Hard Lefschetz Theorem in $p$-adic cohomology

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Abstract

In this short paper, we give a $p$-adic analogue of the Hard Lefschetz Theorem.

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Introduction

The main purpose of this paper is to check a $p$-adic analogue of the Hard Leftschetz Theorem. We have followed the proof in the $l$-adic context written in [KW01, IV.4.1] (compare with [BBD82]). Two main ingredients of the proof are the semi-simplicity of a pure arithmetic $\mathcal{D}$-module (see [AC13b, 4.3.1]) and the construction and the properties of the trace map given in [Abe13, 1.5]. Then, this paper can be considered as a natural continuation of these works. We follow here their terminology and notation.

Let us describe the contents of the paper. In the first chapter, we study the properties of the Serre subcategory consisting of constant objects. In the second chapter, we introduce the $p$-adic analogue of the Brylinsky-Radon transform and use its properties to prove the Hard Leftschetz Theorem. We have tried to write the proofs only when the $p$-adic analogues were not straightforward. Finally, in the last chapter, we check for the sake of completeness the inversion formula satisfied by Radon transform.

In this paper, we fix a complete discrete valuation ring $\mathcal{V}$ of mixed characteristic $(0, p)$. Its residue field is denoted by $k$, and assume it to be perfect. We also fix a lifting $\sigma: \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ of the $s$-th Frobenius automorphism of $k$. We put $q := p^s$, $K := \text{Frac}(\mathcal{V})$. We fix an isomorphism $\iota: \mathbb{Q}_p \xrightarrow{\sim} \mathbb{C}$.

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Notation. We will use the categories defined in [AC13a, 1.5]. We recall in this paragraph the construction. Let $X$ be a realizable variety. Let $\text{Hol}_F(X/K)^s$ be the subset of $\text{Ob(Ovhol}(X/K))$ which can be endowed with some $s'$-th Frobenius structure for some integer $s'$ which is a multiple of $s$, and let $\text{Hol}_F(X/K)$ be the thick abelian subcategory generated by $\text{Hol}_F(X/K)^s$ in $\text{Ovhol}(X/K)$. We denote by $D^b_{\text{hol}, F}(X/K)$ the triangulated full subcategory of $D^b_{\text{vhol}}(X/K)$ such that the cohomologies are in $\text{Hol}_F(X)$. For any integer $n$, we can extend the twist of Tate $(n)$ over $D^b_{\text{hol}, F}(X/K)$: by definition the twist $(n)$ is the identity (and then the forgetful functor $F - D^b_{\text{hol}, F}(X/K) \to D^b_{\text{hol}, F}(X/K)$ commutes with the twist of Tate). For simplicity and if there is no risk of confusion with the notion of holonomicity of Berthelot, we will write $D^b_{\text{hol}}(X/K)$ instead of $D^b_{\text{hol}, F}(X/K)$ and $\text{Hol}(X/K)$ instead of $\text{Hol}_F(X/K)$. With this notation,
we get $F\cdot D^b_{\text{shol}}(X/K) = F\cdot D^b_{\text{hol}}(X/K)$. Be careful that this notation is a bit misleading since in general, we do not know even with Frobenius structures if the notion of holonomicity of Berthelot and the notion of overholonomicity coincides.

## 1 Constant objects with respect to smooth $\mathbb{P}^d$-fibration morphisms

1.1 (Poincaré duality). Let $f$ be an equidimensional smooth morphism of relative dimension $d$ of realizable varieties. T. Abe has checked (see [Abe13, 1.5.13]) that the morphism

$$f^+[d] \to f^+[-d](-d),$$

which is induced by adjunction from the trace map $f_!f^+[2d](d) \to id$, is an isomorphism of t-exact functors. This isomorphism satisfies several compatibility properties (see [Abe13 1.5]), e.g. it is transitive.

1.2. We keep the notation of 1.1. The diagram below

$\xymatrix{f^+(\mathcal{F} \otimes \mathcal{G})[2d](d) \ar[r]^{\sim} & f^+((\mathcal{F}[2d](d)) \otimes f^+((\mathcal{G})) \ar[r]^{\text{adj}} & f^+(\mathcal{F}[2d](d)) \otimes f^+((\mathcal{G})) \ar[r]^{\text{Tr} \otimes id} & f^+(\mathcal{F}) \otimes f^+((\mathcal{G}))}$

is commutative. Indeed, the trapeze is commutative from [Abe13 1.5.1.Var5]. The other parts of the diagram are commutative by definition and functoriality. Hence we get the canonical commutative square:

$$f^+(\mathcal{F} \otimes \mathcal{G})[2d](d) \xrightarrow{\sim} f^+((\mathcal{F}[2d](d)) \otimes f^+((\mathcal{G})) \xrightarrow{\sim} f^+(\mathcal{F}[2d](d)) \otimes f^+((\mathcal{G})) \xrightarrow{\sim} f^+(\mathcal{F}) \otimes f^+((\mathcal{G})).$$

\begin{equation}
\text{(1.1.1)}
\end{equation}

\textbf{Definition 1.3.} Let $f': X \to S$ be a smooth equidimensional morphism of relative dimension $d$,

1. The objects of the essential image of the functor $f^+: (F\cdot)D^b_{\text{shol}}(S/K) \to (F\cdot)D^b_{\text{hol}}(X/K)$ are called constant (with respect to $f$).

2. The objects of the essential image of the functor $f^+[d]: (F\cdot)\text{Hol}(S/K) \to (F\cdot)\text{Hol}(X/K)$ are called constant (with respect to $f$). We denote by $f^+[d]|(F\cdot)\text{Hol}(S/K)$ its essential image.

1.4. Let $X$ be a realizable $k$-variety and $p_X: X \to \text{Spec} k$ be the structural morphism. We denote by $K_X := p_X^+(K)$ the constant coefficient of $X$. The complex $K_X$ is the $p$-adic analogue of the constant sheaf $\mathbb{Q}_l$ over $X$. Let $\mathcal{E} \in D^b_{\text{hol}}(X/K)$. We notice that $K_X \otimes \mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

\textbf{Proposition 1.5.} Let $u: Y \hookrightarrow X$ be a closed immersion of pure codimension $r$ in $X$ of smooth realizable $k$-varieties. Let $\mathcal{E} \in (F\cdot)D^b_{\text{hol}}(X/K)$.

1. There exists a natural functorial morphism of $(F\cdot)D^b_{\text{hol}}(Y/K)$ of the form

$$\mu: u^+(\mathcal{E}) \to u^+(\mathcal{E})[2r](r).$$

\begin{equation}
\text{(1.5.1)}
\end{equation}
2. If (locally on \(X\)) the complex \(\mathcal{E}\) is constant with respect to a smooth equidimensional morphism \(f: X \to S\) of realizable varieties such that \(f \circ u\) is also smooth, then \(\mu\) is an isomorphism.

