NOTES ON INVARIANT MEASURES FOR LOOP GROUPS

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Abstract. Let \( K \) denote a simply connected compact Lie group and let \( G = K^\mathbb{C} \), the complexification. It is known that there exists an \( LK \) bi-invariant probability measure on a natural completion of the complex loop group \( LG \).

It is believed that there exist deformations which are positive line bundle valued and reproduce the unitary structure for (projective) positive energy representations of \( LK \). The purpose of these notes is to publicize a number of conjectures and questions concerning how these measures are characterized, how they are explicitly represented, and how they are relevant to quantum sigma models.

0. Introduction

In [66], and more directly in [68], I proved the following theorem.

Theorem 0.1. Suppose that \( K \) is a simply connected compact Lie group, and let \( G \) denote the complexification. There exists a Borel probability measure \( \mu_0 \) on the formal completion, \( LG \), of the complex loop group \( C^\infty(S^1, G) \) which is bi-invariant with respect to \( L_{\text{fin}}K \), the group of loops in \( K \) with finite Fourier series (in some matrix representation of \( K \)).

The purpose of these notes is to publicize a number of conjectures and questions about the measure \( \mu_0 \) and its deformations (in particular a probability measure \( \mu_l \) parameterized by level \( l \)). I have thought about these matters for a long time, and I have a lot of opinions. It is painful to acknowledge how little I can prove.

0.1. Outline of the Notes. The first half of Section 1 is a review of the definitions of the formal and hyperfunction completions of \( C^\infty(S^1, G) \), denoted by \( LG \) and \( \text{Hyp}(S^1, G) \), respectively. These completions are not groups, and the anti-holomorphic involution corresponding to the unitary form \( C^\infty(S^1, K) \to C^\infty(S^1, G) \) does not continuously extend to the completions. A basic theme of these notes is that the support of the measure \( \mu_0 \) is a kind of surrogate for the missing unitary form of these completions. The second half of Section 1 involves more sophisticated Lie theoretic structure which is needed for example to explain root subgroup factorization (a key tool for understanding Toeplitz determinants and the measures \( \mu_l \)) for higher rank groups. We will frequently specialize to \( K = SU(2) \), to minimize references to the second half of Section 1.

Section 2 concerns uniqueness questions about \( L_{\text{fin}}K \) bi-invariant measures and measure classes on the formal completion \( LG \). An unsurprising but important takeaway: our expectation is that the theory of \( L_{\text{fin}}K \) bi-invariant measures and measure classes on \( LG \) is markedly different from the theory of \( K \) bi-invariant measures and measure classes on \( G \). Nonetheless this is a useful analogy to bear in mind (This is actually more than analogy; these two examples fit into a common
framework involving symmetrizable Kac-Moody groups, as we will describe at the end of the notes).

In Section 3 we discuss how to calculate the measure $\mu_l$. This is the point where the subject comes alive. There are currently two main conjectures - one somewhat fluid, one solid - involving two completely different perspectives. Given a real analytic embedding of $S^1$ into a closed Riemann surface, there is an induced map from the hyperfunction completion to the space of holomorphic $G$ bundles on the surface. This should be viewed as an analogue of mapping a continuous loop to its value at a point. The first (somewhat tenuous) conjecture is that $\mu_0$ pushes forward to a natural normalized (Goldman) symplectic volume element on the subset of stable bundles. The second conjecture is that root subgroup coordinates are independent random variables with respect to $\mu_l$; see the formulas (3.2) and (3.25) in the $SU(2)$ and general cases, respectively. There are other interesting pushforwards and coordinates, and the relevant formulas are simply missing.

Section 4 concerns representation theory. Given a fixed positive integral level $l$, an important problem is to correctly formulate an analytic version of the (holomorphic) Peter-Weyl theorem. There is a well-known algebraic version of the Peter-Weyl theorem for arbitrary symmetrizable Kac-Moody groups due to Kac and Peterson. But this is inadequate for an application we have in mind to sewing, see below. A key ingredient is a diagonal distribution conjecture, which is a corollary of the root subgroup conjectures of the previous section. At least tentatively it appears that the decomposition of $L^2(\mu_0)$ with respect to the bi-invariant action of $L_{fin} K$ is rather dull. On the other hand a corollary of the uniqueness conjectures of Section 2 is that $C^\omega \text{Homeo}(S^1)$ fixes $\mu_0$ and hence acts unitarily on the Hilbert space of half densities. This is highly decomposable, in an interesting way, as a consequence of the first perspective in Section 3. But how to precisely decompose this action is presently unclear.

The other sections involve potential applications to two dimensional sigma and WZW models with target $K$. It is the potential of these applications which has motivated me to persist in thinking about these measures over decades.

In Section 5 I discuss the two dimensional sigma model with target $K$, i.e. the principal chiral model, for which $LK$ is the configuration space. This is a model which is classically conformally invariant and integrable, in the sense that on $\mathbb{R}^{1,1}$, or $\mathbb{R}^2$, there exists a zero curvature representation and a classical Yangian symmetry. At the quantum level it is a ‘theorem of physics’ that on the one hand conformal invariance is broken, while on the other hand a deformed Yangian symmetry survives; this is manifested by the emergence of massive particles. There exist explicit formulas for the masses and scattering matrices in terms of evaluation representations for Yangian quantum groups. This Yangian symmetry is ‘not compatible with periodic boundary conditions’. Consequently it is a longstanding puzzle to rigorously formulate the model in $RS^1 \times \mathbb{R}$ (where $R$ is the radius of the circle), or more generally against a possibly curved Riemannian background, e.g. to construct a model satisfying Segal’s axioms, and to deduce the remarkable scattering solution. In this section I posit that the measure $\mu_0$ is essentially the vacuum, and try to draw some conclusions, based on properties of $\mu_0$ outlined in previous sections. This point of view suggests the possibility that, just as a free quantum field (i.e. the chiral model with target $\mathbb{R}$) is an assembly of harmonic oscillators, when $R$ is finite, the $K$-valued chiral model might be an assembly of harmonic oscillators and
‘spherical harmonic oscillators’. This is an attractive possibility, maybe a fantasy, not a formal conjecture.

In Section 6 we discuss sewing, from a global point of view, for the chiral WZW model (a holomorphic half of the full WZW model for closed strings). It is long known that heuristically the basic sewing mechanism is the conjectural holomorphic Peter-Weyl theorem alluded to above. We spell this out.

It would be great if the analytic point of view of this notes was useful for the full WZW model (involving left and right movers), or even better, the corresponding boundary conformal field theory (for which, to my knowledge, sewing rules have not been proven). But at present I do not see this.

In the last Section there are some comments on generalizations. The basic existence result for invariant measures generalizes to (an appropriate completion of) loops into symmetric spaces. It is a challenge to properly formulate a generalization for loops into more general compact homogeneous or Einstein targets. In another direction, I originally thought the theory might extend to symmetrizable Kac-Moody algebras. This now seems doubtful, and it is enlightening to understand why.

1. Loop Groups and Completions

The polynomial loop groups $L_{fin}K$ and $L_{fin}G := G(\mathbb{C}[z, z^{-1}])$ consist of maps from $S^1$ into $K$ ($G$, respectively) which have finite Fourier series, with pointwise multiplication (relative to a fixed faithful matrix representation). Recall $\mathbb{C}[z]$ is the algebra over $\mathbb{C}$ generated by $z$, and $\mathbb{C}[z, z^{-1}]$ is the algebra of finite Laurent series. Neither of these (skeletal, or minimal) loop groups is a Lie group - they are amenable to algebra (as in [46]), but for purposes of analysis, we need to consider completions (as in [76]).

The analytic loop group, $H^0(S^1, G)$ (loops which are holomorphic in a collar containing $S^1$), is a complex Lie group. Let $D$ ($D^*$) denote the closed unit disk ($\{z : |z| \geq 1\} \cup \{\infty\}$, respectively). A dense open neighborhood of the identity in $H^0(S^1, G)$ consists of those loops which have a unique Riemann-Hilbert factorization

\begin{equation}
  g = g^- \cdot g_0 \cdot g^+
\end{equation}

where $g_- \in H^0(D^*, \infty; G, 1)$, $g_0 \in G$, and $g_+ \in H^0(D, 0; G, 1)$ (thus (1.1) is an equality of holomorphic functions which holds on a thin collar of $S^1$, the collar depending upon $g$). A model for this neighborhood is

$$H^1(D^*, g) \times G \times H^1(D, g)$$

where the linear coordinates are determined by $\theta_+ = g_+^{-1}(\partial g_+)$, $\theta_- = (\partial g_-)g_-^{-1}$. The (left or right) translates of this neighborhood by elements of $L_{fin}K$ cover $H^0(S^1, G)$.

In general there are decompositions

\begin{equation}
G(\mathbb{C}[z, z^{-1}]) = \bigcup_{\lambda \in \text{Hom}(S^1, T)} \Sigma^{G(\mathbb{C}[z, z^{-1}])}_\lambda, \quad \Sigma^{G(\mathbb{C}[z^{-1}])}_\lambda = G(\mathbb{C}[z^{-1}]) \cdot \lambda \cdot G(\mathbb{C}[z])
\end{equation}

and

\begin{equation}
H^0(S^1, G) = \bigcup_{\lambda \in \text{Hom}(S^1, T)} \Sigma^{H^0(S^1, G)}_\lambda, \quad \Sigma^{H^0(S^1, G)}_\lambda = H^0(D^*, G) \cdot \lambda \cdot H^0(D, G)
\end{equation}
where $T$ is a maximal torus of $K$. Note the ‘top stratum’ $\Sigma^{H^0(S^1, G)}_\lambda$ is the dense open neighborhood of the previous paragraph. There are finer decompositions over the affine Weyl group $W \simeq \text{Hom}(S^1, T)$, where for example the ‘top stratum’ consists of $g$ as in (1.1) such that $g_0 \in G$ has a standard LDU factorization. We will loosely refer to all of these decompositions, and generalizations below, as ‘Birkhoff decompositions’.

The global definition of the formal completion is

$$H^1(\Delta^*, g) \times G \times H^1(\Delta, g)$$

where $\Delta$ and $\Delta^*$ denote the open disks centered at 0 and $\infty$, respectively, and the transition functions are obtained by continuously extending the transition functions for the analytic loop space of the preceding paragraph. In the case of $SU(2)$, transition functions are written down explicitly in Section 3 of [68]. The global definition is

$$H^{yp}(S^1, G) = G(H^0(S^1_\pm)) \times H^{0}(S^1, G) G(H^0(S^1_\pm))$$

where $H^0(S^1_\pm)$ denote the direct limits of the spaces $H^0([r < |z| < 1])$ and $H^0([1 < |z| < r])$, as $r \uparrow 1$ and $r \downarrow 1$, respectively.

From the global definition it is clear that $H^0(\Delta, G)$ $(H^0(\Delta^*, G))$ acts naturally from the right (left, respectively) of $H^{yp}(S^1, G)$, hence the analytic loop group $H^{yp}(S^1, G)$ acts naturally from both the left and right of $H^{yp}(S^1, G)$. There is a natural action of $C^\infty\text{Homeo}(S^1)$ on $H^{yp}(S^1, G)$: given a real analytic (orientation-preserving) homeomorphism $\sigma$ and an equivalence class $[g_1, g_2] \in H^{yp}(S^1, G)$ (thus $g_1 : \{1 - \epsilon < |z| < 1\} \to G$ and $g_2 : \{1 < |z| < 1 + \epsilon\} \to G$ are defined and holomorphic for sufficiently small $\epsilon$),

$$\sigma_*([g_1, g_2]) = [g_1 \circ \sigma^{-1}, g_2 \circ \sigma^{-1}]$$

The formal completion is defined in a similar way, where $H^1(\Delta, g)$ is replaced by the corresponding space of formal power series

$$H^1_{\text{formal}}(\Delta, g) = \{\theta_+ = (\theta_1 + \theta_2 z + \ldots) dz, \ \theta_i \in \mathfrak{g}\} \simeq \prod_{i=1}^{\infty} \mathfrak{g}$$

The global definition of the formal completion is

$$\text{LG} = G(\mathbb{C}[[z^{-1}]]) \times G(\mathbb{C}[[z^{-1}]]) G(\mathbb{C}(z))$$

where $\mathbb{C}(z)$ is the field of formal Laurent series $\sum a_n z^n$, $a_n = 0$ for $n << 0$. From the global definition it is clear that $G(\mathbb{C}(z))$ $(G(\mathbb{C}((z^{-1}))))$ acts naturally from right (left, respectively) of $\text{LG}$, and hence $L_{fin} G$ acts naturally from both the left and right of $\text{LG}$. Mokler has clarified the nature of the formal completion from the point of view of algebraic geometry, see [59]. A disadvantage of the formal completion is that only the subgroup $PSU(1, 1)$ (of linear fractional transformations) of $C^\infty\text{Homeo}(S^1)$ acts as in (1.3).

These completions have generalized ‘Birkhoff decompositions’

$$H^{yp}(S^1, G) = \bigsqcup_{\lambda \in \text{Hom}(S^1, T)} \Sigma^h_{\lambda}, \ \Sigma^h = H^0(\Delta^*, G) \cdot \lambda \cdot H^0(\Delta, G),$$

and

$$\text{LG} = \bigsqcup_{\lambda \in \text{Hom}(S^1, T)} \Sigma_{\lambda}^{\text{LG}}, \ \Sigma^{\text{LG}} = G(\mathbb{C}[[z^{-1}]]) \cdot \lambda \cdot G(\mathbb{C}[[z]])$$
where $\mathbb{C}[[\zeta]]$ denotes formal power series in $\zeta$. These decompositions are compatible with the Birkhoff factorizations $[12]$ and $[13]$. It is the existence of these coherent decompositions, corresponding to different smoothness conditions, which imply that the natural inclusions

$$L_{fin}K \subset L_{fin}G \subset H^0(S^1, G) \subset C^0(S^1, G) \subset Hyp(S^1, G) \subset LG$$

(and various other analytic completions) are all homotopy equivalent (see 8.6.6 in $[76]$).

For the measure-theoretic purposes of these notes, we mainly need the top strata corresponding to $\lambda = 1$. In both the formal and hyperfunction cases, the top stratum is open and dense, and for each point $g$ in the top stratum, there is a unique factorization as in $[14]$, where in the hyperfunction case $g_{\pm}$ are $G$-valued holomorphic functions in the open disks $\Delta$ and $\Delta^*$, respectively, and in the formal case $g_{\pm}$ are simply formal power series satisfying the appropriate algebraic equations determined by $G$. We will refer to $g_-, g_0, g_+ (\theta_-, g_0, \theta_+,$ respectively) as the Riemann-Hilbert coordinates (linear Riemann-Hilbert coordinates, respectively) of $g$.

Remark. $C^0(S^1, G)$ is a complex Lie group. To obtain a coordinate neighborhood of the identity, choose a coordinate neighborhood $1 \in U \subset G$ of $1$. Then $C^0(S^1, U)$ is a coordinate neighborhood of $1 \in C^0(S^1, G)$. We cannot evaluation a generic $g \in Hyp(S^1, G)$ at a point in $S^1$. A coordinate neighborhood of $1 \in Hyp(S^1, G)$ is the top stratum with coordinates $\theta_-, g_0, \theta_+$, as in the previous paragraph; the Taylor coefficients of $\theta_{\pm}$ are smeared values of $g$, as in quantum field theory.

Given a finite set of points $V \subset S^1$, there is an evaluation map

$$eval_V : C^0(S^1, G) \times \prod_{s^1} G \to \prod_{v \in V} G : g \to (g(v))_{v \in V}$$

In Subsection 3.1 we will discuss the analogue of these projections for $Hyp(S^1, G)$ - instead of points in $S^1$, we consider analytic loops in Riemann surfaces.

Of central importance, there exists a holomorphic line bundle $\mathcal{L} \to LG$ such that the corresponding $\mathbb{C}^\times$ bundle is a completion of the universal central extension

$$0 \to \mathbb{C}^\times \to C^\infty(S^1, G) \to C^\infty(S^1, G) \to 0$$

This and the unitary form

$$0 \to \mathbb{T} \to C^\infty(S^1, K) \to C^\infty(S^1, K) \to 0$$

are described in chapter 4 of $[76]$ (here we are using the assumption that $g$ is simple). The restrictions of these extensions to $L_{fin}G$ and $L_{fin}K$, respectively, are the ‘minimal untwisted affine Kac-Moody groups’, the group structures of which are elucidated in $[40]$ in terms of generators and relations. The natural maximal unitary universal central extension is

$$0 \to \mathbb{T} \to W^{1/2}(S^1, K) \to W^{1/2}(S^1, K) \to 0$$

where $W^{1/2}(S^1, K)$ is the $(C^0$ Lie) group of equivalence classes of loops with half a derivative in the $L^2$ sense (with the natural Polish topology).

Remark. It is problematic to define a corresponding maximal complex universal central extension. The difficulty is that $W^{1/2}(S^1, G)$ is not a group, because $W^{1/2}$ does not imply boundedness. If one imposes boundedness, then for example the
Birkhoff factorization fails - the factors are not necessarily bounded. I do not know how to resolve this tension, which recurs in a number of slightly different ways.

1.0.1. Appendix: The Abelian Case. In these notes we are assuming that $\mathfrak{k} \subset \mathfrak{g}$ are simple. But it is occasionally useful to contemplate ‘the abelian case’ $i\mathbb{R} \subset \mathbb{C}$.

In this abelian case $\text{Hyp}(S^1, \mathbb{C}^\times)$ is a group; in fact it factors as a product of $\text{Hom}(S^1, S^1)$ and the identity component, which is a quotient of its Lie algebra

$$0 \to 2\pi i\mathbb{Z} \to \text{Hyp}(S^1, \mathbb{C}) \to \text{Hyp}(S^1, \mathbb{C}^\times) \to 0$$

where an ordinary hyperfunction on $S^1$ with linear triangular factorization $f = f_- + f_0 + f_+$ maps to the multiplicative hyperfunction with Riemann-Hilbert factorization $\exp(f) = \exp(f_-) \exp(f_0) \exp(f_+)$. Moreover there is a unitary form, because it makes sense to say that $f$ is real, i.e. $f_0 \in \mathbb{R}$ and $f_+ = -f_-$.

1.1. Supplementary Notation. For the most part these notes should be intelligible using the notation that we have established to this point. For this to be possible, at several points we specialize to $K = SU(2)$. However at some points, e.g. when we discuss root subgroup factorization for higher rank groups, it will be necessary to use more structure.

A venerable source for notational conventions for Kac-Moody algebras is naturally [44]. However, at least in my view, [18] has made a few improvements. One difference between these two sources: Kac employs the convention of denoting a finite dimensional algebra, and the associated structure, using dots, and denoting the associated (untwisted) affine extension, and the associated structure, with the absence of dots, whereas Carter uses $(\cdot)^0$ in place of a dot. I have found it convenient to stick with Kac’s convention. One other note: I like to use $r$ for rank and $l$ for level.

1.1.1. Finite Type Structure. Following Kac’s convention, $\hat{K}$ is a simply connected compact Lie group with simple Lie algebra $\mathfrak{t}$, $\mathfrak{g}$ is the complexification, and $\mathfrak{g} \to \mathfrak{g} : x \to -x^\ast$ is the anticomplex involution fixing $\mathfrak{k}$.

Fix a triangular decomposition

$$(1.5) \quad \hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$$

such that $\mathfrak{i} = \mathfrak{k} \cap \hat{\mathfrak{h}}$ is maximal abelian in $\mathfrak{t}$; this implies $(\hat{\mathfrak{n}}^\ast)^+ = \hat{\mathfrak{n}}^-$. We introduce the following standard notations: for each positive root $\hat{\alpha}$, $\hat{\mathfrak{h}}_{\hat{\alpha}}$ denotes the corresponding coroot; $\{\hat{\alpha}_j : 1 \leq j \leq r\}$ is the set of simple positive roots; $\{\hat{h}_j := \hat{\mathfrak{h}}_{\hat{\alpha}_j}\}$ is the set of simple coroots; $\{\hat{\Lambda}_j\}$ is the set of fundamental dominant integral weights; $\hat{\theta}$ is the highest root; $\hat{W}$ is the Weyl group; $\langle\cdot,\cdot\rangle$ is the ‘standard invariant symmetric bilinear form’, uniquely determined by the condition (for the dual form) $\langle\hat{\theta},\hat{\theta}\rangle = 2$; and

$$(1.6) \quad \hat{\rho} := \frac{1}{2} \sum_{\hat{\alpha} > 0} \hat{\alpha} = \sum_{i=1}^{r} \hat{\Lambda}_i := \sum_{j=1}^{r} \hat{\alpha}_j \hat{\alpha}_j \quad \hat{h}_{\hat{\theta}} = \sum_{j=1}^{r} c_j \hat{h}_j$$

Remark. The integers $\hat{a}_j$ and $c_j$, among other things, are conveniently tabulated in the summary at the end of [18]. However there is a silly error in the summary regarding the relation of the standard form and the Killing form: the correct expression should be

$$\langle\cdot,\cdot\rangle = \frac{1}{2g} \kappa(\cdot,\cdot)$$
where $\hat{g} := 1 + \sum_{j=1}^{\kappa} \hat{a}_j$ is the dual Coxeter number and $\kappa$ is the Killing form.

For each simple root $\gamma$, fix a root homomorphism $i_\gamma : sl(2,\mathbb{C}) \rightarrow \hat{\mathfrak{g}}$ (we denote the corresponding group homomorphism by the same symbol), and let
\[
f_\gamma = i_\gamma\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad e_\gamma = i_\gamma\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \quad \text{and } r_\gamma = i_\gamma\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \in \hat{T} = \exp(\mathfrak{i});
\]
$r_\gamma$ is a representative for the simple reflection $r_\gamma \in \hat{W}$ corresponding to $\gamma$ (we will adhere to the convention that representatives for Weyl group elements will be denoted by bold letters).

Introduce the lattices
\[
\hat{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z}\hat{\Lambda}_i \quad \text{(weight lattice), and } \quad \hat{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z}\hat{\lambda}_i \quad \text{(coroot lattice)}
\]
These lattices and bases are in duality. Recall that the kernel of $\exp : \mathfrak{i} \rightarrow \hat{T}$ is $2\pi i$ times the coroot lattice. Consequently there are natural identifications
\[
\hat{T} \rightarrow Hom(\hat{T}, \mathbb{T})
\]
where a weight $\hat{\Lambda}$ corresponds to the character $\exp(2\pi ix) \rightarrow \exp(2\pi i\hat{\Lambda}(x))$, for $x \in \mathfrak{h}_\mathbb{R}$, and
\[
\hat{T} \rightarrow Hom(\mathbb{T}, \hat{T}),
\]
where an element $h$ of the coroot lattice corresponds to the homomorphism $\mathbb{T} \rightarrow \hat{T} : \exp(2\pi ix) \rightarrow \exp(2\pi ixh)$, for $x \in \mathbb{R}$. Also
\[
\hat{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z}\hat{\alpha}_i \quad \text{(root lattice), and } \quad \hat{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z}\hat{\Theta}_i \quad \text{(coweight lattice)}
\]
where these bases are also in duality. The $\hat{\Theta}_i$ are the fundamental coweights.

The affine Weyl group is the semidirect product $\hat{W} \rtimes \hat{T}$. For the action of $\hat{W}$ on $\mathfrak{h}_\mathbb{R}$, a fundamental domain is the positive Weyl chamber $C = \{ x : \hat{\alpha}_i(x) > 0, i = 1, \ldots, r \}$. For the natural affine action
\[
W \rtimes \hat{T} \times \mathfrak{h}_\mathbb{R} \rightarrow \mathfrak{h}_\mathbb{R}
\]
a fundamental domain is the convex set
\[
C_0 = \{ x \in C : \hat{\theta}(x) < 1 \} \quad \text{(fundamental alcove)}
\]
The set of extreme points for the closure of $C_0$ is $\{ 0 \} \cup \{ \frac{1}{\pi} \hat{\Theta}_1 \}$. 

