IDENTITIES OF CONFORMAL ALGEBRAS AND PSEUDOALGEBRAS

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Abstract. For a given conformal algebra $C$, we write down the correspondence between identities of the coefficient algebra $\text{Coeff } C$ and identities of $C$ itself as of pseudoalgebra. In particular, we write down the defining relations of Jordan, alternative and Mal’cev conformal algebras, and show that the analogue of Artin’s Theorem does not hold for alternative conformal algebras.

1. Conformal algebras

In this note, we present a proof of a technical statement which concerns the relation between identities of a conformal algebra $C$ and its coefficient algebra $\text{Coeff } C$. This relation was mentioned in [8], where some particular cases (associativity, commutativity, Jacobi identity) were considered. Although the approach of [8] is quite general, it is still technically difficult to write down the conformal identities corresponding to a given variety of ordinary algebras.

We propose another approach which uses the language of pseudoproduct [1], in order to obtain the correspondence between identities of $C$ and $\text{Coeff } C$ in a very explicit form. This approach was mentioned in [1], where the most important cases (associativity, commutativity, Jacobi identity) were considered. We prove the general statement for any homogeneous multilinear identity. As an application, we write down the identities of Jordan, alternative and Mal’cev conformal algebras and derive their elementary properties.

Definition 1.1 ([3]). Let $k$ be a field of zero characteristic, and let $k[D]$ be the polynomial algebra in one variable. A conformal algebra $C$ is a unital left $k[D]$-module endowed with a family of $k$-bilinear operations $(\cdot \circ_n \cdot)$ ($n$ ranges the set of non-negative integers) satisfying the following properties:

$$a \circ_n b = 0 \quad \text{for sufficiently large } n, \quad a, b \in C; \quad (1.1)$$

$$Da \circ_n b = -na \circ_{n-1} b, \quad a \circ_n Db = D(a \circ_n b) + na \circ_{n-1} b, \quad n \geq 0. \quad (1.2)$$

The conditions (1.1) and (1.2) are called locality and sesqui-linearity, respectively.

This definition is a formalization of the following structure (appeared in mathematical physics) [4][5]. Let $A$ be an algebra over $k$ (in general, $A$ is non-associative), and let $A[[z, z^{-1}]]$ be the space of formal distributions over $A$: $A[[z, z^{-1}]] = A \otimes k[[z, z^{-1}]]$. An ordered pair of distributions $\langle a(z), b(z) \rangle$ is said to be local, if

$$a(w)b(z)(w - z)^N = 0 \quad (1.3)$$

for some $N \geq 0$. If $\langle a(z), b(z) \rangle$ is a local pair, then the product $a(w)b(z) \in A[[z, z^{-1}, w, w^{-1}]]$ could be presented as a finite sum [3]

$$a(w)b(z) = \sum_{n \geq 0} c_n(z) \frac{1}{n!} \partial^n \delta(w - z), \quad (1.4)$$

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where $\delta(w - z) = \sum_{s \in \mathbb{Z}} w^s z^{-s-1}$ is the formal delta-function. In physics, the relation (1.4) is known as the operator product expansion (OPE) of conformal fields.

Denote $c_n(z)$ by $(a \circ_n b)(z)$. The explicit expression for $c_n$ could be easily derived:

$$
(a \circ_n b)(z) = \text{Res}_w a(w)b(z)(w - z)^n,
$$

where Res$_w$ means the coefficient at $w^{-1}$. The following statement is straightforward.

**Theorem 1.2** (see, e.g., [5, 8]). Let $C$ be a subspace of $A[[z, z^{-1}]]$ such that

- any $a(z), b(z) \in C$ form a local pair;
- for any $a(z), b(z) \in C$ and for any $n \geq 0$ we have $(a \circ_n b)(z) \in C$;
- for any $a(z) \in C$ its derivative $\frac{d}{dz}a(z)$ lies in $C$.

Then $C$ is a conformal algebra with respect to the operations (1.3) and $D = \frac{d}{dz}$.

**Remark 1.3** (Dong’s Lemma, [3]). If $A$ is an associative or Lie algebra, then any subset $B \subset A[[z, z^{-1}]]$ such that $B \times B$ consists of local pairs generates a conformal algebra.

The converse is also true: for any conformal algebra $C$ in the sense of Definition 1.1 one can build an algebra $A$ such that $C$ could be represented as a subspace of $A[[z, z^{-1}]]$ satisfying the conditions of Theorem 1.2. There exists a universal algebra of this kind (it is unique up to isomorphism) called the coefficient algebra $\text{Coeff } C$ [8].

Let us remind the construction of $\text{Coeff } C$ for a given conformal algebra $C$ (see [5] or [8] for details). By the definition, $C$ is a (unital) left module over $H = k[D]$. Consider the vector space $k[t, t^{-1}]$ as a right $H$-module with respect to the action $t^s D = -nt^{n-1}$. The underlying vector space of $\text{Coeff } C$ is $k[t, t^{-1}] \otimes_H C$. Denote $t^n \otimes_H a$ by $a(n), a \in C, n \in \mathbb{Z}$. Define a multiplication on this space via the formula

$$
a(n)b(m) = \sum_{s \geq 0} \binom{n}{s} (a \circ_s b)(n + m - s). \quad (1.6)
$$

This is a well-defined bilinear operation which makes $\text{Coeff } C$ to be an algebra.

**Theorem 1.4** ([8]). Let $C$ be a conformal algebra, and let $A = \text{Coeff } C$. Then

(i) $C$ lies in $A[[z, z^{-1}]]$ as a subspace satisfying the conditions of Theorem 1.2, the embedding is given by $a \mapsto \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$;

(ii) for any algebra $B$ and for any homomorphism of conformal algebras $\varphi : C \rightarrow C' \subset B[[z, z^{-1}]]$ there exists a homomorphism of algebras $\psi : A \rightarrow B$ such that $\varphi(a) = \sum_{n \in \mathbb{Z}} \psi(a(n))z^{-n-1}, a \in C$.

