Malcev algebras corresponding to smooth almost left automorphic Moufang loops

Ramiro Carrillo-Catalán
CONACyT - Universidad Pedagógica Nacional
Unidad 201 Oaxaca
rcarrilloc@conacyt.mx

Marina Rasskazova
Omsk Technic University
Omsk,644050, pr.Mira 15
marinarasskazova@yandex.ru

Liudmila Sabinina
Centro de Investigacion en Ciencias
UAEM, Cuernavaca
liudmila@uaem.mx

June 27, 2017

Abstract

In this note we introduce the concept of an almost left automorphic Moufang loop and study the properties of tangent algebras of smooth loops of this class.

Key words: Malcev algebras, Moufang loops, Alternative loops, Automorphic loops, Local almost left automorphic Moufang loops.

2010 Mathematics Subject Classification: 17D10, 20N05.

1 Preliminaries

In what follows, a loop structure is a universal algebra $\langle Q, \cdot, \backslash, /, 1 \rangle$ of type $(3,0,1)$ such that the identities

$$
(x \cdot y)/y = x = (x/y) \cdot y,
$$

$$
x \cdot (x\backslash y) = y = x\backslash (x \cdot y),
$$

$$
x \cdot 1 = x = 1 \cdot x.
$$

are satisfied for any two elements in the loop. For simplicity we will write $xy$ instead of $x \cdot y$. A Moufang loop is a loop in which any of the following equivalent identities

$$
((xy)x)z = x(y(xz)); \quad ((xy)z)y = x(y(zy)); \quad (xy)(zx) = (x(yz))x.
$$




hold for every three elements of the loop. Moufang loops are diassociative: the subloop generated by every two elements is a group.

For an element \(a\) of \(Q\) the bijections \(L_a : Q \rightarrow Q\) and \(R_a : Q \rightarrow Q\) given by \(L_ax = ax\) and \(R_ax = xa\) will be called, respectively, the left and the right translation by \(a\). The left and right translations generate the multiplication group, \(\text{Mlt}(Q)\). The stabilizer of the neutral element in \(Q\) defines the inner mapping group denoted by \(\text{Inn}(Q)\). This group is actually a subgroup of the multiplication group generated by three different types of elements of \(\text{Mlt}(Q)\), namely

\[
\ell_{x,y} = L_{xy}^{-1} \circ L_x \circ L_y, \quad r_{x,y} = R_{xy}^{-1} \circ R_y \circ R_x \quad \text{and} \quad T_x = L_x^{-1} \circ R_x
\]

for every two elements \(x\) and \(y\) in \(Q\). If the group \(\text{Inn}(Q)\) acts on \(Q\) by automorphisms, the loop \(Q\) is said to be automorphic. Equivalently, the loop is automorphic if the mappings \(\ell_{x,y}, r_{x,y}\) and \(T_x\) are automorphisms of \(Q\). All the left translations by elements in \(Q\) generate the left multiplication subgroup of \(\text{Mlt}(Q)\), denoted by \(\text{LMlt}(Q)\). The stabilizer of the neutral element in \(\text{LMlt}(Q)\) defines the left inner mapping group, \(\text{LInn}(Q)\), generated by the mappings \(\ell_{x,y}\).

If \(\text{LInn}(Q)\) is a group of automorphisms of \(Q\) then \(Q\) will be called a left automorphic loop.

**Definition 1.** A Moufang loop \(L\) with the property that every three elements of \(L\) generate a left automorphic subloop will be called an almost left automorphic Moufang loop.

In the differential geometry framework the notion of a loop being left automorphic is important: a loop with the left automorphic and left power alternative properties completely defines a reductive homogeneous space. [K3], [SLV]. Following ideas of M. Kikkawa [K1] we have studied [CS1] the so-called reductive Kikkawa spaces. There it is shown that smooth left automorphic power alternative loops are Moufang.

