Exact solution for random walks on the triangular lattice with absorbing boundaries

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Abstract. The problem of a random walk on a finite triangular lattice with a single interior source point and zig-zag absorbing boundaries is solved exactly. This problem has been previously considered intractable.

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1. Introduction

The problem of a random walk on a two-dimensional lattice with a single interior source point and absorbing boundaries was first considered by Courant et al. [1] in 1928 where general properties of the solution were discussed. This problem was solved exactly in 1940 [2] for the case of random walks on a square lattice with rectangular absorbing boundaries. The problem on a triangular lattice with finite absorbing boundaries was considered in 1963 [3] where the exact solution was given for an approximation of the problem using straight boundaries rather than the true zig-zag boundaries of the triangular lattice. Indeed the authors of this paper remarked that “An explicit solution of the difference equation can hardly be obtained if these boundary conditions are used.” Other variants of the problem on the square lattice have been solved exactly [3, 4] however the problem on the triangular lattice with true zig-zag triangular lattice boundaries has remained unsolved. In this paper we give the exact solution for this problem.

The problem of random walks on finite lattices with absorbing boundaries is fundamental to the theory of stochastic processes [5] and has numerous applications. These include potential theory [6], electrical networks [7], surface diffusion [8] and Diffusion-Limited Aggregation (DLA) [9]. For example, the exact results for the square lattice problem [2] have been used to expedite the growth of large DLA clusters on the square lattice [10] and the formulae derived in this paper could similarly be used on the triangular lattice.
2. Field equations, boundary conditions and absorption probabilities

A schematic illustration showing random walk pathways (dashed lines) on an equi-angular triangular lattice is shown in Fig. 1. A natural co-ordinate system for the triangular lattice is to label the lattice vertices by the intersection points \((p, q)\) of horizontal straight lines \(p = 0, 1, 2, \ldots, m+1\) and slanted straight lines \(q = 0, 1, 2, \ldots, n+1\) – parallel to the \(p\) axis shown in Fig. 1. In this co-ordinate system the expectation that a random walk on the lattice visits a site \((p, q)\), before exiting at a finite boundary, is given by the non-separable difference equation

\[
F(p, q) = \frac{1}{6} [F(p, q - 1) + F(p, q + 1) + F(p - 1, q) \\
+ F(p + 1, q) + F(p + 1, q - 1) + F(p - 1, q + 1)].
\]

In this paper we adopt a different co-ordinate system, which may at first appear less natural but it has the advantage that the governing equations are separable.

In the co-ordinate system used here we label the vertices of the triangular lattice by the intersection points \((p, q)\) of horizontal straight lines \(p = 0, 1, 2, \ldots, m + 1\) and vertical zig-zag lines \(q = 0, 1, 2, \ldots, n + 1\) (see Fig. 2). In the absence of any source terms the expectation that a random walker visits a lattice site \((p, q)\) is given by the coupled homogeneous difference equations

\[
\hat{F}(p, q) = \frac{1}{6} [\hat{F}(p - 1, q - 1) + \hat{F}(p, q - 1) + \hat{F}(p + 1, q - 1) \\
+ \hat{F}(p + 1, q) + \hat{F}(p + 1, q + 1) + \hat{F}(p - 1, q + 1)], \tag{1}
\]

\[
\hat{F}(p, q) = \frac{1}{6} [F(p - 1, q) + \hat{F}(p, q - 1) + F(p + 1, q) \\
+ F(p + 1, q + 1) + \hat{F}(p, q + 1) + F(p - 1, q + 1)]. \tag{2}
\]
where $F(p, q)$ denotes the field value at an even $p$ co-ordinate and $\hat{F}(p, q)$ denotes the field value at an odd $p$ co-ordinate. This separation of even and odd field equations, which is not necessary for the problem on the square lattice, is central to our solution below.

