Brick Walls on the Brane

by

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ABSTRACT

The so-called “brick-wall model” is a semi-classical approach that has been used to explain black hole entropy in terms of thermal matter fields. Here, we apply the brick-wall formalism to thermal bulk fields in a Randall-Sundrum brane world scenario. In this case, the black hole entity is really a string-like object in the anti-de Sitter bulk, while appearing as a Schwarzchild black hole to observers living on the brane. In spite of these exotic circumstances, we establish that the Bekenstein-Hawking entropy law is preserved. Although a similar calculation was recently considered in the literature, this prior work invoked a simplifying assumption (which we avoid) that can not be adequately justified.
1 Introduction

Much literary attention has recently been directed to the notion that our “physical” universe is really just a 3+1-dimensional submanifold (i.e., three brane) which is embedded in a 4+n-dimensional bulk \[1\]. Particularly interesting proposals along this line have originated from the work of Randall and Sundrum \[2, 3\]. These authors considered a 5-dimensional anti-de Sitter (AdS) spacetime, with the “extra” bulk dimension being related to the 3+1-submanifolds via a “warped” compactification factor. The first of these models, RS1 \[4\], utilizes a pair of branes (one with positive tension and the other, negative) such that the physical universe is embedded on the negative-tension brane. In their second proposal, RS2 \[3\], the universe lives on a single brane of positive tension.\[5\] The phenomenological and cosmological implications of such brane-world scenarios have been considered in a multitude of studies. (See Ref.\[4\] for a review and references.)

One of the more interesting aspects of the RS brane world is how it may influence the physics of black holes. It is clear that the gravitational collapse of matter will result in the formation of a black hole (or, at least, a black hole like object) which, from a brane perspective, must maintain its usual astro-physical properties. However, from a bulk perspective, this black entity must be viewed as a 5-dimensional extended object, as gravitons (unlike most particles) are free to propagate through the extra dimensions of spacetime. In an attempt to resolve these paradoxical implications, CHR (Chamblin, Hawking and Reall \[5\]) have proposed a 5-dimensional black string, whereby the induced metric on the brane reduces to the standard Schwarzschild solution.

As CHR have pointed out themselves, the black string solution suffers from instabilities near the AdS horizon \[5\]. However, they overcame this obstacle by virtue of the following argument. It is known that the black string can also be unstable due to perturbations of wavelength on the order of the horizon radius, \(r_h\). (This effect is commonly known as the Gregory-Laflamme instability \[6\].) However, the AdS bulk geometry can act as a confining box that prevents fluctuations greater than \(l\) from developing (where \(l\) is the usual AdS length parameter). That is, stability will be maintained as long as \(l < r_h\).

\[1\] Alternatively, one can regard RS2 as a dual-brane system in which the negative-tension brane has been moved out to the AdS horizon.
Now consider that, as an artifact of the RS geometry, the black hole effective mass \( (M_e) \) decreases exponentially with transverse proper distance \( (y) \) away from the brane. Hence, \( r_h \sim M_e \) must decrease below \( l \) at some point along the \( y \)-axis. As conjectured by CHR, at this point, the black string will “pinch off” and form into a stabilized “black cigar”. Simple arguments have since verified that the transverse extent of the cigar will be small enough to avoid the unfavorable complications of the AdS horizon \([7, 8]\).

Much work has already been done in generalizing the CHS solution, as well as examining some of its thermodynamic properties. (For an extensive but incomplete list, see Ref.s\([7]-[16]\).) The purpose of our current paper is to further this intriguing topic by way of ’t Hooft’s so-called “brick-wall model” \([17]\).

The underlying premise of the ’t Hooft methodology \([17]\) is that the Bekenstein-Hawking black hole entropy \( S_{BH} = \pi r_h^2 / l_p^2 \) where \( l_p \) is the Planck length \([18, 19]\) can be accounted for via the statistical entropy of thermal fields (particularly, those near the event horizon). For such an approach to be viable, it is necessary to introduce an artificial boundary, or “brick wall”, just outside of the black hole horizon. This wall controls the ultraviolet divergences that are inherent to this type of calculation. Although a seemingly unphysical procedure, the introduction of a wall can be justified as follows: quantum fluctuations prevent events within a Planck length of the horizon from being seen by an external observer.

