Heavy-to-light decays with a two-loop accuracy

Ian Blokland, Andrzej Czarnecki, and Maciej Ślusarczyk
Department of Physics, University of Alberta
Edmonton, AB T6G 2J1, Canada

Fyodor Tkachov
Institute for Nuclear Research, Russian Academy of Sciences
Moscow, 117312, Russian Federation

We present a determination of a new class of three-loop Feynman diagrams describing heavy-to-light transitions. We apply it to find the $O(\alpha_s^3)$ corrections to the top quark decay $t \to bW$ and to the distribution of lepton invariant mass in the semileptonic $b$ quark decay $b \to u \ell \nu$. We also confirm the previously determined total rate of that process as well as the $O(\alpha_s^2)$ corrections to the muon lifetime.

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The determination of higher order corrections in perturbative quantum field theory is notoriously difficult, and with the general tendency towards precision measurements in particle physics, each newly-won class of perturbative integrals expands the possibilities for phenomenological analyses. For instance, quantum corrections to decays of neutral particles, such as a virtual photon or a $Z$ boson into hadrons, are known to sixth order in perturbation theory, $O(\alpha^6)$, and even some $O(\alpha^4)$ effects have been studied. Those results have been very useful in determining a variety of Standard Model parameters such as the $Z$ boson properties, the strong coupling constant, and the running of the electromagnetic coupling constant.\[1\]

Much less is known about radiative corrections to processes with a charged particle in the initial state. Only relatively recently have first results been obtained in fourth order perturbation theory, $O(\alpha^4)$, and only for some kinematic cases. One approach that has been successful consists in expanding Feynman diagrams around the zero recoil limit: when the quark $q$ remains at rest with respect to $Q$. The technical challenge in such calculations is the presence of massive propagators. For example, consider the muon decay. Since the muon is charged, it can emit photons, and the resulting amplitudes will involve propagators of a virtual muon. Its mass sets the energy scale of the process and cannot be treated as a small parameter.

The presence of massive propagators is an obstacle in evaluating the multi-loop diagrams required by precise measurements of heavy quark and lepton decays. So far, genuine $O(\alpha_s^2)$ corrections to heavy quark decays are known only for semileptonic processes, $Q \to q \ell \nu(gg)$, and only for some kinematic cases. One approach that has been successful consists in expanding Feynman diagrams around the zero recoil limit: when the quark $q$ remains at rest with respect to $Q$. The technical challenge in such calculations is the presence of massive propagators. For example, consider the muon decay. Since the muon is charged, it can emit photons, and the resulting amplitudes will involve propagators of a virtual muon. Its mass sets the energy scale of the process and cannot be treated as a small parameter.

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However, the expansion around zero recoil converges slowly near the origin in Fig. 1 that is, when both the quark $q$ and the lepton pair are light; in this case computations become prohibitively expensive. Two other approaches have been used in such cases. First, for the phenomenologically important decays $\mu \to e\nu\bar{\nu}$ and $b \to u\ell\nu$, the total lifetimes have been determined in \[2, 3, 4\] by analytically calculating imaginary parts of four-loop diagrams. That heroic effort is difficult to extend, for example, to differential distributions. The other approach consists in expanding diagrams in an artificial parameter, for example the ratio of muon masses outside and inside loops, and using Padé approximants to sum the expansion for the physical value of this parameter. This approach was used to check the muon and $b \to u$ results \[5, 6\]. It yields results scattered around the exact values which are often sufficient for applications.

The drawback of this method is that it is very difficult to estimate the errors reliably.

The purpose of this study is to extend the method of expansions beyond the zero recoil limit. We start directly at the origin of Fig. 1 which corresponds to the kinematics of a top quark decay into a massless $b$ quark and a massless $W$ boson. We then treat the $W$ mass as a small perturbation and compute several terms of the resulting expansion. We stop when we can smoothly match to the previously obtained expansion around the case of the $W$ boson equally heavy as the decaying quark.\[1\]. The physical top quark decay corresponds to a specific value of the $W$ mass, but in the more general decay $Q \to q +$ leptons, the term “$W$ mass” refers to the invariant mass of the lepton pair, and it is in this context that we can relate $t \to bW$ to $b \to u\ell\nu$ and $\mu \to e\nu\bar{\nu}$.

This is the first time that exact results are available in this limit and we can now address a number of interesting problems. We obtain an accurate value of the $O(\alpha_s^2)$ correction to the top quark lifetime. The combination of
our results with the expansion around the heavy $W$ case allows us to give a complete description of the differential distribution of $b \to u \ell \nu$ decay in the invariant mass of leptons, and thus improves the theoretical description of this decay, important for the determination of $V_{ub}$. We also check the muon and $b \to u$ lifetime corrections with a relative error of about $2 \times 10^{-4}$. In the future, the same method can be employed to improve perturbative corrections to mixing processes such as $B_d \leftrightarrow \overline{B}_d$ and $B_s \leftrightarrow \overline{B}_s$.

