BPS Monopole Equation in $\Omega$-background

Katsushi Ito, Satoshi Kamoshita and Shin Sasaki

Department of Physics
Tokyo Institute of Technology
Tokyo, 152-8551, Japan

Abstract

We study deformed supersymmetries in $\mathcal{N} = 2$ super Yang-Mills theory in the $\Omega$-backgrounds characterized by two complex parameters $\epsilon_1, \epsilon_2$. When one of the $\epsilon$-parameters vanishes, the theory has extended supersymmetries. We compute the central charge of the algebra and obtain the deformed BPS monopole equation. We examine supersymmetries preserved by the equation.
1 Introduction

The Ω-background deformation of $\mathcal{N} = 2$ supersymmetric gauge theories is a useful method to regularize the integrals over the moduli space of instantons [1, 2, 3]. This background is characterized by two anti-symmetric matrices $\Omega^{mn}$ and $\bar{\Omega}^{mn}$ parametrized by two complex numbers $\epsilon_1, \epsilon_2$ and their complex conjugates. The Ω-background induces the $U(1)^2$ vector fields on $\mathbb{R}^4$. This torus action is used to define the supercharge, which is shown to be equivariantly nilpotent by introducing the Wilson line gauge fields. Using the localization theorem [2], the regularized integral (the instanton partition function) leads to the Seiberg-Witten (SW) prepotential [4] in the limit $\epsilon_1, \epsilon_2 \to 0$. In superstring theory the Ω-background is realized as a certain $\mathcal{N} = 2$ supergravity background [5, 6, 7, 8, 9, 10, 11].

Recently it has been pointed out that there is a relation between two-dimensional integrable systems and gauge theories in the Ω-background where one of the $\epsilon$-parameters vanishes [12, 13, 14, 15, 16]. The theory in this Ω-background has two dimensional $\mathcal{N} = 2$ super-Poincaré invariance. In particular it was shown that the instanton partition function in the limit $\epsilon_2 \to 0$ with $\epsilon_1 = \hbar$ is obtained by the deformation of the SW theory [17, 18, 19, 20]. The period integrals of the SW differential are obtained by solving the quantized Toda spectral curve equations, which are evaluated by the exact Bohr-Sommerfeld integrals.

The purpose of this paper is to study the deformed SW theory from a field theoretical point of view. The sum of the period integrals of the SW differential is the central charge of $\mathcal{N} = 2$ supersymmetry algebra [21]. We will study the deformed supersymmetry algebra in the Ω-background and calculate the central charge. Such deformations of the central charge have been known for non(anti)-commutative field theories [22, 23]. The Ω-background deformation of the central charge would give $\hbar$-corrections to the BPS spectrum, with which one can compare the Ω-deformation to the period integral. In this paper we will derive the deformed BPS monopole equation in the Ω-background.

The organization of this paper is as follows. In section 2, we introduce the four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory in the Ω-background and discuss the deformed supersymmetry. In section 3, we calculate the central charges of the supersymmetry algebra and derive the monopole equation in the Ω-background by the Bogomol’nyi
completion of the energy density. We also discuss the supersymmetries preserved by the BPS equation. Section 4 is devoted to conclusions and discussion.

2 Supersymmetries of $\Omega$-deformed $\mathcal{N} = 2$ super Yang-Mills theory

In this section we discuss the deformation of $\mathcal{N} = 2$ $U(N)$ super Yang-Mills theory in the $\Omega$-background [2, 3, 24, 25]. We will define the theory in spacetime with Euclidean signature. The theory contains a gauge field $A_m$ ($m = 1, 2, 3, 4$), Weyl fermions $\Lambda^I_\alpha$, $\bar{\Lambda}^I_\dot{\alpha}$, and complex scalars $\varphi$, $\bar{\varphi}$. They belong to the adjoint representation of $U(N)$ gauge group. The $SO(4) = SU(2)_L \times SU(2)_R$ Lorentz spinor indices are denoted as $\alpha$, $\dot{\alpha} = 1, 2$ while $I = 1, 2$ indicates the $SU(2)_I$ R-symmetry index. These $SU(2)$ indices are raised and lowered by the antisymmetric $\epsilon$-symbol with $\epsilon^{12} = -\epsilon_{12} = 1$. We expand the fields with $U(N)$ basis $T^u$ ($u = 1, 2, \ldots, N^2$) normalized by $\text{Tr}(T^u T^v) = \kappa \delta^{uv}$ with a certain constant $\kappa$. The Lagrangian of the theory in the flat spacetime is given by

$$\mathcal{L}_0 = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F_{mn} F^{mn} - \frac{i \theta g^2}{32 \pi^2} F_{mn} \tilde{F}^{mn} + \Lambda^I \sigma^m D_m \bar{\Lambda}^I + D_m \varphi D^m \bar{\varphi} \right. \right.$$

$$\left. - i \frac{g}{\sqrt{2}} \Lambda^I [\varphi, \Lambda^I] + i \frac{g}{\sqrt{2}} \bar{\Lambda}^I [\bar{\varphi}, \bar{\Lambda}^I] + \frac{g^2}{2} |\varphi, \bar{\varphi}|^2 \right], \quad (2.1)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m + ig [A_m, A_n]$ is the gauge field strength, $g$ is the gauge coupling constant and $D_m = \partial_m + ig[A_m, \ast]$ is the gauge covariant derivative. We also define the Dirac matrices $\sigma_m = (i \tau^1, i \tau^2, i \tau^3, 1)$ and $\bar{\sigma}_m = (-i \tau^1, -i \tau^2, -i \tau^3, 1)$, where $\tau^c$ ($c = 1, 2, 3$) are the Pauli matrices. The constant $\theta$ is the theta-angle and $\tilde{F}_{mn} = \frac{1}{2} \epsilon_{mpq} F^{pq}$ is the dual of $F_{mn}$.

