Practical error estimates for sparse recovery in linear inverse problems

Ignace Loris and Caroline Verhoeven
Mathematics Department, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium
E-mail: igloris@vub.ac.be,cverhoev@vub.ac.be

Abstract. The effectiveness of using model sparsity as a priori information when solving linear inverse problems is studied. We investigate the reconstruction quality of such a method in the non-idealized case and compute some typical recovery errors (depending on the sparsity of the desired solution, the number of data, the noise level on the data, and various properties of the measurement matrix); they are compared to known theoretical bounds and illustrated on a magnetic tomography example.

PACS numbers: 02.30.Zz, 02.60.Cb, 42.30.Wb

AMS classification scheme numbers: 15A29, 65F22, 65F35

Keywords: Inverse problem, sparsity, compressed sensing, regularization, \( \ell_1 \)-norm penalization, magnetoencephalography.
Practical error estimates for sparse recovery in linear inverse problems

1. Introduction

Compressed sensing [1,2] is an area of signal recovery which has recently attracted a great deal of attention thanks to its large potential for applications; it enables reconstruction of sparse signals \( x_0 \in \mathbb{R}^n \) with far fewer linear measurements than traditionally required. In this note we investigate what happens to reconstruction quality when theoretical conditions on the measurement matrix are relaxed and replaced by (less favorable) conditions that are encountered in realistic inverse problems.

When a signal \( x_0 \) is known to be sparse (i.e. has few nonzero components), a practical method to obtain accurate reconstruction results from measurements \( Kx_0 \) is the \( \ell_1 \) minimization technique [1, 3, 4]:

\[
\tilde{x} = \arg \min_{Kx=Kx_0} \| x \|_1, \tag{1}
\]

(\( K \) is a known \( m \times n \) operator). \( \| x \|_1 \) denotes the \( \ell_1 \)-norm of a vector: \( \| x \|_1 = \sum_i |x_i| \). Under certain conditions it can be shown that the \( \ell_1 \) reconstruction \( \tilde{x} \) equals the unknown sparse input signal \( x_0 \) exactly [5]. Data obtained from real measurements are more often than not corrupted by significant quantities of noise; this means that \( Kx_0 \) remains unknown and instead \( y = Kx_0 + \text{noise} \) is acquired. In this case, the ill-posed problem can be regularized by adding a sparsity promoting \( \ell_1 \)-norm penalty to a quadratic misfit functional (assuming Gaussian noise) [6]. In other words, the input signal can then be recovered from the noisy data \( y \) by minimizing the convex functional

\[
\tilde{x}(\lambda) = \arg \min_x \| Kx - y \|_2^2 + 2\lambda \| x \|_1 \tag{2}
\]

for a suitable choice of the penalty parameter \( \lambda \). \( \| \cdot \|_2 \) stands for the \( \ell_2 \)-norm: \( \| a \|_2^2 = \sum_i |a_i|^2 \). Contrary to the noiseless case, the recovered signal (2) will not be exactly equal to the input signal \( x_0 \).

The potential of the \( \ell_1 \) minimization (1) and \( \ell_1 \) penalization method (2) for the reconstruction of sparse signals has already been assessed extensively, both from a theoretical point of view and with numerical simulations [7–12]. Nonetheless, most research focuses almost exclusively on matrices which exhibit mutually incoherent columns or matrices which (likely) satisfy the ‘Restricted Isometry Property’ (RIP) [13], such as e.g. random matrices, structured random matrices (e.g. random rows of a Fourier matrix) or matrices composed of the union of certain bases. In most practical situations it is difficult to verify if a matrix satisfies the RIP. Here we set out to study the quality of reconstruction when the measurement matrix \( K \) does not fit this ideal category. One may e.g. have good a priori control on the sparsity of the desired model (by the choice of a suitable basis), but one may not have sufficient control on the physical measurements to make the matrix satisfy the RIP. Indeed, a single matrix column with small or zero norm will destroy the RIP, but attempting sparse recovery may still be appropriate or worthwhile in practice.

In this paper, we investigate the influence of the noise level in the data and of the singular value spectrum of the measurement matrix \( K \) on the ability of the \( \ell_1 \) method
(2) to faithfully reconstruct a sparse input signal $x_0$. We also discuss the behavior of the recovery as a function of the sparsity of the input signal and as a function of the indeterminacy of the system (number of rows with respect to number of columns in $K$); we provide practical predictions on the relative error of the recovered sparse signal with respect to the sparse ground truth model. In order to accomplish this, a large number of numerical simulations is performed and the results are displayed using the diagrams introduced in [9,14]. We also provide an illustration of the practical use of the method in a magnetic tomography setting.