**Proof.** This can be checked as \([KW01]\ I.11.2\), with the hypotheses of the second part, by using the isomorphism \([1.1.1\]) we construct naturally the isomorphism of the form \(u^+(\mathcal{E}) \to u_!(\mathcal{E})[2r](r)\). In particular, we get \(u^+(K_X) \to u'_!(K_X)[2r](r)\) and then by adjunction \(\text{Hom} \quad u^+ : u^+(K_X) \to K_X[2r](r)\). Then, by using the Künneth formula, we construct \(u_\ast u^+(\mathcal{E}) \to u_\ast u'_!(K_X) \otimes \mathcal{E}[2r](r)\). By adjunction, we obtain from this latter composition the desired morphism \([1.5.1\]) which is build without any hypothesis. Finally, with the hypotheses of the second part of the proposition, it remains to check that the morphism of the form \([1.5.1\]) constructed in general is the same than the canonical isomorphism built from the isomorphisms \([1.1.1\]). This relies on the commutativity of the diagram \([1.2.1\]).

**Remark 1.7.** With the notation of \([1.6\]), from \([Abe13, 3.1.6\]) we put:

\[
\eta_{u, \mathcal{E}} : \mathcal{E} \xrightarrow{\text{adj}} u_+ u^+ \mathcal{E} \quad \xrightarrow{u_+ u^+(\mu)} u_+ u'_! \mathcal{E}[2r](r) \xrightarrow{\text{adj}} \mathcal{E}[2r](r)
\]

\[(1.6.1)\]

is an element of \(\text{Hom}_{\mathcal{D}^b_{\text{hol}}(X/K)}(\mathcal{E}, \mathcal{E}[2r](r))\) (resp. \(\text{Hom}_{\mathcal{D}^b_{\text{hol}}(X/K)}(\mathcal{E}, \mathcal{E}[2r](r))\)). Similarly than \([KW01]\ I.11.3\), we check that \(\eta_{u, \mathcal{E}} = \eta_{u, K_X} \otimes i_! \mathcal{E}\).

**Remark 1.8.** With the notation of \([1.6\]) from \([Abe13, 3.1.6\]), we put \(H^Z_2(X)(r) := \text{Hom}(K_X, u_+ u'K_X[2r](r))\) and \(H^2(X)(r) := \text{Hom}(K_X, K_X[2r](r))\). From \([Abe13, 3.1.6\]), the composition of the first two morphisms \(u_+ (\mu) \circ \text{adj} : K_X \to u_+ u'K_X[2r](r)\) of \([1.6.1\]) is called the cycle class of \(Y\) and is denoted by \(c_Y(Z) \in H^Z_2(X)(r)\). Since \(u_+\) is a left adjoint functor of \(u\), we get a homomorphism \(H^2_2(X)(r) \to H^2(X)(r)\) which sends \(c_Y(Z)\) to \(\eta_{u, K_X}\).

In order to check the theorem \([1.13\]) below we will need the following lemmas:

**Lemma 1.8.** Let

\[
\begin{array}{ccc}
Z \xrightarrow{u} X & & \\
\downarrow & & \downarrow \ \phi \ \\
Z' \xrightarrow{u'} X'
\end{array}
\]

be a cartesian square so that \(u\) and \(u'\) are closed immersions of pure codimension \(r\) of smooth realizable \(k\)-varieties. Let \(\mathcal{E} \in (F-\mathcal{D}^b_{\text{hol}}(X/K)\) and \(\mathcal{E}_Y := f^+(\mathcal{E})\). Let \(\eta_{u, \mathcal{E}_Y} : \mathcal{E}_Y \to \mathcal{E}_Y[2r](r)\) and \(\eta_{u', \mathcal{E}_Y} : \mathcal{E}_Y \to \mathcal{E}_Y[2r](r)\) be the morphisms defined in \([1.6.1\]). Then we get \(f^+(\eta_{u, \mathcal{E}_Y}) = \eta_{u', \mathcal{E}_Y}\).

**Proof.** We can suppose \(\mathcal{E}_Y = K_X\). This comes from \([Abe13, 3.2.6\]) (see also the remark \([1.7\]).

**Lemma 1.9.** Let \(\pi : \mathbb{P}^d \to \text{Spec} k\) be the canonical projection. Let \(\mathcal{E} \in (F-\mathcal{D}^b_{\text{hol}}(\mathbb{P}^d/K)\). Let \(H^X\) be the zero set of a section of the fundamental line bundle \(O_{\mathbb{P}^d}(1)\), and \(u : H^X \to \mathbb{P}^d\) be the closed immersion. The morphism \(\eta_{u, \mathcal{E}} : \mathcal{E} \to \mathcal{E}[2](1)\) of \([1.6.1\]) does not depend on the choice of the hyperplane \(H^X\) and will be denoted by \(\eta_{\pi, \mathcal{E}}\).

**Proof.** We can suppose \(\mathcal{E} = K_X\). Let \(H_1, H_2\) be respectively the zero set of two sections of \(O_{\mathbb{P}^d}(1)\). From \([1.6.1\]) for \(i = 1, 2\), the closed immersions \(u_i : H_i \to \mathbb{P}^d\) induce the morphisms \(\eta_i : K_X \to K_X[2](1)\). For \(i = 1, 2\), we put \(\theta_i : K \xrightarrow{\text{adj}} \pi_i^{-1}(\eta_i) \rightarrow \pi_i \cdot K_X\). By adjunction, \(\eta_1 = \eta_2\) if and only if \(\theta_1 = \theta_2\). There exists an isomorphism \(\sigma : \mathbb{P}^d \xrightarrow{\text{adj}} \mathbb{P}^d\) so that \(\sigma^{-1}(H_1) = H_2\). From \([1.8\]) we get the isomorphism \(\sigma^{-1}(\eta_1) = \eta_2\) and then \(\sigma^{-1}(\eta_2) = \eta_1\). Since \(\pi \circ \sigma = \pi\), we can conclude.

**1.10.** Let \(S\) be a realizable variety, \(\pi : \mathbb{P}^d \to \text{Spec} k\) and \(\pi_S : \mathbb{P}^d_S \to S\), \(f : \mathbb{P}^d \to \mathbb{P}^d\) be the canonical projections. Let \(\mathcal{E} \in (F-\mathcal{D}^b_{\text{hol}}(\mathbb{P}^d_S/K)\). We put \(\eta_{\pi_S, \mathcal{E}} = f^+(\eta_{\pi, \mathcal{E}}) \otimes i_! \mathcal{E} \to \mathcal{E}[2](1)\).

