ON INFINITE DIMENSIONAL QUADRATIC VOLterra OPERATORS

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Abstract
In this paper we study a class of quadratic operators named by Volterra operators on infinite dimensional space. We prove that such operators have infinitely many fixed points and the set of Volterra operators forms a convex compact set. In addition, it is described its extreme points. Besides, we study certain limit behaviors of such operators and give some more examples of Volterra operators for which their trajectories do not converge. Finally, we define a compatible sequence of finite dimensional Volterra operators and prove that any power of this sequence converges in weak topology.

Mathematics Subject Classification: 15A51, 47H60, 46T05, 92B99.
Key words: Volterra operator, infinite dimensional space, quadratic stochastic operator, weak compact, compatibility.

1 Introduction
It is known that the theory of Markov processes is a well-developed field of mathematics which has various applications in physics, biology and so on. But there are some physical models which cannot be described by such processes. One of such models is a model related to population genetics. Namely, consider a biological population, that is, a community of organisms closed with respect to reproduction [13]. Assume that every individual in this population belongs to precisely one of the species 1, 2, · · · , n. The scale of species is such that the species of parents i and j unambiguously determine the probability of every species k for the first generation of direct descendants. We denote this probability (the heredity coefficient) by \( p_{ij,k} \). It is obvious that \( p_{ij,k} \geq 0 \) and \( \sum_{k=1}^{n} p_{ij,k} = 1 \) for all \( i, j \). Assume that the population is so large that frequency fluctuations can be neglected. Then the state of the population can be described by the tuple \( x = (x_1, x_2, \cdots, x_n) \) of species probabilities, that is, \( x_i \) is the fraction of the species \( i \) in the population. In this case of panmictia (random interbreeding), the parent pairs \( i \) and \( j \) arise for a fixed state \( x = (x_1, x_2, \cdots, x_n) \) with probability

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is the total probability of the species \( k \) in the first generation of direct descendants. The set \( S^{n-1} = \{ x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \} \) is an \((n-1)\)-dimensional simplex. Since, \( x'_k \geq 0 \) and \( \sum_{i=1}^{n} x'_i = 1 \), the \textit{quadratic stochastic operator} defined by formula (1) maps \( S^{n-1} \) into itself. In this setting an evolution of the system is described by this operator acting on the simplex. Note that the notion of quadratic operator firstly introduced by Bernstein in [B]. To investigations of such kind operators devoted a lot papers (see [L2] for review). One of the central problem in this theory is to study limit behavior of quadratic operators (see [U]).

In [K], [L1], [SG], [V] the authors investigated limit behavior and ergodic properties of trajectories of the quadratic stochastic operators. But these operators do not occupy quantum systems, so it is natural to investigate quantum quadratic operators. In [GM1], [GM2] a notion of quantum quadratic stochastic operators defined on von Neumann algebra has been introduced. It includes as a particular case of quadratic stochastic operators. In [GM2], [M2] some ergodic and stability properties of such operators were studied. But it would be more interesting to investigate one of the simplest case in which that operators act on infinite dimensional algebras.

In this paper we are going to consider quadratic operators on infinite dimensional commutative algebra. In this setting an infinite dimensional simplex is not weak compact, therefore in general we cannot state that every quadratic operator has at least one fixed point. This is an infinite dimensionality phenomenon. We will study a class of quadratic operators named by Volterra operators. The paper is organized as follows. In section 2 we give some preliminary on quadratic operators defined on von Neumann algebra and describe a form of such operators defined on \( \ell^\infty \). Besides, we demonstrate an example of quadratic operator which has no fixed points. In section 3 we define quadratic Volterra operators and study its certain properties. In particular, we show that such operators have infinitely many fixed points. In section 4 we prove that the set of Volterra operators forms a convex compact set and describe its extreme points. In the next section 5 we study certain limit behaviors of such operators and give some more examples of Volterra operators for which their trajectories do not converge. Finally, in the last section 6 we define a compatible sequence of finite dimensional Volterra operators and prove that any power of this sequence converges in weak topology. It should be noted that finite dimensional Volterra operators were studied in [G].

Note that a part of the results have been announced in [A3].

2 Preliminary and quadratic operators

Let us recall some definitions. Let \( B(H) \) be the algebra of linear bounded operators on a separable Hilbert space \( H \). Let \( M \subset B(H) \) be a von Neumann algebra with unit \( 1 \). By \( M_+ \) we denote the set of all positive elements of \( M \). Weak (operator) closure of algebraic tensor product \( M \odot M \) in \( B(H \otimes H) \) is denoted by \( M \otimes M \), and it is called tensor product of \( M \) into itself. For detail we refer a reader to [BR].
By \( S(M) \) and \( S(M \otimes M) \) it is denoted the set of all normal states on \( M \) and \( M \otimes M \) respectively. Let \( U : M \otimes M \to M \otimes M \) be a linear operator such that \( U(xy) = yx \) for all \( x, y \in M \).

**Definition 2.1.** A linear operator \( P : M \to M \otimes M \) is said to be quantum quadratic stochastic operator (q.q.s.o.) if it is normal and satisfies the following conditions:

(i) \( P \mathbb{1}_M = \mathbb{1}_{M \otimes M} \), where \( \mathbb{1}_M \) and \( \mathbb{1}_{M \otimes M} \) are units of algebras \( M \) and \( M \otimes M \) respectively;

(ii) \( P(M_+) \subset (M \otimes M)_+ \);

(iii) \( UPx = Px \) for every \( x \in M \).

Define an operator \( \tilde{V} : S(M \otimes M) \to S(M) \) as follows

\[
\tilde{V}(\tilde{\varphi})(x) = \tilde{\varphi}(Px), \quad \tilde{\varphi} \in S(M \otimes M), \quad x \in M.
\]

(2)

The operator \( \tilde{V} \) is called conjugate quadratic operator (c.q.o.). Further for the shortness instead of \( \tilde{V}(\varphi \otimes \psi) \) we will write \( \tilde{V}(\varphi, \psi) \), where \( \varphi, \psi \in S(M) \).

Note that the relation (iii) implies that

\[
\tilde{V}(\varphi, \psi) = \tilde{V}(\psi, \varphi).
\]

(3)

In [M2] we have proved that every c.q.o. uniquely defines q.q.s.o. Therefore it is enough to consider c.q.o.

By means of \( \tilde{V} \) one can define an operator \( V : S(M) \to S(M) \) by

\[
V(\varphi) = \tilde{V}(\varphi, \varphi), \quad \varphi \in S(M),
\]

(4)

which is called quadratic operator (q.o.).

**Observation 2.1.** Here we give how linear operator and q.q.s.o. related with each other. Let \( T : M \to M \) be a linear positive normal operator (i.e. \( Tx \geq 0 \) whenever \( x \geq 0 \)) such that \( T\mathbb{1} = \mathbb{1} \). Define a linear operator \( P : M \to M \otimes M \) as follows

\[
P_x = \frac{Tx\mathbb{1}_M + \mathbb{1}_M Tx}{2}, \quad x \in M.
\]

(5)

It is clear that \( P \) is q.q.s.o. Then associated c.q.o. and q.o. have the following form respectively:

\[
\tilde{V}(\varphi, \psi)(x) = \frac{1}{2}(\varphi + \psi)(Tx),
\]

\[
V(\varphi)(x) = \varphi(Tx), \quad x \in M,
\]

(6)

for every \( \varphi, \psi \in S(M) \). Thus linear operator can be viewed as a particular case of q.q.s.o. If \( T \) is the identity operator, then from (6) we can find that the associated q.o. also would be the identity operator of \( S \). The set of all q.q.s.o. associated with linear operators we denote by \( \mathcal{QL}(M) \).

In the paper we are going to consider a case when the von Neumann algebra \( M \) is a infinite-dimensional commutative discrete algebra, i.e.

\[
M = \ell^\infty = \{ x = (x_n) : x_n \in \mathbb{R}, \|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| \},
\]
then the set of all normal functionals defined on $\ell^\infty$ coincides with
\[
\ell^1 = \{x = \{x_n\} : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty\}
\]
(i.e. $\ell^1$ is a pre-dual space to $\ell^\infty$, namely $(\ell^1)^* = \ell^\infty$) and $S(\ell^\infty)$ with
\[
S = \{x = (x_n) \in \ell^1 : x_i \geq 0, \sum_{n=1}^{\infty} x_n = 1\}.
\]
It is known that $S = \text{convh}(\text{Extr}S)$, where $\text{Extr}(S)$ is the extremal points of $S$ and $\text{convh}(A)$ is the convex hall of a set $A$.

