A remark on the contactomorphism group of overtwisted contact spheres

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Abstract

We show the existence of elements of infinite order in some homotopy groups of the contactomorphism group of overtwisted spheres. It follows in particular that the contactomorphism group of some high dimensional overtwisted spheres is not homotopically equivalent to a finite dimensional Lie group.

1 Introduction and statements of the results

Let $(M, \xi)$ be a closed contact manifold. These short notes are concerned with the relationship between the topology of the connected component $\text{Diff}_0 (M)$ of the identity in the group of diffeomorphisms of $M$ and its subgroup $\text{Diff}_0 (M, \xi)$ consisting of contactomorphisms of $(M, \xi)$. More precisely, throughout the notes we will always assume contact structures to be cooriented and contactomorphisms to be coorientation–preserving.

The path components of the group of contactomorphisms of particular contact manifolds have been studied by several authors in the literature; see for instance Dymara [2001], Giroux [2001], Ding and Geiges [2010], Lanzat and Zapolsky [2018], Massot and Niederkrüger [2016], Giroux and Massot [2017], Vogel [2018], Gironella [2018, 2019].

Higher–order homotopy groups have also been studied: for instance, Eliashberg [1992], Casals and Presas [2014], Casals and Spáčil [2016] contain results for the case of the standard tight $(2n+1)$–contact sphere. In this notes, we deal with the case of overtwisted spheres (cf. Borman, Eliashberg, and Murphy [2015]).

Let $(S^{2n+1}, \xi_{ot})$ be any overtwisted sphere, and consider the natural inclusion

$$i : \text{Diff}_0 (S^{2n+1}, \xi_{ot}) \hookrightarrow \text{Diff}_0 (S^{2n+1}).$$

For any $k \in \mathbb{N}$, denote $K^{2n+1}_k$ the kernel of the homomorphism

$$\pi_k (i) : \pi_k (\text{Diff}_0 (S^{2n+1}, \xi_{ot})) \to \pi_k (\text{Diff}_0 (S^{2n+1})).$$

Theorem 1. Let $k \in \mathbb{N}$ be such that $1 \leq 4k + 1 \leq 2n - 1$. The group $K^{2n+1}_{4k+1}$ contains an infinite cyclic subgroup.

Under some conditions on the dimension, Theorem 1 can be improved in the case of the fundamental group and the fifth homotopy group as follows:

Theorem 2. (i) The group $K^3_1$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. 


Let $n \geq 3$. The group $K_{4n+1}^3$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Let $n \geq 6$. The group $K_{2n+1}^3$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

From the methods developed in the paper we are also able to show the following

**Theorem 3.**

(i) Let $n \geq 4$. The group $K_{3n+3}^4$ contains an infinite cyclic subgroup.

(ii) Let $n \geq 2$. The group $K_{3n+7}^4$ contains an infinite cyclic subgroup.

As the even–order higher homotopy groups of a finite dimensional Lie group are finite (see for instance Félix, Oprea, and Tanré [2008, Example 2.51]), Theorem 3 immediately implies:

**Corollary 4.** For $n \geq 2$, $\text{Diff}_0(S^{2n+7},\xi_{\text{ot}})$ is not homotopy equivalent to a finite dimensional Lie group.

The proofs of Theorems 1, 2 and 3 use four main ingredients. The first is the notion of overtwisted group introduced in Casals, del Pino, and Presas [2018], which relies on the flexibility results for overtwisted contact manifolds from Eliashberg [1989], Borman, Eliashberg, and Murphy [2015]. The second is the existence of a long exact sequence relating the homotopy groups of the space of contact structures on $S^{2n+1}$ to those of $\text{Diff}_0(S^{2n+1},\xi_{\text{ot}})$ and of $\text{Diff}_0(S^{2n+1})$; see Section 2.1. The last ingredients are the description of the rational homotopy groups of $\text{Diff}_0(S^{2n+1})$ from Farrell and Hsiang [1978] and the description of some homotopy groups of the homogeneous space $\Gamma_n = \text{SO}(2n)/U(n)$ from Bott [1959], Massey [1961], Harris [1963], Kachi [1978], Mukai [1990].

We point out that these methods could also be applied to the case of any overtwisted contact manifold $(M^{2n+1},\xi)$ such that both the homotopy type of the space of almost contact structures on $M$ and the diffeomorphism group of $M$ can be (at least partially) understood.

