Viscosity Solutions to Path-Dependent HJB Equation and Applications*

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Abstract

In this article, the notion of viscosity solution is introduced for the path-dependent Hamilton-Jacobi-Bellman (PHJB) equations associated with the optimal control problems for path-dependent stochastic differential equations. We identify the value functional of the optimal control problems as unique viscosity solution to the associated PHJB equations. Applications to backward stochastic Hamilton-Jacobi-Bellman equations are also given.

Key Words: Path-dependent Hamilton-Jacobi-Bellman equations; Viscosity solution; Optimal control; Path-dependent stochastic differential equations; Backward stochastic Hamilton-Jacobi-Bellman equations

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1 Introduction

In this paper, we consider the following controlled path-dependent stochastic differential equations:

$$
\begin{cases}
    dX_{\gamma,\tau}^{s,\lambda}(s) = F(X_{s}^{\gamma,\lambda}, u(s))ds + G(X_{s}^{\gamma,\lambda}, u(s))dW(s), \quad s \in [t, T], \\
    X_{t}^{\gamma,\lambda} = \gamma_{t} \in \Lambda_{t}.
\end{cases}
$$

(1.1)

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In the equations above, $\Lambda_t$ denotes the set of all continuous $\mathbb{R}^d$-valued functions defined over $[0, t]$ which vanish at time zero, and $\{W(t), t \geq 0\}$ is an $n$-dimensional standard Wiener process; the unknown $X^{\gamma,t,u}(s)$, representing the state of the system, is an $\mathbb{R}^d$-valued process; the control process $u$ takes values in a compact metric space $U$ and the coefficients $F$ and $G$ are assumed to satisfy Lipschitz conditions with respect to appropriate norms. Under suitable assumptions, there exists a unique adapted process $X^{\gamma,t,u}(s)$, $s \in [t, T]$, solution to (1.1).

We wish to minimize a cost functional of the form:

$$ J(\gamma_t, u) := Y^{\gamma_t,u}(t), \quad t \in [0, T], \quad \gamma_t \in \Lambda, $$

over all admissible controls $U$, where the process $Y^{\gamma_t,u}$ is defined by backward stochastic differential equation (BSDE):

$$ Y^{\gamma_t,u}(s) = \phi(X_T^{\gamma_t,u}) + \int_s^T q(X_{\sigma}^{\gamma_t,u}, Y^{\gamma_t,u}(\sigma), Z^{\gamma_t,u}(\sigma), u(\sigma))d\sigma $$

$$ - \int_s^T Z^{\gamma_t,u}(\sigma)dW(\sigma), \quad a.s., \quad \text{all} \ s \in [t, T], $$

Here $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$, $q$ and $\phi$ are given real functions on $\Lambda \times \mathbb{R} \times \mathbb{R}^d \times U$ and $\Lambda_T$, respectively. We define the value functional of the optimal control problem as follows:

$$ V(\gamma_t) := \text{ess sup}_{u \in U} Y^{\gamma_t,u}(t), \quad \gamma_t \in \Lambda. $$

The goal of this article is to characterize this value functional $V$. We assume that $q$ and $\phi$ satisfy suitable conditions and consider the following path-dependent Hamilton-Jacobi-Bellman (PHJB) equation:

$$ \left\{ \begin{array}{l}
\partial_t V(\gamma_t) + H(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) = 0, \quad t \in [0, T), \quad \gamma_t \in \Lambda, \\
V(\gamma_T) = \phi(\gamma_T), \quad \gamma_T \in \Lambda_T;
\end{array} \right. $$

where

$$ H(\gamma_t, r, p, l) = \sup_{u \in U}[(p, F(\gamma_t, u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr}[lG(\gamma_t, u)G^T(\gamma_t, u)] $$

$$ + q(\gamma_t, r, G^T(\gamma_t, u)p, u)], \quad (\gamma_t, r, p, l) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \Gamma(\mathbb{R}^d). $$

Here we let $G^T$ the transpose of the matrix $G$, $\Gamma(\mathbb{R}^d)$ the set of all $(d \times d)$ symmetric matrices and $(\cdot, \cdot)_{\mathbb{R}^d}$ the scalar product of $\mathbb{R}^d$. The definitions of $\partial_t, \partial_x, \partial_{xx}$ will be introduced in subsequent section.

The primary objective of this article is to develop the concept of the viscosity solution to PHJB equations on the space of continuous paths (see Definition 4.2).
for details). We shall show the value functional $V$ defined in (1.4) is the unique viscosity solution to the PHJB equation given in (1.5).

It is well known that a Markovian backward stochastic differential equation (BSDE) is related with a semi-linear parabolic partial differential equation (PDE), see Pardoux and Peng [29], and a forward and backward stochastic differential equation (FBSDE) is associated to a quasi-linear PDE, see, for example, Ma, Protter and Yong [26], and Pardoux and Tang [30]. Furthermore, the second order BSDE (2BSDE) is associated to a fully nonlinear PDE, see, for example, Cheridito, Soner, Touzi and Victoir [6] and Soner, Touzi and Zhang [38]. For the non-Markovian case, in ICM 2010, Peng [33] pointed out that a non-Markovian BSDE is a path-dependent PDE (PPDE). Dupire in his work [13] introduced horizontal and vertical derivatives in the path space and provided a functional Itô formula under his definition (see Cont and Fournié [7], [8] for a more general and systematic research). In view of Dupire’s functional Itô formula, it is very natural to view a BSDE to a semi-linear PPDE. However, a PPDE, even for the simplest heat equation, rarely has classical solution. We refer to Peng and Wang [35] for some sufficient conditions for a semi-linear PPDE to admit a classical solution. Therefore, generalized solution of general PPDEs are demanding, and it has to be developed.

The notation of viscosity solutions for standard second order Hamilton-Jacobi-Bellman (HJB) equations has been well developed. We refer to the survey paper of Crandall, Ishii and Lions [9] and the monographs of Fleming and Soner [19] and Yong and Zhou [43]. In references [20], [21], [22], [23] and [40], the notion of viscosity solutions were extended to Hilbert spaces by using a limiting argument based on the existence of a countable basis. Because of the lack of local compactness of the path space $\Lambda$, the standard techniques for the proof of the comparison theorem, which rely heavily on compactness arguments, are not applicable in our case.

The notion of viscosity solutions for PPDEs has been studied by several authors in recent years (see [14], [15], [16], [17], [34] and [42]). However, to the best of our knowledge, none of these results are directly applicable to our case. Peng [34] proposes a notion of viscosity solutions for nonlinear PPDEs on right continuous paths. By the left frozen maximization principle and the classical viscosity solution theory for PDE, the comparison principle is proved in the sense of his definition. However, the value functional defined in (1.4) is not guaranteed to be a viscosity solution of (1.5) under their definition. In Ekren et al. [14], the authors introduce a notion of viscosity solutions for semi-linear PPDEs in the space of continuous paths in terms of a nonlinear expectation. The existence and uniqueness of their viscosity solutions is given by Peron’s approach. In the subsequent works, Ekren, Touzi, and Zhang (see for details [15], [16] and [17]) study the fully nonlinear PPDEs under their previous notion of viscosity solution. The results obtained in these
references only apply when the Hamilton function $H$ is uniformly nondegenerate and the diffusion coefficient $G$ is path-invariant. In Tang and Zhang [42], a notion of viscosity solutions for path-dependent Bellman equations is proposed by restricting semi-jet on a space of $\alpha$-Hölder continuous paths, which is a compact subset of $\Lambda$. Using Dupire’s functional Itô calculus, the authors identify the value functional of the optimal control problems as unique viscosity solution to the path-dependent Bellman equations. In this paper, the authors require that the coefficients of both the system and the cost functional are bounded, and the diffusion coefficient $G$ is non-degenerate.

We follow the approach in [42] and define the viscosity solution on the space $C^\alpha_{\mu, M_0, t_0}$ without involving any nonlinear expectation (see Definition 4.2 for details). Taking advantages of this innovation, we succeed at proving the uniqueness of viscosity solution to the PHJB equation given in (1.5). Here we combine the techniques from [34] with the classical methodology of Crandall, Ishii and Lions [9], and it does not include the approximating processes of parameterized state-dependent PDEs, then the Mandatory conditions given in [17] or [42], for example, uniformly nondegenerate, path-invariant and bounded, can be removed.

In the proof procedure of the uniqueness of viscosity solution, our auxiliary function needs to include the term $||\gamma_t||^2_H := \int_0^t |\gamma_t(s)|^2 ds$. Therefore, we only prove uniqueness when the coefficients $F, G$ and $q$ satisfy linear growth condition under norm $|| \cdot ||^2_H$. We hope to overcome this limitation of our approach in future work.

Backward stochastic PDE (BSPDE) is another interesting topic. Peng [31] obtained the existence and uniqueness theorem for the solution to backward stochastic HJB equations in a triple. The relationship between FBSDEs and a class of semi-linear BSPDEs was established in Ma and Yong [27]. For Sobolev and classical solution of BSPDE, we refer to Zhou [44], Tang [41], Du and Meng [10], Du and Tang [11], Du, Tang and Zhang [12] and Qiu and Tang [36]. For viscosity solution of SPDE, we refer to Lions [24], [25], Buckdahn and Ma [2], [3], [4], [5] and Boufoussi et al. [1]. As an application of our results, we given the definition of viscosity solution to stochastic HJB (SHJB) equations, and characterize the value functional of the optimal stochastic control problem as the unique viscosity solution to the associated SHJB equation.

The outline of this article is as follows. In the following section, we introduce the framework of [8] and [13], and preliminary results on BSDEs. In Section 3, we study the path-dependent stochastic optimal control problems and give the dynamic programming principle (DPP), which will be used in the following sections. We define classical and viscosity solutions to our PHJB equations and prove that the value functional $V$ defined by (1.4) is a viscosity solution to the PHJB equations (1.5) in Section 4. The uniqueness of viscosity solution for (1.5) is proven in section 5. Finally in Section 6 applications to SHJB are also given.
2 Preliminary work

Throughout this paper let \( T > 0 \) be a given finite maturity. For each \( t \in [0, T] \), define \( \Lambda_t := D([0, t]; \mathbb{R}^d) \) as the set of càdlàg \( \mathbb{R}^d \)-valued functions on \([0, t] \). For each \( \gamma \in \Lambda_t \), the value of \( \gamma \) at time \( s \in [0, T] \) is denoted by \( \gamma(s) \). The path of \( \gamma \) up to time \( t \) is denoted by \( \gamma_t \), i.e., \( \gamma_t := \{ \gamma(s), s \in [0, t] \} \in \Lambda_t \). We denote \( \Lambda = \bigcup_{t \in [0, T]} \Lambda_t \).

We define a norm and a metric on \( \hat{\Lambda} \) as follows: for any \( 0 \leq t \leq \bar{t} \leq T \) and \( \gamma_t, \bar{\gamma}_{\bar{t}} \in \hat{\Lambda} \),
\[
\| \gamma_t \|_0 := \sup_{0 \leq s \leq t} \left| \gamma_t(s) \right|, \quad d_{\infty}(\gamma_t, \bar{\gamma}_{\bar{t}}) := |t - \bar{t}| + \sup_{0 \leq s \leq \bar{t}} \left| \gamma_t(s) - \bar{\gamma}_{\bar{t}}(s) \right|.
\]
\[
(2.1)
\]

Then \( (\hat{\Lambda}, \| \cdot \|_0) \) is a Banach space, \( (\hat{\Lambda}, d_{\infty}) \) is a complete metric space. Following Dupire \[13\], we define spatial derivatives of \( u : \hat{\Lambda} \to \mathbb{R} \), if exist, in the standard sense: for the basis \( e_i \) of \( \mathbb{R}^d \), \( i = 1, 2, \ldots, d \),
\[
\partial_x u(\gamma_t) := \lim_{h \to 0} \frac{1}{h} \left[ u(\gamma_t^{he_i}) - u(\gamma_t) \right], \quad \partial_{x,x} u := \partial_{x_1}(\partial_{x_j} u), \quad i, j = 1, 2, \ldots, d.
\]
\[
(2.2)
\]

and the right time-derivative of \( u \), if exists, as:
\[
\partial_t u(\gamma_t) := \lim_{h \to 0, h > 0} \frac{1}{h} \left[ u(\gamma_t,t+h) - u(\gamma_t) \right], \quad t < T.
\]
\[
(2.3)
\]

For the final time \( T \), we define
\[
\partial_t u(\gamma_T) := \lim_{t < T, t \uparrow T} \partial_t u(\gamma_t).
\]

We take the convention that \( \gamma_t \) is column vector, but \( \partial_x u \) denotes row vector and \( \partial_{xx} u \) denotes \( d \times d \)-matrix.

**Definition 2.1.** Let \( u : \hat{\Lambda} \to \mathbb{R} \) be given.

(i) We say \( u \in C^0(\hat{\Lambda}) \) if \( u \) is continuous in \( \gamma_t \) under \( d_{\infty} \).