2. Assessment method

We consider an input signal $x_0 \in \mathbb{R}^n$ with $k$ non-zero coefficients, and a data vector $y \in \mathbb{R}^m$, with $m \leq n$, such that $y = Kx_0 + \eta$ for a $m \times n$ matrix $K$ and a noise vector $\eta$. We want to assess the effectiveness of the $\ell_1$ penalization method (2) for recovering $x_0$, using the knowledge of $K$ and of $y$, as a function of the spectrum of the matrix $K$, of the noise level $\epsilon = \|\eta\|/\|Kx_0\|$, of the number of data $m$ and of the sparsity $k$ of $x_0$.

The success of the $\ell_1$ method (2) for the recovery of a sparse signal $x_0$ will depend on the indeterminacy of the linear system and on the sparsity of the input signal. A concise graphical representation of the success rate of $\ell_1$-based compressed sensing was introduced in [9,14]. Likewise, we will use the parameters $\delta = m/n$ and $\rho = k/m$ (not $k/n$) and perform a number of experiments for various values of $\delta$ and $\rho$. More precisely, we will use a cartesian grid in the $\delta - \rho$-plane and, for each grid point, set up input data, a matrix $K$ and a noise vector $\eta$, determine the minimizer $\bar{x}(\lambda)$ and compare it to $x_0$. This experiment is repeated several times over, for different $K$, $x_0$ and $\eta$, but for fixed spectra and values of $\epsilon$. Afterwards the whole experiment is repeated for different spectra and different values of $\epsilon$. Do note that the $\delta - \rho$-plane does not give a uniform description of all possible inverse problems. It is specifically tailored towards the evaluation of sparse recovery.

In this work we use 40 equidistant points for $\delta$ and $\rho$ with values ranging from 0.025 to 1. The matrices $K$ have a fixed number of columns ($n = 800$). The experiment is repeated 100 times in each grid point, for each spectrum and each value of $\epsilon$. The same experiments with a larger number of columns allow us to conclude that the result presented in this paper do not depend on this quantity.

In case of noiseless data, the success rate of the recovery strategy can be measured by simply computing the proportion of successful reconstructions ($\bar{x} = x_0$). In the case of noisy data, one will never have perfect reconstruction ($\bar{x}(\lambda) = x_0$), and therefore ‘good recovery’ needs to be defined in a different way. We measure the success of recovery by calculating the mean of the relative reconstruction error $e = \|\bar{x}(\lambda) - x_0\|/\|x_0\|$ over several trials. It is this number that will be plotted as a function of $\delta = m/n$ and $\rho = k/m$.

When assessing the influence of the spectral properties of $K$ on the success of the method (2), we take into account the change in spectrum of a matrix by the addition of
Practical error estimates for sparse recovery in linear inverse problems

Figure 1. Mean over 100 examples of the normalized spectra of matrices (see section 2) of the following type: random (solid), type 2 (dashed) and type 3 (dotted), $\kappa = 10^4$. The matrices are of size 200 × 800 (a), 400 × 800 (b) and 800 × 800 (c).

extra measurements (rows). We therefore choose the $m \times n$ matrices $K$ as submatrices of $n \times n$ matrices in the following way: We draw an $n \times n$ matrix containing random numbers from the Gaussian distribution with zero mean and unit variance. Three different types of spectral behavior are then considered. “Type 1” is obtained by taking the random matrix itself. For the other types, we first calculate the singular value decomposition and replace the singular values by

$$s_i = s_1 \kappa^{(1-i)/(n-1)} \quad i : 1 \ldots n,$$

(with $s_1 \neq 0$) for “type 2”, and by

$$s_i = s_1 \kappa^{(1-i^2)/(n^2-1)} \quad i : 1 \ldots n,$$

for “type 3”. Several values of $\kappa$ will be chosen in the next section. The singular vectors remain untouched. For all three types, the $m \times n$ matrices $K$ are then found by randomly selecting $m$ different rows from these $n \times n$ matrices. As seen in figure 1, the spectrum changes with $m$. We believe this corresponds to the behavior encountered in real problems.

The position of the $k$ non-zero entries of the input vector $x_0$, as well as their values, were randomly generated from a uniform distribution.

