A characterization and an application of weight-regular partitions of graphs

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Abstract

A natural generalization of a regular (or equitable) partition of a graph, which makes sense also for non-regular graphs, is the so-called weight-regular partition, which gives to each vertex \( u \in V \) a weight which equals the corresponding entry \( \nu_u \) of the Perron eigenvector \( \nu \). This paper contains three main results relating to weight-regular partitions of a graph. Regular partitions are clearly weight-regular, but the converse is not true in general. We investigate when a weight-regular partition of a graph is regular. Inspired by a characterization of regular graphs by Hoffman, we provide a new characterization of weight-regularity by using a Hoffman-like polynomial. As a corollary, we obtain Hoffman’s result for regular graphs. In addition, we show an application of weight-regular partitions to study graphs that attain equality in the classical Hoffman’s lower bound for the chromatic number of a graph, and we show that weight-regularity provides a condition under which Hoffman’s bound can be improved.

Keywords: weight-regular partition, Hoffman polynomial, chromatic number.
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1 Introduction

Let \( G \) be a connected graph with vertex set \( V \), adjacency matrix \( A \), positive eigenvector \( \nu \) and corresponding eigenvalue \( \lambda_1 \). A partition \( \mathcal{P} = \{V_1, \ldots, V_m\} \) of the vertex set \( V \) of a graph is regular (equitable) if, for all \( i, j \), the number of neighbors which a vertex in \( V_i \) has in the set \( V_j \) is independent of the choice of vertex in \( V_i \). Regular partitions have been widely studied in the literature and provide a handy tool for obtaining inequalities and regularity results concerning the structure of regular graphs. In particular, characterizations of
regular partitions and its application to obtain tight bounds for several graph parameters have been previously obtained [12].

A natural generalization of a regular partition, which makes sense also for non-regular graphs, is the so-called weight-regular partition. Its definition is based on giving to each vertex $u \in V$ a weight which equals the corresponding entry $\nu_u$ of $\nu$. Such weights “regularize” the graph, leading to a kind of regular partition that can be useful for general graphs.

Weight partitions have been shown to be a powerful tool used to extend several relevant results for non-regular graphs. Weight partitions were first used by Haemers in 1979 [4] (see Theorem 6) to provide an alternative proof of Hoffman’s spectral lower bound for the chromatic number of a general graph. In [8, 9], Fiol and Garriga formally defined weight partitions, and they used them to obtain several bounds for parameters of non-regular graphs. Examples of such results are an extension of Hoffman’s bound for the chromatic number or a generalization of the Lovász bound for the Shannon capacity of a graph. Moreover, weight-regular partitions have been used to show that a bound for the weight-independence number is best possible [9] and to obtain spectral characterizations of distance-regularity around a set and spectral characterizations of completely regular codes [8].

Note that regular partitions are obviously weight-regular. The converse, however, is not true in general. In this work we investigate when a weight-regular partition is regular in terms of double stochastic matrices. The second part of this work is inspired by the well-known result by Hoffman [13] in which regular graphs are characterized in terms of the Hoffman polynomial. We obtain a new characterization of weight-regularity by using a Hoffman-like polynomial, answering a question of Fiol [16] (see Problem 1.5). Up until now, the only known characterization of weight-regular partitions appears in [9] (see Lemma 2.2 and Lemma 2.3). Finally, we give a new application of weight-regular partitions to study graphs that attain equality in Hoffman’s bound for the chromatic number [13]. As a corollary of the mentioned application, we can show that Hoffman’s bound can be improved for certain class of graphs.

This article is organized as follows. Section 2 recalls some definitions and terminologies about weight partitions. In Section 3 we characterize when weight-regular partitions are regular in terms of double stochastic matrices. Section 4 gives a characterization of weight-regular partitions by using a Hoffman-like polynomial. Finally, in Section 5 we investigate examples of graphs reaching Hoffman’s bound on the chromatic number of a graph and we show its relation to weight-regular partitions.

