Quantum Cosmology

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If the cosmological evolution is followed back in time, we come to the initial singularity where the classical equations of general relativity break down. This led many people to believe that in order to understand what actually happened at the origin of the universe, we should treat the universe quantum-mechanically and describe it by a wave function rather than by a classical spacetime. This quantum approach to cosmology was originated by DeWitt and Misner, and after a somewhat slow start has become very popular in the last decade or so. The picture that has emerged from this line of development is that a small closed universe can spontaneously nucleate out of “nothing”, where by “nothing” I mean a state with no classical space and time. The cosmological wave function can be used to calculate the probability distribution for the initial configurations of the nucleating universes. Once the universe nucleated, it is expected to go through a period of inflation, which is a rapid (exponential) expansion driven by the energy of a false vacuum. The vacuum energy eventually thermalized, inflation ends, and from then on the universe follows the standard hot cosmological scenario. Inflation is a necessary ingredient in this kind of scheme, since it gives the only way to get from the tiny nucleated universe to the large universe we live in today.

In this talk I would like to review where we stand in this programme. Despite the large amount of work in quantum cosmology, we still do not have a “standard model”, and I am sure that you would get a very different picture if this talk were to be given by Stephen
Hawking, Jim Hartle or Jonathan Halliwell. I think my view is close to that of Andrei Linde, although we tend to emphasize different things.

**A SIMPLE MODEL**

First I would like to illustrate how the nucleation of the universe can be described in a very simple model. I assume that the universe is homogeneous and isotropic, so that it is described by the closed Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) d\Omega_3^2.$$  

(The universe should be closed, since otherwise its volume would be infinite and the nucleation probability would be zero). The scale factor $a(t)$ satisfies the evolution equation

$$\dot{a}^2 + 1 = \frac{8\pi G}{3} \rho a^2.$$  

The simplest inflationary model is the one in which $\rho$ is a constant vacuum energy density, $\rho = \rho_v$. Then the solution of (2) is the de Sitter space,

$$a(t) = H^{-1} \cosh(Ht),$$  

where

$$H = (8\pi G \rho_v/3)^{1/2}.$$  

The universe contracts at $t < 0$, reaches the minimum radius $a = H^{-1}$ at $t = 0$ and re-expands at $t > 0$.

This is similar to the behavior of a particle bouncing off a potential barrier, with $a$ playing the role of particle coordinate. Now, we know that in quantum mechanics particles can not only bounce off, but can also tunnel through potential barriers. This suggests the possibility that the negative-time part of the evolution in (3) may be absent, and that the
universe may instead tunnel from \( a = 0 \) directly to \( a = H^{-1} \). To see whether this is indeed the case, we should quantize our simple model.

The quantization amounts to replacing the momentum, \( p_a = -a \dot{a} \), conjugate to the variable \( a \) by an operator \(-id/da\). Then, disregarding the factor-ordering ambiguities, the evolution equation (2) yields the Schrodinger equation

\[
\left[ \frac{d^2}{da^2} - U(a) \right] \psi(a) = 0,
\]

where

\[
U(a) = a^2(1 - H^2a^2). \tag{6}
\]

The “potential” \( U(a) \) has the form of a barrier separating \( a = 0 \) and \( a = H^{-1} \), and it is clear that Eq. (5) should have a tunneling solution. This solution is specified by requiring that \( \psi \) has only an outgoing wave at \( a \rightarrow \infty \). For \( H << m_{\text{pl}} \), the wave function can be found using the semiclassical approximation and can be used to calculate the “tunneling probability”\(^8,9\). However, I shall postpone the discussion of probabilities until we consider a more general model.
WAVE FUNCTION OF THE UNIVERSE

In the general case, the wave function of the universe is defined on superspace, which is the space of all 3-geometrics $g_{ij}(x)$ and matter field configurations $\varphi(x)$,

$$\psi[g_{ij}(x), \varphi(x)].$$

(7)

$\psi$ satisfies the Wheeler-DeWitt (WDW) equation$^1$,

$$\mathcal{H}\psi(g_{ij}, \varphi) = 0,$$

(8)

which can be thought of as expressing the fact that the energy of a closed universe is equal to zero. The WDW equation can be symbolically written in the form

$$(\nabla^2 - U)\psi = 0,$$

(9)

which is similar to the Klein-Gordon equation. Here, $\nabla$ is a functional differential operator on superspace and the function $U(g_{ij}, \varphi)$ can be called “superpotential”.