For a given matrix $K$ and a given sparse input signal $x_0$, synthetic data are constructed by setting

$$y = Kx_0 + \eta$$

where $\eta$ is a $m \times 1$ vector with entries taken from a Gaussian distribution with zero mean. The noise level is determined by the parameter $\epsilon = ||\eta||/||Kx_0||$. We will choose $\epsilon = 0.02, 0.05, 0.10, 0.20$ and 0.50.

The penalty parameter $\lambda$ in (2) is chosen to satisfy Morozov’s discrepancy principle:

$$||K\bar{x}(\lambda) - y|| = ||\eta||.$$  

In other words, we will fit the data up to the level of the noise. In practice, and for these problems sizes, this can be achieved easily by using the (non-iterative) Homotopy/LARS/lasso method [15–17] for the solution of (2).
3. Results

In order to assess the effect of the different variables on the accuracy of the solution of (2), the mean of the reconstruction error $e$ is plotted in the $\delta - \rho$-plane for the three classes of matrices described in section 2 with $\kappa = 10^4$ and noise level of 10% (see figure 2). The quality of the $\ell_1$ penalization reconstruction depends strongly on the spectrum of the matrix $K$. In particular we see that good results for random matrices must be considered as a best case scenario. The results are less favorable for the two other types of matrices.

Secondly, in order to combine results for different spectra of $K$ in one plot, the behavior of the mean relative error $e_{2t} = 2e$ is studied. In figure 3 (a) and (b), the relative accuracy $e_{2t}$ is plotted for various matrices $K$ and $\epsilon = 2\%$ and $10\%$ respectively. Only very sparse vectors $x_0$ are shown in these plots ($0.025 \leq \rho \leq 0.2$) as the results deteriorate rapidly for larger $\rho$. These two $e_{2t}$ curves in the $\delta - \rho$-plane (for the noise levels of $2\%$ and $10\%$) are quite similar in character. We also simulated noise levels of $5\%$, $20\%$ and $50\%$ and observed the same features.

The same sparse setting as figure 3(a,b) was used in figure 3 (c) and (d) where we show the reconstruction errors $e$ for matrices of type 2 with $\kappa = 10^8$ and $\kappa = 10^{12}$ respectively, and $\epsilon = 0.1$.

It is not surprising that the above results depend on the spectrum of $K$. Any change in the spectrum of a matrix $K$ also induces a change in the condition number of the column submatrices of $K$. As the restricted isometry property (RIP) [13] implies that these submatrices are well conditioned, our setting makes it more difficult/impossible for a matrix $K$ to satisfy a RIP. As argued below, even when the RIP is not satisfied, a sparse recovery may still be feasible.

In [15] it is stated that when

$$ (1 - \delta_{2k})\|z\| \leq \|Kz\| \leq (1 + \delta_{2k})\|z\|, \quad \forall z : \|z\|_0 \leq 2k, \tag{5} $$

and $\delta_{2k} \leq 2/(2 + \sqrt{7}/4)$ ($\|z\|_0$ is the number of non-zero coefficients of $z$), any vector $x_0$, with $\|x_0\|_0 \leq k$ can be recovered by $\ell_1$ minimization with an error $\|\bar{x} - x_0\| \leq c\|\eta\|$. The
Figure 3. (a) and (b): Behavior in the $\delta - \rho$-plane of the mean relative error $\epsilon_{2k}$ corresponding to two times the noise level $\epsilon$ with respectively $\epsilon = 2\%$ and $10\%$. The reconstructions were done for the following matrices: Gaussian random (dotted), type 2 (solid) and type 3 (dashed). The value of $\kappa$ is displayed on the curve. Panels (c) and (d): Mean relative errors on reconstructions for matrices of type 2 with respectively $\kappa = 10^{8}$ and $\kappa = 10^{12}$ and $\epsilon = 0.1$. The stars indicate the position in the $\delta - \rho$-plane of the input model of an inverse problem in magnetic tomography described in section 4.

constant $c$ only depends on $\delta_{2k}$ and $\eta$ is the noise vector as defined in section 2. This condition implies that the condition numbers ($\kappa_{2k}$) of all the $2k$ column submatrices of $K$ are bounded from above by:

$$\kappa_{2k} \leq \sqrt{\frac{10 + \sqrt{7}}{2 + \sqrt{7}}} \approx 1.6498,$$

(6)

