Abstract. In this paper, we mainly investigate the $W_{M,2} \otimes W_{L,2}$ system, in which the matter and the Liouville subsystems generate $W_{M,2}$ and $W_{L,2}$ algebras respectively. We first give a brief discussion of the physical states for corresponding $W$ strings. The lower states are given by freezing the spin-2 and spin-$s$ currents. Then, introducing two pairs of ghost-like fields, we give the realizations of $W_{1,2,s}$ algebras. Based on these linear realizations, BRST operators for $W_{2,s}$ algebras are obtained. Finally, we construct new BRST charges of Liouville system for $W_{2,s}$ strings at the specific values of central charges $c$: $c = -\frac{22}{5}$ for $W_{2,3}$ algebra, $c = -24$ for $W_{2,4}$ algebra and $c = -2, -\frac{286}{3}$ for $W_{2,6}$ algebra, at which the corresponding $W_{2,s}$ algebras are singular.

1 Introduction

To well understand the properties of a string theory, one needs to obtain its underlying worldsheet symmetry algebra. Since $W$ algebras [1,2] received considerable attention and application, much work [3,4,5,6,7,8,9,10] has been carried out on the classification of $W$ algebras and the study of $W$ gravities and $W$ strings. In fact, $W$ algebras also imply some underlying symmetry and appear in the quantum Hall effect [11] and black holes [12,13], in lattice models of statistical mechanics at criticality, and in other physical models [14,15] and so on.

In all applications of $W$ algebras, the investigation of $W$ strings is more interesting and important. The idea of building $W$ string theories was first developed in Ref. [16]. Since then, much research on the scalar realization of $W_{2,s}$ strings has been done [17-19,20-21,22-23,24]. In [25-26], the states for $W_{2,3}$ had been studied. The spectrum of $W$ strings also can be found in [27-28,29,30,31,32]. The spinor realizations of $W_{2,s}$ strings were given in [33-34,35-36]. It is known that, when extended to the quantum case, $W_{2,s}$ algebras will become non-linear. Fortunately, some of these algebras could be linearized by the inclusion of a spin-1 current. For the cases of $s = 3$ and 4, the linearizations were obtained in [37,38,39-40]. There exists no linear $W_{1,2,5}$ algebra at the quantum level. However $W_{2,6}$ algebra can be linearized as $W_{1,2,6}$ algebra when the central charge takes the specific value $c = 390$, and a realization for $W_{1,2,6}$ algebra was given in [37]. In this paper, by introducing a pair of bosonic ghost-like fields $(R,S)$ with spins $(6,-5)$ and a pair of fermionic ghost-like fields $(b_1,c_1)$ with spins $(k,1-k)$, we will investigate the realization and the BRST operator of $W_{2,6}$ algebra at the central charge $c = 390$.

Noncritical strings are strings in which the two-dimensional gravitational fields do not decouple after quantization but instead develop an induced kinetic term. The string coordinates

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are called ‘matter’ fields while the non-decoupled gravitational fields are represented by a set of so-called ‘Liouville’ fields. In [9,11], the BRST operators for the Liouville system were obtained. Especially in [37,42], the BRST operators of \( W_{2,s} \otimes W_{2,s'}^L \) were constructed at the classical level, and much valuable results were given. In general, classical \( W_{2,s} \) algebras are regular. But when extended to the quantum case, they will become singular at some values of the central charge, for example, \( c = -22 \frac{2}{3} \) for \( W_{2,3} \) algebra, \( c = -24, \frac{1}{2}, -22 \frac{2}{3} \) for \( W_{2,4} \) algebra, and \( c = -2, -22 \frac{2}{3} \) for \( W_{2,6} \) algebra. But one can rescale the spin-3, spin-4 and spin-6 currents such that the corresponding OPEs have no divergent coefficients. It was proved that at \( c = -22 \frac{2}{3} \) for \( W_{2,3} \), \( c = -24 \) for \( W_{2,4} \), and \( c = -2, -22 \frac{2}{3} \) for \( W_{2,6} \), these algebras satisfy the Jacobi identity [42]. This implies that we can construct the explicit BRST operators for \( W_{2,3}^L, W_{2,4}^L \) and \( W_{2,6}^L \) strings at \( c_L = -22 \frac{2}{3}, c_L = -24 \) and \( c_L = -2, -22 \frac{2}{3} \) respectively. In this paper, we will construct the BRST operators for \( W_{2,s}^L \) string by using grading method.

The present paper is organized as follows. In section 2, we give a brief discussion of the physical state for the \( W_{2,s}^M \otimes W_{2,s}^L \) system. Then in section 3 we build the BRST operators for \( W_{2,s}^M \) algebras by giving the realizations for the linear space of the operator \( Q^M \). The entire result is expressed as

\[
\text{where } T \text{ is the energy-momentum tensor for the matter, and } T^{gh} \text{ the ghost energy-momentum tensor with central charge } -26.
\]

2 Physical states for the \( W_{2,s}^M \otimes W_{2,s}^L \) system

The physical states in string theory can most elegantly be described by using the BRST formalism. For bosonic string, the BRST operator \( Q_B \) is given by

\[
Q_B = \oint dz c(z) \left( T(z) + \frac{1}{2} T^{gh}(z) \right). \tag{1}
\]

There also exists a similar expression for anti-holomorphic part. Here \( T(z) \) is the energy-momentum tensor for the matter, and \( T^{gh} \) the ghost energy-momentum tensor with central charge \(-26\).

