Canonical quantization of the electromagnetic field in an anisotropic polarizable and magnetizable medium

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Abstract

A fully canonical quantization of electromagnetic field is introduced in the presence of an anisotropic polarizable and magnetizable medium. Two tensor fields which couple the electromagnetic field with the medium and have an important role in this quantization method are introduced. The electric and magnetic polarization fields of the medium naturally are concluded in terms of the coupling tensors and the dynamical variables modeling the magnetodielectric medium. In Heisenberg picture, the constitutive equations of the medium together with the Maxwell laws are obtained as the equations of motion of the total system and the susceptibility tensors of the medium are calculated in terms of the coupling tensors. Following a perturbation method the Green function related to the total system is found and the time dependence of electromagnetic field operators is derived.

Key words: Canonical field quantization, Magnetodielectric medium, conductivity tensor, Susceptibility tensor, Coupling tensor, Constitutive equation

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1 introduction

One of the most important quantum dissipative systems is the quantized electromagnetic field in the presence of an absorbing polarizable medium. In this case there are mainly two quantization approaches, the phenomenological method [1]-[7] and the damped polarization model [8, 9]. The phenomenological scheme has been formulated on the basis of the fluctuation- dissipation theorem [10]. In this method by adding a fluctuating noise term, that is the noise polarization field, to the classical constitutive equation of the medium, this equation is taken as the definition of the electric polarization operator. Combination of the Maxwell equations and the constitutive equation in the frequency domain, gives the electromagnetic field operators in terms of the noise polarization field and the classical Green tensor. A set of bosonic operators is associated with the noise polarization which their commutation relations are given in agreement with the fluctuation- dissipation theorem. This quantization scheme has been quite successful in describing some electromagnetic phenomena in the presence of a lossy dielectric medium [11]-[15]. The phenomenological approach has been extended to a lossy magnetic or anisotropic medium [16]-[18]. This formalism has also been generalized to an arbitrary linearly responding medium based on a spatially nonlocal conductivity tensor [19].

The damped polarization model to quantize electromagnetic field in a dispersive dielectric medium [8, 9] is a canonical quantization in which the electric polarization field of the medium applies in the Lagrangian of the total system as a part of the degrees of freedom of the medium. The other parts of the degrees of freedom of the absorbing medium are related to the dynamical variables of a heat bath describing the absorptivity feature of the medium. In this method the dielectric function of the medium is found in terms of the coupling function of the heat bath and the polarization field, so that it satisfies the Kramers-Kronig relations [20]. This quantization method has been generalized to an inhomogeneous medium [21].

In the present work we generalize our previous model [22, 23] to an anisotropic dispersive magnetodielectric medium using a canonical approach. In this formalism the medium is modeled with two independent collections of vector fields. These collections solely constitute the degrees of freedom of the medium and it is not needed the electric and magnetic polarization fields to be included in the Lagrangian of the total system as a part of the degrees of freedom of the medium as in the Huttner-Barnett model [8]. In fact the
dynamical fields modeling the dispersive medium are able to describe both polarizability and absorptivity features of the medium.

This paper is organized as follows. In Sec. 2, a Lagrangian for the total system is proposed and classical electrodynamics in the presence of an anisotropic polarizable and magnetizable medium with spatial-temporal dispersion briefly is discussed. In Sec. 3, applying the Lagrangian introduced in Sec. 2 a fully canonical quantization of both electromagnetic field and the dynamical variable modeling the responding medium is demonstrated. Then in Sec. 4, the constitutive equations of the medium are obtained as the consequences of the Heisenberg equations of the total system and the electric and magnetic susceptibility tensors of the medium are calculated in terms of the parameters applied in the theory. In Sec. 5, it is shown that the Green function of the total system in reciprocal space satisfies an algebraic equation and a perturbation method to obtain the Green function is introduced. Finally in section 6 the model is modified for media which the distinction between polarization and magnetization is not possible. This paper is closed with a summary and some concluding remarks in Sec. 7.

2 Classical electrodynamics in an anisotropic magnetodielectric medium

In order to present a fully canonical quantization of electromagnetic field in the presence of an anisotropic polarizable and magnetizable medium, we model the medium by two independent reservoirs. Each reservoir contains a continuum of three dimensional harmonic oscillators labeled with a continuous parameter $\omega$. We call these two continuous sets of oscillators "$E$ field" and "$M$ field". The $E$ field and $M$ field describe polarizability and magnetizability of the medium, respectively. This means that, in this approach it is not needed the electric and magnetic polarization fields of the medium to be appeared explicitly in the Lagrangian of the total system as a part of the degrees of freedom of the medium, but the contribution of the medium in the Lagrangian of the total system is related only to the Lagrangian of the $E$ and $M$ fields and these fields completely describe the degrees of freedom of the medium. The presence of the "$E$ field" in the total Lagrangian is sufficient for a complete description of both polarizability and the absorption of the
medium due to its electrically dispersive property. Also the "$M$ field" solely is sufficient in order to description of both magnetizability and the absorption of the medium due to its magnetically dispersive property. Therefore, in order to have a classical treatment of electrodynamics in a magnetodielectric medium, we start with a Lagrangian for the total system (medium + electromagnetic field ) which is the sum of three parts

$$L(t) = L_{res} + L_{em} + L_{int}$$  \hspace{1cm} (1)

where $L_{res}$ is the part related to the degrees of freedom of the medium and is the sum of the Lagrangians of the $E$ and $M$ fields

$$L_{res} = L_e + L_m$$  \hspace{1cm} (2)

where

$$L_e = \int_0^\infty d\omega \int d^3r \left[ \frac{1}{2} \dot{\vec{X}}_\omega \cdot \dot{\vec{X}}_\omega - \frac{1}{2} \omega^2 \vec{X}_\omega \cdot \vec{X}_\omega \right]$$  \hspace{1cm} (3)

and

$$L_m = \int_0^\infty d\omega \int d^3r \left[ \frac{1}{2} \dot{\vec{Y}}_\omega \cdot \dot{\vec{Y}}_\omega - \frac{1}{2} \omega^2 \vec{Y}_\omega \cdot \vec{Y}_\omega \right]$$  \hspace{1cm} (4)

Here the fields $\vec{X}_\omega$ and $\vec{Y}_\omega$ are the dynamical variables of the $E$ and $M$ fields, respectively.

