THE BEST M-TERM APPROXIMATION WITH RESPECT TO POLYNOMIALS WITH CONSTANT COEFFICIENTS

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Abstract. In this paper we show that that greedy bases can be defined as those where the error term using m-greedy approximant is uniformly bounded by the best m-term approximation with respect to polynomials with constant coefficients in the context of the weak greedy algorithm and weights.

1. Introduction

Let $(X, \| \cdot \|)$ be an infinite-dimensional real Banach space and let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a normalized Schauder basis of $X$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. Throughout the paper, for each finite set $A \subset \mathbb{N}$ we write $|A|$ for the cardinal of the set $A$, $1_A = \sum_{j \in A} e_j$ and $P_A(x) = \sum_{n \in A} e_n^*(x) e_n$. Given a collection of signs $(\eta_j)_{j \in A} \in \{\pm 1\}$ with $|A| < \infty$, we write $1_{\eta A} = \sum_{n \in A} \eta_j e_j \in X$ and we use the notation $[1_{\eta A}]$ and $[e_n, n \in A]$ for the one-dimensional subspace and the $|A|$–dimensional subspace generated by generated by $1_{\eta A}$ and by $\{e_n, n \in A\}$ respectively. For each $x \in X$ and $m \in \mathbb{N}$, S.V. Konyagin and V.N. Temlyakov defined in [13] the $m$-th greedy approximant $G_m(x)$ by

$$G_m(x) = \sum_{j=1}^{m} e^*_{\rho(j)}(x)e_{\rho(j)},$$

where $\rho$ is a greedy ordering, that is $\rho : \mathbb{N} \rightarrow \mathbb{N}$ is a permutation such that $supp(x) = \{n : e^*_n(x) \neq 0\} \subseteq \rho(\mathbb{N})$ and $|e^*_n(x)| \geq |e^*_m(x)|$ for $j \leq i$. The collection $(G_m)_{m=1}^{\infty}$ is called the Thresholding Greedy Algorithm (TGA).

This algorithm is usually a good candidate to obtain the best m-term approximation with regard to $\mathcal{B}$, defined by

$$\sigma_m(x, \mathcal{B})_X = \sigma_m(x) := \inf \{d(x, [e_n, n \in A]) : A \subset \mathbb{N}, |A| = m\}.$$

The bases satisfying

$$\| x - G_m(x) \| \leq C\sigma_m(x), \quad \forall x \in X, \forall m \in \mathbb{N},$$

(1.1)

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where \( C \) is an absolute constant are called **greedy bases** (see [13]).

The first characterization of greedy bases was given by S.V. Konyagin and V. N. Temlyakov in [13] who established that a basis is greedy if and only if it is unconditional and democratic (where a basis is said to be democratic if there exists \( C > 0 \) so that \( \|1_A\| \leq C\|1_B\| \) for any pair of finite sets \( A \) and \( B \) with \( |A| = |B| \)).

Let us also recall two possible extensions of the greedy algorithm and the greedy basis. The first one consists in taking the \( m \) terms with near-biggest coefficients and generating the Weak Greedy Algorithm (WGA) introduced by V.N. Temlyakov in [15]. For each \( t \in (0, 1] \), a finite set \( \Gamma \subset \mathbb{N} \) is called a \( t \)-greedy set for \( x \in X \), for short \( \Gamma \in G(t, x, N) \), if

\[
\min_{n \in \Gamma} |e^*_n(x)| \geq t \max_{n \in \Gamma} |e^*_n(x)|,
\]

and write \( \Gamma \in G(t, x, t, N) \) if in addition \( |\Gamma| = N \). A \( t \)-greedy operator of order \( N \) is a mapping \( G^t : X \to X \) such that

\[
G^t(x) = \sum_{n \in \Gamma_x} e^*_n(x)e_n, \quad \text{for some } \Gamma_x \in G(t, x, t, N).
\]

A basis is called \( t \)-greedy if there exists \( C(t) > 0 \) such that

\[
\|x - G^t(x)\| \leq C(t)\sigma_m(x) \quad \forall x \in X, \forall m \in \mathbb{N}, \forall G^t \in G(t, x, t, m).
\] (1.2)

It was shown that a basis is \( t \)-greedy for some \( 0 < t \leq 1 \) if and only if it is \( t \)-greedy for all \( 0 < t \leq 1 \). From the proof it follows that greedy basis are also \( t \)-greedy basis with constant \( C(t) = O(1/t) \) as \( t \to 0 \).

