Conformal Cosmology with a Positive Effective Gravitational Constant

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Abstract. The conformal cosmological model presented by Mannheim predicts a negative value for the effective gravitational constant, $G_{\text{eff}}$. It also involves a scalar field, $S$, which is treated classically. In this paper we point out that a classical treatment of $S$ is inappropriate, because the Hamiltonian is non-Hermitean, and the theory must be developed in the way pioneered by Bender and others. When this is done, we arrive at a Hamiltonian with an energy spectrum that is bounded below, and also a $G_{\text{eff}}$ that is positive. The resulting theory closely resembles the conventional cosmology based on Einstein relativity.

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1. INTRODUCTION

Mannheim [6] (henceforth M6) has developed a cosmological model (MM) based on conformal invariance. This model has a number of attractive features, but has not so far been accepted as a viable candidate for the correct theory of gravitation, in large part because it predicts, apparently unambiguously, a negative value for the effective gravitational constant, $G_{\text{eff}}$. From this follows a thermal history markedly different from the usual one, and consequent difficulties in addressing questions such as primordial nucleosynthesis [4].

In this paper we point out a mathematical problem in the formulation of the model, and present a new approach that yields a positive $G_{\text{eff}}$, so that the model closely resembles the conventional one.

2. Basic equations of the Mannheim model

The MM involves a scalar field, $S$, in a Friedmann-Robertson-Walker (FRW) background metric. It is this field $S$ that we will be concentrating on in this paper. The treatment of both $S$ and the gravitational field in M6 is classical. Important subsequent work has been done on the quantization of the fourth-order equations for the gravitational field that result from the conformal Lagrangian [7]. The scalar field, $S$, however, has always been treated classically, a procedure that we will question in this paper.

There are several things that are required of $S$ in a viable model:

• 1 An equation of motion that is conformally invariant.
• 2 An energy-momentum tensor whose 00 component (Hamiltonian) has an energy spectrum that is bounded below.
• 3 A constant vacuum expectation value, $S_0$, derived from the equation of motion. The $S_0^2$ in the action generates the effective gravitational constant, $G_{\text{eff}}$.
• 4 A $S_0^4$ term in the action that gives the correct sign for the cosmological constant.

Let us see whether these criteria are met in the MM. Using (as Mannheim does) a metric signature $-, +, +, +$, and neglecting the coupling to the fermion field, the terms in Mannheim’s Lagrangian involving $S$ are (M6 (61)):

$$\mathcal{L} = -(-g)^{1/2} \left( \frac{1}{2} S^{\mu\nu} S_{\mu\nu} - \frac{1}{12} S^2 R_{\mu}^{\mu} + \lambda_M S^4 \right)$$ (1)

This results in an equation of motion (M6 (63)):

$$S^{\mu}_{\mu,\nu} + \frac{1}{6} S R_{\mu}^{\mu} - 4 \lambda_M S^3 = 0$$ (2)

This equation, as required, is conformally invariant.

$S$ is assumed to acquire a non-zero vacuum expectation value, $S_0$, found by setting the derivative term in (2) equal to zero. This gives

$$S_0^2 = \frac{R_{\mu}^{\mu}}{24 \lambda_M}$$ (3)
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as in [3] (13), with \( h = 0 \).

Further development of the model (M6, section 10) shows that, in order to meet observational criteria at the present time, \( R^\mu_\mu \) and \( \lambda_M \) are both negative, so that \( S^2_0 \) is real and positive. The effective gravitational constant, \( G_{\text{eff}} \), in the MM turns out to be negative (M6 (224)):

\[
G_{\text{eff}} = -\frac{1}{4\pi S^2_0} \quad (4)
\]

choosing units with \( c = 1 \).

