Propagation of chaos for many-boson systems in one dimension with a point pair-interaction

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Abstract

We consider the semiclassical limit of nonrelativistic quantum many-boson systems with delta potential in one dimensional space. We prove that time evolved coherent states behave semiclassically as squeezed states by a Bogoliubov time-dependent affine transformation. This allows us to obtain properties analogous to those proved by Hepp and Ginibre-Velo ([He], [GiVe1, GiVe2]) and also to show propagation of chaos for Schrödinger dynamics in the mean field limit. Thus, we provide a derivation of the cubic NLS equation in one dimension.

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1 Introduction

The justification of the chaos conservation hypothesis in quantum many-body theory is the main concern of the present paper. This well-know hypothesis finds its roots in statistical physics of classical many-particle systems as a quantum counterpart. See, for instance [MS], [Go] and references therein.

Non-relativistic quantum systems of $N$ bosons moving in $d$-dimensional space are commonly described by the Schrödinger Hamiltonian

$$H_N := \sum_{i=1}^N -\Delta_i + \sum_{i<j} V_N(x_i - x_j), \quad x \in \mathbb{R}^d,$$

acting on the space of symmetric square-integrable functions $L^2_s(\mathbb{R}^{dN})$ over $\mathbb{R}^{dN}$. Here $V_N$ stands for an even real pair-interaction potential. The Hamiltonian (1), under appropriate conditions on $V_N$, defines a self-adjoint operator and hence the Schrödinger equation

$$i\partial_t \Psi_N = H_N \Psi_N,$$

admits a unique solution for any initial data $\Psi_N^0 \in L^2(\mathbb{R}^{dN})$. The interacting $N$-boson dynamics (2) are considered in the mean field scaling, namely, when $N$ is large and the pair-potential is given by

$$V_N(x) = \frac{1}{N} V(x),$$

with $V$ independent of $N$. The chaos conservation hypothesis for the $N$-boson system (2) amounts to the study of the asymptotics of the $k$-particle correlation functions $\gamma_{k,N}$ given by

$$\gamma_{k,N}(x_1, \cdots, x_k; y_1, \cdots, y_k) = \int_{\mathbb{R}^{d(N-k)}} \gamma_N(x_1, \cdots, x_k; z_{k+1}, \cdots, z_N; y_1, \cdots, y_k, z_{k+1}, \cdots, z_N) \, dz_{k+1} \cdots dz_N,$$
where $\gamma' = \Psi'_{N}(x_{1}, \cdots, x_{N})\overline{\Psi'_{N}(y_{1}, \cdots, y_{N})}$. More precisely, this hypothesis holds if for an initial datum which factorizes as

$$\Psi'_{N} = \phi_{0}(x_{1}) \cdots \phi_{0}(x_{N})$$

such that $||\phi_{0}||_{L^{2}(\mathbb{R}^{d})} = 1$;

the $k$-particle correlation functions converges in the trace norm

$$\gamma'_{k,N} \rightarrow \phi_{1}(x_{1}) \cdots \phi_{k}(x_{k}) \overline{\phi_{1}(y_{1}) \cdots \phi_{k}(y_{k})},$$

where $\phi_{k}$ solves the nonlinear Hartree equation

$$\left\{ \begin{array}{l}
\imath \partial_{t} \phi = -\Delta \phi + V \ast |\phi|^{2} \\
\phi_{t=0} = \phi_{0}.
\end{array} \right.$$

The convergence of correlation functions (4) for the Schrödinger dynamics (2) is equivalent to the statement below:

$$\lim_{N \rightarrow \infty} \langle \Psi'_{N}, \mathcal{O}_{N} \Psi'_{N} \rangle = \lim_{N \rightarrow \infty} \int_{\mathbb{R}^{2dN}} \gamma'_{N}(x_{1}, \cdots, x_{N}; y_{1}, \cdots, y_{N}) \overline{\gamma'_{N}(y_{1}, \cdots, y_{N}; x_{1}, \cdots, x_{N})} dx_{1} \cdots dx_{N} dy_{1} \cdots dy_{N} = \langle \phi'_{1} \hat{\mathcal{O}}^{\otimes k}, \tilde{\mathcal{O}}_{N} \phi'_{k} \otimes \hat{\mathcal{O}}^{\otimes (N-k)} \rangle,$$

where $\mathcal{O}_{N}$ are observables given by $\mathcal{O}_{N} := \mathcal{O} \otimes 1^{(N-k)}$ acting on $L^{2}(\mathbb{R}^{dN})$ with $\mathcal{O} : L^{2}(\mathbb{R}^{dk}) \rightarrow L^{2}(\mathbb{R}^{dk})$ a bounded operator with kernel $\hat{\mathcal{O}}$ and $k$ is a fixed integer. The relevance of those observables is justified by the fact that $\mathcal{O}_{N}$ are essentially canonical quantizations of classical quantities.

In the recent years, mainly motivated by the study of Bose-Einstein condensates, there is a renewed and growing interest in the analysis of many-body quantum dynamics in the mean field limit (for instance see [ABGT], [BEGMY], [BGM], [ESY], [EY], [FGS], [FKP], [FKS], etc.). For a general presentation on the subject we refer the reader to the reviews [Spe] and [Gal]. Various strategies were developed in order to prove the chaos conservation hypothesis or even stronger statements. One of the oldest approaches is the so-called BBGKY hierarchy (named after Bogoliubov, Born, Green, Kirkwood, and Yvon) which consists in considering the Heisenberg equation,

$$\left\{ \begin{array}{l}
\partial_{t} \rho_{t} = i[\rho_{t}, H_{k}], \\
\rho_{t=0} = |\phi'_{0} \otimes N \rangle \langle \phi'_{0} |^{\otimes N},
\end{array} \right.$$

together with the finite chain of equations arising from (7) by taking partial traces on $0 \leq k \leq N$ variables. Since $\rho_{t}$ are trace class operators one can write the corresponding hierarchy of equations on the $k$-particle correlation functions $\gamma'_{k,N}$:

$$\left\{ \begin{array}{l}
\imath \partial_{t} \gamma'_{k,N} = \sum_{i=1}^{N} [-\Delta_{x_{i}} + \Delta_{y_{i}}] \gamma'_{k,N} + \frac{1}{N} \sum_{1 \leq i < j \leq k} [V(x_{i} - x_{j}) - V(y_{i} - y_{j})] \gamma'_{k,N} \\
+ \frac{1}{N} \sum_{1 \leq i < k, 1 \leq j \leq k} \int_{\mathbb{R}^{(N-k)d}} [V(x_{i} - x_{j}) - V(y_{i} - y_{j})] \gamma'_{k,N} dx_{k+1} \cdots dx_{N} \\
+ \frac{1}{N} \sum_{k+1 \leq i < j \leq N} \int_{\mathbb{R}^{(N-k)d}} [V(x_{i} - x_{j}) - V(y_{i} - y_{j})] \gamma'_{k,N} dx_{k+1} \cdots dx_{N} \\
\gamma'_{0,N} = \phi_{0}(x_{1}) \cdots \phi_{0}(x_{k}) \overline{\phi_{0}(y_{1}) \cdots \phi_{0}(y_{k})}.
\end{array} \right.$$

An alternative approach to the chaos conservation hypothesis uses the second quantization framework (details on this notions are recalled in Section[2]). Consider the Hamiltonian,

$$e^{-1}H_{e} = \int_{\mathbb{R}^{d}} \nabla a^{*}(x) \nabla a(x) \, dx + \frac{e}{2} \int_{\mathbb{R}^{d}} V(x-y) a^{*}(x) a^{*}(y) a(x) a(y) \, dx dy,$$

where $a, a^{*}$ are the usual creation-annihilation operator-valued distributions in the Fock space over $L^{2}(\mathbb{R}^{d})$. Recall that $a$ and $a^{*}$ satisfy the canonical commutation relations

$$[a(x), a^{*}(y)] = \delta(x-y), \; [a^{*}(x), a^{*}(y)] = 0 = [a(x), a(y)].$$
A simple computation leads to the following identity
\[ e^{-1}H_{\varepsilon}(\varepsilon^{-1}H_{\varepsilon}) = H_N, \quad \text{if} \quad \varepsilon = \frac{1}{N}. \]
Thus, the statement on the chaos propagation stated in (6) may be written (up to an unessential factor) as
\[ \lim_{\varepsilon \to 0} \langle e^{-\varepsilon^{-1}H_{\varepsilon}} \Psi_{\varepsilon}^0, b_{\text{Wick}} e^{-\varepsilon^{-1}H_{\varepsilon}} \Psi_{\varepsilon}^0 \rangle = \langle \phi_{\varepsilon}^{\otimes k}, \phi_{\varepsilon}^{\otimes k} \rangle, \]
where \( b_{\text{Wick}} \) denotes \( \varepsilon \)-dependent Wick observables defined by
\[ b_{\text{Wick}} = \varepsilon^k \int_{\mathbb{R}^{2kd}} \prod_{i=1}^{k} a^*(x_i) \tilde{\phi}(x_1, \ldots, x_k; y_1, \ldots, y_k) \prod_{j=1}^{k} a(y_j) \, dx_1 \cdots dx_k \, dy_1 \cdots dy_k, \]
with \( \tilde{\phi}(x_1, \ldots, x_k; y_1, \ldots, y_k) \) the distribution kernel of a bounded operator \( \phi \) on \( L^2(\mathbb{R}^{kd}) \). Therefore, the
mean field limit \( N \to \infty \) for \( H_N \) can be converted to a semiclassical limit \( \varepsilon \to 0 \) for \( H_{\varepsilon} \). The study of the
semiclassical limit of the many-boson systems traces back to the work of Hepp [Hep] and was subsequently improved by Ginibre and Velo [GiVe1] [GiVe2]. The latter analysis are based on coherent states, i.e.,
\[ \Psi_{\varepsilon}^0 = e^{-\frac{i\varepsilon^2}{2\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^{-n/2} \phi_{\varepsilon}^{\otimes n}, \quad \phi \in L^2(\mathbb{R}^d), \]
which have infinite number of particles in contrast to the Hermite states \( \Psi_{\varepsilon}^0 = \phi_{0}^{\otimes N} \). However, a simple
argument in the work of Rodnianski and Schlein [RoSch] shows that the semiclassical analysis is enough
to justify the chaos conservation hypothesis and even provides convergence estimates on the \( k \)-particle
correlation functions. The authors of [RoSch] considered the problem under the assumption of \((-\Delta + 1)^{1/2}
bounded potential \( i.e., V(-\Delta + 1)^{-1/2} \) is bounded). The main purpose of the present paper is to extend the
latter result to more singular potentials using the ideas of Ginibre and Velo [GiVe2].

For the sake of clarity, we restrict ourselves in this paper to the particular example of point interaction
potential in one dimension, i.e.,
\[ V(x) = \delta(x), \quad x \in \mathbb{R}. \]
This example is typical for potentials which are \( -\Delta \)-form bounded \( i.e., (-\Delta + 1)^{-1/2}V(-\Delta + 1)^{-1/2} \) is bounded). Indeed, we believe that such simple example sums up the principal difficulties on the problem.
Moreover, we state in Appendix [C] some abstract results on the non-autonomous Schrödinger equation
which have their own interest and allow to consider a more general setting. We also remark that the results
here can be easily extended to the case \( V(x) = -\delta(x) \) at the price to work locally in time.

The paper is organized as follows. We first recall the basic definitions for the Fock space framework
in Section [2]. Then we accurately introduce the quantum dynamics of the considered many-boson system
and its classical counterpart, namely the cubic NLS equation. The study of the semiclassical limit through
Hepp’s method is carried out in Section [3] where we use results on the time-dependent quadratic approximation derived in Section [5]. Finally, in Section [7] we apply the argument of [RoSch] to prove the chaos propagation result.

2 Preliminaries

Let \( \mathcal{H} \) be a Hilbert space. We denote by \( \mathcal{L}(\mathcal{H}) \) the space of all linear bounded operators on \( \mathcal{H} \). For a linear
unbounded operator \( L \) acting on \( \mathcal{H} \), we denote by \( \mathcal{D}(L) \) \( i.e., \mathcal{D}(L) \) the operator domain (respectively
form domain) of \( L \). Let \( D_{\sigma} \) denotes the differential operator \( -i\partial_{\sigma} \) on \( L^2(\mathbb{R}^n) \) where \( (x_1, \cdots, x_n) \in \mathbb{R}^n \).

In the following we recall the second quantization framework. We denote by \( L^2_2(\mathbb{R}^{nd}) \) the space of
symmetric square integrable functions, i.e.,
\[ \Psi_n \in L^2_2(\mathbb{R}^{nd}) \ \text{iff} \ \Psi_n \in L^2(\mathbb{R}^{nd}) \ \text{and} \ \Psi_n(x_1, \cdots, x_n) = \Psi_n(x_{\sigma_1}, \cdots, x_{\sigma_n}) \ \text{a.e.,} \]
endowed with the inner product

\[ H = \left\{ \Psi_n \right\} \text{ where } \Psi_n = \left( \psi_{n1}, \cdots, \psi_{nN} \right) \in \mathcal{S} \text{ (Schwartz space on } \mathbb{R}^d) \text{.} \]

We will often use the notation

\[ \mathcal{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^{nd}) \]

dowered with the inner product

\[ \langle \Psi, \Phi \rangle = \int_{\mathbb{R}^{nd}} \Psi_n(x_1, \cdots, x_n) \Phi_n(x_1, \cdots, x_n) \, dx_1 \cdots dx_n, \]

where \( \Psi = (\Psi_n)_n \in \mathcal{S} \) and \( \Phi = (\Phi_n)_n \in \mathcal{S} \) are two arbitrary vectors in \( \mathcal{F} \). A convenient subspace of \( \mathcal{F} \) is given as the algebraic direct sum

\[ \mathcal{F} = \mathfrak{alg} \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathbb{R}^{nd}). \]

Most essential linear operators on \( \mathcal{F} \) are determined by their action on the family of vectors

\[ \phi^{\otimes n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} \phi(x_i), \quad \phi \in L^2(\mathbb{R}^d), \]

which spans the space \( L^2(\mathbb{R}^{nd}) \) thanks to the polarization identity,

\[ \mathfrak{S}_n \prod_{i=1}^{n} \phi(x_i) = \frac{1}{2^n n!} \sum_{\varepsilon = \pm 1} \prod_{i=1}^{n} \prod_{j=1}^{n} (\varepsilon_j f(x_j) + f(x_j)), \]

For example, the creation and annihilation operators \( a^\dagger(f) \) and \( a(f) \), parameterized by \( \varepsilon > 0 \), are defined by

\[ a(f)\phi^{\otimes n} = \sqrt{\varepsilon} n \left( f, \phi \right) \phi^{\otimes(n-1)} \]
\[ a^\dagger(f)\phi^{\otimes n} = \sqrt{\varepsilon(n+1)} \mathfrak{S}_{n+1}(f \otimes \phi^{\otimes n}), \quad \forall \phi, f \in L^2(\mathbb{R}^d). \]

They can also be written as

\[ a(f) = \sqrt{\varepsilon} \int_{\mathbb{R}^d} f(x) a(x) \, dx, \quad a^\dagger(f) = \sqrt{\varepsilon} \int_{\mathbb{R}^d} f(x) a^\dagger(x) \, dx, \]

where \( a^\dagger(x), a(x) \) are the canonical creation-annihilation operator-valued distributions. Recall that for any \( \Psi = (\Psi^{(n)})_n \in \mathcal{F} \), we have

\[ a(x)\Psi^{(n)}(x_1, \cdots, x_n) = \sqrt{(n+1)}\Psi^{(n+1)}(x, x_1, \cdots, x_n), \]
\[ a^\dagger(x)\Psi^{(n)}(x_1, \cdots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x-x_j) \Psi^{(n-1)}(x_1, \cdots, \hat{x}_j, \cdots, x_n), \]

where \( \delta \) is the Dirac distribution at the origin and \( \hat{x}_j \) means that the variable \( x_j \) is omitted. The Weyl operators are given for \( f \in L^2(\mathbb{R}^d) \) by

\[ W(f) = e^{\frac{\varepsilon}{2} [a^\dagger(f) + a(f)]}, \]
and they satisfy the Weyl commutation relations,

$$W(f_1)W(f_2) = e^{-\frac{i}{2} \text{Im}(f_1,f_2)} W(f_1 + f_2),$$

with $f_1, f_2 \in L^2(\mathbb{R}^d)$.

Let us briefly recall the Wick-quantization procedure of polynomial symbols.

**Definition 2.1** We say that a function $b : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is a continuous $(p,q)$-homogenous polynomial on $\mathcal{S}(\mathbb{R}^d)$ iff it satisfies:

(i) $b(\lambda z) = \lambda^p b(z)$ for any $\lambda \in \mathbb{C}$ and $z \in \mathcal{S}(\mathbb{R}^d)$,

(ii) there exists a (unique) continuous hermitian form $\Omega : \mathcal{S}_s(\mathbb{R}^{dq}) \times \mathcal{S}_s(\mathbb{R}^{dp}) \to \mathbb{C}$ such that

$$b(z) = \Omega(z^{(p)}, z^{(q)}).$$

We denote by $\mathcal{E}$ the vector space spanned by all those polynomials.

The Schwartz kernel theorem ensures for any continuous $(p,q)$-homogenous polynomial $b$, the existence of a kernel $\tilde{b}(\cdot, \cdot) \in \mathcal{S}'(\mathbb{R}^{d(p+q)})$ such that

$$b(z) = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(k_1', \cdots, k_p'; k_1, \cdots, k_p) \overline{z(k_1') \cdots z(k_p')} z(k_1) \cdots z(k_p) \, dk'dk,$$

in the distribution sense. The set of $(p,q)$-homogenous polynomials $b \in \mathcal{E}$ such that the kernel $\tilde{b}$ defines a bounded operator from $L^2_L(\mathbb{R}^{dp})$ into $L^2_L(\mathbb{R}^{dq})$ will be denoted by $\mathcal{S}_{p,q}(L^2(\mathbb{R}^d))$. Those classes of polynomial symbols are studied and used in [AmNi1, AmNi2].

**Definition 2.2** The Wick quantization is the map which associate to each continuous $(p,q)$-homogenous polynomial $b \in \mathcal{E}$, a quadratic form $b^{\text{Wick}}$ on $\mathcal{S}$ given by

$$\langle \Psi, b^{\text{Wick}} \Phi \rangle = \epsilon^{n+p+q} \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(k', k) \langle a(k_1') \cdots a(k_p') \Psi, a(k_1) \cdots a(k_p) \Phi \rangle \, dk' dk,$$

$$= \sum_{n=p}^{\infty} \epsilon^{n+p+q} \sqrt{n!(n-p+q)!} \int_{\mathbb{R}^{d(n-p)}} dx \int_{\mathbb{R}^{d(p+q)}} dk' dk \tilde{b}(k', k) \Psi(x) \Phi(n-p+q)(k', x),$$

for any $\Phi, \Psi \in \mathcal{S}$.

