Model Representation for Self-Consistent-Field Theory of Isotropic Turbulence

R. V. R. Pandya
Department of Mechanical Engineering
University of Puerto Rico at Mayaguez
Mayaguez, Puerto Rico, PR 00681, USA

December 27, 2021

Abstract

In this paper, Langevin model equation is proposed for Fourier modes of velocity field of isotropic turbulence whose statistical properties are identical to those governed by equations of Self-Consistent-Field (SCF) theory of turbulence [J. R. Herring, Physics of Fluids 9, 2106 (1966)].

1 Introduction

Kraichnan’s seminal and pioneering work on Direct Interaction Approximation (DIA) ([Kraichnan(1958)], [Kraichnan(1959)]) and Lagrangian History Direct Interaction Approximation (LHDIA) ([Kraichnan(1965)]) has been influential in setting the right tone in the field of theory of turbulence and leading to other fundamental renormalized approaches for turbulence closure ([Edwards(1964)], [Herring(1965), Herring(1966)], [McComb(1978)], [Kaneda(1981)], [L’vov(1991)]). Various renormalized approaches are critically reviewed by [Leslie(1973)], [McComb(1990)], [McComb(1995)], [L’vov(1991)], and [Lesieur(1997)]. And the Self-Consistent-Field (SCF) approach of [Herring(1966)] having close relationship with DIA ([Herring & Kraichnan(1972)]) is the central focus in this paper.
Herring (1965) developed the SCF approach for stationary isotropic turbulence and subsequently generalized to the non-stationary isotropic turbulence ([Herring(1966)]). Instead of applying the perturbation technique to the Navier-Stokes equations, following [Edwards(1964)] framework, Herring preferred Livouille equation for probability distribution function of Fourier modes of the velocity field. Then, a self-consistent-field procedure was carried out around the zeroth order probability distribution which is the product of exact single mode distribution. This led [Herring(1966)] to derive equations governing time evolutions of Green's function, single-time velocity correlation and two-time velocity correlation. Yet another method of [Balescu & Senatorski(1970)] yielded the SCF set of equations and thus doubly justified these equations. The equations for Green's function and single-time velocity correlation are identical in form to the corresponding DIA’s equations and generalized fluctuation-dissipation relation represents the equation for two-time velocity correlation in the SCF approach.

Despite being closer to Edwards's theory framework and closer to DIA in terms of the final equations, well justified SCF approach lacks a model representation. Whereas model representations are known to exist for DIA ([Kraichnan(1970)]) and extended Edwards's theory ([Kraichnan(1971)]) associated with the non-stationary turbulence. Also, model representations are available for Kaneda’s theory ([Kaneda(1981)]) and [McComb(1978)] local energy transfer (LET) theory ([Pandya(2004)]). The model representation, if exists, assures the fact that statistical properties predicted by SCF are those of a realizable velocity field and consequently establishes certain consistency properties. The purpose of this paper is to suggest an existence of Langevin model representation for SCF. Consequently, to make sure that SCF does not lag behind other theories when judged from the perspective of realizability and model representation.

2 SCF theory equations

In this section, closed set of equations describing the statistical properties of isotropic turbulence as obtained by SCF approach of [Herring(1966)] are presented. I should be excused for not using the original notations of Herring, rather using the notations of [McComb(1990)] while presenting SCF equations. The Fourier modes $u_i(k, t)$ defined by

$$u_i(x, t) = \int dx^3 k u_i(k, t) \exp(ik \cdot x),$$

(1)
of the velocity field $u_i(x, t)$ of homogeneous, isotropic, incompressible fluid turbulence in space-time $(x - t)$ domain satisfy the following Navier-Stokes equation written in Fourier wavevector ($k$) and time domain:

$$
\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_i(k, t) = M_{ijm}(k) \int d^3p u_j(p, t) u_m(k - p, t). \tag{2}
$$

Here $\nu$ is kinematic viscosity of fluid, inertial transfer operator

$$
M_{ijm}(k) = (2i)^{-1}[k_j P_{im}(k) + k_m P_{ij}(k)], \tag{3}
$$

the projector $P_{ij}(k) = \delta_{ij} - k_i k_j k^{-2}$, $k = |k|$, and $\delta_{ij}$ is the Kronecker delta. The subscripts take the values 1, 2 or 3 along with the usual summation convention over repeated subscript. The two-time velocity correlation $Q_{in}(k, k'; t, t') = \langle u_i(k, t) u_n(k', t') \rangle$, single-time velocity correlation $Q_{in}(k, k'; t, t) = \langle u_i(k, t) u_n(k', t) \rangle$ of the velocity field $u_i(k, t)$ and the Green’s function $G_{in}(k; t, t')$ can be simplified for isotropic turbulence, and written as

$$
Q_{in}(k, k'; t, t') = P_{in}(k) Q(k; t, t') \delta(k + k'), \tag{4}
$$

$$
Q_{in}(k, k'; t, t) = P_{in}(k) Q(k; t, t) \delta(k + k'), \tag{5}
$$

and

$$
G_{in}(k; t, t') = P_{in}(k) G(k; t, t'), \tag{6}
$$

where $\langle \rangle$ represents ensemble average and $\delta$ represents Dirac delta function. The SCF equation for $G(k; t, t')$ may be written as