The corresponding tangent algebra \(A\) of a reductive Kikkawa space is characterized as an anticommutative algebra satisfying the relations

\[
J(xy, z, u) + J(yz, x, u) + J(zx, y, u) = 0, \\
J(x, y, uv) = J(x, y, u)v - J(x, y, v)u, \quad \forall x, y, z, u, v \in A.
\]

where \(J(x, y, z) := (xy)z + (yz)x + (xz)y\).

In ([CS1], p.6, T.2.13) it was also proved that tangent algebras of reductive Kikkawa spaces are a certain type of Malcev algebras (anticommutative algebras satisfying \(J(x, y, xz) = J(x, y, z)x\), see [Sa]). More precisely, we have
Theorem 1. [Carrillo-Catalán, Sabrina] Let $A$ be an anticommutative algebra for which the following identities hold for every five elements $x,y,z,u,v$ of $A$:

\begin{align*}
J(xy, z, u) + J(yz, x, u) + J(zx, y, u) &= 0, \quad (1) \\
J(x, y, uv) &= J(x, y, u)v - J(x, y, v)u, \quad (2)
\end{align*}

If the characteristic of $A$ is not 2 then $A$ is contained in the variety of Malcev algebras defined by the identity:

\[ J(x, y, z)x = J(x, y, xz) = 0, \quad \forall x, y, z \in A. \quad (3) \]

It turns out that the converse of last theorem is not true. In what follows, let us call Malcev algebras of the first type those Malcev algebras which satisfy the identities (1) and (2). In the same way, those Malcev algebras which satisfy (3) will be called Malcev algebras of the second type.

Malcev algebras of the second type form a larger variety of algebras than Malcev algebras of the first type. In 2015 I. Shestakov constructed an algebra of dimension 29 by using the Non Associative Algebra System [Albert] (Jacobs, Muddana and Offutt, 1993), which is a Malcev algebra of the second type, but not a Malcev algebra of the first type, [SIP].

In this paper we construct a new example of Malcev algebra of the second type, but not a Malcev algebra of the first type of dimension 23, moreover, we think that this example has the minimal dimension.

Conjecture 1. Every Malcev algebra the second type of dimension $<23$ over a field of characteristic 0 is an algebra of the first type.

In this work we try to answer the question: What kind of smooth loop corresponds to a Malcev algebra of the second type?

2 Malcev algebras of the first type

It is possible to get equivalent characterizations of Malcev algebras of the first type by using Sagle’s properties of Malcev algebras. The following result gives the equivalences.

Lemma 1. $A$ is an anticommutative algebra for which the identities (1) and (3) hold for every five elements of $A$ if and only if $A$ is a Malcev algebra such that for every four elements of $A$ the identity

\[ J(x, y, uv) = 0 \quad (4) \]

holds, provided the characteristic of $A$ is not 2. In the special case where the characteristic of $A$ is neither 2 nor 3, the former identity is equivalent to having

\[ J(x, y, z)u = 0, \quad (5) \]

for every four elements of $A$. 

3
Proof. From the last result we know that $A$ is a Malcev algebra for which \text{(3)} holds. Now, since the characteristic of $A$ is not 2, the identity \text{(1)} can be expressed, by using Sagle’s properties for Malcev algebras ([Sa], p.429, L. 2.10, 2.14) and some odd permutations, as:

$$-2uJ(x, y, z) = -J(u, x, yz) - J(u, y, zx) - J(u, z, xy)$$
$$= J(yz, x, u) + J(zx, y, u) + J(xy, z, u)$$
$$= 0.$$  

Therefore the identity \text{(1)} can be replaced by the expression $J(x, y, z)u = 0$. Moreover, this identity can be used to rewrite \text{(2)}. Namely, If $J(x, y, z)u = 0$ holds for any four elements of $A$ then $J(x, y, u)v = J(x, y, v)u = 0$ and therefore $J(x, y, uv) = 0$.