We have, for example,

$$a^*(f) = \langle \zeta, f \rangle^{\text{Wick}} \quad \text{and} \quad a(f) = \langle f, \zeta \rangle^{\text{Wick}}.$$

Furthermore, for any self-adjoint operator $A$ on $L^2(\mathbb{R}^d)$ such that $\mathcal{S}(\mathbb{R}^d)$ is a core for $A$, the Wick quantization

$$d \Gamma(A) := \langle \zeta, A \zeta \rangle^{\text{Wick}},$$

defines a self-adjoint operator on $\mathcal{S}$. In particular, if $A$ is the identity we get the $\epsilon$-dependent number operator

$$N := \langle \zeta, \zeta \rangle^{\text{Wick}}.$$

We recall the standard number estimate (see, e.g., [AmNi1, Lemma 2.5]),

$$\left| \langle \Psi, b^{\text{Wick}} \Phi \rangle \right| \leq ||\tilde{b}||_{\mathcal{S}'(\mathbb{R}^{dp} \otimes \mathbb{R}^{dq})} ||N^{q/2} \Psi|| \times ||N^{p/2} \Phi||,$$

which holds uniformly in $\epsilon \in (0,1]$ for $b \in \mathcal{S}_{p,q}(L^2(\mathbb{R}^d))$ and any $\Psi, \Phi \in \mathcal{S}(N^{\max(p,q)}/2)$. 

5
3 Many-boson system

In nonrelativistic many-body theory, boson systems are described by the second quantized Hamiltonian in the symmetric Fock space $\mathcal{F}$ formally given by

$$
-\epsilon \int_{\mathbb{R}^d} a^*(x) \Delta a(x) \, dx + \frac{\epsilon^2}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^*(x) a^*(y) \delta(x-y) a(x) a(y) \, dx \, dy .
$$

(11)

The rigorous meaning of formula (11) is as a quadratic form on $\mathcal{F}$, which we denote by $h^{\text{Wick}}$, obtained by Wick quantization of the classical energy functional bounded from below, which we denote by

$$
\langle h(z) = \int_{\mathbb{R}^d} |\nabla z|^2 \, dx + P(z), \quad \text{where} \quad P(z) = \frac{1}{2} \int_{\mathbb{R}^d} |z(x)|^4 \, dx, \quad z \in \mathcal{H}(\mathbb{R}^d).
$$

(12)

More explicitly, we have for $\Psi \in \mathcal{F}$

$$
\langle \Psi, h^{\text{Wick}} \Psi \rangle = \epsilon \sum_{n=1}^{\infty} \int_{\mathbb{R}^{d n}} \left| \partial_{x_1} \Psi^{(n)}(x_1, \ldots, x_n) \right|^2 \, dx_1 \cdots dx_n
$$

$$
+ \epsilon^2 \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \int_{\mathbb{R}^{d(n-1)}} \left| \Psi^{(n)}(x_2, x_3, \ldots, x_n) \right|^2 \, dx_2 \cdots dx_n.
$$

Moreover, in one dimensional space ($i.e.$, $d = 1$) one can show the existence of a unique self-adjoint operator bounded from below, which we denote by $H_\epsilon$, such that

$$
\langle \Psi, H_\epsilon \Psi \rangle = \langle \Psi, h^{\text{Wick}} \Psi \rangle, \quad \text{for any} \quad \Psi \in \mathcal{F}.
$$

This is proved in Proposition 3.3.

In all the sequel we restrict our analysis to space dimension $d = 1$ and consider the small parameter $\epsilon$ such that $\epsilon \in (0, 1]$. The $\epsilon$-independent self-adjoint operator,

$$
S_\mu \Psi := \Psi + \sum_{n=1}^{\infty} \left[ \mu^n \Psi^{(n)} + \sum_{j=1}^{n} -\Delta_j \Psi^{(n)} \right] = (\epsilon^{-1} d\Gamma(-\Delta) + \epsilon^{-\mu} N^\mu + 1) \Psi,
$$

with $\mu > 0$, defines the Hilbert space $\mathcal{H}_\mu$ given as the linear space $\mathcal{D}(S^{1/2}_\mu)$ equipped with the inner product

$$
\langle \Psi, \Phi \rangle_{\mathcal{H}_\mu} := \langle S^{1/2}_\mu \Psi, S^{1/2}_\mu \Phi \rangle_{\mathcal{D}}.
$$

We denote by $\mathcal{F}_\mu$ the completion of $\mathcal{D}(S^{-1/2}_\mu)$ with respect to the norm associated to the following inner product

$$
\langle \Psi, \Phi \rangle_{\mathcal{F}_\mu} := \langle S^{-1/2}_\mu \Psi, S^{-1/2}_\mu \Phi \rangle_{\mathcal{D}}.
$$

Therefore, we have the Hilbert rigging

$$
\mathcal{H}_\mu \subset \mathcal{F} \subset \mathcal{F}_\mu.
$$

Note that the form domain of the $\epsilon$-dependent self-adjoint operator $d\Gamma(-\Delta) + N^\mu$ with $\mu > 0$ is

$$
\mathcal{D}(d\Gamma(-\Delta) + N^\mu) = \mathcal{F}_\mu \quad \text{for any} \quad \epsilon \in (0, 1].
$$

Lemma 3.1 For any $\Psi, \Phi \in \mathcal{F}$,

$$
\left| \langle \Psi, h^{\text{Wick}} \Phi \rangle \right| \leq \frac{1}{4} \left[ \left\| d\Gamma(-\Delta) + N^{3/2} \right\| \times \left\| [d\Gamma(-\Delta) + N^{3/2}]^{1/2} \right\| \right].
$$

Proof. A simple computation yields for any $\Psi, \Phi \in \mathcal{F}$

$$
\langle \Psi, h^{\text{Wick}} \Phi \rangle = \epsilon^2 \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \int_{\mathbb{R}^{d(n-1)}} \Psi^{(n)}(x_2, x_3, \ldots, x_n) \Phi^{(n)}(x_2, x_3, x_4, \ldots, x_n) \, dx_2 \cdots dx_n.
$$
Observe that we also have
\[ \langle \Psi, P_{Wick} \Phi \rangle \leq \left[ \sum_{n=2}^{\infty} \frac{e^{2} n(n-1)}{2} \int_{\mathbb{R}^{n-1}} |\Psi^{(n)}(x_{2}, x_{3}, \ldots, x_{n})|^{2} dx_{2} \cdots dx_{n} \right]^{1/2} \times \left[ \sum_{n=2}^{\infty} \frac{e^{2} n(n-1)}{2} \int_{\mathbb{R}^{n-1}} |\Phi^{(n)}(x_{2}, x_{3}, \ldots, x_{n})|^{2} dx_{2} \cdots dx_{n} \right]^{1/2}. \]

Using Lemma A.1, we get for any \( \alpha(n) > 0 \)
\[ \left| \langle \Psi, P_{Wick} \Phi \rangle \right| \leq \left[ \sum_{n=2}^{\infty} \frac{e^{2} n(n-1)}{2} \left( \alpha(n) \langle D_{x_{1}}^{2} \Psi^{(n)}, \Psi^{(n)} \rangle + \frac{\alpha(n) - 1}{2} \langle \Psi^{(n)}, \Psi^{(n)} \rangle \right) \right]^{1/2} \times \left[ \sum_{n=2}^{\infty} \frac{e^{2} n(n-1)}{2} \left( \alpha(n) \langle D_{x_{1}}^{2} \Phi^{(n)}, \Phi^{(n)} \rangle + \frac{\alpha(n) - 1}{2} \langle \Phi^{(n)}, \Phi^{(n)} \rangle \right) \right]^{1/2}. \]

Hence, by choosing \( \alpha(n) = \frac{1}{\sqrt{2e(n-1)}} \), it follows that
\[ \left| \langle \Psi, P_{Wick} \Phi \rangle \right| \leq \frac{1}{4} \left[ \sum_{n=2}^{\infty} e^{2} n(n-1) \langle D_{x_{1}}^{2} \Psi^{(n)}, \Psi^{(n)} \rangle + \sum_{n=2}^{\infty} e^{2} n(n-1)^{2} \langle \Psi^{(n)}, \Psi^{(n)} \rangle \right]^{1/2} \times \left[ \sum_{n=2}^{\infty} e^{2} n(n-1) \langle D_{x_{1}}^{2} \Phi^{(n)}, \Phi^{(n)} \rangle + \sum_{n=2}^{\infty} e^{2} n(n-1)^{2} \langle \Phi^{(n)}, \Phi^{(n)} \rangle \right]^{1/2} \leq \frac{1}{4} \sqrt{\langle \Psi, [d\Gamma(-\Delta) + N^{3}]|\Psi \rangle} \times \sqrt{\langle \Phi, [d\Gamma(-\Delta) + N^{3}]|\Phi \rangle}. \]

This leads to the claimed estimate.

**Remark 3.2** Note that, as in Lemma 3.1, the estimate
\[ \left| \langle \Psi, P_{Wick} \Phi \rangle \right| \leq \frac{e^{2}}{4} ||\Psi||_{F_{+}^{3}} ||\Phi||_{F_{+}^{3}} \]
holds true for any \( \Psi, \Phi \in \mathcal{S} \) and \( \epsilon \in (0, 1] \).

We can show that \( h_{Wick} \) is associated to a self-adjoint operator by considering its restriction to each sector \( L_{2}^{+}(\mathbb{R}^{n}) \), however we will prefer the following point of view.

**Proposition 3.3** There exists a unique self-adjoint operator \( H_{e} \) such that
\[ \langle \Psi, h_{Wick} \Phi \rangle = \langle \Psi, H_{e} \Phi \rangle \text{ for any } \Psi \in \mathcal{S}_{+}^{3}, \Phi \in \mathcal{D}(H_{e}) \cap \mathcal{F}_{+}^{3}. \]

Moreover, \( e^{-it/\hbar H_{e}} \) preserves \( \mathcal{F}_{+}^{3} \).

**Proof.** We first use the KLNM theorem ([RS Theorem X17]) and Lemma 3.1 to show that the quadratic form \( h_{Wick} + N^{3} + 1 \) is associated to a unique (positive) self-adjoint operator \( L \) with
\[ \mathcal{D}(L) = \mathcal{D}(d\Gamma(-\Delta) + N^{3}) = \mathcal{F}_{+}^{3}. \]

Observe that we also have
\[ \| [d\Gamma(-\Delta) + N^{3}]^{1/2} \Psi \| \leq \| L^{1/2} \Psi \| \text{ for any } \Psi \in \mathcal{F}_{+}^{3}. \] (14)

Next, by the Nelson commutator theorem (Theorem B.2) we can prove that the quadratic form \( h_{Wick} \) is uniquely associated to a self-adjoint operator denoted by \( H_{e} \) with \( \mathcal{D}(L) \subset \mathcal{D}(H_{e}) \cap \mathcal{F}_{+}^{3} \) and deduce the invariance of \( \mathcal{F}_{+}^{3} \). Indeed, we easily check using Lemma 3.1 and (14) that
\[ \| \langle \Psi, h_{Wick} \Phi \rangle \| \leq \frac{5}{4} \| L^{1/2} \Psi \| \| L^{1/2} \Phi \| \text{ for any } \Psi, \Phi \in \mathcal{F}_{+}^{3}. \] (15)
Furthermore, we have for $\Psi, \Phi \in \mathcal{F}_{\pm}^3$ and $\lambda > 0$

$$
\langle L(\lambda L + 1)^{-1}\Psi, h^{Wick}((\lambda L + 1)^{-1}\Phi) \rangle - \langle (\lambda L + 1)^{-1}\Psi, h^{Wick}L(\lambda L + 1)^{-1}\Phi \rangle = 0.
$$

(16)

The statements (15)-(16) with the help of Lemma B.3 allow to use Theorem B.2

**Remark 3.4** The same argument as in Proposition 3.3 shows that the quadratic form on $\mathcal{F}_{\pm}^3$ given by

$$
G := \varepsilon^{-1}d\Gamma(-\Delta) + \varepsilon^{-2}p^{Wick} + \varepsilon^{-1}N + 1,
$$

is associated to a unique (positive) self-adjoint operator which we denote by the same symbol $G$.

## 4 The cubic NLS equation

Let $H^s(\mathbb{R}^m)$ denote the Sobolev spaces. The energy functional $h$ given by (12) has the associated vector field

$$
X : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R})
$$

$$
z \mapsto X(z) = -\Delta z + \partial_\varepsilon p(z),
$$

which leads to the nonlinear classical field equation

$$
i\partial_t \phi = X(\phi) = -\Delta \phi + |\phi|^2 \phi
$$

(17)

with initial data $\phi_{t=0} = \phi_0 \in H^1(\mathbb{R})$. It is well-known that the above cubic defocusing NLS equation is globally well-posed on $H^s(\mathbb{R})$ for $s \geq 0$. In particular, the equation (17) admits a unique global solution on $C(\mathbb{R}, H^m(\mathbb{R})) \cap C^1(\mathbb{R}, H^{m-2}(\mathbb{R}))$ for any initial data $\phi \in H^m(\mathbb{R})$ when $m = 1$ and $m = 2$ (see [GiVe3] for $m = 1$ and [1] for $m = 2$). Moreover, we have energy and mass conservations i.e.,

$$
h(\phi_t) = h(\phi_0) \quad \text{and} \quad ||\phi_t||_{L^2(\mathbb{R})} = ||\phi_0||_{L^2(\mathbb{R})},
$$

for any initial data $\phi_0 \in H^1(\mathbb{R})$ and $\phi_t$ solution of (17). It is not difficult to prove the following estimates

$$
||\phi||^2_{L^2(\mathbb{R})} \leq 2||\phi||_{L^2(\mathbb{R})}||\partial_\varepsilon \phi||_{L^2(\mathbb{R})} \leq 2||\phi||_{L^2(\mathbb{R})} h(\phi)^{1/2},
$$

$$
||\phi||^p_{L^p(\mathbb{R})} \leq 2^{p-2}||\phi||_{L^2(\mathbb{R})}^{p+2}|||\partial_\varepsilon \phi||_{L^2(\mathbb{R})}^{p-2} \leq 2^{p-2}||\phi||_{L^2(\mathbb{R})}^{p+2} h(\phi)^{p-2},
$$

(18)

for $p \geq 2$ and any $\phi \in H^1(\mathbb{R})$. Furthermore, using Gronwall’s inequality we show for any $\phi_0 \in H^2(\mathbb{R})$ the existence of $c > 0$ depending only on $\phi_0$ such that

$$
||\phi_t||_{H^2(\mathbb{R})} \leq e^{c|t|} ||\phi_0||_{H^2(\mathbb{R})},
$$

(19)

where $\phi_t$ is a solution of the NLS equation (17) with initial condition $\phi_0$.

## 5 Time-dependent quadratic dynamics

In this section we construct a time-dependent quadratic approximation for the Schrödinger dynamics. We prove existence of a unique unitary propagator for this approximation using the abstract results for non-autonomous linear Schrödinger equation stated in the Appendix. This step will be useful for the study of propagation of coherent states in the semiclassical limit in section 6.

The polynomial $P$ has the following Taylor expansion for any $z_0 \in H^1(\mathbb{R})$

$$
P(z + z_0) = \sum_{j=0}^{4} \frac{D^{(j)} P}{j!}(z_0)[z].
$$
Let $\varphi$ be a solution of the NLS equation (17) with an initial data $\varphi_0 \in H^1(\mathbb{R})$. Consider the time-dependent quadratic polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$P_2(t)[z] := \frac{D(2)}{2}(\varphi_t)[z] = \text{Re} \int_{\mathbb{R}} \overline{z(x)^2} \varphi_t(x)^2 \, dx + 2 \int_{\mathbb{R}} |z(x)|^2 |\varphi_t(x)|^2 \, dx.$$ 

Let $\{A_2(t)\}_{t \in \mathbb{R}}$ be the $\varepsilon$-independent family of quadratic forms on $\mathcal{S}$ defined by

$$\varepsilon A_2(t) := d\Gamma(-\Delta) + P_2(t)^{\text{Wick}}.$$ 

Lemma 5.1 For $\varphi_0 \in H^1(\mathbb{R})$ let

$$\vartheta_1 := 16^2(\|\varphi_0\|_{L^2(\mathbb{R})} + 1)^3 (h(\varphi_0) + 1) \quad \text{and} \quad \vartheta_2 := 16^2(\|\varphi_0\|_{L^2(\mathbb{R})} + 1)^{3/2} \sqrt{h(\varphi_0) + 1}.$$ 

The quadratic forms on $\mathcal{S}$ defined by

$$S_2(t) := A_2(t) + \vartheta_1 e^{-1} N + \vartheta_2 1, \quad t \in \mathbb{R},$$

are associated to unique self-adjoint operators, still denoted by $S_2(t)$, satisfying

- $S_2(t) \geq 1$,
- $\mathcal{D}(S_2(t)^{1/2}) = \mathcal{F}_+ \text{ for any } t \in \mathbb{R}.$

Proof. The case $\varphi_0 = 0$ is trivial. By definition of Wick quantization we have for $\Psi, \Phi \in \mathcal{S},$

$$\langle \Phi, P_2(t)^{\text{Wick}} \Psi \rangle = 2 \sum_{n=1}^{\infty} \varepsilon n \int_{\mathbb{R}^n} |\varphi_t(x_1)|^2 |\Phi^{(n)}(x_1, \ldots, x_n)| |\Psi^{(n)}(x_1, \ldots, x_n)| \, dx_1 \cdots dx_n$$

$$+ \sum_{n=0}^{\infty} \varepsilon \sqrt{(n+1)(n+2)} \int_{\mathbb{R}^n} \overline{\Phi^{(n)}(x_1, \ldots, x_n)} \left( \int_{\mathbb{R}} \overline{\varphi_t(x)}^2 |\Psi^{(n+2)}(x, x_1, \ldots, x_n)| \, dx \right) \, dx_1 \cdots dx_n$$

$$+ \sum_{n=0}^{\infty} \varepsilon \sqrt{(n+1)(n+2)} \int_{\mathbb{R}^n} |\Psi^{(n)}(x_1, \ldots, x_n)| \left( \int_{\mathbb{R}} \overline{\varphi_t(x)}^2 |\Phi^{(n+2)}(x, x_1, \ldots, x_n)| \, dx \right) \, dx_1 \cdots dx_n. \quad (21)$$

Therefore, using Cauchy-Schwarz inequality, we show

$$|\langle \Phi, P_2(t)^{\text{Wick}} \Psi \rangle| \leq 2 \|\varphi_t\|^2_{L^2(\mathbb{R})} \|N^{1/2} \Phi\| \times \|N^{1/2} \Psi\|$$

$$+ \|\varphi_t\|^2_{L^2(\mathbb{R})} \|(N + \varepsilon)^{1/2} \Phi\| \times \left[ \sum_{n=0}^{\infty} \varepsilon (n+2) \|\Psi^{(n+2)}(x, x_1, \ldots, x_n)\|_{L^2(\mathbb{R}^{n+1})}^2 \right]^{1/2}$$

$$+ \|\varphi_t\|^2_{L^2(\mathbb{R})} \|(N + \varepsilon)^{1/2} \Psi\| \times \left[ \sum_{n=0}^{\infty} \varepsilon (n+2) \|\Phi^{(n+2)}(x, x_1, \ldots, x_n)\|_{L^2(\mathbb{R}^{n+1})}^2 \right]^{1/2}.$$ 

Now we prove, by Lemma[A.1] the crude estimate

$$|\langle \Phi, P_2(t)^{\text{Wick}} \Psi \rangle| \leq \max(\|\varphi_t\|^2_{L^2(\mathbb{R})}, \|\varphi_t\|^2_{L^2(\mathbb{R})}) \left[ 2 \|N^{1/2} \Phi\| \times \|N^{1/2} \Psi\|$$

$$+ \|(N + \varepsilon)^{1/2} \Phi\| \times \|d\Gamma(-\Delta) + \alpha^{-1} N\|^{1/2}$$

$$+ \|(N + \varepsilon)^{1/2} \Psi\| \times \|d\Gamma(-\Delta) + \alpha^{-1} N\|^{1/2} \right].$$

This yields for any $\alpha > 0$

$$|\langle \Phi, P_2(t)^{\text{Wick}} \Psi \rangle| \leq \alpha \max(\|\varphi_t\|^2_{L^2(\mathbb{R})}, \|\varphi_t\|^2_{L^2(\mathbb{R})})$$

$$\times \left[ \|d\Gamma(-\Delta) + (\alpha^{-1} + 3)\alpha^{-1} N + \alpha^{-1} e 1\|^{1/2} \right] \left[ \|d\Gamma(-\Delta) + (\alpha^{-1} + 3)\alpha^{-1} N + \alpha^{-1} e 1\|^{1/2} \right]. \quad (22)$$
Now, using a similar estimate as (22) we prove norm continuity. Indeed, we have an operator in 
$$L^2(\mathbb{R})$$ and the fact that $$\vartheta$$ are associated to unique self-adjoint operators $$S_2(t)$$ satisfying $$S_2(t) \geq 1$$. Furthermore, we have that the form domains of those operators are time-independent, i.e., 
$$\mathcal{D}(S_2(t)) = \mathcal{F}_+^1$$
for any $$t \in \mathbb{R}$$.

\[ \text{Remark 5.2} \quad \text{The choice of } \vartheta_1, \vartheta_2 \text{ in the previous lemma takes into account the use of KLMN's theorem in the proof of Lemma 6.3} \]

We consider the non-autonomous Schrödinger equation

$$i\partial_t u = A_2(t)u, \quad t \in \mathbb{R},$$

$$u(t = s) = u_s.$$  \hspace{1cm} (24)

Here $$\mathbb{R} \ni t \mapsto A_2(t)$$ is considered as a norm continuous $$\mathcal{L}(\mathcal{F}_+, \mathcal{F}_1)$$-valued map (see Lemma 5.3). We show in Proposition 5.5 the existence of a unique solution for any initial data $$u_s \in \mathcal{F}_+^1$$ using Corollary C.4. Moreover, the Cauchy problem’s features allow to encode the solutions on a unitary propagator mapping $$(t, s) \mapsto U_2(t, s)$$ such that

$$U_2(t, s)u_s = u_t,$$

satisfying Definition C.1 with $$\mathcal{H} = \mathcal{F}, \mathcal{H}_\pm = \mathcal{F}_+^1$$ and $$I = \mathbb{R}$$.

