Almost global existence for the nonlinear Klein-Gordon equation in the nonrelativistic limit

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Abstract

We study the one-dimensional nonlinear Klein-Gordon (NLKG) equation with a convolution potential, and we prove that solutions with small $H^s$ norm remain small for long times. The result is uniform with respect to $c \geq 1$, which however has to belong to a set of large measure.

Keywords: nonrelativistic limit, nonlinear Klein-Gordon

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1 Introduction

In this paper we study the nonlinear Klein-Gordon (NLKG) equation in the nonrelativistic limit, namely as the speed of light $c$ tends to infinity. Formal computations going back to the first half of the last century suggest that, up to corrections of order $O(c^{-2})$, the system should be described by the nonlinear Schrödinger (NLS) equation. Subsequent mathematical results have shown that the NLS describes the dynamics over time scales of order $O(1)$.

The nonrelativistic limit for the Klein-Gordon equation on $\mathbb{R}^d$ has been extensively studied over more then 30 years, and essentially all results only show convergence of the solutions of NLKG to the solutions of the approximate equation fortimes of order $O(1)$. The typical results, due to Masmoudi, Machihara, Nakanishi and Ozawa, ensure convergence locally uniformly in time, either with loss of regularity (see [Tsu84], [Naj90] and [Mac01]) or without loss of regularity (see [MNO02], [MN02]).

We also mention the recent papers [LZ16] and [Pas17], which discuss the long-time convergence of solution of NLKG in the nonrelativistic limit on $\mathbb{R}^d$; however, the results proved in both papers have some limitations, either on the nonlinearity (in [LZ16] the authors studied only the quadratic NLKG) or on the particular form of the solution (in [Pas17]).

Concerning the nonrelativistic limit of the NLKG on the $d$-dimensional torus $\mathbb{T}^d$, we mention the work by Faou-Schratz [FSL], in which the authors were able to justify the approximation of the solutions of NLKG by solutions of the NLS

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over time scales of order $O(1)$ through a variant of Birkhoff Normal Form theory. In [Pas17] Birkhoff Normal Form Theory was exploited in order to recover the result by Faou-Schratz, and to generalize it to all smooth compact Riemannian manifolds.

Actually, [FS14] dealt with the construction of numerical schemes which are robust in the nonrelativistic limit; we refer also to [BD12], [BZ16] and to [BFS17] for the numerical analysis of the nonrelativistic limit of the NLKG equation.

In this paper we generalize the techniques developed in [BG06] in order to prove a long-time existence result for the NLKG with a convolution potential with Dirichlet boundary conditions, uniformly in $c \geq 1$.

An immediate corollary of our result allows us to show that for any $\alpha > 0$ any solution in $H^s$ with initial datum of size $O(c^{-\alpha})$ remains of size $O(c^{-\alpha})$ up to times of order $O(c^{\alpha(r+1/2)})$ for any $r \geq 1$; however, we have to assume that both the parameter $c$ and the coefficients of the potential belong to a set of large measure. The main limitation of such a result is that it holds only for solutions with initial data which are small with respect to $c$.

The new ingredient in the proof with respect to [BG06] is a diophantine type estimate for the frequencies, which holds uniformly when $c \to \infty$.

An aspect that would deserve future work is the study of the nonrelativistic limit of the NLKG without potential. This is expected to be a quite subtle problem since, for $c \neq 0$ the frequencies of NLKG are typically non resonant, while the limiting frequencies are resonant. The issue of long-time existence for small solutions of the NLKG equation without potential on compact manifolds has received a lot of interest; see for example [DS04], [Del09], [FZ10], [FHZ17] and [DI17]. However, all results in the aforementioned papers rely on a nonresonance condition which is not uniform with respect to $c$.

The paper is organized as follows. In sect. 2 we state the results of the paper, together with some examples and comments. In sect. 3 we prove our result for the NLKG with a convolution potential on $I := [0, \pi]$.

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2 Statement of the Main Results

Now consider the following equation:

\[
\frac{1}{c^2} u_{tt} - u_{xx} + c^2 u + V * u + \partial_n f(u) = 0, \quad (1)
\]

with $c \in [1, +\infty)$, $x \in I$, $f \in C^\infty(\mathbb{R})$ a real-valued function with a zero of order four at the origin, with Dirichlet boundary conditions. The potential has the form

\[
V(x) = \sum_{j \geq 1} v_j \cos(jx). \quad (2)
\]
By using the same approach of [BG06], we fix a positive $s$, and for any $M > 0$ we consider the probability space

$$\mathcal{V} := \mathcal{V}_{s,M} = \left\{(v_j)_{j \geq 1} : v_j := M^{-1}j^sv_j \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\},$$

and we endow the product probability measure on the space of $(c, (v'_j)_j)$.

We recall that in this case the frequencies are given by

$$\omega_j := \omega_j(c) = c \sqrt{c^2 + \lambda_j} = c^2 + \frac{\lambda_j}{1 + \sqrt{1 + \lambda_j/c^2}},$$

$$= c^2 + \frac{\lambda_j^2}{2c^2} - \frac{1}{1 + \sqrt{1 + \lambda_j/c^2}},$$

where $\lambda_j = j^2 + v_j$. Now we introduce the following change of coordinates,

$$\psi := \frac{1}{\sqrt{2}} \left[\left(\frac{c^2 - \Delta + \tilde{V})^{1/2}}{c}\right)^{1/2} u - i \left(\frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}}\right)^{1/2} u_t\right],$$

where $\tilde{V}$ is the operator that maps $u$ to $V * u$. The Hamiltonian of (1) now reads

$$H(\psi, \bar{\psi}) = \left\langle \bar{\psi}, c(c^2 - \Delta + \tilde{V})^{1/2} \psi \right\rangle + \int_I f \left(\frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}}\right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} dx.$$

Therefore the Hamiltonian takes the form

$$H(\psi, \bar{\psi}) = H_0(\psi, \bar{\psi}) + N(\psi, \bar{\psi}),$$

where

$$H_0(\psi, \bar{\psi}) = \left\langle \bar{\psi}, c(c^2 - \Delta + \tilde{V})^{1/2} \psi \right\rangle,$$

$$N(\psi, \bar{\psi}) = \int_I f \left(\frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}}\right)^{1/2} (\psi + \bar{\psi}) dx,$$

$$\sim \sum_{l \geq 4} \int_I N_l(x) \left(\frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}}\right)^{1/2} (\psi + \bar{\psi})^l dx,$$

where $N_l \in C^\infty$ for each $l$ (since $V \in C^\infty$), and

$$\left(\frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}}\right)^{1/2} : H^s \to H^s$$

is a smoothing pseudodifferential operator, which can be estimated uniformly in $c \geq 1$. 

