On the oscillation of linear matrix Hamiltonian systems

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Abstract. The Riccati equation method is used to establish new oscillation criteria for linear matrix Hamiltonian systems. New approaches allow to extend and completed a result, obtained by S. Kumary and S. Umamaheswaram. The oscillation problem for linear matrix Hamiltonian systems in a new direction, which is to break the positive definiteness condition, imposed on one of the coefficients of the system, is investigated. Some examples are provided for comparing the obtained results with each other and with the result of S. Kumary and S. Umamaheswaram, as well as to illustrate the applicability of these results.

Key words: Riccati equation, conjoined (prepared, preferred) solutions, Hamiltonian system, comparison theorem.

1. Introduction. Let $A(t)$, $B(t)$ and $C(t)$ be complex-valued locally integrable matrix functions of dimension $n \times n$ on $[t_0, +\infty)$ and let $B(t)$ and $C(t)$ be Hermitian, i.e. $B(t) = B^*(t)$, $C(t) = C^*(t)$, $t \geq t_0$, where $*$ denotes the conjugation sign. Consider the linear matrix Hamiltonian system

$$
\begin{align*}
\Phi' &= A(t)\Phi + B(t)\Psi; \\
\Psi' &= C(t)\Phi - A^*(t)\Psi, \quad t \geq t_0.
\end{align*}
$$

By a solution $(\Phi(t), \Psi(t))$ of this system we mean a pair of absolutely continuous matrix functions $\Phi = \Phi(t)$ and $\Psi = \Psi(t)$ of dimension $n \times n$ on $[t_0, +\infty)$ satisfying (1.1) almost everywhere on $[t_0, +\infty)$.

**Definition 1.1.** A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called conjoined (or prepared, preferred), if $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \geq t_0$.

**Definition 1.2.** The system (1.1) is called oscillatory if for its every conjoined solution $(\Phi(t), \Psi(t))$ the function $\det \Phi(t)$ has arbitrary large zeroes.

The oscillation problem for linear matrix Hamiltonian systems is that of finding explicit conditions on the coefficients of the system, providing its oscillation. This is an important problem of qualitative theory of differential equations and many works are devoted to it (see e.g., [1, 3, 5, 6, 13-18] and cited works therein). Among them notice the following...
result, obtained by S. Kumari and S. Umamaheswaram

**Theorem 1.1** ([11, Theorem 2.1]). Let $A(t)$, $B(t)$ and $C(t)$ be real-valued continuous functions on $[t_0, +\infty)$ and let $B(t)$ be positive definite for all $t \geq t_0$. If there exists a positive linear functional $g$ on the space of matrices of dimension $n \times n$ such that

\[
\lim_{t \to +\infty} \int_{t_0}^{t} \frac{ds}{g[B^{-1}(s)]} = +\infty,
\]

\[
\lim_{t \to +\infty} g \left[ -\int_{t_0}^{t} \left( C(s) + A^*(s)B^{-1}(s)A(s) \right) ds - B^{-1}(t)A(t) \right] = +\infty,
\]

then the system (1.1) is oscillatory.

In this paper we use the Riccati equation method for obtaining some new oscillation criteria for the system (1.1). We show that one of the obtained results is an extension of Theorem 1.1. Traditionally the oscillation problem for the system (1.1) was studied under the restriction that the coefficient $B(t)$ of the system (1.1) is positive definite (a condition, which is essential from the point of view if used methods). New approaches in the papers [5] and [6] allowed to obtain oscillation criteria in the direction of weakening (braking) the positive definiteness of $B(t)$. In this paper we continue to study the oscillation problem for the system (1.1) in the mentioned direction. On examples we compare the obtained oscillation criteria with each other and with Theorem 1.1 and demonstrate their applicability.

2. **Auxiliary propositions.** Let $a(t)$, $b(t)$, $c(t)$, $a_1(t)$, $b_1(t)$ and $c_1(t)$ be real-valued locally integrable functions on $[t_0, +\infty)$. Consider the Riccati equations.

\[
y' + a(t)y^2 + b(t)y + c(t) = 0, \quad t \geq t_0, \quad (2.1)
\]

\[
y' + a_1(t)y^2 + b_1(t)y + c_1(t) = 0, \quad t \geq t_0 \quad (2.2)
\]

and the differential inequalities

\[
\eta' + a(t)\eta^2 + b(t)\eta + c(t) \geq 0, \quad t \geq t_0, \quad (2.3)
\]

\[
\eta' + a_1(t)\eta^2 + b_1(t)\eta + c_1(t) \geq 0, \quad t \geq t_0 \quad (2.4)
\]

Let $[t_1, t_2]$ be an interval in $[t_0, +\infty)$ ($t_0 \leq t_1 < t_2 \leq +\infty$). By a solution of Eq. (2.1) (of Eq. (2.2), inequality (2.3), inequality (2.4)) on $[t_1, t_2]$ we mean an absolutely continuous function on $[t_1, t_2]$, satisfying (2.1) ((2.2), (2.3), (2.4)) almost everywhere on $[t_1, t_2]$. Note that every solution of Eq. (2.1) (Eq. (2.2)) on any interval $[t_1, t_2]$ is also a solution of the
inequality (2.3) ((2.4)) on that interval. Note also that for \(a(t) \geq 0\) \((a_1(t) \geq 0)\), \(t \geq t_0\) the real-valued solutions of the linear equation

\[
\eta' + b(t)\eta + c(t) = 0, \quad t \geq t_0 \quad (\eta' + b_1(t)\eta + c_1(t) = 0, \quad t \geq t_0)
\]

are solutions of the inequality (2.3) ((2.4)) as well. Therefore, for \(a(t) \geq 0\), \((a_1(t) \geq 0)\), \(t \geq t_0\) the inequality (2.3) ((2.4)) has a solution on \([t_0, +\infty)\), satisfying any initial-value condition.

**Theorem 2.1.** Let \(y_0(t)\) be a solution of Eq. (2.1) on \([t_1, t_2]\), \(\eta_0(t)\), \(\eta_1(t)\) be a solution of the inequalities (2.3) and (2.4) respectively with \(\eta_0(t_1) \geq y_0(t_1)\), \(\eta_1(t_1) \geq y_0(t_1)\), and let \(a_1(t) \geq 0\), \(\lambda - y_0(t_1) + \int_{t_1}^t \exp \left\{ \int_{t_1}^\tau [a_1(\xi)(\eta_0(\xi) + \eta_1(\xi)) - b_1(\xi)]d\xi \right\} \times \times [(a(\tau) - a_1(\tau))y_0^2(\tau) + (b(\tau) - b_1(\tau))y_0(\tau) = c(\tau) - c_1(\tau)]d\tau \geq 0\), \(t \in [t_1, t_2]\), for some \(\lambda \in [y_0(t_1), \eta_1(t_1)]\). Then Eq. (2.2) has a solution \(y_1(t)\) on \([t_1, t_2]\) with \(y_1(t_1) \geq y_0(t_1)\), moreover \(y_1(t) \geq y_0(t)\), \(t \in [t_1, t_2]\).

Proof. By analogy with the proof of Theorem 3.1 from [8].

Let \(a_{jk}(t)\), \(j, k = 1, 2\) be real-valued locally integrable functions on \([t_0, +\infty)\). Consider the linear system of ordinary differential equations

\[
\begin{align*}
\phi' &= a_{11}(t)\phi + a_{12}(t)\psi, \\
\psi' &= a_{21}(t)\phi + a_{22}(t)\psi, \quad t \geq t_0.
\end{align*}
\]

and the corresponding Riccati equation

\[
y' + a_{12}(t)y^2 + E(t)y - a_{21}(t) = 0, \quad t \geq t_0,
\]

where \(E(t) \equiv a_{11}(t) - a_{22}(t), \quad t \geq t_0\). By a solution of the system (2.5) we mean an ordered pair \((\phi(t), \psi(t))\) of absolutely continuous functions \(\phi(t), \psi(t)\) on \([t_0, +\infty)\), satisfying for \(\phi = \phi(t)\), \(\psi = \psi(t)\) the system (2.5) almost everywhere on \([t_0, +\infty)\). All solutions \(y(t)\) of Eq. (2.6), existing on any interval \([t_1, t_2] \subset [t_0, +\infty)\) are connected with solutions \((\phi(t), \psi(t))\) of the system (2.5) by relations (see [8])

\[
\phi(t) = \phi(t_1) \exp \left\{ \int_{t_1}^t [a_{12}(\tau)y(\tau) + a_{11}(\tau)]d\tau \right\}, \quad \phi(t_1) \neq 0, \quad \psi(t) = y(t)\phi(t), \quad t \in [t_1, t_2).
\]

**Definition 2.1.** The system (2.2) is called oscillatory if for its every solution \((\phi(t), \psi(t))\) the function \(\phi(t)\) has arbitrary large zeroes.
Remark 2.3. Explicit oscillatory criteria for the system (2.1) (therefore for the system (2.5)) are obtained in [7].

Theorem 2.2. Let the following conditions be satisfied.

\[ a(t) \geq 0, \quad t \geq t_0. \]

\[ \int_{t_0}^{+\infty} a_{12}(t)a_{12}(t) \exp \left\{ -\int_{a}^{t} E(\tau) d\tau \right\} dt = -\int_{t_0}^{+\infty} a_{21}(t) \exp \left\{ \int_{a}^{t} E(\tau) d\tau \right\} dt = +\infty. \]

Proof. By analogy with the proof of Corollary 3.1 from [10] (see also [6, Theorem 2.4]).

