Elimination of All Bad Local Minima in Deep Learning

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Abstract
In this paper, we theoretically prove that we can eliminate all suboptimal local minima by adding one neuron per output unit to any deep neural network, for multi-class classification, binary classification, and regression with an arbitrary loss function. At every local minimum of any deep neural network with added neurons, the set of parameters of the original neural network (without added neurons) is guaranteed to be a global minimum of the original neural network. The effects of the added neurons are proven to automatically vanish at every local minimum. Unlike many related results in the literature, our theoretical results are directly applicable to common deep learning tasks because the results only rely on the assumptions that automatically hold in the common tasks. Moreover, we discuss several limitations in eliminating the suboptimal local minima in this manner by providing additional theoretical results and several examples.

1. Introduction
Deep neural networks have achieved significant practical success in the fields of computer vision, machine learning, and artificial intelligence. However, theoretical understanding of deep neural networks is scarce relatively to its empirical success. One of the major difficulties in theoretically understanding deep neural networks lies in the non-convexity and high-dimensionality of the objective functions used to train the networks. Because of the non-convexity and high-dimensionality, it is often unclear whether a deep neural network will be guaranteed to have a desired property after training, instead of becoming stuck around an arbitrarily poor local minimum. Indeed, it is NP-hard to find a global minimum of a general non-convex function (Blum & Rivest, 1992), which poses a concern regarding high-dimensional problems (Kawaguchi et al., 2015). In the past, such theoretical concerns were considered one of reasons to prefer classical machine learning models (with or without a kernel approach) that require only convex optimization. Given their recent empirical success, a question remains whether deep neural networks can be guaranteed to avoid poor local minima in theory.

There have been numerous recent studies that analyze the landscape and local minima of the objective functions in the training of neural networks. Several studies have provided positive results for deep neural networks under the assumption of significant simplifications (Choromanska et al., 2015; Kawaguchi, 2016; Hardt & Ma, 2017) and strong over-parameterization (Nguyen & Hein, 2017; 2018). For shallow networks with a single hidden layer, there have been many positive results, yet often with strong assumptions, for example, requiring the use of significant over-parameterization, simplification, and Gaussian inputs (Andoni et al., 2014; Sedghi & Anandkumar, 2014; Soltanolkotabi, 2017; Brutzkus & Globerson, 2017; Ge et al., 2017; Soudry & Hoffer, 2017; Goel & Klivans, 2017; Zhong et al., 2017; Li & Yuan, 2017; Du & Lee, 2018).

Beyond the use of strong assumptions, two distinct types of new positive results are emerging for deep neural networks. First, all local minima of certain deep neural networks have been proven to be no worse than the global minima of the corresponding classical machine learning models (Shamir, 2018; Kawaguchi & Bengio, 2018; Kawaguchi et al., 2018), and further improvements have been guaranteed via non-negligible residual representations (Kawaguchi & Bengio, 2018) as well as an increase in the depth and width of the networks, even without the strong over-parameterization (Kawaguchi et al., 2018). Second, it has been shown that adding one neuron can eliminate all suboptimal local minima (i.e., all local minima that are not global minima) for a binary classification with a special type of smoothed hinge loss functions (Liang et al., 2018). This second type of the result has already been noted in several different papers (Nguyen et al., 2018; Wang et al., 2018; Zhang et al., 2018). However, because of the assumption of binary classification with special loss functions, it is currently inapplicable to many common deep

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learning tasks, which has been mentioned as a major limitation, for example, in (Fessler, 2018).

In this paper, we prove, without any strong assumption, that adding one neuron per output unit can eliminate all suboptimal local minima for multi-class classification, binary classification, and regression with an arbitrary loss function. To our knowledge, this is the first result that guarantees no suboptimal local minima for many common deep learning tasks without any typically unsatisfied assumptions. Furthermore, we also show that there is a limitation of removing the suboptimal local minima in this manner.

2. Preliminaries

This section defines the problem setting and presents the most closely related previous results in the literature.

2.1. Problem description

Let \( x \in \mathbb{R}^{d_x} \) and \( y \in \mathbb{R}^{d_y} \) be an input vector and a target vector, respectively. Define \( \{ (x_i, y_i) \}_{i=1}^{m} \) as a training dataset of size \( m \). Given an input \( x \) and parameter \( \theta \), let \( f(x; \theta) \in \mathbb{R}^{d_y} \) be the pre-activation output of the last layer of any arbitrary deep neural network with any structure (e.g., any convolutional neural network with any depth and any width, with or without skip connections). That is, there is no assumption with regard to \( f \) except that \( f(x; \theta) \in \mathbb{R}^{d_y} \). We consider the following standard objective function \( L \) to train an arbitrary neural network \( f \):

\[
L(\theta) = \frac{1}{m} \sum_{i=1}^{m} \ell(f(x_i; \theta), y_i),
\]

where \( \ell : \mathbb{R}^{d_y} \times \mathbb{R}^{d_y} \to \mathbb{R} \) is an arbitrary loss criterion such as cross entropy loss, smoothed hinge loss, or squared loss.

We then consider an auxiliary objective function \( \tilde{L} \), which generalizes that in (Liang et al., 2018) as

\[
\tilde{L}(\tilde{\theta}) = \frac{1}{m} \sum_{i=1}^{m} \ell(\tilde{f}(x_i; \tilde{\theta}) + g(x_i; a, b, W), y_i) + \lambda \| a \|^2_2,
\]

where \( \lambda > 0 \), \( \tilde{\theta} = (\theta, a, b, W) \), \( a, b \in \mathbb{R}^{d_y}, W = [w_1 \ w_2 \ \cdots \ w_{d_y}] \in \mathbb{R}^{d_x \times d_y} \) with \( w_k \in \mathbb{R}^{d_x} \), and

\[
g(x; a, b, W)_k = a_k \exp(w_k^\top x + b_k),
\]

for all \( k \in \{1, \ldots, d_y \} \). Define a modified neural network \( \tilde{f} \) as

\[
\tilde{f}(x; \tilde{\theta}) = f(x; \theta) + g(x; a, b, W),
\]

which is equivalent to adding one neuron \( g(x; a, b, W)_k \) per each output unit \( f(x; \theta)_k \) of the original neural network. Because \( \tilde{L}(\tilde{\theta}) = \frac{1}{m} \sum_{i=1}^{m} \ell(\tilde{f}(x_i; \tilde{\theta}), y_i) + \lambda \| a \|^2_2 \), the auxiliary objective function \( \tilde{L} \) is the standard objective function \( L \) with the modified neural network \( \tilde{f} \) with a regularizer on \( a \).

