UNIFORM HYPERBOLICITY OF NONSEPARATING CURVE GRAPHS OF NONORIENTABLE SURFACES

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ABSTRACT. Let $N$ be a connected finite type nonorientable surface with or without boundary components and punctures. We prove that the graph of nonseparating curves of $N$ is connected and Gromov hyperbolic with a constant which does not depend on the topological type of the surface by using the bicorn curves introduced by Przytycki and Sisto. The proof is based on the argument by Rasmussen on the uniform hyperbolicity of graphs of nonseparating curves for finite type orientable surfaces.

1. Introduction

Let $N = N_{g,p}^f$ be a connected finite type nonorientable surface of genus $g$ with $f$ boundary components and $p$ punctures and $S = S_{g,p}^f$ be a connected finite type orientable surface of genus $g$ with $f$ boundary components and $p$ punctures. When $f = 0$ or $p = 0$, we drop the suffix that denotes 0, excepting $g$, from $N_{g,p}^f$ and $S_{g,p}^f$. For example, $N_{0,0}^g$ is simply denoted as $N_g$. If we are not interested in whether a given surface is orientable or not, we denote the surface by $F$.

A simple closed curve $c$ on $F$ is said to be one-sided if the regular neighborhood of $c$ is a Möbius band, and $c$ is said to be two-sided if the regular neighborhood of $c$ is an annulus. Moreover a simple closed curve $c$ on $F$ is nonseparating if the complement $F \setminus c$ of $c$ in $F$ is connected, and $c$ is separating if the complement $F \setminus c$ of $c$ in $F$ is not connected. We say a curve $c$ is essential if $c$ is nonseparating, or $c$ is separating and does not bound a disk, a disk with a puncture, an annulus, or a Möbius band. In this paper, a curve on $F$ means a simple closed curve on $F$.

The curve graph $\mathcal{C}(F)$ of $F$ is the simplicial graph whose vertex set consists of the homotopy classes of all essential simple closed curves on $F$ and whose edge set consists of all non-ordered pairs of vertices which can be realized disjointly on $F$. We define the nonseparating curve graph $\mathcal{NC}(F)$ of $F$ as the full subgraph of $\mathcal{C}(F)$ consisting of all homotopy classes of nonseparating curves on $F$. We define the distances $d_{\mathcal{C}}(\cdot, \cdot)$ on $\mathcal{C}(F)$ and $d_{\mathcal{NC}}(\cdot, \cdot)$ on $\mathcal{NC}(F)$ by the minimal lengths of edge-paths connecting two vertices in $\mathcal{C}(F)$ and $\mathcal{NC}(F)$, respectively. Thus, we consider $\mathcal{C}(F)$ and $\mathcal{NC}(F)$ as geodesic spaces.

We denote by $i(a, b)$ the geometric intersection number between the homotopy classes of the curves $a$ and $b$ in $F$, that is, the minimal number of intersection points between a representative curve in the homotopy class of $a$ and a representative curve in the homotopy class of $b$. Two curves $a$ and $b$ in $F$ are in minimal position if the
number of intersection points between $a$ and $b$ is minimal in the homotopy classes of $a$ and $b$. Note that two curves are in minimal position in $F$ if and only if they do not bound a bigon on $F$ (see [19] Proposition 2.1 for nonorientable surfaces).

For orientable surfaces, first Masur and Minsky [14] proved that the curve graphs of orientable surfaces are Gromov hyperbolic. Aougab [1], Bowditch [7], Clay, Rafi, and Schleimer [8], and Hensel, Przytycki, and Webb [10] independently proved that the curve graphs of orientable surfaces are uniformly hyperbolic, that is, one can choose hyperbolicity constants which do not depend on the topological types of the surfaces. Moreover, Rasmussen [17] proved that the nonseparating curve graphs of orientable surfaces are also uniformly hyperbolic. We note that by using the result of Rasmussen, Bowden, Hensel, and Webb [5] proved that for orientable surfaces “fine curve graphs” are Gromov hyperbolic and studied the bounded cohomology of orientable surface diffeomorphism groups.

For nonorientable surfaces, Bestvina and Fujiwara [4] proved that the curve graphs of nonorientable surfaces are Gromov hyperbolic (Masur and Schleimer [15] gave another proof), and the author [13] studied the uniform hyperbolicity of curve graphs for nonorientable surfaces by using the unicorn paths argument introduced by Hensel, Przytycki, and Webb [10].

In this paper, we generalize Rasmussen’s result to the nonorientable surfaces, that is, we prove the uniform hyperbolicity of the nonseparating curve graphs of finite type nonorientable surfaces:

**Theorem 1.1.** There exists a constant $\delta > 0$ such that for any finite type nonorientable surface $N$ of genus $g = 1$ and $f + p \geq 3$, of genus $g = 2$ and $f + p \geq 1$, or of genus $g \geq 3$, the nonseparating curve graph $NC(N)$ of $N$ is connected, has infinite diameter, and $\delta$-hyperbolic.

We remark that for nonorientable surfaces of genus 1 and 2, the nonseparating curve graphs are not connected or consists of exactly one isolated vertex. Hence we consider “augmented” nonseparating curve graphs (and we call them also nonseparating curve graphs here) for the nonorientable surfaces of genus 1 and 2 (see Remark 2.2). The nonseparating curve graphs of nonorientable surfaces of genus $g = 1$ and $f + p \geq 3$ or of genus $g = 2$ and $f + p \geq 1$ has infinite vertices. We treat the nonorientable surfaces of genus 1 and 2 in Section 4.

We also note that as an analogy of Bowden, Hensel, and Webb [5], Theorem 1.1 also has an application to study Gromov hyperbolicity for fine curve graphs of nonorientable surfaces and the bounded cohomology of nonorientable surface diffeomorphism groups [12].

We prove Theorem 1.1 by using the bicorn curves defined by Przytycki and Sisto [16] and applying the argument of Rasmussen [17] to nonorientable surfaces, but with some modifications. Here we list some of the differences from the case of orientable surfaces.

- Rasmussen [17] uses the signs of intersection points between two curves in the argument. Although the signs of intersection points between two curves cannot be defined on nonorientable surfaces, the argument by Rasmussen also makes sense by considering the signs of intersection points on each regular neighborhood of an arc including the intersection points.
• While the signs of intersection points between two curves are not enough to calculate the geometric intersection number of the two curves for nonorientable surfaces, additional properties, that is, one-sidedness and two-sidedness of curves allow us to deduce the geometric intersection numbers of the two curves.

• In the proof of Lemma 3.4, we cannot show the same claim as that of orientable surfaces since the signs of intersection points between two curves cannot decide the geometric intersection number between two curves (for the details, see Remark 3.5).

We conclude this section by noting that by combining the result of Rasmussen and the result of Aramayona and Valdez [2], it follows that the nonseparating curve graphs $\mathcal{NC}(S)$ for infinite type orientable surfaces $S$ with finite positive genus are uniformly hyperbolic (see [17, Corollary 1.2]). The author does not know whether the nonseparating curve graphs $\mathcal{NC}(N)$ of infinite type nonorientable surfaces $N$ are uniformly hyperbolic. We hope that Theorem 1.1 has some application to study infinite type nonorientable surfaces.

