THE SEARCH FOR DIFFERENTIAL EQUATIONS FOR ORTHOGONAL POLYNOMIALS BY USING COMPUTERS

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Abstract

We look for differential equations of the form
\[ \sum_{i=0}^{\infty} c_i(x) y^{(i)}(x) = \lambda_n y(x), \]
where the coefficients \( \{c_i(x)\}_{i=0}^{\infty} \) do not depend on \( n \), for the generalized Jacobi polynomials \( \{P_{\alpha,\beta,M,N}^n(x)\}_{n=0}^{\infty} \) found by T.H. Koornwinder in 1984 and for generalized Laguerre polynomials \( \{L_{\alpha,M,N}^n(x)\}_{n=0}^{\infty} \) which are orthogonal with respect to an inner product of Sobolev type.

We introduce a method which makes use of computer algebra packages like Maple and Mathematica and we will give some preliminary results.
1 Introduction

In 1984 T.H. Koornwinder (see [3]) introduced the polynomials $\{P_{n}^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}(1-x)^{\alpha}(1+x)^{\beta} + M\delta(x+1) + N\delta(x-1).$$

These generalized Jacobi polynomials generalize the Legendre type and the Jacobi type polynomials found by H.L. Krall. See for instance [9], [10] and [7].

As a limit case Koornwinder also found generalized Laguerre polynomials $\{L_{n}^{\alpha,M}(x)\}_{n=0}^{\infty}$ which are orthogonal on the interval $[0,\infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha + 1)}x^{\alpha}e^{-x} + M\delta(x).$$

It is well-known that these generalized Jacobi and generalized Laguerre polynomials satisfy a linear second order differential equation with coefficients depending on $n$, but of bounded degree. See for instance [3].

Now we are looking for differential equations of the form

$$\sum_{i=0}^{\infty} c_{i}(x)y^{(i)}(x) = \lambda_{n}y(x),$$

where the coefficients $\{c_{i}(x)\}_{i=0}^{\infty}$ are independent of the degree $n$. In [3], [10] and [7] several special cases were treated. Later more special cases were found by A.M. Krall and L.L. Littlejohn. See [12], [8] and [11].

These differential equations we are looking for are studied in spectral theory. A survey of orthogonal polynomials and spectral theory was given in [1]. A large number of references can be found there.

In [2] J. Koekoek and R. Koekoek found a differential equation of the above type for the generalized Laguerre polynomials $\{L_{n}^{\alpha,M}(x)\}_{n=0}^{\infty}$ for all $\alpha > -1$. This result generalizes the results found for special cases of the Laguerre type polynomials. In fact we showed that this differential equation is of infinite order in general, but reduces to finite order for nonnegative integer values of $\alpha$. See [2] for more details and section 3 of this paper for more results concerning the coefficients of this differential equation.

In [5] R. Koekoek and H.G. Meijer introduced the polynomials $\{L_{n}^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to the following (Sobolev) inner product:

$$< f, g > = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} x^{\alpha}e^{-x} f(x)g(x)dx + Mf(0)g(0) + Nf'(0)g'(0),$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. These polynomials $\{L_{n}^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ generalize the polynomials $\{L_{n}^{\alpha,M}(x)\}_{n=0}^{\infty}$ since $L_{n}^{\alpha,M,0}(x) = L_{n}^{\alpha,M}(x)$.

In this paper we deal with the problem of finding a differential equation of the above type for both the generalized Jacobi polynomials $\{P_{n}^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ and the polynomials $\{L_{n}^{\alpha,M,N}(x)\}_{n=0}^{\infty}$.
The differential equation found in [2] was computed by hand, without the help of computers. This is nearly impossible for these other cases. We need computers to handle the very huge expressions we have to deal with.

2 The method

In this section we will describe the method we use to discover a linear differential equation satisfied by polynomials which are defined as a linear combination of classical orthogonal polynomials and their derivatives. These polynomials depend on $x$, the parameters of the classical orthogonal polynomials and on some extra parameters, say $M$ and $N$.

As an example we look at the polynomials $\{L_{n}^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ which are defined by

$$L_{n}^{\alpha,M,N}(x) = A_{0}L_{n}^{(\alpha)}(x) + A_{1}\frac{d}{dx}L_{n}^{(\alpha)}(x) + A_{2}\frac{d^{2}}{dx^{2}}L_{n}^{(\alpha)}(x),$$

where the coefficients $A_{0}$, $A_{1}$ and $A_{2}$ depend on $n$, $\alpha$, $M$ and $N$. Moreover, $A_{0}$, $A_{1}$ and $A_{2}$ are linear combinations of $1$, $M$, $N$ and $MN$. These generalized Laguerre polynomials are defined in such a way that $L_{n}^{\alpha,0,0}(x) = L_{n}^{(\alpha)}(x)$. See [3] and section 4 of this paper for more details.

Now we start from the well-known (second order) differential equation for the classical orthogonal polynomials, which is

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0$$

in the Laguerre case. If we substitute the new generalized orthogonal polynomials into the left-hand side of this classical differential equation we get an expression which can be seen as a polynomial in $M$ and $N$. Now we add a linear combination of $M$, $N$ and $MN$ depending on $y$, $y'$, $y''$, etc. to the left-hand side of this classical differential equation and set this expression equal to zero. We hope that this is the new differential equation we are looking for.

Our example leads to the differential equation

$$M\sum_{i=0}^{\infty} a_{i}(x)y^{(i)}(x) + N\sum_{i=0}^{\infty} b_{i}(x)y^{(i)}(x) +$$

$$+ MN\sum_{i=0}^{\infty} c_{i}(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0.$$

Now we substitute the new orthogonal polynomials into this new differential equation and then the left-hand side becomes a polynomial in $M$ and $N$ with coefficients involving the classical orthogonal polynomials. Since this differential equation must be valid for all possible values of $M$ and $N$, and $M$ and $N$ are supposed to be independent, all coefficients of this polynomial must be equal to zero. This gives us several equations in terms of unknown coefficients and of the well-known classical orthogonal polynomials. We remark that in case of dependence of $M$ and $N$ we have to deal with a completely different problem, since then we have a polynomial in one variable instead of two. We want the coefficients of $y'$, $y''$, etc. to be independent of the degree $n$. So, if we substitute small values of $n$ into these equations we get rather simple equations for the coefficients we are looking for. In general, these equations are huge expressions and are very difficult to handle. But here we use computers to do these
huge calculations. The mathematics used is very simple, but the formulas are too big to do this by hand. The calculations we did here were done by using Mathematica and sometimes Maple. Both computer algebra packages can do these calculations easily, but they use a lot of computer memory. On a workstation with 8 megabytes of internal memory some calculations took several hours.

3 The infinite order Laguerre differential equation

In [2] the following theorem was proved without the use of computers:

**Theorem 1.** For \( M > 0 \) the polynomials \( \{L_n^{\alpha,M}(x)\}_{n=0}^\infty \) satisfy a unique differential equation of the form

\[
M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,
\]

where \( \{a_i(x)\}_{i=0}^{\infty} \) are continuous functions on the real line and \( \{a_i(x)\}_{i=1}^{\infty} \) are independent of \( n \).

Moreover, the functions \( \{a_i(x)\}_{i=0}^{\infty} \) are polynomials given by

\[
\begin{align*}
  a_0(x) &= \binom{n + \alpha + 1}{n - 1} \\
  a_i(x) &= \frac{1}{i!} \sum_{j=1}^{i} (-1)^{i+j+1} \binom{\alpha + 1}{j - 1} \binom{\alpha + 2}{i - j} (\alpha + 3)_{i-j} x^j, \quad i = 1, 2, 3, \ldots.
\end{align*}
\]

(2)

For \( \alpha \neq 0, 1, 2, \ldots \) we have \( \text{degree}[a_i(x)] = i, \quad i = 1, 2, 3, \ldots. \) This implies that if \( M > 0 \) the differential equation (1) is of infinite order in that case. For nonnegative integer values of \( \alpha \) we have

\[
\begin{align*}
  \text{degree}[a_i(x)] &= i, \quad i = 1, 2, 3, \ldots, \alpha + 2 \\
  \text{degree}[a_i(x)] &= \alpha + 2, \quad i = \alpha + 3, \alpha + 4, \alpha + 5, \ldots, 2\alpha + 4 \\
  a_i(x) &= 0, \quad i = 2\alpha + 5, 2\alpha + 6, 2\alpha + 7, \ldots.
\end{align*}
\]

This implies that for nonnegative integer values of \( \alpha \) and \( M > 0 \) the differential equation (1) is of order \( 2\alpha + 4 \).