The second one consists in replacing \( |A| \) by \( w(A) = \sum_{n \in A} w_n \) and it was considered by G. Kerkyacharian, D. Picard and V.N. Temlyakov in [12] (see also [14, Definition 16]). Given a weight sequence \( \omega = \{\omega_n\}_{n=1}^\infty, \omega_n > 0 \) and a positive real number \( \delta > 0 \), they defined

\[
\sigma^\omega_\delta(x) = \inf\{d(x, [e_n, n \in A]) : A \subset \mathbb{N}, \omega(A) \leq \delta\}
\]

where \( \omega(A) := \sum_{n \in A} \omega_n \), with \( A \subset \mathbb{N} \). They called **weight-greedy bases** (\( \omega \)-greedy bases) to those bases satisfying

\[
\|x - G_m(x)\| \leq C\sigma^\omega_\delta(A_m)(x), \quad \forall x \in X, \forall m \in \mathbb{N},
\] (1.3)

where \( C > 0 \) is an absolute constant and \( A_m = \text{supp}(G_m(x)) \). Moreover, they proved in [12] that \( B \) is a \( \omega \)-greedy basis if and only if it is unconditional and \( \omega \)-democratic (where a basis is \( \omega \)-democratic whenever there exists \( C > 0 \) so that \( \|1_A\| \leq C\|1_B\| \) for any pair of finite sets \( A \) and \( B \) with \( w(A) \leq w(B) \)). This generalization was motivated by the work of A. Cohen, R.A. DeVore and R. Hochmuth in [5] where the basis was indexed by dyadic intervals and \( w_\alpha(I) = \sum_{I \in A} |I|^\alpha \). Later in 2013, similar considerations were considered by E. Hernández and D. Vera to prove some inclusions of approximation spaces (see [9]).
Let us summarize and use the following combined definition.

**Definition 1.1.** Let $\mathcal{B}$ be a normalized Schauder basis in $X$, $0 < t \leq 1$ and weight sequence $\omega = \{\omega_n\}_{n=1}^{\infty}$ with $\omega_n > 0$. We say that $\mathcal{B}$ is $(t, \omega)$-**greedy** if there exists $C(t) > 0$ such that

$$\|x - G^t(x)\| \leq C(t)\sigma^w_{m(t)}(x) \quad \forall x \in X, \forall m \in \mathbb{N}, \forall G^t \in \mathcal{G}(x, t, m)$$

(1.4)

where $A_m(t) = \text{supp}(G^t(x))$ and $m(t) = w(A_m(t))$.

The authors introduced (see [2]) the best $m$-term approximation with respect to polynomials with constant coefficients as follows:

$$D^*_m(x) := \inf \{d(x, [1_{A}]) : A \subset \mathbb{N}, (\eta_n) \in \{\pm 1\}, |A| = m\}.$$ 

Obviously, $\sigma^w_m(x) \leq D^*_m(x)$ but, while $\sigma^w_m(x) \to 0$ as $m \to \infty$ it was shown that for orthonormal bases in Hilbert spaces we have $D^*_m(x) \to \|x\|$ as $m \to \infty$. The following result establishes a new description of greedy bases using the best $m$-term approximation with respect to polynomials with constant coefficients.

**Theorem 1.2.** ([2, Theorem 3.6]) Let $X$ be a Banach space and $\mathcal{B}$ a Schauder basis of $X$.

(i) If there exists $C > 0$ such that

$$\|x - G_m(x)\| \leq CD^*_m(x), \quad \forall x \in X, \forall m \in \mathbb{N},$$

then $\mathcal{B}$ is $C$-suppression unconditional and $C$-symmetric for largest coefficients.

(ii) If $\mathcal{B}$ is $K_s$-suppression unconditional and $C_s$-symmetric for largest coefficients then

$$\|x - G_m(x)\| \leq (K_sC_s)\sigma_m(x), \quad \forall x \in X, \forall m \in \mathbb{N}.$$ 

The concepts of suppression unconditional and symmetric for largest coefficients bases can be found in [2, 3, 4, 6, 13]. We recall here that a basis is $K_s$-suppression unconditional if the projection operator is uniformly bounded, that is to say

$$\|P_A(x)\| \leq K_s\|x\|, \quad \forall x \in X, \forall A \subset \mathbb{N}$$

and $\mathcal{B}$ is $C_s$-symmetric for largest coefficients if

$$\|x + t1_{\varepsilon A}\| \leq C_s\|x + t1_{\varepsilon B}\|,$$

for any $|A| = |B|$, $A \cap B = \emptyset$, $\text{supp}(x) \cap (A \cup B) = \emptyset$, $(\varepsilon_j), (\varepsilon'_j) \in \{\pm 1\}$ and $t = \max\{|e^*_n(x)| : n \in \text{supp}(x)\}$.

In this note we shall give a direct proof of the equivalence between condition (1.1) and (1.5) even in the setting of $(t, w)$-greedy basis.
Let us now introduce our best \( m \)-term approximation with respect to polynomials with constant coefficients associated to a weight sequence and the basic property to be considered in the paper.

**Definition 1.3.** Let \( \mathcal{B} \) be a normalized Schauder basis in \( X \), \( 0 < t \leq 1 \) and a weight sequence \( \omega = \{\omega_n\}_{n=1}^\infty \) with \( \omega_n > 0 \). We denote by

\[
D_\delta(x) := \inf\{d(x,[1,\eta A]) : A \subset \mathbb{N}, (\eta_n) \in \{\pm 1\}, \omega(A) \leq \delta\}.
\]

The basis \( \mathcal{B} \) is said to \((t,\omega)\)-greedy for polynomials with constant coefficients, denoted to have \((t,\omega)\)-PCCG property, if there exists \( D(t) > 0 \) such that

\[
\|x - G^t(x)\| \leq D(t)D^\omega_{m(t)}(x), \forall x \in X, \forall m \in \mathbb{N}, \forall G^t \in \mathcal{G}(x,t,m)
\]

where \( A_m(t) = \text{supp}(G^t(x)) \) and \( m(t) = \omega(A_m(t)) \).

