A New Condition for the Concavity Method of Blow-up Solutions to Semilinear Heat Equations

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Abstract

In this paper, we consider the semilinear heat equations under Dirichlet boundary condition

\begin{align*}
  \begin{cases}
    u_t (x, t) = \Delta u (x, t) + f(u(x, t)), & (x, t) \in \Omega \times (0, +\infty), \\
    u (x, t) = 0, & (x, t) \in \partial \Omega \times [0, +\infty), \\
    u (x, 0) = u_0 \geq 0, & x \in \Omega,
  \end{cases}
\end{align*}

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial \Omega$. The main contribution of our work is to introduce a new condition

\begin{equation}
  (C) \quad \alpha \int_0^u f(s)ds \leq uf(u) + \beta u^2 + \gamma, \quad u > 0
\end{equation}

for some $\alpha, \beta, \gamma > 0$ with $0 < \beta \leq \frac{(\alpha-2)\lambda_0}{2}$, where $\lambda_0$ is the first eigenvalue of Laplacian $\Delta$, and we use the concavity method to obtain the blow-up solutions to the semilinear heat equations. In fact, it will be seen that the condition (C) improves the conditions known so far.

Keywords: Semilinear Heat Equations, Concavity method, Blow-up.

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0. Introduction

In this paper, we discuss the blow-up solutions for the following semilinear heat equations

\begin{align*}
  \begin{cases}
    u_t (x, t) = \Delta u (x, t) + f(u(x, t)), & (x, t) \in \Omega \times (0, +\infty), \\
    u (x, t) = 0, & (x, t) \in \partial \Omega \times [0, +\infty), \\
    u (x, 0) = u_0 (x) \geq 0, & x \in \Omega,
  \end{cases}
\end{align*}

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where \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) with smooth boundary \( \partial \Omega \) and \( f \) is locally Lipschitz continuous on \( \mathbb{R} \), \( f(0) = 0 \) and \( f(u) > 0 \) for \( u > 0 \). Moreover, the initial data \( u_0 \) is a non-negative function in \( C^1(\Omega) \) satisfying that \( u_0(x) = 0 \) on \( \partial \Omega \).

The blow-up solutions to (1) have been studied by many authors. To determine of sufficient conditions for the blow-up of solutions to (1), they assumed some conditions on the nonhomogeneous term \( f \), the \( C_1 \) condition in [5], \( \int_0^\infty \frac{ds}{f(s)} < \infty \) in [3, 6], some homogeneity in [7], the nonnegativity of \( f' \) with convexity in [11], respectively. Commonly, they used a differential inequality technique and a comparison principle to obtain blow-up solutions. On the other hand, Levine and Payne [8, 9, 10] introduced a new and elegant tool for deriving estimates and giving criteria for blow-up, which called concavity method. Afterwards, using concavity method, many authors derived blow-up solution to (1) and whenever they did so, they introduced some conditions for \( f \). For example, in [12], the authors gave the condition for \( f \) as follows: for some \( \epsilon > 0 \),

\[
(2 + \epsilon) \int_0^u f(s) ds \leq uf(u), \; u > 0.
\]

(2)

In [2], the condition (2) was relaxed to

\[
(2 + \epsilon) \int_0^u f(s) ds \leq uf(u) + c^2, \; u > 0.
\]

(3)

Looking into the above conditions (2) and (3) more closely, we can see that there are independent of the eigenvalue which depends on the domain \( \Omega \). However, our main proof consists of a series of inequalities and the Poincaré inequality including the eigenvalue. From this fact, we can expect to develop an improved condition which refines (2) or (3), and depends on the domain \( \Omega \). Being motivated by this point of view, we develop a new condition as follows: for some \( \alpha, \beta, \gamma > 0 \),

\[
(C) \quad \alpha \int_0^u f(s) ds \leq uf(u) + \beta u^2 + \gamma, \; u > 0,
\]

where \( 0 < \beta \leq \left( \frac{\alpha-2}{\alpha} \right) \lambda_0 \) and \( \lambda_0 \) is the first eigenvalue of the Laplacian \( \Delta \).

The main theorem of this paper is as follows:

**Theorem.** Let the function \( f \) satisfy the condition (C). If the initial data \( u_0 \) satisfies

\[
-\frac{1}{2} \int_\Omega \left| \nabla u_0(x) \right|^2 \, dx + \int_\Omega \left[ \int_0^{u_0(x)} f(s) ds - \gamma \right] \, dx > 0,
\]

(4)

then the nonnegative classical solution \( u \) to the equation (1) blows up at finite time \( T^* \) in the sense of

\[
\lim_{t \to T^*} \int_0^t \sum_{x \in S} u^2(x, s) \, ds = +\infty,
\]

where \( \gamma \) is the constant in the condition (C).
Next, in Section 1 we discuss the blow-up solutions using concavity method with the condition (C).

