HYPERBOLIC AX-LINDEMANN THEOREM IN THE COCOMPACT CASE.

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Abstract. We prove an analogue of the classical Ax-Lindemann theorem in the context of compact Shimura varieties. Our work is motivated by J. Pila’s strategy for proving the André-Oort conjecture unconditionally.

1. Introduction.

Pila and Zannier [27] recently gave a new proof of the Manin-Mumford conjecture for abelian varieties. The ideas involved in this proof have been subsequently used by Pila [23] to prove the André-Oort conjecture unconditionally for products of modular curves. From this work emerged a very nice and promising strategy for proving the André-Oort conjecture for general Shimura varieties. One important step in this strategy is a hyperbolic analogue of a theorem of Ax which is a functional version of a classical result of Lindemann. The aim of this paper is to prove this statement for compact Shimura varieties.

Let us first recall the context of the André-Oort conjecture. For notations concerning Shimura varieties and their special subvarieties, we refer to [9] and references contained therein.

Let $(G, X)$ be a Shimura datum and $X^+$ a connected component of $X$. We let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$ and $\Gamma := G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ denotes the stabiliser in $G(\mathbb{Q})$ of $X^+$. Then $S := \Gamma \backslash X^+$ is a connected component of

$$\text{Sh}_K(G, X) := G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K.$$

Conjecture 1.1 (André-Oort). Let $Z$ be an irreducible subvariety of $\text{Sh}_K(G, X)$ containing a Zariski-dense set of special points. Then $Z$ is special.

This conjecture has recently been proved under the assumption of the Generalised Riemann Hypothesis for CM fields (see [33] and [13]).
Part of the strategy consists in establishing a geometric characterisation of special subvarieties of Shimura varieties. This criterion says roughly that subvarieties contained in their images by certain Hecke correspondences are special.

The Ax-Lindemann theorem is a functional transcendence result for the exponential map \( \exp : \mathbb{C} \to \mathbb{C}^* \). The following geometric version is due to Ax [2]. It is the special case of the theorem of Ax which correspond to the Lindemann (or Lindemann-Weierstrass) part of the Schanuel conjecture.

**Theorem 1.2.** Let \( n \) be an integer and \( V \) be an algebraic subvariety of \((\mathbb{C}^*)^n\). Let \( \pi = (\exp, \ldots, \exp) : \mathbb{C}^n \to (\mathbb{C}^*)^n \) be the uniformising map. A maximal complex algebraic subvariety \( W \subset \pi^{-1}(V) \) is a translate of a rational linear subspace.

The Hyperbolic Ax-Lindemann conjecture is an analogue, in the context of Shimura varieties, for the uniformising map \( \pi : X^+ \to S \). Via the Harish-Chandra embedding, \( X^+ \) has a canonical realisation as a bounded symmetric domain in \( \mathbb{C}^n \). Roughly speaking, an algebraic subvariety of \( X^+ \) is defined as the intersection of an algebraic subvariety of \( \mathbb{C}^n \) with \( X^+ \). We will give a precise definition of an irreducible algebraic subset of \( X^+ \) (see section 4.2). A maximal algebraic subvariety of an analytic subset \( Z \) of \( X^+ \) is then an irreducible algebraic subvariety contained in \( Z \) and maximal among irreducible algebraic varieties contained in \( Z \). The main result of this paper is the following theorem.

**Theorem 1.3.** We assume that \( S \) is compact. Let \( \pi : X^+ \to S \) be the uniformising map and let \( V \) be an algebraic subvariety of \( S \). Maximal algebraic subvarieties of \( \pi^{-1}V \) are precisely the components of the preimages of weakly special subvarieties contained in \( V \).

By slight abuse of language, we will refer to a component of a preimage of a weakly special subvariety of \( S \), as a weakly special subvariety of \( X^+ \). The Hyperbolic Ax-Lindemann theorem has the following corollary that we proved without assuming that \( S \) is compact in a recent work [34].

Note that most Shimura varieties are compact. For example, there is only one adjoint Shimura datum defining a non-compact Shimura variety of dimension one, namely \((\text{PGL}_2, \mathbb{H}^\pm)\) where \( \mathbb{H}^\pm \) is the union of upper and lower half planes. This Shimura datum defines modular curves. All other adjoint Shimura data defining Shimura varieties of dimension one are given by \((F^* \backslash B^*, \mathbb{H}^\pm)\) where \( B^* \) is the algebraic group attached to an indefinite quaternion algebra \( B \) over a totally
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The real field $F$ which is split at exactly one real place. There are infinitely many of those and they define compact Shimura curves. A Shimura datum $(G, X)$ defines a compact Shimura variety if and only if the adjoint group $G^{\text{ad}}$ of $G$ is $\mathbb{Q}$-anisotropic.

**Theorem 1.4.** An irreducible subvariety $Z$ of $S$ is weakly special if and only if some (equivalently any) analytic component of $\pi^{-1}Z$ is algebraic in the sense explained above.

The strategy for proving 1.3 is as follows. Let $Y$ be a maximal algebraic subvariety of $\pi^{-1}(Z)$.

First of all, the group $G$ can be assumed to be semisimple of adjoint type and the group $\Gamma$ sufficiently small, hence the variety $S$ is a product $S = S_1 \times \cdots \times S_r$ where the $S_i$s are associated to simple factors of $G$. Without loss of generality, we assume that $V$ and $Y$ are Hodge generic and furthermore their images by projections to the $S_i$s are positive dimensional. It can be seen that we are reduced to proving that $V = S$.

These reductions are done in section 4.2.

The key step is the following result in hyperbolic geometry. Let $F$ be a fundamental domain for the action of $\Gamma$ on $X^+ \subset \mathbb{C}^n$. With our assumption ($S$ is compact) the closure of $F$ is a compact subset of $X^+$. Let $S_F$ be a finite system of generators for $\Gamma$ and $l : \Gamma \to \mathbb{N}$ be the associated word metric (see section 2.1 for details). Let $C$ be an algebraic curve in $\mathbb{C}^n$ such that $C \cap F \neq \emptyset$. We prove the following theorem (see theorem 2.7):

**Theorem 1.5.** There exists a constant $c > 0$ such that for all integers $N$ large enough,

$$|\{\gamma \in \Gamma, C \cap \gamma F \neq \emptyset \text{ and } l(\gamma) \leq N\}| \geq e^{cN}.$$

This theorem is then combined with the Pila-Wilkie counting theorem [24] to obtain information about the stabiliser $\Theta_Y$ of $Y$ in $G(\mathbb{R})$. In what follows ‘definable’ refers to definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ (see section 3 for the relevant background on o-minimal theory). For the main results concerning compact Shimura varieties we could work with the o-minimal structure $\mathbb{R}_{\text{an}}$. The compactness assumption on the Shimura variety is only used in section 2, we therefore work with the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

We define a certain definable subset $\Sigma(Y)$ of $G(\mathbb{R})$ as

$$\Sigma(Y) := \{g \in G(\mathbb{R}), \dim(gY \cap \pi^{-1}(Z) \cap F) = \dim(Y)\}.$$

Using the elementary properties of heights, theorem 1.5, the maximality of $Y$ and the Pila-Wilkie counting theorem (applied to $\Sigma(Y)$) we
show that $\Theta_Y$ contains “many” elements of $\Gamma$. As a consequence we show the following result in section 5.

**Theorem 1.6.** The stabiliser $\Theta_Y$ contains a positive dimensional $\mathbb{Q}$–algebraic subgroup $H_Y$. Moreover $H_Y$ is not a unipotent group.

For the last part of the proof we construct some Hecke operators $T_\alpha$ with $\alpha \in H_Y(\mathbb{Q})$ such that $T_\alpha$ has dense orbits in $S$ and such that $\pi(Y) \subset T_\alpha(Z) \cap Z$. An induction argument finishes the proof.

The idea of using Hecke correspondences already appeared in the proof of the André-Oort conjecture under the assumption of the Generalised Riemann hypothesis mentioned above. The key point is a theorem concerning the monodromy action on $\pi^{-1}(Z)$ due to Deligne [6] and André [1] in the context of variation of polarized Hodge structures. Some simple properties of Hecke correspondences then allow us to conclude.

The assumption that $S$ is a compact Shimura variety is only used in the theorem 1.5 and in some arguments concerning the definability of the restriction of the uniformising map $\pi$ to the fundamental domain $\mathcal{F}$. We believe that the conclusion of the theorem 1.5 could hold even in the non compact case (possibly under certain additional assumptions on the curve $C$). With a suitable analogue of 1.5 in the general case, the hyperbolic Ax Lindemann conjecture can be proved for the moduli space of principally polarised abelian varieties using the main result of Peterzil and Starchenko [21].

Finally, very recently, Pila and Tsimermann (see [26]) announced the proof of the hyperbolic Ax-Lindemann theorem for $A_g$ using a strategy somewhat similar to that used in this article.

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2. **Algebraic curves in hermitian symmetric domains.**

The aim of this section is to prove theorem 1.6.

2.1. **Word metric, Bergman metric.** Let $X$ be a hermitian symmetric domain (we omit the superscript $+$ in this section for simplicity
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of notations) realised as a bounded symmetric domain in \( \mathbb{C}^N \) by the Harish-Chandra embedding (\([18], \text{ch.4}\)). Let \( \overline{X} \) be the closure of \( X \) in \( \mathbb{C}^N \) for the usual topology. Let \( G \) be the the group of holomorphic isometries of \( X \) and \( \Gamma \subset G \) be a cocompact lattice in \( G \).

Let \( x_0 \) be a fixed base point of \( X \) and let \( F \) be a fundamental domain for the action of \( \Gamma \) on \( X \) such that \( x_0 \in F \). We may assume that \( F \) is open, connected and that its closure \( \overline{F} \) is a compact subset of \( X \) (recall that \( \Gamma \) is cocompact). With our conventions \( \Gamma \overline{F} = X \) and \( F \cap \gamma F = \emptyset \) for \( \gamma \in \Gamma \) such that \( \gamma \neq 1 \).

