On dynamical systems approaches and methods in $f(R)$ cosmology

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Abstract. We discuss dynamical systems approaches and methods applied to flat Robertson-Walker models in $f(R)$-gravity. We argue that a complete description of the solution space of a model requires a global state space analysis that motivates globally covering state space adapted variables. This is shown explicitly by an illustrative example, $f(R) = R + \alpha R^2$, $\alpha > 0$, for which we introduce new regular dynamical systems on global compactly extended state spaces for the Jordan and Einstein frames. This example also allows us to illustrate several local and global dynamical systems techniques involving, e.g., blow ups of nilpotent fixed points, center manifold analysis, averaging, and use of monotone functions. As a result of applying dynamical systems methods to globally state space adapted dynamical systems formulations, we obtain pictures of the entire solution spaces in both the Jordan and the Einstein frames. This shows, e.g., that due to the domain of the conformal transformation between the Jordan and Einstein frames, not all the solutions in the Jordan frame are completely contained in the Einstein frame. We also make comparisons with previous dynamical systems approaches to $f(R)$ cosmology and discuss their advantages and disadvantages.

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1 Introduction

The simplest class of fourth order metric gravity theories is based on an action

\[ S = \int \left\{ \frac{f(R)}{2\kappa^2} + \mathcal{L}_m \right\} \sqrt{-\det g} \, d^4x \]  

(1.1)

where \( \kappa^2 = 8\pi G \); the speed of light, \( c \), is set to one; \( \det g \) is the determinant of a Lorentzian 4-dimensional metric \( g \), and \( R \) the associated curvature scalar, while \( \mathcal{L}_m \) is the matter Lagrangian density. General relativity with a cosmological constant \( \Lambda \) is obtained by setting \( f(R) = R - 2\Lambda \).

The vacuum part of these models, i.e., \( \mathcal{L}_m = 0 \), has recently achieved some popularity where certain forms of the function \( f(R) \) have resulted in geometric models of inflation or, more recently, dark energy, see e.g. [1]–[8] and also [9, 10] for a historical background. Although an assessment of cosmological viability requires a study of spatially homogeneous and isotropic Robertson-Walker (RW) models and perturbations thereof, we will restrict the analysis in this paper to flat RW cosmology. The vacuum equations of these models can be written as (see e.g. [3]):

\[ \dot{a} = Ha, \]  

(1.2a)

\[ \dot{H} = -2H^2 + \frac{R}{6}, \]  

(1.2b)

\[ \ddot{R} = -3H\dot{R} - \frac{1}{F_R} \left[ F_{RR} \dot{R}^2 + \frac{1}{3} (2f - FR) \right], \]  

(1.2c)

\[-6H \left( F_{,R} \dot{R} + FH \right) + FR - f = 0.\]  

(1.2d)
In the above equations an overdot represents the Jordan proper time \( t \) derivative, \( a \) is the scale factor of the flat RW metric in the Jordan frame, \( H \) is the Jordan Hubble variable, \( R \) is the Jordan curvature scalar, and

\[
F = \frac{df}{dR}, \quad F_R = \frac{dF}{dR} = \frac{d^2 f}{dR^2}, \quad F_{RR} = \frac{d^2 F}{dR^2} = \frac{d^3 f}{dR^3}.
\]

(1.3)

By regarding \( \dot{R} \) as an independent variable, equation (1.2d) forms a constraint that the evolution equations must satisfy. Furthermore, equation (1.2a) for \( a \) decouples, which leads to a reduced closed system of first order equations for \((H, \dot{R}, R)\), which, due to the constraint (1.2d), yield a dynamical system describing a flow on a 2-dimensional state space. Once the reduced system of first order equations has been solved, the decoupled equation (1.2a) yields 

\[
a \propto \exp(\int dt H).
\]

The above system of equations has some general properties which are worth mentioning. Firstly, the system is invariant under the transformation \((t, H) \rightarrow -(t, H)\), a property we will use below. Secondly, the system is ill-defined whenever \( F_R = 0 \) for some value(s) of \( R \).\(^1\) This is related to pathological properties as regards the characteristics of \( f(R) \) gravity, where, e.g., the properties of gravitational waves will severely constrain the physical viability of such models. It is therefore natural to divide \( f(R) \) gravity into two main classes of models: those for which \( F_R > 0 \) and those for which this is not the case. Thirdly, as it is well known \( F > 0 \) is associated with the one can introduce an Einstein frame. However, \( F = 0 \) is not, in general, an invariant subset in the Jordan frame, since

\[
\dot{F}|_{F=0} = F, \dot{R}|_{F=0} = -\frac{f}{6H}|_{F=0},
\]

(1.4)

where we have used (1.2d) (i.e., \( F = 0 \) is only an invariant subset if \( f \) and \( F \) are simultaneously zero for some value of \( R \)). This suggest that it is also natural to divide \( f(R) \) models into two additional classes, those with \( F > 0 \) everywhere, and those for which \( F \) can change sign. The latter case yields solutions in the Einstein frame that can be conformally extended in the Jordan frame, but which ones depend on the explicit form of \( f(R) \). We will later explicitly illustrate this result in the context of a specific model, which also exemplifies some other general features of \( f(R) \) cosmology.

There are a number of dynamical systems formulations in the literature that are based on transformations from \((H, \dot{R}, R)\) to some other variables (see appendix B for a discussion on several of these formulations). In this context there has been considerable activity concerning fixed points (also called singular points, equilibrium points, critical points) and their linear stability properties. It is therefore of interest to consider the fixed points of the basic reduced state space variables \((H, \dot{R}, R)\), which must satisfy

\[
-2H^2 + \frac{R}{6} = 0, \quad \dot{R} = 0, \quad 2f - FR = 0,
\]

(1.5)
as follows from (1.2b) and (1.2c). Then, (1.2d) takes the form (assuming that \( F, R \) is non-zero)

\[
-6H^2 F + FR - f = 3F(-2H^2 + R/6) = 0,
\]

(1.6)

which is thereby automatically satisfied. Moreover, \( 2f - FR \) is identically zero, if and only if \( f \propto R^2 \), and only in this case there is a line of fixed points for which \( R = 12H^2 \), while all

\(^1\)Of course this is not the case for general relativity for which \( F, R \) is identically zero, a case we will not consider here.
other models have isolated fixed points. When transforming to other variables care has to be taken when it comes to the physical interpretation of fixed point results. As we will see, some fixed points in other formulations simply reflect a break down of those variables, i.e., they correspond to a state space coordinate singularity. It is also essential to note that fixed points do not always give a complete asymptotic description. As will be emphasized in this paper, it is necessary to consider the dynamics on the entire state space of a given model to make an assessment of its physical content.

What is then required in order to obtain a complete description of the solution space and the properties of those solutions for a given $f(R)$ RW model? We will illustrate some of the ingredients that are required to answer this question with a specific example, but for all $f(R)$ RW cosmologies one needs to do the following:

(i) State space analysis.

(ii) A complete state space adapted coordinate cover, including those state space boundaries for which the equations can be extended (this e.g. excludes boundaries for which $F,R$ becomes zero).

(iii) Local and global dynamical systems analysis.

(iv) Physical solution space interpretation.

Let us now comment on the above in a little more detail. (i) A state space analysis entails dimensional and scale considerations, and a study of the algebraic structure of the constraint equation (1.2d), which includes global aspects such as state space topology. (ii) This means that one needs to find state space coordinates that globally cover the state space of a given model, including the boundaries for which the equations can be differentiable extended. This may include limits where $H, \dot{R}$ and $R$ become unbounded, which motivates the introduction of new bounded variables. Note that some models will even in principle require several coordinate patches, but there are classes of models for which one can find common local useful variables. Even in cases where it is possible to find a bounded global state space coordinate system, it might still be useful to consider other variables since it is unlikely that a global system, except under very special circumstances, is the optimal one for all local structures, i.e., there might exist complementary sets of variables. Furthermore, different models have different state space structures, and in general this requires different choices of variables — the common element is instead a state space analysis and an adaption to the structures that the analysis reveals. (iii) To understand the solution space structure of a given model and the asymptotical behaviour of the solutions, which is essential for assessing its physical viability (it is not enough to consider special solutions, e.g., fixed points), one must, in general, apply linear and non-linear fixed point techniques, as well as global dynamical systems analysis. Furthermore, note that fixed points will not in general give a complete asymptotic description, e.g., a problem might naturally give rise to limit cycles. That a global understanding of the solution space is required is illustrated by the fact that if one has found a solution with a desirable evolution, then the models will still only be of interest if this solution is in some sense an ‘attractor solution.’ Even so, this does not exclude that there exists an additional set of solutions that have a different evolution, which leads to issues concerning measures describing how ‘typical’ a solution is. (iv) Solutions, e.g. fixed points, have to be physically interpreted since a solution might be an artifact of the variables one has used. For example, variables that do not cover the entire Jordan state space
result in coordinate singularities, which results in fixed points. Thus fixed points may not correspond to physical phenomena, but may instead show that a formulation breaks down.

To illustrate the above issues (excluding the situation where $F_R$ passes through zero, which we will comment on in the final discussion), as well as allowing us to introduce some dynamical systems methods of quite wide applicability, we will consider a specific example, the vacuum equations for the flat RW metric with

$$f(R) = R + \alpha R^2, \quad \alpha > 0. \tag{1.7}$$

This model has attracted considerable attention in the past, see e.g., [11]–[15], and it still remains as one of the more successful models of inflation [16]. Although some interesting results have been obtained, previous analyses have been severely hampered by formulations that do not give a complete, or sometimes correct, description of the global solution space and its properties. In contrast, we will here give a complete description of the entire solution space of these models, and we will also describe the solutions’ asymptotic behaviour. More importantly though is that this model allows us to explicitly address some aspects about how to obtain useful dynamical systems treatments of RW $f(R)$ models, and to illustrate various dynamical systems methods. For example, we will situate the entire solution space of the Einstein frame in the state space of the Jordan frame, which allows us to explicitly show how some solutions in the Einstein frame for these models correspond to entire solutions in the Jordan frame, while other solutions can be conformally extended in the Jordan frame. In other words, a local dictionary between the two frames does not always entail global equivalence (the curious reader can skip ahead and take a look at figures 3, 5 and 7 below).