2 Preliminaries

In this section we introduce some definitions and properties relating to weight partitions. The all-ones matrix is denoted $J$, and $1$ is the all-ones vector.
Let $G = (V, E)$ be a simple and connected graph on $n = |V|$ vertices, with adjacency matrix $A$, eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and spectrum
\[ spG = spA = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}, \]
where the different eigenvalues of $G$ are in decreasing order $\theta_0^{m_0} > \theta_1^{m_1} > \cdots > \theta_d^{m_d}$, and the superscripts stand for their multiplicities $m_i = m(\theta_i)$. Since $G$ is connected (so $A$ is irreducible), Perron-Frobenius Theorem assures that $\lambda_1$ is simple, positive and has positive eigenvector. If $G$ is non-connected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. Throughout this work, the positive eigenvector associated with the largest (positive and with multiplicity one) eigenvalue $\lambda_1$ is denoted by $\nu = (\nu_1, \ldots, \nu_n)^T$. This eigenvector is normalized in such a way that its minimum entry (in each connected component of $G$) is 1. For instance, if $G$ is regular, we have $\nu = 1$.

Let $P$ be a partition of the vertex set $V = V_1 \cup \cdots \cup V_m$, $1 \leq m \leq n$. Consider the map $\rho : V \to \mathbb{R}^+$ defined by $\rho U := \sum_{u \in U} \rho_u e_u$. In particular, for weight-partitions we consider the map $\rho : \mathcal{P}(V) \to \mathbb{R}^n$ defined by
\[ \rho U := \sum_{u \in U} \nu_u e_u \]
for any $U \neq \emptyset$, where $e_u$ represents the $u$-th canonical (column) vector, $\rho \emptyset = 0$ and $\nu$ is the eigenvector of the largest eigenvalue. Note that, with $\rho U := \rho \{u\}$, we have $||\rho U|| = \nu_u$, so that we can see $\rho$ as a function which assigns weights to the vertices of $G$. In doing so we “regularize” the graph, in the sense that the weight-degree of each vertex $u \in V$ becomes a constant:
\[ \delta^*_u := \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \lambda_1. \quad (1) \]

Given $\mathcal{P} = \{V_1, \ldots, V_m\}$, for $u \in V_i$ we define the weight-intersection numbers as follows:
\[ b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \quad (1 \leq i, j \leq m). \quad (2) \]

Observe that the sum of the weight-intersection numbers for all $1 \leq j \leq m$ gives the weight-degree of each vertex $u \in V_i$:
\[ \sum_{j=1}^m b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \delta^*_u = \lambda_1. \]

A partition $\mathcal{P}$ is called weight-regular whenever the weight-intersection numbers do not depend on the chosen vertex $u \in V_i$, but only on the subsets $V_i$ and $V_j$. In such a case, we denote them by
and we consider the \( m \times m \) matrix \( B^* = (b^*_{ij}) \), called the weight-regular-quotient matrix of \( A \) with respect to \( P \).

A matrix characterization of weight partitions can be done via the following matrix associated with any partition \( P \). The weight-characteristic matrix of \( P \) is the \( n \times m \) matrix \( \tilde{S}^* = (\tilde{s}^*_{uj}) \) with entries

\[
\tilde{s}^*_{uj} = \begin{cases} 
\nu_u & \text{if } u \in V_j, \\
0 & \text{otherwise}
\end{cases}
\]

and, hence, satisfying \( (\tilde{S}^*)^\top \tilde{S}^* = D^2 \), where \( D = \text{diag}(||\rho_{V_1}||, \ldots, ||\rho_{V_m}||) \).

