Canonical Duality Theory for Topology Optimization

David Yang Gao and Ming Li

Abstract

This paper presents a novel canonical duality approach for solving a general topology optimization problem of nonlinear elastic structures. We first show that by using finite element method, this most challenging problem can be formulated as a bi-level mixed integer nonlinear programming problem, i.e. for a given deformation, the upper-level optimization is a typical linear constrained 0-1 programming problem, while for a given structure, the lower-level optimization is a general nonlinear continuous minimization problem in computational nonlinear elasticity. Due to the integer constraint, even the most linear 0-1 programming problems are considered to be NP-hard in global optimization. However, by using canonical duality theory we proved that the upper-level 0-1 programming problem in topology optimization can be solved analytically to obtain exact integer solution. A perturbed canonical dual algorithm is proposed and illustrated by benchmark problems in topology optimization. Numerical results show that the proposed approach produces smaller target compliance, does not produce any gray elements, and the checkerboards issue is much reduced.

1 General Topology Optimization Problem and Challenges

Topology optimization aims to distribute material within a design domain so that the designed product’s physical properties can be optimized subject to certain geometric or physical constraints. Due to its wide potential engineering applications and its intrinsic mathematical challenges, this fundamental engineering problem has attracted researchers’ great interest since the seminal paper by Bendsoe and Kikuch [3]. The challenging problem is mathematically formulated in a discretized design domain of a large number of finite elements, taking as the design space, and the optimization goal is then to decide the void-solid or 0-1 distribution of these elements for
maximal design performance, under volume fraction constraints or other types of constraints.

Let us consider an elastically deformable body that in an undeformed configuration occupies an open domain \( \Omega \subset \mathbb{R}^d \) with (Lipschitz) boundary \( \Gamma = \partial \Omega \). We assume that the body is subjected to a body force \( f \) (per unit mass) in the reference domain \( \Omega \) and a given surface traction \( t(x) \) of dead-load type on the boundary \( \Gamma_t \subset \partial \Omega \), while the body is fixed on the remaining boundary \( \Gamma_u = \partial \Omega \cap \Gamma_t \).

Based on the minimal potential principle in continuum mechanics, the topology optimization of compliance minimization problem of this elastic body can be formulated in the following coupled minimization problem

\[
(P) : \min_{u \in U_a} \min_{\rho \in Z} \left\{ \Pi(u, \rho) = \int_{\Omega} W(\nabla u) \rho \, d\Omega + \int_{\Omega} u \cdot f \rho \, d\Omega - \int_{\Gamma_t} u \cdot t \, d\Gamma \right\},
\]

where the unknown \( u : \Omega \rightarrow \mathbb{R}^d \) is a displacement vector field, the design variable \( \rho(x) \in \{0, 1\} \) is a discrete scalar field, the stored energy per unit reference volume \( W(D) \) is a nonlinear differentiable function of the deformation gradient \( D = \nabla u \). The notation \( U_a \) identifies a \textit{kinematically admissible space} of deformations, in which, certain geometrical/boundary conditions are given, and \( Z = \{ \rho(x) : \Omega \rightarrow \{0, 1\} | \int_{\Omega} \rho(x) \, d\Omega \leq V_c \} \) is a design feasible space, in which, \( V_c > 0 \) is the desired volume.

Mathematically speaking, the topology optimization \((P)\) is a coupled nonlinear-discrete minimization problem in infinite-dimensional space. For large deformation problems, the stored energy \( W(D) \) is usually nonconvex. The criticality condition of this minimization problem leads to a nonlinear system of highly coupled partial differential equations. It is fundamentally difficult to analytically solve this type of problems. Numerical methods must be adopted.