Let

\[ S_F := \{ \gamma \in \Gamma - \{1\}, \gamma F \cap \overline{F} \neq \emptyset \}. \]

Then \( S_F \) is a finite set such that \( S_F = S_F^{-1} \). Moreover \( S_F \) generates \( \Gamma \).

Let \( l : \Gamma \to \mathbb{N} \) be the word metric on \( \Gamma \) relative to \( S_F \). By definition \( l(1) = 0 \) and for \( \gamma \neq 0 \), \( l(\gamma) \) is the minimal number of elements of \( S_F \) required to write \( \gamma \) as a product of elements of \( S_F \). We also define the word distance on \( \Gamma \) by

\[ l(\gamma_1, \gamma_2) = l(\gamma_2^{-1}\gamma_1) \]

for \( \gamma_1, \gamma_2 \) in \( \Gamma \).

We refer to \([18]\) chapter 4.1 for definitions and properties of the Bergman Kernel on a bounded symmetric domain. Let \( K(Z, W) \) be the Bergman kernel on \( X \) and

\[ \omega = \sqrt{-1} \partial \overline{\partial} \log K(Z, Z) \]

be the associated Kähler form. Let \( g \) be the associated hermitian Kähler metric on \( X \) and \( d_{X,h}(\cdot, \cdot) \) be the associated hyperbolic distance on \( X \). Finally let \( d_e(\cdot, \cdot) \) be the Euclidean distance on \( \mathbb{C}^N \) and \( d_{X,e} \) be the restriction of \( d_e \) to \( X \).

The following proposition is a classical result saying in Gromov’s terminology that \((\Gamma, l)\) and \((X, d_{X,h})\) are quasi-isometric. This is a consequence of proposition 8.19 of \([4]\).

**Proposition 2.1.** Choose any map

\[ r : X \to \Gamma \]

such that \( x \in r(x) \overline{F} \).

There exist \( \lambda \geq 1 \) and \( C \geq 0 \) such that for all \( x, y \) in \( X \)

\[ \frac{l(r(x), r(y))}{\lambda} - C \leq d_{X,h}(x, y) \leq \lambda l(r(x), r(y)) + C. \]

It should be noted that the assumption that \( \Gamma \) is cocompact in this proposition is essential. The conclusion of the proposition 2.1 does not hold when \( \Gamma \) is not cocompact: in this case, the distance \( d_{X,h}(x, y) \) is
unbounded for $x$ and $y$ varying within a fixed fundamental domain but
$l(r(x), r(y)) = 0$.

We will also use the following result on the Bergman kernel on $X$.

**Lemma 2.2.** There exists a system of holomorphic coordinates $Z = (z_1, \ldots, z_n)$ in $\mathbb{C}^N$ and a polynomial $Q(Z, \overline{Z})$ in the variables $$(z_1, \ldots, z_N, \overline{z}_1, \ldots, \overline{z}_N)$$ taking real positive values on $X$ and vanishing identically on the boundary $\partial X$ of $X$ such that
$$K(Z, Z) = \frac{1}{Q(Z, \overline{Z})}$$ for $Z$ in $X$.

**Proof.** The Bergman kernel on a product $X = X_1 \times X_2$ of bounded symmetric domains is the product of the Bergman kernels on the $X_i$. We may therefore assume that $X$ is irreducible. The result is then an application of the computation of $K(Z, Z)$. For the classical irreducible domains we refer to ([18] ch. 4.3). For the two exceptional irreducible domains we refer to [36]. We just give the following typical example.

**Example 2.3.** Using the notations of Mok ([18] ch. 4.2), let $p$ and $q$ be two positive integers. Let
$$X = D_{p,q}^I := \{ Z \in M_{p,q}(\mathbb{C}) \simeq \mathbb{C}^{pq}, I_q - t \overline{Z}Z > 0 \}.$$ Then
$$K(Z, Z) = \det(I_q - t \overline{Z}Z)^{-p-q}$$ and we take
$$Q(Z, Z) := \det(I_q - t \overline{Z}Z)^{p+q}.$$

**Lemma 2.4.** There exist positive constants $a_1$, $a_2$ and $\theta$ depending only of $(X, g)$ and the choice of $x_0$ such that for all $x \in X$
$$-a_1 \log d_{X,e}(x, \partial X) - \theta \leq d_{X,h}(x, x_0) \leq -a_2 \log d_{X,e}(x, \partial X) + \theta$$

**Proof.** Let $G = KP$ be the Cartan decomposition associated to $x_0$. Then $K$ is a maximal compact subgroup of $G$ and $Kx_0 = x_0$.

Let $\Delta \subset \mathbb{C}$ be the Poincaré unit disc endowed with the usual Poincaré metric $g_\Delta := \frac{d\overline{z}}{(1-|z|^2)^2}$. Let $r$ be the rank of the hermitian symmetric domain $X$. By the polydisc theorem ([18] ch.5, thm. 1) there exists a totally geodesic complex submanifold $D$ of $X$ such that $(D, g|_D)$ of $X$ is isometric to $(\Delta, g_\Delta)^r$, such that $X = K.D$ and such that $x_0 \in D$.

Let $d_{D,h}(\cdot, \cdot)$ be the hyperbolic distance in $D$. Then $d_{D,h}$ is just the restriction to $D$ of $d_{X,h}$.
Let \( x \in X \), there exists \( k \in K \) such that \( k.x \in D \). Then
\[
d_{X,h}(x_0, x) = d_{X,h}(k.x_0, k.x) = d_{X,h}(x_0, k.x) = d_{D,h}(x_0, k.x).
\]

**Lemma 2.5.** There exist positive constants \( \alpha \) and \( \beta \) such that for all \( y \in D \),
\[
\log(d_e(y, \partial X)) \leq \log(d_e(y, \partial D)) \leq \alpha \log(d_e(y, \partial X)) + \beta
\]

Assuming temporarily the validity of this result, we only need to prove lemma 2.4 in the case where \( X = \Delta^r \subset \mathbb{C}^r \) is a polydisc. As the result is independent of \( x_0 \) we may assume that \( x_0 = 0 \). In this case the boundary \( \partial \Delta^r \) of \( \Delta^r \) is
\[
\partial \Delta^r = \bigcup_{i=1}^{r} \overline{\Delta_i} \times C_i
\]
where \( C_i \) is the unit circle on the \( i \)-th factor and \( \overline{\Delta_i} \) is the product of the closed Poincaré discs of the remaining factors. Therefore for \( x = (x_1, \ldots, x_r) = (\rho_1 e^{i \theta_1}, \ldots, \rho_r e^{i \theta_r}) \in \Delta^r \),
\[
d_e(x, \partial \Delta^r) = \min_{1 \leq k \leq r} (1 - \rho_k).
\]

We recall that for \( x = \rho e^{i \theta} \in \Delta \) we have \( d_{\Delta,h}(0, x) = \log \frac{1+\rho}{1-\rho} \). Therefore
\[
-\log(d_e(x, \partial \Delta)) \leq d_{\Delta,h}(0, x) \leq -\log(d_e(x, \partial \Delta)) + \log 2
\]
The proof of lemma 2.4 is then deduced from the inequalities
\[
\max_k (d_{\Delta,h}(x_k, 0)) \leq d_{\Delta,h}(x_0, 0) \leq r \max_k (d_{\Delta,h}(x_k, 0)).
\]
It remains to prove lemma 2.5.
The first inequality follows from the fact that \( \partial D \subset \partial X \).
We may assume that \( X \) is irreducible. Then \( \partial X \) is a smooth real analytic hypersurface and there is a neighbourhood \( U \) of \( \partial X \) such that \( d_e(z, \partial X) \) is analytic for \( z \) in \( U \) (see [14] theorem 3 and the comments p. 120).
Let \( \theta : \Delta^r \to D \) be the isometry given by the polydisc theorem. The function \( d_e(\theta(z), \partial X)^2 \) is continuous on \( \Delta^r \), analytic in a neighbourhood of \( \partial \Delta^r \) and its zero set is \( \partial \Delta^r \). Let \( C \) be a compact subset of \( \Delta^r \) such that outside of \( C \), \( d_e(\theta(z), \partial X)^2 \) is analytic. We apply Løjasiewicz Vanishing theorem ([13], thm 6.3.4 p. 169) to the function \( d_e(\theta(z), \partial X)^2 \) outside of \( C \). This theorem implies that there exists an integer \( q > 0 \) and a real \( c_1 > 0 \) such that for all \( z \in \Delta^r \setminus C \), we have
\[
d_e(\theta(z), \partial X)^2 \geq c_1 d_e(z, \partial \Delta^r)^q = c_1 d_e(\theta(z), \partial D)^q
\]
On the compact \( C \), the functions \( d_e(\theta(z), \partial X)^2 \) and \( d_e(z, \partial \Delta^r)^q \) are continuous and strictly positive. Therefore there exists \( c_2 > 0 \) such that for \( z \in C \),
\[
d_e(\theta(z), \partial X)^2 \geq c_2 d_e(z, \partial \Delta^r)^q = c_2 d_e(\theta(z), \partial D)^q
\]
Lemma 2.5 follows.

2.1.1. Algebraic curves in bounded symmetric domains. We keep the notations of the previous section. In particular \( X \) is a hermitian symmetric domain in \( \mathbb{C}^N \) via the Harish-Chandra embedding. As in the last section we let \( \overline{X} \) be the closure of \( X \) in \( \mathbb{C}^N \). Let \( C \) be an affine integral algebraic curve in \( \mathbb{C}^N \) such that \( C \cap X \neq \emptyset \).