As an illustration, we consider a matrix $K$ (of type 1, 2, 3 resp.) of size $200 \times 800$ ($\delta = 0.25$) and choose 10000 random 20-column submatrices of $K$. The average condition number $\langle \kappa_{20} \rangle$ of those submatrices is calculated: $\langle \kappa_{20} \rangle \approx 1.8$ for type 1 matrices, $\langle \kappa_{20} \rangle \approx 2.7$ and $\langle \kappa_{20} \rangle \approx 2.0$ for matrices of type 2 and 3 respectively with $\kappa = 10^{4}$. We see that $\langle \kappa_{20} \rangle$ lies above the upper bound (6). This means that for the three types of matrices studied here, there exists some column submatrices with 20 columns, which do not satisfy condition (6). The matrix $K$ (even normalized) will therefore not satisfy (5) for $k = 10$ (which corresponds to $\rho = 0.05$) and therefore [18] does not guarantee the accuracy of the reconstruction of a vector with 10 (or more) nonzero coefficients. However, as seen on figure 2 and 4, the $\ell_{1}$ penalization performs reasonably well, even when the vector to be reconstructed is less sparse than $\rho = 0.05$. We can therefore conclude that the bounds obtained from the RIP theory are quite restrictive.

The same conclusion can be drawn for the results obtained in [10], where the behavior of the solution of (2) with respect to the mutual coherence of the matrix $K$ is studied. Inspired by their analytical results, the authors only studied input signals with a very small (maximum) number of nonzero coefficients ($k = 3$) for a matrix of size $128 \times 256$. This matrix is a concatenation of the identity matrix and the Hadamard matrix, with the columns normalized to unit $\ell_{2}$ norm. For that case we have calculated the value of the mean of the condition number $\langle \kappa_{2k} \rangle$ over $10^{6}$ submatrices with 12
Practical error estimates for sparse recovery in linear inverse problems

We illustrate the previous results on the predicted accuracy of sparse recovery by $\ell_1$ penalization with the help of an inverse problem in magnetic tomography. Our toy problem aims to reconstruct a 2D current distribution $\vec{J}$ on part of a sphere from measurements of the (normal component of the) magnetic field above this surface. The current distribution $\vec{J}$ and the data $\vec{B}$ are assumed to be linked by the formula [19,20]:

$$\vec{B}(\vec{r}) = \mu_0 \frac{1}{4\pi} \int_V \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, dV'.$$

We also assume that the unknown currency distribution $\vec{J}$ is divergence-free: $\text{div} \vec{J} = 0$.

In our toy problem the domain $V$ is part of a thin spherical shell $0.089 \leq r \leq 0.090$ (all units are SI units) which we parametrize by coordinates $\xi$ and $\eta$:

$$x = \frac{r}{s} \tan \xi, \quad y = \frac{r}{s} \tan \eta, \quad z = \frac{r}{s} \quad \text{and} \quad s = \sqrt{1 + \tan^2 \xi + \tan^2 \eta} \quad (8)$$

(with $-\pi/3 \leq \xi, \eta \leq \pi/3$). The coordinate lines of this parametrization are angularly equidistant great circles [21]. The shell covers just over one quarter of the whole sphere. Furthermore we assume that the current distribution has no radial component. The divergence-free angular current distribution $\vec{J}(\xi, \eta)$ is then parametrized by the field $F(\xi, \eta)$: $\vec{J} \equiv \text{curl} \times F(\xi, \eta)\vec{1}_r$ (the dependence on $r$ is not taken into account here).

The domain $V$ is divided into $64^2$ voxels and we choose an input model $F^{\text{in}}(\xi, \eta)$ that is sparse in the CDF 4-2 wavelet basis [22]. The input model has only 60 nonzero coefficients in this basis (out of a possible 4096). Formula (7) is used to calculate the (normal component of the) magnetic field in 1000 randomly distributed points above the patch $-\pi/3 \leq \xi, \eta \leq \pi/3$ at $r = 0.1$ (one centimeter above the patch). 10% Gaussian noise is added to these data. On this $64 \times 64$ grid we therefore have 1000 data versus 4096 unknowns $F(\xi_i, \eta_i)$.