Let \(S' \to S\) be a morphism of realizable varieties and \(a : \mathbb{P}^d_{S'} \to \mathbb{P}^d_S\) be the induced morphism. Then, we remark that

\[
a^+(\eta_{\pi_S, \mathcal{E}}) = \eta_{\pi_{S'}, \mathcal{E}}.
\]

\[(1.10.1)\]
Lemma 1.11. Let $\pi: \mathbb{P}_S^n \to S$ be the canonical projection and $\iota: \mathbb{P}_S^d \hookrightarrow \mathbb{P}_S^n$ be a closed $S$-immersion such that $\pi' := \pi \circ \iota$ is the canonical projection. Let $E \in (\mathcal{F}^{-}) \text{D}_{\text{hol}}^b(\mathbb{P}_S^d/K)$. We have the equality
\[
\eta^+ (\eta_{E, E}) = \eta_{\pi', \pi} (E).
\] (1.11.1)

Proof. We can suppose $S = \text{Spec} \, k$ and $E = K_{pd}$. Then, this comes from [1.8]

Lemma 1.12. Let $S$ be a realizable variety, $q: X = \mathbb{A}_S^d \to S$ be the canonical projection. Let $E \in \text{Hol}(S/K)$.

1. For any $i \neq 0$, we have $\mathcal{H}_{d_i} q^+ (E) = 0$ and $\mathcal{H}_{d_i} q^+ (E) = 0$.

2. We have $\mathcal{H}_{d_i} q^+ (E) \approx E$ and $\mathcal{H}_{d_i} q^+ (E) \approx E$ in $\text{Hol}(S/K)$.

Proof. From [1.1.1] we can only consider the pushforward case. By transitivity of the pushforward, we reduce to the case where $n = 1$. The complex $q^+ (E)$ is isomorphic to the relative de Rham cohomology of $\mathbb{A}_S^d / S$ of $q^+ [d](E) \in \text{Hol}(\mathbb{A}_S^d/K)$. Then, this is an easy computation.

Theorem 1.13. Let $\pi: \mathbb{P}_S^n \to S$ be the canonical projection, $\iota: X \hookrightarrow \mathbb{P}_S^d$ be a closed immersion such that, for any closed point $s$ of $S$, $f^{-1}(s) \hookrightarrow \mathbb{P}_S^d(s)$ where $f := \pi \circ \iota$ (we might call such a morphism $f$ a $\mathbb{P}^d$-fibration morphism). Let $E \in (\mathcal{F}^{-}) \text{D}_{\text{hol}}^b(S/K)$. With the notation of [1.1.1], we put
\[
\eta = \iota^\ast \eta_{\pi', \pi} (E) : f^+ (E)[2](-1) \to f^+ (E).
\]

By composition, for any integer $i \geq 0$, we get $\eta_i^\ast: f^+ (E)[2i](-i) \to f^+ (E)$. By adjunction, this is equivalent to have a morphism of the form $\eta_i^\ast: E[2i](-i) \to f_+ \circ f^+ (E)$. The following map
\[
\oplus_{i=0}^d \eta_i^\ast: \bigoplus_{i=0}^d E[2i](-i) \to f_+ \circ f^+ (E)
\] (1.13.1)
is an isomorphism.

Proof. That [1.1.1] actually is an isomorphism can be checked after pulling back by the closed immersion given by a closed point of $S$. Hence, with [1.10.1] we can suppose that $S = \text{Spec} \, k$ and $X = \mathbb{P}^d$. From [1.11], we can suppose that $d = n$, i.e. $f$ is the canonical projection $\mathbb{P}^d \to \text{Spec} \, k$. From the last property given in [1.6] we check that the diagram

\[
\begin{array}{ccc}
K \otimes E & \xrightarrow{\text{adj} \otimes \text{Id}_E} & f_+ f^+ (K) \otimes E \\
\downarrow \sim & & \downarrow \sim \\
E & \xrightarrow{\text{adj}} & f_+ f^+ (E)
\end{array}
\] (1.13.2)

\[
\begin{array}{ccc}
f_+ (K) \otimes E & \xrightarrow{\eta \otimes \text{Id}_E} & f_+ (K)[2](1) \otimes E \\
\downarrow \sim & & \downarrow \sim \\
f_+ (K) \otimes f^+ E & \xrightarrow{\eta \otimes \text{Id}_E} & f_+ (K)[2](1) \otimes f^+ E
\end{array}
\]

where the vertical arrows of the top are the Künneth isomorphisms, is commutative. Hence, we can suppose $E = K$. We proceed by induction on $d \geq 0$. The case $d = 0$ is obvious. So, we can suppose $d \geq 1$. Let $H$ be the hyperplane at the infinity, $u: H \hookrightarrow X$ the induced closed immersion, $g := f \circ u$ and $q: \mathbb{A}_S^d \to \text{Spec} \, k$ the projection. We put $\eta_i^\ast := u^\ast (\eta_i^\ast): g^+ (K)[2i](-i) \to g^+ (K)$. Consider the commutative diagram

\[
\begin{array}{ccc}
\eta_i^\ast: & K[-2i](-i) & \xrightarrow{\text{adj}} & f_+ f^+ (K)[-2i](-i) & \xrightarrow{\eta_i^\ast} & f_+ f^+ (K) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
\tilde{\eta}_i^\ast: & K[-2i](-i) & \xrightarrow{\text{adj}} & f_+ u_+ u^+ f^+ (K)[-2i](-i) & \xrightarrow{\tilde{\eta}_i^\ast} & f_+ u_+ u^+ f^+ (K).
\end{array}
\] (1.13.3)
This induces the following commutative square
\[ \begin{array}{ccc}
\oplus_{i=0}^{d-1} \eta^i & \xrightarrow{\tau} & \oplus_{i=0}^{d-1} K\{ -2i \}(i) \\
\oplus_{i=0}^{d-1} \eta^i & \xrightarrow{\tau} & \oplus_{i=0}^{d-1} K\{ -2i \}(i)
\end{array} \]
(1.13.4)

By using the induction hypothesis, the arrow of the bottom is an isomorphism. We denote by \( \tau_{\leq 2d-1} \) the truncation functor of the canonical t-structure of \([AC13b]\). Since we have the exact triangle of localization \( q^+ g^+(K) \to f^+ f^+(K) \to g_+ g^+(K) \to +1 \) and the Lemma \([1.12]\), then after having applied the functor \( \tau_{\leq 2d-1} \) to the right morphism of \(1.13.4\) we get an isomorphism (for the degree \( 2d-1 \), we use also the induction hypothesis for \( g_+ g^+(K) \) which implies that \( q^+ g_+ g^+(K) = 0 \)). By considering \(1.13.4\) this implies that the truncation \( \tau_{\leq 2d-1} (\oplus_{i=0}^{d-1} \eta^i) \) of \(1.13.1\) is an isomorphism.

Now, consider the following commutative diagram:
\[
\begin{array}{ccc}
K\{ -2d \}(d) & \xrightarrow{\eta} & f^+ f^+(K)\{ -2d \}(d) \\
K\{ -2d \}(d) & \xrightarrow{\eta} & f^+ f^+(K)\{ -2d \}(d)
\end{array}
\]
(1.13.5)

where the right arrow of the bottom is an isomorphism because of \([1.5.2]\). Hence, by using the induction hypothesis, we check that after having applied the functor \( \tau_{\leq 2d} \) to the diagram \([1.13.5]\), the composition of the arrows of the bottom becomes an isomorphism. A cone of the right morphism of \([1.13.5]\) is isomorphic to \( q^+ g^+(K) \). From \([1.12]\), we get \( \tau_{\geq 2d-1-q} g^+(K) = 0 \). Hence, by applying \( \tau_{\geq 2d} \) to the right morphism of \([1.13.5]\), we get an isomorphism. This implies that \( \tau_{\geq 2d}(\eta^i) \) is an isomorphism. Hence, so is \( \tau_{\geq 2d}(\oplus_{i=0}^{d-1} \eta^i) \).