In (1), $L_{em}$ is the contribution of the electromagnetic field in the Lagrangian of the total system

$$L_{em} = \int d^3r \left[ \frac{1}{2} \varepsilon_0 \vec{E}^2 - \frac{\vec{B}^2}{2\mu_0} \right]$$  \hspace{1cm} (5)

and $L_{int}$ is the part describing the interaction of the electromagnetic field with the medium

$$L_{int} = \int_0^\infty d\omega \int d^3r \int d^3r' f_{ij}(\omega, \vec{r}, \vec{r}') E^i(\vec{r}, t) \vec{X}_\omega^j(\vec{r}', t) + \int_0^\infty d\omega \int d^3r \int d^3r' g_{ij}(\omega, \vec{r}, \vec{r}') B^i(\vec{r}, t) \vec{Y}_\omega^j(\vec{r}', t)$$  \hspace{1cm} (6)

The contributions $L_e$ and $L_m$ in the Lagrangian $L_{res}$ are equivalent to the consequences of diagonalization processes of the matter fields in the Hutten-Barnet model [8]. That is, $L_e$ ( $L_m$ ) is equivalent to the diagonalization
of the contributions of related to three parts in the Huttner-Barnet model: the dynamical variable describing the electric polarization (magnetic polarization) of the medium, a heat-bath $B (B')$ interacting with the electric polarization (magnetic polarization) and the interaction term between the heat-bath $B (B')$ and the electric polarization (magnetic polarization). In the present approach modeling the medium, in a phenomenological way, with two independent set of oscillators the lengthy diagonalization processes have been eliminated in the start of this quantization scheme. Particularly the diagonalization processes may be more tremendous for an anisotropic medium.

In equations (5) and (6) $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ are electric and magnetic fields respectively, where $\vec{A}$ and $\phi$ are the vector and the scalar potentials. The tensors $f$ and $g$ in (6) are called the coupling tensors of the medium with electromagnetic field and for an inhomogeneous medium are dependent on the both position vectors $\vec{r}$ and $\vec{r}'$. The coupling tensors are the key parameters of this theory. As was mentioned above, it is not needed the electric and magnetic polarization fields of the medium explicitly to be appeared in the total Lagrangian (1)-(6) as a part of degrees of freedom of the medium. As we will see, the electric polarization (magnetic polarization) of the medium is obtained in terms of the coupling tensor $f (g)$ and the dynamical variables $\vec{X}_\omega (\vec{Y}_\omega)$. Also the electric susceptibility tensor (magnetic susceptibility tensor) of the medium naturally will expressed in terms of the coupling tensor $f (g)$. The coupling tensors $f$ and $g$ are appeared as common factors in both the noise polarization fields and the susceptibility tensors of the medium, so that for the free space the susceptibility tensors together with the noise polarizations become identically zero and this quantization scheme is reduced to the usual quantization of electromagnetic field in free space. Furthermore when the medium tends to a non-absorbing one, the coupling tensors and the noise polarizations tend also to zero and this quantization method is reduced to the quantization in a non-absorbing medium [22, 24].

In order to prevent some difficulties with a non-local Lagrangian such as in (6), it is the easiest way to work in the reciprocal space and write all the fields and the coupling tensors $f, g$ in terms of their spatial Fourier
transforms. For example the dynamical variable \( \vec{X}_\omega \) can be written as
\[
\vec{X}_\omega(\vec{r}, t) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \, \vec{X}_\omega(\vec{k}, t) \, e^{i\vec{k} \cdot \vec{r}}
\]
(7)

Since we are concerned with real valued fields in the total Lagrangian (1)-(6), we have \( \vec{X}_\omega^*(\vec{k}, t) = \vec{X}_\omega(-\vec{k}, t) \) for the field \( \vec{X}_\omega(\vec{r}, t) \) and the other dynamical fields in this Lagrangian. Similarly the real valued coupling tensors \( f \) and \( g \) can be expressed in reciprocal space as
\[
f_{ij}(\omega, \vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \, f_{ij}(\omega, \vec{k}, \vec{k}') e^{i\vec{k} \cdot \vec{r} - i\vec{k'} \cdot \vec{r}'}
\]
\[
g_{ij}(\omega, \vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \, g_{ij}(\omega, \vec{k}, \vec{k}') e^{i\vec{k} \cdot \vec{r} - i\vec{k'} \cdot \vec{r}'}
\]
(8)

which obey the following conditions
\[
f_{ij}(\omega, \vec{k}, \vec{k}') = f_{ij}^*(\omega, -\vec{k}, -\vec{k}')
\]
\[
g_{ij}(\omega, \vec{k}, \vec{k}') = g_{ij}^*(\omega, -\vec{k}, -\vec{k}')
\]
(9)

The number of independent variables can be recovered by restricting the integrations to the half space \( k_z \geq 0 \). The total Lagrangian (1)-(6) is then obtained as
\[
L(t) = L_{res}(t) + L_{em}(t) + L_{int}(t)
\]
(10)

\[
L_{res}(t) = \int_0^\infty d\omega \int' d^3k \left( |\vec{X}_\omega|^2 - \omega^2 |\vec{X}_\omega|^2 \right) + \int_0^\infty d\omega \int' d^3k \left( |\vec{Y}_\omega|^2 - \omega^2 |\vec{Y}_\omega|^2 \right)
\]
(11)

\[
L_{em}(t) = \int' d^3k \left( \varepsilon_0 |\vec{A}|^2 + \varepsilon_0 |\vec{k} \times \vec{A}|^2 - \frac{|\vec{k} \times \vec{A}|^2}{\mu_0} \right) + \int' d^3k \left( -i\vec{k} \cdot \vec{A} \varphi^* + h.c \right)
\]
(12)

\[
L_{int}(t) =
- \int_0^\infty d\omega \int' d^3q \int' d^3p \left[ \vec{\hat{A}}(\vec{q}, t) + i\vec{q} \cdot \varphi(\vec{q}, t) \right] \cdot \vec{f}(\omega, -\vec{q}, \vec{p}) \cdot \vec{X}_\omega(\vec{p}, t) + h.c]
- \int_0^\infty d\omega \int' d^3q \int' d^3p \left[ \vec{\hat{A}}^*(\vec{q}, t) - i\vec{q} \cdot \varphi^*(\vec{q}, t) \right] \cdot \vec{f}(\omega, \vec{q}, \vec{p}) \cdot \vec{X}_\omega(\vec{p}, t) + h.c]
+ \int_0^\infty d\omega \int' d^3q \int' d^3p \left[ i\vec{q} \times \vec{\hat{A}}(\vec{q}, t) \right] \cdot \vec{g}(\omega, -\vec{q}, \vec{p}) \cdot \vec{Y}_\omega(\vec{p}, t) + h.c]
+ \int_0^\infty d\omega \int' d^3q \int' d^3p \left[ -i\vec{q} \times \vec{\hat{A}}^*(\vec{q}, t) \right] \cdot \vec{g}(\omega, \vec{q}, \vec{p}) \cdot \vec{Y}_\omega(\vec{p}, t) + h.c]
\]
(13)
where \( \int' d^3k \) implies the integration over the half space \( k_z \geq 0 \) (hereafter, we apply the symbol \( \int' d^3k \) for the integration on the half space \( k_z \geq 0 \) and \( \int d^3k \) for the integration on the total reciprocal space). In the reciprocal space the total Lagrangian (10)-(13) do not involve the space derivatives of the dynamical variables of the system and the classical equations of the motion of the system can be obtained using the principle of the Hamilton’s least action, \( \delta \int dt L(t) = 0 \). These equations are the Euler-Lagrange equations.