In Fig. 2 we show three examples of the diagrams that we have to consider in order to calculate $t \to bW$ at $\mathcal{O}(\alpha_s^2)$. We use the optical theorem to connect the imaginary parts of such diagrams with contributions to the decay. Note that we customarily speak about two-loop corrections when what we actually need to compute are the imaginary parts of three-loop diagrams. The various cuts correspond to two-loop virtual corrections or emissions of one or two real quanta.

With $m_b = 0$, there are two scales in the problem: $m_t$ and $m_W$. We define an expansion parameter $\omega = m_W^2/m_t^2$ so that the two scales can be expressed as hard and soft ($\mathcal{O}(1)$ and $\mathcal{O}(\sqrt{\omega})$, using $m_t$ as the unit of energy). Contributions arising from these two scales are identified using asymptotic expansions so that we must consider two regions. In the first region, all the loop momenta are hard and the $W$ propagator can be expanded as a series, in powers of $\omega$, of massless propagators. In the second region, the gluon momenta are hard but the loop momentum flowing through the $W$ is soft. In this region, the diagrams factor into a product of a two-loop self-energy type integral and a one-loop vacuum bubble integral with a scale of $m_W$. The leading contribution from this second region is $\mathcal{O}(\omega^2)$, and the interplay between the two regions gives rise to terms with a large logarithm $\ln \omega$.

All scalar integrals arising in the problem can be expressed in terms of 9 basic topologies. We use differential-algebraic identities to reduce all loop integrals in both regions to a combination of 24 master integrals. The resulting large linear systems can be solved in a few ways. In the traditional method [10], one inspects the structure of the identities and rearranges them manually into the form of recurrence relations for an efficient iterative solution of the
system. This “by inspection” method has proven to be very successful in numerous applications (e.g., [3, 4, 5, 11]) but it requires much human work to implement. Conversely, a straightforward solution of the linear system is much more expensive computationally and was first achieved only recently [12]. In our calculation [13] we used the traditional approach (programmed in FORM [14]) as well as a modified version of the new algorithm for which we implemented a dedicated computer algebra system. In both cases we independently obtained identical results which serve as a check of correctness but also enable us to compare these two methods. Details of the implementation of both methods and the evaluation of master integrals will be presented in a forthcoming technical paper.

The final result for the top quark decay width can be written as

$$\Gamma(t \to bW) = \Gamma_0 \left[ X_0 + \frac{\alpha_s}{\pi} X_1 + \left( \frac{\alpha_s}{\pi} \right)^2 X_2 \right], \quad \Gamma_0 = \frac{G_F m_t^3 |V_{tb}|^2}{8\sqrt{2}\pi}.$$  \hspace*{1cm} (1)

Throughout this paper, we use $\alpha_s \equiv \alpha_s^{MS}(\mu)$, where $\mu$ is the pole mass of the decaying quark. The tree-level and $O(\alpha_s)$ coefficients are already known analytically [15],

$$X_0 = 1 - 3\omega^2 + 2\omega^3,$$

$$X_1 = C_F \left[ \frac{5}{4} - \frac{\pi^2}{3} + \frac{3}{2}\omega + \omega^2 \left( \pi^2 - 6 + \frac{3}{2}L \right) + O(\omega^3L) \right], \quad L \equiv \ln \omega.$$  \hspace*{1cm} (2)

The $O(\alpha_s^2)$ result can be subdivided into four gauge-invariant pieces,

$$X_2 = C_F (T_R N_L X_L + T_R N_H X_H + C_F X_A + C_A X_{NA}),$$  \hspace*{1cm} (4)

where $C_F = 4/3$, $C_A = 3$, and $T_R = 1/2$ are the usual SU(3) color factors and $N_L$ and $N_H$ denote the number of light ($m_q = 0$) and heavy ($m_q = m_t$) quark species. For the coefficients $X_L, X_H, X_A, \text{and} \ X_{NA}$, we have obtained a series to at least $\omega^5$, of which the leading terms are

$$X_L \simeq \left[ \frac{4}{9} + \frac{23\pi^2}{108} + \zeta_3 \right] + \omega \left[ -\frac{19}{6} + \frac{2\pi^2}{9} \right] + \omega^2 \left[ \frac{745}{72} - \frac{31\pi^2}{36} - 3\zeta_3 - \frac{7}{4}L \right],$$

$$X_H \simeq \left[ \frac{12991}{1296} - \frac{53\pi^2}{54} + \frac{1}{3}\zeta_3 \right] + \omega \left[ -\frac{35}{108} - \frac{4\pi^2}{9} + 4\zeta_3 \right] + \omega^2 \left[ -\frac{6377}{432} + \frac{25\pi^2}{18} + \zeta_3 \right],$$

$$X_A \simeq \left[ 5 - \frac{119\pi^2}{48} - \frac{53}{8}\zeta_3 + \frac{19}{4}\pi^2 \ln 2 - \frac{11\pi^4}{720} \right] + \omega \left[ -\frac{73}{8} + \frac{41\pi^2}{8} - \frac{41\pi^4}{90} \right],$$