The outline of the paper is as follows. In the next section we make a state space analysis for the $f(R) = R + \alpha R^2$ models, which is used to produce a new regular unconstrained dynamical system formulation on a compact state space for the Jordan frame. We then use this system to perform a local analysis of the fixed points, focusing on non-linear aspects such as blow ups of nilpotent fixed points. This is followed by a global analysis that gives a complete description of the entire solution space of the models, which is depicted and summarized in figure 3. We emphasize the importance of the global topological structure of the state space for a full understanding of the solution space. In section 3 we present a new regular unconstrained dynamical system formulation on a compact state space for the Einstein frame. We then perform a local analysis of fixed points, again focusing on non-linear aspects such as center manifold analysis. It is also shown that the breakdown of the Einstein frame variables at $F = 0$ leads to fixed points in the Einstein frame state space that correspond to coordinate singularities in the Jordan frame, thereby emphasizing the importance of physical interpretation of fixed points. This is followed by global considerations, which yield a complete description of the solution space in the Einstein frame. The section ends with situating the global Einstein frame state space in the global Jordan frame state space by means of the variable transformations that link the two approaches, given in appendix A. This allows us to identify (a) the solutions in the Einstein frame that can be conformally extended in the Jordan frame, and (b) the solutions in the Einstein frame whose evolution completely describes that in the Jordan frame. In section 4 we comment on the relationship between our global Jordan state space approach and other Jordan state space formulations, which are briefly reviewed in appendix B, where their advantages and disadvantages are discussed. We also give a fairly general discussion of $f(R)$ cosmology, which situates the present models in this more general context.
2 Dynamics in the Jordan frame

In this section we first perform a state space analysis of the \( f(R) = R + \alpha R^2 \), \( \alpha > 0 \) vacuum models with flat RW geometry in the Jordan frame. The result is then used to derive a new regular dynamical systems formulation on a global compactified state space, which, in contrast to other formulations, completely covers the entire physical state space of these models, and its asymptotic boundaries. We then use this state space picture to perform a local fixed points analysis, which includes using blow up techniques and center manifold analysis, followed by global considerations. This yields a complete description of the entire solution space, depicted in figure 3.

2.1 Dynamical systems formulation in the Jordan frame

Specializing \( f(R) \) to \( f(R) = R + \alpha R^2 \), \( \alpha > 0 \), the evolution equations (1.2b) and (1.2c) can be written as:

\[
\dot{H} = -2H^2 + \frac{R}{6}, \quad (2.1a)
\]
\[
\ddot{R} = -3H\dot{R} - \frac{R}{6\alpha}, \quad (2.1b)
\]

while the constraint (1.2d) takes the form

\[
-12H\left(\dot{R} + HR + \frac{H}{2\alpha}\right) + R^2 = 0. \quad (2.1c)
\]

For future reference, note that restricting the general discussion leading to eq. (1.4), which shows that the Einstein frame boundary \( F = 0 \) is not in general an invariant subset, to the present case,

\[
F = 1 + 2\alpha R = 0 \quad \Rightarrow \quad R = -\frac{1}{2\alpha}, \quad (2.2)
\]
yields that

\[
\dot{F}|_{F=0} = 2\alpha\dot{R}|_{F=0} = \frac{1}{24H\alpha}, \quad (2.3)
\]
as also follows from (2.1c). As a consequence there are solutions with \( F > 0 \) that come from the region with \( F < 0 \) (vice-versa if \( H < 0 \)) and pass through the \( F = 0 \) surface in the Jordan state space, i.e., some solutions in the Einstein frame can be conformally extended in the Jordan frame (we will show this explicitly below).

Our first step in the state space analysis is to consider dimensions. The dimensions of \( t, H, R, \dot{R} \), and \( \alpha \) are given by \( L, L^{-1}, L^{-2}, L^{-3}, \) and \( L^2 \), respectively, where \( L \) stands for length (recall that the speed of light has been set to one). In contrast to general relativity, the present models, which reflect a general feature of \( f(R) \) gravity, break scale invariance. As a consequence it will not be possible to use scale invariance to decouple an equation, as is often done in dynamical systems treatments of general relativistic problems. However, we can choose dimensionless variables that eliminate the explicit appearance of \( \alpha \) (in general there can of course exist several dimensional parameters for which one can form dimensionless ratios leaving a single dimensional parameter, where only the explicit appearance of the latter can be eliminated by an appropriate choice of variables).

Our next step in our state space analysis is to study and simplify the constraint (2.1c) as much as possible. For the present case it is possible to globally bring the constraint to
a quadratic canonical form where all variables have the same dimension. First note that if one chooses $\dot{R} + HR + \frac{H^2}{2\alpha}$ as a new variable, then this variable as well as $H$ are seen to be 'state space null variables.' By appropriate scaling them with $\alpha$ so that they obtain the same dimension $L^{-2}$ as $R$, and then making a linear transformation so that the constraint (2.1c) takes a canonical quadratic form, results in

$$H = \sqrt{\frac{\alpha}{12}}(t - x),$$  \hspace{1cm} (2.4a)

$$\dot{R} + HR + \frac{H^2}{2\alpha} = \frac{1}{\sqrt{12\alpha}}(t + x),$$  \hspace{1cm} (2.4b)

with

$$-t^2 + x^2 + R^2 = 0,$$  \hspace{1cm} (2.5a)

where $t, x$ and $R$ all have dimension $L^{-2}$. It is important to note that the variable transformation $(H, \dot{R}, R) \rightarrow (t, x, R)$ is globally valid since the Jacobian determinant is given by $1/6$. Thus the constraint equation (2.5a) makes it explicitly clear that the reduced vacuum state space is a 2-dimensional double cone with a joint apex, see figure 1. The flow on this state space is determined by the following evolution equations:

$$\dot{t} = \frac{1}{2\sqrt{12\alpha}}(R - 2\alpha(t - x)^2),$$  \hspace{1cm} (2.5b)

$$\dot{x} = \frac{1}{2\sqrt{12\alpha}}(-3R + 2\alpha(t - x)^2),$$  \hspace{1cm} (2.5c)

$$\dot{R} = \frac{1}{2\sqrt{12\alpha}}(t + 3x - 2\alpha(t - x)R).$$  \hspace{1cm} (2.5d)

It follows from (2.5b) that the two state space cones, defined by $t > 0$ and $t < 0$, are disconnected invariant subsets with a fixed point $t = x = R = 0$, as their common apex. This fixed point, $M$, represents the Minkowski solution, since $t = x = R = 0 \Rightarrow H = \dot{R} = R = 0$. Note that $M$ is the only fixed point on the physical state space and that it is non-hyperbolic\(^3\) (note that this is consistent with (1.5) when specialized to the present case).

Since the original system is invariant under the transformation $(t, H) \rightarrow -(t, H)$, the system (2.5) is invariant under the transformation $(t, t, x) \rightarrow -(t, t, x)$. It therefore suffices to investigate the dynamics on the invariant future state space light cone with $t > 0$ in order to obtain a complete picture of the dynamics. Furthermore, the definition (2.4a) in combination with the constraint (2.5a) implies that $H \geq 0$ on the future state space light cone, i.e., it is arguably the future state space light cone that is of cosmological interest. For these reasons we will only explicitly describe the dynamics on this part of the global state space. It is clear from the above system that the minimum $H = 0$ on the future state space light cone only holds on the line $t = x, R = 0$, but $H = R = 0$ is not an invariant subset (except at $M$). Indeed, since when $H = R = 0$ and $t > 0$

$$\dot{R} = t/\sqrt{3\alpha} > 0,$$  \hspace{1cm} (2.6)

it follows that when $H = 0$ then $R$ is passing through zero from negative to positive values.

\(^2\) The above state space structure is, of course, particular for the present models, but note that for models with $F, R > 0$ one can make a similar globally valid transformation which brings the constraint to the form $-t^2 + x^2 + g(R) = 0$, where $g(R)$ is determined by $f(R)$.

\(^3\) A non-hyperbolic fixed point is one for which a linearization yields eigenvalues that not all have non-zero real parts.
Figure 1. The state space light cone for $f(R) = R + \alpha R^2$, $\alpha > 0$. The shaded part denotes the state space domain of the Einstein frame, i.e., the state space of the Einstein frame is a (non-invariant) subset of that of the Jordan frame.

To understand the present models it is essential to investigate if there are solutions that come from the future state space null infinity. As a next step we therefore aim at producing a regular system of equations on a compact state space. Furthermore, the variables need to be dimensionless so that we eliminate the specific appearance of $\alpha > 0$, which thereby automatically shows that this parameter is not essential for the solution structure of the present models (this should not come as a surprise since it is possible to use units to set e.g. $\alpha = 1$).

Removing the Minkowski fixed point $t = x = R = 0$ from the analysis, we first introduce two new dimensionless variables

$$ (X, S) = \left( \frac{x}{t}, -\frac{R}{t} \right), $$

which are bounded thanks to the constraint (2.5a). We then use that $\alpha t$ is dimensionless and positive on the future state space null cone and introduce the bounded variable

$$ T = \frac{1}{1 + 2\alpha t}. $$

(2.8)

Next, to obtain a regular dimensionless system of evolution equations we introduce a new dimensionless time variable $\bar{t}$, defined by

$$ \frac{dt}{d\bar{t}} = 2\sqrt{12\alpha} T, $$

(2.9)

which leads to

$$ T' = T (1 - T) \left[ TS + (1 - T)(1 - X)^2 \right], $$

(2.10a)

$$ X' = S \left[ T(3 + X) + (1 - T)(1 - X)S \right], $$

(2.10b)

$$ S' = -X \left[ T(3 + X) + (1 - T)(1 - X)S \right], $$

(2.10c)

where $'$ denotes the derivative with respect to $\bar{t}$, subjected to the constraint

$$ X^2 + S^2 = 1. $$

(2.10d)
Note that the above variable change from \((t, x, R)\) to \((T, X, S)\) amounts to a projection where all circles on the light cone with constant \(t\) now become the unit circle given by \(X^2 + S^2 = 1\), where the different circles are parameterized by the value of \(T\), i.e., the above variables cover all of the future state space light cone.\(^4\) The present state space \(\mathbf{S}\), which is just the future state space light cone, is given by a finite cylinder determined by

\[
0 < T < 1, \quad X^2 + S^2 = 1. \tag{2.11}
\]

Because the state space \(\mathbf{S}\) is relatively compact (i.e., its closure is compact) and the equations are completely regular, we can extend the state space \(\mathbf{S}\) to include the invariant boundaries \(T = 0\), and \(T = 1\) to obtain an extended compact state space \(\bar{\mathbf{S}}\). This turns out to be essential since, as we will see, the asymptotic states for all solutions within the physical state space \(\mathbf{S}\) reside on these invariant boundary subsets, see figure 3 below. Indeed, we will prove that there are no fixed points or periodic orbits in the physical interior state space. Thus all solutions in \(\mathbf{S}\) originate from fixed points on \(T = 0\) (the future null infinity of the future state space light cone), and end at a limit cycle on \(T = 1\) (which describes how all solutions asymptotically approach the Minkowski space-time). Thus the present variables represent a compactification of the future state space light cone, where they blow up the neighborhood of the non-hyperbolic Minkowski fixed point in the dynamical system for \(t, x, R\), thereby yielding a correct description of how all solutions approach the future asymptotic Minkowski state.