In the following two lemmas we check the assumptions in Corollary C.4.

**Lemma 5.3** For any $$\varphi_0 \in H^1(\mathbb{R})$$ and $$t \in \mathbb{R}$$ the quadratic form $$A_2(t)$$ defines a symmetric operator on $$\mathcal{L}(\mathcal{F}_+, \mathcal{F}_1)$$ and the mapping $$t \in \mathbb{R} \mapsto A_2(t) \in \mathcal{L}(\mathcal{F}_+, \mathcal{F}_1)$$ is norm continuous.

\[ \text{Proof. Using (22) we show for any } \Psi, \Phi \in \mathcal{F}_+^1 \]

$$\| \langle \Phi, A_2(t)\Psi \rangle \| \leq \| \langle \Phi, e^{-1}d\Gamma(-\Delta)\Psi \rangle \| + \| \langle \Phi, e^{-1}P_2(t)\text{Wick}\Psi \rangle \|$$

$$\leq \| S_1^{1/2}\Phi \| \| S_1^{1/2}\Psi \| + \frac{\kappa}{4} \vartheta_1 \| S_1^{1/2}\Phi \| \| S_1^{1/2}\Psi \|$$

$$\leq \frac{\kappa}{4} \vartheta_1 \| \Psi \|_{\mathcal{F}_+^1} \| \Phi \|_{\mathcal{F}_+^1},$$

where $$\vartheta_1, \vartheta_2$$ are the parameters introduced in Lemma 5.1. Hence, this allows to consider $$A_2(t)$$ as a bounded operator in $$\mathcal{L}(\mathcal{F}_+, \mathcal{F}_1)$$. Since $$A_2(t)$$ is a symmetric quadratic form it follows that it also symmetric as an operator in $$\mathcal{L}(\mathcal{F}_+, \mathcal{F}_1)$$.

Now, using a similar estimate as (22) we prove norm continuity. Indeed, we have

$$\| \langle \Phi, [A_2(t) - A_2(s)]\Psi \rangle \| = e^{-1}\| \langle \Phi, [P_2(t) - P_2(s)]\text{Wick}\Psi \rangle \|$$

$$\leq 4 \max \left( \| \varphi_t^2 - \varphi_s^2 \|_{L^2(\mathbb{R})}, \| \varphi_t^2 - \varphi_s^2 \|_{L^\infty(\mathbb{R})} \right) \| \Psi \|_{\mathcal{F}_+^1} \| \Phi \|_{\mathcal{F}_+^1}.$$

Note that it is not difficult to prove that

$$\max \left( \| \varphi_t^2 - \varphi_s^2 \|_{L^2(\mathbb{R})}, \| \varphi_t^2 - \varphi_s^2 \|_{L^\infty(\mathbb{R})} \right) \to 0 \quad \text{when } t \to s.$$  \hspace{1cm} (25)

This follows by (18) and the fact that $$\varphi_t \in C^0(\mathbb{R}, H^1(\mathbb{R}))$$.  \hspace{1cm} ■
Lemma 5.4 For any \( \varphi_0 \in H^2(\mathbb{R}) \) there exists \( c > 0 \) (depending only on \( \varphi_0 \)) such that the two statements below hold true.
(i) For any \( \Psi \in \mathcal{F}_+^1 \), we have
\[
|\partial_t \langle \Psi, S_2(t)\Psi \rangle| \leq e^{c(|t|+1)} \|S_2(t)^{1/2}\Psi\|_\mathcal{F}.
\]
(ii) For any \( \Psi, \Phi \in \mathcal{D}(S_2(t)^{3/2}) \), we have
\[
|\langle \Psi, A_2(t)S_2(t)\Phi \rangle - \langle S_2(t)\Psi, A_2(t)\Phi \rangle| \leq c \|S_2(t)^{1/2}\Psi\|_\mathcal{F} \|S_2(t)^{1/2}\Phi\|_\mathcal{F}.
\]
Proof. (i) Let \( \Psi \in \mathcal{F} \), we have
\[
\partial_t \langle \Psi, S_2(t)\Psi \rangle = \varepsilon^{-1} \partial_t \langle \Psi, P_2(t)^{\text{Wick}}\Psi \rangle = \varepsilon^{-1} \langle \Psi, [\partial_t P_2(t)]^{\text{Wick}}\Psi \rangle,
\]
where \( \partial_t P_2(t) \) is a continuous polynomial on \( \mathcal{F}(\mathbb{R}) \) given by
\[
\partial_t P_2(t)[\varepsilon] = 2\text{Re} \int_{\mathbb{R}} \frac{2}{x^2} \varphi_2(x) \partial_t \varphi_2(x) dx + 4\text{Re} \int_{\mathbb{R}} \frac{1}{x^2} \varphi_2(x) \partial_t \varphi_2(x) dx.
\]
A simple computation yields
\[
\langle \Psi, [\partial_t P_2(t)]^{\text{Wick}}\Psi \rangle = 4\text{Re} \sum_{n=1}^{\infty} \varepsilon \sqrt{(n+2)(n+1)} \int_{\mathbb{R}} \Psi^{(n)}(x_1, \ldots, x_n) \int_{\mathbb{R}} \varphi_2(x) \partial_t \varphi_2(x) \Psi^{(n+2)}(x, x_1, \ldots, x_n) dx dx_1 \cdots dx_n + hc.
\]
From (18) we get
\[
|\langle 1 \rangle | \leq \|\varphi_2 \|_{L^1(\mathbb{R})} \sum_{n=1}^{\infty} c \varepsilon \sqrt{(n+2)(n+1)} \left( \sup_{x_1 \in \mathbb{R}^n} \left| \Psi^{(n)}(x_1, \ldots, x_n) \right| \right)^2 dx_2 \cdots dx_n
\]
\[
\leq \|\varphi_2 \|_{L^2(\mathbb{R})} \times \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \left( (1 - \partial^2_{x_1}) \Psi^{(n+1)}(x, x_1, \ldots, x_n) \right) dx_1 \cdots dx_n.
\]
Now we apply Cauchy-Schwarz inequality,
\[
|\langle \Psi, [\partial_t P_2(t)]^{\text{Wick}}\Psi \rangle | \leq 4 \|\varphi_2 \|_{L^2(\mathbb{R})} \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \left( \sum_{n=1}^{\infty} c \varepsilon \sqrt{(n+2)(n+1)} \left[ (1 - \partial^2_{x_1}) \Psi^{(n+1)}(x, x_1, \ldots, x_n) \right] dx_1 \cdots dx_n \right)^{1/2} \times \left( \sum_{n=0}^{\infty} c \varepsilon (n+1) \|\Psi^{(n)}\|_{L^2(\mathbb{R})}^2 \right)^{1/2}.
\]
In the same spirit as in (22), we obtain a rough inequality
\[
|\langle \Psi, [\partial_t P_2(t)]^{\text{Wick}}\Psi \rangle | \leq \max \left\{ \|\varphi_2 \|_{L^2(\mathbb{R})}, \|\varphi_2 \|_{L^2(\mathbb{R})} \right\} \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \|\Psi\|_{L^2(\mathbb{R})} \left[ 4 \|(d\Gamma(-\Delta) + N)^{1/2}\Psi\|^2 + 2 \|(d\Gamma(-\Delta) + N + 1)^{1/2}\Psi\|^2 \right] .
\]
Observe that (23) implies \( S_1 \leq 3 S_2(t) \) for all \( t \in \mathbb{R} \). Hence, we have
\[
\varepsilon^{-1} |\langle \Psi, [\partial_t P_2(t)]^{\text{Wick}}\Psi \rangle | \leq 6 \max \left\{ \|\varphi_2 \|_{L^2(\mathbb{R})}, \|\varphi_2 \|_{L^2(\mathbb{R})} \right\} \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \|\Psi\|_{L^2(\mathbb{R})}^2
\]
\[
\leq 18 \max \left\{ \|\varphi_2 \|_{L^2(\mathbb{R})}, \|\varphi_2 \|_{L^2(\mathbb{R})} \right\} \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \|S_2(t)^{1/2}\Psi\|_{\mathcal{F}}^2.
\]
This proves (i) since (18)-(19) ensure the existence of \( c > 0 \) (depending only on \( \varphi_0 \)) such that
\[
\max \left\{ \|\varphi_2 \|_{L^2(\mathbb{R})}, \|\varphi_2 \|_{L^2(\mathbb{R})} \right\} \|\partial_t \varphi_2 \|_{L^2(\mathbb{R})} \leq e^{c(|t|+1)}.
\]
(ii) If $\Psi, \Phi \in \mathcal{D}(S_2(t)^{3/2})$ the quantity

$$
C := \langle \Psi, A_2(t)S_2(t)\Phi \rangle - \langle S_2(t)\Psi, A_2(t)\Phi \rangle,
$$
is well-defined since $A_2(t) \in \mathcal{L}(\mathcal{F}_+, \mathcal{F}_1^1)$ and $S_2(t) \mathcal{D}(S_2(t)^{3/2}) \subset \mathcal{D}(S_2(t)^{1/2}) = \mathcal{F}_1^1$. Note that $N \in \mathcal{L}(\mathcal{F}_+, \mathcal{F}_1^1)$. Hence, we can write

$$
C = \langle \Psi, [S_2(t) - \vartheta_1 \varepsilon^{-1}N - \vartheta_2 1]S_2(t)\Phi \rangle - \langle S_2(t)\Psi, [S_2(t) - \vartheta_1 \varepsilon^{-1}N - \vartheta_2 1]\Phi \rangle
$$

$$
= \vartheta_1 \left( \langle S_2(t)\Psi, \varepsilon^{-1}N\Phi \rangle - \langle \varepsilon^{-1}N\Psi, S_2(t)\Phi \rangle \right).
$$

Observe that, for $\lambda > 0$, $\varepsilon^{-1}N(\lambda \varepsilon^{-1}N + 1)^{-1} \mathcal{F}_1^1 \subset \mathcal{F}_+$ and that

$$
s - \lim_{\lambda \to 0^+} \varepsilon^{-1}N(\lambda \varepsilon^{-1}N + 1)^{-1} = \varepsilon^{-1}N \text{ in } \mathcal{L}(\mathcal{F}_+, \mathcal{F}_1^1).
$$

Therefore, we have

$$
C = \vartheta_1 \lim_{\lambda \to 0^+} \langle S_2(t)\Psi, \varepsilon^{-1}N(\lambda \varepsilon^{-1}N + 1)^{-1}\Phi \rangle - \langle \varepsilon^{-1}N(\lambda \varepsilon^{-1}N + 1)^{-1}\Psi, S_2(t)\Phi \rangle.
$$

Let $N_\lambda$ denote $\varepsilon^{-1}N(\lambda \varepsilon^{-1}N + 1)^{-1}$. A simple computation yields

$$
\mathcal{E}_\lambda = \langle \Psi, P_2(t)^{\text{Wick}}N_\lambda \Phi \rangle - \langle N_\lambda \Psi, P_2(t)^{\text{Wick}}\Phi \rangle
$$

$$
= \langle \Psi, g(t)^{\text{Wick}}N_\lambda \Phi \rangle - \langle N_\lambda \Psi, g(t)^{\text{Wick}}\Phi \rangle,
$$

where $g(t)$ is the polynomial given by

$$
g(t)[z] = \text{Re} \int_{\mathbb{R}} \overline{z} \phi_0(x)^2 dx.
$$

A similar computation as (21) yields

$$
\mathcal{E}_\lambda = \sum_{n=0}^{\infty} \kappa(n) \int_{\mathbb{R}^n} \Psi^{(n)}(x_1, \ldots, x_n) \left( \int_{\mathbb{R}} \phi_0(x)^2 \Phi^{(n+2)}(x, x, x_1, \ldots, x_n) dx \right) dx_1 \cdots dx_n
$$

$$
- \sum_{n=0}^{\infty} \kappa(n) \int_{\mathbb{R}^n} \Phi^{(n)}(x_1, \ldots, x_n) \left( \int_{\mathbb{R}} \phi_0(x)^2 \Psi^{(n+2)}(x, x, x_1, \ldots, x_n) dx \right) dx_1 \cdots dx_n,
$$

where

$$
\kappa(n) = \frac{(n+2) \sqrt{(n+1)(n+2)}}{\lambda(n+2) + 1} - n \sqrt{(n+1)(n+2)} \frac{\lambda(n+1)}{\lambda n + 1}.
$$

Note that $\kappa(n) \leq 2(n+2)$. Hence, using Cauchy-Schwarz inequality, we show

$$
|\mathcal{E}_\lambda| \leq 2 \|\Psi\|_{L^2(R)}^2 \left[ \sum_{n=0}^{\infty} (n+2) \|\Psi^{(n)}\|_{L^2(R^n)}^2 \right]^{1/2} \left[ \sum_{n=0}^{\infty} (n+2) \|\Phi^{(n+2)}(x, x, x_1, \ldots, x_n)\|_{L^2(R^{n+1})}^2 \right]^{1/2}
$$

$$
+ 2 \|\Psi\|_{L^2(R)}^2 \left[ \sum_{n=0}^{\infty} (n+2) \|\Phi^{(n)}\|_{L^2(R^n)}^2 \right]^{1/2} \left[ \sum_{n=0}^{\infty} (n+2) \|\Psi^{(n+2)}(x, x, x_1, \ldots, x_n)\|_{L^2(R^{n+1})}^2 \right]^{1/2}.
$$

Using Lemma A.1 with $\alpha = \frac{1}{\sqrt{2}}$, we get

$$
\sum_{n=0}^{\infty} (n+2) \|\Psi^{(n+2)}(x, x, x_1, \ldots, x_n)\|_{L^2(R^{n+2})}^2 \leq \frac{1}{2} \sum_{n=0}^{\infty} (n+2) \langle D_{x_1}^2 \Psi^{(n+2)}, \Psi^{(n+2)} \rangle + (n+2) \|\Psi^{(n+2)}\|_{L^2(R^{n+2})}^2
$$

$$
\leq \frac{1}{2} \langle \Psi, S_1 \Psi \rangle,
$$

together with an analogue estimate where $\Psi$ is replaced by $\Phi$. Now, we conclude that there exists $c > 0$ depending only on $\phi_0$ such that

$$
\vartheta_1 |\mathcal{E}_\lambda| \leq c \|\Psi\|_{\mathcal{F}_+^1} \|\Phi\|_{\mathcal{F}_+^1}.
$$

This proves part (ii).
Proposition 5.5 Let \( \varphi_0 \in H^2(\mathbb{R}) \) and \( A_2(t) \) given by (20). Then the non-autonomous Cauchy problem

\[
\begin{cases}
i \partial_t u = A_2(t) u, & t \in \mathbb{R}, \\
u(t = s) = u_s,
\end{cases}
\]

admits a unique unitary propagator \( U_2(t, s) \) in the sense of Definition C.1 with \( I = \mathbb{R} \) and \( \mathcal{H}_\pm = \mathcal{F}_{\pm}^1 \). Moreover, there exists \( c > 0 \) depending only on \( \varphi_0 \) such that

\[
\|U_2(t, 0)\|_{\mathcal{F}(\mathcal{F}_1^\pm)} \leq e^{c|t|}.
\]

Proof. The proof immediately follows using Corollary C.4 with the help of Lemma 5.3-5.4 and the inequality

\[
c_1 S_1 \leq S_2(t) \leq c_2 S_1,
\]

which holds true using (25). 

\[\blacksquare\]

6 Propagation of coherent states

In finite dimensional phase-space, coherent state analysis is a well developed powerful tool, see for instance [CRR]. Here we study, using the ideas of Ginibre and Velo in [GiVe2], the asymptotics when \( \varepsilon \to 0 \) of the time-evolved coherent states

\[
e^{-it/\varepsilon} W\big(\frac{\sqrt{2}}{i\varepsilon} \varphi_0\big) \Psi,
\]

for \( \Psi \) in a dense subspace \( \mathcal{G}_+ \subset \mathcal{F} \) defined below. We consider the following Hilbert rigging

\[
\mathcal{G}_+ \subset \mathcal{F} \subset \mathcal{G}_-,
\]

defined via the \( \varepsilon \)-independent self-adjoint operator (see Remark 3.4) given by

\[
G := \varepsilon^{-1} d\Gamma(-\Delta) + \varepsilon^{-2} P_{\text{Wick}} + \varepsilon^{-1} N + 1,
\]

as the completion of \( \mathcal{D}(G^{1/2}) \) with the respect to the inner product

\[
\langle \Psi, \Phi \rangle_{\mathcal{G}_\pm} := \langle G^{1/2} \Psi, G^{1/2} \Phi \rangle_{\mathcal{F}}.
\]

We have the continuous embedding

\[
\mathcal{F}_3^+ \subset \mathcal{G}_+ \subset \mathcal{F}_1^+.
\]

The main result of this section is the following proposition which describes the propagation of coherent states in the semiclassical limit.

