ON EXTRAPOLATION OF CARLESON MEASURES

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ABSTRACT. Extrapolation of Carleson measures is a technique originated by [LM] which has become an essential step for several important results. See for example [HL], [AHLT], [AHMTT] and more recently [HM], [HMM], and [HMMTZ]. Here we recast the proof of the extrapolation theorems in [AHLT] and [HMM] using the stopping time - generations language from [C1], [C2] or [CG] but no new idea. We give three equivalent versions of the theorem having the same proof. The first is for dyadic cubes in $[0,1]^d$, the second is an abstract formulation of the result on Ahlfors regular sets from [HM], and the third is the version in [AHLT] which follows from the other two.

1. THE DYADIC CASE

In $\mathbb{R}^d$, $d \geq 1$, let $\mathcal{D}$ denote the set of closed dyadic cubes

$$Q = \bigcup_{k=1}^{d} \{ j_k 2^{-n} \leq x_k \leq (j_k + 1)2^{-n} \} \subset Q_0 = [0,1]^d, n \in \mathbb{N}, j_k \in \mathbb{Z}.$$

For $Q \in \mathcal{D}$ write $\ell(Q) = 2^{-n}$ for its side length and $|Q| = 2^{-nd}$ for its measure and define

$$Q^* = Q \times (0, \ell(Q)] \subset \mathbb{R}^{d+1},$$

$$T(Q) = Q \times (\ell(Q)/2, \ell(Q)] = Q^* \setminus \cup \{ Q_1 : Q_1 \subsetneq Q \},$$

and

$$\mathcal{D}(Q) = \{ Q' \in \mathcal{D} : Q' \subset Q \}.$$

A Borel measure $\mu$ on $Q_0^*$ is a Carleson measure if

$$C_1(\mu) = \sup_{Q \in \mathcal{D}} \frac{\mu(Q^*)}{|Q|} < \infty.$$  

Let $\nu$ be another Borel measure on $Q_0^*$ satisfying

$$C_2(\nu) = \sup_{Q \in \mathcal{D}} \frac{\nu(T(Q))}{|Q|} < \infty,$$

a condition clearly necessary for $\nu$ to be a Carleson measure.

Theorem 1.1: Let $\mu$ and $\nu$ be Borel measures on $Q_0^*$ satisfying (1.1) and (1.2) respectively. Assume there exist constants $\delta > 0$ and $C$ such that

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\[
\nu(Q^* \backslash \bigcup_{F} Q'^*) \leq C|Q|
\]
whenever \( Q \in \mathcal{D} \) and \( \mathcal{F} \subset \mathcal{D}(Q) \) is a set of subcubes of \( Q \) with disjoint interiors for which

\[
\sup_{Q' \in \mathcal{D}(Q)} \frac{\mu(Q'^* \backslash \bigcup_{F} Q'^*)}{|Q'|} \leq \delta.
\]

Then \( \nu \) is a Carleson measure with constant

\[
C_1(\nu) \leq \left( C_2(\nu) + \frac{2^d C}{2^d - 1} \right) \frac{C_1(\mu)}{\delta}. \quad (1.5)
\]

Careful reading will show that the proof below is little more than a reformulation of the arguments in Sections 7 and 9 of [HM].

**Proof:** When proving Theorem 1.1. we can assume \( E = \bigcup_{\partial} \partial Q^* \) satisfies \( \mu(E) = \nu(E) = 0 \) by pushing some mass off each \( \partial Q^* \) and thus we can treat all rectangles as open or closed sets when taking unions or intersections.

Fix \( \delta > 0 \) for which (1.4) implies (1.3). For each \( Q \in \mathcal{D} \) we will define a subset \( U(Q) \subset Q^* \) and a pairwise disjoint family \( G_1(Q) \subset \mathcal{D}(Q) \) so that

\[
U(Q) = Q^* \backslash \bigcup_{G_1(Q)} Q'^* . \quad (1.6)
\]