It is of interest to express \((H, \dot{R}, R)\) in terms of \((T, X, S)\):

\[
H = \frac{1}{2\sqrt{12}\alpha} \frac{(1 - T)(1 - X)}{T}, \tag{2.12a}
\]

\[
\dot{R} = \frac{1}{4\alpha\sqrt{12}\alpha} \frac{(1 - T)}{T} \left(1 + 3X + \frac{(1 - T)(1 - X)S}{T}\right), \tag{2.12b}
\]

\[
R = -\frac{1}{2\alpha} \frac{(1 - T)S}{T}. \tag{2.12c}
\]

These equations reveal that \(H\) is not only zero on the invariant boundary \(T = 1\), but also when \(X = 1, S = 0\), which corresponds to the line \(t = x, R = 0\) on the future state space light cone. Since this is not an invariant subset of the dynamical system (2.10), the solution trajectories pass through \(X = 1\), going from positive to negative \(S\). Indeed, we will show later on that all solutions pass through \(X = 1, S = 0\) infinitely many times.

Finally, although (2.10) is a constrained system, the constraint (2.10d) is easily globally solved by introducing

\[
X = \cos \theta, \quad S = \sin \theta, \tag{2.13}
\]

which results in the following unconstrained regular system of equations:

\[
T' = T(1 - T) \left[T \sin \theta + (1 - T)(1 - \cos \theta)^2\right], \tag{2.14a}
\]

\[
\theta' = -T(3 + \cos \theta) - (1 - T)(1 - \cos \theta) \sin \theta. \tag{2.14b}
\]

\(^4\)The sign in the definition of \(S\) has been chosen in order to simplify the comparison with the Einstein frame state space. The reason for defining \(T\) as a monotonically decreasing function of \(t\) instead of a monotonically increasing one, e.g., by setting \(T = 2t/(1 + 2t)\), is that the Minkowski state, as we will prove, is the future asymptotic state of all solutions, and that this definition also makes the transition to the variables we use to describe the Einstein frame state space more convenient.
The above regular global dynamical system form our ‘master equations’ for dealing with the present models in the Jordan frame. However, since the present formulation differs substantially from the ones in the literature it is of interest to take a look at some other formulations for the Jordan frame and make comparisons, which we do in section 4 and in appendix B. Finally, we stress that the above system was possible because we adapted the variables to the particular state space properties of the present models; other models need different variables. However, to find one (or more) set(s) of (differentiably overlapping) useful variables covering the entire state space and its possible infinite limits, one needs to go through the same steps of (i) state space analysis and (ii) state space adapted coordinates as for the present illustrative example. Next we turn to illustrating (iii): local and global dynamical systems analysis.

2.2 Local fixed point analysis in the Jordan frame

In this subsection we perform a local analysis of the fixed points of our new regular dynamical system (2.14) on the compactified global state space, with a focus on necessary non-linear aspects. As we will prove below, all fixed points are located on the boundary subset \( T = 0 \), associated with \( H \to \infty \). Considering this subset, we find that there are two fixed points:

\[
\begin{align*}
\text{R}: & \quad \theta = \pi + 2n\pi, \\
\text{dS}: & \quad \theta = 2n\pi,
\end{align*}
\]

with \( n \) an integer. The motivation for the nomenclature for these fixed points will be made clear below.

The fixed point R is a hyperbolic source, while dS has two zero eigenvalues. More precisely, it is nilpotent of first degree. Such fixed points are dealt with by means of so-called blow up techniques, described in detail in [17] and [18]. In order to bring the problem to standard form for nilpotent fixed points we first scale the variables \( \theta \) and \( T \) and introduce the following notation (without loss of generality, we choose the representation \( \theta = 0 \) for the fixed point dS):

\[
x = -\theta, \quad y = 4T.
\]

This leads to a dynamical system on the form

\[
\begin{align*}
x' &= y + P(x,y); \\
y' &= Q(x,y); \\
P(x,y) &= a(x) + b(x)y, \\
Q(x,y) &= c(x)y + d(x)y^2 + e(x)y^3,
\end{align*}
\]

where

\[
\begin{align*}
a(x) &= -(1 - \cos x) \sin x, & b(x) &= \frac{1}{4} (1 - \cos x) (1 - \sin x), \\
c(x) &= (1 - \cos x)^2, & d(x) &= -\frac{1}{4} (\sin(x) + 2(1 - \cos x)^2), \\
e(x) &= \frac{1}{16} (\sin x + (1 - \cos x)^2).
\end{align*}
\]

Next we introduce a new variable \( Y \) instead of \( y \):

\[
Y = y + P(x,y) = a(x) + (1 + b(x))y,
\]
which leads to
\[ y = \frac{Y + (1 - \cos x) \sin x}{1 - \frac{1}{4}(1 - \cos x)(1 - \sin x)}. \]  
(2.20)

In the neighborhood of the origin this means that the dynamical system takes the form
\[ x' = Y, \]  
(2.21a)
\[ Y' = \frac{x^7}{16} (1 + h(x)) - \frac{3x^2}{2} (1 + g(x)) Y + j(x, Y) Y^2, \]  
(2.21b)

where
\[ h(x) = -\frac{7}{24} x^2 - \frac{1}{4} x^3 + \ldots, \]  
(2.22a)
\[ g(x) = -\frac{1}{3} x^2 - \frac{1}{8} x^3 + \ldots, \]  
(2.22b)
\[ j(x, Y) = -\frac{1}{2} x + \frac{3}{8} x^2 + \ldots + \left( \frac{1}{16} x + \ldots \right) Y. \]  
(2.22c)

We now proceed by making the following so-called blow-up transformation
\[ (x, Y) = (u, u^3 \bar{y}), \]  
(2.23)
and change time variable by dividing the right hand sides by \( u^2 \). This results in
\[ u' = \bar{y} u, \]  
(2.24a)
\[ \bar{y}' = -\frac{3}{2} \bar{y} (1 + 2 \bar{y} + f(u, \bar{y})) + \frac{u^2}{16} (1 + h(u)), \]  
(2.24b)

where the ‘ now refers to the new time variable and where
\[ f(u, \bar{y}) = g(u) - \frac{2u}{3} j(u, \bar{y}) \bar{y}, \]  
(2.25)

which obeys \( f(0, \bar{y}) = 0 \) and \( \frac{\partial f}{\partial u}(0, \bar{y}) = \frac{\partial f}{\partial \bar{y}}(u, 0) = 0 \).

It follows that on the \( u = 0 \) subset there are two fixed points
\[ \text{S: } \bar{y} = -\frac{1}{2}, \]  
(2.26a)
\[ \text{dS: } \bar{y} = 0. \]  
(2.26b)

The fixed point S is a hyperbolic saddle while dS is a non-hyperbolic fixed point with eigenvalues zero and \(-3/2\).

To deal with dS we apply center manifold theory (for examples of center manifold analysis in cosmology, see e.g. [19]–[23]). The center manifold \( W^c \) can be obtained as the graph \( \bar{y} = \varphi(u) \) near \( (u, \bar{y}) = (0, 0) \) (i.e., use \( u \) as an independent variable), where \( \varphi(0) = 0 \) (fixed point condition) and \( \frac{d \varphi}{du}(0) = 0 \) (tangency condition). This leads to
\[ -\frac{3}{2} \varphi(u) [1 + 2 \varphi(u) + f(u, \varphi(u))] + \frac{u^2}{16} (1 + h(u)) - u \varphi(u) \frac{d \varphi}{du} = 0. \]  
(2.27)

This differential equation can be solved approximately by representing \( \varphi(u) \) as the formal power series
\[ \varphi(u) = \sum_{i=2}^{n} a_i u^i + \mathcal{O}(u^{n+1}) \quad \text{as} \quad u \to 0. \]  
(2.28)
Figure 2. Comparisons of the center manifold expansion of dS with the numerically computed solution given by the solid line. The leading-order term in the center manifold expansion is given by the dotted line; the leading-order correction to this by the dashed line, and the next order correction by the long-dashed line.

Solving algebraically for the coefficients we find

$$\varphi(u) = \frac{u^2}{24} \left( 1 - \frac{7}{72} u^2 + \ldots \right) \quad \Rightarrow \quad Y = \frac{u^5}{24} \left( 1 - \frac{7}{72} u^2 + \ldots \right).$$

The present case corresponds to figure 3.16 (a) on p. 112 in [18]. The saddle S is associated with orbits (i.e., solution trajectories) that approach \( \bar{dS} \) from the region \( T \leq 0 \) while the center manifold of dS with \( T > 0 \) corresponds to the only solution from \( dS \) that enters the physical state space. Inserting the above expression for \( Y \) into (2.20) leads to the expression (note that \( \theta < 0 \))

$$T = - \left( \frac{\theta}{2} \right)^3 \left[ 1 - \frac{1}{6} \left( \frac{\theta}{2} \right)^2 + \left( \frac{\theta}{2} \right)^3 + \ldots \right].$$

This is a series expansion that approximates the ‘inflationary attractor solution’ that enters the physical state space from \( dS \). The accuracy of this approximation compared to the numerical solution can be found in figure 2. If one is so inclined, one can obtain further approximation improvements by means of so-called Padé approximants, as described in e.g. [19, 20], and references therein.

