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A remark on the space of metrics having non-trivial harmonic spinors

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Abstract

Let $M$ be a closed spin manifold of dimension $n \equiv 3 \mod 4$. We give a simple proof of the fact that the space of metrics on $M$ with invertible Dirac operator is either empty or it has infinitely many path components.

1 Introduction

Let $M$ be an $n$-dimensional closed spin manifold and let $R(M)$ be the space of all Riemannian metrics on $M$. For any choice of a metric $g \in R(M)$, we can build the associated spinor bundle $\Sigma_g M$ and obtain a natural first order operator $D_g$ acting on sections of $\Sigma_g M$ and which we call the Dirac operator. Elements of $\ker D_g$ are called harmonic spinors and their existence has been studied for a long time. While one can show that on $S^2$ no non-trivial harmonic spinors exist (cf. [Ba92]), it is conjectured that on every closed spin manifold of dimension $n \geq 3$ there exists a Riemannian metric $g$ such that $\ker D_g \neq 0$. The conjecture has been proved by N. Hitchin in [Hi74] if $n \equiv 0, \pm 1 \mod 8$ and by C. Bär in [Ba96] if $n \equiv 3 \mod 4$. As a more general question, one may ask how many metrics exist on $M$ such that the corresponding Dirac operator has non-trivial kernel. A possible way to study this question is to consider the complementary set of metrics $R^{inv}(M)$ consisting of all metrics $g \in R(M)$ such that $\ker D_g = 0$. M. Dahl showed in [Da08] that elements of $R^{inv}(M)$ can be extended to $R^{inv}(W)$ if $W$ is the trace of a surgery of codimension at least 3 on $M$. By using the Atiyah-Singer index theorem and special metrics on the spheres originating from the study of positive scalar curvature, he concluded from this result that $R^{inv}(M)$ is in all dimensions $n \geq 5$ which were considered by Hitchin and Bär either empty or disconnected. Moreover, in the case $n \equiv 3 \mod 4$, he even obtained that, if non-empty, $R^{inv}(M)$ has infinitely many path components. Recently he improved this conclusion in collaboration with N. Grosse to dimension 3 by studying extensions of metrics to attached handles (cf. [DG12]).

The aim of this article is to show that the existence of infinitely many connected components of $R^{inv}(M)$ in all dimensions $n \equiv 3 \mod 4$ can be derived easily from Bär’s results in [Ba96] by using spectral flow and rather elementary homotopy arguments. Finally, we want to mention that Bär improved his theorem in [Ba97] to twisted Dirac operators. Note that for any fixed pair $(F, \nabla)$ of a bundle $F$ over $M$ and a connection $\nabla$ on $F$, we obtain a family $D(F,\nabla)$ of twisted Dirac operators which is again parametrised by the space of Riemannian metrics $R(M)$ on $M$. We believe that one can extend our argument here to this case by using the results from [Ba97] instead of [Ba96]. Accordingly, we conjecture that the corresponding space $R^{inv}_t(F,\nabla)(M) = \{g \in R(M) : \ker D_g^{(F,\nabla)} = 0\}$ is either empty or it has infinitely many path components.

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2 Preliminaries: Dirac operators

In this section we recall briefly the definition of spinor bundles and their Dirac operators. Among the many references for these topics we want to mention [Hij01] and [Am01], on which we base our exposition. In order to simplify the presentation we assume throughout that $M$ is an oriented closed manifold of odd dimension $n \geq 3$.

We denote by $GL^+(M)$ the principal $GL^+(n; \mathbb{R})$-bundle of oriented bases over $M$ and recall that $GL^+(n; \mathbb{R})$ has a unique connected 2-fold covering $\Theta : \tilde{GL}^+(n; \mathbb{R}) \to GL^+(n; \mathbb{R})$ since the fundamental group of $GL^+(n; \mathbb{R})$ is of order two. A spin structure on $M$ is a pair $(GL^+(M), \vartheta)$, where $GL^+(M)$ is a principal $GL^+(n; \mathbb{R})$-bundle over $M$ and $\vartheta : GL^+(M) \to GL^+(M)$ is a 2-fold covering such that

$$\pi \circ \vartheta = \tilde{\pi} \quad \text{and} \quad \vartheta (u \cdot v) = \vartheta (u) \cdot \Theta (v), \quad \text{for all } v \in \tilde{GL}^+(n; \mathbb{R}), u \in \tilde{GL}^+(M),$$

where $\pi$ and $\tilde{\pi}$ denote the corresponding projections of the bundles. Henceforth we assume that $M$ is a spin manifold, that is, $M$ is oriented and a spin structure on $M$ is given. Note that so far we have not required that $M$ is endowed with a Riemannian metric.