The four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory in the $\Omega$ background is obtained by the dimensional reduction of the six-dimensional $\mathcal{N} = 1$ super Yang-Mills theory with the non-trivial metric [24]. We also introduce the R-symmetry Wilson line by gauging...
the SU(2)$_I$ R-symmetry. The Lagrangian of the four-dimensional theory is given by

\[
L_\Omega = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F_{mn} F^{mn} - \frac{i}{32 \pi^2} F_{mn} \tilde{F}^{mn} + (D_m \varphi - g F_{mn} \Omega^n)(D^m \varphi - g F^{mp} \tilde{\Omega}_p) \right. \\
+ \Lambda^I \sigma^m D_m \Lambda_I - \frac{i}{\sqrt{2}} g \Lambda^I [\varphi, \Lambda_I] + \frac{i}{\sqrt{2}} g \Lambda_I [\varphi, \Lambda^I] \\
+ \frac{1}{\sqrt{2}} g \tilde{\Omega}^m \Lambda^I D_m \Lambda_I - \frac{1}{2\sqrt{2}} g \tilde{\Omega}^{mn} \Lambda^I \sigma_{mn} \Lambda_I \\
- \frac{1}{\sqrt{2}} g \Omega^m \Lambda_I D_m \Lambda_I + \frac{1}{2\sqrt{2}} g \Omega^{mn} \Lambda_I \sigma_{mn} \Lambda^I \\
+ \frac{g^2}{2} \left( [\varphi, \tilde{\varphi}] + i \Omega^m D_m \varphi - i \tilde{\Omega}^m D_m \tilde{\varphi} + i g \tilde{\Omega}^m \Omega^n F_{mn} \right)^2 \\
- \frac{1}{\sqrt{2}} g \Lambda^I \Lambda^J \Lambda_I \Lambda_J - \frac{1}{2 \sqrt{2}} g \Lambda^I \Lambda^J \Lambda_I \Lambda_J \right],
\]

(2.2)

where the Lorentz generators $\sigma^{mn}$ and $\bar{\sigma}^{mn}$ are defined by

\[
(\sigma^{mn})_{\alpha \dot{\beta}} = \frac{1}{4} (\sigma^m_{\alpha \dot{\alpha}} \sigma^n_{\dot{\beta} \beta} - \sigma^n_{\alpha \dot{\alpha}} \sigma^m_{\dot{\beta} \beta}), \quad (\bar{\sigma}^{mn})^{\dot{\alpha} \beta} = \frac{1}{4} (\bar{\sigma}^{m \dot{\alpha} \alpha} \sigma^n_{\dot{\beta} \beta} - \bar{\sigma}^{n \dot{\alpha} \alpha} \sigma^m_{\dot{\beta} \beta}).
\]

(2.3)

The R-symmetry Wilson line gauge fields $A_I^I$, $\bar{A}_{\dot{I}}^I$ are constant, $\Omega^m = \Omega^{mn} x_n$, $\tilde{\Omega}^m = \tilde{\Omega}^{mn} x_n$ and the $\Omega$-background is parametrized as follows:

\[
\Omega^{mn} = \frac{1}{2 \sqrt{2}} \begin{pmatrix}
0 & i \epsilon_1 & 0 & 0 \\
-i \epsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \epsilon_2 \\
0 & 0 & i \epsilon_2 & 0
\end{pmatrix}, \quad \tilde{\Omega}^{mn} = \frac{1}{2 \sqrt{2}} \begin{pmatrix}
0 & -i \bar{\epsilon}_1 & 0 & 0 \\
i \bar{\epsilon}_1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \bar{\epsilon}_2 \\
0 & 0 & -i \bar{\epsilon}_2 & 0
\end{pmatrix}.
\]

(2.4)

Here $\epsilon_1$ and $\epsilon_2$ are complex numbers and $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are their complex conjugates.

The undeformed theory defined by (2.1) has $\mathcal{N} = 2$ supersymmetry. The supersymmetry algebra is given by

\[
\{Q^I_{\alpha}, \bar{Q}^{\dot{I}}_{\dot{\beta}}\} = 2 (\sigma^m)_{\alpha \dot{\beta}} P_m \delta^I_J, \\
\{Q^I_{\alpha}, Q^J_{\beta}\} = -2 \sqrt{2} \epsilon_{IJ} \epsilon_{\alpha \beta} Z, \\
\{Q^{\dot{I}}_{\dot{\alpha}}, \bar{Q}^{\dot{J}}_{\dot{\beta}}\} = -2 \sqrt{2} \epsilon^{IJ} \epsilon_{\dot{\alpha} \dot{\beta}} Z,
\]

(2.5)

where $Q^I_{\alpha}$ and $\bar{Q}^{\dot{I}}_{\dot{\alpha}}$ are supercharges, $P^m$ is the four-momentum and $Z$ is the central charge.

The supersymmetry algebra leads to the BPS inequality for mass $M$:

\[
M \geq \sqrt{2} |Z|.
\]

(2.6)
Here the equality holds for the BPS saturated states.

It is convenient to use the topological twist [26], which is defined by identifying the $SU(2)_I$ R-symmetry indices $I$ with the $SU(2)_R$ spinor indices $\dot{\alpha}$. The twisted supercharges are introduced by

$$Q_m = \sigma^I_m Q_m^I, \quad Q = \delta^I_\dot{\alpha} Q^{I\dot{\alpha}}, \quad Q_{mn} = - (\sigma_{mn})^I_\dot{\alpha} Q^{I\dot{\alpha}}. \quad (2.7)$$

The supersymmetry algebra (2.5) becomes

$$\{Q_m, \bar{Q}_p\} = 2P_m, \quad \{Q_m, \bar{Q}_{pq}\} = (\delta^m_p \delta^n_q - \delta^n_p \delta^m_q - \varepsilon^{mnpq}) P_n,$$

$$\{Q_m, Q_n\} = \frac{\sqrt{2}}{4} (\delta^m_p \delta^n_q - \delta^n_p \delta^m_q - \varepsilon^{mnpq}) \bar{Z}. \quad (2.8)$$

For general $\Omega$-background, where $\epsilon_1$ and $\epsilon_2$ are generic, the action defined by (2.2) has one scalar supersymmetry $\bar{Q}$ [21, 25] by choosing the R-symmetry Wilson line gauge fields such as

$$A^I_J = - \frac{1}{2} \Omega_{mn} (\sigma^m)^I_J, \quad \bar{A}^I_J = - \frac{1}{2} \Omega_{mn} (\sigma^m)^I_J. \quad (2.9)$$

If the $\Omega$-background and the Wilson line satisfy special conditions, the theory has further extended supersymmetries.

