COUNTING FINITE ORBITS FOR THE FLIP SYSTEMS OF SHIFTS OF FINITE TYPE

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Abstract. For a discrete system \((X, T)\), the flip system \((X, T, F)\) can be regarded as the action of infinite dihedral group \(D_\infty\) on the space \(X\). Under this action, \(X\) is partitioned into a set of orbits. We are interested in counting the finite orbits in this partition via the prime orbit counting function. In this paper, we prove the asymptotic behaviour of this counting function for the flip systems of shifts of finite type. The proof relies mostly on combinatorial calculations instead of the usual approach via zeta function. Here, we are able to obtain more precise asymptotic result for this \(D_\infty\)-action on shifts of finite type as compared to other group actions on systems available in the literature.

1. Introduction. Let \((X, T)\) be a discrete dynamical system, where \(X\) is a topological space and \(T : X \rightarrow X\) is a homeomorphism. A homeomorphism \(F : X \rightarrow X\) is a flip on \((X, T)\) if it satisfies
\[
F \circ T = T^{-1} \circ F, \quad F^2 = id_X
\]
where \(id_X : X \rightarrow X\) is the identity map. The triplet \((X, T, F)\) is called a flip system. Furthermore, two flip systems \((X, T, F)\) and \((X', T', F')\) are conjugate if there exists a homeomorphism \(\Phi : X \rightarrow X'\) such that
\[
\Phi \circ T = T' \circ \Phi, \quad \Phi \circ F = F' \circ \Phi.
\]
Recall that the infinite dihedral group is defined as
\[
D_\infty = \langle r, s \mid sr = r^{-1}s, \ s^2 = e \rangle
\]
where \(e\) is the identity element. The flip system \((X, T, F)\) can be regarded as the group action of \(D_\infty\) on \(X\), which is defined as
\[
r \cdot x = T(x), \quad s \cdot x = F(x)
\]
for any \(x \in X\). Under this action, \(X\) is partitioned into a collection of orbits which are defined as
\[
Orb_{D_\infty}(x) = \{ g \cdot x \mid g \in D_\infty \}
\]

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for any \( x \in X \). We will be interested in counting the orbits which are finite, and this will be done through a certain function which is called the prime orbit counting function.

To define our counting function, we start by considering an action by a group \( G \) on a space \( X \). From the collection of orbits

\[
\text{Orb}_G(x) = \{ g \cdot x \mid g \in G \}
\]

for \( x \in X \) under this action, we focus on the orbits which are finite. We denote \( \tau_G \) as a finite orbit of size \(|\tau_G|\) under this action. For a periodic element in \( x \in X \), we also write \( \tau_G(x) \) as the finite orbit containing \( x \). Now, the prime orbit counting function is defined as

\[
\pi_G(N) = \sum_{|\tau_G| \leq N} 1
\]

for \( N \in \mathbb{N} \).

The idea of counting finite orbits in this way is analogous to counting primes in number theory. Specifically, the famous Prime Number Theorem (see [8]) proved the asymptotic behaviour of a counting function for primes, which is

\[
\sum_{p \leq N} 1 \sim \frac{N}{\log N}
\]

where \( N \in \mathbb{N} \) and \( p \) runs through primes. Inspired by it, we are interested to obtain an analogous result for our counting function for a given group action.

This problem was initially applied to ordinary discrete dynamical systems. Consider a general discrete system \((X,T)\). The orbit of an element \( x \in X \) under \( T \) is defined as the set

\[
\{ T^k(x) \mid k \in \mathbb{Z} \}.
\]

Recall that an element \( x \in X \) is periodic with period \( n \in \mathbb{N} \) if \( T^n(x) = x \). Furthermore, if \( T^n(x) = x \) but \( T^k(x) \neq x \) for \( k \in \{1, 2, \ldots, n-1\} \), then \( x \) has least period \( n \). In this case, its orbit is finite. Such a finite orbit is called a closed (or periodic) orbit. We denote \( \tau_T \) as a closed orbit of size \(|\tau_T|\).

If \( T \) is a homeomorphism, then \((X,T)\) can be regarded as the group action of \( \mathbb{Z} \) on \( X \), which is defined as \( k \cdot x = T^k(x) \) for \( k \in \mathbb{Z} \) and \( x \in X \). The definitions for the finite orbits under \( \mathbb{Z} \)-action on \( X \) and the closed orbits under \( T \) on \( X \) are equivalent. Therefore, we can define the counting function \( \pi_T(N) \) in similar manner.

Parry and Pollicott tackled this research problem on shifts of finite type and their suspension flows (see [18, 19]). Specifically, for a mixing shift of finite type with Perron value \( \lambda > 1 \) (which is its exponent of topological entropy), it was found that

\[
\pi_T(N) \sim \frac{\lambda^{N+1}}{N(\lambda-1)}.
\]

This result was obtained through a generating function called Artin-Mazur zeta function [5].

Apart from shifts of finite type, there were results for other systems that utilized this approach via their zeta function, such as ergodic toral automorphisms [21], periodic-finite-type shifts [14], and Dyck and Motzkin shifts [16]. In fact, different approaches were also introduced to obtain such results for other systems, such as using estimates of the number of periodic points on Dyck and Motzkin shifts [2, 3, 4], counting in orbit monoids [17] and using orbit Dirichlet series on some algebraic systems [6, 7]. Interested readers may refer to our survey in [15] for a detailed
exposure on this topic. To highlight our point here, there are many results on this research interest for various ordinary discrete systems. However, similar results involving other group actions are relatively scarce. Miles & Ward [13] obtained such results for a finitely-generated, torsion-free, nilpotent group $G$ acting on the full $G$-shift over alphabet of $k$ symbols. It was obtained that

$$C_1 N^{d(G)-2} \leq \frac{\pi_G(N)}{k^N} \leq C_2 N^{r(G)-1}$$

where $r(G)$ and $d(G)$ are ranks of $G$ and abelianization of $G$ respectively, and $C_1$ and $C_2$ are some positive constants. In case of $G = \mathbb{Z}^d$ for $d \geq 2$, it was shown further that

$$C_1 N^{d-2} \leq \frac{\pi_{\mathbb{Z}^d}(N)}{k^N} \leq C_2 N^{d-2}(\log N)^{d-1}.$$  

The proof used some results from group theory to obtain certain estimates in the calculation.