However for \( W_{2,s}^M \otimes W_{2,s}^L \) algebra, which has two series systems, i.e., the matter and the Liouville systems, the BRST operator for this case is given by

\[
Q_B = Q_B^M + Q_B^L, \tag{2}
\]

\[
Q_B^M = \oint dz \left[ c^M(z) T^M(z) + \gamma^M(z) W^M(z) \right], \tag{3}
\]

\[
Q_B^L = \oint dz \left[ c^L(z) T^L(z) + \gamma^L(z) W^L(z) \right], \tag{4}
\]

where \( T^M \) and \( W^M \) are the currents of matter system for \( W_{2,s}^M \) algebras and have spin-2 and spin-6 respectively, while \( T^L \) and \( W^L \) are the currents of Liouville system for \( W_{2,s}^L \) algebras. Note that the ghost sector is included in \( T^M \) and \( W^M \), and also in \( T^L \) and \( W^L \). The fermionic ghosts \( (b,c) \) and \( (\beta, \gamma) \) are introduced for the currents \( T \) and \( W \). The up index \( 'M' \) denotes matter and \( 'L' \) Liouville. In fact, we ignore the anti-holomorphic sector and just focus on the holomorphic part.

Physical states are defined to be states in the cohomology of BRST operator \( Q_B \). The entire linear space \( H = H^M \oplus H^L \) of the operator \( Q_B = Q_B^M + Q_B^L \) can also be decomposed, with respect to a grading naturally associated with the underlying affine algebra, as \( H = H_+ \oplus H_- \) [13,14]. \( H_- \) and \( H_+ \) have negative and non-negative grading, respectively. The matter subspace \( H^M \) and Liouville subspace \( H^L \) can also be decomposed as \( H^M = H^M_- \oplus H^M_+ \) and \( H^L = H^L_- \oplus H^L_+ \) with \( H^M_+ \) and \( H^L_+ \) the matter and Liouville subspaces with non-negative gradings. Then the physical state condition is expressed as

\[
Q_B^M |\psi\rangle = Q_B^L |\psi\rangle = 0, \quad |\psi\rangle \in H^M_+ \otimes H^L_+, \tag{5}
\]
and
\[
|\psi\rangle \neq Q_B^M |\varphi\rangle + Q_B^L |\phi\rangle, \quad |\varphi\rangle, |\phi\rangle \in H_+^M \otimes H_+^L,
\]
(6)

In this system, physical states have the form
\[
|\psi\rangle = |\text{phys}\rangle_M \otimes |\downarrow\rangle_M + |\text{phys}\rangle_L \otimes |\downarrow\rangle_L,
\]
(7)

where $|\text{phys}\rangle_M$ involves operators built from the matter system. The matter system ghost vacuum $|\downarrow\rangle_M$ is built from ghost fields $(b^M, c^M)$ and $(\beta^M, \gamma^M)$.

For the latter, we just consider the Liouville system and ignore the matter one which is equivalent to the Liouville system. Here, we write down these currents in Laurent modes
\[
T^L(z) = \sum_{n=-\infty}^{+\infty} L_n^L z^{-n-2},
\]
(8)
\[
W^L(z) = \sum_{n=-\infty}^{+\infty} W_n^L z^{-n-s},
\]
(9)

where $s$ is the conformal spin of the current $W^L$. The modes $L_n^L$ generate the Virasoro algebra:
\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0},
\]
(10)

where $c$ is the central charge. For critical strings, by introducing the ghost fields $(b, c)$ for the spin-2 current, the central charge $c$ will vanish and (10) will reduce to the classical case.

The ghost fields $(b, c)$ and $(\beta, \gamma)$ in Laurent modes are
\[
b^L(z) = \sum_{n=-\infty}^{+\infty} b_n^L z^{-n-2}, \quad c^L(z) = \sum_{n=-\infty}^{+\infty} c_n^L z^{-n+1},
\]
(11)
\[
\beta^L(z) = \sum_{n=-\infty}^{+\infty} \beta_n^L z^{-n-s}, \quad \gamma^L(z) = \sum_{n=-\infty}^{+\infty} \gamma_n^L z^{-n-(1-s)}.
\]
(12)

It is easy to see that the conformal spin of $(b, c)$ and $(\beta, \gamma)$ are $(2, -1)$ and $(s, 1-s)$ respectively. The ghost vacuum $|\downarrow\rangle_L$ is given by
\[
|\downarrow\rangle_L = c_1^L c_2^L \cdots c_{s-1}^L |0\rangle,
\]
(13)

which is obtained from the $SL(2, C)$-invariant vacuum $|0\rangle$, which satisfies
\[
b_n^L |0\rangle = 0, \quad n \geq -1, \quad c_n^L |0\rangle = 0, \quad n \geq 2,
\]
(14)
\[
\beta_n^L |0\rangle = 0, \quad n \geq -s + 1, \quad \gamma_n^L |0\rangle = 0, \quad n \geq s.
\]
(15)

For the states of the form (7), the condition of BRST invariance becomes
\[
L_n^L |\text{phys}\rangle = \Delta |\text{phys}\rangle,
\]
(16)
\[
W_0^L |\text{phys}\rangle = \sigma_s |\text{phys}\rangle,
\]
(17)
\[
L_n^L |\text{phys}\rangle = W_n^L |\text{phys}\rangle = 0, \quad n \geq 1,
\]
(18)

where the constants $\Delta$ and $\sigma_s$ are the intercepts for the zero modes of the spin-2 and spin-$s$ currents respectively. For $W_{2,4}$ string $\Delta = 4$, and for bosonic string $\Delta = 1$. The intercepts $\sigma_s$ of the spin-$s$ currents are given in [29].
Using Eqs. (8)-(12), the BRST operator can be written in Laurent modes

\[ Q^L_B = \sum_{n=-\infty}^{+\infty} \left( c^L_n L^L_{-n} + \gamma^L_n W^L_{-n} \right). \]  

(19)