Let now $g$ be a Riemannian metric on $M$ and denote by $SO(M,g)$ the associated principal $SO(n)$-bundle of positively oriented orthonormal bases. Then $Spin(M,g) := \vartheta^{-1}(SO(M,g))$ is a principal $Spin(n)$-bundle over $M$, where $Spin(n) := \Theta^{-1}(SO(n))$ is the unique connected 2-fold covering of $SO(n)$. Let $\rho : \mathbb{C}l_n \to \text{End}(\Sigma_n)$ denote the usual irreducible representation of the complex Clifford algebra, where $\Sigma_n$ is the space of complex spinors. We fix an inner product $(\cdot, \cdot)$ on $\Sigma_n$ such that $\langle \rho(x)\sigma_1, \rho(x)\sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle$ for all $x \in \mathbb{R}^n$, $\|x\| = 1$, and $\sigma_1, \sigma_2 \in \Sigma_n$. If now $\rho' : \text{Spin}(n) \to \text{Aut}(\Sigma_n)$ denotes the complex spinor representation of $\text{Spin}(n)$, which is obtained by restricting $\rho$ to $\text{Spin}(n) \subset \mathbb{C}l_n$, then the spinor bundle $\Sigma_g M$ of $M$ with respect to $g$ is defined as the associated vector bundle $\text{Spin}(M,g) \times_{\rho'} \Sigma_n$.

The representation $\rho$ induces a Clifford multiplication on $\Sigma_g M$, that is, a complex linear vector bundle homomorphism

$$m : T^* M \otimes \Sigma_g M \to \Sigma_g M, \quad X^\flat \otimes \varphi \mapsto X \cdot \varphi$$

such that $X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X,Y)\varphi$ for all $X,Y \in TM$ and $\varphi \in \Sigma_g M$. Moreover, the inner product on $\Sigma_n$ gives rise to an Hermitian structure on the bundle $\Sigma_g M$ such that $\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle$ for all $X \in TM$ and $\varphi, \psi \in \Sigma_g M$. Finally, the Levi-Civita connection on $TM$ induces a connection on $SO(M,g)$ and this connection lifts in a canonical way to a connection on $\text{Spin}(M,g)$. The associated covariant derivative $\nabla : C^\infty (M, \Sigma_g M) \to C^\infty (M, T^*M \otimes \Sigma_g M)$ on the spinor bundle has the properties

$$X \langle \varphi, \psi \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle \quad \text{and} \quad \nabla_X (Y \cdot \varphi) = (\nabla^T_X Y) \cdot \varphi + Y \cdot (\nabla_X \varphi)$$

for vector fields $X,Y$ and a spinor field $\varphi$.

Now the Dirac operator with respect to the metric $g$ is defined by
\( D_g = m \circ \nabla : C^\infty(M, \Sigma_g M) \to C^\infty(M, \Sigma_g M) \)

and is an elliptic, essentially selfadjoint differential operator of first order.

3 The Proof

We assume from now on that \( M \) is a closed spin manifold of dimension \( n \equiv 3 \mod 4 \). We denote by \( R(M) \) the space of all Riemannian metrics on \( M \) with the \( C^1 \)-topology and note that it is obviously contractible. Moreover, we define

\[
R^{\text{inv}}(M) = \{ g \in R(M) : \ker D_g = 0 \} \subset R(M)
\]

and recall that our aim is to show that this set has infinitely many path components if it is not empty. Accordingly, we assume henceforth that \( R^{\text{inv}}(M) \neq \emptyset \) and now we conclude in three steps the announced disconnectedness of this space.

Step 1: The spectral flow

Since our operators \( D_g, g \in R(M) \), are essentially selfadjoint, they have real spectra. Moreover, by ellipticity their spectra are discrete and consist entirely of eigenvalues of finite multiplicity. We define for any compact interval \([a, b] \subset \mathbb{R}\) a non-negative integer by

\[
m(g, [a, b]) = \sum_{\lambda \in [a, b]} \dim \ker(D_g - \lambda \cdot \text{id}).
\]

Next we quote the following stability result for the spectra of the operators \( D_g \) that can be found in [Ba96, Prop. 7.1].