For flip systems, Miles [12] showed that for an expansive algebraic flip system with topological entropy $h > 0$, it was found that

$$\pi_{D^\infty}(N) \asymp e^{hN}.$$  

His idea was to use estimates of the number of periodic points of the system and some algebraic tools to lead to the result.

Kim et al [9] initiated the study of the flip systems in general, specifically on their dynamical zeta function, periodic points and finite orbits. For shifts of finite type, they obtained the general form of their flip systems and also the formula for their zeta function. Kim & Ryu [10] then extended the results for the flip systems of sofic shifts. Despite such extensive results, the asymptotic behaviour of the prime orbit counting function for these flip systems was left undetermined.

Therefore, the aim of this paper is to obtain the asymptotic behaviour of the prime orbit counting function for the flip systems of shifts of finite type. We will utilize the results in [9] and some tools in matrix theory to obtain precise estimates on the number of finite orbits. Our approach will focus on combinatorial calculations instead of analytical tools such as zeta function.

Note that for the group actions in [12] and [13], the results for $\pi_G(N)$ were obtained in the form of asymptotic bounds. However, for our flip systems, we will obtain the asymptotic equivalence of $\pi_G(N)$, which is more precise.

In Section 2, we review some properties of flip systems in general, especially on the finite orbits. Next, we proceed with some facts on shifts of finite type and their flip systems. Finally in Section 3, we obtain the asymptotic result of the prime orbit counting function for the flip systems of shifts of finite type. Our main result can be found in Theorem 3.1.

2. Background and setting.

2.1. Finite orbits in a flip system. Here, we discuss the properties of the finite orbits in a flip system $(X, T, F)$ based on [9]. We will determine the forms of the finite orbits, and how these finite orbits relate to the closed orbits under $T$. Later, we will see how to count these finite orbits.

Consider the corresponding $D^\infty$-action on $X$. Recall that the stabilizer of $x \in X$ under this action is the subgroup

$$Stab_{D^\infty}(x) = \{ g \in D^\infty \mid g \cdot x = x \}.$$  

Denote the set of (left) cosets induced by $Stab_{D_{\infty}}(x)$ as

$$D_{\infty}/Stab_{D_{\infty}}(x) = \{g \cdot Stab_{D_{\infty}}(x) \mid g \in D_{\infty}\}.$$  

The orbit-stabilizer theorem states that $D_{\infty}/Stab_{D_{\infty}}(x)$ and $Orb_{D_{\infty}}(x)$ have the same cardinality.

Recall also that the index of a subgroup is the cardinality of its induced set of cosets. Note that $Orb_{D_{\infty}}(x)$ is finite if and only if $Stab_{D_{\infty}}(x)$ has a finite index. Therefore, to determine the forms of the finite orbits, we need to find the subgroups of $D_{\infty}$ with finite index that can be the corresponding stabilizers.

It is easy to check that the only subgroups of $D_{\infty}$ with finite index are of the forms

(i) if $H(i, j)$ is its stabilizer, then

$$\tau_{D_{\infty}}(x) = \{x, T(x), \ldots, T^{i-1}(x)\}.$$ 

We denote $\tau_1$ as a finite orbit of this form. Moreover, we denote $O_1(n)$ as the set of finite orbits $\tau_1$ of size $n$, and $O_1(n)$ as its cardinality. Note that the finite orbit of this form is a closed orbit under $T$;

(ii) if $H(i)$ is its stabilizer, then

$$\tau_{D_{\infty}}(x) = \{x, T(x), \ldots, T^{i-1}(x)\} \cup \{F(x), F \circ T(x), \ldots, F \circ T^{i-1}(x)\}.$$ 

We denote $\tau_2$, $O_2(n)$ and $O_2(n)$ accordingly as above for finite orbits of this form. Note that the size of a finite orbit of this form is even.

Now, let $O_T(n)$ be the set of closed orbits of size $n$ under $T$, and $O_T(n)$ as its cardinality. As mentioned above, we know that $O_1(n) \subseteq O_T(n)$. So, for a closed orbit $\tau_T \in O_T(n)$,

(i) either $\tau_T \in O_1(n)$, or

(ii) $\tau_T \in O_T(n) \setminus O_1(n)$. In this case, the set $F(\tau_T)$ is another closed orbit under $T$ and $F(\tau_T) \in O_T(n) \setminus O_1(n)$. Furthermore, $\tau_T$ and $F(\tau_T)$ are disjoint, and $\tau_T \cup F(\tau_T) \in O_2(2n)$.

Conversely, for a finite orbit $\tau_2 \in O_2(2n)$, we can easily find a closed orbit $\tau_T \in O_T(n) \setminus O_1(n)$ such that $\tau_2 = \tau_T \cup F(\tau_T)$ (see the proof of Theorem 2.7 in [9]).

Overall, every finite orbit in $O_2(2n)$ corresponds uniquely to two distinct closed orbits in $O_T(n) \setminus O_1(n)$, and vice versa. This implies that

$$\left|O_T(n) \setminus O_1(n)\right| = 2 \cdot O_2(2n).$$

Based on the argument above, we deduce that

$$O_T(n) = O_1(n) + 2 \cdot O_2(2n).$$  

(1)

Next, we will count the finite orbits of the first form based on the number of fixed points under the group action. Let $P_{i,j}$ be the set of points in $X$ that are fixed by $H(i,j)$, i.e.

$$P_{i,j} = \{x \in X \mid g \cdot x = x \forall g \in H(i,j)\}. $$  

(2)

Denote $p_{i,j}$ as its cardinality. It is proved that

$$\sum_{m=1}^{\infty} \left(2^m \cdot z^{2m-1} + \frac{p_{2m,0} + p_{2m,1}}{2} z^{2m}\right) = \sum_{\tau_1} \frac{z^{\left|\tau_1\right|}}{1 - z^{\left|\tau_1\right|}}$$  

(3)
for some \( z \in \mathbb{C} \) (see Theorem 2.2 and Corollary 2.8 in [9]).

