Quantum secret-sharing protocols involving \(N\) partners (NQSS) are key distribution protocols in which Alice encodes her key into \(N-1\) qubits, in such a way that all the other partners must cooperate in order to retrieve the key. On these protocols, several eavesdropping scenarios are possible: some partners may want to reconstruct the key without the help of the other ones, and consequently collaborate with an Eve that eavesdrops on the other partners’ channels. For each of these scenarios, we give the optimal individual attack that the Eve can perform. In case of such an optimal attack, the authorized partners have a higher information on the key than the unauthorized ones if and only if they can violate a Bell’s inequality.

1 Introduction

In the rapidly growing field of quantum information, the first protocol that has almost reached the level of application is quantum cryptography, a beautiful solution to the important problem of secure communication. The authorized partners Alice and Bob can establish an absolutely secure communication provided that they share a common sequence of bits (the key), unknown to anybody else: this is the very principle of the so-called secret-key cryptographic schemes. In 1984, Bennett and Brassard proposed a way of distributing the key in a physically secure way by using quantum physics: their protocol bears the acronym BB84, and was the first protocol of quantum cryptography — from now on, we shall use the more precise name of quantum key distribution (QKD). In the intuition of Bennett and Brassard, security is provided by the well-known feature of quantum mechanics: "measurement perturbs the system"; or, under a different viewpoint which is equivalent, by the no-cloning theorem. In 1991, Ekert proposed a QKD protocol that uses entangled particles, and stated that the violation of Bell’s inequality might be the physical principle that ensures security. This view was challenged by Bennett, Brassard and Mermin, who showed that...
Ekert’s protocol is actually equivalent to the BB84 protocol, that involves single particles. A link between security of QKD and Bell’s inequalities was nevertheless noticed in further studies [3, 4].

The plan of this paper is as follows. In Section 2, we consider two-partners QKD. The material of this section is not new in itself, but the approach is; moreover, it is a useful introduction to the following Sections. In Section 3, we define the N-partners protocols that we consider. Several eavesdropping scenarios can be imagined on these protocols, and we give Eve’s optimal individual attack in each case. In Section 4, we introduce a family of M-qubit Bell’s inequalities (Mermin-Klyshko inequalities), and we discuss the link between the violation of these inequalities and the security of the N-partners protocol. Section 5 is a conclusion.

2 QKD involving two partners

2.1 The BB84 protocol

The BB84 protocol of quantum key distribution between two partners, Alice and Bob, is characterized by the fact that two complementary bases are used to encode the bits. In the original version of the BB84 protocol [2], Alice prepares a qubit into a randomly chosen eigenstate of $\sigma_x$ or of $\sigma_y$, and sends it to Bob. Since we want to discuss Bell’s inequalities, we consider the following preparation method [4]: Alice has an EPR source that produces a maximally entangled state, say $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where $|0\rangle$ and $|1\rangle$ are the eigenstates of $\sigma_z$. On her side, Alice measures randomly $\sigma_x$ or $\sigma_y$ on one qubit. The moment at which Alice performs her measurement is irrelevant; in particular, she can measure her qubit immediately after it leaves the source. This way, Alice’s measurement acts as a preparation of the second qubit, which goes to Bob through a quantum channel. Bob also measures either $\sigma_x$ or $\sigma_y$. If he measures the same observable as Alice, his result is perfectly correlated to hers, since $\langle \sigma_x \otimes \sigma_x \rangle_{\Phi^+} = -\langle \sigma_y \otimes \sigma_y \rangle_{\Phi^+} = 1$; if he measures the other observable, he has no information on Alice’s result, since $\langle \sigma_x \otimes \sigma_y \rangle_{\Phi^+} = \langle \sigma_y \otimes \sigma_x \rangle_{\Phi^+} = 0$. At the end of the transmission, for each qubit Alice and Bob reveal publicly the measurement that they performed (but of course not its result). They simply discard those cases where they have measured different observables, and they end up with two identical lists of random bits. Bob’s information on Alice’s bits is measured by the mutual information $I(A : B)$, defined as $H(A) + H(B) - H(AB) = H(A) - H(A|B)$, where $H(\{p_i\}) = -\sum_i p_i \log_2 p_i$ is the Shannon entropy. Therefore, in the absence of eavesdropping, $H(A|B) = 0$ since knowing the list of B is equivalent to knowing the list of A; and since Alice is supposed to choose her measurement randomly, $H(A) = 1$; whence $I(A : B) = 1$, as it should.

2.2 Security and mutual information

The security of a key-distribution protocol based on quantum mechanics comes from the no-cloning theorem. Suppose that Eve tries to eavesdrop on the quantum channel linking Alice and Bob: she cannot get information on the state that is sent on the channel without introducing perturbations, that should reveal her presence to the authorized partners. If A and B observe the presence of the spy, in most cases they can
still perform some operations that ultimately lead them to share a secret key. More precisely, A and B can run a one-way protocol known as *privacy amplification* if and only if

\[ I(A : B) > \min[I(A : E), I(B : E)] . \tag{1} \]

This is the condition that we ask for *security*. In fact, it has been shown that this condition is not strictly necessary: even if it does not hold, there exist a protocol allowing the extraction of a secret key. But this protocol, called *advantage distillation*, is a two-way protocol, much less efficient than one-way privacy amplification.

