Paranilpotency in uncountable groups

MARCO TROMBETTI

Abstract. The aim of this paper is to provide a contribution to the theory of uncountable groups and to that of paranilpotent groups. Extending the structural results in Franciosi and de Giovanni (Ricerche Mat 40:321–333, 1991) and de Giovanni et al. (Comm Algebra 49:3020–3033, 2021), we prove that locally soluble minimal non-paranilpotent groups, i.e. non-paranilpotent groups whose proper subgroups are paranilpotent, are soluble. It is also shown that the class of paranilpotent groups is countably recognizable and, as an application of these results, that a soluble uncountable group whose proper uncountable subgroups are paranilpotent is itself paranilpotent.

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1. Introduction. A group $G$ is said to have finite rank if there is a positive integer $r$ such that every finitely generated subgroup of $G$ can be generated by $r$ elements; if such an integer does not exist, the group $G$ is said to have infinite rank. As shown in many papers (see for instance [10], [11], [14], [13], [18], [6], [27]), the structure of (generalized soluble) groups of infinite rank is strongly influenced by that of its proper subgroups of infinite rank. In fact, it turns out that if one requires that all proper subgroups of infinite rank satisfy a given property (such as solubility or nilpotency), then the group itself has that property, or, at least, all its proper subgroups have that property; the latter possibility is beautifully exemplified by Heineken and Mohamed groups (see for instance [1]), which are metabelian primary groups of infinite rank with a trivial centre and all proper subgroups nilpotent. This type of problems is a special case of more general ones concerning the influence of “large” proper
substructures on the whole structure. Requiring that all subgroups of infinite rank of a group satisfy certain conditions has usually a strong influence on the structure of the group itself, and this is essentially due to the fact that the rank is defined mostly in group theoretical terms only. On the other hand, what seems to be in general a more natural concept of “largeness”, i.e. cardinality, is not directly (especially in the infinite case) connected with the pure group theoretical structure. In 2016, F. de Giovanni and the author [7] began the investigation of the influence of proper subgroups of large cardinality on the whole group. This investigation and subsequent ones (see [2,3,9,15,18,19], ...) show that, although in a mild form (compared with that of the rank), the influence is still there.

In [6], it is shown that if $G$ is a group of infinite rank with a nilpotent commutator subgroup and in which all the proper subgroups of infinite rank are paranilpotent, then all proper subgroups of $G$ are paranilpotent (see the next section for definitions); of course, due to the Heineken and Mohamed groups, one cannot say much more here. The analogous problem for uncountable groups has been for many years impossible to solve for me, and the reasons were essentially two: (1) a key ingredient in these kinds of proof is usually the countable recognizability of the class in question, and, for what I know, it has been up to now an open problem if paranilpotent groups constitute a countably recognizable group class or not (see for instance the survey [5]); (2) the structure of locally soluble groups whose proper subgroups are paranilpotent is described in [20], but there nothing is said about solubility of these groups (in fact, there is not much information about infinite locally soluble groups whose proper subgroups are soluble).

Recently, probably because of the research concerning similar problems in the linear case (see [4]), I have been able to solve the above two problems. Actually, I have first proved that locally graded groups whose proper subgroups are paranilpotent are soluble (see Theorem 2.8); then I moved to the proof of the countable character of paranilpotent groups (see Theorem 2.6); and finally I proved the “uncountable character” of these same groups (see Theorem 2.10).

Most of our notation is standard and can be found in [23].

2. Paranilpotent groups. A group $G$ is called paranilpotent if it admits a paranilpotent series, i.e. if there exists a finite series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_t = G$$

of normal subgroups of $G$ such that, for $i = 0, \ldots, t - 1$, the factor group $G_{i+1}/G_i$ is abelian and each of its subgroups is normal in $G/G_i$ (the smallest possible length of a paranilpotent series is called the paranilpotent series length, or the paraheight of $G$). These groups were usually referred to in literature as parasoluble groups (see [25], where these groups were introduced), but here, we prefer to follow [4] and speak of paranilpotent groups. Clearly, paranilpotent groups are locally supersoluble and, actually, hypercyclic, i.e. they admit an ascending normal series with cyclic factors; moreover, the commutator subgroup of a paranilpotent group is nilpotent. Groups with many paranilpotent proper subgroups have been studied in [4,20]. In particular, it is proved in [4] that linear groups whose
proper subgroups are paranilpotent are either Černikov or paranilpotent, while in [20] the following result, which generalizes P. Hall’s well known nilpotency criterion, is established.

