Glivenko’s theorem, finite height, and local tabularity

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Abstract

Glivenko’s theorem states that a formula is derivable in classical propositional logic \( CL \) iff under the double negation it is derivable in intuitionistic propositional logic \( IL \): \( CL \vdash \varphi \) iff \( IL \vdash \neg \neg \varphi \). Its analog for the modal logics \( S5 \) and \( S4 \) states that \( S5 \vdash \varphi \) iff \( S4 \vdash \neg \Box \neg \varphi \). In Kripke semantics, \( IL \) is the logic of partial orders, and \( CL \) is the logic of partial orders of height 1. Likewise, \( S4 \) is the logic of preorders, and \( S5 \) is the logic of equivalence relations, which are preorders of height 1. In this paper we generalize Glivenko’s translation for logics of arbitrary finite height.

Keywords: Glivenko’s translation, modal logic, intermediate logic, finite height, pre-transitive logic, local tabularity, local finiteness, top-heavy frame

1 Introduction

For a modal or intermediate logic \( L \), let \( L[h] \) be its extension with the formula restricting the height of a Kripke frame by finite \( h \). In the intermediate case, such formulas are defined as \( B_0^i = \bot \), \( B_h^i = p_h \lor (p_h \to B_{h-1}^i) \), and in the modal transitive case as \( B_0 = \bot \), \( B_h = p_h \to \Box (\Diamond p_h \lor B_{h-1}) \). In particular, classical logic \( CL \) is the extension of intuitionistic logic \( IL \) with the formula \( p \lor \neg p \), that is \( CL = IL[1] \). Similarly, \( S5 = S4[1] \).

Glivenko’s translation \cite{13} and its analog for the modal logics \( S5 \) and \( S4 \) \cite{20} can be formulated as follows:

\[
\begin{align*}
IL[1] \vdash \varphi & \quad \text{iff} \quad IL \vdash \neg \neg \varphi, \quad (1) \\
S4[1] \vdash \varphi & \quad \text{iff} \quad S4 \vdash \Diamond \Box \varphi. \quad (2)
\end{align*}
\]

For finite variable fragments of \( IL \) and \( S4 \), the above equivalences can be generalized for arbitrary finite height. A \( k \)-formula is a formula in variables \( p_0, \ldots, p_{k-1} \). Let \((W,R)\) be

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the $k$-generated canonical frame of $S4$ (that is, $W$ is the set of maximal $S4$-consistent sets of $k$-formulas). It follows from \cite{24} (see also \cite{25}, \cite{9}, \cite{1}, \cite{2}) that there exist formulas $B_{h,k}$ (and their intuitionistic analogs $B_{i,h,k}$) such that for every $x \in W$, $B_{h,k} \in x$ iff the depth of $x$ in $W$ is less than or equal to $h$. We observe that for all finite $k$, for all $k$-formulas $\varphi$,

$$IL[h + 1] \vdash \varphi \text{ iff } IL \vdash \bigwedge_{i \leq h} ((\varphi \rightarrow B_{i,k}^i) \rightarrow B_{i,k}^i),$$

(3)

$$S4[h + 1] \vdash \varphi \text{ iff } S4 \vdash \bigwedge_{i \leq h} (\Box(\Box \varphi \rightarrow B_{i,k}) \rightarrow B_{i,k}).$$

(4)

In particular, for $h = 0$ we have equivalences (1) and (2), since the formulas $B_{0,k}$ and $B_{i,0,k}$ are $\bot$ for all $k$.

Sometimes, analogs of the formulas $B_{h,k}$ exist for unimodal logics smaller than $S4$ and for polymodal logics. A modal logic $L$ is pretransitive (or weakly transitive, in another terminology), if the transitive reflexive closure modality $\Diamond^*$ is expressible in $L$. Namely, for the language with $n$ modalities $\Diamond_i$ ($i < n$), put $\Diamond^0 \varphi = \varphi$, $\Diamond^{m+1} \varphi = \Diamond^m \bigvee_{i < n} \Diamond_i \varphi$, $\Diamond^{\leq m} \varphi = \bigvee_{i \leq m} \Diamond^i \varphi$. A logic $L$ is pretransitive if it contains $\Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p$ for some finite $m$. In this case $\Diamond^{\leq m}$ plays the role of $\Diamond^*$. The height of a polymodal frame $(W, (R_i)_{i < n})$ is the height of the preorder $(W, (\bigcup_{i < n} R_i)^*)$. In the pretransitive case, formulas of finite height can be defined analogously to the transitive case.

$L$ is said to be $k$-tabular if, up to the equivalence in $L$, there exist only finitely many $k$-formulas. $L$ is locally tabular (or locally finite) if it is $k$-tabular for every finite $k$.

We show (Theorems 5 and 6) that if $L$ is a pretransitive logic, $h, k < \omega$, and $L[h]$ is $k$-tabular, then:

1. For every $i \leq h$, there exists a formula $B_{i,k}$ such that $B_{i,k} \in x$ iff the depth of $x$ in the $k$-generated canonical frame of $L$ is less than or equal to $i$.