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Theorem 2.1. Consider the equation (1) and fix \( \gamma > 0 \), and \( r > 1 \). Then for any \( r \geq 1 \) there exists \( s^* > 0 \) and, for any \( s > s^* \), there exists a set \( \mathcal{R}_\gamma := \mathcal{R}_{\gamma,r,s} \subset [1, +\infty) \times \mathcal{V} \) satisfying

\[
|\mathcal{R}_\gamma \cap ([n, n + 1] \times \mathcal{V})| = O(\gamma) \quad \forall n \in \mathbb{N}_0,
\]

and there exists \( R_s > 0 \) such that for any \((c, (v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma \) and for any \( R < R_s \) there exist \( N := N(r, R) > 0 \), and a canonical transformation

\[
\mathcal{T}_c := \mathcal{T}_c^{(r)} : B_s(R/3) \to B_s(R)
\]

such that

\[
H_c := H \circ \mathcal{T}_c^{(r)} = H_0 + Z^{(r)} + R^{(r)},
\]

where \( Z^{(r)} \) is a polynomial of degree (at most) \( r + 2 \), such that

\[
Z^{(r)}(\psi, \bar{\psi}) = \sum_{n \in \mathbb{N}^d} Z_n \psi^n \bar{\psi}^n,
\]

\[
Z_n \neq 0 \implies \sum_{i \geq N + 1} n_i \leq 2,
\]

and such that

\[
\sup_{B_s(R/3)} \|X_{R^{(r)}}(\psi, \bar{\psi})\|_{H^s} \leq K_s R^{s + 3/2},
\]

\[
\sup_{B_s(R/3)} \|\mathcal{T}_c^{(r)} - id\|_{H^s} \leq K_s R^2.
\]

and we have that \( Z^{(r)} \) depends on the actions \( I = \psi \bar{\psi} \) only. Moreover, there exist \( K_s^* > 0 \) and \( K' > 0 \) such that if the initial datum satisfies

\[
\|(\psi_0, \bar{\psi}_0)\|_{H^s} \leq K R
\]

with \( K < K_s^* \), then

\[
\|(\psi(t), \bar{\psi}(t))\|_{H^s} \leq 2K R, \quad |t| \leq K' R^{-(r+1/2)}
\]

\[
\|(I(t), \bar{I}(t))\|_{H^s} \leq K R^3, \quad |t| \leq K' R^{-(r+1/2)}.
\]

Finally, there exists a smooth torus \( I_c \) such that for any \( s_1 < s - 1/2 \) there exists \( K_{s_1} > 0 \) such that

\[
d_{s_1}(((\psi(t), \bar{\psi}(t)), I_c) \leq K_{s_1} R^{\frac{s_1}{r+1}}, \quad |t| \leq K' R^{-(r-r_1+1/2)},
\]

where \( r_1 \leq r \), and \( d_{s_1} \) is the distance in \( H^{s_1} \).

Remark 2.2. The fact that \( Z \) depends only on the actions is a direct consequence of the non-resonance property established in Theorem 3.3.

Remark 2.3. It would also be interesting to study the dependence of \( I_c \) on \( c \). One could expect that it should converge to an invariant torus of the NLS with a convolution potential. We expect this fact to be true, but it needs further investigation for a proof. This is due to the fact that the NLS is the singular limit of NLKG and to the fact that \( c \) is only allowed to vary in Cantor like sets, so that one can only expect a Whitney-smooth dependence.
By exploiting the same argument used to prove Theorem 2.1 one can immediately deduce the following almost global existence result for solutions with small (with respect to \( c \)) initial data.

**Corollary 2.4.** Fix \( \alpha > 0 \), \( \gamma > 0 \), and \( \tau > 1 \). Then for any \( r \geq 1 \) there exists \( s^* > 0 \) and, for any \( s > s^* \), there exists a set \( \mathcal{R}_{s^*, r, s, \alpha, \tau} \subset [1, +\infty) \times \mathcal{V} \) such that there exists \( c^* > 0 \) such that for any \( (c, (v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_{s^*, r, s, \alpha, \tau} \) with \( c > c^* \) the following holds: there exist \( K^*_s > 0 \) and \( K' > 0 \) such that if the initial datum satisfies

\[
\| (\psi_0, \tilde{\psi}_0) \|_{H^s} \leq \frac{K}{c^s}
\]

with \( K < K^*_s \), then

\[
\| (\psi(t), \tilde{\psi}(t)) \|_{H^s} \leq \frac{2K}{c^{r^s}}, \quad |t| \leq K' e^{\alpha(t + 1/2)},
\]

\[
\| (\tilde{I}(t), \tilde{I}(t)) \|_{H^s} \leq \frac{K}{c^{r^s}}, \quad |t| \leq K' e^{\alpha(t + 1/2)}.
\]

### 3 Proof of Theorem 2.1

In order to prove Theorem 2.1 we need to show some nonresonance properties of the frequencies \( \omega = (\omega_j)_{j \geq 0} \): it will be crucial that these properties hold uniformly (or at least, up to a set of small probability) in \((1, +\infty) \times \mathcal{V}\), since this will allow us to deduce a result which is valid in the nonrelativistic limit regime.

We use the notation “\( a \lesssim b \)” (resp. “\( a \gtrsim b \)” to mean “there exists a constant \( K > 0 \) independent of \( c \) such that \( a \leq K b \)” (resp. \( a \geq K b \)).

**Proposition 3.1.** Let \( r \geq 1, \ e \geq 1 \) be fixed. Then \( \forall \gamma > 0 \ \exists \mathcal{V}^s_{s, M, \gamma} \subset \mathcal{V} \) with \( |\mathcal{V} \setminus \mathcal{V}^s_{s, M, \gamma}| = O(\gamma) \), and \( \exists \tau > 1 \) s.t. \( \forall (v_j)_j \in \mathcal{V}^s_{s, M, \gamma} \) and \( \forall N \geq 1 \)

\[
|\omega \cdot k + n| \geq \frac{\gamma}{N^r}
\]

for \( 0 < |k| \leq r \), \( \text{supp}(k) \subseteq \{1, \ldots, N\} \), and \( \forall \ n \in \mathbb{Z} \).

**Proof.** Let \( p_{k, n}((v_j)_j_{j \geq 1}) := \sum_{j=1}^{N} \omega_j k_j + n \), and assume that \( k_h \neq 0 \) for some \( h \). Then

\[
\frac{\| \partial p_{k, n} \|}{\| \partial x_h^\gamma \|} = \left| \frac{k_h h^{-s}}{2\sqrt{1 + \lambda_h/c^2}} \right| \gtrsim \frac{1}{2h^s\sqrt{1 + \overline{h}^{\max(s, 2)}}} \geq \frac{1}{2N^s\sqrt{1 + N^{\max(s, 2)}}} > 0,
\]

hence by Lemma 17.2 of [Ris01]

\[
\left| \left\{ (v_j')_{j \geq 1} : |p_{k, n}((v_j)_j)_{j \geq 1}| < r_0 \right\} \right| \leq r_0 N^{r+s+\max(s, 2)/2}.
\]

\[
\left| \bigcup_{|n| \leq rN} \bigcup_{\text{supp}(k) \subseteq \{1, \ldots, N\}} \left\{ (v_j')_{j \geq 1} : |p_{k, n}((v_j)_j)_{j \geq 1}| < r_0 \right\} \right| \leq r_0 N^{r+1+s+\max(s, 2)/2},
\]

and by choosing \( r_0 = \frac{1}{\alpha} \) with \( \tau > s + r + 2 + \max(s, 2) \) we get that the left hand side of (20) is bounded, and therefore the thesis holds. □
Proposition 3.2. Let \( r \geq 1 \) be fixed. Then \( \forall \gamma > 0 \) there exists a set \( \mathcal{R}_\gamma := \mathcal{R}_{\gamma, r} \subset [1, +\infty) \times \mathcal{V} \) satisfying
\[
|\mathcal{R}_\gamma \cap ([n, n+1] \times \mathcal{V})| = O(\gamma) \quad \forall n \in \mathbb{N}_0,
\]
and \( \exists \tau > 1 \) such that \( \forall (c, (v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma \) and \( \forall N \geq 1 \)
\[
\left| \sum_{j=1}^{N} \omega jk_j + \sigma \omega \right| \geq \frac{\gamma}{N^\tau}
\] (21)
for \( 0 < |k| \leq r \), \( \text{supp}(k) \subseteq \{1, \ldots, N\} \), \( \sigma = \pm 1, l \geq N \).