We denote the physical ground as \(| \downarrow, k \rangle\) which is characterized by level number \(\ell = 0\) and ghost number \(G = 0\). Then one will get excited states by the action of descendent operators like \(c^L_{-1}, b^L_{-1}, \beta^L_{-1}, \gamma^L_{-1}, L^L_{-n}\) and \(W^L_{-n}\) \((n > 0)\) on \(| \downarrow, k \rangle\). In fact, in order to get the physical states, the known of detail \(W^L_{2,s}\) algebra is necessary. But we can ‘freeze’ the spin-2 current \(T^L\) and spin-\(s\) current \(W^L\), which means no creation operators from them can appear in physical states. Then the physical states up to level \(\ell = 2\) are as follows:

\[ \begin{align*}
\ell &= 0 : \quad | \downarrow, k \rangle, \\
\ell &= 1 : \quad (p_1 c^{L}_{-1} + p_2 \gamma^{L}_{-1}) | \downarrow, k \rangle, \\
\ell &= 2 : \quad (p_3 c^{L}_{-2} + p_4 \gamma^{L}_{-2} + p_5 c^{L}_{-1} \gamma^{L}_{-1}) | \downarrow, k \rangle,
\end{align*} \]

(20)

where \(p_1 - p_5\) are constants.

Now, we would like to end this section with some comments. It is worth to note that the sum of down-index in each term in (19) is zero. This lead to a special property, i.e., the level of an arbitrary physical state does not change when acting it with \(Q_B\). The physical states we given above are based on the condition of ‘freezing’ the spin-2 and spin-\(s\) currents. The more detail of the physical states needs the explicit OPEs of \(W^L_{2,s}\) algebra.

### 3 BRST operators for \(W^L_{2,s}\) algebras from \(W^L_{1,2,s}\) algebras

After a brief investigation of the physical states, we would like to construct BRST operators for \(W^L_{2,s}\) algebras from the linear \(W^L_{1,2,s}\) algebras.

#### 3.1 BRST operators for \(W^L_{2,3}\) and \(W^L_{2,4}\) algebras

At the classical level, \(W^L_{2,s}\) algebras exist for all positive integer values of \(s\). These algebras are generated by the spin-2 energy-momentum tensor \(T\) and a primary spin-\(s\) current \(W\), which satisfy the OPEs

\[ \begin{align*}
T(z)T(\omega) &\sim \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \\
T(z)W(\omega) &\sim \frac{sW}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \\
W(z)W(\omega) &\sim \frac{2T^{s-1}}{(z-\omega)^2} + \frac{\partial T^{s-1}}{z-\omega}.
\end{align*} \]

(21)

The corresponding BRST operator is given by

\[ Q^L_B = \oint dz \left[ c(T - s\beta \partial \gamma - (s-1)b \partial c) + \gamma W - \partial \gamma b T^{s-2} \right], \]

(22)

where \((b, c)\) and \((\beta, \gamma)\) are the ghost and anti-ghost fields for the currents \(T\) and \(W\) respectively.

When extending these algebras to the quantum case, the BRST operator \(Q_B\) will not be the form of (22) since these algebras are not linear anymore. In fact the OPE of two currents with spins \(s\) and \(s'\) produces terms, at leading order, with spin \(s + s' - 2\). For example, there will be terms with spin 4 in the OPEs of \(W^L_{2,3}\) algebra. But these terms with spin \(s + s' - 2\)
can be interpreted as composite fields built from the products of the fundamental currents with spin $s$ and $s'$. The OPE $W(z)W(\omega)$ for $W_{2,3}$ algebra is given by

$$W(z)W(\omega) \sim \frac{c/3}{(z-w)^3} + \frac{2T}{(z-w)^4} + \frac{\partial T}{(z-w)^3}$$

$$+ \frac{1}{(z-w)^2} \left( 2\Theta A + \frac{3}{10} \partial^2 T \right) + \frac{1}{(z-w)} \left( \Theta \partial A + \frac{1}{15} \partial^3 T \right),$$

where

$$\Theta = \frac{16}{22 + 5c}, \quad A = T^2 - \frac{3}{10} \partial^2 T.$$  \hspace{1cm} (23)

For the case $W_{2,4}$, the OPE $W(z)W(\omega)$ takes the form

$$W(z)W(\omega) \sim \left\{ \begin{array}{l} \frac{2T}{(z-w)^6} + \frac{\partial T}{(z-w)^5} + \frac{3}{10} \frac{\partial^2 T}{(z-w)^4} \\ + \frac{1}{15} \frac{\partial^3 T}{(z-w)^3} + \frac{1}{84} \frac{\partial^4 T}{(z-w)^2} + \frac{1}{560} \frac{\partial^5 T}{(z-w)} \end{array} \right\}$$

$$+ \sigma_1 \left\{ \begin{array}{l} \frac{U}{(z-w)^4} + \frac{\partial U}{2(z-w)^3} + \frac{5}{36} \frac{\partial^2 U}{(z-w)^2} + \frac{1}{36} \frac{\partial^3 U}{(z-w)^3} \\ + \frac{W}{(z-w)^4} + \frac{1}{2} \frac{\partial W}{(z-w)^3} + \frac{5}{36} \frac{\partial^2 W}{(z-w)^2} + \frac{1}{36} \frac{\partial^3 W}{(z-w)^3} \end{array} \right\}$$

$$+ \sigma_2 \left\{ \begin{array}{l} \frac{G}{(z-w)^3} + \frac{\partial G}{2(z-w)^2} + \frac{A}{2(z-w)^2} + \frac{1}{2} \frac{\partial A}{(z-w)^3} \\ + \frac{B}{(z-w)^3} + \frac{1}{2} \frac{\partial B}{(z-w)^2} + \frac{c/4}{(z-w)^3} \end{array} \right\}.$$

where the composites $U$ (spin 4), and $G$, $A$ and $B$ (all spin 6), are defined by

$$U = (TT) - \frac{3}{10} \partial^2 T, \quad G = (\partial^2 TT) - \partial(\partial TT) + \frac{2}{9} \partial^2 (TT) - \frac{1}{42} \partial^4 T,$$  \hspace{1cm} (25)

$$A = (TU) - \frac{1}{6} \partial^2 U, \quad B = (TW) - \frac{1}{6} \partial^2 W,$$

with normal ordering of products of currents understood. The coefficients $\sigma_i (i = 1 - 5)$ are given by

$$\sigma_1 = \frac{42}{5c + 22}, \quad \sigma_2 = \sqrt{\frac{54(c + 24)(c - 172c + 196)}{(5c + 22)(7c + 68)(2c - 1)}},$$

$$\sigma_3 = \frac{3(19c - 524)}{(10(7c + 68)(2c - 1))}, \quad \sigma_4 = \frac{24(72c + 13)}{(5c + 22)(7c + 68)(2c - 1)},$$

$$\sigma_5 = \frac{28}{3(5c + 24)} \sigma_2.$$  \hspace{1cm} (26)

