Relevance of stochasticity for the emergence of quantization

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Abstract The theories of stochastic quantum mechanics and stochastic electrodynamics bring to light important aspects of the quantum dynamics that are concealed in the standard formalism. Here, we take further previous work regarding the connection between the two theories, to exhibit the role of stochasticity and diffusion in the process leading from the originally classical + zpf regime to the quantum regime. Quantumlike phenomena present in other instances in which a mechanical system is subject to an appropriate oscillating background that introduces stochasticity, may point to a more general appearance of quantization under such circumstances.

1 Introduction

In a recent paper [1], we have discussed the connections between stochastic quantum mechanics (SQM) and stochastic electrodynamics (SED), two statistical theories that study the dynamics of (otherwise classical) particles when embedded in a stochastic environment. In essence, both theories are shown to provide support in favor of a stochastic process underlying quantum mechanics.\textsuperscript{1}

The results of SQM and SED suggest that for a more satisfactory understanding of the mechanism of quantization it is essential to consider that matter is in permanent interaction with a physical medium that brings about a stochastic behavior of the system. While SQM does not specify the nature of such medium, SED in particular identifies it with the zero-point radiation field (ZPF). This is an ubiquitous random electromagnetic radiation field with energy per mode $\hbar \omega/2$, which accounts for the appearance of Planck’s constant and the wave element in quantum mechanics, as well as Born’s rule associated with it [15,16]. The ZPF has been shown to play an essential role in producing quantum features such as the so-called quantum indeterminism [17], entanglement [12,16,18] and others.

The present paper is devoted to a more in-depth discussion of the transit from the originally classical + ZPF regime to the quantum regime, and of the crucial role played by diffusion in bringing about such a qualitative change in the dynamics. The discussion suggests investigating other instances in which the permanent interaction of a particle with a vibrating field introduces a stochastic element into the dynamics, to look for possible signs pointing to the generality of quantization under such circumstances.

The structure of the paper is as follows. To set the framework, in Sect. 2, we succinctly recall the SED (statistical) treatment of a particle subject to an external potential, leading to a description in configuration space. In Sect. 3, we briefly introduce the basic equations of SQM and, by linking with the SED description, we complete the equations of SQM with the inclusion of the radiative corrections. We further show how the SED and SQM equations connect with the Schrödinger equation, and relate the quantum momentum operator with the local average velocities that are central elements in the SED and SQM approaches. This allows to stress the role of the ZPF-induced diffusion in driving the system to its non-classical behavior. In Sect. 4, a clue to understand the mechanism leading to a (statistical) description of the quantum regime in terms of operators and state vectors is put forward. The paper ends with a discussion on the relevance of the wave element associated with the source of stochasticity, which is also present in the walking-droplet systems that exhibit a quantum-like behavior [19,20].

\textsuperscript{1}There are of course a variety of theories containing a stochastic element, aimed to explain or reproduce quantum mechanics. These may widely differ from one another in their method, purpose or philosophy; see, e. g., [2–14]. It is not our intention to review such theories, but to draw on the specific connections between SQM and SED for a better understanding of the role of stochasticity in the emergence of quantization.

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2 The underlying equations of SED

SED provides a statistical description of the dynamics of a charged particle (typically an atomic electron) in interaction with the zpf, subject to an external potential and possibly to an external electromagnetic field. The conventional starting point of the theory is the Langevin equation, also known in SED as Bragford–Marshall equation, which is the nonrelativistic dipole approximation of the (stochastic) Abraham–Lorentz equation (for simplicity, we consider the case in which there is no external radiation field)

\[ m \ddot{x} = f(x) + m \tau \dot{x} + eE, \]

where \( m \tau \dot{x} \) represents the radiation reaction force with \( \tau = 2e^2/3mc^3 \) (~10^{-23} s for the electron), \( f = -\nabla V(x) \), and \( E \) represents the electric component of the (random) zpf. The latter is usually—but not necessarily—taken in the dipole approximation, \( E(t) \).