2. For all $k$-formulas $\varphi$,

$$L[h + 1] \vdash \varphi \text{ iff } L \vdash \bigwedge_{i \leq h} (\Box^* (\Box^* \varphi \rightarrow B_{i,k}) \rightarrow B_{i,k}).$$

(5)

The equivalence (5) generalizes (4). Recall that a unimodal transitive logic is locally tabular iff it is of finite height iff it is 1-tabular (\cite{22}, \cite{18}). In the non-transitive case the situation is much more complicated. It follows from \cite{24} that every locally tabular (even 1-tabular) logic is a pretransitive logic of finite height; however, it follows from \cite{17} that there exists a pretransitive $L$ such that none of the logics $L[h]$ are 1-tabular.

In Section 5 we discuss how $k$-tabularity of $L[h]$ depends on $h$ and $k$. In particular, we construct the first example of a modal logic which is 1-tabular but not locally tabular.

2 Preliminaries

Fix a finite $n > 0$; $n$-modal formulas are built from a countable set $\{p_0, p_1, \ldots\}$ of proposition letters, the classical connectives $\rightarrow, \bot$, and the modal connectives $\Diamond_i$, $i < n$;
the other Boolean connectives are defined as standard abbreviations; $\Box_i$ abbreviates $\neg\bigcirc_i\neg$. We omit the subscripts on the modalities when $n = 1$. By a logic we mean a propositional $n$-modal normal logic, that is a set of $n$-modal formulas containing all classical tautologies, the formulas $\bigcirc_i(p \lor q) \rightarrow \bigcirc_i p \lor \bigcirc_i q$ and $\neg\bigcirc_i \bot$ for all $i < n$, and closed under the rules of Modus Ponens, Substitution, and Monotonicity (if $\varphi \rightarrow \psi$ is in the logic, then so is $\bigcirc_i \varphi \rightarrow \bigcirc_i \psi$).

For a logic $L$ and a set of formulas $\Psi$, the smallest logic containing $L \cup \Psi$ is denoted by $L + \Psi$. For a formula $\varphi$, the notation $L + \varphi$ abbreviates $L + \{\varphi\}$. In particular, $K4 = K + \bigcirc \bigcirc p \rightarrow \bigcirc p$, $S4 = K4 + p \rightarrow \bigcirc p$, $S5 = S4 + p \rightarrow \bigcirc \Box p$, where $K$ denotes the smallest unimodal logic. $L \vdash \varphi$ is a synonym for $\varphi \in L$.

The truth and the validity of modal formulas in Kripke frames and models are defined as usual, see, e.g., [6]. By a frame we always mean a Kripke frame $(W, (R_i)_{i<n})$. We put $R_F = \bigcup_{i<n} R_i$. The transitive reflexive closure of a relation $R$ is denoted by $R^*$; the notation $R(x)$ is used for the set $\{y \mid xRy\}$. The restriction of $F$ onto its subset $V$, $F|V$ in symbols, is $(V, (R_i \cap (V \times V))_{i<n})$. In particular, we put $F\langle x \rangle = F[R_F^*(x)]$.

For $k \leq \omega$, a $k$-formula is a formula in proposition letters $p_i$, $i < k$.

Let $L$ be a consistent logic. For $k \leq \omega$, the $k$-canonical model of $L$ is built from maximal $L$-consistent sets of $k$-formulas; the relations and the valuation are defined in the standard way, see e.g. [5]. Recall the following fact.

**Proposition 1** (Canonical model theorem). Let $M$ be the $k$-canonical model of a logic $L$, $k \leq \omega$. Then for all $k$-formulas $\varphi$ we have:

1. $M, x \models \varphi$ iff $\varphi \in x$, for all $x$ in $M$;
2. $M \models \varphi$ iff $L \vdash \varphi$.

A logic $L$ is said to be $k$-tabular if, up to the equivalence in $L$, there exist only finitely many $k$-formulas. $L$ is locally tabular (or locally finite) if it is $k$-tabular for every finite $k$. The following proposition is straightforward from the definitions.

**Proposition 2.** Let $L$ be a logic, $k < \omega$. The following are equivalent:

- $L$ is $k$-tabular.
- The $k$-generated Lindenbaum-Tarski algebra of $L$ is finite.
- The $k$-canonical model of $L$ is finite.

A unimodal logic is transitive if it contains the formula $\bigcirc \bigcirc p \rightarrow \bigcirc p$; recall that this formula expresses transitivity of a binary relation. Below we consider a weaker property, pretransitivity of (polymodal) logics and frames.

For a binary relation $R$ on a set $W$, put $R^{\leq m} = \bigcup_{i\leq m} R^i$, where $R^0 = Id(W)$, $R^{i+1} = R \circ R^i$. $R$ is called $m$-transitive if $R^{\leq m} = R^*$. $R$ is pretransitive if it is $m$-transitive for some $m$. A frame $F$ is $m$-transitive if $R_F$ is $m$-transitive.

Let $\bigcirc^0 \varphi = \varphi$, $\bigcirc^{i+1} \varphi = \bigcirc^i (\bigcirc \varphi \lor \ldots \lor \bigcirc \bigcirc \varphi)$, $\bigcirc^{\leq m} \varphi = \bigvee_{i \leq m} \bigcirc^i \varphi$, $\Box^{\leq m} \varphi = \neg \bigcirc^{\leq m} \neg \varphi$. 3
Proposition 3. Let F be a frame. The following are equivalent:

- F is \( m \)-transitive;
- \( R_{F}^{m+1} \subseteq R_{F}^{\leq m} \);
- \( F \models \Diamond^{m+1}p \rightarrow \Diamond^{\leq m}p \).