Finite element method is the most popular numerical approach for topology optimization, by which, the domain \( \Omega \) is divided into \( n \) disjointed elements \( \{\Omega_e\} \) and in each element, the unknown fields can be numerically discretized as

\[
u(x) = N_e(x)u_e, \quad \rho(x) = \rho_e \in \{0, 1\} \quad \forall x \in \Omega_e,
\]

where \( N_e \) is an interpolation matrix, \( u_e \) is a nodal displacement vector, the binary design variable \( \rho_e \in \{0, 1\} \) is used for determining whether the element \( \Omega_e \) is a void (\( \rho_e = 0 \)) or a solid (\( \rho_e = 1 \)). Thus, by substituting (2) into \( \Pi(u, \rho) \) and let \( \mathcal{A}_a \subset \mathbb{R}^m \) be an admissible nodal displacement space,

\[
\mathcal{A}_a = \left\{ \rho = \{\rho_e\} \in \{0, 1\}^n | V(\rho) = \sum_{e=1}^n \rho_e \Omega_e \leq V_c \right\},
\]

the variational problem \((P)\) can be numerically reformulated the following global optimization problem
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\[
(\mathcal{P}_h) : \min_{u \in \mathbb{R}^m} \min_{\rho \in \mathcal{X}_d} \{ \Pi_h(u, \rho) = C(\rho, u) - u^T f(\rho) \},
\]

where

\[
C(\rho, u) = \rho^T c(u), \quad c(u) = \left\{ \int_{\Gamma^e} W(\nabla N(x)u_e)d\Omega \right\} \in \mathbb{R}^n,
\]

\[
f(\rho) = \left\{ \int_{\Gamma^e} \rho_e N_e(x)^T b_e(x)d\Omega \right\} + \left\{ \int_{\Omega^t} \nabla(x)^T t(x)d\Gamma \right\} \in \mathbb{R}^m.
\]

Clearly, this discretized topology optimization involves both the continuous variable \( u \in \mathbb{R}^m \) and the integer variable \( \rho \in \mathcal{X}_d \), it is the so-called mixed integer nonlinear programming problem (MINLP) in mathematical programming. Since \( \rho^p = \rho_e \quad \forall \rho_e \in \{0, 1\}, \quad \forall p \in \mathbb{R} \), we have

\[
C_p(\rho, u) := \sum_{e=1}^{n} \rho^p_e c_e(u) = (\rho \circ \ldots \circ \rho)^T c(u) = C(\rho, u) \quad \forall p \in \mathbb{R},
\]

where \( \rho \circ c = \{\rho_e c_e\} \) represents the Hadamard product. Particularly, for \( p = 2 \), we write

\[
C_2(\rho, u) := \frac{1}{2} \rho^T A(u) \rho, \quad A(u) = 2 \text{Diag}\{c(u)\}.
\]

Clearly, \( C_2(\rho, u) \) is a convex function of \( \rho \) since \( A(u) \geq 0 \quad \forall u \in \mathbb{R}^m \). By the facts that \( \rho \in \mathcal{X}_d \) is the main design variable and the displacement \( u \) depends on each given domain \( \Omega \), the problem \( (\mathcal{P}_h) \) is actually a so-called bi-level programming problem:

\[
(\mathcal{P}_{bl}) : \min_{\rho \in \mathcal{X}_d, u \in \mathbb{R}^m} C_p(\rho, u) \quad \text{s.t.} \quad u = \arg \min_{v \in \mathbb{R}^m} \Pi(v, \rho).
\]

In this formulation, \( C_p(\rho, u) \) represents the upper-level cost function and the total potential energy \( \Pi(u, \rho) \) represents the lower-level cost function. For large deformation problems, the total potential energy \( \Pi \) is usually a nonconvex function of \( u \). Therefore, this bi-level optimization could be the most challenging problem in global optimization.

For linear elastic structures, the total potential energy \( \Pi \) is a quadratic function of \( u \)

\[
\Pi(u, \rho) = \frac{1}{2} u^T K(\rho) u - u^T f(\rho)
\]

where \( K(\rho) = \{\rho_e K_e\} \in \mathbb{R}^{m \times m} \) is the overall stiffness matrix, which is obtained by assembling the sub-matrix \( \rho_e K_e \) for each element \( \Omega_e \). In this case, the lower-level optimization (10) is a convex minimization and for each given upper-level design variable \( \rho \), the lower-level solution is simply governed by the linear equilibrium equation \( K(\rho) u = f(\rho) \). Therefore, the topology optimization for linear elasticity is mathematically an linear constrained integer programming programming problem:
\[(\mathcal{P}_{le}) : \min_{\rho \in \mathbb{R}^N} \min_{\mathbf{u} \in \mathbb{R}^m} \{ C_p(\rho, \mathbf{u}) | \mathbf{K}(\rho) \mathbf{u} = f(\rho) \}. \]  

(12)

Particularly, it is a mixed integer linear programming if \( p = 1 \). However, due to the integer constraint, even this most simple linear programming problem has been considered as one of the most challenging tasks in topology optimization. Only problems in very small scale have been tackled to global optimality [18].