The stress tensor in the MM model is obtained by taking the variation of the action with respect to the metric, in the usual way (M6 (64); a similar expression is given in [3]). Retaining just the terms involving \( S \) we get

\[
T^\mu_\nu = \frac{2}{3} S^\mu S_\nu - \frac{1}{6} g^\mu_\nu S S^\alpha_\alpha - \frac{1}{3} SS^\mu_\nu + \frac{1}{3} g^\mu_\nu SS^\alpha_\alpha \\
- \frac{1}{6} S^2 \left( R^\mu_\nu - \frac{1}{2} g^\mu_\nu R^\alpha_\alpha \right) - g^\mu_\nu \lambda_M S^4 \quad (5)
\]

The Lagrangian (1) is analogous to the Minkowski space Lagrangian for a \( \phi^4 \) field theory ([8] (7.2.14), with \( H(\phi) = \mu^2 \phi^4 \)):

\[
L_{\text{mink}} = \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - m^2 \phi^2 - \mu^2 \phi^4 \quad (6)
\]

The \( S^2 \) term in (1) has the “right” sign for an \( m^2 \) term. But with \( \lambda_M < 0 \), the \( S^4 \) term has the “wrong” sign, and, in a conventional treatment, will lead to a spectrum that is not bounded below. A theory of this sort has to be treated by methods appropriate to non-Hermitean Hamiltonians [1]. At the level of quantum mechanics, “wrong sign” \( \phi^4 \) theory is well developed. Nevertheless, even in Minkowski space, constructing a corresponding quantum field theory is a difficult problem, still incompletely understood [2]. Conformal cosmology will remain flawed until we can make progress in understanding the scalar field.

3. A new approach to non-Hermitean \( \phi^4 \) theory in Minkowski space

A conformally invariant theory with a single scalar field has a unique action, M6 (61), provided we use the familiar techniques appropriate for Hermitean Lagrangians. Once we recognize that our theory involves a non-Hermitean Lagrangian, however, a new approach is suggested, that we introduce in this section. We begin by working in Minkowski space, but retain \( g^\mu_\nu \) and \((-g)^{1/2}\) in formulae to simplify the transition to a FRW space.

A \( \phi^4 \) theory with a “wrong sign” \( \phi^4 \) term is non-Hermitean but is nevertheless \( \mathcal{PT} \) symmetric, and can be treated by the methods outlined in [1]. The distinctive feature of this approach is the use of the \( \mathcal{CPT} \) norm in place of the usual Dirac norm; for a quantized field, \( S \), we write this norm as

\[
N(S) = \langle |S^{\mathcal{CPT}}S| \rangle \quad (7)
\]
Here $P$ and $T$ represent the usual parity and time-reversal operations, while $C$ represents a special operation designed to ensure the norm is real and positive definite and the theory is unitary. The $C$ operator has to be specifically calculated for each Hamiltonian.

Our cosmological model is written in terms of classical fields (expectation values, $S(x)$), which we take to be real. We assume $S^{\text{CPT}}(x)$ can then be expressed as

$$S^{\text{CPT}}(x) \equiv \int \! d^4 y \, C(x^\mu - y^\mu) S^*(-y)$$

(compare [1] (78)), so that

$$N(S) = \int \! d^4 x \int \! d^4 y \, C(x^\mu - y^\mu) [S^*(z)]_{\nu \rho = -y^\rho} S(x)$$

In an expression of this sort, $S(x)$ and $S^*(-y)$ describe the field $S$ at the same physical point, but use different coordinate systems to refer to that point.

Take the complex conjugate of (9), and let $x^\mu \rightarrow -y^\mu$ and $y^\mu \rightarrow -x^\mu$:

$$N(S) = \int \! d^4 x \int \! d^4 y \, C^*(x^\mu - y^\mu) [S(z)]_{\nu \rho = x^\rho} [S^*(u)]_{\nu \rho = -y^\rho} S(x)$$

showing that $C(x^\mu - y^\mu)$ must be real.

