ON \textit{k}-POINT CONFIGURATION SETS WITH NONEMPTY INTERIOR

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Abstract. We give conditions for \textit{k}-point configuration sets of thin sets to have nonempty interior, applicable to a wide variety of configurations. This is a continuation of our earlier work [13] on 2-point configurations, extending a theorem of Mattila and Sjöglin [27] for distance sets in Euclidean spaces. We show that for a general class of \textit{k}-point configurations, the configuration set of a \textit{k}-tuple of sets, \( E_1, \ldots, E_k \), has nonempty interior provided that the sum of their Hausdorff dimensions satisfies a lower bound, dictated by optimizing \( L^2 \)-Sobolev estimates of associated generalized Radon transforms over all nontrivial partitions of the \( k \) points into two subsets. We illustrate the general theorems with numerous specific examples. Applications to 3-point configurations include areas of triangles in \( \mathbb{R}^2 \) or the radii of their circumscribing circles; volumes of pinned parallelepipeds in \( \mathbb{R}^3 \); and ratios of pinned distances in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Results for 4-point configurations include cross-ratios on \( \mathbb{R} \), triangle area pairs determined by quadrilaterals in \( \mathbb{R}^2 \), and dot products of differences in \( \mathbb{R}^d \).

1. Introduction

A classical result of Steinhaus [34] states that if \( E \subset \mathbb{R}^d, d \geq 1 \), has positive Lebesgue measure, then the difference set \( E - E \subset \mathbb{R}^d \) contains a neighborhood of the origin. \( E - E \) can interpreted as the set of two-point configurations, \( x - y \), of points of \( E \) modulo the translation group.

Similarly, in the context of the Falconer distance set problem, a theorem of Mattila and Sjöglin [27] states that if \( E \subset \mathbb{R}^d, d \geq 2 \), is compact, then the distance set of \( E, \Delta(E) := \{ |x - y| : x, y \in E \} \subset \mathbb{R} \), contains an open interval, i.e., has nonempty interior, if the Hausdorff dimension \( \dim_H(E) \geq \frac{d+1}{2} \). This represented a strengthening of Falconer’s original result [7], from \( \Delta(E) \) merely having positive Lebesgue measure to having nonempty interior, for the same range of \( \dim_H(E) \). This was generalized to distance sets with respect to norms on \( \mathbb{R}^d \) with positive curvature unit spheres in Iosevich, Mourgoglou and Taylor [22].

These latter types of results, for two-point configurations in \textit{thin sets}, i.e., \( E \) allowed to have Lebesgue measure zero but satisfying a lower bound on \( \dim_H(E) \), were

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extended by the current authors to more general settings in [13]: (i) configurations in $E$ as measured by a general class of $\Phi$-configurations, which can be vector-valued and nontranslation-invariant; and (ii) asymmetric configurations, i.e., between points in sets $E_1$ and $E_2$ lying in different spaces, for example between points and circles in $\mathbb{R}^2$, or points and hyperplanes in $\mathbb{R}^d$.

We point out that there are a number of other results that are explicitly, or can be interpreted as being, concerned with establishing conditions under which configuration sets of thin sets, have nonempty interior, including [3, 6, 12, 23] in the continuous setting and [5, 33, 31, 32, 4] in finite field analogues.

The purpose of the current paper is to extend our results in [13] from 2-point to quite general $k$-point configuration sets, for $k \geq 3$, using that paper’s Fourier integral operator (FIO) approach, making use of linear $L^2$-Sobolev estimates, but now optimizing over all possible nontrivial partitions of the $k$ points into two subsets. The FIO method we describe works in the absence of symmetry and on general manifolds, and indeed, exploring that generality, rather than sharpness of the lower bounds on the Hausdorff dimensions, is the focus of the current work.

However, we will start by illustrating the variety of what can be obtained via this approach with configurations defined by classical geometric quantities in low dimensional Euclidean spaces. We describe a number of concrete examples, but emphasize that the choice of these specific configurations is arbitrary; our general results can be applied to other configurations of interest, with the Hausdorff dimension threshold guaranteeing that the configuration set has nonempty interior depending on the outcome of optimizing over a family of FIO estimates; see Thms. 2.1 and 5.2 for exact statements. Our first example is the following.

**Theorem 1.1.** (Areas and circumradii of triangles) If $E \subset \mathbb{R}^2$ is compact with $\dim_H (E) > 5/3$, then

(i) the set of areas of triangles determined by triples of points of $E$,

\[
\left\{ \frac{1}{2} |\det [x - z, y - z]| : x, y, z \in E \right\} \subset \mathbb{R},
\]

contains an open interval; and

(ii) the set of radii of circles determined by triples of points in $E$ contains an open interval.

We will see in Remark 4.1 that the FIO method does not yield a pinned version of Theorem 1.1, i.e., says nothing about two-point configuration sets,

\[
\left\{ \frac{1}{2} |\det [x - z, y - z]| : x, y \in E \right\},
\]
for a fixed \( z \in E \). However, in dimension \( d \geq 3 \) it does yield a result for all \( z \in E \) (or, indeed, any \( z \in \mathbb{R}^d \)), and we have the following \( d \)-point configuration result.

**Theorem 1.2.** *(Strongly pinned volumes)* Let \( d \geq 3 \). If \( E \subset \mathbb{R}^d \) is compact, then for any \( x^0 \in \mathbb{R}^d \), the set of volumes of parallelepipeds determined by \( x^0 \) and \( d \)-tuples of points of \( E \),

\[
V^x_0(E) := \left\{ \left\lvert \det \left[ x^1 - x^0, x^2 - x^0, \ldots, x^d - x^0 \right] \right\rvert : x^1, x^2, \ldots, x^d \in E \right\},
\]

has nonempty interior in \( \mathbb{R} \) if \( \dim_H(E) > d - 1 + (1/d) \).

**Remark 1.3.** For \( d = 3 \), this improves upon an earlier result of the first two authors and Mourgoglou [11], which was that if \( \dim_H(E) > 13/5 \), then \( V^0_3(E) \) has positive Lebesgue measure.

Returning to three-point configurations, our method also yields a result about ratios of distances. S. Mkrtchyan and the first two named authors studied in [10] the existence of similarities of \( k \)-point configurations in thin sets. They posed the question of whether, under some lower bound restriction on \( \dim_H(E) \), for every \( r > 0 \) there exist \( x, y, z \in E \) such that \( |x - z| = r|y - z| \). We can partially address this, showing that the set of such \( r \) at least contains an interval.

To put this in perspective, note that an immediate consequence of the result of Mattila and Sjölin [27] is that if \( \dim_H(E) > (d + 1)/2 \), then

\[
\text{int} \left( \left\{ \frac{|w - z|}{|x - y|} : x, y, z, w \in E \right\} \right) \neq \emptyset.
\]

(See also [20] for a finite field analogue.) On the other hand, Peres and Schlag [30] showed that if \( \dim_H(E) > (d + 2)/2 \), a stronger property holds:

\[
\text{there exists an } x \in E \text{ s.t. } \text{int} \left( \left\{ \frac{|x - z|}{|x - y|} : y, z \in E \right\} \right) \neq \emptyset.
\]

(See also [24, 21] for extensions of this.) Here we prove a result for a property of intermediate strength, one which implies (1.3) but is in turn implied by (1.4); however, in dimensions \( d = 2, 3 \) our result is proved for lower \( \dim_H(E) \) than the known range for (1.4):

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1The term pinned is often used to refer to estimates for the supremum over \( x^0 \in E \) of expressions such as (1.2). Here we obtain a result valid for all \( x^0 \), hence our adoption of strongly pinned for lack of a better term.
Theorem 1.4. (Ratios of pairs of pinned distances) Let $d \geq 2$ and $E \subset \mathbb{R}^d$ compact. Then, if $\dim_H(E) > (2d + 1)/3,$

$$\text{(1.5)} \quad \text{int} \left( \left\{ \frac{|x - z|}{|x - y|} : x, y, z \in E, x \neq y \right\} \right) \neq \emptyset.$$ 

We now turn from three-point configurations to a pair of results concerning four-point configurations, in $\mathbb{R}$ and $\mathbb{R}^2,$ resp.

Theorem 1.5. (Cross ratios) Let $E \subset \mathbb{R}$ be compact with $\dim_H(E) > 3/4.$ Then the set of cross ratios of four-tuples of points of $E,$

$$\text{Cross}(E) = \left\{ [x_1, x_2; x_3, x_4] = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} : x_1, x_2, x_3, x_4 \in E \right\} \subset \mathbb{R},$$

contains an open interval.

So far, all of the configurations described have been measured by scalar-valued functions. Returning to $d = 2,$ an example of a vector-valued configuration is a variation of Theorem 1.1, where one takes four points in the plane, say $x, y, z, w,$ and considers the quadrilateral they generate. Pick one of the two diagonals, say $yw;$ this splits the quadrilateral into two triangles, and we study the vector-valued configuration consisting of their areas.

Theorem 1.6. (Pairs of areas of triangles) If $E \subset \mathbb{R}^2$ is a compact set and has $\dim_H(E) > 7/4,$ then the set of pairs of areas of triangles determined by 4-tuples of points of $E,$

$$\text{(1.6)} \quad \left\{ \left( \frac{1}{2} |\det [x - w, y - w]|, \frac{1}{2} |\det [y - w, z - w]| \right) : x, y, z, w \in E \right\} \subset \mathbb{R},$$

has nonempty interior in $\mathbb{R}^2.$

Finally, we give two more applications of the FIO method, this time to configurations with a more additive combinatorics flavor. There has been considerable work on products of differences in the discrete or finite field setting; just a few references are $[16, 1, 2, 28, 29].$ An analogue of some of these results in the continuous setting is the following

Theorem 1.7. For $d \geq 1$ and $E \subset \mathbb{R}^d$ compact, the set of dot products of differences of points in $E,$

$$\left\{ (x - y) \cdot (z - w) : x, y, z, w \in E \right\} \subset \mathbb{R},$$

has nonempty interior if $\dim_H(E) > \frac{d}{2} + \frac{1}{3}.$
Another result of sum-product type is

**Theorem 1.8.** (Generalized sum-product sets) Let $Q_1, \ldots, Q_l$ be nondegenerate, symmetric bilinear forms on $\mathbb{R}^d$. Suppose that $E_i \subset \mathbb{R}^d$ are compact sets with $\dim_H(E_i) > \frac{d}{2} + \frac{1}{2l}$ for all $1 \leq i \leq 2l$. Then the set of values

$$\Sigma_{Q}(E_1, \ldots, E_{2l}) := \left\{ \sum_{j=1}^{l} Q_j(x^{2j-1}, x^{2j}) : x^i \in E_i, 1 \leq i \leq 2l \right\} \subset \mathbb{R}$$

has nonempty interior. In particular, taking all of the $Q_j(x, y) = x \cdot y$, under the same conditions on the $\dim_H(E_i)$, the sum-(Euclidean inner) product set of the $E_i$,

$$\left\{ (x^1 \cdot x^2) + (x^3 \cdot x^4) + \cdots + (x^{2l-1} \cdot x^{2l}) : x^i \in E_i, 1 \leq i \leq 2l \right\},$$

has nonempty interior.

**Remark 1.9.** This follows from a more general result allowing the forms to be on spaces of different dimensions; see Thm. 6.1.

### 2. Three-point configurations

To describe a general class of $k$-point configurations which includes the examples above, we start by recalling the framework of $\Phi$-configuration sets, introduced by Grafakos, Palsson and the first two authors in [8]. We used this approach to establish nonempty interior results for 2-point configuration sets in the current article’s prequel, [13]. To minimize the notation, we initially describe these for 3-point configurations, introducing the basic method and results, which will be extended to higher $k$ in Sec. 5.

A 3-point configuration function is initially a smooth $\Phi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^p$ (with $p \leq d$); we use the notation $\Phi(x^1, x^2, x^3)$, $x^j \in \mathbb{R}^d$, $j = 1, 2, 3$. Since, for many problems of interest there are points, often corresponding to degenerate configurations, where $\Phi$ has critical points or fails to be smooth, it is useful to restrict the domain of $\Phi$. Anticipating the extension to $k$-point configurations later, we label the three copies of $\mathbb{R}^d$ (or open subsets of $\mathbb{R}^d$) as $X^1, X^2, X^3$. Furthermore, for some applications it is useful to allow the $X^j$ to be manifolds, of possibly different dimensions, $d_j$, $j = 1, 2, 3$. Thus, in general we define a 3-point configuration function to be a mapping $\Phi : X^1 \times X^2 \times X^3 \to T$, where $T \subset \subset \mathbb{R}^p$, or even a $p$-dimensional manifold, containing the range of $\Phi$ on the compact sets of interest. Function spaces on the $X^j$ are with respect to smooth densities, which do not play a significant role and therefore are suppressed in the notation.

