After a brief overview of the realization of chiral symmetry in hot and/or dense medium, the implications of the existence of a light scalar-isoscalar meson (which we call $\sigma$ throughout this report) are discussed in the vacuum and in the medium. Special emphasis is put on its relation to the fluctuation of the chiral order parameter $\langle \bar{q}q \rangle$. In the vacuum, the $\sigma$-meson is an elusive resonance corresponding to a pole deep in the second Riemann sheet of the $\pi$-$\pi$ scattering matrix in the $I=J=0$ channel. This is because the the amplitude fluctuation of $\langle \bar{q}q \rangle$ (corresponding to $u_r\sigma$) and the phase fluctuation of $\langle \bar{q}q \rangle$ (2 pion states) mix strongly in the vacuum. As the temperature and/or density of the system increase, however, there arises a softening of this complex pole due to the partial restoration of chiral symmetry. Such soft modes play a key role to understand the QCD phase structure. We demonstrate that even a slight softening of the $\sigma$ mode could induce strong spectral enhancement and strong $\pi$-$\pi$ correlation near the $2m_\pi$ threshold in the $I=J=0$ channel, which thereby can be a signature of the partial restoration of chiral symmetry. Such spectral enhancement may be seen not only in hot matter created by the relativistic heavy ion collisions but also in cold matter (heavy nuclei) probed by photons and hadrons. Relevance of the partial restoration of chiral symmetry in heavy nuclei with the recent data by the CHAOS and CB collaborations as well as with on-going and future experiments are also discussed.
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VI. Summary and concluding remarks
I. INTRODUCTION

One of the most intriguing phenomena in quantum chromodynamics (QCD) is the dynamical breaking of chiral symmetry (DBCS). This explains the existence of the pion and governs most of the low energy phenomena in hadron physics. DBCS is associated with the condensation of quark-anti-quark pairs in the QCD vacuum, $\langle \bar{q}q \rangle$, which is analogous to the condensation of Cooper pairs in the theory of superconductivity [1]. As the temperature ($T$) and/or the baryon density ($n_B$) increase, the QCD vacuum undergoes a phase transition to the chirally symmetric phase where $\langle \bar{q}q \rangle$ vanishes. Studying the mechanism of DBCS, exploring the phase structure of the QCD ground state, and finding possible signatures of the QCD phase transition are the central issues in modern hadron physics [2].

Various phases of QCD are characterized not only by $T$ and $n_B$, but also by the current masses of light quarks ($m_{u,d,s}$). In the real world, the masses of $u$ and $d$ quarks are much lighter than the $s$ quark [3],

$$
\frac{m_s}{m_d} = 18.9 \pm 0.8, \quad \frac{m_u}{m_d} = 0.553 \pm 0.043,
$$

where $m_s(2\text{GeV}) \sim 100$ MeV is comparable to $\Lambda_{QCD}$ (the QCD scale parameter) and also to the critical temperature (density $^{1/3}$) of the quark-hadron transition. Therefore, studying the phase structure in the $m_s$-$T$-$n_B$ space by neglecting $m_{u,d}$ as a first approximation would give us a good insight into the real world.

Shown in Fig.1 is a possible phase structure projected onto the $m_s$-$T$ plane at $n_B=0$ (the left panel) and onto $m_s$-$n_B$ plane with $T=0$ (the right panel). $m_s=0$ ($m_s=\infty$) corresponds to the limit of $SU_L(3) \otimes SU_R(3)$ ($SU_L(2) \otimes SU_R(2)$) chiral symmetry.

![Fig. 1. Possible phases in the $m_s$-$T$ plane at $n_B=0$ (left panel) and the $m_s$-$n_B$ plane at $T=0$. $m_{u,d}=0$ is assumed. Relevant degrees of freedom in each phase are also shown. $n_0 = 0.17\text{fm}^{-3}$ is the normal nuclear matter density.](image)
The chiral phase transition for $T \neq 0$ and $n_B = 0$ has been extensively studied on the lattice for recent years with $\langle \bar{u}u \rangle (= \langle \bar{d}d \rangle)$ as the chiral order parameter [4]. The analysis based on the renormalization group and the universality hypothesis indicates that the chiral transition at finite $T$ is of second (first) order for large (small) $m_s$ [5]. This has been confirmed by the lattice QCD simulations with dynamical quarks. The critical temperature for massless 2 and 3 flavors has been also studied on the lattice and the results are summarized as

\[ T_{pc}(N_f = 3) = (154 \pm 8)\text{MeV} \quad \text{(staggered fermion)}, \]
\[ T_{pc}(N_f = 2) = (173 \pm 8)\text{MeV} \quad \text{(staggered fermion)}, \]
\[ T_{pc}(N_f = 2) = (171 \pm 4)\text{MeV} \quad \text{(Wilson fermion)}, \]  

where $T_{pc}$ is a pseudo-critical temperature extracted from the chiral susceptibility.

The second order line and the first order line in Fig.1 (left panel) is expected to meet at some intermediate value of $m_s$ at the so-called tricritical point [6]. However, the precise location of this tricritical point in $m_s$-$T$ plane is not identified yet. If we have finite but small $u, d$ quark masses, the second order line turns into a smooth crossover, and the tricritical point becomes an end point of the first order line. Therefore, in the real world where two light quarks + one medium-heavy quark exist, the chiral transition at $T \neq 0$ is either the first order or the crossover.

The chiral phase transition at finite baryon density is not well understood. One of the reasons is the first principle lattice QCD simulation for finite chemical potential ($\mu$) has a severe sign problem originating from the complex fermionic determinant [4]. Analyses based on simple models such as the Nambu-Jona-Lasinio (NJL) model show that the chiral transition may be first order for $n_B \neq 0$ and $T \ll T_c$ [7]. If this is the case, the quark-hadron mixed phase is expected in the $m_s$-$n_B$ plane as schematically shown in the right panel of Fig.1. The system at finite baryon density, which is intrinsically quantum, is obviously richer in physics than that at finite $T$ and has various phases such as the baryon superfluidity, meson condensations, and the color superconductivity (see the reviews, [9]).

For sufficiently large $m_s$ with $m_{u,d} = 0$, there will be also a tricritical point in the $T$-$\mu$ plane. For small but finite $m_{u,d}$, the tricritical point again becomes an end point of the first order line. Very recently, a new attempt on the lattice has been made to locate this end point by looking for the Lee-Yang zero of the partition function numerically, where the 2-dimensional reweighting method is employed to obtain the partition function at finite $T$ and $\mu$ [10]. It is found that $T_{end}=(160\pm3.5) \text{MeV}$ and $\mu_{end}=(725\pm35) \text{MeV}$ for $(2+1)$-flavors using the data on $4^3 \times 4, 6^3 \times 4$ and $8^3 \times 4$ lattices. Although obtaining a clear signal of Lee-Yang zeros may require much larger lattice size, this approach provides a new way of analyzing the physics near the critical line in the $T$-$\mu$ plane.

\[ \text{1The possible vector coupling between quarks in the NJL model may alter the order of the chiral transition at finite density with low temperatures. [8]} \]
II. THE LIGHT SCALAR-ISOSCALAR MESON IN QCD

The order parameter \( \langle \bar{q}q \rangle \) of the chiral transition is determined as the value at which the effective potential (free energy) \( V(\sigma) \) takes the minimum. The light scalar-isoscalar meson, which we call \( \sigma \) in the following, is a particle representing the amplitude fluctuation of the order parameter around the minimum of \( V(\sigma) \). In this sense, the \( \sigma \) meson is analogous to the Higgs particle \( H \) in the Glashow-Salam-Weinberg model. However, the properties of \( \sigma \) and \( H \) or, more precisely, the physical modes in the respective channels, could be quite different because of the following reason: The Nambu-Goldstone (NG) boson, which is a phase fluctuation of the order parameter, appears as the pion in QCD, while the NG boson in the electro-weak theory is absorbed into the gauge bosons. Therefore, \( \sigma \) can have a strong width decaying into two pions, while such process for \( H \) does not exist. In other words, the physical \( \sigma \) state in QCD can be at most a broad resonance and is represented by a linear superposition of the ur-\( \sigma \) (genuine amplitude-fluctuation of the order parameter) and the two-pion state.

