Quiver Yangians and Crystal Meltings: A Concise Summary

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The goal of this short article is to summarize some of the recent developments in the quiver Yangians and crystal meltings. This article is based on a lecture delivered by the author at International Congress on Mathematical Physics (ICMP), Geneva, 2021.

I. INTRODUCTION

One of the fascinating aspects of integrable models is that they are located at the intersection of many different topics in mathematics and physics. This comment applies very well to the contents of this paper—even within the limited pages, we will encounter branches of mathematics such as geometry, algebra, representation theory, and combinatorics, and those of physics such as gauge theory and string theory.

The goal of this note is to discuss (1) a class of newly-introduced algebras called the (shifted) quiver Yangians, and (2) their representations in terms of the statistical-mechanical model of crystal melting. Our primary task here is to give a leisurely introduction to the main ideas and results in the recent developments.\textsuperscript{1} This inevitably means that our presentation will omit many important details and our references will be incomplete; for full treatments readers are encouraged to consult original papers\textsuperscript{2–4} on the subject.

II. (SHIFTED) QUIVER YANGIANS

A. Quiver and Superpotential

Yangians\textsuperscript{6,7} are often defined from a Lie algebra $\mathfrak{g}$, which is determined by a (generalized) Cartan matrix. We here instead start with a quiver $Q$ and a superpotential $W$. Here a quiver $Q$ consists of a set of vertices $Q_0$ and that of arrows $Q_1$, and a superpotential $W$ is a formal linear sum of oriented cycles of the quiver. Physically, this is the defining data for an $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics.

While the definition of the quiver Yangian in itself works for a general choice of $Q$ and $W$, in the following we are interested in the cases where $(Q,W)$ originates from a toric Calabi-Yau three-fold: the $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics defined from $(Q,W)$ reproduces the toric Calabi-Yau three-fold as the vacuum moduli space. While there have been a lot of discussions on identifying the quiver/superpotential from the toric data\textsuperscript{8}, it is enough for the purposes of this exposition to look at a few examples shown in Figure 1.

We assign equivariant parameters (to be called charges) $h_I$ to each arrow $I \in Q_1$ such that the total charge of (any monomial term of) the superpotential $W$ is zero. (For example, for $\mathbb{C}^3$ in Figure 1 we associate parameters $h_X, h_Y, h_Z$ to $X, Y, Z$ with one constraint $h_X + h_Y + h_Z = 0$.) This gives a parameterization of the flavor symmetries of the theory.
\( Q = \begin{array}{c}
X \\
\Upsilon \\
Z 
\end{array} \)

\( W = \text{Tr}(XYZ - XZY) \)

\( (\text{CY}_3 = \mathbb{C}^3) \)

\( Q = \begin{array}{c}
A_1, A_2 \\
B_1, B_2
\end{array} \)

\( W = \text{Tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1) \)

\( (\text{CY}_3 = \text{conifold}) \)

\( Q = \begin{array}{c}
D_1, D_2 \\
B_1, B_2 \\
C_1, C_2
\end{array} \)

\( W = \text{Tr}(A_1B_1C_1D_1 + A_2B_2C_2D_2 - A_1B_2C_1D_2 - A_2B_1C_2D_1) \)

**FIG. 1.** Examples of quivers \( Q \) and superpotentials \( W \) describing toric Calabi-Yau three-folds, \( \mathbb{C}^3 \), resolved conifold and the canonical bundle over \( \mathbb{P}^1 \times \mathbb{P}^1 \).

### B. Generators and Relations

We are now ready to define the \( K \)-shifted quiver Yangian \( \mathcal{Y}(Q, W) \) associated with \( (Q, W) \), in terms of generators and relations:

- **(Generators)**
  - For each vertex \( a \) of the quiver we have a triple \( (e^{(a)}(z), f^{(a)}(z), \psi^{(a)}(z)) \), whose mode expansion gives an infinite set of generators \( (e^{(a)}_n, \psi^{(a)}_n, f^{(a)}_n) \):
    
    \[
    e^{(a)}_n(z) \equiv \sum_{n=0}^{+\infty} c_n^{(a)} z^{n+1}, \quad \psi^{(a)}_n(z) \equiv \sum_{n=K}^{+\infty} \psi^{(a)}_n z^{n+1}, \quad f^{(a)}_n(z) \equiv \sum_{n=0}^{+\infty} f^{(a)}_n z^{n+1},
    \]

    with \( K \) being the “shift”.