**Lemma 2.1.** Let $N$ be a nilpotent normal subgroup of a group $G$. If $G/N'$ is paranilpotent, then $G$ is paranilpotent.

Another relevant result is the following one, a proof of which can be found in [6].

**Lemma 2.2.** Let $G$ be a group whose commutator subgroup $G'$ is nilpotent. If $X$ and $Y$ are paranilpotent normal subgroups of $G$, then the product $XY$ is also paranilpotent.

While studying paranilpotent groups, (the structure of) the group of all power automorphisms of abelian groups often comes in handy. In fact, if $G_{i+1}/G_i$ is a factor of a paranilpotent series of a paranilpotent group $G$, then $G/G_i$ is a subgroup of the power automorphism group of the abelian group $G_{i+1}/G_i$. For what concerns the well known structure of the group of all power automorphisms of an abelian group, we refer to [24], but we just mention that, using such a structure, one easily sees that every paranilpotent group which is a $p$-group for some prime $p$ is nilpotent-by-finite, i.e. contains a normal nilpotent subgroup of finite index.

Before we start proving our three main results, we need some preliminary steps. Let $G$ be a group and let

$$S : \{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

be a fixed series of $G$. We define the **power stability group of the pair** $(G, S)$ as the subgroup $\text{Pow}(G, S)$ of $\text{Aut}(G)$ made by all automorphisms $\alpha$ such that, for all $i = 0, \ldots, n - 1$,

- $G_i^\alpha = G_i$, and
- $\alpha$ acts as a power automorphism on $G_{i+1}/G_i$.

Clearly, if $n = 1$, then $\text{Pow}(G, S)$ is just the usual power automorphism group. The stability group of the pair $(G, S)$ is defined as the subgroup $\text{Stab}(G, S)$ of $\text{Aut}(G)$ made by all automorphisms stabilizing the series $S$; clearly, $\text{Stab}(G, S)$ is a subgroup of $\text{Pow}(G, S)$. It is well known that the only non-trivial power automorphism of a non-periodic abelian group is the inversion, and our next technical lemma generalizes this fact to power stability groups of torsion-free abelian groups.

**Lemma 2.3.** Let $G$ be a torsion-free abelian group and let

$$S : \{1\} = G_0 \leq G_1 \leq G_2 = G$$

be a series of $G$ such that $G_1$ is torsion-free and $G/G_1$ is periodic. Put $P = \text{Pow}(G, S)$. Then $C_P(G_1) = \{1\}$ and in particular $|P| \leq 2$.

**Proof.** The factor group $P/C_P(G_1)$ is cyclic of order at most 2 (since the only non-trivial power automorphism of a non-trivial torsion-free abelian group is the inversion), so we only need to show that $C_P(G_1) = \{1\}$. To this aim, let $x \in C_P(G_1)$. 

Let $a \in G$. Of course, there is a positive integer $m$ such that $c = a^m$ lies in $G_1$. Since $x$ and $x^{-1}$ act as power automorphisms on $G/G_1$, there are positive integers $n_1$ and $n_2$, and $b_1, b_2 \in G_1$ such that $a^{x^{-1}} = a^{n_1}b_1$ and $a^x = a^{n_2}b_2$. On the other hand, 

$$c = c^{x(-1)^i} = (a^m)^{x(-1)^i} = (a^{x(-1)^i})^m = (a^{n_1}b_1)^m = a^{n_1mb_1^m} = c^{n_1b_1^m}$$

for $i = 1, 2$, and so we have that $\langle a, b_1, b_2, c \rangle$ is cyclic, generated by a certain $d$, say, and certainly $\langle x \rangle$-invariant. But then $dx = d \pm 1$ and since a power of $d$ lies in $G_1$, it follows that $d^x = d$; in particular, we get $a^x = a$. This shows that $x$ is the trivial automorphism of $G$ and completes the proof of the lemma. □

**Corollary 2.4.** Let $G$ be a torsion-free abelian group and let 

$$S : \{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

be a series of $G$ such that if, for some $i$, the factor group $G_i/G_{i+1}$ is non-periodic, then $G/G_i$ is torsion-free. Then $\text{Pow}(G, S)/\text{Stab}(G, S)$ is a finite elementary abelian 2-group.