Definition 2.2. An interval \([t_1, t_2) \subset [t_0, +\infty)\) is called the maximum existence interval for a solution \(y(t)\) of Eq. (2.1), if \(y(t)\) exists on \([t_1, t_2)\) and cannot be continued to the right from \(t_2\) as a solution of Eq. (2.1).

Lemma 2.1. Let \(y(t)\) be a solution of Eq. (2.1) on \([t_1, t_2) \subset [t_0, +\infty)\), and let \(t_2 < +\infty\). If \(a(t) \geq 0, \quad t \in [t_1, t_2)\) and the function \(F(t) \equiv \int_{t_1}^{t} a(\tau)y(\tau)d\tau, \quad t \in [t_1, t_2)\) is bounded from below on \([t_1, t_2)\), then \([t_1, t_2)\) is not the maximum existence interval for \(y(t)\).

Proof. By analogy with the proof of Lemma 2.1 from [9].

Let \(e(t)\) and \(e_1(t)\) be real-valued functions on \([t_0, +\infty)\) and let \(e(t)\) be locally integrable and \(e_1(t)\) be absolutely continuous on \([t_0, +\infty)\). Consider the Riccati integral equations

\[ y(t) + \int_{t_0}^{t} a(\tau)y^2(\tau)d\tau + e(t) = 0, \quad t \geq t_0, \quad (2.8) \]

\[ y(t) + \int_{t_0}^{t} a(\tau)y^2(\tau)d\tau + e_1(t) = 0, \quad t \geq t_0, \quad (2.9) \]

Lemma 2.2. Let \(y_0(t)\) be a solution of Eq. (2.8) on \([t_0, t_1)\). If \(a(t) \geq 0, \quad e(t) > e_1(t) > 0, \quad t \in [t_0, t_1)\), then Eq. (2.9) has a solution \(y_1(t)\) on \([t_0, t_1)\) and

\[ y_1(t) > y_0(t), \quad t \in [t_0, t_1). \quad (2.10) \]

Proof. Since \(a(t) \geq 0, \quad e(t) > 0, \quad t \in [t_0, t_1)\) by (2.8)

\[ y_0(t) < 0, \quad t \in [t_0, t_1). \quad (2.11) \]
Let $y_1(t)$ be a solution of Eq. (2.9) and let $[t_0, t_2)$ be its maximum existence interval. Show that
\[ t_2 \geq t_1. \] (2.12)
Suppose
\[ t_2 < t_1. \] (2.13)
Show that
\[ y_1(t) > y_0(t), \quad t \in [t_0, t_2). \] (2.14)
Suppose that this is false. By (2.8) and (2.9) from the conditions $e(t) > e_1(t) > 0$, $t \in [t_0, t_1)$ of the lemma it follows that $y_1(t_0) = -e_1(t_0) > -e(t_0) = y_0(t_0)$. Then there exists $t_3 \in (t_0, t_2)$ such that
\[ y_1(t) > y_0(t), \quad t_0 \leq t < t_3, \] (2.15)
\[ y_1(t_3) = y_0(t_3). \] (2.16)
On the other hand by (2.8) and (2.9) we have
\[ y_1(t_3) - y_0(t_3) = \int_{t_0}^{t_3} a(\tau)[y_0^2(\tau) - y_1^2(\tau)]d\tau + e(t_3) - e_1(t_3). \]
This together with (2.11), (2.15) and the condition $a(t) \geq 0$, $t \in [t_0, t_1)$ of the lemma implies that $y_1(t_3) > y_0(t_1)$, which contradicts (2.16). The obtained contradiction proves (2.14). Obviously $y_1(t)$ is a solution of the Riccati equation (recall that $e_1(t)$ is absolutely continuous)
\[ y' + a(t)y^2 + e_1'(t) = 0, \quad t \geq t_0 \]
on $[t_0, t_1)$. Then by Lemma 2.1 and (2.7) from the condition $a(t) \geq 0$, $t \in [t_0, t_1)$ of the lemma and from (2.6) it follows that $[t_0, t_2)$ is not the maximum existence interval for $y_1(t)$, which contradicts our supposition. The obtained contradiction proves (2.12). From (2.12) and (2.14) it follows existence $y_1(t)$ on $[t_0, t_1)$ and the inequality (2.10). The lemma is proved.

**Lemma 2.3.** For any two square matrices $M_k \equiv (m^l_{i,j})^n_{i,j=1}$, $l = 1, 2$ he equality
\[ tr(M_1M_2) = tr(M_2M_1) \]
is valid.

Proof. We have
\[ tr(M_1M_2) = \sum_{j=1}^{n} (\sum_{k=1}^{n} m^1_{jk}m^2_{kj}) = \sum_{k=1}^{n} (\sum_{j=1}^{n} m^1_{jk}m^2_{kj}) = \sum_{k=1}^{n} (\sum_{j=1}^{n} m^2_{kj}m^1_{jk}) = tr(M_2M_1). \]
The lemma is proved.

In the system (1.1) substitute
\[ \Psi = Y\Phi. \] (2.17)
We obtain
\[
\begin{cases}
\Phi' = [A(t) + B(t)Y]\Phi, \\
[Y' + YB(t)Y + A^*(t)Y + YA(t) - C(t)]\Phi = 0, \quad t \geq t_0.
\end{cases}
\]

It follows from here and from (2.17) that all solutions of the matrix Riccati equation
\[
Y' + YB(t)Y + A^*(t)Y + YA(t) - C(t) = 0, \quad t \geq t_0,
\]
(2.18)
existing on any interval \([t_1, t_2] \subset [t_0, +\infty]\), are connected with solutions \((\Phi(t), \Psi(t))\) of the system (1.1) by relations.

\[
\Phi'(t) = [A(t) + B(t)Y(t)]\Phi(t), \quad \Psi(t) = Y(t)\Phi(t), \quad t \in [t_1, t_2]
\]
(2.19)
(by a solution of Eq. (2.18) on \([t_1, t_2] \subset [t_0, +\infty]\) we mean an absolutely continuous matrix function on \([t_1, t_2]\), satisfying (2.18) almost everywhere on \([t_1, t_2]\)).

For any matrix \(M\) of dimension \(n \times n\) denote by \(\lambda_1(M), \ldots, \lambda_n(M)\) the eigenvalues of \(M\), and if they are real-valued then we will assume that they are ordered by
\[
\lambda_1(M) \leq \ldots \leq \lambda_n(M).
\]
The nonnegative (positive) definiteness of any Hermitian matrix will be denoted by \(H \geq 0\) \((> 0)\). By \(I\) it will be denoted the identity matrix of dimension \(n \times n\).

**Lemma 2.4.** For a matrix \(S\) of dimension \(n \times n\) and any Hermitian matrix \(H \geq 0\) of the same dimension the inequality
\[
tr(SSH^*) \geq \frac{\lambda_1(H)}{n} \left\{ \left[ tr \left( \frac{S + S^*}{2} \right) \right]^2 + \left[ tr \left( \frac{S - S^*}{2i} \right) \right]^2 \right\}
\]
is valid.

Proof. Let \(U_H\) be a unitary matrix such that
\[
U_HHU_H^* = diag\{\lambda_1(H), \ldots, \lambda_n(H)\} \overset{def}{=} H_0.
\]
and let
\[
S_H = U_HS_H^* = (s_{jk})_{j,k=1}^n.
\]
Then
\[
tr(SSH^*) = tr(S_HH_0S_H) = \sum_{j,k=1}^n \lambda_k(H)s_{jk}s_{jk}^*.
\]
(2.20)
Since $H \geq 0$ we have $\lambda_n(H) \geq \ldots \lambda_1(H) \geq 0$. This together with (2.20) implies

$$
\text{tr}(SHS^*) \geq \lambda_1(H) \sum_{j,k=1}^{n} s_{jk}s_{jk} \geq \lambda_1(H) \sum_{j=1}^{n} s_{jj}s_{jj} =
$$

$$
= \lambda_1(H) \left[ \sum_{j=1}^{n} \left( \frac{s_{jj} + s_{jj}}{2} \right)^2 + \sum_{j=1}^{n} \left( \frac{s_{jj} - s_{jj}}{2i} \right)^2 \right] \geq
$$

$$
\geq \frac{\lambda_1(H)}{n} \left\{ \left[ \sum_{j=1}^{n} \frac{s_{jj} + s_{jj}}{2} \right]^2 + \sum_{j=1}^{n} \left\{ \left[ \frac{s_{jj} - s_{jj}}{2i} \right]^2 \right\} \right\} =
$$

$$
= \lambda_1(H) \left\{ \left[ \text{tr} \left( \frac{S + S^*}{2} \right) \right]^2 + \left[ \text{tr} \left( \frac{S - S^*}{2i} \right) \right]^2 \right\}.
$$

The lemma is proved.

Let $g$ be a positive linear functional on the space of matrices of dimension $n \times n$. For any matrix $M \geq 0$ of dimension $n \times n$ set

$$
\nu_g(M) \equiv \begin{cases} 
0, & \text{if } \det M = 0, \\
\{g(M^{-1})\}^{-1}, & \text{if } \det M \neq 0.
\end{cases}
$$

**Lemma 2.5.** For any matrix $M$ and any Hermitian matrix $H \geq 0$ of dimension $n \times n$ the inequality

$$
g(M^*HM) \geq \nu_g(H)[g(M)]^2 \tag{2.21}
$$

is valid.

