ADDITIVE-QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES
IN FUZZY NORMED SPACES

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Abstract. Let
\[ M_1 f(x, y) := \frac{3}{4} f(x + y) - \frac{1}{4} f(-x - y) + \frac{1}{4} f(x - y) + \frac{1}{4} f(y - x) - f(x) - f(y), \]
\[ M_2 f(x, y) := 2 f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) - f(x) - f(y). \]

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequalities
\[ N(M_1 f(x, y), t) \geq N(\rho M_2 f(x, y), t) \tag{0.1} \]
where $\rho$ is a fixed real number with $|\rho| < 1$, and
\[ N(M_2 f(x, y), t) \geq N(\rho M_1 f(x, y), t) \tag{0.2} \]
where $\rho$ is a fixed real number with $|\rho| < \frac{1}{2}$.

1. Introduction and Preliminaries

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 16, 38]. In particular, Bag and Samanta [3], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 19, 20] to investigate the Hyers-Ulam stability of additive $\rho$-functional inequalities in fuzzy Banach spaces.
Definition 1.1 ([3, 19, 20, 21]). Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

1. $N(x, t) = 0$ for $t \leq 0$;
2. $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
3. $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
4. $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
5. $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.
6. for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed vector space.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [15, 19].

Definition 1.2 ([3, 19, 20, 21]). Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n \to \infty} x_n = x$.

Definition 1.3 ([3, 19, 20, 21]). Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$ (see [4]).

The stability problem of functional equations originated from a question of Ulam [37] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [2] for additive mappings and by Rassias [29] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the
unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

The functional equation \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a \textit{quadratic mapping}. The stability of quadratic functional equation was proved by Skof [36] for mappings \( f : E_1 \to E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 6, 7, 10, 17, 18, 22, 25, 26, 27, 30, 31, 32, 33, 34, 35, 39, 40]).

Park [23, 24] defined additive \( \rho \)-functional inequalities and proved the Hyers-Ulam stability of the additive \( \rho \)-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that \( X \) is a real vector space and \( (Y, N) \) is a fuzzy Banach space.

2. ADDITVE-QUADRATIC \( \rho \)-FUNCTIONAL INEQUALITY (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.1) in fuzzy Banach spaces. Let \( \rho \) be a real number with \(|\rho| \leq 1\).

We need the following lemma to prove the main results.

\textbf{Lemma 2.1.}

(i) If an odd mapping \( f : X \to Y \) satisfies

\begin{equation}
N(M_1 f(x, y), t) \geq N(\rho M_2 f(x, y), t)
\end{equation}

for all \( x, y \in X \) and all \( t > 0 \), then \( f \) is the Cauchy additive mapping.

(ii) If an even mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and (2.1), then \( f \) is the quadratic mapping.

\textit{Proof.} (i) Letting \( y = x \) in (2.1), we get \( N(f(2x) - 2f(x), t) = 1 \) for all \( t > 0 \) and so
\( f(2x) = 2f(x) \) for all \( x \in X \). Thus

\[
\frac{f(x)}{2} = \frac{1}{2}f(x)
\]

for all \( x \in X \).

It follows from (2.1) and (2.2) that

\[
N(f(x + y) - f(x) - f(y), t) = N(\rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right), t)
\]

for all \( t > 0 \) and so

\[
f(x + y) = f(x) + f(y)
\]

for all \( x, y \in X \) by \( (N_3) \).

(ii) Letting \( y = x \) in (2.1), we get \( N \left( \frac{1}{2}f(2x) - 2f(x), t \right) = 1 \) for all \( t > 0 \) and so \( f(2x) = 4f(x) \) for all \( x \in X \). Thus

\[
\frac{f(x)}{4} = \frac{1}{4}f(x)
\]

for all \( x \in X \).

It follows from (2.1) and (2.3) that

\[
N \left( \frac{1}{2}f(x + y) + \frac{1}{2}f(x - y) - f(x) - f(y), t \right)
\]

\[
= N \left( \rho \left( 2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y) \right), t \right)
\]

\[
= N \left( \rho \left( \frac{1}{2}f(x + y) + \frac{1}{2}f(x - y) - f(x) - f(y) \right), t \right)
\]

for all \( t > 0 \) and so

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

for all \( x, y \in X \) by \( (N_3) \). \( \square \)

**Theorem 2.2.** Let \( \varphi : \mathbb{X}^2 \to [0, \infty) \) be a function such that

\[
\sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty
\]

for all \( x, y \in \mathbb{X} \).