For compact sets $E_j \subset X^j$, $j = 1, 2, 3$, define the 3-point $\Phi$-configuration set of $E_1, E_2, E_3$,

$$\Delta_{\Phi}(E_1, E_2, E_3) := \left\{ \Phi(x^1, x^2, x^3) : x^j \in E_j, j = 1, 2, 3 \right\} \subset T.$$
The goal is to find conditions on \( \dim_H(E_j) \) ensuring that \( \text{int}(\Delta_{\Phi}(E_1, E_2, E_3)) \neq \emptyset \).

If the full differential \( D_{x^1, x^2, x^3} \Phi \) has maximal rank (= \( p \)) everywhere, i.e., \( \Phi \) is a submersion, then \( \Phi \) is a defining function for a family of smooth surfaces in \( X := X^1 \times X^2 \times X^3 \), and for each \( t \in T \), the level set

\[
Z_t := \{(x^1, x^2, x^3) \in X : \Phi(x^1, x^2, x^3) = t\}
\]

(2.2) is smooth and of codimension \( p \) in \( X \), and \( Z_t \) depends smoothly on \( t \). (For \( p = 1 \), we denote \( t \) by simply \( t \).

If \( s_j < \dim_H(E_j) \), let \( \mu_j \) be a Frostman measure on \( E_j \) with finite \( s_j \)-energy. (See the discussion in Sec. 3 below.) The choice of the \( \mu_j \) induces a configuration measure, \( \nu \), on \( T \), having various equivalent definitions, e.g., for \( g \in C_0(T) \),

\[
\int_T g(t) \, d\nu(t) = \int \int \int_{E_1 \times E_2 \times E_3} g(\Phi(x^1, x^2, x^3)) \, d\mu_1(x^1) \, d\mu_2(x^2) \, d\mu_3(x^3).
\]

(2.3) If one can show that \( \nu \) is absolutely continuous with respect to Lebesgue measure, \( dt \); its density function is continuous; and \( \Delta_{\Phi}(E_1, E_2, E_3) \) is nonempty, then it follows that \( \text{int}(\Delta_{\Phi}(E_1, E_2, E_3)) \neq \emptyset \).

Following the general approach of [13], but now exploiting the fact that we are studying 3-point, rather than 2-point, configurations, we will derive the continuity of the density of \( d\nu \) (denoted \( \nu(t) \)) from \( L^2 \)-Sobolev mapping properties of any of three different families of generalized Radon transforms associated to \( Z_t \), as follows.

Write a nontrivial partition of \( \{1, 2, 3\} \) as \( \sigma = (\sigma_L|\sigma_R) \), grouping the variable(s) \( x^i \) corresponding to \( i \in \sigma_L \) on the left and the variable(s) corresponding to \( i \in \sigma_R \) on the right. Due to the symmetry of \( L^2 \)-Sobolev estimates for FIOs under adjoints, we may assume that \( |\sigma_L| = 2 \), \( |\sigma_R| = 1 \); furthermore, permutation within \( \sigma_L \) is irrelevant, so up to interchange of those two indices, there are three such partitions, \( \sigma = (12|3), (13|2) \) and \( (23|1) \). Corresponding to each of these, for each \( t \in T \), partitioning and permuting the variables according to \( \sigma \), the surface \( Z_t \) defines incidence relations,

\[
Z_{t}^{12|3} := \{(x^1, x^2; x^3) : (x^1, x^2, x^3) \in Z_t \} \subset (X^1 \times X^2) \times X^3,
\]

\[
Z_{t}^{13|2} := \{(x^1, x^3; x^2) : (x^1, x^3, x^2) \in Z_t \} \subset (X^1 \times X^3) \times X^2,
\]

\[
Z_{t}^{23|1} := \{(x^2, x^3; x^1) : (x^2, x^3, x^1) \in Z_t \} \subset (X^2 \times X^3) \times X^1.
\]

Each \( Z_t^{\sigma} \), with \( \sigma = (ij|k) \), defines an incidence relation from \( X^k \) to \( X^i \times X^j \), and to this is associated a generalized Radon transform, \( R_{t}^{\sigma} \); all the \( R_{t}^{\sigma} \) have the ‘same’ Schwartz kernel, namely the singular measure supported on \( Z_t \),

\[
\lambda_{t} := \chi(x^1, x^2, x^3) \cdot \delta \left( \Phi(x^1, x^2, x^3) - t \right),
\]
except that the order and grouping of the variables are dictated by $\sigma$. I.e., the kernel of $\mathcal{K}_{ij|k}^t(x^i, x^j, x^k) := \lambda(x^1, x^2, x^3)$. (Here $\chi$ is a fixed cutoff function $\equiv 1$ on $E_1 \times E_2 \times E_3$ which plays no further role.)

For each $\sigma$, we can formulate the double fibration condition, $(DF)_{\sigma}$, standard in the theory of generalized Radon transforms and originating in the works of Gelfand; Helgason [17]; and Guillemin and Sternberg [14, 15], namely that the two spatial projections from $Z_{\sigma}^t$ have maximal rank, namely

$$ (DF)_{\sigma} \quad \pi_{ij} : Z_{\sigma}^t \to X^i \times X^j \text{ and } \pi_k : Z_{\sigma}^t \to X^k \text{ are submersions.} $$

This implies that not only does $\mathcal{R}_t^\sigma : D(X^k) \to \mathcal{E}(X^i \times X^j)$, but also

$$ \mathcal{R}_t^\sigma : \mathcal{E}'(X^k) \to D'(X^i \times X^j), $$

defined weakly by

$$ \mathcal{R}_t^\sigma f(x^i, x^j) = \int_{\{x^k : \Phi(x^1, x^2, x^3) = t\}} f(x^k), $$

where the integral is with respect to the surface measure induced by $\lambda_t$ on the codimension $p$ surface $\{x^k : \Phi(x^1, x^2, x^3) = t\} \subset X^k$.

An alternate description of the configuration measure defined by (2.3) is in terms of the $\mathcal{R}_t^\sigma$; this was stated and proved in the case of 2-point configuration measures in [13, Sec. 3]. However, the proof there goes over with minor modifications to the case of $k$-point configurations, and for completeness we give the argument for $k = 3$ in Sec. 3.4 below. Namely, as long as the terms in the two arguments of the $\langle \cdot, \cdot \rangle$ pairing below belong to Sobolev spaces on which the bilinear pairing is continuous, $\nu$ has a density given by

$$ \nu(t) = \left\langle \mathcal{R}_t^{i|j}(\mu_k), \mu_i \times \mu_j \right\rangle. $$

Now, under the double fibration condition $(DF)_{\sigma}$, the generalized Radon transform $\mathcal{R}_t^\sigma$ is a Fourier integral operator (FIO) associated with a canonical relation

$$ C_t^\sigma \subset \left(T^* (X^i \times X^j) \setminus 0\right) \times \left(T^* X^k \setminus 0\right), $$

where $C_t^\sigma = (N^* Z_t^\sigma)'$, the (twisted) conormal bundle of $Z_t^\sigma$ (see Sec. 3). All three of $\mathcal{R}_t^{(1|2)(3)}, \mathcal{R}_t^{(1|3)(2)}, \mathcal{R}_t^{(2|3)(1)}$ are Fourier integral operators of the same order,

$$ m = 0 + \frac{1}{2}p - \frac{1}{4}(d_1 + d_2 + d_3) = \frac{p}{2} - \frac{1}{4}d_{\text{tot}}, \quad d_{\text{tot}} := d_1 + d_2 + d_3. $$

However, due to the (possibly) different dimensions, in order to understand the optimal estimates for the operators $\mathcal{R}_t^\sigma$, one knows from standard FIO theory that the estimates are conveniently expressed in terms of what we will call their effective
orders, $m_{\text{eff}}^\sigma$. These are defined by writing $m$ in three different ways, accounting for the dimension differences $|\dim (X^i \times X^j) - \dim (X^k)|$:

$$m = m_{\text{eff}}^{(12|3)} - \frac{1}{4}|d_1 + d_2 - d_3|,$$

$$m = m_{\text{eff}}^{(13|2)} - \frac{1}{4}|d_1 + d_3 - d_2|,$$

$$m = m_{\text{eff}}^{(23|1)} - \frac{1}{4}|d_2 + d_3 - d_1|.$$  

In terms of the $m_{\text{eff}}^\sigma$, the mapping properties of the operators $\mathcal{R}_t^\sigma$ can be described as

$$\mathcal{R}_t^\sigma : L_r^2 \rightarrow L_{r - m_{\text{eff}}^\sigma - \beta_t^\sigma}, \forall r \in \mathbb{R},$$

for certain (possible) losses $\beta_t^\sigma \geq 0$. If, for some value $t_0 \in T$, $C_{t_0}^\sigma$ is nondegenerate, i.e., one of its two natural projections to the left or right, $\pi_L$ or $\pi_R$, is of maximal rank (which implies that the other is as well), then $\beta_{t_0}^\sigma = 0$, and by structural stability of submersions this is also true for all $t$ near $t_0$ (see Sec. 3.) Our basic assumption is that, for at least one $\sigma$, there is a known $\beta^\sigma \geq 0$ such that $\mathcal{R}_t^\sigma : L_r^2 \rightarrow L_{r - m_{\text{eff}}^\sigma - \beta^\sigma}$ uniformly for $t \in T$.

To simplify the arithmetic, assume that for all the $\sigma = (ij|k)$, we have $d_i + d_j \geq d_k$, which includes the equidimensional case, $d_1 = d_2 = d_3$. Then $m_{\text{eff}}^{(ij|k)} = (p - d_k)/2$, and thus our basic boundedness assumption is that, for at least one of the $\sigma$,  

$$\mathcal{R}_t^\sigma : L_r^2(X^k) \rightarrow L_{r + \frac{1}{2}(d_k - p) - \beta_t^\sigma}(X^i \times X^j) \text{ uniformly in } t.$$  

At the start of the argument, the $s_j$, $j = 1, 2, 3$, were chosen to be any values such that $\dim(H(E_j)) > s_j$, and each $\mu_j$ has finite $s_j$ energy, so that $\mu_j \in L_{(s_j - d_j)/2}(X^j)$. An easy calculation with Sobolev norms shows that if $u_j \in L_{r_j}^2(\mathbb{R}^{d_j})$ with $r_j \leq 0$, $j = 1, 2$, then $u_1 \otimes u_2 \in L_{r_1 + r_2}^2(\mathbb{R}^{d_1 + d_2})$, and this extends to compactly supported distributions on manifolds (see Prop. 3.2 below). Thus,

$$\mu_1 \times \mu_2 \in L_{(s_1 + s_2 - d_1 - d_2)/2}, \quad \mu_1 \times \mu_3 \in L_{(s_1 + s_3 - d_1 - d_3)/2}, \quad \text{and} \quad \mu_2 \times \mu_3 \in L_{(s_2 + s_3 - d_2 - d_3)/2}.$$

Combining all of these considerations, and focusing on $\sigma = (12|3)$ for the moment, we see that the bilinear pairing in the expression (2.6) for $\nu(t)$ is continuous if

$$(s_3 - d_3)/2 + (d_3 - p)/2 - \beta^{(12|3)} + (s_1 + s_2 - d_1 - d_2)/2 \geq 0,$$

i.e.,

$$s_1 + s_2 + s_3 \geq d_1 + d_2 + p + 2\beta^{(12|3)}.$$  

The analogous calculation holds for whichever of the $\mathcal{R}_t^\sigma$ one knows estimates for, and the minimum over $\sigma$ of the right hand sides gives a sufficient condition for $\nu(t)$.
to be continuous. Thus, the set where \( \nu(t) > 0 \) is an open set; to conclude that \( \text{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset \), it suffices to show that \( \Delta_\Phi(E_1, E_2, E_3) \) itself is nonempty.

As in \cite{13}, this follows by noting that what we have done above already implies the Falconer-type conclusion that \( \Delta_\Phi(E_1, E_2, E_3) \subset \mathbb{R}^p \) has positive Lebesgue measure. In fact, if \( \{B(t_j, \epsilon_j)\} \) is any cover of \( \Delta_\Phi(E_1, E_2, E_3) \), one has

\[
1 = \mu_1(E_1) \cdot \mu_2(E_2) \cdot \mu_3(E_3) = (\mu_1 \times \mu_2 \times \mu_3)(E_1 \times E_2 \times E_3)
\]

\[
\leq (\mu_1 \times \mu_2 \times \mu_3)\left(\Phi^{-1}\left(\bigcup_j B(t_j, \epsilon_j)\right)\right)
\]

\[
\leq \sum_j (\mu_1 \times \mu_2 \times \mu_3)\left(\Phi^{-1}(B(t_j, \epsilon_j))\right)
\]

\[
(2.8) = \sum_j \nu(B(t_j, \epsilon_j)) \leq C_\Phi \sum_j \epsilon_j^p
\]

by (3.6) below, so that \( \sum_j |B(t_j, \epsilon_j)|_p \geq C'_\Phi \) is bounded below. Hence \( \Delta_\Phi(E_1, E_2, E_3) \) has positive \( p \)-dimensional Lebesgue measure and is therefore nonempty; by the continuity of \( \nu(t) \), it in fact has nonempty interior.