**Corollary 1.14.** We keep the geometrical notation of \([1.13]\) and we suppose \( f \) smooth. Let \( E \in F.D^b_{\text{hol}}(S/K)^{\leq 0} \) and \( F \in F.D^b_{\text{hol}}(S/K)^{\geq 0} \). Let \( \Xi \in D^b_{\text{hol}}(S/K)^{\leq 0} \) and \( \mathcal{F} \in D^b_{\text{hol}}(S/K)^{\geq 0} \). Then
\[
\text{Hom}_{F.D^b_{\text{hol}}(S/K)}(f^+ \Xi, f^+ F) = \text{Hom}_{F.D^b_{\text{hol}}(S/K)}(\Xi, \mathcal{F})
\]
(1.14.1)

**Proof.** Since \( E \in F.D^b_{\text{hol}}(S/K)^{\leq 0} \) and \( F \in F.D^b_{\text{hol}}(S/K)^{\geq 0} \), then
\[
\text{Hom}_{F.D^b_{\text{hol}}(S/K)}(E, \mathcal{F}) = \text{Hom}_{F.D^b_{\text{hol}}(S/K)}(\Xi, \mathcal{F})
\]
(1.14.2)

Since \( f \) is smooth, then the functor \( f^+ f^+ \) preserves \( F.D^b_{\text{hol}}(S/K)^{\geq 0} \). Hence, by adjunction, we get
\[
\text{Hom}_{F.D^b_{\text{hol}}(X/K)}(f^+ E, f^+ \mathcal{F}) = \text{Hom}_{F.D^b_{\text{hol}}(S/K)}(E, f^+ f^+(\mathcal{F})) = \text{Hom}_{F.D^b_{\text{hol}}(S/K)}(\Xi, f^+ f^+ (\mathcal{F}))
\]
With \( \Xi \xrightarrow{\tau} \mathcal{F} \) \( \xrightarrow{\tau} \mathcal{F} \circ f^+(\mathcal{F}) \) \( \xrightarrow{\tau} \mathcal{F} \circ f^+(\mathcal{F}) \), then we obtain the last equality of \(1.14.1\). The proof without Frobenius is identical.

**Proposition 1.15.** We keep the notation and hypotheses of \([1.14]\).
\[ \begin{enumerate}
\item The functor \( f^+[d] : (F-)\text{Hol}(S/K) \to (F-)\text{Hol}(X/K) \) is t-exact and fully faithful.
\item For any \( E, \mathcal{F} \in \text{Hol}(S/K) \), the functor \( f^+[d] \) induces the equality
\[ \text{Ext}^1_{\text{Hol}(S/K)}(E, \mathcal{F}) = \text{Ext}^1_{\text{Hol}(X/K)}(f^+[d](E), f^+[d](\mathcal{F})). \]
\[ \text{Ext}^1_{\text{Hol}(X/K)}(f^+[d](E), f^+[d](\mathcal{F})). \]
\end{enumerate} \]
(1.15.1)

**Proof.** The first part comes from \([1.14]\). Since the canonical morphism \( \mathcal{F}[1] \to \tau_{\leq 0} f^+ (\mathcal{F}[1]) \) is an isomorphism, we get the second assertion (e.g. see the proof of \(1.14\)).
Remark 1.16. We keep the notation and hypotheses of [1.14]. Let \( \mathcal{E} \in \text{Hol}(S/K) \). Since the pull back under Frobenius commutes with the functor \( f^+[d] \), then we get a bijection between Frobenius structures on \( \mathcal{E} \) and Frobenius structures on \( f^+[d](\mathcal{E}) \). Moreover, let \( \mathcal{F}, \mathcal{G} \in F\text{-Hol}(S/K) \) and \( \phi: \mathcal{F} \to \mathcal{G} \) a morphism of \( \text{Hol}(S/K) \). Then \( \phi \) commutes with Frobenius if and only if so is \( f^+[d](\phi) \).

Proposition 1.17. We keep the notation and hypotheses of [1.14]. We suppose furthermore that the morphism \( f \) has locally a section.

1. The functor \( f^+[d]: (F-)\text{Hol}(S/K) \to (F-)\text{Hol}(X/K) \) sends simple objects to simple objects.

2. The functor \( f^+[d]: (F-)\text{Hol}(S/K) \to (F-)\text{Hol}(X/K) \) has the right adjoint functor \( 3\mathcal{C}_f \) of \( f^+_+: (F-)\text{Hol}(X/K) \to (F-)\text{Hol}(S/K) \) and the left adjoint functor \( 3\mathcal{C}_f \) of \( f_-(d) : (F-)\text{Hol}(X/K) \to (F-)\text{Hol}(S/K) \).

Proof. Let \( \mathcal{E} \) be a simple object of \((F-)\text{Hol}(S/K)\). From [AC13b, 1.4.9.(i)] (by using [Car11a, 3.7]), we notice that this is still available without Frobenius structures, there exist an open dense smooth subscheme \( S' \) of \( S \), an irreducible object \( \mathcal{E}' \in (F-)\text{Hol}(S'/K) \) which is also an object of \((F-)\text{Isoc}\) \((S'/K)\) (the category \( \text{Isoc} \) \((S'/K)\) is constructed in [Car11b]) such that \( \mathcal{E} \rightarrow u_+ (\mathcal{E}') \) when \( u: S' \rightarrow S \) is the inclusion. Put \( X := f^{-1}(S') \), \( f^+: X' \rightarrow S' \), \( v: X' \rightarrow X \).

By adjunction, we remark that the canonical morphism \( u_+ (\mathcal{E}') \rightarrow u_+ (\mathcal{E}) \) is the only one so that we get the identity over \( S' \). With this remark, since \( f_+^+[d] = u_+ (\mathcal{E}') \rightarrow v_+ \circ f^+_+[d](\mathcal{E}') \), and \( f_+^+[d] = u_+ (\mathcal{E}') \rightarrow f^+_-[d](\mathcal{E}') \rightarrow v_+ \circ f^+_+[d](\mathcal{E}') \), then we get the isomorphism \( f^+_+[d] = u_+ (\mathcal{E}') \rightarrow f^+_-[d](\mathcal{E}') \). Since the functor \( v_+ \) preserve the irreducibility, then we reduce to the case where \( S' \) is affine, smooth, irreducible, and where we have moreover \( S \in (F-)\text{Isoc}(S/K) \). Hence \( f^{-1}(d)(\mathcal{E}) \in (F-)\text{Isoc}(X/K) \cap (F-)\text{Hol}(X/K) \). Let 0 \( \neq \mathcal{G} \in (F-)\text{Hol}(X/K) \) be a subobject of \( f^{-1}(d)(\mathcal{E}) \). Since \((F-)\text{Isoc}(X/K) \cap (F-)\text{Hol}(X/K) \) is a Serre subcategory of \((F-)\text{Ohol}(X/K) \), then \( \mathcal{G} \in (F-)\text{Isoc}(X/K) \). Since the rank of an overconvergent isocrystal is preserved under pull-backs, since \( f \) locally has a section, since \( X \) is irreducible (because the fibers of \( f \) are irreducible) then we can conclude that \( \mathcal{G} = f^{-1}(d)(\mathcal{E}) \) and hence \( f^+[d](\mathcal{E}) \) is a simple object.

