A Morita Characterisation for Algebras and Spaces of Operators on Hilbert Spaces

G. K. Eleftherakis and E. Papapetros

Abstract. We introduce the notion of $\Delta$ and $\sigma\Delta$– pairs for operator algebras and characterise $\Delta$– pairs through their categories of left operator modules over these algebras. Furthermore, we introduce the notion of $\Delta$-Morita equivalent operator spaces and prove a similar theorem about their algebraic extensions. We prove that $\sigma\Delta$-Morita equivalent operator spaces are stably isomorphic and vice versa. Finally, we study unital operator spaces, emphasising their left (resp. right) multiplier algebras, and prove theorems that refer to $\Delta$-Morita equivalence of their algebraic extensions.

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1. Introduction

In what follows, we denote by $\mathcal{B}(H_1, H_2)$ the space of all linear and bounded operators from the Hilbert space $H_1$ to the Hilbert space $H_2$. If $\mathcal{X}$ is a subset of $\mathcal{B}(H_1, H_2)$ and $\mathcal{Y}$ is a subset of $\mathcal{B}(H_2, H_3)$, then we denote by $[\mathcal{Y}\mathcal{X}]$ the norm-closure of the linear span of the set

$$\{yx \in \mathcal{B}(H_1, H_3), y \in \mathcal{Y}, x \in \mathcal{X}\}.$$

Similarly, if $\mathcal{Z}$ is a subset of $\mathcal{B}(H_3, H_4)$, we define the space $[\mathcal{Z}\mathcal{Y}\mathcal{X}]$.

A linear subspace $M \subseteq \mathcal{B}(H, K)$ is called a ternary ring of operators (TRO) if $MM^* M \subseteq M$. It then follows that $M$ is an $A-B$ equivalence bimodule in the sense of Rieffel for the $C^*$-algebras $A = [MM^*]$ and $B = [M^*M]$. 
We call a norm closed ternary ring of operators \( M, \sigma\)-TRO if there exist sequences \( \{ m_i \in M, i \in \mathbb{N} \} \) and \( \{ n_j \in M, j \in \mathbb{N} \} \) such that
\[
\lim_{t} \sum_{i=1}^{t} m_i^* m = m, \lim_{t} \sum_{j=1}^{t} m n_j^* n_j = m, \forall m \in M
\]
and
\[
\left\| \sum_{i=1}^{t} m_i m_i^* \right\| \leq 1, \left\| \sum_{j=1}^{t} n_j^* n_j \right\| \leq 1, \forall t \in \mathbb{N}.
\]
Equivalently, a TRO \( M \) is a \( \sigma\)-TRO if and only if the \( \mathcal{C}^*\)-algebras \([M^* M]\), \([MM^*]\) have a \( \sigma\)-unit.

At the beginning of the 1970s, M. A. Rieffel introduced the idea of Morita equivalence of \( \mathcal{C}^*\)-algebras. In particular, he gave the following definitions:

(i) Two \( \mathcal{C}^*\)-algebras, \( A \) and \( B \), are said to be Morita equivalent if they have equivalent categories of \( \star\)-representations via \( \star\)-functors.

(ii) The same algebras are said to be strongly Morita equivalent if there exists an \( A - B \) module of equivalence or if there exists a TRO \( M \) such that the \( \mathcal{C}^*\) algebras \([M^* M]\) and \( A \) (resp. \([MM^*]\) and \( B \)) are \( \star\)-isomorphic. We write \( A \sim_R B \). If \( A \sim_R B \), then \( A \) and \( B \) have equivalent categories of representations. The converse does not hold. For further details, see [15–17].

Brown, Green and Rieffel proved the following fundamental theorem for \( \mathcal{C}^*\)-algebras ([7,8]).

**Theorem 1.1.** If \( A, B \) are \( \mathcal{C}^*\)-algebras with \( \sigma\)-units, then \( A \sim_R B \) if and only if they are stably isomorphic, which means that the algebras \( A \otimes K, B \otimes K \) are \( \star\)-isomorphic. Here, \( K \) is the algebra of compact operators acting on \( \ell^2(\mathbb{N}) \), and \( \otimes \) is the minimal tensor product.

The next step in this theory came from Blecher, Muhly and Paulsen. They defined the notion of strong Morita equivalence \( \sim_{BMP} \) for operator algebras, self-adjoint or not, and they proved that if \( A \sim_{BMP} B \), their categories of left operator modules are equivalent ([6]). Later, Blecher proved that the converse is also true ([4]). Therefore, he proved that two \( \mathcal{C}^*\)-algebras \( A, B \) have equivalent categories of left operator modules if and only if \( A \sim_R B \).

A third notion of Morita equivalence was introduced by the first author of this article. According to this theory, two operator algebras, \( A, B \), are said to be \( \Delta\)-equivalent and we write \( A \sim_{\Delta} B \) if they have completely isometric representations \( \alpha : A \to \alpha(A) \subseteq \mathcal{B}(H) \), \( \beta : B \to \beta(B) \subseteq \mathcal{B}(K) \) and there exists a TRO \( M \subseteq \mathcal{B}(H,K) \) such that
\[
\alpha(A) = [M^* \beta(B) M], \quad \beta(B) = [M \alpha(A) M^*]
\]
([10]). If \( M \) is a \( \sigma\)-TRO, we write \( A \sim_{\sigma \Delta} B \).
G. K. Eleftherakis proved that \( A \sim_{\sigma} B \) if and only if \( A, B \) are stably isomorphic ([11]). If we define \( C = [M^* M] \), \( D = [M M^*] \), then the spaces 
\[
A_0 = \alpha(A) + C, \quad B_0 = \beta(B) + D
\]
are operator algebras with contractive approximate identities, even if \( A, B \) do not have, and they are also \( \Delta \)-equivalent since
\[
A_0 = [M^* B_0 M], \quad B_0 = [M A_0 M^*].
\]
Also observe that
\[
A_0 = \bar{A_0} C = \overline{C A_0}, \quad B_0 = \overline{B_0 D} = \overline{D B_0}
\]
and that \( \alpha(A) \) (resp. \( \beta(B) \)) is an ideal of \( A_0 \) (resp. \( B_0 \)).

Generally, if \( A_0 \) is an operator algebra and \( C \subseteq A_0 \) is a \( C^* \)-algebra satisfying relation (1), we call \((A_0, C)\) a \( \Delta \)-pair. Furthermore, if \( C \) has a \( \sigma \)-unit, we call \((A_0, C)\) a \( \sigma \Delta \)-pair.

In Sect. 2, we characterise the \( \Delta \)-equivalence and stable isomorphism of \( \Delta \)-pairs under the notion of equivalence of categories of their left operator modules. In Sect. 3, using the above theory, we characterise the \( \Delta \)-equivalence and stable isomorphism of the operator spaces \( X \) and \( Y \) through the equivalence of the categories of left operator modules of operator algebras \( A_X, A_Y \), on which \( X \) and \( Y \) naturally embed completely isometrically. If \( X \) and \( Y \) are unital operator spaces, we get stronger results using the algebras \( \Omega_X, \Omega_Y \) generated by \( X, Y \) and the diagonals of their multiplier algebras (see Sect. 4).

If \( X \) is an operator space, then \( X \otimes K \) is completely isometrically isomorphic with the space \( K_\infty(\mathcal{X}) \), which is the norm closure of the finitely supported matrices in \( M_\infty(\mathcal{X}) \). Here, \( M_\infty(\mathcal{X}) \) is the space of \( \infty \times \infty \) matrices, which define bounded operators. Also, by \( \mathcal{X} \otimes_h \mathcal{Y} \), we denote the Haagerup tensor product of the operator spaces \( \mathcal{X} \) and \( \mathcal{Y} \). If \( \mathcal{A} \) is an operator algebra, \( \mathcal{X} \) is a right \( \mathcal{A} \)-module and \( \mathcal{Y} \) is a left \( \mathcal{A} \)-module, we denote by \( \mathcal{X} \otimes_h \mathcal{Y} \) the balanced Haagerup tensor product of \( \mathcal{X} \) and \( \mathcal{Y} \) over \( \mathcal{A} \) ([6]).

For further details about operator spaces, operator algebras, Morita theory and category theory, we refer the reader to [1,5,9,13,14,18].

2. \( \Delta \)-Morita Equivalence of Operator Algebras

Definition 2.1. Let \( \mathcal{A} \subseteq \mathbb{B}(H), \mathcal{B} \subseteq \mathbb{B}(K) \) be operator algebras. We call them TRO-equivalent (resp. \( \sigma \)-TRO equivalent) if there exists a TRO (resp. \( \sigma \)-TRO) \( M \subseteq \mathbb{B}(H, K) \) such that
\[
\mathcal{A} = [M^* B M], \quad \mathcal{B} = [M A M^*].
\]
We write \( \mathcal{A} \sim_{TRO} \mathcal{B}, \) resp. \( \mathcal{A} \sim_{\sigma TRO} \mathcal{B}. \)

Definition 2.2. Let \( \mathcal{A}, \mathcal{B} \) be operator algebras. We call them \( \Delta \)-equivalent (resp. \( \sigma \Delta \)- equivalent) if there exist completely isometric homomorphisms \( a : \mathcal{A} \to \mathbb{B}(H) \) and \( \beta : \mathcal{B} \to \mathbb{B}(K) \) such that \( a(\mathcal{A}) \sim_{TRO} \beta(\mathcal{B}) \) (resp. \( a(\mathcal{A}) \sim_{\sigma TRO} \beta(\mathcal{B}) \)). We write \( \mathcal{A} \sim_{\Delta} \mathcal{B} \) (resp. \( \mathcal{A} \sim_{\sigma \Delta} \mathcal{B} \))
Definition 2.3. Let $\mathcal{A}$ be an operator algebra and $C$ be a $C^*$-algebra such that $C \subseteq \mathcal{A}$. If $\mathcal{A} = [\mathcal{A}C] = [C\mathcal{A}]$, we call the pair $(\mathcal{A}, C)$ a $\Delta$-pair. If $C$ has a $\sigma$-unit, we call $(\mathcal{A}, C)$ a $\sigma\Delta$-pair.