The natural problem is now: for a given error rate that is introduced on Bob’s information, that is, for a given value of \( I(A : B) \), find the attack of Eve that optimizes her information on Alice’s key, \( I(A : E) \). The answer is not known in all generality; but it is, if we restrict the analysis to *individual attacks* with \( p \).

It has been shown that Eve can perform the optimal individual attack having a single qubit as resource, and Eve acts separately on each qubit that is sent on the quantum channel, i.e., she does not perform coherent measurements of subsequent qubits. To date, it is not known whether a more general attack would be more efficient, only bounds are known — as for the experimental state-of-the-art, even the implementation of individual attacks would be a great challenge.

It has been shown that Eve can perform the optimal individual attack having a single qubit as resource, by implementing the following unitary transformation affecting her and Bob’s qubits (by convention, we suppose that Eve prepares her qubit in the state \(|0\rangle\)):

\[
U_{BE}|00\rangle = |00\rangle, \quad U_{BE}|10\rangle = \cos \phi |10\rangle + \sin \phi |01\rangle.
\]

where \(|00\rangle\) etc. are shorthand for \(|0\rangle_E \otimes |0\rangle_E\) etc.; \(\phi \in [0, \frac{\pi}{2}]\) characterizes the strength of Eve’s attack. Note that the roles of B and E are symmetric under the exchange of \(\phi\) with \(\frac{\pi}{2} - \phi\). Due to eavesdropping, the three-qubit state of A, B and E reads

\[
|\Psi_{21}(\phi)\rangle = \frac{1}{\sqrt{2}} (|000\rangle + \cos \phi |110\rangle + \sin \phi |101\rangle).
\]

(3)

Note that both statistics are the same, whatever the pair. Moreover, \(p(M = 0, N = 0) = p(M = 1, N = 1)\) and \(p(M = 0, N = 1) = p(M = 1, N = 0)\). Writing \(D_{MN} = p(M = 0, N = 1) + p(M = 1, N = 0)\) the probability that the bit of \(M\) is different from the bit of \(N\), we find finally

\[ I(M : N) = 1 - H(\{D_{MN}, 1 - D_{MN}\}) \tag{4} \]

with

\[
D_{AB} = \frac{1 - \cos \phi}{2}, \quad D_{AE} = \frac{1 - \sin \phi}{2}, \quad D_{BE} = \frac{1 - \sin 2\phi}{2}.
\]

In figure 3, we plotted \(I(A : B)\) versus \(\min[I(A : E), I(B : E)]\); we see that the condition for security (1) is fulfilled if and only if \(\phi \leq \frac{\pi}{4}\).
2.3 Violation of a Bell’s inequality

Having the three density matrices $\rho_{AB}$, $\rho_{AE}$ and $\rho_{BE}$ derived from the three-qubit state (3), we can also investigate whether one or more pairs violate a Bell’s inequality for a given value of $\phi$. Given a set of four unit vectors $\vec{a} = \{\vec{a}_1, \vec{a}_1', \vec{a}_2, \vec{a}_2'\}$, we build the two-qubit Bell operator

$$B_2(a) = (\sigma_{a_1} + \sigma_{a_1'}) \otimes \sigma_{a_2} + (\sigma_{a_1} - \sigma_{a_1'}) \otimes \sigma_{a_2'}$$

(6)

with $\sigma_a = \vec{a} \cdot \vec{\sigma}$. The CHSH inequality [11] reads $S_2 = \max_{a} \text{Tr}(\rho B_2(a)) \leq 2$, while the maximal value allowed by QM is $S_2 = 2\sqrt{2}$ [12]. The calculation of $S_2$ using the Horodecki criterion [13] can be carried out explicitly for the three pairs, and we find

$$S_{AB} = 2\sqrt{2} \cos \phi, \quad S_{AE} = 2\sqrt{2} \sin \phi, \quad S_{BE} = \sqrt{2} \sin 2\phi.$$  

(7)

Therefore the pair A-B violate the inequality if and only if the pair A-E does not violate it, and the curves cross at $\phi = \frac{\pi}{4}$, exactly were the security condition ceases to be fulfilled (fig. 1). As for the pair B-E, it never violates the inequality. The fact that the curves $S_{AB}$ and $S_{AE}$ cross at $\phi = \frac{\pi}{4}$ is an immediate consequence of the symmetry of the attack (2); but what is interesting is that they cross precisely for $S_{AB} = S_{AE} = 2$. In other words: simply using the symmetry, we could have guessed that $I(A : B) > I(A : E)$ if and only if $S_{AB} > S_{AE}$; but here we found that Eve’s optimal attack is such that A and B can establish a secret key using one-way privacy amplification iff $S_{AB} > 2$, i.e., iff they violate the CHSH inequality. This coincidence was already stressed in [5, 6].

It has recently been shown that the analysis of the BB84 protocol holds unchanged even if we suppose that Eve controls the source [14]. Also, in the case of the six-state protocol for QKD [15], the violation of CHSH is still a sufficient, but no longer a necessary condition. In conclusion, a tour d’horizon of the two-partners QKD protocols with qubits shows that the violation of the CHSH inequality is a sufficient condition for security. In Section 4, we generalize this statement to protocols in which the key is distributed among more than two partners. The next section is devoted to the definition of these protocols.