Proposition 6.1 For any \( \varphi_0 \in H^2(\mathbb{R}) \) there exists \( c > 0 \) depending only on \( \varphi_0 \) such that

\[
\left\| e^{-it/\varepsilon} W\big(\frac{\sqrt{2}}{i\varepsilon} \varphi_0\big) \Psi - e^{i\omega(t)/\varepsilon} W\big(\frac{\sqrt{2}}{i\varepsilon} \varphi_t\big) U_2(t, 0) \Psi \right\|_{\mathcal{F}} \leq e^{c|t|} \varepsilon^{1/8} \|\Psi\|_{\mathcal{G}_+},
\]

holds for any \( t \in \mathbb{R} \) and \( \Psi \in \mathcal{G}_+ \) where \( \varphi_t \) solves the NLS equation (17) with the initial condition \( \varphi_0 \) and \( \omega(t) = \int_0^t P(\varphi_s) \, ds \). Here \( U_2(t, s) \) is the unitary propagator given by Proposition 5.5.

To prove this proposition we need several preliminary lemmas.

Lemma 6.2 The following three assertions hold true.

(i) For any \( \xi \in L^2(\mathbb{R}) \) and \( k \in \mathbb{N} \), the Weyl operator \( W(\xi) \) preserves \( \mathcal{D}(N^{k/2}) \). If in addition \( \xi \in H^1(\mathbb{R}) \) then \( W(\xi) \) preserves also \( \mathcal{F}_{\mu}^\pm \) when \( \mu \geq 1 \).
(ii) For any \( \xi \in H^1(\mathbb{R}) \), we have in the sense of quadratic forms on \( \mathcal{F}_+^2 \),
\[
W(\frac{\sqrt{2}}{i\varepsilon} \xi)^* h^{\text{Wick}} W(\frac{\sqrt{2}}{i\varepsilon} \xi) = h(\cdot + \xi)^{\text{Wick}}.
\]

(iii) Let \((\mathbb{R} \ni t \mapsto \phi_t) \in C^1(\mathbb{R}, L^2(\mathbb{R}))\), then for any \( \Psi \in \mathcal{D}(N^{1/2}) \) we have in \( \mathcal{F} \)
\[
i \varepsilon \partial_t W(\frac{\sqrt{2}}{i\varepsilon} \phi_t)\Psi = W(\frac{\sqrt{2}}{i\varepsilon} \phi_t) \left[ \text{Re} \langle \phi_t, i\partial_t \phi_t \rangle + 2\text{Re} \langle z, i\partial_t \phi_t \rangle^{\text{Wick}} \right] \Psi = \left[ -\text{Re} \langle \phi_t, i\partial_t \phi_t \rangle + 2\text{Re} \langle z, i\partial_t \phi_t \rangle^{\text{Wick}} \right] W(\frac{\sqrt{2}}{i\varepsilon} \phi_t)\Psi.
\]

**Proof.** (i) Let \( \mathcal{F}_0 \) be the linear space spanned by vectors \( \Psi \in \mathcal{F} \) such that \( \Psi^{(n)} = 0 \) for any \( n \) except for a finite number. It is known that for any \( \xi \in L^2(\mathbb{R}) \) and \( \Psi \in \mathcal{F}_0 \)
\[
\hat{N}\Psi := W(\frac{\sqrt{2}}{i\varepsilon} \xi)^* N W(\frac{\sqrt{2}}{i\varepsilon} \xi)\Psi = \left( N + 2\text{Re} \langle z, \xi \rangle^{\text{Wick}} + ||\xi||^2 \right)\Psi.
\]

For a proof of the latter identity see [AmNi1, Lemma 2.10 (iii)]. Hence, by Cauchy-Schwarz inequality it follows that
\[
||N^{1/2} W(\frac{\sqrt{2}}{i\varepsilon} \xi)\Psi||^2 = \langle \Psi, \left[ N + 2\text{Re} \langle z, \xi \rangle^{\text{Wick}} + ||\xi||^2 \right] \Psi \rangle = \langle \Psi, (N + ||\xi||^2)^{\mathcal{F}_0(1)} \Psi \rangle
+ \sum_{n=0}^{\infty} \sqrt{\varepsilon(n+1)} \int_{\mathbb{R}^n} \Psi(n)(y) \left( \int_{\mathbb{R}} \xi(x) \Psi^{(n+1)}(x, y) dx \right) dy + hc
\leq (1 + ||\xi||_{L^2(\mathbb{R})}^2) ||N + 1||_{1/2}\Psi||^2.
\]

Now, for \( k \geq 1 \) we show the existence of an \( \varepsilon \)-independent constant \( C_k > 0 \) depending only on \( k \) and \( ||\xi||_{L^2(\mathbb{R})} \) such that
\[
||N^{k/2} W(\frac{\sqrt{2}}{i\varepsilon} \xi)\Psi||^2 = \langle \Psi, \hat{N}^k \Psi \rangle \leq C_k ||(N + 1)^{k/2}\Psi||^2.
\]

This is a consequence of the number operator estimate (10) and the fact that \( \hat{N}^k \) is a Wick polynomial in \( \sum_{0 \leq r, s \leq k} \mathcal{P}_{rs}(L^2(\mathbb{R})) \) (see, e.g., [AmNi1, Prop. 2.7 (i)]). Thus, we have proved the invariance of \( \mathcal{D}(N^{1/2}) \) since \( \mathcal{F}_0 \) is a core of \( N^{1/2} \).

Now the invariance of \( \mathcal{F}_+^\mu, \mu \geq 1 \), follows by Faris-Lavine Theorem [B.1] where we take the operator
\[
A = 2\text{Re} \langle z, \xi \rangle^{\text{Wick}} \quad \text{and} \quad S = S_\mu = \varepsilon^{-1} d\Gamma(-\Delta) + \varepsilon^{-\mu} N^\mu + 1,
\]
and remember that
\[
W(\xi) = e^{i\sqrt{2\text{Re} \langle z, \xi \rangle^{\text{Wick}}}}.
\]

In fact, assuming \( \xi \in H^1(\mathbb{R}) \) we have to check assumptions (i)-(ii) of Theorem [B.1] For any \( \Psi \in \mathcal{F}_+^\mu \), we have by Wick quantization
\[
2\text{Re} \langle z, \xi \rangle^{\text{Wick}}\Psi = \sum_{n=0}^{\infty} \sqrt{\varepsilon(n+1)} \int_{\mathbb{R}^n} \xi(x) \Psi(n+1)(x, x_1, \ldots, x_n) dx
+ \sum_{n=1}^{\infty} \frac{\sqrt{\varepsilon}}{n} \sum_{j=1}^{n} \xi(x_j) \Psi(n-1)(x_1, \ldots, x_j, \ldots, x_n).
\]

Therefore, it is easy to show
\[
||2\text{Re} \langle z, \xi \rangle^{\text{Wick}}\Psi|| \leq \sqrt{\varepsilon} ||\xi||_{L^2(\mathbb{R})} ||(\varepsilon^{-1} N + 1)^{1/2}\Psi||
\leq \sqrt{\varepsilon} ||\xi||_{L^2(\mathbb{R})} ||S_1\Psi||
\]

\[\]
and hence we obtain that $\mathcal{D}(S_\mu) \subset \mathcal{D}(A)$. Let $\Psi \in \mathcal{D}(S_\mu)$, a standard computation yields

$$\sqrt{2} \left( \langle a(-\Delta)^{\frac{1}{2}} \Psi, \Psi \rangle - \langle S_\mu \Psi, a \Psi \rangle \right) = \langle a(-\Delta^2) \Psi, \Psi \rangle - \langle \Psi, a(-\Delta^2) \Psi \rangle + \langle (\frac{1}{2} - \frac{1}{2})^\mu - (\frac{1}{2})^\mu \lambda \Psi, a^*(\xi) \Psi \rangle - hc. \quad (29)$$

Each two terms in the same line of (29) are similar and it is enough to estimate only one of them. We have by Cauchy-Schwarz inequality

$$|\langle a(-\Delta^2) \Psi, \Psi \rangle| \leq \left\| \sum_{n=0}^\infty \sqrt{\epsilon(n+1)} \int_{\mathbb{R}^n} \Psi^{(n)}(y) \left( \int_{\mathbb{R}} (-\Delta^2)^{\frac{1}{2}}(x) \Psi^{(n+1)}(x,y) dx \right) dy \right\| \leq \left\| \xi \right\|_{L^2(\mathbb{R})} \left\| S_\mu^{1/2} \Psi \right\|^2,$$

and for $1 \leq \theta \leq \mu - 1$

$$|\langle e^{-\theta} N^\theta \Psi, a^*(\xi) \Psi \rangle| \leq \left\| \sum_{n=0}^\infty \sqrt{\epsilon(n+1)(n+1)^\theta} \int_{\mathbb{R}^n} \Psi^{(n)}(y) \left( \int_{\mathbb{R}} \xi(x) \Psi^{(n+1)}(x,y) dx \right) dy \right\| \leq 2^{\mu} \left\| \xi \right\|_{L^2(\mathbb{R})} \left\| S_\mu^{1/2} \Psi \right\|^2.$$

This shows for any $\Psi \in \mathcal{D}(S_\mu)$,

$$\pm i \langle \Psi, [A, S_\mu] \Psi \rangle \leq C \left\| S_\mu^{1/2} \Psi \right\|^2.$$

Part (ii) follows by a similar argument as [AmNi1, Lemma 2.10(iii)] and part (iii) is a well-known formula, see [GiVe1] Lemma 3.1 (3)].

Set

$$\mathcal{H}(t) = W\left( \frac{\sqrt{c}}{i \epsilon} \phi_0 \right)^* e^{-i\omega(t) / \epsilon} e^{-it / \epsilon} W\left( \frac{\sqrt{c}}{i \epsilon} \phi_0 \right).$$

**Lemma 6.3** For any $\phi_0 \in H^2(\mathbb{R})$ there exists $c > 0$ such that the inequality

$$\left\| \mathcal{H}(t) \right\|_{\mathcal{D}(S_\mu, \mathbb{F}^3_1)} \leq e^{ce^{|t|}}$$

holds for $t \in \mathbb{R}$ uniformly in $\epsilon \in (0, 1]$.

**Proof.** Observe that the subspace $\mathcal{D}_+$ given as the image of $\mathcal{D}(H_\mu) \cap \mathcal{F}_+^3$ by $W(\sqrt{c} / i \epsilon \phi_0)^*$ is dense in $\mathcal{F}_+$. Let $\Psi \in \mathcal{D}_+$ and $\Phi \in \mathcal{D}_+$, then differentiating the quantity $\langle \Phi, \mathcal{H}(t) \Psi \rangle$ with the help of Lemma [6.2] and Proposition [3.3], we obtain

$$\begin{align*}
\imath \epsilon \partial_t \langle \Phi, \mathcal{H}(t) \Psi \rangle &= \langle \Phi, [P(\phi_0) - \Re(\phi_0), i \partial_t \phi_0) - 2 \Re(\zeta, i \partial_t \phi_0)^{\text{Wick}} \mathcal{H}(t) \Psi \rangle \\
&+ \langle \Phi, W(\sqrt{c} / i \epsilon \phi_0)^* e^{-i \omega(t) / \epsilon} H_\infty \left( \frac{\sqrt{c}}{i \epsilon} \phi_0 \right)^T \Psi \rangle. \quad (30)
\end{align*}$$

Let $R_v := 1_{\{0, v\}}(e^{-1} N)$ and remark that $s - \lim_{v \to \infty} R_v = 1$. Furthermore, we have that $R_v \mathcal{D}_+ \subset \mathcal{F}_+^3$ since it easily holds that

$$\left\| R_v \Phi \right\|_{\mathcal{F}_+^3} \leq v^3 \left\| \Phi \right\|_{\mathcal{D}_+^3}.$$

Therefore, since $W(\sqrt{c} / i \epsilon \phi_0) R_v \Phi$ and $W(\sqrt{c} / i \epsilon \phi_0) \Psi$ belong to $\mathcal{F}_+^3$, we have

$$\begin{align*}
(1) &= \lim_{v \to \infty} \langle R_v \Phi, W(\sqrt{c} / i \epsilon \phi_0)^* e^{-i \omega(t) / \epsilon} H_\infty \left( \frac{\sqrt{c}}{i \epsilon} \phi_0 \right)^T \Psi \rangle \\
&= \lim_{v \to \infty} \langle R_v \Phi, h(\zeta) \Psi \rangle^{\text{Wick}} \mathcal{H}(t) \Psi \rangle.
\end{align*}$$
So, we get
\[
\imath \varepsilon \partial_t \langle \Phi, \mathcal{W}(t) \Psi \rangle = (1 + \lim_{V \to \infty} (R_\varepsilon \Phi \left[ P(\phi) - \Re \langle \phi, i \partial_x \phi \rangle - 2 \Re \langle z, i \partial_x \phi \rangle \right] \mathcal{W}(t) \Psi) = \lim_{V \to \infty} (R_\varepsilon \Phi \left( \varepsilon A_2(t) + P(t) \right) + P(t) \mathcal{W}(t) \Psi),
\]
where we denote
\[
P(t)[z] := \frac{D(3)P}{3!}(\phi)[z] = 2 \Re \int_R \phi \langle x, \bar{z} \rangle |z(x)|^2 \, dx \quad \text{and} \quad P(z) = \frac{D(4)P}{4!}(\phi)[z] = \frac{1}{2} \int_R |z(x)|^4 \, dx.
\]
A simple computation yields
\[
\langle \Phi, P(t) \mathcal{W} \Psi \rangle = \sum_{n=1}^\infty \sqrt{n^2(n+1)} \varepsilon^3 \int_{\mathbb{R}^{n+1}} \left( \int_R \phi \langle x, \bar{z} \rangle \Phi(n)[x,y] \Psi(n+1)[x,y] \, dx \right) \, dy
\]
\[
+ \sum_{n=1}^\infty \sqrt{n^2(n+1)} \varepsilon^3 \int_{\mathbb{R}^{n+1}} \left( \int_R \phi \langle x, \bar{z} \rangle \Phi(n+1)[x,y] \Psi(n)[x,y] \, dx \right) \, dy.
\]
Using Cauchy-Schwarz inequality and Lemma 5.1, we obtain
\[
\left| \langle \Phi, P(t) \mathcal{W} \Psi \rangle \right| \leq 2 \sqrt{2} \frac{|\phi|_{L^2(\mathbb{R})}}{\sqrt{2}} \sqrt{\langle \Phi, |\varepsilon^{-1} P \mathcal{W} + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 | \Phi \rangle}
\times \sqrt{\langle \Psi, |\varepsilon^{-1} P \mathcal{W} + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 | \Psi \rangle},
\]
where \( \varepsilon_1, \varepsilon_2 \) are the parameters in Lemma 5.1. Hence, \( \Theta(t) \) extends to a bounded operator in \( \mathcal{L}(\mathcal{G}_+; \mathcal{G}_-) \). As an immediate consequence we obtain
\[
\imath \varepsilon \partial_t \langle \Phi, \mathcal{W}(t) \Psi \rangle = \langle \Phi, \varepsilon \Theta(t) \mathcal{W}(t) \Psi \rangle.
\]
Now, we consider the quadratic form \( \Lambda(t) \) on \( \mathcal{G}_+ \) given by
\[
\Lambda(t) := \Theta(t) + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 1.
\]
It is easily follows, by (10) and (29), that
\[
\left| \langle \Phi, P(t) \mathcal{W} \Psi \rangle \right| \leq \frac{1}{2} \left| \left( \varepsilon^{-1} d \Gamma(-\Delta) + \varepsilon^{-1} P \mathcal{W} + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 1 \right)^{1/2} \Phi \right|
\times \left| \left( \varepsilon^{-1} d \Gamma(-\Delta) + \varepsilon^{-1} P \mathcal{W} + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 1 \right)^{1/2} \Psi \right|.
\]
Therefore, using (23) and (33) we show that
\[
\varepsilon^{-1} \left[ \frac{D(2)P}{2}(\phi)[z] + \frac{D(3)P}{3!}(\phi)[z] \right] ^{\mathcal{W}}
\]
is form bounded by \( \varepsilon^{-1} d \Gamma(-\Delta) + \varepsilon^{-1} P \mathcal{W} + \varepsilon_1 \varepsilon^{-1} N + \varepsilon_2 1 \) with a form-bound less than 1 uniformly in \( \varepsilon \in (0,1] \). Hence, by the KLMN Theorem [RS, Thm. X17], the quadratic form \( \Lambda(t) \) is associated to a unique self-adjoint operator which we still denote by \( \Lambda(t) \), satisfying \( \mathcal{D}(\Lambda(t)) = \mathcal{G}_+ \) and \( \Lambda(t) \geq 1 \). Moreover, it is not difficult to show the existence of \( c_1, c_2 > 0 \) such that
\[
c_1 S_1 \leq \Lambda(t) \leq c_2 G
\]
uniformly in \( \varepsilon \in (0,1] \) for any \( t \in \mathbb{R} \). Now, we consider the non-autonomous Schrödinger equation
\[
i \varepsilon \partial_t u = \Theta(t) u,
\]
with initial data \( u_0 \in \mathcal{G}_+ \). Next, we prove existence and uniqueness of a unitary propagator \( \mathcal{V}(t,s) \) of the Cauchy problem (35). This will be done if we can check assumptions of Corollary C.4 with \( \mathcal{G}_+ = \mathcal{H}_+, \Lambda(t) = \Theta(t) \) and \( S(t) = \Lambda(t) \). Thus, we will conclude that
\[
||\Lambda(t)^{1/2} \mathcal{V}(t,0) \Psi||_{\mathcal{F}} \leq e^{c_2 t} ||\Lambda(0)^{1/2} \Psi||_{\mathcal{F}}.
\]
Observe that $\mathbb{R} \ni t \mapsto \Theta(t) \in \mathcal{L}(\mathcal{F}_+, \mathcal{F}_-)$ is norm continuous since
\[
|\langle \Phi, (\Theta(t) - \Theta(s)) \Psi \rangle| \leq \|\Phi\|_{\mathcal{F}_+} |A_2(t) - A_2(s)| \|\Psi\|_{\mathcal{F}_-} + |\langle \Phi, e^{-1}(P_3(t) - P_3(s))^{\text{Wick}} \Psi \rangle|,
\]
and an estimate similar to (31) yields
\[
|\langle \Phi, e^{-1}(P_3(t) - P_3(s))^{\text{Wick}} \Psi \rangle| \leq 2\sqrt{2}\|\Phi - \Phi_s\|_{L^2(\mathbb{R})} \|\Phi\|_{\mathcal{F}_+} \|\Psi\|_{\mathcal{F}_-}.
\]
Let us check assumption (i) of Corollary C.4. We have for $\Psi \in \mathcal{F}_+ \subset \mathcal{F}_+$,
\[
\partial_t \langle \Psi, \Lambda(t) \Psi \rangle = \partial_t \langle \Psi, S_2(t) \Psi \rangle + \partial_t \langle \Psi, e^{-1}P_3(t)^{\text{Wick}} \Psi \rangle.
\]
A simple computation yields
\[
\partial_t \langle \Psi, e^{-1}P_3(t)^{\text{Wick}} \Psi \rangle = 2\text{Re} \left[ \sum_{n=1}^{\infty} \sqrt{n^2(n+1)} \psi \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}} \partial_t \psi(x) \psi^{(n)}(x,y) \psi^{(n+1)}(x,y) dx \right) dy \right].
\]
So, by Cauchy-Schwarz inequality and Lemma A.1, we get
\[
\left| \partial_t \langle \Psi, e^{-1}P_3(t)^{\text{Wick}} \Psi \rangle \right| \leq 2\|\partial_t \psi\|_{L^2(\mathbb{R})} \left[ \sum_{n=1}^{\infty} \frac{(n+1)}{n^2} \|\psi^{(n)}(x,y)\|_{L^2(\mathbb{R}^{n+1})} \right]^{1/2} \leq 2\sqrt{2}\|\partial_t \psi\|_{L^2(\mathbb{R})} \|\Lambda(t)^{1/2} \Psi\|^2.
\]
The latter estimate with Lemma 5.4(i) and (18)-(19) give us
\[
|\partial_t \langle \Psi, \Lambda(t) \Psi \rangle| \leq e^{\rho(t+1)}\|\Lambda(t)^{1/2} \Psi\|^2.
\]
Now, we check assumption (ii) of Corollary C.4. We follow the same lines of the proof of Lemma 5.4(ii) by replacing $S_2(t)$ by $\Lambda(t)$ and $A_2(t)$ by $\Theta(t)$. So, we arrive at the step where we have to estimate for $\Psi, \Phi \in \mathcal{G}(\Lambda(t)^{1/2})$ and $\lambda > 0$, the quantity
\[
\mathcal{E}_\lambda[g(t)] := \langle \Psi, e^{-1}g(t)^{\text{Wick}} N_\lambda \Phi \rangle - \langle N_\lambda \Psi, e^{-1}g(t)^{\text{Wick}} \Phi \rangle,
\]
where $N_\lambda := e^{-1}N(\lambda e^{-1}N + 1)^{-1}$ and $g(t)$ is the continuous polynomial on $\mathcal{G}(\mathbb{R})$ given by
\[
g(t)[z] = P_2(t)[z] + P_3(t)[z].
\]
Note that the part $\mathcal{E}_\lambda[P_2(t)]$ involving only the symbol $P_2(t)$ is already bounded by (26). Thus, we need only to consider $\mathcal{E}_\lambda[P_3(t)]$. A simple computation yields
\[
\mathcal{E}_\lambda[P_3(t)] = \sum_{n=1}^{\infty} \kappa(n) \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}} \phi(x) \Phi^{(n+1)}(x,y) \psi^{(n)}(x,y) dx \right) dy \]
\[
- \sum_{n=1}^{\infty} \kappa(n) \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}} \phi(x) \Phi^{(n)}(x,y) \psi^{(n+1)}(x,y) dx \right) dy,
\]
where
\[
\kappa(n) = \frac{(n+1)\sqrt{n^2(n+1)} - n\sqrt{n^2(n+1)}}{(\lambda + 1)(n+1)}
\]
satisfying $|\kappa(n)| \leq \sqrt{n^2(n+1)}$ uniformly in $\epsilon \in (0,1]$ and $\lambda > 0$. So, using a similar estimate as (31), we obtain
\[
|\mathcal{E}_\lambda[P_3(t)]| \leq \frac{1}{\sqrt{2}} \|\Phi\|_{L^2(\mathbb{R})} \|\Lambda(t)^{1/2} \Psi\| \|\Lambda(t)^{1/2} \Phi\|.
\]
This proves assumption (ii) of Corollary 6.4. Now, we check that
\[ W(t) = V(t, 0). \]
In fact, for \( \Phi \in \mathcal{D}_+ \) and \( \Psi \in \mathcal{D}_+ \) we have
\[
i \partial_t \langle \Phi, \mathcal{W}(0, r) \mathcal{W}(r) \Psi \rangle = - \langle \Theta(r) \mathcal{W}(r, 0) \Phi, \mathcal{W}(r) \Psi \rangle + i \lim_{s \to 0} \langle \mathcal{W}(r + s) - \mathcal{W}(r) \rangle \Phi, \frac{\mathcal{W}(r + s) - \mathcal{W}(r)}{s} \rangle \Psi,
\]
and since by (30) we know that \( \lim_{s \to 0} \frac{\mathcal{W}(r + s) - \mathcal{W}(r)}{s} \Psi \) exists in \( \mathcal{F} \), we conclude using (32) that
\[
i \partial_t \langle \Phi, \mathcal{W}(0, r) \mathcal{W}(r) \Psi \rangle = 0.\]
This identifies \( \mathcal{W}(t) \) as the unitary propagator of the non-autonomous Schrödinger equation (35). Therefore, by (33)–(36) we get
\[
\sqrt{c_1} ||W(t)\Psi||_{\mathcal{F}_1^+} \leq ||(\Lambda(1/2)W(t)\Psi)\leq e^{c_2t}||\Lambda(0)1/2\Psi||_{\mathcal{F}_1^+} \leq \sqrt{c_2}e^{c_2t}||\Psi||_{\mathcal{F}_1^+},
\]
for any \( t \in \mathbb{R} \) uniformly in \( \varepsilon \in (0, 1] \).