If $\mathcal{A}$ is an operator algebra, then $\mathcal{A}OMOD$ is the category with objects the essential left $\mathcal{A}$-operator modules, namely operator spaces $U$ such that there exists a completely contractive bilinear map $\theta : \mathcal{A} \times U \to U$ such that $U = [\mathcal{A}U]$, where $\mathcal{A}U = \{\theta(a, x) \in U : a \in \mathcal{A}, x \in U\}$. For our convenience, we write $ax$ instead of $\theta(a, x)$. If $U_1, U_2 \in \mathcal{A}OMOD$, the space of homomorphisms between $U_1$ and $U_2$ is the space of completely bounded maps, which are left operator maps over $\mathcal{A}$, and we denote this space by $\mathcal{A}CB(U_1, U_2)$. Observe that if $(\mathcal{A}, C)$ is a $\Delta$-pair, then $\mathcal{A}OMOD$ is a subcategory of $COMOD$.

A functor $F : \mathcal{A}OMOD \to \mathcal{B}OMOD$ is called completely contractive if for all $U_1, U_2 \in \mathcal{A}OMOD$ the map

$$F : \mathcal{A}CB(U_1, U_2) \to \mathcal{B}CB(F(U_1), F(U_2))$$

is completely contractive.

Definition 2.4. Let $(\mathcal{A}, C), (\mathcal{B}, D)$ be $\Delta$-pairs. We call them $\Delta$-Morita equivalent if there exist completely contractive functors $F : COMOD \to DOMOD$ and $G : DOMOD \to COMOD$ such that

$$G \circ F \cong Id_{COMOD}, F \circ G \cong Id_{DOMOD}$$

and

$$G|_{{\mathcal{B}OMOD}} \circ F|_{\mathcal{A}OMOD} \cong Id_{\mathcal{A}OMOD}, F|_{\mathcal{A}OMOD} \circ G|_{{\mathcal{B}OMOD}} \cong Id_{\mathcal{B}OMOD}.$$ 

Here, $\cong$ is the natural equivalence.

If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are operator algebras such that $\mathcal{C} \subseteq \mathcal{A}, \mathcal{D} \subseteq \mathcal{B}$ and $\mathcal{A} \sim_{\Delta} \mathcal{B}, \mathcal{C} \sim_{\Delta} \mathcal{D}$, we say that $\Delta$-equivalence is implemented in both cases by the same TRO if there exist completely isometric homomorphisms $\alpha : \mathcal{A} \to \mathcal{B}, \beta : \mathcal{B} \to \mathcal{C}$ such that $\alpha(\mathcal{A}) \subseteq \mathcal{B}(H), \beta : \mathcal{B} \to \mathcal{C}$ and a TRO $M \subseteq \mathcal{B}(H, K)$ such that

$$\alpha(\mathcal{A}) = [M \beta(\mathcal{B}) M], \beta(\mathcal{B}) = [M \alpha(\mathcal{A}) M^*]$$

and

$$\alpha(\mathcal{C}) = [M \beta(\mathcal{D}) M], \beta(\mathcal{D}) = [M \alpha(\mathcal{C}) M^*].$$

We now prove our main theorem for operator algebras.

Theorem 2.5. Let $(\mathcal{A}, C), (\mathcal{B}, D)$ be $\Delta$-pairs. The following are equivalent:

(i) $\mathcal{A} \sim_{\Delta} \mathcal{B}, \mathcal{C} \sim_{\Delta} \mathcal{D}$, where $\Delta$-equivalence is implemented in both cases by the same TRO.

(ii) The pairs $(\mathcal{A}, C), (\mathcal{B}, D)$ are $\Delta$-Morita equivalent.

Proof. We start with the proof of $(i) \implies (ii)$.

Assume that $\mathcal{A} = [M \mathcal{B} M], \mathcal{B} = [M \mathcal{A} M^*]$ and also $C = [M \mathcal{D} M], D = [M \mathcal{C} M^*]$ for the same TRO $M \subseteq \mathcal{B}(H, K)$.

Let $U \in \mathcal{A}OMOD$ and $E = [M \mathcal{M} M^*]$. We notice that

$$[EU] = [M^* M U] = [M^* M A U] = [M^* M M^* B M U] \subseteq [M^* B M U] = [A U] = U$$
(so $U$ is a left $E$-operator module).

We set $\mathcal{F}(U) = M \otimes_E^h U$. We fix

$$v = \sum_{i=1}^r m_i a_i n_i^* \in [M \mathcal{A} M^*].$$

We define the bilinear map

$$f_v : M \times U \to M \otimes_E^h U, f_v(\ell, x) = \sum_{i=1}^r m_i \otimes_E a_i n_i^* \ell x$$

and then there exists a linear map denoted again by $f_v : M \otimes U \to M \otimes_E^h U$ such that

$$f_v(\ell \otimes x) = \sum_{i=1}^r m_i \otimes_E a_i n_i^* \ell x, \ell \in M, x \in U.$$

Let

$$u = \sum_{j=1}^k \ell_j \otimes x_j \in M \otimes U.$$

Since $M$ is a TRO, there exists a net $m_\lambda = (m_{1,\lambda}, ..., m_{n,\lambda})^t \in \mathbb{M}_{n,1}(M^*)$ such that $\|m_\lambda\| \leq 1, \forall \lambda \in \Lambda$ and also $m_\lambda \in M^* \rightarrow \ell, \forall \ell \in M,$ (see [5]).

Let $\epsilon > 0$. We choose $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$ holds

$$\|f_v(u)\| - \epsilon = \left\| \sum_{i=1}^r \sum_{j=1}^k m_i \otimes_E a_i n_i^* \ell_j x_j \right\| \leq \left\| \sum_{i=1}^r \sum_{j=1}^k m_\lambda m_\lambda^* \otimes_E a_i n_i^* \ell_j x_j \right\|$$

Using now the fact that $\|y \otimes C b\| \leq \|y\| \|b\|, y \in M_{p,q}(M^*), b \in M_{q,s}(U), p, q, s \in \mathbb{N}$, we get

$$\|f_v(u)\| - \epsilon \leq \left\| \sum_{i=1}^r \sum_{j=1}^k m_\lambda \otimes_E m_\lambda^* m_i a_i n_i^* \ell_j x_j \right\| = \left\| m_\lambda \otimes_E \sum_{i=1}^r m_i a_i n_i^* \sum_{j=1}^k m_\lambda^* \ell_j x_j \right\|$$

$$\leq ||m_\lambda|| \left\| \sum_{i=1}^r m_i a_i n_i^* \right\| \left\| \sum_{j=1}^k m_\lambda^* \ell_j x_j \right\|$$

$$\leq ||v|| \left\| \sum_{j=1}^k m_\lambda^* \ell_j \otimes x_j \right\|_h \leq ||v|| ||(m_\lambda^* \ell_1, ..., m_\lambda^* \ell_k)|| \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \right\|$$

$$\leq ||v|| ||(\ell_1, ..., \ell_k)|| \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \right\|$$

We have shown that the above procedure is independent of $\lambda$, so if $\epsilon \to 0^+$, and by taking infimum over all representations of $u$, we get $\|f_v(u)\|_h \leq ||v||_h ||u||_h$. Therefore, $f_v$ is continuous and contractive, since $||f_v|| \leq ||v||_h$.

Let $n \in \mathbb{N}$ and the corresponding map

$$(f_v)_n : \mathbb{M}_n(M \otimes U) \to \mathbb{M}_n(M \otimes_E^h U).$$
We have to prove that \((f_v)_n\) is contractive, that is, \(f_v\) is completely contractive with respect to the Haagerup norm. This statement is true since
\[
\mathbb{M}_n(M) \otimes_{\mathbb{M}_n(E)}^h \mathbb{M}_n(U) \cong \mathbb{M}_n(M \otimes_E^h U), \quad n \in \mathbb{N}.
\]
For more details check [5].

