EXISTENCE RESULTS FOR A CLASS OF CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS AND CAPUTO FRACTIONAL DERIVATIVES IN BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for a fractional boundary value problem supplemented with nonlocal Riemann-Liouville fractional integral and Caputo fractional derivative boundary conditions. Our results are based on some known tools of fixed point theory. Finally, some illustrative examples are included to verify the validity of our results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. Fractional differential equations have recently proved to be valuable tools in many fields, such as viscoelasticity, engineering, physics, chemistry, mechanics, and economics, see [19], [22], [21], [23], [12]. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [12], Miller and Ross [20], [24], and the papers [5, 17, 31, 32, 13, 10, 5] and the references therein.

Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering, cellular systems, underground water flow, heat transmission, plasma physics, thermoelasticity, etc. Nonlocal conditions come up when values of the function on the boundary is connected to values inside the domain.

Nonlocal conditions are found to be more plausible than the standard initial conditions for the formulation of some physical phenomena in certain problems of thermodynamics, elasticity and wave propagation. Further details can be found in the work by Byszewski [7, 8].

In recent years, boundary value problems of fractional differential equations with Riemann-Liouville fractional integral and Caputo fractional derivative in boundary conditions have achieved great deal of interest and attention of several researchers. Many authors have studied the existence of solution of the fractional boundary value problems under various boundary conditions and by different approaches. We refer the readers to the papers [11, 2, 3, 11, 14, 15, 16, 18, 26, 27, 28, 29, 30].

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Very recently, Agarwal et al. [1] studied the following fractional order boundary value problem

\[ cD^qx(t) = f(t, x(t)), \quad 1 < q \leq 2, \quad t \in [0, 1], \]

supplemented by boundary conditions, of the form

\[ x(0) = \delta x(\sigma), \quad a cD^px(\zeta_1) + b cD^px(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i), \quad 0 < p < 1. \]

Together with the above fractional differential equation they also investigated the boundary conditions

\[ x(0) = \delta_1 \int_0^\sigma x(s)ds, \quad a cD^px(\zeta_1) + b cD^px(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i), \quad 0 < p < 1, \]

where \( cD^q, cD^p \) denote the Caputo fractional derivatives of orders \( q, p \) and \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is a given continuous function and \( \delta, \delta_1, a, b, \alpha_i \in \mathbb{R} \), with \( 0 < \sigma < \zeta_1 < \beta_1 < \beta_2 < ... < \beta_{m-2} < \zeta_2 < 1 \).

The existence and uniqueness results were proved via some well known tools of the fixed point theory.

In [4], the authors studied the existence and uniqueness of solutions to the fractional differential equation with four-point nonlocal Riemann-Liouville fractional integral boundary conditions of different order given by

\[ cD^qx(t) = f(t, x(t)), \quad 1 < q \leq 2, \quad t \in [0, 1], \]

\[ x(0) = \delta \int_0^\eta x(s)ds, \quad a cD^px(\zeta_1) + b cD^px(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i), \quad 0 < p < 1, \]

where \( cD^q, cD^p \) denote the Caputo fractional derivatives of order \( q, p \), \( f \) is a given continuous function, and \( a, b, \eta, \sigma \) are real constants with \( 0 < \eta, \sigma < 1 \).

The existence results are obtained with the aid of some classical fixed point theorems.

Bashir et al. in [3] discusses the existence and uniqueness of solutions of a new class of fractional boundary value problems

\[ cD^qx(t) = f(t, x(t)), \quad t \in [0, 1], \quad q \in (1, 2], \]

\[ x(0) = 0, \quad x(\xi) = a \int_{\eta}^{1} x(s)ds, \]

where \( cD^q \) denotes the Caputo fractional derivative of order \( q \), \( f \) is a given continuous function, and \( a \) is a positive real constant, \( \xi \in (0, 1) \) with \( \xi < \eta < 1 \). The existence results are obtained with the aid of some classical fixed point theorems.
In [27], Sudsutad and Tariboon studied the existence and uniqueness of solutions for a boundary value problem of fractional order differential equation with three-point fractional integral boundary conditions given by

\[ c^D_q x(t) = f(t, x(t)), \quad t \in [0, 1], \quad q \in (1, 2], \]

\[ x(0) = 0, \quad x(1) = \alpha \int_0^\eta \frac{(\eta - s)^{p-1}}{\Gamma(p)} x(s) ds, \quad 0 < \eta < 1, \quad p > 0, \]

where \( c^D_q \) denotes the Caputo fractional derivative of order \( q \), \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \alpha \in \mathbb{R} \) is such that \( \alpha \neq \Gamma(p + 2)/\eta^{p+1} \).