**Definition 2.6.** We define the following counting functions
\[
N_C(n) := |\{ \gamma \in \Gamma, \text{ such that } \dim(\gamma.F \cap C) = 1 \text{ and } l(\gamma) \leq n \}|
\]
and
\[
N'_C(n) := |\{ \gamma \in \Gamma, \text{ such that } \dim(\gamma.F \cap C) = 1 \text{ and } l(\gamma) = n \}|.
\]

The main result we have in view in this section is the following theorem.

**Theorem 2.7.** There exists a positive constant \( c \) such that for all \( n \) big enough
\[
N_C(n) \geq e^{cn}
\]
As \( C \) is algebraic, \( C \) is not contained in a compact subset of \( \mathbb{C}^N \). Therefore, there exists \( b \in C \cap \partial X \) such that for all neighbourhoods \( V_b \) of \( b \) in \( \mathbb{C}^N \), \( V_b \cap C \) is not contained in \( \overline{X} \). As \( \partial X \) is a real-analytic hypersurface of \( \mathbb{C}^N \cong \mathbb{R}^{2N} \), there exists a neighborhood \( V_b \) of \( b \) such that
\[
C \cap \partial X \cap V_b
\]
is a real analytic curve.

Let \( \alpha \) and \( \beta \) two real numbers such that \( 0 \leq \alpha < \beta \leq 2\pi \). Let \( \Delta_{\alpha,\beta} \) be the subset of the unit disc \( \Delta \) defined as
\[
\Delta_{\alpha,\beta} := \{ z = re^{i\theta}, 0 \leq r < 1, \alpha \leq \theta \leq \beta \}.
\]
Let \( \overline{\Delta} \) be the closure of \( \Delta \) in \( \mathbb{C} \) and \( C_{\alpha,\beta} \) the subset of \( \partial \Delta \)
\[
C_{\alpha,\beta} := \{ z = e^{i\theta}, \alpha \leq \theta \leq \beta \}.
\]

We may find \( \alpha \) and \( \beta \) with the previous properties and a real analytic map
\[
\psi : \Delta_{\alpha,\beta} \to C \cap X
\]
such that \( \psi \) extends to a real analytic map from a neighborhood of \( \Delta_{\alpha,\beta} \) to \( C \) such that \( \psi(C_{\alpha,\beta}) \subset C \cap \partial X \).
Lemma 2.8. Let $K(Z,W)$ be the Bergman kernel on $X$ and
\[ \omega = \sqrt{-1} \partial \overline{\partial} \log K(Z, Z) \]
be the associated Kähler form. Let $\omega_\Delta = \sqrt{-1} \frac{dz \wedge d\overline{z}}{1-|z|^2}$ be the Kähler form on $\Delta$ associated to the Poincaré metric.

(a) There exists a positive integer $s$ such that up to changing $\alpha$ and $\beta$
\[ \psi^* \omega = s\omega_\Delta + \eta \]
for a $(1,1)$-form $\eta$ smooth in a neighbourhood of $C_{\alpha,\beta}$.

(b) Let $d_e(\cdot, \cdot)$ be the Euclidean distance on $\Delta$. Changing $\alpha$ and $\beta$ if necessary, there exists $\lambda' > 0$ and $C' > 0$ such that for all $z \in \Delta_{\alpha,\beta}$
\[ |\log d_e(\psi(z), \partial X) - \lambda' \log d_{\Delta, e}(z, \partial \Delta)| \leq C'. \]

Let $Q(Z, \overline{Z})$ be the polynomial defined in lemma 2.2. Then
\[ \psi^* \omega = -\sqrt{-1} \partial_z \overline{\partial}_{\overline{z}} \log Q(\psi(z), \overline{\psi(z)}). \]
For $z \in \Delta_{\alpha,\beta}$, $Q(\psi(z), \overline{\psi(z)})$ is a real analytic function taking positive real values on $\Delta_{\alpha,\beta}$ and vanishing identically on
\[ C_{\alpha,\beta} = \{ z = re^{i\theta} \in \overline{\Delta}, 1 - \overline{zz} = 0, \alpha \leq \theta \leq \beta \}. \]
Therefore
\[ Q(\psi(z), \overline{\psi(z)}) = (1 - \overline{zz})^s Q_1(z) \]
for a positive integer $s$ and a real analytic function $Q_1$ taking positive real values on $\Delta_{\alpha,\beta}$ and non vanishing identically on $C_{\alpha,\beta}$. By changing $\alpha$ and $\beta$ we may assume that $Q_1$ doesn’t vanish in a neighbourhood of $C_{\alpha,\beta}$. We then take $\eta = -\sqrt{-1} \partial_z \overline{\partial}_{\overline{z}} \log Q_1(z)$ and use the fact that $\omega_\Delta = -\sqrt{-1} \partial_z \overline{\partial}_{\overline{z}} \log(1 - \overline{zz})$ to conclude the proof of the part (a) of lemma.

The proof of (b) is similar. As $\psi$ is real analytic, $d_e(\psi(z), \partial X)^2$ is a real analytic function positive on $\Delta_{\alpha,\beta}$ and vanishing uniformly on $C_{\alpha,\beta}$. Therefore
\[ d_e(\psi(z), \partial X)^2 = (1 - \overline{zz})^{2\lambda'} \psi_1(z) \]
for a real analytic function $\psi_1$ positive on $\Delta_{\alpha,\beta}$ which doesn’t vanish uniformly on $C_{\alpha,\beta}$. Changing $\alpha$ and $\beta$ if necessary we may assume that $\psi_1$ doesn’t vanish on $C_{\alpha,\beta}$. Therefore $\log(\psi_1(z))$ is a bounded function on $\Delta_{\alpha,\beta}$. This finishes the proof of part (b) of lemma.

As a consequence of part (a) of the lemma we obtain the following result.
Corollary 2.9. Let $\gamma \in \Gamma$ such that $\dim(\gamma.F \cap \psi(\Delta_{\alpha,\beta})) = 1$. Then
\[
(2) \quad \int_{\gamma.F \cap C} \omega = s \int_{\psi^{-1}(\gamma.F \cap C)} \omega_\Delta + \int_{\psi^{-1}(\gamma.F \cap C)} \eta.
\]

Lemma 2.10. There exists a constant $B$ such that for all $\gamma \in \Gamma$ such that $\dim(\gamma.F \cap C) = 1$,
\[
(3) \quad \int_{\gamma.F \cap C} \omega \leq B.
\]

Proof. Let $X_c$ be the compact dual of $X$. Then $X_c$ is a projective algebraic variety. Let $L$ be the dual of the canonical line bundle endowed with the $G(C)$-invariant metric $\| \|_{FS}$. Then $L$ is ample. Let $\omega_{FS}$ be the curvature form of $(L, \| \|_{FS})$.

By the Harish-Chandra embedding theorem ([18], ch4.2, thm. 1) there is a biholomorphism $\lambda$ from $\mathbb{C}^N$ onto a dense open subset of $X_c$. This $\lambda$ is in fact algebraic. To see this let’s recall its definition. We refer to [35] p 281 for the following facts. There is a commutative subalgebra $m^+$ of Lie($G_C$) consisting of nilpotent elements such that $\mathbb{C}^N = m^+$. Let $M^+ = \exp(m^+)$. As $m^+$ is a nilpotent Lie algebra, the group $M^+$ is algebraic (see for example [17], Cor 4.8, p 276) The map $\exp : m^+ \to M^+$ is therefore algebraic. Then $X_c = G_C/P$ for some parabolic subgroup of $G_C$ and as $\lambda(m) = \exp(m).P$, the morphism $\lambda$ is algebraic.

We recall that the degree of a subvariety $V$ of dimension $d$ of $X_c$ with respect to $L$ can be computed as $\deg_L(V) = \int_V \omega_{FS}^d$. For a subvariety $V$ of $\mathbb{C}^N$ we may define $\deg_L(V)$ as the degree of the Zariski closure in $X_c$ of $\lambda(V)$.

On the compact set $\overline{F}$, $\omega$ and $\lambda^* \omega_{FS}$ are two smooth positive $(1,1)$-forms. There exists therefore a constant $B_1$ such that
\[
\omega \leq B_1 \lambda^* \omega_{FS}
\]
on $\mathcal{F}$.

Therefore
\[
\int_{\gamma.F \cap C} \omega = \int_{\gamma^{-1}(\gamma.F \cap C)} \omega \leq B_1 \int_{\gamma^{-1}(\gamma.F \cap C)} \lambda^* \omega_{FS}.
\]

Moreover,
\[
\int_{\gamma^{-1}(\gamma.F \cap C)} \lambda^* \omega_{FS} \leq \int_{\gamma^{-1}\lambda(C)} \omega_{FS} = \deg_L(\gamma^{-1}.\lambda(C)) = \deg_L(\lambda(C)).
\]

This finishes the proof of the lemma. $\square$
Lemma 2.11. There exist positive constants $\lambda_1$, $\lambda_2$ and $D$ such that for all $z \in \Delta_{\alpha,\beta}$ such that $z \in \psi^{-1}(\gamma.F \cap C)$,

$$\lambda_1 l(\gamma) - D \leq -\log(1 - zz) \leq \lambda_2 l(\gamma) + D$$

Proof. This is a combination of proposition 2.1, lemma 2.4 and the part (b) of the lemma 2.8. □

We now prove theorem 2.7. Let $n$ be an integer and

$$I_n := \{ z \in \Delta_{\alpha,\beta}, e^{-(n+1)} \leq 1 - |z| \leq e^{-n} \}.$$

Then there exists a $\delta_1 > 0$ such that for all $n$ big enough,

$$\text{vol}_{\Delta,h}(I_n) = \int_{I_n} \sqrt{-1} \frac{dz \wedge d\overline{z}}{(1 - |z|^2)^2} \geq \delta_1 e^n.$$

By lemma 2.11 there exist positive constants $c_1 < c_2$ such that for all $n$ big enough and for all $z \in I_n$, $\psi(z) \in \gamma.F$ for some $\gamma$ such that $c_1 n \leq l(\gamma) \leq c_2 n$.

