Interacting vector fields in Relativity without Relativity

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Abstract

Barbour, Foster and Ó Murchadha have recently developed a new framework, called here the \textit{3-space approach}, for the formulation of classical bosonic dynamics. Neither time nor a locally Minkowskian structure of spacetime are presupposed. Both arise as emergent features of the world from geodesic-type dynamics on a space of 3-dimensional metric–matter configurations. In fact gravity, the universal light cone and Abelian gauge theory minimally coupled to gravity all arise naturally through a single common mechanism. It yields relativity – and more – without presupposing relativity. This paper completes the recovery of the presently known bosonic sector within the 3-space approach. We show, for a rather general ansatz, that 3-vector fields can interact among themselves only as Yang–Mills fields minimally coupled to gravity.

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1 Introduction

This paper develops further the 3-space approach to relativity and gauge theory introduced by Barbour, Foster and Ó Murchadha [1]. In Sec. 2, we recall briefly the principles of Dirac’s generalized Hamiltonian dynamics [2], on which the 3-space approach is based. We then give Arnowitt, Deser and Misner’s (ADM) [3] (3 + 1) reformulation of 4-dimensional general relativity (GR) as an example. This recasts GR as geometrodynamics [4]: the dynamics of Riemannian 3-geometries. The 3-space approach proceeds in the opposite direction, recovering GR coupled to the matter fields of nature from 3-dimensional dynamical principles alone. Some work has been done on this by Hojman, Kuchar and Teitelboim (HKT) [5] and by Teitelboim [6], but they presupposed that the dynamics unfolds in spacetime. This is unnecessary.

To set the scene, Sec. 3 outlines the 3-space approach, which consists largely of a systematic examination of Baierlein–Sharp–Wheeler (BSW) [7] geodesic-type actions. We identify the two key principles of such actions: best matching (a universal method to implement 3-dimensional diffeomorphism invariance) and a local square root (taken at each space point before integration over 3-space). These two principles replace Einstein’s assumptions of 4-dimensional diffeomorphism invariance (general covariance) and a locally Minkowskian structure of spacetime. The 3-space principles have the following consequences.

First, they essentially single out pure GR as the theory that arises from considering 3-geometries alone. Second, if a scalar field on the 3-geometries is included, they force it to obey the same light-cone structure as gravity. Both minimal and Brans–Dicke couplings are allowed [1]. Third, and even more remarkably, if a 3-vector field is included, it must not only respect the gravitational light-cone but also be massless and satisfy the Maxwell equations of electromagnetism minimally-coupled to gravity. Finally they show in a forthcoming paper that such a 3-vector field can couple to scalar fields only through Abelian U(1) gauge theory. The new approach leads to these sharp predictions because it uses a spare ontology (3-space replaces spacetime) and employs Dirac’s powerful generalized Hamiltonian dynamics [2] to construct geodesic-type actions on the arena of configuration space, not spacetime. We note that quantum mechanics also unfolds on configuration space.

In Sec. 4, we extend the existing 3-space results to 3-vector fields allowed to interact amongst themselves. We find that Yang–Mills theory [8] minimally coupled to GR is the only possibility allowed for quite a general ansatz for the 3-vector fields’ potential. More precisely, the 3-vector fields must again respect the gravitational light-cone, be fundamentally massless, and have a Yang–Mills type mutual interaction. Our current formalism can not predict how many vector fields there are in nature, nor what their gauge groups are.

In Sec. 5 we conclude that, within the bosonic sector, the 3-space approach yields the key features of the observed world. Gravity, the universal light cone, and gauge theory all arise in essentially the same manner through the single mechanism of consistent Dirac-type constraint propagation applied to the interplay of best matching with the local square root. We finally consider whether classical topological terms can be accommodated in our formalism. These would usually play a part in the interpretation of quantum chromodynamics (QCD) [3, 10, 11, 12].
2 Hamiltonian Dynamics and General Relativity

