On the radius of Gaussian free field excursion clusters

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Abstract
We consider the Gaussian free field $\varphi$ on $\mathbb{Z}^d$, for $d \geq 3$, and give sharp bounds on the probability that the radius of a finite cluster in the excursion set $\{\varphi \geq h\}$ exceeds a large value $N$ for any height $h \neq h_*$, where $h_*$ refers to the corresponding percolation critical parameter. In dimension 3, we prove that this probability is sub-exponential in $N$ and decays as $\exp\{-\frac{\pi}{6}(h - h_*)^2 \frac{N}{\log N}\}$ as $N \to \infty$ to principal exponential order. When $d \geq 4$, we prove that these tails decay exponentially in $N$. Our results extend to other quantities of interest, such as truncated two-point functions and the two-arms probability for annuli crossings at scale $N$.

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1 Introduction

This article investigates the percolative properties of excursion sets \( \{ \varphi \geq h \} \) of the Gaussian free field \( \varphi \) on \( \mathbb{Z}^d \) in dimensions \( d \geq 3 \), for varying height parameter \( h \in \mathbb{R} \). This model, the rigorous study of which was initiated in [1], and more recently re-instigated in [39], serves as a benchmark example of a (non-planar) percolation model with strong, algebraically decaying correlations. One of its appealing features is the rich interplay with potential theory for the underlying random walk, which is beneficial to its study. A central role is thus played by electrostatic notions such as capacity, see e.g. [2], [10], and more recently [32], [34], [33], [9], [43], [8], which also feature prominently in the present work.

Our main focus concerns the radii of finite clusters in the excursion sets \( \{ \varphi \geq h \} \), which we set out to introduce. Under a suitable probability \( \mathbb{P} \), the field \( \varphi \) is the centered Gaussian field with covariance \( \mathbb{E}[\varphi_x \varphi_y] = g(x, y) \), for \( x, y \in \mathbb{Z}^d \), where \( g(\cdot, \cdot) \) denotes the Green function of the simple random walk on \( \mathbb{Z}^d \), see (2.1). The critical parameter for percolation of the associated excursion sets \( \{ \varphi \geq h \} := \{ x \in \mathbb{Z}^d : \varphi_x \geq h \} \) is defined as

\[
(1.1) \quad h_* = h_* (d) := \inf \{ h \in \mathbb{R} : \lim_{N \to \infty} \mathbb{P} [0 \xrightarrow{\varphi \geq h} \partial B_N] = 0 \},
\]

where, with hopefully obvious notation, the event in (1.1) refers to a (nearest-neighbor) path in \( \{ \varphi \geq h \} \) connecting 0 and \( \partial B_N \), where \( B_N = B_N (0) \), \( B_N (x) := \{ y \in \mathbb{Z}^d : \|y - x\|_\infty \leq N \} \) for all \( x \in \mathbb{Z}^d \), \( N \geq 1 \), and \( \partial K \) refers to the inner (vertex) boundary of a set \( K \subset \mathbb{Z}^d \). It is known that \( 0 < h_* < \infty \) for all \( d \geq 3 \), see [3], [39], [14], and that the infinite cluster, when existing, is almost surely unique. Auxiliary parameters \( \bar{h} \) and \( h_{**} \) satisfying \( \bar{h} \leq h_* \leq h_{**} \) were frequently used in the past, respectively characterizing a phase of ‘well-behavedness’ for the infinite cluster and a strongly subcritical regime, in which connectivities decay rapidly. Recently it was proved in [19] that

\[
(1.2) \quad \bar{h} = h_* = h_{**}.
\]

As a consequence, one knows the following: there exist \( c_1 = c_1 (d, h) > 0 \) and \( c = c (h, d) > 0 \) such that, for all \( N \geq 1 \)

\[
(1.3) \quad \mathbb{P} [0 \xrightarrow{\varphi \geq h} \partial B_N] \leq e^{-c N c_1}, \quad \text{if } h > h_* (d) \quad (= h_{**} (d)),
\]

\[
(1.4) \quad \mathbb{P} [\text{LocUniq} (N, h)^c] \leq e^{-c N c_1}, \quad \text{if } h \leq h_* (d) \quad (= \bar{h} (d)),
\]

where the ‘local uniqueness’ event in (1.4) is defined as

\[
(1.5) \quad \text{LocUniq} (N, h) = \left\{ \{ \varphi \geq h \} \text{ has a unique connected component crossing } B_{2N} \setminus B_N \right\}.
\]

Here and in the sequel, a set \( S \subset \mathbb{Z}^d \) is said to cross \( V \setminus U \), for \( U \subset V \subset \mathbb{Z}^d \) if \( S \) has a connected component intersecting both \( U \) and \( \partial V \). The estimate (1.4) is inherited from the bounds for the ‘existence’ and ‘uniqueness’ events usually appearing in the definition of \( \bar{h} \), see e.g. (1.10)–(1.11) in [13], which assert that for all \( h < \bar{h} \quad (= h_*) \) and \( N \geq 1 \),

\[
\mathbb{P} \left[ \text{there exists a connected component in } \{ \varphi \geq h \} \cap B_N \text{ with diameter at least } N/5 \right] \geq 1 - e^{-c N c_1}
\]

and

\[
\mathbb{P} \left[ \text{any two clusters in } \{ \varphi \geq h \} \cap B_N \text{ having diameter at least } N/10 \text{ are connected to each other in } \{ \varphi \geq h \} \cap B_{2N} \right] \geq 1 - e^{-c N c_1},
\]
where the diameter of a set is with respect to the sup-norm. Indeed, \((1.4)\) follows by straightforward gluing arguments, combining the events in \((1.6)\) at a fixed number of scales commensurate with \(N\). Stretched exponential bounds such as \((1.3), (1.4)\) and \((1.6)\) typically arise as a byproduct of certain static renormalization methods, see e.g. \([39]\) regarding \((1.3)\), which exemplifies this phenomenon. Little is otherwise known about the true order of decay for the probabilities in \((1.3)\) and \((1.4)\). To date, the best available results are due to \([37], [38]\), which solely concern the subcritical regime and yield that \((1.3)\) holds with \(c_1(d, h) = 1\) if \(d \geq 4\) and \(h > h_*\), along with an upper bound for \(\mathbb{P}[0 \leftarrow N \partial B_N] \) of exponential order \(N/(\log N)^{3+\varepsilon}\), for any \(\varepsilon > 0\), when \(d = 3\) and \(h > h_*\).

Our findings address these matters. Our main results are most easily formulated in terms of a ‘truncated one-arm event’. We refer to the discussion following the statement of Theorem 1.2 below regarding extensions of \((1.7)\) and \((1.8)\) to other quantities of interest. Upper bounds in the spirit of those obtained below for \(h > h_*\) have also been derived in \([12]\) for the so-called metric graph associated to \(Z^d\). Contrary to what is suggested in Section 1.3 of \([12]\), the logarithmic factor in dimension three is not an artefact.

**Theorem 1.1.** For \(d = 3\) and all \(h \in \mathbb{R}\),

\[
\lim_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[0 \leftarrow N \partial B_N, 0 \leftarrow \infty] = -\frac{\pi}{6} (h - h_*)^2.
\]

In higher dimensions, we have the following:

**Theorem 1.2.** For every \(d \geq 4\) and \(h \neq h_*\), there exist \(C = C(d, h), c = c(d, h) \in (0, \infty)\) such that

\[
e^{-CN} \leq \mathbb{P}[0 \leftarrow N \partial B_N, 0 \leftarrow \infty] \leq e^{-cN}.
\]

Theorems 1.1 and 1.2 follow immediately by combining the results of Theorems 3.1 and 5.1 below, which separately deal with the corresponding lower and upper bounds, respectively. In fact, as asserted in these two theorems, \((1.7)\) and \((1.8)\) continue to hold when \(h \leq h_*\) if one replaces the event in question by \(\text{LocUniq}(N, h)^c\), see \((1.5)\). We also have the similar asymptotic for a slightly modified version of the event \(\text{LocUniq}(N, h)^c\), namely

\[
\lim_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[\text{2-arms}(N, h)] = -\frac{\pi}{6} (h - h_*)^2, \text{ if } h \leq h_* \text{ and } d = 3,
\]

along with a statement similar to \((1.8)\) when \(d \geq 4\), where ‘2-arms’ refers to the existence of two disjoint crossing clusters of \(\{ \varphi \geq h\} \cap (B_{2N} \setminus B_N)\).

Concerning the truncated two-point function, defined as

\[
\tau_{tr}^h(x, y) = \mathbb{P}[x \leftarrow y, x \leftarrow \infty], \text{ for } x, y \in \mathbb{Z}^d \text{ and } h \in \mathbb{R},
\]

one readily sees that this is symmetric in \(x\) and \(y\) and satisfies \(\tau_{tr}^h(x, y) = \tau_{tr}^h(0, y - x)\) by translation invariance of the set \(\{ \varphi \geq h\}\). Our results readily imply (cf. the proof of Theorem 3.1 below) that the asymptotics \((1.7)\) and \((1.8)\) also hold for \(\tau_{tr}^h(0, N e_1)\), with \(h \in \mathbb{R}\) and \(h \neq h_*\) respectively, where \(e_1\) denotes the unit vector in a coordinate direction of \(\mathbb{Z}^d\). More generally,
with $|\cdot|$ denoting the Euclidean distance, one may also expect that for arbitrary $x, y \in \mathbb{Z}^d$ and $h \in \mathbb{R}$,

$$\lim_{|x-y| \to \infty} \frac{\log |x-y|}{|x-y|} \log \tau_h^{tr}(x,y) = -\frac{\pi}{6}(h-h^*)^2, \text{ when } d = 3.$$  \tag{1.11}

We refer to Remarks 2.3 and 5.17 (2) below regarding the (technical) modifications to our argument needed to prove (1.11) and compelling evidence for its truthfulness. Finally, let us emphasize that, while (1.2) leads to a form of Theorem 1.1 indicating that $h_*$ is approached at the same rate both as $h \downarrow h_*$ and $h \nearrow h_*$, a version of our findings could be stated in terms of $\hat{h}$ and $h_*$ only, much as in [12], [33], [9], thus yielding (1.7) and (1.8) upon applying (1.2).

This is the sole place where (1.2) is used.

Next, we compare the findings of Theorems 1.1 and 1.2 to related existing results. For Bernoulli percolation (i.e. taking $\varphi$ to be a field of independent standard Gaussians, traditionally parametrized using $p = \mathbb{P}[\varphi_0 \geq h]$, or the analogue bond percolation problem), exponential decay of the radius function at a leading rate $e^{-N^c(\varphi)}$ can be obtained for all $p \neq p_c$ [32], [1], [7], [23], see also [22] Sections 6.2 and 8.4]. Little is known apart from existence and qualitative properties of $\xi(\cdot)$ in intermediate dimensions. One can show polynomial lower bounds of the form $\xi(p) \geq |p - p_c|^{-c(d)}$ for $p < p_c$, see [20] Section 8 and refs. therein. The currently best available upper bounds [20], valid for all $p \neq p_c$, do not witness a polynomial scaling. One also has estimates similar to (1.11) for the two-point function $\tau_p$, including improved Ornstein-Zernike asymptotics for $p < p_c$ [40, 8] or $p$ close to $1$ [5], but the relevant norm $\xi_p(x) := -\lim N^{-1}\log \tau_p(0, Nx)$ in the leading order term is implicit. In particular, the possible isotropy of $\xi_p(\cdot)$ (near $p_c$), cf. (1.11), see also [31] Chap. XII.5] regarding the Ising model in dimension $d = 2$ and [35] Fig. 2.2 for related simulations, is not clear. In this regard (1.11) hints at the rotational symmetry of a possible scaling limit at criticality.

The situation improves dramatically in high dimensions or when $d = 2$. Indeed, one knows by [24], [25], that $\xi(p) \sim |p - p_c|^{-\nu}$ with $\nu = \frac{3}{4}$ as $p \nearrow p_c$ when $d$ is large enough. As far as we understand, the case of $p \searrow p_c$, i.e. the behavior of truncated observables, is completely open in high dimensions. For site percolation on the planar triangular lattice, $\nu = \frac{4}{3}$ and $\xi_p(x)$ is independent of the direction of $x$ near $p_c$, see [40] and refs. therein.

The percolation problem (1.1) has a close cousin, involving the excursion sets of the free field $\tilde{\varphi}$ on the corresponding cable system $\mathbb{Z}^d$, which can be understood as a purely discrete bond percolation problem involving additional disorder on the edges: given $\varphi$, each edge $e = \{x,y\}$ is closed independently with probability $\exp\{-2(\varphi_x - h)(\varphi_y - h)\}$, cf. the discussion around [15] (1.6)]; see also [29] or [16] Section 2] for further details. Variants of this observation have been recently used in several related contexts [29], [30], [18].

For percolation of the excursion sets $\{\tilde{\varphi} \geq h\}$, one knows that the associated critical parameter $\tilde{h}_*$ corresponding to (1.1) satisfies $\tilde{h}_* = 0$ in great generality [16] Corollary 1.2, including all transitive graphs (and in particular $\mathbb{Z}^d$, $d \geq 3$). This is due to the behavior of the cluster capacity observable, which was recently realized to play a central role in this context; to wit, this observable naturally emerges as part of the differential identities derived in [15] Section 2] which correspond to the classical Margulis-Russo formula [22] Section 2.4] for independent percolation. Essentially, the fact that $\tilde{h}_* = 0$ on most graphs is a direct consequence of such a formula and the fact that the capacity of clusters of $\{\tilde{\varphi} > 0\}$ is finite on all transient graphs [16] Theorem 1.1.1]). Following [15], [16], and somewhat in the spirit of Theorems 1.1 and 1.2 above, the leading exponential order for the tail of the corresponding capacity observable for $\varphi$
on $\mathbb{Z}^d$, $d \geq 3$, was subsequently derived in [36] Theorem 1.3 for $h \neq h_*$, thus mirroring parts of [16] Theorem 3.7 for $\tilde{\varphi}$; see also [15] (1.14) for corresponding (near-)critical tails.

With regards to connectivity functions, the result of Theorem 1.1 (as well as 1.11) can be drastically improved when considering the percolation problem for $\{\tilde{\varphi} \geq h\}$. Indeed, as recently shown in [15] Theorem 1.4, abbreviating by $\tilde{\tau}^\nu_h (\cdot, \cdot)$ the quantity in 1.11 with $\tilde{\varphi}$ in place of $\varphi$ and $\tilde{\xi} (h) = |h - h_*|^{-2} = |h|^{-2}$, one has when $d = 3$ that

\begin{equation}
(1.12) \quad c' \tilde{\tau}_0^\nu (0, x) \exp \left\{ - \frac{C |x| / \tilde{\xi} (h)}{\log \left( \frac{|x|}{\tilde{\xi} (h)} \vee 2 \right)} \right\} \leq \tilde{\tau}_0^\nu (0, x) \exp \left\{ - \frac{c |x| / \tilde{\xi} (h)}{\log (|x| \vee 2)} \right\},
\end{equation}

which is valid for all $h \in \mathbb{R}$ and $x \in \mathbb{Z}^3$, where the lower bound requires that $\frac{|x|}{\tilde{\xi} (h)} \geq (\log \xi (h))^\delta$ for some $\delta \in (0, 1)$, along with analogous results for the radius observable of $(1.7)$. In particular, the bounds $(1.12)$ witness the onset of the leading exponential asymptotics of Theorem 1.1 as occurring at scale $\xi$, which justifies its use as a correlation length. Together with the scaling behavior of $\tilde{\tau}_0^\nu (0, x)$, which follows readily from the Markov property, as observed in [29], the bounds $(1.12)$ almost display the full scaling behavior of $\tilde{\tau}$, and are sufficient to yield Fisher’s scaling law, see [16] Corollary 1.5; see also [26] regarding recent findings displaying analogous scaling to that of $\tilde{\tau}_0^\nu$ for the critical two-point function of independent long-range percolation on the hierarchical lattice.

Noteworthily, whereas upper bounds will be the main difficulty in proving Theorems 1.1 and 1.2 below, the lower bounds in $(1.12)$ are far more difficult. These bounds require understanding connection mechanisms up to the critical scale $\xi$ and they rely on different ideas, which will not be discussed here; see [15]. On the other hand, the upper bound in $(1.12)$ follows readily by comparison with the aforementioned cluster capacity observable, see [15] Section 4; see also [12] Theorem 4 for related if somewhat weaker bounds.

Results corresponding to Theorem 1.2 for $\tilde{\varphi}$ in dimensions $d \geq 4$ follow by combining [16] (1.26) for $\nu = d - 2$ and an adaptation of the argument leading to (3.3) below to the cables. This produces both Gaussian upper and lower bounds in $h$ when $|h|$ is at least of the order of local fluctuations, i.e. $\text{var}(\varphi_0)^{1/2}$ or so, see also [15] Remark 8.3. However, currently available upper and lower bounds in the spirit of $(1.12)$ for $d \geq 4$ are far from matching, cf. [15] Theorem 1.4 and Proposition 6.1 for known results. Improving on this, i.e. obtaining the correct analogue of (1.12) in higher dimensions for $|h| \ll \text{var}(\varphi_0)^{1/2}$ along with the correct choice of $\xi$, is presently open; see also [16] Theorem 1.7 for some results in this direction when $d = 4$ (corresponding to $\nu = 2 = \frac{d}{2}$ in [16]).

We now highlight some ideas behind our proofs of Theorems 1.1 and 1.2. One is immediately struck by the discrepancy in the strength of the above results. This is closely related to the fact that the random walk does not ‘see’ one-dimensional sets (such as bounded off-critical percolation clusters) when $d \geq 4$. Our proofs witness this structural difference between the cases $d = 3$ and $d \geq 4$ very clearly. To see this, first observe that (see Lemma 2.2 for precise statements) as $N \to \infty$,

\begin{equation}
(1.13) \quad \text{cap} (([0, N] \cap \mathbb{Z}) \times \{0\}^{d-1}) \sim \frac{\pi}{3 \log N}, \quad \text{when } d = 3, \text{ whereas} \quad \text{cap} (([0, N] \cap \mathbb{Z}) \times \{0\}^{d-1}) \asymp N, \quad \text{when } d \geq 4.
\end{equation}

Now, the coarse-graining described in more detail below (from which we eventually deduce the upper bounds in $(1.7), (1.8)$), yields a sum of two terms for the probability in question. One of
them corresponds to a truncated version of \( \varphi \) (a local field, independent at large scales), for which a corresponding one-arm event decays exponentially in \( N \), regardless of the dimension \( d \). The other term, which carries the long-range dependence, stems from the behavior of the harmonic field in a collection of well-separated boxes, and will turn out to behave in a manner proportional to \( \text{cap}(([0,N] \cap \mathbb{Z}) \times \{0\}^{d-1}) \) to leading exponential order. In view of (1.13), this means that the harmonic term clearly dominates in dimension 3, whereas the two terms live at the same exponential scale in dimension four and higher (and in fact the local term is typically larger).

The lower bounds derived in Section 3 further reflect this disparity. For \( d = 3 \), in the subcritical regime, we use a change of measure argument in order to draw a finite path in \( \{ \varphi \geq h \} \) in a thin horizontal tube. The supercritical regime requires a more delicate treatment, as discussed below. Intuitively, the field shifts itself by the right amount in a suitable region as to make the event in question typical, cf. Lemma 3.2 for a general result in this direction, which is of independent interest. The limit on the right-hand side of (1.7) thereby emerges in the corresponding Radon-Nikodym derivative as half of the leading order pre-factor for the capacity of the shifted region, which is close to that of a line of length \( N \), see (2.12), times the square of the height gap. Similar arguments have been used in the study of hard wall conditions for \( \varphi \), see [2], and disconnection probabilities for supercritical excursion sets, see [42]. Importantly, the monotonicity of the events in question (common to these references) is absent for the one in (1.7) when \( h < h_* \), which requires that we ‘insulate’ the path, i.e. build an interface in \( \{ \varphi < h \} \) to shield it away from \( \infty \). This makes the implementation of our lower bound strategy relatively involved in the supercritical regime and forces us to introduce Dirichlet boundary conditions to decorrelate constituents of opposite monotonicity. In sharp contrast, the lower bounds in (1.8) follow by ‘FKG-type’ arguments, which do not witness the critical parameter \( h_* \) at all, see (3.3).

Most of our work goes into proving the upper bounds required for Theorems 1.1 and 1.2, summarized in Theorem 5.1 below. A stepping stone towards this is a certain coarse-graining scheme for paths, developed in Section 4 (see in particular Proposition 4.3 below), which we now briefly describe. Roughly speaking, for a path \( \gamma \) of linear size \( N \), the coarse-graining of \( \gamma \), formalized in Definition 4.2, only retains the trace of \( \gamma \) in a system of ‘well-separated’ boxes at scale \( L \ll N \). Importantly, the scheme walks the fine line of operating at a preferential entropic cost (parametrized by a function \( \Gamma(\cdot) \), see (4.13), (4.14)), while retaining a sufficiently ‘large’ piece of path when measured in terms of capacity. This latter property, ensured by Proposition 4.3, see (4.16), is crucial for the precise estimates we aim at.

In the subcritical phase \( h > h_* \), the above scheme is used to cascade a connection event such as \( \{ 0 \xrightarrow{\varphi \geq h} \partial B_N \} \) from scale \( N \) down to scale \( L(\ll N) \). For each of the boxes at scale \( L \) in the resulting collection, the occurrence of a crossing in that box is split into a similar event for a localized field with good decorrelation properties as the box is varied, and the occurrence of an atypical behavior for the corresponding harmonic average, see (5.5)-(5.8). The leading-order contribution is thereby carried by the harmonic field in all but a small fraction of \( L \)-boxes, which we control by means of state-of-the-art estimates developed in [42], cf. Lemma 4.1 below. The strength of these estimates hinges on a suitable capacity lower bound for the underlying collection of boxes, which Proposition 4.3 provides.

The resulting two-scale estimate for the one-arm event can then be applied iteratively, see Proposition 5.2 below, to boost an a-priori bound such as (1.3) (but see Remarks 5.3 and 5.6 below to accommodate much weaker a-priori bounds) to the desired decay in a finite number of steps, if \( L \) is carefully chosen as a function of \( N \) (as will turn out, \( L \) needs to grow polylogarithmically in \( N \)). In fact, two steps suffice if one starts from (1.3).
The derivation of the desired upper bounds in the supercritical regime, see (5.2) and (5.4) in Theorem 5.1, is considerably more involved. When \( h < h^* \), connections become typical and the cost displayed in (1.7) and (1.8) measures the difficulty to avoid the infinite cluster. Our approach revolves around an event \( G_N \), see (5.32), ensuring, roughly speaking, that any macroscopic path at scale \( N \) will have \( a_N \) ‘contact points’ in each of \( b_N \) interfaces, all of which are connected to infinity in \( \{ \varphi \geq h \} \). These contact points are in fact local areas at a microscopic scale \( L_0 \) in which a certain insertion tolerance property holds (which the model does not possess as such due to the strength of the correlations), thus yielding a small i.i.d. cost to avoid connecting to the infinite cluster. This property is conveniently defined in terms of a ‘mid-point’ extension of \( \varphi \) that was used in [13], see (2.37) and (5.30). Incidentally, we also take advantage of this extension to deal with competing monotonicity properties of the path and the insulating interface when deriving the lower bounds for \( d \geq 4 \) and \( h < h^* \).

An upper bound on the key quantity \( P[G_N] \) is then derived using a bootstrapping scheme, see Proposition 5.14 below, which works roughly as follows. Starting from a certain (localized) good event \( G_z \) at base scale \( L \gg L_0 \), comprising a local uniqueness property at that scale and a number \( a_L (=1 \) to begin with) of contact points to the ambient cluster for any large path, see Definition 5.12 for which a suitable a-priori estimate is available (cf. Lemma 5.16), the scheme does one of two things: i) in intermediate steps, it re-produces the same event \( G_z \) at larger scale \( N \), improving on both its likelihood and the number \( a_N \) of contact points (eventually we need \( a_Nb_N \) to grow linearly with \( N \) when \( d \geq 4 \) and sub-linearly but with \( a_Nb_N \gg N/\log N \) when \( d = 3 \)); ii) in the final step, the scheme generates the target event \( G_N \), creating multiple interfaces by stacking good boxes at scale \( L \). In either case, the scheme witnesses this improvement on a certain event, see (5.51), defined in terms of the coarse-graining from Proposition 4.3, and for which a dichotomy (involving local fields and harmonic averages) holds, see (5.65)-(5.67). The proofs of the desired upper bounds then follow somewhat similarly as in the subcritical case.

We now briefly describe the organization of this article. Section 2 gathers several preliminary results that will be used in subsequent sections. Section 3 proves the lower bounds corresponding to Theorems 1.1 and 1.2, see Theorem 3.1. Section 4 supplies the coarse-graining scheme for paths, see Proposition 4.3, which will be instrumental in deriving the upper bounds. The proof differs depending on whether \( d = 3 \) or \( d \geq 4 \), which are dealt with separately in Sections 4.1 and 4.2. The desired upper bounds are then derived in Section 5. The sub- and supercritical phases are considered separately in Sections 5.1 and 5.2.

Our convention regarding constants is the following. Throughout, \( c, c', C, C', \ldots \) denote positive constants that may change from place to place. Numbered constants are defined the first time they appear and remain fixed thereafter. All constants may depend implicitly on the dimension \( d \). Their dependence on other parameters will be made explicit.

2 Preliminaries and capacity estimates for tubes

In this section, we gather several ingredients that will be used in the sequel. We first introduce some more notation and state a topological condition on paths yielding the existence of blocking interfaces, see Lemma 2.1 below. We proceed to recall certain aspects of potential theory for the random walk on \( \mathbb{Z}^d \) and supply suitably precise capacity estimates for ‘tubular’ sets, including ‘porous’ versions thereof, see Lemmas 2.2-2.5 below. Finally, we discuss important properties of the free field \( \varphi \), including a certain mid-point extension of \( \varphi \).

We consider \( \mathbb{Z}^d, d \geq 3 \), endowed with the usual nearest-neighbor graph structure. We write
$x \sim y$ if $x$ and $y$ are neighbors, i.e. if $x, y \in \mathbb{Z}^d$ and $|x - y| = 1$. We use $|\cdot|$ to denote the Euclidean and $|\cdot|_\infty$ the $\ell^\infty$-norm in $\mathbb{Z}^d$ as well as $d(\cdot, \cdot)$ and $d_\infty(\cdot, \cdot)$ to denote the corresponding distances between sets. Recall that $B_N(x)$ denotes the box of radius $N$ around $x$ with respect to $|\cdot|_\infty$, and let $B_N(U) := \bigcup_{x \in U} B_N(x)$ for $U \subset \mathbb{Z}^d$. For $U \subset \mathbb{Z}^d$, $\partial U := \{ x \in U : \exists y \notin U \text{ s.t. } y \sim x \}$ is the inner (vertex) boundary of $U$ and $U^c = \mathbb{Z}^d \setminus U$ is the complement of $U$ in $\mathbb{Z}^d$. We also define the outer boundary of a set $U \subset \mathbb{Z}^d$ as $\partial_{\text{out}} U = \partial(U^c)$. For $U, V \subset \mathbb{Z}^d$, we write $U \subset V$ to indicate that $U$ has finitely many elements. A path $\gamma$ in $\mathbb{Z}^d$ is a map $\gamma : \{0, \ldots, k\} \to \mathbb{Z}^d$ for some integer $k \geq 0$ such that $|\gamma(i+1) - \gamma(i)| = 1$ for all $0 \leq i < k$. A $*$-path is defined similarly, with $|\cdot|_\infty$ replacing $|\cdot|$. A ($*$-)connected set $U \subset \mathbb{Z}^d$ is a set such that any points $x, y \in U$ can be joined by a ($*$-)path whose range is contained in $U$. Throughout, we use the words connected component and cluster interchangeably to refer to maximal connected sets.

We now state a useful criterion for the existence of 'dual' surfaces separating two sets, which is interesting in its own right. In the sequel for any $U \subset \subset \mathbb{Z}^d$, let $U^c_\infty$ denote the (unique) connected component of $U^c$ having infinite cardinality, and define $\partial_{\text{ext}} U = \partial(U^c_\infty)$, the exterior boundary of $U$. For any two finite sets $U_1, U_2 \subset \mathbb{Z}^d$, we say $U_1$ is surrounded by $U_2$, denoted as $U_1 \lessdot U_2$, if $U_1$ is contained in some finite connected component of $\mathbb{Z}^d \setminus U_2$. Notice that the relation '$\lessdot$' is in fact a partial order.

