mathcal{B}$ where $\text{gr}_j \mathcal{B} = \mathcal{B}_j / \mathcal{B}_{j-\frac{1}{2}}$. Let $\mathfrak{p}_j : \mathcal{B}_j \rightarrow \text{gr}_j \mathcal{B}$ be the natural projection. Suppose we have, for all $j, k, \in \frac{1}{2}\mathbb{N}$,

$$[\mathcal{B}_j, \mathcal{B}_k] \subseteq \mathcal{B}_{j+k-1} \quad (3.1)$$

Then $\text{gr} \mathcal{B}$ is commutative and moreover $\text{gr} \mathcal{B}$ is a graded Poisson algebra with Poisson bracket given by $\{ \mathfrak{p}_j(b), \mathfrak{p}_k(c) \} = \mathfrak{p}_{j+k-1}(bc - cb)$ where $b \in \mathcal{B}_j$ and $c \in \mathcal{B}_k$.

**Definition 3.1.1.** Let $\mathcal{A} = \bigoplus_{p \in \frac{1}{2}\mathbb{N}} \mathcal{A}^p$ be a graded Poisson algebra as in Definition 2.2.1 with Hamiltonian $\mathfrak{g}$-symmetry given by a Lie algebra embedding $\phi : \mathfrak{g} \rightarrow \mathcal{A}^1$, $x \mapsto \phi^x$. A noncommutative model of $(\mathcal{A}, \phi)$ is a triple $(\mathcal{B}, \gamma, \psi)$ where $\mathcal{B}$ is a noncommutative filtered algebra $\mathcal{B} = \bigcup_{j \in \frac{1}{2}\mathbb{N}} \mathcal{B}_j$ satisfying (3.1), $\gamma : \text{gr} \mathcal{B} \rightarrow \mathcal{A}$ is a graded Poisson algebra isomorphism, and $\psi : \mathfrak{g} \rightarrow \mathcal{B}_1$, $x \mapsto \psi^x$, is a Lie algebra homomorphism such that $\gamma(\mathfrak{p}_1(\psi^x)) = \phi^x$ for all $x \in \mathfrak{g}$.

We have representations of $\mathfrak{g}$ on $\mathcal{B}$ and $\mathcal{A}$ given by the operators $b \mapsto [\psi^x, b]$ and $a \mapsto \{ \phi^x, a \}$; the former induces a $\mathfrak{g}$-representation on $\text{gr} \mathcal{B}$. Clearly $\gamma$ is $\mathfrak{g}$-equivariant and we have

**Lemma 3.1.2.** Suppose $\mathcal{A}^j$ is finite-dimensional for each $j \in \frac{1}{2}\mathbb{N}$. Then $\mathcal{B}$ is isomorphic to $\mathcal{A}$ as a $\mathfrak{g}$-representation with $\mathcal{B}_j \cong \bigoplus_{k=0}^j \mathcal{A}^k$.

If $\mathcal{A}$ and $\mathcal{B}$ are graded and filtered over $\mathbb{N}$, then we may form a $\frac{1}{2}\mathbb{N}$-grading of $\mathcal{A}$ and a $\frac{1}{2}\mathbb{N}$-filtration of $\mathcal{B}$ by putting $\mathcal{A}^{p+\frac{1}{2}} = 0$ and $\mathcal{B}_{p+\frac{1}{2}} = \mathcal{B}_p$ for $p \in \mathbb{N}$. In this way the $\mathbb{N}$-graded/filtrated theory is subsumed in the $\frac{1}{2}\mathbb{N}$-graded/filtrated theory.

We often speak of a noncommutative model of $\mathcal{A}$ where $\phi$ is implicitly understood. In particular, if $\mathcal{A}$ is $R(O)$ or $R(M)$, then we always take the Hamiltonian symmetry to be the one defined in (2.7).

If we identify $\mathfrak{g}$ with the diagonal in $\mathfrak{g} \oplus \mathfrak{g}$, then our $\mathfrak{g}$-representation on $\mathcal{B}$ extends to the representation

$$\Pi : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \text{End} \mathcal{B}, \quad \Pi^{(x,y)}(b) = \pi^x b - b \pi^y \quad (3.2)$$

This is a key aspect of noncommutative models.

3.2. The algebras $\mathfrak{D}^\lambda(X)$ of twisted differential operators. The canonical bundle $\mathcal{N}$ on $X$ is the algebraic holomorphic complex line bundle given by the top exterior power of $T^*X$. Every $G$-homogeneous line bundle over $X$ is a (rational) tensor power of $\mathcal{N}$; this follows by Lemma 2.3.2.

We can construct the sheaf $\mathfrak{D}^\lambda = \mathfrak{D}_{X, \mathcal{N}^\lambda}$ of $\mathcal{N}^\lambda$-twisted differential operators on $X$ where $\lambda$ is an arbitrary complex number. This is a sheaf of noncommutative algebras. When $\lambda$ is
an integer, the line bundle $\mathbb{N}\lambda$ exists and then $\mathcal{D}_{X,\mathbb{N}\lambda}$ is the usual sheaf constructed using $\mathbb{N}\lambda$. In particular $\mathcal{D} = \mathcal{D}^0$ is the usual sheaf of differential operators.

For the theory of twisted differential operators on flag varieties (and more generally on algebraic manifolds) with applications to representation theory, see e.g., [Be-Be], [Bo-Br], [Bj], [Ka], [MT], [Vog1], [Vog4].

We have a sheaf filtration $\mathcal{D}^\lambda = \bigcup_{p\in \mathbb{N}} \mathcal{D}_p^\lambda$ where $\mathcal{D}_p^\lambda$ is the subsheaf of differential operators of order at most $p$. We have $[\mathcal{D}_p^\lambda, \mathcal{D}_q^\lambda] \subseteq \mathcal{D}_{p+q-1}^\lambda$ and so $\text{gr} \mathcal{D}^\lambda$ is a sheaf of graded Poisson algebras which is isomorphic by the symbol map $\eta\lambda : \mathfrak{g} \to \mathcal{D}^\lambda_1(X)$, where $\mathcal{D}^\lambda_1(X)$ is the usual sheaf of differential operators.

The symbol map defines a graded Poisson algebra inclusion

$$s_\lambda : \text{gr} \mathcal{D}^\lambda(U) \to \Pi(U, S(\mathfrak{T})) = R(T^*U) \quad (3.3)$$

where $U$ is Zariski open in $X$; we omit the subscript $U$ when the context is clear. We have a natural Lie algebra homomorphism

$$\eta_\lambda : \mathfrak{g} \to \text{Vec}_X(X) \to \mathcal{D}^\lambda_1(X), \quad x \mapsto \eta^x_\lambda \quad (3.4)$$

where $\eta^x_\lambda$ is the Lie derivative $L_x^\lambda$ acting on $\lambda$-twisted forms. We say $\eta_\lambda^x$ is a twisted vector field on $X$. Let $\mathfrak{v}^\lambda$ denote the composite map $\text{gr} \mathcal{D}^\lambda(X) \xrightarrow{s_\lambda^\lambda} R(T^*X) \xrightarrow{\mathfrak{v}} R(\mathcal{O})$. Now we know (see [Vog1]):

**Proposition 3.2.1.** $(\mathcal{D}^\lambda(X), \mathfrak{v}^\lambda, \eta_\lambda)$, is a noncommutative model of $R(\mathcal{O})$.

**Proof.** As $X$ is a generalized flag variety, the sheaf cohomology $H^1(X, S^p(\mathfrak{T}))$ vanishes and it follows that $s_\lambda^\lambda$ is an isomorphism – see [Bo-Br] §1, Lemma 1.4. Hence $(\mathcal{D}^\lambda(X), s_\lambda^\lambda, \eta_\lambda)$ is a noncommutative model of $R(T^*X)$. This implies the result for $R(\mathcal{O})$.

We will use later the following result (true for any flag variety $X$).

**Proposition 3.2.2.** $\mathcal{D}^\lambda(X)$ is a simple ring if $\lambda$ satisfies $2\lambda \notin \mathbb{Z} - \{1\}$.

### 3.3. Relations with the enveloping algebra $\mathcal{U}(\mathfrak{g})$

Now (3.4) extends to an algebra homomorphism

$$\eta_\lambda : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}^\lambda(X) \quad (3.5)$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. We have the standard algebra filtration $\mathcal{U}(\mathfrak{g}) = \bigcup_{p\in \mathbb{N}} \mathcal{U}_p(\mathfrak{g})$ and then $\eta_\lambda$ is a filtered map, i.e., $\eta_\lambda(\mathcal{U}_p(\mathfrak{g})) \subseteq \mathcal{D}_p^\lambda(X)$.

Let $J^\lambda$ be the kernel of $\eta_\lambda$ so that $J^\lambda$ is a two-sided ideal in $\mathcal{U}(\mathfrak{g})$. According to [Bo-Br] (which applies since the moment map $\mu : T^*X \to \mathcal{O}(\mathcal{C})$ is birational with normal image), the map (3.5) is surjective in each filtration degree. Although only the untwisted case was treated in [Bo-Br], their method of proof (symbols), and hence their result, extends immediately to the twisted case. Thus we find

**Proposition 3.3.1.** The algebra $\mathcal{D}^\lambda(X)$ is generated by the twisted vector fields $\eta^x_\lambda$, $x \in \mathfrak{g}$. Moreover, $\eta_\lambda$ induces a filtered algebra isomorphism

$$\eta^x_\lambda : \mathcal{U}(\mathfrak{g}) / J^\lambda \to \mathcal{D}^\lambda(X) \quad (3.6)$$

Hence $\text{gr} J^\lambda = I$ and $\text{gr} \eta^x_\lambda : S(\mathfrak{g}) / I \to R(T^*X)$ coincides with the isomorphism $\mu^*$. 


Corollary 3.3.2. $J^\lambda$ is completely prime. Moreover if $2\lambda \not\in \mathbb{Z} - \{1\}$ then $J^\lambda$ is maximal.

Proof. The first statement follows as $I$ is prime and the second by Proposition 3.2.2. 

3.4. The anti-symmetry $\lambda \mapsto (1 - \lambda)$. The following anti-symmetry will be important throughout the paper.

Proposition 3.4.1. There is a unique map $\theta : \mathcal{D}^\lambda(X) \to \mathcal{D}^{1-\lambda}(X)$ such that $\theta$ is an algebra anti-isomorphism and $\theta(\eta^\lambda_X) = -\eta^{1-\lambda}_X$.

Proof. Let $U$ be any Zariski open affine in $X$. Then $\mathcal{D}^\lambda(U)$ is generated by the multiplication operators $f \in R(U)$ and the order 1 operators $\mathcal{L}_\eta$ where $\eta \in \text{Vect}(U)$. We obtain an algebra anti-isomorphism $\theta_U : \mathcal{D}^\lambda(U) \to \mathcal{D}^{1-\lambda}(U)$ by assigning $\theta_U(f) = f$ and $\theta_U(\mathcal{L}_\eta) = -\mathcal{L}_{\eta^{1-\lambda}}$. This follows by checking the relations among our generators of $\mathcal{D}^\lambda(U)$; see \cite{A-B, proof of Prop. 5.6.2}. In particular, $\theta_U(\eta^\lambda_X) = -\eta^{1-\lambda}_X$.

Now it is easy to see that the maps $\theta_U$ patch together to define an anti-isomorphism $\theta : \mathcal{D}^\lambda \to \mathcal{D}^{1-\lambda}$ of sheaves of algebras. Then $\theta$ evaluated on global sections gives $\theta$ and we have $\theta(\eta^\lambda_X) = -\eta^{1-\lambda}_X$. Finally, $\theta$ is unique since the vector fields $\eta^\lambda_X$ generate $\mathcal{D}^\lambda(X)$.

Let $\tau : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be the algebra anti-automorphism of $\mathcal{U}(\mathfrak{g})$ such that $\tau(x) = -x$ if $x \in \mathfrak{g}$; $\tau$ is called the principal anti-automorphism.

Corollary 3.4.2. We have $\tau(J^\lambda) = J^{1-\lambda}$. Thus $\tau$ induces $\theta$ according to the commutative square:

$$
\begin{array}{ccc}
\mathcal{U}(\mathfrak{g})/J^\lambda & \xrightarrow{\eta^\lambda_X} & \mathcal{D}^\lambda(X) \\
\downarrow \tau & & \downarrow \theta \\
\mathcal{U}(\mathfrak{g})/J^{1-\lambda} & \xrightarrow{\eta^{1-\lambda}_X} & \mathcal{D}^{1-\lambda}(X)
\end{array}
$$

(3.7)

3.5. Embedding $\mathcal{D}^\lambda(X)$ into $\mathcal{D}(\mathfrak{p}^+)$. Let $\mathcal{D}(Y)$ denote the algebra of differential operators, in the sense of Grothendieck, on a variety $Y$. For $Y$ affine (or even quasi-affine), $\mathcal{D}(Y)$ coincides with the algebra $\mathcal{D}(R(Y))$ of differential operators, in the sense of non-commutative algebra, on the ring $R(Y)$ of regular functions.

We can identify $\mathfrak{p}^+$ with a “big cell” $X^o$ in $X$ by means of the Zariski open embedding $\mathfrak{p}^+ \hookrightarrow X$, $v \mapsto (\exp v)Q^--Q^-$. The canonical line bundle $\mathcal{N}$ on $X$ trivializes over $X^o$; let $\sigma$ be a nowhere vanishing section. Then we have the algebra embedding

$$
\mathcal{D}^\lambda(X) \hookrightarrow \mathcal{D}^\lambda(X^o) \xrightarrow{h_\lambda} \mathcal{D}(X^o) = \mathcal{D}(\mathfrak{p}^+)
$$

(3.8)

where $h_\lambda$ is the isomorphism defined by $(h_\lambda D)(f)\sigma^\lambda = D(f\sigma^\lambda)$ for $f \in R(X^o)$. Notice that $h_\lambda$ is independent of the choice of $\sigma$ since $\sigma$ is unique up to scaling.

We will regard (3.8) an inclusion. In particular the twisted vector fields $\eta^\lambda_X$ now give a realization of $\mathfrak{g}$ inside $\mathcal{D}(\mathfrak{p}^+)$ where $\eta^\lambda_X(f) = \mathcal{L}_{\eta^\lambda_X}(f\sigma^\lambda)/\sigma^\lambda$. Using the familiar rules for the Lie derivative we find

$$
\eta^\lambda_X = \eta^\sigma + \lambda \left( \frac{\mathcal{L}_{\eta^\sigma}(\sigma)}{\sigma} \right)
$$

(3.9)

Thus the “twisting” of $\eta^\sigma$ amounts to adding a “quantum correction term” $\lambda \mathcal{L}_{\eta^\sigma}(\sigma)/\sigma$ which is just a function. We will see later that this sort of correction is necessary for in order to quantize $\mathcal{M}$.
Example 3.5.1. The infinitesimal action $\mathfrak{g} \to \text{Vect}(\mathfrak{p}^+)$, $x \mapsto \eta^x$, integrates to a rational action of $G$ on $\mathfrak{p}^+$. Given $Z \in \mathfrak{p}^+$, this rational action is then well-defined at $Z$ for some neighborhood of the identity in $G$. In the example of §2.3, this rational action of $G = SL(2r, \mathbb{C})$ is given by $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \cdot Z = (AZ + B)(CZ + D)^{-1}$. Using this, it is easy to explicitly write down the vector fields $\eta^x$.

3.6. Working in coordinates. In this subsection we set up coordinate systems on $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Using these we will explicitly write out realizations of $\mathcal{U}(\mathfrak{g})$ in the Weyl algebras $\mathcal{D}(\mathfrak{p}^+)$ and $\mathcal{D}(\mathfrak{p}^-)$; see §3.7 and §4.3.

We have a unique $K$-invariant bilinear pairing $\langle \cdot, \cdot \rangle : \mathfrak{p}^+ \times \mathfrak{p}^- \to \mathbb{C}$ such that $\langle e, \overline{e} \rangle = r$. We can extend $\langle \cdot, \cdot \rangle$ canonically to a non-singular pairing of the symmetric algebras $S(\mathfrak{p}^+)$ and $S(\mathfrak{p}^-)$. Then $f \in S^d(\mathfrak{p}^\pm)$ defines a homogeneous degree $d$ polynomial function on $\mathfrak{p}^\pm$.

Let $v_1, \ldots, v_n$ and $z_1, \ldots, z_n$ be dual vector space bases of $\mathfrak{p}^-$ and $\mathfrak{p}^+$, respectively. These bases form coordinate systems on $\mathfrak{p}^+$ and $\mathfrak{p}^-$ respectively and we can identify

$$R(\mathfrak{p}^+) = S(\mathfrak{p}^-) = \mathbb{C}[v_1, \ldots, v_n] \quad \text{and} \quad R(\mathfrak{p}^-) = S(\mathfrak{p}^+) = \mathbb{C}[z_1, \ldots, z_n]$$

We get algebra embeddings $S(\mathfrak{p}^\pm) \to \mathcal{D}(\mathfrak{p}^\pm)$, $A \mapsto \partial_A$, defined by $\partial_{v_j} = \frac{\partial}{\partial v_j}$ and $\partial_{z_j} = \frac{\partial}{\partial z_j}$. Then we can identify

$$\mathcal{D}(\mathfrak{p}^+) = \mathbb{C}[\partial_{v_1}, \ldots, \partial_{v_n}, v_1, \ldots, v_n] \quad \text{and} \quad \mathcal{D}(\mathfrak{p}^-) = \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_n}, z_1, \ldots, z_n]$$

We also have the intrinsic algebra embeddings $S(\mathfrak{p}^\pm) \to \mathcal{D}(\mathfrak{p}^\pm)$, $A \mapsto \partial^A$, where $x \in \mathfrak{p}^\pm$ defines the constant coefficient vector field $\partial^x$ on $\mathfrak{p}^\pm$.