Tariboon et al. [29] have also studied the following fractional boundary value problem with three-point nonlocal Riemann-Liouville integral boundary conditions

\[ D^\alpha x(t) = f(t, x(t)), \quad 0 < t < T, \quad \alpha \in (1, 2], \]

\[ x(\eta) = 0, \quad I^\nu \left[ x(T) \right] = \int_0^T \frac{(T - s)^{\nu-1}}{\Gamma(\nu)} x(s) ds = 0, \]

where \( D^\alpha \) denotes the Riemann-Liouville fractional derivative of order \( \alpha > 0 \), \( \eta \in (0, T) \) is a given constant. The existence and uniqueness results were proved via the Banach contraction principle, the Banach’s fixed point theorem and H"older’s inequality, the Krasnoselskii fixed point theorem and the Leray-Schauder nonlinear alternative.

In this paper, we introduce a new class of boundary value problems of fractional differential equations supplemented with nonlocal Riemann-Liouville fractional integral and Caputo fractional derivative boundary conditions. In precise terms, we consider the following nonlocal problems:

\[ c^D_q x(t) = f(t, x(t)), \quad t \in [0, 1], \quad \alpha \in (1, 2], \]

\[ c^D_\mu x(\xi) = \beta \ c^D_\nu x(\eta), \quad x(1) = \alpha [I^p x](\eta), \]

where \( c^D_\mu \) is the Caputo fractional derivative of order \( \mu \in \{q, \nu\} \) such that \( 1 < q \leq 2, \ 0 < \nu \leq 1, \ I^p \) is the Riemann-Liouville fractional integral of order \( p > 0 \), and \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a given continuous function, \( 0 \leq \xi < \eta < 1 \) and \( \alpha, \beta \) are appropriate real constants.

The boundary conditions in (1.2) implies that the value of the derivative of the unknown function at the nonlocal position \( \xi \) is proportional to the value of the fractional derivative of the unknown function at the nonlocal position \( \eta \), while the value of the unknown function at the right end point \( (t = 1) \) of the interval \( [0, 1] \) is proportional to the value of the fractional integral of the unknown function at the nonlocal position \( \eta \).

Motivated by the above recent works, the aim of this paper is to investigate the existence and uniqueness of solutions for the problem (1.1)-(1.2).

Our analysis relies on some known fixed point theorems.
2. Preliminaries

In this section, we recall some basic definitions of fractional calculus and an auxiliary lemma to define the solution for the problem (1.1)-(1.2) is presented.

**Definition 2.1.** The Riemann-Liouville fractional integral of order \( q \) for a continuous function \( f \) is defined as

\[
I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0,
\]

provided the integral exists, where \( \Gamma(\cdot) \) is the gamma function, which is defined by \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \).

**Definition 2.2.** For at least \( n \)-times continuously differentiable function \( f : [0, \infty) \to \mathbb{R} \), the Caputo derivative of fractional order \( q \) is defined as

\[
^C D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds, \quad n-1 < q < n, \quad n = [q] + 1,
\]

where \( [q] \) denotes the integer part of the real number \( q \).

**Lemma 2.3** ([12]). For \( q > 0 \), the general solution of the fractional differential equation \( ^C D^q x(t) = 0 \) is given by

\[
x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},
\]

where \( c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, n-1 \quad (n = [q] + 1) \).

According to Lemma 2.3 it follows that

\[
I^q \ ^C D^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},
\]

for some \( c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, n-1 \quad (n = [q] + 1) \).

**Lemma 2.4** ([24], [12]). If \( \beta > \alpha > 0 \) and \( x \in L_1[0, 1] \), then

(i) \( ^C D^\alpha I^\beta x(t) = I^{\beta-\alpha} x(t) \), holds almost everywhere on \([0, 1]\) and it is valid at any point \( t \in [0, 1] \) if \( x \in C[0, 1] \); \( ^C D^\alpha I^\beta x(t) = x(t) \), for all \( t \in [0, 1] \).

(ii) \( ^C D^\alpha t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} t^{\lambda-\alpha-1}, \lambda > [\alpha] \) and \( ^C D^\alpha t^{\lambda-1} = 0, \lambda < [\alpha] \).