Since the inception of the brick-wall model, many authors have applied it to various black hole geometries. (For yet another extensive, incomplete list, see Ref.s\([20]-[30]\).) This model has also endured much constructive criticism; see Ref.\([29]\) for an interesting discussion and references. A pair of relatively recent papers, however, have significantly improved the status of brick-wall calculations.

Firstly, Mukohyama and Israel \([29]\) have resolved many of the critical issues; including massive energy densities arising near the horizon, the unphysical implications of an artificial wall, and the back reaction of this effective boundary on the black hole geometry. They accomplished this by identifying the ground state of the brick-wall model as a “topped-up” Boulware state \([31]\) (i.e., the Boulware vacuum plus thermal excitations). It can be consequently argued that the presence of a brick wall with thermal excitations is an alternative, equivalent description of the Hartle-Hawking vacuum state \([32]\). That is, the entropy of the thermal fields just outside of the wall can be iden-
tified with the geometrical entropy that arises out of the Gibbons-Hawking “instanton” \[33\].

Secondly, Winstanley \[30\] has rigorously demonstrated that, at least for large black holes, the brick-wall entropy can be entirely accounted for by renormalizing the coupling constants of the “complete” one-loop effective action.\(^2\) Not only was this demonstrated for the divergent entropy terms (i.e., terms that diverge as the brick wall coincides with the horizon), but for the finite terms as well. Although the procedure broke down with regard to small black holes, this failure can be attributed to quantum gravity corrections, which become important in this limiting case.

This brief discussion on branes and brick walls leads us to the focus of the current study. Namely, the contribution of thermal bulk fields to the entropy of a black hole on a Randall-Sundrum brane or, alternatively, a black cigar in the bulk. (Note that the contribution of thermal brane fields would proceed as in any number of prior publications, starting with Ref.\[17\].) Our particular interest is to see if the Bekenstein-Hawking area law \[18, 19\] is preserved in the leading-order divergent term(s). A failure in this regard could jeopardize the renormalization process as discussed directly above.

The rest of this paper thus proceeds as follows. Section 2 considers the necessary formalism for the calculations of interest. In Section 3, we evaluate the free energy associated with a thermal bulk field. As shown in Section 4, it is then straightforward to extract the corresponding entropy. In Section 5, we consider the thermal energy and use this result to touch base with an earlier study on thermal fields in the Randall-Sundrum brane world \[37\]. The paper ends in Section 6 with a summary and discussion of the results.

Before concluding this introductory section, we point out that a similar study (to ours) has been carried out by Kim et al. \[16\]. However, their analysis relied on a unreasonable assumption in order to simplify the calculations. This point will be elaborated on at an appropriate interval in our paper.

\(^2\)This gravitational action is complete in the sense that it includes terms that are quadratic in the curvature. We also note that such a program of renormalization was originally proposed in Refs.\[34, 35\] and had already been demonstrated via a conical-singularity method of renormalization \[36\].
2 The Setup

We begin here by considering a dual-brane Randall-Sundrum scenario in 5-dimensional AdS. Without loss of generality, we place the positive-tension brane at $y = 0$ (with $y$ denoting the extra bulk dimension). This would be just like RS1 \cite{RS1}, except that we assume the physical universe to be living on the positive-tension brane. Hence, the model of interest is more in the “spirit” of RS2 \cite{RS2}; however, in this study, it is necessary to cut off the bulk spacetime at some point $y_c$ as will be explained below.

If we further assume Poincare invariance on the branes, then the general solution can be written as follows \cite{RS1, RS2}:

$$ds^2 = e^{-2ky} [g_{\mu\nu} dx^\mu dx^\nu] + dy^2,$$

where the inverse AdS length parameter, $k$, has been appropriately fixed in terms of the cosmological constant and brane tensions, and where $g_{\mu\nu}$ describes a 3+1-dimensional Ricci-flat spacetime.