3.7. The twisted vector fields $\eta^x_\lambda$. Let $\nu : \mathfrak{q}^\pm \to \mathbb{C}$ be the weight obtained by differentiating the character $\chi : Q^\pm \to \mathbb{C}^\ast$. Recall $r$ and $n = \dim \mathfrak{p}^\pm$ from §2.3. We now introduce the scalar

$$m = \frac{n}{r} \quad (3.10)$$

Lemma 3.7.1. The twisted vector fields $\eta^x_\lambda \in \mathcal{D}_1(\mathfrak{p}^\pm)$ are given in coordinates by:

$$\eta^x_\lambda = -\partial^x \quad \text{if} \quad x \in \mathfrak{p}^+$$

$$\eta^x_\lambda = -\left(\sum_i v_i \partial^{[x, z_i]}\right) - 2m \lambda \nu(x) \quad \text{if} \quad x \in \mathfrak{k}$$

$$\eta^x_\lambda = -\frac{1}{2} \left(\sum_{i,j} v_i v_j \partial^{[x, z_i], [z_j]}\right) + 2m \lambda x \quad \text{if} \quad x \in \mathfrak{p}^-$$

Proof. Using the geometry of the big cell we get the coordinate expressions for the vector fields $\eta^x$ and then we work out the twisting correction (3.9) by choosing $\sigma = dv_1 \wedge \cdots \wedge dv_n$. We use the fact that $\sigma$ is $K$-semi-invariant of weight $\chi_2^{2m}$. See [Tor], [Tan]; there are minor variations in the final answers owing to different normalizations of $\langle \cdot, \cdot \rangle$. □

We worked out these particular formulas for $\eta^x_\lambda$ with Aravind Asok in our project on quantizing $K$-orbits in $\mathfrak{p}^-$. So after twisting, $\mathfrak{p}^+$ acts by constant coefficient vector fields, $\mathfrak{k}$ acts by homogeneous linear vector fields corrected by adding a constant, and $\mathfrak{p}^-$ acts by homogeneous quadratic vector fields corrected by adding a homogeneous linear function.

In particular $\pi^x_\lambda$ is the constant coefficient differential operator $\partial^P$ if $P$ lies in $\mathcal{U}(\mathfrak{p}^+) = S(\mathfrak{p}^+)$ (equality since $\mathfrak{p}^+$ is abelian). 13
4. Building a noncommutative model of $R(\mathcal{M})$

4.1. The need for a square root. We want to try to extend our noncommutative models $\mathcal{D}^\lambda(X)$ of $R(\mathcal{O})$ to noncommutative models of $R(\mathcal{M})$. To begin with, we observe

**Lemma 4.1.1.** Suppose $(C, \gamma, \psi)$ is a noncommutative model of $R(\mathcal{M})$. Extend $\psi$ to an algebra homomorphism $\psi: U(g) \to C$, $P \mapsto \psi^P$. Then $\psi^F$ is a square in $C$ so that $\psi^F = \varrho^2$ where $\varrho \in C^\times$.

**Proof.** By Lemma 3.1.2, $C \simeq R(\mathcal{M})$ as $g$-representations with $C_j \simeq \bigoplus_{k=0}^j R^k(\mathcal{M})$. Then $C$ is multiplicity-free by Proposition 2.4.1. It follows that $C^\times$ contains a unique copy of $V$; let $\varrho$ be a highest weight vector in that copy so that $[\psi^F, \varrho] = \nu(x) \varrho$ for all $x \in q^+$. Then $\varrho^2$ and $\psi^F$ are highest weight vectors in $C$ of the same weight, and so they are equal up to scaling.

This says that we need to embed $\mathcal{D}^\lambda(X)$ into some bigger algebra where $\eta^F_\lambda$ becomes a square. But $\eta^F_\lambda = \partial^F$ is a constant coefficient differential operator and it is uncomfortable to try to take its square root. It is not clear what $\sqrt{\partial^F}$ could operate on. To remedy this, we perform a Fourier transform in §4.2 following Goncharov in [Gon].

4.2. Fourier transform of $\mathcal{D}^\lambda(X)$. The Fourier transform is the anti-isomorphism

$$\mathcal{F}: \mathcal{D}(p^+) \longrightarrow \mathcal{D}(p^-)$$

of algebras defined by $\mathcal{F}(v) = \partial^v$ for $v \in p^-$ and $\mathcal{F}(\partial^z) = z$ for $z \in p^+$ (see §3.6 for notations). In §3.3 we embedded $\mathcal{D}^\lambda(X)$ into $\mathcal{D}(p^+)$. 

**Definition 4.2.1.** Let $B^\lambda = \mathcal{F}(\mathcal{D}^\lambda(X))$ with $B^\lambda_p = \mathcal{F}(\mathcal{D}^\lambda_p(X))$ for $p \in \mathbb{N}$. Put $\pi^x_\lambda = -\mathcal{F}(\eta^x_\lambda)$ for $x \in g$.

The operators $\pi^x_\lambda$ define a Lie algebra homomorphism $g \to B^\lambda_1$ which then extends to a filtered algebra homomorphism

$$\pi_\lambda : U(g) \longrightarrow B^\lambda, \quad u \mapsto \pi^u_\lambda$$

**Lemma 4.2.2.** $B^\lambda$ is the subalgebra of $\mathcal{D}(p^-)$ generated by the operators $\pi^x_\lambda$ for $x \in g$. Then $B^\lambda = \cup_{p \in \mathbb{N}} B^\lambda_p$ is an algebra filtration over $\mathbb{N}$. The kernel of $\pi_\lambda$ is $J^{1-\lambda}$ and we get an induced filtered algebra isomorphism $\pi^*_\lambda : U(g)/J^{1-\lambda} \longrightarrow B^\lambda$.

**Proof.** Immediate from Proposition 3.3.1 and Corollary 3.4.2 since $\mathcal{F}(\eta^x_\lambda) = \pi^x(\lambda)$ for $u \in U(g)$.

Let $\alpha : R(\mathcal{O}) \to R(\mathcal{O})$ be the Poisson algebra anti-isomorphism defined by $\alpha(\phi) = (-1)^p \phi$ if $\phi \in R^p(\mathcal{O})$. Now we define $\gamma_\lambda$ by the commutative square

$$\begin{array}{ccc}
\text{gr } \mathcal{D}^\lambda(X) & \longrightarrow & \text{gr } B^\lambda \\
\downarrow & & \downarrow \\
R(T^*X) & \longrightarrow & R(\mathcal{O})
\end{array}$$

(4.3)

Proposition 3.2.1 gives

**Corollary 4.2.3.** $(B^\lambda, \gamma_\lambda, \pi_\lambda)$ is a noncommutative model of $R(\mathcal{O})$. 

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4.3. **The operators** $\pi^x_\lambda$. To see what is going on, and for future use, we need to explicitly write out the $\pi^x_\lambda$.

**Proposition 4.3.1.** The operators $\pi^x_\lambda \in \mathcal{D}(p^-)$ are given in coordinates by:

\[
\begin{align*}
\pi^x_\lambda &= x & \text{if } x \in p^+ \\
\pi^x_\lambda &= (\sum_i [x, z_i]\partial_{z_i}) + 2m\lambda \nu(x) & \text{if } x \in \mathfrak{k} \\
\pi^x_\lambda &= \tfrac{1}{2} \left( \sum_{i,j} [[x, z_i], z_j] \partial_{z_i}\partial_{z_j} \right) - 2m\lambda \partial^x & \text{if } x \in p^-
\end{align*}
\]

**Proof.** Immediate from Lemma 3.7.1 since $F(v_i) = \partial_{z_i}$ and $F(\partial_{v_i}) = z_i$. \qed

So now $p^+$ acts on $R(p^-) = S(p^+)$ by multiplication operators, $\mathfrak{k}$ acts by order 1 differential operators and $p^-$ acts by order 2 differential operators. In particular $\pi^x_p = P$ if $P$ lies in $\mathcal{U}(p^+) = S(p^+)$. 

**Corollary 4.3.2.** The representation $\mathfrak{g} \to \text{End } S(p^+)$, $x \mapsto \pi^x_\lambda$, is a familiar geometric model of the generalized Verma module for $q^-$ with lowest weight $\zeta = 2m\lambda\nu$; see e.g., [Ian]. The $\mathfrak{g}$-isomorphism is $S(p^+) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(q^-)} \mathbb{C}_\zeta$, $u \mapsto u \otimes 1$. The annihilator of $S(p^-)$ in $\mathcal{U}(\mathfrak{g})$ is $J^{1-\lambda}$.

Let $\mathfrak{h} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$ and so of $\mathfrak{g}$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of the positive roots with respect to a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{h} \oplus p^+ \subseteq \mathfrak{b} \subseteq q^+$.

**Corollary 4.3.3.** $J^{1-\lambda}$ has infinitesimal character $-2m\lambda\nu + \rho$.

4.4. **Extracting a square root of** $\pi^F_\lambda$. Our aim now is to try to extend $B^\lambda$ to a non-commutative model of $R(M)$. According to Lemma 4.1.1, we need to extract a square root of $\pi^F_\lambda$. According to §4.3, this operator is simply the function

\[\pi^F_\lambda = F\] (4.7)

Thus we are now in a nice geometric situation, as we need to extract a square root of the function $F$. To do this, we replace $p^-$ by its Zariski open dense set

\[Z = \{q \in p^- \mid F(q) \neq 0\}\] (4.8)

Then $Z$ is affine and we may identify $R(Z) = S(p^+)[F^{-1}]$. Clearly $Z$ is $K$-stable; in fact, $Z = K \cdot \tau$. Now $F$ is not a square in $R(Z)$; this follows for instance from Lemma 2.3.2. In the next result we construct the covering of $Z$ defined by “extracting a square root of $F$”.

**Lemma 4.4.1.** The complex algebraic manifold

\[\tilde{Z} = \{(q, t) \mid F(q) = t^2\} \subset Z \times \mathbb{C}^*\] (4.9)

is a non-trivial $K$-equivariant 2-fold covering of $Z$ where the covering map is $(q, t) \mapsto q$ and $K$ acts on $\tilde{Z}$ by $a \cdot (q, t) = (a \cdot q, \chi(a) t)$. The formula $w(q, t) = t$ defines a function $w \in R(\tilde{Z})$ such that

\[w^2 = F\] (4.10)

Up to isomorphism, $\tilde{Z}$ is the unique double cover of $Z$ such that $F$ becomes a square.
Notice that $\tilde{Z}$, being closed in $\mathbb{C}^*$, is affine and

$$R(\tilde{Z}) = S(p^+[w^{-1}]) = \mathbb{C}[z_1, \ldots, z_n][w^{-1}]$$

The square of the $\mathbb{C}^*$-action on $Z$ lifts to the $\mathbb{C}^*$-action on $\tilde{Z}$ given by $s \cdot (q, t) = (s^2 q, s^t t)$. Here $-1$ interchanges points in the fibers of the cover $\tilde{Z} \to Z$ if $r$ is odd, or acts trivially if $r$ is even. The $\mathbb{C}^*$-action gives the algebra grading

$$R(\tilde{Z}) = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} R^j(\tilde{Z})$$

where $R^j(\tilde{Z})$ is the subspace of homogeneous functions of degree $2j$. Then $w$ lies in $R^{\frac{1}{2}}(\tilde{Z})$.

We have the $g$-representation

$$\Pi_{\lambda} : g \to \text{End} \mathfrak{D}(\tilde{Z}), \quad \Pi_{\lambda}^s(D) = [\pi^r_{\lambda}, D]$$

(4.12)

This extends to an algebra homomorphism $\Pi_{\lambda} : U(g) \to \text{End} \mathfrak{D}(\tilde{Z})$, $u \mapsto \Pi_{\lambda}^s$.

Let $\mathcal{G}$ be the Galois group of the cover $\tilde{Z} \to Z$. Then $\mathcal{G}$ induces the algebra $\mathbb{Z}_2$-gradings

$$R(\tilde{Z}) = R(\tilde{Z})^\uparrow \oplus R(\tilde{Z})^\downarrow \quad \text{and} \quad \mathfrak{D}(\tilde{Z}) = \mathfrak{D}(\tilde{Z})^\uparrow \oplus \mathfrak{D}(\tilde{Z})^\downarrow$$

(4.13)

where $R(\tilde{Z})^\uparrow = R(\tilde{Z})^\mathcal{G} = R(Z)$ and $\mathfrak{D}(\tilde{Z})^\uparrow = \mathfrak{D}(\tilde{Z})^\mathcal{G} = \mathfrak{D}(Z)$. These gradings are $g$-stable in the representations $\pi_{\lambda}$ and $\Pi_{\lambda}$.

We now have algebra inclusions $\mathcal{B}^\lambda \subset \mathfrak{D}(p^-) \subset \mathfrak{D}(Z) \subset \mathfrak{D}(\tilde{Z})$. Our plan for constructing a noncommutative model of $R(\mathcal{M})$ is to look inside $\mathfrak{D}(\tilde{Z})$ for a suitable overring of $\mathcal{B}^\lambda$.

4.5. The example $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. The simplest case occurs when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. This the case $r = 1$ in §2.3 and Example 3.5.1 and so $X = \mathbb{C}P^1$ and $\mathcal{O} = \{(a, c) \ | a^2 + bc = 0\} - \{(0, 0)\}$. Then $\mathcal{M}$ identifies with $\mathbb{C}^2 - \{0\}$ and the covering is given by

$$\kappa(\zeta, \xi) = \begin{pmatrix} \zeta & \xi \\ \zeta^2 & -\xi \end{pmatrix}$$

(4.14)

Then $R(\mathcal{M}) = \mathbb{C}[\zeta, \xi]$ with Poisson bracket $\{\phi, \psi\} = \frac{\partial \phi}{\partial \zeta} \frac{\partial \psi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial \zeta}$ and $R(\mathcal{O}) = \mathbb{C}[\zeta^2, \zeta \xi, \xi^2]$. The functions $\zeta$ and $\xi$ are each homogeneous of degree $\frac{1}{2}$ and $R^j(\mathcal{M})$ is the space of homogeneous polynomials in $\zeta$ and $\xi$ of ordinary degree $2j$.

Now $v = \bar{v}$ and $z = e$ are dual bases of $p^-$ and $p^+$. Twisting transforms $\eta = v^k \partial_v$ into $\eta_{\lambda} = v^k \partial_v + k\lambda v^{k-1}$ and so we find

$$\eta_{\lambda} = -\partial_v, \quad \eta_{\lambda}^h = -2v \partial_v - 2\lambda, \quad \eta_{\lambda}^f = v^2 \partial_v + 2\lambda v$$

The Fourier transform converts these into

$$\pi_{\lambda}^c = z, \quad \pi_{\lambda}^h = 2z \partial_z + 2\lambda, \quad \pi_{\lambda}^f = -z \partial_z^2 - 2\lambda \partial_z$$

We have $F = z$ and so $Z = \mathbb{C}^* v$ and $w = \sqrt{z}$. Then $\mathfrak{D}(\tilde{Z}) = \mathbb{C}[w, w^{-1}, \partial_w]$ where $\partial_w = \frac{\partial}{\partial w}$. Extending $\mathfrak{D}(Z)$ to $\mathfrak{D}(\tilde{Z})$ amounts to making the change of variables from $z$ to $w$. We find $\partial_z = \frac{1}{2w} \partial_w$ and

$$\pi_{\lambda}^c = w^2, \quad \pi_{\lambda}^h = w \partial_w + 2\lambda, \quad \pi_{\lambda}^f = -\frac{1}{4} \partial_w^2 - (\lambda - \frac{1}{4}) \frac{1}{w} \partial_w$$

Looking at these formulas, we see that the value $\lambda = \frac{1}{4}$ is special as it eliminates the unpleasant non-polynomial term in $\pi_{\lambda}^c$. 