**Theorem 3.1.** Let \((M, g)\) be a closed Riemannian spin manifold with Dirac operator \( D_g \). Let \( \varepsilon > 0 \) and let \( \Lambda > 0 \) such that \( -\Lambda, \Lambda \notin \sigma(D_g) \). Write

\[
\sigma(D_g) \cap (-\Lambda, \Lambda) = \{ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \}.
\]

Then there exists a neighbourhood of \( g \) in the \( C^1 \)-topology such that for any metric \( \tilde{g} \) in this neighbourhood with Dirac operator \( D_{\tilde{g}} \) the following holds:

- \( \sigma(D_g) \cap (-\Lambda, \Lambda) = \{ \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k \} \),
- \( |\lambda_i - \mu_i| < \varepsilon, i = 1, \ldots, k \).

The eigenvalues \( \lambda_i \) and \( \mu_i \) are repeated according to their multiplicities.

We obtain immediately the following corollary.

**Corollary 3.2.** For all \( g_0 \in R(M) \) and \( \Lambda > 0 \) such that \( \pm\Lambda \notin \sigma(D_{g_0}) \) there exists an open neighbourhood \( N(g_0, \Lambda) \subset R(M) \) such that \( \pm\Lambda \notin \sigma(D_g) \) and \( m(g, [-\Lambda, \Lambda]) = m(g_0, [-\Lambda, \Lambda]) \) for all \( g \in N(g_0, \Lambda) \).
Let now $\gamma : I \rightarrow R(M)$ be a path of metrics. Because of corollary 3.2 we can find a decomposition $0 = t_0 < t_1 < \ldots < t_N = 1$ and positive numbers $a_1, \ldots, a_N$ such that the functions $[t_{i-1}, t_i] \ni t \mapsto m(\gamma(t), [-a_i, a_i])$ are constant. We define

$$
\Gamma(\gamma) = \sum_{i=1}^{N} m(\gamma(t_i), [0, a_i]) - m(\gamma(t_{i-1}), [0, a_i]) \in \mathbb{Z}
$$

(1)

and note that, roughly speaking, $\Gamma(\gamma)$ counts the number of negative eigenvalues of $D_{\gamma(0)}$ that become positive as the parameter $t$ travels from 0 to 1 minus the number of positive eigenvalues of $D_{\gamma(0)}$ that become negative; i.e., the net number of eigenvalues which cross zero. The formula (1) corresponds precisely to the definition of the spectral flow for paths of selfadjoint Fredholm operators acting on a fixed Hilbert space which can be found for example in [Phi96] and [BLP05]. Accordingly, one can show verbatim as in [Phi96] that $\Gamma(\gamma)$ indeed does only depend on the path $\gamma$ and not on the choices of the $t_i, a_i, i = 1, \ldots N$. Moreover, if $\gamma, \tilde{\gamma} : I \rightarrow R(M)$ are two paths of metrics, then the following properties hold:

i) $\Gamma(\gamma) = 0$ if $\gamma(t) \in R_{\text{inv}}(M)$ for all $t \in [0, 1]$,

ii) $\Gamma(\gamma \ast \tilde{\gamma}) = \Gamma(\gamma) + \Gamma(\tilde{\gamma})$, whenever the concatenation $\gamma \ast \tilde{\gamma}$ exists,

iii) $\Gamma(\gamma^{-1}) = -\Gamma(\gamma)$, where $\gamma^{-1}(t) = \gamma(1-t), t \in I$,

iv) $\Gamma(\gamma) = \Gamma(\tilde{\gamma})$ if $\gamma \simeq \tilde{\gamma}$ through a homotopy having ends in $R_{\text{inv}}(M)$.

Note that the first three properties are immediate consequences of the definition. The homotopy invariance can be obtained again verbatim as in [Phi96].

Step 2: The range of $\Gamma$

Our argument in this section is based on results from [Ba96] which we introduce before we proceed with the proof. At first, we need the existence of the following metrics on the sphere $S^n$, that were constructed in [Ba96, §3].

Proposition 3.3. For $n \equiv 3 \mod 4$ and any integer $m > 0$, there exists a path of metrics $g^m_t$, $t \in [0, 1]$, on $S^n$ such that the following holds for the associated Dirac operators $D^m_t$:

- there is $\lambda(t) \in \sigma(D^m_t)$ such that $\lambda(0) = -1$ and $\lambda(1) = 1$,
- $\lambda(t)$ depends linearly on $t$,
- the multiplicity of $\lambda(t)$ is constant in $t$ and greater than $m$,
- $\lambda(t)$ is the only eigenvalue of $D^m_t$ in the interval $[-2, 2]$.