This differential equation was found by setting \( y(x) = L_n^{\alpha,M}(x) \) and by substituting small values of \( n \) in (1). Since the coefficients \( \{a_i(x)\}_{i=1}^{\infty} \) are independent of \( n \) this gives us \( a_1(x), a_2(x), a_3(x), \ldots \) explicitly. Then the general form of \( a_i(x) \) was guessed and the result was proved. Note that \( a_0(x) \) does not depend on \( x \), but does depend on \( n \). From the three special cases (\( \alpha = 0, \alpha = 1 \) and \( \alpha = 2 \)) found by A.M. Krall and L.L. Littlejohn the general form of this coefficient could be guessed rather easily and be proved too.

Later we discovered that the coefficients \( \{a_i(x)\}_{i=1}^{\infty} \) have the following interesting property.
Theorem 2. The coefficients \( \{a_i(x)\}_{i=1}^{\infty} \) of the differential equation given by (1) and (2) satisfy
\[
\sum_{i=1}^{\infty} a_i(x) = -\frac{\sin \pi \alpha}{\pi} \frac{x}{(\alpha + 2)(\alpha + 3)} \ _2F_1 \left( \frac{1}{\alpha + 4} \left| -x \right. \right), \ \alpha > -1. \quad (3)
\]

For nonnegative integer values of \( \alpha \) we have :
\[
\sum_{i=k}^{\infty} \binom{i}{k} a_i(x) = (-1)^{a+k} a_k(-x), \ k = 1, 2, 3, \ldots. \quad (4)
\]

Note that this theorem implies for nonnegative integer values of \( \alpha \) :
\[
\sum_{i=1}^{\infty} a_i(x) = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} ia_i(x) = (-1)^{\alpha+1} x.
\]

Proof. First we prove (3). Changing the order of summation we find
\[
\sum_{i=1}^{\infty} a_i(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{i+j+1}}{i!} \binom{\alpha + 1}{j-1} \binom{\alpha + 2}{i-j} (\alpha + 3)_{i-j} x^j
\]
\[
= \sum_{j=1}^{\infty} (-1)^{j+1} \binom{\alpha + 1}{j-1} x^j \sum_{i=j}^{\infty} \frac{(-1)^i}{i!} \binom{\alpha + 2}{i-j} (\alpha + 3)_{i-j}
\]
\[
= \sum_{j=1}^{\infty} (-1)^{j+1} \binom{\alpha + 1}{j-1} x^j \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+j)!} \binom{\alpha + 2}{i} (\alpha + 3)_i
\]
\[
= -\sum_{j=1}^{\infty} \binom{\alpha + 1}{j-1} x^j \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+j)!} \binom{\alpha + 2}{i} (\alpha + 3)_i.
\]

Now we use the well-known summation formula
\[
\ _2F_1 \left( \frac{a, b}{c} \left| 1 \right. \right) = \frac{\Gamma(c - a - b) \Gamma(c)}{\Gamma(c - a) \Gamma(c - b)} c - a - b > 0, \ c \neq 0, -1, -2, \ldots \quad (5)
\]
to find
\[
\sum_{i=0}^{\infty} \frac{(-1)^i}{[i+j]!} \binom{\alpha + 2}{i} (\alpha + 3)_i = \frac{1}{j!} \sum_{i=0}^{\infty} \frac{(-\alpha - 2)_i (\alpha + 3)_i}{(j+1)_i i!}
\]
\[
= \frac{1}{j!} \ _2F_1 \left( \frac{-\alpha - 2, \alpha + 3}{j + 1} \left| 1 \right. \right)
\]
\[
= \frac{\Gamma(j)}{\Gamma(j + \alpha + 3) \Gamma(j - \alpha - 2)}, \ j = 1, 2, 3, \ldots.
\]

Hence
\[
\sum_{i=1}^{\infty} a_i(x) = -\sum_{j=1}^{\infty} \binom{\alpha + 1}{j-1} \frac{\Gamma(j)}{\Gamma(j + \alpha + 3) \Gamma(j - \alpha - 2)} x^j
\]
\[
= -x \sum_{j=0}^{\infty} \binom{\alpha + 1}{j} \frac{\Gamma(j + 1)}{\Gamma(j + \alpha + 4) \Gamma(j - \alpha - 1)} x^j
\]
\[
\begin{align*}
&= -x \sum_{j=0}^{\infty} (-1)^j \frac{(-\alpha - 1)_j x^j}{\Gamma(\alpha + 1)\Gamma(\alpha + 4) \Gamma(-\alpha - 1)(-\alpha - 1)_j} \\
&= -x \frac{\Gamma(\alpha + 4)\Gamma(-\alpha - 1)}{\Gamma(\alpha + 1)} \sum_{j=0}^{\infty} \frac{(-x)^j}{(\alpha + 4)_j} \\
&= -x \frac{\Gamma(\alpha + 4)\Gamma(-\alpha - 1)}{\Gamma(\alpha + 1)} \, _1F_1 \left( 1 \left| \frac{\alpha + 4}{-x} \right. \right).
\end{align*}
\]

Finally we use
\[
\frac{1}{\Gamma(z)\Gamma(1 - z)} = \frac{\sin \pi z}{\pi}
\]

to obtain
\[
\frac{1}{\Gamma(\alpha + 4)\Gamma(-\alpha - 1)} = \frac{1}{(\alpha + 2)(\alpha + 3)} \frac{\Gamma(\alpha + 2)\Gamma(-\alpha - 1)}{\Gamma(\alpha + 1)} \frac{\sin \pi(\alpha + 2)}{(\alpha + 2)(\alpha + 3)} \frac{\sin \pi \alpha}{\pi}.
\]

This proves (3).

To prove (4) we take \( \alpha \in \{0, 1, 2, \ldots\} \) and start with
\[
\sum_{i=k}^{\infty} \frac{i!}{(i - k)!} a_i(x) = \sum_{i=k}^{\infty} \sum_{j=1}^{i} \frac{(-1)^{i+j+1}}{(i - k)!} \binom{i}{j+1} \binom{i+1}{j} \binom{i+2}{i-j} x^j.
\]

Now we use
\[
\sum_{i=k}^{\infty} \sum_{j=1}^{i} \frac{i!}{(i - k)!} a_i(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{i=\max(j,k)}^{\infty} \frac{(-1)^{i+j+1}}{(i - k)!} \binom{i+1}{j} \binom{i+2}{i-j} (\alpha + 3)_{i-j} x^j.
\]
For $j = 1, 2, \ldots, k$ we obtain
\[
\sum_{i=k}^{\infty} \frac{(-1)^i}{(i-k)!} \binom{\alpha + 2}{i-j} (\alpha + 3)_{i-j} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \binom{\alpha + 2}{i} (\alpha + 3)_{i+k-j} = (-1)^j \sum_{i=0}^{\infty} \frac{(-\alpha - 2)_i}{i!} (\alpha + 3)_{i+k-j} \]
and for $j = k + 1, k + 2, \ldots$ we find by using the summation formula (3)
\[
\sum_{i=j}^{\infty} \frac{(-1)^i}{(i-k)!} \binom{\alpha + 2}{i-j} (\alpha + 3)_{i-j} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+j-k)!} \binom{\alpha + 2}{i} (\alpha + 3)_i = (-1)^j \sum_{i=0}^{\infty} \frac{(-\alpha - 2)_i}{i!} (\alpha + 3)_i = (\alpha + 1) \Gamma(j-k) \frac{\Gamma(j-k+\alpha+3)}{\Gamma(j-k+\alpha+3)\Gamma(j-k-\alpha-2)}.
\]

Hence
\[
\sum_{i=k}^{\infty} \frac{i!}{(i-k)!} a_i(x) = -\sum_{j=1}^{k} \binom{\alpha + 1}{j-1} x^j \sum_{i=0}^{\infty} \frac{(-\alpha - 2)_i}{i!} (\alpha + 3)_{i+k-j} + \sum_{j=k+1}^{\infty} \binom{\alpha + 1}{j-1} x^j \frac{\Gamma(j-k)}{\Gamma(j-k+\alpha+3)\Gamma(j-k-\alpha-2)}.
\]