In order to overcome the combinatorics complexity of (12), various approximations were proposed in the last decades to resolve it, including homogenization [3], density-based [2], level set [19], topological derivative [17] and so on. These approaches essentially tries to reformulate the problem into a continuous parameter optimization problem using size, density or shape, which can be solved efficiently based on gradient-based optimization algorithms. A very good and recent survey on topology optimization approaches is also referred to [16]. The approaches have produced various excellent results in the last decades using the approaches, and have also found industrial applications.

The so-called Simplified Isotropic Material with Penalization (SIMP) is one of the most popular approaches in topology optimization:

\[(SIMP) : \min_{\rho \in \mathbb{R}^N} C_p(\rho, \mathbf{u}(\rho)) \]  

\( s.t. \quad \mathbf{K}(\rho^p) \mathbf{u} = f(\rho), \ V(\rho) \leq V_c, \)  

\( 0 < \rho_e \leq 1, \ e = 1, \ldots, n \)  

(13)

(14)

(15)

where \( p \) is the so-called penalization parameter in topology optimization.

The SIMP formulation has been studied extensively in topology optimization and numerous research papers have been produced during the past decades. By the fact that \( \rho^p = \rho \ \forall p \in \mathbb{R}, \ \forall \rho \in \{0,1\}^n \), we can see that the integer constraint \( \rho \in \{0,1\}^n \) in \( (\mathcal{P}_{le}) \) is simply replaced by the box constraint \( \rho \in (0,1]^n \). Although it was discovered by engineers that the “magic number” \( p = 3 \) can ensure good convergence to almost 0-1 solutions, the SIMP formulation \( (SIMP) \) is not mathematically equivalent to the topology optimization problem \( (\mathcal{P}_{le}) \). Actually, in many real-world applications, most SIMP solutions \( \{\rho_e\} \) are only approximate to 0 or 1 but never be exactly 0 or 1. Correspondingly, these elements are in gray scale which have to be filtered or interpreted physically. Additionally, this method suffers some key limitations such as global optimization not ensured, gray scale element produced, and lacking rigorous mathematical verification, which will be discussed during the following part of the paper.

2 Canonical Dual Problem and Analytical Solution

In order to overcome these issues mentioned above, a novel discrete approach to resolve the topology optimization is proposed here using the canonical duality-triality theory, CDT theory for short. Canonical dual finite element methods for
solving elasto-plastic structures and large deformation problems have been studied since 1988 [5, 6]. Applications to nonconvex mechanics are given recently for post-buckling problems [1, 4, 13]. Our attention in this paper mainly focuses on the canonical duality theory for solving the challenging integer programming problem in \((P_{bl})\).

Let \(\text{a} = \{a_e = \text{Vol}(\Omega_e)\} \in \mathbb{R}^n\), where \(\text{Vol}(\Omega_e)\) represents the volume of each element \(\Omega_e\). Then we have \(\mathcal{Z}_a = \{\rho \in \{0, 1\}^n \mid \rho^T \text{a} \leq V_c\}\). For a given optimal solution \(u\) to the lower-level programming problem, the primal problem for the upper-level optimization in \((P_{bl})\) can be written in the following form (\((P)\) for short):

\[
(P): \min \{P(\rho) = C_p(\rho, u) \mid \rho \in \mathcal{Z}_a\}.
\]

This is a typical quadratic integer programming problem for \(p = 2\). The canonical duality theory for solving this type of problems was first proposed by Gao in 2007 [9]. The key idea of this theory is the introducing of a canonical measure

\[
\xi = \Lambda(\rho) = \{\rho \circ \rho - \rho, \rho^T \text{a} - V_c\} : \mathbb{R}^n \rightarrow \mathcal{E} = \mathbb{R}^{n+1}.
\]