Bär combined in [Ba96, proposition 3.3] and a general gluing theorem for Dirac operators [Ba96, theorem B] to conclude the existence of non-trivial harmonic spinors in dimensions $n \equiv 3 \mod 4$. Actually, in order to find the spinors he just needed a special case of his gluing theorem which reads as follows.
Theorem 3.4. Let $(M, g)$ be a closed Riemannian spin manifold of odd dimension $n \geq 3$. Let $D_g$ be the corresponding Dirac operator and let $\mathcal{D}$ denote the Dirac operator on $S^n$ with respect to some Riemannian metric. Finally, let $\Lambda > 0$ be such that $\pm \Lambda \notin \sigma(D_g) \cup \sigma(\mathcal{D})$. Write

$$(\sigma(D_g) \cup \sigma(\mathcal{D})) \cap (-\Lambda, \Lambda) = \{ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \}.$$ 

Then for any $\varepsilon > 0$ there exists a Riemannian metric $\tilde{g}$ on $M$ such that the corresponding Dirac operator $D_{\tilde{g}}$ has the following properties:

i) $\pm \Lambda \notin \sigma(D_{\tilde{g}})$,

ii) $\sigma(D_{\tilde{g}}) \cap (-\Lambda, \Lambda) = \{ \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k \}$

iii) $|\lambda_j - \mu_j| < \varepsilon$ for $j = 1, \ldots, k$.

The eigenvalues $\lambda_i$ and $\mu_i$ are repeated according to their multiplicities.

We now take some metric $g_0 \in R^{\text{inv}}(M)$. Because of the conformal covariance of the Dirac operator (cf. [Hij01, Prop. 5.13]), we can assume that $[-2, 2] \cap \sigma(D_{g_0}) = \emptyset$ simply by rescaling the metric if necessary.

Let $m > 0$ be an integer and consider the operators $\mathcal{D}_t^{m_n}$ on $S^n$ from proposition 5.3. Recall that we denote by $\Lambda(t)$ the unique eigenvalue of $\mathcal{D}_t^{m_n}$ in the interval $[-2, 2]$ and that $\Lambda(t)$ depends linearly on $t$ with $\Lambda(0) = -1, \Lambda(1) = 1$.

We now apply theorem 3.4 for $\Lambda = 2$ and $\varepsilon = \frac{1}{2}$ to $D_{g_0}$ and the operators $\mathcal{D}_t^{m_n}, t \in [0, 1], on S^n$. We obtain for any $t \in [0, 1]$ a metric $\tilde{g}_t$ on $M$ such that each eigenvalue of $D_{\tilde{g}_t}$ in the interval $[-2, 2]$ is of distance less then $\frac{1}{2}$ to $\Lambda(t)$. In particular, $D_{g_0}$ and $D_{\tilde{g}_t}$ are invertible and hence $\{ \tilde{g}_t \}_t \in [0, 1]$ defines a path $\gamma : (I, \partial I) \to (R(M), R^{\text{inv}}(M))$. Moreover, the function $t \mapsto m(\gamma(t); [-2, 2])$ is constant on the whole interval $[0, 1]$. Hence we finally obtain from the definition of $\Gamma$

$$\Gamma(\gamma) = m(\tilde{g}_1, [0, 2]) - m(\tilde{g}_0, [0, 2]) = m(\tilde{g}_1, [0, 2]) = \dim \ker(\mathcal{D}_t^{m_n} - \text{id}) > m.$$ 

To sum up, we have shown that the set

$$\{ \Gamma(\gamma) : \gamma : (I, \partial I) \to (R(M), R^{\text{inv}}(M)) \text{ continuous} \} \subset \mathbb{Z}$$

is not bounded from above.

Step 3: The final argument

We fix some $g_0 \in R^{\text{inv}}(M)$. Our first aim of this final step is to construct inductively a sequence of paths $\gamma_k : (I, \partial I) \to (R(M), R^{\text{inv}}(M)), k \in \mathbb{N}$, such that $\gamma_k(0) = g_0$ for all $k \in \mathbb{N}$ and $\Gamma(\gamma_i) \neq \Gamma(\gamma_j)$ for all $i \neq j$.