Since the zpf is an intrinsic component of the SED system, the canonical momentum is defined simply as \( p = m \dot{x} \), whence the equation of motion becomes

\[ \dot{p} = f + m \tau \dot{x} + eE(t). \]

In the absence of the zpf, we would have a purely classical electrodynamic problem. The presence of the term \( eE \) renders the problem stochastic and therefore amenable to a statistical treatment only. A standard procedure (see e.g. [21], [16, Ch. 4]) leads to a generalized Fokker–Planck equation (GFPE) for the phase-space probability distribution \( Q(x,p,t) \) with a memory term,

\[ \left( \hat{L}_c + e^2 \hat{L}_t \right) Q(x,p,t) = 0, \]

where

\[ \hat{L}_c = \frac{\partial}{\partial t} + \frac{1}{m} \nabla \cdot p + \nabla \cdot f \]

stands for the classical Liouvillian and

\[ \hat{L}_t = \nabla \cdot \left( \frac{2}{3c^2} \dot{x} - \hat{D} \right) \]

for the radiative and diffusive terms. The latter contains the integro-differential operator, written here to lowest order in \( e^2 \),

\[ \hat{D} = \int_{-\infty}^{t} dt' \varphi(t-t')\nabla \varphi', \quad \varphi(t) = \frac{2\hbar}{3\pi c^3} \int_0^{\infty} d\omega \omega^3 \cos \omega t, \]

where \( p' = p(t') \) is the value of the momentum at \( t' < t \), such that it evolves towards \( p = p(t) \), and \( \varphi(t) \) stands for the zpf covariance.

Equation (3) describes the evolution of the phase-space probability density at all times. It is virtually impossible to construct its general solution, yet an approximate method has been developed that leads to a good (approximate) description for asymptotic times, when the average effects of the radiative and diffusive terms compensate each other, and they become then small radiative corrections (see [16] and references therein). As shown in detail in [22] (or [16, Ch. 4]), it is found that the corresponding (asymptotic) regime can be identified with the quantum regime (see Sect. 3.2).

To connect with SQM, we first reduce the SED description to the configuration space by integrating over the momentum. The local mean value of a dynamical variable \( G(x,p) \) is given by

\[ \langle G \rangle_x \equiv \frac{1}{\rho} \int dp \ G(x,p) Q(x,p,t), \]

where \( \rho = \rho(x,t) = \int dp \ Q(x,p,t) \) stands for the probability density of particles in \( x \)-space. The equation of evolution for \( \langle G \rangle_x \) is obtained by left-multiplying the GFPE by \( G \) before integrating over \( p \). For \( G = 1 \), integration of (3) gives the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \]

with \( v = v(x,t) \) the flux (or current) velocity,

\[ v = \langle \dot{x} \rangle_x = \frac{1}{m} \langle p \rangle_x. \]

For \( G = p \) one gets, summing over repeated indices

\[ \frac{\partial}{\partial t} \left( m v \rho + m^2 \partial_j \langle \dot{x} \rangle_x \right) \rho - \langle f \rangle_x \rho \]

\[ = e^2 \left( \frac{2}{3c^3} \langle \dot{x}^2 \rangle_x - \langle \dot{D} \rangle_x \right) \rho \equiv R_x. \]

3 Connecting with SQM

SQM describes the dynamics of a particle undergoing a stochastic motion, without further inquiry about the source of the stochasticity. The general equations of motion correspond to the time-inversion invariant and non-invariant stochastic extension, respectively, of Newton’s equation of motion, namely [1,15]

\[ m \left( \dot{\mathcal{D}}_c v - \lambda \mathcal{D}_x u \right) = f_+, \quad m \left( \dot{\mathcal{D}}_c u + \mathcal{D}_x v \right) = f_-, \]

where \( \lambda \) is a real parameter that can be taken as \( \lambda^2 = 1 \),

\[ \mathcal{D}_c = \frac{\partial}{\partial t} + v \cdot \nabla, \quad \mathcal{D}_x = u \cdot \nabla + D \nabla^2 \]
are the so-called systematic and stochastic derivatives, respectively, \( \mathbf{v} \) is the flux velocity, and \( \mathbf{u} \) is the diffusive (or stochastic) velocity

\[
\mathbf{u}(\mathbf{x}, t) = D \frac{\nabla \rho}{\rho},
\]