In the case \( t = 1 \) and \( w(A) = |A| \) we simply call it the PCCG property.

Of course \( \sigma^\omega_{\mathcal{B}}(x) \leq D^\omega_{\mathcal{B}}(x) \) for all \( \delta > 0 \), hence if the basis is \((t,\omega)\)-greedy then \((1.6)\) holds with the \( D(t) = C(t) \). We now formulate our main result which produces a direct proof of the result in [2] and give the extension to \( t \)-greedy and weighted greedy versions.

**Theorem 1.4.** Let \( \mathcal{B} \) be a normalized Schauder basis in \( X \) and let \( \omega = \{\omega_n\}_{n=1}^\infty \) be a weight sequence with \( \omega_n > 0 \) for all \( n \in \mathbb{N} \). The following are equivalent:

(i) There exist \( 0 < s \leq 1 \) such that \( \mathcal{B} \) has the \((s,w)\)-PCCG property.

(ii) \( \mathcal{B} \) is \((t,\omega)\)-greedy for all \( 0 < t \leq 1 \).

**Proof.** Only the implication (i) \( \implies \) (ii) needs a proof. Let us assume that \((1.6)\) holds for some \( 0 < s \leq 1 \). Let \( 0 < t \leq 1 \), \( x \in X \), \( m \in \mathbb{N} \) and \( G^t \in \mathcal{G}(x,t,m) \). We write \( G^t(x) = P_{A_m(t)}(x) \) with \( A_m(t) \in \mathcal{G}(x,t,m) \). For each \( \varepsilon > 0 \) we choose \( z = \sum_{n \in B} e^*_n(x)e_n \) with \( \omega(B) \leq \omega(A_m(t)) \) and \( \|x - z\| \leq \sigma^\omega_{\omega(A_m)}(x) + \varepsilon \).

We write

\[
x - P_{A_m(t)}(x) = x - P_{A_m(t) \cup B}(x) + P_{B \setminus A_m(t)}(x).
\]

Taking into account that \( P_{B \setminus A_m(t)}(x) \in co(\{S\delta(\eta(B \setminus A_m(t)) : |\eta_j| = 1\}) \) for any \( S \geq \max_{j \in B \setminus A_m(t)} |e^*_j(x)| \), it suffices to show that there exists \( R \geq 1 \) and \( C(t) > 0 \) such that

\[
\|x - P_{A_m(t) \cup B}(x) + R\gamma 1_{\eta(B \setminus A_m(t))}\| \leq C(t)\|x - z\| \tag{1.7}
\]

for any choice of signs \( (\eta_j)_{j \in B \setminus A_m(t)} \) where \( \gamma = \max_{j \in B \setminus A_m(t)} |e^*_j(x)| \).

Let us assume first that \( t \geq s \). We shall show that

\[
\|x - P_{(A_m(t) \cap B)}(x) + \frac{t}{s}\gamma 1_{\eta(B \setminus A_m(t))}\| \leq D(s)\|x - P_B(x)\| \tag{1.8}
\]

for any choice of signs \( (\eta_j)_{j \in B \setminus A_m(t)} \).
Given \((\eta)_{j \in B \setminus A_m(t)}\) we consider
\[
y_{\eta} = x - P_B(x) + \frac{t}{s} \gamma_{1_{\eta(B \setminus A_m(t))}} + \sum_{n \notin B} e_n^*(x)e_n + \sum_{n \in B \setminus A_m(t)} t \frac{1}{s} \gamma_{\eta_n} e_n.
\]
Note that
\[
\min_{n \in A_m(t) \setminus B} |e_n^*(y_{\eta})| = \min_{n \in A_m(t) \setminus B} |e_n^*(x)| \geq \min_{n \in A_m(t)} |e_n^*(x)|
\]
and
\[
s \max_{n \in (A_m(t), B)} |e_n^*(y_{\eta})| = \max\{s \max_{n \notin A_m(t)} |e_n^*(x)|, t \gamma\}.
\]
Therefore, since \(t \geq s\), we conclude that
\[
\min_{n \in A_m(t) \setminus B} |e_n^*(y_{\eta})| \geq s \max_{n \in (A_m(t), B)} |e_n^*(y_{\eta})|.
\]
Hence \(A_m(t) \setminus B \in G(y_{\eta}, s, N)\) with \(N = |A_m(t) \setminus B|\). We write \(G^s(y_{\eta}) = P_{A_m(t) \setminus B}(x)\) and notice that
\[
y_{\eta} - G^s(y_{\eta}) = x - P_{A_m(t) \setminus B}(x) + \frac{t}{s} \gamma_{1_{\eta(B \setminus A_m(t))}}.
\]
Since \(\omega(B) \leq \omega(A_m(t))\) we have also that \(\omega(B \setminus A_m(t)) \leq \omega(A_m(t) \setminus B)\). Hence for \(N(s) = \omega(A_m(t) \setminus B)\) we conclude
\[
\|x - P_{A_m(t) \setminus B}(x) + \frac{t}{s} \gamma_{1_{\eta(B \setminus A_m(t))}}\| \leq D(s) \mathcal{D}_s^N(s)(y_{\eta})
\]
\[
\leq D(s) \|y_{\eta} - \frac{t}{s} \gamma_{1_{B \setminus A_m(t)}}\|
\]
\[
= D(s) \|x - P_B(x)\|.
\]
Now, let \(y = x - z + \mu 1_B\) for \(\mu = s \max_{j \notin B} |e_j^*(x - z)| + \max_{j \in B} |e_j^*(x - z)|\).