Furthermore, for every \(\ell \in M, z^\ast w \in M^* M, x \in U\) holds
\[
f_v(\ell z^\ast w \otimes x) = \sum_{i=1}^r m_i \otimes_E a_i n_i^\ast \ell z^\ast w x
\]
\[
= \sum_{i=1}^r m_i \otimes_E a_i n_i^\ast \ell (z^\ast w x)
\]
\[
= f_v(\ell \otimes z^\ast w x)
\]

Since \(f_v\) is continuous and linear and \(E = [M^* M]\), we get \(f_v(\ell e \otimes x) = f_v(\ell \otimes e x)\) for every \(\ell \in M, e \in E, x \in U\). Therefore, \(f_v\) extends to a linear and completely contractive map
\[
\hat{f}_v : M \otimes_E^h U \rightarrow M \otimes_E^h U
\]
with the property
\[
\hat{f}_v(\ell \otimes_E x) = \sum_{i=1}^r m_i \otimes_E a_i n_i^\ast \ell x, \ell \in M, x \in U
\]

So, we have the map \(\hat{f} : [M \mathcal{A} M^*] \rightarrow CB(\mathcal{F}(U)), v \mapsto \hat{f}_v\), which is completely contractive and therefore extends to a completely contractive map denoted again by \(\hat{f} : \mathcal{B} \rightarrow CB(\mathcal{F}(U))\), where \(CB(\mathcal{F}(U))\) is the space of all linear and completely bounded maps of \(\mathcal{F}(U)\) to itself. The algebra \(\mathcal{B}\) acts to \(\mathcal{F}(U)\) via the map
\[
\hat{\theta} : \mathcal{B} \times \mathcal{F}(U) \rightarrow \mathcal{F}(U), \hat{\theta}(b, y) = \hat{f}(b)(y),
\]
such that \([\mathcal{B} \mathcal{F}(U)] = \mathcal{F}(U)\) and thus \(\mathcal{F}(U) = M \otimes_E^h U \subset \mathcal{B}OMOD\).

Therefore, we have a correspondence between the objects
\[
\mathcal{F} : \mathcal{A}OMOD \rightarrow \mathcal{B}OMOD, U \mapsto \mathcal{F}(U) = M \otimes_E^h U.
\]

Let \(U_1, U_2 \in \mathcal{A}OMOD\). We fix \(f \in \mathcal{A}CB(U_1, U_2)\) and we define the map
\[
\mathcal{F}(f) : M \times U_1 \rightarrow M \otimes_E^h U_2 = \mathcal{F}(U_2), \mathcal{F}(f)(\ell, x) := \ell \otimes_E f(x)
\]

The map \(\mathcal{F}(f)\) is linear, completely contractive and \(E\)-balanced, so we denote again by \(\mathcal{F}(f)\) the linear and completely contractive map
\[
\mathcal{F}(f) : M \otimes_E^h U_1 = \mathcal{F}(U_1) \rightarrow M \otimes_E^h U_2 = \mathcal{F}(U_2)
\]
with the property
\[
\mathcal{F}(f)(\ell \otimes_E x) = \ell \otimes_E f(x), \ell \in M, x \in U
\]

Furthermore,
\[
\mathcal{F}(f)(mn^* \cdot \ell \otimes_E x) = \mathcal{F}(f)(m \otimes_E an^* \ell x) = m \otimes_E f(an^* lx)
\]
\[
= m \otimes_E an^* \ell f(x) = m a n^* \cdot \mathcal{F}(f)(\ell \otimes_E x), m, n, \ell \in M, x \in U_1, a \in \mathcal{A}
\]
and since 
\[ \mathcal{B} = [M A M^*], \]
we have 
\[ \mathcal{F}(f)(b \cdot y) = b \cdot \mathcal{F}(f)(y), b \in \mathcal{B}, y \in M \otimes^h_E U_1. \]

We proved that \( \mathcal{F}(f) \in \mathcal{B} CB(\mathcal{F}(U_1), \mathcal{F}(U_2)) \)

Therefore, we have a completely contractive map 
\[ \mathcal{F} : \mathcal{A} CB(U_1, U_2) \to \mathcal{B} CB(\mathcal{F}(U_1), \mathcal{F}(U_2)), f \mapsto \mathcal{F}(f) \]

Similarly, we have a functor \( \mathcal{G} : \mathcal{B} OMOD \to \mathcal{A} OMOD \) defined as 
\[ \mathcal{G}(V) = M^* \otimes^h_{E'} V, V \in \mathcal{B} OMOD, \]

where \( E' = [M M^*] \) and the corresponding functor 
\[ \mathcal{G} : \mathcal{B} CB(V_1, V_2) \to \mathcal{A} CB(G(V_1), G(V_2)) \]

for every \( V_1, V_2 \in \mathcal{B} OMOD. \)

We are going to prove that \( \mathcal{G} \) is the natural inverse of \( \mathcal{F}. \) If \( U \in \mathcal{A} OMOD, \) we have that 
\[ (\mathcal{G} \mathcal{F})(U) = \mathcal{G}(M \otimes^h_E U) \]
\[ = M^* \otimes^h_{E'} (M \otimes^h_E U) \cong (M^* \otimes^h_{E'} M) \otimes^h_E U \cong E \otimes^h_E U \]
\[ \cong U = \text{Id}_{\mathcal{A} OMOD}(U) \]

(Similarly, \( (\mathcal{F} \mathcal{G})(V) \cong V, \forall V \in \mathcal{B} OMOD). \)

We note that if \( U \in \mathcal{A} OMOD, \) then there exists an isometry 
\[ f_U : (\mathcal{G} \mathcal{F})(U) = M^* \otimes^h_{E'} (M \otimes^h_E U) \to U = \text{Id}_{\mathcal{A} OMOD}(U) \]
such that 
\[ f_U(m^* \otimes^h_{E'} (\ell \otimes^h_E x)) = m^* \ell x, m, \ell \in M, x \in U. \]

We have to prove that for every \( U_1, U_2 \in \mathcal{A} OMOD, f \in \mathcal{A} CB(U_1, U_2), \)
the following diagram 
\[ \begin{array}{ccc} 
(G \mathcal{F})(U_1) & \xrightarrow{f_{U_1}} & U_1 \\
\downarrow & & \downarrow f \\
(G \mathcal{F})(U_2) & \xrightarrow{f_{U_2}} & U_2 
\end{array} \]
is commutative, or equivalently, the following diagram is commutative 
\[ \begin{array}{ccc} 
M^* \otimes^h_{E'} (M \otimes^h_E U_1) & \xrightarrow{f_{U_1}} & U_1 \\
\downarrow \mathcal{G}(\mathcal{F}(f)) & & \downarrow f \\
M^* \otimes^h_{E'} (M \otimes^h_E U_2) & \xrightarrow{f_{U_2}} & U_2 
\end{array} \]
So, we have to prove that \( f \circ f_U = f_{U_2} \circ G(\mathcal{F}(f)) \).

Indeed,

\[
(f_{U_2} \circ G(\mathcal{F}(f))(m^* \otimes_{E'} (\ell \otimes_E x)) = f_{U_2}(G(\mathcal{F}(f))(m^* \otimes_{E'} (\ell \otimes_E x)) = f_{U_2}(m^* \otimes_{E'} \mathcal{F}(f)(\ell \otimes_E x)) = f_{U_2}(m^* \otimes_{E'} \ell \otimes_E f(x)) = m^* \ell f(x) = f(m^* \ell x)
\]

and on the other hand

\[
(f \circ f_U_1)(m^* \otimes_{E'} (\ell \otimes_E x)) = f(f_U_1(m^* \otimes_{E'} (\ell \otimes_E x)) = f(m^* \ell x)
\]

for every \( m, \ell \in M, x \in U_1 \).

The functor \( \mathcal{F} \) extends to a functor \( \mathcal{F}^\delta \) to the category \( cOMOD \) in the same sense that is \( \mathcal{F}^\delta(U) = M^* \otimes_C^h U, U \in cOMOD \) and \( \mathcal{F}^\delta|_{AOMOD} = \mathcal{F} \) (similarly for \( G^\delta \)). In conclusion, we have proved that the pairs \((A, C), (B, D)\) are \( \Delta \)-Morita equivalent.

We are now going to complete the remaining proof of (ii) \( \implies \) (i). Suppose that the pairs \((A, C), (B, D)\) are \( \Delta \)-Morita equivalent. We fix an equivalence functor \( \mathcal{F} : cOMOD \to dOMOD \) with inverse \( G : dOMOD \to cOMOD \) such that

\[
\mathcal{F}(AOMOD) = BOMOD, G(BOMOD) = AOMOD
\]

Let \( \mathcal{F}(\mathcal{C}) = \mathcal{Y}_0, G(\mathcal{D}) = \mathcal{X}_0 \). By [2] we have that \( \mathcal{Y}_0 \) is a TRO and \( \mathcal{X}_0 \cong \mathcal{Y}_0^* \). Also, \( \mathcal{C} \cong \mathcal{X}_0 \otimes_C^h \mathcal{Y}_0 \), \( \mathcal{D} \cong \mathcal{Y}_0 \otimes_C^h \mathcal{X}_0 \). We also assume that \( \mathcal{F}(A) = \mathcal{Y}, G(B) = \mathcal{X} \), and by [4] we get

\[
A \cong \mathcal{X} \otimes_B^h \mathcal{Y}, B \cong \mathcal{Y} \otimes_A^h \mathcal{X}.
\]

From both the above papers, we have that

\[
\mathcal{F}(U) \cong \mathcal{Y} \otimes_A^h U, \forall U \in AOMOD, \mathcal{F}(U) \cong \mathcal{Y}_0 \otimes_C^h U, \forall U \in cOMOD
\]

and then we get

\[
\mathcal{Y} \otimes_A^h U \cong \mathcal{Y}_0 \otimes_C^h U, \forall U \in AOMOD.
\]

Similarly, \( \mathcal{X} \otimes_B^h V \cong \mathcal{X}_0 \otimes_D^h V, \forall V \in BOMOD \).