The singular value spectrum of this $1000 \times 4096$ matrix (in the wavelet basis) is plotted in figure 4, where it is compared to the spectrum of the matrices used in sections 2 and 3. Its spectrum is quite similar to the spectrum of a matrix of type 2 with $\kappa = 10^{12}$. Although the spectra are alike, it is important to point out that some of the (wavelet) basis functions lie close to the null space of the matrix. In other words, the norms of the columns of the toy problem matrix are quite different from those of the matrices used in the setting of the preceding sections (see figure 4). The average reconstruction errors
Practical error estimates for sparse recovery in linear inverse problems

Figure 4. Left: The spectrum of the matrix used in the toy magnetic tomography experiment. Center: The singular value spectra of matrices of type 2 (solid) with $\kappa = 10^8$ (top) and $\kappa = 10^{12}$ (bottom) and a rescaled spectrum of the matrix of the toy problem (dotted). Right: A histogram of columns norms of the toy problem matrix (blue) and the matrix of (b) with $\kappa = 10^{12}$ (grey). They are both normalized by their respective matrix spectral norms.

Figure 4 found in section 3 should therefore be understood as a lower bound: If one or more of the basis functions (used to express the sparsity of the model) lies more or less in the null space of the operator, the reconstruction quality may be further reduced.

In our simulation we attempt to reconstruct the input model using an $\ell_1$ penalization approach (to exploit the sparsity of the model in the wavelet basis), and a traditional $\ell_2$ penalty approach for comparison. For the former we use the so-called ‘Fast iterative soft-thresholding algorithm’, for the latter we use the conjugate gradient method (the variational equations are linear in that case). The $\ell_1$ reconstruction of the field $F(\xi, \eta)$ has a relative reconstruction error of 31%, the generic $\ell_2$ method of 87%. Comparing this to the results on figure 3 we see that, for $\delta = 1000/4096 = 0.2441$ and $\rho = 60/1000 = 0.06$, the predicted relative reconstruction error also lies around 30% for the $\ell_1$ penalty method. A similar experiment for the $\ell_2$ penalty method predicts an error around 95%. The input model and the two reconstructions are shown in figure 5.

5. Conclusion

Sparse recovery can be a very powerful tool for inverse problems especially when (structured) random matrices or other matrices that satisfy the Restricted Isometry Property (RIP) are concerned. However these cases constitute a theoretical best-case scenario. In this note we investigate the reconstruction performance when such conditions are loosened and this ‘compressed sensing’ framework is abandoned. Indeed, most matrices discussed here do not satisfy the RIP for submatrices of non-negligible size, but reasonably good reconstruction results can still be obtained.

By means of extensive numerical simulations we assessed the performance of the $\ell_1$ recovery method (2) for realistic, non-idealized, linear systems with noisy data. Various levels of sparsity of the model, different numbers of data, and different noise levels and spectral behaviors of the measurement matrices were studied. It was shown that the mean relative reconstruction error $e$ grows sharply when the framework of random
Practical error estimates for sparse recovery in linear inverse problems

Figure 5. The current density model $\vec{J}$ used in the synthetic sparse magnetic tomography experiment. Arrows represent the field $\vec{J}$ whereas color indicates the field $F$. Left: the input model. Center: the $\ell_1$ penalized reconstruction. Right: the $\ell_2$ penalized reconstruction. The $\ell_1$ reconstruction is less noisy and its amplitude is more faithful to the input model than the $\ell_2$ reconstruction.

matrices is abandoned. This means that accurate reconstruction results can only be achieved for problems with a more sparse solution. Nonetheless the known (RIP) bounds from [18] seem too restrictive in practice. These bounds guarantee an accurate reconstruction for all vectors that are sufficiently sparse. But, although loosening this condition will sometimes provide bad reconstructions, we showed that it is often possible to obtain reconstructions of acceptable quality of less sparse vectors.

Figures 2 and 3 allow the reader to estimate the relative reconstruction error based on the shape of the measurement matrix (number of rows and columns), on the expected sparsity of the model and on the spectral properties of the measurement matrix. These predicted errors unfortunately are not an upper bound but a lower bound, in the sense that the simulations did not take into account all factors that could potentially negatively influence $\ell_1$ penalized reconstructions. Other factors that play an important role on the quality of reconstructions are e.g. the distribution of the matrix column norms (i.e. whether basis vectors lie in the null space of the matrix, in contradiction to the RIP), and the presence of (nearly) identical columns. In the latter case, the $\ell_1$ penalty will not impose a unique minimizer to functional (2) and such a penalty is therefore from the outset not a suitable regularization method for that inverse problem.