Let

\[
\mathcal{E}_a := \{\xi = \{\varepsilon, \nu\} \in \mathbb{R}^{n+1} \mid \varepsilon \leq 0, \nu \leq 0\}
\]

be a convex cone in \(\mathbb{R}^{n+1}\). Its indicator \(\Psi(\xi)\) is defined by

\[
\Psi(\xi) = \begin{cases} 0 & \text{if} \ \xi \in \mathcal{E}_a, \\ +\infty & \text{otherwise} \end{cases}
\]

which is a convex and lower semi-continuous (l.s.c) function in \(\mathbb{R}^{n+1}\). By this function, the primal problem can be relaxed in the following unconstrained minimization form:

\[
\min \{ \Phi(\rho) = C_p(\rho, u) + \Psi(\Lambda(\rho)) \mid \rho \in \mathbb{R}^n \}.
\]

Due to the convexity of \(\Psi(\xi)\), its conjugate function can be defined uniquely by the Fenchel transformation:

\[
\Psi^*(\zeta) = \sup_{\xi \in \mathbb{R}^{n+1}} \{\xi^T \zeta - \Psi(\xi)\} = \begin{cases} 0 & \text{if} \ \zeta \in \mathcal{E}_a^* \\ +\infty & \text{otherwise} \end{cases}
\]

where \(\mathcal{E}_a^* = \{\zeta = \{\sigma, \varsigma\} \in \mathbb{R}^{n+1} \mid \sigma \geq 0, \varsigma \geq 0\}\) is the dual space of \(\mathcal{E}_a\). Thus, by using the Fenchel-Young equality \(\Psi(\xi) + \Psi^*(\zeta) = \xi^T \zeta\), the function \(\Phi(\rho)\) can be written in the Gao-Strang total complementary function [11]

\[
\Xi(\rho, \zeta) = C_p(\rho, u) + \Lambda(\rho)^T \zeta - \Psi^*(\zeta).
\]

Based on this function, the canonical dual of \(\Phi(\rho)\) can be defined by

\[
\Phi^d(\zeta) = \text{sta} \{\Xi(\rho, \zeta) \mid \rho \in \mathbb{R}^n\} = C^A_p(\zeta, u) - \Psi^*(\zeta)
\]
where sta \( \{ f(x) \mid x \in X \} \) stands for finding a stationary value of \( f(x) \), \( \forall x \in X \), and

\[
C_p^A(\zeta, u) = \text{sta} \{ \Lambda(p)^T \zeta + C_p(p, u) \}
\]

(23)
is the \( \Lambda \)-conjugate of \( C_p(p, u) \).

Its format depends on \( p \in \mathbb{R} \) and, particularly, for \( p = 1, 2 \), we have

\[
C_1^A(\zeta, u) = -\frac{1}{4} \tau_1^T(\zeta, u)G_1^{-1}(\zeta)\tau_1(\zeta, u) - \zeta V_c
\]

(24)

\[
C_2^A(\zeta, u) = -\frac{1}{4} \tau_2^T(\zeta)G_2^{-1}(\zeta, u)\tau_2(\zeta) - \zeta V_c,
\]

(25)

where

\[
G_1(\zeta) = \text{Diag}\{\sigma\}, \quad \tau_1(\zeta, u) = \sigma - \zeta a - c(u);
\]

\[
G_2(\zeta, u) = \text{Diag}\{c(u) + \sigma\}, \quad \tau_2(\zeta) = \sigma - \zeta a.
\]

Clearly, \( C_p^A(\zeta) \) is well-defined if \( \det G_p \neq 0 \), i.e. \( \sigma \neq 0 \in \mathbb{R}^n \) for \( p = 1 \) and \( \sigma \neq -c(u) \) for \( p = 2 \). Let \( \mathcal{A}_p = \{ \zeta \in \mathbb{R}^n \mid \det G_p \neq 0 \} \). We have the following standard result in the canonical duality theory:

**Theorem 1 (Complementary-Dual Principle).** For any given \( p \in \mathbb{R} \) and \( u \in \mathcal{A}_p^m \), if \( (\bar{p}, \bar{\zeta}) \) is a KKT point of \( \Xi \), then \( \bar{p} \) is a KKT point of \( \Phi \), \( \bar{\zeta} \) is a KKT point of \( \Phi^d \), and

\[
\Phi(\bar{p}) = \Xi(\bar{p}, \bar{\zeta}) = \Phi^d(\bar{\zeta}).
\]

(26)

