The mixed-phase version of moist-air entropy.

by Pascal Marquet (WGNE Blue-Book 2016).

Météo-France. CNRM/GMAP. Toulouse. France. E-mail: pascal.marquet@meteo.fr

1 Motivations

The specific (per unit mass of moist-air) entropy defined in Marquet (2011, M11) by

\[ s = s_{\text{ref}} + c_{pd} \ln(\theta_s), \]

where \( s_{\text{ref}} \) and \( c_{pd} \) are two constants. The first- and second-order approximations \( (\theta_s)_1 \) and \( (\theta_s)_2 \) of the moist-air entropy potential temperature \( \theta_s \) have been more recently derived in Marquet (2015, M15).

The aim of this note is to derive the mixed-phase version of \( \theta_s \), \( (\theta_s)_1 \) and \( (\theta_s)_2 \), namely if liquid water and ice are allowed to coexist, with possible under- or super-saturations, with possible supercooled water and with possible different temperatures for dry air and water vapour, on the one hand, condensed water and ice, on the other hand.

2 The mixed-phase definition of \( \theta_s \)

The specific (per unit mass of moist-air) entropy given by (B.1) in M11 is equal to the sum

\[ s = q_d s_d + q_v s_v + q_l s_l + q_i s_i, \quad (1) \]

where specific contents in dry-air, water vapor, liquid water and ice \( (q_d, q_v, q_l, q_i) \) act as weighting factors. The common temperature \( T \) for the dry air and water vapour entropies \( (s_d, s_v) \) is possibly different from those \( T_l \) or \( T_i \) for liquid water or ice entropies \( (s_l, s_i) \), respectively.

Without lost of generality, the moist-air entropy given by (1) can be rewritten in a way similar to (B.2) in M11, leading to

\[ s = q_d s_d + q_v s_v + q_l (s_l^* - s_v) + q_i (s_i^* - s_v) + q_l (s_l - s_i^*) + q_i (s_i - s_i^*), \quad (2) \]

where \( q_l = q_v + q_l + q_i \) is the total water content.

The first difference from the result derived in M11 is due to \( s_d \) and \( s_l \) which must be computed in the second line of (2) at temperatures \( T_l \) and \( T_i \), respectively, whereas \( s_l^* \) and \( s_i^* \) are computed at the common temperature \( T \) for the two gaseous species. The second line of (2) can thus be computed with \( s_l - s_i^* = c_l \ln(T_i/T) \) and \( s_l^* = c_l \ln(T_i/T) \), where the reference entropies \( (s_l)_r \) and \( (s_i)_r \) have no impact.

The other difference concerns the bracketed terms in (B.7) in M11, namely the term \( R_v[qt \ln(H_l) + q_l \ln(H_i)] \), where \( H_l = e/e_d \) and \( H_i = e/e_i \) are the relative humidities with respect to liquid water and ice, respectively. These bracketed terms no longer cancel out if liquid water and ice are allowed to coexist, and/or with possible under- or super-saturations.

These differences with respect to non-mixed phase results of M11 lead to the following mixed-phase generalisation of \( \theta_s \):

\[ \theta_s = \left[ \theta \exp\left(-\frac{L_v q_l + L_s q_i}{c_{pd} T}ight) \right] \exp(A_r q_i) \]

\[ \left( T_i \right)^{1+\gamma_i/\kappa} \left( \frac{p_r}{\rho_r} \right)^{-\gamma_i q_i} \left( T_i \right)^{1+\gamma_q/\kappa} \left( \frac{c_i q_i/c_{pd} q_i}{T_i} \right)^{1+\gamma_q/\kappa} \exp(1+\gamma_i q_i) \]

\[ (H_l)^{\gamma_q} (H_i)^{\gamma_i} \left( T_i \right)^{c_i q_i/c_{pd} q_i} \left( \frac{T_i}{T} \right)^{c_i q_i/c_{pd} q_i} \cdot (3) \]

The bracketed terms in the first line of (3) is the ice-liquid version of the Betts’ potential temperature \( \theta_l \), where the latent heats \( L_v \) and \( L_s \) depend on \( T \). The whole first line of (3), including the term \( \exp(\Lambda_r q_i) \) which depends on the Third-Law reference values \( (s_v)_r \) and \( (s_d)_r \), forms the first-order approximation \( (\theta_s)_1 \). Some of the terms in the second line of (3) are used in M15 to derive the second-order approximations \( (\theta_s)_2 \).