2.2. Background

Liang et al. (2018) proved their main result, Proposition 1, under the assumptions PA1, PA2, PA3, and PA4:

**PA1.** The output dimension \( d_y = 1 \) and \( y \in \{-1, 1\} \) (binary classification).

**PA2.** The loss criterion \( \ell(f, y) \) is in the form of \( \ell(f, y) = \ell(-y f) \) such that the function \( \ell : \mathbb{R} \to \mathbb{R} \) is monotonically non-decreasing, and every global minimum \( q \) of \( \ell \) satisfies \( q < 0 \).

**PA3.** The function \( \tilde{\ell} \) in PA2 is twice continuously differentiable, and every critical point of \( \tilde{\ell} \) in PA2 is a global minimum of \( \ell \).

**PA4.** There exists a \( \theta \) such that \( f(x_i; \theta) = y_i \) for all \( i \in \{1, 2, \ldots, m\} \) (realizability).

Here, assumption PA4 may not be satisfied if the target labels \( y \) are chosen randomly given \( x \) such that \( y_i \neq y_j \) for some \( x_i = x_j \). Accordingly, Liang et al. (2018) also proved Proposition 2 by using the alternative assumption PA5:

**PA5.** The function \( \tilde{\ell} \) in PA2 is twice continuously differentiable and convex.

Proposition 2. (Liang et al., 2018) Let assumptions PA1, PA2, and PA5 hold. Then, at every local minimum \( \tilde{\theta} = (\theta, a, b, W) \) of \( \tilde{L} \), the following statements hold:

(i) \( \tilde{\theta} \) is a global minimum of \( L \),

(ii) \( \tilde{f}(x; \theta, a, b, W) = f(x; \theta) \) for all \( x \in \mathbb{R}^{d_x} \) and \( \tilde{L}(\theta, a, b, W) = L(\theta) \), and

(iii) \( \text{sign}(f(x_i; \theta)) = y_i \) for all \( i \in \{1, \ldots, m\} \).

All the results of Liang et al. (2018) require assumptions PA1 and PA2, as well as the assumption that \( \tilde{\ell} \) is twice continuously differentiable (assumption PA3 or PA5). These assumptions prevent us from applying these results to multi-class classification, regression, or even binary classification with standard loss criteria (e.g., cross entropy loss and differentiable, yet not twice differentiable, smoothed hinge loss).
3. Main results

Our main results are presented in Section 3.1 for arbitrary datasets, and in Section 3.2 for realizable datasets.

3.1. Result for arbitrary datasets

Under only a mild assumption (Assumption 1), Theorem 1 states that at every local minimum \((\theta, a, b, W)\) of the modified objective function \(\tilde{L}\), the parameter \(\theta\) achieves a global minimum of the original objective function \(L\), and the modified neural network \(\tilde{f}\) automatically becomes the original neural network \(f\).

**Assumption 1.** (Use of common loss criteria) For any \(i \in \{1, \ldots, m\}\), the function \(\ell_{y_i} : q \mapsto \ell(q, y_i)\) is differentiable and convex (e.g., the squared loss, cross entropy loss, or polynomial hinge loss satisfies this assumption).

**Theorem 1.** Let Assumption 1 hold. Then, at every local minimum \((\theta, a, b, W)\) of \(L\), the following statements hold:

(i) \(\theta\) is a global minimum of \(L\), and

(ii) \(\tilde{f}(x; \theta, a, b, W) = f(x; \theta)\) for all \(x \in \mathbb{R}^d\), and \(\tilde{L}(\theta, a, b, W) = L(\theta)\).

When compared with Proposition 2, Theorem 1 only requires Assumption 1, which is weaker than the set of assumptions PA1, PA2, and PA5; i.e., Assumption 1 is implied by the assumptions PA1, PA2, and PA5, but not vice versa.

Assumption 1 is satisfied by simply using a common loss criterion, including the squared loss as \(\ell(q, y) = \|q - y\|^2\) or \(\ell(q, y) = (1 - y)^2\) (the latter with \(d_y = 1\)), cross entropy loss as \(\ell(q, y) = \sum_{k=1}^{d_y} y_k \log \frac{\exp(q_k)}{\sum_{k'} \exp(q_{k'})}\), or smoothed hinge loss as \(\ell(q, y) = (\max\{0, 1 - yq\})^p\) with \(p \geq 2\) (the hinge loss with \(d_y = 1\)). Although the objective function \(L : \theta \mapsto L(\theta)\) used to train a neural network is non-convex in \(\theta\), the loss criterion \(\ell_{y_i} : q \mapsto \ell(q, y_i)\) is usually convex in \(q\).

Therefore, Theorem 1 is directly applicable to most common deep learning tasks in practice. This means that, in practice, one can eliminate all suboptimal local minima by simply adding one neuron per output unit. Furthermore, the added neurons automatically vanish at every local minimum and thus do not affect the output of the neural network.

3.2. Result for realizable datasets

Theorem 2 makes a statement similar to Theorem 1 under a weaker assumption on the loss criterion (Assumption 2) but with an additional assumption on the training dataset (Assumption 3).

**Assumption 2.** (On the loss) For any \(i \in \{1, \ldots, m\}\), the function \(\ell_{y_i} : q \mapsto \ell(q, y_i)\) is differentiable, and \(q \in \mathbb{R}^{d_y}\) is a global minimum of \(\ell_{y_i}\), if \(\nabla \ell_{y_i}(q) = 0\).

**Assumption 3.** (On the label consistency) There exists a function \(f^*\) such that \(f^*(x_i) = y_i\), for all \(i \in \{1, \ldots, m\}\).

**Theorem 2.** Let Assumptions 2 and 3 hold. Then, at every local minimum \((\theta, a, b, W)\) of \(\tilde{L}\), the following statements hold:

(i) \(\theta\) is a global minimum of \(L\),

(ii) \(\tilde{f}(x; \theta, a, b, W) = f(x; \theta)\) for all \(x \in \mathbb{R}^d\), and \(\tilde{L}(\theta, a, b, W) = L(\theta)\), and

(iii) \(f(x_i; \theta)\) is a global minimum of \(\ell_{y_i} : q \mapsto \ell(q, y_i)\) for all \(i \in \{1, \ldots, m\}\).