Any extremal point $\varphi$ of $S$ has the following form
\[
\varphi = (0, 0, ..., 1, 0, ...),
\]
for some $n \in \mathbb{N}$. Such elements will be denoted as $e^{(n)}$.

The following Theorem describes c.q.o. when $M = \ell^\infty$.

**Theorem 2.1.** Every c.q.o. $\tilde{V}$ defines an infinite dimensional matrix $(p_{ij,k})_{i,j,k \in \mathbb{N}}$ such that
\[
p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \sum_{k=1}^{\infty} p_{ij,k} = 1, \quad i,j \in \mathbb{N}. \tag{7}
\]
Conversely, every such matrix defines c.q.o. $\tilde{V}$ as follows:
\[
(\tilde{V}(x,y))_k = \sum_{i,j=1}^{\infty} p_{ij,k} x_i y_j, \quad k \in \mathbb{N}, \quad x = (x_i), y = (y_i) \in S. \tag{8}
\]

**Proof.** Let $\tilde{V}$ be a c.q.o. For every $e^{(n)}, e^{(m)} \in \text{Extr}(S)$ put
\[
p_{mn,k} = (\tilde{V}(e^{(m)}, e^{(n)}))_k, \quad m,n,k \in \mathbb{N}.
\]
According to positivity of $e^{(n)}, n \in \mathbb{N}$ and (ii) (see def.1.1) we get $p_{mn,k} \geq 0$. It follows from that $\tilde{V}(e^{(m)}, e^{(n)}) = \tilde{V}(e^{(n)}, e^{(m)})$, which implies that $p_{mn,k} = p_{nm,k}$. Since $\tilde{V}(e^{(m)}, e^{(n)}) \in S$ we find $\sum_{k=1}^{\infty} p_{mn,k} = 1$. Note that we have
\[
(\tilde{V}(x,y))_k = \sum_{i,j=1}^{\infty} p_{ij,k} x_i y_j, \quad k \in \mathbb{N}.
\]
for every $x = (x_i), y = (y_i) \in S$.

Conversely, let $(p_{ij,k})$ be a matrix satisfying (7). Define $P : \ell^\infty \to \ell^\infty \otimes \ell^\infty$ as follows
\[
(Pf)_{ij} = \sum_{k=1}^{\infty} p_{ij,k} f_k, \quad i,j \in \mathbb{N},
\]
for every $f = (f_k) \in \ell^\infty$. The condition (7) implies that $P$ is a q.q.s.o. In particular, we have
\[
P e^{(k)} = \sum_{i,j \in \mathbb{N}} p_{ij,k} e^{(i)} \otimes e^{(j)}. \tag{9}
\]
Let \( \tilde{V} \) be the c.q.o. associated with \( P \). Take arbitrary \( x, y \in S \). Then using (9) we find

\[
(\tilde{V}(x, y))_k = x \diamond y(P_{e(k)}) = \sum_{k=1}^{\infty} p_{ij,k} x_i y_j,
\]

here \( x \diamond y = (x_i y_j) \in S(\ell^\infty \otimes \ell^\infty) \).

Thus the theorem is proved.

We note that in this case q.o. \( V \) defined by (4) has the following form:

\[
(V(x))_k = \sum_{i,j=1}^{\infty} p_{ij,k} x_i x_j \quad k \in \mathbb{N}, \quad x = (x_i) \in S.
\]

Observation 2.2. Let \( T : \ell^\infty \to \ell^\infty \) be a positive identity preserving operator. Then it is easy to see that this operator can be represented as infinite dimensional stochastic matrix \( (p_{ij})_{i,j} \), i.e. \( p_{ij} \geq 0, \sum_{j=1}^{\infty} p_{ij} = 1 \) for every \( i, j \in \mathbb{N} \).

Then the determining matrix \( (p_{ij,k})_{i,j,k} \) corresponding to q.o. given by (10) is defined as

\[
p_{ij,k} = \frac{p_{ik} + p_{jk}}{2}, \quad i,j,k \in \mathbb{N}.
\]

Observation 2.3. It is known that the set \( S \) is not compact in norm topology of \( \ell^1 \), even in \( \sigma(\ell^1, \ell^\infty) \)-topology. This is the difference between finite and infinite dimensional cases. In finite dimensional case every q.o. \( V : S^{n-1} \to S^{n-1} \) has at least one fixed point (i.e. \( V(x) = x, x \in S^{n-1} \)). In the infinite dimensional setting, not every q.o. has fixed points. Indeed, define a linear operator \( T : \ell^\infty \to \ell^\infty \) as follows

\[
T(x_1, x_2, \ldots, x_n, \ldots) = (x_2, \ldots, x_{n+1}, \ldots),
\]

\( (x_n) \in \ell^\infty \). It is clear that \( T \) is positive and \( T_1 = 1 \). Now consider q.q.s.o. defined by (9). Then by Observation 2.1 q.o. \( V \) acts as follows

\[
V(\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots) = (0, \varphi_1, \varphi_2, \ldots, \varphi_n, \ldots)
\]

where \( (\varphi_n) \in S \). It is easy to see that this operator has no fixed points belonging to \( S \).

3 Volterra operators

In this section we define Volterra operators and give some their properties.

Recall that a convex set \( C \subseteq S \) is called face, if \( lx + (1-l)y \in C \), where \( x, y \in S \quad l \in (0,1) \), implies that \( x, y \in C \). For \( \varphi, \psi \in S \) denote \( \Gamma(\varphi, \psi) = \{l\varphi + (1-l)\psi : l \in [0,1]\} \).

Definition 3.1. An operator \( V \) defined by (4) is called Volterra operator if \( \tilde{V}(\varphi, \psi) \in \Gamma(\varphi, \psi) \) is valid for every \( \varphi, \psi \in \text{Extr}(S) \).

By \( \mathcal{QV} \) we denote the set of all quadratic operators defined on \( S \), and the set of all Volterra operators is denoted by \( \mathcal{V} \).

Proposition 3.1. Let \( V \in \mathcal{QV} \) be a q.o. Then \( V \) is Volterra if and only if the determining matrix \( (p_{ij,k}) \) of this operator satisfies the following property:

\[
p_{ij,k} = 0, \quad \text{if} \quad k \notin \{i,j\}.
\]
Proof. Let \( V \) be a Volterra operator. Then from definition 3.1 we infer that
\[
\tilde{V}(e^{(i)}, e^{(j)}) = p_{ij,i}e^{(i)} + p_{ij,j}e^{(j)}.
\]
This yields that \( p_{ij,i} + p_{ij,j} = 1 \), so (11) is valid. The converse implication easily follows from Theorem 2.1. The proposition is proved.

Note that the condition (11) biologically means that each individual can inherit only the species of the parents.

From Theorem 2.1 and Proposition 3.1 we immediately get the following

**Proposition 3.2.** Let \( V_1, V_2 \in \mathcal{V} \) be two Volterra operators such that for every \( e^{(i)}, e^{(j)}, i, j \in \mathbb{N} \) the equality holds \( \tilde{V}_1(e^{(i)}, e^{(j)}) = \tilde{V}_2(e^{(i)}, e^{(j)}) \), then \( V_1 = V_2 \).

**Theorem 3.3.** Let \( V \in \mathcal{QV} \) be a q.o. Then \( V \) is Volterra operator if and only if it can be represented as follows:
\[
(V(x))_k = x_k(1 + \sum_{i=1}^{\infty} a_{ki}x_i), \quad k \in \mathbb{N},
\]
where
\[
a_{ki} = -a_{ik}, \quad |a_{ki}| \leq 1 \quad \text{for every} \quad k, i \in \mathbb{N}
\]
(13).

Proof. From Definition 3.1 and Proposition 3.1, one gets \( p_{kk,k} = 1 \), \( k \in \mathbb{N} \). Then from (10) we obtain
\[
(V(x))_k = \sum_{i,j=1}^{\infty} p_{ij,k}x_i x_j = p_{kk,k}x_k^2 + \sum_{i=1,i\neq k}^{\infty} p_{ik,k}x_i x_k + \sum_{j=1,j\neq k}^{\infty} p_{kj,k}x_k x_j, \quad k \in \mathbb{N},
\]
whence keeping in mind \( p_{ij,k} = p_{ji,k} \) we infer that
\[
(V(x))_k = x_k(1 + 2 \sum_{i=1,i\neq k}^{\infty} p_{ik,k}x_i), \quad k \in \mathbb{N}.
\]
Using \( \sum_{i=1}^{\infty} x_i = 1 \) we have
\[
(V(x))_k = x_k(1 + \sum_{i=1,i\neq k}^{\infty} (2p_{ik,k} - 1)x_i), \quad k \in \mathbb{N}.
\]
Setting \( a_{ki} = 2p_{ik,k} - 1 \) while \( i \neq k \), and \( a_{kk} = 0 \), it yields (12). The inequality \( 0 \leq p_{ik,k} \leq 1 \) implies that \( |a_{ki}| \leq 1 \). Taking into account \( p_{ik,k} + p_{ik,i} = 1 \), we have
\[
a_{ki} + a_{ik} = 2p_{ik,k} - 1 + 2p_{ki,i} - 1 = 2(p_{ik,k} + p_{ik,i} - 1) = 0.
\]
Therefore \( a_{ki} = -a_{ik} \).