From such a weight-characteristic matrix we define the weight-quotient matrix of \( A \) with respect to \( P \), as \( \tilde{B}^* := (\tilde{S}^*)^\top A \tilde{S}^* = (\tilde{b}^*_{ij}) \). Notice that this matrix is symmetric and has entries

\[
\tilde{b}^*_{ij} = \sum_{u,v \in V} \tilde{s}^*_{ui}a_{uv}\tilde{s}^*_{vj} = \sum_{u \in V_i, v \in V_j} a_{uv}\nu_u
\nu_v = \sum_{u \in V_i, v \in V_j} \nu_u
\nu_v = \tilde{b}^*_{ji}
\]

where \( E(V_i, V_j) \) stands for the set of edges with ends in \( V_i \) and \( V_j \) (when \( V_i = V_j \) each edge counts twice). Also, in terms of the weight-intersection numbers,

\[
\tilde{b}^*_{ij} = \sum_{u \in V_i} \sum_{v \in V_j} \nu_u \nu_v = \sum_{u \in V_i} \nu_u \nu_v = \sum_{u \in V_i} \nu_u \nu_v = \tilde{b}^*_{ji}.
\]

In this article we will use the normalized weight-characteristic matrix of \( P \), which is the \( n \times m \) matrix \( \bar{S}^* = (\bar{s}^*_{uj}) \) with entries obtained by just normalizing the columns of \( \tilde{S}^* \), that is, \( \bar{S}^* = \tilde{S}^* D^{-1} \). Thus,

\[
\bar{s}^*_{uj} = \begin{cases} 
\frac{\nu_u}{||\rho_{V_j}||} & \text{if } u \in V_j, \\
0 & \text{otherwise}
\end{cases}
\]

and it holds that \( (\bar{S}^*)^\top \bar{S}^* = I \). We define the normalized weight-quotient matrix of \( A \) with respect to \( P \), \( \bar{B}^* = (\bar{b}^*_{ij}) \), as

\[
\bar{B}^* = (\bar{S}^*)^\top A \bar{S}^* = D^{-1}(\tilde{S}^*)^\top \tilde{S}^* D^{-1} = D^{-1}\tilde{B}^* D^{-1},
\]
and hence \( b_{ij} = \frac{\tilde{b}_{ij}}{||p_{V_i}|| \cdot ||p_{V_j}||} \).

The following result was partially stated in [8].

**Lemma 2.1.** A \( \mathcal{P} = \{V_1, V_2, \ldots, V_m\} \) partition of a graph \( G \) is regular if and only if it is weight-regular and the map on \( V \), denoted \( \rho : u \mapsto \nu_u \), is constant over each \( V_k \), say \( \nu_k \). Then, it holds that the quotient matrix entries of the regular partition \( (b_{ij}) \) and the quotient matrix entries of the weight-regular partition \( (b_{ij}^*) \) satisfy

\[
\tilde{b}_{ij} = \frac{\nu_j}{\nu_i} b_{ij}.
\]

**Proof.** If \( \mathcal{P} \) is a regular partition, we know that the value of \( \nu_u \) is constant on each \( V_k \) (say \( \nu_k \)). By direct calculation, we obtain that for any \( u \in V_i \) it holds

\[
b_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v = \frac{\nu_j}{\nu_i} b_{ij}, \quad 1 \leq i, j \leq m.
\]

Hence,

\[
b_{ij}^* = \frac{\nu_j}{\nu_i} b_{ij} \quad (1 \leq i, j \leq m),
\]

which means that the partition \( \mathcal{P} \) is weight-regular.

Conversely, if \( \mathcal{P} \) is a weight-regular partition and it holds that \( \nu_u \) is constant for \( u \in V_k \) (say \( \nu_k \)), then it follows that for any \( u \in V_i \)

\[
b_{ij}(u) = |G(u) \cap V_j| = \frac{1}{\nu_k} \sum_{v \in G(u) \cap V_j} \nu_v = \frac{\nu_u}{\nu_j} \nu_j \nu_i \sum_{v \in G(u) \cap V_j} \nu_v = \frac{\nu_j}{\nu_j} b_{ij}^*, \quad 1 \leq i, j \leq m.
\]

Hence,

\[
b_{ij} = \frac{\nu_j}{\nu_j} b_{ij}^* \quad (1 \leq i, j \leq m),
\]

which means that the partition \( \mathcal{P} \) is regular.

\( \blacksquare \)

| Regular partition | Weight-regular partition |
|-------------------|--------------------------|
| \( m = 1 \)      | \( \iff \) \( G \) regular | always                  |
| \( m = 2 \)      | \( \iff \) \( G \) biregular | \( \iff \) \( G \) bipartite |
| \( m = n \)      | always                   | \( \iff \) \( G \) regular |

Table 1: Some particular cases of trivial partitions.
We will also need the following two results, which relate eigenvalue interlacing and weight-
regular partitions.