**Lemma 6.4** For any \( \Phi_0 \in H^2(\mathbb{R}) \) and \( \Psi \in \mathcal{D}_+ \) we have
\[
||W(t)\Psi - U_2(t, 0)\Psi||_{\mathcal{F}_1^+}^2 = 2\langle \Psi, (1 - R_v)\Psi \rangle - 2\text{Re}(W(t)\Psi, (1 - R_v)U_2(t, 0)\Psi)
\]
\[
+2\text{Im} \int_0^t (W(s)\Psi, [\Theta(s)R_v - R_vA_2(s)]U_2(s, 0)\Psi) \, ds,
\]
where \( R_v := \sigma(\frac{v^2}{\nu}) \) with \( \sigma \) any bounded Borel function on \( \mathbb{R}_+ \) with compact support and here
\[
\Theta(s) = A_2(s) + \varepsilon^{-1}Q_s(z)_{\text{wick}},
\]
with \( Q_s(z) \) the continuous polynomial on \( \mathcal{S}(\mathbb{R}) \) given by
\[
Q_s(z) = D^{(3)}P \frac{3!}{3!}(\Phi_0)[z] + D^{(4)}P \frac{4!}{4!}(\Phi_3)[z].
\]

**Proof.** We have
\[
||W(t)\Psi - U_2(t, 0)\Psi||_{\mathcal{F}_1^+}^2 = 2||\Psi||_{\mathcal{F}_1^+}^2 - 2\text{Re}(W(t)\Psi, U_2(t, 0)\Psi)
\]
\[
+2\text{Re}(\Psi, (1 - R_v)\Psi) - 2\text{Re}(W(t)\Psi, (1 - R_v)U_2(t, 0)\Psi)
\]
\[
+2\text{Im} \int_0^t (W(s)\Psi, [\Theta(s)R_v - R_vA_2(s)]U_2(s, 0)\Psi) \, ds. \tag{37}
\]
Hence to prove the lemma it is enough to show that
\[
\mathbb{R} \ni s \mapsto \text{Re}(W(s)\Psi, R_vU_2(s, 0)\Psi) \in C^1(\mathbb{R}) \tag{38}
\]
and compute its derivative. Recall that the propagator \( U_2(s, 0) \in C^0(\mathbb{R}, \mathcal{L}(\mathcal{F}_1^+)) \), by Proposition 5.5 and that \( W(s) \in C^0(\mathbb{R}, \mathcal{L}(\mathcal{D}_+)) \) since it is the unitary propagator of the Cauchy problem (35). It is easily seen that
\[
s \mapsto R_vU_2(s, 0)\Psi,
\]
are in \( C^0(\mathbb{R}, \mathcal{D}_+) \) since \( R_v \) maps continuously \( \mathcal{F}_1^+ \) into \( \mathcal{D}_+ \). We also have that
\[
s \mapsto W(s)\Psi \in C^1(\mathbb{R}, \mathcal{D}_+) \quad \text{and} \quad s \mapsto U_2(s, 0)\Psi \in C^1(\mathbb{R}, \mathcal{D}_+^1).
\]
This proves the statement (38). Therefore, we have
\[
2\text{Re}(\Psi, R_v\Psi) - 2\text{Re}(W(t)\Psi, R_vU_2(t, 0)\Psi) = -2\varepsilon^{-1}\int_0^t i\varepsilon\partial_s \langle W(s)\Psi, R_vU_2(s, 0)\Psi \rangle \, ds. \tag{39}
\]
Proof of Proposition 6.1. We are now ready to prove Proposition 6.1. First observe that we have

\[ \left\| e^{-it/\varepsilon} \mathcal{W}(\frac{\sqrt{\varepsilon}}{i} \phi_0) \Psi - e^{it(1/\varepsilon)} \mathcal{W}(\frac{\sqrt{\varepsilon}}{i} \phi_1) U_2(t,0) \Psi \right\|^2 \geq 2 = \| \mathcal{W}(t) \Psi - U_2(t,0) \Psi \|^2. \]

Now, using Lemma 6.4 one obtains for \( t > 0 \) (the case \( t < 0 \) is similar) the estimate

\[ \| \mathcal{W}(t) \Psi - U_2(t,0) \Psi \|^2 \leq 2 \| \mathcal{W}(t) \Psi, (1 - R_y) U_2(t,0) \Psi \| + 2 \| \mathcal{W}(t) \Psi, (1 - R_y) U_2(t,0) \Psi \| + 2 \int_0^t \| \mathcal{W}(s) \Psi, [\Theta(s) R_y - R_y A_2(s), U_2(s,0) \Psi] \| \, ds. \]

Here we consider \( \sigma \) to be in the class \( C^1(\mathbb{R}_+) \), decreasing and satisfying \( \sigma(s) = 1 \) if \( s \leq 1 \) and \( \sigma(s) = 0 \) if \( s \geq 2 \). We have for \( \nu \) positive integer,

\[ \langle \Psi, (1 - R_y) \Psi \rangle \leq \frac{1}{\nu} \sum_{n=\nu+1}^{\infty} n(\Psi^{(n)}, (D_z^2 + 1) \Psi^{(n)}) \leq \frac{1}{\nu} \| \Psi \|^2_{F_+}. \]

Hence, we easily check with the help of Proposition 5.5 and Lemma 6.3 that

\[ \| \langle \mathcal{W}(t) \Psi, (1 - R_y) U_2(t,0) \Psi \rangle \| \leq \frac{1}{\nu} \| U_2(t,0) \Psi \|_{F_+}^2 \| \mathcal{W}(t) \Psi \|_{F_+} \leq \frac{1}{\nu} e^{2\epsilon t} \| \Psi \|_{F_+} \| \Psi \|_{F_+} \leq \frac{1}{\nu} e^{2\epsilon t} \| \Psi \|^2_{F_+}. \]

Next, we show that there exists \( C > 0 \) depending only on \( \phi_0 \) such that

\[ \left\| e^{-1/\varepsilon} P_2(z) \mathcal{W}(\frac{\sqrt{\varepsilon}}{i} \phi_0) \Psi \right\|_{F_+} \leq C(\nu \epsilon^{1/2} + \nu^2 \epsilon). \]

The latter bound follows by Cauchy-Schwarz inequality, Lemma A.1 and (18),

\[ \left\| \langle \Phi, P_2(\frac{\sqrt{\varepsilon}}{i} \phi_0) \Psi \rangle \right\|_{L^1(\mathbb{R}_+)} \leq \sqrt{\nu} \| \Phi \|_{L^2(\mathbb{R}_+)} \left[ \sum_{n=\nu+1}^{\infty} (n+1) \| \Phi^{(n)} \|^2_{L^2(\mathbb{R}_+)} \right]^{1/2} \| \Phi \|^2_{\mathcal{L}^2(\mathbb{R}_+)} \leq 2 \nu \sqrt{\nu} \| \Phi \|_{L^2(\mathbb{R}_+)} \| (e^{-1/\varepsilon} N + 1)^{1/2} \Phi \|_{F_+} \| \Psi \|_{F_+}. \]

and a similar estimate for \( P_{2}^{\text{Wick}} \),

\[ \left\| \langle \Phi, P_{2}^{\text{Wick}}(\frac{\sqrt{\varepsilon}}{i} \phi_0) \Psi \rangle \right\| \leq \nu^2 \| \Phi \|^2_{F_+} \| \Psi \|^2_{F_+}. \]

Hence we can check that

\[ \int_0^t \left\| \langle \mathcal{W}(s) \Psi, e^{-1/\varepsilon} P_2(z) \mathcal{W}(\frac{\sqrt{\varepsilon}}{i} \phi_0) U_2(s,0) \Psi \rangle \right\| \, ds \leq C(\nu \epsilon^{1/2} + \nu^2 \epsilon) \int_0^t \| \mathcal{W}(s) \Psi \|_{F_+} \| U_2(s,0) \Psi \|_{F_+} \, ds. \]
Now, by Lemma 6.3 and Proposition 5.5 we obtain

\[
\int_0^t \|W(s)\Psi\|_{\mathcal{F}_+} \|U_2(s,0)\Psi\|_{\mathcal{F}_+} \, ds \leq \int_0^t e^{\varepsilon_2 s^2} \|\Psi\|_{\mathcal{F}_+} \|U_2(s,0)\Psi\|_{\mathcal{F}_+} \, ds
\leq \int_0^t e^{2s^2} \|\Psi\|_{\mathcal{F}_+} \|\Psi\|_{\mathcal{F}_+} \, ds
\leq e^{\varepsilon_2 s} \|\Psi\|_{\mathcal{F}_+}^2.
\]

A simple computation yields

\[
A_2(s)R_v - R_vA_2(s) = \frac{1}{2} \left[ \sigma\left(\frac{e^{-1}N + 2}{v}\right) - \sigma\left(\frac{e^{-1}N - 2}{v}\right) \right] \left( \int_{\mathbb{R}} \phi(x)^2 z(x)^2 \, dx \right)^{\text{Wick}}
+ \frac{1}{2} \left[ \sigma\left(\frac{e^{-1}N - 2}{v}\right) - \sigma\left(\frac{e^{-1}N}{v}\right) \right] \left( \int_{\mathbb{R}} \phi(x)^2 z(x)^2 \, dx \right)^{\text{Wick}}.
\]

We easily check that

\[
\left\| \sigma\left(\frac{e^{-1}N + 2}{v}\right) - \sigma\left(\frac{e^{-1}N - 2}{v}\right) \right\|_{\mathcal{F}_+} \leq \frac{2}{v} \left\| \sigma' \right\|_{L^1(\mathbb{R})},
\]

since \(e^{-1}d\Gamma(-\Delta) + e^{-1}N\) commute with \(e^{-1}N\). Thus, using (23) there exists \(c_0, c > 0\) such that

\[
\int_0^t \left\| \langle W(s)\Psi, [A_2(s), R_v]U_2(s,0)\Psi \rangle \right\| \, ds \leq \frac{c_0}{c} \int_0^t \left\| W(s)\Psi \right\|_{\mathcal{F}_+} \left\| U_2(s,0)\Psi \right\|_{\mathcal{F}_+} \, ds
\leq \frac{1}{v} e^{c_1 t} \left\| \Psi \right\|_{\mathcal{F}_+}^2.
\]

Finally, the claimed inequality in Proposition 6.1 follows by collecting the previous estimates and letting \(v = \varepsilon^{-1/4}\).

We have the following two corollaries.

**Corollary 6.5** For any \(\phi_0 \in H^2(\mathbb{R})\) and any \(\xi \in L^2(\mathbb{R})\) we have the strong limit

\[
s - \lim_{\varepsilon \to 0} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) e^{it/\varepsilon H_e} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \left( W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \right) = e^{i\sqrt{2} Re(\xi) \phi_0} \psi.
\]

where \(\psi\) solves the NLS equation (17) with initial data \(\phi_0\).

**Proof.** It is enough to prove for any \(\Psi, \Phi \in \mathcal{F}_+\) the limit:

\[
\lim_{\varepsilon \to 0} \langle e^{-it/\varepsilon H_e} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \Psi, W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \rangle e^{-it/\varepsilon H_e} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \Phi \rangle = e^{i\sqrt{2} Re(\xi) \phi_0} \langle \Psi, \Phi \rangle.
\]

Indeed, using Proposition 6.1 we show

\[
\langle e^{-it/\varepsilon H_e} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \Psi, W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \rangle e^{-it/\varepsilon H_e} W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) \Phi \rangle = \langle W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) U_2(t,0) \Psi, W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) U_2(t,0) \Phi \rangle
+ O(\varepsilon^{1/8}).
\]

Therefore by Weyl commutation relations we have

\[
\langle W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) U_2(t,0) \Psi, W\left(\frac{\sqrt{2}}{i\varepsilon} \phi_0\right) U_2(t,0) \Phi \rangle = \langle U_2(t,0) \Psi, U_2(t,0) \Phi \rangle e^{i\sqrt{2} Re(\xi) \phi_0},
\]

Thus the limit is proved since \(s - \lim_{\varepsilon \to 0} W(\xi) = 1\).

Recall that \(\mathcal{F}_0\) is the subspace of \(\mathcal{F}\) spanned by vectors \(\Psi \in \mathcal{F}\) such that \(\Psi^{(n)} = 0\) for any index \(n \in \mathbb{N}\) except for finite number. Note that \(\mathcal{F}_0 \cap \mathcal{F}_+\) is dense in \(\mathcal{F}\).
Corollary 6.6 For any $\varphi_0 \in H^2(\mathbb{R})$ and any $\Psi, \Phi \in \mathcal{F}_0 \cap \mathcal{G}_+$ and $b \in \mathcal{P}_{p,q}(L^2(\mathbb{R}))$, we have

$$
\lim_{\epsilon \to 0} \langle W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Psi, e^{it/\hbar} b_{\text{Wick}} e^{-it/\hbar} W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Phi \rangle = b(\varphi_0) \langle \Psi, \Phi \rangle,
$$

where $\varphi_0$ solves the NLS equation (17) with initial data $\varphi_0$.