For the vector potential \( \vec{A}(\vec{k}, t) \) and the scalar potential \( \varphi(\vec{k}, t) \) we find

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta (\vec{A}^*(\vec{k}, t))} \right) - \frac{\delta L}{\delta (\vec{A}^*(\vec{k}, t))} = 0 \quad i = 1, 2, 3
\]

\[
\Rightarrow \quad \mu_0 \varepsilon_0 \ddot{\vec{A}}(\vec{k}, t) + \mu_0 \varepsilon_0 \vec{k} \varphi(\vec{k}, t) - \vec{k} \times \left( \vec{k} \times \vec{A}(\vec{k}, t) \right) =
\mu_0 \dot{\vec{P}}(\vec{k}, t) + \mu_0 \vec{k} \times \vec{M}(\vec{k}, t) \quad (14)
\]

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta (\varphi^*(\vec{k}, t))} \right) - \frac{\delta L}{\delta (\varphi^*(\vec{k}, t))} = 0
\]

\[
\Rightarrow \quad -\varepsilon_0 \vec{k} \cdot \vec{A}(\vec{k}, t) + \varepsilon_0 |\vec{k}|^2 \varphi(\vec{k}, t) = -i\vec{k} \cdot \vec{P}(\vec{k}, t) \quad (15)
\]

for any wave vector \( \vec{k} \) in the half space \( k_z \geq 0 \) where

\[
\vec{P}(\vec{k}, t) = \int_0^\infty d\omega \int d^3p \ f(\omega, \vec{k}, \vec{p}) \cdot \vec{X}_\omega(\vec{p}, t) \quad (16)
\]

\[
\vec{M}(\vec{k}, t) = \int_0^\infty d\omega \int d^3p \ g(\omega, \vec{k}, \vec{p}) \cdot \vec{Y}_\omega(\vec{p}, t) \quad (17)
\]

are respectively the spatial Fourier transforms of the electric and magnetic polarization densities of the medium and it has been used from the relations (9). Therefore in this method the polarization fields of the medium are naturally concluded in terms of the coupling tensors \( f, g \) and the dynamical variables of the \( E \) and \( M \) fields. Similarly the Euler-Lagrange equations for
the fields $\vec{X}_\omega$ and $\vec{Y}_\omega$ for any vector $\vec{k}$ in the half space $k_z \geq 0$ are easily obtained as

$$\frac{d}{dt} \left( \frac{\delta L}{\delta (\dot{X}^*_\omega(k, t))} \right) - \frac{\delta L}{\delta (X^*_\omega(k, t))} = 0 \quad i = 1, 2, 3$$

$$\Rightarrow \ddot{X}_\omega(k, t) + \omega^2 \vec{X}_\omega(k, t) = -\int d^3q f^\dagger(\omega, \vec{q}, \vec{k}) \cdot \left( \vec{A}(\vec{q}, t) + i\vec{q}_\omega(\vec{q}, t) \right)$$

(18)

$$\frac{d}{dt} \left( \frac{\delta L}{\delta (\dot{Y}^*_\omega(k, t))} \right) - \frac{\delta L}{\delta (Y^*_\omega(k, t))} = 0 \quad i = 1, 2, 3$$

$$\Rightarrow \ddot{Y}_\omega(k, t) + \omega^2 \vec{Y}_\omega(k, t) = \int d^3q g^\dagger(\omega, \vec{q}, \vec{k}) \cdot \left( i\vec{q} \times \vec{A}(\vec{q}, t) \right)$$

(19)

where $f^\dagger$ and $g^\dagger$ are the hermitian conjugate of the tensors $f$ and $g$, respectively.

### 3 Canonical quantization

Following the standard approach, we choose the Coulomb gauge $\vec{k} \cdot \vec{A}(\vec{k}, t) = 0$ to quantize electromagnetic field. In this gauge the vector potential $\vec{A}$ is a purely transverse field and can be decomposed along the unit polarization vectors $\vec{e}_\lambda\vec{k}$ $\lambda = 1, 2$ which are orthogonal to each other and to the wave vector $\vec{k}$.

$$\vec{A}(\vec{k}, t) = \sum_{\lambda=1}^{2} A_\lambda(k, t)\vec{e}_\lambda\vec{k}$$

(20)

Although the vector potential is purely transverse, but the dynamical fields $\vec{X}_\omega$ and $\vec{Y}_\omega$ may have both transverse and longitudinal parts and can be expanded along the three mutually orthogonal unit vectors $\vec{e}_\lambda\vec{k}$ $\lambda = 1, 2$
and \( \vec{e}_{\lambda k} = \frac{\vec{k}}{|\vec{k}|} \) as

\[
\begin{align*}
X_{\omega}(\vec{k}, t) &= \sum_{\lambda=1}^{3} X_{\omega\lambda}(\vec{k}, t) \vec{e}_{\lambda k} \\
Y_{\omega}(\vec{k}, t) &= \sum_{\lambda=1}^{3} Y_{\omega\lambda}(\vec{k}, t) \vec{e}_{\lambda k}
\end{align*}
\] (21)

Furthermore in Coulomb gauge the Euler- Lagrange equation (15) can be used to eliminate the extra degree of freedom \( \varphi(\vec{k}, t) \) from the Lagrangian of the system

\[
\varphi(\vec{k}, t) = -\frac{i\vec{k} \cdot \vec{P}(\vec{k}, t)}{\varepsilon_0 |\vec{k}|^2}
\] (22)

The total Lagrangian (10)-(13) can now be rewritten in terms of the independent dynamical variables \( A_{\lambda} \lambda = 1, 2 \) and \( X_{\omega\lambda} \lambda = 1, 2, 3 \) which constitute completely the coordinates of the total system

\[
\begin{align*}
L(t) &= \int_0^\infty d\omega \int' d^3k \sum_{\lambda=1}^{3} \left( |\dot{X}_{\omega\lambda}|^2 - \omega^2 |X_{\omega\lambda}|^2 + |\dot{Y}_{\omega\lambda}|^2 - \omega^2 |Y_{\omega\lambda}|^2 \right) \\
&\quad + \int' d^3k \sum_{\lambda=1}^{2} \left( \varepsilon_0 |\dot{A}_{\lambda}|^2 - \frac{|\vec{k} \cdot \vec{A}_{\lambda}|^2}{\mu_0} \right) - \frac{1}{\varepsilon_0} \int' d^3k \frac{|\vec{k} \cdot \vec{P}|^2}{|\vec{k}|^2} \\
&\quad + \int' d^3k \left[ -\sum_{\lambda=1}^{2} \dot{A}_{\lambda} \vec{e}_{\lambda k} \cdot \vec{P}^* + \left( \vec{k} \times \sum_{\lambda=1}^{2} A_{\lambda \lambda} \vec{e}_{\lambda k} \right) \cdot \vec{M}^* + h.c \right]
\end{align*}
\] (23)

where the polarizations \( \vec{P} \) and \( \vec{M} \) have been defined previously in terms of the dynamical variables of the \( E \) and \( M \) fields in equations (16) and (17). The Lagrangian (23) can now be used to define the canonical conjugate momenta of the system. For any wave vector \( \vec{k} \) in half space \( k_z \geq 0 \) this momenta are
defined as

\[- \mathcal{D}_\lambda(\vec{k}, t) = \frac{\delta L}{\delta (\dot{\mathcal{A}}^*_\lambda(\vec{k}, t))} = \varepsilon_0 \dot{\mathcal{A}}^*_\lambda(\vec{k}, t) - \vec{e}_\lambda \cdot \vec{F}(\vec{k}, t)\]

\[Q_{\omega\lambda}(\vec{k}, t) = \frac{\delta L}{\delta (\dot{X}^{\omega\lambda}(\vec{k}, t))} = \dot{X}^{\omega\lambda}(\vec{k}, t)\]

\[\Pi_{\omega\lambda}(\vec{k}, t) = \frac{\delta L}{\delta (\dot{Y}^{\omega\lambda}(\vec{k}, t))} = \dot{Y}^{\omega\lambda}(\vec{k}, t)\]