Now, we have that

\[
\mathcal{X}_0 \otimes_D^h B \otimes_D^h \mathcal{X}_0^* \cong \mathcal{X}_0 \otimes_D^h (\mathcal{Y} \otimes_A^h \mathcal{X}) \otimes_B^h \mathcal{Y}_0 \\
\cong (\mathcal{X}_0 \otimes_D^h \mathcal{Y}) \otimes_A^h (\mathcal{X} \otimes_D^h \mathcal{Y}_0) \\
\cong (\mathcal{X} \otimes_B^h \mathcal{Y}) \otimes_A^h (\mathcal{X} \otimes_B^h \mathcal{Y}) \\
\cong A \otimes_A^h A \\
\cong A
\]

Similarly, \( \mathcal{X}^*_0 \otimes_C^h A \otimes_C^h \mathcal{X}_0 \cong B \). The following lemma implies that \( A \sim_\Delta B \) and \( C \sim_\Delta D \), where \( \Delta \)-equivalence is implemented in both cases by the same TRO. The proof of Theorem 2.5 is complete. \( \square \)

**Lemma 2.6.** Suppose that \( A, B \) are operator algebras and \( D \subseteq B \) be a \( C^* \)-algebra such that \( [D, B] = [B, D] = B \). Let \( M \subseteq \mathbb{B}(K, H) \) be a TRO such that \( [M, M] \cong D \) (as \( C^* \) algebras) and assume that \( A \cong M \otimes_D^h B \otimes_D^h M^* \). Then \( A \sim_\Delta B \).
Proof. We fix a completely isometric homomorphism $\beta : B \to B(K)$, and we have that $\beta_{|D}$ is also a $\ast$-homomorphism. We can consider that the operator space $M \otimes^h_D K$ is a Hilbert space with the inner product given by

$$\langle n \otimes_D x, \ell \otimes_D y \rangle := \langle \beta(\phi(\ell^* n))(x), y \rangle, n, \ell \in M, x, y \in K,$$

where $\phi : [M^* M] \to D$ is a $\ast$-isomorphism ([6]). Instead of $\phi(\ell^* n)$, we may write

$$\langle n \otimes_D x, \ell \otimes_D y \rangle := \langle \beta(\ell^* n)(x), y \rangle, n, \ell \in M, x, y \in K,$$

and for the action of $D$ on $K$, we denote $\ell^* nx$ instead of $\phi(\ell^* n)x$ where $\ell, n \in M, x \in K$.

For each $m \in M$, we define $r_m : K \to M \otimes^h_D K$ by $r_m(x) = m \otimes_D x$.

Obviously, $r_m$ is a linear map and $r_m \in B(K, M \otimes^h_D K)$.

We can easily see that $r_{m_1}, r_{m_2}, r_{m_3} = r_{m_1} m_2 m_3 \in r(M)$, therefore, $r(M)$ is a TRO. Also, with similar arguments, we have that $\beta(D) = [r(M)^* r(M)]$ (since $r(M)^* r(M) = \beta(M^* M))$. We also claim that $r$ is completely isometric. By Lemma 8.3.2 (Harris-Kaup) of [5], it is sufficient to prove that $r$ is one-to-one. Indeed, for every $m \in M$ holds

$$(r^*_m r_m)(x) = r^*_m (m \otimes_D x) = \beta(m^* m)(x), \forall x \in K$$

so $||r_m||^2 = ||r^*_m r_m|| = ||\beta(m^* m)|| = ||m^* m|| = ||m||^2$, which means that $r$ is isometric and also one-to-one. Therefore, $M \cong r(M)$, and using Lemma 5.4 in [12], we get

$$M \otimes^h_D B \otimes^h_D M^* \cong M \otimes^h_D [\beta(B) r(M)^*] \cong [r(M) \beta(B) r(M)^*].$$

Therefore, there exists a completely isometric map $a : A \to a(A)$ such that $a(A) = [r(M) \beta(B) r(M)^*]$ (4), so

$$[r(M)^* a(A) r(M)] = [r(M)^* r(M) \beta(B) r(M)^* r(M)]$$

$$= [\beta(D) \beta(B) \beta(D)]$$

$$= \beta(D) (5)$$

By (4), (5), we get $a(A) \sim_{TRO} \beta(B) \implies A \sim_{\Delta} B$. □

Corollary 2.7. The relation $\sim_{\Delta}$ is an equivalence relation for $\Delta$-pairs.

Remark 2.8. We consider that the $\Delta$-pairs $(A, C), (B, D)$ are equivalent in the sense of Theorem 2.5, and $\mathcal{F}$ is the functor defined in its proof. For every $U_1, U_2 \in \mathcal{A}MOD$, the map $\mathcal{F} : \mathcal{A}CB(U_1, U_2) \to \mathcal{B}CB(\mathcal{F}(U_1), \mathcal{F}(U_2))$ is a complete isometry.

Proof. For every $g \in \mathcal{B}CB(\mathcal{F}(U_1), \mathcal{F}(U_2))$, we define

$$\theta = f_{U_2} \circ g \circ f_{U_1}^{-1} \in \mathcal{A}CB(U_1, U_2).$$
So, for any \( f \in _A CB(U_1, U_2) \), we have that \( \mathcal{F}(f) \in _B CB(\mathcal{F}(U_1), \mathcal{F}(U_2)) \) and

\[
(\theta \circ G)(\mathcal{F}(f)) = \theta(G(\mathcal{F}(f))) = f_{U_2} \circ G \mathcal{F}(f) \circ f_{U_1}^{-1} = f \circ f_{U_1} \circ f_{U_1}^{-1} = f
\]

so, \( ||f||_{cb} = ||(\theta \circ G)(\mathcal{F}(f))||_{cb} \leq ||\mathcal{F}(f)||_{cb} \) (since \( \theta \circ G \) is completely contractive)

**Theorem 2.9.** Let \((A, C), (B, D)\) be \( \sigma \Delta \)-pairs. The following are equivalent:

(i) \( A \sim_\sigma \Delta B, C \sim_\sigma \Delta D \), where \( \sigma \Delta \)-equivalence is implemented in both cases by the same \( \sigma \)-TRO.

(ii) The pairs \((A, C), (B, D)\) are \( \Delta \)-Morita equivalent.

(iii) There exists a completely isometric isomorphism \( \phi : A \otimes K \to B \otimes K \) such that \( \phi(C \otimes K) = D \otimes K \), where \( K \) is the algebra of compact operators of \( \ell^2(\mathbb{N}) \).

**Proof.** (i) \( \iff \) (ii) It is obvious according to the main Theorem 2.5.

(i) \( \implies \) (iii). We may consider \( A = [M^*BM], B = [MAM^*] \) and also \( C = [M^*DM], D = [MCM^*] \). Since \( C, D \) have a \( \sigma \)-unit by Lemma 3.4 of [10], \( M \) is a \( \sigma \)-TRO. By Theorem 3.2 in the same article, there exists a completely isometric onto map \( \phi : A \otimes K \to B \otimes K \) such that \( \phi(C \otimes K) = D \otimes K \).

(iii) \( \implies \) (i) We have that \((A, C) \sim_\Delta (A \otimes K, C \otimes K)\), so \((A, C) \sim_\Delta (B \otimes K, D \otimes K)\), but also \((B, D) \sim_\Delta (B \otimes K, D \otimes K)\). Since \( \sim_\Delta \) is an equivalence relation for \( \Delta \)-pairs, we get \((A, C) \sim_\Delta (B, D)\). \( \square \)

In the rest of this section, we consider that the \( \Delta \)-pairs \((A, C), (B, D)\) are equivalent in the sense of Theorem 2.5, and \( \mathcal{F} \) is the functor defined in its proof.

We consider the subcategory of representations of \( A \) denoted by \( A_{HMOD} \). If \( H' \in A_{HMOD} \), there exists a completely contractive morphism \( \pi : A \to \mathbb{B}(H') \) such that \( \pi(A)(H') = \{ \pi(a)(h) \in H' : a \in A, h \in H' \} \). The space \( \mathcal{F}(H') = M \otimes_E h' \) is also a Hilbert space. Its inner product is given by

\[
\langle m \otimes E \xi, \ell \otimes E w \rangle := \langle \pi(\ell^*) m(\xi), w \rangle_{H'}, m, \ell \in M, \xi, w \in H'
\]

(for more details check [6,15,17]).

Also, the map \( \mathcal{F}(\pi) : A \to \mathbb{B}(\mathcal{F}(H')) \) given by

\[
\mathcal{F}(\pi)(m b n^*)(\ell \otimes E \xi) = m \otimes E \pi(b n \ell^*)(\xi)
\]

is completely contractive.