1. Blow-Up: Concavity Method

In this section, we discuss the blow-up phenomena of the solution to (1) by using concavity method, which is the main part of this paper. In fact, it is well known that the equation (1) has a classical solution. (See [1] for more details).

The following lemma is used to prove the main theorem.

**Lemma 1.1** (See [4]). There exist \( \lambda_0 > 0 \) and \( \phi_0(x) \in C^2(\Omega) \) such that

\[
\begin{align*}
-\Delta \phi_0(x) &= \lambda_0 \phi_0(x), & x \in \Omega, \\
\phi_0(x) &= 0, & x \in \partial \Omega,
\end{align*}
\]

Moreover, \( \lambda_0 \) is given by

\[
\lambda_0 = \min_{u \in \mathcal{A}, u \neq 0} \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} |u(x)|^2 \, dx},
\]

where \( \mathcal{A} := \{ u : u \in C^2(\Omega), u \neq 0, u = 0 \text{ for } x \in \partial \Omega \} \).

In the above, the number \( \lambda_0 \) is the first eigenvalue of \( \Delta \) and \( \phi_0 \) is a corresponding eigenfunction.

Now, we state and prove our main result.

**Theorem.** Let the function \( f \) satisfy the condition (C). If the initial data \( u_0 \) satisfies

\[
-\frac{1}{2} \int_{\Omega} |\nabla u_0(x)|^2 \, dx + \int_{\Omega} \left[ \int_0^{u_0(x)} f(s) \, ds - \gamma \right] \, dx > 0,
\]

then the nonnegative classical solution \( u \) to the equation (1) blows up at finite time \( T^* \) in the sense of

\[
\lim_{t \to T^*} \int_0^t \sum_{x \in S} u^2(x, s) \, ds = +\infty,
\]

where \( \gamma \) is the constant in the condition (C).

**Proof.** Now, we define a functional \( J \) by

\[
J(t) := -\frac{1}{2} \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\Omega} [F(u(x, t)) - \gamma] \, dx, \quad t \geq 0,
\]

where \( F(u) := \int_0^u f(s) \, ds \).
Then by (5),

\[ J(0) = -\frac{1}{2} \int_{\Omega} |\nabla u_0(x)|^2 \, dx + \int_{\Omega} [F(u_0(x)) - \gamma] \, dx > 0. \]

and we can see that

\[ J(t) = J(0) + \int_0^t \frac{d}{dt} J(s) \, ds = J(0) + \int_0^t \int_{\Omega} u_t(x,s) \, dx \, ds. \tag{6} \]

Now, we introduce a new function

\[ I(t) = \int_0^t \int_{\Omega} u^2(x,s) \, dx \, ds + M, \quad t \geq 0, \tag{7} \]

where \( M > 0 \) is a constant to be determined later. Then it is easy to see that

\[ I'(t) = \int_{\Omega} u^2(x,t) \, dx = \int_{\Omega} \int_0^t 2u(x,s) u_t(x,s) \, ds \, dx + \int_{\Omega} u_0^2(x) \, dx. \tag{8} \]

Then we use integration by parts, the condition (C), Lemma 1.1 and (6) in turn to obtain

\[ I''(t) = \frac{d}{dt} \int_{\Omega} u^2(x,t) \, dx \]

\[ = \int_{\Omega} 2u(x,t) u_t(x,t) \, dx \]

\[ = \int_{\Omega} 2u(x,t) [\Delta u(x,t) + f(u(x,t))] \, dx \]

\[ = -2 \int_{\Omega} |\nabla u(x,t)|^2 \, dx + \int_{\Omega} 2u(x,t) f(u(x,t)) \, dx \]

\[ \geq -2 \int_{\Omega} |\nabla u(x,t)|^2 \, dx + \int_{\Omega} 2\alpha F(u(x,t)) - \beta u^2(x,t) - \alpha \gamma \, dx \]

\[ = 2\alpha \left[ -\frac{1}{2} \int_{\Omega} |\nabla u(x,t)|^2 \, dx + \int_{\Omega} [F(u(x,t)) - \gamma] \, dx \right] + (\alpha - 2) \int_{\Omega} |\nabla u(x,t)|^2 \, dx - 2\beta \int_{\Omega} u^2(x,t) \, dx \]