Finally, note that the deceleration parameter \( q \) in the Jordan frame, defined by \( dH/dt = -(1 + q)H^2 \), is given by

$$q = 1 + 4 \left( \frac{T}{1 - T} \right) \frac{\sin \theta}{(1 - \cos \theta)^2}.$$

It follows that, except at \( \bar{dS} \), where \( \cos \theta = 1 \) and (2.31) is ill-defined, the deceleration parameter takes the value \( q = 1 \) on the invariant boundary \( T = 0 \), including the fixed point R. The reason for choosing this notation for the fixed point is due to the fact that this value of \( q \) corresponds to a Universe filled with radiation in general relativity. On the other hand, the fixed point dS does not describe the asymptotic features of the solution that originates from it into the physical state space, since the right hand side of (2.31) diverges for \( \theta = 2n\pi \). However, inserting the asymptotic expression (2.30) for the center manifold of dS into (2.31) leads to the following expansion in \( \theta \):

$$q = -1 + \frac{1}{12} \theta^2 + \frac{1}{120} \theta^4 + \ldots$$
which reveals that the center manifold solution has \( q = -1 \) asymptotically, i.e., ‘the inflationary attractor solution’ originates from a (quasi) de-Sitter state. Furthermore, just as for the solutions originating from \( R \), this state is associated with \( H \to \infty \), as follows from (2.12a) and (2.30).

In the next section we will prove that the one-parameter set of solutions that enter the physical state space from \( R \), and the single solution that comes from \( \overline{dS} \), all of them originating from a singular state at \( H \to \infty \) (i.e., at future null infinity with respect to the light cone state space), constitute all solutions in the physical state space.

### 2.3 Global analysis in the Jordan frame

Consider the function

\[
J = \frac{(1 - T)(3 + \cos \theta)}{T} > 0,
\]

which obeys the equation

\[
J' = -2 \frac{(1 - T)^2(1 - \cos \theta)^2}{T}.
\]

It follows that \( J \) is monotonically decreasing when \( 0 < T < 1 \) and \( \theta \neq 2n\pi \). Furthermore, since

\[
J''|_{\theta = 2n\pi} = 0, \quad J'''|_{\theta = 2n\pi} = 0, \quad J''''|_{\theta = 2n\pi} = -4(1 - T)^2 T,
\]

it follows that \( \cos \theta = 1 \) only represents an inflection point in the evolution of \( J \). As a consequence \( J \to \infty \) when \( \ell \to -\infty \), which implies that all orbits in the physical state space \( S \) with \( 0 < T < 1 \) originate from the subset \( T = 0 \), while \( J \to 0 \) when \( \ell \to +\infty \), which implies that all orbits in the physical state space \( 0 < T < 1 \) end at the subset \( T = 1 \). There are thereby no fixed points or periodic orbits in the physical state space \( S \).

The analysis of the subset \( T = 0 \) is trivial and our previous investigation of the fixed points on \( T = 0 \) shows that there is a single orbit that enters the physical state space from \( dS \) while there is a 1-parameter set that originates from \( R \). The above global considerations based on \( J \) proves that these local fixed point results describe the origins of all solutions in the physical state space \( S \).

The invariant subset \( T = 1 \) yields the equation

\[
\theta' = -(3 + \cos \theta),
\]

as follows from (2.14). Since \( 3 + \cos \theta > 0 \) it follows that \( T = 1 \) represents a periodic orbit where \( \theta \) is monotonically decreasing. From our considerations of the function \( J \), this proves that this periodic orbit is a limit cycle that describes the future asymptotic behaviour of all solutions in the physical state space \( S \), i.e., it constitutes the \( \omega \)-limit set of all solutions with \( 0 < T < 1 \). As an aside, this provides a simple cosmological example that it is often not sufficient to just do fixed point analysis.

We end this section by depicting representative solutions describing the entire solution space in the Jordan frame in figure 3. Note that there is an open set of solutions that are not attracted to the inflationary attractor solution until the oscillatory regime at late times, where all solutions approach the future attractor, i.e., the limit cycle at \( T = 1 \). Thus, to argue that the inflationary attractor solution is in some sense an attractor requires the introduction of some measure. In this context we refer to the recent interesting discussion about scales and measures given in [24], and references therein.
Figure 3. Two representations of the Jordan frame state space. All solutions in the Jordan frame state space end at the periodic orbit at $T = 1$, and they all originate from the fixed point $R$, except for ‘the inflationary attractor solution’ (solid line) that comes from $dS$. The space-dotted lines in figure (b) depict constant values of the monotone function $J$.

As a final remark, we note that heuristic approximations in the Jordan frame for the inflationary attractor solution and for the oscillatory ‘reheating’ regime at late times have been given in [12], and later reproduced in [3]. Next we will deal with Einstein frame dynamics, and then we will present rigorous approximations schemes for the oscillatory regime at late times. Such methods can also be applied to the Jordan frame, or one can translate the approximations in the Einstein frame to the Jordan frame by means of the relations given in appendix A, but for brevity we will refrain from doing this.

3 Dynamics in the Einstein frame

The analysis in the Einstein frame will serve as an illustrative example of (iv): physical solution space interpretation. It exemplifies the situation where a state space only covers part of the Jordan frame state space, which, e.g., leads to coordinate singularities in the form of fixed points. Thus fixed points in a given formulation may not correspond to physical phenomena, but may instead reflect that the formulation breaks down, thus necessitating a physical interpretation. (We will see several other examples of this in appendix B.) When dealing with the Einstein frame we will introduce a new regular unconstrained dynamical system on a compact state space in the Einstein frame, which gives a complete description of the solution space in this frame. This will enable us to situate the entire solution space in the Einstein frame in the state space of the Jordan frame.

3.1 Dynamical systems formulation in the Einstein frame

The Einstein frame formulation of $f(R)$ gravity is based on the following conformal transformation of the Jordan metric $g_{\mu\nu}$ to the Einstein frame metric $\tilde{g}_{\mu\nu}$ (see e.g. [3] and [25]):

$$\tilde{g}_{\mu\nu} = F g_{\mu\nu}, \quad F = \frac{df}{dR},$$

(3.1)
which thereby assumes that $F > 0$. Thus $F = 0$ constitutes the boundary between the Einstein and the Jordan frame state spaces, of which it in general will be a subset, but not an invariant subset in the Jordan frame, as shown in eq. (2.3). As a consequence, as we will see, there are solutions with $F > 0$ that come from the region with $F < 0$ and pass through the $F = 0$ surface in the Jordan state space, i.e., some solutions in the Einstein frame can be (conformally) extended in the Jordan frame.

Introducing
\[ \kappa \phi = \sqrt{\frac{3}{2}} \ln F, \quad V(\phi) = RF - f (4.2) \]
the action in the Jordan frame (1.1) transforms to an action with the Einstein-Hilbert form for a scalar field minimally coupled to gravity,
\[ S = \int \left\{ \frac{\tilde{R}}{2 \kappa^2} - \frac{\tilde{g}^{\mu\nu}}{2} \left( \nabla_\mu \phi \right) \left( \nabla_\nu \phi \right) - V(\phi) + F^{-2}(\phi) \tilde{L}_m \right\} \sqrt{-\det \tilde{g}} d^4x, \quad (3.3) \]
where $\tilde{R}$ is the curvature scalar of the Einstein frame metric $\tilde{g}_{\mu\nu}$.

Specializing to the present $f(R) = R + \alpha R^2$ vacuum models leads to (see e.g. [26])
\[ \kappa \phi = \sqrt{\frac{3}{2}} \ln(1 + 2\alpha R), \quad V(\phi) = V_0 \left( 1 - e^{-\sqrt{\frac{2}{3}} \kappa \phi} \right)^2, \quad (3.4) \]
where
\[ V_0 = \frac{1}{8\alpha \kappa^2} > 0. \quad (3.5) \]

The above potential is depicted in figure 4.

In a flat RW geometry, the present models in the Einstein frame yield the following evolution equations
\[
\begin{align*}
\frac{d\tilde{a}}{dt} &= \tilde{H} \tilde{a}, \quad \text{(3.6a)} \\
\frac{d\tilde{H}}{dt} &= \frac{\kappa^2}{2} \left( \frac{d\phi}{dt} \right)^2, \quad \text{(3.6b)} \\
\frac{d^2\phi}{dt^2} &= -3H \frac{d\phi}{dt} - 2\sqrt{\frac{2}{3}} \kappa V_0 \left( 1 - e^{-\sqrt{\frac{2}{3}} \kappa \phi} \right) e^{-\sqrt{\frac{2}{3}} \kappa \phi}, \quad \text{(3.6c)}
\end{align*}
\]
and the constraint
\[ 3 \tilde{H}^2 = \kappa^2 \left[ \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V_0 \left( 1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi} \right)^2 \right], \tag{3.6d} \]

where \( \tilde{t} \) denotes the Einstein frame proper time variable, while \( \tilde{a} \) and \( \tilde{H} \) are the Einstein frame scale factor and Hubble variable, respectively.

As in the Jordan frame, the Einstein frame scale factor \( \tilde{a} \) decouples, leaving a reduced closed system of first order evolution equations for \( (\tilde{H}, \frac{d\phi}{d\tilde{t}}, \phi) \) determined by (3.6b) and (3.6c), which, due to the constraint (3.6d), yield a dynamical system describing a flow on a 2-dimensional state space. Once the reduced system of first order equations has been solved, equation (3.6a) yields \( \tilde{a} \propto \exp(\int d\tilde{t} \tilde{H}) \).