Let $A$ and $B$ be two square matrices having only real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$, respectively ($m \leq n$). If for all $1 \leq i \leq m$ we have $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$, then we say that the eigenvalues of $B$ interlace the eigenvalues of $A$. The interlacing is called tight if there exists an integer $k \in [0, \ldots, m]$ such that $\lambda_i = \mu_i$ for $1 \leq i \leq k$ and $\lambda_{n-m+i} = \mu_i$ for $k+1 \leq i \leq m$.

**Lemma 2.2.** [Interlacing] Let $S$ be a complex $n \times m$ matrix such that $S^*S = I$. Let $A$ be a hermitian $n \times n$ matrix. Then the eigenvalues of $S^*AS$ interlace the eigenvalues of $A$.

The following lemma is a direct consequence of Interlacing:

**Lemma 2.3.** [Interlacing] Let $G = (V, E)$ be a graph with adjacency matrix $A$ and positive eigenvector $\nu$, and consider a vertex partition $P$ of $V$ inducing the normalized weight-quotient matrix $B^*$. Then the following holds:

(i) The eigenvalues of $B^*$ interlace the eigenvalues of $A$.

(ii) If the interlacing is tight, then the partition $P$ is weight-regular.

### 3 Double stochastic matrices and weight-regularity

In [8] an example of a weight-regular partition which is not regular is given. Many other examples arise, for instance, from the bipartition of any connected bipartite graph (see Table 1), which is always weight-regular but does not always define a regular partition. Thus, weight-regular partitions are not necessarily regular. Therefore it is natural to inquire when a weight-partition is regular. In fact, Godsil presented the analogous result for classical partitions being regular in [11]. In this section we give a characterization of weight-regular partitions being regular in terms of double stochastic matrices.

A matrix is **double stochastic** if it is nonnegative and each of its rows and each of its columns sums up to one. If $A$ is the adjacency matrix of a graph $G$, we denote by $\Omega(A)$ the set of all double stochastic matrices which commute with $A$. Note that $\Omega(A)$ is a convex polytope since it consists of all matrices $X$ such that

$$XA = AX, \quad X1 = 1X, \quad X \geq 0.$$ 

Note that any permutation matrix is a doubly stochastic matrix having integral entries only. For more details on double stochastic matrices, see [5].

The following results extend the characterization of regular partitions given in [11] to weight-regular partitions.
Lemma 3.1. Let $A$ be the adjacency matrix of a graph $G$, and let $\mathcal{P}$ be a weight-regular partition of the vertex set with normalized weight-characteristic matrix $\overline{S}^\ast$. Then $\mathcal{P}$ is regular if and only if $A$ and $\overline{S}^\ast \overline{S}^\ast\top$ commute.

Proof. Using Lemma 2.2 from [9] we know that $\mathcal{P}$ is weight-regular if and only if there exist a $m \times m$ normalized weight-quotient matrix $\overline{B}^\ast$ such that

$$
\overline{S}^\ast \overline{B}^\ast = A \overline{S}^\ast
$$

where $\overline{S}^\ast$ is the normalized weight-characteristic matrix of $\mathcal{P}$. If $\mathcal{P}$ is regular, equation (8) implies that

$$
\overline{B}^\ast = \overline{S}^{\ast\top} A \overline{S}^\ast,
$$

hence $\overline{B}^\ast$ is symmetric. Using equation (8) again, we obtain

$$
A \overline{S}^\ast \overline{S}^{\ast\top} = \overline{S}^\ast \overline{B}^\ast \overline{S}^\ast\top
$$

which implies that $A \overline{S}^\ast \overline{S}^{\ast\top}$ is symmetric. Then, it follows that $A$ and $\overline{S}^\ast \overline{S}^{\ast\top}$ commute.