**Proof.** By the convexity of \( \Psi^d(\zeta) \), we have the following canonical duality relations:

\[
\zeta \in \partial \Psi^d(\xi) \iff \xi \in \partial \Psi^*(\zeta) \iff \Psi^d(\xi) + \Psi^*(\zeta) = \xi^T\zeta,
\]

(27)

where

\[
\partial \Psi^d(\xi) = \begin{cases} 0 & \text{if } \xi \in \mathcal{A}_p \\ \emptyset & \text{otherwise} \end{cases}
\]

is the sub-differential of \( \Psi^d \). Thus, in terms of \( \xi = \Lambda(p) \) and \( \zeta = \{\sigma, \zeta\} \), the canonical duality relations (27) can be equivalently written as

\[
\rho \circ \rho - \rho \leq 0 \iff \sigma \geq 0 \iff \sigma^T(\rho \circ \rho - \rho) = 0
\]

(28)

\[
\rho^T a - V_c \leq 0 \iff \zeta \geq 0 \iff \zeta(\rho^T a - V_c) = 0.
\]

(29)

These are exactly the KKT conditions for the inequality constraints \( \rho \circ \rho - \rho \leq 0 \) and \( \rho^T a - V_c \leq 0 \). Thus, \( (\bar{p}, \bar{\zeta}) \) is a KKT point of \( \Xi \) if and only if \( \bar{\zeta} \) is a KKT point of \( \Phi \). \( \bar{\zeta} \) is a KKT point of \( \Phi^d \). The equality (26) holds due to the canonical duality relations in (27). \( \Box \)

Indeed, on the effective domain \( \mathcal{A}_p^* \) of \( \Psi^d(\zeta) \), the total complementary function \( \Xi \) can be written as

\[
\Xi(\rho, \sigma, \zeta) = C_p(\rho, u) + \sigma^T(\rho \circ \rho - \rho) + \zeta(\rho^T a - V_c),
\]

(30)
which can be considered as the Lagrangian of \(\mathcal{P}\) for the canonical constraint \(\Lambda(\rho) \leq 0 \in \mathbb{R}^{a+1}\). The Lagrange multiplier \(\zeta = \{\sigma, \zeta\} \in \mathcal{E}_a^+\) must satisfy the KKT conditions in (28) and (29). By the complementarity condition \(\sigma^T (\rho \circ \rho - \rho) = 0\) we know that \(\rho \circ \rho = \rho\) iff \(\sigma > 0\). Let

\[
\mathcal{N}_a^+ = \{\zeta = \{\sigma, \zeta\} \in \mathcal{E}_a^+ | \sigma > 0\}.
\]

Then for any given \(\zeta = \{\sigma, \zeta\} \in \mathcal{N}_a^+\), the function \(\Xi(\cdot, \zeta) : \mathbb{R}^m \to \mathbb{R}\) is strictly convex and for a given \(\rho = 1, 2\), the canonical dual function of \(P\) can be well-defined by

\[
P^d(\zeta) = \min_{\rho \in \mathbb{R}^m} \Xi(\rho, \zeta) = C^A_\rho(\zeta, u) \forall \zeta \in \mathcal{N}_a^+.
\]

Thus, the canonical dual problem of \(\mathcal{P}\) can be proposed as the following:

\[
(\mathcal{P}^d) : \max \{P^d(\sigma, \zeta) | (\sigma, \zeta) \in \mathcal{N}_a^+\}.
\]

**Theorem 2 (Analytical Solution).** For any given \(u \in \mathbb{R}_a^m\) and either \(p = 1\) or \(p = 2\), if \(\bar{\zeta}\) is a solution to \(\mathcal{P}^d\), then

\[
\bar{\rho} = \begin{cases} 
\frac{1}{2} G_1^{-1}(\bar{\zeta}) \tau_1(\bar{\zeta}, u) & \text{for } p = 1,
\frac{1}{2} G_2^{-1}(\bar{\zeta}) \tau_1(\bar{\zeta}) & \text{for } p = 2
\end{cases}
\]

is a global optimal solution to \(\mathcal{P}\) and

\[
P(\bar{\rho}) = \min_{\rho \in \mathbb{R}^m} P(\rho) = \max_{\zeta \in \mathcal{N}_a^+} P^d(\zeta) = P^d(\bar{\zeta}).
\]