2. Preliminaries

Let $N$ be a connected finite type nonorientable surface. At the beginning, $N$ may have boundary components and punctures. However, since the difference between boundary components and punctures will not relevant for us, we will assume that $N$ has no punctures from now on.

In this section, we show the connectedness and the infinite diameter of $\mathcal{NC}(N)$, i.e. the first and the second parts of Theorem 1.1. Then, we recall the criterion for hyperbolicity and the definition of bicorns, as well as describe the notation for the proof of Theorem 1.1.

Firstly, we prove that the nonseparating curve graphs are connected:

**Proposition 2.1.** (cf. [9, Theorem 4.4]) Let $N$ be a nonorientable surface of genus $g \geq 3$ with $f \geq 0$ boundary components. Then, the nonseparating curve graph $\mathcal{NC}(N)$ of $N$ is connected.

**Proof.** The proof follows the argument by Farb and Margalit [9, Theorem 4.4] for orientable surfaces.

Note that, from Ivanov [11] (see also Szepietowski [20, Theorem 6.1]), we know that the usual curve graph $\mathcal{C}(N)$ is connected. We will prove the proposition by induction on the number of boundary components $f$. If $f \leq 1$, let $a, b$ be any pair of vertices of $\mathcal{NC}(N)$. Since $\mathcal{C}(N)$ is connected, we can take a geodesic path $a = c_0, c_1, \ldots, c_{n-1}, c_n = b$ in $\mathcal{C}(N)$. Suppose that $c_i$ is a separating curve in $N$, and $N \setminus c_i = F' \cup F''$. As we take a geodesic path, the vertices $c_i-1$ and $c_i+1$ must intersect, and they are included in the same connected component $F'$. Both $F'$ and $F''$ have positive genus because $f \leq 1$. Hence, we can take a nonseparating curve $d$ in $F''$ and replace the separating curve $c_i$ with it. By repeating this argument for every separating curve in the geodesic, we obtain a path in $\mathcal{NC}(N)$ that connects $a$ and $b$.

For the induction on $f$, we assume that $f \geq 2$ and proceed as above. The only problem is that a separating curve $c_i$ may bound a subsurface $F''$ of genus 0. In this case, the other subsurface $F'$ containing $c_{i-1}$ and $c_{i+1}$ is a nonorientable surface of genus $g$ that has at most $f - 2$ boundary components. By induction, we
can find a path connecting \(c_{i-1}\) and \(c_{i+1}\), and we replace \(c_i\) by the corresponding sequence.

\[
\square
\]

Remark 2.2. For \(g = 1\) and \(f \leq 1\), the nonseparating curve graphs consist of an isolated vertex, for \(g = 1\) and \(f = 2\), the nonseparating curve graphs consist of two isolated vertices, and for \(g = 1\) and \(f \geq 3\), the nonseparating curve graphs consist of an infinite number of isolated vertices. For \(g = 2\) and \(f \geq 0\), the nonseparating curve graphs are not connected. In fact, the complement of any nonseparating two-sided curve in a nonorientable surface of genus 2 with any number of boundary components is always orientable (and so we cannot take any one-sided curve in the complement). (See also [3] Section 2.4 for the curve graphs in low dimensions for nonorientable surfaces.) Therefore, we see that a nonseparating curve graph \(\mathcal{NC}(N)\) which has at least two vertices is connected if and only if the genus of \(N\) is at least three. We will discuss the hyperbolicity of the “augmented” nonseparating curve graphs for a nonorientable surface of genus 2 with one boundary component in Section [4].

Next, we prove that the nonseparating curve graphs have infinite diameters:

Proposition 2.3. For any finite type nonorientable surface \(N\) of genus \(g \geq 3\), the nonseparating curve graph \(\mathcal{NC}(N)\) of \(N\) has infinite diameter.

Proof. Let \(N = N^f_2\) be a nonorientable surface and \(S\) the orientation double cover of \(N\). Let \(c\) be a vertex of the curve graph \(\mathcal{C}(N)\) of \(N\). We take a pseudo-Anosov element \(\varphi\) of the mapping class group \(\text{Mod}(N)\) of \(N\), and fix any \(n \in \mathbb{Z}\). We write \(\mathcal{G}\) as a geodesic which connects \(c\) and \(\varphi^n(c)\) in \(\mathcal{C}(N)\). We denote the consecutive vertices in \(\mathcal{G}\) by \(\delta_0, \delta_1, \ldots, \delta_m\), where \(\delta_0 = c\) and \(\delta_m = \varphi^n(c)\). Since \(\delta_i\) and \(\delta_{i+1}\) can be realized disjointly on \(N\) for any \(i = 0, \ldots, m - 1\), any pair of lifts of \(\delta_i\) and \(\delta_{i+1}\) as vertices of \(\mathcal{C}(S)\) can also be realized disjointly on \(S\). We take a lift \(\gamma \in \mathcal{C}(S)\) of \(c = \delta_0\). Let \(\gamma^1, \ldots, \gamma^{m-1}\) be lifts of \(\delta_1, \ldots, \delta_{m-1}\), respectively. Moreover, we choose the lift \(\tilde{\varphi}^n(\gamma) \in \mathcal{C}(S)\) of \(\varphi^n(c)\), where \(\tilde{\varphi}\) is a lift of \(\varphi\) which is orientation-preserving, and put \(\gamma^m = \tilde{\varphi}^n(\gamma)\). Then, \(\{\gamma^i\}_{i=0}^m\) is a path in \(\mathcal{C}(S)\) which connects \(\gamma\) and \(\tilde{\varphi}^n(\gamma)\), and the length is the same as that of \(\mathcal{G}\). Therefore, we have \(d_{\mathcal{C}(S)}(\gamma, \tilde{\varphi}^n(\gamma)) \leq d_{\mathcal{C}(N)}(c, \varphi^n(c))\). By Masur and Minsky [14] a pseudo-Anosov element \(\tilde{\varphi} \in \text{Mod}(S)\) acts on \(\mathcal{C}(S)\) loxodromically (see also Przytycki and Sisto [16]), and so it follows that there exists a constant \(\mathcal{E} > 0\) such that \(\mathcal{E}|n| \leq d_{\mathcal{C}(S)}(\gamma, \tilde{\varphi}^n(\gamma))\). Thus we have \(\mathcal{E}|n| \leq d_{\mathcal{C}(N)}(c, \varphi^n(c))\) (and we see that any pseudo-Anosov element of \(\text{Mod}(N)\) acts on \(\mathcal{C}(N)\) loxodromically). Since the nonseparating curve graph \(\mathcal{NC}(N)\) is a full subgraph of the curve graph \(\mathcal{C}(N)\), for any \(c \in \mathcal{NC}(N)\), there exists \(\mathcal{E} > 0\) such that

\[
\mathcal{E}|n| \leq d_{\mathcal{C}(N)}(c, \varphi^n(c)) \leq d_{\mathcal{NC}(N)}(c, \varphi^n(c)),
\]

and we see that the nonseparating curve graph \(\mathcal{NC}(N)\) has infinite diameter.

\[
\square
\]

We say that a geodesic space is \(\delta\)-hyperbolic if, in every geodesic triangle, each side lies in a \(\delta\)-neighborhood of the union of the other two. The following criterion for hyperbolicity is used in the argument of Rasmussen [17] to prove uniform hyperbolicity of nonseparating curve graphs \(\mathcal{NC}(S)\) for finite type orientable surfaces. We also use it here to prove the uniform hyperbolicity of nonseparating curve graphs \(\mathcal{NC}(N)\) for finite type nonorientable surfaces \(N\).
Proposition 2.4. ([7, Proposition 3.1], [15, Theorem 3.15]) Let $X$ be a graph and $a$ and $b$ two distinct vertices of $X$. Suppose that $A(a, b)$ is a connected subgraph of $X$ containing $a$ and $b$. Suppose also that there exists a constant $D > 0$ such that

(i) if $a$ and $b$ are joined by an edge, then the diameter of $A(a, b)$ is at most $D$,
(ii) for any triple $a, b, c$ of the vertices of $X$, $A(a, c)$ is contained in the $D$-neighborhood of $A(a, b) \cup A(b, c)$.