Conversely, we know that $\mathcal{P}$ is regular if and only if each vertex set induces a regular subgraph of $G$ and the edges joining any two vertex sets form a semiregular bipartite graph, which holds if and only if $A \overline{S}^\ast \overline{S}^{\ast\top} = \overline{S}^\ast \overline{S}^{\ast\top} A$. $\square$

The above result yields to the following corollary.

Corollary 3.2. Let $\mathcal{P}$ be a weight-regular partition of the vertices of $G$ with normalized weight-characteristic matrix $\overline{S}^\ast$. Then $\mathcal{P}$ is regular if and only if $\overline{S}^\ast \overline{S}^{\ast\top} \in \Omega(A)$.

4 Polynomials and weight-regularity

In [13], Hoffman proved that a (connected) graph $G$ is regular if and only if $H(A) = J$, in which case $H$ becomes the Hoffman polynomial. An analogous of Hoffman’s result for biregular graphs was given in [1]. In [10] [13] a generalization of Hoffman’s characterization for nonregular graphs is given.

The following result proves a natural extension of Hoffman’s result for weight-regular partitions of a graph. Recall that $b^\ast_{ij}$ denote the entries of the weight-quotient matrix defined in Section 2.

Theorem 4.1. Let $G$ be a connected graph with a partition of its vertices into $m$ sets, $\mathcal{P} = \{V_1, \ldots, V_m\}$, such that $n = n_1 + \cdots + n_m$ and such that the map on $V$, denoted by $\rho: u \to \nu_u$, is constant over each $V_k$. Then there exists a polynomial $H \in \mathbb{R}_d[x]$ such that
$H(A) = \begin{pmatrix}
  b_{11}^* & b_{12}^* & \cdots & b_{1m}^* \\
  b_{21}^* & b_{22}^* & \cdots & b_{2m}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{m1}^* & b_{m2}^* & \cdots & b_{mm}^*
\end{pmatrix}$

(9)

if and only if $\mathcal{P}$ is a weight-regular partition of $G$.

**Proof.** Assume that $G$ has a weight-regular partition of its vertices. Let $A$ be the adjacency matrix of $G$ and $B^*$ its weight-regular quotient matrix. By Perron-Frobenius Theorem we know that the maximum eigenvalue $\theta_0$ of $A$ has algebraic and geometric multiplicity one, and also that there is an eigenvector $\nu$ belonging to $\theta_0$ with all coordinates positive. Note that $\text{ev } B^* \subseteq \text{ev } A$. In a weight-regular partition, this eigenvector is $\nu = (\nu_1 \mathbf{1} | \cdots | \nu_m \mathbf{1})^T$, with the $\mathbf{1}$'s being all 1-vectors with appropriate lengths, depending on the size of $n_i$, $i = 1, \ldots, m$. This leads to a partition of $A$ with quotient matrix

$$B^* = \begin{pmatrix}
  b_{11}^* & b_{12}^* & \cdots & b_{1m}^* \\
  b_{21}^* & b_{22}^* & \cdots & b_{2m}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{m1}^* & b_{m2}^* & \cdots & b_{mm}^*
\end{pmatrix}.$$ 

By the spectral decomposition theorem we can write $A = \sum_{i=0}^d \theta_i E_i = \theta_0 E_0 + \cdots + \theta_d E_d$. We have that the weight-Hoffman polynomial can be computed as $H = \alpha \prod_{i=1}^d (x - \theta_i)$ for some non-zero constant $\alpha$. Using the fact that $p(A) = \sum_{i=0}^d p(\theta_i) E_i$ for any polynomial $p \in \mathbb{R}_d[x]$, then

$$H(A) = H(\theta_0) E_0 + H(\theta_1) E_1 + \cdots + H(\theta_d) E_d = H(\theta_0) E_0,$$

where $H(\theta_0) = \alpha \prod_{i=1}^d (\theta_0 - \theta_i) = \alpha \pi_0$.

