WEAK COMPACTNESS CRITERIA IN NON-COMMUTATIVE ORLICZ SPACES

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ABSTRACT. In this paper, we provide a direct proof for the equivalence of K.M. Chong’s and De la Vallée Poussin’s criteria of weak compactness of a subset $K$ of $L_1(0,1)$ in terms of some Orlicz function. Furthermore, we discuss the equivalence in $L_1(0,\infty)$. We show that the analogous result remains valid in the non-commutative setting. We also obtain non-commutative analogues of M. Nowak’s and K.M. Chong’s weak compactness criteria in non-commutative Orlicz spaces. In addition, we verify Pelczyński’s property (V) in non-commutative Orlicz spaces. Finally, we obtain a non-commutative analogue of Kolmogorov’s compactness criterion in terms of conditional expectations.

1. Introduction

It is well known that for a subset $K \subset L_1(\Omega,\Sigma,\nu)$, where $(\Omega,\Sigma,\nu)$ is a finite measure space, the following conditions are equivalent

(i) $K$ is relatively weakly compact set;
(ii) $K$ is bounded and uniformly integrable (Dunford’s criterion, see [12, Theorem 15, p.76], [19], [37, Theorem 23, p.20]);
(iii) there exists an $N$-function $F$ (see Definitions 2.1 and 2.2) such that
\[ \sup \left\{ \int F(f)d\nu : f \in K \right\} < \infty \]
(De la Vallée Poussin’s criterion, see [37, Theorem 22, p.19-20], see also [41, Theorem 2, p.3]);
(iv) $K$ is contained in the orbit of some positive integrable function (in the sense of the Hardy-Littlewood-Pólya submajorization) (K.M. Chong’s criterion, see [10, Theorem 4.2]).

The concept of uniform integrability can be easily generalized to any Banach lattice $X$ of measurable functions over a measure space $(\Omega,\Sigma,\nu)$. We shall say that a set $K \subset X$ has equi-absolutely continuous norms in $X$ if (see, for example, [4])
\[ \lim_{\delta \to 0} \sup_{\nu(E) < \delta} \sup_{x \in K} ||x\chi_E||_X = 0. \]

The study of weak compactness criteria in Orlicz spaces was of interest to W. Orlicz himself, who proved that each Orlicz space $L_G = L_G(0,1)$ such that
\[ \lim_{t \to \infty} \frac{G^*(2t)}{G^*(t)} = \infty, \]

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where $G^*$ is the complementary (see Chapter 1, formula (2.9)] to $G$, satisfies Dunford-Pettis criterion of weak compactness [39, assertion 1.5] (see also [1]), that is, every relatively weakly compact subset of $L_G$ has equi-absolutely continuous norms in $L_G$.

These characterisations of weak compactness have been shown time and time again to be powerful tools in functional analysis, and have served as sources of inspiration for much subsequent research (see [1, 3, 4, 5, 9, 10, 18, 29, 33, 38, 41, 42, 50, 51]).

As noted in [41, p.1] “The uniform integrability concept through its equivalence with a condition discovered by De la Vallée Poussin in 1915 has given a powerful inducement for the study of Young’s functions and the corresponding function spaces.”

What is likely the most vivid example of such a study indeed delivers Orlicz spaces (see e.g. [41, Chapter 1].) There is a substantial literature devoted to the study of weak compactness in both Orlicz function and sequence spaces, see, for example [1, 3, 4, 5, 9, 10, 18, 29, 33, 38, 41, 50, 51], and references therein.

Our objective in this paper is to study criteria listed above and their applications in the class of non-commutative (operator) Orlicz spaces.

Our first result is of mostly pedagogical value: in Section 3 (see Theorem 3.5), we prove directly the equivalence of K.M. Chong’s and De la Vallée Poussin’s criteria of weak compactness of a subset $K$ of the space $L_1(0,1)$. Our proof does not refer to Dunford–Pettis criterion and provides a clear demonstration of powerful methods from the general theory of symmetric function spaces. We also prove there that any function from $L_1(0,\infty)$ belongs to some Orlicz space, different from $L_1(0,\infty)$ itself (see Lemma 3.1). For the case of an integrable function from a finite measure space, this result is known (see [31, Chapter II, p.60]). In fact, the proof is similar to the case of finite measure space, however, we need to choose another partition of $(0,\infty)$, different from the partition of $(0,1)$ in [31, Chapter II, p.60]. Using Lemma 3.1, it is straightforward to show that the Chong’s condition implies the condition of De la Vallée Poussin in $L_1(0,\infty)$. We also show that the converse statement is also true under some additional condition (see Remark 3.8).

In 1962, T. Ando (see [3]) described weak compactness in Orlicz spaces using Köthe duality. The results of T. Ando were extended from the setting of finite measure spaces to the setting of $\sigma$-finite measure spaces in the work of M. Nowak in 1986, see [38].

In Section 4 we obtain non-commutative analogues of T. Ando and M. Novak’s results concerning weak relative compactness of a bounded subset $K$ of a non-commutative Orlicz space (see Theorem 4.1) as well as a non-commutative analogue of K.M. Chong’s criterion (see Theorem 4.5) and its equivalence to De la Vallée Poussin’s criterion in non-commutative $L_1$-spaces (see Theorem 4.3).

In Section 5, we extend known results concerning Pelczyński’s property (V) of Orlicz function spaces from [33, 20, 34] to the non-commutative setting. Our methods here are based on the recent study of M-ideals in [23].

Section 6 is devoted to the extension of a characterization of compact sets, originally due to Kolmogorov (see [30]) in the case of reflexive $L_p$-spaces over a bounded measurable set in $\mathbb{R}^d$, to non-commutative separable symmetric spaces. To separable Orlicz function spaces, Kolmogorov’s result was extended by Takahashi [44]. Further, an analogue of Kolmogorov’s compactness criterion for separable Orlicz
spaces $L_G$ in terms of Steklov functions may be found in [31, Theorem 11.1, p. 97]. In Section 6, we present its non-commutative analogue in terms of conditional expectations for general separable symmetric spaces over semifinite von Neumann algebras, see Theorem 6.1.

2. Preliminaries

Recall that a subset $K$ of a space $L_1(\nu)$ is called uniformly integrable if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup \left\{ \int_E |f|d\nu : f \in K \right\} < \varepsilon$ whenever $\nu(E) < \delta$. In particular, every bounded subset of $L_2$ is uniformly integrable. Alternatively, $K$ is bounded and uniformly integrable if and only if, for any $\varepsilon > 0$, there is $N > 0$ such that $\sup \left\{ \int_{|f|>c} |f|d\nu : f \in K \right\} < \varepsilon$ whenever $c \geq N$ (see [11, p.2]).

2.1. Singular value functions. Let $(I, m)$ denote the measure space, where $I = (0, \infty)$ (resp. $(0, 1)$), equipped with Lebesgue measure $m$. Let $L(I, m)$ be the space of all measurable real-valued functions on $I$ equipped with Lebesgue measure $m$. Define $S(I, m)$ to be the subset of $L(I, m)$, which consists of all functions $f$ such that $m\{t : |f(t)| > s\} < \infty$ for some $s > 0$. Note that if $I = (0, 1)$, then $S(I, m) = L(I, m)$.

For $f \in S(I, m)$, we denote by $\mu(f)$ the decreasing rearrangement of the function $|f|$. That is,

$$\mu(t, f) = \inf\{s \geq 0 : m\{|f| > s\} \leq t\}, \quad t > 0.$$  

We say that $f$ is submajorized by $g$ in the sense of Hardy–Littlewood–Pólya (written $f \prec\prec g$) if

$$\int^t_0 \mu(s, f)ds \leq \int^t_0 \mu(s, g)ds, \quad t \geq 0.$$  

Also, we say that $f$ is majorized by $g$ on $I$ in the sense of Hardy–Littlewood–Pólya (written $f \prec g$) if in addition to $f \prec\prec g$, we have

$$\int^t_I \mu(s, f)ds = \int^t_I \mu(s, g)ds.$$  

For a positive function $g \in L_1(X, m)$ we define the following set

$$C_g := \{f : f \in L_1(X, m), |f| \prec\prec g\},$$

which is called the orbit of a function $g$.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a semifinite faithful normal trace $\tau$. For a given von Neumann algebra $\mathcal{M}$ we set $\mathcal{M}_+ := \{A \in \mathcal{M} : A \geq 0\}$, which is called the positive part of $\mathcal{M}$. Let $Proj(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$.

