N-dimensional geometries and Einstein equations from systems of PDEs

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Abstract

The aim of the present work is twofold: first, we show how all the n-dimensional Riemannian and Lorentzian metrics can be constructed from a certain class of systems of second-order PDE’s which are in duality to the Hamilton-Jacobi equation and second we impose the Einstein equations to these PDE’s.
1 Introduction

Towards the end of the 19th century and early 20th century Tresse, Wünschmann, Lie, Cartan and Chern ([1]-[9]), studied the classification of second- and third-order ODE’s according to their equivalence classes under a variety of transformations and the resulting induced geometries on the solution spaces. In particular, Cartan ([5]-[8]) and Chern [9] found that for a certain subclass of third-order ODE’s, a unique Lorentzian conformal metric can be constructed in a natural way on the solution space. This subclass was defined by the vanishing of a specific relative invariant, defined from the differential equation and first obtained by Wünschmann [3]. It is now referred to as the Wünschmann invariant. In a more recent work Tod [10] showed how the Einstein-Weyl spaces can be obtained from a particular class of third-order differential equations.

In a more recent series of papers Frittelli, Kozameh, Newman, Kamran and Nurowski, ([11]-[18]) were able to generalize this result. They showed that all four-dimensional conformal Lorentzian geometries were encoded in equivalence classes (under contact transformations) of pairs of second-order PDE’s that were characterized by the vanishing of an analogous (generalized) Wünschmann invariant, referred to as the metricity conditions. In this approach, referred to as the null surface formulation of general relativity, the metric of the space-time is a derived concept. The fundamental objects are two functions, $Z(x^a, s, s^*)$ and $\Omega(x^a, s, s^*)$, of the space-time points $x^a$ and parametrized by points on the sphere; that is, by functions defined on
\( M \times S^2 \) (the sphere bundle over the space-time). The first of the functions, 
\( Z(x^a, s, s^*) \), which encodes all the conformal information of the space-time, describes a sphere’s worth of surfaces through each space-time point. It is from these surfaces that a conformal metric can be constructed. The second function, \( \Omega(x^a, s, s^*) \), which plays the role of a conformal factor, converts it into any metric in the conformal class. The level surfaces of \( Z(x^a, s, s^*) \) in \( M \), for each fixed value of \((s, s^*)\), are null hypersurfaces with respect to this metric. As \((s, s^*)\) take different values on \( S^2 \) at a fixed point \( x^a \) in \( M \), the normals to the null hypersurfaces sweep out the null-cone at \( x^a \).

To establish this new approach to general relativity, these authors began with a four-dimensional Lorentzian manifold, already containing a metric \( g_{ab} \) and a complete integral to the eikonal equation

\[ g^{ab}(x^a)\nabla_a Z \nabla_b Z = 0. \]  

(1)

A complete integral, expressed as,

\[ u = Z(x^a, s, s^*), \]  

(2)

contains the space-time coordinates, \( x^a \), and the needed (for a complete integral) two-parameters \((s, s^*)\). By constructing the four functions,

\[ \theta^i \equiv (u, \omega, \omega^*, R) \equiv (Z, \partial_s Z, \partial_{s^*} Z, \partial_{s^* s} Z), \]  

(3)

from Eq. (2) and its derivatives, and by eliminating \( x^a \), via the algebraic inversion

\[ x^a = X^a(s, s^*, \theta^i), \]  

(4)
they found that \( u = Z(x^a, s, s^*) \) satisfies in addition to Eq. (2) the pair of second-order partial differential equations in \( s, s^* \), of the form

\[
\begin{align*}
\partial_{s s} Z &= \Lambda(Z, \partial_s Z, \partial_{s*} Z, \partial_{s s*} Z, s, s^*), \\
\partial_{s^* s^*} Z &= \Lambda^*(Z, \partial_s Z, \partial_{s*} Z, \partial_{s s*} Z, s, s^*). 
\end{align*}
\]

The \( x^a \), in the solution of Eq. (5), appear now as constants of integration. The roles of \( x^a \) and \( (s, s^*) \) are thereby interchanged. Note that the metric has disappeared from the equations.

