On the functional equation $f(p(z)) = g(q(z))$, where $p, q$ are “generalized” polynomials and $f, g$ are meromorphic functions

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Abstract

We find all the solutions to the equation $f(p(z)) = g(q(z))$, where $p, q$ are polynomials and $f, g$ are transcendental meromorphic functions in $\mathbb{C}$. In fact, a more general algebraic problem is solved.

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Introduction

0.1 Motivation

This paper and the previous one [3] were partially motivated by the following result by L. Flatto [1]:
Let \( p, q \in \mathbb{C}[z] \) be polynomials of equal degree. Let

\[
f \circ p = g \circ q
\]

where \( f \) and \( g \) are nonconstant entire functions on \( \mathbb{C} \). Then one of the following is true:

(i) \( p(z) = \lambda q(z) + a \), with \( \lambda, a \in \mathbb{C} \);

(ii) \( p(z) = r(z)^2 + a, q(z) = br(z)^2 + cr(z) + d \), where \( r \) is a polynomial in \( z \) and \( a, b, c, d, \in \mathbb{C}, b \neq 0 \).

L. Flatto asked [1] question 5 whether there is an analog of his theorem if \( \deg p \neq \deg q \). One can also ask what happens if \( f \) and \( g \) are not entire but meromorphic functions (on the whole \( \mathbb{C} \) or only in neighborhood of infinity). Partial results related to Flatto’s question were obtained in [1], [2], [12], [3] ([2] contains a survey of most of these results). The goal of this paper is to describe all pairs \( p, q \) for which there exist nonconstant meromorphic \( f \) and \( g \) satisfying (i) and there exist no rational \( f \) and \( g \) with this property (actually we consider a more general problem; see [1] and Section [1]).

Our interest to equation (1) is also motivated by its relation to the following problem which seems interesting: describe equivalence relations \( R \) on \( \mathbb{C} \) such that 1) \( R \) considered as a subset of \( \mathbb{C}^2 \) is a union of a sequence of algebraic curves, 2) there exists a nonconstant meromorphic function on \( \mathbb{C} \) whose restriction to each equivalence class of \( R \) is constant.

Such equivalence relations can be considered as generalizations of discrete subgroups of biholomorphic automorphisms of \( \mathbb{C} \) (a discrete subgroup \( \Gamma \) defines the following equivalence relation : \( z \sim w \) iff \( z = \gamma(w) \) for some \( \gamma \in \Gamma \); clearly, this equivalence relation satisfies conditions 1) and 2)). Notice that a solution to (1) is to describe all pairs \( p, q \) for which there exist nonconstant meromorphic \( f \) and \( g \) satisfying (i) and there exist no rational \( f \) and \( g \) with this property (actually we consider a more general problem; see [1] and Section [1]).

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### 0.2 Formulation of the problem

Let \((X, \infty_X), (Y, \infty_Y)\) be compact Riemann surfaces with marked points \( \infty_X \in X, \infty_Y \in Y \). By abuse of notation we write \( \infty \) instead of \( \infty_X \) and \( \infty_Y \). Let \( p : (X, \infty) \to (Y, \infty) \) be a holomorphic map. We say that \( p \) is a generalized polynomial if \( p^{-1}(\{\infty\}) = \{\infty\} \).

We will study equation (1), where \( p : (X, \infty_X) \to (Y, \infty_Y) \) and \( q : (X, \infty_X) \to (Z, \infty_Z) \) are generalized polynomials and \( f, g \) are meromorphic functions in punctured neighborhoods of \( \infty_Y \) and \( \infty_Z \) respectively. By rational function on a compact Riemann surface we shall mean a meromorphic function on it (this agrees with the usage of the term “rational function” in algebraic geometry). It is required to find all pairs \( p, q \) such that there exists nonconstant \( f, g \) satisfying (1) and there exist no rational \( f, g \) with this property. In fact, we solve a more general algebraic problem, which is explained in Section 1.
0.3 Main result

There are several standard solutions to (1).

Example 1 Let \( p(z) = z^n \), \( q(z) = (z + 1)^m \) with \( n, m, \text{lcm}(n, m) \in \{2, 3, 4, 6\} \). Then there exist nonconstant functions \( f, g \) meromorphic in \( \mathbb{C} \) and satisfying (4). There exist no rational \( f, g \) with this property.

Remark 1 Suppose we are given a diagram of generalized polynomials

\[
\begin{array}{ccc}
(X, \infty) & \xrightarrow{p} & (Z, \infty) \\
\downarrow & & \downarrow q \\
(Y, \infty) & & 
\end{array}
\]

such that \( \gcd(\deg p, \deg q) = 1 \). Then there exists a diagram of generalized polynomials

\[
\begin{array}{ccc}
(W, \infty) & \xrightarrow{p_1} & (Z, \infty) \\
\downarrow q_1 & & \downarrow q \\
(X, \infty) & \xrightarrow{p} & (Y, \infty) \\
\end{array}
\]

such that \( \deg p_1 = \deg p \), \( \deg q_1 = \deg q \). It is unique up to canonical isomorphism. In fact, \( W = W_0 \), where \( W_0 \) is the normalization (=desingularization) of the analytic curve \( X \times_Y Z = \{(x, z) \in X \times Z \mid p(x) = q(z)\} \) (Let us explain that if \( \deg p \) and \( \deg q \) are coprime then \( W_0 \) has only one point over \( \infty_Y \), which implies that \( W_0 \) is connected; so the maps \( W_0 \to X \) and \( W_0 \to Y \) are generalized polynomials). Notice that if \( X, Y, Z \) are of genus 0 then, as a rule, \( W \) is of genus greater than 0.

Example 2 Let \( \tilde{p}, \tilde{q} \) be the pair of polynomials from Example 1, \( \deg \tilde{p} = n \), \( \deg \tilde{q} = m \). Let \( h : (Y, \infty) \to (\mathbb{C}P^1, \infty) \), \( r : (Z, \infty) \to (\mathbb{C}P^1, \infty) \) be generalized polynomials, \( \deg h = \alpha \), \( \deg r = \beta \). Suppose that \( \gcd(\alpha, n) = \gcd(\beta, m) = \gcd(\alpha, \beta) = 1 \). Using Remark 1, we get the following commutative diagram of generalized polynomials

\[
\begin{array}{ccc}
W & \xrightarrow{h_2} & Z_1 & \xrightarrow{q_1} & Z \\
\downarrow r_2 & & \downarrow r_1 & & \downarrow r \\
Y_1 & \xrightarrow{h_1} & \mathbb{C}P^1 & \xrightarrow{\tilde{q}} & \mathbb{C}P^1 \\
\downarrow p_1 & & \downarrow \tilde{p} & & \\
Y & \xrightarrow{h} & \mathbb{C}P^1 \\
\end{array}
\]

with \( \deg r_1 = \deg r_2 = \beta \), \( \deg h_1 = \deg h_2 = \alpha \), \( \deg p_1 = n \), \( \deg q_1 = m \). Let \( s : (X, \infty) \to (W, \infty) \) be a generalized polynomial. Put \( p = p_1 \circ r_2 \circ s \), \( q = q_1 \circ h_2 \circ s \). The pair \( p, q \) is a solution of the problem under consideration.

Our main result is that Example 2 provides all the solutions of our problem (in the special case \( \deg p = \deg q \) it means that \( h, h_1, h_2 \) and \( r, r_1, r_2 \) are isomorphisms, so actually \( p = \tilde{p} \circ s \), \( q = \tilde{q} \circ s \)).
0.4 Organization

In Section 1 we replace our problem by a more general algebraic one, which is actually treated, and formulate the corresponding results. We introduce the concept of irreducible pair of generalized polynomials and separate the results in two parts. First, we reduce our problem to that with irreducible pairs of generalized polynomials (Section 3). Secondly, we study the irreducible pairs (Section 4). In Section 4 we formulate the Main group-theoretic lemma, which plays a central role in the proof of our main result. This lemma is proved in Section 5.

0.5 Conventions

All Riemann surfaces are supposed to be connected. Recall that the following three concepts are equivalent: a compact Riemann surface, a nonsingular connected projective algebraic curve over \( \mathbb{C} \), a finitely generated field over \( \mathbb{C} \) of transcendence degree 1. We shall identify a point of a curve and the corresponding place of the field of rational functions on this curve.

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1 Formulation of results

Denote by \( J \) the group of (all) germs of conformal mappings: \( (\mathbb{CP}^1, \infty) \to (\mathbb{CP}^1, \infty) \).

Definition. Suppose \( \Gamma \) is a subgroup of \( J \). We say that \( \Gamma \) is discrete if there exists a nonconstant function \( F \) meromorphic in a punctured neighborhood of infinity in \( \mathbb{C} \) such that \( F(g(z)) = F(z) \) for all \( g \in \Gamma \).