The total system are quantized canonically in a standard method by imposing equal-time commutation relations between the coordinates of the system and their conjugates variables as follows

\[
\begin{align*}
[A^*_\lambda(\vec{k}, t), -\mathcal{D}_{\lambda'}(\vec{k}', t)] &= i\hbar \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \\
[X^{\omega\lambda}(\vec{k}, t), Q^{\omega\omega'}_{\lambda'}(\vec{k}', t)] &= i\hbar \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\vec{k} - \vec{k}') \\
[Y^{\omega\lambda}(\vec{k}, t), \Pi^{\omega\omega'}_{\lambda'}(\vec{k}', t)] &= i\hbar \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\vec{k} - \vec{k}')
\end{align*}
\]

Using the Lagrangian (23) and the conjugates momenta introduced in (24) the Hamiltonian of the total system can be written in the form

\[
H(t) = \int d^3 k \sum_{\lambda=1}^{2} \left( \frac{|\mathcal{D}_\lambda - \vec{e}_\lambda \cdot \vec{F}|^2}{\varepsilon_0} + \frac{|\vec{k} A^*_\lambda|^2}{\mu_0} \right) + \int' d^3 k [\frac{|i\vec{k} \times \sum_{\lambda=1}^{2} A^*_\lambda \vec{e}_\lambda|^2}{\varepsilon_0} \vec{M}^* + h.c] \\
- \int' d^3 k \left[ \left( i\vec{k} \times \sum_{\lambda=1}^{2} A^*_\lambda \vec{e}_\lambda \right) \cdot \vec{M}^* + h.c \right] + \int_0^\infty d\omega \int' d^3 k \sum_{\lambda=1}^{3} \left( |Q_{\omega\lambda}|^2 + \omega^2 |X_{\omega\lambda}|^2 + |\Pi_{\omega\lambda}|^2 + \omega^2 |Y_{\omega\lambda}|^2 \right)
\]

(26)
### 3.1 Maxwell equations

Using Heisenberg equations for the operators $\mathbf{D}_\lambda$ and $\mathbf{A}_\lambda$, that is

$$
\dot{\mathbf{A}}_\lambda(\vec{k}, t) = \frac{i}{\hbar} \left[ H, \mathbf{A}_\lambda(\vec{k}, t) \right] = -\frac{\mathbf{D}_\lambda(\vec{k}, t)}{\varepsilon_0} - \vec{e}_{\lambda\vec{k}} \cdot \vec{P}(\vec{k}, t)
$$

$$
\dot{\mathbf{D}}_\lambda(\vec{k}, t) = \frac{i}{\hbar} \left[ H, \mathbf{D}_\lambda(\vec{k}, t) \right] = \frac{|\vec{k}|^2}{\mu_0} \mathbf{A}_\lambda(\vec{k}, t) - \vec{e}_{\lambda\vec{k}} \cdot \left( i\vec{k} \times \vec{M}(\vec{k}, t) \right)
$$

Maxwell equations can be obtained as is expected. Multiplying both sides of Eqs (27) in the polarization unit vector $\vec{e}_{\lambda\vec{k}}$ and then summation on polarization index $\lambda = 1, 2$ and using (22) clearly give us

$$
\mathbf{D}(\vec{k}, t) = \varepsilon_0 \mathbf{E}(\vec{k}, t) + \vec{P}(\vec{k}, t)
$$

$$
\dot{\mathbf{D}}(\vec{k}, t) = i\vec{k} \times \mathbf{H}
$$

where $\mathbf{D} = \sum_{\lambda=1}^{2} \dot{\mathbf{D}}_\lambda \vec{e}_{\lambda\vec{k}}$ plays the role of the displacement field, $\mathbf{E} = -\dot{\mathbf{A}} - i\vec{k} \varphi = -\dot{\mathbf{A}} - \frac{\vec{k}(\vec{k} \cdot \vec{P})}{\varepsilon_0}$ is the total electric field and $\mathbf{H}(\vec{k}, t) = \frac{i\vec{k} \times \vec{A}}{\mu_0} - \vec{M}$ is the magnetic induction field.

### 4 The constitutive equations of the medium

It is well known that the constitutive equations of a responding medium are the consequences of the interaction of the medium with electromagnetic field. Any quantization method such as this theory, in which the medium enters directly in the process of quantization and the interaction of the medium with the vacuum field is explicitly given, must be able to give the constitutive equations of the medium using the Heisenberg equations of the total system. Also the susceptibility tensors of the medium which are a measure for the polarizability of the medium should be specified in terms of the parameters describing the interaction of the medium with electromagnetic field. In this section, we find the constitutive equations of the medium using the Heisenberg equations of the dynamical fields modeling the medium. As we do this,
the electric and magnetic susceptibility tensors of the medium are naturally found in terms of the coupling tensors $f$ and $g$. In the Heisenberg picture, using the commutation relations (25) and the total Hamiltonian (26), the equations of motion for the canonical variables $X_{\omega\lambda}$, $Q_{\omega\lambda}$, $\lambda = 1, 2, 3$ for each wave vector in the half space $k_z \geq 0$ follow as

\[
\begin{align*}
X_{\omega\lambda}(\vec{k}, t) &= \frac{i}{\hbar} [H, X_{\omega\lambda}(\vec{k}, t)] = Q_{\omega\lambda}(\vec{k}, t) \\
\dot{Q}_{\omega\lambda}(\vec{k}, t) &= [H, Q_{\omega\lambda}(\vec{k}, t)] = -\omega^2 X_{\omega\lambda}(\vec{k}, t) + \int d^3q f^*_i(\omega, \vec{q}, \vec{k}) E^j(\vec{q}, t) e^j_{\lambda k}
\end{align*}
\]  

(30)

where $\vec{E}(\vec{q}, t) = -\dot{\vec{A}}(\vec{q}, t) - \hat{\vec{q}} \cdot (\hat{\vec{q}} \cdot \vec{P}(\vec{q}, t))$ is the Fourier transform of the total electric field. Multiplying both sides of Eqs. (30) in the mutually orthogonal unit vectors $e^i_{\lambda k}$ and then summation on $\lambda = 1, 2, 3$ and using the completeness relations $\sum_{\lambda=1}^3 e^i_{\lambda k} e^j_{\lambda k} = \delta_{ij}$ we find

\[
\ddot{\vec{X}}(\vec{k}, t) + \omega^2 \vec{X}(\vec{k}, t) = \int d^3q \ f^i(\omega, \vec{q}, \vec{k}) \cdot \vec{E}(\vec{q}, t)
\]  