**Proof.** It is easy to prove that for any given \(u \in \mathbb{R}_a^m\), the canonical dual function \(P^d(\zeta)\) is concave on the open convex set \(\mathcal{N}_a^+\). If \(\bar{\zeta}\) is a KKT point of \(P^d(\zeta)\), then it must be a unique global maximizer of \(P^d(\zeta)\) on \(\mathcal{N}_a^+\). By Theorem 1 we know that if \(\bar{\zeta} = \{\bar{\sigma}, \bar{\zeta}\} \in \mathcal{N}_a^+\) is a KKT point of \(\Phi^d(\zeta)\), then \(\bar{\rho} = \rho(\bar{\zeta})\) defined by (34) must be a KKT point of \(\Phi(\rho)\). Since \(\Xi(\rho, \zeta)\) is a saddle function on \(\mathbb{R}^n \times \mathcal{N}_a^+\), we have

\[
\min_{\rho \in \mathbb{R}^n} \Phi(\rho) = \min_{\rho \in \mathbb{R}^n} \max_{\zeta \in \mathcal{N}_a^+} \Xi(\rho, \zeta) = \max_{\zeta \in \mathcal{N}_a^+} \min_{\rho \in \mathbb{R}^n} \Xi(\rho, \zeta)
\]

\[
= \max_{\zeta \in \mathcal{N}_a^+} \Phi^d(\zeta) = \max_{\zeta \in \mathcal{N}_a^+} P^d(\zeta).
\]

Since \(\bar{\sigma} > 0\), the complementarity condition in (28) leads to

\[
\bar{\rho} \circ \bar{\rho} - \bar{\rho} = 0 \quad \text{i.e. } \bar{\rho} \in \{0, 1\}^n.
\]

Thus, we have

\[
P(\bar{\rho}) = \min_{\rho \in \mathcal{N}_a} P(\rho) = \max_{\zeta \in \mathcal{N}_a^+} P^d(\zeta) = P^d(\bar{\zeta})
\]

as required. \(\square\)
Remark 1. The indicator function of a convex set and its sub-differential were first introduced by J.J. Moreau in 1968 in his study on unilateral constrained problems in contact mechanics [12]. His pioneering work laid a foundation for modern analysis and the canonical duality theory. In solid mechanics, the indicator of a plastic yield condition is also called a super-potential. Its sub-differential leads to a general constitutive law and a unified pan-penalty finite element method in plastic limit analysis [5]. In mathematical programming, the canonical duality leads to a unified framework for nonlinear constrained optimization problems in multi-scale systems [8, 7, 10].

3 Perturbed Canonical Duality Method and Algorithm

Numerically speaking, although the global optimal solution of the integer programming problem \((\mathcal{P})\) can be obtained by solving the canonical dual problem \((\mathcal{P}_d)\), the rate of convergence is very slow since \(P_d^\beta(\sigma, \zeta)\) is nearly a linear function of \(\sigma \in \mathcal{S}_a^+\) when \(\sigma\) is far from its origin. In order to overcome this problem, a so-called \(\beta\)-perturbed canonical dual method has been proposed by Gao and Ruan in integer programming [10], i.e. by introducing a perturbation parameter \(\beta > 0\), the problem \((\mathcal{P}_d)\) is replaced by

\[
(\mathcal{P}_d^\beta): \max \left\{ P_d^\beta(\sigma, \zeta) = P_d(\sigma, \zeta) - \frac{1}{4} \beta^{-1} \sigma^T \sigma \mid \{\sigma, \zeta\} \in \mathcal{S}_a^+ \right\}
\]

(36)

which is strictly concave on \(\mathcal{S}_a^+\). For \(p = 1\), this perturbed canonical dual function reads

\[
P_d^\beta(\sigma, \zeta) = - \sum_{e=1}^n \left[ \frac{1}{4} (\sigma_e - c_e(u) - \zeta a_e)^2 \sigma_e^{-1} - \frac{1}{4} \beta^{-1} \sigma_e^2 \right] - \zeta V_c.
\]

(37)

Theorem 3. For a given \(u \neq 0 \in \mathbb{R}^m\) and \(V_c > 0\), there exists a \(\beta_c > 0\) such that for any given \(\beta \geq \beta_c\), the problem \((\mathcal{P}_d^\beta)\) has a unique solution \(\zeta^\beta \in \mathcal{S}_a^+\). If \(\rho^\beta = \frac{1}{2} G^{-1}(\zeta^\beta) \tau_1(\zeta^\beta, u) \in (0, 1]^n\), then \(\rho^\beta\) is a global optimal solution to \((\mathcal{P})\).