Proof. Without loss of generality, we can choose \( \alpha \) and \( \exists \tau > 1 \) such that \( \forall (c, (v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma \) and \( \forall N \geq 1 \)
\[
\left| \sum_{j=1}^{N} \omega jk_j + \sigma \omega \right| \geq \frac{\gamma}{N^\tau}
\]
\[
\sum_{j=1}^{N} \omega jk_j + \sigma \omega
\]
\( \leq \frac{\gamma}{N^\tau}
\]
Now fix \( k \in \mathbb{Z}^N \) with \( 0 < |k| \leq r \), and fix \( l \geq N \). Set \( p_{k,l}(c, (v_j)_j) := \sum_{j=1}^{N} \omega jk_j - \omega \). We can rewrite the function \( p_{k,l} \) in the following way:
\[
p_{k,l}(c, (v_j)_j) = \alpha c^2 + \sum_{j=1}^{N} \frac{k_j\lambda_j}{1 + \sqrt{1 + \lambda_j/c^2}} - \frac{\lambda_l}{1 + \sqrt{1 + \lambda_l/c^2}},
\]
where \( \alpha := (\sum_{j=1}^{N} k_j) - 1 \in \{-r - 1, \ldots, r - 1\} \). Now we have to distinguish some cases:

Case \( \alpha = 0 \): in this case we have that \( p_{k,l} \) can be small only if \( l^2 \leq 3(N^2 + N^*)^2r^2 \). So to obtain the result we just apply Proposition 5.1 with \( N' := \sqrt{3}(N^2 + N^*)r, r' := r + 1 \).

Case \( \alpha \neq 0 \), \( c \leq \lambda_N^{1/2}r^{1/2} \): we have that
\[
\sum_{j=1}^{N} c\sqrt{c^2 + \lambda_jk_j} \leq r\sqrt{c^4 + c^2\lambda_N} \leq \sqrt{2}r^2\lambda_N,
\]
so \( |\sum_{j=1}^{N} \omega jk_j - \omega| \) can be small only for \( l^2 < rN^2 \). Therefore, in order to get the thesis we apply Proposition 3.1 with \( N' := \sqrt{r}N, r' := r + 1 \).

Case \( \alpha > 0 \), \( c > \lambda_N^{1/2}r^{1/2} \): first notice that if we set \( f(x) := \frac{x^2}{2\sqrt{1+x(1+\sqrt{1+x})^2}} \), and we put \( x_j := \lambda_j/c^2 \), in this regime we get
\[
\left| \sum_{j=1}^{N} k_jf(x_j) \right| \leq \frac{r}{2} f(x_N) \leq \frac{1}{2}.
\]
Now define \( \tilde{p}_{k,l}(c^2) := \alpha c^2 - \frac{\lambda_l}{1 + \sqrt{1 + \lambda_l/c^2}} \). One can verify that
\[
\tilde{p}_{k,l}(c^2) = 0;
\]
\[
c^2 = c^2_{l,\alpha} := \frac{\lambda_l}{\alpha(\alpha + 2)},
\]
and that
\[
\frac{\partial \tilde{p}_{k,l}}{\partial(c^2)}(c^2_{l,\alpha}) = \alpha - \frac{\alpha^2(\alpha + 2)^2}{2\sqrt{1 + \alpha(\alpha + 2)}(1 + \sqrt{1 + \alpha(\alpha + 2)})^2} > 0.
\]
Besides, in an interval \( \left[ c_{l,\alpha}^2 - \frac{\rho}{\alpha(\alpha+2)}, c_{l,\alpha}^2 + \frac{\rho}{\alpha(\alpha+2)} \right] \) we have that
\[
\frac{\partial \tilde{p}_{k,l}}{\partial (c^2)}(c^2) > \left( \frac{1}{2} + \frac{1}{2(r+1)} \right) \alpha.
\]
Then, by exploiting Lemma 17.2 of [Rüs01], we get that
\[
\left\{ c^2 \in B \left( c_{l,\alpha}^2, \frac{\rho}{\alpha(\alpha+2)} \right) : \left| \tilde{p}_{k,l}(c^2) \right| \leq \gamma \right\} \leq \frac{2(r+1)}{(r+2)\alpha}.
\]
for any \( \gamma > 0 \) s.t. \( \gamma \frac{2(r+1)}{(r+2)\alpha} < \frac{\rho}{\alpha(\alpha+2)} \); \( \gamma < \frac{(r+2)\rho}{2(\alpha(\alpha+2))} \).

Now, since in this regime \( \left| \frac{\partial \tilde{p}_{k,l}}{\partial c} \right| \leq \frac{1}{2} \), we can deal with \( p_{k,l} \) in a similar way as before, and we can conclude that
\[
\lim_{\gamma \to 0} \left| \bigcup_{0 < |k| \leq r} \bigcup_{l \geq N} \{ c^2 \in [c_{l,\alpha}^2, c_{l,\alpha}^2] : |p_{k,l}(c^2)| \leq \gamma \} \right| = 0. \quad (22)
\]

Case \( \alpha < 0, c > \lambda_N^{1/2}r^{1/2} \): since
\[
\left| \sum_{j=1}^{N} \frac{k_j \lambda_j}{1 + \sqrt{1 + \lambda_j/c^2}} \right| \leq \frac{r\lambda N}{2} \leq \frac{c^2}{2},
\]
we have that \( p_{k,l} \) can be small only if \( \lambda_N < r\lambda_N \). So, in order to get the result, we apply Proposition 3.1 with \( N' := r^{1/2}N, r' := r + 1 \). □

**Theorem 3.3.** Let \( r \geq 1 \) be fixed. Then \( \forall \gamma > 0 \) there exists a set \( \mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,r} \subset [1, +\infty) \times \mathcal{V} \) satisfying
\[
|\mathcal{R}_\gamma \cap ([n, n+1] \times \mathcal{V})| = O(\gamma) \quad \forall n \in \mathbb{N}_0,
\]
and \( \exists \tau > 1 \) such that \( \forall (c, v_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma \) and \( \forall N \geq 1 \)
\[
\left| \sum_{j=1}^{N} \frac{\omega_j k_j}{1 + \sqrt{1 + \lambda_j/c^2}} \right| \geq \frac{\gamma}{N^\tau} \quad (23)
\]
for \( 0 < |k| \leq r, \text{supp}(k) \subseteq \{1, \ldots, N\}, \sigma_1, \sigma_2 \in \{\pm 1\}, m > l \geq N \).

**Proof.** If \( \sigma_i = 0 \) for \( i = 1, 2 \), then we can conclude by using Proposition 3.2.

Now, consider the case \( \sigma_1 = -1, \sigma_2 = 1, \) and denote
\[
p_{k,l,m}(c^2) := \sum_{j=1}^{N} \omega_j(c^2)k_j - \omega_l(c^2) + \omega_m(c^2).
\]

Now fix \( \delta > 3 \). If \( m \leq N^\delta \), then we can conclude by applying Proposition 3.1 and 3.2. So from now on we will assume that \( m, l > N^\delta \).