**Figure 2.** Co-ordinate system, represented by dashed lines, adopted in this paper for random walks on the triangular lattice with true zig-zag lattice boundaries. In this example $m = 7, n = 7$. Nearest neighbour sites, highlighted with open circles, are shown surrounding a source point, highlighted by an open box, at $a = 4, b = 4$. The absorbing boundary sites are connected by solid lines.

Now consider a source term at $(a, b)$ and absorbing boundaries at $p = 0, m + 1$ and $q = 0, n + 1$. In Section 3 we give the solution for the case where $m > 1$ is odd. The same methods can be used with slightly altered equations to obtain the solution for $m$ even. The special case of the strip with $m = 1$ has been detailed elsewhere [11]. Following McCrea and Whipple [4] we construct separate solutions, $F_I(p, q), \hat{F}_I(p, q)$ for $q \leq b$ and $F_{II}(p, q), \hat{F}_{II}(p, q)$ for $q \geq b$. The absorbing boundary conditions are thus given by

\[
\begin{align*}
F(0, q) &= 0, \\
F(m + 1, q) &= 0, \\
F_I(p, 0) &= 0, \\
\hat{F}_I(p, 0) &= 0, \\
F_{II}(p, n + 1) &= 0, \\
\hat{F}_{II}(p, n + 1) &= 0.
\end{align*}
\]

The omission of a subscript $I$ or $II$ in the above indicates that the same equations are satisfied by both $F_I$ and $F_{II}$.

At $q = b$ we have the matching conditions

\[F_I(p, b) = F_{II}(p, b),\]
Finally, with the inclusion of the source term, we have the coupled inhomogeneous field equations at $q = b,$

\[
6 F_I(p, b) = 6 \delta_{p,a} ((a + 1) \mod 2) \\
+ \left[ \hat{F}_I(p - 1, b - 1) + F_I(p, b - 1) + \hat{F}_I(p + 1, b - 1) \\
+ \hat{F}_I(p + 1, b) + F_{II}(p, b + 1) + \hat{F}_I(p - 1, b) \right],
\]

(11)

\[
6 \hat{F}_I(p, b) = 6 \delta_{p,a} (a \mod 2) \\
+ \left[ F_I(p - 1, b) + \hat{F}_I(p, b - 1) + F_I(p + 1, b) \\
+ F_{II}(p + 1, b + 1) + \hat{F}_{II}(p, b + 1) + F_{II}(p - 1, b + 1) \right].
\]

(12)

In Section III we present the solutions to Eqs. (1)–(12).

The probabilities for the random walking particle to be absorbed at any specified point on one of the four boundaries, $p = 0, p = m + 1, q = 0, q = n + 1,$ are readily obtained by averaging over the nearest neighbour expectation values subject to the boundary conditions, Eqs. (3)–(8). For example,

\[
P(2k - 1, 0) = \frac{1}{6} \left[ \hat{F}_I(2k - 1, 1) + F_I(2k - 2, 1) + F_I(2k, 1) \right]; \\
k = 1, \ldots \frac{m + 1}{2}.
\]

(13)

3. Solution

3.1. Homogeneous equations – Separation of variables

We begin by solving the homogeneous equations, Eqs. (1),(2), subject to the absorbing boundary conditions, Eqs. (3),(4). The homogeneous field equations separate, on using

\[
F(p, q) = P(p)Q(q),
\]

(14)

\[
\hat{F}(p, q) = P(p)\hat{Q}(q),
\]

(15)

into

\[
6 Q(q) - Q(q + 1) - Q(q - 1) \\
\hat{Q}(q - 1) + Q(q) = \frac{\hat{P}(p - 1) + \hat{P}(p + 1)}{\hat{P}(p)} = \lambda,
\]

(16)

\[
6 \hat{Q}(q) - \hat{Q}(q + 1) - \hat{Q}(q - 1) \\
Q(q + 1) + Q(q) = \frac{P(p - 1) + P(p + 1)}{\hat{P}(p)} = \kappa,
\]