The proof is straightforward, details can be found, e.g., in [15].

Definition 1. A logic L is said to be \( m \)-transitive if \( L \vdash \Diamond^{m+1}p \rightarrow \Diamond^{\leq m}p \). L is pretransitive if it is \( m \)-transitive for some \( m \geq 0 \).

If L is pretransitive, then there exists the least \( m \) such that L is \( m \)-transitive; in this case we write \( \Diamond^{*} \varphi \) for \( \Diamond^{\leq m} \varphi \), and \( \Box^{*} \varphi \) for \( \Box^{\leq m} \varphi \).

For a unimodal formula \( \varphi \), \( \varphi^{*} \) denotes the formula obtained from \( \varphi \) by replacing \( \Diamond \) with \( \Diamond^{*} \) and \( \Box \) with \( \Box^{*} \).

Proposition 4. For a pretransitive logic L, the set \( \{ \varphi \mid L \vdash \varphi^{*} \} \) is a logic containing S4.

Proof. Follows from [12, Lemma 1.3.45]. \( \square \)

A poset is of height \( h < \omega \) if it contains a chain of \( h \) elements and no chains of cardinality \( > h \).

A cluster in a frame F is an equivalence class with respect to the relation \( \sim_{F} = R_{F}^{*} \cap R_{F}^{-1} \). For clusters C, D, put \( C \leq_{F} D \) iff \( xR_{F}^{*}y \) for some \( x \in C, y \in D \). The poset \((W/\sim_{F}, \leq_{F})\) is called the skeleton of F. The height of a frame F, in symbols \( \text{ht}(F) \), is the height of its skeleton.

Put
\[
B_{0} = \bot, \quad B_{i+1} = p_{i+1} \rightarrow \Box^{*} (\Diamond^{*} p_{i+1} \lor B_{i}).
\]

In the unimodal transitive case, the formula \( B_{h} \) expresses the fact that the height of a frame \( \leq h \) [22]. In the case when \( F = (W, (R_{i})_{i<n}) \) is \( m \)-transitive, the operator \( \Diamond^{*} = \Diamond^{\leq m} \) relates to \( R_{F}^{*} \). Since the height of F is the height of the preorder \((W, R_{F}^{*})\), we have \( F \models B_{h} \) iff \( \text{ht}(F) \leq h \).

Definition 2. A pretransitive logic is of finite height if it contains \( B_{h} \) for some \( h < \omega \). For a pretransitive L, we put
\[
L[h] = L + B_{h}.
\]

Example 1. Unimodal examples of 1-transitive logics are S4, wK4 = K + \( \Diamond \Diamond p \rightarrow \Diamond p \lor p \). The logic S5 and the difference logic DL = wK4 + p \rightarrow \Box \Diamond p \) are examples of logics of height 1.

A well-known logic K5 = K + \( \Diamond p \rightarrow \Box \Diamond p \) is a 2-transitive logic of height 2. To show this, recall that K5 is Kripke complete and its frames are those that validate the

\[\text{Pretransitive logics sometimes are called weakly transitive. However, in the other terminology, the term ‘weakly transitive’ is used for logics containing the formula } \Diamond \Diamond p \rightarrow \Diamond p \lor p.\]
∀x∀y∀z(xRy ∧ xRz → yRz). Every K5-frame is 2-transitive. Indeed, suppose that aRbRcRd for some elements of a K5-frame. Then bRb; we also have bRc, thus cRb; from cRb and cRd we infer that bRd. Thus aR^2d. It is not difficult to see that if a K5-frame F has an irreflexive serial point, then the height of F is 2; otherwise F is a disjoint sum of S5-frames and irreflexive singletons, so its height is 1.

**Theorem 1** ([22][18]). A unimodal transitive logic is locally tabular iff it is of finite height.

In [23], it was shown that every locally tabular unimodal logic is a pretransitive logic of finite height; in fact, the proof yields the following stronger formulation.

**Theorem 2.** If a logic is 1-tabular, then it is a pretransitive logic of finite height.

**Proof.** Let L be 1-tabular. Then its 1-canonical frame is finite. Every finite frame is m-transitive for some m. Thus L is m-transitive.

By Proposition 4, the set *L = {φ | L ⊢ □^∗ *φ} is a logic containing S4. Since L is 1-tabular, *L is 1-tabular. In [18], it was shown that for transitive logics 1-tabularity implies local tabularity. Thus *L is of finite height. It follows that L is of finite height too.

Thus, all locally tabular logics are pretransitive of finite height. However, unlike the transitive case, the converse is not true in general even for unimodal logics. Let Tr_m be the smallest m-transitive unimodal logic. For m ≥ 2, h ≥ 1, none of the logics Tr_m[h] are locally tabular [7]; moreover, they are not 1-tabular [17].