Proof. If $\det M \neq 0$, then $H > 0$ (since $H \geq 0$), and the inequality (2.21) is proved in [11] (see [11, p. 178]). Assume $\det M = 0$. Then since $H \geq 0$ for arbitrary small $\varepsilon > 0$ the Hermitian matrix $H_\varepsilon \equiv \varepsilon I + H$ is positive definite ($H_\varepsilon > 0$). Therefore according to the already established fact we have

$$
g(M^*H_\varepsilon M) \geq \nu_g(H_\varepsilon)[g(M)]^2 \geq 0.
$$

Therefore

$$
g(M^*HM) = -\varepsilon g(M^*M) + g(M^*H_\varepsilon M) \geq -\varepsilon g(M^*M).
$$

From here it follows (2.21). The lemma is proved.

It is known that for every Hermitian matrix $D \geq 0$ of dimension $n \times n$ the estimates

$$
\lambda_1(D) \leq g(D) \leq \lambda_n(D) \tag{2.22}
$$
are valid for every positive linear functional $g$ (see [18]). Then from the relation $g(B^{-1}(t)) \geq \lambda_1(B(t))$, $B(t) > 0$ it follows that

$$\nu_g(B(t)) \leq \lambda_1(B(t)) \leq trB(t), \quad t \geq t_0. \tag{2.23}$$

provided $B(t) \geq 0$, $t \geq t_0$. Hence, $\nu_g(B(t))$, $t \geq t_0$ is always locally integrable for $B(t) \geq 0$, $t \geq t_0$.

3. Oscillation criteria. Hereafter by the satisfiability of a relation $\mathcal{P}$ (equality, inequality) on any interval we will mean (if it is necessary) the satisfiability of $\mathcal{P}$ almost everywhere on that interval.

Consider the linear matrix equation

$$B(t)X = A(t), \quad t \geq t_0. \tag{3.1}$$

This equation has always a unique solution when $B(t) > 0$, $t \geq t_0$ ($X = X(t) \equiv B^{-1}(t)A(t)$, $t \geq t_0$). In the general case it has a solution iff (the Kronecker-Capelli theorem [12, p. 77])

$$\text{rank} B(t) = \text{rank}(B(t)A(t)), \quad t \geq t_0.$$ 

For any matrix function $P(t)$, $t \geq t_0$ of dimension $n \times n$ set

$$J_P(t) \equiv - \int_{t_0}^{t} [C(\tau) + A^*(\tau)P(\tau)]d\tau - P(t), \quad t \geq t_0.$$ 

Denote by $\mathcal{M}_\mathbb{R}$ the set of matrices of dimension $n \times n$ with real entries.

**Theorem 3.1.** Let $A(t)$, $B(t)$, $C(t) \in \mathcal{M}_\mathbb{R}$, $t \geq t_0$, Eq. (3.1) have a solution $F(t)$ such that $A^*(t)F(t)$ is locally integrable on $[t_0, +\infty)$, and let the following conditions be satisfied.

1) $B(t) \geq 0$, $t \geq t_0$.

2) $\int_{t_0}^{+\infty} \nu_g(B(t))dt = +\infty$.

3) $\lim_{t \to +\infty} g(J_F(t)) = +\infty$.

Then the system (1.1) is oscillatory.

**Proof.** Suppose the system (1.1) is not oscillatory. Then it has a prepared solution $(\Phi(t), \Psi(t))$ such that $\det \Phi(t) \neq 0$, $t \geq t_1$ for some $t_1 \geq t_0$. By (2.18) and (2.19) it follows from here that for the Hermitian matrix function $Y(t) \equiv \Psi(t)\Phi^{-1}(t)$, $t \geq t_1$ the equality

$$Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \geq t_1$$

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is fulfilled. Integrate this equality from \( t_1 \) to \( t \). We obtain

\[
Y(t) - Y(t_1) + \int_{t_1}^{t} [Y(\tau)B(\tau)Y(\tau) + A^*(\tau)Y(\tau) + Y(\tau)A(\tau) - C(\tau)]d\tau = 0, \quad t \geq t_1. \tag{3.2}
\]

Set \( Z(t) \equiv Y(t) + F(t), \quad t \geq t_1 \). Then since by the condition of the theorem \( A^*(t)F(t) \) is locally integrable from (3.2) we obtain

\[
Z(t) - Y(t_1) + \int_{t_1}^{t} [C(\tau) + A^*(\tau)F(\tau)]d\tau + \int_{t_1}^{t} Z^*(\tau)B(\tau)Z(\tau)d\tau + J_F(t) = 0, \tag{3.3}
\]

\( t \geq t_1 \) (since by (3.1) \( Z^*(\tau)B(\tau)Z(\tau) = Y(\tau)B(\tau)Y(\tau) + F^*(\tau)B(\tau)Y(\tau) + Y(\tau)B(\tau)F(\tau) + F^*(\tau)B(\tau)F(\tau) = Y(\tau)B(\tau)Y(\tau) + A^*(\tau)Y(\tau) + Y(\tau)A(\tau) + F^*(\tau)B(\tau)F(\tau), \quad \tau \geq t_1 \)). By Lemma 2.5 from the condition I) it follows

\[
g(Z^*(\tau)B(\tau)Z(\tau)) \geq \nu_y(B(\tau))[g(Z(\tau))]^2, \quad \tau \geq t_1.
\]

This together with (3.3) implies

\[
g(Z(t)) - g \left[ Y(t_1) - \int_{t_0}^{t_1} [C(\tau) + A^*(\tau)F(\tau)]d\tau \right] +
\]

\[
+ \int_{t_1}^{t} \nu_y(B(\tau))[g(Z(\tau))]^2d\tau + g(J_F(t)) \leq 0, \quad t \geq t_1. \tag{3.4}
\]

Without loss of generality on the basis of the condition III) we can take that \( t_1 \) is so large that

\[
-g \left[ Y(t_1) - \int_{t_0}^{t_1} [C(\tau) + A^*(\tau)F(\tau)]d\tau \right] + g(J_F(t)) \geq 2, \quad t \geq t_1.
\]

Then from (3.4) we obtain

\[
g(Z(t)) \leq -2, \quad t \geq t_1. \tag{3.5}
\]

Set \( f(t) \equiv -g(Z(t)) - \int_{t_1}^{t} \nu_y(B(\tau))[g(Z(\tau))]^2d\tau, \quad f_1(t) \equiv f(t) - 1, \quad t \geq t_1 \). It follows from (3.5) that

\[
f(t) > f_1(t) > 0, \quad t \geq t_1. \tag{3.6}
\]
Moreover \( f_1(t) \) is absolutely continuous on \([t_1, +\infty)\). Consider the integral Riccati equations

\[
y(t) + \int_{t_1}^{t} \nu g(B(\tau)) y^2(\tau) d\tau + f(t) = 0, \quad t \geq t_1, \tag{3.7}
\]

\[
y(t) + \int_{t_1}^{t} \nu g(B(\tau)) y^2(\tau) d\tau + f_1(t) = 0, \quad t \geq t_1. \tag{3.8}
\]

It follows from (3.4) that \( f_1(t) \geq -1 - g(Z(t)) - ilt_1 t \nu g(B(\tau)) [g(Z(\tau))]^2 d\tau \geq -1 - g \left[ Y(t_1) - \int_{t_0}^{t_1} [C(\tau) + A^*(\tau) F(\tau)] d\tau \right] + g(J_F(t)), \quad t \geq t_1. \) This together with the condition II) implies that

\[
\lim_{t \to +\infty} f_1(t) = +\infty. \tag{3.9}
\]

Obviously \( y(t) \equiv g(Z(t)), \quad t \geq t_1 \) is a solution of Eq. (3.7) on \([t_1, +\infty)\). Then in virtue of Lemma 2.2 it follows from (3.5) that Eq. (3.8) has a solution \( y_1(t) \) on \([t_1, +\infty)\). Note that \( y_1(t) \) is a solution of the Riccati equation

\[
y'(t) + \nu g(B(t)) y^2(t) = f_1'(t) = 0, \quad t \geq t_1.
\]

(recall that \( f_1(t) \) is absolutely continuous on \([t_1, +\infty)\)). Then by (2.7) the linear system

\[
\begin{cases}
\phi' = \nu g(B(t)) \psi, \\
\psi' = -f_1'(t) \phi, \quad t \geq t_1
\end{cases}
\]

is not oscillatory. On the other hand since according to (3.9) \( \int_{t_1}^{+\infty} f_1'(\tau) d\tau - \lim_{t \to +\infty} [f_1(t) - f_1(t_1)] = +\infty \) by Theorem 2.2 from the conditions I), II) it follows that the system (3.10) is oscillatory. We have obtained a contradiction, which completes the proof of the theorem.

Note that in the case when \( A(t), B(t) \) and \( C(t) \) are continuous and \( B(t) > 0, \quad t \geq t_0 \) the conditions of Theorem 3.1 become the conditions of Theorem 1.1. Therefore, Theorem 3.1 is a extension of Theorem 1.1. It should be noted here also that by analogy of this extension of Theorem 1.1 it can be extended Theorem 2.2 of work [11] (to do this it is needs to substitute \( Y = \alpha V, \quad \alpha > 0, \quad \alpha \in \mathbb{C}^1 \) in Eq. (2.18)).

