\section*{1. Introduction}

In 1882, Čebyšev \cite{1} gave the following inequality

\begin{equation}
|T(f, g)| \leq \frac{1}{2n} (b - a)^2 \|f\|_\infty \|g\|_\infty^\prime ,
\end{equation}

where \( f, g : [a, b] \rightarrow \mathbb{R} \) are absolutely continuous function, whose first derivatives \( f' \) and \( g' \) are bounded and

\begin{equation}
T(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b-a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b-a} \int_a^b g(x) \, dx \right),
\end{equation}

and \( \|\cdot\|_\infty \) denotes the norm in \( L_\infty [a, b] \) defined as \( \|f\|_\infty = \text{ess sup}_{t \in [a,b]} |f(t)| \).

During the past few years, many researchers have given considerable attention to the inequality (1). Various generalizations, extensions and variants have been appeared in the literature \cite{2–6}.

Recently, Guezane-Lakoud and Aissaoui \cite{2} gave the analogue of the functional (2) for functions of two variables and established the following Čebyšev type inequalities for functions whose mixed derivatives are co-ordinated quasi-convex and \( \alpha \)-quasi-convex and \( s \)-quasi-convex functions.

\begin{equation}
|T(f, g)| \leq \frac{4g}{8(b-a)^2} \|f_{,a}\|_\infty \|g_{,a}\|_\infty ,
\end{equation}

and

\begin{equation}
|T(f, g)| \leq \frac{1}{8(b-a)^2} \int_a^b \left[ (g(x,y)) \|f_{,a}\|_\infty + |f(x,y)| \|g_{,a}\|_\infty \right] \left[ ((x-a)^2 + (b-x)^2) ((y-c)^2 + (d-y)^2) \right] \, dy dx,
\end{equation}

where

\begin{equation}
T(f, g) = \frac{1}{k} \int_a^b \int_c^d f(x, y) g(x, y) \, dy dx - \frac{d-c}{k^2} \int_a^b \int_c^d g(x, y) \left( \int_a^b f(t, y) \, dt \right) \, dy dx
\end{equation}

\begin{equation}
- \frac{b-c}{k^2} \int_a^b \int_c^d g(x, y) \left( \int_c^d f(x, v) \, dv \right) \, dy dx + \frac{1}{k^2} \left( \int_a^b \int_c^d f(x, y) \, dy dx \right) \left( \int_a^b \int_c^d g(t, v) \, dv dt \right).
\end{equation}
Motivated by the existing results, in this paper we establish some new Čebyšev type inequalities for functions whose mixed derivatives are co-ordinates quasi-convex and co-ordinates \((a, QC)\) and \((s, QC)\)-convex.

2. Preliminaries

Throughout this paper, we denote by \(\Delta\), the bidimensional interval in \([0, \infty)^2\), \(\Delta := [a, b] \times [c, d]\) with \(a < b\) and \(c < d\), \(k = (b - a) (d - c)\) and \(\frac{\partial^2 f}{\partial x \partial w}\) by \(f_{\lambda w}\).

Definition 1. [7] A function \(f: \Delta \to \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\) if

\[
f(\lambda x + (1 - \lambda) t, \omega y + (1 - \omega) v) \leq \lambda w f(x, y) + \lambda (1 - \omega) f(x, v) + (1 - \lambda) w f(t, y) + (1 - \lambda) (1 - \omega) f(t, v)
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

Definition 2. [8] A function \(f: \Delta \to \mathbb{R}\) is said to be quasi-convex on the co-ordinates on \(\Delta\) if

\[
f(\lambda x + (1 - \lambda) t, \omega y + (1 - \omega) v) \leq \max \{ f(x, y) + f(x, v) + f(t, y) + f(t, v) \}
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

Definition 3. [9] For some \(a \in (0, 1]\), a function \(f: \Delta \to \mathbb{R}\) is said to be \((a, QC)\)-convex on the co-ordinates on \(\Delta\), if

\[
f(\lambda x + (1 - \lambda) t, \omega y + (1 - \omega) v) \leq \lambda^a \max \{ f(x, y) + f(x, v) \} + (1 - \lambda)^a \max \{ f(t, y) + f(t, v) \}
\]

holds for all \(\lambda, \omega \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

Definition 4. [10] For some \(s \in [-1, 1]\), a function \(f: \Delta \to [0, \infty)\) is said to be \((s, QC)\)-convex on co-ordinates on \(\Delta\), if

\[
f(\lambda x + (1 - \lambda) t, \omega y + (1 - \omega) v) \leq \lambda^s \max \{ f(x, y) + f(x, v) \} + (1 - \lambda)^s \max \{ f(t, y) + f(t, v) \}
\]

holds for all \(\lambda \in (0, 1), \omega \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