**Lemma 2.5.** Let \( \alpha \neq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \) and \( \beta \neq \frac{\Gamma(2-\nu)}{\Gamma(\nu)} \). Then, for any \( h \in C([0, 1], \mathbb{R}) \), the linear fractional boundary value problem

\[
^C D^q x(t) = h(t), \quad 0 < t < 1, \quad 1 < q \leq 2,
\]

\[
x'(\xi) = \beta \, ^C D^\nu x(\eta), \quad x(1) = \alpha[I^\nu x(\eta)], \quad 0 < \nu \leq 1,
\]

has an integral solution given by

\[
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds - \frac{\Gamma(p+1)}{\Delta_1} \int_0^t \frac{(1-s)^{q-1}}{\Gamma(q)} h(s) ds
\]
\[+ \frac{\Gamma(2-\nu)(p+1)\Delta_1 t + \Delta_2}{(p+1)\Delta_1 \Delta_3} \int_0^\xi \frac{(\xi-s)^{q-2}}{\Gamma(q-1)} h(s) ds
\]
\[- \frac{\beta \Gamma(2-\nu)(p+1)\Delta_1 t + \Delta_2}{(p+1)\Delta_1 \Delta_3} \int_0^\eta \frac{(\eta-s)^{q-\nu-1}}{\Gamma(q-\nu)} h(s) ds
\]
\[+ \frac{\alpha p}{\Delta_1} \int_0^\eta \int_0^s \frac{(s-\tau)^{p-1}(s-\tau)^{q-1}}{\Gamma(q)} h(\tau) d\tau ds,
\]

(2.3)
where
\[
\Delta_1 = \Gamma(p + 1) - \alpha \eta^p \\
\Delta_2 = \alpha \eta^{p+1} - \Gamma(p + 2) \\
\Delta_3 = \beta \eta^{1-\nu} - \Gamma(2 - \nu).
\] (2.4)

Proof. From Lemma 2.3, we may reduce (2.1) to an equivalent integral equation
\[
x(t) = I^q h(t) - c_0 - c_1 t,
\] (2.5)
where \(c_0, c_1 \in \mathbb{R}\) are arbitrary constants. Consequently, the general solution of equation (2.1) is
\[
x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds - c_0 - c_1 t,
\] (2.6)
and
\[
x'(t) = I^{q-1} h(t) - c_1.
\] (2.7)
Now, in view of Lemma 2.4, by taking the Caputo fractional derivative of order \(\nu\) to both sides of (2.6), we get
\[
c D^\nu x(t) = I^{q-\nu} h(t) - c_1 \frac{t^{1-\nu}}{\Gamma(2 - \nu)}.
\] (2.8)
From (2.7) and (2.8), the boundary condition \(x'(\xi) = \beta c D^\nu x(\eta)\) implies that
\[
1 = \frac{\Gamma(2 - \nu)}{\Delta_3} \left( \beta \int_0^\eta (\eta-s)^{q-\nu-1} h(s) ds - c_1 \frac{\eta^{1-\nu}}{\Gamma(2 - \nu)} \right),
\] which, on solving, yields
\[
c_1 = \frac{\Gamma(2 - \nu)}{\Delta_3} \left( \beta \int_0^\eta (\eta-s)^{q-\nu-1} h(s) ds - \int_0^\xi (\xi-s)^{q-2} h(s) ds \right).
\]
Using the Riemann-Liouville fractional integral of order \(p\) for (2.6), we obtain
\[
I^p x(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left( \frac{1}{\Gamma(q)} \int_0^s (s-\tau)^{q-1} h(\tau) d\tau - c_0 - c_1 s \right) ds
\]
\[
= \frac{1}{\Gamma(p+1)\Gamma(q)} \int_0^t (t-s)^{p-1} (s-\tau)^{q-1} h(\tau) d\tau ds - c_0 \frac{\alpha \eta}{\Gamma(p+1)} - c_1 \frac{\alpha \eta^{p+1}}{\Gamma(p+2)}.
\]
The second condition of (1.1) implies that
\[
\frac{\alpha}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{p-1}(s-\tau)^{q-1} h(\tau) d\tau ds - c_0 \frac{\alpha \eta^p}{\Gamma(p+1)} - c_1 \frac{\alpha \eta^{p+1}}{\Gamma(p+2)} = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s) ds - c_0 - c_1,
\] (2.9)
which, on inserting the value of \(c_1\) in (2.9), we obtain
\[ c_0 = \frac{1}{\Gamma(q)\Delta_1} \left\{ \Gamma(p + 1) \int_0^1 (1 - s)^{q-1} h(s) \, ds ight. \\
- \alpha \int_0^1 \int_0^s (\eta - s)^{p-1} \Gamma(\eta - \tau)^{q-1} h(\tau) \, d\tau \, ds \\
+ \frac{\Gamma(2 - \nu)}{(p + 1)\Delta_1\Delta_3} \left\{ \beta \int_0^\eta (\eta - s)^{q - \nu - 1} \Gamma(q - \nu) \, h(s) \, ds \\
- \int_0^\xi (\xi - s)^{q - 2} \Gamma(q - 1) \, h(s) \, ds \right\}. \] (2.10)

Substituting the values of \( c_0 \) and \( c_1 \) in (2.6), we obtain the solution (2.3). This completes the proof. \( \square \)

3. Existence results

In this section, we establish sufficient conditions for the existence of solutions to the fractional order boundary value problem (1.1)-(1.2) using certain fixed point theorems.