**Definition 1.5** ([8]). Let $\Omega$ be a variety of algebras. A conformal algebra $C$ is said to be an $\Omega$-conformal algebra if and only if $\text{Coeff } C$ belongs to $\Omega$.

It was shown in [8] how to convert an identity of a coefficient algebra $\text{Coeff } C$ into the corresponding series of identities of $C$. In this way, one should proceed with a routine computation, and it is difficult to predict the final result. The most common identities have the following form [5, 8]:

**Associativity**

$$
(a \circ_n b) \circ_m c = \sum_{t \geq 0} (-1)^t \binom{n}{t} a \circ_{n-t} (b \circ_{m+t} c); \quad (1.7)
$$
(Anti-)Commutativity

\[ a \circ_n b + \sum_{s \geq 0} \frac{(-1)^{n+s}}{s!} D^s (b \circ_{n+s} a) = 0; \quad (1.8) \]

Jacobi identity

\[ (a \circ_n b) \circ_m c = \sum_{s \geq 0} (-1)^s \left( \binom{n}{s} (a \circ_{n-s} (b \circ_m c) - b \circ_m (a \circ_{n-s} c)) \right) \quad (1.9) \]

2. PSEUDOALGEBRAS

Theorems 1.2 and 1.3 show that any conformal algebra could be considered as an algebraic structure on a formal distribution space. But there is a more formal approach to the theory of conformal algebras related with the notion of a pseudo-tensor category [2].

Consider \( H = k[D] \) as a Hopf algebra with respect to the usual coproduct \( \Delta(D) = 1 \otimes D + D \otimes 1 \), counit \( \varepsilon(D) = 0 \), and antipode \( S(D) = -D \). We will use the standard notation

\[ \Delta(f) = \sum_i \frac{D^i}{i!} \otimes f^{(i)} = f^{(1)} \otimes f^{(2)}, \]

\[ (\Delta \otimes \text{id}) \Delta(f) = (\text{id} \otimes \Delta) \Delta(f) = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}, \quad \text{etc}. \]

Let us denote by \( \Delta^{(n)} \), \( n \geq 1 \), the iterated coproduct: \( \Delta^{(1)} = \text{id}_H \), \( \Delta^{(n+1)} = (\text{id}_H \otimes \Delta^{(n)}) \Delta \).

The algebra \( H \) acts on its tensor power \( H^\otimes n \) as follows:

\[ (f_1 \otimes \cdots \otimes f_n) h = f_1 h_{(1)} \otimes \cdots \otimes f_n h_{(n)}, \quad f_i, h \in H. \]

For a given conformal algebra \( C \), define the operation

\[ * : C \otimes C \to (H \otimes H) \otimes_H C, \]

\[ a * b = \sum_{s \geq 0} \frac{(-D)^s}{s!} \otimes 1 \otimes_H (a \circ_s b), \quad a, b \in C, \quad (2.1) \]

called pseudoproduct (the axiom (1.1) implies this sum to be finite). It follows from (1.2) that the pseudoproduct is \( H \)-bilinear:

\[ f(D) a * g(D) b = (f(D) \otimes g(D) \otimes_H 1)(a * b). \quad (2.2) \]

**Definition 2.1** (1). Let \( H \) be a bialgebra. A left unital \( H \)-module \( P \) endowed with an \( H \)-bilinear map \( * : P \otimes P \to (H \otimes H) \otimes_H P \) is called an \( H \)-pseudoalgebra.

**Remark 2.2** (1). If \( H = k \), then \( H \)-pseudoalgebra is just an ordinary algebra over the field \( k \). If \( H = k[D] \), then we obtain the notion of a conformal algebra.

One of the main features of a pseudo-tensor category is the composition of multi-operations (2). Any \( H \)-pseudoalgebra is just an algebra in the pseudo-tensor category \( \mathcal{M}(H) \) (see, e.g., [1] for details) associated with \( H \). In this category, an arbitrary composition of \( H \)-bilinear maps could be described by the following structure.

**Definition 2.3.** Let \( P \) be an \( H \)-pseudoalgebra with a pseudoproduct \( * \). The expanded pseudoproduct is an \( H^{\otimes(n+m)} \)-linear map

\[ * : (H^{\otimes n} \otimes_H P) \otimes (H^{\otimes m} \otimes_H P) \to H^{\otimes(n+m)} \otimes_H P, \quad n, m \geq 1, \]

defined as

\[ (F \otimes_H a) * (G \otimes_H b) = (F \otimes G \otimes_H 1)(\Delta^{(n)} \otimes \Delta^{(m)} \otimes_H \text{id}_P)(a * b), \quad (2.3) \]

\( F \in H^{\otimes n}, \; G \in H^{\otimes m}, \; a, b \in P. \)
Now, fix $H = \mathbb{k}[D]$ (an $H$-pseudoalgebra is the same as conformal algebra). The series of identities \cite{1, 2, 3} could be expressed in terms of the expanded pseudoproduct $\otimes$ as follows \cite{4}:

**Associativity**

$$(a * b) * c = a * (b * c); \quad (2.4)$$

**(Anti-)Commutativity**

$$a * b \pm (\tau_{12} \otimes_H \text{id}_P)(b * a) = 0; \quad (2.5)$$

**Jacobi identity**

$$(a * b) * c = a * (b * c) - (\tau_{12} \otimes_H \text{id}_P)(b * (a * c)). \quad (2.6)$$

Here $\tau_{12}$ means the permutation of two tensor factors in $H \otimes H$ or $H \otimes H \otimes H$ (it is a well-defined $H$-module automorphism provided that $H$ is cocommutative).

It is easy to see that the expressions \cite{1, 2, 4} are similar in some sense to the ordinary associativity, (anti-)commutativity, and Jacobi identity, respectively. It is natural to suppose that the similarity holds for an arbitrary identity. In the next section, we prove the correspondence.