Let $L(q, \dot{q})$ be a Lagrangian provisionally adopted for some general theory of tensor fields $q(x, \lambda)$ on a Lorentzian manifold $\mathcal{M}$. If not all the conjugate momenta $p = \frac{\partial L}{\partial \dot{q}}$ can be inverted to give the $\dot{q}$ in terms of the $p$, then the theory has primary constraints $C_{\Pi}(q, p) = 0$ solely by virtue of the form of $L$. As Dirac noted, in such a case a theory described by a Hamiltonian $H(q, p)$ could just as well be described by a Hamiltonian

$$H_{\text{Total}} = H + N_{\Pi}C_{\Pi}$$

for arbitrary functions $N_{\Pi}$. Moreover, one needs to check that the primary constraints and any further secondary constraints $C_{\Gamma}(q, p)$ (obtained as true variational equations $C_{\Gamma} = 0$) are propagated by the evolution equations. If they are, then the constraint algebra indexed by $\Delta_{(1)} = \Pi \cup \Gamma$ closes, and a classically-consistent theory is obtained. This happens when $\dot{C}_{\Delta_{(1)}}$ vanishes either due to the Euler–Lagrange equations alone or additionally due to the vanishing of $C_{\Delta_{(1)}}$, which is Dirac’s notion of weak vanishing, denoted by $\dot{C}_{\Delta_{(1)}} \approx 0$.

If $C_{\Delta_{(1)}}$ does not vanish weakly, then it must contain further independently-vanishing expressions $C_{\Sigma_{(1)}}(q, p)$ in order for the theory to be consistent. One must then enlarge the indexing set to $\Delta_{(2)} = \Delta_{(1)} \cup \Sigma_{(1)}$ and see if $\dot{C}_{\Delta_{(2)}} \approx 0$. In principle, this becomes an iterative process by which one may construct a full constraint algebra indexed by $\Delta_{(\text{final})} = \Pi \cup \Gamma$ by successive enlargements $\Delta_{(i+1)} = \Delta_{(i)} \cup \Sigma_{(i)}$. In practice, however, the process cannot continue for many steps since $\#\Delta_{(i+1)} > \#\Delta_{(i)}$, $\#\Theta$ is a small number, and we need the true number of degrees of freedom to satisfy $\#\Theta - \#\Delta_{(\text{final})} > 0$ to have any nontrivial theory. It should be emphasized that there is no guarantee that a given Lagrangian will give rise to any consistent theory.

In the case of 4-dimensional GR, $H$ is zero, but the $(3+1)$ ADM split yields the gravitational case of (2), $H_{\text{Total}} = NH + N^iH_i \approx 0$, where $H$ is the Hamiltonian constraint, $H_i$ is the momentum constraint and the arbitrary functions $N$ and $N^i$ are the lapse and the shift. The Bianchi identities then ensure that $H_i \approx 0$ and $H \approx 0$ so that $H$ and $H_i$ form a closed constraint algebra: the Dirac algebra. The first propagation corresponds to the invariance of the ADM action under 3-diffeomorphisms. The second propagation corresponds to a remarkable hidden symmetry of GR, foliation invariance, which is the invariance under local reparametrization of the time label. The $(3+1)$ split can also be done in the presence of matter fields indexed by $\Psi$. We will denote the Hamiltonian and momentum constraints obtained in this case by $\Psi H$ and $\Psi H_i$.

Whereas ADM decomposed 4-dimensional spacetime in a $(3+1)$ split, the work of HKT and of Teitelboim goes in the opposite direction. They reconstruct 4-dimensional GR with matter fields from geometrodynamics by requiring that the constraint algebra of $\Psi H$ and $\Psi H_i$ closes to reproduce the Dirac algebra.

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1 In this paper, $x$ is a 3-dimensional spatial argument, $\lambda$ is a time label and $\dot{\lambda}$ is denoted by a dot. We use capital Greek letters as indexing sets for the fields; the number of fields in the set indexed by $\Theta$ is denoted by $\#\Theta$. We use lower-case Greek letters for spacetime indices and lower-case Latin letters for spatial indices. We use capital Latin letters for internal indices; no significance is attached to whether these are raised or lowered but their order will be important. We use round brackets to denote symmetrization of indices, and square brackets to denote antisymmetrization. Indices unaffected by the (anti)symmetrization are set between vertical lines.
3 Relativity without Relativity

Barbour, Foster and Ó Murchadha (BF ´O)[1] have recently used Dirac’s general procedure under the much weaker assumption that the constraint algebra merely closes. The requirement that the Dirac algebra be reproduced imports the foliability of 4-dimensional spacetime. BF ´O showed that this is largely unnecessary if one wishes to derive GR from 3-dimensional principles. They use truly 3-dimensional principles alone, so we can call it the 3-space approach. It gives new insights into the origin of both special and general relativity, and furthermore, of Abelian gauge theory.