$$
\left( \frac{\partial}{\partial t} + \nu k^2 \right) G(k; t, t') + \int d^3p L(k, p) \int_{t'}^{t} ds G(p; t, s) Q(|k - p|; t, s) G(k; s, t') = 0 \forall t > t' \tag{7}
$$

and $G(k; t', t') = 1$. The SCF equation for $Q(k; t, t)$ may be written as

$$
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) Q(k; t, t) + 2 \int d^3p L(k, p) \int_{0}^{t} ds G(p; t, s) Q(|k - p|; t, s) Q(k; t, s) = 2 \int d^3p L(k, p) \int_{0}^{t} ds G(k; t, s) Q(|k - p|; t, s) Q(p; t, s) \tag{8}
$$

where

$$
L(k, p) = \frac{\mu(k^2 + p^2) - kp(1 + 2\mu^2))(1 - \mu^2)kp}{k^2 + p^2 - 2kp\mu} \tag{9}
$$

3
and $\mu$ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{p}$. These equations (7) and (8) have form identical to the corresponding equations obtained by DIA theory. In SCF approach, the equation for $Q(k; t, t')$ is associated with generalized fluctuation-dissipation relation

$$Q(k; t, t') = G(k; t, t')Q(k; t', t'), \ \forall t \geq t'.$$  \hspace{1cm} (10)

We write the equation for $Q(k; t, t')$, by using equations (7) and (10), in the following form convenient for further use:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)Q(k; t, t') + \int_0^t ds G(p; t, s)Q(|\mathbf{k} - \mathbf{p}|; t, s)Q(k; s, t')$$

$$= \int d^3pL(k, p)\int_0^{t'} ds G(p; t, s)Q(|\mathbf{k} - \mathbf{p}|; t, s)Q(k; s, t').$$ \hspace{1cm} (11)

Thus equations (7), (8) and (11) form a closed set of final equations of SCF approach of [Herring(1966)]. Now the goal is to obtain model equation for $u_i(k, t)$ which would have statistical properties identical to those as predicted by this closed set of equations. And a Langevin equation as a model representation for SCF is presented in the section to follow.

3 Langevin model equation for SCF

Similar to Langevin model representation for DIA, consider a Langevin equation for $u_i(k, t)$ written as

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)u_i(k, t) + \int_0^t ds \eta(k; t, s)u_i(k, s) = f_i(k, t) + b_i(k, t)$$ \hspace{1cm} (12)

where $\eta(k; t, s)$ is statistically sharp damping function, $f_i(k, t)$ is a forcing term with zero mean and $b_i(k, t)$ is white noise forcing term having zero mean. It should be noted that $b_i(k, t)$ is an additional new forcing term that is not present in DIA’s Langevin model representation. We consider these two different forcing terms to be statistically independent

$$\langle f_i(k, t)b_n(k', t') \rangle = 0 \ \forall k', t'$$ \hspace{1cm} (13)
and their statistical properties for isotropic turbulence, written as

\[ \langle f_i(k, t) f_n(k', t') \rangle = P_{in}(k) F(k, t, t') \delta(k + k') \] (14)

and

\[ \langle b_i(k, t) b_n(k', t') \rangle = P_{in}(k) B(k, t) \delta(k + k') \delta(t - t'). \] (15)

For particular choice of \( \eta(k; t, s) \), \( F(k, t, t') \) and \( B(k, t) \), the Langevin equation (12) would recover the closed set of SCF equations (7), (8) and (11). Now we obtain that particular choice.

For isotropic turbulence, the Green’s function of the Langevin equation (12) satisfies

\[ \left( \frac{\partial}{\partial t} + \nu k^2 \right) G(k; t, t') + \int_t^{t'} ds \eta(k; t, s) G(k; s, t') = 0 \quad \forall \quad t > t' \] (16)

and \( G(k; t', t') = 1 \). The expression for \( \eta \) given by

\[ \eta(k; t, s) = \int d^3p L(k, p) G(p; t, s) Q(|k - p|; t, s) \] (17)

would make equation (16) identical to SCF equation (7) for the Green’s function. To obtain \( F(k, t, s) \) we compare equation (11) with the equation for \( Q(k; t, t') \) obtained from Langevin equation (12), written as

\[ \left( \frac{\partial}{\partial t} + \nu k^2 \right) Q(k; t, t') + \int_0^{t'} ds \eta(k; t, s) Q(k; s, t') = \int_0^{t'} ds G(k; t', s) F(k, t, s). \] (18)