Let \( C \) be the Banach space of all continuous functions from \([0, 1]\) into \( \mathbb{R} \) equipped with the norm: \( \|x\| = \sup \{ |x(t)|, t \in [0, 1] \} \). We define the operator \( \mathcal{S} : C \to C \) by

\[
(\mathcal{S}x)(t) = \int_0^t (t-s)^{q-1} f(s, x(s)) \, ds - \frac{\Gamma(p+1)}{\Gamma(q)} \int_0^1 (1-s)^{q-1} \Gamma(q) f(s, x(s)) \, ds \\
+ \frac{\Gamma(2 - \nu)}{(p + 1)\Delta_1\Delta_3} \left\{ \beta \int_0^\eta (\eta - s)^{q - \nu - 1} \Gamma(q - \nu) \, h(s) \, ds \\
- \int_0^\xi (\xi - s)^{q - 2} \Gamma(q - 1) \, h(s) \, ds \right\}.
\]

(3.1)

Obviously, the fixed points of the operator \( \mathcal{S} \) are the solutions of the fractional order boundary value problem (1.1)-(1.2).

In order to prove our main results, the following well known fixed point theorems are needed.

**Theorem 3.1** ([9]). Let \( (X, d) \) be a complete metric space and \( T : X \to X \) be a contraction. Then \( T \) has a unique fixed point in \( X \).

**Theorem 3.2** ([25]). Let \( X \) be a Banach space. Assume that \( F : X \to X \) is a completely continuous operator and the set \( V = \{ x \in X : x = \lambda Fx, \ 0 < \lambda < 1 \} \) is bounded. Then \( F \) has a fixed point in \( X \).

**Theorem 3.3** ([25]). Let \( E \) be a closed convex, bounded and nonempty subset of a Banach space \( X \). Let \( A, B \) be the operators such that

1. \( Ax + By \in E \), for any \( x, y \in E \);
2. \( A \) is a completely continuous operator;
3. \( B \) is a contraction operator.

Then there exists at least one fixed point \( z \in E \) such that \( z = Az + Bz \).
In the following, for computational convenience, we set
\[
\Omega = \frac{1}{\Gamma(q+1)} + \frac{\Gamma(2-\nu)((p+1)|\Delta_1| + |\Delta_2|)}{(p+1)|\Delta_1\Delta_3|} \left( \frac{\beta \eta^{q-\nu}}{\Gamma(q-\nu+1)} + \frac{\xi^{q-1}}{\Gamma(q)} \right) + \frac{\Gamma(p+1)}{|\Delta_1|} \left( \frac{1}{\Gamma(q+1)} + \frac{\alpha \eta^{p+q}}{\Gamma(p+q+1)} \right). \tag{3.2}
\]

Now, we are in a position to present the main results of this paper. The first one existence result is based on Banach’s contraction mapping principle 3.1.

**Theorem 3.4.** Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying the Lipschitz condition:

\[(H_1)\ |f(t, x) - f(t, y)| \leq L|x - y|, \quad L > 0, \quad \forall t \in [0,1], \quad x, y \in \mathbb{R}.
\]

Then the problem \((1.1), (1.2)\) has a unique solution on \([0,1]\) provided that \(L \Omega < 1\), where \(\Omega\) is given by \(3.2\).

**Proof.** Setting \(\sup\{|f(t,0)|, t \in [0,1]\} = M < \infty\) and define \(B_\rho = \{x \in C : \|x\| \leq \rho\}\), where

\(\rho \geq \Omega M (1 - \Omega L)^{-1}\).

As a first step, we show that \(\mathcal{S} B_\rho \subset B_\rho\). From \((H_1)\), for \(x \in B_\rho\), and \(t \in [0,1]\), we get

\[
|f(t, x(t))| \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
\leq L \|x\| + M \\
\leq L \rho + M.
\]