**Example 3.1.** Assume \( A(t) \equiv 0, \quad B(t) = \sin^2 t B_0, \quad C(t) \in \mathcal{M}_\mathbb{R}, \quad t \geq t_0, \)

\[
\int_{t_0}^{+\infty} -g(C(t)) dt = +\infty, \quad \text{where } B_0 \in \mathcal{M}_\mathbb{R} \text{ is a positive definite Hermitian matrix.}
\]

Obviously...
with such $A(t)$, $B(t)$ and $C(t)$ Theorem 1.1 is not applicable to the system (1.1). For this case of $B(t)$ we have $\nu(B(t)) = \sin^2 t \frac{1}{g(B(t))}$, $t \geq t_0$. Moreover $F(t) \equiv 0$ is a locally integrable solution of Eq. (3.1). If we set

$$\nu_g(B(t)) = \frac{1}{g(B(t))} \int_{t_0}^{+\infty} \sin^2 t \, dt = +\infty$$

and

$$\int_{t_0}^{+\infty} (-g(C(t))) \, dt = +\infty$$

by Theorem 3.1 the system (1.1) for the considered case of its coefficients is oscillatory.

**Theorem 3.2.** Let $A(t)$, $B(t)$, $C(t) \in \mathcal{M}_R$, $B(t) \geq 0$, $t \geq t_0$, and let $F(t)$ be an absolutely continuous solution of Eq. (3.1). If for some positive linear functional $g$ the scalar system

$$\begin{align*}
\dot{\phi} &= \nu_g(B(t)) \psi, \\
\psi' &= -g[C(t) + A^*(t)F(t) + F'(t)] \phi, \quad t \geq t_0
\end{align*}$$

(3.11)

is oscillatory, then the system (1.1) is also oscillatory.

Proof. Suppose the system (1.1) is not oscillatory. Then there exists its a prepared solution $(\Phi(t), \Psi(t))$ such that $\det \Phi(t) \neq 0$, $t \geq t_1$, for some $t_1 \geq t_0$. By (2.18) and (2.19) it follows from here that for the Hermitian matrix function $Y(t) \equiv \Psi(t) \Phi^{-1}(t)$, $t \geq t_1$ the equality

$$Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \geq t_1$$

is fulfilled. If we set $Z(t) = Y(t) + F(t)$, $t \geq t_1$, then from the above equality we obtain

$$Z'(t) + Z^*(t)B(t)Z(t) - C(t) - A^*(t)F(t) - F'(t) = 0, \quad t \geq t_1.$$  (3.12)

Since $B(t) \geq 0$, $t \geq t_0$ by virtue of Lemma 2.5 we have

$$g[Z^*(t)B(t)Z(t)] \geq \nu_g(t)|g(Z(t))|^2, \quad t \geq t_1.$$  

This together with (3.12) implies

$$[g(Z(t))]' + \nu_g(B(t))[g(t)]^2 - g[C(t) + A^*(t)F(t) + F'(t)] \leq 0, \quad t \geq t_1.$$  (3.13)

Set $f(t) \equiv -g[Z(t)]' - \nu_g(B(t))[g(t)]^2$, $t \geq t_1$. It follows from (3.13) that

$$f(t) \geq -g[C(t) + A^*(t)F(t) + F'(t)], \quad t \geq t_1.$$  (3.14)

Consider the scalar Riccati equations

$$y' + \nu_g(B(t))y^2 - g[C(t) + A^*(t)F(t) + F'(t)] = 0, \quad t \geq t_1.$$  (3.15)
Obviously \( y(t) \equiv g(Z(t)), \ t \geq t_1 \) is a solution of Eq. (3.16) on \([t_1, +\infty)\). Then applying Theorem 2.1 to the pair of equations (3.15) and (3.16), and taking into account (3.14) we conclude that Eq. (3.15) has a solution on \([t_1, +\infty)\). By (2.7) it follows from here that the system (3.11) is not oscillatory, which contradicts the condition of the theorem. The obtained contradiction completes the proof of the theorem.

Note that if under the restriction, that \( F(t) \) is absolutely continuous on \([t_0, +\infty)\), the conditions I) - III) are satisfied, then by Theorem 2.2 the system (3.11) is oscillatory. Hence, Theorem 3.2 is a complement to Theorem 3.1 (therefore to Theorem 1.1).

Denote

\[
J(t) \equiv \text{tr} \left[ \frac{A(t) + A^*(t)}{2} B^{-1}(t) \right] - \int_{t_0}^{t} \text{tr} \left[ A(\tau) B^{-1}(\tau) A^*(\tau) + C(\tau) \right] d\tau + \\
+ \int_{t_0}^{t} \frac{\lambda_1(B(\tau))}{n} \left[ \text{tr} \left( \frac{A(\tau) - A^*(\tau)}{2i} \right) \right]^2 d\tau, \ t \geq t_0.
\]

**Theorem 3.3.** Let the functions \( \text{tr} \left[ A(t) B^{-1}(t) A^*(t) \right], \ \lambda_1(B(t)) \left[ \text{tr}(A(t) - A^*(t)) \right]^2 \), \( t \geq t_0 \) be locally integrable and let the following conditions be satisfied.

I') \( B(t) > 0, \ t \geq t_0 \).

IV) \( \int_{t_0}^{+\infty} \lambda_1(B(\tau)) d\tau = \lim_{t \to +\infty} J(t) = +\infty. \)

Then the system (1.1) is oscillatory.

Proof. Suppose the system (1.1) is not oscillatory. Then by (2.19) Eq. (2.18) has a Hermitian solution \( Y(t) \) on \([t_1, +\infty)\) for some \( t_1 \geq t_0 \). Then using I') we can write

\[
Y''(t) + \frac{1}{2} \left\{ [Y(t) + A(t) B^{-1}(t)] B(t) [Y(t) + B^{-1}(t) A(t)] + \\
+[Y(t) + A^*(t) B^{-1}(t)] B(t) [Y(t) + B^{-1}(t) A(t)] \right\} + \\
+ \frac{A^*(t) - A(t)}{2} Y(t) + Y(t) \frac{A(t) - A^*(t)}{2} - \\
- \frac{1}{2} [A(t) B^{-1}(t) A^*(t) + A^*(t) B^{-1}(t) A(t)] - C(t) = 0, \ t \geq t_1.
\] (3.17)
Since $Y(t)$ and $B^{-1}(t)$ are Hermitian by Lemma 2.4 we have

$$tr\frac{1}{2}\left\{ [Y(t) + A(t)B^{-1}(t)]B(t)[Y(t) + B^{-1}(t)A(t)] + \right.$$\n\n$$+ [Y(t) + A^*(t)B^{-1}(t)]B(t)[Y(t) + B^{-1}(t)A(t)] \right\} \geq \frac{\lambda_1(B(t))}{2n}\left\{ \left[ tr\left( Y(t) + \frac{A(t)B^{-1}(t) + B^{-1}(t)A^*(t)}{2} \right) \right]^2 + \left[ tr\left( \frac{A(t)B^{-1}(t) - B^{-1}(t)A^*(t)}{2i} \right) \right]^2 \right\},$$\n\n$t \geq t_1.$ By Lemma 2.3 from here we obtain

$$tr\frac{1}{2}\left\{ [Y(t) + A(t)B^{-1}(t)]B(t)[Y(t) + B^{-1}(t)A(t)] + \right.$$\n\n$$+ [Y(t) + A^*(t)B^{-1}(t)]B(t)[Y(t) + B^{-1}(t)A(t)] \right\} \geq \frac{\lambda_1(B(t))}{n}\left\{ \left[ tr\left( Y(t) + \frac{A(t) + A^*(t)}{2}B^{-1}(t) \right) \right]^2 + \left[ tr\left( \frac{A(t) - A^*(t)}{2i}B^{-1}(t) \right) \right]^2 \right\}, \quad t \geq t_1.$$

This together with (3.17) implies

$$trY'(t) + \frac{\lambda_1(B(t))}{n}\left\{ \left[ tr\left( Y(t) + \frac{A(t) + A^*(t)}{2}B^{-1}(t) \right) \right]^2 - \right.$$\n\n$$- \frac{1}{2} tr\left\{ [A(t)B^{-1}(t)A^*(t) + A^*(t)B^{-1}(t)A(t)] - C(t) \right\} + \left[ tr\left( \frac{A(t) - A^*(t)}{2i}B^{-1}(t) \right) \right]^2 \right\} \leq 0,$$
\n$t \geq t_1.$ If we substitute $Z(t) \equiv Y(t) + \frac{A(t) + A^*(t)}{2}B^{-1}(t), \quad t \geq t_1$ in the above inequality and integrate (by taking into account the condition of local integrability of the functions $tr[A(t)B^{-1}(t)A^*(t)], \quad tr\lambda_1(B(t))[tr(A(t) - A^*(t))]^2$) from $t_1$ to $t$ we obtain

$$trZ(t) + \int_{t_1}^{t} \frac{\lambda_1(B(\tau))}{n}[trZ(\tau)]^2 d\tau + J(t) + c \leq 0, \quad t \geq t_1,$$

where $c = Y(t_1) + \int_{t_0}^{t_1} [tr(C(\tau) + A(\tau)B^{-1}(\tau)A^*(\tau)]d\tau - \int_{t_0}^{t_1} \left[ tr\left( \frac{A(\tau) - A^*(\tau)}{2i}B^{-1}(\tau) \right) \right]^2 d\tau = const.$ Further as in the proof of Theorem 3.1. The theorem is proved.
Set
\[ \nu_0(B(t)) \equiv \begin{cases} 0, & \text{if } \det B(t) = 0, \\ \frac{1}{\text{tr}(B^{-1}(t))}, & \text{if } \det B(t) \neq 0, \end{cases} \quad t \geq t_0. \]

It is not difficult to verify that
\[ \frac{1}{\text{tr}(B^{-1}(t))} \leq \lambda_1(B(t)) \leq \frac{n}{\text{tr}(B^{-1}(t))}, \]
for all \( t \geq t_0 \), for which \( B(t) > 0 \). \hspace{1cm} (3.18)

Therefore Theorem 3.3 remains valid if we replace the condition \( \int_{t_0}^{+\infty} \lambda_1(B(t))dt = +\infty \) of
Theorem 3.3 by the following one \( \int_{t_0}^{+\infty} \nu_0(B(t))dt = +\infty \). Moreover by (2.22) in the case
\[ -\int_{t_0}^{t} \left( C(s) + A^*(s)B^{-1}(s)A(s) \right) ds - B^{-1}(t)A(t) \geq 0, \quad t \geq T, \]
for some \( T \geq t_0 \), the functional \( g \) in Theorem 1.1 is equivalent to the functional \( \text{tr} \). Hence due to (3.18) Theorem 3.3 is a
complement to Theorem 1.1.