It is known that if the $\Omega$-background are self-dual $\epsilon_1 + \epsilon_2 = 0$ or anti-self-dual $\epsilon_1 - \epsilon_2 = 0$ and there are no Wilson lines, the theory has $\mathcal{N} = (4, 0)$ or $\mathcal{N} = (0, 4)$ supersymmetries [27]. The deformed supersymmetry transformations are given by

$$\delta A_m = - (\xi^I \sigma^I_m \bar{\Lambda}_I + \xi_I \sigma_I \Lambda^I),$$

$$\delta \Lambda^I = \sigma^m \xi^I F_{mn} + \sqrt{2} \sigma^m \xi^I D_m \varphi + ig \xi^I [\varphi, \bar{\varphi}]$$

$$- \sqrt{2} g F_{mn} \Omega^m \sigma^m \xi^I + g \xi^I (\bar{\Omega}^m D_m \varphi - \Omega^m D_m \bar{\varphi}) + g^2 \Omega^m \bar{\Omega}^n F_{mn} \xi^I,$$

$$\delta \bar{\Lambda}_I = \bar{\sigma}^m \xi_I F_{mn} - \sqrt{2} \bar{\sigma}^m \xi_I D_m \bar{\varphi} - ig \xi_I [\varphi, \bar{\varphi}]$$

$$- \sqrt{2} g F_{mn} \Omega^m \xi_I \sigma^m - g \xi_I (\bar{\Omega}^m D_m \varphi - \Omega^m D_m \bar{\varphi}) + g^2 \bar{\Omega}^m \Omega^n F_{mn} \bar{\xi}_I,$$

$$\delta \varphi = \sqrt{2} \xi^I \Lambda_I - g \Omega^m \left( (\xi^I \sigma^m \bar{\Lambda}_I + \xi_I \sigma_I \Lambda^I) \right),$$

$$\delta \bar{\varphi} = \sqrt{2} \bar{\xi}^I \bar{\Lambda}_I - g \bar{\Omega}^m \left( (\xi^I \sigma^m \bar{\Lambda}_I + \xi_I \sigma_I \Lambda^I) \right).$$

\[\text{We denote } \mathcal{N} = (p,q) \text{ by supersymmetry with } p \text{ chiral and } q \text{ anti-chiral supercharges.}\]
Here, we have set $\xi^I = 0$ for the self-dual and $\bar{\xi}^I = 0$ for the anti-self-dual $\Omega$-background. We note that in these cases, there are no translational symmetries and only the chiral- or anti-chiral-sector of the supersymmetry remains.

We now discuss supersymmetries of the theory with the Wilson line (2.9) in the cases that one of the deformation parameters $\epsilon_1, \epsilon_2$ is zero. In these cases, the translational invariance in the $(1, 2)$-plane or the $(3, 4)$-plane is restored.

As we have mentioned before, at least one supersymmetry $\bar{Q}$ corresponding to the transformation with $\bar{\xi} = \delta^A_\beta \bar{\xi}_\beta^A$ in (2.10) is conserved. We examine invariance of the action under other transformations generated by supercharges $\bar{Q}^{mn}$ and $Q^m$ associated with the deformed supersymmetry transformations (2.10). We find that the action is not invariant by the transformation generated by $\bar{Q}^{mn}$. The variation of the Lagrangian under the transformation generated by $Q^m$ is

$$\delta_{\xi_m} \mathcal{L} = \xi_m \left( \Omega^{+mn} + \Omega^{-mn} \right) \left\{ \sqrt{2} g \delta_{np} F^+ pq \Lambda_q - \frac{1}{\sqrt{2}} i g^2 [\varphi, \bar{\varphi}] \Lambda_n \right.$$ 

$$+ \frac{1}{\sqrt{2}} g^2 \Lambda_n \left( \Omega^p D_p \bar{\varphi} - \bar{\Omega}^p D_p \varphi \right) - \frac{1}{\sqrt{2}} g^3 \bar{\Omega}^q \bar{\Omega}^p \Lambda_p \Lambda_n \right\}$$

$$+ \xi_m \left( \Omega^{+mn} + \Omega^{-mn} \right) \left\{ g \left( D^p \varphi \right) \left( 2 \bar{\Lambda}_p \delta_{np} \bar{\Lambda} + g^2 \bar{\Omega}^l \left( 2 F_{pq} \bar{\Lambda}^p + F_{npq} \bar{\Lambda} \right) \right) \right\} \right \}, \quad (2.11)$$

where $\Lambda_m, \bar{\Lambda}_mn, \bar{\Lambda}$ are the fermions obtained by the twist (2.7). The (anti-)self-dual part of an antisymmetric tensor $A_{mn}$ is defined by $A^\pm_{mn} = \frac{1}{2} (A_{mn} \pm \bar{A}_{mn})$.

From (2.11), we find that the variation by the $Q^m$-transformation vanishes under the conditions:

$$\xi_m \left( \Omega^{+mn} + \Omega^{-mn} \right) = 0, \quad \xi_m \left( \bar{\Omega}^{+mn} + \bar{\Omega}^{-mn} \right) = 0. \quad (2.12)$$

When $\epsilon_2 = 0$, these conditions are satisfied for $\xi^m = (0, 0, \xi^3, \xi^4)$. Therefore for $\epsilon_2 = 0$, $\mathcal{N} = (0, 1)$ supersymmetry is enhanced to $\mathcal{N} = (2, 1)$ generated by supercharges $Q^3, Q^4, \bar{Q}$. When $\epsilon_1 = 0$, these conditions are satisfied for $\xi^m = (\xi^1, \xi^2, 0, 0)$. Therefore for $\epsilon_1 = 0$, the theory has $\mathcal{N} = (2, 1)$ supersymmetry generated by supercharges $Q^1, Q^2, \bar{Q}$.

