COUNTING ALL EQUILATERAL TRIANGLES IN \( \{0, 1, ..., n\}^3 \)

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Abstract. We describe a procedure of counting all equilateral triangles in the three dimensional space whose coordinates are allowed only in the set \( \{0, 1, ..., n\} \). This sequence is denoted here by \( ET(n) \) and it has the entry A102698 in “The On-Line Encyclopedia of Integer Sequences”. The procedure is implemented in Maple and its main idea is based on the results in [3]. Using this we calculated the values \( ET(n) \) for \( n=1..55 \) which are included here. Some facts and conjectures about this sequence are stated. The main of them is that \( \lim_{n \to \infty} \frac{\ln ET(n)}{\ln n + 1} \) exists.

1. INTRODUCTION

If we restrict the vertices of an equilateral triangle to be in \( \mathbb{Z}^3 \) we obtain a typical element in \( ET(\mathbb{Z}) \). It is not that hard to see that there are no such triangles whose vertices are contained in the coordinate planes or any other plane parallel to one of them. Also, the sides of a triangle in \( ET(\mathbb{Z}) \) cannot be of an arbitrary length. If one such triangle is considered a whole family in \( ET(\mathbb{Z}) \) can be generated from it that have vertices in the same plane. Moreover, we have shown in [3] the following theorems that we are going to use in our construction here.

**Theorem 1.1.** If the triangle \( \triangle OPQ \in ET(\mathbb{Z}) \) with \( O \) the origin and \( l = ||\vec{OP}|| \) then:

(i) the points \( P \) and \( Q \) are contained in a plane of equation \( ax + by + cz = 0 \), where

\[
\begin{align*}
a^2 + b^2 + c^2 &= 3d^2, \quad a, b, c, d \in \mathbb{Z} \\
\end{align*}
\]

and \( l^2 = 2q \);

(ii) the side length, \( l \), is of the form \( \sqrt{2(m^2 - mn + n^2)} \) with \( m, n \in \mathbb{Z} \).

It is important to be able to generate all the solutions of (1):

**Theorem 1.2.** The following formulae give a three integer parameter solution of (1):

\[
\begin{align*}
a &= -x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 \\
b &= x_1^2 - x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 \\
c &= x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 - 2x_2x_3 \\
d &= x_1^2 + x_2^2 + x_3^2 \\
\end{align*}
\]

Moreover, every solutions of (1), \( a, b, c, d \) can be found with (3), simplifying if necessary by a common divisor of \( a, b, c \) and \( d \) with \( x_1, x_2, x_3 \) no bigger in absolute value than \( \sqrt{\frac{3-\sqrt{3}}{2}}q \).
We include some general remarks about the solutions of (1) which are discussed in [3]:

- if we assume that \( \gcd(a, b, c) = 1 \) then all \( a, b, c, d \) must be odd integers
- for every \( d \) odd there exist at least one solution which is not trivial (i.e. \( a = b = c = q \))
- \([\text{Gauss}]\) positive integer \( n \) can be written as a sum of three squares iff \( n \) is not of the form \( 4^k(8l + 7) \) with \( k, l \in \mathbb{Z} \) (see [1] for an elementary proof)

Our construction depends essentially on a particular solution, \( (r, s) \in \mathbb{Z}^2 \), of the equation:

\[
2(a^2 + b^2) = s^2 + 3r^2.
\]

It turns out that this Diophantine equation has always solutions if \( a, b, c \) and \( d \) are integers satisfying (1). The family we have mentioned can be described as another parametrization.

**Theorem 1.3.** Let \( a, b, c, d \) be odd positive integers such that \( a^2 + b^2 + c^2 = 3d^2 \), with \( \gcd(d, c) = 1 \). Then a triangle \( \triangle OPQ \in \mathcal{ET}(\mathbb{Z}) \) has its vertices in the plane of equation \( a\alpha + b\beta + c\gamma = 0 \) iff \( P(u, v, w) \) and \( Q(x, y, z) \) are given by

\[
\begin{align*}
\begin{cases}
  u = m_u m - n_u n, \\
v = m_v m - n_v n, \\
w = m_w m - n_w n,
\end{cases}
\quad \text{and} \quad 
\begin{cases}
  x = m_x m - n_x n, \\
y = m_y m - n_y n, \\
z = m_z m - n_z n,
\end{cases}
\end{align*}
\]

with

\[
\begin{align*}
\begin{cases}
  m_x = -\frac{1}{2}[db(3r + s) + ac(r - s)]/q, \\
m_y = \frac{1}{2}[da(3r + s) - bc(r - s)]/q, \\
m_z = (r - s)/2,
\end{cases}
\quad \text{and} \quad 
\begin{cases}
  n_x = -(rac + dbs)/q, \\
n_y = (das - bcr)/q, \\
n_z = r,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  m_u = -(rac + dbs)/q, \\
m_v = (das - rbc)/q, \\
m_w = r,
\end{cases}
\quad \text{and} \quad 
\begin{cases}
  n_u = -\frac{1}{2}[db(s - 3r) + ac(r + s)]/q, \\
n_v = \frac{1}{2}[da(s - 3r) - bc(r + s)]/q, \\
n_w = (r + s)/2
\end{cases}
\end{align*}
\]

where \( q = a^2 + b^2 \) and \( (r, s) \) is a suitable solution of (3) and \( m, n \in \mathbb{Z} \).