### 3.1. The action

We define our action by

$$I \equiv \int \! d^4 x \, (-g)^{1/2} \left\{ \frac{\sigma_k}{2} g^{\mu \nu} \left[ \frac{\partial S(x)}{\partial x^\mu} \right]^{\text{CPT}} \frac{\partial S(x)}{\partial x^\nu} \right\}$$

$$+ \frac{\sigma_m}{2} m^2 S^{\text{CPT}}(x) S(x) + \sigma_{\mu \nu} \left[ S^{\text{CPT}}(x) S(x) \right]^{2} \right\}$$

(11)

where $\sigma_k$, $\sigma_m$ and $\sigma_{\mu \nu}$ are simply “sign factors”, each of which can be equal to $\pm 1$. We will determine their actual values as we proceed.

### 3.2. Energy-momentum tensor

The energy-momentum tensor is obtained in the usual way by varying the action with respect to the metric:

$$T^{\mu \nu}(x) \equiv \frac{2}{(-g)^{1/2}} \frac{\delta I}{\delta g_{\mu \nu}(x)}$$

$$= \frac{2}{(-g)^{1/2}} \left( \frac{1}{2(-g)^{1/2}} (-g)^{\mu \nu} \left\{ \frac{\sigma_k}{2} g^{\alpha \beta} \left[ \frac{\partial S(x)}{\partial x^\alpha} \right]^{\text{CPT}} \frac{\partial S(x)}{\partial x^\beta} \right\} \right.$$ 

$$+ \frac{\sigma_m}{2} m^2 S^{\text{CPT}}(x) S(x) + \sigma_{\mu \nu} \left[ S^{\text{CPT}}(x) S(x) \right]^{2} \right\}$$

$$- (-g)^{1/2} \frac{\sigma_k}{2} g^{\alpha \beta} g^{\nu \beta} \left[ \frac{\partial S(x)}{\partial x^\alpha} \right]^{\text{CPT}} \frac{\partial S(x)}{\partial x^\beta} \right)$$

$$- \frac{1}{2} g^{\mu \nu} \left[ S^{\text{CPT}}(x) S(x) \right]^{2} \right\}$$

$$- \frac{1}{2} g^{\mu \nu} \left[ S^{\text{CPT}}(x) S(x) \right]^{2} \right\}$$
\[ g^{\mu\nu} \left\{ \frac{\sigma_k}{2} g^{\alpha\beta} \left[ \frac{\partial S(x)}{\partial x^\alpha} \right]^{CPT} \frac{\partial S(x)}{\partial x^\beta} \right. \\
+ \frac{\sigma_m}{2} m^2 S^{CPT}(x) S(x) + \sigma_\mu \mu^2 \left[ S^{CPT}(x) S(x) \right]^2 \left. \right\} \\
- \sigma_k g^{\mu\alpha} g^{\nu\beta} \left[ \frac{\partial S(x)}{\partial x^\alpha} \right]^{CPT} \frac{\partial S(x)}{\partial x^\beta} \] 

(12)

Setting \( g_{\mu\nu} = \text{diag} \ (-1, +1, +1, +1) \), the Hamiltonian is given by

\[ \mathcal{H} \equiv T^{00} \]

\[ = - \left( \frac{\sigma_k}{2} g^{\alpha\beta} \left[ \frac{\partial S(x)}{\partial x^\alpha} \right]^{CPT} \frac{\partial S(x)}{\partial x^\beta} \right) + \frac{\sigma_m}{2} m^2 [S(x)]^{CPT} S(x) \]

\[ + \sigma_\mu \mu^2 \left\{ [S(x)]^{CPT} S(x) \right\}^2 - \sigma_k \left[ \frac{\partial S(x)}{\partial x^0} \right]^{CPT} \frac{\partial S(x)}{\partial x^0} - \frac{\sigma_k}{2} \left[ \nabla S(x) \right]^{CPT} \nabla S(x) \]

\[ - \frac{\sigma_m}{2} m^2 [S(x)]^{CPT} S(x) - \sigma_\mu \mu^2 \left\{ [S(x)]^{CPT} S(x) \right\}^2 \] 

(13)

All four terms in (13) include a (positive definite) \( CPT \) norm. By analogy with ordinary \( \phi^4 \) theory, the first two terms and the last must be positive if we are to have an energy spectrum that is bounded below. We therefore set \( \sigma_k = \sigma_\mu = -1 \).