### 3. Identities of conformal algebras

Let us consider a conformal algebra $C$ and its coefficient algebra $\text{Coeff} C = \mathbb{k}[t, t^{-1}] \otimes_H C$ with the multiplication \cite{1}. In this section, we denote $x \otimes_H a \in \text{Coeff} C$ by $a(x)$, $x \in \mathbb{k}[t, t^{-1}]$, and $H = \mathbb{k}[D]$ as before. Note that the dual algebra $X = H^*$ is isomorphic to $\mathbb{k}[[t]]$, where $(t, D^n) = \delta_{n,1}$.

Consider the associative topological algebra $X^{(1)} = \mathbb{k}[t, t^{-1}] \otimes \mathbb{k}[t]$, where the basic neighborhoods of zero are of the form \{(1 $\otimes t^n)f \mid f \in X^{(1)}\}, n \geq 0$. Denote by $X^{(1)}$ the completion of $X^{(1)}$. The algebra $X^{(1)}$ consists of series like $\sum_{s \geq 0} f_s(t) \otimes t^s$, $f_s(t) \in \mathbb{k}[t, t^{-1}]$.

The standard coproduct $\Delta : \mathbb{k}[t] \rightarrow \mathbb{k}[t][\otimes^2]$, $\Delta(t) = t \otimes 1 + 1 \otimes t$, could be continued to the homomorphism $\Delta : \mathbb{k}[t, t^{-1}] \rightarrow X^{(1)}$ via

$$\Delta(t^n) = \sum_{s \geq 0} \binom{n}{s} t^{n-s} \otimes t^s, \quad n \in \mathbb{Z}. \quad (3.1)$$

Let us denote $\Delta(x) = x^{(1)} \otimes x^{(2)}$, as before.

By the same way, we can define the natural topology on $X^{(r)} = X^{(r-1)} \otimes \mathbb{k}[t]$, $r > 1$. Namely, let the basic neighborhoods of zero be of the form

\[ \{(1 \otimes t^{n_1} \otimes \cdots \otimes t^{n_r})f \mid f \in X^{(r)}, n_1 + \cdots + n_r = n\}, \quad n \geq 0. \]

By $X^{(r)}$ we denote the completion of $X^{(r)}$. Then the map $\text{id} \otimes \Delta : X^{(1)} \rightarrow X^{(2)}$ is a continuous homomorphism, as well as $\Delta \otimes \text{id} : X^{(1)} \rightarrow X^{(2)}$. Hence, $\Delta \otimes \text{id}$ and $\text{id} \otimes \Delta$ are defined on $X^{(1)}$; it is easy to check that they coincide on $\Delta(\mathbb{k}[t, t^{-1}])$. So the “expanded” homomorphism $\Delta$ is cocommutative. By $S$ we denote the standard antipode of $\mathbb{k}[t]$ given by $S(t) = -t$.

It is easy to see that \cite{1} is equivalent to

\[ a(x)(y) = (a \circ_{x^{(2)}} b)(x^{(1)}y), \quad x, y \in \mathbb{k}[t, t^{-1}], \quad (3.2) \]

so

\[ (a \circ x)(y) = a(x^{(2)})b(S(x^{(1)})y), \quad x \in \mathbb{k}[t], \quad y \in \mathbb{k}[t, t^{-1}], \quad (3.3) \]

for any $a, b \in C$. 


Let $f(a_1, \ldots, a_n)$ be a (non-associative) homogeneous multilinear polynomial with coefficients in $k$. Any polynomial of this kind could be written as

$$f(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} t_\sigma (a_{1\sigma}, \ldots, a_{n\sigma}), \quad (3.4)$$

where each of the terms $t_\sigma (b_1, \ldots, b_n)$ is a linear combination of non-associative words obtained from $b_1 \ldots b_n$ by some bracketings. For any (non-associative) monomial $m(b_1, \ldots, b_n)$ in $(3.4)$, replace the usual multiplication with the (expanded) pseudoproduct $\ast$. We obtain an expression $m^\ast (b_1, \ldots, b_n)$ which has sense in a pseudoalgebra. Denote by $f^\ast (a_1, \ldots, a_n)$ the result of this operation:

$$f^\ast (a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (\tau \otimes_H \id_C) t_\sigma^\ast (a_{1\sigma}, \ldots, a_{n\sigma}). \quad (3.5)$$

**Theorem 3.1.** Let $C$ be a conformal algebra such that $\Coeff C$ satisfies homogeneous multilinear identity $f = 0$. Then $C$ as a pseudoalgebra satisfies the (pseudo-) identity $f^\ast = 0$.

**Proof.** Let us denote by $\theta$ the following map $H^\otimes n \otimes_H C \to H^\otimes (n-1) \otimes C$:

$$\theta((h_1 \otimes \cdots \otimes h_{n-1} \otimes 1) \otimes_H c) = h_1 \otimes \cdots \otimes h_{n-1} \otimes c.$$  

For any $x_i \in k[t] \subset X = k[[t]]$, $i = 1, \ldots, n-1$, $n \geq 1$, define the linear operator

$$P^{x_1, \ldots, x_{n-1}} = (\langle S(x_1), \cdot \rangle \otimes \cdots \otimes \langle S(x_{n-1}), \cdot \rangle \otimes \id_C) \theta : H^\otimes n \otimes_H C \to C, \quad (3.6)$$

Therefore,

$$P : k[t]^{n-1} \to \Hom(H^\otimes n \otimes_H C, C)$$

provided by $(x_1, \ldots, x_{n-1}) \mapsto P^{x_1, \ldots, x_{n-1}}$ is a multilinear map which could be defined on $k[t]^{\otimes n-1}$. Since $P$ is continuous (with respect to the finite topology on $\Hom(H^\otimes n \otimes_H C, C)$), one may build a map

$$X^{(n-1)} \to \Hom(H^\otimes n \otimes_H C, \Coeff C)$$

which sends $(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \in X^{(n-1)}$ to

$$P^{x_1 \otimes \cdots \otimes x_n} : A \mapsto \langle P^{x_2 \otimes \cdots \otimes x_n}(A) \rangle (x_1), \quad A \in H^\otimes n \otimes_H C.$$

It is clear that an element $A \in H^\otimes n \otimes_H C$ is zero if and only if $P^{\bar{x}}(A) = 0$ for any $\bar{x} \in k[t]^{\otimes n-1}$. Also, an element $a \in C$ is equal to zero if and only if $a(y) = 0$ for any $y \in k[t, t^{-1}]$, see Theorem [1.4].