Using corollary 2.9 and lemma 2.10 we see that there exists a constant $B_1$ such that for all $z \in \Delta_{\alpha,\beta}$ with $\psi(z) \in \gamma.F$ for some $\gamma \in \Gamma$

$$\text{vol}_{\Delta,h}(\psi^{-1}(\gamma.F \cap C)) \leq B_1$$

Combining these two results we see that there exists a positive constant $\delta$ such that for all $n$ big enough

$$\sum_{c_1 n \leq k \leq c_2 n} N'_C(k) \geq \delta e^n.$$

This finishes the proof of theorem 2.7.

Question 2.12. A natural extension of theorem 2.7 would be that for all arithmetic lattices $\Gamma$ of $X$ (a priori not cocompact) and for any algebraic curve $C$ in $\mathbb{C}^n$ such that $C \cap X \neq \emptyset$,

$$N_C(n) \geq e^{cn}$$

for some positive constant $c$. We have not been able to prove this but we believe that this statement could be true.

3. O-MINIMALITY.

3.1. Preliminaries. In this section we briefly review the notions of o-minimal structures and their properties that we will use later on. For details we refer to [7] and [30] as well as references therein.
Definition 3.1. We recall that a subset of $\mathbb{R}^n$ is semi-algebraic if it is a finite boolean combinations of sets of the form
\[ \{(x_1, \ldots, x_n) \in \mathbb{R}^n, P(x_1, \ldots, x_n) \geq 0\} \]
for some polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$. A subset of $\mathbb{C}^n$ is semi-algebraic if it is a semi-algebraic subset of $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$ via the real and imaginary parts of the coordinates.

Definition 3.2. A structure over $\mathbb{R}$ is a collection $\mathcal{S}$ of subsets of $\mathbb{R}^n$ (with $n \in \mathbb{N}$) which contains all semi-algebraic subsets and which is closed under Cartesian products, Boolean operations and coordinate projections. A structure is called o-minimal (standing for ‘order-minimal’) if subsets of $\mathbb{R}$ belonging to $\mathcal{S}$ are finite unions of points and intervals. Given a structure $\mathcal{S}$, subsets of $\mathbb{R}^n$ belonging to $\mathcal{S}$ are called definable. If $A$ is a subset of $\mathbb{R}^n$ and $B$ a subset of $\mathbb{R}^m$, a function $f : A \rightarrow B$ is called definable if its graph in $A \times B$ is definable.

In this paper we consider the structure $\mathbb{R}_{\text{an,exp}}$. This structure contains the graph of $\exp : \mathbb{R} \rightarrow \mathbb{R}$ and ‘an’ indicates that this structure contains the graphs of all restricted analytic functions. A restricted analytic function is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(x) = 0$ for $x \notin [-1,1]^n$ and $f$ coincides on $[-1,1]^n$ with an analytic function $\tilde{f}$ defined on a neighborhood $U$ of $[-1,1]^n$. It was proved by Van Den Dries and Miller [8] that the structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal. Throughout the paper, by definable we mean definable in the structure $\mathbb{R}_{\text{an,exp}}$.

We will use the Pila-Wilkie counting theorem and its refinement by Pila. For a set $X \subset \mathbb{R}^n$ which is definable in some o-minimal structure, we define the algebraic part $X_{\text{alg}}$ of $X$ to be the union of all positive dimensional semi-algebraic subsets of $X$.

Pila ([25], 3.4) gives the following definition:

Definition 3.3. A semialgebraic block of dimension $w$ in $\mathbb{R}^n$ is a connected definable set $W \subset \mathbb{R}^n$ of dimension $w$, regular at every point, such that there exists a semialgebraic set $A \subset \mathbb{R}^n$ of dimension $w$, regular at every point with $W \subset A$.

For $x = (x_1, \ldots, x_n) \in \mathbb{Q}^n$, we define the height of $x$ as
\[ H(x) = \max(H(x_1), \ldots, H(x_n)) \]
where for $a, b \in \mathbb{Z}\{0\}$ coprime $H(\frac{a}{b}) = \max(|a|, |b|)$, and $H(0) = 1$.

For a subset $Z$ of $\mathbb{R}^n$ and a positive real number $t$, we define the set
\[ \Theta(Z, t) := \{x \in Z \cap \mathbb{Q}^n, H(x) \leq t\} \]
and the counting function
\[ N(Z, t) := |\Theta(Z, t)|. \]
The following theorem is the main result of Pila-Wilkie [24] and its refinement by Pila ([23], thm. 3.6).

**Theorem 3.4.** Let $X \subset \mathbb{R}^n$ be a definable set in some o-minimal structure. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$N(X \setminus X^{\text{alg}}, t) < C_\epsilon t^\epsilon$$

and the set $\Theta(X, t)$ is contained in the union of at most $C_\epsilon t^\epsilon$ semialgebraic blocks contained in $X$.

This theorem was extended by Pila to points with coordinates in arbitrary number fields. A version for definable families is also available. However we will not use these generalised versions of the theorem.

## 4. The setup.

In this section we set the relevant notations and prove some preliminary lemmas.

Let $(G, X)$ be a Shimura datum and $X^+$ a connected component of $X$. We let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$ and $\Gamma := G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ denotes the stabiliser in $G(\mathbb{Q})$ of $X^+$. We let $S := \Gamma \setminus X^+$. Then $S$ is a connected component of the Shimura variety $\text{Sh}_K(G, X) := G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K$. We will assume that $\Gamma$ is a cocompact lattice. In this situation $S$ is a projective variety. We recall that this is the case if and only if $G$ is $\mathbb{Q}$-anisotropic (see [5] thm. 8.4).

Via a faithful rational representation, we view $G_{\mathbb{Q}}$ as a closed subgroup of some $\text{GL}_n(\mathbb{Q})$. We assume that $K$ is neat (see [28] sec. 0.6) and contained in $\text{GL}_n(\hat{\mathbb{Z}})$. With these assumptions $\Gamma$ is contained in $\text{GL}_n(\mathbb{Z})$.

Using Deligne’s interpretation of symmetric spaces in terms of Hodge theory we obtain variations of polarised $\mathbb{Q}$-Hodge structures on $X^+$ and $S$. We refer to ([19] sec. 2) for the definitions of the Hodge locus on $X^+$ or $S$. An irreducible analytic subvariety $M$ of $X^+$ or $S$ is said to be Hodge generic if $M$ is not contained in the Hodge locus. If $M$ is not assumed to be irreducible we say that $M$ is Hodge generic if all the irreducible components of $M$ are Hodge generic.

### 4.1. Definability of the uniformization map in the cocompact case.

In this section we will for simplicity of notations write $X$ instead of $X^+$. 
A realisation $\mathcal{X}$ of $X$ is a real or complex analytic subset of a real or complex algebraic variety $\hat{\mathcal{X}}$ such that there is a transitive $G(\mathbb{R})^+$-action on $\mathcal{X}$ such that for $x_0 \in \mathcal{X}$ the map

$$\psi_{x_0} : G(\mathbb{R})^+ \to \mathcal{X}$$

$$g \mapsto g \cdot x_0$$

is semi-algebraic and such that $\mathcal{X}$ is a hermitian symmetric space associated to $G$. In particular $\mathcal{X}$ is endowed with a $G(\mathbb{R})^+$-invariant complex structure and the action of $G(\mathbb{R})^+$ on $\mathcal{X}$ is given by holomorphic maps ([12] ch. 8-4 prop. 4.2). We will say that the realisation $\mathcal{X}$ is real or complex if $\hat{\mathcal{X}}$ is real or complex.

The realisation of Borel $X_B$ of $X$ as a subset of the compact dual $X^\vee$ of $X$ or the realisation $D$ of $X$ as a bounded domain are complex realisations in the previous sense. For $X_B$ we recall ([18] 3.3 thm 1 p. 52) that there is an algebraic subgroup $P_C$ of $G_C$ such that $X^\vee = G(\mathbb{C})/P(\mathbb{C})$ and $X_B = G(\mathbb{R})^+ \cdot P(\mathbb{C})$. The action of $G(\mathbb{R})^+$ on $X_B$ is the restriction to $G(\mathbb{R})^+$ of the algebraic action of $G(\mathbb{C})$ on $X^\vee$ by left multiplication and is therefore semi-algebraic.

For details on the bounded realisation $D$ of $X$ we refer to [3] 1.4. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then we have the decompositions $\mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_C$ and $\mathfrak{p}_C = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are commutative nilpotent Lie algebras. Then $D$ is realised as a subset of $\mathfrak{p}^+$. Let $P^+$, $P^-$ and $K_C$ be the algebraic subgroups of $G(\mathbb{C})$ corresponding to $\mathfrak{p}^+$, $\mathfrak{p}^-$ and $\mathfrak{k}_C$ respectively. Then there is a Zariski open subset $\Omega$ of $G(\mathbb{C})$ containing $G(\mathbb{R})^+ \cdot P(\mathbb{C})$ such that $\Omega = P^+ K_C P^-$. We usually write $g = g^+ k g^-$ the associated decomposition of an element of $\Omega$. The map

$$\zeta : \Omega \to \mathfrak{p}^+$$

$$g = g^+ k g^- \mapsto \log(g^+)$$

is algebraic as $\mathfrak{p}^+$ is nilpotent. The action of $G(\mathbb{R})^+$ on $D \subset \mathfrak{p}^+$ is given by

$$g.p = \zeta(g \cdot \exp(p))$$

and is therefore semi-algebraic as again $\mathfrak{p}^+$ is nilpotent. This shows that $D$ is a realisation of $\mathcal{X}$ in the previous sense.