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So let us choose \( \lambda_0 = \frac{1}{4} \). Then

\[
\pi^e_{\lambda_0} = w^2, \quad \pi^h_{\lambda_0} = w\partial_w + \frac{1}{2}, \quad \pi^f_{\lambda_0} = -\frac{1}{4}\partial_w^2
\]

We recognize these operators from Weyl quantization. They generate the even part \( \mathcal{B}^{\lambda_0} = \mathbb{C}[w^2, w\partial_w, \partial_w^2] \) of the Weyl algebra \( \mathcal{E} = \mathbb{C}[w, \partial_w] \). So \( \mathcal{E} \) is the obvious candidate inside \( \mathcal{D}(\tilde{Z}) \) for an overring of \( \mathcal{B}^{\lambda_0} \) which is a noncommutative model of \( R(\mathcal{M}) \). It is easy to see that this candidate works. Indeed we introduce the filtration \( \mathcal{E} = \bigcup_{j \in \mathbb{N}} \mathcal{E}_j \) where \( \mathcal{E}_j \) is the span of the operators \( w^a\partial_w^b \) for \( a + b \leq 2j \). Then \( \mathcal{E} \) is a graded Poisson algebra and we obtain the commutative square

\[
\begin{array}{ccc}
gr \mathcal{B}^{\lambda_0} & \longrightarrow & gr \mathcal{E} \\
\downarrow \gamma_{\lambda_0} & & \downarrow \gamma_{\lambda_0} \\
\mathbb{C}[\zeta^2, \zeta, \xi^2] & \longrightarrow & \mathbb{C}[\zeta, \xi]
\end{array}
\]

where \( \gamma_{\lambda_0} \) maps the image of \( w^a\partial_w^b \) in \( gr \frac{1}{2}(a+b) \tilde{B} \) to \( \zeta^a \xi^b \). Now \((\mathcal{E}, \gamma, \pi)\) is a noncommutative model of \( R(\mathcal{M}) \) which extends \((\mathcal{B}^{\lambda_0}, \gamma_{\lambda}, \pi_{\lambda_0})\). See also Remark 3.1.3.

### 4.6. The algebras \( \mathcal{E}^\lambda \)

Our plan is to look inside \( \mathcal{D}(\tilde{Z}) \) for an extension of \( \mathcal{B}^\lambda \) to a noncommutative model \( \mathcal{C}^\lambda \) of \( R(\mathcal{M}) \). Fortunately, there is a very simple way to narrow our search. Recall that a vector \( v \) in a \( \mathfrak{g} \)-representation \( \mathcal{V} \) is called \( \mathfrak{g} \)-finite if the \( \mathcal{U}(\mathfrak{g}) \)-submodule generated by \( v \) is finite dimensional. The set \( \mathcal{V}^{\mathfrak{g}-\text{fin}} \) of \( \mathfrak{g} \)-finite vectors in \( \mathcal{V} \) is a \( \mathfrak{g} \)-stable subspace which we call the \( \mathfrak{g} \)-finite part of \( \mathcal{V} \). Now Lemma 3.1.2 says in particular that \( \mathcal{C}^\lambda \), if it exists, must lie in the \( \mathfrak{g} \)-finite part of \( \mathcal{D}(\tilde{Z}) \). So we make

**Definition 4.6.1.** Let \( \mathcal{E}^\lambda \) be the \( \mathfrak{g} \)-finite part of \( \mathcal{D}(\tilde{Z}) \) in the representation (\( \mathcal{I}.12 \)).

Then \( \mathcal{E}^\lambda \) is a subalgebra of \( \mathcal{D}(\tilde{Z}) \) and the action of \( \mathcal{S} \) defines an algebra \( \mathbb{Z}_2 \)-grading

\[
\mathcal{E}^\lambda = (\mathcal{E}^\lambda)^\uparrow \oplus (\mathcal{E}^\lambda)^\downarrow \quad (4.15)
\]

The purpose of our next two results, Proposition 4.6.2 and Theorem 5.1.1, is to determine the size of \( \mathcal{E}^\lambda \). We will show that \( \mathcal{E}^\lambda \) is “smaller than or equal to \( R(\mathcal{M}) \)” in size, and moreover, if \( \mathcal{C}^\lambda \) exists, then \( \mathcal{C}^\lambda = \mathcal{E}^\lambda \).

**Proposition 4.6.2.** For each \( \lambda \in \mathbb{C} \), the algebra \( (\mathcal{E}^\lambda)^\uparrow = \mathcal{D}(Z)^{\mathfrak{g}-\text{fin}} \) is equal to \( \mathcal{B}^\lambda \).

**Proof.** Let \( \mathfrak{frD}(\mathfrak{p}^\pm) \) be the fraction field of the Weyl algebra \( \mathfrak{D}(\mathfrak{p}^\pm) \) and consider the \( \mathfrak{g} \)-representations on \( \mathfrak{frD}(\mathfrak{p}^+) \) and \( \mathfrak{frD}(\mathfrak{p}^-) \) given respectively by \( x \mapsto [\gamma_{\lambda}, \cdot] \) and \( x \mapsto [\pi_{\lambda}, \cdot] \). The Fourier transform (\( \mathfrak{I}.1 \)) extends uniquely to an algebra isomorphism \( \mathcal{F} : \mathfrak{frD}(\mathfrak{p}^+) \to \mathfrak{frD}(\mathfrak{p}^-) \). Moreover \( \mathcal{F} \) identifies the \( \mathfrak{g} \)-finite parts of \( \mathfrak{frD}(\mathfrak{p}^+) \) and \( \mathfrak{frD}(\mathfrak{p}^-) \).

**Lemma 4.6.3.** \( \mathcal{D}^\lambda(X) \) is the \( \mathfrak{g} \)-finite part of \( \mathfrak{frD}(\mathfrak{p}^+) \).

**Proof.** In \( \mathfrak{S}.3 \) we have identified \( \mathfrak{p}^+ \) with a big cell in \( X \) and the map (\( \mathfrak{I}.8 \)) embeds \( \mathcal{D}^\lambda(X) \) into \( \mathcal{D}(\mathfrak{p}^+)^{\mathfrak{g}-\text{fin}} \). To see that \( \mathcal{D}^\lambda(X) \) is all of \( \mathfrak{frD}(\mathfrak{p}^+) \), we consider the algebra filtration \( \mathfrak{frD}(\mathfrak{p}^+) = \bigcup_{j \in \mathbb{Z}} \mathfrak{frD}_j(\mathfrak{p}^+) \) where \( \mathfrak{frD}_j(\mathfrak{p}^+) \) is the subspace spanned by the quotients \( D_1/D_2 \) where \( D_i \in \mathcal{D}(\mathfrak{p}^+) \) and \( \text{ord}(D_1) - \text{ord}(D_2) \leq j \). (Notice that \( \mathfrak{frD}(\mathfrak{p}^+) \) is \( \mathbb{Z} \)-filtered while \( \mathcal{D}^\lambda(X) \) is \( \mathbb{N} \)-filtered.) Then \( \mathfrak{frD}(\mathfrak{p}^+) \) is commutative and embeds, as a graded Poisson
algebra, into the function field $\mathbb{C}(T^*p^+)$. This embedding is $\mathfrak{g}$-linear with respect to the $\mathfrak{g}$-representation on $\mathbb{C}(T^*p^+)$ given by $x \mapsto \{\mu^x, \cdot\}$.

Clearly $R(T^*X)$ lies in $\mathbb{C}(T^*p^+)^{\mathfrak{g}\text{-}\text{fin}}$ and in fact $R(T^*X)$ is all of $\mathbb{C}(T^*p^+)^{\mathfrak{g}\text{-}\text{fin}}$. To see this we recall that, by Proposition 2.1.1, $N = \mathfrak{j}(O)$ is a Zariski open $G$-orbit in $T^*X$ and $R(N) = R(T^*X)$. So $\mathbb{C}(T^*p^+)^{\mathfrak{g}\text{-}\text{fin}} = \mathbb{C}(T^*X)^{\mathfrak{g}\text{-}\text{fin}} = \mathbb{C}(N)^{\mathfrak{g}\text{-}\text{fin}} = R(N) = R(T^*X)$ where the third equality is automatic since $N$ is a $G$-orbit.

Thus our embedding $\mathfrak{D}\lambda(X) \to \mathfrak{D}(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}}$ induces an isomorphism on the associated graded rings. In particular then, $\text{gr}_j \mathfrak{D}(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}}$ vanishes for $j < 0$ and so $\mathfrak{D}_j(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}} = \mathfrak{D}_{j-1}(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}}$ if $j < 0$. But $\cap_{j<0} \mathfrak{D}_j(\mathfrak{p}^+) = 0$ and so $\mathfrak{D}_j(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}} = 0$ if $j < 0$. It follows now that $\mathfrak{D}\lambda(X) = \mathfrak{D}_0(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}}$ for all $j \in \mathbb{Z}$. 

Thus $\mathfrak{D}(\mathfrak{p}^-)^{\mathfrak{g}\text{-}\text{fin}} = \mathcal{F}(\mathfrak{D}(\mathfrak{p}^+)^{\mathfrak{g}\text{-}\text{fin}}) = \mathcal{F}(\mathfrak{D}\lambda(X)) = B^\lambda$. But also $\mathfrak{D}(\mathfrak{p}^-) \supset \mathfrak{D}(Z) \supset B^\lambda$. So $\mathfrak{D}(Z)^{\mathfrak{g}\text{-}\text{fin}} = B^\lambda$. 

5. **The algebras $\mathcal{E}\lambda$ and symplectic geometry of $\mathcal{M}$**

5.1. **Filtration theorem for $\mathcal{E}\lambda$.** We regard our $\mathbb{N}$-filtration of $B^\lambda$ as a $\frac{1}{2}\mathbb{N}$-filtration by the recipe given in §4.1. Recall $\mathcal{E}\lambda = B^\lambda \oplus (\mathcal{E}\lambda)^\dagger$ by Proposition 4.6.2.

**Theorem 5.1.1.** Pick $\lambda \in \mathbb{C}$. There is a unique $\mathfrak{g}$-stable algebra filtration

$$\mathcal{E}\lambda = \bigcup_{p \in \frac{1}{2}\mathbb{N}} \mathcal{E}\lambda_p$$

(5.1)

extending our filtration of $B^\lambda$ such that $\text{gr} \mathcal{E}\lambda$ has no zero-divisors. This satisfies $[\mathcal{E}\lambda_p, \mathcal{E}\lambda_q] \subseteq \mathcal{E}\lambda_{p+q-1}$ and so $\text{gr} \mathcal{E}\lambda$ is a graded Poisson algebra.

The map $\gamma_\lambda : \text{gr} B^\lambda \to R(O)$ defined in (1.3) extends, uniquely up to the $\mathbb{Z}_2$-actions defined by the Galois groups $\mathfrak{G}$ and $\mathfrak{S}$, to a $\mathbb{Z}_2$-equivariant Poisson algebra homomorphism

$$\widetilde{\gamma}_\lambda : \text{gr} \mathcal{E}\lambda \longrightarrow R(\mathcal{M})$$

(5.2)

In fact $\widetilde{\gamma}_\lambda$ is 1-to-1. Thus, if we identify $\text{gr} \mathcal{E}\lambda$ with its image, we get

$$R(O) \subseteq \text{gr} \mathcal{E}\lambda \subseteq R(\mathcal{M})$$

(5.3)

§5.2 is devoted to proving Theorem 5.1.1.

5.2. **Proof of Theorem 5.1.1.** Suppose we have extended our filtration of $B^\lambda$ to an algebra filtration (5.1) such that $\text{gr} \mathcal{E}\lambda$ has no nilpotents. Say $S \in (\mathcal{E}\lambda)^\dagger$. Then $S^2$ lies in $B^\lambda$ and so has some known filtration degree $p$. Since $\text{gr} \mathcal{E}\lambda$ has no zero divisors, it follows that the filtration degree of a product is equal to the sum of the filtration degrees of the factors. Hence $S$ has filtration degree $\frac{1}{2}p$. This proves uniqueness.

To prove existence, we will construct an algebra filtration of $\mathfrak{D}(\tilde{Z})$ and then restrict it to $\mathcal{E}\lambda$. We start with the vector space isomorphism

$$\mathfrak{m} : R(T^*\tilde{Z}) = R(\tilde{Z}) \otimes S(\mathfrak{p}^-) \longrightarrow \mathfrak{D}(\tilde{Z}), \quad \mathfrak{m}(f \otimes P) = f\partial^P$$

(5.4)

Here $\partial^P \in \mathfrak{D}(\mathfrak{p}^-)$ defines a differential operator on $\tilde{Z}$ by first restricting it to $Z$ and then lifting it to $\tilde{Z}$. Then $P$ defines a function on $T^*\tilde{Z}$, namely the symbol of $\partial^P$. 

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For \( j \in \frac{1}{2} \mathbb{Z} \), we put
\[
D^j_\lambda(\widetilde{Z}) = m \left( R^{\leq j}(\widetilde{Z}) \otimes S(p^-) \right)
\]
Then \( D(\widetilde{Z}) = \bigcup_{j \in \frac{1}{2} \mathbb{Z}} D^j_\lambda(\widetilde{Z}) \) is an \( \mathfrak{g} \)-stable algebra filtration. This induces a filtration \( D(\widetilde{Z}) = \bigcup_{j \in \frac{1}{2} \mathbb{Z}} D^j_\lambda(\widetilde{Z}) \) which extends our filtration of \( B^\lambda \). We have \([D^j_\lambda(\widetilde{Z}), D^k_\lambda(\widetilde{Z})] \subseteq D^\delta_{j+k-\frac{1}{2}}(\widetilde{Z})\). Put \( \text{gr}^j_\lambda D(\widetilde{Z}) = D^j_\lambda(\widetilde{Z})/D^j_{\lambda-1}(\widetilde{Z}) \). Let \( p_j : D^j_\lambda(\widetilde{Z}) \to \text{gr}^j_\lambda D(\widetilde{Z}) \) be the natural projection. Then \( \text{gr}^j_\lambda D(\widetilde{Z}) = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \text{gr}^j_\lambda D(\widetilde{Z}) \) is commutative and acquires a Poisson bracket induced by commutator of operators.

In this way, \( \text{gr}^j_\lambda D(\widetilde{Z}) \) is a graded Poisson algebra in the sense of Definition 2.2.1 with \( \frac{1}{2} \mathbb{Z} \) replacing \( \frac{1}{2} \mathbb{N} \). Although this \( \mathfrak{z} \)-filtration of \( D(\widetilde{Z}) \) is not the usual (order) filtration, fortunately \( \text{gr}^j_\lambda D(\widetilde{Z}) \) is very nice. It is easy to check

**Lemma 5.2.1.** The map \( m^{-1} : D(\widetilde{Z}) \to R(T^* \widetilde{Z}) \) induces an \( \mathfrak{g} \)-invariant Poisson algebra isomorphism \( t : \text{gr}^j_\lambda D(\widetilde{Z}) \to R(T^* \widetilde{Z}) \).

For \( D \in D^j_\lambda(\widetilde{Z}) \), we say that \( t(p^j_j(D)) \in R(T^* \widetilde{Z}) \) is the degree \( j \) \( \mathfrak{z} \)-symbol of \( D \). What is happening here is that the two associated graded algebras of \( D(\widetilde{Z}) \), for the \( \mathfrak{z} \)-filtration and the order filtration, are the same as Poisson algebras but of course different as graded algebras. We call the grading \( R(T^* \widetilde{Z}) = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} t(\text{gr}^j_\lambda D(\widetilde{Z})) \) the \( \mathfrak{z} \)-gradation; this corresponds to homogeneity along the base \( \widetilde{Z} \).

**Example 5.2.2.** In Example 4.3, \( w^a \partial_x^b = z^c z^d \partial_x^b \) has \( \mathfrak{z} \)-filtration degree equal to \( \frac{1}{2}(a + b) \).

For \( j \in \frac{1}{2} \mathbb{Z} \), we put
\[
\mathcal{E}_\lambda = \mathcal{E} \cap D^j_\lambda(\widetilde{Z}) \tag{5.5}
\]
Then \( \text{gr} \mathcal{E}_\lambda \) is a Poisson subalgebra of \( \text{gr}^j_\lambda D(\widetilde{Z}) \) and this has no zero divisors by Lemma 5.2.1. We see now that \( \mathcal{E}_\lambda = \bigcup_{j \in \mathbb{Z}} \mathcal{E}_\lambda \) is an algebra filtration which has all the desired properties, except that we still need to prove \( \mathcal{E}_\lambda = 0 \) for \( j < 0 \). This will be easy once we analyze \( \text{gr} \mathcal{E}_\lambda \).

The \( \mathfrak{g} \)-representation (4.12) induces a \( \mathfrak{g} \)-representation on \( R(T^* \widetilde{Z}) \) and this is given by \( x \mapsto \{ b^x, \cdot \} \) where \( b^x \in R(T^* Z) \) is the degree 1 \( \mathfrak{z} \)-symbol of \( \pi^x \in \mathcal{D}_1^x(Z) \). Now \( \text{gr} \mathcal{E}_\lambda \) lies inside \( R(T^* \widetilde{Z})^{\mathfrak{g} \text{-fin}} \) (recall Definition 4.6.1) and we get the commutative diagram
\[
\begin{array}{ccc}
\text{gr} B^\lambda & \longrightarrow & \text{gr} \mathcal{E}_\lambda \\
\downarrow t & & \downarrow t \\
R(T^* Z)^{\mathfrak{g} \text{-fin}} & \longrightarrow & R(T^* \widetilde{Z})^{\mathfrak{g} \text{-fin}}
\end{array}
\tag{5.6}
\]

The problem now is to recognize the functions \( b^x \) and then compute \( R(T^* \widetilde{Z})^{\mathfrak{g} \text{-fin}} \). Let us identify \( R(T^* p^\pm) = S(p^+) \otimes S(p^-) \) and let \( x : R(T^* p^+) \longrightarrow R(T^* p^-) \) be the linear map such that \( x(f \otimes g) = g \otimes f \).
**Lemma 5.2.3.** For each \( x \in \mathfrak{g} \), \( b^x \) lies in \( R(T^*p^-) \) and is the image under \( x \) of the usual order 1 symbol of the twisted vector field \( \eta_{\lambda}^{-x} \) on \( p^+ \).