Then
\[
\min_{j \in B} |\mu + e_n^*(x - z)| \geq s \max_{j \notin B} |e_n^*(x - z)|,
\]
which gives that \(B \in G(y, s, |B|)\) and we obtain \(G^s(y) = P_B(x - z) + \mu 1_B\). Hence
\[
\|x - P_B(x)\| = \|y - G^s(y)\| \leq D(s) \|y - \mu 1_B\| = D(s) \|x - z\|. \tag{1.9}
\]
Therefore, by (1.8) and (1.9) we obtain
\[
\|x - P_{A_m(t) \setminus B}(x) + \frac{t}{s} \gamma_{1_{\eta(B \setminus A_m(t))}}\| \leq D(s)^2 \|x - z\|.
\]
Then, for \(s \leq t\) we obtain that \(\mathcal{B}\) is \((t, w)\)-greedy with constant \(C(t) \leq D(s)^2\).

We now consider the case \(s > t\). We use the following estimates:
\[
\|x - P_{A_m(t) \setminus B}(x) + \gamma_{1_{\eta(B \setminus A_m(t))}}\| \leq \|x - P_B(x)\| + \|P_{A_m(t) \setminus B}(x)\| + \gamma \|1_{\eta(B \setminus A_m(t))}\|.
\]
Arguing as above, using now
\[
\bar{y}_{\eta} = P_{A_m(t) \setminus B}(x) + \frac{t}{s} \gamma_{1_{\eta(B \setminus A_m(t))}},
\]
we conclude that \( \frac{4}{5} \gamma \| 1_{\eta B \setminus A_m(t)} \| \leq D(s) \| P_{A_m(t) \setminus B} (x) \| \).

The argument used to show (1.9) gives \( \| z - P_C z \| \leq D(s) \| z \| \) for all \( z \in \mathbb{X} \) and finite set \( C \). Therefore

\[
\| P_{A_m(t) \setminus B} (x) \| = \| P_{A_m(t)} (x - P_B x) \| \leq (1 + D(s)) \| x - P_B x \|. 
\]

Putting all together we have

\[
\| x - P_{(A_m(t) \cup B)} (x) + \gamma 1_{\eta B \setminus A_m(t)} \| \leq (2 + \frac{t + s}{t} D(s)) \| x - P_B x \|,
\]

and therefore \( \mathcal{B} \) is \((t, w)\) -greedy with constant \( C(t) \leq (2 + \frac{t + s}{t} D(s)) D(s) \). \( \square \)

**Corollary 1.5.** If \( t = 1 \) and \( \omega(A) = |A| \), then \( \mathcal{B} \) has the PCCG property if and only if \( \mathcal{B} \) is greedy.

**Corollary 1.6.** If \( \omega(A) = |A| \), then \( \mathcal{B} \) has the \( t \)-PCCG property if and only if \( \mathcal{B} \) is \( t \)-greedy.

### 2. A Remark on the Haar System

Throughout this section \(|E|\) stands for the Lebesgue measure of a set in \([0, 1]\), \( \mathcal{D} \) for the family of dyadic intervals in \([0, 1]\) and \( \text{card}(\Lambda) \) for the number of dyadic elements in \( \Lambda \). We denote by \( \mathcal{H} := \{ H_I \} \) the Haar basis in \([0, 1]\), that is to say

\[
H_{[0,1]}(x) = 1 \text{ for } x \in [0, 1],
\]

and for \( I \in \mathcal{D} \) of the form \( I = [(j - 1)2^{-n}, j 2^{-n}], j = 1, \ldots, 2^n, n = 0, 1, \ldots \) we have

\[
H_I(x) = \begin{cases} 
2^{n/2} & \text{if } x \in [(j - 1)2^{-n}, (j - \frac{1}{2})2^{-n}), \\
-2^{n/2} & \text{if } x \in [(j - \frac{1}{2})2^{-n}, j2^{-n}), \\
0 & \text{otherwise.}
\end{cases}
\]