Moreover, the side-lengths of this triangle are equal to \( d\sqrt{2(m^2 - mn + n^2)} \).

Let us observe that one can use Theorem 1.3 as long as \( \min(\gcd(q, c), \gcd(q, a), \gcd(d, b)) = 1 \).

We have found the following counterexample to this property: \( a = 55063, b = 2396393, c = 5(71)(2017)(1694953) \) and \( d = 3(41)(3361)(1694953) \) with \( \gcd(a, d) = 41, \gcd(b, d) = 3361 \) and \( \gcd(c, d) = 1694953 \). However we have calculated that this property holds true for all solutions \( a, b, c, d \) of (1) such that \( \gcd(a, b, c) = 1 \) and with all odd \( d \leq 4096 \). This allows us to calculate \( ET(n) \) for \( n = 1, \ldots, 64 \) as we will see later.
2. Description of the procedure and some ingredients

The idea is based on the facts above and a few other results. One would like to first find the side lengths of the triangles in $\mathcal{ET}(\mathbb{Z}) \cap C_n$. This will partition these triangles into clear classes. For this purpose we will use the Proposition 1.2. Then for a given side-length, $l$, we need to find all the possible planes that contain triangles of sides $l$. This gives another criteria of sub-partition even further these triangles. Using the parametrizations given in Theorem 1.3 then one finds the smallest such triangle within a given plane and which can fit in $C_n$ after a translation. Once that is obtained we have to rotate and translate it in all possible ways, but in a pairwise disjoint manner to fill out $C_n$. A formula for the number of all these will be given. Finally all these numbers will be added up to make up $ET(n)$.

The first fact that we will use is the following geometric observation.

**Proposition 2.1.** The largest side length of an equilateral whose vertices are contained in a cube of side lengths $r$ is $r\sqrt{2}$.

**Proof.** If an equilateral triangle of side lengths $l$, has its vertices in the cube $[0,r]^3$, we denote by $A_1$, $A_2$, and $A_3$ the areas of the three projections of this triangle on the three coordinate planes. It is easy to see that $A^2 = A_1^2 + A_2^2 + A_3^2$ where $A$ is the area of the equilateral triangle given. The maximum of the area of a triangle inscribed in a square of side lengths $r$ is easy to see that is at most $r^2/2$. Hence $A^2 = 3l^4/16 \leq 3r^4/4$. This gives $l \leq r\sqrt{2}$. Certainly this happens when the vertices are at the corners diagonally opposite on each face of the cube. 

Let us work out a concrete example example: $n = 4$. Using Proposition 2.1 and the part (ii) of Theorem 1.1 the side lengths can be only $\sqrt{2}$, $\sqrt{6}$, $2\sqrt{2}$, $\sqrt{14}$, $3\sqrt{2}$, $2\sqrt{3}$, $\sqrt{26}$ or $4\sqrt{2}$. The $d$ values here are 1 or 3. Since $3(1^2) = 1^2 + 1^2 + 1^2$ and $3(3^2) = 1^2 + 1^2 + 5^2$ are the only solutions of (1), the parametrizations we need in this case are, as shown in [3]:

$$\mathcal{T}_{1,1,1} = \{((0,0,0),(m,-n,n-m),(m-n,-m,n)) : m,n \in \mathbb{Z}, m^2 + n^2 \neq 0\}$$

and

$$\mathcal{T}_{1,1,5} = \{((0,0,0),(4m-3m,m+3n,-m),(3m+n,-3m+4n,-n)) : m,n \in \mathbb{Z}, m^2 + n^2 \neq 0\},$$

here we used the notation $\mathcal{T}_{a,b,c}$ standing for all triangles in $\mathcal{ET}(\mathbb{Z})$ having a vertex the origin and the other two in the plane $\{(\alpha,\beta,\gamma) : a\alpha + b\beta + c\gamma = 0\}$.