In the paper [3] a necessary condition for a group \( \Gamma \) to be discrete was obtained using the results from analytic local dynamics [4]. This condition will serve as the main tool in the proofs of our theorems. Let us formulate it here. For any \( g \in \mathbb{C}((\frac{1}{z})) \) we write \( \text{ord}_\infty g = n \) if \( g = \sum_{k=n}^{\infty} a_k z^{-k} \), \( a_n \neq 0 \). Put \( J_k = \{ g \in J \mid \text{ord}_\infty (g(z) - z) \geq 1 - k \} \) for \( k \leq 1 \). We have \( J \supset J_1 \supset J_0 \supset J_{-1} \supset \ldots \). Here \( J_k \) is a normal subgroup of \( J \). If \( \Gamma \subset J \) is a subgroup, then we write \( \Gamma_k = \Gamma \cap J_k \), \( k \leq 1 \).

Theorem 1 ([3]) Suppose \( \Gamma \subset J \) is a discrete subgroup; then

1. at most one of the quotient groups \( \Gamma_k/\Gamma_{k-1} \) (\( k \leq 1 \)) is nontrivial,

2. for all \( k \leq 1 \) the subgroup \( \Gamma_k/\Gamma_{k-1} \subset J_k/J_{k-1} \simeq (\mathbb{C},+) \) is discrete.
Definition. A subgroup $\Gamma \subset J$ is formally discrete if it satisfies the conditions \([1], [2]\) from Theorem \([1]\).

Remarks. 1. The quotient group $\Gamma/\Gamma_1$ is ignored here.

2. Theorem \([1]\) can be partially proved using the result of Scherbakov \([14]\).

Suppose $X$ is a Riemann surface, $\infty$ is a point of $X$. Denote by $J(X, \infty)$ the group of germs of conformal mappings $:(X, \infty) \to (X, \infty)$. One can identify $J(X, \infty)$ with $J$ by choosing a local parameter at $\infty$ on $X$. Let $Y$ be another Riemann surface and $f$ a holomorphic map from a punctured neighborhood of infinity in $X$ to $Y$. Then we define a group $T_f$ by the formula $T_f = \{g \in J(X, \infty) | f \circ g = f\}$.

Consider the following diagram of generalized polynomials:

$$
\begin{array}{ccc}
(X, \infty) & \xrightarrow{p} & (Y, \infty) \\
\downarrow & & \downarrow \\
(Z, \infty) & \xrightarrow{q} & (Y, \infty)
\end{array}
$$

Let $f$ and $g$ be meromorphic functions in punctured neighborhoods of infinity in $Y$ and $Z$ respectively. Suppose we have

$$f \circ p = g \circ q$$

Assume that $f$ and $g$ are nonconstant. Obviously, then $T_p$ and $T_q$ generate a discrete subgroup in $J(X, \infty)$. Conversely, if $T_p$ and $T_q$ generate a discrete group, then there exist nonconstant functions $f$ and $g$ as above such that \((3)\) holds.

In fact, in some sense, we find all pairs of generalized polynomials $p$ and $q$ such that $T_p$ and $T_q$ generate a formally discrete group.

Remark 2 Denote by $\bar{J}$ the group of all formal diffeomorphisms : $(\mathbb{CP}^1, \infty) \to (\mathbb{CP}^1, \infty)$, i.e., $\bar{J} = \{z \mapsto a_1 z + a_0 + a_{-1} z^{-1} + \ldots | a_i \in \mathbb{C}, a_1 \neq 0\}$ with respect to superposition. The subgroups $\bar{J}_k \subset J$ are defined in the same way as $J_k \subset J$. We have $J \subset \bar{J}$. If $\Gamma \subset J$ is a subgroup such that $\Gamma \not\subset J$, then the discreteness property does not make sense for $\Gamma$, whereas the formal discreteness is an algebraic property.

In what follows we denote by $\mathcal{M}(X)$ the field of meromorphic functions on a Riemann surface $X$.

Let $p : (X, \infty) \to (Y, \infty)$ be a generalized polynomial; then we have $\mathcal{M}(Y) \subset \mathcal{M}(X)$. Let $F$ be a field such that $\mathcal{M}(Y) \subset F \subset \mathcal{M}(X)$, $W$ its model, i.e., the compact Riemann surface such that $F$ is isomorphic to $\mathcal{M}(W)$ over $\mathbb{C}$. We get a commutative diagram :
Put $\infty_W = p_1(\infty_X)$. Then $p_1 : (X, \infty) \to (W, \infty)$ and $p_2 : (W, \infty) \to (Y, \infty)$ are generalized polynomials.

The following theorem may be considered as a description of rational solutions to the functional equation (3).

**Theorem 2** Suppose we are given a diagram (3). Then there is an alternative:

1. There exists a commutative diagram of generalized polynomials

   $$(X, \infty) \xrightarrow{p} (Y, \infty) \xrightarrow{q_1} (Z, \infty)$$

   such that $deg f = \gcd(deg p, deg q)$, $deg g = (deg p_1) \cdot (deg q_1)$. Such a diagram is unique up to isomorphism. The groups $T_p$ and $T_q$ generate $T_{qf}$, $\mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathcal{M}(W)$, $\mathcal{M}(V)$ is the composite of the fields $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$.

2. $T_p$ and $T_q$ generate an infinite nonabelian subgroup of $J(X, \infty)$. In this case $\mathcal{M}(Y) \cap \mathcal{M}(Z) = C$.

The following theorems describe the pairs of generalized polynomials $p, q$ for which $T_p$ and $T_q$ generate an infinite formally discrete subgroup of $J(X, \infty)$.

**Proposition 1** Suppose we are given a diagram (3). Then there exists a unique field $F_{q,p}$ with the following property. First, $\mathcal{M}(Z) \subset F_{q,p} \subset \mathcal{M}(X)$, $F_{q,p} \cap \mathcal{M}(Y) \neq C$. Secondly, given a field $F$ such that $\mathcal{M}(Z) \subset F \subset \mathcal{M}(X)$ and $F \cap \mathcal{M}(Y) \neq C$, we have $F_{q,p} \subset F$.

**Remark 3** From the geometrical point of view it means that there exists a commutative diagram of generalized polynomials:

$$X \xrightarrow{q_1} X_{q,p} \xrightarrow{q_2} Z$$

$$\downarrow \quad \downarrow$$

$$Y \quad \longrightarrow \quad Y_{q,p}$$

such that $q_2 \circ q_1 = q$ and the following universal property holds. Given a commutative diagram of generalized polynomials:

$$X \xrightarrow{g} X' \xrightarrow{h} Z$$

$$\downarrow \quad \downarrow$$

$$Y \quad \longrightarrow \quad Y'$$

such that $h \circ g = q$, there exists a unique holomorphic $f : X' \to X_{q,p}$ such that $f \circ g = q_1$. 
**Definition.** We say that the pair \( p, q \) in diagram (2) is *irreducible* if \( F_{q,p} = F_{p,q} = \mathcal{M}(X) \).

**Remark 4** Suppose the pair \( p, q \) in (4) is irreducible and \( \deg p, \deg q > 1 \); then \( \mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C} \).

**Example.** Put \( X = \mathbb{C}P^1 \), \( p(z) = z^n, q(z) = (z + 1)^m \), where \( n, m \) are positive integers.

Then the pair of polynomials \( p, q \) is irreducible.

To each pair of generalized polynomials (2) we assign an irreducible pair as follows. Put \( F = F_{p,q} \cap F_{q,p}, F_1 = \mathcal{M}(Y) \cap F, F_2 = \mathcal{M}(Z) \cap F \). Let \( K \) be the composite of \( F_{p,q} \) and \( F_{q,p} \). The diagram of fields commutes:

\[
\begin{array}{cccc}
\mathcal{M}(X) & \supset & K & \supset \mathcal{M}(Z) \\
\cup & & \cup & \\
F_{p,q} & \supset & F & \supset F_2 \\
\cup & & \cup & \\
\mathcal{M}(Y) & \supset & F_1
\end{array}
\]

To this diagram there corresponds the following diagram of generalized polynomials:

\[
\begin{array}{cccc}
X & \rightarrow & V & \xrightarrow{h_2} & X_{q,p} & \xrightarrow{q_1} & Z \\
\downarrow r_2 & & \downarrow r_1 & & \downarrow r & \\
X_{p,q} & \xrightarrow{h_1} & W & \xrightarrow{\tilde{q}} & W_2 \\
\downarrow p_1 & & \downarrow \tilde{p} & \\
Y & \xrightarrow{h} & W_1
\end{array}
\] (5)

Diagram (5) will be referred to as *the canonical diagram.*

From the definition of \( F_{p,q} \) (see Proposition 3) it follows that \( F_{p,q} \) is the composite of \( F \) and \( \mathcal{M}(Y) \). Similarly, \( F_{q,p} \) is the composite of \( F \) and \( \mathcal{M}(Z) \). From Theorem 2 it follows that \( \deg h = \deg h_1 = \deg h_2, \deg r = \deg r_1 = \deg r_2, \deg p_1 = \deg \tilde{p}, \deg q_1 = \deg \tilde{q}, \gcd(\deg h, \deg \tilde{p}) = \gcd(\deg r, \deg \tilde{q}) = \gcd(\deg h, \deg r) = 1.\)

**Proposition 2** The pair \( \tilde{p}, \tilde{q} \) is irreducible.

**Proposition 3** Suppose we are given a diagram (3) such that \( \mathcal{M}(Y) \cap \mathcal{M}(Z) = \mathbb{C} \). Let \( \tilde{p}, \tilde{q} \) be the corresponding irreducible pair of generalized polynomials. Then the following conditions are equivalent:

- \( T_p \) and \( T_q \) generate a discrete subgroup of \( J(X, \infty) \)
- \( T_{\tilde{p}} \) and \( T_{\tilde{q}} \) generate a discrete subgroup of \( J(W, \infty) \)

The assertion remains valid if we replace discreteness by formal discreteness.
Theorem 3 Suppose we are given a diagram (3) such that the pair \( p, q \) is irreducible and \( \deg p, \deg q > 1 \). Suppose that \( T_p \) and \( T_q \) generate a formally discrete group. Then there exist a commutative diagram:

\[
\begin{array}{ccc}
(Y, \infty) & \xleftarrow{p} & (X, \infty) \xrightarrow{q} (Z, \infty) \\
\downarrow & & \downarrow \\
(CP^1, \infty) & \xleftarrow{p_1} & (CP^1, \infty) \xrightarrow{q_1} (CP^1, \infty)
\end{array}
\]

where the vertical arrows are isomorphisms and \( p_1, q_1 \) is the following standard pair of polynomials : \( p_1(z) = z^n \), \( q_1(z) = (z + 1)^m \) with \( n, m, \text{lcm}(n, m) \in \{2, 3, 4, 6\} \). Conversely, the pair \( p_1, q_1 \) is irreducible, \( T_{p_1} \) and \( T_{q_1} \) generate a discrete subgroup of \( J, C(p_1) \cap C(q_1) = C \).

Our main result, which was formulated in the Introduction, follows from Theorems 1, 2, 3 and Propositions 1, 2, 3.

2 An algebraic set-up

Suppose \( X \) is a compact Riemann surface, \( \infty \) is a point of \( X \). The place of \( M(X) \) corresponding to \( \infty \) will be denoted by the same symbol \( \infty \). We denote by \( M(X)_{\infty} \) the completion of \( M(X) \) at \( \infty \). Let \( p : (X, \infty) \to (Y, \infty) \) be a generalized polynomial. Denote the restriction of \( \infty \) to \( M(Y) \) by the same letter. It is known that \( M(X)_{\infty} \) is a cyclic Galois extension of \( M(Y)_{\infty} \) of order \( \deg p \). To each \( g \in J(X, \infty) \) assign the automorphism of \( M(X)_{\infty} \) given by \( (gf)(x) = f(g^{-1}x) \), \( f \in M(X)_{\infty} \). We get an embedding of \( J(X, \infty) \) into the group of automorphisms of the topological field \( M(X)_{\infty} \) over \( C \). In what follows \( J(X, \infty) \) will be considered as a subgroup of the latter group. This embedding induces also a canonical isomorphism between \( T_p \) and \( Gal(M(X)_{\infty}/M(Y)_{\infty}) \). These two groups will be identified as well.

Lemma 1 Let \( X \) and \( Y \) be compact Riemann surfaces, \( f : X \to Y \) a holomorphic n-sheeted covering. Let \( g : W \to Y \) be the least Galois covering that can be factorized as follows:

\[
\begin{array}{ccc}
W & \xrightarrow{g} & X \\
\downarrow & & \downarrow f \\
Y & & \end{array}
\]

Let \( y_0 \in Y, f^{-1}(y_0) = \{x_1, \ldots, x_k\} \subset X \). Suppose the multiplicity of \( f \) at \( x_i \) is equal to \( l_i \), \( w_0 \in W, g(w_0) = y_0 \); then the multiplicity of \( g \) at \( w_0 \) is equal to \( \text{lcm}(l_1, \ldots, l_k) \).

The following explicit construction of \( W \) is useful to prove this lemma. Let \( A \subset Y \) be the set of critical values of \( f \). Put \( Y' = Y \setminus A, X' = X \setminus f^{-1}(A) \). Denote by \( Z' \) the set of pairs \( (y, \varphi) \), where \( y \in Y' \) and \( \varphi \) is bijection : \( f^{-1}(y) \to \{1, \ldots, n\} \). Let \( g \) be the map from \( Z' \) to \( Y' \) such that \( g(y, \varphi) = y \). A Riemann surface structure on \( Z' \) is defined in the natural way. The group \( S_n \) acts on \( Z' \) by biholomorphic transformations and this action is transitive on the fibres of \( g \). Let \( Z \) be the smooth compactification of \( Z' \) and \( W \) a connected component of \( Z \). Then \( g : W \to Y \) is the desired Galois covering.

The rest of the proof is omitted.
Corollary. Let \( p : (X, \infty) \rightarrow (Y, \infty) \) be a generalized polynomial, \( K \) a finite Galois extension field of \( \mathcal{M}(X) \) such that \( K \) is not ramified over \( \infty \in X \). Let \( L \) be the least Galois extension of \( \mathcal{M}(Y) \) such that \( K \subset L \). Then \( L \) is not ramified over \( \infty \in X \).

Now we consider a diagram (I) of generalized polynomials. Fix an algebraic closure \( \overline{\mathcal{M}(X)} \). We construct a tower of fields \( k_p^m, k_q^m \subset \overline{\mathcal{M}(X)}, \ m \geq 0 \) as follows. Put \( k_p^0 = k_q^0 = \mathcal{M}(X) \). Let \( k_p^m \) be the least Galois extension of \( \mathcal{M}(Y) \) containing \( k_{q}^{m-1} \). Let \( k_{q}^m \) be the least Galois extension of \( \mathcal{M}(Z) \) containing \( k_{p}^{m-1} \). One proves by induction that \( k_p^m \supset k_{p}^{m-1}, \ k_q^m \supset k_{q}^{m-1} \). By definition, \( k_p^m \supset k_{q}^{m-1}, \ k_q^m \supset k_{p}^{m-1} \). Put \( E = \bigcup m k_p^m = \bigcup m k_q^m \). \( E \) is a field containing \( \mathcal{M}(X) \) and normal over both \( \mathcal{M}(Y) \) and \( \mathcal{M}(Z) \). Actually \( E \) is the smallest subfield of \( \overline{\mathcal{M}(X)} \) with this property (if \( \mathcal{M}(X) \subset E' \subset \overline{\mathcal{M}(X)} \) and \( E' \) is normal over both \( \mathcal{M}(Y) \) and \( \mathcal{M}(Z) \) then one shows by induction that \( E' \supset k_p^m \) and \( E' \supset k_q^m \) for all \( m \)).

Lemma 2 For all \( m \geq 0 \) the fields \( k_p^m \) and \( k_q^m \) are not ramified over \( \infty \in X \). Therefore \( E \) is not ramified over \( \infty \in X \).

Proof. This follows immediately from the previous corollary. □

Fix a place \( \infty' \) of \( E \) over \( \infty \in X \). The choice of \( \infty' \) provides an embedding \( E \rightarrow \mathcal{M}(X)_{\infty} = E_{\infty} \) over \( \mathcal{M}(X) \). Since \( E \) is normal over \( \mathcal{M}(X) \), the image of \( E \) in \( \mathcal{M}(X)_{\infty} \) does not depend on the choice of \( \infty' \).

Put \( G_p = Gal(E/\mathcal{M}(Y)), \ G_q = Gal(E/\mathcal{M}(Z)), \ U = Gal(E/\mathcal{M}(X)) \). Let \( G \) be the subgroup of \( Aut E \) generated by \( G_p \) and \( G_q \). It is well known that for any place \( \omega \) of \( \mathcal{M}(X) \) (trivial on \( C \)) the action of \( U \) on the set of places of \( E \) over \( \omega \) is transitive.

Denote the set of places of \( E \) over \( \infty \) by \( S \). Clearly, \( S \) is invariant with respect to \( G_p \) and \( G_q \). So \( G \) acts on \( S \). The action of \( U \) on \( S \) is free because \( E \) is not ramified over \( \infty \). As explained above this action is transitive.

It is well known that assotiating to \( \sigma \in Gal(\mathcal{M}(X)_{\infty}/\mathcal{M}(Y)_{\infty}) \) its restriction to \( E \subset \mathcal{M}(X)_{\infty} = \mathcal{M}(X)_{\infty} \) one obtains an isomorphism:

\[ T_p = Gal(\mathcal{M}(X)_{\infty}/\mathcal{M}(Y)_{\infty}) \rightarrow \{ g \in G_p \mid g\infty' = \infty' \} \]

Similarly, we have an isomorphism:

\[ T_q = Gal(\mathcal{M}(X)_{\infty}/\mathcal{M}(Z)_{\infty}) \rightarrow \{ g \in G_q \mid g\infty' = \infty' \} \]

Let \( \Gamma \) be the subgroup of \( Aut \mathcal{M}(X)_{\infty} \) generated by \( T_p \) and \( T_q \).

Lemma 3 1. For any \( g \in G_p \) (resp. \( g \in G_q \)) there exist unique \( h \in T_p \) (resp. \( h \in T_q \)), \( \sigma \in U \) such that \( g = h\sigma \).