3 QKD involving N partners: definition, and eavesdropping

3.1 The N-partners secret-sharing protocol

The QKD protocol can be generalized to more than two partners in several ways. For instance, one may think of a protocol in which Alice sends information to Bob and Charlie so that she can choose a posteriori with whom a secret key will be established. Here we consider another family of protocols, based on the following idea: Alice sends information to her $N-1$ partners $B_1, ..., B_{N-1}$ in such a way that all of them must cooperate in order to retrieve the secret key, and any smaller subset of Bobs has no information on the key. More formally, this means that the bipartite mutual information $I(A : B_k)$ must be 0 for $k < N - 1$, and 1 for $k = N - 1$. Such protocols exist, and are called secret-sharing protocols [16].

Now, take $k+1$ of the partners and divide the partners into three non-empty groups $A, B$ and $C$. In general, it holds $I(A : BC) = I(AB : C) + H(B|C) - H(B|A)$. But in the protocols that we are considering, if we
know only \( A \), we have no information on \( B \), since we lack the information of \( C \); thus \( H(B \mid A) = H(B) \). Similarly, \( H(B \mid C) = H(B) \). Consequently for secret-sharing protocols we have \( I(A : BC) = I(AB : C) = I(A : B_1, ..., B_k) \).

The quantum version of a secret sharing protocol involving \( N \) partners (NQSS) goes as follows: Alice’s source produces the \( N \)-qubit GHZ state \( |GHZ_N \rangle = \frac{1}{\sqrt{2}}(|0^N \rangle + i|1^N \rangle) \), with \(|0^N \rangle_z = |0 \rangle \otimes ... \otimes |0 \rangle_z \), and \(|1^N \rangle_z = |1 \rangle \otimes ... \otimes |1 \rangle_z \). Alice measures \( \sigma_x \) or \( \sigma_y \) on one of the qubits, and sends the other qubits to her partners \( B_1, ..., B_{N-1} \). Each Bob also measures either \( \sigma_x \) or \( \sigma_y \). At the end of the transmission, all partners communicate publicly their measurements. Each time that an even number of partners have measured \( \sigma_y \), the results exhibit the desired correlation. In fact, consider a measurement where all partners measured \( \sigma_x \): each partner has one bit \( s_A, s_{B_1}, ..., s_{B_{N-1}} \), where \( s = \pm 1 \). Since \( (\sigma_x \otimes ... \otimes \sigma_x)_{GHZ} = 1 \), these \( N \) bits must satisfy \( s_A s_{B_1} \cdots s_{B_{N-1}} = 1 \). Consequently, if all the Bobs cooperate, they know Alice’s bit \( s_A = s_{B_1} \cdots s_{B_{N-1}} \); and if one or more of the Bobs refuses to cooperate, then the other Bobs have strictly no information on what Alice has sent.

Before studying Eve’s optimal individual attacks on NQSS, we introduce the useful notations
\[
|0^N \rangle_z = |0 \rangle \otimes ... \otimes |0 \rangle_z, \quad |1^N \rangle_z = |1 \rangle \otimes ... \otimes |1 \rangle_z
\]
\[
|0^N \rangle_x = \frac{1}{\sqrt{2}}(|0^N \rangle_z + |1^N \rangle_z), \quad |1^N \rangle_x = \frac{1}{\sqrt{2}}(|0^N \rangle_z - |1^N \rangle_z)
\]
\[
|0^N \rangle_y = \frac{1}{\sqrt{2}}(|0^N \rangle_z + i|1^N \rangle_z), \quad |1^N \rangle_y = \frac{1}{\sqrt{2}}(|0^N \rangle_z - i|1^N \rangle_z).
\]

These notations may seem misleading, since \(|0^N \rangle_z, |1^N \rangle_z, |0^N \rangle_y \) and \(|1^N \rangle_y \) are not product states like \(|0^N \rangle_z \) and \(|1^N \rangle_z \), but GHZ states. It will become evident in the following why such notations are indeed suited to our analysis.

### 3.2 Optimal eavesdropping for 3QSS

For clarity, we discuss in all detail Eve’s attacks in the case \( N = 3 \), that is, on the quantum secret sharing protocol proposed in [16]. Alice’s authorized partners are called Bob and Charlie. Two scenarios for eavesdropping can be imagined.

**Scenario 1.** An external Eve tries to eavesdrop on both channels A-B and A-C, in order to gain as much information as possible on Alice’s message. For the analysis of this scenario, it is convenient to suppose that Alice measures immediately \( \sigma_x \) or \( \sigma_y \) on her qubit. This way, she prepares the two-qubit state that is sent to Bob and Charlie, according to Table 1. If we forget the difference in the *physical realization* of the flying bit and stick to the information content of what is being transmitted, this eavesdropping scenario is identical to

| Measure of A | Result | State of BC |
|-------------|--------|-------------|
| \( \sigma_x \) | \( \equiv 0 \) | \(|0^2 \rangle_x \) |
|            | \( \equiv 1 \) | \(|1^2 \rangle_x \) |
| \( \sigma_y \) | \( \equiv 0 \) | \(|0^2 \rangle_y \) |
|            | \( \equiv 1 \) | \(|1^2 \rangle_y \) |

Table 1: Preparation of the state of BC by measurements of A.
the eavesdropping on the BB84 protocol. Therefore we know an individual attack that maximizes $I(A : E)$ for a given $I(A : BC)$: it is given by (3), replacing $|0\rangle$ with $|0^N\rangle_z$ on Bob’s side. Note that this attack is "coherent", in the sense that Eve attacks coherently the two qubits flying to B and C; but is nevertheless an "individual" attack, since each pair of qubits is attacked separately form the other pairs. This concludes the study of scenario 1.