(13)

with \( \rho(\mathbf{x}, t) \) the probability density of particles and \( D \) the diffusion constant. The forces \( \mathbf{f}_- \) and \( \mathbf{f}_+ \) in Eqs. (11) do and do not change sign, respectively, under a time inversion. The Newtonian limit (the classical Hamiltonian description) corresponds to \( D = 0 \), hence \( \mathbf{u} = \mathbf{0} \), which means no diffusion at all.

Equations (11) can be further combined into the single, compact equation

\[
\hat{D}_\kappa \mathbf{p}_\kappa = \mathbf{f}_\kappa,
\]

(14)

with

\[
\mathbf{p}_\kappa = m \mathbf{w}, \quad \mathbf{w} = \mathbf{v} - \sqrt{-\lambda} \mathbf{u}, \quad \mathbf{f}_\kappa = \mathbf{f}_+ - \sqrt{-\lambda} \mathbf{f}_-,
\]

(15)

\[
\hat{D}_\kappa = \hat{D}_\kappa - \sqrt{-\lambda} \hat{\mathbf{D}}_\kappa = \frac{\partial}{\partial t} + \frac{1}{m} \mathbf{p}_\kappa \cdot \nabla - \sqrt{-\lambda} D \nabla^2.
\]

(16)

The sign of \( \lambda \) serves to distinguish between the two basic stochastic processes in the Markovian approximation (see e.g. [23]): \( \lambda = -1 \) corresponds to the classical case (Brownian motion), and \( \lambda = 1 \) to the quantum case. Therefore, in what follows we take \( \lambda = 1 \). This means, in particular, that (see Eq. (15))

\[
\mathbf{w} = \mathbf{v} - i \mathbf{u}.
\]

(17)

We shall come back to this important equation below.

### 3.1 From SED to SQM

The link between the two theories is established by introducing the SQM expressions for the velocity \( \mathbf{u} \), Eq. (13), and the coefficient [15]

\[
D = \frac{\hbar}{2m},
\]

(18)

into the SED equation (10), and combining this with (8) to obtain

\[
mv_i \left( \frac{2m}{\hbar} \mathbf{u} \cdot \nabla + \nabla \cdot \mathbf{v} \right) + m \left( \frac{2m}{\hbar} \mathbf{u} \cdot (\dot{\mathbf{x}}_i)_x + \nabla \cdot (\dot{\mathbf{x}}_i)_x \right) = f_i + R_{xi}.
\]

(19)

for every Cartesian component \( i \). This suggests introducing the tensor \( T_{ij} \), given by the (local) correlation between the components of the vector \( \dot{\mathbf{x}} \),

\[
T_{ij} = -\frac{2m}{\hbar} \left( \langle \dot{x}_i \dot{x}_j x \rangle - v_i v_j \right)
= -\frac{2m}{\hbar} \left( \langle \dot{x}_i \dot{x}_j x \rangle - \langle \dot{x}_i x \rangle \langle \dot{x}_j x \rangle \right).
\]

(20)

Equation (19) takes then the simpler form (summing over repeated indices)

\[
m \left( \frac{\partial v_i}{\partial t} + v_j \partial_j v_i - T_{ij} u_j - \frac{\hbar}{2m} \partial_j T_{ij} \right) = f_i + R_{xi}.
\]

(21)