For the last sum we find
\[
\sum_{j=k+1}^{\infty} \binom{\alpha + 1}{j-1} x^j \frac{\Gamma(j-k)}{\Gamma(j-k+\alpha+3)\Gamma(j-k-\alpha-2)} = \sum_{j=0}^{\infty} \binom{\alpha + 1}{j+k} x^{j+k+1} \frac{\Gamma(j+1)}{\Gamma(j+\alpha+4)\Gamma(j+\alpha-1)} = (-1)^k x^{k+1} \sum_{j=0}^{\infty} \frac{(-\alpha - 1)_{j+k}}{(j+k)!} \frac{\Gamma(j+1)}{\Gamma(j+\alpha+4)\Gamma(j+\alpha-1)} (-x)^j
\]
which equals zero since
\[
\frac{1}{\Gamma(j-\alpha-1)} = 0 \text{ for } j = 0, 1, 2, \ldots, \alpha + 1
\]
and
\[
(-\alpha - 1)_{j+k} = 0 \text{ for } j = \alpha + 2, \alpha + 3, \ldots \text{ and } k = 1, 2, 3, \ldots.
\]
So we have
\[
\sum_{i=k}^{\infty} \frac{i!}{(i-k)!} a_i(x) = -\sum_{j=1}^{k} \binom{\alpha + 1}{j-1} x^j \sum_{i=0}^{\infty} \frac{(-\alpha - 2)_i}{i!} (\alpha + 3)_{i+k-j}.
\]
The inner sum equals zero if \( k - j > \alpha + 2 \). Hence

\[
\sum_{i=k}^{\infty} \frac{i!}{(i-k)!} a_i(x) = - \sum_{j=\max(1,k-\alpha-2)}^{k} \binom{\alpha+1}{j-1} (-\alpha-2)_{k-j}(\alpha+3)_{k-j} x^j \frac{(-\alpha-x)_{k-j}}{(k-j)!} \\
\times \binom{\alpha+2}{k-j} (\alpha+3)_{k-j} (-x)^j \\
\times \binom{\alpha+3}{k-j} (\alpha+3)_{k-j} x^j \\
\times 2F1 \left( \begin{array}{c} -\alpha - 2 + k - j \alpha + 3 + k - j \\ k - j + 1 \end{array} \right) 1) \\
= (-1)^{k+1} \sum_{j=\max(1,k-\alpha-2)}^{k} \binom{\alpha+1}{j-1} \binom{\alpha+2}{k-j} (\alpha+3)_{k-j} (-x)^j \\
\times \binom{\alpha+3}{k-j} (\alpha+3)_{k-j} x^j \\
\times 2F1 \left( \begin{array}{c} -\alpha - 2 + k - j \alpha + 3 + k - j \\ k - j + 1 \end{array} \right) 1) .
\]

Now we use the Vandermonde summation formula

\[
2F1 \left( \begin{array}{c} -n, b \\ c \end{array} \right) 1) = \frac{(c-b)n}{(c)_n}, \quad n = 0, 1, 2, \ldots
\]

to find

\[
2F1 \left( \begin{array}{c} -\alpha - 2 + k - j \alpha + 3 + k - j \\ k - j + 1 \end{array} \right) 1) = \frac{(-\alpha - 2)_{a+2-k+j}}{(k-j+1)_{a+2-k+j}} = (-1)^{a+2-k+j} .
\]

Hence

\[
\sum_{i=k}^{\infty} \frac{i!}{(i-k)!} a_i(x) = (-1)^{a+1} \sum_{j=\max(1,k-\alpha-2)}^{k} \binom{\alpha+1}{j-1} \binom{\alpha+2}{k-j} (\alpha+3)_{k-j} x^j \\
= (-1)^{a+1} \sum_{j=1}^{k} \binom{\alpha+1}{j-1} \binom{\alpha+2}{k-j} (\alpha+3)_{k-j} x^j \\
= (-1)^{a+k} k! a_k(-x) .
\]

This proves (4).

4 Some preliminary results for the Sobolev Laguerre polynomials

In this section we look for a differential equation of the form

\[
M \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x) y^{(i)}(x) + \\
+ MN \sum_{i=0}^{\infty} c_i(x) y^{(i)}(x) + xy''(x) + (\alpha + 1 - x) y'(x) + ny(x) = 0 \\
\]

for the polynomials

\[
y(x) = L_n^{\alpha,M}(x) = A_0 L_n^{(\alpha)}(x) + A_1 \frac{d}{dx} L_n^{(\alpha)}(x) + A_2 \frac{d^2}{dx^2} L_n^{(\alpha)}(x) ,
\]

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where the coefficients $A_0$, $A_1$ and $A_2$ are defined by

$$
A_0 = 1 + M \left( \frac{n + \alpha}{n - 1} \right) + \frac{n(\alpha + 2) - (\alpha + 1)}{(\alpha + 1)(\alpha + 3)} N \left( \frac{n + \alpha}{n - 2} \right) + \frac{MN}{(\alpha + 1)(\alpha + 2)} \left( \frac{n + \alpha}{n - 1} \right) \left( \frac{n + \alpha + 1}{n - 2} \right)
$$

$$
A_1 = M \left( \frac{n + \alpha}{n} \right) + \frac{(n - 1)}{(\alpha + 1)} N \left( \frac{n + \alpha}{n - 1} \right) + \frac{2MN}{(\alpha + 1)^2} \left( \frac{n + \alpha}{n} \right) \left( \frac{n + \alpha + 1}{n - 2} \right)
$$

$$
A_2 = \frac{N}{(\alpha + 1)} \left( \frac{n + \alpha}{n - 1} \right) + \frac{MN}{(\alpha + 1)^2} \left( \frac{n + \alpha}{n} \right) \left( \frac{n + \alpha + 1}{n - 1} \right).
$$

For details concerning these generalized Laguerre polynomials and their definition the reader is referred to [5] and [3].

Of course, since $L_n^{\alpha,0}(x) = L_n^{\alpha}(x)$, the coefficients $\{a_i(x)\}_{i=0}^{\infty}$ are given by [3].

Although the general form is still an open problem so far, we know that the differential equation given by (3) is not unique as in the case of the differential equation (2). We introduce the notation

$$
\begin{align*}
    b_0(0, \alpha, x) &= 0 \\
    b_0(n, \alpha, x) &= b_0(1, \alpha, x) + \beta_0(n, \alpha, x), \ n = 1, 2, 3, \ldots \\
    b_i(\alpha, x) &= b_0(1, \alpha, x)b_i^*(\alpha, x) + \beta_i(\alpha, x), \ i = 1, 2, 3, \ldots 
\end{align*}
$$

and

$$
\begin{align*}
    c_0(0, \alpha, x) &= 0 \\
    c_0(n, \alpha, x) &= b_0(1, \alpha, x) + \gamma_0(n, \alpha, x), \ n = 1, 2, 3, \ldots \\
    c_i(\alpha, x) &= b_0(1, \alpha, x)c_i^*(\alpha, x) + \gamma_i(\alpha, x), \ i = 1, 2, 3, \ldots 
\end{align*}
$$

Now we can prove the following theorem:

**Theorem 3.** The polynomials $\left\{ L_n^{\alpha,M,N}(x) \right\}_{n=1}^{\infty}$ satisfy the following infinite order differential equation:

$$
\sum_{i=0}^{\infty} b_i^*(\alpha, x)y^{(i)}(x) + M \sum_{i=0}^{\infty} c_i^*(\alpha, x)y^{(i)}(x) = 0,
$$

where

$$
b_i^*(\alpha, x) = \frac{1}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (\alpha + 1)i-j x^j, \ i = 0, 1, 2, \ldots
$$

and

$$
c_i^*(\alpha, x) = \frac{(-1)^i}{i!} x^i, \ i = 0, 1, 2, \ldots
$$

**Proof.** The proof is very easy and is based on the observation that

$$
\sum_{i=0}^{\infty} b_i^*(\alpha, x)D^{i+k}L_n^{(\alpha)}(x) = \frac{(-n)_k}{n\Gamma(k)}, \ n \geq 1, \ k = 0, 1, 2
$$

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and
\[ \sum_{i=0}^{\infty} c_i^* (\alpha, x) D^{i+k} L_n^{(\alpha)} (x) = \left( n + \frac{\alpha}{n} \right) \frac{(-n)_k}{(\alpha+1)_k}, \quad k = 0, 1, 2. \tag{10} \]

To prove (10) we change the order of summation to obtain
\[
\sum_{i=0}^{\infty} b_i^* (\alpha, x) D^{i+k} L_n^{(\alpha)} (x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \binom{i}{j} (\alpha+1)_{i-j} x^j D^{i+k} L_n^{(\alpha)} (x)
\]
\[
= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \frac{(-1)^j}{j!} \binom{i}{j} (\alpha+1)_{i-j} x^j D^{i+k} L_n^{(\alpha)} (x)
\]
\[
= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j}{(i+j)!} \binom{i+j}{j} (\alpha+1)_{i+j} x^i D^{i+j+k} L_n^{(\alpha)} (x)
\]
\[
= \sum_{i=0}^{\infty} (\alpha+1)_i \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^i D^{i+j+k} L_n^{(\alpha)} (x).
\]