Then, $X$ is hyperbolic with a hyperbolicity constant depending only on $D$.

In this paper, for a curve $a$ on $N$, we denote by $[a]$ the homology class represented by $a$ in $H_1(N, \partial N; \mathbb{Z}_2)$; the $\mathbb{Z}_2$-coefficient relative homology group of $N$. Useful facts are that

- a simple closed curve $a$ is separating if and only if $[a] = 0$, and
- a simple closed curve $a$ is nonseparating if and only if there exists a curve $b$ such that $i(a, b) = 1$. We remark that one-sided curves are always nonseparating.

For an oriented curve or an oriented arc $a$, we write $xya$ for a subarc of $a$ from $x$ to $y$ according to the orientation of $a$. If we reverse the orientation of the arc $xya$, we denote by $-xya$ the reversed oriented arc.

Definition 2.5. Let $a$ and $b$ be two curves (including arcs) on $N$ which intersect at least two times. Choose $x, y \in a \cap b$. Let $\alpha$ and $\beta$ be subarcs of $a$ and $b$, respectively whose endpoints are $x$ and $y$. If $\alpha$ and $\beta$ intersect at their endpoints $x$ and $y$ and nowhere in their interiors, we say that $\alpha \cup \beta$ is a bicorn curve or simply a bicorn between $a$ and $b$, and we call $\alpha$ and $\beta$, respectively the $a$-arc and the $b$-arc of the bicorn, and $x$ and $y$ the corners of the bicorn. We also consider $a$ and $b$ themselves to be bicorns between $a$ and $b$.

In this paper, for a subset $M$ of a nonorientable surface $N$, we denote by $\text{nbd}(M)$ a regular neighborhood of $M$ in $N$.

3. Nonseparating curve graphs of nonorientable surfaces are uniformly hyperbolic

Let $N$ be any finite type nonorientable surface whose genus is at least 3. We recall that since the difference between boundary components and punctures is no longer relevant for us, we assume that nonorientable surfaces have no punctures. Unless it causes confusion, abusing the notation, we realize vertices on nonseparating curve graphs as nonseparating curves which are mutually in minimal position from now on.

The goal of this section is to prove the second half of Theorem 1.1, that is, the uniform hyperbolicity of nonseparating curve graphs for nonorientable surfaces. From this section, we denote by $\mathcal{NC}(N)$ the graph of nonseparating curves defined as follows: the vertices are the homotopy classes of nonseparating curves, and an edge joins two vertices if we can choose representatives of vertices which intersect at most twice. Moreover, we denote by $\mathcal{NC}'(N)$ the nonseparating curve graph defined in the usual way, that is, the vertices are the homotopy classes of nonseparating curves, and the edges correspond to the pairs of vertices which can be realized disjointly. Let $d_{\mathcal{NC}}(\cdot, \cdot)$ and $d_{\mathcal{NC}'}(\cdot, \cdot)$ be the metrics on $\mathcal{NC}(N)$ and $\mathcal{NC}'(N)$, respectively. We can show that $\mathcal{NC}(N)$ and $\mathcal{NC}'(N)$ are quasi-isometric:
Lemma 3.1. (cf. [7, Lemma 3.1]) Let \( a \) and \( b \) be any pair of vertices of \( NC'(N) \). Then, we have \( d_{NC'}(a, b) \leq 2i(a, b) + 1 \).

Proof of Lemma 3.1. We will prove the lemma by induction on the intersection number \( i(a, b) \) between \( a \) and \( b \). When \( i(a, b) = 1 \), the regular neighborhood \( nbd(a \cup b) \) of \( a \cup b \) is any one of \( N_{1,2}, N_{2,1}, \) or \( S_{1,1} \). Since the genus of \( N \) is at least 3, at least one component of \( N - \text{nbd}(a \cup b) \) has positive genus. We can take a nonseparating curve \( c \) on \( N - \text{nbd}(a \cup b) \), and \( c \) is disjoint from both \( a \) and \( b \). Hence, we have \( d_{NC'}(a, b) \leq d_{NC'}(a, c) + d_{NC'}(c, b) = 2 \leq 2i(a, b) + 1 \), and we are done.

When \( i(a, b) \geq 2 \), we choose an orientation of \( b \). We take an intersection point \( x \in a \cap b \). Let \( y \) be the first point of \( a \cap b \) after \( x \) along \( b \) according to the orientation of \( b \). We put \( \beta = \pi_{y}b \), and so \( \beta \) does not intersect \( a \) on its interior. We fix a regular neighborhood \( nbd(\beta) \) of \( \beta \). We note that \( nbd(\beta) \) is homeomorphic to a disk, and so it is orientable. We have two cases where the signs in \( nbd(\beta) \) at the intersection points \( x \) and \( y \) between \( a \) and \( b \) are the same and different (see Figure 1). We define two curves \( c_{1} \) and \( c_{2} \) as shown in Figure 2.

![Figure 1](image1.png)

**Figure 1.** Two points \( x \) and \( y \) of \( a \cap b \) consecutive along \( b \) whose signs in \( nbd(\beta) \) are the same (left-hand side) and different (right-hand side).

![Figure 2](image2.png)

**Figure 2.** The way to make new curves \( c_{1} \) and \( c_{2} \) when the signs in \( nbd(\beta) \) at the intersection points \( x \) and \( y \) between \( a \) and \( b \) are the same (left) and different (right).