We note that when we exchange $\Omega^{mn}$ with $\bar{\Omega}^{mn}$ in the definition of the Wilson line (2.9), we find that certain linear combination of $\bar{Q}^{mn}$ is conserved. In this case, the Wilson line becomes

$$\mathcal{A}^I J = -\frac{1}{2} \Omega_{mn} (\bar{\sigma}^{mn})^I J, \quad \bar{\mathcal{A}}^I J = -\frac{1}{2} \bar{\Omega}_{mn} (\sigma^{mn})^I J. \quad (2.13)$$
We find that the action is no longer invariant by the transformation generated by $\bar{Q}$. The variations of the Lagrangian by $Q^m$ and $\bar{Q}^{mn}$ become

$$\delta_{\xi_m} L = \xi_m (\Omega^+_{mn} + \Omega^{- mn}) \left\{ \sqrt{2} g \delta_{np} F^{mp} \Lambda_q - \frac{1}{\sqrt{2}} i g^2 [\varphi, \bar{\varphi}] \Lambda_n + \frac{1}{\sqrt{2}} g^2 \Lambda_n \left( \Omega^p D_p \bar{\varphi} - \bar{\Omega}^p D_p \varphi \right) - \frac{1}{\sqrt{2}} g^3 \Omega^p \bar{\Omega}^q F^{pq} \Lambda_n \right\}$$

$$+ \xi_m \left( \Omega^+_mn + \Omega^-_{mn} \right) \left\{ g \left( D^p \varphi \right) (2\bar{\Lambda}^-_m + \delta_{np} \bar{\Lambda}) + g^2 \Omega^q (2F^-_{pq} \bar{\Lambda}^- + F^-_{nq} \bar{\Lambda}) \right\}, \quad (2.14)$$

$$\delta_{\bar{\xi}_{mn}} L = \bar{\xi}_{mn} \left( \Omega^-_{pm} + \bar{\Omega}^-_{pm} \right) \frac{1}{\sqrt{2}} g \left\{ 2F^-_{pq} \bar{\Lambda}^-_n - F^-_{np} \bar{\Lambda}^- + i g \bar{\Lambda}^-_n [\varphi, \bar{\varphi}] \right. \right.$$ 

$$+ g \left( \Omega^p D_q \bar{\varphi} - \bar{\Omega}^p D_q \varphi \right) \bar{\Lambda}^-_n + g^2 \Omega^p \bar{\Omega}^q F^-_{pq} \bar{\Lambda}^- - 2\sqrt{2} g \Omega^q F^-_{pq} \bar{\Lambda}_n + 2\sqrt{2} D_p \varphi \Lambda_n \right\}$$

$$+ \bar{\xi}_{mn} \Omega^-_{mn} \left( \frac{1}{\sqrt{2}} g F^-_{pq} \bar{\Lambda}^-_n + g \left( D^p \varphi \right) \Lambda_n \right) + \bar{\xi}_{mn} g^2 \Omega^-_{mn} F^-_{pq} \Omega^q \Lambda^p$$

$$+ \bar{\xi}_{mn} \left( \Omega^-_{mn} - \bar{\Omega}^-_{mn} \right) \frac{1}{2\sqrt{2}} g^2 \left\{ - i [\varphi, \bar{\varphi}] \bar{\Lambda} + \left( \Omega^p D_q \bar{\varphi} - \bar{\Omega}^p D_q \varphi \right) \bar{\Lambda} + g \bar{\Omega}^p \Omega^q F^-_{pq} \bar{\Lambda} \right\}. \quad (2.15)$$

These vanish for

$$\bar{\xi}^{mn} (\Omega^-_{pm} + \bar{\Omega}^-_{pm}) = 0, \quad \bar{\xi}^{mn} \Omega^-_{mn} = 0,$$

$$\xi^m (\Omega^-_{mn} + \bar{\Omega}^+_mn) = 0, \quad \xi^m (\Omega^-_{mn} + \bar{\Omega}^+_mn) = 0. \quad (2.16)$$

When $\epsilon_2 = 0$, these conditions are satisfied for real $\epsilon_1$, $\xi^m = (\xi^1, \xi^2, 0, 0)$ and

$$\bar{\xi}^{mn} = \begin{pmatrix} 0 & 0 & \xi_{13} & \xi_{14} \\ 0 & 0 & -\bar{\xi}_{14} & \bar{\xi}_{13} \\ -\xi_{13} & \bar{\xi}_{14} & 0 & 0 \\ -\bar{\xi}_{14} & -\bar{\xi}_{13} & 0 & 0 \end{pmatrix}. \quad (2.17)$$

Therefore for $\epsilon_2 = 0$, the theory has $\mathcal{N} = (2, 2)$ supersymmetry generated by supercharges $Q^1$, $Q^2$, $\bar{Q}^{13}$, $\bar{Q}^{14}$. When $\epsilon_1 = 0$, these conditions are satisfied for real $\epsilon_2$, $\xi^m = (0, 0, \xi^3, \xi^4)$, and $\bar{\xi}^{mn}$ given by $(2.17)$. Therefore for $\epsilon_1 = 0$, the theory has $\mathcal{N} = (2, 2)$ supersymmetry and the supercharges $Q^3$, $Q^4$, $\bar{Q}^{13}$, $\bar{Q}^{14}$ are conserved.