The last part comes from the left t-exactness of \( f_+[-d] \) and the right t-exactness of \( f_-[d] \), from the fact that the couples \((f^{-1}[d], f_+[-d]) \) and \((f_-[d], f^+[-d]) \) are adjoint functors and from the isomorphism \( f^+_+[d] \rightarrow f^+_-[d] \).

Proposition 1.18. We keep the notation and hypotheses of [1.14]. Let \( \mathcal{E} \in (F-)\text{Hol}(X/K) \).

1. The category of constant objects with respect to \( f \) is a thick subcategory of \((F-)\text{Hol}(X/K)\).

2. The object \( 3\mathcal{C} \circ f^+_+[d](\mathcal{E}) = f^+[-d](\mathcal{E}) \) is the largest constant with respect to \( f \) subobject of \( \mathcal{E} \) in \((F-)\text{Hol}(X/K)\).

3. The object \( 3\mathcal{C} \circ f^+[-d](\mathcal{E}(d)) = f^+[d](\mathcal{E}(d)) \) is the largest constant with respect to \( f \) quotient object of \( \mathcal{E}(d) \in (F-)\text{Hol}(X/K) \).

Proof. The thickness of the category of constant objects without Frobenius structures comes from the equality [1.15.1]. With the remark [1.16] we get the thickness with Frobenius structures. The rest is similar to the proof of [KW01, III.11.3] i.e. this comes from the general fact [KW01, III.11.1] and from [1.17].

2 The Brylinski-Radon transform and the Hard Lefschetz Theorem

Let \( d \geq 1 \) be an integer and \( \mathbb{P}^d \) be the \( d \)-dimensional projective space defined over \( k \), let \( \mathbb{P}^d \) be the dual projective space over \( k \), which parameterizes the closed subvarieties of \( \mathbb{P}^d \times \mathbb{P}^d \) so that \((x, h) \in H \) if and only if the point \( x \in h \). Let \( Y \) be a realizable \( k \)-variety. We denote by \( i: H \times Y \hookrightarrow \mathbb{P}^d \times \mathbb{P}^d \) the canonical immersion and \( p_1: \mathbb{P}^d \times \mathbb{P}^d \times Y \rightarrow \mathbb{P}^d \times Y \), \( p_2: \mathbb{P}^d \times \mathbb{P}^d \times Y \rightarrow \mathbb{P}^d \times Y \), \( \tilde{p}_1: \mathbb{P}^d \times Y \rightarrow Y \), \( \tilde{p}_2: \mathbb{P}^d \times Y \rightarrow Y \) the canonical projections and \( \pi_1 := p_1 \circ i, \pi_2 := p_2 \circ i \).

Definition 2.1. We define the Brylinski-Radon transform:

\[
\text{Rad}: F-D\text{hol}(\mathbb{P}^d \times Y/K) \to F-D\text{hol}(\mathbb{P}^d \times Y/K) \tag{2.1.1}
\]

by posing \( \text{Rad}(\mathcal{E}) := \pi_2^+ \pi_1^+ (\mathcal{E})(d-1), \) for any \( \mathcal{E} \in F-D\text{hol}(\mathbb{P}^d \times Y/K) \). For any \( n \in \mathbb{Z} \), put \( \text{Rad}^n := 3\mathcal{C}^n \circ \text{Rad} \).
Definition 2.2. Let $U$ be the open complement of the closed subvariety $H \times Y$ in $\mathbb{P}^d \times \mathbb{P}^d \times Y$. Let $j: U \hookrightarrow \mathbb{P}^d \times \mathbb{P}^d \times Y$ be the open immersion and $q_1 := p_1 \circ j$, $q_2 := p_2 \circ j$. We define the modified Radon transform $Rad_1: F-D^b_{\text{hol}}(\mathbb{P}^d \times Y / K) \to F-D^b_{\text{hol}}(\mathbb{P}^d \times Y / K)$ by posing, for any $E \in F-D^b_{\text{hol}}(\mathbb{P}^d \times Y / K)$,

$$Rad_1(E) := q_2! \circ q_1^+ [d](E).$$

(2.2.1)

For any integer $n \in \mathbb{Z}$, we put $Rad_n(E) := \mathcal{H}^n \circ Rad_1(E)$.

2.3. 1. Since the functor $q_1^+ [d]$ is t-exact and the functor $q_2!$ is left t-exact (because $q_2$ is affine, e.g. see [ACT13, 1.3.13]) but this is obvious here since $q_2$ is moreover smooth) then $Rad_1$ is left t-exact.

2. The exact triangle $i_+ \circ i^+ [-1] \to j_* j^+ \to id \to i_+ \circ i^+$ induces for any $E \in F-D^b_{\text{hol}}(\mathbb{P}^d \times Y / K)\text{ the exact triangle}$

$$Rad_1(E) \to Rad_1(E) \to p_2^* p_2^+ [d](E) \to Rad_1(E)[1].$$

(2.3.1)

Lemma 2.4. Let $E \in F-D^b_{\text{hol}}(\mathbb{P}^d \times Y / K)$.

1. We have the isomorphism $E[-2d](−d) \xrightarrow{\sim} q_1!q_1^+(E)$.

2. We have the isomorphism $\bar{p}_1 + (Rad_1(E)) \xrightarrow{\sim} \bar{p}_2(E)[-d](-d)$.

3. If $E \in (F-)D^{≥0}(\mathbb{P}^d \times Y / K)$, then $\bar{p}_1_+(Rad_1(E)) \in (F-)D^{≥0}(Y / K)$.

Proof. We put $\eta = \eta_{p_1q_1}(E)[−2](−1)$ and $\bar{\eta} := i^+(\eta)$, in order to check the first isomorphism, by using the Künneth isomorphism, we can suppose $E = K_{\mathbb{P}^d \times Y}$. Then, by using a base change theorem, we can suppose $Y = \text{Spec} k$. We put $\mathcal{G} = K_{\mathbb{P}^d}[-d] \in F-\text{Hol}(\mathbb{P}^d / K)$. Consider the commutative diagram below

$$
\begin{array}{c}
q_1!q_1^+(\mathcal{G}) \\
\xrightarrow{\text{adj}} \\
p_1 + p_1^+ (\mathcal{G}) \\
\xrightarrow{\sim} \oplus_{i=0}^{d} \mathcal{G}[-2i](-i) + \xrightarrow{\sim} \oplus_{i=0}^{d-1} \mathcal{G}[-2i](-i) + \xrightarrow{\sim} \mathcal{G}[-2d](-d)
\end{array}
$$