It is a common practice to take $g_{\mu\nu}$ to be the 3+1-Minkowski metric. However, since our interest is in black holes, we follow CHR \cite{CHR} and incorporate a 3+1-Schwarzschild geometry. That is:

$$ds^2 = e^{-2ky} \left[ -U(r) dt^2 + U^{-1}(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] + dy^2,$$

where $U(r) = 1 - 2MG^{(4)}/r = 1 - r_h/r$. This solution describes a black string in 5-dimensional AdS.

As discussed in Section 1 (and see Ref.\cite{CHR}), the black string is expected to pinch off (on account of the Gregory-Laflamme instability \cite{GL}) and form into a stable “black cigar” well before reaching the AdS horizon. That is, Eq.\cite{GL} can be considered an approximate solution with validity over some finite region $|y| < y_p$, where $y_p$ represents the “pinching-off” point.\cite{GL} Hence, we will effectively restrict the bulk spacetime by placing a second brane at $y = y_c$, where $y_c < y_p$ is to be assumed. Let us further assume $Z_2$ symmetry, and so considerations may be limited to the region $0 \leq y \leq y_c$.

Since we are following the brick-wall program of ’t Hooft \cite{tHooft}, it is appropriate to consider a matter field propagating in the relevant spacetime.

\textsuperscript{3}A precise evaluation of $y_p$ remains an unresolved problem. However, “ballpark” estimates have put it at $y_p \sim k^{-1} \ln(kr_h)$ \cite{RS1, RS2}.
For simplicity, let us assume a minimally coupled (massive) scalar field that satisfies the Klein-Gordon equation:

$$\Box \Psi - m^2 \Psi = 0.$$  \hfill (3)

Rewriting this expression in terms of Eq. (2), we have:

$$\frac{e^{2ky}}{r^2} \partial_r \left[ r^2 U(r) \partial_r \Psi \right] - \frac{e^{2ky}}{U(r)} \partial_t^2 \Psi + \frac{e^{2ky}}{r^2 \sin \theta} \partial_\theta [\sin \theta \partial_\theta \Psi] + \frac{e^{2ky}}{r^2 \sin^2 \theta} \partial_\phi^2 \Psi$$

$$+ \frac{1}{e^{-4ky}} \partial_y \left[ e^{-4ky} \partial_y \Psi \right] - m^2 \Psi = 0. \hfill (4)$$

In compliance with the ’t Hooft prescription, it is also necessary to introduce the following boundary conditions:

$$\Psi = 0 \quad \text{for} \quad r \leq r_h + \epsilon,$$  \hfill (5)

$$\Psi = 0 \quad \text{for} \quad r \geq L. \hfill (6)$$

Here, $\epsilon \ll r_h$ represents the brick-wall cutoff that eliminates the ultraviolet divergences; whereas the boundary at $L >> r_h$ eliminates the infrared divergences.

Given the spherical symmetry of the 4-dimensional brane world and the existence of a timelike Killing vector, the scalar field can be decomposed as follows:

$$\Psi = e^{-iEt} Y_{l,m_l}(\theta, \phi) f(y) R(r).$$  \hfill (7)

$Y_{l,m_l}(\theta, \phi)$ is the usual spherical harmonic function, which is known to satisfy:

$$\frac{1}{\sin \theta} \partial_\theta [\sin \theta \partial_\theta Y_{l,m_l}(\theta, \phi)] + \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_{l,m_l}(\theta, \phi) = -l(l + 1) Y_{l,m_l}(\theta, \phi). \hfill (8)$$

Let us now define an “effective mass” $m_n$, where $n$ labels the various modes of the function $f(y)$, such that:

$$\frac{1}{e^{-4ky}} \partial_y \left[ e^{-4ky} \partial_y f_n(y) \right] - m^2 f_n(y) = -e^{2ky} m^2_n f_n(y). \hfill (9)$$