**Scenario 2.** Bob does not want to cooperate with Charlie in order to retrieve Alice’s message. Consequently, he collaborates with an Eve that tries to eavesdrop on the line A-C. In this scenario, two triples come into play: A-B-C and A-B-E, and the meaningful information measures are $I(A : BC)$ and $I(A : BE)$. Just as in the analysis of scenario 1, it is useful to recast the protocol in the following form: by measuring $\sigma_x$ or $\sigma_y$, $A$ and $B$ prepare the state of the qubit that is sent to $C$. The preparation is given in Table 2. The bits flying on the channel A-C are exactly in the same physical state as in a BB84 protocol. The conclusion here is not as straightforward as for scenario 1 however, because the individual attack (2) optimizes for $I(AB : E)$ with respect to $I(AB : C)$, while we need the attack that optimizes $I(A : BE)$ with respect to $I(A : BC)$. But $I(AB : C) = I(A : BC)$ and $I(AB : E) = I(A : BE)$ hold even in the presence of the eavesdropper: in fact, $B$ and $C$ are not correlated before the eavesdropping, and $E$ is not correlated with $A$ and $B$; therefore $H(B|A) = H(B|C) = H(B|E) = H(B)$.

### Table 2: Preparation of the state of $C$ by measurements of $A$ and $B.$

| Meas. of $A$ and $B$ | Result | State of $C$ | Meas. of $A$ and $B$ | Result | State of $C$ |
|----------------------|--------|-------------|----------------------|--------|-------------|
| $\sigma_x \otimes \sigma_x$ | ++ | $|0\rangle_x$ | $\sigma_y \otimes \sigma_x$ | ++ | $|0\rangle_y$ |
| ++ | $|1\rangle_x$ |
| -- | $|0\rangle_x$ |
| $\sigma_x \otimes \sigma_y$ | ++ | $|0\rangle_y$ |
| ++ | $|1\rangle_y$ |
| -- | $|0\rangle_y$ |
| $\sigma_y \otimes \sigma_x$ | ++ | $|1\rangle_x$ |
| ++ | $|0\rangle_x$ |
| -- | $|0\rangle_x$ |

3.3 Optimal eavesdropping for NQSS

The same argument can be worked out for any eavesdropping scenario on NQSS, for arbitrary $N$. One the one side, as discussed, even under eavesdropping, it holds that $I(AB_1...B_{N-n-1} : B_{N-n}...B_{N-1}) = I(A : B_1...B_{N-1})$. On the other side, conditioned to the measurements of $\sigma_x$ and $\sigma_y$ by $N-n$ partners, the $n$ other partners can only receive one of the four states $|0^n\rangle_x, |1^n\rangle_x, |0^n\rangle_y, |1^n\rangle_y$. In fact, take as an example the case where all the $N-n$ partners measure $\sigma_x$. The N-qubit GHZ state $|\text{GHZ}_N\rangle$ can be rewritten as (we neglect normalization)

$$
|0^N\rangle_z + |1^N\rangle_z \simeq \left(|0\rangle_x + |1\rangle_x\right)^{\otimes (N-n)} |0^n\rangle_z + \left(|0\rangle_x - |1\rangle_x\right)^{\otimes (N-n)} |1^n\rangle_z = \text{ (even number of } 1_x \text{'s)} \otimes |0^n\rangle_x + \text{ (odd number of } 1_x \text{'s)} \otimes |1^n\rangle_x.
$$
Then, conditioned on the result of the measurement of $\sigma_x$ on the first $N - n$ qubits, either $|0^n\rangle_x$ or $|1^n\rangle_x$ is sent to the remaining $n$ partners. The case where some of the $N - n$ partners measure $\sigma_y$ is analogous; the states that are prepared are $|0^n\rangle_x$ or $|1^n\rangle_x$ if an even number of partners measure $\sigma_y$, $|0^n\rangle_y$ or $|1^n\rangle_y$ otherwise.