2. The restriction to \( E \) induces an isomorphism:

\[ \Gamma \rightarrow \{ g \in G \mid g\infty' = \infty' \} \]

3. For any \( g \in G \) there exist unique \( h \in \Gamma, \ \sigma \in U \) such that \( g = h\sigma \).
Proof. 1). It follows from the fact that \( S \) is identified with \( G_p/T_p \) or \( G_q/T_q \) and the action of \( U \) on \( S \) is free and transitive.

2) and 3). Clearly, we have a homomorphism \( f : \Gamma \to \{ g \in G \mid g\infty = \infty' \} \). Since \( E \) is dense in \( \mathcal{M}(X)_{\infty} \), it follows that \( f \) is injective.

From 1) it follows that for every \( g \in G \) there exist \( h \in \Gamma \), \( \sigma \in U \) such that \( g = f(h)\sigma \). These \( h, \sigma \) are unique because the action of \( U \) on \( S \) is free. If \( g\infty = \infty' \) then \( \sigma = 1 \), so \( f \) is surjective. \( \Box \)

Remarks. 1. We shall consider the groups \( T_p, T_q \) and \( \Gamma \) as subgroups of \( G \). At the same time \( \Gamma \) can be considered as the subgroup of \( J(X, \infty) \) generated by \( T_p \) and \( T_q \). By Lemma 3 we get a bijection \( G/U \leftrightarrow \Gamma \). The group \( U \) acts on \( G/U \) by left translations, so \( U \) acts on the set \( \Gamma \) without preserving the group structure of \( \Gamma \). This action plays a critical role in this paper. It has the following analytical meaning. The elements of \( \Gamma \) can be regarded as germs of algebraic functions at \( \infty \in X \). The analytic continuation around closed paths provides the monodromy action of \( H \) on \( \Gamma \), where \( H \) is inverse limit of \( \pi_1(X\setminus S, \infty), S \subset X\setminus\{\infty\}, #S < \infty \).

There is a canonical homomorphism \( f : H \to U \) with dense image, and the monodromy action of \( H \) on \( \Gamma \) comes from the action of \( U \) on \( \Gamma \). So the action of \( U \) on \( \Gamma \) is an algebraic version of the monodromy action used in §4 from [3].

2. Notice that \( E \) is of transcendence degree 1 over \( C \) and, in general, \( E \) is not generated over \( C \) by a finite numbers of elements. At the same time there exists a finite subset \( A \subset E \) such that every subfield \( E' \) of \( E \) containing \( A \) and invariant with respect to \( \text{Aut}_C E \) coincides with \( E \). (Let \( A \) be the set of generators of \( \mathcal{M}(X) \) over \( C \). Then \( E \) is generated by \( \bigcup_{g \in \Gamma} gA \) over \( C \)). Fields of this form were studied in [3].

Let us consider the following situation. Let \( G \) be a group, \( U \) and \( \Gamma \) its subgroups. Suppose that \( G = \Gamma \cdot U \), \( \Gamma \cap U = 1 \) (then \( U\Gamma = (\Gamma U)^{-1} = G \)). We get a bijection \( \Gamma \leftrightarrow G/U \). The group \( G \) acts on \( G/U \) by left translations, so \( G \) acts on \( \Gamma \).

Lemma 4 Let \( A \) and \( B \) be subsets of \( \Gamma \).

1. if \( A \) is invariant with respect to \( U \), then \( A^{-1} \) is also invariant;

2. \( A \) is invariant \( \iff \) \( UA \subset AU \iff AU \subset UA \iff UA = AU \);

3. if \( A \) and \( B \) are invariant, then \( AB \) is also invariant.

Proof. Clearly \( A \) is invariant \( \iff \) \( UA \subset AU \) and \( A^{-1} \) is invariant \( \iff \) \( UA^{-1} \subset A^{-1}U \iff AU \subset UA \). So to prove 1) and 3) it suffices to show that if \( UA \subset AU \) then \( AU \subset UA \). Let \( a \in A, \sigma \in U \). Then \( a\sigma = \sigma' a' \) for some \( \sigma' \in U, a' \in \Gamma \). Let us show that \( a' \in A \). Indeed, \( a' = (\sigma')^{-1} a\sigma \in UAU \subset AUU = AU \). So \( a' \in AU \cap \Gamma = A \).

To prove 3) notice that if \( UA \subset AU \) and \( UB \subset BU \) then \( UAB \subset AUB \subset ABU \). \( \Box \)

Lemma 5 If \( \Delta \subset \Gamma \) is a subgroup then the following conditions are equivalent:
1. $U \cdot \Delta$ is a subgroup;

2. $\Delta \cdot U$ is a subgroup;

3. $\Delta$ is invariant with respect to the action of $U$.

**Proof.** 1) and 2) are equivalent because $\Delta U = (U \Delta)^{-1}$. 3) means that $U \Delta U \Delta \subset U \Delta$ and $(U \Delta)^{-1} \subset U \Delta$. Each of these inclusions is equivalent to the inclusion $\Delta U \subset U \Delta$, i.e., to 3). □

In Section 3 we shall need the following definition. Let $A_1, \ldots, A_k, B_1, \ldots, B_n \subset \Gamma$ be $U$-invariant subsets.

**Definition.** A relation of the form

$$A_1 \cdot \ldots \cdot A_k = B_1 \cdot \ldots \cdot B_n$$

is a $(k + n)$-tuple $(a_1, \ldots, a_k, b_1, \ldots, b_n) \in A_1 \times \ldots \times A_k \times B_1 \times \ldots \times B_n$ such that $a_1 \cdot \ldots \cdot a_k = b_1 \cdot \ldots \cdot b_n$.

Let us define an action of $U$ on the set of relations of the form (6). Let $\sigma \in U$ and $(a_1, \ldots, a_k, b_1, \ldots, b_n)$ be a relation of the form (6). There exist unique $a'_1 \in A_1$, $\sigma_1 \in U$ such that $\sigma a_1 = a'_1 \sigma_1$. There exist unique $a'_2 \in A_2$, $\sigma_2 \in U$ such that $\sigma_1 a_2 = a'_2 \sigma_2$, and so on. Thus we obtain $a'_1 \in A_1, a'_k \in A_k$ and $\sigma_1, \ldots, \sigma_k \in U$ such that $\sigma a_1 \ldots a_k = a'_1 \ldots a'_k \sigma_k$. Similarly, we get $b'_1 \in B_1, \ldots, b'_n \in B_n$ and $\bar{\sigma}_1, \ldots, \bar{\sigma}_n \in U$ such that $\sigma b_1 \ldots b_n = b'_1 \ldots b'_n \bar{\sigma}_n$. Since $a_1 \ldots a_k = b_1 \ldots b_n$, we have $a'_1 \ldots a'_k \sigma_k = b'_1 \ldots b'_n \bar{\sigma}_n$. Therefore $a'_1 \ldots a'_k = b'_1 \ldots b'_n$. So $(a'_1, \ldots, a'_k, b'_1, \ldots, b'_n)$ is a relation of the form (6). The action of $U$ on the set of relations of the form (6) is defined as follows: $\sigma \in U$ maps $(a_1, \ldots, a_k, b_1, \ldots, b_n)$ to $(a'_1, \ldots, a'_k, b'_1, \ldots, b'_n)$. To show that this is really an action notice that for any $i \leq k$ and $j \leq n$ the product $a'_1 \ldots a'_i$ is the result of the action of $\sigma$ on $a_1 \ldots a_i \in \Gamma = G/U$ and $b'_1 \ldots b'_j$ is the result of the action of $\sigma$ on $b_1 \ldots b_j$.

## 3 The canonical diagram

We shall need the following slight generalization of Artin’s theorem in Galois theory.

**Lemma 6** Let $K$ be a Galois extension field of $F$, $U = Gal(K/F)$. Let $G$ be a subgroup of $\text{Aut} K$ such that $G \supset U$ and $[G : U] = n$. Put $k = K^G$. Then $K$ is a Galois extension of $k$, $[F : k] = n$, $Gal(K/k) = G$.

This can be proved repeating word-for-word the arguments of Artin [3, ch. VII, Theorem 2].

**Proof of Theorem 2.** The diagram (4) with $\deg f = \gcd(\deg p, \deg q)$, $\deg g = (\deg p_1) \cdot (\deg q_1)$ is unique if it exists, because $\mathcal{M}(W) = \mathcal{M}(Y) \cap \mathcal{M}(Z)$ and $\mathcal{M}(V)$ is the composite of $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. Indeed, $\deg p_1 = \deg p/\gcd(\deg p, \deg q)$ and $\deg q_1 = \deg q/\gcd(\deg p, \deg q)$ are coprime, so $\mathcal{M}(V)$ is the composite of $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$. Since
\([\mathcal{M}(Y) : \mathcal{M}(W)] = \deg g / \deg p_1 = \deg q_1\) and \([\mathcal{M}(Z) : \mathcal{M}(W)] = \deg p_1\) are coprime it follows that \(\mathcal{M}(W) = \mathcal{M}(Y) \cap \mathcal{M}(Z)\).

In the rest of the proof we use the notation of Section 2. If \(\Gamma\) is infinite, then \(\Gamma\) is non-abelian (if \(T_p\) and \(T_q\) commute, then \(\Gamma\) is finite). Obviously, in that case \(\mathcal{M}(Y) \cap \mathcal{M}(Z) = C\).