Notice that when \( R_{xi} \) is neglected, this equation is equivalent to the first equation in (11) with \( \lambda = 1 \), \( f_+ = \mathbf{f} \), and \( T_{ij} \) given by

\[
T_{ij} = \partial_j u_i = D \partial_j \partial_i \ln \rho = T_{ji},
\]

(22)

hence \( T_{ij} \) plays the role of a stress rate tensor associated with the local mean changes of \( \mathbf{u} \). By combining this with Eqs. (9), (13), (18) and (20) we get

\[
\langle p_i p_j \rangle_x - \langle p_i \rangle_x \langle p_j \rangle_x = \frac{\hbar^2}{4} \partial_i \partial_j \ln \rho = m^2 DT_{ij},
\]

(23)

which points to the significance of \( T_{ij} \), in particular of its trace \( \sum_i T_{ii} = \langle (m^2 D)^{-1} \sigma^2_\rho(x) \rangle \). Notice that the local dispersion of the momentum, \( \sigma^2_\rho(x) \), is determined by the (divergence of the) diffusive velocity alone.

Furthermore, the SED approach has provided us the means to arrive at the complete dynamical equation of SQM with the radiative terms included,

\[
\hat{D}_\kappa \mathbf{p}_\kappa = \mathbf{f}_\kappa + \mathbf{R}_x.
\]

(24)

### 3.2 Final step: the quantum description

Connecting with the Schrödinger equation is now essentially an algebraic exercise, which is accomplished by introducing the complex function \( \psi \) such that

\[
\mathbf{p}_\kappa = m (\mathbf{v} - i \mathbf{u}) = \hbar \left( \text{Im} \frac{\nabla \psi}{\psi} - i \text{Re} \frac{\nabla \psi}{\psi} \right) = -i \hbar \frac{\nabla \psi}{\psi},
\]

(25)

into Eq. (24) in the radiationless regime (i.e., without the term \( \mathbf{R}_x \)) and integrating once, to obtain

\[
\frac{1}{2m} (-i \hbar \nabla)^2 \psi + V \psi = i \hbar \frac{\partial \psi}{\partial t}.
\]

(26)

Note that according to Eq. (25), the momentum operator \( \hat{\mathbf{p}} = -i \hbar \nabla \) is directly related with the (complex)
velocity \( \mathbf{w} \) of SQM,

\[
\hat{p} \psi = -i\hbar \nabla \psi = m(\mathbf{v} - i\mathbf{w})\psi = m\mathbf{w} \psi,
\]

which shows that \( \mathbf{w} \) is the relevant velocity, or rather, that the two velocity components \( \mathbf{v} \) and \( \mathbf{u} \) play an equally important role in the dynamics. The fact that they both contribute to the average energy can be utilized to derive the time-independent Schrödinger equation from a variational principle. Indeed, from Eq. (27), the average kinetic energy can be written as

\[
\langle T \rangle = \frac{1}{2} m \int (\mathbf{v}^2 + \mathbf{u}^2) \rho \, d\mathbf{x} = \frac{1}{2} m \int (\mathbf{w} \psi) \cdot (\mathbf{w}^* \psi^*) \, d\mathbf{x}
\]

The total average energy is then

\[
E = \int \left[ \frac{\hbar^2}{2m} (\nabla \psi^*) \cdot (\nabla \psi) + V \psi^* \psi \right] \, d\mathbf{x},
\]

and by varying both \( \psi^* \) and \( \psi \), subject to the constraint imposed by the normalization condition, i.e.,

\[
\delta N = \int (\psi^* \delta \psi + \psi \delta \psi^*) \, d\mathbf{x} = 0,
\]

one obtains after an integration by parts, assuming a bounded system,

\[
\delta E = \int \left\{ \left[ -\frac{\hbar^2}{2m} \nabla^2 \psi^* + (V - \gamma) \psi \right] \delta \psi \right. \\
\left. + \left[ -\frac{\hbar^2}{2m} \nabla^2 \psi + (V - \gamma) \psi \right] \delta \psi^* \right\} \, d\mathbf{x},
\]

with \( \gamma \) a free real parameter. Since \( \delta N = 0 \), one may add a term \( (\gamma - E) (\psi^* \delta \psi + \psi \delta \psi^*) \) to the integrand of (30). The condition \( \delta E = 0 \) implies thus the (stationary) Schrödinger equation

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = E \psi
\]

and its complex conjugate, with \( E \) given by Eq. (29).

