Irreducible $\varphi$-Verma modules for hyperelliptic Heisenberg algebras

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Abstract. We defined the hyperelliptic Heisenberg algebra as the Heisenberg subalgebra of a hyperelliptic Krichever-Novikov algebra. Then we gave an explicit irreducibility criteria for $\varphi$-Verma modules for these algebras.

Introduction

If $R$ is a commutative $\mathbb{C}$-algebra and $g$ is a simple Lie $\mathbb{C}$-algebra, it is well known from the work of Kassel and Loday (see [KL82], and [Kas84]) that the universal central extension $G$ of $g \otimes R$ is the vector space $(g \otimes R) \oplus \Omega^1_R/dR$ where $\Omega^1_R/dR$ is the space of Kähler differentials modulo exact forms. The space $G$ is made into a Lie algebra by defining

$$[x \otimes f, y \otimes g] = [xy] + (x, y)fdg, \quad [x \otimes f, \omega] = 0$$

for $x, y \in g$, $f, g \in R$, $\omega \in \Omega^1_R/dR$, $(\cdot, \cdot)$ denotes the Killing form on $g$, and $\bar{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient $\Omega^1_R/dR$.

Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and fix a set composed by $n > 2$ distinct points on this Riemann sphere $P = \{p_1, p_2, ..., p_n\}$. Bremner gave $G$ type algebras the name $n$ point algebras when $R$ is the ring of rational functions with poles allowed only at $P$. The $n$ point Lie algebras are examples of Krichever-Novikov algebras.

Probably the simplest example of Krichever-Novikov algebra beyond an affine Kac-Moody algebra is the three point algebra. If we look at the three point algebra case when $g = \mathfrak{sl}(2, \mathbb{C})$, Cox and Jurisich showed at [CJ13] that the three point ring is isomorphic to $R_1 = \mathbb{C}[t, t^{-1}, u]$ with $u^2 = t^2 + 4t$ and gave a free field realization for the three point affine Lie algebra. Furthermore, [CJ13] defined the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \otimes R_1 \oplus \Omega^1_R/dR_1$ as the three point Heisenberg algebra through generators and relations.

The four point algebra was studied by Bremner at [Bre95] and, when $g = \mathfrak{sl}(2, \mathbb{C})$, by Cox at [Cox08]. Bremner showed that the four point ring is isomorphic to $R_2 = \mathbb{C}[t, t^{-1}, u]$ where $u^2 = t^2 - 2bt + 1$ with $b \in \mathbb{C} \setminus \{\pm 1\}$ and gave a free field realization of the four point affine Lie algebra in terms of the ultraspherical Gegenbauer polynomials. Cox gave a realization for the four point algebra where the center acts nontrivially, furthermore [Cox08] defined the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \otimes R_2 \oplus \Omega^1_R/dR_2$ as the four point Heisenberg algebra.

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We can find another examples of Krichever-Novikov algebras beyond the point algebras such as the elliptic algebra and Date-Jimbo-Kashiwara-Miwa algebra. Bueno, Cox and Futorny showed that two realizations of the elliptic algebra $\mathfrak{sl}(2, \mathbb{C}) \otimes R^3 \oplus \Omega R^3 \frac{1}{dR^3}$ where $R^3 = \mathbb{C}[t, t^{-1}, u]$ with $u^2 = t^3 - 2bt + t$ and defined the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \otimes R \oplus \Omega R^{1/3}$ as the elliptic Heisenberg algebra.

The DJKM algebra introduced by Date, Jimbo, Kashiwara and Miwa in [DJKM83] are nothing but a one dimensional central extension of $g \otimes R^3$ with $R^3 = \mathbb{C}[t, t^{-1}, u]$ with $u^2 = (t^2 - b^2)(t^2 - c^2)$ where $b \neq \pm c$ are complex constants and $g$ is a simple finite dimensional Lie $\mathbb{C}$-algebra. Cox, Futorny and Martins constructed a realization of the DJKM algebra when $g = \mathfrak{sl}(2, \mathbb{C})$ in terms of sums of partial differential operators in [CFM14], furthermore defined the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \otimes R^3 \oplus \Omega R^{1/3}$ as the DJKM Heisenberg algebra.