We will use the following notation in order to simplify computations. For $\bar{x} \in k[t]^{\otimes n-1}$ set $\Pi \bar{x}$ to be the product of its components: $\Pi (x_1 \otimes \cdots \otimes x_{n-1}) = x_1 \ldots x_{n-1} \in k[t]$, and denote by $\Delta : \bar{x} \mapsto \bar{x}_{(1)} \otimes \bar{x}_{(2)}$ the componentwise coproduct. For any $\bar{a} \in C^{\otimes n}$, $\bar{x} \in k[t, t^{-1}]^{\otimes n}$ set $\bar{a}(\bar{x}) \in (\Coeff C)^{\otimes n}$ in the obvious way.

**Lemma 3.2.** Let $A \in H^\otimes s \otimes_H C$, $B \in H^\otimes n-s \otimes_H C$, $\bar{x} \in k[t]^{\otimes s-1}$, $\bar{z} \in k[t]^{\otimes n-s-1}$, $y \in k[t]$. Then

$$P^{\bar{z} \otimes y} (A \ast B) = P^{y \Pi \bar{z}} (P^{\bar{z} \otimes y} (A) \ast P^{\bar{z} \otimes y} (B)). \quad (3.7)$$

**Proof.** We may assume $A = F \otimes 1 \otimes_h a$, $B = G \otimes 1 \otimes_h b$, and let $a \ast b = \sum \alpha_i h_i \otimes 1 \otimes_h c_i$.

Then

$$A \ast B = \sum \alpha_i F h_{i(1)} \otimes h_{i(2)} \otimes G \otimes 1 \otimes_h c_i,$$
in accordance with \( \Box \). The left-hand side of (3.7) could be expressed as
\[
\sum_i \langle S(\bar{x}), F h_i(1) \rangle \langle S(y), h_i(2) \rangle \langle S(\bar{z}), G \rangle c_i \\
= \sum_i \langle S(\bar{x}(2)), F \rangle \Pi S(\bar{x}(1)), h_i(1) \rangle \langle S(y), h_i(2) \rangle \langle S(\bar{z}), G \rangle c_i \\
= \sum_i \langle S(\bar{x}(2)), F \rangle \langle S(y \Pi \bar{x}(1)), h_i \rangle \langle S(\bar{z}), G \rangle c_i, \tag{3.8} \]
which is equal to the right-hand side of (3.7).

\[\Box\]

**Lemma 3.3.** Let \( (a_1, \ldots, a_n) \) be a non-associative monomial obtained from \( a_1 \ldots a_n \) by some bracketing.

(i) For any \( \bar{x} \in k[t] \otimes_{\sigma} n-1, y \in k[t, t^{-1}], \bar{a} \in C^{\otimes n-1} \), we have
\[
P^\bar{x}_y(t^*(\bar{a}, a_n)) = t(\bar{a}(\bar{x}(2)), a_n(S(y \Pi \bar{x}(1)))) \tag{3.9} \]
(ii) For any \( \bar{a} \in C^{\otimes n} \) and for any \( \bar{x} = x_1 \otimes \cdots \otimes x_n \in k[t, t^{-1}] \otimes n \) we have
\[
t(\bar{a}(\bar{x})) = P^\bar{x}(a_1(1) \otimes \cdots \otimes a_{n-1}(1) t^*(\bar{a}) \tag{3.10} \]

**Proof.** (i) If \( n = 2 \), then \( \Box \). If \( n > 2 \) then we may assume that there is a non-trivial decomposition
\[
t = t_1(\bar{a}, a_s) t_2(\bar{b}, b_m),
\]
where \( \bar{a} = (a_1, a_2, \ldots, a_{s-1}) \), \( \bar{b} = (b_1, b_2, \ldots, b_{m-1}) \). Let \( \bar{x} \in k[t] \otimes_{\sigma} s-1, x_s \in k[t], \bar{z} \in k[t] \otimes_{\sigma} m-1, y \in k[t, t^{-1}] \). Then by Lemma 3.3 and by (3.2) we have
\[
P^\bar{x} \times \bar{z} (t_1^*(\bar{a}, a_s) \ast t_2^*(\bar{b}, b_m)) = P^\bar{x}(\Pi \bar{x}(1)) \left( P^\bar{x}(a_s) t_1(\bar{a}, a_s) \ast P^\bar{z}(\Pi \bar{z}(1)) t_2(\bar{b}, b_m) \right) \\
= \left( P^\bar{x}(t_1(\bar{a}, a_s)) \left( P^\bar{z}(y S(x_s(1)) S(\Pi \bar{x}(1))) t_2(\bar{b}, b_m) \right) \right). \tag{3.11} \]
The inductive assumption allows to proceed as follows:
\[
P^\bar{x} \times \bar{z} (t_1^*(\bar{a}, a_s) \ast t_2^*(\bar{b}, b_m)) = t_1(\bar{a}(\bar{x}(4)), a_s(x_s(2) \Pi \bar{x}(2)) S(\Pi \bar{x}(3))) \\
\times t_2(\bar{b}(\bar{x}(2)), b_m(y S(x_s(1)) S(\Pi \bar{x}(1))) S(\bar{z}(1))) = t_1(\bar{a}(\bar{x}(2)), a_s(x_s(2))) \\
\times t_2(\bar{b}(\bar{x}(2)), b_m(y S(\Pi \bar{x}(1)) x_s(1) \Pi \bar{x}(1))) \tag{3.12} \]
so we obtain \( \Box \).