The converse implication is obvious. This completes the proof.

**Corollary 3.4.** Let \( V \in \mathcal{QV} \) be a q.o. Then \( V \) is Volterra operator if and only if \( \tilde{V} \) can be represented as follows:
\[
(\tilde{V}(x,y))_k = \frac{1}{2} \left( x_k(1 + \sum_{i=1}^{\infty} a_{ki}y_i) + y_k(1 + \sum_{i=1}^{\infty} a_{ki}x_i) \right), \quad k \in \mathbb{N}.
\]
(14)
Recall that an element $x \in S$ is called fixed point of $V$ if $V(x) = x$. The set of all fixed points of $V$ is denoted by $\text{Fix}(V)$. For given subset $K$ of $\mathbb{N}$ set

$$S^K = \{ x \in S : x_i = 0, \forall i \in \mathbb{N} \setminus K \}.$$

**Corollary 3.5.** For every Volterra operator $V$ the following assertions hold:

(i) every face of $S$ invariant with respect to $V$;

(ii) $\text{Extr}(S) \subset \text{Fix}(V)$.

The proof immediately follows from Theorem 3.3 since every face of $S$ is $S^K$ for some $K \subset \mathbb{N}$ and $\{ e^{(i)} \} = S^{(i)}$ for every $e^{(i)} \in \text{Extr}(S)$.

Put

$$riS^K = \{ x \in S^K | x_i > 0, \forall i \in K \}.$$

**Corollary 3.6.** Let $V$ be a Volterra operator, then the relation holds $V(riS^K) \subset riS^K$ for every $K \subset \mathbb{N}$.

**Proof.** Let $x_k > 0, k \in K$, then according to the equality $a_{kk} = 0$ and (12) we have

$$V(x) = x_k(1 + \sum_{i=1}^{\infty} a_{ki} x_i + ... + a_{k,k-1} x_{k-1} + a_{k,k+1} x_{k+1} + ...)
\geq x_k(1 - x_1 - ... - x_{k-1} - x_{k+1} - ...) = x_k^2 > 0.$$

The corollary is proved.

**Remark 3.1.** From Theorem 3.3 we see that the identity operator $\text{Id} : S \rightarrow S$, i.e.

$$(\text{Id}(x))_k = x_k, \text{ } k \in \mathbb{N}$$

is Volterra operator. From Proposition 3.2, Observations 2.1 and 2.2 we infer that $\mathcal{Q}(\mathcal{L}(\mathbb{R})) \cap V = Id$.

**Theorem 3.7.** Let $V \in \mathcal{V}$ be a Volterra operator, then it is a bijection of $S$.

**Proof.** Let us first show that $V$ is injective. Assume that there are two elements $x, y \in S(x \neq y)$ such that

$$V(x) = V(y)$$

(15)

Without loss of generality we may assume that $x_i > 0, y_i > 0, \forall i \in \mathbb{N}$. If it is not true, then there is a face $S^K$, for some subset $K \subset \mathbb{N}$, of $S$ such that $x, y \in riS^K$, i.e. $x_i > 0, y_i > 0, \forall i \in K$. According to Corollaries 3.5 and 3.6 we have $V(S^K) \subset S^K$, therefore we may restrict $V$ to $S^K$. From (15) one gets that

$$x_k(1 + \sum_{i=1}^{\infty} a_{ki} x_i) = y_k(1 + \sum_{i=1}^{\infty} a_{ki} y_i),$$

or

$$(x_k - y_k)(1 + \sum_{i=1}^{\infty} a_{ki} y_i) = -x_k \sum_{i=1}^{\infty} a_{ki}(x_i - y_i).$$

(16)

We have

$$1 + \sum_{i=1}^{\infty} a_{ki} y_i \geq 1 - y_1 - y_2 - ... - y_{k-1} - y_{k+1} - ... = y_k > 0,$$
whence \( x_k > 0 \) with (16) implies that
\[
\text{sgn}(x_k - y_k) = -\text{sgn} \sum_{i=1}^{\infty} a_{ki}(x_i - y_i).
\]  
(17)

Hence
\[
(x_k - y_k) \sum_{i=1}^{\infty} a_{ki}(x_i - y_i) \leq 0, \ k \in \mathbb{N},
\]
whence
\[
\sum_{k=1}^{\infty} (x_k - y_k) \sum_{i=1}^{\infty} a_{ki}(x_i - y_i) \leq 0.
\]

Note that the last series absolutely converges, since
\[
\left| \sum_{k=1}^{\infty} (x_k - y_k) \sum_{i=1}^{\infty} a_{ki}(x_i - y_i) \right| \leq \sum_{k=1}^{\infty} |x_k - y_k| \sum_{i=1}^{\infty} |a_{ki}| |x_i - y_i| \leq \sum_{k=1}^{\infty} (x_k + y_k) \sum_{i=1}^{\infty} (x_i + y_i) = 4 < \infty.
\]

According to \( a_{ki} = -a_{ik} \) we find
\[
\sum_{k=1}^{\infty} (x_k - y_k) \sum_{i=1}^{\infty} a_{ki}(x_i - y_i) = 0.
\]

Consequently,
\[
(x_k - y_k) \sum_{i=1}^{\infty} a_{ki}(x_i - y_i) = 0, \ k \in \mathbb{N}.
\]

The equality (17) with the last equality imply that \( x = y \). Thus, \( V : S \to S \) is injective.

Now let us show that \( V \) is onto. Denote
\[
A_1 = \{[1, n] \subset \mathbb{N} : n \in \mathbb{N}\}, \ A_2 = \{a \subset [1, n] : |[1, n] \setminus a| \geq 2, n \in \mathbb{N}\},
\]
\[
A_3 = \{b \subset \mathbb{N} : a \subset b, \ a \in A_1 \cup A_2, \ |\mathbb{N} \setminus b| < \infty\},
\]
\[
A = A_1 \cup A_2 \cup A_3.
\]

Order \( A \) by inclusion, i.e. \( a \leq b \) means that \( a \subset b \) for \( a, b \in A \). It is clear that \( A \) is a completely ordering set. To prove that \( V \) is surjective we will use transfer induction method with respect to the set \( A \). Obviously, that for the first element \( \{1\} \) of the set \( A \), the operator \( V \) on \( S^{\{1\}} \) is surjective (see Corollary 3.5 and [G]). Assume that for an element \( a \in A \) the operator \( V \) is surjective on \( S^b \) for every \( b < a \). Let us show that it is surjective on \( S^a \). Suppose that \( V(S^a) \neq S^a \). For the boundary \( \partial S^a \) of \( S^a \) we have \( \partial S^a = \bigcup_{c \in A : c < a} S^c \). According to the assumption of the induction one gets
\[
V(\partial S^a) = \partial S^a.
\]  
(18)

On the other hand, there exist \( x, y \in riS^a \) such that \( x \in V(S^a), \ y \notin V(S^a) \). The segment \([x, y]\) contains at least one boundary point \( z \) of the set \( V(S^a) \). Since \( V : S^a \to V(S^a) \) is continuous and bijection, then the boundary point goes to boundary one. Therefore for \( z \in riS^a \), \( V^{-1}(z) \in \partial S^a \), which contradicts to (18). Thus the theorem is proved.
4 The set of Volterra operators

In this section we will prove that the set $V$ is compact.

Now endow $QV$ with a topology which is defined by the following system of semi-norms:

$$p_{\varphi,\psi,k}(V) = |(V(\varphi,\psi))_k|, \quad V \in QV,$$

where $\varphi,\psi \in S$ and $k \in \mathbb{N}$. This topology is called \textit{weak topology} and is denoted by $\tau_w$.

A net $\{V_\nu\}$ of quadratic operators converges to $V$ with respect to the defined topology if for every $\varphi,\psi \in S$ and $k \in \mathbb{N}$

$$(V_\nu(\varphi,\psi))_k \rightarrow (V(\varphi,\psi))_k$$

is valid.