A linear closed and densely defined operator $A$ affiliated with $\mathcal{M}$ is called $\tau$-measurable if $\tau(E_{(s,\infty)}(|A|)) < \infty$ for sufficiently large $s \geq 0$, where $E_{(s,\infty)}(|A|)$ is the spectral projection of $|A|$ corresponding to the interval $(s,\infty)$. For a $\tau$-measurable operator $A$ and $s \geq 0$, $\tau(E_{(s,\infty)}(|A|)) < \infty$ is called the distribution function of $|A|$ and denoted by (see [23, Definition 1.3])

$$\lambda_s(A) = \tau(E_{(s,\infty)}(|A|)), \quad s \geq 0.$$  

We denote the set of all $\tau$-measurable operators by $S(\mathcal{M}, \tau)$. For every $A \in S(\mathcal{M}, \tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu(t, A) = \inf\{\|A(1-P)\|_\mathcal{M} : P \in Proj(\mathcal{M}), \tau(P) \leq t\}, \quad t > 0.$$
For more details on generalised singular value functions, we refer the reader to [25] and [35]. If \( A, B \in S(M, \tau) \), then we say that \( A \) is submajorized by \( B \) (in the sense of Hardy–Littlewood–Pólya), denoted by \( A \prec\prec B \), if
\[
\int_0^t \mu(s, A)ds \leq \int_0^t \mu(s, B)ds, \quad t \geq 0.
\]
For \( 1 \leq p < \infty \), we set
\[
L^p(M) = \{ A \in S(M, \tau) : \| A \|_{L^p(M)} = \| \mu(A) \|_p < \infty \}, \quad \| \mu(A) \|_p^p := \int_0^\infty \mu(s, A)^p ds.
\]
Such Banach spaces \((L^p(M), \| \cdot \|_{L^p(M)}) (1 \leq p < \infty)\) are separable. We denote by \( S_0(M, \tau) \) the subspace of \( S(M, \tau) \) which consists of all elements in \( S(M, \tau) \) whose singular value functions vanish at infinity. For more information on these spaces see [35, Chapter 2, p.60], and the handbook [40].

2.2. Marcinkiewicz spaces. Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing concave function such that \( \psi(0^+) = 0 \). For any such function \( \psi \) the Marcinkiewicz space \( M_\psi(I) \) is defined by setting
\[
M_\psi(I) = \{ f \in S(I) : \| f \|_{M_\psi(I)} < \infty \}
\]
equipped with the norm
\[
\| f \|_{M_\psi(I)} = \sup_{t \in I} \frac{1}{\psi(t)} \cdot \int_0^t \mu(s, f) dm.
\]
For more details on Marcinkiewicz spaces of functions, we refer the reader to [8, Chapter II.5] and [32, Chapter II.5].

2.3. Orlicz spaces.

**Definition 2.1.** A continuous and convex function \( G : [0, \infty) \to [0, \infty) \) is called an \( N \)-function if
1. \( G(0) = 0 \),
2. \( G(\lambda) > 0 \) for \( \lambda > 0 \),
3. \( \frac{2G(\lambda)}{\lambda} \to 0 \) as \( \lambda \to 0 \),
4. \( G(\lambda) \to \infty \) as \( \lambda \to \infty \).

**Definition 2.2.** A function \( G : [0, \infty) \to [0, \infty] \) is said to be an Orlicz function if (see [24] p.258)
1. \( G(0) = 0 \),
2. \( G \) is not identically equal to zero,
3. \( G \) is convex,
4. \( G \) is continuous at zero.

It follows from the definitions that every \( N \)-function is also an Orlicz function. The converse, however, does not hold. For example, the function \( G(t) = t \) is an Orlicz function but not an \( N \)-function. In what follows, unless otherwise specified, we always denote by \( G \) an \( N \)-function. For such a function we shall consider an (extended) real valued functional \( G(f) \) (also called the modular defined by an \( N \)-function \( G \)) defined, on the class of all measurable functions \( f \) on \( I \), by
\[
G(f) = \int_I G(|f(t)|) dt.
\]
The set
\[ L_G = \{ f \in S(I, m) : \|f\|_{L_G} < \infty \}, \]
where
\[ \|f\|_{L_G} = \inf \left\{ c > 0 : \int_I G \left( \frac{|f|}{c} \right) dm \leq 1 \right\} \]
is called an Orlicz space defined by the Orlicz function \( G \) (equipped with Orlicz norm).

We will denote by \( G^* \) the function complementary (or conjugate) to \( G \) in the sense of Young, defined by (see [21 Chapter 1, p.11])
\[ G^*(t) = \sup \{ s | t - G(s) : s \geq 0 \}. \]

We notice that \( G^* \) is again an \( N \)-function (see [23 p.258]).

For any \( A \in S(M, \tau) \), by means of functional calculus applied to the spectral decomposition of \( |A| \), we have
\[ \tau(G(|A|)) = \int_0^{\tau(1)} \lambda_s(|A|) dG(s) = \int_0^{\tau(1)} G(\mu(s, A)) ds. \]

For an Orlicz function \( G \), the non-commutative Orlicz space \( L_G(M, \tau) \) (or simply \( L_G(M) \)) is defined as the space of all \( \tau \)-measurable operators \( A \) affiliated with \( M \) such that
\[ \tau \left( G \left( \frac{|A|}{c} \right) \right) < \infty \]
for some \( c > 0 \). The space \( L_G(M) \), equipped with the norm
\[ \|A\|_{L_G(M)} = \inf \left\{ c > 0 : \tau \left( G \left( \frac{|A|}{c} \right) \right) \leq 1 \right\} , \]
is a Banach space. Observe that if \( \tau(1) = \infty \), then \( 1 \notin L_G(M) \). Otherwise, \( \tau(G((\frac{1}{c})) = \infty \) for any \( c \). Hence, for any \( N \)-function \( G \), \( L_G(0, \infty) \) is a subspace of \( S_0(0, \infty) \) and hence, \( L_G(M, \tau) \subset S_0(M, \tau) \). Note if \( G(t) = t^p \) with \( 1 \leq p < \infty \) then \( L_G(M) = L_p(M) \). For more details on non-commutative Orlicz spaces see, for example, [6], [32], and [33].

2.4. Symmetric Banach Function and Operator Spaces. For the general theory of symmetric Banach function spaces, we refer the reader to [8], [32], and [33].

**Definition 2.3.** Let \( \mathcal{E} \) be a linear subset in \( S(M, \tau) \) equipped with a complete norm \( \| \cdot \|_{\mathcal{E}} \). We say that \( \mathcal{E} \) is a non-commutative symmetric space (or symmetric operator space) (on \( M \), or in \( S(M, \tau) \)) if for every \( A \in \mathcal{E} \) and for every \( B \in S(M, \tau) \) with \( \mu(B) \leq \mu(A) \), we have \( B \in \mathcal{E} \) and \( \|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}} \).

A symmetric function space is the term reserved for a symmetric operator space when \( M = L_\infty(I, m) \), where \( I = (0, \infty) \) (or \( I = (0, 1) \)).

Recall the construction of a non-commutative symmetric (operator) space \( \mathcal{E}(M, \tau) \). Let \( E \) be a symmetric Banach function space on \((0, \infty)\). Set
\[ \mathcal{E}(M, \tau) = \left\{ A \in S(M, \tau) : \mu(A) \in E \right\} . \]
We equip \( \mathcal{E}(M, \tau) \) with a natural norm
\[ \|A\|_{\mathcal{E}(M, \tau)} := \|\mu(A)\|_E, \quad A \in \mathcal{E}(M, \tau). \]
For brevity, we shall frequently omit \( \tau \) in the notation above and simply write \( \|A\|_\mathcal{E}(\mathcal{M}) \). The space \( (\mathcal{E}(\mathcal{M}), \|\cdot\|_\mathcal{E}(\mathcal{M})) \) is a Banach space with the norm \( \|\cdot\|_\mathcal{E}(\mathcal{M}) \) and is called the non-commutative symmetric (operator) space associated with \((\mathcal{M}, \tau)\) corresponding to \((E, \|\cdot\|_E)\) \cite{28}. An extensive discussion of the various properties of such spaces can be found in \cite{28,35}. Furthermore, the following fundamental theorem was proved in \cite{28} (see also \cite{35} Question 2.5.5, p. 58).

**Theorem 2.4.** Let \((E, \|\cdot\|_E)\) be a symmetric Banach function space on \((0, \infty)\) and let \(\mathcal{M}\) be a semifinite von Neumann algebra. Set

\[
\mathcal{E}(\mathcal{M}) = \left\{ A \in S(\mathcal{M}, \tau) : \mu(A) \in E \right\}, \quad \|A\|_{\mathcal{E}(\mathcal{M})} := \|\mu(A)\|_E.
\]

So defined \((\mathcal{E}(\mathcal{M}), \|\cdot\|_{\mathcal{E}(\mathcal{M})})\) is a non-commutative symmetric space.