(2.4.1)

where the rows are exact triangles and the vertical arrows are isomorphisms thanks to [1.13.1]. By applying the functor $\tau_{≥2d}$ to the diagram (2.4.1) we get $q_1!q_1^+(\mathcal{G}) \xrightarrow{\sim} \tau_{≥2d}q_1!q_1^+(\mathcal{G}) \xleftarrow{\sim} \tau_{≥2d}p_1 + p_1^+(\mathcal{G}) \xleftarrow{\sim} \tau_{≥2d} \oplus_{i=0}^{d} \mathcal{G}[-2i](-i) \xleftarrow{\sim} \mathcal{G}[-2d](-d)$, which finishes the proof of the first isomorphism. We get the second isomorphism by composition:

$$\bar{p}_1_+(Rad_1(E)) = \bar{p}_1 q_2! \circ q_1^+ [d](E) \xrightarrow{\sim} \bar{p}_2 q_1! \circ q_1^+ [d](E) \xrightarrow{\sim} \bar{p}_2 \mathcal{G}[-d](-d).$$

We conclude since the third point is obvious from the second one.

Lemma 2.5. Let $E \in (F-)D^{≥0}(\mathbb{P}^d \times Y / K)$. Then $Rad_1^n(E)$ is left reduced with respect to $\bar{p}_1$, i.e. does not have any nontrivial constant with respect to $\bar{p}_1$ subobject.

Proof. The proof is the same than [KOW01, IV.2.7]: from (1.18) we have to prove that $\mathcal{H}^{−d} \bar{p}_1_+(Rad_0(E)) = 0$. Since $Rad_1(E) \in (F-)D^{≥0}(\mathbb{P}^d \times Y / K)$ (see 2.3), since $\mathcal{H}^{−d} \bar{p}_1_+$ is left t-exact, then $\mathcal{H}^{−d} \bar{p}_1_+(Rad_1(E)) \xrightarrow{\sim} \mathcal{H}^{−d} \bar{p}_1_+(Rad_1(E))$. We conclude by using 2.4

2.6. Let $f: X \to Y$ be a projective morphism of realizable varieties. Let $E \in F-D^b_{\text{hol}}(X / K)$. A morphism $\eta: E \to \mathcal{E}[2][1]$ is called a Chern class of a relative hyperplane for the projective morphism $f$ if there exists a closed immersion $\iota: X \hookrightarrow \mathbb{P}^d_Y$ so that $f = \pi \circ \iota$, where $\pi: \mathbb{P}^d_Y \to Y$ is the canonical projection, and so that $\eta = id_{\mathcal{E}} \otimes t^+(\eta_{\pi,K_{\mathbb{P}^d_Y}})$ (see the notation 1.10).
Theorem 2.7 (Hard Lefschetz Theorem). Let \( f : X \to Y \) be a projective morphism of realizable varieties. Let \( E \in F\text{-Hol}(X/K) \) be a \( t \)-pure module and \( \eta : E \to \mathcal{E}(2)[1] \) be a Chern class of a relative hyperplane for \( f \) (see 2.6). For any positive integer \( r \), we obtain by composition \( \eta^r : E \to \mathcal{E}(2r)[r] \). We get the homomorphism
\[
\mathcal{H}^r\tau_+ f_+ (\eta^r) : \mathcal{H}^r\tau_+ f_+ (E) \to \mathcal{H}^r\tau_0 f_+ (E)(r).
\]
(2.7.1)

The homomorphism 2.7.1 is an isomorphism.

Proof. We follow the proof of the Hard Lefschetz Theorem of [KW01, IV.4.1] which is similar to that of [BBD82].

0. Since the assertion is local on \( Y \), one can suppose \( Y \) affine and smooth. We reduce to the case where \( f \) is the projection \( \mathbb{P}^d_f \to Y \). Then we keep the notation of the section.

1. We treat the case \( r = 1 \). We put \( \mathcal{G} = p_1^*[d](E) \in F\text{-Hol}(\mathbb{P}^d \times \hat{\mathbb{P}}^d \times Y/K) \). From 1.6.1, we get from the closed immersion \( i : H \times Y \to \mathbb{P}^d \times \hat{\mathbb{P}}^d \times Y \) the morphism \( \theta := \eta\circ \mathcal{G} \to \mathcal{G}(2)[1] \).

a) Let \( \text{Spec} k \to \hat{\mathbb{P}}^d \) be a rational section, \( r : Y \to \mathbb{P}^d \times Y \) and \( s : \mathbb{P}^d \times Y \to \mathbb{P}^d \times \hat{\mathbb{P}}^d \times Y \) the induced closed immersions. Since \( s^* (H \times Y) \) is an hyperplane of \( \mathbb{P}^d \times Y \) times \( Y \), from 1.3, we get that \( s^* [-d](\theta) = \eta \). Since \( p_{2+} \circ p_1^+ \to \tilde{\mathcal{P}}_{P2+} \) and \( \tilde{\mathcal{P}}_{P2+} \to t^+ p_{2+} \), from the functor \( s^* [-d] \) (resp. \( t^* [-d] \)) is acyclic for the constant objects with respect to \( p_1 \) (resp. \( p_{2+} \)), we get that
\[
t^+ [-d] = \mathcal{H}^r\tau_0 p_{2+} (\theta) \to \mathcal{H}^r\tau_0 \mathcal{P}_{P2+} [-d](\theta) = \mathcal{H}^r\tau_0 \mathcal{P}_{P2+} (\eta).
\]
Hence, this is enough to check that \( \mathcal{H}^r\tau_0 p_{2+} (\theta) : \mathcal{H}^r\tau_0 p_{2+} (\mathcal{E}) \to \mathcal{H}^r\tau_0 p_{2+} (\mathcal{E})(1) \) is an isomorphism. For simplicity, we denote this morphism \( \theta \).

