VOLUME AND LATTICE POINTS COUNTING FOR THE CYCLOPERMUTOHEDRON

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Abstract. The face lattice of the permutohedron realizes the combinatorics of linearly ordered partitions of the set \([n] = \{1, \ldots, n\}\). Similarly, the cyclopermutohedron is a virtual polytope that realizes the combinatorics of cyclically ordered partitions of \([n]\).

It is known that the volume of the standard permutohedron equals the number of trees with \(n\) labeled vertices multiplied by \(\sqrt{n}\). The number of integer points of the standard permutohedron equals the number of forests on \(n\) labeled vertices.

In the paper we prove that the volume of the cyclopermutohedron also equals some weighted number of forests, which eventually reduces to zero. We also derive a combinatorial formula for the number of integer points in the cyclompermutohedron.

Another object of the paper is the configuration space of a polygonal linkage \(L\). It has a cell decomposition \(K(L)\) related to the face lattice of cyclopermutohedron. Using this relationship, we introduce and compute the volume \(\text{Vol}(K(L))\).

1. Introduction

The standard permutohedron \(\Pi_n\) is defined (see [11]) as the convex hull of all points in \(\mathbb{R}^n\) that are obtained by permuting the coordinates of the point \((1, 2, \ldots, n)\). It has the following properties:

(1) \(\Pi_n\) is an \((n - 1)\)-dimensional polytope.
(2) The \(k\)-faces of \(\Pi_n\) are labeled by ordered partitions of the set \([n] = \{1, 2, \ldots, n\}\) into \((n - k)\) non-empty parts.
(3) A face \(F\) of \(\Pi_n\) is contained in a face \(F'\) iff the label of \(F\) refines the label of \(F'\).

Here and in the sequel, we mean the order-preserving refinement. For instance, the label \((\{1, 3\}, \{5, 6\}, \{4\}, \{2\})\) refines the label \((\{1, 3\}, \{5, 6\}, \{2, 4\})\), but does not refine \((\{1, 3\}, \{2, 4\}, \{5, 6\})\).
(4) The permutohedron is a zonotope, that is, Minkowski sum of line segments \(q_{ij}\), whose defining vectors are \(\{e_i - e_j\}_{i < j}\), where \(e_i\) are the standard basis vectors.
(5) The permutohedron splits into the union of bricks (that is, some elementary parallelepipeds) labeled by all possible trees on \(n\).
vertices. The volume of each of the bricks equals $\frac{1}{\sqrt{n}}$, so

$$Vol(\Pi_n) = \frac{1}{\sqrt{n}} \cdot \text{number of trees on } n \text{ labeled vertices} = \sqrt{n} \cdot n^{n-3}.$$ 

(6) The number of integer points of the standard permutohedron equals the number of forests on $n$ labeled vertices. This fact comes from some more delicate splitting of $\Pi_n$ into bricks: unlike volume computing, we have to take into account pieces of all dimensions, so we deal with *semiopen bricks* (details are given in Section 5).

Similarly, the *cyclopermutohedron* $\mathcal{CP}_{n+1}$ realizes the combinatorics of cyclically ordered partitions of $[n+1] = \{1, \ldots, n, n+1\}$: all the $k$-faces of the cyclopermutohedron are labeled by (all possible) cyclically ordered partitions of the set $[n+1]$ into $(n+1-k)$ non-empty parts, where $(n+1-k) > 2$. The incidence relation in $\mathcal{CP}_{n+1}$ corresponds to the refinement: a cell $F$ contains a cell $F'$ whenever the label of $F'$ refines the label of $F$.

The cyclopermutohedron is defined explicitly, as a weighted Minkowski sum of line segments.

In the paper we prove that the volume of the cyclopermutohedron equals some weighted number of forests. Making use of the theory of Abel polynomials, we eventually reduce the expression to zero. We also give a combinatorial formula for the number of integer points in the cyclopermutohedron.

Another object of the paper is the *configuration space*, or *moduli spaces of a polygonal linkage* $L$. One of the motivations for introducing the cyclopermutohedron is that $\mathcal{CP}_{n+1}$ is a "universal" polytope for moduli spaces of polygonal linkages. Namely, given a flexible polygon $L$, the space of its planar shapes (that is, the configuration space) has a cell decomposition $\mathcal{K}(L)$, whose combinatorics embeds in the combinatorics of the face poset of cyclopermutohedron. Using this relationship we introduce and compute the volume $Vol(\mathcal{K}(L))$.

The paper is organized as follows. In Section 2 we give all necessary information about virtual polytopes, and also the definition and properties of the cyclopermutohedron. Abel polynomials are also sketched in the section.

In Section 3 we explain the meaning of the "volume of cyclopermutohedron", and prove that it equals zero. In Section 4 we explain the relationship with polygonal linkages and give a formula for the volume of the configuration space (Theorem 3).
Finally, in Section 5 we compute the number of integer points in the cyclopermutohedron (Theorem 4).

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2. Theoretical backgrounds

2.1. Virtual polytopes. Virtual polytopes appeared in the literature as useful geometrization of Minkowski differences of convex polytopes. A detailed discussion can be found in [4, 6, 8]; below we give just a brief sketch. As a matter of fact, in the paper (except for Section 4) we need no geometrization. Even for volume and integer point counting, it is sufficient to know that virtual polytopes form the Grothendieck group associated to the semigroup of convex polytopes.

More precisely, a convex polytope is the convex hull of a finite, non-empty point set in the Euclidean space $\mathbb{R}^n$. Degenerate polytopes are also included, so a closed segment and a point are polytopes, but not the empty set. We denote by $\mathcal{P}^+$ the set of all convex polytopes.

Let $K$ and $L \in \mathcal{P}^+$ be two convex polytopes. Their Minkowski sum $K + L$ is defined by:

$$K + L = \{x + y : x \in K, y \in L\}.$$

Minkowski addition turns the set $\mathcal{P}^+$ to a commutative semigroup whose unit element is the convex set containing exactly one point $E = \{0\}$.

**Definition 1.** The group $\mathcal{P}$ of virtual polytopes is the Grothendieck group associated to the semigroup $\mathcal{P}^+$ of convex polytopes under Minkowski addition.