The sign of the \( \sigma_m \) term will be determined by the requirement of conformal invariance when we go to FRW space.

3.3. Equation of motion

The equation of motion for \( S \) is obtained in the usual way, by requiring the action to be stationary under small variations, \( \delta S \). The appearance of \( S^{CPT} \) in the action is unusual, and the variation requires some care; details are given in the appendix. The equation of motion is given in (A.16). Writing \( \sigma_k = \sigma_\mu = -1 \), this becomes

\[ g^{\mu\nu} \frac{\partial^2 S(x)}{\partial x^\mu \partial x^\nu} - \sigma_m m^2 S(x) + 4\mu^2 \left[ S(x) S^{CPT}(x) \right] S(x) = 0 \] 

(14)

Comparing this with Mannheim’s equation of motion, (2), we infer that we go over to FRW space by setting \( \sigma_m m^2 = -R^{\mu}_{\mu}/6 \). Since \( R^{\mu}_{\mu} < 0 \), we must set \( \sigma_m = +1 \). To maintain Mannheim’s notation as far as possible, we will here define \( \lambda_M = -\mu^2 < 0 \), giving the equation of motion

\[ g^{\mu\nu} \frac{\partial^2 S(x)}{\partial x^\mu \partial x^\nu} + \frac{1}{6} R^{\mu}_{\mu} S(x) - 4\lambda_M \left[ S(x) S^{CPT}(x) \right] S(x) = 0 \] 

(15)

Following Mannheim, we assume that \( S \) develops a constant vacuum expectation value, calculated by setting the derivative term in (15) equal to zero. The result is the same as (3), \( S_0^2 = R^{\mu}_{\mu}/24\lambda_M \), where, as before, \( R^{\mu}_{\mu} < 0 \) and \( \lambda_M < 0 \).
3.4. The action and energy-momentum tensor revisited

We can now use (11) to write the action in FRW space:

$$I \equiv \int d^4x \left( -g \right)^{1/2} \left\{ -\frac{1}{2} g^{\mu\nu} [S(x)]^{CPT}_{\mu} [S(x)]_{\nu} - \frac{R^\sigma}{12} S^{CPT}(x)S(x) + \lambda_M \left[ S^{CPT}(x)S(x) \right]^2 \right\}$$

(16)

From this we can infer the energy-momentum tensor analogous to (5). For a cosmological model comparable to Mannheim’s only the last two terms are of interest, which are

$$T^{\mu\nu} \approx \frac{1}{6} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_\alpha^{\alpha} \right) \left[ S^{CPT}(x)S(x) \right] + g^{\mu\nu} \lambda_M \left[ S^{CPT}(x)S(x) \right]^2$$

(17)

Replacing $S(x)$ by its vacuum expectation value, $S_0$, we get

$$T^{\mu\nu} \approx \frac{1}{6} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_\alpha^{\alpha} \right) S_0^2 + g^{\mu\nu} \lambda_M S_0^4$$

(18)

The important point is that the signs of both these terms are reversed in comparison to (5).

4. Cosmological implications

Mannheim points out (M6, section 10) that because the FRW space is conformally flat, the cosmological equation of motion (CEM) reduces to

$$T^{\mu\nu}_{total} = 0$$

(19)

$T^{\mu\nu}_{total}$ is just the sum of (18) and $T^{\mu\nu}_{kin}$, the contribution from ordinary matter (fermion fields, electromagnetic radiation, etc.). So our CEM, analogous to M6 (222), becomes

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{kin} + \frac{1}{6} S_0^2 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_\alpha^{\alpha} \right) + g^{\mu\nu} \lambda_M S_0^4 = 0$$

(20)

or, as in M6 (223)

$$\frac{1}{6} S_0^2 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_\alpha^{\alpha} \right) = -T^{\mu\nu}_{kin} - g^{\mu\nu} \Lambda_M$$

(21)

with $\Lambda_M \equiv \lambda_M S_0^4 < 0$.