When compared with Proposition 1, Theorem 2 only requires the set of Assumptions 2 and 3, which is weaker than the set of assumptions PA1, PA2, PA3, and PA4; i.e., the former is implied by the latter, but not vice versa. By using an appropriate loss criterion for classification, Theorem 2 (iii) implies that the trained neural network \(f(\cdot; \theta)\) at every local minimum correctly classifies all training data points, implying Proposition 1 (iii).

Assumption 2 is weaker than Assumption 1 in the sense that the former is implied by the latter but not vice versa. However, as discussed above, Assumption 1 already accommodates most common loss criteria. Assumption 3 is automatically satisfied if a target \(y\) given an input \(x\) is not random, but the non-randomness is not necessary to satisfy Assumption 3. Even if the targets are generated at random, as long as all \(x_1, x_2, \ldots, x_m\) are distinct (i.e., \(x_i \neq x_j\) for all \(i \neq j\)), Assumption 3 is satisfied.

Therefore, although Theorem 2 might be less applicable in practice when compared to Theorem 1, Theorem 2 can still be applied to many common deep learning tasks with the additional guarantee, as stated in Theorem 2 (iii).

4. Proofs of main results

Our proofs differ from those of Liang et al. (2018) because the assumptions are significantly weakened. For example, because we do not assume twice differentiability, our proofs do not use second-order Taylor expansions. Furthermore, in the proofs of Theorems 1 and 2, we let \(\theta\) be arbitrary so that we can prove a limitation of eliminating the suboptimal local minima in the next section (Theorem 3) based on these proofs. We present two lemmas in Section 4.1, and the proofs of our main results in Section 4.2.

4.1. Lemmas

The following two lemmas are used in the proofs of the main theorems in the next section. Let \(\ell_y(q) = \ell(q, y)\), and
\[ \nabla \ell_y(\varphi(q)) = (\nabla \ell_y)(\varphi(q)) \] be the gradient \( \nabla \ell_y \) evaluated at an output \( \varphi(q) \) of a function \( \varphi \).

**Lemma 1.** Assume that for any \( i \in \{1, \ldots, m\} \), the function \( \ell_{y_i} : q \mapsto \ell(q, y_i) \) is differentiable. Then, for any \( (\theta, W) \), if \((a, b)\) is a stationary point of \( \tilde{L}_{|\theta, W}(a, b) := \tilde{L}(\theta, a, b, W) \), it holds that \( a = 0 \).

**Proof.** Since \( \ell_y : q \mapsto \ell(q, y) \) is assumed to be differentiable, \( \tilde{L}_{|\theta, W} \) is also differentiable (because a sum of differentiable functions is differentiable, and a composition of differentiable functions is differentiable). From the definition of a stationary point of a differentiable function \( \tilde{L}_{|\theta, W} \), for all \( k \in \{1, 2, \ldots, d_y\} \),

\[
\frac{\partial \tilde{L}(\theta, a, b, W)}{\partial a_k} = \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta) + g(x_i; a, b, W)))_{ka} \exp(w^*_k x + b_k) = \frac{\partial \tilde{L}(\theta,a,b,W)}{\partial \theta} = 0
\]

\[ + 2m \lambda a^2_k = 2m \lambda a^2_k = 0, \]

which implies that \( a_k = 0 \) for all \( k \in \{1, 2, \ldots, d_y\} \), since \( 2m \lambda \neq 0 \). \( \Box \)

**Lemma 2.** Assume that for any \( i \in \{1, \ldots, m\} \), the function \( \ell_{y_i} : q \mapsto \ell(q, y_i) \) is differentiable. Then, for any \( \theta \), if \((a, b, W)\) is a local minimum of \( \tilde{L}_{|\theta, W}(a, b, W) := \tilde{L}(\theta, a, b, W) \), it holds that for all \( k \in \{1, 2, \ldots, d_y\} \), all \( u_k \in \{u_k \in \mathbb{R}^{d_x} : ||u_k||_2 = 1\} \), and all \( p \in \mathbb{N}_0 \),

\[
\sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_{k} \exp(w^*_k x_i + b_k) (u^*_k x_i)^p = 0.
\]

**Proof.** Let \( \theta \) be fixed. Let \((a, b, W)\) be a local minimum of \( \tilde{L}_{|\theta} \). Then, from Lemma 1, we have \( a = 0 \), yielding

\[ \tilde{L}(\theta, a, b, W) = \tilde{L}(\theta). \]

Note that, among other equivalent definitions, a function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) is said to be differentiable at \( q \in \mathbb{R}^d \) if there exist a vector \( \nabla h(q) \) and a function \( \varphi(q; \cdot) \) (with its domain being a deleted neighborhood of the origin \( 0 \in \mathbb{R}^d \)) such that \( \lim_{\Delta q \to 0} \varphi(q; \Delta q) = 0 \), and

\[ h(q + \Delta q) = h(q) + \nabla h(q)^\top \Delta q + ||\Delta q|| \varphi(q; \Delta q), \]

for any non-zero vector \( \Delta q \in \mathbb{R}^d \) that is sufficiently close to \( 0 \in \mathbb{R}^d \) (e.g., see fundamental increment lemma and the definition of differentiability for multivariable functions). Thus, with sufficiently small perturbations \( \Delta a \in \mathbb{R}^{d_y} \) and \( \Delta W = [\Delta w_1 \Delta w_2 \ldots \Delta w_{d_y}] \in \mathbb{R}^{d_x \times d_y} \), there exists a function \( \varphi \) such that

\[
\tilde{L}(\theta, a + \Delta a, b, W + \Delta W) = \frac{1}{m} \sum_{i=1}^{m} \ell_{y_i}(f(x_i; \theta) + \Delta g_i) + \lambda ||\Delta a||^2_2
\]

\[
= \frac{1}{m} \sum_{i=1}^{m} \ell_{y_i}(f(x_i; \theta)) + \nabla \ell_{y_i}(f(x_i; \theta))^\top \Delta g_i + \lambda ||\Delta a||^2_2,
\]

where \( \lim_{\Delta q \to 0} \varphi(f(x_i; \theta); \Delta q) = 0 \) and \( \Delta g_i = g(x_i; a, b, W + \Delta W) \). Here, the last line follows the definition of the differentiability of \( \ell_{y_i} \), since \( g(x_i; \Delta a, b, W + \Delta W) \). Combining the above two equations, since \((a, b, W)\) is a local minimum, we have that, for any sufficiently small \( \Delta a \) and \( \Delta w_i \),

\[
\tilde{L}(\theta, a + \Delta a, b, W + \Delta W) - \tilde{L}(\theta, a, b, W)
\]

\[
= \frac{1}{m} \sum_{i=1}^{m} \nabla \ell_{y_i}(f(x_i; \theta))^\top \Delta g_i + \frac{1}{m} \sum_{i=1}^{m} ||\Delta g_i||^2_2 \varphi(f(x_i; \theta); \Delta g_i) + \lambda ||\Delta a||^2_2
\]

\[
\geq 0.
\]