Let us now follow the ideas presented in [19, 20] for scalar fields and introduce new variables, which give a global dynamical system formulation on the reduced Einstein state space:

\[
\left( \tilde{T}, \tilde{X}, \Sigma_{\phi} \right) = \left( \frac{M}{M + \tilde{H}}, \frac{M \left( 1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi} \right)}{2\tilde{H}}, \frac{\kappa^2 \frac{d\phi}{d\tilde{t}}}{\sqrt{6}\tilde{H}} \right), \tag{3.7a}
\]

\[
\left( \tilde{H}, \kappa\phi, \kappa \frac{d\phi}{d\tilde{t}} \right) = \left( M\tilde{t}^{-1}, -\sqrt{\frac{3}{2}} \ln \left( 1 - 2\tilde{t}^{-1} \tilde{X} \right), \sqrt{6} M\tilde{t}^{-1}\Sigma_{\phi} \right), \tag{3.7b}
\]

where
\[
\tilde{t} = \left( \frac{\tilde{T}}{1 - \tilde{T}} \right), \quad M = 2\kappa \sqrt{\frac{V_0}{3}} = \frac{1}{\sqrt{6}\alpha}, \tag{3.8}
\]

and a new time variable
\[
\frac{d\tilde{\tau}}{dt} = M\tilde{T}^{-1}, \tag{3.9}
\]

which takes into account the different asymptotic scales of the model, as described in [19, 20].

This leads to the following evolution equations

\[
\frac{d\tilde{T}}{d\tilde{\tau}} = 3\tilde{T}(1 - \tilde{T})^2\Sigma_{\phi}^2, \tag{3.10a}
\]

\[
\frac{d\tilde{X}}{d\tilde{\tau}} = \Sigma_{\phi} \left[ 3(1 - \tilde{T})\tilde{X}\Sigma_{\phi} + \tilde{T}F \right], \tag{3.10b}
\]

\[
\frac{d\Sigma_{\phi}}{d\tilde{\tau}} = -\tilde{X} \left[ 3(1 - \tilde{T})\tilde{X}\Sigma_{\phi} + \tilde{T}F \right], \tag{3.10c}
\]

subjected to the constraint
\[
1 = \Sigma_{\phi}^2 + \tilde{X}^2, \tag{3.10d}
\]

where
\[
\tilde{T}F = \tilde{T} - 2(1 - \tilde{T})\tilde{X}. \tag{3.11}
\]

Note that, since
\[
\frac{d(\tilde{T}F)}{d\tilde{\tau}} = (1 - \tilde{T})\Sigma_{\phi} \left[ 3(1 - \tilde{T})\Sigma_{\phi} - 2 \right] (\tilde{T}F), \tag{3.12}
\]

\[5\text{In the present scalar field context, this paper can be regarded as one in a series about scalar field inflation models [19, 20], and quintessence models [27, 28], with the aim of showing how one can produce useful dynamical systems and apply various dynamical systems and approximation techniques.}\]
$\tilde{T}F = 0$ is an invariant boundary subset in the Einstein frame, but not in the Jordan frame. This difference is due to the fact that the relation between the time variables in the two frames is singular at $F = 0$.

The relatively compact Einstein state space $\tilde{\mathcal{S}}$ is defined by the cylinder with $0 < \tilde{T} < 1$ and with the region $\tilde{T}F = \tilde{T} - 2(1 - \tilde{T})\tilde{X} \leq 0$ cut out from it. The state space can then be regularly extended to include the invariant boundary subsets $\tilde{T} = 1$, $\tilde{T}F = 0$, and $\tilde{T} = 0$ when $\tilde{X} \leq 0$, yielding the extended state space $\tilde{\mathcal{S}}$ (see figure 5). Note that constant $\tilde{T}$ surfaces in the state space $\tilde{\mathcal{S}}$ correspond to constant values of $\tilde{H}$, while the invariant boundaries are associated with the asymptotic limits $\tilde{H} \to 0$ ($\tilde{T} = 1$), $\phi \to +\infty$ ($F = 0$), and $\tilde{H} \to \infty$ ($\tilde{T} = 0$).

The constraint (3.10d) can be globally solved by introducing $\tilde{X} = \cos \tilde{\theta}$, $\Sigma \phi = \sin \tilde{\theta}$, which leads to the unconstrained 2-dimensional dynamical system

$$\frac{d\tilde{T}}{d\tilde{\tau}} = 3\tilde{T}(1 - \tilde{T}^2)\sin^2 \tilde{\theta},$$  
(3.14a)
$$\frac{d\tilde{\theta}}{d\tilde{\tau}} = \left(2 - 3\sin \tilde{\theta}\right)(1 - \tilde{T})\cos \tilde{\theta} - \tilde{T}.\quad (3.14b)$$

Finally, the above changes of independent and dependent variables yield $d\tilde{a}/d\tilde{\tau} = (1 - \tilde{T})\tilde{a}$, which leads to a quadrature for $\tilde{a}$ once $\tilde{T}$ has been found. If one wants to express the results in terms of the Einstein frame proper time variable $\tilde{\tau}$, one also needs to integrate eq. (3.9).

### 3.2 Local fixed point analysis in the Einstein frame

The dynamical system (3.14) admits 4 fixed points on $\tilde{\mathcal{S}}$, all located on the boundaries $\tilde{T} = 0$ and $\tilde{T}F = 0$:

- **M$_\pm$:** $\tilde{T} = 0$, $\Sigma \phi = \pm 1$, $\tilde{X} = 0$ \quad $\tilde{\theta} = 2n\pi \pm \frac{\pi}{2}$, (3.15a)
- **PL:** $\tilde{T} = 0$, $\Sigma \phi = \frac{2}{3}$, $\tilde{X} = -\frac{\sqrt{5}}{3}$ \quad $\tilde{\theta} = \arccos \left(-\frac{\sqrt{5}}{3}\right) + 2n\pi$, (3.15b)
- **dS:** $\tilde{T} = \frac{2}{3}$, $\Sigma \phi = 0$, $\tilde{X} = 1$ \quad $\tilde{\theta} = 2n\pi$, (3.15c)

where $n$ is an integer.

The Einstein frame deceleration parameter $\tilde{q}$, defined by $d\tilde{H}/d\tilde{\tau} = -(1 + \tilde{q})\tilde{H}^2$, is given by

$$\tilde{q} = -1 + 3\Sigma_\phi^2 = -1 + 3\sin^2 \tilde{\theta}.\quad (3.16)$$

It follows that M$_+$ and M$_-$ have $\tilde{q} = 2$, which for the minimally coupled scalar field interpretation corresponds to a massless state ($V(\phi) = 0$), while dS corresponds to a (quasi) de Sitter state in the Einstein frame (associated with $\phi \to +\infty$), since $\tilde{q} = -1$ for the solution that originates from this fixed point asymptotically.$^6$ The notation PL stands for power law, since the asymptotic behaviour of the solutions that originate from this fixed point in

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$^6$For further discussion on (quasi) de Sitter states, see [19, 20].
the Einstein frame are described by the self-similar power-law solution associated with an exponential potential.

The fixed points $M_{\pm}$ are hyperbolic sources with eigenvectors tangential to the invariant subsets $\tilde{T} = 0$ and $\tilde{T}\tilde{F} = 0$. The fixed point PL is a hyperbolic saddle with a single solution entering the state space $\tilde{S}$. Finally the fixed point dS has one negative eigenvalue and a zero eigenvalue, where the latter is associated with the center manifold of dS. The center manifold in turn corresponds to the inflationary attractor solution in both the Einstein and Jordan state spaces, which is the single solution that enters $\tilde{S}$ from dS.

We now use center manifold analysis to establish that there is only a single solution that enters the state space $\tilde{S}$ from dS, and to obtain an approximation for this solution in the vicinity of dS in $\tilde{S}$. Linearizing the equations in the neighborhood of dS yields the following stable, $E^s$, and center, $E^c$, tangential subspaces, respectively:

$$E^s = \left\{ (\tilde{T}, \tilde{\theta}) | \tilde{T} = \frac{2}{3} \right\}, \quad (3.17a)$$

$$E^c = \left\{ (\tilde{T}, \tilde{\theta}) | \tilde{T} - \frac{2}{3} + \frac{\tilde{\theta}}{3} = 0 \right\}. \quad (3.17b)$$

To investigate the center manifold $W^c$ associated with the tangent space $E^c$, we adapt the variables to the location of dS and the tangent space and replace $\tilde{T}$ and $\tilde{\theta}$ with

$$u = \tilde{T} - \frac{2}{3}, \quad (3.18a)$$

$$v = \tilde{T} - \frac{2}{3} + \frac{\tilde{\theta}}{3}. \quad (3.18b)$$

so that dS is located at $(u, v) = (0, 0)$. The center manifold $W^c$ can be obtained as the graph $v = h(u)$ near $(u, v) = (0, 0)$, where $h(0) = 0$ (fixed point condition) and $\frac{dh}{du}(0) = 0$ (tangency condition). Inserting these relationships into eq. (3.14) and using $u$ as the independent variable leads to

$$3\tilde{T}(u)(1 - \tilde{T}(u))^2 \sin^2 \tilde{\theta}(u) \left( \frac{dh(u)}{du} - 1 \right)$$

$$- \frac{1}{3} \left[ (2 - 3 \sin \tilde{\theta}(u)) (1 - \tilde{T}(u)) \cos \tilde{\theta}(u) - \tilde{T}(u) \right] = 0,$$

where $\tilde{T}(u) = u + 2/3$ and $\tilde{\theta}(u) = 3(h(u) - u)$. As before, we can solve the equation approximately by representing $h(u)$ as a formal power series truncated at some chosen order $n$. Inserting this into eq. (3.19) and algebraically solving for the coefficients leads to

$$h(u) = -2u^2 - 6u^3 + O(u^4). \quad (3.20)$$

It follows that the single solution that originates from dS into $\tilde{S}$ (the ‘inflationary attractor solution’) is described by the approximate expansion

$$\tilde{\theta}(u) = -3u \left\{ 1 + 2u + 6u^2 + O(u^3) \right\}. \quad (3.21)$$

---

This was also established by means of center manifold analysis in [21], although in other variables. Furthermore, together with [29], which we will comment on further later on, this paper gives the most complete description of the present models that we have found in the literature. Even so, in contrast to the present paper, it does not give a global state space picture, and it is restricted to the Einstein frame formulation.
Figure 5. The extended state space $\tilde{\mathbf{S}}$ of the Einstein frame, consisting of a finite cylinder with a removed non-physical $\tilde{TF} < 0$ region, and representative solutions.