(i) Let \( f : \mathbb{X} \to \mathbb{Y} \) be an odd mapping satisfying

\[
N(M_1 f(x, y), t) \geq \min \left\{ N(\rho M_2 f(x, y), t), \frac{t}{t + \varphi(x, y)} \right\}
\]

for all \( x, y \in \mathbb{X} \).
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := N \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N \left( f(x) - A(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Psi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \), where \( \Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.5). Then \( Q(x) := N \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N \left( f(x) - Q(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Phi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \) for all \( x, y \in X \).

Proof. (i) Letting \( y = x \) in (2.5), we get

\[
N \left( f(2x) - 2f(x), t \right) \geq \frac{t}{t + \varphi(x, x)}
\]

and so

\[
N \left( f \left( \frac{x}{2} \right) - 2f(x), t \right) \geq \frac{t}{t + \varphi \left( \frac{x}{2}, \frac{x}{2} \right)}
\]

for all \( x \in X \). Hence

\[ \tag{2.9} \]

\[
N \left( 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right) \geq \min \left\{ N \left( 2^l f \left( \frac{x}{2^l} \right) - 2^{l+1} f \left( \frac{x}{2^{l+1}} \right), t \right), \ldots, \right.
\]

\[
\cdots, N \left( 2^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right) \left. \right\}
\]

\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 2f \left( \frac{x}{2^{l+1}} \right), t \right), \ldots, N \left( f \left( \frac{x}{2^m} \right) - 2f \left( \frac{x}{2^{m-1}} \right), \frac{t}{2^{m-1}} \right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{2^l + \varphi \left( \frac{x}{2^l+1}, \frac{x}{2^{l+1}} \right)}, \ldots, \frac{t}{2^m + \varphi \left( \frac{x}{2^m}, \frac{x}{2^m} \right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 2^l \varphi \left( \frac{x}{2^l+1}, \frac{x}{2^{l+1}} \right)}, \ldots, \frac{t}{t + 2^m - \varphi \left( \frac{x}{2^m}, \frac{x}{2^m} \right)} \right\}
\]

\[
\geq \frac{t}{t + \frac{1}{2} \sum_{j=l+1}^{m} 2^j \varphi \left( \frac{x}{2^j}, \frac{x}{2^j} \right)}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \) and all \( t > 0 \). It follows from (2.4) and (2.9) that the sequence \( \{2^n f \left( \frac{x}{2^n} \right)\} \) is a Cauchy sequence for all
Let $x \in X$. Since $Y$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.6).

By (2.5),

$$N \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \right) - \rho \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) \right), 2^n t) \geq \frac{t}{t + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \right) - \rho \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) \right), t) \geq \frac{t}{\frac{t}{2^n} + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$A(x+y) - A(x) - A(y) = \rho \left( 2A \left( \frac{x+y}{2} \right) - A(x) - A(y) \right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = x$ in (2.5), we get

$$N \left( \frac{1}{2} f(2x) - 2f(x), t \right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.10}$$

and so

$$N \left( f(x) - 4f \left( \frac{x}{2} \right), t \right) \geq \frac{t}{\frac{t}{2} + \varphi \left( \frac{x}{2}, \frac{x}{2} \right)} = \frac{t}{t + 2 \varphi \left( \frac{x}{2}, \frac{x}{2} \right)}$$

for all $x \in X$. Hence
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\[
N \left( 4^l f \left( \frac{x}{2^l} \right) - 4^m f \left( \frac{x}{2^m} \right), t \right) \geq \min \left\{ \begin{array}{c}
N \left( 4^l f \left( \frac{x}{2^l} \right) - 4^{l+1} f \left( \frac{x}{2^{l+1}} \right), t \right), \\
\cdots, N \left( 4^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 4^m f \left( \frac{x}{2^m} \right), t \right) \end{array} \right\}
\]

