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reliable conclusions. These contain no adjustable parameters, and the Monte Carlo (which requires up to $10^8$ Monte Carlo steps for each system) [13]. A diagonalization of this matrix produces essentially exact results.

A remark is in order regarding another method that has been used previously to treat interactions between composite fermions in a partially filled AL. In this method, the 2-body inter-CF interaction is determined [9, 15, 16] by considering two composite fermions in the relevant AL, and then making the assumption that the state of many composite fermions is only approximate. Importantly for our purposes, the CFD spectra accurately reproduce the exact spectra including the very fine differences between them, in spite of the large dimensions of the Fock spaces in the relevant sectors (shown in Fig. 1). These results show that: (i) the physics of the 3/8 state is indeed described in terms of composite fermions; and (ii) the CFD produces essentially exact results.

In Fig. 1 we compare the CFD spectra with those obtained from an exact diagonalization of the Coulomb interaction in the full lowest LL space for $N = 14$ at the Pf flux and $N = 12$ at APf flux. The following features can be noted. The exact spectrum contains a band of low energy states which has a complete one to one correspondence with the low energy band of non-interacting fermions at $Q^*$. The remarkable similarity between the low energy bands of the two spectra is thus nicely explained by the observation that, for non-interacting composite fermions these two would be related by an exact particle-hole symmetry for composite fermions in the second AL. The splitting of states in these bands, which arises due to inter-CF interactions, is not identical, however, indicating that the particle hole symmetry for composite fermions is only approximate. Importantly for our purposes, the CFD spectra accurately reproduce the exact spectra including the very fine differences between them, in spite of the large dimensions of the Fock spaces in the relevant sectors (shown in Fig. 1). These results show that: (i) the physics of the 3/8 state is indeed described in terms of composite fermions; and (ii) the CFD produces essentially exact results.

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CFD allows us to obtain the spectra for larger systems, shown in Fig. 2. An incompressible state manifests itself through an $L = 0$ (spatially uniform) ground state that is separated from others by a robust gap. The fact that all of the APf flux values produce incompressible states (but not all of the Pf flux values do) suggests that an incompressible state indeed occurs at 3/8 at the APf flux. The system sizes are still not large enough to be able to estimate the gap reliably, but we note that the gap to the lowest neutral excitation for the two largest systems is $\sim 0.002 \varepsilon_c / \varepsilon_L$, which we take as a measure of the energy scales associated with this state. This is a factor of 50 smaller than the ideal gap of the nearby 1/3 state, and a factor of 10 smaller than the gap at 5/2, thus indicating the fragile nature of the state.

For a further confirmation that the actual state is indeed described by the APf wave function, we construct the following trial wave functions, labeled 1 and 2, at the Pf and the APf flux values:

$$\Psi_{3/8}^{\text{trial}} = P_{\text{LLL}} \prod_{j<k} (u_j v_k - v_j u_k)^2 \Phi_{3/2}^{\text{Pf/APf}}$$
\[ \Psi_{3/8}^{\text{trial}} = \rho_{\text{LLL}} \prod_{j<k} (v_j v_k - v_j u_k)^2 \Phi_{3/2}^{\text{Coulomb}} \]

Here, \( \Phi_{3/2}^{\text{Pf/APf}} \) is the Pf or APf wave function at 3/2, which refers to the state in which the lowest LL is fully occupied and the electrons in the second LL form a Pf or an APf state. (We produce the Pf state in the lowest LL by diagonalizing the 3-body interaction Hamiltonian \( V_3 = \sum_{i<j<k} P_{ijk}^3 (3Q - 3) \), where \( P_{ijk}^3(L) \) projects the state of the three particles \((i,j,k)\) into the subspace of total orbital angular momentum \(L\); the APf state is obtained by its particle hole conjugation; we then elevate the Pf/APf to the second LL and fill the lowest LL fully to obtain \( \Phi_{3/2}^{\text{Pf/APf}} \).) The wave function \( \Phi_{3/2}^{\text{Coulomb}} \) is the exact Coulomb eigenstate at the relevant \( Q^* \) at \( \nu' = 3/2 \). Composite-fermionization of these wave functions gives two trial wave functions at 3/8. Tables I and II compare the energies of these trial wave functions with the CFD energies, and also give the overlaps of these

| \( N \) | \( O_1 \) | \( O_2 \) | \( E_{3/8}^{\text{trial}} \) | \( E_{3/8}^{\text{CFD}} \) |
|-----|-----|-----|-------------|-------------|
| 14  | 0.726(1) | 0.973(2) | -0.44153(8) | -0.44372(9) |
| 20* | 0.379(1) | 0.434(1) | -0.43418(2) | -0.43515(8) |
| 26  | 0.271(1) | 0.526(1) | -0.43021(9) | -0.43146(6) |

| \( N \) | \( O_1 \) | \( O_2 \) | \( E_{3/8}^{\text{trial}} \) | \( E_{3/8}^{\text{CFD}} \) |
|-----|-----|-----|-------------|-------------|
| 12  | 0.816(1) | 0.994(1) | -0.4303(2)  | -0.44076(6) |
| 18  | 0.587(2) | 0.622(2) | -0.43168(9) | -0.43225(7) |
| 24  | 0.503(1) | 0.781(1) | -0.42845(9) | -0.42948(8) |