Summarizing, we have established the following method for proving that 3-point configuration sets have nonempty interior:

**Theorem 2.1.** (i) With the notation and assumptions as above, define

\[
s_\Phi = p + \min \left( d_1 + d_2 + 2\beta^{(12|3)}, d_1 + d_3 + 2\beta^{(13|2)}, d_2 + d_3 + 2\beta^{(23|1)} \right),
\]

where the min is taken over those of the partitions \( \sigma = (ij|k) \) for which

(a) the double fibration condition \((DF)_\sigma \) (2.5) holds, and

(b) one has uniform boundedness of the generalized Radon transforms \( \mathcal{R}_\sigma^t \) with loss of \( \leq \beta^\sigma \) derivatives (2.7).

Then, if \( E_j \subset X^j \) are compact sets with \( \text{dim}_\mathcal{H}(E_1) + \text{dim}_\mathcal{H}(E_2) + \text{dim}_\mathcal{H}(E_3) > s_\Phi \), it follows that \( \text{int}(\Delta_\Phi(E_1, E_2, E_3)) \neq \emptyset \).

(ii) In particular, suppose that \( X^1 = X^2 = X^3 = X \), with \( \text{dim}(X) = d \), and there is a partition \( \sigma = (ij|k) \) such that (a) holds and the canonical relations \( C_\sigma^t \) are nondegenerate (so that \( \beta^\sigma = 0 \)). It follows that, if \( E \subset X \) is compact with \( \text{dim}_\mathcal{H}(E) > (2d + p)/3 \), then \( \text{int}(\Delta_\Phi(E, E, E)) \neq \emptyset \).
3. Background material

We give a brief survey of the relevant facts needed in the paper, referring for more background and further details to Hörmander [18, 19] for Fourier integral operator theory, Mattila [25, 26] for geometric measure theory, and [13] for the case of 2-point configurations.

3.1. Fourier integral operators. Let $X$ and $Y$ be smooth manifolds of dimensions $n_1, n_2$, resp. Then $T^*X$, $T^*Y$ are each symplectic manifolds, with canonical two-forms denoted $\omega_{T^*X}$, $\omega_{T^*Y}$, resp. Equip $T^*X \times T^*Y$ with the difference symplectic form, $\omega_{T^*X} - \omega_{T^*Y}$. For our purposes, a canonical relation will mean a submanifold, $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ (hence of dimension $n_1 + n_2$), which is conic Lagrangian with respect to $\omega_{T^*X} - \omega_{T^*Y}$.

For some $N \geq 1$, let $\phi : X \times Y \times (\mathbb{R}^N \setminus 0) \to \mathbb{R}$ be a smooth phase function which is positively homogeneous of degree 1 in $\theta \in \mathbb{R}^N$, i.e., $\phi(x, y, \tau \theta) = \tau \cdot \phi(x, y, \theta)$ for all $\tau \in \mathbb{R}_+$. Let $\Sigma_\phi$ be the critical set of $\phi$ in the $\theta$ variables,

$$\Sigma_\phi := \{(x, y, \theta) \in X \times Y \times (\mathbb{R}^N \setminus 0) : d_\theta \phi(x, y, \theta) = 0\},$$

and

$$C_\phi := \{(x, d_x \phi(x, y, \theta); y, -d_y \phi(x, y, \theta)) : (x, y, \theta) \in \Sigma_\phi\},$$

both of which are conic sets. If we impose the first order nondegeneracy conditions

$$d_x \phi(x, y, \theta) \neq 0 \text{ and } d_y \phi(x, y, \theta) \neq 0, \forall (x, y, \theta) \in \Sigma_\phi,$$

then $C_\phi \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$. If in addition one demands that

$$\text{rank}[d_{x,y,\theta} d_\theta \phi(x, y, \theta)] = N, \forall (x, y, \theta) \in \Sigma_\phi,$$

then $\Sigma_\phi$ is smooth, $\dim(\Sigma_\phi) = n_1 + n_2$, and the map

$$(3.1) \quad \Sigma_\phi \ni (x, y, \theta) \to (x, d_x \phi(x, y, \theta); y, -d_y \phi(x, y, \theta)) \in C_\phi$$

is an immersion, whose image is an immersed canonical relation; the phase function $\phi$ is said to parametrize $C_\phi$.

For a canonical relation $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ and $m \in \mathbb{R}$, one defines $I^m(X,Y; C) = I^m(C)$, the class of Fourier integral operators $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ of order $m$, as the collection of operators whose Schwartz kernels are locally finite sums of oscillatory integrals of the form

$$K(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(x, y, \theta) d\theta,$$

where $a(x, y, \theta)$ is a symbol of order $m - N/2 + (n_1 + n_2)/4$ and $\phi$ is a phase function as above, parametrizing some relatively open $C_\phi \subset C$. 
The FIO relevant for this paper are the generalized Radon transforms \( \mathcal{R}_t \) determined by defining functions \( \Phi : X \times Y \to \mathbb{R}^p \) satisfying the double fibration condition that \( D_x \Phi \) and \( D_y \Phi \) have maximal rank. The Schwartz kernel of each \( \mathcal{R}_t \) is a smooth multiple of \( \delta_p(\Phi(x,y) - t) \), where \( \delta_p \) is the delta distribution on \( \mathbb{R}^p \). From the Fourier inversion representation of \( \delta_p \), we see that \( \mathcal{R}_t \) has kernel

\[
K_t(x,y) = \int_{\mathbb{R}^k} e^{i(\Phi(x,y) - t) \cdot \theta} b(x,y) \cdot 1(\theta) d\theta,
\]

where \( b \in C_0^\infty \). Since the amplitude is a symbol of order 0, \( \mathcal{R}_t \) is an FIO of order \( 0 + p/2 - (n_1 + n_2)/4 = -(n_1 + n_2 - 2p)/4 \) associated with the canonical relation parametrized as in (3.1) by \( \phi(x,y,\theta) = (\Phi(x,y) - t) \cdot \theta \), which is the twisted conormal bundle of the incidence relation \( Z_t \),

\[
C_t = N^*Z_t := \left\{ \left( x, \sum_{j=1}^{k} d_x \Phi_j(x,y) \theta_j, y, -\sum_{j=1}^{k} d_x \Phi_j(x,y) \theta_j \right) : (x,y) \in Z_t, \theta \in \mathbb{R}^k \setminus 0 \right\}.
\]

For \( T \)-valued defining functions \( \Phi \), as in the general formulation of our results, this discussion is easily modified by introducing local coordinates on \( T \).

For a general canonical relation, \( C \), the natural projections \( \pi_L : T^*X \times T^*Y \to T^*X \) and \( \pi_R : T^*X \times T^*Y \to T^*Y \) restrict to \( C \), and by abuse of notation we refer to the restricted maps with the same notation. One can show that, at any point \( c_0 = (x_0,\xi_0; y_0,\eta_0) \in C \), one has \( \text{corank}(D\pi_L)(c_0) = \text{corank}(D\pi_R)(c_0) \); we say that the canonical relation \( C \) is nondegenerate if this corank is zero at all points of \( C \), i.e., if \( D\pi_L \) and \( D\pi_R \) are of maximal rank. If \( \dim(X) = \dim(Y) \), then \( C \) is nondegenerate iff \( \pi_L, \pi_R \) are local diffeomorphisms, and then \( C \) is a local canonical graph, i.e., locally near any \( c_0 \in C \) is equal to the graph of a canonical transformation. If \( \dim(X) = n_1 > n_2 = \dim(Y) \), then \( C \) is nondegenerate iff \( \pi_L \) is an immersion and \( \pi_R \) is a submersion. To describe the \( L^2 \)-Sobolev estimates for FIOs, it is convenient to normalize the order and consider \( A \in I^{m_{\text{eff}} - \lfloor m_{\text{eff}}/4 \rfloor}(C) \). One has

**Theorem 3.1.** [18, 19] Suppose that \( C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0) \) is a canonical relation, where \( \dim(X) = n_1, \dim(Y) = n_2 \), and \( A \in I^{m_{\text{eff}} - \lfloor m_{\text{eff}}/4 \rfloor} \) has a compactly supported Schwartz kernel.

(i) If \( C \) is nondegenerate, then \( A : L^2_s(Y) \to L^2_{s-m_{\text{eff}}}(X) \) for all \( s \in \mathbb{R} \). Furthermore, the operator norm depends boundedly on a finite number of derivatives of the amplitude and phase function.

(ii) If the spatial projections from \( C \) to \( X \) and to \( Y \) are submersions and, for some \( l \), the corank of \( D\pi_L \) (and thus that of \( D\pi_R \)) is \( \leq l \) at all points of \( C \), then \( A : L^2_s(Y) \to L^2_{s-m_{\text{eff}}-(l/2)}(X) \).
3.2. **Frostman measures and s-energy.** Also recall (see Mattila [25, 26]) that if $E \subset \mathbb{R}^d$ is a compact set and $0 < s < d$ satisfies $s < \dim_H(E)$, then there exists a Frostman measure on $E$ relative to $s$: a probability measure $\mu$, supported on $E$, satisfying the ball condition

$$\mu(B(x, \delta) \lesssim \delta^s, \forall x \in \mathbb{R}^d, 0 < \delta < 1,$$

and of finite $s$-energy,

$$\int_E \int_E |x - y|^{-s} d\mu(x) d\mu(y) < \infty,$$

or equivalently,

$$\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \cdot |\xi|^{s-d} d\xi < \infty.$$

Since $\mu$ is of compact support, $\hat{\mu} \in C^\infty$ and thus (3.3) implies that

$$\mu \in L^2_{(s-d)/2}(\mathbb{R}^d).$$

This also holds in the general setting of $E \subset X$, a compact subset of a $d$-dimensional manifold $X$ with $\dim_H(E) > s$.

3.3. **Tensor products of Sobolev spaces.** We need an elementary result on the tensor products of Sobolev spaces of negative order:

**Proposition 3.2.** For $1 \leq j \leq k$, let $X^j$ be a $C^\infty$ manifold of dimension $d_j$, and suppose that $u_j \in L^2_{r_j, \text{comp}}(X^j)$, $1 \leq j \leq k$, with each $r_j \leq 0$. Then the tensor product $u_1 \otimes \cdots \otimes u_k$ belongs to $L^2_{r, \text{comp}}(X^1 \times \cdots \times X^k)$, for $r = \sum_{j=1}^k r_j$.

**Proof.** Due to the compact support assumption, we can localize to a coordinate patch on each manifold, reducing the problem to showing that

$$L^2_{r_1}(\mathbb{R}^{d_1}) \otimes \cdots \otimes L^2_{r_k}(\mathbb{R}^{d_k}) \hookrightarrow L^2_r(\mathbb{R}^{\sum d_j}),$$

and this follows from the fact that each $\hat{u}_j(\xi^j) \cdot (\xi^j)^{r_j} \in L^2(\mathbb{R}^{d_j})$, together with the lower bound $\Pi_{j=1}^k (\xi^j)^{r_j} \geq c \langle \xi^1, \ldots, \xi^k \rangle^r$ on $\mathbb{R}^{\sum d_j}$. Q.E.D.

3.4. **Justification of density formula.** To justify (2.6), we argue as follows, restricting for simplicity the analysis to the case $k = 3$ discussed in Sec. 2, when $\Phi : X^1 \times X^2 \times X^3 \to \mathbb{R}^p$. The proof extends to $\Phi$ with codomain a general $T$ of dimension $p$ using local coordinates on $T$, and also extends in a straightforward way to general $k$.

Without loss of generality, we consider $\sigma = (123)$. For a $\chi \in C^\infty_0(\mathbb{R}^p)$ supported in a sufficiently small ball, $\chi \equiv 1$ near $0$, and with $\int \chi dt = 1$, set $\chi_\epsilon(t) := \epsilon^{-p} \chi(t^\epsilon)$ the
associated approximation to the identity, which converges to $\delta(t)$ weakly as $\epsilon \to 0^+$.