The Lie algebra $\hat{\mathfrak{g}}$ is graded by height, where the height of a simple positive root vector is one. In general, for a positive root $\hat{\alpha}$, $\hat{\alpha}(\hat{h}_\mathbb{R}) = \text{height}(\hat{\alpha})$. When $\hat{\mathfrak{g}}$ is simply laced, i.e. all roots have the same length, this is equal to $\hat{\delta}(\hat{h}_\mathbb{R})$

Let $\hat{N}^\pm = \exp(\hat{h}_\mathbb{R})$ and $\hat{A} = \exp(\hat{\mathfrak{h}}_\mathbb{R})$. An element $g \in \hat{N}^- \hat{T} \hat{A} \hat{N}^+$ has a unique triangular decomposition
\[
g = \hat{1}(g)\hat{d}(g)\hat{u}(g), \quad \text{where } \hat{d} = \hat{r}\hat{a} = \prod_{j=1}^{r} \hat{\sigma}_j(g)^{\hat{h}_j},
\]
and $\hat{\sigma}_i(g) = \phi_{\hat{\Lambda}_i}(\pi_{\hat{\Lambda}_i}(g)\nu_{\hat{\Lambda}_i})$ is the fundamental matrix coefficient for the highest weight vector corresponding to $\hat{\Lambda}_i$. 

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1.1.2. Affine Algebra Structure. Let \( L_\hat{g} = C^\infty(S^1, \hat{g}) \), viewed as a Lie algebra with pointwise bracket. There is a universal central extension

\[
0 \to \mathbb{C}c \to \tilde{L}_\hat{g} \to L_\hat{g} \to 0,
\]

where as a vector space \( \tilde{L}_\hat{g} = L_\hat{g} \oplus \mathbb{C}c \), and in these coordinates

\[
[X + \lambda c, Y + \lambda' c]|_{\tilde{L}_\hat{g}} = [X, Y]_{L_\hat{g}} + \frac{i}{2\pi} \int_{S^1} (X \wedge dY)c.
\]

The smooth completion of the untwisted affine Kac-Moody Lie algebra corresponding to \( \hat{g} \) is

\[
\tilde{L}_\hat{g} = \mathbb{C}d \rtimes \hat{L}_\hat{g} \quad \text{(the semidirect sum),}
\]

where the derivation \( d \) acts by \( d(X + \lambda c) = \frac{d}{d\theta} X \), for \( X \in L_\hat{g} \), and \([d,c] = 0 \). The algebra generated by \( \hat{k} \) and \( d \) is the central extension

\[
0 \to i\mathbb{R}c \to \hat{L}k \to \hat{L}k \to 0
\]

and a real form \( \hat{L}k = i\mathbb{R}d \rtimes \hat{L}k \) for \( \tilde{L}_\hat{g} \).

We identify \( \hat{g} \) with the constant loops in \( \tilde{L}_\hat{g} \). Because the extension is trivial over \( \hat{g} \), there are embeddings of Lie algebras

\[
\hat{g} \to \tilde{L}_\hat{g} \to \tilde{L}_\hat{g}.
\]

There are triangular decompositions

\[
(1.10) \quad \tilde{L}_\hat{g} = n^- \oplus h \oplus n^+ \quad \text{and} \quad \tilde{L}_\hat{g} = n^- \oplus (Cd + h) \oplus n^+,
\]

where \( h = \hat{h} + \mathbb{C}c \) and \( n^+ \) is the smooth completion of \( \hat{n}^+ + \hat{g}(z^{\pm 1}C[z^{\pm 1}]) \), respectively. The simple roots for \( (\tilde{L}_\hat{g}, Cd + h) \) are \( \{\alpha_j : 0 \leq j \leq r\} \), where

\[
\alpha_0 = \delta - \hat{\theta}, \quad \alpha_j = \hat{\alpha}_j, \quad j > 0,
\]

\( \delta(d) = 1, \delta(c) = 0, \delta(h) = 0 \), and the \( \hat{\alpha}_j \) are extended to \( \mathbb{C}d + h \) by requiring \( \hat{\alpha}_j(c) = \hat{\alpha}_j(d) = 0 \). We also crucially introduce a linear functional \( \gamma \) which vanishes on \( \hat{h} \) and satisfies \( \gamma(c) = 1, \gamma(d) = 0 \). The simple coroots are \( \{h_j : 0 \leq j \leq r\} \), where

\[
h_0 = c - \hat{h}_\theta, \quad h_j = \hat{h}_j, \quad j > 0.
\]

Note that \( c = h_0 + \sum_{j=1}^r c_j \hat{h}_j \) (which explains Carter’s convention that \( \hat{h}_\theta = \sum_{j=1}^r c_j \hat{h}_j \)).

For a dominant integral weight \( \Lambda \), \( \Lambda(c) = l(\Lambda) \) is the level. In general

\[
(1.11) \quad \Lambda = \sum_{i=0}^r \Lambda(h_j)\Lambda_j = l\gamma + \hat{\Lambda}
\]

where the \( \Lambda_j, j = 0, ..., r, \) are the fundamental dominant integral weights, and given the level \( l, \Lambda \) is uniquely determined by \( \hat{\Lambda} \), a dominant integral weight on \( h_\mathbb{R} \) (the condition that \( \Lambda(h_0) = l - \hat{\Lambda}(h_\theta) \geq 0 \) is equivalent to \( \Lambda(h_\theta) \leq l \)).

For \( i > 0 \), the root homomorphism \( i_{\alpha_i} \) is \( i_{\hat{\alpha}_i} \), followed by the inclusion \( \hat{g} \subset \tilde{L}_\hat{g} \).

For \( i = 0 \)

\[
(1.12) \quad i_{\alpha_0}\left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right) = e_\theta z^{-1}, \quad i_{\alpha_0}\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) = f_\theta z,
\]

where \( \{f_\theta, \hat{h}_j, e_\theta\} \) satisfy the \( sl(2, \mathbb{C}) \)-commutation relations, and \( e_\theta \) is a highest root vector for \( \hat{g} \). The fundamental dominant integral functionals on \( h \) are \( \Lambda_j, j = 0, ..., r, \) where \( \Lambda_i(h_j) = \delta(i - j) \) and \( \Lambda_i(d) = 0 \).
Also set \( t = iRc \oplus i \) and \( a = hR = Rc \oplus hR \). Finally
\[
\rho := \sum_{i=0}^{r} \Lambda_i = \hat{\delta} + \hat{\gamma}
\]
where (to repeat) the dual Coxeter number \( \hat{\gamma} = 1 + \sum_{i=1}^{r} \hat{a}_j \).

1.1.3. Loop Groups and Extensions. Let \( \Pi : \tilde{L}G \to LG \) (\( \Pi : \tilde{L}K \to LK \)) denote the universal central \( \mathbb{C}^* \) (\( \mathbb{T} \)) extension of the smooth loop group \( LG \) (\( LK \), respectively), as in [76]. Let \( N^\pm \) denote the subgroups corresponding to \( n^\pm \). Since the restriction of \( \Pi \) to \( N^\pm \) is an isomorphism, we will always identify \( N^\pm \) with its image, e.g. \( l \in N^+ \) is identified with a smooth loop having a holomorphic extension to \( \Delta \) satisfying \( l(0) \in \hat{N}^+ \). Also set \( T = \exp(t) \) and \( A = \exp(a) \).

As in the finite dimensional case, for \( \tilde{g} \in N^+ \cdot TA \cdot N^+ \subset \tilde{L}G \), there is a unique triangular decomposition
\[
\tilde{g} = l \cdot d \cdot u, \quad \text{where} \quad d = ma = \prod_{j=0}^{r} \sigma_j(\tilde{g})^{h_j},
\]
and \( \sigma_j = \sigma_{\Lambda_j} \) is the fundamental matrix coefficient for the highest weight vector corresponding to \( \Lambda_j \). If \( \Pi(\tilde{g}) = g \), then because \( \sigma_0^{h_0} = \sigma_0^{-h_0} \) projects to \( \sigma_0^{h_0} \), \( g = l \cdot \Pi(d) \cdot u \), where
\[
\Pi(d)(g) = \sigma_0(\tilde{g})^{-h_0} \prod_{j=1}^{r} \sigma_j(\tilde{g})^{h_j} = \prod_{j=1}^{r} \left( \frac{\sigma_j(\tilde{g})}{\sigma_0(\tilde{g})^{a_j}} \right)^{h_j},
\]
and the \( \hat{a}_j \) are positive integers such that \( \hat{h}_0 = \sum \hat{a}_j \hat{h}_j \) (these numbers are also compiled in Section 1.1 of [76]).

If \( \tilde{g} \in \tilde{L}K \), then \( |\sigma_j(\tilde{g})| \) depends only on \( g = \Pi(\tilde{g}) \). We will indicate this by writing
\[
|\sigma_j(\tilde{g})| = |\sigma_j(g)| \quad \text{and} \quad a(\tilde{g}) = a(g),
\]
where \( a(\tilde{g}) \) is defined as in (1.14).

2. Characterizing Measures and Measure Classes

The first conjecture states that there is a very simple characterization of the measure \( \mu_0 \) and various deformations:

**Conjecture 1.** There exists a unique Borel probability measure \( \mu_0 \) on \( LG \) which is bi-invariant with respect to \( L_{fin}K \).

More generally there exists an essentially unique Borel measure (denoted \( \mu_{|L|^{2l}} \)) with values in the positive line bundle \( |L|^{2l} := (L \otimes \mathcal{L})^{\otimes l} \to LG \) which is bi-invariant with respect to \( L_{fin}K \), for \( l > -1 \).

**Remarks.** (a) Fix \( l \in \mathbb{N} \). At a heuristic level, given sections \( \sigma_i \) of \( \mathcal{L}^{\otimes l} \)
\[
\sigma_1 \otimes \sigma_2 d\mu_{|L|^{2l}} = \frac{1}{3} \langle \sigma_1, \sigma_2 \rangle d\mu_0
\]
where \( \langle \cdot, \cdot \rangle \) is a unitarily invariant Hermitian structure for \( \mathcal{L}^{\otimes l} \). This structure is only defined on unitary loops of order \( W^{1/2} \), which has \( \mu_0 \) measure zero. So we have to think of \( d\mu_{|L|^{2l}} \) as having values in \( |L|^{2l} \).
(b) In place of \( \mu |L|^2 \), it is often convenient to consider the coordinate expression
\[
d\mu_l := |\sigma_0|^2 \mu |L|^2
\]
where \( \sigma_0 \) is the ‘fundamental matrix coefficient’; \( \mu_l \) is an ordinary Borel probability measure - an implicit normalization - which is bi-quasi-invariant with respect to translations by \( L_{\text{fin}}K \), and having a Radon-Nikodym derivative with a specific form determined by the line bundle. We will clarify this when we write down explicit formulas in the next section.

(c) To put this conjecture into perspective, compare this with the finite dimensional analogue of \( K \) acting from the left and right of \( G \). For this action the \( K \) Haar measure is far from the unique \( K \times K \) invariant probability measure on \( G \). At one extreme the \( K \) Haar measure has minimal support. At the opposite extreme there are \( K \) bi-invariant measures which are absolutely continuous with respect to the Haar measure for \( G \). In general \( K \) bi-invariant probability measures can be classified using the spherical (or Harish-Chandra) transform; see chapter IV of [40].

If the conjecture is false, then there must be an affine analogue of the spherical transform which classifies the possibilities, see Subsection 5.4 for a possible analogue.

(d) For some perspective on the centrality of this conjecture, see Subsection 7.3 at the very end of these notes.

One practical reason this conjecture is important is that there are multiple approaches to constructing the measure \( \mu_0 \) and its deformations. It is not a priori clear they yield the same measure. For example in both [66] and [68], the basic strategy is to show that Wiener measure on \( C^0(S^1, K) \subset LG \) has weak limits in a coordinate system for \( LG \) as the inverse temperature parameter \( \beta \to 0 \). It is not known that the limit is unique. A crucial ingredient in the proof of the bi-invariance of these limits is the asymptotic invariance of Wiener measure (as \( \beta \to 0 \)), established earlier by Marie and Paul Malliavin (see [58] and section 4.1 of Part III of [66]). Conjecturally one can also use heat kernel measures (denoted \( \nu_t(s) \)) which are parameterized by a degree of smoothness \( s > 1/2 \) and temperature (or time) \( t \), as \( t \to \infty \) (see [70]). Asymptotic invariance has not been proven for this limit (It is possible that one can also use the limit \( s \downarrow 1/2 \), but this is a more subtle question which I will return to later). In the next section we will suggest another possible construction, using explicit formulas; again, it is not clear this will yield the same measure.

Conjecture 1 implies a number of statements which will probably have to be resolved first, e.g.

**Conjecture 2.** There exists a unique \( L_{\text{fin}}K \)-invariant Borel probability measure on the fundamental ‘homogeneous space’
\[
LG/G(C[[z]]) = G(C((z^{-1}))/G(C[z])
\]

There is a similar statement involving level \( l \) and line bundles.

To clarify the statement, the space in question is an equivariant quotient of \( LG \), for the left action of \( L_{\text{fin}}K \), hence there does exist an invariant measure as in the first part of the statement (\( \mu_0 \) can be pushed forward). To prove Conjecture 1 it is necessary, in linear Riemann-Hilbert coordinates, to show that invariance determines the joint distribution for \( \theta_-, g_0 \) and \( \theta_+ \); in Conjecture 2 it suffices to
show that invariance determines the distribution for $\theta_-$. One can argue that this is more plausible than Conjecture 1 because the finite dimensional analogue, $K$ acting on $K/T = G/B^+$, does have a unique invariant measure, whereas this is not true for $K \times K$ acting on $G$.

I made an attempt to prove uniqueness of $\mu_0$ on the basic homogeneous space (in the case $K = SU(2)$) in section 2 of [68], using a series of computable distributions for $\mu_0$. I do not know if I failed for lack of persistence, or if this approach is doomed.

Here is a more technical version of Conjecture 2, involving line bundle valued measures.

Suppose that $\Lambda$ is a dominant integral functional and consider the Borel subgroup $B^+ := \{ \hat{g}_+ \in \hat{G}(\mathbb{C}[z]) : g_+(0) \in \hat{B}^+ \}$

$\Lambda$ defines a character $B^+$, and in turn there is a holomorphic line $L_\Lambda := \hat{G}(\mathbb{C}(z^{-1})) \times B^+ \mathbb{C}$

Conjecture 3. (a) There exists a unique $L_{fin}K$-invariant measure $\mu_{|L\Lambda|^2}$ having values in the positive line bundle $|L\Lambda|^2$ which is normalized such that

$$
\mu_\Lambda := \sigma_\Lambda \otimes \sigma_{L\Lambda}|L\Lambda|^2
$$

is a probability measure.

(b) When $\Lambda = I\Lambda_0 = I\gamma$, then $\mu_\Lambda = \mu_I$ (this is trivial).

(c) If $\Lambda$ has level $l$, then $[\mu_\Lambda] = [\mu_l]$, i.e. the measure class depends only on the level.

The following refers to the natural action of analytic homeomorphisms of $S^1$ on the hyperfunction completion of the loop group, [13].

Conjecture 4. The measure $\mu_0$ is supported on the hyperfunction completion $\text{Hyp}(S^1, G)$. It is bi-invariant with respect to $H^0(S^1, K)$ and invariant with respect to $C^\omega\text{Homeo}(S^1)$.

More generally the measure $\mu_{|L\Lambda|^2}$ is supported on the hyperfunction completion. It is bi-invariant with respect to the natural actions of $C^\omega(S^1, K)$ and $C^\omega\text{Homeo}(S^1)$.

These measures have the property that the associated unitary representations extend continuously to $W^{1/2}(S^1, K) \times W^{1/2}(S^1, K)$ and $W^{1+1/2}\text{Homeo}(S^1)$.

In reference to the first claim, we will gain more insight into the support of the measure $\mu_0$ when we compute in the next section. The use of the hyperfunction completion is akin to asserting that typical Brownian paths are continuous - we can say more, but this is enough for many purposes.

Note that $PSU(1,1) \subset C^\omega\text{Homeo}(S^1)$, and $PSU(1,1)$ acts in a natural way on $\theta_\pm$, in linear Riemann-Hilbert coordinates. Thus we are asserting that the distributions for $\theta_\pm$ are conformally invariant, a very strong statement that we can potentially check against specific calculations.

A measure having values in a line bundle determines a measure class on the base.

Conjecture 5. The measure classes arising in the previous conjecture exhaust all ergodic $L_{fin}K$ bi-invariant measure classes on $LG$ with the property that the natural associated unitary representation of $L_{fin}K \times L_{fin}K$ (realized naturally, on half-densities for the measure class) extends continuously to a strong operator continuous unitary representation of $W^{1/2}(S, K) \times W^{1/2}(S, K)$.
The continuity requirement is essential. For example Wiener measures and heat kernel measures determine $L^\infty K$ bi-invariant measure classes which are disjoint from the measure classes in the previous conjectures. $W^{1/2}$ loops are akin to the Cameron-Martin Hilbert space associated to a Gaussian measure on a topological vector space.

This conjecture implies that any such invariant measure class, which a priori has nothing to do with line bundles, has a representative which is invariant in a natural sense related to the universal central extension of $LK$.

**Question 1.** Is $\mu_0$ the unique probability measure on $Hyp(S^1, G)$ which is fixed by $K \times C^\omega Homeo(S^1) \times K$?

### 3. Computing Invariant Measures

#### 3.1. Pushforwards to Moduli Spaces.

A first problem: The uniqueness conjectures of the previous section lead to a test. Given any compact manifold $X$ and a strong operator continuous representation for $Diff(X)$, Shimomura ([80]) showed that the space of smooth vectors is dense. Assuming the truth of Conjectures 4 and 5, there is a unitary representation of the group of smooth homeomorphisms on $L^2(\mu_0)$ (or more invariantly, on half densities). Hence there should be smooth vectors (it does not make sense to talk about analytic vectors; diffeomorphisms of a manifold is not an analytic Lie group, see [53]).

The measure $\mu_0$ is a limit of Wiener measures. One way to think about Wiener measure $\nu_\beta$ is the following. Suppose that $V$ is a finite set of vertices around $S^1$.

There is then a group evaluation homomorphism

$$eval_V : C^0(S^1, K) \to \prod_v K : g \to (g(v))_{v \in V}$$

and

$$(eval_V)_*(\nu_\beta) = \frac{1}{p_{2\pi t}(1)} \prod_{e \in E} p_t(l(e)) (g_0e) \prod_{v \in V} d\lambda_K(g_v)$$

where $t = 1/\beta$, $l(e)$ denotes the length of an edge $e$, $p_t$ is the heat kernel on $K$, and $d\lambda_K$ is normalized Haar measure on $K$. These projections are coherent (as $V$ varies), and Wiener measure is determined and essentially defined by these projections. The pullbacks of smooth functions are smooth vectors.

A second problem: When we pass to the limit $\beta \downarrow 0$, it is no longer true that we can evaluate generalized loops at points, almost surely relative to $\mu_0$. Instead we look at the natural functions which are defined on the hyperfunction completion. This will solve both of our problems.

Suppose that $\Sigma$ is a compact Riemann surface, and let $c : S^1 \to \Sigma$ be an embedded analytic loop in $\Sigma$. The map $c$ extends uniquely to a bi-holomorphic embedding $c : \{1 - \epsilon < |z| < 1 + \epsilon\} \to \Sigma$ for some $\epsilon > 0$. Given a pair $(g, h)$ representing $[g, h] \in Hyp(S^1, G)$, we obtain a holomorphic bundle on $\Sigma$ by using $g$ as a transition function on an $\epsilon'$-collar to the left of $c$ and $h$ as a transition function on an $\epsilon'$-collar to the right of $c$, for some $\epsilon' < \epsilon$, depending upon the pair $(g, h)$. The isomorphism class of this bundle is independent of the choice of $\epsilon'$, and depends only upon $[g, h] \in Hyp(S^1, G)$. This can be summarized as follows:

**Proposition 1.** (a) There is an induced map

$$P(c) : Hyp(S^1, G) \to Bun_G(\Sigma)$$
where a $G$ valued hyperfunction $g$ maps to the holomorphic $G$ bundle defined by using $g$ for transition functions, and $Bun_G(\Sigma)$ denotes the set (or stack in algebraic geometry) of holomorphic $G$ bundles.

(b) There is a holomorphic action

$$H^0(\Sigma \setminus \text{Im}(c), G) \times \text{Hyp}(S^1, G) \to \text{Hyp}(S^1, G) : f, [g, h] \mapsto [f|_{S^1} \circ cg, hf|_{S^1}^{-1} \circ c],$$

and the mapping $P(c)$ induces an isomorphism of sets

$$\text{Hyp}(S^1, G)/c^*H^0(\Sigma \setminus \text{Im}(c), G) \to Bun_G(\Sigma)$$

(c) If $\sigma \in C^\infty\text{Homeo}(S^1)$, then the induced map

$$\text{Hyp}(S^1, G) \xrightarrow{\sigma} \text{Hyp}(S^1, G) \xrightarrow{P(c)} Bun_G(\Sigma)$$

equals $P(c \circ \sigma^{-1})$.

A fundamental theorem of Narasimhan and Seshadri, and Ramanathan, asserts that the open dense subset $Bun_G^0(\Sigma)$ of stable $G$ bundles can be identified real analytically with the set of irreducible homomorphisms $\pi_1(\Sigma) \to K$, modulo conjugation by $K$ (see [39] for an alternate proof and references). Via this identification, $Bun_G^0(\Sigma)$ supports a unique mapping class group invariant Borel probability measure, the normalized Goldman symplectic volume element (see [67]).

**Conjecture 6.** (a) The pushforward $P(c)_*\mu_0$ is equal to the unique mapping class group invariant probability measure on $Bun_G^0(\Sigma)$.

(b) Given a smooth function $f$ with compact support in $Bun_G^0$, $P(c)^*f$ is a smooth vector for the action of $C^\infty\text{Homeo}(S^1)$ on $L^2(\mu_0)$.

(c) There are similar conjectures involving $\mu|_{C^\infty(\Sigma)}$ and the $l$th power of a positive line bundle on the moduli space.

**Remark.** (i) One can reformulate (a) in the following way. Given $c$, project $\nu_{\beta}$ to the moduli space. It seems obvious, but is not so easy to prove, that $P(c)_*\nu_{\beta}$ is in the Lebesgue class. Now let $\beta \downarrow 0$, i.e. increase the entropy. The limit should be as information free as possible. It seems clear, but I do not have a proof, that the limit should be in the Lebesgue class and invariant with respect to the subgroup of the mapping class group fixing the homotopy class of $c$. This would imply that the density is a central function of $(\rho, [c])$, the pairing of $\rho \in H^1(\Sigma, K)$ and the homotopy class of $c$. Thus the crux of the matter seems to be the question, is this density equal to one, as I have conjectured?

(ii) This form of (a) in the first remark makes sense for an abelian group such as the circle $T$ in place of $K$. In this abelian case $P(c)$ (restricted to the identity component, see [140, 0.1]) is a group homomorphism

$$\text{Hyp}(S^1, C^\times)_0 \to H^1(\mathcal{O}_{C^\times})_0$$

onto a compact torus (here and in what follows, $\mathcal{O}$ denotes the sheaf of holomorphic functions). The Lie algebra covering map

$$\text{Hyp}(S^1, C) = H^0(S^1)^* \to H^1(\mathcal{O}) = H^{0,1}(\Sigma) \cong H^{1,0}(\Sigma)^*$$

is the dual of the restriction map

$$H^{1,0}(\Sigma) \to \mathcal{O}(S^1) : \theta \mapsto c^*\theta \frac{dz}{dz}$$

where $z$ is the usual complex coordinate along $S^1$ (see Section 1 of [39]).
Asymptotic invariance of Wiener measure implies that the pushforward of measures is asymptotically translation invariant. Hence in this reformulation, in this abelian context (and we restrict attention to the identity component of the loop group), the limit as $\beta \downarrow 0$ is the normalized Haar measure for the image of $P(c)$. This image is all of the identity component for the moduli space precisely when the image of $c$ is not a ‘straight line’; see the appendix to [9].

Part (b) of the conjecture raises a critical completeness question:

**Question 2.** Do functions as in (b) of Conjecture 4 span a dense subspace of $L^2(\mu_0)$? Below we will note there are similar smooth vectors associated to parabolic bundles, and perhaps this generalization is essential to obtain a dense subspace.