It is worth to point out that, at the quantum level, $W_{2,3}$ algebra with central charge $c = -\frac{22}{5}$ and $W_{2,4}$ algebra with $c = -24, \frac{1}{2}, -\frac{22}{5}$ or $-\frac{68}{5}$, are singular.

By introducing a spin-1 current $J_0$, nonlinear $W_{2,3}$ and $W_{2,4}$ algebras can be linearized as $W_{1,2,3}$ and $W_{1,2,4}$ algebras, respectively. The linear $W_{1,2,s}$ algebras for $s = 3, 4$ take the form

$$T_0(z)T_0(\omega) \sim \frac{c/2}{(z-w)^4} + \frac{2T}{(z-w)^4} + \frac{\partial T}{z-w}, \quad T_0(z)W_0(\omega) \sim \frac{sW}{(z-w)^2} + \frac{\partial W}{z-w},$$

$$J_0(z)T_0(\omega) \sim \frac{c_1}{(z-w)^3} + \frac{J_0}{(z-w)^3} + \frac{\partial J_0}{z-w}, \quad J_0(z)J_0(\omega) \sim \frac{-1}{(z-w)^2},$$

$$J_0(z)W_0(\omega) \sim \frac{hW_0}{z-w}, \quad W_0(z)W_0(\omega) \sim 0.$$  \hspace{1cm} (27)
From these OPEs, it is clear that the current \( W_0 \) is a primary field, but \( J_0 \) not. The coefficients \( c, c_1 \) and \( h \) are given by

\[
c = 50 + 24t^2 + \frac{24}{t^2}, \quad c_1 = -\sqrt{6}(t + \frac{1}{t}), \quad h = \sqrt{\frac{3}{2}} t, \quad (s = 3)
\]

\[
c = 86 + 30t^2 + \frac{60}{t^2}, \quad c_1 = -3t - \frac{4}{t}, \quad h = t. \quad (s = 4)
\]

The bases \( T \) and \( W \) of \( W_{2,s} \) algebras are constructed by the linear bases of the \( W_{1,2,s} \) algebras in our previous paper [35]:

\[
T = T_0, \quad (30)
\]

\[
W = W_0 + \zeta_1 \partial^2 J_0 + \zeta_2 \partial J_0 J_0 + \zeta_3 J_0^3 + \zeta_4 \partial T_0 + \zeta_5 T_0 J_0, \quad (s = 3) \quad (31)
\]

\[
W = W_0 + \eta_1 \partial^3 J_0 + \eta_2 \partial^2 J_0 + \eta_3 (\partial J_0)^2 + \eta_4 \partial J_0 (J_0)^2 + \eta_5 (J_0)^4 + \eta_6 \partial^2 T_0 + \eta_7 (T_0)^2 + \eta_8 \partial T_0 J_0 + \eta_9 T_0 (J_0)^2, \quad (s = 4) \quad (32)
\]

where the realization of \( T_0, J_0 \) and \( W_0 \) as well as the coefficients \( \zeta_i \) and \( \eta_i \) can be found in [35]. Note that this linearization does not contain the case \( c = -\frac{22}{5} \) for \( W_{2,3} \), and the cases \( c = -24, -\frac{1}{2}, -\frac{22}{5} \) and \( -\frac{68}{7} \) for \( W_{2,4} \), for which the algebra are singular. After careful calculation, we find that there exists no linearization for these singular algebras. However, for these specific values of central charge, we can rescale the spin-3 and spin-4 currents such that the OPEs have no divergent coefficients. This will be discussed in detail in next section.

The BRST operator for a \( W_{2,s} \) algebra is given by

\[
Q_B = \oint dz [c(z)T(z) + \gamma(z)W(z)], \quad (33)
\]

Substituting (30)-(32) into (33), we will obtain the explicit forms of the BRST operators for \( W_{2,3} \) and \( W_{2,4} \) algebras.

### 3.2 BRST operator for \( W_{2,6} \) algebra

For the higher spin case of \( W_{2,5} \), there is no such linearization as \( W_{2,3} \) and \( W_{2,4} \). However, the spin-6 current \( W \) can be linearized as [37]

\[
W = W_0 - \frac{1}{8} J_0^6 - \frac{1}{2} T_0 J_0^4 - \frac{4921}{114718} T_0^3 - \frac{3}{8} T_0^2 J_0^2 + \frac{9}{8} T_0^2 \partial J_0 - \frac{15}{2} T_0 J_0 J_0^2 - \frac{21}{2} T_0 (\partial J_0)^2
\]

\[
- \frac{41}{4} T_0 \partial^2 J_0 J_0 + \frac{21}{4} T_0 \partial^3 J_0 + \frac{11}{2} \partial J_0 J_0^3 - \frac{315}{8} (\partial J_0)^2 J_0^2 + \frac{277}{8} (\partial J_0)^3 + \frac{7}{4} \partial T_0 J_0^3
\]

\[
+ \frac{3}{2} \partial T_0 T_0 J_0 - \frac{57}{4} \partial T_0 \partial J_0 J_0 - \frac{190257}{229436} (\partial T_0)^2 + \frac{43}{4} \partial T_0 \partial^2 J_0 - \frac{157}{12} \partial^2 J_0 J_0^3
\]

\[
+ \frac{409}{4} \partial^2 J_0 \partial J_0 J_0 - \frac{1763}{48} (\partial J_0)^2 + \frac{108753}{114718} (\partial^2 T_0) T_0 - \frac{45}{16} \partial^2 T_0 J_0^2 + \frac{135}{16} \partial^2 T_0 \partial J_0
\]

\[
+ \frac{273}{16} \partial^2 J_0 J_0^3 - \frac{787}{16} \partial J_0 \partial J_0 J_0 - \frac{5}{2} \partial^3 T_0 J_0 - \frac{197}{16} \partial^4 J_0 J_0 - \frac{440915}{458872} \partial^4 T_0 + \frac{383}{96} \partial^5 J_0.
\]