b) The diagram
\[
\begin{array}{ccc}
\mathcal{H}^d\tau_0 p_{2+} (E) & \to & \mathcal{H}^d\tau_0 p_{2+} (\mathcal{G})(1) \\
\downarrow \text{adj} & \to & \downarrow \text{adj} \\
\mathcal{H}^d\tau_0 p_{2+} (i^+ (\mathcal{G})) & \to & \mathcal{H}^d\tau_0 p_{2+} (i^+ (\mathcal{G}))(1)
\end{array}
\]
\[
\begin{array}{cc}
\tilde{\mathcal{H}}^d\tau_0 p_{2+} (\tilde{\mathcal{G}}) & \to \mathcal{H}^d\tau_0 p_{2+} (\tilde{\mathcal{G}})(1) \\
\downarrow \text{adj} & \downarrow \text{adj} \\
\mathcal{H}^d\tau_0 p_{2+} (i^+ (\tilde{\mathcal{G}})) & \to \mathcal{H}^d\tau_0 p_{2+} (i^+ (\tilde{\mathcal{G}}))(1)
\end{array}
\]
where \( \mathcal{G} = p_1^*[d](\mathcal{D}(\mathcal{E})) \in F\text{-Hol}(\mathbb{P}^d \times \hat{\mathbb{P}}^d \times Y/K) \) and the right square is the dual of the left square used for \( \mathcal{D}(\mathcal{E}) \) instead of \( \mathcal{E} \), is commutative. Moreover, the second left arrow of the bottom row is indeed an isomorphism because of 1.5.2. Since \( \pi_1 \) is smooth of relative dimension \( d - 1 \) and \( \pi_2 \) is proper, we get \( \mathcal{D}\mathcal{R}^{d-1}(\mathcal{D}(\mathcal{E})) \to \mathcal{D}\mathcal{R}^{d-1}(\mathcal{D}(\mathcal{E}))(1) \) from 2.3 we get the first exact sequence
\[
0 \to \mathcal{H}^{d-1}\tau_0 p_{2+} (E) \to \mathcal{D}\mathcal{R}^{d-1}(\mathcal{D}(\mathcal{E})) \to \mathcal{D}\mathcal{R}^{d-1}\tau_0 p_{2+} (\mathcal{D}(\mathcal{E})) \to 0,
\]
(2.7.3)
\[
\mathcal{D}\mathcal{R}^{d-1}(\mathcal{D}(\mathcal{E})) \to \mathcal{D}\mathcal{R}^{d-1}(\mathcal{E}) \to \mathcal{D}\mathcal{R}^{d-1}\tau_0 p_{2+} (\mathcal{D}(\mathcal{E})) \to 0,
\]
(2.7.4)
the second one is induced by duality. By construction, the first morphism of 2.7.3 and the last one of 2.7.4 are respectively the left vertical arrow and the right vertical arrow of 2.7.2. From 2.3 \( \mathcal{R}^{d-1}(\mathcal{E}) \) (resp. \( \mathcal{D}\mathcal{R}^{d-1}(\mathcal{D}(\mathcal{E})) \)) is left (resp. right) reduced with respect to \( \tilde{\mathcal{P}}_{P1} \), i.e. does not have any nontrivial constant with respect to \( \tilde{\mathcal{P}}_{P1} \) subobject (resp. quotient). This implies that \( \mathcal{H}^{d-1}\tau_0 p_{2+} (\tilde{\mathcal{E}}) \) (resp. \( \mathcal{D}\mathcal{R}^{d-1}\tau_0 p_{2+} (\mathcal{D}(\mathcal{E})) \)) is the maximal constant with respect to \( \tilde{\mathcal{P}}_{P1} \) subobject (resp. quotient) of \( \mathcal{R}^{d-1}(\mathcal{E}) \).

Since \( \pi_1 \) is smooth, \( \pi_2 \) is proper and \( \mathcal{E} \) is \( t \)-pure then so is \( \mathcal{R}^{d-1}(\mathcal{E}) \) and hence is semi-simple in the category \( \text{Hol}(\mathbb{P}^d \times Y/K) \) (see [AC13a, 4.3.1]). From the diagram 2.7.2, this implies that the morphism \( \theta : \mathcal{H}^{d-1}\tau_0 p_{2+} (\mathcal{E}) \to \mathcal{H}^{d-1}\tau_0 p_{2+} (\mathcal{E})(1) \) is an isomorphism in \( \text{Hol}(\mathbb{P}^d \times Y/K) \) and then in \( F\text{-Hol}(\mathbb{P}^d \times Y/K) \).

2. We proceed by induction on \( r \). Suppose \( r \geq 2 \). We put \( \mathcal{G} := i^+ (\mathcal{E})[1] \to \pi_1^*[d-1](\mathcal{E}) \). The morphism \( \theta \) induces by pull-back \( \tilde{\mathcal{G}} := \mathcal{G}[2](1) \). We notice that \( \tilde{\mathcal{G}} \) is the Chern class of a relative ample hyperplane for the projective morphism \( \pi_2 \). Consider the commutative diagram:
\[
\begin{array}{ccc}
\mathcal{H}^{r-1}\tau_0 p_{2+} (\mathcal{E}) & \to & \mathcal{H}^{r-1}\tau_0 p_{2+} (\mathcal{E})(1) \\
\downarrow \text{adj} & \to & \downarrow \text{adj} \\
\mathcal{H}^{r-1}\tau_0 p_{2+} (i^+ (\mathcal{G})) & \to & \mathcal{H}^{r-1}\tau_0 p_{2+} (i^+ (\mathcal{G}))(1)
\end{array}
\]
\[
\begin{array}{cc}
\mathcal{H}^{r-1}\tau_0 p_{2+} (\tilde{\mathcal{G}}) & \to \mathcal{H}^{r-1}\tau_0 p_{2+} (\tilde{\mathcal{G}})(1) \\
\downarrow \text{adj} & \downarrow \text{adj} \\
\mathcal{H}^{r-1}\tau_0 p_{2+} (i^+ (\tilde{\mathcal{G}})) & \to \mathcal{H}^{r-1}\tau_0 p_{2+} (i^+ (\tilde{\mathcal{G}}))(1)
\end{array}
\]
(2.7.5)
where the middle arrow of the middle row is an isomorphism because of \[1.3.2\]. By considering the long exact sequence induced by \[2.3.1\] since \(Rad\) is left exact and \(r \geq 2\), we check that the adjunction morphism \(\Delta_{r}^{-1}p_{2+}(\mathcal{S}) \to \Delta_{r}^{-1}p_{2+}i^{+}(\mathcal{S})\) is an isomorphism. Similarly, we get that the right vertical arrow of \[2.7.5\] is an isomorphism. By using the induction hypothesis, the arrow of the bottom of the diagram \[2.7.5\] is an isomorphism. Then so is \(\theta^{r} \). □

3 The dual Brylinski-Radon and the inversion formula

We keep the notation of the chapter 2.

**Definition 3.1.** We define the dual Brylinski-Radon transform:

\[Rad^{i}: F \cdot D^{b}_{hol}(\mathbb{P}^{d} \times Y/K) \to F \cdot D^{b}_{hol}(\mathbb{P}^{d} \times Y/K)\]  \hspace{1cm} (3.1.1)

by posing \(Rad^{i}(\mathcal{E}) := \pi_{1+}\pi_{+}^{i}(\mathcal{E})|d-1}\].