Further define by induction

\[
G_n(Q) = \bigcup \{ G_1(Q') : Q' \in G_{n-1}(Q) \}
\]

and \( G_0(Q) = \{ Q \} \). Then the family \( \bigcup_{n=0}^{\infty} U(Q') : Q' \in G_n(Q) \) is pairwise disjoint and

\[
\nu(Q^*) = \sum_{n=0}^{\infty} \sum_{G_n(Q)} \nu(U(Q')). \quad (1.7)
\]

For each \( Q \) and \( n \geq 1 \) write

\[
\mathcal{E}_n(Q) = \{ Q_n \subset Q : \ell(Q_n) = 2^{-n}\ell(Q) \}.
\]

Also define

\[
\mathcal{B} = \{ Q : \mu(T(Q)) \geq \delta|Q| \}. \quad (1.8)
\]

Fix \( Q \in \mathcal{D} \). For \( Q' \subset Q \) the definitions of \( U(Q') \) and \( G_1(Q') \) depend on whether or not \( Q' \in \mathcal{B} \). When \( Q \in \mathcal{B} \) take \( G_1(Q) = \mathcal{E}_1(Q) \) and \( U(Q) = T(Q) \). Then for any \( Q \) (1.2), (1.8) and disjointness yield
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\[ \sum_{n=0}^{\infty} \sum_{B \cap G_n(Q)} \nu(U(Q')) \leq \frac{C_2}{\delta} \sum_{B \cap D(Q)} \mu(T(Q')) \leq \frac{C_1 C_2}{\delta} |Q|, \]

so that when estimating (1.7) we need only study \( Q' \notin B \).

When \( Q \in D \setminus B \) we define by induction sets \( \mathcal{F}_n(Q) \subset \bigcup_{k=1}^{n} \mathcal{E}_k(Q) \) so that

\[ \mathcal{F}_n(Q) \subset \mathcal{F}_{n+1}(Q) \]

and so that (1.4) holds for \( Q \) and

\[ \tilde{\mathcal{F}}_n(Q) = \mathcal{F}_n(Q) \cup \mathcal{B}_{n+1}(Q) \]

where

\[ \mathcal{B}_{n+1}(Q) = \mathcal{E}_{n+1}(Q) \setminus \bigcup_{Q' \in \mathcal{F}_n(Q)} \mathcal{D}(Q'). \]

Then (1.4) will hold for \( Q \) and

\[ \mathcal{F}(Q) = \bigcup_{n} \mathcal{F}_n(Q). \]

**Step I:** \( n = 1 \). Include \( B \cap \mathcal{E}_1(Q) \subset \mathcal{F}_1(Q) \) and then consider the family of subsets \( \mathcal{I} \subset \mathcal{E}_1(Q) \setminus B \) such that (1.4) holds for \( Q \) and \( \tilde{\mathcal{F}} \) where \( \tilde{\mathcal{F}} = \mathcal{I} \cup (\mathcal{E}_1(Q) \setminus B) \). This family is non-empty because it contains \( \mathcal{E}_1(Q) \setminus B \) since \( Q \notin B \). Order this set family by inclusion, let \( \mathcal{I}_1(Q) \) be a minimal element and define

\[ \tilde{\mathcal{F}}_1(Q) = \mathcal{I}_1(Q) \cup (B \cap \mathcal{E}_1(Q)). \]

Then (1.4) holds for \( \tilde{\mathcal{F}}_1(Q) \).

**Step II:** Now assume \( n \geq 1 \) and \( \mathcal{F}_n(Q) \) has been constructed. If \( \bigcup \mathcal{F}_n(Q) Q' = Q \) stop the construction for \( Q \). If not, include \( B \cap \mathcal{B}_n(Q) \subset \mathcal{F}_{n+1}(Q) \) and consider the family of subsets \( \mathcal{I} \subset \mathcal{B}_n(Q) \setminus B \) such that when

\[ \mathcal{F}(Q) = \mathcal{F}_n(Q) \cup \mathcal{I} \cup (B \cap \mathcal{B}_{n+1}(Q)) \]

(1.4) holds for \( Q \) and

\[ \tilde{\mathcal{F}}_{n+1}(Q) = \mathcal{F}(Q) \cup (\mathcal{E}_{n+2}(Q) \setminus \bigcup_{Q' \in \mathcal{F}(Q)} \mathcal{D}(Q')). \]

This family is non-empty because by induction (1.4) holds for \( Q \) and \( \tilde{\mathcal{F}} \) when \( \mathcal{I} = \mathcal{B}_{n+1}(Q) \). Order this finite set family by inclusion, let \( \mathcal{I}_{n+1}(Q) \) be a minimal member, and define \( \mathcal{F}_{n+1}(Q) \) by (1.6) with \( \mathcal{I} = \mathcal{I}_{n+1}(Q) \). Then (1.4) holds for \( Q \) and \( \tilde{\mathcal{F}}_{n+1}(Q) \).