If \(\Gamma\) is finite, then \(\Gamma_1\) is trivial because \(J_1\) is torsion-free. Therefore \(\Gamma = \Gamma / \Gamma_1\) is cyclic of order \(d = \text{lcm}(\deg p, \deg q)\). By Lemma 3, we have \([G : U] = d\). Put \(F = E^G\). Using Lemma 3 we obtain \([\mathcal{M}(X) : F] = d\). Clearly, \(\mathcal{M}(Y) \cap \mathcal{M}(Z) = F\). Let \(W\) be the model of \(F\). We get the diagram:

\[
\begin{array}{ccc}
X & \overset{p}{\longrightarrow} & Y \\
\downarrow & & \downarrow q \\
\downarrow p_0 & & \downarrow r \\
W & \overset{q_0}{\longrightarrow} & Z
\end{array}
\]

(7)

Put \(\infty_W = r(\infty_X)\). Passing to the completions, we get:

\[
\begin{aligned}
\mathcal{M}(X)_\infty & \supset \mathcal{M}(Z)_\infty \\
\cup & \quad \cup \\
\mathcal{M}(Y)_\infty & \supset F_\infty
\end{aligned}
\]

Since \(\deg r = d\), it follows that \([\mathcal{M}(X)_\infty : F_\infty] \leq d\). Since \([\mathcal{M}(X)_\infty : \mathcal{M}(Y)_\infty]\) and \([\mathcal{M}(X)_\infty : \mathcal{M}(Z)_\infty]\) divide \([\mathcal{M}(X)_\infty : F_\infty]\), it follows that \(d\) divides \([\mathcal{M}(X)_\infty : F_\infty]\), hence \(d = \lcm(\deg(X)_\infty, \deg(Z)_\infty)\). So \(r : (X, \infty) \to (W, \infty)\) is a generalized polynomial. Clearly, \(p_0\) and \(q_0\) are generalized polynomials too. Besides, \(T_p\) and \(T_q\) generate a subgroup of \(T_r\) of order \(d\), i.e., the group \(T_r\).

Notice that \(\deg p_0 = \deg r / \deg p = \lcm(\deg p, \deg q) / \deg p\) and \(\deg q_0 = \lcm(\deg p, \deg q) / \deg q\) are coprime. So applying Remark 4 to the diagram \((Y, \infty) \to (W, \infty) \leftarrow (Z, \infty)\), one obtains the commutative diagram

\[
\begin{array}{ccc}
(V, \infty) & \overset{g}{\longrightarrow} & (Z, \infty) \\
\downarrow p_1 & & \downarrow q_1 \\
(Y, \infty) & \overset{g}{\longrightarrow} & (Z, \infty) \\
\downarrow p_0 & & \downarrow q_0 \\
(W, \infty) & \overset{g}{\longrightarrow} & (Z, \infty)
\end{array}
\]

(8)

where \(V\) is the normalization of \(Y \times_W Z\), \(\deg q_1 = \deg p_0\), \(\deg p_1 = \deg q_0\). From the definition of \(Y \times_W Z\) it follows that the diagrams (7) and (8) can be included into a diagram of the form (3). Clearly, this is the desired diagram. □

**Remark.** We have two equivalence relation on \(X\): \(R_p = \{(x_1, x_2) \in X \times X \mid p(x_1) = p(x_2)\}\), \(R_q = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\}\). Denote by \(R\) the equivalence relation generated by them. If \(\Gamma\) is finite it is easy to show that \(R\) is an algebraic curve on \(X \times X\). The essential part of the proof of Theorem 3 is the construction of the
quotient $X/R$ as a Riemann surface. We have done it using Lemma 3 (in fact, we have constructed the field $\mathcal{M}(X/R)$). One can also construct the Riemann surface $X/R$ directly using Theorem G from [4, Appendix A]. Besides, one can construct the algebraic curve $X/R$ in the framework of algebraic geometry using Theorem 4.1 from [3, exposé V, p.262].

**Corollary.** Let $p_i : (X, \infty) \to (Y_i, \infty)$ be generalized polynomials, $i \in \{1, 2, 3\}$. We have the following diagram of fields:

$$
\mathcal{M}(Y_1) \subset \mathcal{M}(X) \supset \mathcal{M}(Y_3) \\
\cup \\
\mathcal{M}(Y_2)
$$

Suppose $\mathcal{M}(Y_i) \cap \mathcal{M}(Y_j) \neq \mathbb{C}$ for any $i, j$; then $\mathcal{M}(Y_1) \cap \mathcal{M}(Y_2) \cap \mathcal{M}(Y_3) \neq \mathbb{C}$.

**Proof.** By Theorem 2 the elements $T_{p_1}$ and $T_{p_2}$ commute, and we have a diagram of generalized polynomials

\[
\begin{array}{ccc}
X & \xrightarrow{p_1} & Y_1 \\
& \searrow \quad \swarrow & \ \\
& r & \ \\
& \downarrow & \ \\
W & \searrow & Y_2 \\
& & \ \\
& \nwarrow & \ \\
& & \ \\
& & \downarrow & \ \\
& & Z
\end{array}
\]

such that $\mathcal{M}(W) = \mathcal{M}(Y_1) \cap \mathcal{M}(Y_2)$, $T_{p_1}$ and $T_{p_2}$ generate $T_r$. The elements of $T_r$ and $T_{p_3}$ commute. Applying Theorem 2 to the pair $r, p_3$, we get $\mathcal{M}(Y_1) \cap \mathcal{M}(Y_2) \cap \mathcal{M}(Y_3) \neq \mathbb{C}$. □

Proposition 1 follows immediately from this corollary. So this proposition is also proved.

Suppose we are given a commutative diagram of generalized polynomials:

\[
\begin{array}{ccc}
X & \xrightarrow{h_1} & Y \\
& \searrow \quad \swarrow & \ \\
& h_2 & \ \\
& \downarrow & \ \\
Z & \searrow & Y_{q,p}
\end{array}
\]

Then $T_{h_1}$ is a subgroup of $T_{h_2}$. A right factor $h_1$ of $h_2$ can be reconstructed from the group $T_{h_1}$ as follows: $\mathcal{M}(Y) = \mathcal{M}(X) \cap \mathcal{M}(X)_{T_{h_1}^\infty}$. Thus certain subgroups of $T_{h_2}$ corresponds to right factors of $h_2$ (not necessarily all the subgroups!).

Consider a diagram (3) again. In Section 3 the intermediate field $\mathcal{M}(Z) \subset F_{q,p} \subset \mathcal{M}(X)$ was introduced. The following theorem indicates the subgroup of $T_q$ corresponding to this field. There exists a commutative diagram of generalized polynomials:

\[
\begin{array}{ccc}
X & \xrightarrow{q_1} & X_{q,p} & \xrightarrow{q_2} & Z \\
\downarrow^p & & \downarrow & & \\
Y & \xrightarrow{q} & Y_{q,p}
\end{array}
\]
such that \( q_2 \circ q_1 = q \), \( \mathcal{M}(X_{q,p}) = F_{q,p} \), \( \mathcal{M}(Y_{q,p}) = F_{q,p} \cap \mathcal{M}(Y) \). Put \( H_{q,p} = \{ \sigma \in T_q \mid \tau \sigma = \sigma \tau \text{ for all } \tau \in T_p \} \).

**Theorem 4** \( T_{q_1} = H_{q,p} \).

To prove the theorem we need several lemmas.

**Lemma 7** Let \( H \) be a finite subgroup of \( J \). Then \( H \) is a cyclic group.

**Proof.** Since \( J_1 \) is torsion-free, it follows that \( H \cap J_1 = 1 \). So \( H \cong H/H_1 \hookrightarrow J/J_1 \cong C^* \). A finite subgroup of \( C^* \) is cyclic. \( \square \)

We use the constructions and notation of Section 3.

**Lemma 8** Let \( L \subset \Gamma \) be a subset invariant under the action of \( U \), i.e., \( uLU = LU \) for all \( u \in U \). Put

\[
N_L = \{ \sigma \in T_q \mid L \sigma = L \}
\]

Then \( N_L \) is a subgroup of \( T_q \) invariant under the action of \( U \).

**Remark.** The subgroups \( T_p \) and \( T_q \) of \( \Gamma \) are invariant under the action of \( U \).

**Proof.** It is easily checked that \( N_L \) is a subgroup. According to Lemma 4, it remains to show that \( UN_L \subset N_L \cdot U \). Suppose \( \sigma \in N_L, \gamma \in U, \gamma \sigma = \sigma' \gamma' \), where \( \sigma' \in T_q, \gamma' \in U \). We must prove that \( L \sigma' = L \). Using Lemma 4, we have \( LU \subset UL \), hence \( LUN_L \subset ULN_L \subset UL \subset LU \). Therefore \( L \sigma' = L \gamma \sigma (\gamma')^{-1} \subset LU \). On the other hand, \( L \sigma' \subset \Gamma \) and \( \Gamma \cap LU = L \), hence \( L \sigma' \subset L \). Since \( \sigma' \) is of finite order, it follows that \( L \sigma' = L \). \( \square \)

**Proof of Theorem 4.** Put \( L = T_q T_p \). By 3) of Lemma 4, \( L \) is invariant under the action of \( U \). By Lemma 8, \( N \cdot U \) is a subgroup of \( G \), \([NU : U] = \#N \). Further, we have \( \mathcal{M}(Z) \subset E^{NU} \subset \mathcal{M}(X) \), \([\mathcal{M}(X) : E^{NU}] = \#N \). This means that \( N \) corresponds to a right factor of \( q \). Obviously, \( H_{q,p} \subset N \). Let us show that \( H_{q,p} = N \).