**Lemma 3.2.** Let \(1: X := (H \times \mathbb{P}^{d}) \times Y \hookrightarrow \mathbb{P}^{d} \times \mathbb{P}^{d} \times \mathbb{P}^{d} \times Y\) be the canonical embedding, \(p_{13}: \mathbb{P}^{d} \times \mathbb{P}^{d} \times \mathbb{P}^{d} \times Y \to \mathbb{P}^{d} \times \mathbb{P}^{d} \times Y\) be the projection and \(\pi = p_{13} \circ 1\). Let \(\Delta: \mathbb{P}^{d} \times Y \to \mathbb{P}^{d} \times \mathbb{P}^{d} \times Y\) be the diagonal immersion (and the identity over \(Y\)). Let \(F \in F \cdot D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d} \times Y/K)\). We have the isomorphism of \(F \cdot D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d} \times Y/K)\) of the form

\[\Delta_{+} \circ \Delta^{+}(F)[2-2d](1-d) \bigoplus \oplus_{i=0}^{d-2} F[-2i](-i) \xrightarrow{i} \pi_{+}\pi^{+}(F)\].

**Proof.** By using Künneth isomorphisms, we can suppose \(F = K_{\mathbb{P}^{d} \times \mathbb{P}^{d} \times Y}\). Hence, by using some base change theorems (induced by the projection \(Y \to \text{Spec} k\)), then we can suppose \(Y = \text{Spec} k\). We put \(\overline{K} := K_{\mathbb{P}^{d} \times \mathbb{P}^{d}}\). Consider the following cartesian squares

\[
\begin{array}{cccc}
X & \xleftarrow{1} & \mathbb{P}^{d} \times \mathbb{P}^{d} & \xrightarrow{p_{13}} \mathbb{P}^{d} \\
\nwedge & & \nwedge & \\
\overline{X} & \xleftarrow{\overline{1}} & \mathbb{P}^{d} \times \mathbb{P}^{d} & \xrightarrow{p_{1}} \mathbb{P}^{d} \\
\end{array}
\]

Put \(\overline{\eta} := p_{1}^{-1} \eta \), \(\eta := t^{+} \eta p_{13}K_{\mathbb{P}^{d} \times \mathbb{P}^{d}}\), \(\overline{\eta} := (\overline{1} \eta) p_{13}K_{\mathbb{P}^{d} \times \mathbb{P}^{d}}\). From \[1.8\] we check \(\overline{\Delta}^{+}(\eta) = \overline{\eta}\). By following \[1.13\] we get the morphism

\[\oplus_{i=0}^{d-2} \eta^{i}: \oplus_{i=0}^{d-2} \overline{K}[-2i](-i) \to \pi_{+}\pi^{+}(\overline{K})\]  \hspace{1cm} (3.2.2)

Since \(\pi\) is outside \(\Delta(\mathbb{P}^{d})\) a \(\mathbb{P}^{d-2}\)-fibration, from \[1.13\] the morphism \[3.2.2\] is an isomorphism outside \(\Delta(\mathbb{P}^{d})\). Hence, a cone of \[3.2.2\] is in the essential image of \(\Delta_{+}\). Since \(\Delta_{+}\pi_{+}\pi^{+}(K) \xrightarrow{i} \pi_{+}\Delta^{+}\pi^{+}(K) \xrightarrow{\Delta} \pi_{+}\pi^{+}\Delta^{+}(K)\), by applying \(\Delta^{+}\) to the morphism \[3.2.2\] we get

\[\oplus_{i=0}^{d-2} \eta^{i}: \oplus_{i=0}^{d-2} \Delta^{+}\overline{K}[-2i](-i) \to \pi_{+}\pi^{+}\Delta^{+}(\overline{K})\]  \hspace{1cm} (3.2.3)

Since \(\overline{\eta}\) is \(\mathbb{P}^{d-1}\)-fibration, from \[1.13\] we get the morphism \(\Delta_{+}\pi_{+}\pi^{+}(\overline{K}) \to \Delta^{+}(\overline{K})[2-2d](1-d)\) and then by adjunction the second morphism of the sequence in \(F \cdot D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)\):

\[\oplus_{i=0}^{d-2} \overline{K}[-2i](-i) \to \pi_{+}\pi^{+}(\overline{K}) \to \Delta_{+} \circ \Delta^{+}(\overline{K})[2-2d](1-d)\]  \hspace{1cm} (3.2.4)

Since \(\pi_{+}\pi^{+}(\overline{K})\) is \(t\)-pure, then \(\pi_{+}\pi^{+}(\overline{K})\) is semisimple in \(D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)\) (see \[AC13\] 4.3.1)). Then, we get the isomorphism \(\Delta_{+}\circ\Delta^{+}(\overline{K})[2-2d](1-d) \bigoplus \oplus_{i=0}^{d-2} \overline{K}[-2i](-i) \xrightarrow{i} \pi_{+}\pi^{+}(\overline{K})\) in \(D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)\) which induces the morphisms of \[3.2.4\]. For any \(i = 0, \ldots, d-2\), we have

- \(\text{Hom}_{D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)}(\Delta_{+} \circ \Delta^{+}(\overline{K})[2-2d], \overline{K}[-2i]) \xrightarrow{i} \text{Hom}_{D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)}(\Delta^{+}(\overline{K})[2-2d], \Delta^{+}\overline{K}[-2i])\)

\[\xrightarrow{1.5.1} \text{Hom}_{D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)}(\Delta^{+}(\overline{K})[2-2d], \Delta^{+}\overline{K}[-2i]) = 0\]  \hspace{1cm} (3.2.5)

- \(\text{Hom}_{D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)}(\overline{K}[-2i], \Delta_{+} \circ \Delta^{+}(\overline{K})[2-2d]) \xrightarrow{i} \text{Hom}_{D^{b}_{hol}(\mathbb{P}^{d} \times \mathbb{P}^{d}/K)}(\Delta^{+}(\overline{K})[2-2d]) = 0\].  \hspace{1cm} (3.2.6)

Hence, we get the compatibility with Frobenius. □
Proposition 3.3 (Radon Inversion Formula). Let $E \in F^{-D}_{hol}(\mathbb{P}^d \times Y / K)$. Then the following formula holds

$$Rad^\vee \circ Rad(E) \sim E(1-d) \oplus \tilde{p}_{2+}^{-2}[d]\phi(E), \quad (3.3.1)$$

where $\phi(E) := \bigoplus_{i=0}^{d-2} \tilde{p}_{2+}(E)[d-2-2i](-i)$

Proof. With the notation of 3.2, let respectively $u, v: \mathbb{P}^d \times \mathbb{P}^d \times Y \to \mathbb{P}^d \times Y$ be the left and right projection. Then, by using the base change theorem (more precisely, the left projection $X \to H$ by the right projection $X \to H$) we get $Rad^\vee \circ Rad(E) \sim v_+ \pi_+ \pi_+ u^+(E)[2d-2]$. Hence we obtain:

$$Rad^\vee \circ Rad(E) \sim v_+ \Delta_+ \circ \Delta_+(u^+(E))(1-d) \bigoplus \bigoplus_{i=0}^{d-2} v_+ u^+(E)[2d-2i](-i)$$

$$\sim E(1-d) \bigoplus \bigoplus_{i=0}^{d-2} \tilde{p}_{2+}^{-2}[d]\tilde{p}_{2+}(E)[d-2i-2](-i) \quad (3.3.2)$$

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