It is a natural question how to classify irreducible modules for these Heisenberg subalgebras defined above. Inspired by [BBFK13], in this paper we give an irreducibility criteria for $\varphi$-Verma modules over the hyperelliptic Heisenberg algebras.

Let $c$ be a Lie algebra. The central extension of $g$ by $c$ is an exact sequence of Lie algebras

$$0 \longrightarrow c \longrightarrow e \xrightarrow{\pi} g \longrightarrow 0$$

such that $c$ is the centre of $e$.

A morphism from one existing central extension to another central extension

$$0 \longrightarrow e' \xrightarrow{\phi'} c \xrightarrow{\pi'} g \longrightarrow 0$$

a pair $(\phi, \phi_0)$ of Lie algebra homomorphisms such that the diagram
The central extension is said to be a covering of \( \mathfrak{g} \) in case \( \mathfrak{c} \) is perfect. A covering of \( \mathfrak{g} \) is said to be universal central extension if for every central extension of \( \mathfrak{g} \) there exists a unique morphism from the covering to the central extension.

2. The hyperelliptic Lie algebra

Our goal in this section is to recall, following [Bre95] and [CI17], the universal central extension of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{R} \) with \( \mathbb{R} = \mathbb{C}[t^\pm 1, u : u^2 = p(t)] \), where \( p(t) = t(t - \alpha_1) \cdots (t - \alpha_r) = \sum_{i=1}^{r+1} a_i t^i \in \mathbb{C}[t] \), \( \alpha_i \)'s are pairwise distinct nonzero complex numbers with, \( a_i \)'s are complex numbers and \( a_{r+1} = 1 \). Using a classical result by C. Kassel [Kas84], Bremner described the universal central extension of \( \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{R} \) in [Bre95]. Before state this, we recall the module of Kähler differentials module.

Let \( F = \mathbb{R} \otimes \mathbb{R} \) be the left \( \mathbb{R} \)-module with action \( f(g \otimes h) = fg \otimes h \) for \( f, g, h \in \mathbb{R} \). Let \( K \) be the submodule generated by the elements \( 1 \otimes fg - f \otimes g - g \otimes f \). Then \( \Omega^1_R = F/K \) is the module of Kähler differentials. We denote the element \( f \otimes g + K \) of \( \Omega^1_R \) by \( fdg \). We define a map \( d : \mathbb{R} \rightarrow \Omega^1_R \) by \( d(f) = df = 1 \otimes f + K \) and we denote the coset of \( fdg \) modulo \( d \mathbb{R} \) by \( dfg \).

**Theorem 2.1 (KL82).** Let \( \mathfrak{g} \) be a simple finite-dimensional complex Lie algebra and let \( R \) be any commutative associative \( \mathbb{C} \)-algebra. The universal central extension of \( \mathfrak{g} \otimes R \) is linearly isomorphic to \( (\mathfrak{g} \otimes R) \oplus \Omega^1_R/d\mathbb{R} \), where \( \Omega^1_R/d\mathbb{R} \) is the space of Kähler differentials of \( R \) modulo exact differentials.

The commutation relations of \( \hat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \Omega^1_R/d\mathbb{R} \) are

\[
[x \otimes f, y \otimes g] = [xy] \otimes fg + (x, y)\overline{fg},
\]

\[
[x \otimes f, \omega] = 0,
\]

where \( x, y \in \mathfrak{g}, f, g \in R, \omega \in \Omega^1_R/d\mathbb{R} \), \((\cdot, \cdot)\) denotes the Killing form on \( \mathfrak{g} \).