**Lemma 2.1** (Existence of blocking interfaces). Let $V \subset \mathbb{Z}^d$ be a box and $U \subset V$. Also let $\Sigma \subset V \setminus U$ be such that any $*$-path between $U$ and $\partial V$ intersects $\Sigma$ in at least $k \geq 1$ points. Then there exist $*$-connected subsets $O_1, \ldots, O_k$ of $\Sigma$ such that $U \lessdot O_1 \lessdot \ldots \lessdot O_k$.

*Proof of Lemma 2.1.* An obvious consequence of the hypothesis of the lemma is that $U$ is not $*$-connected to $\partial V$ in $V \setminus \Sigma \supset U$. It then follows e.g. by [11, Lemma 2.1] that the exterior boundary of the $*$-connected component $C^*_U$ of $U$ in $V \setminus \Sigma$ is itself $*$-connected, which we pick as $O_1$. Notice that $U \lessdot O_1$ and $O_1 \subset \Sigma$ by definition. Now observe that the hypothesis of the lemma still holds with $k - 1$, $C^*_U \cup O_1$ – which is a $*$-connected set – and $\Sigma \setminus (C^*_U \cup O_1)$ substituting for $k$, $U$ and $\Sigma$ respectively. Thus, by iterating the same argument $k$ times we deduce the lemma. □

We now review various aspects of potential theory on $\mathbb{Z}^d$ which will be used in the sequel. We denote by $P_x$ the canonical law of the discrete-time (symmetric) simple random walk on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$. We write $(X_n)_{n \geq 0}$ for the corresponding canonical process and $(\theta_n)_{n \geq 0}$ for the canonical time shifts. For $U \subset \mathbb{Z}^d$, we introduce the following stopping times: the entrance time $H_U := \inf\{n \geq 0 : X_n \in \bar{U}\}$ in $U$, the exit time $T_U := H_{\mathbb{Z}^d \setminus U}$ from $U$ and the hitting time $\tilde{H}_U := \inf\{n \geq 1 : X_n \in U\}$ of $U$. We write

\begin{equation}
(2.1) \quad g_U(x, y) := \sum_{n \geq 0} P_x[X_n = y, n < T_U], \quad \text{for } x, y \in \mathbb{Z}^d
\end{equation}

for the Green function of the walk killed outside $U$. By [28, Theorem 1.5.4], with $g = g_{\mathbb{Z}^d}$, one has the asymptotic formula

\begin{equation}
(2.2) \quad g(x) := g(0, x) \sim c_2|x|^{2-d}, \quad \text{as } |x| \to \infty,
\end{equation}

(\text{where } \sim \text{ means that the ratio of both sides tends to } 1 \text{ in the given limit}), for an explicit constant $c_2 = c_2(d) \in (0, \infty)$ with $c_2(3) = \frac{\alpha}{2\pi}$. For $K \subset \subset U \subset \mathbb{Z}^d$, we introduce the equilibrium measure of $K$ relative to $U$,

\begin{equation}
(2.3) \quad e_{K,U}(x) := P_x[\tilde{H}_K > T_U]1_{x \in \partial K}
\end{equation}
and its total mass

\[(2.4) \quad \text{cap}_U(K) := \sum_x e_{K,U}(x),\]

the capacity of \(K\) (relative to \(U\)). We will omit \(U\) from all notation whenever \(U = \mathbb{Z}^d\). One has the last-exit decomposition, see, e.g. [28, Lemma 2.1.1] for a proof, valid for all \(K \subset \subset U \subset \mathbb{Z}^d\),

\[(2.5) \quad P_x[H_K < T_U] = \sum_y g_U(x,y)e_{K,U}(y), \text{ for all } x \in \mathbb{Z}^d.\]

Summing (2.5) over \(x \in K\), one immediately sees that

\[(2.6) \quad |K| \max_{x \in K} \sum_{y \in K} g_U(x,y) \leq \text{cap}_U(K) \leq \min_{x \in K} |K| \sum_{y \in K} g_U(x,y).\]

One also has the following sweeping identity (see for instance (1.12) of [42] when \(U = \mathbb{Z}^d\)):

\[(2.7) \quad e_{K,U}(y) = P_{e_{K',U}}[H_K < T_U, X_{H_K} = y], \text{ for every } K \subset K' \subset \subset U \text{ and } y \in \mathbb{Z}^d,\]

where \(P_\mu := \sum_x \mu(x)P_x\) for any measure \(\mu\) on \(\mathbb{Z}^d\). Summing over \(y\) in (2.7) gives

\[(2.8) \quad \text{cap}_U(K) = \text{cap}_U(K')P_{e_{K',U}}[H_K < T_U],\]

where \(e_{K,U}(\cdot) = e_{K',U}(\cdot)/\text{cap}_U(K')\) is the normalized equilibrium measure. In particular, it follows immediately from (2.8) that \(\text{cap}_U(K)\) is increasing in \(K\). Note also that \(\text{cap}_U(K)\) is decreasing in \(U\) for fixed \(K\). We will also use the following variational characterization of the capacity, see e.g. [42, (1.10)] and [41, Prop. 1.9] for the proof of a similar statement: for \(K \subset \subset U \subset \mathbb{Z}^d\),

\[(2.9) \quad \text{cap}_U(K) = \frac{1}{\inf_\nu E_U(\nu)}, \text{ where } E_U(\nu) = \sum_{x,y} \nu(x)g_U(x,y)\nu(y)\]

and the infimum runs over all probability measures supported on \(K\).

We now give precise bounds on the capacity of certain sets of interest. The capacity of a ball classically satisfies

\[(2.10) \quad cN^{d-2} \leq \text{cap}(B_N) \leq CN^{d-2}, \text{ for all } N \geq 0,\]

see, e.g., [28, (2.16)]. We are typically going to work in certain (cylindrical) ‘tube domains’, which we introduce now. Given \(L \leq N\), the tube of length \(N\) and width \(L\), which we denote by \(T_N(L)\), is defined as the \(L\)-neighborhood of the \(N\)-line segment \([0,N] \times \{0\}^{d-1}\). Formally,

\[(2.11) \quad T_N(L) := ([-L,N + L] \cap \mathbb{Z}) \times ([-L,L] \cap \mathbb{Z})^{d-1}.\]

We abbreviate \(T_N(0) = T_N\), which is a line of length \(N\), and routinely omit the intersection with \(\mathbb{Z}\) from our notation below. We now derive certain capacity estimates for tube domains which will be useful in the sequel. We start with the line.
Lemma 2.2 (Capacity of lines). For $d = 3$, one has

\begin{equation}
\text{cap}(T_N) \sim \frac{\pi}{3} \frac{N}{\log N}, \quad \text{as } N \to \infty,
\end{equation}

whereas for $d \geq 4$, there exists $c_3(d) \in (0, 1)$ such that for all $N \geq 1$,

\begin{equation}
c_3 N \leq \text{cap}(T_N) \leq c_3^{-1} N.
\end{equation}

Proof. Using (2.2) with the precise value of $c_2(3)$, we obtain

\[ \sum_{y \in T_N} g(x, y) \leq 2 \sum_{y \in x + T_N} g(x, y) \sim \frac{3}{\pi} \log N, \text{ for all } x \in T_N. \]

Substituting this into (2.6) with the choice $K = T_N$ and $U = \mathbb{Z}^d$ yields the asserted lower bound in (2.12). By a similar argument, using (2.6) and noting that the Green function is summable along one-dimensional sets when $d \geq 4$, one obtains both upper and lower bound in (2.13).

It remains to show the upper bound in (2.12). For $\delta \in (0, 1)$, letting $T_N^\delta := T_N \setminus ([0, N^{1-\delta}] \cup [N - N^{1-\delta}, N]) \times \{0\}$,

\begin{equation}
\text{cap}(T_N) \leq 2(1 + N^{1-\delta}) + \inf_{y \in T_N^\delta} \sum_{x \in T_N} g(x, y).
\end{equation}

To take care of the sum over $T_N^\delta$ on the right-hand side, one sums (2.5) for $K = T_N$ and $U = \mathbb{Z}^d$ over $x \in T_N$ and foregoes the terms with $y \in T_N \setminus T_N^\delta$. Together with (2.14) this yields

\begin{equation}
\text{cap}(T_N) \leq 2(1 + N^{1-\delta}) + \frac{N + 1}{\inf_{y \in T_N^\delta} \sum_{x \in T_N} g(x, y)}.
\end{equation}

Now by definition of $T_N^\delta$ and using (2.2) again we obtain for any $x \in T_N^\delta$,

\[ \sum_{y \in T_N} g(x, y) \geq \sum_{y \in x + T_N^{1-\delta}} g(x, y) \sim (1 - \delta) \frac{3}{\pi} \log N. \]

Plugging this into (2.15), we get that $\limsup_{N \to \infty} \frac{\text{cap}(T_N)}{\frac{3}{\pi} \log N} \leq \frac{1}{1 - \delta}$, whereupon the upper bound in (2.12) follows by taking $\delta \to 0$. \qed

Remark 2.3 (Rotational invariance of asymptotic capacity for lines). Let $u \in \mathbb{R}^3$ with $|u| = 1$ be any unit vector. Then the asymptotic expression in (2.12) remains valid if one replaces $T_N$ by the line segment joining 0 and $Nu$ discretized in the following manner. For any $x \in \mathbb{R}^3$, let $[x]$ denote a point in $\mathbb{Z}^3$ achieving the minimum distance between $x$ and $\mathbb{Z}^3$. Now let $T_{N,u} \subset \mathbb{Z}^3$ consist of the points $[j\sqrt{3}u]$ for all integers $j$ between 0 and $\lceil N/\sqrt{3} \rceil$. Notice that it is always possible to choose the points in such a way that they are distinct. By this construction and the triangle inequality we have, for any $x, y \in \mathbb{R}^3$ such that $[x], [y] \in T_{N,u}$,

\[ |x - y| - \sqrt{3} \leq |[x] - [y]| \leq |x - y| + \sqrt{3}. \]
and consequently $g(x, y) \sim g(x, y) - |x - y| \to \infty$. The asymptotics on the right-hand side of (2.12) now follow for $\text{cap}(T_{N, u})$ by the exact same arguments as in the proof of Lemma 2.2. Indeed, the additional $1/\sqrt{3}$ factor appearing in the numerator in (2.6) owing to reduced cardinality compared to $T_N$ gets canceled by the $1/\sqrt{3}$ factor appearing in the denominator because of the increased separation between successive points in $T_{N, u}$. In fact, the asymptotics (2.12) should hold for any ‘reasonable’ discretisation of the line segment between 0 and $Nu$.

We will need the following upper bound on the escape probability from a sufficiently dense subset of the line in order to derive capacity estimates for thicker tube regions. We will also use this result in Section 4 while proving Lemma (4.6), which will involve porous versions of these sets (i.e. containing holes).

**Lemma 2.4** (Visibility of (porous) lines, $d = 3$). For all $N \geq 1$, $T \subset T_N$ and $x \in \mathbb{Z}^3$ such that $d_\infty(x, T) < N/10^3$, the following holds. If, for some $\gamma > 0$,

\[
|B_r(x) \cap T| \geq \gamma |B_r(x) \cap T_N|, \quad \text{for all } r \text{ satisfying } 2d_\infty(x, T) \leq r < N,
\]

then for some $c(\gamma) > 0$ and $C \in (1, \infty)$, one has

\[
P_x[H_T = \infty] \leq C \left[\frac{\log(1 + d_\infty(x, T))}{\log N}\right]^{c(\gamma)}.
\]

**Proof.** Let $k_0, k_1$ be two integers with $k_0$ smallest so that $2d_\infty(x, T) \leq 20^{k_0}$ and $k_1$ largest such that $20^{k_1} \leq N$. Notice that $k_0 = k_0(x)$ and that $k_1 \geq k_0$ when $N \geq C$, which we may assume. Consider the boxes $U_k := B_{20^k}(x)$, for $0 \leq k \leq k_1$. By (2.16), one knows that

\[
|U_k \cap T| \geq \gamma |B_{20^k}(x) \cap T_N| \geq c(\gamma)20^k, \quad \text{for all } 0 \leq k \leq k_1.
\]

Since

\[
\sum_{z'} g(z, z') \leq \sum_{z' \in U_k \cap T_N} g(z, z') \leq Ck
\]

uniformly in $z \in U_k \cap T$ by (2.2), it follows from (2.6) and (2.18) that $\text{cap}(U_k \cap T) \geq c'(\gamma)20^k/k$ for all $0 \leq k \leq k_1$. Therefore, fixing $L \geq 1$ such that

\[
2^{-1} \leq c_2^{-1}(g(x)|x|) \leq 2, \quad \text{if } |x|_\infty \geq L
\]

it follows that for all $x$ such that $d_\infty(x, T) \geq L$, all $k_0(= k_0(x)) \leq k < k_1$ and $y = \partial_{\text{out}}U_k,

\[
P_y[H_{U_k \cap T} < T_{U_{k+1}}] \geq \inf_{z, z' \in U_k} g_{U_{k+1}}(z, z') \text{cap}(U_k \cap T) \geq 20^{-k} c'20^k/k \geq \frac{c}{k},
\]

for suitable $c, c'$ depending on $\gamma$. In obtaining (2.20), we also used the fact that $g_{U_{k+1}}(z, z') \geq cg(z, z')$ for $z, z' \in U_k$. Indeed, by the Markov property we have $g(z, z') = g_{U_{k+1}}(z, z') + E_z[g(X_{U_{k+1}} \cap T)]$, and on the other hand, the definition of $(U_k)_{k_0 \leq k \leq k_1}$ readily implies that $|y - z'| \geq 5(|z - z'| \vee L)$, for all $y \in \partial_{\text{out}}U_{k+1}$, which together with (2.19) gives $E_z[g(X_{U_{k+1}} \cap T)] \leq \frac{4}{5}g(z, z')$.

Now consider the process $\{Z_k : k \geq 0\}$ on $\mathbb{N} \cup \{\Delta\}$ defined by $Z_0 = 0$ and for $k \geq 1$, conditionally on $Z_0, \ldots, Z_{k-1}$,

\[
Z_k := \begin{cases} 
        k, & \text{if } Z_{k-1} \neq \Delta \text{ and } H_{U_{k-1} \cap T} \circ \theta_{U_{k-1}} > T_{U_k}, \\
        \Delta, & \text{otherwise.}
\end{cases}
\]
Using the strong Markov property we get that $Z$ is a Markov chain under $P_x$ and (2.20) implies that $P_x[Z_k \neq \Delta | Z_{k-1} \neq \Delta] \leq 1 - \frac{c(\gamma)}{k}$ for all $k_0 < k \leq k_1$. It follows that for all $x$ with $d_{\infty}(x, T) \geq L$,

$$P_x[H_T = \infty] \leq P_x[Z_k \neq \Delta, k_0 \leq k \leq k_1] \leq \prod_{k_0 < k \leq k_1} \left(1 - \frac{c(\gamma)}{k}\right) \leq \exp \left\{ -c(\gamma) \sum_{k_0 < k \leq k_1} \frac{1}{k} \right\} \leq \left(\frac{k_0}{k_1}\right)^{c'(\gamma)},$$

which yields (2.17) for such $x$, as $k_0 \leq C \log d_{\infty}(x, T)$ and $k_1 \geq c \log N$ (see above (2.18)) and we can assume $c'(\gamma) < 1$. To handle the case $d_{\infty}(x, T) \leq L$, the strong Markov property at the time of first exit from $T_N(L)$, see (2.11), with $L$ given by (2.19), implies that $P_x[H_T = \infty] \leq \sup_y P_y[H_T = \infty]$, with the supremum ranging over $y \in \partial_{\text{out}} T_N(L)$ and (2.18) still follows from (2.22) as $d_{\infty}(y, T) \geq L$. This completes the proof.

We now move on to the capacities of tubes whose width is a fractional power of their length. In the sequel, let $T_N^\delta = T_N(N^\delta)$ for $\delta > 0$.

**Lemma 2.5** (Capacity of tubes, $d = 3$). Let $T_N^\delta = T_N(N^\delta)$ for $\delta > 0$ (cf. (2.11)). There exist $C_1 \in [1, \infty)$ and $c_4 \in (0, 1)$ such that for every $\delta \in (0, \frac{1}{2})$ and $N \geq C(\delta)$,

$$\cap(T_N^\delta) \leq (1 + C_1 \delta^{c_4}) \cap(T_N),$$

$$\cap(T_N^\delta) \leq C_1 \delta^{-1} \cap(T_N).$$

**Proof.** We claim that for every $x \in \partial T_N^\delta$ and $y \in \partial_{\text{out}} T_N^\delta$, one has

$$P_x[H_{TN} = \infty] \leq C \delta^c \quad \text{and} \quad P_y[H_{TN} = \infty] \geq c \delta, \quad \text{for all } N \geq C(\delta).$$

Before proving (2.25) let us deduce the lemma from it. By the sweeping identity (2.7),

$$\cap(T_N) = P_{e_{T_N^\delta}}[H_{TN} < \infty] = \cap(T_N^\delta) P_{e_{T_N^\delta}}[H_{TN} < \infty],$$

and (2.23) follows directly from the upper bound in (2.25). Also, by decomposing on the first exit time of $T_N^\delta$, one finds that for all $x \in T_N^\delta$,

$$P_x[H_{TN} = \infty] = \sum_{y \in \partial_{\text{out}} T_N^\delta} P_x[H_{TN} > H(T_N^\delta), X_H(T_N^\delta) = y] P_y[H_{TN} = \infty],$$

which combined with the lower bound in (2.25) implies that $e_{T_N^\delta}(x) \geq c \delta e_{T_N^\delta}(x)$. Summing over $x \in T_N^\delta$ yields (2.24). We proceed to the proof of the bounds in (2.25).

The upper bound in (2.25). For all $\delta \in (0, 1/2)$ the hypotheses of Lemma 2.4 hold with $T = T_N$ and $\gamma = \frac{1}{10}$ for any $x \in \partial T_N^\delta$ and $N \geq C(\delta)$, whence the upper bound in (2.25) follows from (2.17).

The lower bound in (2.25). Below we will use $B_L^\prime$ to denote the two-dimensional box $[0] \times [-L, L]^2$. First of all, notice that for any $y = (y_1, y_2, y_3) \in \partial_{\text{out}} T_N^\delta$, either $(y_2, y_3) \in \partial B^\prime_{[N^{2\delta}]}$ or $y \in \{-[N^{2\delta}] - 1, N + [N^{2\delta}] + 1\} \times [-N^{2\delta}, N^{2\delta}]^2$. We deal with the former case first. To this end let us consider the projection $X^\prime = (X_n^\prime)_{n \geq 0}$ of $(X_n)_{n \geq 0}$ onto its last two coordinates,
which has the law of a (lazy) simple random walk in $\mathbb{Z}^2$. Let $H'_{U}$ denote the entrance time in $U$ for $X'$ and abbreviate $H'_n = H'_{B'_{N,\delta}}$ and $H'_{out} = H'_{\partial B'_{10N}}$. Applying Exercise 1.6.8 in [28], we get

\begin{equation}
(2.26) \quad P_y[H'_{out} < H'_{in}] \geq \frac{\log(N^{2\delta}) - \log(N\delta) - C}{\log(10N) - \log(N\delta) + C} \geq c\delta
\end{equation}

whenever $N \geq C(\delta)$ and $|y|' \geq N^{2\delta}$. Notice that the errors are of constant order as opposed to $O(N^{-\delta})$ as in [28] since we are working with $\ell^\infty$-balls instead of $\ell^2$-balls. Now since $\{(x_2, x_3) : x \in T_{N,\delta} \} \subset B'_{N,\delta}$, the inclusion

\begin{equation}
\{H'_{out} < H'_{in}, H'_{T_N^\delta} \circ \theta \rightarrow \text{infinite}\} \subset \{H_{T_N^\delta} = \text{infinite}\}
\end{equation}

holds and consequently, by the strong Markov property, we have

\begin{equation}
(2.27) \quad P_y[H_{T_N^\delta} = \text{infinite}] \geq P_y[H'_{out} < H'_{in}] \inf_{z:d_{\infty}(z,T_N) \geq 10N} P_z[H_{T_N^\delta} = \text{infinite}].
\end{equation}

However, by the last-exit decomposition [25] and [22],

\begin{equation}
P_z[H_{T_N^\delta} < \text{infinite}] \leq \text{cap}(T_N^\delta) \max_{z' \in T_N^\delta} g(z, z') \leq \frac{\text{cap}(T_N^\delta)}{N}, \text{ when } d_{\infty}(z,T_N) \geq 10N.
\end{equation}

In view of Lemma [22] and [2.23] which, let us recall, requires only the upper bound in [2.25], the right-hand side in the previous display is bounded by $\frac{\text{cap}(T_N^\delta)}{N}$. Plugging this and the bound [2.26] into [2.27], we deduce the lower bound in [2.26] in the case $(y_2, y_3) \in \partial B'_{N,2\delta}$.

To deal with the case when $y \in \{-1, N + 1\} \times [-N^{2\delta}, N^{2\delta}]$ let us assume without loss of generality that $y_1 = -1$. Notice that, by means of the strong Markov property and the previous case, it suffices to show that

\begin{equation}
(2.28) \quad P_y[H' < H_{T_N^\delta}] \geq c, \text{ where } H' := H'_{\partial B'_{N,2\delta}}.
\end{equation}

To this end, let $(X_n)_{n \geq 0}$ denote the projection of $(X_n)_{n \geq 0}$ onto its first coordinate and consider the event

\begin{equation}
G := \{H_{-1} < H_{-\delta} \} \text{ which has a constant positive probability under } P_y \text{ by the standard gambler’s ruin estimate}
\end{equation}

(one could even afford to replace $[\frac{1}{2}, N^{2\delta}]$ by $[\frac{1}{2}, N^{2\delta}]$ on the right-hand side). It follows from the definition of $T_N^\delta$ that $H'_{T_N^\delta} > H_{-1} < H_{-\delta} \Rightarrow H' < H_{\partial B'_{N,2\delta}}$ on $G$ and hence

\begin{equation}
(2.29) \quad G \cap \{H' \geq H'_{T_N^\delta}\} \subset \{H' < H_{T_N^\delta}\}.
\end{equation}

On the other hand, we have

\begin{equation}
G \cap \{H' < H'_{T_N^\delta}\} \subset \{X_{H'_{T_N^\delta}} \subset \partial B'_{N,2\delta}\}.
\end{equation}

An implication of the condition on the right hand side above is that $B'_{4N,2\delta}(X_{H'}) \subset T_N^\delta = \emptyset$, whereas $B'_{N,2\delta} \subset B'_{4N,2\delta}(X_{H'})$ for all $N \geq C(\delta)$. Denoting $H := H'_{\partial B'_{4N,2\delta}(X_{H'})} \circ \theta_1$, we therefore have

\begin{equation}
(2.30) \quad \{G \cap \{H' < H'_{T_N^\delta}\}\} \cap \{X_{H} < \partial B'_{4N,2\delta}(X_{H'})\} \subset \{H' < H_{T_N^\delta}\}.
\end{equation}
However, since all two-dimensional axial projections of $(X_n)_{n \geq 0}$ have the same law, we can
deduce via a union bound that $P_x[H_{\partial B_{4N^{2d}}(x)} \in \partial B_{4N^{2d}}(x)] \geq c$ uniformly for all $x \in \mathbb{Z}^d$.

Hence, in view of (2.30), we get, applying the strong Markov property at time $H^1$,

$$P_y[H^1 < H_{\partial N} \mid G, H^1 < H^1] \geq c,$$

with $y$ as in (2.28). Combined with (2.29) and the fact that $P_y[G] \geq c$, this yields (2.28) and consequently the lower bound in (2.25) in this case. 

We conclude this section by reviewing some important features of Gaussian free fields. For $U \subset \mathbb{Z}^d$, we write $\mathbb{P}_U$ for the law of the centered Gaussian process with covariance $g_U(\cdot, \cdot)$, with $g_U$ as given by (2.1) (in particular $\mathbb{P} = \mathbb{P}_{\mathbb{Z}^d}$ following our above convention). Notice that under $\mathbb{P}_U$, the field $\varphi$ is almost surely 0 on $\mathbb{Z}^d \setminus U$. For $U \subset \mathbb{Z}^d$, we further introduce the Gaussian fields (functions of $\varphi$)

$$\xi^U_x := E_x[\varphi_{X_{T_U}}] = \sum_y P_x[X_{T_U} = y] \varphi_y, \quad \psi^U_x := \varphi_x - \xi^U_x, \quad \text{for } x \in \mathbb{Z}^d.$$

The field $\xi^U$ will be referred to as the harmonic average of $\varphi$ in $U$ and $\psi^U$ as the local field in $U$. Plainly, $\xi^U_x = \varphi_x$ (and therefore $\psi^U_x = 0$) for all $x \in \mathbb{Z}^d \setminus U$. As in [39, Lemma 1.2], one observes that $\xi^U$ is independent of $\psi^U$ and that $(\psi^U_x)_{x \in \mathbb{Z}^d}$ has law $\mathbb{P}_U$ under $\mathbb{P}$.

It will be convenient at times to consider a certain extension of the above setup. Let $\mathbb{M}^d$ denote the set of mid-points of the edges of $\mathbb{Z}^d$. We regard $\hat{\mathbb{Z}}^d := \mathbb{Z}^d \cup \mathbb{M}^d$ as the graph obtained from $\mathbb{Z}^d$ by splitting every edge of $\mathbb{Z}^d$ into two (and adding the corresponding mid-point to the vertex set). Let $\hat{X} = (\hat{X}_n)_{n \geq 0}$ be the discrete-time random walk on $\hat{\mathbb{Z}}^d$, which at each step jumps with uniform probability to one of its neighboring vertices in $\hat{\mathbb{Z}}^d$. Let $\hat{P}_x$ denote the canonical law of $\hat{X}$ with starting point $\hat{X}_0 = \hat{x} \in \hat{\mathbb{Z}}^d$. By suitable extension of $\mathbb{P}$, one defines a centered Gaussian field $\hat{\varphi} = (\hat{\varphi}_{\hat{x}})_{\hat{x} \in \hat{\mathbb{Z}}^d}$ such that

$$\mathbb{E}[\hat{\varphi}_{\hat{x}} \hat{\varphi}_{\hat{y}}] := \frac{1}{2} \sum_{n \geq 0} \hat{P}_{\hat{x}}[\hat{X}_n = \hat{y}], \quad \text{for } \hat{x}, \hat{y} \in \hat{\mathbb{Z}}^d.$$

Indeed, it follows from (2.32) and (2.1) that $\mathbb{E}[\hat{\varphi}_{\hat{x}} \hat{\varphi}_{\hat{y}}] = g(x, y)$ whenever $x, y \in \mathbb{Z}^d$, whence

$$\hat{\varphi}|_{\mathbb{Z}^d} = \varphi.$$

The decomposition (2.31) also extends and one obtains that

$$\hat{\varphi} = \hat{\xi}^V + \hat{\psi}^V, \quad \text{for } V \subset \hat{\mathbb{Z}}^d,$$

where $\hat{\xi}^V := E_{\hat{x}}[\hat{\varphi}_{\hat{X}_{T_V}}]$; here $\hat{\xi}^V$ and $\hat{\psi}^V$ are independent Gaussian fields and

$$\mathbb{E}[\hat{\psi}_{\hat{x}}^V \hat{\psi}_{\hat{y}}^V] = \frac{1}{2} \sum_{n \geq 0} \hat{P}_{\hat{x}}[\hat{X}_n = \hat{y}, n < T_V], \quad \text{for } \hat{x}, \hat{y} \in \hat{\mathbb{Z}}^d,$$

where with hopefully obvious notation, $T_V = T_V(\hat{X})$ denotes the exit time of $\hat{X}$ from $V$. The analogue of the restriction property (2.33) for the harmonic extension is then the following. For
Define \( \tilde{U} := U \cup \{ \tilde{x} \in \tilde{Z}^d : \exists x \in U \text{ s.t. } |x - \tilde{x}| = \frac{1}{2} \} \), noting that \( X_{T_U} \) under \( P_x \) has the same law as \( \tilde{X}_{T_{\tilde{U}}} \) under \( \tilde{P}_x \) for any \( x \in \tilde{Z}^d \) and using (2.33), one sees that

\[
(2.36) \quad \tilde{\xi}_{\tilde{U}}|_{\tilde{Z}^d} = \xi_U.
\]

We conclude with a particular instance of (2.34), which will prove useful on several occasions. Let

\[
(2.37) \quad \tilde{\varphi} = \tilde{\xi} + \tilde{\psi}
\]

be the decomposition (2.34) corresponding to the choice \( V := \tilde{Z}^d(\subset \tilde{Z}^d) \), i.e. \( \tilde{\xi}_x = \tilde{\xi}_{\tilde{z}^d} = \frac{1}{2d} \sum_{m \in \mathbb{Z}^d, m \sim x} \tilde{\varphi}_m \) if \( x \in \mathbb{Z}^d \) (where \( \sim \) refers to neighbors in \( \tilde{Z}^d \) ), \( \tilde{\xi}_m = \tilde{\xi}_{\tilde{z}^d} = \tilde{\varphi}_m \) if \( m \in \mathbb{M}^d \), and by (2.35),

\[
(2.38) \quad (\tilde{\psi}_x)_{x \in \mathbb{Z}^d} \text{ is a field of i.i.d. centered Gaussian variables with variance } 1/2 \text{ each.}
\]

### 3 Lower bounds

The main result of this section is the following

**Theorem 3.1 (Lower bounds).** The following holds:

i) If \( d = 3 \), then

\[
(3.1) \quad \liminf_{N \to \infty} \frac{1}{N} \log P[0 < \varphi_{\geq h} \to \partial B_N] \geq -\frac{\pi}{6} (h - h_*)^2,
\]

ii) If \( d \geq 4 \), then for all \( h \in \mathbb{R} \),

\[
(3.3) \quad \liminf_{N \to \infty} \frac{1}{N} \log P[0 < \varphi_{\geq h} \to \partial B_N, 0 \not< h \to \infty] > -\infty.
\]

Moreover, the bounds (3.2) and (3.3) also hold for the events \( \text{LocUniq}(N,h)^c \) and \( 2\text{-arms}(N,h) \) (see (1.5) and (1.9) in place of \( \{0 \not< h \to \partial B_N, 0 \not< h \to \infty\} \)).