**Proof.** Put $P = \text{Pow}(G, S)$. Let $i$ be such that $G_{i+1}/G_i$ is torsion-free. If $j(i)$ is the largest integer such that $G_{j(i)}/G_{i+1}$ is periodic, it follows from a repeated application of Lemma 2.3 that $P/C_P(G_{j(i)}/G_i)$ has order at most 2 (here you need to notice that, if $j(i) \geq i + 1$, then $C_P(G_{j(i)}/G_i) = C_P(G_{i+1}/G_i)$ by Lemma 2.3). Consequently, since 

$$\bigcap_{i \in I} C_P(G_{j(i)}/G_i) = \text{Stab}(G, S), \quad I = \{i : 0 \leq i < n, G_{i+1}/G_i \text{is torsion-free}\},$$

the result is proved. □

An immediate consequence of the above corollary is the following one.

**Corollary 2.5.** Let $G$ be a paranilpotent group containing a normal torsion-free abelian subgroup $M$ such that $G/M$ is abelian. Then $G$ is nilpotent-by-finite.

**Proof.** First we construct a paranilpotent series 

$$S : \{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = M \leq G$$

of $G$ such that if $M_{i+1}/M_i$ is non-periodic, then $M/M_i$ is torsion-free. This can be easily achieved and we only describe the procedure. We start with a paranilpotent series $\mathcal{U}$ of $G$ of the following type 

$$\{1\} = M_{0,0} < M_{1,0} < \cdots < M_{i_0,0} = M < G.$$ 

Put $M_1 = M_{1,0}$ and let $T/M_1$ be the periodic part of $M/M_1$. Since $G/M_1$ is paranilpotent, there is a paranilpotent series $\mathcal{T}$ of $G/M_1$ that goes through $T/M_1$. We take the numerators of the series $\mathcal{T}$ as further members of our series-to-be $S$. Then we continue this procedure in $G/T$ using the series $\mathcal{U}T/T$. After finitely many steps, we add in $S$ the subgroup $M_{i_0,0} = M$ and the procedure stops.

Since $G/C_G(M)$ is a group of power automorphisms of a series $S$ of $M$, it follows from Corollary 2.4 that there exists a subgroup $L/C_G(M)$ of finite
index in $G/C_G(M)$ stabilizing the series $S$. But then, since $M \leq C_G(M)$, there is a positive integer $m$ such that $M \leq \zeta_m(L)$. Now, the result follows from the fact that $L/M$ is abelian.

Now, we are ready to prove the three main results of the paper. We start with the countable character of paranilpotent groups.

A group class $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group $G$ belong to $\mathfrak{X}$, then $G$ itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by Reinhold Baer and it represents a very relevant tool for studying uncountable groups through their “small” subgroups. It is clear that soluble and nilpotent groups both form countably recognizable group classes and, in recent years, many other relevant classes of groups were shown to be countably recognizable (see for instance [8, 12, 15–17, 19], and their bibliography, or the recent survey [5]). As one can see from [5], there are two open problems in this area, and these concern the class $SJ$ (see [23] for the definition) and the class of paranilpotent groups. Our first main result settles the latter question.

**Theorem 2.6.** Let $G$ be a group whose countable subgroups are paranilpotent. Then $G$ is paranilpotent.

**Proof.** We break the proof up into steps.

(1) $G'$ is nilpotent.

- Suppose $G'$ is not nilpotent. Then, since the class of nilpotent groups is countably recognizable, it follows that there is a countable subgroup $H$ of $G'$ which is not nilpotent. Of course, any element of $H$ is a product of finitely many commutators and so there is a countable subgroup $L$ of $G$ such that $L' \geq H$. But $L$ is paranilpotent, and so $L' \geq H$ is nilpotent, a contradiction.

(2) We can assume $G$ is metabelian and in particular $G'$ is abelian.

- Since $G'$ is nilpotent, Lemma 2.1 yields that it is enough to prove that $G/G''$ is paranilpotent, so we may assume $G$ is metabelian.