The conserved supercharges are summarized in table II. In all cases, the theory has supersymmetries including both the chiral- and anti-chiral-sectors. So far we have identified the $SU(2)_T$ R-symmetry indices with the $SU(2)_R$ spinor indices. When we identify them with the $SU(2)_L$ spinor indices $\alpha$ and replace $\bar{\sigma}^{mn}$ in the Wilson lines by $\sigma^{mn}$, the theory has $\mathcal{N} = (1, 2)$ or $\mathcal{N} = (2, 2)$ supersymmetry.
\[ \epsilon_1 \neq 0, \epsilon_2 = 0 \quad \epsilon_1 = 0, \epsilon_2 \neq 0 \]

| Wilson line (2.9) | \( Q^2, Q^1, \bar{Q} \) | \( Q^1, Q^2, \bar{Q} \) |
| Wilson line (2.13) | \( Q^1, Q^2, Q^{13}, \bar{Q}^{14} \) | \( Q^3, Q^1, \bar{Q}^{13}, \bar{Q}^{14} \) |

Table 1: Classification of conserved supercharges.

3 Central charges and BPS monopole equation

In this section, we will derive the Noether currents and evaluate the central charge of the algebra. We will obtain the BPS monopole equation and classify supersymmetries preserved by the equation, which depend on the \( \Omega \)-background and the \( SU(2)_I \) Wilson line. From the deformed supersymmetry transformations (2.10), we can calculate the Noether currents \( J^m_I \) and \( \bar{J}^\alpha_m \) associated with them. We find

\[
J^m_{I\alpha} = \frac{1}{\kappa} \text{Tr} \left[ \sqrt{2} (D_m \bar{\varphi}) \Lambda_{I\alpha} + \left\{ -ig [\varphi, \bar{\varphi}] \delta_{mn} - F_{mn} + \left( \frac{i\theta g^2}{8\pi^2} + 1 \right) \tilde{F}_{mn} \right\} \sigma_{\alpha\dot{\alpha}}^n \bar{\Lambda}_{I\dot{\alpha}} 
- 2\sqrt{2} (D_n \bar{\varphi}) \sigma_{mn}^n \beta \Lambda_{I\beta} - g \left( \varphi \bar{\Omega}^n - \bar{\varphi} \bar{\Omega}^m \right) \sigma_{\alpha\dot{\alpha}}^n D_n \bar{\Lambda}_{I\dot{\alpha}} - g \left( \varphi \bar{\Omega}_{mn} - \bar{\varphi} \Omega_{mn} \right) \sigma_{\alpha\dot{\alpha}}^n \bar{\Lambda}_{I\dot{\alpha}} 
+ g \left( \varphi \bar{\Omega}_n - \bar{\varphi} \Omega_n \right) \sigma_{\alpha\dot{\alpha}}^n D^n \bar{\Lambda}_{I\dot{\alpha}} - g \left( \Omega^n D^n \bar{\varphi} + \bar{\Omega}^m D^n \varphi \right) \sigma_{\alpha\dot{\alpha}}^n \bar{\Lambda}_{I\dot{\alpha}} 
- \sqrt{2} g F_{mn} \bar{\Omega}^n \Lambda_{I\alpha} - i\sqrt{2} g^2 \bar{\Omega}^n [\varphi, \bar{\varphi}] \Lambda_{I\alpha} + \sqrt{2} g^2 \bar{\Omega}^n \left( \Omega^n D_n \bar{\varphi} - \bar{\Omega}^n D_n \varphi \right) \Lambda_{I\alpha} 
+ 2\sqrt{2} g F_{np} \bar{\Omega}^n \sigma_{mn}^n \alpha \beta \Lambda_{I\beta} + \sqrt{2} g F_{np} \bar{\Omega}^n \sigma_{np}^m \alpha \beta \Lambda_{I\beta} 
+ 2g^2 F_{np} \Omega^n \bar{\Omega}^n \sigma_{\alpha\dot{\alpha}}^n \bar{\Lambda}_{I\dot{\alpha}} - g^2 \bar{\Omega}^n \Omega^n \sigma_{\alpha\dot{\alpha}}^n \bar{\Lambda}_{I\dot{\alpha}} + \sqrt{2} g^3 \bar{\Omega}^n \bar{\Omega}^n \Omega^n F_{np} \Lambda_{I\alpha} \right]. \tag{3.1} \]

The complex conjugate \( \bar{J}^\dot{\alpha}m \) may be calculated in a similar way. The anti-commutation relations of the supercharges, which are defined by the spatial integration of \( J^m_{I\alpha} \) is evaluated by using canonical anti-commutation relations of the fermions, which is given by

\[
\left\{ \bar{\Lambda}_{I\dot{\alpha}}(\bar{x}, x^4), \Lambda^I_{\alpha}(\bar{x}', x'^4) \right\} = \delta_I^J \sigma^{4\dot{\alpha} \alpha} \delta^4(\bar{x} - \bar{x}'). \tag{3.2} \]

We then obtain [21]

\[
\left\{ Q_{I\alpha}, Q_{J\beta} \right\} = -2\sqrt{2} \epsilon_{\alpha\beta} \epsilon_{IJ} Z, \tag{3.3} \]

7
where $Z$ is the deformed central charge:

$$Z = \int d^3x \frac{1}{\kappa} \text{Tr} \left[ ig (D_4 \bar{\varphi}) [\varphi, \bar{\varphi}] + (D^a \bar{\varphi}) \left\{ F_{4n} - \left( \frac{i\theta g^2}{8\pi^2} + 1 \right) \bar{F}_{4n} \right\} + g (D_4 \bar{\varphi}) \left( \Omega^n D_n \varphi - \Omega^n D_n \bar{\varphi} \right) + g \left\{ -ig [\varphi, \bar{\varphi}] \delta_{4n} - F_{4n} + \left( \frac{i\theta g^2}{8\pi^2} + 1 \right) \bar{F}_{4n} \right\} F_{np} \bar{\Omega}^p \right].$$

(3.4)

The complex conjugate $\bar{Z}$ can be calculated in a similar way. In this paper, we focus on the magnetic monopole configuration such that the fields that depend on the three-dimensional space spanned by $(x^1, x^2, x^3)$ and are independent of $x^4$. The scalar field is taken to be in the Cartan subalgebra so that $[\varphi, \bar{\varphi}] = 0$ and we fix the gauge $A_4 = 0$ so that the electric field $F_{4i}$ vanishes. We find that $Z$ for the monopole configuration is given by,

$$Z = -\left( \frac{i\theta g^2}{8\pi^2} + 1 \right) \frac{1}{\kappa} \int d^3x \partial^i \text{Tr}[B_i \varphi],$$

(3.5)

where we have defined the magnetic field $B_i = \frac{1}{2} \varepsilon_{ijk} F^{jk}$. This is the same as the undeformed case. However, we will find that the monopole equation is deformed by the $\Omega$-background as discussed below. Then the central charge could depend on the $\epsilon$-parameter.