Define $R^{(12|3)}_{t,\epsilon}$ to be the operator with Schwartz kernel

$$K^\epsilon_t(x^1, x^2; x^3) := \chi_\epsilon(\Phi(x^1, x^2, x^3) - t).$$

Then $R^{(12|3)}_{t,\epsilon}(\mu_3) \in C^\infty(X^1 \times X^2)$ and depends smoothly on $t$, and thus we can represent the measure $\nu$ in (2.6) as the weak limit of absolutely continuous measures with smooth densities,

$$\nu(t) = \lim_{\epsilon \to 0^+} \nu'(t) := \lim_{\epsilon \to 0^+} \langle R^{(12|3)}_{t,\epsilon}(\mu_3), \mu_1 \times \mu_2 \rangle,$$

with $\nu$ having a density, which is in fact continuous in $t$, if the integral represented by the pairing converges. Now, the operators $R^{(12|3)}_{t,\epsilon}$ are in $\mathcal{I}^{-\infty}(C^{(12|3)}_t)$, with symbols which converge in the Fréchet topology on the space of symbols as $\epsilon \to 0$ to the symbol of $R_t$. Since the singular limits $R^{(12|3)}_{t,\epsilon}$ satisfy (2.7) (for $\sigma = (12|3)$), so do the $R^{(12|3)}_{t,\epsilon}$ uniformly in $\epsilon$. Hence, $\nu(t)$, being the uniform limit of smooth functions of $t$, is continuous. Furthermore, since $\epsilon^p \cdot \chi_\epsilon$ is bounded below by a constant times the characteristic function of the ball of radius $\epsilon$ in $\mathbb{R}^p$, we have that

$$\nu(B(t, \epsilon)) := (\mu_1 \times \mu_2 \times \mu_3) \left( \{ (x^1, x^2, x^3) : |\Phi(x^1, x^2, x^3) - t| < \epsilon \} \right) \leq C_\Phi \epsilon^p,$$

with constant $C_\Phi$ uniform in $t$, which was used in (2.8) above.

4. Proofs of theorems on 3-point configurations

We are now able to prove the theorems stated in the Introduction that concern 3-point configurations: Thm. 1.1 about the areas of triangles in $\mathbb{R}^2$ and the radii of their circumscribing circles; the three-dimensional case of Thm. 1.2 regarding strongly pinned volumes of parallelepipeds; and Thm. 1.4 on ratios of pinned distances. For all of these, we will show that Thm. 2.1 (ii) applies for appropriate choice of $\sigma$.

4.1. Areas of triangles in $\mathbb{R}^2$. We start with part (i) of Thm. 1.1, on areas. The absolute value of the determinant is irrelevant for the conclusion of nonempty interior; this will also be true for the other results where the configuration measurements have absolute values. Additionally, the $1/2$ can be ignored. So, we start with the scalar-valued configuration function,

$$\Phi(x^1, x^2, x^3) = \det [x^1 - x^3, x^2 - x^3]$$

on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. Here, $d = 2$, $p = 1$, and we will show that, using $\sigma = (12|3)$, that the canonical relations $C^{\sigma}_t$ are (after localizing) nondegenerate, so that Thm. 2.1 (ii) yields a result for $\dim_{\mathcal{H}}(E) > (2 \cdot 2 + 1)/3 = 5/3$.

We compute the gradient of $\Phi$ by noting that, on $\mathbb{R}^2 \times \mathbb{R}^2$,

$$d_{u,v}(\det [u, v]) = (-v^\perp, u^\perp),$$
Thus, if we define 
\[ \tilde{\mu}(F) = \mu(F) \int_{E} \int_{E} x_0 \, dx_1 \, dx_2 \, dx_3 \]
where \( u^\perp = (-u_2, u_1) \) for \( u = (u_1, u_2) \). Hence,
\begin{align*}
\det [x_1^j - x_0^j, x_2^j - x_3^j] &\neq 0 \\
\mu (B(x_0^j, \delta')) &> 0, j = 1, 2, 3, \forall \delta' < \delta.
\end{align*}

To verify this, suppose not. Then, for every \( x^1, x^2, x^3 \in E \) and any \( \delta > 0 \), either
(i) \( \Phi(x^1, x^2, x^3) = 0 \), i.e., \( x^1, x^2, x^3 \) are collinear, or
(ii) for some \( j = 1, 2 \) or 3, and some \( \delta' < \delta, \mu(B(x^j, \delta')) = 0 \).

Now, \( Z_0 = \{ x \in \mathbb{R}^6 : \Phi(x^1, x^2, x^3) = 0 \} \) is a five-dimensional algebraic variety. Since \( \mu \times \mu \times \mu \) has finite \( 3s \)-energy, and \( 3s > 5 \), it follows that \( (\mu \times \mu \times \mu)(Z_0) = 0 \), and hence \( (\mu \times \mu \times \mu)(E \times E \times E \setminus Z_0) = 1 \). We can in fact make this quantitative: Assuming without loss of generality that \( E \) is contained in the unit square centered at the origin, for \( \epsilon > 0 \), let \( Z_\epsilon := \{ x \in \mathbb{R}^6 : |x| < 2 \) and \( |\Phi(x)| < \epsilon \} \). Then, since \( Z_0 \) is a rigid motion in \( \mathbb{R}^6 \) of the Cartesian product of \( \mathbb{R}^2 \) with a quadratic cone in \( \mathbb{R}^4 \), one sees that \( Z_\epsilon \) is covered by \( \simeq \epsilon^{-5} \) balls of radius \( \epsilon \) (away from the conical points), together with \( \simeq \epsilon^{-1} \) balls of radius \( \epsilon^{1/2} \) (covering a tubular neighborhood of the conical points). Since \( \mu \) satisfies the ball condition (3.2) on \( \mathbb{R}^2 \), \( \mu \times \mu \times \mu \) satisfies the corresponding condition on \( \mathbb{R}^6 \) with exponent \( 3s \) and is thus dominated by \( 3s \)-dimensional Hausdorff measure (up to a multiplicative constant). Thus,
\[ (\mu \times \mu \times \mu)(Z_\epsilon) \lesssim \epsilon^{-5} \cdot \epsilon^{3s} + \epsilon^{-1} \cdot \epsilon^{3s/2} \lesssim \epsilon^{3s-5} \to 0 \text{ as } \epsilon \to 0. \]

Thus, if we define \( F_\epsilon := E \times E \times E \setminus Z_\epsilon \), which is compact, and \( \tilde{\mu}_\epsilon := (\mu \times \mu \times \mu)|_{F_\epsilon} \), then \( \tilde{\mu}_\epsilon(F_\epsilon) > 1/2 \) for \( \epsilon \) sufficiently small. By (ii) above, every \( x \in F_\epsilon \) is in a \( (\mu \times \mu \times \mu) \)-null set which is also relatively open, the intersection of \( F_\epsilon \) with a set of one of the three forms,
\[ B(x^1, \delta') \times \mathbb{R}^2 \times \mathbb{R}^2, \quad \mathbb{R}^2 \times B(x^2, \delta') \times \mathbb{R}^2 \text{ or } \mathbb{R}^2 \times \mathbb{R}^2 \times B(x^3, \delta'). \]

Since \( F_\epsilon \) is compact, it is covered by a finite number of these, and hence it follows that \( (\mu \times \mu \times \mu)(F_\epsilon) = \tilde{\mu}_\epsilon(F_\epsilon) = 0 \). Contradiction. Hence, there exists an \( x_0 = (x_0^1, x_0^2, x_0^3) \in E \times E \times E \) such that (4.2), (4.3) hold. We now show that localizing near this base point allows us to apply Thm. 2.1.

\footnote{Note that (4.3) just says that the \( x_0^j \) belong to \( \text{supp}(\mu) \) (which by Frostman’s Lemma is \( \subseteq E \) but can be a proper subset); however, for our purposes, it is useful to express this as (4.3).}
Set \( t_0 = \Phi(x_0^1, x_0^2, x_0^3) \neq 0 \); by continuity of \( \Phi \) and relabelling there is a \( \delta > 0 \) with
\( \Phi(x, x^1, x^2) \neq 0 \) for \( x \in X^j := B(x_0^j, \delta), \ j = 1, 2, 3. \) We claim that for \( \Phi|_{X^1 \times X^2 \times X^3} \) and \( t \) close to \( t_0 \), \((DF)_{(12)(3)} \) is satisfied and and \( C_t^{(12)(3)} \) is nondegenerate, so that \( \beta^{12,3} = 0 \) and the last statement of Thm. 2.1 applies with \( d = 2 \) and \( p = 1 \); hence, if
\( \dim_{R}(E) > 5/3 \), then \( \text{int}(A_3(E)) \neq \emptyset \).

That \((DF)_{(12)(3)} \) is satisfied is immediate, since all three components of \( d_{x^1x^2x^3}\Phi \) are nonzero on \( X^1 \times X^2 \times X^3 \) (the linear independence of the first two by \((4.2) \) and the nonvanishing of third by that linear independence), which is an even stronger condition. As for the canonical relations, one computes
\[
C_t^{(12)(3)} = \left\{ \left( x^1, x^2, \theta (x^3 - x^2)^{\perp}, \theta (x^1 - x^3)^{\perp}; x^3, \theta (x^2 - x^1)^{\perp} \right) : (x^1, x^2, x^3) \in Z_t^{(12)(3)}, \theta \neq 0 \right\}.
\]

Now, shrinking \( \delta \) if necessary, for \( (x^1, x^2, x^3) \in X^1 \times X^2 \times X^3 \) and \( t \) near \( t_0 \), for a smooth, \( X^1 \)-valued function \( y^1(x^2, x^3, t) \) we can parametrize \( Z_t^{(12)(3)} \) by
\[
( y^1(x^2, x^3, t) + u(x^2 - x^3), x^2, x^3), \quad (x^2, x^3) \in X^2 \times X^3, \ u \in \mathbb{R};
\]
for example, one can take
\[
y^1(x^2, x^3, t) = t |x^2 - x^3|^2 \cdot (x^2 - x^3)^{\perp}.
\]
Thus, \( (x^2, x^3, u, \theta) \) form coordinates on \( C_t^{12,3} \), with respect to which
\[
\pi_R(x^2, x^3, u, \theta) = \left( x^3, \theta (x^2 - x^1 (x^2, x^3, u))^{\perp} \right),
\]
from which we see that
\[
D_{x^1, u, \theta}^R = \begin{bmatrix} I & 0 \\ \star & \theta(x^2 - x^3)^{\perp} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\theta(x^2 - x^3)^{\perp} & (x^2 - x^1)^{\perp} \end{bmatrix},
\]
which is of maximal rank since the last two columns are linearly independent. Thus, \( \pi_R \) is a submersion; by the general properties of canonical relations from Sec. 3.1, \( \pi_L \) is an immersion and \( C_t^{(12)(3)} \) is nondegenerate. Q.E.D.

4.2. Circumradii of triangles in \( \mathbb{R}^2 \). We now turn to the proof of Thm. 1.1 (ii). Changing the notation to denote the vertices of the triangle as \( x, y, z \in \mathbb{R}^2 \), the circumradius \( R(x, y, z) \) of \( \triangle xyz \) is the distance from \( x \) to the intersection point of the perpendicular bisectors of \( \overline{yz} \) and \( \overline{zx} \). For computational purposes, we work with \((4.4)\)
\[
\Phi(x, y, z) := 2R^2(x, y, z) = \frac{1}{2} \left( |y - x|^2 + |z - x|^2 + \frac{|z - x|^2 ((y - x) \cdot (z - x))^2}{((y - x)^{\perp} \cdot (z - x))^2} \right),
\]
with the homeomorphism \( r \to 2r^2 - 2 \) of \( \mathbb{R}_+ \) of course preserving nonempty interior. With the same hypersurface \( Z_0 \) as in the proof of part (i) corresponds to degenerate triangles, \( 5/3 < s < \dim_H(E) \) and Frostman measure \( \mu \), as in (i) we can find \( x_0, y_0, z_0 \in \text{supp}(\mu) \) such that all of the components of \( d\Phi \) are nonzero. Picking \( \sigma = (13|2) \), the incidence relation \( \mathcal{I}_\sigma = \{(x, z; y) : \Phi(x, y, z) = t\} \) satisfies the double fibration condition \((DF)_\sigma \) for \( t \) near \( t_0 = \Phi(x_0, y_0, z_0) \) and \( x \in X^1 \), \( y \in X^2 \), \( z \in X^3 \), neighborhoods of \( x_0, y_0, z_0 \), resp. We can thus assume that \( x_2 \in \mathbb{R}, y \in \mathbb{R}^2, z \in \mathbb{R}^2 \) form coordinates on \( \mathcal{I}_\sigma \), with \( x_1 \) a function of \( x_2, y, z \), and rotating in \( x \) if necessary, can further assume that \( d_z x_1 = 0 \) at \( x_0, y_0, z_0 \) and is therefore small nearby. On \( \mathcal{I}_\sigma = (N^*\mathcal{I}_\sigma)' \), these together with the radial phase variable \( \theta \in \mathbb{R} \setminus \{0\} \) are coordinates.