We have to distinguish several cases:
Case $c < \lambda_l^2$: we point out that, since

$$c\sqrt{c^2 + \lambda_l} = c\lambda_l^{1/2} \sqrt{1 + \frac{c^2}{\lambda_l}} = c\lambda_l^{1/2} \left(1 + \frac{c^2}{2\lambda_l} + O\left(\frac{1}{\lambda_l^2}\right)\right),$$

we get (denote $m = l + j$)

$$\omega_m - \omega_l = j c + \frac{1}{2} \left(v_m - v_l\right) + \frac{c^2}{2\lambda_l^{1/2}} + \frac{c^3}{2\lambda_m^{1/2}} + O\left(\frac{1}{m^2}\right) + O\left(\frac{1}{r^2}\right),$$

that is, the integer multiples of $c$ are accumulation points for the differences between the frequencies as $l, m \to \infty$, provided that $\alpha < \frac{1}{6}$.

Case $c > \lambda_m$: in this case we have (again by denoting $m = l + j$) that

$$\lambda_m - \lambda_l = j(j + 2l) + (v_m - v_l) = 2jl + j^2 + a_{lm},$$

with $|a_{lm}| \leq C_l$, so that

$$p_{k,l,m} = \sum_{h=1}^{N} \omega_h k_h \pm 2jl \pm j^2 \pm a_{lm}.$$

If $l > 2CN^\tau/\gamma$ then the term $a_{lm}$ represents a negligible correction and therefore we can conclude by applying Proposition 3.1. On the other hand, if $l \leq 2CN^\tau/\gamma$, we can apply the same Proposition with $N' := 2CN^\tau/\gamma$ and $r' := r + 2$.

Case $\lambda_l^{1/6} \leq c \lesssim \lambda_l^{1/2}$: if we rewrite the quantity to estimate

$$p_{k,l,m}(c^2) = \alpha c^2 + \sum_{h=1}^{N} \frac{\lambda_h k_h}{1 + \sqrt{1 + \lambda_h/c^2}} + \omega_m - \omega_l,$$

where $\alpha := \sum_{h=1}^{N} k_h$, we distinguish three cases:

- if $\alpha > 0$, then we notice that

$$\left|\sum_{h=1}^{N} \frac{\lambda_h k_h}{1 + \sqrt{1 + \lambda_h/c^2}}\right| \leq \frac{r\lambda_N}{1 + \sqrt{1 + \lambda_N/c^2}} \leq \frac{r\lambda_N}{1 + \sqrt{1 + \lambda_N/\lambda_l}} \leq \frac{r\lambda_N}{2},$$

$$|\omega_m - \omega_l| = c \frac{\lambda_m - \lambda_l}{\sqrt{c^2 + \lambda_m + \sqrt{c^2 + \lambda_l}}} \leq \frac{c\lambda_l^{1/2}}{\sqrt{c^2 + \lambda_m + \sqrt{c^2 + \lambda_l}}} \geq \frac{\lambda_l}{\sqrt{N^{2/3} + \lambda_m^{1/2}} + \sqrt{N^{2/3} + \lambda_l^{1/2}}} > 0,$$

thus $|p_{k,l,m}| > |\lambda_l^{1/3} - \frac{x}{2} \lambda_N| > 0$, since $l > N^3$;

- if $\alpha = 0$, then we just notice that

$$|\omega_m - \omega_l| \geq \gamma (\lambda_m - \lambda_l) \geq \gamma_0 \lambda_l^{1/2},$$

which is greater than $\gamma_0/N^\tau$ for $\tau > -1$, since $l > N^3$;
• if $\alpha < 0$, then we just recall that $|\omega_m - \omega_l| > \gamma_0 \lambda_l^{1/2}$, and by choosing $\gamma_0$ sufficiently small (actually $\gamma_0 \leq |\alpha|$) we get that also in this case $p_{k,l,m}$ is bounded away from zero.

The proof is based on the method of Lie transform. Let $s > s^*$ be fixed.

Given an auxiliary function $\chi$ analytic on $H^s$, we consider the auxiliary differential equation

$$\dot{\psi} = i\nabla \bar{\psi} \chi(\psi, \bar{\psi}) =: X(\psi, \bar{\psi})$$

and denote by $\Phi^t$ its time-$t$ flow. A simple application of Cauchy inequality gives

**Lemma 3.4.** Let $\chi$ and its symplectic gradient be analytic in $B_s(\rho)$. Fix $\delta < \rho$, and assume that

$$\sup_{B_s(\rho)} \|X(\psi, \bar{\psi})\| \leq \delta.$$ 

Then, if we consider the time-$t$ flow $\Phi^t$ of $X$ we have that for $|t| \leq 1$

$$\sup_{B_s(R-\delta)} \|\Phi^t(\psi, \bar{\psi}) - (\psi, \bar{\psi})\|_s \leq \sup_{B_s(R)} \|X(\psi, \bar{\psi})\|_s.$$ 

The map $\Phi := \Phi^1$ will be called the Lie transform generated by $\chi$.

Given a homogeneous polynomial $f$ of degree $m$, we denote, following [BG06], its modulus

$$|f|(\psi, \bar{\psi}) := \sum_{|j|=m} |f_j| z^j,$$

where $f_j$ is given by

$$f(\psi) = \sum_{|j|=m} f_j z^j,$$

$$z^j := \cdots z^{j-1} z^{-1}^{\dagger} \cdots z^{j-1} z^{\dagger} \cdots z^{\dagger} \cdots z^{-1} = (\psi, e^{-it}), \ z^{-1} = (\bar{\psi}, e^{-it}).$$

Furthermore, given a multivector

$$\phi := (\phi^{(1)}, \ldots, \phi^{(r)}) = (\psi^{(1)}, \bar{\psi}^{(1)}, \ldots, \psi^{(r)}, \bar{\psi}^{(r)})$$

we introduce the following norm

$$\|\phi\|_{s,1} := \frac{1}{r} \sum_{l=1}^r \|\phi^{(l)}\|_1 \cdots \|\phi^{(l-1)}\|_1 \|\phi^{(l)}\|_s \|\phi^{(l+1)}\|_1 \cdots \|\phi^{(r)}\|_1.$$ (26)

**Definition 3.5.** Let $X : H^s \oplus H^s \to H^s \oplus H^s$ be a homogeneous polynomial of degree $r$,

$$X(\psi, \bar{\psi}) = \sum_{l \in \mathbb{Z}_{\{0\}}} X_l(\psi, \bar{\psi}) e^{itl}.$$
Consider the $r$-linear symmetric form $\tilde{X}$ such that $\tilde{X}(\psi, \bar{\psi}, \ldots, \psi, \bar{\psi}) = X_l(\psi, \bar{\psi})$, and set
\[
\tilde{X} := \sum_{l \in \mathbb{Z}\setminus\{0\}} \tilde{X}_l(\psi, \bar{\psi}) e^{il \cdot},
\]
so that $\tilde{X}(\psi, \bar{\psi}, \ldots, \psi, \bar{\psi}) = X_l(\psi, \bar{\psi})$.

Let $s \geq 1$, then we say that $X$ is an $s$-tame map if there exists $K_s > 0$ such that
\[
\|\tilde{X}(\phi(1), \ldots, \phi(r))\|_s \leq K_s \sum_{l=1}^{r} \|\phi(1)\|_1 \ldots \|\phi(l-1)\|_1 \|\phi(l)\|_s \|\phi(l+1)\|_1 \ldots \|\phi(r)\|_1,
\]
(27)
\[
\forall \phi(1), \ldots, \phi(r) \in H^s \oplus H^s.
\]

If a map is $s$-tame for any $s \geq 1$, then it will be said to be tame.