**Theorem 3.4.** Let the following conditions be satisfied.
I' \( B(t) > 0, \quad t \geq t_0 \).
V) \( \int_{t_0}^{+\infty} \frac{dt}{\text{tr}(B^{-1}(t))} = +\infty \).
VI) the function \( \text{tr}[(A(t) + A^*(t))B^{-1}(t)(A(t) + A^*(t))] \), \( t \geq t_0 \) is locally integrable on \([t_0, +\infty)\) and \( \lim_{t \to +\infty} -\text{tr} \left[ 2(A(t) + A^*(t))B^{-1}(t) + \right. \]
\[ \left. + \int_{t_0}^{t} \left( A(\tau) + A^*(\tau) \right) B^{-1}(\tau)(A(\tau) + A^*(\tau)) + 4G(\tau) \right)d\tau \right] = +\infty. \]

Then the system (1.1) is oscillatory.

Proof. Suppose the system (1.1) is not oscillatory. Then by (2.19) Eq. (2.18) has a
solution \( Y(t) \) on \([t_1, +\infty)\) for some \( t_1 \geq t_0 \). Hence,
\[ Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \geq t_1. \]

From here it follows
\[ \text{tr} \left\{ Y'(t) + \left[ Y(t) + \frac{A(t) + A^*(t)}{2} B^{-1}(t) \right] B(t) \left[ Y(t) + B^{-1}(t) \frac{A(t) + A^*(t)}{2} \right] + \right. \]
\[ + \frac{A^*(t) - A(t)}{2} Y(t) + Y(t) \frac{A(t) - A^*(t)}{2} - A^*(t) + A(t) \frac{B^{-1}(t) A^*(t) + A(t)}{2} - C(t) \right\} = 0, \]
\( t \geq t_1 \). Substitute \( Z(t) \equiv Y(t) + \frac{A(t) + A^*(t)}{2} \), \( t \geq t_1 \) in the obtained equality and integrate from \( t_1 \) to \( t \). Taking into account the fact that the function \( tr[(A(t) + A^*(t))B^{-1}(t)(A(t) + A^*(t))] \) is locally integrable, we obtain

\[
tr \left\{ \int_{t_1}^{t} Z(\tau)B(\tau)Z^*(\tau) d\tau + \int_{t_1}^{t} \left[ \frac{A^*(\tau) - A(\tau)}{2} Y(\tau) + Y(\tau) \frac{A(\tau) - A^*(\tau)}{2} \right] d\tau + J_1(t) \right\} = 0, \quad t \geq t_1, \quad (3.19)
\]

where

\[
J_1(t) \equiv -Y(t_1) - \frac{A(t) + A^*(t)}{2} B^{-1}(t) - \int_{t_1}^{t} \left[ \frac{A(\tau) + A^*(\tau)}{2} B^{-1}(\tau) \frac{A(\tau) + A^*(\tau)}{2} + C(\tau) \right] d\tau, \quad t \geq t_1.
\]

By Lemma 2.3 we have

\[
tr \left[ \int_{t_1}^{t} \left[ \frac{A^*(\tau) - A(\tau)}{2} Y(\tau) + Y(\tau) \frac{A(\tau) - A^*(\tau)}{2} \right] d\tau \right] = 0, \quad t \geq t_1. \quad (3.20)
\]

Since \( Y(t) \), \( A(t) + A^*(t) \) and \( B^{-1}(t) \) are Hermitian we have also \( tr(Z(t) - Z^*(t)) = 0, \quad t \geq t_1 \). By Lemma 2.4 from here we obtain

\[
tr \int_{t_1}^{t} Z(\tau)B(\tau)Z^*(\tau) d\tau \geq \int_{t_1}^{t} \frac{\lambda_1(B(\tau))}{n} \left[ trZ(\tau) + Z^*(\tau) \right]^2 d\tau = \int_{t_1}^{t} \frac{\lambda_1(B(\tau))}{n} \left[ trZ(\tau) \right]^2 d\tau,
\]

\( t \geq t_1 \). This together with (3.19) and (3.20) implies that

\[
trZ(t) - trZ(t_1) + \int_{t_1}^{t} \frac{\lambda_1(B(\tau))}{n} \left[ trZ(\tau) \right]^2 d\tau + trJ_1(t) \leq 0, \quad t \geq t_1.
\]

Further as in the proof of Theorem 3.1 one can show that, if the conditions I'), VI) and the condition
This equation has always a solution when \( B \) is a real-valued matrix of dimension \( n \times n \) has a solution if and only if the equations
\[
\text{rank}(V') = 0
\]
have solutions (see [4], p. 23). Hence, Eq. (3.21) has a solution if and only if
\[
\lim_{t \to +\infty} g\left[ -\int_{t_0}^{t} \left( C(\tau) + A^*(\tau)B^{-1}(\tau)A(\tau) \right) d\tau - B^{-1}(t)A(t) \right] = \lim_{t \to +\infty} g[-A_0] \neq +\infty.
\]
Therefore, for this particular case Theorem 1.1 is not applicable to the system (1.1). It is not difficult to verify that
\[
\text{tr}C(t) = -\text{tr}(A_0^*A_0) < 0, \quad t \geq t_0.
\]
Then since \( A_0 + A_0^* = 0 \) using Theorem 3.4 to the system (1.1) we conclude that for this particular case the system (1.1) is oscillatory.

Example 3.2. Let \( B(t) \equiv I, \) \( A(t) \equiv A_0, \) \( C(t) \equiv -A_0^*A_0, \) \( t \geq t_0, \) \( A_0 = \text{const} \) is a real-valued matrix of dimension \( n \times n \). Then
\[
\lim_{t \to +\infty} g\left[ -\int_{t_0}^{t} \left( C(\tau) + A^*(\tau)B^{-1}(\tau)A(\tau) \right) d\tau - B^{-1}(t)A(t) \right] = \lim_{t \to +\infty} g[-A_0] \neq +\infty.
\]
This equation has always a solution when \( B(t) \geq 0, \) \( t \geq t_0. \) But it can have also a solution when \( B(t) \) is not invertible for all (for some) \( t \geq t_0 \) (see [5]). In the general case Eq. (2.21) has a solution if and only if the equations
\[
\sqrt{B(t)}X(A(t)\sqrt{B(t)} - \sqrt{B(t)}) = A(t)\sqrt{B(t)} - \sqrt{B(t)}, \quad t \geq t_0.
\]
This equation has always a solution when \( B(t) \geq 0, \) \( t \geq t_0. \) But it can have also a solution when \( B(t) \) is not invertible for all (for some) \( t \geq t_0 \) (see [5]). In the general case Eq. (2.21) has a solution if and only if
\[
\text{rank}(\sqrt{B(t)}) = \text{rank}(\sqrt{B(t)}(A(t)\sqrt{B(t)} - \sqrt{B(t)})), \quad t \geq t_0.
\]
Let \( F(t) \) be a solution of Eq. (3.21). We set:
\[
A_F(t) \equiv F(t)(A(t)\sqrt{B(t)} - \sqrt{B(t)}), \quad J_2(t) \equiv -\frac{1}{2}\text{tr}(A_F(t) + A_F^*(t)) -
\]
\[
-\int_{t_0}^{t} \text{tr}\left[A_F(\tau)A_F^*(\tau) + B(\tau)C(\tau)\right] d\tau + \frac{1}{n} \int_{t_0}^{t} \left[\text{tr}\left(\frac{A_F(\tau) - A_F^*(\tau)}{2t}\right)\right]^2 d\tau, \quad t \geq t_0.
\]
**Theorem 3.5.** Let \( \sqrt{B(t)} \) be absolutely continuous on \([t_0, +\infty)\) and let \( F(t) \) be a solution of Eq. (3.10) such that the functions \( tr(A_F(t)A^*_F(t) + B(t)C(t))\), \( [tr(A_F(t) - A^*_F(t))]^2 \), \( t \geq t_0 \) are locally integrable on \([t_0, +\infty)\). If

\[
\lim_{t \to +\infty} J_2(t) = +\infty
\]

then the system (1.1) is oscillatory.