- **(Relations)**
  - The generators satisfy the relations
    
    \[
    \begin{align*}
    \psi^{(a)}(z) \psi^{(b)}(w) &= \psi^{(b)}(w) \psi^{(a)}(z), \\
    \psi^{(a)}(z) e^{(b)}(w) &= \varphi^{b \Rightarrow a}(z-w) e^{(b)}(w) \psi^{(a)}(z), \\
    e^{(a)}(z) e^{(b)}(w) &\sim (-1)^{|a||b|} \varphi^{b \Rightarrow a}(z-w) e^{(b)}(w) e^{(a)}(z), \\
    \psi^{(a)}(z) f^{(b)}(w) &= \varphi^{b \Rightarrow a}(z-w)^{-1} f^{(b)}(w) \psi^{(a)}(z), \\
    f^{(a)}(z) f^{(b)}(w) &\sim (-1)^{|a||b|} \varphi^{b \Rightarrow a}(z-w)^{-1} f^{(b)}(w) f^{(a)}(z), \\
    \left[ e^{(a)}(z), f^{(b)}(w) \right] &\sim -\delta^{a,b} \frac{\psi^{(a)}(z) - \psi^{(b)}(w)}{z-w}.
    \end{align*}
    \]
Here the relations are understood to hold in the expansions in powers of \( z \) and \( w \), with “\( \sim \)” meaning equality up to \( z^n w^m \geq 0 \) terms, and “\( \approx \)” meaning equality up to \( z^n \geq 0 \) and \( w^m \geq 0 \) terms. The bracket \( \{e^{(a)}(z), f^{(b)}(w)\} \) represents a supercommutator as determined by a \( \mathbb{Z}_2 \)-grading

\[
|a| = \begin{cases} 
0 & \text{(if there exists } I \in Q_1 \text{ s.t. } I \text{ begins and ends at } a) \\
1 & \text{(otherwise)}
\end{cases}
\]  

(3)

The “bond factor” \( \varphi^a \rightarrow b (z) \) is defined to be

\[
\varphi^a \rightarrow b (u) \equiv \pm \prod_{I \in \{b \rightarrow a\}} (u + h_I) \prod_{I \in \{a \rightarrow b\}} (u - h_I),
\]  

(4)

where \( \{a \rightarrow b\} \) denotes the set of edges from vertex \( a \) to vertex \( b \).

As shown in Ref. 2, the quiver Yangians contain interesting algebras as special examples:

- For \( C^3 \) we get \( Y(\widehat{gl}_1) \), the affine Yangian of \( gl_1 \).
- For conifold, we get \( Y(\widehat{gl}_{1|1}) \), the affine Yangian of \( gl_{1|1} \).
- More generally, for a toric Calabi-Yau three-fold \( xy = z^n w^m \) (often called a generalized conifold) we get an affine Yangian of \( gl_{m|n} \).\(^{10}\) Note that for Lie superalgebras there are ambiguities in the choice of the Cartan matrix, but the affine Yangian is independent of such a choice up to an isomorphism.\(^{11}\)
- For \( (Q, W) \) corresponding to a general toric Calabi-Yau manifold there is no associated Lie algebra \( g \), and the algebra seems to be new. In general, we have multiple quiver/superpotential pairs for the same toric Calabi-Yau geometry, and it was conjectured in Ref. 2 that their quiver Yangians are all isomorphic.

III. REPRESENTATION FROM CRYSTAL MELTING

A. Crystal Melting

The concept of crystal melting is best explained in the simplest example of \( C^3 \),\(^{12,13}\) whose quiver and superpotential are shown in Figure 2.

We can describe the moduli space by associating three variables \( X, Y, Z \) to the three edges of the quiver. The superpotential constraints give \( \partial W / \partial X = YZ - ZY = 0 \), as well as its cyclic permutations. This means that we have a polynomial ring \( \mathbb{C}[X, Y, Z] \), which reproduces the coordinate ring of \( C^3 \). Now, to construct the crystal we consider all paths starting at the vertex modulo the superpotential constraints, so that we obtain \( X^I Y^J Z^K \) with \( i, j, k \geq 0 \). Let us call such an equivalence class of paths to be an atom. By placing the atoms at the locations \( (i, j, k) \) (namely, according to their charges\(^{14}\), we obtain a three-dimensional arrangement of atoms, which we call the vacuum crystal. The excited states of the crystal are obtained by removing atoms from the corner of the crystal.