**Definition 3.6.** Let us consider a vector field $X : H^s \oplus H^s \to H^s \oplus H^s$, and denote by $X_l$ its $l$-th component. We define its modulus by
\[
[X](\psi, \bar{\psi}) := \sum_{l \in \mathbb{Z}\setminus\{0\}} [X_l](\psi, \bar{\psi}) e^{il \cdot}.
\]

A polynomial vector field $X$ is said to have $s$-tame modulus if its modulus $[X]$ is an $s$-tame map. The set of polynomial functions $f$, whose Hamiltonian vector fields have $s$-tame modulus will be denoted by $T_s M$. If $f \in T_s M$ for any $s > 1$, we will write $f \in T_m$, and say that $f$ has tame modulus.

**Remark 3.7.** The property of having tame modulus depends on the coordinate system.

**Definition 3.8.** Let $X$ be an $s$-tame vector field homogeneous polynomial of degree $r$. The infimum of the constants $K_s$ such that the inequality
\[
\|\tilde{X}(\phi(1), \ldots, \phi(r))\|_s \leq K_s \|\phi(1)\|_1 \ldots \|\phi(l-1)\|_1 \|\phi(l)\|_s \|\phi(l+1)\|_1 \ldots \|\phi(r)\|_1
\]
\[
\forall \phi(1), \ldots, \phi(r) \in H^s \oplus H^s
\]
holds will be called tame $s$ norm of $X$, and will be denoted by $|X|^T_s$.

The tame $s$ norm of a polynomial Hamiltonian $f$ of degree $r+1$ is given by
\[
|f|_s := \sup \frac{\|\tilde{X}_lf(\phi)\|_s}{\|\phi\|_{s,1}},
\]
where the sup is taken over all multivectors $\phi = (\phi(1), \ldots, \phi(r))$ such that $\phi(j) \neq 0$ for any $j$.

**Definition 3.9.** Let $f \in T_s^k$ be a non-homogeneous polynomial, and consider its Taylor expansion
\[f = \sum_m f_m,\]
where $f_m$ is homogeneous of degree $m$. Let $R > 0$, then we denote

$$\langle |f| \rangle_{s,R} := \sum_{m \geq 2} |f_m|_s R^{m-1}. \quad (29)$$

Such a definition extends naturally to analytic functions such that $\langle |f| \rangle$ is finite. The set of functions of class $T_M^s$ for which $\langle |f| \rangle$ is finite will be denoted by $T_{s,R}$.

With the above definitions,

$$\sup_{B_s(R)} \| X_f(\psi, \bar{\psi}) \|_s \leq \langle |f| \rangle_{s,R}. \quad (29)$$

It is easy to check that the set $T_{s,R}$ endowed with the norm $(29)$ is a Banach space.

Now we introduce the Fourier projection

$$\Pi_N \psi(x) := \int_{|k| \leq N} \hat{\psi}(k)e^{ik\cdot x} dk,$$

and we split the variables $(\psi, \bar{\psi})$ into

$$(\psi_l, \bar{\psi}_l) := (\Pi_N \psi, \Pi_N \bar{\psi}), \quad (\psi_h, \bar{\psi}_h) := ((id - \Pi_N) \psi, (id - \Pi_N) \bar{\psi}).$$

The use of Fourier projection is important in view of the following result, whose proof can be found in Appendix A of [BG06].

**Lemma 3.10.** Fix $N$, and consider the decomposition $\psi = \psi_l + \psi_h$ as above. Let $f \in T_M^s$ be a polynomial of degree less or equal than $r + 2$. Assume that $f$ has a zero of order three in the variables $(\psi_h, \bar{\psi}_h)$, then one has

$$\sup_{B_s(R)} \| X_f(\psi, \bar{\psi}) \|_s \lesssim \langle |f| \rangle_{s,R} N^{-1}. \quad (30)$$

**Lemma 3.11.** Let $f, g \in T_M^s$ be homogeneous polynomial of degrees $n + 1$ and $m + 1$ respectively. Then one has $\{f, g\} \in T_M^s$, and

$$\|\{f, g\}\|_s \leq (n + m)|f|_s |g|_s. \quad (31)$$

The proof of this lemma can be found again in Appendix A of [BG06].

**Remark 3.12.** Given $g$ analytic on $H^s \oplus H^s$, consider the differential equation

$$\dot{\psi} = X_g(\psi, \bar{\psi}), \quad (32)$$

where by $X_g$ we denote the vector field of $g$. Now define

$$\Phi^g(\phi, \bar{\phi}) := g \circ \Phi(\psi, \bar{\psi}).$$

In the new variables $(\phi, \bar{\phi})$ defined by $(\psi, \bar{\psi}) = \Phi(\phi, \bar{\phi})$ equation $(32)$ is equivalent to

$$\dot{\phi} = X_{\Phi^g}(\phi, \bar{\phi}). \quad (33)$$
Using the relation
\[
\frac{d}{dt}(\Phi_t^*) g = (\Phi_t^*)^\ast \{\chi, g\},
\]
we formally get
\[
\Phi_t^* g = \sum_{l=0}^{\infty} g_l,
\]
(34)

\[
g_0 := g,
\]
(35)

\[
g_l := \frac{1}{l} \{\chi, g_{l-1}\}, \quad l \geq 1.
\]
(36)

In order to estimate the terms appearing in (34) we exploit the following results

**Lemma 3.13.** Let \(h, g \in T_{s,R}\), then for any \(d \in (0, R)\) one has that \(\{h, g\} \in T_{s,R-d}\), and
\[
\langle |\{h, g\}| \rangle_{s,R-d} \leq \frac{1}{d} \langle |h| \rangle_{s,R} \langle |g| \rangle_{s,R}.
\]
(37)

**Proof.** Write \(h = \sum_j h_j\) and \(g = \sum_k g_k\), with \(h_j\) homogeneous of degree \(j\) and similarly for \(g\). Then we have
\[
\{h, g\} = \sum_{j,k} \{h_j, g_k\},
\]
where \(\{h_j, g_k\}\) has degree \(j + k - 2\). Therefore by (31) in Lemma 3.11
\[
\langle |\{h, g\}| \rangle_{s,R-d} = \langle |\{h_j, g_k\}| \rangle_{s,(R-d)^{j+k-3}}
\]
\[
\leq |h_j|_s |g_k|_s (j + k - 2)(R - d)^{j+k-3}
\]
\[
\leq |h_j|_s |g_k|_s \frac{1}{d} R^{j+k-2} = \frac{1}{d} \langle |h| \rangle_{s,R} \langle |g| \rangle_{s,R},
\]
where we exploited the inequality \(k(R - d)^{k-1} < R^k/d\), which holds for any positive \(R\) and \(d \in (0, R)\). \(\square\)

**Lemma 3.14.** Let \(g, \chi \in T_{s,R}\) be analytic functions; denote by \(g_l\) the functions defined recursively by (34); then for any \(d \in (0, R)\) one has that \(g_l \in T_{s,R-d}\), and
\[
\langle |g_l| \rangle_{s,R-d} \leq \langle |g| \rangle_{s,R} \left(\frac{c}{d} \langle |\chi| \rangle_{s,R}\right)^l.
\]
(38)

**Proof.** Fix \(l\), and denote \(\delta := d/l\). We look for a sequence \(C^{(l)}_m\) such that
\[
\langle |g_m| \rangle_{s,R-\delta m} \lesssim C^{(l)}_m, \quad \forall m \leq l.
\]
By (37) in Lemma 3.13 we can define the sequence
\[
C^{(l)}_0 := \langle |g| \rangle_{s,R},
\]
\[
C^{(l)}_m = \frac{2}{\delta m} C^{(l)}_{m-1} \langle |\chi| \rangle_{s,R}
\]
\[
= \frac{2l}{\delta m} C^{(l)}_{m-1} \langle |\chi| \rangle_{s,R}.
\]
One has

\[ C_l^{(1)} = \frac{1}{l!} \left( \frac{2l}{d} \langle |\chi| |g] \rangle_{s,R} \right) \langle |g| |s,R \rangle, \]

and using the inequality \( l! < l^d \) one can conclude.