Moreover we have:

**Lemma 1.2:** If \( Q' \in \mathcal{F}(Q) \setminus B \) there exists \( \bar{Q}' \) such that \( Q' \subset \bar{Q}' \subset Q \) and
\[ \mu(\tilde{Q}'^* \backslash \bigcup_{F(Q)} Q'^*) \geq \left(1 - \frac{1}{2^d}\right)\delta|\tilde{Q}'|. \]

**Proof:** We have \( Q' \in \mathcal{F}_n \) for some first \( n \) and by the minimality of \( S_n \) there exists \( \tilde{Q}' \supseteq Q' \) such that

\[ \mu(\tilde{Q}'^* \bigcup \{Q''^* : Q'' \in \mathcal{F}_n(Q) \cup \mathcal{E}_{n+1}(Q)\}) + \mu(T(Q')) > \delta|\tilde{Q}'| \]

while \( \mu(T(Q')) < \frac{\delta}{2^d}|\tilde{Q}'| \) since \( Q' \notin \mathcal{B} \). Therefore (1.11) holds.

When \( Q' \notin \mathcal{B} \), (1.4) holds for \( Q' \) and \( \mathcal{F}(Q') \) so that by (1.3) \( \nu(U(Q')) \leq C|Q'| \) and therefore

\[ \sum_{n} \sum_{G_n(Q) \notin \mathcal{B}} \nu(U(Q')) \leq C \sum_{n=0}^{\infty} \sum_{G_n(Q) \notin \mathcal{B}} |Q'|. \]

By Lemma 1.2 we can cover \( \bigcup_{\mathcal{F}(Q) \notin \mathcal{B}} Q' \) by the pairwise disjoint family \( \mathcal{H}(Q) \) of maximal dyadic cubes \( \tilde{Q}' \) that satisfy (1.11). Then since both families \( \mathcal{H}(Q) \) and \( \mathcal{F}(Q) \backslash \mathcal{B} \) are pairwise disjoint, (1.11) yields

\[ \sum_{\mathcal{F}(Q) \backslash \mathcal{B}} |Q'| \leq \sum_{\mathcal{H}(Q)} |\tilde{Q}'| \leq \frac{2^d}{2^d - 1} \frac{1}{\delta} \mu(U(Q)), \]

and (1.13), (1.12) and (1.1) give

\[ \sum_{n=1}^{\infty} \sum_{G_n(Q) \notin \mathcal{B}} \nu(Q') \leq \frac{2^d}{2^d - 1} \frac{C C_1}{\delta} |Q|, \]

and together (1.14) and (1.9) establish the estimate (1.5) and prove Theorem 1.1.

### 2. A GENERAL CASE

Let \( d \) be a positive integer and let \((X, \rho)\) be a metric space on which there exists a positive Borel measure \( \sigma \) that is \( d \) Ahlfors regular: there exists \( c_1 > 0 \) such that

\[ \frac{1}{c_1} R^d \leq \sigma(B(x, R)) \leq c_1 R^d \]

for all \( x \in X \) and all \( 0 < R \leq \text{diam}(X) \). We assume for simplicity that \( X \) is compact and \( \sigma(X) = 1 \). Following Christ [Ch], there exists a positive integer \( N \) and a family

\[ \mathcal{D} = \bigcup_{j=0}^{\infty} \mathcal{D}_j \]

of Borel subsets of \( X \) satisfying (2.2) - (2.6) below:
(2.2) \[ \text{diam } Q \sim 2^{-Nj} \text{ if } Q \in \mathcal{D}_j; \]

(2.3) \[ X = \bigcup_{\mathcal{D}_j} Q, \text{ for all } j; \]