Then Eq. (4) conveniently reduces to the following radial equation:

$$\frac{1}{r^2} \partial_r \left[ r^2 U(r) \partial_r R(r) \right] + \frac{E^2}{U(r)} R(r) - \frac{l(l + 1)}{r^2} R(r) - m^2_n R(r) = 0. \hfill (10)$$
To further simplify the separated wave equations (9,10), we can invoke the WKB approximation. That is, we now assume that each of $R(r)$ and $f_n(y)$ can be expressed as the product of a slowly varying amplitude and an exponent with a rapidly varying phase. To leading order, one need only consider the derivatives of the phase functions, and this leads to the following expressions:

$$-\frac{1}{r^2 U(r)} \partial_r \left[ r^2 U(r) \partial_r R(r) \right] = K_r^2 R(r), \quad (11)$$

$$-\frac{1}{e^{-4ky}} \partial_y \left[ e^{-4ky} \partial_y f_n(y) \right] = K_n^2 f_n(y); \quad (12)$$

where the wave numbers (each corresponding to the derivative of the appropriate phase) are given as follows:

$$K_r = \frac{1}{U(r)} \left[ E^2 - U(r) \left( \frac{l^2 + l}{r^2} + m_n^2 \right) \right]^{\frac{1}{2}}, \quad (13)$$

$$K_n = \left[ m_n^2 e^{2ky} - m^2 \right]^{\frac{1}{2}}. \quad (14)$$

What will be particularly useful is the degeneracy of modes ($n_i$) for any given wave number ($K_i$). According to the semi-classical quantization rule, we have [17]:

$$n_r = \frac{1}{\pi} \int_{r_{n+\epsilon}}^L dr K_r, \quad (15)$$

$$n_n = \frac{1}{\pi} \int_0^{y_c} dy K_n. \quad (16)$$

From Eqs. (14,16), it is straightforward to obtain the following useful result:

$$\frac{dn_n}{dm_n} = \frac{1}{\pi k m_n} \left[ \sqrt{m_n^2 e^{2ky_c} - m^2} - \sqrt{m_n^2 - m^2} \right]. \quad (17)$$

3 The Free Energy

Let us now proceed to evaluate the free energy ($F$) of a thermal bath of bulk scalars at temperature $\beta^{-1}$. We begin by considering the standard thermodynamic definition:

$$e^{-\beta F} = \sum \tau e^{-\beta E_\tau}, \quad (18)$$
where $E_\tau$ is the thermal energy corresponding to quantum state $\tau$. Since the analysis is for bosons (whose occupation number can take on any positive integral value or zero), the following is an equivalent relation:

$$e^{-\beta F} = \prod_{n_r,n_l,m_l} \left(1 - e^{-\beta E}\right)^{-1}. \quad \text{(19)}$$

Solving for $F$, we have:

$$F = \frac{1}{\beta} \sum_{n_r,n_l,m_l} \ln \left(1 - e^{-\beta E}\right) \quad \text{(20)}$$

or in the continuum limit:

$$F = \frac{1}{\beta} \int dl (2l + 1) \int dn \int d\tau \ln \left(1 - e^{-\beta E}\right). \quad \text{(21)}$$

The factor of $2l + 1$ is, of course, due to the degeneracy of the quantum number $m_l$ for a given value of $l$.

After some additional manipulation, including an integration by parts, we find:

$$F = -\int dl (2l + 1) \int dm n \int d\tau \frac{1}{e^{\beta E} - 1} n_r. \quad \text{(22)}$$

The substitution of Eqs.(13,15,17) then yields:

$$F = -\frac{1}{\pi^2 k} \int dl (2l + 1) \int dm n \frac{1}{m_n} \left[\sqrt{m_n e^{2\beta E} - m^2} - \sqrt{m_n^2 - m^2}\right] \times \int dE \frac{1}{e^{\beta E} - 1} \int_{r_n+c}^{L} dr \frac{1}{U(r)} \sqrt{E^2 - U(r) \left(\frac{l^2 + l}{r^2} + m_n^2\right)}. \quad \text{(23)}$$