The main result of \cite{28} (see also \cite{35} Section 3) shows that the correspondence

\[
(E, \|\cdot\|_E) \longleftrightarrow (\mathcal{E}(\mathcal{M}), \|\cdot\|_{\mathcal{E}(\mathcal{M})})
\]

is a one-to-one correspondence between the set of all symmetric operator spaces in \(S(\mathcal{M}, \tau)\) and the set of all symmetric function spaces in \(S(I, m)\) whenever \((\mathcal{M}, \tau)\) does not contain any minimal projections or is atomic and all minimal projections have equal trace. Of course, depending on \((\mathcal{M}, \tau)\) the symmetric function space \(E \subset S(I, m)\) is considered either on \((0, 1)\), or on \((0, \infty)\).

3. Equivalence of Chong’s and De la Vallée Poussin’s criteria

In this section, we discuss the equivalence of K.M. Chong’s and De la Vallée Poussin’s criteria of relative weak compactness of a subset \(K \subset L_1(0, 1)\) in terms of an Orlicz function \(G\). The following result presents an extension of \cite{31} Chapter II, p.60 to \(\sigma\)-finite measure spaces.

**Lemma 3.1.** For any integrable function \(f\) on \(I = (0, \infty)\), there exists an \(N\)-function \(G\) such that \(G(|f|)\) is integrable on \(I\). Moreover, \(G(t) \rightarrow \infty\) as \(t \rightarrow \infty\).

**Proof.** Note that if \(f = 0\) on a set \(\Omega\), then \(G(f) \equiv 0\) on \(\Omega\). Hence \(\int_\Omega G(f(t))dt = 0\), so \(G(f)\) is integrable on \(\Omega\). We set

\[
\text{supp} \ f = \{ t \in [0, \infty) : f(t) \neq 0 \}.
\]

Consider the family of pairwise disjoint sets

\[
I_n = \{ t \in \text{supp} \ f : 2^n \leq |f(t)| < 2^{n+1} \}, \quad n \in \mathbb{Z}.
\]

Then \((0, \infty) = \bigcup_{n=-\infty}^{\infty} I_n\), and \(f\) is integrable on \(I_n\) for all \(n \in \mathbb{Z}\). Hence,

\[
\sum_{n=-\infty}^{\infty} 2^n \cdot m(I_n) \leq \int_0^\infty |f(t)|dt < \infty.
\]

By Lemma \cite{74} in the Appendix below there exists an increasing sequence of real numbers \(\{\alpha_n\}_{n=-\infty}^{\infty}\) with \(\alpha_n = 0\) for all \(n \leq 0\) such that \(\lim_{n \rightarrow \infty} \alpha_n = \infty\) and

\[
\sum_{n=-\infty}^{\infty} \alpha_{n+1} \cdot 2^n \cdot m(I_n) < \infty.
\]

We set
$p(t) = \begin{cases} 
t & \text{if } 0 \leq t < 1, 
\alpha_n & \text{if } 2^{n-1} \leq t < 2^n \ (n = 1, 2, \ldots). 
\end{cases}$

Without loss of generality we may assume $\alpha_1 \geq 1$. Since $p(t)$ is nondecreasing and right-continuous, $p(0) = 0$, $p(t) > 0$ whenever $t > 0$, and $\lim_{t \to \infty} p(t) = \infty$ we may define an $N$–function $G$ (see [11, Definition 1.1, p.3]) by

$$G(x) = \int_{0}^{x} p(t) dt, \quad x \geq 0.$$ 

Since

$$G(2^n) = \int_{0}^{2^n} p(t) dt \leq \int_{0}^{2^n} \alpha_n dt = 2^n \cdot \alpha_n, \quad n = 1, 2, \ldots,$$

it follows, in virtue of (2), that

$$\int_{0}^{\infty} G(|f(t)|) dt = \sum_{n=-\infty}^{\infty} \int_{I_n} G(|f(t)|) dt$$

$$\leq \sum_{n=-\infty}^{\infty} G(2^{n+1}) m(I_n) \leq \sum_{n=-\infty}^{\infty} 2^{n+1} \cdot \alpha_{n+1} \cdot m(I_n) < \infty.$$ 

Hence, $G(|f|)$ is integrable on $(0, \infty)$. The condition $\frac{G(t)}{t} = \frac{\int_{0}^{t} p(s) ds}{t} \to \infty$ as $t \to \infty$ follows immediately by applying the L’Hôpital’s rule.

Remark 3.2. Observe, that if we had asked in Lemma 3.1 for an Orlicz function $G$, then there would be nothing to prove. Indeed, for any integrable function $f$ on $I = (0, \infty)$, there exists an Orlicz function $G$ such that $G(|f|)$ is integrable on $I$, given by $G(t) = t$ for all $t \in [0, \infty)$. However, the function $G(t) = t$ is not an $N$–function.

Recall, in [10, Lemma 4.1] K.M. Chong proved that a weakly compact set in $L_1$ associated with finite measure space is a subset of the orbit of some positive integrable function.

Another characterization of uniform integrability (relative weak compactness) is given in a theorem of De la Vallée Poussin [37, Theorem 22, p.19-20], which states the following: A subset $K$ of $L_1(I, m)$ (with $m(I) < \infty$) is bounded and uniformly integrable if and only if there is an Orlicz function $G$ such that $\frac{G(t)}{t} \to \infty$ as $t \to \infty$ so that

$$\sup \left\{ \int_{I} G(|f|) dm : f \in K \right\} < \infty.$$ 

Remark 3.3. In the theorem of De la Vallée Poussin above, we may omit boundedness as uniform integrability implies boundedness.

The following lemma may be found in [21, Proposition 2.3], [22, Proposition 1.2] for an infinite measure space or in [24, Proposition 2.4] for a finite measure space (see also [36, p. 22, Theorem D.2] and [45]).
Lemma 3.4. Assume that \( f = \mu(f) \) and \( g = \mu(g) \) are integrable functions on \((0, \infty)\). If \( \int_0^t f(s) ds \leq \int_0^t g(s) ds \) for every \( 0 < t < \infty \), then for every increasing continuous convex function \( \varphi \) on \((0, \infty)\), we have \( \int_0^t \varphi(f(s)) ds \leq \int_0^t \varphi(g(s)) ds \) for every \( 0 < t < \infty \).

The following theorem, the main result of this section, provides the direct proof of the equivalence of K.M. Chong’s and De la Vallée Poussin’s criteria of weak compactness of a subset \( K \) of \( L_1(0, 1) \).

Theorem 3.5. Let \( K \) be a bounded subset of \( L_1(0, 1) \). Then the following two conditions are equivalent

(a) there exists an Orlicz function \( G \) with \( \frac{G(t)}{t} \to \infty \) as \( t \to \infty \) so that

\[
\sup \left\{ \int_0^1 G(|f|) ds : f \in K \right\} < \infty;
\]

(b) there exists a positive function \( g \in L_1(0, 1) \) such that \( |f| \prec \prec g \) for all \( f \in K \).

Proof. (a) \( \Rightarrow \) (b). Suppose (a) holds and \( K \) is a bounded subset of \( L_G \). Without loss of generality, we may assume that \( K \) is the unit ball of \( L_G \).

Let \( \varphi \) be a fundamental function of \( L_G \). The function \( \varphi \) is quasiconcave. Let \( \psi \) be its least concave majorant, so \( \frac{1}{t} \psi \leq \varphi \leq \psi \) (see e.g. [31, p. 71, Proposition 5.10]). The Marcinkiewicz space \( M_\psi \) contains the Orlicz space \( L_G \) (see [31, Theorem II. 5.13, p.72, see also [3] Corollary II. 5.14, p.73]). By Theorem II.5.7 from [32] we know that \( K \) lies in a unit ball of \( M_\psi \). Hence, by (2.12) in [32, p.64], we have

\[
\int_0^t \mu(s, f) ds \leq \|f\|_{M_\psi} \int_0^t \psi'(s) ds \leq \|f\|_{M_\psi} \cdot \int_0^t \mu(s, \psi') ds \leq \int_0^t \mu(s, \psi') ds
\]

for all \( f \in K \) and \( t \in (0, 1) \), i.e. \( |f| \prec \prec \psi' \) for all \( f \in K \). Thus, the assertion (b) holds with \( g = \psi' \).