Conversely, suppose $A$ is a Malcev algebra such that for every four elements \text{(4)} holds. Then, being a Malcev algebra, the linearized form of the Malcev property ([Sa], p.429, 2.7):

$$J(x, y, wz) = J(x, y, z)w + J(w, y, z)x - J(w, y, xz)$$

is satisfied for any $x, y, z, w$ in $A$. In particular, if \text{(4)} holds, then

$$J(x, y, v)u = -J(u, y, v)x$$

for any four elements in $A$. Then, after an odd permutation, for any four elements in the algebra:

$$J(x, y, u)v - J(x, y, v)u = -J(v, y, u)x - (-J(u, y, v))x = 2J(u, y, v)x.$$  

If $A$ is a Malcev algebra with characteristic different from 2, then $2J(u, y, v)x = 0$ ([Sa], p.429, 2.14). Hence the identity \text{(2)} holds trivially:

$$J(x, y, uv) - J(x, y, u)v + J(x, y, v)u = 0.$$  

For a Malcev algebra of characteristic different from 2 or 3 the identities $J(x, y, uv) = 0$ and $J(x, y, z)u = 0$ are equivalent. To see this, suppose that $J(x, y, uv) = 0$ for every four elements of the algebra. Since the characteristic of the algebra is not 2 and due to Malcev algebra properties ([Sa], p.429, 2.14) we should have:

$$-2uJ(x, y, z) = J(yz, x, u) + J(zx, y, u) + J(xy, z, u) = 0,$$

hence $J(x, y, z)u = 0$. Conversely, assume the characteristic of the algebra is different from 3, then once again due to Malcev algebra properties ([Sa], p.430, 2.15) we know that

$$3J(wx, y, z) = J(x, y, z)w - J(y, z, w)x - 2J(z, w, x)y + 2J(w, x, y)z.$$
Therefore, if \( J(x, y, z)u = 0 \) then \( J(wx, y, z) = 0 \) for any four elements of the algebra. Hence \( J(y, z, wx) = 0 \).

Finally, to prove (4), suppose that for every four elements in \( A \) the identities (1) and (2) hold. Then by Theorem 1, \( A \) is a Malcev algebra. As before, the identity (1) and Sagle’s properties for Malcev algebras ([Sa], p.429, 2.14) can be combined to get (5). Conversely, we know that (5) holds in \( A \) if and only if (1) is satisfied for every four elements of the algebra. But the identity (1) directly implies (1) and (2) as shown above.

From all of the above, we conclude that an anticommutative algebra with characteristic different from 2 and 3 and satisfying the identities (1) and (2) is equivalent to a Malcev algebra for which (1) or (5) holds for every four elements in the algebra. In other words, Malcev algebras of the first type can be characterized as algebras for which (1) and (2) hold or as Malcev algebras satisfying (4) or as Malcev algebras satisfying (3) or as anticommutative algebra satisfying (4) and (3), provided the characteristic of the algebra is not 2 nor 3.

In what follows we will use the definition of Malcev algebra of the first type as a Malcev algebra with the identity (4). It is known that the tangent algebra of a smooth diassociative loop is a Binary Lie algebra, which can be defined by the identity \( J(x, y, xy) = 0 \) (see [GA])

3 Malcev algebras of second type

Let \( A \) be a Malcev algebras of the second type. In this section we will prove that such algebras turn out to be tangent of a local almost left automorphic Moufang loops.

We begin with statements which are direct consequences of Malcev algebras properties:

**Proposition 1.** If \( y \) and \( z \) are two elements of \( A \) such that \( J(y, z, A) = 0 \) then \( yz \) belongs to the Lie kernel of \( A \), that is, \( yz \in N(A) = \{ x \in A \mid J(x, A, A) = 0 \} \).