First, consider the case where the signs in \( nbd(\beta) \) at the intersection points \( x \) and \( y \) between \( a \) and \( b \) are the same. It follows that \( [a] = [c_{1}] + [c_{2}] \) in \( H_{1}(N, \partial N; \mathbb{Z}_{2}) \) (again, note that we are considering the \( \mathbb{Z}_{2} \)-coefficient relative homology group of \( N \)). Since \( a \) is nonseparating, we have \( [a] \neq 0 \), so at least one of \( [c_{1}] \) and \( [c_{2}] \) is not 0. Without loss of generality we may assume that \( [c_{1}] \neq 0 \), and we put \( c = c_{1} \). Accordingly, we see that \( i(a, c) = 1 \) if \( c \) is two-sided and \( i(a, c) = 0 \) if \( c \) is one-sided. Moreover, we have \( i(b, c) \leq i(a, b) - 2 \) in both cases where \( c \) is two-sided and where \( c \) is one-sided. By induction, we have

\[
d_{NC'}(a, b) \leq d_{NC'}(a, c) + d_{NC'}(c, b) \leq 2 + 2i(c, b) + 1 \leq 2 + 2(i(a, b) - 1) + 1 \leq 2i(a, b) + 1.
\]
Second, consider the case where the signs in \(\text{ubd}(\beta)\) at the intersections \(x\) and \(y\) between \(a\) and \(b\) are different. Here, it also follows that \([a] = [c_1] + [c_2]\). Similarly, we may assume that \([c_1] \neq 0\) and put \(c = c_1\). Then, we see that \(i(a, c) = 0\) if \(c\) is two-sided and \(i(a, c) = 1\) if \(c\) is one-sided. Moreover, we have \(i(b, c) \leq i(a, b) - 2\), particularly \(i(b, c) \leq i(a, b) - 1\) in both cases where \(c\) is two-sided and where \(c\) is one-sided. By induction, we have
\[
d_{\mathcal{NC}'}(a, b) \leq d_{\mathcal{NC}'}(a, c) + d_{\mathcal{NC}'}(c, b) \leq 2 + 2i(c, b) + 1 \leq 2i(a, b) + 1.
\]

□

**Corollary 3.2.** Let \(N\) be a finite type nonorientable surface whose genus is at least 3. Then, \(\mathcal{NC}(N)\) and \(\mathcal{NC}'(N)\) are quasi-isometric.

**Proof of Corollary 3.2.** We show the natural inclusion \(i : \mathcal{NC}'(N) \to \mathcal{NC}(N), i(a) = a\) is a quasi-isometry. Let \(a\) and \(b\) be any vertices in \(\mathcal{NC}'(N)\). Since \(\mathcal{NC}'(N) \subset \mathcal{NC}(N)\), it follows that \(d_{\mathcal{NC}}(i(a), i(b)) \leq d_{\mathcal{NC}'}(a, b)\). We consider the opposite direction of the inequality. We assume that \(d_{\mathcal{NC}}(i(a), i(b)) = n\). Then there exists a sequence \(a = c_0, c_1, \ldots, c_n = b\) such that \(i(c_i, c_{i+1}) \leq 2\) for any \(i = 0, 1, \ldots, n - 1\). By Lemma 3.1 we have
\[
d_{\mathcal{NC}'}(c_i, c_{i+1}) \leq 2i(c_i, c_{i+1}) + 1 \leq 5d_{\mathcal{NC}}(c_i, c_{i+1}).
\]

Thus we obtain
\[
d_{\mathcal{NC}'}(a, b) \leq 5d_{\mathcal{NC}}(i(a), i(b)).
\]

Therefore we see that \(i\) is a quasi-isometric embedding.

For any vertex \(a \in \mathcal{NC}(N)\), we choose \(a \in \mathcal{NC}'(N)\). Then we have
\[
d_{\mathcal{NC}}(a, i(a)) = d_{\mathcal{NC}'}(a, a) = 0.
\]

Therefore we see that \(i\) is quasi-dense. □

For vertices \(a\) and \(b\) in \(\mathcal{NC}(N)\), we define \(A(a, b)\) to be the full subgraph of \(\mathcal{NC}(N)\) spanned by the homotopy classes of the nonseparating bicorn between \(a\) and \(b\).

From Corollary 3.2 it is enough to prove the uniform hyperbolicity of \(\mathcal{NC}(N)\) for Theorem 1.1. Through the following steps, we will check that \(\mathcal{NC}(N)\) and the full subgraphs \(A(a, b)\) of \(\mathcal{NC}(N)\) satisfy the conditions of Proposition 2.4:

- find a uniform bound on the diameter of \(A(a, b)\) whose vertices \(a\) and \(b\) are connected by an edge (Lemma 3.3),
- prove the connectedness of \(A(a, b)\) (Lemma 3.4),
- show the slim triangle properties (Lemma 3.6).

First, we prove Lemma 3.3.

**Lemma 3.3.** (cf. [17] Proposition 3.2) Let \(a\) and \(b\) be any pair of vertices of \(\mathcal{NC}(N)\) that are connected by an edge. Then, \(A(a, b)\) has a diameter at most two.

**Proof of Lemma 3.3.** If \(i(a, b) \leq 1\), then \(A(a, b)\) consists of two vertices \(a\) and \(b\), and exactly one edge connects \(a\) and \(b\), so we are done.

If \(i(a, b) = 2\), we fix an orientation of \(b\). We show that any vertex \(c \neq b\) of \(A(a, b)\) satisfies \(i(a, c) \leq 1\). Take any bicorn \(c \neq b\) between \(a\) and \(b\), and put \(c = \alpha \cup \beta\), where \(\alpha\) is the \(a\)-arc and \(\beta\) is the \(b\)-arc of \(c\). According to the orientation of \(b\), we assume that \(\beta\) starts at \(x \in a \cap b\) and ends at \(y \in a \cap b\) (We note that now \(a\) and \(b\) satisfies \(i(a, b) = 2\), and so it follows that \(a \cap b = \{x, y\}\)).
First, consider the case where the signs in \( \text{nbd}(\beta) \) at \( x \) and \( y \) between \( a \) and \( b \) are the same. We see that \( i(a, c) = 1 \) if \( c \) is two-sided, and \( i(a, c) = 0 \) if \( c \) is one-sided (see Figure 3). Second, consider the case where the signs in \( \text{nbd}(\beta) \) at \( x \) and \( y \) between \( a \) and \( b \) are different. We see that \( i(a, c) = 0 \) if \( c \) is two-sided, and \( i(a, c) = 1 \) if \( c \) is one-sided (see Figure 4), and we are done.

**Lemma 3.4.** (cf. [17, Claim 3.4]) For any pair \( a \) and \( b \) of vertices in \( \mathcal{N}\mathcal{C}(N) \), the graph \( A(a, b) \) is connected.

**Proof of Lemma 3.4.** If \( i(a, b) \leq 1 \), then \( A(a, b) \) consists of exactly two vertices \( a \) and \( b \) and a single edge connects \( a \) and \( b \), and hence \( A(a, b) \) is connected. So, we assume that \( i(a, b) \geq 2 \).

We define a partial order on the vertices of \( A(a, b) \) as follows: \( c < c' \) if the \( b \)-arc of \( c' \) properly contains the \( b \)-arc of \( c \). It is enough to prove that given any \( c \in A(a, b) \), if \( c \neq b \), we can find \( c' \in A(a, b) \) so that \( c < c' \) and \( i(c, c') \leq 2 \).