For the classical bounded domains an explicit description is given in [18] ch.4. As an example let us mention (using the the notations of loc. cit.)

$$D^f_{p,q} := \{ Z \in M_{p,q}(\mathbb{C}) \simeq \mathbb{C}^{pq} : I_q - i\mathbb{Z}Z > 0 \}$$

which can be seen to be semialgebraic by a direct use of the definition 3.1. The associated group is $\text{SU}(p,q)$ and the semi-algebraic action of
a matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) of \( \text{Su}(p,q) \) on \( D^I_{p,q} \) is given by
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}.Z = (AZ + B)(CZ + D)^{-1}
\]

The other cases are similar. For explicit descriptions of the two exceptional bounded symmetric domains we refer to [36].

The theory of Cayley transforms ([3] 1.6) shows that the unbounded realisations of \( X \) in \( p^+ \) including the classical Siegel domains of type I, II and III are complex realisations as defined above.

We recall that we fixed a faithful representation \( G \hookrightarrow \text{GL}_n \) over \( \mathbb{Q} \).

Let \( x_0 \in X \) and \( K_{x_0} \) be the stabiliser of \( x_0 \) in \( G(\mathbb{R}) \). This induces an identification of \( X \) with \( G(\mathbb{R})/K_{x_0} \). Let \( K_\infty \) be a maximal compact subgroup of \( \text{GL}_n(\mathbb{R}) \) containing \( K_{x_0} \). Notice that \( \text{GL}_n(\mathbb{R})/K_\infty \) can be identified with the set \( P_n \) of positive definite symmetric matrices and is therefore a semi-algebraic set. Let \( S_n \) be the algebraic set of symmetric matrices.

The inclusion \( X \hookrightarrow \text{GL}_n(\mathbb{R})/K_\infty \) is a real realisation of \( X \) as a subset \( X_{ps} \) (\( ps \) stands for 'positive symmetric') of \( \text{GL}_n(\mathbb{R})/K_\infty = P_n \subset S_n \) as the map
\[
\phi_{x_0} : \text{GL}_n(\mathbb{R}) \to P_n
\]
\[
g \mapsto g \cdot x_0 = g^t g
\]
is semi-algebraic and \( X_{ps} = \phi_{x_0}(G(\mathbb{R})) \).

Let \( X_1 \) and \( X_2 \) be two realisations of \( X \). Then an isomorphism of realisations is an analytic \( G(\mathbb{R})^+ \)-equivariant analytic diffeomorphism \( \psi : X_1 \to X_2 \). Two realisations of \( X \) are always isomorphic as they are isomorphic to \( X \).

**Proposition 4.1.** Let \( \mathcal{X} \) be a realisation of \( X \). Then \( \mathcal{X} \) is semi-algebraic. The map
\[
\phi : G \times X \to X
\]
given by \( \phi(g,x) = g \cdot x \) is semi-algebraic. Let \( \psi : X_1 \to X_2 \) be an isomorphism of realisations of \( X \). Then \( \psi \) is a semi-algebraic map.

**Proof.** Fix \( x_0 \in X \), then \( X \) is defined by the formula
\[
\mathcal{X} := \{ g \cdot x_0, g \in G(\mathbb{R})^+ \}
\]
for a semi-algebraic action of \( G(\mathbb{R})^+ \) so is semi-algebraic.

Let \( Gr(\phi) \subset G \times \mathcal{X} \times \mathcal{X} \) be the graph of \( \phi \). Let
\[
\phi_1 : G \times G \to G \times G \times G
\]
be the map $\phi_1(g, \alpha) = (g, \alpha, g\alpha)$. Then $\phi_1$ is semi-algebraic as $G$ is an algebraic group.

Let

$$\phi_2 : G \times G \times G \to G \times X \times X$$

be the map $\phi_2(h_1, h_2, h_3) = (h_1, h_2, x_0, h_3, x_0)$. The maps $\phi_2$ is semi-algebraic by our definition of a realisation. Therefore

$$Gr(\phi) = \phi_2 \phi_1(G \times G)$$

is semi-algebraic hence $\phi$ is semi-algebraic.

As $\psi$ is $G(\mathbb{R})^+$-equivariant the graph $Gr(\psi)$ of $\psi$ is

$$Gr(\psi) := \{(g, x_0, g, \psi(x_0)) \in X_1 \times X_2, g \in G(\mathbb{R})\}$$

which is semi-algebraic as the actions of $G(\mathbb{R})^+$ on $X_1$ and $X_2$ are semi-algebraic. □

Let $X^+$ and $\Gamma$ be as before, recall that $\Gamma$ is cocompact. There are many choices for the fundamental domain in $X^+$ for the action of $\Gamma$. We recall that a fundamental set $\Omega$ in $X^+$ for the action of $\Gamma$ is by definition a subset of $X^+$ such that $\Gamma \Omega = X^+$ and such that

$$\{\gamma \in \Gamma, \gamma \Omega \cap \Omega \neq \emptyset\}$$

is a finite set. A fundamental set for the action of $\Gamma$ on $G(\mathbb{R})$ is defined in the same way. A fundamental domain in $X^+$ is as before an open set $F$ such that $\Gamma F = X^+$ and such that for all $\gamma \in \Gamma$ with $\gamma \neq 1$ we have $F \cap \gamma F = \emptyset$.

For our purposes, we make the following choice.

**Proposition 4.2.** Let $\mathcal{X}$ be any realisation of $X^+$. There exists a semi-algebraic fundamental set $\Omega$ in $\mathcal{X}$ for the action of $\Gamma$ such that $\overline{\Omega}$ is compact. There is a connected fundamental domain $F$ contained in $\Omega$ which is definable in $\mathbb{R}_{an}$.

**Proof.** As two realisations of $X^+$ are always isomorphic the proposition is independent of the realisations of $\mathcal{X}$ by the proposition 4.1. We will use the realisation $X_{ps}$ of $X^+$.

By Theorem 4.8 of [29], we can choose $b_1, \ldots, b_r \in \text{GL}_n(\mathbb{Z})$ and a semi-algebraic subset $\Sigma$ of $\text{GL}_n(\mathbb{R})$ (as one can check using it’s definition p. 177 of [29]), such that $\Sigma K_\infty = \Sigma$ and such that

$$\Omega := \phi_{x_0}((\cup_{i=1}^r b_i \Sigma) \cap G(\mathbb{R}))$$

is a fundamental set for $\Gamma$ acting on $X_{ps}$. Note that the choice of $a$ in the theorem 4.8 of [29] correspond to our choice of $x_0$. As $\Gamma$ is cocompact, $\overline{\Omega}$ is compact.
Let us now define
\[
\mathcal{F} = \{ x \in X_{ps}, \ d_{X_{ps},h}(x, x_0) < d_{X_{ps},h}(\gamma x, x_0), \forall \gamma \in \Gamma; \gamma \neq 1 \}.
\]

One can show that \( \mathcal{F} \) is connected in the following way. For all \( \gamma \in \Gamma \) with \( \gamma \neq 1 \) we define
\[
\mathcal{F}_\gamma = \{ x \in X_{ps}, \ d_{X_{ps},h}(x, x_0) < d_{X_{ps},h}(\gamma x, x_0) \}.
\]
Then as \( x_0 \in \mathcal{F}_\gamma \) and \( \mathcal{F} = \bigcap_{\gamma \in \Gamma - \{1\}} \mathcal{F}_\gamma \) we just need to show that \( \mathcal{F}_\gamma \) is connected. A simple geometric argument shows that if \( x \in \mathcal{F}_\gamma \) then the geodesic arc joining \( x_0 \) to \( x \) is contained in \( \mathcal{F}_\gamma \).

We may enlarge \( \Sigma \) and choose the \( b_i \) such that \( F \subset \Omega \). We assume therefore that \( \mathcal{F} \subset \Omega \). The set \( \mathcal{F} \) is a fundamental domain for the action of \( \Gamma \) on \( X_{ps} \). We claim that \( \mathcal{F} \) is definable in \( \mathbb{R}_{an} \). As \( \Gamma \) is cocompact and \( \Omega \) is a semi-algebraic set containing \( \mathcal{F} \), we only need to check the inequalities for a finite number of elements \( \gamma \). The function \( x \mapsto d_{X_{ps},h}(x, x_0) \) is a real analytic function, it follows that the set \( \mathcal{F} \) is definable in \( \mathbb{R}_{an} \). \( \square \)

We’ll need the following statement which is probably well-known to the experts:

**Proposition 4.3.** Let \( \mathcal{X} \) be any realisation of the hermitian symmetric space \( G(\mathbb{R})/K_{x_0} \). Let
\[
\pi_1 : \mathcal{X} \longrightarrow S := \Gamma \backslash \mathcal{X}
\]
be the uniformising map. Then there exists a definable fundamental relatively compact domain \( \mathcal{F}_1 \) of \( \mathcal{X} \) (in \( \mathbb{R}_{an} \)) such that the restriction of \( \pi_1 \) to \( \mathcal{F}_1 \) is definable in \( \mathbb{R}_{an} \) (hence in particular in \( \mathbb{R}_{an,exp} \)).

**Proof.** Using proposition 4.1 we see that the result is independent of the choice of the realisation \( \mathcal{X} \).

We prove the result for the bounded realisation \( \mathcal{D} \). We let \( \pi : \mathcal{D} \rightarrow S = \Gamma \backslash \mathcal{D} \) be the uniformising map. In our situation \( S \) is a smooth projective variety. By the main result of [6], we have an algebraic embedding
\[
\Psi = (\psi_0, \psi_1, \ldots, \psi_n) : S \longrightarrow \mathbb{P}^n \subseteq \mathbb{C}^n.
\]
As such \( \Psi \) is definable in \( \mathbb{R}_{an} \).