The elements of $\mathcal{P}$ are called virtual polytopes.

More instructively, $\mathcal{P}$ can be explained as follows.

1. A virtual polytope is a formal difference $K - L$.
2. Two such expressions $K_1 - L_1$ and $K_2 - L_2$ are identified, whenever $K_1 + L_2 = K_2 + L_1$.
3. The group operation is defined by

$$(K_1 - L_1) + (K_2 - L_2) := (K_1 + K_2) - (L_1 + L_2).$$

It is important that the notions of ”volume” and ”number of integer points” extend nicely to virtual polytope. We explain these constructions in the subsequent sections.
2.2. **Cyclopermutohedron.**

Assuming that \( \{e_i\}_{i=1}^n \) are standard basic vectors in \( \mathbb{R}^n \), define the points

\[
R_i = \sum_{i=1}^n (e_j - e_i) = (-1, \ldots, -1, n-1, -1, \ldots, -1, -1, -1) \in \mathbb{R}^n,
\]

and the following two families of line segments:

\[
q_{ij} = [e_i, e_j], \quad i < j
\]

and

\[
r_i = [0, R_i].
\]

We also need the point \( e = (1, 1, \ldots, 1) \in \mathbb{R}^n \).

The **cyclopermutohedron** is a virtual polytope defined as the Minkowski sum:

\[
CP_{n+1} := \bigoplus_{i<j} q_{ij} + e - \bigoplus_{i=1}^n r_i.
\]

Here and in the sequel, the sign ”\( \bigoplus \)” denotes the Minkowski sum, whereas the sign ”\( \bigcup \)” is reserved for the sum of numbers.

The cyclopermutohedron \( CP_{n+1} \) lies in the hyperplane

\[
x_1 + \ldots + x_n = \frac{n(n+1)}{2},
\]

so its actual dimension is \( (n-1) \).

**Remark 1.** The Minkowski sum

\[
\bigoplus_{i<j} q_{ij} + e
\]

is known to be equal to the standard permutohedron \( \Pi_n \) (see [11]). Therefore we can write

\[
CP_{n+1} = \Pi_n - \bigoplus_{i=1}^n r_i.
\]

The face poset of \( CP_{n+1} \) encodes cyclically ordered partitions of the set \( [n+1] = \{1, \ldots, n+1\} \):

1. For \( k = 0, \ldots, n-2 \), the \( k \)-dimensional faces of \( CP_{n+1} \) are labeled by (all possible) cyclically ordered partitions of the set \( [n+1] \) into \( (n-k+1) \) non-empty parts.
2. A face \( F' \) is a face of \( F \) whenever the label of \( F' \) refines the label of \( F \). Here we mean order preserving refinement.
2.3. Abel polynomial and rooted forests. [10]

A rooted forest is a graph equal to a disjoint union of trees, where each of the trees has a marked vertex.

The Abel polynomials form a sequence of polynomials, where the $n$-th term is defined by

$$ A_{n,a}(x) = x(x - an)^{n-1}. $$

A special case of the Abel polynomials with $a = -1$ counts rooted labeled forests. Namely, if $A_n(x) := A_{n,-1}(x) = x(x + n)^{n-1}$ is the $n$-th Abel polynomial, then

$$ A_n(x) = \sum_{k=0}^{n} t_{n,k} \cdot x^k, $$

where $t_{n,k}$ is the number of forests on $n$ labeled vertices consisting of $k$ rooted trees.

3. Volume of cyclopermutohedron equals zero

The notion of volume extends nicely from convex polytopes to virtual polytopes. We explain below the meaning of the volume of a virtual zonotope.

Assume we have a convex zonotope $Z \subset \mathbb{R}^n$, that is, the Minkowski sum of some linear segments $\{s_i\}_{i=1}^m$:

$$ Z = \bigoplus_{i=1}^m s_i. $$

For each subset $I \subset [m]$ such that $|I| = n$, denote by $Z_I$ the elementary parallelepiped, or the brick spanned by $n$ segments $\{s_i\}_{i \in I}$, provided that the defining vectors of the segments are linearly independent. In other words, the brick equals the Minkowski sum

$$ Z_I = \bigoplus_I s_i. $$

It is known that $Z$ can be partitioned into the union of all such $Z_I$, which implies immediately

$$ Vol(Z) = \sum_{I \subset [m], |I| = n} Vol(Z_I) = \sum_{I \subset [m], |I| = n} |\text{Det}(S_I)|, $$

where $S_I$ is the matrix composed of defining vectors of the segments from $I$.

Now take positive $\lambda_1, ..., \lambda_m$ and sum up the dilated segments $\lambda_i s_i$. Clearly, we have
\[ \text{Vol} \left( \bigoplus_{i=1}^{m} \lambda_i s_i \right) = \sum_{I \subseteq [m], |I| = n} \prod_{i \in I} \lambda_i \cdot |\text{Det}(S_I)|. \]

For fixed \( s_i \), we get a polynomial in \( \lambda_i \), which counts not only the volume of convex zonotope (which originates from positive \( \lambda_i \)), but also the volume of a virtual zonotope, which originates from any real \( \lambda_i \), including negative ones, see [4, 8].

So, one can use the above formula as the definition of the volume of a virtual zonotope.

An almost immediate consequence is:

**Lemma 1.** Let \( E = E_n \) be the set of edges of the complete graph \( K_n \). The \((n - 1)\)-volume of the cyclopermutohedron can be computed by the formula:

\[ \text{Vol}(\mathcal{CP}_{n+1}) = \text{Vol} \left( \bigoplus_{i<j} q_{ij} - \bigoplus_{i=1}^{n} r_i \right) = \frac{1}{\sqrt{n}} \sum_{|I|+|M|=n-1} (-1)^{|M|} |\text{Det}(q_{ij}, r_k, e)_{|(ij)| \in I}, k \in M|. \]

Here \( I \) ranges over subsets of \( E \), whereas \( M \) ranges over subsets of \([n]\). The matrix under determinant is composed of defining vectors of the segments \( q_{ij} \) and \( r_k \), and also of the vector \( e = (1, 1, \ldots, 1, 1) \).