If \((X,T)\) is a subshift such as a shift of finite type, then the radius of convergence for each series in (3) is at least \( \exp \left( -\frac{h}{2} \right) \), where \( h \) is the topological entropy of \((X,T)\). So, for sufficiently small \(|z|\),

\[
\sum_{\tau_1} \frac{z^{\left|\tau_1\right|}}{1 - z^{\left|\tau_1\right|}} = \sum_{\tau_1} \left( \sum_{k=1}^{\infty} z^{k\left|\tau_1\right|} \right) = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} O_1(l) z^{k l} = \sum_{n=1}^{\infty} \left( \sum_{d \mid n} O_1(d) \right) z^n \tag{4}
\]

where \( d \) runs through divisors of \( n \). By using (4) and comparing coefficients in (3), it is obtained that

\[
c_n = \sum_{d \mid n} O_1(d) \tag{5}
\]

where

\[
c_n = \begin{cases} p_{n,0} & \text{if } n \text{ is odd}, \\ \frac{p_{n,0} + p_{n,1}}{2} & \text{if } n \text{ is even}. \end{cases} \tag{6}
\]

By Möbius inversion formula, (5) can be rearranged as

\[
O_1(n) = \sum_{d \mid n} \mu \left( \frac{n}{d} \right) c_d \tag{7}
\]

where \( \mu : \mathbb{N} \to \{-1, 0, 1\} \) is the Möbius function.

To sum up, both equations (1) and (7) allow us to count the finite orbits of both forms, given that we can count the fixed points under \( D_\infty \)-action and also the closed orbits under \( T \). We will demonstrate this idea later for the flip systems of shifts of finite type.

2.2. Shifts of finite type and their flip systems. We discuss some facts on shifts of finite type, and then proceed to their flip systems based on [9]. Many details on shifts of finite type can be found in [11].

Let \( \mathcal{A} \) be a non-empty finite set equipped with the discrete topology. Define the set

\[
\mathcal{A}^\mathbb{Z} = \{ x = (x_i)_{i \in \mathbb{Z}} \mid x_i \in \mathcal{A} \ \forall i \in \mathbb{Z} \}
\]

i.e. it is a collection of bi-infinite sequences made from elements of \( \mathcal{A} \). The set \( \mathcal{A}^\mathbb{Z} \) is equipped with the product topology.

The set \( \mathcal{A} \) is called the alphabet, and its element is called a symbol. A sequence \( x \in \mathcal{A}^\mathbb{Z} \) is called a point, and a finite sequence \( w \in \mathcal{A}^k \) for some \( k \in \mathbb{N} \) is called a word of length \( k \). A word \( w = a_1 a_2 \ldots a_k \) is said to occur in \( x \in \mathcal{A}^\mathbb{Z} \) if there exists \( i \in \mathbb{Z} \) such that \( x_i x_{i+1} \ldots x_{i+k-1} = a_1 a_2 \ldots a_k \). Denote it as \( w \prec x \).

Let \( \mathcal{F} \) be a set of words. A word \( w \in \mathcal{F} \) is called a forbidden word. Define the set

\[
\mathcal{X}_\mathcal{F}^\mathbb{Z} = \{ x \in \mathcal{A}^\mathbb{Z} \mid w \not\prec x \ \forall w \in \mathcal{F} \}.
\]

Define also the shift map \( \sigma : \mathcal{X}_\mathcal{F}^\mathbb{Z} \to \mathcal{X}_\mathcal{F}^\mathbb{Z} \) as follows: for \( x = (x_i)_{i \in \mathbb{Z}} \), the image under this map is given by \( \sigma(x) = (x_{i+1})_{i \in \mathbb{Z}} \). The pair \( (\mathcal{X}_\mathcal{F}^\mathbb{Z}, \sigma) \) is called a subshift. The map \( \sigma \) is indeed a homeomorphism.

If \( \mathcal{F} \) is finite, then \( (\mathcal{X}_\mathcal{F}^\mathbb{Z}, \sigma) \) is called a shift of finite type. In [11], it is shown that any shift of finite type is topologically conjugate to another shift of finite type over alphabet \( \mathcal{A} = \{1, 2, \ldots, m\} \) for some \( m \in \mathbb{N} \) such that the set \( \mathcal{F} \) is either empty or has all words of length 2. This is known as 1-step shift of finite type. From now
on, we denote \((\mathcal{X}, \sigma)\) as a shift of finite type in this form. For \((\mathcal{X}, \sigma)\), we define an \(m \times m\) transition matrix \(A\) with entries as follows: for indices \(i\) and \(j\),

(i) \(A_{ij} = 1\) if and only if \(ij \notin \mathcal{F}\), and

(ii) \(A_{ij} = 0\) if and only if \(ij \in \mathcal{F}\).

Recall that the matrix \(A\) is said to be irreducible if for any pair of indices \(i\) and \(j\), there exists \(k \in \mathbb{N}\) such that \((A^k)_{ij} > 0\). Moreover, for an irreducible matrix \(A\), its period is the greatest common divisor of the set \(\{k \in \mathbb{N} \mid (A^k)_{ii} > 0\}\), regardless of the choice of index \(i\). The irreducible matrix \(A\) is said to be aperiodic if its period is 1.

A shift of finite type is irreducible with period \(p \in \mathbb{N}\) if its transition matrix \(A\) is irreducible with period \(p\). Furthermore, the shift is mixing if \(A\) is aperiodic.

For this paper, we are dealing with irreducible (including mixing) shifts of finite type. This is because any reducible shift of finite type can be decomposed into several irreducible shifts of finite type, which can be studied separately. From now on, we assume that \((\mathcal{X}, \sigma)\) is irreducible.

Recall Perron-Frobenius theory as follows: for an irreducible matrix \(A\) with period \(p\), there exists a positive eigenvalue \(\lambda\) such that

(i) \(|\mu| \leq \lambda\) for other eigenvalues \(\mu\) of \(A\), and

(ii) there are exactly \(p\) eigenvalues with modulus \(\lambda\), and they are in the form

\[
\exp \left(\frac{2k\pi i}{p}\right) \cdot \lambda \quad \text{for}\ k \in \{0, 1, 2, \ldots, p-1\}.
\]

Each is a simple eigenvalue.

The eigenvalue \(\lambda\) is called the Perron value of \(A\). Furthermore, if \(A\) is aperiodic, then

(i) \(\lambda\) has a right eigenvector \(r = (r_1, r_2, \ldots, r_m)^T\) and a left eigenvector \(l = (l_1, l_2, \ldots, l_m)\) that are positive i.e. each entry is positive, and

(ii) \(\lim_{k \to \infty} \frac{A^k}{\lambda^k} = rl\)

where \(r\) and \(l\) are normalised so that \(l^t r = 1\).