Using the first parametrization we find the $m,n$ such that the triangle obtained after a translation fits in $C_4$ and the side lengths are $\sqrt{2}$: $T_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$. This triangle can be translated in various ways inside of $C_4$, and together with all its cube symmetries and
translations contribute with a total of 512 in $ET(4)$. We will prove a formula that gives the total of all these triangles generated by $T_1$ inside of $C_n$. This parametrization has to be used for all the side lengths $\sqrt{6}$, $2\sqrt{2}$, $\sqrt{14}$, $3\sqrt{2}$, $2\sqrt{6}$, $\sqrt{26}$ and $4\sqrt{2}$: the corresponding triangles respectively are $T_2 := \{(1, 0, 2), (2, 1, 0), (0, 2, 1)\}$, $2T_1$, $T_3 := \{(2, 0, 3), (0, 3, 2), (3, 2, 0)\}$, $3T_1$, $2T_2$, $T_4 := \{(1, 4, 0), (4, 0, 1), (0, 1, 4)\}$, $4T_1$. Using the same formula we will see the contribution of all these to $ET(4)$ is respectively: 216, 216, 128, 64, 8, 16 and 8.

There is need to use the second parametrization too since one can take $d = 3$ to obtain the side length $3\sqrt{2}$. This gives still a new triangle $T_5 := \{(0, 0, 1), (1, 4, 0), (4, 1, 0)\}$ with a total of 96 other generated by it in $C_4$. Tallying all these we get $ET(4) = 1264$.

As we can see from this example, one has to derive a way of finding how many other triangles can a given one, say $T$, generate inside of $C_n$ under all possible translations and cube symmetries roughly speaking. We need to make this a little more precise. We are going to assume that the given triangle, $T$, that is inside $C_t$ is minimal in the sense that at least one of the coordinates of the vertices in $T$ is zero and $t$ is the smallest dimension $k$ of a cube $C_k$ containing $T$.

Let us denote by $O(T)$ the orbit generated by $T$ within $C_t$ under all translations and cube symmetries. We also need to introduce the standard unit vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

It is actually surprising that in order to compute the number $f$, of all distinct triangles generated by $T$ (union of all translations of $O(T)$) within $C_n$ in terms of the following five variables:

(i) $n$ -the dimension of the cube,
(ii) $t$ -the maximum of all the coordinates in $T$,
(iii) $\alpha(T)$ -the cardinality of $O(T)$,
(iv) $\beta(T)$ -the cardinality of $O(T) \cap [O(T) + e_1]$,
(v) $\gamma(T)$ -the cardinality of $[O(T) + e_1] \cap [O(T) + e_2]$.

**Theorem 2.2.** The function $f(T, n)$ described above is given by

$$f(T, n) = (n + 1 - t)^2 \alpha - 3(n + 1 - t)^2(n - t)\beta + 3(n + 1 - t)(n - t)^2\gamma,$$

for all $n \geq t$.

**Proof.** Let us consider the cube $C_s = \{0, 1, ..., s\}^3$ where $s = n - t$. Clearly the number of points in this set is $(s + 1)^3$. Each point $p$ in the set $C_s$ is considered in here as a vector. So, $f = \{ \bigcup_{p \in C_s} O(T) + p \}$. One essential observation here is that $||O(T) + p|| \cap [O(T) + q|| = 0$ for every $p, q$ such that $||p - q|| \geq 1$, where $||p - q|| = \min_{i=1,2,3} (|p_i - q_i|)$. This is due to the minimality of $T$.

Let us write the elements of $C_s$ in lexicographical order: $p_1, p_2, ..., p_k$ where $k = (s + 1)^3$. We look now at $C_s$ as the three dimensional grid graph. Faces in this graph are simply unit squares formed
with vertices from \( C_s \). One can look at the cardinality of \( \bigcup_{i=1}^{j} O(T) + p_i \) and show by induction on \( j \) that this is equal to

\[
j\alpha - (\#\text{edges}(C_s(j)))\beta + (\#\text{faces}(C_s(j)))\gamma
\]

where \( C_s(j) \) is the graph induced in \( C_s \) by the vertices \( p_1, p_2, ..., p_j \). Hence we just need to compute the number of edges and faces in \( C_s \). There are eight vertices in this graph that have degree 3 (the corners), there are \((s-1)^3\) vertices of degree 6, also \(6(s-1)^2\) vertices with degree 5 and finally \(12(s-1)\) of degree 4. This gives a total of

\[
\frac{1}{2}[24 + 6(s-1)^3 + 30(s-1)^2 + 48(s-1)] = 3s(s+1)^2
\]
edges. The number of faces is equal to \( \frac{1}{2}[6s^3 + 6s^2] = 3s^2(s+1) \).

**Example:** Suppose we take \( T = T_5 \). Then clearly \( t = 4 \). One can use a symbolic calculator to find \( \alpha(T) = 96 \), \( \beta(T) = 24 \) and \( \gamma(T) = 0 \). So, the contribution of \( T_5 \) to a cube \( C_n \) if \( f(T_5, n) = 96(n-3)^3 - 72(n-3)^2(n-4) = 24n(n-3)^2 \).