We now want to expand on this to propose a direct construction of $\mu_0$, in analogy with the Wiener measure construction above.

Suppose that $C$ is a finite collection of analytically parameterized embedded loops $c : S^1 \to \Sigma_c$. There is then a map

$$P(C) : Hyp(S^1, G) \rightarrow \prod_{c \in C} \text{Bun}_G(\Sigma_c) : g \rightarrow (P(c)(g))$$

**Conjecture 7.** Suppose that there does not exist a Riemann surface isomorphism of $\Sigma_{c_1}$ and $\Sigma_{c_2}$ which maps $\text{Im}(c_1)$ to $\text{Im}(c_2)$ for distinct $c_1, c_2 \in C$. Then the pushforward $P(C)_* \mu_0$ is in the Lebesgue class for $\prod_{c \in C} \text{Bun}_G(\Sigma_c)$.

In the cases which we have excluded in this conjecture, the map $Hyp(S^1, G) \rightarrow \prod_{c \in C} \text{Bun}_G(\Sigma_c)$ is not surjective, hence $[\mu_0]$ will not project to the Lebesgue class. It is possible that a stronger hypothesis is needed, e.g. maybe it is necessary to require that the free homotopy classes of the $c_i$ are distinct.

Assuming the truth of this conjecture, the basic task is to compute the density $\delta_{\mathcal{C}}(\rho_1, \ldots, \rho_n)$ on $H^1(\Sigma, K) \times \ldots \times H^1(\Sigma, K)$, where we are identifying stable bundles with irreducible $K$ representations of the fundamental group, using Narasimhan-Seshadri-Ramanathan, and we use the product of of the mapping class group invariant measures as background.

**Question 3.** Suppose that the surfaces $\Sigma_c$ are topologically distinct. Is $\delta_{\mathcal{C}} = 1$, implying that the distributions $c* \mu_0$ are independent, $c \in C$? More generally if $C$ is partitioned into subsets $C_1, \ldots, C_n$ corresponding to topologically distinct surfaces, are the $P(C_i)_* \mu_0$ independent?

In general $\delta_{\mathcal{C}}$ will depend on some kind of interaction of the loops, but what is the nature of this interaction? For example a priori the density $\delta_{\mathcal{C}}$ depends on how the loops are parameterized. If the measure $\mu_0$ is invariant with respect to reparameterizations, then one can simultaneously reparameterize all of them, but not necessarily individually. The simplest hypothesis is the following: If $C = \{c_1, \ldots, c_n\}$, then we obtain $n$ random $\rho_i \in \text{Hom}(\pi_1(\Sigma), K)$ (modulo conjugation by $K$) for each $i$, where $\rho_i$ corresponds to $P(c_i)(g)$, $g \in Hyp(S^1, G)$ distributed according to $\mu_0$. Maybe the density only depends on the $n^2$ elements of $K$, $\rho_i([c_j])$, where $[c_j]$ denotes the homotopy class of $c_j$. This would be a stunning simplification. However I do not see some natural formula for the density emerging.

Suppose that $l$ is positive level. One can pose the analogous question for $\mu_l^{[\mathcal{C}]}$, although the basic intuition seems to be lacking for a measure having values in a line bundle.
3.1.1. Parabolic Reductions. There is an obvious, but technically demanding, generalization of these conjectures in which the \( \Sigma \) are replaced by punctured surfaces with parabolic markings. This may be essential to know that we are actually obtaining a dense subspace of functions. It also leads to some consistency checks. We refer to [83] for a clear exposition of the background.

Fix a triangular decomposition for \( g \). A parabolic subgroup is a Lie subgroup \( P \subset G \) such that \( B^+ \subset P \) (A theorem of Tits asserts that parabolic subgroups of the triple \( (G, B^+, H) \) are in bijective correspondence with subsets of the positive simple roots).

Assume that \( \Sigma \) is a closed Riemann surface, and that \( \{z_1, ..., z_n\} \) is a finite set of points, and to each point \( z_i \), there is an associated parabolic subgroup \( P_i \). If we are given an analytic embedding \( c : S^1 \to \Sigma \setminus \{z_i\} \), then there are projections

\[
Hyp(S^1, G) \to Hyp(S^1, G)/H^0(\Sigma \setminus Im(c), z_1, ..., z_n; G, P_1, ..., P_n)
\]

\[
\to Hyp(S^1, G)/H^0(\Sigma \setminus Im(c), z_1, ..., z_{n-1}; G, P_1, ..., P_{n-1})
\]

\[
\to ... \to Hyp(S^1, G)/H^0(\Sigma \setminus Im(c), G) = Bun_G(\Sigma)
\]

The measure \( \mu_0 \) has to push forward to a coherent family of probability measures on these quotients, and our aim is to identify these measures. This will incidentally generate some consistency checks.

Recall that given \( (\Sigma, z_1, ..., z_n) \) and associated parabolic subgroups \( P_i \), a parabolic bundle is a \( G \)-bundle \( P \to \Sigma \) together with, for each \( i \), a fixed isomorphism \( P_{z_i}/P_i \to G/P_i \). Two parabolic bundles are isomorphic if there is an isomorphism of holomorphic bundles, i.e. a \( G \)-equivariant holomorphic bijection \( F : P \to P' \), such that for each \( i \) there are commuting diagrams

\[
P_{z_i}/P_i \quad \downarrow \quad G/P_i
\]

\[
P_{z_i}/P_i \quad \uparrow
\]

Proposition 2. Using a hyperfunction as a (pair of) transition functions (as at the top of this section) induces a natural identification

\[
Hyp(S^1, G)/H^0(\Sigma \setminus Im(c), z_1, ..., z_n; G, P_1, ..., P_n) \to Bun_G(\Sigma, z_1, ..., z_n; P_1, ..., P_n)
\]

the set of isomorphism classes of parabolic bundles with the given marked points.

In order to define stable, and semistable, for parabolic bundles, it is necessary to introduce more structure. Given \( P_i \) we assume that there is a fixed conjugacy class \( C_i \) with the property that it is related to \( P_i \) in the following way. There are bijective correspondences

\[
K/conj \leftrightarrow T/\hat{W} \leftrightarrow t/W
\]

where \( \hat{W} \) is the Weyl group of \( (K, T) \) and \( W = \hat{W} \ltimes \hat{T} \) is the affine Weyl group. The last space is the closure of the fundamental positive alcove. \( C_i \) represents a point in these identified spaces, and we additionally fix a representative \( x_i \) for \( C_i \) in \( t \). We require that the inclusion \( K/C_K(x_i) \to G/P_i \) is an isomorphism. Using this one can define stable and semistable for parabolic bundles; see [83]. But we will bypass this.
Suppose that we are given a representation
\[ g : \pi_1(\Sigma \setminus \{z_i\}, z_0) \to K \]
with the property that for fixed closed loops \( \gamma_i \) surrounding the points \( z_i, g(\gamma_i) \in C_i \).

The inclusion
\[ \Sigma \setminus \{z_i\}, z_0 \to (\Sigma, z_0) \]
induces a map
\[ \pi_1(\Sigma \setminus \{z_i\}, z_0) \to \pi_1(\Sigma, z_0) \]
Using the homomorphism \( g \), there is an induced holomorphic bundle
\[ \tilde{\Sigma} \setminus \{z_i\} \times \pi_1(\Sigma \setminus \{z_i\}, z_0) G \]
(the tilde indicates the universal covering) together with a reduction
\[ \tilde{\Sigma} \setminus \{z_i\} \times \pi_1(\Sigma \setminus \{z_i\}, z_0) K \]
that defines a unitary connection which has prescribed holonomy around the marked points.

**Proposition 3.** The map
\[ H^1(\Sigma, z_1, ..., z_n; C_1, ..., C_n) \to \text{Bun}_G(\Sigma, z_1, ..., z_n; P_1, ..., P_n) \]
induces a bijection of the set of irreducible representations, up to conjugation, with the set of stable parabolic bundles.

If \( \text{genus}(\Sigma) > 0 \) then there is a unique Lebesgue class probability measure on \( H^1(\Sigma, z_i; K, C_i) \) which is invariant with respect to the mapping class group \( \text{MCG}(\Sigma, z_1, ..., z_n) \).

**Conjecture 8.** Suppose that \( c \) is an analytic embedding of \( S^1 \) in \( \Sigma \setminus \{z_i\} \). The pushforward \( P(c)_* \mu_0 \) is absolutely continuous with respect to the unique mapping class group invariant probability measure on the set of stable bundles, \( \text{Bun}_G^0(\Sigma, z_1, ..., z_n; P_1, ..., P_n) \). The density is invariant with respect to the subgroup of the mapping class group which fixes the homotopy class of \( c \).

One can now ask if it might be possible to reconstruct the measure \( \mu_0 \) from these pushforwards. Unfortunately there is something missing. Even if one could prove these measures are coherent (which I have not done), relative to the maps \( B_n \), this would not give an independent method of constructing \( \mu_0 \), because it is necessary somehow vary the surface.

There is a relatively straightforward generalization of all of this to the measures with values in line bundles from the previous section.

### 3.2. Root Subgroup Factorization: The \( SU(2) \) Case.

For simplicity suppose that \( K = SU(2, \mathbb{C}) \) and \( G = SL(2, \mathbb{C}) \). From a technical point of view, this is a dramatic simplification.

A triangular factorization for \( g \in LG \) is a multiplicative factorization of the form
\[ g = l \cdot m \cdot a \cdot u, \]
where
\[ l = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, G), \quad l(\infty) = \begin{pmatrix} 1 \\ l_{21}(\infty) \end{pmatrix}, \]
l has appropriate boundary values on $S^1$ (depending on the smoothness properties of $g$), $m = \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix}$, $m_0 \in S^1$, $a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$, $a_0 > 0$,

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in H^0(\Delta, G), \quad u(0) = \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix},$$

and $u$ has appropriate boundary values on $S^1$, where $\Delta$ ($\Delta^*$) is the open unit disk centered at $z = 0$ ($z = \infty$, respectively), and $H^0(U)$ denotes holomorphic functions in a domain $U \subset \mathbb{C}$. The basic fact is that for $g \in LK$ having a triangular factorization, there is a second unique ‘root subgroup factorization’

$$g(z) = k_1^*(z) \begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)} \end{pmatrix} k_2(z), \quad |z| = 1,$$

where

$$k_1(z) = \begin{pmatrix} a_1(z) & b_1(z) \\ -b_1^*(z) & a_1^*(z) \end{pmatrix} = \lim_{n \to \infty} a(\eta_n) \begin{pmatrix} 1 & -\eta_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \cdot a(\eta_0) \begin{pmatrix} 1 & -\eta_0 \\ \eta_0 & 1 \end{pmatrix},$$

$$\chi(z) = \sum \chi_j z^j$$

is a $i\mathbb{R}$-valued Fourier series (modulo $2\pi i\mathbb{Z}$),

$$k_2(z) = \begin{pmatrix} d_2^*(z) & -c_2^*(z) \\ c_2(z) & d_2(z) \end{pmatrix} = \lim_{n \to \infty} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\zeta_n^* z^n & 1 \end{pmatrix} \cdot a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ \zeta_1 & 1 \end{pmatrix},$$

$a(\cdot) = (1+|\cdot|^2)^{-1/2}$, and it is understood that if $g \in C^\infty(S^1, K)$, then the coefficients are rapidly decreasing, and similarly for other function spaces; conversely a root subgroup factorization as in (3.3) implies that $g$ has a triangular factorization (see [71]).

For $l > -1$ (minus half the dual Coxeter number, as in Conjecture [1], let $\tilde{\mu}_l$ denote the product probability measure

$$\left( \prod_{j=0}^{\infty} \frac{1 + (l + 2)i}{\pi} \frac{d\lambda(\eta_j)}{(1 + |\eta_j|^2)^{(l+2)i}} \right) \times \left( \prod_{j=1}^{\infty} \frac{2j(l+2)}{\pi} e^{2ji(l+2)|\eta_j|^2} d\lambda(\chi_j) \right) \times d\lambda(e^{\chi_0}) \times \left( \prod_{k=1}^{\infty} \frac{(l+2)k - 1}{\pi} \frac{d\lambda(\zeta_k)}{(1 + |\zeta_k|^2)^{(l+2)k}} \right),$$

where $d\lambda$ denotes Lebesgue measure for $\mathbb{C}$, or normalized Haar measure for $S^1$. The following is technically useful, because it reduces $0 - 1$ issues to questions about Gaussians.

**Lemma 1.** *In the sense of measures, $\tilde{\mu}_l$ is equivalent to the Gaussian measure*

$$\left( \prod_{j=0}^{\infty} \frac{2 + (l + 2)i}{\pi} e^{-(2+(l+2)i)|\eta_j|^2} d\lambda(\eta_j) \right) \times \left( \prod_{j=1}^{\infty} \frac{2j(l+2)}{\pi} e^{-2j(l+2)|\chi_j|^2} d\lambda(\chi_j) \right) \times d\lambda(e^{\chi_0}) \times \left( \prod_{k=1}^{\infty} \frac{(l+2)k - 1}{\pi} e^{-(l+2)k|\zeta_k|^2} d\lambda(\zeta_k) \right)$$

(3.7)
This follows from Kakutani’s criterion for equivalence of product measures.

Our basic heuristic claim is that for \( l = 0 \), \( \tilde{\mu}_l \) is a coordinate expression for the invariant (ordinary) measure \( \mu_0 \) for \( LK \) (more generally \( \tilde{\mu}_l \) is a coordinate expression for the positive line bundle valued measure \( \mu|C|^{\mu} \)). To give rigorous meaning to this claim, it is necessary to consider a completion of the loop group, as in the statement of Theorem [11.1]. The basic difficulty is that the measure \( \tilde{\mu}_l \) is supported on sequences for which the sum \( \chi \) and the products [3.4] and [3.5] (marginally) fail to converge (the behavior of the random sum \( \sum_{n>0} \chi_n z^n \) is analyzed in Chapter 13 of [45]; because of Lemma [11.1] the same qualitative analysis applies to the products). Consequently it is not possible to view \( \tilde{\mu}_l \) as a countably additive measure on any kind of pointwise defined, or even measurable, loop space of \( K \) or \( G \).

Initially, following standard practice, we consider a relatively thick completion, the formal completion \( LG \). A generic point in \( LG \) has a unique formal triangular factorization

\[
g = l \cdot m \cdot a \cdot u
\]

where \( m \in T \) (the diagonal torus in \( SU(2) \)), \( a \in A := \exp(\mathbb{R}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})) \), \( l \in \mathcal{N}^- \), the (profinite nilpotent) group consisting of formal power series in \( z^{-1} \),

\[
l = \left( 1 + \sum_{j=1}^{\infty} A_j z^{-j} \right) \left( \sum_{j=0}^{\infty} C_j z^{-j} \right) \left( 1 + \sum_{j=1}^{\infty} D_j z^{-j} \right),
\]

with \( \det(l) = 1 \) (as a formal power series in \( z^{-1} \)), and \( u \in \mathcal{N}^+ \), the group consisting of formal power series in \( z \),

\[
u = \left( 1 + \sum_{j=1}^{\infty} a_j z^j \right) \left( \sum_{j=0}^{\infty} b_j z^j \right) \left( 1 + \sum_{j=1}^{\infty} d_j z^j \right),
\]

with \( \det(u) = 1 \). The Fourier coefficients of the factors in the triangular factorization of a generalized loop are well-defined random variables (as is typically the case for fields that are relevant to quantum field theory).

The formal completion \( LG \) is not a group, but it is a standard Borel space which is acted on naturally from the left and right by the complex polynomial loop group \( L_{\text{fin}}G := G(\mathbb{C}[z, z^{-1}]) \).

The first claim is that the composite mapping

\[
\{(\eta, \chi, \zeta)\} \rightarrow LG
\]

\[
(\eta, \chi, \zeta) \rightarrow g = k_1(\eta)^* \begin{pmatrix} e^{\sum z^j} & 0 \\ 0 & e^{-\sum z^j} \end{pmatrix} k_2(\zeta) \rightarrow l(g) \cdot m(g) \cdot a(g) \cdot u(g)
\]

which is a priori only defined for rapidly decreasing sequences, extends to a measurable mapping which is defined \( \tilde{\mu}_l \)-almost surely. To be more precise, if \( \eta, \chi \) and \( \zeta \) are \( l^2 \) sequences, then the matrix products defining \( g \) and the triangular factors are deterministically defined as Lebesgue measurable functions of \( z \in S^1 \) (see section 2 of [11]). But the measure [3.2] is supported on sequences with a \( l^2 \) logarithmic divergence; the composite mapping (jumping over \( g \)), where the ‘products’ are now understood to be formal, can be extended in an almost sure sense. As a consequence \( \tilde{\mu}_l \) can be pushed forward to a probability measure \( \mu_l \) on \( LG \), where \( L_{\text{fin}}K \times L_{\text{fin}}K \) acts. This is somewhat analogous to Ito’s uniformization of Brownian motion on nonlinear targets, using stochastic differential equations.
In the case of $K = SU(n)$ the line bundle $\mathcal{L} \to C^\infty(S^1, G)$ can be realized using Toeplitz operators: the line bundle $\mathcal{L}^{-1} = Det(A)$ is the the pullback of a Fredholm determinant line bundle, where $A(g)$ is the Toeplitz operator corresponding to $g$; see chapter 6 of [76].

**Conjecture 9.** The measure $\mu_0$ is bi-invariant with respect to $L_{fin}K$. More generally $\mu_l$ is bi-quasi-invariant with respect to $L_{fin}K$, and the Radon-Nikodym derivative can be read off from the heuristic expression

$$d\mu_l(g) = \frac{1}{Z} det(A(g)^* A(g))^l d\mu_0(g),$$

where $A(g)$ is the (block) Toeplitz operator associated to $g : S^1 \to K$. In more sophisticated terms, the measure $\mu_l$ is a coordinate expression for a bi-invariant measure with values in the positive line bundle $|\mathcal{L}|^2$.

**Remark.** There are multiple ways to think about this conjecture. For example suppose that $g \in C^0(S^1, K)$ is distributed according to Wiener measure $\nu_\beta$. The following is known: (1) $g$ has a triangular factorization $\nu_\beta$-almost surely, (2) the coefficients of the triangular factors, and also the variables $\eta_i, \chi_j$ and $\zeta_k$ (which can be viewed as functions defined on the top stratum of the formal completion) are tight with respect to the measures $\nu_\beta$ as $\beta \to 0$, and hence (3) for each $i$, the family of measures $(\eta_i)_* (\nu_\beta)$ has finite limits as $\beta \to 0$ (i.e. the mass does not escape to infinity), and similarly for $\chi_j$ and $\zeta_k$. When the level $l = 0$, the conjecture predicts the values of these limits. Some of these are known. For example it is known that both $(\eta_0)_* (\nu_\beta)$ and $(\zeta_1)_* (\nu_\beta)$ converge to the rotationally invariant distribution on a sphere, because these variables are equivariant with respect to appropriate (root subgroup) actions of $SU(2)$ and Wiener measure is asymptotically invariant.

What justifies this speculation that $\mu_l$ might factor in root subgroup coordinates? The utility of root subgroup coordinates (3.3) is manifested by the (Plancherel-esque) identities

$$\det(A(g)^* A(g)) = \left(\prod_{i=0}^{\infty} \frac{1}{(1 + |\eta|^2)^i}\right) \times \left(\prod_{j=1}^{\infty} \frac{1}{e^{-2j|x_i|^2}}\right) \times \left(\prod_{k=1}^{\infty} \frac{1}{(1 + |\chi|^2)^k}\right)$$

(3.12)

$$\det(A_1(g)^* A_1(g)) = \left(\prod_{i=0}^{\infty} \frac{1}{(1 + |\eta|^2)^{i+1}}\right) \times \left(\prod_{j=1}^{\infty} \frac{1}{e^{-2j|x_i|^2}}\right) \times \left(\prod_{k=1}^{\infty} \frac{1}{(1 + |\zeta|^2)^{k-1}}\right)$$

(3.13)

where $A_1$ is the shifted Toeplitz operator

$$a_0(g)^2 = \frac{\det(A_1(g)^* A_1(g))}{\det(A(g)^* A(g))} = \left(\prod_{i=0}^{\infty} \frac{1}{(1 + |\eta|^2)}\right) \times \left(\prod_{k=1}^{\infty} \frac{1}{(1 + |\zeta|^2)}\right)$$

These formulas are valid for $g \in W^{1/2}(S^1, K)$, and generalize venerable identities of Szego and Widom, see [71]. This fits seamlessly with the continuity claim in Conjecture 4.

A corollary of this product expression is the following result, which expresses the diagonal distribution for $\mu_l$ in terms of an affine analogue of Harish-Chandra’s $c$-function (see Section 4.4 of Part III of [66]).
Conjecture 10. For \( \lambda \in \mathbb{R} \),

\[
\int a_0(g)^{-2\sqrt{-1}\lambda}d\mu_l(g) = \frac{\sin\left(\frac{\pi}{2+l}\right)}{\sin\left(\frac{\pi}{2+l}(1-\sqrt{-1}\lambda)\right)}
\]

The ‘2’ is included in the exponent to match up with the general diagonal distribution conjecture below (or because \( \dot{\alpha}_1(h_{\dot{\alpha}_1}) = 2 \)).

**Heuristic Proof of the Conjecture** In this (heuristic) proof we will write \( g = k_1^* e^{\chi h_1} k_2 \) as in root subgroup factorization. This could be dispensed with. The main point is that (conjecturally) the root subgroup coordinates \( \eta_i, \chi_j, \zeta_k \) are independent random variables relative to \( \mu_l \) and \( a_0 \) factors in terms of these variables.

Suppose that \( \lambda \in \mathbb{R} \). Then, using (3.13)

\[
\int a_0(k_1^* e^{\chi h_1} k_2)^{-2\sqrt{-1}\lambda}d\mu_l(a_0(k_1^*)^{-2\sqrt{-1}\lambda}d\mu_l)
\]

The important point here: there is no dependence on imaginary roots, i.e. the \( \chi_j \).

In root subgroup coordinates this

\[
\prod_{i=0}^{\infty} \frac{1 + (l + 2)i}{\pi (1 + |\eta_i|^2)^{l+2} + (l+2)i} d\lambda(\eta_i) \prod_{k=1}^{\infty} \frac{1}{1 + (l + 2)k} d\lambda(\zeta_k)
\]

\[
= \prod_{i=0}^{\infty} \frac{(l + 2)i + 1}{(l + 2)i + (1 - \sqrt{-1}\lambda)} \prod_{k=1}^{\infty} \frac{l + 2 k - 1}{(l + 2)k - (1 - \sqrt{-1}\lambda)}
\]

\[
= \frac{1}{1 - \sqrt{-1}\lambda} \prod_{i>0} \frac{1 + \frac{1}{l+2}i}{1 + \frac{1}{l+2}i^{l+2}} \prod_{k>0} \frac{1 - \frac{1}{l+2}k}{1 - \frac{1}{l+2}k^{l+2}}
\]

\[
= \frac{\sin\left(\frac{\pi}{2+l}\right)}{\sin\left(\frac{\pi}{2+l}(1-\sqrt{-1}\lambda)\right)}
\]

This ‘proves’ that the conjecture follows from the product expression for root subgroup coordinates and (3.13).

**Remark.** In this calculation there are individual products that do not make sense; this reflects the fact that \( a_1 := a_0(k_1) \) and \( a_2 := a_0(k_2) \) are not individually well-defined random variables. But the product \( a_1 a_2 \) is a well-defined random variable, so in the end the formula is valid. We are gliding over a number of subtle issues of this sort.

At a heuristic level this formula can be viewed as an infinite dimensional example of the Duistermaat-Heckman exact stationary phase formula (see section 7 of [68]). Conjecturally root subgroup coordinates are essentially action angle variables for the homogeneous Poisson structure that is in the background.