We are going to prove that the functor \( \mathcal{F} \) maintains the complete isometric representations. So, let \( \pi : A \to \mathbb{B}(H') \) be a homomorphism and also a complete isometry. We set \( \rho = \mathcal{F}(\pi) : \mathcal{B} \to \mathbb{B}(\mathcal{F}(H')) \), where \( \mathcal{F}(H') = M \otimes_E h' \) and

\[
\rho(m a n^*)(\ell \otimes E h) = m \otimes E \pi(a n \ell^*)(h), m, n, \ell \in M, a \in A, h \in H'
\]
We define the unitary operator
\[ U : G \mathcal{F}(H') \to H', U(k^* \otimes_{E'} (n \otimes_E h)) := \pi(k^* n)(h), \]
and we consider \( \phi = G(\rho) : A \to \mathcal{B}(G \mathcal{F}(H')) \) given by
\[ \phi(m^* b n)(\ell^* \otimes_{E'} x) = m^* \otimes_{E'} \rho(b n \ell^*)(x), m, n, \ell \in M, b \in B, x \in \mathcal{F}(H') \]

**Lemma 2.10.** It holds that \( U \phi(a) U^* = \pi(a), \forall a \in A \).

**Proof.** For every \( m, k, s, n, t, \ell \in M, a \in A, h \in H' \), we have that
\[ \phi(m^* k a s^* n)(\ell^* \otimes_{E'} (t \otimes_E h)) = m^* \otimes_{E'} \rho(k a s^* n \ell^*)(t \otimes_E h) \]
\[ = m^* \otimes_{E'} k \otimes_E \pi(a s^* n \ell^* t)(h) \]
Therefore,
\[ U(\phi(m^* k a s^* n)(\ell^* \otimes_{E'} (t \otimes_E h))) = U(m^* \otimes_{E'} k \otimes_E \pi(a s^* n \ell^* t)(h)) \]
\[ = \pi(m^* k)(\pi(a s^* n \ell^* t)(h)) \]
\[ = \pi(m^* k a s^* n)(\ell^* t)(h) \]
\[ = \pi(m^* k a s^* n) U(\ell^* \otimes_{E'} (t \otimes_E h)) \]
So, \( U(\phi(m^* k a s^* n)) = \pi(m^* k a s^* n) U \), but since \( A = [M^* M A M^* M] \), we get that \( U \phi(a) U^* = \pi(a) \). \( \square \)

We conclude that since \( \pi \) is an isometry and \( U \phi(a) U^* = \pi(a), a \in A \), where \( U \) is unitary, \( \phi \) is also an isometry. Observe now that if \( (m_i)_{i \in I} \) is a net of \( M_{n_i,1}(M^*) \) such that \( ||m_i|| \leq 1 \forall i \in I \) and also \( m_i m_i^* m \to m, \forall m \in M \), then we have that
\[ \phi(m_i^* b m_i)(\ell^* \otimes_{E'} x) = m_i^* \otimes_{E'} \rho(b m_i \ell^*)(x) \]
\[ = m_i^* \otimes_{E'} \rho(b) V(m_i \otimes_E (\ell^* \otimes_{E'} x)) \]
for every \( b \in B, x \in \mathcal{F}(H') \), \( \ell \in M \) where \( V \) is the unitary operator
\[ V : M \otimes_E^h (M^* \otimes_E^h K') \to K', \quad K' = \mathcal{F}(H'). \]

Therefore, for all \( w \in M^* \otimes K' \) holds
\[ \phi(m_i^* b m_i)(w) = m_i \otimes_E \rho(b) V(m_i \otimes_E w). \]
So,
\[ ||\phi(m_i^* b m_i)(w)|| \leq ||m_i|| ||\rho(b)|| ||m_i^*|| ||w|| \leq ||\rho(b)|| ||w||, \]
which means that \( ||\phi(m_i^* b m_i)|| \leq ||\rho(b)|| \), but \( \phi \) is an isometry, and we conclude that \( ||m_i^* b m_i|| \leq ||\rho(b)||, \forall b \in B \) (3).

Since \( \lim_i m_i m_i^* b m_i m_i^* = b \), we have that
\[ \sup_{i \in I} ||m_i m_i^* b m_i m_i^*|| = ||b||. \]
On the other hand, \( ||m_i m_i^* b m_i m_i^*|| \leq ||m_i^* b m_i|| \leq ||b|| \), therefore,
\[ \sup_{i \in I} ||m_i^* b m_i|| = ||b||, \forall b \in B. \]

We conclude from (3) that \( ||b|| \leq ||\rho(b)||, \forall b \in B \), but also that \( \rho \) is completely contractive and therefore \( ||\rho(b)|| = ||b||, \forall b \in B \), so \( \rho \) is an isometry.
Similarly, \( \rho \) is a complete isometry. Using the above facts, we can prove that \( \mathcal{F} \) restricts to an equivalence functor from \( _A \text{HMOD} \) to \( _B \text{HMOD} \). This functor maps completely isometric representations to completely isometric representations.

3. \( \Delta \)-Morita Equivalence of Operator Spaces

**Definition 3.1.** Let \( \mathcal{X} \subseteq \mathbb{B}(H_1, H_2) \), \( \mathcal{Y} \subseteq \mathbb{B}(K_1, K_2) \) be operator spaces. We call them TRO-equivalent (resp. \( \sigma \)-TRO equivalent) if there exist TROs (resp. \( \sigma \)-TROs) \( M_i \subseteq \mathbb{B}(H_i, K_i), i = 1, 2 \) such that

\[
\mathcal{X} = [M_2 Y M_1], \mathcal{Y} = [M_2 X M_1^*]
\]

We write \( \mathcal{X} \sim_{TRO} \mathcal{Y} \), resp. \( \mathcal{X} \sim_{\sigma TRO} \mathcal{Y} \).

**Definition 3.2.** Let \( \mathcal{X}, \mathcal{Y} \) be operator spaces. We call them \( \Delta \)-equivalent (resp. \( \sigma \Delta \)-equivalent) if there exist completely isometric maps \( \phi : \mathcal{X} \to \mathbb{B}(H_1, H_2), \psi : \mathcal{Y} \to \mathbb{B}(K_1, K_2) \) such that \( \phi(\mathcal{X}) \sim_{TRO} \psi(\mathcal{Y}) \) (resp. \( \phi(\mathcal{X}) \sim_{\sigma TRO} \psi(\mathcal{Y}) \)). We write \( \mathcal{X} \sim_{\Delta} \mathcal{Y} \), resp. \( \mathcal{X} \sim_{\sigma \Delta} \mathcal{Y} \).

**Definition 3.3.** Let \( \mathcal{X} \) be an operator space and \( D_1, D_2 \) be \( C^* \)-algebras (resp. \( \sigma \) unital \( C^* \)-algebras) such that

\[
\mathcal{X} = [D_1 \mathcal{X}] = [D_2 \mathcal{X}]
\]

Then, the space

\[
\mathcal{A}_\mathcal{X} = \begin{pmatrix} D_2 & \mathcal{X} \\ 0 & D_1 \end{pmatrix}
\]

is an operator algebra, which we call an algebraic \( \Delta \)-extension of \( X \) (resp. \( \sigma \Delta \)-extension of \( \mathcal{X} \)).

In what follows, if \( \mathcal{A} \) is an operator algebra, then \( \Delta(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^* \) is it’s diagonal.

**Definition 3.4.** Let \( \mathcal{X}, \mathcal{Y} \) be operator spaces. We call them \( \Delta \)-Morita equivalent (resp. \( \sigma \Delta \)-Morita equivalent) if they have algebraic \( \Delta \)-extensions (resp. \( \sigma \Delta \)-extensions) \( \mathcal{A}_\mathcal{X}, \mathcal{A}_\mathcal{Y} \) such that the \( \Delta \)-pairs \( (\mathcal{A}_\mathcal{X}, \Delta(\mathcal{A}_\mathcal{X})), (\mathcal{A}_\mathcal{Y}, \Delta(\mathcal{A}_\mathcal{Y})) \) to be \( \Delta \)-Morita equivalent.

**Lemma 3.5.** The TRO-equivalence (resp. \( \sigma \)-TRO) of operator spaces is an equivalence relation.

**Proof.** The fact TRO-equivalence is an equivalence relation has been proved in [12]. The proof that \( \sigma \)-TRO-equivalence is an equivalence relation is similar. \( \square \)

**Theorem 3.6.** Let \( \mathcal{X}, \mathcal{Y} \) be operator spaces. The following are equivalent:

(i) \( \mathcal{X} \sim_{\Delta} \mathcal{Y} \)
(ii) \( \mathcal{X} \) and \( \mathcal{Y} \) are \( \Delta \)-Morita equivalent.
Proof. (i) $\implies$ (ii) We may assume that $\mathcal{X} = [M_2^* {\mathcal{Y}} M_1^*], \mathcal{Y} = [M_2 {\mathcal{X}} M_1^*]$ for TROs $M_1 \subseteq \mathbb{B}(H_1, K_1)$ and $M_2 \subseteq \mathbb{B}(H_2, K_2)$. If we consider the $C^*$-algebras

$$D_1 = [M_1^* M_1], D_2 = [M_2^* M_2], E_1 = [M_1 M_1^*], E_2 = [M_2 M_2^*],$$

we get

$$\mathcal{X} = [D_2 \mathcal{X}] = [\mathcal{X} D_1], \mathcal{Y} = [E_2 \mathcal{Y}] = [\mathcal{Y} E_1].$$

So, the operator algebras

$$A_\mathcal{X} = \begin{pmatrix} D_2 & \mathcal{X} \\ 0 & D_1 \end{pmatrix} \subseteq \mathbb{B}(H_2 \oplus H_1), A_\mathcal{Y} = \begin{pmatrix} E_2 & \mathcal{Y} \\ 0 & E_1 \end{pmatrix} \subseteq \mathbb{B}(K_2 \oplus K_1)$$

are $\Delta$-algebraic extensions of $\mathcal{X}, \mathcal{Y}$, respectively, such that

$$\Delta(A_\mathcal{X}) = \begin{pmatrix} D_2 & 0 \\ 0 & D_1 \end{pmatrix}, \Delta(A_\mathcal{Y}) = \begin{pmatrix} E_2 & 0 \\ 0 & E_1 \end{pmatrix}.$$ 

Clearly

$$M = \begin{pmatrix} M_2 & 0 \\ 0 & M_1 \end{pmatrix} \subseteq \mathbb{B}(H_2 \oplus H_1, K_2 \oplus K_1)$$

is a TRO. Furthermore, $[M^* A_\mathcal{Y} M] = A_\mathcal{X}$ and $A_\mathcal{Y} = [M A_\mathcal{X} M^*]

\Delta(A_\mathcal{X}) = [M^* \Delta(A_\mathcal{Y}) M], \Delta(A_\mathcal{Y}) = [M \Delta(A_\mathcal{X}) M^*]$ so the pairs $(A_\mathcal{X}, \Delta(A_\mathcal{X}))$, $(A_\mathcal{Y}, \Delta(A_\mathcal{Y}))$ are $\Delta$-Morita equivalent. That is, $\mathcal{X}$ and $\mathcal{Y}$ are $\Delta$-Morita equivalent.