\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 4f \left( \frac{x}{2^{l+1}} \right), \frac{t}{4^l} \right), \cdots, N \left( f \left( \frac{x}{2^{m-1}} \right) - 4f \left( \frac{x}{2^m} \right), \frac{t}{4^{m-1}} \right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{4^l + 2\varphi \left( \frac{x}{2^l}, \frac{x}{2^{l+1}} \right)}, \cdots, \frac{t}{4^{m-1} + 2\varphi \left( \frac{x}{2^{m-1}}, \frac{x}{2^m} \right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 2 \cdot 4^l \varphi \left( \frac{x}{2^l}, \frac{x}{2^{l+1}} \right)}, \cdots, \frac{t}{t + 2 \cdot 4^{m-1} \varphi \left( \frac{x}{2^{m-1}}, \frac{x}{2^m} \right)} \right\}
\]

\[
\geq \frac{t}{t + \frac{1}{2} \sum_{j=l+1}^{m} 4^j \varphi \left( \frac{x}{2^j}, \frac{x}{2^{j+1}} \right)}
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.4) and (2.11) that the sequence $\{4^n f \left( \frac{x}{2^n} \right)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f \left( \frac{x}{2^n} \right)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

\[
Q(x) := N-\lim_{n \rightarrow \infty} 4^n f \left( \frac{x}{2^n} \right)
\]

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.11), we get (2.7).

The rest of the proof is similar to the above additive case. \hfill \Box

**Corollary 2.3.** Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\| \cdot \|$. 

(i) Let $f : X \rightarrow Y$ be an odd mapping satisfying

\[
(2.12) \quad N \left( M_1 f(x, y), t \right) \geq \min \left\{ N \left( \rho M_2 f(x, y), t \right), \frac{t}{t + \theta \left( \| x \|^{p} + \| y \|^{p} \right)} \right\}
\]

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N-\lim_{n \rightarrow \infty} 2^n f \left( \frac{x}{2^n} \right)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

\[
N \left( f(x) - A(x), t \right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta \| x \|^{p}}
\]

for all $x \in X$ and all $t > 0$. 

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and \((2.12)\). Then \( Q(x) := N\lim_{n \to \infty} A^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N \left( f(x) - Q(x), t \right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 4\theta \|x\|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. The proof follows from Theorem 2.2 by taking \( \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

**Theorem 2.4.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that

\[
\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi \left( 2^j x, 2^j y \right) < \infty
\]

for all \( x, y \in X \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying \((2.5)\). Then

\[
A(x) := N - \lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right)
\]

exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N \left( f(x) - A(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Phi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi \left( 2^j x, 2^j y \right) \) for all \( x, y \in X \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and \((2.5)\). Then \( Q(x) := N\lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N \left( f(x) - Q(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Psi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \), where \( \Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi \left( 2^j x, 2^j y \right) \) for all \( x, y \in X \).

Proof. (i) It follows from \((2.8)\) that

\[
N \left( f(x) - \frac{1}{2} f(2x), \frac{1}{2} t \right) \geq \frac{t}{t + \varphi(x, x)}
\]

and so

\[
N \left( f(x) - \frac{1}{2} f(2x), t \right) \geq \frac{2t}{2t + \varphi(x, x)} = \frac{t}{t + \frac{1}{2} \varphi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \).
(ii) It follows from (2.10) that
\[ N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x)} \]
and so
\[ N\left(f(x) - \frac{1}{4}f(2x), t\right) \geq \frac{2t}{2t + \varphi(x,x)} = \frac{t}{t + \frac{1}{2}\varphi(x,x)} \]
for all \( x \in X \) and all \( t > 0 \).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Corollary 2.5. Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying (2.12). Then \( A(x) := N\lim_{n \to \infty} \frac{1}{2^n}f(2^n x) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[ N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta\|x\|^p} \]
for all \( x \in X \) and all \( t > 0 \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.12). Then \( Q(x) := N\lim_{n \to \infty} \frac{1}{2^n}f(2^n x) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that
\[ N\left(f(x) - Q(x), t\right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 4\theta\|x\|^p} \]
for all \( x \in X \) and all \( t > 0 \).

Proof. The proof follows from Theorem 2.4 by taking \( \varphi(x,y) := \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

3. Additive-Quadratic \( \rho \)-functional Inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.2) in fuzzy Banach spaces. Let \( \rho \) be a real number with \( |\rho| \leq \frac{1}{2} \).

Lemma 3.1.
(i) If an odd mapping \( f : X \to Y \) satisfies
\[ N(M_2 f(x,y), t) \geq N(\rho M_1 f(x,y), t) \]
for all \( x, y \in X \) and all \( t > 0 \), then \( f \) is the Cauchy additive mapping.
(ii) If an even mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and (3.1), then \( f \) is the quadratic mapping.

Proof. (i) Letting \( y = 0 \) in (3.1), we get \( N \left( 2f \left( \frac{x}{2} \right) - f(x), t \right) = 1 \) for all \( t > 0 \). So

\[
f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)
\]

for all \( x \in X \).