### 3 Translation for logics of height 1

For a pretransitive logics L, L[1] = L + B_1, that is L[1] is the smallest logic containing L ∪ {p → □^∗ □^∗ p}. It is known that S4[1] = S5 ⊢ φ iff S4 ⊢ □φ, and S5 ⊢ □ψ → □φ iff S4 ⊢ □□ψ → □φ [20][21]. In [16], it was shown that in the pretransitive unimodal case we have L[1] ⊢ φ iff L ⊢ □ □^∗ φ. In this section we generalize these facts to the polymodal case using the maximality property of pretransitive canonical frames (see Proposition 6 below).

**Proposition 5.** Let F be the k-canonical frame of a pretransitive logic L, k ≤ ω. For all x, y in F, we have

\[ xR^*_F y \text{ iff } \forall \varphi (\varphi \in y \Rightarrow \Box^* \varphi \in x). \]

The proof is straightforward; for details see, e.g., Proposition 5.9 and Theorem 5.16 in [8].

Consider a frame F and its subset V. We say that x ∈ V is a maximal element of V, if for all y ∈ V, xR^*_F y implies yR^*_F x.

It is known that in canonical transitive frames every non-empty definable subset has a maximal element [9]; the next proposition shows that this property holds in the pretransitive case as well.
Proposition 6 (Maximality lemma). Suppose that \( F \) is the \( k \)-canonical frame of a pretransitive \( L, k \leq \omega \). Let \( \varphi \in x \) for some \( x \in F \) and some formula \( \varphi \). Then \( R^k_F(x) \cap \{ y \mid \varphi \in y \} \) has a maximal element.

**Proof.** For a formula \( \alpha \), put \( \| \alpha \| = \{ y \mid \alpha \in y \} \). Since \( \varphi \in x \), \( R^k_F(x) \cap \| \varphi \| \) is non-empty.

Let \( \Sigma \) be an \( R^k_F \)-chain in \( R^k_F(x) \cap \| \varphi \| \). The family \( \{ R^k_F(y) \cap \| \varphi \| \mid y \in \Sigma \} \) has the finite intersection property (indeed, if \( \Sigma_0 \) is a non-empty finite subset of \( \Sigma \), then for some \( y_0 \in \Sigma_0 \) we have \( y R^k_F y_0 \) for all \( y \in \Sigma_0 \); so \( y_0 \in R^k_F(y) \cap \| \varphi \| \) for all \( y \in \Sigma_0 \)). By Proposition 5, \( R^k_F(y) = \bigcap \{ \| \alpha \| \mid \Box \alpha \in y \} \). It follows that all sets \( R^k_F(y) \cap \| \varphi \| \) are closed in the Stone topology on \( F \) (see, e.g., [14, Theorem 1.9.4]). By the compactness, \( \bigcap \{ R^k_F(y) \cap \| \varphi \| \mid y \in \Sigma \} \) is non-empty. Thus \( \Sigma \) has an upper bound in \( \| \varphi \| \). By Zorn’s lemma, \( R^k_F(x) \cap \| \varphi \| \) contains a maximal element. \( \square \)

Proposition 7. A pretransitive logics \( L \) is consistent iff \( L[1] \) is consistent.

**Proof.** Easily follows from Proposition 4 and the fact that if a logic containing S4 is consistent, then its extension with the formula \( p \rightarrow \Box \Diamond p \) is consistent. \( \square \)

Since \( L[1] \supseteq L[2] \supseteq L[3] \supseteq \ldots \), it follows that if \( L \) is consistent, then \( L[h] \) is consistent for any \( h > 0 \).

For a frame \( F \) and a point \( x \) in \( F \), the depth of \( x \) in \( F \) is the height of the frame \( F(x) \). Let \( F[h] \) denote the restriction of \( F \) onto the set of its points of depth less than or equal to \( h \), i.e., \( F[h] = F[\{ x \mid \text{ht}(F(x)) \leq h \}] \).

Proposition 8. Let \( F \) be the \( k \)-canonical frame of a pretransitive logic \( L, k \leq \omega \).

1. For all \( x \in F, 0 \leq h < \omega \),

the depth of \( x \) in \( F \) is \( \leq h \) iff \( B_h(\psi_1, \ldots, \psi_h) \in x \) for all \( k \)-formulas \( \psi_1, \ldots, \psi_h \).

2. For \( 0 < h < \omega \), the frame \( F[h] \) is the canonical frame of \( L[h] \).

**Proof.** 1. If \( \text{ht}(F(x)) \leq h \), then \( B_h \) is valid at \( x \) in \( F \); by the Canonical model theorem, \( B_h(\psi_1, \ldots, \psi_h) \in x \) for all \( k \)-formulas \( \psi_1, \ldots, \psi_h \).

By induction on \( h \), let us show that if \( \text{ht}(F(x)) > h \), then \( B_h(\psi_1, \ldots, \psi_h) \notin x \) for some \( \psi_1, \ldots, \psi_h \). The basis is trivial, since there are no points containing \( B_0 = \perp \) in \( F \). Suppose \( \text{ht}(F(x)) > h + 1 \). Then there exists \( y \) such that \( \text{ht}(F(y)) > h \), \( (x, y) \in R^k_F \), and \( (y, x) \notin R^k_F \). By induction hypothesis, \( B_h(\psi_1, \ldots, \psi_h) \notin y \) for some \( \psi_1, \ldots, \psi_h \). By Proposition 5 for some \( \psi_{h+1} \) we have \( \psi_{h+1} \in x \) and \( \Diamond^* \psi_{h+1} \notin y \). It follows that \( B_{h+1}(\psi_1, \ldots, \psi_{h+1}) \notin x \).