Since $V \subset QV$, therefore on $V$ we consider the induced topology by $QV$.

We note that in [M1] we have proved that the set of all quantum quadratic stochastic operators defined on semi-finite von Neumann algebra, without normality condition, forms is a weak compact convex set. In the present situation we cannot apply the mentioned result since our q.q.s.o. are normal. In general, the set of all normal q.q.s.o. is not weak compact\textsuperscript{5}. Therefore, here we use another method to prove that $V$ is weak compact.

Denote the set of all matrices $(a_{ki})$ satisfying (13) by $A$. It is clear that $A$ is convex. The set $A$ can be considered as a subset of the space

$$\ell^\infty(N \times N) = \{x = (x_{n,m}) : n,m \in \mathbb{N}, \|x\|_{\infty} = \sup_{n,m \in \mathbb{N}} |x_{n,m}| \}.$$

It is well-known [BR] that the space

$$\ell^1(N \times N) = \{x = (x_{n,m}) : n,m \in \mathbb{N}, \|x\|_{1} = \sum_{n,m \in \mathbb{N}} |x_{n,m}| < \infty \}$$

is pre-dual to $\ell^\infty(N \times N)$, i.e. $\ell^1(N \times N)^* = \ell^\infty(N \times N)$. Therefore on $\ell^\infty(N \times N)$ we can consider $\sigma(\ell^\infty(N \times N),\ell^1(N \times N))$-topology. In the sequel we will denote it as $\tau$. According to Alaoglu-Banach theorem the set $A$ is $\sigma(\ell^\infty(N \times N),\ell^1(N \times N))$-weak compact in $\ell^\infty(N \times N)$. From Theorem 3.3 we conclude that every $(a_{ki})$ matrix with the property (13) defines a Volterra operator $V$ of the form (12) (see also [L3]). So, it is defined a map $T : A \rightarrow V$. It is clear that Theorem 3.3 and Proposition 3.2 imply that this map is bijection and convex.

\textbf{Theorem 4.1.} The map $T : (A,\tau) \rightarrow (V,\tau_w)$ is continuous.

\textbf{Proof.} Let a net $(a_{ki}^{(\nu)}) \subset A$ converge to $(a_{ki})$ in the weak topology. This means that for an arbitrary $\varepsilon > 0$ and every $k,i \in \mathbb{N}$ there is $\nu_0(k,i)$ such that $|a_{ki}^{(\nu)} - a_{ki}| < \varepsilon$ for every $n \geq \nu_0(k,i)$. Denote $V^{(\nu)} = T((a_{ki}^{(\nu)}))$ and $V = T((a_{ki}))$.

Take any $x,y \in S$. Then there is a number $N_0 \in \mathbb{N}$ such that

$$\sum_{i=N_0+1}^{\infty} x_i < \varepsilon, \quad \sum_{i=N_0+1}^{\infty} y_i < \varepsilon. \quad (19)$$

Now consider two separate cases.

\textsuperscript{5}Each state $\omega \in S(M)$ defines a linear positive operator as $T(x) = \omega(x)$. So according to Observation 2.1 the set of all normal states can be included to $QV$. Therefore, we can consider the induced weak topology (defined as above) on $S(M)$. It is clear that this topology coincides with $*$-topology on $S(M)$, but in this topology $S(M)$ is not compact. Hence, $QV$ is not weak compact.
Case (i). In this case we assume that $1 \leq k \leq N_0$. Then according to Corollary 3.4 and using (13), (19) we infer that

$$\left| (\tilde{V}^{(\nu)}(x, y))_k - (\tilde{V}(x, y))_k \right| \leq \frac{1}{2} \left( \sum_{i \in \mathbb{N}} (y_k x_i + x_k y_i) | a^{(\nu)}_{ki} - a_{ki} | \right)$$

$$\leq \frac{1}{2} \left( \sum_{i=1}^{N_0} (x_i + y_i) | a^{(\nu)}_{ki} - a_{ki} | \right) + \sum_{i=N_0+1}^\infty (x_i + y_i) < 3\varepsilon$$

for every $\nu \geq \max\{\nu_0(ki) : k, i \leq N_0\}$. Here we have used that $\sum_{i=1}^{N_0} (x_i + y_i) \leq \sum_{i=1}^\infty (x_i + y_i) = 2$.

Case (ii). Now assume that $k \geq N_0 + 1$. Using the above argument we have

$$\left| (\tilde{V}^{(\nu)}(x, y))_k - (\tilde{V}(x, y))_k \right| \leq \frac{1}{2} \left( \sum_{i \in \mathbb{N}} (y_k x_i + x_k y_i) | a^{(\nu)}_{ki} - a_{ki} | \right)$$

$$\leq y_k + x_k \leq \sum_{i=N_0+1}^\infty (x_i + y_i) < 2\varepsilon$$

for every $\nu \geq \max\{\nu_0(ki) : k, i \leq N_0\}$. Thus the map $T$ is continuous. The theorem is proved.

**Corollary 4.2.** The set $V$ is weak convex compact.

The proof immediately comes from that $A$ is compact and $T$ is continuous.

We say that q.o. $V \in QV$ is pure if for every $\varphi, \psi \in S$ the relation holds

$$\tilde{V}(\varphi, \psi) \in Extr\Gamma(\varphi, \psi) = \{\varphi, \psi\}.$$  

It is clear that pure q.o. are Volterra.

**Proposition 4.3.** The set $V$ is convex. Moreover, $V$ is extreme point of $V$ if and only if it is pure.

**Proof.** Convexity of $V$ is obvious. Let $V$ be a pure q.o. Let us assume that there exits $i \in (0, 1)$ and operators $V_1, V_2 \in V$ such that $V = lV_1 + (1 - l)V_2$.

Let $\varphi, \psi \in Extr(S)$, then we have

$$\tilde{V}(\varphi, \psi) = l\tilde{V}_1(\varphi, \psi) + (1 - l)\tilde{V}_2(\varphi, \psi).$$

Without loss of generality we may suppose that $\tilde{V}(\varphi, \psi) = \varphi$, since $V$ is pure. Therefore, the extremity of $\varphi$ with (20) implies that $\tilde{V}_i(\varphi, \psi) = \varphi, \ i = 1, 2$. Hence, $V_1(\varphi, \psi) = V_2(\varphi, \psi)$ for every $\varphi, \psi \in Extr(S)$. According to Proposition 3.2 one gets $V = V_1 = V_2$. Thus $V \in Extr(V)$.

Now let $V \in Extr(V)$. Show that $V$ is pure. Assume that $V$ is not pure, i.e. there is $\varphi_0, \psi_0 \in Extr(S)$ and a number $l \in (0, 1)$ such that $\tilde{V}(\varphi_0, \psi_0) = l\varphi_0 + (1 - l)\psi_0$. Define q.o. $V_1$ and $V_2$ as follows:

$$\tilde{V}_1(\varphi_0, \psi_0) = \varphi_0, \ \tilde{V}_2(\varphi_0, \psi_0) = \psi_0$$

$$V_i(\varphi, \psi) = V(\varphi, \psi) \ \forall \varphi, \psi \in Extr(S), \varphi, \psi \notin \{\varphi_0, \psi_0\}.$$  

Then again using Proposition 3.2 we get $V = lV_1 + (1 - l)V_2$, which contradicts to the extremity of $V$. This completes the proof.
We note that Proposition 4.3 can be also proved by means of Theorem 3.3 and Corollary 3.4.

From Corollary 4.2 and Proposition 4.3 we have the following

**Corollary 4.4.** A Volterra operator \( V \in V \) is extremal if and only if for the associated skew-symmetric matrix \( (a_{ki}) \) the equality holds \( |a_{ki}| = 1 \), for every \( k, i \in \mathbb{N} \).

The proof comes from that the extremal points of \( A \) satisfy the last condition and the map \( T \) is convex and bijection.

## 5. A limit behavior of Volterra operators

In this section we give some limit theorems concerning trajectories of Volterra operators.

Let \( V : S \to S \) be a Volterra operator. Then according to Theorem 3.3 it has the form (12).

Denote
\[
Q = \{ y \in S : \sum_{i=1}^{\infty} a_{ki}y_i \leq 0, \; k \in \mathbb{N} \}. \tag{21}
\]

It is clear that \( Q \) is convex subset of \( S \).