Proof. Consider a $(p, q)$-homogenous polynomial $b \in \mathcal{P}_{p,q}(L^2(\mathbb{R}))$. We have

$$
\mathcal{A} := \langle W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Psi, e^{it/\hbar} b_{\text{Wick}} e^{-it/\hbar} W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Phi \rangle
$$

$$
= \langle (N + 1)^q W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Psi, e^{it/\hbar} B_{\epsilon} e^{-it/\hbar} ((N + 1)^p W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)\Phi),
$$

where $B_{\epsilon} := (N + 1)^{-\eta} b_{\text{Wick}} (N + 1)^{-p}$. The number estimate (10) yields

$$
\| B_{\epsilon} \| \leq \| \tilde{b} \|_{L^2(\mathbb{R})},
$$

uniformly in $\epsilon \in (0, 1)$. Let $\tilde{N}_{\epsilon}$ be the positive operator given by

$$
\tilde{N}_{\epsilon} = N + 2 \text{Re} \langle z, \Phi \rangle_{\text{Wick}} + \| \varphi \|^2_{L^2(\mathbb{R})}.
$$

By (27), we get

$$
\mathcal{A} = \langle W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)(\tilde{N}_{\epsilon} + 1)^q \Psi, e^{it/\hbar} B_{\epsilon} e^{-it/\hbar} W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)(\tilde{N}_{\epsilon} + 1)^p \Phi \rangle.
$$

Now, observe that

$$
\lim_{\epsilon \to 0} \langle \tilde{N}_{\epsilon} + 1 \rangle^p \Phi = (1 + \| \varphi \|_{L^2(\mathbb{R})})^p \varphi \quad \text{and} \quad \lim_{\epsilon \to 0} \langle \tilde{N}_{\epsilon} + 1 \rangle^q \Psi = (1 + \| \varphi \|_{L^2(\mathbb{R})})^q \Psi.
$$

So, using Proposition 6.1 we obtain

$$
\mathcal{A} = (1 + \| \varphi \|^2_{L^2(\mathbb{R})})^{p+q} \langle W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)U_2(t, 0)\Psi, B_{\epsilon} W(\sqrt{\frac{\tau}{i\epsilon}} \varphi_0)U_2(t, 0)\Phi \rangle + O(\epsilon^{1/8})
$$

$$
= \langle U_2(t, 0)\Psi, (\tilde{N}_{\epsilon} + 1)^{-\eta} b \tilde{b}(., \Phi)_{\text{Wick}} (\tilde{N}_{\epsilon} + 1)^{-p} U_2(t, 0)\Phi \rangle + O(\epsilon^{1/8}).
$$

We set $\Psi_{\epsilon} = (N + 1)^q \tilde{N}_{\epsilon} + 1)^{-q} U_2(t, 0)\Psi$ and $\Phi_{\epsilon} = (N + 1)^p (\tilde{N}_{\epsilon} + 1)^{-p} U_2(t, 0)\Phi$ and remark that we can show for $\varphi_0 \neq 0$ and $\mu$ a positive integer the following strong limit

$$
(s - \lim_{\epsilon \to 0} (N + 1)^\mu (\tilde{N}_{\epsilon} + 1)^{-\mu} = \frac{1}{(1 + \| \varphi \|^2_{L^2(\mathbb{R})})^\mu}.
$$

This holds since we have by explicit computation

$$
\left| \langle (a(\varphi_0) + a^*(\varphi_0))(N + \| \varphi \|^2 + 1)^{-1} \rangle \right| \leq \frac{| \varphi_0 |}{2 \sqrt{| \varphi_0 |^2 + 1}} + \frac{| \varphi_0 |}{2 \sqrt{| \varphi_0 |^2 + 1 - \epsilon}} < 1,
$$

for $\epsilon$ sufficiently small and hence we can write

$$
(N + 1)(\tilde{N}_{\epsilon} + 1)^{-1} = (N + 1)(N + \| \varphi \|^2 + 1)^{-1} \left[ (a(\varphi_0) + a^*(\varphi_0))(N + \| \varphi \|^2 + 1)^{-1} + 1 \right]^{-1}.
$$

This proves (42) for $\mu = 1$ since $s - \lim_{\epsilon \to 0} \mathcal{A}_{\epsilon} = 0$. Now, we proceed by induction on $\mu$ using a commutator argument

$$
(N + 1)^{\mu+1}(\tilde{N}_{\epsilon} + 1)^{-1} = (N + 1)^\mu (\tilde{N}_{\epsilon} + 1)^{-\mu} (N + 1)(\tilde{N}_{\epsilon} + 1)^{-1}
$$

$$
+ (N + 1)^{\mu}(\tilde{N}_{\epsilon} + 1)^{-\mu} [(\tilde{N}_{\epsilon} + 1)^\mu, N](\tilde{N}_{\epsilon} + 1)^{-1}.\mu+1,
$$

21
Proposition 6.7 Let $\Phi_0 \in H^{2}(\mathbb{R})$ and consider the propagator $U_2(t,0)$ given in Proposition 5.5. For a given $s \in \mathbb{R}$ let $\xi_s \in H^{2}(\mathbb{R})$, we have

$$U_2(t,s)W\left(\frac{\xi_t}{i\sqrt{\xi}}\right)U_2(s,t) = W\left(\frac{\beta(t,s)\xi_t}{i\sqrt{\xi}}\right)$$

where $\beta(t,s)$ is the symplectic propagator on $L^2(\mathbb{R})$, solving the equation

$$\begin{cases}
i\partial_t \xi_t(x) = [-\Delta + 2|\phi_t(x)|^2] \xi_t(x) + \phi_t(x)^2 \overline{\xi_t(x)}, \\
\xi_{t\lesssim s} = \xi_s,
\end{cases} \quad (43)$$

such that $\beta(t,s)\xi_t = \xi_s$.

Proof. Observe that if $\Phi_0 \in H^{2}(\mathbb{R})$ then the solution $\phi_t$ of the NLS equation (17) with initial condition $\Phi_0$ satisfies $\phi_t \in C^{0}(\mathbb{R}, L^{\infty}(\mathbb{R}))$. Hence, by standard arguments the equation (43) admits a unique solution $\xi_t \in C^{0}(\mathbb{R}, H^{2}(\mathbb{R})) \cap C^{1}(\mathbb{R}, L^{2}(\mathbb{R}))$ for any $\xi_s \in H^{2}(\mathbb{R})$. Moreover, the propagator

$$\beta(t,s)\xi_t = \xi_s,$$

defines a symplectic transform on $L^2(\mathbb{R})$ for any $t, s \in \mathbb{R}$. This follows by differentiating

$$\text{Im}(\beta(t,s)\xi_t, \beta(t,s)\eta),$$

with respect to $t$ for $\xi, \eta \in H^{2}(\mathbb{R})$. Furthermore, $\beta$ satisfies the laws

$$\beta(s,s) = 1, \quad \beta(t,s)\beta(s,r) = \beta(t,r) \quad \text{for} \quad t, r, s \in \mathbb{R}.$$ 

Now, we differentiate with respect to $t$ the quantity

$$U_2(s,t)W\left(\frac{\xi_t}{i\sqrt{\xi}}\right)U_2(t,s)$$

in the sense of quadratic forms on $\mathcal{F}_+^1$, with $\xi_t$ solution of (43). Hence, using Lemma 6.2(ii), we get

$$\frac{\partial_t}{\partial t} \left[ U_2(s,t)W\left(\frac{\sqrt{7}}{i\sqrt{\xi}}\xi_t\right)U_2(t,s) \right] = U_2(s,t)W\left(\frac{\sqrt{7}}{i\sqrt{\xi}}\xi_t\right) \left[ W\left(\frac{\sqrt{7}}{i\sqrt{\xi}}\xi_t\right)\text{Im}A_2(t)W\left(\frac{\sqrt{7}}{i\sqrt{\xi}}\xi_t\right) - iA_2(t) \right]
$$

$$- i \left( \text{Re}(\xi_t, i\partial_t \xi_t) + \frac{2}{\sqrt{7}}\text{Re}(z, i\partial_z \xi_t)^{\text{Wick}} \right) U_2(t,s). \quad (44)$$

with the observation that the second term of (r.h.s.) converges strongly to 0. Therefore, we obtain

$$\lim_{\epsilon \to 0} \Psi_{\epsilon} = \frac{1}{(1 + ||\xi||^2_{L^2(\mathbb{R})})^q} U_2(t,0)\Psi \quad \text{and} \quad \lim_{\epsilon \to 0} \Phi_{\epsilon} = \frac{1}{(1 + ||\xi||^2_{L^2(\mathbb{R})})^p} U_2(t,0)\Phi.$$

It is also easy to show by explicit computation that

$$w - \lim_{\epsilon \to 0} (N + 1)^{-q} b_{r,s}^{\text{Wick}} (N + 1)^{-p} = 0,$$

for any $b_{r,s} \in \mathcal{P}_{r,s}(L^2(\mathbb{R}))$ such that $0 < r \leq p$ and $0 < s \leq q$. Hence, letting $\epsilon \to 0$, we get

$$\lim_{\epsilon \to 0} \epsilon = \left(1 + ||\phi_0||^2_{L^2(\mathbb{R})}\right)^{p+q} \lim_{\epsilon \to 0} \langle \Psi_{\epsilon}, (N + 1)^{-q}b(\phi_0) (N + 1)^{-p}\Phi_{\epsilon}\rangle$$

$$= b(\phi_0) \langle U_2(t,0)\Psi, U_2(t,0)\Phi \rangle = b(\phi_0) \langle \Psi, \Phi \rangle,$$

since $||\phi||_{L^2(\mathbb{R})} = ||\phi_0||_{L^2(\mathbb{R})}$ and $s - \lim_{\epsilon \to 0} (N + 1)^{-\mu} = 1$ for $\mu > 0.$
Now, by [AmNi1, Lemma 2.10], we obtain
\[
W(\sqrt{\frac{i}{\epsilon}} \xi)^* A_2(t) W(\sqrt{\frac{i}{\epsilon}} \xi) = \epsilon^{-1} m(t)[z + \sqrt{\epsilon} \xi]^\text{Wick},
\]
where \( m(t)[z] \) is the continuous polynomial on \( \mathcal{S}(\mathbb{R}) \) given by
\[
m(t)[z] = (z, -\Delta z) + P_2(t)[z].
\]
Therefore, the (r.h.s.) of (44) is null if we show that
\[
m(t)[z + \sqrt{\epsilon} \xi] - m(t)[z] - (\epsilon \text{Re} \langle \xi, i\partial_x \xi \rangle + 2\sqrt{\epsilon} \text{Re} \langle z, i\partial_x \xi \rangle) = 0.
\]
This follows by straightforward computation.

7 Propagation of chaos

Propagation of chaos for a many-boson system with point pair-interaction in one dimension was studied in \([ABGT]\) (see also the related work \([AGT]\)). Here we prove this conservation hypothesis for such quantum system using the method in \([RoSch]\). Thus, we are led to study the asymptotics of time-evolved Hermite states
\[
e^{-it/\epsilon H_{\text{bos}} \otimes^n} \varphi_0^\otimes_n \quad \text{with} \quad \varphi_0 \in H^2(\mathbb{R}), \quad \|\varphi_0\|_{L^2(\mathbb{R})} = 1,
\]
when \( n \to \infty \) with \( n\epsilon_n = 1 \). We denote the coherent states by
\[
E(\varphi_0) := W(\sqrt{\frac{i}{\epsilon}} \varphi_0) \Omega_0,
\]
where \( \Omega_0 = (1, 0, \cdots) \) is the vacuum vector in the Fock space \( \mathcal{F} \). To pass from coherent states to Hermite states we use the integral representation proved in \([RoSch]\),
\[
\varphi_0^\otimes_n = \frac{\gamma_n}{2\pi} \int_0^{2\pi} e^{-i\theta n} E(e^{i\theta} \varphi_0) \, d\theta, \quad \text{where} \quad \gamma_n := \frac{e^{1/2n} \sqrt{n!}}{\epsilon_n^{-n/2}}.
\]
Asymptotically, the factor \( \gamma_n \) grows as \((2\pi n)^{1/4}\) when \( n \to \infty \).

In the following proposition we prove the chaos conservation hypothesis.

**Proposition 7.1** For any \( \varphi_0 \in H^2(\mathbb{R}) \) such that \( \|\varphi_0\|_{L^2(\mathbb{R})} = 1 \) and any \( b \in \mathcal{B}_{p,p}(L^2(\mathbb{R})) \), we have
\[
\lim_{n \to \infty} \langle \varphi_0^\otimes_n, e^{it/\epsilon H_{\text{bos}}} b^\text{Wick} e^{-it/\epsilon H_{\text{bos}}} \varphi_0^\otimes_n \rangle = b(\varphi_0),
\]
where \( n\epsilon_n = 1 \) and \( \varphi_0 \) solves the NLS equation (17) with initial data \( \varphi_0 \).

**Proof.** It is known that if a sequence of positive trace-class operators \( \rho_n \) on \( L^2(\mathbb{R}) \) converges in the weak operator topology to \( \rho \) such that \( \lim_{n \to \infty} \text{Tr}[\rho_n] = \text{Tr}[\rho] < \infty \) then \( \rho_n \) converges in the trace norm to \( \rho \) (see, for instance \([DA]\)). This argument reduces the proof to the case
\[
b(z) = \prod_{i=1}^p \langle z, f_i \rangle \langle g_i, z \rangle,
\]
where \( f_i, g_i \in L^2(\mathbb{R}) \). For shortness, we set
\[
E_\theta = E(e^{i\theta} \varphi_0) \quad \text{and} \quad E'_\theta = e^{-it/\epsilon H_{\text{bos}}} E_\theta.
\]
Using formula (45), we get
\[
\Gamma_n := \langle \varphi_0^\otimes_n, e^{it/\epsilon H_{\text{bos}}} b^\text{Wick} e^{-it/\epsilon H_{\text{bos}}} \varphi_0^\otimes_n \rangle = \frac{\gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} e^{-in(\theta - \theta')} \langle E'_{\theta'}, b^\text{Wick} E_{\theta'} \rangle \, d\theta d\theta'.
\]
It is easily seen that 
\[(N + 1)^{-p} e^{-\frac{\theta}{\epsilon_0 H_{0}}} \phi_0^{\otimes n} = 2^{-p} e^{-\frac{\theta}{\epsilon_0 H_{0}}} \phi_0^{\otimes n}.\]

Therefore, we write 
\[
\Gamma_n = \frac{4p N^2}{(2\pi)^2} \int_{[0,2\pi]^2} e^{-i(n(\theta - \theta'))(E_{\theta'}^t, (N + 1)^{-p} \prod_{i=1}^{p} a^*(f_i) \prod_{j=1}^{p} a(g_j) (N + 1)^{-p} E_{\theta}^t)} d\theta d\theta'.
\]

Now, we use the decomposition 
\[
\prod_{i=1}^{p} a^*(f_i) \prod_{j=1}^{p} a(g_j) = \sum_{I,J \subseteq \mathcal{N}_p} \prod_{i \in I} a^*(f_i) - \langle \phi_i, f_i \rangle \prod_{j \in J} a(g_j) - \langle g_j, \phi_i \rangle \left| e^{-i(\theta' - \theta)(\theta - \theta')} \right|
\times \prod_{i \in I} \langle f_i, \phi_i \rangle \prod_{j \in J} \langle g_j, \phi_i \rangle,
\]

where the sum runs over all subsets \(I, J\) of \(\mathcal{N}_p := \{1, \cdots, p\}\). Thus, we can write 
\[
\Gamma_n - b(\phi_i) = \sum_{I,J \subseteq \mathcal{N}_p \cap [0,2\pi]^2} \left| \langle \phi_i, f_i \rangle \prod_{j \in J} \langle g_j, \phi_i \rangle \left( \prod_{i \in I} \langle f_i, \phi_i \rangle \prod_{j \in J} \langle g_j, \phi_i \rangle \right) \right|
\]

where \(\tilde{E}_{\theta}^t := (N + 1)^{-p} E_{\theta}^t\) and \(B_{I,J}(z)\) are sums of homogenous polynomials such that 
\[
\langle E_{\theta'}^t, B_{I,J}^{\text{Wick}} E_{\theta}^t \rangle = \prod_{i \in I} \langle \phi_i, f_i \rangle \prod_{j \in J} \langle g_j, \phi_i \rangle \times \left( \prod_{i \in I} \langle a(f_i) - \langle f_i, \phi_i \rangle \rangle \right) E_{\theta'}^t, \prod_{j \in J} \langle a(g_j) - \langle g_j, \phi_i \rangle \rangle E_{\theta}^t.
\]

We have, for \(0 \leq \#I, \#J < p\), by Cauchy-Schwarz inequality 
\[
\left| \langle E_{\theta'}^t, B_{I,J}^{\text{Wick}} E_{\theta}^t \rangle \right| \leq \prod_{i \in I, j \in J} \| g_j \|_{L^2(\mathbb{R})} \| f_i \|_{L^2(\mathbb{R})}
\times \left| \prod_{i \in I} \langle a(f_i) - \langle f_i, \phi_i \rangle \rangle \tilde{E}_{\theta'}^t \right| \times \left| \prod_{j \in J} \langle a(g_j) - \langle g_j, \phi_i \rangle \rangle \tilde{E}_{\theta}^t \right|
\]

In the following we make use of the positive self-adjoint operator 
\[
\tilde{N} := N + 2\text{Re}(z, \phi_i)^{\text{Wick}} + \|\phi_i\|^2 1.
\]

Observe that we have for any \(\theta' \in [0,2\pi]\) and \(r \geq 1\), 
\[
\left| \prod_{i=1}^{r} \langle a(f_i) - \langle f_i, \phi_i \rangle \rangle \tilde{E}_{\theta'}^t \right| \leq \left| \prod_{i=1}^{r} a(f_i)(\tilde{N} + 1)^{-p} \mathcal{W}(t) \Omega_0 \right|_{\mathcal{F}}
\leq \left| \prod_{i=1}^{r-1} a(f_i)(\tilde{N} + 1)^{-p} a(f_r) \mathcal{W}(t) \Omega_0 \right|_{\mathcal{F}}
+ \left| \prod_{i=1}^{r-1} a(f_i) a(f_r), (\tilde{N} + 1)^{-p} \mathcal{W}(t) \Omega_0 \right|_{\mathcal{F}}.
\]

We easily show that 
\[
\| a(f_r) \mathcal{W}(t) \Omega_0 \|_{\mathcal{F}} \leq ||f_r||_{L^2(\mathbb{R})} \sqrt{\epsilon_n} \| \mathcal{W}(t) \|_{\mathcal{L}(\mathcal{F}_{\mathcal{H}^\perp})}.
\]

Furthermore, we have 
\[
\| a(f_r) (\tilde{N} + 1)^{-p} (\tilde{N} + 1)^{-p} \|_{\mathcal{L}(\mathcal{F})} \leq C \epsilon_n.
\]
Thus, we conclude that \( \lim \) in terms of Wigner measures, introduced in \([\text{AmNi1}, \text{AmNi2}]\), Proposition 7.1 says that the sequence \( (\phi_i) \) is a Wick polynomial where we gained \( \epsilon_n \) in its symbol, see \([\text{AmNi1} \text{ Proposition } 2.7 \text{ (ii)}]\). Recall also that we have by the number estimate \([10] \text{ and } (28)\),

\[
\left\| \prod_{i=1}^{n-1} a(f_i) (\hat{N} + 1)^{-p} \right\| \leq C,
\]

uniformly in \( n \) and \( \theta' \in [0, 2\pi] \). Therefore, we have

\[
\left| \sum_{0 \leq \#f \leq p} \frac{4^p \gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} e^{-\frac{n}{2} (n-p)\theta - n \theta'} \langle \phi^{t}_f, \psi^{t}_f \rangle \, d\theta \, d\theta' \right| \leq C \gamma_n^2 \epsilon_n \frac{2^{p-\#f} n^{\frac{p-\#f}{2}}}{\pi} \rightarrow 0. \tag{47}
\]

It still to control the terms \( \#f = p, \#f - 1 \) and \( \#f = p - 1, \#f = p \) which are similar. In fact, remark that we have

\[
\frac{4^p \gamma_n^2}{2\pi} \int_{[0,2\pi]^2} e^{-\frac{n}{2} (n-p)\theta - n \theta'} \langle \phi^{t}_f, \psi^{t}_f \rangle \, d\theta \, d\theta' =
\]

\[
\frac{4^p \gamma_n^2}{2\pi} \int_{0}^{2\pi} e^{\frac{n}{2} (n-p+1)\theta} \langle \phi^{t}_f, \psi^{t}_f \rangle \, d\theta \, d\theta' =
\]

\[
\frac{4^p \gamma_n^2}{2\pi} \int_{0}^{2\pi} e^{\frac{n}{2} (n-p+1)\theta} \langle \phi^{t}_f, \psi^{t}_f \rangle \, d\theta \, d\theta' =
\]

\[
\leq C \gamma_n \sqrt{\epsilon_n} \rightarrow 0. \tag{47}
\]

Thus, we conclude that \( \lim_{n \to \infty} \Gamma_n - b(\phi_i) = 0 \). \( \square \)

**Remark 7.2**

1) Let \( \gamma_{k,n} \) be the \( k \)-particle correlation functions, defined by \([3] \), associated to the states \( e^{-\mu/\epsilon_n} \phi_0^\otimes n \). Then Proposition 7.1 implies the following convergence in the trace norm

\[
\lim_{n \to \infty} \gamma_{k,n} = \phi(x_1) \cdots \phi(x_k) \phi(y_1) \cdots \phi(y_k).
\]

2) In terms of Wigner measures, introduced in \([\text{AmNi1}, \text{AmNi2}]\), Proposition 7.1 says that the sequence \( \langle e^{-\mu/\epsilon_n} \phi_0^\otimes n \rangle_{n \in \mathbb{N}} \) admits a unique (Borel probability) Wigner measure \( \mu_t \) given by

\[
\mu_t = \frac{1}{2\pi} \int_{0}^{2\pi} \delta_{e^{\theta} \phi_i} \, d\theta,
\]

where \( \delta_{e^{\theta} \phi_i} \) is the Dirac measure on \( L^2(\mathbb{R}) \) at the point \( e^{\theta} \phi_i \).