Rearranging with \( \Delta a = \epsilon v \) such that \( \epsilon > 0 \) and \( ||v||^2_2 = 1 \), and with \( \Delta \tilde{g}_i = g(x_i; v, b, W + \Delta W) \),

\[
\frac{\epsilon}{m} \sum_{i=1}^{m} \nabla \ell_{y_i}(f(x_i; \theta))^\top \Delta \tilde{g}_i
\]

\[
\geq -\frac{\epsilon}{m} \sum_{i=1}^{m} ||\Delta \tilde{g}_i||^2_2 \varphi(f(x_i; \theta); \epsilon \Delta \tilde{g}_i) - \epsilon^2 ||v||^2_2,
\]

since \( \Delta g_i = \epsilon \Delta \tilde{g}_i \). By multiplying \( 1/\epsilon \) on both sides,

\[
\frac{1}{m} \sum_{i=1}^{m} \nabla \ell_{y_i}(f(x_i; \theta))^\top \Delta \tilde{g}_i
\]

\[
\geq -\frac{1}{m} \sum_{i=1}^{m} ||\Delta \tilde{g}_i||^2_2 \varphi(f(x_i; \theta); \epsilon \Delta \tilde{g}_i) - \epsilon ||v||^2_2.
\]

Since \( \varphi(f(x_i; \theta); \epsilon \Delta \tilde{g}_i) \to 0 \) and \( \epsilon ||v||^2_2 \to 0 \) as \( \epsilon \to 0 \),

\[
\sum_{i=1}^{m} \nabla \ell_{y_i}(f(x_i; \theta))^\top g(x_i; v, b, W + \Delta W) \geq 0.
\]

For any \( k \in \{1, 2, \ldots, d_y\} \), by setting \( v_{k'} = 0 \) for all \( k' \neq k \), we have that

\[
v_k \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_{k} \exp(w^*_k x_i + \Delta w^*_k x_i + b_k) \geq 0,
\]
for any $v_k \in \mathbb{R}$ such that $|v_k| = 1$. Hence,
\[
m \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k) \exp(\Delta w_k x_i) = 0.
\]

By setting $\Delta w_k = \bar{c}_k u_k$ such that $\bar{c}_k > 0$ and $\|u\|_2 = 1$,
\[
\lim_{T \rightarrow \infty} \sum_{t=0}^{T} \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k)(u_k^T x_i)^t = 0,
\]
since $\exp(q) = \lim_{T \rightarrow \infty} \sum_{t=0}^{T} q^t$ and a finite sum of limits of convergent sequences is the limit of the finite sum. Rewriting this using $z_t = \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k)(u_k^T x_i)^t$,
\[
\lim_{T \rightarrow \infty} \sum_{t=0}^{T} \frac{\bar{c}_k}{t!} z_t = 0. \tag{1}
\]

We now show that $z_p = 0$ for all $p \in \mathbb{N}_0$ by induction. Consider the base case with $p = 0$. Equation (1) implies that
\[
\lim_{T \rightarrow \infty} \left( z_0 + \sum_{t=1}^{T} \frac{\bar{c}_k}{t!} z_t \right) = z_0 + \lim_{T \rightarrow \infty} \sum_{t=1}^{T} \frac{\bar{c}_k}{t!} z_t = 0
\]
since $\lim_{T \rightarrow \infty} \sum_{t=1}^{T} \frac{\bar{c}_k}{t!} z_t$ exists (which follows that $\lim_{T \rightarrow \infty} \sum_{t=0}^{T} \frac{\bar{c}_k}{t!} z_t = 0$ exists). Here, $\lim_{T \rightarrow \infty} \sum_{t=1}^{T} \frac{\bar{c}_k}{t!} z_t \rightarrow 0$ as $\bar{c} \rightarrow 0$, and hence $z_0 = 0$. Consider the inductive step with the inductive hypothesis that $z_t = 0$ for all $t \leq p - 1$. Similarly to the base case, Equation (1) implies
\[
\sum_{t=0}^{p-1} \frac{\bar{c}_k}{t!} z_t + \frac{\bar{c}_k^p}{p!} z_p + \lim_{T \rightarrow \infty} \sum_{t=p+1}^{T} \frac{\bar{c}_k}{t!} z_t = 0.
\]
Multiplying $p!/\bar{c}_k^p$ on both sides, since $\sum_{t=0}^{p-1} \frac{\bar{c}_k}{t!} z_t = 0$ from the inductive hypothesis,
\[
z_p + \lim_{T \rightarrow \infty} \sum_{t=p+1}^{T} \frac{\bar{c}_k^p}{p!} t! z_t = 0.
\]
Since $\lim_{T \rightarrow \infty} \sum_{t=p+1}^{T} \frac{\bar{c}_k^p}{p!} t! z_t \rightarrow 0$ as $\bar{c} \rightarrow 0$, we have that $z_p = 0$, which finishes the induction. Therefore, for any $k \in \{1, 2, \ldots, d_y\}$ and any $p \in \mathbb{N}_0$,
\[
\sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k)(u_k^T x_i)^p = 0.
\]

\[\square\]