We fix an orientation of \( b \). First, consider the case where \( c = a \). Choose an intersection point \( x \in a \cap b \). Let \( y \) be the first point of \( a \cap b \) after \( x \) along \( b \) according to the orientation of \( b \). Put \( \beta = xy_b \subset b \) so that the interior of \( \beta \) is disjoint from \( a \). The points \( x \) and \( y \) bound the two subarcs \( \alpha \) and \( \alpha' \) of \( a \) with disjoint interiors such that \( a = \alpha \cup \alpha' \). Accordingly, \( c'_1 = \alpha \cup \beta \) and \( c'_2 = \alpha' \cup \beta \) are bicorns between \( a \) and \( b \). For any \( i = 1, 2 \), we see that \( i(c'_i, c) \leq 1 \). In Fact, first consider the case where the signs in \( \text{nbd}(\beta) \) at the intersection points \( x \) and \( y \) between \( a \) and \( b \) are the same. Then, we see that \( i(c'_i, c) = 1 \) if \( c'_i \) is two-sided, and \( i(c'_i, c) = 0 \) if \( c'_i \) is one-sided. Second consider the case where the signs in \( \text{nbd}(\beta) \) at the intersection points \( x \) and \( y \) between \( a \) and \( b \) are different. we see that \( i(c'_1, c) = 0 \) if \( c'_1 \) is two-sided, and \( i(c'_2, c) = 1 \) if \( c'_2 \) is one-sided. Moreover, we have \( [a] = [c'_1] + [c'_2] \). Since \( a \) is nonseparating, at least one of \( c_1 \) and \( c_2 \) is nonseparating.
Without loss of generality we may assume that \([c_1'] \neq 0\) and put \(c' = c_1' \in A(a, b)\). Then, it follows that \(c < c'\), and \(c\) and \(c'\) are joined by an edge in \(A(a, b)\).

Second, consider the case where \(c \neq a\). We put \(c = \alpha \cup \beta\), where \(\alpha\) is a subarc of \(a\) and \(\beta\) is a subarc of \(b\), and we assume that \(\beta\) starts at \(x\) and ends at \(y\) according to the orientation of \(b\). In this case, we extend \(\beta\) past both of its endpoints until it intersects the interior of \(\alpha\) for the first time on each side and name the intersection points \(x'\) and \(y'\), respectively. We note that when we extends \(\beta\) past \(x\) along \(b\) we consider the opposite orientation of \(b\). In the case where the extended \(b\)-arc does not intersect with \(\alpha\) any more, we find that \(i(c, b) \leq 1\) and we are done. In the case where \(x' = y'\), we see that \(i(c, b) \leq 2\) and we are also done.

Therefore, we may assume that \(x' \neq y'\). Let \(c_1\) be a bicorn between \(a\) and \(b\) whose corners are \(x, y',\) and \(c_2\) a bicorn between \(a\) and \(b\) whose corners are \(x', y\). We claim that for any \(i = 1, 2\), we have \(i(c, c_i) \leq 1\). We prove this claim from now on. Without loss of generality we put \(c' = c_1 = \alpha' \cup \beta'\), where \(\alpha'\) is a subarc of \(a\) and \(\beta'\) is a subarc of \(b\), and we put \(z = y'\). We note that \(\alpha' \subset \alpha\) and \(\beta' \subset \beta\), hence \(c < c'\), and \(\beta'\) intersects with \(\alpha\) exactly once, that is at \(z\), in its interior. We will prove the claim by examining the cases below (8 cases).

- the signs in \(nbd(\alpha)\) of the intersection points \(y, z\) between \(a\) and \(b\) are the same or different,
- the signs in \(nbd(\alpha)\) of the intersection points \(x, y\) between \(a\) and \(b\) are the same or different, and
- \(c\) is two-sided or one-sided.

First we consider the case where the signs in \(nbd(\alpha)\) of the intersection points \(y, z\) between \(a\) and \(b\) are the same, and the signs in \(nbd(\alpha)\) of the intersection points \(x, y\) between \(a\) and \(b\) are the same. Then we see that \(i(c, c') = 1\) if \(c\) is two-sided and \(i(c, c') = 0\) if \(c\) is one-sided (Figure 5). We will proceed with this discussion in all cases. When the signs in \(nbd(\alpha)\) of the intersection points \(y, z\) between \(a\) and \(b\) are the same and the signs in \(nbd(\alpha)\) of the intersection points \(x, y\) between \(a\) and \(b\) are different, we see that \(i(c, c') = 1\) if \(c\) is two-sided, and \(i(c, c') = 0\) if \(c\) is one-sided. When the signs in \(nbd(\alpha)\) of the intersection points \(y, z\) between \(a\) and \(b\) are different and the signs in \(nbd(\alpha)\) of the intersection points \(x, y\) between \(a\) and \(b\) are the same, we see that \(i(c, c') = 0\) if \(c\) is two-sided, and \(i(c, c') = 1\) if \(c\) is one-sided. When the signs in \(nbd(\alpha)\) of the intersection points \(y, z\) between \(a\) and \(b\) are different and the signs in \(nbd(\alpha)\) of the intersection points \(x, y\) between \(a\) and \(b\) are different, we see that \(i(c, c') = 0\) if \(c\) is two-sided, and \(i(c, c') = 1\) if \(c\) is one-sided. Hence, we conclude that \(i(c, c') \leq 1\). (We note that these cases can be rephrased in this order if the signs in \(nbd(\alpha)\) of the intersection points \(x, y,\) and \(z\) are the same, if the signs in \(nbd(\alpha)\) of \(x\) is different from those of \(y\) and \(z\), if the sign in \(nbd(\alpha)\) of \(z\) is different from those of \(x\) and \(y\), and if the sign in \(nbd(\alpha)\) of \(y\) is different from those of \(x\) and \(z\), respectively.)

If \(c_i\) for some \(i\) is nonseparating, we may take \(c' = c_i\). Then \(c'\) is a nonseparating curve which is adjacent to \(c\) in \(A(a, b)\) and satisfies \(c < c'\).

Hence, we assume that both \(c_1\) and \(c_2\) are separating. In this case, there exists a bicorn \(e_2\) between \(a\) and \(b\) such that \([c_2] + [e_2] = [c]\) when all three are appropriately oriented. Figure 6 illustrates an example how to take \(e_2\). Since \(e_2\) is separating and \(c\) is nonseparating, the curve \(e_2\) is nonseparating. We define a bicorn \(c'\) between \(a\) and \(b\) with corners \(x'\) and \(y'\) such that \([c'] = [c_1] + [e_2]\). Then \([c'] \neq 0\) and \(c'\) satisfies
Figure 5. Intersections between $c$ and $c'$ when the signs in $\text{nbd}(\alpha)$ of the intersection points $y, z$ between $a$ and $b$ are the same and the signs in $\text{nbd}(\alpha)$ of the intersection points $x, y$ between $a$ and $b$ are the same, and $c$ is two-sided (left-hand side) and one-sided (right-hand side).

c < c'. We see $i(c, c') \leq 2$ by similar argument as before drawn in Figures 3, 4, and 5. □

Figure 6. A bicorn $e_2$ between $a$ and $b$.

Remark 3.5. We note a difference from the case of orientable surfaces ([17]) in the proof of Lemma 3.4. For the orientable surfaces, if the signs of the intersection points $y, z$ between $a$ and $b$ are the same, then we can conclude that the geometric intersection number of $c$ and $c'$ is exactly 1, that is, $i(c, c') = 1$. And this implies that $c'$ is a nonseparating curve. On the other hand for nonorientable surfaces, even the case where the signs of the intersection points $y, z$ between $a$ and $b$ are the same, we have both $i(c, c') = 0$ and $i(c, c') = 1$, and we cannot conclude that $c'$ is nonseparating. And so, we treat the both cases where the signs of the intersection points $y, z$ between $a$ and $b$ are the same and different at the same time in our proof.