**Proof.** Using \( \text{Lemma } [5.2.1] \) we find the commutative diagram
\[
\begin{array}{cccc}
\text{gr } \mathcal{D}^\lambda(X) & \xrightarrow{\text{gr } F} & \text{gr } \mathcal{D}(p^-) & \xrightarrow{\text{gr } F} & \text{gr } \mathcal{D}(Z) \\
\downarrow{s^\lambda} & & \downarrow{s^\lambda} & & \downarrow{t} \\
R(T^*X) & \xrightarrow{x} & R(T^*p^+) & \xrightarrow{x} & R(T^*Z)
\end{array}
\]
(5.7)
Then commutativity of the middle square gives \( x(s^\lambda(\eta_{\lambda}^x)) = t(\pi_{-x}^{-}) = b^{-x} \).

Let \( b : T^*Z \to \mathfrak{g} \) be the moment map defined by \( (b(m), x)_\mathfrak{g} = b^x(m) \).

**Proposition 5.2.4.** The moment map \( b \) is a symplectic Zariski open embedding of \( T^*\widetilde{Z} \) into \( \mathcal{O} \). Moreover \( b \) lifts to a \( \mathbb{Z}_2 \)-equivariant symplectic Zariski open embedding \( \tilde{b} \) of \( T^*\widetilde{Z} \) into \( \mathcal{M} \) so that we get the commutative square
\[
\begin{array}{ccc}
T^*\widetilde{Z} & \xrightarrow{\tilde{b}} & \mathcal{M} \\
\downarrow & & \downarrow \\
T^*Z & \xrightarrow{b} & \mathcal{O}
\end{array}
\]
(5.8)

**Proof.** To prove this it is easiest to start with the geometry of \( \mathcal{O} \) and \( \mathcal{M} \). In \( \text{Lemma } [1.4] \), we introduced \( Z \) and \( \widetilde{Z} \) expressly for the purpose of extracting a square root of the function \( F \). In fact, \( Z \) occurs naturally in the geometry of \( \mathcal{O} \). To begin with, \( Z = K \cdot \tau = \mathcal{O} \cap p^- \) and so \( Z \) is a smooth Lagrangian submanifold of \( \mathcal{O} \) with respect to the KKS symplectic form \( \omega \). Then \( F \in R(Z) \) is simply the restriction of \( \phi^F \in R(\mathcal{O}) \), i.e.,
\[
\phi^F|_Z = F
\]
(5.9)

The composite map \( \mathcal{O} \xrightarrow{j} T^*X \to X \) makes \( \mathcal{O} \) into a \( G \)-equivariant fiber bundle over \( X \) with typical fiber \( Z \). Indeed, the cotangent bundle \( T^*X \to X \) identifies with the contracted product bundle \( G \times_{Q^-} p^- \to G/Q^- \to X \) and then \( \mathcal{O} \) identifies with \( G \times_{Q^-} Z \). We will treat the map \( j \) as an inclusion.

The cotangent bundle \( T^*X \to X \), and hence the subbundle \( \mathcal{O} \to X \), trivializes over the big cell \( X^\circ \). We have identified \( X^\circ \) with \( p^+ \). Now we get the following commutative diagram:
\[
\begin{array}{cccc}
T^*X & \supset T^*p^+ & = p^+ \times p^- & \xrightarrow{\text{permute}} & p^- \times p^+ = T^*p^- \\
\uparrow{j} & & & & \uparrow & \uparrow \\
\mathcal{O} & \supset \mathcal{O} \cap T^*p^+ & = p^+ \times Z & \xrightarrow{\text{permute}} & Z \times p^+ = T^*Z
\end{array}
\]
(5.10)

Here all maps are birational symplectomorphisms, except the two permutation maps are anti-symplectic. The mathematical content of the left part of (5.10) is that \( \mathcal{O} \cap T^*p^+ \) is a trivial bundle over \( p^+ \) and moreover the standard trivialization \( T^*p^+ = p^+ \times p^- \) induces the trivialization \( \mathcal{O} \cap T^*p^+ = p^+ \times Z \). This is true because \( p^+ \) is abelian.

The bottom row of (5.10) read right to left defines an anti-symplectic Zariski open embedding \( T^*Z \to \mathcal{O} \). Using \( \text{Lemma } [5.2.3] \) we see that the composition of this embedding
with the map \( \mathcal{O} \to \mathfrak{g}, \ w \mapsto -w \), is the moment map \( \mathbf{b} : T^*Z \to \mathfrak{g} \). This proves the first statement.

The covering \( \kappa : \mathcal{M} \to \mathcal{O} \) induces a covering \( \kappa^{-1}(Z) \to Z \). This identifies with the covering \( \widetilde{Z} \to Z \) we constructed in Lemma 4.4.4 (on account of (5.2) for instance) in such a way that we get the commutative square

\[
\begin{array}{ccc}
\widetilde{Z} & \to & \mathcal{M} \\
\downarrow & & \downarrow \kappa \\
Z & \to & \mathcal{O}
\end{array}
\]  
(5.11)

where \( \zeta|_{\tilde{Z}} = w \) (cf. (2.8) and (4.10)). Then the inclusion of \( \tilde{Z} \) into \( \mathcal{M} \) is \( K \)-invariant, the Galois groups \( \mathcal{S} \) and \( \mathfrak{S} \) identify naturally and \( \tilde{Z} \) is Lagrangian in \( \mathcal{M} \).

Now let \( \mathcal{N} = \kappa^{-1}(T^*Z) \). We can lift the projection \( \tau : \mathcal{N} \overset{\kappa}{\to} T^*Z \to Z \) to a map \( \tilde{\tau} : \mathcal{N} \to \tilde{Z} \) in the following way. Notice that if \( p, p' \in \tilde{Z} \) lie above \( q \in Z \), then \( \kappa^{-1}(T_q^*Z) \) breaks into two connected components \( \mathcal{N}_p \) and \( \mathcal{N}_{p'} \) which contain \( p \) and \( p' \) respectively. Then \( \mathcal{N} = \bigcup_{p \in \tilde{Z}} \mathcal{N}_p \). Now we define \( \tilde{\tau} \) by \( \tilde{\tau}(\mathcal{N}_p) = p \). Then \( \mathcal{N} \) identifies naturally with \( T^*Z \times_Z \tilde{Z} \simeq T^*\tilde{Z} \) and the rest of the result follows. \( \Box \)

**Corollary 5.2.5.** We have the commutative diagram

\[
\begin{array}{ccc}
R(T^*\tilde{Z})^{gr-fin} & \overset{\mathbf{b}^*}{\leftarrow} & R(\mathcal{M}) \\
\uparrow & & \uparrow \kappa^* \\
R(T^*Z)^{gr-fin} & \overset{\mathbf{b}^*}{\leftarrow} & R(\mathcal{O})
\end{array}
\] 
(5.12)

The horizontal maps are \( \mathfrak{g} \)-linear graded Poisson algebra isomorphisms, where \( R(T^*\tilde{Z}) \) and \( R(T^*Z) \) have the \( \sharp \)-gradations.

**Proof.** The only point that is not immediate is the surjectivity of \( \mathbf{b}^* \) and \( \mathbf{b}^* \) in (5.12). But this follows since, as \( \mathcal{M} \) and \( \mathcal{O} \) are \( G \)-orbits, \( R(\mathcal{M}) \) and \( R(\mathcal{O}) \) are the \( \mathfrak{g} \)-finite parts of the function fields \( \mathbb{C}(\mathcal{M}) \) and \( \mathbb{C}(\mathcal{O}) \). \( \Box \)

Corollary 5.2.3 says in particular that the \( \sharp \)-gradation of \( R(T^*\tilde{Z})^{gr-fin} \), and hence our gradation of \( \text{gr} \mathcal{E}^\lambda \), vanishes in negative degrees. So \( \mathcal{E}^\lambda_j = \mathcal{E}^\lambda_{-1/2} \) if \( j < 0 \). But \( \cap_{j < 0} \mathcal{E}^\lambda_j = 0 \) since each operator \( D \in \mathfrak{D}(\tilde{Z}) \) has finite \( \sharp \)-filtration degree. Thus \( \mathcal{E}^\lambda_j = 0 \) for all \( j < 0 \).

Let \( \tilde{\gamma}_\lambda \) be the composite map

\[
\text{gr} \mathcal{E}^\lambda \overset{t}{\longrightarrow} R(T^*\tilde{Z})^{gr-fin} \overset{\mathbf{b}^*}{\longrightarrow} R(\mathcal{M})
\]
(5.13)

Clearly \( \tilde{\gamma}_\lambda \) extends \( \gamma_\lambda \) and \( \tilde{\gamma}_\lambda \) is a \( \mathbb{Z}_2 \)-equivariant embedding of Poisson algebras.

Finally suppose that \( \alpha : \text{gr} \mathcal{E}^\lambda \to R(\mathcal{M}) \) is some map enjoying the same properties as \( \tilde{\gamma} = \tilde{\gamma}_\lambda \). We want to to show that \( \alpha \) is either \( \tilde{\gamma} \) or \( \zeta \tilde{\gamma} \) where \( \zeta \) is the non-trivial automorphism of \( R(\mathcal{M}) \) defined by the \( \mathcal{S} \)-action. Since \( \alpha \) is 1-to-1 on \( \text{gr} \mathcal{B}^\lambda \) it follows easily that \( \alpha \) is 1-to-1 on \( \text{gr} \mathcal{E}^\lambda \). Then by considering the fraction field of \( \alpha(\text{gr} \mathcal{E}^\lambda) \) we find \( \alpha = \tilde{\gamma} \) or \( \alpha = \zeta \tilde{\gamma} \).

This completes the proof of Theorem 5.1.1.
Remark 5.2.6. (i) If \( x \) lies in one of \( \mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^- \), then \( b^x \) is the symbol (in the usual sense) of \( \pi^x_1 \). (ii) the map \( \tilde{b} \) in Proposition 5.2.4 is unique up to the \( \mathbb{Z}_2 \)-action.

5.3. Simplicity of \( \mathcal{E}^\lambda \). A nice fact which will be important later (see Corollary 7.1.1 and Proposition 7.6.2) is

**Corollary 5.3.1.** \( \mathcal{E}^\lambda \) is a simple ring if \( \mathcal{D}^\lambda(X) \) is a simple ring.

**Proof.** Let \( L \) be a non-zero 2-sided ideal in \( \mathcal{E}^\lambda \). Then \( L \) is in particular \( \mathfrak{g} \)-stable with respect to the representation \((1.12)\). Let \( L_0 \subset L \) be a subspace carrying a non-zero \( \mathfrak{g} \)-irreducible representation. Then \( L_0 \) lies in \( (\mathcal{E}^\lambda)^\dagger \) or \( (\mathcal{E}^\lambda)^\ddagger \). This follows since, by \((5.3)\), \( \mathcal{E}^\lambda \) is isomorphic as a \( G \)-representation to a subspace of \( \mathcal{R}(\mathcal{M}) \) and so, by Proposition 2.4.1, \( \mathcal{E}^\lambda \) is multiplicity-free. So \( L_0^\alpha \) lies in \( (\mathcal{E}^\lambda)^\dagger \) which is equal to \( \mathcal{B}^\lambda \) by Proposition 1.6.2. So \( L \cap \mathcal{B}^\lambda \neq 0 \). But \( \mathcal{B}^\lambda \) is anti-isomorphic to \( \mathcal{D}^\lambda(X) \) and so is simple by hypothesis. Thus \( L \) contains \( \mathcal{B}^\lambda \) and so \( L \) contains 1. \( \square \)

5.4. The anti-symmetry \( \lambda \mapsto (1 - \lambda) \). We define an anti-isomorphism of noncommutative models from \( (\mathcal{E}^\lambda, \tilde{\gamma}_\lambda, \pi_\lambda) \) to \( (\mathcal{E}^\lambda, \tilde{\gamma}_\lambda', \pi_\lambda) \) to be a filtered algebra anti-isomorphism \( \delta : \mathcal{E}^\lambda \rightarrow \mathcal{E}^\lambda \) such that we have \( \delta(\pi^\lambda) = -\pi^\lambda \) and commutativity in

\[
\begin{align*}
\text{gr } \mathcal{E}^\lambda & \xrightarrow{\text{gr } \delta} \text{gr } \mathcal{E}^\lambda' \\
\tilde{\gamma}_\lambda & \xrightarrow{\alpha} \tilde{\gamma}'_\lambda \\
\mathcal{R}(\mathcal{M}) & \xrightarrow{\alpha} \mathcal{R}(\mathcal{M})
\end{align*}
\]

(5.14)

where \( \alpha \) is the automorphism of \( \mathcal{R}(\mathcal{M}) \) defined by \( \alpha(\phi) = i^{2j} \phi \) if \( \phi \in \mathcal{R}^j(\mathcal{M}) \). (So we are extending the involution \( \alpha \) of \( \mathcal{R}(\mathcal{O}) \) defined before \((1.3)\).) Notice that then \( \delta \) is \( \mathfrak{g} \)-linear, and so \( G \)-equivariant, with respect to the representations \( \Pi_\lambda \) and \( \Pi_\lambda' \). Now Proposition 3.4.1 gives

**Corollary 5.4.1.** The anti-isomorphism \( \mathcal{F}\theta\mathcal{F}^{-1} \) extends, uniquely up the action of \( \mathfrak{g} \), to a \( (G \times \mathfrak{g}) \)-invariant anti-isomorphism \( \delta \) so that we get the commutative diagram

\[
\begin{align*}
\mathcal{D}^\lambda(X) & \xrightarrow{\mathcal{D}^\lambda} \mathcal{B}^\lambda & \xrightarrow{\mathcal{D}^\lambda} \mathcal{E}^\lambda \\
\mathcal{D}^{1-\lambda} & \xrightarrow{\mathcal{D}^{1-\lambda}} \mathcal{B}^{1-\lambda} & \xrightarrow{\mathcal{D}^{1-\lambda}} \mathcal{E}^{1-\lambda}
\end{align*}
\]

(5.15)

We can specify \( \delta \) by \( \delta(w) = iw \) and then \( \delta \) gives an \( \mathfrak{g} \)-invariant anti-isomorphism of noncommutative models from \( (\mathcal{E}^\lambda, \tilde{\gamma}_\lambda, \pi_\lambda) \) to \( (\mathcal{E}^{1-\lambda}, \tilde{\gamma}_{1-\lambda}, \pi_{1-\lambda}) \).

**Proof.** Define \( \delta : \mathcal{B}^\lambda \rightarrow \mathcal{B}^{1-\lambda} \) by \( \delta = \mathcal{F}\theta\mathcal{F}^{-1} \). Then \( \delta \) extends to an algebra anti-involution \( \delta : \mathcal{D}(\mathfrak{p}^-) \rightarrow \mathcal{D}(\mathfrak{p}^-) \) where \( \delta(z_i) = -z_i \), \( \delta(\partial_j) = \partial_j \). This follows from the definition of \( \theta \) in the proof of Proposition 3.4.1. Now \( \delta \) naturally extends to an anti-involution of \( \mathcal{D}(\mathcal{Z}) \) and then to an \( \mathfrak{g} \)-invariant anti-automorphism \( \delta \) of \( \mathcal{D}(\tilde{\mathcal{Z}}) \) such that \( \delta(w) = iw \). Then \( \delta \) preserves the \( \tilde{\mathfrak{z}} \)-filtration of \( \mathcal{D}(\tilde{\mathcal{Z}}) \). The relation \( \theta(\eta^x_\lambda) = -\eta^x_{1-\lambda} \) implies \( \delta(\pi^x_\lambda) = -\pi^x_{1-\lambda} \). So \( \delta(\mathcal{E}^\lambda) = \mathcal{E}^{1-\lambda} \). The rest is now clear. \( \square \)
5.5. \(g\)-finiteness of \(w\). Let \(W^\lambda = \mathcal{U}(g) \cdot w\) be the \(\mathcal{U}(g)\)-submodule of \(\mathfrak{D}(\tilde{Z})\) generated by \(w\) in the representation \((4.12)\). So \(w\) is \(g\)-finite \(\iff\) \(W^\lambda\) is finite-dimensional. Theorem 5.1.1 gives

**Corollary 5.5.1.** Pick \(\lambda \in \mathbb{C}\). The following are equivalent:

(i) \(w \in \mathcal{E}^\lambda\), i.e., \(w\) is \(g\)-finite in the representation \((4.12)\).

(ii) \(\tilde{\gamma}_\lambda(\text{gr} \mathcal{E}^\lambda) = R(\mathcal{M})\).

(iii) The triple \((\mathcal{E}^\lambda, \tilde{\gamma}_\lambda, \pi_\lambda)\) is a noncommutative model of \(R(\mathcal{M})\).

(iv) \(\mathcal{E}^\lambda\) is generated as an algebra by \(B^\lambda\) and \(w\).

(v) We have \(\mathcal{E}^\lambda = B^\lambda \oplus W^\lambda B^\lambda\).