4.2. Proofs of theorems

We now present the proofs of the theorems, using the following additional notation. Let $x \otimes x$ be the tensor product of the vectors $x$ and $x^{\otimes p} = x \otimes \cdots \otimes x$ where $x$ appears $p$ times. For a $p$-th order tensor $M \in \mathbb{R}^{d_1 \times \cdots \times d}$ and $p$ vectors $u^{(1)}, u^{(2)}, \ldots, u^{(p)} \in \mathbb{R}^d$, defines
\[
M(u^{(1)}, u^{(2)}, \ldots, u^{(p)}) = \sum_{1 \leq i_1 \cdots i_p \leq d} M_{i_1 \cdots i_p} u^{(1)}_{i_1} \cdots u^{(p)}_{i_p}.
\]
Define $\{I_1, \ldots, I_{m'}\}$ as a partition of the set $\{1, \ldots, m\}$ (i.e., $I_1 \cup \cdots \cup I_{m'} = \{1, \ldots, m\}$, $I_j \cap I_{j'} = \emptyset$ for all $j \neq j'$, and $I_j \neq \emptyset$ for all $j \in \{1, \ldots, m'\}$) such that for any $x \in I_j$ and $x' \in I_{j'}$, $x = x'$ if $j = j'$, and $x \neq x'$ if $j \neq j'$. Let $\bar{x}_j := x$ with a representative $x \in I_j$.

**Proof of Theorem 1.** Let $\theta$ be fixed. Let $(a, b, W)$ be a local minimum of $\tilde{L}|_{\theta}(a, b, W) := \tilde{L}(\theta, a, b, W)$. Let $\xi_{i,k} = (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k)$. Then, for any $k \in \{1, 2, \ldots, d_y\}$ and any $p \in \mathbb{N}_0$,
\[
\max_{\|u^{(1)}\|_2 = \cdots = \|u^{(p)}\|_2 = 1} \left( \sum_{i=1}^{m} \xi_{i,k} x_i^{\otimes p} \right) (u^{(1)}, \ldots, u^{(p)}) = \max_{\|u\|_2 = 1} \left( \sum_{i=1}^{m} \xi_{i,k} x_i \right) (u, u, \ldots, u) = \max_{\|u\|_2 = 1} \sum_{i=1}^{m} \xi_{i,k} (u^T x_i)^p = 0.
\]
where the second line follows theorem 2.1 in (Zhang et al., 2012), and the third line follows Lemma 2. This implies that
\[
\sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k) \exp(x_i^{\otimes p}) = 0.
\]
Using Equation (2), we now prove statement (i). For any $\theta'$, there exist $p$ and $u_{i,k}$ (for $t = 1, \ldots, p$ and $k = 1, \ldots, d_y$) such that
\[
\sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta'))) \exp(x_i^{\otimes p}) = 0
\]
\[
\sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta)))_k \exp(w_k^T x_i + b_k) \exp(x_i^{\otimes p}) = 0
\]
\[
\sum_{t=1}^{p} \sum_{i=1}^{d_y} u_{i,k}^T \nabla \ell_{y_i}(f(x_i; \theta)) \exp(w_k^T x_i + b_k) \exp(x_i^{\otimes p}) = 0
\]
\[
= 0 \quad \text{(from Equation (2))}
\]
\[
= 0
\]
where the second line follows from the assumption that $\ell_{y_i}$ is convex and differentiable, and the fourth line follows from the fact that $\bar{x}_j = x$ for all $x \in I_j$. The fifth line follows from the fact that the vector $\vec{v}(x_{i})$ contains all monomials in $x_i$ of degree $t$, and $m'$ input points $\bar{x}_1, \ldots, \bar{x}_{m'}$ are distinct, which allows the basic existence (and construction) result of a polynomial interpolation of the finite $m'$ points; i.e., with $p$ sufficiently large ($p = m' - 1$ is sufficient), for each $k$, there exists $u_{t,k}$ such that $\sum_{t=1}^{p} u_{t,k} \vec{v}(x_{i}) = a_{j,k}$ for any $j,k \in \mathbb{R}$ for all $j \in \{1, \ldots, m'\}$ (e.g., see equation (1.9) in Gasca & Sauer 2000), in particular, including $a_{j,k} = (f(\bar{x}_j; \theta') - f(\bar{x}_j; \theta)) \exp(-w_{k}' \bar{x}_j - b_k)$.

Therefore, we have that, for any $\theta'$, $L(\theta') \geq L(\theta)$, which proves statement (i). Statement (ii) directly follows from Lemma 1. □

Proof of Theorem 2. Let $\theta$ be fixed. Let $(a, b, W)$ be a local minimum of $\bar{L}\theta(a, b, W) := L(\theta, a, b, W)$. Then, for any $k \in \{1, 2, \ldots, d_y\}$, there exist $p$ and $u_{t,k}$ (for $t = 1, \ldots, p$) such that

$$
\begin{align*}
\sum_{i=1}^{m'} (\nabla \ell_{y_i}(f(x_i; \theta)))^2_k \\
= \sum_{j=1}^{m} |I_j| (\nabla \ell_{f_j(x_j)}(f(\bar{x}_j; \theta)))^2_k \\
= \sum_{t=1}^{p} u_{t,k} \sum_{i=1}^{m} (\nabla \ell_{y_i}(f(x_i; \theta))) \exp(w_{k}' x_i + b_k) \vec{v}(x_{i}) \\
= 0,
\end{align*}
$$

where the second line utilizes Assumption 3. The third line follows from the fact that since $m'$ input points $\bar{x}_1, \ldots, \bar{x}_{m'}$ are distinct, with $p$ sufficiently large ($p = m' - 1$ is sufficient), for each $k$, there exist $u_{t,k}$ for $t = 1, \ldots, p$ such that $\sum_{t=1}^{p} u_{t,k} \vec{v}(x_{i}) = (\nabla \ell_{f_j(x_j)}(f(\bar{x}_j; \theta))) \exp(-w_{k}' \bar{x}_j - b_k) |I_j|^{-1}$ (similarly to the proof of Theorem 1). The fourth line follows from Equation (2). Here, Equation (2) still holds because it is obtained in the proof of Theorem 1 under only the assumption that the function $\ell_{y_i}$: $q \mapsto \ell(q, y_i)$ is differentiable for any $i \in \{1, \ldots, m\}$, which is still satisfied by Assumption 2.

This implies that for all $i \in \{1, \ldots, m\}$, $\nabla \ell_{y_i}(f(x_i; \theta)) = 0$, which proves statement (iii) because of Assumption 2. Statement (i) directly follows from Statement (iii). Statement (ii) directly follows from Lemma 1. □

5. On the limitation of eliminating bad local minima

Our theoretical results in the previous sections have shown that all suboptimal local minima can be removed by simply adding one neuron per output unit for a wide range of deep learning tasks. This might be surprising given the fact that dealing with the suboptimal local minima in general is known to be challenging in theory.