Note the unspecified limits of integration in the above expression. We must integrate over all values of phase space for which the reality of the square roots is preserved. For the implied order of integration, this condition leads to the following limits:

$$0 \leq l \leq \frac{1}{2} \left[-1 + \sqrt{1 + 4r^2 \left(\frac{1}{U(r)} E^2 - m_n^2\right)}\right], \quad \text{(24)}$$
\[ m \leq m_n \leq \frac{E}{\sqrt{U(r)}}, \quad (25) \]
\[ m\sqrt{U(r)} \leq E \leq \infty. \quad (26) \]

Before proceeding any further, let us consider the integration over all permissible values of the mass parameter \( m_n \). This is the primary difference between our calculation and that of a prior study [16]. The authors of Ref. [16] fixed the effective 4-dimensional mass \( m_n \) (which they called \( \mu \)) equal to the 5-dimensional mass \( m \). Such a simplification is contrary to the results of prior works that have studied the decomposition of bulk scalar fields. (See, for instance, Ref. [38].) It has been amply demonstrated that the bulk field manifests itself, to a 4-dimensional observer, as an infinite tower of scalars; each of which has an associated, distinct value of mass. (Note that these masses are obtainable, in principle, by solving the relevant eigenvalue problem.) Hence, for a reliable calculation, there can be no justification in singling out any one particular value of \( m_n \) as the preferred one.

The \( l \) integration of Eq. (23) can be done explicitly to yield:

\[
F = -\frac{2}{3\pi^2k} \int_m^{E/\sqrt{U(r)}} \frac{dm_n}{m_n} \left[ \frac{m_n^2e^{2ky_c} - m^2}{\sqrt{m_n^2e^{2ky_c} - m^2}} - \frac{m_n^2 - m^2}{\sqrt{m_n^2 - m^2}} \right] \times \int_{m\sqrt{U(r)}}^{\infty} dE \frac{1}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{L} dr \left[ E^2 - U(r)m_n^2 \right]^{\frac{3}{4}}. \quad (27)
\]

Let us now consider the integration over \( m_n \). This cannot be done exactly, except in the trivial case \( m = 0 \). However, we can obtain a very reasonable approximation by first considering the following argument.

Ultimately, we are only interested in contributions to the entropy (and, hence, free energy) arising from the proximity of the wall. The remaining contribution, for which \( r >> r_h \), is well understood to be the entropy of a quantum field in flat space [17] and can be neglected for our purposes. With this, as well as \( U(r) \approx 0 \) if \( r \approx r_h \), in mind, the following are valid approximations for quantities in the above integrand:

\[ E^2 - U(r)m_n^2 \approx E^2 \quad \text{unless} \quad m_n \approx E/\sqrt{U(r)}, \quad (28) \]

\(^{4}\text{More accurately, they fixed } m_n \text{ (or } \mu \text{) equal to } me^{-ky} \text{ and then used } y = 0 \text{ on the brane.}\)
\[
\frac{1}{m_n} \left[ \sqrt{m_n^2 e^{2kyc} - m^2} - \sqrt{m_n^2 - m^2} \right] \approx e^{kyc} - 1
\]

unless \( m_n \approx m \). \hspace{1cm} (29)

By virtue of the above observations, it is not difficult to show that:

\[
\int_{m}^{E/\sqrt{U(r)}} dm_n \frac{1}{m_n} \left[ \sqrt{m_n^2 e^{2kyc} - m^2} - \sqrt{m_n^2 - m^2} \right] \left[ E^2 - U(r)m_n^2 \right]^{3/2} \approx m \Upsilon(kyc)E^3 + \frac{3\pi (e^{kyc} - 1)}{16 \sqrt{U(r)}} E^4, \hspace{1cm} (30)
\]

where:

\[
\Upsilon(kyc) \equiv 2 \tan^{-1} \left( e^{kyc} + \sqrt{e^{2kyc} - 1} \right) - \frac{\pi}{2} - \sqrt{e^{2kyc} - 1}. \hspace{1cm} (31)
\]

Again, we note that Eq.(30) is exact when \( m = 0 \).