**Proposition 5.1.** For every Volterra operator \( V \) the relation holds \( Q \subset \text{Fix}(V) \).

**Proof.** Let \( y \in Q \) then
\[
(V(y))_k = y_k(1 + \sum_{i=1}^{\infty} a_{ki}y_i) \leq y_k, \; k \in \mathbb{N}. \tag{22}
\]

According to the equality \( \sum_{i=1}^{\infty} y_i = \sum_{i=1}^{\infty} (V(y))_i = 1 \), from (22) we find \( (V(y))_k = y_k \) for every \( k \in \mathbb{N} \), i.e. \( Vy = y \).

**Theorem 5.2.** Let \( V \) be a Volterra operator such that \( Q \neq \emptyset \). Suppose \( x^0 \in riS \) (i.e. \( x^0_i > 0, \forall i \in \mathbb{N} \)) such that \( Vx^0 \neq x^0 \) and the limit \( \lim_{n \to \infty} V^n x^0 \) exits. Then \( \lim_{n \to \infty} V^n x^0 \in Q \).

**Proof.** Let \( x^0 \in riS \) and \( \lim_{n \to \infty} x^{(n)} = \hat{x} \), where \( x^{(n)} = V^n x^0, n \in \mathbb{N} \). Denote \( \hat{x} = (q_1, q_2, ..., q_n, ...) \). It is clear that \( V\hat{x} = \hat{x} \). Hence
\[
q_k = g_k(1 + \sum_{i=1}^{\infty} a_{ki}q_i), \; k \in \mathbb{N}. \tag{23}
\]

Set \( I_+ = \{ i \in \mathbb{N} | q_i > 0 \} \), \( I_0 = \{ i \in \mathbb{N} | q_i = 0 \} \). If \( k \in I_+ \), then from (23) we get
\[
\sum_{i=1}^{\infty} a_{ki}q_i = 0, \; k \in I_+.
\]

Assume that there is \( k_0 \in I_0 \) such that
\[
\sum_{i=1}^{\infty} a_{k_0 i}q_i > 0.
\]
Since \( x^{(m)}_k \rightarrow q_k \), then there is \( m_0 \in \mathbb{N} \) such that
\[
\sum_{i=1}^{\infty} a_{kij} x^{(m)}_i > 0, \quad \text{for every } m \geq m_0.
\] (24)

According to Corollary 3.6 we have \( x^{(m)} \in riS, \forall m \in \mathbb{N} \), i.e. \( x^{(m)}_k > 0, \forall m, k \in \mathbb{N} \). The inequality (24) with one
\[
x^{(m+1)} = x^{(m)}_k (1 + \sum_{i=1}^{\infty} a_{kij} x^{(m)}_i) > x^{(m)}_k, \quad \forall m \geq m_0
\]
implies that \( x^{(m+1)}_k > x^{(m)}_k \), which contradicts to \( x^{(m)}_k \rightarrow q_k = 0 \). Therefore if
\( k \in \mathcal{I}_0 \), then \( \sum_{i=1}^{\infty} a_{kij} q_i \leq 0 \). Thus \( \bar{x} \in \mathcal{Q} \). The theorem is proved.

Given \( V \) Volterra operator and \( K \subset \mathbb{N} \). Set \( V_K = V|_{S^K} \). Let \( \mathcal{Q}_K \) be the set \( \mathcal{Q} \) corresponding to \( V_K \). Then from Theorem 5.2 and Corollary 3.6 we immediately get

**Corollary 5.3.** Let \( \mathcal{Q}_K \neq \emptyset \) and \( x^0 \in riS^K \) (i.e. \( x^0 > 0, \forall i \in K \)) such that \( V x^0 \neq x^0 \) and the limit \( \lim_{n \rightarrow \infty} V^n x^0 \) exists. Then \( \lim_{n \rightarrow \infty} V^n x^0 \in \mathcal{Q}_K \).

**Corollary 5.4.** If a Volterra operator \( V \) has an isolated fixed point \( x^0 \in Fix(V) \) (i.e. there is a weak neighbor \( U(x^0) \subset S \) of \( x^0 \) such that \( U(x^0) \cap Fix(V) = \{x^0\} \)) such that \( x^0 \in riS \). Then for any \( x \in riS, x \notin Fix(V) \) the limit \( \lim_{n \rightarrow \infty} V^n x \) does not exists.

**Proof.** Assume that \( \lim_{n \rightarrow \infty} V^n x = \bar{x} \) exists. Then according to Theorem 5.2 we have \( \bar{x} \in \mathcal{Q} \). Since \( x^0 \in Fix(V) \), \( x^0 \in riS \) imply that \( x^0 \in \mathcal{Q} \). Convexity of \( \mathcal{Q} \) yields that \( \bar{x} (1 - l) x^0 \in \mathcal{Q} \) for every \( l \in [0, 1] \). But this contradicts the fact that \( x^0 \) is isolated. This completes the proof.

**Remark 5.1.** It is known \( \mathcal{Q} \) that the set \( \mathcal{Q} \) is not empty for any Volterra operator in finite dimensional setting. But unfortunately, in our situation \( \mathcal{Q} \) can be empty.

Let us give some more examples of q.o. for which \( \mathcal{Q} \) is empty and non empty.

**Example 5.1.** Let us consider a Volterra operator defined as follows:
\[
\begin{cases}
(Vx)_{2k-1} = x_{2k-1}(1 - a(k)x_{2k}), \\
(Vx)_{2k} = x_{2k}(1 + a(k)x_{2k-1}), \quad k \in \mathbb{N}
\end{cases}
\]
\( a^{(k)} > 0 \quad |a^{(k)}| \leq 1 \).

Let us describe \( \mathcal{Q} \) for defined \( V \). To this end we should find solutions of the system:
\[
\begin{cases}
-a^{(k)} x_{2k} \leq 0, \\
-a^{(k)} x_{2k-1} \leq 0, \quad k \in \mathbb{N}
\end{cases}
\]

One easily gets that \( \mathcal{Q} = \{ x \in S : x_{2k-1} = 0, \quad k \in \mathbb{N} \} \). So \( \mathcal{Q} \neq \emptyset \).

Let \( x \in riS \), then the trajectory of \( x \) is defined as the following recurrent relations
\[
\begin{cases}
x^{(m+1)}_{2k-1} = x^{(m)}_{2k-1}(1 - a^{(k)}x^{(m)}_{2k}), \\
x^{(m+1)}_{2k} = x^{(m)}_{2k}(1 + a^{(k)}x^{(m)}_{2k-1}),
\end{cases}
\]
\( k \in \mathbb{N}, m \in \mathbb{N} \).
According to $a^{(k)} > 0$ we find $1 + a^{(k)}x_{2k+1}^{(m)} > 0$, hence we have $x_{2k}^{(m+1)} \geq x_{2k}^{(m)}$, therefore $\{x_{2k}^{(m)}\}$ is non-decreasing sequence. From $0 \leq 1 - a^{(k)}x_{2k}^{(m)} \leq 1$ it follows that $\{x_{2k}^{(m)}\}$ is non-increasing sequence, such that $0 \leq x_{2k}^{(m)}, x_{2k-1}^{(m)} \leq 1$. So the limits
\[
\lim_{m \to \infty} x_{2k-1}^{(m)} = \alpha_{2k-1}, \quad \lim_{m \to \infty} x_{2k}^{(m)} = \beta_{2k}
\]
exists.

According to Theorem 5.2 we infer that $\alpha_{2k-1} = 0$ for every $k \in \mathbb{N}$.

Now let $x \notin rI_S$, then denote $I_x = \{k \in \mathbb{N} : x_k = 0\}$. Then using Corollary 3.5 we find $V(S^{\mathbb{N}\setminus I_x}) = S^{\mathbb{N}\setminus I_x}$. The restriction of $V$ to $S^{\mathbb{N}\setminus I_x}$ is denoted by $V_{\mathbb{N}\setminus I_x}$. From definition of $S^{\mathbb{N}\setminus I_x}$ we find that $x \in rS^{\mathbb{N}\setminus I_x}$, whence according to Corollary 5.3 we obtain
\[
\lim_{m \to \infty} x_{2k-1}^{(m)} = 0, \quad \lim_{m \to \infty} x_{2k}^{(m)} = \begin{cases} \beta_{2k}, & 2k \in \mathbb{N} \setminus I_x, \\ 0, & 2k \in I_x. \end{cases}
\]

**Example 5.2.** Let us define a Volterra operator as follows:
\[
(V(x))_k = x_k(1 + \sum_{i=1}^{\infty} a_{ki}x_i), \quad k \in \mathbb{N},
\]
where $a_{ki} = (-1)^i$, $a_{ik} = -a_{ki}$ at $i \geq k + 1$.