(31)

This equation can be integrated formally as

\[
\vec{X}_\omega(\vec{k}, t) = \frac{\sin \omega t}{\omega} \vec{X}_\omega(\vec{k}, 0) + \vec{Q}_\omega(\vec{k}, 0) \cos \omega t + \int_0^t dt' \sin \omega (t-t') \int d^3p \ f^i(\omega, \vec{q}, \vec{p}) \cdot \vec{E}(\vec{q}, t')
\]

(32)

Now, by substituting $\vec{X}_\omega(\vec{k}, t)$ from (32) in the definition of electric polarization density given by (16), we find the constitutive equation of the medium in reciprocal space relating the electric polarization to the electric field

\[
\vec{P}(\vec{k}, t) = \vec{P}_0(\vec{k}, t) + \varepsilon_0 \int_0^{|t|} dt' \int d^3q \chi^e(\vec{k}, \vec{q}, |t|-t') \cdot \vec{E}(\vec{q}, \pm t')
\]

(33)

where the upper (lower) sign corresponds to $t > 0$ ($t < 0$). Here the $\chi^e$ is the spatial Fourier transform of the electric susceptibility tensor

\[
\chi^e(\vec{k}, \vec{q}, t) = \begin{cases} 
\frac{1}{\varepsilon_0} \int_0^\infty d\omega \sin \omega t \int d^3p \ f(\omega, \vec{k}, \vec{p}) \cdot \overline{f^*(\omega, \vec{q}, \vec{p})} & t > 0 \\
0 & t \leq 0
\end{cases}
\]

(34)
and is guaranteed to possess solutions for the coupling tensor \( f \) in terms of the temporal Fourier transform of \( \chi^e \) using a type of eigenvalue problem [19].

It must be pointed out that for a given \( \chi^e \) the coupling tensor \( f \) satisfying the relation (34) is not unique. In fact if \( f \) satisfy (34), for a given susceptibility tensor, then

\[
\mathbf{f}'(\omega, \mathbf{k}, \mathbf{q}) = \int d^3 p' f(\omega, \mathbf{k}, \mathbf{p}') \cdot \mathbf{A}(\omega, \mathbf{q}, \mathbf{p}')
\]  

also satisfy the (34) where \( \mathbf{A} \) is a tensor with orthogonality condition

\[
\int d^3 p A(\omega, \mathbf{p}, \mathbf{p}') \cdot A^\dagger(\omega, \mathbf{p}, \mathbf{p}'') = I \delta^3(\mathbf{p}' - \mathbf{p}'')
\]

Although for a given \( \chi^e \) various choices of the tensor \( f \) satisfying (34) affect the space-time dependence of electromagnetic field operators, but all of these choices are equivalent and the commutation relations between electromagnetic field operators remain unchanged [24]. That is, the commutation relations between electromagnetic operators and the physical observables finally are dependent only on the given susceptibility tensor and not on the coupling functions \( f, g \).

In (33), \( \mathbf{P}_N \) is the spatial Fourier transform of the noise electric polarization density which is expressed in terms of the the dynamical variables of the "E field" at \( t = 0 \)

\[
\mathbf{P}_N(\mathbf{k}, t) = \int_0^\infty d\omega \int d^3 p f(\omega, \mathbf{k}, \mathbf{p}) \cdot \left( \mathbf{Q}_\omega(\mathbf{p}, 0) \sin \omega t \omega + \mathbf{X}_\omega(\mathbf{p}, 0) \cos \omega t \right)
\]  

In a similar fashion the constitutive equation relating the magnetic polarization density of the medium to the magnetic field \( \mathbf{B}(\mathbf{q}, t) = \mathbf{i} \mathbf{q} \times \mathbf{A}(\mathbf{q}, t) \) can be obtained straightforwardly using the Heisenberg equations for the conjugate dynamical variables \( \chi_{\omega\lambda} \) and \( \Pi_{\omega\lambda} \) as the form

\[
\overline{\mathbf{M}}(\mathbf{k}, t) = \overline{\mathbf{M}}_N(\mathbf{k}, t) + \frac{1}{\mu_0} \int_0^{[|t|} dt' \int d^3 q \chi^m(\mathbf{k}, \mathbf{q}, |t| - t') \cdot \mathbf{B}(\mathbf{q}, \pm t')
\]  

where the \( \chi^m \) is the spatial Fourier transform of the magnetic susceptibility tensor which is written in terms the coupling tensor \( g \) as

\[
\chi^m(\mathbf{k}, \mathbf{q}, t) = \left\{ \begin{array}{ll}
\mu_0 \int_0^\infty d\omega \omega \sin \omega t \int d^3 p g(\omega, \mathbf{k}, \mathbf{p}) \cdot \mathbf{q}^\dagger(\omega, \mathbf{q}, \mathbf{p}) & t > 0 \\
0 & t \leq 0
\end{array} \right.
\]
and

\[
\vec{M}_N(\vec{k}, t) = \int_0^\infty d\omega \int d^3p \ g(\omega, \vec{k}, \vec{p}) \cdot \left( \vec{\Pi}_\omega(\vec{p}, 0) \frac{\sin \omega t}{\omega} + \vec{Y}_\omega(\vec{p}, 0) \cos \omega t \right)
\]

is the noise magnetic polarization density. Therefore modeling an anisotropic magnetodielectric medium with two independent collections of harmonic oscillators, that is, the $E$ and $M$ field, we successfully have constructed a fully canonical quantization of electromagnetic field in the presence of such a medium, so that the Maxwell equations together with the constitutive equations of the medium have been obtained from the Heisenberg equations of the total system. In this method the susceptibility tensors of the medium are concluded in terms of the coupling tensors which are the key parameters of this theory and are describing the coupling of the electromagnetic field with the medium.

The Equations (34) and (39) relate the parameters of this theory, that is the coupling tensors $f$ and $g$, to the physical quantities $\chi^e$ and $\chi^m$. If the coupling tensors $f$ and $g$ are specified, so that the right hands of (34) and (39) become identical to the electric and magnetic susceptibility tensors of the medium, then according to the constitutive equations (33) and (38) the operators $\vec{P}$ and $\vec{M}$ defined by (16) and (17) are respectively the polarization and magnetization of the medium.

There are media for which the polarization (the magnetization) is dependent on both the electric and magnetic field. This quantization scheme can be generalized for such media if in the interaction Lagrangian (6) each of the dynamical variables $\vec{X}_\omega$ and $\vec{Y}_\omega$ is interacted with both the electric and magnetic field.