Since we are interested in the description of \( \hat{\mathfrak{g}} \), it is important to remember that:

**Theorem 2.2 ([Bre95], Theorem 3.4).** Let \( R = \mathbb{C}[t^\pm, u : u^2 = p(t)] \). The set

\[
\{t^{-i}dt, t^{-i}udt, \ldots, t^{-i}udt\},
\]

forms a basis of \( \Omega^1_R/d\mathbb{R} \).

We set the following notation

\[
\omega_0 := \overline{t^{-1}dt} \quad \text{and} \quad \omega_k := \overline{t^{-k}udt}
\]

for \( 1 \leq k \leq r \).

Following [CI17], we defined some polynomials in order to give a description of \( \hat{\mathfrak{g}} \). With \( m = 2 \) and \( a_0 = 0 \), we let \( P_{k,i} := P_{k,i}(a_1, ..., a_r), k \geq -r, -r \leq i \leq -1 \) be the polynomials in the \( a_i \) satisfying the recursion relations:
we will denote it by \( \hat{b} \) with generators

\[
(3.2)
\]

where

\[
(3.1)
\]

of \( Q \) with initial condition

\[
(2.10)
\]

for \( k \geq 0 \) with the initial condition \( P_{i,i} = \delta_{l,i}, -r \leq i, l \leq -1 \). Furthermore, we set \( Q_{m,i} \) satisfying

\[
(2.4)
\]

with initial condition \( Q_{m,i} = \delta_{m,-i} \) for \( 1 \leq m \leq r \) and \( -r \leq i \leq -1 \).

**Theorem 2.3** ([CL17, Theorem 5.1]). Let \( a_1 \neq 0 \). Let \( g \) be a simple finite dimensional Lie algebra over the complex numbers with Killing form \((\cdot,\cdot)\) and for \( a = (a_1, \ldots, a_r) \) define \( \psi_{ij}(a) \in \Omega_R^1/dR \) by

\[
(2.5)
\]

The universal central extension of the hyperelliptic Krichever-Novikov algebra \( g \otimes R \) is the \( \mathbb{Z}_2 \)-graded Lie algebra

\[
(2.6)
\]

where

\[
(2.7)
\]

with braket

\[
(2.8)
\]

\[
(2.9)
\]

\[
(2.10)
\]

We call this algebra hyperelliptic Lie algebra.

3. The hyperelliptic Heisenberg subalgebra

**Definition 3.1** (hyperelliptic Heisenberg algebra). The Cartan subalgebra \( h \) of \( \mathfrak{sl}(2, \mathbb{C}) \) tensored with \( R \) generates a Heisenberg subalgebra of \( \hat{g} \). The Lie algebra with generators \( b_m, b_m^1, m \in \mathbb{Z}, 1, i \in \{0, 1, \ldots, r\} \), and relations

\[
(3.1)
\]

\[
(3.2)
\]

\[
(3.3)
\]

\[
(3.4)
\]

where \( i, j \in \{0, \ldots, r\} \) and \( k \in \mathbb{Z} \), is called the hyperelliptic Heisenberg algebra and we will denote it by \( \hat{h} \).
Let $\varphi$ be an function from $\mathbb{Z}$ to $\{\pm\}$ such that $\varphi(n) = + \iff \varphi(-n) = -$. Setting

$$\hat{b}_\varphi^\pm = \left( \sum_{n \in \mathbb{Z}_{<0} \varphi(n) = \mp} (\mathbb{C}b_n + \mathbb{C}b_n^1) \right) \oplus \left( \sum_{m \in \mathbb{Z}_{>0} \varphi(m) = \pm} (\mathbb{C}b_m + \mathbb{C}b_m^1) \right)$$

and

$$\hat{b}_0 = \mathbb{C}b_0 + \mathbb{C}b_0^1 + \sum_{i=0}^r 1_i \mathbb{C}.$$  

we introduce a Borel type subalgebra $\hat{b}_\varphi := \hat{b}_0^0 \oplus \hat{b}_0^+$. Due to the defining relations above one can see that $\hat{b}_\varphi$ is a subalgebra.