Using \((3.1)\) and \((3.3)\), we obtain

\[
\|\mathcal{S} x\| \leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + \frac{\Gamma(p+1)}{|\Delta_1|} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right. \\
+ \frac{\Gamma(2-\nu)((p+1)|\Delta_1| + |\Delta_2|)}{(p+1)|\Delta_1\Delta_3|} \int_0^\xi \frac{(\xi-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\\n+ \frac{\beta \Gamma(2-\nu)((p+1)|\Delta_1| + |\Delta_2|)}{(p+1)|\Delta_1\Delta_3|} \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q-\nu)} |f(s, x(s))| ds \\\n+ \frac{\alpha \eta^{p+q}}{|\Delta_1|} \int_0^\xi \int_0^\eta \frac{(\eta-s)^{p+q-(s-\tau)^{q-1}}}{\Gamma(q)} |f(\tau, x(\tau))| d\tau ds \right\} \\
\leq (L \rho + M) \sup_{t \in [0,1]} \left\{ \frac{\eta^q}{\Gamma(q+1)} + \frac{\Gamma(2-\nu)((p+1)|\Delta_1| + |\Delta_2|)}{(p+1)\Gamma(q)|\Delta_1\Delta_3|} \xi^{q-1} \right. \\
+ \frac{\beta \Gamma(2-\nu)((p+1)|\Delta_1| + |\Delta_2|)}{(p+1)\Gamma(q-\nu+1)|\Delta_1\Delta_3|} \eta^{q-\nu} + \frac{\Gamma(p+1)}{\Gamma(q+1)|\Delta_1|} \\
+ \frac{\alpha \Gamma(p+1)}{\Gamma(p+q+1)|\Delta_1|} \xi^{p+q} \right\} \\
\leq (L \rho + M) \Omega \leq \rho.
\]

Thus, \(\mathcal{S} B_\rho \subset B_\rho\). Now, for \(x, y \in C\), we have
According to the condition $L \Omega < 1$, it follows that the operator $\mathcal{S}$ is a contraction. Therefore, by Theorem 3.1 (Banach’s contraction principle), there exists a unique fixed point in $B_\rho$ for the operator $\mathcal{S}$ which is a unique solution for the problem (1.1)-(1.2). This completes the proof.

Now, we establish another existence result for BVP (1.1)-(1.2) by applying Schaefer’s fixed point Theorem 3.2.

**Theorem 3.5.** Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the assumption $(H_1)$. In addition, it is assumed that $(H_2)$ there exists a constant $L > 0$ such that $|f(t, x)| \leq L$, for all $t \in [0, 1]$, $x \in \mathbb{R}$.

Then the problem (1.1)-(1.2) has at least one solution on $[0, 1]$.

**Proof.** We prove that the operator $\mathcal{S}$ defined by (3.1) has a fixed point by utilizing Schaefer’s fixed point theorem. The proof consists of several steps. Firstly, we show that the operator $\mathcal{S}$ is continuous.

Let $x_n$ be a sequence such that $x_n \rightarrow x$ in $C$. Then for each $t \in [0, 1]$, we have

\[
\|(\mathcal{S}x_n)(t) - (\mathcal{S}x)(t)\| \leq \left\{ \begin{array}{l}
\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{\Gamma(p+1)}{|\Delta_1|} \int_0^1 \frac{1-s)^{q-1}}{\Gamma(q)} ds \\
+ \frac{\Gamma(2-\nu)}{(p+1)|\Delta_1|} \int_0^1 \frac{(\zeta-s)^{q-2}}{\Gamma(q)} ds \\
+ \frac{\beta\Gamma(2-\nu)}{(p+1)|\Delta_1|} \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} ds \\
+ \frac{\alpha p}{|\Delta_1|} \int_0^\eta \frac{(\eta-s)^{p-1}(s-\tau)^{q-1}}{\Gamma(q)} d\tau ds \right\} \times \|f(., x_n(.)) - f(., x(.))\| \\
\leq L \Omega \|x_n - x\|.
\]

Since $f$ is continuous, then $\|\mathcal{S}x_n - \mathcal{S}x\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\mathcal{S}$ is continuous.

Now, it will be shown that $\mathcal{S}$ maps bounded sets into bounded sets in $C$. For $\rho > 0$, let $B_\rho = \{x \in C : \|x\| \leq \rho\}$ be bounded set in $C$. In view of the condition $(H_2)$, it is easy to establish that $\|\mathcal{S}x\| \leq L \Omega = M, x \in B_\rho$. 

Thus $\mathcal{S}$ is uniformly bounded on $B_\rho$. Moreover, for $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and $x \in B_\rho$, we get the following estimates

$$
|\mathcal{S}x(t_2) - \mathcal{S}x(t_1)| \leq \int_0^{t_1} \frac{(t_2 - s)^{q-1} - (t_1 - s)^{q-1}}{\Gamma(q)}|f(s, x(s))|ds
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)}|f(s, x(s))|ds
+ \frac{\beta \Gamma(2-\nu)(t_2 - t_1)}{\Gamma(q-\nu)|\Delta_3|} \int_0^{\eta} (\eta - s)^{q-\nu-1}|f(s, x(s))|ds
+ \frac{\Gamma(2-\nu)(t_2 - t_1)}{\Gamma(q-1)|\Delta_3|} \int_0^{\xi} (\xi - s)^{-2}|f(s, x(s))|ds
\leq L \left\{ \frac{1}{\Gamma(q+1)}[(t_2^2 - t_1^2) + 2(t_2 - t_1)^2] + \frac{\Gamma(2-\nu)}{|\Delta_3|} \times \left[ \frac{\beta \eta^{q-\nu}}{\Gamma(q-\nu + 1)} + \frac{\xi^{q-1}}{\Gamma(q)} \right] \right\}
$$

As $t_2 \to t_1$, the right-hand side tends to zero independently of $x \in B_\rho$. Thus, by the Arzelà-Ascoli theorem, the operator $\mathcal{S}$ is completely continuous.