### 3.3. Root Subgroup Factorization: The General Case

There is a generalization of root subgroup coordinates (see [71]) and Conjecture [9] for a general simply connected \( K \) (with simple Lie algebra).
3.3.1. The Short Version. In heuristic terms, the first step is to choose a reduced factorization for the (fictitious) ‘longest Weyl group element’ for the affine Weyl group. One uses this choice to ‘order’ the positive affine roots for the affine Lie algebra \( g(\mathbb{C}[z, z^{-1}]) \). The ordering has the form

\[
\tau_1, \tau_2, \ldots; \text{imaginary roots}; \ldots \tau'_0, \ldots, \tau'_{-N}
\]

where the \( \tau_k \) is a listing of the real positive affine roots of the form \( q\delta - \hat{\alpha} \), \( q > 0 \) and \( \hat{\alpha} \) is a positive root of \( g \), the ordering of the imaginary roots is irrelevant (the corresponding root subgroup elements all commute), and the \( \tau'_i \) list the other positive affine roots; see [74]. Let \( \hat{g} \) denote the dual Coxeter number, and \( \rho = \sum_{i=0}^{r} \Lambda_i \), the sum of the fundamental affine positive weights. The general form of the invariant measure \( d\mu \)

\[
\prod_{k=0}^{1} \frac{1}{3} \frac{d\lambda(\eta_k)}{(1 + |\eta_k|^2)^{1+\rho(h, \tau_k)}} \prod_{k=0}^{1} \frac{1}{3} \frac{d\lambda(\xi_k)}{(1 + |\xi_k|^2)^{1+\rho(h, \tau_k)}}
\]

(times Haar measure for \( T \)). We will further parse this, and the corresponding expression for \( \mu_t \), in the Longer Version below.

Remarks. (a) A note on possible notational confusion: \( \chi_j \in i\mathfrak{h}_\mathbb{R}; \) in the \( SU(2) \) case we were writing this element as \( \begin{pmatrix} \chi_j & 0 \\ 0 & -\chi_j \end{pmatrix} \). This explains the non-appearance of a 2 in the Gaussian measure involving \( \chi_j \).

(b) This formula is a Kac-Moody analogue of a formula known to be valid in finite dimensions.

There is a similar product expression for the measure \( d\mu_t \),

\[
d\mu_t = |\sigma_0|^2 |d\mu| |L|^{2l}
\]

where \( \sigma_0 \) is the matrix coefficient corresponding to \( \Lambda_0 \), the basic fundamental positive weight (in the case of \( SL(n, \mathbb{C}) \), \( \sigma_0(g) = \det(A)(\hat{g}) \), where \( A(g) \) is the Toeplitz operator corresponding to \( g : S^1 \to SL(n, \mathbb{C}) \), and the determinant is really a holomorphic function on the central extension, as in [76]). This product expression emerges because \( \sigma_0 \) (and other fundamental matrix coefficients) factor completely in terms of root subgroup coordinates. This reduces to our previous expression in the case \( K = SU(2) \).

3.3.2. The Detailed Version. In this detailed version we will use the supplementary notation in Subsection 1.1. Thus we will be replacing \( K \) with \( \hat{K} \). The following is a synopsis of the main results in [74].

The Weyl group \( W \) for \( (\hat{L\mathfrak{g}}, \mathbb{C}d + \mathfrak{h}) \) acts by isometries of \( (\mathbb{R}d + \mathfrak{h}_\mathbb{R}, \langle \cdot, \cdot \rangle) \). The action of \( W \) on \( \mathbb{R}c \) is trivial. The affine plane \( d + \mathfrak{h} \) is \( W \)-stable, and this action identifies \( W \) with the affine Weyl group and its affine action [147] (see Chapter 5 of [76]). In this realization

\[
r_{\alpha_0} = \hat{h}_\delta \circ r_{\delta}, \quad \text{and} \quad r_{\alpha_i} = r_{\alpha_i}, \quad i > 0.
\]

Definition 1. (a) A sequence of simple reflections \( r_1, r_2, \ldots \) in \( W \) is called reduced if \( w_n = r_n r_{n-1} \cdots r_1 \) is a reduced expression for each \( n \).

(b) A reduced sequence of simple reflections \( \{r_j\} \) is affine periodic if, in terms of the identification of \( W \) with the affine Weyl group, (1) there exists \( l \) such that
positive roots \( \gamma \)

**Theorem 3.1.** (a) There exists an affine periodic reduced sequence \( \tilde{r} \) such that the span of the corresponding root spaces is \( \hat{n}^{-}(z\mathbb{C}[z]) \). The period can be chosen to be any point in \( \mathbb{C} \setminus \mathbf{T} \).

(b) Given a reduced sequence as in (a), and a reduced expression for \( \tilde{w}_{0} = r_{-N} \cdots r_{0} \) (where \( \tilde{w}_{0} \) is the longest element of \( \tilde{W} \)), the sequence

\[
\tilde{r}_{-N}, \ldots, \tilde{r}_{0}, \tilde{r}_{1}, \ldots
\]

is another reduced sequence. The corresponding set of positive roots mapped to negative roots is

\[
\{ q\delta + \alpha : \alpha > 0, q = 0, 1, \ldots \},
\]

i.e. the span of the corresponding root spaces is \( \hat{n}^{+}(\mathbb{C}[z]) \).

From now on we fix a reduced sequence \( \{r_{j}\} \) as in Theorem 3.1 and a reduced expression \( \tilde{w}_{0} = r_{-N} \cdots r_{0} \). We set

\[
i_{r_{n}} = w_{n-1}^{-1}i_{\gamma_{n}}w_{n-1}^{-1}, \quad n = 1, 2, \ldots
\]

\[
i_{r_{-N}} = i_{\gamma_{-N}}, \quad i_{\tilde{r}_{(N-1)}} = r_{-N}i_{\gamma_{-(N-1)}}r_{-N}^{-1}, \ldots, \quad i_{\tilde{r}_{0}} = w_{0}i_{\gamma_{0}}w_{0}^{-1}
\]

and for \( n > 0 \)

\[
i_{\tilde{r}_{n}} = w_{0}w_{n-1}^{-1}i_{\gamma_{n}}w_{n-1}^{-1}w_{0}^{-1}.
\]

Also for \( \zeta \in \mathbb{C} \), let \( a(\zeta) = (1 + |\zeta|^{2})^{-1/2} \) and

\[
(3.17) \quad k(\zeta) = a(\zeta) \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a(\zeta) & 0 \\ 0 & a(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\zeta \\ 0 & 1 \end{pmatrix} \in SU(2).
\]

**Theorem 3.2.** Suppose that \( \tilde{k}_{1} \in \tilde{L}K \) and \( \Pi(\tilde{k}_{1}) = k_{1} \). The following are equivalent:

(1.1) \( m(k_{1}) = 1 \), and for each complex irreducible representation \( V(\pi) \) for \( \tilde{G} \), with lowest weight vector \( \phi \in V(\pi) \), \( \pi(k_{1})^{-1}(\phi) \) has holomorphic extension to \( \Delta \), is nonzero at all \( z \in \Delta \), and is a positive multiple of \( v \) at \( z = 0 \).

(1.2) \( k_{1} \) has a factorization of the form

\[
\tilde{k}_{1} = \lim_{n \to -\infty} i_{\tilde{r}_{n}}(k(\eta_{n})) \cdots i_{\tilde{r}_{-N}}(k(\eta_{-N})),
\]

for a rapidly decreasing sequence \( \{\eta_{j}\} \).

(1.3) \( \tilde{k}_{1} \) has triangular factorization of the form \( \tilde{k}_{1} = l_{1}a_{1}u_{1} \) where \( l_{1} \in H^{0}(\Delta^{*}, \tilde{N}^{\ast}) \) has smooth boundary values.

Moreover, in the notation of (1.2),

\[
a_{1} = \prod_{j=-N}^{\infty} a(\eta_{j})^{h_{j}}.
\]

Similarly, the following are equivalent: for \( \tilde{k}_{2} \in \tilde{L}K \),
(II.1) $m(\tilde{k}_2) = 1$; and for each complex irreducible representation $V(\pi)$ for $\mathcal{G}$, with highest weight vector $v \in V(\pi)$, $\pi(k_2)^{-1}(v) \in H^0(\Delta; V)$ has holomorphic extension to $\Delta$, is nonzero at all $z \in \Delta$, and is a positive multiple of $v$ at $z = 0$.

(II.2) $\tilde{k}_2$ has a factorization of the form

$$\tilde{k}_2 = \lim_{n \to \infty} i_{\tau_n}(k(\zeta_n)) \cdots i_{\tau_1}(k(\zeta_1))$$

for some rapidly decreasing sequence $(\zeta_j)$.

(II.3) $\tilde{k}_2$ has triangular factorization of the form $\tilde{k}_2 = l_2 a_2 u_2$, where $l_2 \in H^0(\Delta^*, \infty; N^+, 1)$ has smooth boundary values.

Also, in the notation of (II.2),

$$a_2 = \prod_{j=1}^{\infty} a(\zeta_j)^{h_j}.$$

**Theorem 3.3.** Suppose $\tilde{g} \in \tilde{L}K$ and $\Pi(\tilde{g}) = g$.

(a) The following are equivalent:

(i) $\tilde{g}$ has a triangular factorization $\tilde{g} = lmau$, where $l$ and $u$ have $C^\infty$ boundary values.

(ii) $\tilde{g}$ has a factorization of the form $\tilde{g} = \tilde{k}_1^* \exp(\chi) \tilde{k}_2$

where $\chi \in \tilde{L}A$, and $\tilde{k}_1$ and $\tilde{k}_2$ are as in Theorem 1.3.

(b) In reference to part (a),

$$a(\tilde{g}) = a(g) = a(k_1) a(\exp(\chi)) a(k_2), \quad \Pi(a(g)) = \Pi(a(k_1)) \Pi(a(k_2))$$

and

$$a(\exp(\chi)) = |\sigma_0|(\exp(\chi))^{h_0} \prod_{j=1}^{r} |\sigma_0|(\exp(\chi))^{a_j h_j}.$$

**Remarks.** (a) As in the $\tilde{K} = SU(2)$ case, we expect that if $\eta, \chi$ and $\zeta$ are $l^2$ sequences, then the matrix products defining $g$ and the triangular factors are deterministically defined as Lebesgue measurable functions of $z \in S^1$.

(b) There is a technical challenge lurking here: In the $SU(2)$ case there is a known algorithm for finding $\eta, \chi, \zeta$, given $g = g_0 g_+$. In higher rank cases any algorithm will depend on the choice of ‘ordering of the positive roots’, i.e. the choice of reduced sequence in Theorem 1.1.

3.3.3. RSF and Measures. Now we want to discuss measures in connection with root subgroup factorization. Because the $\tilde{K} = SU(2)$ case is far more explicit, there are a number of technical things which we understand for $\tilde{K} = SU(2)$ and which currently elude us for general $\tilde{K}$.

Recall that $d\mu_t$ is to have the official expression

$$d\mu_t = (\sigma_0 \otimes \sigma_0)^{\otimes t} d\mu \mid_{L^{2t}}$$

and the heuristic expression

$$d\mu_t = \frac{1}{3} (\sigma_0, \sigma_0)^{t} d\mu_0$$

(3.21)
Consider the measure $\tilde{\nu}_l$ on sequences $\eta, \chi, \zeta$ given by

$$\prod_{i=-N}^{\infty} \frac{1}{3} \frac{d\lambda(\eta_i)}{1 + |\eta_i|^2} \prod_{j=1}^{\infty} \frac{1}{3} \frac{d\lambda(\chi_j)}{1 + |\chi_j|^2} \prod_{k=1}^{\infty} \frac{1}{3} \frac{d\lambda(\zeta_k)}{1 + |\zeta_k|^2} \prod_{j=1}^{\infty} \frac{1}{3} \frac{d\lambda(\lambda(\chi_j))}{1 + |\chi_j|^2} \prod_{k=1}^{\infty} \frac{1}{3} \frac{d\lambda(\lambda(\chi_j))}{1 + |\chi_j|^2}$$

(times Haar measure for $e^{x\lambda} \in \hat{T}$), where $h_\tau$ denotes the coroot corresponding to the real root $\tau$, $\tau = q(\tau)\delta + \hat{\alpha}$ (Note: $\delta$ is a function of the root $\tau$, but it will be convenient to occasionally write it as a function of the coroot $h_\tau$), and $\rho := \sum_{l=0}^{\infty} \Lambda_l$.

**Remark.** Where does this formula come from? When $l = 0$ this is the exact analogue of a corresponding formula in finite dimensions for $\hat{K}$. When the level is nonzero, we are adding in a known formula for $|\sigma_0|^{2l}$.

To make this formula more explicit, we use the following formulas: Given a positive root $\hat{\alpha}$

$$h_\delta \pm \hat{\alpha} = qc \pm h_\delta, \quad \rho = \delta \gamma + \hat{\rho}, \quad \lambda(h_\delta) = 2 \frac{\lambda(\delta \gamma + \hat{\rho})}{\lambda(\delta \gamma) + \lambda(\hat{\rho})}$$

Then (3.22) equals

$$\prod_{i=-N}^{\infty} \frac{1}{3} \frac{d\lambda(\eta_i)}{1 + |\eta_i|^2} \prod_{j=1}^{\infty} \frac{1}{3} \frac{d\lambda(\chi_j)}{1 + |\chi_j|^2} \prod_{k=1}^{\infty} \frac{1}{3} \frac{d\lambda(\zeta_k)}{1 + |\zeta_k|^2} \prod_{j=1}^{\infty} \frac{1}{3} \frac{d\lambda(\lambda(\chi_j))}{1 + |\chi_j|^2}$$

(times Haar measure for $e^{x\lambda} \in \hat{T}$)

$$\prod_{\alpha > 0} \left( \prod_{q' = 0}^{\infty} \frac{1}{3} \frac{d\lambda(\eta_{q'\delta + \hat{\alpha}})}{1 + |\eta_{q'\delta + \hat{\alpha}}|^2} \prod_{q = 0}^{\infty} \frac{1}{3} \frac{d\lambda(\zeta_{q\delta - \hat{\alpha}})}{1 + |\zeta_{q\delta - \hat{\alpha}}|^2} \prod_{j=1}^{\infty} \frac{1}{3} \frac{d\lambda(\lambda(\chi_j))}{1 + |\chi_j|^2} \right)$$

(times Haar measure for $e^{x\lambda} \in \hat{T}$), where in this last expression we are using the roots $q\delta - \hat{\alpha}$ to index the zeta variables, rather than $i$, and similarly for the eta variables.

**Conjecture 11.** For general $\hat{K}$ the composite mapping

$$(\eta, \chi, \zeta) \rightarrow (k_1(\eta)^* \begin{pmatrix} e^{\sum \chi_j z^j} & 0 \\ 0 & e^{-\sum \chi_j z^j} \end{pmatrix} k_2(\zeta) \rightarrow l(g) \cdot m(g) \cdot a(g) \cdot u(g)$$

which is a priori only defined for rapidly decreasing sequences as in Theorem 3.3 extends to a measurable mapping which is defined $\hat{\nu}_l$-almost surely.
To clarify the notation, $\sum \chi_j$ is shorthand for $\sum_{j=-\infty}^{\infty} \chi_j$, where $\chi_{-j} = -\chi_j$.

As a consequence of Conjecture 11, $\hat{\mu}_t$ can be pushed forward to a probability measure $\mu_t$ on $LG$, where $L_{fin}K \times L_{fin}K$ acts.

**Conjecture 12.** The measure $\mu_0$ is bi-invariant with respect to $L_{fin}K$. More generally $\mu_l$ is bi-quasi-invariant with respect to $L_{fin}K$, and the Radon-Nikodym derivative can be read off from its heuristic expression \((3.21)\). (In the case $K = SU(n)$

$$d\mu_l(g) = \frac{1}{2} \det(A(g)^* A(g))^{l} d\mu_0(g),$$ \[(3.27)\]

where $A(g)$ is the (block) Toeplitz operator associated to $g : S^1 \to \hat{K}$).

As in the $SU(2)$ case, there are (at least) two possible strategies for proving this conjecture. One is to prove that $\mu_l$ is invariant with respect to $LG \to LG : g \to g^*$ (this restricts to inversion on unitary loops). In the $SU(2)$ case, $g \to g^*$ can be written down explicitly, but this does not seem to help in proving invariance. If we could push this through, this first strategy would work for any level $l$.

The second strategy only applies when $l = 0$. The following is known: If $g \in C^0(S^1, K)$ is distributed according to Wiener measure $\nu_\beta$, then (1) $g$ has a triangular factorization $\nu_\beta$-almost surely, (2) the coefficients (in a coordinate system) of the triangular factors are tight with respect to the measures $\nu_\beta$ as $\beta \to 0$. This together with asymptotic invariance guarantees the existence of an invariant measure.

**Conjecture 13.** The variables $\eta_i, \chi_j$ and $\zeta_k$ can be expressed as algebraic functions (in fact using only roots and rational expressions) in terms of coefficients of the triangular factors for $g$. Consequently the variables $\eta_i, \chi_j$ and $\zeta_k$ are also tight with respect to the measures $\nu_\beta$ as $\beta \to 0$, and hence (3) for each $i$, the family of measures $(\eta_i)_*(\nu_\beta)$ has finite limits as $\beta \to 0$ (i.e. the mass does not escape to infinity), and similarly for $\chi_j$ and $\zeta_k$.

From this second point of view, conjecture 11 is predicting these $\beta \downarrow 0$ limits. This strikes me as very plausible, but I could easily be suffering from a form of Stockholm syndrome.

3.4. The Diagonal Distribution Conjecture. A corollary is the following diagonal distribution conjecture. The statement that follows is slightly different from the way in which I have stated the conjecture in the past. The crucial point is the inclusion of the factor $\frac{2}{\langle \alpha, \alpha \rangle}$, which is simply unity in simply laced cases.

**Conjecture 14.**

$$\int a(g)^{-\sqrt{-1} \lambda} d\mu_l(g) = \prod_{\alpha > 0} \frac{\sin(\frac{\pi}{1+g \langle \alpha, \alpha \rangle}(2\hat{\rho}, \dot{\alpha}))}{\sin(\frac{\pi}{1+g \langle \alpha, \alpha \rangle}(2\hat{\rho} - \sqrt{-1} \lambda, \alpha))}$$

where $\dot{\gamma}$ is the dual Coxeter number, $2\hat{\rho}$ is the sum of the positive roots, $\lambda \in h^*_R$, and $\langle \cdot, \cdot, \cdot \rangle$ is any positive multiple of the Killing form.

A standing assumption in these notes is that $\langle \cdot, \cdot, \cdot \rangle$ is normalized so that the length squared of a long root is 2. The point we are making is that, in our revised conjecture, that assumption is not needed. (Of course this is a conjecture, so I should be careful to not be so dogmatic on this point).
Heuristic proof of the Conjecture

\[
\int a(k_1^* e^\lambda k_2)^{-\sqrt{-1} \lambda} d\mu_l = \int a(k_1^*)^{-\sqrt{-1} \lambda} a(k_2)^{-\sqrt{-1} \lambda} d\mu_l
\]

It follows from (3.28) that this equals

(3.29)

\[
\prod_{\alpha > 0} \left( \prod_{q > 0} \frac{1}{1 + \frac{1}{(l+g)q} \langle \alpha, \alpha \rangle} \frac{2(2\delta, \dot{\alpha})}{(2\delta - \sqrt{-1} \lambda, \dot{\alpha})} \prod_{q > 0} \frac{1}{1 - \frac{1}{(l+g)q} \langle \alpha, \alpha \rangle} \frac{2(2\delta, \dot{\alpha})}{(2\delta - \sqrt{-1} \lambda, \dot{\alpha})} \right)
\]

The standard product expansion for \(\text{sine}\) now (at least heuristically) implies the Conjecture!

For \(l = 0\) this formula should be compared with the known formula of Harish-Chandra,

\[
\int_K a(g)^{-\sqrt{-1} \lambda} d\lambda(g) = c(\rho - \sqrt{-1} \lambda) = \prod_{\alpha > 0} \frac{\langle \rho, \alpha \rangle}{\langle \rho - \sqrt{-1} \lambda, \alpha \rangle}
\]

When we incorporate the level \(l\), the generalization of the conjecture is

\[
\int a^{-i \lambda} d\mu_l = \prod_{\alpha > 0} \frac{\sin(2\pi \alpha + i(\rho, \alpha))}{\sin(2\pi \alpha - i(\rho - i \lambda, \alpha))}
\]

As \(l \to \infty\), we recover Harish-Chandra’s formula. This is consistent with the standard intuition that \(l \to \infty\) is a ‘classical limit’. The original heuristic derivation of these conjectures is in Part III of [66]. There is a stationary phase interpretation in [68]. And the formulas follow directly from the factorization of the measures in terms of root subgroup coordinates (and the usual product expression for the sine function); the upshot is that the factorization of the measures has a much deeper significance.

A triangular factorization for \(g \in Hyp(S^1, G)\) implies a Riemann-Hilbert factorization

\[
g = g_+ \cdot g_0 \cdot g_
\]

The diagonal distribution conjecture determines the distribution for \(g_0\). This is most neatly expressed in terms of the Harish-Chandra transform

\[
\mathcal{H}(\mu_l)(\lambda) = \prod_{\alpha} \Gamma(1 + \frac{i \pi}{l + g} \langle \lambda, \dot{\alpha} \rangle)
\]

(this follows from 4.4.27 of Part II of [66]).

3.5. Many Missing Formulas. Let’s return to root subgroup factorization and the \(SU(2)\) case. For a sufficiently regular loop \(g : S^1 \to SU(2)\), the invertibility of \(A(g)\) and \(\bar{A}(g)\) is equivalent to a factorization

(3.29)

\[
g(z) = \begin{pmatrix} a_1(z) & b_1(z) \\ b_1^*(z) & a_1^*(z) \end{pmatrix} \begin{pmatrix} e^{x(z)} & 0 \\ 0 & e^{-x(z)} \end{pmatrix} \begin{pmatrix} d_2^*(z) & -c_2^*(z) \\ -c_2(z) & d_2(z) \end{pmatrix}
\]

where in particular \(c_2, d_2\) are holomorphic in \(\Delta\) and do not simultaneously vanish (and are subject to some mild normalizations which we ignore). One can say more: \(k_2\) has a triangular factorization

(3.30)

\[
k_2 = \begin{pmatrix} d_2^* & -c_2^* \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^* \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}
\]

where \(x = x(z)\) is holomorphic in \(\Delta\).
Now suppose that \( \eta, \chi, \zeta \) are distributed according to \( \mu_i \). As we already remarked, \( k_2 \) is not well-defined on the circle. For reasons which are obscure to me at a conceptual level, it is important to rewrite the triangular factorization of \( k_2 \) as

\[
(3.31) \quad k_2 = \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},
\]

where \( X = a_2^{-2}x \). It turns out that \( x(z) \) is not a well-defined random function, but \( X(z) \) is. It also easy to see that \( a_2 \) is not a well-defined random variable, but the product \( a_1a_2 \) is.

Most intriguing, \( \mathbb{P}(c_2, d_2) : \Delta \to \mathbb{P}^1 \) is a well-defined random holomorphic function, i.e.

\[
c_2/d_2 = \gamma_2/\delta_2 = (g_+)_21/(g_+)_22
\]

is a well-defined random meromorphic function in the disk. This function has a Taylor series \( \sum_{n=1}^{\infty} \xi_n z^n \) where \( \xi_n \) is the sum of terms

\[
(-1)^s(-\overline{\zeta}_s) \left( \zeta_{j_1}(-\overline{\zeta}_{i_1}) \right) \ldots \left( \zeta_{j_r}(-\overline{\zeta}_{i_r}) \right)
\]

where \( j_s < i_s \) and \( j_s \leq i_{s-1} \) for \( s = 1, \ldots, r \), and \( \sum_{s=1}^{r+1} i_s - \sum_{s=1}^{r} j_s = n \); in particular

\[
\xi_n = (-\overline{\zeta}_n) \prod_{s=1}^{n-1} (1 + |\zeta_s|^2) + \text{polynomial}(\zeta_s, \overline{\zeta}_s, s < n)
\]

For example

\[
c_2/d_2 = (-\overline{\zeta}_1)z + (-\overline{\zeta}_2)(1 + |\zeta_1|^2)z^2 + \left( (-\overline{\zeta}_3)(1 + |\zeta_1|^2)(1 + |\zeta_2|^2) + (-\overline{\zeta}_1\zeta_2^2)(1 + |\zeta_1|^2) \right) z^3
\]

\[
+ ((-\overline{\zeta}_4)(1 + |\zeta_1|^2)(1 + |\zeta_2|^2)(1 + |\zeta_3|^2) + (1 + |\zeta_1|^2)\zeta_2\zeta_3^2(1 + |\zeta_2|^2)
\]

\[
+ 2\zeta_1\zeta_2\zeta_3(1 + |\zeta_2|^2) + \overline{\zeta}_1\zeta_2^3)z^4 + \ldots
\]

How to go back and forth from the distribution for \( \zeta \) and the distribution for \( \mathbb{P}(c_2, d_2) \) is a mystery.