Lemma 1. [11] Let \(f: \Delta \to \mathbb{R}\) be a partial differentiable mapping on \(\Delta\) in \(\mathbb{R}^2\). If \(f_{\lambda w} \in L_1(\Delta)\) then for any \((x, y) \in \Delta\), we have the equality;

\[
f(x, y) = \frac{1}{b-a} \int_a^b f(t, y) dt + \frac{1}{d-c} \int_c^d f(x, v) dv - \frac{1}{k} \int_0^k \int_a^b f(t, v) dv dt + \frac{1}{k} \int_0^k \int_0^d (x - t) (y - v) \times \left( \int_0^1 \int_0^1 f_{\lambda w}(\lambda x + (1 - \lambda) t, \omega y + (1 - \omega) v) d\lambda d\omega \right) dv dt.
\] (6)

3. Main result

Theorem 1. Let \(f, g: \Delta \to \mathbb{R}\) be partially differentiable functions such that their second derivatives \(f_{\lambda w}\) and \(g_{\lambda w}\) are integrable on \(\Delta\). If \(|f_{\lambda w}|\) and \(|g_{\lambda w}|\) are co-ordinated quasi-convex on \(\Delta\), then

\[
|T(f, g)| \leq \frac{49}{3600} MNk^2,
\] (7)

where \(T(f, g)\) is defined as in (5), \(M = \max_{x, t, \omega \in [a, b], v \in [c, d]} |f_{\lambda w}(x, y)| + |f_{\lambda w}(x, v)| + |f_{\lambda w}(t, y)| + |f_{\lambda w}(t, v)|\), and \(N = \max_{x, t, \omega \in [a, b], v \in [c, d]} |g_{\lambda w}(x, y)| + |g_{\lambda w}(x, v)| + |g_{\lambda w}(t, y)| + |g_{\lambda w}(t, v)|\), and \(k = (b - a) (d - c)\).
Proof. From Lemma 1, we have
\[
f(x, y) - \frac{b}{\pi \xi} \int_a^b f(t, y) dt - \frac{1}{\pi \xi} \int_c^d f(x, \nu) d\nu + \frac{b}{\pi \xi} \int_a^b f(t, \nu) d\nu dt = \frac{1}{k} \int_a^b \int_c^d (x - t) (y - \nu) \left( \int_0^1 \lambda x + (1 - \lambda) t, \nu y - (1 - \omega) y \right) d\lambda d\nu dt,
\]
and
\[
g(x, y) - \frac{1}{\pi \xi} \int_a^b g(t, y) dt - \frac{1}{\pi \xi} \int_c^d g(x, \nu) d\nu + \frac{b}{\pi \xi} \int_a^b g(t, \nu) d\nu dt = \frac{1}{k} \int_a^b \int_c^d (x - t) (y - \nu) \left( \int_0^1 \lambda x + (1 - \lambda) t, \nu y - (1 - \omega) y \right) d\lambda d\nu dt.
\]
Multiplying (8) by (9), and then integrating the resulting equality with respect to \(x\) and \(y\) over \(\Delta\), using modulus and Fubini’s Theorem, and multiplying the result by \(\frac{1}{k}\), we get
\[
|T(f, g)| \leq \frac{1}{k} \int_a^b \int_c^d \int_a^b \left| f(x - t) (y - \nu) \right| \left( \int_0^1 \lambda x + (1 - \lambda) t, \nu y - (1 - \omega) y \right) d\lambda d\nu dt \times \left( \int_0^1 \lambda x + (1 - \lambda) t, \nu y - (1 - \omega) y \right) d\lambda d\nu dt
\]
Since \(|f_{\lambda x}|\) and \(|g_{\lambda x}|\) are co-ordinated quasi-convex, we deduce
\[
|T(f, g)| \leq \frac{1}{k} MN \int_a^b \int_c^d \left( \int_a^b \int_a^b \left| x - t \right| (y - \nu) d\nu dt \right)^2 d\lambda dx = \frac{49}{3600} k^2 MN,
\]
where we have used the fact that
\[
\int_a^b \int_a^b \left( \int_a^b \int_a^b \left| x - t \right| (y - \nu) d\nu dt \right)^2 d\lambda dx = \frac{49}{3600} k^5.
\]
The proof is completed. \(\Box\)

Theorem 2. Under the assumptions of Theorem 1, we have
\[
|T(f, g)| \leq \frac{1}{k^2} \int_a^b \int_c^d \left[ M |g(x, y)| + N |f(x, y)| \right] \left( (x - a)^2 + (b - x)^2 \right) \times \left( (y - c)^2 + (d - y)^2 \right) d\nu dx,
\]
where \(T(f, g)\) is defined as in (5), \(M, N\), and \(k\) are as in Theorem 1.