To find the BPS equation corresponding to the monopole configuration, we perform the Bogomol'nyi completion of the energy functional by combining the kinetic and potential terms. In the monopole configuration, we obtain the energy

$$E = \frac{1}{\kappa} \int d^3x \text{Tr} \left[ \frac{1}{2} B_i^2 + (D_i \varphi + g \Omega^j F_{ji})(D_i \bar{\varphi} + g \bar{\Omega}^k F_{ki}) + \frac{g^2}{2} \left( [\varphi, \bar{\varphi}] + i\Omega^i D_i \varphi - i\bar{\Omega}^i D_i \bar{\varphi} - ig \Omega^i \bar{\Omega}^j F_{ij} \right)^2 \right],$$

(3.6)

where we have taken $\theta = 0$ for simplicity. The vacuum configurations of the theory are given by

$$D_i \varphi = D_i \bar{\varphi} = 0, \quad A_i = 0, \quad [\varphi, \bar{\varphi}] = 0.$$

(3.7)

Hence, the Higgs field $\varphi$ takes value in the Cartan subalgebra $U(1)^N$ and the vacuum moduli space becomes $U(N)/U(1)^N \cong SU(N)/U(1)^{N-1}$, which is the same as the undeformed theory.
To find the energy bound, we use the phase transformation of $\varphi$ and $\Omega$ to set those to the values which are consistent with the first equation in (2.16):

$$\varphi = -\bar{\varphi}, \quad \Omega_{mn} = -\bar{\Omega}_{mn}. \quad (3.8)$$

Then the third term in (3.6) vanishes and the energy is rewritten in the perfect square form,

$$E = \frac{1}{\kappa} \int d^3x \, \text{Tr} \left[ \frac{1}{2} \left( B_i \pm (D_i \phi + g\hat{\Omega}^j F_{ji}) \right) \right]^2 \pm \frac{1}{\kappa} \int d^3x \, \partial_i \text{Tr} [B_i \phi] \quad \geq \quad \pm \frac{1}{\kappa} \int d^3x \, \partial_i \text{Tr} [B_i \phi], \quad (3.9)$$

where we have defined $\phi = i\sqrt{2}\varphi$, $\hat{\Omega}^{ij} = i\sqrt{2}\Omega^{ij}$ and $\hat{\Omega}^i = \hat{\Omega}^{mn} x_n$. The energy bound is saturated if the following deformed BPS equation is satisfied:

$$B_i \pm (D_i \phi + g\hat{\Omega}^j F_{ji}) = 0. \quad (3.10)$$

From (3.9), we find that the energy at the lower bound is the same as the central charges (3.5) derived from the supersymmetry algebra.

When $\epsilon_1 \neq 0$ and $\epsilon_2 = 0$, the BPS equation is deformed by $\epsilon_1$ and its solutions depend on the parameter $\epsilon_1$. On the other hand, when $\epsilon_1 = 0$ and $\epsilon_2 \neq 0$, the BPS equation depends on $x^4$ since the third term in the equation (3.10) contains $x^4$. In this case, we can consider the monopole configuration which depends on $(x^2, x^3, x^4)$ and is independent of $x^1$, which gives the similar deformed BPS equation.

In the following, we will investigate supersymmetries that are preserved by the BPS state. Substituting the BPS equation (3.10) into the supersymmetry transformations of the fermions in (2.10), we obtain the condition on the supersymmetry parameters that are preserved by the BPS configurations:

$$\pm i(\sigma^m)^I_\beta \varepsilon^{\beta\alpha} \xi_m \pm i\varepsilon^{I\alpha} \xi + \xi_m (\sigma^m)_{\alpha J} \varepsilon^{IJ} = 0, \quad (\alpha = \hat{\alpha} = 1, 2). \quad (3.11)$$

As we have discussed in section 2, only part of the $\xi_m, \xi, \xi_{mn}$ symmetries exist in the theory in the $\Omega$-background. We will examine the condition (3.11) for the cases where $\Omega^{mn}$ is (anti-)self-dual or one of the $\epsilon$-parameters is zero.
When $\Omega^m$ and $\bar{\Omega}^m$ are (anti-)self-dual, the action is invariant under $\xi_{\alpha I}$ (anti-self-dual case) or $\bar{\xi}_{\dot{\alpha} I}$ (self-dual case) transformations. In both cases, we find that all the supersymmetry is broken by the BPS conditions (3.11).

On the other hand, when $\Omega^m$ and $\bar{\Omega}^m$ are not (anti-)self-dual, the action is invariant under the $\bar{\xi}$ transformation if one considers the Wilson line (2.9). The condition (3.11) implies that the $\bar{\xi}$-supersymmetry is broken and the BPS configuration does not preserve any supersymmetries.

In the cases that one of the $\epsilon$-parameters vanishes, the supersymmetries that are preserved by the BPS condition are classified as follows:

(i) $\epsilon_1 \neq 0$, $\epsilon_2 = 0$ and Wilson line (2.9)

In this case, the action is invariant under the $\bar{\xi}, \xi_3, \xi_4$ supersymmetries and there are translational symmetries in the (3,4)-plane. The BPS condition (3.11) becomes

\[
\begin{pmatrix}
0 & i\xi_3 - i\xi_4 \pm i\bar{\xi} \\
-i\xi_1 + i\xi_2 & 0
\end{pmatrix} = 0,
\]

(3.12)

where we have written the condition as the $2 \times 2$ matrix form with respect to the spinor and R-symmetry indices. Therefore the BPS state preserves one supersymmetry specified by a linear combination of $Q_4$ and $\bar{Q}$.