In conclusion, we have shown that, for all possible eavesdropping scenarios on NQSS protocols, Eve can perform the optimal individual attack having a single qubit as resource. The interaction that describes the optimal individual attack on $n$ channels is

$$
\begin{align*}
|0^n\rangle_z \otimes |0\rangle_E &\rightarrow |0^n\rangle_z \otimes |0\rangle_E \\
|1^n\rangle_z \otimes |0\rangle_E &\rightarrow \cos \phi |1^n\rangle_z \otimes |0\rangle_E + \sin \phi |0^n\rangle_z \otimes |1\rangle_E
\end{align*}
$$

(8)

where $\phi \in [0, \pi]$ measures the strength of the interaction. Of course, this interaction is presumably more complicated when she eavesdrops on several channels (like in scenario 1 for 3QSS), since she must have her qubit interacting coherently with all flying qubits. For the attack (8), the mutual information for the authorized and for the unauthorized partners, $I_a$ and $I_u$ respectively, can be calculated explicitly; it is not astonishing that the result is

$$
I_a = I(A : B_1, ..., B_{N-1}) = I_{N=2}(A : B) = 1 - H(D_{AB}, 1 - D_{AB})
$$

(9)

$$
I_u = I(A : B_1, ..., B_{N-n-1}, E) = I_{N=2}(A : E) = 1 - H(D_{AE}, 1 - D_{AE})
$$

(10)

with $D_{AB}$ and $D_{AE}$ given by (8). So again by symmetry $I_a > I_u$ if and only if $\phi < \frac{\pi}{4}$. To our knowledge, privacy amplification has not been studied in secret-sharing protocols; in particular, it is not clear if there is still a huge difference in efficiency between "one-way" and "two-way" protocols. It seems however obvious that the set of partners having the highest mutual information can run some protocol to extract a secret key.

4 Violations of Bell’s inequalities

4.1 Multiqubit Bell’s inequalities

In the case $N = 2$, we saw that $I_a > I_u$ if and only if the authorized partners violate the CHSH inequality, that is if $S_a > 2 > S_u$. We extend this result to all NQSS protocols.

The number of inequivalent BI grows rapidly with $M$, the number of qubits. We restrict to the family of inequalities obtained when only two measurement are performed on each qubit, which are the natural generalization of the CHSH inequality. Even with this restriction, the number of possible inequalities grows as $2^{2^M}$; but recently, the inequalities in this family have been completely classified by Werner and Wolf [17]. In particular, these authors have shown that in this family one can find some inequalities that are "optimal" under several respects. These optimal inequalities are nothing but the so-called Mermin-Klyshko inequalities (MKI) proposed some years ago [18, 19]. These are the $M$-qubit BI that we are going to consider in this work.
Let $\mathbf{u}$ be a set of $2M$ unit vectors. The Bell operator that enters the MKI for $M$ qubits is defined recursively as

$$B_M(\mathbf{u}) \equiv B_M = \frac{1}{2}(\sigma_{a_M} + \sigma_{a'_M}) \otimes B_{M-1} + \frac{1}{2}(\sigma_{a_M} - \sigma_{a'_M}) \otimes B'_{M-1}$$

where $B'_m$ is obtained from $B_m$ by exchanging all the $\mathbf{a}_k$ and $\mathbf{a}'_k$. The maximal value for product states is $\langle B_M \rangle = 2$; quantum correlations allow $\langle B_M \rangle > 2$, up to $\langle B_M \rangle = 2^{\frac{M+1}{2}}$, obtained for M-qubit GHZ states.

It is important to stress another property of MKIs [19, 20]. Let $\rho$ be a $M$-qubit state, and suppose that you can find a decomposition $\rho = \sum_i \rho_i \rho_i$ such that in all $\rho_i$ at most $m < M$ qubits are entangled (not necessarily the same ones in each $\rho_i$): then $\langle B_M \rangle_{\rho} \leq 2^{\frac{m+1}{2}}$. In other words, if $2^{\frac{m}{2}} < \langle B_M \rangle_{\rho} \leq 2^{\frac{m+1}{2}}$, the inequality for product states is violated, but this violation is weak, in the sense that it can be achieved with $m$-qubit entanglement. Now, for MQSS to work, $\rho$ must be ”close” to $|\text{GHZ}_M\rangle$, that is, must exhibit ”strong” M-qubit entanglement. Thus in all that follows we shall say that a M-qubit state $\rho$ violates the MKI if the violation is higher than the one that could be achieved with M-1 qubits, that is, if $\langle B_M \rangle_{\rho} > 2^{\frac{m}{2}}$.

### 4.2 Violation of MKI in NQSS

We consider the state that is generated in an eavesdropping scenario on NQSS. As usual, Alice is the sender. Some of the receivers would like to retrieve Alice’s message without the other $n$ partners to know it; they ask then Eve to spy on those lines. We call Bobs $B_1, ..., B_{N-n-1}$ the partners that collaborate with Eve, and Charlie $C_1, ..., C_n$ those that are spied. In the quantum protocol, each partner has a qubit, so we consider a system of $N + 1$ qubits. We write the Hilbert space as $\mathcal{H}_{AB} \otimes \mathcal{H}_C \otimes \mathcal{H}_E$. We have demonstrated in the previous section that under Eve’s optimal attack the state shared by the $N + 1$ partners becomes (we drop the subscript $z$)

$$|\Psi_{Nn}\rangle = \frac{1}{\sqrt{2}} \left( |0^{N-n}\rangle|0^n\rangle|0\rangle + \cos \phi |1^{N-n}\rangle|1^n\rangle|0\rangle + \sin \phi |1^{N-n}\rangle|0^n\rangle|1\rangle \right).$$