A. Chiral symmetry, analyticity, crossing symmetry and complex \( \sigma \)-pole in \( \pi^-\pi^+ \) scattering

As mentioned above, a tricky point with the \( \sigma \) meson is that it couples to two pions to acquire a large width \( \Gamma_\sigma \sim \text{Re } m_\sigma \) and make itself elusive \cite{11}. Nevertheless, recent careful phase shift analyses of the \( \pi^-\pi^+ \) scattering in the \( I=J=0 \) channel have come to claim a \( \sigma \) pole in the complex energy plane with the real part \( \text{Re } m_\sigma \approx 500-800 \text{ MeV} \) and the imaginary part \( \text{Im } m_\sigma \approx 500 \text{ MeV} \) \cite{12,11,13}.

One of the crucial points to deduce the poles in the \( \pi^-\pi^+ \) scattering matrix is to construct the invariant amplitude so that it satisfies the chiral symmetry constraints, the analyticity, the unitarity and the (approximate) crossing symmetry. For this purpose, some reliable resummation method such as the \( N/D \) method should be employed. This point has been emphasized e.g. in \cite{14} and in \cite{15,16}. In the construction of \cite{17}, the ur-\( \sigma \) strongly coupled to \( 2\pi \) is favored to have a good overall agreement with the phase shifts in \( I=J=0, I=J=1 \) and \( (I=2, J=0) \) phase shifts. In \cite{15,16}, the unitalization of the strong \( 2\pi \) correlation in the \( I=J=0 \) channel is enough to reproduce the phase shifts. Although it is not clear at the moment whether one really needs to introduce the ur-\( \sigma \) or not to reproduce the phase shifts, the existence of the light \( \sigma \) pole with a large imaginary part in the 2nd Riemann sheet is the common feature in all modern analyses. This is seen in Fig.2 where a recent summary (taken from \cite{17}) of the complex \( \sigma \) pole is shown.
FIG. 2. The $\sigma$-pole of the $\pi-\pi$ scattering matrix in the complex energy plane in various analyses. This figure is taken from [17].

B. Hadron phenomenology and the $\sigma$ meson

If the location of the $\sigma$ pole in the 2nd Riemann sheet has $\text{Re } m_\sigma \sim 500$ MeV, many experimental facts which otherwise are mysterious can be nicely accounted for in a simple way. Some examples in hadron phenomenology where the scalar-isoscalar fluctuations are relevant include the following. More extensive discussions can be see in [11].

- The scalar-isoscalar resonance with $\text{Re } m_\sigma \sim m_K$ may have a sizable contribution to the enhancement of the $\Delta I = 1/2$ process in $K \to \pi\pi$ decay [18].

- The state-independent attraction, which is responsible for the nuclear binding, originates from the two-pion exchange potential. The two-pion exchange includes the ladder, the cross and the rescattering diagrams with the $\Delta(1232)$ being incorporated in the intermediate states. The $\sigma$ plays a role through the rescattering contribution as well as through the possible direct $\sigma$-$N$ coupling. The former should be constructed consistently with the $\pi$-$\pi$ phase shift in the $I=J=0$ channel. For the latter, systematic analysis of $N$-$N$, hyperon-$N$ and hyperon-hyperon interactions may be useful [19].

- The $\pi$-$N$ sigma term and the generalized scalar-charge of baryons can be defined as $\Sigma_{\pi N} = \hat{m} \langle N | \bar{u}u + \bar{d}d | N \rangle$ and $\langle B | \bar{q}q | B \rangle$ ($q = u, d, s, ...$), respectively. Effective charges are usually enhanced (suppressed) due to collective excitations generated by the attractive (repulsive) forces. The enhancement is caused by the polarization of the vacuum in the $I=J=0$ channel and can indeed account for the empirical value of $\Sigma_{\pi N} \simeq 45$ MeV without introducing a large OZI violation in the nucleon [20].

- There are some effective theories of QCD which predict the light ur-$\sigma$ such as the Schwinger-Dyson approach with the rainbow-ladder approximation [21], the Nambu-Jona-Lasinio model [22], and an approach based on the mended symmetry [23].
The above list implies that the correlation in the scalar-isoscalar channel is indeed important in the hadron dynamics. This is in a sense natural because the dynamics which is responsible for the correlations in this channel is nothing but the one which drives the chiral symmetry breaking.

A fundamental question is, then, whether the complex $\sigma$-pole observed in the $\pi$-$\pi$ phase shift can be interpreted as a quantum fluctuation of the order parameter associated with the dynamical breaking of chiral symmetry or its remnant. In condensed matter physics, such a question can be answered by changing the ground state properties of the system with temperature, pressure, doping etc as external controlling parameters [24]. If the frequency of some collective mode has characteristic change near the critical point, it provides us with a clear signature of a direct link between the mode and the phase transition. The life is, unfortunately, not that easy in QCD from the experimental point of view. Nevertheless the heavy nuclei provide us with finite baryon-density environment which allows us to study the partial restoration of chiral symmetry and associated change of the meson properties. Heavy-ion collisions also provide us with not only high baryon density but also high temperature although the system stays in such environment only in a transient time.

Some years ago, one of the present authors [25] proposed several nuclear experiments including one using electro-magnetic probes for producing the $\sigma$ meson in nuclei to see a clearer evidence of the $\sigma$ pole and to explore possible restoration of chiral symmetry in nuclear medium. When a particular meson is put in a nucleus, it may dissociate into complicated excitations to loose its identity in the medium; for example, $\sigma \leftrightarrow 2\pi, p-h, \pi+p-h, \Delta-h, \pi+\Delta-h, \cdots$. Therefore it is not appropriate to talk about the “mass” and “width” of the particle, but one should study the spectral function which contains all those information.

III. CHIRAL RESTORATION AND SOFT MODE AT FINITE $T$ AND DENSITY

The basic observation on which the whole discussions in this report are based is that the dynamical breaking of chiral symmetry is a phase transition of the QCD vacuum with an order parameter $\langle \bar{q}q \rangle$: Then there may exist collective excitations corresponding to the quantum fluctuations of the order parameter. The phase (amplitude) fluctuation of the order parameter corresponds to the pion ($\pi$-$\sigma$). As we have mentioned before, they couple strongly and mix through the process $\sigma \leftrightarrow 2\pi$.

Now let us first show how the magnitude of the condensate $\langle \bar{q}q \rangle$ decreases at finite $T$ and/or density $n_B$ in an analytic way.

