A Link Between The Continuous And
The Discrete Logistic Equations

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Abstract

Two types of population models are well known – the continuous and the discrete types. The two have very different characteristics and methods of solutions and analysis. In this note, we point out that an iterative technique when applied to the continuous case mimics, surprisingly the discrete theory. The implication is that techniques and conclusions of the latter theory can now be applied to the former case (and vice versa).

1 Discrete Logistic Equation

Population growth in nature is seldom as smoothly continuous as a classical logistic curve suggests [1]. In a species with a short annual breeding season whose members live for several breeding seasons and die at any time of the year, a continuous record of population size would undoubtedly show seasonal undulations. For many species in fact population growth is markedly discontinuous. These are, for example species whose members reproduce only once in their lifetime and die before their descendants’ lives begin, that is each generation dies before the eggs are hatched, or the seeds are germinated, to form the successive generations. A continuous-time differential equation is then inappropriate as a representation of population growth. We have to consider instead difference equations. As this will be needed in the sequel, we touch upon the salient features.

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1.1 First-Order Difference Equation

One of the simplest systems an ecologist can study is a seasonal breeding population in which generations do not overlap. Many natural populations, particularly among temperate zone insects (including the many economically important crop and orchard pests) are of this kind. In this situation, the observed data will consist of information about the maximum or the average, or the total population in each generation [2]. The studies try to form a relation between the magnitude of the population in the generation $n$ represented by $x_n$ and the magnitude of the population in generation $n+1$ represented by $x_{n+1}$ such a relation may be expressed in the general form

$$x_{n+1} = F(x_n)$$ (1)

The function $F(x_n)$ will be what the biologist calls “density dependant”, and a mathematician calls non-linear [3]. Equation (1) is thus a first order, non-linear difference equation. Equation (1) also describes many other examples in biology apart from population growth.

For instance:

1. In genetics, where the equation describes the change in gene frequency in time.

2. In epidemiology where $x_n$ represents the fraction of population infected in time $n$.

3. In economics where the relationships between commodity quantity and price are studied.

4. In social sciences to study the propagation of rumors where $x_n$ could be the number of people to have heard the rumor after time $t$.

In many of these contexts, and for biological populations in particular, there is a tendency for the variable $x_n$ to increase from one generation to the next when it is small, and for it to decrease when it is large. That is, the nonlinear function $F(x_n)$ often has the following properties: $F(0) = 0$ and $F(x)$ increases monotonically as $x$ increases through the range $0 < x < A$ with $F(x)$ attaining its maximum value at $x = A$, and $F(x)$ decreases monotonically as $x$ increases beyond $x = A$. A specific example is afforded by the equation

$$F(x_n) = x_{n+1} = x_n(a - bx_n)$$ (2)
This is sometimes referred to as the logistic difference equation. Using the substitution \( x_n = b/ax_n \) Equation (2) can be written as

\[
F(x_n) = x_{n+1} = ax_n(1 - x_n)
\]  

(3)

The behavior of the solutions of (3) is a function of the parameter ‘a’. Equation (3) has meaningful solutions for

\[
0 \leq a \leq 4
\]

with \( x_n \) measuring a non-negative quantity.

Studies have shown [3] that the very simple nonlinear difference equation can possess an extraordinarily rich spectrum of dynamical behavior, from stable points, stable cycles to ultimately chaotic behavior. Thus the problem is far richer than the continuous case seen earlier. In this form equation (3) is a simple nonlinear difference equation.

2 Examples of chaotic dynamical systems:

The logistic map

The discrete logistic map described by the single difference equation

\[
x_{n+1} = ax_n(1 - x_n)
\]  

(4)

as mentioned earlier, determines the future value of the variable \( x_{n+1} \) at time-step \( n+1 \) from the past value at time-step \( n \). The time evolution of \( x_n \) generated by this algebraic equation exhibits an extraordinary transformation from order to chaos as the parameter \( a \), which measures the strength of nonlinearity is increased [4].

Although nonlinear difference equations of this type have been studied extensively as simple models for turbulence in fluids, they also arise naturally in the study of evolution of biological populations.

For the purpose of illustration we consider the population of gypsy moths in the northern United States, which exhibits wild and unpredictable fluctuations from year to year. However we could equally well consider the evolution of economic prices determined by a nonlinear web model.

Writing (4) in a slightly different form

\[
x_{n+1} = ax_n - ax_n^2
\]
we see that it is a simple quadratic equation, with the first term linear and the second term nonlinear. If the parameter $a > 1$, the population increases, if $a < 1$, the population decreases. If $a > 1$, the population will eventually grow to a large enough value for the nonlinear term $-ax^2_n$ to become important. Since this term is negative, it represents a nonlinear death rate which dominates when the population is too large. Biologically this nonlinear death rate could be due to the depletion of food supplies or the outbreak of diseases in an over-crowded environment. The dynamics of this map and the dependence on the parameter $a$ which measures the rate of linear growth and the size of the nonlinear term, are best understood using graphical analysis.