(ii) We say \( u \in C^0_b(\hat{\Lambda}) \subset C^0(\hat{\Lambda}) \) if \( u \) is bounded.

(iii) We say \( u \in C^{1,2}(\hat{\Lambda}) \subset C^0(\hat{\Lambda}) \) if \( \partial_t u, \partial_x u, \partial_{xx} u \) exist and are in \( C^0(\hat{\Lambda}) \).

(iv) We say \( u \in C^{1,2}_p(\hat{\Lambda}) \subset C^{1,2}(\hat{\Lambda}) \) if \( u \) and all of its derivatives grow in a polynomial way.
Let $\Lambda_t := \{ \gamma \in C([0, t], R^d) : \gamma(0) = 0 \}$ be the set of all continuous $R^d$-valued functions defined over $[0, t]$ which vanish at time zero, and $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$. Here and in the sequel, for notational simplicity, we use $0$ to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. Clearly, $\Lambda \subset \hat{\Lambda}$, and each $\gamma \in \Lambda$ can also be viewed as an element of $\hat{\Lambda}$. $(\Lambda_T, \| \cdot \|_0)$ is a Banach space, and $(\Lambda, d_{\infty})$ is a complete metric space. $u : \Lambda \to R$ and $\hat{u} : \hat{\Lambda} \to R$ are called consistent on $\Lambda$ if $u$ is the restriction of $\hat{u}$ on $\Lambda$. For every $\alpha \in (0, 1]$, $\mu > 0$, $t_0 \in [0, T]$ and $M_0 > 0$, we also define $C_{\mu, M_0}^{\alpha, t_0}$ by

$$C_{\mu, M_0}^{\alpha, t_0} := \left\{ \gamma_t \in \Lambda : t_0 \leq t, \| \gamma_t \|_0 \leq M_0 \text{ and } \sup_{0 \leq s < r \leq t} \frac{|\gamma_t(s) - \gamma_t(r)|}{|s - r|^\alpha} \leq \mu \right\}.$$  

We let $C_{\mu, M_0}^\alpha$ denote $C_{\mu, M_0}^{\alpha, t_0}$ when $t_0 = 0$.

**Definition 2.2.** Let $u : \Lambda \to R$ be given.

(i) We say $u \in C^0(\Lambda)$ if $u$ is continuous in $\gamma_t$ under $d_{\infty}$.

(ii) We say $u \in C^0_0(\Lambda) \subset C^0(\Lambda)$ if $u$ is bounded.

(iii) We say $u \in C^{1, 2}(\Lambda) \subset C^0(\Lambda)$ if there exists $\hat{u} \in C^{1, 2}(\hat{\Lambda})$ which is consistent with $u$ on $\Lambda$.

(iv) We say $u \in C^{1, 2}_p(\Lambda) \subset C^{1, 2}(\Lambda)$ if there exists $\hat{u} \in C^{1, 2}_p(\hat{\Lambda})$ which is consistent with $u$ on $\Lambda$.

Now we introduce the filtration of $\Lambda_T$. Let $\mathcal{F}_T := \mathcal{B}(\Lambda_T)$, the smallest Borel $\sigma$-field generated by Banach space $(\Lambda_T, \| \cdot \|_0)$. For any $t \in [0, T]$, define $\mathcal{F}_t := \sigma(\theta_t^{-1}(\mathcal{F}_T))$, where $\theta_t = \Lambda_T \to \Lambda_T$ is defined by

$$(\theta_t \gamma)(s) = \gamma(t \land s), \quad 0 \leq s \leq T, \gamma \in \Lambda_T.$$  

Then $\mathcal{F} := \{ \mathcal{F}_t, t \in [0, T] \}$ is a filtration. A map $H : [0, T] \times \Lambda_T \to E$ is called a functional process, where $E$ is a Banach space. Moreover, we say a process $H$ is adapted to the filtration $\mathcal{F}$, if $H(t, \cdot)$ is $\mathcal{F}_t$-measurable for any $t \in [0, T]$. Clearly, an adapted process $H$ has the property that, for any $\gamma_1, \gamma_2 \in \Lambda_T$ satisfying $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0, t]$, $H(t, \gamma_1) = H(t, \gamma_2)$. Hence $H(t, \gamma_T)$ can be viewed as $H(\gamma_t)$.

By Dupire [13] and Cont & Fournie [8], we have the following functional Itô formula.

**Theorem 2.1.** Suppose $X$ is a continuous semi-martingale and $u \in C^{1, 2}(\Lambda)$. The for any $t \in [0, T]$:

$$u(X_t) = u(X_0) + \int_0^t \partial_t u(X_s) ds + \frac{1}{2} \int_0^t \partial_{xx} u(X_s) d\langle X \rangle(s).$$  

6
Lemma 2.1. Under assumptions (A1) and (A2), for any random variable \( s \), the BSDE
\[
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_s^T Z(s) dW(s), \quad 0 \leq t \leq T, \tag{2.5}
\]
has a unique adapted solution \((Y, Z) \in \mathcal{S}^2(0, T) \times \mathcal{H}^2(0, T)\).

Lemma 2.2. Assume \( f \) satisfies (A1) and (A2). \( f^i(s, y, z) = f(s, y, z) + \varphi^i(s) \), \( \varphi^i(s) \in \mathcal{H}^2(0, T) \), \( \xi^i \in L^2(\Omega, \mathcal{F}_T, P) \), \( i = 1, 2 \). Then the difference of the solutions \((Y^1, Z^1)\) and \((Y^2, Z^2)\) of BSDE (2.5) with the data \((\xi^1, f^1)\) and \((\xi^2, f^2)\), respectively, satisfies the following estimate:
\[
\begin{align*}
|Y^1(t) - Y^2(t)|^2 &+ \frac{1}{2} E \left[ \int_t^T (|Y^1(s) - Y^2(s)|^2 + |Z^1(s) - Z^2(s)|^2) e^{\beta(s-t)} ds \right] \leq E[|\xi^1 - \xi^2|^2 e^{\beta(T-t)} |\mathcal{F}_t] + E \left[ \int_t^T |\varphi^1(s) - \varphi^2(s)|^2 e^{\beta(s-t)} ds \right] \mathcal{F}_t, \quad a.s., \tag{2.6}
\end{align*}
\]
for all \( t \in [0, T] \), where \( \beta \geq 2(2L^2 + L + 1) \).

We also shall recall the following comparison theorem on BSDEs (see El Karoui, Peng and Quenez [18]).
Lemma 2.3. (Comparison Theorem) Let two BSDEs of data $(\xi_1, f_1)$ and $(\xi_2, f_2)$ satisfy all the assumptions of Lemma 2.1. We denote by $(Y^1, Z^1)$ and $(Y^2, Z^2)$ their respective adapted solutions. If

$$\zeta_1 \geq \zeta_2, \text{P-a.s. and } f_1(t, Y^2(t), Z^2(t)) \geq f_2(t, Y^2(t), Z^2(t)) \text{ } dP \otimes dt \text{ a.s.}$$

Then we have that $Y^1(t) \geq Y^2(t), \text{ a.s., for all } t \in [0, T]$. And if, in addition, we also assume that $P(\xi_1 > \xi_2) > 0$, then $P(Y^1(t) > Y^2(t)) > 0$ for any $t \in [0, T]$, and in particular, $Y^1(0) > Y^2(0)$.

3 A DPP for stochastic optimal control problems

First, we introduce the setting for stochastic optimal control problems that we will consider later. Let the set of admissible control processes $U$ be the set of all $F_t$-predictable processes valued in some compact metric space $(U, d)$. For any $t \in [0, T], \geq 1, L^p(\Omega, F_t; \Lambda_t, \mathcal{F}_t)$ is the set of all $F_t/\mathcal{F}_t$-measurable maps $\Gamma_t: \Omega \rightarrow \Lambda_t$ satisfying $E\|\Gamma_t\|_p^p < \infty$.

For given $t \in [0, T), F_t/\mathcal{F}_t$-measurable map $\Gamma_t: \Omega \rightarrow \Lambda_t$ and admissible control $u \in U$, the corresponding orbit is defined by the solution of the following SDE:

$$\begin{aligned}
&dX^{\Gamma_t,u}(s) = F(X^{\Gamma_t,u}_s, u(s))ds + G(X^{\Gamma_t,u}_s, u(s))dW(s), \quad s \in [t, T], \\
&X^{\Gamma_t,u}_t = \Gamma_t \in \Lambda_t,
\end{aligned}$$

where the mappings $F: \Lambda \times U \rightarrow \mathbb{R}^d$, $G: \Lambda \times U \rightarrow \mathbb{R}^{d \times n}$ satisfy the following assumptions.

Hypothesis 3.1. There exists a constant $L > 0$ such that for all $(t, \gamma, u), (t', \gamma', u') \in [0, T] \times \Lambda_T \times U$ it holds that

$$|F(\gamma_t, u)| \|G(\gamma_t, u)| \leq L\|\gamma_t\|_0;$$

and

$$|F(\gamma_t, u) - F(\gamma'_t, u')| \|G(\gamma_t, u) - G(\gamma'_t, u')| \leq L(d_\infty(\gamma_t, \gamma'_t) + d(u, u')).$$

Lemma 3.1. Let $0 \leq t \leq T$ be fixed and assume that Hypothesis 3.1 holds. Then for every $u \in U$, $\Gamma_t \in L^p(\Omega, F_t; \Lambda_t, \mathcal{F}_t)$ and $p \geq 2$, there exists a unique solution $X^{\Gamma_t,u}$ to equation (3.1) such that $X^{\Gamma_t,u}_s: \Omega \rightarrow \Lambda_s$ is $\mathcal{F}_s/\mathcal{F}_s$ measurable for
all \( s \in [t, T] \), and \( E\|X_{t, u}^{\Gamma, u}\|_p^p < \infty \). Moreover, for any \( t \leq s \leq r \leq T, \Gamma_t, \Gamma_t' \in L^p(\Omega, F_t; \Lambda_t, \mathcal{F}_t) \), we have the following estimates, \( P \)-a.a.,

\[
E[\|X_{T, u}^{\Gamma, u} - X_{T, u}^{u'}\|_p^p | F_t] \leq C_p \left( \|\Gamma_t - \Gamma_t'\|_p^p + E \left[ \int_t^T d^p(u(r) - u'(r))dr | F_t \right] \right),
\]

\[
E[\|X_{T, u}^{\Gamma, u}\|_p^p | F_t] \leq C_p \left( 1 + \|\Gamma_t\|_p^p \right),
\]

\[
E[\|X_{T, u}^{\Gamma, u} - X_{T, u}^{\Gamma, u'}\|_p^p | F_t] \leq C_p \left( 1 + E[\|\Gamma_t\|_p^p] (r - s)^\frac{p}{2} \right).
\]

The constant \( C_p \) depending only on \( \tau, p, T \) and \( L \).

Now let there be given two mappings

\[
q : \Lambda \times R \times R^d \times U \rightarrow R, \quad \phi : \Lambda_T \rightarrow R
\]

that satisfy the following condition.