Then it is not hard to check that the set $Q$ consists of the solutions the following system
\[
\begin{align}
\sum_{k=2}^{\infty} (-1)^{k+1}x_k & \leq 0, \\
x_1 + \sum_{k=3}^{\infty} (-1)^kx_k & \leq 0, \\
-x_1 + x_2 + \sum_{k=4}^{\infty} (-1)^{k+1}x_k & \leq 0, \\
\cdots & \\
\sum_{k=2}^{n-1} (-1)^{n+k}x_k + \sum_{k=n+1}^{\infty} (-1)^{n+k+1}x_k & \leq 0.
\end{align}
\]
Whence one gets $x_n \leq x_{n+1}$ for every $k \in \mathbb{N}$. Since $x_1 \geq 0$ and $x_n \to 0$ at $n \to \infty$, we obtain $x_n = 0$, $\forall n \in \mathbb{N}$, which is impossible, because of $\sum_{k=1}^{\infty} x_k = 1$.

Consequently, $Q = \emptyset$.

Now let us look for the set $Fix(V)$. Let $x^0 \in rS$, i.e. $x^0_k > 0, \forall k \in \mathbb{N}$, be a fixed point of $V$. It follows from (25) that
\[
x_1^0 = x_2^0 = \ldots, x_k^0 = \ldots, \quad k \in \mathbb{N},
\]
but this equality is impossible since $x_1^0 \neq 0$ and $x_n^0 \to 0$. Hence, inner fixed points for $V$ does not exist. So there is a subset $I \subset \mathbb{N}$ such that $I = \{k \in \mathbb{N} | x_k^0 = 0\}$. The set $\mathbb{N} \setminus I$ is finite. Indeed, assume that $|\mathbb{N} \setminus I| = \infty$, then consider a face $S^{\mathbb{N}\setminus I}$. Then according to Corollary 3.5 $V_{\mathbb{N}\setminus I}$ is a Volterra operator. It is clear that a point $x^{0,\mathbb{N}\setminus I} = \{x_k^0 | k \in \mathbb{N} \setminus I\}$ is a fixed point of $V_{\mathbb{N}\setminus I}$. From (25) and using the same argument as above we find that the set $J = \{k \in \mathbb{N} | x_k^{0,\mathbb{N}\setminus I} = 0\}$ is non-empty, which contradicts to the choice of $I$. 

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Consequently, we infer that all fixed points of \( V \) lie on the faces \( S^I \) such that \( |N \setminus I| < \infty \). Thus we conclude that the set \( Q \) turns out to be empty while the set \( \text{Fix}(V) \) is not. Therefore, Theorem 5.2 implies that if \( x \in riS \) then the limit \( \lim_{n \to \infty} V^n x \) does not exist. Now let \( x \notin riS \), then for the set \( I_z \) there are two possibilities. The first case. Let \( |N \setminus I_z| = \infty \), then \( x \in riS^{N \setminus I_z} \). From condition \( |N \setminus I_z| = \infty \) analogously reasoning as above one can show that the set \( Q_I \) is empty. According to Corollary 5.3 we infer that the limit \( \lim_{n \to \infty} V^n x \) does not exist. The second case. In this setting \( |N \setminus I_z| < \infty \), then the operator \( V_I \) reduces to finite dimensional operator, therefore the set \( Q_I \) is not empty (see \([G]\)). So the limit \( \lim_{n \to \infty} V^n x \) exists since \( |a_{ik}| = 1 \) (see \([G]\)).

Now we will give a sufficient condition for \( V \) which ensures that the set \( Q \) is not empty.

Let \( V : S \to S \) be a Volterra operator which has the form \((12)\). Let \( A = (a_{ki}) \) be the corresponding skew-symmetric matrix. Further we will assume that \( A \) acts on \( \ell^1 \). A matrix \( A \) is called finite dimensional if \( A(\ell^1) \) is finite dimensional. We say that \( A \) is finitely generated if there are a sequence of finite dimensional matrices \( \{A_n\} \) such that \( \sup_n \|A_n\| < \infty \) and

\[
A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots
\]

**Proposition 5.5.** Let \( A = (a_{ki}) \) be the skew-symmetric matrix corresponding to a Volterra operator (see \([G]\)), is finitely generated. Then the system

\[
\sum_{i=1}^{n} a_{ki} y_i \geq 0, \quad k \in \mathbb{N}
\]

has at least one element belonging to \( S \).

**Proof.** First assume that \( A \) is finite-dimensional, i.e. there is \( n \in N \) such that \( A(\ell^1) = \mathbb{R}^n \). According to skew-symmetricity of \( A \) we find that \( a_{ij} = 0 \) at \( i, j \geq n + 1 \). Therefore we may assume that \( A \) acts on \( \mathbb{R}^n \). Then \((27)\) is rewritten as follows

\[
\sum_{j=1}^{n} a_{kj} y_j \geq 0, \quad k = 1, \ldots, n.
\]

According to \([G]\) this system has a solution \( y = \{y_k\}_{k=1}^{n} \in S^{n-1} \) such that \((28)\) holds. Now define an element \( \tilde{y} = \{\tilde{y}_k\}_{k=1}^{n} \in S \) as follows

\[
\tilde{y}_k = \begin{cases} 
  y_k, & \text{if } 1 \leq k \leq n \\
  0, & \text{if } k \geq n + 1
\end{cases}
\]

It is evident that \( A \tilde{y} \geq 0 \).

Now let us assume that \( A \) is finitely generated, i.e. \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots \). Since operators \( A_n \) are finite dimensional, therefore suppose that for every \( n \in N \) there is \( m_n \in N \) such that \( A_n \) acts on \( \mathbb{R}^{m_n} \), i.e. \( A_n : \mathbb{R}^{m_n} \to \mathbb{R}^{m_n} \). Consider the system

\[
A_n y^{(n)} \geq 0, \quad n \in \mathbb{N}.
\]

According to the above argument, for every \( n \in \mathbb{N} \), there is an element \( z^{(n)} \in S^{m_n-1} \) such that \( A_n z^{(n)} \geq 0 \). Define \( z = (z_k)_{k=1}^{\infty} \) by

\[
z = \frac{1}{2} z^{(1)} \oplus \frac{1}{2^2} z^{(2)} \oplus \cdots \oplus \frac{1}{2^n} z^{(n)} \oplus \cdots .
\]
From
\[
\sum_{k=1}^{\infty} z_k = \frac{1}{2} \sum_{k=1}^{m_1} z_k^{(1)} + \frac{1}{2^2} \sum_{k=1}^{m_2} z_k^{(2)} + \ldots + \frac{1}{2^n} \sum_{k=1}^{m_n} z_k^{(n)} + \ldots =
\]
\[
= \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots = 1.
\]
we see that \( z \in S \). The element \( z \) is a solution of (27). Since
\[
A z = \frac{1}{2} A_1 z^{(1)} + \ldots + \frac{1}{2^n} A_n z^{(n)} + \ldots \geq 0.
\]
The proposition is proved.

**Corollary 5.6.** Let the condition of the previous proposition is valid. Then
the set \( Q \) is not empty.

The proof immediately comes from Proposition 5.5 by changing the matrix \( A \) to \(-A\), since \(-A\) is also skew-symmetric.

### 6 Extension of finite dimensional Volterra operators

In this section we are going to construct infinite dimensional Volterra operators
by means of finite dimensional ones.

Let \( K_n = [1, n] \cap \mathbb{N} \) for every \( n \in \mathbb{N} \). Consider a sequence \( V_n : S^{K_n} \to S^{K_n} \)
of finite dimensional Volterra operators, i.e.
\[
(V_n)(x)_k = x_k (1 + \sum_{i=1}^{n} a_{ki}^{[n]} x_i) \quad k = 1, \ldots, n, \quad n \in \mathbb{N},
\]  
(29)

here \((a_{ki}^{[n]})\) is a skew-symmetric matrix.