Let $\rho_{ABC} = \rho_a$ and $\rho_{ABE} = \rho_u$ be the density matrices of the authorized and of the unauthorized partners that are derived from $|\Psi_{Nn}\rangle$. We have $S_u = \max_a \text{Tr}(B_{N}(\mathbf{u})\rho_a)$ and $S_a = \max_a \text{Tr}(B_{N-n+1}(\mathbf{u})\rho_u)$. In the absence of a criterion like Horodeckis’ [3], it is difficult to perform the optimization that gives $S$, even for the particular state that we consider. We found an explicit result when $N$ and $n$ have different parity, and relied on numerical optimization for the other cases. These results are given in Appendix A. Within these warnings, we can safely state that the following holds: in the NQSS protocol, whatever the number $n$ of honest partners that are eavesdropped by Eve:

$$I_a > I_b \quad \text{if and only if} \quad S_a > 2^{\frac{N}{2}};$$

and in this case $S_u < 2^{\frac{N-n+1}{2}}$. This is the exact analog of the result obtained for $N = 2$: in case of optimal attack by Eve, the authorized partners have a higher information than the unauthorized ones if and only if they violate the MKI. In other words, to within the warnings above, we have proved the Conjecture put forward in a previous work [21].
4.3 MKI and the structure of the Hilbert space

The main feature of the link between optimal eavesdropping and the violation of MKIs is that the authorized partners violate the inequality if and only if the unauthorized partners don’t. Of course, it is trivial to loosen this link: non-optimal attacks can easily be found in which neither set of partners violate an inequality. Thus so far we have met only states of qubits characterized by the following property: if a set $\mathcal{M}$ of $M$ qubits violate a MKI, then all other sets of qubits having an overlap with $\mathcal{M}$ do not violate a MKI. A natural question is: is this property true for all possible states of qubits? If the answer were positive, then for any given state the violation of MKIs would define a unique partition of the set of qubits, into subsets of ”strongly entangled” qubits. This would provide an astonishing link between the violation of MK inequalities and the structure of the Hilbert space.

However, the answer to this question turns out to be negative. The simplest counterexample is provided by a system of four qubits A,B,C and D, where it is possible that both triples (A,B,C) and (B,C,D) violate the Mermin’s inequality. For example, for $\alpha \approx 0.955$, the state $\cos \alpha (|0011\rangle + |100\rangle + i|0101\rangle + i|1010\rangle)/2 + \sin \alpha (i|1001\rangle + |1111\rangle)/\sqrt{2}$ gives $S_{ABC} = S_{BCD} = 3$, obviously higher than $2\sqrt{2}$ which is the bound for a three-qubit violation. We have found numerically several more examples in which a violation of MKIs by two overlapping sets is allowed; these results are listed in Appendix B. Interestingly, there are also some cases in which a double violation is not possible: thus there is indeed a link between the violation of MKIs and the structure of the Hilbert space, although this link may be difficult to unravel. Possibly a stronger link could be found by using more general inequalities.

In the meantime, the link between violation of MKI and security is strengthened by these remarks. In fact, even though there exist states in the Hilbert space that would allow double violations of MKI, these states can never be produced in any eavesdropping scenario [21].

5 Conclusion

We have demonstrated a link between the security of some quantum key distribution protocols and the violation of some Bell’s inequalities. Precisely: in a secret-sharing protocol, the authorized partners have a higher mutual information than the unauthorized ones if and only if they violate a Mermin-Klyshko inequality. Whether this result is valid for other protocols, or for other inequalities, is an open question worth investigating.

All the protocols described in this paper can be implemented using qubits. It is a current field of investigation whether higher security can be achieved using higher-dimensional quantum systems [22]. Now, for such systems, no satisfactory Bell inequality has been found yet; the link with cryptography may provide a pathway to some new advances in this direction.
Acknowledgements

We acknowledge partial financial support from the Swiss FNRS and the Swiss OFES within the European project EQUIP (IST-1999-11053).

Appendix A

We want to calculate $S_a = \max_a \operatorname{Tr}(B_N(a)\rho_a)$ and $S_u = \max_u \operatorname{Tr}(B_{N-n+1}(a)\rho_a)$, where $\rho_a = \rho_{ABE}$ and $\rho_u = \rho_{ABE}$ derived from the state $|\Psi_{N,n}\rangle$ defined in (12). We discuss first the calculation of $S_a$.

Let $\mathcal{B} = B_N(a) \otimes 1_E$. Then we have $S_a = \max_a \langle \Psi_{N,n} | \mathcal{B} | \Psi_{N,n} \rangle$. Now:

$$\langle \Psi_{N,n} | \mathcal{B} | \Psi_{N,n} \rangle = \frac{1}{2} \left[ (0^N | \mathcal{B}_N | 0^N) + \cos^2 \phi \langle 1^N | \mathcal{B}_N | 1^N \rangle + \sin^2 \phi \langle 1^{N-n}0^n | \mathcal{B}_N | 1^{N-n}0^n \rangle + \cos \phi (\langle 1^N | \mathcal{B}_N | 0^N \rangle + \text{c.c.}) \right].$$