Next, we need to show that the set $\mathcal{V} = \{ x \in C : x = \lambda \mathcal{S}x, \ 0 < \lambda < 1 \}$ is bounded.

Let $x \in \mathcal{V}$ and $t \in [0, 1]$. Then

$$
x(t) = \lambda \left\{ \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \frac{\Gamma(p + 1)}{\Delta_1} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \right. \\
+ \frac{\Gamma(2-\nu)(p + 1)\Delta_1 t + \Delta_2}{(p + 1)\Delta_1 \Delta_3} \int_0^\xi \frac{(\xi - s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds \\
- \frac{\beta \Gamma(2-\nu)(p + 1)\Delta_1 t + \Delta_2}{(p + 1)\Delta_1 \Delta_3} \int_0^{\eta} \frac{(\eta - s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s))ds \\
+ \frac{\alpha p \Delta_1}{\Delta_1} \int_0^{s} \frac{(\eta - s)^{p-1}(s - \tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau))d\tau d\tau \\
\right\},
$$

which implies using $\lambda < 1$ that

$$
\|x\| = \sup_{t \in [0,1]} |\lambda(\mathcal{S}x)(t)| \leq L\Omega = M.
$$

Therefore, $\mathcal{V}$ is bounded. By Schaefer’s fixed point Theorem 3.2, we conclude that the operator $\mathcal{S}$ has a fixed point which is a solution of the fractional order boundary value problem (1.1)-(1.2). This completes the proof.

Our next result on existence and uniqueness is based on Krasnoselskii’s fixed point Theorem 3.3.

**Theorem 3.6.** Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption $(H_1)$. Moreover, it is assumed that

$(H_3) \ |f(t, x)| \leq \sigma(t), \ \forall (t, x) \in [0, 1] \times \mathbb{R}$, where $\sigma \in C([0, 1], \mathbb{R}^+)$. Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0, 1]$ if

$$
L \left( \Omega - \frac{1}{\Gamma(q + 1)} \right) < 1,
$$

where $\Omega$ is defined by (3.21).
where $\Omega$ is given by (3.2).

Proof. If we denote $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$, where $\rho \geq \Omega\|\sigma\|$ with $\|\sigma\| = \sup_{t \in [0,1]} |\sigma(t)|$, and $\Omega$ is given by (3.2). Then $B_\rho$ is a bounded closed convex subset of $\mathcal{C}$.

For $t \in [0,1]$, we define two operators on $B_\rho$ as

$$
(\mathcal{S}_1x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds
$$

and

$$
(\mathcal{S}_2x)(t) = \frac{\Gamma(2-\nu)(p+1)|\Delta_1 t + \Delta_2|}{(p+1)|\Delta_1|\Delta_3} \int_0^\xi \frac{(\xi-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds
$$

$$
+ \frac{\alpha p}{\Delta_1} \int_0^\eta f(s) (s-t)^{p-1}(s-t)^{q-1} \frac{d\tau ds}{\Gamma(q)}
$$

$$
- \frac{\Gamma(p+1)}{\Delta_1} \int_0^1 (1-s)^{q-1} f(s, x(s)) ds
$$

$$
- \frac{\beta \Gamma(2-\nu)(p+1)|\Delta_1 t + \Delta_2|}{(p+1)|\Delta_1|\Delta_3} \int_0^\eta \frac{(\eta-s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s)) ds.
$$

For $x, y \in B_\rho$, we find that $\|\mathcal{S}_1x + \mathcal{S}_1y\| \leq \Omega\|\sigma\| \leq \rho$, which implies that $\mathcal{S}_1x + \mathcal{S}_2y \in B_\rho$.