**Acknowledgments** The authors are extremely grateful to Fran Presas for explaining them the construction of the overtwisted group and encouraging them to write down this note, as well as to Javier Martínez Aguinaga for very interesting discussions on the problem. The first author is supported by the Spanish Research Projects SEV–2015–0554, MTM2016–79400–P, and MTM2015–72876–EXP as well as by a Beca de Personal Investigador en Formación UCM. The second author is supported by the grant NKFIH KKP 126683.

## 2 Preliminaries

### 2.1 A long exact sequence of homotopy groups

Let $(M,\xi)$ be a closed contact manifold. In this section, the spaces $\text{Diff}_0(M)$ and $\text{Diff}_0(M,\xi)$ are be considered as pointed spaces, with base point $\text{Id}$. Similarly, $\text{Cont}(M,\xi)$ is considered with base point $\xi$.

As shown for instance in Giroux and Massot [2017] (and, more in detail, in Massot [2015]), the natural map
\[
\text{Diff}_0 (M) \longrightarrow \text{Cont} (M, \xi) \\
\varphi \longmapsto \varphi \ast \xi
\]
is a locally–trivial fibration with fiber \(\text{Diff}_0 (M, \xi)\); see also Geiges and Gonzalo Perez [2004] for a proof of the fact that the map is a Serre fibration (which is enough for what follows). In particular, it induces a long exact sequence of homotopy groups
\[
\ldots \rightarrow \pi_{k+1} (\text{Cont} (M, \xi)) \rightarrow \pi_k (\text{Diff}_0 (M, \xi)) \rightarrow \pi_k (\text{Diff}_0 (M)) \rightarrow \pi_k (\text{Cont} (M, \xi)) \rightarrow \ldots
\]  

(1)

2.2 Almost contact structures on \(\mathbb{S}^{2n+1}\)

Recall that, given an oriented smooth manifold \(M^{2n+1}\), an almost contact structure is a triple \((\xi, J, R)\), where \(\xi \subseteq TM\) is a cooriented hyperplane distribution, \(J : \xi \rightarrow \xi\) is a complex structure on \(\xi\), \(R = \langle v \rangle \subseteq TM\) is a trivial line sub–bundle defining the coorientation of \(\xi\) and \(\xi \oplus R \cong TM\) as oriented vector bundles. Fixing an auxiliary Riemannian metric \(g\) on \(M\) which is adapted to \(J\) and such that \(w\) is of norm 1 and orthogonal to \(\xi\), one can see that \((\xi, J, R)\) is equivalent to a reduction of the structure group \(\text{SO}(2n+1)\) of the principal bundle \(\text{Fr}_{\text{SO}}(M)\) of orthonormal oriented frames of \(TM\) to its subgroup \(\text{U}(n) = \text{U}(n) \times 1 \subseteq \text{SO}(2n+1)\). The space of such reductions is the space of sections \(\Gamma(M; X)\) of a fiber bundle \(\pi : X = \text{Fr}_{\text{SO}}(M)/ \text{U}(n) \rightarrow M\), with typical fiber \(\text{SO}(2n+1)/ \text{U}(n)\). Such space \(\Gamma(M; X)\) is naturally identified with the space of almost contact structures on \(M\), which we denote \(\text{AlmCont}(M)\).

Recall also (see Geiges [2008, Lemma 8.2.1]) that there is an identification
\[
\Gamma_{n+1} := \text{SO}(2n+2)/\text{U}(n+1) \simeq \text{SO}(2n+1)/\text{U}(n) .
\]

(2)

In particular, the fiber bundle \(\pi\) can also be seen as a fibration
\[
\begin{array}{ccc}
\Gamma_{n+1} & \hookrightarrow & X \\
\pi & \downarrow & \\
M & & 
\end{array}
\]

(3)

Denote the trivial real line bundle over \(M\) by \(\varepsilon = \langle w \rangle\). Then, the Riemannian metric \(g\) on \(M\) naturally extends to a metric on \(TM \oplus \varepsilon\), still denoted \(g\), by declaring the vector \(w\) to be orthogonal to \(TM\) and of norm 1. Let now \(\text{Complex}(TM \oplus \varepsilon)\) be the space of complex structures on the oriented bundle \(TM \oplus \varepsilon\), which are compatible with the metric \(g\) (i.e. \(g(J, J) = g(\langle ., . \rangle)\)). Notice that this space can be identified with the space of sections of a fiber bundle over \(M\) with fiber the space of complex structures on \(\mathbb{R}^{2n+2}\) compatible with the standard metric, i.e. \(\Gamma_{n+1}\).