**Proof.** The equivalence (ii) \(\iff\) (iii) is immediate from Theorem 5.1.1. The implication (i) \(\implies\) (ii) follows easily from (2.9). We get (iii) \(\implies\) (i) as follows. Given (iii), we know by Lemma 4.1.1 that \(\pi^F_\lambda = F\) admits a square root \(D \in \mathcal{E}^\lambda\). But the equation \(D^2 = F\) in \(\mathfrak{D}(\tilde{Z})\) forces \(D \in \mathfrak{D}_0(\tilde{Z}) = R(\tilde{Z})\). So \(D = \pm w\). So \(w \in \mathcal{E}^\lambda\). Finally, the equivalences with (iv) and (v) follow from Corollary 2.4.4 and (2.9).

The problem now is to determine which, if any, values of \(\lambda\) satisfy (i)-(v); we call these critical values. We solve this in Theorem 6.1.1 below.

6. Critical values of \(\lambda\)

6.1. **Critical values theorem.** We find exactly two critical values of \(\lambda\). Recall the number \(m\) defined by (3.10) and Table I.

**Theorem 6.1.1.** There are exactly two values of \(\lambda\), namely \(\lambda = \frac{1}{2} \pm \frac{1}{4m}\), such that the triple \((\mathcal{E}^\lambda, \tilde{\gamma}_\lambda, \pi_\lambda)\) constructed in \(\S\)4-5 is a noncommutative model of \(R(\mathcal{M})\).

The proof occupies \(\S\)6.2-\(\S\)6.4. From now on we write

\[
\lambda_0 = \frac{1}{2} - \frac{1}{4m} \quad \text{and} \quad \lambda'_0 = \frac{1}{2} + \frac{1}{4m}
\]

(6.1)

**Remark 6.1.2.** The two algebras \(\mathcal{E}^{\lambda_0}\) and \(\mathcal{E}^{\lambda'_0}\) are anti-isomorphic by Corollary 5.4.1 since \(\lambda_0 + \lambda'_0 = 1\). The difference \(d = \lambda_0 - \lambda'_0 = -\frac{1}{2m}\) has the property that \(N^{-d}\) is the G-homogeneous line bundle over \(X\) with \(\Gamma(X, N^{-d}) \simeq V\). With a little thought we see that this is what we would expect. It might interesting to interpret the fact that \(\lambda_0\) and \(\lambda'_0\) tend to \(\frac{1}{2}\) as the rank of \(\mathfrak{g}\) tends to infinity.

Here is an overview of the proof of Theorem 6.1.1. By Corollary 5.5.1, Theorem 6.1.1 amounts to

\[
w \text{ is } g\text{-finite in } \mathfrak{D}(\tilde{Z}) \iff \lambda = \frac{1}{2} \pm \frac{1}{4m}
\]

(6.2)

where we are considering \(g\)-finiteness of \(w\) with respect to the representation \((4.12)\) which of course depends on \(\lambda\).

In \(\S\)6.2, we reduce proving (6.2) to verifying that a certain double commutator vanishes in \(\mathfrak{D}(\tilde{Z})\). In \(\S\)6.4 we carry out the computations. To do this, we exploit the fact that \(p^-\) is a Jordan algebra and \(Z \subset p^-\) is the subset of Jordan invertible elements. A key point is that the Jordan theory gives us explicit formulas for the first and second partial derivatives.
of \( w \) and we explain this in §6.3. To aid the reader, we give several explicit references to [F-K]; see also [Sat].

We are using Jordan algebra theory as a tool. It would be very interesting to find some deeper connections, which we feel surely exist.

**Remark 6.1.3.** In the example of §4.5, Theorem 6.1.1 gives the two critical values \( \lambda = \frac{1}{4}, \frac{3}{4} \).

In hindsight, we see that the “new” value \( \lambda = \frac{3}{4} \) arises by writing our differential operators in the form \( \partial_w^4 w^q \). Indeed, then \( \lambda = \frac{3}{4} \) becomes the value for which an unpleasant term goes away.

### 6.2. Strategy.

In the representation (4.12) we have \([\pi^*_x, w] = \nu(x)w\) for all \( x \in q^+ \) and so \( W^\lambda = S(p^-) \cdot w \). Similarly, in the representation (2.11), we have \( \{\phi^x, \zeta\} = \nu(x)\zeta \) for all \( x \in q^+ \) and so \( V = S(p^-) \cdot \zeta \). Let \( \mathcal{J}_\lambda \subset S(p^-) \) and \( \mathcal{I} \subset S(p^-) \) be the annihilators of \( w \in W^\lambda \) and \( \zeta \in V \) respectively. Since \( V \) is finite-dimensional, it follows by highest weight theory that, as a \( \mathcal{U}(g) \)-module, \( W^\lambda \) admits a unique finite-dimensional quotient and this is isomorphic to \( V \). Hence \( \mathcal{J}_\lambda \subset \mathcal{I} \) and \( W^\lambda \) is finite-dimensional if and only if \( \mathcal{J}_\lambda = \mathcal{I} \), i.e.,

\[
w \text{ is } g\text{-finite} \iff \mathcal{J}_\lambda = \mathcal{I} \tag{6.3}
\]

Using the action of \( \text{ad } h \), we see that \( \mathcal{I} \) and \( \mathcal{J}_\lambda \) are graded ideals in \( S(p^-) \) so that \( \mathcal{I} = \oplus_{p \geq 0} \mathcal{I}^p \) and \( \mathcal{J}_\lambda = \oplus_{p \geq 0} \mathcal{J}_\lambda^p \). We can give a precise description of \( \mathcal{I} \) once we recall the structure of \( S(p^-) \) as a \( K \)-representation.

To do this, we need to set up some structure to write down lowest weights of \( K \)-representations. We can find \( r \) commuting Lie subalgebras \( s_1, \ldots, s_r \) of \( g \), such that (i) each \( s_i \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \), (ii) the sum \( s_1 + \cdots + s_r \) is direct and contains \( s = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}e \), (iii) the decompositions \( e = \sum_{i=1}^r e_i \) and \( h = \sum_{i=1}^r h_i \), where \( e_i, h_i \in s_i \), satisfy \( e_i \in \mathfrak{p}^+ \) and \( h_i \in \mathfrak{t} \), and (iv) each \( s_i \) is stable under complex conjugation and under the complex Cartan involution of \( (g, \mathfrak{t}) \). Then \( h_1, \ldots, h_r \) span an \( r \)-dimensional abelian subalgebra \( a \subset \mathfrak{t} \) and \( e_i \) is a weight vector of \( a \) of weight \( 2t_i \) where \( t_i(\sum_{p=1}^r c_p h_p) = c_i \).

The pair \( (\mathfrak{t}, \mathfrak{e}) \) is a complex symmetric pair; we have \( \mathfrak{e} = \mathfrak{t}^e \). We can embed \( a \) in a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{t} \) such that \( \mathfrak{h} = a \oplus (\mathfrak{h} \cap \mathfrak{e}) \). Now we extend \( t_1, \ldots, t_r \) to weights of \( \mathfrak{h} \) by having them vanish on \( \mathfrak{h} \cap \mathfrak{e} \). We set \( \gamma_i = -2(t_1 + \cdots + t_i) \). The weights \( \gamma_1, \ldots, \gamma_r \) are (for an appropriate system of positive roots) the lowest weights of \( r \) distinct finite-dimensional irreducible \( \mathfrak{t} \)-representations. In particular, \(-2\nu = \gamma_r \).

**Theorem 6.2.1.** **S** \( S(p^-) \) is a multiplicity-free \( K \)-representation and the set of lowest weights occurring in \( S(p^-) \) is \( \{ \sum_{i=1}^r c_i \gamma_i \mid c_i \geq 0 \} \). We have

\[
S^q(p^-) = \bigoplus_{c_1+2c_2+\cdots+rc_r=q} L_{c_1\gamma_1+\cdots+c_r\gamma_r} \tag{6.4}
\]

where the subspace \( L_\psi \) carries the \( K \)-representation of lowest weight \( \psi \).

**Lemma 6.2.2.** The ideal \( \mathcal{I} \subset S(p^-) \) is generated by its degree 2 component \( \mathcal{I}^2 \) and \( \mathcal{I}^2 = L_{2\gamma_1} \). Then: \( w \) is \( g \)-finite \( \iff \) \( L_{2\gamma_1} \subset \mathcal{J}_\lambda \).

**Proof.** By decomposing \( V \simeq S(p^-)/\mathcal{I} \) as a \( K \)-representation, we find that the (unique) \( K \)-stable direct sum complement to \( \mathcal{I} \) in \( S(p^-) \) is \( \bigoplus_{i=0}^r L_{\gamma_i} \). So in particular, \( L_{\gamma_2} \) is the complement to \( \mathcal{I}^2 \) in \( S^2(p^-) \). But \( S^2(p^-) = L_{2\gamma_1} \oplus L_{\gamma_2} \) and so \( \mathcal{I}^2 = L_{2\gamma_1} \). We find that \( \mathcal{I}^2 \) generates \( \mathcal{I} \), and so \( \mathcal{J}_\lambda = \mathcal{I} \iff \mathcal{I}^2 \subset \mathcal{J}_\lambda \). \( \square \)
We can simplify the criterion in Lemma 6.2.2 considerably.

**Lemma 6.2.3.** Let \( y = \overset{\sim}{e} \). Then: \( w \) is \( g \)-finite \( \iff \) \( [\pi^y_\lambda, [\pi^y_\lambda, w]] = 0 \).

**Proof.** Since \( u = \overset{\sim}{e} \) is a lowest weight vector in \( L_{2\eta} \) and \( w \) is \( K \)-semi-invariant, it follows that \( L_{2\eta} \subset J_\lambda \iff \Pi_\lambda(w) = 0 \). But \( \Pi_\lambda(w) = [\pi^y_\lambda, [\pi^y_\lambda, w]] \). \( \square \)

### 6.3. Calculus on the coupled Jordan algebras \( p^\pm \).

By TKK theory, \( p^+ \) and \( p^- \) are (isomorphic) complex simple Jordan algebras with Jordan products defined by \( [x, e] \circ [y, e] = [x, [y, e]] \) and \( [x, \overset{\sim}{e}] \circ [y, \overset{\sim}{e}] = [x, [y, \overset{\sim}{e}]] \) where \( x, y \in \mathfrak{r} \) and \( r \) is the orthogonal complement in \( \mathfrak{k} \) to \( \mathfrak{r}^* \). The Jordan identity elements are \( e \in p^+ \) and \( \overset{\sim}{e} \in p^- \). In this subsection, we explain some basic formulas from Jordan theory that we will use throughout §6.4. See [F-K, Table on page 160] for the list of complex Jordan algebras carried by \( p^\pm \).

(Notation warning: our algebra \( \mathfrak{g} \) is called “\( g \)” in [F-K].)

From now on, we usually omit the symbol “\( \circ \)” and write the Jordan product \( a \circ b \) as \( ab \). We adopt this convention: if \( D \) is an operator and \( f \) is a function then \( Df \) is the composition of operators (where \( f \) is regarded as a multiplication operator) and \( [Df] \) is the function obtained by applying \( D \) to \( f \).

The polynomial function \( F \), normalized so that \( F(\overset{\sim}{e}) = 1 \), is the Jordan norm of \( p^- \). Thus by (6.8), \( Z \) is the set of Jordan invertible elements in \( p^- \). The partial derivatives of \( F \) are (see [F-K, Prop. III.4.2, page 52]), where \( v \in p^- \),

\[
\frac{\partial^v F}{\partial x} = F \text{ tr}(v^{-1}) \tag{6.5}
\]

Here \( q \) is an arbitrary point in \( p^- \) so that the RHS of (6.5) is the function \( q \mapsto F(q) \text{ tr}(v^{-1}) \) where \( q^{-1} \) is the Jordan inverse and \( \text{tr} \) is the Jordan trace. A quick definition is \( \text{tr}(x) = \frac{1}{m} \text{ Tr} L_x \), where \( L_x : p^\pm \to p^\pm \) is Jordan multiplication by \( x \in p^\pm \) and \( \text{Tr} \) is the usual trace. Then \( \text{tr}(e) = \text{tr}(\overset{\sim}{e}) = r \). See [F-K, Prop. III.4.2].

Since \( w = \sqrt{F} \), (6.3) gives

\[
[\partial^u w] = \frac{1}{2} w \text{ tr}(v^{-1}) \tag{6.6}
\]

To get \( [\partial^u \partial^v w] \), where \( v, u \in p^- \), we recall ([F-K, Proposition II.3.3(i), page 33])

\[
[\partial^u \text{ tr}(vq^{-1})] = - \text{ tr}(v\{q^{-1}, u, q^{-1}\}) \tag{6.7}
\]

Here \( \{a, b, c\} = a(bc) + b(ac) - (ab)c \) is the Jordan triple product. Then

\[
[\partial^u \partial^v w] = \frac{1}{4} w \text{ tr}(uq^{-1}) \text{ tr}(v^{-1}) - \frac{1}{2} w \text{ tr}(v\{q^{-1}, u, q^{-1}\}) \tag{6.8}
\]

The coupling of our Jordan algebras is achieved by the transpose maps \( p^\pm \to p^\pm \), \( x \mapsto x' \), defined by \( \langle x, y \rangle = \text{tr}(xy') = \text{tr}(x'y) \). These maps are inverse Jordan algebra isomorphisms and \( \text{tr}(u) = \text{tr}(u') \). From now on, we assume that our basis \( v_1, \ldots, v_n \) of \( p^- \) introduced in (6.6) is orthonormal with respect to \( \text{tr} \). Then \( v_i' = z_i \) and \( z = \text{tr}(z'q) \).

Our realization \( x \mapsto \eta^x \) of \( g \) inside \( \mathfrak{D}(p^\pm) \) is the TKK construction (see [Sat]). In that language (6.3) becomes, for \( y \in p^- \),

\[
\pi^y_\lambda \chi = - \left( \sum_{i,j} \{z_i, y_j', z_j\} \partial_{z_i} \partial_{z_j} \right) - 2 m \lambda \partial^y
\]

\[
= - \left( \sum_{i,j} \text{tr}(v_j, y, \partial_{\eta^v}) \partial^v \partial^w \right) - 2 m \lambda \partial^y \tag{6.9}
\]
since \( \{a, b, c\} = -\frac{1}{2}[b', a], c \) if \( a, b, c \in \mathfrak{p}^\pm \).

**Example 6.3.1.** We will write out everything for the case \( \mathfrak{g}_R = \mathfrak{su}(r, r) \). We began this example in §2.3 and continued it in Example 3.5.1.

Now \( \mathfrak{p}_+ \) identifies, in the obvious way, with the complex Jordan algebra \( M(r, \mathbb{C}) \) of \( r \times r \) matrices with Jordan product \( A \circ B = \frac{1}{2}(AB + BA) \) where \( AB \) is the ordinary matrix product. The Jordan triple product is \( \{A, B, C\} = \frac{1}{2}(ABC + CBA) \). We have \( \text{tr}(A) = \text{Tr}(A) \) and \( F(A) = \text{Det}(A) \) where \( \text{Tr} \) and \( \text{Det} \) are the usual matrix trace and determinant. Also \( ((0_{A, B}), (0_{C, 0})) = \text{Tr}(BC) \) and \( (0_{B, 0})^t = (0_{B, 0}) \).

To actually write out our calculations for this case with matrices, it is convenient to use the basis \( \{E_{i,j}\} \) of \( M(r, \mathbb{C}) \) by elementary matrices (even though it is not orthonormal). For instance, \( (6.3) \) becomes the familiar formula \( \frac{\partial |z|}{\partial z_{ij}} = |z|(z^{-1})_{ji} \) where \( |z| = \text{Det} z \).

### 6.4. Computing \([\pi^y_\lambda, [\pi^y_\lambda, w]]\). We now compute a bracket relation in \( \mathfrak{D}(\tilde{Z}) \).

**Lemma 6.4.1.** Let \( y \in \mathfrak{p}^- \). Then
\[
[\pi^y_\lambda, w] = -\partial^y w - 2m(\lambda - \lambda_0)[\partial^y w] \tag{6.10}
\]

**Proof.** The commutator \([\pi^y_\lambda, w]\) is a differential operator on \( \tilde{Z} \) of order at most 1 and so we can write it uniquely as the sum of a vector field \( \xi \) and a function \( g \). It is convenient to compute these parts individually. Using \( (5.3) \) and \( (5.6) \) we find
\[
\begin{align*}
\xi &= -\sum_{i,j} \text{tr}(\{v_j, y, v_i\}q) (\partial^w v_j)[\partial^y w] + [\partial^y w][\partial^w v_j] \\
&= -w \sum_{i,j} \text{tr}(\{v_j, y, v_i\}q) \text{tr}(v_i q^{-1}) \partial^w v_j \\
&= -w \sum_{j} \text{tr}(\{v_j, y, q^{-1}\}q) \partial^w v_j \\
&= -w \sum_{j} \text{tr}(v_j y) \partial^w v_j = -w \partial^w v
\end{align*}
\]
The fourth equality follows from Jordan identities. Indeed, (i) the operator \( \mathcal{P}_{a,c} \) defined by \( \mathcal{P}_{a,c}(b) = \{a, b, c\} \) is self-adjoint and (ii) \( \{a, b, c\} = a \). Hence \( \text{tr}(\{v_j, y, q^{-1}\}q) = \text{tr}(y\{v_j, q, q^{-1}\}) = \text{tr}(yv_j) \).