**Proof.** The cyclopermutohedron \( \mathcal{CP}_{n+1} \) lies in the hyperplane

\[ x_1 + \cdots + x_n = \frac{n(n+1)}{2}, \]

so its dimension equals \( n - 1 \). That is, we deal with \((n - 1)\)-volume, which reduces to the \( n \)-volume by adding \( e = (1, 1, \ldots, 1, 1) \) and dividing by \(|e| = \sqrt{n}|. \]

**Remark 2.** The formula for the volume of a virtual zonotope also has a geometrical meaning which we briefly sketch here. Due to Brianchon-Gram decomposition of virtual polytopes (see [8] or [4]), any virtual polytope can be viewed as a codimension one homological cycle, and therefore possesses a well-defined (algebraic) volume.

For a virtual zonotope, the associated cycle decomposes into homological sum of elementary bricks, but the latter should be understood also as homological cycles coming with different orientations. More precisely, if the number of negative \( \lambda_i \) in the sum \( \bigoplus_{i=1}^{n} \lambda_i s_i \) is even, then the corresponding elementary brick equals the boundary of elementary
parallelepiped $\partial \left( \bigoplus_{i=1}^{n} |\lambda_i| s_i \right)$ with the positive orientation (that is, cooriented by the outer normal vector). If the number of negative $\lambda_i$ is odd, we have the same cycle with the negative orientation.

**Theorem 1.** $\text{Vol}(\mathcal{CP}_{n+1}) = 0$.

Proof. Keeping in mind Lemma 1, let’s first fix $I$ and $M$ with $|I| + |M| = n - 1$, and compute one single summand $|\text{Det}(q_{ij}, r_k, e)|_{(ij) \in I, k \in M}$.

If $M = \emptyset$, the determinant equals 1 iff the set $I$ gives a tree. Otherwise it is zero. (This is the reason for the volume formula of the permutohedron.)

Assume now that $M$ is not empty.

$$|\text{Det}(q_{ij}, r_k, e)| = \begin{vmatrix} 0 & 0 & \ldots & -1 & \ldots & 1 \\ \vdots & \vdots & \ddots & -1 & \ldots & 1 \\ -1 & 0 & \ldots & -1 & \ldots & 1 \\ \vdots & -1 & \ldots & -1 & \ldots & 1 \\ 1 & 0 & \ldots & -1 & \ldots & 1 \\ \vdots & \vdots & \ddots & -1 & \ldots & 1 \\ 0 & 0 & \ldots & n - 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & -1 & \ldots & 1 \\ 0 & 1 & \ldots & -1 & \ldots & 1 \\ \vdots & \vdots & \ddots & -1 & \ldots & 1 \\ 0 & 0 & \ldots & -1 & \ldots & 1 \end{vmatrix} = n^{\lvert M \rvert} \cdot (*)$$

Adding $e$ to all the columns $r_i$, we get:

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & 0 & \ldots & 1 \\ -1 & 0 & \ldots & 0 & \ldots & 1 \\ \vdots & -1 & \ldots & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & 0 & \ldots & 1 \\ 0 & 1 & \ldots & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{vmatrix} = n^{\lvert M \rvert} \cdot (*)$$
We wish to proceed in a similar way, that is, add the columns containing the unique entry 1 to other columns chosen in an appropriate way. To explain this reduction let us give two technical definitions.

**Definition 2.** A decorated forest $F = (G, M)$ is a graph $G = ([n], I)$ without cycles on $n$ labeled vertices together with a set of marked vertices $M \subseteq [n]$ such that the following conditions hold:

1. Number of marked vertices $|M| +$ number of edges $|I|$ equals $n - 1$.
2. Each connected component of $G$ has at most one marked vertex.

Immediate observations are:

Each decorated forest has exactly one connected component with no vertices marked. We call it a free tree. Denote by $N(F)$ the number of vertices of the free tree.

Each decorated forest is a disjoint union of the free tree and some rooted forest. The number of rooted trees equals $|M|$.

Each decorated forest $F$ yields a collection of $\{e_{ij}, r_k\}_{(ij) \in I, k \in M}$, whose above determinant $(\ast)$ we denote by $|\text{Det}(F)|$ for short.

For instance, for the first decorated forest in Figure 1, we have $N(F) = 2$, $|M| = 1$.

Now we define the *reduction of a decorated forest* (see Figure 1 for example). It goes as follows.

Assume we have a decorated forest. Take a marked vertex $i$ and an incident edge $(ij)$. Remove the edge and mark the vertex $j$. Repeat until is possible.

Roughly speaking, a marked vertex $i$ kills the edge $(ij)$ and generates a new marked vertex $j$.

![Figure 1](image_url)  

**Figure 1.** Reduction process for a forest with $N(F) = 2$, $|M|(F) = 1$. Grey balls denote the marked vertices.

An obvious observation is:

**Lemma 2.** (1) The free tree does not change during the reduction.
(2) The reduction brings us to a decorated forest with a unique free tree. All other trees are one-vertex trees, and all these vertices are marked.

(3) The reduction can be shortened: take the connected components one by one and do the following.
   (a) If a connected component has no marked vertices, leave it as it is.
   (b) If a connected component has a marked vertex, eliminate all its edges and mark all its vertices.

(4) The reduction does not depend on the order of the marked vertices we deal with.

Before we proceed with the proof of Theorem 1, prove the lemma:

Lemma 3.  
(1) For each decorated forest $F$,
   $$|\text{Det}(F)| = N(F).$$

(2) If a collection $\{e_{ij}, r_k\}$ does not come from a decorated forest, that is, violates condition (2) from Definition 2, then
   $$|\text{Det}(e_{ij}, r_k)| = 0.$$

Proof of the lemma.  (1) For a decorated forest, we manipulate with the columns according to the reduction process. We arrive at a matrix which (up to a permutation of the columns and up to a sign) is:

$$\begin{pmatrix} A & O & 1 \\ O & E & 1 \end{pmatrix}.$$  

Here $A$ is the matrix corresponding to the free tree, $E$ is the unit matrix, and the very last column is $e$. Its determinant equals 1.

(2) If the collection of vectors does not yield a decorated forest, that is, there are two marked vertices on one connected component, the analogous reduction gives a zero column.