(b) \( \Rightarrow \) (a). Suppose there is a positive function \( g \in L_1(0, 1) \) such that \( |f| \prec \prec g \) for all \( f \in K \). Then by Lemma 3.1 (see also [31, Chapter II, p.60]) there exists an \( N \)-function \( G \) (hence an Orlicz function) with \( \frac{G(t)}{t} \to \infty \) as \( t \to \infty \) such that \( \int_0^1 G(g(s)) ds < \infty \). In other words, \( g \in L_G(0,1) \). We have \( \int_0^t |f(s)| ds \leq \int_0^t \mu(s, f) ds \) for all \( t \in (0, 1) \) (see e.g. [32, (2.12), p.64]). By the assumption, we have \( \int_0^t \mu(s, f) ds \leq \int_0^t \mu(s, g) ds \) for all \( f \in K \) and for all \( t \in (0, 1] \) and so, by Lemma 3.4 and Lemma 2.5 (iv)], we have

\[
\int_0^t G(|f(s)|) ds \leq \int_0^t G(\mu(s, f)) ds \leq \int_0^t G(\mu(s, g)) ds < \infty.
\]

This completes the proof. \( \square \)

Remark 3.6. Recall that the classical Dunford’s criterion identifies bounded and uniformly integrable subsets of \( L_1(I) \) with \( m(I) < \infty \) with relatively weakly compact sets ([12, Theorem 15, p.76], [37, Theorem 23, p.20]). Note, however, that this criterion of weak compactness is no longer valid in \( L_1(0, \infty) \) as the following example illustrates.

Let \( M = \{ f_n(x) = \frac{1}{n} \chi_{[n,2n]} \}_{n=1}^\infty \). Clearly, \( M \) is norm bounded in \( L_1(0, \infty) \) and uniformly integrable. However, \( M \) is not relatively weakly compact in \( L_1(0, \infty) \).
Remark 3.7. Neither De la Vallée Poussin’s criterion (condition (a) in Theorem 3.5), nor Chong’s criterion (condition (b) in Theorem 3.5) describe relatively weakly compact subsets in $L_1(0, \infty)$.

For example, let $K = \{f_n(x) = \chi_{[n,n+1]}(x)\}_{n=0}^{\infty}$. Obviously, $K$ is a bounded subset of $L_1(0, \infty)$, which is not relatively weakly compact in $L_1(0, \infty)$. However, $|f_n| \prec \prec g$ for all $f_n \in K$, where $g(x) = \chi_{[0,1]}(x) + \frac{1}{x} \chi_{(1,\infty)}(x)$, where $\alpha > 1$.

Also, taking $G(x) = x^\alpha$, $\alpha > 1$, we obtain an Orlicz function $G$ with $G(t) \to \infty$ as $t \to \infty$ such that

$$\sup \left\{ \int_0^1 G(|f|)ds : f \in K \right\} < \infty.$$ 

Remark 3.8. A quick analysis of the proof of the implication $(b) \implies (a)$ in Theorem 3.5 shows that it holds verbatim for bounded subsets $K$ in $L_1(0, \infty)$. Now, we show that the implication $(a) \implies (b)$ in Theorem 3.5 also holds in this setting under an additional assumption that

$$\sup_{f \in K} \int_N^\infty |f|ds \to 0, \quad N \to \infty.$$ 

Proof. We define concave function $\psi$ on $(0, \infty)$ analogously as in the proof of the Theorem 3.5, that is, we have $\int_0^t \mu(s, f)ds \leq \int_0^t \mu(s, \psi')ds$ for all $f \in K$ and $t \in (0, \infty)$.

Fix $\varepsilon > 0$. Due to [3] there exists a real number $N_2 > 1$ such that

$$\sup_{f \in K} \int_{N_2}^\infty \mu(s, f)ds < \varepsilon.$$ 

We define

$$g(s) := \begin{cases} 
\psi'(s) + \varepsilon & \text{if } 0 \leq s \leq N_2, \\
1/s^\alpha & \text{if } s > N_2,
\end{cases}$$

where $\alpha > 1$. Clearly, $g \in L_1(0, \infty)$ is a positive function and $|f| \prec \prec g$ for all $f \in K$.

4. Weak compactness criteria in Orlicz spaces of $\tau$–measurable operators.

In this section, we obtain non-commutative analogues of Ando–Nowak’s and Chong’s theorems as well as a non-commutative version of the Theorem 3.5.

Let $L_G(M)$ be the non-commutative Orlicz space. Recall that the Köthe dual (or associate) space, denoted by $L_G^\times(M)$, is defined by setting (see [13, Definition 5.1], [14], [17])

$$L_G^\times(M) := \{A \in S(M) : AB \in L_1(M) \text{ for all } B \in L_G(M)\}$$

with the norm defined by setting

$$\|A\|_{L_G^\times(M)} := \sup\{|\tau(AB)| : B \in L_G(M), \|B\|_{L_G(M)} \leq 1\}.$$ 

The following criterion extends results obtained in [3] Theorem 1 and [38] Theorem 1.1.
Theorem 4.1. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Let $L_0$ be an Orlicz space of functions on $(0, \tau(1))$ and $L_G(\mathcal{M})$ be the corresponding non-commutative Orlicz space such that $L_G(\mathcal{M}) \subset S_0(\mathcal{M}, \tau)$. A bounded subset $K$ of $L_G(\mathcal{M})$ is relatively $\sigma(L_G(\mathcal{M}), L_G(\mathcal{M}))$-compact if and only if the following condition holds

\[
\sup_{A \in K} \frac{\tau(G(\lambda \cdot |A|))}{\lambda} \to 0 \quad \text{as} \quad \lambda \downarrow 0.
\]

Proof. Let $K$ be a bounded subset of $L_G^\infty(\mathcal{M})$. Assume that $K$ is relatively $\sigma(L_G^\infty(\mathcal{M}), L_G(\mathcal{M}))$-compact. Then by [18, Theorem 5.4] $\mu(K) := \{\mu(A) : A \in K\}$ is relatively $\sigma(L_G^\infty, L_G)$-compact. Hence $\mu(K)$ satisfies the conditions of the Nowak's theorem (see [38, Theorem 1.1]), which in turn implies

\[
\sup_{A \in K} \frac{G(\lambda \mu(A))}{\lambda} \to 0 \quad \text{as} \quad \lambda \downarrow 0.
\]

Appealing to formula (1), we obtain (4).

Conversely, suppose that (4) holds and hence by (1) we obtain (5). By [18, Theorem 1.1] the set $\mu(K)$ is relatively $\sigma(L_G^\infty, L_G)$-compact. Applying [38, Theorem 5.4] $\mu(K) \implies (i)$, we obtain that $K$ is relatively $\sigma(L_G^\infty(\mathcal{M}), L_G(\mathcal{M}))$-compact. $\square$

Remark 4.2. When $\tau(1) = \infty$, we note that $L_G^\infty(\mathcal{M})$ cannot be $L_1(\mathcal{M})$ since $L_\infty(\mathcal{M}) \not\subset S_0(\mathcal{M}, \tau)$, and $L_G(\mathcal{M}) \subset S_0(\mathcal{M}, \tau)$ in Theorem 4.1. Recall that $A \subset L_1(\mathcal{M})$ is said to be of uniformly absolutely continuous norms in $L_1(\mathcal{M})$ if $\mathcal{A}$ is a bounded set and

\[
\sup_{x \in \mathcal{A}} \|e_\alpha e_\alpha^*\|_{L_1(\mathcal{M})} \to 0
\]

for every downwards directed system $e_\alpha \downarrow 0$ of projections in $\mathcal{M}$. If $\mathcal{A} \subset L_1(\mathcal{M})$ is bounded, then $\mathcal{A}$ is relatively weakly compact if and only if $\mathcal{A}$ is of uniformly absolutely continuous norms (see e.g. [10, Theorem 4.4]).

The following theorem is the non-commutative version of the Theorem 5.3. It extends [50, Theorem 3.10] (see also [51, Lemma 3.4]).

Theorem 4.3. Let $\mathcal{M}$ be a non-atomic finite von Neumann algebra and $L_1(\mathcal{M})$ be the corresponding non-commutative space of $\tau$-measurable operators. Let $K$ be a bounded subset of $L_1(\mathcal{M})$. Then the following conditions are equivalent
(a) there exists an Orlicz function $G$ with $\frac{G(t)}{t} \to \infty$ as $t \to \infty$ so that

\[
\sup_{A \in K} \tau(G(|A|)) < \infty;
\]
(b) there exists a positive operator $B \in L_1(\mathcal{M})$ such that $\mu(|A|) \prec \mu(B)$ for all $A \in K$.

Proof. $(a) \implies (b)$. The condition $(a)$ implies that $K$ is a bounded subset of $L_G(\mathcal{M})$. Without loss of generality, we may assume that $K$ is the unit ball in $L_G(\mathcal{M})$.