But $\langle 1^N | \mathcal{B}_N | 1^N \rangle = (-1)^N \langle 0^N | \mathcal{B}_N | 0^N \rangle$ and $\langle 1^{N-n}0^n | \mathcal{B}_N | 1^{N-n}0^n \rangle = (-1)^{N-n} \langle 0^N | \mathcal{B}_N | 0^N \rangle$, as can be easily verified from the definition of $\mathcal{B}_N$. Therefore

$$S_a = \max_a \left( f_{N,n}(\phi) B_{00}(a) + \cos \phi B_{10}(a) \right)$$

(14)

where $B_{00} = \langle 0^N | \mathcal{B}_N | 0^N \rangle$, $B_{10} = \text{Re}(\langle 1^N | \mathcal{B}_N | 0^N \rangle)$, and where the function $f_{N,n}(\phi)$ is positive and depends on the parity of $N$ and $n$. Before discussing it in detail let’s see why the calculation of $S_a$ is not trivial. We know that there are sets $a$ of unit vectors that saturate the bound $B_{10}(a) = 2^{N+1}$; but for these we find $B_{00}(a) = 0$. Similarly, the sets $a$ that saturate the bound $B_{00}(a) = 2$ give $B_{10}(a) = 0$. Thus to calculate $S_a$ we cannot optimize both $B_{00}$ and $B_{10}$: we must know whether it is better to optimize one of the two and letting the other go to zero, or if we must find an intermediate value. Numerical estimates suggest that the first strategy is the good one. However, by considering all possible values of $f_{N,n}(\phi)$ we can get some more insight

- For $N$ odd and $n$ even, $f_{N,n}(\phi) = 0$. Consequently the maximization is immediate: $S_a = 2^{N+1} \cos \phi$, that goes below the limit $2^{N+1}$ precisely for $\phi = \frac{\pi}{4}$.

- For $N$ even and $n$ odd, $f_{N,n}(\phi) = \cos^2 \phi$. Therefore $S_a \leq \cos \phi \max_a \left( B_{00}(a) + B_{10}(a) \right)$. But since $N$ is even, $B_{00}(a) + B_{10}(a) = \langle \text{GHZ} | \mathcal{B}_N(a) | \text{GHZ} \rangle$, that can reach $2^{N+1}$. Consequently the bound can be achieved, and we have again $S_a = 2^{N+1} \cos \phi$.

- For both $N$ and $n$ odd, $f_{N,n}(\phi) = \sin^2 \phi$; and for both $N$ and $n$ even, $f_{N,n}(\phi) = 1$. For these cases, we did not find any argument leading to a simple estimate of $S_a$. However, several numerical estimates strongly suggest that $S_a = \max \left[ 2^{N+1} \cos \phi, 2 f_{N,n}(\phi) \right]$. In particular, the boundary $S = 2^{N+1}$ is once again crossed for $\phi = \frac{\pi}{4}$.

The same discussion can be made for $S_u$, replacing $\phi$ by $\frac{\pi}{4} - \phi$, $N$ by $N' = N-n+1$ and $n$ by $n' = 1$. Therefore $n'$ is always odd. If $N$ and $n$ have different parities, then $N'$ is even, and we have certainly
$S_u = 2^{N-\frac{n+1}{2}} \sin \phi$. If $N$ and $n$ have the same parity, then $N'$ is odd as $n'$, and we are left with numerical arguments.

In conclusion, the condition for security (13) has been rigorously demonstrated for $N$ and $n$ of different parity. Note that this case includes the case where $n = N - 1$, that is the case of an external Eve and no dishonest Bob. For the cases where $N$ and $n$ have the same parity, we did not find a conclusive demonstration. However, both numerical arguments [23] and formal analogies (the structure of the states is identical) strongly suggest that (13) holds in these cases too.

Appendix B

Consider a set $K$ of $k$ qubits, and let $M$ and $N$ be two different but overlapping subsets of $K$ containing respectively $m$ and $n$ qubits. For definiteness, take $m \leq n$. For a given $|\Psi\rangle \in \mathcal{H}_K$, we write $\rho_N$ and $\rho_M$ the density matrices for the qubits in the two subsets obtained from $|\Psi\rangle\langle\Psi|$ by partial traces. We ask if it is possible to find a state $|\Psi\rangle \in \mathcal{H}_K$ such that

$$S_N = \max_{a} \text{Tr}(\rho_N B_n(a)) > 2^{n/2} \quad \text{and} \quad S_M = \max_{A} \text{Tr}(\rho_M B_m(A)) > 2^{m/2}$$

where $a$ and $A$ are sets of, respectively, $2^n$ and $2^m$ unit vectors. To tackle this question, we define the observable

$$V_{knm} = B_n(a) \otimes 1_{k-n} + 2^{n-m} 1_{k-m} \otimes B_m(A).$$

The computer program maximizes the highest eigenvalue of $V_{knm}$ over all possible choices of $a$ and $A$. If the highest eigenvalue does not exceed $2 \times 2^{n/2}$, then it is impossible to find a state that allows both $S_N > 2^{n/2}$ and $S_M > 2^{m/2}$. The results of the numerical calculations that we performed are listed here (for clarity, we print in boldface the common qubits):