However, eliminating all suboptimal local minima is not sufficient to guarantee the global optimality in polynomial time without taking advantage of additional assumptions or structures of neural networks. This can be inferred from the fact that finding a global minimum of a general objective function to train an arbitrary neural network is NP-hard. The major reason for this might include the possibility that adding such neurons does not solve the issue of the possible discrete nature of the space of $\theta$, and of bad saddle points (e.g., see Kawaguchi 2016). In particular, there can be more bad saddle points for $\tilde{L}$ than for $L$. The issue of ill-conditioning also remains a concern, particularly for a first-order optimization algorithm, such as (stochastic) gradient descent.

The following theorem suggests another possible problem: an iterative optimization algorithm might converge to a suboptimal local minimum $\theta$ of $\tilde{L}$ and diverge in terms of $(\theta, a, b, W)$ of $L$. The proof of Theorem 3 is presented in Appendix A. Note that Theorem 3 also holds true for the problem setting in Propositions 1 and 2 because their assumptions are stronger.

Theorem 3. Let Assumption 1 hold, or let Assumptions 2 and 3 hold. Then, for any $\theta$, if $\theta$ is not a global minimum of $L$, there is no local minimum $(\theta, a, b, W) \in \mathbb{R}^{d_y} \times \mathbb{R}^{d_y} \times \mathbb{R}^{d_x} \times \mathbb{R}_+$ of $L\theta(a, b, W) := L(\theta, a, b, W)$. Furthermore, there exist a tuple $(\ell, f, \{(x_i, y_i)\}_{i=1}^{m})$ and a suboptimal stationary point $\theta$ of $L$ such that $\frac{\partial L(\theta, a, b, W)}{\partial \theta} = 0$ for all $(a, b, W) \in \mathbb{R}^{d_y} \times \mathbb{R}^{d_y} \times \mathbb{R}^{d_x} \times \mathbb{R}_+$.

Figure 1 illustrates the limitation suggested by Theorem 3. Here, the loss criterion $\ell$ is set to be a squared loss, but the function $f$ is differentiable for any $i \in \{1, \ldots, m\}$, which is still satisfied by Assumption 2.

Example 1. Let $m = 1$, $d_y = 1$, and $x_1 = 0$. In addition, let $L(\theta) = \ell(f(x_1; \theta), y_1) = (f(x_1; \theta) - y_1)^2$. Because of the function $f$ in our results (Theorems 1 - 3) and in the previous results (Propositions 1 and 2) can be of any form, let $f(x_1; \theta) = 5(-0.3e^{-166(\theta - 0.2)^2} - 0.7e^{-32(\theta - 0.8)^2} + 0.5)$ for a simple illustration. Because $x_1 = 0$, we can think of this function as a model with an extra parameter $\theta'$, the effect of which disappears as $\theta'x_1 = 0$ (e.g., $f(x_1; \theta') = f(x_1; \theta') = 5(-0.3e^{-166(\theta' + 0.2)^2} - 0.7e^{-32(\theta' + 0.8)^2} + 0.5)$). Let $y_1 = f(x_1; 0.8)$.

To further clarify this limitation in an analytical manner, we now present several simple examples without specify-
Example 2. Let \( m = 1 \) and \( d_y = 1 \). In addition, let \( L(\theta) = \ell(f(x_1; \theta), y_1) = (f(x_1; \theta) - y_1)^2 \). Accordingly, \( \tilde{L}(\theta, a, b, W) = (f(x_1; \theta) + a \exp(w^\top x_1 + b) - y_1)^2 + \lambda a^2 \).

Let \( \theta \) be a non-global minimum of \( L \) as \( f(x_1; \theta) \neq y_1 \). In particular, let us first consider the case of \( f(x_1; \theta) = 2 \) and \( y_1 = 1 \). Then, \( L(\theta) = 1 \) and

\[
\tilde{L}(\theta, a, b, W) = 1 + 2a \exp(w^\top x_1 + b) + a^2 \exp(2w^\top x_1 + 2b) + \lambda a^2.
\]

If \((a, b, W)\) is a local minimum, from the stationary point conditions of \( \tilde{L}_{\theta a} = 0 \) and \( \tilde{L}_{\theta b} = 0 \), we must have \( a = 0 \), resulting in \( \tilde{L}(\theta, a, b, W) = 1 \). However, a point with \( a = 0 \) is clearly not a local minimum (with finite \((b, w)\)), because with \( a < 0 \) and \( |a| > 0 \) being sufficiently small,

\[
\tilde{L}(\theta, a, b, W) = 1 - 2|a| \cdot \exp(w^\top x_1 + b) + |a|^2 \exp(2w^\top x_1 + 2b) + \lambda |a|^2 < 0
\]

with \( |a| > 0 \) being sufficiently small.

Hence, there is no local minimum \((a, b, W) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d\) of \( \tilde{L}_{\theta a} \). Indeed, if we set \( a = -\exp(-1/\epsilon) \) and \( b = 1/\epsilon - w^\top x_1 \),

\[
\tilde{L}(\theta, a, b, W) = \lambda \exp(-2/\epsilon) \to 0
\]
as \( \epsilon \to 0 \), and hence as \( a \to 0^- \) and \( b \to \infty \), illustrating the case in which \((a, b)\) does not attain a solution in \( \mathbb{R} \times \mathbb{R} \). The identical conclusion holds with the general case of \( f(x_1; \theta) \neq y_1 \) by following the same steps of reasoning.

Example 3 illustrates the same phenomena as those in Example 2, but uses a smoothed hinge loss that satisfies the assumptions of Proposition 1.

Example 3. Let \( m = 1 \) and \( d_y = 1 \). In addition, \( L(\theta) = \ell(f(x_1; \theta), y_1) = (\max(0, 1 - y_1 f(x_1; \theta)) \} \). Accordingly, \( \tilde{L}(\theta, a, b, W) = (\max(0, 1 - y_1 f(x_1; \theta) - y_1 a \exp(w^\top x_1 + b)) \} + \lambda a^2 \).