**Hypothesis 3.2.** There exists a constant \( L > 0 \) such that, for all \( (t, \gamma_T, y, z, u), (t', \gamma_T', y', z', u') \in [0, T] \times \Lambda_T \times R \times R^d \times U \),

\[
|q(\gamma_t, y, z, u) - q(\gamma_t', y', z', u')| \leq L(d_\infty(\gamma_t, \gamma_t') + |y - y'| + |z - z'| + d(u, u')),
\]

\[
|\phi(\gamma_T) - \phi(\gamma_T')| \leq L|\gamma_T - \gamma_T'|_0.
\]

Combing Lemmas 2.1 and 3.1, we obtain

**Lemma 3.2.** Let \( 0 \leq t \leq T \) be fixed and assume that Hypotheses 3.1 and 3.2 hold. Then for every \( u \in \mathcal{U}, \Gamma_t \in L^p(\Omega, F_t; \Lambda_t, \mathcal{F}_t) \) and \( p \geq 2 \), The BSDE

\[
Y_{t, u}^{\Gamma}(s) = \phi(X_{s, u}^{\Gamma}(s)) + \int_s^T q(X_{s, u}^{\Gamma}(s), Y_{s, u}^{\Gamma}(s), Z_{s, u}^{\Gamma}(s), u(s)) ds
\]

\[
- \int_s^T Z_{s, u}^{\Gamma}(s) dW(s), \quad a.s.-\omega, \quad all \ s \in [t, T], \quad (3.2)
\]

has a unique solution \( (Y_{t, u}^{\Gamma}, Z_{t, u}^{\Gamma}) \in S^2(0, T) \times \mathcal{H}^2(0, T) \). Furthermore, there is a constant \( C_p \) such that for any \( t \in [0, T], \Gamma_t, \Gamma_t' \in L^p(\Omega, F_t; \Lambda_t, \mathcal{F}_t) \), and \( u, u' \in \mathcal{U}, P\)-a.a.,

\[
E \left[ \sup_{t \leq s \leq T} |Y_{t, u}^{\Gamma}(s) - Y_{t, u'}^{\Gamma}(s)|^p | F_t \right] \leq C_p \left( ||\Gamma_t - \Gamma_t'||_p^p + E \left[ \int_t^T d^p(u(r) - u'(r))dr | F_t \right] \right),
\]

\[
E \left[ \sup_{t \leq s \leq T} |Y_{t, u}^{\Gamma}(s)|^p | F_t \right] \leq C_p \left( 1 + ||\Gamma_t||_p^p \right),
\]

\[
E \left[ \sup_{t \leq s \leq r} |Y_{t, u}^{\Gamma}(s) - Y_{t, u}^{\Gamma}(t)|^p | F_t \right] \leq C_p \left( 1 + ||\Gamma_t||_p^p (r - t)^\frac{p}{2} \right).
\]
Given a control process \( u \in \mathcal{U} \), we introduce an associated cost functional:

\[
J(\gamma_t, u) := Y^{\gamma_t, u}(t), \quad t \in [0, T], \quad \gamma_t \in \Lambda,
\]

where the process \( Y^{\gamma_t, u} \) is defined by BSDE (3.2). The value functional of the optimal control is defined by

\[
V(\gamma_t) := \text{ess sup}_{u \in \mathcal{U}} Y^{\gamma_t, u}(t), \quad \gamma_t \in \Lambda.
\]  

(3.3)

We easily show that, for any \( t \in [0, T] \) and \( \Gamma_t \in L^2(\Omega, \mathcal{F}_t; \Lambda_t, \mathbb{P}) \),

\[
J(\Gamma_t, u) = Y^{\Gamma_t, u}(t), \quad \text{a.a.}
\]

Obviously, under the assumptions Hypotheses 3.1 and 3.2, the value functional \( V(\gamma_t) \) is well defined and \( \mathcal{F}_t \)-measurable. But it turns out that \( V(\gamma_t) \) is even deterministic. We have

**Theorem 3.1.** (See Proposition 3.4 in [32]) The value functional \( V(\gamma_t) \) is a deterministic.

The first property of the value function \( V \) which we present is an immediate consequence of Lemma 3.2.

**Lemma 3.3.** Assume that Hypotheses 3.1 and 3.2 hold, there exists a constant \( C > 0 \) such that, for all \( 0 \leq t \leq T \), \( \gamma_t, \gamma'_t \in \Lambda_t \),

\[
|V(\gamma_t) - V(\gamma'_t)| \leq C||\gamma_t - \gamma'_t||_0; \quad |V(\gamma_t)| \leq C(1 + ||\gamma_t||_0).
\]  

(3.4)

We now discuss a DPP for the optimal control problem (3.1), (3.2) and (3.3). For this purpose, we define the family of backward semigroups associated with BSDE (3.2), following the idea of Peng [32].

Given the initial path \( \gamma_t \in \Lambda \), a positive number \( \delta \leq T - t \), an admissible control \( u(\cdot) \in \mathcal{U} \) and a real-valued random variable \( \eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R}) \), we put

\[
G_{s,t+\delta}^{\gamma_t, u}[\eta] := \tilde{Y}^{\gamma_t, u}(s), \quad s \in [t, t + \delta],
\]

(3.5)

where \((\tilde{Y}^{\gamma_t, u}(s), \tilde{Z}^{\gamma_t, u}(s))_{t \leq s \leq t + \delta}\) is the solution of the following BSDE with the time horizon \( t + \delta \):

\[
\begin{cases}
 d\tilde{Y}^{\gamma_t, u}(s) = -q(X^{\gamma_t, u}_s, \tilde{Y}^{\gamma_t, u}(s), \tilde{Z}^{\gamma_t, u}(s), u(s))ds + \tilde{Z}^{\gamma_t, u}(s)dW(s), \\
 \tilde{Y}^{\gamma_t, u}(t + \delta) = \eta.
\end{cases}
\]  

(3.6)

and \( X^{\gamma_t, u}(\cdot) \) is the solution of SDE (3.1). Then, obviously, for the solution \((Y^{\gamma_t, u}(s), Z^{\gamma_t, u}(s))_{t \leq s \leq T}\) of BSDE (3.2), the uniqueness of the BSDE yields

\[
J(\gamma_t, u) = Y^{\gamma_t, u}(t) = G_{t,T}^{\gamma_t, u}[\phi(X^{\gamma_t, u}_T)] = G_{t,t+\delta}^{\gamma_t, u}[Y^{\gamma_t, u}(t + \delta)] = G_{t,t+\delta}^{\gamma_t, u}[J(X^{\gamma_t, u}_{t+\delta}, u)].
\]
Theorem 3.2. (See Theorem 3.6 in [43]) Assume Hypotheses 3.1 (i) and 3.2 (i) hold true, the value functional $V$ obeys the following DPP: for any $\gamma_t \in \Lambda$ and $0 \leq t < t + \delta \leq T$,

$$V(\gamma_t) = \operatorname{esssup}_{u \in \mathcal{U}} G_{t,t+\delta}^{\gamma_t,u}[V(X_{t+\delta}^{\gamma_t,u})].$$

(3.7)

We are now ready to state the following lemma which will be used to prove the uniqueness of viscosity solution in next section. For simplicity, we define

$$\mathcal{G} = \{ g \in C^0(\Lambda) \mid \exists g_0 \in C^1([0,T] \times \mathbb{R}) \text{ and } a_t \in \Lambda_t \text{ such that } g(\gamma_t) = g_0(t, ||\gamma_t - a_t||_H^2) \text{ for all } t \geq \hat{t} \text{ and } \gamma_t \in \Lambda, \text{ and } g'_0 \text{ is bounded} \},$$

where $g'_0$ denotes first derivative with respect to the second variable of $g_0$, and

$$||\gamma_t||_H^2 = \int_0^t |\gamma_t(s)|^2 ds, \quad \gamma_t \in \Lambda.$$

Lemma 3.4. If $g \in \mathcal{G}$, then the following holds:

$$\partial \varepsilon g(\gamma_t) = (g_0)_t(t, ||\gamma_t - a_t||_H^2) + g'_0(t, ||\gamma_t - a_t||_H^2) ||\gamma_t(t) - a_t(\hat{t})||^2,$$

(3.8)

where $(g_0)_t$ denote first derivative with respect to the first variable $t$ of $g_0$.

Proof. Since $g \in \mathcal{G}$, then by Taylor’s Theorem we get that, for every $t \in [\hat{t}, T - \varepsilon]$,

$$g(\gamma_{t+\varepsilon}) - g(\gamma_t) = g_0(t + \varepsilon, ||\gamma_{t+\varepsilon} - a_{t+\varepsilon}||_H^2) - g_0(t, ||\gamma_t - a_t||_H^2)$$

$$= g'_0(t + \varepsilon, ||\gamma_t - a_t||_H^2 + s\varepsilon||\gamma_t(t) - a_t(\hat{t})||^2) |\gamma_t(t) - a_t(\hat{t})|^2$$

$$+ g_0(t + \varepsilon, ||\gamma_t - a_t||_H^2) - g_0(t, ||\gamma_t - a_t||_H^2).$$

(3.9)

By $g_0 \in C^1([0,T] \times \mathbb{R})$, we have that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (g_0(t + \varepsilon, ||\gamma_t - a_t||_H^2) - g_0(t, ||\gamma_t - a_t||_H^2)) = (g_0)_t(t, ||\gamma_t - a_t||_H^2).$$

(3.10)

Thus we have

$$\partial \varepsilon g(\gamma_t) = \lim_{\varepsilon \to 0^+} \frac{g(\gamma_{t+\varepsilon}) - g(\gamma_t)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} g'_0(t + \varepsilon, ||\gamma_t - a_t||_H^2 + s\varepsilon||\gamma_t(t) - a_t(\hat{t})||^2) |\gamma_t(t) - a_t(\hat{t})|^2$$

$$+ (g_0)_t(t, ||\gamma_t - a_t||_H^2)$$

$$= g'_0(t, ||\gamma_t - a_t||_H^2) |\gamma_t(t) - a_t(\hat{t})|^2 + (g_0)_t(t, ||\gamma_t - a_t||_H^2).$$
4 Viscosity solution to the HJB equation: Existence theorem.

In this section, we consider the following second order path-dependent HJB (PHJB) equation:

\[
\begin{aligned}
&\frac{\partial}{\partial t} V(\gamma_t) + H(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) = 0, \quad t \in [0, T), \quad \gamma_t \in \Lambda, \\
&V(\gamma_T) = \phi(\gamma_T), \quad \gamma_T \in \Lambda_T;
\end{aligned}
\]  

(4.1)

where

\[
H(\gamma_t, r, p, l) = \sup_{u \in U} [(p, F(\gamma_t, u))_{R^d} + \frac{1}{2} \text{tr}[G(\gamma_t, u)G^\top(\gamma_t, u)] \\
+ q(\gamma_t, r, G^\top(\gamma_t, u)p, u)], \quad (\gamma_t, r, p, l) \in \Lambda \times R \times R^d \times \Gamma(R^d).
\]

Here we let \(G^\top\) the transpose of the matrix \(G\), \(\Gamma(R^d)\) the set of all \((d \times d)\) symmetric matrices and \((\cdot, \cdot)_{R^d}\) the scalar product of \(R^d\).

**Definition 4.1.** (Classical solution) A functional \(v \in C^{1,2}(\Lambda)\) is called a classical solution to the path-dependent HJB equation (4.1) if it satisfies the path-dependent HJB equation point-wisely.

In this section we will prove that the value functional \(V\) defined by (3.3) is a viscosity solution of PHJB equation (4.1). Before giving the definition of the viscosity solution, let us introduce the following key lemma for use in the proof of the uniqueness of the viscosity solution.

**Lemma 4.1.** (See Proposition 2.9 in [42]) For \(\alpha \in (0, 1/2), M_0 > 0\) and \(t_0 \in [0, T]\), \(C_{\mu, M_0, t_0}^\alpha\) is a compact subset of \((\Lambda, d_{\infty})\).

Throughout the rest of this paper, we fix \(\alpha \in (0, \frac{1}{2})\). Now we can give the following definition for the viscosity solution.

**Definition 4.2.** \(w \in C^0(\Lambda)\) is called a viscosity subsolution (supersolution) to (4.1) if the terminal condition, \(w(\gamma_T) \leq \phi(\gamma_T)\) (resp. \(w(\gamma_T) \geq \phi(\gamma_T)\)), \(\gamma_T \in \Lambda_T\) is satisfied, and for every \(M_0 > 0, t_0 \in [0, T]\) and \(\varphi \in C_p^{1,2}(\Lambda)\), whenever the function \(w - \varphi\) (resp. \(w + \varphi\)) satisfies

\[
0 = (w - \varphi)(\hat{\gamma}_s) = \sup_{\gamma_t \in C_{\mu, M_0, t_0}^\alpha} (w - \varphi)(\gamma_t),
\]

(respectively,

\[
0 = (w + \varphi)(\hat{\gamma}_s) = \inf_{\gamma_t \in C_{\mu, M_0, t_0}^\alpha} (w + \varphi)(\gamma_t).
\]

12
where \( \hat{\gamma}_s \in C^\alpha_{\mu,M_0,t_0} \), \( s \in [t_0, T) \) and \( |\hat{\gamma}_s(s)| < M_0 \), we have

\[
\lim_{\mu \to +\infty} \left[ \partial_t \varphi(\hat{\gamma}_s) + H(\hat{\gamma}_s, \varphi(\hat{\gamma}_s), \partial_x \varphi(\hat{\gamma}_s), \partial_{xx} \varphi(\hat{\gamma}_s)) \right] \geq 0,
\]

(respectively, \( \lim_{\mu \to +\infty} \left[ -\partial_t \varphi(\hat{\gamma}_s) + H(\hat{\gamma}_s, -\varphi(\hat{\gamma}_s), -\partial_x \varphi(\hat{\gamma}_s), -\partial_{xx} \varphi(\hat{\gamma}_s)) \right] \leq 0 \)).

\( w \in C(\Lambda) \) is said to be a viscosity solution to (4.1) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 4.1.**

(i) A viscosity solution \( V \) of the PHJB equation (4.1) is a classical solution (See Definition 4.1) if it further lies in \( C^{1,2}(\Lambda) \).

(ii) In the classical uniqueness proof of viscosity solution to second order HJB equation in infinite dimensions, the weak compactness of closed balls in separable Hilbert spaces is used (See [40]). In our case, the path-dependent HJB equation is defined on space \( \Lambda \), which does not have weak compactness. For the sake of the uniqueness, our new concept of viscosity solution is enhanced.