Basing on Lemmata 3 and 1 we conclude:

$$\text{Vol}(\mathcal{CP}_{n+1}) = \frac{1}{\sqrt{n}} \sum_{F} (-n)^{|M(F)|} \cdot N(F) =,$$

where the sum extends over all decorated forests $F$ on $n$ vertices. (Remind that $M(F)$ is the set of marked vertices, $N(F)$ is the number of vertices of the free tree.)

Next, we group the forests by the number $N = N(F)$ and write
\[ = \frac{1}{\sqrt{n}} \sum_{N=1}^{n} \left( \binom{n}{N} \right) N^{N-2} \cdot N \sum_{f} (-n)^{C(f)} = \]
\[ = \frac{1}{\sqrt{n}} \sum_{N=1}^{n} \left( \binom{n}{N} \right) N^{N-1} \sum_{f} (-n)^{C(f)} = \frac{1}{\sqrt{n}} \cdot (**) , \]

where the second sum ranges over all rooted forest on \((n-N)\) labeled vertices, \(C(\cdot)\) is the number of connected components.

Let us explain this in more details.

1. \(N\) ranges from 1 to \(n\). We choose \(N\) vertices in \(\binom{n}{N}\) different ways and place a tree on these vertices in \(N^{N-2}\) ways.
2. On the rest of the vertices we place a rooted forest.

Recalling that \(t_{n-N,k}\) the number of forests on \((n-N)\) labeled vertices of \(k\) rooted trees, we write:

\[ (**) = \sum_{N=1}^{n} \left( \binom{n}{N} \right) N^{N-1} \sum_{k=1}^{n-N} (-n)^{k} \cdot t_{n-N,k}. \]

Section 2.3 gives us:

\[ \sum_{k=0}^{n} t_{n,k} x^{k} = A_{n}(x), \]

where \(A_{n}(x) = x(x+n)^{n-1}\) is the Abel polynomial.

Setting \(-n = x\), we get

\[ \sum_{k=1}^{n-N} (-n)^{k} \cdot t_{n-N,k} = A_{n-N}(-n). \]

Thus (**) converts to

\[ \sum_{N=1}^{n} \left( \binom{n}{N} \right) N^{N-1} A_{n-N}(-n) =: Q_{n}. \]

Applying the definition of \(A_{n-N}(-n)\), we get

\[ Q_{n} = \sum_{N=1}^{n} \left( \binom{n}{N} \right) N^{N-1} (-n) (-n+n-N)^{n-N-1} = (-1)^{n} n \sum_{N=1}^{n}(-1)^{N} \left( \binom{n}{N} \right) N^{n-2}. \]

Introduce the following polynomial:

\[ p(x) := \sum_{N=0}^{n} N^{n-2} \left( \binom{n}{N} \right) x^{N}, \]
for which we have $Q_n = p(-1)$. Set also

$$p_0(x) := (1 + x)^n = \sum_{N=0}^{n} \binom{n}{N} x^N,$$

$$p_i(x) := x \cdot p'_{i-1}(x) = \sum_{N=0}^{n} N^i \binom{n}{N} x^N.$$

We clearly have $p(x) = p_{n-2}(x)$. Besides, $(1 + x)^{n-k}$ divides $p_k(x)$, which implies $Q_n = 0$. □

4. Polygonal linkages: volume of the configuration space

4.1. Definitions and notation. A flexible $(n+1)$-polygon, or a polygonal $(n+1)$-linkage is a sequence of positive numbers $L = (l_1, \ldots, l_{n+1})$. It should be interpreted as a collection of rigid bars of lengths $l_i$ joined consecutively in a closed chain by revolving joints. We always assume that the triangle inequality holds, that is,

$$\forall j, \ l_j < \frac{1}{2} \sum_{i=1}^{n+1} l_i$$

which guarantees that the chain of bars can close.

We also assume that the last bar is the longest one:

$$\forall j \ l_{n+1} \geq l_j.$$

A planar configuration of $L$ is a sequence of points

$$P = (p_1, \ldots, p_{n+1}), \ p_i \in \mathbb{R}^2$$

with $l_i = |p_ip_{i+1}|$, and $l_{n+1} = |p_{n+1}p_1|$. As follows from the definition, a configuration may have self-intersections and/or self-overlappings.

The moduli space, or the configuration space $M(L)$ is the space of all configurations modulo orientation preserving isometries of $\mathbb{R}^2$.

Equivalently, we can define $M(L)$ as

$$M(L) = \{(u_1, \ldots, u_{n+1}) \in (S^1)^{n+1} : \sum_{i=1}^{n+1} l_i u_i = 0\}/SO(3).$$

The (second) definition shows that $M(L)$ does not depend on the ordering of $\{l_1, \ldots, l_{n+1}\}$; however, it does depend on the values of $l_i$.

Let us comment on this dependance. Consider $(l_1, \ldots, l_{n+1})$ as a point in the parameter space $\mathbb{R}^{n+1}$. The hyperplanes in $\mathbb{R}^{n+1}$ defined by all
possible equations
\[ \sum_{i=1}^{n+1} \varepsilon_i l_i = 0 \] with \( \varepsilon_i = \pm 1 \)
are called \textit{walls}. Throughout the section we assume that the point \( \{l_1, ..., l_{n+1}\} \) belongs to none of the walls. This genericity assumption implies that the moduli space \( M(L) \) is a closed \((n-2)\)-dimensional manifold.

The walls dissect \( \mathbb{R}^{n+1} \) into a number of \textit{chambers}; the topology of \( M(l_1, ..., l_{n+1}) \) depends only on the chamber containing \( \{l_1, ..., l_{n+1}\} \) (see [1]).

The manifold \( M(L) \) is already well studied. In this paper we make use of the described below cell structure on the space \( M(L) \).

\subsection*{4.2. The complex \( K(L) \).}
Assume that \( (l_1, ..., l_{n+1}) \) is fixed.

A set \( I \subset [n+1] = \{1, 2, ..., n+1\} \) is called \textit{short}, if
\[ \sum_{i \in I} l_i < \frac{1}{2} \sum_{i=1}^{n+1} l_i. \]
Otherwise \( I \) is a \textit{long} set.

A partition of the set \( [n+1] \) is called \textit{admissible} if all the sets in the partition are short.