(3) There is a bound on the paraheights of the countable subgroups of $G$.

- Suppose the claim is false. Then, for each positive integer $n$, there is a countable subgroup $X_n$ of $G$ whose paraheight is $n$. The subgroup $X$ generated by all the $X_n$’s is countable, so it is paranilpotent of paraheight $m$, say. However, $X$ contains $X_{m+1}$, which has paraheight $m + 1$, a contradiction.

- Let $T$ be the periodic part of the abelian group $G'$ and put $W = G/G'$. Moreover, let $\ell$ be the largest paraheight of a countable subgroup of $G$.

(4) The natural semi-direct product $G_1 = W \ltimes T$ is paranilpotent.

- Let $\mathcal{E}$ be the set of all finitely generated subgroups of $G_1$. For each $E \in \mathcal{E}$, consider the set $\tau(E)$ of all $E$-invariant series

$$\{1\} = U_0 \leq U_1 \leq \cdots \leq U_\ell = E \cap T$$

such that $E$ acts as a group of power automorphisms on each factor; of course, $\tau(E)$ is finite since $G_1$ (actually $G$) is locally supersoluble (so in particular $E \cap T$ is finite). If $E_2 \geq E_1$ are in $\mathcal{E}$, then we can naturally associate to each element of $\tau(E_2)$ an element of $\tau(E_1)$ through intersection with $E_1$; denote by $\alpha_{E_2, E_1}$ such a mapping. The family of the sets $\tau(E)$, with $E \in \mathcal{E}$, and
the mappings $\alpha_{E_2,E_1}$, with $E_2 \geq E_2$ in $E$, constitute an inverse system. A well known theorem of Kuroš yields that the inverse limit of such an inverse system is non-empty. Let $(y_E)_{E \in E}$ be an element of this inverse limit; here, for each $E \in E$,

$$y_E : \{1\} = Y_{0,E} \subseteq Y_{1,E} \subseteq \cdots \subseteq Y_{\ell,E} = E \cap T$$

is an $E$-invariant series such that $E$ acts as a group of power automorphisms on each factor. Moreover, $Y_{i,E_1} = Y_{i,E_2} \cap E_1$ for each $E_2 \geq E_1$ in $E$, and $i = 0, \ldots, \ell$.

For each $i = 0, \ldots, \ell$, let

$$T_i = \langle Y_{i,E} : E \in E \rangle = \bigcup_{E \in E} Y_{i,E}.$$ 

Let $w \in W$ and $x \in T_1$. Then there is $E_x$ such that $x$ belongs to $Y_{1,E_x}$, and so even to $Y_{1,(E_x,w)}$. Therefore $\langle x \rangle$ is normalized by $w$. It follows that $G_1$ acts as a group of power automorphisms on $T_1$. Repeating this argument, we see that $G_1$ acts as a group of power automorphisms on each factor $T_{i+1}/T_i$ for $i = 0, \ldots, \ell$. Since $T_\ell = T$, we have that $G_1$ is parnilpotent. •

A direct consequence of (4) is the following one.

(5) We can assume $G'$ is torsion-free.

(6) Every countable subgroup $E$ of $G$ is nilpotent-by-finite.

• Since $E$ is parnilpotent, the claim is a consequence of Corollary 2.5. •

(7) We can assume that $G$ contains an abelian normal subgroup $M$ such that $G/M$ is finite and abelian.

• It follows from (6) and [8, Corollary 3.3] that $G$ is nilpotent-by-finite. Let $N$ be a normal nilpotent subgroup $N$ of finite index. Then, by Fitting’s theorem and (2), $M = NG'$ is nilpotent. By Lemma 2.1, we can assume $M' = \{1\}$. •

(8) We can assume $G/M$ is cyclic (of prime power order) and that $M$ is torsion-free.

• Since $G'$ is abelian by (2) and $G/M$ is finite and abelian, the first part of the claim is an obvious consequence of Lemma 2.2. The second part of the claim follows using the same argument employed in (4); alternatively, we can regard point (4) as a general lemma. •

Let $x$ be an element of $G$ such that $G = \langle x, G' \rangle$.

(9) $G$ is hypercyclic.