(iii) Assume that the coefficients \( F(\gamma_t, u) = \overline{F}(t, \gamma_t(t), u) \), \( G(\gamma_t, u) = \overline{G}(t, \gamma_t(t), u) \),

\[
qu(t, \gamma_t(t), y, z, u) = \overline{q}(t, \gamma_t(t), y, z, u), \quad \phi(\gamma_T) = \overline{\phi}(\gamma_T(T)), \quad (\gamma_t, y, z, u) \in \Lambda_T \times \mathbb{R} \times \mathbb{R}^d \times U.
\]

Let the function \( V : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be a viscosity solution to (4.1) as a functional of \( V(\gamma_t) : \Lambda \to \mathbb{R} \). Then, \( V \) is also a classical viscosity solution as a function of the state.

For every \( \mu > 0 \), \( M_0 > 0 \), \( 0 \leq t_0 \leq T \), \( \varepsilon \in (0, \mu) \) and \( \gamma_t \in C^\alpha_{\mu,M_0,t_0} \), define a perturbation of \( \gamma_t \) in the following manner:

\[
\gamma^\varepsilon_t(s) := \begin{cases} 
\gamma_t(s), & |\gamma_t(s) - \gamma_t(t)| \leq (\mu - \varepsilon)|s - t|^{\alpha}; \\
\gamma_t(t) + (\mu - \varepsilon)(t - s)^\alpha \frac{\gamma_t(s) - \gamma_t(t)}{|\gamma_t(s) - \gamma_t(t)|}, & |\gamma_t(s) - \gamma_t(t)| > (\mu - \varepsilon)|s - t|^{\alpha}.
\end{cases}
\]

**Lemma 4.2.** (See Lemma 5.1 in [43]) For every \( \gamma_t \in C^\alpha_{\mu,M_0,t_0} \) and \( \varepsilon \leq \frac{\mu}{2} \), we have

(i) \( ||\gamma^\varepsilon_t - \gamma_t||_0 \leq 2M_0\varepsilon(\mu - \varepsilon)^{-1} \leq 4M_0\varepsilon\mu^{-1} \).

(ii) \( \gamma^\varepsilon_t \in C^\alpha_{\mu,M_0,t_0} \).

(iii) There exists a constant \( C > 0 \), independent of \( \mu \) and \( u \in \mathcal{U} \), such that, for some \( p, p(\frac{1}{2} - \alpha) > 1 \) and for all \( \delta < T - t \),

\[
P \left\{ \sup_{0 \leq s < r \leq t + \delta} \frac{|X^{\gamma^\varepsilon_t, u}_t(s) - X^{\gamma^\varepsilon_t, u}_t(t)|}{|s - r|^{\alpha}} > \mu \right\} \leq C\delta^{p\left(\frac{1}{2} - \alpha\right)}\varepsilon^{-p}.
\]

We conclude this section with the existence result on viscosity solution.
Theorem 4.1. Suppose that Hypotheses 3.1 and 3.2 hold. Then the value functional $V$ defined by (3.3) is a viscosity solution to (4.1).

Proof. Let $M_0 > 0$, $\mu > 1$, $t_0 \in [0, T)$ and $\varphi \in C^{1,2}_p(\Lambda)$ such that

$$0 = (V - \varphi)(\hat{\gamma}_t) = \sup_{\gamma_s \in C^{1,2}_{\mu,M_0,t_0}} (V - \varphi)(\gamma_s),$$

where $\hat{\gamma}_t \in C^{1,2}_{\mu,M_0,t_0}$, $t \in [t_0, T)$ and $\delta_1 := M_0 - |\hat{\gamma}_t(t)| > 0$.

For any $\mu > 1$ and $\varepsilon < \frac{1}{2} \land \frac{\delta_1}{M_0}$, from Assertion (i) of Lemma 4.2, we have

$$||\hat{\gamma}_t^{\varepsilon} - \gamma_t||_0 \leq 4M_0 \varepsilon \mu^{-1} < \frac{\delta_1}{2}.$$  

For any $u \in \mathcal{U}$, we define an $\mathcal{F}$-stopping time

$$\tau^{\varepsilon} := \inf \{s > t : ||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_\alpha > \mu\} \land \inf \{s > t : ||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_0 > M_0\}.$$  

Obviously, $X^{\hat{\gamma}_t^{\varepsilon},u}_s \in C^{1,2}_{\mu,M_0,t_0}$ and

$$\{\tau^{\varepsilon} < t + \delta\} \subset \{||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_\alpha > \mu\} \cup \{||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_0 > M_0\}.$$  

Therefore, from Lemma 3.1 and Assertion (iii) of Lemma 4.2, we have

$$P\{||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_\alpha > \mu\} \leq C\delta^p \left(\frac{1}{2} - \alpha\right) \varepsilon^{-p};$$

and

$$P\{||X^{\hat{\gamma}_t^{\varepsilon},u}_s||_0 > M_0\} \leq P\{||X^{\hat{\gamma}_t^{\varepsilon},u}_s - \hat{\gamma}_t^{\varepsilon}||_0 > \frac{\delta_1}{2}\} \leq C(1 + M_0^{6}) \frac{\delta_3}{\delta_1^{6}}.$$  

Hence

$$P\{\tau^{\varepsilon} \leq t + \delta\} \leq C\delta^p \left(\frac{1}{2} - \alpha\right) \varepsilon^{-p} + C(1 + M_0^{6}) \frac{\delta_3}{\delta_1^{6}} \downarrow 0 \text{ as } \delta \to 0,$$

uniformly with respect to $(u, \mu)$, where $p\left(\frac{1}{2} - \alpha\right) > 2$. For $t < t + \delta \leq T$, by the DPP (Theorem 3.2), we obtain the following result:

$$V(\hat{\gamma}_t^{\varepsilon}) - \varphi(\hat{\gamma}_t^{\varepsilon}) = \sup_{u \in \mathcal{U}} G_{t,t+\delta}^{\hat{\gamma}_t^{\varepsilon},u}[V(\hat{\gamma}_t^{\varepsilon})] - \varphi(\hat{\gamma}_t^{\varepsilon}). \quad (4.3)$$

Then, for any $\varepsilon_1 > 0$ and $0 < \delta \leq T - t$, we can find a control $u^{\varepsilon_1}(\cdot) \equiv u^{\varepsilon_1,\delta}(\cdot) \in \mathcal{U}$ such that the following result holds:

$$- \varepsilon_1 \delta + V(\hat{\gamma}_t^{\varepsilon}) - \varphi(\hat{\gamma}_t^{\varepsilon}) \leq G_{t,t+\delta}^{\hat{\gamma}_t^{\varepsilon},u^{\varepsilon_1}}[V(\hat{\gamma}_t^{\varepsilon})] - \varphi(\hat{\gamma}_t^{\varepsilon}). \quad (4.4)$$
We note that $G_{s,t+\delta}^{\hat{\gamma},u^\delta} [V(X_t^{\hat{\gamma},u^\delta})]$ is defined in terms of the solution of the BSDE:

$$
\begin{aligned}
\begin{cases}
    dY^{\hat{\gamma},u^\delta}(s) &= -q(s, X_t^{\hat{\gamma},u^\delta}(s), Y_t^{\hat{\gamma},u^\delta}(s), Z_t^{\hat{\gamma},u^\delta}(s), u^\delta(s)) ds \\
    &\quad + Z_t^{\hat{\gamma},u^\delta}(s) dW(s), \quad s \in [t, t+\delta],
    \\
    Y^{\gamma^0, u^\delta}(t+\delta) &= V(X_t^{\hat{\gamma},u^\delta}),
\end{cases}
\end{aligned}
$$

(4.5)

by the following formula:

$$
G_{s,t+\delta}^{\hat{\gamma},u^\delta} [V(X_t^{\hat{\gamma},u^\delta})] = Y^{\hat{\gamma},u^\delta}(s), \quad s \in [t, t+\delta].
$$

Applying functional Itô formula (2.4) to $\varphi(X_t^{\hat{\gamma},u^\delta})$, we get that

$$
\varphi(X_t^{\hat{\gamma},u^\delta}) = \varphi(\gamma_t) + \int_t^s (\mathcal{L}\varphi)(X^{\hat{\gamma},u^\delta}_\sigma, u^\delta(\sigma)) d\sigma - \int_t^s q(X^{\hat{\gamma},u^\delta}_\sigma, \varphi(X^{\hat{\gamma},u^\delta}_\sigma)), u^\delta(\sigma)) d\sigma \\
\quad + \int_t^s [\partial_x \varphi(X^{\hat{\gamma},u^\delta}_\sigma)]^\top G(X^{\hat{\gamma},u^\delta}_\sigma, u^\delta(\sigma)) dW(\sigma),
$$

(4.6)

where

$$
(\mathcal{L}\varphi)(\gamma_t, u) = \partial_t \varphi(\gamma_t) + (\partial_x \varphi(\gamma_t), F(\gamma_t, u))_{R^d} + \frac{1}{2} \text{tr} \left[ \partial_{xx} \varphi(\gamma_t) G(\gamma_t, u) G(\gamma_t, u)^\top \right] \\
+ q(\gamma_t, \varphi(\gamma_t), \partial_x \varphi(\gamma_t))^\top G(\gamma_t, u), u), \quad (\gamma_t, u) \in \Lambda \times U.
$$

Set

$$
Y^{2,\hat{\gamma},u^\delta}(s) := \varphi(X^{\hat{\gamma},u^\delta}_s) - Y^{\hat{\gamma},u^\delta}(s),
$$

$$
Z^{2,\hat{\gamma},u^\delta}(s) := [\partial_x \varphi(X^{\hat{\gamma},u^\delta}_s)]^\top G(X^{\hat{\gamma},u^\delta}_s, u^\delta(s)) - Z^{\hat{\gamma},u^\delta}(s).
$$

Comparing (4.5) and (4.6), we have, P-a.s.,

$$
\begin{aligned}
dY^{2,\hat{\gamma},u^\delta}(s) &= [(\mathcal{L}\varphi)(X^{\hat{\gamma},u^\delta}_s, u^\delta(s)) - q(X^{\hat{\gamma},u^\delta}_s, \varphi(X^{\hat{\gamma},u^\delta}_s)), u^\delta(s)) \\
&\quad + q(X^{\hat{\gamma},u^\delta}_s, Y^{\hat{\gamma},u^\delta}(s), Z^{\hat{\gamma},u^\delta}(s), u^\delta(s))] ds + Z^{2,\hat{\gamma},u^\delta}(s) dW(s) \\
&= [(\mathcal{L}\varphi)(X^{\hat{\gamma},u^\delta}_s, u^\delta(s)) - A(s)Y^{2,\hat{\gamma},u^\delta}(s) - (\bar{A}(s), Z^{2,\hat{\gamma},u^\delta}(s))_{R^m}] ds \\
&\quad + Z^{2,\hat{\gamma},u^\delta}(s) dW(s),
\end{aligned}
$$

where $|A| \vee |\bar{A}| \leq C$ ($C$ only depends on Lipschitz constant of $q$). Therefore, we obtain (see Proposition 2.2 in [18])

$$
Y^{2,\hat{\gamma},u^\delta}(t) = E \left[ Y^{2,\hat{\gamma},u^\delta}(t+\delta) \Gamma^t(t+\delta) \right].
$$
\[
- \int_{t}^{t+\delta} \Gamma'(\sigma)(\mathcal{L}\varphi)(X_{\sigma}^{\gamma_{\epsilon},u_{\epsilon}}, u_{\epsilon}(\sigma))d\sigma \bigg| \mathcal{F}_{t},
\]
where \(\Gamma'(\cdot)\) solves the linear SDE
\[
d\Gamma'(s) = \Gamma'(s)(A(s)ds + \bar{A}(s)dW(s)), \quad s \in [t, t+\delta]; \quad \Gamma'(t) = 1.
\]
Obviously, \(\Gamma' \geq 0\). Combining (4.4) and (4.7), we have
\[
-\varepsilon_{1} + \frac{1}{\delta}[V(\gamma_{\epsilon}) - \varphi(\gamma_{\epsilon})]
\leq \frac{1}{\delta}E\left[ - Y^{2,\gamma_{\epsilon},u_{\epsilon}}(t+\delta)\Gamma'(t+\delta) + \int_{t}^{t+\delta} \Gamma'(\sigma)(\mathcal{L}\varphi)(X_{\sigma}^{\gamma_{\epsilon},u_{\epsilon}}, u_{\epsilon}(\sigma))d\sigma \right]
\leq -\varepsilon_{1} + \frac{1}{\delta}E\left[ \int_{t}^{t+\delta} (\mathcal{L}\varphi)(\gamma_{\epsilon}, u_{\epsilon}(\sigma))d\sigma \right]
\leq -\varepsilon_{1} + \frac{1}{\delta}E\left[ \int_{t}^{t+\delta} (\Gamma'(\sigma) - 1)(\mathcal{L}\varphi)(X_{\sigma}^{\gamma_{\epsilon},u_{\epsilon}}, u_{\epsilon}(\sigma))d\sigma \right]
:= I + II + III + IV.
\]
Since the coefficients in \(\mathcal{L}\) are Lipschitz continuous, combining the regularity of \(\varphi \in C_{p}^{1,2}(\Lambda)\), there exist a integer \(q \geq 1\) and a constant \(C > 0\) independent of \(u \in \mathcal{U}\) such that
\[
E|\varphi(X_{\sigma}^{\gamma_{\epsilon},u_{\epsilon}}, u(\sigma))| \vee E|\mathcal{L}\varphi(X_{\sigma}^{\gamma_{\epsilon},u_{\epsilon}}, u(\sigma))| \vee E|\mathcal{L}\varphi(\gamma_{\epsilon}, u(\sigma))|
\leq C(1 + ||\gamma_{\epsilon}||_{0}), \quad \sigma \in [t, T], \quad \gamma_{\epsilon} \in \Lambda.
\]
In view of Lemma 3.1, we also have
\[
\sup_{u \in \mathcal{U}} E\left[ \sup_{t \leq s \leq t+\delta} |X_{s}^{\gamma_{\epsilon},u}(s) - \gamma_{\epsilon}(t)|^{2} \right] \leq C\delta,
\sup_{u \in \mathcal{U}} E\left[ \sup_{t \leq s \leq t+\delta} |\Gamma'(s) - 1|^{2} \right] \leq C\delta.
\]
Thus we have
\[
\frac{1}{\delta}|V(\gamma_{\epsilon}) - \varphi(\gamma_{\epsilon})|
= \frac{1}{\delta}|(V - \varphi)(\gamma_{\epsilon}) + V(\gamma_{\epsilon}) - V(\gamma_{\epsilon}) + \varphi(\gamma_{\epsilon}) - \varphi(\gamma_{\epsilon})|
\[ \lim_{\varepsilon \to 0} \left| I \right| \leq \frac{1}{\delta} E \left[ \int_{t}^{t+\delta} \sup_{u \in U} |(\mathcal{L}\varphi)(X_{\sigma}^{\varepsilon,u}, u(\sigma)) - (\mathcal{L}\varphi)(\hat{\gamma}_t, u(\sigma))| \, d\sigma \right] \]