\textbf{Theorem 2.} [6] \textit{There is a structure of a regular CW-complex} \( K(L) \) \textit{on the moduli space} \( M(L) \). \textit{Its complete combinatorial description reads as follows:}

\begin{enumerate}
\item \textit{k-cells of the complex} \( K(L) \) \textit{are labeled by cyclically ordered admissible partitions of the set} \( [n+1] \) \textit{into} \((n+1-k)\) \textit{non-empty parts.}
\item \textit{A closed cell} \( C \) \textit{belongs to the boundary of some other closed cell} \( C' \) \textit{iff the partition} \( \lambda(C) \) \textit{is finer than} \( \lambda(C') \).
\end{enumerate}

\textbf{A remark on notation.} We write a cyclically ordered partition as a (linearly ordered) string of sets where the set containing the entry "n" stands on the last position.

We stress that the order of the sets matters, whereas there is no ordering inside a set. For example,
\[ (\{1\}\{3\}\{4, 2, 5, 6\}) \neq (\{3\}\{1\}\{4, 2, 5, 6\}) = (\{3\}\{1\}\{2, 4, 5, 6\}). \]

\textbf{Example 1.} Assume that
\[ l_{n+1} = \sum_{i=1}^{n} l_i - \varepsilon, \]
where $\varepsilon$ is small. In this case the moduli space $M(L)$ is the sphere $S^{(n-2)}$, see [1], and the complex $K(L)$ is isomorphic to the boundary complex of the permutohedron $\Pi_n$.

For any $(n + 1)$-linkage $L$, the complex $K(L)$ automatically embeds in the face complex of cyclopemutohedron $\mathcal{CP}_{n+1}$, and therefore can be realized by a polyhedron which we denote by $\mathcal{P}(L)$. Vividly speaking, the polyhedron $\mathcal{P}(L)$ is patched of those faces of the cyclopemutohedron, whose labels are admissible partitions.

Example.
For $L$ as in Example 4, $\mathcal{P}(L)$ equals the boundary of the permutohedron $\Pi_n$.

Example.
Let $n = 5$, $L = (1.2, 1, 1, 0.8, 2.2)$. Then $\mathcal{P}(L)$ is the cylinder depicted in Fig. 2. The two shadowed faces are labeled by $\{(123\{4\})$ and $(\{4\}123\})$. Since the partitions $(\{123\}1\{4\})$ and $(\{4\}123\{5\})$ are non-admissible, these faces of permutohedron are removed, whereas all other faces of the permutohedron survive. There are also six “diagonal” rectangular faces. They are labeled by $(\{12\}3\{45\})$, $(\{1, 3\}2\{45\})$, $(\{2\}13\{45\})$, $(\{23\}1\{45\})$, $(\{3\}12\{45\})$, and $(\{1\}23\{45\})$.

![Figure 2](image)

**Figure 2.** The complex $K(L)$ for the 5-linkage $L = (1, 2; 1; 1; 0.8; 2, 2)$. We remove from the permutohedron the two shaded facets and patch in the cylinder.

4.3. **Volume of the complex $K(L)$**. Following the ideology of Remark 2, $\mathcal{P}(L)$ can be viewed as a codimension one homological cycle (or as a generalization of closed piecewise linear oriented manifold) in
the Euclidean space. Therefore it makes sense to speak of the volume of the part of the space bounded by \( \mathcal{P}(L) \). Since \( \mathcal{P}(L) \) may have many self-intersections, the volume means the algebraic volume, that is, multiplicities (which can be also negative) are taken into account.

Let us explain this in more details. For each point \( x \in \mathbb{R}^n \), denote by \( \text{ind}_x(\mathcal{P}(L)) \) the index of the cycle with respect to the point \( x \). Then by the volume of the configuration space we mean

\[
\text{Vol}(M(L)) := \text{Vol}(\mathcal{P}(L)) := \int_{\mathbb{R}^n} \text{ind}_x(\mathcal{P}(L)) dx.
\]

**Definition 3.** For an \((n + 1)\)-linkage \( L \), a decorated forest \( F \) on \( n \) labeled vertices is called non-admissible, if the vertex set of the free tree is a long set.

In notation of Section 3, the following lemma holds:

**Lemma 4.** For a \((n + 1)\)-linkage \( L \), we have:

\[
\text{Vol}(M(L)) = \frac{1}{\sqrt{n}} \sum_{\text{non-admissible } F} (-n)^{|M(F)|} \cdot N(F),
\]

where the sum ranges over all non-admissible decorated forests on \( n \) labeled vertices.

We remind that \(|M(F)|\) denotes the number of marked vertices, \( N(F) \) is the number of vertices of the free tree.

Proof. Let us take the linkage \( L_0 = (l_1, ..., l_n, \lambda) \) assuming that the value of \( \lambda \) continuously and monotonely changes from

\[
\sum_{i=1}^{n} l_i - \varepsilon \quad \text{to} \quad l_{n+1}.
\]

In the beginning we have the permutohedron \( \Pi_n \), whose volume we already know. At the end, we have \( \mathcal{P}(L) \), whose volume we wish to calculate. In between we have a (finite) number of Morse surgeries, and we can control the behavior of the volume at each of the surgeries.

Prove first that the formula holds true for \( \lambda = \sum_{i=1}^{n} l_i - \varepsilon \). Indeed, for this particular \( \lambda \), a ”decorated non-admissible forest on \( n \) vertices” means just ”a free tree on \( n \) vertices”, so the statement of the theorem reduces to the formula for the volume of the standard permutohedron, see Section 1.

Now we start changing \( \lambda \). This means that we have a path in the parameter space \( \mathbb{R}^{n+1} \), which crosses some of the walls. We can assume that the walls are crossed one by one; if this is not the case, we perturb generically the original lengths \( l_i \).
Once we cross a wall, the complex $K$, and its polytopal realization change by a surgery which we describe below. Denote by $Pol_{Old}$ and by $Pol_{New}$ the polyhedra that realize $K$ before and after the surgery respectfully.