**Remark:** These facts give a way to find lower bounds for \( ET(n) \). For instance, if we put the contribution of \( T_1 \) and \( T_2 \) we obtain \( ET(n) \geq 8(2n-1)(n^2 - n + 1) \) for all \( n \geq 2 \).

To generate the side lengths we would like use a well-known result due to Euler (see [4], pp. 568 and [2], pp. 56).

**Proposition 2.3. [Euler’s 6k + 1]** An integer \( t \) can be written as \( m^2 - mn + n^2 \) for some \( m, n \in \mathbb{Z} \) if and only if in the prime factorization of \( t \), 2 and the primes of the form \( 6k - 1 \) appear to an even exponent.

3. **The code**

Using Proposition 2.3 and Proposition 2.1 we have the following procedure in Maple to compute the side lengths modulo a factor of two and the square root:

- \( \text{sides} := \text{proc}(n) \)
- \( \text{local i, j, k, L, a, m, p, q, r, ms;} \)
- \( L := \{1\}; ms := n^2; \)
- \( \text{for i from 2 to ms do} \)
  - \( a := \text{ifactors}(i); k := \text{nops}(a[2]); r := 0; \)
  - \( \text{for j from 1 to k do} \)
    - \( m := a[2][j][1]; p := m \text{ mod } 6; q := a[2][j][2] \text{ mod } 2; \)
    - \( \text{if } r = 0 \text{ and } (m = 2 \text{ or } p = 5) \text{ and } q = 1 \) then \( r := 1 \) fi;
  - \( \text{if } r = 0 \) then \( L := L \cup \{i\} \) fi;
- \( od; L := \text{convert}(L, \text{list}); \text{end;} \)
This procedure gives for \( n = 10 \): \( 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39, 43, 48, 49, 52, 57, 61, 63, 64, 67, 73, 75, 76, 79, 81, 84, 91, 93, 97, 100 \]. This gives the corresponding side-lengths

\[
\sqrt{2}, \sqrt{6}, \sqrt{2}, \sqrt{38}, \sqrt{42}, 3\sqrt{2}, 2\sqrt{6}, \sqrt{26}, 4\sqrt{2}, 2\sqrt{14}, 3\sqrt{6}, \sqrt{74}, \sqrt{78}, \sqrt{86}, 4\sqrt{2}, \sqrt{38}, \sqrt{42}, 5\sqrt{2}, 3\sqrt{6}, 2\sqrt{14}, \sqrt{62}, \sqrt{78}, \sqrt{86}, \sqrt{93}, 9\sqrt{2}, 2\sqrt{42}, \sqrt{82}, \sqrt{100}, \sqrt{104}, 10\sqrt{2}
\]

We need a procedure that will give the odd values of \( d \) that “divide” a certain side length in the sense it is possible to write it as \( d\sqrt{m^2 - mn + n^2} \) with \( m, n \in \mathbb{Z} \):

\[
dkl := \text{proc}(\text{side})
\]

- local i, x, noft, div, y, y1, z;
- \( x := \text{convert(divisors(\text{side}), list)}; \) noft := nops(x); div := {};
- for i from 1 to noft do z := x[i] mod 2;
- if \( z = 1 \) then \( y := \text{side}/x[i]^2; \) y1 := floor(y);
- if \( y = y1 \) then div := div union \{x[i]\}; fi;
- od;
- convert(div, list); end:

For instance, if \( \text{side} = \sqrt{882} \) this procedure gives \( [1, 3, 7, 21] \) which means we have at least four possible parametrizations that we can use to find minimal equilateral triangles.

Next we need to find all the nontrivial solutions \([a, b, c]\) of (1), given and odd positive integer \( d \), with the property \( \gcd(a, b, c) = 1 \), \( 0 < a \leq b \leq c \) which is based on an internal procedure to solve Diophantine equation \( A = X^2 + Y^2 \):

\[
\text{abcso} := \text{proc}(q) \text{ local i, j, k, u, x, y, sol, cd; } \text{sol} := \{\};
\]

- for i from 1 to q do
- \( u := [\text{isolve}(3d^2 - i^2 = x^2 + y^2)]; \) k := nops(u);
- for j from 1 to k do
- if \( \text{rhs}(u[j][1]) >= i \) and \( \text{rhs}(u[j][2]) >= i \) then
- \( \text{cd} := \gcd(\gcd(i, \text{rhs}(u[j][1])), \text{rhs}(u[j][2])); \)
- if \( \text{cd} = 1 \) then \( \text{sol} := \text{sol} \cup \{\text{sort([i, rhs(u[j][1]), rhs(u[j][2])])}\} ); fi;
- od;
- od;
- convert(sol, list); end:

For \( d = 17 \), \( \text{ancsol} \) finds four different solutions, \([11, 11, 25], [13, 13, 23], [1, 5, 29], [7, 17, 23] \], and in a few seconds sends out 333 solutions for \( d = 2007 \). One interesting solution in this last case is

\[
1937^2 + 1973^2 + 2107^2 = 3(2007)^2.
\]

Now based on the Theorem \([1,3]\) we take a solution of (1) as given by the procedure above and calculate the general parametrization:
\begin{itemize}
  \item \texttt{findpar:=proc(a,b,c,m,n)}
  \item \texttt{local i,j,r,s,sol,mx,nx,my,nu,mv,mz,nw,q,d,u,v,w,x,y,z,ef,ns,om,l;}
  \item \texttt{q := a^2 + b^2; sol := convert(isolve(2q = x^2 + 3y^2),list); ns := nops(sol); d := \sqrt{\frac{a^2+b^2+c^2}{3};}
    ef := 0;
  \item for i from 1 to ns do
    \item if ef = 0 then \texttt{r := rhs(sol[i][1]); s := rhs(sol[i][2]);}
    \item if \texttt{s^2 + 3r^2 = 2q} then \texttt{mz := (r - s)/2; nz := r; nw := (r + s)/2; mx := -(db(3r + s) + ac(r - s))/2q; nx := -(rac + dbs)/q; my := (da(3r + s) - bc(r - s))/2q; ny := -(rbc - das)/q; mu := -(rac + dbs)/q; nu := -(db(s - 3r) + ac(r + s))/2q; mv := (das - rbc)/q; nv := -(da(3r - s) + bc(r + s))/2q;}
    \item if \texttt{mx = floor(mx)} and \texttt{nx = floor(nx)} and \texttt{my = floor(my)} and \texttt{ny = floor(ny)} and \texttt{mu = floor(mu)} and \texttt{nu = floor(nu)} and \texttt{mv = floor(mv)} and \texttt{nv = floor(nv)} then
      \item \texttt{u := (mu)m - (nu)n; v := (mv)m - (nv)n; w := (mw)m - (nv)n; x := (mx)m - (nx)n; y := (my)m - (ny)n; z := (mz)m - (nz)n; om := [u, v, w, x, y, z];}
    \item ef := 1; fi; fi; od; od; om: end;
\end{itemize}

For the solution, \texttt{[1, 5, 29]}, found earlier for the case \texttt{d = 17}, \texttt{findpar} gives
\[
\langle [−11m − 13n, −21m + 20n, 4m − 3n], [−24m + 11n, −m + 21n, m − 4n]\rangle.
\]

Next, using this parametrization we would like to find if there is any equilateral triangle in \(\mathcal{ET}(\mathbb{Z})\) which after a translation fits inside \(\mathcal{C}_{\text{stop}}\).

1: \texttt{minimaltr:=proc(s,a,b,c,stop)}
2: \texttt{local i,z,u,nt,d,m,n,T,α,β,γ,tri,not,tri,orb,avb,length,lengthn;}
3: \texttt{d := \sqrt{\frac{a^2+b^2+c^2}{3}; z := \frac{s}{\sqrt{d}}; u := convert(isolve(z = q^2 - qr + r^2),list); nt := nops(u);}
4: \texttt{for i from 1 to nt do}
5: \texttt{T := findpar(a, b, c, rhs(u[i][1]), rhs(u[i][2]));}
6: \texttt{α := min(T[1][1], T[2][1], 0); β := min(T[1][2], T[2][2], 0); γ := min(T[1][3], T[2][3], 0);}
7: \texttt{tri[i] := \{T[1][1] − α,T[1][2] − β,T[1][3] − γ\}, T[2][1] − α,T[2][2] − β,T[2][3] − γ\};}
8: \texttt{out[i] := max(tri[i][1], tri[i][2], tri[i][3]); tri[i] := tri[i][3]; tri[i] := tri[i][2]; tri[i] := tri[i][1];}
9: \texttt{od; L := sort(seq(out[i], i = 1..nt)); tri := \{\};}
10: \texttt{for i from 1 to nt do if out[i] <= stop then tri := tri union \{tri[i]\}; fi; od;}
11: \texttt{tri := convert(tri, list); tria := \{\};}
12: \texttt{if nops(tri) > 0 then noft := nops(tri); tria := \{tri[1]\}; orb := transl(tri[1]);}
13: \texttt{for i from 1 to noft do avb := evalb(tri[i] in orb);}
14: \texttt{if avb = false then orb := orb union transl(tri[i]); tria := tria union \{tri[i]\};}
The minimal triangle given by this procedure for \( s = 17\sqrt{2}, a = 1, b = 5, c = 29, \) \( \text{stopp} = 30: \) \{[11, 21, 0], [24, 1, 3], [0, 0, 4]\}. The last part of the procedure is actually searching for a set of triangles that generate all the triangles in \( \mathcal{ET}(\mathbb{Z}) \) that lay in planes of normal \((a, b, c)\) or all other 23 possibilities obtained by permuting the coordinates and changing signs. The next procedure is used above and later in order to compute the parameters \( \alpha(T), \beta(T) \) and \( \gamma(T) \).