The $S_0^2$ term defines the effective gravitational constant in the theory. We find

$$G_{eff} = \frac{3}{4\pi S_0^2}$$

(22)

analogous to M6 (224), but with a positive sign.

Of particular interest, since it permits an analytic solution, is a model containing radiation only (M6, section 10.2). A corresponding solution exists for the present model also; let us see how it differs. The equation analogous to M6 (226) is (with $H \equiv \dot{R}/R$):

$$\dot{R}^2(t) = \dot{R}^2(t) \left( \Omega_M(t) + \Omega_\Lambda(t) + \Omega_k(t) \right)$$

(23)
\[ \Omega_M(t) = \frac{8\pi G_{\text{eff}} \rho}{3H^2} \]  
\[ \Omega_\Lambda(t) = -\frac{8\pi G_{\text{eff}} \Lambda_M}{3H^2} \]  
\[ \Omega_k(t) = \frac{k}{R^2(t)} \]

\[ \Omega_M(t) + \Omega_\Lambda(t) + \Omega_k(t) = 1 \]  

Since \( k < 0 \) for the open geometry of the MM, all three terms in (27) are positive, in contrast to M6, where \( \Omega_M \) is negative.

Write the solution as in M6 (230), with \( \rho_M(t) = A/R^4 \):

\[ R^2(t) = -\frac{k(\beta - 1)}{2\alpha} - \frac{k\beta \sinh^2(\alpha^{1/2}t)}{\alpha} \]  
\[ \alpha = -\frac{8\pi G_{\text{eff}} \Lambda_M}{3} > 0 \]  
\[ \beta = \left(1 + \frac{16A\lambda_M}{k^2}\right)^{1/2} < 1 \]  

The first term on the right of (28) is negative, not positive as in the MM. This means that \( R \) must start from zero, as in the standard cosmology, not from some finite value, as in the MM. This is illustrated in figure [1]. Both curves were drawn using the formula (28), but in the upper graph \( \beta > 1 \), while in the lower graph \( \beta < 1 \). The origin of \( t \) is conventionally shifted in the lower graph so that \( t = 0 \) is at point B, where \( R = 0 \).
5. How does the new model differ from conventional cosmology?

The present model has two features that are not present in conventional cosmology based on Einstein’s equations.

First, the model includes a scalar field that is essentially massless. The non-zero vacuum expectation value of this field is essential, but we have ignored any excitations. This may be permissible because the field couples very weakly to normal matter, and is difficult to observe, or it may undergo a spontaneous transition that renders it massive.

Second, Mannheim uses (19) for his basic cosmological equation, rather than the more complete one that results from the conformal action, M6 (188):

$$T_{\text{total}}^{\mu\nu} = 4\alpha g W^{\mu\nu}$$

(31)

where $W^{\mu\nu}$ is the Weyl tensor, defined in M6 (107), (108) and (185). Mannheim justifies the neglect of $W^{\mu\nu}$ by observing that a FRW metric is conformally flat, and in such a space $W^{\mu\nu} = 0$. This is true, but if we are to include perturbations to the metric (as, for example, in the study of anisotropies of the CMB) then this neglected term may become important.

6. Conclusion

We have pointed out two flaws in Mannheim’s conformal cosmological model.

• 1 The model predicts, apparently unambiguously, a negative value for the effective gravitational constant, $G_{\text{eff}}$.

• 2 The model involves a scalar field, $S(x)$, that satisfies a conformally invariant equation of motion and develops a vacuum expectation value, $S_0$. The values of the parameters that are needed to satisfy observations lead to a “wrong sign $S^4$ theory”, with a Hamiltonian that has a spectrum that is unbounded below.