The question then was, could this procedure be reversed? Could one start with a pair of equations of the form, (5), and then find the eikonal equation, (1), with a metric \( g^{ab}(x^a) \)?

It was shown that when the functions \( (\Lambda, \Lambda^*) \) satisfy an integrability condition, a weak inequality and a certain set of differential conditions (the metricity or generalized Wünschmann conditions), the procedure can be reversed. The solutions to the pair do determine a conformal four-dimensional Lorentzian metric and, in fact, all conformal Lorentzian metrics can be obtained from equivalence classes of equations of the form Eq. (5). When certain specific conditions \([19, 20]\), in addition to the Wünschmann condition, are imposed on the \( (\Lambda, \Lambda^*) \), the metrics, determined by the solutions, are in the vacuum conformal Einstein class.

In a recent work we presented a new approach to 4-dimensional general relativity, which is similar to the null surface formulation, but now instead of using a complete integral to the eikonal equation, we used a complete integral to the Hamilton-Jacobi equation. The aim of the present work is to
generalize our previous results.

In section 2 we begin with an \( n \)-dimensional manifold, \( \mathcal{M} \), with no further structure and then investigate arbitrary \((n-1)\)-parameter families of surfaces on \( \mathcal{M} \) given by

\[
u = \text{constant} = Z(x^a, s^i). \tag{6}\]

The \( x^a \) are local coordinates on \( \mathcal{M} \) and \( s^i \) parametrize the families and can take values on an open neighborhood of a manifold \( \mathcal{N} \) of dimension \((n-1)\). More specifically, we then ask when do such families of surfaces define an \( n \)-dimensional metric, \( g_{ab}(x^a) \), such that

\[
g^{ab}\nabla_a Z(x^a, s^i)\nabla_b Z(x^a, s^i) = 1. \tag{7}\]

We have here either taken the mass in the H-J equation to be 1 or put it into the \( g^{ab} \) as a factor.

By taking \((s^i)\) derivatives of Eq. (6) and eliminating the \( x^a \), we will show that the \( u = Z(x^a, s^i) \) must also satisfy a system of \( \frac{n(n-1)}{2} \) second-order PDE’s

\[
\partial_{s^i s^j} Z = \Lambda_{ij}(u, w^i, s^j) = \Lambda_{ij}(u, \partial_{s^i} Z, s^j), \tag{8}\]

where \( \Lambda_{ij} \), are restricted to satisfy certain metricity or ‘Wünschmann-like’ conditions.

Here \( \partial_{s^i} \), denotes the partial derivative with respect to the parameter \( s^i \). Observe that in the solutions of Eqs. (8) \( u = Z(x^a, s^i) \), the \( x^a \) are \( n \) constants.
of integration for Eqs. (8) while the ‘$s^i$’ are $n - 1$ integration constants for Eq. (7). Observe that $u = Z(x^a, s^i)$ is a complete integral to the Hamilton-Jacobi equation (7).

In this section we also remark that the $n$-dimensional metric $g_{ab}(x^a)$ associated with the system of partial differential equations, (8), is invariant under a subset of contact transformations of the differential equations.

In section 3, we present our new formulation of $n$-dimensional general relativity. For this purpose we substitute the $n$-dimensional metric already obtained in section 2 into the Einstein equations. From our results we conclude that the Einstein equations in $n$ dimensions can be reformulated as equations for families of $(n - 1)$-dimensional surfaces given by the level surfaces of $u = Z(x^a, s^i)$.

As in four dimensions, this new point of view can be given in either of two versions. In the first version, the variables are the $\frac{n(n-1)}{2}$ functions, $\Lambda_{ij}$, of the $2n - 1$ variables $(u, w^i, s^i)$, i.e., the right-side of Eqs. (8). These functions must satisfy three sets of equations; the integrability conditions, the Wünschmann-like conditions and a further condition obtained from the Einstein equations. The metric, on a $n$-manifold, can be written down directly in terms of these $\frac{n(n-1)}{2}$ functions and their derivatives. There is no need to use the set, Eqs. (8). In the second version one uses the same set of $\Lambda_{ij}$ in the right-side of Eqs. (8) and solves for the $Z(x^a, s^i)$. The metric is then written in terms of the $Z(x^a, s^i)$ and its derivatives. The advantage of the
first version is that one does not need to solve Eqs. (8), but one has to extract (algebraically) the $n$-manifold from the $(u, \partial_i u)$ while in the second version the four-manifold is explicitly given by the four constants of integration, $x^a$.