The natural generalization of \( \mathbb{P}(c_2, d_2) \) to a higher rank group involves a holomorphic map into the flag space of \( \mathcal{G} \) in place of \( \mathbb{P}^1 \). It is far more complicated to recover the \( \zeta \) variables because this depends on a choice of ordering of roots. In any event I do not have heuristic formulas that might lead to reasonable conjectures for these distributions (or, in the simplest \( SU(2) \) case, the point processes that would describe their zeroes and poles).

Now let’s return to Riemann-Hilbert factorization, \( g = g_0g_+ \). We have a conjecture for the \( g_0 \) distribution. There is a heuristic expression for the distribution for \( \theta_+ := g_+^{-1}\partial g_+ \in H^1(\Delta, g) \) (which I refer to as ‘time ordered exponential coordinates’):

\[
(3.32) \quad (\theta_+)_{\mu_i} = \lim_{n \to \infty} \frac{1}{Z} \det(1 + W^*W)^{-\gamma^{-1}} dm(P_n\theta_+)
\]

where \( W = W(g_+) = A(g_+^{-1})B(g_+) \) (the graph operator, following the notation in [76]), \( g_+ \) corresponds to \( P_n\theta_+ \), and \( P_n \) projects \( \theta_+ \) to its first \( n \) coefficients (so that it is an orthogonal projection for \( H^1(\Delta, g) \). The heuristic expression [34,32] is an analogue of yet another formula of Harish-Chandra from finite dimensions. This heuristic expression is manifestly \( PSU(1, 1) \) (conformally) invariant, which is consistent with the uniqueness conjectures. Unfortunately I do not know how
to turn this heuristic expression for the $\theta_+$ distribution into a precisely stated conjecture - this is a major source of frustration. In the $SU(2)$ case, if

$$\theta_+ = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & -\alpha(z) \end{pmatrix}$$

then it is possible that $\beta$ and $\gamma$ have standard Cauchy type distributions, or are built of Cauchy type distributions in some simple way. In general we use the root space decomposition

$$\theta_+ = \sum_\alpha \theta_+(\dot{\alpha}) + \sum_{j=1}^r \theta_+(\dot{h}_j)\dot{h}_j$$

Each term is conformally invariant. This should tell us something. Ultimately we have to understand how they are coupled.

It would be a dream to compute the $\mu_l$ distribution for the random $G$-valued holomorphic function $g_0 g_+$. In principle it should be possible to determine this distribution just using knowledge of the distribution of $\theta_+$ and invariance properties; this is implied by the first uniqueness conjecture. How this determines the distribution for $g_0$ is a mystery to me.

Remark. Understanding the meaning of (3.32) is important because there are analogous formulas in other contexts, such as unitarizing measures for the Virasoro group (sometimes called Malliavin measures) and the Kontsevich-Suhov generalization of Werner’s measure on self-avoiding loops on Riemann surfaces.

4. Harmonic Analysis

4.1. The Loop Group Action on $L^2(\mu_0)$. Let $\nu_\beta$ denote Wiener measure on $C^0(S^1, K)$ with inverse temperature $\beta$. This family of measures interpolates between Haar measure $\delta_\beta$ for $K$ and $\mu_0$ for $LK$:

$$\delta_K = \nu_\infty \xleftarrow{\beta \to \infty} \nu_\beta \xrightarrow{\beta \downarrow 0} \mu_0$$

Let $L_\beta$ ($R_\beta$), denote the left (the right, respectively) regular representation of $L_{\text{fin}} K$ on half-densities for the measure class of $\nu_\beta$, $0 \leq \beta \leq \infty$.

For $\beta = \infty$, the finite dimensional case, the Peter-Weyl theorem implies that the Von Neumann algebras $L_\beta''$ and $R_\beta''$ are commutants, each is a sum (over positive weights) of matrix algebras, and the intersection is equivalent to (a completion of) the convolution algebra of central functions on $K$.

In infinite dimensions the situation appears to be rather different:

Conjecture 15. (a) Suppose that $0 < \beta < \infty$. Then $L_\beta''$ and $R_\beta''$ are commutants, each is a factor of type III, and the intersection is trivial, i.e. the action of $L_{\text{fin}} K \times L_{\text{fin}} K$ is irreducible.

(b) When $\beta = 0$, i.e. for $\mu_0$, the same statements hold for the complement of the invariant subspace of constants.

Part (a) is stated in [6], although this paper appears to be incomplete. I think the expectation is the factors in (a) are of type $\text{III}_1$. The obvious strategy is to first prove that the constant 1 is a cyclic vector for the left and right representations, in order to concretely realize the modular automorphism group. This appears to be open; see section 3 of [7] for the case of paths (as opposed to loops).
In the case $\beta = 0$, i.e. for $\mu_0$, there does not appear to be a convenient choice of cyclic vector (in the complement of the invariant vector $1$). However there does appear to be an obvious candidate for the modular group (which is well-defined up to inner automorphisms):

**Conjecture 16.** Suppose that $\beta = 0$. In this case the modular group is realized by the action of the standard hyperbolic one parameter subgroup of $PSU(1,1)$ acting on $Hyp(S^1,G)$.

In general we can consider the (conjectural) unitary action

$$W^{1/2}(S^1,K) \times \mathcal{H}^{1/2}([\mu_l]) \times W^{1/2}(S^1,K) \to \mathcal{H}^{1/2}([\mu_l])$$

for $l > -1$. Presumably this is irreducible for $l \neq 0$.

If this is correct, the basic conclusion is that it is essentially not possible to decompose the loop group actions on half-densities. As we will argue below, the more productive thing to look at is the action of $Diff(S^1)$.

### 4.2. Holomorphic Actions.

Suppose that the level $l$ is a positive integer. In this case the previous discussion can be modified by considering the action on $L$ as opposed to the action on $|L|$.

There are natural holomorphic actions of the Kac-Moody extension $\hat{L}_{fin}G$ on the holomorphic line bundle $L \to LG$ covering the action of $L_{fin}G$ on $LG$ from the left and the right. Although it is a long story, Kac and Peterson proved an algebraic version of the holomorphic Peter-Weyl theorem, and of the Borel-Weil theorem. Their work can be expressed in terms of the action of $\hat{L}_{fin}G \times \hat{L}_{fin}G$ acting on holomorphic sections of $L^{*\otimes l} \to LG$; see section 1.7 of Part I of [66] (which applies to all symmetrizable Kac-Moody groups). However, as we will see in the next section, it is important to have an analytic version.

For this reason consider the natural holomorphic actions of the Kac-Moody extension $C^\omega(S^1,G)$ on the holomorphic line bundle $L \to Hyp(S^1,G)$ covering the actions of $C^\omega(S^1,G)$ on $Hyp(S^1,G)$ from the left and the right. Assuming the truth of our conjectures regarding the measure $\mu_{C^\omega}$, there are now unitary representations $L_l$ and $R_l$ for $C^\infty(S^1,K)$ acting on sections of $L^{*\otimes l}$ which are square integrable relative to $\mu_{C^\omega}$. The subspace of holomorphic square integrable sections is an invariant subspace, which we provisionally denote by $L^2H^{0}(Hyp(S^1,G))$. Unfortunately, whereas the space of all square integrable sections is complete, because the base $Hyp(S^1,G)$ is quite thick, it will not be the case that the subspace of holomorphic sections is complete. The conjectural way to resolve this is to identify a core subspace such that each element of the completion of $L^2H^{0}(Hyp(S^1,G))$ will be holomorphic on this core (this is modeled on how I approached the same issue for measures on infinite rank Grassmannians in [66]). Let $\Phi$ denote the finite set of integrable highest weight representations of level $l$ for the affine Lie algebra $\mathfrak{g}([z,z^{-1}])$. Each of these representations can be globalized and unitarized, as in [76], using purely algebraic methods (following Garland, [31]).

The following has the general shape of the result we are looking for.
Conjecture 17. (a) The algebraic Peter-Weyl isomorphism of Kac and Peterson extends to an equivariant Hilbert space isomorphism

\[ \bigoplus_{\pi \in \Phi} H(\pi) \otimes H(\pi)^* \to L^2 \widehat{Hyp}(S^1, G) \]

where the overline indicates the Hilbert space completion.

(b) Given \( g \in L^\infty W^{1/2}(S^1, G) \), there exists a continuous evaluation map

\[ L^2 \widehat{Hyp}(S^1, G) \to L^* \otimes |g| : s \to s(g) \]

so that the completion consists of square integrable sections which are holomorphic on the subspace \( L^\infty W^{1/2}(S^1, G) \).

This statement is flawed because in (b) we need to be able to say precisely how to estimate the value of a holomorphic section at \( g \in L^\infty W^{1/2}(S^1, G) \) in terms of its \( L^2 \) norm. Note that it is essential that \( g \in L^\infty W^{1/2}(S^1, G) \) as opposed to \( g \in W^{1/2}(S^1, G) \), because in the absence of the boundedness condition (which guarantees the associated Toeplitz operator is bounded and Fredholm), it is not the case that \( g \in H_{hyp}(S^1, G) \).

It may be that the complement of \( L^2 \widehat{Hyp}(S^1, G) \) in the space of all square integrable sections is similar to the \( \mu_0 \) situation: the left and right actions are commutants, the combined action is irreducible, and the left and right Von Neumann algebras are type \( III_1 \) factors.

4.3. Reparametrization Actions. Assuming the truth of Conjecture 17, there is an action of \( K \times C^\omega \text{Homeo}(S^1) \times K \) on \( H^{1/2}(\mu_0) \) (half-densities). Of course we can identify \( H^{1/2}(\mu_0) \) with \( L^2(\mu_0) \).

Conjecture 18. Assuming the truth of Conjecture 17 and an affirmative answer to Question 4, for the action of \( K \times C^\omega \text{Homeo}(S^1) \times K \), \( L^2(\mu_0) \) can be regarded as a completion of the tensor product

\[ L^2(\mu_0) = \otimes_{g \geq 0} L^2(\mu_0)_g \]

where \( L^2(\mu_0)_g \) is generated by functions of the form \( P(c)^* f, c : S^1 \to \Sigma, f \in C^\infty(Bun^0_G(\Sigma)), \) and \( g = \text{genus}(\Sigma) \) (see Subsection 3.1 for the notation).

Fix the genus. There is a decomposition

\[ L^2(\mu_0)_g = \bigoplus_{\lambda, \nu} \lambda \otimes W(\lambda, \nu)_g \otimes \nu^* , \]

where \( \lambda \) and \( \nu \) denote irreducible representations of \( K \). This seems to exhaust the obvious discrete ways of decomposing the representation.

Remark. The natural functions on \( H^1(\Sigma, K) \) involve holonomy around nontrivial loops. These are very far from compactly supported. They are giving us generalized functions, although they are not defined everywhere and not smooth on all of \( Bun_G \).

Question 4. How does one decompose the action of \( C^\omega \text{Homeo}(S^1) \) on \( W(\lambda, \nu)_g \)?

More generally for any positive integral level \( l \), there will be a similar decomposition for \( L^2 \) sections of \( L^* \otimes l \).
Conjecture 19.
\[ L^2 \Omega^0_1(\hat{H}y^p) = \otimes_{g \geq 0} L^2 \Omega^0_1(\hat{H}y^p)_g \]
where \( \Omega_1(\hat{H}y^p)_g \) is generated by sections of the form \( P(c)^* s, c : S^1 \to \Sigma, s \in \Omega_1(Bun_0^H(\Sigma)) \) has compact support or is otherwise reasonably well-behaved, and \( g = \text{genus}(\Sigma) \).

In the holomorphic sector, the action of \( C^\omega \text{Homeo}(S^1) \) is the diagonal action as a subgroup of \( \text{Homeo}(S^1) \times \text{Homeo}(S^1) \) where these factors are acting on the tensor product of a highest weight representation and its dual.

To fix ideas, we will consider the holomorphic sector and suppose that \( G = SL(2, \mathbb{C}) \) and \( l = 1 \). In this case there are two irreducible highest weight representations of level 1. Hence as a representation of \( \hat{L}G \times \hat{L}G \)
\[ H^0_{l=1} = H(0) \otimes H(0)^* \oplus H(1/2) \otimes H(1/2)^* \]
It is known that the decompositions of \( H(0) \) and \( H(1/2) \) with respect to \( SU(2, \mathbb{C}) \times Diff(S^1) \) are multiplicity free, and more precisely,
\[ H(0) = \bigoplus_{n=0}^{\infty} V(2n+1) \otimes L(c = 1, h = (\frac{n}{2})^2) \]
and
\[ H(1/2) = \bigoplus_{n=1}^{\infty} V(2n) \otimes L(c = 1, h = (\frac{n}{2})^2) \]
where \( V(N) \) is the unique irreducible representation of \( SU(2, \mathbb{C}) \) of dimension \( N \).
Thus \( H(0) \otimes H(0)^* \) equals
\[ \bigoplus_{n,m \geq 0} V(2n+1) \otimes L(1, (\frac{n}{2})^2) \otimes L(1, (\frac{m}{2})^2)^* \otimes V(2m+1)^* \]
There is a similar decomposition for \( H(1/2) \otimes H(1/2)^* \) with \( V(2n) \) in place of \( V(2n+1) \). The structure of a representation of the form \( L \otimes L^* \) for \( Diff(S^1) \) is not known; it is possible that it might be irreducible.

Given a closed Riemann surface of genus \( g \) and a real analytic embedding \( c : S^1 \to \Sigma \), there is an inclusion (of a conformal block) induced by pullback
\[ P(c)^* H^0_{l=1} \left( Bun_G(\Sigma) \right) \subset H^0_{l=1} \]

**Question 5.** Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) have distinct genuses and \( c : S^1 \to \Sigma_i \) are real analytic embeddings. are the subspaces (generated by conformal blocks)
\[ P(c_1)^* H^0_{l=1} \left( Bun_G(\Sigma_1) \right) \perp P(c_2)^* H^0_{l=1} \left( Bun_G(\Sigma_2) \right) \]

When the level is zero, as in Conjecture 19, there is some probabilistic intuition which suggests the corresponding subspaces are perpendicular. But when we are thinking about bundle valued measures, this seems less clear. But let’s assume perpendicularity.

On the one hand there is a decomposition
\[(4.1) \quad H^0_{l=1} = \left( \bigoplus_{n,m \geq 0} V(2n+1) \otimes L(1, (\frac{n}{2})^2) \otimes L(1, (\frac{m}{2})^2)^* \otimes V(2m+1)^* \right) \oplus \]
On the other hand there is a decomposition

\[
H^0_{l=1} = \bigoplus_{g \geq 0} \bigoplus_{N,M \geq 1} V(N) \otimes W(N,M) \otimes V(M)^*.
\]

How are these related? Suppose that the genus \( g = 0 \). In this case \( W(N,M) \) vanishes unless \( N = M = 1 \), and

\[
W(1,1) \subset L(c = 1, h = 0) \otimes L(c = 1, h = 0)^*
\]

These might actually be equal. In any event we can see how this genus zero piece fits into the decomposition \( 4.1 \). How about higher genus?

4.4. Locality. The Haar measure

\[
\prod_{v \in S^1} d\lambda_K(g_v) \text{ for } \prod_{v \in S^1} K
\]

is ultralocal in the sense that if \( S^1 = E \sqcup E^c \), then the measure splits as a product. Here the structure of the circle is essentially irrelevant.

Wiener measure is heuristically of the form

\[
d\nu^\beta = \frac{1}{Z} e^{-\beta E(g)} \prod_{v \in S^1} d\lambda_K(g_v)
\]

where \( E(g) = \frac{1}{2} \int_{S^1} (dg \wedge * dg) \) is the standard energy (see \( 25 \) and references for a thorough justification of this point of view, for mathematicians). This is local, in the sense that (1) it is possible to define Wiener measure on intervals with Dirichlet boundary condition, and (2) if \( S^1 = I \sqcup I^c \), where \( I \) is an interval, then it is possible to express \( d\nu^\beta \) as a convex combination of local Wiener type measures as in (1).

Heat kernel measures \( \nu_t^{(s)} \) are not local in any sense that I can imagine. It might be possible to formulate heat kernel measures for an interval with appropriate boundary conditions, but it is not at all clear how to split up \( \nu_t^{(s)} \) as in the case of Wiener measures.

If \( s = 1 \), then \( \nu_1^{(1)} = [\nu_\beta], t = 1/\beta \). Hence from a representation theoretic point of view (as above), it does not matter whether we think about \( \nu_1^{(1)} \) or \( \nu_\beta \). But from a heuristic point of view it does not seem possible to express \( \nu_t^{(s)} \) in terms of a local functional against the ultralocal background \( 4.2 \). The upshot seems to be that heat kernel measures are nonlocal.

In the next section (see Subsubsection \( 5.2.8 \)) we will see that it is important to identify locality characteristics for \( \mu_0 \). The situation here could be similar to what Jones and Wassermann discovered for loop groups acting on highest weight representations; see \( 56 \). Loops supported in \( I \) and loops supported in \( I^c \) commute, hence their actions on \( L^2(\mu_0) \) commute (A major complication is that we have not proven that \( C^\infty \) loops - as opposed to \( C^\omega \) loops - fix \( \mu_0 \). We will put this aside). In the case of highest weight representations (which in some respects is more complicated, because of the central extension), the corresponding Von Neumann algebras are commutants. Is this true for the natural representation associated to \( \mu_0 \) (one should perhaps first investigate this for Wiener measure \( w_T \) and let \( T \uparrow \infty \))? In the next section we will toy with the idea of thinking of \( \mu_0 \) as having a kind of
tautological density relative to the ultralocal Haar measure for $\prod S_1 K$, namely the characteristic function for its support (which we also think of as a surrogate for the missing unitary form for $Hyp(S^1,G)$). But this is clearly too vague to be useful. We clearly need to understand $L^2(\mu_0)$ from the Jones-Wassermann point of view, to get started.

5. The Chiral Models $CM_l$

In the remaining sections I will discuss the relevance of the measures $\mu_l$, and their conjectural structure, to 2D sigma models with target $K$. We will mainly focus on the case $K = SU(2)$.

If $\Sigma$ is a closed Riemann surface, we would like to make sense of the heuristic Feynman measure on maps (or fields) $g : \Sigma \to K$,

$$\exp(-\int_{\Sigma} \left( \frac{1}{2T} (g^{-1}dg \wedge *g^{-1}dg) + 2\pi il\Gamma(g) \right) \prod_{v \in \Sigma} d\lambda(g_v))$$

where $\Gamma$ is the (multi-valued) WZW term, $T > 0$ is a dimensionless parameter, and the multi-valuedness of $\Gamma$ forces $l$ to be integral. To emphasize, at a heuristic level, this Feynman measure is conformally invariant. In principle the meaning of ‘to make sense of’ is to construct a quantum field theory satisfying the axioms in (for example) [52]. If $l = 0$, then this is the sigma model with target $K$, i.e. the principal chiral model, denoted $CM = CM_0$ (or sometimes $PCM$). It is not known how to mathematically construct the quantum chiral model in finite volume. However the action is local, and the model should be a kind of central limit for a class of statistical mechanical models. Renormalization group heuristics suggest that conformal invariance is broken at the quantum level, and the dimensionless parameter $T$ transmutes into a mass parameter $M$. In $\mathbb{R}^{1,1}$ there is a remarkable Yangian symmetry (or to put it another way, the scattering is assumed to be elastic), and consequently there is a conjectural description of the scattering theory, see e.g. [88], [63] and [14]. On the other hand the chiral model is believed to be asymptotically free. This suggests that for the Minkowski type space $RS^1 \times \mathbb{R}$, if the radius $R$ is small (or maybe just finite), then in some sense to be made precise, the theory should behave like a free massless theory. Below we will speculate that the conjectured factorization of $\mu_0$ in root subgroup coordinates might lead to an explanation for this.

If $l \in \mathbb{N}$, then the situation is murkier. The action is not local, and there apparently does not exist a tangible connection with statistical mechanics. Nonetheless renormalization group ideas are relevant. The special value $T = 1/l$ corresponds to a renormalization group fixed point, the $WZW_l$ conformally invariant model. One possible interpretation of $[89]$ is that there should be a one parameter family of massless integrable theories which interpolates between a massless theory $CM_l$ and the conformally invariant $WZW_l$ model. The parameter $T$ presumably transmutes into the flow parameter, the nature of which is unclear. I am confused by the interpretation in [13], which seems to imply that $CM_l$ is conformally invariant, and which seems to suggest that inversion invariance is broken, yielding a left right Yangian symmetry, analogous to the left right symmetry for the WZW model (the recent preprint [65] is also relevant to $CM_l$).

The $WZW_l$ model, as a conformal field theory involving closed strings only, has apparently been constructed in the sense of Segal (see [50] and [51]), which is a
major achievement. Construction of the corresponding boundary conformal field theory is apparently open. One also wonders in the closed case if there is some more elementary approach which might shed some light on the flow from $CM_l$ to $WZW_l$.

In this section we consider the ordinary sigma model ($l = 0$), initially with target $X$. The first subsection, on the classical theory, is included to motivate our point of view, nothing more.

We will then discuss some possible conjectures about the quantum theory, suggested by the structure of $\mu_0$.

**Question 6.** Something that I do not understand is whether the fusion algebra, for fixed positive level $l$, is a topological quantum field theory. If so, how does this fit into this picture?

5.1. The Classical Euclidean Perspective. The fields for the sigma model are maps $x : \Sigma \rightarrow X$, where $\Sigma$ is a space time and $X$ (the target) is a Riemannian manifold. The action is given by

$$S : W^1(\Sigma, X) \rightarrow \mathbb{R} : x \rightarrow \frac{1}{2} \int_{\Sigma} \langle dx \wedge *dx \rangle$$

where the derivative $dx$ is regarded as a one form on $\Sigma$ with values in the tangent bundle of the target and $* = *_{\Sigma}$ is the star operator. When $\dim(\Sigma) = 2$, this action depends only on the conformal structure of $\Sigma$, because the star operator is conformally invariant on one forms in dimension 2.

The traditional Hamiltonian point of view is developed for example in [14] or section 2 of [81]. From the Hamiltonian point of view one considers a Minkowskian type space time $\Sigma = \mathbb{R}^{1,1}$ or $\Sigma := (S^1 \times \mathbb{R}, dt^2 - d\theta^2)$. The classical solutions are so called wave maps $\Sigma \rightarrow X$, and the initial value problem is known to be globally well-posed if the target $X$ is compact.

Suppose that $X = K$. If $\Sigma = \mathbb{R}^{1,1}$, then the wave map system is integrable in the sense that there exist a zero curvature representation of the equations and a classical Yangian symmetry (see e.g. [14]). If $\Sigma = S^1 \times \mathbb{R}$, with nontrivial topology, the Yangian symmetry does not exist, and it is perhaps unlikely that the system is completely integrable in the sense of the existence of action-angle variables (on the complement of some ‘small subset’).