2. Since \( L \subseteq L[h] \), the \( k \)-canonical frame of \( L[h] \) is a generated subframe of \( F \). Now the statement follows from the first statement of the proposition. \( \square \)

A logic \( L \) is \( k \)-canonical if it is valid in its \( k \)-canonical frame.

Proposition 9. If a pretransitive \( L \) is \( k \)-canonical \(( k \leq \omega )\), then \( L[h] \) is \( k \)-canonical for all \( 0 < h < \omega \).
Proof. Follows from Proposition 8. □

Theorem 3. Let L be a pretransitive logic. Then for all formulas \( \varphi, \psi \) we have

\[ L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \text{ iff } L \vdash \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi. \]

Proof. By Proposition 7, we may assume that both L and L[1] are consistent. Let F be the \( \omega \)-canonical frame of L, and G the \( \omega \)-canonical frame of L[1].

Suppose \( L \vdash \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi \). Consider an element \( x \) of G. By Proposition 4, \( \{ \varphi \mid L[1] \vdash \varphi[\Box^*] \} \) is a logic containing S5. Thus \( x \) contains formulas \( \Box^* \psi \rightarrow \Box^* \varphi \) and \( \Diamond^* \Box^* \varphi \rightarrow \Box^* \varphi \). Since \( L \subseteq L[1] \), \( x \) also contains \( \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi \). It follows that if \( x \) contains \( \Box^* \psi \), then \( x \) contains \( \Box^* \varphi \). By the Canonical model theorem, \( L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \).

Now suppose \( L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \). Assume that \( \Diamond^* \Box^* \psi \in x \) for some element \( x \) of F. Then for some \( y \) we have \( \Box^* \psi \in y \) and \( x R^*_F y \). The set \( R^*_F(y) \) has a maximal element \( z \) by Proposition 6. It follows that \( \text{ht}(F[z]) = 1 \). By Proposition 8, \( G = F[1] \). Thus \( z \) is in \( G \) and hence \( \Box^* \psi \rightarrow \Box^* \varphi \) is in \( z \). Since \( y R^*_F z \), we have \( \Box^* \psi \in z \), which implies that \( \Box^* \varphi \in z \). Hence \( \Diamond^* \Box^* \varphi \) is in \( x \). It follows that \( L \vdash \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi \). □

Theorem 4. Let L be a pretransitive logic.

1. For all \( \varphi \), we have \( L[1] \vdash \varphi \) iff \( L \vdash \Diamond^* \Box^* \varphi \).
2. If L is decidable, then so is L[1].
3. If L has the finite model property, then so does L[1].

Proof. By Theorem 3, we have

\[ L[1] \vdash \Box^* \top \rightarrow \Box^* \varphi \text{ iff } L \vdash \Diamond^* \Box^* \top \rightarrow \Diamond^* \Box^* \varphi. \]

By Proposition 4 we have \( \top \leftrightarrow \Box^* \top \) and \( \top \leftrightarrow \Diamond^* \Box^* \top \) in every pretransitive logic; also, we have \( \Box^* \varphi \in L[1] \) iff \( \varphi \in L[1] \). Now the first statement follows.

The second statement is an immediate consequence of the first one.

Suppose L has the finite model property. Consider a formula \( \varphi \notin L[1] \). Then \( \Diamond^* \Box^* \varphi \notin L \). Then \( \Diamond^* \Box^* \varphi \) is refuted in some finite L-frame F. If follows that \( \varphi \) is refuted in F at some point in a maximal cluster \( C \). The restriction \( F[C] \) is a generated subframe of F. Thus F[C] refutes \( \varphi \) and validates L. The height of this restriction is 1, so \( F[C] \models L[1] \). Thus L[1] has the finite model property. □

Example 2. Important examples of pretransitive frames are birelation al frames \((W, \leq, R)\) with transitive \( R \). Recall that \((W, \leq, R)\) is a birelational frame, if \( \leq \) is a partial order on \( W \), \( R \subseteq W^2 \), and

\[ (R \circ \leq) \subseteq (\leq \circ R), \quad (R^{-1} \circ \leq) \subseteq (\leq \circ R^{-1}). \]

Consider the class of all birelational frames \((W, \leq, R)\) with transitive reflexive \( R \). Its modal logic L is the smallest bimodal logic containing the axioms of S4 for modalities.
and the formulas □₀ □₁ p → □₀ □₁ p and □₀ □₁ p → □₁ □₀ p \[11\] (recall that in the semantics of modal intuitionistic logic, the logic of this class is known to be IS₄ \[10\], one of the “most prominent logics for which decidability is left open” \[26\]). In this case, □₀ □₁ plays the role of the master modality, and the formula B₁ says that ≤ ◦ R is an equivalence. The decidability and the finite model property of the logic L, as well as of the logic IS₄, is an open question. By the above theorem, we have L ⊩ □₀ □₁ ϕ iff L[1] ⊩ ϕ.