We write

\[
c_I(f) := \langle f, H_I \rangle = \int_0^1 f(x) H_I(x) \, dx \quad \text{and} \quad c_I(f, p) := \| c_I(f) H_I \|_p, \quad 1 \leq p < \infty.
\]

It is well known that \( \mathcal{H} \) is an orthonormal basis in \( L^2([0, 1]) \) and for \( 1 < p < \infty \) we can use the Littlewood-Paley’s Theorem which gives

\[
c_p \left( \left\| \sum_I |c_I(f, p) H_I| \frac{1}{\| H_I \|_p^2} \right\|^{1/2} \right)_p \leq \left\| f \right\|_p \leq C_p \left( \left\| \sum_I |c_I(f, p) H_I| \frac{1}{\| H_I \|_p^2} \right\|^{1/2} \right)_p
\]

(2.1)

to conclude that \( \left( \frac{H_I}{\| H_I \|_p} \right)_I \) is an unconditional basis in \( L^p([0, 1]) \). Denoting \( f \ll_p g \) whenever \( c_I(f, p) \leq c_I(g, p) \) for all dyadic intervals \( I \) we obtain from (2.1) the existence of a constant \( K_p \) such that

\[
\| f \|_p \leq K_p \| g \|_p \quad \forall f, g \in L^p([0, 1]) \text{ with } f \ll_p g,
\]

(2.2)
and also
\[ \|P_\Lambda g\| \leq K_p\|g\|_p \quad \forall g \in L^p \quad \forall \Lambda \subset \mathcal{D}. \tag{2.3} \]

Regarding the greedyness of the Haar basis it was V. N. Temlyakov the first one who proved (see [15]) that the every wavelet basis \( L^p \)-equivalent to the Haar basis is \( t \)-greedy in \( L_p([0,1]) \) with \( 1 < p < \infty \) for any \( 0 < t \leq 1 \).

Let \( \omega : [0,1] \to \mathbb{R}^+ \) be a measurable weight and, as usual, we denote \( \omega(I) = \int_I \omega(x)dx \) and \( m_I(\omega) = \frac{\omega(I)}{|I|} \) for any \( I \in \mathcal{D} \). In the space \( L^p(\omega) = L^p([0,1],\omega) \) we denote \( \|f\|_{p,\omega} = \left( \int_0^1 |f(x)|^p \omega(x)dx \right)^{1/p} \) and
\[ c_I(f,p,\omega) := \|c_I(f)H_I\|_{p,\omega} = |c_I(f)| \frac{\omega(I)^{1/p}}{|I|^{1/2}}. \]

Recall that \( \omega \) is said to be a dyadic \( A_p \)-weight (denoted \( \omega \in A_p^d \)) if
\[ A_p^d(\omega) = \sup_{I \in \mathcal{D}} m_I(\omega) \left( m_I(\omega^{-1/(p-1)}) \right)^{p-1} < \infty. \tag{2.4} \]

As one may expect, Littlewood-Paley theory holds for weights in the dyadic \( A_p \)-class.

**Theorem 2.1.** (see [1] [10] for the multidimensional case) If \( \omega \in A_p^d \) then
\[ \|f\|_{p,\omega} \approx \left( \sum_I |c_I(f,p,\omega)| \frac{H_I}{\|H_I\|_{p,\omega}} \right)^{1/2} \|H_I\|_{p,\omega}. \tag{2.5} \]

In particular \( \left( \frac{H_I}{\|H_I\|_{p,\omega}} \right)_I \) is an unconditional basis in \( L^p(\omega) \) for \( 1 < p < \infty \).

The greedyness of the Haar basis in \( L^p(\omega) \) goes back to M. Izuki (see [10] [11]) who showed that this holds for weights in the class \( A_p^d \). We shall use the ideas in these papers to show that the Haar basis satisfies the PCCG property for certain spaces defined using the Littlewood-Paley theory.

**Definition 2.2.** Let \( \omega : [0,1] \to \mathbb{R}^+ \) be a measurable weight and \( 1 \leq p < \infty \). For each finite set of dyadic intervals \( \Lambda \) we define \( f_\Lambda = \sum_{I \in \Lambda} c_I(f)H_I = \sum_{I \in \Lambda} c_I(f,p,\omega) \frac{H_I}{\|H_I\|_{p,\omega}} \)
and write
\[ \|f\|_{X_p(\omega)} = \left( \sum_{I \in \Lambda} |c_I(f,p,\omega)| \frac{H_I}{\|H_I\|_{p,\omega}} \right)^{1/2} \|H_I\|_{p,\omega}. \]

The closure of \( \text{span}(f_\Lambda : \text{card}(\Lambda) < \infty) \) under this norm will be denoted \( X^p(\omega) \).