Lemma 3.6. (cf. [17, Claim 3.5]) There exists a uniform constant $D > 0$ such that for any triple $a, b, d$ of vertices in $\mathcal{N}(N)$, the graph $A(a, b)$ is contained in the $D$-neighborhood of $A(a, d) \cup A(b, d)$.

Proof of Lemma 3.6. Let $c \in A(a, b)$ be any nonseparating bicorn between $a$ and $b$ and put $c = \alpha \cup \beta$, where $\alpha$ is the $a$-arc of $c$ and $\beta$ is the $b$-arc of $c$. Let $s$ and $t$ be the corners of the bicorn $c$ between $a$ and $b$. If $d$ intersects both $\alpha$ and $\beta$ at most once, we have $i(c, d) \leq 2$ and hence $c$ is at distance 1 from $d \in A(a, d)$. Otherwise, $d$ intersects at least one of $\alpha$ and $\beta$ at least twice. We assume that $d$ intersects $\beta$ at least twice in this proof. We fix an orientation of $d$, and let $x_1, x_2, \ldots, x_m$ ($m \geq 2$) be the intersection of $d$ and $\beta$ along $d$ in this order according to the orientation of
For any \( i = 1, \cdots, m \), let \( \delta_i = \overline{x_i x_{i+1}}_d \) be the unique subarc of \( d \) bounded by \( x_i \) and \( x_{i+1} \) whose orientation agrees with the orientation of \( d \) (indices are taken modulo \( m \)). Also, let \( \beta_i \) be the unique subarc of \( \beta \) such that \( c_i' = \beta_i \cup \delta_i \) is a bicorn between \( b \) and \( d \) (we orient \( \beta \) from \( x_{i+1} \) to \( x_i \)). If we give \( c_i' \) the orientation induced by \( \delta_i \), we have

\[
[d] = [c_1'] + [c_2'] + \cdots + [c_m'].
\]

Since \([d] \neq 0\), there is some \( u \) such that \([c_u'] \neq 0\). Put \( c' = c_u' \). We note that \( c' \in A(b, d) \). Let \( \beta' \) be the \( b \)-arc of \( c' \). If \( c' \) intersect with \( \alpha \) at most once, then it follows that \( i(c, c') \leq 2 \), and \( c \) is at distance 1 from \( c' \in A(b, d) \) in \( \mathcal{NC}(N) \).

If \( c' \) intersects with \( \alpha \) at least twice, we denote by \( y_1, y_2, \cdots, y_n \) (\( n \geq 2 \)) the points of \( c' \cap \alpha \) appearing in this order along \( c' \) (i.e., \( d \)) according to the orientation of \( d \) from one of the endpoints of \( \beta' \) (we refer to Figure 7 as a figure which describes this situation). For any \( i = 1, \cdots, n - 1 \), the points \( y_i \) and \( y_{i+1} \) bound a unique arc \( \alpha_i \) of \( \alpha \) and a unique arc \( \gamma_i \) of \( c' \) not containing the \( b \)-arc \( \beta' \). Then, \( c_i' = \alpha_i \cup \gamma_i \) is a bicorn between \( a \) and \( d \). It follows that \( i(c, c_i') \leq 1 \). Actually, the interior of \( \gamma_i \) does not intersect with \( c \), and hence, the intersection occurs only on \( \alpha_i \). Therefore, if there exists \( c_i'' \) (\( i = 1, 2, \cdots, n - 1 \)) which is nonseparating, then \( c'' = c_i'' \) is an element of \( A(a, d) \) and \( d_{\mathcal{NC}}(c, c'') = 1 \).

Thus let us assume that all \( c_i'' \) are separating. In this case, we claim that \( c \) is a uniformly bounded distance from \( c' \in A(b, d) \). To show this, we orient the regular neighborhood of \( \alpha \) (note that the regular neighborhood of \( \alpha \) is homeomorphic to a disk) and make the following two observations:

(I) In the oriented regular neighborhood of \( \alpha \), for each \( i = 1, \cdots, n - 1 \), \( d \)-arc \( \gamma_i \) joins the left-hand side (resp. right-hand side) of \( \alpha \) to the left-hand side (resp. right-hand side) of \( \alpha \) following the orientation of the regular neighborhood. Otherwise, \( c_i'' \) intersects with \( c \) exactly once, and it indicates that \( c_i'' \) is nonseparating.

(II) For each \( 1 \leq i < j \leq n - 1 \), if in the oriented regular neighborhood of \( \alpha \) the \( d \)-arcs of \( c_i'' \) and \( c_j'' \), namely, \( \gamma_i \) and \( \gamma_j \), join the left-hand side of \( \alpha \) to the left-hand side of \( \alpha \) following the orientation of the regular neighborhood of \( \alpha \), then the \( a \)-arcs of \( c_i'' \) and \( c_j'' \), namely, \( \alpha_i \) and \( \alpha_j \), are either nested or disjoint. In fact, we suppose that \( \gamma_i \) has the endpoints \( x, z \) and \( \gamma_j \) has the endpoints \( y, w \) with \( x < y < z < w \) in the orientation of \( \alpha \). Then the bicorns \( c_i'' \) and \( c_j'' \) have \( i(c_i'', c_j'') = 1 \). This contradicts that \( c_i'' \) and \( c_j'' \) are separating.

Now enumerate all of the \( d \)-arcs \( \gamma_j \) which join the left-hand side of \( \alpha \) to the left-hand side of \( \alpha \) in the oriented regular neighborhood of \( \alpha \) such that the corresponding \( a \)-arcs \( \alpha_j \) are maximal with respect to the inclusion: \( \gamma_1', \cdots, \gamma_r' \). Note that \( \gamma_j' \in \{ \gamma_{i_1}, \cdots, \gamma_{i_r} \} \) for \( i = 1, \cdots, r \). Let \( \alpha_1', \cdots, \alpha_r' \) be the corresponding \( a \)-arcs. For each \( i = 1, \cdots, r \), we put \( c_i''' = \alpha_i' \cup \gamma_i' \). Note that \( c_i''' \in \{ c_1'', c_2'', \cdots, c_{n-1}'' \} \) for \( i = 1, \cdots, r \), and \( c_i''' \) is a bicorn between \( a \) and \( d \). For each \( i = 1, \cdots, r \), we replace the subarc \( \alpha_i' \) of \( c \) with \( \gamma_i' \) and call the resulting curve \( c_0 \). We see that \( [c_0] = [c] + [c_1'''] + [c_2'''] + \cdots + [c_m'''] \) in \( H_1(N, \partial N; \mathbb{Z}_2) \). Since \( [c_i'''] = 0 \) for all \( i = 1, \cdots, r \), we have \([c_0] = [c] \). Thus, \( c_0 \) is nonseparating, that is, \( c_0 \in \mathcal{NC}(N) \).