- \( \text{transl} := \text{proc}(T) \)
- local \( S, Q, i, j, k, a_2, b_2, c_2, a, b, c, d; \)
- \( Q := \text{convert}(T, \text{list}); a := \max(Q[1][1], Q[2][1], Q[3][1]); b := \max(Q[1][2], Q[2][2], Q[3][2]); c := \max(Q[1][3], Q[2][3], Q[3][3]); d := \max(a, b, c); a2 := d - a; b2 := d - b; c2 := d - c; S := \text{orbit}(T); \)
- for \( i \) from 0 to \( a_2 \) do
- for \( j \) from 0 to \( b_2 \) do
- for \( k \) from 0 to \( c_2 \) do
- \( S := S \cup \text{orbit(addvect}(T, [i, j, k])); \)
- od; od; od; \( S; \) end:

Here the procedure \( \text{addvect} \) and \( \text{orbit} \) are:

1: \( \text{addvect} := \text{proc}(T, v) \) local \( Q, a, b, c; \)
2: \( Q := \text{convert}(T, \text{list}); a := \text{v}[1]; b := \text{v}[2]; c := \text{v}[3]; \)
3: \( \{[Q[1][1]+a, Q[1][2]+b, Q[1][3]+c], [Q[2][1]+a, Q[2][2]+b, Q[2][3]+c], [Q[3][1]+a, Q[3][2]+b, Q[3][3]+c]\}; \)
4: end;

and

- \( \text{orbit} := \text{proc}(T) \)
- local \( S, Q, T1; \)
- \( Q := \text{convert}(T, \text{list}); \)
- \( T1 := \{[Q[1][3], Q[1][2], Q[1][1]], [Q[2][3], Q[2][2], Q[2][1]], [Q[3][3], Q[3][2], Q[3][1]]\}; \)
- \( S := \text{orbit1}(T) \cup \text{orbit1}(T1); S; \)
- end:

The \( \text{orbit1} \) takes care of the cube symmetries:

- \( \text{orbit1} := \text{proc}(T) \) local
- \( i, k, T1, a, b, c, x, T2, T3, T4, T5, T6, T7, T8, T9, T10, T11, T12, T13, T14, T15, T16, T17, T18, \)
- \( T19, T20, T21, T22, T23, T24, S, Q, d, a1, b1, c1; \) \( Q := \text{convert}(T, \text{list}); \)
COUNTING ALL EQUILATERAL TRIANGLES IN $\{0, 1, \ldots, n\}^3$

- $a := \max(Q[1][1], Q[2][1], Q[3][1]); a1 := \min(Q[1][1], Q[2][1], Q[3][1]);$
- $b := \max(Q[1][2], Q[2][2], Q[3][2]); b1 := \min(Q[1][2], Q[2][2], Q[3][2]);$
- $c := \max(Q[1][3], Q[2][3], Q[3][3]); c1 := \min(Q[1][3], Q[2][3], Q[3][3]);$
- $d := \max(a, b, c); T1 := T;$
- $T2 := [Q[1][2], Q[1][3], Q[1][1]], [Q[2][2], Q[2][3], Q[2][1]], [Q[3][2], Q[3][3], Q[3][1]];$$
- $T3 := [Q[1][1], Q[1][3], Q[1][2]], [Q[2][1], Q[2][3], Q[2][2]], [Q[3][1], Q[3][3], Q[3][2]];$$
- $T4 := [Q[1][2], Q[1][3], d-Q[1][1]], [Q[2][2], Q[2][3], d-Q[2][1]], [Q[3][2], Q[3][3], d-Q[3][1]];$$
- $T5 := [Q[1][1], Q[1][3], d-Q[1][2]], [Q[2][1], Q[2][3], d-Q[2][2]], [Q[3][1], Q[3][3], d-Q[3][2]];$$
- $T6 := [Q[1][2], Q[1][3], d-Q[1][1]], [Q[2][2], Q[2][3], d-Q[2][2]], [Q[3][2], Q[3][3], d-Q[3][2]];$$
- $T7 := [Q[1][2], Q[1][3], d-Q[1][1]], [Q[2][2], Q[2][3], d-Q[2][2]], [Q[3][2], Q[3][3], d-Q[3][2]];$$
- $T8 := [Q[1][1], Q[1][3], d-Q[1][1]], [Q[2][1], Q[2][3], d-Q[2][2]], [Q[3][1], Q[3][3], d-Q[3][2]];$$
- $T9 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T10 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T11 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T12 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T13 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T14 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T15 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T16 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T17 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T18 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T19 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T20 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T21 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T22 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T23 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $T24 := [d-Q[1][1], Q[1][2], Q[1][3]], [d-Q[2][1], Q[2][2], Q[2][3]], [d-Q[3][1], Q[3][2], Q[3][3]];$$
- $S := \{T1, T2, T3, T4, T5, T6, T7, T8, T9, T10, T11, T12, T13, T14, T15, T16, T17, T18, T19, T20, T21, T22, T23, T24\};$
- $S; end:$