From the definition \( \left( \frac{H_I}{\|H_I\|_{p,\omega}} \right)_I \) is an unconditional basis with constant 1 in \( X^p(\omega) \) and due to (2.5) \( X^p(\omega) = L^p(\omega) \) whenever \( \omega \in A_p^d \). Our aim is to analyze conditions on the weight \( \omega \) for the basis to be greedy. For such a purpose we do not need the weight to belong to \( A_p^d \). In fact analyzing the proof in [10] [11] one notices that only the dyadic reverse doubling condition (see [8] p. 141) was used. Recall that a weight
\( \omega \) is said to satisfy the **dyadic reverse doubling condition** if there exists \( \delta < 1 \) such that

\[
\omega(I') \leq \delta \omega(I), \forall I, I' \in \mathcal{D} \text{ with } I' \subset I.
\] (2.6)

Let us introduce certain weaker conditions.

**Definition 2.3.** Let \( \alpha > 0 \) and \( \omega \) be a measurable weight. We shall say that \( \omega \) satisfies the **dyadic reverse Carleson condition of order \( \alpha \)** with constant \( C > 0 \) whenever

\[
\sum_{I \in \mathcal{D}, J \subseteq I} \omega(I)^{-\alpha} \leq C \omega(J)^{-\alpha}, \forall J \in \mathcal{D}.
\] (2.7)

**Definition 2.4.** Let \( \alpha > 0 \) and two sequences \( (w_I)_{I \in \mathcal{D}} \) and \( (v_I)_{I \in \mathcal{D}} \) of positive real numbers. We say that the pair \( (w_I)_{I \in \mathcal{D}}, (v_I)_{I \in \mathcal{D}} \) satisfies the **\( \alpha \)-DRCC** with constant \( C > 0 \) whenever

\[
\sum_{I \in \mathcal{D}, J \subseteq I} w_I^{-\alpha} \leq C v_J^{-\alpha}, \forall J \in \mathcal{D}.
\] (2.8)

**Remark 2.5.** (i) If \( \omega \in \bigcup_{p>1} A_p^w \) then \( \omega \) satisfies the dyadic reverse doubling condition (see [8, p 141]).

(ii) If \( \omega \) satisfies the dyadic reverse doubling condition then \( \omega \) satisfies the dyadic reverse Carleson condition of order \( \alpha \) with constant \( \frac{1}{1-\delta^\alpha} \) for any \( \alpha > 0 \).

Indeed,

\[
\sum_{J \subseteq I} \omega(I)^{-\alpha} \leq \omega(J)^{-\alpha} + \omega(J)^{-\alpha} \sum_{m=1}^\infty \delta^{ma} \leq \frac{1}{1-\delta^\alpha} \omega(J)^{-\alpha}.
\]

(iii) If \( \omega \) satisfies the dyadic reverse Carleson condition of order \( \alpha \) and \( w_I = \omega(I) \) for each \( I \in \mathcal{D} \) then \( (w_I)_{I \in \mathcal{D}}, (\omega(I))_{I \in \mathcal{D}} \) satisfies \( \alpha \)-DRCC.

We need the following lemmas, whose proofs are essentially included in [5, 10, 11].

**Lemma 2.6.** Let \( \omega \) be a weight and \( (v_I)_{I \in \mathcal{D}} \) be a sequence of positive real numbers such that \( (v_I)_{I \in \mathcal{D}}, (\omega(I))_{I \in \mathcal{D}} \) satisfies \( 1 \)-DRCC with constant \( C \). Then

\[
\left( \sum_{I \in \Lambda} \omega(I) \right)^{1/p} \leq C \left( \sum_{I \in \Lambda} \frac{H_I}{p, \omega} \right)^{1/p}, \forall 1 \leq p < \infty.
\] (2.9)

**Proof.** We first write

\[
\left( \sum_{I \in \Lambda} \frac{H_I}{p, \omega} \right)^{1/p} = \left( \int_0^1 \left( \sum_{I \in \Lambda} \omega(I)^{-2/p} \chi_I \right)^{p/2} \omega(x)dx \right)^{1/p}.
\] (2.10)

Let \( I(x) \) denote the minimal dyadic interval in \( \Lambda \) with regard to the inclusion relation that contains \( x \). Now we use that

\[
\sum_{I \in \mathcal{D}, I(x) \subseteq I} v_I^{-1} \leq C \omega(I(x))^{-1}
\]
to conclude that

\[ \left( \sum_{I \in \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p} = \left( \sum_{I \in \Lambda} \int v_I^{-1} \omega(x) \, dx \right)^{1/p} = \left( \int_0^1 \left( \sum_{I \in \Lambda} v_I^{-1} \chi_I(x) \right) \omega(x) \, dx \right)^{1/p} \]

\[ \leq C \left( \int_0^1 \omega(I(x))^{-1} \omega(x) \, dx \right)^{1/p} \leq C \left( \int_0^1 \left( \sum_{I \in \Lambda} \omega(I)^{-2/p} \chi_I(x) \right)^{p/2} \omega(x) \, dx \right)^{1/p} \]

\[ = C \left\| \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right\|_{X^p(\omega)}. \]

The proof is complete. □

**Lemma 2.7.** Let \( 1 < p < \infty \), \( \omega \) be a weight and \((v_I)_{I \in \mathcal{D}}\) of positive real numbers. If \((\omega(I))_{I \in \mathcal{D}}, (v_I)_{I \in \mathcal{D}}\) satisfies \(2/p\)-DRCC with constant \(C > 0\) then

\[ \left\| \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right\|_{X^p(\omega)} \leq C \left( \sum_{I \in \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p} \]

(2.11)

for all finite family \(\Lambda\) of dyadic intervals.