**Lemma 3.2.** Let $\mathcal{V} = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ be a two-dimensional representation of $\hat{b}$, where $\hat{b}_0^i v_i = 0$ for $i = 0, 1$. Suppose $\lambda, \mu, \nu, \chi, \gamma, \kappa \in \mathbb{C}$ for $j \in \{1, \ldots, r\}$ are such that

$$b_0 \cdot v_0 = \lambda v_0 , \quad b_0^1 \cdot v_0 = \mu v_0 + \nu v_1 , \quad 1_j v_i = \chi v_i ,$$

$$b_0 \cdot v_1 = \lambda v_1 , \quad b_0^1 \cdot v_1 = \gamma v_0 + \mu v_1 , \quad 1_0 v_i = \kappa v_i ,$$

for $i \in \{0, 1\}$. Then

$$\sum_{k=1}^r (P_{m+n-1,-k}) \chi_k = 0 , \text{ if } m + n \geq -r + 1 , \text{ and}$$

$$\sum_{k=1}^r (Q_{-m-n+1,-k}) \chi_k = 0 \text{ otherwise.}$$

**Proof.** Since $b_m$ acts by scalar multiplication for $m, n \in \mathbb{Z}$, the first defining relation $[3.1]$ is satisfied. The second relation $[3.2]$ is also satisfied. If $n = 0$, then since $b_0$ acts by a scalar, the relation $[3.3]$ leads to no condition on $\lambda, \mu, \nu, \chi, \gamma, \kappa \in \mathbb{C}$, the third relation gives the condition on $\chi_j$ as

$$0 = b_0^1 b_n v_i - b_n b_0^1 v_i = [b_0^1, b_n]v$$

$$= 2n \sum_{k=1}^r ((\delta_{m+n \geq -r+1})P_{m+n-1,-k} + (\delta_{m+n < -r+1})Q_{-m-n+1,-k}) 1_k v$$

\[ \square \]

4. $\varphi$-Verma modules for the hyperelliptic Heisenberg subalgebra

Now consider the following induced $\varphi$-Verma $\hat{g}$-module

$$M_{\hat{g}, \varphi} = \mathfrak{u}(\hat{g}) \otimes_{\hat{b}_\varphi} \mathbb{C}v.$$  

Given an integer $n$, we consider that

$$\text{sgn}(n) = \begin{cases} + & \text{if } n > 0, \\ - & \text{if } n < 0. \end{cases}$$
LEMMA 4.1. If \( m, n \in \mathbb{Z} \setminus \{0\} \), \( \varphi(m) = \text{sgn}(m) \) and \( \varphi(n) \neq \text{sgn}(n) \), then

\begin{align*}
(4.3) \quad & b_m(b_n)^{v} = (2ln\kappa_0)\delta_{m+n,0}(b_n)^{l-1}v, \\
(4.4) \quad & b_m(b_n^l)^{v} = 0, \\
(4.5) \quad & b_m(b_n^l)^{v} = (1/2)(l\kappa_0) ((n-m)a_{-(m+n)}) ((b_n^{l})^{-1})v, \\
(4.6) \quad & b_m(b_n^l)^{v} = 0.
\end{align*}

PROOF. Using the Lemma 3.2 and the relations in Definition 3.1 we have that

\[ b_m(b_n)^{v} = [b_m, (b_n)^{v}] + (b_m)^{v}b_n^{v} = 0 = \] 

\[ = (2ln\kappa_0)\delta_{m+n,0}(b_n)^{l-1}v. \]

\[ b_m(b_n^l)^{v} = [b_m, (b_n^l)^{v}] + (b_m^l)^{v}b_n^{v} = 0 \]

\[ = (1/2)(l\kappa_0) ((n-m)a_{-(m+n)}) ((b_n^{l})^{-1})v. \]

\[ b_m(b_n^l)^{v} = [b_m, (b_n^l)^{v}] + (b_m^l)^{v}b_n^{v} = 0 \]

\[ = 0 \] 

Here we state our main result.