We will adopt a Euclidean perspective which is more akin to the point of view of [52]. In this view we think of the classical two dimensional sigma model as a functor from Segal’s category of compact Riemann surfaces, where the objects are compact oriented 1-manifolds, and the morphisms are compact Riemann surfaces, to (an infinite dimensional version of) the symplectic category of Guillemin/Sternberg and Weinstein, where the objects are symplectic manifolds and the morphisms are Lagrangian submanifolds (see [61] for a tutorial on this point of view). More precisely a compact oriented 1-manifold $S$ maps to the cotangent bundle of the configuration space, $W^{1/2}(S, X)$ (closed strings in $X$ parameterized by $S$), and a compact Riemann surface $\Sigma$ with boundary $S$ maps to the Lagrangian submanifold of $T^*W^{1/2}(S, X)$ defined by

$$W^1Harm(\Sigma, X) \subset T^*W^{1/2}(S, X) : x \rightarrow *dx|_{S}$$

where $W^1Harm(\Sigma, X)$ is the space of harmonic maps from $\Sigma$ to $X$. Note $*dx|_{S}$ remembers the boundary values and outward pointing normal derivative of the
harmonic map $x$, i.e. the Dirichlet and Neumann data associated to the map. If $X$ is linear, then this Lagrangian submanifold is a graph, because the Neumann data is a function of the Dirichlet data. In general the geometry of these Lagrangian submanifolds is highly complex, reflecting the nonlinearity of the harmonic map equations.

**Proposition 4.** Consider a composition $\Sigma_2 \circ \Sigma_1$ in Segal’s category. Then

$$\text{Harm}(\Sigma_2, X) \circ \text{Harm}(\Sigma_1, X) = \text{Harm}(\Sigma_2 \circ \Sigma_1, X)$$

**Remarks.**

(a) On the left hand side of this proposition, $\circ$ denotes composition in the infinite dimensional generalization of the symplectic category, while on the right hand side $\circ$ denotes composition in Segal’s category of Riemann surfaces.

(b) When all of the abstraction is unwound, this simply says that given harmonic maps $x_1 : \Sigma_1 \to X$ and $x_2 : \Sigma_2 \to X$, to put them together to obtain a harmonic map $\Sigma_2 \circ \Sigma_1 \to X$, it is necessary and sufficient that the values of the maps agree along the common boundary and that the normal derivatives match up.

This is how sewing is understood classically. We want to understand sewing in the quantum realm.

(c) Roughly as in the Minkowskian setting, when $\Sigma = \mathbb{P}^1$ is simply connected, harmonic maps into $K$ are in some sense known (or at least expressible in terms of a holomorphic map into the basic homogeneous space for $LK$ and a Gram-Schmidt process), and for other surfaces a ‘classification’ is elusive (see e.g. [84], [38] and [41]).

5.2. **On the Hamiltonian for $CM_0$.** Our main goal in this subsection is to formulate a possible conjecture for the $CM_0$ Hamiltonian. This will involve the introduction of three hypotheses. At least in my view, the first two hypotheses are quite plausible.

For the classical chiral sigma with target $X$, in the Euclidean framework, a compact one manifold $S$ is mapped to the phase space, the cotangent bundle of $W^{1/2}(S, X)$. For the quantum sigma model, in the Euclidean framework, $S$ should naively map to a quantization of the cotangent bundle, e.g. to the Hilbert space of half-densities associated to the configuration space $W^{1/2}(S, X)$. In general this is meaningless. The basic problem is that to define a space of half-densities, we need a measure class, and interesting measure classes tend to live on completions, or a thickening, of $W^{1/2}(S, X)$. From a physics perspective, because of the uncertainty principle, one cannot expect that the time zero quantum fields will have values in the possibly curved space $X$. A solution to this problem, at least in principle, is to embed $X$ into $\mathbb{R}^n$, use ‘$\mathbb{R}^n$-valued’ generalized functions defined on $S$, and use a potential to coax fields to have some affinity for having values near $X$. In practice a renormalization group process is necessary, which modifies the potential at each scale, and what one ends up with is invariably obscure.

Now suppose that $X = K$. In this case $Hyp(S, G)$ is a natural thickening of $TW^{1/2}(S, K) \sim W^{1/2}(S, G)$ which is homotopically faithful and equivariant with respect to real analytic $K$ valued loops acting from the left and right. There is a natural measure class associated to $W^{1/2}(S, K)$, namely the measure class of $\mu_0$ (assuming $S$ is a simple circle; otherwise we interpret this as the product measure for the loop groups corresponding to the different components of $S$). As we have stressed previously, a basic theme is that the support of the measure class of $\mu_0$ is
a kind of surrogate for the classical configuration space, $W^{1/2}(S, K)$.

5.2.1. First Hypothesis. $S$ maps to the Hilbert space of half-densities of the measure class, $H^{1/2}([\mu_0])$. In other words we are substituting the measure class of $\mu_0$ for the configuration space. We can and will identify the state space with $L^2(\mu_0)$ whenever this is convenient. As we will see below, the fact that there is a canonical measure representing the measure class $[\mu_0]$ is important.

5.2.2. Comments. Any two separable infinite dimensional Hilbert spaces are isomorphic. The question is whether this specific realization of the quantum state space has some natural meaning. The fundamental problems are to associate a vector $Z \in H^{1/2}([\mu_0])$ to a morphism $\Sigma$ with boundary $S$, and to prove the Segal sewing axioms are satisfied. The standard heuristic prescription (as in [69]) is to consider the Feynman measure associated to the double $\hat{\Sigma} = \Sigma^* \circ \Sigma$ and set

$$Z(\Sigma) := (Eval_S)_* \left( \exp \left( -\frac{1}{T} S(g) \right) \prod_{v \in \hat{\Sigma}} d\lambda_K(g_v) \right)^{1/2} \in H^{1/2}([\mu_0])$$

where $T$ is a coupling constant.

We will initially focus on the flat Euclidean cylinder $RS^1 \times \mathbb{R}$. The nature of the dependence on the radius $R$ is important, but we will temporarily assume $R = 1$. There will be an associated one parameter semigroup of contractions, which we can interpret as a homogeneous Markov process (if we analytically continue to the Minkowski point of view, we obtain a one parameter group of isometries, and the generator, the Hamiltonian $H$, has positive energy).

Remark. As an aside, I will briefly mention some standard heuristics - which admittedly are of limited utility.

For general target $X$, on the cylinder the action is given by

$$(5.1) \quad S(x : S^1 \times \mathbb{R} \to X) = \frac{1}{2} \int_{S^1} \left| \frac{\partial x}{\partial t} \right|^2 + \left| \frac{\partial x}{\partial \theta} \right|^2 \, d\theta \, dt$$

The time zero fields constitute the loop space $Map(S^1, X)$. The tangent space to the loop space at $x$ is naturally identified with $\Omega^0(T^* X)$, the space of vector fields along the loop $x$. There is a $W^0$ Riemannian metric on this tangent space, given by

$$(5.2) \quad \langle v, w \rangle_x = \int_{S^1} \langle v(\theta), w(\theta) \rangle_{x(\theta)} \, d\theta$$

where $v(\theta), w(\theta) \in TX|_{x(\theta)}$, and $\langle \cdot, \cdot \rangle_x(\theta)$ denotes the inner product (Riemannian metric) for $X$ at the point $x(\theta)$. In this way we can view $Map(S^1, X)$ as a Riemannian manifold, which we denote by $W^0(S^1, X)$.

In the second expression in (5.1) for the action, the first term is the usual kinetic energy for a path in the Riemannian manifold $W^0(S^1, X)$, and the second term represents a potential energy term, corresponding to the energy function on the finite energy loop space $W^1(S^1, X)$,

$$(5.3) \quad E(x : S^1 \to X) = \frac{1}{2} \int_{S^1} \langle dx \wedge * dx \rangle = \frac{1}{2} \int \left| \frac{\partial x}{\partial \theta} \right|^2 d\theta$$
Note that the Riemannian metric $E$ and $E$ depend upon the radius of $S^1$.

From this we heuristically deduce that the quantum Hamiltonian for the sigma model is of the form

$$H = \Delta_{W^0} + E$$

where $\Delta_{W^0}$ is the Laplacian for the Riemannian manifold $W^0(S^1, X)$, and $E$ is viewed as a (extremely singular) multiplication operator.

Admittedly, this point of view does not at all suggest why $W^{1/2}(S^1, X)$ is the natural analytic configuration space.

5.2.3. Second Hypothesis. As above, our first hypothesis is that the state space is $H_{1/2}(\mu_0)$, which we will identify with $L^2(d\mu_0)$. Our second hypothesis is that the ground state is $\mu_{1/2}^{0} \in H_{1/2}(\mu_0)$, or in other words the characteristic function of the support of $\mu_0$ in $L^2(d\mu_0)$.

5.2.4. Comments. Heuristically, if we consider $H_0 = \Delta_{W^0}$ (the Laplacian from Remark 5.2.2), then we would classically be considering geodesics on the product group $\prod_{S^1} K$, and the ground state would be the Haar measure $\prod_{S^1} d\lambda_K$ (or more precisely its square root). For the chiral model the energy function is added as a potential. Our second hypothesis is essentially asserting that the ground state is now shaped more like the characteristic function for ‘the support of $\mu_0$’, our quantum surrogate for $W^{1/2}(S^1, K)$. In particular the restriction of a field to space is marginally distributional in nature; we can only make sense of such a field in terms of, for example, its root subgroup (or nonlinear Fourier series) coefficients.

In combination with Segal’s axioms, this second hypothesis has a surprising consequence. Segal’s axioms mimic properties of the path integral, and when we assume the vacuum has a measure theoretic realization, it basically means that the path integral actually exists. We will discuss this below.

5.2.5. Third Hypothesis. Our objective now is to use what we have learned (or at least conjectured) about the measure $\mu_0$ to make an educated guess at the dynamics for the chiral model. For a free field (essentially the chiral model with target $\mathbb{R}$ in place of $K$), the quantum field is an assembly of harmonic oscillators, which is apparent in terms of the Fourier transform. This is true independent of the dimension of space, and whether space is compact or otherwise. For the nonabelian $K$ valued chiral model dimension and compactness matter a great deal.

We are now focusing on the chiral model restricted to $S^1 \times \mathbb{R}$, i.e. space is compact and one dimensional ($R = 1$). The analogue of the Fourier transform for a loop in $K$ is root subgroup factorization. As it happens the vacuum in our second hypothesis factors in root subgroup coordinates. This should tell us something.

Our third hypothesis is that the infinitesimal generator $H$ also factors in root subgroup coordinates. At first sight this must seem nuts. We will first spell this out in a precise way, then discuss why this might actually be possible. Even if this is naive, it might be useful.
Given a $g \in W^{1/2}(S^1, K)$ having a triangular factorization, there is a unique root subgroup factorization (or multiplicative Fourier series)

(5.4)

$$g(z) = \prod_{i \geq 0} a(\eta_i) \begin{pmatrix} 1 & \eta_i z^i \\ -\eta_i z^{-i} & 1 \end{pmatrix} \begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)} \end{pmatrix} \prod_{k \geq 1} a(\zeta_k) \begin{pmatrix} 1 & \zeta_k z^{-k} \\ -\zeta_k z^k & 1 \end{pmatrix}, \quad |z| = 1,$$

where $\chi(z) = \sum \chi_j z^j$ is an $i\mathbb{R}$-valued Fourier series (modulo $2\pi i\mathbb{Z}$).

Remark. It is useful to recall that the $\eta_i$ and $\zeta_j$ variables should be viewed as something similar to affine coordinates for spheres. This is because the limit points are of the form $(z, \zeta)$ which potentially leads to problems when we consider all the modes at once (because we are subtracting infinity).

$$\chi(z) = \sum \chi_j z^j$$

Then we are subtracting infinity.

Remark. It is useful to recall that the $\eta_i$ and $\zeta_j$ variables should be viewed as something similar to affine coordinates for spheres. This is because the limit points are of the form $(z, \zeta)$ which potentially leads to problems when we consider all the modes at once (because we are subtracting infinity).

We should emphasize that this is contingent on subtracting the ground state energy, which potentially leads to problems when we consider all the modes at once (because then we are subtracting infinity).

$$\text{trace}(e^{-tH_{Re(\chi_j)}}) = \sum_{n=0}^{\infty} e^{-\omega_n} = \frac{1}{1 - e^{-4jt}}$$

We should emphasize that this is contingent on subtracting the ground state energy, which potentially leads to problems when we consider all the modes at once (because then we are subtracting infinity).
Naively, the partition function for the $\chi$ Hamiltonian is the product

$$\text{trace}(e^{-tH_\chi}) = \prod_{j=1}^{\infty} \prod_{n=0}^{\infty} \frac{1}{1 - e^{-jt}} = \frac{1}{\phi(e^{-4t})^2}$$

If we remember that we subtracted $\sum_{n=1}^{\infty} n$, then we should correct this using a zeta function regularization and write

$$\text{trace}(e^{-tH_\chi}) = \frac{1}{\eta(4\tau)^2} \quad \text{where } \tau = \frac{it}{2\pi}$$

One can arrive at this same formula by using the standard zeta regularization of the determinant for the Laplace operator on a torus (see sections 1 and 2 of [33]), or for path integral heuristics, see page 341 of [22]. At least in my view, the justification for using zeta function regularization is that it is consistent with Segal’s axioms for quantum field theory (this is one point of [69]).

In the general case of $K$, using the same reasoning, the partition function is, in the first form,

$$\text{trace}(e^{-tH_\chi}) = \frac{1}{\phi(e^{-2gt})^{2r}}$$

where $r$ is the rank.

**Remark.** Note this is not modular invariant by itself, because our abelian loops are not winding around the torus.

Now consider (in the $SU(2)$ case) one of the modes corresponding to a real root $\tau$, say corresponding to the parameter $\eta_i$. The corresponding measure has a density proportional to

$$\frac{1}{(1 + |\eta|^2)^{2+2i}}$$

**Remark.** Recall from Remark 5.2.5 that one should actually think of $\eta_i$ as an affine parameter for a sphere. Let $\kappa$ denote the canonical bundle (cotangent bundle) for $\mathbb{P}^1$. For the bundle $\kappa^{-1/2}$ on $\mathbb{P}^1$ (which can be identified with the dual of the tautological bundle) the norm of the canonical section is

$$\frac{1}{(1 + |\eta|^2)}$$

So there is a geometric interpretation lurking here.

Our basic observation is that this density corresponds a kind of spherical harmonic oscillator.

Consider the rotationally invariant metric on $\mathbb{P}^1$ with unit Gaussian curvature. In the standard affine coordinate $z$, the metric is

$$ds^2 = \frac{4}{(1 + z\bar{z})^2}(dx^2 + dy^2)$$

Let $\Delta$ denote the corresponding nonnegative Laplace operator.

Fix a parameter $\omega \geq 0$. We now consider what we will call the spherical harmonic oscillator

$$L_\omega = \Delta + \omega^2 r^2 = \frac{1}{4}(1 + r^2)^2 \left( (\frac{\partial}{\partial r})^2 + \frac{1}{r} \frac{\partial}{\partial r} + r^{-2} \left( \frac{\partial}{\partial \theta} \right)^2 \right) + \omega^2 r^2$$

where $z = re^{i\theta}$. 
Theorem 5.1. (a) The ground state is 

\[(1 + r^2)^{-\omega}\]

with corresponding eigenvalue \(\omega\).

(b) If \(\omega = 0\), then the spectrum is \(n(n + 1)\) with multiplicity \(2n + 1\), \(n = 0, 1, \ldots\).

The partition function

\[\text{tr}(e^{-tH}) = \sum_{m=0}^{\infty} (2m + 1)q^m = \sum_{m \geq 0, n=0}^{\infty} +2 \sum_{m \geq 0, n \geq 1}^{\infty} q^{(n+m)(n+m+1)}\]

(c) If \(\omega > 0\), then the spectrum is \(\omega\), with multiplicity one, and

\[\lambda_{m,n} = m^2 + m + m(n + \sqrt{n^2 + \omega^2}) + (n + 1) \frac{n + \sqrt{n^2 + \omega^2}}{2}\]

where generically the multiplicity is 2, but it is possible there could be exceptional cases, related to Pythagorean triples.

The partition function is

\[\text{tr}(e^{-tH}) = (\sum_{m \geq 0, n=0}^{\infty} +2 \sum_{m \geq 0, n \geq 1}^{\infty} q^{\lambda_{m,n}}\]

where \(q = e^{-t} = e^{2\pi i r/\tau}\), \(\tau = \frac{it}{2\pi}\).

We write down the corresponding eigenfunctions in the proof.

Proof. It is straightforward to check that the function in (a) is an eigenfunction. It is positive, hence it is the groundstate (We will add an abstract argument in Remark 5.2.5 following the proof. This is how we were led the operator from the groundstate).

To find eigenvalues and eigenfunctions, we use separation of variables. Suppose we consider a eigenfunction of the form \(g(r)f(\theta)\). Then

\[-\frac{1}{4}(1 + r^2)^2(g''(r)f(\theta) + \frac{1}{r}g'(r)f(\theta) + \frac{1}{r^2}g(r)f''(\theta)) + \omega r^2 g(r)f(\theta) = \lambda g(r)f(\theta)\]

Then

\[-(1 + r^2)^2\left(\frac{g''(r)}{g} + \frac{1}{r}g'(r) + \frac{1}{r^2}f''(\theta)\right) + 4\omega r^2 = 4\lambda\]

and

\[\frac{r^2}{(1 + r^2)^2}\left(-(1 + r^2)^2\left(\frac{g''(r)}{g} + \frac{1}{r}g'(r) + \frac{1}{r^2} + 4\omega r^2 - 4\lambda\right) = f''(\theta)\right)\]

This implies that \(f(\theta)\) is a combination of \(\cos(n\theta)\) and \(\sin(n\theta)\), where \(n\) is a non-negative integer, and

\[-(1 + r^2)^2\left(g''(r) + \frac{1}{r}g'(r) - \frac{n^2}{r^2} g(r)\right) + 4\omega r^2 g(r) = 4\lambda g(r)\]

or

\[r^2 g''(r) + rg'(r) + (-n^2 + 4(1 + r^2)^{-2}(-\omega r^4 + \lambda r^2))g(r) = 0\]

This has regular singular points and is equivalent to the hypergeometric equation. The indicial equation is \(\alpha^2 - n^2 = 0\) and \(g\) has the form

\[g(r) = r^{|\alpha|} \sum_{l=0}^{\infty} g_l r^l\]
The relevant solution involves \( +n \), so that \( g \) is regular at \( r = 0 \). One finds
\[
g(r) = r^n (1 + r^2)^{(1 - \sqrt{4 \omega^2 + 4 \lambda + 1})/2} F(a, b, c, z)
\]
where
\[
a = \frac{1}{2} (-\sqrt{4 \omega^2 + 4 \lambda + 1 + n + 1} + \sqrt{4 \omega^2 + n^2})
\]
\[
b = \frac{1}{2} (-\sqrt{4 \omega^2 + 4 \lambda + 1 + n + 1} - \sqrt{4 \omega^2 + n^2})
\]
\[
c = n + 1, \text{ and } z = -r^2.
\]

The condition for \( \lambda \) to be an eigenvalue is that \( g(r) \) is square integrable on \((0, \infty)\) with respect to \((1 + r^2)^{-1} r dr\). A necessary condition is that \( a \) or \( b \) is a nonpositive integer.

Suppose that \( a \) or \( b \) equals \(-l\), a nonpositive integer, so that \( F(a, b, c, -r^2) \) will be a polynomial of degree \( l \) in \( z = -r^2 \). Then
\[
\lambda_{l,n} = l^2 + (n + 1)l + \frac{1}{2}(n^2 + n) + \frac{2l + n + 1}{2}\sqrt{4 \omega^2 + n^2}
\]
\[
= l^2 + l + l(n + \sqrt{n^2 + 4 \omega^2}) + (n + 1)\frac{n + \sqrt{n^2 + 4 \omega^2}}{2}
\]
We prefer the plus sign for the square root, because we know that the ground state has to correspond to \( \lambda_{l=0,n=0} = \omega \) we are off by a 2 in above formula for eigenvalue. This is the smallest possibility if we choose the plus sign.

Examples. As a check, suppose that \( \omega = 0 \). If \( n = 0 \), then \( \lambda_{m,n=0} = m^2 + m = m(m+1) \). This corresponds to a unique spherical harmonic eigenfunction. Now fix \( \lambda \) corresponding to \( m_0 \) and \( n_0 = 0 \). To find the multiplicity of this eigenvalue, we ask how many \( m, n > 0 \) are there so that
\[
m_0(m_0 + 1) = m^2 + (n + 1)m + \frac{1}{2}(n^2 + n) - \frac{1}{2}n(2m + n + 1)
\]
\[
= m^2 + m + 2mn + n(n + 1) = (m + n)^2 + m + n = (m + n)(m + n + 1)
\]
Thus
\[
\lambda m, n = \lambda_{m+n}
\]
This number is clearly \( m_0 \), because in the last expression \( m \) can be anything between \( 1 \) and \( m_0 \). Thus the multiplicity is \( 1 + 2m_0 \). (because when \( n = 0 \) we can only choose cos and when \( n > 0 \) we can choose cos or sin).

We return to the general case.

Question 7. Suppose \( \omega > 0 \). Given \( \lambda \), does there exist a unique pair \( m, n \) such that \( \lambda = \lambda_{m,n} \)

Before trying to answer this, let’s see if there is another necessary condition for \( \lambda_{m,n} \) to be an eigenvalue. For \( g \) to be square integrable, we need
\[
\int_1^\infty r^{2\lambda - \frac{1}{2}(\sqrt{4 \omega^2 + 4 \lambda + 1} - 1) + 2m} \frac{rdr}{(1 + r^2)^2} < \infty
\]

or
\[
\int_1^\infty u^{\lambda - \frac{1}{2}(\sqrt{4 \omega^2 + 4 \lambda + 1} + 1) + m} \frac{du}{(1 + u)^2} < \infty
\]
This implies
\[ 2(n - \frac{1}{2}(\sqrt{4\omega^2 + 4\lambda + 1} - 1) + 2m) - 3 < -1 \]
Suppose that we fix the value of \( \lambda \). Then we can conclude
\[ 2(n + 2m) < 1 + \sqrt{4\omega^2 + 4\lambda + 1} \]
This bounds the possible \( m, n \) producing a given value of \( \lambda \), in case there is some multiplicity.

**Question 8.** Given \( \lambda = \lambda_{m,n} \), do \( m, n \) necessarily satisfy this inequality, i.e. is it true that
\[ (2(n+2m)-1)^2 < 1 + 4\omega^2 + 4(m^2 + m + m(n + \sqrt{n^2 + 4\omega^2}) + (n+1)(n + \sqrt{n^2 + 4\omega^2})^2 \]

It suffices to assume \( \omega = 0 \), and then this is easy.

\[ \square \]

**Remark.** This is a continuation of Remark 5.2.5. The ground state calculation can be cast in more abstract terms. Suppose that \( L \) is a holomorphic hermitian line bundle with canonical connection \( \nabla \), where \( \nabla(0,1) = \partial \). There is an associated nonnegative Laplace operator
\[ \Delta \]
If \( s \) is a holomorphic section, then
\[ \partial(s,s) = \theta(s,s) \]
(see page 73 of Griffiths and Harris) and
\[ \overline{\partial}\partial(s,s) = \overline{\theta}(s,s) - \theta \wedge \overline{\theta}(s,s) \]
Thus
\[ \overline{\partial}\theta(s,s) + \theta \wedge \overline{\theta}(s,s) = \Theta(s,s) \]
where \( \Theta \) is the curvature, a \((1,1)\) form. As we noted in Remark 5.2.5, in our case the line bundle is the kth power of the dual of the canonical bundle (or the 2kth power of the generating positive line bundle on \( \mathbb{P}^1 \), the dual of the tautological bundle).

5.2.6. **The Partition Function for CM₀, assuming K = SU(2).** The conjectural full partition function is an infinite product. We have already evaluated the free massless part of the product. Let
\[ \lambda_{k,m,n} := m^2 + m + m(n + \sqrt{n^2 + 16k^2}) + (n+1)(n + \sqrt{n^2 + 16k^2})^2 \]
The full partition function is
\[ \text{trace}(e^{-tH}) = \left( e^t \prod_{k=1}^\infty (\sum_{m \geq 0, n=0} + 2 \sum_{m \geq 0, n \geq 1}) e^{-t\lambda_{k,m,n}} \right)^2 \]
\[ \eta(e^{-4t})^2 \]
We should check that this actually converges.
Lemma 2. Suppose $q < 1$. Then the product

$$\prod_{k=1}^{\infty} \left( \sum_{m \geq 0, n=0}^{\infty} q^{m^2+m+m(n+\sqrt{n^2+16k^2})+(n+1)\frac{n+\sqrt{n^2+16k^2}}{2}-2k} \right)$$

converges and is nonzero.