(17)

where $\kappa$ and $\lambda$ are separation constants. The two coupled equations for $P$ and $\hat{P}$ can be readily decoupled into separate equations

\[
P(2k + 2) + (2 - \lambda \kappa)P(2k) + P(2k - 2) = 0,
\]

(18)

\[
\hat{P}(2k + 3) + (2 - \lambda \kappa)\hat{P}(2k + 1) + \hat{P}(2k - 1) = 0,
\]

(19)

where $k$ is an integer. This second order linear difference equation for $P, (\hat{P})$ on a lattice of even (odd) integers has the solution

\[
P(2k) = A\mu^k + B\mu^{-k},
\]

(13)
where
\[ \mu = \frac{-(2 - \lambda \kappa) + \sqrt{(2 - \lambda \kappa)^2 - 4}}{2}, \]
and \( A \) and \( B \) are arbitrary. It follows that the solution of \( P(p) \) (with \( p \) even) corresponding to the boundary conditions, Eqs. (3),(4), is
\[ P(p) = c \sin\left(\frac{\pi jp}{m + 1}\right) \]  
with
\[ \lambda \kappa = 4 \cos^2\left(\frac{\pi j}{m + 1}\right). \]  
The solution for \( \hat{P}(p) \) (with \( p \) odd) now follows from Eq. (17) as
\[ \hat{P}(p) = 2c \kappa \cos\left(\frac{\pi j}{m + 1}\right) \sin\left(\frac{\pijp}{m + 1}\right). \]  
Equations (16) and (17) can also be decoupled for \( Q(q) \) and \( \hat{Q}(q) \) resulting in the same fourth order linear difference equation in each case;
\[ Q(q + 4) - (12 + \lambda \kappa) Q(q + 3) + (38 - 2\lambda \kappa)Q(q + 2) - (12 + \lambda \kappa)Q(q + 1) + Q(q) = 0. \]  
The zeroes of the corresponding characteristic quartic polynomial are in reciprocal pairs leading to the solutions
\[ Q(q) = c_1 e^{\alpha q} + c_2 e^{-\alpha q} + c_3 e^{\beta q} + c_4 e^{-\beta q}, \]  
\[ \hat{Q}(q) = \hat{c}_1 e^{\alpha q} + \hat{c}_2 e^{-\alpha q} + \hat{c}_3 e^{\beta q} + \hat{c}_4 e^{-\beta q}, \]  
where
\[ \cosh \alpha = 3 + \frac{\lambda \kappa}{4} + \frac{1}{4} \sqrt{(\lambda \kappa)^2 + 32\lambda \kappa}, \]  
\[ \cosh \beta = 3 + \frac{\lambda \kappa}{4} - \frac{1}{4} \sqrt{(\lambda \kappa)^2 + 32\lambda \kappa}. \]  
The coefficients \( c_1, c_2, c_3, c_4 \) may be chosen arbitrarily with the coefficients \( \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4 \) then determined by substituting Eqs. (24), (25) into Eq. (16) or Eq. (17). From Eq. (17) we obtain
\[ \hat{c}_1 = c_1 f(\alpha), \hat{c}_2 = c_2 f(-\alpha), \hat{c}_3 = c_3 f(\beta), \hat{c}_4 = c_4 f(-\beta) \]  
where
\[ f(\omega) = \frac{\kappa(e^{\omega} + 1)}{2(3 - \cosh \omega)}. \]  
By combining Eqs. (14),(15),(20),(22),(24),(25),(28),(29) we can write the solutions to the field equations, Eqs. (1), (2), that satisfy the boundary conditions, Eqs. (3),(4), in the form
\[ F(p, q; A, B, C, D) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m + 1}\right) \times \left[ A_j e^{\alpha q}(3 - \cosh \alpha) + B_j e^{-\alpha q}(3 - \cosh \alpha) \right] \]
\[ +C_j e^{\beta q}(3 - \cosh \beta) + D_j e^{-\beta q}(3 - \cosh \beta) \]\]

\[
\hat{F}(p, q; A, B, C, D) = \sum_{j=1}^{m} \sin \left( \frac{\pi j p}{m + 1} \right) \cos \left( \frac{\pi j}{m + 1} \right) \\
\times \left[ A_j e^{\alpha q}(e^\alpha + 1) + B_j e^{-\alpha q}(e^{-\alpha} + 1) \\
+ C_j e^{\beta q}(e^\beta + 1) + D_j e^{-\beta q}(e^{-\beta} + 1) \right],
\]

where \( A_j, B_j, C_j, D_j \) are arbitrary.