THEOREM 4.2. \( M_{\mathfrak{g},\varphi} \) is irreducible if and only if \( \kappa_0 \neq 0 \).

PROOF. Set

\[ (M_{\mathfrak{g},\varphi})_n = \left\{ w \in M_{\mathfrak{g},\varphi} : w = \left( \sum_{(\bar{\alpha},\bar{\beta})} \xi(\bar{\alpha},\bar{\beta}) \prod_{i \in \mathbb{Z} \setminus \{0\}, \varphi(i) \neq \text{sgn}(i)} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}, \varphi(j) \neq \text{sgn}(n)} (b_j^l)^{\beta_j} \right) v, \right. \]

\[ \left. \sum_i \alpha_i + \sum_j \beta_j = n, \forall (\bar{\alpha},\bar{\beta}) \right\} . \]

We say that \( \text{deg}(w) = n \) if \( w \in (M_{\mathfrak{g},\varphi})_n \). We suppose \( \kappa_0 \neq 0 \) and proceed by induction in \( \text{deg}(w) \).
(1) Suppose that \( \deg(w) = 1 \), then

\[
w = \left( \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i b_i + \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i^1 b_i^1 \right) v
\]

where only finitely many of \( \xi_i, \xi_i^1 \in \mathbb{C} \) are nonzero.

(a) If \( \xi_i^1 \)'s are all zero, then there is \( \xi_m^1 \neq 0 \) for some \( m \in \mathbb{Z} \) such that \( \varphi(m) \neq \text{sgn}(m) \). Let be \( x \in \hat{h}_\varphi^+ \) such that \( x = b_{-m} \). We have that

\[
xw = x \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i^1 b_i^1 v = (1/2) \kappa_0 \sum_{k \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i^1 (k - m) a_{-(k+m)} v.
\]

(b) If there is some \( \xi_k \) nonzero, let \( x \in \hat{h}_\varphi^+ \) such that \( x = b_{-k} \). We have that

\[
xw = x \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i b_i v = 2 \kappa_0 k \xi_k v.
\]

Then in both cases there is \( x \in \hat{h}_\varphi^+ \) such that \( xw \neq 0 \) and \( \deg(xw) = 0 \).

(2) Suppose that for all \( v \in M_{\hat{h}, \varphi} \) with degree \( n \), there is \( y \in \hat{h}_\varphi^+ \) such that \( yw \neq 0 \) and \( \deg(yw) = 0 \). Suppose that \( \deg(w) = n + 1 \), so an arbitrary element \( M_{\hat{h}, \varphi} \) with this degree is

\[
w = \left( \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i b_i^{(n+1)} + \sum_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} \xi_i^1 (b_i^1)^{(n+1)} \right. \\
\left. + \sum_{(\alpha, \beta)} \xi(\tilde{\alpha}, \tilde{\beta}) \prod_{i \in \mathbb{Z}, \varphi(i) \neq \text{sgn}(i)} b_i^{a_i} \prod_{j \in \mathbb{Z}, \varphi(j) \neq \text{sgn}(j)} (b_j^1)^{\beta_j} \right) v
\]

where, \( \overline{a} := \{a_0, a_2, ..., a_l\} \) with \( a_i \in \mathbb{Z} \), only finitely many of \( a_i, \xi_i, \xi_i^1 \) and \( \xi(\tilde{\alpha}, \tilde{\beta}) \) are nonzero.