Proof. It is easy to show that for any given \(\beta > 0\), \(P_d^\beta(\zeta)\) is strictly concave on the open convex set \(\mathcal{S}_a^+\), i.e. \((\mathcal{P}_d^\beta)\) has a unique solution. Particularly, for \(p = 1\), the criticality condition \(\nabla P_d^\beta(\zeta) = 0\) leads to the following canonical dual algebraic equations:

\[
2 \beta^{-1} \sigma_e^3 + \sigma_e^2 = (\zeta a_e + c_e)^2, \quad e = 1, \ldots, n,
\]

(38)

\[
\sum_{e=1}^n \frac{1}{2} \frac{a_e}{\sigma_e} (\sigma_e - a_2 \zeta - c_e) - V_c = 0.
\]

(39)

It has been proved in [8] that for any given \(\beta > 0\) and \(\theta_e = \zeta a_e + c_e(u_e) \neq 0\), \(e = 1, \ldots, n\), the canonical dual algebraic equation (38) has a unique positive real solu-
The canonical dual algebraic equation (39) has a unique solution
\[ \zeta = \sum_{e=1}^{n} \frac{a_e (1 - c_e / \sigma_e)}{\sum_{e=1}^{n} a_e^2 / \sigma_e}. \] (41)
This shows that the perturbed canonical dual problem \((P_{\beta})\) has a unique solution
in \(S^+_n\), which can be analytically obtained by (40) and (41). The rest proof of this theorem is similar to that given in [10]. □

Theoretically speaking, for any given \(V_c < V_o\), the perturbed canonical duality method can produce global optimal solution to the integer constrained problem \((P)\). However, if \(V_c \ll V_o\), to reduce the initial volume \(V_o\) directly to \(V_c\) by solving the bi-level topology optimization problem \((P_{bl})\) may lead to unreasonable solutions. In order to resolve this problem, a volume decreasing control parameter \(\mu \in (V_c / V_o, 1)\) is introduced to slowly reduce the volume in the iteration. Thus, based on the above strategies, the canonical duality algorithm (CDA) for solving the general topology optimization problem \((P_{bl})\) can be proposed below.

**Algorithm 1 (Canonical Dual Algorithm)**

(I) Initialization. Let \(\rho^0 = \{1\} \in \mathbb{R}^n\). Find \(u^0\) by solving the sub-level optimization problem
\[ u^0 = \arg\min \{\Pi(u, \rho^0) | \ u \in \mathbb{R}^n\}. \] (42)
Compute \(c^0 = c(u^0)\). Define an initial value \(\zeta_0 > 0\) and an initial volume \(V_\gamma \in [V_c, V_o]\). Let \(\gamma = 0\), \(k = 1\).

(II) Find \(\sigma_k = \{\sigma_e^k\}\) by
\[ \sigma_e^k = \frac{1}{6} \beta [-1 + \phi(\zeta^{k-1}, u^{k-1}) + \phi^c(\zeta^{k-1}, u^{k-1})], \ e = 1, \ldots, n. \]

(III) Find \(\zeta^k\) by
\[ \zeta^k = \frac{\sum_{e=1}^{n} a_e (1 - c_e^k / \sigma_e^k) - 2V_\gamma}{\sum_{e=1}^{n} a_e^2 / \sigma_e^k}. \]

(IV) Find \(\rho^k\) by
\[ \rho_e^k = \frac{1}{2} [1 - (\zeta^k a_e + c_e^k) / \sigma_e^k], \ e = 1, \ldots, n. \]

(V) If
\[ |C(\rho^k, \mathbf{u}^\gamma) - C(\rho^{k-1}, \mathbf{u}^{\gamma-1})| \leq \omega_1, \]

and \( \sum_{e=1}^n \rho_e^k a_e \leq V_\gamma \), let \( \rho^\gamma = \rho^k \), go to (VI); otherwise, let \( k = k + 1 \), go to (II).