To show that the canonical relation \( \mathcal{I}_\sigma \) is nondegenerate, it suffices to show that \( \pi_R : \mathcal{I}_\sigma \to T^*X^2 \) is a submersion. Since the \( y \) coordinate of \( \pi_R(x_2, y, z, \theta) = y \), it suffices to show that

\[
\text{rank} \left[ \frac{D\eta}{D(x_2, z, \theta)} \right] = 2.
\]

Setting

\[
a = |y - x|^2, \quad b = (y - x) \cdot (z - x), \quad c = |z - x|^2, \quad d = (y - x) \perp (z - y),
\]

one calculates

\[
(4.5) \quad \frac{D\eta}{Dz} = \frac{cb}{d^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I - \frac{b}{d} J \\ 1 + \frac{cb}{d^2} - \frac{cb^2}{d^3} \end{pmatrix} \begin{bmatrix} 0 \\ d_z x_1 \end{bmatrix} ,
\]

where \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is the standard \( 2 \times 2 \) symplectic matrix, representing the \( \perp \) map. The operator pencil \( I - \lambda J \lambda \in \mathbb{R} \), is nonsingular, while the second term in (4.5) is small near \( x_0, y_0, z_0 \), and thus \( D\eta/Dz \) is nonsingular, and \( \mathcal{I}_\sigma \) is nondegenerate. Thus, as for areas, Thm. 2.1 (ii) applies for \( s > 5/3 \). Q.E.D.

4.3. **Volumes of strongly pinned parallelepipeds in \( \mathbb{R}^3 \).** For the proof of the \( d = 3 \) case of Thm. 1.2, the configuration function \( \Phi \) on \( \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \) is

\[
\Phi(x^1, x^2, x^3) = \det [x^1, x^2, x^3] = x^1 \cdot (x^2 \times x^3) = -x^2 \cdot (x^1 \times x^3) = x^3 \cdot (x^1 \times x^2).
\]

We will show that Thm. 2.1 (ii) applies for \( \sigma = (12|3) \), with \( k = 3, p = 1 \) and \( d = 3 \), giving a positive result for \( \dim_H(E) > (2 \cdot 3 + 1)/3 = 7/3 \). One computes

\[
d\Phi_{x^1, x^2, x^3} = \left( x^2 \times x^3, -x^1 \times x^3, x^1 \times x^2 \right).
\]

As in the previous proofs, given a compact \( E \subset \mathbb{R}^3 \), contained in the unit cube and with \( \dim_H(E) > 7/3 \), pick \( s \) with \( 7/3 < s < \dim_H(E) \) and let \( \mu \) be a Frostman measure of finite \( s \)-energy. We claim there exist \( x_0^1, x_0^2, x_0^3 \in E \) and \( \delta > 0 \) such that
Together, these span all of the $\mathbb{R}^3$.

As before, we proceed with a proof by contradiction: suppose not. Then for every $x = (x^1, x^2, x^3) \in \mathbb{R}^9$ and $\delta > 0$, either

(i) at least one of $x^i \times x^j = 0$, for some $1 \leq i < j \leq 3$, or

(ii) for some $j = 1, 2$ or $3$, and some $\delta' < \delta$, $\mu(B(x^j, \delta')) = 0$.

On $\mathbb{R}^9$, $\mu \times \mu \times \mu$ has finite $3s$-energy and satisfies the ball condition with exponent $3s > 7$. For each $1 \leq i < j \leq 3$ and $\epsilon > 0$, $W^{ij}_\epsilon := \{ x \in \mathbb{R}^9 : |x^i \times x^j| < \epsilon \}$ is a tubular neighborhood of $\{ x : x^i \times x^j = 0 \}$, a codimension two quadratic variety in $\mathbb{R}^9$ which is a rigid motion in $\mathbb{R}^9$ of the Cartesian product of $\mathbb{R}^3$ with a 4-dimensional cone in $\mathbb{R}^6$. Following the analysis in the previous proof, each of the $W^{ij}_\epsilon$ can be covered by $\simeq \epsilon^{-7}$ balls of radius $\epsilon$ and $\epsilon^{-3}$ balls of radius $\epsilon^{1/2}$, and thus

$$(\mu \times \mu \times \mu)(W^{ij}_\epsilon) \lesssim \epsilon^{3s-7} + \epsilon^{3s/2-3} \lesssim \epsilon^{3s-7} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Hence, if we let $F_\epsilon = E \times E \times E \setminus (\cup_{i,j} W^{ij}_\epsilon)$ and $\tilde{\mu}_\epsilon = (\mu \times \mu \times \mu)|_{F_\epsilon}$, then $F_\epsilon$ is compact and $\tilde{\mu}_\epsilon(F_\epsilon) > 1/2$ for $\epsilon$ sufficiently small. On the other hand, $F_\epsilon$ is covered by $\tilde{\mu}_\epsilon$-null and relatively open sets which are intersections of $F_\epsilon$ with sets of the three forms

$$B(x^1, \delta') \times \mathbb{R}^3 \times \mathbb{R}^3, \quad \mathbb{R}^3 \times B(x^2, \delta') \times \mathbb{R}^3 \text{ or } \mathbb{R}^3 \times \mathbb{R}^3 \times B(x^3, \delta'),$$

and the compactness of $F_\epsilon$ leads to a contradiction. Hence, we can find $x^1_0, x^2_0, x^3_0$ and $\delta$ such that (4.6), (4.7) hold. Further restricting $\delta$ if necessary, we can assume that $x^1 \times x^2, x^1 \times x^3$ and $x^2 \times x^3$ are $\neq 0$. for all $x^j \in B(x^j_0, \delta) =: X^j, j = 1, 2, 3$.

Restricting $\Phi$ to $X^1 \times X^2 \times X^3$, $(DF)_{(12|3)}$ is satisfied; in fact all three components of $d\Phi$ are nonzero. On the incidence relation $Z^{(12|3)}_t$, we can take as coordinates $x^2, x^3$ and $\bar{u} = (u_2, u_3) \in \mathbb{R}^2$, solving for $x^1$ with

$$x^1 = y^1(x^2, x^3, t) + u_2 x_2 + u_3 x_3,$$

for some smooth function $y^1$. Thus,

$$C^{(12|3)}_t = \{ (x^1, x^2, \theta(x^2 \times x^3), -\theta(x^1 \times x^3); x^3, -\theta(x^1 \times x^3)) : (x^2, x^3) \in X^2 \times X^3, \bar{u} \in \mathbb{R}^2, \theta \neq 0 \}.$$

Then $\pi_R$ is a submersion since, with $\xi^3 = -\theta(x^1 \times x^2)$, $D_{x^2, u_3, \xi^3}$ is surjective: one has $D\xi^3(\partial_{u_3}) = x^1 \times x^2, D\xi^3(\partial_{u_3}) = \theta(x^2 \times x^3)$ and the range of $D_{x^2, \xi^3}$ is $(x^1)^\perp$; together, these span all of the $\partial_{\xi^3}$ directions.

Since $\pi_R$ is a submersion, $C^{(12|3)}_t$ is nondegenerate, and Thm. 2.1 (ii) applies, this time with $d = 3$ and $p = 1$; hence, if dim$_R(E) > 7/3$, then int $(V^0_3(E)) \neq \emptyset$. Q.E.D.
Remark 4.1. The proof for $d \geq 4$ will be presented in Sec. 6.2 below. On the other hand, Thm. 1.2 does not give a positive result for pinned volumes (areas) in two dimensions,

$$V^0_2(E) := \left\{ \det \begin{bmatrix} x^1 & x^2 \\ x^1 & x^2 \end{bmatrix} : x^1, x^2 \in E \right\} .$$

In fact, since this concerns a 2-point configuration, it would already fall under the framework of [13]; however, the projections $\pi_L, \pi_R$ from the canonical relation to $T^*\mathbb{R}^2$ both drop rank by 1 everywhere, resulting in a loss of $\beta^{(1|2)} = 1/2$ derivatives. Hence, $\mathcal{R}_t^{(1|2)} \in I^{-\frac{1}{2}} \left( C_t^{(1|2)} \right)$ is not smoothing on $L^2$-based Sobolev spaces, and the FIO approach to configuration problems does not imply a result in this case.

4.4. Ratios of pinned distances. For $E \subset \mathbb{R}^d$, $d \geq 2$, we prove Thm. 1.4 concerning the set defined in (1.5). On $(\mathbb{R}^d)^3$, let

$$\Phi(x^1, x^2, x^3) = \frac{|x^1 - x^3|}{|x^1 - x^2|} .$$

We show that, after suitable localization, Thm. 2.1 (ii), with $k = 3$, $p = 1$, applies for $\sigma = (12|3)$, implying a nonempty interior result when $\dim H(E) > (2d + 1)/3$.

One computes,

$$d\Phi(x^1, x^2, x^3) = \left| x^1 - x^2 \right|^{-2} \left( (x^2 - x^3), -(x^1 - x^2), -(x^1 - x^3) \right) .$$

Let $\dim H(E) > (2d + 1)/3$ and $\mu$ be a Frostman measure on $E$ of finite $s$-energy for some $(2d + 1)/3 < s < \dim H(E)$. Then $\mu \times \mu \times \mu$ is dominated by $3s$-dimensional Hausdorff measure, and $3s > 2d + 1 > 2d$. Since $\{(x^1 - x^2)(x^1 - x^3)(x^2 - x^3) = 0\}$ is a union of three $2d$ dimensional planes, as above one can show that there exist $x^i_0, x^j_0, x^k_0 \in \text{supp}(\mu)$ such that $x^i_0 - x^j_0 \neq 0, i \neq j$. Taking $X^j = B(x^j_0, \delta)$ for suitably small $\delta$, all three components of $d\Phi$ are nonzero and, setting $t_0 = \Phi(x^i_0, x^j_0, x^k_0)$, the double fibration condition $(DF)_\sigma$ is satisfied by $Z^\sigma_t$ for $t$ close to $t_0$.

On $Z^\sigma_t$, solving for $x^3 = x^1 - t|x^1 - x^2|\omega$ we can take as coordinates $(x^1, x^2, \omega) \in \mathbb{R}^d \times \mathbb{R}^2 \times S^{d-1}$. Then, the canonical relation of $Z^\sigma_t$ is

$$C^\sigma_t = (N^*Z^\sigma_t)' = \left\{ (x^1, x^2, \omega, t|x^1 - x^2|\omega, \theta t|x^1 - x^2|\omega) : x^1, x^2 \in \mathbb{R}^d, \omega \in S^{d-1}, \theta \neq 0 \right\} ,$$

where we have suppressed the $T^* (X^1 \times X^2)$ components as irrelevant for analyzing $\pi_R$. One easily sees that the projection from $C^\sigma_t$ to $T^*X^3$ is a submersion, so that Thm. 2.1 (ii) applies as claimed.
5. $k$-POINT CONFIGURATION SETS, GENERAL $k$

To describe results for general $k$-point configurations, let $X^i, 1 \leq i \leq k$, and $T$, be smooth manifolds of dimensions $d_i$ and $p$, resp. We sometimes denote $X^1 \times \cdots \times X^k$ by $X$, and set $d_{\text{tot}} := \dim(X) = \sum_{i=1}^k d_i$.

**Definition 5.1.** Let $\Phi \in C^\infty(X, T)$. Suppose that $E_i \subset X^i, 1 \leq i \leq k$, are compact sets. Then the $k$-configuration set of the $E_i$ defined by $\Phi$ is

$$\Delta_\Phi(E_1, E_2, \ldots, E_k) := \{ \Phi(x^1, \ldots, x^k) : x^i \in E_i, 1 \leq i \leq k \} \subset T.$$

We want to find sufficient conditions on $\dim_H(E_i)$ ensuring that $\Delta_\Phi(E_1, E_2, \ldots, E_k)$ has nonempty interior. To this end, now suppose that $\Phi : X \to T$ is an submersion, so that for each $t \in T$, $Z_t := \Phi^{-1}(t)$ is a smooth, codimension $p$ submanifold of $X$, and these vary smoothly with $t$. For each $t$, the measure

$$\lambda_t := \delta(\Phi(x^1, \ldots, x^k) - t)$$

is a smooth density on $Z_t$; using local coordinates on $T$, one sees that this can be represented as an oscillatory integral of the form

$$\int_{\mathbb{R}^p} e^{i[\sum_{l=1}^p (\Phi_l(x^1, \ldots, x^k) - t_l)\theta_l]} a(t)1(\theta) d\theta,$$

where the $a(\cdot)$ belongs to a partition of unity on $T$. Thus, $\lambda_t$ is a Fourier integral distribution on $X$; in Hörmander’s notation [18, 19],

$$\lambda_t \in I^{(2p-d_{\text{tot}})/4}(X; N^*Z_t),$$

where $N^*Z_t \subset T^*X \setminus 0$ is the conormal bundle of $Z_t$ and the value of the order follows from the amplitude having order zero and the numbers of phase variables and spatial variables being $p$ and $d_{\text{tot}}$, resp., so that the order is $m := 0 + p/2 - d_{\text{tot}}/4$.