A. Chiral condensate at low $T$

The condensate at $T \neq 0$ ($n_B = 0$) can be written as

$$\langle \bar{q}q \rangle = \frac{1}{Z} \text{Tr} \left[ \bar{q}q \ e^{-H_{QCD}/T} \right] \frac{\partial f(T)}{\partial m_q} = -\frac{\partial P(T)}{\partial m_q}, \quad (3)$$

where $Z$ is the QCD partition function, $f$ is the free energy density and $P$ is the total pressure of the system including vacuum pressure and the thermal pressure. This is because
the current quark masses enters \( H_{\text{QCD}} \) only in the form \( \sum_i m_q \bar{q} q \); hence this formula is valid for each flavor \( (q = u, d, s, c, b, t) \).

At high \( T \gg \Lambda_{\text{QCD}} \), one can calculate \( P(T) \) using the high \( T \) QCD perturbation theory and \( \langle \bar{q} q \rangle \) is shown to be exactly zero for \( m_q = 0 \). This is because there is no vertex which flips chirality in the QCD Lagrangian except for the mass term. On the other hand, there is a dynamical breaking of chiral symmetry at low \( T \) and \( \langle \bar{q} q \rangle \) does not necessarily vanish. For \( T \) low enough, the system may be approximated by a dilute and weakly interacting gas of pions. Its pressure can be calculated using the chiral perturbation theory [26]:

\[
Z = \text{Tr} \left[ e^{-H_{\text{QCD}}/T} \right] = \int [dU] \exp \left[ -\int_0^{1/T} d\tau \int d^3 x \, \mathcal{L}_{\text{eff}}(U) \right],
\]

where \( \mathcal{L}_{\text{eff}}(U) \) is expanded as \( \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \cdots \) in powers of \( \partial / 4 \pi f_\pi \) and \( m_\pi / 4 \pi f_\pi \). Therefore, the quark mass dependence of \( P(T) \) for \( N_f = 2 \) at low \( T \) in (3) can be extracted from its \( m_\pi \) dependence through the Gell-Mann-Oakes-Renner relation

\[
m_\pi^2 = -2 \hat{m} \frac{\langle \bar{q} q \rangle_0}{f_\pi^2} + O(m_{u,d}^2),
\]

where \( \hat{m} = (m_u + m_d)/2 \), and \( \langle \bar{q} q \rangle_0 \) and \( f_\pi \) are the vacuum condensate and the pion decay constant in the chiral limit, respectively.

Writing \( P(T) = P_{\text{pion}}(T) + P_{\text{vac}} \), and using the above relation, one thus finds the quark-condensate in the chiral limit for \( N_f = 2 \) as [26]

\[
\frac{\langle \bar{q} q \rangle}{\langle \bar{q} q \rangle_0} = 1 + \frac{1}{f_\pi^2} \frac{\partial P_{\text{pion}}(T)}{\partial m_\pi^2} \bigg|_{m_\pi \to 0} = 1 - \frac{T^2}{8 f_\pi^2} - \frac{1}{6} \left( \frac{T^2}{8 f_\pi^2} \right)^2 - \frac{16}{9} \left( \frac{T^2}{8 f_\pi^2} \right)^3 \ln \left( \frac{\Lambda_\pi}{T} \right) + O(T^8),
\]

where \( \Lambda_\pi (= 470 \pm 110 \text{ MeV}) \) is a parameter extracted from the experimental \( \pi^-\pi^- \) scattering length in the isoscalar \( D \)-wave channel. The \( T^2 \) corrections in the above expansion originate solely from the free pion-gas contribution. Therefore, it can be alternatively obtained by applying the soft pion theorem to the free pion gas. The negative contribution of the \( O(T^2) \) term is due to the fact that the pion has positive scalar-charge \( \langle \pi(p)|\bar{q} q|\pi(p) \rangle \), which can be seen from either by the soft-pion theorem or by the Feynman-Hellman theorem. On the other hand, the interaction among pions are relevant for the terms \( O(T^{n \geq 4}) \).

The above result of the low \( T \) expansion suggests that the chiral condensate decreases uniformly as \( T \) increases and the symmetry restoration takes place somewhere around 100-200 MeV. However, the behavior of \( \bar{q} q \) near the critical point is out of reach of the low \( T \) expansion, because the system is not any more the dilute gas of pions near the critical point. In such a region, non-perturbative methods such as the the lattice QCD simulations are necessary to make a reliable estimate of the chiral condensate.

B. Chiral condensate at finite baryon density

\( \langle \bar{q} q \rangle \) at finite baryon density obeys an exact formula in QCD directly obtained from the Feynman-Hellman theorem.
\[ \langle \Psi | \bar{q}q | \Psi \rangle = \frac{\partial \langle \Psi | H_{QCD} | \Psi \rangle}{\partial m_q}, \]  
(7)

which is valid for any eigenstates \(|\Psi\rangle\) of the QCD Hamiltonian \(H_{QCD} = \int d^3 x H_{QCD}\) with the normalization \(\langle \Psi | \Psi \rangle = 1\). For cold nuclear matter \(|\Psi\rangle = |nm\rangle\), the energy density is given as
\[ \langle nm | H_{QCD} | nm \rangle = \mathcal{E}_{\text{vac}} + n_B [M_N + E_{\text{bin}}], \]  
(8)

where \(\mathcal{E}_{\text{vac}}\) is the vacuum energy density, \(M_N\) is the rest mass of the nucleon and \(E_{\text{bin}}\) is the binding energy of nuclear matter per particle. Applying this and the similar formula for the nucleon to (7), one arrives at
\[ \langle \bar{u}u + \bar{d}d \rangle \langle \bar{u}u + \bar{d}d \rangle_0 = 1 - \frac{n_B}{f_\pi^2 m_\pi^2} \left[ \Sigma_{\pi N} + \frac{d \dot{m}}{dn} E_{\text{bin}}(n_B) \right], \]  
(9)

where \(\Sigma_{\pi N} = 45 \pm 10 \text{ MeV}\) is the pion-nucleon sigma term. The density-expansion of the r.h.s. in the lowest order gives a reduction of almost 35% of \(\langle \bar{q}q \rangle\) already at the nuclear matter density \(n_0 = 0.17 \text{fm}^{-3}\): The physical origin of this reduction is similar with the pion gas case discussed above; the nucleon has the positive scalar charge \(\langle N | \bar{q}q | N \rangle > 0\), hence the nucleon gas gives the positive contribution to the quark condensate. The above estimate suggests that heavy nuclei can be a good laboratory to study the partial restoration of chiral symmetry in nuclear medium; 20 to 30% reduction of the chiral condensate might be large enough to manifest typical phenomena of partial restoration of chiral symmetry. We shall argue that this is actually the case.

C. Hadronic correlations and symmetry restoration

The wisdom of many-body theory tells us that the structure of the ground state is closely related with and reflected in the excitation spectra of the system. In the case of chiral symmetry restoration, a model-independent statement on this connection goes as follows. Consider the hadronic correlations in a same chiral multiplet, such as \(S\) (scalar) – \(P^a\) (pseudoscalar) and \(V^a_\mu\) (vector) – \(A^a_\mu\) (axial-vector). Then they must be degenerate when \(\langle \bar{q}q \rangle \to 0\), namely,
\[ \langle S(x) S(y) \rangle \to \langle P^a(x) P^a(y) \rangle, \quad \langle A^a_\mu(x) A^b_\mu(y) \rangle \to \langle V^a_\mu(x) V^b_\mu(y) \rangle. \]  
(10)

To see such chiral degeneracy by lattice QCD simulations, it is useful to define the thermal susceptibilities for hadronic operators \(\mathcal{O}\) defined in the Euclidean space,
\[ \chi_\mu = \int_0^{1/T} d\tau \int d^3 x \langle \mathcal{O}^\dagger(\tau, \vec{x}) \mathcal{O}(0, \vec{0}) \rangle. \]  
(11)