**Lemma 3.15.** Let \( f \in T_M^s \) be a polynomial which is at most quadratic in the variables \((\psi_h, \bar{\psi}_h)\).

Then \( \forall \gamma > 0 \) there exists a set \( R_\gamma \subset [1, +\infty) \times V \) satisfying

\[ |R_\gamma \cap ([n, n+1] \times V)| = O(\gamma) \quad \forall n \in \mathbb{N}_0, \]

and \( \exists \tau > 1 \) such that \( \forall (c, (v_j)) \in ([1, +\infty) \times V) \setminus R_\gamma \) and \( \forall N \geq 1 \) the following holds: there exist \( \chi, Z \in T_{s,R} \) such that

\[ \{H_0, \chi\} + Z = f, \tag{39} \]

and such that \( Z \) depends only on the actions and satisfies (12). Moreover, \( \chi \) and \( Z \) satisfy the following estimates

\[ \langle |\chi| \rangle_{s,R} \leq \frac{N^r}{\gamma} \langle |f| \rangle_{s,R}, \tag{40} \]

\[ \langle |Z| \rangle_{s,R} \leq \langle |f| \rangle_{s,R}. \tag{41} \]

**Proof.** Expanding \( f \) in Taylor series, namely \( f(\psi, \bar{\psi}) = \sum_{j,l} f_{j,l} \psi^j \bar{\psi}^l, \) and similarly for \( \chi \) and \( Z, \) equation (39) becomes an equation for the coefficients of \( f, \) \( \chi \) and \( Z, \)

\[ i\omega \cdot (j-l)\chi_{j,l} + Z_{j,l} = f_{j,l}. \]

Then we define

\[ Z_j := Z_{j,j} = f_{j,j}, \tag{42} \]

\[ \chi_{j,l} := \frac{f_{j,l}}{i\omega \cdot (j-l)}, \quad \text{when } j \neq l, \quad |\omega \cdot (j-l)| \geq \frac{\gamma}{N^r}. \tag{43} \]

By construction and by Theorem 3.3 we get estimates (40) and (41). Furthermore, since \( f \) is at most quadratic in \((\psi_h, \bar{\psi}_h), \) we obtain that \( \sum_{k>N} j_k \leq 2, \) and thus \( Z \) satisfies (12).

**Remark 3.16.** Let \( s > s^*, \) and assume that \( \chi, F \) are analytic on \( B_s(R). \) Fix \( d \in (0, R), \) and assume also that

\[ \sup \limits_{B_s(R)} \|X_\chi(\psi, \bar{\psi})\|_s \leq d/3, \]

Then by Lemma 3.4 for \( |t| \leq 1 \)

\[ \sup \limits_{B_s(R-d)} \|X_{(\phi^*_\chi) \cdot F - F}(\psi, \bar{\psi})\|_s = \sup \limits_{B_s(R-d)} \|X_{F \phi^*_\chi - F}(\psi, \bar{\psi})\|_s \leq \frac{5}{d} \sup \limits_{B_s(R)} \|X_\chi(\psi, \bar{\psi})\|_s \sup \limits_{B_s(R)} \|X_F(\psi, \bar{\psi})\|_s \leq 2 \sup \limits_{B_s(R)} \|X_F(\psi, \bar{\psi})\|_s. \tag{44} \]

13
Lemma 3.17. Let $\chi \in T_{s,R}$ be the solution of the equation (39), with $f \in T_{s,M}$. Denote by $H_{0,l}$ the functions defined recursively via (34) from $H_0$. Then for any $d \in (0,R)$ one has that $H_{0,l} \in T_{s,R-d}$, and

$$
\langle |H_{0,l}| \rangle_{s,R-d} \leq 2 \langle |f| \rangle_{s,R-d} \left(\frac{c}{d} \langle |\chi| \rangle_{s,R} \right)^1.
$$

(47)

Proof. Using (39) one gets $H_{0,1} = Z - f \in T_{s,M}$. Then, arguing as for (38), one can conclude. \qed

The main step of the proof of Theorem 2.1 is the following result, that allows to increase by one the order of the perturbation. As a preliminary step, we take the Taylor series of $N(\psi, \bar{\psi})$ up to order $r + 2$,

$$
N(\psi, \bar{\psi}) = \sum_{l=1}^{r} \hat{N}_l(x, \psi, \bar{\psi})
$$

(48)

$$
+ N(\psi, \bar{\psi}) - \sum_{l=1}^{r} \hat{N}_l(x, \psi, \bar{\psi})
$$

(49)

$$
=: N^{(1)}(\psi, \bar{\psi}) + N^{(1,r)}(\psi, \bar{\psi}),
$$

(50)

where $\hat{N}_l$ is a homogeneous polynomial in $\psi$ and $\bar{\psi}$ of degree $l + 2$ with variable $C^\infty$-coefficients (since $V \in C^\infty$).

Now we consider the analytic Hamiltonian

$$
H^{(0)} := H_0 + N^{(1)}.
$$

(51)

Then for $R$ sufficiently small one has that

$$
\langle |N^{(1)}| \rangle_{s,R} \lesssim R^2,
$$

(52)

$$
\langle |N^{(1,r)}| \rangle_{s,R} \lesssim R^{r+2}.
$$

(53)

Lemma 3.18. Consider the Hamiltonian (51), and fix $s > s^*$. Then $\forall \gamma > 0$ there exists a set $\mathcal{R}_{\gamma} \subset [1, +\infty) \times V$ satisfying

$$
|\mathcal{R}_{\gamma} \cap ([n, n+1] \times V)| = O(\gamma) \quad \forall n \in \mathbb{N}_0,
$$

and $\exists \tau > 1$ such that $\forall (c, (v_j)_j) \in ([1, +\infty) \times V) \setminus \mathcal{R}_{\gamma}$ and $\forall N \geq 1$ the following holds: for any $m \leq r$ there exists $R_m^* \ll 1$ and, for any $N > 1$ there exists an analytic canonical transformation

$$
\mathcal{T}^{(m)} : B_s \left(\frac{(2r - m)}{2N^r} R_m^{*,2} \right) \to H^s
$$

such that

$$
H^{(m)} := H^{(0)} \circ \mathcal{T}^{(m)} = H^{(0)} + Z^{(m)} + f^{(m)} + R_N^{(m)} + \mathcal{R}_T^{(m)},
$$

(54)

where for any $R < R_m^*/N^r$ the following properties are fulfilled
We argue by induction. The theorem is trivial for the case\footnote{We refer to (12) for more details.} $m = 0$, by setting $f^{(0)} = 0$. Moreover, \( f^{(m)} \) is a polynomial of degree (at most) \( r + 2 \).