As in the analysis of 3-point configurations in Sec. 2, we separate the variables $x^1, \ldots, x^k$ into groups on the left and right, associating to $\Phi$ a collection of families of generalized Radon transforms indexed by the nontrivial partitions of $\{1, \ldots, k\}$, with each family then depending on the parameter $t \in T$. Write such a partition as $\sigma = (\sigma_L | \sigma_R)$, with $|\sigma_L|, |\sigma_R| > 0$, $|\sigma_L| + |\sigma_R| = k$, and let $\mathcal{P}_k$ denote the set of all such partitions. We will use $i$ and $j$ to refer to elements of $\sigma_L$ and $\sigma_R$, resp. Define $d_{\sigma_L}^p = \sum_{i \in \sigma_L} d^i$ and $d_{\sigma_R}^p = \sum_{i \in \sigma_R} d^i$, so that $d_{\sigma_L}^p + d_{\sigma_R}^p = d_{\text{tot}}$.

For each $\sigma \in \mathcal{P}_k$, $\sigma_L = \{i_1, \ldots, i_{|\sigma_L|}\}$ and $\sigma_R = \{j_1, \ldots, j_{|\sigma_R|}\}$, where we may assume that $i_1 < \cdots < i_{|\sigma_L|}$ and $j_1 < \cdots < j_{|\sigma_R|}$. with a slight abuse of notation we still refer to as $x$ the permuted version as $x$,

$$x = (x_L; x_R) := (x^{i_1}, \ldots, x^{i_{|\sigma_L|}}; x^{j_1}, \ldots, x^{j_{|\sigma_R|}}).$$
Write the corresponding reordered Cartesian product as
\[ X_L \times X_R := \left( X^{i_1} \times \cdots \times X^{i_{|\sigma|}} \right) \times \left( X^{j_1} \times \cdots \times X^{j_{|\sigma'|}} \right); \]
again by abuse of notation, we sometimes still refer to this as \( X \). The dimensions of the two factors are \( \dim(X_L) = d_L^\sigma \) and \( \dim(X_R) = d_R^\sigma \), resp. The choice of \( \sigma \) also defines a permuted version of each \( Z_t^\sigma \),
\[ \pi^\sigma \] with spatial projections to the left and right, \( \pi^\sigma_{X_L} : Z^\sigma_t \rightarrow X_L \) and \( \pi^\sigma_{X_R} : Z^\sigma_t \rightarrow X_R \).
The integral geometric double fibration condition extending (2.5) to general \( k \) is the requirement,
\[ (DF)^\sigma \] (Note that \( (DF)^\sigma \) can only hold for a given \( \sigma \) if \( p \leq \min(d_L^\sigma, d_R^\sigma) \).)
If \( (DF)^\sigma \) holds, then the generalized Radon transform \( \mathcal{R}^\sigma_t \), defined weakly by
\[ \mathcal{R}^\sigma_t f(x_L) = \int_{\{x_R : \Phi(x_L,x_R) = t\}} f(x_R), \]
where the integral is with respect to the surface measure induced by \( \lambda_t \) on the codimension \( p \) submanifold \( \{x_R : \Phi(x_L,x_R) = t\} = \{x_R : (x_L,x_R) \in Z^\sigma_t \} \subset X_R \),
extends from mapping \( \mathcal{D}(X_R) \rightarrow \mathcal{E}(X_L) \) to
\[ \mathcal{R}^\sigma_t : \mathcal{E}'(X_R) \rightarrow \mathcal{D}'(X_L). \]
Furthermore,
\[ C^\sigma_t := (N^*Z_t^\sigma)' = \{(x_L,\xi_L; x_R,\xi_R) : (x_L, x_R) \in Z^\sigma_t, (\xi_L, -\xi_R) \perp TZ^\sigma_t \} \]
is contained in \( (T^*X_L \setminus 0) \times (T^*X_R \setminus 0) \). Thus, \( \mathcal{R}^\sigma_t \) is an FIO, \( \mathcal{R}^\sigma_t \in I^m(X_L, X_R; C^\sigma_t) \),
where the order \( m \) is determined by Hörmander’s formula, \( m = 0 + p/2 - d_{\text{tot}}/4 \).
Given the possible difference in the dimensions of \( X_L \) and \( X_R \), due to the clean intersection calculus it is useful to express \( m \) as
\[ m = m_{\text{eff}}^\sigma - \frac{1}{4} |d_L^\sigma - d_R^\sigma|, \]
where the effective order of \( \mathcal{R}^\sigma_t \) is defined to be
\[ m_{\text{eff}}^\sigma := \left( 2p - d_{\text{tot}} + |d^\sigma - d^\sigma'| \right) / 4 = (p - \min(d_L^\sigma, d_R^\sigma)) / 2. \]
As recalled in Sec. 3.1, if \( C^\sigma_t \) is a nondegenerate canonical relation, i.e., the cotangent space projections \( \pi^\sigma_{X_L} : C^\sigma_t \rightarrow T^*X_L \) and \( \pi^\sigma_{X_R} : C^\sigma_t \rightarrow T^*X_R \) have differentials of maximal rank\(^3\), then
\[ \mathcal{R}^\sigma_t : L^2_r(X_R) \rightarrow L^2_{r-m_{\text{eff}}^\sigma}(X_L). \]
\[^3\]This is a structurally stable condition, so that if \( C^\sigma_{t_0} \) is nondegenerate, then \( C^\sigma_t \) is nondegenerate for all \( t \) in some neighborhood of \( t_0 \).
As in the result concerning 3-point configurations in Thm. 2.1, it is natural to express the estimates for possibly degenerate FIO in terms of possible losses relative to the optimal estimates. As for \( k = 3 \), our basic assumption is that, for at least one \( \sigma \), the double fibration condition (5.5) is satisfied and there is a known \( \beta^\sigma \geq 0 \) such that, for all \( r \in \mathbb{R} \),

\[
R^\sigma_t : L^2_r(X_R) \to L^2_{r-m^\text{eff}_\sigma - \beta^\sigma}(X_L) \text{ uniformly for } t \in T.
\]

Now suppose that, for \( 1 \leq i \leq k \), \( E_i \subset X_i \) are compact sets. Our goal is to find conditions on the \( \dim_H(E_i) \) ensuring that \( \Delta_\Phi(E_1, E_2, \ldots, E_k) \) has nonempty interior in \( T \). For each \( i \), fix an \( s_i < \dim_H(E_i) \) and a Frostman measure \( \mu_i \) on \( E_i \) of finite \( s_i \)-energy. Define measures

\[
\mu_L := \mu_{i_1} \times \cdots \times \mu_{i_{|\sigma|}} \text{ on } X_L \text{ and } \mu_R := \mu_{j_1} \times \cdots \times \mu_{j_{|\sigma|}} \text{ on } X_R.
\]

By (3.4), each \( \mu_i \in L^2_{(s_i-d_i)/2}(X_i) \), so that Prop. 3.2 implies that \( \mu_L \in L^2_{r_L}(X_L) \) and \( \mu_R \in L^2_{r_R}(X_R) \), where \( r_L = \frac{1}{2} \sum_{i=1}^{|\sigma|} (s_i - d_i) \) and \( r_R = \frac{1}{2} \sum_{j=1}^{|\sigma|} (s_j - d_j) \), resp. The analogue of the representation formula (3.5) for \( \nu(t) \), justified by a minor modification of the \( k = 3 \) case in Sec. 3.4, is

\[
\nu(t) = \langle R^\sigma_t(\mu_R), \mu_L \rangle.
\]

Our basic assumption, that (5.8) holds for the \( \sigma \) in question, then implies that \( R^\sigma_t(\mu_R) \in L^2_{r_R-m^\text{eff}_\sigma - \beta^\sigma}(X_L) \). Since \( \mu_L \in L^2_{r_L}(X_L) \), the pairing in (5.9) is bounded, and yields a continuous function of \( t \), if

\[
r_R - m^\text{eff}_\sigma - \beta^\sigma + r_L \geq 0.
\]

Noting that

\[
r_L + r_R = \frac{1}{2} \left[ \left( \sum_{i=1}^k s_i \right) - d_{\text{tot}} \right],
\]

and using (5.7), we see that (5.10) holds iff

\[
\sum_{i=1}^k s_i \geq d_{\text{tot}} + 2 (m^\text{eff}_\sigma + \beta^\sigma)
\]

\[
= d_{\text{tot}} + p - \min(d_L, d_R) + 2 \beta^\sigma
\]

\[
= \max(d_L, d_R) + p + 2 \beta^\sigma.
\]

Optimizing over all \( \sigma \in \mathcal{P}_k \), we obtain the analogue of Thm. 2.1 for \( k \)-point configuration sets:

**Theorem 5.2.** (i) With the notation and assumptions as above, define

\[
s_\Phi = \min \left( \max(d_L, d_R) + p + 2 \beta^\sigma \right),
\]
where the min is taken over those $\sigma \in P_k$ for which both the double fibration condition (5.5) and the uniform boundedness of the generalized Radon transforms $R_\sigma t$ with loss of $\leq \beta^\sigma$ derivatives (5.8) hold.

Then, if $E_i \subset X^i$, $1 \leq i \leq k$, are compact sets with $\sum_{i=1}^k \dim_H(E_i) > s_\Phi$, it follows that $\operatorname{int}(\Delta_\Phi(E_1, E_2, \ldots, E_k)) \neq \emptyset$.

(ii) In particular, if $X^1 = \cdots = X^k = X_0$, with $\dim(X_0) = d$, and if $E \subset X_0$ is compact with

$$\dim_H(E) > \frac{1}{k} \min (\max (d_L, d_R) + p),$$

where the minimum is taken over all $\sigma \in P_k$ such that the canonical relations $C_\sigma^t$ are nondegenerate, then $\operatorname{int}(\Delta_\Phi(E, E, \ldots, E)) \neq \emptyset$.

6. Proofs of theorems for $k$-point configurations, $k \geq 4$

We now use Thm. 5.2 to prove Thm. 1.5 on cross ratios of four-tuples of points in $\mathbb{R}$, Thm. 1.2 concerning strongly pinned volumes of parallelepipeds generated by $d$-tuples of points in $\mathbb{R}^d$ for $d \geq 4$, Thm. 1.6 on pairs of areas of triangles in $\mathbb{R}^2$, Thm. 1.7 on dot products of differences and Thm. 1.8 on generalized sum-product sets.

6.1. Cross ratios on $\mathbb{R}$. We prove Thm. 1.5 on the set of cross ratios of four-tuples of points in a set $E \subset \mathbb{R}$. This will be an application of the second part of Thm. 5.2, with $d_j = 1$, $1 \leq j \leq 4$. Since each of the variables in one dimensional, we use subscripts rather than superscripts. Thus, set $\Phi : \mathbb{R}^4 \to \mathbb{R}$,

$$\Phi(x_1, x_2, x_3, x_4) = [x_1, x_2; x_3, x_4] = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)},$$

and introduce the notation

$$x_{ij} = x_i - x_j, \quad 1 \leq i < j \leq 4.$$

Then one computes

$$d\Phi(x_1, x_2, x_3, x_4) = (x_{14}x_{23})^{-2} \left(x_{23}x_{24}x_{34}, -x_{13}x_{14}x_{34}, x_{14}x_{24}x_{34}, -x_{12}x_{13}x_{23}\right).$$

Given a compact $E \subset \mathbb{R}$ with $\dim_H(E) > 3/4$, let $s$ be such that $3/4 < s < \dim_H(E)$ and $\mu$ be a Frostman measure on $E$ of finite $s$-energy. We claim that

$$\text{(6.1) } (\exists x^0_1, x^0_2, x^0_3, x^0_4 \in \operatorname{supp}(\mu)) \text{ s.t. } x^0_i - x^0_j \neq 0, \quad \text{for all } 1 \leq i < j \leq 4,$$
so that all four components of $d\Phi$ are nonzero at $x^0 := (x_1^0, x_2^0, x_3^0, x_4^0)$. Arguing as in the proofs for 3-point configurations in Sec. 4, set

$$Z = \left\{ x \in \mathbb{R}^4 : \prod_{1 \leq i < j \leq 4} x_{ij} = 0 \right\},$$

on the complement of which all of the components of $d\Phi$ are nonzero. Noting that $Z$ is a union of hyperplanes, $\dim_H(Z) = 3$; thus, since $\mu \times \mu \times \mu \times \mu$ is dominated by $4s$-dimensional Hausdorff measure and $4s > 3$, $(\mu \times \mu \times \mu \times \mu)(Z) = 0$. By a slight variant of the reasoning in Thms. 1.1 and Thm. 1.2, one obtains (6.1). This actually shows that the conditions in (6.1) hold for a set of full $\mu \times \mu \times \mu \times \mu$ measure, which we use below.