For \((\mathcal{X}, \sigma)\) whose transition matrix \(A\) has Perron value \(\lambda\), its topological entropy is

\[h = \log \lambda.\]  

(8)

The asymptotic behaviour of the prime orbit counting function for irreducible \((\mathcal{X}, \sigma)\) can be deduced from the mixing case in [18].

**Theorem 2.1** ([18]). Let \((\mathcal{X}, \sigma)\) be an irreducible shift of finite type of period \(p\) and topological entropy \(h = \log \lambda > 0\). Then,

\[
\pi_\sigma(N) \sim \frac{p}{N} \cdot \frac{\lambda^p(\lfloor \frac{N}{p} \rfloor + 1)}{\lambda^p - 1}.
\]  

(9)

Now, we are looking at the flip systems of shifts of finite type. First, we introduce a type of flip systems for the shifts. Let \((\mathcal{X}, \sigma)\) be a shift of finite type with transition matrix \(A\). Suppose that there exists an \(m \times m\) \(\{0, 1\}\)-matrix \(P\) such that

\[AP = PAT, \quad P^2 = 1.\]  

(10)

\(P\) is a symmetric permutation matrix. Intuitively, \(P\) defines a bijection of the alphabet \(\mathcal{A}\) onto itself as follows:

(i) each \(a \in \mathcal{A}\) corresponds uniquely to an \(a^* \in \mathcal{A}\),

(ii) if \(a \in \mathcal{A}\) corresponds to \(a^* \in \mathcal{A}\), then \(a^*\) corresponds to \(a\), and
(iii) for any \(a, b \in \mathcal{A}\), if \(A_{ab} = 1\), then \(A_{b^*a^*} = 1\). In other meaning, if the word \(ab\) occurs in some points in \(X\), then so is the word \(b^*a^*\).

Define the map \(\rho : X \to X\) as follows: for \(x = (x_i)_{i \in \mathbb{Z}}\), the image under this map is given by \(\rho(x) = (x_{i-1}^*)_{i \in \mathbb{Z}}\). It is easy to check that \(\rho\) is a flip on \((X, \sigma)\), thus \((X, \sigma, \rho)\) is a flip system.

The following theorem states that every flip system of shifts of finite type can be represented in this way.

**Theorem 2.2 ([9])**. Let \((X, T, F)\) be a flip system of a shift of finite type. Then, there exists 1-step shift of finite type \((X, \sigma)\) with transition matrix \(A\) and a symmetric permutation matrix \(P\) satisfying (10) such that \((X, T, F)\) is conjugate to \((X, \sigma, \rho)\).

Since conjugacy defines a bijection between the finite orbits in both flip systems, it is sufficient to consider the flip systems in the form \((X, \sigma, \rho)\) only.

### 3. Prime Orbit Theorem

In this section, we determine the asymptotic behaviour of \(\pi_{D_{\infty}}(N)\) for the flip system \((X, \sigma, \rho)\).

Consider the counting function \(\pi_\sigma(N)\) for the shift of finite type \((X, \sigma)\). Define the counting functions for the finite orbits under \(D_{\infty}\)-action as

\[
\pi_1(N) = \sum_{|\tau_1| \leq N} 1, \quad \pi_2(N) = \sum_{|\tau_2| \leq N} 1
\]

for \(N \in \mathbb{N}\). Equation (1) implies that

\[
\pi_\sigma(N) = \sum_{n=1}^{N} O_\sigma(n) = \sum_{n=1}^{N} O_1(n) + 2 \sum_{n=1}^{N} O_2(2n) = \sum_{n=1}^{N} O_1(n) + 2 \sum_{n=1}^{2N} O_2(n) = \pi_1(N) + 2 \cdot \pi_2(2N)
\]

(11)

since \(O_2(n) = 0\) for odd \(n\). Using (11), we obtain that

\[
\pi_{D_{\infty}}(N) = \pi_1(N) + \pi_2(N) = \pi_1(N) - \frac{1}{2} \pi_1 \left( \left\lfloor \frac{N}{2} \right\rfloor \right) + \frac{1}{2} \pi_\sigma \left( \left\lfloor \frac{N}{2} \right\rfloor \right).
\]

(12)

In (12), \(\pi_{D_{\infty}}(N)\) is expressed in terms of \(\pi_\sigma(N)\) and \(\pi_1(N)\). Since we have the result for \(\pi_\sigma(N)\) in (9), we are only left to determine \(\pi_1(N)\). Fortunately, equation (7) implies that

\[
\pi_1(N) = \sum_{n=1}^{N} \sum_{d|n} \mu\left(\frac{n}{d}\right) c_d,
\]

(13)

so we can use (13) to obtain \(\pi_1(N)\).

#### 3.1. Asymptotic behaviour of prime orbit counting function

Let \((X, \sigma)\) be our shift of finite type over alphabet \(\mathcal{A}\). Based on the flip system \((X, \sigma, \rho)\), define the following subsets of \(\mathcal{A}\):

\[
\mathcal{R} = \{ a \in \mathcal{A} \mid a^* = a \}, \quad \mathcal{S} = \{ a \in \mathcal{A} \mid A_{aa^*} = 1 \}, \quad \mathcal{T} = \{ a \in \mathcal{A} \mid A_{a^*a} = 1 \}.
\]

Consider the set \(P_{n,0}\) for odd \(n\) as defined in (2). Let \(x = (x_i)_{i \in \mathbb{Z}} \in P_{n,0}\). This means that \(x\) is fixed by the subgroup \(H(n,0)\), or more specifically, fixed by the maps \(\sigma^n\) and \(\rho\). Observe that \(\sigma^n(x) = x\) implies that \(x_i + n = x_i\) for all \(i \in \mathbb{Z}\), so \(x\) is determined by the initial word \(x_0x_1 \ldots x_{n-1}\). Furthermore, \(\rho(x) = x\) implies that \(x_{i+1}^* = x_i\) for all \(i \in \mathbb{Z}\), so the previous word can also be written as
\(x_0^*x_{n-1}^*x_{n-2}^* \cdots x_2^*x_1^*.\) By comparing both representations, the word can be further written as \(x_0x_1 \cdots x_{n-1}x_n \cdots x_{n-2}x_{n-1} \cdots x_2^*x_1^*\). Hence, we obtain the following conditions:

(i) \(x\) is determined completely by the initial word \(x_0x_1 \cdots x_{n-1}\), since this word will determine the subsequent word \(x_{n-1}^*x_{n-2}^* \cdots x_2^*x_1^*\). The concatenation of both words will be made periodic to form the whole \(x\);

(ii) \(x_0\) must satisfy \(x_0^* = x_0\), i.e. \(x_0 \in \mathcal{R}\);

(iii) \(x_{n-1}\) must satisfy \(A_{x_{n-1}x_n} = 1\), i.e. \(x_{n-1} \in \mathcal{S}\).