Now we use the definition of the classical Laguerre polynomial
\[
L_n^{(\alpha)} (x) = \binom{n + \alpha}{n} I_n^1 \left( \frac{-n}{\alpha+1} \middle| x \right)
\]
to obtain
\[
\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j D^{i+j+k} L_n^{(\alpha)} (x) = \binom{n + \alpha}{n} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j \sum_{m=i+j+k}^{\infty} \frac{(-n)_m}{(\alpha+1)_m} \frac{x^{m-i-j-k}}{(m-i-j-k)!}
\]
\[
= \binom{n + \alpha}{n} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(-n)_{i+j+k}}{(\alpha+1)_{i+j+k}} \frac{x^m}{(m-j)!}
\]
\[
= \binom{n + \alpha}{n} \sum_{m=0}^{\infty} \frac{(-n)_{i+j+k}}{(\alpha+1)_{i+j+k}} \frac{x^m}{m!} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \binom{m}{j}
\]
\[
= \binom{n + \alpha}{n} \frac{(-n)_{i+j+k}}{(\alpha+1)_{i+j+k}}.
\]

Hence, by using the summation formula (10) we find
\[
\sum_{i=0}^{\infty} b_i^* (\alpha, x) D^{i+k} L_n^{(\alpha)} (x) = \binom{n + \alpha}{n} \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \frac{(-n)_{i+k}}{(\alpha+1)_{i+k}}
\]
\[
= \binom{n + \alpha}{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{2F_1}{(\alpha+1)_k} \left( \frac{-n+k, \alpha+1}{\alpha+k+1} \middle| 1 \right)
\]
\[
= \binom{n + \alpha}{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{\Gamma(n)\Gamma(\alpha+k+1)}{\Gamma(n+\alpha+1)\Gamma(k)} = \frac{(-n)_k}{n\Gamma(k)}
\]
\[
which proves (10). The proof of (11) is much shorter:
\[
\sum_{i=0}^{\infty} c_i^* (\alpha, x) D^{i+k} L_n^{(\alpha)} (x) = \binom{n + \alpha}{n} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i \sum_{m=i+k}^{\infty} \frac{(-n)_m}{(\alpha+1)_m} \frac{x^{m-i-k}}{(m-i-k)!}
\]
\[
= \left( \frac{n + \alpha}{n} \right) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{m=i}^{\infty} \frac{(-n)_{m+k}}{(\alpha + 1)_{m+k} (m - i)!} x^{m-i}
\]

\[
= \left( \frac{n + \alpha}{n} \right) \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{(-1)^i}{i!} \frac{(-n)_{m+k}}{(\alpha + 1)_{m+k} (m - i)!} x^{m}
\]

\[
= \left( \frac{n + \alpha}{n} \right) \sum_{m=0}^{\infty} \frac{(-n)_{m+k}}{(\alpha + 1)_{m+k} m!} \sum_{i=0}^{m} \frac{(-1)^i}{i!} \binom{m}{i}
\]

\[
= \left( \frac{n + \alpha}{n} \right) \frac{(-n)_k}{(\alpha + 1)_k}.
\]

By using these formulas (9) and (10) and by using the definition (7) of the coefficients \(A_0, A_1\) and \(A_2\) we find

\[
A_0 \sum_{i=0}^{\infty} b_i^*(\alpha, x) D^i L_n^{(\alpha)}(x) + A_1 \sum_{i=0}^{\infty} b_i^*(\alpha, x) D^{i+1} L_n^{(\alpha)}(x) + A_2 \sum_{i=0}^{\infty} b_i^*(\alpha, x) D^{i+2} L_n^{(\alpha)}(x) +
\]

\[
+ MA_0 \sum_{i=0}^{\infty} c_i^*(\alpha, x) D^i L_n^{(\alpha)}(x) + MA_1 \sum_{i=0}^{\infty} c_i^*(\alpha, x) D^{i+1} L_n^{(\alpha)}(x) +
\]

\[
+ MA_2 \sum_{i=0}^{\infty} c_i^*(\alpha, x) D^{i+2} L_n^{(\alpha)}(x)
\]

\[
= -A_1 + (n-1)A_2 + M \left[ \left( \frac{n + \alpha}{n} \right) A_0 - \left( \frac{n + \alpha}{n - 1} \right) A_1 + \left( \frac{n + \alpha}{n - 2} \right) A_2 \right] = 0.
\]

This proves (8).

In the special cases \(\alpha = 0, \alpha = 1\) and \(\alpha = 2\) differential equations of the form (6) are found too. In these three special cases of integer values of the parameter \(\alpha\) we find a linear differential equation of formal order \(4\alpha + 10\). By formal order we mean that for special cases \((M = 0\) or \(N = 0)\) the true order might be lower. We give the results, but we will not give any proofs here.

4.1 The special case \(\alpha = 0\)

If we take \(\alpha = 0\) in (2) we find

\[
a_0(n, 0, x) = \frac{1}{2} n(n + 1), \quad n = 0, 1, 2, \ldots
\]

and

\[
a_1(0, x) = -x
\]

\[
a_2(0, x) = 3x - \frac{1}{2} x^2
\]

\[
a_3(0, x) = -2x + x^2
\]

\[
a_4(0, x) = -\frac{1}{2} x^2
\]

\[
a_i(0, x) = 0, \quad i = 5, 6, 7, \ldots
\]
For the coefficients \( \{ \beta_i(x) \}_{i=0}^{\infty} \) we find in this case
\[
\beta_0(n, 0, x) = \frac{1}{12} n^2(n^2 - 1), \ n = 1, 2, 3, \ldots
\]
and
\[
\begin{align*}
\beta_1(0, x) &= 0 \\
\beta_2(0, x) &= 1 - \frac{1}{2} x^2 \\
\beta_3(0, x) &= -3 - x + \frac{9}{2} x^2 - \frac{1}{2} x^3 \\
\beta_4(0, x) &= 2 + 3x - \frac{25}{2} x^2 + \frac{17}{6} x^3 - \frac{1}{12} x^4 \\
\beta_5(0, x) &= -2x + \frac{27}{2} x^2 - \frac{11}{2} x^3 + \frac{1}{3} x^4 \\
\beta_6(0, x) &= -5x^2 + \frac{9}{2} x^3 - \frac{1}{2} x^4 \\
\beta_7(0, x) &= -\frac{4}{3} x^3 + \frac{1}{3} x^4 \\
\beta_8(0, x) &= -\frac{1}{12} x^4 \\
\beta_i(0, x) &= 0, \ i = 9, 10, 11, \ldots
\end{align*}
\]
Finally, the coefficients \( \{ \gamma_i(x) \}_{i=0}^{\infty} \) turn out to be
\[
\gamma_0(n, 0, x) = \frac{1}{120} n(n^2 - 1)(n + 2)(2n + 1), \ n = 1, 2, 3, \ldots
\]
and
\[
\begin{align*}
\gamma_1(0, x) &= 0 \\
\gamma_2(0, x) &= -\frac{1}{2} x^2 \\
\gamma_3(0, x) &= 5x^2 - \frac{2}{3} x^3 \\
\gamma_4(0, x) &= -\frac{35}{2} x^2 + 5x^3 - \frac{5}{24} x^4 \\
\gamma_5(0, x) &= 28x^2 - 14x^3 + \frac{5}{4} x^4 - \frac{1}{60} x^5 \\
\gamma_6(0, x) &= -21x^2 + \frac{56}{3} x^3 - \frac{35}{12} x^4 + \frac{1}{12} x^5 \\
\gamma_7(0, x) &= 6x^2 - 12x^3 + \frac{10}{3} x^4 - \frac{1}{6} x^5 \\
\gamma_8(0, x) &= 3x^3 - \frac{15}{8} x^4 + \frac{1}{6} x^5 \\
\gamma_9(0, x) &= \frac{5}{12} x^4 - \frac{1}{12} x^5 \\
\gamma_{10}(0, x) &= \frac{1}{60} x^5 \\
\gamma_i(0, x) &= 0, \ i = 11, 12, 13, \ldots
\end{align*}
\]
Note that we have in this case:

\[ \sum_{i=1}^{4} a_i(0, x) = \sum_{i=1}^{8} \beta_i(0, x) = \sum_{i=1}^{10} \gamma_i(0, x) = 0. \]