Before imposing the remaining absorbing boundary conditions, Eqs. (5)–(8), we represent the solutions in the two regions: \( I : q \leq b; \ II : q \geq b \) by

\[
F_I(p, q) = F(p, q; A, B, C, D),
\]

\[
\hat{F}_I(p, q) = \hat{F}(p, q; A, B, C, D),
\]

\[
F_{II}(p, q) = F(p, q; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}),
\]

\[
\hat{F}_{II}(p, q) = \hat{F}(p, q; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}).
\]

The absorbing boundary conditions, Eqs. (5)–(8), can be used to eliminate \( C, D, \tilde{C}, \tilde{D} \).

After an appropriate re-normalization of the coefficients \( A, B, \tilde{A}, \tilde{B} \) this yields

\[
F_I(p, q) = \sum_{j=1}^{m} \sin \left( \frac{\pi j p}{m + 1} \right) \\
\times \left[ A_I(q, \alpha, \beta)A_j + A_I(q, -\alpha, \beta)B_j \right],
\]

\[
\hat{F}_I(p, q) = \sum_{j=1}^{m} \sin \left( \frac{\pi j p}{m + 1} \right) \cos \left( \frac{\pi j}{m + 1} \right) \\
\times \left[ \hat{A}_I(q, \alpha, \beta)A_j + \hat{A}_I(q, -\alpha, \beta)B_j \right],
\]

\[
F_{II}(p, q) = \sum_{j=1}^{m} \sin \left( \frac{\pi j p}{m + 1} \right) \\
\times \left[ A_{II}(q, \alpha, \beta)\tilde{A}_j + A_{II}(q, -\alpha, \beta)\tilde{B}_j \right],
\]

\[
\hat{F}_{II}(p, q) = \sum_{j=1}^{m} \sin \left( \frac{\pi j p}{m + 1} \right) \cos \left( \frac{\pi j}{m + 1} \right) \\
\times \left[ \hat{A}_{II}(q, \alpha, \beta)\tilde{A}_j + \hat{A}_{II}(q, -\alpha, \beta)\tilde{B}_j \right],
\]

where

\[
A_I(q, \alpha, \beta) = (3 - \cosh \alpha)(3 - \cosh \beta)2 \sinh \beta e^{\alpha q} \\
- (3 - \cosh \beta)\gamma(-\alpha, -\beta)e^{\beta q} \\
+ (3 - \cosh \beta)\gamma(-\alpha, \beta)e^{-\beta q},
\]

\[
\hat{A}_I(q, \alpha, \beta) = 2e^{\alpha q}(e^\alpha + 1)(3 - \cosh \beta) \sinh \beta
\]
Random walks with absorbing boundaries

\[-\gamma(-\alpha, -\beta)(e^\beta + 1)e^{\beta q} + \gamma(-\alpha, \beta)(e^{-\beta} + 1)e^{-\beta q}, \quad (37)\]

\[A_{II}(q, \alpha, \beta) = (3 - \cosh \alpha)(3 - \cosh \beta)2 \sinh \beta e^{\alpha q} \]
\[- e^{(\alpha - \beta)(n+1)}\gamma(-\alpha, -\beta)(3 - \cosh \beta)e^{\beta q} \]
\[+ e^{(\alpha + \beta)(n+1)}\gamma(-\alpha, \beta)(3 - \cosh \beta)e^{-\beta q}, \quad (38)\]