- $\max(S_{AB} + S_{AC}) = 4$: double violation impossible (Theorem 1 in [21]).
- $\max(S_{ABC} + S_{ADE}) = 4\sqrt{2}$: impossible (Theorem 2 in [21]).
- $\max(S_{ABC} + S_{ABD}) = 6.0945 > 4\sqrt{2}$: possible (Theorem 3 in [21]; see main text for a state that gives a double violation).
- $\max(S_{ABC} + \sqrt{2}S_{AD}) = 4\sqrt{2}$: impossible.
- $\max(S_{ABC} + \sqrt{2}S_{AB}) = 6.9282 > 4\sqrt{2}$: possible, e.g. for the state $\frac{1}{\sqrt{2}}(|000\rangle + \cos \alpha|111\rangle + \sin \alpha|110\rangle)$, $\alpha \in [0, \frac{\pi}{2}]$.
- $\max(S_{ABCD} + S_{ADEF}) = 8$: impossible.
- $\max(S_{ABCD} + S_{ABEF}) = 8.612 > 8$: possible.
- $\max(S_{ABCD} + S_{ABCE}) = 8$: impossible.
- $\max(S_{ABCD} + 2S_{AE}) = 8$: impossible.
\[
\begin{align*}
&\max(S_{ABCD} + 2S_{AB}) = 9.7566 > 8: \text{possible.} \\
&\max(S_{ABCD} + \sqrt{2}S_{AEF}) = 8: \text{impossible.} \\
&\max(S_{ABCD} + \sqrt{2}S_{ABE}) = 6\sqrt{2} > 8: \text{possible.} \\
&\max(S_{ABCD} + \sqrt{2}S_{ABC}) = 9.6566 > 8: \text{possible.} \\
&\max(S_{ABCD} + S_{ABFGH}) = 12.088 > 8\sqrt{2}: \text{possible.} \\
&\max(S_{ABCD} + S_{ABC}) = 8\sqrt{2}: \text{impossible.} \\
&\max(S_{ABCD} + S_{ABCD}) = 8\sqrt{2}: \text{impossible.} \\
&\max(S_{ABCD} + 2S_{AB}) = 12.18 > 8\sqrt{2}: \text{possible.}
\end{align*}
\]

In these examples, double violations appear to be possible when one set is completely contained into the other one i.e. \(\mathcal{M} \subset \mathcal{N}\), or when \(\text{Card}(\mathcal{M} \cap \mathcal{N}) = 2\). Of course, it is difficult to guess general rules from these observations, since we have explored only the cases \(n, m = 2, 3, 4\) and some cases with \(n, m = 5\).

References

[1] For a recent review article, see: N. Gisin, G. Ribordy, W. Tittel, H. Zbinden, e-print quant-ph/0101098 (2001)

[2] C. Bennett, G. Brassard, in: Proceedings of the Int. Conf. on Computer, System and Signal Processing, Bangalore, India (IEEE, New York, 1984).

[3] A. Ekert, Phys. Rev. Lett. 67 (1991) 661

[4] C. Bennett, G. Brassard, N.D. Mermin, Phys. Rev. Lett. 68 (1992) 557

[5] B. Huttner, N. Gisin, Phys. Lett. A 228 (1997) 13

[6] C. Fuchs, N. Gisin, R.B. Griffiths, C.-S. Niu, A. Peres, Phys. Rev. A 56 (1997) 1163

[7] I. Csiszár, J. Körner, IEEE Trans. Inf. Theory IT-24 (1978) 339

[8] N. Gisin, S. Wolf, Phys. Rev. Lett. 83 (1999) 4200

[9] P.W. Shor, J. Preskill, Phys. Rev. Lett. 85 (2000) 441; and ref. therein

[10] C.-S. Niu, R.B. Griffiths, Phys. Rev. A 60(1999) 2764

[11] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett. 23 (1969) 880

[12] B.S. Cirel’son, Lett. Math. Phys. 4 (1980) 83

[13] M. Horodecki, P. Horodecki, M. Horodecki, Phys. Lett. A 200 (1995) 340
[14] H. Inamori, L. Rallan, V. Vedral, e-print quant-ph/0103058 (2001)

[15] H. Bechmann-Pasquinucci, N. Gisin, Phys. Rev. A 59 (1999) 4238

[16] M. Hillery, V. Buzek, A. Berthiaume, Phys. Rev. A 59 (1999) 1829; A. Karlsson, M. Koashi, N. Imoto, Phys. Rev. A 59 (1999) 162

[17] R.F. Werner, M.M. Wolf, e-print quant-ph/0102024 (2001).

[18] N.D. Mermin, Phys. Rev. Lett. 65 (1990) 1838; A.V. Belinskii, D.N. Klyshko, Phys. Usp. 36 (1993) 653

[19] N. Gisin, H. Bechmann-Pasquinucci, Phys. Lett. A 246 (1998) 1

[20] R.F. Werner, M.M. Wolf, Phys. Rev. A 61 (2000) 062102

[21] V. Scarani, N. Gisin, e-print quant-ph/0101110 (2001)

[22] H. Bechmann-Pasquinucci, A. Peres, Phys. Rev. Lett. 85 (2000) 3313; H. Bechmann-Pasquinucci, W. Tittel, Phys. Rev. A 61 (2000) 062308-1; M. Bourrenane, A. Karlsson, G. Björn, N. Gisin, in preparation

[23] The particular case $N = 3, n = 1$ was discussed in detail in [21], theorem 3. Other numerical verifications were made for $N = 4$ and $N = 5$. 
Figure 1: Bell parameter $S$ and mutual information $I$ for the two-partners QKD protocol BB84. The security criterion is satisfied if and only if $S_{AB} > 2$. 