Hence, we have found the following linear differential equation of formal order 10:

\[
\frac{1}{60} MN x^5 y^{(10)}(x) + \frac{1}{12} MN(5x^4 - x^5)y^{(9)}(x) + \\
+ \left[ \frac{1}{24} MN(72x^3 - 45x^4 + 4x^5) - \frac{1}{12} N x^4 \right] y^{(8)}(x) + \\
+ \left[ \frac{1}{6} MN(36x^2 - 72x^3 + 20x^4 - x^5) - \frac{1}{3} N(4x^3 - x^4) \right] y^{(7)}(x) + \\
+ \left[ \frac{1}{12} MN(-252x^2 + 224x^3 - 35x^4 + x^5) + \frac{1}{2} N(-10x^2 + 9x^3 - x^4) \right] y^{(6)}(x) + \\
+ \left[ \frac{1}{60} MN(1680x^2 - 840x^3 + 75x^4 - x^5) + \frac{1}{6} N(-12x + 81x^2 - 33x^3 + 2x^4) \right] y^{(5)}(x) + \\
+ \left[ \frac{1}{24} MN(-420x^2 + 120x^3 - 5x^4) + \\
\quad + \frac{1}{12} N(24 + 36x - 150x^2 + 34x^3 - x^4) - \frac{1}{2} Mx^2 \right] y^{(4)}(x) + \\
+ \left[ \frac{1}{3} MN(15x^2 - 2x^3) + \frac{1}{2} N(-6 - 2x + 9x^2 - x^3) + M(-2x + x^2) \right] y^{(3)}(x) + \\
+ \left[ -\frac{1}{2} MNx^2 + \frac{1}{2} N(2 - x^2) + \frac{1}{2} M(6x - x^2) + x \right] y''(x) + \left[ 1 - (M + 1)x \right] y'(x) + \\
+ \frac{1}{120} n \left[ MN(n^2 - 1)(n + 2)(2n + 1) + 10Nn(n^2 - 1) + 60M(n + 1) + 120 \right] y(x) = 0,
\]

for the polynomials

\[ y(x) := L_{n}^{0,M,N}(x) = A_0 L_n(x) + A_1 L'_n(x) + A_2 L''_n(x) \]

with

\[ L_n(x) := L_n^{(0)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k, \quad n = 0, 1, 2, \ldots \]

and

\[
\begin{cases}
A_0 = 1 + Mn + \frac{1}{6} Nn(n-1)(2n-1) + \frac{1}{12} MNn^2(n^2 - 1) \\
A_1 = M + Nn(n-1) + \frac{1}{3} MNn(n^2 - 1) \\
A_2 = Nn + \frac{1}{2} MNn(n+1).
\end{cases}
\]

### 4.2 The special case \( \alpha = 1 \)

In the special case that \( \alpha = 1 \) we find successively

\[ a_0(n, 1, x) = \frac{1}{6} n(n+1)(n+2), \quad n = 0, 1, 2, \ldots \]
\[
\begin{align*}
a_1(1, x) &= -x \\
a_2(1, x) &= 6x - x^2 \\
a_3(1, x) &= -10x + 4x^2 - \frac{1}{6}x^3 \\
a_4(1, x) &= 5x - 5x^2 + \frac{1}{2}x^3 \\
a_5(1, x) &= 2x^2 - \frac{1}{2}x^3 \\
a_6(1, x) &= \frac{1}{6}x^3 \\
a_i(1, x) &= 0, \ i = 7, 8, 9, \ldots ,
\end{align*}
\]

\[
\beta_0(n, 1, x) = \frac{1}{240}n(n^2 - 1)(n + 2)(3n - 1), \ n = 1, 2, 3, \ldots ,
\]

\[
\begin{align*}
\beta_1(1, x) &= 0 \\
\beta_2(1, x) &= \frac{3}{2} - \frac{1}{4}x^2 \\
\beta_3(1, x) &= -8 - \frac{3}{2}x + 4x^2 - \frac{5}{12}x^3 \\
\beta_4(1, x) &= \frac{25}{2} + 8x - \frac{77}{4}x^2 + \frac{47}{12}x^3 - \frac{7}{48}x^4 \\
\beta_5(1, x) &= -6 - \frac{25}{2}x + \frac{77}{2}x^2 - \frac{51}{4}x^3 + \frac{23}{24}x^4 - \frac{1}{80}x^5 \\
\beta_6(1, x) &= 6x - 34x^2 + \frac{227}{12}x^3 - \frac{19}{8}x^4 + \frac{1}{16}x^5 \\
\beta_7(1, x) &= 11x^2 - \frac{79}{6}x^3 + \frac{17}{6}x^4 - \frac{1}{8}x^5 \\
\beta_8(1, x) &= \frac{7}{2}x^3 - \frac{79}{48}x^4 + \frac{1}{8}x^5 \\
\beta_9(1, x) &= \frac{3}{8}x^4 - \frac{1}{16}x^5 \\
\beta_{10}(1, x) &= \frac{1}{80}x^5 \\
\beta_i(1, x) &= 0, \ i = 11, 12, 13, \ldots ,
\end{align*}
\]

\[
\gamma_0(n, 1, x) = \frac{n(n^2 - 1)(n + 2)(n + 3)(5n^2 + 10n + 2)}{10080}, \ n = 1, 2, 3, \ldots
\]

and

\[
\begin{align*}
\gamma_1(1, x) &= 0 \\
\gamma_2(1, x) &= -\frac{1}{4}x^2 \\
\gamma_3(1, x) &= 5x^2 - \frac{2}{3}x^3 \\
\gamma_4(1, x) &= -35x^2 + \frac{115}{12}x^3 - \frac{23}{48}x^4 \\
\gamma_5(1, x) &= 119x^2 - \frac{105}{2}x^3 + \frac{65}{12}x^4 - \frac{31}{240}x^5 \\
\end{align*}
\]
\[\gamma_6(1, x) = -\frac{441}{2} x^2 + 147x^3 - \frac{49}{2} x^4 + \frac{29}{24} x^5 - \frac{1}{72} x^6\]
\[\gamma_7(1, x) = 228x^2 - 232x^3 + \frac{117}{2} x^4 - \frac{14}{3} x^5 + \frac{1}{9} x^6 - \frac{1}{2016} x^7\]
\[\gamma_8(1, x) = -\frac{495}{4} x^2 + \frac{837}{4} x^3 - \frac{1289}{16} x^4 + \frac{467}{48} x^5 - \frac{3}{8} x^6 + \frac{1}{288} x^7\]
\[\gamma_9(1, x) = \frac{55}{2} x^2 - \frac{605}{6} x^3 + \frac{129}{2} x^4 - \frac{571}{48} x^5 + \frac{25}{36} x^6 - \frac{1}{192} x^7\]
\[\gamma_{10}(1, x) = \frac{121}{6} x^3 - \frac{671}{24} x^4 + \frac{257}{30} x^5 - \frac{55}{72} x^6 + \frac{5}{288} x^7\]
\[\gamma_{11}(1, x) = \frac{61}{12} x^4 - \frac{27}{8} x^5 + \frac{1}{2} x^6 - \frac{5}{288} x^7\]
\[\gamma_{12}(1, x) = \frac{9}{16} x^5 - \frac{13}{72} x^6 + \frac{1}{96} x^7\]
\[\gamma_{13}(1, x) = \frac{1}{36} x^6 - \frac{1}{192} x^7\]
\[\gamma_{14}(1, x) = \frac{1}{2016} x^7\]
\[\gamma_i(1, x) = 0, \ i = 15, 16, 17, \ldots.\]

Hence, in this case we have
\[
\sum_{i=1}^{6} a_i(1, x) = \sum_{i=1}^{10} \beta_i(1, x) = \sum_{i=1}^{14} \gamma_i(1, x) = 0.
\]

This implies that we have found a linear differential equation of formal order 14 for the polynomials \(\{L_n^{1,M,N}(x)\}_{n=0}^{\infty}\).

### 4.3 The special case \(\alpha = 2\)

If we take \(\alpha = 2\) we find successively
\[
a_0(n, 2, x) = \frac{1}{24} n(n + 1)(n + 2)(n + 3), \ n = 0, 1, 2, \ldots,
\]
\[
a_1(2, x) = -x
\]
\[
a_2(2, x) = 10x - \frac{3}{2} x^2
\]
\[
a_3(2, x) = -30x + 10x^2 - \frac{1}{2} x^3
\]
\[
a_4(2, x) = 35x - \frac{45}{2} x^2 + \frac{5}{2} x^3 - \frac{1}{24} x^4
\]
\[
a_5(2, x) = -14x + 21x^2 - \frac{9}{2} x^3 + \frac{1}{6} x^4
\]
\[
a_6(2, x) = -7x^2 + \frac{7}{2} x^3 - \frac{1}{4} x^4
\]
\[
a_7(2, x) = -x^3 + \frac{1}{6} x^4
\]
\[
a_8(2, x) = -\frac{1}{24} x^4
\]
\[
a_i(2, x) = 0, \ i = 9, 10, 11, \ldots,
\]
\[ \beta_0(n, 2, x) = \frac{1}{1080} n(n^2 - 1)(n + 2)(n + 3)(2n - 1), \quad n = 1, 2, 3, \ldots, \]