5 Space-time dependence of the electromagnetic field operators

Our goal for this section is to determine the complete expressions of the electromagnetic field operators for both negative and positive times. The Heisenberg equations of the total system, that is the Maxwell equation (29) together with the constitutive equations (28), (33) and (38), constitute a
set of coupled linear equations which can be solved in terms of the initial conditions by the temporal Laplace transformation. This technique has been used previously in the damped polarization model[21]. For an arbitrary time-dependent operator $\Gamma(t)$ the backward and forward Laplace transformations, denoted respectively, by $\Gamma^b(s)$ and $\Gamma^f(\rho)$ are defined as

$$\Gamma^b(s) = \int_0^\infty dt \ e^{-st} \Gamma(-t)$$
$$\Gamma^f(\rho) = \int_0^\infty dt \ e^{-\rho t} \Gamma(t)$$

Carrying out the backward and forward Laplace transformations of Eqs.(28), (29), (33) and $\vec{k} \times \vec{E}(\vec{k}, t) = -\dot{\vec{B}}(\vec{k}, t)$ and then their combination, for the electric field operator we find

$$\int d^3q \left[ k \times \tilde{\mu}(\vec{k}, \vec{q}, s) \left( \vec{q} \times \vec{E}^b(\vec{q}, s) \right) \right] - \frac{s^2}{c^2} \int d^3q \ \tilde{\varepsilon}(\vec{k}, \vec{q}, s) \vec{E}^b(\vec{q}, s) = \vec{J}^b_N(\vec{k}, s)$$
$$\int d^3q \left[ k \times \tilde{\mu}(\vec{k}, \vec{q}, s) \left( \vec{q} \times \vec{E}^f(\vec{q}, s) \right) \right] - \frac{s^2}{c^2} \int d^3q \ \tilde{\varepsilon}(\vec{k}, \vec{q}, s) \vec{E}^f(\vec{q}, s) = \vec{J}^f_N(\vec{k}, s)$$

where

$$\tilde{\varepsilon}(\vec{k}, \vec{q}, s) = \delta(\vec{k} - \vec{q}) \ I_s + \tilde{\chi}(\vec{k}, \vec{q}, s)$$
$$\tilde{\mu}(\vec{k}, \vec{q}, s) = \delta(\vec{k} - \vec{q}) \ I_m - \tilde{\chi}(\vec{k}, \vec{q}, s)$$

are respectively the permittivity and permeability tensors of the medium in Laplace language. The source terms in the right hand of the inhomogeneous Eqs.(42) are the backward and forward Laplace transformations of the noise current density and are expressed in terms of initial conditions of the dynamical variables of the total system at $t = 0$ as

$$\vec{J}^b_N(\vec{k}, s) = \mu_0 s^2 \vec{P}^b_N(\vec{k}, s) - \mu_0 s \ i \vec{k} \times \vec{M}^b_N(\vec{k}, s) + i \vec{k} \times \int d^3q \ \tilde{\mu}(\vec{k}, \vec{q}, s) \ \vec{B}(\vec{q}, 0) - \mu_0 s \vec{D}(\vec{k}, 0)$$
$$\vec{J}^f_N(\vec{k}, s) = \mu_0 s^2 \vec{P}^f_N(\vec{k}, s) + \mu_0 s \ i \vec{k} \times \vec{M}^f_N(\vec{k}, s) - i \vec{k} \times \int d^3q \ \tilde{\mu}(\vec{k}, \vec{q}, s) \ \vec{B}(\vec{q}, 0) - \mu_0 s \vec{D}(\vec{k}, 0)$$
Eqs. (42) can be rewritten in a compact form as
\[\int d^3 q \: \Lambda(k, q, s) \cdot \vec{E}^b(q, s) = \vec{J}^b_N(k, s)\]
\[\int d^3 q \: \Lambda(k, q, s) \cdot \vec{E}^f(q, s) = \vec{J}^f_N(k, s)\] (45)
with
\[\Lambda_{ij}(k, q, s) = \varepsilon_{i\delta\gamma} \varepsilon_{\alpha\beta j} k^\alpha \bar{\mu}^\gamma_\alpha(k, q, s) - s^2 \frac{c^2}{\varepsilon} \bar{\epsilon}_{ij}(k, q, s)\] (46)

From (45) we see that the backward and forward Laplace transformations of the electric field operator satisfy an inhomogeneous equation with a source term. To solve such equations it is common to use the Green function method. The suitable Green function associated to the Eq. (45) is defined as the solution of an algebraic equation as follows
\[\int d^3 q \: \Lambda(k, q, s) \cdot G(q, p, s) = I \delta(k - p)\] (47)

Accordingly the solution of Eqs. (43) in terms of the Green function defined in (47) can be written as
\[\vec{E}^b(k, s) = \int d^3 p \: G(k, p, s) \cdot \vec{J}^b_N(p, s)\]
\[\vec{E}^f(k, s) = \int d^3 p \: G(k, p, s) \cdot \vec{J}^f_N(p, s)\] (48)

In the special case of a homogeneous medium for which the electric and magnetic susceptibility tensors are translationally invariant, that is when the tensors \(\chi^e(\vec{r}, \vec{r}', t)\) and \(\chi^m(\vec{r}, \vec{r}', t)\) are a function of the difference \(\vec{r} - \vec{r}'\) and so is then, the permittivity and permeability tensors \(\varepsilon(\vec{r}, \vec{r}', t)\) and \(\mu(\vec{r}, \vec{r}', t)\), we deduce
\[\bar{\epsilon}(k, q, s) = \left( I + \bar{\chi}^e(k, s) \right) \delta(k - q)\]
\[\bar{\mu}(k, q, s) = \left( I - \bar{\chi}^m(k, s) \right) \delta(k - q)\] (49)
and therefore from the Eqs. (46) and (47) we find
\[G(k, p, s) = T^{-1}(k, s)\delta(k - p)\]
\[T_{ij} = \left[ \varepsilon_{i\delta\gamma} \varepsilon_{\alpha\beta j} k_\delta k_\beta \left( \delta_{\gamma\alpha} - \bar{\chi}^m_{\gamma\alpha}(k, s) \right) - s^2 \frac{c^2}{\varepsilon} \left( \delta_{ij} + \bar{\epsilon}_{ij}(k, s) \right) \right]\] (50)
where it should be summed on the indices $\alpha, \beta, \gamma, \delta$ in the first term in the bracket. For a general medium it may be impossible to solve Eq. (47) exactly to obtain the Green function $G$. However one can use a suitable iteration method and find the Green function up to an arbitrary accuracy which may be useful for media with considerably weak polarizability. Let us write the tensor $\mathbf{\Lambda}(\vec{k}, \vec{q}, s)$ as the sum of two parts

$$
\mathbf{\Lambda}(\vec{k}, \vec{q}, s) = \mathbf{\Lambda}^{(0)}(\vec{k}, \vec{q}, s) + \mathbf{\Lambda}^{(1)}(\vec{k}, \vec{q}, s) \tag{51}
$$

where

$$
\mathbf{\Lambda}_{ij}^{(0)}(\vec{k}, \vec{q}, s) = \left(k_i k_j - \delta_{ij} k^2 - \frac{s^2}{c^2} \delta_{ij}\right) \delta(\vec{k} - \vec{q}) \equiv \mathbf{L}(\vec{k}, s) \delta(\vec{k} - \vec{q}) \tag{52}
$$
is the tensor $\Lambda$ in free space, that is when there is no medium and