The third line of (3) is made of the four new mixed-phase correction terms. These terms are clearly equal to unity for the non-mixed phase conditions retained in M11, namely if \( T_l = T_i = T \), \( H_l = 1 \) for \( q_l \neq 0 \) and \( H_i = 1 \) for \( q_i \neq 0 \).

3 Some Numerical results

The impact of the two new mixed-phase terms \( (H_l)^{\gamma_q} \) and \( (H_i)^{\gamma_i} \) in (3) are evaluated by using SHEBA/FIRE-ACE vertical profiles for \( (\theta_l, q_l, q_i, q_i) \) depicted in Figure 7 of Morisson et al. (2011).

The profiles of \( (q_l, q_v, q_l, q_i) \) and \( (H_l, H_i) \) are shown in Fig.1. The contents in liquid water and ice are small (mind the factor 10!), but they are associated with relative humidities mostly different from 100%. One may thus expect the factors \( (H_l)^{\gamma_q} \) and \( (H_i)^{\gamma_i} \) to be (slightly) different from unity. The vertical profiles \( H_i(z) \) and \( H_i(z) \) shown in Fig.1 are similar to those described in Fig.2 for Arctic mixed-phase clouds, with liquid and ice water content typical of updrafts and relative humidities typical of downdrafts.
Figure 1: The vertical profile of water species contents and relative humidities corresponding to the yellow curve (SAM-SBM) in Figure 7 of Morisson et al. (2011).

Figure 2: The conceptual model depicted in Shupe et al. (2008) showing typical values for water species contents, θ and relative humidities in autumn Arctic mixed-phase stratiform clouds (for updraft and downdraft regions).

The paradigm for describing and simulating mixed-phase cloud is to consider that the ice-liquid potential temperature $\theta_{il}$ is a conservative variable, where $\theta_{il}$ defined in Tripoli and Cotton (1981) is similar to the bracketed terms in the first line of (3), except that the latent heats $L_v(T_0)$ and $L_s(T_0)$ are computed at the triple-point temperature $T_0 = 273.16$ K (not at $T$).

The conserved (namely constant) feature observed for $\theta_{il}$ in the PBL of Fig.3 is likely due to the choice of the ice-liquid water static energy $h_L$ as a prognostic variable in the SAM-SBM runs, where $h_L = c_{pd} T + g z - L_v(T_0) q_l - L_s(T_0) q_s$ is clearly a proxy for $\theta_{il}$.

Differently, it is shown in Fig.3 that the mixed-phase moist-air entropy value $\theta_s$ given by (3) is not conserved (with $(\theta_s)_2$ being indeed a good approximations of $\theta_s$).

This may be interpreted as an impact of the term $\exp(\Lambda_r q_t)$ in the first line of (3) and due to changes in $q_t$ shown in Fig.1 close to the ground (below 50 m).

This impact of $q_t$ was missing in the definition of $\theta_{il}$ and in the approximate integration of the first and the second laws of thermodynamics derived in Dutton (1976, see before Eq.30, p.284, in the 1986 edition).

The “equivalent” version $\theta_{eil}$ defined in Tripoli and Cotton (1981) includes a factor $\exp[(L_v(T_0)/c_{pd} T)]$ which depends on $q_t$, where $L_v(T_0)/c_{pd} T \approx 9$. This factor is however different from the one $\exp(\Lambda_r q_t)$ appearing in $\theta_s$ given by (3), where $\Lambda_r \approx 6$ depends on the Third-Law reference values $(s_v)_r$ and $(s_d)_r$. Only $\theta_s$ with $\Lambda_r \approx 6$ is an equivalent of the moist-air entropy.

4 Conclusions

The search for “conserved” variables based on approximations of the moist-air entropy (function or equation) should be replaced by the use of the conservative variables $\theta_s$ given by (3) which is a true equivalent variable.

A model using the mixed-phase version (3) for $\theta_s$ as a prognostic variable, including for turbulent and mass-flux mixing processes, could lead to more accurate results. The impacts of the last two terms of (3) are to be investigated (ex. for supercooled water).

References

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