\[ \beta_1(2, x) = 0 \]
\[ \beta_2(2, x) = 2 - \frac{1}{6} x^2 \]
\[ \beta_3(2, x) = -\frac{50}{3} - 2x + \frac{25}{6} x^2 - \frac{7}{18} x^3 \]
\[ \beta_4(2, x) = 45 + \frac{50}{3} x - \frac{92}{3} x^2 + \frac{11}{2} x^3 - \frac{5}{24} x^4 \]
\[ \beta_5(2, x) = -49 - 45x + \frac{290}{3} x^2 - 27x^3 + \frac{49}{24} x^4 - \frac{13}{360} x^5 \]
\[ \beta_6(2, x) = \frac{56}{3} + 49x - \frac{887}{6} x^2 + \frac{188}{3} x^3 - \frac{91}{12} x^4 + \frac{97}{360} x^5 - \frac{1}{540} x^6 \]
\[ \beta_7(2, x) = -\frac{56}{3} x + \frac{217}{6} x^2 - \frac{451}{3} x^3 + \frac{169}{12} x^4 - \frac{29}{36} x^5 + \frac{1}{90} x^6 \]
\[ \beta_8(2, x) = -\frac{92}{3} x^2 + \frac{271}{6} x^3 - \frac{337}{24} x^4 + \frac{5}{4} x^5 - \frac{1}{36} x^6 \]
\[ \beta_9(2, x) = -\frac{97}{9} x^3 + \frac{173}{24} x^4 - \frac{77}{72} x^5 + \frac{1}{27} x^6 \]
\[ \beta_{10}(2, x) = -\frac{3}{2} x^4 + \frac{173}{360} x^5 - \frac{1}{36} x^6 \]
\[ \beta_{11}(2, x) = -\frac{4}{45} x^5 + \frac{1}{90} x^6 \]
\[ \beta_{12}(2, x) = -\frac{1}{540} x^6 \]
\[ \beta_3(2, x) = 0, \quad i = 13, 14, 15, \ldots, \]

\[ \gamma_0(n, 2, x) = \frac{n(n^2 - 1)(n + 2)(n + 3)(n + 4)(2n + 3)(7n^2 + 21n + 2)}{1088640}, \quad n = 1, 2, 3, \ldots, \]

and

\[ \gamma_1(2, x) = 0 \]
\[ \gamma_2(2, x) = -\frac{1}{6} x^2 \]
\[ \gamma_3(2, x) = \frac{35}{6} x^2 - \frac{13}{18} x^3 \]
\[ \gamma_4(2, x) = -70 x^2 + \frac{35}{2} x^3 - \frac{7}{8} x^4 \]
\[ \gamma_5(2, x) = 413 x^2 - 161 x^3 + \frac{49}{3} x^4 - \frac{13}{30} x^5 \]
\[ \gamma_6(2, x) = -1386 x^2 + \frac{2317}{3} x^3 - \frac{245}{2} x^4 + \frac{119}{18} x^5 - \frac{73}{720} x^6 \]
\[ \gamma_7(2, x) = 2827 x^2 - 2189 x^3 + 497 x^4 - 42 x^5 + \frac{21}{16} x^6 - \frac{59}{5040} x^7 \]
\[ \gamma_8(2, x) = -3575 x^2 + 3872 x^3 - \frac{9779}{8} x^4 + 148 x^5 - \frac{29}{4} x^6 + \frac{19}{144} x^7 - \frac{11}{17280} x^8 \]
\[ \gamma_9(2, x) = \frac{16445}{6} x^2 - \frac{77935}{18} x^3 + \frac{11473}{6} x^4 - \frac{5797}{18} x^5 + \frac{3257}{144} x^6 - \frac{31}{48} x^7 + \]
In this section we will deal with the problem of finding a differential equation for the generalized Jacobi polynomials $L_{n}^{2,M,N}(x)$. Since the well-known second order differential equation for the classical Jacobi polynomials

$$
(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + n(n + \alpha + \beta + 1)y(x) = 0,
$$

it is clear that we look for a differential equation of the form

$$
M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) +
+ (1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + n(n + \alpha + \beta + 1)y(x) = 0,
$$

hence, we have

$$
\sum_{i=1}^{8} a_i(2, x) = \sum_{i=1}^{12} b_i(2, x) = \sum_{i=1}^{18} \gamma_i(2, x) = 0.
$$

This implies that the polynomials $\left\{L_{n}^{2,M,N}(x)\right\}_{n=0}^{\infty}$ satisfy a linear differential equation of formal order 18.

### 5 The generalized Jacobi polynomials

In this section we will deal with the problem of finding a differential equation for the generalized Jacobi polynomials $\left\{P_{n}^{\alpha,\beta,M,N}(x)\right\}_{n=0}^{\infty}$.

Since the well-known second order differential equation for the classical Jacobi polynomials $\left\{P_{n}^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$ is given by

$$
(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + n(n + \alpha + \beta + 1)y(x) = 0,
$$

it is clear that we look for a differential equation of the form

$$
M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) +
+ (1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + n(n + \alpha + \beta + 1)y(x) = 0,
$$

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where

\[ y(x) = P^{\alpha, \beta, M, N}_n(x) = A_0 P^{(\alpha, \beta)}_n(x) + [A_1 (1-x) - A_2 (1+x)] \frac{d}{dx} P^{(\alpha, \beta)}_n(x) \]

and

\[
\begin{align*}
A_0 &= 1 + M \left( \frac{n + \beta}{n - 1} \right) \left( \frac{n + \alpha + \beta + 1}{n} \right) + N \left( \frac{n + \alpha}{n - 1} \right) \left( \frac{n + \alpha + \beta + 1}{n} \right) + MN \frac{(\alpha + \beta + 2)^2}{(\alpha + 1)(\beta + 1)} \left( \frac{n + \alpha + \beta + 1}{n - 1} \right)^2 \\
A_1 &= M \left( \frac{n + \beta}{n} \right) \left( \frac{n + \alpha + \beta}{n} \right) + MN \left( \frac{n + \alpha + \beta}{n - 1} \right) \left( \frac{n + \alpha + \beta + 1}{n} \right) \\
A_2 &= N \left( \frac{n + \alpha}{n} \right) \left( \frac{n + \alpha + \beta}{n} \right) + MN \left( \frac{n + \alpha + \beta}{n - 1} \right) \left( \frac{n + \alpha + \beta + 1}{n} \right).
\end{align*}
\]

Here we used the same definition as in [6], but in a slightly different notation.

In this case the differential equation will be unique in its general form if it exists. We introduce the notation

\[
\begin{align*}
a_0(x) &= a_0(n, \alpha, \beta, x) \\
a_i(x) &= a_i(\alpha, \beta, x), \ i = 1, 2, 3, \ldots, \\
b_0(x) &= b_0(n, \alpha, \beta, x) \\
b_i(x) &= b_i(\alpha, \beta, x), \ i = 1, 2, 3, \ldots
\end{align*}
\]

and

\[
\begin{align*}
c_0(x) &= c_0(n, \alpha, \beta, x) \\
c_i(x) &= c_i(\alpha, \beta, x), \ i = 1, 2, 3, \ldots
\end{align*}
\]

Since the polynomials \( \{P^{\alpha, \beta, M, N}_n(x)\}_{n=0}^{\infty} \) satisfy the symmetry relation

\[ P^{\alpha, \beta, M, N}_n(-x) = (-1)^n P^{\beta, \alpha, N, M}_n(x) \]

we have

\[
\begin{align*}
a_0(n, \alpha, \beta, -x) &= b_0(n, \beta, \alpha, x) \\
a_i(\alpha, \beta, -x) &= (-1)^i b_i(\beta, \alpha, x), \ i = 1, 2, 3, \ldots
\end{align*}
\]
and
\[
\begin{cases}
  c_0(n, \alpha, \beta, -x) = c_0(n, \beta, \alpha, x) \\
  c_i(\alpha, \beta, -x) = (-1)^i c_i(\beta, \alpha, x), \ i = 1, 2, 3, \ldots.
\end{cases}
\]

The general form is still an open problem, but the symmetric case \(\alpha = \beta\) and \(M = N\) seems to be much less difficult. All special cases known so far (see for instance [1]) point out that if \(\alpha = \beta\) and \(M = N\) we can choose \(c_i(x) = 0\) for all \(i = 0, 1, 2, \ldots\). Therefore, we will consider this special case first.