3.3 Global analysis in the Einstein frame

In this case $\tilde{H}$ is monotonically decreasing, except when $\tilde{q} = -1$, which corresponds to the following monotonicity properties of $\tilde{T}$ in our Einstein frame state space setting. From its evolution equation (3.14a) we see that $\tilde{T}$ is monotonically increasing in $\tilde{S}$ when $\tilde{q} \neq -1$. Since by (3.16) $\tilde{q} = -1$ corresponds to $\tilde{\theta} = 2n\pi$, we have

$$\frac{d\tilde{T}}{d\tilde{\tau}} \bigg|_{\tilde{\theta}=2n\pi} = 0, \quad \frac{d^2\tilde{T}}{d\tilde{\tau}^2} \bigg|_{\tilde{\theta}=2n\pi} = 0, \quad \frac{d^3\tilde{T}}{d\tilde{\tau}^3} \bigg|_{\tilde{\theta}=2n\pi} = 18\tilde{T} \left( 1 - \tilde{T} \right)^2 \left( \frac{2}{3} - \tilde{T} \right),$$

(3.22)

and thus $\tilde{q} = -1$ only represents an inflection point in the evolution of $\tilde{T}$ in $\tilde{S}$ (there are no invariant sets at $\tilde{T} = 2/3$ in $\tilde{S}$). The monotonicity of $\tilde{T}$ in combination with the expression for $d^3\tilde{T}/d\tilde{\tau}^3|_{\tilde{\theta}=2n\pi}$, shows that solutions in $\tilde{S}$ either come from $\tilde{T} = 0$ or from dS. Combining this with the previous local analysis show that there are two one-parameter sets of solutions entering $\tilde{S}$ from $M_+$ and $M_-$, respectively, a single solution entering from PL, and one from dS.

At the boundary subset $\tilde{T} = 1$ we obtain

$$\frac{d\tilde{\theta}}{d\tilde{\tau}} = -1$$

(3.23)

from (3.14). This shows that $\tilde{T} = 1$ is a periodic orbit with monotonically decreasing $\tilde{\theta}$. Furthermore, the monotonicity of $\tilde{T}$ shows that this is the limit cycle of all orbits in $\tilde{S}$, thus constituting their $\omega$-limit set. The solution space in the Einstein frame is depicted in figure 5.

We now present some approximation schemes for solutions at late times close to $\tilde{T} = 1$, which corresponds to $\tilde{H} \to 0$, i.e., we will give approximations for the oscillatory phase at late times. We first consider an averaging technique used in [19]. We thereby take the average
(a) The averaged solution at late times.  
(b) An oscillatory approximation for the oscillatory late time regime

**Figure 6.** The plots show the behavior at late times of the numerically computed inflationary attractor solution (solid line), the averaged solution, and the oscillatory late time approximation in the Einstein frame state space (the dotted lines in the two figures).

with respect to $\tilde{\theta}$ of the right hand side of (3.14) (since $-\tilde{\theta} \to \tilde{\tau} \propto \tilde{t} \to \infty$ while $\tilde{T}$ slowly approaches one), which leads to

$$\frac{d\tilde{T}}{d\tilde{\tau}} = \frac{3}{2} \tilde{T}(1 - \tilde{T})^2,$$

$$\frac{d\tilde{\theta}}{d\tilde{\tau}} = -\tilde{T}.$$  

(3.24a)

(3.24b)

It follows that

$$\tilde{\theta} = -\frac{2}{3(1 - \tilde{T})} + C,$$

(3.25)

where $C = \tilde{\theta}_i + 2/3(1 - \tilde{T}_i)$, where $(\tilde{\theta}_i, \tilde{T}_i)$ is some initial point for the trajectory. This approximation is valid for all solutions when $\tilde{T}$ approaches one, including the center manifold attractor solution, see figure 6(a).

In [30] Rendall gave rigorous results for late time behavior for scalar field models with a potential with a minimum that asymptotically can be described by a $\phi^2$ potential. Since this covers the present models, we can translate the results in [30], which yields the following asymptotic approximation:

$$\tilde{\theta} = -\tilde{t} - \frac{3 + 2 \cos(2\tilde{t})}{4\tilde{t}}, \quad \tilde{T} = \left(1 + \frac{2}{3(\tilde{t} - \tilde{t}_0)} \left(1 + \frac{\sin(2\tilde{t})}{2\tilde{t}}\right)\right)^{-1},$$

(3.26)

where $\tilde{t}_0$ is a constant and $\tilde{t}$ is proper time in the Einstein frame (with $M$ normalized to one). The relations given in eq. (3.26) describe a parameterized curve in the global state space $\tilde{S}$, which is plotted in figure 6(b); note that the oscillatory approximation becomes increasingly accurate toward the future, reflecting that it describes the asymptotic evolution at late times.

### 3.4 Situating the Einstein frame state space in that of the Jordan frame

The Einstein frame is characterized by the conformal factor $F$ which has to satisfy $F > 0$, and therefore the Einstein frame state space is characterized by a boundary $F = 0$ in the
Jordan frame state space. In the present case the conformal factor $F = 1 + 2\alpha R$ is given in terms of our Jordan frame state space variables by

$$F(\theta, T) = 1 - \frac{(1 - T) \sin \theta}{T}$$

(3.27)

The curve $F = 0$ intersects the invariant boundary $T = 0$ at $\theta = 2n\pi$ and $\theta = \pi + 2n\pi$, i.e., at the fixed points dS and R. The variable $T$ has a maximum along the curve $F = 0$ at $T = 1/2$, which happens when $\theta = \pi/2 + 2n\pi$. Moreover, at $F = 0$ it follows that

$$\left.\frac{dF}{dT}\right|_{F=0} = 2T(1 + \cos \theta)$$

(3.28)

which is everywhere positive except at $\theta = \pi + 2n\pi$, i.e., at the fixed point R on the boundary of the Jordan frame state space where it vanishes (see figure 7). This implies that there are no orbits that pass through $F = 0$ from the region $F > 0$ into the state space where $F < 0$.

The relation between the solutions originating from the fixed points in the Jordan and Einstein frame state spaces can be obtained by noting that

$$q = 1 - \left(\frac{\tilde{T}}{1 - \tilde{T}}\right) \frac{\tilde{X}}{(1 - \Sigma \phi)^2}.$$  

(3.29)

By inserting the fixed point dS into the above expression, it follows that the asymptotic (quasi) de Sitter state dS in the Einstein frame state space is also the asymptotic (quasi) de Sitter state in the Jordan frame state space. Thus $\tilde{H}$ is asymptotically finite for the inflationary attractor solution in the Einstein frame, while $H \to \infty$ in the Jordan frame.

We also note that $q = 1$ for PL and M$, while the above expression for $q$ is ill-defined for $M^+$. However, since $M^+$ is hyperbolic and the dynamical system (3.14) is analytic we can insert the solution of the linearized equations in the neighbourhood of $M^+$ to find a suitable approximation for $q$. It turns out that near $M^+$ $q$ becomes a constant with the value determined by the ratio of the two arbitrary constants associated with the two eigenvectors. This describes a 1-parameter set of solutions passing through $F = 0$ from negative to positive $F$ in the Jordan frame state space. These results establish that PL and M$ yield the solutions that originate from the fixed point R in the Jordan state space (where the solution from PL corresponds to the solution that initially is tangential to $F = 0$ in the Jordan frame state space) while the solutions that originate from $M^+$ correspond to a coordinate singularity associated with the breakdown of the Einstein frame at $F = 0$.

The solution that originates from PL divides the orbits that originate from R in the Jordan frame into two classes: (a) Orbits, like itself and the single orbit from dS (which is the same orbit as that coming from dS in the Einstein frame), that throughout their evolution have $F > 0$ (i.e. their evolution in the Jordan frame is entirely covered by that in the Einstein frame), and (b) orbits that begin with $F < 0$ in the Jordan frame and then pass through $F = 0$ and subsequently have $F > 0$ throughout their remaining evolution. The last class, therefore, consists of solutions in the Einstein frame that are past conformally extendible in the Jordan frame; see figure 7 where the shaded region corresponds to the region in the Jordan state space that is conformal to the Einstein frame (cf. figure 1; note that is easy to translate the present results to the original state space picture by means of the figures). If one is so inclined, one can obtain further details of the above nature by inserting approximate asymptotic solutions into eq. (A.1) in appendix A, which describes the transition between the Jordan and Einstein frame state space variables.
Figure 7. As in figure 1, the shaded region in the figure above depicts the domain of the Jordan state space that is conformal to the Einstein frame, given by $F = 1 + 2\alpha R > 0$. Note that the special space-dashed solution, originating from $R$ in the direction transverse to the invariant boundary $T = 0$, divides the solutions into those that enter $S$ from the $F < 0$ region (exemplified by the dotted and dash-dotted lines) and those that are always in the region $F > 0$ (exemplified by the dashed and long-dashed lines and the inflationary attractor solution originating from $dS$).

4 Discussion

The global regular system we have presented for the Jordan frame naturally conveys the global properties of the models at hand, as illustrated by figure 3. Nevertheless, it is by no means an optimal dynamical system for all aspects one might want to investigate: there are other dynamical systems that have complementary properties. This is already exemplified by the fact that our dynamical system for the Einstein frame, among other things, simplifies the asymptotic analysis of the inflationary attractor solution originating from the dS fixed point and offers various approximation schemes for the oscillatory regime at late times, thereby complementing other heuristic Jordan frame methods [3, 12]. Another useful system is discussed in appendix B (where its close relationship to the works in [29, 31] is also commented on). It is based on a variable transformation from $(H, R)$ to the variables $(z, q)$, defined by

$$z = \frac{1}{12\alpha H^2}, \quad q = 1 - \frac{R}{6H^2}, \quad (4.1)$$

and the time variable $N = \ln a$, which leads to the following simple regular system of unconstrained equations:

$$\frac{dz}{dN} = 2(1 + q)z, \quad (4.2a)$$

$$\frac{dq}{dN} = z - \frac{3}{2}(1 - q^2). \quad (4.2b)$$

As discussed in appendix B, the variable transformation breaks down at $H = 0$, which for $z$ and $q$ are located at infinity, i.e., $z$ and $q$ are unbounded. Straightforward compactifications
such as a Poincaré compactification of $z$ and $q$ are inappropriate, since such compactifications result in an erroneous state space topology, which may result in wrong conclusions about the properties of the solutions. For example, such a compactification ruins, or at least complicates, a treatment of the oscillatory regime at late times. This illustrates that it is necessary to take into account the global topological properties of the physical state space in order to obtain a correct description of the solution space and its properties, which illustrates a non-local aspect (apart from fixed points reflecting coordinate singularities) concerning the relationship between dynamical systems formulations and (iv): physical solution space interpretation. Nevertheless, the above system has local advantages. The system admits two fixed points, both located on the invariant boundary $z = 0$ ($H \to +\infty$):

$$
\text{R: } z = 0, \quad q = 1,
$$

(4.3a)

$$
\text{dS: } z = 0, \quad q = -1.
$$

(4.3b)

The fixed point R is a hyperbolic source and corresponds to R in our global system, while dS is non-hyperbolic with one negative eigenvalue and one zero eigenvalue. This fixed point corresponds to the fixed point dS in the global system, obtained after a blow up. A center manifold analysis associated with the zero eigenvalue of dS yields the following approximation for the inflationary attractor solution (see appendix B):

$$
q(z) = -1 + \frac{z}{3} \left[1 + \frac{z}{3} - \frac{z^3}{3^3} + \ldots \right].
$$

(4.4)

Note that obtaining an approximation for the inflationary attractor solution that comes from $q = -1$ when $H \to \infty$ is considerably easier in these variables than for the $T, \theta$ variables, and the expansion is given as $q(H^{-2})$ since $z \propto H^{-2}$, which might be regarded as preferable.