We have \( T_p N \subset T_q T_p \). Let \( \tau \sigma = \sigma' \tau' \), where \( \tau, \tau' \in T_p, \sigma \in N, \sigma' \in T_q \). Then

\[
T_q T_p \sigma' = T_q T_p (\sigma'^{-1}) = T_q T_p (\tau')^{-1} = T_q T_p .
\]

Hence \( \sigma' \in N \). We get the property \( T_p N \subset NT_p \). This implies that \( N \cdot T_p \) is a subgroup of \( \Gamma \). By Lemma 4, \( N \cdot T_p \) is abelian. Therefore \( N \subset H_{q,p} \).

We have proved that \( H_{q,p} \) corresponds to a right factor of \( q \). Using Theorem 4, it is not hard to check that \( H_{q,p} \) corresponds to \( F_{q,p} \). \( \square \)

By definition, the pair \( p, q \) in (2) is irreducible if \( F_{q,p} = F_{p,q} = \mathcal{M}(X) \). By Theorem 4, this is equivalent to the property \( H_{p,q} = H_{q,p} = 1 \).

**Example.** Consider the following pair of polynomials: \( p(z) = z^n, q(z) = (z + 1)^m \). We have \( T_p = \{ z \mapsto \varepsilon z \mid \varepsilon^n = 1 \} \), \( T_q = \{ z \mapsto \delta z + (\delta - 1) \mid \delta^m = 1 \} \). Now it is easy to check that \( H_{p,q} = H_{q,p} = 1 \). So this pair is irreducible.

**Lemma 9** Consider a diagram \( (3) \) such that \( \mathcal{M}(Y) \cap \mathcal{M}(Z) \neq C \). Let \( K \) be an intermediate field: \( \mathcal{M}(Y) \cap \mathcal{M}(Z) \subset K \subset \mathcal{M}(Y) \). Let \( \tilde{K} \) be the composite of \( K \) and \( \mathcal{M}(Z) \). Then \( \tilde{K} \cap \mathcal{M}(Y) = K \). 

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Proof. Let $F = \mathcal{M}(Y) \cap \mathcal{M}(Z)$. We get the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(X) & \supset & \tilde{K} & \supset & \mathcal{M}(Z) \\
\cup & & \cup & & \cup \\
\mathcal{M}(Y) & \supset & K & \supset & F
\end{array}
$$

By Theorem 2, $[\mathcal{M}(Y) : F]$ and $[\mathcal{M}(Z) : F]$ are coprime. Therefore $[K : F]$ and $[\mathcal{M}(Z) : F]$ are coprime, hence $[\tilde{K} : K] = [\mathcal{M}(Z) : F]$. It follows that $[\mathcal{M}(Y) : K]$ and $[\tilde{K} : K]$ are coprime, so $K = \tilde{K} \cap \mathcal{M}(Y)$. □

Proof of Proposition 3. It follows immediately from the previous lemma. □

Proof of Proposition 5. The assertion is nontrivial for the formal discreteness property. Taking into account Theorem 2, it suffices to prove the following result.

Lemma 10 Suppose we are given the following diagram of generalized polynomials:

\[ X \xrightarrow{r} Y \xleftarrow{p} Z \xleftarrow{q} W \]

Let $\Gamma$ be the group generated by $T_p$ and $T_q$, $\Gamma'$ the group generated by $T_{por}$ and $T_{qor}$. Then $\Gamma$ is formally discrete iff $\Gamma'$ is formally discrete.

Proof. Put $J(X, Y, \infty) = \{(g_X, g_Y) \mid g_X \in J(X, \infty), g_Y \in J(Y, \infty), g_Y \circ r = r \circ g_X\}$. $J(X, Y, \infty)$ is a subgroup of $J(X, \infty) \times J(Y, \infty)$. Consider the projections $\pi : J(X, Y, \infty) \to J(Y, \infty)$, $j : J(X, Y, \infty) \to J(X, \infty)$. It is easy to see that $\pi$ is surjective, $\ker \pi = T_r$, $j$ is injective, $T_{por} = j(\pi^{-1}(T_p))$, $T_{qor} = j(\pi^{-1}(T_q))$. So $\Gamma' = j(\pi^{-1}(\Gamma))$.

Choose a meromorphic function $z$ in a neighborhood of $\infty \in Y$ with a pole of order 1 at $\infty$ (so $z^{-1}$ is a local coordinate at $\infty$). Choose a similar function $\zeta$ in a neighborhood of $\infty \in X$ so that $r^*(z) = \zeta^n$, $n = \deg r$ (i.e., in terms of $z$ and $\zeta$ the mapping $r$ is described by $z = \zeta^n$). Then we can write $g_X \in J(X, \infty)$ and $g_Y \in J(Y, \infty)$ as $g_X(\zeta) = \sum_{j=1}^{\infty} a_j \zeta^{-j}$, $g_Y(z) = \sum_{k=1}^{\infty} b_k z^{-k}$, $a_{-1} \neq 0$, $b_{-1} \neq 0$. Further, the relation $g_Y \circ r = r \circ g_X$ can be written as

$$
\left(\sum_{j=-1}^{\infty} a_j \zeta^{-j}\right)^n = \sum_{k=-1}^{\infty} b_k \zeta^{-kn} \quad \text{or} \quad \sum_{j=-1}^{\infty} a_j \zeta^{-j} = \zeta(b_{-1} + b_0 \zeta^{-n} + b_1 \zeta^{-2n} + \ldots)^{\frac{1}{n}}. \quad \text{Now it is clear that a subgroup $\Gamma \subset J(Y, \infty)$ is formally discrete iff $j(\pi^{-1}(\Gamma))$ is formally discrete. □}

Proposition 5 is also proved. □

4 Irreducible pairs of generalized polynomials
**Theorem 5** Suppose we are given a diagram (2) such that the pair \( p, q \) is irreducible and \( \deg p > 1, \deg q > 1 \). Let \( \Gamma \) be the group generated by \( T_p \) and \( T_q \). Suppose \( \Gamma_1 \) is abelian. Put \( n = \deg p, m = \deg q \). Put \( \tilde{p}(z) = z^n, \tilde{q}(z) = (z + 1)^m \), \( \tilde{p}, \tilde{q} \in \mathbb{C}[z] \). Let \( \tilde{\Gamma} \) be the group generated by \( T_{\tilde{p}} \) and \( T_{\tilde{q}} \). Then there exists an isomorphism \( \varphi : \Gamma \to \tilde{\Gamma} \) such that \( \varphi|T_p : T_p \to T_{\tilde{p}}, \varphi|T_q : T_q \to T_{\tilde{q}} \), and \( \varphi|\Gamma_1 : \Gamma_1 \to \tilde{\Gamma}_1 \) are isomorphisms. Besides, \( \Gamma \) is formally discrete iff \( \tilde{\Gamma} \) is formally discrete.

To prove the theorem we need the following result.

Let \( k \geq 1 \) be an integer. Denote by \( g_{z, k+1}^t \) the germ \( (C, 0) \to (C, 0) \) of a time \( t \) map for the flow of the holomorphic vector field \( G \). The set of germs \( G(\kappa) = \{ \lambda g_{z, k+1}^t \mid \lambda \in C^*, t \in C \} \) is a group with respect to superposition. For brevity, denote \( \lambda g_{z, k+1}^t \) by \( (\lambda, t) \).

The multiplication table for \( G(\kappa) \) has the following form:

\[
(\lambda, t) \times (\mu, s) = (\lambda \mu, t\mu^k + s).
\]

The subgroup \( C(\kappa) = \{ \lambda \in C \mid \lambda^k = 1 \} \) is the center of \( G(\kappa) \). Put \( G_d(\kappa) = \{ \lambda g_{z, k+1}^t \in G(\kappa) \mid \lambda^d = 1 \} \). Then \( G_d(\kappa) \) is a subgroup of \( G(\kappa) \). It is easy to see that if \( h \in G_k(k) \) is an element of finite order, then \( h \in C(k) \).

**Theorem A.** (\[9, Theorem 2.2, p.66\]) A finitely generated nonabelian solvable group of germs of conformal mappings \( (C, 0) \to (C, 0) \) is formally equivalent to a finitely generated subgroup of the group \( G(\kappa) \) for some \( k \).