Using $(H_1)$ and (3.2), for $x, y \in \mathcal{C}$, we obtain

$$
\|\mathcal{S}_1x - \mathcal{S}_2y\| \leq \sup_{t \in [0,1]} \left\{ \frac{\Gamma(2-\nu)(p+1)|\Delta_1 t + |\Delta_2|}{(p+1)|\Delta_1|\Delta_3} \int_0^\xi \frac{(\xi-s)^{q-2}}{\Gamma(q-1)} ds
$$

$$
+ \frac{\alpha p}{|\Delta_1|} \int_0^\eta f(s) (s-t)^{p-1}(s-t)^{q-1} \frac{d\tau ds}{\Gamma(q)}
$$

$$
+ \frac{\Gamma(p+1)}{|\Delta_1|} \int_0^1 (1-s)^{q-1} ds
$$

$$
+ \frac{\beta \Gamma(2-\nu)(p+1)|\Delta_1 t + |\Delta_2|}{(p+1)|\Delta_1|\Delta_3} \int_0^\eta \frac{(\eta-s)^{q-\nu-1}}{\Gamma(q-\nu)} ds \right\} L\|x - y\|
$$

$$
\leq L (\Omega - 1/\Gamma(q+1)) \|x - y\|,
$$

which shows that the operator $\mathcal{S}_2$ is a contraction since $L(\Omega - 1/\Gamma(q+1)) < 1$.

For $x \in B_\rho$, we have

$$
\|\mathcal{S}_1x\| \leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right\} \leq \frac{\|\sigma\|}{\Gamma(q+1)}.
$$

Therefore, $\mathcal{S}_2$ is uniformly bounded on $B_\rho$. Now, we prove the compactness of the operator $\mathcal{S}_1$. 

Let \( t_1, t_2 \in [0; 1] \) with \( t_1 < t_2 \) and \( x \in B_\rho \). Then, we obtain
\[
|\mathcal{G}_1(x)(t_2) - \mathcal{G}_1(x)(t_1)| \leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} |f(s, x(s))| ds \\
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} |f(s, x(s))| ds \\
\leq \frac{\hat{f}}{\Gamma(\alpha + 1)} \left( (t_2^\alpha - t_1^\alpha) + 2(t_2 - t_1)^\alpha \right),
\]
where \( \sup_{(t,x) \in [0,1] \times B_\rho} |f(t, x)| = \hat{f} \). Obviously, the right-hand side of the above inequality tends to zero independently of \( x \in B_\rho \) as \( t_2 \to t_1 \). So \( \mathcal{G}_1 \) is relatively compact on \( B_\rho \). Hence, by the Arzelà-Ascoli theorem, \( \mathcal{G}_1 \) is compact on \( B_\rho \). Continuity of \( f \) implies that the operator \( \mathcal{G}_1 \) is continuous. Therefore, \( \mathcal{G}_1 \) is completely continuous. Thus all the hypothesis of Theorem 3.3 are satisfied and consequently the problem [10, 12] has at least one solution on \([0, 1]\). This completes the proof. \( \Box \)

4. Examples

Example 4.1. Consider the following fractional boundary value problem
\[
\begin{align*}
\left\{ \begin{array}{l}
cD^{\frac{\alpha}{\eta}} x(t) = e^{-\cos^2 t} \sin x + \sqrt{\frac{\pi}{2}} t^{\frac{\alpha}{\eta}} - t^{\frac{\alpha}{\eta}}, \quad t \in [0, 1], \\
x'(\frac{1}{2}) = cD^{\frac{\alpha}{\eta}} x(\frac{1}{2}), \quad x(1) = 3[I^{\alpha} x](\frac{1}{2}).
\end{array} \right.
\end{align*}
\]
(4.1)

Here, \( \alpha = 3, \quad \beta = 1, \quad \eta = \frac{1}{2}, \quad \xi = \frac{1}{3}, \quad q = \frac{3}{4}, \quad p = \frac{3}{4}, \quad \nu = \frac{1}{2}, \) and \( f(t, x) = \frac{e^{-\cos^2 t}}{(3^{\xi} + 1)\sqrt{t+\frac{1}{2}}} \sin x + \frac{\sqrt{\pi}}{2} t^\alpha - t^\alpha \). With the given values, it is easy to see that \( \alpha = 3 \neq \frac{\Gamma(p+1)}{\eta^{\frac{\alpha}{\eta}}} = \sqrt{\frac{\pi}{2}}, \beta = 1 \neq \frac{\Gamma(2-\nu)}{\eta^{\frac{\alpha}{\eta}}} = \frac{1}{\sqrt{\eta}}, \Delta_1 = \frac{3}{2}|\Delta_3| \) with \( |\Delta_3| = \frac{1}{2}(\sqrt{\pi} - \sqrt{2}), \Delta_2 = \frac{3}{8}(5\sqrt{\pi} - \sqrt{2}) \), and \( \Omega \approx 0.4684 \).