It follows from (3.1) and (3.2) that

\[
N(f(x + y) - f(x) - f(y), t) = N \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y), t \right)
\]

\[
= N(\rho(f(x + y) - f(x) - f(y)), t)
\]

for all \( t > 0 \) and so

\[
f(x + y) = f(x) + f(y)
\]

for all \( x, y \in X \) by \((N_5)\).

(ii) Letting \( y = 0 \) in (3.1), we get \( N(4f \left( \frac{x}{2} \right) - f(x), t) \) for all \( t > 0 \). So

\[
f \left( \frac{x}{2} \right) = \frac{1}{4} f(x)
\]

for all \( x \in X \).

It follows from (3.1) and (3.3) that

\[
N \left( \frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y), t \right)
\]

\[
= N \left( 2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y), t \right)
\]

\[
= N \left( \rho \left( \frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y) \right), t \right)
\]

for all \( t > 0 \) and so

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

for all \( x, y \in X \) by \((N_5)\). \(\square\)

**Theorem 3.2.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that

\[
\sum_{j=0}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty
\]

(3.4)

for all \( x, y \in X \).
(i) Let $f : X \to Y$ be an odd mapping satisfying

\[
N(M_2 f(x, y), t) \geq \min \left\{ N(\rho M_1 f(x, y), t), \frac{t}{t + \varphi(x, y)} \right\}
\]

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

\[
N(f(x) - A(x), t) \geq \frac{t}{t + \Phi(x, 0)}
\]

for all $x \in X$ and all $t > 0$, where $\Phi(x, y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$ for all $x, y \in X$.

(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then $Q(x) := N \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

\[
N(f(x) - Q(x), t) \geq \frac{t}{t + \Psi(x, 0)}
\]

for all $x \in X$ and all $t > 0$, where $\Psi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$ for all $x, y \in X$.

Proof. (i) Letting $y = 0$ in (3.5), we get

\[
N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)}
\]

for all $x \in X$. Hence

\[
N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)
\]

\[
\geq \min \left\{ N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \ldots, \right. \right.
\]

\[
\left. \left. \ldots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \right\}
\]

\[
= \min \left\{ N\left(f\left(\frac{x}{2^l}\right) - 2 f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \ldots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2 f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \ldots, \frac{t}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 2^l \varphi\left(\frac{x}{2^l}, 0\right)}, \ldots, \frac{t}{t + 2^{m-1} \varphi\left(\frac{x}{2^{m-1}}, 0\right)} \right\}
\]

\[
\geq \frac{t}{t + \sum_{j=1}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right)}
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.4) and (3.9) that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all
Since \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f(x/2^n)\} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{n \to \infty} 2^n f(x/2^n)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get (3.6).

By (3.5),

\[
N \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) - \rho \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \right), t \right) \geq \frac{t}{t + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}
\]

for all \( x, y \in X \), all \( t > 0 \) and all \( n \in \mathbb{N} \). So

\[
N \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) - \rho \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \right), t \right) \geq \frac{t}{t + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}\]

for all \( x, y \in X \), all \( t > 0 \) and all \( n \in \mathbb{N} \). Since \( \lim_{n \to \infty} \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)} = 1 \) for all \( x, y \in X \) and all \( t > 0 \),

\[
2A \left( \frac{x+y}{2} \right) - A(x) - A(y) = \rho \left( A(x+y) - A(x) - A(y) \right)
\]

for all \( x, y \in X \). By Lemma 3.1, the mapping \( A : X \to Y \) is Cauchy additive.

(ii) Letting \( y = 0 \) in (3.1), we get

\[
(3.10) \quad N \left( f(x) - 4f \left( \frac{x}{2} \right), t \right) = N \left( 4f \left( \frac{x}{2} \right) - f(x), t \right) \geq \frac{t}{t + \varphi(x,0)}
\]

for all \( x \in X \). Hence

\[
(3.11) \quad N \left( 4^i f \left( \frac{x}{2^i} \right) - 4^i f \left( \frac{x}{2^i} \right), t \right)
\]
\[
\geq \min \left\{ N \left( 4^l f \left( \frac{x}{2^l} \right) - 4^{l+1} f \left( \frac{x}{2^{l+1}} \right) ; t \right), \ldots , N \left( 4^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 4^m f \left( \frac{x}{2^m} \right) ; t \right) \right\}
\]