We have attempted to apply the techniques appropriate for such Hamiltonians to this cosmological problem, restricting ourselves to the classical limit of field equations that are still imperfectly understood. In this limit, using assumptions that appear reasonable, we find both a positive value for $G_{\text{eff}}$ and a positive definite spectrum for the Hamiltonian.

Our derivation depends on one simple observation, the change of sign as we go from (A.10) to (A.11).

The derivation presented here will remain conjectural until progress is made in two main directions:

• 1 The techniques that have been successfully applied to $\phi^4$ quantum mechanics will have to be developed to cover the corresponding quantum field theory; this seems not to have been achieved at this time [2]. In particular, the form [3] for $S^{\text{CPT}}(x)$ must be shown to be appropriate at the classical level.
2 The various manipulations we have employed are suitable for Minkowski space, but more detailed investigations are needed to show whether they can legitimately be extended to a FRW space.

If, on the other hand, we can accept the present model as viable, without first filling in these important gaps in our understanding, then we have to face the difficult question: how can we conclusively distinguish this model from the conventional one?

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Appendix: variation of the action

The action is defined in (11). We will start with the simplest term,

\[
I_m \equiv \int d^4 x \left( -g \right)^{1/2} \frac{\sigma_m}{2} m^2 S^{CPT}(x) S(x)
\]

\[
= \int d^4 x \int d^4 y \left( -g \right)^{1/2} \frac{\sigma_m}{2} m^2 C(x^\mu - y^\mu) \left[ S^*(z) \right]_{\nu \rho \nu \rho = -y^\rho} S(x)
\]

Let \( S(x) \) vary by a small amount \( \delta S(x) \). Then \( S^*(-x) \) will vary by \( \delta [S^*(-x)] \). The variation of \( I_m \) will be the sum of two terms, \( \delta I_m(1) \) and \( \delta I_m(2) \):

\[
\delta I_m(1) = \int d^4 x \int d^4 y \left( -g \right)^{1/2} \frac{\sigma_m}{2} m^2 C(x^\mu - y^\mu) \left[ S^*(z) \right]_{\nu \rho \nu \rho = -y^\rho} \delta S(x)
\]

\[
\delta I_m(2) = \int d^4 x \int d^4 y \left( -g \right)^{1/2} \frac{\sigma_m}{2} m^2 C(x^\mu - y^\mu) \delta [S^*(z)]_{\nu \rho \nu \rho = -y^\rho} S(x)
\]

Take the complex conjugate of (A.3), and let \( x^\mu \rightarrow -y^\mu \), \( y^\mu \rightarrow -x^\mu \):

\[
\delta I_m^*(2) = \int d^4 x \int d^4 y \left( -g \right)^{1/2} \frac{\sigma_m}{2} m^2 C(x^\mu - y^\mu) \delta S(x) S^*(y)
\]

\[
= \delta I_m(1)
\]

so that

\[
\delta I_m = 2 \text{Re} [\delta I_m(1)]
\]

\[
= \int d^4 x \int d^4 y \left( -g \right)^{1/2} \sigma_m m^2 C(x^\mu - y^\mu) \text{Re} \left\{ [S^*(z)]_{\nu \rho \nu \rho = -y^\rho} \delta S(x) \right\}
\]

The \( \mu^2 \) term in the action can be treated in the same way, to give

\[
\delta I_\mu = 4 \int d^4 x \int d^4 y \left( -g \right)^{1/2} \sigma_\mu \mu^2 C(x^\mu - y^\mu)
\]

\[
\times \left[ S^{CPT}(x) S(x) \right] \text{Re} \left\{ [S^*(z)]_{\nu \rho \nu \rho = -y^\rho} \delta S(x) \right\}
\]

Calculating the variation of the kinetic term starts just as with conventional Lagrangians. We imagine a variation \( \delta (\partial S(x)/\partial x^\nu) \), and convert this to a variation
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of $S(x)$ by an integration by parts, discarding a surface term. Recalling that we are working in Minkowski space, where the metric tensor is constant, we get

$$\delta I_k(1) = - \int d^4x (-g)^{1/2} \frac{\sigma_k}{2} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \left[ \frac{\partial S(x)}{\partial x^\mu} \right]^{\text{CPT}} \delta S(x)$$  \hspace{1cm} (A.7)