**Proof.** Since \( A \) is a Malcev algebra, it follows from Sagle ([Sa], p.431, 2.26) that the identity

\[
J(wx, y, z) = wJ(x, y, z) + J(w, y, z)x + 2J(yz, w, x),
\]

holds for any four elements of the algebra. Now, if \( J(y, z, A) = 0 \) then \( 2J(yz, A, A) = 0 \).

**Proposition 2.** \( A^4 \subseteq N(A) \).
\textbf{Proof.} By the last result, since the identity \textcolor{red}{[3]} holds in $A$, we know that $x(xz) \in N(A)$ for any two elements of the algebra. Then the quotient algebra $A/N(A)$ must satisfy $x(xy) = 0$. As proved in ([E]A, Prop 3.1, p.1-5) we can conclude that $(A/N(A))^2 = 0$ or, equivalently, $A^2 \subseteq N(A)$. \hfill \Box

The above ideas can be generalized by using the fact that the Jacobian is a skew-symmetric multilinear map.

\textbf{Lemma 2.} The multilinear maps $\xi, \zeta, \varsigma : A \times A \times A \rightarrow A$ given by

\begin{align*}
\xi(x_1, x_2, x_3, x_4) &= J(x_1, x_2, x_3x_4) \\
\zeta(x_1, x_2, x_3, x_4) &= J(x_1, x_2, x_3)x_4 \\
\varsigma(x_1, x_2, x_3, x_4, x_5) &= J(x_1x_2, x_3x_4, x_5)
\end{align*}

are skew-symmetric.

\textbf{Proof.} Since $A$ is anticommutative and due to the properties of the Jacobian it is clear that $\xi(x_1, x_2, x_3, x_4) = J(x_1, x_2, x_3x_4)$ is skew-symmetric in $x_1$ and $x_2$ as well as in $x_3$ and $x_4$. If \textcolor{red}{[3]} holds for $A$, the map $\xi$ is also skew-symmetric for $x_1$ and $x_3$ since $J((x_1 + x_3), x_2, (x_1 + x_3)x_4) = 0$ can be rewritten by linearity as

\[ J(x_1, x_2, x_1x_4) + J(x_3, x_2, x_3x_4) + J(x_1, x_2, x_3x_4) + J(x_3, x_2, x_3x_4) = 0. \]

Using \textcolor{red}{[3]} we obtain:

\[ J(x_1, x_2, x_3x_4) = -J(x_3, x_2, x_1x_4). \]

Notice that if $\xi$ is skew-symmetric in $x_1$ and $x_3$ it is also skew-symmetric in $x_1$ and $x_4$:

\[ J(x_1, x_2, x_3x_4) = -J(x_1, x_2, x_4x_3) = J(x_4, x_2, x_3x_1) = -J(x_4, x_2, x_1x_3). \]

The proof that the map $\zeta$ is skew-symmetric is similar: due to the properties of the Jacobian $\zeta$ is skew-symmetric in $x_1$ and $x_2$ as well as in $x_1$ and $x_3$. Now by linearity and applying \textcolor{red}{[3]}, the fact that $J((x_1 + x_4), x_2, x_3)(x_1 + x_4) = 0$ implies the skew-symmetry in $x_1$ and $x_4$.

As above, $\zeta$ is skew-symmetric in $x_1$ and $x_2$ as well as in $x_1$ and $x_4$. The skew-symmetry of $\xi$ can be used to prove that $\varsigma$ is skew-symmetric in $x_1$ and $x_5$ for $i = 1, 2, 3, 4$. Notice that $J(x_1x_2, x_3x_4, x_5) = J(x_5, x_1x_2, x_3x_4)$ due to an even permutation. Since $\xi$ is skew-symmetric in $x_1$ and $x_3$, $\varsigma$ is skew-symmetric in $x_5$ and $x_3$: $J(x_5, x_1x_2, x_3x_4) = -J(x_3, x_1x_2, x_5x_4)$, as well as in $x_5$ and $x_4$: $J(x_5, x_1x_2, x_3x_4) = -J(x_4, x_1x_2, x_3x_5)$. To verify that $\varsigma$ is skew-symmetric in $x_5$ and $x_2$ we also use the skew-symmetry of $\xi$, suitable permutations and the properties of the Jacobian:

\[ J(x_5, x_1x_2, x_3x_4) = -J(x_5, x_3x_4, x_1x_2) = -J(x_1, x_5x_2, x_3x_4). \]
and similarly for \( x_5 \) and \( x_1 \). The result follows since the permutations \((i, 5)\) for \( i = 1, 2, 3, 4 \) generate the symmetric group in \( \{1, 2, 3, 4, 5\} \).

**Theorem 2.** Let \( A \) be an anticommutative algebra satisfying (3). Then

A. \( A^3 \) belongs to the Lie Kernel \( N(A) \). In particular \( A/N(A) \) is a nilpotent Lie algebra.

B. \( A^2 \) is a Lie algebra.

C. \( A \) satisfies the identity:

\[ 2wJ(x, y, z) = 3J(w, x, yz). \]

D. The following hold:

i. \( J(A^i, A^j, A^k) = 0 \) if \( i + j + k \geq 5 \).

ii. \( J(A^i, A^j, A^k)A^r = 0 \) if \( i + j + k + r \geq 5 \).

iii. \( (J(A, A, A))A = 0 \).

E. \( J(A, A, A)^2 = 0 \).

**Proof.** Notice that due to anticommutativity we know that

\[ J(x_1x_2, x_3x_4, x_5) = -J(x_3x_4, x_1x_2, x_5) \]

for any elements \( x_1, x_2, x_3, x_4, x_5 \) of \( A \). In the previous Lemma, the map \( \varsigma \) was proved to be skew-symmetric. Then we can write

\[ J(x_1x_2, x_3x_4, x_5) = -J(x_3x_2, x_1x_4, x_5) = J(x_3x_4, x_1x_2, x_5). \]

We conclude that \( J(A^2, A^2, A) = 0 \). Now, due to the skew-symmetry of the map \( \xi \), for every \( J(xy, ab, c) \in J(A^2, A^2, A) \) we have

\[ J(xy, ab, c) = -J(x(ab), y, c). \]

In the same way, by using the skew-symmetry of \( \varsigma \) followed by the skew-symmetry of the map \( \xi \) we get:

\[ J(xy, ab, c) = -J(xy, ac, b) = J(x(ac), y, b). \]

Hence

\[ 0 = J(A^2, A^2, A) = J(A^3, A, A), \]

which implies that \( A^3 \subseteq N(A) \). On the other hand, \( A/N(A) \) is a Lie algebra since \( J(A, A, A) \subset A^3 \subseteq N(A) \) and \( N(A) \) is zero in \( A/N(A) \). This proves (A),
(B) and part (i) of (D) of the Theorem.

On the other hand, since $A$ is a Malcev algebra, once again according to Sagle’s properties for Malcev algebras ([Sa], p.429, L. 2.10, 2.14), for every $x,y,z,w$ of $A$ we should have

$$2wJ(x,y,z) = J(w,x,yz) + J(w,y,zx) + J(w,z,xy).$$

It is enough to use this identity and the skew-symmetry of the map $\xi$ to prove (C): each of the terms can be written as

$$J(w,y,zx) = -J(w,y,xz) = J(w,x,yz)$$

and

$$J(w,z,xy) = -J(w,x,zy) = J(w,x,yz).$$

The last part of (A) follows from (C) and (i): from (C) we know that

$$2AJ(A, A, A) = 3J(A, A, A^2).$$

This implies that $2AJ(A, A, A^2) = 3J(A, A, A^3) = 0$ due to (i). We conclude that $AJ(A, A, A^2) = 0$. Therefore

$$(2AJ(A, A, A))A = (3J(A, A, A^2))A = 3J(A, A, A^2)A = 0.$$ Since $(2AJ(A, A, A))A = -2(J(A, A, A)A)A$ we conclude that $(J(A, A, A)A)A = 0$.