We say that this sequence of Volterra operators is *compatible* if
\[
V_{n+1} \mid S^{K_n} = V_n
\]  
(30)

for every \( n \in \mathbb{N} \). The compatibility condition with (29) implies that
\[
a_{ki}^{n+1} = a_{ki}^{[n]} \quad \forall k, i \in \{1, \ldots, n\}.
\]  
(31)

Denote
\[
S^{[n]} = \{ x = (x_n, x_{n+1}, \ldots) : k \geq 0, \forall k \geq n, \sum_{k=n}^{\infty} x_k = 1 \}, \quad n \in \mathbb{N}.
\]

Let \( \{ W_n : S^{[n]} \to S^{[n]} : n \in \mathbb{N} \} \) be a sequence of Volterra operators
\[
(W_n(x))_k = x_k (1 + \sum_{i=n}^{\infty} a_{ki}^{[n]} x_i) \quad k \geq n, \quad n \in \mathbb{N}.
\]  
(32)

Define a sequence \( \{ W_n : S \to S, \quad n \in \mathbb{N} \} \) of infinite dimensional operators
as follows
\[
(W_n(x))_k = \begin{cases} 
(V_n(x))_k, & \text{if } n \leq k, \\
(W_{n+1}(x))_k, & \text{if } k \geq n + 1, 
\end{cases} \quad n \in \mathbb{N}.
\]  
(33)
According to Theorem 3.3 the defined operators are Volterra.

**Theorem 6.1.** The sequence \( \{W_n\} \) of Volterra operators weakly converges to a Volterra operator \( W \). Moreover, if \( V_n \) are pure then \( W \) is so.

**Proof.** Let \( x \in S \). If there is a finite subset \( K \) of \( \mathbb{N} \) such that \( x \in S^K \), then according to the compatibility condition \( 30 \) we get \( W(x) = W_n(x) \) for all \( n \geq \max\{m : m \in K\} \).

Now assume that \( x_i > 0 \) for all \( i \in \mathbb{N} \). Let us prove that \( \{W_n(x)\} \) is a Cauchy sequence with respect to weak topology. Let \( \varepsilon > 0 \) be an arbitrary number. Since \( x \in S \) there is a number \( n_0 \in \mathbb{N} \) such that

\[
\sum_{j=n+1}^{\infty} x_j < \varepsilon, \quad \forall n \geq n_0
\]  

(34)

Consider several cases:

**Case (i).** Suppose that \( 1 \leq k \leq n \). Using \( 22 \), \( 21 \), \( 12 \), \( 13 \) and \( 31 \) we have

\[
\|(W_n(x))_k - (W_{n+p}(x))_k\| = \|(V_{n+1}(x))_k - (V_{n+p}(x))_k\|
\]

\[
= \left| x_k \left( 1 + \sum_{i=1}^{n} a_{ki} x_i \right) - x_k \left( 1 + \sum_{j=1}^{n+p} a_{kj} x_j \right) \right|
\]

\[
\leq x_k \left( \sum_{j=n+1}^{n+p} x_j \right) \leq \sum_{j=n+1}^{\infty} x_j < \varepsilon
\]  

(35)

for all \( n \geq n_0 \).

**Case (ii).** Assume \( n+1 \leq k \leq n+p \). It then follows from \( 24 \), \( 23 \) that

\[
\|(W_n(x))_k - (W_{n+p}(x))_k\| = \|(W_{n+1}(x))_k - (W_{n+p}(x))_k\|
\]

\[
= \left| x_k \left( 1 + \sum_{j=n+1}^{\infty} a_{kj} x_j \right) - x_k \left( 1 + \sum_{j=1}^{n+p} a_{kj} x_j \right) \right|
\]

\[
\leq x_k \sum_{j=1}^{\infty} |\gamma_{kj}| x_j \leq 2x_k < 2\varepsilon
\]  

(36)

for all \( n \geq n_0 \). Here

\[
\gamma_{kj} = \begin{cases} 
    a_{kj}^{n+p} & \text{if } j \leq n, \\
    a_{kj}^{n+p} + a_{kj}^n & \text{if } n+1 \leq j \leq n+p \\
    a_{kj}^{n+1} & \text{if } j \geq n+p+1
\end{cases}
\]

**Case (iii).** Now assume that \( k \geq n + p + 1 \) then from \( 33 \) we have

\[
\|(W_n(x))_k - (W_{n+p}(x))_k\| = \left| x_k \left( 1 + \sum_{j=n+1}^{\infty} a_{kj}^{n+1} x_j \right) - x_k \left( 1 + \sum_{j=n+p+1}^{\infty} a_{kj}^{n+p+1} x_j \right) \right|
\]

\[
\leq 2x_k \sum_{j=n+1}^{\infty} x_j < 2\varepsilon^2
\]  

(37)
Hence the sequence \((W_n(x))\) is Cauchy, therefore \(W_n(x) \rightarrow W(x)\). By the same way we can show that \(W_n(x, y) \rightarrow W(x, y)\). Because of \(W_n(e^{(i)}, e^{(j)}) \in \Gamma(e^{(i)}, e^{(j)})\) and the compatibility condition we find that \(W\) is Volterra. According to (33) for every \(e^{(i)}\) and \(e^{(j)}\) there is \(n_0 \in \mathbb{N}\) such that \(W(e^{(i)}, e^{(j)}) = \tilde{V}_n(e^{(i)}, e^{(j)})\) for all \(n \geq n_0\). Now if \(V_n\) is pure for all \(n \in \mathbb{N}\) then \(W\) is also pure. The theorem is proved.

Let \(\{V_n : S \rightarrow S, n \in \mathbb{N}\}\) be a sequence of operators associated with (see 33)
\[
(W_n(x))_k = x_k, \quad k \geq n, \quad n \in \mathbb{N}.
\]
According to Theorem 6.1 the defined sequence \(\{V_n\}\) converges to a Volterra operator \(V\).

Now naturally comes a question: are the operators \(V\) and \(W\) equal? Next theorem gives an affirmative answer to this question.

**Theorem 6.2.** The operators \(V\) and \(W\) are equal.

**Proof.** Let \(\varepsilon > 0\) be an arbitrary number and \(x \in S\) be fixed. To prove the assertion it is enough to show for every \(k \in \mathbb{N}\) the relation holds
\[
|(W_n(x))_k - (V_n(x))_k| < \varepsilon.
\]

There is a number \(n_0 \in \mathbb{N}\) such that (34) holds. Consider two cases.

**case (i).** Let \(1 \leq k \leq n\). Then (33) implies that
\[
|(W_n(x))_k - (V_n(x))_k| = 0,
\]
for every \(n \geq n_0\).

**case (ii).** Let \(k \geq n + 1\), then it follows from (33) that
\[
|(W_n(x))_k - (V_n(x))_k| = \left|x_k \left(1 + \sum_{j=n+1}^{\infty} a_{k,j} x_j\right) - x_k\right| \leq x_k \sum_{j=n+1}^{\infty} x_j < \varepsilon.
\]

Hence, we have proved the desired relation. This completes the proof.

Thus according to the last Theorem we will consider only the sequence \(\{V_n\}\).

Now we are interested about the convergence of powers of the sequence \(\{V_n\}\).

Let \(V\) be an arbitrary Volterra operator. By \(V^m\) we will denote \(m\)-th iteration of \(V\), i.e. \(V^m(x) = V(V \cdots V(V(x)) \cdots)\). Before going to formulate the result we need the following

**Lemma 6.3.** Let \(V\) be an arbitrary Volterra operator. Then \((V^m(x))_k \leq 2^m x_k\), for every \(k, m \in \mathbb{N}\) and \(x \in S\).

**Proof.** According to Theorem 3.3 we have
\[
(V^m(x))_k = (V^{m-1}(x))_k \left(1 + \sum_{i=1}^{\infty} a_{ki}(V^{m-1}(x))_i\right) \leq 2(V^{m-1}(x))_k \leq \cdots \leq 2^m x_k
\]
this is the required relation.

**Theorem 6.4.** For every \(m \in \mathbb{N}\) the sequence \(\{V^m_n\}\) converges.

**Proof.** To show the convergence it is enough to prove that \(\{V^m_n(x)\}\) is a Cauchy sequence for every \(x \in S\). Without loss of generality we may assume that \(x_k > 0\) for all \(k \in \mathbb{N}\).