Shown in Fig. is the lattice QCD simulation of \(\sqrt{1/\chi_\mu}\) with \(N_f = 2\) dynamical quarks. One can see the degeneracy between the \(\sigma\)-channel \((I=0, J^P=0^+)\) and the \(\pi\)-channel \((I=1, J^P=0^-)\) above the critical point. Also, \(\sqrt{1/\chi_\mu}\) for \(\sigma\) is smaller than that for \(\delta\) at low \(T\), which indicates that \(\sigma\) could be light. There remains a splitting between the \(\delta\)-channel.
\(I=1, J^P=0^+\) and the \(\pi\)-channel even above the critical point, which reflects the breaking of \(U_A(1)\) symmetry in the 2-point correlation function for \(N_f=2\); in other words, the \(U_A(1)\) anomaly survives the chiral restoration. In general, the effect of the \(U_A(1)\) symmetry breaking appears for \(N(\geq N_f)\)-point function for \(N_f\)-quarks as shown in [28].

\[
I = 1, J^P = 0^+ + \pi\text{-channel even above the critical point, which reflects the breaking of } U_A(1) \text{ symmetry in the 2-point correlation function for } N_f = 2; \text{ in other words, the } U_A(1) \text{ anomaly survives the chiral restoration. In general, the effect of the } U_A(1) \text{ symmetry breaking appears for } N(\geq N_f)\text{-point function for } N_f\text{-quarks as shown in [28].}
\]

To make a direct connection of the chiral restoration with the experimental observables, the Euclidean correlations such as (11) are not useful enough; one needs to extract the information of the spectral distribution as we mentioned before. Recently, a new approach has been proposed to extract hadronic spectral functions (SPFs) from lattice QCD data by using the maximum entropy method (MEM) [29]. MEM has been successfully applied for similar problems in image reconstruction in crystallography and astrophysics, and in quantum Monte Carlo simulations in condensed matter physics [30].

The Euclidean correlation function \(D(\tau)\) of an operator \(\mathcal{O}(\tau, \vec{x})\) and its spectral decomposition at zero three-momentum read

\[
D(\tau > 0) = \int \langle \mathcal{O}^\dagger(\tau, \vec{x})\mathcal{O}(0, \vec{0})\rangle d^3x \equiv \int_0^\infty K(\tau, \omega)A(\omega)d\omega, \tag{12}
\]

where \(\omega\) is a real frequency, and \(A(\omega)\) is the spectral function (or sometimes called the image), which is positive semi-definite. \(K(\tau, \omega)\) is a known integral kernel (it reduces to Laplace kernel \(e^{-\tau\omega}\) at \(T=0\).) Monte Carlo simulation provides \(D(\tau_i)\) on the discrete set of temporal points \(0 \leq \tau_i/a \leq N_\tau\). From this data with statistical noise, we need to reconstruct the spectral function \(A(\omega)\) with a continuous variable \(\omega\). This is a typical ill-posed problem, where the number of data is much smaller than the number of degrees of freedom to be reconstructed. MEM is a method to circumvent this difficulty through Bayesian statistical inference of the most probable image together with its reliability. There are three important aspects of MEM: (i) it does not require a priori assumptions or parameterizations of SPFs, (ii) for given data, a unique solution is obtained if it exists, and (iii) the statistical significance of the solution can be quantitatively analyzed.

In Fig.4, shown is an example of the spectral function in the pion and the rho meson channels at \(T=0\) extracted from the quenched lattice QCD data [29]. Correct resonance
peaks at low energies and the continuum structure at high energies are well reproduced. The long-standing problem of in-medium spectral functions of vector mesons ($\rho$, $\omega$, $\phi$, $J/\psi$, $\Upsilon$, · · ·, etc) and scalar/pseudo-scalar mesons ($\sigma$, $\pi$, · · ·, etc) can be studied using MEM combined with finite $T$ lattice simulations.

FIG. 4. Spectral functions $\rho_{\text{out}}(\omega) \equiv A(\omega)/\omega^2$ obtained by MEM using the quenched lattice QCD data. The lattice size is $20^3 \times 24$ with $a = 0.0847$ fm. 12 data points in the temporal direction ($1 \leq \tau_i/a \leq 12$) are used for the MEM analysis. (a) is for the pion channel and (b) is for the rho-meson channel. The figures are taken from [29].

IV. SCALAR CORRELATION IN THE MEDIUM

A. Spectral enhancement in hot matter

The fluctuation of the order parameter becomes large as the system approaches the critical point of a second order or to the end point of the first order phase transition. This is easily seen for the second order case by looking at the Landau free-energy of a double-well type written in terms of the order parameter $\sigma \sim \langle \bar{q}q \rangle$,

$$V(\sigma) = -\frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4, \quad (13)$$

where $a(\propto T_c - T)$ changes sign at the critical point, while $b$ remains positive. The minimum value and the curvature at the minimum of the effective potential $\sigma_0$ read

$$\sigma_0 = \sqrt{\frac{a}{b}}, \quad \frac{1}{2} \left. \frac{d^2V(\sigma)}{d\sigma^2} \right|_{\sigma=\sigma_0} = a, \quad (14)$$

respectively. As $a$ becomes small, not only the order parameter $\sigma_0$ but also the curvature decrease. This is nothing but the softening of the oscillational mode of the order parameter associated with the second order phase transition.
1. Soft mode in the Wigner phase

Historically, the dynamical fluctuation of the order parameter near the chiral critical point was first considered in the Wigner phase above $T_c$ by using the Nambu-Jona-Lasinio model \[31\]-\[33\]. It was shown that such collective oscillation becomes soft and narrow as the system approaches to the critical point. Since this is the collective mode in the chirally symmetric phase, it may be called “para-pion” or “para-sigma” in analogy to the para-magnon in the electron gas and liquid $^3$He \[34\] and para-superconductivity in low dimensional superconductors \[35\]. In nuclear physics, the soft mode for the superconductivity is called the pairing vibration, and the $2^+$ phonon is the soft mode for the quadrupole deformation of nuclei. \[36\] The soft mode for the pion condensation \[9\] should be the longitudinal spin-dependent isovector modes with finite momentum \[37,9\]. In Fig.5, shown is the spectral function of the para-pion defined by

$$S^{ab}(\omega, \mathbf{p}; T) \propto \text{Im} \int d^4x \ e^{ipx} \langle R \bar{q} \Gamma^a q(x)q \Gamma^b q(0) \rangle,$$

(15)

where $\Gamma^a = (1, i\gamma_5 \vec{\tau})$ and $R$ denotes the retarded product \[32\].

FIG. 5. The spectral function of degenerate “para-pion” and “para-sigma” at finite $T$ in the Wigner phase calculated by the 2-flavor Nambu-Jona-Lasinio model in the chiral limit. Critical temperature of the chiral transition is 164 MeV and the spectral enhancement is seen as $T$ approaches to the critical point. Taken from \[32\].

2. Soft mode in the Nambu-Goldstone phase

In the chirally broken phase just below $T_c$, there arises two types of soft modes, namely the phase and amplitude fluctuations of the chiral order parameter $\langle \bar{q} q \rangle$ \[32,33\]. Because of the Nambu-Goldstone theorem, the pion stays massless in the chiral limit below $T_c$, while one can expect sizable softening (the red-shift) of $\sigma$. In the real world, the situation is not that simple, since $\sigma$ has a large width from the decay $\sigma \rightarrow 2\pi$. Nevertheless, $\sigma$ may appear as a narrow resonance near $T_c$ because of the phase space suppression of the above decay channel, although it is at best a broad resonance in the free space. This idea has been put
forward in [38] for the first time and various phenomenological applications in relation to the relativistic heavy ion collisions have been discussed later [39].