Next using \( (6.9), (6.8) \) and self-adjointness of \( \mathcal{P}_{a,c} \) we find
\[
\begin{align*}
g &= -\left(\sum_{i,j} \text{tr}(\{v_j, y, v_i\}q)[\partial^w v_j][\partial^w v_i] w\right) - 2m\lambda [\partial^y w] \\
&= -\frac{1}{4}w \text{tr}(\{q^{-1}, y, q^{-1}\}q) + \frac{1}{2}w \sum_i \text{tr}(\{q^{-1}, v_i, q^{-1}\}q, v_i) - m\lambda w \text{tr}(yq^{-1}) \\
&= -(\frac{1}{4} + m\lambda) w \text{tr}(yq^{-1}) + \frac{1}{2}w \sum_i \text{tr}(yq^{-1}v_i^2) \\
&= -(\frac{1}{4} + m\lambda - \frac{1}{2}m) w \text{tr}(yq^{-1})
\end{align*}
\]
For the third equality we used the identity \( \{\{q^{-1}, v, q^{-1}\}, q, v\} = q^{-1}v^2 \), and for the fourth we used \( \sum_i v_i^2 = \mu \mathcal{Z} \) (see [K-Ka], page 117).

Notice that we can rewrite \( (6.10) \) as
\[
[\pi^y_\lambda, w] = -\partial^y w - 2m(\lambda - \lambda_0)[\partial^y w] \tag{6.11}
\]

**Lemma 6.4.2.** Put \( y = \pi_1 \). Then \([\pi^y_\lambda, \partial^y] = (\partial^y)^2 \).
Proof. Starting from (6.9) we find
\[
[\pi^y_\lambda, \partial^y] = \sum_{i,j} \text{tr}(\{v_j, y, v_i\}y)\partial^{v_i}\partial^{v_j} = \sum_{i,j} \text{tr}(yv_i) \text{tr}(yv_j)\partial^{v_i}\partial^{v_j} = (\partial^y)^2 \tag{6.12}
\]
We will explain the second equality. We start from the fact that $y = e_1$ is a primitive idempotent in the Jordan algebra $p^-$. Indeed, $e = \sum_{i=1}^r e_i$ is a decomposition of $e$ into orthogonal primitive idempotents.

For any primitive idempotent $y$, then the map $x \mapsto \{y, x, y\}$ is the orthogonal projection onto $C_y$ and so $\{y, x, y\} = y \text{tr}(xy)$. Then
\[
y \text{tr}(\{v_j, y, v_i\}y) = \{\{y, v_j, y\}, v_i, y\} = y \text{tr}(yv_j) \text{tr}(yv_i)
\]
because of the Jordan identity \(\{a, \{b, a\}, c\}, a\} = \{\{a, b\}, c, a\}\) (see [F-K, Ex. 8, page 40]). This proves the second equality in (6.12). \[
\]
Lemma 6.4.3. Put $y = e_1$. Then
\[
[\pi^y_\lambda, [\pi^y_\lambda, w]] = -m^2(\lambda - \lambda_0) (\lambda - \lambda_0') w \text{tr}(yq^{-1})^2 \tag{6.13}
\]
Hence $[\pi^y_\lambda, [\pi^y_\lambda, w]] = 0$ if and only if $\lambda$ equals $\lambda_0$ or $\lambda_0'$.

Proof. If $\lambda = \lambda_0$ then (6.10) gives $[\pi^y_\lambda, w] = -w\partial^y$ and using Lemma 6.4.2 we find $[\pi^y_\lambda, [\pi^y_\lambda, w]] = w(\partial^y)^2 - (w\partial^y)\partial^y = 0$. If $\lambda = \lambda_0'$ then (6.11) gives $[\pi^y_\lambda, w] = -\partial^yw$ and we find $[\pi^y_\lambda, [\pi^y_\lambda, w]] = -(\partial^y)^2w + \partial^y(\partial^yw) = 0$. Thus (6.13) is true when $\lambda$ equals $\lambda_0$ or $\lambda_0'$.

Now we can prove (6.13) for all values of $\lambda$ without further calculation by simply examining the form of $[\pi^y_\lambda, [\pi^y_\lambda, w]]$. Let $\lambda$ be arbitrary. Then $\pi^y_\lambda = \pi^y_{\lambda_0} - 2c_\lambda\partial^y$ where $c_\lambda = m(\lambda - \lambda_0)$. Since $[\pi^y_{\lambda_0}, [\pi^y_{\lambda_0}, w]] = 0$ we get
\[
[\pi^y_\lambda, [\pi^y_\lambda, w]] = -2c_\lambda[\pi^y_{\lambda_0}, [\partial^y w]] - 2c_\lambda[\partial^y, [\pi^y_{\lambda_0}, w]] + 4c_\lambda^2[\partial^y[\partial^yw]] \tag{6.14}
\]
The RHS is the sum of a vector field which in linear in $\lambda$ and a function which is quadratic in $\lambda$. But the RHS vanishes for the two distinct values $\lambda_0$ and $\lambda_0'$. So the vector field vanishes identically and the function is of the form $(\lambda - \lambda_0)(\lambda - \lambda_0')g$ where $g \in R(\tilde{Z})$. Comparing coefficients of $\lambda^2$ we find $g = 4m^2[\partial^y, [\partial^y, w]]$. Using (6.6), (6.7) and the fact that $y$ is a primitive idempotent we find $4[\partial^y, [\partial^y, w]] = -w \text{tr}(yq^{-1})^2$.

Lemmas 6.4.3 and 6.2.3 give (5.2). This concludes the proof of Theorem 6.1.1.

Remark 6.4.4. We can also write down $Q^-$-semi-invariant (lowest weight) vectors $T \in W^\lambda_0$ and $T' \in W^\lambda_0'$ using these methods. We find that $T = w\partial_F$ and $T' = \partial_Fw$. This agrees with §4.5 since there $\lambda_0 = \frac{1}{4}$ and $w\partial_4 = \frac{1}{2}\partial_0$.

6.5. Comparison of critical values. While it was easy to see that $(\mathcal{E}^{\lambda_0}, \tilde{\gamma}_{\lambda_0}, \pi_{\lambda_0})$ and $(\mathcal{E}^{\lambda_0'}, \tilde{\gamma}_{\lambda_0'}, \pi_{\lambda_0'})$ are anti-isomorphic (see Remark 6.1.2), a more subtle fact is that they are isomorphic. We define an isomorphism of noncommutative models from $(\mathcal{E}^\lambda, \tilde{\gamma}_\lambda, \pi_\lambda)$ to $(\mathcal{E}^\gamma, \tilde{\gamma}_\gamma, \pi_\gamma)$ to be a filtered algebra isomorphism $\sigma : \mathcal{E}^\lambda \to \mathcal{E}^\gamma$ such that $\sigma(\pi_\lambda^x) = \pi_\gamma^x$ and $\gamma_\lambda = \gamma_\gamma(\text{gr} \, \sigma)$. Then $\sigma$ is $g$-linear, and so $G$-equivariant, with respect to the representations $\Pi_\lambda$ and $\Pi_\gamma$.

Proposition 6.5.1. The inner automorphism $D \mapsto wDw^{-1}$ of $\mathcal{O}(\tilde{Z})$ maps $\mathcal{E}^{\lambda_0}$ onto $\mathcal{E}^{\lambda_0}$. The induced map $\text{Inn}_w : \mathcal{E}^{\lambda_0'} \to \mathcal{E}^{\lambda_0}$ is an $G$-invariant isomorphism of noncommutative models from $(\mathcal{E}^{\lambda_0'}, \tilde{\gamma}_{\lambda_0'}, \pi_{\lambda_0'})$ to $(\mathcal{E}^{\lambda_0}, \tilde{\gamma}_{\lambda_0}, \pi_{\lambda_0})$. 27
**Proof.** The result is clear once we prove that $w \pi^x_{\lambda_0} w^{-1} = \pi^x_{\lambda_0}$ where $x \in \mathfrak{g}$. It suffices to check this for $x \in \mathfrak{p}^+$ and $x \in \mathfrak{p}^-$. Clearly $\text{Inn}_w$ is the identity on $R(\tilde{Z})$. So for $x \in \mathfrak{p}^+$ we find $w \pi^x_{\lambda_0} w^{-1} = w \pi^x w^{-1} = \pi^x = \pi^x_{\lambda_0}$. Next suppose $y \in \mathfrak{p}^-$. Then

$$w \pi^y_{\lambda_0} w^{-1} = \pi^y_{\lambda_0} - [\pi^y_{\lambda_0}, w] w^{-1} = \pi^y_{\lambda_0} + \partial^y = \pi^y_{\lambda_0} - \frac{1}{2} m = \pi^y_{\lambda_0}$$

The second equality follows by (5.10) because $c_{\lambda_0} = \frac{1}{2}$ and so $[\pi^y_{\lambda_0}, w] = -\partial^y w$. \[\Box\]

We will find a sort of explanation for $\text{Inn}_w$ later in §7.4. Notice that $\text{Inn}_w$ induces a filtered anti-isomorphism $\mathcal{B}^{\lambda_0} \to \mathcal{B}^{\lambda_0}$ which is then the restriction of the outer automorphism $D \mapsto wDw^{-1}$ of $\mathcal{D}(Z)$. It would be interesting to give a direct geometric description of the corresponding anti-isomorphism $\mathcal{D}^{\lambda_0}(X) \to \mathcal{D}^{\lambda}(X)$ obtained by Fourier transform.

7. **The noncommutative model $\mathcal{E}^{\lambda_0}$**

7.1. **Algebraic structure of $\mathcal{E}^{\lambda_0}$**. To begin with, we have

**Corollary 7.1.1.** $\mathcal{E}^{\lambda_0}$ and $\mathcal{B}^{\lambda_0}$ are simple rings.

**Proof.** $\mathcal{D}^{\lambda_0}(X)$ is simple by Proposition 3.2.2 since $2\lambda_0 = 1 - \frac{m}{n}$ does not lie in $\mathbb{Z} - \{1\}$. (Note $m \geq 1$ by Table III.) The result follows by Corollary 5.3.1. \[\Box\]

**Corollary 7.1.2.** We have $J^{\lambda_0} = \tau(J^{\lambda_0}) = J^{\lambda_0}$. Moreover $J^{\lambda_0}$ is a maximal 2-sided ideal in $\mathcal{U}(\mathfrak{g})$ and its infinitesimal character is given by the weights $(-m \pm \frac{1}{2})\nu + \rho$.

**Proof.** $J^{\lambda_0} = J^{\lambda_0}$ follows by Proposition 6.5.1 while $J^{\lambda_0} = \tau(J^{\lambda_0})$ follows by Corollary 3.4.2. $J^{\lambda_0}$ is maximal since $\mathcal{B}^{\lambda_0} \simeq \mathcal{U}(\mathfrak{g}) / J^{\lambda_0}$ is simple. The infinitesimal character follows by Corollary 4.3.3: the two weights are then Weyl group conjugate. \[\Box\]

**Remark 7.1.3.** We checked that our infinitesimal character is the same as the one given by McGovern in [McG4, Tables 5-10]. Moreover if the “root multiplicity” $d$ given by $n = r + \binom{r}{2} d$ is equal to 2, which happens exactly when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(r, r)$, then our infinitesimal character coincides with $\frac{1}{2} \rho$.

**Corollary 7.1.4.** Viewed as $\mathcal{U}(\mathfrak{g})$-bimodules, both $\mathcal{B}^{\lambda_0}$ and $(\mathcal{E}^{\lambda_0})^\perp$ have left annihilator and right annihilator equal to $J^{\lambda_0}$.

**Proof.** Both $\mathcal{B}^{\lambda_0}$ and $(\mathcal{E}^{\lambda_0})^\perp$ are faithful as right or left modules over $\mathcal{B}^{\lambda_0}$ since the algebra $\mathcal{E}^{\lambda_0}$ has no zero-divisors. Since $\mathcal{B}^{\lambda_0}$ identifies with $\mathcal{U}(\mathfrak{g}) / J^{\lambda_0}$, the left and right annihilators in $\mathcal{U}(\mathfrak{g})$ are $J^{\lambda_0}$ and $\tau(J^{\lambda_0}) = J^{\lambda_0}$. \[\Box\]

7.2. **A simple module for $\mathcal{E}^{\lambda_0}$**. Our construction of $\mathcal{E}^{\lambda_0}$ gives us a natural module for it, namely $R(\tilde{Z})$. Our next result produces a simple submodule.

Let $\mathcal{H}$ be the $\mathcal{E}^{\lambda_0}$-submodule of $R(\tilde{Z})$ generated by the function 1.

**Proposition 7.2.1.** $\mathcal{H}$ is a faithful simple $\mathcal{E}^{\lambda_0}$-module. We have

$$\mathcal{H} = S(\mathfrak{p}^+) \oplus wS(\mathfrak{p}^+)$$

(7.1)

The maximal Poisson abelian subalgebra $\mathcal{G}$ (from Corollary 2.4.3) identifies with $\mathcal{H}$ under the restriction homomorphism $R(\mathcal{M}) \to R(\tilde{Z})$. 28
The proof requires two lemmas.

Lemma 7.2.2. \( H \) is the subalgebra of \( R(\tilde{Z}) \) generated by \( S(p^+) \) and \( w \). The action of \( \mathfrak{g} \) on \( R(\tilde{Z}) \) induces an algebra \( \mathbb{Z}_2 \)-grading \( H = H^\uparrow \oplus H^\downarrow \) where \( H^\uparrow = S(p^+) \) and \( H^\downarrow = w S(p^+) \) (recall \( w^2 = F \)).

Proof. Let \( A \) be the algebra generated by \( S(p^+) \) and \( w \). Then \( A = S(p^+) \oplus w S(p^+) \) and this is the \( \mathfrak{g} \)-grading. Now \( A \), regarded as a space of multiplication operators, lies in \( E_{\tilde{Z}}^{\lambda_0} \). Consequently \( A \), regarded as a space of functions, lies in \( H \). The problem then is to show that \( A \) is \( E_{\lambda_0} \)-stable. We know (Corollary [5.3.1]) that \( E_{\lambda_0} \) is generated by \( B_{\lambda_0} \) and \( w \), and \( B_{\lambda_0} \) is generated by \( \pi_{x_0}^z \), \( x \in \mathfrak{g} \). The multiplication operator \( w \) certainly preserves \( A \), and the operators \( \pi_{x_0}^z \), \( x \in \mathfrak{g} \), preserve \( S(p^+) \). So we need to check that the operators \( \pi_{x_0}^z \), \( x \in \mathfrak{q}^+ \), this clear. For \( x \in \mathfrak{p}^- \), this follows using the bracket relations \( [\pi_{x_0}^z, w] = -w \partial_x \) from Lemma [3.4.1]. \( \square \)

Remark 7.2.3. Lemma 7.2.2 implies that \( H \) is the full subalgebra of all order zero differential operators (multiplication operators) in \( E_{\lambda_0} \), i.e., \( H = R(\tilde{Z}) \cap E_{\lambda_0} \). This suggests that \( H \) might be maximal abelian in \( E_{\lambda_0} \), we prove this in Corollary 8.5.3.

Now \( H^\uparrow \) and \( H^\downarrow \) are the \( B_{\lambda_0} \)-modules generated by \( 1 \) and \( w \) respectively. Let \( \tilde{K} \) be the double cover of \( K \) which admits \( \sqrt{\chi} \) as a character.

Lemma 7.2.4. \( H^\uparrow \) and \( H^\downarrow \) are each lowest weight \( \mathfrak{g} \)-representations and generalized Verma modules for \( \mathfrak{q}^- \). The lowest weight vectors \( 1 \in H^\uparrow \) and \( w \in H^\downarrow \) are \( \tilde{K} \)-semi-invariant of weights \( \chi^{m - \frac{1}{2}} \) and \( \chi^{m + \frac{1}{2}} \) respectively.

\( H^\uparrow \) and \( H^\downarrow \) are irreducible \( (\mathfrak{g}, \tilde{K}) \)-modules with the same annihilator \( J_{\lambda_0} \) in \( \mathcal{U}(\mathfrak{g}) \).

Proof. By Corollary [1.3.2] we know that \( H^\uparrow = S(p^+) \) is a Verma module for \( \mathfrak{q}^- \) with lowest weight vector \( 1 \) of weight \( 2m_{\lambda_0} \nu = (m - \frac{1}{2}) \nu \). Similarly \( H^\downarrow = w S(p^+) \) is a Verma module for \( \mathfrak{q}^- \) with lowest weight vector \( w \) of weight \( 2m_{\lambda_0} \nu = (m + \frac{1}{2}) \nu \). This follows because for \( x \in \mathfrak{t} \) we have \( \pi_{\lambda_0}^x (w) = \nu(x) + 2m_{\lambda_0} \nu(x) = (m + \frac{1}{2}) \nu(x) \), and for \( y \in \mathfrak{p}^- \) we have (by Lemma [3.4.1]) \( \pi_{\lambda_0}^y (w) = ([\pi_{\lambda_0}^y, w] + w \pi_{\lambda_0}^y)(1) = 0 \).