4 On the emergence of the quantum behavior

The above results highlight the importance of diffusion in eliciting a non-classical behaviour of the system. In what follows we delve into the physical mechanism by which the ZPF-induced diffusion leads to the quantum regime.

4.1 The equilibrium condition: effect of diffusion on the dynamics

Looking back at the GFPE, Eq. (3), one can see that what makes the system behave nonclassically are the two terms contained in \( \tilde{L}_t \), which means they deserve closer inspection. For this purpose, we multiply (3) by any constant of the motion \( G(x, p) = \xi \) and integrate over \( p \). The terms associated with the classical Liouvillian, \( \tilde{L}_c \xi \), cancel out, and those associated with \( \tilde{L}_e \xi \) must balance each other on average by virtue of the equilibrium condition \( \langle d\xi / dt \rangle = 0 \), i.e., (here \( g(x, p) = \nabla_\mathbf{p} \xi(x, p) \))

\[
- \langle \hat{x} \cdot \hat{g} \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \omega^3 \int_{-\infty}^t dt \cos(\omega(t - t')) \langle \nabla_{\mathbf{p}'} \cdot \mathbf{g} \rangle,
\]

where, as said before, \( \mathbf{p}' = \mathbf{p}(t') \) is the value of the momentum at \( t' < t \), such that it evolves towards \( \mathbf{p} = \mathbf{p}(t) \). This constitutes a strong condition on the dynamics; it implies that only those solutions of the radiationless (zero-order) part of the GFPE that satisfy this condition, are valid solutions in the equilibrium regime.

Although Eq. (32) holds only under stationarity, each side of it can be analyzed separately at all times. The l.h.s. term is due to radiation reaction, and therefore, represents the dissipative part. Take for instance the case in which \( \xi \) represents the energy. Initially (at \( t = -\infty \), when particles and ZPF start to interact), the dissipative term obviously dominates over the diffusive one. Were it not for the r.h.s. term, the particles would eventually exhaust their energy and come to a complete standstill. As time progresses, however, the diffusion of the momentum increases thanks to the action of the ZPF, until it reaches a point where the r.h.s. term does not depend on the time variable; this is the Markovian limit, well described by the equations of SQM with the second-order derivative term containing a constant diffusion coefficient \( D \).

The factor \( \nabla_{\mathbf{p}'} \cdot \mathbf{g} \) is at the core of the mechanism of evolution towards the balance regime; it signals the effects of the diffusion of \( \mathbf{p} \) due to the ZPF. Let us consider the upper time limit of the integral, \( t \), close to the asymptotic time. For small values of \( t' \) (close to the lower limit), the behavior of \( \mathbf{p}(t') \) is largely classical (not diffusive) and differs markedly from that of \( \mathbf{p}(t) \); yet this difference is blurred as time progresses. In classical mechanics (in the absence of diffusion), the quantity \( \nabla_{\mathbf{p}'} \cdot \mathbf{g} \) can be expressed in terms of the Poisson bracket

\[
\frac{\partial g_i(t)}{\partial p_j(t')} = \{x_i(t'), g_i(t)\}.
\]