Now the observation of Ref. 15 (see also Ref. 16) is that this story generalizes to an arbitrary toric Calabi-Yau manifold. We refer to Refs. 15 and 17 for detailed explanation; we here instead show an example of a crystal, since a picture is worth a thousand words (Figure 3):

The crystal shown in Figure 3 shows the “vacuum” of the model. The excited states are given by “molten crystals”, which are obtained by removing a finite set of atoms \( \Lambda \) from the top of the crystal. This gives a generalization of the \( C^3 \)-crystal melting, where the connectivities of the atoms are dictated by the edges of the quiver, and the atoms are colored according to the vertices of the quiver (the end-point of the path corresponding to the atom).
FIG. 2. The crystal melting model for $\mathbb{C}^3$.

$Q = \begin{pmatrix} X & h_X \\ Y & h_Y \end{pmatrix}$

$W = \text{Tr}(XYZ - XZY)$

(a path from $\bullet$) /($\partial W$) = (an atom in crystal)

FIG. 3. An example of the vacuum crystal associated with the Calabi-Yau geometry $xy = zw^2$. The origin/top of the crystal is shown by the blue dot in the center. (Figures reproduced and modified from Ref. 15.)

FIG. 4. An example of a configuration of the molten crystal (right) and the complement, the crystal configuration (left), for the crystal of Figure 3. (Figures reproduced from Ref. 15.)
B. Representations

We can now describe a representation of the quiver Yangian. The representation space is spanned by the set of the complements of molten crystal configurations: $|\Lambda\rangle$. The generators of the quiver Yangian act on the states by the following formulas:\(^{18}\)

\[
\psi^{(a)}(z)|\Lambda\rangle = \Psi^{(a)}_\Lambda(z)|\Lambda\rangle ,
\]

\[
e^{(a)}(z)|\Lambda\rangle = \sum_{a \in \text{Add}(\Lambda)} \frac{E^{(a)}(\Lambda \to \Lambda + [a])}{z - h([a])} |\Lambda + [a]\rangle ,
\]

\[
f^{(a)}(z)|\Lambda\rangle = \sum_{a \in \text{Rem}(\Lambda)} \frac{F^{(a)}(\Lambda \to \Lambda - [a])}{z - h([-a])} |\Lambda - [a]\rangle .
\]

While $\psi^{(a)}(z)$ acts diagonally on the crystal basis, $e^{(a)}(z)$ ($f^{(a)}(z)$) adds (removes) an atom from the crystal: here $[a] \in \text{Add}(\Lambda)$ ($[-a] \in \text{Rem}(\Lambda)$) means that we consider an atom of color $a$ which can be added to (removed from) the crystal $\Lambda$. Note that for each atom $[a]$ which can be added to (removed from) the crystal there is a corresponding pole at $z = h([a])$ on the right-hand side.

The expressions $\Psi^{(a)}_\Lambda(u)$, $E^{(a)}(\Lambda \to \Lambda + [a])$ and $F^{(a)}(\Lambda \to \Lambda - [a])$ are given by

\[
\Psi^{(a)}_\Lambda(u) = \psi^{(a)}_0(z) \prod_{b \in Q_\Lambda} \prod_{b \in \Lambda} \varphi^{b \rightarrow a}(u - h([b])) ,
\]

\[
E^{(a)}(\Lambda \to \Lambda + [a]) = \pm \frac{\pm \text{Res}_{u=h([a])} \Psi^{(a)}_\Lambda(u)}{\Psi^{(a)}_{\Lambda + [a]}(u)} ,
\]

\[
f^{(a)}(\Lambda \to \Lambda - [a]) = \pm \frac{\pm \text{Res}_{u=h([-a])} \Psi^{(a)}_\Lambda(u)}{\Psi^{(a)}_{\Lambda - [a]}(u)} ,
\]

where we here do not discuss subtle sign choices, for which readers are referred to Ref. 2. ($E^{(a)}$ and $F^{(a)}$ are essentially the same expressions except for sign differences.) Note that $\varphi^{a \rightarrow b}(u)$ was defined previously in (4).