Now let $\sigma = (12|34)$. Setting $t^0 = \Phi(x^0)$, and taking the $X^j$ to be sufficiently small neighborhoods of $x^0_j, 1 \leq j \leq 4$, it follows that for $t$ close to $t^0$ the double fibration condition (5.5) holds on

$$Z^\sigma_t := \{ x : \Phi(x) = t \} \subset X := (X^1 \times X^2) \times (X^3 \times X^4) =: X_L \times X_R.$$

Since $d_{x^0} \Phi \neq 0$, on $Z^\sigma_t$ we can solve for $x_1$ as a function of $x_2, x_3, x_4$ and the parameter $t$ (possibly again reducing the size of the neighborhoods of the $x^0_j$),

$$x_1 = y^1(x_2, x_3, x_4, t).$$

Furthermore, using the fact that the factor $(x_{14}x_{23})^{-2}$ in all of the terms of $d\Phi$ can be absorbed into the radial scaling factor, as in (5.6) we define

$$C^\sigma_t := (N^*Z^\sigma_t)^! = \left\{ (*, *, x_3, x_4, \theta x_{14}x_{24}x_{34}, -\theta x_{12}x_{13}x_{23}) \right\} \subset X^2 \times X^3 \times X^4, \theta \neq 0 \right\},$$

where we have suppressed the $T^*(X^1 \times X^2)$ components on the left. We claim that this is a local canonical graph, so that the family of FIOs lose no derivatives ($\beta^\sigma = 0$); the desired result then follows from Thm. 5.2 (ii), using $d_L = d_R = 2, k = 4$ and $p = 1$ in (5.11).

To see that $C^\sigma_t$ is a local canonical graph, it suffices to show that the differential of $\pi_R : C^\sigma_t \to T^*X_L$ has rank 4. Due to the $(x_3, x_4)$ in the spatial variables, this is equivalent to showing that

$$\frac{D(\xi_3, \xi_4)}{D(\theta, x_2)} = \begin{bmatrix} x_{14}x_{24}x_{34}, & \theta \left(x_{34} \left(x_{14} + x_{24}y^1_{x_2} \right) \right) \\ -x_{12}x_{13}x_{23}, & -\theta \left(-x_{13}x_{23} + x_{12}x_{13} + x_{23} \left(x_{13}y^1_{x_2} + x_{12}y^1_{x_2} \right) \right) \end{bmatrix}$$

is nonsingular. However, the determinant of this is an algebraic function, not identically vanishing on $Z^\sigma_t$, so its zero variety is three-dimensional and thus a null set with respect to $4s$-Hausdorff measure, and hence with respect to $\mu \times \mu \times \mu \times \mu$. Thus,
choosing our basepoint \( x^0 \), and then shrinking the \( X^j \) suitably, to avoid this, ensures that \( C^r_\sigma \) is a local canonical graph, finishing the proof of Thm. 1.5.

6.2. **Strongly pinned volumes in** \( \mathbb{R}^d, d \geq 4 \). With Thm. 5.2 in hand, we now prove Thm. 1.2 concerning pinned volumes in \( \mathbb{R}^d \) for \( d \geq 4 \), following the lines of the proof for \( d = 3 \) in Sec. 4. On \( (\mathbb{R}^d)^d \), let \( \Phi(x^1, \ldots, x^d) = \det [x^1, x^2, \ldots, x^d] \). We will show that for \( \sigma = (12 \ldots (d - 1)|d) \), some \( t_0 \neq 0 \) and with the domain of \( \Phi \) suitably localized, condition \((DF)_\sigma\) is satisfied and the canonical relation \( C^r_\sigma \) is nondegenerate; these conditions then hold for all \( t \) near \( t_0 \) by structural stability of submersions. Using just this \( \sigma \), applying (5.11) with \( d_L = d(d - 1) > d_R = d, p = 1 \) and \( \beta_\Phi = 0 \) shows that if \( \dim_H(E) > (1/d)(d(d - 1) + 1) = d - 1 + (1/d) \) then \( \text{int}(\Delta_\Phi(E, \ldots, E)) \neq \emptyset \), proving Thm. 1.2.

To verify the claims for \( \sigma \), we start by noting that
\[
d\Phi = \begin{pmatrix} x^{(1)} & -x^{(2)} & \ldots & (-1)^{d-1}x^{(d)} \end{pmatrix},
\]
where
\[
x^{(j)} := * (x^1 \wedge x^2 \wedge \cdots \wedge x^{j-1} \wedge x^{j+1} \wedge \cdots \wedge x^d),
\]
where \( * \) is the Hodge star operator, which is an isomorphism \( \Lambda^{d-1} \mathbb{R}^d \to \mathbb{R}^d \). As in the proof for \( d = 3 \), note that if \( d - 1 + (1/d) < s < \dim_H(E) \) and \( \mu \) is a Frostman measure on \( E \) of finite \( s \)-energy, one can find \( x_0^1, x_0^2, \ldots, x_0^d \in \text{supp}(\mu) \) and \( \delta > 0 \) such that \( x^{(j)} \neq 0, 1 \leq j \leq d \), whenever \( x^j \in B(x_0^j, \delta) =: X^j, 1 \leq j \leq d \). This follows by a straightforward modification of the argument in Sec. 4.3.

For \( 1 \leq j \leq d \), each variety \( \mathcal{W}^{(j)} := \{ x \in \mathbb{R}^d : x^{(j)} = 0 \in \mathbb{R}^d \} \) is codimension \( (d - 1) \), and thus their union is a null set with respect to \( \otimes^d \mu \), since \( sd > d^2 - (d - 1) \).

Restricting \( \Phi \) to \( X^1 \times \cdots \times X^d \), \((DF)_\sigma\) is satisfied. In fact, each of the components of \( d\Phi \),
\[
d_{\mu}\Phi|_{x_0} = (-1)^j x^{(j)},
\]
is nonzero, since when paired against \( x_0^j \) it gives \( \Phi(x_0^1, \ldots, x_0^d) \neq 0 \).

Thus, for \( \sigma = (12 \ldots (d - 1)|d) \) and \( t \) close to \( t_0 \), as coordinates on the incidence relations \( Z^r_\sigma \) we can take \( (x^2, \ldots, x^d) \in X^2 \times \cdots \times X^d \) and \( \vec{u} = (u_2, \ldots, u_d) \in \mathbb{R}^{d-1} \), with \( x^1 \) determined by
\[
x^1 = y^1(x^2, \ldots, x^d, t) + u_2 x^2 + \cdots + u_d x^d,
\]
for some smooth function \( y^1 \), since the perturbations of any specific \( x^1 \) that preserve \( \det [x^1, x^2, \ldots, x^d] \) are arbitrary translates in the directions spanned by \( x^2, \ldots, x^d \). Furthermore, by translating by a constant in the \( s \) variables, we can assume that at the base point,
\[
(6.2) \quad D_{x^2} y^1(x_0^1, t_0) = 0.
\]
Thus, in $T^*(X^1 \times \cdots \times X^{d-1}) \times T^*X^d$, 
\[ C^\sigma_t = \{(\cdots ; x^d, \pm \theta \left[ (y^1(x^2, \ldots, x^d, t) \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge x^d) \right] ) : (x^2, \ldots, x^d) \in X^2 \times \cdots \times X^d, \nu \in \mathbb{R}^{d-1}, \theta \in \mathbb{R} \setminus 0 \}, \]
where the first entries, giving the coordinates in $T^*(X^1 \times \cdots \times X^{d-1})$, have been suppressed because they are not needed to study $\pi_R$. In the last, i.e., $\xi^d$, entry, we have used 
\[ (y^1(x^2, \ldots, x^d, t) + u_2 x^2 + \cdots + u_d x^d) \wedge x^2 \wedge \cdots \wedge x^{d-1} \]
\[ = y^1(x^2, \ldots, x^d, t) \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge x^d. \]

We claim that $\pi_R : C^\sigma_t \to T^*X^d$ is a submersion, which, as described in Sec. 3.1, then implies that $C^\sigma_t$ is nondegenerate and thus $\sigma$ is one of the competitors in (5.11). Note that 
\[ D_{x^d} \pi_R = I_d \oplus (\pm \theta \left[ D_{x^d} y^1 \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge x^{d-1} \wedge I_d \right]) \]
while, for $2 \leq j \leq d - 1$, 
\[ D_{x^j} \pi_R = 0 \oplus (\pm \theta \left[ D_{x^j} y^1 \wedge x^2 \wedge \cdots \wedge x^{d-1} + (-1)^d u_d x^2 \wedge \cdots \wedge I_d \wedge \cdots \wedge x^d \right]) . \]

Due to the form of $D_{x^d} \pi_R$, it suffices to show that $D_{x^d} \pi_R$ has rank equal to $d$. Since $\theta \neq 0$ and $\sigma$ is an isomorphism, we can ignore the $\pm \theta$ and work directly in the $d$-dimensional vector space $\Lambda^{d-1} \mathbb{R}^d$. At $x_0$, the expression in square brackets in (6.3) equals $(-1)^d u_d I_d \wedge (x^3 \wedge \cdots \wedge x^d)$ due to (6.2). Since $x^3 \wedge \cdots \wedge x^d$ in $\Lambda^{d-2} \mathbb{R}^d - \{0\}$, this last map is an isomorphism $\mathbb{R}^d \to \Lambda^{d-1} \mathbb{R}^d$, thus has rank $d$, finishing the proof.

### 6.3. Pairs of areas in $\mathbb{R}^2$. We now prove Thm. 1.6 concerning the set of pairs of areas of triangles generated by 4-tuples of points in a compact $E \subset \mathbb{R}^2$. Here, $d = 2$, $k = 4$ and $p = 2$. On $(\mathbb{R}^2)^3$, let 
\[ \Phi (x^1, x^2, x^3, x^4) = \left( \det \left[ x^1 - x^4, x^2 - x^4 \right], \det \left[ x^2 - x^4, x^3 - x^4 \right] \right) \]
\[ = \left( (x^1 - x^4) \cdot (x^2 - x^4)^\perp, (x^2 - x^4) \cdot (x^3 - x^4)^\perp \right) \]
We will show that, for $\sigma = (13/24)$, although $C^\sigma_t$ is degenerate, the projections $\pi_L, \pi_R$ drop rank by at most 1 everywhere, and therefore, by Thm. 3.1(ii), there is a loss of at most $\beta^\sigma = 1/2$ derivative. Here, $d_L = d_R = 4$, so Thm. 5.2 implies that for 
\[ 4 \dim_H(E) > \max(d_L, d_R) + p + 2 \beta_\Phi = 4 + 2 + 1 = 7, \]
i.e., for $\dim_H(E) > 7/4$, one has $\text{int}(\Delta_\Phi(E, E, E)) \neq \emptyset$. 

To verify \((DF)_\sigma\) for \(\sigma = (13|24)\), we calculate
\[
D\Phi = \begin{bmatrix}
(x^2 - x^4)^\perp & (x^4 - x^1)^\perp & 0 & (x^1 - x^2)^\perp \\
0 & (x^3 - x^4)^\perp & (x^4 - x^2)^\perp & (x^2 - x^3)^\perp
\end{bmatrix}
\]
and note that the first and third columns form a matrix of rank two if \(x^2 \neq x^4\), as do the second and fourth columns under the same condition.

Pick any \(s\) with \(7/4 < s < \dim_R(E)\) and let \(\mu\) be a Frostman measure on \(E\) of finite \(s\)-energy. Arguing as in the earlier proofs, we can pick a four-tuple \(x_0 = (x_0^1, x_0^2, x_0^3, x_0^4)\) with each \(x_0^i \in \text{supp}(\mu)\) such that

(i) \(x_0^1 - x_0^4 \neq 0\); and
(ii) \(x_0^1 - x_0^4\) and \(x_0^3 - x_0^4\) are linearly independent;

Let \(X = B(x_0, \delta)\), with \(\delta\) chosen small enough so that (i) and (ii) hold with \(x_0\) replaced by any \(x \in X^1 \times X^2 \times X^3 \times X^4\).