#### 3.1. General entropic lower bound.

The following lemma will be used in the course of proving Theorem 3.1, but is of independent interest. The lower bound it asserts in (3.6) will follow by a change of measure argument, see e.g. the proof of Theorem 2.1 in [42], or Lemma 2.3 in [2], for results of a similar flavor. Given an event \( A \in \mathcal{B}(\mathbb{R}^K), K \subset \mathbb{Z}^d \) and a height parameter \( h \in \mathbb{R} \), we define

\[
(3.4) \quad A^h = A^h(\varphi) = \{\varphi|_{\tilde{K}} - h \in A\},
\]

where, with hopefully obvious notation \( \varphi|_{\tilde{K}} - h \) refers to the field (restricted to \( K \) ) shifted by \( -h \) coordinatewise.
Lemma 3.2 (Entropic lower bound). Let $K_N \subset U_N \subset \mathbb{Z}^d$ be subsets with $\text{cap}_{U_N}(K_N) \to \infty$. Let $A_N \in \mathcal{B}(\mathbb{R}^{K_N})$ and $I \subset \mathbb{R}$ be an interval such that, for every $h' \in I$,

\begin{equation}
\mathbb{P}_{U_N}[A^h_N] \to 1.
\end{equation}

Then for every $h \notin I$,

\begin{equation}
\liminf_{N \to \infty} \frac{1}{\text{cap}_{U_N}(K_N)} \log \mathbb{P}_{U_N}[A^h_N] \geq \frac{1}{2}d(h, I)^2.
\end{equation}

Proof. Recall the following fact, which is a consequence of Jensen’s inequality, see e.g. the discussion following (2.7) in [2] for a proof. Given two probability measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$ such that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, and an event $A$ with positive $\mathbb{P}$-probability, one has

\begin{equation}
\mathbb{P}[A] \geq \tilde{\mathbb{P}}[A]e^{-\frac{1}{2}H(\tilde{\mathbb{P}}|\mathbb{P})}.
\end{equation}

where $H(\tilde{\mathbb{P}}|\mathbb{P}) := \mathbb{E}_\mathbb{P} \left[ \log \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right]$ is the relative entropy of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Abbreviate $K = K_N$ and $U = U_N$ in the sequel. Pick $h \notin I$ and $h' \in I$. Using (2.5), it follows by an application of the Cameron-Martin theorem, see e.g. [27, Theorem 14.1] with the choice (in the notation of [27]) $\xi := (h - h')(e_{K,U}, \varphi) \in L^2(\mathbb{P})$, that $\tilde{\mathbb{P}}_U$ defined by

\begin{equation}
\frac{d\tilde{\mathbb{P}}_U}{d\mathbb{P}_U} = \exp \left\{ \frac{(h - h')(e_{K,U}, \varphi) - \frac{(h - h')^2}{2} \text{cap}(K)}{2} \right\}
\end{equation}

is a probability such that $\varphi$ has the same law under $\tilde{\mathbb{P}}_U$ as $\varphi + f$ under $\mathbb{P}_U$, where

\begin{equation}
f(x) = (h - h')P_x[H_K < T_U], \quad x \in \mathbb{Z}^d.
\end{equation}

(Indeed observe to this effect that in (14.3) of [27], one obtains $\rho_\xi(\varphi.) = \varphi + \mathbb{E}_U[\xi\varphi.] = \varphi + f$, by (2.5)). In particular, $f = h - h'$ on $K$, whence $\tilde{\mathbb{P}}_U[A^h(\varphi)] = \mathbb{P}_U[A^h(\varphi + f)] = \mathbb{P}_U[A^{h'}(\varphi)]$ which tends to 1 as $N \to \infty$ by (3.5). Moreover, by (3.8) and (3.9), noting that $\tilde{\mathbb{E}}_U[\langle e_{K,U}, \varphi \rangle] = \langle e_{K,U}, f \rangle$, one sees that

\begin{equation}
H(\tilde{\mathbb{P}}_U|\mathbb{P}_U) = \langle h - h' \rangle \mathbb{E}_{\mathbb{P}_U}[\langle e_{K,U}, \varphi \rangle] - \frac{(h - h')^2}{2} \text{cap}(K) = \frac{(h - h')^2}{2} \text{cap}(K).
\end{equation}

Applying (3.7), taking logarithms and letting $N \to \infty$ now readily yields (3.6), since $h' \in I$ was arbitrary.

3.2. Lower bounds for $d = 3$. In this section, we show the lower bounds (3.1) and (3.2), which will both follow from an application of Lemma 3.2 with carefully chosen events $A^h$ in (3.4) as to implement sufficiently cost-effective strategies for connection. The asserted bound (3.2) bears the additional difficulty that the event in question is not monotone, which makes its proof more involved than (3.1).

Proof of (3.1) and (3.2). We begin with the proof of (3.1). Recalling the notation from (2.11), define the thin cylinder $K_N := T_N(L)$, with

\begin{equation}
L := \lfloor N^{\delta} \rfloor, \quad \delta \in (0, \frac{1}{2})
\end{equation}
and let $F_N^- = \{0\} \times [-L, L]^{d-1}$, $F_N^+ = \{N\} \times [-L, L]^{d-1}$. Note that $F_N^\pm \subset K_N$ and that $0 \not\in F_N^-$, while $F_N^+ \subset \partial B_N$. For $h \in \mathbb{R}$, consider the event

$$A_h^N := A_{h,L}^N := \{F_N^h \text{ and } F_N^+\}$$

are connected by a path in $\{\varphi \geq h\} \cap K_N$, which is of the form (3.4). Let us first check that

$$\lim_{N \to \infty} \mathbb{P}[A_h^N] = 1 \text{ for all } h' < h_*.$$  

In view of (1.5), one readily sees that

$$\liminf_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[A_h^N] \geq -\frac{\pi}{6} (h-h_*)^2 (1 + C_1 \delta^{c_1}),$$

using that $\text{cap}(K_N) \approx (1 + C_1 \delta^{c_1}) \frac{N}{\log N}$ as $N \to \infty$, as follows directly from (2.18) and (2.12).

Next, by the FKG-inequality for $\varphi$, one knows that $\mathbb{P}[F_N^- \subset \{\varphi \geq h\}] \geq \exp \{-C(h)L^{d-1}\}$.

In particular, in view of (3.10), it follows that

$$\lim_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[F_N^- \subset \{\varphi \geq h\}] = 0, \text{ for all } h \in \mathbb{R}.$$  

Recalling $A_h^N$ from (3.11) and the fact that $0 \not\in F_N^-$, one observes that $A_h^N \cap \{F_N^- \subset \{\varphi \geq h\}\}$ implies $\{0 \overset{\varphi \geq h}{\leftrightarrow} \partial B_N\}$. Hence, (3.1) follows directly from (3.13), (3.14) and the FKG-inequality upon letting $\delta \to 0$.

We now show (3.2). For arbitrary $\delta \in \left(0, \frac{1}{N}\right)$, let $D_N \subset U_N \subset K_N$ be defined as

$$D_N := \text{int}(N\delta), \quad U_N := \text{int}(N^{2\delta}), \quad K_N := \text{int}(N^{3\delta}) \quad \text{(see (2.11) for notation)} \text{ and abbreviate } \xi^N = \xi^{U_N}, \psi^N = \psi^{U_N}, \text{ cf. (2.31).} \quad \text{For an additional parameter } \epsilon > 0 \text{ and } h < h_* + \epsilon, \text{ consider the event}$$

$$C_h^N := \{\partial_{\text{out}} U_N \overset{\varphi \geq h}{\leftrightarrow} \partial K_N \} \cap \{\inf_{D_N} \xi^N \geq -(h_* + \epsilon - h)\},$$

and note that $C_h^N$ is of the form (3.4) with $K = K_N$; observe to this effect that due to (2.31), the condition $\xi^N \geq -(h_* + \epsilon - h)$ can be recast as $\sum_{y \in U_N} P_x[X_{U_N} = y](\varphi_y - h) \geq -(h_* + \epsilon)$, i.e. as a condition on the field $\varphi - h$ restricted to $U_N (\subset K_N)$.

We now argue that $C_h^N$ is typical as $N \to \infty$ for every $h' \in (h_*, h_* + \epsilon)$. For such $h'$, the probability of $\{\partial_{\text{out}} U_N \overset{\varphi \geq h'}{\leftrightarrow} \partial K_N\}$ vanishes as $N \to \infty$ by (1.3) and a union bound. Next, for $x \in D_N$ one has

$$\mathbb{E}[(\xi^N)^2] = \sum_{y,z} P_x[X_{U_N} = y] P_{x(z)} = z] g(y,z) = \sum_y P_x[X_{U_N} = y] E_x[g(X_{U_N},y)]$$

$$= \sum_y \mathbb{P}_x[X_{U_N} = y] g(x,y) \leq C \text{ dist}(x,U_N)^{-d-2} \leq C N^{-2\delta(d-2)},$$

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where the equality in the second line is obtained by applying the strong Markov property at time $T_{U_N}$, noting that $x \in U_N$ whereas $y \notin U_N$. It then follows from a union bound over $D_N$ and a standard Gaussian tail estimate that $\mathbb{P} [ \inf_{D_N} \xi_N \geq -(h_* + \varepsilon - h')] \to 1$ as $N \to \infty$ for all $h' \in (h_*, h_* + \varepsilon)$ (alternatively one could also use Lemma 4.1 below to deduce this). All in all, in view of (3.15), one obtains

$$\mathbb{P} [C_N^h] \to 1$$

as $N \to \infty$ for every $h' \in (h_*, h_* + \varepsilon)$.

Using (3.16) and applying Lemma 3.2, one infers that for every $h \leq h_*$ (and all $\delta, \varepsilon > 0$, implicit in the definition of $C_N^h$),

$$\liminf_{N \to \infty} \frac{1}{\text{cap}(K_N)} \log \mathbb{P} [C_N^h] \geq -\frac{1}{2}(h_* - h)^2.$$  

Next, recall the event $A_N^h := \tilde{A}_{N,h}^h$ from (3.11). In words, $A_N^h$ is the event that the cylinder $D_N$ contains a crossing in $\{ \varphi \geq h \}$ intersecting both ‘slices’ $F_{N,h}^\pm$ as defined below (3.10), with $L = [N^4]$. Combining the occurrence of $A_N^h$ and the insulating property of the disconnection event in (3.15), one deduces that

$$A_N^h \cap C_N^h \subset \bigcup_{x \in F_N^-} \{ x \leftrightarrow^{\varphi \geq h} \partial B_N(x), x \leftrightarrow^{\varphi \geq h} \infty \}$$

for all $h \in \mathbb{R}$, and hence by a union bound and translation invariance, that

$$\mathbb{P} [0 \leftrightarrow^{\varphi \geq h} \partial B_N, 0 \leftrightarrow^{\varphi \geq h} \infty] \geq \frac{1}{|F_N^-|} \mathbb{P} [A_N^h \cap C_N^h].$$

Since the event $C_N^h$ in (3.15) is $\mathcal{F}_{U_N} = \sigma(\varphi_x; x \in U_N^c)$-measurable, applying the decomposition (2.31), one sees that for all $h \leq h_*$,

$$\mathbb{P} [A_N^h \cap C_N^h] = \mathbb{E} [\mathbb{P} [A_N^h | \mathcal{F}_{U_N}^h] 1_{C_N^h}] = \mathbb{E} [\mathbb{P}_{U_N} [A_N^h (\cdot + \xi_N)] 1_{C_N^h}] \geq \mathbb{P}_{U_N} [A_N^{h_0+\varepsilon}] \mathbb{P} [C_N^h],$$

where in the last step, we used monotonicity of the event $A_N^h (\cdot)$ along with the control on $\xi_N$ supplied by (3.15). A meaningful lower bound on $\mathbb{P}_{U_N} [A_N^{h_0+\varepsilon}]$ is obtained as follows. First recall that $\mathbb{P} [A_N^h] \to 1$ for all $h < h_*$ by (3.12). Then one writes, for all $h < h_*$, with $\varepsilon_0 = \frac{1}{2} (h_* - h)$,

$$\mathbb{P}_{U_N} [A_N^h] = \mathbb{P} [A_N^h (\psi_N)] \geq \mathbb{P} [A_N^h (\psi_N), \sup_{D_N} \xi_N \leq \varepsilon_0] \geq \mathbb{P} [A_N^{h_0+\varepsilon} (\varphi)] - \mathbb{P} [\sup_{D_N} \xi_N > \varepsilon_0].$$

By (3.12) and since $h + \varepsilon_0 = \frac{1}{2} (h_* + h_*) < h_*$, the first term on the right-hand side tends to 1 as $N \to \infty$. By similar considerations as above (3.16), one sees that $\mathbb{P} [\sup_{D_N} \xi_N > \varepsilon_0] \to 0$, yielding that $\mathbb{P}_{U_N} [A_N^h] \to 1$ as $N \to \infty$ for every $h < h_*$. Thus, Lemma 3.2 applies and gives, for all $\delta, \varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{\text{cap}_{U_N} (D_N)} \log \mathbb{P}_{U_N} [A_N^{h_0+\varepsilon}] \geq -\frac{1}{2} \varepsilon^2$$

(note also that $\text{cap}_{U_N} (D_N) \geq \text{cap} (D_N) \to \infty$, as required for Lemma 3.2 to apply). Finally, we put all these estimates together to arrive at (3.2): to see this, first plug (3.20)
and (3.17) into (3.19) to get, for all \( h \leq h_* \) and \( \varepsilon, \delta > 0 \), with \( \rho_1 := \liminf_N \frac{\cap(K_N)}{\cap(T_N)} \) and 
\( \rho_2 := \liminf_N \frac{\cap(U_N(D_N))}{\cap(T_N)} \),

\[
\liminf_{N \to \infty} \frac{1}{\cap(T_N)} \log \mathbb{P}[A_N^h \cap C_N^h] \geq -\frac{1}{2} ((h_* - h)^2 \rho_1 + \varepsilon^2 \rho_2).
\]

Now using (2.23) and (2.24) to bound \( \rho_1 \) and \( \rho_2 \) respectively, and subsequently applying (3.18) while using the fact that \(|F_{N^-}| \leq CN^{\delta(d-1)}\), one obtains for all \( h \leq h_* \) and \( \varepsilon, \delta > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{\cap(T_N)} \log \mathbb{P}[0 \not\leftrightarrow h \partial B_N, 0 \not\leftrightarrow \infty] \geq \frac{1}{2} ((h_* - h)^2 (1 + C_1 \delta^2) + C_1 \delta^{-1} \varepsilon^2).
\]

The result (3.2) now follows from (3.21) by first letting \( \varepsilon \to 0 \) and then \( \delta \to 0 \) (recall (2.12)).

It remains to argue that (3.2) continues to hold for the event \( 2\text{-arms}(N,h) \). Then the claim automatically follows for \( \text{LocUniq}(N,h)^c \) as the former is contained in the latter (recall their definitions in (1.9) and (1.5)). To this end, let \( H^- = \{ x \in B_{2N} : x^1 \leq -N \} \) and

\[
A_{N,\text{left}}^h = \{ B_N \not\leftrightarrow^{(h \geq h)} \partial B_{2N} \}, \quad A_{N,\text{right}}^h = (C_N^h \cap A_N^h) \circ \tau_{Ne_1}
\]

with \( A_N^h \) and \( C_N^h \) given by (3.11) and (3.15) respectively. Note that the latter implicitly depends on \( \delta \), cf. (3.10), (3.11) and above (3.15). Since \( A_{N,\text{left}}^h \supset A_N^h \circ \tau_{-2Ne_1} \), combining (3.12), Lemma 3.2 and translation invariance of \( \mathbb{P} \), one deduces in particular that

\[
\liminf_{N \to \infty} \frac{1}{\cap(T_N)} \log \mathbb{P}[A_{N,\text{left}}^h] \geq -\frac{1}{2} ((h_* - h)^2) ^2, \text{ for all } h \in \mathbb{R}.
\]

The choices (3.22) imply that

\[
(A_{N,\text{left}}^h \cap A_{N,\text{right}}^h) \subset 2\text{-arms}(N,h).
\]

Indeed, the events \( A_{N,\text{left}}^h \) and \( A_N^h \circ \tau_{Ne_1} \) each yield a crossing cluster in \( B_{2N} \setminus B_N \), and \( C_N^h \circ \tau_{Ne_1} \) implies that these are not connected in \( \{ \varphi \geq h \} \), whence (3.24). Now, with a similar decoupling argument as around (3.19), one obtains that for all \( \varepsilon > 0 \),

\[
P[A_{N,\text{left}}^h \cap A_{N,\text{right}}^h] \geq P[A_{N,\text{left}}^{h,\varepsilon}] P[A_{N,\text{right}}^{h,\varepsilon}] - 2e^{-\varepsilon^2 N^{d-2}},
\]

where, in the notation of (2.31), the exponential error term corresponds to twice the probability that \( |\xi_x^{H^+}| > \frac{\varepsilon}{2} \) for some \( x \in H^- \), with \( H^+ := \{ x \in B_{2N} : x^1 \geq 0 \} \). Taking logarithms on the both sides of (3.22), multiplying by \( \frac{\log N}{N} \) and letting \( N \to \infty \), the desired lower bound for \( \mathbb{P}[2\text{-arms}(N,h)] \) with \( h \leq h_* \) follow on account of (3.24), (3.23) and the second line of (3.21) upon letting \( \delta, \varepsilon \to 0^+ \).

### 3.3. Lower bound for \( d \geq 4 \)

We now supply the proof of (3.3). A straightforward strategy consists of opening all the vertices in the line segment \( T_N = T_N(0) \) (recall (2.11) for notation) and closing all vertices in its outer boundary \( \partial_{\text{out}}T_N \). However, the Gaussian free field does not satisfy a uniform finite-energy property, which would make this easy to implement. Moreover, unlike in the subcritical case, one cannot immediately apply the FKG-inequality as the event in question is not monotonic. In order to deal with these issues, we use the midpoint extension \( \tilde{\varphi} \) introduced at the end of Section 2 see in particular (2.22) and (2.33).
Proof of (3.3). Recall the decomposition (2.37) of \( \hat{\varphi} \). Now let \( T_N := T_N \cup \partial_{\text{out}} T_N \) (where \( T_N \) is viewed as a subset of \( \mathbb{Z}^d \), whence \( \overline{T}_N \subset \mathbb{Z}^d \)) and let \( M_N := \{ m \in M^d : m \sim x, \text{ for some } x \in \overline{T}_N \} \). Using the fact that the absolute value of the Gaussian free field on any transient graph (and thus in particular of \( \hat{\varphi} \) on \( \mathbb{Z}^d \)) satisfies the FKG-inequality, see e.g. (1.3) and Corollary 1.3 in [21], one obtains that

\[
\Pr[|\hat{\xi}_x| \leq 1, \forall x \in \overline{T}_N] \geq \Pr[|\hat{\varphi}_m| \leq 1, \forall m \in M_N] \geq \Pr[|\hat{\varphi}_{m_0}| \leq 1]^{|M_N|} \geq e^{-CN},
\]

where the first inequality follows because \( \hat{\xi}_x \) is the mean of \( \hat{\varphi} \) evaluated at its neighbors and \( m_0 \) refers to an arbitrary reference point in \( M^d \). As a consequence, one has, for all \( h \in \mathbb{R} \),

\[
\Pr[0 \leftrightarrow h, 0 \leftrightarrow \infty] \geq \Pr[|\hat{\xi}_x| \leq 1, \forall x \in \overline{T}_N, \hat{\psi}_x \geq h + 1, \forall x \in T_N, \hat{\psi}_y < h - 1, \forall y \in \partial_{\text{out}} T_N] \geq \Pr[|\hat{\xi}_x| \leq 1, \forall x \in \overline{T}_N] \Pr[X \geq h + 1]^{\overline{T}_N} \Pr[X < h - 1]^{\partial_{\text{out}} T_N} \geq e^{-C(h)N},
\]

where in the penultimate step, we also used independence of \( \hat{\psi} \) and \( \hat{\xi} \) and \( X \) is a \( N(0,1/2) \)-distributed random variable. One easily adapts the above argument in order to create two ‘insulated’ paths in \( \{ \varphi \geq h \} \) joining \( B_N \) to \( \partial B_{2N} \) at an exponential cost in \( N \), thus obtaining the lower bound (3.3) for the event \( \text{LocUniq}(N,h)^c \) instead. This completes the proof of (3.3) and with it that of Theorem 3.1. \( \square \)

4 Coarse-graining

We now prepare the ground for the upper bounds that will be derived in the next section. The main result of this section is a certain coarse-graining scheme for paths, see Proposition 4.3 below. Its proof is split over Sections 4.1 and 4.2 which deal with the case \( d = 3 \) and \( d \geq 4 \), respectively. The key effect of the scheme is to keep the capacity of the ‘coarse-grained path’ above a certain threshold, see (4.15), (4.16), while maintaining good control on the entropy factor for its possible choices. The notion of ‘coarse-grained path’ is formalized in Definition 4.2. It depends on a function \( \Gamma(\cdot) \) parametrizing this entropy, see (4.13) and (4.14).

We now describe the precise setup. We consider positive integers \( L \geq 1 \) and \( K \geq 100 \) and introduce the lattice

\[
L = L(L) := \mathbb{Z}^d
\]

along with the boxes

\[
C_z := z + [0,L)^d, \ D_z := z + [-3L,4L)^d, \ U_z := z + [-KL,1,L + KL - 1)^d
\]

as well as

\[
\tilde{C}_z := z + [-L,2L)^d, \ \tilde{D}_z := z + [-2L,3L)^d
\]

attached to \( z \in L \). Notice that \( C_z \subset \tilde{C}_z \subset \tilde{D}_z \subset D_z \subset U_z \). When considering more than one scale in a given context, we will sometimes explicitly refer to the associated length scale \( L \) by
writing \( C_{z,L} = C_z, \tilde{C}_{z,L} = \tilde{C}_z \) etc. Using the notation of [42, Section 4], for any \( z \in \mathbb{L} \), we introduce the decomposition
\[
(4.4) \quad \varphi = \xi^z + \psi^z
\]
where \( \xi^z := \xi_{z,x}^U = E_x[\varphi_{X^U}] \) for all \( x \in \mathbb{Z}^d \), cf. (2.31), and \( \psi^z = \psi_{U_z} \), with \( U_z \) as in (4.2). Recall that \( \psi^z \) has law \( P_{U_z} \). Letting \( \tilde{U}_z := (U_z \cap \mathbb{Z}^d) \cup \{ \tilde{x} \in \mathbb{Z}^d : \exists x \in (U_z \cap \mathbb{Z}^d) \text{ s.t. } |x - \tilde{x}| = \frac{1}{2} \} \), we further define for \( z \in \mathbb{L} \) the extended harmonic average
\[
(4.5) \quad \tilde{\xi}^z := \tilde{\xi}_{U_z}(\mathbb{E}, \tilde{E}) [\varphi_{X^U}], \quad \tilde{\psi}^z := \tilde{\varphi} - \tilde{\xi}^z,
\]
where \( \tilde{\varphi} \) refers to the extension of \( \varphi \) to the graph \( \mathbb{Z}^d \), see the discussion leading to (2.32). On account of (2.33) and (2.36), one thus has \( \tilde{\xi}^z \mid_{\mathbb{Z}^d} = \xi^z \) and \( \tilde{\psi}^z \mid_{\mathbb{Z}^d} = \psi^z \). Moreover, if \( \mathcal{C} \subset \mathbb{L} \) is a non-empty collection of sites with mutual \( \cdot \mid_{\mathbb{L}} \)-distance at least \( 2KL + L \), then denoting by \( \tilde{\psi}^z = (\tilde{\psi}^z)_{\tilde{x} \in \tilde{U}_z} \), for \( z \in \mathcal{C} \), one has that
\[
(4.6) \quad \mathcal{C} \subset \mathbb{L} \text{ is a non-empty collection of sites with mutual } | \cdot |_{\mathbb{L}} \text{-distance at least } 2KL + L.
\]
the Gaussian fields \( \{ \xi, \tilde{\psi}^z, z \in \mathcal{C} \} \), where \( \tilde{\xi} = (\tilde{\xi}^z)_{z \in \mathcal{C}, \tilde{x} \in \tilde{U}_z} \), are independent.

Given \( \mathcal{C} \) as in (4.6), we write
\[
(4.8) \quad \Sigma := \Sigma(\mathcal{C}) := \bigcup_{z \in \mathcal{C}} C_z.
\]
The following precise result will be useful.

**Lemma 4.1 (Control of harmonic average).** There exist \( C_2(d) \in [1, \infty) \) as well as, for all \( K \geq 100, \alpha(K) > 1 \) with \( \lim_{K \to \infty} \alpha(K) = 1 \) such that for every \( a > 0 \), one has
\[
(4.9) \quad \limsup_L \sup_{\mathcal{C}} \left\{ \log P\left[ \bigcap_{\tilde{z} \in \mathcal{C}} \{ \sup_{D_{\tilde{z}}} \tilde{\xi}^z \geq a \} \right] + \frac{1}{2} \left( a - \frac{C_2}{K} \sqrt{\frac{|\mathcal{C}|}{\text{cap}(\Sigma)}} \right)^2 \frac{\text{cap}(\Sigma)}{\alpha(K)} \right\} \leq 0,
\]
where the supremum over \( \mathcal{C} \) runs over the sets satisfying (4.6), \( \sup_{D_{\tilde{z}}} \) refers to the supremum over all points in \( D_{\tilde{z}} \cap \tilde{\mathbb{Z}}^d \) and \( r_+ = r \vee 0 \) for any \( r \in \mathbb{R} \).