**Question.** Is the logic L[1] decidable? Does it have the fmp?

### 4 Translation for logics of arbitrary finite height

In the proof of Theorem 3 we used the following property of a canonical frame F of L: every point in F is below (w.r.t. to the preorder \(R^*_F\)) a maximal point; maximal points form F[1], the canonical frame of L[1]. To describe translations from L[h] to L for \(h > 1\), we shall use the following analog of this property.

**Definition 3.** Let 0 < \(h < ω\). A frame F is said to be \(h\)-heavy if for its every element \(x\) which is not in \(F[h]\) there exists \(y\) such that \(x R^*_F y\) and \(ht(F(y)) = h\).

F is said to be top-heavy if it is \(h\)-heavy for all positive finite \(h\).

**Proposition 10.** The \(k\)-canonical frame of a consistent pretransitive logic is 1-heavy for every \(k ≤ ω\).

**Proof.** In the Maximality lemma (Proposition 6), put \(ϕ = ⊤\).\[\Box\]

It is known that \(k\)-canonical frames of unimodal transitive logics are top-heavy for all finite \(k\) \(([24], [9], [1])\).\[2\] This can be generalized for the pretransitive case as follows.

**Theorem 5.** Let L be a consistent pretransitive logic, \(h, k < ω\). If L[h] is \(k\)-tabular, then:

1. For every \(i ≤ h\), there exists a formula \(B_{i,k}\) such that \(B_{i,k} ∈ x\) iff the depth of \(x\) in the \(k\)-canonical frame of L is less than or equal to \(i\).
2. The \(k\)-canonical frame of L is \((h + 1)\)-heavy.

**Proof.** The case \(h = 0\) follows from Proposition 10. Suppose \(h > 0\).

Let F = \((W, (R_i)_{i<n})\) be the \(k\)-canonical frame of L. By Proposition 8, the frame F[h] = \((\overline{W}, (\overline{R_i})_{i<n})\) is the \(k\)-canonical frame of L[h]. Since L[h] is \(k\)-tabular, it follows that \(\overline{W}\) is finite and for every \(a \in \overline{W}\) there exists a \(k\)-formula \(α(a)\) such that

\(∀b ∈ \overline{W} (α(a) ∈ b ⇔ b = a)\). \[6\]

Without loss of generality we may assume that \(α(a)\) is of the form

\[2\]The term ‘top-heavy’ was introduced in [9].
\[ p_0^+ \land \ldots \land p_k^+ \land \varphi, \]  
\text{(7)}

where \( p_i^+ \in \{ p_i, \neg p_i \} \).

For \( a \in \overline{W} \) let \( \beta(a) \) be the following Jankov-Fine formula:

\[ \beta(a) = \alpha(a) \land \gamma, \]  
\text{(8)}

where \( \gamma \) is the conjunction of the formulas

\[ \Box^* \bigwedge \{ \alpha(b_1) \rightarrow \Diamond_i \alpha(b_2) \mid (b_1, b_2) \in \overline{R}_i, \ i < n \} \]  
\text{(9)}

\[ \Box^* \bigwedge \{ \alpha(b_1) \rightarrow \neg \Diamond_i \alpha(b_2) \mid (b_1, b_2) \in W^2 \setminus \overline{R}_i, \ i < n \} \]  
\text{(10)}

\[ \Box^* \lor \{ \alpha(b) \mid b \in \overline{W} \} \]  
\text{(11)}

For all \( x, y \in W, i < n \) we have

\[ \text{if } \gamma \in x \text{ and } xR_i y, \text{ then } \gamma \in y. \]  
\text{(12)}

We claim that

\[ \forall a \in \overline{W} \forall x \in W (\beta(a) \in x \iff x = a). \]  
\text{(13)}

To prove this, by induction on the length of formulas we show that for all \( k \)-formulas \( \varphi \), all \( a \in \overline{W} \), and all \( x \in W \),

\[ \text{if } \beta(a) \in x, \text{ then } \varphi \in a \iff \varphi \in x. \]  
\text{(14)}

The basis of induction follows from (7). The Boolean cases are trivial.

Assume that \( \varphi = \Diamond_i \psi \).

First, suppose \( \Diamond_i \psi \in a \). We have \( \psi \in b \) for some \( b \) with \( a \overline{R}_i b \). Since \( \beta(a) \in x \), by (9) we have \( \Diamond_i \alpha(b) \in x \). Then we have \( \alpha(b) \in y \) for some \( y \) with \( xR_i y \); by (12), \( \beta(b) \in y \).

Hence \( \psi \in y \) by induction hypothesis. Thus \( \Diamond_i \psi \in x \).

Now let us show that \( \Diamond_i \psi \in a \) whenever \( \Diamond_i \psi \in x \). In this case we have \( \psi \in y \) for some \( y \) with \( xR_i y \). By (11) we infer that \( \alpha(b) \in y \) for some \( b \in \overline{W} \). Thus \( \Diamond_i \alpha(b) \in x \). Since \( \alpha(a) \in x \), it follows from (10) that \( a \overline{R}_i b \). By (12) we have \( \gamma \in y \), thus \( \beta(b) \in y \); by induction hypothesis \( \psi \in b \). Hence \( \Diamond_i \psi \in a \), as required.

Thus (14) is proved and (13) follows.