• In order to do this, it is enough to prove that every homomorphic image of $G$ contains a non-trivial normal cyclic subgroup. For the sake of clarity, we prove this only for $G = G/\{1\}$, but the argument works in any homomorphic image of $G$. Let $1 \neq y \in G'$ and consider the countable subgroup $K = \langle x, y \rangle$. By hypothesis, $K$ is parnilpotent, so it is hypercyclic and hence contains a non-trivial normal cyclic subgroup $C$ contained in its normal (non-trivial) subgroup $G' \cap K$. Of course, $C$ is normalized by $\langle x \rangle$ and by $G'$ (which is abelian). Thus $C$ is normal in $G$ and we have proved that $G$ is hypercyclic. •

Finally, an application of [26, Lemma 11.2] to $M$ and $G/M = \langle x \rangle M/M$ shows that $G$ is paranilpotent.
Now, we move to prove that locally soluble minimal non-paranilpotent groups are soluble-by-finite. Recall that, if $\mathcal{X}$ is a class of groups, then a group $G$ is said to be minimal non-$\mathcal{X}$ if it is not an $\mathcal{X}$-group but all its proper subgroups have the $\mathcal{X}$-property. Recall also that a group is locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index.

**Lemma 2.7.** Let $G$ be a paranilpotent group and assume that the largest periodic normal subgroup of $G$ is trivial. Then $G$ is nilpotent-by-finite.

**Proof.** We use induction on the paraheight $n$ of $G$. The result is obvious if $n = 1$, so assume $n > 1$ and that the result is true for groups satisfying the hypothesis and with a smaller paraheight. Let

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

be a paranilpotent series of $G$. Let $C = C_G(G_1)$, so $G/C$ is finite since $G_1$ is torsion-free. Let $T/G_1$ be the largest periodic normal subgroup of $C/G_1$. Since $G_1$ is contained in the centre of $T \leq C$, we have that $T'$ is locally finite (Schur’s theorem) and so trivial. Thus $T$ is abelian, so torsion-free and we apply Lemma 2.3 to deduce that $T$ lies in the centre of $C$. Now, the induction hypothesis applies to $C/T$, so there is a nilpotent subgroup $N/T$ of finite index in $C/T$. But then $N$ is a nilpotent subgroup of finite index of $G$ and the statement is proved. $\square$

**Theorem 2.8.** Let $G$ be a locally graded minimal non-paranilpotent group. Then $G$ is either finite or soluble.

**Proof.** Suppose $G$ is infinite and perfect. Since all proper subgroups of $G$ are locally supersoluble, it follows that $G$ itself is locally supersoluble (see [6, Lemma 3.1]). This shows in particular that $G$ is not finitely generated and that it coincides with the subgroup generated by all its proper normal subgroups.

Now, it is easy to construct by transfinite induction an ascending series

$$\{1\} = G_0 \leq G_1 \leq \cdots G_\alpha \leq G_{\alpha+1} \leq \cdots \quad (\alpha + 1 < \lambda)$$

made by proper (so paranilpotent) normal subgroups of $G$ such that $\bigcup_{\alpha<\lambda} G_\alpha = G$ (here $\lambda$ is an ordinal number). Of course, for any $\alpha < \lambda$, $G_\alpha'$ is nilpotent. But $G$ is perfect, so $G = \bigcup_{\alpha<\lambda} G_\alpha'$ and hence $G$ is locally nilpotent.

Let $T$ be the periodic part of $G$ and assume first $T < G$; clearly, it is enough to suppose $G$ is torsion-free in this case. Since torsion-free paranilpotent groups are nilpotent-by-finite (Lemma 2.7), it follows from [7, Lemma 4.3] that $G$ contains a normal nilpotent subgroup $N$ such that $G/N$ is Černikov. Then $G \neq G'$ and we have a contradiction.

It is therefore possible to assume that $G$ is periodic and, actually, that $G$ is a $p$-group for some prime $p$. Now, any paranilpotent $p$-group is again nilpotent-by-finite and hence again [7, Lemma 4.3] yields a contradiction. $\square$

Before proving our final main theorem, we need the following lemma whose proof is essentially the same as that of [7, Lemma 4.5], so we omit it.