\[ \leq \frac{1}{\delta} \int_{t}^{t+\delta} \left( E(\Gamma^t(\sigma) - 1) \right) \left( E((L\varphi)(X_{\sigma}^{\varepsilon,u}, u(\sigma))) + \frac{\delta^2}{\delta^3} \right) d\sigma \]

\[ \leq C(1 + M_0^2)(\delta^{2(\frac{1}{2} - \frac{3}{2})} + \frac{\delta^2}{\delta^3}); \quad (4.10) \]

Now we estimate higher order terms III and IV.

\[ \lim_{\delta \to 0} \int_{t}^{t+\delta} \sup_{u \in U} E|\mathcal{L}\varphi(X_{\sigma}^{\varepsilon,u}, u(\sigma)) - (\mathcal{L}\varphi)(\hat{\gamma}_t, u(\sigma))| \, d\sigma \]

\[ \leq \frac{1}{\delta} \int_{t}^{t+\delta} \left( E(\Gamma^t(\sigma) - 1) \right) \left( E((L\varphi)(X_{\sigma}^{\varepsilon,u}, u(\sigma))) + \frac{\delta^2}{\delta^3} \right) d\sigma \]

\[ \leq C(1 + M_0^2)(\delta^{2(\frac{1}{2} - \frac{3}{2})} + \frac{\delta^2}{\delta^3}); \quad (4.10) \]

uniformly with respect to u and \( \mu \), and

\[ |IV| \leq \frac{1}{\delta} \int_{t}^{t+\delta} E|\mathcal{L}\varphi(X_{\sigma}^{\varepsilon,u}, u(\sigma)) - (\mathcal{L}\varphi)(\hat{\gamma}_t, u(\sigma))| \, d\sigma \]

\[ \leq \frac{1}{\delta} \int_{t}^{t+\delta} \left( E(\Gamma^t(\sigma) - 1) \right) \left( E((L\varphi)(X_{\sigma}^{\varepsilon,u}, u(\sigma))) + \frac{\delta^2}{\delta^3} \right) d\sigma \]

\[ \leq C(1 + M_0^2)(\delta^{2(\frac{1}{2} - \frac{3}{2})} + \frac{\delta^2}{\delta^3}); \quad (4.13) \]

Substituting (4.9), (4.10), (4.11), (4.12) and (4.13) into (4.8), we have

\[ -\varepsilon_1 \leq 4CM_0\varepsilon\mu^{-1} \delta^{-1} + |\varphi(\hat{\gamma}_t) - \varphi(\hat{\gamma}_t)| + C(1 + M_0^2)(\delta^{2(\frac{1}{2} - \frac{3}{2})} + \frac{\delta^2}{\delta^3}) + \frac{1}{\delta} \int_{t}^{t+\delta} \sup_{u \in U} E|\mathcal{L}\varphi(X_{\sigma}^{\varepsilon,u}, u(\sigma)) - (\mathcal{L}\varphi)(\hat{\gamma}_t, u(\sigma))| \, d\sigma \]

17
For any $\varepsilon_2 > 0$, we can let $\delta > 0$ be small enough such that

$$-\varepsilon_1 - \varepsilon_2 \leq 4CM_0 \varepsilon^{-1} + |\varphi(\hat{\gamma}_t) - \varphi(\hat{\gamma}_t')| +$$

$$+ \partial_t \varphi(\hat{\gamma}_t) + H(\hat{\gamma}_t, \varphi(\hat{\gamma}_t), \partial_x \varphi(\hat{\gamma}_t), \partial_{xx} \varphi(\hat{\gamma}_t))$$

$$+ \sup_{u \in U} E[(L\varphi)(\hat{\gamma}_t', u) - (L\varphi)(\hat{\gamma}_t, u)].$$

(4.15)

Taking $\mu \to +\infty$, we have

$$-\varepsilon_1 - \varepsilon_2 \leq \lim_{\mu \to +\infty} \partial_t \varphi(\hat{\gamma}_t) + H(\hat{\gamma}_t, \varphi(\hat{\gamma}_t), \partial_x \varphi(\hat{\gamma}_t), \partial_{xx} \varphi(\hat{\gamma}_t)).$$

By the arbitrariness of $\varepsilon_1$ and $\varepsilon_2$, we show $V$ is a viscosity subsolution to (4.1).

In a symmetric (even easier) way, we show that $V$ is also a viscosity supsolution to (4.1). This step completes the proof. $\square$

## 5 Viscosity solution to the HJB equation: Uniqueness theorem.

This section is devoted to a proof of uniqueness of the viscosity solution to (4.1). This result, together with the results from the previous section, will be used to characterize the value functional defined by (3.3).

We assume without loss of generality that, there exists a constant $K > 0$, such that, for all $(\gamma_t, p, l) \in \Lambda \times R^d \times \Gamma(R^d)$ and $r_1, r_2 \in R$ such that $r_1 < r_2$,

$$H(\gamma_t, r_1, p, l) - H(\gamma_t, r_2, p, l) \geq K(r_2 - r_1).$$

(5.1)

To obtain the uniqueness of the viscosity solution to HJB equation (4.1), we make the following assumptions.

**Hypothesis 5.1.** There exists a constant $L > 0$ such that for all $(t, \gamma_T, u) \in [0, T] \times \Lambda_T \times U$ it holds that

$$|F(\gamma_t, u)| \vee |G(\gamma_t, u)| \vee |q(\gamma_t, u)| \leq L(1 + |\gamma_t|) + ||\gamma_t||_H).$$

We now state the main result of this section.

**Theorem 5.1.** Suppose Hypotheses 3.1, 3.2 and 5.1 hold. Let $W \in C^0(\Lambda)$ be a viscosity solution to (4.1) and let there exist a constant $L > 0$ such that for $\gamma_t, \gamma'_t \in \Lambda$,

$$|W(\gamma_t)| \leq L(1 + ||\gamma_t||_0) \quad \text{and} \quad |W(\gamma_t) - W(\gamma'_t)| \leq L||\gamma_t - \gamma'_t||_0.$$  

(5.2)

Then $W = V$, where $V$ is the value functional defined in (3.3).
Theorems 4.1 and 5.1 lead to the result (given below) that the viscosity solution to the PHJB equation given in (4.1) corresponds to the value functional $V$ of our optimal control problem given in (3.1) and (3.3).

**Theorem 5.2.** Let Hypotheses 3.1, 3.2 and 5.1 hold. Then the value functional $V$ defined by (3.3) is the unique viscosity solution to (4.1) in the class of functions satisfying (3.4).

**Proof.** Theorem 4.1 shows that $V$ is a viscosity solution to (4.1). By Lemma 3.3, it is easy to show $V \in C^0(\Lambda)$. Thus, our conclusion follows from Theorem 5.1.

Next, we prove Theorem 5.1. Let $W$ be a viscosity solution of PHJB equation (4.1). We note that it is sufficient to show $W \leq V$ because the inverse of which can be proved in a similar way. We also note that for $\delta > 0$, the function defined by $\tilde{W} := W - \frac{\delta}{T}$ is a subsolution for

$$\begin{cases}
\partial_t \tilde{W}(\gamma_t) + H(\gamma_t, \tilde{W}(\gamma_t), \partial_x \tilde{W}(\gamma_t), \partial_{xx} \tilde{W}(\gamma_t)) = \frac{\delta}{T}, & \gamma_t \in \Lambda, \\
\tilde{W}(\gamma_T) = \phi(\gamma_T).
\end{cases}$$

As $W \leq V$ follows from $\tilde{W} \leq V$ in the limit $\delta \downarrow 0$, it suffices to prove $W \leq V$ under the additional assumption given below:

$$\partial_t W(\gamma_t) + H(\gamma_t, W(\gamma_t), \partial_x W(\gamma_t), \partial_{xx} W(\gamma_t)) \geq c, \quad c := \frac{\delta}{T^2}, \quad \gamma_t \in \Lambda.$$

**Proof of Theorem 5.1** The proof of this theorem is rather long. Thus, we split it into several steps.

**Step 1. Definitions of auxiliary functions.**

We only need to prove that $W(\gamma_t) \leq V(\gamma_t)$ for all $(t, \gamma_t) \in [T - \check{a}, T) \times \Lambda$. Here,

$$\check{a} = \frac{1}{16(1 + L)^2}.$$

Then, we can repeat the same procedure for the case $[T - i\check{a}, T - (i - 1)\check{a})$. Thus, we assume the converse result that $\bar{\tilde{m}} := W(\bar{\gamma}_T) - V(\bar{\gamma}_T) > 0$. Because $\bigcup_{\mu > 0, M_0 > 0} C^{0,\alpha}_{\mu, M_0}$ is dense in $\Lambda$, by (5.2) there exists $\bar{\gamma}_T \in \bigcup_{\mu > 0, M_0 > 0} C^{0,\alpha}_{\mu, M_0}$ such that $W(\bar{\gamma}_T) - V(\bar{\gamma}_T) > \bar{m}$.

Consider that $\varepsilon > 0$ is a small number such that

$$W(\bar{\gamma}_T) - V(\bar{\gamma}_T) - 2\varepsilon \frac{\nu T - \bar{t}}{\nu T}(||\bar{\gamma}_T||^2_H + |\bar{\gamma}_T(\bar{t})|^2) > \frac{\bar{m}}{2},$$

and

$$\frac{\varepsilon}{\nu T} \leq \frac{c}{4}. \quad (5.3)$$
where
\[ \nu = 1 + \frac{1}{16T(1+L)^2}, \]

Next, we define for any \( \gamma_1^1, \gamma_2^1 \in \Lambda, \)
\[
\Psi(\gamma_1^1, \gamma_2^1) = W(\gamma_1^1) - V(\gamma_2^1) - \frac{\alpha}{2}d(\gamma_1^1, \gamma_2^1) - \epsilon \frac{\nu T - t}{\nu T} (||\gamma_1^1||_H^2 + ||\gamma_2^1||_H^2 + |\gamma_1^1(t)|^2 + |\gamma_2^1(t)|^2),
\]
where
\[ d(\gamma_1^1, \gamma_2^1) = |\gamma_1^1(t) - \gamma_2^1(t)|^2 + ||\gamma_1^1 - \gamma_2^1||_H^2, \]
and
\[ ||\gamma_1||_H^2 = \int_0^t |\gamma(s)|^2 ds, \quad \gamma \in \Lambda. \]

Finally, for every \( \mu > 0 \) and \( M_0 > 0 \) satisfying \( \tilde{\gamma}_t \in C_{\mu,M_0,\tilde{t}}^\alpha, \) we can apply Lemma 4.1 to find \( (\tilde{t}, \tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \in [\tilde{t}, T] \times C_{\mu,M_0,\tilde{t}}^\alpha \times C_{\mu,M_0,\tilde{t}}^\alpha \) such that
\[ \Psi(\tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \geq \Psi(\gamma_t^1, \gamma_t^2), \quad t \geq \tilde{t}, \gamma_t^1, \gamma_t^2 \in C_{\mu,M_0,\tilde{t}}^\alpha. \]