This represents an abridged description of the classical evolution, which is purely deterministic, as opposed to that described by the GFPE, which is statistically deterministic, meaning that although the motion of individual particles follows deterministic rules, the evolution of the ensemble is defined only in a statistical sense. The r.h.s. of Eq. (32)—and with it the entire equation—ceases to follow classical Hamiltonian laws as soon as the diffusion enters into force. The new dynamics should reflect as a fundamental property the role played by dif-
fusion (which in usual quantum mechanics is considered under the notion of indeterminism). In the asymptotic limit, the expression $\nabla_{\mathbf{p}'} \cdot \mathbf{g}$, and the corresponding symplectic structure represented in the classical case by the Poisson bracket, must capture this fundamental change in the dynamics. This, in essence, is what justifies in SED the (otherwise pragmatic) transition from the Poisson bracket to the corresponding commutator, which serves to express in a language proper of a statistical treatment the meaning of the quantity $\nabla_{\mathbf{p}'} \cdot \mathbf{g}$:

$$
\frac{\partial g(t)}{\partial p_j(t')} \rightarrow \beta [\hat{x}_i(t'), \hat{g}_j(t)] ,
$$

(34)

with the operators acting on the state function $\psi$ that represents the ensemble under consideration. The value of the parameter $\beta$ is to be determined by the balance condition (32). For this purpose, we apply consistently in Eq. (32) the substitution rule (34), which for $t' = t$ and $\hat{g}_j = \hat{p}_j$ means

$$
\{x_i, p_j\} = \delta_{ij} \rightarrow \beta [\hat{x}_i, \hat{p}_j] = \delta_{ij} ,
$$

(35)

and take an average over the ensemble of systems in the ground state, viz. the (sole) state in equilibrium with the ZPF. For the sake of clarity, we restrict here the calculations to one dimension. With the matrix elements $x_{kn}(t) = x_{kn} \exp(\mathbf{i} \omega_{kn} t)$, $g_{kn} = -\beta(\xi_k - \xi_0) x_{kn}$, and taking into account that $\xi_{kn} = \xi_0 \delta_{kn}$ for any constant $\xi$, we obtain for the l.h.s. of (32),

$$
- \langle \mathbf{x} \cdot \mathbf{g} \rangle_0 = -i \beta \sum_k (\xi_k - \xi_0) \omega_{k0}^3 |x_{k0}|^2 ,
$$

(36)

which does not explicitly contain Planck’s constant. For the r.h.s. we get

$$
\frac{\hbar}{\pi} \int_0^\infty d\omega \omega^3 \int_{-\infty}^{\infty} dt \cos(\omega(t - t')) \langle \nabla_{\mathbf{p}'} \cdot \mathbf{g} \rangle_0 \\
= \hbar \beta^2 \sum_k (\xi_k - \xi_0) \omega_{k0}^3 |x_{k0}|^2 .
$$

(37)

The balance condition reads, therefore,

$$
- i \beta \sum_k (\xi_k - \xi_0) \omega_{k0}^3 |x_{k0}|^2 = \hbar \beta^2 \sum_k (\xi_k - \xi_0) \omega_{k0}^3 |x_{k0}|^2 ,
$$

(38)

whence $\beta = -i / \hbar$, and consequently,

$$
[\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij} .
$$

(39)

This result encapsulates in a most remarkable form the profound effect of the ZPF on the dynamics. SED endows thus the formal rule $[f, g] \rightarrow \frac{1}{i} [\hat{f}, \hat{g}]$ with a deep physical sense: $\hbar$ is the hallmark of the ZPF. The new dynamics—an extension of Hamiltonian dynamics that embodies the effects of fluctuations—becomes expressed in terms of operators, and refers no more to trajectories of particles moving in ordinary space, but to a statistical ensemble of them in a given state.\(^4\)

The physical description leaps from ordinary space into an abstract Hilbert space; from a transparent visualization into a formal representation. Here, the descriptions of Schrödinger and Heisenberg converge; the ensuing description is statistical (although neither of these authors knew it at their time), with wave functions living in the Hilbert space on which the operators act.\(^5\)