For all \( 0 \leq i \leq n \) the map \( \psi_i \pi \) is holomorphic therefore its restriction to the compact semi-algebraic set \( \Omega \) (as in 4.2) is definable in \( \mathbb{R}_{an} \). As \( \mathcal{F} \) is a definable subset of \( \Omega \), \( \psi_i \pi \) is definable when restricted to \( \mathcal{F} \). As a consequence the restriction of \( \Psi \pi \) to \( \mathcal{F} \) is definable in \( \mathbb{R}_{an} \) and we may conclude that the restriction of \( \pi \) to \( \mathcal{F} \) is definable in \( \mathbb{R}_{an} \). \( \square \)

**Remark 4.4.** The generalisation of proposition 4.3 (definability in \( \mathbb{R}_{an,exp} \) instead of \( \mathbb{R}_{an} \)) for the moduli space \( \mathcal{A}_g \) of principally polarised...
abelian varieties of dimension $g$ is given by Peterzil and Starchenko [21]. We believe that the natural extension of their results to Shimura varieties of abelian type could be deduced from the result for $A_g$. The exceptional Shimura varieties would require new ideas.

4.2. Algebraic subsets of bounded symmetric domains. We refer to section 2 of [34] for relevant definitions and properties of weakly special subvarieties. Let $X \subset \widetilde{X}$ be a complex realisation of $X^+$. Let $\hat{Y}$ be an algebraic subvariety of $\widetilde{X}$. Proposition [4.1] implies that $\hat{Y} \cap X$ is semi-algebraic (as a variety over the reals) and complex analytic. Using [10] (section 2), we see that $\hat{Y} \cap X$ has only finitely many irreducible analytic components and that these components are semi-algebraic.

An irreducible algebraic subvariety of $X$ is then defined as an irreducible analytic component of the intersection of $X$ with a closed algebraic subvariety of $\widetilde{X}$. An algebraic subvariety of $X$ is defined to be a finite union of irreducible algebraic subvarieties of $X$.

Let $Z$ be a closed analytic subset of $X$. We define the complex algebraic locus $Z^{ca}$ of $Z$ as the union of the positive dimensional algebraic subvarieties of $X$ which are contained in $Z$. We define the semi-algebraic locus $Z^{sa}$ of $Z$ as the union of the connected positive dimensional semi-algebraic sets which are contained in $Z$. Then by the previous discussion $Z^{ca} \subset Z^{sa}$. Lemma 4.1 of [25] implies in fact that $Z^{ca} = Z^{sa}$.

Let $V$ be an irreducible algebraic subvariety of $S$, let $\hat{V}$ be the preimage of $V$ in $\widetilde{X}$. An irreducible algebraic subvariety $Y$ of $\hat{V}$ is said to be maximal if $Y$ is maximal among irreducible algebraic subvarieties of $\hat{V}$.

Let $Y$ be a maximal irreducible algebraic subvarieties of $\hat{V}$. We let $\mathcal{F}$ be a fundamental set for the action of $\Gamma$ on $X$ as in section [4.1] such that $\dim(Y \cap \mathcal{F}) = \dim(Y)$. In our situation $\mathcal{F}$ is an open subset of $X$ and by our assumption on $\Gamma$, the closure $\overline{\mathcal{F}}$ is a compact subset of $X$.

We can now state the general hyperbolic Ax-Lindemann conjecture. Using that isomorphisms of realisations are given by semi-algebraic maps (see [4.1]) and equality [4] we see that this statement is independent of the complex realisation of $X^+$.

**Conjecture 4.5.** Let $S$ be a Shimura variety and $X$ a complex realisation of $X^+$. Let $\pi : X \to S$ be the uniformisation map and let $V$ be an algebraic subvariety of $S$. Maximal algebraic subvarieties of $\pi^{-1}V$ are weakly special subvarieties.
Remark 4.6. The Hyperbolic Ax-Lindemann conjecture for a not necessary compact Shimura variety would essentially be a consequence of a positive answer to the question 2.12 and a generalisation of the definability of the uniformisation map to non compact situations.

As noted before, a proof of the Ax-Lindemann conjecture for $A_g$ has been announced by Pila and Tsimerman [26]. They provide a positive answer to a variant of the question 2.12.

Some applications of the hyperbolic Ax-Lindemann conjecture to the set of positive dimensional special subvarieties of subvarieties of Shimura varieties and an unconditional proof of the André-Oort conjecture for subvarieties of projective Shimura varieties in $A^n_6$ are given in the recent preprint by the first named author in [32].

For the purposes of proving Theorem 1.3, we can make the following simplifying assumptions.

Lemma 4.7. Without loss of generality we can assume that $G$ is semisimple of adjoint type and that $Y$ (and hence $V$) is Hodge generic.

Proof. Let us first check that we can assume that $V$ and $Y$ are Hodge generic. To assume that $V$ is Hodge generic, it suffices to replace $S$ by the smallest Shimura variety containing $V$. This amounts to replacing $X^+$ by a certain symmetric subspace and hence does not alter the property of $Y$ being algebraic. Suppose $\pi(Y)$ is contained in a smaller special subvariety $S' \subset S$. Then one can replace $S$ by $S'$ and $V$ by an irreducible component of $V \cap S'$ and hence assume that $Y$ is Hodge generic. Notice that the conclusion of 1.3 is independent of the choice of the compact open subgroup $K$. Indeed, replacing $K$ by a smaller subgroup does not change $Y$.

Let $G^{ad}$ be the quotient of $G$ by its centre. Let $\pi^{ad}: G \rightarrow G^{ad}$ be the natural morphism. Recall that $X^+$ is a connected component of the $G(\mathbb{R})$-conjugacy class of a morphism $x: S \rightarrow G_{\mathbb{R}}$. Let $X^{ad}$ be the $G^{ad}(\mathbb{R})$-conjugacy class of $\pi^{ad} \circ x$. Then $(G^{ad}, X^{ad})$ is a Shimura datum and $\pi^{ad}$ induces an isomorphism between $X^+$ and $X^{+ad}$, a connected component of $X^{ad}$. A choice of a compact open subgroup $K^{ad}$ containing $\pi^{ad}(K)$ induces a finite morphism of Shimura varieties $\text{Sh}_K(G, X) \rightarrow \text{Sh}_{K^{ad}}(G^{ad}, X^{ad})$ and a subvariety is special if and only if its image is special. It follows that we can replace $V, \tilde{V}, Y$ by their images and, respectively $G, X, K$ by $G^{ad}, X^{ad}, K^{ad}$ and hence assume that $G$ is semisimple of adjoint type.

Since the group $G$ is now assumed to be adjoint, we can write

$$G = G_1 \times \cdots \times G_r$$
where the $G_i$s are $\mathbb{Q}$-simple factors of $G$. The group $K$ is assumed to be neat and to be a product $K = \prod_p K_p$ where $K_p$s are compact open subgroups of $G(\mathbb{Q}_p)$. Furthermore, each $K_p$ is assumed to be the product

$$K_p = K_{p,1} \times \cdots \times K_{p,r}$$

with $K_{p,i}$s compact open subgroups of $G_i(\mathbb{Q}_p)$. The group $K$ is a direct product $K = K_1 \times \cdots \times K_r$ where $K_i = \prod_p K_{i,p}$. The group $\Gamma$ is a direct product

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$$

where $\Gamma_i = K_i \cap G_i(\mathbb{Q})^+$. This induces a decomposition of $S$ as a direct product

$$S = S_1 \times \cdots \times S_r$$

where $S_i = \Gamma_i\backslash X_i^+$.

From now on we assume that $V$ and $Y$ are Hodge generic. We now make the next simplifying assumption.

**Lemma 4.8.** Without loss of generality, we may assume that $V$ is not of the form $S_1 \times \cdots \times S_t \times V_1$ where $t \geq 1$ and $V_1$ is a subvariety of $S_{t+1} \times \cdots \times S_r$.

Without loss of generality, we may assume that $Y$ is not of the form $\{x\} \times Y_1$ where $x$ is a point of a product $X_1^+ \times \cdots \times X_t^+$ for $t \geq 1$.

**Proof.** Suppose that $V$ is of this form. Then $\tilde{V}$ is of the form $X_1^+ \times \cdots \times X_t^+ \times \tilde{V}_1$. As $Y$ is a maximal algebraic subset of $\tilde{V}$, $Y$ is of the form $Y = X_1^+ \times \cdots \times X_t^+ \times Y_1$ where $Y_1$ is a maximal algebraic subset of $\tilde{V}_1$. Hence we are reduced to proving that $Y_1$ is weakly special. Notice also that for similar reasons, we can assume that $V$ is not of the form $\{x\} \times V_1$ where $x$ is a point of a product of a certain number of $S_i$s.

As for the second claim, if $Y$ is of this form, then proving that $Y$ is weakly special is equivalent to proving that $Y_1$ is weakly special. \(\Box\)

**Remark 4.9.** Note that with these assumptions we need in fact to prove that $Y = X^+$. As $Y$ should be both prespecial and Hodge generic, a simple argument shows that $Y$ should be of the form

$$Y = X_1^+ \times \cdots \times X_t^+ \times \{x_{t+1}\} \times \cdots \times \{x_r\}$$

for some Hodge generic points $x_i$ of $X_i^+$ for $t+1 \leq i \leq r$. By the assumption of the previous lemma we see that we should have $Y = X^+$.

We’ll prove in fact that $\tilde{V} = X^+$, but this implies that $\tilde{V}$ is algebraic and as $Y$ is maximal algebraic in $\tilde{V}$ we can conclude that $Y = \tilde{V} = X^+$. 