**Lemma 2.9.** Let $\aleph$ be a cardinal number whose cofinality is strictly larger than $\omega$. Let $P$ be a group of type $p^\infty$ for some prime number $p$, and let $A$ be a
$P$-module whose additive group has prime exponent $q$ and cardinality $\aleph$. Then $A$ contains a proper $P$-submodule of cardinality $\aleph$.

**Theorem 2.10.** Let $\aleph$ be a cardinal number such that $\text{cf}(\aleph) > \omega$ and let $G$ be a locally graded group of cardinality $\aleph$ such that all proper subgroups of cardinality $\aleph$ of $G$ are paranilpotent. If every homomorphic image of $G$ of cardinality $\aleph$ is not simple, then $G$ is paranilpotent.

**Proof.** Suppose by contradiction that $G$ is not paranilpotent, so it admits a countable subgroup $X$ which is not paranilpotent by Theorem 2.6. Clearly, $X$ cannot be contained in any proper subgroup of $G$ of cardinality $\aleph$.

Assume first that $G$ has no proper normal subgroup of cardinality $\aleph$. Since $G$ has no simple homomorphic image of cardinality $\aleph$, it follows that $G$ is the join of its proper normal subgroups. Now, for each element $x$ of $X$, there is a proper normal subgroup $N_x$ (of cardinality strictly smaller than $\aleph$) such that $x$ lies in $N_x$. Since the cofinality of $\aleph$ is strictly bigger than $|X|$, we have that the subgroup

$$N = \langle N_x : x \in X \rangle$$

is a proper normal subgroup of $G$ containing $X$. Now $G/N$ has cardinality $\aleph$ and contains a proper subgroup $Y/N$ of cardinality $\aleph$ (see for instance [7, Corollary 2.6]). Thus $Y$ is paranilpotent and consequently also $X$ is such, a contradiction. Therefore $G$ admits a proper normal subgroup $M$ of cardinality $\aleph$.

Obviously, $M$ is soluble, so $G/M$ is locally graded by [21]. Since all proper subgroups of $G/M$ are paranilpotent, $G/M$ is either paranilpotent or minimal non-paranilpotent. In both cases, by Lemma 2.8, we have that $G/M$ (and so $G$) is soluble.

Now, if $|G/G'| = \aleph$, then [7, Lemma 2.4] shows that $G/G'$ contains a proper subgroup $L/G'$ of cardinality $\aleph$ such that $|G/L| = \aleph$. Since $LX$ is a proper subgroup of $G$ of cardinality $\aleph$ and contains $X$, we get a contradiction. Therefore $|G/G'| < \aleph$, so $|G'| = \aleph$ and in particular $G = G'X$. Let $K$ be the smallest term of the derived series of $G$ such that $|K| = \aleph$. Then $K \leq G'$, $G = KX$, and $|K'| < \aleph$. In order to derive a contradiction, we replace $G$ by $G/K'$, so we may assume $K$ is abelian. Now, $K \cap X$ is normal in both $K$ and $X$, so even in $G$ and has countable cardinality; we may therefore replace $G$ by $G/K \cap X$ and assume that $G = X \times K$.

Notice that, if $H$ is any proper $G$-invariant subgroup of $K$, then $|H| < \aleph$ (otherwise $XH$ is a proper subgroup of cardinality $\aleph$ containing $X$).

Let $T$ be the torsion part of $K$. If $|T| < \aleph$, then we may assume $T = \{1\}$ replacing $G$ by $G/T$. Assume $|T| = \aleph$, so $T = K$. If no Sylow $p$-subgroup of $T$ has cardinality $\aleph$, we can clearly find a set of primes $\pi$ such that the Hall $\pi$-subgroup $T_\pi$ of $T$ is a proper subgroup of $T$ of cardinality $\aleph$, thus obtaining a contradiction. Therefore there is a prime $p$ such that the Sylow $p$-subgroup $T_p$ of $T$ has cardinality $\aleph$. Since $T_p$ is an abelian $p$-group, it follows that also its socle $S$ has cardinality $\aleph$ (see for instance [7]). Thus $S = K$. In any case, we have that $T$ can be assumed to be of finite exponent.
Clearly, $C_X(K)$ is a normal subgroup of $G$ of countable cardinality. Replacing $G$ by $G/C_X(K)$, we can assume that $C_X(K) = \{1\}$; in particular, the action of $X$ on $K$ is faithful.