Since $A \in K$, we have $\mu(A) \in \mu(K) \subset L_1(0, \tau(1))$, where $\mu(K) := \{\mu(A) : A \in K\}$. Proceeding as in the commutative case (see Theorem 5.3) we obtain a positive function $g \in L_1(0, \tau(1))$ such that $\mu(|A|) \prec \mu(g)$ for all $A \in K$. Since $\mu(g) \in L_1(0, \tau(1))$ it follows from [35, Theorem 2.5.3 (b), p. 57] that there exists a positive operator $B \in L_1(\mathcal{M})$ such that $\mu(g) = \mu(B)$. The fact that $B$ may be
chosen positive is explained in [11, Proposition 3.0.3] for which we refer the reader for additional references and comments. Hence $\mu(|A|) \prec \prec \mu(B)$ for all $A \in K$.

$(b) \implies (a)$. Suppose there is a positive operator $B \in \mathcal{L}_1(\mathcal{M})$ such that $\mu(|A|) \prec \prec \mu(B)$ for all $A \in K$, i.e. $\int_0^1 \mu(s, A) ds \leq \int_0^t \mu(s, B) ds < \infty$ for all $t \in (0, \tau(1))$. Then, by Theorem 3.5 ((b) $\implies$ (a)), we have

$$\sup_{\mu(A) \in K} \left\{ \int_0^{\tau(1)} G(\mu(s, A)) ds \right\} < \infty.$$ 

By [11] it is equivalent to

$$\sup_{A \in K} \tau(G(|A|)) < \infty.$$ 

The following result shows that the statement of Theorem 4.3 holds when $\mathcal{M}$ is a non-atomic von Neumann algebra such that $\tau(1) = \infty$, provided a non-commutative counterpart of condition (3) holds.

**Theorem 4.4.** Let $\mathcal{M}$ be a non-atomic von Neumann algebra equipped with a faithful normal trace $\tau$ such that $\tau(1) = \infty$. Let $K$ be a bounded subset of $\mathcal{L}_1(\mathcal{M})$.

If

$$\sup_{A \in K} \int_N^{\infty} \mu(s, A) ds \to 0, \quad N \to \infty,$$

then the following conditions are equivalent

(a) there exists an Orlicz function $G$ with $\frac{G(t)}{t} \to \infty$ as $t \to \infty$ so that

$$\sup_{A \in K} \tau(G(|A|)) < \infty;$$

(b) there exists a positive operator $B \in \mathcal{L}_1(\mathcal{M})$ such that $\mu(|A|) \prec \prec \mu(B)$ for all $A \in K$.

**Proof.** By Remark 3.8 the argument for the implication $(b) \implies (a)$ follows the same line as in the proof of Theorem 4.3 and is therefore omitted.

Let us show that $(a) \implies (b)$. Suppose there exists an Orlicz function $G$ so that

$$\sup_{A \in K} \tau(G(|A|)) < \infty.$$

In other words, $K \subset \mathcal{L}_G(\mathcal{M})$. Without loss of generality, we may assume that $K$ is the unit ball in $\mathcal{L}_G(\mathcal{M})$. For all $A \in K$, we have that $\mu(A) \in \mu(K) \subset L_1(0, \infty)$, where $\mu(K) := \{\mu(A) : A \in K\}$. Since

$$\sup_{A \in K} \int_N^{\infty} \mu(s, A) ds \to 0, \quad N \to \infty,$$

by Remark 3.8 there exists a positive function $g \in L_1(0, \infty)$ such that $\mu(|A|) \prec \prec \mu(g)$ for all $A \in K$. Since $\mu(g) \in L_1(0, \infty)$, it follows from [35] Theorem 2.5.3 (a), p. 57 and [11] Proposition 3.0.3 that there exists a positive operator $B \in \mathcal{L}_1(\mathcal{M})$ such that $\mu(g) = \mu(B)$. Hence $\mu(|A|) \prec \prec \mu(B)$ for all $A \in K$, thereby completing the proof.

The following theorem is the non-commutative analogue of the Chong’s criterion [10, Lemma 4.1].
Theorem 4.5. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Let $\mathcal{K} \subset \mathcal{L}_1(\mathcal{M})$ be any family of operators. Then $\mathcal{K}$ satisfies

(i) $\sup \{ \tau(|A|) : A \in \mathcal{K} \} < \infty$;
(ii) $\sup \left\{ \int_0^\infty (\mu(s,|A|) - u)^+ \, ds : A \in \mathcal{K} \right\} \to 0$ as $u \to \infty$;

if and only if there exists a positive operator $B \in \mathcal{L}_1(\mathcal{M})$ such that $\mu(|A|) \prec \prec \mu(B)$ for all $A \in \mathcal{K}$.

Proof. Let (i) and (ii) hold. By [35, Theorem 2.6.3, p. 61, formula (2.4)], we have

$$\sup \{ \tau(|A|) : A \in \mathcal{K} \} = \sup \left\{ \int_0^\infty \mu(s,|A|) \, ds : A \in \mathcal{K} \right\} < \infty.$$ 

Then, there exists a positive function $f \in L_1(I)$, where $I = (0, \tau(1))$, such that $\mu(|A|) \prec \prec f$ (see [10, Lemma 4.1]). Since $\mu(f) \in L_1(I)$, it follows from [35, Theorem 2.5.3, p. 57] (see also [11, Proposition 3.0.3]) that there exists a positive operator $B \in \mathcal{L}_1(\mathcal{M})$ such that $\mu(f) = \mu(B)$, and hence $\mu(|A|) \prec \prec \mu(B)$ for all $A \in \mathcal{K}$.

Conversely, let $B$ be a positive operator in $\mathcal{L}_1(\mathcal{M})$ such that $\mu(|A|) \prec \prec \mu(B)$ for all $A \in \mathcal{K}$ or, equivalently

$$\int_0^t \mu(s,|A|) \, ds \leq \int_0^t \mu(s,B) \, ds, \quad t \in [0, \infty), \quad \forall \ A \in \mathcal{K}.$$ 

Since $B \in \mathcal{L}_1(\mathcal{M})$, it follows immediately that

$$\sup_{A \in \mathcal{K}} \int_0^\infty \mu(s,|A|) \, ds < \infty$$

which implies that the condition (i) of the theorem holds.

By [35, Theorem 4], for every $A \in \mathcal{K}$ and all $u \geq 0$, we have

$$\int_0^\infty (\mu(s,|A|) - u)^+ \, ds \leq \int_{\mu(A)>u} (\mu(s,|A|) - u) \, ds \leq \int_{\mu(B)>u} (\mu(s,B) - u) \, ds \leq \int_{\mu(B)>u} \mu(s,B) \, ds.$$ 

As $\int_{\mu(B)>u} \mu(s,B) \, ds \to 0$ as $u \to \infty$, we conclude that

$$\sup \left\{ \int_0^\infty (\mu(s,|A|) - u)^+ \, ds : A \in \mathcal{K} \right\} \to 0$$

so the condition (ii) of the theorem holds. \qed

5. Pełczyński’s property (V) of Orlicz spaces

In this section, we consider the Pelczyński’s property (V) of non-commutative Orlicz spaces. For the treatment of the property $\mu^*$ in symmetric non-commutative (operator) spaces, we refer the reader to [13].

Definition 5.1. A Banach space $X$ is said to have the Pelczyński’s property (V) if every subset $F$ of $X^*$ is relatively weakly compact whenever it has the following property

$$\lim_{n \to \infty} \sup_{x^* \in F} |x^*(x_n)| = 0$$
for every weakly unconditionally Cauchy sequence \( \{x_n\}_{n \geq 1} \) in \( X \) (i.e. such that \( \sum_{n \geq 1} |x^*(x_n)| < \infty \) for any \( x^* \in X^* \)). Equivalently, \( X \) has the Pełczyński’s property (V) if and only if for every Banach space \( Z \) and for every non-weakly compact operator \( T : X \to Z \), there exists a subspace \( X_0 \), isomorphic to \( c_0 \), such that \( T \) is an isomorphism between \( X_0 \) and \( T(X_0) \).

The property (V) of Orlicz function spaces has been considered in \([20, 33, 34]\). We characterize below the Orlicz functions such that the corresponding non-commutative Orlicz spaces have the property (V).