(a) If there is \( \xi(\tilde{\alpha}, \tilde{\beta}) \neq 0 \), so let \( \xi(\tilde{k}, \tilde{l}) \neq 0 \) and let \( m \) be the greatest index with \( k_m \neq 0 \) and \( x \in \hat{h}_\varphi^+ \) such that \( x = b_{-m} \). Furthermore, reorganize
the monomials in the way that with the appropriate constants

\[
\begin{align*}
w &= \left( \sum_{i \in \mathbb{Z} \setminus \{0\}} \xi_i b_i^{(n+1)} + \sum_{i \in \mathbb{Z} \setminus \{0\}} \xi_i^1 (b_j^1)^{(n+1)} \right) \\
&\quad + \sum_{(\bar{\alpha}, \bar{\beta})} \xi(\bar{\alpha}, \bar{\beta}) \prod_{i \in \mathbb{Z} \setminus \{0\}} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}} (b_j^1)^{\beta_j} \\
&= \left( \sum_{i \in \mathbb{Z} \setminus \{0\}} \tilde{\xi}_i b_i^{(n+1)} + \sum_{i \in \mathbb{Z} \setminus \{0\}} \tilde{\xi}_i^1 (b_j^1)^{(n+1)} \right) \\
&\quad + \sum_{(\bar{\alpha}, \bar{\beta})} \tilde{\xi}(\bar{\alpha}, \bar{\beta}) b_{m_i}^{\alpha_m} \prod_{i \in \mathbb{Z} \setminus \{0\}} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}} (b_j^1)^{\beta_j} \bigg) v.
\end{align*}
\]

Applying \( x \) to \( w \) we get

\[
xw = \tilde{\xi}_m 2^{\kappa_0(n+1)m} b_m^{\alpha_m} v + \sum_{(\bar{\alpha}, \bar{\beta})} \tilde{\xi}(\bar{\alpha}, \bar{\beta}) b_{-m} b_{m_i}^{\alpha_m} \prod_{i \in \mathbb{Z} \setminus \{0\}} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}} (b_j^1)^{\beta_j} v
\]

\[
+ \sum_{(\bar{\alpha}, \bar{\beta})} \xi(\bar{\alpha}, \bar{\beta}) b_{-m} b_{m_i}^{\alpha_m} \prod_{i \in \mathbb{Z} \setminus \{0\}} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}} (b_j^1)^{\beta_j} v
\]

\[
= \tilde{\xi}_m 2^{\kappa_0(n+1)m} b_m^{\alpha_m} v
\]

\[
+ \sum_{(\bar{\alpha}, \bar{\beta})} \xi(\bar{\alpha}, \bar{\beta}) a_m (2^{m \kappa_0}) b_{m_i}^{\alpha_m-1} \prod_{i \in \mathbb{Z} \setminus \{0\}} b_i^{\alpha_i} \prod_{j \in \mathbb{Z} \setminus \{0\}} (b_j^1)^{\beta_j} v.
\]

Then \( xw \neq 0 \) and \( \deg(xw) = n \).

(b) If \( \xi(\bar{\alpha}, \bar{\beta}) = 0 \), \( \forall(\bar{\alpha}, \bar{\beta}) \).

(b.i) If there is \( \xi_k \neq 0 \) let \( x = b_{-k} \) and \( xw = 2^{\kappa_0(n+1)k} b_k^{\alpha_k} v \).

(b.ii) If all the \( \xi_k \)'s are zero, then there is \( \chi_m \) for some \( m \in \mathbb{Z} \) such that \( \varphi(m) \neq \text{sgn}(m) \). Let \( x = b_{-m}^1 \). We have that

\[
xw = (1/2)(n+1)\kappa_0 \sum_{k \in \mathbb{Z}} \xi_k^1 (k-m) a_{-(k+m)} (b_k)^n v.
\]

Since \( \kappa_0 \neq 0 \) implies that the submodule generated by \( w \) contains \( v \) so is all \( M_{\bar{\alpha}, \varphi} \). But \( w \) was an arbitrary nonzero element, so \( M_{\bar{\alpha}, \varphi} \) is irreducible in this case.

If \( \kappa_0 = 0 \) then \( N_{\bar{\alpha}, \varphi} := \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} (M_{\bar{\alpha}})_n \) is a proper submodule. \( \square \)
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