(ii) $\implies$ (i) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are $\Delta$-Morita equivalent. There exist $C^*$-algebras $D_i, E_i, i = 1, 2$ such that $\mathcal{X} = [D_2 \mathcal{X}] = [\mathcal{X} D_1], \mathcal{Y} = [E_2 \mathcal{Y}] = [\mathcal{Y} E_1]$ and the pairs $(A_\mathcal{X}, \Delta(A_\mathcal{X}))$, $(A_\mathcal{Y}, \Delta(A_\mathcal{Y}))$ are $\Delta$-Morita equivalent, where

$$A_\mathcal{X} = \begin{pmatrix} D_2 & \mathcal{X} \\ 0 & D_1 \end{pmatrix}, A_\mathcal{Y} = \begin{pmatrix} E_2 & \mathcal{Y} \\ 0 & E_1 \end{pmatrix}.$$ 

so

$$\Delta(A_\mathcal{X}) = \begin{pmatrix} D_2 & 0 \\ 0 & D_1 \end{pmatrix}, \Delta(A_\mathcal{Y}) = \begin{pmatrix} E_2 & 0 \\ 0 & E_1 \end{pmatrix}.$$ 

Let $N$ be a TRO such that

$$A_\mathcal{X} = [N^* A_\mathcal{Y} N^*], A_\mathcal{Y} = [N A_\mathcal{X} N^*]$$

\Delta(A_\mathcal{X}) = [N^* \Delta(A_\mathcal{Y}) N], \Delta(A_\mathcal{Y}) = [N \Delta(A_\mathcal{X}) N^*].$$

We define $M = [\Delta(A_\mathcal{Y}) N] = [N \Delta(A_\mathcal{X})]$, then $M$ is a TRO since

$$[M M^* M] = [\Delta(A_\mathcal{Y}) N N^* \Delta(A_\mathcal{Y}) \Delta(A_\mathcal{Y}) N]$$

$$= [\Delta(A_\mathcal{Y}) N N^* \Delta(A_\mathcal{Y}) N] = [\Delta(A_\mathcal{Y}) N \Delta(A_\mathcal{X})] = M$$

Using the fact that $[\Delta(A_\mathcal{Y}) A_\mathcal{Y} \Delta(A_\mathcal{Y})] = A_\mathcal{Y}$, we get

$$[M^* A_\mathcal{Y} M] = [N^* \Delta(A_\mathcal{Y}) A_\mathcal{Y} \Delta(A_\mathcal{Y}) N] = [N^* A_\mathcal{Y} N^*] = A_\mathcal{X}$$

and with similar arguments we have that

$$A_\mathcal{Y} = [M A_\mathcal{X} M^*], \Delta(A_\mathcal{X}) = [M^* M], \Delta(A_\mathcal{Y}) = [M M^*].$$
We define the TROs
\[ M_2 = \begin{bmatrix} E_2 & 0 \\ 0 & D_2 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & E_1 \\ 0 & 0 & D_1 \end{bmatrix} \]

Since \( Y = [E_2 Y E_1] \), we have that
\[ \begin{bmatrix} 0 & Y' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} A_Y \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} M A_X M^* \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} \]
\[ = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} \left( M \Delta(A_X) M^* + M \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} \]

But \( [M \Delta(A_X) M^*] = \Delta(A_Y) = \begin{bmatrix} E_2 & 0 \\ 0 & E_1 \end{bmatrix} \), so it holds that
\[ \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} M \Delta(A_X) M^* \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} = 0, \]
and thus
\[ \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

Similarly, \( \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) and therefore
\[ \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & 0 \end{bmatrix} \sim_{TRO} \begin{bmatrix} 0 & Y' \\ 0 & 0 \end{bmatrix}. \]

Since \( (\mathbb{C}, 0) \begin{bmatrix} 0 & \mathcal{X} \\ 0 & \mathcal{X} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & 0 \end{bmatrix} = \mathcal{X} \) and \( \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & \mathcal{X} \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & \mathcal{X} \end{bmatrix} \)

we have that \( \mathcal{X} \sim_{TRO} \begin{bmatrix} 0 & \mathcal{X}^* \\ 0 & 0 \end{bmatrix} \). Similarly, \( \mathcal{Y} \sim_{TRO} \begin{bmatrix} 0 & Y' \\ 0 & 0 \end{bmatrix} \). Therefore, according to Lemma 3.5, we get \( \mathcal{X} \sim_{\Delta} \mathcal{Y} \).

Theorem 3.6 and Theorem 3.11 in [12] imply the following corollary:

**Corollary 3.7.** \( \Delta \)-Morita equivalence of operator spaces is an equivalence relation.

**Theorem 3.8.** Let \( \mathcal{X}, \mathcal{Y} \) be operator spaces. The following are equivalent:

(i) \( \mathcal{X} \sim_{\sigma \Delta} \mathcal{Y} \)

(ii) \( \mathcal{X} \) and \( \mathcal{Y} \) are \( \sigma \Delta \)-Morita equivalent

(iii) \( \mathcal{X} \) and \( \mathcal{Y} \) are stably isomorphic.
Proof. \((i) \iff (ii)\) Check the proof of the Theorem 3.6.

\((i) \implies (iii)\) See \([12, \text{Theorem } 4.6]\).

\((iii) \implies (i)\) Since \(X \sim_{st} Y\), we have that \(K_\infty(X) \cong K_\infty(Y)\), but since \(X \sim_{\sigma TRO} K_\infty(X)\), we get \(X \sim_{\sigma \Delta} K_\infty(Y)\). Also, \(Y \sim_{\sigma TRO} K_\infty(Y)\), so \(Y \sim_{\sigma \Delta} K_\infty(Y)\), and due to the fact that \(\sim_{\sigma \Delta}\) is an equivalence relation, we have \(X \sim_{\sigma \Delta} Y\). \(\square\)

4. \(\Delta\)-Morita Equivalence of Unital Operator Spaces

**Definition 4.1.** We call an operator space \(X\) unital if there exists a completely isometric map \(\phi : X \to B(H)\) such that \(I_H \in \phi(X)\).

If \(Y\) is an operator space that is bimodule over the \(C^*\) algebra \(A\), we say that the map

\[(\pi, \psi, \pi) : \mathcal{A}Y \mathcal{A} \to B(H)\]

is a completely contractive bimodule map if \(\psi : Y \to B(H)\) is a completely contractive map and \(\pi : \mathcal{A} \to B(H)\) is a \(*\)-homomorphism such that

\[\phi(asb) = \pi(a)\phi(s)\pi(b), \quad \forall a, b \in \mathcal{A}, \ s \in Y.\]

**Lemma 4.2.** Let \(X, Y\) be operator spaces and \(M\) be a TRO such that

\[X = \begin{bmatrix} MYM^* \end{bmatrix}, \ Y = \begin{bmatrix} MYM^* \end{bmatrix}.\]

We denote \(A = \begin{bmatrix} M^*M \end{bmatrix}, B = \begin{bmatrix} MM^* \end{bmatrix}\). For every completely isometric bimodule map

\[(\pi, \psi, \pi) : \mathcal{A}Y \mathcal{A} \to B(H)\]

there exists a completely isometric bimodule map

\[(\sigma, \phi, \sigma) : B \mathcal{A} B \to B(K)\]

and a TRO \(N \subseteq B(H, K)\) such that

\[\psi(Y) = \begin{bmatrix} N^*\phi(X)N \end{bmatrix}, \ \phi(X) = \begin{bmatrix} N\psi(Y)N^* \end{bmatrix}, \ \pi(A) = \begin{bmatrix} N^*N \end{bmatrix}, \ \sigma(B) = \begin{bmatrix} NN^* \end{bmatrix}.\]

Proof. Suppose that \(K = M \otimes_A^h H\) is the Hilbert space with the inner product given by

\[\langle m \otimes \xi, n \otimes \omega \rangle = \langle \pi(n^*m)\xi, \omega \rangle \quad m, n \in M, \ \xi, \omega \in H.\]

By the usual arguments, we can define a completely isometric map \(\phi : X \to B(K)\) given by

\[\phi(msn^*)(l \otimes \xi) = m \otimes \psi(sn^*l)(\xi), \ m, n, l \in M, \ s \in S, \xi, \omega \in H\]

and the \(*\)-homomorphism \(\sigma : B \to B(K)\) given by

\[\sigma(mn^*)(l \otimes \xi) = m \otimes \pi(n^*l)(\xi).\]

We also define the map \(\mu : M \to B(L, K)\) given by

\[\mu(m)(\pi(n^*l)(\xi)) = (mn^*l) \otimes \xi\]

and the map \(\nu : M^* \to B(K, L)\) given by

\[\nu(m^*)(l \otimes \xi) = \pi(m^*l)(\xi).\]
We can easily see that \( \nu(m^*) = \mu(m)^* \) for all \( m \), and \( N = \mu(M) \) is a TRO satisfying \( \psi(\mathcal{Y}) = [N^* \phi(\mathcal{X}) N^*] \), \( \phi(\mathcal{X}) = [N \psi(\mathcal{Y}) N^*] \) and also \( \pi(A) = [N^* N] \), \( \sigma(B) = [NN^*] \).