Let us look at the surgery in more details. Once a wall is crossed, some maximal by inclusion short set $T \subseteq \{1, ..., n\}$ turns to a long set, whereas its complement $\overline{T} = [n+1] \setminus T$ becomes short. We conclude that the new complex $K$ can be obtained from the old complex by removing some of the cells and adding some new cells. The cells that get removed after crossing the wall are labeled by $(\ast, T, \ast)$. Here whereas the new cells that appear are labeled by $(\ast, [n] \setminus T)$. Here "$\ast$" means just any ordered partition of the complement assuming that altogether we have at least three parts.

The cells that get removed form a subcomplex isomorphic to the boundary of the permutohedron $\Pi_{n-|T|}$ multiplied by a $|T| - 1$)-ball. The cell structure of $K$ converts this ball to the permutohedron $\Pi_{|T|}$. So, we have the following Morse surgery: we cut out the cell subcomplex $C_1 = (\partial \Pi_{n-|T|}) \times \Pi_{|T|}$, and patch instead the cell complex $C_2 = \Pi_{n-|T|} \times \partial \Pi_{|T|}$ along the identity mapping on their common boundary $\partial \Pi_{n-|T|} \times \partial \Pi_{|T|}$.

Denote by $\mathcal{C} := C_1 \cup C_2$ the union of these complexes. Combinatorially, we have $\mathcal{C} = \partial \left( \Pi_{n-|T|} \times \Pi_{|T|} \right)$.

$\mathcal{C}$ (taken with an appropriate orientation) relates the old and new polyhedra. Namely, we have a homological sum:

$$Pol_{New} = Pol_{Old} + \mathcal{C}.$$  

This means that the new and old volumes are related by

$$Vol(Pol_{New}) = Vol(Pol_{Old}) + Vol(\mathcal{C}).$$

After geometrically realizing these complexes, we decompose the realization of $\Pi_{n-|T|} \times \Pi_{|T|}$ into the homological sum of bricks $P_i \times P_j$, where $P_i$ is an elementary brick from $\Pi_{n-|T|}$, and $P_j$ is an elementary brick from $\Pi_{|T|}$. The first elementary brick $P_i$ corresponds to a tree on $T$, whereas $P_j$ corresponds to a tree on $[n+1] \setminus T$, or, equivalently, to a rooted forest on $[n] \setminus T$. In other words, each such pair $(P_i, P_j)$ gives us a rooted forest $F$ whose free tree is non-admissible.

The brick $P_i \times P_j$ has a geometrical realization as the Minkowski sum of corresponding line segments. It contributes $(-n)^{|\mathcal{M}(F)|}$ to $Vol(\mathcal{C})$. 

Therefore, if the statement of the theorem is true for \( P_{\text{Old}} \), it is also true for \( P_{\text{New}} \). \qed

**Theorem 3.** For a flexible \((n + 1)\)-polygon \( L \), we have:

\[
\text{Vol}(M(L)) = \sqrt{n} \sum_{k=0}^{n} (-1)^k \cdot a_k \cdot (n-k)^{n-2},
\]

where \( a_k \) is the number of \((k + 1)\)-element short subsets of \([n + 1]\) containing the entry \((n + 1)\).

**Proof.** Using Lemma 4, we first fix a number \( k \) and choose a long \( k \)-element subset of \([n]\). This can be done in \( a_{n-k} \) ways. We put a tree on these vertices in \( k^{k-2} \) ways and arrive at

\[
\text{Vol}(M(L)) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_{n-k} \cdot k^{k-2} \sum_{g \text{ is a rooted forest on } (n-k) \text{ vertices}} \frac{(-n)^{C(g)} \cdot N(F)}{C(g)} =
\]

\[
= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_{n-k} \cdot k^{k-1} \sum_{g \text{ is a rooted forest on } (n-k) \text{ vertices}} (-n)^{C(g)} =
\]

By the identity from Section 2.3

\[
\sum_{g \text{ is a rooted forest on } m \text{ vertices}} x^{C(g)} = x \cdot (x + m)^{m-1},
\]

we get

\[
= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_{n-k} (-k)^{k-1} \cdot (-n) \cdot k^{n-k-1}
\]

\[
= -\sqrt{n} \sum_{k=1}^{n} a_{n-k} k^{k-1} \cdot (-k)^{n-k-1} =
\]

\[
= \sqrt{n} \sum_{k=1}^{n} a_{n-k} k^{n-2} \cdot (-1)^{n-k}.
\]

Interchanging \( k \) and \( n-k \), we get the desired. \qed

**Remark.** Betty numbers \( \beta_k = \beta_k(M(L)) \) are expressed in terms of \( a_k \), see [2]:

\[
\beta_k = a_k + a_{n-k-3}.
\]
Corollary 1. Assume $n+1 = 2m+1$. For the equilateral $(n+1)$-linkage $L = (1, 1, ..., 1)$ we have:

$$Vol(M(L)) = \sqrt{2} m \sum_{k=0}^{m} (-1)^k \cdot \binom{2m}{k} \cdot (2m - k)^{2m-2}.$$ 

Proof. Indeed, for the equilateral linkage, ”a short set” means ”a set with cardinality $\leq m$”. Therefore

$$a_k = \begin{cases} \binom{n-1}{k}, & \text{if } k \leq m; \\ 0, & \text{otherwise}. \end{cases}$$

5. INTEGER POINTS COUNTING FOR CYCLOPERMUTOHEDRON

5.1. Integer points counting for cyclopermutohedron: theoretical backgrounds. The first leading idea for integer points enumeration in a zonotope is to decompose it into elementary bricks, as we did in Section 1. However, unlike volume computation, we have to take into account the ”pieces” of all dimensions, including points. By this reason we introduce semiopen bricks. The latter are Minkowski sums of semiopen segments, see Figure 3.

![Figure 3](image_url)

**Figure 3.** A semiopen segment and a semiopen rectangle. The dashed lines and white points are missing.

![Figure 4](image_url)

**Figure 4.** The permutohedron $\Pi_3$ splits into three semiopen parallelograms, three semiopen segments, and one point.
A zonotope decomposes in a disjoint union of semiopen bricks of dimensions ranging from 0 to $n$.