Given any almost contact structure \((\xi, J, R)\), one can naturally extend \(J\) to a complex structure \(\tilde{J} : TM \oplus \varepsilon \rightarrow TM \oplus \varepsilon\) on \(TM \oplus \varepsilon\), by defining \(Jv = -w\). This gives an inclusion \(j : \text{AlmCont}(M) \hookrightarrow \text{Complex}(TM \oplus \varepsilon)\).

In fact, Equation (2) says that \(i\) is a diffeomorphism. More precisely, denoting the projection on the first factor by \(\text{pr} : TM \oplus \varepsilon \rightarrow TM\), the map
\[
\Phi : \text{Complex}(TM \oplus \varepsilon^1) \longrightarrow \text{AlmCont}(M) \\
J \longmapsto (TM \cap J(TM), J|_{TM \cap J(TM)}, \langle \text{pr}(Jw) \rangle)
\]
is the inverse of $i$. As a consequence:

**Lemma 5.** If the vector bundle $TM$ is stably trivial of type 1 over $\mathbb{R}$, i.e. $TM \oplus \varepsilon$ is trivializable (as real vector bundle), the fiber bundle $\pi: X \to M$ is trivializable.

For the rest of the section we focus on the case of almost contact structures on $S^{2n+1}$.

According to Lemma 5, the fiber bundle $\pi: X \to S^{2n+1}$ is trivial. Once fixed any trivialization, one can then identify $\text{AlmCont}(S^{2n+1}) = \text{Map}(S^{2n+1}, \Gamma_{n+1})$.

**Remark 6.** The homotopy groups $\pi_k(\Gamma_{n+1})$, in the stable range $1 \leq k \leq 2n$, were computed in Bott [1959]: they are of period 8 and the first eight groups are, in order, $0, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2$. Moreover, some of the first unstable groups $\pi_{2n+1+k}(\Gamma_{n+1})$ were computed in Massey [1961], Harris [1963], Kachi [1978], Mukai [1990]. More precisely, we will use the fact that the following unstable homotopy groups contain a cyclic subgroup: $\pi_{4n+3}(\Gamma_{2n+1}), \pi_{4n+7}(\Gamma_{2n+1}), \pi_{4n+7}(\Gamma_{2n+2})$ and $\pi_{8n+12}(\Gamma_{4n+4})$.

**Lemma 7.** All the path connected components of the space $\text{AlmCont}(S^{2n+1})$ are homeomorphic.

**Proof.** Let $J_0 \in \Gamma_{n+1}$ be the standard (almost) complex structure on $\mathbb{R}^{2n+2}$, and denote

$$\xi_0: S^{2n+1} \to \Gamma_{n+1}$$

$$z \mapsto J_0$$

the almost contact structure with constant value $J_0$. Consider then any other almost contact structure $\xi: S^{2n+1} \to \Gamma_{n+1}$. Because $\Gamma_{n+1}$ is path-connected, up to homotopy, we can moreover assume that $\xi(N) = J_0$, where $N$ denotes the north pole of $S^{2n+1}$.

Denote by $\text{AlmCont}_{\xi_0}(S^{2n+1})$ and $\text{AlmCont}_{\xi}(S^{2n+1})$ the path connected components of $\xi_0$ and $\xi$, respectively. Consider the $U(n+1)$-principal bundle $p: SO(2n+2) \to \Gamma_{n+1}, A \mapsto A \cdot J_0 \cdot A^{-1}$. By Bott periodicity, $\pi_{2n}(U(n+1)) = 0$. In particular, the homomorphism

$$\pi_{2n+1}(p): \pi_{2n+1}(SO(2n+2)) \to \pi_{2n+1}(\Gamma_{n+1})$$

is surjective, so that there exists a lift $\tilde{\xi}: S^{2n+1} \to SO(2n+2)$ of $\xi$ such that $\tilde{\xi}(N) = \text{Id}$.