It is important to remark that no claim is made that this approach to the $n$-dimensional Einstein equations has any obvious advantage over the usual metric approach. It however does give certain mathematical insights into the differential geometry associated with general relativity in $n$ dimensions.

2 N-D metrics and the metricity or Wünschmann-like conditions

We start with an $n$-dimensional manifold $\mathcal{M}$ (with local coordinates $x^a = (x^0, ..., x^{n-1})$) and assume we are given an $(n-1)$-parameter set of functions $u = Z(x^a, s^i)$. As we said the parameters $s^i$ can take values on an open neighborhood of a manifold $\mathcal{N}$ of dimension $(n - 1)$. We also assume that for fixed values of the parameters $s^i$ the level surfaces

$$u = \text{constant} = Z(x^a, s^i), \quad (9)$$

locally foliate the manifold $\mathcal{M}$ and that $u = Z(x^a, s^i)$ satisfies the H-J equation

$$g^{ab}(x^a)\nabla_a Z(x^a, s^i)\nabla_b Z(x^a, s^i) = 1, \quad (10)$$

for some unknown metric $g_{ab}(x^a)$. 
The basic idea now is to solve Eq. (10) for the components of the metric in terms of $\nabla_a Z(x^a, s^i)$. To do so, we will consider a number of parameter derivatives of the condition (10), and then by manipulation of these derivatives, obtain both the $n$-dimensional metric and the system of partial differential equations defining the surfaces plus the conditions these PDE’s must satisfy. They will be referred to as the metricity or Wünschmann-like conditions.

**Remark 1:** The notation is as follows: there will be two types of differentiation, one is with respect to the local coordinates, $x^a$, of the manifold $\mathcal{M}$, denoted by $\nabla_a$ or “comma a,” the other is with respect to the parameters $s^i$ denoted by $\partial_s^i \equiv \partial_i$.

From the assumed existence of $u = Z(x^a, s^i)$, we define $n$ parameterized scalars $\theta^A$ in the following way

$\theta^A = (Z, w^i) \equiv (Z, \partial_i Z)$. \hfill (11)

**Remark 2:** For each value of $s^i$, Eqs. (11) can be thought of as a coordinate transformation between the $x^a$’s and $(u, w^i)$.

We also define the following $\frac{n(n-1)}{2}$ important scalars

$\tilde{\Lambda}_{ij} = \partial_{ij} Z(x^a, s^i)$. \hfill (12)

In what follows we will assume that Eqs. (11) can be inverted, i.e., solved for the $x^a$’s;

$x^a = X^a(u, w^i, s^i)$. 8
Eqs. (12) can then be rewritten as

\[ \partial_{ij}Z = \Lambda_{ij}(u, w^i, s^i). \]  

(13)

This means that the \((n-1)\)-parameter family of level surfaces, Eq. (9), can be obtained as solutions to the system of \(\frac{n(n-1)}{2}\) second-order PDE’s (13). Note that \(\Lambda_{ij}\) satisfy the integrability conditions

\[ D_s\Lambda_{ij} = D_s\Lambda_{kj} = D_s\Lambda_{ki}, \]  

(14)

where

**Definition 1:** The total \(s^i\) derivative of a function \(F = F(u, w^i, s^i)\) is defined by

\[ D_s F = F_s + F_{w^i} \Lambda_{li}. \]  

(15)

The solution space of Eqs. (13) is \(n\)-dimensional. This can be seen in the following way. The system of PDE’s (13) is equivalent to the vanishing of the \(n\) one-forms, \(\omega^A = (\omega^0, \omega^i)\)

\[ \omega^0 \equiv du - w^i ds^i, \]

\[ \omega^i \equiv dw^i - \Lambda_{im} ds^m. \]  

(16)

A simple calculation, using the integrability conditions on \(\Lambda_{ij}\), leads to

\(d\omega^A = 0 \ (\text{modulo } \omega^A)\) from which, via the Frobenius Theorem, we conclude that the solution space of Eqs. (13) is \(n\)-dimensional.