**Proof.** Let \( E = \bigcup_{I \in \Lambda} I \). As above \( I(x) \) stands for the minimal dyadic interval in \( \Lambda \) with regard to the inclusion relation that contains \( x \). From (2.8) we have that

\[ \sum_{I \in \Lambda} \omega(I)^{-2/p} \chi_I(x) \leq C v_I^{-2/p}, \quad x \in E. \]

(2.12)

Now denote for each \( I \in \Lambda \), \( \tilde{I} = \{x \in E : I(x) = I \} \). Clearly \( \tilde{I} \subseteq I \) and \( E = \bigcup_{I \in \Lambda} \tilde{I} \). Hence applying (2.10) and (2.12) we obtain

\[ \left\| \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right\|_{X^p(\omega)} \leq C \left( \int_E v_I^{-1} \omega(x) \, dx \right)^{1/p} = C \left( \int_{\bigcup_{I \in \Lambda} \tilde{I}} v_I^{-1} \omega(x) \, dx \right)^{1/p} \]

\[ \leq C \left( \sum_{I \in \Lambda} \int_{\tilde{I}} v_I^{-1} \omega(x) \, dx \right)^{1/p} \leq C \left( \sum_{I \in \Lambda} v_I^{-1} \int_I \omega(x) \, dx \right)^{1/p} \]

\[ = C \left( \sum_{I \in \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p}. \]

The proof is now complete. □

Combining Remark 2.5 and Lemmas 2.6 and 2.7 we obtain the following corollary.

**Corollary 2.8.** Let \( 1 < p < \infty \), \( \omega \) be a weight satisfying the dyadic reverse doubling condition then

\[ \left\| \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right\|_{X^p(\omega)} \approx \text{card}(\Lambda)^{1/p} \]

(2.13)

for all finite family \(\Lambda\) of dyadic intervals.
Corollary 2.9. Let $1 < p < \infty$, $\omega$ be a weight and $(v_I)_{I \in \mathcal{D}}$ of positive real numbers. If $\left( (\omega(I))_{I \in \mathcal{D}}, (v_I)_{I \in \mathcal{D}} \right)$ satisfies $2/p'\text{-DRCC}$ with constant $C > 0$ then

$$
\left( \sum_{I \in \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p} \leq C \left( \max_{I \in \Lambda} \frac{\omega(I)}{v_I} \right) \left( \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right) \leq X^{p(\omega)} (2.14)
$$

for all finite family $\Lambda$ of dyadic intervals.

**Proof.** Note that, using Lemma 2.7, we have

$$
\sum_{I \in \Lambda} \frac{\omega(I)}{v_I} = \int_0^1 \left( \sum_{I \in \Lambda} v_I^{-1} \chi_I(x) \omega(x) \right) dx
$$

$$
\leq \int_0^1 \left( \sum_{I \in \Lambda} (\omega(I)^{-2/p} \chi_I)^{1/2} \langle \sum_{I \in \Lambda} v_I^{-2} \omega(I)^{2/p} \chi_I(x) \rangle^{1/2} \omega(x) \right) dx
$$

$$
\leq \left( \int_0^1 \left( \sum_{I \in \Lambda} (\omega(I)^{-2/p} \chi_I)^{p/2} \omega(x) \right) dx \right)^{1/p} \left( \int_0^1 \left( \sum_{I \in \Lambda} v_I^{-2} \omega(I)^{2/p} \chi_I(x) \right)^{p'/2} \omega(x) dx \right)^{1/p'}
$$

$$
\leq \left( \max_{I \in \Lambda} \frac{\omega(I)}{v_I} \right) \left( \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \right) \left( \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p',\omega}} \right) \left( \sum_{I \in \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p'}
$$

The result now follows. \qed

Taking into account that dyadic reverse Carleson condition of order $\alpha$ implies dyadic reverse Carleson condition of order $\beta$ for $\beta > \alpha$ we obtain the following fact.

Corollary 2.10. Let $1 < p < \infty$, $\omega$ be a weight satisfying the dyadic reverse Carleson condition of order $\min\{2/p', 2/p\}$ then

$$
\| \sum_{I \in \Lambda} \frac{H_I}{\|H_I\|_{p,\omega}} \|_{X^{p(\omega)}} \approx \text{card}(\Lambda)^{1/p} (2.15)
$$

for all finite family $\Lambda$ of dyadic intervals.