Now using the formulas (5), for \( i \leq h \) we can define the formulas \( B_{i,k} \) such that for all \( x \) in \( F \),

\[ \text{the depth of } x \text{ in } F \text{ is } \leq i \iff B_{i,k} \in x. \]  
\text{(15)}

For this, we put

\[ B_{i,k} = \bigvee \{ \beta(a) \mid \text{ht}(F\langle a \rangle) \leq i \}. \]  
\text{(16)}

This proves the first statement of the theorem.

In particular, it follows that \( W \setminus \overline{W} \) is definable in the \( k \)-canonical model of \( L \):

\[ x \in W \setminus \overline{W} \iff \neg B_{h,k} \in x. \]
Now by Proposition 6 we have that if $x$ is not in $\overline{W}$, then there exists a maximal $y$ in $R^*_F(x) \setminus \overline{W}$. Hence if $(y, z) \in R^*_F$ and $(z, y) \notin R^*_F$ for some $z$, then $z$ belongs to $\overline{W}$, which means $\text{ht}(F\langle z \rangle) \leq h$. Thus $\text{ht}(F\langle y \rangle) \leq h + 1$. On the other hand, $y \notin \overline{W}$. It follows that $\text{ht}(F\langle y \rangle) = h + 1$, as required.

The logic $L[0]$ is inconsistent, so it is $k$-tabular. Hence Proposition 10 can be considered as a particular case of the above theorem.

Note that formulas (8) define atoms in the Lindenbaum-Tarski (i.e., free) $k$-generated algebra of $L$.

**Theorem 6.** Let $L$ be a pretransitive logic, $h, k < \omega$. If $L[h]$ is $k$-tabular, then for all $k$-formulas $\varphi$ we have

$$L[h + 1] \vdash \varphi \text{ iff } L \vdash \bigwedge_{i \leq h} (\square_i^* (\square_i^* \varphi \rightarrow B_{i,k}) \rightarrow B_{i,k}).$$

**Proof.** We may assume that both $L$ and $L[h + 1]$ are consistent (Proposition 7). Let $F$ be the $k$-canonical frame of $L$.

Suppose $L[h + 1] \vdash \varphi$. We claim that for all $i \leq h$, $\square_i^*(\square_i^* \varphi \rightarrow B_{i,k}) \rightarrow B_{i,k}$ is true at every point $x$ in the $k$-canonical model of $L$. Let $\neg B_{i,k}$ be in $x$. Let us show that $\neg B_{i,k} \wedge \square_i^* \varphi \in y$ for some $y$ with $xR^*_F y$. First, assume that $x$ is in $F[h + 1]$. By Proposition 8, $x$ contains $L[h + 1]$. Since $\varphi \in L[h + 1]$, we have $\square_i^* \varphi \in L[h + 1]$. Thus $\square_i^* \varphi \in x$. Since $R^*_F$ is reflexive, in this case we can put $y = x$. Second, suppose $x$ is not in $F[h + 1]$. By Theorem 5, there exists $y$ such that $xR^*_F y$ and $\text{ht}(F\langle y \rangle) = h + 1$. We have $\square_i^* \varphi \in y$ and $B_{i,k} \notin y$. This proves the “only if” part.

Now suppose that $L \vdash \square_i^*(\square_i^* \varphi \rightarrow B_{i,k}) \rightarrow B_{i,k}$ for all $i \leq h$. Assume $\text{ht}(F\langle x \rangle) = i \leq h + 1$. In this case $B_{i-1,k} \notin x$. Since $\square_i^*(\square_i^* \varphi \rightarrow B_{i-1,k}) \rightarrow B_{i-1,k}$ is in $x$, it follows that $\neg B_{i-1,k} \wedge \square_i^* \varphi$ is in $y$ for some $y$ with $xR^*_F y$. The first conjunct says that $y$ is not in $F[i - 1]$. Since $y$ is in $F[i]$, it follows that $\text{ht}(F\langle y \rangle) = i$. Hence $y$ and $x$ belong to the same cluster. Since $\square_i^* \varphi \in y$, we obtain $\varphi \in x$. It follows that $\varphi \in x$ for all $x$ in $F[h + 1]$. By Proposition 6, $L[h + 1] \vdash \varphi$.

Note that $B_{0,k}$ is $\perp$ for all $k < \omega$. Thus, (17) generalizes the translation described in Theorem 3.

Theorem 6 provides translations for the case when $L$ is a unimodal transitive logic (recall that transitive logics of finite height are locally tabular [22]). It should be noted that in this case Theorem 3 the key ingredient of the proof of Theorem 6 has been known since 1970s: formulas $B_{i,k}$ in transitive canonical frames were described in [24] (see also [9], [1]).

An analog of Theorem 6 can be formulated for intermediate logics. Formulas $B_{i,k}$ defining points of finite depth in finitely generated intuitionistic canonical frames were described in [20] (see also [2]). Similarly to the proof of Theorem 6 it can be shown that

$$IL[h + 1] \vdash \varphi \text{ iff } IL \vdash \bigwedge_{i \leq h} ((\varphi \rightarrow B_{i,k}) \rightarrow B_{i,k})$$

for all finite $h$ and for all $k$-formulas $\varphi$.  