\[\hat{A}_{II}(q, \alpha, \beta) = (e^\alpha + 1)(3 - \cosh \beta)2 \sinh \beta e^{\alpha q} \]
\[- e^{(\alpha - \beta)(n+1)}\gamma(-\alpha, -\beta)(e^\beta + 1)e^{\beta q} \]
\[+ e^{(\alpha + \beta)(n+1)}\gamma(-\alpha, \beta)(e^{-\beta} + 1)e^{-\beta q}, \quad (39)\]

and the function

\[\gamma(\alpha, \beta) = 4 \cosh \alpha - 4 \cosh \beta - (3 - \cosh \beta) \sinh \alpha \]
\[- (3 - \cosh \alpha) \sinh \beta. \quad (40)\]

The coefficients defined in Eq. (36) which are shown as implicit functions of \(\alpha\) and \(\beta\) are also functions of \(j\) via Eqs. \((21),(26),(27)\). Equations \((32)-(35)\), are general solutions to the coupled homogeneous field equations, Eqs. \((1),(2)\), that satisfy all of the absorbing boundary conditions, Eqs. \((3)-(8)\).

3.2. Inhomogeneous equations – Matching conditions

By using the two matching conditions, Eqs. \((9),(10)\), the four arbitrary constants, \(A, B, \hat{A}, \hat{B}\), can be reduced to two arbitrary constants, \(A, B\) say. The solutions in Region II can then be written as

\[F_{II}(p, q) = \sum_{j=1}^{m} \sin \left(\frac{\pi j p}{m + 1}\right) \left[ A^*_I(q, \alpha, \beta) A_j + A^*_I(q, -\alpha, \beta) B_j \right], \quad (41)\]

\[\hat{F}_{II}(p, q) = \sum_{j=1}^{m} \sin \left(\frac{\pi j p}{m + 1}\right) \cos \left(\frac{\pi j}{m + 1}\right) \]
\[\times \left[ \hat{A}^*_I(q, \alpha, \beta) A_j + \hat{A}^*_I(q, -\alpha, \beta) B_j \right], \quad (42)\]

where

\[A^*_I(q, \alpha, \beta) = A_{II}(q, \alpha, \beta) \Gamma_1 + A_{II}(q, -\alpha, \beta) \Gamma_2, \quad (43)\]

\[\hat{A}^*_I(q, \alpha, \beta) = \hat{A}_{II}(q, \alpha, \beta) \Gamma_1 + \hat{A}_{II}(q, -\alpha, \beta) \Gamma_2, \quad (44)\]

and

\[\Gamma_1 = \frac{\hat{A}_{II}(b, -\alpha, \beta) A_I(b, \alpha, \beta) - A_{II}(b, -\alpha, \beta) \hat{A}_I(b, \alpha, \beta)}{A_{II}(b, \alpha, \beta) A_I(b, -\alpha, \beta) - A_{II}(b, \alpha, \beta) \hat{A}_I(b, -\alpha, \beta)}, \quad (45)\]

\[\Gamma_2 = \frac{\hat{A}_{II}(b, \alpha, \beta) A_I(b, \alpha, \beta) - A_{II}(b, \alpha, \beta) \hat{A}_I(b, \alpha, \beta)}{A_{II}(b, -\alpha, \beta) A_I(b, \alpha, \beta) - A_{II}(b, \alpha, \beta) \hat{A}_I(b, -\alpha, \beta)}, \quad (46)\]
Finally the remaining arbitrary constants $A, B$ are determined from the requirement that the solutions satisfy the coupled inhomogeneous field equations, Eqs. (11),(12). This step is facilitated using the identity
\[ \delta_{\mu,a} = \frac{2}{m+1} \sum_{j=1}^{m} \sin \left( \frac{\pi ja}{m+1} \right) \sin \left( \frac{\pi jp}{m+1} \right). \]  

(47)