**Proof of Theorem 5.** Put \( d = \text{lcm}(n, m) \). Choose a local parameter \( z \) at \( \infty \in X \) and identify the group \( \tilde{J}(X, \infty) \) with the group of germs of conformal mappings : \( (C, 0) \to (C, 0) \). By Remark \[4], \( \mathcal{M}(Y) \cap \mathcal{M}(Z) = C \). By Theorem \[4] \( \Gamma \) is nonabelian. One the other hand, \( \Gamma_1 \) is abelian, hence \( \Gamma \) is solvable. By Theorem A, \( \Gamma \) is formally equivalent to a subgroup of \( G_d(\kappa) \) for some \( k \). We have \( T_p \cap G_k(\kappa) \subset C(\kappa), T_q \cap G_k(\kappa) \subset C(\kappa) \). By Theorem \[4], \( H_{p,q} = H_{q,p} = 1 \). It follows that \( T_p \cap G_k(\kappa) = T_q \cap G_k(\kappa) = 1 \). Therefore \( \text{gcd}(n, k) = \text{gcd}(m, k) = 1 \), so \( \text{gcd}(d, k) = 1 \). Let the map \( f : G_d(\kappa) \to G_d(1) \) be given by \( \lambda g_{z, k+1}^t \to \lambda g_{z}^{t\delta} \). Since \( \text{gcd}(d, k) = 1 \), it follows that \( f \) is bijective. The multiplication table \( \[4 \] \) shows that \( f \) is an isomorphism.

Let \( h_p \) (resp. \( h_q \)) be a generator of \( T_p \) (resp. \( T_q \)). Let \( \varepsilon g_{z_1}^{t_1}, \delta g_{z_2}^{t_2} \) be the elements of \( G_d(1) \) corresponding to \( h_p \) and \( h_q \) respectively (Then \( \varepsilon \) (resp. \( \delta \)) is an primitive \( n \)-th (resp. \( m \)-th) root of unity.) Using the conjugation in \( G_d(1) \), we may assume that \( t_1 = 0 \). Then \( t_2 \neq 0 \). Notice that the map given by \( \lambda g_{z_2}^{t_2} \to \lambda g_{z_2}^{t_2} \) (\( c \in C^* \)) is an automorphism of \( G_d(1) \). Finally, the pair of generators becomes \( \varepsilon, \delta g_{z_2}^{t_2} \). Clearly, \( \Gamma' \) is formally discrete. \( \square \)

**Lemma 11** Let \( p(z) = z^n, q(z) = (z + 1)^m; n, m \geq 2 \). Let \( \Gamma \) be the group generated by \( T_p \) and \( T_q \). The group \( \Gamma \) is formally discrete iff \( \text{lcm}(n, m) \in \{2, 3, 4, 6\} \).

**Proof.** Let \( \varepsilon \) (resp. \( \delta \)) be a primitive \( n \)-th (resp. \( m \)-th) root of unity. We have \( \varepsilon \Gamma_1 = \Gamma_1, \delta \Gamma_1 = \Gamma_1 \). (Here \( \Gamma_1 \) is considered as a subgroup of \( C \)). Therefore \( e^{2\pi i/d} \Gamma_1 \subset \Gamma_1 \), where \( d : = \text{lcm}(n, m) \). So if \( \Gamma_1 \) is discrete then \( d \in \{2, 3, 4, 6\} \). On the other hand, \( \Gamma_1 \subset \mathbb{Z}[e^{2\pi i/d}] \), so if \( d \in \{2, 3, 4, 6\} \) then \( \Gamma_1 \) is discrete and \( \Gamma \) is formally discrete. \( \square \)
Corollary. Under the conditions of Theorem 3, the group $\Gamma$ is formally discrete iff $\text{lcm}(n, m) \in \{2, 3, 4, 6\}$.

Lemma 12 (Main group-theoretic lemma) Put $p(z) = z^n$, $q(z) = (z + 1)^m$, $n, m \geq 2$. Let $\Gamma$ be the group generated by $T_p$ and $T_q$. Let $G$ be an abstract group and $U$ its subgroup. Suppose $\Gamma$ is embedded into $G$ as a subgroup and $\Gamma U = G$, $\Gamma \cap U = 1$. Suppose $T_pU$ and $T_qU$ are subgroups of $G$. Suppose that $(n, m) \in P_1 \cup P_2 \cup P_3$, where $P_1 = \{(n, m) \mid n = m\}$, $P_2 = \{(n, m) \mid n = 2 \text{ or } m = 2\}$, and $P_3$ consists of $(3, 6)$ and $(6, 3)$. Then there exists a subgroup $U'$ of $U$ such that $[U : U'] < \infty$ and $U'$ is a normal subgroup of $G$.

This lemma will be proved in the following Section.

Remark. If $\text{lcm}(n, m) \in \{2, 3, 4, 6\}$ and $n, m \geq 2$ then $(n, m) \in P_1 \cup P_2 \cup P_3$.

Theorem 6 Suppose we are given a diagram (3) such that the pair $p, q$ is irreducible and \(\deg p > 1, \deg q > 1\). Let $\Gamma$ be the group generated by $T_p$ and $T_q$. Suppose $\Gamma_1$ is abelian. Put $n = \deg p$, $m = \deg q$, $\bar{p}(z) = z^n$, $\bar{q}(z) = (z + 1)^m$. If $(n, m)$ belongs to the set $P_1 \cup P_2 \cup P_3$ from the main group-theoretic lemma then there exists a commutative diagram:

\[
\begin{array}{ccc}
(Y, \infty) & \xleftarrow{p} & (X, \infty) \\
\downarrow & & \downarrow \\
(CP^1, \infty) & \xleftarrow{\bar{p}} & (CP^1, \infty)
\end{array}
\quad
\begin{array}{ccc}
& & \xrightarrow{q} \\
\downarrow & & \downarrow \\
& & (Z, \infty)
\end{array}
\quad
\begin{array}{ccc}
& & \xrightarrow{\bar{q}} \\
& & \downarrow \\
& & (CP^1, \infty)
\end{array}
\]

where the vertical arrows are isomorphisms.

Proof. We use the notation of Section 3. By Theorem 3, we can apply the main group-theoretic lemma. We get a subgroup $U'$ of $U$ such that $[U : U'] < \infty$ and $U'$ is a normal subgroup of $G$. The field $E^{U'}$ is normal over both $\mathcal{M}(Y)$ and $\mathcal{M}(Z)$, hence $E^{U'} = E$, i.e., $U' = 1$. So $\#U < \infty$ and we get a diagram:

\[
\begin{array}{ccc}
W & & \\
\downarrow r & & \\
X & & \\
\downarrow p & & \downarrow q \\
Y & & Z
\end{array}
\]

where $W$ is a compact Riemann surface, $\mathcal{M}(W) = E$, $r$ is nonconstant holomorphic, $p \circ r$ and $q \circ r$ are Galois coverings. Since $G \subset \text{Aut} W$, we have $\# \text{Aut} W = \infty$. Therefore, $W$ is of genus 0 or 1. $G$ acts on the finite set $S = r^{-1}(\infty) \subset W$. If $W$ is of genus 1, then for any $w \in W$ the group $\{g \in \text{Aut} W \mid gw = w\}$ is finite. Therefore the group $\{g \in \text{Aut} W \mid gS = S\}$ is finite too. So $W$ is of genus 0. Put $G_0 = \{g \in G \mid \forall s \in S : g(s) = s\}$. Then $[G : G_0] < \infty$, hence $\#G_0 = \infty$. It means that $\#S \leq 2$. By Lemma 3, $r$ is not ramified over $\infty \in X$. Suppose
\#S = 2. By (2) of Lemma 3, \( \Gamma = G_0 \). It can be assumed that \( W = \mathbb{C}P^1 \), \( S = \{0, \infty\} \). We have \( \Gamma \subset \{ g \in \text{Aut} \mathbb{C}P^1 \mid g(0) = 0, g(\infty) = \infty \} \simeq \mathbb{C}^* \), hence \( \Gamma \) is abelian, which is impossible. Thus \( \#S = 1 \), i.e., \( r \) is an isomorphism, \( p \) and \( q \) are Galois coverings. This completes the proof. \( \square \)

Theorem 3 is a special case of Theorem 6.

5 Proof of the main group-theoretic lemma

The main group-theoretic lemma was formulated in Section 4.

Remark. The basic idea of our proof is to study the action of \( U \) on the set of relations of some form in \( \Gamma \) (this action was defined at the end of Section 2). Put \( A = T_p \setminus \{id\} \), \( B = T_q \setminus \{id\} \). We use the relations of the form

\[ B \cdot A = A \cdot B \]  \hspace{1cm} (10)

and of the form

\[ A \cdot B \cdot A = B \cdot A \cdot B. \]  \hspace{1cm} (11)

Proposition 4 The main group-theoretic lemma holds for \( n = m \).

Proof. Put \( G_p = T_p U \), \( G_q = T_q U \), \( U_p = \{ \sigma \in U \mid \sigma \tau U = \tau U \text{ for all } \tau \in T_p \} \), \( U_q = \{ \sigma \in U \mid \sigma \tau U = \tau U \text{ for all } \tau \in T_q \} \). Clearly, \( U_p \) is the kernel of the left action of \( G_p \) on \( G_p/U \). So \( U_p \) is a normal subgroup of \( G_p \) and \( [U : U_p] < \infty \). Since \( U_q \) is a normal subgroup of \( G_q \), it suffices to show that \( U_p = U_q \) (this will imply that \( U_p \) is a normal subgroup of \( G \)).