**Proof.** With \( \xi^z \) in place of \( \tilde{\xi}^z \), (4.9) is proved in [42, Corollary 4.4]. To extend the result to \( \tilde{\xi}^z \), observe that, for any mid-point \( \tilde{x} \in (D_z \cap \tilde{\mathbb{Z}}^d) \setminus \mathbb{Z}^d \), the neighbors \( x_1 \) and \( x_2 \) of \( \tilde{x} \) (in \( \tilde{\mathbb{Z}}^d \)) both belong to \( D_z \cap \mathbb{Z}^d \). Furthermore, by harmonicity of \( \tilde{\xi}^z \) in \( D_z \cap \tilde{\mathbb{Z}}^d \), cf. (4.2) and (4.5), one has \( \tilde{\xi}^z = \frac{1}{2}(\tilde{\xi}^z_{x_1} + \tilde{\xi}^z_{x_2}) = \frac{1}{2}(\xi^z_{x_1} + \xi^z_{x_2}) \), whence \( \tilde{\xi}^z \leq a \) whenever \( \xi^z_{x_1} \leq a \) for \( i = 1, 2 \). Together, these observations yield that
\[
\{ \sup_{D_z \cap \tilde{\mathbb{Z}}^d} \tilde{\xi}^z \geq a \} \subset \{ \sup_{D_z \cap \mathbb{Z}^d} \xi^z \geq a \},
\]
and the claim follows. \( \square \)

We will be interested in families of collections \( \mathcal{C} \) satisfying (4.6) with certain finer properties. In what follows, let \( \Lambda_N \) be any of the elements in
\[
(4.10) \quad \mathcal{S}_N := \{ B_N, B_{2N} \setminus B_N, D_{0,N} \setminus C_{0,N} \}, \text{ for } N \geq 1 \text{ (see below (4.3) for notation)}.
\]
In line with the wording below (1.5), for \( U \subset V \subset \subset \mathbb{Z}^d \), we say that a \( \ast \)-path \( \gamma \) in \( \mathbb{Z}^d \) (see Section 2 for its definition) crosses \( V \setminus U \) if \( \gamma \) intersects both \( U \) and \( \partial V \). If \( U = \{ 0 \} \) we omit the reference to \( U \); e.g. when \( \gamma \) crosses \( B_N \) we mean that \( \gamma \) intersects both 0 and \( \partial B_N \).
Definition 4.2. \((K \geq 100, L \geq 1, N \geq 10KL)\). Let \(\Gamma : [1, \infty) \mapsto [0, \infty)\). A family \(A = A_{NL}(\Lambda_N)\) of collections \(C \subset \mathbb{L}\) is called \(\Gamma\)-admissible if, for some \(c_5(d) \in (0,1)\),

\[
\begin{align*}
\text{all } C \in A \text{ have equal cardinality } n := |C| \text{ satisfying } n \in \left[\frac{c_5 N}{u(KL)}, \frac{N}{u(KL)}\right], \text{ where } \\
u(x) = x \text{ if } d = 3 \text{ and } u(x) = x(\log x)^2 \text{ if } d \geq 4, \quad (4.6) \\
\text{holds for all } C \in A \\
\text{and } \tilde{D}_z = \tilde{D}_{z,L} \subset \Lambda_N \text{ for all } z \in C, \\
\end{align*}
\]

(4.11) for any \(*\)-path \(\gamma\) crossing \(\Lambda_N\), there exists \(C \in A\) such that \(\gamma\) crosses \(\tilde{D}_z \setminus C_z\) (with \(C_z = C_{z,L}, \tilde{D}_z = \tilde{D}_{z,L}\) as in (4.2) \(4.3\) ) for all \(z \in C\),

(4.12) \[\log |A| \leq \Gamma(N/L).\] 

The main result of this section is the following:

Proposition 4.3 (Coarsening of crossing paths). There exists \(C_3(d) \in [1, \infty)\) such that, for all \(K \geq 100, L \geq 1, N \geq 10KL\) and \(\Lambda_N \in S_N\), one can construct a \(\Gamma\)-admissible collection \(A = A_{NL}(\Lambda_N)\) with the following properties.

\(i)\) If \(d = 3\),

\[
\Gamma(r) = C_3 K^{-1} r \log(r \vee 2), \quad (4.14)
\]

and there exist positive numbers \(\lambda(K) \in (0,1]\) satisfying

\[
\lim_{K \to \infty} \lambda(K) = 1, \quad (4.15)
\]

such that, for all \(\rho \in (0,1)\),

\[
\liminf_{N \to \infty} \inf_{L_0(N) \leq L \leq L_1(N)} \inf_{C \in A} \inf_{C \subset C} \frac{\text{cap}(\tilde{\Sigma})}{\text{cap}(T_N)} \geq \lambda(K) (1 - \rho) \quad (4.16)
\]

with \(\tilde{\Sigma} = \Sigma(\tilde{C})\) (see (4.8) for \(\Sigma(\cdot)\)) and \((L_0(N), L_1(N))\) any sequence with \(\lim_N L_0(N) = \infty\) and \(L_1(N) \leq N/10K\) satisfying \(\lim_N \frac{\log L_1(N)}{\log N} = 0\).

\(ii)\) If \(d \geq 4\),

\[
\Gamma(r) = C_3 r \quad (4.14')
\]

and (4.16) remains valid with \(L_0(N) = 1, L_1(N) = N/10K\) and for some \(\lambda(K) \in (0,1]\).

Moreover, the above conclusions continue to hold with \(\Lambda_N = B_N \setminus B_{\varepsilon N}\), for any \(\varepsilon \in (0, 1/3)\), upon replacing \(T_N\) by \(T_{(1-\varepsilon)N}\) in (4.16).

Remark 4.4. 1) The statement of Proposition 4.3 could even be generalized to the case \(\Lambda_N = B_N \setminus B_{\varepsilon N}\), for any \(\varepsilon \in (0,1)\), with suitable modifications. Specifically, the condition in (4.16) relating \(L\) and \(N\) would involve \(K\) and \(\varepsilon\), and \(c_5\) in (4.11) would depend on \(\varepsilon\). We refrain from such extensions since we will only be interested in the limit \(\varepsilon \downarrow 0\) in the sequel.
2) The differing complexities \( \Gamma(r) \) in (4.11), (4.12), reflect different procedures in implementing the coarse-graining depending on whether \( d = 3 \) or \( d \geq 4 \). An interesting question this brings about is the following: can one devise a coarsening scheme in dimension 3 (for instance the strategy employed for \( d \geq 4 \), or a variation thereof), which would be more cost-effective, i.e. reduce \( \Gamma(r) \), ideally getting rid of the logarithmic factor in (4.11), while retaining the controls (4.15), (4.16) on the capacity?

3) The coarse-graining scheme used in case \( d = 3 \) can also be employed in dimensions \( d \geq 4 \) in such a way that (4.16) continues to hold. The resulting higher combinatorial complexity, see (4.14), already yields near-optimal asymptotic upper bounds for the quantities of interest. We refer to Remarks 4.7 and 5.10 below for further details.

The proof of Proposition 4.3 is split over Sections 4.1 and 4.2, which deal separately with the cases \( d = 3 \) and \( d \geq 4 \).

4.1. Proof of Proposition 4.3 for \( d = 3 \). Let \( L \geq 1 \) and \( N \geq 10KL \) be integers. We consider the case \( \Lambda_N = B_N \). The small adaptations needed to account for the remaining cases in \( S_N \), see (4.10), as well as for \( \Lambda_N = B_N \setminus B_{2N} \), are indicated at the end of the proof. For each \( i = 1, \ldots, n \) with \( n := \lfloor N/3KL \rfloor - 1 \) (note that \( n \geq 1 \)), define the concentric shells \( S_i := \partial B_{3KLi} \).

By paving \( S_i \) with boxes of the form \( C_z = z + [0, L]^d \) for \( z \in \mathbb{L} \), and considering the successive first exits of the path \( \gamma \) from each of the sets enclosed by \( S_i \), one finds for each \( i = 1, \ldots, n \) a point \( z_i \in \mathbb{L} \) such that

\[
C_{z_i} \cap S_i \neq \emptyset \text{ and } \gamma \text{ crosses } \hat{D}_{z_i} \setminus C_{z_i}.
\]

Note that \( \hat{D}_{z_i} \subset B_N \) for all \( i = 1, \ldots, n \). We then define \( A = A_{N,L}(B_N) \) as the family consisting of all collections \( C := \{z_i : 1 \leq i \leq n\} \) that can be obtained in this way.

We proceed to verify the conditions of Definition 4.12 for \( \Gamma(\cdot) \) as in (4.14) (with \( d = 3 \)). Properties (4.11) and (4.12) are immediate by construction. Regarding the cardinality of \( A \), one notes that the number of choices for \( z_i \) is bounded by \( (C_{N,T})^{d-1} \), whence \( |A| \leq (C_{N,T})^{n(d-1)} \), from which (4.13) follows with \( \Gamma(\cdot) \) given by (4.14), for suitable choice of \( C_3 \).

It remains to argue that (4.16) holds, with \( \lambda(K) \) satisfying (4.15). This will be done in two steps, corresponding to Lemmas 4.5 and 4.6 below. We begin with a reduction step (Lemma 4.5), consisting of replacing the set \( \tilde{\Sigma} \) appearing in (4.10) (recall (4.8)) by the ‘porous line’

\[
(4.18) \quad \tilde{T}_N := \bigcup_{i=1}^{\lfloor (1-\rho)n \rfloor} (\tilde{z}_i + [0, L] \times \{0\}^2), \quad \text{with} \quad \tilde{z}_i := S_i \cap ([0, \infty) \times \{0\}^2) = (3KL) + 0, 0, 0).
\]

Lemma 4.6 then compares the capacity of porous and non-porous lines.

**Lemma 4.5.** For all \( \rho \in (0, 1) \),

\[
(4.19) \quad \liminf_{N \to \infty} \inf_{L_0(N) \leq L \leq N/10K} \inf_{C \subset \mathcal{A}} \inf_{\tilde{C} \subset \tilde{C}} \frac{\text{cap}(\tilde{\Sigma})}{\text{cap}(\tilde{T}_N)} \geq \lambda(K),
\]

for suitable \( \lambda(K) > 0 \) satisfying (4.15) and any \( L_0(N) \) with \( \lim_N L_0(N) = \infty \).

We prove (4.19) using a projection argument, first by rigidly displacing the boxes in \( \tilde{\Sigma} \) onto \( \mathbb{R} \times \{0\}^2 \), then by ‘packing them together’ along this axis to obtain the ‘homogenous porosity’ of \( \tilde{T}_N \). Since these operations essentially reduce the \( (\ell^2) \) -distances between pairs of points in \( \tilde{\Sigma} \), the capacity expectedly decreases. This intuition is formalized below.
Indeed, by construction, (4.21) holds with 
\[ |(4.21) \]
\[ (4.17) \]
Let \( \tau \)
\[ S \]
\[ \tau \]
\[ \ell \]
\[ \kappa \]
\[ \tau \]
\[ \tau \]
\[ \tau \]
Importantly, \( \tau \) is an \((\ell^2)\)-projection, in that 
\[ |(4.24) \] \[ \cap(\tilde{\Sigma}) \]
\[ (\tilde{\Sigma}) \]
\[ \cup \]
\[ \kappa \]
\[ \mu \]
\[ \tilde{\Sigma}_r := \{ \tau(z) : z \in \tilde{\Sigma} \} \]
\[ (4.22) \]
\[ \mu'(x)g(\tau^{-1}(x),\tau^{-1}(y)) \mu'(y) \leq \kappa \cdot E(\tau \circ \mu) \]
\[ (4.23) \]
\[ \kappa := \sup_{x,y \in \tilde{\Sigma}_r} \frac{g(\tau^{-1}(x),\tau^{-1}(y))}{g(x,y)} \]
The quantity \( \kappa \) is suitably bounded as follows. By translation invariance of the Green’s function, if \( x, y \) belong to the same box \( C_{\tau(z)} \) for some \( z \in \tilde{C} \), then \( g(\tau^{-1}(x),\tau^{-1}(y)) = g(x,y) \). Otherwise, i.e. if \( x \in C_{\tau(z)} \) and \( y \in C_{\tau(z')} \) for \( z, z' \in \tilde{C} \) with \( z \neq z' \), using the triangle inequality and (4.21), one readily infers that \( |x-y| \leq |\tau^{-1}(x) - \tau^{-1}(y)| + \sqrt{3}L \), for all \( x, y \in \tilde{\Sigma}_r \). With this observation, returning to (4.23) and using (4.20), which is in force as \(|x-y| \leq |\tau^{-1}(x) - \tau^{-1}(y)| \geq KL \geq L_0 \) whenever \( x \) and \( y \) belong to different boxes, one finds that 
\[ \kappa \leq 1 + \frac{1 + C/K}{(1-\varepsilon)^2}, \]
using in the last step that \(|\tau^{-1}(x) - \tau^{-1}(y)| \geq KL \). Substituting this bound into (4.22), taking an infimum over \( \mu \), noting that \( \mu \to \tau \circ \mu \) is a bijection between probability measures with support on \( \tilde{\Sigma} \) and \( \tilde{\Sigma}_r \), and applying the variational formula (2.49), one obtains the lower bound 
\[ (4.24) \]
\[ \text{cap}(\tilde{\Sigma}) \geq (1-C/K)(1-\varepsilon)^2\text{cap}(\tilde{\Sigma}_r). \]
In view of (4.19), in order to produce the set \( \tilde{T}_N \), cf. (4.18), one trims \( \tilde{\Sigma}_r \) as follows. First, let \( \tilde{T}'_N := \bigcup_{i=1}^{\lceil (1-\rho)n \rceil} C_{\tilde{z}_i} \) and note that \( \tilde{T}_N \subset \tilde{T}'_N \), whence \( \text{cap}(\tilde{T}_N) \leq \text{cap}(\tilde{T}'_N) \). Observe that the elements of \( \tilde{\mathcal{C}} \), and hence of \( \tilde{C} \), are naturally ordered according to increasing index \( i \), cf. below (4.17). Now one only retains the boxes in \( \tilde{\Sigma}_r \) corresponding to the first \( \lceil (1-\rho)n \rceil \) elements of \( \tilde{\mathcal{C}} \) in this ordering (recall that \( \tilde{\mathcal{C}} \) has at least this many elements), and only keeps the intersection of the resulting set of boxes with the line \( \mathbb{Z} \times \{0\}^2 \), thus obtaining overall a smaller set \( \tilde{\Sigma}_r' \subset \tilde{\Sigma}_r \), whence \( \text{cap}(\tilde{\Sigma}_r') \leq \text{cap}(\tilde{\Sigma}_r) \). The resulting set \( \tilde{\Sigma}_r' \) is in natural bijection with \( \tilde{T}'_N \), essentially
by removing the gaps, one by one by rigidly shifting all the intervals to the (say) right of the gap by a suitable constant amount to the left. This operation either leaves the relative position between two points \( x, y \in \Sigma_r \) unchanged or reduces their Euclidean norm, but in the latter case never as to fall below \( KL \geq \tilde{L}_0 \) (a lower bound on the gap size in \( \tilde{T}_N \)). With this observation, a similar argument as above, using \((2.9)\) and \((4.20)\) yields that

\[
(4.25) \quad \text{cap}(\bar{\Sigma}_r) \geq \text{cap}(\bar{\Sigma}_r) \geq (1 - \varepsilon)^2 \text{cap}(\bar{T}_{N}) \geq (1 - \varepsilon)^2 \text{cap}(\bar{T}_N).
\]

By letting \( \varepsilon \to 0 \) (and therefore \( L \geq \tilde{L}_0 \to \infty \) as well as \( N \geq 10KL \to \infty \)), \((4.24)\) and \((4.25)\) imply \((4.19)\) with \( \lambda(K) = 1 - C/K \).

We proceed with

**Lemma 4.6.** For all \( K \geq 100, \rho \in (0, 1) \),

\[
(4.26) \quad \liminf_{\log N / \log k \to \infty} \frac{\text{cap}(\bar{T}_{N})}{\text{cap}(T_N)} \geq 1 - \rho,
\]

where the \( \liminf \) regards any sequence \((N_k, L_k)_{k\geq 0}\) satisfying \(\log N_k / \log L_k \to \infty\) as \( k \to \infty \).

**Proof.** Let \( \hat{T}_N := T_{[(1-\rho)n]3KL+L} \) (recall that \( n = [N/3KL] - 1 \) and \((2.11)\) for notation). On account of \((2.12)\), it suffices to show that \( \liminf_{N/L \to \infty} \frac{\text{cap}(\bar{T}_{N})}{\text{cap}(T_N)} \geq 1 \) and \((4.20)\) directly follows.

By definition, see \((4.18)\), \( \tilde{T}_N \subset \hat{T}_N \) and

\[
(4.27) \quad \text{for all } x \in \hat{T}_N, d(x, \tilde{T}_N) \leq 3KL.
\]

Hence, by \((2.8)\), \((4.26)\) follows at once if we argue that

\[
(4.28) \quad \liminf_{N/L \to \infty} \inf_{x \in \hat{T}_N} P_x[H_{\tilde{T}_N} < \infty] = 1.
\]

In view of \((4.27)\), Lemma \ref{lem:1.2} applied to \( T = \tilde{T}_N \subset \hat{T}_N \) and \( x \in \hat{T}_N \) with the choices \( r = 6KL \) and \( \gamma = \frac{1}{3KL} \), by which \((2.16)\) is satisfied, yields that

\[
P_x[H_{\tilde{T}_N} = \infty] \leq C \left( \frac{\log L}{\log N} \right)^{\varepsilon(K)} \to 0 \text{ as } \log N / \log L \to \infty.
\]

Thus, \((4.28)\) follows, which completes the proof. \( \square \)

With Lemmas \ref{lem:4.5} and \ref{lem:4.6} at hand, we can conclude the proof of \((4.16)\) (and \((4.15)\)) for \( \Lambda_N = B_N \). The remaining cases in \( \mathcal{S}_N \), see \((1.10)\), i.e. \( \Lambda_N = B_{2N} \setminus B_N \), resp. \( \Lambda_N = D_{0,N} \setminus \bar{C}_{0,N} \) are dealt with by considering instead \( S_i = \partial B_{N+i3KL} \), resp. \( S_i = \partial([-3KL - N, 2N + 3KL]) \) (cf. \((4.3)\)), for \( 1 \leq i \leq n \), and adapting the subsequent arguments accordingly. In particular, the bound for \( |A| \) remains valid for these choices. The set \( \tilde{T}_N \) in \((4.18)\) changes accordingly whence \( \tilde{z}_i = (N + 3LK, 0) \), resp. \( \tilde{z}_i = (2N + 3LK, 0, 0) \). The statements and proofs of Lemmas \ref{lem:4.5} and \ref{lem:4.6} then remain valid.

In the case \( \Lambda_N = B_N \setminus B_{\varepsilon N} \) for a given \( \varepsilon \in (0, \frac{1}{3}) \), one considers the shells \( S_i := \partial B_{[\varepsilon N]+i3KL} \), for \( 1 \leq i \leq n := [(1 - \varepsilon)N/3KL] - 1 \) (note that \( n \geq 1 \) when \( N \geq 10KL \)). Then, defining \( \tilde{T}_N \) as in \((4.18)\) (whence \( \tilde{z}_i = ([\varepsilon N] + 3LK, 0, 0) \)), Lemma \ref{lem:4.5} remains valid and Lemma \ref{lem:4.6} as well upon replacing \( T_N \) by \( T_{(1-\varepsilon)N} \) in \((4.20)\). The lower bound \((4.16)\) with \( T_{(1-\varepsilon)N} \) instead of \( T_N \) then follows as above. This concludes the proof of Proposition \ref{prop:4.3} in the case \( d = 3 \). \( \square \)
Remark 4.7. The above coarse-graining scheme also applies when \( d \geq 4 \). As a result, one obtains a \( \Gamma \)-admissible collection \( \mathcal{A}' = \mathcal{A}_{N,L}(\Lambda_N)' \), with \( \Gamma(\cdot) \) as defined in (4.14) for \( d = 3 \), i.e. \( \Gamma(r) = \frac{C}{r} \log r \), and with \( u(x) = x \) in (4.11), so that the statement (4.10) for \( d \geq 4 \) holds with \( \mathcal{A}' \) in place of \( \mathcal{A} \). Only (4.16) requires an explanation. Repeating the steps leading up to (4.18), one shows an analogue of Lemma 4.5 without the condition (4.15). The proof essentially remains the same except that one simply uses \( c \leq |x|^{d-2}g(0,x) \leq C \) for all \( x \in \mathbb{Z}^d \) instead of (4.20), which is sufficient since one only aims at a \( \lambda(K) > 0 \) in (4.19). Then Lemma 4.6 gets replaced by the statement that \( \frac{\text{cap}(\widetilde{T}_N)}{\text{cap}(T_N)} \geq cK^{-1}(1 - \rho) \) for all \( L \geq 1 \) and \( N \geq 10KL \), which follows by covering \( T_N \) with at most \( CK \) shifted copies of the set \( \widetilde{T}_N \) and using monotonicity and subadditivity of \( \text{cap}(\cdot) \) (see e.g. Proposition 2.2.1(b) in [28] regarding the latter). The slightly higher combinatorial complexity of the collection \( \mathcal{A}' \), reflected by the logarithmic factor in \( \Gamma(\cdot) \), cf. (4.14), only yields near-optimal upper bounds for \( d = 4 \), see Remark 5.10 below. The presence of the additional logarithm is remedied by the approach of Section 4.2.

4.2. Proof of Proposition 4.3 for \( d \geq 4 \). We introduce the length scales

\[(4.29) \quad L_0 := 1, \quad L_{m+1} := \lceil 2(1 + \varepsilon_m)L_m \rceil, \quad \text{with} \quad \varepsilon_m := (m + 1)^{-2}, \quad m \geq 0.\]

Note that \( 2^m \leq L_n \leq C2^m \) for all \( m \geq 0 \), which will be used frequently below. Throughout the proof we use \( C_{z,k}, \tilde{C}_{z,k}, D_z, k \) and \( U_{z,k} \) to denote the boxes \( C_z, \tilde{C}_z, D_z \) and \( U_z \), see (4.2), corresponding to the length \( L = L_k \) for any \( z \in \mathbb{Z}^d \). Also let \( \tilde{C}_{z,k} := z + \lfloor -L_k + L_{k-1}, 2L_k - L_{k-1} \rfloor^d \)

and observe that \( C_{z,k} \subseteq \tilde{C}_{z,k} \subseteq \tilde{C}_{z,k} \).

We first define a certain coarse graining of paths crossing generic shapes, see Lemma 4.8 below, which will later be applied inductively to define an admissible collection with the desired properties. Roughly speaking, the idea is to implement a cascading scheme on the path \( \gamma \) of diameter \( N \), thus only retaining its trace in a system of well-separated boxes at scale \( L \) (naturally indexed by the leaves of a binary tree of depth \( \approx \log_2(N/L) \)), see for instance [17], Section 8.1 for a gentle introduction to this circle of ideas in a related model.

The precise recursive scheme underlying the proof of Lemma 4.8 builds upon the ideas from [39, 38]. In particular, we modify the sub-division scheme of [38] in order to ensure that the smaller annuli obtained after each iteration lie inside \( \Lambda_N \) (see the proof of Lemma 4.8 below). This is to guarantee that the annuli obtained at the very end are also contained in \( \Lambda_N \) as required by Definition 4.2. However, doing so causes the annuli to overlap and we need to use more restricted regions than full annuli in order to avoid that. A general notion of regions with which we can accomplish this is that of shapes which we introduce now.

A shape at level \( k \) anchored at \( z \in \mathbb{Z}^d \) is any \( * \)-connected subset \( \mathcal{S} = \mathcal{S}_z, k \) of \( \tilde{C}_{z,k} \setminus C_{z,k}^{-} \), where \( C_{z,k}^{-} = C_{z,k} \setminus \partial C_{z,k} \), intersecting both \( \partial C_{z,k} \) and \( \partial \tilde{C}_{z,k} \). The collection of all shapes at level \( k \) anchored at \( z \) will be denoted by \( \mathcal{S}_z, k \) and \( \mathcal{S}_k := \bigcup_{z \in \mathbb{Z}^d} \mathcal{S}_z, k \) is the set of all shapes at level \( k \). For all practical purposes except the one mentioned in the previous paragraph, the reader can safely picture a shape in \( \mathcal{S}_k \) as an annulus \( \tilde{C}_{z,k} \setminus C_{z,k}^{-} \).

In what follows, let \( \mathbb{T}_n, n \geq 0 \), denote the (rooted) complete binary tree of depth \( n \) and let \( \mathcal{L}(\mathbb{T}_n) = \{0, 1\}^n \) (with \( \mathcal{L}(\mathbb{T}_0) = \{\emptyset\} \), the root) be its set of leaves. The leaves of \( \mathbb{T}_n \) provide a natural indexing set due to the recursive dyadic manner in which the coarsening scheme operates, cf. (4.32).
Lemma 4.8 (Coarse-graining of shapes). For any integers \( n \geq k > 0 \), \( z \in \mathbb{Z}^d \) and all shapes \( S \in S_{z,n} \), there exists a family \( A_S = A_{S,k} \) with

\[
\log |A_S| \leq C L_{n-k}
\]

of collections \( D = \{(z(\ell),S(\ell)) : \ell \in L(T_{n-k}) \} \subset \mathbb{Z}^d \times S_k \) satisfying the following three properties:

\[
S(\ell) \subset S \text{ and } S(\ell) \in S_{z(\ell),k} \text{ for all } \ell \in L(T_{n-k}),
\]

if \( n > k \), the sub-collections \( D_i := \{(z(\ell),S(\ell)) : \ell \in L(T_{n-k,i}) \} \), \( i = 1, 2 \), of \( D \), where \( T_{n-k,i}, i = 1, 2 \), denote the two binary sub-trees of \( T_{n-k} \) with depth \( n-k-1 \), belong to \( A_{S_1} \) and \( A_{S_2} \) for some shapes \( S_1, S_2 \in S_{n-1} \) such that \( d_\infty(S_1, S_2) \geq 2\varepsilon_{n-1} L_{n-1} \),

\[
\text{for any } \ast\text{-path } \gamma \subset S \text{ crossing } \hat{C}_{z,n} \setminus C_{z,n}, \text{ there exists } D \in A_S \text{ such that for all } \ell \in L(T_{n-k}), \gamma \text{ induces a } \ast\text{-path } \gamma' \subset S(\ell) \text{ crossing } \hat{C}_{z(\ell),k} \setminus C_{z(\ell),k}.
\]

Proof. Fix \( k > 0 \). We proceed by induction over \( n \geq k \). When \( n = k \), we simply choose \( A_S \) to be the singleton set consisting of \( D := \{(z,S)\} \) whence (4.31), (4.32) and (4.33) hold.

Suppose now that for some \( n \geq 0 \) and each \( S \in S_n \), there exists a family \( A_S \) satisfying (4.31)-(4.33), and such that, for some \( b_{n-k} \in (0, \infty) \),

\[
\sup_{S \in S_n} \log |A_S| \leq b_{n-k} L_{n-k}
\]

(note that (4.34) holds for \( n = k \) with \( b_0 = 0 \)). Consider a shape \( S \in S_{z,n+1} \) for some \( z \in \mathbb{Z}^d \). Choose a fixed set of vertices \( T_1 \subset C_{z,n+1} \) of cardinality \( |T_1| \leq C_4 \) for suitable \( C_4 \in [1, \infty) \) such that \( C_{w,n} \subset C_{z,n+1} \) for all \( w \in T_1 \) and the faces of the boxes \( \{C_{w,n} : w \in T_1\} \) form a cover of the faces of \( C_{z,n+1} \). Now, since \( C_{w,n} \subset C_{z,n+1} \) for all \( w \in T_1 \), given any \( \ast\)-path \( \gamma \) crossing \( C_{z,n+1} \setminus C_{z,n+1} \), one finds \( z_1 \in T_1 \) such that \( \gamma \) induces a path \( \gamma_1 \) crossing \( \hat{C}_{z_1,n} \setminus C_{z_1,n} \). Furthermore if \( \gamma \subset S \), it follows that \( \gamma_1 \subset S_1 \) where \( S_1 \subset S \) is the \( \ast \)-connected component of \( S \cap (\hat{C}_{z_1,n} \setminus C_{z_1,n}) \) containing \( \gamma_1 \). It is clear from this definition that \( S_1 \subset S_{z,n} \).

By a similar reasoning, one finds a set \( T_2 \subset C_{z,n+1} \), see below (4.29) for notation, with \( |T_2| \leq C_5 \) for suitable \( C_5 \in [1, \infty) \) such that for some \( z_2 \in T_2 \), \( \gamma \) exits \( C_{z,n+1} \) for the last time through \( C_{z_2,n} \subset C_{z,n+1} \). Since \( z_2 \in C_{z,n+1} \) and consequently \( C_{z_2,n} \subset C_{z,n+1} \), we deduce from these definitions that \( \gamma \) induces a path \( \gamma_2 \) crossing \( \hat{C}_{z_2,n} \setminus C_{z_2,n} \). Now, let \( S_2 \in S_{z_2,n} \) be defined as the \( \ast \)-connected component of \( S \cap (\hat{C}_{z_2,n} \setminus C_{z_2,n}) \) containing \( \gamma_2 \). Note that \( \gamma_2 \subset S_2 \subset S \setminus \hat{C}_{z_2,n} \). Now recalling that \( S_1 \subset \hat{C}_{z_1,n} \subset z + [-L_n, L_{n+1} + L_n]^d \), it follows that

\[
d_\infty(S_1, S_2) \geq d_\infty([-L_n, L_{n+1} + L_n]^d, \partial_{\text{out}} \hat{C}_{0,n+1}) \geq L_{n+1} - 2L_n \geq 2\varepsilon_n L_n.
\]

Therefore, upon defining \( A_S \) to be the collection of all \( D \) such that the restriction of \( D \) to the leaves of the left and right sub-trees of \( T_{n+1-k} \) of depth \( n-k \) correspond to some \( D_1 \in A_{S_1} \) and \( D_2 \in A_{S_2} \), for some \( z_1 \in T_1 \) and \( z_2 \in T_2 \) respectively, then the properties (4.31)-(4.33) follow as an immediate consequence of the above construction and the induction hypothesis. In particular, the distance constraint in (4.32) (at level \( n+1 \)) is exactly (4.35).