Explicitly we find:

i) $a$ even,
\[ A_j = \frac{T_1(j)}{T_2(j)} B_j, \]  
\[ B_j = \frac{12}{m+1} \sin \left( \frac{\pi ja}{m+1} \right) \frac{T_2(j)}{T_1(j)T_3(j) - T_3(j)T_2(j)}; \]  

(48)

(49)

ii) $a$ odd,
\[ A_j = \frac{T_4(j)}{T_3(j)} B_j, \]  
\[ B_j = \frac{12}{m+1} \sin \left( \frac{\pi ja}{m+1} \right) \frac{T_3(j)}{T_2(j)T_4(j) - T_4(j)T_2(j)}, \]  

(50)

(51)

and
\[ T_1(j) = 2A_I(b, -\alpha, \beta) + 2A_I^*(b + 1, -\alpha, \beta) + \hat{A}_I(b - 1, -\alpha, \beta) \]  
\[ + \hat{A}_I^*(b + 1, -\alpha, \beta) - 6 \hat{A}_I(b, -\alpha, \beta), \]  

(52)

\[ T_2(j) = 6\hat{A}_I(b, \alpha, \beta) - 2A_I(b, \alpha, \beta) - 2A_I^*(b + 1, \alpha, \beta) \]  
\[ - \hat{A}_I(b - 1, \alpha, \beta) - \hat{A}_I^*(b + 1, \alpha, \beta), \]  

(53)

\[ T_3(j) = 6A_I(j, b) - 2\cos^2 \left( \frac{\pi j}{m+1} \right) \hat{A}_I(b - 1, \alpha, \beta) \]  
\[ - 2\cos^2 \left( \frac{\pi j}{m+1} \right) \hat{A}_I(b, \alpha, \beta) - A_I(b - 1, \alpha, \beta) \]  
\[ - A_I^*(b + 1, \alpha, \beta), \]  

(54)

\[ T_4(j) = 2\cos^2 \left( \frac{\pi j}{m+1} \right) \hat{A}_I(b - 1, -\alpha, \beta) \]  
\[ + 2\cos^2 \left( \frac{\pi j}{m+1} \right) \hat{A}_I(b, -\alpha, \beta) + A_I(b - 1, -\alpha, \beta) \]  
\[ + A_I^*(b + 1, -\alpha, \beta) - 6A_I(b, -\alpha, \beta). \]  

(55)

Our solution for the triangular lattice site expectation values with absorbing boundary conditions and a source point is finally given by Eqs. (32),(33) in Region I and Eqs. (41),(42) in Region II. The relevant quantities appearing in these equations are defined through the series of equations, Eqs. (21),(26),(27),(36)-(40),(43)-(46),(48)-(50).
Random walks with absorbing boundaries

4. Example

Consider the case of a triangular lattice with $m = 7, n = 7$ and a source at $a = 4, b = 4$ (Fig. 2). In the approximation to this problem using straight edge boundaries and the $(p, q)$ co-ordinate system of Keberle and Montet [3] this problem corresponds to; $m = 16, n = 7, a = 8, b = 4$.

We have calculated expectation values at the nearest neighbour lattice sites around the source and absorption probabilities at the boundaries; (a) using our exact results above, and (b) using the results of Keberle and Montet [3] for the approximation to the problem – their equations $(7a), (7b), (7c)$. In this small model lattice system we found that the straight line boundary approximation provides reasonable results for expectation values near the source (accurate to within a few percent) but provides poor results for the absorption probabilities (Table 1). Note that the ‘absorption probabilities’ for the zig-zag boundary co-ordinates which are inside the straight line boundary of Keberle and Montet [3] are not true probabilities in their solution.

5. Conclusion

Although our solution is somewhat unwieldy, we have nevertheless solved the underlying field equations for random walks on a finite triangular lattice with a single interior source point and zig-zag absorbing boundaries and have thus calculated the associated absorption probabilities. This problem was previously considered to be intractable [3].
We hope that our result will inspire further work in this area.

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