Substituting Eq.(30) into Eq.(27), we can see that the remaining integrations, over \( E \) and \( r \), have been separated. To perform the integration over \( E \), it is useful to note that the lower bound tends to zero when near the horizon; cf. Eq.(26). Hence, standard formulas \[39\] can be applied to obtain the following:

\[
\int_{0}^{\infty} dE \frac{E^4}{e^{\beta E} - 1} = \frac{24\zeta(5)}{\beta^5}, \hspace{1cm} (32)
\]

\[
\int_{0}^{\infty} dE \frac{E^3}{e^{\beta E} - 1} = \frac{\pi^4}{15\beta^4}. \hspace{1cm} (33)
\]

With the above set of results, Eq.(27) simplifies as follows:

\[
F_\epsilon \approx -\frac{3\zeta(5)(e^{kyc} - 1)}{\pi k\beta^5} \int_{r_h + \epsilon} dr \frac{r^2}{[U(r)]^{5/2}} - \frac{\pi^2 m \Upsilon(kyc)}{45k\beta^4} \int_{r_h + \epsilon} dr \frac{r^2}{[U(r)]^2}, \hspace{1cm} (34)
\]

where the subscript \( \epsilon \) indicates a near-horizon form.

Because of our near-horizon considerations, the following approximation can be used:

\[
U(r) \approx (r - r_h) \left. \frac{dU}{dr} \right|_{r=r_h} = \frac{(r - r_h)}{r_h}. \hspace{1cm} (35)
\]

The integration of Eq.(34) is now straightforward and yields:

\[
F_\epsilon \approx -\frac{2\zeta(5)(e^{kyc} - 1)}{\pi k\beta^5} r_h^{9/2} - \frac{m\pi^2 \Upsilon(kyc) r_h^4}{45k\beta^4} \epsilon. \hspace{1cm} (36)
\]
4 The Entropy

We finish off the analysis by calculating the near-horizon contribution to the entropy. The first law of thermodynamics tells us:

\[ S_\epsilon = \beta^2 \frac{\partial F_\epsilon}{\partial \beta}. \] (37)

From a thermodynamic perspective, it is most appropriate to make an off-shell evaluation and then consider the on-shell limit \([23]\). Hence, we directly apply Eq.(37) to Eq.(36) to obtain:

\[ S_\epsilon \approx \frac{10\zeta(5)(e^{ky_c} - 1)}{\pi k\beta^4} \frac{r_h^{9/2}}{\epsilon^{3/2}} + \frac{4m\pi^2\Upsilon(ky_c) r_h^4}{45k\beta^3} \frac{1}{\epsilon}. \] (38)

and then use the well-known on-shell Schwarzschild relation\(^5\) of \( \beta = 4\pi r_h \) \([33]\). This yields:

\[ S_\epsilon \approx \frac{5\zeta(5)(e^{ky_c} - 1)}{128\pi^5 k} \frac{r_h^{1/2}}{\epsilon^{3/2}} + \frac{m\Upsilon(ky_c)}{720\pi k} \frac{r_h}{\epsilon}. \] (39)

To make sense of this entropy result, it is necessary to re-express \( \epsilon \) in terms of the invariant distance from the horizon to the brick wall \([17]\). Denoting this invariant distance as \( \tilde{\epsilon} \), we can write:

\[ \tilde{\epsilon} = \int_{r_h}^{r_h + \epsilon} dr \frac{1}{\sqrt{U(r)}}. \] (40)

With the help of Eq.(33), this becomes:

\[ \tilde{\epsilon}^2 = 4r_h \epsilon. \] (41)

So, in terms of invariant quantities only, the near-horizon entropy \([39]\) takes on the form:

\[ S_\epsilon \approx \left[ \frac{5\zeta(5)(e^{ky_c} - 1)}{16\pi^6 k} \frac{1}{\tilde{\epsilon}^3} + \frac{m\Upsilon(ky_c)}{180\pi^2 k} \frac{1}{\tilde{\epsilon}^2} \right] \frac{A_h}{4}. \] (42)

\(^5\)The Schwarzschild relation is appropriate given that the surface gravity (and, hence, temperature) is constant at all points along the event horizon; whether on the brane or in the bulk \([9]\).
where $A_h = 4\pi r_h^2$ is the horizon area on the brane. Notice that the square-bracket quantity appears to be independent of the black hole geometry. Thus, we have successfully verified the black hole area law with regard to contributions from bulk scalar fields. We further interpret this result in the Section 6.