### 5.1 The symmetric generalized Jacobi polynomials

All known examples seem to point out that there might be a differential equation of the form
\[
M \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + (1 - x^2)y''(x) - 2(\alpha + 1)xy'(x) + n(n + 2\alpha + 1)y(x) = 0, \tag{11}
\]
where
\[
y(x) = P_n^{\alpha,\alpha,M,M}(x) = C_0 P_n^{(\alpha,\alpha)}(x) - C_1 x \frac{d}{dx} P_n^{(\alpha,\alpha)}(x) \tag{12}
\]
and
\[
\begin{cases}
  C_0 = 1 + M \frac{2n}{(\alpha + 1)} \binom{n + 2\alpha + 1}{n} + 4M^2 \binom{n + 2\alpha + 1}{n - 1}^2 \\
  C_1 = \frac{2M}{(\alpha + 1)} \binom{n + 2\alpha}{n} + \frac{2M^2}{(\alpha + 1)} \binom{n + 2\alpha}{n - 1} \binom{n + 2\alpha + 1}{n}.
\end{cases} \tag{13}
\]

This differential equation turns out not to be unique. We write
\[
\begin{cases}
  a_0(x) := a_0(n, \alpha, x), \ n = 0, 1, 2, \ldots \\
  a_i(x) := a_i(\alpha, x), \ i = 1, 2, 3, \ldots
\end{cases}
\]
If we substitute [12] in the differential equation [11] then we finally find three equations for the coefficients \(\{a_i(x)\}_{i=0}^{\infty}\) which are equivalent to the following two :
\[
\sum_{i=0}^{\infty} a_i(x) D^i P_n^{(\alpha,\alpha)}(x) = 4 \left(\frac{n + 2\alpha}{\alpha + 1}\right) \binom{n + 2\alpha}{n} \frac{d^2}{dx^2} P_n^{(\alpha,\alpha)}(x)
\]
and
\[
\sum_{i=0}^{\infty} i a_i(x) D^i P_n^{(\alpha,\alpha)}(x) + x \sum_{i=0}^{\infty} a_i(x) D^{i+1} P_n^{(\alpha,\alpha)}(x) = 4 \left(\frac{n + 2\alpha + 1}{n - 1}\right) \frac{d^2}{dx^2} P_n^{(\alpha,\alpha)}(x).
\]

We introduce the notation
\[
\begin{cases}
  a_0(n, \alpha, x) = a_0(1, \alpha, x) b_0(n, \alpha, x) + c_0(n, \alpha, x), \ n = 0, 1, 2, \ldots \\
  a_i(\alpha, x) = a_0(1, \alpha, x) b_i(\alpha, x) + c_i(\alpha, x), \ i = 1, 2, 3, \ldots
\end{cases}
\]
Now we can prove the following theorem.
Theorem 4. The polynomials \( \{ P_n^{\alpha,M,M}(x) \} \) satisfy a linear infinite order differential equation of the form:
\[
\sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) = 0,
\]
where
\[
\begin{cases}
  b_0(x) := b_0(n, \alpha, x) = \frac{1}{2} [1 - (-1)^n], & n = 0, 1, 2, \ldots \\
  b_i(x) := b_i(\alpha, x) = \frac{2^{i-1}}{i!} (-x)^i, & i = 1, 2, 3, \ldots .
\end{cases}
\]

Proof. To prove this theorem we have to show that
\[
\begin{align*}
  &\sum_{i=0}^{\infty} b_i(x) D^i P_n^{(\alpha,\alpha)}(x) = 0 \\
  &\sum_{i=0}^{\infty} i b_i(x) D^i P_n^{(\alpha,\alpha)}(x) + x \sum_{i=0}^{\infty} b_i(x) D^{i+1} P_n^{(\alpha,\alpha)}(x) = 0.
\end{align*}
\]
To do this we note that the polynomials \( \{ P_n^{(\alpha,\alpha)}(x) \} \) are defined by
\[
P_n^{(\alpha,\alpha)}(x) = \left( \frac{n + \alpha}{n} \right) 2F_1 \left( -n, n + 2\alpha + 1 \middle| \frac{1 - x}{2} \right).
\]
Further we write
\[
\sum_{i=0}^{\infty} b_i(x) D^i P_n^{(\alpha,\alpha)}(x) = \frac{1}{2} [1 - (-1)^n] P_n^{(\alpha,\alpha)}(x) + \sum_{i=1}^{\infty} b_i(x) D^i P_n^{(\alpha,\alpha)}(x)
\]
and
\[
\sum_{i=0}^{\infty} b_i(x) D^{i+1} P_n^{(\alpha,\alpha)}(x) = \frac{1}{2} [1 - (-1)^n] \frac{d}{dx} P_n^{(\alpha,\alpha)}(x) + \sum_{i=1}^{\infty} b_i(x) D^{i+1} P_n^{(\alpha,\alpha)}(x).
\]
Now we easily obtain
\[
\sum_{i=1}^{\infty} b_i(x) D^i P_n^{(\alpha,\alpha)}(x) = \frac{1}{2} \left( \frac{n + \alpha}{n} \right) \sum_{i=1}^{\infty} \frac{x^i}{i!} \sum_{k=1}^{\infty} \frac{(-n)_k (n + 2\alpha + 1)_k}{(\alpha + 1)_k (k - i)!} \left( \frac{1 - x}{2} \right)^{k-i}
\]
\[
= \frac{1}{2} \left( \frac{n + \alpha}{n} \right) \sum_{k=1}^{\infty} \frac{(-n)_k (n + 2\alpha + 1)_k}{(\alpha + 1)_k k!} \sum_{i=1}^{k} \frac{k}{i} \frac{x^i}{i!} \left( \frac{1 - x}{2} \right)^{k-i}
\]
\[
= \frac{1}{2} \left( \frac{n + \alpha}{n} \right) \sum_{k=1}^{\infty} \frac{(-n)_k (n + 2\alpha + 1)_k}{(\alpha + 1)_k k!} \left[ \left( \frac{1 + x}{2} \right)^k - \left( \frac{1 - x}{2} \right)^k \right]
\]
\[
= \frac{1}{2} \left( P_n^{(\alpha,\alpha)}(-x) - P_n^{(\alpha,\alpha)}(x) \right) = -\frac{1}{2} [1 - (-1)^n] P_n^{(\alpha,\alpha)}(x),
\]

\[ \sum_{i=0}^{\infty} ib_i(x) D^i P_n^{\alpha, \alpha}(x) = \frac{1}{2} \left( \frac{n + \alpha}{n} \right) \sum_{i=1}^{\infty} \frac{x^i}{(i-1)!} \sum_{k=i}^{\infty} \frac{(-n)_k(n+2\alpha+1)_k}{(\alpha+1)_k(k-i)!} \left( \frac{1-x}{2} \right)^{k-i} \]

\[ = \frac{1}{2} \left( \frac{n + \alpha}{n} \right) \sum_{k=1}^{\infty} \frac{(-n)_k(n+2\alpha+1)_k}{(\alpha+1)_k(k-1)!} \sum_{i=1}^{k} \left( \frac{k-1}{i-1} \right) x^i \left( \frac{1-x}{2} \right)^{k-i} \]

\[ = \frac{1}{2} \left( \frac{n + \alpha}{n} \right) x \sum_{k=1}^{\infty} \frac{(-n)_k(n+2\alpha+1)_k}{(\alpha+1)_k(k-1)!} \left( 1 + x \right)^{k-1} \]

\[ = (-1)^n x \frac{d}{dx} P_n^{\alpha, \alpha}(x) \]

and

\[ \sum_{i=1}^{\infty} b_i(x) D^{i+1} P_n^{\alpha, \alpha}(x) \]

\[ = -\frac{1}{4} \left( \frac{n + \alpha}{n} \right) \sum_{i=1}^{\infty} \frac{x^i}{i^2} \sum_{k=1}^{\infty} \frac{(-n)_{k+1}(n+2\alpha+1)_{k+1}}{(\alpha+1)_{k+1}(k-i)!} \left( \frac{1-x}{2} \right)^{k-i} \]

\[ = -\frac{1}{4} \left( \frac{n + \alpha}{n} \right) \sum_{k=1}^{\infty} \frac{(-n)_{k+1}(n+2\alpha+1)_{k+1}}{(\alpha+1)_{k+1}k!} \sum_{i=1}^{k} \left( \frac{k}{i} \right) x^i \left( \frac{1-x}{2} \right)^{k-i} \]

\[ = -\frac{1}{4} \left( \frac{n + \alpha}{n} \right) \sum_{k=1}^{\infty} \frac{(-n)_{k+1}(n+2\alpha+1)_{k+1}}{(\alpha+1)_{k+1}k!} \left[ \left( \frac{1+x}{2} \right)^{k} - \left( \frac{1-x}{2} \right)^{k} \right] \]

\[ = -\frac{1}{4} \left( \frac{n + \alpha}{n} \right) \sum_{k=0}^{\infty} \frac{(-n)_{k+1}(n+2\alpha+1)_{k+1}}{(\alpha+1)_{k+1}k!} \left[ \left( \frac{1+x}{2} \right)^{k+1} - \left( \frac{1-x}{2} \right)^{k+1} \right] \]

\[ = -\frac{1}{2} \left[ 1 + (-1)^n \right] \frac{d}{dx} P_n^{\alpha, \alpha}(x). \]

This proves the theorem.