Put \( h_p(z) = \varepsilon z \), \( h_q(z) = \varepsilon z + (\varepsilon - 1) \). Here \( h_p \in T_p \), \( h_q \in T_q \), \( \varepsilon \) is a primitive \( n \)-th root of unity. Let us consider two cases.

Case 1 \( n \) is even.

(a) If \( n = 2 \), then \( U = U_p = U_q \), there is nothing to prove.

(b) Assume \( n \geq 4 \). Consider the set of relations of the form (10). If \( h_q^{l_1} h_p^{l_2} = h_q^{k_1} h_p^{k_2} \), then \( \varepsilon^{l_1 + l_2} = \varepsilon^{k_1 + k_2} \) and \( \varepsilon^{l_1} - 1 = \varepsilon^{k_1} (\varepsilon^{k_2} - 1) \), so it is easily checked that the relation of the form (10) are precisely the following ones:

\[ h_q^{l_1} h_p^{l_2} = h_q^{k_1} h_p^{k_2}, \]  \hspace{1cm} (12)

where \( 2l \not\equiv 0 \text{ mod } n \). Notice that all \( h_p^s \ (s \not\equiv 0 \text{ mod } \frac{n}{2}) \) occur in the right hand side of (12). Let us show that \( U_p = U_q \). Let \( \sigma \in U_p \). Then \( \sigma \) preserves each relation (12), hence \( \sigma \) preserves \( h_p^i \) for every \( i \not\equiv 0 \text{ mod } \left(\frac{n}{2}\right) \). Since \( \sigma \) preserves \( T_p \) and \( id \), \( \sigma \) preserves \( h_p^{\frac{n}{2}} \). So \( \sigma \in U_p \). Similarly, \( U_p \subset U_q \).
Case 2 $n$ is odd.

Since $\Gamma_1$ is abelian, $h_q^{-l}h_p^l$ and $h_q^{-s}h_p^s$ commute. This provides the following relations:

$$h_p^{l_1}h_q^{l_2}h_p^{l_3} = h_q^{l_3}h_p^{l_1}h_q^{l_2},$$

(13)

where $l_i \not\equiv 0 \mod n$, $l_1 + l_2 + l_3 \equiv 0 \mod n$. The relations of the form (11) contain the relations (13). We shall not find all the relations of the form (11), but prove the following lemma.

Lemma 13 Suppose that

$$h_p^{l_1}h_q^{l_2}h_p^{l_3} = h_p^{k_1}h_q^{k_2}h_p^{k_3}, \quad l_i, k_i \not\equiv 0 \mod n.$$  

(14)

Then $l_1 \not\equiv k_1 \mod n$ (recall that $n$ is odd!).

Deduction of Case 2 from Lemma [13]. From (13) it follows that for any $l_1 \not\equiv 0 \mod n$, $k_1 \not\equiv 0 \mod n$, $k_1 \not\equiv l_1 \mod n$ there exists a relation of the form (14) with these $l_1, k_1$. Let $\sigma \in U$, $\sigma h_p^sU = h_t^rU$ for some $s, t$. Then $\sigma$ takes the set of relations (11) such that $h_p^{l_1} = h_p^s$ to the set of relations that $h_p^{l_1} = h_p^s$. Therefore $\sigma$ takes the set $\{h_p^j | j \not\equiv s \mod n\}$ to the set $\{h_q^j | j \not\equiv t \mod n\}$, hence $\sigma h_p^sU = h_q^tU$. Finally, for $\sigma \in U$ we have $\sigma h_p^sU = h_q^tU$ iff $\sigma h_p^tU = h_q^sU$. This implies $U_p = U_q$. □

It remains to prove Lemma [13].

Put $\alpha_t = h_q^t h_p^{-l_1}$. Then $\alpha_t(z) = z + (\varepsilon^t - 1)$. Suppose $h_p^{l_1}h_q^{l_2}h_p^{l_3} = h_p^{k_1}h_q^{k_2}h_p^{k_3}$, $l_i, k_i \not\equiv 0 \mod n$. Then $k_2 + k_3 = l_2 + l_3$. Therefore $(h_q^{-l_1}h_p^{l_1}) \cdot (h_q^{-l_2}h_p^{l_2}) = (h_q^{-l_2}h_p^{l_2}) \cdot (h_q^{-l_1}h_p^{l_1})$. We get:

$$(\varepsilon^{-l_1} - 1) + (\varepsilon^{l_2} - 1) = (\varepsilon^{k_2+k_3} - 1) - (\varepsilon^{k_2} - 1).$$

So it suffices to prove the following lemma.

Lemma 14 Let $\varepsilon$ be a primitive $n$-th root of unity. Let $n$ be odd. Then the equation

$$\varepsilon^{\gamma_1} + \varepsilon^{\gamma_2} + \varepsilon^{\gamma_3} = \varepsilon^\mu + 2$$

(15)

has no solutions such that $\gamma_1, \gamma_2, \gamma_3 \not\equiv 0 \mod n$ (here $\gamma_i, \mu_i \in \mathbb{Z}$ are unknowns).

Proof. If $\mu \equiv 0 \mod n$, then $\varepsilon^{\gamma_1} = \varepsilon^{\gamma_2} = \varepsilon^{\gamma_3} = 1$, i.e., a contradiction. Therefore $\mu \not\equiv 0 \mod n$.

The idea is to average by the action of $Gal(Q(\varepsilon)/Q)$.

Put $K = \bigcup_{m} Q(\sqrt[m]{1})$. For any $m \in \mathbb{N}$ define a $Q$-linear functional $T_m : Q(\sqrt[m]{1}) \rightarrow Q$ by $T_m(z) = \frac{1}{\#H} \sum_{h \in H} h(z)$, where $H = Gal(Q(\sqrt[m]{1})/Q)$. If $m' | m$, then $T_m|_{Q(\sqrt[m']{1})} = T_{m'}$. Therefore a $Q$-linear functional $T : K \rightarrow Q$ is well defined by $T|_{Q(\sqrt[m]{1})} = T_m$. Recall that if $\delta$ is a primitive $m$-th root of unity, then $Tr_Q^Q(\delta)(\delta) = \mu(m)$, where $\mu$ is the M"{o}bius function. Therefore, $T\delta = \frac{\mu(m)}{\varphi(m)}$. One has $\varphi(1) = 1 = \varphi(2), \varphi(3) = \varphi(4) = \varphi(6) = 2, \varphi(m) > 2$ for $m \neq 1, 2, 3, 4, 6$. So if $m > 1$ is odd then $-\frac{1}{2} \leq T\delta < \frac{1}{2}$.

Applying $T$ to (13), we obtain

$$\frac{3}{2} \leq T(\varepsilon^\mu) = T(\varepsilon^{\gamma_1}) + T(\varepsilon^{\gamma_2}) + T(\varepsilon^{\gamma_3}) < \frac{3}{2},$$

i.e., a contradiction. □

Proposition 3 is proved.
Proposition 5 The main group-theoretic lemma holds for \( n = 3, \ m = 6 \).

Proof. Let \( \omega \) be a primitive 6-th root of unity. Put \( h_p(z) = \omega^2 z, \ h_q(z) = \omega z + (\omega - 1) \). It is easy to show that there are exactly two relations of the form (16)

\[
h_qh_p = h_p^2h_q^5, \quad h_q^2h_p^2 = h_p^5h_q.
\]

The group \( U \) acts on the set of these relations. Therefore the set \( A = \{ h_q, h_q^5 \} \) is invariant with respect to \( U \). By (3) of Lemma 4, \( A \cdot A = \{ h_q^2, h_q^4, h_q^6 = id \} \) is also invariant. Let \( \Gamma' \) be the subgroup of \( \Gamma \) generated by \( T_p \) and \( \{ id, h_q^2, h_q^4 \} \). Then \( \Gamma' \) is \( U \)-invariant, hence \( G' = \Gamma'U \) is a subgroup of \( G \). Further, \( [G : G'] = [\Gamma : \Gamma'] < \infty \). According to Proposition 4 the main group-theoretic lemma holds for \( G' \) (in this case \( m = n = 3 \)). We get a subgroup \( U' \) of \( U \) such that \( [U : U'] < \infty \) and \( U' \) is a normal subgroup of \( G' \). Now \( \bigcap_{g \in G/G'} gU'g^{-1} \) is the desired subgroup of \( U \). \( \square \)

Proposition 6 The main group-theoretic lemma holds for \( n = 2 \) and arbitrary \( m \).

Proof. Let \( \sigma \in T_p, \ \sigma \neq 1 \). The group \( U \) preserves \( \sigma \). Therefore \( \sigma T_q \sigma \) is invariant with respect to \( U \). Let \( \Gamma' \) be the subgroup of \( \Gamma \) generated by \( T_q \) and \( \sigma T_q \sigma \). Then \( G' = \Gamma'U \) is a subgroup of \( G \). We have \( [G : G'] = [\Gamma : \Gamma'] < \infty \). According to Proposition 4 the main group-theoretic lemma holds for \( G' \), so one can obtain the desired subgroup of \( U \) just as in the proof of Proposition 3. \( \square \)

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