Since the character of $\sigma$ changes from a very broad resonance to a sharp resonance, it is not at all enough to just talk about the mass and width but is necessary to study the spectral function. By using the linear $\sigma$-model, the spectral functions in $\pi$ and $\sigma$ channels have been studied at finite $T$ and it was found that there is a significant spectral enhancement just above the $2m_\pi$ threshold as $T$ increases [40].

To describe the general features of this spectral enhancement, let us consider the propagator of the scalar-isoscalar $\sigma$-meson at rest in the medium:

$$D_{\sigma}^{-1}(\omega) = \omega^2 - m_\sigma^2 - \Sigma_\sigma(\omega; T),$$

where $m_\sigma$ is the mass of the $\sigma$ in the tree-level, and $\Sigma_\sigma(\omega; T)$ is the loop corrections in the vacuum as well as in the hot medium. The corresponding spectral function is given by

$$\rho_\sigma(\omega) = -\frac{1}{\pi} \text{Im} D_{\sigma}(\omega).$$

Near the $2m_\pi$ threshold, the imaginary part in the one-loop order reads

$$\text{Im} \Sigma_\sigma \propto \theta(\omega - 2m_\pi) \sqrt{1 - \frac{4m_\pi^2}{\omega^2}}.$$  \hspace{1cm} (17)

When chiral symmetry is being restored, $m_\sigma^*$ ("effective mass" of $\sigma$ defined as a zero of the real part of the propagator $\text{Re} D_{\sigma}^{-1}(\omega = 2m_\pi) = 0$) approaches to $m_\pi$. Therefore, there exists a temperature at which $\text{Re} D_{\sigma}^{-1}(\omega = 2m_\pi) = 0$ even before the complete $\sigma$-$\pi$ degeneracy is realized; $\text{Re} D_{\sigma}^{-1}(\omega = 2m_\pi) = [\omega^2 - m_\sigma^2 - \text{Re} \Sigma_\sigma]_{\omega = 2m_\sigma} = 0$. At this point, the spectral function can be solely represented by the imaginary part of the self-energy;

$$\rho_\sigma(\omega \simeq 2m_\pi) = -\frac{1}{\pi} \text{Im} \Sigma_\sigma \propto \frac{\theta(\omega - 2m_\pi)}{\sqrt{1 - \frac{4m_\pi^2}{\omega^2}}},$$

which shows an enhancement of the spectral function at the $2\pi$ threshold. One should note that this enhancement is due to the partial restoration of chiral symmetry and hence generic [40].

To make the argument more quantitative, let us evaluate $\rho_\sigma(\omega)$ in the $O(4)$ linear $\sigma$-model:

$$L_M = \frac{1}{4} \text{tr} [\partial M \partial M^\dagger - \mu^2 MM^\dagger - \frac{2\lambda}{4!} (MM^\dagger)^2 - h(M + M^\dagger)],$$

where tr is for the flavor index and $M = \sigma + i\vec{\tau} \cdot \vec{\pi}$. Although the model has only a limited number of parameters and is not a precise low energy representation of QCD at zero $T$, it can describe the pion dynamics qualitatively well up to 1GeV [41]. It also provides a good description of the static critical phenomena of QCD owing to the universality argument [42]. The coupling constants $\mu^2, \lambda$ and $h$ have been determined in the vacuum to reproduce $f_\pi = 93$ MeV, $m_\pi = 140$ MeV as well as the s-wave $\pi$-$\pi$ scattering phase shift in the one-loop order: one of such parameter sets is $-\mu^2 = (284\text{MeV})^2$, $\lambda/4\pi = 5.81$ and $h^{1/3} = 123\text{MeV}$ [40]. This will be used throughout this report.

In Fig.6, shown are the spectral functions $\rho_{\pi(\sigma)}$ in the $\pi$ ($\sigma$)-channel at finite $T$ calculated in the $O(4)$ linear $\sigma$ model: Two characteristic features are the broadening of the pion peak
(Fig.3(A)) and the spectral concentration at the $2m_{\pi}$ threshold ($\omega \simeq 2m_{\pi}$) in the $\sigma$-channel (Fig.3(B)). The latter may be measured by the $2\gamma$ spectrum from the hot plasma created in the relativistic heavy ion collisions \[41,42\].

![Graph showing spectral functions in the $\pi$ channel (A) and in the $\sigma$ channel (B) for $T=50$, 120, and 145 MeV with the parameter set shown in the text \[40\]. $\rho_\sigma(\omega)$ in (B) shows only a broad bump at low $T$ (a), while the spectral concentration is developed as $T$ increases, (a)$\rightarrow$(b)$\rightarrow$(c).]

**B. Spectral enhancement in nuclear matter**

The idea of the spectral enhancement near the $2m_{\pi}$ threshold can be also applied to the finite density system \[43\]. A surprise was that such enhancement can be seen even near the nuclear matter density if the chiral condensate decreases by 30 \% or so. To see this explicitly, let us consider again the O(4) linear $\sigma$ model to demonstrate the essential idea of the near-threshold enhancement. The Lagrangian adopted in \[43\] is the standard Gell-Mann-Levy model with a minor modification

$$L_{MN} = L_M + \bar{\psi}(i\not\partial - gM_5)\psi + \cdots,$$

where $M = \sigma + i\vec{\tau} \cdot \vec{\pi}$, $M_5 = \sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}$, and $\psi$ is the nucleon field. In the mean-field level, medium effect originating from the nucleons can be incorporated by making the following replacement;

$$\langle \sigma \rangle_0 \rightarrow \langle \sigma \rangle = \Phi(\rho)\langle \sigma \rangle_0,$$

where $\langle \cdot \rangle_0 (\langle \cdot \rangle)$ denotes the vacuum (nuclear matter) expectation value, and $\rho (\rho_0)$ is the baryon density (nuclear matter saturation density) \[\frac{\Phi(\rho)}{\Phi(\rho_0)}\] may be parameterized as $\Phi(\rho) = 1 - C\rho/\rho_0$ with $C = 0.1 - 0.3$ at low density. The resultant spectral function in the $\sigma$ channel

\[\rho \] in the following corresponds to $n_B$ in previous sections.
is shown in Fig. 7 (left panel). The spectral enhancement just above the $2m_\pi$ threshold similar to the case at finite $T$ (Fig. 5(B)) can be seen when $\langle \sigma \rangle$ changes by 25% from its vacuum value. This mean-field treatment was later improved by including the $p$-$h$ and $\Delta$-$h$ contributions to the pion propagator \cite{44}. In this case, the spectral strength spreads into the energy region even below $2m_\pi$, but the qualitative feature of the enhancement at $2m_\pi$ still remains as shown in Fig. 7 (right panel) where $\alpha \equiv 1 - m_\sigma^*(\rho = \rho_0)/m_\sigma^*(\rho = 0)$.

FIG. 7. Left panel: The spectral function in the $\sigma$ channel for several values of $\Phi = \langle \sigma \rangle/\langle \sigma_0 \rangle$ with the same parameter set of the linear $\sigma$ model as Fig. 5. This figure is taken from \cite{43}. Right panel: The phenomenological spectral function in the $\sigma$ channel with the p-wave effects such as $p$-$h$ and $\Delta$-$h$ excitations. The solid line corresponds to the vacuum case and the dashed lines correspond to the in-medium case at $\rho = \rho_0$ with several values of $\alpha(\equiv 1 - m_\sigma^*(\rho = \rho_0)/m_\sigma^*(\rho = 0))$, \cite{44}.