(ii) The proof is analogous to the one of (i).

For a given \( \bar{x} \in k[t] \otimes_{\sigma} n-1, y \in k[t, t^{-1}] \), and for any permutation \( \sigma \in S_n \), define \( \bar{\zeta}(\bar{x}, y, \sigma) \in k[t, t^{-1}] \otimes n \) as follows: \( \bar{\zeta} = \zeta_1 \otimes \cdots \otimes \zeta_n \), where \( \zeta_i = x_{i \sigma(2)} \) for \( i \sigma \neq n \), \( \zeta_n = y S(\Pi \bar{x}(1)) \), for \( s = n \sigma^{-1} \). The following statement generalizes Lemma 3.3

**Lemma 3.4.** For any \( \bar{x} \in k[t] \otimes_{\sigma} n-1, y \in k[t, t^{-1}] \), \( \bar{a} \in C^n \), and for any \( \sigma \in S_n \) we have
\[
P^\bar{x}_y((\sigma \otimes \text{id}_C) t^*(\bar{a}, \sigma)) = t(\bar{a}_{\sigma}(\bar{\zeta}(\bar{x}, y, \sigma))), \tag{3.13} \]
where \( \bar{a}_{\sigma}(\bar{\zeta}) = a_{\sigma}(\zeta_1) \otimes \cdots \otimes a_{n \sigma}(\zeta_n) \).

**Proof.** First, let us assume \( n \sigma = n \). Then it is straightforward to check that for any \( A \in H^{\otimes n} \otimes H C \) and for any \( \bar{x} = x_1 \otimes \cdots \otimes x_{n-1} \in k[t] \otimes_{\sigma} n-1 \) we have
\[
P^\bar{x}(B(1) \otimes \text{id}_C) A = P^\bar{x} B, \tag{3.14} \]
where \( \bar{x}_\sigma = x_{1 \sigma} \otimes \cdots \otimes x_{(n-1) \sigma} \).

Second, assume \( \sigma \) to be a transposition \( \sigma = (s n), s \in \{1, \ldots, n-1\} \). Then for any \( A \in H^{\otimes n} \otimes H C, \bar{x} \in k[t] \otimes_{\sigma} s-1, x_s \in k[t], \bar{z} \in k[t] \otimes_{\sigma} n-1, y \in k[t, t^{-1}] \), we have
\[
P^\bar{x} \times \bar{z} ((\sigma \otimes \text{id}_C) A) = P^\bar{x}(B(1) \otimes S(\Pi \bar{x}(1) x_s(1) \Pi \bar{x}(1) x_s(1)) B). \tag{3.15} \]
To obtain the last relation, one needs the equality 
\( \langle x, h(1) \rangle (yh(2)) = \langle xS(y(2)), h \rangle y(1) \),
which is straightforward to check for any \( h \in H, \ x \in \mathbb{k}[t], \ y \in \mathbb{k}[t, t^{-1}] \).

An arbitrary permutation \( \sigma \in S_n, \ n\sigma = s \), could be presented as \( \sigma = (sn)\sigma_1 \),
where \( n\sigma_1 = n \). It follows from (3.14), (3.15) that for \( n \neq s \) we have
\[
P_{y}^{\bar{x}^1 \otimes x_s \otimes \bar{x}^2}((\sigma \otimes_H \text{id}_C)A) = P_{y}^{\bar{x}^1 \otimes S(\Pi_{x_s^1}^{(1)} \Pi_{x_s^2}^{(2)} x_s) \otimes \bar{x}^2}((\sigma_1 \otimes_H \text{id}_C)A) = P_{y}^{\bar{\zeta}'(y(2)) A}, \tag{3.16}
\]
where \( \bar{x} = \bar{x}^1 \otimes x_s \otimes \bar{x}^2, \ \bar{x}^1 \in \mathbb{k}[t]^{\otimes s-1}, \ \bar{x}^2 \in \mathbb{k}[t]^{\otimes n-s-1}, \) and
\[
\zeta'(y) = (\bar{x}(2) \otimes S(\Pi_{x_1^1}^{(1)} \Pi_{x_1^2}^{(2)} x_1) y \otimes \bar{x}^2_1(2))\sigma_1.
\]

Now, compute the right-hand side of (3.13) via (3.10). For \( n\sigma = n \) or \( n\sigma = s \neq n \),
it coincides with the right-hand side of (3.14) or (3.16), respectively, if we substitute \( A = t^*(a_{1\sigma}, \ldots, a_{n\sigma}) \).

Let us complete the proof of Theorem 3.1. Lemmas 3.3 and 3.4 imply that if Coeff \( C \) satisfies the identity (3.1),
\[
P_y^{x_1 \otimes \cdots \otimes x_{n-1}}(\sigma \otimes_H \text{id}_C) t^*_\sigma(a_{1\sigma}, \ldots, a_{n\sigma}) = 0
\]
for any \( x_1, \ldots, x_{n-1} \in \mathbb{k}[t], \ y \in \mathbb{k}[t^{-1}, t] \). Hence, \( C \) satisfies (3.6) as a pseudoalgebra.

**Theorem 3.5.** Let \( C \) be a conformal algebra. If \( C \) as a pseudoalgebra satisfies a (pseudo-) identity \( f^* = 0 \) of the form (3.3), then Coeff \( C \) satisfies the corresponding identity \( f = 0 \).

**Proof.** It is sufficient to show that
\[
t(a_{1\sigma}(x_{1\sigma}), \ldots, a_{n\sigma}(x_{n\sigma})) = P^{x_{\bar{x}_1(1)} \otimes \cdots \otimes x_{\bar{x}_{n-1}(1)}}(\sigma \otimes_H \text{id}_C) t^*(a_{1\sigma}, \ldots, a_{n\sigma}) \tag{3.17}
\]
for any \( x_i \in \mathbb{k}[t, t^{-1}], \ \sigma \in S_n \).