## Appendix

### A Elementary estimate

**Lemma A.1** For any \( \alpha > 0 \) and any \( \psi^{(n)} \in \mathcal{S}_i(\mathbb{R}^n) \), we have

\[
\int_{\mathbb{R}^n} |\psi^{(n)}(x_1, x_2, \ldots, x_n)|^2 \, dx_1 \cdots dx_n \leq \frac{\alpha}{\sqrt{2}} \langle D^2_x \psi^{(n)}, \psi^{(n)} \rangle_{L^2(\mathbb{R}^n)} + \frac{\alpha^{-1}}{2\sqrt{2}} \langle \psi^{(n)} \rangle_{L^2(\mathbb{R}^n)}. \tag{48}
\]

**Proof.** Let \( x', \xi' \in \mathbb{R}^{n-1} \) and \( g \in \mathcal{S}(\mathbb{R}^n) \). Let us denote the Fourier transform of \( g \) by

\[
\hat{g}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} g(x) \, dx.
\]

We have

\[
g(0, x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix\xi'} \left( \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi_1, \xi') \, d\xi_1 \right) d\xi'.
\]
Cauchy-Schwarz inequality yields
\[
\left| \int_{\mathbb{R}} \hat{g}(\xi_1, \xi') \, d\xi_1 \right|^2 \leq \int_{\mathbb{R}} |\hat{g}(\xi_1, \xi')|^2 (\alpha^{-1} + \alpha \xi_1^2) \, d\xi_1 \times \int_{\mathbb{R}} \frac{d\xi_1}{\alpha^{-1} + \alpha \xi_1^2}.
\]

Therefore, we get
\[
\int_{\mathbb{R}^{n-1}} |g(0,x')|^2 \, dx' = \frac{1}{4\pi^2 (2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} \hat{g}(\xi_1, \xi') \, d\xi_1 \right|^2 \, d\xi'
\leq \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} |\hat{g}(\xi_1, \xi')|^2 (\alpha^{-1} + \alpha \xi_1^2) \, d\xi_1 \, d\xi'.
\]

Set \( g(x_1, \cdots, x_n) = \Psi^{(n)}(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_2 - x_1}{\sqrt{2}}, x_3, \cdots, x_n) \), we obtain
\[
\int_{\mathbb{R}^{n-1}} |\Psi^{(n)}(x_2, x_2, \cdots, x_n)|^2 \, dx_2 \cdots dx_n = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^{n-1}} |g^{(n)}(0, x_2, \cdots, x_n)|^2 \, dx_2 \cdots dx_n
\leq \frac{(2\pi)^{-n}}{2\sqrt{2}} \int_{\mathbb{R}^n} |g^{(n)}(\xi_1, \xi')|^2 (\alpha^{-1} + \alpha \xi_1^2 + \alpha \xi_2^2) \, d\xi_1 \, d\xi'.
\]

Thus, by Plancherel’s identity we obtain
\[
\int_{\mathbb{R}^{n-1}} |\Psi^{(n)}(x_2, x_2, \cdots, x_n)|^2 \, dx_2 \cdots dx_n \leq \frac{\alpha}{2\sqrt{2}} \frac{\langle (D_2^2 + D_2^2) \Psi^{(n)}, \Psi^{(n)} \rangle_{L^2(\mathbb{R}^n)} + \alpha^{-1} \|\Psi^{(n)}\|_{L^2(\mathbb{R}^n)}^2}{\alpha^{-1} (\Psi^{(n)})_{L^2(\mathbb{R}^n)}}.
\]

Thanks to the symmetry of \( \Psi^{(n)} \), it is easy to see that
\[
\langle (D_2^2 + D_2^2) \Psi^{(n)}, \Psi^{(n)} \rangle = 2 \langle D_2 \Psi^{(n)}, \Psi^{(n)} \rangle.
\]

Hence, we arrive at the claimed estimate \((48)\).  \(\square\)

**B Commutator theorems**

Here we first recall an abstract regularity argument from Faris-Lavine work \([FL, \text{ Theorem 2}]\).

**Theorem B.1** Let \( A \) be a self-adjoint operator and let \( S \) be a positive self-adjoint operator satisfying
\[
\bullet \quad \mathcal{D}(S) \subset \mathcal{D}(A),
\]
\[
\bullet \quad \pm i [\langle A \Psi, S \Psi \rangle - \langle S \Psi, A \Psi \rangle] \leq c \|S^{1/2} \Psi\|^2 \text{ for all } \Psi \in \mathcal{D}(S).
\]

Then \( \mathcal{D}(S) \) is invariant by \( e^{-tA} \) for any \( t \in \mathbb{R} \) and the inequality
\[
\|S^{1/2} e^{-tA} \Psi\| \leq e^{ct} \|S^{1/2} \Psi\|
\]
holds true.

Next we recall the Nelson commutator theorem (see, e.g., \([RS, \text{ Theorem X.36'}]\),\([N]\)) with a useful regularity property added as a consequence of Faris-Lavine’s Theorem [B.1].

**Theorem B.2** Let \( S \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) such that \( S \geq 1 \). Consider a quadratic form \( a(\cdots) \) with \( \mathcal{D}(a) = \mathcal{D}(S^{1/2}) \) and satisfying:
\[
\begin{align*}
(i) \ |a(\Psi, \Phi)| & \leq c_1 \|S^{1/2} \Psi\| \|S^{1/2} \Phi\| \text{ for any } \Psi, \Phi \in \mathcal{D}(S^{1/2}); \\
(ii) \ |a(\Psi, S \Phi) - a(S \Psi, \Phi)| & \leq c_2 \|S^{1/2} \Psi\| \|S^{1/2} \Phi\| \text{ for any } \Psi, \Phi \in \mathcal{D}(S^{1/2}).
\end{align*}
\]
Then the linear operator \( A : \mathcal{D}(A) \to \mathcal{H}, \mathcal{D}(A) = \{ \Phi \in \mathcal{D}(S^{1/2}) : \mathcal{H} \ni \Psi \mapsto a(\Psi, \Phi) \text{ continuous} \} \) associated to the quadratic form \( a(\cdot, \cdot) \) through the relation

\[
\langle \Psi, A \Phi \rangle_{\mathcal{H}} = a(\Psi, \Phi) \text{ for all } \Psi \in \mathcal{D}(S^{1/2}), \Phi \in \mathcal{D}(A)
\]
is densely defined and satisfies:

1. \( \mathcal{D}(S) \subset \mathcal{D}(A) \) and \( \| A \Psi \| \leq c \| S \Psi \| \) for any \( \Psi \in \mathcal{D}(S) \);
2. \( A \) is essentially self-adjoint on any core of \( S \);
3. \( e^{-itA} \) preserves \( \mathcal{D}(S^{1/2}) \) with the inequality

\[
\| S^{1/2} e^{-it\hat{A}} \Psi \| \leq e^{c|t|} \| S^{1/2} \Psi \|
\]

where \( \hat{A} \) denotes the self-adjoint extension of \( A \).

**Proof.** The point (3) follows from Theorem \[B.1\] since its assumptions:

- \( \mathcal{D}(S) \subset \mathcal{D}(A) \),
- \( \pm i[\langle A \Psi, S \Psi \rangle - \langle S \Psi, A \Psi \rangle] \leq c_2 \| S^{1/2} \Psi \|^2 \), for any \( \Psi \in \mathcal{D}(S) \),

hold true using items 1), 2) and hypothesis (ii).

We naturally associate to a self-adjoint operator \( S \geq 1 \) acting on a Hilbert space \( \mathcal{H} \), a Hilbert rigging \( \mathcal{H}_{1} \) where \( \mathcal{H}_{1} \) is defined as \( \mathcal{D}(S^{1/2}) \) endowed with the inner product

\[
\langle \psi, \phi \rangle_{\mathcal{H}_{1}} := \langle S^{1/2} \psi, S^{1/2} \phi \rangle_{\mathcal{H}},
\]

and \( \mathcal{H}_{-1} \) is the completion of \( \mathcal{D}(S^{-1/2}) \) with respect to the inner product

\[
\langle \psi, \phi \rangle_{\mathcal{H}_{-1}} := \langle S^{-1/2} \psi, S^{-1/2} \phi \rangle_{\mathcal{H}}.
\]

Assumption (ii) of Theorem \[B.2\] can be reformulated in some other slightly different ways.

**Lemma B.3** Consider a self-adjoint operator \( S \) satisfying \( S \geq 1 \) with the associated Hilbert rigging \( \mathcal{H}_{\pm 1} \) defined above. Let \( A \) be a symmetric bounded operator in \( \mathcal{L}(\mathcal{H}_{+1}, \mathcal{H}_{-1}) \), then the three following statements are equivalent,

1. There exists \( c > 0 \) such that for any \( \Psi, \Phi \in \mathcal{D}(S^{1/2}) \),

\[
\| \langle S \Psi, A \Phi \rangle - \langle A \Psi, S \Phi \rangle \| \leq c \| \Psi \|_{\mathcal{H}_{+1}} \| \Phi \|_{\mathcal{H}_{-1}},
\]

2. There exists \( c > 0 \) such that for any \( \Psi, \Phi \in \mathcal{D}(S^{1/2}) \) and \( \lambda > 0 \),

\[
\| (\lambda S + 1)^{-1} S \Psi A (\lambda S + 1)^{-1} \Phi - A (\lambda S + 1)^{-1} S \Psi (\lambda S + 1)^{-1} S \Phi \| \leq c \| \Psi \|_{\mathcal{H}_{+1}} \| \Phi \|_{\mathcal{H}_{-1}},
\]

3. There exists \( c > 0 \) such that for any \( \Psi, \Phi \in \mathcal{D}(S^{1/2}) \) and \( \lambda > 0 \),

\[
\| (\lambda S + 1)^{-1} S \Psi A \Phi - A \Psi (\lambda S + 1)^{-1} S \Phi \| \leq c \| \Psi \|_{\mathcal{H}_{+1}} \| \Phi \|_{\mathcal{H}_{-1}}.
\]

**Proof.** (1) \( \iff \) (2):

Observe that if \( \lambda > 0 \) then \( (\lambda S + 1)^{-1} \mathcal{D}(S^{1/2}) \subset \mathcal{D}(S^{3/2}) \). Assume (1) and let us prove (2) for \( \Psi, \Phi \in \mathcal{D}(S^{1/2}) \). Using (1) with \( \Psi = (\lambda S + 1)^{-1} \Psi \in \mathcal{D}(S^{3/2}) \) and \( \Phi = (\lambda S + 1)^{-1} \Phi \in \mathcal{D}(S^{3/2}) \), we obtain

\[
\| \langle S \Psi, A \Phi \rangle - \langle A \Psi, S \Phi \rangle \| \leq c \| (\lambda S + 1)^{-1} \Psi \|_{\mathcal{H}_{+1}} \times \| (\lambda S + 1)^{-1} \Phi \|_{\mathcal{H}_{-1}}. \tag{49}
\]

It is easy to see that the right hand side of (49) is bounded by \( c \| \Psi \|_{\mathcal{H}_{+1}} \| \Phi \|_{\mathcal{H}_{-1}} \). Thus, we obtain (2). Now, to prove (2) \( \Rightarrow \) (1), we observe that \( (\lambda S + 1) \mathcal{D}(S^{3/2}) \subset \mathcal{D}(S^{1/2}) \) and use (2) with \( \Psi_{\lambda} = (\lambda S + 1) \Psi \in \mathcal{D}(S^{1/2}) \), \( \Phi_{\lambda} = (\lambda S + 1) \Phi \in \mathcal{D}(S^{1/2}) \) such that \( \Psi, \Phi \in \mathcal{D}(S^{3/2}) \). Therefore, we get for \( \lambda > 0 \)

\[
\| \langle S \Psi, A \Phi \rangle - \langle A \Psi, S \Phi \rangle \| \leq c \| \Psi_{\lambda} \|_{\mathcal{H}_{+1}} \times \| \Phi_{\lambda} \|_{\mathcal{H}_{-1}}. \tag{50}
\]
We aim to solve the following abstract non-autonomous Schrödinger equation
\[ A(\lambda S + 1)(\lambda S + 1)^{-1} = A\lambda S(\lambda S + 1)^{-1} + A(\lambda S + 1)^{-1}, \]
since \(\lambda S(\lambda S + 1)^{-1} \in L(\mathcal{H}_+, \mathcal{H}_-)\) and \((\lambda S + 1)^{-1} \in L(\mathcal{H}_+, \mathcal{H}_+)\). Therefore, since \((\lambda S + 1)^{-1}\Psi, \Phi \in \mathcal{H}_+\) and \((\lambda S + 1)^{-1}\Phi \in \mathcal{H}_+, \mathcal{H}_-\), the following computation is justified
\[ \langle (\lambda S + 1)^{-1}\Psi, A\Phi \rangle - \langle A\Psi,(\lambda S + 1)^{-1}\Phi \rangle = \langle (\lambda S + 1)^{-1}\Psi, A(\lambda S + 1)(\lambda S + 1)^{-1}\Phi \rangle - \langle A(\lambda S + 1)(\lambda S + 1)^{-1}\Psi,(\lambda S + 1)^{-1}\Phi \rangle. \]

So, this shows the equivalence of the statements (2) and (3). ■

C Non-autonomous Schrödinger equation

Consider the Hilbert rigging
\[ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-. \]
This means that \(\mathcal{H}\) is a Hilbert space with an inner product \((.,.)_{\mathcal{H}}\) and \(\mathcal{H}_+\) is a dense subspace of \(\mathcal{H}\) which is itself a Hilbert space with respect to another inner product \((.,.)_{\mathcal{H}_+}\) such that
\[ ||u||_{\mathcal{H}} := \sqrt{(u,u)_{\mathcal{H}}} \leq ||u||_{\mathcal{H}_+} := \sqrt{(u,u)_{\mathcal{H}_+}} \quad \forall u \in \mathcal{H}_+. \]
The Hilbert space \(\mathcal{H}_-\) is defined as the completion of \(\mathcal{H}\) with respect to the norm
\[ ||u||_{\mathcal{H}_-} := \sup_{f \in \mathcal{H}_+, ||f||_{\mathcal{H}_+} = 1} |(f,u)_{\mathcal{H}}|. \]
This extends by continuity the inner product \((.,.)_{\mathcal{H}}\) to a sesquilinear form on \(\mathcal{H}_- \times \mathcal{H}_+\) satisfying
\[ |(u,\xi)_{\mathcal{H}}| \leq ||u||_{\mathcal{H}_-} ||\xi||_{\mathcal{H}_+} \quad \forall u \in \mathcal{H}_+, \forall \xi \in \mathcal{H}_-. \]
Furthermore, we have
\[ ||u||_{\mathcal{H}_-} = \sup_{\xi \in \mathcal{H}_-, ||\xi||_{\mathcal{H}_+} = 1} |(u,\xi)_{\mathcal{H}}|. \]

Let \(I\) be a closed interval of \(\mathbb{R}\) and let \((A(t))_{t \in I}\) denote a family of self-adjoint operators on \(\mathcal{H}\) such that \(\mathcal{D}(A(t)) \cap \mathcal{H}_+\) is dense in \(\mathcal{H}_-\) and \(A(t)\) are continuously extendable to bounded operators in \(L(\mathcal{H}_+, \mathcal{H}_-\)) and satisfies the following abstract non-autonomous Schrödinger equation
\[
\begin{cases}
  i\partial_t u = A(t)u, & t \in I \\
  u(t=0) = u_0,
\end{cases}
\]
where \(u_0 \in \mathcal{H}_+\) is given and \(t \mapsto u(t) \in \mathcal{H}_+\) is the unknown. This is a particular case of the more general topic of solving non-autonomous Cauchy problems where \(-iA(t)\) is infinitesimal generators of \(C_0\)-semigroups (see [Si], [Ki]). We provide here a useful result (Theorem C.2) which follows from the work of Kato [Ka].

Definition C.1 We say that the map
\[ I \times I \ni (t,s) \rightarrow U(t,s) \]
is a unitary propagator of the problem (53) iff:
(a) \(U(t,s)\) is unitary on \(\mathcal{H}_+\),
(b) \(U(t,t) = 1\) and \(U(t,s)U(s,r) = U(t,r)\) for all \(t,s,r \in I\),
(c) The map \(t \mapsto U(t,s)\) belongs to \(C^1(I, L(\mathcal{H}_+)) \cap C^1(I, L(\mathcal{H}_+, \mathcal{H}_-))\) and satisfies
\[ i\partial_t U(t,s)\psi = A(t)U(t,s)\psi, \quad \forall \psi \in \mathcal{H}_+, \forall t,s \in I. \]
Here $C^k(I, \mathcal{B})$ denotes the space of $k$-continuously differentiable $\mathcal{B}$-valued functions where $\mathcal{B}$ is endowed with the strong operator topology.

**Theorem C.2** Let $I$ be a compact interval and let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be a Hilbert rigging with $(A(t))_{t \in I}$ a family of self-adjoint operators on $\mathcal{H}$ as above satisfying:

(i) $I \ni t \mapsto A(t) \in L(\mathcal{H}_+, \mathcal{H}_-)$ is norm continuous.
(ii) $\mathbb{R} \ni t \mapsto e^{itA(t)} \in L(\mathcal{H}_+)$ is strongly continuous.
(iii) There exists a family of Hilbertian norms $\{\|\cdot\|_{\mathcal{H}}\}_{t \in I}$ on $\mathcal{H}$ such that:

$$\exists c > 0, \forall \psi \in \mathcal{H}_+ : \|\psi\|_t \leq e^{c|t-s|}\|\psi\|_s \quad \text{and} \quad \|e^{itA(t)}\psi\|_t \leq e^{c|t|}\|\psi\|_t.$$  

Then the non-autonomous Cauchy problem (53) admits a unique unitary propagator $U(t,s)$. Moreover, the following estimate holds

$$\forall \psi \in \mathcal{H}_+, \quad \|U(t,s)\psi\|_t \leq e^{2c|t-s|}\|\psi\|_s.$$  

**Proof.** We follow the same strategy as in [Ka] and split the proof into three steps. We assume, for reading convenience, that the interval $I$ is of the form $[0,T]$, $T > 0$ however the proof works exactly in the same way for any compact interval. Remark also that there is no restriction if we assume that $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_0$.