Proof. From Lemma 1, (8) and (9) are valid. Let \(G(x, y) = \frac{1}{k} g(x, y)\) and \(F(x, y) = \frac{1}{k} f(x, y)\). Multiplying \(G(x, y)\) by \(F(x, y)\), then integrating the resultant equalities with respect to \(x\) and \(y\) over \(\Delta\), and by using the modulus, we get
\[ |T(f, g)| \leq \frac{1}{MN} \left[ \int_a^b \int_c^d |g(x,y)| \left( \int_a^b |x-t| |y-v| \, dv \right) \, dx \right] \]

Substituting (16) in (15), we get the desired result. \( \square \)

Theorem 3. Let \( f, g : \Delta \rightarrow \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda w} \) and \( g_{\lambda w} \) are integrable on \( \Delta \). If \( |f_{\lambda w}| \) and \( |g_{\lambda w}| \) are co-ordinated \( \alpha \)-quasi-convex on \( \Delta \), for some \( \alpha \in (0,1] \), then

\[ |T(f, g)| \leq \frac{49}{500} MNk^2, \]

where \( T(f, g) \) is defined as in (5), \( M, N, \) and \( k \) are as in Theorem 1.

Proof. Clearly the inequalities (8)-(10) are valid, using the co-ordinated \( \alpha \)-quasi-convexity of \( |f_{\lambda w}| \) and \( |g_{\lambda w}| \), (10) gives

\[ |T(f, g)| \leq \frac{1}{MN} \left[ \int_a^b \int_c^d |g(x,y)| \left( \int_a^b |x-t| |y-v| \, dv \right) \, dx \right] \]

Substituting (16) in (15), we get the desired result. \( \square \)
Theorem 5. Let \( f, g : \Delta \to \mathbb{R} \) be partially differentiable functions such that their second derivatives \( f_{,u} \) and \( g_{,u} \) are integrable on \( \Delta \), and let \( s \in (-1, 1) \) fixed. If \( |f_{,u}| \) and \( |g_{,u}| \) are co-ordinated s-quasi-convex on \( \Delta \), then

\[
|T(f, g)| \leq \frac{49}{900(s+1)^2} MNk^2,
\]

where \( T(f, g) \) is defined as in (5) and \( M, N, \) and \( k \) are as in Theorem 1.

Proof. Clearly inequalities (8)-(10) are satisfied. Using second definition of the co-ordinated \( s \)-quasi-convex of \( |f_{,u}| \) and \( |g_{,u}| \), (10) gives;
|T(f,g)| \leq \frac{1}{2^{(s+1)^2}}\int_a^b \int_c^d \left[ \int_a^b \int_c^d |x-t| |y-v| \int_0^1 \int_0^1 \left[ \lambda^s \max \{|f_{\lambda,w}(x,y)| + |f_{\lambda,w}(x,v)|\} + (1-\lambda)^s \max \{|f_{\lambda,w}(t,y)| + |f_{\lambda,w}(t,v)|\} \right] d\lambda \right] d\lambda dt dt \\
\times \left[ \int_a^b \int_c^d |x-t| |y-v| \int_0^1 \int_0^1 \left[ \lambda^s \max \{|g_{\lambda,w}(x,y)| + |g_{\lambda,w}(x,v)|\} + (1-\lambda)^s \max \{|g_{\lambda,w}(t,y)| + |g_{\lambda,w}(t,v)|\} \right] d\lambda \right] d\lambda dt dy dx \\
\times \left[ \int_a^b \int_c^d |x-t| |y-v| \int_0^1 \int_0^1 \left[ \lambda^s \max \{|f_{\lambda,w}(x,y)| + |f_{\lambda,w}(x,v)|\} + (1-\lambda)^s \max \{|f_{\lambda,w}(t,y)| + |f_{\lambda,w}(t,v)|\} \right] d\lambda \right] d\lambda dt dy dx \\
\leq \frac{1}{2^{(s+1)^2}}\int_a^b \int_c^d \left[ \left( \int_a^b \int_c^d |x-t| |y-v| \left( \frac{M}{s+1} + \frac{M}{s+1} \right) d\lambda \right) \right] d\lambda dt dy dx \\
\times \left[ \left( \int_a^b \int_c^d |x-t| |y-v| \left( \frac{N}{s+1} + \frac{N}{s+1} \right) d\lambda \right) \right] d\lambda dt dy dx \\
\leq \frac{4MN}{(s+1)^2}\int_a^b \int_c^d \left( \int_a^b \int_c^d |x-t| |y-v| \right) dy dx.