TABLE II: Comparing the CFD state at “APf flux” \( 2Q = (8N - 9)/3 \) with two trial wave functions, \( \Psi_{3/8}^{\text{trial}} \) and \( \Psi_{3/8}^{\text{CFD}} \), obtained by composite fermionization of the APf and the exact Coulomb states at 3/2. Other symbols have the same meaning as in Table I.
non-Abelian braid statistics of the quasiparticles. The excess charge associated with an excitation can be seen to be \( e/16 \) by one of many methods. The non-Abelian braid statistics of the excitations will have similar signatures as those predicted for 5/2 \([19, 20]\). (iii) The above features do not distinguish between whether the state is Pf or APf. Proposals have been made on how the Pf and the APf states at 5/2 can be distinguished experimentally through their different edge structures \([6, 7, 21, 22]\), and these analyses carry over to the 3/8 state with appropriate modifications. The Pf and APf states at 3/2 have edge structures (disregarding the possibility of edge reconstruction) 3/2(Pf)-1-0 and 3/2(APf)-2-1-0, respectively, which translate, upon composite-fermionization, into 3/8(Pf)-1/3-0 and 3/8(APf)-3/5-1/3-0 at 3/8. An immediate consequence is that the Pf will necessarily contain counter-propagating edge modes, including an up-stream neutral Majorana mode, which can have experimental signatures, e.g., in noise measurements in an upstream voltage contact \([23]\). Observation of such modes would not constitute a proof of APf, because the Pf state can also have backward moving modes due to edge reconstruction. However, we expect that the physics of edge reconstruction at 3/8 should not be too different from that at the nearby fractions 1/3 or 2/5, so an observation of counter-propagating modes at 3/8 concurrent with an absence of such modes at 1/3 and 2/5 can be taken as a substantial evidence for APf state at 3/8.

The thermal Hall conductivity \( K_H = \frac{\partial J_Q}{\partial T} \), where \( J_Q \) is the thermal energy current and \( \partial T \) is the “Hall” temperature difference, can also in principle distinguish the Pf and the APf \([6]\). In units of \( (\pi^2 k_B^2 / 3ht^4)T \), each chiral boson edge mode contributes one unit and the Majorana fermion mode 1/2 unit \([24, 22]\), with the sign depending on the direction of propagation. The boundary 3/8(Pf)-1/3 supports a chiral boson and a Majorana mode; the boundary 3/8(APf)-2/5 also supports a chiral boson and a Majorana mode, but moving in the upstream direction. This produces thermal Hall conductivity of \( 1 + 1/2 + 1 = 5/2 \) for the Pf and \( -1 - 1/2 + 1 + 1 = 1/2 \) for the APf at 3/8. This result is believed to be robust against interactions, disorder or edge reconstruction. One may also consider various tunneling exponents, following Wen \([21, 22]\). The exponent describing the long distance decay of the propagator of the charge 1/16 non-Abelian quasiparticles can be shown \([20]\) to be \( q = 7/13 \) for the 3/8 Pf; this exponent appears in the prediction \([22]\), assuming absence of edge reconstruction, that the current from one edge of the sample to the opposite edge near a quantum point contact satisfies \( I \sim V^{2q-1} \) and the tunnel conductance has a temperature dependence \( \sigma \sim T^{2q-2} \). For the APf state, on the other hand, the presence of up-stream neutral modes renders the various exponents non-universal even for an unreconstructed edge.

There have been previous studies of 3/8 within a model that treats the effective interaction between composite fermions through a two-body term. Ref. \([16]\) found, by comparing several variational wave functions, that for a fully polarized state the stripe phase has lower energy than the Pfaffian. Using the same model interaction, Ref. \([27]\) studied the 3/8 state, but the model does not distinguish between Pf and APf and also does not produce incompressible states at all even \( N \). While we believe that our current treatment is more reliable for reasons mentioned above, we obviously cannot rule out that the system sizes considered here may not capture the true nature of the thermodynamic phase, and the ultimate resolution to this issue will likely come from experiments. An earlier study \([28]\) considered composite fermions in the spin reversed \( n = n_A \) AL, also using a 2-body interaction model for composite fermions, and pointed toward a partially spin polarized paired FQHE state (without distinguishing between Pf or APf); that physics is not relevant at very high magnetic fields where the electrons are fully spin polarized.

Before closing, we note that we have not included in our work the effect of finite thickness, LL mixing, and disorder. While these will surely make quantitative corrections, we do not see any reason why they should change the qualitative physics of the state.

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billion, and splits into sectors counting $>6$ million states for $L = 0$, $>18$ million states for $L = 1$, etc. For $N = 20$ at $2Q = 49$ (Pf) the $L_z = 0$ subspace has dimensions greater than 368 billion, with $>69$ million independent states in the $L = 0$ sector, $>208$ million states in the $L = 1$ sector, and so on. These and larger systems are clearly out of the reach of exact diagonalization.

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