(ii) $\epsilon_1 = 0$, $\epsilon_2 \neq 0$ and Wilson line (2.9)

In this case, the action is invariant under the $\bar{\xi}, \xi_1, \xi_2$ supersymmetries and the translational symmetry in the (1,2)-plane. The BPS condition (3.11) becomes

\[
\begin{pmatrix}
-i\xi_1 - \xi_2 & \pm i\bar{\xi} \\
\mp i\bar{\xi} & i\xi_1 - i\xi_2
\end{pmatrix} = 0.
\]

(3.13)

This condition implies that $\xi_3 = \xi_4 = \bar{\xi} = 0$. Therefore the BPS state does not preserve any supersymmetries.

(iii) $\epsilon_1 \neq 0$, $\epsilon_2 = 0$ and Wilson line (2.13)

In this case, the action is invariant under the $\xi_1, \xi_2, \bar{\xi}_3, \bar{\xi}_4$ supersymmetries and the translational symmetry in the (3,4)-plane. The BPS condition (3.11) becomes

\[
\begin{pmatrix}
\pm i\bar{\xi}_3 \mp \bar{\xi}_4 - i\xi_1 - \xi_2 & 0 \\
0 & \pm i\bar{\xi}_3 \pm \bar{\xi}_4 + i\xi_1 - \xi_2
\end{pmatrix} = 0.
\]

(3.14)
This implies the conditions,
\[ \pm i\bar{\xi}_{13} - \xi_2 = 0, \mp i\bar{\xi}_{14} + \xi_1 = 0. \quad (3.15) \]
Therefore the BPS state preserves two supersymmetries specified by linear combinations of \( Q_2, \bar{Q}_{13} \) and \( Q_1, \bar{Q}_{14} \).

(iv) \( \epsilon_1 = 0, \epsilon_2 \neq 0 \) and Wilson line (2.13)
In this case the action is invariant under the \( \bar{\xi}_{13}, \xi_{14}, \xi_3, \xi_4 \) supersymmetries and the translational symmetry in the (1,2)-plane. The BPS condition (3.11) becomes
\[
\begin{pmatrix}
\pm i\bar{\xi}_{13} \mp i\bar{\xi}_{14} \\
i\xi_3 - i\xi_4 \\
i\xi_3 + i\xi_4 \\
\pm i\bar{\xi}_{13} \pm i\bar{\xi}_{14}
\end{pmatrix}
= 0. \quad (3.16)
\]
This condition implies \( \bar{\xi}_{13} = \xi_{14} = \xi_3 = \xi_4 = 0 \). Thus the BPS state does not preserve any supersymmetries.

The supercharges that are preserved by the BPS condition are summarized in Table 2.

### Table 2: Supercharges preserved by the BPS state.

| Wilson line (2.9) | \( iQ^4 \mp i\bar{Q} \) no supercharge |
| Wilson line (2.13) | \( \pm i\bar{Q}_{13} - Q^2, \mp i\bar{Q}_{14} + Q^1 \) no supercharge |

4 Conclusions and discussion

In this paper, we have studied the deformed supersymmetries of \( \mathcal{N} = 2 \) super Yang-Mills theories in the \( \Omega \)-background and the Wilson lines. For general \( \Omega \)-background, there is one scalar supersymmetry. When one of the \( \epsilon \)-parameters is zero, we have found the theories with \( \mathcal{N} = (2, 1) \) or \( (2, 2) \) or \( (1, 2) \) supersymmetry by choosing the appropriate Wilson line gauge fields. We have also calculated the central charge of the deformed supersymmetry algebra. For the monopole configurations the formula for the central charge does not contain the \( \epsilon \)-parameter. We have performed the Bogomol’nyi completion of the energy density and obtained the deformed BPS monopole equation. We have examined the supersymmetries preserved by the monopole equation.
The central charge for the monopole could receive \( \epsilon \)-corrections through the deformed BPS monopole solution, where this situation occurs for \( \epsilon_2 = 0 \). In this case, the deformed BPS equation has the axial symmetry around the \( z \)-axis. We may use Manton’s ansatz [28] for the fields:

\[
A_i^a = \left\{ \eta_1 \hat{\rho}^a + \left( \eta_2 + \frac{1}{2g\rho} \right) \hat{z}^a \right\} \hat{\phi}^i + W_1 \hat{\rho}^i \hat{\phi}^a + W_2 \hat{z}^i \hat{\phi}^a, \\
\phi^a = \phi_1 \hat{\rho}^a + \phi_2 \hat{z}^a,
\]

(4.1) where \((\rho, \varphi, z)\) are the cylindrical coordinates. \(\eta_\alpha, W_\alpha\) and \(\phi_\alpha (\alpha = 1, 2)\) are functions of \((\rho, z)\) and

\[
\hat{\rho} = (\cos \varphi, \sin \varphi, 0), \quad \hat{\varphi} = (-\sin \varphi, \cos \varphi, 0), \quad \hat{z} = (0, 0, 1).
\]

(4.2)