Proof. We must show that

$$\sum_{k=0}^{\infty} \left( \sum_{m \geq 0, n=0}^{\infty} q^{m^2+m+m(n+\sqrt{n^2+16k^2})+(n+1)\frac{n+\sqrt{n^2+16k^2}}{2}-2k} \right)$$

converges. This can be split into two sums. The first sum is

$$\sum_{k=0}^{\infty} q^{m^2+m+4mk}$$

$$= \sum_{m=1}^{\infty} q^{m^2+m}(1-q^{4m})^{-1}$$

Since $(1-q^{4m})$ converges to 1 as $m \uparrow \infty$, this sum is convergent. The second sum is 2 times

$$\sum_{k=0}^{\infty} \sum_{m \geq 0, n \geq 1} q^{m^2+m+m(n+\sqrt{n^2+16k^2})+(n+1)\frac{n+\sqrt{n^2+16k^2}}{2}-2k}$$

$$\leq \sum_{k=0}^{\infty} \sum_{m \geq 0, n \geq 1} q^{m^2+m+m(n+4k)+(n+1)(n+2k)-2k}$$

$$\leq \sum_{k=0}^{\infty} \sum_{m \geq 0, n \geq 1} q^{m^2+m+m(n+4k)+(n+1)n+2nk}$$

$$= \sum_{m \geq 0, n \geq 1} \sum_{k=0}^{\infty} q^{m^2+m+n^2+n+mn} (1-q^{4m+2k})^{-1}$$

Since $(1-q^{4m+2k})$ tends to 1 as $m, n \uparrow \infty$, this will converge iff

$$\sum_{m \geq 0, n \geq 1} q^{m^2+m+n^2+n+mn}$$

converges. This does converge it would be convenient to have $2mn$ in places of $mn$.

This partition function is not modular invariant.
5.2.7. The Feynman Measure for $S^1 \times \mathbb{R}$. By abstract nonsense there is a path integral, or Feynman measure on the path space $C^0((-\infty, \infty), H yp(S^1, G))$. Given any time $t_1$, the projection of the measure by the evaluation map $g \to g(t_1)$ is the vacuum $d\mu_0$. Given times $t_1 < t_2 < ... < t_n$ the corresponding projected measure is

$$\prod_{j=2}^{n} \langle g_{j-1}|e^{-tH}|g_j \rangle \prod_{k=1}^{n} d\mu_0(g_k)$$

where $\langle g_1|e^{-tH}|g_2 \rangle$ denotes the kernel (it might be possible to evaluate this kernel exactly, similar to the Mehler kernel for the standard harmonic oscillator).

Just as we can define Wiener measure on $C^0(S^1, K)$ using evaluation projections

$$(eval)_*(w_t) = \prod_E p_{\mu(e)}(g_{\partial e}) \prod_V d\lambda_K(g_v)$$

we can, modulo some technical issues, define a Feynman type measure $W_t$ on $C^0(S^1, H yp(S^1, G))$ using evaluation projections

$$(eval)_*(W_t) = \prod_E P_{\mu(e)}(g_{\partial e}) \prod_V d\mu_0(g_v)$$

where now

$$P_t(g_{\partial e}) = \langle g_1|e^{-tH}|g_2 \rangle$$

denotes the kernel for $e^{-tH}$.

The total integral of the Wiener measure is $p_{2\pi i}(1)$. What is the analogue for $W_t$? It is $\text{trace}(e^{2\pi i H})$.

5.2.8. The Flat Torus, and the Dependence of $H$ on the Radius $R$. Consider the flat torus with projections

$$R_1S^1 \times R_2S^1 \leftarrow \begin{array}{c} \nearrow \times \searrow \\ R_1S^1 \searrow \nearrow R_2S^1 \end{array}$$

For each of these projections, there is a disintegration formula for a candidate for the Feynman measure corresponding to this flat torus. By calculating the total measure, using the previous subsections, we obtain

$$\text{trace}(e^{-2\pi R_1 H(R_2)}) = \text{trace}(e^{-2\pi R_2 H(R_1)})$$

In conformal field theory, at least according to page 337 of [22], $H(R) = \frac{1}{R} H(1)$. This would imply the consistency of (5.6). However so far as consistency of (??) is concerned, one could just as well assume that $H(R) = RH(1)$.

If one graphs the function $\text{trace}(e^{-tH(1)})$, one sees that it is not modular invariant, i.e. invariant with respect to $t \to 1/t$. The chiral model is believed to be massive, hence this is to be expected.

If there does exist a Feynman measure for the chiral model, hence for fields on this flat torus, then there must exist a more coordinate free definition of this measure. This is the point where it is important to know something about the locality properties of $\mu_0$, see Subsection 4.4.

Showing that these two disintegrations are consistent would be strong evidence that we are on the right track.
5.2.9. *Commentary.* Are these hypotheses correct?

The most obvious justification would be to show that this is part of a theory in the sense of Segal. Going from a flat torus and cylinder to a general Riemannian surface in a coherent way is daunting.

Secondly we need to establish asymptotic freedom in the limit $R \downarrow 0$. Thirdly we need to show that the infinite volume limit is compatible with the scattering predictions of \[65\]. For this we need to include the radius of the circle as a parameter.

5.2.10. *The $R \downarrow 0$ Limit and Asymptotic Freedom.* The chiral model is expected to be asymptotically free, vaguely meaning that as $R \downarrow 0$, the rank($K$) elementary particles should be approximately free massless and independent. In the case of 3D Yang-Mills, there is a technical formulation of what this might mean in \[17\]. If our hypotheses are correct, then nothing nearly as sophisticated as \[17\] is needed.

To properly address this issue, we probably need a theory which enables us to talk about observables.

Superficially it might seem that we actually have dim($K$) independent free fields, one for each positive root of $g$ and $r = \text{rank}(K)$ for the torus. But this is misleading. The apparent field that corresponds to a given positive root of $g$ is not really an independent quantum field. There is a fundamental difference for the vacua for the real roots versus the imaginary roots. In the infinite volume limit the spherical harmonic oscillators do not converge to

5.2.11. *The Limit as $R \uparrow \infty$.* We have to recover the Yangian symmetry and the scattering solution. We have not made any progress on this.

5.3. *On the Hamiltonian for the Chiral Model $CM_l$, $l = 1, 2, ..., K = SU(2)$.* This is completely parallel to the case $l = 0$. We will introduce parallel hypotheses, without much discussion.

Various aspects of the classical theory have been considered by many authors; for a sampling, see e.g. \[3\], \[13\], \[13\], \[85\]. The relevance of this to the Hamiltonian is that we need to consider sections of a line bundle, indexed by the level $l$, over the phase space, rather than simply functions.

5.3.1. *First Hypothesis.* Space $S$ maps to the Hilbert space of square integrable sections of $\Omega_l^0(Hyp(S^1, G))$. Note that the Hilbert space for $WZW_l$ is the subspace of holomorphic functions, and in turn this contains the ‘primary fields’:

$$\Omega_l^0(Hyp(S^1, G)) \supset H^0_l(Hyp(S^1, G)) \supset \langle \otimes_{\text{level} (\Lambda) = l} C \sigma_\Lambda$$

(in this case there is only one primary field, $\sigma_0$, heuristically the determinant of the Toeplitz operator associated to a loop).

5.3.2. *Second Hypothesis.* The ground state is $\sigma_l^0$. Recall that

$$d\mu_l := \sigma_0 \otimes \overline{\sigma_0} d\mu|^{il}$$

We think of this groundstate as a square root.
5.3.3. Third Hypothesis. As in the case \( l = 0 \), the measure \( \mu_l \), and also the square root \( \sigma_0^l \), factors in root subgroup coordinates. The one complication is that for \( \sigma_0 \) we have to use root subgroup coordinates for the central extension.

Our third hypothesis is that the infinitesimal generator \( H \) also factors in root subgroup coordinates.

Remark. It would make absolutely no sense to introduce a hypothesis like this for the WZW model. We know what the infinitesimal generator is in the WZW case: \( H = L_0 + \frac{\partial}{\partial y} \), which is only defined relative to the left right factorization of the Hilbert space for the WZW model:

\[
L^2 H^0_l(Hyp(S^1, G)) = \bigoplus_{\text{level}(\Lambda) = l} H(\Lambda) \otimes \overline{H(\Lambda)}
\]

In some sense we are trying to understand if there is some kind of flow from root subgroup coordinates to the left-right factorization for the WZW model. This could be completely nuts.

As before the form of the vacuum suggests a form for the Hamiltonian. The density for \( \chi \) (and asymptotic freedom) suggests that \( \chi \) is a massless free field, with the zero mode compactified.

Now consider (in the \( SU(2) \) case) one of the modes corresponding to a real root \( \tau \), say corresponding to the parameter \( \eta_i \). The corresponding measure has a density proportional to

\[
\frac{1}{(1 + |\eta_i|^2)^{2+(2+l)}}
\]

Fix a parameter \( \omega \geq 0 \). We now consider what we will call the spherical harmonic oscillator

\[
L_{\omega} = \Delta + \omega^2 r^2 = \frac{1}{4}(1 + r^2)^2 \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} + r^{-2} \left( \frac{\partial}{\partial \theta} \right)^2 + \omega^2 r^2
\]

where \( z = re^{i\theta} \) and now \( \Delta \) denotes the Laplacian acting on sections of \( T^* \otimes (2 + l) \), the \((2 + l)\) power of the dual of the tautological line bundle on \( \mathbb{P}^1 \).

We can calculate the spectrum exactly as before. The upshot is \( \dot{g} = 2 \), the dual Coxeter number, is replaced by \( \dot{g} + l \) in the formulas.

As in the case \( l = 0 \), the partition function will not be modular invariant. This is not consistent with [13].

5.4. Symmetric Space Targets. Consider the sigma model with target \( U/K \), a compact Riemannian symmetric space. As we will note in Section 7 there exists an analogue of \( \mu_0 \). Consequently there are analogues of the first two hypotheses in our discussion of the chiral model. However it seems unlikely that there exists an analogue of root subgroup coordinates.

6. Sewing Rules for WZW Modular Functors

It is well-known that at least heuristically the sewing rules for the WZW modular functor (the holomorphic part of the theory) should be a consequence of a holomorphic level \( l \) Peter-Weyl theorem, as in Conjecture [17]. The point of this section is to explain that if Conjectures [5] and [17] are true, then the sewing rules can be proved using Peter-Weyl. Whether the measure-theoretic point of view we are advocating is useful for proving Segal’s axioms for the full theory will be discussed in the next section.
In this section we will use, without comment, the general definition of a modular functor, and representation of a modular functor (or weakly conformal field theory), which is given in Section 5 of [78]. We will begin by recalling Segal’s construction of the level $l$ WZW modular functor and its representation. We will then discuss the sewing property.

6.1. Construction of the chiral WZW theory. Let $\Phi_l$ denote the finite set of dominant integral functionals $\Lambda$ of level $l$, or what is the same thing, the set of integrable highest weight representations of the Kac-Moody group $L_{\text{fin}}G$ of level $l$. A dominant integral functional $\Lambda$ of level $l$ is determined by a dominant integral functional $\tilde{\Lambda}$ on $\mathfrak{h}$ satisfying $\tilde{\Lambda}(h_\theta \leq l)$; see (1.11). These are the labels of the WZW$_l$ modular functor (or as a boundary conformal field theory).

Suppose that $\Lambda \in \Phi_l$. We will realize the dual Hilbert space representation, $H(\Lambda)^*$, using the Borel-Weil theorem, in the following way:

$$L^2H^0(\mathcal{L}_\Lambda^* \to \mathcal{F}_{\text{hyp}}) \subset H(\Lambda)^* \subset H^0(\mathcal{L}_\Lambda^* \to \mathcal{F}_{\text{an}})$$

We will realize $H(\Lambda)$ analogously, using the right coset flag space.

Suppose that we are given a collection, $\coprod C$, of labeled oriented circles, together with a positive parameterization,

$$p_C : S^1 \to C,$$

for each circle $C$ in the collection. To this data (or object) we associate the Hilbert space

$$H(\coprod C) = \bigotimes_C H(\Lambda_C)^{\epsilon_C},$$

where $\Lambda_C$ is the label for the component $C$, and $\epsilon_C$ is vacuous if $C$ is positively parameterized and the dual otherwise.

As in (6.1) we can also associate to this object a Borel-Weil realization of the Hilbert space,

$$L^2H^0(\mathcal{L}^* \to \prod \mathcal{F}_{C,\text{hyp}}) \subset H(\coprod C) \subset H^0(\mathcal{L}^* \to \prod \mathcal{F}_{C,\text{an}}).$$

Here $\mathcal{F}_{C,\text{an}}$ is the right, resp. left, coset flag space if $C$ is positively, resp. negatively, parameterized, and the line bundle over this product is the product of the line bundles corresponding to the factors.

The final space in (6.1) is acted upon by the oriented Baer product of the extensions corresponding to the components, i.e. the extension

$$0 \to \mathbb{C}^* \to \tilde{G}(H^0(\Pi C)) \to G(H^0(\Pi C)) \to 0,$$

where

$$0 \to ker\chi \to \prod \tilde{G}(H^0(C)) \to \tilde{G}(H^0(\coprod C)) \to 0,$$

and

$$\chi : \prod \mathbb{C}^* \to \mathbb{C}^* : (\lambda_C) \to \prod \lambda_C^C.$$
parameterizations of each of the boundary components, and (3) to each boundary component an assignment of a label from $\Phi_l$.

By Segal reciprocity, there is a canonical global cross-section

$$G(H^0(\partial \Sigma)) \to \tilde{G}(H^0(\partial \Sigma))$$

(because the induced Lie algebra cocycle vanishes on $g(H^0(\Sigma))$, by Cauchy’s theorem, and the group $G(H^0(\Sigma))$ is simply connected). There is a continuous extension of this cross-section to $H^0(\Sigma^0, G)$, which maps into the multicomponent analogue for the groups $G(H^0(S^1_1))$.

**Lemma 3.** The natural inclusions

$$L^2 H^0(\mathcal{L}^* \otimes l \to \mathcal{F}_{\partial \Sigma, \text{hyp}}) \to H(\partial \Sigma) H^0(\Sigma, G) \to H^0(\mathcal{L}^* \otimes l \to \mathcal{F}_{\partial \Sigma, \text{hyp}})$$

are isomorphisms.

**Idea of Proof** Suppose that $\sigma$ is an $H^0(\Sigma, G)$-invariant section of

$$\mathcal{L}^* \otimes l \to \mathcal{F}_{\partial \Sigma, \text{hyp}}$$

The action of $H^0(\Sigma, G)$ on $\mathcal{L}^* \to \mathcal{F}_{\partial \Sigma, \text{hyp}}$ extends continuously to a holomorphic action by $H^0(\Sigma^0, G)$. Since

$$\mathcal{F}_{\partial \Sigma, \text{hyp}}/H^0(\Sigma^0, G) \cong F_{\partial \Sigma, \text{hyp}} / H^0(\Sigma^0, G),$$

it follows that $\sigma$ extends to a holomorphic section of $\mathcal{L}^*$ over the hyperfunction flag space. It is automatically $L^2$ because the integral is performed over a compact space by Conjecture 5.

Returning to the definition of the level $k$ modular functor, we associate to the morphism $\Sigma$ the space in (6.6), which we denote by $E_l(\Sigma)$.  

6.2. The sewing property. Suppose that $\Sigma$ has two distinguished boundary components $C_{in}$ and $C_{out}$ which are negatively and positively parameterized, respectively. Suppose that $C_{in}$ and $C_{out}$ have the same label $\Lambda$. To indicate this, we will write $\Sigma = \Sigma_{\Lambda}$. Let $\tilde{\Sigma}$ denote the morphism obtained by sewing along these components (we assume that each component of $\tilde{\Sigma}$ has nonempty boundary). Then

$$H(\partial \Sigma) = H(\Lambda)^* \otimes H(\Lambda) \otimes H(\partial \tilde{\Sigma})$$

**Lemma 4.**

$$E_l(\Sigma_{\Lambda}) \subset \text{Domain}(\text{trace}).$$

There is a natural map

$$H^0(\Sigma, G) \leftarrow H^0(\tilde{\Sigma}, G)$$

This together with the lemma will imply that there is a natural map induced by trace,

$$E_l(\Sigma_{\Lambda}) \to E_l(\tilde{\Sigma}).$$

We will verify the lemma in the course of proving the following

**Proposition 5.** The natural map

$$\bigoplus_{\Lambda \in \Phi} E_l(\Sigma_{\Lambda}) \to E_l(\tilde{\Sigma})$$

is an isomorphism of vector spaces.
Idea of Proof} Given a pre-Hilbert space $V$, we will denote the completion by $V^{\text{complete}}$. We claim that

$$(6.14) \bigoplus_{\Lambda} E_{\ell}(\Sigma_{\Lambda}) = \left( L^{2} H^{0}(\tilde{L}_{hyp}^{\otimes l} \rightarrow Hyp(S^{1}, G)) \right)^{\text{complete}} \otimes H(\partial \tilde{\Sigma})^{H^{0}(\Sigma, G)}.$$ 

The second equality follows from the level $l$ Peter-Weyl theorem. In the third, resp. fourth, line, the notation means that we are considering sections having values in $H(\partial \tilde{\Sigma})$, resp. $H^{0}(\Sigma^{0} \rightarrow F_{\partial \tilde{\Sigma}, hyp})$. The third and fourth equalities follow from (??).

If we view a holomorphic section of

$$(6.15) \tilde{L}_{hyp}^{\otimes l} \otimes H^{0}(\mathcal{L}^{*} \rightarrow F_{\partial \tilde{\Sigma}, hyp}) \rightarrow Hyp(S^{1}, G)$$

as a section-valued function on $\tilde{L}_{hyp} G$, then the contraction map in (6.11) is given simply by

$$(6.16) F \rightarrow F(1),$$

where $1 \in \tilde{L}_{an} G \subset \tilde{L}_{hyp} G$. The action of $H^{0}(\Sigma, G)$ on such functions extends to $H^{0}(\Sigma^{0}, G)$; for $g \in H^{0}(\Sigma^{0}, G)$

$$(6.17) (g \cdot F)(\tilde{g}_{hyp}) = \tilde{g}_{\partial \tilde{\Sigma}} \cdot F(\tilde{g}_{-}^{-1} \cdot \tilde{g}_{hyp} \cdot \tilde{g}_{+}),$$

where $g_{\pm} \in H^{0}(S^{1}_{\pm})$ are the restrictions of $g$ to collars adjacent to $C_{in}$ and $C_{out}$, respectively, and $\tilde{g}_{\partial \tilde{\Sigma}}$ denotes the analogous restriction for $\partial \tilde{\Sigma}$. We also have chosen $\tilde{g}_{\pm} \in H^{0}(S^{1}_{\pm})$ and $\tilde{g}_{\partial \tilde{\Sigma}}$, so that

$$(6.18) [\tilde{g}_{\pm}, \tilde{g}_{\partial \tilde{\Sigma}}] = g$$

with respect to Segal reciprocity.

Because each component of $\tilde{\Sigma}$ has nonempty boundary, hence is a Stein manifold, it follows that the map

$$(6.19) H^{0}(\Sigma^{0}, G) \rightarrow Hyp(S^{1}, G) : g \rightarrow [g_{-}^{-1}, g_{+}]$$

is surjective. Setting $\tilde{g}_{hyp} = \lambda \in \mathbb{C}^{x}$ in (??), we now see that we can reconstruct $F$ in (6.19) from $F(1)$. Thus the map in the proposition is certainly injective.

It remains to show that $F$ is $L^{2}$, if we construct $F$ using (6.17) from $F(1)$. Because of the invariance properties that $F$ inherits, $F$ is the pullback of a section over a compact space, so that this is automatic.

6.3. WZW as a Boundary Conformal Field Theory. So far as I know it is an open question to rigorously formulate the WZW model as a boundary conformal field theory. There is a nice description of this from a heuristic point of view in [60].
7. Remarks on Generalizations

7.1. Symmetric Space Target. Suppose that $X$ is a simply connected compact symmetric space with a fixed basepoint. From this we obtain a diagram of groups, 

![Diagram](https://example.com/diagram.png)

(7.1)

where $U$ is the universal covering of the identity component of the isometry group of $X$, $X \simeq U/K$ (using the basepoint), $G$ is the complexification of $U$, and $X_0 = G_0/K$ is the noncompact type symmetric space dual to $X$; and a diagram of equivariant totally geodesic (Cartan) embeddings of symmetric spaces:

![Diagram](https://example.com/diagram.png)

(7.2)

Examples. There are two general examples which are of particular interest.

The first example of (7.2), using standard matrix groups for concreteness, is

![Diagram](https://example.com/diagram.png)

(7.3)

This is embedded in the second example, the ‘group case’,

![Diagram](https://example.com/diagram.png)

(7.4)

These two examples can be generalized in the following way. Fix a complex group $G$ (e.g. $SL(n, \mathbb{C})$, as above). Let $G_0$ be the normal real form (e.g. $SL(n, \mathbb{R})$), and let $U$ and $K$ be maximal compact subgroups of $G$ and $G_0$, respectively (e.g. $SU(n)$ and $SO(n)$, respectively).

We have focused exclusively on the group case in these notes. We will mention why we are interested in the first set of examples in the next subsection.

There is a prolongation of the diagram (7.2) to loop spaces. The formal completion of the loop space of $G/G_0$ is defined in terms of the inclusion

$\mathbf{L}(G/G_0) := \{ [g_1, g_2] \in \mathbf{L}G : [\Theta(g_2^*), \Theta(g_1^*)] = [g_1, g_2] \}

where $\Theta$ is the complex linear extension to $G$ of the Cartan involution which fixes $K$ inside $U$ (see [65]). The hyperfunction completion is similarly defined.
Remark. (a) $U/K \subset G/G_0$ and $C^0(S^1, U/K) \subset Hyp(S^1, G/G_0) \subset LG$ are homotopy equivalences, i.e. these completions faithfully remember the original topologies.

(b) From the point of view of qft, the uncertainty principle requires completions - a nonlinear quantum field will not (obviously) remember that its classical cousin has values in $U/K$.

(c) Given a general compact simply connected Riemannian $X$, it is not clear how to form similar completions. In qft, for example for the sigma model with target a sphere, it is standard to simply consider a $\phi^4$ theory, which is believed to be in the same universality class as the sigma model, hence should yield the same qft in an appropriate critical limit.

We will persist, because we aspire to realize the limit in an elegant way.

We now discuss unitarily invariant measures on these completions. There is an existence result (see [68]):

**Theorem 7.1.** For the natural action

$$L_{fin} U \times L(G/G_0) \to L(G/G_0)$$

there exists an invariant probability measure.

**Conjecture 20.** The invariant measure is unique.

As in the group case, uniqueness, and proof that the measure is supported on the hyperfunction completion, would imply invariance with respect to $L^\infty Homeo(S^1)$.

In the group case we formulated a conjecture which purports to characterize invariant measure classes having appropriate symmetry. The generalization of this to the symmetric space context is uncertain; the issue is whether there is a deformation involving something like a determinant line.

It is instructive to reflect on what we might hope to compute in this more general context:

1. A generic point in $L(G/G_0)$ has Riemann-Hilbert factorization $g = g_+ g_0 g_-$, where $g_+ : \Delta \to G$ and $g_- : \Delta^* \to G$ are holomorphic, $g_0 \in G/G_0 \subset G$ and $g_- = g_+^\Theta$. The analogue of linear Riemann-Hilbert coordinates for $Hyp(S^1, G/G_0)$ is the pair $(g_0, \theta_+ := g_0^{-1} \partial g_+) \in G/G_0 \times H^1(\Delta, g)$. There is a conjecture for the distribution of $g_0$, see 3. below. It is not obvious how to compute the distribution for $\theta_+$. As noted previously, I have so far failed to find a reasonable conjecture even in the group case.