\[ \geq 2\alpha J(t) + (\alpha - 2) \lambda_0 - 2\beta \int_{\Omega} u^2(x,t) \, dx \]

\[ \geq 2\alpha \left[ J(0) + \int_0^t \int_{\Omega} u_t^2(x,s) \, dx \, ds \right]. \tag{9} \]
Using the Schwarz inequality, we obtain
\[ I'(t)^2 \leq 4(1 + \delta) \left[ \int \int_0^t u(x,s) u_t(x,s) \, ds \, dx \right]^2 + \left( 1 + \frac{1}{\delta} \right) \left[ \int u_0^2(x) \, dx \right]^2 \]
\[ \leq 4(1 + \delta) \left[ \int \left( \int_0^t u^2(x,s) \, ds \right)^{\frac{1}{2}} \left( \int_0^t u_t^2(x,s) \, ds \right)^{\frac{1}{2}} \, dx \right]^2 \]
\[ + \left( 1 + \frac{1}{\delta} \right) \left[ \int u_0^2(x) \, dx \right]^2 \]
\[ \leq 4(1 + \delta) \left( \int \int_0^t u^2(x,s) \, ds \, dx \right) \left( \int \int_0^t u_t^2(x,s) \, ds \, dx \right) \]
\[ + \left( 1 + \frac{1}{\delta} \right) \left[ \int u_0^2(x) \, dx \right]^2, \]
where \( \delta > 0 \) is arbitrary. Combining the above estimates (7), (9), and (10), we obtain that for \( \xi = \delta = \sqrt{\frac{\alpha}{2m}} - 1 > 0 \),
\[ I''(t) I(t) - (1 + \xi) I'(t)^2 \geq 2\alpha \left[ J(0) + \int_0^t \int_0^s u_t^2(x,s) \, ds \, dx \right] \left[ \int_0^t \int u^2(x,s) \, ds \, dx + M \right] \]
\[ - 4(1 + \xi)(1 + \delta) \left[ \int \int_0^t u^2(x,s) \, ds \, dx \right] \left[ \int \int_0^t u_t^2(x,s) \, ds \, dx \right] \]
\[ - (1 + \xi) \left( 1 + \frac{1}{\delta} \right) \left[ \int u_0^2(x) \, dx \right]^2 \]
\[ \geq 2\alpha M : J(0) - (1 + \xi) \left( 1 + \frac{1}{\delta} \right) \left[ \int u_0^2(x) \, dx \right]^2 . \]
Since \( J(0) > 0 \) by assumption, we can choose \( M > 0 \) to be large enough so that
\[ I''(t) I(t) - (1 + \xi) I'(t)^2 > 0. \] (11)
This inequality (11) implies that for \( t \geq 0 \),
\[ \frac{d}{dt} \left[ \frac{I'(t)}{I^\xi(t)} \right] > 0 \text{ i.e. } I'(t) \geq \left[ \frac{I'(0)}{I^{\xi+1}(0)} \right] I^\xi(t). \]
Therefore, it follows that \( I(t) \) cannot remain finite for all \( t > 0 \). In other words, the solutions \( u(x,t) \) blow up in finite time \( T^* \). \( \square \)

**Remark 1.2.** The above blow-up time can be estimated roughly. Taking
\[ M := \frac{\alpha - 2}{2\alpha} \left( 1 + \sqrt{\frac{\alpha}{2m}} \right) \left[ \int u_0^2(x) \, dx \right]^2 \]
\[ - \frac{1}{2} \int \int |\nabla u_0(x)|^2 \, dx + \int \int [F(u_0(x)) - \gamma] \, dx \].
we see that
\[
\begin{align*}
I'(t) & \geq \left[ \frac{\int_{\Omega} u_0^2(x) \, dx}{M^\xi} \right] I^{\xi+1}(t), \quad t > 0, \\
I(0) &= M,
\end{align*}
\]
which implies
\[
I(t) \geq \left[ \frac{1}{M^\xi} - \frac{\xi \int_{\Omega} u_0^2(x) \, dx}{M^{\xi+1}} \right]^{-\frac{1}{\xi}}
\]
where \(\xi = \sqrt{\frac{2}{\alpha}} - 1 > 0\). Then the blow-up time \(T^*\) satisfies
\[
0 < T^* \leq \frac{M}{\xi \int_{\Omega} u_0^2(x) \, dx}.
\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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