Now consider the truncated Hamiltonian \( H \). Let \( \chi \) be the analytic Hamiltonian generating \( T \) of the truncated Hamiltonian. Let \( \chi_m \) be the analytic Hamiltonian generating \( T_m \) of the truncated Hamiltonian. Let \( \chi_m \) be the analytic Hamiltonian generating \( T_m \).

Using (34) we have

\[
\sup_{B_s((1-m/2r)R)} \| X_{f^{(m)}(\psi, \tilde{\psi})} \| \lesssim R^{m+2} N^{m}, \quad \forall m \geq 1;
\]

3. the remainder terms \( R^{(m)}_N \) and \( R^{(m)}_T \) satisfy

\[
\sup_{B_s((1-m/2r)R)} \| X_{R^{(m)}_N(\psi, \tilde{\psi})} \| \lesssim R^{m+2} N^{m},
\]

\[
\sup_{B_s((1-m/2r)R)} \| X_{R^{(m)}_T(\psi, \tilde{\psi})} \| \lesssim R^{m+2} N^{m}, \quad \forall m \geq 1;
\]

Proof. We argue by induction. The theorem is trivial for the case \( m = 0 \), by setting \( T^{(0)} = id \), \( Z^{(0)} = 0 \), \( f^{(0)} = 0 \), \( R^{(m)}_N = R^{(m)}_T = 0 \).

Then we split \( f^{(m)} \) into two parts, an effective one and a remainder. Indeed, we perform a Taylor expansion of \( f^{(m)} \) only in the variables \( (\psi, \tilde{\psi}) \), namely we write

\[
f^{(m)} = f^{(m)}_N + f^{(m)}_N,
\]

where \( f^{(m)}_N \) is the truncation of such a series at second order, and \( f^{(m)}_N \) is the remainder. Since both \( f^{(m)}_N \) and \( f^{(m)}_N \) are truncations of \( f^{(m)} \), one has that

\[
\langle \| f^{(m)}_N \| \rangle \leq \langle \| f^{(m)} \| \rangle.
\]

Now consider the truncated Hamiltonian \( H_0 + Z^{(m)} + f^{(m)}_0 \); we look for a Lie transform \( T_m \) that eliminates the non-normalized part of order \( m + 4 \) of the truncated Hamiltonian. Let \( \chi_m \) be the analytic Hamiltonian generating \( T_m \).

Using (34) we have

\[
(H_0 + Z^{(m)} + f^{(m)}_0) \circ T_m = H_0 + Z^{(m)} + f^{(m)}_N + \{ \chi_m, H_0 \} + \sum_{l \geq 1} Z^{(m)}_l + \sum_{l \geq 1} f^{(m)}_{0,l} + \sum_{l \geq 2} H_{0,l},
\]
are already normalized, that the term in the \(60\) is the non-normalized part of order \(m + 3\) that will vanish through the choice of a suitable \(\chi_m\), and that the last lines contains all the terms having a zero of order \(m + 4\) at the origin.

Now we want to determine \(\chi_m\) in order to solve the so-called “homological equation”

\[
\{\chi_m, H_0\} + f_0^{(m)} = Z_{m+1},
\]

with \(Z_{m+1}\) depending only on the actions and satisfying (12). The existence of \(\chi_m\) and \(Z_{m+1}\) is ensured by Lemma 3.15 and by applying (41) and (56) we get

\[
\langle |\chi_m| \rangle_{s,(1-m/(2r))R} \leq N^r R^2 (N^r R)^m, \quad (62)
\]

\[
\langle |Z_{m+1}| \rangle_{s,(1-m/(2r))R} \leq R^2 (N^r R)^m. \quad (63)
\]

In particular, in view of (46), we can deduce (55) at level \(m + 1\). Now define \(Z^{(m+1)} := Z^{(m)} + Z_{m+1}\), and \(f_{c}^{(m+1)} := f^{(m+1)}\). By (62) recalling that \(R < R_m^*/N^r\), we can deduce (56) at level \(m + 1\). Moreover, provided that \(R_m^* < 2^{-(m+1)/2}\), one has

\[
\delta := e^{2r} \frac{2r}{R} \langle |\chi_m| \rangle_{s,(1-m/(2r))R} \leq (N^r R)^{m+1} \frac{1}{2}.
\]

By (38) and (56) one thus gets

\[
\langle |f_{c}^{(m+1)}| \rangle_{s,(1-(m+1)/(2r))R} \leq \sum_{l \geq 1} R^2 \delta^l + \sum_{l \geq 1} R^2 \delta^l (N^r R)^m + \sum_{l \geq 2} R^2 \delta^{l-1} (N^r R)^m \leq R^2 (N^r R)^{m+1}.
\]

Write now \(f_{c}^{(m+1)} = f^{(m+1)} + R_{m,T}\), where \(f^{(m+1)}\) is the Taylor polynomial of order \(r + 2\) of \(f_{c}^{(m+1)}\), and where \(R_{m,T}\) has a zero of order \(r + 3\) at the origin. Clearly \(f^{(m+1)}\) satisfies (57) at level \(m + 1\), since it is a truncation of \(f_{c}^{(m+1)}\).

The remainder may be bounded by using Lagrange and Cauchy estimates,

\[
\sup_{B_c((1-m/(2r))R)} \|X_{R_{m,T}}(\psi, \psi)\|_s \leq \frac{1}{(r+2)!} R^{r+2} \sup_{B_c(R_m^*/(2N^r))} \|\partial^r X_{f_{c}^{(m+1)}}(\psi, \psi)\|_s \leq R^{r+2} \frac{2N^r}{R_m^*} \sup_{B_c(R_m^*/N^r)} \|X_{f_{c}^{(m+1)}}(\psi, \psi)\|_s \leq (N^r R)^{r+2}.
\]

Now define \(R_T^{(m+1)} := R_T^{(m)} \circ T_m + R_{m,T}\). By (40) we can deduce (58) at level \(m + 1\). Then set \(R_N^{(m+1)} := (R_N^{(m)} + f_N^{(m)}) \circ T_m\). By (57) and (59), together with (40) and (30) in Lemma 3.10 we obtain (59) at level \(m + 1\).

Now we conclude the proof of Theorem 2.1.

By taking the canonical transformation \(T^{(r)}\) defined in the iterative Lemma 5.15 we have that

\[
H^{(r)} = H_0 + Z^{(r)} + R_N^{(r)} + R_T^{(r)} + N^{(1,r)} \circ T^{(r)}, \quad (64)
\]
with $Z^{(r)}$ depending only on the actions and satisfying (12), and for any $R < R_n/N^7$ the following holds
\[
\sup_{B_{s}(R)} \|T^{(r)}(\psi, \bar{\psi}) - (\psi, \bar{\psi})\| < N^{2r} R^3,
\]
\[
\sup_{B_{s}(R)} \|X^{(r)}_{\mathcal{K}}(\psi, \bar{\psi})\| < R^2 \frac{s}{N^{s-1}},
\]
\[
\sup_{B_{s}(R)} \|X^{(r)}_{\mathcal{R}}(\psi, \bar{\psi})\| < (N^7 R)^{r+2},
\]
\[
\sup_{B_{s}(R)} \|X^{(1)}_{\mathcal{R}}(\psi, \bar{\psi})\| < (N^{7} R)^{r+2}.
\]
To conclude we have just to choose $N$ and $s$ such that $\mathcal{R}_{N}^{(r)}$ and $\mathcal{R}_{T}^{(r)}$ are of the same order of magnitude. First take $N = R^{-a}$, with $a$ still to be determined; then, in order to obtain that $\mathcal{R}_{N}^{(r)}$ is of order $O(R^{r+3/2})$ we choose $a := \frac{1}{2(r+\beta)}$.