We still need to describe the function $\psi^{(a)}_0(z)$, which determines the action of the $\psi^{(a)}(z)$ generator on the vacuum crystal:

\[
\psi^{(a)}(z)|\emptyset\rangle = \psi^{(a)}_0(z)|\emptyset\rangle .
\]

The simplest choice of $\psi^{(a)}_0(z)$ is to allow for a single-order pole for one of the vertices $a$, so that we have $\psi^{(a)}_0(z) = 1/(z - z_1)$. This corresponds to starting the growth of the crystal from the vertex $a$, at the location $z = z_1$ of the “starter”. We can consider a more complicated choice, for example $\psi^{(a)}_0(z) = (z - z_3)/(z - z_1)(z - z_2)$. In this case, the growth of the crystal starts at two separate locations, at $z = z_1$ and $z = z_2$; the two crystals then begin to overlap at $z = z_3$, and the zero of $\psi^{(a)}_0(z)$ ensures that the two copies of the crystal tails should be identified beyond $z = z_3$ (the location of the “pauser”), so that we have a single crystal of the combined shape shown in Figure 5. We can also delete portions of the crystal as in Figure 5 (c), by considering the $\psi^{(a)}_0(z) = (z - z_4)/(z - z_1)$ where $z = z_4$ (the “stopper”) is located inside the crystal. In general, we can choose $\psi^{(a)}_0(z)$ to be a rational function—the poles of $\psi^{(a)}_0$ specify the “starters” of the crystals, while its zeros specify the “pauser/stoppers” (which can be regarded as starters for “negative crystals”); we obtain a representation of the $K$-shifted quiver Yangian, where $K$ is determined by a net degree of $\psi^{(a)}_0$ (degree of the numerator minus that of the denominator when we express $\psi^{(a)}_0$ as a ratio of two polynomials.).\(^{19}\)
Generalizing this argument, we can obtain a rather general class of representations using a combination of starter/pauser/stoppers, and a general shape of the crystal can be used for a representation. This includes, for example, open/closed BPS state countings and their wall crossings. In Figure 6 we show examples of crystals relevant for wall crossings in the resolved conifold.

One should keep in mind, however, that almost all of the crystals representations have no known counterparts in the geometric discussions of Donaldson-Thomas invariants, see Figure 7 for examples of such crystals.

Finally, let us mention that the representations discussed above can become reducible for non-generic values of the equivariant parameters. In this case, we can truncate the algebra (namely, consider the quotient of the algebra by the ideal annihilating the null state), so that the representation is irreducible again. This method gives rise to a zoo of interesting algebras. For $\mathbb{C}^3$ this gives vertex algebras discussed in Ref. 26.
C. Derivations from Supersymmetric Quiver Quantum Mechanics

We have introduced quiver Yangians and their representations by a top-down approach. One can, however, derive the algebra by assuming the crystal representation (5) as an ansatz. Alternatively, we can derive the algebras and their representations by an equivariant localization of supersymmetric quiver quantum mechanics. In this section we explain the latter approach.

Let us here quickly summarize the salient features of the derivation (see Refs. 3 and 4 for details). One starts with an $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics defined from the quiver $Q$ and the superpotential $W$. By applying an equivariant localization to the Higgs branch of the moduli space, the BPS state counting problem of the quiver quantum mechanics reduces to the counting of the fixed points, which are captured by the configurations $\Lambda$ of crystal melting. We can moreover identify the associated effective wavefunction $\Psi_{\Lambda}$ as the Euler class of the moduli space $\mathcal{M}_{\Lambda}$. Now, the action of the raising operator $e$ is to bring in an extra atom from infinity, causing the change of the wavefunction $\Psi_{\Lambda}$ into $\Psi_{\Lambda+\Box}$. This operation—the “Hecke modification” of the associated sheaf—is described by a Fourier-Mukai transformation over $\mathcal{M}_{\Lambda} \times \mathcal{M}_{\Lambda+\Box}$, whose kernel is given by the incidence relation. What is important in practice is that one can then explicitly compute the matrix elements of the generators, and verify the relations satisfied by the generators. This line of logic gives a first-principle derivation of the results and makes the connection with geometry more manifest. In practice, the match with quiver Yangians involves miraculous-looking cancellations as in Appendix of Ref. 3.

IV. SUMMARY

Integrable models have a rather long history with so many sophisticated results in the literature. However, the recent developments described in this paper demonstrate (rather convincingly in my opinion) that there are still new integrable structures yet to be discovered, even at the level of algebras generalizing Yangians. I am optimistic that such a research direction will continue to prosper in the future, and that dialogues between physics and mathematics will be the crucial driving force in such an endeavor.

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1See also Ref. 28 for another introduction on related topics.
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9The signs in front are chosen such that $\varphi^{a\to b}(u)\varphi^{b\to a}(-u) = 1$, which is needed for the consistency of the relations; see Ref. 30 for details. Alternatively we can disregard the signs by choosing an ordering between the vertices.
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