\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 4 f \left( \frac{x}{2^{l+1}} \right) ; \frac{t}{4^l} \right), \ldots , N \left( f \left( \frac{x}{2^{m-1}} \right) - 4 f \left( \frac{x}{2^m} \right) ; \frac{t}{4^{m-1}} \right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{4^l + \varphi \left( \frac{x}{2^l}, 0 \right)}, \ldots , \frac{t}{4^{m-1} + \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 4^l \varphi \left( \frac{x}{2^l}, 0 \right)}, \ldots , \frac{t}{t + 4^{m-1} \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]

\[
\geq \frac{t}{t + \sum_{j=l}^{m-1} 4^j \varphi \left( \frac{x}{2^j}, 0 \right)}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \) and all \( t > 0 \). It follows from (3.4) and (3.11) that the sequence \( \left\{ 4^n f \left( \frac{x}{2^n} \right) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ 4^n f \left( \frac{x}{2^n} \right) \right\} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := N_{\lim_{n \to \infty}} 4^n f \left( \frac{x}{2^n} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.11), we get (3.7).

The rest of the proof is similar to the above additive case. \( \square \)

**Corollary 3.3.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying

\[
(3.12) \quad N \left( M_2 f(x, y), t \right) \geq \min \left\{ N \left( \rho M_1 f(x, y), t \right), \frac{t}{t + \theta \left( \| x \|^p + \| y \|^p \right)} \right\}
\]

for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := N_{\lim_{n \to \infty}} 2^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N \left( f(x) - A(x), t \right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p \theta \| x \|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (3.12). Then \( Q(x) := N_{\lim_{n \to \infty}} 4^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping.
Q : X → Y such that

\[ N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p \theta \|x\|^p} \]

for all x ∈ X and all t > 0.

**Proof.** The proof follows from Theorem 3.2 by taking ϕ(x, y) := θ(∥x∥^p + ∥y∥^p) for all x, y ∈ X, as desired. □

**Theorem 3.4.** Let ϕ : X^2 → [0, ∞) be a function such that

\[ \sum_{j=1}^{∞} \frac{1}{2^j} ϕ (2^j x, 2^j y) < ∞ \]

for all x, y ∈ X.

(i) Let f : X → Y be an odd mapping satisfying (3.5). Then A(x) := N-lim_n→∞ (1/2^p) f (2^n x) exists for each x ∈ X and defines an additive mapping A : X → Y such that

\[ N(f(x) - A(x), t) \geq \frac{t}{t + Φ(x, 0)} \]

for all x ∈ X and all t > 0, where Φ(x, y) := ∑_{j=1}^{∞} \frac{1}{2^j} ϕ (2^j x, 2^j y) for all x, y ∈ X.

(ii) Let f : X → Y be an even mapping satisfying f(0) = 0 and (3.5). Then Q(x) := N-lim_n→∞ (1/4^p) f (2^n x) exists for each x ∈ X and defines a quadratic mapping Q : X → Y such that

\[ N(f(x) - Q(x), t) \geq \frac{t}{t + Ψ(x, 0)} \]

for all x ∈ X and all t > 0, where Ψ(x, y) := ∑_{j=1}^{∞} \frac{1}{4^j} ϕ (2^j x, 2^j y) for all x, y ∈ X.

**Proof.** (i) It follows from (3.8) that

\[ N\left(f(x) - \frac{1}{2} f(2x), \frac{t}{2}\right) \geq \frac{t}{t + ϕ(2x, 0)} \]

and so

\[ N\left(f(x) - \frac{1}{2} f(2x), t\right) \geq \frac{2t}{2t + ϕ(2x, 0)} = \frac{t}{t + \frac{1}{2} ϕ(2x, 0)} \]

for all x ∈ X and all t > 0.

(ii) It follows from (3.10) that

\[ N\left(f(x) - \frac{1}{4} f(2x), \frac{t}{4}\right) \geq \frac{t}{t + ϕ(2x, 0)} \]

and so

\[ N\left(f(x) - \frac{1}{4} f(2x), t\right) \geq \frac{4t}{4t + ϕ(2x, 0)} = \frac{t}{t + \frac{1}{4} ϕ(2x, 0)} \]
for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Corollary 3.5.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\| \cdot \|$. 

(i) Let $f : X \to Y$ be an odd mapping satisfying (3.12). Then $A(x) := N \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N (f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p \theta \|x\|^p}$$

for all $x \in X$.

(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.12). Then $Q(x) := N \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N (f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p \theta \|x\|^p}$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

**ACKNOWLEDGMENTS**

S. Yun was supported by Hanshin University Research Grant.

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