**Example 2.** Permutohedron $\Pi_n$ decomposes in a disjoint union of semiopen bricks that are in a one-to-one correspondence with forests on $n$ labeled vertices. Each of the bricks contributes exactly one integer point, so for the number of integer points $\Lambda$, we have:

$$\Lambda(\Pi_n) = \text{number of forests on } n \text{ labeled vertices}.$$ 

Below we almost literally repeat the arguments from Section 3. Assume we have a convex zonotope $Z \subset \mathbb{R}^n$, that is, the Minkowski sum of linear segments $\{s_i\}_{i=1}^m$:

$$Z = \bigoplus_{i=1}^m s_i.$$ 

For each subset $I \subset [m]$ with $|I| \leq n$, which gives linearly independent $\{s_i\}_{i \in I}$, denote by $Z_I$ the semiopen brick spanned by segments $\{s_i\}_{i \in I}$. It is well-known that $Z$ can be partitioned into the union of all such $Z_I$, which implies immediately

$$\Lambda(Z) = \sum_{I \subset [m]} \sharp(Z_I),$$ 

where $\sharp(\cdot)$ denotes the number of integer points in a semiopen brick provided that the brick is spanned by linearly independent vectors. For linearly dependent vectors we set $\sharp := 0$.

For positive integer numbers $\lambda_1, \ldots, \lambda_n$ let us sum up the dilated segments $\lambda_i s_i$. Clearly, we have

$$\Lambda\left( \bigoplus_{i=1}^m \lambda_i s_i \right) = \sum_{I \subset [m]} \sharp(Z_I) \cdot \prod_{i \in I} \lambda_i.$$ 

For fixed $s_i$, $\Lambda$ is a polynomial in $\lambda_i$, which counts not only the number of integer points in a convex zonotope (which originates from positive $\lambda_i$), but also the number of integer points in a virtual zonotope, (which originates from any integer $\lambda_i$, including negative ones), see [4, 8].

**Remark.** According to Khovanskii’s and Pukhlikov’s construction [4], given a lattice virtual polytope, each lattice point has a weight, which is some (possibly, negative) integer number. The above defined $\Lambda(\cdot)$ for virtual zonotopes counts the sum of weights. This fact generalizes the Erhart’s reciprocity law and has many other interpretations, such as Riemann-Roch Theorem for toric varieties.
We immediately have:

**Lemma 5.** Let $E = E_n$ be the set of edges of the complete graph $K_n$. For the cyclopermutohedron we have:

$$\Lambda(CP_{n+1}) = \sum_{(I,M) : |I|+|M| \leq n-1} (-1)^{|M|} \cdot \#( \bigoplus_I q_{ij} + \bigoplus_M r_k )$$

Here $I$ ranges over subsets of $E$, whereas $M$ ranges over subsets of $[n]$. □

Our next aim is to give a formula for one single summand.

**Definition 4.** A partial decorated forest $F = (G,M)$ is a graph $G = ([n], I)$ without cycles on $n$ labeled vertices together with a set of marked vertices $M \subset [n]$ such that the following conditions hold:

1. Number of marked vertices $|M|$ + number of edges $|I|$ is smaller or equal than $n$.
2. Each connected component of $G$ has at most one marked vertex.

We already know that decorated forests are in a bijection with linearly independent $(n-1)$-tuples of $\{q_{ij}, r_k\}$ (see Section 3). Therefore, partial decorated forests are in a bijection with linearly independent collections of segments $\{q_{ij}, r_k\}$.

From now on, we fix one particular partial decorated forest $F$ and work with the associated segments.

**Notation:** $F$ splits into a disjoint union of two forests: (1) a forest $T = T(F)$ without marked vertices, which is called the free forest, and (2) a rooted forest $R(F)$. In turn, $T$ is a disjoint union of trees $T_j(F)$.

As in the previous sections, $C(\cdot)$ denotes the number of connected components of a forest. In particular, $C(R(F)) = |M|$ is the number of marked vertices.

In this notation we have:

**Lemma 6.** For the number $\#$ of integer points in the semiopen brick spanned by $\{q_{ij}, r_k\}$, we have:

1. If the segments in question do not come from a partial decorated forest, then $\# = 0$.
2. If the segments in question come from a decorated partial forest $F$ with at least one marked vertex, then

$$\# = n^{|M|-1} \cdot \gcd[V(T_1), \ldots, V(T_{C(T)})],$$
where $|M|$ is the number marked vertices in $F$, $T_i$ are the connected components of the free forest $T$, $V(T_i)$ is the number of vertices in $T_i$.

(3) If the segments in question come from a decorated partial forest $F$ with no marked vertices, then

$$\sharp = 1.$$ 

For the proof of the lemma, see Section 6.

Basing on the lemma, we obtain:

**Theorem 4.** Define

$$\Phi(v) = \sum_T \gcd\{\{V(T_i)\}\},$$

where the sum ranges over all (non-rooted) forests $T$ on $v$ labeled vertices, $T_i$ are the trees in the forest $T$, and $V(\cdot)$ is the number of vertices.

Then

$$\Lambda(CP_{n+1}) = \varphi(n) - \sum_{v=1}^{n-1} \binom{n}{v} (-v)^{n-v-1} \cdot \Phi(v) =$$

$$= \Lambda(\Pi_n) - \sum_{v=1}^{n-1} \binom{n}{v} (-v)^{n-v-1} \cdot \Phi(v),$$

where $\varphi(n)$ is the number of (non-rooted) forests on $n$ labeled vertices.

Proof.

(1) We count partial decorated forests with no marked points separately. Altogether they contribute $\varphi(n) = \Lambda(\Pi_n)$.

(2) Next we choose $v$ vertices of the free forest. This can be done in $\binom{n}{v}$ ways.

(3) Each of the forests gives us its own $\gcd$. Altogether they give us $\Phi(v)$.

(4) Next, we count rooted forests on the remaining $n - v$ vertices. Each forest is counted with multiplicity $(-n)^{C(F)}$. The equality

$$\sum_{f \text{ is a rooted forest on } m \text{ vertices}} x^{C(f)} = x \cdot (x + m)^{m-1}$$

(see Section 2.3) completes the proof.

**Examples:**

$\Lambda(CP_3) = 1,$

$\Lambda(CP_4) = 18.$
5.2. Proof of Lemma 6. We fix a partial decorated forest and the corresponding semiopen brick spanned by \{q_{ij}, r_k\}. The vectors \(r_i\) will be called long vectors, whereas \(q_{ij}\) will be called short vectors.