**Propagator approximation:**

Let $(t_0, \cdots, t_n)$ be a regular partition of the interval $I$ with

$$t_j = \frac{jT}{n}, \quad j = 0, \cdots, n.$$  

Consider the sequence of operator-valued step functions defined by

$$A_n(t) := A(T)1_{[t]}(t) + \sum_{j=0}^{n-1} A(t_j)1_{[t_j, t_{j+1}]}(t),$$  

for any $n \in \mathbb{N}^+$ and $t \in I$. Assumption (i) ensures that

$$\lim_{n \to \infty} \|A_n(t) - A(t)\|_{L(\mathcal{H}_+, \mathcal{H}_-)} = 0,$$  

uniformly in $t \in I$. We now construct an approximating unitary propagator $U_n(t,s)$ as follows:

$$U_n(t,s) = \begin{cases} 
\left\{ \begin{array}{ll}
    e^{i\int_{t_j}^{t} A(t) dt} & \text{if } t_j \leq t, s \leq t_{j+1} \\
    e^{i\int_{t_j}^{s} A(t) dt} & \text{if } t_j < s \leq t_{j+1} \leq t < t_{j+1} \\
    e^{i\int_{t_j}^{t} A(t) dt} & \text{if } t < s \leq t \leq t_{j+1} \text{ and } j < l
\end{array} \right. & \text{if } j = 0, \cdots, n - 1 \text{ and } l = 1, \cdots, n \text{ with } j < l.
\end{cases}$$

By definition, the operators $U_n(t,s)$ are unitary on $\mathcal{H}$ for $t,s \in I$ and satisfy

$$U_n(t,t) = 1, \quad U_n(t,s)^* = U_n(s,t).$$

Moreover, one can first check that

$$U_n(t,s)U_n(s,r) = U_n(t,r) \text{ for } r \leq s \leq t, \text{ with } t,s,r \in I$$

and then extend it for any $(t,s,r) \in I^3$ with the help of (55). Therefore, $U_n(t,s)$ satisfy the properties (a)-(b) of Definition C.1. Again by (54) and assumptions (i)-(ii) we have

$$i\partial_t U_n(t,s)\psi = A_n(t)U_n(t,s)\psi \quad \text{and} \quad -i\partial_s U_n(t,s)\psi = U_n(t,s)A_n(s)\psi,$$  

for any $\psi \in \mathcal{H}_+$ and any $t,s \neq t_j, j = 0, \cdots, n$.

**Convergence of the approximation:**

Assumption (iii) implies that

$$\|e^{-it_0A(t_0)} \cdots e^{-it_1A(t_1)}\|_I \leq e^{cT} e^{c|t_1 + \cdots + t_n|} \|\psi\|_0.$$  

29
and
\[ \|e^{-i\lambda(\alpha_1)\cdots e^{-i\lambda(\alpha_n)}\psi}\|_0 \leq e^{\epsilon T} e^{(\alpha_1 + \cdots + \alpha_n)}\|\psi\|_r, \]
for any \( s_j \geq 0, j = 1, \ldots, n \). Hence, using the equivalence of the norms \( \|.\|_0 = \|.\|_{\mathcal{H}_0} \) and \( \|.|\|_T \) one shows the existence of \( M > 0 \) (\( M = e^{2\epsilon T} \)) such that
\[ \|U_n(t,s)\|_{\mathcal{L}^{\infty(H)}} \leq M e^{\epsilon|t-s|}. \] (57)

Furthermore, the same argument above yields
\[ \|U_n(t,s)\|_t \leq e^{2(\epsilon|t-s|+T/n)}\|\psi\|_s. \]  

Using \( \mathcal{L}^{\infty(H)} \) we obtain for any \( \psi \in \mathcal{H}_+ \)
\[ \partial_t [U_n(t,r)U_m(r,s)\psi] = i U_n(t,r)[A_n(r) - A_m(r)]U_m(r,s)\psi, \] 
for \( r \neq \frac{T}{m}, r \neq \frac{T}{m} \) with \( j = 1, \ldots, \max(n,m) \). Integrating \( \mathcal{L}^{\infty(H)} \) we get the identity
\[ U_m(t,s)\psi - U_n(t,s)\psi = i \int_s^t U_n(t,r)[A_n(r) - A_m(r)]U_m(r,s)\psi dr. \]

Now \( \mathcal{L}^{\infty(H)} \) yields
\[ \|U_m(t,s) - U_n(t,s)\|_{\mathcal{L}^{\infty(H)}} \leq M^2|t-s|e^{2\epsilon|t-s|} \sup_{r \in I} \|A_n(r) - A_m(r)\|_{\mathcal{L}^{\infty(H)}}. \]  

Therefore, for any \( t,s \in I \), the sequence \( U_n(t,s) \) converges in norm to a bounded linear operator \( U(t,s) \in \mathcal{L}^{\infty(H)} \). Since \( U_n(t,s) \) are norm bounded operators on \( \mathcal{H}_+ \), uniformly in \( n \), it follows by \( \mathcal{L}^{\infty(H)} \) that they converge strongly to an operator in \( \mathcal{L}^{\infty(H)} \) continuously extending \( U(t,s) \). Moreover, this strong convergence yields
\[ \lim_{n \to \infty} (\phi, U_n(t,s)\psi)_{\mathcal{H}} = (\phi, U(t,s)\psi)_{\mathcal{H}} \quad \forall \psi \in \mathcal{H}_+, \forall \phi \in \mathcal{H}_+. \]

where \((.,.)_{\mathcal{H}}\) is the continuous extension of the inner product of \( \mathcal{H}_+ \) to the rigged Hilbert spaces \( \mathcal{H}_{\pm} \). Thus, using \( \mathcal{L}^{\infty(H)} \), we obtain
\[ \|(\phi, U(t,s)\psi)_{\mathcal{H}}\|_{\mathcal{H}} \leq Me^{\epsilon|t-s|}\|\phi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}_+}. \]

Hence, it is easy to see by \( \mathcal{L}^{\infty(H)} \) that
\[ \|U(t,s)\|_{\mathcal{L}^{\infty(H)}} \leq Me^{\epsilon|t-s|}. \]  

A similar argument yields
\[ \|U(t,s)\|_{\mathcal{L}^{\infty(H)}} \leq 1. \]  

Now, since \( U_n(t,s) \) satisfy part (b) of Definition \( \mathcal{C.1} \) we easily conclude that
\[ U(t,t) = 1, \quad U(t,r)U(r,s) = U(t,s), \quad t,s,r \in I, \] 
by strong convergence in \( \mathcal{L}^{\infty(H)} \). Furthermore, combining \( \mathcal{C.1} \) and \( \mathcal{L}^{\infty(H)} \) we show the unitarity of \( U(t,s) \) on \( \mathcal{H}_+ \). Thus, we have proved that \( U(t,s) \) satisfy (a)-(b) of Definition \( \mathcal{C.1} \).

For any \( \psi \in \mathcal{H}_+ \), the continuity of the map \( I \ni t \mapsto U_n(t,s)\psi \in \mathcal{H}_- \) follows from the definition of \( U_n(t,s) \). Now, we prove
\[ \lim_{n \to \infty}(\phi, U(t,s)\psi)_{\mathcal{H}} = (\phi, \psi)_{\mathcal{H}} \quad \forall \psi \in \mathcal{H}_+, \forall \phi \in \mathcal{H}_+, \]
by applying an \( \epsilon/3 \) argument when writing
\[ \|(\phi, U(t,s)\psi) - (\phi, \psi)_{\mathcal{H}}\|_{\mathcal{H}} \leq \|\phi - \phi_{\alpha}\|_{\mathcal{H}} \|U(t,s)\psi\|_{\mathcal{H}_+} + \|\phi_{\alpha} [U(t,s) - U_n(t,s)] \psi\|_{\mathcal{H}_+} \]
\[ + \|\phi_{\alpha} [U_n(t,s) - 1] \psi\|_{\mathcal{H}_+} + \|\phi - \phi_{\alpha}\|_{\mathcal{H}_+} \|\psi\|_{\mathcal{H}_+}, \]
where $\phi_K \to \phi$ in $\mathcal{H}_-$. Therefore, by the duality $(\mathcal{H}_+)' \simeq \mathcal{H}_-$, we get the weak limit
\[ w-lim_{t-s} U(t,s) = 1, \]
in $\mathcal{L}(\mathcal{H}_+)$. Now, observe that when $t \to s$ we can show by (57) that
\[ \limsup_{t-s} \| U(t,s) \psi \|^2 \leq \| \psi \|^2. \]
So, we conclude that
\[ \limsup_{t-s} \| U(t,s) \psi - \psi \|^2 \leq \limsup_{t-s} \left( \| \psi \|^2 + \| U(t,s) \psi \|^2 - 2\text{Re}(\psi, U(t,s) \psi) \right) = 0. \]
This gives the continuity of $I \ni t \mapsto U(t,s) \psi \in \mathcal{H}_+$ since we have in $\mathcal{H}_+$
\[ s - \lim_{t-r} U(t,s) = s - \lim_{t-r} U(t,r)U(r,s) = U(r,s). \]
Now, we have for $\psi \in \mathcal{H}_+$ as identity in $\mathcal{H}_-$
\[ e^{-itA(s)} \psi = \psi - iA(s) \int_0^t e^{-irA(s)} dr, \] (63)
since this holds first for $\psi \in \mathcal{D}(A(s)) \cap \mathcal{H}_+$ and then extends by density of $\mathcal{D}(A(s)) \cap \mathcal{H}_+$ in $\mathcal{H}_+$. By (63) we have
\[ \| e^{-itA(s)} \psi - \psi + iA(s) \int_0^t e^{-irA(s)} dr \| \leq \frac{1}{t} \| A(s) \| \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-) \int_0^t \| e^{-irA(s)} \psi - \psi \| \| A(s) \| \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-) dr \]
and hence using assumption (ii), we show the differentiability of $\tau \mapsto e^{-i\tau A(s)} \psi$ for $\psi \in \mathcal{H}_+$. By differentiating $e^{-i(t-r)A(s)} U_m(r,s) \psi$ with $\psi \in \mathcal{H}_+$ and then integrating w.r.t. $r$, we get
\[ U_m(t,s) \psi - e^{-i(t-s)A(s)} \psi = i \int_s^t e^{-i(t-r)A(s)} [A(s) - A_m(r)] U_m(r,s) \psi dr. \]
Letting $m \to \infty$ in the latter identity and estimating as in (60), one obtains
\[ \| U(t,s) \psi - e^{-i(t-s)A(s)} \psi \| \| \mathcal{H}_- \leq M^2 e^{2c|t-s|} \int_s^t \| A(s) - A(r) \| \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-) dr \| \psi \| \| \mathcal{H}_-. \]
Using the fact that
\[ \lim_{t \to s} \frac{1}{t-s} \int_s^t \| A(s) - A(r) \| \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-) dr = 0 \] and \[ \lim_{t \to s} e^{-i(t-s)A(s)} \psi - \psi = -iA(s) \psi \]
it holds that
\[ \lim_{t \to s} \left\| \frac{U(t,s) \psi - \psi}{t-s} + iA(s) \psi \right\| \| \mathcal{H}_- = 0. \]
Thus, we obtain with the help of (62)
\[ i\partial_t U(s,r) \psi = \lim_{t \to s} U(t,s) U(s,r) \psi - U(s,r) \psi = A(s) U(s,r) \psi, \]
for any $\psi \in \mathcal{H}_+$ and any $s, r \in I$. Hence we have proved the existence of a unitary propagator $U(t,s) \psi$ for the non-autonomous Cauchy problem (53).

**Uniqueness:**
Suppose that $V(t,s)$ is a unitary propagator for (53). By differentiating $U_n(t,r)V(r,s) \psi, \psi \in \mathcal{H}_+$ with respect to $r$ we get
\[ V(t,s) \psi - U_n(t,s) \psi = i \int_s^t U_n(t,r) A_n(r) - A(r) V(r,s) \psi. \]
Using a similar estimate as (60) we obtain
\[ \|V(t,s)\psi - U_n(t,s)\psi\|_{\mathcal{H}_+} \leq M e^{[t-s]} \sup_{r \in [t,s]} \|V(r,s)\|_{\mathcal{L}(\mathcal{H}_+)} \int_s^t \|A(r) - A_n(r)\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_+)} dr \|\psi\|_{\mathcal{H}_+} \]
and since the r.h.s. vanishes when \( n \to \infty \) we conclude that \( V(t,s) = U(t,s) \).
Finally, the uniform boundedness principle, equivalence of norms \( \|\cdot\|_t, \|\cdot\|_{\mathcal{H}_+} \) and the inequality (58) give us the claimed estimate,
\[ \forall \psi \in \mathcal{H}_+, \quad \|U(t,s)\psi\|_t \leq \liminf_{n \to \infty} \|U_n(t,s)\psi\|_t \leq e^{2[t-s]}\|\psi\|_s . \]

**Remark C.3** It also follows that \( (t,s) \mapsto U(t,s) \in \mathcal{L}(\mathcal{H}_+) \) is jointly strongly continuous.

In the following we provide a more effective formulation of the above result (Theorem C.2) which appears as a time-dependent version of the Nelson commutator theorem (see, e.g., [NI, RS] and Theorem B.2).

We associate to each family of self-adjoint operators \( \{S(t)\}_{t \in I} \) on \( \mathcal{H} \) such that \( S \geq 1, S(t) \geq 1 \) and \( \mathcal{D}(S(t)^{1/2}) = \mathcal{D}(S^{1/2}) \) for any \( t \in I \), a Hilbert rigging \( \mathcal{H}_{\pm 1} \) defined as the completion of \( \mathcal{D}(S^{\pm 1/2}) \) with respect to the inner product
\[ \langle \psi, \phi \rangle_{\mathcal{H}_{\pm 1}} = \langle S^{1/2} \psi, S^{1/2} \phi \rangle_{\mathcal{H}} . \]

**Corollary C.4** Let \( I \subset \mathbb{R} \) be a closed interval and let \( \{S(t)\}_{t \in I} \) be a family of self-adjoint operators on a Hilbert space \( \mathcal{H} \) such that:
- \( S \geq 1 \) and \( S(t) \geq 1 \), \( \forall t \in I \),
- \( \mathcal{D}(S(t)^{1/2}) = \mathcal{D}(S^{1/2}) \), \( \forall t \in I \), and consider the associated Hilbert rigging \( \mathcal{H}_{\pm 1} \) given by (64).

Let \( \{A(t)\}_{t \in I} \) be a family of symmetric bounded operators in \( \mathcal{L}(\mathcal{H}_{\pm 1},\mathcal{H}_{\pm 1}) \) satisfying:
- \( t \in I \mapsto A(t) \in \mathcal{L}(\mathcal{H}_{\pm 1},\mathcal{H}_{\pm 1}) \) is norm continuous.

Assume that there exists a continuous function \( f : I \to \mathbb{R}_+ \) such that for any \( t \in I \), we have:
- (i) for any \( \psi \in \mathcal{D}(S(t)^{1/2}), \)
  \[ |\partial_t \langle \psi, S(t)\psi \rangle| \leq f(t) \|S(t)^{1/2}\psi\|^2 ; \]
- (ii) for any \( \Phi, \Psi \in \mathcal{D}(S(t)^{3/2}), \)
  \[ |\langle S(t)^{3/2} \Phi, A(t)\Psi \rangle - \langle A(t)\Phi, S(t)^{3/2} \Psi \rangle| \leq f(t) \|S(t)^{1/2}\Psi\| \|S(t)^{1/2}\Phi\| . \]

Then the non-autonomous Cauchy problem (53) admits a unique unitary propagator \( U(t,s) \). Moreover, we have
\[ \|S(t)^{1/2}U(t,s)\psi\| \leq e^{2\int_s^t f(\tau) d\tau} \|S(s)^{1/2}\psi\| . \]

In addition, if we have \( c_1, c_2 > 0 \) such that \( c_1 S \leq S(t) \leq c_2 S \) for \( t \in I \), then there exists \( c > 0 \) such that
\[ \|U(t,s)\|_{\mathcal{L}(\mathcal{H}_{\pm 1})} \leq e^{2\int_{s}^{t} f(\tau) d\tau} , \quad \forall t \in I . \]

**Proof.** First observe that the operator \( A(t) \) satisfies the hypothesis of Nelson’s commutator theorem (Theorem B.2) for any \( t \in I \), hence, we conclude that \( A(t) \) is essentially self-adjoint on \( \mathcal{D}(S(t)^{3/2}) \) which is dense in \( \mathcal{H}_{\pm 1} \). We keep the same notation for its closure. Moreover, the unitary group \( e^{itA(t)} \) preserves \( \mathcal{H}_{\pm 1} \) and we have the estimate
\[ \|S(t)^{1/2}e^{itA(t)}\psi\|_{\mathcal{H}} \leq e^{f(t)|\tau|} \|\psi\|_{\mathcal{H}} . \]
Now, observe that \( t \mapsto e^{-\eta A(t)} \psi \in \mathcal{H}_+ \) is weakly continuous for any \( \psi \in \mathcal{H}_+ \). This holds using a \( \eta/3 \)-argument with the help of the estimate
\[
\left| \langle f, (e^{-\eta A(s)} - 1) \psi \rangle \right| \leq (1 + e^{c(|t|+1)}) \| f - f \xi \|_{\mathcal{H}_-} \| \psi \|_{\mathcal{H}_+} + \left| \langle (e^{-\eta A(s)} - 1) f \xi, \psi \rangle \right|
\]
where \( f \xi \in \mathcal{H}_+ \) is a sequence convergent to \( f \) in \( \mathcal{H}_- \) and \( t \) is near 0. Since strong and weak continuity of the group of bounded operators \( e^{-\eta A(t)} \) in \( \mathcal{L}(\mathcal{H}_-) \) are equivalent, we conclude that assumption (ii) of Theorem C.2 holds true.

By assumption (ii), we also have
\[
\frac{d}{dt} \| S(t) \|_{\mathcal{H}_+}^2 \leq f(t) \| S(t) \|_{\mathcal{H}_+}^2.
\]

Hence, by Gronwall’s inequality we have
\[
\| S(t) \|_{\mathcal{H}_+}^2 \leq e^{\int_0^t f(\tau) d\tau} \| S(0) \|_{\mathcal{H}_+}^2, \quad \forall t, s \in I.
\]

Now, we use Theorem C.2 with the Hilbert rigging
\[
\mathcal{H}_+ = \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- = \mathcal{H}_-
\]
and the family of equivalent norms on \( \mathcal{H}_+ \) given by
\[
\| \psi \|_t := \| S(t) \|_{\mathcal{H}_+}^2.
\]

Indeed, assumptions (i)-(iii) of Theorem C.2 are satisfied in any compact subinterval of \( I \) with the help of (67)-(66). Therefore, we obtain existence and uniqueness of a unitary propagator \( U(t,s) \) of the Cauchy problem (55) in the whole interval \( I \) with the following estimate
\[
\| U(t,s) \|_t \leq e^{2|t-s| \max_{t \in \Delta(t,s)} f(\tau)} \| \psi \|_s,
\]
for any \( t, s \in I \) and where \( \Delta(t,s) \) stands for the interval of extremities \( t, s \).

Using the multiplication law of the propagator, we obtain for any partition \( (t_0, \cdots, t_n) \) of the interval \( \Delta(t,s) \) the inequality
\[
\| U(t,s) \|_t \leq \prod_{j=0}^{n-1} e^{2|\tau_{j+1}-\tau_j| \max_{t \in \Delta_j} f(\tau)} \| \psi \|_s,
\]
where \( \Delta_j \) are the subintervals \([t_j, t_{j+1})\). Since \( f \) is continuous, by letting \( n \to \infty \), we get
\[
\| U(t,s) \|_t \leq e^{2 \int_{\Delta(t,s)} f(\tau) d\tau} \| \psi \|_s.
\]

Finally, the assumption \( c_1 S \leq S(t) \leq c_2 S \) for \( t \in I \), allows to involve the norm \( \| \cdot \|_{\mathcal{H}_+} \). Thus we have
\[
\| U(t,s) \|_{\mathcal{H}_+} \leq \frac{1}{\sqrt{c_1}} \| U(t,s) \|_t \leq \frac{1}{\sqrt{c_1}} e^{2 \int_{\Delta(t,s)} f(\tau) d\tau} \| \psi \|_s \leq \sqrt{\frac{c_2}{c_1}} e^{2 \int_{\Delta(t,s)} f(\tau) d\tau} \| \psi \|_{\mathcal{H}_+}.
\]

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