Let \( \theta \) be a non-global minimum of \( L \) as \( f(x_1; \theta) \neq y_1 \), in particular, by setting \( f(x_1; \theta) = -1 \) and \( y_1 = 1 \). Then, \( L(\theta) = 8 \). If \((a, b, W)\) is a local minimum, we must have \( a = 0 \) similarly to Example 2, yielding that \( \tilde{L}(\theta, a, b, W) = 8 \). However, a point with \( a = 0 \) is clearly not a local minimum because with \( a > 0 \) being sufficiently small,

\[
\tilde{L}(\theta, a, b, W) = (2 - a \exp(w^\top x_1 + b)) + \lambda a^2 < 8.
\]

because the first order term in \( a \) becomes dominant with \( a > 0 \) being sufficiently small. Hence, there is no local minimum \((a, b, W) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d\) of \( \tilde{L}_{\theta a} \). Indeed, if we set \( a = -\exp(-1/\epsilon) \) and \( b = 1/\epsilon - w^\top x_1 \),

\[
\tilde{L}(\theta, a, b, W) = \lambda \exp(-2/\epsilon) \to 0
\]
as \( \epsilon \to 0 \), and hence as \( a \to 0^- \) and \( b \to \infty \), illustrating the case in which \((a, b)\) does not attain a solution in \( \mathbb{R} \times \mathbb{R} \). The identical conclusion holds with the general case of \( f(x_1; \theta) \neq y_1 \) by following the same logic.

Examples 4 and 5 are the versions of Examples 2 and 3 with two data points instead one. Unlike Examples 2 and 3, Examples 4 and 5 show that the value of \( \tilde{L} \) can also approach a suboptimal value.

Example 4. Let \( m = 2 \) and \( d_y = 1 \). In addition, \( L(\theta) = (f(x_1; \theta) - y_1)^2 + (f(x_2; \theta) - y_2)^2 \). Let us consider the case of \( f(x_1; \theta) = f(x_2; \theta) = 0, y_1 = 1, \) and
y_2 = -1. Then, \( L(\theta) = 2 \). If \((a, b, W)\) is a local minimum, we must have \( a = 0 \) similarly to Example 2, yielding that \( \tilde{L}(\theta, a, b, W) = 2 \). On the other hand,

\[
\tilde{L}(\theta, a, b, W) = 2 - 2a(\exp(w^\top x_1 + b) - \exp(w^\top x_2 + b)) + \varphi(a^2),
\]

where \( \varphi(a^2) = a^2 \exp(2w^\top x_1 + 2b) + a^2 \exp(2w^\top x_2 + 2b) + \lambda a^2 \). Note that, with a sufficiently small \(|a| > 0\), the term \( \varphi(a^2) \) becomes negligible. Let \( x_1 \neq x_2 \). In this case, our \( \theta \) with \( f(x_1; \theta) = f(x_2; \theta) = 0 \) is not a global minimum. Therefore, there is no local minimum.

Example 5. Let \( m = 2 \) and \( d_y = 1 \). In addition, \( L(\theta) = (\max(0, 1-y_1 f(x_1; \theta))^3) + (\max(0, 1-y_2 f(x_2; \theta))^3 \). Moreover, let \( x_1 \neq x_2 \). Finally, let \( f(x_1; \theta) = -1, f(x_2; \theta) = 1, y_1 = 1, \) and \( y_2 = -1 \). If \((a, b, W)\) is a local minimum, we must have \( a = 0 \) similarly to Example 2, yielding \( \tilde{L}(\theta, a, b, W) = 16 \). However, a point with \( a = 0 \) is not a local minimum, which follows from the perturbations of \((a, W)\) in the same manner as in Example 4. Therefore, there is no local minimum \((a, b, W)\) of \( \tilde{L}(\theta, a, b, W) \). Indeed, if we set \( a = 2 \exp(-1/\epsilon), b = 1/\epsilon - w^\top x_1 \), and \( w = -\frac{1}{4}(x_2 - x_1) \),

\[
\tilde{L}(\theta, a, b, W) = (2 + 2 \exp(-\|x_2 - x_1\|^2/\epsilon^2))\lambda \exp(-2/\epsilon) \rightarrow 8
\]

as \( \epsilon \rightarrow 0 \), and hence as \( a \rightarrow 0^- \), \( b \rightarrow \infty \) and \( \|w\| \rightarrow \infty \), illustrating the case in which \((a, b, W)\) does not attain a solution in \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \).

As can be seen through Examples 1 - 5, an essential issue is that the solution may not be attained in any bounded subspace. Indeed, a classical proof using the Weierstrass theorem to guarantee the existence of the optimal solutions in a (nonempty) subspace \( S \subseteq \mathbb{R}^d \) requires a lower semi-continuity of the objective function \( \tilde{L} \) and the existence of a \( q \in S \) for which the set \( \{ q' \in S : \tilde{L}(q') \leq \tilde{L}(q) \} \) is compact (e.g., see Bertsekas 1999 for more discussion on the existence of optimal solutions). In the above examples, given a suboptimal \( \theta \), the former condition of lower semi-continuity is satisfied, whereas the latter condition of compactness is not.

Although our results as well as those of Liang et al. (2018) are subject to this limitation, these results are not vacuous in the following sense: there exists a local minimum of \( \tilde{L} \) if there exists a global minimum \( \theta \) of \( L \) such that \( f(x_1; \theta) \) achieves a global minimum for each \( f(x_2; \theta) \) for \( i \in \{1, \ldots, m\} \). This is because, given such a \( \theta \), any point with \( a = 0 \) is a local minimum of \( \tilde{L} \).

Example 6. Consider the same example as in Example 4, with the exception that \( x_1 = x_2 \). In this case, Assumption 3 does not hold, and a \( \theta \) with \( f(x_1; \theta) = f(x_2; \theta) = 0 \) is a global minimum unlike in Example 4. A point with \( a = 0 \) is indeed a local minimum, which can be seen in Equation (3) as \( -2a \exp(w^\top x_1 + b) + 2a \exp(w^\top x_2 + b) = 0 = \varphi(a^2) \).