We now recall some properties of the monodromy attached to a Hodge generic subvariety of $S$. Let $Z$ be an irreducible Hodge generic algebraic subvariety of $S$ all of whose projections to $S_i$’s are positive dimensional.

Fix a Hodge generic point $s$ of $Z$ and let $x$ be a point of $\pi^{-1}(s)$.

Fix a faithful rational representation $\rho: G \hookrightarrow \text{GL}_n$. The choice of a $\Gamma$-invariant lattice induces a variation of polarisable Hodge structures on $S$. We then obtain the monodromy representation

$$\rho_s^{\text{mon}}: \pi_1(Z^{\text{sm}}, s) \rightarrow \text{GL}_n(Z)$$

where $Z^{\text{sm}}$ is the smooth locus of $Z$. Let $\Gamma'$ be the image of $\rho_s^{\text{mon}}$. Let $H^{\text{mon}}$ be the neutral connected component of the Zariski closure of $\rho_s^{\text{mon}}$ in $\text{GL}_n\mathbb{Q}$. By a theorem of Deligne-André (see for example [19], theorem 1.4), the group $H^{\text{mon}}$ is a normal subgroup of $G$. In our case ($G$ of adjoint type), $H^{\text{mon}}$ is a product of simple factors of $G$ and, after reordering, we can write

$$G = H^{\text{mon}} \times G_1$$

Proposition 3.7 of [19] then shows that $V = S^1 \times V'$ where $S^1$ is the product of the $S_i$’s corresponding to $H^{\text{mon}}$ and $V'$ is a subvariety of the remaining factors. According to 4.8, we have

$$H^{\text{mon}} = G$$

Via $\rho$, we view $\Gamma'$ is a subgroup of $\Gamma$. Let $C(Z)$ be the maximum of the constants $n$ from theorems 5.1 and 6.1 of [9]. We call such a constant $C(Z)$ the Nori constant of $Z$. The next two propositions recall the main properties of the constant $C(Z)$.

**Proposition 4.10.** Let $g \in G(\mathbb{Q})^+$ and $p > C(Z)$ a prime such that for any prime $l \neq p$, the image $g_l$ of $g$ in $G(\mathbb{Q}_l)$ is contained in $K_l$, then $T_gZ$ is irreducible.

**Proof.** This is a direct application of the theorem 5.1 of [9]. $\square$

**Proposition 4.11.** Let $g$ be an element of $G(\mathbb{Q})^+$ such that the following holds. The images of $g_p$ in $G_i(\mathbb{Q}_p)$ for $i = 1, \ldots, t$ are not contained in compact open subgroups while the images of $g_p$ in $G_{t+1}(\mathbb{Q}_p), \ldots, G_r(\mathbb{Q}_p)$ are contained $K_{p,1}$.

Suppose that $T_gZ = Z$ and $T_{g^{-1}}Z = Z$. Then $Z = S_1 \times \cdots \times S_i \times Z'$ where $Z'$ is a subvariety of $S_{t+1} \times \cdots \times S_r$.

**Proof.** For every $x \in Z$, the $T_g + T_{g^{-1}}$ orbit of $x$ is contained in $Z$. Let $(x_1, \ldots, x_r)$ be a point of $Z$ (where for all $i$, $x_i$ is a point of $S_i$). By theorem 6.1 of [9], the closure of the $T_g + T_{g^{-1}}$ orbit of $x$ is

$$S_1 \times \cdots \times S_t \times \{x_{t+1}\} \times \cdots \times \{x_r\}.$$
Hence \( Z = S_1 \times \cdots \times S_t \times Z' \) where \( Z' \) is the image of \( Z \) in \( S_{t+1} \times \cdots \times S_r \).

\[ \square \]

5. Stabilisers of maximal algebraic subsets.

We keep the notations of the previous section. View \( G(\mathbb{R}) \) as an algebraic (hence definable) subset of some \( \mathbb{R}^m \) and \( X \) as a semi-algebraic (in particular definable) subset of some \( \mathbb{R}^k \).

**Lemma 5.1.** Let \( Y \) be a maximal irreducible algebraic subset of \( \tilde{V} \). Define

\[
\Sigma(Y) = \{ g \in G(\mathbb{R}) : \dim(gY \cap \tilde{V} \cap \mathcal{F}) = \dim(Y) \}
\]

(a) The set \( \Sigma(Y) \) is definable and for all \( g \in \Sigma(Y) \), \( gY \subset \tilde{V} \).

(b) For all \( \gamma \in \Sigma(Y) \cap \Gamma \), \( \gamma.Y \) is a maximal algebraic subvariety of \( \tilde{V} \).

**Proof.** The set \( \mathcal{F} \cap \tilde{V} \) is definable by proposition 4.3. For each \( g \in G(\mathbb{R}) \), the set \( gY \cap \tilde{V} \cap \mathcal{F} \) is a definable subset of \( \mathbb{R}^n \). The definability of \( \Sigma(Y) \) is now a consequence of the second part of proposition 4.1 and proposition 1.5 of [7].

Let \( g \in \Sigma(Y) \), by definition of \( \Sigma(Y) \)

\[
gY \cap \mathcal{F} \subset \tilde{V}.
\]

Both \( gY \) and \( \tilde{V} \) are analytic varieties, therefore the above inclusion implies that \( gY \subset \tilde{V} \). This finishes the proof of the first part of lemma.

Let \( \gamma \in \Sigma(Y) \cap \Gamma \). Then \( \gamma.Y \subset \tilde{V} \) by the previous result. Let \( Y' \subset \tilde{V} \) be an algebraic set containing \( \gamma.Y \). Then \( \gamma^{-1}.Y' \subset \tilde{V} \). As \( Y \subset \gamma^{-1}.Y' \), \( \gamma^{-1}Y' \subset \tilde{V} \) and by maximality of \( Y \), \( Y' = \gamma.Y \).

\[ \square \]

**Lemma 5.2.** Let \( Y \) be a maximal algebraic subset of \( \tilde{V} \). Define

\[
\Sigma'(Y) = \{ g \in G(\mathbb{R}) : g^{-1}\mathcal{F} \cap Y \neq \emptyset \}
\]

Then

\[
\Sigma(Y) \cap \Gamma = \Sigma'(Y) \cap \Gamma
\]

**Proof.** The inclusion \( \Sigma(Y) \subset \Sigma'(Y) \) is obvious.

Let \( \gamma \in \Sigma'(Y) \cap \Gamma \). Then \( \gamma^{-1}\mathcal{F} \cap Y \neq \emptyset \). The set \( \gamma^{-1}\mathcal{F} \) is an open subset of \( \mathbb{R}^n \), hence \( \dim(\gamma^{-1}\mathcal{F} \cap Y) = \dim(Y) \). Translating by \( \gamma \), we get \( \dim(\gamma Y \cap \mathcal{F}) = \dim(Y) \). Therefore

\[
\dim(\gamma Y \cap \tilde{V} \cap \mathcal{F}) = \dim(Y)
\]
i.e. \( \gamma \in \Sigma(Y) \cap \Gamma \). Note that that we have used in an essential way that \( \tilde{V} \) is \( \Gamma \)-invariant and hence \( \gamma Y \subset \tilde{V} \) for any \( \gamma \in \Gamma \). In particular \( \gamma Y \cap \mathcal{F} = \gamma Y \cap \tilde{V} \cap \mathcal{F} \) for any \( \gamma \in \Gamma \). \( \square \)

We recall that we defined in section 2.1 a finite set \( S_F \) generating \( \Gamma \) and the associated word metric \( l : \Gamma \to \mathbb{N} \).

**Proposition 5.3.** Let \( N_{Y,\Gamma}(N) := |\{ \gamma \in \Gamma \cap \Sigma(Y), l(\gamma) \leq N \} | \). There exist a constant \( c > 0 \) such that for all \( N \) big enough

\[
R_{Y,\Gamma}(N) \geq e^{cN}.
\]

**Proof.** As \( Y \) is semi-algebraic and maximal in \( \tilde{V} \), \( Y \) contains a component of the intersection of an algebraic curve \( C \) with \( X \). For \( \gamma \in \Gamma \) the condition \( \gamma \mathcal{F} \cap C \neq \emptyset \) implies that \( \gamma \mathcal{F} \cap Y \neq \emptyset \). The result is then a consequence of the previous lemma and the part b of theorem 2.7. \( \square \)

Let \( \Theta_Y \) be the stabilizer of \( Y \) in \( G(\mathbb{R}) \). The main result of this part is the following statement.

**Theorem 5.4.** Let \( H_Y \) be the neutral component of the Zariski closure of \( \Gamma \cap \Theta_Y \). Then \( H_Y \) is a non-trivial reductive \( \mathbb{Q} \)-algebraic group.

The proof will use the following two lemmas.

**Lemma 5.5.** Let \( W \) be a semi-algebraic block of \( \Sigma(Y) \) containing some \( \gamma \in \Sigma(Y) \cap \Gamma \). Then

\[
W \subset \gamma \Theta_Y.
\]

**Proof.** By the part (b) of lemma 5.1 we know that \( \gamma \cdot Y \) is a maximal algebraic subvariety of \( \tilde{V} \). Let \( U_\gamma \) be an open connected semi-algebraic subset of \( W \) containing \( \gamma \cdot Y \). Then \( U_\gamma \cdot Y \) is semi-algebraic containing \( \gamma \cdot Y \). By maximality of \( \gamma \cdot Y \) and the equality \( \tilde{V} \) applied to \( U_\gamma \cdot Y \), we have

\[
U_\gamma \cdot Y = \gamma \cdot Y
\]

Let \( w \in W \). We claim that there exists a connected semi-algebraic open subset \( U \) of \( W \) which contains \( w \) and \( \gamma \). Indeed, \( W \) is path connected, we choose a compact path between \( w \) and \( \gamma \). We can cover this path with a finite number of open semi-algebraic subsets of \( W \). Their union is the desired subset \( U \). Then, by the previous argument,

\[
U \cdot Y = \gamma \cdot Y = w \cdot Y
\]

Therefore \( \gamma \cdot Y = W \cdot Y \) and \( W \subset \gamma \Theta_Y \). \( \square \)
We define the height $H(\alpha)$ of an element $\alpha$ of $\text{GL}_n(Z)$ as the maximum of the absolute values of the coefficients of $\alpha$. The triangle inequality shows that if $\alpha, \beta \in \text{GL}_n(Z)$, then
\begin{equation}
H(\alpha \beta) \leq nH(\alpha)H(\beta)
\end{equation}
and
\begin{equation}
H(\alpha^{-1}) \leq c_n H(\alpha)^{-1}
\end{equation}
for a positive constant $c_n$ depending only on $n$.