The above brings the inflationary attractor solution into focus. Usually the inflationary regime is understood in terms of slow-roll approximations. In the Jordan frame this approximation can be found in [3, 12] and reads $\dot{H} = -1/36\alpha$. Since $\dot{H} = -(1 + q)H^2$, this leads to $1 + q = 1/36\alpha H^2$, which gives $q = -1 + z/3$, i.e., it just gives the leading order term in the center manifold expansion (4.4) for the inflationary attractor solution in the variables $z, q$. A comparison with the global variables shows that to leading order $z = (\theta/2)^2$. The above illustrates that not only are the $z, q$ variables a useful complement to the global variables $T, \theta$, since they more straightforwardly give approximations for the inflationary regime, but they are also intimately linked to the Hubble slow-roll approach.

Next we consider the Einstein frame and the usual slow-roll approximation. In this setting the slow-roll approximation is obtained by inserting $\tilde{H} = \kappa \sqrt{V(\phi)/3}$ into

$$
\kappa \frac{d\phi}{dt} = -2 \frac{\partial \tilde{H}}{\partial \phi},
$$

(4.5)

which for the present scalar field potential gives

$$
\kappa \frac{d\phi}{dt} \approx -\sqrt{\frac{2}{3}} Me^{-\sqrt{\frac{2}{3}} \kappa \phi}.
$$

(4.6)

Expressed in terms of the variables $\Sigma_\phi$ and $\tilde{X}$, this results in

$$
\Sigma_\phi \approx -\frac{1}{3} \left[ \left( \frac{T}{1 - T} \right) - 2 \tilde{X} \right].
$$

(4.7)
In the neighborhood of dS, represented by \( \tilde{\theta} = \tilde{T} = 0 \), this yields
\[
\tilde{\theta} \approx -3 \left( \tilde{T} - \frac{2}{3} \right),
\] (4.8)
which is the tangency condition for the center submanifold of dS, given by the leading order expression in (3.21). The slow-roll approximation is therefore just an approximation for the center manifold in the vicinity of dS in our Einstein frame state space formulation. In this context it should be pointed out that we can of course use variable relationships, given in eq. (A.1) in appendix A, to translate the various approximations from the Einstein to the Jordan frames and vice versa, and their series expansions can be improved by taking Padé approximants, as discussed in e.g. [19].

We end this discussion by emphasizing once more that the main purpose of the presently studied models was to specifically illustrate some general aspects of \( f(R) \) cosmology with a simple example, namely (a) the ingredients (i) – (iv) in the introduction, and (b) some dynamical systems methods with a broad range of applicability, even though the particulars have been tailored to the specific properties of the \( f = R + \alpha R^2 \) RW models. Although quite special, these models also capture very clearly some central issues in \( f(R) \)-gravity beyond the above methodological aspects. For example, as stated in the introduction, one way of classifying \( f(R) \)-gravity models is according to if \( F > 0 \) for all \( R \) or not. In the latter case the correspondence between the original \( f(R) \) model and its Einstein frame formulation, or its Brans-Dicke (\( \omega_{BD} = 0 \)) version (see e.g. [3, 8] for a description of this correspondence), only holds locally for the range of \( R \) where \( F > 0 \). For such models the evolution in the Jordan frame of some solutions are incompletely described in these formulations, i.e., a local formulation correspondence does not entail a global correspondence, as is clearly illustrated in figure 7.

In this context, note that if \( F = 0 \) was an invariant subset in the Jordan frame, then the \( F < 0 \) part of the Jordan state space would constitute an invariant subset. In this case, one could perhaps argue that the solutions associated with this part of the state space could be discarded on some claimed physical grounds, thus leading to a global physical correspondence between the solution spaces of the different frame formulations. However, as we have shown, \( F = 0 \) is not in general an invariant subset, nor is therefore \( F < 0 \). Thus if one wants to argue that a global physical correspondence exists for the different formulations, one is forced to come up with some arguments for why part of some solutions in the Jordan frame should be discarded (note that the existence of such solutions is ensured by that \( F = 0 \) is not an invariant subset; again, see figure 7 as an illustrative example).

There are, of course, some things that the present models cannot address. In particular this holds for models where the condition \( F_R > 0 \) is broken (leading to e.g. tachyonic instabilities, see e.g. [3] and references therein). The change of sign of \( F_R \) is particularly problematic from a mathematical point of view since the constraint (1.2d) becomes degenerate when \( F_R = 0 \), and, moreover, the causal properties of the field equations change when \( F_R \) changes sign (hence the tachyonic instability). There has been some work in \( f(R) \) cosmology to extend solutions when the equations are ill-defined, notably [29]. However, we here point out that the existence of \( F_R = 0 \) state space boundaries mathematically resemble sonic shock waves for fluids. It is therefore worthwhile to note that such problems have been dealt with in e.g. the context of spherically symmetric self-similar perfect fluid models [32]–[35], where it was shown how to extend solutions through sonic shock wave surfaces. Incidentally, these models also provide examples where it is useful to cover the state space
with several coordinate patches in order to exploit special structures in different parts of the state space, a problem one will inevitably will have to deal with when it comes to most \( f(R) \) cosmological models.

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A Jordan and Einstein state space relationships

To translate between our global dynamical systems formulations for the Jordan and Einstein frame state spaces, we need the explicit relationship between our global dynamical systems variables, which are as follows:

\[
H = M \left( 1 - \frac{T}{\tilde{T}} \right) (1 - \Sigma \phi) F^{-1/2}, \quad 2\alpha t = \frac{1}{\sqrt{2}} \left( 1 - \frac{T}{\tilde{T}} \right) F^{-3/2} G_+, \quad (A.1a)
\]
\[
\dot{R} = \frac{M}{\alpha} \left( 1 - \frac{T}{\tilde{T}} \right) \Sigma \phi F^{-3/2}, \quad X = G_- G_-^{-1}, \quad (A.1b)
\]
\[
R = \frac{1}{\alpha} \left( 1 - \frac{T}{\tilde{T}} \right) \tilde{X} F^{-1}, \quad S = -2\sqrt{2} \tilde{X} F^{1/2} G_-^{-1}, \quad (A.1c)
\]

and

\[
\dot{H} = \frac{M}{4\sqrt{2}} \left( 1 - \frac{T}{\tilde{T}} \right) H_+ F^{-3/2}, \quad \tilde{t} = 4\sqrt{2} \left( \frac{T}{1 - T} \right) F^{3/2} H_+^{-1}, \quad (A.1d)
\]
\[
\kappa \frac{d\phi}{dt} = \frac{1}{4\sqrt{2} \alpha} \left( 1 - \frac{T}{\tilde{T}} \right) H_- F^{-3/2}, \quad \Sigma \phi = H_- H_-^{-1}, \quad (A.1e)
\]
\[
\kappa \phi = \sqrt{\frac{3}{2}} \ln F, \quad \tilde{X} = -2\sqrt{2} SF^{1/2} H_+^{-1}, \quad (A.1f)
\]

where we have defined

\[
G_+ = 1 + \Sigma \phi \pm 2(1 - \Sigma \phi) F, \quad (A.2a)
\]
\[
H_\pm = 2(1 + X) \pm (1 - X) F, \quad (A.2b)
\]

and where we recall that

\[
M = \frac{1}{\sqrt{6} \alpha}, \quad \tilde{T} = \frac{\tilde{t}}{1 + \tilde{t}}, \quad T = \frac{1}{1 + 2\alpha t},
\]
\[
F = 1 - 2 \left( \frac{1 - \tilde{T}}{\tilde{T}} \right) \tilde{X} = 1 - \frac{(1 - T)}{T} S. \quad (A.3)
\]
As expected it follows that the variable transformation from the future light cone Jordan state space to the Einstein state space breaks down at the boundary $\tilde{T} F = 0$, since the Jacobian determinant for the variable transformation $(H, \dot{R}, R) \to (\tilde{T}, \Sigma_\phi, \tilde{X})$ is given by $-(1 - \tilde{T})^2(\tilde{T} F)^{-4}/6\alpha^3$.

### B Various dynamical systems formulations

Consider the dynamical systems formulation in the Jordan frame based on the variable transformation from $(H, R)$ to the variables $(z, q)$, defined in (4.1) and which obey the evolution equation given in (4.2), while the constraint is used to solve for $\dot{R}$. This system is manifestly invariant under the discrete symmetry $(t, H) \to - (t, H)$, and it is remarkably simple. So why not use this system of equations instead of the previous ones?