We should note that the point \( (\tilde{t}, \tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \) depends on \( \tilde{t}, \alpha, \mu, M_0. \)

**Step 2.** For fixed \( \mu, \) the following inequality holds true:
\[ ad(\tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \leq |W(\tilde{\gamma}_t^1) - W(\tilde{\gamma}_t^2)| + |V(\tilde{\gamma}_t^1) - V(\tilde{\gamma}_t^2)| \to 0 \text{ as } \alpha \to +\infty. \tag{5.4} \]

Let us show the above. By the definition of \( (\tilde{\gamma}_t^1, \tilde{\gamma}_t^2), \) we have
\[ 2\Psi(\tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \geq \Psi(\gamma_t^1, \gamma_t^2) + \Psi(\gamma_t^2, \gamma_t^1) \tag{5.5} \]
This implies that
\[ ad(\tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \leq |W(\tilde{\gamma}_t^1) - W(\tilde{\gamma}_t^2)| + |V(\tilde{\gamma}_t^1) - V(\tilde{\gamma}_t^2)|. \tag{5.6} \]

By the local boundedness of \( V \) and \( W, \) we see that \( d(\tilde{\gamma}_t^1, \tilde{\gamma}_t^2) \to 0 \text{ as } \alpha \to +\infty. \) Noting that \( \tilde{\gamma}_t^1, \tilde{\gamma}_t^2 \in C_{\mu,M_0,\tilde{t}}^\alpha, \) we also have
\[ ||\tilde{\gamma}_t^1 - \tilde{\gamma}_t^2||_0 \to 0 \text{ as } \alpha \to +\infty. \]

Then combining (5.2) and (5.6), we see that (5.4) holds. Moreover, from \( \tilde{\gamma}_t^1, \tilde{\gamma}_t^2 \in C_{\mu,M_0,\tilde{t}}^\alpha \) and (5.4) it follows that
\[ \frac{\alpha}{2}||\gamma_t^1 - \gamma_t^2||_0^2 \to 0 \text{ as } \alpha \to +\infty. \tag{5.7} \]
Step 4. There exists $M_0 > 0$ such that $\hat{t} \in [\tilde{t}, T)$, $\hat{\gamma}_1^1, \hat{\gamma}_2^1 \in C_{\mu,M_0,\hat{t}}^\alpha$ and $|\hat{\gamma}_1^1(\hat{t})| \vee |\hat{\gamma}_2^1(\hat{t})| < M_0$ for all $\mu > 0$ satisfying $\tilde{\gamma}_i \in C_{\mu,M_0,\hat{t}}^\alpha$.

First, noting $\nu$ is independent of $\mu$, by the definition of $\Psi$, there exists an $M_0 > 0$ that is sufficiently large that $\Psi(\gamma_1^1, \gamma_2^1) < 0$ for all $t \in [0, T]$ and $|\gamma_1^1| = M_0$ or $|\gamma_2^1| = M_0$. Thus, we have $|\hat{\gamma}_1^1(\hat{t})| \vee |\hat{\gamma}_2^1(\hat{t})| < M_0$.

Next, noting $\Psi(\gamma_1^1, \gamma_2^1) \leq 0 < \frac{m}{2} \leq \Psi(\tilde{\gamma}_i, \tilde{\gamma}_i)$ for every $\gamma_1^1, \gamma_2^1 \in \Lambda$, we have $\hat{t} \in [\tilde{t}, T)$.

Step 5. Completion of the proof.

From above all, for the fixed $M_0 > 0$ in step 4, we find $\hat{\gamma}_1^1, \hat{\gamma}_2^1 \in C_{\mu,M_0,\hat{t}}^\alpha$ satisfying $\hat{t} \in [\tilde{t}, T)$ and $|\hat{\gamma}_1^1(\hat{t})| \vee |\hat{\gamma}_2^1(\hat{t})| < M_0$ with $\Psi(\hat{\gamma}_1^1, \hat{\gamma}_2^1) \geq \Psi(\tilde{\gamma}_i, \tilde{\gamma}_i)$ such that

$$\Psi(\hat{\gamma}_1^1, \hat{\gamma}_2^1) \geq \Psi(\gamma_1^1, \gamma_2^1), \quad \hat{t} \leq t < T, \quad \gamma_1^1, \gamma_2^1 \in C_{\mu,M_0,\hat{t}}^\alpha.$$  

(5.8)

We put, for $(t, \gamma_1^1, \gamma_2^1) \in [\tilde{t}, T] \times \Lambda \times \Lambda$,

$$W_1(\gamma_1^1) = W(\gamma_1^1) - \varepsilon \frac{\nu T - t}{\nu T}(|\gamma_1^1|_H^2 + |\gamma_1^1(t)|^2) - \varepsilon(|t-\hat{t}|^2 + |\gamma_1^1 - \gamma_1^1|_H^2),$$

$$V_1(\gamma_2^1) = V(\gamma_2^1) + \varepsilon \frac{\nu T - t}{\nu T}(|\gamma_2^1|_H^2 + |\gamma_2^1(t)|^2) + \varepsilon(|t-\hat{t}|^2 + |\gamma_2^1 - \gamma_2^1|_H^2).$$

Moreover, we define, for $(t, x_0, y_0) \in [\tilde{t}, T] \times R^d$,

$$\tilde{W}_1(\gamma_1^1, x_0) = \sup_{\xi_1^1 \in C_{\mu,M_0,\hat{t}}^\alpha, \xi_1^1(t) = x_0} \left[ W_1(\xi_1^1) - \alpha ||\gamma_1^1 - \xi_1^1||_H^2 \right],$$

$$\tilde{V}_1(\gamma_2^1, y_0) = \inf_{\xi_2^1 \in C_{\mu,M_0,\hat{t}}, \xi_2^1(t) = y_0} \left[ V_1(\xi_2^1) + \alpha ||\gamma_2^1 - \xi_2^1||_H^2 \right].$$

Then we obtain that

$$\tilde{W}_1(\xi_1, \gamma_1^1(t)) - \tilde{V}_1(\xi_1, \gamma_2^1(t)) - \frac{\alpha}{2}||\gamma_1^1(t) - \gamma_2^1(t)||^2$$

$$= \sup_{\gamma_1^1, \gamma_2^1 \in C_{\mu,M_0,\hat{t}}^\alpha} \left[ W_1(\gamma_1^1) - \alpha |\gamma_1^1 - \xi_1^1||_H^2 - V_1(\gamma_2^1) - \alpha |\gamma_2^1 - \xi_2^1||_H^2 - \frac{\alpha}{2}||\gamma_1^1(t) - \gamma_2^1(t)||^2 \right]$$

$$\leq \sup_{\gamma_1^1, \gamma_2^1 \in C_{\mu,M_0,\hat{t}}^\alpha} \left[ W_1(\gamma_1^1) - V_1(\gamma_2^1) - \frac{\alpha}{2}d(\gamma_1^1, \gamma_2^1) \right]$$

$$\leq W_1(\hat{\gamma}_1^1) - V_1(\hat{\gamma}_2^1) - \frac{\alpha}{2}d(\hat{\gamma}_1^1, \hat{\gamma}_2^1),$$

(5.9)

where the last inequality becomes equality if and only if $t = \hat{t}$ and $\gamma_1^1 = \hat{\gamma}_1^1, \gamma_2^1 = \hat{\gamma}_2^1$.

The previous inequality becomes equality if and only if $\xi_1 = \frac{\hat{\gamma}_1^1 + \hat{\gamma}_2^1}{2}$. Then we obtain that

$$\tilde{W}_1(\xi_1, \gamma_1^1(t)) - \tilde{V}_1(\xi_1, \gamma_2^1(t)) - \frac{\alpha}{2}||\gamma_1^1(t) - \gamma_2^1(t)||^2$$

21
\[
W_1(\hat{x}_t) - V_1(\hat{\gamma}_t^2) - \frac{\alpha}{2}d(\hat{\gamma}_t^1, \hat{\gamma}_t^2), \quad t \in [\hat{t}, T],
\]
and the equality only holds at \( \hat{t}, \hat{\gamma}_t^1(\hat{t}), \hat{\gamma}_t^2(\hat{t}), \hat{\xi}_t = \frac{\gamma_t^1 + \gamma_t^2}{2} \).

We define, for \( t \in [\hat{t}, T], x_0, y_0 \in R^d \)
\[
W_1(t, x_0) = W_1(\hat{\xi}_{t,t}, x_0), \quad V_1(t, y_0) = V_1(\hat{\xi}_{t,t}, y_0).
\]

Set \( r = \frac{1}{T}(|T - \hat{t}| \wedge (M_0 - |\gamma_t(\hat{t})|))) \), for a given \( L > 0 \), let \( \varphi \in C^{1,2}([\hat{t}, T] \times R^d) \) be a function such that \( W_1(t, x_0) - \varphi(t, x_0) \) has a maximum at \((\hat{t}, \bar{x}_0) \in (\hat{t}, T) \times R^d \), moreover, the following inequalities hold true:
\[
|\tilde{t} - t| + |\bar{x}_0 - \tilde{\gamma}_1(\bar{t})| < r,
|\bar{W}_1(\tilde{t}, \bar{x}_0)| + |\nabla_x \varphi(\tilde{t}, \bar{x}_0)| + |\nabla^2_x \varphi(\tilde{t}, \bar{x}_0)| \leq L.
\]

Here and in the sequel, \( \nabla_x \) and \( \nabla^2_x \) are the classical partial derivatives in the state variable \( x \). We can modify \( \varphi \) such that \( \varphi \in C^{1,2}([\hat{t}, T] \times R^d) \) and \( \bar{W}_1(t, x_0) - \varphi(t, x_0) \) has a strict positive maximum at \((\tilde{t}, \bar{x}_0) \in (\hat{t}, T) \times R^d \) and the above two inequalities hold true. Now we consider the function
\[
\Upsilon(\gamma_t) = W_1(\gamma_t) - \alpha ||\gamma_t - \hat{\xi}_{t,t}||^2_H - \varphi(t, \gamma_t(t)), \quad (t, \gamma_t) \in [\hat{t}, T] \times \Lambda.
\]

By Lemma 4.1 and the definition of \( \bar{W}_1 \), we may assume that
\[
\Upsilon(\hat{\gamma}_t) \geq \Upsilon(\gamma_t), \quad (t, \gamma_t) \in [\hat{t}, T] \times C^{a}_{\mu, M_0, \hat{t}},
\]
with equality at \((\tilde{t}, \bar{\gamma}_t)\) for some \( \bar{\gamma}_t \in C^{a}_{\mu, M_0, \hat{t}} \) satisfying \( \bar{\gamma}_t(\tilde{t}) = \bar{x}_0 \).

Since \( \tilde{t} \leq \hat{t} + \frac{T - \hat{t}}{2} \) and \( |\bar{x}_0| \leq |\tilde{\gamma}_t(\tilde{t})| + \frac{M_0 - |\gamma_t(\tilde{t})|}{2} \), we get \( \tilde{t} < T \) and \( |\gamma_t(\tilde{t})| = |\bar{x}_0| < M_0 \). Thus, the definition of the viscosity subsolution can be used to obtain the following result:
\[
\begin{align*}
&\lim_{\mu \to +\infty} [\varphi(t, \gamma_t(t)) - \frac{\varepsilon}{\nu T}(|\gamma_t||^2_H + |\gamma_t(t)|^2) + \varepsilon \frac{\nu T - \tilde{t}}{\nu T}||\gamma_t||^2 + 2\varepsilon(\tilde{t} - \hat{t}) \nonumber \\
&+ \varepsilon |\gamma_t(\tilde{t}) - \hat{\gamma}_t(\tilde{t})|^2 + \alpha |\gamma_t(\tilde{t}) - \hat{\xi}_t(\tilde{t})|^2 + H(\gamma_t, W(\gamma_t), \nabla_x \varphi(\tilde{t}, \gamma_t(\tilde{t})), 2\varepsilon \frac{\nu T - \tilde{t}}{\nu T} ||\gamma_t(\tilde{t})||^2 + 2\varepsilon \frac{\nu T - \tilde{t}}{\nu T})] \geq 0. \quad (5.10)
\end{align*}
\]

By the definition of \( H \), it follows that there exists a constant \( \bar{c} \) such that \( b = \varphi_t(\tilde{t}, \bar{x}_0) \geq \bar{c} \). Then by the Theorem 8.3 in [9] to obtain sequences \( t_k, s_k \in (\hat{t}, T), \)
\( x_0^k, y_0^k \in R^d \) such that \((t_k, x_0^k) \rightarrow (\hat{t}, \bar{x}_0^k)\), \((s_k, y_0^k) \rightarrow (\hat{t}, \bar{\gamma}_t(\hat{t}))\) as \( k \to +\infty \) and the sequences of functions \( \varphi_k, \psi_k \in C^{1,2}((\hat{t}, T) \times R^d) \) such that
\[
\bar{W}_1(t, x_0) - \varphi_k(t, x_0) \leq 0, \quad \bar{V}_1(t, x_0) + \psi_k(t, x_0) \geq 0.
\]
equalities hold true at \((t_k, x^k_0), (s_k, y^k_0)\), respectively,

\[
(\varphi_k)_t(t_k, x^k_0) \rightarrow b_1, \quad (\psi_k)_s(s_k, y^k_0) \rightarrow b_2,
\]
\[
\nabla_x \varphi_k(t_k, x^k_0) \rightarrow 2\alpha(\gamma^1_k(t) - \dot{\gamma}^2_k(t)), \quad -\nabla_x \psi_k(s_k, y^k_0) \rightarrow 2\alpha(\hat{\gamma}^1_k(t) - \dot{\gamma}^2_k(t)),
\]
\[
\nabla^2_x \varphi_k(t_k, x^k_0) \rightarrow X, \quad \nabla^2_x \psi_k(s_k, y^k_0) \rightarrow Y,
\]

where \(b_1 + b_2 = 0\) and \(X, Y\) satisfy the following inequality:

\[
-12\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \preceq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \preceq 4\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\] (5.11)

We may without loss of generality assume that \(\varphi_k, \psi_k\) grow quadratically at \(\infty\).