One can also study the in-medium $\pi$-$\pi$ amplitude using the same model \cite{45}. A simple unitarized $\pi$-$\pi$ amplitude in the $I=J=0$ channel reads

\begin{equation}
T(s) = \frac{1}{T_{\text{tree}}^{-1}(s) - i\Theta(s)},
\end{equation}

where $T_{\text{tree}}(s)$ is the in-medium tree-level amplitude with the mean-field replacement \cite{21}, and $\Theta(s)$ is the phase space factor defined as \cite{15}

\begin{equation}
\Theta(s) = \theta(s - 4m_\pi^2) \frac{1}{16\pi} \sqrt{1 - \frac{4m_\pi^2}{s}}.
\end{equation}

In Fig. 8 (left panel), the in-medium $\pi$-$\pi$ cross section $\sigma_{\pi\pi}(s; \rho)$ is shown in the arbitrary unit, in which the same parameter sets as in Fig. 5 and Fig. 7 are used. Again a large enhancement of the cross section near the threshold is seen as the baryon density increases.

In Fig. 8 (right panel), shown is the $\sigma_{\pi\pi}(s; \rho)$ relative to its vacuum value defined as

\begin{equation}
\sigma_{\pi\pi}(s; \rho) = \frac{\sigma_{\pi\pi}(s; \rho)}{\sigma_{\pi\pi}(s; 0)}.
\end{equation}

Here we have changed the definition of $T(s)$ by factor 2 from that in \cite{15}.

\end{document}
\[ R = \frac{\sigma_{\pi\pi}(s; \rho)}{\sigma_{\pi\pi}(s; \rho = 0)} \quad (24) \]

A large enhancement near \(2m_\pi\) threshold in Fig. 8 can be understood as follows: As the partial restoration of chiral symmetry takes place and \(\langle \sigma \rangle\) decreases, \(m_\sigma^*\) approaches \(2m_\pi\), which implies that \(T_{\text{tree}}^{-1}(s \simeq 2m_\pi)\) tends to be suppressed. Hence the \(s\)-dependence of the full inverse amplitude \(T^{-1}(s)\) just above the \(2\pi\) threshold is governed by the imaginary part \(\Theta(s)/2\), which causes the near-threshold enhancement of \(T(s)\).

![Graph](https://example.com/graph.png)

**FIG. 8.** Left panel: In-medium \(\pi\pi\) cross section calculated in the linear \(\sigma\) model in the mean-field approximation [45]. The same parameter set with Fig.(6,7) are taken. Right panel: In-medium enhancement of the cross section relative to its vacuum value in the range \(2m_\pi \leq \sqrt{s} \leq 500\) MeV [45].

At this point, a natural question to ask is whether the near-threshold enhancement obtained in the \(O(4)\) linear \(\sigma\) model arises also in the non-linear models or in the models which do not have \(\sigma\) explicitly [45]. Furthermore, if it is the case, what kind of vertices in the non-linear chiral Lagrangian are responsible for the enhancement? To study these problems, it is best to start with the standard polar parameterization of the chiral field [45], \(M = \sigma + i\vec{\tau} \cdot \vec{\pi} = (\langle \sigma \rangle + S)U\) with \(U = \exp(i\vec{\tau} \cdot \vec{\phi}/f_\pi)\). Here, \(\langle \sigma \rangle\) is the chiral condensate in nuclear matter as before, while \(f_\pi^*\) is an “in-medium pion decay constant”. The nucleon field is also transformed from \(\psi\) to the chiral invariant field \(N\).

Now let us take a heavy-scalar (\(S\)) limit and heavy-baryon (\(N\)) limit simultaneously. These limits can be achieved by taking \(\lambda, g \to \infty\) with \(g/\lambda\) and \(\langle \sigma \rangle_0 = f_\pi^*\) being fixed. In this limit, the heavy scalar field \(S\) may be integrated out and the following effective Lagrangian is obtained:

\[
\mathcal{L}_{\text{eff}} = \left(\frac{f_\pi^2}{4} - \frac{gf_\pi}{2m_\sigma^2}\right)\bar{N}N \left(\text{Tr}[\partial U \partial U^\dagger] - \frac{h}{f_\pi} \text{Tr}[U^\dagger + U]\right) + \mathcal{L}_{\pi N}^{(1)} + \cdots ,
\]

where all the constants take their vacuum values: \(f_\pi = \langle \sigma \rangle_0\), \(m_\sigma^2 = \lambda\langle\sigma\rangle_0^2/3 + m_\pi^2\), and \(m_N = g\langle\sigma\rangle_0\). Note that \(gf_\pi/2m_\sigma^2\) in front of \(\bar{N}N\) approaches to a finite value \(3g/2\lambda f_\pi\) in the heavy limit, thus it cannot be neglected. In (25), \(\mathcal{L}_{\pi N}^{(1)}\) is the standard \(p\)-wave \(\pi-N\) coupling and \(\cdots\) denotes other higher dimensional operators which are not relevant for the present discussion.
In the uniform nuclear matter, $\bar{N}N$ in eq.(25) may be replaced by $\rho$. This leads to a reduction of the vacuum condensate; $f_\pi = \langle \sigma \rangle_0 \rightarrow \langle \sigma \rangle_0 (1 - g\rho/f_\pi m_N^2) = f_\pi^*$. Then the proper normalization of the pion field in nuclear matter should be $\phi' = (\phi/f_\pi) \cdot f_\pi^*$. Thus, the origin of the near-threshold enhancement in the heavy limit can be ascribed to the following new vertex:

$$\mathcal{L}_{\text{new}} = -\frac{3g}{2\lambda f_\pi} \bar{N}N\text{Tr}[\partial U\partial U^\dagger].$$

(26)

Because this vertex is proportional to the scalar-isoscalar density of the nucleon, it affects not only the pion propagator but also the interaction among pions in nuclear matter. In Fig.9, 4$\pi$-$N$-$N$ vertex generated by $\mathcal{L}_{\text{new}}$ is shown as an example. Note that the vertex in Fig.9 acts to enhance the $\pi\pi$ attraction in the $I=J=0$ channel in nuclear matter. Despite its important role, this vertex has not been considered so far in the calculations of the $\pi\pi$ scattering amplitudes in nuclear matter in the non-linear approaches.

FIG. 9. An example of the new 4$\pi$-$N$-$N$ vertex generated by $\mathcal{L}_{\text{new}}$. The solid line with arrow and the dashed line represent the nucleon and the pion, respectively.

Here it should be pointed out that the vertex eq.(26) has been known to be one of the next-to-leading order terms in the non-linear chiral Lagrangian in the heavy-baryon formalism [46]. In fact, Thorsson and Wirzba [47] have derived a general in-medium chiral Lagrangian by taking the mean-field approximation for the nucleon field:

$$\langle \mathcal{L} \rangle = \left(\frac{f_\pi^2}{4} + \frac{c_3}{2}\rho \right) \text{Tr}[\partial U \partial U^\dagger] + \left(\frac{c_2}{2} - \frac{g_A^2}{16m_N} \right) \rho \text{Tr}[\partial_0 U \partial_0 U^\dagger]$$

$$+ \left(\frac{f_\pi^2}{4} + \frac{c_1}{2}\rho \right) \text{Tr}(U^\dagger \chi + \chi^\dagger U).$$

(27)

Note that $\mathcal{L}_{\pi N}^{(1)}$ disappears in (27) in the mean-field level because of the isospin symmetry in nuclear matter, although it may contribute beyond the mean-field approximation as pointed out in [48]. As is shown in [46], the coefficients $c_{1,2,3}$ are related to the $\pi N$ sigma term, the isospin even $S$-wave $\pi N$ scattering length, and the nucleon axial polarizability, respectively. We just quote here the numbers although they have large potential uncertainties: $c_1 = -0.87\text{GeV}^{-1}$, $c_2 = 3.34\text{GeV}^{-1}$, and $c_3 = -5.25\text{GeV}^{-1}$.