Here we have considered the \(SU(2)\) gauge group for simplicity and the superscript \(a\) denotes the \(SU(2)\) index. Substituting (4.1) into the deformed BPS equation (3.10), we obtain

\[
- \partial_3 \eta_1 + g \eta_2 W_2 = \frac{1}{1 + g^2 \epsilon_2 \rho^2} \left\{ \partial_\rho \phi_1 - g W_1 \phi_2 + g \epsilon \rho (\partial_3 \phi_1 - g W_2 \phi_2) \right\}, \\
- \partial_3 \eta_2 - g \eta_1 W_2 = \frac{1}{1 + g^2 \epsilon_2 \rho^2} \left\{ \partial_\rho \phi_2 + g W_1 \phi_1 + g \epsilon \rho (\partial_3 \phi_2 + g W_2 \phi_1) \right\}, \\
\partial_\rho \eta_1 + \frac{\eta_1}{\rho} - g W_1 \eta_2 = \frac{1}{1 + g^2 \epsilon_2 \rho^2} \left\{ \partial_3 \phi_1 - g W_2 \phi_2 - g \epsilon \rho (\partial_\rho \phi_1 - g W_2 \phi_2) \right\}, \\
\partial_\rho \eta_2 + \frac{\eta_2}{\rho} + g W_1 \eta_1 = \frac{1}{1 + g^2 \epsilon_2 \rho^2} \left\{ \partial_3 \phi_2 + g W_2 \phi_1 - g \epsilon \rho (\partial_\rho \phi_2 + g W_1 \phi_1) \right\}, \\
\partial_\rho W_2 - \partial_3 W_1 = -g \eta_1 \phi_2 + g \eta_2 \phi_1,
\]

(4.3)

where \(\epsilon = -\frac{1}{2} \text{Re} \epsilon_1\) and we have chosen the minus sign in the BPS equation. These equations are invariant under the gauge transformations [29]

\[
W_1' = W_1 + \frac{1}{g} \partial_\rho \Lambda, \quad \quad W_2' = W_2 + \frac{1}{g} \partial_3 \Lambda, \\
\phi_1' = \cos \Lambda \phi_1 + \sin \Lambda \phi_2, \quad \quad \phi_2' = \cos \Lambda \phi_2 - \sin \Lambda \phi_1, \\
\eta_1' = \cos \Lambda \eta_1 + \sin \Lambda \eta_2, \quad \quad \eta_2' = \cos \Lambda \eta_2 - \sin \Lambda \eta_1,
\]

(4.4)

where \(\Lambda\) is a function of \((\rho, z)\). There are five differential equations for the six unknown functions, which have one gauge degree of freedom. For \(\epsilon = 0\), the solution for (4.3) has been found in [30, 31]. It is an interesting problem to find solutions to the deformed
equation by perturbations around the exact solution and calculate the $\epsilon$-corrections to the central charges. It would also be interesting to study the Nahm construction of monopoles [32] and other BPS solitons such as vortices and domain walls. These subjects will be discussed elsewhere.

Acknowledgment

The work of S. S. is supported by the Japan Society for the Promotion of Science (JSPS) Research Fellowship.

References

[1] G. W. Moore, N. Nekrasov and S. Shatashvili, Commun. Math. Phys. 209 (2000) 97 [arXiv:hep-th/9712241].

[2] N. A. Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831 [arXiv:hep-th/0206161].

[3] A. S. Losev, A. Marshakov and N. A. Nekrasov, arXiv:hep-th/0302191.

[4] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19 [Erratum-ibid. B 430 (1994) 485] [arXiv:hep-th/9407087].

[5] M. Billo, M. Frau, F. Fucito and A. Lerda, JHEP 0611 (2006) 012 [arXiv:hep-th/0606013].

[6] I. Antoniadis, S. Hohenegger, K. S. Narain and T. R. Taylor, Nucl. Phys. B 838 (2010) 253 [arXiv:1003.2832 [hep-th]].

[7] Y. Nakayama, JHEP 1007 (2010) 054 [arXiv:1004.2986 [hep-th]].

[8] A. Iqbal, C. Kozcaz and C. Vafa, JHEP 0910 (2009) 069 [arXiv:hep-th/0701156].

[9] M. Taki, JHEP 0803 (2008) 048 [arXiv:0710.1776 [hep-th]].

[10] H. Awata and H. Kanno, Int. J. Mod. Phys. A 24 (2009) 2253 [arXiv:0805.0191 [hep-th]].
[11] M. x. Huang and A. Klemm, arXiv:1009.1126 [hep-th].

[12] N. A. Nekrasov and S. L. Shatashvili, arXiv:0908.4052 [hep-th].

[13] N. Nekrasov and E. Witten, JHEP 1009 (2010) 092 [arXiv:1002.0888 [hep-th]].

[14] K. Maruyoshi and M. Taki, Nucl. Phys. B 841 (2010) 388 [arXiv:1006.4505 [hep-th]].

[15] D. Orlando and S. Reffert, JHEP 1010 (2010) 071 [arXiv:1005.4445 [hep-th]].

[16] R. Poghossian, arXiv:1006.4822 [hep-th].

[17] A. Mironov and A. Morozov, JHEP 1004 (2010) 040 [arXiv:0910.5670 [hep-th]].

[18] A. Mironov and A. Morozov, J. Phys. A 43 (2010) 195401 [arXiv:0911.2396 [hep-th]].

[19] A. Mironov, A. Morozov and S. Shakirov, JHEP 1002 (2010) 030 [arXiv:0911.5721 [hep-th]].

[20] A. Popolitov, arXiv:1001.1407 [hep-th].

[21] E. Witten and D. I. Olive, Phys. Lett. B 78 (1978) 97.

[22] C. S. Chu and T. Inami, Nucl. Phys. B 725 (2005) 327 [arXiv:hep-th/0505141].

[23] K. Ito and H. Nakajima, Phys. Lett. B 633 (2006) 776 [arXiv:hep-th/0511241].

[24] N. Nekrasov and A. Okounkov, arXiv:hep-th/0306238.

[25] K. Ito, H. Nakajima, T. Saka and S. Sasaki, JHEP 1011 (2010) 093 [arXiv:1009.1212[hep-th]].

[26] E. Witten, Commun. Math. Phys. 117 (1988) 353.

[27] K. Ito, H. Nakajima and S. Sasaki, JHEP 0812 (2008) 113 [arXiv:0811.3322 [hep-th]].

[28] N. S. Manton, Nucl. Phys. B135 (1978) 319.

[29] P. Forgács, Z. Horváth and L. Palla Nucl. Phys. B221 (1983) 235.

[30] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
[31] E. B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449 [Yad. Fiz. 24 (1976) 861].

[32] W. Nahm, Phys. Lett. B 90 (1980) 413.