$$
\mathbf{\Lambda}_{ij}^{(1)}(\vec{k}, \vec{q}, s) = -\varepsilon_i \delta_{ij} \varepsilon_{\alpha \beta} k_\delta q_\beta \chi^{m}_{\gamma \alpha}(\vec{k}, \vec{q}, s) - \frac{s^2}{c^2} \chi^{e}_{ij}(\vec{k}, \vec{q}, s) \tag{53}
$$
is the part due to the presence of the medium, that is the effect of the medium in the total tensor $\Lambda$. Then, following the Born approximation method previously used in the scattering theory [25], the Green function $G$ can be expressed as a series as follows

$$
\mathbf{G}(\vec{k}, \vec{p}, s) = \mathbf{G}^{(0)}(\vec{k}, \vec{p}, s) + \mathbf{G}^{(1)}(\vec{k}, \vec{p}, s) + \mathbf{G}^{(2)}(\vec{k}, \vec{p}, s) + \cdots \tag{54}
$$

where

$$
\mathbf{G}^{(0)}(\vec{k}, \vec{p}, s) = \mathbf{L}^{-1}(\vec{k}, s) \ \delta(\vec{k} - \vec{p}) \tag{55}
$$
is the Green function for the free space and

$$
\mathbf{G}^{(n)}(\vec{k}, \vec{p}, s) \quad n = 1, 2, 3, \cdots \quad \text{are found from the following recurrence relation}
$$

$$
\int d^3 q \ \mathbf{\Lambda}^{(0)}(\vec{k}, \vec{q}, s) \cdot \mathbf{G}^{(n)}(\vec{q}, \vec{p}, s) = -\int d^3 q \ \mathbf{\Lambda}^{(1)}(\vec{k}, \vec{q}, s) \cdot \mathbf{G}^{(n-1)}(\vec{q}, \vec{p}, s)
$$

$$
\implies \mathbf{G}^{(n)}(\vec{k}, \vec{p}, s) = -\mathbf{L}^{-1}(\vec{k}, s) \int d^3 q \ \mathbf{\Lambda}^{(1)}(\vec{k}, \vec{q}, s) \cdot \mathbf{G}^{(n-1)}(\vec{q}, \vec{p}, s) \tag{56}
$$

where we have used Eq. (52). Using the series (54) and the recurrence relation (56) one can obtain the Green function $G$ up to an arbitrary accuracy. In $n$'th order approximation it can be neglected from the terms $G^{(j)} \quad j \geq n + 1$ in the series (54) which may be useful for media with considerably weak polarizability.
polarizability.
Now, having the Green function $G$, the time dependence of electric field operator for negative and positive times can be obtained by inverse Laplace transformations of $\vec{E}^b$ and $\vec{E}^f$, respectively. Carrying out the inverse Laplace transformation of Eq. (48) for $t < 0$ we deduce
\[
\vec{E}(\vec{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{-i\omega t} \int d^3p \ G(\vec{k}, \vec{p}, \omega + 0^+) \cdot \vec{J}_N^b(\vec{p}, \omega + 0^+) \quad (57)
\]
where $0^+$ is an arbitrarily small positive number. For $t > 0$ the inverse Laplace transformation of $\vec{E}^f$ is
\[
\vec{E}(\vec{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{-i\omega t} \int d^3p \ G(\vec{k}, \vec{p}, -\omega + 0^+) \cdot \vec{J}_N^f(\vec{p}, -\omega + 0^+) \quad (58)
\]
Because the Laplace transformations of the susceptibility tensors are analytic functions respect to the variable $s$ in any point of the half-plane $Re[s] \geq 0$ then, clearly the $\omega$-dependence integrand in Eq. (57) also is analytic in the half-plane $Im[\omega] \leq 0$. Accordingly, the integral over $\omega$ in (57) vanishes for the positive times. Likewise the integral in (58) is zero for $t < 0$. Therefore we can combine the two expressions (57) and (58) into a single one and write the full time-dependence of electric field for all times as
\[
\vec{E}(\vec{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{-i\omega t} \int d^3p \ \left[ G(\vec{k}, \vec{p}, \omega + 0^+) \cdot \vec{J}_N^b(\vec{p}, \omega + 0^+) + G(\vec{k}, \vec{p}, -\omega + 0^+) \cdot \vec{J}_N^f(\vec{p}, -\omega + 0^+) \right] \quad (59)
\]
It is remarkable that using the asymptotic behavior of the Green function $G$ and the susceptibility tensors for large $|\omega|$ it can be verified straightforwardly, the integral in (59) reduces to $\vec{E}(\vec{k}, 0)$ at $t = 0$ and hence this equation give us correctly the time dependence of the electric field for all times. Finally having the electric field operator, we can obtain the magnetic field using $i\vec{k} \times \vec{E}(\vec{k}, t) = \vec{B}(\vec{k}, t)$ and then the polarization densities employing the constitutive Eqs. (33) and (38).
6 The magnetodielectric media with only one response equation

In the previous sections we have assumed the magnetodielectric medium under consideration is one for which the distinction between polarization and magnetization physically is possible and the total current induced in medium can be separated into electric and magnetic parts. It is remarkable that the use of polarization and magnetization, separately, to describe the response of a medium to electromagnetic field is useful for media which are not spatially dispersive or which have a particular form of spatial dispersion \[26\]. There are in general media for which it is not clear how the separation of the total current induced in medium is to be made into electric and magnetic parts. The disturbances, caused by the electromagnetic field, in such media are described completely in terms of only a response equation relating the total current induced in medium to the electric field \[26\]. The quantization method outlined in the previous sections can cover such cases applying the Lagrangian

\[
L(t) = \int_0^\infty d\omega \int d^3r \left[ \frac{1}{2} \dot{\vec{X}}_\omega \cdot \dot{\vec{X}}_\omega - \frac{1}{2} \omega^2 \vec{X}_\omega \cdot \vec{X}_\omega \right] \\
+ \int d^3r \left[ \frac{1}{2} \varepsilon_0 \vec{E}^2 - \frac{\vec{B}^2}{2\mu_0} \right] \\
+ \int_0^\infty d\omega \int d^3r \int d^3r' f_{ij}(\omega, \vec{r}, \vec{r}') E^i(\vec{r}, t) \vec{X}_\omega^j(\vec{r}, t) \tag{60}
\]

which is the same as the Lagrangian (1)-(6) with the exceptional that, now the medium is modeled by a single set of three dimensional harmonic oscillators \(\vec{X}_\omega\). In this case the Euler- Lagrange equation (14) takes the form