While writing this equation we have made use of equations (13)-(15). On comparison and making use of expression for \( \eta \) and generalized fluctuation-dissipation relation (10), we obtain

\[ F(k, t, s) = \eta(k; t, s) Q(k; s, s) \] (19)

which would make (18) identical to SCF equation (11) for \( Q(k; t, t') \). Now the equation for \( Q(k; t, t) \) as obtained from the Langevin equation (12) and using equations (13)-(15) can be written as

\[ \left( \frac{\partial}{\partial t} + 2\nu k^2 \right) Q(k; t, t) + 2 \int_0^t ds \eta(k; t, s) Q(k; t, s) = 2 \int_0^t ds G(k; t, s) F(k, t, s) + B(k, t). \] (20)
Comparison of this equation with SCF equation (8) for $Q(k; t, t)$, making use of expressions for $\eta(k; t, s)$, $F(k, t, s)$ given by equations (17) and (19) respectively and using equation (10) we obtain

$$B(k, t) = 2 \int d^3p L(k, p) \int_0^t ds G(k; t, s)Q(|k−p|; t, s)Q(p; t, s)−2 \int_0^t ds \eta(k; t, s)Q(k; t, s)$$
(21)

which makes equation (20) identical to equation (8). Thus, the Langevin equation (12) along with the expression for $\eta$ given by (17) and statistical properties of the two forcing functions $F(k, t, s)$ and $B(k, t)$ given by (19) and (21) respectively, is the required model representation for SCF approach of [Herring(1966)].

4 Concluding remarks

A long awaited model representation for self-consistent-field approach of [Herring(1966)] has been suggested in this paper. This model is in the form of a Langevin equation having two statistically independent forcing terms in contrary to one forcing term present in DIA’s Langevin model representation. The proposed model assures that the closed set of equations of SCF approach generates statistical properties of the velocity field that is realizable. It should be noted that the expression for $\eta(k; t, s)$ is identical to that present in DIA’s Langevin model representation ([Kraichnan(1970)]). It is worth mentioning here the reason for different type of model representations for SCF approach, LET theory and extended Edwards’s theory despite the fact that the generalized fluctuation-dissipation relation is central to all of them. The SCF approach has been modelled here by Langevin equation whereas LET and extended Edwards’s theories have an almost-Markovian model representations ([Pandya(2004)], [Kraichnan(1971)]). This difference is mainly due to an additional condition for Green’s function $i.e.$ $G(k; t, t') = G(k; t, s)G(k; s, t')$ which is the property of only LET and extended Edwards’s theories and is satisfied by an almost-Markovian equation and not satisfied by the Langevin equation and SCF approach.

Acknowledgement

I acknowledge the financial support provided by the University of Puerto Rico at Mayaguez, Puerto Rico, USA.
References

[Balescu & Senatorski(1970)] BALESCU, R. & SENATORSKI, A. 1970 A new approach to the theory of fully developed turbulence. Ann. Phys. 58, 587–624.

[Edwards(1964)] EDWARDS, S. F. 1964 The statistical dynamics of homogeneous turbulence. J. Fluid Mech. 18, 239–273.

[Herring(1965)] HERRING, J. R. 1965 Self-consistent-field approach to turbulence theory. Phys. Fluids 8, 2219–2225.

[Herring(1966)] HERRING, J. R. 1966 Self-consistent-field approach to non-stationary turbulence. Phys. Fluids 9, 2106–2110.

[Herring & Kraichnan(1972)] HERRING, J. R. & KRAICHNAN, R. H. 1972 Comparison of some approximations for isotropic turbulence. In Lecture Notes in Physics, Vol. 12, Statistical Models and Turbulence. Springer, Berlin.

[Kaneda(1981)] KANEDA, Y. 1981 Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function. J. Fluid Mech. 107, 131–145.

[Kraichnan(1958)] KRAICHNAN, R. H. 1958 Irreversible statistical mechanics of incompressible hydromagnetic turbulence. Phys. Rev. 109, 1407–1422.

[Kraichnan(1959)] KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. J. Fluid Mech. 5, 497–543.

[Kraichnan(1965)] KRAICHNAN, R. H. 1965 Lagrangian-history closure approximation for turbulence. Phys. Fluids 8, 575–598.

[Kraichnan(1970)] KRAICHNAN, R. H. 1970 Convergents to turbulence functions. J. Fluid Mech. 41, 189–217.

[Kraichnan(1971)] KRAICHNAN, R. H. 1971 An almost-Markovian Galilean-invariant turbulence model. J. Fluid Mech. 47, 513–524.

[Lesieur(1997)] LESIEUR, M. 1997 Turbulence in Fluids, 3rd edn. Dordrecht: Kluwer.

[Leslie(1973)] LESLIE, D. C. 1973 Developments in the Theory of Turbulence. Oxford: Clarendon Press.
[L’vov(1991)] L’vov, V. S. 1991 Scale invariant-theory of fully-developed hydrodynamic turbulence - Hamiltonian approach. *Phys. Rep.* **207**, 1–47.

[McComb(1978)] McComb, W. D. 1978 A theory of time-dependent isotropic turbulence. *J. Phys. A* **11**, 613–632.

[McComb(1990)] McComb, W. D. 1990 *The Physics of Fluid Turbulence*. New York, NY: Oxford University Press.

[McComb(1995)] McComb, W. D. 1995 Theory of turbulence. *Rep. Prog. Phys.* **58**, 1117–1206.

[Pandya(2004)] Pandya, R. V. R. 2004 Model representation for local energy transfer theory of isotropic turbulence. *Submitted to Journal of Fluid Mechanics*. 