Quantum cosmology is based on quantum gravity and shares all of its problems, in particular the uncontrollable infinities. In addition it has some extra problems which arise when one tries to quantize a closed universe. The first problem stems from the fact that $\psi$ is independent of time. This can be understood$^1$ in the sense that the wave function of the universe should describe everything, including the clocks which show time. In other words, time should be defined intrinsically in terms of the geometric or matter variables. However, no general prescription has yet been found that would give a function $t(g_{ij}, \varphi)$ that would be, in some sense, monotonic. A related problem is the definition of probability. Given a wave function $\psi$, how can we calculate probabilities? One can try to use the conserved current$^{1,2}$

$$J = i(\psi^*\nabla\psi - \psi\nabla\psi^*), \quad \nabla \cdot J = 0.$$  

(10)
The conservation is a useful property, since we want probability to be conserved. But one runs into the same problem as with Klein-Gordon equation: the probability defined in this way is not positive-definite. Although we do not know how to solve these problems in general, they can both be solved in the semiclassical domain. In fact, it is possible that this is all we need.

**SEMICLASSICAL UNIVERSES**

Let us consider the situation when some of the variables \( \{ c \} \) describing the universe behave classically, while the rest of the variables \( \{ q \} \) must be treated quantum-mechanically. Then the wave function of the universe can be written as

\[
\psi = \sum A(c) e^{iS(c)} \chi(c, q) \equiv \Sigma \psi_c \chi,
\]

(11)

where the classical variables are described by the WKB wave functions \( \psi_c = Ae^{iS} \). In the semiclassical approximation, \( \nabla S \) is large, and substitution of (11) into the WDW equation (9) yields the Hamilton-Jacobi equation for \( S(c) \),

\[
\nabla S \cdot \nabla S + U = 0.
\]

(12)

Each solution of (12) is a classical action describing a congruence of classical trajectories (which are essentially the gradient curves of \( S \)). Hence, a semiclassical wave function \( \psi_c = Ae^{iS} \) describes an ensemble of classical universes evolving along the trajectories of \( S(c) \). A probability distribution for these trajectories can be obtained using the conserved current (10). Since the variables \( c \) behave classically, the probabilities do not change in the course of evolution and can be thought of as probabilities for various initial conditions. The time variable \( t \) can be defined as any monotonic parameter along the trajectories, and it can be shown\(^{1,10} \) that in this case the corresponding component of the current \( J \) is non-negative, \( J_t \geq 0 \). Moreover, one finds\(^{11,12,13} \), that the “quantum” wave function \( \chi \)
satisfies the usual Schrodinger equation,

\[ i\partial \chi / \partial t = H \chi \]  

with an appropriate hamiltonian \( H_\chi \). Hence, all the familiar physics is recovered in the semiclassical regime.

This semiclassical interpretation of the wave function \( \psi \) is valid to the extent that the WKB approximation for \( \psi_c \) is justified and the interference between different terms in (11) can be neglected. Otherwise, time and probability cannot be defined, suggesting that the wave function has no meaningful interpretation. In a universe where no object behaves classically (that is, predictably), no clocks can be constructed, no measurements can be made, and there is nothing to interpret.

Suppose for a moment that the cosmological wave function \( \psi \) is known, the semiclassical domain is identified, and the probability distribution for the ensemble of universes is calculated. How can we test this distribution observationally? Strictly speaking, one needs an observer who would survey the universes in the ensemble. We do not know how to get in touch with such an observer and can make predictions only if we assume that we live in a “typical” universe. If the probability distribution has a strong peak, we make a prediction.

It should be noted that the semiclassical approach to time and probability outlined in this section has been implemented only in the context of minisuperspace, which includes only a finite number of degrees of freedom, or perturbative superspace in which all but a few degrees of freedom are treated as small perturbations. An extension to the general case may be non-trivial. For an up to date discussion see the review articles by Kuchar\textsuperscript{14} and Isham\textsuperscript{15}. 