(VI) Find \( \mathbf{u}^{\gamma} \) by solving

\[ \mathbf{u}^{\gamma} = \arg\min \{ \Pi(\mathbf{u}, \rho^\gamma) \mid \mathbf{u} \in \mathcal{U}_a \} \tag{43} \]

(VII) Convergence test: If

\[ |C(\rho^\gamma, \mathbf{u}^\gamma) - C(\rho^{\gamma-1}, \mathbf{u}^{\gamma-1})| \leq \omega_2, \quad V_\gamma \leq V_c \]

then stop; otherwise, let \( V_{\gamma+1} = \mu V_\gamma \geq V_o \) and computing \( c^{\gamma+1} = c(\mathbf{u}^\gamma) \), Let \( \gamma = \gamma + 1, \ k = 1 \), go to (II).

Note: the parameters \( \mu < 1 \), \( \beta > 10 \) and the initial data \( \varsigma_0 \) are important for the iteration. Please make sure chose correct ones and have tests.

For linear elastic materials, the lower-level optimization (43) in the algorithm (CDA) can be simply replaced by

\[ \mathbf{u}^\gamma = K^{-1}(\rho^\gamma)\mathbf{f}(\rho^\gamma). \]

4 Numerical Examples for Linear Elastic Structures

The proposed approach is also implemented in Matlab. The obtained results are compared with those obtained via the classical SIMP approach computed via [14], at aspects on the value of the target compliance, the appearance of the gray elements, and the checkerboard issues [15]. Basically, various density filters have applied in the SIMP to avoid the checkerboard issues and to improve the optimization convergence. However, the SIMP approach applied here does not take any filter in order to explore the numerical aspects of the optimization approach. Note also that the proposed optimization approach does not use any filter. Comparisons between the two approaches in cases of applying filters will be done in our future work.

4.1 MBB Beam Problem

The well-known benchmark Messerschmitt-Bölkow-Blohm (MBB) beam problem in topology optimization is selected as the first test example (see Fig. 1). The design domain is discretized with \( 180 \times 60 \) quad mesh elements.
Fig. 1 The design domain, boundary conditions and external loads for the optimization of a MBB

For the purpose of illustration, the material, load and geometry data are chosen as dimensionless. Young’s modulus and Poisson’s ratio of the solid material are taken as $E = 1$ and $\nu = 0.3$, respectively. The lower bound of the Young’s modulus variable is $E_{min} = 10^{-9}$ in numerical implementation. We test performance of the example in case of with or without filter as shown in Tables 1 and 2, respectively. The volume fraction is 0.6 and the radius of filter is 2. The volume evolution rate of our approach is 0.2. As it can be seen from the results, in case of without filter, SIMP has a large range of checkerboard patterns and gray elements, while our approach does not have. On the other hand, in case of using filter, the number of iteration steps of our approach are much less than that of SIMP.

| Method | Structures | Steps | Compliance |
|--------|------------|-------|-------------|
| SIMP   |            | 41    | 169.2908    |
| CDT    |            | 29    | 165.0746    |

Table 1 The comparison between SIMP and ours with $\nu = 0.6$ without filter.

| Method | Structures | Steps | Compliance |
|--------|------------|-------|-------------|
| SIMP   |            | 225   | 166.5211    |
| CDT    |            | 39    | 166.1418    |

Table 2 The comparison between SIMP and ours with $\nu = 0.6$ with a filter of radius $r = 2$. 
4.2 Cantilever Beam

The second test example is the classical Cantilever problem (see Figure 2). The example consists of $180 \times 60$ quad meshes and the target volume fraction is 0.5. The model is fixed along its left side, and a downward traction is applied at its right middle point. The obtained results of the proposed approach and of the SIMP approach are respectively shown in Figure 3(a) and (d) together with their close-up at two different regions in Figures 3(b),(c) and (e),(f). Their computed target compliances are respectively 184.3573 and 173.1323, the proposed approach showing a much smaller value. In addition using the proposed approach the checkerboard issue is almost avoided completely and no gray elements are generated. As a comparison, the SIMP approach, without the usage of any filter, has a large range of checkerboards and gray elements.

Fig. 2 A test example of the benchmark Cantilever problem at volume fraction of 0.5.

Fig. 3 Comparison between results obtained by the proposed approach (CDT) with those obtained by the SIMP approach for the problem defined in Figure 2 on a mesh of size $180 \times 60$. 
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