A subspace \( X \) of a Banach space \( Y \) is called an \( M \)-ideal of \( Y \) if there is an \( L \)-projection \( P \) on \( Y^* \) whose kernel is \( X^* \), the annihilator of \( X \); that is, we have

\[
\|y^*\| = \|Py^*\| + \|y^* - Py^*\|, \quad y^* \in Y^*.
\]

In particular, when \( Y = X^{**} \), \( X \) is called an \( M \)-embedded space (see e.g. \([20\text{ Chapter III, Definition 1.1}])\). The theory of \( M \)-embedded spaces has been developed intrinsically since it was introduced by Alfsen and Effros \([2]\) in 1972. Apart from the intrinsic mathematical beauty of the theory in its own right, the interest to the theory of \( M \)-embedded spaces has been maintained by its numerous applications in diverse areas of mathematics such as \( C^* \)-algebras, ordered Banach spaces and \( L^1 \)-preduals (see e.g. \([20]\)). Examples of \( M \)-embedded spaces are given by special examples of Orlicz sequence and function spaces, by the predual space of Lorentz function space \( L_{p,1}(0,\infty) \), \( 1 < p < \infty \), and by the set \( K(H) \) of all compact operators on a Hilbert space \( H \), see e.g. \([20\text{ Chapter III, Example 1.4}]\), \([47]\) and \([33]\). Furthermore, Werner \([47\text{ Proposition 4.1}]\) proved that, under some mild additional conditions imposed on a symmetric sequence space \( E \), the property of being an \( M \)-embedded space carries over to its non-commutative counterpart \( E \), the symmetric ideal of bounded operators on a separable Hilbert space associated with \( E \). This result was recently extended to the setting of arbitrary semifinite von Neumann algebra \([23\text{ Theorem 3.3}]\).

**Theorem 5.2.** Assume that \( E(0,\infty) \) is a fully symmetric function space having order continuous norm (i.e. separable), which fails to be a superset of \( C_0(0,\infty) \), where \( C_0(0,\infty) \) is the space of all bounded vanishing functions. If \( E(0,\infty) \) is an \( M \)-embedded space, then \( E(M) \) is an \( M \)-embedded space, too.

In fact, an \( M \)-embeddedness is a stronger property than the Pełczyński’s property (V).

**Theorem 5.3.** (see e.g. \([33\text{ Theorem 1}])\) Every Banach space which is an \( M \)-embedded space has the property (V).

It has been proved in \([23]\) that the separable part \( L_{q,\infty}^0(M) \) of \( L_{q,\infty}(M) \) is \( M \)-embedded when \( 1 < p < \infty \). Hence, it has the property (V).

We define \( H_{G}(0,\infty) \) by setting \([20\text{ p. 103}]\)

\[
H_{G}(0,\infty) = \left\{ \int G \left( \frac{|f(s)|}{\rho} \right) ds < \infty \text{ for all } \rho > 0 \right\}.
\]

Let \( G^* \) be the complementary (in the sense of Young) function to \( G \). We say that \( G^* \) satisfies the \( \Delta_2 \)-condition if

\[
\limsup_{t \to 0} \frac{G^*(2t)}{G^*(t)} < \infty \quad \text{and} \quad \limsup_{t \to \infty} \frac{G^*(2t)}{G^*(t)} < \infty.
\]
Theorem 5.4. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. Let $G : [0, \infty) \to [0, \infty)$ be a continuous convex function such that $G(0) = 0$ and $G(t) > 0$, $t > 0$. If $G^*$ satisfies the $\Delta_2$-condition but $G$ fails it, then $\mathcal{H}_G(\mathcal{M})$ has the Property (V).

Proof. When $G^*$ satisfies the $\Delta_2$-condition while $G$ fails it, $H_G(0, \infty)$ is an $\mathcal{M}$-embedded space [20, p. 105, Example 1.4] (see also [47]). Moreover, since the decreasing rearrangement of any element $L_G(0, \infty)$ vanishes at infinity and

$$H_G(0, \infty) \times \times = L_G(0, \infty)$$

[20, p. 103], it follows that $H_G(0, \infty) \not\supset C_0(0, \infty)$ [23 Proposition 2.4]. Hence, by Theorem 5.2, $\mathcal{H}_G(\mathcal{M})$ is an $\mathcal{M}$-embedded space and therefore, by Theorem 5.3 it has the property (V). □

Remark 5.5. The proof in [33 Theorem 2] works only for finite measure spaces and could not be adjusted for infinite measure spaces. On the other hand, the result of Theorem 5.4 holds for an arbitrary semifinite von Neumann algebra.

6. Non-commutative analogue of Kolmogorov’s compactness criterion in terms of conditional expectations

This section is devoted to the extension of a well-known Kolmogorov’s criterion of compactness (see [30], see also [34 Theorem 11.1, p. 97]). This criterion found non-trivial generalizations in several directions, see, for example [14], [31 Theorem 11.1, p. 97]. Also, in [16 Section 5] relatively compact sets in symmetrically normed spaces characterized completely in terms of sets of uniformly absolutely continuous norms, that is, $\sup_{E \in A} \|e_n x e_n\|_E \to_n 0$ for all mutually disjoint sequences $\{e_n\}_{n \geq 1}$ of projections in $\mathcal{M}$. In particular, if $\mathcal{M}$ is atomic von Neumann algebra such that $\tau(1) < \infty$ and $\mathcal{E}(\mathcal{M})$ is a symmetrically normed space with the order continuous norm, then a bounded set $A \subset \mathcal{E}(\mathcal{M})$ is relatively compact if and only if $A$ is of uniformly (equi-) absolutely continuous norms (see [16 Corollary 5.3]).

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. An increasing sequence $(\mathcal{M}_n)_{n \geq 0}$ of von Neumann subalgebras of $\mathcal{M}$ such that the union $\bigcup_{n \geq 0} \mathcal{M}_n$ is weak* dense in $\mathcal{M}$ is called a filtration of $\mathcal{M}$. Assume that for every $n \geq 0$, the restriction $\tau|_{\mathcal{M}_n}$ is semifinite. Then there exists a map $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ satisfying the following properties:

(i) $\mathcal{E}_n$ is a normal contractive positive projection from $\mathcal{M}$ onto $\mathcal{M}_n$;
(ii) $\mathcal{E}_n(abx) = a\mathcal{E}_n(x)b$ for any $x \in \mathcal{M}$ and $a, b \in \mathcal{M}_n$;
(iii) $\tau \circ \mathcal{E}_n = \tau$.

The map $\mathcal{E}_n$ satisfying above conditions is called the conditional expectation with respect to $\mathcal{M}_n$. For more information we refer to [16, 49]. Since each $\mathcal{E}_n$ preserves the trace, it may be extended to a contractive projection from $\mathcal{L}_p(\mathcal{M})$ onto $\mathcal{L}_p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$.

In this section, we characterize relatively compact sets in separable symmetric spaces of $\tau$-measurable operators associated with a hyperfinite von Neumann algebra from the perspective of conditional expectations. In particular, we obtain relative compactness criterion in separable non-commutative Orlicz spaces. Observe that when $\mathcal{M}$ acts on a separable Hilbert space, the space $\mathcal{E}(\mathcal{M})$ is separable provided that the space $E(I)$ is separable (see [15 Corollary 6.10] and detailed discussion in [11, p.54]).
Theorem 6.1. Let $\langle M, \tau \rangle$ be a hyperfinite non-commutaive probability space and let $(M_n)_{n \geq 0}$ be a filtration which consists of finite dimensional von Neumann algebras. Let $E(M)$ be the corresponding separable symmetric Banach space of $\tau-$measurable operators. If $F \subset E(M)$ is a bounded set, then the following conditions are equivalent

(a) $F$ is relatively compact;
(b) 
\[ \sup_{x \in F} \| x - E_n x \|_{E(M)} \to 0, \quad n \to \infty. \]

Lemma 6.2. Let $E(M)$ be a separable symmetric Banach function space. Let $(M, \tau)$ be a non-commutative probability space and let $(M_n)_{n \geq 0}$ be a filtration. For every $y \in E(M)$, we have
\[ \| y - E_n y \|_{E(M)} \to 0, \quad n \to \infty. \]

Proof. Let $\psi$ be the fundamental function of $E(M)$. If $\| z \|_{L_\infty(M)} = \alpha$ and $\| z \|_{L_1(M)} = \beta$, then $z < \alpha \chi_{(0, \frac{1}{2})}$. Thus,
\[ \| z \|_{E(M)} \leq \| \alpha \chi_{(0, \frac{1}{2})} \|_{E(M)} = \alpha \psi \left( \frac{\beta}{\alpha} \right) = \| z \|_{L_\infty(M)} \psi \left( \frac{\| z \|_{L_1(M)}}{\| z \|_{L_\infty(M)}} \right). \]

Take $u \in L_\infty(M)$. Let $z_n = u - E_n u$ and note that $\| z_n \|_{L_\infty(M)} \leq 2 \| u \|_{L_\infty(M)}$. Thus,
\[ \| z_n \|_{E(M)} \leq \| z_n \|_{L_\infty(M)} \psi \left( \frac{\| z_n \|_{L_1(M)}}{\| z_n \|_{L_\infty(M)}} \right) \leq 2 \| u \|_{L_\infty(M)} \psi \left( \frac{2 \| z_n \|_{L_1(M)}}{2 \| u \|_{L_\infty(M)}} \right). \]