Proof. Suppose the system (1.1) is not oscillatory. Then by (2.19) Eq. (2.18) has a solution \( Y(t) \) on \([t_1, +\infty)\) for some \( t_1 \geq t_0 \). Hence,

\[
Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \geq t_1.
\]

Multiply both sides of this equality at left and at right by \( \sqrt{B(t)} \). Taking into account the equality

\[
(\sqrt{B(t)}Y(t)\sqrt{B(t)})' = \sqrt{B(t)}Y(t)\sqrt{B(t)} + \sqrt{B(t)}Y'(t)\sqrt{B(t)} + \sqrt{B(t)}Y(t)\sqrt{B(t)},
\]

\( t \geq t_1 \) we obtain

\[
(\sqrt{B(t)}Y(t)\sqrt{B(t)})' + (\sqrt{B(t)}Y(t)\sqrt{B(t)})^2 + (\sqrt{B(t)}A^*(t) - \sqrt{B(t)})Y(t)\sqrt{B(t)} + \\
+ \sqrt{B(t)}Y(t)(A(t)\sqrt{B(t)} - \sqrt{B(t)}) - \sqrt{B(t)}C(t)\sqrt{B(t)}, \quad t \geq t_1. \tag{3.22}
\]

Since \( F(t) \) is a solution of Eq. (3.21) we have \( \sqrt{B(t)}A^*(t) - \sqrt{B(t)} = (\sqrt{B(t)}A^*(t) - \sqrt{B(t)})F^*(t)\sqrt{B(t)} = A^*_F(t)\sqrt{B(t)} = A^*_F(t)\sqrt{B(t)}, \quad A(t)\sqrt{B(t)} - \sqrt{B(t)} = A_F(t)\sqrt{B(t)}, \quad t \geq t_1. \) From here and from (3.22) it follows

\[
tr\{V(t)\}' + tr\{V^2(t) + A^*_F(t)V(t) + V(t)A_F(t) - \sqrt{B(t)}C(t)\sqrt{B(t)}\} = 0, \quad t \geq t_1, \tag{3.23}
\]

where \( V(t) \equiv \sqrt{B(t)}Y(t)\sqrt{B(t)}, \quad t \geq t_1. \) By Lemma 2.3

\[
tr[\sqrt{B(t)}C(t)\sqrt{B(t)] = tr[B(t)C(t)]}, \quad t \geq t_0.
\]

Then if we substitute \( V(t) \equiv Z(t) - \frac{A_F(t) + A^*_F(t)}{2}, \quad t \geq t_1 \) in (3.23) (except in the expression \( tr\{V(t)\}' \)) and take into account the condition of local integrability of \( tr[A_F(t)A^*_F(t) + B(t)C(t)] \) and \( [tr(A_F(t) - A^*_F(t))]^2 \) we can, as in the proof of Theorem 3.3, to derive the inequality

\[
[trZ(t)] + \int_{t_1}^{t} [trZ(\tau)]^2 + J_2(t) + c_1 \leq 0,
\]

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where \( c_1 \equiv -\text{tr} V(t_1) + \int_{t_0}^{t_1} \text{tr} \left[ A_F(t)A_F^*(t) + B(t)C(t) \right] - \frac{1}{4n} \int_{t_0}^{t_1} [\text{tr} (A_F(t) - A_F^*(t))]^2 \) is a constant.

Further as in the proof of Theorem 3.1. The theorem is proved.

Example 3.3. Assume \( B(t) \equiv \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \), \( A(t) \equiv \begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} \), \( C(t) \equiv I \), \( \text{rank} A_2(t) \not= 0 \), \( t \geq t_0 \), where \( I_m \) is an identity matrix of dimensions \( m \times m \) \((m < n)\), \( \theta_{11} \), \( \theta_{21} \), \( \theta_{21} \) and \( \theta_{22} \) are null matrices of dimensions \( n \times m \), \( k \times (n - k) \) and \( (m - k) \times (n - k) \) respectively. Obviously \( \text{rank} B(t) \not= \text{rank}(B(t)|A(t)) \), \( t \geq t_0 \).

Therefore Eq. (3.1) has no solution, which means that for this particular case Theorems 3.1 and 3.2 are not applicable to the system (1.1). Obviously for this case \( F(t) \equiv 0 \), \( t \geq t_0 \) is a solution for Eq. (3.21). Then \( A_F(t) \equiv 0 \), \( t \geq t_0 \), \( J_2(t) = (t - t_0)m \rightarrow +\infty \) for \( t \rightarrow +\infty \). By Theorem 3.5 it follows from here that in this particular case the system (1.1) is oscillatory.

Denote by \( \Omega_n \) the set of \( n \times n \) dimensional matrices \( M \) for which

\[
\text{Re} \lambda_1(M) = \cdots = \text{Re} \lambda_n(M) .
\]

Let \( \Lambda(t) \in \Omega_n \), \( t \geq t_0 \) be a complex-valued locally integrable matrix function on \( [t_0, +\infty) \).

Consider the linear matrix equation

\[
B(t)X + XB(t) = \Lambda(t) + \Lambda^*(t) + A(t) + A^*(t), \quad t \geq t_0 .
\]

(3.24)

Note that if \( X(t), \ t \geq t_0 \) is any solution of this equation, then \( H(t) \equiv \frac{X(t) + X^*(t)}{2} \), \( t \geq t_0 \) is its a Hermitian solution. Indicate some particular cases, when Eq. (3.24) has a solution.

I) \( B(t) > 0 \), \( t \geq t_0 \). In this case \( B(t) \) and \( -B(t) \) have no common eigenvalues. Then (see [4], pp. 203, 207) Eq. (3.24) has a unique (therefore Hermitian) solution, which can be given in the following closed form (see[2], p. 212, Theorem 6)

\[
H_{\Lambda}(t) \equiv \int_{0}^{+\infty} \exp \left\{ -\tau B(t) \right\} \left[ \Lambda(t) + \Lambda^*(t) + A(t) + A^*(t) \right] \exp \left\{ -\tau B(t) \right\} d\tau , \quad t \geq t_0 .
\]

Note that this integral converges and gives a hermitian solution for Eq. (3.24) not only for the case \( B(t) > 0 \), \( t \geq t_0 \), but also for a more general case, when \( B(t) \geq 0 \), \( t \geq t_0 \) and \( \Lambda(t) + \Lambda^*(t) + A(t) + A^*(t) = 0 \) for all \( t \geq t_0 \) for which \( \lambda_1(B(t)) = 0 \).

II) \( \text{rank} B(t) \geq n - 1 \), \( t \geq t_0 \). Show that there exists a real-valued locally integrable function \( \mu(t) \), \( t \geq t_0 \) such that if for some \( \Lambda(t) \in \Omega_n \), \( t \geq t_0 \) \( \Lambda(t) + \Lambda^*(t) = \mu(t)I \), \( t \geq t_0 \), then Eq. (3.24) has a solution. If \( \text{rank} B(t) = n \), then we have the considered case I).
Suppose \( \text{rank}B(t) = n - 1 \) (\( t \) is fixed). Let \( U(t) \) be a \( n \times n \) dimensional unitary matrix such that

\[
U(t)B(t)U^*(t) = \text{diag}\{b_1(t), \ldots, b_n(t)\} \overset{\text{def}}{=} B_0(t), \quad 0 = b_1(t) < b_2(t) \leq \cdots \leq b_n(t).
\]

Then Eq. (3.24) is equivalent to the following

\[
B_0(t)V + VB_0(t) = \mathcal{A}(t),
\]

where \( V \equiv U(t)XU^*(t), \quad \mathcal{A}(t) \equiv U(t)[\Lambda(t) + \Lambda^*(t) + A(t) + A^*(t)]U^*(t) \). If we write

\[
B_0(t) = \begin{pmatrix} 0 & \theta \\ \theta^T & B_1(t) \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_{12} \\ v_{21} & V_{22} \end{pmatrix}, \quad \mathcal{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & A_{22}(t) \end{pmatrix}, \quad t \geq t_0,
\]

where \( \theta \equiv (0, \ldots, 0) \) is a null vector of dimension \( n - 1 \), \( \theta^T \) is the transpose of \( \theta \), \( v_{12} \) and \( a_{12}(t) \) are matrices of dimension \( 1 \times n \), \( v_{21} \) and \( a_{21}(t) \) are matrices of dimension \( n \times 1 \), \( V_{22} \) and \( A_{22}(t) \) are matrices of dimension \( (n - 1) \times (n - 1) \), then Eq. (3.25) can be written equivalently in the form

\[
\begin{aligned}
&v_{12}B_1(t) = a_{12}(t), \\
&B_1(t)v_{21}(t) = a_{21}(t), \\
&B_1(t)V_{22} + V_{22}B_1(t) = A_{22}(t), \\
&2\Re \mu(t) + a_{11}(t) = 0.
\end{aligned}
\]

Since \( \text{rank}B(t) = n - 1 \) we have \( B_1(t) > 0 \). Moreover since \( A(t) \) is Hermitian \( A_{11}(t) \) is real valued. Then, obviously, for \( \mu(t) \equiv -\frac{1}{2}a_{11}(t), \quad t \geq t_0 \) the system (3.26) has a solution. Thus Eq. (3.24) has always a solution with this \( \mu(t) \) and the appropriate chosen \( \Lambda(t) \).