Finally, observe that

\[
|A_S| \leq |T_1| \cdot |T_2| \sup_{S' \in S_n} |A_{S'}|^2 \leq C_4 C_5 \sup_{S' \in S_n} |A_{S'}|^2.
\]
Together, (4.34), (4.36) and (4.29) readily imply that a bound similar to (4.34) holds with $n + 1$ in place of $n$ and $b_{n+1-k} := b_{n-k} + \frac{\log(C_0b)}{L^{n-k}}$, from which (4.30) follows with the choice $C = \lim_{n \to \infty} b_n(< \infty)$.

The next result entails a capacity estimate which will be key in deducing (4.16). For a given shape $S \in S_n$ and a collection $D = \{ (z(\ell),S(\ell)) : \ell \in \mathcal{L}(T_{n-k}) \} \in \mathcal{A}_{S,k}$ (with $\mathcal{A}_{S,k}$ as given by Lemma 4.8), consider a collection $B = \{ B(\ell) : \ell \in \mathcal{L}(T_{n-k}) \}$ of boxes such that, for all $\ell \in \mathcal{L}(T_{n-k}),$

$B(\ell) = z + [0,r]^d$, for some $z \in \mathbb{Z}^d$ s.t. $B(\ell) \cap S(\ell) \neq \emptyset$ and $1 \leq r \leq \frac{1}{2} \varepsilon_k L_k$.

Now, define the set

$$S_B(\tilde{D}) := \bigcup_{\ell \in \mathcal{L}(T_{n-k}): (z(\ell),S(\ell)) \in \tilde{D}} B(\ell), \text{ for } \tilde{D} \subset D$$

and

$$\kappa_{n,k} := \inf_{\rho \in \mathcal{O}(1)} \inf_{S \in S_n} \inf_{D \in A_{S,k}} \inf_{B \subset \mathcal{D} \subset \mathcal{D}_c} \rho^{-1} \text{cap}(S_B(\tilde{D})), \quad \text{ (4.39)}$$

with the infimum over $B$ ranging over all collections of boxes satisfying (4.37). The following lemma supplies suitable lower bounds on the quantity $\kappa_{n,k}$.

**Lemma 4.9.** ($d \geq 4$) For any $n \geq k > 0$, one has

$$\kappa_{n+1,k} \geq 2^{n+1} \land \frac{2\kappa_{n,k}}{1+C\frac{2^{n+1}}{\varepsilon_n L_n^{d-2}}}. \quad \text{ (4.40)}$$

As a consequence, for all for all $n \geq k > 0$, one has

$$\kappa_{n,k} \geq 2^n \land c_6 2^{n-k} \kappa_{k,k}. \quad \text{ (4.41)}$$

**Proof.** Consider $S \in S_{n+1}$, $D \in A_S = A_{S,k}$, a collection $B$ satisfying (4.37) and $\tilde{D} \subset D$ with $|\tilde{D}| \geq \rho D$ for some $\rho \in (0,1)$. Consider the sub-collections $D_1$ and $D_2$ of $D$ given by (4.32), define $\tilde{D}_1 := \tilde{D} \triangle D_1$ as well as the sets $S_B(\tilde{D}_i)$ for $i = 1,2$ in similar fashion as (4.38), so that $S_B(\tilde{D}) = S_B(\tilde{D}_1) \cup S_B(\tilde{D}_2)$. By (4.32) and (4.37) and since $n \geq k$, one has

$$d_2(S_B(\tilde{D}_1),S_B(\tilde{D}_2)) \geq \varepsilon_n L_n. \quad \text{ (4.42)}$$

Now, since $S_B(\tilde{D}) \supset S_B(\tilde{D}_i)$ for $i = 1,2$, using the sweeping identity (2.27), one bounds

$$\text{cap}(S_B(\tilde{D}_1)) \leq \sum_{z \in S_B(\tilde{D}_1)} e_{S_B(\tilde{D}_1)}(z) + \max_{z \in S_B(\tilde{D}_{3-i})} P_z[H_{S_B(\tilde{D}_i)} < \infty] \sum_{z \in S_B(\tilde{D}_{3-i})} e_{S_B(\tilde{D}_i)}(z). \quad \text{ (4.43)}$$

Using (2.28), one finds, for $i = 1,2$,

$$\max_{z \in S_B(\tilde{D}_{3-i})} P_z[H_{S_B(\tilde{D}_i)} < \infty] \leq \text{cap}(S_B(\tilde{D}_i)) \max_{z \in S_B(\tilde{D}_{3-i}), y \in S_B(\tilde{D}_i)} g(z,y). \quad \text{ (4.44)}$$
The maximum of Green’s functions on the right-hand side is bounded by $C(\varepsilon_n L_n)^{2-d}$ in view of (4.42) and (2.2). Substituting the resulting bound into (4.43) yields the estimates

$$\text{cap}(S_B(\tilde{D}_i)) \leq \sum_{z \in S_B(\tilde{D}_i)} e_{S_B(\tilde{D})}(z) + C \frac{\text{cap}(S_B(\tilde{D}_1)) \vee \text{cap}(S_B(\tilde{D}_2))}{(\varepsilon_n L_n)^{d-2}} \sum_{z \in S_B(\tilde{D}_{3-i})} e_{S_B(\tilde{D})}(z),$$

valid for $i = 1, 2$. Adding these and solving for $\text{cap}(S_B(\tilde{D}))$ gives, in view of (4.39),

$$(4.44) \quad \text{cap}(S_B(\tilde{D})) \geq \frac{\text{cap}(S_B(\tilde{D}_1)) + \text{cap}(S_B(\tilde{D}_2))}{1 + C \frac{\text{cap}(S_B(\tilde{D}_1)) \vee \text{cap}(S_B(\tilde{D}_2))}{(\varepsilon_n L_n)^{d-2}}} \frac{2\kappa_{n,k}}{|D|} \frac{2^n \wedge (2^{n+1} \vee (\varepsilon_n L_n)^{d-2})}{1 + C \frac{2^{n+1}}{(\varepsilon_n L_n)^{d-2}}},$$

However, since $\text{cap}(S_B(\tilde{D})) \geq \text{cap}(S_B(\tilde{D}_1)) \vee \text{cap}(S_B(\tilde{D}_2)) =: \Xi$ due to the monotonicity of $\text{cap}(\cdot)$, see (2.28), distinguishing whether $\Xi \geq 2^{n+1}$, in which case $\text{cap}(S_B(\tilde{D}))$ inherits this lower bound, or $\Xi < 2^{n+1}$, in which case one applies (4.44), it follows that

$$\text{cap}(S_B(\tilde{D})) \geq \frac{|D|}{|D|} \left( 2^{n+1} \wedge \frac{2^{n+1}}{1 + C \frac{2^{n+1}}{(\varepsilon_n L_n)^{d-2}}} \right),$$

yielding (4.40). The lower bound (4.41) follows from (4.40) and a straightforward induction argument, with $c_0 = \prod_{n=0}^{\infty} (1 + C \frac{2^{n+1}}{(\varepsilon_n L_n)^{d-2}})^{-1} > 0$ (see (4.29)).

We now complete the proof of Proposition 4.3 for $d \geq 4$. Let $L \geq 1$ $K \geq 100$ and $N \geq 10KL$. We first introduce the collection $A = A_{N,L}^K(\Lambda_N)$ for any $\Lambda_N \in S_N$ (recall its definition from (4.10)) and verify that it is $\Gamma$-admissible. Let $n_0$ be maximal such that $L_{n_0} \leq N/10$, which by (4.29) implies that

$$(4.45) \quad N/30 \leq L_{n_0} \leq N/10,$$

and let $k_0$ be such that $L_{k_0-1} \leq 5L \leq L_{k_0}$. Note that $k_0 \leq n_0$. The set $A$ will be defined in terms of the coarse grainings $A_{S,k_0}$ of a fixed number of shapes $S \in S_{n_0}$, which we now introduce. Let $V^0 \subset \mathbb{Z}^d$ consist of 0 and any point $z \in L_{n_0} \mathbb{Z}^d$ such that $C_{z,n_0} = z + [0, L_{n_0})^d$ intersects $\partial B_{3N/2}$ or $\partial [-N, N, N]^d$. Define $S^0 = \{ C_{z,n_0} \setminus C_{z,n_0} : z \in V^0 \} \subset S_{n_0}$. Note that $|S^0| \leq C$ by choice of $n_0$ in (4.45). Moreover, in view of (4.10), any $\gamma$-path $\gamma$ crossing $\Lambda_N$ induces a $\gamma$-path crossing $S$ for some $S \in S^0$.

Let us briefly pause to describe in plain words how $A$ will be constructed. With $L_{n_0} \approx N$ and to a given shape $S \in S_{n_0}$ among the finite set $S^0$, we can associate by means of Lemma 4.8 (applied with $n = n_0$ and $k = k_0$) a collection of shapes at scale $L_{k_0} \approx L$ indexed by the leaves of a binary tree of depth $n_0 - k_0$. We can attach to this setup a system of $L$-boxes which are simply the translates of $[0, L)^d$ anchored at a point $y \in \Lambda$ intersecting the resulting shapes. The important requirement (4.12) that these boxes witness the macroscopic crossing will essentially follow from (4.33). One issue with this is that the resulting system of boxes is only separated by a distance roughly $\varepsilon_{k_0} L_{k_0} \ll L$, which does not satisfy (4.10). To fix this, we ‘trim’ the collection by going down the tree until reaching an intermediate scale $L_k$ of order $KL$, and then only retain a fixed leave (say the smallest one in lexicographic order) in each subtree of depth $k - k_0$ attached to a vertex at depth $n_0 - k$. We now proceed to formalize this and subsequently verify $\Gamma$-admissibility (see Definition 4.2 and (4.14)) of the resulting collection of boxes along with the required estimate (4.10) on their capacity.
Let $k \in [k_0, n_0]$ be such that

\[(4.46)\quad 2\varepsilon_{k-1}L_{k-1} \leq 2KL + 3L \leq 2\varepsilon_{k-1}L_{k-1} \]

and consider the subset of leaves $\ell' \in \mathcal{L}(T_{n_0-k_0})$ of the form $\ell' = \ell \times (0, \ldots, 0) =: \ell_0$, where $\ell \in \mathcal{L}(T_{n_0-k_0})$ is arbitrary and $(0, \ldots, 0) \in \mathcal{L}(T_{k-k_0})$ is fixed. For a given shape $S \in S^0$ and a collection $D_0 \in A_{S,k_0}$, ‘prune’ $D_0 = \{ (z(\ell'), S(\ell')) : \ell' \in \mathcal{L}(T_{n_0-k_0}) \}$ (and forget the anchor point $z(\ell')$) to obtain the collection $\mathcal{D} = \{ S(\ell_0) : \ell \in \mathcal{L}(T_{n_0-k_0}) \}$. Let $A'$ denote the collections $\mathcal{D}$ thereby obtained as $\mathcal{D}_0$ ranges over $A_{S,k_0}$ and $S \in S^0$. The collection $\mathcal{A} = A'_{S,L}(A_N)$ is then defined as (recall $\mathcal{L}$ from (4.1))

\[(4.47)\quad \mathcal{A} = \{ C = (y(\ell'))_{\ell' \in \mathcal{L}(T_{n_0-k})} : y(\ell') \in \mathcal{L}, \exists \mathcal{D} = \{ S(\ell_0) : \ell \in \mathcal{L}(T_{n_0-k}) \} \in A' \}
\]

\[\text{s.t. } C_{y(\ell),L} \cap S(\ell_0) = \emptyset \text{ for all } \ell \in \mathcal{L}(T_{n_0-k}) \} \}

We now verify that $\mathcal{A}$ defined in (4.47) is $\Gamma$-admissible. The fact that $|\mathcal{A}|$ satisfies (4.13) with $\Gamma(\cdot)$ as in (4.14) follows from (4.30), the fact that $|S^0| \leq C$, whence $\log |\mathcal{A}| \leq CL_{n_0-k_0}$, and since the choice of points in $C$ for a given $D \in A'$ is bounded by $C|T_{n_0-k}|$. Overall, this gives $\log |\mathcal{A}| \leq CL_{n_0-k_0}$, as desired (note that $L_{n_0-k_0} \leq CN/L$ by (4.21)).

The crossing property (4.12) can be seen as follows. Let $\gamma$ be a $\ast$-path crossing $\Lambda_N$. As noted above, $\gamma$ induces a crossing for one of the annuli shapes $S \in S^0$. By (4.33) there exists $D \in A_{S,k_0}$ such that the following holds for every $\ell \in \mathcal{L}(T_{n_0-k})$. The $\ast$-path $\gamma$ induces a $\ast$-path $\gamma'$ crossing $\overline{C}(\ell), \overline{C}(\ell)$ with $\gamma' \subset S(\ell_0)$. In particular, by paving the part of $\partial S(\ell_0)$ adjacent to $\mathcal{C}(\ell)$ by boxes $C_{y,L}$, for $y \in \mathcal{L}$ and by choice of $L_{k_0}$ in (4.45), one finds a point $y(\ell)$ such that $\overline{D}(y(L)) \cap C_{y,L}$ is crossed by $\gamma'$. The resulting collection $\mathcal{C}$ belongs to $A$ and (4.12) follows.

Regarding (4.11), observe that $n = |\mathcal{C}| = |\mathcal{L}(T_{n_0-k})|$, whose logarithm is comparable to $L_{n_0}/L_k$, hence to $N/(KL (\log KL)^2)$ using (4.45), (4.46) and the fact that $c (\log KL)^2 \leq \varepsilon_k \leq C (\log KL)^{-2}$. The required separation property (4.16) then follows from (4.31) and (4.32). Indeed, the latter (applied inductively) implies that any two shapes $S(\ell_0), S(\ell'_0)$ with $\ell \neq \ell' \in \mathcal{L}(T_{n_0-k})$ are each subsets of two shapes $S(\ell), S(\ell') \in S_k$ separated by $2\varepsilon_{k-1}L_{k-1} \geq 2KL + 3L$, see (4.36). Hence $S(\ell_0), S(\ell'_0)$ inherit this separation. On account of (4.47), the resulting points $y(\ell), y(\ell')$ are then at $\varepsilon_k$-distance at least $2KL + L$.

Thus $\mathcal{A}$ is $\Gamma$-admissible. To see that the capacity lower bound (4.16) holds, first observe that $L \geq \frac{c_7}{K} \varepsilon_k L_k$ by (4.46) upon choosing $c_7$ small enough and choose $r \geq 1$ satisfying

\[(4.48)\quad \frac{c_7}{2K} \leq \frac{r}{\varepsilon_k L_k} \leq \left( \frac{1}{2} \wedge \frac{c_7}{K} \right).
\]

In particular $r \leq L$. Now consider an arbitrary collection $\mathcal{C} \in \mathcal{A}$ and note that

\[(4.49)\quad \text{each box } C_{y(L),L}, y(\ell) \in \mathcal{C}, \text{ contains a box } B(\ell) \text{ satisfying (4.37) with } n = n_0; \]

indeed this follows immediately by construction of $C_{y(L),L}$, which intersects $S(\ell)$ by definition, see (4.47), and the fact that $r \leq L$.

Together, (4.49), (4.8) and (4.33) imply that for any $\mathcal{C} \in \mathcal{A}$ and any sub-collection $\mathcal{C} \subset \mathcal{C}$ with $|\mathcal{C}| \geq (1-\rho)|\mathcal{C}|$ for some $\rho \in (0,1)$, $\operatorname{cap}(\Sigma(\mathcal{C})) \geq \operatorname{cap}(\Sigma(B))$, for some family $\mathcal{B}$ satisfying (4.37), where $\mathcal{D}$ refers to the collection generating $\mathcal{C}$, see (4.47), and $\mathcal{D} \subset \mathcal{D}$ is the sub-collection of $\mathcal{D}$ corresponding to the indices $\ell \in \mathcal{L}(T_{n_0-k})$ appearing in $\mathcal{C} \subset \mathcal{C}$. It follows that

\[(4.50)\quad (1-\rho)^{-1} \operatorname{cap}(\Sigma(\mathcal{C})) \geq \kappa_{n_0,k} \geq cK^{-d} N,
\]
where the last inequality is obtained by combining the fact that \(2^0 \geq cN\) due to (4.45) (see also the note following (4.29)) and observing that

\[
2^{-k} \kappa_k, k \geq cL^{-1} \cap(B_r) \geq cK^{-(d-2)} \inf_{m \geq 0} L^{-1} \cap(B_{\epsilon_m} L_m)
\]

(4.51)

\[
\geq cK^{-(d-2)} \inf_{m \geq 0} L_{m-3} \epsilon_{m-2}^{-1} \geq c' K^{-(d-2)}.
\]

The bound (1.16) follows immediately from (4.50). If \(\Lambda_N = B_N \setminus B_{\varepsilon N}\) for some \(\varepsilon \in (0, \frac{1}{3})\), one simply sets \(A_{K_N, L}(\Lambda_N) := A_{K_N, L}(B_N \setminus B_{\varepsilon N})\), which has the desired properties. This completes the proof of Proposition 4.3 for \(d \geq 4\).

\[\square\]

5 Upper bounds

Using the coarse-graining scheme developed in the last section, see in particular Proposition 4.3, we now derive companion upper bounds to the lower bounds obtained in Theorem 3.1. The main result of this section is:

**Theorem 5.1** (Upper bounds).

i) If \(d = 3\), then

\[
\limsup_{N \to \infty} \log N \log \mathbb{P}(0 \leq \varphi \geq h \leftarrow \partial B_N) \leq -\frac{\pi}{6} (h - h^*)^2,
\]

(5.1)

for all \(h > h^*\),

\[
\limsup_{N \to \infty} \log N \log \mathbb{P}(0 \leq \varphi \geq h \leftarrow \partial B_N, 0 \varphi \geq h \leftarrow \partial B_N) \leq -\frac{\pi}{6} (h^* - h)^2.
\]

(5.2)

for all \(h < h^*\),

ii) If \(d \geq 4\), then

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(0 \leq \varphi \geq h \leftarrow \partial B_N) < 0,
\]

(5.3)

for all \(h > h^*\),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(0 \leq \varphi \geq h \leftarrow \partial B_N, 0 \varphi \geq h \leftarrow \partial B_N) < 0.
\]

(5.4)

for all \(h < h^*\),

Moreover, the bounds (5.2) and (5.4) also hold for the events LocUniq\((N, h)\) and 2-arms\((N, h)\) (see (1.5) and (1.9)) in place of \(\{0 \varphi \geq h \leftarrow \partial B_N, 0 \varphi \geq h \leftarrow \partial B_N\}\).

In spite of a common thread, the treatment of the subcritical \((h > h^*)\) and supercritical \((h < h^*)\) regimes involve significantly different ideas. The supercritical case is more involved, mostly due to the additional disconnection constraint present in the events. Correspondingly, the upper bounds of Theorem 5.1 are furnished separately in two subsections. Section 5.1 contains the proof of (5.1) and (5.3), Section 5.2 that of (5.2) and (5.4).

5.1 Upper bounds for the subcritical phase. We start by giving an overview of the proof strategy leading to (5.1) and (5.3). To any path connecting 0 to \(\partial B_N\) in \(\{\varphi \geq h\}\), one associates, in view of Proposition 4.3, a collection of well-separated boxes of carefully chosen size \(L \ll N\), each containing a box-to-box crossing at scale \(L\). By the decomposition (4.4) of \(\varphi\) into the sum of
ψ and ξ within each such box, it follows that when \( h > h^* \), either ξ is atypical for all but a small proportion of the boxes, or the localized version of the event, involving crossings in \( \{ \psi \geq h^* + \varepsilon \} \), behaves atypically for the remaining boxes. The corresponding events \( E_{N,L} \) and \( F_{N,L} \) (for ψ and ξ, respectively), are defined in (5.7) below. Together, they yield the central estimate (5.8), which drives the subsequent upper bounds. The key control on the event \( E_{N,L} \) is derived in Lemma 5.5 for \( d = 3 \) and Lemma 5.9 for \( d \geq 4 \), is obtained by combining Lemma 4.1 and the capacity estimates of (4.16). The resulting bound ends up carrying the proportion of the boxes, or the localized version of the event, involving crossings in (4.1)–(4.4), given \( L \leq 100 \) and a vertex \( z \in \mathbb{L} = \mathbb{L}(L) \), we introduce the events

\[
\{ z \text{ is } \psi\text{-bad} \} := \{ C_z \ni \psi^+ \geq h^* + \frac{\varepsilon}{2} \partial \tilde{C}_z \}, \quad \text{and} \quad \{ z \text{ is } \xi\text{-bad} \} := \{ \sup_{D_z} \xi^+ \geq h - h^* - \frac{\varepsilon}{2} \}.
\]

We also refer to the box \( C_z \) as \( \psi/\xi\text{-bad} \) whenever \( z \) is \( \psi/\xi\text{-bad} \). Next, for any \( N \geq 4KL \) and \( \rho \in (0,1) \), consider the events

\[
E_{N,L} = E_{N,L}^K(\rho, h, h^*, \varepsilon) := \begin{cases} \exists \mathcal{C} \in \mathcal{A}_{N,L} \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = [\rho|\mathcal{C}|] & \text{such that all the sites } z \in \tilde{\mathcal{C}} \text{ are } \psi\text{-bad} \\ \end{cases},
\]

\[
F_{N,L} = F_{N,L}^K(\rho, h, h^*, \varepsilon) := \begin{cases} \exists \mathcal{C} \in \mathcal{A}_{N,L} \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = [\mathcal{C}] - [\rho|\mathcal{C}|] & \text{such that all the sites } z \in \tilde{\mathcal{C}} \text{ are } \xi\text{-bad,} \\ \end{cases}
\]

where \( \mathcal{A}_{N,L} = \mathcal{A}^K_{N,L}(B_N) \) is the admissible collection given by Proposition 4.3. In applications below, the events \( E_{N,L} \) and \( F_{N,L} \) will typically be ‘bad’, i.e. have low probability, and \( \rho \) will be close to 0. Since \( \varphi = \xi^+ + \psi^+ \) on \( U_z \supset \tilde{C}_z \) and \( D_z \supset \tilde{C}_z \), it is then a consequence of the property (4.12) of \( \mathcal{A}_{N,L} \) and (5.5), (5.6), (5.7) that

\[
\mathbb{P}[0 \leftrightarrow h^* \partial B_N] \leq \mathbb{P}[E_{N,L}] + \mathbb{P}[F_{N,L}].
\]

The following (a-priori) bound will be useful in dealing with (5.5) and the event \( E_{N,L} \) in (5.7). It will also apply to a different notion of \( \psi\text{-badness} \) in the next subsection, hence the general formulation. Consider an arbitrary increasing set \( A \in \mathcal{B}(\mathbb{R}^{\mathbb{C}_z}) \). Then, in the notation of (3.4), for all \( \varepsilon > 0 \) there exists \( c_8(\varepsilon) \in (0,1) \) increasing in \( \varepsilon \) with the property that, if for some \( h' \in \mathbb{R} \) and \( L \geq 1 \),

\[
\mathbb{P}[A^{h'}(\varphi)] \leq e^{-2f(L)} \text{ with } \log 2 \leq f(L) \leq c_8(\varepsilon)L,
\]

then

\[
\mathbb{P}[A^{h^* + \frac{\varepsilon}{2}}(\psi^+)] \leq e^{-f(L)}.
\]
Indeed, (5.10) follows immediately from the decomposition $\varphi = \xi^z + \psi^z$ valid on $U_z$, whence
\[ \mathbb{P}[A^{h'+\xi}(\psi^z)] \leq \mathbb{P}[A^{h'}(\varphi)] + \mathbb{P}[\inf_{D_z} \xi^z \leq -\varepsilon/4] \leq \mathbb{P}[A^{h'}(\varphi)] + e^{-2c_4(\varepsilon)L} \leq e^{-f(L)}, \]
where in the penultimate step we used Lemma 4.1 for the singleton $\mathcal{C} := \{z\}$ along with the lower bound $\text{cap}(C_z) \geq cL$ from (2.10) (valid for all $d \geq 3$).

At this point we consider the cases $d = 3$ and $d \geq 4$ separately.

**Upper bound for $d = 3$.** Recall from (1.3) that the quantity $\mathbb{P}[0 \xleftarrow{\varphi \geq h} \partial B_N]$ decays stretched exponentially in $N$ for every $h > h_*$ with some exponent $\beta = c_1(h) \in (0, 1)$. In what follows, we will bootstrap this decay to the one asserted by (5.1) in – as will soon turn out to be necessary – two steps. This is encapsulated in the following proposition, from which the upper bound (5.1) will quickly follow.

**Proposition 5.2 (Bootstrap).** Let $h' \in \mathbb{R}$ and $\beta' \in (0, 1)$ be such that
\[ \lim_{N \to \infty} \sup \frac{1}{N^{\beta'}} \log \mathbb{P}[0 \xleftarrow{\varphi \geq h'} \partial B_N] < 0. \]
Then for all $h > h'$, the following improved bounds hold, depending on the value of $\beta'$. If $\beta' \leq 1/2$, then
\[ \lim_{N \to \infty} \sup \frac{1}{N^{\beta}} \log \mathbb{P}[0 \xleftarrow{\varphi \geq h} \partial B_N] < 0 \quad \text{for every } \beta < 1, \]
whereas if $\beta' > 1/2$, then
\[ \lim_{N \to \infty} \sup \frac{\log N}{N} \log \mathbb{P}[0 \xleftarrow{\varphi \geq h} \partial B_N] \leq \frac{\pi}{6} (h - h')^2. \]

Assuming Proposition 5.2 to hold, we first give the short:

**Proof of (5.1).** Let $h > h_*$. By (1.3) we have (5.11) at any height $h' > h_*$ with exponent $\beta' = c_1(h') > 0$. Therefore, by (5.12), we obtain that (5.11) holds for every $h' > h_*$ and $\beta' < 1$ – in particular for $\beta' = 3/4$ (say). Consequently, we obtain the bound in (5.13) for any $h' \in (h_*, h)$. The result now follows by sending $h' \to h_*$. \qed

**Remark 5.3.** Proposition 5.2 highlights in a transparent form the paradigm underlying our strategy to obtain sharp upper bounds. Indeed, a similar (but considerably more involved) bootstrapping mechanism is at work in the supercritical regime; see Section 5.2. The choice (5.11) as a starting point for the bootstrap reflects the fact that stretched exponential estimates naturally come out of the static renormalization arguments leading to the existence of a non-trivial subcritical regime, see [39]. One could forego one step in deducing (5.1) as (5.12) is implied by the strongest available results [37, 38], but our findings do not rely on these. Moreover, we will face similar issues in the supercritical regime, where such results are not available a-priori. In fact, one could even deduce the desired bound (5.13) from a much weaker a-priori estimate than a stretched-exponential bound by bootstrapping a few more times, see Remark 5.6 below.

We now aim at showing Proposition 5.2. Its proof combines individual estimates for $\mathbb{P}[E_{N,L}]$ and $\mathbb{P}[F_{N,L}]$, cf. (5.7) and (5.8), which are supplied in the following two lemmas.
Lemma 5.4. \((\rho \in (0, 1), K \geq 100, h > h', \varepsilon \in (0, h - h'))\). If \((5.9)\) holds with \(A^h(\varphi) = \{C_0 \leftrightarrow \varphi \in h' \in \partial C_0\} \) for some \(L \geq 1\), then with \(E_{N,L} = E_{N,L}^K(\rho, h, h', \varepsilon)\), for all \(N \geq 10KL\) one has
\[
\log \mathbb{P}[E_{N,L}] \leq n(C \log(nK) - \rho f(L)),
\]
where \(n = \mid C \mid\) for any \(C \in \mathcal{A}\) (cf. \((4.11)\) in Definition \((4.2)\)).

Proof. On account of \((5.7)\) and by a union bound, one obtains
\[
\mathbb{P}[E_{N,L}] \leq \sum_{C \in \mathcal{A}_{N,L}} \mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \hat{C}]
\]
\[
\leq e^{C \log(nK)} \sum_{C \in \mathcal{A}_{N,L}} \mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \hat{C}]
\]
\[
\leq e^{C \log(nK)} \sum_{C \in \mathcal{A}_{N,L}} \mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \hat{C}] \leq e^{-\rho f(L)},
\]
which together with \((5.15)\) gives \((5.14)\).