As the main tool, we shall use the following lemma, whose proof comes from elementary linear algebra.

**Lemma 7.**

1. The number of integer points \(\sharp(\{v_i\})\) doesn’t change if we replace any \(v_j\) by the vector \(v_j + \sum_{i \neq j}(\pm v_i)\).

2. For an integer \(\lambda\), we have:

\[
\sharp(\{\lambda \cdot v_1, v_2, v_3, \ldots, v_k\}) = \lambda \cdot \sharp(\{v_1, v_2, v_3, \ldots, v_k\}).
\]

3. Suppose there exists a coordinate \(x_j\) such that among vectors \(\{v_i\}\) only one vector (say, \(v_1\)) has nonzero \(j^{th}\) coordinate which equals \(\pm 1\). This will be called the free coordinate. Then we can remove \(v_1\) from the collection of segments without changing the number of integer points:

\[
\sharp(\{v_i\}) = \sharp(\{v_i\}_{i \neq 1}).
\]

4. Given a partial decorated forest, replace all the trees by path trees, keeping for each tree the set of its vertices. This manipulation does not change the value \(\sharp(F)\).

5. Given one vector \(v = (V_1, \ldots, V_n)\),

\[
\sharp(v) = \gcd([V_i]),
\]

where \(\gcd\) denotes the greatest common divisor. □

Reduction of a partial decorated forest (see Figure 5) goes as follows: Assume we have a partial decorated forest \(F\).

1. Choose a marked vertex. We shall call it the principal marked vertex.
2. Join the principal marked vertex with each of the other marked vertices by an edge.
3. Remove all marks from the marked vertices that are not principal.
4. Replace the tree with the marked vertex by a path tree on the same vertices. We arrive at a partial decorated forest \(\overline{F}\).

Lemma 7 implies:
Lemma 8. For a partial decorated forest $F$ and its reduction $\overline{F}$, we have:

$$\#(F) = n^{\#M} \cdot \#(\overline{F}),$$

where $|M| = |M(F)|$ is the number of marked vertices in $F$.  \hfill \Box

Figure 5. Reduction of a partial decorated forests. Grey balls denote the marked vertices.

Now we are ready to calculate one single summand from Lemma 5. We arrange the column vectors in a matrix: first come all the $q_{ij}$, after them come all the $r_k$. The main idea is that the reduction process encodes the way of manipulating with the columns in the matrix. Using Lemma 7, (1), we can assume that all the $P_i$ are path trees.

1. Assume that the collection contains some long vector. The algorithm runs as follows: first, we take the long vector which corresponds to the principal marked vertex and subtract it from all the other long vectors. Each of the long vectors (except for the first one) yields a multiple $n$ and a new short vector.

Next, we subtract the short vectors from the (unique that survived) long vector aiming at killing its coordinates. Finally, we get a matrix which allows to remove vectors using Lemma 7 (3). Eventually we arrive at

$$n^{\#M} \cdot \# \left( \begin{array}{c} -V(T_1) \\ \vdots \\ -V(T_{C(T)}) \\ V(T_1) + \cdots + V(T_{C(T)}) \end{array} \right) = n^{\#M} \cdot \gcd[V(T_1), \ldots, V(T_{C(T)})].$$

2. If there are no long vectors in the collection, we remove the vectors one by one using Lemma 7 (3), and arrive at $\# = 1$.  \hfill \Box
Examples. We exemplify below the reduction for three collections of vectors. Corresponding partial decorated forests are depicted in Fig. 6.

(1) Two free trees with \( V = 2 \) and \( V = 3 \), \( |M| = 1 \).

\[
\begin{pmatrix}
1 & 0 & 0 & -1 \\
-1 & 1 & 0 & -1 \\
0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & -3 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 5
\end{pmatrix} = 5 \cdot
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & -3 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & -2 \\
0 & 0 & 0 & 5
\end{pmatrix} = 1\]

(2) One free tree with \( V = 4 \), \( |M| = 1 \).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & -1 & -1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -4 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix} =\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -4 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -4 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 5 \\
0 & -1 & 0 & 0 & -4 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
=\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 \\
0 & -1 & 0 & 0 & -4 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix} = \begin{pmatrix}
0 \\
4 \\
-4 \\
0 \\
0 \\
0
\end{pmatrix} = 4.
\]

(3) One free tree with \( V = 2, |M| = 1. \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 5 \\
0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -2 \\
0 & -1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 5 \\
0 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix} = 2.
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -2 \\
0 & -1 & 0 & -1 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -2 \\
-1 & 0 & 0 \\
1 & -1 & 2 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} = 2.
\]

\textbf{References}

[1] M. Farber, \textit{Invitation to topological robotics}, European Mathematical Society, 2008.

[2] M. Farber and D. Schütz, \textit{Homology of planar polygon spaces}, Geom. Dedicata, 125 (2007), 75-92.

[3] A. Khovanskii, \textit{Newton polyhedra and toroidal varieties}, Functional Analysis and Its Applications, 11(4):289–296, 1977.

[4] A. Khovanskii and A. Pukhlikov, \textit{Finitely additive measures of virtual polytopes}, St. Petersburg Math. J., Vol. 4, 2 (1993), 337-356.

[5] G. Panina, \textit{Virtual polytopes and some classical problems}, St. Petersburg Math. J., Vol. 14, 5 (2003), 823-834.

[6] G. Panina, \textit{Moduli space of a planar polygonal linkage: a combinatorial description}, [arXiv:1209.3241](https://arxiv.org/abs/1209.3241).

[7] G. Panina, \textit{Cyclopermutohedron}, Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 288, 132-144.

[8] G. Panina and I. Streinu, \textit{Virtual polytopes}, Oberwolfach preprint OWP 2015-02.

[9] A. Postnikov, \textit{Permutohedra, Associahedra, and Beyond}, Int. Math. Res. Not. Vol. 6 (2009), 1026-1106.

[10] B. Sagan, \textit{A note on Abel polynomials and rooted labeled forests}, Discrete Mathematics, Vol. 44, 3 (1983), 293-298.

[11] G.M. Ziegler, \textit{Lectures on polytopes}, Graduate Texts in Mathematics. (Springer, Berlin 1995).