\( \text{III}^c \) \( B(t) \equiv \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & B_{22}(t) \end{pmatrix}, \quad \Lambda(t) + A(t) \equiv \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & A_{21}(t) \end{pmatrix} \) \( \overset{\text{def}}{=} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & A_{21}(t) \end{pmatrix}, \quad t \geq t_0 \), where \( \theta_{11}, \theta_{12} \) and \( \theta_{21} \) are null matrices of dimensions \( m \times m, (n - m) \times m \) and \( (n - m) \times (n - m) \) respectively, \( A_{21}(t) \) is a matrix function of dimension \( m \times (n - m) \), \( B_{22}(t) \) and \( A_{22}(t) \) are matrix functions of dimension \( (n - m) \times (n - m) \), \( 0 < m < n \). If we write \( X \equiv \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \), then the system (3.24) we can rewrite in the form of the following system

\[
\begin{aligned}
x_{11} &= 0, \\
x_{12}B_{22}(t)A_{21}^*(t), \\
B_{22}(t)x_{21} &= A_{21}(t), \\
B_{22}(t)x_{22} + x_{22}B_{22}(t) &= \Lambda(t) + \Lambda^*(t) + A(t) + A^*(t), \\
&\quad \text{t} \geq t_0.
\end{aligned}
\]
Then if \( B_{22}(t) > 0, \ t \geq t_0, \) the Hermitian solution \( H(t) \) of Eq. (3.24) can be given in the closed form \( H(t) = \begin{pmatrix} 0 & H_{12}(t) \\ H_{12}^*(t) & H_{22}(t) \end{pmatrix}, \ t \geq t_0, \) where \( H_{12}(t) = A_{21}(t)B_{22}^{-1}(t), \)

\[
H_{22}(t) \equiv \int_0^\infty \exp\left\{ -\tau B_{22}(t) \right\} \left[ \Lambda(t) + \Lambda^*(t) + A(t) + A^*(t) \right] \exp\left\{ -\tau B_{22}(t) \right\} d\tau, \ t \geq t_0.
\]

For any absolutely continuous matrix function \( F(t) \) of dimension \( n \times n \) on \( [t_0, +\infty) \) set

\[
\mathcal{D}_F(t) \equiv -F'(t) + F(t)B(t)F(t) - F(t)A(t) - A^*(t)F(t) - C(t), \quad t \geq t_0.
\]

**Theorem 3.6.** Let the following conditions be satisfied.

I) \( B(t) \geq 0, \ t \geq t_0 \)

VII) For some locally integrable \( \Lambda(t) \in \Omega_n, \ t \geq t_0 \) Eq. (3.24) has a solution \( F(t) \) such that \( \mathcal{D}_F(t) \) is locally integrable on \( [t_0, +\infty) \).

VIII) The scalar system

\[
\begin{cases}
\frac{1}{n}\text{tr}[\Lambda(t) + \Lambda^*(t)]\phi + \frac{\lambda_1(B(t))}{n}\psi, \\
\psi' = -[\text{tr}\mathcal{D}_F(t)]\phi, \ t \geq t_0
\end{cases}
\]

is oscillatory.

*Then the system (1.1) is oscillatory.*

Proof. Suppose the system (1.1) is not oscillatory. Then it has a conjoined solution \((\Phi(t), \Psi(t))\) such that \( \det \Phi(t) \neq 0, \ t \geq t_1 \) for some \( t_1 \geq t_0 \). By (2.19) form here it follows that the Hermitian matrix function \( Y(t) \equiv \Psi(t)\Phi^{-1}(t), \ t \geq t_1 \) is a solution for Eq. (2.18) on \([t_1, +\infty)\). Then it is not difficult to verify that for the matrix function \( Z(t) \equiv Y(t) + F(t), \ t \geq t_1 \) the equality

\[
Z'(t) + Z(t)B(t)Z(t) + (A^*(t) - F(t)B(t))Z(t) + Z(t)(A(t) - B(t)F(t)) + \mathcal{D}_F(t) = 0, \quad (3.27)
\]

\( t \geq t_1. \) Since \( \text{tr} \) is a positive linear functional we have

\[
\text{tr}\left[Z(t)B(t)Z(t)\right] \geq \frac{\lambda_1(B(t))}{n}[\text{tr}Z(t)]^2, \quad t \geq t_1.
\]

By virtue of Lemma 2.3 we have

\[
\text{tr}\left[(A^*(t) - F(t)B(t))Z(t) + Z(t)(A(t) - B(t)F(t))\right] =
\]

\[
\frac{1}{n}\text{tr}[\Lambda(t) + \Lambda^*(t)]\phi + \frac{\lambda_1(B(t))}{n}\psi, \quad t \geq t_0.
\]
\[ + \text{tr}(Z(t)(A(t) + A^{*}(t) - B(t)F(t) - F(t)B(t))), \quad t \geq t_1. \]

Then by (3.24) from here we obtain
\[ \text{tr}[(A^{*}(t) - F(t)B(t))Z(t) + Z(t)(A(t) - B(t)F(t))] = \text{tr}(Z(t)(\Lambda(t) + \Lambda^{*}(t))), \quad t \geq t_1. \quad (3.29) \]

Since \( \Lambda(t) \in \Omega_n, \quad t \geq t_0 \) it is not difficult to verify that
\[ \text{tr}(Z(t)(\Lambda(t) + \Lambda^{*}(t))) = \frac{1}{n} \text{tr}[\Lambda(t) + \Lambda^{*}(t)] \text{tr}Z(t), \quad t \geq t_1. \]

This together with (3.27)-(3.29) implies that
\[ [\text{tr}Z(t)]' + \frac{\lambda_1(B(t))}{n} [\text{tr}Z(t)]^2 + \frac{1}{n} \text{tr}[\Lambda(t) + \Lambda^{*}(t)] \text{tr}Z(t) + \text{tr}D_F(t) \leq 0, \quad t \geq t_1. \]

Further as in the proof of Theorem 3.2. The theorem is proved.

**Remark 3.1.** Using (2.23) on the basis of comparison Theorem 2.1 one can show that in Theorem 3.6 \( \lambda_1(B(t)) \) can be replaced by a more explicit function \( v_0(B(t)) \).

If for some locally integrable matrix function \( \Lambda(t) \in \Omega_n, \quad t \geq t_0 \) the matrix function \( \Lambda(t) + A(t), \quad t \geq t_0 \) is skew symmetric \( (\Lambda(t) + A(t) = -\Lambda^{*}(t) - A^{*}(t), \quad t \geq t_0) \), then \( F(t) \equiv 0, \quad t \geq t_0 \) is a Hermitian solution of Eq. (3.24) and, hence, \( D_F(t) = -C(t), \quad t \geq t_0 \). Therefore combining Theorem 3.6 with Theorem 2.2 and taking into account Remark 3.1 we obtain immediately

**Corollary 3.1.** Let the following conditions be satisfied
I) \( B(t) \geq 0, \quad t \geq t_0 \).
IX) There exists a locally integrable on \( [t_0, +\infty) \) matrix function \( \Lambda(t) \in \Omega_n, \quad t \geq t_0 \) such that \( \Lambda(t) + A(t), \quad t \geq t_0 \) is skew symmetric.

\[ X) \quad \int_{t_0}^{+\infty} v_0(B(t)) \exp \left\{ -\frac{1}{n} \int_{t_0}^{t} \text{tr}[\Lambda(\tau) + \Lambda^{*}(\tau)] d\tau \right\} dt = \]
\[ = -\int_{t_0}^{+\infty} \text{tr}[C(t)] \exp \left\{ \frac{1}{n} \int_{t_0}^{t} \text{tr}[\Lambda(\tau) + \Lambda^{*}(\tau)] d\tau \right\} dt = +\infty. \]

Then the system (1.1) is oscillatory.

**Remark 3.2** Corollary 3.1 is a generalization of Lighton’s oscillation criterion (see [15, Theorem 2.24]).
For any matrix $L \equiv (l_{jk})_{j,k=1}^n$ denote $Sum(L) \equiv \sum_{j,k=1}^n l_{jk}$. Define the Hermitian matrix 

$$H = H_L = (h_{jk})_{j,k=1}^n$$

by elements of matrix $L = (l_{jk})_{j,k=1}^n$ by formulae:

$$h_{kk} = -\sum_{j=1}^n \Re l_{jk}, \ k = 1, n;$$

$$h_{nk} = \overline{h_{kn}} = -i \sum_{j=1}^n \Im l_{jk} + \frac{i}{n} \sum_{m,s=1}^n \Im l_{ms}, \ k = 1, n - 1;$$

$$h_{jk} = 0, \ j \neq k, j \neq n, k \neq n, j, k = 1, n.$$ 

The matrix $H_L$ is called separator of $L$ and is denoted by $\text{Sep}(L)$.

Let $\alpha(t), \beta(t)$ and $\gamma(t)$ be real-valued locally integrable functions on $[t_0, +\infty)$ such that $\alpha(t) + \beta(t) \equiv 1$, $t \geq t_0$. Set $A_{\alpha,\beta,\gamma}(t) \equiv \alpha(t)A(t) + \beta(t)A^*(t) + \gamma(t)I$, $t \geq t_0$. Consider the linear matrix equation

$$B(t)X = \text{Sep}(A_{\alpha,\beta,\gamma}(t)), \ t \geq t_0. \quad (3.30)$$

We are interested in whether this equation has a Hermitian solution. Indicate some particular cases, when it has a Hermitian solution.

- **IV** $B(t) = \sigma(t)\text{Sep}(A_{\alpha,\beta,\gamma}(t)), \ \sigma(t) \neq 0, t \geq t_0$ Then $H(t) \equiv \frac{1}{\sigma(t)}I$.