Now we consider the function, for \(t, s \in [\bar{t}, T], \gamma^1, \gamma^2 \in \Lambda\),

\[
\Gamma(\gamma^1, \gamma^2) = W_1(\gamma^1) - V_1(\hat{\gamma}^2) - \alpha(\|\gamma^1 - \hat{\xi}_{t,s}\|_H^2 + \|\gamma^2 - \hat{\xi}_{t,s}\|_H^2)
- \varphi_k(t, \gamma^1(t)) - \psi_k(s, \gamma^2(s)).
\]

It has a maximum at some point \((\bar{t}, \bar{s}, \gamma^1_{\bar{t}}, \gamma^2_{\bar{s}})\) in \([\bar{t}, T] \times [\bar{t}, T] \times C^\alpha_{\mu,M_0,t} \times C^\alpha_{\mu,M_0,t}\).

Then by the definitions of \(W_1, V_1\), we get that, for every \(\gamma^1, \gamma^2 \in C^\alpha_{\mu,M_0,t}\),

\[
W_1(\bar{t}, \gamma^1_{\bar{t}}(\bar{t})) - V_1(\hat{\gamma}^2) - \alpha(\|\gamma^1_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2 + \|\gamma^2_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2) - \varphi_k(\bar{t}, \gamma^1_{\bar{t}}(\bar{t})) + \psi_k(\bar{s}, \gamma^2(\bar{s}))
\]

\[
\geq W_1(\gamma^1_{t_k}, x^k_0) - V_1(\gamma^2_{s_k}, y^k_0) - \alpha(\|\gamma^1_{t_k} - \hat{\xi}_{t_k,s_k}\|_H^2 + \|\gamma^2_{s_k} - \hat{\xi}_{t_k,s_k}\|_H^2) - \varphi_k(t_k, \gamma^1_{t_k}(t_k)) + \psi_k(s_k, \gamma^2(s_k)).
\]

Taking the supremum over \(\gamma^1, \gamma^2 \in C^\alpha_{\mu,M_0,t}\), we have that

\[
W_1(\bar{t}, \gamma^1_{\bar{t}}(\bar{t})) - V_1(\hat{\gamma}^2) - \alpha(\|\gamma^1_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2 + \|\gamma^2_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2)
- \varphi_k(\bar{t}, \gamma^1_{\bar{t}}(\bar{t})) + \psi_k(\bar{s}, \gamma^2(\bar{s}))
\]

\[
\geq W_1(t_k, x^k_0) - V_1(s_k, y^k_0) - \varphi_k(t_k, \gamma^1_{t_k}(t_k)) + \psi_k(s_k, \gamma^2(s_k)).
\]

This shows that

\[
\bar{t} = t_k, \bar{s} = s_k, \gamma^1_{\bar{t}} = x^k_0, \gamma^2_{\bar{s}} = y^k_0,
\]

and

\[
W_1(\gamma^1_{\bar{t}}) - V_1(\gamma^2_{\bar{s}}) - \alpha(\|\gamma^1_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2 + \|\gamma^2_{\bar{s}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2) = W_1(t_k, x^k_0) - V_1(s_k, y^k_0).
\] (5.12)

Letting \(k \rightarrow \infty\), by (5.9) we show that

\[
\lim_{k \rightarrow \infty} \left[ W_1(\gamma^1_{\bar{t}}) - V_1(\gamma^2_{\bar{s}}) - \alpha(\|\gamma^1_{\bar{t}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2 + \|\gamma^2_{\bar{s}} - \hat{\xi}_{\bar{t},\bar{s}}\|_H^2) \right]
\]

23
Now from the definition of viscosity solution it follows that

\[ \bar{\lim}_{k \to \infty} \left[ W_1(\gamma^1_i) - V_1(\gamma^2_i) - \alpha(||\gamma^1_i - \hat{\xi}_i, t||_H^2 + ||\gamma^2_i - \hat{\xi}_i, t||_H^2) \right] \leq W_1(\gamma^1_i) - V_1(\gamma^2_i) - \frac{\alpha}{2}||\gamma^1_i - \gamma^2_i||_H^2. \]

On the other hand, since \( \gamma^1_i, \gamma^2_i \in C^\alpha_{\mu, M_0}, t \), we may assume \( \gamma^1_i \to \gamma^1_i, \gamma^2_i \to \gamma^2_i \) for some \( \gamma^1_i, \gamma^2_i \in C^\alpha_{\mu, M_0}, t \). Then we have that

\[ W_1(\gamma^1_i) = V_1(\gamma^2_i) - \frac{\alpha}{2}||\gamma^1_i - \gamma^2_i||_H^2. \]

Therefore, \( \gamma^1_i = \gamma^1_i, \gamma^2_i = \gamma^2_i \). From \( (t_k, x_k^k) \to (\hat{t}, \gamma^1_i(\hat{t})), (s_k, y_k^k) \to (\hat{t}, \gamma^2_i(\hat{t})) \) as \( k \to +\infty, \hat{t} < T \) and \( |\gamma^1_i(\hat{t})| \vee |\gamma^2_i(\hat{t})| < M_0 \), it follows that a constant \( K_0 > 0 \) exists such that

\[ |t_k| \vee |s_k| < T, \quad |x_k^k| \vee |y_k^k| < M_0, \quad \text{for all} \quad k \geq K_0. \]

Now from the definition of viscosity solution it follows that

\[ \lim_{\mu \to +\infty} [(\varphi_k)_{\hat{t}}(\hat{t}, \gamma^1_i(\hat{t})) - \frac{\varepsilon}{\nu T} (||\gamma^1_i(t)||_H^2 + |\gamma^1_i(\hat{t})|^2)] + \varepsilon \frac{\nu T - \hat{t}}{\nu T} |\gamma^1_i(\hat{t})|^2 \]

\[ +2\varepsilon(\hat{t} - \hat{t}) + \varepsilon|\gamma^1_i(\hat{t}) - \hat{\xi}_i(\hat{t})|^2 + \alpha|\gamma^1_i(\hat{t}) - \hat{\xi}_i(\hat{t})|^2 + H(\gamma^1_i, W(\gamma^1_i)), \]

\[ \nabla_x(\varphi_k)_{\hat{t}}(\hat{t}, \gamma^1_i(\hat{t})) + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma^1_i(\hat{t}), \nabla^2_x(\varphi_k)_{\hat{t}}(\hat{t}, \gamma^1_i(\hat{t})) + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma^1_i(\hat{t}) \geq c; \quad (5.13) \]

and

\[ \lim_{\mu \to +\infty} [-(\psi_k)_{\hat{s}}(\hat{s}, \gamma^2_i(\hat{s})) + \frac{\varepsilon}{\nu T} (||\gamma^2_i(s)||_H^2 + |\gamma^2_i(\hat{s})|^2)] - \varepsilon \frac{\nu T - \hat{s}}{\nu T} |\gamma^2_i(\hat{s})|^2 \]

\[ -2\varepsilon(\hat{s} - \hat{t}) - \varepsilon|\gamma^2_i(\hat{s}) - \hat{\xi}_i(\hat{s})|^2 - \alpha|\gamma^2_i(\hat{s}) - \hat{\xi}_i(\hat{s})|^2 + H(\gamma^2_i, V(\gamma^2_i)), \]

\[ -\nabla_x(\psi_k)_{\hat{s}}(\hat{s}, \gamma^2_i(\hat{s})) - 2\varepsilon \frac{\nu T - \hat{s}}{\nu T} \gamma^2_i(\hat{s}), -\nabla^2_x(\psi_k)_{\hat{s}}(\hat{s}, \gamma^2_i(\hat{s})) - 2\varepsilon \frac{\nu T - \hat{s}}{\nu T} \gamma^2_i(\hat{s}) \leq 0. \quad (5.14) \]

Set \( \mu > 0 \) be large enough and let \( k \to \infty \), by (5.13) and (5.14) we obtain

\[ \begin{aligned}
& b_1 - \frac{\varepsilon}{\nu T} (||\gamma^1_i||_H^2 + |\gamma^1_i(\hat{t})|^2) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} |\gamma^1_i(\hat{t})|^2 + \frac{\alpha}{4} |\gamma^1_i(\hat{t}) - \gamma^2_i(\hat{t})|^2 + H(\gamma^1_i), \\
& W(\gamma^1_i), 2\alpha(\gamma^1_i(\hat{t}) - \gamma^2_i(\hat{t})), 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma^1_i(\hat{t}), X + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma^1_i(\hat{t}) \geq \frac{3c}{4}, \quad (5.15)
\end{aligned} \]

and

\[ \begin{aligned}
& -b_2 + \frac{\varepsilon}{\nu T} (||\gamma^2_i||_H^2 + |\gamma^2_i(\hat{t})|^2) - \varepsilon \frac{\nu T - \hat{t}}{\nu T} |\gamma^2_i(\hat{t})|^2 - \frac{\alpha}{4} |\gamma^1_i(\hat{t}) - \gamma^2_i(\hat{t})|^2 + H(\gamma^2_i), \\
& -b_2 + \frac{\varepsilon}{\nu T} (||\gamma^2_i||_H^2 + |\gamma^2_i(\hat{t})|^2) - \varepsilon \frac{\nu T - \hat{t}}{\nu T} |\gamma^2_i(\hat{t})|^2 - \frac{\alpha}{4} |\gamma^1_i(\hat{t}) - \gamma^2_i(\hat{t})|^2 + H(\gamma^2_i),
\end{aligned} \]
Combining (5.15) and (5.16), we obtain

\[ V(\hat{\gamma}_t^2), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^2(i), -Y + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \leq \frac{c}{4}. \]  

(5.16)

Combining (5.15) and (5.16), we obtain

\[
\begin{align*}
\frac{c}{2} + \frac{\varepsilon}{\nu T} & \left(||\hat{\gamma}_t^1||^2_H + ||\hat{\gamma}_t^1(i)||^2\right) \\
& \leq ||\hat{\gamma}_t^1||^2_H + ||\hat{\gamma}_t^2(i)||^2 + \varepsilon \frac{\nu T - \hat{t}}{\nu T} + \frac{\alpha}{2} ||\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)||^2 (||\hat{\gamma}_t^1(i)||^2 + ||\hat{\gamma}_t^2(i)||^2) \\
+ H(\hat{\gamma}_t^1, W(\hat{\gamma}_t^1), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^1(i), X + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \\
- H(\hat{\gamma}_t^2, V(\hat{\gamma}_t^2), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^2(i), -Y - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T}).
\end{align*}
\]  

(5.17)

On the other hand, by (5.11) and a simple calculation we obtain

\[
\begin{align*}
H(\hat{\gamma}_t^1, W(\hat{\gamma}_t^1), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^1(i), X + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \\
- H(\hat{\gamma}_t^2, V(\hat{\gamma}_t^2), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^2(i), -Y - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T}) \\
& \leq \sup_{u \in U}(J_1 + J_2 + J_3),
\end{align*}
\]

(5.18)

where

\[
\begin{align*}
J_1 &= (F(\hat{\gamma}_t^1, u), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^1(i))_{R^d} \\
& - (F(\hat{\gamma}_t^2, u), 2\alpha(\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)) - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} \gamma_t^2(i))_{R^d} \\
& \leq 2\alpha L ||\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)|| \times ||\hat{\gamma}_t^1(i) - \hat{\gamma}_t^2(i)||_0 + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} L ||\hat{\gamma}_t^1(i)|| (1 + ||\hat{\gamma}_t^1(i)|| + ||\hat{\gamma}_t^1||_H) \\
& + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} L ||\hat{\gamma}_t^2(i)|| (1 + ||\hat{\gamma}_t^2(i)|| + ||\hat{\gamma}_t^2||_H); \\
J_2 &= \frac{1}{2} \text{tr}[(X + 2\varepsilon \frac{\nu T - \hat{t}}{\nu T})G(\gamma_t^1, u)G^\top(\gamma_t^1, u)] \\
& - \frac{1}{2} \text{tr}[-Y - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T})G(\gamma_t^2, u)G^\top(\gamma_t^2, u)]
\]