The latter inequality follows from the fact that the mapping $t \to \psi(t)$ is decreasing. Since $\| z_n \|_{L_1(M)} = \| u - E_n u \|_{L_1(M)} \to 0$ (see Theorem 2 in [15]), it follows that $\| z_n \|_{E(M)} \to 0$. In other words,
\[ \| u - E_n u \|_{E(M)} \to 0, \quad n \to \infty \quad u \in L_\infty(M). \]

Now, let $y \in E(M)$ and fix $\epsilon > 0$. By the separability of $E(M)$, one can find $u \in L_\infty(M)$ such that $\| y - u \|_{E(M)} \leq \epsilon$. By triangle inequality, we have
\[ \| y - E_n y \|_{E(M)} \leq \| u - E_n u \|_{E(M)} + \| u - y \|_{E(M)} + \| E_n y - E_n u \|_{E(M)} \leq \| u - E_n u \|_{E(M)} + 2 \| u - y \|_{E(M)}. \]

By the preceding paragraph, we have
\[ \limsup_{n \to \infty} \| y - E_n y \|_{E(M)} \leq 2 \| u - y \|_{E(M)} \leq 2 \epsilon. \]

Since $\epsilon > 0$ is arbitrary, the assertion follows.

Proof of Theorem 6.1. $(a) \implies (b)$. $F$ is relatively compact and, therefore, is totally bounded. Fix $\epsilon > 0$ and choose natural number $m = m(\epsilon)$ and points $(y_k)_{1 \leq k \leq m}$ in $E(M)$ such that
\[ \min_{1 \leq k \leq m} \| x - y_k \|_{E(M)} \leq \epsilon, \quad \text{for all} \quad x \in F. \]

By triangle inequality, we have
\[ \| x - E_n x \|_{E(M)} \leq \| y_k - E_n y_k \|_{E(M)} + \| x - y_k \|_{E(M)} + \| E_n x - E_n y_k \|_{E(M)} \leq \| y_k - E_n y_k \|_{E(M)} + 2 \| x - y_k \|_{E(M)} \quad \text{for all} \quad 1 \leq k \leq m. \]
Hence,
\[
\|x - E_n x\|_{\mathcal{E}(\mathcal{M})} \leq \min_{1 \leq k \leq m} (\|y_k - E_n y_k\|_{\mathcal{E}(\mathcal{M})} + 2 \|x - y_k\|_{\mathcal{E}(\mathcal{M})})
\]
\[
\leq \max_{1 \leq k \leq m} \|y_k - E_n y_k\|_{\mathcal{E}(\mathcal{M})} + 2 \min_{1 \leq k \leq m} \|x - y_k\|_{\mathcal{E}(\mathcal{M})}
\]
\[
\leq \max_{1 \leq k \leq m} \|y_k - E_n y_k\|_{\mathcal{E}(\mathcal{M})} + 2 \epsilon \leq \sum_{1 \leq k \leq m} \|y_k - E_n y_k\|_{\mathcal{E}(\mathcal{M})} + 2 \epsilon.
\]
Thus,
\[
\limsup_{n \to \infty} \sup_{x \in X} \|x - E_n x\|_{\mathcal{E}(\mathcal{M})} \leq \sum_{1 \leq k \leq m} \limsup_{n \to \infty} \|y_k - E_n y_k\|_{\mathcal{E}(\mathcal{M})} + 2 \epsilon.
\]
By Lemma 6.2, we have
\[
\limsup_{n \to \infty} \sup_{x \in F} \|x - E_n x\|_{\mathcal{E}(\mathcal{M})} \leq 2 \epsilon.
\]
Since \(\epsilon > 0\) is arbitrary, the assertion follows.

(b) \(\implies\) (a). Fix \(\epsilon > 0\). Choose \(n = n(\epsilon)\) such that
\[
\|x - E_n x\|_{\mathcal{E}(\mathcal{M})} \leq \epsilon, \quad x \in F.
\]
The set
\[
\{E_n x, \quad x \in F\}
\]
is bounded and finite dimensional since it is a subset of \(\mathcal{M}_n\), which is assumed to be finite dimensional. Thus, it is relatively compact and, therefore, totally bounded. Hence, there exist a natural number \(m = m(\epsilon)\) and points \((y_k)_{1 \leq k \leq m}\) such that
\[
\min_{1 \leq k \leq m} \|E_n x - y_k\|_{\mathcal{E}(\mathcal{M})} \leq \epsilon, \quad x \in F.
\]
By triangle inequality, we have
\[
\min_{1 \leq k \leq m} \|x - y_k\|_{\mathcal{E}(\mathcal{M})} \leq 2 \epsilon, \quad x \in F.
\]
Since \(\epsilon > 0\) is arbitrary, it follows that \(F\) is totally bounded. Therefore, \(F\) is relatively compact. \(\Box\)

**Remark 6.3.** Note that the proof of necessity ((a) \(\implies\) (b)) works without the assumption that filtration is finite dimensional. The latter condition is only used in the proof of sufficiency.

The following corollary is a direct consequence of Theorem 6.1

**Corollary 6.4.** Let \(\mathcal{L}_G(\mathcal{M})\) be a separable Orlicz space. Let \((\mathcal{M}, \tau)\) be a hyperfinite non-commutative probability space and let \((\mathcal{M}_n)_{n \geq 0}\) be a filtration which consists of finite dimensional von Neumann algebras. Let \(F \subset \mathcal{L}_G(\mathcal{M})\) be a bounded set. Then the following conditions are equivalent

(a) \(F\) is relatively compact;

(b) 
\[
\sup_{x \in F} \inf c > 0 : \tau \left( G \left( \frac{|x - E_n x|}{c} \right) \right) \leq 1 \rightarrow 0, \quad n \to \infty.
\]
Appendix

The following lemma is, most probably, well known. However, since we could not find any suitable reference, we include its proof here for the sake of convenience.

**Lemma 7.1.** Let \( \{x_n\}_{n=1}^\infty \) be a sequence of real numbers such that the series \( \sum_{n=1}^\infty |x_n| \) is convergent. Then there exists a sequence of real numbers \( \{y_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} y_n = \infty \) and the series \( \sum_{n=1}^\infty |x_n y_n| \) is convergent.

**Proof.** Let us construct a (strictly) increasing sequence of natural numbers \( \{n_l\}_{l=1}^\infty \) as follows. By the Cauchy’s theorem we can find \( n_1 \in \mathbb{N} \) such that
\[
\sum_{k=n_1}^n |x_k| < 1, \quad \text{for any} \quad n > n_1.
\]
Similarly, we can find \( n_2 > n_1 \) such that
\[
\sum_{k=n_2}^n |x_k| < \frac{1}{2}, \quad \text{for any} \quad n > n_2.
\]
Continuing this procedure we construct the sequence \( \{n_l\}_{l=1}^\infty \) such that \( n_{l+1} > n_l \) for all \( l \in \mathbb{N} \), and
\[
\sum_{k=n_l}^n |x_k| < \frac{1}{2^{l-1}}, \quad \text{for any} \quad n > n_l,
\]
and for any \( l \geq 1 \).

Now we construct a nondecreasing sequence \( \{y_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} y_n = \infty \). Put \( y_n = 1 \) for any \( 1 \leq n \leq n_1 \) and
\[
y_n = l - 1 \quad \text{for any} \quad n_{l-1} < n \leq n_l, \quad l \geq 2.
\]
It is easy to see that \( \{y_n\}_{n=1}^\infty \) is nondecreasing. Moreover, \( \lim_{n \to \infty} y_n = \sup_{n \in \mathbb{N}} y_n = \infty \).

Now we prove that the series \( \sum_{n=1}^\infty |x_n y_n| \) is convergent by using the Cauchy’s theorem.

Let \( \varepsilon > 0 \). Since the series \( \sum_{k=1}^\infty \frac{k}{2^k} \) is convergent we can choose \( l_0 = l_0(\varepsilon) \in \mathbb{N} \) such that \( \sum_{k=l_0}^\infty \frac{k}{2^k} < \varepsilon \).