Let $R$ be any proper normal subgroup of $X$, so $W = RK$ is paranilpotent. Then it is easy to see that $K$ admits a $W$-invariant series

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_s = T \leq \cdots \leq K_t = K$$
on whose factors $W$ acts as a group of power automorphisms, and such that if $K_{i+1}/K_i$ is non-periodic for $i \geq s$, then $K/K_i$ is torsion-free. Put

$$C = C_R(K_1) \cap C_R(K_2/K_1) \cap \cdots \cap C_R(K_t/K_{t-1}).$$

Since the group of all power automorphisms of a periodic abelian group of finite exponent is finite, it follows from Corollary 2.4 that $R/C$ is finite. Now, $C$ is nilpotent since it stabilizes the finite series of the $K_i$’s, and $CK$ is nilpotent as well; in particular, the Fitting subgroup $F$ of $RK$ is nilpotent, has finite index in $RK$, and contains $K$. Now, $|F| = \aleph$, so $|F/F'| = \aleph$ as well. Since $R$ is countable, it follows that $F' \cap K$ is strictly contained in $K$; so $|F'| < \aleph$ (notice that $F$ is normal in $G$). Thus, $U(R) := (R \cap F)F'$ is a normal subgroup of $G$ of cardinality strictly smaller than $\aleph$, and $R/U(R)$ is finite.

If $X$ contains a proper normal subgroup $Y$ of finite index, then, replacing $G$ by $G/U(Y)$, we may assume $X$ is finite (in this step we may lose the assumption on the periodic part of $K$ and that on $C_X(K)$, but these can be assumed again using the same arguments as above). Now, [7, Lemma 4.2] yields that $K$ is periodic (and, as we saw above, necessarily an elementary abelian $q$-group for some prime $q$). Let $H$ be a proper subgroup of $K$ such that $|K:H|$ is finite. Then $H$ has finite index in $G$, so it contains a normal subgroup $H_G$ of finite index. Clearly, $XH_G$ is a proper subgroup of $G$ of cardinality $\aleph$ containing $X$, a contradiction. Thus, $X$ does not contain proper subgroups of finite index.

Since $X$ is soluble, $X/X'$ is divisible, so there is a subgroup $L/X'$ of $X/X'$ such that $X/L$ is a locally cyclic $p$-group for some prime $p$. As above, we can factor out $U(L)$ and assume that $X$ contains a finite normal subgroup $L$ such that $X/L$ is a locally cyclic $p$-group. However, $X$ does not contain proper subgroups of finite index, so $L \leq \zeta(X)$ and hence $X$ is abelian. It follows that $G'$ is abelian, so by Lemma 2.2 that $G$ cannot be the product of two proper subgroups containing $K$; in particular, $X$ is a locally cyclic $p$-group. Again by [7, Lemma 4.2], we may assume $K$ is an elementary abelian $q$-group for some prime $q$.

Assume first $p = q$. If $p$ is odd, then [20, Lemma 2.3] shows that all proper subgroups of $G$ of cardinality $\aleph$ are nilpotent groups and so [15, Corollary 2.5] shows that $G$ is nilpotent, but then $|G/G'| = \aleph$, a contradiction. If $p = 2$ and $Y$ is any paranilpotent subgroup of $G$, then one easily sees that there is some positive integer $k$ such that $Y \cap K \leq \zeta_k(Y)$, so $Y$ is nilpotent. Again [15, Corollary 2.5] shows that $G$ is nilpotent, and gives a contradiction.

Thus $p \neq q$. Now, an application of Lemma 2.9 gives the final contradiction.
Finally, we briefly describe how to remove the cofinality assumption in the above theorem if the generalized continuum hypothesis (GCH) holds. This assumption is used in three points in the previous proof. The first time in the second paragraph: this whole paragraph can be replaced by an application of [15, Lemma 4.1]. In the second-last paragraph there is [15, Corollary 2.5] (which uses the assumption), but it is shown in [15, Section 4] that, under GCH, [13, Corollary 2.5] works even without the assumption on the cofinality. Finally, in the last paragraph of the proof, we may employ [22, Proposition 26].

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