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta (A^*_i(\vec{k}, t))} \right) - \frac{\delta L}{\delta (A^*_i(\vec{k}, t))} = 0 \quad i = 1, 2, 3
\]

\[
\Rightarrow \quad \mu_0 \varepsilon_0 A_\omega(\vec{k}, t) + \mu_0 \varepsilon_0 i\vec{k} \cdot \vec{\phi}(\vec{k}, t) - \vec{k} \times \left( \vec{k} \times A(\vec{k}, t) \right) = \mu_0 \vec{P}(\vec{k}, t) \tag{61}
\]

where now

\[
\vec{P}(\vec{k}, t) = \int_0^\infty d\omega \int d^3p f(\omega, \vec{k}, \vec{p}) \cdot \vec{X}_\omega(\vec{p}, t) \tag{62}
\]
can no longer be interpreted as the electric polarization, but is related to
the total current induced in the medium by \( j(\vec{k}, t) = \vec{P}(\vec{k}, t) \). Following the
canonical quantization similar to the previous sections we obtain the only
response equation of the medium as
\[
\vec{P}(\vec{k}, t) = \vec{P}_N(\vec{k}, t) + \varepsilon_0 \int_0^{|t|} dt' \int d^3q \chi(\vec{k}, \vec{q}, |t| - t') \cdot \vec{E}(\vec{q}, \pm t')
\]
(63)
instead of the two constitutive equations (33) and (40), where \( \vec{P}_N(\vec{k}, t) \) is
given by (37) and the response tensor
\[
\chi(\vec{k}, \vec{q}, t) = \begin{cases} 
\frac{1}{\varepsilon_0} \int_0^\infty d\omega \frac{\sin \omega t}{\omega} \int d^3p \ f(\omega, \vec{k}, \vec{p}) \cdot f^\dagger(\omega, \vec{q}, \vec{p}) & t > 0 \\
0 & t \leq 0
\end{cases}
\]
(64)
is related to the conductivity tensor of the medium by \( \frac{\partial \chi}{\partial t} = \sigma \). This is one
of the two alternative descriptions of the response of a magnetodielectric
medium for which the distinction between polarization and magnetization is
not possible [26]. Finally following the same calculations as before sections
one can obtain the space-time dependence of electromagnetic field operators
in terms of the response tensor \( \chi \) and the dynamical variables \( \vec{X}_ω \) at
\( t = 0 \).
The calculations in this case are precisely the same as before sections except
that the quantities \( \vec{Y}_ω, \vec{Π}_ω, \vec{M}_ω, \chi^m, g \) are identically zero.

7 Summary and conclusion

By modeling an anisotropic polarizable and magnetizable medium with two
continuous collections of three dimensional vector fields, we have presented a
fully canonical quantization for electromagnetic field in the presence of such
a medium. In this model the polarization fields of the medium did not enter
directly in the Lagrangian of the total system as a part of the degrees of
freedom of the medium, but the space-dependent oscillators modeling the
medium solely was sufficient to describe both polarizability and the absorp-
tion of the medium. The interaction of the medium with the electromagnetic
field explicitly was given and both the Maxwell’s laws and the constitutive
equations of the medium were obtained as the consequences of the Heisen-
berg equations of total system. The electric and magnetic polarization fields

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of the medium could be deduced naturally in terms of the dynamical fields modeling the medium. Some space dependent real valued tensor coupling the medium with electromagnetic field were introduced. The coupling tensor had an important role in this quantization scheme, so that the electric and magnetic susceptibility tensors of the medium were determined in terms of the coupling tensor. Also the noise polarizations were expressed in terms of the coupling tensor and the dynamical variables describing the degrees of freedom of the medium at $t = 0$. In the free space and in the case of a non-absorbing medium the coupling tensor and the noise polarizations tend to zero and this quantization method is reduced to the usual quantization in these limiting cases. As a tool, we have used the reciprocal space to quantize the total system. It was shown the temporal backward and forward Laplace transforms of the electric field obey some algebraic equations with source terms that could be determined in terms of the dynamical variables of the system at $t = 0$. By introducing the Green function of these algebraic equations we were able to determine the Green function up to an arbitrary accuracy using a perturbation method which may be useful in some real cases with weak polarizability. The time-dependence of electric field was derived for both negative and positive times. Finally the model was modified to the case of a medium for which the distinction between the polarization and magnetization is not possible.

References

[1] T. Gruner, D. G. Welsch, Phys. Rev. A 51, 3246 (1995).
[2] R. Matloob, R. Loudon, S. M. Barnett, J. Jeffers, Phys. Rev. A 52, 4823 (1995).
[3] T. Gruner, D. G. Welsch, Phys. Rev. A 53, 1818 (1996).
[4] R. Matloob, R. Loudon, Phys. Rev. A 53, 4567 (1996).
[5] T. Gruner, D. G. Welsch, Phys. Rev. A 54, 1661 (1996).
[6] H. T. Dung, L. Knöll, D. G. Welsch, Phys. Rev. A 57, 3931 (1998).
[7] S. Scheel, L. Knöll, D. G. Welsch, Phys. Rev. A 58, 700 (1998).
[8] B. Huttner, S. M. Barnett, Phys. Rev. A 46, 4306 (1992).
[9] B. Huttner, S. M. Barnett, Europhys. Lett. 18, 487 (1992).

[10] U. Weiss, Dissipative Quantum Systems (World Scientific 1993).

[11] S. M. Barnett, B. Huttner, R. Loudon, R. Matloob, J. Phys. B 29, 3763 (1996).

[12] S. Scheel, L. Knöll, D. G. Welsch, S. M. Barnett, Phys. Rev. A 60, 1590 (1999).

[13] S. Scheel, L. Knöll, D. G. Welsch, Phys. Rev. A 60, 4094 (1999).

[14] H. T. Dung, L. Knöll, D. G. Welsch, Phys. Rev. A 62, 053804 (2000).

[15] H. T. Dung, L. Knöll, D. G. Welsch, Phys. Rev. A 66, 063810 (2002).

[16] L. Knöll, S. Scheel, D. G. Welsch, in Coherence and Statistics of Photons and Atoms, edited by J. Peřina (Wiley, New York, 2001).

[17] A. Lukš, V. Peřinová, in Progress in Optics, Edited by E. Wolf (North-Holland, Amsterdam, 2002 Vol. 43, p. 295).

[18] H. T. Dung, S. Y. Buhmann, L. Knöll, D. G. Welsch, Phys. Rev. A 68, 043816 (2003).

[19] C. Raabe, S. Scheel, D. G. Welsch, Phys. Rev. A 75, 053813 (2007).

[20] L. D. Landau, E. M. Lifshitz, Electrodynamics of Continuous Media (Oxford: Pergamon, 1977).

[21] L. G. Suttorp, M. Wubs, Phys. Rev. A 70, 013816 (2004).

[22] F. Kheirandish, M. Amooshahi, Phys. Rev. A 74, 042102 (2006).

[23] M. Amooshahi, F. Kheirandish, Phys. Rev. A 76, 062103 (2006).

[24] M. Amooshahi, F. Kheirandish, Modd. Phys. Lett. A 23, No. 26 (2008).

[25] J. J. Sakurai, Modern Quantum Mechanics, (Addison-Wesley 1985).

[26] D. B. Melrose, R. C. McPhedran, Electromagnetic in Dispersive Media, (Cambridge, University Press 1971).