From the \(n\) scalars, \(\theta^A\), we have their associated gradient basis \(\theta^A_a\) given by

\[ \theta^A_a = \nabla_a \theta^A = \{Z_a, w^i_a\}, \]  

(17)
and its dual vector basis $\theta^A {}_a$, so that
\[ \theta^A {}_a \theta^B {}_a = \delta^B {}_a, \quad \theta^A {}_a \theta^A {}_b = \delta^a {}_b. \quad (18) \]

It is easier to search for the components of the $n$-dimensional metric in the gradient basis rather than in the original coordinate basis. Furthermore, it is preferable to use the contravariant components rather than the covariant components of the metric; i.e., we want to determine
\[ g^{AB}(x^s, s^i) = g^{ab}(x^a) \theta^A {}_a \theta^B {}_b. \quad (19) \]

The metric components and the Wünschmann-like conditions are obtained by repeatedly operating with $\partial_i$ on Eq. (10), which, by definition, is
\[ g^{00} = g^{ab} Z {}_a Z {}_b = 1. \quad (20) \]
Applying $\partial_i$ to Eq. (20) yields $\partial_i g^{00} = 2g^{ab} \partial_i Z {}_a Z {}_b = 0$, i.e.,
\[ g^{i0} = 0. \quad (21) \]
A direct computation shows that
\[ \partial_{ji}(g^{00}/2) = g^{ab} \partial_{ji} Z {}_a Z {}_b + g^{ab} \partial_i Z {}_a \partial_j Z {}_b = g^{ab} \Lambda_{ij,a} Z {}_b + g^{ij} = 0. \quad (22) \]
Since, by the assumed linear independence of $(Z {}_a, \partial_i Z {}_a)$,
\[ \Lambda_{ij,a} = \Lambda_u Z {}_a + \Lambda_{ij,w} \partial_k Z {}_a, \quad (23) \]
Eq. (22), using Eqs. (20)-(23), is equivalent to
\[ g^{ij} = -\Lambda_{ij,u}. \quad (24) \]
Therefore, the final result is

\[
(g^{AB}) = \begin{pmatrix}
1 & 0 \\
0 & -\Lambda_{ij,u}
\end{pmatrix}.
\] (25)

**Remark 3:** We require that \( \det(g^{ij}) = \Delta \) be different from zero, with

\[\Delta \equiv \det(-\Lambda_{ij,u}).\] (26)

Finally the metricity or Wünschmann-like conditions are obtained from the third derivatives, i.e., from \( \partial_{iji}g^{00} = 0 \). By a direct computation we obtain that

\[
D_{sk}[\Lambda_{mn,u}] = \Lambda_{ln,u}\Lambda_{km,w} + \Lambda_{lm,u}\Lambda_{kn,w}.
\] (27)

In \( n \) dimensions, with \( n \geq 2 \), there will be \( \frac{n(n^2-1)}{6} \) Wünschmann-type conditions. For example for \( n = 2 \) we have a second-order ODE and one Wünschmann condition, for \( n = 3 \) we have a system of three second-order PDE’s and four Wünschmann conditions and for \( n = 4 \) we have a system of six second-order PDE’s and ten Wünschmann conditions.

Summarizing:

a) If we start from a complete integral, \( u = Z(x^a, s^i) \) to the H-J equation, (10), then it satisfies the system of \( \frac{n(n-1)}{2} \) second-order PDE’s (13), with \( \Lambda_{ij} \) satisfying Eqs. (14) and the Wünschmann-like conditions (27); In other words, in the solution space of Eqs. (13) there is the naturally defined metric

\[
g^{ab} = g^{AB}\theta_A^a\theta_B^b,
\] (28)
where $g^{AB}$ is given by Eq. (25).

b) If we start with a system of $\frac{n(n-1)}{2}$ second-order PDE’s (13), where $\Lambda_{ij}$ satisfy Eqs. (27) and the integrability conditions, (14), then in its solution space there exist a natural $n$-dimensional metric given by Eq. (25). Though it might appear as if the metric components depend on the parameters $s^i$, the Wünschmann-like conditions guarantees that they do not. Furthermore, the solutions $u = Z(x^a, s^i)$ satisfy the H-J equation

$$g^{ab} \nabla_a Z(x^a, s^i) \nabla_b Z(x^a, s^i) = 1,$$

with the just determined metric, Eq. (28).