Finally, we claim that \( i(c, c_0) \leq 1 \) and \( i(c_0, c') \leq 3 \). First, we show the first half. Recall that \( c = \alpha \cup \beta \). Note that \( \beta \) is properly contained in \( c_0 \). Then, by similar argument as before drawn in Figures 6, 2, and 5, we can show \( i(c, c_0) \leq 1 \) (The intersection occurs only on \( \beta \). Figure 7 is a possible intersection pattern of \( c \) and
c_0. (Note that in the case of orientable surfaces [17, Claim 3.5], the intersection number is exactly \( i(c, c_0) = 0 \).)

Now we show the second half. Recall that \( c_0 \) consists of \( \beta \) and several \( \alpha \)-arcs and \( d \)-arcs. In particular, \( \beta \subseteq c_0 \). Moreover, \( c' \) is a bicorn between \( b \) and \( d \) whose corners are \( x_u \) and \( x_{u+1} \). In particular, \( x_u, x_{u+1} \in \beta \). Then we see that the number of intersection points between \( c_0 \) and \( c' \) on \( \beta \) is at most one by similar argument as before drawn in Figures 3, 4, and 5. Any other intersection points of \( c_0 \) with \( c' \) occur between \( \alpha \)-arcs of \( c_0 \) and the \( d \)-arc of \( c' \). By the definition of \( c_0 \), we see that such intersection points occur only on \( y_1 \) and \( y_n \). This concludes that \( i(c_0, c') \leq 3 \).

Figure 8 is a possible intersection pattern of \( c_0 \) and \( c' \).

\[ \square \]

![Figure 7](image_url)

**Figure 7.** A curve \( c_0 \) which intersects with \( c \) once, where the antipodal points of a crosscap are identified.

![Figure 8](image_url)

**Figure 8.** Curves \( c_0 \) and \( c' \) intersecting three times.

**Remark 3.7.** We can take the uniform constant \( D = 10 \) in Lemma 3.6. Indeed, let \( c, c', \) and \( c_0 \) be the nonseparating curves in the proof of Lemma 3.6. At the last of the proof of Lemma 3.6 we showed that \( i(c, c_0) \leq 1 \) and \( i(c_0, c') \leq 3 \). By
Lemma 3.1 it follows that
\[ d_{\mathcal{NC}'}(c, c') \leq d_{\mathcal{NC}'}(c, c_0) + d_{\mathcal{NC}'}(c_0, c') \]
\[ \leq 2i(c, c_0) + 1 + 2i(c_0, c') + 1 \]
\[ \leq 2 + 1 + 6 + 1 \]
\[ \leq 10. \]

Thus, we see that it is enough to take 10 as \( D \) for the constant in Lemma 3.6.

4. THE CASES OF GENUS 1 AND 2

As mentioned in Remark 2.2, for nonorientable surfaces of genus 1 and 2, the nonseparating curve graphs \( \mathcal{NC}'(N) \) are finite or not connected. In the cases of genus 1 and 2, we modify the definition of \( \mathcal{NC}'(N) \) so that two vertices are joined by an edge if they are represented by curves which intersect at most once. Then we see that \( \mathcal{NC}'(N) \) are connected (Proposition 4.1). We also denote by \( \mathcal{C}'(F) \) the curve graph whose vertices are the homotopy classes of essential simple closed curves, and two vertices are joined by an edge if they are represented by curves which intersect at most once. In this section, we denote by \( \mathcal{C}_u(F) \) and \( \mathcal{NC}_u(F) \) the usual curve graph and nonseparating curve graph, respectively, that is, the vertices are homotopy classes of essential simple closed curves and nonseparating curves, respectively, and the edges are the pairs of vertices which can be realized disjointly. We also denote by \( d_{\mathcal{C}_u(F)}(\cdot, \cdot), d_{\mathcal{NC}_u(F)}(\cdot, \cdot), d_{\mathcal{C}_u(F)}(\cdot, \cdot), \) and \( d_{\mathcal{NC}_u(F)}(\cdot, \cdot) \) the distances of \( \mathcal{C}'(F), \mathcal{NC}'(F), \mathcal{C}_u(F), \) and \( \mathcal{NC}_u(F) \), respectively.

Firstly, we prove the connectedness of the nonseparating curve graphs \( \mathcal{NC}'(N) \) of genus 1 and 2:

Proposition 4.1. Let \( N = N_g^f \) be a nonorientable surface of \( g = 1 \) and \( f \geq 3 \) or \( g = 2 \) and \( f \geq 1 \). Then, \( \mathcal{NC}'(N) \) is connected.

Proof. At the first we consider nonorientable surfaces of genus \( g = 1 \). For \( f = 0, 1 \), the nonseparating curve graph \( \mathcal{NC}'(N) \) consists of exactly one isolated vertex. For \( f \geq 2 \), each nonseparating curve on \( N \) goes through the unique crosscap of \( N \), and so for any pair of vertices \( a, b \in \mathcal{NC}'(N) \) the geometric intersection number \( i(a, b) \) is larger than 0. We prove by the induction on \( i(a, b) \). When \( i(a, b) = 1 \), then \( a \) and \( b \) are connected by an edge of \( \mathcal{NC}'(N) \). When \( i(a, b) \geq 2 \), we choose an orientation of \( b \). We take an intersection point \( x \in a \cap b \). Let \( y \) be the first point of \( a \cap b \) after \( x \) along \( b \) according to the orientation of \( b \). Then, we can construct two curves \( c_1 \) and \( c_2 \) as shown in Figure 2 as the proof of Lemma 3.1. By the same argument as the proof of Lemma 3.1, we see that at least one of \( c_1 \) and \( c_2 \) is nonseparating, and denote it by \( c \). Then it follows that \( i(a, c) = 1 \) and \( i(b, c) \leq i(a, b) - 1 \), and we can connect \( a \) and \( c \) by an edge and connect \( c \) and \( b \) by a path in \( \mathcal{NC}'(N) \) by induction. Hence \( a \) and \( b \) are connected by a path in \( \mathcal{NC}'(N) \).

Second, we consider nonorientable surfaces of genus \( g = 2 \). For \( f = 0, 1 \), the nonseparating curve graph \( \mathcal{NC}'(N) \) is connected by [3] Section 2.4 and our definition of \( \mathcal{NC}'(N) \). For \( f \geq 2 \), we can prove the nonseparating curve graph \( \mathcal{NC}'(N) \) is connected by the same argument as the case of \( g = 1 \) and \( f \geq 2 \) above. \( \square \)

If \( g = 1 \) and \( f \leq 2 \) or \( g = 2 \) and \( f = 0 \), \( \mathcal{NC}'(N_g^f) \) is a finite graph, and hence it is Gromov hyperbolic. If \( g = 1 \) and \( f \geq 3 \) or \( g = 2 \) and \( f \geq 1 \), \( \mathcal{NC}'(N_g^f) \) has infinite vertices and we see the following:
Proposition 4.2. Let $N = N^f_g$ be a nonorientable surface of $g = 1$ or $g = 2$. Then, $\mathcal{NC}'(N)$ has infinite diameter.