The desired homeomorphism is then given by

$$\Phi_{\xi}: \text{AlmCont}_{\xi_0}(S^{2n+1}) \to \text{AlmCont}_{\xi}(S^{2n+1})$$

$$\eta \mapsto \tilde{\xi} \cdot \eta$$

where

$$\tilde{\xi} \cdot \eta: S^{2n+1} \to \Gamma_{n+1}$$

$$z \mapsto \tilde{\xi}(z) \cdot \eta(z)$$

is defined by using the left action of $SO(2n+2)$ on $\Gamma_{n+1}$.

\[\square\]
Proposition 8. For each $k \in \mathbb{N}$ there is an isomorphism

$$\pi_k \left( \text{AlmCont}(S^{2n+1}) \right) \cong \pi_k (\Gamma_{n+1}) \oplus \pi_{2n+k+1} (\Gamma_{n+1})$$

Proof. For $k = 0$ we argue as follows. Recall that $[S^n, X] = \pi_n (X, x) / \pi_1 (X, x)$, for any pointed topological space $(X, x)$. Hence,

$$\pi_0 \left( \text{AlmCont}(S^{2n+1}) \right) = [S^{2n+1}, \Gamma_{n+1}] = \pi_{2n+1} (\Gamma_{n+1}) / \pi_1 (\Gamma_{n+1}) = \pi_{2n+1} (\Gamma_{n+1}) ,$$

and the statement follows from the fact that, according to Remark 6, $\Gamma_{n+1} = \text{SO}(2n+2)/\text{U}(n+1)$ is simply connected.

We now prove the statement for $\pi_k$ with $k \geq 1$. According to Lemma 7, we can consider $\text{AlmCont}_0 (S^{2n+1})$ as space pointed at $\xi_0 \equiv J_0 : S^{2n+1} \to \Gamma_{n+1}$. Similarly, we consider $\Gamma_{n+1}$ as space pointed at $J_0$. There is then a natural Serre fibration (of pointed spaces)

$$ev_N : \text{AlmCont}(S^{2n+1}) \to \Gamma_{n+1}$$

$$\xi \mapsto \xi(N)$$

The fiber over $J_0$ is the space $F := \text{AlmCont}_{\xi(N)=J_0}(S^{2n+1})$ of almost contact structures which evaluate at $J_0$ on the north pole, which is naturally considered as pointed at $\xi_0$. In particular, $F = \text{Map}(S^{2n+1}, (\Gamma_{n+1}, J_0))$, so that $\pi_k (F) = \pi_{2n+k+1} (\Gamma_{2n+1})$.

Moreover, the map

$$s : \Gamma_{n+1} \to \text{AlmCont}_{\xi_0}(S^{2n+1})$$

$$J \mapsto \xi_J$$

where $\xi_J \equiv J$, defines a section of the fibration. In particular, the boundary map in the long exact sequence of homotopy groups associated to the Serre fibration $ev_N$ is trivial, and every obtained short exact sequence of groups splits. In other words,

$$\pi_k \left( \text{AlmCont}(S^{2n+1}) \right) \cong \pi_k (\Gamma_{n+1}) \oplus \pi_k (F) = \pi_k (\Gamma_{n+1}) \oplus \pi_{2n+k+1} (\Gamma_{2n+1}) .$$

2.3 The overtwisted group

Let $M$ be a $(2n + 1)$–dimensional manifold. We denote in this section the subspaces of contact and almost contact structures on $M$ with a fixed overtwisted disk $\Delta_0 \subset M$ respectively by $\text{Cont}_{\text{OT}} (M, \Delta_0) \subseteq \text{Cont} (M)$ and $\text{AlmCont} (M, \Delta_0) \subseteq \text{AlmCont} (M)$. 

Theorem 9 (Eliashberg [1989], Borman, Eliashberg, and Murphy [2015]). The following forgetful map induces a weak homotopy equivalence:

$$\text{Cont}_{\text{OT}} (M, \Delta_0) \to \text{AlmCont} (M, \Delta_0) ,$$

Notice that the overtwisted disk is not allowed to move in this results. However, an easy corollary is the fact that the forgetful map

$$\text{Cont}_{\text{OT}} (M) \to \text{AlmCont} (M)$$

(4)
induces a bijection at $\pi_0$–level, where $\text{Cont}_{\text{OT}} (M)$ denotes the space of overtwisted contact structures on $M$. This can be seen by introducing an overtwisted disk in a neighborhood of a (properly chosen) point of $M$, and using Theorem 9.

To deal with the higher–order homotopy groups, one needs the existence of a continuous choice of overtwisted disks in order to run the same argument.

**Definition 1** (Casals, del Pino, and Presas [2018]). Let $0 \leq k \leq 2n$. The overtwisted $k$–group of $M$, denoted $\text{OT}_k(M)$, is the subgroup of $\pi_k(\text{Cont}_{\text{OT}} (M))$ made of those classes that admit a representative $\xi : S^k \to \text{Cont}_{\text{OT}} (M)$ for which there is a certificate of overtwistedness, i.e. a continuous map

$$\Delta : S^k \to \text{Emb}_{\text{PL}} (B^{2n}, M) := \{ \psi : B^{2n} \to M \text{ piece–wise linear embedding} \}$$

such that, for each $p \in S^k$, $\Delta(p)$ is overtwisted for $\xi(p)$.