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5 Corollaries, examples, and open problems

The translation (17) holds for all finite $h, k$ in the case when $L$ is a transitive unimodal logic. Indeed, by the Segerberg – Maksimova criterion (Theorem 1), a transitive logic is locally tabular iff it is of finite height. This criterion was recently generalized to a wide family of pretransitive logics [23]. For example, if a unimodal $L$ contains the formula $\Diamond^{m+1}p \rightarrow \Diamond p \lor p$ for some $m > 0$, then $L$ is locally tabular iff it is of finite height. Thus, (17) holds for all finite $h, k$ in this case too.

However, in general $k$-tabularity of $L[h]$ depends both on $h$ and on $k$.

Example 3. Consider the smallest reflexive 2-transitive logic $K + \{ p \rightarrow \Diamond p, \Diamond^3 p \rightarrow \Diamond^2 p \}$ and its extension $L$ with the McKinsey formula for the master modality, $\Box^2 \Diamond^2 p \rightarrow \Diamond^2 \Box^2 p$. Maximal clusters in the canonical frames of $L$ are reflexive singletons, so $L[1] = K + p \leftrightarrow \Box p$ by Proposition 8. Clearly, $L[1]$ is locally tabular. It follows that we have the translation (17) from $L[2]$ to $L$ for all finite $k$.

However, $L[2]$ is not even 1-tabular. To see this, consider the frame $F_0 = (\omega, R_0)$, where
\[ xR_0y \iff x \neq y + 1 \text{ and } y \neq x + 1. \]
Let $F = (\omega + 1, R)$, where $xRy$ iff $xR_0y$ or $y = \omega$. Clearly, $F \models L[2]$. Consider a model $M$ on $F$ such that $x \models p_0$ iff $x = 0$ or $x = \omega$. Put $\alpha_0 = p_0 \land \Diamond \neg p_0$, $\alpha_1 = \neg \Diamond \alpha_0 \land \neg p_0$, and $\alpha_{i+1} = (\Diamond \alpha_i \lor \alpha_{i-1}) \land \neg p_0$ for $i > 0$. By an easy induction, in $M$ we have for all $i$: $x \models \alpha_i$ iff $x = i$. Thus if $i \neq j$, then $\alpha_i \leftrightarrow \alpha_j \notin L$.

It is not difficult to construct other examples of this kind for arbitrary finite $h$: there are pretransitive logics such that $L[h]$ is locally tabular, and $L[h+1]$ is not one-tabular.

With the parameter $k$, the situation is much more intriguing. The following result was proved in [18]:

A unimodal transitive logic is locally tabular iff it is 1-tabular. \hfill (18)

The recent results [23] show that this equivalence also holds for many non-transitive logics. For example, if a unimodal $L$ contains $\Diamond^{m+1}p \rightarrow \Diamond p \lor p$ for some $m > 0$, then it is locally tabular iff it is 1-tabular. The question whether this equivalence holds for every modal logic has been open since 1970s.

Theorem 7. There exists a unimodal 1-tabular logic $L$ which is not locally tabular.

Proof (sketch). Let $L$ be the logic of the frame $(\omega + 1, R)$, where
\[ xRy \iff x \leq y \text{ or } x = \omega. \]

First, we claim that $L$ is not locally tabular.

The following fact follows from Theorem 4.3 and Lemma 5.9 in [23]: if the logic of a frame $F$ is locally tabular, then the logic of an arbitrary restriction $F[V]$ of $F$ is locally tabular.

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The restriction of \((\omega + 1, R)\) onto \(\omega\) is the frame \((\omega, \leq)\), which is not locally tabular (it is of infinite height). Thus \(L\) is not locally tabular.

To show that \(L\) is 1-tabular, we need the following observation. If every \(k\)-generated subalgebra of an algebra \(A\) contains at most \(m\) elements for some fixed \(m < \omega\), then the free \(k\)-generated algebra in the variety generated by \(A\) is finite; see \([19]\).

Consider the complex algebra \(A\) of the frame \((\omega + 1, R)\). One can check that every 1-generated subalgebra of \(A\) contains at most 8 elements. By the above observation, \(L\) is 1-tabular.

It is unknown whether 2-tabularity of a modal logic implies its local tabularity. At least, does \(k\)-tabularity imply local tabularity, for some fixed \(k\) for all unimodal logics? The same questions are open in the intuitionistic case \([5\text{, Problem 2.4}]\).

Finite height is not a necessary condition for local tabularity of intermediate logics. What can be an analog of Gliveko’s translation in the case of a locally tabular intermediate logic with no finite height axioms? Another generalization can probably be found in the area of modal intuitionistic logics. In \([3]\), Glivenko type theorems were proved for extensions of the logic MIPC; in \([4]\), local tabularity of these extensions was considered. What can be an analog of Theorem \([6]\) for modal intuitionistic logics?

In \([21]\), Glivenko’s theorem was used to obtain decidability (and the finite model property) for extensions of S4 with \(\Box\Diamond\)-formulas (such formulas are built from literals \(\Box\Diamond p_i\)). An analog of this result can be obtained for extensions of a pretransitive logic \(L\) in the case when \(L\) is decidable (or has the finite model property) and \(L[1]\) is locally tabular.

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