Consider the graphs of $x_n$ versus $x_{n+1}$ displayed in the Fig. (1) for four different values of $a$. Equation (1) defines an inverted parabola with intercepts at

$$x_n = 0 \text{ and } 1$$

and a maximum value of

$$x_{n+1} = a/4$$
Using these maps we can get a quantitative understanding of the dynamics of the logistic map in a quick way. Briefly, this graphical analysis tells us that if the normalized population starts out larger than 1, then it immediately goes negative, becoming extinct in one time-step. Moreover if $a > 4$, the peak of the parabola will exceed 1, which makes it possible for initial populations near 0.5 to become extinct in two time-steps. We will therefore restrict our analysis to values of $a$ between 0 and 4.

For values of $a < 1$, the population always decreases to 0, as shown for $a = 0.95$ in Fig. 1. The intersection of the parabola with the 45° line at $x_n = 0$ represents a stable fixed point on the map. Because $a$ is small a perturbation can be used to verify that almost all initial populations are attracted to this fixed point and become extinct. However for $a > 1$ this fixed point becomes unstable. Instead the parabola now intersects the 45° line at

$$x = \frac{a - 1}{a}$$

which corresponds to a new fixed point.

For values of $a$ between 1 and 3 almost all initial populations evolve to this equilibrium population. Then, as $a$ is increased between 3 and 4, the dynamics change in remarkable ways. First the fixed point becomes unstable and the population evolves to a dynamic steady state in which it alternates between a large and small population. A time sequence converging to such a period-2 cycle is displayed in Fig. 1 for $a = 3.2$: the population cycles between two points on the parabola, $x_n \sim 0.5$ and $x_n \sim 0.8$, in alternate years. For somewhat larger values of $a$, this period-2 cycle becomes unstable and is replaced by a period-4 cycle in which the population alternates high-low, returning to its original value every four time-steps. As $a$ is increased the longtime motion converges to period – 8,16,32,64, cycles, finally accumulating to a cycle of infinite period $a = a_{inf} \sim 3.57$.

Having observed a period doubling sequence in numerical experiments Feigenbaum was able to prove that the intervals over which a cycle is stable decreases at a geometric rate of $\sim 4.6692016$. The tremendous significance
of this work is that this rate and other properties of the period-doubling bifurcation are universal in the sense that they appear in the dynamics of any system which can be approximately modelled by a nonlinear map with a quadratic extremum. Feigenbaum’s theory has subsequently been confirmed by a wide variety of physical systems such as turbulent fluids, oscillating chemical reactions, nonlinear electrical circuits, and ring lasers.

The investigation of period doubling in nonlinear dynamical systems provides a superb example of the interplay between numerical experiments and analytical theory. However, this sequence of regular periodic orbits is only the precursor to chaos. Included below is a bifurcation diagram, showing the beginning of Chaos:
3 A Remarkable Mathematical Equivalence

We would now like to deduce a mathematical equivalence between the continuous and discrete cases and will comment on this. Let us consider the equation from the continuous growth

\[ \frac{dP}{dt} = f(P), \]

set \( P \equiv x \) \hspace{1cm} (5)

which gives

\[ x = \int f(x(t))dt \]

where the integral is over suitable limits. Here \( f \) generalizes the dependence of the right side of

\[ \frac{d(P(t))}{dt} = r(M - P(t))P(t) \]

where \( P(t) \) is the population at given time \( t \) and \( M \) is the maximum sustainable population \([5, 6]\) on the population \( P \) at that time. Let us solve equation \( (6) \) by the method of successive approximation i.e., we try on the right side of \( (6) \) a tentative solution \( x_0(t) \). \( (6) \) can then be written in the form

\[ x_1 = F(x_0) \equiv \int f(x_0)dt \]

where \( x_1 \) gives the next level of approximation. Before proceeding further, we remark that \( (5) \) is in the form of the initial value problem where the Lipschitz condition is necessary for the convergence of the iterative procedure \([7]\). In a similar manner we get from \( (8) \) the more general equation

\[ x_{n+1} = F(x_n). \] \hspace{1cm} (9)

\( (9) \) can immediately identified with the discrete logistic equation for example \( (4) \). It must be stressed however that the discrete equation is based on a completely different foundation that is the subscript \( n \) in the discrete case represents the population at the \( n^{th} \) generation, whereas in \( (9) \) \( x_n \) represents the \( n^{th} \) iteration or approximation of the population \( x \equiv P \) of the continuous case. Nevertheless, this mathematical equivalence enables us to apply the conclusions of the discrete case including the domain of chaos. Thus chaotic
behaviour of \( x_n \) of the discrete case would represent the lack of convergence of the iterates of the continuous case. In this specific example if

\[
F(x_n) = ax_n(1 - x_n),
\]

then for \( a = 3.57 \) the above iterative procedure breaks down. In this case, using (11), it follows from (8) that

\[
\int f(x)dx = ax(1 - x).
\]

Finally we remark that for a more conventional approach to the above problem reference can be made to Krempasky, [8].

References

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