Conversely, it is easy to check that if a point \(x \in \mathcal{X}\) satisfies all conditions above, then \(x \in P_{n,0}\). Overall, we have found all points in \(P_{n,0}\).

Now, we can calculate its cardinality \(p_{n,0}\). Let \(A\) be the transition matrix for \((\mathcal{X}, \sigma)\). Observe that for \(A^k\) where \(k \in \mathbb{N}\), the entry \((A^k)_{ij}\) is the number of words of length \(k+1\) that begin with \(i \in \mathcal{A}\) and end with \(j \in \mathcal{A}\). So, the number of words of the form \(x_0x_1 \cdots x_{n-1}\) satisfying the conditions above (hence, the cardinality \(p_{n,0}\)) is

\[p_{n,0} = \sum_{i,j} (A^{n-1})_{ij} \cdot \sum_{i,j} (A^{n-1})_{ij}\]

The expression above remains valid for \(n = 1\) by taking \(A^0\) as the identity matrix.

We can follow similar argument to obtain \(p_{n,0}\) and \(p_{n,1}\) for even \(n\). We state the following result briefly: \(x \in P_{n,0}\) for even \(n\) if and only if

(i) \(x\) is determined completely by the initial word \(x_0x_1 \cdots x_{n-1}\), since this word will determine the subsequent word \(x_{n-1}^*x_{n-2}^* \cdots x_2x_1\). The concatenation of both words will be made periodic to form the whole \(x\);

(ii) \(x_0\) must satisfy \(x_0^* = x_0\), i.e. \(x_0 \in \mathcal{R}\);

(iii) \(x_{n-1}\) must satisfy \(A_{x_{n-1}x_n} = 1\), i.e. \(x_{n-1} \in \mathcal{S}\).

Similarly, \(x \in P_{n,1}\) for even \(n\) if and only if

(i) \(x\) is determined completely by the initial word \(x_0x_1 \cdots x_{n-1}\), since this word will determine the subsequent word \(x_{n-1}^*x_{n-2}^* \cdots x_2x_1\). The concatenation of both words will be made periodic to form the whole \(x\);

(ii) \(x_0\) must satisfy \(A_{x_0x_n} = 1\), i.e. \(x_0 \in \mathcal{T}\);

(iii) \(x_{n-1}\) must satisfy \(A_{x_{n-1}x_n} = 1\), i.e. \(x_{n-1} \in \mathcal{S}\).

We summarize our finding in the next proposition.

**Proposition 1.** Let \(p_{i,j}\) be the number of points in \(\mathcal{X}\) fixed by the subgroup \(H(i,j)\). Then,

(a) for odd \(n\),

\[p_{n,0} = \sum_{i,j} (A^{n-1})_{ij} \cdot \sum_{i,j} (A^{n-1})_{ij}\]

It is valid for \(n = 1\) by taking \(A^0\) as the identity matrix;

(b) for even \(n\),

\[p_{n,0} = \sum_{i,j} (A^{n})_{ij}, \quad p_{n,1} = \sum_{i,j} (A^{n-1})_{ij}\]

For the latter, it is valid for \(n = 2\) by taking \(A^0\) as the identity matrix.
Next, our task is to estimate the entries of the matrix power $A^k$ for $k \in \mathbb{N}$.

**Proposition 2.** Let $A$ be the transition matrix of $(X, \sigma)$. Suppose that $A$ is irreducible with period $p$ and has Perron value $\lambda > 1$. Let

$$\hat{\lambda} = \max \{|\mu| \mid \mu \text{ is eigenvalue of } A \text{ with } |\mu| < \lambda \} \cup \{1\}.$$ 

Then, for any pair of indices $i$ and $j$, there exist constants $a_{ij}^{(0)}, a_{ij}^{(1)}, \ldots, a_{ij}^{(p-1)} \in \mathbb{C}$ such that for any $k \in \mathbb{N}_0$,

$$(A^k)_{ij} = \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \exp \left( \frac{2l\pi i}{p} \cdot \lambda \right) \right)^k + \epsilon_{ij}(k)$$

where $\epsilon_{ij}(k)$ is the error term and $\epsilon_{ij}(k) = O\left( \hat{\lambda}^k \right)$.

**Proof.** Recall that $A$ can be expressed into its Jordan normal form as follows: $A = BJB^{-1}$ where $B$ is some invertible matrix and $J$ is the Jordan matrix. The matrix $J$ has the form

$$J = \begin{pmatrix}
J(\mu_1) & 0 & \cdots & 0 \\
0 & J(\mu_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & J(\mu_s)
\end{pmatrix}$$

where $J(\mu_r)$ is the Jordan block for the eigenvalue $\mu_r$ for $r \in \{1, 2, \ldots, s\}$, and other entries of $J$ are 0. The Jordan block $J(\mu_r)$ has the form

$$J(\mu_r) = \begin{pmatrix}
\mu_r & 1 & 0 & \cdots & 0 \\
0 & \mu_r & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \mu_r & 1 \\
0 & \cdots & 0 & 0 & \mu_r
\end{pmatrix}$$

where other entries are 0. Let the dimension of $J(\mu_r)$ be $m_r \times m_r$. For $k \in \mathbb{N}_0$, the block $J(\mu_r)^k$ has the form

$$J(\mu_r)^k = \begin{pmatrix}
\mu_r^k & 0 & \cdots & 0 \\
\frac{\mu_r^k-1}{1} & \mu_r^{k-1} & \cdots & 0 \\
\frac{\mu_r^k-2}{2} & \frac{\mu_r^{k-1}}{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \frac{\mu_r^{k-m_r+1}}{m_r-1} & (m_r-1) \mu_r^{k-m_r+2} \\
0 & \cdots & \frac{\mu_r^{k-m_r+2}}{m_r-2} & (m_r-2) \mu_r^{k-m_r+3} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \mu_r^k
\end{pmatrix}$$

where other entries are 0. (Here, we follow the convention that if $b < a$, then $\binom{b}{a} = 0$.)