Let \(t_0 = (t_0^1, t_0^2) = \Phi(x_0)\). Then, we claim that the projections \(\pi_L : C_{t_0}^\sigma \to T^*X_L\) and \(\pi_R : C_{t_0}^\sigma \to T^*X_R\) drop rank by 1 everywhere; as described in Thm. 3.1 (ii), it suffices to show this for one of projections, say \(\pi_L\). By (ii) above, we can parametrize \(Z_{t_0}^\sigma\) by \((x^1, x^3, x^4)\), with \(x^2\) determined by the nonsingular linear system
\[
(x^1 - x^4) \cdot (x^2 - x^4)^\perp = t_0^1, \quad (x^2 - x^4) \cdot (x^3 - x^4)^\perp = t_0^1,
\]
whose unique solution we can describe by \(x^2 = X^2(x^1, x^3, x^4)\). Then
\[
C_{t_0}^\sigma = \left\{ \left( x^1, x^3, \theta_1 \left( X^2 - x^4 \right)^\perp, -\theta_2 \left( X^2 - x^4 \right)^\perp ; \ldots, \ldots \right) : (x^1, x^3, x^4) \in X^1 \times X^3 \times X^4, (\theta_1, \theta_2) \in \mathbb{R}^2 \setminus \{0\} \right\},
\]
where the \(T^*X^\delta\) components on the right are suppressed because they are not needed for the analysis. One easily sees that \(D\pi_L\) drops rank by 1 everywhere, i.e., has constant rank equal to \(7\), with the image of \(\pi_L\) being contained in the hypersurface \(\{(x^1, x^3, \xi^1, \xi^3) : \xi^1 \wedge \xi^3 = 0\}\). By a fact valid for general canonical relations, \(D\pi_R\) also drops rank by 1 everywhere, as well, and by semicontinuity of the rank, \(C_{t_0}^\sigma\) drops rank by \(k \leq 1\) for all \(t\) close to \(t_0\). (Thus, \(\delta\) above is chosen small enough that all of the values in \(\Phi(X)\) are sufficiently close to \(t_0\).) By Thm. 3.1(ii), the \(R_{t_0}^\sigma\) lose at most \(\beta_\Phi \leq 1/2\) derivatives, and we are done.

6.4. Dot products of differences. To prove Thm. 1.7, define
\[
\Phi : (\mathbb{R}^d)^4 \to \mathbb{R}, \Phi(x, y, z, w) = (x - y) \cdot (z - w).
\]
We will show that using \(\sigma = (13|24)\) results in \(C_t^\sigma\) which are local canonical graphs, so that Thm. 5.2(ii) applies (with \(\beta^\sigma = 0\)) to yield nonempty interior of the set of dot products of differences for \(\dim_H(E) > (d/2) + (1/4)\).
One computes
\[ d\Phi(x, y, z, w) = (z - w, - (z - w), x - y, - (x - y)), \]
so that \((DF)_\sigma\) is satisfied away from \(\mathcal{W} := \{x - y = z - w = 0\}\), which is a codimension \(2d\) plane in \(\mathbb{R}^{4d}\). If \((d/2) + (1/4) < s < \dim_H(E)\) and \(\mu\) is a Frostman measure on \(E\) of finite \(s\)-energy then, arguing as we have above, \(\otimes^4 \mu\) is dominated by \(4s\)-dimensional Hausdorff measure, and \(4s > 2d + 1\). Since \(\mathcal{W}\) is a subspace of dimension \(2d\), \((\otimes^4 \mu) (\mathcal{W}) = 0\); repeating previous arguments, we can find base points \(x^0, y^0, z^0, w^0 \in \text{supp}(\mu)\) and \(\epsilon, \delta > 0\) such that \(|x - y| + |z - w| > \epsilon\) for \(x \in X^1 := B(x^0, \delta), y \in X^2 := B(y^0, \delta), z \in X^3 := B(z^0, \delta)\) and \(w \in X^4 := B(w^0, \delta)\), resp. Thus, \((DF)_\sigma\) is satisfied on \(X_L \times X_R\). Furthermore, by relabelling and rotating if necessary, we can assume that \(|z_1 - w_1| \neq 0\) on \(X_L \times X_R\), so that \(d_{z_1} \Phi \neq 0\).

Thus, letting \(t^0 = \Phi(x^0, y^0, x^0, w^0)\), for \(t\) close to \(t^0\), on the hypersurface \(Z^\sigma_t\) we can solve for \(x_1\) as a smooth function of the other variables: \(x_1 = x_1(x', y, z, w)\), defined for \(x'\) in a small ball \(B \subset \mathbb{R}^{d-1}\), and then parametrize
\[ C^\sigma_t = \{ (\cdot, \cdot, \cdot, \cdot; y, w, \theta(z - w), \theta((x_1, x') - y)) : y, z, w \in X^2 \times X^3 \times X^4, x' \in B, \theta \neq 0 \}, \]
where we have suppressed the \(T^*X_L\) entries as irrelevant for the analysis of \(\pi_R : C^\sigma_t \rightarrow T^*X_R = T^* (X^2 \times X^4)\). Due to the simple dependence of the \(T^*X^2\) and \(X^4\) entries on the coordinates \(y, z\) and \(w\) on \(C^\sigma_t\), and denoting elements of \(T^*X^4\) by \((w, \omega)\), we see that
\[ \text{rank}(D\pi_R) = 3d + \text{rank}\left( \frac{D\omega}{D(\theta, x')} \right) = 4d. \]

Thus, \(C^{(13, 24)}_t\) is a local canonical graph for \(t\) close to \(t_0\), and Thm. 5.2 (ii) applies with \(k = 4, p = 1, d_L = d_R = 2d\), so that for \(\dim_H(E) > (1/4)(2d + 1) = (d/2) + (1/4)\),
\[ \text{int} \{ ((x - y) \cdot (z - w) : x, y, z, w \in E \} \neq \emptyset. \]

6.5. **Sum-product sets for bilinear forms.** We now state and prove a more general version of Thm. 1.8 on sum-product sets associated to families of bilinear forms.

**Theorem 6.1.** Let \(\bar{Q} = (Q_1, \ldots, Q_l)\), with the \(Q_j\) nondegenerate, symmetric bilinear forms on \(\mathbb{R}^{n_j}\), \(1 \leq j \leq l\). Define \(d_1, \ldots, d_{2l}\) by \(d_{2j-1} = d_{2j} = n_j, 1 \leq j \leq l\). Suppose that \(E_i \subset \mathbb{R}^{d_i}\) are compact, \(1 \leq i \leq 2l\), with
\[ \sum_{i=1}^{2l} \dim_H(E_i) > 1 + \frac{1}{2} \sum_{i=1}^{2l} d_i = 1 + \sum_{j=1}^{l} n_j. \]
Then the generalized sum-product set,

\[
\Sigma_Q(E_1, \ldots, E_{2l}) := \left\{ \sum_{j=1}^l Q_j \left( x^{2j-1}, x^{2j} \right) : x^i \in E_i, \ 1 \leq i \leq 2l \right\} \subset \mathbb{R},
\]

has nonempty interior.

Define

\[
\Phi \left( x^1, \ldots, x^{2l} \right) = \sum_{j=1}^l Q_j \left( x^{2j-1}, x^{2j} \right) \text{ on } \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{2l}}.
\]

We show that Thm. 6.1 follows from Thm. 5.2 (ii), using \( \sigma = (13 \ldots (2l-1) \mid 24 \ldots (2l)) \), so that \( d_L = d_R = n := \sum_{j=1}^l n_j \), and with \( p = 1 \). Since we may write \( Q_j \left( x^{2j-1}, x^{2j} \right) = A^j x^{2j-1} \cdot x^{2j} \) for nonsingular, symmetric \( A^j \in \mathbb{R}^{n_j \times n_j} \),

\[
d_{x^{2j-1}} \Phi = A^j x^{2j} \text{ and } d_{x^{2j}} \Phi = A^j x^{2j-1}.
\]

Since the \( A^j \) are nonsingular, all of these are nonzero, and thus the double fibration condition (5.5) is satisfied if all \( x^{2j-1}, x^{2j} \neq 0 \). Letting \( X^i = \mathbb{R}^{d_i} \setminus 0, \ 1 \leq i \leq 2l \), it follows that \( Z^\sigma \subset X := \prod_i X^i \) is a smooth hypersurface, and we need to analyze the canonical relation in \((T^* X_L \setminus 0) \times (T^* X_R \setminus 0)\),

\[
C^\sigma_i = \left\{ \left( x^1, x^3, \ldots, x^{2l-3}, x^{2l-1}, \theta A^1 x^2, \theta A^2 x^4, \ldots, \theta A^l x^{2l}, \ldots \right) : x \in Z^\sigma, \theta \neq 0 \right\},
\]

where the entries on the right, in \( T^* X_R \), are the even variants of the entries on the left and have been suppressed.

For each of the \( 2l \) sets \( E_i \), let \( s_i < \dim_H(E_i) \) and \( \mu_i \) be a Frostman measure on \( E_i \) with finite \( s_i \)-energy. Let \( E := E_1 \times E_1 \times \cdots \times E_l \times E_l \) and pick a base point \( x_0 := (x_0^1, \ldots, x_0^{2l}) \in E \), which we can assume has all of its components nonzero and thus belongs to \( X \), and a \( 0 < \delta_i < |x_0^i| \) such that \( \mu_i (B (x_0^i, \delta_i)) > 0 \).

Set \( t_0 = \Phi(x_0) \). By rotations in \( x^1 = (x_1^1, \ldots, x_{d_1}) =: (x_1^1, (x_1^i)^\prime) \) and \( x^2 \) if necessary, we can assume that \( d_{x_1^1} \Phi(x_0) \neq 0 \), so that near \( x_0 \), \( Z^\sigma_{t_0} \) is the graph of a function, \( x_1^1 = f ((x_1^1)^\prime, x^2, \ldots, x^{2l}) \), with \( d_{x_1^1} f = 0 \). Hence, we can compute the projection \( \pi_L : C^\sigma_{t_0} \to T^* X_L \) with respect to coordinates \((x_1^1)^\prime, x^2, \ldots, x^{2l}, \theta \). Since \( A^1 \) is nonsingular and \( \theta \neq 0 \), one sees that the map \((x_1^1)^\prime, x^2, \theta \) into the \( T^* X^1 \) entries has full rank, as do all of the maps \( x^{2j-1}, x^{2j} (\text{with } \theta \text{ fixed}) \) to \( T^* X^{2j-1} \), so that \( D_{\pi_L} \) has full rank, and \( C^\sigma_{t_0} \) is a local canonical graph. Hence, \( \beta^\sigma = 0 \) and Thm. 5.2(ii) applies, yielding int \( \left( \Sigma_Q \left( E_1, \ldots, E_{2l} \right) \right) \neq \emptyset \) if \( \sum_i s_i > n + 1 \). i.e., if \( \sum_i \dim_H(E_i) > 1 + (1/2) \sum d_i \).
7. Final comments

It would be interesting to know whether the Hausdorff dimension thresholds in any of these theorems are sharp. However, it is worth remarking that the results on pinned volumes and sum-products at least have the correct asymptotic behavior as the dimension or the number of quadratic forms tend to infinity, even for the weaker Falconer problem of positive Lebesgue measure:

In Thm. 1.2, since all of the volumes are zero if $x^0$ and $E$ both lie in a hyperplane, one can not take $\dim_H(E) \leq d - 1$, and so the restriction $\dim_H(E) > d - 1 + (1/d)$ cannot be improved by more than $1/d$.

Similarly, in Thms. 1.8 and 6.1, if we take $E_{2j-1}$ and $E_{2j}$ to be in $Q_j$-orthogonal subspaces of $\mathbb{R}^d$ (in the notation of Thm. 1.8), then $\Sigma_Q(E_1, \ldots, E_{2l}) = \{0\}$. Thus, it is necessary that $\dim_H(E_{2j-1}) + \dim_H(E_{2j}) > d$, $1 \leq j \leq l$, so that the $1/l$ in $\dim_H(E_{2j-1}) + \dim_H(E_{2j}) > d + (1/l)$ cannot be reduced by more than $1/l$.

Finally, we observe that the results here are obtained by extracting as much as possible from standard estimates for linear Fourier integral operators. A number of previous results on translation-invariant Falconer-type configuration problems, such as [9, 8, 11], are based on genuinely bilinear or multi-linear estimates for generalized Radon transforms and FIOs, in settings where the Fourier transform is an effective tool. One can ask whether the thresholds in this paper (and in [13] for 2-point configurations), where the families $\mathcal{R}^c_x$ are typically nontranslation-invariant, can be lowered by obtaining truly multi-linear estimates for FIOs.

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