Next we present the relevant bound for \(\mathbb{P}[F_{N,L}]\). Fix any function \(L_1(N)\) such that \((\log L_1(N) / \log N) \rightarrow \infty\) and \((\log L_1(N) / \log N) \rightarrow 0\) (cf. below \((4.10)\) in Proposition \((4.3)\)). As will become clear, as soon as \(L\) grows fast enough with \(N\), the ‘energy term’ stemming from \(F_{N,L}\) will dominate the ‘entropy term’ arising from the relevant union bound (similar to \((5.15)\) above).

Lemma 5.5. \((\rho \in (0, 1), K \geq 100, h > h', \varepsilon \in (0, h - h'))\). For any \(\theta > 0\), one has
\[
\limsup_{N \rightarrow \infty} \frac{\log N}{N} \log \mathbb{P}[F_{N,L}] \leq -\frac{\pi \lambda(K)(1 - \rho)}{6 \alpha(K)}(h - h' - \varepsilon/4)^2,
\]
with \(\alpha(K)\) and \(\lambda(K)\) as appearing in Lemma \((4.7)\) and Proposition \((4.3)\), respectively.

Proof. Recalling \((5.9)\) and \((5.10)\), and proceeding as in \((5.15)\), one finds that
\[
\log \mathbb{P}[F_{N,L}] \leq Cn \log(nK) + \sum_{C \in \mathcal{A}_{N,L}} \sum_{\hat{C} \subseteq C} \log \mathbb{P}\left[\bigcap_{z \in \hat{C}} \left\{ \sup_{x \in D_z} \xi_x^\sharp \geq h - h' - \varepsilon/4 \right\} \right].
\]
Denoting the event on the right hand side above by \(F(\hat{C})\), we get, combining Lemma \((4.1)\) the capacity lower bound \((4.16)\) from Proposition \((4.3)\), and the fact that \(n \leq N/LK\) by \((4.11)\),
\[
\sup_{L \in [(\log N)^{2+\theta}, L_1(N)\]} \sup_{C \in \mathcal{A}_{N,L}} \sum_{\hat{C} \subseteq C} \log \mathbb{P}[F(\hat{C})]
\]
\[
\leq -\frac{1}{2}(h - h' - \varepsilon/4 - \delta)^2 \frac{\lambda(K)(1 - \rho)}{\alpha(K)} \text{cap}(T_N)(1 + o_N(1)),
\]

where
\[ \delta = \delta(N, K, \rho, \theta) := C \sqrt{\frac{N}{\lambda(K)(1-\rho)K^3 \lambda(\log N)^{2+\theta} \mathrm{cap}(T_N)}} \rightarrow 0, \text{ as } N \to \infty, \]

using the asymptotics for \( \mathrm{cap}(T_N) \) from (2.12) in the last step. Finally notice that
\[ n \log(nK) \leq N K (\log N)^{1+\theta} \]
for \( L \geq (\log N)^{2+\theta} \).

The lemma follows by combining the previous displays along with the asymptotic of \( \mathrm{cap}(T_N) \).

With Lemmas 5.4 and 5.5 at hand, we proceed to the

**Proof of Proposition 5.2**

Let \( h' \in \mathbb{R} \) be such that (5.11) holds and consider \( h > h' \) and \( \epsilon \in (0, h-h') \). First choose \( K \geq 100 \) large enough and \( \rho \in (0, 1) \) close enough to 0, both depending on \( h, h' \) and \( \epsilon \), such that, applying Lemma 5.5, one obtains, for all \( \theta > 0 \), \( N \geq C(\epsilon, h, h', \theta) \) and \( L \in [(\log N)^{2+\theta}, L_1(N)] \),

\[ \log P[F_{N,L}] \leq -\frac{\pi}{6} (h-h'-\epsilon/2)^2 \frac{N}{\log N} \]

(recall to that effect that both \( \alpha(K) \) and \( \lambda(K)(1-\rho) \) converge to 1 in the limit \( \rho \to 0 \) and \( K \to \infty \) by Lemma 4.1 and (4.15) in Proposition 4.3 respectively). Now for any \( N \geq 10^3 \), let

\[ L = L(N, \theta) := (\log N)^{2+\theta}, \]

for some \( \theta > 0 \) to be chosen. Notice that with (5.15) and by (5.11), the condition (5.9) holds with \( A^K(\varphi) = \{ C_0 \varphi \leq h' \} \) and \( f(L) = c(h', \beta') L^{\beta'} \) whenever \( N \geq C(\epsilon, h, h', \theta) \). With this choice of \( f(\cdot) \) and since \( nK \leq N/L \) by (4.11), it follows that

\[ C \log(nK) - \rho f(L) \leq -\rho f(L)/2, \] for all \( N \geq C(\epsilon, h, h', \theta) \)
as soon as \( \theta \) is chosen such that

\[ (2+\theta)\beta' > 1. \]

Hence applying Lemma 5.4 and using the lower bound on \( n \) from (4.11), one gets (with \( L \) as in (5.18))

\[ \log P[E_{N,L}] \leq -\frac{c(h', \beta') \rho N}{K(\log N)^{(2+\theta)(1-\beta')}}, \]

provided (5.19) is satisfied and \( N \) is sufficiently large. Plugging the bounds from (5.17) and (5.20) into (5.8) we immediately deduce, letting \( N \to \infty \) and then \( \epsilon \to 0 \), that

\[ \lim_{N \to \infty} \frac{(\log N)^\beta}{N} \log P[0 \varphi \geq h \partial B_N] \leq -\frac{\pi}{6} (h-h')^2 \]

for any value of \( \beta \) satisfying

\[ \beta > (2+\theta)(1-\beta'), \quad \text{if } (2+\theta)(1-\beta') \geq 1 \]

\[ \beta = 1, \quad \text{otherwise} \]
and any choice of $\theta > 0$ such that (5.19) is satisfied. If $\beta' \leq 1/2$, the conditions $\theta > 0$ and $(2 + \theta)(1 - \beta') < 1$ cannot simultaneously hold. Hence, in this case, choosing for example $\theta = \theta'(\beta')$ so that $(2 + \theta)\beta' = 2$, whence (5.19) is satisfied. (5.21) yields the bound (5.12). On the other hand when $\beta' > 1/2$, the conditions (5.19) and $(2 + \theta)(1 - \beta') < 1$ can be recast as

$$\frac{1}{\beta'} - 2 < \theta < \frac{1}{1 - \beta'} - 2$$

(note that the interval of admissible values for $\theta$ is non-degenerate because $\beta' > 1/2$). So choosing for instance $\theta = \frac{1}{2\beta'(1 - \beta') - 2}$, we obtain (5.21) with $\beta = 1$ (since $(2 + \theta)(1 - \beta') < 1$ holds), which is (5.13).

\[\square\]

Remark 5.6. A careful examination of the proof of Proposition 5.2 reveals that a stretched exponential a-priori bound such as (5.11) is not required to arrive at (5.1). Indeed one could for instance obtain the same result by means of a few additional bootstrapping steps starting from a much weaker estimate of the type $\limsup_{L} \frac{P[C_{0,L}, e^{\frac{x}{\rho}} \partial \tilde{C}_{0,L}]}{\log L} < 0$ for some $\beta'(h) > 0$ and all $h > h_*$ (or even a $k$-fold composition of log, for some fixed integer $k \geq 1$). Combining with other existing methods, see e.g. [37], one would further obtain that (5.1) holds as soon as $P[C_{0,L}, e^{\frac{x}{\rho}} \partial \tilde{C}_{0,L}]$ is bounded from above by a suitable $c(d) \in (0, 1)$ uniformly along a diverging subsequence of scales $L$. Similar conclusions could be drawn in the supercritical regime, cf. Remark 5.17(1).

Upper bound for $d \geq 4$. We now supply the proof of (5.3). Throughout the remainder of Section 5.1, for an arbitrary level $h > h_*$ (as appearing in (5.3)), we simply fix $h' = (h_* + h)/2$, $\varepsilon = (h - h')/8$, $\rho = 1/2$ and $K = 100$ in (5.5)–(5.7). The events $E_{N,L}$ and $F_{N,L}$ thus effectively depend on the sole parameter $h$. The following two results replace Lemmas 5.3 and 5.5, respectively.

**Lemma 5.7.** ($d \geq 4$, $h > h_*$) For all $L \geq C(h)$ and $N \geq 10^3L$ one has

$$P[E_{N,L}] \leq e^{-c(h)N/L}. \tag{5.23}$$

**Proof.** The proof mimics that of Lemma 5.1 with small modifications. Proceeding as in (5.15), using (4.13), (4.14) and the bound $e^{C_{L}(\log L)^2} \leq n$ from (4.11) to bound $|A_{N,L}^{100}|$, one finds,

$$P[E_{N,L}] \leq e^{C_n(\log L)^2} P[0 \text{ is } \psi\text{-bad}]^{[n/2]} \leq e^{n(C(\log L)^2 - c(h)L^{\varepsilon_1})}, \tag{5.24}$$

for all $L \geq 1$, $N \geq 10^3L$, where the first inequality also relies on the independence property (4.7) and the second one on the fact that (1.3) and (5.9)–(5.10) combine to give a suitable bound on $P[0 \text{ is } \psi\text{-bad}]$. Using the lower bound on $n$ yet again, (5.23) readily follows from (5.24). \[\square\]

Remark 5.8. The conclusions of Lemma 5.7 would remain unaltered if one replaced (1.3) by the (weaker) assumption that $P[C_{0,L}, e^{\frac{x}{\rho}} \partial \tilde{C}_{0,L}] \leq e^{-c(h)(\log L)^{2+\varepsilon}}$, for some $\varepsilon = \varepsilon(h) > 0$ and all $L \geq 1$, $h > h_*$. This is related to the power in the definition of $\varepsilon_m$ in (4.29), and could be relaxed to a $1 + \varepsilon$-condition by suitable modification of (4.29) and the subsequent arguments of Section 4.2 which would lead to a corresponding improvement of the lower bound on $n$ in (4.11).

The analogue of Lemma 5.5 is

**Lemma 5.9.** ($d \geq 4$, $h > h_*$) For some $C(h) > 0$,

$$\limsup_{N \to \infty} \sup_{L \in [C(h), N/10K]} \frac{1}{N} \log P[F_{N,L}] < 0. \tag{5.25}$$
Proof. For arbitrary $L \geq 1$, $N \geq 10^3L$ and any given collection $\mathcal{C} \in \mathcal{A} = \mathcal{A}_{N,L}^{100}$ and $\tilde{\mathcal{C}} \subset \mathcal{C}$ with $|\tilde{\mathcal{C}}| = n - \lfloor n/2 \rfloor$, one obtains by virtue of (4.16) and (2.13) that $\mathrm{cap}(\tilde{\Sigma}) \geq cN$. Together with Lemma 4.1, this is seen to imply that,

$$ (5.26) \quad \mathbb{P} \left[ \bigcap_{z \in \tilde{\mathcal{C}}} \left\{ \sup_{x \in D_z} \xi_z^x \geq h - h' - \frac{3}{4} \right\} \right] \leq e^{-c(h)N} $$

(with $h'$, $\varepsilon$ as defined above Lemma 5.7) whenever $L \geq C(h)$ and $N \geq 10^3L$, noting that $\frac{|\tilde{\mathcal{C}}|}{\mathrm{cap}(\tilde{\Sigma})} \leq cL^{-1}$ becomes suitably small for such $L$, cf. (4.9). In view of (5.7), applying a union bound over the choices of $\mathcal{C} \in \mathcal{A}_{N,L}$ and $\tilde{\mathcal{C}} \subset \mathcal{C}$, (5.26) is easily seen to follow since $\log |\mathcal{A}| \leq Cn(\log L)^2 \leq C\frac{N}{L}$, see (5.24) and the upper bound on $n$ from (1.11). Thus, the resulting combinatorial complexity doesn’t spoil the upper bound in (5.26) whenever $L \geq C'(h)$. \hfill \Box

Proof of (5.3). The upper bound (5.3) follows immediately by combining (5.8), (5.23) and (5.25) upon choosing $L = C(h)$ large enough to ensure both Lemmas 5.7 and 5.9 are in force. \hfill \Box

Remark 5.10. Following up on Remark 4.7, we describe which upper bounds can be derived for $d \geq 4$ using the collection $\mathcal{A}'$ (obtained by following the coarse-graining scheme used for $d = 3$). For $h > h_*$, fix $h'$, $\varepsilon$, $K$ and $\rho$ as above (5.7) and define $E'_{N,L} = E'_{N,L}^{K}(\rho, h, h', \varepsilon)'$, $F'_{N,L}$ as in (5.7), but with $\mathcal{A}'$ in place of $\mathcal{A}$. Similarly as in (5.8), by admissibility of $\mathcal{A}'$, one sees that $\mathbb{P}[0 \xrightarrow{\varphi \geq h} \partial B_N] \leq \mathbb{P}[E'_{N,L}] + \mathbb{P}[F'_{N,L}]$. Using (1.11), (5.26) and recalling the larger combinatorial complexity of $|\mathcal{A}'|$ (as in (5.15) for instance), one finds that

$$ (5.27) \quad \mathbb{P}[F'_{N,L}] \leq e^{Cn\log(nK)}e^{-c(h)N} \leq e^{CN\log N - cN} \leq e^{-c'(h)N}, \quad \text{if } C(h) \log N \leq L \leq cN. $$

Regarding $\mathbb{P}[E'_{N,L}]$, observe that the bound (5.11) from Lemma 5.4 holds with $E'_{N,L}$ instead of $E_{N,L}$. This crucially uses the fact that $n = |\mathcal{C}|$ for $\mathcal{C} \in \mathcal{A}'$ satisfies the lower bound (4.11) with $u(x) = x$, which is used in the proof of Lemma 5.4. As a consequence, one obtains

$$ (5.28) \quad \mathbb{P}[E'_{N,L}] \leq e^{-CL(L)N/L}, \quad \text{if } f(L) \geq C \log N, \quad L \leq cN, $$

assuming (5.9) holds with $A'(\varphi) = \{C_0 \xrightarrow{\varphi \geq h'} \partial \bar{C}_0\}$. As we now explain, one can deduce from (5.27) and (5.28) that for any integer $k \geq 1$

$$ (5.29) \quad \mathbb{P}[0 \xrightarrow{\varphi \geq h} \partial B_N] \leq e^{-c(h)N/(\log(h)N)^{C_6}}, \quad \text{for all } N \geq 1, \ h > h_*, $$

where $C_0 = C_0(d) \in [1, \infty)$ and $\log^{(k)}(\cdot)$ denotes the $k$-fold composition of $\log(\cdot)$. To obtain (5.29) one proceeds as follows: in a separate (first) step, starting from (1.3), one chooses $f(L) = c(h)L^{c_1}$ and $L = C(h)(\log N)^{1+c_1^{-1}}$, whence (5.27) and (5.28) apply and yield (5.29) for $k = 1$ with $C_0 = c_1^{-1} - c_1$. Now, assuming (5.29) to hold for some $k \geq 1$, one chooses $f(L) = c(h)L/(\log(h)N)^{C_6}$ and $L = C(h)\log N(\log^{(k)}N)^{C_6}$, whence (5.27) and (5.28) apply (in particular (5.9) holds with this choice of $f(\cdot)$ due to (5.29)), and the conditions on $L$ and $f(L)$ in (5.27), (5.28) are met) and readily yield (5.29) with $k + 1$ instead of $k$.

Note that (5.29) is nearly (5.3) and that the obstruction to obtaining the desired result comes from competition between the entropy of $\mathcal{A}'$ and the local fields leading to (5.28), not the contribution (5.27) from the harmonic average, which exhibits the desired exponential decay.

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5.2. Supercritical phase. We now proceed to the proofs of (5.2) and (5.4) in Theorem 5.1 along with the corresponding statements for LocUniq\((N,h)^c\), \(h < h_\ast\), which we will actually prove first. In all cases, our argument revolves around a notion of good event \(G_N\), see (5.32), which will allow us to construct ambient clusters with certain desirable properties. The bottom line is that it will be costly for any large connected set to avoid connecting to any such ambient cluster. This is quantified in Lemma 5.11. The desired estimate for \(\mathbb{P}[G_N]\) is then arrived to by means of a renormalization scheme, starting from a certain localized good event \(G_2\), see Definition 5.12 which satisfies a suitable a-priori estimate, see Lemma 5.16. Importantly, the scheme, whose essential features are captured by Proposition 5.14, improves not only only probabilistic estimates but also the number \(a\) of contact points inherent to the definition of \(G_2\), see (5.44), in each step of the iteration. The proofs of (5.2) and (5.4) for all events of interest follow by combining Lemma 5.11 Proposition 5.14 and Lemma 5.16 and are presented at the end of this section.

For \(\tilde{f} : \mathbb{Z}^d \to \mathbb{R}\) (cf. above (2.32) regarding \(\mathbb{Z}^d \supset \mathbb{Z}^d\)), we define the local average \((A\tilde{f})_x = (2d)^{-1} \sum_{m \in \mathcal{M} : m \sim x} f_m\), for \(x \in \mathbb{Z}^d\). For integers \(L_0 \geq 1\) and \(M > 1\), we then introduce the set
\[
(5.30) \quad \mathcal{M}(\tilde{f}) := \{x \in \mathbb{Z}^d : (A\tilde{f})_y \geq -M \text{ for all } y \in B_{2L_0}(x)\}, \quad \mathcal{M} := \mathcal{M}(\tilde{f})
\]
(note that \(B_{2L_0}(x) \subseteq \mathbb{Z}^d\) by definition and recall the extension \(\tilde{f}\) from (2.33)). Due to the decomposition \(\tilde{f} = \tilde{\xi} + \tilde{\psi}\) from (2.37), this means that \(\mathcal{M} = \{x \in \mathbb{Z}^d : \tilde{\xi}_y \geq -M; y \in B_{2L_0}(x)\}\). The condition used in the definition (5.30) provides us with a uniform insertion tolerance bound on the set \(\mathcal{M}\) which will be used in the proof of Lemma 5.11 below (see around the display (5.37)) and which — as already noted in the beginning of Section 3.3 — is not otherwise true.

We now introduce a key (good) event, involving various parameters, which roughly ensures the existence of ambient clusters with desirable properties. In the sequel, \(N, L\) and \(L_0\) will denote three relevant length scales, \(M > 1\) controls the midpoint averages from below in (5.30), and \(b\) will count a certain number of ‘interfaces’, to be introduced shortly, each containing \(a\) (so-called) ‘contact points’. Throughout the rest of this section, we assume that
\[
(5.31) \quad N, L, L_0 \text{ are integers with } N \geq L > 2L_0 \geq 1, \quad M > 1 \text{ and } a, b \in \mathbb{N} > 0,
\]
and that \(\Lambda_N \in \{B_{2N} \setminus B_N, B_N \setminus B_{\sigma N}, \sigma \in (0, \frac{1}{4})\}\) is arbitrary, unless specified otherwise. To avoid clumsy notation, we will keep dependence on the quantities appearing in (5.31) and \(\Lambda_N\) implicit in the sequel, except for the ones that are subject to change in any given context.

The intermediate scale \(L\) will first appear in Definition 5.12 below. For \(h' \in \mathbb{R}\), the event \(G_N = G_N(L_0, a, b, h', M)\) is defined as
\[
(5.32) \quad G_N = \left\{ \begin{array}{l}
\exists \text{ disjoint connected sets } \mathcal{C}_i \subset (\Lambda_N \cap \{\varphi \geq h'\}), \text{ for } 1 \leq i \leq b, \text{ such that any } * \text{-path } \gamma \text{ crossing } \Lambda_N \text{ contains points } x_{i,j} = x_{i,j}(\gamma) \text{ satisfying } \\
B_{L_0}(x_{i,j}) \cap \mathcal{C}_i \cap \mathcal{M} \neq \emptyset, \text{ for all } 1 \leq i \leq b \text{ and } 1 \leq j \leq a.
\end{array} \right\}
\]
(for concreteness, the indexing of the sets \(\mathcal{C}_i\) can be induced by a fixed deterministic ordering of the vertices of \(\mathbb{Z}^d\), with \(\mathcal{C}_1\) containing the smallest vertex in this ordering etc.) Notice that the sets \(\mathcal{C}_i\) in (5.32) may very well be connected in \(\Lambda_N \cap \{\varphi \geq h'\}\). In the sequel, we will refer to the sets \(\mathcal{C}_i\) as ‘interfaces’ and to \(x_{i,j}\), for \(1 \leq i \leq b\) and \(1 \leq j \leq a\), as the ensuing ‘contact points’. Thus, there will be a total of \(a \cdot b\) contact points whenever \(G_N\) occurs.

For \(h' \in \mathbb{R}\), consider the sigma-algebra \(\mathcal{F}_{h'} := \sigma(\tilde{\varphi}_y, 1\{\varphi_x \geq h'\}; x \in \mathbb{Z}^d, y \in \mathbb{M}^d)\) and notice for later reference that \(\mathcal{M}\), the event \(G_N\) as well as the sets \(\mathcal{C}_1, \ldots, \mathcal{C}_b\) are all \(\mathcal{F}_{h'}\)-measurable. Our interest in \(G_N\) stems from the following lemma. For later reference, note that the contribution of \(\mathbb{P}[G_N] \) in (5.33) and (5.34) will eventually turn out to be dominant in generating (1.7).
Lemma 5.11. \((h < h', (5.31))\). There exists \(c = c(h, h', L_0, M) > 0\) such that, with \(G_N = G_N(L_0, a, b, h')\) as defined in (5.32), the following holds when \(a \geq C(L_0, h, h', M)\) and \(b \geq C(L_0)\). If \(\Lambda_N = B_N \setminus B_{\sigma N}\),

\[
\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N, 0 \xleftrightarrow{\varphi \geq h'} \infty] \leq \mathbb{P}[G_N] + \mathbb{P}[B_{\sigma N} \xleftrightarrow{\varphi \geq h'} \infty] + e^{-cba}.
\]

and if \(\Lambda_N = B_{2N} \setminus B_N\),

\[
\mathbb{P}[\text{LocUniq}(N, h'c)] \leq \mathbb{P}[G_N] + \mathbb{P}[B_N \xleftrightarrow{\varphi \geq h'} \partial B_{2N}] + e^{-cba}.
\]

Before delving into the proof, let us briefly give some intuition for (5.33) and (5.34) and describe the salient features of the argument. We will focus on (5.33) (the case of (5.34) is similar). In essence, (5.33) asserts that, whenever the event \(G_N\) occurs and \(B_{\sigma N}\) connects to infinity at level \(h'\) slightly below \(h\), the event of interest on the left-hand side of (5.33) comes at a uniform cost in each of the areas around the \(a \cdot b\) contact points generated by the event \(G_N\). Indeed, around each of those contact points, the cluster of 0 in \(h\) in infinite cluster at level \(h\) to hold in the area around contact points by definition of \(G_N\). The range correlations, which is dealt with in the proof by exploring the contact points one by one, and working with two sprinkled levels \(h\) and \(h'\). The controls given by (5.30), which are required to hold in the area around contact points by definition of \(G_N\), see (5.32) (in particular, note that every contact point has a nearby point in \(M\)) will be used to implement this decoupling.

**Proof.** We only give the proof of (5.33). The proof of (5.34) follows by straightforward modifications of the argument. We begin by introducing an auxiliary event \(G_N'\) as follows. With \(h'' = (h + h')/2\) and for \(\lambda > 0\), the event \(G_N' (\lambda)\) occurs if all of the following hold:

i) the set \(\{\varphi \geq h''\}\) contains an infinite cluster \(C\) which intersects \(B_{\sigma N}\).

ii) Letting \(S = B_N \cap C \cap M\), there exists — for any \(*\)-path \(\gamma\) crossing \(B_N \setminus B_{\sigma N}\) — a set \(S \subset \gamma\) with \(|S| \geq \lambda ba\) such that \(B_{L_0}(x) \cap S \neq \emptyset\) for every \(x \in S\).

The parameter \(\lambda\), chosen below in (5.38), will ensure a certain well-separatedness property. Clearly the event \(G_N' (\lambda)\) is measurable relative to \(F_{h''}\) (cf. below (5.32)). We will argue that for suitable \(c, \lambda > 0\) each depending on \(h, h', L_0\) only,

\[
\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N, 0 \xleftrightarrow{\varphi \geq h'} \infty | F_{h''}] 1_{G_N'(\lambda)} \leq e^{-cba},
\]

and

\[
\mathbb{P}[G_N'(\lambda) \cap G_N \cap \{B_{\sigma N} \xleftrightarrow{\varphi \geq h'} \infty\}] \leq e^{-cba},
\]

from which (5.33) readily follows.

The proofs of (5.35) and (5.36) both hinge on the following representation of the conditional distribution of \(\{\varphi \geq h\}\) under \(\mathbb{P}[\cdot | F_{h''}]\). In what follows, given \(p = (p_x)_{x \in \mathbb{Z}^d}\) with \(p_x \in [0, 1]\), let \(P_p\) denote the corresponding product measure on \([0, 1]^{\mathbb{Z}^d}\), with canonical coordinates \(Y = \cdots\)
\((Y_x)_{x \in \mathbb{Z}^d}\), so that \(P_p[Y_x = 1] = p_x\) for all \(x \in \mathbb{Z}^d\). By means of (2.37) and (2.38), one infers that for all \(h > h^\prime\),

\[
\text{the law of } (1\{\varphi_x \geq h\})_{x \in \mathbb{Z}^d} \text{ under } \mathbb{P}[\cdot | \mathcal{F}_h^0] \text{ is } P_p \text{ with } p_x = \mathbb{P}[\varphi_x \geq h | \mathcal{F}_h^0], \quad x \in \mathbb{Z}^d,
\]

\[
p_x = \begin{cases} 1 & \text{if } x \in \{\varphi \geq h\} \\ \geq c_0 = c_0(h, h^\prime, M) & \text{if } x \in B_{L_0}(M). \end{cases}
\]

Indeed, regarding the last part, on the event \(\{\varphi_x < h^\prime\}\), one has by (2.37) and (2.38) that \(\mathbb{P}[\varphi_x \geq h | \mathcal{F}_h^0] = 1 - \frac{\Phi(\sqrt{h - \xi_x})}{\Phi(\sqrt{h^\prime - \xi_x})}\), where \(\Phi(\cdot)\) denotes the distribution function of a standard Gaussian random variable. In case \(x \in B_{L_0}(M)\), one obtains the desired lower bound using that \(\xi_x \geq -M\) by (5.30) and \(h > h^\prime\) whilst noting that \(\lim_{s \to \infty} \frac{\Phi(s - t)}{\Phi(s)} = 0\) for any \(s, t \in \mathbb{R}\) with \(s < t\).

With (5.37) at hand, we proceed to show (5.35), (5.36), starting with (5.35) and choosing

\[
\lambda := 1/(3|B_{3L_0}|)^2 \text{ (so that } 2\sqrt{\lambda b} \leq b/|B_{3L_0}| \text{ when } b \geq C(L_0)).
\]

The choice of \(\lambda\) in (5.33) will soon become clear. Let \(C(0)\) denote the cluster of 0 in \(B_{L_0} \cap \{\varphi \geq h\}\), under the measure \(\mathbb{P}[\cdot | \mathcal{F}_h^0] = P_p\). We explore \(C(0)\) vertex by vertex starting from 0 in a canonical way, i.e. checking at each step the state of the smallest (in a fixed deterministic ordering of the points in \(\mathbb{Z}^d\)) unexplored vertex in the exterior neighborhood of the currently explored piece of \(C(0)\). We do so until the first time we discover a vertex \(x_1 \in C(0)\) (there may not be any) which is in the exterior neighborhood of some \(B_{L_0}(y_1)\) with \(y_1 \in \mathcal{S}\) (note that \(\mathcal{S}\) as defined in ii) above is \(\mathcal{F}_{h^\prime}\)-measurable). At this point, we explore the state of every vertex in \(B_{L_0}(y_1)\). By definition, \(\mathcal{S} \subset \mathcal{C}\) so \(\mathcal{C}\) intersects \(B_{L_0}(y_1)\). We stop the exploration if some vertex of \(\mathcal{C} \cap B_{L_0}(y_1)\) lies in \(C(0)\), which occurs for instance if all the vertices of \(B_{L_0}(y_1)\) belong to \(C(0)\). Otherwise we continue exploring \(C(0)\) until we discover a vertex \(x_2 \in C(0)\) in the exterior neighborhood of \(B_{L_0}(y_2)\) for some \(y_2 \in \mathcal{S} \setminus \{y_1\}\) which was not visited by the exploration yet. As before, we then explore the state of every vertex in \(B_{L_0}(y_2)\), stopping the exploration if \(C(0)\) intersects \(\mathcal{C}\) in that box and continuing otherwise. We proceed like this until either stopping or discovering the whole cluster \(C(0)\). By construction, if the algorithm stops prior to discovering \(C(0)\), the event \(\{0 \not\xrightarrow{\mathcal{G}_N} h^\prime \\infty\}\) occurs.