Observe that $A^k = BJkB^{-1}$, where

$$J^k = \begin{pmatrix}
J(\mu_1)^k & 0 & \cdots & 0 \\
0 & J(\mu_2)^k & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & J(\mu_s)^k
\end{pmatrix}.$$ 

Since $B$ and $B^{-1}$ are constant matrices, the entry $(A^k)_{ij}$ is a linear combination of the set

$$\bigcup_{r=1}^{s} \left\{ \mu_r^k, \frac{\mu_r^k-1}{1}, \ldots, \frac{\mu_r^{k-m_r+1}}{m_r-1} \right\}.$$
over \( \mathbb{C} \). Furthermore, the coefficients in the linear combination are fixed regardless of the power \( k \).

From Perron-Frobenius theory, each eigenvalue \( \exp \left( \frac{2 \pi i}{p} \right) \cdot \lambda \) is simple for any \( l \in \{0, 1, \ldots, p-1\} \). Therefore, for \( \exp \left( \frac{2 \pi i}{p} \right) \cdot \lambda \), there is exactly one Jordan block of size \( 1 \times 1 \) whose single entry is \( \exp \left( \frac{2 \pi i}{p} \right) \cdot \lambda \). So, the entry \( (A^k)_{ij} \) is a linear combination of the set

\[
\bigcup_{l=0}^{p-1} \left\{ \left( \frac{2 \pi i}{p} \right) \cdot \lambda \right\}^k \cup \bigcup_{r=p+1}^s \left\{ \mu_r \cdot \left( \frac{k}{1} \right) \mu_r^{k-1}, \ldots, \left( \frac{k}{m_r-1} \right) \mu_r^{k-m_r+1} \right\}.
\]

For each entry \( (A^k)_{ij} \), we have now obtained the coefficient \( a_{ij}^{(l)} \) for the term \( \left( \frac{2 \pi i}{p} \right) \cdot \lambda \) from the linear combination.

For the error term \( \epsilon_{ij}(k) \), we need to find a real \( M_{ij} > 0 \) such that

\[
\left| (A^k)_{ij} - \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \frac{2 \pi i}{p} \right) \cdot \lambda \right|^k \leq M_{ij} \cdot \tilde{\lambda}^k.
\]

If there are no other non-zero eigenvalues \( \mu \) with \( |u| < \lambda \), then \( \tilde{\lambda} = 1 \) and

\[
(A^k)_{ij} - \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \frac{2 \pi i}{p} \right) \cdot \lambda = 0,
\]

so we can set \( M_{ij} = 1 \). Otherwise, there exist \( b_{ij}^{(0,r)}, b_{ij}^{(1,r)}, \ldots, b_{ij}^{(m_r-1,r)} \in \mathbb{C} \) for each \( r \in \{p+1, p+2, \ldots, s\} \) such that

\[
(A^k)_{ij} - \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \frac{2 \pi i}{p} \right) \cdot \lambda = \sum_{r=p+1}^s \sum_{l=0}^{m_r-1} b_{ij}^{(r,l)} \left( \frac{k}{l} \right) \mu_r^{k-l}.
\]

Note that \( |\mu_r| \leq \tilde{\lambda} \) and \( \left( \frac{k}{l} \right) \mu_r^{k-l} \) is a polynomial in \( k \) of degree \( l \). So, the sequence \( \left\{ \frac{k}{l} \mu_r^{k-l} \right\}_{k=1}^{\infty} \) converges to 0, thus is bounded by some \( K_r > 0 \). Hence, we can set

\[
M_{ij} = \sum_{r=p+1}^s K_r \sum_{l=0}^{m_r-1} |b_{ij}^{(r,l)}|.
\]

\( \square \)

**Remark 1.** (a) In the definition of \( \tilde{\lambda} \), we add the set \( \{1\} \) to include the case where there are no other non-zero eigenvalues \( \mu \) with \( |\mu| < \lambda \).
(b) We allow \( k = 0 \) to include the case where \( A^0 \) (which is taken to be the identity matrix).
(c) We can set further \( M = \max_{i,j} \{M_{ij}\} \), so that

\[
\left| (A^k)_{ij} - \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \frac{2 \pi i}{p} \right) \cdot \lambda \right|^k \leq M \cdot \tilde{\lambda}^k
\]

for any pair of indices \( i \) and \( j \), and matrix power \( k \). Because of this, we can do summation (including infinite sum) involving the error terms later. We can use similar idea for any error term onwards.
(d) In this paper, we prefer to use an error term instead of Landau notation directly whenever there is summation involved. This is to avoid abuse of notation and also confusion.

The previous proposition allows us to estimate $p_{n,0}$ for odd $n$, and $p_{n,0}$ and $p_{n,1}$ for even $n$.

**Proposition 3.** Let $A$ be the transition matrix of $(\mathcal{X}, \sigma)$. Suppose that $A$ is irreducible with period $p$ and has Perron value $\lambda > 1$. Let $p_{i,j}$ be the number of points in $\mathcal{X}$ fixed by the subgroup $H(i,j)$.

(a) If $p$ is odd, then there exist real $\alpha_1, \beta_1, \gamma_1 \geq 0$ such that for any $k \in \mathbb{N}$,

$$p_{(2k-1)p,0} = \alpha_1 \lambda^{kp} + \delta_1(k), \quad p_{2kp,0} = \beta_1 \lambda^{kp} + \delta_2(k),$$

where $\delta_1(k), \delta_2(k)$ and $\delta_3(k)$ are error terms, and each is equal to $O(\tilde{\lambda}^{kp})$.