The currents \( T_0, J_0 \), and \( W_0 \) have spin 2, 1, and 6 respectively. They generate the linear \( W_{1,2,6} \) algebra which also takes the form of (28), but the coefficients are given by

\[
c = 390, \quad c_1 = 11, \quad h = -1, \quad s = 6.
\]

Note that this is different from the cases of \( W_{2,3} \) and \( W_{2,4} \), for these constants take specific values.

To obtain a new realization for the linear \( W_{1,2,6} \) algebra, we introduce a pair of bosonic ghost-like fields \((R, S)\) with spins \((6, -5)\) and a pair of fermionic ghost-like fields \((b_1, c_1)\) with
spins \((k, 1 - k)\) to construct the linear bases of it. The realization for the \(W_{1,2,6}\) algebra is given by

\[
T_0 = T_{\text{eff}} - T_g,
J_0 = \rho RS + \lambda b_1 c_1,
W_0 = R,
\]

where

\[
T_g = 6 R \partial S + 5 \partial RS + k b_1 \partial c_1 + (k - 1) \partial b_1 c_1,
\]

and \(T_g\) and \(T_{\text{eff}}\) have central charges \(c_g\) and \(c_{\text{eff}}\), respectively. By making use of the OPEs \(J_0(z)J_0(\omega)\) and \(J_0(z)W_0(\omega)\) in (35), we can solve the coefficients \(\rho\) and \(\lambda\). And the value of \(k\) is determined from the OPE relation of \(T_0\) and \(J_0\). Substituting this value into (37), we get the value of \(c_g\). Since the central charges \(c_g\) and \(c_{\text{eff}}\) satisfy the condition \(c_{\text{eff}} + c_g = 390\), the value of \(c_{\text{eff}}\) can be obtained. All the coefficients are listed as follows:

\[
k = 1, \quad \lambda = 0, \quad \rho = -1, \quad c_g = 360, \quad c_{\text{eff}} = 30.
\]

Substituting (38) into (34), one can obtain the spin-6 current \(W\), which together with the \(T = T_0\) generate the \(W_{2,6}\) algebra. Substituting \(T\) and \(W\) into (33), we will obtain the explicit form of the BRST operator \(Q_B\) for \(W_{2,6}\) algebra.

4 BRST operators of Liouville system for \(W_{2,s}\) strings

As shown in section 3 at \(c_L = -\frac{22}{5}\) for \(W_{2,3}\) algebra and \(c_L = -24, \frac{1}{25}, -\frac{22}{5}, -\frac{68}{7}\) for \(W_{2,4}\) algebra, these algebras will become singular. But one can rescale the spin-3 and spin-4 currents such that their OPEs have no divergent coefficients [42]. One can prove that at \(c_L = -\frac{22}{5}\) for \(W_{2,3}\) and \(c_L = -24\) for \(W_{2,4}\), these algebras satisfy the Jacobi identity. In this section, we will give the explicit BRST operators for the corresponding \(W_{2,s}\) strings.

Now, we turn our attention to the grading method. The BRST operator can be rewritten in the grading form

\[
Q_B = Q_0 + Q_1,
\]

\[
Q_0 = \oint dz c T(\phi, b, c, \beta, \gamma, T_L),
\]

\[
Q_1 = \oint dz \gamma W(\phi, \beta, \gamma, T_L, W_L),
\]

where we introduce the \((b, c)\) ghost system for the spin-2 current \(T_L\), and the \((\beta, \gamma)\) ghost system for the spin-\(s\) current \(W_L\). The ghost fields \(b, c, \beta, \gamma\) are all fermionic and anticommuting. They satisfy the OPEs

\[
b(z)c(\omega) \sim \frac{1}{z - \omega}, \quad \beta(z)\gamma(\omega) \sim \frac{1}{z - \omega}.
\]

In other cases the OPEs vanish. The nilpotency conditions are

\[
Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0.
\]

One could see that, with this grading method, the construction of BRST operators will become easy. However, it imposes some restrained conditions on the BRST operators.

Next, we would like to construct the explicit BRST operators of Liouville system for \(W_{2,s}\) strings for \(s = 3, 4, 6\) by using the grading method. For simplify, we ignore the index ‘\(L\)’ in \(b^L, c^L, \beta^L\) and \(\gamma^L\) which denotes the Liouville system.
4.1 BRST operator for $W_{2,3}^L$ string

At $c_L = -\frac{22}{5}$, the quantum $W_{2,3}^L$ algebra will become singular. But after rescaling the spin-3 current, the OPE of $W_L(z)W_L(\omega)$ reads [42]

$$W_L(z)W_L(\omega) \sim \frac{2(T_L^2 - \frac{1}{m}\partial^2 T_L)}{(z - \omega)^2} + \frac{\partial T_L^2 - \frac{1}{m}\partial^3 T_L}{z - \omega}. \quad (44)$$

It can be verified that $T_L$ and $W_L$ define a consistent algebra which satisfies the Jacobi identity. This implies that it can be used to construct the non-critical BRST operator for $W_{2,3}^M \otimes W_{2,3}^L$ at $c_L = -\frac{22}{5}$.