Let \(\varepsilon > 0\) be an arbitrary number. Since \(x \in S\) there is a number \(n_0 \in \mathbb{N}\) such that (33) holds. Consider several cases.
Case (i). Let \(1 \leq k \leq n\) and \(p \in \mathbb{N}\) be an arbitrary number. For the sake of brevity we will denote
\[
a^{(s)}_k = (V^n_s(x))_k, \quad b^{(s)}_k = (V^n_{n+p}(x))_k,
\]
where \(s, k \in \mathbb{N}\). Then from (31),(33) and (38) we have
\[
|(V^n_m(x))_k - (V^n_{m+p}(x))_k| = |a^{(m)}_k - b^{(m)}_k| = \left| a^{(m-1)}_k \left( 1 + \sum_{i=1}^{\infty} a^{n}_{ki} a_{i}^{(m-1)} \right) - b^{(m-1)}_k \left( 1 + \sum_{i=1}^{\infty} a^{n+p}_{ki} b_{i}^{(m-1)} \right) \right|
\]
\[
\leq |a^{(m-1)}_k - b^{(m-1)}_k| + \left( \sum_{i=1}^{n} a^{n}_{ki} \left( a_k^{(m-1)} a_i^{(m-1)} - b_k^{(m-1)} b_i^{(m-1)} \right) \right)
\]
\[
+ b^{(m-1)}_k \sum_{j=n+1}^{n+p} |a^{n+p}_{kj} b_{j}^{(m-1)}|
\]
\[
\leq |a^{(m-1)}_k - b^{(m-1)}_k| + \left( \sum_{i=1}^{n} a^{n}_{ki} \left( a_k^{(m-1)} - b_k^{(m-1)} \right) + b^{(m-1)}_k \sum_{j=n+1}^{n+p} b_{j}^{(m-1)} \right)
\]
\[
\leq 2 |a^{(m-1)}_k - b^{(m-1)}_k| + b^{(m-1)}_k \sum_{j=n+1}^{n+p} b_{j}^{(m-1)}.
\]
(39)

Now we need the following

**Lemma 6.5.** For every \(m \in \mathbb{N}\) the following inequality holds
\[
|a^{(m)}_k - b^{(m)}_k| \leq \alpha_m x_k \sum_{j=n+1}^{n+p} x_j,
\]
(40)

where
\[
\alpha_1 = 1, \quad \alpha_m = \alpha_{m-1} (2 + 2^{m-1}) + 2^{2(m-1)}, \quad m \geq 2.
\]

**Proof.** Let us firstly consider the case \(m = 1\). We have
\[
|a^{(1)}_k - b^{(1)}_k| = |(V^n_1(x))_k - (V^n_{n+p}(x))_k|
\]
\[
= \left| x_k \left( \sum_{j=n+1}^{n+p} a_{kj}^{n+p} x_j \right) \right| \leq x_k \sum_{j=n+1}^{n+p} x_j.
\]

This shows that \(\alpha_1 = 1\). Now assume that (40) is valid for \(m - 1\). Show that it is true for \(m\). Indeed, it follows from (39) and Lemma 6.3 that
\[
|a^{(m)}_k - b^{(m)}_k| \leq 2 \alpha_{m-1} x_k \sum_{j=n+1}^{n+p} x_j + \alpha_{m-1} 2^{m-1} x_k \sum_{i=1}^{n} x_i \sum_{j=n+1}^{n+p} x_j
\]
\[
= \alpha_m x_k \sum_{j=n+1}^{n+p} x_j.
\]
\[ +2^{2(m-1)}x_k \sum_{j=n+1}^{n+p} x_j \]
\[ \leq (\alpha_{m-1}(2 + 2^{m-1}) + 2^{2(m-1)})x_k \sum_{j=n+1}^{n+p} x_j \]

which proves the lemma.

Now continue the proof of Theorem 6.4. According to Lemma 6.5 we find that 
\[ |(V_m^n(x))_k - (V_{n+p}^m(x))_k| < \varepsilon, \text{ for every } n \geq n_0. \]

Case (ii). Let \( n + 1 \leq k \leq n + p. \) We have

\[
|(V_m^n(x))_k - (V_m^{n+p}(x))_k| = \left| x_k - b_k^{(m-1)} \left( 1 + \sum_{i=1}^{\infty} a_{ki}^{n+p} b_i^{(m-1)} \right) \right|
\]
\[ \leq |x_k - b_k^{(m-1)}| + b_k^{(m-1)} \sum_{i=1}^{n+p} b_i^{(m-1)} \]
\[ \leq |x_k - b_k^{(m-1)}| + 2^{m-1}x_k. \quad (41) \]

Now consider

\[
|x_k - b_k^{(m-1)}| \leq |x_k - b_k^{(m-2)}| + b_k^{(m-2)} \sum_{i=1}^{n+p} b_i^{(m-2)}
\]
\[ \leq \cdots \leq |x_k - b_k^{(1)}| + \sum_{j=1}^{m-2} b_k^{(j)} \sum_{i=1}^{n+p} b_i^{(j)} \]
\[ \leq x_k \sum_{i=1}^{n+p} x_i + x_k \sum_{j=1}^{m-2} 2^j \leq x_k \sum_{j=0}^{m-2} 2^j. \]

Hence, from (41) we infer that

\[
|(V_m^n(x))_k - (V_m^{n+p}(x))_k| \leq \left( \sum_{j=0}^{m-1} 2^j \right) \varepsilon
\]

for every \( n \geq n_0. \)

Now let \( k \geq n + p \) then \((V_m^n(x))_k = (V_m^{n+p}(x))_k. \)

Thus we have proved that \( \{V_m^n(x)\} \) is a Cauchy sequence. The limit of this sequence we denote as \( W_m^n(x). \) The theorem is proved.

From this theorem naturally arises a question: whether does the equality \( W_m = V_m \) hold?

Before answer to this question we should prove the following an auxiliary

**Lemma 6.6.** Let \( V \) be an arbitrary Volterra operator. Then the following inequality holds

\[ ||V(x) - V(y)||_1 \leq 3||x - y||_1 \]

for every \( x, y \in S. \)

**Proof.** We have

\[ ||V(x) - V(y)||_1 = \sum_{k=1}^{\infty} |(V(x))_k - (V(y))_k| \]
\[ \leq \sum_{k=1}^{\infty} \left( 1 + \sum_{i=1}^{\infty} |a_{ki}|x_i \right) |x_k - y_k| + x_k \sum_{i=1}^{\infty} |a_{ki}| |x_i - y_i| \]
\[ \leq 2 \sum_{k=1}^{\infty} |x_k - y_k| + \sum_{i=1}^{\infty} |x_i - y_i| = 3\|x - y\|_1 \]

Lemma is proved.

**Theorem 6.7.** For every \( m \in \mathbb{N} \) the equality \( W_m = V^m \) is valid.

**Proof.** Let \( x \in S \) be an arbitrary element. Then given \( \varepsilon > 0 \) there is a number \( n \in \mathbb{N} \) and \( y \in S^{K^n} \) such that \( \|x - y\|_1 < \varepsilon \). According to the compatibility condition \( |(y_n)_{k}\| \) we have \( V(y) \in S^{K^n} \) and hence \( V^m(y) = V^m_n(y) \) therefore \( W_m(y) = V^m(y) \). Using this we have

\[ |(W_m(x))_k - (V^m(x))_k| \leq |(W_m(x))_k - (W_m(y))_k| + |(V^m(x))_k - (V^m(y))_k| \]
\[ (42) \]

for every \( k \in \mathbb{N} \).

According to Theorem 6.4 we know that there is \( n_0 \in \mathbb{N} \) such that

\[ |(W_m(x))_k - (V^m_n(x))_k| < \varepsilon \]
\[ (43) \]

for every \( n \geq n_0 \).

Using Lemma 6.6 one gets

\[ |(V^m_n(x))_k - (V^m_n(y))_k| \leq \|V^m_n(x) - V^m_n(y)\|_1 \leq 3^m\|x - y\|_1 < 3^m\varepsilon. \]
\[ (44) \]

It follows from \( (43),(44) \) that

\[ |(W_m(x))_k - (W_m(y))_k| \leq |(W_m(x))_k - (V^m_n(x))_k| + |(V^m_n(x))_k - (V^m_n(y))_k| + |(V^m_n(y))_k - (W_m(y))_k| \leq (1 + 3^m)\varepsilon, \]
\[ (45) \]

here we have used the equality \( W_m(y) = V^m_n(y) \).

Now again using Lemma 6.6 we find

\[ |(V^m_n(x))_k - (V^m_n(y))_k| \leq 3^m\|x - y\|_1 < 3^m\varepsilon. \]
\[ (46) \]

Consequently, the inequalities \( (45),(46) \) with \( (42) \) imply that

\[ |(W_m(x))_k - (V^m(x))_k| < (1 + 2 \cdot 3^m)\varepsilon. \]

As \( \varepsilon \) has been an arbitrary, so this completes the proof.

This Theorem gives us some how to investigate limit behaviors of infinite dimensional Volterra operators by means of finite dimensional ones. This would be a theme of our next investigations.

**Acknowledgements.** The first named author (F.M.) thanks NATO-TUBITAK for providing financial support and Harran University for kind hospitality and providing all facilities. The work is also partially supported by Grant Φ-1.1.2 of Rep. Uzb.

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