By taking $s > 2\pi r + 2 + 1$ we get that also $N^{(1,r)}$ is of the same order of magnitude.

Now take $K^* = 1/24$, and construct the canonical transformation $(\psi, \bar{\psi}) = T^{(r)}(\psi', \bar{\psi}')$. Denote by $I'$ the actions expressed in the variable $(\psi', \bar{\psi}')$, and define the function $\mathcal{N}(\psi', \bar{\psi}') := \|I'\|^2_r$. By (13) one has that $\mathcal{N}(\psi'_0, \bar{\psi}'_0) \lesssim \frac{32}{9} R^2$, provided that $R$ is sufficiently small. Since
\[
\frac{\partial \mathcal{N}}{\partial t}(\psi', \bar{\psi}') = \{R^{(r)}, \mathcal{N}\}(\psi', \bar{\psi}'),
\]
and therefore, as far as $\mathcal{N}(\psi', \bar{\psi}') < \frac{64}{9} R^2$,
\[
\left| \frac{\partial \mathcal{N}}{\partial t}(\psi', \bar{\psi}') \right| \leq K'_s R^{r+5/2}.
\] (65)

Denote by $T_f$ the escape time of $(\psi', \bar{\psi}')$ from $B_{s}(R/3)$; observe that for all times smaller than $T_f$, (65) holds. So one has
\[
\frac{64}{9} R^2 = \mathcal{N}(\psi'(T_f), \bar{\psi}'(T_f)) \leq \mathcal{N}(\psi'_0, \bar{\psi}'_0) + K'_s R^{r+5/2}T_f,
\]
which shows that $T_f$ should be of order (at least) $R^{r+1/2}$. Going back to the original variables one gets (16). To show (17), one has to recall that
\[
|I(t) - I(0)| \leq |I(t) - I'(t)| + |I'(t) - I'(0)| + |I'(0) - I(0)|,
\]
and that by (14) and (16) one can estimate the first and the third term; the second term can be bounded by computing the time derivative of $\|I'\|^2_r$ with the Hamiltonian, and observing that it is of order $O(R^{r+5/2})$.

Now, consider the initial actions $(I_0, \bar{I}_0) := (I(0), \bar{I}(0))$. By passing to the Fourier transform,
\[
I_j(t) := \tilde{I}(t)(j), \quad j \geq 1,
\]
we have that for any $r_1 \leq r$
\[
|\langle I_j(t), \bar{I}_j(t) \rangle - (I_j(0), \bar{I}_j(0))| \lesssim \frac{R^{2r_1}}{j^{2s}}, \quad |t| \lesssim R^{-(r-r_1+1/2)}.
\] (66)
If we define the torus
\[ I_c := \{(\psi, \bar{\psi}) \in H^s : (I_j(\psi, \bar{\psi}), \bar{I}_j(\psi, \bar{\psi}) = (I_j(0), \bar{I}_j(0)), \text{ for any } j \geq 1\}, \]
we get
\[ d_{s_1}(((\psi(t), \bar{\psi}(t)), I_c) \leq \left[ \sum_j j^{2s_1} \left( |\sqrt{I_j(t)} - \sqrt{I_j(0)}|^2 + |\sqrt{\bar{I}_j(t)} - \sqrt{\bar{I}_j(0)}|^2 \right) \right]^{1/2}, \]
and by using \[ \text{(66)} \] we obtain
\[ d_{s_1}(((\psi(t), \bar{\psi}(t)), I_c)^2 \leq \left( \sup_j j^{2s_1}|I_j(t) - I_j(0)|^2 + j^{2s_1}|\bar{I}_j(t) - \bar{I}_j(0)|^2 \right) \sum_j \frac{1}{j^{2(s-s_1)}}, \]
which is convergent for \( s_1 < s - 1/2 \).

References

[BD12] Weizhu Bao and Xuanchun Dong. Analysis and comparison of numerical methods for the Klein–Gordon equation in the nonrelativistic limit regime. Numerische Mathematik, 120(2):189–229, 2012.

[BFS17] Simon Baumstark, Erwan Faou, and Katharina Schratz. Uniformly accurate exponential-type integrators for Klein-Gordon equations with asymptotic convergence to the classical NLS splitting. Mathematics of Computation, 2017.

[BG06] Dario Bambusi and Benoît Grébert. Birkhoff normal form for partial differential equations with tame modulus. Duke Mathematical Journal, 135(3):507–567, 2006.

[BZ16] Weizhu Bao and Xiaofei Zhao. A uniformly accurate (UA) multiscale time integrator Fourier pseudospectral method for the Klein–Gordon–Schrödinger equations in the nonrelativistic limit regime. Numerische Mathematik, pages 1–41, 2016.

[Del09] J-M Delort. On long time existence for small solutions of semi-linear Klein-Gordon equations on the torus. Journal d’Analyse Mathématique, 107(1):161–194, 2009.

[DI17] Jean-Marc Delort and Rafik Imekraz. Long time existence for the semi-linear Klein-Gordon equation on a compact boundaryless Riemannian manifold. Communications in Partial Differential Equations, 42(3):388–416, 2017.

[DS04] J-M Delort and Jeremie Szeftel. Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres. International Mathematics Research Notices, 2004(37):1897–1966, 2004.

[FHZ17] Daoyuan Fang, Zheng Han, and Qidi Zhang. Almost global existence for the semi-linear Klein–Gordon equation on the circle. Journal of Differential Equations, 262(9):4610–4634, 2017.
Erwan Faou and Katharina Schratz. Asymptotic preserving schemes for the Klein–Gordon equation in the non-relativistic limit regime. *Numerische Mathematik*, 126(3):441–469, 2014.

Daoyuan Fang and Qidi Zhang. Long-time existence for semi-linear Klein–Gordon equations on tori. *Journal of Differential Equations*, 249(1):151–179, 2010.

Yong Lu and Zhifei Zhang. Partially strong transparency conditions and a singular localization method in geometric optics. *Archive for Rational Mechanics and Analysis*, 222(1):245–283, 2016.

Shuji Machihara. The nonrelativistic limit of the nonlinear Klein-Gordon equation. *Funkcialaj Ekvacioj Serio Internacia*, 44(2):243–252, 2001.

Nader Masmoudi and Kenji Nakanishi. From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations. *Mathematische Annalen*, 324(2):359–389, 2002.

Shuji Machihara, Kenji Nakanishi, and Tohru Ozawa. Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations. *Mathematische Annalen*, 322(3):603–621, 2002.

Branko Najman. The nonrelativistic limit of the nonlinear Klein-Gordon equation. *Nonlinear Analysis: Theory, Methods & Applications*, 15(3):217–228, 1990.

Stefano Pasquali. Dynamics of the nonlinear Klein-Gordon equation in the nonrelativistic limit, I. *arXiv preprint arXiv:1703.01609*, 2017.

Helmut Rüssmann. Invariant tori in non-degenerate nearly integrable Hamiltonian systems. *Regul. Chaotic Dyn.*, 6(2):119–204, 2001.

Masayoshi Tsutsumi. Nonrelativistic approximation of nonlinear Klein-Gordon equations in two space dimensions. *Nonlinear Analysis: Theory, Methods & Applications*, 8(6):637–643, 1984.