(2.4) \[ Q \cap Q' = \emptyset \text{ if } Q, Q' \in \mathcal{D}_j \text{ and } Q' \neq Q; \]

(2.5) \[ \text{if } j < k, Q_j \in \mathcal{D}_j \text{ and } Q_k \in \mathcal{D}_k, \text{ then } Q_k \subset Q_j \text{ or } Q_k \cap Q_j = \emptyset. \]

There exists constant \( c_0 \) such that for all \( Q \in \mathcal{D} \) there exists \( x_Q \in Q \) with

(2.6) \[ B(x_Q, c_0 \ell(Q)) \cap \partial \Omega \subset Q. \]

Note that by (2.1), (2.2) and (2.6) there is a constant \( c_2 \) so that for all \( Q \in \mathcal{D} \),

(2.7) \[ \frac{1}{c_2} \ell(Q)^d \leq \sigma(Q) \leq c_2 \ell(Q)^d. \]

Now let \( \mu \) and \( \nu \) be positive discrete measures on the countable set \( \mathcal{D} \), so that there exist \( \alpha_Q \geq 0 \) and \( \beta_Q \geq 0 \) such that for any \( E \subset \mathcal{D} \)

\[ \mu(E) = \sum_{Q \in E} \alpha_Q \]

and

\[ \nu(E) = \sum_{Q \in E} \beta_Q. \]

Analogous to (1.1) and (1.2) we assume \( \mu \) is a discrete Carleson measure,

(2.8) \[ C_1(\mu) = \sup_{Q \in \mathcal{D}} \frac{\mu(Q^*)}{\sigma(Q)} < \infty, \]

where we write \( Q^* = \{ Q' : Q' \subseteq Q \} \), and we assume

(2.9) \[ C_2(\nu) \sup_{Q \in \mathcal{D}} \frac{\beta_Q}{\sigma(Q)} < \infty, \]

which is necessary for \( \nu \) to be a discrete Carleson measure. Then we have the following abstract version of Theorem 1.1:

**Theorem 2.1**: Let \( \mu \) and \( \nu \) be measures on \( \mathcal{D} \) satisfying (2.8) and (2.9) respectively. Assume there exist constants \( \delta > 0 \) and \( C \) such that
(2.10) \[ \nu(Q^* \setminus \bigcup_{F} Q') \leq C \sigma(Q) \]

whenever \( Q \in \mathcal{D} \) and \( \mathcal{F} \subset \mathcal{D}(Q) \) is a set of subcubes of \( Q \) for which

(2.11) \[ \sup_{Q' \subset Q} \frac{\mu(Q^* \setminus \bigcup_{F} Q'')}{\sigma(Q')} \leq \delta. \]

Then \( \nu \) is a Carleson measure with constant \( C_1(\nu) \) depending only on \( c_2, C, \delta, C_1(\mu) \) and \( C_2(\nu) \).

But for the choices of constants the proof of Theorem 2.1 is a repetition of the proof of Theorem 1.1 and we omit the details.

Theorem 2.1 can be stated more abstractly as a result about the tree \((\mathcal{D}, \subset)\) with transition probabilities \( \sigma(Q')/\sigma(Q) \) for \( Q' \subset Q \). However for the important applications in [HM], [HMM] and [HMMTZ] \( \Omega \) is a domain in \( \mathbb{R}^{d+1} \), \( E = \partial \Omega \) is uniformly rectifiable, and \( \alpha_Q \) and \( \beta_Q \) depend critically on the uniform rectifiability properties of \( E \) and on the elliptic differential equation whose solutions are being estimated. Thus the real difficulty lies in finding suitable functions \( \alpha_Q \) and \( \beta_Q \).

3. A THIRD VERSION

The “extrapolation lemma” in [AHLT] assumes \( \mu \) is a Carleson measure in \( \mathbb{R}^{d+1} \), i.e. \( \mu \) satisfies (1.1) for all cubes \( Q \subset \mathbb{R}^d \) and \( \nu \) is another Borel measure on \( \mathbb{R}^{d+1} \) satisfying

(3.1) \[ \nu(T(Q)) \leq C_2|Q|, \]

where now \( T(Q) = Q \times [\ell(Q), \ell(Q)] \subset \mathbb{R}^{d+1} \).