We recall that we have fixed an embedding of $\Gamma$ in some $\text{GL}_n(Z)$. Let $A$ be the maximum of the heights of elements of $S_F$. Then an element $\gamma \in \Gamma$ with $l(\gamma) \leq N$ satisfies $H(\gamma) \leq (nA)^N$.

We recall that for any positive real $T$ we defined
\[ \Theta(\Sigma_Y, T) := \{ g \in G(Q) \cap \Sigma(Y), \ H(g) \leq T \} \]
and $N(\Sigma(Y), T) = |\Theta(\Sigma_Y, T)|$.

**Lemma 5.6.** There exists a constant $c_1 > 0$ such that for all $T$ big enough
\[ \{ \gamma \in \Gamma \cap \Sigma(Y), H(\gamma) \leq T \} \geq T^{c_1}. \]
As a consequence
\begin{equation}
N(\Sigma(Y), T) \geq T^{c_1}.
\end{equation}

**Proof.** Let $N$ be an integer. The previous discussion shows that the set
\[ \{ \gamma \in \Gamma \cap \Sigma(Y), l(\gamma) \leq N \} \]
is contained in $\Theta(\Sigma_Y, (AN)^N)$. The result is therefore an application of proposition 5.3. \qed

We can now give the proof of theorem 5.4. Let $c_1$ be the constant from lemma 5.6. Using the theorem of Pila and Wilkie 3.4 with $\epsilon = \frac{1}{2n}$, we know that for all $T$ big enough, $\Theta(\Sigma_Y, T^{\frac{1}{2n}})$ is contained in at most $T^{\frac{c_1}{2}}$ semi-algebraic blocks. Using lemma 5.6 we see that there exists a semi-algebraic block $W$ of $\Sigma(Y)$ containing at least $T^{\frac{c_1}{2n}}$ elements $\gamma \in \Sigma(Y) \cap \Gamma$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$. Using lemma 5.5 we see that there exists $\sigma \in \Sigma(Y)$ such that $\sigma \Theta_Y$ contains at least $T^{\frac{c_1}{2n}}$ elements $\gamma \in \Sigma(Y) \cap \Gamma$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Let $\gamma_1$ and $\gamma_2$ be two elements of $\sigma \Theta_Y \cap \Gamma$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$. Then using the equations (6) and (7) we see that $\gamma := \gamma_2^{-1}\gamma_1$ is an element of $\Gamma \cap \Theta_Y$ such that $H(\gamma) \leq nc_n T^{1/2}$. Therefore for all $T$ big enough $\Theta_Y$ contains at least $T^{\frac{c_1}{2n}}$ elements $\gamma \in \Gamma$ such that $H(\gamma) \leq T$. As $Y$ is algebraic, the stabilizer $\Theta_Y$ is an algebraic group and the group $H_Y$ generated by the Zariski closure of $\Gamma \cap \Theta_Y$ is a $Q$-algebraic subgroup of $G$ of positive dimension contained in $\Theta_Y$. As $G_Q$ is $Q$-anisotropic,
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$G(\mathbb{Q})$ doesn’t contain unipotent elements and $H_Y$ is therefore a reductive $\mathbb{Q}$-group.

6. END OF THE PROOF OF THE THEOREM \[1.3\] USING HECKE CORRESPONDENCES.

In this section we prove theorem \[1.3\]

Let us recall the situation. We have a Hodge generic subvariety $V$ of $S$ and $Y$ a maximal algebraic subvariety of $V = \pi^{-1}V$. We assume that $Y$ is Hodge generic.

The group $G$ is semisimple of adjoint type:

$$G = G_1 \times \cdots \times G_r$$

and, accordingly,

$$X^+ = X_1^+ \times \cdots \times X_r^+.$$ 

Let $p_i : G \to G_i$ be the projection on $G_i$. Suppose that $V \neq S$ (otherwise there is nothing to prove). By lemma 4.8 we assume that $V$ is not of the form $S_1 \times \cdots \times S_k \times V'$ where $V'$ is a subvariety of $S_{k+1} \times \cdots \times S_r$. Recall that $H_Y$ is a reductive group. After, if necessary, reordering the factors, we let $G_1, \ldots, G_t$ be the factors to which $H_Y$ projects non-trivially.

**Lemma 6.1.** Let $C(V)$ be the Nori constant of $V$. There exists $p > C(V)$ and $g \in H_Y(\mathbb{Q})$ such that for any $l \neq p$, $g_l \in K_l$ and for $i = 1, \ldots, t$, the element $p_i(g_p)$ is not contained in compact subgroups.

**Proof.** Let $T$ be a maximal $\mathbb{Q}$-torus in $H_Y$. Pick a prime $p > C(V)$ such that $T_{\mathbb{Q}_p}$ is a split torus. For $i = 1, \ldots, t$, the torus $T_i := p_i(T)$ is a maximal in $G_i$. Moreover, each $T_{\mathbb{Q}_p}$ is split. Let $X_*(T)$ be the cocharacter group of $T$. As $T_{\mathbb{Q}_p}$ is split, we have $T(\mathbb{Q}_p) = X_*(T) \otimes \mathbb{Q}_p$ and the valuation map $v_p : \mathbb{Q}^* \to \mathbb{Z}$ induces an isomorphism

$$\psi : T(\mathbb{Q}_p)/K^m_{T,p} \cong X_*(T)$$

where $K^m_{T,p}$ is the maximal compact open subgroup of $T(\mathbb{Q}_p)$. Let $X_*(p_i)$ be the map $X_*(T) \to X_*(T_i)$ induced by $p_i$. The kernels of $X_*(p_i)$ give $t$ proper subgroups of $X_*(T)$. We choose an element $\alpha_p \in T(\mathbb{Q}_p)$ such that the $\psi(\alpha_p)K^m_{T,p} \in X_*(T)$ avoids these subgroups. Let $\alpha$ be the element of $T(\mathbb{A}_f)$ such that $\alpha_p = \alpha_p$ and for all $l \neq p \alpha_l = 1$. For all $l \neq p$, $\alpha_l \in K_l$ and for $i = 1, \ldots, t$ the images of $\alpha_p$ in $T_i(\mathbb{Q}_p)$ are not contained in compact subgroups.

Let $K^m_T$ be the maximal compact open subgroup of $T(\mathbb{A}_f)$. As the class group $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K^m_T$ is finite there exist an integer $s$, such that
\( \alpha^g = gk \) where \( g \in T(\mathbb{Q}) \) and \( k \in K^m_T \). For \( l \neq p \), \( g_l \in K^m_T \), and for \( i = 1, \ldots, t \), the images of \( g_p \) in \( G_i(\mathbb{Q}_p) \) are not contained in compact open subgroups. For \( i > t \), \( p_i(T) \) is trivial. This finishes the proof of lemma 6.1.

\[ \square \]

By properties of the Nori constant \( C(V) \), the element \( g \) of 6.1 satisfies the assumptions of propositions 4.10 and 4.11. As \( Y = gY \),

\[ \pi(Y) \subset V \cap T_yV. \]

By the choices we made and proposition 4.10 the varieties \( T_yV \) and \( T_{g^{-1}}V \) are irreducible. Suppose that \( V = T_yV \). Then \( V = T_{g^{-1}}V \).

Indeed, \( V = T_yV \) if and only if \( \gamma_1 g \gamma_2 \cdot \tilde{V} = \tilde{V} \) for all \( \gamma_1, \gamma_2 \in \Gamma \). This is equivalent to \( \gamma_1 g^{-1} \gamma_2 \cdot \tilde{V} = \tilde{V} \) for all \( \gamma_1, \gamma_2 \in \Gamma \). By proposition 4.11, we conclude that

\[ V = S_1 \times \cdots S_t \times V' \]

where \( V' \) is a subvariety of \( S_{t+1} \times \cdots \times S_r \). This finishes the proof in the case where \( V = T_yV \).

Suppose now that the intersection \( V \) and \( T_yV \) is not proper. As \( T_yV \) is irreducible, the intersection \( V \cap T_yV \) is not proper. Let \( Z = V \cap T_yV \). Then \( Y \subset \pi^{-1}Z \). As \( Y \) and \( \pi^{-1}Z \) are analytic varieties, there exists an analytic component containing \( Y \). We let \( V_1 \) be the image of this component by \( \pi \). As \( V_1 \) contains \( \pi(Y) \), it is Hodge generic. Notice that the fact that the images of all projections of \( Y \) are positive dimensional implies that the projections of \( V_1 \) are positive dimensional (in particular the monodromy is maximal). We then choose a prime \( p > C(V_1) \) and reiterate the process. Notice that we may, by lemma 4.8 again assume that \( V_1 \) is not of the form \( S_1 \times \cdots \times S_k \times V' \). This way, we construct a decreasing sequence \( V_i \) of subvarieties. A dimension argument shows that we end up with \( V_k = S_1 \times \cdots \times S_t \times V' \) as above. This gives a contradiction, hence \( V_k = S \). This finishes the proof.

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