Furthermore, adding the exponential neurons to a neural network may help gradient-based optimization algorithms when \( \frac{\partial L}{\partial \theta} \) gives better information than \( \frac{\partial L}{\partial w} \). For example, in Figure 1 (a), there is a steep hill in \( L \) when we move from \( \theta = 0.2 \) towards \( \theta = 0.8 \). This might prevent \( \theta \) from escaping the region near \( \theta = 0.2 \), even with random noises in the search direction of \( \theta \). In Figure 1 (b), as \( b \) increases, the hill becomes flatter in \( \tilde{L} \), which might make it easier to escape the region. The proof of Theorem 3 also reveals a scenario in which adding the exponential neurons to a neural network would help the optimization by creating first-order decreasing directions (see Appendix A for further details).

6. Conclusion

In this paper, we proved that all suboptimal local minima (which are not global minima) can be eliminated by simply adding one unit per output unit to any deep neural network with commonly used loss criteria. Furthermore, we discussed several limitations with regard to this strategy for eliminating all bad local minima. In particular, Theorem 3 together with Examples 1 - 5 shows that an iterative optimization algorithm can converge to a suboptimal local minimum \( \theta \) (i.e., a local minimum that is not a global minimum), and diverge in terms of \( \theta, a, b, W \) of \( \tilde{L} \).

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### Appendix

Given a function \( \varphi(q) \in \mathbb{R}^d \) and a vector \( v \in \mathbb{R}^{d'} \), let \( \frac{\partial \varphi(q)}{\partial v} \) be a \( d \times d' \) matrix with each entry \( \left( \frac{\partial \varphi(q)}{\partial v} \right)_{i,j} = \frac{\partial \varphi(q)}{\partial v} \).

#### A. Proof of Theorem 3

The proofs of Theorems 1 and 2 are designed such that the proof of Theorem 3 is simple, as shown below.

**Proof of Theorem 3.** Let Assumption 1 hold (instead of Assumptions 2 and 3). In our proof of Theorem 1 (as well as the proofs of Lemmas 1 and 2), \( \theta \) was arbitrary and \((a, b, W)\) was an arbitrary local minimum of \( L|_\theta(a, b, W) := L(\theta, a, b, W) \). Thus, the same proof proves that, for any \( \theta \), at every local minimum \((a, b, W) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \times \mathbb{R}^{d_z \times d_y} \) of \( L|_\theta \), \( \theta \) is a global minimum of \( L \). Thus, based on the logical equivalence \((p \rightarrow q) \equiv \neg q \rightarrow \neg p\), if \( \theta \) is not a global minimum of \( L \), then there is no local minimum \((a, b, W) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \times \mathbb{R}^{d_z \times d_y} \) of \( L|_\theta \), proving the first statement in the case of using Assumption 1. Instead of Assumption 1, if Assumptions 2 and 3 hold, then the exact same proof as above (with Theorem 1 being replaced by Theorem 2) proves the first statement.

Example 1 suffices to prove the second statement. However, to obtain better theoretical insight, let us consider a more general construction of the desired tuples \((\ell, f, \{(x_i, y_i)\}_{i=1}^m)\) to prove the second statement. Let \( \theta \in \mathbb{R}^{d_{\theta}} \). In addition, let \( A[\theta] = \frac{1}{m} \left( \left( \frac{\partial f(x_i; \theta)}{\partial \theta} \right) \cdot \ldots \cdot \left( \frac{\partial f(x_n; \theta)}{\partial \theta} \right) \right) \in \mathbb{R}^{d_{\theta} \times m (d_{\theta})} \), be a matrix, and \( r[\varphi] = \left[ \nabla_{\theta_1} \varphi(x_1) \right] \cdot \ldots \cdot \left[ \nabla_{\theta_n} \varphi(x_n) \right] \right)^\top \in \mathbb{R}^{m (d_{\theta})} \) be a column vector given a function \( \varphi : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_y} \). Then,

\[
\frac{\partial L(\theta)}{\partial \theta} = \frac{1}{m} \sum_{i=1}^m \nabla_{y_i} f(x_i; \theta) \left( \frac{\partial f(x_i; \theta)}{\partial \theta} \right)^\top,
\]

and

\[
\frac{\partial \tilde{L}(\theta, a, b, W)}{\partial \theta} = (A[\theta] r[f(\cdot; \theta)] + g(\cdot; a, b, W))^\top.
\]

Here, the equality \( A[\theta] r[f(\cdot; \theta)] = 0 \) is equivalent to \( r[f(\cdot; \theta)] \in \text{Null}(A[\theta]) \), where \( \text{Null}(A[\theta]) \) is the null space of the matrix \( A[\theta] \). Therefore, any tuple \((\ell, f, \{(x_i, y_i)\}_{i=1}^m)\) such that \( r[f(\cdot; \theta)] \in \text{Null}(A[\theta]) \Rightarrow r[f(\cdot; \theta)] + g(\cdot; a, b, W) \in \text{Null}(A[\theta]) \) at a suboptimal \( \theta \) suffices to provide a proof for the second statement. An (infinite) set of tuples \((\ell, f, \{(x_i, y_i)\}_{i=1}^m)\) such that there exists a suboptimal \( \theta \) with \( A[\theta] = 0 \) (e.g., Example 1) satisfies this condition, which proves the second statement.

The construction of worst-case type examples in the proof of Theorem 3 also shows a construction of best-case type examples, where adding the exponential neurons to a neural network would help the optimization. Let \( \theta \in \mathbb{R}^{d_{\theta}} \) be a suboptimal stationary point as \( r[f(\cdot; \theta)] \in \text{Null}(A[\theta]) \). Then, if the added function \( g(\cdot; a, b, W) \) moves \( r[f(\cdot; \theta) + g(\cdot; a, b, W)] \) out from the linear subspace \( \text{Null}(A[\theta]) \), first-order decreasing directions occur in \( \theta \) for \( \tilde{L} \), although not for \( L \). Because there is no local minimum for \((a, b, W) \) unless \( \theta \) is a global minimum, \( g(\cdot; a, b, W) \) may continue changing during an optimization search, and \( g(\cdot; a, b, W) \) might prevent \( r[f(\cdot; \theta) + g(\cdot; a, b, W)] \) from moving too close to \( \text{Null}(A[\theta]) \) with a suboptimal \( \theta \).

### B. Illustration of the limitation with a smoothed hinge loss

Figure 2 illustrates the limitation suggested by Theorem 3 with a smoothed hinge loss. Here, the setting is exactly same as that in Figure 1 except that \( \ell(f(x_1; \theta), y_1) = (\max(0, 1 - y_1 f(x_1; \theta))^3 \) and \( y_1 = -1 \).