Homotopy classes in $\text{OT}_k(M)$ are called overtwisted. A homotopy class which is not overtwisted is called tight.

In these terms, Equation (4) says that the map $\text{OT}_0(M) \to \pi_0(\text{AlmCont}(M))$ is a bijection. For higher–order homotopy groups one then has the following:

**Proposition 10** (Casals, del Pino, and Presas [2018, Proposition 33]). Let $(M, \xi_{\text{ot}})$ be any closed overtwisted contact manifold. For each $0 \leq k \leq 2n$, the inclusion $\text{Cont}_{\text{OT}} (M) \hookrightarrow \text{AlmCont} (M)$ induces an isomorphism

$$\text{OT}_k(M) \sim \pi_k(\text{AlmCont}(M)).$$

Moreover, $\text{OT}_k(M) < \pi_k(\text{Cont}_{\text{OT}} (M), \xi_{\text{ot}}) = \pi_k(\text{Cont} (M), \xi_{\text{ot}}) / \text{OT}_k(M)$ is a normal subgroup for $k > 0$ and, thus, the set of tight classes $\text{Tight}_k(M) = \pi_k(\text{Cont}(M), \xi_{\text{ot}}) / \text{OT}_k(M)$ has group structure. In particular, for any $1 \leq k \leq 2n$ there is an isomorphism

$$\pi_k(\text{Cont}(M), \xi_{\text{ot}}) \cong \text{OT}_k(M) \oplus \text{Tight}_k(M).$$

To the authors’ knowledge, the only known example of a non–trivial tight class is contained in Vogel [2018], where the author exhibits an order 2 loop of overtwisted contact structures on $S^3$, based at the only overtwisted structure on $S^3$ having Hopf invariant $-1$, which does not admit a certificate of overtwistedness. It follows that this tight loop cannot come from a loop of diffeomorphisms in the long exact sequence in Equation (1). In particular, its image via the boundary map is a non–trivial element (of order 2) in the contact mapping class group.

## 3 Proofs of the statements

We start by recalling some known facts in algebraic topology. Recall the following standard homotopy equivalence (see for instance Antonelli, Burghielea, and Kahn [1972, Lemma 1.1.5] for a proof):

$$\text{Diff}_0 (S^{2n+1}) \xleftarrow{\cong} \text{Diff}_0 (B^{2n+1}, \partial) \times \text{SO}(2n + 2). \quad (5)$$
Here, the group $\text{Diff}_0 (\mathbb{D}^{2n+1}, \partial)$ of diffeomorphisms of the disk relative to its boundary which are smoothly isotopic to the identity is understood as the subgroup of $\text{Diff}_0 (\mathbb{S}^{2n+1})$ of diffeomorphisms which fixes (a neighborhood of) the north hemisphere, and the arrow is the natural inclusion map. Moreover, some of the rational homotopy groups of the first factor of the right-hand side of Equation (5) are completely characterized (see also Weiss and Williams [2001, Section 6]):

**Theorem 11** (Farrell and Hsiang [1978]). Let $0 \leq k < \min\{\frac{2n-3}{3}, n-3\}$. Then

$$
\pi_k (\text{Diff}_0 (\mathbb{D}^{2n+1}, \partial)) \otimes \mathbb{Q} = \begin{cases} 
0 & \text{if } k \not\equiv 3 \text{ mod } 4, \\
\mathbb{Q} & \text{if } k \equiv 3 \text{ mod } 4.
\end{cases}
$$

Let’s now go back to contact topology and prove the statements announced in the introduction.

**Proof (Theorem 1).** Let $\xi_{\text{ot}}$ be any overtwisted structure on $\mathbb{S}^{2n+1}$, and $k \in \mathbb{N}$ such that $1 \leq 4k+1 \leq 2n-1$. The relevant part of the long exact sequence in Equation (1) is the following:

$$
\pi_{4k+2}(\text{Diff}_0 (\mathbb{S}^{2n+1})) \longrightarrow \pi_{4k+2}(\text{Cont} (\mathbb{S}^{2n+1})) \longrightarrow K_{4k+1}^{2n+1}
$$

According to Propositions 8 and 10, there is an isomorphism

$$
\pi_{4k+2} (\text{Cont} (\mathbb{S}^{2n+1}, \xi_{\text{ot}})) \cong \pi_{4k+2} (\Gamma_{n+1}) \oplus \pi_{2n+4k+3} (\Gamma_{n+1}) \oplus \text{Tight}_k (\mathbb{S}^{2n+1}).
$$