Indeed, let us rewrite the left-hand side of (3.17) by (3.10): in the obvious notations, we obtain
\[
t(\bar{a}_\sigma(x_{\bar{x}_\sigma})) = P^{\bar{y}(2)}_{x_s \Pi_{\bar{y}(1)}} t^*(\bar{a}_\sigma), \tag{3.18}
\]
where \( \bar{x}_\sigma = (\bar{y} \otimes x_s) \), for \( s = n\sigma \).

The right-hand side of (3.17) could be rewritten by (3.14) or (3.16). It is easy to note that the expression obtained coincides with (3.18).

The final statement follows directly from (3.10) and (3.17).

We will need the identities analogous to (3.5) for the expanded pseudoproduct. Let \( n \geq 1 \), and let \( \pi = \{m_i \mid i = 1, \ldots, n\} \) be a family of positive integers. For a given \( \sigma \in S_n \), define \( \sigma \pi \in S_{m_1 + \cdots + m_n} \) in such a way that
\[
\sigma \pi(\Delta^{(m_1)} \otimes \cdots \otimes \Delta^{(m_n)})(F) = (\Delta^{(m_1)} \otimes \cdots \otimes \Delta^{(m_n)})\sigma(F) \tag{3.19}
\]
for any \( F \in H^{\otimes n} \).

**Proposition 3.6.** Let \( C \) be a conformal algebra satisfying a homogeneous multilinear identity
\[
\sum_{\sigma \in S_n}(\sigma \otimes_H \text{id}_C) t^*_\sigma(a_{1\sigma}, \ldots, a_{n\sigma}) = 0.
\]
Then for the expanded pseudoproduct \( \mathfrak{P} \), we have
\[
\sum_{\sigma \in \Sigma_n} (\sigma_H \otimes H \text{id}_C) t^*_\sigma(A_{1\sigma}, \ldots, A_{n\sigma}) = 0, \quad (3.20)
\]
\[A_i = G_i \otimes H a_i, \quad a_i \in C, \quad G_i \in H \otimes m_i, \quad m_i \geq 1, \quad i = 1, \ldots, n.
\]

**Proof.** It follows from the definition of expanded pseudoproduct \( \mathfrak{P} \) that for a (non-associative) homogeneous multilinear term \( t \) we have
\[t^*(A_1, \ldots, A_n) = (G_1 \otimes \cdots \otimes G_n \otimes H 1)(\Delta^{m_1} \otimes \cdots \otimes \Delta^{m_n} \otimes H \text{id}_C) t^*(a_1, \ldots, a_n).
\]
Hence,
\[t^*(A_{1\sigma}, \ldots, A_{n\sigma}) = (G_{1\sigma} \otimes \cdots \otimes G_{n\sigma} \otimes H 1)(\Delta^{m_{1\sigma}} \otimes \cdots \otimes \Delta^{m_{n\sigma}} \otimes H \text{id}_C) t^*(a_{1\sigma}, \ldots, a_{n\sigma}).
\]

Since (3.19), we have
\[
\sum_{\sigma \in \Sigma_n} (\sigma_H \otimes H \text{id}_C) t^*_\sigma(A_{1\sigma}, \ldots, A_{n\sigma}) = (G_1 \otimes \cdots \otimes G_n \otimes H 1) \times (\Delta^{m_1} \otimes \cdots \otimes \Delta^{m_n} \otimes H \text{id}_C) \sum_{\sigma \in \Sigma_n} (\sigma_H \otimes H \text{id}_C) t^*_\sigma(a_{1\sigma}, \ldots, a_{n\sigma}) = 0, \quad (3.21)
\]
and (3.20) holds. \( \square \)

**Remark 3.7.** Note that an analogue of Theorem 3.5 also holds for an arbitrary pseudoalgebra over a cocommutative Hopf algebra \( H \), if a pseudoalgebra \( P \) satisfies an identity \( f^* = 0 \) of the form (3.14), then its annihilation algebra \( A(P) \) satisfies \( f = 0 \).

### 4. Jordan, alternative and Mal’cev conformal algebras

Theorems 3.1 and 3.5 together with Remark 3.7 provide a foundation for the following generalization of Definition 1.3.

**Definition 4.1.** Let \( \Omega \) be a variety of ordinary algebras defined by a family of homogeneous multilinear identities \( \{f_i(x_1, \ldots, x_{n_i}) = 0\}_{i \in I} \), and let \( P \) be a pseudoalgebra over a cocommutative Hopf algebra \( H \). If \( P \) satisfies the identities \( \{f_i(x_1, \ldots, x_{n_i}) = 0\}_{i \in I} \), then \( P \) is said to be an \( \Omega \)-pseudoalgebra.

In this section, we write down the identities of Jordan, alternative and Mal’cev conformal algebras obtained by Theorems 3.1 and 3.5.

Let \( A \) be an algebra over a field \( k \) with bilinear multiplication \( \cdot : A \times A \to A \).

**Definition 4.2** (see, e.g., 10). A commutative algebra \( (A, \cdot) \) is said to be **Jordan**, if it satisfies
\[
((a \cdot a) \cdot b) \cdot a = (a \cdot a) \cdot (b \cdot a). 
\]
In the multilinear form (remind that \( \text{char} k = 0 \)), the Jordan identity 4.1 could be rewritten as follows 10:
\[a \cdot (b \cdot (c \cdot d)) + (b \cdot (a \cdot c)) \cdot d + c \cdot (b \cdot (a \cdot d)) = (a \cdot b) \cdot (c \cdot d) + (a \cdot c) \cdot (b \cdot d) + (c \cdot b) \cdot (a \cdot d). \quad (4.2)
\]

Now, let \( C \) be a conformal algebra with \( n \)-products \((\cdot_0 \cdot \cdots \cdot_0)\), \( n \geq 0 \), and let \( * : C \otimes C \to H^{\otimes n} \otimes C \) be the pseudoproduct (2.1). Denote \( h_n = (-1)^n / n! \). It is clear that \( h_n h_m = (n + m)_n h_{n+m} \).