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BOUNDARY CONDITIONS

Thus, to explain the initial conditions of the universe, all we need to do is find the wave function $\psi$ from the WDW equation (9). However, as any differential equation, it has an infinite number of solutions. To get a unique solution, one has to specify some boundary conditions for $\psi$. In ordinary quantum mechanics, the boundary conditions for the wave function are determined by the physical setup external to the system under consideration. In quantum cosmology, there is nothing external to the universe, and a boundary condition should be added to eq. (9) as an independent physical law.

Several candidates for this law of boundary condition have been proposed. One of these is the tunneling boundary condition\textsuperscript{16}, which was inspired by the picture of the universe tunneling from “nothing”. It requires that at the boundaries of superspace $\psi$ should include only outgoing waves. It is not clear whether or not incoming and outgoing waves can be rigorously defined in general, but they certainly can be defined in the semiclassical approximation. As we discussed in the previous section, the WKB wave function (11) describes a congruence of classical trajectories, and the tunneling boundary condition requires that these trajectories can end, but cannot begin at the boundaries. The boundaries of superspace correspond to singular geometries and matter fields, and a typical cosmological trajectory will both begin and end at the boundary. The tunneling condition selects the trajectories which begin at the “barrier”, where some components of $\nabla S$ vanish and the semiclassical approximation breaks down.

A different proposal has been made by Hartle and Hawking\textsuperscript{7,17}, who require that the wave function $\psi(g_{ij}, \varphi)$ should be given by a Euclidean (imaginary-time) path integral over compact 4-geometries bounded by the 3-geometry $g_{ij}$ with the field configuration $\varphi$,

$$\psi = \int [dg_{ij}] [d\varphi] e^{-S_E}.$$  \hfill (14)

This proposal is motivated by mathematical elegance and simplicity. However, as it stands,
The integral (14) is badly divergent, because the Euclidean action $S_E$ is not positive-definite. Attempts to define it by analytic continuation were unsuccessful\textsuperscript{18}, but again the Hartle-Hawking wave function can be well defined in the semiclassical approximation. Linde\textsuperscript{8} has also used a path integral prescription, which in the simple models that have been studied so far, gives the same results as the tunneling wave function.

Of course, these two boundary conditions are not the only possible ones, but I will concentrate on them because they are relatively well studied.

**COSMOLOGICAL PREDICTIONS FROM $\psi$**

To see what kinds of cosmological predictions we can get from different boundary conditions, I would like to consider a somewhat more realistic model than the one I discussed at the beginning. Instead of a constant vacuum energy $\rho_v$, I introduce a scalar field $\varphi$ with a potential $V(\varphi)$. Since vacuum energy is zero (or very small) in our part of the universe, $V(\varphi)$ should have a minimum with $V = 0$. The WDW equation for this two-dimensional model can be solved assuming that $V(\varphi)$ is a slowly-varying function, $|V'/V| \ll m_{\text{pl}}^{-1}$, which is well below the Planck density, $\rho_{\text{pl}} = m_{\text{pl}}^4$, for all values of $\varphi$. A slowly-varying $V(\varphi)$ helps to simplify the equation, but is also necessary for the inflationary scenario. If the condition $V(\varphi) \ll \rho_{\text{pl}}$ is violated, then the semiclassical approximation is not valid and higher-order corrections to quantum gravity are important. Potentials used in particle physics are often unbounded from above; however, one can also consider sigma-model-type theories in which $\varphi$ is defined on a compact manifold and $V(\varphi)$ is bounded. Another possibility would be to allow an infinite range of $\varphi$ with $V(\varphi)$ unbounded from below, but bounded from above.

The tunneling boundary condition supplemented by the condition of regularity, $|\psi| < \infty$, defines a unique wave function $\psi$. The corresponding probability distribution for the
initial values of $\varphi$ in nucleating universes is\textsuperscript{19}
\[
\mathcal{P}_T \propto \exp \left( -\frac{3\rho_{pl}}{8V(\varphi)} \right).
\]
(15)
This probability is strongly peaked at the value $\varphi = \varphi_*$ where $V(\varphi)$ has a maximum. Thus, the tunneling wave function “predicts" that the universe nucleates with the largest possible vacuum energy. This is just the right initial condition for inflation. The high vacuum energy drives the inflationary expansion, while the field $\varphi$ gradually “rolls down" the potential hill, and ends up at the minimum with $V(\varphi) = 0$, where we are now. The predicted initial size of the universe is $a_{\text{min}} = [3/(8\pi GV(\varphi_*)^{1/2})$.