$$X_{NA} \simeq \left[ \frac{521}{576} + \frac{505\pi^2}{864} + \frac{9}{16}\zeta_3 - \frac{19}{8}\pi^2 \ln 2 + \frac{11\pi^4}{1440} \right] + \omega \left[ \frac{91}{48} + \frac{329\pi^2}{144} - \frac{13\pi^4}{60} \right],$$

$$+ \omega^2 \left[ -\frac{12169}{576} + \frac{2171\pi^2}{576} + \frac{377}{64}\zeta_3 + \frac{27}{32}\pi^2 \ln 2 - \frac{77\pi^4}{288} + \frac{73}{16} - \frac{3\pi^4}{32} \right].$$  \hspace*{1cm} (5)

The leading term, $O(\omega^0)$, of these results can be compared with the numerical estimates obtained with the zero recoil expansions in Eq. (14) of [16]; all of our results agree within their error estimations. Our result can also be compared with a numerical study of the top decay rate obtained by means of Padé approximations up to $O(\omega^2)$ [8]. In many cases we find agreement. However, there are also instances where the numerical estimates in [8] differ from our analytic expressions [3] by a few error bar lengths, illustrating limitations of the Padé approximation in this problem. For example, the coefficient of the $O(\omega)$ term of the nonabelian part $X_{NA}$ of Eq. (3) is 3.3398 whereas the value cited in [8] reads 3.356(3), corresponding to a $5\sigma$ discrepancy. Similarly in $X_A$, the $O(\omega)$ term is off by $3\sigma$.

With a sufficient number of terms, the present expansion can be smoothly matched with the one around the $\omega = 1$ limit studied previously [3] in the context of semileptonic $b$ quark decays. The result of such a matching procedure is depicted in the graph in Fig. 3. Although strict matching of the two expansions in the entire interval $0 \leq \omega \leq 1$ would require a very large number of terms from each side, a wide overlap region arises even when only a few terms are taken into account.

The most obvious application of the above result is the precise determination of second order QCD corrections to the top quark decay rate. An estimation of this effect is already known, both from numerical studies and from an extrapolation of the zero recoil limit. However, for the measured ratio of $W$ and top masses, $\omega \approx 0.213$ [17], the present expansion is the best way to calculate an accurate value of this contribution with a reliable error estimate.
Our expansion gives $X_2 = -15.5(1)$ where the uncertainty is almost entirely due to the experimental uncertainty of $m_t$. The theoretical error, which originates from taking a finite number of terms in our expansion, is 20 times smaller and can be still easily reduced if needed. Using $\alpha_s(m_t) = 0.11$, we find that the two-loop correction decreases the tree level decay rate by about 2%, in agreement with earlier expectations.

Our result also provides a check of the total lifetime calculations carried out for $\mu \to e\nu\bar{\nu}$ and $b \to ul\nu$ decays. In these processes the expansion parameter $\omega$ corresponds to the invariant mass of leptons produced in the decay and our matching procedure allows us to obtain a differential width $d\Gamma/d\omega$ valid in the full range of $\omega$ with desired accuracy. The inclusive semileptonic decay rate $b \to ul\nu$ can be calculated by integrating over $\omega$ within the kinematical boundaries. Taking $N_L = 4$ and $N_H = 1$, we end up with $\int_0^1 d\omega X_2(\omega) = -10.644$, which almost perfectly reproduces the $-10.648$ given in [6]. Analogously, the two-photon correction to the muon lifetime emerges from an integration of the abelian contribution $X_A$. We find $\int_0^1 d\omega X_A(\omega) = 1.7797$, which is in excellent agreement with the exact result 1.7794 [5].

To summarize, we have presented a new analytic $O(\alpha_s^2)$ result for the decay $t \to bW$ in terms of a parameter $\omega = m_W^2/m_t^2$ and in the limit of $m_b = 0$, corresponding to the last remaining kinematic region in which the $O(\alpha_s^2)$ heavy quark decay rates were not analytically known. This result has enabled us to confirm or modify slightly the corresponding results of previous numerical calculations. Our formulas are readily applicable to other physical processes such as muon decay and the semileptonic $b$ quark decay $b \to ul\nu$.

Our results depend on the imaginary parts of a novel class of three-loop integrals, which we have obtained using two independent paradigms for the solution of large systems of recurrence relations. To the best of our knowledge, this is the first time that both approaches have been used simultaneously to obtain a new result, and an objective analysis of the strengths and weaknesses of each approach will increase the efficiency of other large calculations in the future. This augurs well for the increasingly difficult physical problems that lie ahead. In particular, the top quark decay problem considered here has laid the foundation for $O(\alpha_s^2)$ perturbative calculations of mixing processes such as $B_d \leftrightarrow \overline{B}_d$ and $B_s \leftrightarrow \overline{B}_s$. Since the recently found $O(\alpha_s)$ effects are large and suffer from strong scale dependence, such improvement will help use those processes as a probe for new physics.

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