By comparing eq.(23) with eq.(27), we observe that the new vertex of the form $\bar{N}N\text{Tr}[\partial U\partial U^\dagger]$ appear in both cases with the same sign. In eq.(23), we have only the Lorentz invariant terms because we have integrated out only the scalar meson $S$. On the other hand, eq.(27) contains more general $O(3)$ invariant terms because it potentially contains the effect of heavy excited baryons [46]. It is of great importance to make an extensive analysis of the $\pi\pi$ interaction in nuclear matter with the new 4$\pi$-$N$-$N$ vertex.
C. Behavior of the σ pole in 2nd Riemann sheet

We emphasize here the relation of the near-threshold enhancement and the softening of the fluctuating mode in the σ channel in nuclear matter. We shall show that such a fluctuating mode can be characterized by a complex pole of the unitarized amplitude $T(s)$ in eq.(22).

To see this in a simplest way, let us first take the heavy scalar limit ($m_\sigma \to \infty$) and the chiral limit ($m_\pi \to 0$). In this case, the $\pi\pi$ correlation is generated only by the standard lowest order chiral amplitude in eq.(22) with $T_{\text{tree}}(s) = s/f_{\pi}^2$. Then one readily finds a pole of $T(s)$ in the lower half plane as

$$\sqrt{s_{\text{pole}}} = \sqrt{8\pi \langle \sigma \rangle } (1 - i),$$

$$= (466 - 466i) \langle \sigma \rangle_0 \text{ MeV.} \quad (28)$$

This complex pole represents a broad damping-mode dynamically generated by the $\pi\pi$ multiple scattering in the $I=J=0$ channel. Since $\langle \sigma \rangle$ decreases in nuclear matter, the pole moves toward the origin in the complex $\sqrt{s}$-plane and thus causes an enhancement in the $\pi\pi$ cross section as shown in Fig. 8. The existence of such softening in the complex plane and its connection with the near-threshold enhancement of the spectral function was first shown in [31,33] using the Nambu-Jona-Lasinio model.

We shall now see that these qualitative features, i.e., the existence and the softening of a complex pole do not change in the more realistic case with the real part of the rescattering amplitude included properly. We start again with the non-linear model with $m_\sigma \to \infty$. The $T$ matrix resumed in the inverse amplitude method [50] reads

$$T(s) = T_2/(1 + T_2 g(s)), \quad (29)$$

where $T_2$ is the lowest order chiral amplitude (= $s/f_{\pi}^2$ in the chiral limit) and $g(s)$ is the pion loop integral;

$$g(s) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_\pi^2 + i\epsilon} \frac{1}{q^2 - m_\pi^2 + i\epsilon}, \quad (30)$$

with $q_\pm = q \pm p/2$ and $s = p^2$. We notice that $\text{Im}g(s) = -\Theta(s)$, which is given in (23). The real part of $g(s)$ is logarithmically divergent, so one needs a renormalization or use the one-subtracted dispersion relation, which introduce an arbitrary constant. Actually, this program can be nicely formulated in the $N/D$ method with the dimensional regularization [15]. However, as is also shown in [15], the resultant formula can be obtained by a simple cut-off method provided that the cut-off value is properly chosen. The result in the chiral limit is given by [15,49]

$$g(z) = \frac{1}{16\pi^2} \{ \tilde{a}^{SL} - \log(-\mu^2/z) \}, \quad (31)$$

for $z$ in the first sheet, and

$$g(z) = \frac{1}{16\pi^2} \{ \tilde{a}^{SL} - \log(-\mu^2/z) + 2\pi i \}, \quad (32)$$
for \( z \) in the second sheet (\( \text{Im} \, z < 0 \)). Here \( \text{Log} \, z \) is the principal value of the logarithm; it has a cut along the real negative axis and \(-\pi < \text{Arg}[\text{Log} \, z] < \pi \). The constant \( \tilde{a}^{SL} \) is a cut-off dependent constant and is determined so that the empirical phase shift in the scalar channel is reproduced \( [51] \): \( \tilde{a}^{SL} \simeq -0.75 \pm 0.2 \) with \( \mu = 760 \) MeV. The pole position is given by the following dispersion equation

\[
T_2^{-1}(z) = -g(z). \quad (33)
\]

The result is shown in Fig.5: When \( f_\pi = 93 \) MeV, the pole exists at \( \sqrt{s} = 461 - 341i \) MeV. As \( f_\pi = \langle \sigma \rangle \) is decreased, the pole moves toward the origin. It means that the mass and the width of the \( \sigma \) mode decreases as the chiral symmetry is restored; thus we confirm that the \( \sigma \) is the soft mode of the chiral transition.

![Graph showing the shift of the pole position of the T matrix in the chiral limit as \( f_\pi = \langle \sigma \rangle \) decreases toward 0.]

**V. POSSIBLE EXPERIMENTAL SIGNATURE**

A. Softening in the scalar channel

The CHAOS collaboration measured the \( \pi^+ \pi^\pm \) invariant mass distribution \( M^A_{\pi^+ \pi^\pm} \) in the reaction \( A(\pi^+, \pi^+ \pi^\pm)A' \) with the incident pion momentum \( p_{\pi^+} = 399 \) MeV and the targets \( A=^2\text{H}, ^{12}\text{C}, ^{40}\text{Ca}, ^{208}\text{Pb} \) \( [52] \). They observed that the yield for \( M^A_{\pi^+ \pi^-} \) near the \( 2m_\pi \) threshold is close to zero for \( A = 2 \), but increases with increasing \( A \). They identified that the \( \pi^+ \pi^- \) pairs in this range of \( M^A_{\pi^+ \pi^-} \) is in the \( I=J=0 \) state.

Although this experiment was originally motivated by the possible strong \( \pi-\pi \) correlation in nuclei due to many-body effects \( [53] \), state of the art calculations based on the conventional many-body theoretical approach without incorporating the effect of partial restoration of chiral symmetry do not reproduce the sufficient enhancement in the near-threshold region in the \( I=J=0 \) channel \( [54] \). However, once the effect of partial chiral restoration in nuclei suggested in \( [43] \) is incorporated together with the conventional approach, the near threshold enhancement enough to explain the experimental data may be obtained \( [44] \). So far such an analysis is still in the qualitative level. Quantitative calculations based on both the linear and non-linear approach discussed in the previous section are called for.
For the experimental confirmation of the threshold enhancement seen in [52], measurements of $2\pi^0$ and $2\gamma$ final states with hadron/photon beams off the heavy nuclear targets are necessary. Those channels are free from the $\rho$ meson background inherent in the $\pi^+\pi^-$ measurement. One of such experiments measuring $2\pi^0$ from the reaction $A(\pi^-, \pi^0\pi^0)A'$ with $p_{\pi^-} = 408$ MeV and $A=H, ^2H, C, Al, Cu$ has been done by the Crystal Ball Collaboration [55]. They report some qualitative difference of the $\pi-\pi$ invariant mass distribution from the CHAOS data. For making a better comparison between the data from different detectors with different acceptance, the following “combined ratio” is useful [56],

$$C_{\pi\pi}^A(M_{\pi\pi}) = \frac{\sigma^A(M_{\pi\pi})/\sigma^A_T}{\sigma^N(M_{\pi\pi})/\sigma^N_T},$$

(34)

where $\sigma^A_T (\sigma^N_T)$ is the measured total cross section of the $(\pi, 2\pi)$ process in nuclei (nucleon): This ratio represents the net effect of nuclear matter on the interacting $(\pi\pi)_{I=J=0}$ system. As shown in Fig.11, it has recently been argued that both the CB data and the CHAOS data are consistent and gives an enhancement near the $2m_\pi$ threshold [57], although the statistics in the CB data is relatively poor. For more detailed discussions, see the contributions to this Proceedings from CHAOS and CB groups.