Finally, we are ready to calculate the parameters in Theorem 2.2. We have $\alpha(T) = transl(T), \beta(T) = \text{inters}(T)$ where

- $\text{inters} := \text{proc}(T) \text{ local } a, b, c, d, s, m, i, S1, S2; \text{ Q := convert(T, list)};$
- $a := \max(Q[1][1], Q[2][1], Q[3][1]);$
\[ b := \max(Q[1][2], Q[2][2], Q[3][2]) ; \]
\[ c := \max(Q[1][3], Q[2][3], Q[3][3]) ; \]
\[ d := \max(a, b, c) ; \]
\[ S2 := \text{transl}(T) ; S := \text{convert}(S2, \text{list}) ; m := \text{nops}(S) ; S1 := ; \]
\[ \text{for} \ i \ \text{from} \ 1 \ \text{to} \ m \ \text{do} \]
\[ S1 := S1 \cup \{ \text{addvect}(S[i], [0, 0, 1]) \} ; \]
\[ \text{end} ; \]
\[ S2 \ \text{intersect} \ S1 ; \]
\[ \end : \]

\[ \gamma(T) = \text{intersch}(T) \] where

\[ \text{intersch} := \text{proc}(T) \ \text{local} \ a, b, c, Q, d, S, m, i, S1, S2, S3, S4 ; \]
\[ Q := \text{convert}(T, \text{list}) ; \]
\[ S2 := \text{transl}(T) ; S := \text{convert}(S2, \text{list}) ; m := \text{nops}(S) ; S1 := ; \]
\[ \text{for} \ i \ \text{from} \ 1 \ \text{to} \ m \ \text{do} \]
\[ S1 := S1 \cup \{ \text{addvect}(S[i], [0, 0, 1]) \} ; \]
\[ \text{od} ; \]
\[ S3 := \{ \} ; \]
\[ \text{for} \ i \ \text{from} \ 1 \ \text{to} \ m \ \text{do} \]
\[ S3 := S3 \cup \{ \text{addvect}(S[i], [1, 0, 0]) \} ; \]
\[ \text{od} ; \]
\[ \text{nops}(S1 \ \text{intersect} \ S3) ; \]
\[ \end : \]

The Theorem 2.2 is then implemented in

\[ f := (n, d, \alpha, \beta, \gamma) \rightarrow (n - d + 1)^3 \alpha - 3(n - d + 1)(n - d) \beta + 3(n - d + 1)(n - d)^2 \gamma \]

\[ \text{notrincn} := \text{proc}(T,n) \]
\[ \text{local} \ Q, a, b, c, x, a2, b2, c2, d, y, z, w ; \]
\[ Q := \text{convert}(T, \text{list}) ; \]
\[ a2 := \max(Q[1][1], Q[2][1], Q[3][1]) ; \]
\[ b2 := \max(Q[1][2], Q[2][2], Q[3][2]) ; \]
\[ c2 := \max(Q[1][3], Q[2][3], Q[3][3]) ; \]
\[ d := \max(a2, b2, c2) ; \]
\[ x := \text{nops}(\text{transl}(T)) ; y := \text{nops}(\text{inters}(T)) ; w := \text{intersch}(T) ; \]
\[ z := f(n, d, x, y, w) ; \]
\[ \end : \]

In the end one has to put together all these procedures and add the number of triangles together.