Substituting (12) in (22), we get the desired result. □

**Theorem 6.** Under the assumptions of Theorem 5, we have

|T(f,g)| \leq \frac{1}{4(s+1)^2} \left[ \int_a^b \int_c^d \left( M |g(x,y)| + N |f(x,y)| \right) (x-a)^2 + (b-x)^2 (y-c)^2 + (d-y)^2 \right] dy dx, \quad (23)

where T(f,g) is defined as in (5) and M, N, and k are as in Theorem 1.

**Proof.** By the same argument given in Theorem 2, we easily obtain the inequality (14), using the second definition of s-quasi-convexity on the co-ordinates of |f_{\lambda,w}| and |g_{\lambda,w}|, we get

|T(f,g)| \leq \frac{1}{2^{(s+1)^2}} \left[ \int_a^b \int_c^d |g(x,y)| + \int_a^b \int_c^d |x-t| |y-v| \times \left( \int_0^1 \int_0^1 \lambda^s d\lambda + \int_0^1 \int_0^1 (1-\lambda)^s d\lambda \right) d\lambda \right] dy dx \\
\times \left[ \int_a^b \int_c^d |f(x,y)| + \int_a^b \int_c^d |x-t| |y-v| \times \left( \int_0^1 \int_0^1 \lambda^s d\lambda + \int_0^1 \int_0^1 (1-\lambda)^s d\lambda \right) d\lambda \right] dy dx.

\leq \frac{1}{(s+1)^2} \int_a^b \int_c^d \left( M |g(x,y)| + N |f(x,y)| \right) \int_a^b \int_c^d |x-t| |y-v| d\lambda \right] dy dx. \quad (24)
Substituting (16) in (24), we get the desired result. □

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**References**

[1] Čebyšev, P. L. (1882). Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites. *In Proc. Math. Soc. Charkov*, 2, 93-98.

[2] Guezane-Lakoud, A., & Aissaoui, F. (2011). New Čebyšev type inequalities for double integrals. *Journal of Mathematical Inequalities*, 5(4), 453-462.

[3] Meftah, B., & Boukerrioua, K. (2015). New Čebyšev type inequalities for functions whose second derivatives are \((s_1, m_1) - (s_2, m_2)\) convex on the co-ordinates. *Theory and Applications of Mathematics & Computer Science*, 5(2), 116-125.

[4] Meftah, B., & Boukerrioua, K. (2015). Čebyšev inequalities whose second derivatives are \((s, r)\)–convex on the co-ordinates. *Journal of Advanced Research in Applied Mathematics*, 7(3), 76-87.

[5] Meftah, B., & Boukerrioua, K. (2015). On some Čebyšev type inequalities for functions whose second derivative are \((h_1; h_2)\)-convex on the co-ordinates. *Konuralp Journal of Mathematics*, 3(2), 77-88.

[6] Sarikaya, M. Z., Budak, H., & Yaldiz, H. (2014). Čebyšev type inequalities for co-ordinated convex functions. *Pure and Applied Mathematics Letters*, 2, 44-48.

[7] Dragomir, S. S. (2001). On the Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics*, 5(4), 775-788.

[8] Latif, M. A., Hussain, S., & Dragomir, S. S. (2012). Refinements of Hermite–Hadamard type inequalities for co-ordinated quasi–convex functions. *International Journal of Mathematical Archive*, 3(1), 161-171.

[9] Xi, B. Y., Sun, J., & Bai, S. P. (2015). On some Hermite-Hadamard type integral inequalities for co-ordinated \((a, QC)\) and \((a, C)\) –convex functions. *Tbilisi Mathematical Journal*, 8(2), 75-86.

[10] Wu, Y., & Qi, F. (2016). On some Hermite–Hadamard type inequalities for \((s, QC)\)–convex functions. *SpringerPlus*, 5(1), 1-13.

[11] Sarikaya, M. Z. (2014). On the Hermite–Hadamard type inequalities for co-ordinated convex function via fractional integrals. *Integral Transforms and Special Functions*, 25(2), 134-147.

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