![Graph](image)

**FIG. 11.** The composite ration as a function of the $\pi-\pi$ invariant mass for the CB data (filled diamonds) and CHAOS data (open diamonds). The target is $^{12}$C in both cases. This figure is taken from [57].

We list some of the possible experiments related to the in-medium $\sigma$ in the following.

- Measuring of $2\gamma$’s from the electro-magnetic decay of the $\sigma$ or $(\pi\pi)_{I=J=0}$ in nuclear matter, although the branching ratio is small, is an interesting alternatives to the $2\pi$ measurement because of the small final state interactions. In such an experiment, one needs to fight with large background of photons mainly coming from $\pi^0$s. Nevertheless, if the enhancement is prominent, there is a chance to find the signal.
• When $\sigma$ has a finite three momentum, one can also detect dileptons through the scalar-vector mixing in matter owing to the violation of the charge conjugation symmetry at finite density \(^{25}\); $\sigma \rightarrow \gamma^* \rightarrow e^+e^-$. (See also, \(^{28}\).)

• The inverse process $\gamma + A \rightarrow 2\pi + X$ can be also used to produce the $\sigma$ or $(\pi\pi)_{I=J=0}$ system by the electro-magnetic probes. Such experiments with electro-magnetic probes have been planned and are being performed in SPring-8 \(^{59}\).

• We remark that the nuclear reactions such as (d, \(^3\)He) and (d, t) are also useful to explore the spectral enhancement and/or the formation of the $\sigma$-meson nuclei \(^{10}\) as in the case of the deeply bound pionic nuclei and the possible production of $\eta$- or $\omega$-mesic nuclei \(^{51}\). The incident kinetic energy $E$ of the deuteron in the laboratory system is estimated to be $1.1\text{GeV} < E < 10 \text{ GeV}$, to cover the spectral function in the range $2m_\pi < \omega < 750 \text{ MeV}$.

B. Medium effect in other channels

The dilepton spectrum measured in the heavy ion collisions such as Pb-Au collisions both at high energy (158 GeV/A) and at low energy (40 GeV/A) show a sizable enhancement of the $e^+e^-$ yield below the $\rho$-meson peak \(^{62}\). This may or may not be related to the partial chiral restoration in nuclear medium originally proposed in \(^{63}\) (for the recent review, see \(^{64}\)). In Fig.12, the recent CERES/NA45 data for 40 GeV/A collision are shown. The E325 experiment at KEK \(^{57}\) measured $e^+e^-$ pairs from the p-A collision at 12 GeV. The similar enhancement over the known source and combinatorial background as CERES is seen in the mass range of about 200 MeV below the $\rho$-$\omega$ peak for $A = Cu$ (Fig.13).

The deeply bound pionic atom has proved to be a good probe of the properties of the hadronic interaction deep inside of heavy nuclei \(^{66}\). It is suggested that the anomalous energy shift of the pionic atoms (pionic nuclei) owing to the strong interaction could be attributed to the decrease of the effective pion decay constant $f_\pi^*(\rho)$ at finite density $\rho$ which may imply that the chiral symmetry is partially restored deep inside the nuclei \(^{66,57}\).
FIG. 12. Normalized $e^+e^-$ invariant mass spectrum in the Pb-Au collision at 40 A GeV. Each contribution from the known hadronic sources are shown by thin solid lines, and their sum is shown by thick solid line. This figure is taken from the second reference in [62].

FIG. 13. $e^+e^-$ invariant mass spectrum in the $p$-$A$ reaction. (a) is for the C and polyethylene targets and (b) is for the Cu target. The solid lines in both figures are the result of the best-fit including known dilepton sources and the combinatorial background. Excess over the solid line is seen below the $\rho$-$\omega$ peak in the case (b).

VI. SUMMARY AND CONCLUDING REMARKS

This report may be summarized as follows.

The QCD phase diagram should be considered in $m_i$-$T$-$n_B$ space where $m_i$ ($i = u, d, s$) being the current quark mass, $T$ ($n_B$) is the temperature (baryon density).
extensive studies based on the lattice QCD simulations and effective theories of QCD indeed shows it quite probable that the chiral phase transition takes place for $T_c \simeq 170 \text{MeV}$ and $n_B \sim \text{several} \times n_0$, although the precise order of the transition in the real world is not clear yet.

The order parameter of the chiral symmetry breaking is $\langle \bar{q} q \rangle$. Furthermore, its amplitude and phase fluctuations are the ur$_\sigma$ and the pion, respectively, which play a central role in hadronic world. They mix strongly through the process $\sigma \leftrightarrow 2\pi$, which may account for the fact that the physical $\sigma$ is elusive in the real world. Irrespective of the introduction of the ur-$\sigma$, it has become more and more plausible in recent years that the strong $\pi\pi$ attraction produces a light $\sigma$ pole in the 2nd sheet of the $\pi\pi$ scattering amplitude in the $I=J=0$ channel. Constraints from chiral symmetry, analyticity and crossing symmetry are key ingredients for the confirmation of the light $\sigma$-pole. Such a scalar-isoscalar fluctuation has also phenomenological importance in modern hadron physics.

If $T$ and/or $n_B$ of the system increases, the intimate connection between the QCD vacuum structure and the fluctuation modes becomes even more apparent. In fact, the light $\sigma$ meson acts as a soft mode and the complex $\sigma$-pole in the second sheet has characteristic shift toward the real axis in association with the partial restoration of chiral symmetry $\langle \bar{q} q \rangle \rightarrow 0$.

In cold nuclear matter, a sizable decrease of the quark condensate, as large as 30 % at normal nuclear matter density, may take place according to the low-density expansion of $\langle \bar{q} q \rangle$. If this is the case, probing the scalar-isoscalar fluctuation in cold nuclear matter using heavy nuclei is promising for exploring the partial restoration of chiral symmetry. The softening of the complex $\sigma$-pole leads to a strong enhancement of the spectral function and the $\pi\pi$ cross section in the $I=J=0$ channel near the $2m_\pi$ threshold. This phenomena can be explicitly demonstrated e.g. in the linear and nonlinear realization of chiral symmetry provided that the possible reduction of the quark condensate or $f_\pi$ is taken into account.

The possible relevance of the above theoretical considerations and the $(\pi^+\pi^-)_{I=J=0}$ data (the CHAOS collaboration) and the $(\pi^0\pi^0)$ data (the Crystal Ball collaboration) is suggestive and should be studied further. Independent tests using the electro-magnetic probes as well as the mesic-nuclei will be also useful to unravel the realization of chiral symmetry in hadronic matter.

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