Using the above algorithm, one deduces the following. Let \(\tau := |\{y_1, y_2, \ldots, \}|\) denote the number of points \(y_i\) discovered until the algorithm stops (possibly \(\tau = 0\). Then in view of property ii) above, with \(\lambda^\prime = \lambda/|B_{L_0+1}|\),

\[
G_N^\prime(\lambda) \cap \{0 \not\xrightarrow{\mathcal{G}_N} h^\prime \\infty\} \subset \{\tau \geq \lambda^\prime b a\}
\]

(in particular, the right-hand side implies \(\{\tau \geq 1\}\) whenever \(a \geq C(h, h^\prime, L_0)\), as stipulated above (5.33)). Moreover, by means of (5.37) and (5.38), one sees that under \(\mathbb{P}[\cdot | \mathcal{F}_{h^\prime}^0]\) and on the event \(G_N^\prime(\lambda)\), conditionally on \(\{\tau > n\}\) for some \(0 \leq n < \lambda^\prime b a\), the event \(\{\tau = n + 1\}\) occurs with probability at least \(c_0^\prime\), where \(c_0^\prime(h, h^\prime, L_0, M) := c_0(h, h^\prime, M)^{|B_{L_0+1}|}\) (with \(c_0\) as in (5.37)). Together with (5.39), this readily implies that the left-hand side of (5.35) is bounded by \((1 - c_0^\prime)^{\lambda^\prime b a}\), as claimed.

We now turn to the proof of (5.36). Recall that \(G_N \cap \{B_{\sigma N} \not\xrightarrow{\mathcal{F}_{h^\prime}} h^\prime \\infty\} \text{ is } \mathcal{F}_{h^\prime}\)-measurable. We will in fact show that

\[
\mathbb{P}[G_N^\prime(\lambda)^c | \mathcal{F}_{h^\prime}] 1\{G_N, B_{\sigma N} \not\xrightarrow{\mathcal{F}_{h^\prime}} h^\prime \\infty\} \leq e^{-c_b a},
\]

39
which implies (5.36). Under $\mathbb{P}[\cdot \mid \mathcal{F}_h']$ and on the event $G_N \cap \{B_{\sigma_N} \xleftarrow{\varphi \geq h'} \infty \}$, fix a path $\gamma_0$ in the infinite cluster of $\{\varphi \geq h'\}$ crossing $\Lambda_N$. Notice in particular that $\gamma_0 \subset \mathcal{C}$, cf. i) above. By definition of $G_N$, see (5.32), and by suitable labeling (using for instance the above ordering of $\mathbb{Z}^d$), one finds disjoint connected sets $\mathcal{C}_i \subset (\Lambda_N \cap \{\varphi \geq h'\})$, for $1 \leq i \leq [2\sqrt{\lambda}b]$ (with $\lambda$ as in (5.33)) and corresponding points $x_{i,j}(\gamma_0), 1 \leq j \leq [\lambda], \gamma$, satisfying $B_{L_0}(x_{i,j}(\gamma_0)) \cap \mathcal{C}_i \cap M \neq \emptyset$ and $|x_{i,j}(\gamma_0) - x_{i',j'}(\gamma_0)| \geq 2L_0$ for any $(i,j) \neq (i',j')$ and all $1 \leq i,i' \leq [2\sqrt{\lambda}b], 1 \leq j,j' \leq [\lambda a]$. Thus, letting

$$I := \{i : x_{i,j}(\gamma_0) \xleftarrow{\varphi \geq h''} \mathcal{C}_i \text{ for some } 1 \leq j \leq [\sqrt{\lambda}a]\}$$

it follows that under $\mathbb{P}[\cdot \mid \mathcal{F}_h']$ and on the event $\{B_{\sigma_N} \xleftarrow{\varphi \geq h'} \infty, G_N\}$,

$$G_N(\lambda)^c \subset \{|I| < \sqrt{\lambda}b\};$$

indeed for an arbitrary $\gamma$-path $\gamma$ crossing $\Lambda_N$, the occurrence of $G_N$ guarantees the existence of vertices $x_{i,j}(\gamma)$ such that $B_{L_0}(x_{i,j}(\gamma)) \cap \mathcal{C}_i \cap M \neq \emptyset$, for every $1 \leq i \leq [2\sqrt{\lambda}b]$ and $1 \leq j \leq [\sqrt{\lambda}a]$. Moreover, if $i \in I$, then in fact $\mathcal{C}_i \subset \mathcal{C}$ on account of (5.41). Hence, if $|I| \geq \sqrt{\lambda}b$, the set $S \{x_{i,j}(\gamma) : i \in I, 1 \leq j \leq [\sqrt{\lambda}a]\}$ satisfies the properties listed in ii) above and (5.42) follows (the occurrence of $i$ is ensured by that of $B_{\sigma_N} \xleftarrow{\varphi \geq h'} \infty$, which is conditioned on).

Finally, using (5.37), (5.33), (5.41) and the separation of the points $x_{i,j}(\gamma_0), 1 \leq i \leq [2\sqrt{\lambda}b], 1 \leq j \leq [\sqrt{\lambda}a]$, one infers that $|I|$ stochastically dominates (under $\mathbb{P}[\cdot \mid \mathcal{F}_h']$ and on $G_N$) a binomial random variable with $[2\sqrt{\lambda}b]$ trials and success probability $1 - (1 - c_{10})[\sqrt{\lambda}a]$, where $c_{10} := c_9(h,h',L_0,M)$. Thus,

$$\mathbb{P}[|I| < \sqrt{\lambda}b \mid \mathcal{F}_h'] \leq 2[2\sqrt{\lambda}b](1 - c_{10})[\sqrt{\lambda}a]\sqrt{\lambda}b \leq e^{-c_{10}b}$$

for suitable $c = c(L_0,h,h',M)$, as soon as $\sqrt{\lambda}a \geq C(L_0,h,h',M)$, whence (5.40) readily follows from (5.42). This completes the proof of (5.36) and with it that of Lemma 5.11.

The upper bounds (5.2) and (5.4), as well as the corresponding bounds for LocUniq$(h,N)^c$, will eventually follow from Lemma 5.11. As mentioned above that lemma, all estimates will ultimately turn out to be carried by the upper bound for $\mathbb{P}[G_N^c]$. Our next step is thus to derive suitable upper bounds on $\mathbb{P}[G_N^c]$, which we achieve by a renormalization scheme operating as follows. In a single step, the scheme goes from a base scale $L$ to a larger scale $N$, cf. (5.31). Whereas $b$ as appearing in (5.32) remains effectively fixed through the iteration (as will turn out $b$ will grow roughly like $N/u(KL)$, cf. (4.11)), the scheme is designed to simultaneously improve on both the growth of $a := a_L$ in (5.32) and the strength of the estimate on $\mathbb{P}[G_N^c]$ in each step. Roughly speaking, this will boost $a_L$ from being of order 1 to growing faster than $L/(\log L)^C$ when $d = 3$ and linearly in $L$ when $d \geq 4$, whence the error terms in (5.33), (5.34) become sufficiently small.

The starting point of the argument is a certain good event $\mathcal{G}_z$, which we now introduce, and for which we will later supply a suitable a-priori estimate, see Lemma (5.16). In the sequel, $\tilde{\chi} = (\tilde{\chi}^z)_{z \in \mathcal{L}}$, where $\tilde{\chi}^z$ refers to either $\tilde{\phi}$ or $\tilde{\psi}^z$, see (4.5). We write $\chi^z$ for the restriction of $\tilde{\chi}^z$ to $\mathbb{Z}^d$. Recall (4.1)-(4.4), (5.30) and (5.31) regarding notation.
Definition 5.12. (Good event). For \( h_1 \leq h_2 \leq h_3 \in \mathbb{R} \) and \( z \in \mathbb{L} \), the event \( \mathcal{G}_z(\tilde{x}) = \mathcal{G}_z(\tilde{x}, L_0, L, a, h_1, h_2, h_3, M) \) occurs if all of the following hold:

\[
C_z \text{ is connected to } \partial D_z \text{ in } \{ \chi^z \geq h_3 \} \text{ and for all } z' \in \mathbb{L} \text{ such that } |z - z'|_\infty \leq L, \text{ all clusters of } \{ \chi^{z'} \geq h_2 \} \text{ crossing } \tilde{D}_z \setminus \tilde{C}_z \text{ are connected inside } D_z \cap \{ \chi^z \geq h_1 \}.
\]

(5.43)

Letting \( \mathcal{S}_z = \tilde{D}_z \cap \mathcal{C}_z \cap \mathcal{M}(\tilde{\chi}^z) \) where \( \mathcal{C}_z \) denotes the cluster of \( C_z \)

(5.44)

in \( \{ \chi^z \geq h_1 \} \), for any \( \gamma \)-path \( \gamma \) crossing \( \tilde{D}_z \setminus \tilde{C}_z \) there exists a set \( S \subset \gamma \) with \( |S| \geq a \) such that \( B_{L_0}(x) \cap \mathcal{S}_z \neq \emptyset \) for every \( x \in S \).

Henceforth we routinely suppress the dependence of \( \mathcal{G}_z \) on parameters which stay fixed. Note that \( \mathcal{G}_z \) indeed depends on \( \tilde{\varphi} \), the extension of \( \varphi \) to \( \tilde{Z}^d \), through \( \mathcal{M}(\tilde{\chi}^z) \) in (5.44), see (5.30). We now introduce, abbreviating \( \tilde{\psi} = (\tilde{\psi}^z)_{z \in \mathbb{L}} \) and where, as in (1.9), \( \sup_{D_z}, \inf_{D_z} \) refer to the suprema/infima over all points in \( D_z \cap \tilde{Z}^d \).

The events introduced in Definition 5.12 will be the fundamental building blocks of most constructions to follow. Lemma 5.16 will soon witness these events to be typical in the supercritical regime when \( a = 1 \). The goal will then roughly consist of improving both their likelihood and the value of \( a \) simultaneously, a feature which will be achieved by means of the bootstrapping scheme in Proposition 5.14 below.

The intuition for the above definition is as follows. Typically, neighboring points \( z, z' \in \mathbb{L} \) for which \( \mathcal{G}_z \) and \( \mathcal{G}_{z'} \) occur roughly give rise to corresponding macroscopic cluster at scale \( L \) which must communicate (i.e. be glued together) due to (5.43). Moreover, by means of (5.44), the resulting joint cluster has a desirable number of contact point(s) (parametrized by \( S \)) for any macroscopic annulus crossing \( \gamma \) in either of the constituent boxes. Continuing in this way by stacking neighboring (good) boxes in which these events occur thus generates large clusters with these properties. Here, we are deliberately passing over various technical details, including the necessity to work with both the full and localized fields, whence \( \tilde{\chi} \), and the fact that our construction must operate simultaneously at various sprinkled levels \( h_1 - h_3 \), essentially to permit going back and forth between the two fields, whenever the relevant harmonic average \( \tilde{\psi} \) is under control, as ensured by (5.45) or (5.46).

We now make these heuristics precise. Let \( \varepsilon_0 := \frac{1}{4}(1 \wedge (h_2 - h_1) \wedge (h_3 - h_2)) \). We start by considering two points. The key features of (5.43)-(5.46) are summarized in the following

Lemma 5.13 (Gluing; \( h_1 < h_2 < h_3, \varepsilon \in (0, \varepsilon_0), (5.31) \)).

For \( z \in \mathbb{L} \), with \( (\tilde{x}_1, \tilde{x}_2) \in \{ (\tilde{\varphi}, \tilde{\psi}), (\tilde{\psi}, \tilde{\varphi}) \} \) and \( \mathcal{G}^1 := \mathcal{G}_z(\tilde{x}_1, h_1, h_2, h_3, M), \mathcal{G}^2 := \mathcal{G}_z(\tilde{x}_2, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon) \), one has

\[
\mathcal{G}^1 \cap \{ \sup_{D_{z'}} \sup_{z''} |\tilde{\chi}^{z''}| \leq \varepsilon \} \subset \mathcal{G}^2
\]

(5.47)

with \( z' \) ranging over all points in \( \mathbb{L} \) with \( |z' - z|_\infty \leq L \). In particular,

\[
\mathbb{P}[\mathcal{G}^2] \geq 1 - e^{-f(L)} \text{ whenever } \mathbb{P}[\mathcal{G}^1] \geq 1 - e^{-2f(L)} \text{ with } 4 \log(d) \leq 4f(L) \leq c_8(4\varepsilon)L
\]

(5.48)
where \( c_8 \) is as in (5.9). Moreover, for \( z, z' \in \mathbb{L} \) with \( |z' - z|_\infty = L \), if \( \mathcal{H}_z^i \cap \mathcal{H}_{z'}^i \) occurs for some \( i \in \{1, 2\} \),

\[
\text{all clusters of } \{ \Xi_i^y \geq h_2 + \varepsilon \} \text{ crossing } \tilde{D}_y \setminus \tilde{C}_y, \text{ for } y \in \{z, z'\}, \text{ belong to a single connected subset } \mathcal{C}_{z, z'}^i \text{ of } \{ \varphi \geq h_1 - \varepsilon \} \cap (D_z \cup D_{z'}), \text{ which crosses both } D_z \setminus C_z \text{ and } D_{z'} \setminus C_{z'},
\]

(5.49)  

where \( \Xi_i^y = \varphi, \varepsilon_1 = \varepsilon \) and \( \Xi_i^y = \psi^y, \varepsilon_2 = 0 \), and

(5.50)  

for any \( * \)-path \( \gamma \) crossing \( \tilde{D}_y \setminus \tilde{C}_y \), for \( y \in \{z, z'\} \), there exists a set \( S \subset \gamma \) with \( |S| \geq 1 \) such that \( B_{L_0}(x) \cap \tilde{D}_y \cap \mathcal{C}_{z, z'}^i \cap \mathcal{M}_{M+\varepsilon}(\tilde{\varphi}) \neq \emptyset \), for all \( x \in S \).

**Proof.** We write \( \mathcal{M} = \mathcal{M}_M \) throughout the proof to make the dependence on \( M \) explicit, cf. (5.30). The two-sided control on the harmonic average in (5.47) and (4.1) imply that for all \( h \in \mathbb{R} \) and \( i \in \{1, 2\} \), any connected subset of \( \{ \chi_i \geq h \} \cap D_{z'} \) is contained in a connected subset of \( \{ \chi_{3-i} \geq h - \varepsilon \} \cap D_{z'} \), for any \( z' \in \mathbb{L} \) with \( |z' - z|_\infty \leq L \). Applying this repeatedly in the context of (5.48) readily yields the corresponding property for the event on the right-hand side of (5.47). Regarding (5.44), recalling the local averaging operator \( A \) from above (5.30), on \( B_{2L_0}(\tilde{D}_z)(\subset \tilde{\Xi}_i^z) \) one has that \( A\tilde{\chi}_2 = A(\tilde{\chi}_1 \pm \tilde{\xi}^z) \geq A\tilde{\chi}_1 - |A\tilde{\xi}^z| \geq A\tilde{\chi}_1 - \varepsilon \) by linearity and using that \( |A\tilde{\xi}^z| \leq A\tilde{\xi}^z = \varepsilon \) on \( B_{2L_0}(\tilde{D}_z) \), whence \( \mathcal{M}_M(\tilde{\chi}_1^i) \cap \tilde{D}_z \) is contained in \( \mathcal{M}_{M+\varepsilon}(\tilde{\xi}_2^z) \cap \tilde{D}_z \).

The implication (5.48) follows readily from (5.47) upon applying e.g. (4.9) for a single box (along with the bound on the capacity of a box as given by (2.10)) to control the tails of the harmonic average.

To obtain (5.49), first observe that regardless of the choice of \( i \in \{1, 2\} \), by (5.43), (5.45), (5.46) and the fact that any path crossing \( D_{z'} \setminus C_{z'} \) also crosses \( D_z \setminus C_z \) (see (4.2) and (4.3)), the following holds: all clusters of \( \{ \psi^y \geq h_2 \} \) crossing \( \tilde{D}_y \setminus \tilde{C}_y \), for \( y \in \{z, z'\} \), belong to a single connected component \( \mathcal{C}_{z, z'}^i \) of \( \{ \psi^z \geq h_1 \} \cap D_z \cup \{ \psi^{z'} \geq h_1 \} \cap D_{z'} \), which crosses both \( D_z \setminus C_z \) and \( D_{z'} \setminus C_{z'} \) (the last part is due to the first item in (5.43), which guarantees the existence of such a crossing for \( \psi^y \) above level \( h_3 > h_2 \)). The control on the lower tail of \( \xi^z, \xi^{z'} \) present in (5.46) (and (5.45)) implies that \( \mathcal{C}_{z, z'}^i \) belongs to a connected component of \( \{ \varphi \geq h_1 - \varepsilon \} \), thus yielding (5.49) in case \( i = 2 \). The upper bound on \( \xi^y, y \in \{z, z'\} \), in (5.45) further implies that any cluster in \( \{ \varphi \geq h_2 + \varepsilon \} \) crossing \( \tilde{D}_y \setminus \tilde{C}_y \) is part of a crossing cluster of \( \{ \psi^y \geq h_2 \} \), whence (5.49) follows for \( i = 1 \).

Finally, the property (5.50) is inherited from (5.44) when \( \mathcal{H}_z^i \cap \mathcal{H}_{z'}^i \) occurs, regardless of \( i \in \{1, 2\} \). By construction, the cluster of \( \chi_y \) in \( \mathcal{C}_{z, z'}^i \) contains the cluster of \( \chi_y \) in \( \{ \psi^y \geq h_1 \} \) for \( y \in \{z, z'\} \), and \( (\mathcal{M}_M(\tilde{\psi}_i^y) \cap \tilde{D}_y) \subset (\mathcal{M}_{M+\varepsilon}(\tilde{\varphi}) \cap \tilde{D}_y) \), since \( A\tilde{\varphi} = A(\tilde{\psi}_i^y + \tilde{\xi}_i^y) \geq A\tilde{\psi}_i^y - \varepsilon \) in \( B_{2L_0}(\tilde{D}_y) \) if the event \( \mathcal{H}_y^i \) occurs (this only uses the lower bound on \( \tilde{\xi}_i^y \)). \( \square \)

We now set up the bootstrapping scheme that will lead to a suitable control of \( \mathbb{P}[\mathcal{G}_N^e] \). The index \( i \in \{1, 2\} \) in (5.45), (5.46) and below reflects the fact that intermediate steps (corresponding to \( i = 1 \)) and the final step (corresponding to \( i = 2 \)) of the argument need to be dealt with in distinct ways. The coarse-graining scheme developed in Section 4 now enters the picture. Referring to Proposition 1.3 for integers \( K \geq 100, N \geq 10KL \), we let \( A^1 = \mathcal{A}^{K}_{N,L}(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}) \) and \( A^2 = \mathcal{A}^{K}_{N,L}(\Lambda_N) \) (with \( \Lambda_N \in \{ B_N \setminus B_{\sigma_N}, B_{2N} \setminus B_N \} \), \( \sigma \in (0, 1/3) \), as below (5.31), and define, for \( \rho \in (0, 1) \),

\[
H^{i}_{N,L} := \bigcap_{c \in A^i} \bigcup_{\tilde{C} \subset C} \bigcup_{z \in \tilde{C}} \mathcal{H}_z^i.
\]
The event $H_{N,L}^i$ will typically be ‘good’, i.e. likely, and $\rho$ small, i.e. close to 0. We elaborate a bit more on the central role of the event $H_{N,L}^i$ in Remark 5.15 below. The next result is at the heart of our argument. It shows that $H_{N,L}^i$ typically reproduces the event from Definition 5.12 (implicit in $H_{L}^i$) at a higher scale $N$ and with an improved choice of $a$ (for $i = 1$), as well as the target event $G_N$ from (5.32) (for $i = 2$).

**Proposition 5.14** (Bootstrap; $h_1 < h_2 < h_3$, (5.31), $\rho \in (0,1), K \geq 100, i = 1, 2$). With $H_{N,L}^i = H_{N,L}^i(a, h_1, h_2, h_3, \varepsilon, M, \rho)$, whenever $\frac{\rho N}{u(KL)} \geq C$, one has the inclusion

\[(5.52) \quad (H_{N,L}^i \cap \Omega_N^i) \subseteq \left\{ \begin{array}{ll} G_0(\tilde{\varphi}, N, a', h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon), & \text{if } i = 1, \text{ for } \varepsilon \in (0, \varepsilon_0), \\
G_N(a, b, h_1 - \varepsilon, M + \varepsilon), & \text{if } i = 2, \text{ for all } \varepsilon > 0, \end{array} \right.\]

where $\Omega_N^1 = \{C_{0,N} \left\{ \varphi \overset{h_3 - \varepsilon}{\longrightarrow} \partial D_{0,N} \right\}, \Omega_N^2 = \mathbb{R}^d \}$ and

\[(5.53) \quad b := \left\lfloor \frac{c_5}{10} \frac{\rho N}{u(KL)} \right\rfloor, \quad a' := ba\]

(Here $c_5$ is as in (4.11)).

**Remark 5.15.**

1) In (5.52), one also has the inclusion $H_{N,L}^1 \subset G_N(a, b, h_1 - \varepsilon)$ since $H_{N,L}^1 \subset H_{N,L}^2$. The weaker condition on the harmonic field entering the definition of $H_{N,L}^2$, cf. (5.45), (5.46), will play a role in obtaining the sharp bound (5.32) for $d = 3$. The asserted result (5.31) for $d \geq 4$ could be obtained by means of $H_{N,L}^1$ (and $H_{N,L}^1$) alone.

2) (The role of $\rho$). In view of (5.34), (5.31) and (5.52), our goal in bounding $\mathbb{P}[G_N]$ becomes to control the probability of $(H_{N,L}^i)^c$, which by definition, see (5.51), entails the existence of at least one collection $C \in \mathcal{A}_i$ containing a large fraction $(1 - \rho)$ of (bad) points $z$ for which $H_{L}^i$ does not occur. The sacrifice of a small fraction $\rho > 0$ of (good) points inherent in (5.51) is utilized in the proof below in order to create $b$ interfaces (growing linearly with $\rho$) with certain good properties, implied locally by the occurrence of $H_{L}^i$. This eventually leads to the desired improvement (5.53) over $a$.

Let us outline the main lines of the proof of Proposition 5.14 for, say, $i = 1$, which is the more complicated case, because this is where we witness the improvement $a \rightarrow a'$ in the number of contact points. This will also shed some light on (5.52) and (5.53). A key feature of the event $H_{N,L}^1$ defined in (5.51) is the following: in combination with Lemma 2.1 applied on the coarse-grained lattice $L$, the definition of $H_{N,L}^1$ will be seen to imply the existence of $b$ interfaces made of $L$-boxes, with $b$ as in (5.53) that must all be traversed by any macroscopic ‘radial’ path $\gamma$, and having roughly the following two properties. First, each interface has an associated cluster which is ‘present in all of its boxes’. Second, whenever the path $\gamma$ traverses one interface, it picks up a contact points with the corresponding cluster. Now, whenever the event $\Omega_N^1$ occurs, all of these clusters actually communicate (much in spirit as in the proof of Lemma 5.11 see also the discussion following (5.34)). Actually this happens at sprinkled levels but let us forego these details. All in all, this mechanism generates a macroscopic cluster at scale $N$, now with $a \cdot b = a'$ contact points for any crossing path, whence (5.53).

**Proof.** For any $A \subset \mathbb{Z}^d$, we use $\mathbb{L}_A$ to denote the set of all $z \in \mathbb{L} = \mathbb{L}(L)$ such that $A \cap C_{z,L} \neq \emptyset$ (see (1.1) and (1.2) for notation). In what follows, let $U = \mathbb{L}_{C_{0,N}}$ if $i = 1$ and $U = \mathbb{L}_{B_{a,N}}$
(resp. $U = \mathbb{L}_{B_N}$) if $i = 2$ and $\Lambda_N = B_N \setminus B_{\sigma N}$ (resp. $\Lambda_N = B_{2N} \setminus B_N$). In a similar vein, let $V = \mathbb{L}_{D_{0N}}$ if $i = 1$ and $V = \mathbb{L}_{B_N}$ or $\mathbb{L}_{B_{2N}}$ if $i = 2$.

Now let $\Sigma \subset \mathbb{L}$ denote the collection of all points $z \in \mathbb{L} \setminus U$ such that $\mathcal{H}_z^i$ occurs and $\tilde{D}_{z,L} \subset D_{0N} \setminus \tilde{C}_{0N}$ if $i = 1$, resp. $\tilde{D}_{z,L} \subset B_N \setminus \tilde{C}_{0N}$ if $i = 2$. As we now explain, on $H_{N,L}^i$,

\begin{equation}
\Sigma \subset \mathbb{L} \text{ satisfies the hypotheses of Lemma 2.1 with } U, V \text{ as above and } k = \lfloor \frac{c_0 \rho^N}{n(A_k)} \rfloor - 1
\end{equation}

(it is understood that Lemma 2.1 is applied here to the lattice $\mathbb{L} = \mathbb{L}(L)$ rather than $\mathbb{Z}^d$). To see (5.54), consider any $*$-path $\gamma_L$ in $\mathbb{L}$ crossing $V \setminus U$. By suitably interpolating between successive vertices of $\gamma_L$, one creates a $*$-path $\gamma$ in $\mathbb{Z}^d$ such that $|\gamma|_L = |\gamma_L|$. Let $\tilde{C}$ be the admissible collection corresponding to $\gamma$, i.e. such that (4.12) holds. Then by definition, see (5.51), if $H_{N,L}^i$ occurs, there exists $\tilde{C} \subset \Sigma$ such that $\mathcal{H}_{\tilde{C}}^i$ occurs for all $z \in \tilde{C}$, hence $\tilde{C} \subset \Sigma$, and (5.54) follows since $|\tilde{C}| \geq k$ on account of (5.51) and (4.11).

With (5.54) in force, applying Lemma 2.1 we deduce the existence of disjoint $*$-connected subsets $O_1 \subset \ldots \subset O_k$ (each part of $L$) of all of which surround $U$. By definition of $\Sigma$, it follows that $(O_i + D_{0L}) \cap (O_j + D_{0L}) = \emptyset$ as soon as $|i - j| \geq 7$. Consequently we can extract from $\{O_i : 1 \leq i \leq k\}$ a subcollection $\{O_i^* : 1 \leq i \leq k'\}$ with $k' \geq k/8$ such that $(O_i^* + D_{0L})$ are pairwise disjoint subsets of $A$ with $V = L_A$. Now for each $j \in \{1, \ldots, k'\}$, by $*$-connectedness of $O_j^*$, the fact that $\mathcal{H}_{O_j^*}^i$ occurs for every $z \in O_j^*(\subset \Sigma)$ and using property (5.49), one finds that the connected sets $\mathcal{C}_{z,z'}^j$, for $z, z' \in O_j^*$ with $|z - z'|_\infty = L$, are contained in a single connected subset $\mathcal{C}_j$ of $\{z \geq h_1 - \varepsilon\} \cap (O_j^* + D_{0L})$. The sets $\mathcal{C}_j$, $1 \leq j \leq k'$, are disjoint by construction. Moreover by (5.50),

\begin{equation}
\text{for any } *\text{-path } \gamma \text{ crossing } \tilde{D}_{z,L} \setminus \tilde{C}_{z,L}, \text{ for some } z \in O_j^*, \text{ there exists a set } S \subset \gamma \text{ with } |S| \geq \alpha \text{ such that } B_{L_\alpha}(x) \cap \tilde{D}_{z,L} \cap \mathcal{C}_j \cap \mathcal{M}_{M+\varepsilon} \neq \emptyset, \text{ for all } x \in S.
\end{equation}

In addition, if $i = 1$, (5.49) yields that

\begin{equation}
\text{all conn. subsets of } \{z \geq h_2 + \varepsilon\} \text{ crossing } \tilde{D}_{z,L} \setminus \tilde{C}_{z,L} \text{ for some } z \in O_j^* \text{ intersect } \mathcal{C}_j.
\end{equation}

We now explain how the inclusions (5.52) follow from this, and first consider the case $i = 1$. In view of Definition 5.12, this amounts to verifying (5.43) for $\chi^2, \chi^2 = \varphi$, with $N$ in place of $L$ and at the heights given by (5.52), as well as (7.41) with $a'$ in place of $a$. First, the connection required in (5.43) is ensured by $\Omega_N^1$.