4.2 The undulatory element

Objections have been raised by some authors\(^{[29,30]}\) to the conceptualization of quantum mechanics as a classical Brownian-motion-type stochastic process, and rightly so. As has been demonstrated in Sect. 3.2, to arrive at the quantum description it is necessary to assign the value $+1$ to the parameter $\lambda$, whereas with $\lambda = -1$ one remains in the classical domain. Even if the two stochastic processes described by Eq. (14) are treated here in the same Markov (second order) approximation, they are of a totally different nature: one is classical, and is generated by a totally uncorrelated white noise. Quite on the contrary, the quantum one (corresponding to a positive $\lambda$, which gives rise to the appearance of complex quantities for its full description), is due to the action of a highly colored stochastic field, and follows its proper (quantum) laws.

The wavelike nature of the ZPF and the strong self-correlation of its modes, as manifest in the field covariance (6), are features that distinguish the quantum case from Brownian motion. As discussed above, initially the dynamics of the SED system is irreversible, until a balance is reached between the average effects of diffusion and dissipation; at that point the dynamics has become reversible and can be described in terms of stationary solutions. This suggests that it should be possible to imagine other instances in which a (material) system is acted on by a permanent oscillatory background field that induces a stochastic response; does this always imply that the system acquires wavelike

\[This is not a unique situation in theoretical physics; there are several (although related) examples in which stochasticity brings about a qualitative change in the dynamics. The title of section 2.4 of the forerunner paper by Chandrasekhar [27] reads “The Fokker–Planck Equation. The Generalization of Liouville’s Theorem.” The generalization at issue is just an extension of the Hamiltonian dynamics of Liouville’s theorem, so as to embed fluctuations and dissipation into the scheme. A more recent example, closer to our case, is the discovery by Nelson [3] of the need of two velocities for an appropriate description of the dynamics of a stochastic system, just as discussed above. However. Nelson continued to call his theory Newtonian.\]

\[In a separate work [28], it been shown that under conditions of ergodicity, the dynamical variables describing the statistical properties of an ensemble in a given (pure) state are expressed by the corresponding quantum operators; this represents a complementary derivation of quantum mechanics à la Heisenberg.\]
properties and eventually reaches a regime characterized by a quantum-like behavior? It remains to investigate to what extent such a qualitative change in the dynamics can be reproduced (or observed) in other instances where an otherwise ‘classical’ system is subject to a similar combination of the undulatory and the stochastic element, giving rise to quantization.

This question gains relevance in light of the recent series of remarkable experimental, theoretical and numerical work carried out in hydrodynamics, with droplets bouncing on the surface of a vibrating fluid and describing trajectories guided by their accompanying surface waves [19], which thus constitute a sort of hydrodynamic de Broglie waves. A number of phenomena have been observed with one or more walking droplets, which show clear signs of interference effects, nonlocal interaction between droplets, and quantization of orbits ([19,20,31,32] and references therein). The detailed equations that govern such systems have all imaginable complexities; yet in the end, the smoothed-out, averaged trajectories of the droplets are observed to form regular patterns that strongly suggest an analogy with quantum mechanics. It should be possible, therefore, to find some way of transiting from the detailed description of the coupled (droplet+surface) system to an approximate, averaged description of the droplet motion that serves to determine the extent of the quantum analogy and the conditions under which it manifests itself.

In this regard, it is pertinent to remark that Eq. (38) is satisfied frequency by frequency; it expresses a detailed balance [16]. The balance condition can therefore be considered as a kind of fluctuation-dissipation theorem. Whether an equivalent balance condition can be established in the hydrodynamic case is a matter for further investigation. By pursuing this analogy, one should be able to learn more about the process of quantization. In particular, the fact that in the macroscopic hydrodynamic case the trajectories can be visualized and recorded, should help in investigating the transient phase, when the ‘quantum regime’ has not yet taken over. In the context of the quantum analogy, this would be equivalent to testing the SED predictions by taking the system out of the quantum regime.

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