In Ref. [45], we obtained four solutions for the first grading BRST operator $Q_0$. Here we choose the following solution:

$$Q_0 = \oint dz \frac{1}{z} \left( T_{\text{eff}} + T_L + T_\phi + \frac{1}{2} T_{bc} + T_{\beta\gamma} \right), \quad (45)$$

where $T_{\text{eff}}$ is an effective energy-momentum tensor with central charge $c_{\text{eff}}$. The other energy-momentum tensors are given by

$$T_\phi = -\frac{1}{2} (\partial \phi)^2 - q \partial^2 \phi, \quad (46)$$
$$T_{bc} = -2b \partial c - \partial bc, \quad (47)$$
$$T_{\beta\gamma} = -3b \partial_\gamma - 2\partial_\beta \gamma, \quad (48)$$

where $q$ is the background charge of $T_\phi$. The first nilpotency condition $Q_0^2 = 0$ requires that the total central charge vanishes, i.e.,

$$-\frac{517}{60} + \frac{c_{\text{eff}}}{12} + q^2 = 0. \quad (49)$$

Next, we will construct the explicit form of $Q_1$. The most extensive combinations can be constructed as follows:

$$Q_1 = \oint dz \frac{1}{z} \left( f_1 W_L + f_2 \partial T_L + f_3 T_L \partial \phi + f_4 \partial \beta \partial \gamma \right. \right.$$  

$$\left. + f_5 \partial^3 \phi + f_6 \partial^2 \phi \partial \phi + f_7 (\partial \phi)^3 + f_8 \partial \phi \partial \beta \gamma \right). \quad (50)$$

Then considering the last two conditions in [43], we obtain two solutions:

- Solution 1

  $$f_1 = f_2 = f_3 = 0, \quad f_4 = 147m_1, \quad f_5 = 38m_1 q, \quad f_6 = 294m_1, \quad f_7 = 16m_1 q, \quad f_8 = 72m_1 q, \quad (51)$$

  where $m_1$ is a non-zero constant. One may note that in this solution, $Q_1$ has no terms containing $T_L$ or $W_L$. If choose $m_1 = \frac{1}{16q^4}$, we will get the analogous results as Ref. [31] for $W_{2,3}$ string. The main difference is that the ghost fields $(b, c)$ and $(\beta, \gamma)$ here are introduced for Liouville system.

- Solution 2

  $$f_1 = 24\sqrt{5}m_2, \quad f_2 = -96m_2, \quad f_3 = -30m_2 q, \quad f_4 = 192m_2, \quad f_5 = 53m_2 q, \quad f_6 = 384m_2, \quad f_7 = 20m_2 q, \quad f_8 = 90m_2 q, \quad (52)$$

  where $h^2 = 1$ and $m_2$ is a non-zero constant. Different from the Solution [31], all coefficients here are non-vanishing.

In Table I we give a list of background charges and central charges of various fields. It is clear that all the central charges of $T_{\text{eff}}$ and $T_\phi$ are fractional for both solutions, which is different from the matter system.
Table 1. Background charges and central charges of various fields for $W_{2,3}^L$ string.

| field          | background charge | central charge |
|----------------|-------------------|----------------|
| $T_L$          |                   | $-\frac{27}{5}$|
| $T_{e,f}(solution \: 1)$ |                   | $\frac{399}{10}$|
| $T_{e,f}(solution \: 2)$ |                   | $\frac{153}{5}$|
| $\phi(solution \: 1)$ | $\pm \frac{7}{27\sqrt{2}}$ | $\frac{149}{8}$|
| $\phi(solution \: 2)$ | $\pm 4\sqrt{\frac{7}{5}}$ | $\frac{389}{20}$|
| $(b, c)$       |                   | $-26$          |
| $(\beta, \gamma)$ |                   | $-74$          |

4.2 BRST operator for $W_{2,4}^L$ string

From the OPE $W_L(z)W_L(\omega)$ for $W_{2,4}^L$, one will find that there are four values of the central charge at which the $W_{2,4}^L$ algebra becomes singular, namely $c_L = -24, \frac{1}{2}, -\frac{14}{5}$ and $-\frac{24}{5}$. After rescaling the spin-4 current, it is shown that only at $c_L = -24$, the $W_{2,4}^L$ algebra is consistent and satisfies the Jacobi identity, this case was found in [42] and neglected in [46,47]. Then the OPE relation provides us with a way to construct the BRST operator for the Liouville system of $W_{2,4}^L$ string. Next, we will give an explicit BRST operator for $W_{2,4}^L$ string under the grading form [39].

First, $Q_0$ takes the form of (54), where $T_\phi$ and $T_{bc}$ are the same as $W_{2,3}^L$, while $T_{\beta\gamma}$ is given by

$$T_{\beta\gamma} = -4\partial\beta\gamma - 3\partial\beta\gamma. \quad (54)$$

Considering the first nilpotency condition $Q_0^2 = 0$, we obtain that the total central charge vanishes, i.e.,

$$- \frac{65}{4} + \frac{c_{e,f}}{12} + q^2 = 0. \quad (55)$$

This offers us a relation between the central charge $c_{e,f}$ of energy-momentum tensor $T_{e,f}$ and the background charge $q$ of scalar field $\phi$. Once obtain the value of $q$, the central charge $c_{e,f}$ can also be obtained through this equation.