Theorems 3.1 and 3.5 imply that \( C \) is a Jordan conformal algebra if and only if \( C \) satisfies the identities
\[a \ast b = (\sigma_{12} \otimes H \text{id}_C)(b \ast a) \quad (4.3)
\]
and
\[(a \ast (b \ast (c \ast d))) + (\sigma_{12} \otimes_H \text{id}_C)((b \ast (a \ast c)) \ast d) + (\sigma_{13} \otimes_H \text{id}_C)(c \ast (b \ast (a \ast d)))
= ((a \ast b) \ast (c \ast d)) + (\sigma_{23} \otimes_H \text{id}_C)((a \ast c) \ast (b \ast d))
+ (\sigma_{13} \otimes_H \text{id}_C)((c \ast b) \ast (a \ast d)), \tag{4.4}\]
where \(\sigma_{ij} = (i \ j) \in S_4\).

Identity (4.3) is equivalent to the conformal commutativity (1.8), so let us proceed with (4.4). For example,

\[(b \ast (a \ast c)) \ast d) = \left(b \ast \sum_{n \geq 0} h_n \otimes 1 \otimes_H (a \circ_n c)\right) \ast d
= \left(\sum_{n,m \geq 0} h_m \otimes h_n \otimes 1 \otimes_H (b \circ_m (a \circ_n c))\right) \ast d
= \sum_{n,m,l \geq 0} (h_m \otimes h_n \otimes 1 \otimes 1)(\Delta^{(3)}(h_l) \otimes 1) \otimes_H (b \circ_m (a \circ_n c)) \circ_l d.
\]

Hence,

\[(\sigma_{12} \otimes_H \text{id}_C)((b \ast (a \ast c)) \ast d) = \sum_{n,m,l \geq 0} \sum_{s_1,s_2 \geq 0} \binom{n}{s_1} \binom{m}{s_2} (b \circ_{m-s_2} (a \circ_{n-s_1} c) \circ_{l+s_1+s_2} d). \tag{4.5}\]

By the same way, one may proceed with other monomials in (4.4) and get the equivalent relation (more precisely, this is a system of relations) in terms of conformal operations:

\[a \circ_n (b \circ_m (c \circ_l d)) + \sum_{s_1,s_2 \geq 0} \binom{n}{s_1} \binom{m}{s_2} (b \circ_{m-s_2} (a \circ_{n-s_1} c) \circ_{l+s_1+s_2} d)
+ c \circ_l (b \circ_m (a \circ_n d)) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} (c \circ_l d)
+ \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} c) \circ_{l+s} (b \circ_m d) + \sum_{s \geq 0} \binom{l}{s} (c \circ_{l-s} b) \circ_{m+s} (a \circ_n d) \tag{4.6}\]
for any \(n,m,l \geq 0\).

**Proposition 4.3.** A commutative conformal algebra \(C\) is Jordan if and only if \(C\) satisfies the identities (4.6).

**Definition 4.4** (see, e.g., [3, 7, 10]). (i) An algebra \((A, \cdot)\) is said to be left or right alternative, if for any \(a, b \in A\) we have

\[a^2 \cdot b = a \cdot (a \cdot b), \tag{4.7}\]
\[a \cdot b^2 = (a \cdot b) \cdot b. \tag{4.8}\]

(ii) An anti-commutative algebra \((A, \cdot)\) is said to be a Mal’cev algebra, if it satisfies the identity

\[J(a,b,a \cdot c) = J(a,b,c) \cdot a, \tag{4.9}\]
where \(J(a,b,c) = (a \cdot b) \cdot c - a \cdot (b \cdot c) + b \cdot (a \cdot c), a,b,c \in A\).

The following statements are just corollaries of Theorems 3.1 and 3.5.
Proposition 4.5. A conformal algebra \( C \) is left or right alternative if and only if \( C \) as a pseudoalgebra satisfies the relations

\[
a \ast (b \ast c) - (a \ast b) \ast c = (\sigma_{12} \otimes_H \text{id}_C)((b \ast a) \ast c - b \ast (a \ast c)) \tag{4.10}
\]

or

\[
a \ast (b \ast c) - (a \ast b) \ast c = (\sigma_{23} \otimes_H \text{id}_C)((a \ast c) \ast b - a \ast (c \ast b)), \tag{4.11}
\]

respectively.

Proposition 4.6. An anti-commutative conformal algebra \( C \) is a Mal’cev conformal algebra if and only if it satisfies the identity

\[
J^*(a, b, c \ast d) - J^*(a, b, c) \ast d = (\sigma_{23} \otimes_H \text{id}_C)(a \ast J^*(c, b, d) - J^*(a \ast c, b, d)), \tag{4.12}
\]

where \( J^*(a, b, c) = (a \ast b) \ast c - a \ast (b \ast c) + (\sigma_{12} \otimes_H \text{id}_C)(b \ast (a \ast c)) \).

Remark 4.7. The “pseudo”-Jacobian \( J^* \) in (4.12) is mentioned to be defined with respect to Proposition 4.5.

The following statement describes some elementary relations between the considered varieties of pseudoalgebras. To check these properties, one should perform exactly the same computations as for ordinary algebras, using Proposition 4.6.

Proposition 4.8 (c.f. [115]). Let \( H \) be a commutative Hopf algebra and let \( P \) be an \( H \)-pseudoalgebra with a pseudoproduct \( \ast \). Define the following \( H \)-bilinear maps on \( P \otimes P \):

\[
[a \ast b]_+ = a \ast b + (\sigma_{12} \otimes_H \text{id}_P)(b \ast a), \quad [a \ast b]_- = a \ast b - (\sigma_{12} \otimes_H \text{id}_P)(b \ast a).
\]

Denote by \( P^{(\pm)} \) the same \( H \)-module \( P \) endowed with the pseudoproduct \([\ast , \ast ]^{(\pm)}\).