The weights \( (m + \frac{1}{2}) \nu \) exponentiate to characters of \( \tilde{K} \) and then \( H^\uparrow \) and \( H^\downarrow \) are \( (\mathfrak{g}, \tilde{K}) \)-modules. A theorem of Wallach ([Wal]) says in particular that \( \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{q}^-) \mathbb{C}_{s\nu} \) is irreducible as a \( \mathfrak{g} \)-module if \( s > m - 1 \). Our values \( s = m \pm \frac{1}{2} \) satisfy this bound. \( \square \)

Remark 7.2.5. The same theorem of Wallach says that \( H^\uparrow \) and \( H^\downarrow \) are unitarizable as \( \mathfrak{g}_R \)-representations. We can regard them as quantizations of the real nilpotent orbit \( O_R \).

Proof. of Proposition [7.2.1]. \( H^\uparrow \) and \( H^\downarrow \) are faithful simple \( B_{\lambda_0} \)-modules by Lemma 7.2.4. Since \( H^\uparrow \) and \( H^\downarrow \) carry different \( \mathfrak{g} \)-representations, they are the only non-trivial \( B_{\lambda_0} \)-submodules of \( H \). Neither \( H^\uparrow \) nor \( H^\downarrow \) is \( E_{\lambda_0} \)-stable, since multiplication by \( w \) moves each into the other. Thus \( H \) is simple for \( E_{\lambda_0} \). Faithfulness is automatic as \( E_{\lambda_0} \) is a simple ring. Our descriptions of \( \mathcal{G} \) and \( H \) in Corollary [2.4.3] and Lemma 7.2.2 imply that \( \mathcal{G} \) maps isomorphically onto \( H \). \( \square \)
7.3. **Using \( \mathcal{H} \) to realize \( \mathcal{E}^{\lambda_0} \).** Since \( \mathcal{E}^{\lambda_0} \) acts faithfully on \( \mathcal{H} \) we have an algebra embedding

\[
\mathcal{E}^{\lambda_0} \subset \text{End}_{g-fin}(\mathcal{H})
\]  

(7.2)

where the representation of \( g \) on \( \text{End} \mathcal{H} \) is still given by the operators \( \Pi^{\lambda_0}_g \). The algebra \( \text{End}_{g-fin}(\mathcal{H}) \) is much larger that \( \mathcal{E}^{\lambda_0} \); in particular it contains \( \text{End}_{g-fin}(\mathcal{H}^\uparrow) \oplus \text{End}_{g-fin}(\mathcal{H}^\downarrow) \).

**Corollary 7.3.1.** \( \mathcal{E}^{\lambda_0} \) is the \( g \)-finite part of \( \mathcal{D}(\mathcal{H}) \) for the representation \( \Pi^{\lambda_0} \).

*Proof.* Clearly \( \mathcal{E}^{\lambda_0} \) lies in \( \mathcal{D}(\mathcal{H})^{g-fin} \). The converse follows since \( R(\tilde{Z}) \) is a localization of \( \mathcal{H} \) and so any differential operator on \( \mathcal{H} \) extends to one on \( R(\tilde{Z}) \).

We can also recover \( \mathcal{E}^{\lambda_0} \) as a vector space in the following way.

**Corollary 7.3.2.** The natural map \( \mathcal{E}^{\lambda_0} \to \text{Hom}_{g-fin}(\mathcal{H}^\uparrow, \mathcal{H}) \) is a vector space isomorphism.

*Proof.* The map is injective because any differential operator on \( \tilde{Z} \) is uniquely determined by its values on \( S(p^+) \). This is true since any vector space basis of \( p^+ \) is a set of local \( \acute{e}tale \) coordinates on \( \tilde{Z} \). To prove surjectivity we need to show that if \( L \in \text{Hom}_{g-fin}(\mathcal{H}^\uparrow, \mathcal{H}) \) then \( L \) extends to a differential operator \( P \) on \( \tilde{Z} \).

We may write \( L = L^\uparrow + L^\downarrow \) where \( L^\uparrow \in \text{Hom}_{g-fin}(\mathcal{H}^\uparrow, \mathcal{H}^\downarrow) \). (Read \( \uparrow \) like \( \pm \).) Since \( L \) is \( g \)-finite, its components \( L^\uparrow \) and \( L^\downarrow \) are each \( g \)-finite, and so in particular are \( p^+ \)-finite. Now \( x \in p^+ \) acts by commutator with multiplication by \( x \), i.e., \( \Pi^{\lambda_0}_g(D) = [\pi^{\lambda_0}_g(D), D] = [x, D] \). It follows that \( L^\uparrow \) lies in \( \mathcal{D}(p^-) \). But also \( \Pi^{\lambda_0}_g(w^{-1}L^\downarrow) = w^{-1}\Pi^{\lambda_0}_g(L^\downarrow) \) and so \( w^{-1}L^\downarrow \) is \( p^+ \)-finite and thus lies in \( \mathcal{D}(p^-) \). Let \( P_1 \) and \( P_2 \) be the differential operators on \( Z \) defined by restriction of \( L^\uparrow \) and \( w^{-1}L^\downarrow \) respectively. Then \( P = P_1 + wP_2 \) is the operator we wanted.

**7.4. Comparing \( \mathcal{E}^{\lambda_0} \) with \( \mathcal{E}^{\lambda_0} \).** Similarly, we can let \( \mathcal{H}' \subset R(\tilde{Z}) \) be the \( \mathcal{E}^{\lambda_0} \)-submodule generated by 1. Then \( \mathcal{H}' = (\mathcal{H}')^\uparrow \oplus (\mathcal{H}')^\downarrow \) where \( (\mathcal{H}')^\uparrow = S(p^+) \) and \( (\mathcal{H}')^\downarrow = w^{-1}S(p^+) \).

Then \( 1 \in (\mathcal{H}')^\uparrow \) and \( w^{-1} \in (\mathcal{H}')^\downarrow \) are \( \tilde{K} \)-semi-invariant lowest weight vectors of weights \( \chi^{m+\frac{1}{2}} \) and \( \chi^{m-\frac{1}{2}} \) respectively. (Notice that \( \mathcal{H}' \) is not a subalgebra of \( R(\tilde{Z}) \) and \( \mathcal{H}' \) does not lie in \( \mathcal{E}^{\lambda_0} \).)

Now we get a nice way to derive the isomorphism \( \text{Inn}_{w} \) found in Proposition 6.5.1. Indeed we have an isomorphism of \( (g, \tilde{K}) \)-modules

\[
\mathcal{H}' \to \mathcal{H}, \quad f \mapsto w f
\]

(7.3)

which carries \( \mathcal{H}' \) to \( (\mathcal{H}')^\downarrow \) and \( \mathcal{H}' \) to \( (\mathcal{H}')^\uparrow \). Now \( \text{Inn}_{w} \) is simply the induced isomorphism \( \text{End}_{g-fin}(\mathcal{H}') \to \text{End}_{g-fin}(\mathcal{H}) \) and this sends \( \mathcal{E}^{\lambda_0} \) and \( \mathcal{E}^{\lambda_0} \).

**7.5. The algebra anti-automorphism \( \beta \).** Let \( \beta \) be the composition

\[
\mathcal{E}^{\lambda_0} \xrightarrow{\delta} \mathcal{E}^{\lambda_0} \xrightarrow{\text{Inn}_{w}} \mathcal{E}^{\lambda_0}
\]

(7.4)

Corollary 5.4.1 and Proposition 6.5.1 give
Corollary 7.5.1. $\beta$ is an $\mathcal{G}$-invariant anti-automorphism of the noncommutative model $(\mathcal{E}^{\lambda_0}, \gamma^{\lambda_0}, \pi^{\lambda_0})$. In particular $\beta$ is $G$-invariant we have two commutative squares

$$
\begin{array}{ccc}
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\pi_{\lambda_0}} & \mathcal{E}^{\lambda_0} \\
\downarrow{\tau} & & \downarrow{\beta} \\
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\pi_{\lambda_0}} & \mathcal{E}^{\lambda_0}
\end{array}
\quad
\begin{array}{ccc}
\text{gr} \mathcal{E}^{\lambda_0} & \xrightarrow{\text{gr} \beta} & \text{gr} \mathcal{E}^{\lambda_0} \\
\downarrow{\gamma^{\lambda_0}} & & \downarrow{\gamma^{\lambda_0}} \\
\mathcal{R}(\mathcal{M}) & \xrightarrow{\alpha} & \mathcal{R}(\mathcal{M})
\end{array}
$$

(7.5)

$\beta$ has order 2 or 4; in fact $\beta^2 = 1$ if $r$ is even while $\beta^2 = \varsigma$ if $r$ is odd where $\varsigma$ is the non-trivial element of $\mathcal{G}$.

Proof. The first part is clear. Now $\beta^2$ is a filtered $G$-invariant algebra automorphism of $\mathcal{E}^{\lambda_0}$ which is trivial on $\mathcal{B}^{\lambda_0}$. By considering the induced action of $\beta^2$ on $\text{gr} \mathcal{E}^{\lambda_0} \cong \mathcal{R}(\mathcal{M})$, we see easily that $\beta^2$ lies in $\mathcal{G}$. We have $\beta(w) = \iota^r w$ and so $\beta^2 = \varsigma^r$. \qed

Remark 7.5.2. In the sense of (7.5), (i) $\beta$ is the unique algebra anti-automorphism of $\mathcal{E}^{\lambda_0}$ which extends $\tau$, and (ii) $\beta$ is the unique $\mathfrak{g}$-linear endomorphism of $\mathcal{E}^{\lambda_0}$ such that $\text{gr} \beta$ corresponds to $\alpha$.

Corollary 7.5.3. Here is a simple description of $\beta$: if $\mathcal{L}$ is a subspace carrying an irreducible $G$-representation and $\mathcal{L} \subset \mathcal{E}^{\lambda_0}$ with $j$ as small as possible, then $\beta(D) = i^j D$ for all $D \in \mathcal{L}$.

This description is nice but does not reveal why $\beta$ is an anti-automorphism.

7.6. $\mathcal{E}^{\lambda_0}$ is a superalgebra with supertrace. The constant functions in $\mathcal{E}^{\lambda_0}$ are the only $G$-invariants. (This is true for a noncommutative model of any coadjoint orbit cover.) So there is a unique $G$-linear map

$$T : \mathcal{E}^{\lambda_0} \to \mathbb{C} \quad (7.6)$$

such that $T(1) = 1$. The restriction of $T$ to $\mathcal{B}^{\lambda_0}$ is a trace, i.e., $T(ab) = T(ba)$. This is immediate since $\mathcal{B}^{\lambda_0}$ is a quotient of $\mathcal{U}(\mathfrak{g})$. Indeed $T(\pi^{\lambda_0}_x b - b \pi^{\lambda_0}_x) = 0$ by $\mathfrak{g}$-invariance and so $T(\pi^{\lambda_0}_x b - b \pi^{\lambda_0}_x) = T(b \pi^{\lambda_0}_x - \pi^{\lambda_0}_x b)$.

We can ask now if $T$ is a trace on $\mathcal{E}^{\lambda_0}$. The answer is no even for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We will show this is remedied by introducing a superalgebra structure on $\mathcal{E}^{\lambda_0}$, which is filtered in the sense that each filtration piece is the sum of its even and odd subspaces. Corollary 7.5.1 gives

Lemma 7.6.1. Let $\mathcal{E} = \mathcal{E}^{\lambda_0}$. Then $\mathcal{E} = \mathcal{E}^{\text{even}} \oplus \mathcal{E}^{\text{odd}}$ where $\mathcal{E}^{\text{even}}$ and $\mathcal{E}^{\text{odd}}$ are the $\pm 1$-eigenspaces of $\beta^2$. This makes $\mathcal{E}$ into a $G$-equivariant filtered superalgebra.

If $r$ is odd then $\mathcal{E}^{\text{even}} = \mathcal{E}^{+}$ and $\mathcal{E}^{\text{odd}} = \mathcal{E}^{-}$, but if $r$ is even then $\mathcal{E}^{\text{even}} = \mathcal{E}$ and $\mathcal{E}^{\text{odd}} = 0$.

An element $a \in \mathcal{E}$ is called superhomogeneous if $a$ lies in $\mathcal{E}^{\text{even}}$ or $\mathcal{E}^{\text{odd}}$. The parity of $a$ is then $|a| = 0$ or $|a| = 1$ respectively.

Proposition 7.6.2. Our projection $T$ is a supertrace on $\mathcal{E}^{\lambda_0}$, i.e.,

$$T(ab) = (-1)^{|a||b|} T(ba) \quad (7.7)$$

when $a$ and $b$ have same parity, while $T(ab) = 0$ when $a$ and $b$ have different parity.
Proof. Let \( \mathcal{E} = \mathcal{E}^{\lambda_0} \). Since \( \mathcal{E} \simeq R(\mathcal{M}) \) is multiplicity-free as a \( G \)-representation, there is a unique \( G \)-stable complement, call it \( \mathcal{E}^j \), to \( \mathcal{E}_{j-\frac{1}{2}} \) in \( \mathcal{E}_j \). Then \( \mathcal{E} = \oplus_{j \in \mathbb{Z}} \mathcal{E}^j \) is a \( \beta \)-stable vector space grading and \( \beta \) acts on \( \mathcal{E}^j \) by multiplication by \( i^{2j} \). So

\[
\mathcal{E}^{\text{even}} = \oplus_{j \in \mathbb{N}} \mathcal{E}^j \quad \text{and} \quad \mathcal{E}^{\text{odd}} = \oplus_{j \in \mathbb{N} + \frac{1}{2}} \mathcal{E}^j
\]  

(7.8)

The pairing \( \mathcal{P}(a, b) = T(ab) \) is \( G \)-invariant. Suppose \( j \neq k \). We know by Proposition 2.4.1 that \( \mathcal{E}^j \) and \( \mathcal{E}^k \) contain no common \( G \)-types and all \( G \)-types appearing are self-dual. It follows by Schur’s Lemma that \( \mathcal{P} \) pairs \( \mathcal{E}^j \) and \( \mathcal{E}^k \) trivially. Now suppose \( a, b \in \mathcal{E}^j \). Notice \( T(\beta(c)) = T(c) \) for any \( c \in \mathcal{E} \). So

\[
T(ab) = T(\beta(ab)) = T((\beta b)(\beta a)) = (-1)^{2j}T(ba)
\]  

(7.9)

This proves \( T \) is a supertrace.

\[\square\]

Corollary 7.6.3. The bilinear pairing \( \mathcal{P}(a, b) = T(ab) \) on \( \mathcal{E}^{\lambda_0} \) is \((G \times \mathfrak{S})\)-invariant, supersymmetric and non-degenerate.

The pairing \( \mathcal{P} \) is supersymmetric in the sense that \( \mathcal{P} \) is symmetric on \( \mathcal{D}^{\text{even}} \) and is anti-symmetric on \( \mathcal{D}^{\text{odd}} \), while \( \mathcal{P}(a, b) = 0 \) if \( a \) and \( b \) have different parity. Consequently, for any \( \beta^2 \)-stable subspace \( L \) in \( \mathcal{E} \), the right and left \( \mathcal{P} \)-orthogonal subspaces coincide, thus giving us a notion of the \( \mathcal{P} \)-orthogonal subspace \( L^\perp \). Now we say \( \mathcal{P} \) is non-degenerate on \( L \) if \( L^\perp \cap L = 0 \).

Proof. Everything is immediate except non-degeneracy. Now \( \mathcal{E}^\perp \cap \mathcal{E} = 0 \) because \( \mathcal{E} \) is simple (Corollary 7.1.1) and \( \mathcal{E}^\perp \cap \mathcal{E} \) is a 2-sided ideal in \( \mathcal{E} \) which does not contain \( 1 \).

\[\square\]

Remark 7.6.4. The multiplicity-free decomposition of \( \mathcal{E} = \mathcal{E}^{\lambda_0} \) into \( G \)-types is orthogonal for \( \mathcal{P} \). Let \( L \) be a \( G \)-type which lies in \( \mathcal{E}^j \). If \( j \in \mathbb{N} \) then \( \mathcal{P} \) is symmetric non-degenerate on \( L \) and so \( L \) is an orthogonal \( G \)-representation; if \( j \in \mathbb{N} + \frac{1}{2} \) then \( \mathcal{P} \) is anti-symmetric non-degenerate on \( L \) and so \( L \) is a symplectic \( G \)-representation.

7.7. \( \mathcal{U}(\mathfrak{g}) \)-bimodule structure of \( \mathcal{E}^{\lambda_0} \). As a first application of Proposition 7.6.2, we get a quick proof of another algebraic fact about \( \mathcal{E}^{\lambda_0} \).

Corollary 7.7.1. \((\mathcal{E}^{\lambda_0})^\dagger\), like \((\mathcal{E}^{\lambda_0})^+ = \mathcal{B}^{\lambda_0}\), is a simple bimodule over \( \mathcal{U}(\mathfrak{g}) \).