Clearly, \( L = \frac{1}{4^{\frac{1}{\eta}}} \) as \( |f(t, x) - f(t, y)| \leq \frac{1}{4^{\frac{1}{\eta}}} |x - y| \). Furthermore, upon computation, we get
\[
L \Omega \approx \frac{1}{4^{\frac{1}{\eta}}} \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{\Gamma(2-\nu)}{(p+1)|\Delta_3|} \right] ^\frac{\alpha q^{\frac{\alpha}{\eta} - q}}{\alpha q^{\frac{\alpha}{\eta} - q}}
\]
\[
\leq \frac{1}{\Gamma(\alpha + 1)} + \frac{\Gamma(2-\nu)}{(p+1)|\Delta_3|} \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{\Gamma(2-\nu)}{(p+1)|\Delta_3|} \right] \approx 0.2810 < 1.
\]

Thus, for the given boundary value problem (4.1), all the conditions of Theorem 3.4 are satisfied. So, by Theorem 3.4 there exists a unique solution for the problem (4.1) on \([0, 1]\).

Example 4.2. Consider a fractional boundary value problem given by
\[
\begin{align*}
\left\{ \begin{array}{l}
cD^{\frac{\alpha}{\eta}} x(t) = e^{-2[2+\sin(t^2-t)]} \sin x + \sqrt{\frac{\pi}{2}} t^\alpha - t^\alpha, \quad t \in [0, 1], \\
x'(\frac{1}{2}) = cD^{\frac{\alpha}{\eta}} x(\frac{1}{2}), \quad x(1) = \frac{1}{2}[I^{\alpha} x](\frac{1}{2}).
\end{array} \right.
\end{align*}
\]
(4.2)

where, \( \alpha = \frac{1}{2}, \quad \beta = \frac{1}{4}, \quad \eta = \frac{3}{4}, \quad \xi = \frac{1}{5}, \quad q = \frac{3}{2}, \quad p = \frac{4}{3}, \quad \nu = \frac{2}{3}, \) and \( f(t, x) = \frac{e^{-2[2+\sin(t^2-t)]}}{(3^{\xi} + 1)\sqrt{t+\frac{1}{2}}} \sin x + \frac{\sqrt{\pi}}{2} t^\alpha - t^\alpha \). By simple calculations, we find that \( \frac{\Gamma(2-\nu)}{\eta^{\frac{\alpha}{\eta}}} = \frac{1}{\Gamma(1/3)} \Gamma(1/3), \quad \frac{\Gamma(p+1)}{\eta^{\frac{\alpha}{\eta}}} = \frac{1}{\Gamma(4/3)} \Gamma(4/3), \quad \frac{\Gamma(p+1)}{\eta^{\frac{\alpha}{\eta}}} = \frac{1}{\Gamma(4/3)} \Gamma(4/3) \).
As a third example we consider the fractional boundary value problem

\[ \left\{ \begin{array}{l}
\frac{d^\alpha}{dt^\alpha} x(t) = -\frac{e^{-t} \cos(t \sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{1+25}} + \frac{t}{t+1}, \quad t \in [0, 1], \\
\phi(x) = \frac{3}{2} \frac{c}{c} D^{\phi} x(\frac{3}{2}), \quad x(1) = \frac{2}{1+1} D^{\phi} x(\frac{3}{2}).
\end{array} \right. \]  

(4.3)

where, \( \alpha = 2, \beta = \frac{3}{2}, \eta = \frac{3}{2}, \xi = \frac{3}{2}, q = \frac{3}{2}, p = \frac{3}{2}, \nu = \frac{1}{4}, \) and \( f(t, x) = -\frac{e^{-t} \cos(t \sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{1+25}} + \frac{t}{t+1}. \) With the given values, it is found that \( \alpha = 2 \neq \frac{\Gamma(p+1)}{\eta^p} = \frac{3}{2} \frac{c}{c} D^{\phi} x(\frac{3}{2}) \approx 2.6361, \beta = \frac{3}{2} \neq \frac{\Gamma(2-\nu)}{\eta^{1-\nu}} = \frac{(3/4)(3/4)}{(\frac{1}{4})^2} \approx 1.1829, \) \( \Delta_1 = 0.3631, \) \( |\Delta_2| = 3.1968, \Delta_3 = 0.2464, \) and \( \Omega - \frac{1}{\Gamma(q+1)} \approx 35.5044. \) Since \( |f(t, x) - f(t, y)| \leq \frac{c}{c} |x - y|, \) then \( (H_1) \) is satisfied with \( L = \frac{1}{45}. \) Further,

\[ |f(t, x)| = \left| -\frac{e^{-t} \cos(t \sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{1+25}} + \frac{t}{t+1} \right| \leq \frac{e^{-t}}{45} + \frac{t}{t+1}. \]

Obviously all the conditions of Theorem 3.6 are satisfied with \( L(\Omega - 1/\Gamma(q+1)) \approx 0.7890 < 1. \) Hence, by Theorem 3.7 the fractional order boundary value problem \((4.3)\) has at least one solution on \([0, 1].\)

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