**Lemma 4.3.** Let \( \mathcal{X} \subseteq \mathcal{B}(K') \), \( \mathcal{Y} \subseteq \mathcal{B}(H') \) be unital operator spaces and \( M \subseteq \mathcal{B}(H', K') \) be a TRO such that

\[
\mathcal{X} = [M \mathcal{Y} M^*], \quad \mathcal{Y} = [M^* \mathcal{X} M].
\]

We denote \( A = [M^* M], B = [MM^*] \). For every completely isometric bimodule map

\[
(\pi, \psi, \pi) : A \mathcal{Y} A \to \mathcal{B}(H)
\]

there exists a unital completely isometric bimodule map

\[
(\sigma, \phi, \sigma) : B \mathcal{X} B \to \mathcal{B}(K)
\]

and a TRO \( N \subseteq \mathcal{B}(H, K) \) such that

\[
\psi(\mathcal{Y}) = [N^* \phi(\mathcal{X}) N^*], \quad \phi(\mathcal{X}) = [N \psi(\mathcal{Y}) N^*], \quad \pi(A) = [N^* N], \quad \sigma(B) = [NN^*].
\]

**Proof.** Suppose that \( K, \psi, \mu, \sigma, N \) are as in the proof of Lemma 4.2. We can see that

\[
\phi(msn^*) = \mu(m) \psi(s) \mu(n)^*, \quad \forall m, n \in M, \ s \in \mathcal{X}.
\]

Since \( \mathcal{Y} \) is unital, we have that

\[
\phi(mn^*) = \mu(m) \psi(I_{H'}) \mu(n)^*, \quad \forall m, n \in M.
\]

Assume that

\[
I_{K'} = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} m_i^{\lambda} (m_i^{\lambda})^*.
\]

If \( l \in M, \xi \in H \), we have

\[
\phi(I_{K'})(l \otimes \xi) = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} \mu(m_i^{\lambda}) \psi(I_{H'}) \mu((m_i^{\lambda})^*) (l \otimes \xi) \\
= \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} \mu(m_i^{\lambda}) \psi(I_{H'}) \pi((m_i^{\lambda})^*) (l) (\xi) = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} \mu(m_i^{\lambda}) \psi((m_i^{\lambda})^*) (l) (\xi).
\]

We can easily see that \( \psi|_{M^* M} = \pi \), thus

\[
\phi(I_{K'})(l \otimes \xi) = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} \mu(m_i^{\lambda}) \pi((m_i^{\lambda})^*) (l) (\xi) = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} m_i^{\lambda} (m_i^{\lambda})^* l \otimes \xi = l \otimes \xi.
\]

Therefore, \( \phi(I_{K'}) = I_K \). □

**Lemma 4.4.** Let \( \mathcal{X}, \mathcal{Y} \) be unital operator spaces such that \( \mathcal{X} \sim_\Delta \mathcal{Y} \). Then, there exist completely isometric maps

\[
\phi : \mathcal{X} \to \mathcal{B}(H), \quad \psi : \mathcal{Y} \to \mathcal{B}(K)
\]

such that \( I_H \in \phi(\mathcal{X}), I_K \in \psi(\mathcal{Y}) \) and a \( \sigma \)-TRO \( L \subseteq \mathcal{B}(K, H) \) such that

\[
\psi(\mathcal{Y}) = [L^* \phi(\mathcal{X}) L], \quad \phi(\mathcal{X}) = [L \psi(\mathcal{Y}) L^*].
\]
Proof. We have that \( \mathcal{Y} \sim_{TRO} K_{\infty}(\mathcal{Y}) \), and the TRO equivalence is implemented by one TRO. Since \( \mathcal{X} \) and \( \mathcal{Y} \) are unital, by \cite{12} \( K_{\infty}(\mathcal{Y}) \cong K_{\infty}(\mathcal{X}) \) as \( K_{\infty} \)-operator modules. Lemma 4.2 implies that there exists a completely isometric map \( \zeta : \mathcal{Y} \to \mathcal{Y}(\mathcal{Y}) \) such that \( \zeta(\mathcal{Y}) \sim_{TRO} K_{\infty}(\mathcal{X}) \), and this TRO equivalence is implemented by one TRO. Since \( \mathcal{X} \sim_{TRO} K_{\infty}(\mathcal{X}) \) with one TRO as in the proof of Theorem 2.1 in \cite{10}, we have that \( \zeta(\mathcal{Y}) \sim_{TRO} \mathcal{X} \) with one TRO. From Lemma 4.2, given the complete isometry \( \zeta^{-1} : \zeta(\mathcal{Y}) \to \mathcal{Y} \), there exists a complete isometry \( \phi : \mathcal{X} \to \phi(\mathcal{X}) \) and a TRO \( M \) such that

\[
\mathcal{Y} = [M\phi(\mathcal{X})M^*], \quad \phi(\mathcal{X}) = [M^*\mathcal{Y}M].
\]

By Lemma 4.9 in \cite{12}, the algebra \( [M^*M] \) is unital, thus \( \phi(\mathcal{X}) \) is a unital operator space. The map \( \phi^{-1} : \phi(\mathcal{X}) \to \mathcal{X} \) is a complete isometry, thus by Lemma 4.3 there exists a unital complete isometry \( \psi : \mathcal{Y} \to \psi(\mathcal{Y}) \) and a TRO \( L \) such that \( \psi(\mathcal{Y}) = [L^*\mathcal{X}L], \quad \mathcal{X} = [L\psi(S)L^*] \).

If \( \mathcal{X} \) is an operator space, we denote by \( M_\ell(\mathcal{X}) \) (resp. \( M_r(\mathcal{X}) \)) the left (resp. right) multiplier algebra of \( \mathcal{X} \). We also denote

\[
\mathcal{A}_\ell(\mathcal{X}) = \Delta(M_\ell(\mathcal{X})), \quad \mathcal{A}_r(\mathcal{X}) = \Delta(M_r(\mathcal{X})).
\]

Remark 4.5. If we consider \( \mathcal{X} \) as unital subspace of its \( C^* \)-envelope, \( C^*_{\text{env}}(\mathcal{X}) \), then by Proposition 4.3 in \cite{3}, we have

\[
M_\ell(\mathcal{X}) = \{ a \in C^*_{\text{env}}(\mathcal{X}) : a \mathcal{X} \subseteq \mathcal{X} \}
\]

and

\[
M_r(\mathcal{X}) = \{ a \in C^*_{\text{env}}(\mathcal{X}) : \mathcal{X} a \subseteq \mathcal{X} \}
\]

Lemma 4.6. If \( \mathcal{X}, \mathcal{Y} \) are \( \Delta \)-equivalent unital operator spaces, we can consider that \( \mathcal{X} \subseteq C^*_{\text{env}}(\mathcal{X}) \subseteq B(H) \), \( \mathcal{Y} \subseteq C^*_{\text{env}}(\mathcal{Y}) \subseteq B(K) \) and there exists a TRO \( M \subseteq B(H, K) \) such that \( \mathcal{X} = [M^*\mathcal{Y}M], \quad \mathcal{Y} = [M\mathcal{X}M^*] \) and also

\[
C^*_{\text{env}}(\mathcal{X}) = [M^* C^*_{\text{env}}(\mathcal{Y}) M], \quad C^*_{\text{env}}(\mathcal{Y}) = [M C^*_{\text{env}}(\mathcal{X}) M] \]

\[
M_\ell(\mathcal{X}) = [M^* M_\ell(\mathcal{Y}) M], \quad M_\ell(\mathcal{Y}) = [M M_\ell(\mathcal{X}) M^*] \]

\[
M_r(\mathcal{X}) = [M^* M_r(\mathcal{Y}) M], \quad M_r(\mathcal{Y}) = [M M_r(\mathcal{X}) M^*] \]

Proof. From Lemma 4.4, we may assume that \( \mathcal{X} \) and \( \mathcal{Y} \) have TRO equivalent completely isometric representations whose images are TRO equivalent by one TRO. Using this fact and the proof of Theorem 5.10 in \cite{12}, we may consider that there exists a TRO \( M \) such that

\[
C^*_{\text{env}}(\mathcal{X}) = [M^* C^*_{\text{env}}(\mathcal{Y}) M], \quad C^*_{\text{env}}(\mathcal{Y}) = [M C^*_{\text{env}}(\mathcal{X}) M].
\]

Let us prove that \( M_\ell(\mathcal{X}) = [M^* M_\ell(\mathcal{Y}) M] \). Let \( a \in M_\ell(\mathcal{Y}) \), that is \( a \in C^*_{\text{env}}(\mathcal{Y}) \) and \( a \mathcal{Y} \subseteq \mathcal{Y} \). For all \( m, n \in M \), we have that \( m^* a n \in C^*_{\text{env}}(\mathcal{Y}) \) and

\[
m^* a n \mathcal{X} = m^* a n M^* \mathcal{Y} M \subseteq m^* a \mathcal{Y} M \subseteq M^* \mathcal{Y} M = \mathcal{X},
\]

and
so \( m^* a n \in M_l(\mathcal{X}) \), that is \( M^* M_l(\mathcal{Y}) M \subseteq M_l(\mathcal{X}) \). Similarly, \( M M_l(\mathcal{X}) M^* \subseteq M_l(\mathcal{Y}) \), so

\[
M^* M M_l(\mathcal{X}) M^* M \subseteq M^* M_l(\mathcal{Y}) M \subseteq M_l(\mathcal{X}).
\]

Since, \([M^* M C_{env}(\mathcal{X})] = C_{env}^*(\mathcal{X})\), we have that \([M^* M M_l(\mathcal{X}) M^* M] = M_l(\mathcal{X})\). Therefore, \( M_l(\mathcal{X}) = [M^* M_l(\mathcal{Y}) M] \). The proofs of the other assertions are similar.