Let \( n \in \mathbb{N} \) be such that \( n > n_{l_0} \), where \( l_0 \) is defined above. Let \( m > n \), consider the sum
\[
\sum_{k=n}^m |x_k y_k| = \sum_{k=n}^m |x_k| y_k.
\]
Define \( s > l_0 \) by condition \( n_{s-1} < m \leq n_s \). Then,
\[
\sum_{k=n}^m |x_k y_k| \leq \sum_{k=n_{l_0}+1}^{n_s} |x_k| y_k = \sum_{i=l_0}^{s-1} \sum_{k=n_i+1}^{n_{i+1}} |x_k| y_k.
\]
Since \( y_k = i \) for any \( n_i < k \leq n_{i+1} \), we obtain
\[
\sum_{i=l_0}^{s-1} \sum_{k=n_i+1}^{n_{i+1}} |x_k| y_k = \sum_{i=l_0}^{s-1} \sum_{k=n_i+1}^{n_{i+1}} |x_k|.
\]
By the definition of the sequence \( \{n_i\} \) and inequality (6), we have

\[
\sum_{i=l_0}^{s-1} i \sum_{k=n_i+1}^{n_{i+1}} |x_k| \leq \sum_{i=l_0}^{s-1} \frac{i}{2^i-1} \leq \sum_{i=l_0}^{\infty} \frac{i}{2^i-1} \leq 2\varepsilon.
\]

Therefore, for any \( \varepsilon > 0 \) there exists \( n_0 = n_0(\varepsilon) = n_{l_0} \) such that for any \( n > n_0 \) and any \( m > n \)

\[
\sum_{k=n}^{m} |x_k y_k| \leq 2\varepsilon.
\]

\( \square \)

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References

[1] J. Alexopoulos, *De La Vallée Poussin’s theorem and weakly compact sets in Orlicz spaces*, Quaestiones Math. (1994), 231–248.

[2] E. Alfsen, E. Effros, *Structure in real Banach spaces. Part I and Part II*, Ann. of Math. 96 (1972), 98–173.

[3] T. Ando, *Weakly compact sets in Orlicz spaces*, Canad.J.Math. 14 (1962), 170–176.

[4] S.V. Astashkin, *Rearrangement invariant spaces satisfying Dunford-Pettis criterion of weak compactness*, Contemporary Mathematics, 733, (2019), 45–59.

[5] D. Barcenas, C.E. Finol, *On Vector Measures, Uniform Integrability and Orlicz Spaces*, Operator theory: Advances and Applications, 201, (2009) 51–57.

[6] T.N. Bekjan, *Φ−inequalities of noncommutative martingales*, Rocky Mountain J. Math, 36 (2006), 401–412.

[7] T.N. Bekjan, Z.Q. Chen, *Interpolation of Φ−moment inequalities of noncommutative martingales*, Probab. Theory Related Fields, 152 (2012), 179–206.

[8] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, 129, Academic Press, 1988.

[9] K.M. Chong, *Doubly stochastic operators and rearrangement theorems*, J. Math. Anal. Appl., 56 (1976), 309–316.

[10] K.M. Chong, *Spectral orders, uniform integrability and Lebesgue’s dominated convergence theorem*, Trans. Amer. Math. Soc., 191 (1974), 395–404.

[11] M. M. Czerwińska, A. Kamińska, *Geometric properties of noncommutative symmetric spaces of measurable operators and unitary matrix ideals*, Comment. Math. 57 (2017), no. 1, 45–122.

[12] J. Diestel, J. J., Jr. Uhl, *Vector measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[13] P.G. Dodds, T.K. Dodds, B. de Pagter, *Noncommutative Köthe duality*, Trans. Amer. Math. Soc., 339 (1993), 717–750.

[14] P.G. Dodds, B. de Pagter, *Normed Köthe spaces: A non-commutative viewpoint*, Indag. Math., 25 (2012), 206–249.

[15] P.G. Dodds, B. de Pagter, *Properties (u) and (V∗) of Pelczynski in symmetric spaces of τ-measurable operators*, Positivity 15 (2011), no. 4, 571–594.
[16] P.G. Dodds, B. de Pagter, F. Sukochev, *Sets of uniformly absolutely continuous norm in symmetric spaces of measurable operators*, Trans. Amer. Math. Soc. **368**(6) (2016), 4315–4355.

[17] P.G. Dodds, B. de Pagter, F. Sukochev, *Theory of Noncommutative integration*, Unpublished manuscript.

[18] P.G. Dodds, F. Sukochev, G. Schlüchtermann, *Weak compactness criteria in symmetric spaces of measurable operators*, Math. Proc. Camb. Phil. Soc. (2001), **131**, 363–384.

[19] N. Dunford, B.J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.

[20] P. Harmand, D. Werner, W. Werner, *M-ideals in Banach spaces and Banach algebras*, Springer-Verlag, Berlin/New York, 1993.

[21] F. Hiai, *Majorization and stochastic maps in von Neumann algebras*, J. Math. Anal. Appl. **127** (1987), 18–48.

[22] F. Hiai, Y. Nakamura, *Majorizations for generalized s-numbers in semifinite von Neumann algebras*, Math. Z. **195** (1987), no. 1, 17–27.

[23] J. Huang, G. Levitina, F. Sukochev, *M-embedded symmetric operator spaces and the derivation problem*, Math. Proc. Camb. Phil. Soc. (in press).

[24] J. Huang, F. Sukochev, D. Zanin, *Logarithmic submajorization and order-preserving isometries*, J. Funct. Anal. **278**:4 (2020), 108352.

[25] T. Fack, H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math., **123**(2) (1986), 269–300.

[26] G. Godefroy, P. Saab, *Quelques espaces de Banach ayant les propriétés (V) ou (V∗) de A. Pelczyński*, C. R. Acad. Sci. Paris Ser. A. **303** (1986), 503–506.

[27] G. Godefroy, P. Saab, *Weakly unconditionally convergent series in M-ideals*, Math. Scand. **64** (1989), 307–318.

[28] N. Kalton, F. Sukochev, *Symmetric norms and spaces of operators*, J. Reine Angew. Math. **621** (2008), 81–121.

[29] A. Kamińska, M. Mastylo, *The Schur and (weak) Dunford-Pettis properties in Banach Lattices*, J. Austral. Math. Soc. **73** (2002), 251–278.

[30] A.N. Kolmogorov, *Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel*, Nachr. Ges. Wiss. Göttingen, I (1931), 60–63.

[31] M.A. Krasnosel’skii, Ya.B. Rutickii, *Convex functions and Orlicz spaces*, translated from russian by Leo F.Boron, Noorhoff Ltd., Groningen, 1961.

[32] S. Krein, Y. Petunin, and E. Semenov, *Interpolation of linear operators*, Amer. Math. Soc., Providence, R.I., 1982.

[33] P. Lefèvre, D. Li, H. Queffélec, L. Rodriguez-Piazza, *Weak compactness and Orlicz spaces*, Colloquium Math., 2008, **112** (1), 23–32.

[34] D. Leung, *Weak∗ convergence in higher duals of Orlicz spaces*, Proc. Amer. Math. Soc. **103** (1988), 797–800.

[35] S. Lord, F. Sukochev, D. Zanin, *Singular traces. Theory and applications*, De Gruyter Studies in Mathematics, **46**, De Gruyter, Berlin, 2013.

[36] A. Marshall, I. Olkin, B. Arnold, *Inequalities: theory of majorization and its applications*, second edition, Springer series in statistics, Springer, New York, 2011.

[37] P. Meyer, *Probability and Potentials*, Blaisdell Publishing Co., 1966.

[38] W. Orlicz, *Über Raüme (Lϕ, Lϕ∗)*, Bull. Acad. Polon. Sci. Ser. A (1936), 93–107.

[39] G. Pisier, Q. Xu, *Noncommutative Lp-spaces*, Handbook of the geometry of Banach spaces, **2** (2003), 1459–1517.

[40] M. Nowak, *A characterization of the Mackey topology τ(Lp, Lp∗) on Orlicz spaces*, Bulletin of the Polish Academy of Sciences, Mathematics, **34**:9-10, (1986), 577–583.

[41] M. Takaizumi, *On the compactness of the function-set by the convergence in mean of general type*, Studia Math. **5**, (1935), 141–150.

[42] F. Sukochev, D. Zanin, *Majorization and stochastic maps in von Neumann algebras*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 537–540.
[46] Umegaki H. (1954). Conditional expectation in an operator algebra. *Tohoku Math. J. (2)* 6, 177–181.

[47] D. Werner, *New classes of Banach spaces which are M-ideals in their biduals*, Math. Proc. Camb. Phil. Soc. 111 (1992), 337–354.

[48] H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Natl. Acad. Sci. USA. 35 (1949), 408–411.

[49] Q. Xu, *Noncommutative Lp-spaces and martingale inequalities*. Book manuscript. (2007).

[50] C. Zhang, Y.L. Hou, *Convergence of weighted averages of martingales*, Sci China Math, 56, (2013), 823–830.

[51] C. Zhang, Y.L. Hou, *Convergence of weighted averages of martingales in noncommutative Banach function spaces*, Acta Mathematica Scientia, 32B(2), (2012), 735–744.

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