Using (8) we write this as

$$\delta I_k(1) = - \int d^4x \int d^4y (-g)^{1/2} \frac{\sigma_k}{2} g^{\mu\nu} \frac{\partial}{\partial y^\nu} C(x^\mu - y^\mu)$$

$$\times \left[ \frac{\partial S^*(z)}{\partial z^\mu} \right]_{\forall \rho: z^\rho = -y^\rho} \delta S(x)$$  \hspace{1cm} (A.8)

$$= \int d^4x \int d^4y (-g)^{1/2} \frac{\sigma_k}{2} g^{\mu\nu} \frac{\partial}{\partial y^\nu} C(x^\mu - y^\mu)$$

$$\times \left[ \frac{\partial S^*(z)}{\partial z^\mu} \right]_{\forall \rho: z^\rho = -y^\rho} \delta S(x)$$  \hspace{1cm} (A.9)

Now integrate by parts and discard a surface term:

$$\delta I_k(1) = - \int d^4x \int d^4y (-g)^{1/2} \frac{\sigma_k}{2} g^{\mu\nu} C(x^\mu - y^\mu)$$

$$\times \frac{\partial}{\partial y^\nu} \left[ \frac{\partial S^*(z)}{\partial z^\mu} \right]_{\forall \rho: z^\rho = -y^\rho} \delta S(x)$$  \hspace{1cm} (A.10)

$$= \int d^4x \int d^4y (-g)^{1/2} \frac{\sigma_k}{2} g^{\mu\nu} C(x^\mu - y^\mu) \left[ \frac{\partial^2 S^*(z)}{\partial z^\mu \partial z^\nu} \right]_{\forall \rho: z^\rho = -y^\rho} \delta S(x)$$  \hspace{1cm} (A.11)

The passage from (A.10) to (A.11) is best illustrated by an example. Suppose

$$\frac{\partial S^*(x)}{\partial x^\mu} = (k_\sigma x^\sigma)^n k_\mu$$  \hspace{1cm} (A.12)

where $k_\mu$ is some fixed vector. Then

$$\frac{\partial}{\partial x^\nu} \left[ \frac{\partial S^*(y)}{\partial y^\mu} \right]_{\forall \rho: y^\rho = -x^\rho} = n(-1)^n (k_\sigma x^\sigma)^{n-1} k_\mu k_\nu$$  \hspace{1cm} (A.13)

and

$$\left[ \frac{\partial^2 S^*(y)}{\partial y^\mu \partial y^\nu} \right]_{\forall \rho: y^\rho = -x^\rho} = n(-1)^{n-1} (k_\sigma x^\sigma)^{n-1} k_\mu k_\nu$$  \hspace{1cm} (A.14)

Note the change in sign from (A.13) to (A.14); it implies that the variation of the kinetic term results in

$$\delta I_k = \Re \left\{ \int d^4x \int d^4y (-g)^{1/2} \sigma_k g^{\mu\nu} C(x^\mu - y^\mu) \left[ \frac{\partial^2 S^*(z)}{\partial z^\mu \partial z^\nu} \right]_{\forall \rho: z^\rho = -y^\rho} \delta S(x) \right\}$$  \hspace{1cm} (A.15)

Now use $\delta L_m + \delta I_\mu + \delta I_k = 0$ for arbitrary (possibly complex) variations $\delta S(x)$ to get the equation of motion

$$\sigma_k g^{\mu\nu} \frac{\partial^2 S(x)}{\partial x^\mu \partial x^\nu} + \sigma_m m^2 S(x) + 4\sigma_\mu^2 \left[ S^{\text{CPT}}(x) S(x) \right] S(x) = 0$$  \hspace{1cm} (A.16)
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