Proof. We need to show that \((\mathcal{E}^{\lambda_0})^\dagger\) is a simple bimodule over \( \mathcal{B}^{\lambda_0} = \mathcal{U}(\mathfrak{g})/J^{\lambda_0} \); we already know that \( \mathcal{B}^{\lambda_0} \) is a simple ring. Suppose \( L \) is a \( \mathcal{B}^{\lambda_0} \)-bimodule of \((\mathcal{E}^{\lambda_0})^\dagger\) and \( a \in L \) with \( a \neq 0 \). Then \( wa \in \mathcal{B}^{\lambda_0} \). Now \( \mathcal{P} \) is non-degenerate on \( \mathcal{B}^{\lambda_0} \) and so there exists \( b \in \mathcal{B}^{\lambda_0} \) such that \( \mathcal{P}(b, \omega a) = T(b \omega a) = 1 \). Then \( \mathcal{P}(ab, w) = T(abw) = -1 \) since \( T \) is a supertrace. Thus \( L \) is not \( \mathcal{P} \)-orthogonal to \( \mathcal{W}^\lambda \) (defined in 3.5). It follows, since \( \mathcal{W}^\lambda \) is self-dual and appears only once in \((\mathcal{E}^{\lambda_0})^\dagger\), that \( L \) contains \( \mathcal{W}^\gamma \). But \( \mathcal{W}^\lambda \) generates \((\mathcal{E}^{\lambda_0})^\dagger\) as a bimodule over \( \mathcal{B}^{\lambda_0} \) by Corollary 7.5.1(v). Thus \( L = (\mathcal{E}^{\lambda_0})^\dagger \).

\[\square\]
8. Dixmier product on \( R(\mathcal{M}) \)

8.1. Constructing the Dixmier product. Suppose we have a noncommutative model \((\mathcal{E}, \gamma, \pi)\) of a graded Poisson algebra \( R \) with Hamiltonian symmetry \( g \rightarrow R^1, x \mapsto \phi^x \); see Definitions 2.2.1 and 3.1.1. We say that \( q : R \rightarrow \mathcal{E} \) is an associated quantization map if (i) \( q \) is \( g \)-linear, (ii) \( q \) is filtered, and (iii) the induced map \( \text{gr} q : R \rightarrow \text{gr} \mathcal{E} \) is inverse to \( \gamma \). Here (i) means that \( q(\{\phi^x, \psi\}) = [\pi^x, q(\psi)] \) and (ii) means that \( q(R^j) \subseteq \mathcal{E}_j \). So \( q \) induces a vector space grading \( \mathcal{E} = \bigoplus_{j \in \frac{1}{2}} R^j \). In this way, we get a bijection between choices for \( q \) and \( g \)-linear gradings \( \mathcal{E} = \bigoplus_{j \in \frac{1}{2}} E^j \) which satisfy \( E_k = \bigoplus_{j \leq k} E^j \).

Our noncommutative model \((\mathcal{E}^{\lambda_0}, \tilde{\gamma}_{\lambda_0}, \pi_{\lambda_0})\) of \( R(M) \) admits a unique quantization map \( q : R(M) \rightarrow \mathcal{E}^{\lambda_0} \) because there is only one choice for the corresponding grading as \( \mathcal{E}^{\lambda_0} \) is multiplicity-free (cf. proof of Proposition 7.6.2). We now get a new associative noncommutative product \( \circ \) on \( R(M) \) defined by

\[
\phi \circ \psi = q^{-1}(q(\phi)(q(\psi)))
\]

We say that \( \circ \) is a Dixmier product because it makes \( R(M) \) into a Dixmier algebra for \( M \).

Example 8.1.1. In Example 4.3, \( q \) is Weyl symmetrization map and the circle product is the Moyal star product specialized at \( t = 1 \).

The Euler grading defines a filtration of \( R(M) \) and also the projection

\[
\mathcal{T} : R(M) \rightarrow \mathbb{C}
\]

Then \( \mathcal{T}(\phi) \) is the constant term of \( \phi \). There is a supergrading on \( R(M) \) given by

\[
R(M)^{\text{even}} = \bigoplus_{j \in \mathbb{N}} R^j(M) \quad \text{and} \quad R(M)^{\text{odd}} = \bigoplus_{j \in \mathbb{N} + \frac{1}{2}} R^j(M)
\]

These are the \( \pm 1 \)-eigenspaces of \( \alpha^2 \).

8.2. Main theorem. We can now deduce

Theorem 8.2.1. The Dixmier product \( \circ \) is \((G \times S)\)-invariant and makes \( R(M) \) into a filtered superalgebra where \((8.4)\) defines the supergrading. With respect to \( \circ \), \( \alpha \) is an anti-automorphism and \( \mathcal{T} \) is a supertrace. The bilinear pairing \( Q(\phi, \psi) = \mathcal{T}(\phi \circ \psi) \) on \( R(M) \) is \((G \times S)\)-invariant, supersymmetric, non-degenerate and orthogonal for the Euler grading.

Let \( R^j = R^j(M) \). Then, for all \( j, k \in \frac{1}{2} \mathbb{N} \),

\[
R^j \circ R^k \subseteq R^{j+k} \oplus R^{j+k-1} \oplus \cdots \oplus R^{|j-k|}
\]

Suppose \( \phi \in R^j \) and \( \psi \in R^k \) so that \( \phi \circ \psi = \sum_p C_p(\phi, \psi) \) where \( C_p(\phi, \psi) \) lies in \( R^{j+k-p} \). Then

\[
\phi \circ \psi \equiv \phi \psi + \frac{1}{2} \{\phi, \psi\} \mod R^{\leq j+k-2}
\]

\[
C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi)
\]
Proof. The map \( q \) is \( G \)-invariant, equivariant with respect to \( S \) and \( \mathcal{S} \), and also \( q \) intertwines \( \alpha \) and \( \beta \). So our results on \( E^{\lambda_0} \) transfer over to \( R(M) \) via \( q \). This proves the first paragraph.

Since \( \circ \) is a filtered superalgebra product, we have \( R^j \circ R^k \subseteq \bigoplus_{p \in \mathbb{N}} R^{j+k-p} \). Now proving (8.3) reduces to showing that if \( R^j \circ R^k \) is not \( Q \)-orthogonal to \( R^s \) then \( s \geq |j-k| \). Showing this is easy since the hypothesis means that there exist \( a \in R^j \), \( b \in R^k \) and \( c \in R^s \) such that \( T(abc) = 1 \). Then \( bc \) has a component in \( R^j \) and so \( k+s \geq j \). But also \( T(bca) = \pm 1 \) (since \( T \) is a supertrace) and so similarly \( s+j \geq k \). Hence \( s \geq |j-k| \).

Since \( \alpha \) is an anti-automorphism we find
\[
\alpha(\phi \circ \psi) = (\alpha \psi) \circ (\alpha \phi) = i^{2j+2k} \psi \circ \phi \quad (8.8)
\]

Then \( i^{-2p} C_p(\phi, \psi) = C_p(\psi, \phi) \) and this proves (8.7). Next the relations \( C_0(\phi, \psi) = \phi \psi \) and \( C_1(\phi, \psi) - C_1(\psi, \phi) = \{\phi, \psi\} \) follow since \( \text{gr} \ q : R \to \text{gr} E \) is inverse to \( \tilde{\gamma}_{\lambda_0} \). But \( C_1(\phi, \psi) = -C_1(\psi, \phi) \) and so \( C_1(\phi, \psi) = \frac{1}{2} \{\phi, \psi\} \). This proves (8.4). \[\square\]

Notice that (8.5) implies that if \( \phi \in R^j \) and \( \psi \in R^k \) then
\[
T(\phi \circ \psi) = \delta_{jk} C_{2j}(\phi, \psi) \quad (8.9)
\]

Remark 8.2.2. For \( \phi \in R^j \) and \( \psi \in R^k \) we have
\[
q(\{\phi, \psi\}) \equiv (q\phi)(q\psi) - (q\psi)(q\phi) \mod E_{j+k-2} \quad (8.10)
\]

This means that \( q \) approximately satisfies the Dirac rule that quantization converts Poisson brackets into commutators, thus \( q \) deserves to be called a “quantization map”.

Corollary 8.2.3. If \( \phi \in R^1(M) \), for instance if \( \phi = \phi^x \) where \( x \in \mathfrak{g} \), then for all \( \psi \in R(M) \) we have \( \{\phi, \psi\} = \phi \circ \psi - \psi \circ \phi \).

Proof. Let \( \psi \in R^k(M) \). Then we have \( \phi \circ \psi = \phi \psi + \frac{1}{2} \{\phi, \psi\} + C_2(\phi, \psi) \) by (8.5). Now the result is immediate because of (8.7). \[\square\]

8.3. Underlying star product. The properties of the Dixmier product given in Theorem 8.2.1 reveal an underlying star product. Here we mean star product in the usual sense, except that we drop the requirement of locality and work with regular functions (rather than all \( \mathbb{C} \)-valued smooth functions). This point of view is known in star product theory. See e.g. [C-G], [ABC], [A-B] for this and also for what it means for a star product to be graded or strongly invariant.

Corollary 8.3.1. The Dixmier product \( \circ \) is the specialization at \( t = 1 \) of a unique graded strongly \( \mathfrak{g} \)-invariant star product \( \star \) on \( R(M) \).

Proof. The graded star product is defined by \( \phi \star \psi = \sum_{p \in \mathbb{N}} C_p(\phi, \psi)t^p \) where \( \phi \) and \( \psi \) are Euler homogeneous. This is strongly \( \mathfrak{g} \)-invariant by Corollary 8.2.3. \[\square\]

Remark 8.3.2. In [B2], we lift the complex conjugation automorphism \( \sigma \) of \( O \) (induced by a Cartan involution of \( \mathfrak{g} \) which exchanges \( p^+ \) and \( p^- \)) to an antiholomorphic automorphism \( \tilde{\sigma} \) of \( M \) (of order 2 or 4 according to whether \( r \) is even or odd) such that (i) \( \tilde{\sigma} \) induces a \( \mathbb{C} \)-antilinear \( \sigma \)-algebra automorphism of \( R(M) \) and (ii) the pairing \( (\phi|\psi) = T(\phi \circ \psi^\sigma) \) is Hermitian positive-definite. (In fact \( (\cdot|\cdot) \) is Hermitian precisely because \( T \) is a supertrace.)
Then (|·|) is invariant under the Lie algebra \( \{ (x, x^\sigma) \mid x \in \mathfrak{g} \} \) and \( R(\mathcal{M}) \) becomes a unitary representation of \( G \).

8.4. The operators \( \Lambda^x \). A natural first step in understanding the Dixmier product (or the corresponding star product) is to compute the products \( \phi^x \circ \psi \) where \( x \in \mathfrak{g} \). We get a neat form for the answer because of our additional structure given by the supertrace etc.

Since our pairing \( \mathcal{Q} \) on \( R(\mathcal{M}) \) is supersymmetric and non-degenerate, it makes sense to talk about the \( \mathcal{Q} \)-adjoint of a linear endomorphism of \( R(\mathcal{M}) \). Theorem 8.2.1 gives

**Corollary 8.4.1.** Let \( \Lambda^x \) be the \( \mathcal{Q} \)-adjoint of ordinary multiplication by \( \phi^x \) where \( x \in \mathfrak{g} \). Then for every \( \psi \in R(\mathcal{M}) \) we have

\[
\phi^x \circ \psi = \phi^x \psi + \frac{1}{2} \{ \phi^x, \psi \} + \Lambda^x(\psi)
\]

(8.11)

The linear operators \( \Lambda^x \) satisfy:

(i) \( \Lambda^x \) is graded of degree \(-1\), i.e., \( \Lambda^x(R^j(\mathcal{M})) \subseteq R^{j-1}(\mathcal{M}) \).

(ii) If \( x \neq 0 \) and \( j \) is positive, then \( \Lambda^x \) is non-zero somewhere on \( R^j(\mathcal{M}) \).

(iii) The operators \( \Lambda^x \) commute, i.e., \( [\Lambda^x, \Lambda^y] = 0 \).

(iv) The operators \( \Lambda^x \) transform in the adjoint representation of \( \mathfrak{g} \), i.e., \([\Phi^x, \Lambda^y] = \Lambda^{[x,y]} \).

(v) The operators \( \Lambda^x \) commute with the action of \( \mathcal{S} \).

**Proof.** Define \( \Lambda^x \) by \( \Lambda^x(\psi) = C_2(\phi^x \psi) \). Then (8.5) and (8.6) imply (8.11). We claim \( \mathcal{Q}(\phi^x \psi_1, \psi_2) = \mathcal{Q}(\psi_1, \Lambda^x(\psi_2)) \). We may assume \( \psi_1 \in \mathcal{R}^j \) and \( \psi_2 \in \mathcal{R}^{j+1} \). Then (8.11) gives \( \mathcal{Q}(\phi^x \psi_1, \psi_2) = \mathcal{Q}(\phi^x \circ \psi_1, \psi_2) \) since the grading of \( \mathcal{R} \) is \( \mathcal{Q} \)-orthogonal. Now

\[
\mathcal{Q}(\phi^x \circ \psi_1, \psi_2) = T(\phi^x \circ \psi_1 \circ \psi_2) = T(\psi_1 \circ \psi_2 \circ \phi^x) = \mathcal{Q}(\psi_1, \psi_2 \circ \phi^x)
\]

since \( T \) is a supertrace and \( \phi^x \) and \( \psi_1 \circ \psi_2 \) are even. But \( \mathcal{Q}(\psi_1, \psi_2 \circ \phi^x) = \mathcal{Q}(\psi_1, C_2(\psi_2, \phi^x)) \) and \( C_2(\psi_2, \phi^x) = \Lambda^x(\psi_2) \) by the parity relation (8.7). This proves our claim.

Now the properties (i)-(v) of \( \Lambda^x \) follow immediately from the corresponding properties of their \( \mathcal{Q} \)-adjoints, the operators \( \psi \mapsto \phi^x \psi \). This works because of the properties of \( \mathcal{Q} \). \( \square \)

In [B2] we give a formula for the operators \( \Lambda^x \).

**Remark 8.4.2.** We conjecture that there exist commuting homogeneous degree \(-1\) algebraic differential operators \( D^x \) on \( \mathcal{O} \) and a diagonalizable \((G \times \mathcal{S})\)-invariant algebraic differential operator \( L \) on \( \mathcal{M} \) with positive real spectrum such that \( \Lambda^x = L^{-1} D^x \) as operators on \( R(\mathcal{M}) \).

In the simplest case, where \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \), the operators \( \Lambda^x \) are differential. Indeed using the results in Examples 4.3 and 3.1.1 we find that \( \mathcal{Q}(\xi^p, \xi^q) = \delta_{pq} 2^{-p} \). The operators \( \Lambda^x \) corresponding to the functions \( \xi^2, \xi^1 \) and \( \xi^2 \) are \( \frac{1}{4} \frac{\partial^2}{\partial \xi^2}, \frac{1}{4} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \) and \( \frac{1}{4} \frac{\partial^2}{\partial \xi^2} \).

If we identify \( R(\mathcal{M}) \) with \( \mathcal{E}^{\lambda_0} \) via \( q \), then the representation (3.2) becomes

\[
\Pi : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \text{End}_\mathcal{S} R(\mathcal{M}), \quad \Pi^{(x,y)}(\psi) = \phi^x \circ \psi - \psi \circ \phi^y
\]

(8.12)

Then \( \Pi^{(x,x)} = \eta^x \) and Corollary 8.4.1 gives

**Corollary 8.4.3.** For \( x \in \mathfrak{g} \) we have \( \Pi^{(x,-x)} = 2 \phi^x + 2 \Lambda^x \).
8.5. Dixmier product collapses on $\mathcal{G}$. Recall the maximal Poisson abelian subalgebra $\mathcal{G}$ of $R(\mathcal{M})$ from Corollary 2.4.3.

**Proposition 8.5.1.** If $\psi$ and $\psi'$ lie in $\mathcal{G}$ then $\psi \circ \psi' = \psi \psi'$.

**Proof.** We can easily compute the restriction to $\mathcal{G}$ of $q : R(\mathcal{M}) \to \mathcal{E}^{\lambda_0}$. We find

$$q(\phi P^b) = Pw^b$$

(8.13)

where $P \in S(p^+)$, $b \in \mathbb{N}$ and $\zeta$ was defined in (2.8). Now if $\psi = \phi P^b \xi^b$ and $\psi' = \phi' P' w^{b'}$ then

$$\psi \circ \psi' = \zeta^{-1} (PW^b P' w^{b'}) = \zeta^{-1} (PP' w^{b+b'}) = \phi PP' \xi^{b+b'} = \psi \psi'.$$

\[\square\]

**Corollary 8.5.2.** $\mathcal{G}$ is a maximal $\circ$-abelian subalgebra of $R(\mathcal{M})$.

**Proof.** We have $\psi \circ \psi' = \psi \psi'$ and so $\mathcal{G}$ is $\circ$-abelian. Suppose $\phi \in R(\mathcal{M})$ and $\phi \circ \psi = \psi \circ \phi$ for all $\psi \in \mathcal{G}$. We can write $\phi = \sum_{j=0}^{p} \phi_j$ where $\phi_j \in R^j(\mathcal{M})$ and $\phi_p \neq 0$. Since $\mathcal{G}$ is graded it follows easily that $\{\phi_p, \psi\} = 0$ for all $\psi \in \mathcal{G}$. But then $\phi_p \in \mathcal{G}$ since $\mathcal{G}$ is maximal Poisson abelian. It follows by induction on $p$ that $\phi \in \mathcal{G}$.

\[\square\]

**Corollary 8.5.3.** We have $\mathcal{H} = q(\mathcal{G})$ and so $\mathcal{H}$ is a maximal abelian subalgebra of $\mathcal{E}^{\lambda_0}$.

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