To proceed further, we need the following observation. For a $*$-path $\gamma$ on $\mathbb{Z}^d$, define the trace $\gamma^L$ of $\gamma$ on $L$ as follows: $\gamma^L(0)$ is the unique point in $L$ such that $\gamma(0) \in C_{\gamma^L(0),L}$ (recall that these boxes partition $\mathbb{Z}^d$, see (4.12)). Set $n_0 = 0$. Given $\gamma^L(0), \ldots, \gamma^L(k - 1)$ and $n_0, \ldots, n_{k - 1}$ for some $k \geq 1$, set $n_k = \inf\{n > n_{k - 1} : \gamma(n) \notin C_{\gamma^L(k - 1),L}\}$ and $\gamma^L(k) \in L$ is such that $\gamma(n_k) \in C_{\gamma^L(k),L}$. By construction $\gamma^L$ is a $*$-path on $L$. Moreover, if $\gamma$ crosses $D_{0N} \setminus \tilde{C}_{0N}$, then $\gamma^L$ crosses $V \setminus U$. As the sets $\mathcal{C}_j$, $1 \leq j \leq k'$, each surround $U$, it follows that for each $*$-path $\gamma$ crossing $\tilde{D}_{0N} \setminus \tilde{C}_{0N},$

\begin{equation}
\text{there exists } \{z_j : 1 \leq j \leq k'\} \subset \gamma^L \text{ such that dist}_{\ell^\infty(L)}(z_j, O_j^*) \leq 1.
\end{equation}

Indeed, (5.57) follows for instance by extending $\gamma^L$ to a nearest-neighbor path $\gamma^L$ on $L$, which only requires adding vertices at unit $\ell^\infty(L)$-distance from $\gamma^L$. The path $\gamma^L$ crossing $V \setminus U$ in turn intersects $O_j^*$ for all $1 \leq j \leq k'$ by [11] Lemma 2.1 and the surrounding property of each $O_j^*$. 

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Now, returning to the verification of (5.43), consider a cluster of \( \{ \varphi \geq h_2 + \varepsilon \} \) crossing \( \tilde{D}_{0,N} \setminus \tilde{C}_{0,N} \). Extracting a crossing path \( \gamma \) from this cluster, it follows by definition of \( \gamma^L \) that \( \gamma \) induces a crossing in \( D_{z_1,L} \setminus C_{z_2,L} \), with \( z_1 \) as in (5.57) (recall that \( O'_1 + D_{0,L} \subset \tilde{D}_{0,N} \)). Hence, \( \gamma \) induces a crossing in \( D_{y_1,L} \setminus \tilde{C}_{y_1,L} \) for some \( y_1 \in O'_1 \) with \( |z_1 - y_1|_{\infty(L)} \leq 1 \). Thus, \( \gamma \) intersects \( \mathscr{C}_1 \) by (5.56). All in all, each cluster of \( \{ \varphi \geq h_2 + \varepsilon \} \) crossing \( \tilde{D}_{0,N} \setminus \tilde{C}_{0,N} \) intersects \( \mathscr{C}_1 \subset \{ \varphi \geq h_1 - \varepsilon \} \). Since \( \mathscr{C}_1 \) is connected, the second part of (5.43) follows.

To deduce (5.44) relative to the event \( G_0(\tilde{\varphi}, N, a', h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon) \) in (5.52), one proceeds as follows. Repeating the above argument for all \( j \in \{1, \ldots, k'\} \) using (5.56) and (5.57), one first observes that the cluster of \( \{ \varphi \geq h_3 - \varepsilon \} \) crossing \( \tilde{D}_{0,N} \setminus C_{0,N} \) stipulated by \( \Omega_N^1 \), intersects each \( \mathscr{C}_j \), and therefore

\[
\mathscr{C}_0, \text{ the cluster of } C_{0,N} \text{ in } \{ \varphi \geq h_1 - \varepsilon \}, \text{ contains } \mathscr{C}_j \text{ for all } 1 \leq j \leq k'.
\]

Now, still by the same argument, every \(*\)-path \( \gamma \) crossing \( \tilde{D}_{0,N} \setminus \tilde{C}_{0,N} \) induces a crossing in \( \tilde{D}_{y_j,L} \setminus \tilde{C}_{y_j,L} \) for some \( y_j \in O'_j \) and all \( 1 \leq j \leq k' \) (in fact \( \gamma \) is also connected to every \( \mathscr{C}_j \) but we won’t use this). By (5.55), there exist sets \( S_j \subset \gamma \) for all \( 1 \leq j \leq k' \), each of cardinality at least \( a \), such that \( B_{L_1}(x) \cap \tilde{D}_{y_j} \cap \mathscr{C}_j \cap \mathcal{M}_{M + \varepsilon} \neq \emptyset \) for all \( x \in S_j \) and one can replace \( \mathscr{C}_j \) by \( \mathcal{G}_0 \) in the previous intersection due to (5.58). By construction the sets \( S_j \) are disjoint. Thus letting \( S := \bigcup_j S_j \), one obtains by (5.59) that \( |S| \geq k' a \geq (k/8) a \), whence \( |S| \geq a' \) on account of (5.44) and (5.53) whenever \( \frac{\rho_1}{a(KL)} \geq C \). All in all, \( S \) has all the properties required by (5.44).

We now verify (5.52) in case \( i = 2 \). Consider a \(*\)-path \( \gamma \) crossing \( \Lambda_N \). By (5.57), \( \gamma \) induces crossings in \( \tilde{D}_{y_j,L} \setminus \tilde{C}_{y_j,L} \) for suitable \( y_j = y_j(\gamma) \in O'_j \) and all \( 1 \leq j \leq k' \). Applying (5.55), we obtain for every \( j \in \{1, \ldots, k'\} \) a set of points \( \{x_{j,k} = x_{j,k}(\gamma) \} : 1 \leq k \leq a \} \) such that \( B_{L_1}(x_{j,k}) \cap \tilde{D}_{y_j} \cap \mathscr{C}_j \cap \mathcal{M}_{M + \varepsilon} \neq \emptyset \). In view of (5.32), the inclusion (5.52) for \( i = 2 \) follows since the sets \( \mathscr{C}_j \) are connected subsets of \( \Lambda_N \cap \{ \varphi \geq h_1 - \varepsilon \} \) and \( k' \geq b \).

Our last missing ingredient needed prior to proceeding to the proofs of (5.2) and (5.4) is an a-priori bound for the event \( \mathcal{G}_2 \) from Definition 5.12 at levels below \( h_* \), which is available by current methods and which we supply next. This a-priori bound will play a role akin to (5.11) in the subcritical case and enable us to initiate the bootstrap argument in Proposition 5.14.

**Lemma 5.16** (A-priori estimate). \( h_1 < h_2 < h_3 < h_* \), \( \varepsilon \in (0, \varepsilon_0 \wedge \frac{1}{2} (h_* - h_3)) \). There exist \( L_0 \geq 1 \), \( M > 1 \) and \( c_{11} > 0 \), each depending on \( h = (h_1, h_2, h_3) \) and \( \varepsilon \) only, such that for \( \mathcal{G}_0 = \mathcal{G}_0(\tilde{\varphi}, L_0, L_n, a = 1, h, M) \) with \( \tilde{\varphi} \in \{ \tilde{\varphi}, \tilde{\psi} \} \) and \( \frac{L_{n+1}}{L_{n+1}} = c(d) \), one has

\[
\lim_{n \to \infty} \frac{1}{L_{n+1}^{c_{11}}} \log \mathbb{P}(\mathcal{G}_0) < 0.
\]

**Proof.** It suffices to consider the case \( \tilde{\varphi} = \tilde{\varphi} \). The case \( \tilde{\varphi} = \tilde{\psi} \) then follows by applying (5.48). The bound (5.59) (with \( \tilde{\varphi} = \tilde{\varphi} \)) will follow by applying results of [13] to the graph \( \tilde{Z}^d \) (with unit weights). For \( x \in \mathbb{Z}^d \), we consider the events (at scale \( L_0 \), see (4.2) for notation)

\[
A_1^x := \{ C_{x,L_0} \leftarrow_{\varphi \geq h_2 + \varepsilon} \partial D_{x,L_0} \}
\]

\[
A_2^x := \{ \text{all clusters of } \{ \varphi \geq h_2 - \varepsilon \} \text{ crossing } \tilde{D}_{x,L_0} \setminus \tilde{C}_{x,L_0} \}
\]

\[
A_3^x := \{ \mathcal{M} \supset D_{x,L_0} \}
\]
with \( \mathcal{M} = \mathcal{M}(\tilde{\varphi}) \) as defined in \((5.30)\) and \( M := (\log L_0)^2 \). For \( \lambda = \lambda(d) \geq 100 \) sufficiently large—the choice of \( \lambda \) corresponds to the constant 20c_{18}C_{10} \) appearing e.g. in \((8.3)\) of \([13]\), and in the present case \( C_{10} = 1 \) and \( c_{18} \) is determined by the isoperimetric constant on \( \mathbb{Z}^d \)—one then sets, for \( x \in \mathbb{Z}^d \), with \( \tilde{\lambda} = 1.1\lambda, \ell_0 = 3^d \vee 12\tilde{\lambda} \) and \( L_n = c_0^{\lambda}L_0 \), for \( 1 \leq k \leq 3 \),

\[
A_{x,0}^k = \bigcap_{y \in B_{\lambda L_0}(x)} A_y^k \\
A_{x,n}^k = \bigcap_{y,z \in (L_n-1)^d \cap B_{\lambda L_n}(x) : d(y,z) \geq L_n} (A_{y,n-1}^k \cup A_{z,n-1}^k), \quad \text{for } n \geq 1.
\]

(5.63)

Since \( h_3 + \varepsilon < h_* \), combining the bounds in \((1.6)\) with \((5.60)-(5.62)\) this event is readily seen to imply \( L_0\), \( \tilde{\rho} \) follows that for any \( \varepsilon, h \), \( \tilde{\lambda} \) and \( h_3 \) as \( \varepsilon, h \) and \( h_3 \) in \((5.63)\), \((5.64)\) in \([13]\) are satisfied, whence Proposition 7.1 therein applies and yields that

\[
\mathbb{P}(\tilde{A}_{x,n}^{k'}) \leq 2^{-2^n}, \quad \text{for all } 1 \leq k \leq 3 \quad \text{and } n \geq 0,
\]

where the primed events \( \tilde{A}_{x,n}^{k'} \) refer to those defined in \((5.63)\), but with sprinkled parameters \((h_1, h_2, h_3)\) in place of \((h_1 + \varepsilon, h_2 - \varepsilon, h_3 + \varepsilon)\) in \((5.60)-(5.62)\). Note to this effect that the event \( \tilde{A}_{x,0}^{k'} \) is measurable with respect to the restriction of \( \tilde{\varphi} \) to \( \tilde{\mathbb{Z}}^d \cap B_{\lambda L_0}(x) \), as required for Proposition 7.1 in \([13]\) to apply, and that a slight extension (of the underlying decoupling inequality \((2.20)\)) is required where \( k = 2 \), cf. \((5.61)\), in order to take care of the two opposite directions of monotonicity. To conclude, one applies Lemma 8.6 in \([13]\), which implies that whenever \( \bigcap_k \tilde{A}_{x,n}^{k'} \) occurs, any two connected sets in \( B_{\lambda L_n} \) of diameter at least \((\lambda/20)L_n\) each, are connected by a path \( \gamma \subset B_{2\lambda L_n} \) such that \( \bigcap_k \tilde{A}_{x,0}^{k'} \) occurs for all \( x \in \gamma \). Due to the (primed versions of the) choices \((5.60)-(5.62)\), this event is readily seen to imply \( G_0 \) with \( L = \lfloor (\lambda/10)L_n \rfloor \).

The bound \((5.59)\) then follows from \((5.64)\). \( \blacksquare \)

We are now ready to assemble the pieces and prove \((5.2)\) and \((5.4)\). In view of \((5.52)\), this entails probing into the complements of the events \( H_{i,L}^i, i = 1,2, \) from \((5.51)\). In the spirit of \((5.5), (5.6)\), for \( z \in L = L(L) \) we say that

\[
\{ z \text{ is } \psi \text{-bad} \} := G_z^\psi(\tilde{\psi}), \quad \text{and}
\]

(5.65)

\[
\{ z \text{ is } (\xi,i) \text{-bad} \} := \begin{cases} \{ \sup_{D_z} |\tilde{\xi}_z| > \varepsilon \}, & \text{if } i = 1 \\ \{ \inf_{D_z} |\tilde{\xi}_z| < -\varepsilon \}, & \text{if } i = 2, \end{cases}
\]

(5.66)

whence \( z \) is either \( \psi \)-bad or \((\xi,i)\)-bad whenever \( H_z^i \) occurs, cf. \((5.45)-(5.46)\). By \((5.51)\), it then follows that for any \( \rho' \in (0,1-\rho) \) and \( i \in \{1,2\} \),

\[
(H_{i,L}^i)^c \subset E_{N,L}^i \cup F_{N,L}^i,
\]

(5.67)

where

\[
E_{N,L}^i := \left\{ \exists C \in A^i \text{ and } \tilde{C} \subset C \text{ with } |\tilde{C}| = \lfloor \rho' |C| \rfloor \quad \text{such that all the sites in } \tilde{C} \text{ are } \psi \text{-bad} \right\},
\]

\[
F_{N,L}^i := \left\{ \exists C \in A^i \text{ and } \tilde{C} \subset C \text{ with } |\tilde{C}| = |C| - \lfloor \rho' |C| \rfloor - \lfloor \rho |C| \rfloor \quad \text{such that all the sites in } \tilde{C} \text{ are } (\xi,i)\text{-bad} \right\}.
\]
At this point we consider the cases $d = 3$ and $d \geq 4$ separately.

**Upper bound for $d = 3$**. Combining Proposition 5.14, Lemmas 5.16 and 5.11 with a bootstrap argument (similar in spirit to the one leading to the corresponding subcritical upper bound), we proceed to give the

**Proof of (5.2)**. Let $h < h_\ast$. We assume in the sequel that $\varepsilon \in (0, \frac{h_\ast - h}{16})$ and set $h_1 = h_\ast - 12\varepsilon$, $h_2 = h_\ast - 8\varepsilon$ and $h_3 = h_\ast - 4\varepsilon$, whence $\varepsilon \in (0, \varepsilon_0 \wedge \frac{1}{2}(h_\ast - h_3))$ (cf. below (5.46) regarding $\varepsilon_0$).

In the first step, we take the bound given by Lemma 5.16 as our input and improve it (along with the parameter $a$) via Proposition 5.14 applied with $i = 1$. To this end, we first choose $L_0$ and $M$, both depending on $h, \varepsilon$ only, such that Lemma 5.16 is in force. Then, applying Proposition 5.14 with these choices for $L_0$ and $M$, as well as $K = 100$, $\rho = 1/2$ (see (5.31)) and $a = 1$, the inclusion (5.52) and (5.53) yield that

$$\mathbb{P}[(\mathcal{G}_{0,N})^c] \leq \mathbb{P}[(H_{N,L}^1)^c] + \mathbb{P}[C_{0,N} \xrightarrow{\varepsilon \to 0} \partial D_{0,N}].$$

for all $L > 2L_0$ and $N \geq CL$, where $\mathcal{G}_{0,N} = \mathcal{G}_0(\hat{\varphi}, L_0, N, a = \lfloor \frac{cN}{2} \rfloor, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)$ and $H_{N,L}^1 = H_{N,L}^1(L_0, a = 1, h_1, h_2, h_3, \varepsilon, M, \rho = \frac{1}{2})$. Incorporating (5.65) with the choice $\rho' = \frac{1}{2}$ and the upper bound on disconnection probability given by Theorem 5.5 in [42], we obtain from the previous display, under the same assumptions on $L$ and $N$ that

$$(5.68) \quad \mathbb{P}[(\mathcal{G}_{0,N})^c] \leq \mathbb{P}[E_{N,L}^1] + \mathbb{P}[F_{N,L}^1] + e^{-c(h,\varepsilon)N}.$$  

In view of (5.66), adapting the argument used in the proof of Lemma 5.5 to the present case where $\mathcal{A}^1 = A_{N,L}^{100}(\tilde{D}_{0,N} \cap C_{0,N})$, using symmetry and applying a union bound (costing an inconsequential factor $\binom{n}{\lfloor n/2 \rfloor}$ where $n = |C|, C \in \mathcal{A}^1$) to get rid of absolute values in (5.66) and $F_{N,L}^1$, we deduce that

$$\mathbb{P}[E_{N,L}^1] \leq e^{-c(h,\varepsilon)\frac{N}{\log N}}, \text{ for all } N \geq C(h,\varepsilon) \text{ and } (\log N)^3 \leq L \leq L_1(N)$$

(with $L_1(N)$ as fixed above Lemma 5.5). On the other hand, in view of (5.65), retracing the steps that led to the proof of Lemma 5.4 and (5.20), replacing the input bound (5.9) by (5.59) for $\tilde{\chi} = \hat{\varphi}$ (whence $f(L) = c(h,\varepsilon)L^{c_1}$), one finds that for all $L = L_n$ as appearing in Lemma 5.16 satisfying $L \in [C(h,\varepsilon)(\log N)^{2/c_1}, cN]$ and $N \geq C(h,\varepsilon)$,

$$\mathbb{P}[F_{N,L}^1] \leq e^{-c(h,\varepsilon)\frac{N}{L^{1/2}}}.$$  

Substituting the estimates for $\mathbb{P}[E_{N,L}^1]$ and $\mathbb{P}[F_{N,L}^1]$ into (5.68) and choosing $L = \lambda_0$ with $n_0 = n_0(N, h, \varepsilon) := \inf\{n \geq 0 : L_n \geq C(h,\varepsilon)(\log N)^{\max(3,2/c_1)}\}$, we obtain for all $N \geq C(h,\varepsilon)$,

$$(5.69) \quad \mathbb{P}[\mathcal{G}_0(\hat{\varphi}, L_0, N, a_N, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)^c] \leq e^{-f'(N)},$$

where $a_N := \lfloor \frac{N}{(\log N)^{12}} \rfloor$ and $f'(N) := \frac{c(h,\varepsilon)N}{(\log N)^{2/c_1}}$. This yields the desired improvement, both in terms of $a$ and the probabilistic bound, over the a-priori estimate from Lemma 5.16.

In the second step, we start with the improved bound (5.69) and feed it to Proposition 5.14 (in case $i = 2$) to derive an estimate for the event $G_N$ in (5.32), for a suitable choice of the parameters. Thus applying (5.52) with the height parameters from (5.69) in place of $(h_1, h_2, h_3)$,
\( \epsilon' := h_1 - h - 2\epsilon (> 0) \) in place of \( \epsilon \) and \( L_0 = L_0(h, \epsilon) \) as in the previous step, we obtain, for all \( L \geq C(h, \epsilon), K \geq 100, \rho \in (0, 1) \) and \( N \) such that \( \frac{\rho N}{u(KL)} \geq C \), using the decomposition \( 5.67 \)\).

\[
P[G_N(a_L, b, h + \epsilon, M')] \leq P[E^2_{N,L}] + P[F^2_{N,L}],
\]
with \( b \) as given by \( 5.33 \), \( M' = M + h_1 - h - \epsilon \) and where the events \( E^2_{N,L}, F^2_{N,L} \) inherit the parameters from \( H^2_{N,L} = H^2_{N,L}(a_L, h_1 - h, h_3 - h, M + \epsilon, \rho) \), and depend on an additional \( \rho' \in (0, 1 - \rho). \) Now, mimicking the arguments of the previous step to bound the probabilities on the right-hand side in \( 5.70 \), but this time using \( f' \) instead of \( f \) (as implied by \( 5.69 \)) when estimating \( P[E^2_{N,L}] \), and with a view to \( 5.66 \) (compare with the definition of \( F_{N,L} \) in \( 5.7 \) and the proof of \( 5.10 \)) when dealing with \( P[F^2_{N,L}] \), one obtains the following for the choice \( L := L_{n_1} \) where \( n_1(N, h, \epsilon) := \inf\{n \geq 0 : L_n \geq (\log N)^3 \} \), whence \( (\log N)^3 \leq L_{n_1} \leq C(h, \epsilon) (\log N)^3 \): For \( K \) large enough, \( \sigma \) (in the case \( \Lambda_N = B_N \setminus B_{\sigma N} \)) and \( \rho \) close enough to 0, all depending on \( h \) and \( \epsilon \) and \( \rho' = 1 - 2\rho, \)

\[
P[G_N(d'_N, b'_N, h + \epsilon, M')] \leq e^{-\frac{1}{2}(h_1 - h - 2\epsilon)^2 N \log N}, \quad \text{for all } N \geq C(h, \epsilon),
\]
where \( d'_N = a_{L_{n_1}} \) and \( b'_N = \lfloor \frac{c(h, \epsilon) N}{L_{n_1}} \rfloor \).

With \( 5.71 \) at hand, we now deduce \( 5.2 \) and first consider the finite-volume event \( \text{LocUniq}(N, h) \). Plugging \( 5.71 \) (with \( \Lambda_N = B_N \setminus B_{\sigma N} \)) into \( 5.34 \) with \( h' = h + \epsilon \), we obtain for \( N \geq C(h, \epsilon) \),

\[
P[\text{LocUniq}(N, h)] \leq e^{-\frac{1}{2}(h_1 - h - 2\epsilon)^2 N \log N} + P[B_N \not\leftrightarrow \partial B_{2N}] + e^{-cN^{-a_N}}
\leq 2e^{-\frac{1}{2}(h_1 - h - 2\epsilon)^2 N \log N} + e^{-cN^{-a_N}},
\]
where the second line follows by Theorem 5.5 in \( 12 \) and since \( b'_N d'_N \geq c(h, \epsilon) \frac{N}{(\log \log N)^{\sigma(h, \epsilon)}} \).

The claim readily follows from \( 5.72 \) by taking logarithms, multiplying by \( \frac{\log N}{N} \) on both sides, letting \( N \to \infty \) and then \( \epsilon \downarrow 0 \). The same upper bound then automatically holds for the event \( 2\text{-arms}(N, h)(\subset \text{LocUniq}(N, h)) \).

The upper bound in \( 5.2 \) for the truncated one-arm event will follow similarly from \( 5.33 \) and \( 5.71 \) upon supplying a suitable upper bound for the probability of disconnecting \( B_{\sigma n} \) from infinity above level \( h' = h + \epsilon \) (here \( \sigma = \sigma(h, \epsilon) > 0 \) refers to the choice that leads to \( 5.71 \)), which is not readily available for us to use. To circumvent this issue, we combine the upper bound \( 5.2 \) for the local uniqueness event derived above and the disconnection upper bound from \( 12 \) as follows: consider the sequence of events, for some integer \( M > 1 \),

\[
\mathcal{E}_1 := \{ B_N \not\leftrightarrow \partial B_{4M} \},
\]
\[
\mathcal{E}_{2k} := \text{LocUniq}(h', 2^k 2M) \land \mathcal{E}_{2k+1} := \{ B_{2^k 2M} \not\leftrightarrow \partial B_{6^k 2M} \}, \quad \text{for } k \geq 1.
\]

It readily follows from the definition of \( \text{LocUniq} \), see \( 13 \), that \( \{ B_N \not\leftrightarrow \partial B_{\infty} \} \subset \bigcap_{k \geq 1} \mathcal{E}_k \).

Therefore, applying a union bound and subsequently using the bounds from \( 12 \) and \( 5.72 \) for the respective probabilities, we get for any \( M > 1 \) and \( N \geq C(h, \epsilon, M) \),

\[
P[B_N \not\leftrightarrow \partial B_{\infty}] \leq \sum_{k \geq 1} P[\mathcal{E}_k] \leq \sum_{k \geq 0} e^{-c(h') 2^k 2M} + \sum_{k \geq 1} e^{-c(h_1 - h') 2^k 2M} \leq e^{-c(h_1 - h') \frac{2^k M}{\log(2^k M)}} \leq e^{-c(h_1 - h') \frac{2^k M}{\log(2^k M)}}.
\]

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Since $\mathcal{M}$ is arbitrary, (5.73) implies that

\[(5.74) \lim_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[B_N \leftrightarrow \infty] = -\infty.\]

for all $h < h_*$. Similarly as with (5.72), (5.2) readily follows from (5.74), (5.71) and (5.33). \(\square\)

**Upper bound for $d \geq 4$.** Similarly as with the subcritical phase, the proof of upper bounds simplifies in higher dimensions.

**Proof of** (5.4). Let $h < h_*$. We choose $h_1, h_2, h_3$ as in the beginning of the proof of (5.2) and simply fix $\varepsilon := (h_* - h)/20$. Following the same line of reasoning that led to the bound (5.69), except for applying Proposition 5.14 with $i = 2$ directly (instead of $i = 1$) with $\sigma = 1/4$ in case $\Lambda_N = B_N \setminus B_{\sigma N}$, and using analogues of Lemmas 5.7 and 5.9 in place of Lemmas 5.4 and 5.5 respectively, to bound $\mathbb{P}[E_{N,L}]$ and $\mathbb{P}[F_{N,L}]$, thereby choosing $L = C(h)$ large enough, one finds that for all $N \geq C(h)$,

\[(5.75) \mathbb{P}[G_N(a = 1, b = [c(h)N, h_1 - \varepsilon, M + \varepsilon]) \leq e^{-c'(h)N}.\]

Here $M = M(h)$ and $L_0 = L_0(h)$, implicit in the definition $G_N$, cf. (5.30) and (5.32), are chosen as in the proof of (5.2) when applying Lemma 5.10. Plugging (5.75) for the choice $\Lambda_N = B_2N \setminus B_N$ into (5.34) with $h' = h_1 = \varepsilon(> h)$ and using Theorem 5.5 in [42] in order to bound the disconnection probability, we get

\[(5.76) \lim sup_{N \to \infty} \frac{1}{N} \log \mathbb{P}[\text{LocUniq}(N, h)^c] < 0,\]

along with a similar bound for 2-arms($N, h$). Regarding (5.4) for the truncated one-arm event, proceeding similarly as in the case $d = 3$, we first derive from (5.76) and the disconnection upper bound from [42] that $\frac{1}{N} \log \mathbb{P}[B_N \leftrightarrow \infty]$ tends to $-\infty$ as $N \to \infty$, from which (5.4) follows upon applying (5.33) and using (5.75) with $\Lambda_N = B_N \setminus B_{N/4}$.

We conclude with a few comments.

**Remark 5.17.** 1) Analogues of Remarks 5.6 and 5.10 also hold in the supercritical regime. Namely, at the expense of iterating a few more times, initializing the above scheme does not require the full strength of the stretched exponential a-priori bound provided by Lemma 5.16. Moreover, when $d \geq 4$, replacing the admissible collection $\mathcal{A}$ inherent to the definition of $H_{N,L}$ in (5.51) by the corresponding collection $(\mathcal{A}')^i$ (cf. Remark 4.7), which employs the coarse-graining strategy used for $d = 3$, one can derive analogues of (5.29) for LocUniq($N, h)^c$ or the event in (5.4) when $h < h_*$. The proof essentially follows the line of argument of Remark 5.10 applying Proposition 5.14 with $i = 1$ for all but the $k$-th step (where $k \geq 1$ refers to any target number of iterated logarithms, cf. (5.29)). The important thing to notice is that the number $a$ of contact points, albeit always growing sublinearly in the macroscopic scale due to the choice of $L$, improves suitably along with the bound on $G_0$ through intermediate steps of the iteration.

2) (Two-point functions). We now briefly discuss the amendments to our methods that would be required to obtain the asymptotics (1.11) for the truncated two-point function. Regarding lower bounds, in the arguments leading to (5.1) and (5.2), one would need
to ‘tilt’ the construction, thus following the $\ell^2$-geodesic between $x$ and $y$) rather than a horizontal line, in order to force a connection between $x$ and $y$. An asymptotic capacity estimate for such a discretized $\ell^2$-geodesic similar to (2.12) (when $|x - y| = N$) is given by Remark 2.3. Corresponding analogues of (2.23) and (2.24) would also be required. The former relies on the visibility Lemma 2.4, which is robust with respect to ‘tilting’ of the above kind. To adapt the proof of (2.24), one could compare to a random walk on $\mathbb{R}^d$ with Gaussian increments, whose law inherits the symmetries of $\mathbb{R}^d$ and to which the arguments leading to the lower bound in (2.25) can be extended, and then rely on the results of [44] for comparison with $X$. The relevant upper bound for (1.11) would naturally follow by adapting our coarse-graining and bootstrapping scheme to a framework with Euclidean balls replacing $\ell^\infty$-ones (although, since $L \ll N$ in practice, one may in fact get away by coarse-graining using $\ell^\infty$-boxes at scale $L$).

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