(5.19)
In this section, we let Ω := Λ^t and define a norm on Ω:

\[ ||\omega||_0 := \sup_{0\leq t\leq T} |\omega(t)|. \]

\[ \leq 2\alpha |G(\hat{\gamma}_t^1, u) - G(\hat{\gamma}_t^2, u)|^2 + \frac{\epsilon}{\nu T - \hat{t}}\left(2 + |\hat{\gamma}_t^1(\hat{t})|^2 + |\hat{\gamma}_t^2(\hat{t})|^2 + ||\hat{\gamma}_t^1||_H^2 + ||\hat{\gamma}_t^2||_H^2\right) 
+ 2\alpha L^2 ||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_0^2 \tag{5.20} \]

\[ J_3 = q(\hat{\gamma}_t^1, V(\hat{\gamma}_t^2), G(\hat{\gamma}_t^2, u)(2\alpha(\hat{\gamma}_t^1(\hat{t}) - \hat{\gamma}_t^2(\hat{t}))) + 2\epsilon\frac{\nu T - \hat{t}}{\nu T} ||\hat{\gamma}_t^1(\hat{t})||_0 
- q(\hat{\gamma}_t^2, V(\hat{\gamma}_t^2), G(\hat{\gamma}_t^2, u)(2\alpha(\hat{\gamma}_t^1(\hat{t}) - \hat{\gamma}_t^2(\hat{t}))) + 2\epsilon\frac{\nu T - \hat{t}}{\nu T} ||\hat{\gamma}_t^2(\hat{t})||_0 
\leq L||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_0 + 2\alpha L^2 ||\hat{\gamma}_t^1(\hat{t}) - \hat{\gamma}_t^2(\hat{t})||_0 \times ||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_0 
+ 2\epsilon\frac{\nu T - \hat{t}}{\nu T} L^2 ||\hat{\gamma}_t^1(\hat{t})||_0 + ||\hat{\gamma}_t^1||_H^2 \tag{5.21} \]

Combining (5.17)-(5.21), we obtain

\[ \frac{c}{2} \leq \frac{-\epsilon}{\nu T}(||\hat{\gamma}_t^1||_H^2 + ||\hat{\gamma}_t^1(\hat{t})||_2^2 + ||\hat{\gamma}_t^2||_H^2 + ||\hat{\gamma}_t^2(\hat{t})||_2^2) + (\frac{\alpha}{2} + \alpha L + \alpha L^2)||\hat{\gamma}_t^1(\hat{t}) - \hat{\gamma}_t^2(\hat{t})||_0^2 
+ (3\alpha L + \alpha L)||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_0^2 + L||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_H^2 + 4\epsilon(1 + L + L^2)\frac{\nu T - \hat{t}}{\nu T} \times (1 + ||\hat{\gamma}_t^1||_H^2 + ||\hat{\gamma}_t^1(\hat{t})||_2^2 + ||\hat{\gamma}_t^2||_H^2 + ||\hat{\gamma}_t^2(\hat{t})||_2^2) \leq \alpha(2L + 1)^2(||\hat{\gamma}_t^1(\hat{t}) - \hat{\gamma}_t^2(\hat{t})||_2^2 + ||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_H^2) + L||\hat{\gamma}_t^1 - \hat{\gamma}_t^2||_H^2 \]

\[ -\frac{\epsilon}{2\nu T}\left[\left(\frac{||\hat{\gamma}_t^1||_0}{2} - \frac{1}{2}\right)^2 + \left(\frac{||\hat{\gamma}_t^2||_0}{2} - \frac{1}{2}\right)^2\right] + \frac{\epsilon}{\nu T} \]

Letting α → +∞, the following contradiction is induced:

\[ \frac{c}{2} \leq \frac{c}{4}. \]

The proof is now complete.  

6 Application to stochastic HJB equations.

In this section, we let Ω := Λ^t and define a norm on Ω:

\[ ||\omega||_0 := \sup_{0\leq t\leq T} |\omega(t)|. \]
Then \((\Omega, ||\cdot||_0)\) is a Banach space. We let \(W\) the canonical process, \(P_0\) the Wiener measure, \(\mathcal{B} := \mathcal{B}(\Omega)\) the smallest Borel \(\sigma\)-field generated by \((\Omega, ||\cdot||_0)\) and \(\bar{\mathcal{B}} := \mathcal{B}^{P_0}\) the completeness of \(\mathcal{B}\) with respect to \(P_0\). Then \((\Omega, \mathcal{B}, P_0)\) is a complete space. We let \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) the natural filtration of \(W\), augmented with the family \(\mathcal{N}\) of \(P_0\)-null of \(\bar{\mathcal{B}}\). It is easy to obtain that \(\mathcal{F}_t\) coincides with \(\mathfrak{F}_t \vee \mathcal{N}\).

We consider the controlled state equation:

\[
\dot{X}^{t,x,u}(s) = x + \int_t^s \bar{F}(W_\sigma, \dot{X}^{t,x,u}(\sigma), u(\sigma))d\sigma + \int_t^s \bar{G}(W_\sigma, \dot{X}^{t,x,u}(\sigma), u(\sigma))dW(\sigma), \quad s \in [t, T],
\]

and the associated BSDE:

\[
\bar{Y}^{t,x,u}(s) = \bar{\phi}(W_T, \dot{X}^{t,x,u}(T)) + \int_s^T \bar{q}(W_\sigma, \dot{X}^{t,x,u}(\sigma), \bar{Y}^{t,x,u}(\sigma), \bar{Z}^{t,x,u}(\sigma), u(\sigma))d\sigma - \int_s^T \bar{Z}^{t,x,u}(\sigma)dW(\sigma), \quad s \in [t, T],
\]

with \(F : \Lambda \times R^m \times U \to R^m\), \(G : \Lambda \times R^m \times U \to R^{m \times d}\), \(q : \Lambda \times R^m \times R \times R^d \times U \to R\) and \(\phi : \Lambda_T \times R^m \to R\). The value functional of the optimal control is defined by

\[
\bar{V}(t, x) := \text{ess sup}_{u \in \mathcal{U}} \bar{Y}^{t,x,u}(t), \quad (t, x) \in [0, T] \times R^m.
\]

This problem is path dependent on \(\omega\) and state dependent on \(X(t)\). Now we transform this problem into the path dependent case.

For each \(t \in [0, T]\), define \(\Lambda_{d+m}^t\) as the set of continuous \(R^{d+m}\)-valued functions on \([0, t]\). We denote \(\Lambda_{d+m} = \bigcup_{t \in [0, T]} \Lambda_{d+m}^t\). For any \((\omega_t, \xi_t), (\omega_T, \xi_T) \in \Lambda_{d+m}\), \((y, z) \in R \times R^d\) and \(u \in U\), we define \(F : \Lambda_{d+m} \times U \to R^{d+m}\), \(G : \Lambda_{d+m} \times U \to R^{(d+m) \times d}\), \(q : \Lambda_{d+m} \times R \times R^d \times U \to R\) and \(\phi : \Lambda_{d+m} \to R\) as

\[
F((\omega_t, \xi_t), u) := \begin{pmatrix} 0 \\ \bar{F}(\omega_t, \xi_t(t), u) \end{pmatrix}, \quad G((\omega_t, \xi_t), u) := \begin{pmatrix} 1 \\ \bar{G}(\omega_t, \xi_t(t), u) \end{pmatrix},
q((\omega_t, \xi_t), y, z, u) := \bar{q}(\omega_t, \xi_t(t), y, z, u), \quad \phi(\omega_T, \xi_T) := \bar{\phi}(\omega_T, \xi_T(T)).
\]

We assume \(F, G, g, \phi\) satisfy Hypotheses 3.1, 3.2 and 5.1 then following (3.1), (3.2) and (3.3), for any \((\omega_t, \xi_t) \in \Lambda_{d+m}\) and \(u(\cdot) \in \mathcal{U}\) we can define \(X^{(\omega_t, \xi_t), u}, Y^{(\omega_t, \xi_t), u}\) and \(V(\omega_t, \xi_t) := \text{ess sup}_{u \in \mathcal{U}} Y^{(\omega_t, \xi_t), u}(t)\). Noting \(V(\omega_t, \xi_t)\) only depends on the state \(x = \xi_t(t)\) of the path \(\xi_t\) at time \(t\), we can rewrite \(X^{(\omega_t, \xi_t), u}, Y^{(\omega_t, \xi_t), u}\) and \(V(\omega_t, \xi_t)\) into \(X^{\omega_t, x, u}, Y^{\omega_t, x, u}\) and \(V(\omega_t, x)\), respectively. Then, in view of Theorem 5.2.
Theorem 6.1. Thus we obtain that $V(\bar{\omega}_t, x)$ is a unique viscosity solution to the stochastic HJB equation:

$$
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t V(\omega_t, x) + \sup_{u \in U} [(\nabla_x V(\omega_t, x), F(\omega_t, x, u))]_{\mathbb{R}^d} \\
+ \frac{1}{2} \text{tr}(\nabla^2_x V(\omega_t, x) G(\omega_t, x, u) \bar{G}(\omega_t, x, u)) + \frac{1}{2} \text{tr} D_{\gamma\gamma} V(\omega_t, x) \\
+ \bar{G}(\omega_t, x, u) D_x \gamma(\omega_t, x, v(\omega_t, x), D_\gamma V(\omega_t, x) \\
+ \bar{G}(\omega_t, x, u) \nabla_x \bar{V}(\omega_t, x, u)] = 0, \quad t \in [0, T), \quad \omega \in \Omega,
\end{array} \right.
\end{aligned}
$$

(6.4)

Here, $D_\gamma$ and $D_{\gamma\gamma}$ are the spatial derivatives in $\gamma_t \in \Lambda$, and $\nabla_x$ and $\nabla^2_x$ are the classical partial derivatives in the state variable $x$.

If $V(\omega_t, x)$ is smooth enough, applying functional Itô formula to $V(W_t, x)$, we obtain

$$
dV(W_t, x) = [\partial_t V(W_t, x) + \frac{1}{2} \text{tr} D_{\gamma\gamma} V(W_t, x)]dt + D_\gamma V(W_t, x)dW(t), \quad P_0\text{-a.s.},
$$

Define the pair of $\mathcal{F}_t$-adapted processes

$$
(\bar{V}(t, x), p(t, x)) := (V(W_t, x), D_\gamma V(W_t, x)),
$$

(6.5)

and combine with (6.4), we have

$$
\begin{aligned}
d\bar{V}(t, x) = -\sup_{u \in U} [(\nabla_x \bar{V}(t, x), F(W_t, x, u))]_{\mathbb{R}^d} \\
+ \frac{1}{2} \text{tr}(\nabla^2_x \bar{V}(t, x) G(W_t, x, u) \bar{G}(W_t, x, u)) \\
+ \bar{G}(W_t, x, u) D_x \gamma(W_t, x, \bar{V}(t, x), p(t, x) \\
+ \bar{G}(W_t, x, u) \nabla_x \bar{V}(t, x, u)] + p(t, x)dW(t), \quad t \in [0, T), P_0\text{-a.s.},
\end{aligned}
$$

(6.6)

Thus we obtain that

**Theorem 6.1.** If the value functional $V \in C^{1,2}(\Lambda^{d+m})$, then the pair of $\mathcal{F}_t$-adapted processes $(\bar{V}(t, x), p(t, x))$ defined by (6.4) is a classical solution to (6.6).

Notice that $(\bar{V}(t, x), p(t, x))$ is only dependent on $V$ which is a unique viscosity solution to stochastic HJB equation (6.4), then we can give the definition of viscosity solution to backward stochastic HJB equation (6.6).

**Definition 6.1.** If $V \in C(\Lambda^{d+m})$ is a viscosity solution to stochastic HJB (6.4), then we call $\mathcal{F}_t$-adapted process $\bar{V}(t, x) := V(W_t, x)$ defined by (6.4) is a viscosity solution to backward stochastic HJB equation (6.6).

**Remark 6.1.** (i) In view of Theorem 5.2, $V(\omega_t, x)$ is a unique viscosity solution to the stochastic HJB equation (6.4), then $\bar{V}(t, x)$ defined by (6.5) is a unique viscosity solution to backward stochastic HJB equation (6.6).
If the coefficients in (6.6) are independent of $x$ and $u$, the backward stochastic HJB equation (6.6) reduces to a BSDE:

$$
\begin{cases}
    d\bar{V}(t) = -\bar{q}(W_t, \bar{V}(t), p(t))dt + p(t)dW(t), & t \in [0, T), \\
    \bar{V}(T) = \bar{\phi}(W_T),
\end{cases}
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
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