Next, we give a mostly extensive combinations of BRST operator $Q_1$ for $W_{2,4}^L$ string:

$$Q_1 \equiv \oint dz\gamma (g_1 W_L + g_2 T_L^2 + g_3 \partial^2 T_L + g_4 \partial T_L \partial \phi + g_5 T_L \partial^2 \phi + g_6 T_L (\partial \phi)^2 + g_7 T_L \partial \beta\gamma + g_8 \partial^4 \phi + g_9 (\partial^2 \phi)^2 + g_{10} (\partial \phi)^4 + g_{11} \partial^3 \phi \partial \phi + g_{12} \partial^2 \phi (\partial \phi)^2 + f g_{13} (\partial \phi)^2 \beta \partial \gamma + g_{14} \partial^2 \phi \beta \partial \gamma + g_{15} \partial \phi \beta \partial^2 \gamma + g_{16} \partial^2 \beta \partial \gamma + g_{17} \beta^3 \gamma). \quad (56)$$

Then considering the last two conditions in (53), we also obtain two solutions:

- Solution 1

$$\begin{align*}
g_i &= 0 \quad (i = 1 - 7), \quad g_8 = -779 m_3, \quad g_9 = -7590 m_3 q, \quad g_{10} = -900 m_3 q, \\
g_{11} &= -7020 m_3 q, \quad g_{12} = -43320 m_3, \quad g_{13} = -7200 m_3 q, \quad g_{14} = 14820 m_3, \\
g_{15} &= 37620 m_3, \quad g_{16} = -5220 m_3 q, \quad g_{17} = 1560 m_3 q, \\
q^2 &= \frac{361}{80}, \quad c_{e,f} = \frac{253}{5},
\end{align*} \quad (57)$$

where $m_3$ is a non-zero constant.
– Solution 2

\[ g_i = 0 \ (i = 1 - 7) \]
\[ g_8 = -22356m_4, \quad g_9 = -44280m_4q, \quad g_{10} = -54004m_4q, \]
\[ g_{11} = -44640m_4q, \quad g_{12} = -262440m_4q, \]
\[ g_{13} = -4320m_4q, \quad g_{14} = 116640m_4q, \]
\[ g_{15} = 233280m_4q, \quad g_{16} = -28800m_4q, \quad g_{17} = 4320m_4q, \]
\[ q_i = \frac{231}{3}, \quad c_{eff} = \frac{246}{5} \]

where \( m_4 \) is a non-zero constant.

Both solutions have \( g_i = 0 \) for \( i = 1 - 7 \). This leads to the vanishing of \( T_L \) and \( W_L \) in \( Q_1 \). One can also see that these coefficients \( g_i \) here is more larger and complicated than the case of \( W_{2,5}^2 \) string. Choose the exact values for \( m_3 \) and \( m_4 \), i.e. \( m_3 = -\frac{1}{10830}q \) and \( m_4 = -\frac{1}{165610}q \), these two solutions is analogous the results in Ref. [21] for \( W_{2,4}^4 \) strings. The background charges and central charges for these fields can be found in Table 2. In [42], the ghost fields \((b, c)\) together with \((\beta, \gamma)\) were used to construct the BRST operator, and the result for this \( W_{2,4}^4 \otimes W_{2,4}^L \) at central charge \( c_L = -24 \) is given by

\[ Q = Q_0 + Q_1 - \oint dz \gamma \left( \frac{167}{22} W_L + \frac{27889}{484} W_L b \partial \gamma \right), \]  

where the first two terms are for the matter system.

| field         | background charge | central charge |
|---------------|-------------------|---------------|
| \( T_L \)     |                   | -24           |
| \( T_{eff} \) (solution 1) |                   | \( \frac{231}{3} \) |
| \( T_{eff} \) (solution 2) |                   | \( \frac{246}{3} \) |
| \( \phi \) (solution 1) | \( \pm \frac{19}{\sqrt{30}} \) | \( \frac{27}{10} \) |
| \( \phi \) (solution 2) | \( \pm \frac{9\sqrt{5}}{22} \) | \( \frac{367}{15} \) |
| \( b, c \)    |                   | -26           |
| \( \beta, \gamma \) |                   | -146          |

### 4.3 BRST operator for \( W_{2,6}^L \) string

It is worth to point that \( W_{2,5}^2 \) algebra does not exist at the quantum level for the value of central charge required by the criticality. But one can obtain the nilpotent quantum BRST operator by adding \( \hbar \)-dependent corrections for it. The explicit result was given in [42] and we will not discuss it in detail.

Next, we consider the case of \( W_{2,6}^L \). We expect that it has the same graded form. At \( c_L = -2 \) and \( c_L = -\frac{286}{3} \), the \( W_{2,6}^L \) algebra becomes degenerate. At \( c_L = -2 \), the OPE of the spin-6 current with itself is of the following form [42]:

\[ W_L(z)W_L(\omega) \sim \frac{2A}{(z-\omega)^2} + \frac{\partial A}{z-\omega}, \]  

where

\[ A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
31 & 32 & 33 & 34 & 35 & 36
\end{pmatrix} \]
where \( A = T^2_L W_L - \frac{5}{3} T_L \partial^2 W_L + \frac{9}{10} \partial T_L \partial W_L - \frac{12}{5} \partial^2 T_L W_L + \frac{1}{36} \partial^4 W_L \). At \( c_L = -\frac{286}{3} \), the OPE is given by

\[
W_L(z) W_L(\omega) \sim \frac{W_L}{(z - \omega)^6} + \frac{1}{2} \frac{\partial W_L}{(z - \omega)^4} + \frac{7}{52} \frac{\partial^2 W_L}{(z - \omega)^3} + \frac{1}{39} \frac{\partial^3 W_L}{(z - \omega)^2} + \frac{1}{260} \frac{\partial^4 W_L}{(z - \omega)} \\
+ \frac{1}{2080} \frac{\partial^5 W_L}{z - \omega} - \frac{9}{35} \frac{\partial^6 W_L}{(z - \omega)^2} - \frac{9}{70} \frac{\partial^7 W_L}{(z - \omega)^3} - \frac{81}{2380} \frac{\partial^8 W_L}{(z - \omega)^2} - \frac{3}{476} \frac{\partial^9 W_L}{z - \omega}.
\]  

(61)

where \( A_1 \) and \( A_2 \) have spins 8 and 10 respectively, they are given by

\[
A_1 = T_L W_L - \frac{3}{26} \partial^2 W_L, \quad (62)
\]

\[
A_2 = T^2_L W_L - \frac{35}{153} T_L \partial^2 W_L - \frac{2}{153} \partial T_L \partial W_L - \frac{29}{102} \partial^2 T_L W_L + \frac{7}{612} \partial^4 W_L.
\]  

(63)

It is clear that every term on the right hand side of the OPEs (60) and (61) has \( W_L \), so we can consistently set it to zero, and the BRST operators will continue to be nilpotent.