Theorem 2.11. Let $1 < p < \infty$, $0 < t \leq 1$, $(w_I)_{I \in \mathcal{D}}$ be a sequence of real numbers such that

$$
0 < m_0 = \inf_{I \in \mathcal{D}} w_I \leq \sup_{I \in \mathcal{D}} w_I = M_0 < \infty
$$

and let $\omega$ be a weight satisfying the dyadic reverse Carleson condition of order $\min\{1, 2/p\}$ with constant $C > 0$. Then the Haar basis has the $(t, w_I)$-PCCG property in $X^{p(\omega)}$. 
Proof. Let \( f \in X^p(\omega) \) and let \( \Lambda_m \) be a set of \( m \) dyadic intervals where

\[
\min_{I \in \Lambda_m} c_I(f, p, \omega) \geq t \max_{I \notin \Lambda_m} c_I(f, p, \omega).
\]

For each \( \alpha \in \mathbb{R} \), \( (\varepsilon_n) \in \{ \pm 1 \} \) and \( \Lambda \) with \( \sum_{J \in \Lambda} w_J \leq \sum_{I \in \Lambda_m} w_I \) we need to show that \( \| f - P_{\Lambda_m}(f) \|_{X^p(\omega)} \leq C(t) \| f - \alpha 1_{\varepsilon \Lambda} \|_{X^p(\omega)} \) for some constant \( C(t) > 0 \). From triangular inequality

\[
\| f - P_{\Lambda_m}(f) \|_{X^p(\omega)} \leq \| P_{(\Lambda_m \cup \Lambda)^c}(f - \alpha 1_{\varepsilon \Lambda}) \|_{X^p(\omega)} + \| P_{\Lambda \setminus \Lambda_m}(f) \|_{X^p(\omega)}
\]

and the fact \( \| P_{\Lambda}(f - \alpha 1_{\varepsilon B}) \|_{X^p(\omega)} \leq \| f - \alpha 1_{\varepsilon B} \|_{X^p(\omega)} \) for any \( \Lambda \) we only need to show that there exists \( C > 0 \) such that

\[
\| P_{\Lambda \setminus \Lambda_m}(f) \|_{X^p(\omega)} \leq C \| f - \alpha 1_{\varepsilon \Lambda} \|_{X^p(\omega)}.
\]

Set \( v_I = \frac{\omega(I)}{m} \) and observe that \( (\omega(I))_{I \in \mathcal{D}}, (v_I)_{I \in \mathcal{D}} \) satisfies 2/\( p \)-DRCC with constant \( M_0 C \) and \( (v_I)_{I \in \mathcal{D}}, (\omega(I))_{I \in \mathcal{D}} \) satisfies 1-DRCC with constant \( C/m_0 \). Note that \( \sum_{J \in \Lambda \setminus \Lambda_m} w_J \leq \sum_{I \in \Lambda_m} w_I \) implies that

\[
\sum_{J \in \Lambda \setminus \Lambda_m} \frac{\omega(J)}{v_J} \leq \sum_{I \in \Lambda_m \setminus \Lambda} \frac{\omega(I)}{v_I}
\]

and then, invoking Lemma 2.7 and Lemma 2.6 we get the estimates

\[
\| P_{\Lambda \setminus \Lambda_m}(f) \|_{X^p(\omega)} \leq \| \max_{I \in \Lambda \setminus \Lambda_m} c_I(f, p, \omega) 1_{\Lambda \setminus \Lambda_m} \|_{X^p(\omega)}
\]

\[
\leq CM_0 \max_{I \in \Lambda_m \setminus \Lambda} c_I(f, p, \omega) \left( \sum_{J \in \Lambda \setminus \Lambda_m} \frac{\omega(J)}{v_J} \right)^{1/p}
\]

\[
\leq t^{-1} CM_0 \min_{I \in \Lambda_m \setminus \Lambda'} c_I(f, p, \omega) \left( \sum_{I \notin \Lambda_m \setminus \Lambda} \frac{\omega(I)}{v_I} \right)^{1/p}
\]

\[
\leq \frac{C^2 M_0}{tm_0} \min_{I \in \Lambda_m \setminus \Lambda'} c_I(f, p, \omega) 1_{\Lambda_m \setminus \Lambda'} \|_{X^p(\omega)}
\]

\[
\leq \frac{C^2 M_0}{tm_0} \| P_{\Lambda_m \setminus \Lambda}(f) \|_{X^p(\omega)}
\]

\[
= \frac{C^2 M_0}{tm_0} \| P_{\Lambda_m \setminus \Lambda}(f - \alpha 1_{\varepsilon B}) \|_{X^p(\omega)}
\]

\[
\leq \frac{C^2 M_0}{tm_0} \| f - \alpha 1_{\varepsilon \Lambda} \|_{X^p(\omega)}.
\]

This completes the proof with \( C(t) = 1 + \frac{C^2 M_0}{tm_0} \).

\[\square\]

Corollary 2.12. (i) If \( \omega \in A^d \) then the Haar basis has the t-PCCG property (and hence is t-greedy) in \( L^p(\omega) \) with \( 1 < p < \infty \).

(ii) The Haar basis has the \((t, w_I)\)-PCCG property (and hence is \((t, w_I)\)-greedy) in \( L^p([0,1]) \) for any sequence \((w_I)_{I \in \mathcal{D}}\) with \( 0 < \inf w_I \leq \sup w_I < \infty \).

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