(i) If \( P \) is associative, then \( P^{(-)} \) is Lie and \( P^{(+)} \) is Jordan.

(ii) If \( P \) is alternative, then \( P^{(-)} \) is Mal’cev and \( P^{(+)} \) is Jordan.

It is also easy to write down the identities (4.10), (4.11) and (4.12) in terms of conformal products, as it was done for Jordan identity. But since the “conformal” form of identities is more complicated than the “pseudoalgebraic” one (e.g., compare (4.14) with (4.16)), the language of pseudoproduct seems to be more adequate even in the case of conformal algebras.

Definition 4.9. Let \( H \) be a Hopf algebra. An algebra \( A \) (non-associative, in general) endowed with homomorphism of algebras \( \Delta_A : A \to H \otimes A, \Delta_A(a) = a_{(1)} \otimes a_{(2)} \), is said to be an \( H \)-comodule algebra, if

\[
(\text{id}_H \otimes \Delta_A)\Delta_A(a) = (\Delta \otimes \text{id}_A)\Delta_A(a) = a_{(1)} \otimes a_{(2)} \otimesachi(3), \tag{4.13}
\]

\[
\varepsilon(a_{(1)})a_{(2)} = a. \tag{4.14}
\]

The following statement shows how to construct conformal algebras satisfying homogeneous multilinear identities.

Proposition 4.10. Let \( H \) be a commutative and cocommutative Hopf algebra and let \( A \) be an \( H \)-comodule algebra. Then the free \( H \)-module \( P(A) = H \otimes A \) with the pseudoproduct \( \ast \) given by

\[
(h \otimes a) \ast (g \otimes b) = hb_{(1)} \otimes ga_{(1)} \otimes_H (1 \otimes a_{(2)}b_{(2)}), \quad h, g \in H, \ a, b \in A, \tag{4.15}
\]

is an \( H \)-pseudoalgebra. If \( A \) satisfies an identity \( f(A) \) then the pseudoalgebra \( P(A) \) satisfies \( \tilde{f}(A) \).
Proof. Let \( t(a_1, \ldots, a_n) \) be a non-associative word obtained from \( a_1 \ldots a_n \) by some bracketing. It is sufficient to prove that

\[
t^*(a_1, \ldots, a_n) = \left( \bigoplus_{k=1}^{n} a_{1(k-1)} \ldots a_{k-1(k-1)} a_{k+1(k)} \ldots a_{n(k)} \right) \otimes H (1 \otimes t(a_1(n), \ldots, a_n(n))).
\]

It could be easily done by induction on \( n \).

In particular, let \( H = k[D] \), and let \( A \) be an algebra. Denote by \( A[t] \) the tensor product \( A \otimes k[t] \). If \( A \) satisfies a homogeneous multilinear identity \( f = 0 \) of type (3.4), then \( A[t] \) endowed with \( \Delta_A[t] : a \otimes t^n \rightarrow \sum_{s \geq 0} \binom{n}{s} D^s \otimes (a \otimes t^{n-s}) \)

is an \( H \)-comodule algebra satisfying the same identity. Hence, the pseudo algebra \( P(A[t]) \) constructed in Proposition 4.10 satisfies the corresponding \( f^* = 0 \) of type (3.5).

There is a well-known fact (Artin’s Theorem) in the theory of alternative algebras. It states that any two elements of an alternative algebra \( A \) generate an associative subalgebra of \( A \). Let us show that this statement does not hold for alternative conformal algebras.

Let \( A \) be an alternative conformal algebra which is not associative. Then \( P(A[t]) \) constructed above is an alternative pseudoalgebra. Let us choose \( a, b, c \in A \) such that \( \{a, b, c\} \equiv (ab)c - a(bc) \neq 0 \), and fix \( x, y \in P(A[t]) \) as follows: \( x = 1 \otimes (a \otimes t + b \otimes 1), \ y = 1 \otimes (c \otimes 1) \). Direct computation shows that

\[
(x * y) * x - x * (y * x) = (D \otimes 1 \otimes 1) \otimes_H (1 \otimes \{b, c, a\} \otimes 1) + (1 \otimes 1 \otimes D) \otimes_H (1 \otimes \{a, c, b\} \otimes 1) \neq 0.
\]

Therefore, \( x \) and \( y \) do not generate an associative conformal subalgebra of \( P(A[t]) \).

REFERENCES

[1] Bakalov B., D’Andrea A., Kac V. G. Theory of finite pseudoalgebras, Adv. Math. 162 (2001), no. 1.
[2] Beilinson A. A., Drinfeld V. G. Chiral algebras, Amer. Math. Soc. Colloquium Publications, 51. AMS, Providence, RI, 2004.
[3] Kac V. G. Vertex algebras for beginners, Univ. Lect. Series 10. AMS, 1996.
[4] Kac V. G. The idea of locality, In: H.-D. Doebner et al. (eds.) Physical applications and mathematical aspects of geometry, groups and algebras, Singapore: World Scientific (1997), 16–32.
[5] Kac V. G. Formal distribution algebras and conformal algebras, XII-th International Congress in Mathematical Physics (ICMP’97) (Brisbane), Internat. Press: Cambridge, MA (1999) 80–97.
[6] Kuz’min E. N. Mal’tsev algebras and their representations (Russian), Algebra i Logika, 7 (1968), no. 4, 48–69.
[7] Sagle A. A. Malcev algebras, Trans. Amer. Math. Soc., 101 (1961), 426–458.
[8] Roittman M. On free conformal and vertex algebras, J. Algebra 217 (1999), no. 2, 496–527.
[9] Zelmanov E. I. On the structure of conformal algebras, Intern. Conf. on Combinatorial and Computational Algebra, May 24–29, 1999, Hong Kong. Cont. Math. 264 (2000), 139–153.
[10] Zhevlovakov K. A., Slin’ko A. M., Shestakov I. P., Shirshov A. I. Rings that are nearly associative, Academic Press, New York, 1986.

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