Moreover, under this isomorphism, the projection on the first factor

$$
\pi_{4k+2} (\text{Cont} (\mathbb{S}^{2n+1}, \xi_{\text{ot}})) \to \pi_{4k+2} (\Gamma_{n+1})
$$

is just the map induced by the evaluation at the north pole $ev_N$. As $\text{Diff}_0 (\mathbb{D}^{2n+1}, \partial) \subset \text{Diff}_0 (\mathbb{S}^{2n+1})$ is the subgroup of diffeomorphisms fixing the north hemisphere, it follows that the following composition is trivial:

$$
\pi_{4k+2}(\text{Diff}_0 (\mathbb{D}^{2n+1}, \partial)) \overset{\pi_{4k+2}(i)}{\longrightarrow} \pi_{4k+2}(\text{Diff}_0 (\mathbb{S}^{2n+1})) \overset{\pi_{4k+2}(ev_N)}{\longrightarrow} \pi_{4k+2} (\Gamma_{n+1})
$$

Moreover, according to Bott periodicity, $\pi_{4k+2} (\text{SO}(2n+2)) = 0$. In particular, the following composition is also trivial:

$$
\pi_{4k+2}(\text{Diff}_0 (\mathbb{S}^{2n+1})) \longrightarrow \pi_{4k+2}(\text{Cont} (\mathbb{S}^{2n+1})) \overset{\pi_{4k+2}(ev_N)}{\longrightarrow} \pi_{4k+2} (\Gamma_{n+1})
$$

Now, according to Remark 6, $\pi_{4k+2} (\Gamma_{n+1})$, hence $\pi_{4k+2} (\text{Cont} (\mathbb{S}^{2n+1}))$, contains a subgroup $\mathbb{Z}$. It then follows from the exact sequence that $K_{4k+1}^{2n+1}$ must have at least one element of infinite order, as desired. □

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Proof (Theorem 2). According to Hatcher [1983], the Smale Conjecture holds for $S^3$; in particular, $\pi_2\left(\text{Diff}_0(S^3)\right) = 0$. Moreover, since $\Gamma_2 = \text{SO}(4)/\text{U}(2) = S^2$ it follows from Propositions 8 and 10 that the group

$$\text{OT}_2(S^3) \cong \pi_2(S^2) \oplus \pi_5(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

is a subgroup of $\pi_2\left(\text{Cont}(S^3), \xi_{st}\right)$. Item (i) then follows from the exact sequence in Equation (1).

Since $\pi_2(\text{SO}(4n+1)) = \pi_6(\text{SO}(4n+1)) = 0$, Theorem 11 implies that $\pi_2\left(\text{Diff}_0(S^{4n+1})\right) \otimes \mathbb{Q} = 0$ for $n \geq 3$, and $\pi_6\left(\text{Diff}_0(S^{4n+1})\right) \otimes \mathbb{Q} = 0$ for $n \geq 6$. Moreover, according to Remark 6, each of the following homotopy groups contain a cyclic subgroup: $\pi_2(\Gamma_{2n+1})$ for $n \geq 1$, $\pi_6(\Gamma_{2n+1})$ for $n \geq 2$, $\pi_{4n+3}(\Gamma_{2n+1})$ and $\pi_{4n+7}(\Gamma_{2n+1})$. Items (ii) and (iii) then follow from the exact sequence in Equation (1) and from Propositions 8 and 10.

Proof (Theorem 3). Since $\pi_4(\text{SO}(4n+4))$ is trivial, it follows from the identification in Equation (5) and from Theorem 11 that $\pi_4\left(\text{Diff}_0(S^{4n+3})\right) \otimes \mathbb{Q} = 0$ for $n \geq 4$. Moreover, according to Remark 6, $\pi_{4n+7}(\Gamma_{2n+2})$ contains a subgroup $\mathbb{Z}$.

Similarly, $\pi_5(\text{SO}(8n+8)) = 0$ thus Equation (5) and Theorem 11 imply that $\pi_5\left(\text{Diff}_0(S^{8n+7})\right) \otimes \mathbb{Q} = 0$ for $n \geq 2$. According to Remark 6, $\pi_{8n+12}(\Gamma_{4n+4}) \cong \mathbb{Z}$.

The statement then follow from the exact sequence in Equation (1) and from Propositions 8 and 10. \qed

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