It only remains to find the idempotent $E_0$, which can be calculated as follows:

$$E_0 = \frac{1}{||\nu||^2} \nu \nu^T = (\nu_1 \mathbf{1} | \cdots | \nu_m \mathbf{1})(\nu_1 \mathbf{1} | \cdots | \nu_m \mathbf{1})^T$$

$$= \frac{1}{||\nu||^2} \begin{pmatrix}
  \nu_1 \nu_1 \mathbf{J} & \cdots & \nu_1 \nu_m \mathbf{J} \\
  \vdots & \ddots & \vdots \\
  \nu_m \nu_1 \mathbf{J} & \cdots & \nu_m \nu_m \mathbf{J}
\end{pmatrix}$$

where $\mathbf{J}$'s are the all 1-matrix with appropriate sizes. If we denote $b_{ij}^* = \nu_i \nu_j$ for $i, j = 1, \ldots, m$, and we consider that $\alpha = \frac{||\nu||^2}{\pi_0}$, it follows that
Conversely, suppose that (9) is valid. We know that rows and columns of $A$ are partitioned according to $P$ as follows

$$A = \begin{bmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{m1} & \cdots & A_{mm}
\end{bmatrix}.$$ 

Then, by (9) it follows that each block $A_{ij}$ has constant row (and column) sum $(i, j = 1, \ldots, m)$. Since by assumption $\rho : u \to \nu_u$ is constant over each $V_k$ (say $\nu_k$) and we know that $b^*_ij(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v$, it follows that $P$ is weight-regular.

It is worth noting that the condition on the map on $V$ ($\rho : u \to \nu_u$ is constant over each $V_k$) is necessary to obtain an iff characterization, otherwise only the left direction would hold.

Observe that weight-regular partitions maintain the structure of the Perron eigenvector $\nu$ corresponding to the largest eigenvalue $\lambda_1$. Also, recall that regularity happens when all vector components are 1. As a corollary to Theorem 4.1 we obtain Hoffman’s result [13] (take $m = n$ and recall that for a regular graph $\nu = 1$).

**Corollary 4.2.** $G$ is a regular connected graph if and only if $H(A) = J$.

5 Chromatic number and weight-regularity

The aim of this section is to show that weight-regular partitions can be used to improve the well-known Hoffman’s bound for the chromatic number of a graph.

A proper coloring of $G$ is a partition of the vertex set of $G$ into cocliques (i.e., independent sets of vertices). Such cocliques are called color classes. The chromatic number $\chi(G)$ of $G$ is the minimum number of color classes in a proper coloring.

For general graphs, Hoffman [14] proved the following well-known lower bound for the chromatic number, which only involves the maximum and minimum eigenvalues of the adjacency matrix:

**Theorem 5.1.** [14] If $G$ has at least one edge, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$
When equality holds we call the coloring a Hoffman coloring. Recently, there has been some studies on finding reasonable lower bounds of $\chi(G)$ and on extending Hoffman’s bound \cite{2} \cite{4} \cite{17}.

If $\chi(G) \leq 2$, then $G$ is bipartite. Bipartite graphs are easily recognized and there is a characterization in terms of the eigenvalues \cite{6}:

**Proposition 5.2.** $\chi(G) \leq 2$ if and only if $\lambda_i = -\lambda_{n-i+1}$ for $i = 1, \ldots, n$.

Note that by Proposition 5.2 all 2-chromatic graphs have a Hoffman coloring. Moreover, as mentioned in earlier, such bipartition is always weight-regular. For graphs with a given chromatic number greater than 2, there are not many characterizations in terms of the spectrum. In \cite{3} the authors give some necessary conditions for a graph to be 3-chromatic in terms of the spectrum of the adjacency matrix. On the other hand, not many infinite graph families having a Hoffman coloring are known and it appears rather difficult to find them. The next result shows that if a graph has a Hoffman coloring, the partition defined by the color classes must be weight-regular:

**Proposition 5.3.** If $G$ has chromatic number $\chi(G)$ and a Hoffman coloring, then the following holds:

(i) The partition defined by the color classes is weight-regular.

(ii) The multiplicity of $\lambda_n$ is $\chi(G) - 1$ and $G$ has a unique coloring with $\chi(G)$ colors (up to permutation of the colors).