- **V** $B(t) = \sigma(t)\sqrt{\text{Sep}(A_{\alpha,\beta,\gamma}(t))}$. Then $H(t) \equiv \frac{1}{\sigma(t)}\sqrt{\text{Sep}(A_{\alpha,\beta,\gamma}(t))}$.

- **VI**

$$\text{Sep}(A_{\alpha,\beta,\gamma}(t)) = \text{diag}\left\{\nu_1(t), \ldots, \nu_1(t), \ldots, \nu_p(t), \ldots, \nu_p(t), 0, \ldots, 0\right\}$$

$n_1 + \ldots + n_p + m = n$, $B(t) \equiv \text{Bloc}\{B_1(t), \ldots, B_p(t), \Theta_m\}$ is a bloc diagonal matrix, $B_k(t) > 0, t \geq t_0$ is a Hermitian matrix of dimension $n_k \times n_k, k = 1, p$, $\Theta_m$ is a null matrix of dimension $m \times m$. Then $H(t) = \text{Bloc}\{\nu_1(t)B_1^{-1}(t), \ldots, \nu_p(t)B_p^{-1}(t), \Theta_m\}, t \geq t_0$.

- **VII** $\text{Sep}(A_{\alpha,\beta,\gamma}(t)) \equiv 0, t \geq t_0$. Then $H(t) \equiv 0, t \geq t_0$ is a Hermitian solution of Eq. (3.30).

For any $n \times n$ dimensional absolutely continuous matrix function $X(t)$ set

$$K_X(t) \equiv X'(t) + X(t)B(t)X(t) + A^*(t)X(t) + X(t)A(t) - C(t), \ t \geq t_0.$$

**Theorem 3.7.** Let $H(t)$ be an absolutely continuous Hermitian solution of Eq. (3.30). If the scalar system

$$\begin{cases} 
\phi' = \gamma(t)\phi + \frac{\lambda_1(B(t))}{n}\phi, \\
\psi' = -K_H(t)\phi - \gamma(t)\psi, \ t \geq t_0
\end{cases}$$

is oscillatory, then the system (1.1) is also oscillatory.
Proof. In Eq. (2.18) substitute
\[ Y = Z + H(t), \quad t \geq t_0. \]  
(3.31)

We obtain
\[ Z' + ZB(t)Z + (A^*(t) + H(t)B(t))Z + Z(A(t) + B(t)H(t)) + K_H(t) = 0, \quad t \geq t_1. \]  
(3.32)

Suppose the system (1.1) is not oscillatory. Then by (2.19) and (3.31) Eq. (3.32) has a solution \( Z(t) \) on \([t_1, +\infty)\) for some \( t_1 \geq t_0 \). Hence,
\[ Z'(t) + ZB(t)Z(t) + (A^*(t) + H(t)B(t))Z(t) + Z(t)(A(t) + B(t)H(t)) + K_H(t) = 0, \quad t \geq t_1. \]  
(3.33)

It was shown in [5], that for any two \( n \times n \) dimensional matrices \( L \) and \( U \) the equality
\[ \text{Sum}([L + \text{Sep}L]U) = \frac{i \text{Im} \left( \text{Sum}(L) \right)}{n} \text{Sum}(U). \]
is valid. Then
\[ \text{Sum}\{(A^*(t) + \text{Sep}(A(t)))Z(t) + Z(t)(A(t) + \text{Sep}(A(t)))\} = 0. \]
\[ \text{Sum}\{(A^*(t) + \text{Sep}(A^*(t)))Z(t) + Z(t)(A(t) + \text{Sep}(A^*(t)))\} = 0. \]

This together with (3.33) and the inequality (see [5])
\[ \text{Sum}(Y(t)B(t)Y(t)) \geq \frac{\lambda(B(t))}{n} (\text{Sum}(Y(t)))^2, \quad t \geq t_1 \]
implies that
\[ \text{Sum}(Z(t))' + \frac{\lambda(B(t))}{n} (\text{Sum}(Z(t))^2 + 2\gamma(t)\text{Sum}(Z(t)) + K_H(t) \leq 0, \quad t \geq t_1. \]

Further as in the proof of Theorem 3.2. The theorem is proved.

**Remark 3.3.** Using (2.23) on the basis of comparison Theorem 2.1 one can show that in Theorem 3.7 \( \lambda_1(B(t)) \) can be replaced by a more explicit function \( \nu_0(B(t)) \).

**Corollary 2.2.** If
\[ \text{Sep}(A_{\alpha,\beta,\gamma}(t)) \equiv 0, \quad t \geq t_0 \]  

(3.34)

and

\[
\int_{t_0}^{+\infty} \exp\left\{-\int_{t_0}^{t} \gamma(\tau) d\tau\right\} \nu_0(B(t)) dt = \int_{t_0}^{+\infty} \exp\left\{\int_{t_0}^{t} \gamma(\tau) d\tau\right\} \left(-\sum C(t)\right) dt = +\infty,
\]

then the system (1.1) is oscillatory.

Proof. According to IV) if \( \text{Sep}(A_{\alpha,\beta,\gamma}(t)) \equiv 0 \) then \( H(t) \equiv 0 \) is a Hermitian solution of Eq. (3.30) on \([t_0, +\infty)\). In this case \( K_H(t) = -C(t) \). Then Corollary 2.2 follows from Theorems 2.2, 3.7 and Remark 3.3. The corollary is proved.

**Remark 3.2.** Corollary 2.2 is another generalization of Leighton's oscillation criterion.

**Example 3.4.** Let \( Q_j, \quad j = 1, 3 \) be measurable sets such that \( Q_j \cap Q_k = \emptyset, \quad j \neq \emptyset, \quad j, k = 1, 3 \) and \( Q_1 \cup Q_2 \cup Q_3 = [t_0, +\infty) \). Let \( a_{jk}(t), \quad a_{nk}(t), \quad a_{jn}(t), \quad j, k = 1, n - 1 \) be real valued locally integrable functions on \([t_0, +\infty)\). Set

\[
A_{Q_1}(t) \equiv \begin{pmatrix}
  a_{11}(t) & \ldots & a_{1n}(t) \\
  \ldots & \ldots & \ldots \\
  -\sum_{j=1}^n a_{j1}(t) & \ldots & -\sum_{j=1}^n a_{jn}(t)
\end{pmatrix},
\]

\[
A_{Q_2}(t) \equiv \begin{pmatrix}
  a_{11}(t) & \ldots & a_{1,n-1}(t) & -\sum_{j=1}^{n-1} a_{1j}(t) \\
  \ldots & \ldots & \ldots & \ldots \\
  a_{n,1}(t) & \ldots & a_{n,n-1}(t) & -\sum_{j=1}^{n-1} a_{nj}(t)
\end{pmatrix},
\]

\[
A_{Q_3}(t) \equiv \begin{pmatrix}
  a_{11}(t) & \ldots & a_{1,n-1}(t) & -\sum_{j=1}^{n-1} a_{1j}(t) \\
  \ldots & \ldots & \ldots & \ldots \\
  a_{n-1,1}(t) & \ldots & a_{n-1,n-1}(t) & -\sum_{j=1}^{n-1} a_{nj}(t) \\
  -\sum_{j=1}^{n-1} a_{j1}(t) & \ldots & -\sum_{j=1}^{n-1} a_{jn}(t) & \sum_{j,k=1}^{n-1} a_{jk}(t)
\end{pmatrix},
\]

\[
A(t) \equiv \begin{cases}
  A_{Q_j}(t), & t \in Q_j, \quad j = 1, 2 \\
  A_0(t) + A_{Q_3}(t), & t \in Q_3
\end{cases}
\]

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where $A_0(t)$ is a locally integrable skew symmetric matrix ($A_0^*(t) = -A_0(t)$). Then it is not difficult to verify that for $\alpha(t) \equiv \begin{cases} 
1, & t \in Q_1, \\ 0, & t \in Q_2, \\ 1/2, & t \in Q_3 \end{cases}$, the condition (3.34) is fulfilled.

**Example 3.5.** Let $B_0 > 0$ be a Hermitian matrix of dimension $n \times n$. Consider the linear matrix Hamiltonian system

$$
\begin{align*}
\Phi' &= \frac{t(t\sqrt{B_0})^{-1} - 1}{\sqrt{1+t^2}} \Phi + \frac{B_0}{1+t^2} \Psi, \\
\Psi' &= -t\Phi - \frac{t(t\sqrt{B_0})^{-1}}{\sqrt{1+t^2}} \Psi, \\
&\text{for } t \geq t_0.
\end{align*}
$$

(3.35) 

It is not difficult to verify that for this system $F(t) \equiv 0$ is a solution of Eq. (3.21) (since for this system $A(t) \sqrt{B(t) + (\sqrt{B(t)})'} \equiv 0$, $t \geq t_0$). Hence, $J_2(t) = \int_{t_0}^{t} \frac{\text{tr}B_0\text{d}t}{1+t^2} \to +\infty$ for $t \to +\infty$. By Theorem 3.5 from here it follows that the system (3.25) is oscillatory. Since $\int_{t_0}^{+\infty} \frac{\text{tr}B_0}{1+t^2} \text{d}t < +\infty$ by (2.23) Theorems 3.1 and 3.3 are not applicable to the system (3.35). By (3.18) the condition $V$ of Theorem 3.4 for the system (3.35) is not fulfilled. Therefore Theorem 3.4 is not applicable to the system (3.35) as well.

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