(b) If $p$ is even, then there exist real $\beta_2, \gamma_2, \gamma_3, \gamma_4 \geq 0$ such that for any $k \in \mathbb{N}$,

$$p_{(2k-1)p,0} = \beta_2 \lambda^{kp} + \Delta_1(k), \quad p_{(2k-1)p,1} = \gamma_2 \lambda^{kp} + \Delta_2(k)$$

$$p_{2kp,0} = \beta_3 \lambda^{kp} + \Delta_3(k), \quad p_{2kp,1} = \gamma_3 \lambda^{kp} + \Delta_4(k)$$

where $\Delta_1(k), \Delta_2(k), \Delta_3(k)$ and $\Delta_4(k)$ are error terms, and each is equal to $O(\tilde{\lambda}^{kp})$.

**Proof.** By definition of period $p$, observe that for $x \in \mathcal{X}$, if $\sigma^n(x) = x$, then $n$ must be a multiple of $p$. Therefore, we only consider $p_{i,j}$ where $i$ is a multiple of $p$.

For $p_{(2k-1)p,0}$, Propositions 1 and 2 imply that

$$p_{(2k-1)p,0} = \sum_{i \in \mathcal{R}} \left( A^{kp-\frac{p+1}{2}} \right)_{ij} = \sum_{i \in \mathcal{R}} \sum_{l=0}^{p-1} a_{ij}^{(l)} \left( \exp \left( \frac{2l\pi i}{p} \cdot \lambda \right) \right)^{kp-\frac{p+1}{2}} + \delta_1(k)$$

$$= \lambda^{kp-\frac{p+1}{2}} \sum_{i \in \mathcal{R}} \sum_{l=0}^{p-1} a_{ij}^{(l)} \exp \left( -\frac{(p+1)l\pi i}{p} \right) + \delta_1(k) = \alpha_1 \lambda^{kp} + \delta_1(k)$$

where

$$\alpha_1 = \frac{1}{\tilde{\lambda}^{kp}} \sum_{i \in \mathcal{R}} \sum_{j \in \mathcal{S}} \sum_{l=0}^{p-1} a_{ij}^{(l)} \exp \left( -\frac{(p+1)l\pi i}{p} \right), \quad \delta_1(k) = \sum_{i \in \mathcal{R}} \sum_{j \in \mathcal{S}} \epsilon_{ij} \left( kp - \frac{p+1}{2} \right).$$

A simple calculation shows that $\delta_1(k) = O(\tilde{\lambda}^{kp})$. On the other hand, observe that

$$\alpha_1 = \lim_{k \to \infty} \frac{p_{(2k-1)p,0}}{\lambda^{kp}} = \lim_{k \to \infty} \sum_{j \in \mathcal{S}} \left( A^{kp-\frac{p+1}{2}} \right)_{ij} / \lambda^{kp}.$$

Since $A$ is a non-negative matrix and so does its power, we conclude that $\alpha_1$ is non-negative real.

The calculations for the rest are similar. \qed
Corollary 1. If the transition matrix $A$ is aperiodic, then $\alpha_1$, $\beta_1$ and $\gamma_1$ are expressed as

$$
\alpha_1 = \frac{1}{\lambda} \sum_{i \in R} r_i l_j, \quad \beta_1 = \sum_{i,j \in R} r_i l_j, \quad \gamma_1 = \frac{1}{\lambda} \sum_{i \in T} r_i l_j
$$

where $r = (r_1, r_2, \ldots, r_m)^T$ and $l = (l_1, l_2, \ldots, l_m)$ are a right and left eigenvector of the Perron value $\lambda$, normalised to $l^T r = 1$.

Proof. From the proof above, we obtain that

$$
\alpha_1 = \lim_{k \to \infty} \frac{1}{\lambda^k} \sum_{i \in R} (A^{k-1})_{ij} \lambda_k = \frac{1}{\lambda} \sum_{i \in R} r_i l_j,
$$

by using Perron-Frobenius theory. The calculations for $\beta_1$ and $\gamma_1$ are similar. \qed

We are ready to obtain the asymptotic behaviours of $\pi_1(N)$, and consequently, $\pi_{D_\infty}(N)$.

Proposition 4. For the flip system $(X, \sigma, \rho)$,

$$
\pi_1(N) = \eta(N) \lambda^\frac{p}{2} \left\lfloor \frac{N}{p} \right\rfloor + o \left( \frac{\lambda^p \left\lfloor \frac{N}{p} \right\rfloor}{N} \right)
$$

where

(i) for odd period $p$,

$$
\eta(N) = \begin{cases} 
\frac{\lambda^p}{\lambda^p - 1} \left( \frac{\alpha_1 \lambda^\frac{p}{2} + \beta_1 + \gamma_1}{2 \lambda^\frac{p}{2}} \right) & \text{if } \left\lfloor \frac{N}{p} \right\rfloor \text{ is odd,} \\
\frac{\lambda^p}{\lambda^p - 1} \left( \alpha_1 + \frac{\beta_1 + \gamma_1}{2} \right) & \text{otherwise;}
\end{cases}
$$

(ii) for even period $p$,

$$
\eta(N) = \begin{cases} 
\frac{\lambda^p}{\lambda^p - 1} \left( \frac{(\beta_2 + \gamma_2) \lambda^p + \beta_3 + \gamma_3}{2 \lambda^\frac{p}{2}} \right) & \text{if } \left\lfloor \frac{N}{p} \right\rfloor \text{ is odd,} \\
\frac{\lambda^p}{\lambda^p - 1} \left( \beta_2 + \frac{\gamma_2 + \beta_3 + \gamma_3}{2} \right) & \text{otherwise.}
\end{cases}
$$

Proof. From (13), note that

$$
\pi_1(N) = \sum_{n=1}^{N} c_n + \sum_{n=1}^{N} \sum_{d|n} \mu \left( \frac{n}{d} \right) c_d.
$$

From (6) and Proposition 3, it is easy to deduce that there exists real $\delta > 0$ such that for any $d \in \mathbb{N}$,

$$
0 \leq c_d \leq \delta \lambda^\frac{d}{2}.
$$

So,

$$
\left| \sum_{n=1}^{N} \sum_{d|n} \mu \left( \frac{n}{d} \right) c_d \right| \leq \sum_{n=1}^{N} \sum_{d=1}^{\left\lfloor \frac{N}{d} \right\rfloor} c_d \leq \delta \sum_{n=1}^{N} \lambda^\frac{n}{d} \leq \delta' \lambda^\frac{N}{2}
$$