5 The Thermal Energy

In this brief section, we compare our results with an earlier study by Brevik et al. [37]. These authors considered thermal quantum fields in a “conventional” (i.e., non-black hole) Randall-Sundrum setting. In this prior work, the thermal energy was evaluated for both high and low temperature limits. Naturally, the low-temperature regime is most suitable for comparison with a semi-classical black hole. We thus quote their thermal-energy result (as applicable to scalar fields) for a large value of inverse temperature $\beta$ [37]:

$$E = \sum_n V_3 m_n^3 \frac{e^{-\beta m_n}}{(2\pi \beta)^{3/2}}, \quad (43)$$

where the summation is over all possible “Kaluza-Klein-like” modes and $V_3$ is the 3-dimensional brane volume of interest.

Because of the rapidly vanishing exponential factor, it is a suitable approximation to set $m_n$ equal to its lower-bound value. (This is what was effectively done in Ref. [37].) Hence, by way of Eq.(25), the above can be simplified as follows:

$$E \approx V_3 m_\epsilon^3 \frac{e^{-\beta m_\epsilon}}{(2\pi \beta)^{3/2}}. \quad (44)$$

Let us now consider the thermal energy in our black hole model. This is directly obtainable from the free energy (36) with application of a standard thermodynamic relation: $E = \partial(\beta F)/\partial \beta$. This yields:

$$E_\epsilon \approx \frac{8\zeta(5)(e^{k y_c} - 1)}{\pi k/\beta^5} r_h^{9/2} \frac{e^{3/2}}{\epsilon^{3/2}} + \frac{m \pi^2 \Upsilon(k y_c) r_h^4}{15 k \beta^4} \epsilon. \quad (45)$$

6 A possible exception to this statement may be the parameter $y_c$, depending on its physical interpretation. This point is elaborated on in Section 5.

7 That is, modes that arise in the decomposition of the bulk scalar field. Hence, $m_n$ is the “effective mass” as defined by Eq.(9).
Next using $\varepsilon^2 = 4r_h \varepsilon$, $\beta = 4\pi r_h$ and appropriately setting $V_3 = 4\pi r_h^2 \varepsilon$, we find:

$$
E_\varepsilon \approx \frac{V_3}{\beta} \left[ \frac{\zeta(5)(e^{ky_c} - 1)}{16\pi^6 k \varepsilon^4} + \frac{m \Upsilon(k y_c)}{960\pi^2 k \varepsilon^3} \right].
$$

A comparison of Eq.(46) with Eq.(44) reveals quite a contrast in thermodynamics for the two different scenarios. For instance, only the black hole thermal energy has an explicit dependence on the bulk parameter $y_c$ in this low-temperature regime. (Although, one would expect this to change when considering generic values of temperature.) Also, it is evident that the black hole thermal energy diverges much more slowly as $\beta \to \infty$. This behavior can likely be attributed to the black hole horizon being a surface of infinite “red-shifting”. That is, only modes of arbitrarily small wavelength can exist close to the wall [17, 29].

6 Conclusion

In the preceding paper, we have considered a bulk scalar field propagating in the background spacetime of a black hole on a Randall-Sundrum brane [2, 3] (or black cigar in the bulk [3]). A semi-classical quantization procedure, along the lines of 't Hooft's original brick-wall model [17], allowed us to calculate the thermal energy due to this field. The near-horizon contribution to this thermal energy lead directly to an evaluation of the corresponding entropy. This result (see Eq.(42)) was found to satisfy the Bekenstein-Hawking area law of black hole entropy [18, 19]. We also considered the thermal energy, and how it compared with that found in an earlier study [37].