Finally (C) implies that $J(A, A, A)A^3 = 0$. Since $J(A, A, A) \subseteq A^3$, we get:

$$J(A, A, A)^2 \subseteq J(A, A, A)A^3 = 0.$$ 

\[\square\]

**Corollary 1.** If $A$ is a semiprime algebra, then $A$ is a Lie algebra.

**Proof.** Suppose that $A$ is a semiprime algebra. Since, by the previous Theorem, the ideal $J(A, A, A)$ satisfies $J(A, A, A)^2 = 0$ it follows that $J(A, A, A) = 0$, namely, $A$ is a Lie algebra. \[\square\]

**Theorem 3.** A 3-generated Malcev algebra $A$ of the second type is a Malcev algebra of the first type.

**Proof.** Let $A$ be a Malcev algebra of the second type generated by the elements $a,b,c$. Let $\omega$ be an arbitrary element of $A$. Since $A$ is a 3-generated algebra, $\omega$ is a formal linear combination of products of the generators $a,b,c$.

Let us show that the identity \[4\] holds in $A$. First, consider $J(x,y,c\omega)$, where $c$ is an arbitrary generator of $A$. Since the Jacobian is a multilinear operator, $J(x,y,c\omega)$ can be simplified using (4, i) of Theorem 2.

$$J(x,y,c\omega) = \ell_1 J(a,b,ca) + \ell_2 J(a,b,c) + \ell_4 J(a,b,c(ab)) + \ell_5 J(a,b,c(ab)) + \ell_6 J(a,b,c(bc)) + \ell_7 J(a,b,c((ab)c)) + \ell_8 J(a,b,c(a(bc))) + \cdots$$

Using again the fact that $A^3 \subseteq N(A)$ and $J(a,b,c) = 0$, we conclude that
\[ J(x, y, c\omega) = 0 \text{ for any } \omega \text{ in } A. \]

The general case, \( J(x, y, uv) = 0 \) where \( x, y, u \) and \( v \) are arbitrary elements of \( A \), can be handled by analogous considerations.

Due to Malcev-Kuzmin theory [Kuz], we have

**Corollary 2.** A Malcev algebra of the second type is, in fact, a tangent algebra of a local almost left automorphic Moufang loop. The tangent algebra of every smooth almost left automorphic Moufang loop is a Malcev algebra of the second type.

**Remark** The question of the existence of global almost left automorphic Moufang loop, which corresponds to the given Malcev algebra of the second type is solved positively in the article [CGRS].

4 Example

In this section we discuss an example of a 23-dimensional algebras of second type which is not an algebra of first type.

Let \( F \) be a free noncommutative algebra generated by \( X = \{x_1, x_2, x_3, x_4\} \), nilpotent of class 4, it means \( F^4 = 0 \). Let \( I \) be a subspace with a basis of all \( X \)-words, \( w = w(x_1, x_2, x_3, x_4) \), such that some letter \( x_i \) appears in \( w \) two or more times. It is clear that \( I \) is an ideal of \( F \). Let’s denote by \( A \) the factor algebra \( F/I \). Then a basis of \( A \) has 22 elements: \( B = \cup_{i=1}^3 B_i \) with

\[
\begin{align*}
B_1 &= X, \\
B_2 &= \{[x_i, x_j] \mid 1 \leq i < j \leq 4\}, \\
B_3 &= \{[x_i, x_j, x_k] \mid 1 \leq i < j \leq 4, 1 \leq k \leq 4, k \neq i, j\}.
\end{align*}
\]

The algebra \( A \) is a Malcev algebra.