When the same procedure is repeated for the Hartle-Hawking boundary condition, one finds a probability distribution similar to (15), but with a crucial difference in sign,
\[
\mathcal{P}_H \propto \exp \left( +\frac{3\rho_{pl}}{8V(\varphi)} \right).
\]
(16)
This distribution is peaked at $V(\varphi) = 0$, and thus the Hartle-Hawking wave function appears to predict an empty universe with $\rho \approx 0$. Such initial condition does not lead to inflation and is therefore inconsistent with observations.

An attempt to rescue the Hartle-Hawking wave function has been made by Grischuk and Rozhansky\textsuperscript{20}. They assert that Lorentzian classical trajectories for the nucleating universes begin at the caustics formed by a class of Euclidean trajectories. Analysis of the caustics shows that they do not extend into the dangerous region where $V(\varphi)$ is very small, and the authors conclude that no trajectories begin in that region of superspace. I disagree with this picture. It is true that the under-barrier evolution can sometimes be represented as a motion in imaginary time. When this is possible, Euclidean and Lorentzian trajectories can be matched only at points where $\nabla S = 0$, and it is clear from eq.(12) that such points must lie on the surface $U(a, \varphi) = 0$. This “barrier" does not coincide with the caustics of Ref. 20 and extends all the way to $V(\varphi) = 0$. In a more general case, the under-barrier
action is complex \textsuperscript{21}, and no conclusion can be reached on the basis of purely Euclidean trajectories.\textsuperscript{b)}

It appears that the only escape route for the Hartle-Hawking wave function is to relax the condition on $V(\varphi)$ and allow it to reach over-Plankian values\textsuperscript{22}. It is quite possible that Planckian densities were reached or exceeded at the origin of the universe, but we have no idea how to handle this case, and cannot make any predictions.

\textbf{ETERNAL INFATION}

The problem of “eternal inflation” is not strictly a quantum cosmological problem, but it is very important for quantum cosmology, and I think it is appropriate to discuss it here. Once started, inflation never ends completely\textsuperscript{23,24,25,26,27}. Thermalization of the false vacuum energy is a stochastic process and does not occur simultaneously in all space. Regions of false vacuum constantly undergo thermalization, and these thermalized regions expand at the speed approaching the speed of light. However, the inflating regions between them expand even faster and do not allow them to merge and fill the entire universe. Hence, there are large parts of the universe in which inflation continues even at this time.

This picture suggests the possibility that inflation can also be extended to the infinite past, avoiding in this way the problem of initial singularity. The universe would then be in a steady state of eternal inflation which does not have a beginning. It appears, however, that such an extension is impossible and that eternal inflation must have a beginning in time.

The impossibility of inflation without a beginning can be easily understood in the simple de Sitter model (1), (3) with a constant vacuum energy. The metric (3) describes

\textsuperscript{b)} It should be noted that the assumption of slow variation of $V(\varphi)$ breaks down near $V(\varphi) = 0$. Although the probability $\mathcal{P}(\varphi)$ is expected to be large in this region, the analytic approximation (16) cannot be used.
contracting and re-expanding universe. Thermalized regions are driven apart during the expanding phase, but they would easily merge and fill the entire universe during the contraction. The universe would then collapse to a singularity without ever getting to the expanding phase. Hence, deSitter spacetime cannot be used to describe eternal inflation.

In more realistic models, the spacetime is locally approximately de Sitter, but can be globally quite different. To study the problem in the general case, I formulated conditions that a spacetime should satisfy in order to describe an eternally inflating universe. I was then able to prove that these conditions cannot be satisfied in a two-dimensional spacetime and gave a plausibility argument for the more difficult four-dimensional case.

We are thus led to an unexpected and somewhat bizarre conclusion that the universe had a beginning, but will have no end. As inflation continues, new regions of thermalization are formed, and we live in one of these regions. Our region is likely to be at a very large (but finite) separation from the “moment of creation”.

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