Proof. Let $S$ be the orientation double cover of $N$. We note that the vertex sets of $C_u(S)$ and $C'(S)$ are the same. Firstly we show that any pseudo-Anosov element $\varphi$ of $\text{Mod}(S)$ acts on $C'(S)$ loxodromically. By Bowditch [6, Lemma 1.1] (also Schleimer [13, Lemma 1.21]), we see that for any vertices $a$ and $b$ of $C_u(S)$, $d_{C_u(S)}(a, b) \leq \log_2(i(a, b)) + 2$. Then since two vertices $a$ and $b$ of $C'(S)$ are connected by an edge if $i(a, b) \leq 1$, we see that for any vertices $a$ and $b$ of $C_u(S)$, $d_{C_u(S)}(a, b) \leq 2d_{C'(S)}(a, b)$. Let $\varphi$ be any pseudo-Anosov element of $\text{Mod}(S)$. By Masur and Minsky [14], $\varphi$ acts on $C_u(S)$ loxodromically. Hence, for any vertex $c$ of $C'(S)$, there exists a constant $E > 0$ such that for any $n \in \mathbb{Z}$, we have $E|n| \leq d_{C_u(S)}(c, \varphi^n(c)) \leq d_{C'(S)}(c, \varphi^n(c))$. Therefore any pseudo-Anosov element $\varphi$ of $\text{Mod}(S)$ acts on $C'(S)$ loxodromically.

From now on let $\varphi$ be any pseudo-Anosov element of $\text{Mod}(N)$ and $c$ a vertex of $C'(N)$. We fix any $n \in \mathbb{Z}$. By applying the same argument in the proof of Proposition 2.3 to $C'(N)$ and $C'(S)$, it follows that there exists a constant $E > 0$ such that for any $n \in \mathbb{Z}$, we have $E|n| \leq d_{C'(N)}(c, \varphi^n(c))$. We remark that if $\delta_i$ and $\delta_{i+1}$ are vertices of $C'(N)$ which are connected by an edge in $C'(N)$, then we have $i(\delta_i, \delta_{i+1}) \leq 1$. Let $\gamma^i$ and $\gamma^{i+1}$ be lifts of $\delta_i$ and $\delta_{i+1}$ to $S$, respectively. If $i(\delta_i, \delta_{i+1}) = 0$, then $i(\gamma^i, \gamma^{i+1}) = 0$, and if $i(\delta_i, \delta_{i+1}) = 1$, then $i(\gamma^i, \gamma^{i+1}) \leq 1$. Hence $\gamma^i$ and $\gamma^{i+1}$ are also connected by an edge in $C'(S)$, and we can use the same argument in the proof of Proposition 2.3. Since the nonseparating curve graph $\mathcal{NC}'(N)$ is a full subgraph of the curve graph $C'(N)$, for any $c \in \mathcal{NC}'(N)$, there exists $E > 0$ such that for any $n \in \mathbb{Z}$, $E|n| \leq d_{C'(N)}(c, \varphi^n(c)) \leq d_{\mathcal{NC}'(N)}(c, \varphi^n(c))$, and we see that the nonseparating curve graph $\mathcal{NC}'(N)$ has infinite diameter.

Moreover, we obtain a modification of Lemma 3.1 as below:

Lemma 4.3. Let $N = N^f_g$ be a nonorientable surface of $g = 1$ and $f \geq 3$ or $g = 2$ and $f \geq 1$. Let $a$ and $b$ be any pair of vertices of $\mathcal{NC}'(N)$. Then, we have $d_{\mathcal{NC}'(N)}(a, b) \leq 2i(a, b) + 1$.

Proof of Lemma 4.3. We prove the lemma by induction on $i(a, b)$. In the base case where $i(a, b) \leq 1$, we have $d_{\mathcal{NC}'(N)}(a, b) \leq 1$ by the definition. The rest of the proof goes through unmodified (see Proof of Lemma 3.1).

By Lemma 4.3 we can see $\mathcal{NC}'(N)$ and $\mathcal{NC}(N)$ are quasi-isometric, and so we see the hyperbolicity of the augmented nonseparating curve graph for $N = N^f_g$ of $g = 1$ and $f \geq 3$ or $g = 2$ and $f \geq 1$ by the same argument as we did in Section 3.

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HYPERBOLICITY OF NONSEPARATING CURVE GRAPHS 15

References

[1] T. Aougab, Uniform hyperbolicity of the graphs of curves, Geom. Topol. 17 (2013), no. 5, 2855–2875.
[2] J. Aramayona and F. Valdez, On the geometry of graphs associated to infinite-type surfaces, Math. Z. 289 (2018), no. 1–2, 309–322.
[3] F. Atalan and M. Korkmaz, Automorphisms of curve complexes on nonorientable surfaces, Groups Geom. Dyn. 8 (2014), no. 1, 39–68.
[4] M. Bestvina and K. Fujiwara, Quasi-homomorphisms on mapping class groups, Glas. Mat. Ser. III, 42 (62) (2007), no. 1, 213–236.
[5] J. Bowden, S. W. Hensel, and R. Webb, Quasi-morphisms on surface diffeomorphism groups, J. Amer. Math. Soc. 35 (2022), no.1, 211–231.
[6] B. H. Bowditch, Intersection numbers and the hyperbolicity of the curve complex, J. Reine Angew. Math. 598 (2006), 105–129.
[7] B. H. Bowditch, Uniform hyperbolicity of the curve graphs, Pacific J. Math. 269 (2014), no. 2, 269–280.
[8] M. T. Clay, K. Rafi, and S. Schleimer, Uniform hyperbolicity of the curve graph via surgery sequences, Algebr. Geom. Topol. 14 (2014), 3325–3344.
[9] B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, 49, Princeton University Press, Princeton, NJ, 2012.
[10] S. Hensel, P. Przytycki, and R. C. H. Webb, J-slim triangles and uniform hyperbolicity for arc graphs and curve graphs, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 4, 755–762.
[11] N. V. Ivanov, Complexes of curves and Teichmüller modular groups, Uspekhi Mat. Nauk 42 (1987), 49–91, English transl.: Russ. Math. Surv. 42, (1987) 55–107.
[12] M. Kimura and E. Kuno, Quasimorphisms on nonorientable surface diffeomorphism groups, arXiv:2111.05540 [math.GT].
[13] E. Kuno, Uniform hyperbolicity for curve graphs of non-orientable surfaces, Hiroshima Math. J. 46 (2016), no. 3, 343–355.
[14] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, Invent. Math. 138 (1999), no. 3, 103–149.
[15] H. A. Masur and S. Schleimer, The geometry of the disk complex, J. Amer. Math. Soc. 26 (2013), no. 1, 1–62.
[16] P. Przytycki and A. Sisto, A note on acylindrical hyperbolicity of mapping class groups, Hyperbolic geometry and geometric group theory, 255–264, Adv. Stud. Pure Math., 73, Math. Soc. Japan, Tokyo, 2017.
[17] A. Rasmussen, Uniform hyperbolicity of the graphs of nonseparating curves via bicorn curves, Proc. Amer. Math. Soc. 148 (2020), no. 6, 2345–2357.
[18] S. Schleimer, Notes on the complex of curves, available at https://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf.
[19] M. Stukow, Subgroups generated by two Dehn twists on a nonorientable surface, Topology Proc. 50 (2017), 151–201.
[20] B. Szepietowski, A presentation for the mapping class group of a non-orientable surface from the action on the complex of curves, Osaka J. Math. 45 (2008), no. 2, 283–326.

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