We also conclude that the $\ell_1$ method responds well to an increased amount of noise: When the noise level increases, less sparse vectors can be recovered to the same relative accuracy $e_2e$. The magnetic tomography example shows the practical usefulness of the numerical study, and there is good agreement between that toy problem and the numerical simulations.
Acknowledgments

This research was supported by VUB GOA-062 and by the FWO-Vlaanderen grant G.0564.09N. The authors thank Massimo Fornasier and Francesca Pitolli for discussion concerning the magnetic tomography problem.

References

[1] D L Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52:1289–1306, 2006.
[2] A M Bruckstein, D L Donoho, and M Elad. From sparse solutions of systems of equations to sparse modeling of signals and images. *SIAM Review*, 51:34–81, 2009.
[3] E J Candès, J Romberg, and T Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Transactions On Information Theory*, 52:489–509, 2006.
[4] S S Chen, D L Donoho, and M A Saunders. Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing*, 20:33–61, 1998.
[5] E J Candès. The restricted isometry property and its implications for compressed sensing. *Comptes Rendus de l’Académie des sciences, Série I*, 346:589–592, 2008.
[6] I Daubechies, M Defrise, and C De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on Pure and Applied Mathematics*, 57:1413–1457, 2004.
[7] D L Donoho and M Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via $\ell_1$ minimization. *Proceedings Of The National Academy Of Sciences Of The United States Of America*, 100:2197–2202, 2003.
[8] D L Donoho. For most large underdetermined systems of linear equations the minimal $\ell_1$-norm solution is also the sparest solution. *Communications on Pure and Applied Mathematics*, 59:797–829, 2006.
[9] D L Donoho and Y Tsaig. Fast solution of $\ell_1$-norm minimization problems when the solution may be sparse. *IEEE Transactions on Information Theory*, 54:4789–4812, 2008.
[10] D L Donoho, M Elad, and V Temlyakov. Stable recovery of sparse overcomplete representations in the presence of noise. *IEEE Transaction on Information Theory*, 52:6–18, 2006.
[11] M Elad and A M Bruckstein. A generalized uncertainty principle and sparse representation in pairs of bases. *IEEE Transactions on Information Theory*, 48:2558–2567, 2002.
[12] J J Fuchs. Recovery of exact sparse representations in the presence of noise. In 2004 IEEE International Conference on Acoustics, Speech, and Signal Processing, Vol II, Proceedings - Sensor Array and Multichannel Signal Processing Signal Processing Theory and Methods, International Conference on Acoustics Speech and Signal Processing (ICASSP), pages 533–536, 2004.
[13] E J Candès and T Tao. Decoding by linear programming. *IEEE Transactions On Information Theory*, 51:4203–4215, 2005.
[14] D L Donoho, Y Tsaig, I Drori, and J-L Starck. Sparse solution of underdetermined linear equations by stagewise orthogonal matching pursuit. *IEEE Transactions On Information Theory*, 2010. submitted.
[15] B Efron, T Hastie, I Johnstone, and R Tibshirani. Least angle regression. *Annals of Statistics*, 32:407–499, 2004.
[16] M R Osborne, B Presnell, and B A Turlach. A new approach to variable selection in least squares problems. *IMA Journal of Numerical Analysis*, 20:389–403, 2000.
[17] K Sjöstrand. Matlab implementation of LASSO, LARS, the elastic net and SPCA, 2005. Version 2.0.
Practical error estimates for sparse recovery in linear inverse problems

[18] H Rauhut. Compressive sensing and structured random matrices. In M. Fornasier, editor, *Theoretical Foundations and Numerical Methods for Sparse Recovery*, volume 9 of *Radon Series Comp. Appl. Math*. De Gruyter, 2010.

[19] M Fornasier and F Pitolli. Adaptive iterative thresholding algorithms for magnetoencephalography (MEG). *Journal of Computational and Applied Mathematics*, 221:386–395, 2008.

[20] S P van den Broek, H Zhou, and M J Peters. Computation of neuromagnetic fields using finite-element method and Biot-Savart law. *Med. & Biol. Eng. & Comput*, 34:21–26, 1996.

[21] C Ronchi, R Iacono, and P S Paolucci. The “cubed sphere”: A new method for the solution of partial differential equations in spherical geometry. *Journal of Computational Physics*, 124:93–114, 1996.

[22] A Cohen, I Daubechies, and J Feauveau. Bi-orthogonal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, 45:485–560, 1992.

[23] A Beck and M Teboulle. A fast iterative shrinkage-threshold algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2:183–202, 2009.