\[ \text{main} := \text{proc}(p, \text{lastside}, \text{nuptols}) \]
\[ \text{local} \ i, j, k, s, \text{nos}, \text{div}, \text{nod}, \text{sol}, x, \text{netr}, \text{noft}, l, z ; \]
\[ \text{netr} := \text{nuptols} ; \]
\[ s := \text{sides}(p) ; \text{nos} := \text{nops}(s) ; \text{print}(s) ; \]
\[ \text{for} \ i \ \text{from} \ \text{lastside} \ \text{to} \ \text{nos} \ \text{do} \]
\[ \text{div} := \text{dkl}(s[i]) ; \text{nod} := \text{nops}(\text{div}) ; \]
\[ \text{for} \ j \ \text{from} \ 1 \ \text{to} \ \text{nod} \ \text{do} \]
\[ \text{sol} := \text{abcsol}(\text{div}[j]) ; \text{nop} := \text{nops}(\text{sol}) ; \]
\[ \text{for} \ k \ \text{from} \ 1 \ \text{to} \ \text{nop} \ \text{do} \]
\[ x := \text{minimaltr}(s[i], \text{sol}[k][1], \text{sol}[k][2], \text{sol}[k][3], p) ; \]
The values $ET(n)$ for $n = 1...55$ computed with main are in given below in increasing order:
8, 80, 368, 1264, 3448, 7792, 16176, 30696, 54216, 90104, 143576, 220328, 326680, 471232, 664648, 916344, 1241856, 1655208, 2172584, 2812664, 3598664, 4553800, 5702776, 7075264, 8705088, 10628928, 12880056, 15496616, 18523472, 22003808, 26000584, 30567400, 35756776, 41631672, 48278136, 55753272, 64134536, 73495760, 83924408, 95513248, 108379264, 122661840, 138315720, 155613408, 174622488, 195478424, 218279240, 243170376, 270288064, 299790968, 331832248, 366610560, 404253120, 444911712, and 488902856.

4. Some facts and conjectures

If we look at the sequence $a_n = \frac{\ln ET(n)}{\ln(n+1)}$, $n \in \mathbb{N}$ it seems like it is increasing. This sequence is clearly bounded from above since the number of all triangles in $\{0,...,n\}^3$ is not more than $(n+1)^9$ and so $a_n \leq 9$. Numerically, the best upper-bound for $a_n$ seems to be 5 which is equivalent to saying that $ET(n) \leq (n+1)^5$ for all $n \in \mathbb{N}$.

From what we have seen before each class of triangles determined by $a, b, c, d$, a solutions of (1), brings in a contribution that is a polynomial in terms of $n$. If we add these polynomials together, we get a polynomial which can be expressed is in the variable $\zeta = n - 1$ $(n = \zeta + 1)$ as follows:

\[
\begin{align*}
n &= 1: \quad p_1(\zeta) &= 8\zeta^3 + 24\zeta^2 + 24\zeta + 8, \quad ET(1) = p_1(0) = 8; \\
n &= 2: \quad p_2(\zeta) &= p_1(\zeta) + 16 + 48(\zeta - 1) + 16(\zeta - 1)^3 + 48(\zeta - 1)^2, \quad ET(2) = p_2(1) = 80; \\
n &= 3: \quad p_3(\zeta) &= p_2(\zeta) + 24 + 72(\zeta - 2) + 24(\zeta - 2)^3 + 72(\zeta - 2)^2, \quad ET(3) = p_3(2) = 368; \\
n &= 4: \quad p_4(\zeta) &= p_3(\zeta) + 128 + 312(\zeta - 3) + 56(\zeta - 3)^3 + 240(\zeta - 3)^2, \quad ET(4) = p_4(3) = 1264; \\
n &= 5: \quad p_5(\zeta) &= p_4(\zeta) + 400 + 120(\zeta - 4) + 120(\zeta - 4)^2, \quad ET(5) = p_5(4) = 3448; \\
n &= 6: \quad p_6(\zeta) &= p_5(\zeta) + 48 + 144(\zeta - 5) + 144(\zeta - 5)^2, \quad ET(6) = p_6(5) = 7792; \\
n &= 7: \quad p_7(\zeta) &= p_6(\zeta) + 776 + 1392(\zeta - 5) + 128(\zeta - 6)^3 + 744(\zeta - 6)^2, \quad ET(7) = p_7(6) = 16176; \\
n &= 8: \quad p_8(\zeta) &= p_7(\zeta) + 232 + 552(\zeta - 7) + 408(\zeta - 7)^2, \quad ET(8) = p_8(7) = 30606; \\
n &= 9: \quad p_9(\zeta) &= p_8(\zeta) + 360 + 840(\zeta - 8) + 120(\zeta - 8)^3 + 600(\zeta - 8)^2, \quad ET(9) = p_9(8) = 54216; \\
n &= 10: \quad p_{10}(\zeta) &= p_9(\zeta) + 80 + 80(\zeta - 9)^3 + 240(\zeta - 9)^2 + 240(\zeta - 9), \quad ET(10) = p_{10}(9) = 90104;
\end{align*}
\]

We conjecture that in general

\[
(7) \quad p_n(\zeta) = p_{n-1}(\zeta) + u_n(\zeta - n + 1)^3 + v_n(\zeta - n + 1)^2 + w_n(\zeta - n + 1) + s_n, \quad n \in \mathbb{N},
\]
with $u_n, v_n, w_n$, and $s_n$ non-negative integers.

As the graph above of $n \to \frac{\ln ET(n)}{\ln(n+1)}$ suggests, the second conjecture is that the following limit exists

$$
\lim_{n \to \infty} \frac{\ln ET(n)}{\ln(n+1)} = C.
$$

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