For the coefficients \( \{c_i(x)\}_{i=0}^{\infty} \) we find by using packages like Maple or Mathematica:

\[
\begin{align*}
    c_0(0, \alpha, x) &= 0 \\
    c_0(1, \alpha, x) &= 0 \\
    c_0(2, \alpha, x) &= 4(2\alpha + 3) \\
    c_0(3, \alpha, x) &= 4(2\alpha + 3)(2\alpha + 5) \\
    c_0(4, \alpha, x) &= 2(2\alpha + 3)(2\alpha + 5)(2\alpha + 6) \\
    c_0(5, \alpha, x) &= \frac{2}{3}(2\alpha + 3)(2\alpha + 5)(2\alpha + 6)(2\alpha + 7).
\end{align*}
\]

Hence, we might guess that

\[
c_0(n, \alpha, x) = \frac{2n(n-1)}{(\alpha + 2)} \binom{n+2\alpha+2}{n} = 4(2\alpha + 3) \binom{n+2\alpha+2}{n-2}, \quad n = 0, 1, 2, \ldots \quad (14)
\]

If we write

\[
c_i(\alpha, x) = (2\alpha + 3)(1 - x^2)c_i^*(\alpha, x), \quad i = 1, 2, 3, \ldots \quad (15)
\]
we obtain
\[
\begin{align*}
c_1^*(\alpha, x) & = 0 \\
c_2^*(\alpha, x) & = 2 \\
c_3^*(\alpha, x) & = \frac{4}{3}(\alpha + 1)x \\
c_4^*(\alpha, x) & = \frac{1}{6}(\alpha + 1) \left(2\alpha + 1)x^2 - 1\right) \\
c_5^*(\alpha, x) & = \frac{1}{45}\alpha(\alpha + 1)x \left(2\alpha + 1)x^2 - 3\right) \\
c_6^*(\alpha, x) & = \frac{1}{1080}\alpha(\alpha + 1) \left(2\alpha - 1)(2\alpha + 1)x^4 - 6(2\alpha - 1)x^2 + 3\right) \\
c_7^*(\alpha, x) & = \frac{1}{18900}(\alpha - 1)\alpha(\alpha + 1)x \left(2\alpha - 1)(2\alpha + 1)x^4 - 10(2\alpha - 1)x^2 + 15\right) \\
c_8^*(\alpha, x) & = \frac{1}{907200}(\alpha - 1)\alpha(\alpha + 1) \left(2\alpha - 3)(2\alpha - 1)(2\alpha + 1)x^6 + \\
&\quad - 15(2\alpha - 3)(2\alpha - 1)x^4 + 45(2\alpha - 3)x^2 - 15\right) \\
c_9^*(\alpha, x) & = \frac{1}{2857600}(\alpha - 2)(\alpha - 1)\alpha(\alpha + 1)x \left(2\alpha - 3)(2\alpha - 1)(2\alpha + 1)x^6 + \\
&\quad - 21(2\alpha - 3)(2\alpha - 1)x^4 + 105(2\alpha - 3)x^2 - 105\right).
\end{align*}
\]

So we might guess that
\[
\begin{align*}
c_1^*(\alpha, x) & = 0 \\
c_2^*(\alpha, x) & = \frac{4(-1)^{i+1}}{(2i)!}(\alpha + 1)i \left(-i + 1, \alpha + \frac{5}{2} - i \left|x^2\right.\right)_{2F1}, i = 1, 2, 3, \ldots \\
c_{2i}^*(\alpha, x) & = \frac{8(-1)^{i+1}}{(2i + 1)!}\left(\alpha + 1\right)i \left(-i + 1, \alpha + \frac{5}{2} - i \left|x^2\right.\right)_{2F1}, i = 1, 2, 3, \ldots .
\end{align*}
\]

Hence, we have found the following conjecture.

**Conjecture.** The polynomials \(P_n^{\alpha,\alpha,M,M}(x)\) satisfy a linear differential equation of the form
\[
M \sum_{i=0}^{\infty} c_i(x) y^{(i)}(x) + (1 - x^2)y''(x) - 2(\alpha + 1)xy'(x) + n(n + 2\alpha + 1)y(x) = 0,
\]
where the coefficients \(\{c_i(x)\}_{i=0}^{\infty}\) are given by (14), (15) and (16).

We remark that this conjecture would imply that for nonnegative integer values of \(\alpha\) the polynomials \(P_n^{\alpha,\alpha,M,M}(x)\) satisfy a linear differential equation of order \(2\alpha + 4\).

Finally, we note that since
\[
2F1\left(-i + 1, \alpha + \frac{5}{2} - i \left|1\right\right) = \frac{(-\alpha - 2 + i)_{i-1}}{(\frac{1}{2})_{i-1}}, i = 1, 2, 3, \ldots
\]
and
\[ 2F_1 \left( \begin{array}{c} -i + 1, \alpha + \frac{5}{2} - i \\ \frac{3}{2} \end{array} \right) = \frac{(-\alpha - 1 + i)_{i-1}}{(\frac{3}{2})_{i-1}}, \quad i = 1, 2, 3, \ldots \]

we have
\[
\sum_{i=1}^{\infty} c_{2i}(\alpha, -1) = \sum_{i=1}^{\infty} \frac{4}{(\frac{3}{2})_{i}(1)^{4i}} \frac{(-\alpha - 1)_{i-1} (-\alpha - 2 + i)_{i-1}}{(1)^{4i}} (i-1)! (\frac{3}{2})_{i-1} = 2 \sum_{i=0}^{\infty} \frac{(-\alpha - 1)_{i} (-\alpha - 1 + i)_{i}}{i!(\frac{3}{2})_{i}(2)!} \frac{1}{4} 2^i = 2 \sum_{i=0}^{\infty} (-\alpha - 1)_{2i} 2^{2i}.
\]

and
\[
\sum_{i=0}^{\infty} c_{2i+1}(\alpha, -1) = -\sum_{i=1}^{\infty} \frac{8(\alpha + 1)}{(1)^{4i}} \frac{(-\alpha)_{i-1} (-\alpha - 1 + i)_{i-1}}{(1)^{4i}} (i-1)! (\frac{3}{2})_{i-1} = -\frac{4}{3}(\alpha + 1) \sum_{i=0}^{\infty} \frac{(-\alpha)_{i} (-\alpha + i)_{i}}{i!(\frac{3}{2})_{i}(\frac{3}{2})_{i}} \frac{1}{4} 2^{2i} = -\frac{4}{3}(\alpha + 1) \sum_{i=0}^{\infty} \frac{(-\alpha)_{2i}}{(2i + 1)!} \frac{1}{4} (3)_{2i+1} 2^{2i+1} = 2 \sum_{i=1}^{\infty} (-\alpha - 1)_{2i+1} 2^{2i+1}.
\]

In the same way we find
\[
\sum_{i=1}^{\infty} c_{2i}(\alpha, 1) = 2 \sum_{i=0}^{\infty} (-\alpha - 1)_{2i} 2^{2i}
\]

and
\[
\sum_{i=0}^{\infty} c_{2i+1}(\alpha, 1) = -2 \sum_{i=0}^{\infty} (-\alpha - 1)_{2i+1} 2^{2i+1}.
\]

This implies that
\[
\sum_{i=1}^{\infty} c_{i}(\alpha, -1) = \sum_{i=1}^{\infty} c_{2i}(\alpha, -1) + \sum_{i=0}^{\infty} c_{2i+1}(\alpha, -1) = 2 \sum_{i=0}^{\infty} \frac{(-\alpha - 1)_{i} 2^{i}}{i!(3)_{i}} = 2 \text{ } \frac{1}{F_1} \left( \begin{array}{c} -\alpha - 1 \\ 3 \end{array} \right) 2
\]

and
\[
\sum_{i=1}^{\infty} c_{i}(\alpha, 1) = \sum_{i=1}^{\infty} c_{2i}(\alpha, 1) + \sum_{i=0}^{\infty} c_{2i+1}(\alpha, 1) = 2 \sum_{i=0}^{\infty} \frac{(-\alpha - 1)_{i} (-2)_{i}}{i!(3)_{i}} = 2 \text{ } \frac{1}{F_1} \left( \begin{array}{c} -\alpha - 1 \\ 3 \end{array} \right) - 2.
\]

**Remark.** The proof of the conjecture has been found and will be given in a forthcoming paper 4.

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