Let $\gamma_1$ be the constant path $\gamma_1 \equiv g_0 \in R^{\text{inv}}(M)$. Assume that we have already constructed $\gamma_1, \ldots, \gamma_k : (I, \partial I) \to (R(M), R^{\text{inv}}(M))$ such that $\gamma_i(0) = g_0, i = 1, \ldots, k$, and $\Gamma(\gamma_i) \neq \Gamma(\gamma_j)$ for all $i \neq j$.

According to the second step of our proof we can find a path $\tilde{\gamma} : (I, \partial I) \to (R(M), R^{\text{inv}}(M))$ such that

$$\tilde{\gamma}(0) = \gamma_k(0).$$

Now we can conclude that $\tilde{\gamma}(t)$ is a path in $R^{\text{inv}}(M)$ for all $t \in [0, 1]$, and that $\Gamma(\tilde{\gamma}(t))$ is constant on the whole interval $[0, 1]$. Hence we finally obtain from the definition of $\Gamma$

$$\Gamma(\tilde{\gamma}) = \dim \ker(D_{\tilde{g}}^{m_n} - \text{id}) > m.$$
\[ \Gamma(\tilde{\gamma}) > \max_{1 \leq i,j \leq k} |\Gamma(\gamma_i) - \Gamma(\gamma_j)|. \] (2)

Moreover, we choose a path \( \hat{\gamma} : (I, \partial I) \to (R(M), R^{inv}(M)) \) such that \( \hat{\gamma}(0) = g_0 \) and \( \hat{\gamma}(1) = \tilde{\gamma}(0) \). Then \( \hat{\gamma} \ast \tilde{\gamma} : (I, \partial I) \to (R(M), R^{inv}(M)) \) and we set \( \gamma_{k+1} = \hat{\gamma} \ast \tilde{\gamma} \) if \( \Gamma(\hat{\gamma} \ast \tilde{\gamma}) \neq \Gamma(\gamma_j) \) for all \( j = 1, \ldots, k \).

If, on the other hand, \( \Gamma(\hat{\gamma} \ast \tilde{\gamma}) = \Gamma(\gamma_j) \) for some \( j = 1, \ldots, k \), then we set \( \gamma_{k+1} = \hat{\gamma} \). In order to justify this choice, assume that also \( \Gamma(\hat{\gamma}) = \Gamma(\gamma_i) \) for some \( 1 \leq i \leq k \). Then we obtain

\[ \Gamma(\gamma_j) = \Gamma(\hat{\gamma} \ast \tilde{\gamma}) = \Gamma(\hat{\gamma}) + \Gamma(\tilde{\gamma}) = \Gamma(\gamma_i) + \Gamma(\tilde{\gamma}), \]

which contradicts (2). Hence we indeed obtain a sequence \( \{\gamma_k\}_{k \in \mathbb{N}} \) with the required properties.

We now finish our proof by claiming that the metrics \( \gamma_k(1) \), \( k \in \mathbb{N} \), all lie in different path components of \( R^{inv}(M) \). Assume on the contrary that we can find \( i, j \in \mathbb{N}, i \neq j \), and a path \( \tilde{\gamma} : I \to R^{inv}(M) \) such that \( \tilde{\gamma}(0) = \gamma_i(1) \) and \( \tilde{\gamma}(1) = \gamma_j(1) \). Then \( \gamma_i \ast \tilde{\gamma} \ast \gamma_j^{-1} \) is a closed path with initial point \( g_0 \in R^{inv}(M) \). Since \( R(M) \) is contractible, \( \gamma_i \ast \tilde{\gamma} \ast \gamma_j^{-1} \) is homotopic to the constant path \( \gamma_1 \equiv g_0 \) through a \( g_0 \)-preserving homotopy. We obtain from the properties of \( \Gamma \)

\[ 0 = \Gamma(\gamma_1) = \Gamma(\gamma_i \ast \tilde{\gamma} \ast \gamma_j^{-1}) = \Gamma(\gamma_i) + \Gamma(\tilde{\gamma}) + \Gamma(\gamma_j^{-1}) = \Gamma(\gamma_i) + \Gamma(\gamma_j^{-1}) \]

and hence \( \Gamma(\gamma_i) = \Gamma(\gamma_j) \) contradicting the construction of the sequence \( \{\gamma_k\}_{k \in \mathbb{N}} \).

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