But instead of a cube family \( \mathcal{F} \) one now works with a nonnegative Lipschitz function \( \psi \) such that

(3.2) \[ ||\nabla \psi||_{\infty} \leq 1 \]

and the region

\[ \Omega_{\psi} = \{(x, t) \in \mathbb{R}^d \times (0, \infty) : t \geq \psi(x)\} \]

and instead of (1.4) one tests \( \mu(T_Q \setminus \Omega_{\psi}) \) where \( T_Q \) is the tent

\[ T_Q = \{(x, t) : x \in Q, 0 \leq t \leq \text{dist}(x, \mathbb{R}^d \setminus Q)\}. \]

**Theorem 3.1:** Let \( \mu \) and \( \nu \) be Borel measures on \( \mathbb{R}^{d+1} \) satisfying (1.1) and (3.1) respectively. Assume there are constants \( \delta > 0 \) and \( C > 0 \) such that whenever \( Q \subset \mathbb{R}^d \) is a cube and \( \psi : Q \to \mathbb{R} \) is a nonnegative Lipschitz function satisfying (3.2)

(3.3) \[ \nu(Q^* \cap \Omega_{\psi}) \leq C|Q| \]
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holds provided

\[(3.4) \quad \sup_{Q' \subset Q} \frac{\mu(TQ' \cap \Omega_\psi)}{|Q'|} \leq \delta \]

where the supremum is taken over the decomposition of \(Q\) into cubes having side \(2^{-n}\ell(Q), n \in \mathbb{N}\). Then \(\nu\) is a Carleson measure, i.e. for all \(Q\)

\[(3.5) \quad \nu(Q^*) \leq C|Q| \]

**Proof:** First note that (3.1) implies for all \(Q\)

\[(3.6) \quad \nu(Q^* \setminus TQ) \leq (C_2 + d)|Q| \]

so that to prove (3.4) for a fixed cube \(Q\) we may assume

\[(3.7) \quad \nu(Q^* \setminus TQ) = 0 \]

and as before we can assume \(\nu(\partial TQ' \cup T(Q')) = 0\) for all \(Q' \subset Q\). For each \(Q'\) write \(p_{Q'} = (c(Q'), \frac{\ell(Q')}{2})\) for the center of \(Q^*\) which is also the vertex of \(TQ'\) and define the discrete measures

\[\tilde{\nu} = \sum_{Q' \subset Q} \nu(T(Q'))\delta_{p_{Q'}}, \text{ and } \tilde{\mu} = \sum_{Q' \subset Q} \mu(T(Q'))\delta_{p_{Q'}}\]

where \(\delta_p\) is the unit point mass at \(p\). Then by (3.7)

\[(3.8) \quad \tilde{\nu}(Q^*) = \nu(Q^*) \text{ and } \tilde{\mu}(Q^*) = \mu(Q^*) \]

for all \(Q' \subset Q\).

To prove Theorem 3.1 we show that if (3.4) \(\implies\) (3.3) for \(\mu\) and \(\nu\) then (1.4) \(\implies\) (1.3) for \(\tilde{\mu}\) and \(\tilde{\nu}\), so that by (3.8) and Theorem 1.1, (3.5) holds for \(\nu\). To that end let \(\mathcal{F} = \{Q_1, \ldots\}\) be a pairwise disjoint family of subcubes of \(Q\) satisfying (1.4) with \(\tilde{\nu}\). Set

\[\psi_{\mathcal{F}}(x) = \sum_{\mathcal{F}} \frac{2}{\ell(Q_j)} \chi_{Q_j}(x) \text{dist}(x, \partial Q_j).\]

Then

\[Q^* \cap \Omega_\psi = (Q^* \setminus \bigcup_{\mathcal{F}} Q_j^*) \cup \bigcup_{\mathcal{F}} (Q_j^* \setminus TQ_j)\]

so that by (1.4) and (3.8) we obtain (3.4) for \(\mu\) and (3.3) for \(\nu\) and therefore (1.3) for \(\tilde{\nu}\), again by (3.8).
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