The complete black hole entropy in this RS brane world is obtainable, in principle, by summing over the thermal contributions of all relevant fields. This summation should include bulk fields, as well as those restricted to the brane (i.e., the Standard Model fields). The leading-order divergent term (or terms) could then be absorbed in a renormalization of Newton’s gravitational constant to yield the standard form of $A_h/4G^{(4)}$ [34]. Meanwhile, the sub-leading divergent terms could (in principle) be absorbed by renormalizing the coupling constants of action terms that are quadratic in curvature [30].

With the above discussion in mind, it is interesting to compare the leading-order contribution from the bulk fields ($S_{bu} \sim A_h \varepsilon^{-2}$) with the analogous result for brane fields of $S_{br} \sim A_h \varepsilon^{-2}$ [17]. This comparison would
imply that the bulk fields dominate the entropy, but this need not be the case. To see this, first consider that the bulk-field contribution also contains a factor of $k$ in the denominator (i.e., the inverse AdS parameter). Because of hierarchical arguments, it is believed that $k^{-1}$ is on the order of the Planck length $\tilde{l}_{pl}$. One would also expect the ultraviolet cutoff $\tilde{\epsilon}$ to be of this order, as quantum fluctuations prevent events closer (to the horizon) than $l_{pl}$ from being observed. Hence, the leading-order contributions from bulk and brane fields should be of the same order; namely, $S_i \sim r_{h}^2 l_{pl}^{-2}$.

It is interesting to note that the sub-leading term in the bulk thermal entropy is likely negative, as can been seen by carefully examining the defining relation for $\Upsilon(ky_c)$; cf. Eqs.(31,42). If $m \sim m_{pl}$, then this term is also of order $r_{h}^2 l_{pl}^{-2}$ and may effectively cancel out the $\tilde{\epsilon}^{-3}$ contribution. It is worth pointing out, however, that the bulk fields (for instance: gravitons, three-form tensors, moduli scalars) are likely to be predominantly massless; thus, negating this negative contribution.

For illustrative purposes, we have focused attention on a relatively simple brane world scenario. However, the brick-wall formalism should be applicable to other, (perhaps) more interesting cases. For instance, one might apply these techniques to a model where the brane is realized dynamically out of a higher-derivative gravity bulk [40, 41]. Significantly to this case, the bulk can be interpreted as a 5-dimensional Schwarzschild-anti-de Sitter black hole. In Ref.[41], Nojiri et al. calculated the associated entropy (by geometric arguments) and found that, in general, there was a discrepancy between this entropy and that deduced from holographic considerations in 4-dimensions. The authors then interpreted this discrepancy as a measure of the deviation from a broken AdS/CFT correspondence [43]. It would be interesting to see if this discrepancy is resilient in a brick-wall context, and we hope to address this issue in a future work. We do, however, anticipate that the discrepancy perseveres, given the observed breakdown in the area law for a 5-dimensional brick-wall calculation [24] (noting that a similar breakdown was found in Ref.[41]).

Finally, a brief comment regarding the position of the negative-tension brane (i.e., $y_c$) is in order. If we assume an explicit RS2 model (where the neg-

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8This “holographic” entropy was obtained [41] by identifying the brane dynamics as a Friedmann-Robertson-Walker cosmological equation and then applying an analogue of the Cardy formula [42].
ative brane tends to the AdS horizon), then the parameter $y_c$ (as it appears in our formalism) no longer represents the location of the negative brane. Rather, it represents a transverse limit in the bulk due to instabilities in the black string solution $[3, 8]$. If this is the case, $y_c$ would not be determined by a stabilization mechanism (such as those of Ref.$[44, 45]$); instead, it should probably be regarded as an implicit function of the black hole geometry. So in this event, the area law breaks down, as both terms in Eq.$[12]$ depend explicitly on $y_c$. However, one could argue that the multiplicity of bulk fields is small compared to that of the brane fields (which includes all particles prescribed by the Standard Model); thus, suppressing the bulk contribution to the entropy. That is to say, observers living on the brane may not readily detect such a breakdown.

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