2. Given a real analytic embedding $c : S^1 \to \Sigma_c$, one can consider the composition

$$Hyp(S^1, G/G_0) \to Hyp(S^1, G) \xrightarrow{p(c)} Bun_G(\Sigma_c)$$

For a generic $c$ this composition is probably surjective, and I have no guess for the pushforward of the invariant measure. However suppose that $c$ is a parameterization of the waist for a Riemann surface with reflection symmetry, i.e. $\Sigma_c$ is a double $\Sigma^* \circ \Sigma$. In this case the image of the composition, in terms of the parameterization $Bun^0_G(\Sigma_c) \sim Hom_{irred}(\pi_1, U)/U$, is identified with a submanifold of homomorphisms $\rho$ which satisfy the reality condition $\rho \circ R_* = \Theta \circ \rho$, where $R$ denotes reflection symmetry. This submanifold has a natural probability measure,
akin to the normalized symplectic volume of $\text{Hom}_{\text{irred}}(\pi_1, U)/U$, which conjecturally can be characterized in terms of its invariance with respect to the subgroup of the mapping class group compatible with reflection symmetry. Conjecturally this is the image of the unitarily invariant measure on $\text{Hyp}(S^1, G/G_0)$. This restriction on $c$ seemingly precludes the possibility of constructing the invariant measure in analogy with the construction of Wiener measure that we outlined in the group case.

3. Although it is more involved to formulate than in the group case, there is a diagonal distribution conjecture for $\mu_0$ which determines the distribution for the zero mode $g_0$ (see [68]). As we previously pointed out, in the group case this is a consequence of the existence of root subgroup coordinates. There does exist a homogeneous Poisson structure on $L(U/K)$. Consequently one can ask whether it is possible to formulate some analogue of root subgroup coordinates that would serve as action-angle variables. Whether this is possible is unknown to me.

7.2. Homogeneous Space Targets. There has been recent interest in sigma models with homogeneous space targets, see the review [1], or [64]. A generic flag manifold is of the form $U/T$, where $T$ is a maximal torus in $U$. In general there is not an essentially unique invariant metric, which introduces a family of parameters. The approach advocated in [1] and references is to consider a differential geometric (not necessarily isometric) embedding of $U/T$ into a product of symmetric spaces. This approach does suggest an extrinsic way to realize a kind of hyperfunction completion of the loop space of $U/T$, as a submanifold of the hyperfunction completions of the symmetric space factors. I have not made any progress in understanding how to formulate an existence result for an invariant measure that would have some intrinsic meaning. One obstacle is to understand the intrinsic meaning of the zero mode from this point of view (in the symmetric space context above, the zero mode is $g_0 \in G/G_0$, for which there is a conjecture for the distribution).

In [87] there is a speculative discussion concerning 2D sigma models with moduli space targets.

7.3. Symmetrizable Kac-Moody Groups. As evident from the title of [66], I once believed that the proper framework for this work was symmetrizable Kac-Moody Lie algebras. I am now skeptical. The following discussion might help to put these notes into perspective.

Given a symmetrizable generalized Cartan matrix $A$, there is a formal completion $G(A)$ of the Kac-Peterson group $G(A)$. The inclusion $G(A) \subset G(A)$ is a homotopy equivalence, and the formal completion is a natural algebro-geometric framework for the Kac-Peterson version of the Peter-Weyl theorem (see [66]). In affine and indefinite cases, this completion is not a group and there does not exist a unitary form. In affine cases (ignoring the automorphism group $\exp(\mathfrak{cL}_0)$), this completion is a $\mathbb{C}^\times$ bundle over a completion $L\hat{G}$ (or a twisted version of this) in the sense of Section [1].

**Question 9.** Suppose that $A$ is a generalized symmetrizable Cartan matrix. Does there exist an invariant probability measure for the action

$$K(A) \times G(A) \times K(A) \to G(A)$$
If \( A \) is of finite type, then the answer to the question is yes, and there are many bi-invariant measures which are classified using the Harish-Chandra transform. For example there is the Haar measure of \( K(A) \), the unique bi-invariant measure on the unitary form. At the other extreme, when we consider Riemann-Hilbert factorization for \( g \in \mathbf{L}G(A), \ g = g_0 g_+ \), the \( \mu_0 \) distribution for \( g_0 \) is a bi-invariant measure which we conjecture is absolutely continuous with respect to Haar measure for \( G(A) \), hence has a support which is very spread out (and of course it is very far from ergodic).

**Question 10.** Haar measure for \( K(A) \) can be realized as a product measure using root subgroup coordinates, and Haar measure for \( G(A) \) can be realized as a product measure using a complex analogue of root subgroup coordinates (see \([72]\)). It is not known (to me) if there are \( K(A) \) bi-invariant measures on \( G(A) \) which are absolutely continuous with respect to Haar measure for \( G(A) \) and can be realized in a simple way using root subgroup factorization. This might be at least heuristically relevant below.

Suppose that \( A \) is of (for simplicity untwisted) affine type, and \( \breve{G} \) is the corresponding finite dimensional complex group. In this case Conjecture \([1]\) asserts that there is a unique \( K(A) \) bi-invariant probability measure \( \mu_0 \) on the completion \( \mathbf{L}\breve{G} \), and we have emphasized that the support of \( \mu_0 \) is a kind of surrogate for a unitary real form for \( \mathbf{L}\breve{G} \). A \( K(A) \) bi-invariant probability measure \( \tilde{\mu}_0 \) on \( G(A) \) would, according to Conjecture \([1]\) necessarily project to \( \mu_0 \). I do not know how to prove the non-existence of this lift. However the existence of such a lift is inconsistent with basically everything we know about invariant measures. For example in order to integrate matrix coefficients, we have to fix a level and consider a bundle-valued measure on \( \mathbf{L}\breve{G} \), and these bundle-valued measures have disjoint support, which is not consistent with the existence of one lift.

As we have emphasized in these notes, there are multiple ways in which \( \mu_0 \) can conjecturally be realized: using root subgroup coordinates, using projections to moduli spaces of \( G \)-bundles on Riemann surfaces, using time ordered exponential coordinates (i.e. \( \theta_+ = g_+^{-1} \partial g_+ \) as a coordinate), and using holomorphic maps to flag spaces (we did not present concrete proposals in the latter two cases). There are lifts of these structures to the appropriate central extensions, but the corresponding lifted formulas for measures simply do not make any sense. For example a key point in Section \([3]\) is that for triangular factorization for a loop in \( SL(2, \mathbb{C}) \),

\[
g = l(ma)^{h_1} u
\]

the scalar \( ma \) is a (meromorphic) quotient \( ma = \sigma_1(\breve{g}) / \sigma_0(\breve{g}) \). The conjectural distribution for the quotient makes sense, but the numerator and denominator are simply not well-defined random variables.

I now want to explain why the existence of a unitarily bi-invariant measure in indefinite cases seems unlikely, or at least is definitely inconsistent with Conjecture \([1]\).

A preliminary comment: It is not clear whether there are interesting analogues of root subgroup coordinates, and so on, in indefinite cases. The existence of root subgroup coordinates is related to presently intractable questions about imaginary roots for \( g(A) \), such as their multiplicities. To produce root subgroup coordinates it is necessary to ‘order’ the roots in some generalized sense, as in the affine case, see
In the affine case the ordering of imaginary roots is unimportant because the root vectors commute. This is not true in general in the indefinite cases. It is possible that commutativity of imaginary root vectors is a characterization of the finite and affine cases.

A central indefinite example is the maximal hyperbolic Kac-Moody algebra $E_{10}$. In this case one can exploit the inclusion of $E_9 \subset E_{10}$ in a way which is analogous to the inclusion $\hat{\mathfrak{g}} \subset \hat{L}\hat{\mathfrak{g}}$ (see [45], following ideas of Feingold and Frenkel). $E_9$ is the untwisted affine algebra corresponding to $E_8$,

$$g(E_9) = \mathbb{C}L_0 \ltimes \hat{\mathfrak{g}}, \text{ where } \hat{\mathfrak{g}} = g(E_8)$$

The $E_{10}$ adjoint action of the center of $E_9$ induces an ‘affine grading’ (by $\mathbb{Z}$) for $g(E_9)$ (see [45]), and in turn this produces an analogue of the linear Riemann-Hilbert decomposition for loop algebras,

$$g(E_{10}) = g(E_{10})_0 \oplus g(E_9) \oplus g(E_{10})_+$$

For the formal group completion there is a corresponding analogue of (generic) Riemann-Hilbert factorization for loop groups,

$$g = g_0 g_+$$

where now (and this is the crucial point) $g_0$ is in the formal completion for the Kac-Moody group corresponding to $E_{10}$. According to the Uniqueness Conjecture $\dagger$ the $g_0$ distribution would have to be an invariant probability measure which pushes down to the measure $\mu_0$ on $Lg(E_8)$. As we have argued in the affine case above, the existence of such a lift seems unlikely.

Conclusions: The truth of Conjecture $\dagger$ implies a negative answer to the question of whether there (always) exists a unitarily bi-invariant measure in indefinite cases. It is at least conceivable that one could use the complex version of root subgroup factorization (see [10]) to produce a counterexample to Conjecture $\dagger$ and in turn it might be possible to use root subgroup factorization to produce an example of a unitarily bi-invariant measure in some indefinite case.

There has been considerable speculation in the physics literature about a connection between $E_{10}$ and M Theory. For example in [10] it is conjectured that (one aspect of) M Theory is related (in a necessarily heuristic way) to a Laplace type operator on the ‘non-compact type symmetric space’ $G_0(E_{10})/K$, where $G_0(E_{10})$ denotes the group associated to the normal real form of $E_{10}$, as in Example 7.1.

The proposals in these notes do not apply to the ‘non-compact type symmetric spaces $G_0/K$ in Example 7.1. It is commonplace to instead consider a double coset space $\Gamma \backslash G_0/K$, but it is not clear if we have anything interesting to say about this. In any event one wonders if the ‘dual symmetric space’, $U(E_{10})/K(E_{10})$, or more simply $U(E_{10})$, the group corresponding to the unitary real form of $E_{10}$, might also be relevant.

This suggests the following questions. In our musings about the chiral model with compact simply connected target $K$, we emphasized the natural role of the completions

$$L_{fin}G \subset C^0(S^1, G) \subset Hyp(S^1, G) \subset LG$$
For $E_{10}$ we only have analogues of the first group (essentially the Kac-Peterson group) and the last space

$$G(E_{10}) \subset G(E_{10})$$

A first question is whether there exists a Lie group completion of the Kac-Peterson group $G(E_{10})$ analogous to $C^0(S^1, G)$, or $C^\infty(S^1, G)$. I suspect the answer to this question is ‘no’; see [73] for some evidence. A second question is whether there exists some analogue of $Hyp(S^1, G)$, which we have argued is a kind of universal version of $Bun_G$ for Riemann surfaces.

This second question is clearly intertwined with the long standing puzzle of how to think about the transition from $E_9$, the affine case, to $E_{10}$.

8. Appendix: The Topology of $Hyp(S^1, G)$

Suppose that $\Sigma$ is a closed Riemann surface. There a natural quotient topology for $Bun_G(\Sigma)$:

$$Bun_G(\Sigma) = \Omega^{0,1}(\Sigma, g)/\Omega^0(\Sigma, G)$$

Presumably this is the same as the topology used in [27].

Let’s assume this has been taken care of.

Suppose we define the topology on $Hyp(S^1, G)$ to be the weakest topology such that all the projections

$$Hyp(S^1, G)\to \prod_{c\in C}Bun_G(\Sigma_c)$$

are continuous.

**Lemma 5.** The sets $\{g \in Hyp(S^1, G) : g = g_0 \cdot g_+\}$ and $\{g \in Hyp(S^1, G) : g = lmau\}$ are open.

This follows by considering the standard inclusion of $S^1$ into $\mathbb{P}^1$. A problem: I think that $Bun^0_G(\mathbb{P}^1)$, set of stable bundles, is empty. Inside $Bun_G(\mathbb{P}^1)$ there are the semistable bundles. Is this actually open?

Question: What is the closure of $C^\infty(S^1, K)$ inside of the hyperfunction completion?

**References**

[1] I. Affleck, D. Bykov, and K. Wamer, Flag manifold sigma models, spin chains and integrable theories, ArXiv2101.11638.

[2] S. Aida, B. Driver, Equivalence of heat kernel measure and pinned Wiener measure on loop groups, C. R. Acad. Sci. Paris Ser. I Math. 331, no. 9 (2000) 709-712.

[3] J. Andersen, J. Mattes, and N. Reshetikhin, The Poisson structure on the moduli space of flat connections and chord diagrams, Topology 35 no. 4 (1996) 1069-1083.

[4] Asorey and Mitter

[5] M. Atiyah and R. Bott, The Yang-Mills equations on Riemann surfaces, Phil. Trans. R. Soc. Lond. A 308 (1982) 523-615.

[6] S. Albeverio, R. Hoegh-Krohn, D. Testard, and A. Vershik, Factorial representations of path groups, J. Funct. Anal. 51 (1983) 115-131.

[7] S. Albeverio, B. Driver, M. Gordina, A. M. Vershik, Equivalence of the Brownian and energy representations, J. Math. Sci. (N.Y.) 219, no. 5 (2016) 612-630.

[8] G. Baverez, C. Guillarmou, A. Kupiainen, R. Rhodes, V. Vargas, The Virasoro structure and the scattering matrix for Liouville conformal field theory, arXiv:2204.02745

[9] E. Basor and D. Pickrell, Loops in SU(2), Riemann Surfaces, and Factorization, I. SIGMA 12 (2016).
10. E. Basor and D. Pickrell, Loops in $SL(2, \mathbb{C})$ and factorization, Random Matrices and Applications (2019)
11. E. Basor and D. Pickrell, Loops in $SU(2)$ and factorization, II, The Mathematical Legacy of Harold Widom, Birkhauser (2022) (a long version is available at arXiv:2009.14267)
12. A. Beauville, Conformal blocks, fusion rules, and the Verlinde formula, Israeli Mathematical Proceedings, Vol. 9 (1996) 75-96
13. D. Bernard, On Symmetries of Some Massless 2D Field Theories, arXiv:hep-th/9201006
14. D. Bernard, An introduction to Yangian symmetries, hep-th/9211133 There is also a book on integrable field theories (2003)
15. V. Bogachev, Gaussian Measures, Mathematical Surveys and Monographs 62, Providence, RI: Amer. Math. Soc. (1998).
16. J. Brown, O. Ganor, and C. Helfgott, M-theory and E10: Billiards, Branes, and Imaginary Roots, hep-th/0401053
17. S. Cao and S. Chatterjee, A state space for 3D Euclidean Yang-Mills theories, ArXiv2111.12813
18. R. Carter, Lie Algebras of Finite and Affine Type, Cambridge Univ. Press (2005)
19. V. Chari and A. Pressley, Fundamental representations of Yangians and singularities of R-matrices, J. reine angew. Math. 417 (1991) 87-128.
20. V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press (1994)
21. T.S. Cubitt, D. Perez-Garcia, and M. Wolf, Undecidability of the Spectral Gap, ArXiv1502.04135.
22. P. Di Francesco, P. Mathieu and D. Senechal, Conformal Field Theory, Springer (1996).
23. S. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, Journal of Differential Geometry, 18 (1983) 269-278.
24. B. Driver, Integration by parts and quasi-invariance for heat kernel measures on loop groups. J. Funct. Anal. 149 (1997), no. 2, 470-547.
25. B. Driver, Analysis of Wiener measure on path and loop groups, In: Finite and Infinite Dimensional Analysis in Honor of Leonhard Gross, Contemp Math. 317, Amer. Math. Soc., 57-85.
26. B. Driver and M. Gordina, Integrated Harnack inequalities on Lie groups. J. Differential Geom. 83 (2009), no. 3, 501-550.
27. Etingof, E. Frenkel, and D. Kazhdan, Geometric Langlands proposal
28. L. Faddeev and N. Reshetikhin, Integrability of the principal chiral model in 1+1 dimensions, Annals of Physics 167 (1986) 227-256.
29. G. Faltings, A proof for the Verlinde formula, J. Alg. Geom. 3 (1994) 347-374.
30. L. Faddeev, Beyond affine Lie algebras, Proc. I.C.M. Berkeley (1986) 821-839.
31. H. Garland, The arithmetic theory of loop algebras, J. Algebra, 53 (1978) 511-551.
32. K. Gawedzki, SU(2) WZW theory at higher genera, Comm. Math. Phys. 169 (1995) 329-.
33. K. Gawedzki, Introduction to CFT, Quantum Fields and Strings: A Course for Mathematicians, Vol. 2, AMS-IAS, (1998).
34. W. Goldman, Ergodic theory on moduli spaces, Ann. Math., Volume 146, Issue 3 (1997) 475-507.
35. P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley (1978)
36. C. Guillarmou, A. Kupiainen, R. Rhodes, V. Vargas, Segal’s axioms and bookstrap for Liouville theory, arXiv:2112.14859
37. M. Guest, Geometry of maps between generalized flag manifolds, J. Diff. Geom. 25 (1987) 223-247.
38. M. Guest, An update on harmonic maps of finite unitor number, via the zero curvature equation. Integrable systems, topology, and physics (Tokyo, 2000), 85-113, Contemp. Math., 309, Amer. Math. Soc., Providence, RI (2002).
39. S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press (1978).
40. S. Helgason, Groups and Geometric Analysis, Academic Press (1984).
41. N. Hitchin, Harmonic maps from a 2-torus into a 3-sphere, J. Diff. Geom. 31 (1990) 627-710
42. N. Hitchin, The Wess-Zumino term for a harmonic map, arXiv:math/0008038
43. Y. Inahama, Logarithmic Sobolev inequality on free loop groups for heat kernel measures associated with the general Sobolev space, J. Funct. Anal. 179, no. 1 (2001) 170-213
[44] V. Kac, Constructing groups from infinite dimensional Lie algebras, Infinite Dimensional Groups with Applications, edited by V Kac, MSRI publication, Springer-Verlag (1985) 167-216.

[45] V. Kac, R. Moody and M. Wakimoto, On E10, Proceedings of the 1987 conference on differential-geometrical methods in physics, Kluwer (1988), 102-128.

[46] V. Kac and D. Peterson, Defining relations of certain infinite dimensional groups, Elie Cartan et les mathematiques d’aujourd’hui - Lyon, 25-29 juin 1984, Asterisque, no. S131 (1985), 44 p.

[47] V. Kac and M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, Adv. Math. 70 (1988), 156-236.

[48] J.P. Kahane, Some Random Series of Functions, Cambridge Texts in Advanced Mathematics 5, Cambridge University Press (1985).

[49] J. Kilgore, Weyl's law for singular algebraic varieties, U. Mich. dissertation (2019).

[50] A. Kirillov Jr., On an inner product in modular tensor categories. J. Amer. Math. Soc. 9, no. 4 (1996) 1135-1169.

[51] A. Kirillov Jr., On inner product in modular tensor categories. II. Inner product on conformal blocks and affine inner product identities. Adv. Theor. Math. Phys. 2 no. 1 (1998) 155-180.

[52] M. Kontsevich and G. Segal, Wick rotation and the positivity of energy in quantum field theory ArXiv: 2105.10161.

[53] T. Lam and P. Pylyavskyy, Total positivity in loop groups II: Chevalley generators, arXiv:0906.0610 (2009).

[54] A. LeClair, TT-deformation of the Ising model and its ultraviolet completion arXiv:2107.02230 (2021).

[55] S. Lukyanov and A. B. Zamolodchikov, Integrability in 2D fields theory/sigma-models, in Integrability: From Statistical Systems to Gauge Theory: Lecture Notes of the Les Houches Summer School: Volume 106. P. Dorey (ed.) et al. (2016).

[56] J. Milnor, Remarks on infinite dimensional Lie groups, Part of Relativity, groups and topology: Proceedings, 40th Summer School of Theoretical Physics - Session 40: Les Houches, France, June 27 - August 4, 1983, vol. 2, 1007-1057.

[57] A.V. Mikhailov and V.E. Zakharov, Sov. Phys. JETP (Engl. Transl.), 47 (1978), p. 1017.

[58] G. Papadopoulos and E. Witten, Scale and Conformal Invariance in 2d Sigma Models, with an Application to N=4 Supersymmetry, arXiv:2404.19520 (2021).

[59] G. Moore, K-Theory from a physical perspective, Geometry and Quantum Field Theory, Proc. of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal, Ed. U. Tillmann, London Math. Soc. Lect. Note Ser. 308, Cambridge U. Press (2004).

[60] D. Pickrell, Harmonic maps and the symplectic category, arXiv:1408.5402 (2014).

[61] D. Pickrell, Invariant measures for unitary forms of Kac-Moody Lie groups, Memoirs of the AMS, Vol 146, No 693 (2000).

[62] D. Pickrell and E. Xia, Ergodicity of mapping class group actions on representation varieties, I. Closed Surfaces, Comment. Math. Helv. 77 (2002) 339-362.

[63] D. Pickrell, An invariant measure for the loop space of a simply connected compact symmetric space, J. Funct. Anal. 234 (2006) 321-363.

[64] D. Pickrell, F(φ)2 quantum field theories and Segal’s axioms, Comm. Math. Phys. 280 (2008) 403-425.

[65] D. Pickrell, Heat kernel measures and critical limits, in Developments and Trends in Infinite-Dimensional Lie Theory, Ed K-H Neeb and A. Pianzola, Progress in Mathematics, Volume 288, Birkhauser (2010).
D. Pickrell, Loops in SU(2) and factorization, J. Funct. Anal. (2011).
D. Pickrell, Complex groups and root subgroup factorization, J. Lie Th. 28 (4) (2017).
D. Pickrell, Conversations with Flaschka: Kac-Moody groups and Verblunsky coefficients, Physica D (2023)
B. Pittman-Polletta and D. Pickrell, Unitary loop groups and factorization, J. Lie Theory 20, no. 1 (2010) 93-112.
K. Pohlmeyer, Integrable Hamiltonian systems and interactions through constraints Comm. Math. Phys, 46 (1976) 207-221.
A. Pressley and G. Segal, Loop groups, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, New York (1986).
G. Segal, Unitary representations of some infinite dimensional groups, Comm. Math. Phys. (1981).
G. Segal, The definition of conformal field theory, Topology, Geometry and Quantum Field Theory, Proc. of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal, Ed. U. Tillmann, London Math. Soc. Lect. Note Ser. 308, Cambridge U. Press (2004)
Y. Shimada, On irreducibility of the energy representation of the gauge group and the white noise distribution theory, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 8, no. 2 (2005) 153-177.
H. Shimomura, Quasi-invariant Measures on the Group of Diffeomorphisms and Smooth Vectors of Unitary Representations, J. F. A., Volume 187, Issue 2 (2001) 406-441
C-L. Terng and K. Uhlenbeck, 1 + 1 wave maps into symmetric spaces, Comm. Anal. Geom. 12 (2004), no. 1-2, 345-388
A. Tsuchiya, K. Ueno, and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, Advanced Studies in Pure Math. 19 (1989) 459-566.
C. Teleman and C. Woodward, Parabolic bundles, products of conjugacy classes, and Gromov-Witten invariants, Annales de L’Institut Fourier, Tome 53, No. 3 (2003) 713-749.
K. Uhlenbeck, Harmonic maps into Lie groups, J. Diff. Geom. 30 (1989) 1-50
K. Uhlenbeck, On the connection between harmonic maps and the self-dual Yang-Mills and the sine-Gordon equations. J. Geom. Phys. 8 (1992), no. 1-4, 283-316.
A. Wassermann, Operator algebras and conformal field theory, III. Fusion of positive energy representations of LSU(N) using bounded operators. Invent. Math. 133, no. 3 (1998) 467-538.
E. Witten, Instantons and the Large N=4 Algebra, [arXiv:2407.20964]
A.B. Zamolodchikov and Al.B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, Annals of Physics 120 (1979) 253-291.
A.B. Zamolodchikov and Al.B. Zamolodchikov, Massless factorized scattering and sigma models with topological terms, Nuclear Physics B379 (1992) 602-623.
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