The proof of the previous Lemma implies the following corollary:

**Corollary 4.7.** If \( \mathcal{X}, \mathcal{Y} \) are \( \Delta \)-equivalent unital operator spaces, then \( M_l(\mathcal{X}) \sim_\Delta M_l(\mathcal{Y}) \), and thus \( M_l(\mathcal{X}) \) and \( M_l(\mathcal{Y}) \) are stably isomorphic. The same assertion holds for \( M_r(\mathcal{X}) \) and \( M_r(\mathcal{Y}) \).

**Definition 4.8.** If \( \mathcal{X} \) is an operator space, then we define the operator algebra

\[
\Omega_\mathcal{X} = \left( \begin{array}{cc} A_l(\mathcal{X}) & \mathcal{X} \\ 0 & A_r(\mathcal{X}) \end{array} \right)
\]

**Theorem 4.9.** If \( \mathcal{X}, \mathcal{Y} \) are unital operator spaces, the following are equivalent:

(i) \( \mathcal{X} \) and \( \mathcal{Y} \) are stably isomorphic.

(ii) \( \mathcal{X} \sim_\sigma \Delta \mathcal{Y} \).

(iii) \( \mathcal{X} \sim_\Delta \mathcal{Y} \)

(iv) \( \Omega_\mathcal{X} \) and \( \Omega_\mathcal{Y} \) are stably isomorphic.

(v) \( \Omega_\mathcal{X} \sim_\sigma \Delta \Omega_\mathcal{Y} \)

(vi) \( \Omega_\mathcal{X} \sim_\Delta \Omega_\mathcal{Y} \).

**Proof.** We have proved the equivalence \( (i) \iff (ii) \) at the Theorem 3.8. Also, \((ii) \implies (iii)\) is obvious.

\((iii) \implies (ii)\) Suppose that \( \phi(\mathcal{X}) = [M^* \psi(\mathcal{Y}) M], \psi(\mathcal{Y}) = [M \phi(\mathcal{X}) M^*] \) for some TRO \( M \). By Lemma 4.9 in [12], the \( C^* \)-algebras \([M^* M], [M M^*]\) are unital, so it follows that \( \mathcal{X} \sim_\sigma \Delta \mathcal{Y} \).

Similarly, we have the equivalence \( (iv) \iff (v) \iff (vi) \). It remains to prove that \( (iii) \iff (vi) \).

\((iii) \implies (vi)\) If \( \mathcal{X} \sim_\Delta \mathcal{Y} \), then by Lemma 4.6, there exists a TRO \( M \) such that

\[
\mathcal{X} = [M^* \mathcal{Y} M], \mathcal{Y} = [M \mathcal{X} M^*]
\]

\[
M_l(\mathcal{X}) = [M^* M_l(\mathcal{Y}) M], M_l(\mathcal{Y}) = [M M_l(\mathcal{X}) M^*]
\]

\[
M_r(\mathcal{X}) = [M^* M_r(\mathcal{Y}) M], M_r(\mathcal{Y}) = [M M_r(\mathcal{X}) M^*]
\]

Since \( A_l(\mathcal{X}) = \Delta(M_l(\mathcal{X})), A_l(\mathcal{Y}) = \Delta(M_l(\mathcal{Y})), A_r(\mathcal{X}) = \Delta(M_r(\mathcal{X})), A_r(\mathcal{Y}) = \Delta(M_r(\mathcal{Y})) \), we get

\[
A_l(\mathcal{X}) = [M^* A_l(\mathcal{Y}) M], A_l(\mathcal{Y}) = [M A_l(\mathcal{X}) M^*]
\]

\[
A_r(\mathcal{X}) = [M^* A_r(\mathcal{Y}) M], A_r(\mathcal{Y}) = [M A_r(\mathcal{X}) M^*]
\]
\[ \begin{align*}
\Omega_X &= \left( A_l(\mathcal{X}) \quad \mathcal{X} \right) = \left[ \begin{pmatrix} M^* A_l(\mathcal{Y}) & M^* \mathcal{Y} M \ 
\end{align*}\]

so

\[
\Omega_X = \left( A_l(\mathcal{X}) \quad \mathcal{X} \right) = \left[ \begin{pmatrix} M^* & 0 \ 
\end{align*}\]

where \( \begin{pmatrix} M & 0 \\
0 & M \end{pmatrix} \) is TRO. Similarly,

\[
\Omega_Y = \left[ \begin{pmatrix} M & 0 \\
0 & M \end{pmatrix} \right]^{*}
\]

and we conclude that \( \Omega_X \sim_{\Delta} \Omega_Y \) (Theorem 2.5).

\((vi) \implies (iii)\) Let \( \Omega_X \sim_{\Delta} \Omega_Y \). The operator algebras \( \Omega_X, \Omega_Y \) are \( \Delta \)-algebraic extensions of \( \mathcal{X}, \mathcal{Y} \), respectively, so \( X, Y \) are \( \Delta \)-Morita equivalent. According to Theorem 3.6, we conclude that \( X \sim_{\Delta} Y \). \(\square\)

**Corollary 4.10.** If \( \mathcal{X}, \mathcal{Y} \) are unital operator spaces, the following are equivalent:

1. \( \mathcal{X} \sim_{\Delta} \mathcal{Y} \)
2. \( \mathcal{X} \) and \( \mathcal{Y} \) are \( \Delta \)-Morita equivalent.
3. The \( \Delta \)-pairs \( (\Omega_X, \Delta(\Omega_X)), (\Omega_Y, \Delta(\Omega_Y)) \) are \( \Delta \)-Morita equivalent.

**Proof.** (i) \( \iff \) (ii) It has been proven previously at Theorem 3.6.

(iii) \( \implies \) (ii) It is obvious since \( \Omega_X, \Omega_Y \) are algebraic \( \Delta \)-extensions of \( \mathcal{X}, \mathcal{Y} \), respectively.

(i) \( \implies \) (iii) We may consider, using again the Lemma 4.6, that there exists a TRO \( M \) such that

\[
\mathcal{X} = \begin{pmatrix} M^* \mathcal{Y} M \\
0 & \mathcal{X} \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} M \mathcal{X} M^* \\
0 & \mathcal{Y} \end{pmatrix}
\]

\[
A_l(\mathcal{X}) = \begin{pmatrix} M^* A_l(\mathcal{Y}) M \\
0 & A_l(\mathcal{X}) \end{pmatrix}, \quad A_l(\mathcal{Y}) = \begin{pmatrix} M A_l(\mathcal{X}) M^* \\
0 & A_l(\mathcal{Y}) \end{pmatrix}
\]

\[
A_r(\mathcal{X}) = \begin{pmatrix} M^* A_r(\mathcal{Y}) M \\
0 & A_r(\mathcal{X}) \end{pmatrix}, \quad A_r(\mathcal{Y}) = \begin{pmatrix} M A_r(\mathcal{X}) M^* \\
0 & A_r(\mathcal{Y}) \end{pmatrix}
\]

Using the TRO \( N = \begin{pmatrix} M & 0 \\
0 & M \end{pmatrix} \), we have that \( \Omega_X = \begin{pmatrix} N^* \Omega_Y N \\
\end{pmatrix}, \Omega_Y = \begin{pmatrix} N \Omega_X N^* \end{pmatrix} \). Also,

\[
\Delta(\Omega_X) = \begin{pmatrix} A_l(\mathcal{X}) & 0 \\
0 & A_r(\mathcal{X}) \end{pmatrix}, \quad \Delta(\Omega_Y) = \begin{pmatrix} A_l(\mathcal{Y}) & 0 \\
0 & A_r(\mathcal{Y}) \end{pmatrix}
\]

and it is obvious that \( \Delta(\Omega_X) = \begin{pmatrix} N^* \Delta(\Omega_Y) N \\
\end{pmatrix}, \Delta(\Omega_Y) = \begin{pmatrix} N \Delta(\Omega_X) N^* \end{pmatrix} \), so \( \Omega_X \sim_{\Delta} \Omega_Y \) and \( \Delta(\Omega_X) \sim_{\Delta} \Delta(\Omega_Y) \) with the same TRO, which means that the \( \Delta \)-pairs \( (\Omega_X, \Delta(\Omega_X)), (\Omega_Y, \Delta(\Omega_Y)) \) are \( \Delta \)-Morita equivalent. \(\square\)
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G. K. Eleftherakis and E. Papapetros
Department of Mathematics, Faculty of Sciences
University of Patras
26500 Patras
Greece
e-mail: gelefth@math.upatras.gr

E. Papapetros
e-mail: e.papapetros@upatras.gr

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