Let’s define an antisymmetric bilinear function \( \psi : A \times A \to k \) given by the following values:

\[
\begin{align*}
\psi([x_1, x_2], [x_3, x_4]) &= 2, & \psi([x_1, x_3], [x_2, x_4]) &= -2, \\
\psi([x_1, x_4], [x_2, x_3]) &= 2, & \psi([x_2, x_3, x_1], x_4) &= -3, \\
\psi([x_2, x_4, x_1], x_3) &= 3, & \psi([x_2, x_4, x_3], x_1) &= -1, \\
\psi([x_2, x_4, x_1], x_2) &= -3, & \psi([x_3, x_4, x_2], x_1) &= 1,
\end{align*}
\]

and \( \psi(v, w) = 0 \) for all other values.
Consider a space $\tilde{A} = A \oplus kv$ and define a product on $\tilde{A}$ as follows:

$$[(a, \alpha v), (b, \beta v)] = ([a, b], \psi(a, b))$$  \hspace{1cm} (7)

Direct computations show that $\tilde{A}$ is a second type Malcev algebra. On the other hand, if we set $x_i = (x_i, 0)$, then

$$[J(x_1, x_2, x_3), x_4] = [x_1, x_2, x_3, x_4] + [x_2, x_3, x_1, x_4] - [x_1, x_3, x_2, x_4]$$

$$= (0, -3v) \neq 0.$$

Hence $\tilde{A}$ is not an algebra of the first type.

**Acknowledgments**

The authors thank Alberto Elduque, Alexander Grishkov and Ivan Shestakov for useful comments.

**Funding**

The first author thanks CONACYT and Universidad Pedagógica Nacional Unidad 201 Oaxaca for supporting the Cátedras CONACYT project 1522. The second author thanks CNPq (Brasil), grant 308221/2012-5 and the third author thanks SNI and FAPESP grant process 2015/07245-4 for support.
References

[CS1] Carrillo-Catalán R., Sabinina L. *On smooth power-alternative loops*, Communications in algebra, Vol. 32, No. 8, pp. 2969 - 2976, (2004).

[CPS] Chein O., Pflugfelder H.O. and Smith J.D.H., *Quasigroups and loops: Theory and applications*, Sigma Series in Pure Mathematics, 8, Heldermann Verlag, (1990).

[EA] Elduque A., *Quadratic Alternative Algebras*, J. Math. Phys.31, 1, 1-5, 17A45 (17D05), (1990).

[Ga] Gainov A.T. *Binary Lie algebras of characteristic two*. Algebra and Logic , 8 : 5 (1969) pp. 287 - 297 Algebra i Logika , 8 : 5 pp. 505 - 522, (1969).

[CGRS] Grishkov A.,Carrillo Catalan R.,Rasskazova M.,Sabinina L. *Nilpotent by Lie center Malcev algebras and corresponding analytic Moufang loops*. Preprint.

[Ker] Kerdman F.S. *Analytic Moufang loops in the large*, Algebra and Logic (1979) 18: 325. Translated from Algebra i Logika, Vol. 18, No. 5, pp. 523 - 555, September -October, (1979).

[K1] Kikkawa M., *On local loops in affine manifolds*, J.Sci. Hiroshima Univ. Ser. A-I, 28, 199 - 207. (1964).

[K3] Kikkawa M., *Geometry of homogeneous Lie loops*, Hiroshima Math. J. 5:141-179, (1975).

[Kuz] Kuz’min, E.N. *On the relation between Mal’tsev algebras and analytic Moufang groups*. Algebra and Logic (1971) 10: 1. Translated from Algebra i Logika, Vol. 10, No. 1, pp. 3 -22, January -February, (1971).

[SLV] Sabinin Lev V. *Smooth Quasigroups and Loops* Kluver Academic Publishers. Dordrecht/Boston/London 1999.

[Sa] Sagle A. *Malcev Algebras*. Trans. Amer. Math. Soc.,101 (1961), 3, 426 - 458 MR 26 1343.(1963).

[SIP] Shestakov I.P. *Private Communication*. 

11