(15)
for some $\delta' > 0$. Hence,
\[
\sum_{n=1}^{N} \sum_{d|n \text{ and } d < n} \mu\left(\frac{n}{d}\right) c_d = O\left(\lambda^N\right). \tag{16}
\]

Now, suppose that $p$ is odd. Proposition 3 implies that
\[
\sum_{n=1}^{N} c_n = \sum_{k=1}^{\lfloor N/2 \rfloor} c_{kp} = \sum_{k=1}^{\lfloor N/2 \rfloor} p_{(2k-1)p,0} + \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{p_{2kp,0} + p_{2kp,1}}{2} = \alpha_1 \lambda^p \lambda^{\lfloor \frac{N}{2} \rfloor} + \beta_1 \lambda^p \lambda^{\lfloor \frac{N}{2} \rfloor} + \gamma_1 \lambda^p \lambda^{\lfloor \frac{N}{2} \rfloor} + \theta(N)
\]

where $\eta(N)$ is defined above, and $\theta(N)$ is the error term given by
\[
\theta(N) = \sum_{k=1}^{\lfloor N/2 \rfloor} \delta_1(k) + \sum_{k=1}^{\lfloor N/2 \rfloor} \delta_2(k) + \delta_3(k) - \frac{\lambda^p}{\lambda^p - 1} \left(\alpha_1 + \beta_1 + \gamma_1 + \frac{1}{2}\right).
\]

A simple calculation shows that $\theta(N) = O\left(\lambda^N\right)$. Since $\frac{\chi(\lfloor \frac{N}{2} \rfloor)}{N}$ dominates both $\lambda^N$ and $\tilde{\lambda}^N$, (16) and (17) imply the desired result.

The case for even $p$ is done in similar manner.

**Theorem 3.1.** Let $(\mathcal{X}, \sigma)$ be a 1-step shift of finite type. Let $A$ be its transition matrix which is irreducible with period $p$ and has Perron value $\lambda > 1$. Let $(\mathcal{X}, \sigma, \rho)$ be its flip system defined by a symmetric permutation matrix $P$. Then,
\[
\pi_{D_{\infty}}(N) = \eta(N) \lambda^{\lfloor \frac{N}{2} \rfloor} + \frac{1}{\lambda^p} \pi_{\sigma}\left(\lfloor \frac{N}{2} \rfloor\right) + o\left(\lambda^N\right)
\]

where $\eta(N)$ is defined as in Proposition 4 and
\[
\pi_{\sigma}\left(\lfloor \frac{N}{2} \rfloor\right) \sim \frac{2p}{N} \lambda^p \left(\frac{\lfloor \frac{N}{2} \rfloor + 1}{N}\right).
\]

**Proof.** This is straightforward from (9), (12) and Proposition 4, and also the fact that $\pi_1\left(\lfloor \frac{N}{2} \rfloor\right) = O\left(\lambda^N\right)$.

It is possible for $\eta(N) = 0$ e.g. if the sets $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are empty. In fact, $\eta(N)$ determines the final asymptotic behaviour of $\pi_{D_{\infty}}(N)$.

**Corollary 2.** Let $(\mathcal{X}, \sigma, \rho)$ be the flip system of the shift of finite type in Theorem 3.1.

(a) If $\eta(N) = 0$, then
\[
\pi_{D_{\infty}}(N) \sim \frac{p}{N} \lambda^p \left(\frac{\lfloor \frac{N}{2} \rfloor + 1}{N}\right).
\]

(b) If $\eta(N)$ is not zero, then
\[
\pi_{D_{\infty}}(N) \sim \eta(N) \lambda^{\lfloor \frac{N}{2} \rfloor}.
\]
Example 1. (a) Let $A = \{1,2,3,4\}$, and let $(X,\sigma,\rho)$ be the flip system of a shift of finite type with transition matrix $A$ and symmetric permutation matrix $P$, where
\[
A = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
i.e. $1^* = 2$ and $3^* = 4$. Note that $R = S = T = \emptyset$, and hence, $\eta(N) = 0$. So,
\[
\pi_{D_\infty}(N) \sim \frac{3\left(\frac{N}{2}\right) + 1}{2N}.
\]
(b) Let $A = \{1,2,\ldots,m\}$ for $m \geq 2$ and let $(X,\sigma)$ be the full $m$-shift i.e. its transition matrix $A$ is the matrix of ones. Any symmetric permutation matrix $P$ will induce a flip system $(X,\sigma,\rho)$. For example, if $P$ is the identity matrix, then $R = S = T = A$. In this case, we obtain that
\[
\eta(N) = \begin{cases}
\sqrt{m}(3m + 1) & \text{if } N \text{ is odd,} \\
\frac{m(m + 3)}{2(m - 1)} & \text{otherwise.}
\end{cases}
\]
So,
\[
\pi_{D_\infty}(N) \sim \eta(N)m^{\frac{N}{2}}.
\]

4. Additional Comment. As pointed by the referee, the flip system of a shift of finite type can be regarded as a symbolic analogue of the time-reversal symmetry that is manifested by geodesic flows and certain other Hamiltonian systems. For the geodesic flows $\phi_t : T^1M \to T^1M$ on the unit tangent bundle of a manifold $M$, the time-reversal symmetry is given by $F(x,v) = (x,-v)$, where $x \in M$ and $v \in T^1_x M$. Furthermore, if this flow is of Anosov type, then the symbolic dynamics can be chosen to reflect this symmetry (see [1, 20]). More precisely, one can model this flow with time-reversal symmetry by a suspension flow over a mixing shift of finite type $(X,\sigma)$. Additionally, $(X,\sigma)$ supports a fixed point-free involution $a \mapsto a^*$ on its alphabet $A$, and its transition matrix $A$ satisfies $A_{ab} = A_{b^*a^*}$ and $A_{aa^*} = A_{a^*a} = 0$ for any $a,b \in A$. This corresponds to the case $R = S = T = \emptyset$ (such as Example 1(a)) where the counting function is particularly simple, which is
\[
\pi_{D_\infty}(N) = \frac{1}{2}\pi_\sigma \left(\left\lfloor \frac{N}{2} \right\rfloor \right).
\]
In general, for the flip system $(X,\sigma,\rho)$, the involution on $A$ under the flip $\rho$ is not assumed to be fixed-point free, but the fixed point-free case is especially interesting.

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