Firstly, note that the Jacobian determinant of the variable transformation $(z, q)$ to $(H, R)$ is given by $1/(36\alpha H^5)$, i.e., the variable transformation breaks down at $H = 0$, which for $(z, q)$ is located at infinity. Further insight is obtained by using our definitions to express $z$ and $q$ in terms of $T$ and $\theta$:

\[
\begin{align*}
   z &= \left( \frac{T}{1 - T} \right) \left( \frac{1 + \tan^2(\theta/2)}{\tan^2(\theta/2)} \right)^2, \\
   q &= 1 + 2 \left( \frac{T}{1 - T} \right) \left( \frac{1 + \tan^2(\theta/2)}{\tan^2(\theta/2)} \right).
\end{align*}
\]  

(B.1)

As can be seen, both $z$ and $q$ diverges when $\theta = 2n\pi$, i.e., when $H = 0$, which is where the variables break down. Furthermore, all solutions pass through $H = 0$ (where $q$ blows up) infinitely many times during the oscillating era at late times. Our Jordan state space formulation in the main text has also the advantage of clearly showing that $q \to \pm \infty$ is not associated with any spacetime singularity, but instead reflects the fact that $H$ becomes zero during the cosmic evolution.

Secondly, all the present variables are unbounded, although $z$ is positive. In addition, it is possible to use the natural extension of the state space that includes the $z = 0$ invariant subset boundary, since all interior orbits originate from fixed points on this boundary, which corresponds to $H \to \infty$. One can then use that $z = 0$ is an invariant boundary and that $z$ is non-negative to produce a new bounded variable, defined by $z/(1 + z)$, and change the time variable appropriately so that the right hand sides of the equations become polynomial in the dependent variables. However, a similar procedure is not possible with $q$. Of course one can replace $q$ with a bounded variable in a number of ways, but there does not seem to exist a physical structure which one can tie to such a compactification, except by essentially going back to our original bounded system. For example, a Poincaré compactification of $z, q$ would enforce an erroneous topology on the state space, which would have resulted in wrong conclusions about the properties of the solutions, since such a compactification would compromise a treatment of the oscillatory regime at late times.

Although the dynamical system (4.2) is inappropriate for global considerations, we have seen that it still has advantages, as illustrated by its simplicity and desirable local fixed point properties. The system has two fixed points located on the $z = 0$ boundary, given in (4.3).

The fixed point R is a hyperbolic, while dS is non-hyperbolic with one negative eigenvalue and one zero eigenvalues. To deal with the zero eigenvalue of dS, we apply center manifold theory. The negative eigenvalue corresponds to a stable subspace $W^s$ given by the invariant

---

8 The present variables $z$ and $q$ are closely related to those in [29] and [31]. Indeed, they are affinely related to $B$ and $Q$ in [29], while $z$ is proportional to $A^{-1}$ and $q$ is affinely related to $Q$ in [31].

9 For situating these fixed points in a broader context, see [29].
subset \( z = 0 \), which thereby coincides with the tangential stable subset \( E^s \), while the center manifold \( W^c \) has a tangential subspace \( E^c \), described by:

\[
E^s = \{ (z, q) \mid z = 0 \}, \\
E^c = \{ (z, q) \mid z - 3(1 + q) = 0 \}.
\]  

To investigate the center manifold \( W^c \) we proceed as in the main text. We adapt the variables to the tangent space \( E^c \) by introducing

\[
v = z - 3(1 + q),
\]  

which implies that in a neighborhood of dS the center manifold is described by the graph \( v = h(z) \). From (4.2) it follows that \( h(z) \) obeys the first order differential equation

\[
\frac{2}{3}z(h(z)) \left( \frac{dh}{dz} - 1 \right) + 3h(z) + (z - h(z))^2 = 0,
\]  

which is solved approximately for \( h \) by a formal series expansion, which results in

\[
q(z) = -1 + \frac{z}{3} \left[ 1 + \frac{z}{3} - \frac{z^3}{3^3} + \ldots \right],
\]  

as \( z \to 0 \).

Let us now consider an example of another dynamical systems treatment of \( f(R) \) flat RW cosmology which can be found in [3], where the following variables (when restricted to the present vacuum case) are defined:

\[
x_1 = -\frac{2\alpha\dot{R}}{H(1 + 2\alpha R)}, \quad x_2 = -\frac{R(1 + \alpha R)}{6H^2(1 + 2\alpha R)}, \quad x_3 = \frac{R}{6H^2} = 1 - q.
\]  

Viewing this as a variable transformation from \((x_1, x_2, x_3)\) to \((H, \dot{R}, R)\) leads to the Jacobian determinant

\[
\frac{\alpha^2 R^2}{9H^6(1 + 2\alpha R)^3}.
\]  

The variables thereby break down at \( R = 0 \), \( H = 0 \), and \( F = 1 + 2\alpha R = 0 \), i.e., at the boundary of the state space of the Einstein frame. It brings parts of the future null infinity of the state space to finite values of the variables, but other regions, such as the generic one close to the Minkowski fixed point, are now shifted to infinity in these unbounded variables. In the present case it follows that the auxiliary quantity \( m \) in [3] is given by \( 2(x_2 + x_3)/x_3 \), which when using a time variable \( N = \ln a \) leads to the evolution equations

\[
\frac{dx_1}{dN} = -1 - x_3 - 3x_2 + x_1^2 - x_1x_3,
\]  

\[
\frac{dx_2}{dN} = \frac{x_1x_3^2}{2(x_2 + x_3)} - x_2(2x_3 - 4 - x_1),
\]  

\[
\frac{dx_3}{dN} = -\frac{x_1x_3^2}{2(x_2 + x_3)} - 2x_3(x_3 - 2),
\]  

subjected to the constraint

\[
1 = x_1 + x_2 + x_3.
\]
which can be solved globally for one of the variables. It is clear that not only has the system an unbounded incomplete state space, but the equations are also irregular as the right hand side blows up on the line \( x_2 + x_3 = 0, x_1 = 1 \). To conclude, the system is inappropriate for a global analysis of the problem, but it is still possible to do some local analysis. To accomplish this, let us solve for one of the variables, e.g., \( x_1 = 1 - x_2 - x_3 \) (which variable we solve for does not change any conclusions), and look for fixed points. Since \( x_2 + x_3 \) appears in the denominator (writing the right hand sides of \( dx_2/dN \) and \( dx_3/dN \) with \( x_2 + x_3 \) as a common denominator shows that it is impossible to get rid of this denominator), the equations are not defined for \( x_2 + x_3 = 0 \) and hence fixed points must have \( x_2 + x_3 \neq 0 \).

There are two fixed points: \( P_1: (x_2, x_3) = (-1, 2) \) and \( P_4: (x_2, x_3) = (5, 0) \). The vacuum fixed point \( P_3 \) in \([3]\) is not defined for the present models, since it is associated with numerators and denominators in the equations that simultaneously are zero. Furthermore, \( P_6 \) in \([3]\) only exists when it coincides with \( P_1 \), for which \( m = 1 \). Note that the definitions \((B.6)\) implies that \( P_1 \) corresponds to the asymptotic limit \( H \to \infty \) and \( R \to \infty \). In fact

\[
-1 = x_2 = -x_3 \frac{1 + \alpha R}{1 + 2\alpha R}, \quad 2 = x_3 = \frac{R}{6H^2}, \tag{B.10}
\]

which implies \( R \to \infty \) and \( H \to \infty \). The fixed point \( P_4 \) on the other hand implies that \( H \to \infty \) and \( R = -1/2\alpha \). This is because

\[
5 = x_2 = -x_3 \frac{1 + \alpha R}{1 + 2\alpha R}, \quad 0 = x_3 = \frac{R}{6H^2}, \tag{B.11}
\]

which implies that \( 1 + 2\alpha R \to 0^- \) and hence \( H \to \infty \). Thus both fixed points correspond to part of future null infinity of the state space light cone. Note that \( P_4 \) is associated with the boundary of the Einstein frame.

The fixed point \( P_1 \) corresponds to \( q = -1 \) and is a non-hyperbolic fixed point with one negative value and one zero eigenvalue with an associated center manifold, corresponding to the dS fixed point for \( z \) and \( q \). It therefore gives similar results, but the more complicated dynamical system leads to unnecessary technical complications. The situation for \( q = 1 \) when \( H \to \infty \), is, however, worse. In this case \( P_4 \) is a hyperbolic saddle, which is associated with a coordinate singularity due to the break down of \((B.7)\) at \( F = 1 + 2\alpha R = 0 \). Because of this breakdown, \( P_4 \) yields a solution that comes from a particular direction from future null infinity of the physical state, namely \( R \to -1/2\alpha \), thereby missing the one-parameter set of solutions that originates from there into the physical state space. Thus a fixed point analysis in these variables does not show that there actually is a one-parameter set of solutions that originate from the limit \( q = 1 \) and \( H \to \infty \), which covers all solutions except the single solution from dS.

Next we comment on a previous attempt to provide a compact state space, given in \([36]\). In this work it was assumed that \( R > 0 \) and \( F = df/dR > 0 \), where the latter follows from the first condition in our case. The variables the authors introduced were given by

\[
x = \frac{3\dot{f}}{2fD}, \quad y = \frac{3f}{2D^2}, \quad z = \frac{3R}{2D^2}, \quad Q = \frac{3H}{D}, \tag{B.12}
\]

where \( D = \left[ 3\left(H + \frac{\dot{f}}{f}\right)^2 + \frac{3f}{f}\right]^{1/2} \). For the vacuum case one can solve for \( y \) and \( z \) to obtain a system of evolution equations for \( x \) and \( Q \). The authors also introduce an auxiliary quantity

\[ -27 \]
Γ, which for the present case can be written as
\[ \Gamma = 1 - \frac{x^2}{2Q(Q + 2x)} . \]  
(B.13)

Explicitly inserting this into the equations in [36] (which we refrain from giving because of their considerable complexity) shows that they have \( Q(Q + 2x) \) in the denominator, which means that the equations are non-regular and break down at \( Q = 0 \) and at \( Q + 2x = 0 \). This result, in combination with the fact that the variables only compactify the \( R > 0 \) part of the state space, and that \( R = 0 \) is not an invariant subset on the physical state space (except for at the Minkowski fixed point), unfortunately leads to complications and some erroneous conclusions (due to a breakdown of the time variable in [36]). For example, as we have proven, all solutions pass through \( R = 0 \) infinitely many times, in contrast to what is claimed in [36]. This example illustrates that compactifications must respect the structure of the state space; if one chooses to compactify only part of it there will be coordinate singularities associated with the boundary one chooses for the compactification, unless it is associated with an invariant subset in the original Jordan state space for \( (H, R, R) \).

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