Remark 4: From the results presented above we conclude that solving the $n$-dimensional H-J equation, in an $n$-dimensional background space-time, is equivalent to solving a system of $\frac{n(n-1)}{2}$ second-order PDE’s.

In some of the earlier work on the eikonal equation in three- and four-dimensional Lorentzian spaces, it was proved that the conformal Lorentzian metrics associated with third-order ODE’s and pairs of second-order PDE’s satisfying the Wünschmann condition and generalized Wünschmann condition, is preserved when the differential equation is transformed by a contact transformation. For our present case, there is an analogous result given by the following:

Theorem 1: Let Eqs. (13) be a system of $\frac{n(n-1)}{2}$ second-order PDE’s,
with $\Lambda_{ij}$ satisfying the conditions (14) and (27), and let

$$\overline{\partial}_{ij}Z = \overline{\Lambda}_{ij}(\overline{u}, \overline{w}, \overline{s}),$$

be a second system of $n(n-1)/2$ second-order PDE’s locally equivalent to Eqs. (13) under the subset of contact transformations generated by the generating function

$$H(s, s^*, \gamma, u, \bar{s}, \bar{s}^*, \bar{\gamma}, \bar{u}) = \bar{u} \mp u - G(s^i, \bar{s}^j).$$

Then under this subset of contact transformations the metric given by Eq. (28) is preserved.

The proof of this theorem is exactly as that presented in Ref. [16] for a system of two second-order PDE’s such that on its space of solutions there is a unique four-dimensional conformal Lorentzian metric, $g^{ab}$, such that $g^{ab}u_{,a}u_{,b} = 0$. The justification of the form of the generating function (30) can be done as in Refs. [21, 22, 23]. For a definition of contact transformation see Ref. [24]

Before closing this section we remark that for, $n = 2$ and 3, our general results reduce to that reported in Refs. [21, 22, 25]. For $n = 4$ we have discovered that in Ref. [23] is missing one Wünschmann condition [26].
3 The Einstein equations

We now adopt a new point of view towards geometry on an $n$-dimensional manifold. Instead of a Lorentzian metric $g^{ab}(x^a)$ on $\mathcal{M}$, as the fundamental variable we consider as the basic variables a family of surfaces on $\mathcal{M}$ given by $u = constant = Z(x^a, s^i)$ or preferably its second derivatives with respect to $s^i$. From this new point of view these surfaces are basic and the metric is a derived concept. Now we will find the conditions on $u = Z(x^a, s^i)$ or more accurately on the second order system such that the $n$-dimensional metric, Eq. (25), be a solution to the Einstein equations.

We start with the Einstein equations in $n$ dimensions, which are given by (see for example [27, 28]).

$$ R_{ab} = 8\pi G \left( T_{ab} - \frac{1}{n-2} g_{ab} T \right), $$

(31)

with the Ricci tensor given by

$$ R_{ab} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^c} (\Gamma^c_{ab} \sqrt{-g}) - \frac{\partial^2}{\partial x^a \partial x^b} \ln \sqrt{-g} - \Gamma^c_{ad} \Gamma^d_{bc}, $$

(32)

$g = \det(g_{ab})$ and

$$ \Gamma^c_{ab} = \frac{1}{2} g^{cd} \left( \frac{\partial g_{da}}{\partial x^b} + \frac{\partial g_{db}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right), $$

(33)

are the Christoffel symbols.