Now, using the OPEs relations (60) and (61), we would like to construct the BRST operator of Liouville system for \( W_{2,6} \) string at \( c_L = -2 \) and \( c_L = -\frac{286}{3} \). The BRST operator are given in the grading form

\[
Q_B = Q_0 + Q_1,
\]

(64)

\[
Q_0 = \int dz c(T_{eff} + T_L - 6 \beta \partial \gamma - 5 \partial \beta \gamma - b \partial c).
\]

(65)

\[
Q_1 = \int dz \gamma (h_1 W_L + h_2 T^3_L + h_3 \partial^4 T_L + h_4 \partial^2 T_L T_L + h_5 (\partial T_L)^2 + h_6 T_L \partial^2 \beta \partial \gamma \\
+ h_7 T_L \beta \partial^3 \gamma + h_8 T_L \beta \partial^2 \partial \gamma + h_9 T^2_L \beta \partial \gamma + h_{10} \partial^3 \beta \partial^2 \gamma + h_{11} \partial^2 \beta \partial^3 \gamma \\\n+ h_{12} \partial \beta \partial^4 \gamma + h_{13} \beta \partial^5 \gamma + h_{14} \beta \partial^2 \gamma \partial \gamma).
\]

(66)

Considering the nilpotency condition \( Q_0^2 = 0 \), we get

\[
c_L + c_{eff} = 388.
\]

(67)

This implies that \( c_{eff} = 390 \) and \( \frac{1450}{3} \) at \( c_L = -2 \) and \( -\frac{286}{3} \), respectively. Using the nilpotency conditions \( Q_1^2 = 0 \) and \( \left\{ Q_0, Q_1 \right\} = 0 \), we could determine the coefficients \( h_i \). For both cases of \( c_L = -2 \) and \( -\frac{286}{3} \), these coefficients are

\[
h_i = 0 \quad (i = 1 - 9, 13, 14), \quad h_{10} = m_5, \quad h_{11} = 2m_5, \quad h_{12} = m_5,
\]

(68)

where \( m_5 \) is a non-zero constant.

Then \( Q_1 \) is given by

\[
Q_1 = \int dz \gamma (m_5 \partial^3 \beta \partial^2 \gamma + 2m_5 \partial^2 \beta \partial^3 \gamma + m_5 \partial \beta \partial^4 \gamma).
\]

(69)

This result shows that \( W_L \) and \( T_L \) don’t appear in the final expression of \( Q_1 \). One of the main reasons may be that more strict conditions are required in the grading form.

## 5 Conclusion

In this paper, we mainly investigate the \( W_{2,6}^M \otimes W_{2,6}^L \) system in which these two sub-systems generate two different \( W_{2,s} \) algebras. We first give a brief discussion on the physical states of
$W_{2,s}$ strings. These physical states are characterized by level number $\ell$ and ghost number $G$. The ground state is denoted by $\ell = 0$, and $G = 0$. The higher states can be obtained by acting on the ground state $|\downarrow, k\rangle$ with descendents. Using the condition $Q_B|\psi\rangle = 0$ and the non-trivial condition $|\psi\rangle \neq Q_B|\varphi\rangle$, one can get the explicit form of higher level states. The physical states up to $\ell = 2$ are given in [20], where the spin-2 and spin-3 currents are ‘frozen’.

Then the realizations of non-linear quantum $W_{2,s}$ algebras from linear $W_{1,2,s}$ algebras are discussed. The BRST operators of $W_{2,s}$ algebras are also obtained through the realizations of the linear $W_{1,2,s}$ algebras. For the case of $s = 3$ and 4, the realizations of these linear algebras are got by introducing ghost-like fields. In fact, this is not the only way to construct the linear $W_{1,2,s}$ algebras. When $s = 5$, there exists no such linear algebra. Fortunately, the linear $W_{1,2,6}$ algebra exists, but the central charge takes an exact value, i.e., $c = 390$. By introducing a pair of bosonic ghost-like fields $(R, S)$ with spins $(6, -5)$ and a pair of fermionic ghost-like fields $(b_1, c_1)$ with spins $(k, 1 - k)$, we obtain a new realization of $W_{1,2,6}$ algebra. A new BRST operator for $W_{2,6}$ algebra can be given out by making an invert of the basis.

We also obtain the BRST operators for the Liouville system of $W_{2,s}^M \otimes W_{2,s}^L$. We find that the coefficients appeared in the $W_{1,2,3}$ algebra will become divergent and the algebras will be singular at $c_L = -\frac{24}{5}$. But one can rescale the spin-3 current such that its OPE with itself has no divergent coefficients. This provides us a way to construct the $W_{2,4}^L$ string at $c_L = -\frac{22}{5}$, and we obtained two solutions for this case. For the $W_{2,4}^L$ algebra, the singularity appears at $c_L = -24, \frac{1}{2}, -\frac{27}{5}$, and $-\frac{68}{5}$. After rescaling the spin-4 current, the algebra will satisfy the Jacobi identity only at $c_L = -24$, and the two explicit forms of BRST operators for $W_{2,4}^L$ string at $c_L = -24$ are given. At last, we construct the BRST operator for $W_{2,6}^L$ at $c_L = -2$ and $c_L = -\frac{286}{3}$. It is worth to point that in all these constructions, the BRST grading method is used.

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