**Proof.**

(i) Let $A$ be the adjacency matrix of $G$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $P = \{C_1, C_2, \ldots, C_\chi\}$ represent the partitioning of the vertices of $G$ according to the $\chi$ different color classes of a colouring. Let $\nu$ be a real eigenvector belonging to $\lambda_1$. Denote by $S^*$ the normalized weight-characteristic matrix of $P$, and let $B^* = S^* A S^*$ be the normalized weight-quotient matrix with eigenvalues $\mu_1 \geq \cdots \geq \mu_\chi$. Then $S^*^\top S^* = I$ and Interlacing (Lemma 2.3(i)) implies:

(1) The eigenvalues of $B^*$ interlace the eigenvalues of $A$.

From the definition of $S^*$ it is clear that:

(2) All diagonal entries of $B^*$ are zero.

Moreover, since $S^*^\top A S^* D^\frac{1}{2} 1 = S^* A \nu = \lambda_1 S^* \nu = \lambda_1 D^\frac{1}{2} 1$ ($D$ is a diagonal matrix with positive entries), it follows that

(3) $\lambda_1$ is an eigenvalue of $B^*$.

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\chi$ be the eigenvalues of $B^*$. Then (1) and (3) imply $\lambda_1 = \mu_1$.

Furthermore, if $G$ has chromatic number $\chi$ and has a Hoffman coloring, then $\lambda_1 = -(\chi - 1)\lambda_n = \mu_1$. Using (2), (3) and $\text{trace}(B^*) = \mu_1 + \mu_2 + \cdots + \mu_\chi = 0$, we obtain $\mu_2 + \cdots + \mu_\chi = -\mu_1 = -\lambda_1 = (\chi - 1)\lambda_n$.

By Interlacing (Lemma 2.3(i)), we know $\mu_2 + \cdots + \mu_\chi + \mu_\chi = (\chi - 1)\lambda_n \leq (\chi - 1)\mu_\chi$, thus $\mu_2 + \cdots + \mu_\chi + \mu_\chi \leq (\chi - 2)\mu_\chi$. We also know that $\mu_1 \geq \cdots \geq \mu_\chi$.
\[(\chi - 2)\mu_{\chi - 1} \leq \mu_2 + \cdots + \mu_{\chi - 1} \leq (\chi - 2)\mu_\chi,\text{ which implies } (\chi - 2)\mu_{\chi - 1} \leq (\chi - 2)\mu_\chi,\text{ hence } \mu_{\chi - 1} \leq \mu_\chi. \text{ But since by assumption } \mu_{\chi - 1} \geq \mu_\chi, \text{ it follows that } \mu_{\chi - 1} = \mu_\chi. \text{ We do it recursively until we obtain } \mu_2 = \cdots = \mu_{\chi - 1} = \mu_\chi = \lambda_n, \text{ which means there is tight interlacing and the multiplicity of } \lambda_n \text{ is } \chi(G) - 1. \text{ Finally, by using Lemma 2.3(ii) it follows that } \mathcal{P} \text{ is weight-regular.}

(ii) The first part has already been shown in (i), and the second part follows from Proposition 2.3 in [3]. \qed

The above result implies that if a graph does not have a weight-regular partition, then it cannot have a Hoffman coloring. Such result may be useful for obtaining contradictions to the existence of non-regular graphs having a Hoffman coloring, and to find families of non-regular graphs for which the Hoffman bound could be improved. Actually, the following corollary is a straight-forward consequence of Proposition 2.3(i) and shows that Hoffman bound can be improved for certain classes of graphs:

**Corollary 5.4.** If \( G \) has at least one edge and the vertex partition defined by the \( \chi \) color classes is not weight-regular, then

\[
\chi(G) \geq 2 - \frac{\lambda_1}{\lambda_n}.
\]

Finally, we propose the following open problems.

**Problem 5.5.** Find new conditions on the graph, besides the one of Corollary 5.4, under which Hoffman’s lower bound on the chromatic number can be improved by a factor larger than 1.

**Problem 5.6.** Find other examples of tight interlacing for weight-regular partitions.

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