As in the null surface formulation of general relativity, in the present case the Einstein equations are given by
\[ R^{ab}Z_{a}Z_{b} = 8\pi G \left( T^{ab}Z_{a}Z_{b} - \frac{T}{n-2} \right). \] (34)

That is, to obtain the Einstein equation in this case, we need to compute 
\[ R^{00} \equiv R^{ab}Z_{a}Z_{b}, \] which is one of the components of \( R^{AB} \equiv R^{ab}\theta^{A}_{a}\theta^{B}_{b}. \) From Eq. (25) we have that \( R^{00} = R_{00}. \) Using the metric given by Eq. (25) with coordinates \( \theta^{A} = (\theta^{0}, \theta^{i}) \) in Eq. (32) to compute \( R_{00}, \) we find that Eq. (34) is equivalent to

\[ \frac{1}{2\Delta} \Delta_{uu} - \frac{1}{2\Delta^2} \Delta^{2}_{u} - \frac{1}{4} \Lambda_{ch,u}\Lambda_{dl,u}g_{hd,u}g_{lc,u} = 8\pi G \left( T^{ab}Z_{a}Z_{b} - \frac{T}{n-2} \right), \] (35)

where the covariant components of the metric \( g_{hc} = g_{hc}[\Lambda_{mn,u}] \) are obtained as functions of \( \Lambda_{ij} \)'s from Eq. (25).

At first glance it appears that Eq. (35) cannot be equivalent to the \( \frac{n(n+1)}{2} \) components of the Einstein equations. However, Eq. (35) is valid for any value of the \( s^{i} \)'s. Thus if we add to Eq. (35) the metricity or Wünschmann-like conditions, we obtain a set of consistent equations equivalent to the standard Einstein equations in \( n \) dimensions. The final equations read

\[ \frac{1}{2\Delta} \Delta_{uu} - \frac{1}{2\Delta^2} \Delta^{2}_{u} - \frac{1}{4} \Lambda_{ch,u}\Lambda_{dl,u}g_{hd,u}g_{lc,u} = 8\pi G \left( T^{ab}Z_{a}Z_{b} - \frac{T}{n-2} \right), \]

\[ D_{s}^{b}[\Lambda_{mn,u}] = \Lambda_{ln,u}\Lambda_{km,u^{i}} + \Lambda_{lm,u}\Lambda_{kn,u^{i}} , \] (36)

plus the integrability conditions, Eq.(14).
As we said in the introduction, we can now view the Einstein equations in either of the two closely related fashions:

We can consider Eqs. (36) as differential equations (of high order) for the single function $Z$. In this case the integrability conditions are not relevant. Alternatively, the Einstein equations can be considered as the equations, Eqs. (36), for the independent variables $\Lambda_{ij}$. In this case, the integrability conditions, Eq. (14), must be added but the order of the equations is much lower.

**Conclusions**

In the first part of this work, we have shown that the ideas and procedures developed in our recent papers [21, 22, 23, 25], on the H-J equation can be generalized to the $n$-dimensional H-J equation on an arbitrary manifold $\mathcal{M}$. That is, we have shown that on an $n$-dimensional manifold $\mathcal{M}$, a definite or indefinite metric, $g_{ab}$, is equivalent to a family of foliations of $\mathcal{M}$, depending on $(n-1)$ parameters $s^i$, described by $u = Z(x^a, s^i)$ that satisfies the Wünschmann-like conditions, Eqs. (27). Furthermore, from Eqs. (27) we observe that one can adopt other points of view, where the $\Lambda_{ij}$ are the basic variables and $u = Z(x^a, s^i)$ is an auxiliary variable. From this second point of view, Eqs. (27), are simpler but require that we add the integrability conditions (14) so that a $Z$ does exist.

In the second part of this work we have reformulated the Einstein equations in $n$ dimensions as equations for families of surfaces. If $Z$ is taken as
the basic variable then the Einstein equations are equivalent to Eqs. (35). But if the $\Lambda_{ij}$ are the basic variables the Einstein equations are equivalent to Eqs. (35) and (14). In both cases the Wünschmann equations are needed.

To establish our main results we have used a complete integral to the H-J equation on an $n$-dimensional manifold. We point out that preliminary computations suggest that a similar program can be carry out for the Eikonal equation in $n$ dimensions. In a future paper we will present these results.

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[26] By using the notation if Ref.[23] the missing Wünschmann condition is given by

\[ D_u \Psi^*_u = \Psi_u \Lambda^*_w + \Psi^*_u \Lambda^*_w + \Gamma_u \Lambda^*_R + \Phi_u \Psi_w + \Lambda^*_u \Psi^*_w + \Psi^*_u \Psi_R. \]

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