Their point of departure was a paper of Baierlein, Sharp and Wheeler [7], in which it is shown that the Einstein–Hilbert action

\[ S_{BSW} = \int d\lambda \int d^3x \sqrt{g} \sqrt{R} \sqrt{T_g}, \]

(2)

where \( g \) is the determinant of the spatial 3-metric \( g_{ij} \) induced on spacelike hypersurfaces by the 4-metric \( g^{(4)}_{\mu\nu} \), and \( R \) is the 3-dimensional Ricci scalar formed from \( g_{ij} \). In principle this action defines a measure on the quotient space

\[ \{\text{Superspace}\} = \{\text{Riem}\}/\{\text{Diffeomorphisms}\}, \]

(3)

where \{Riem\} is the space of all Riemannian 3-geometries on some given topology; we will work throughout this paper with compact topologies without boundary. The label \( \lambda \) parametrizes some chosen curve of 3-metrics, which is a primary object in superspace. The gravitational kinetic term \( T_g \) is given by

\[ T_g = G^{abcd}(\dot{g}_{ab} - \nabla_a \xi_b - \nabla_b \xi_a)(\dot{g}_{cd} - \nabla_c \xi_d - \nabla_d \xi_c), \]

(4)

where \( G^{abcd} = g^{ac}g^{bd} - g^{ab}g^{cd} \) is the DeWitt supermetric [14], \( \xi_a = g_{ab}N^b \) and \( \nabla_a \) (or \( ;a \)) denotes the covariant derivative with respect to \( g_{ij} \).

To compensate for the possible coordinate change in going between neighbouring 3-geometries, correction terms have been added to each of the bare velocities \( \dot{g}_{ab} \) in \( T_g \). One can think of this as Bertotti and Barbour’s method for achieving 3-diffeomorphism invariance [10, 1]. It is appropriately called best matching because the variation with respect to \( \xi_i \) can be seen as implementing a best matching of two infinitesimally-differing 3-dimensional configurations on a compact manifold; the aim of this is to bring the two configurations as close as possible to congruence and then define the residual difference between them as a measure on superspace. Three-diffeomorphism invariance is achieved by the prescription, valid and uniquely defined for any bosonic field \( B \),

\[ \text{bare velocity} \; \dot{B} \rightarrow \text{best-matched velocity} \; \dot{B} - \mathcal{L}_\xi B, \]

(5)

where \( \mathcal{L}_\xi \) is the Lie derivative with respect to the 3-diffeomorphism-generating auxiliary field \( \xi_i \). Three-diffeomorphism invariance is an example of a gauge symmetry leading to a constraint that is homogeneously linear in the momenta. Whenever a theory has this form of gauge symmetry, some corresponding form of best matching occurs. Because 3-diffeomorphisms must, in a Machian

\[ ^2 \xi_i \text{ is formally a velocity, so } T_g \text{ is homogeneously quadratic in its velocities. The interpretation of } \xi_i \text{ and its importance in gauge theory are explained in [10].} \]
framework, be applied to all fields, both metric and material, the best matching that implements them has far-reaching universal consequences, as we shall see.

The BSW action resembles Jacobi’s action principle for the (timeless) dynamical orbit of a Newtonian N-body system in its 3N-dimensional configuration space, but differs from it in that the latter contains a single square root, whereas the former has one square root at each space point, after which these are integrated over all space. We call the latter choice the local square root. The presence of the square root means that the Lagrangian is homogeneous of degree 1 in the velocities, so that the canonical momenta must be homogeneous of degree 0. As Dirac noted, such canonical momenta must satisfy at least one primary constraint as an identity.

For the BSW Lagrangian, the canonical momenta (defined at each space point) are

\[ p_{ij} = \frac{\partial L}{\partial \dot{g}_{ij}} = \sqrt{gRT} \left( g^{ic} g^{jd} - g^{ij} g^{cd} \right) (\dot{g}_{cd} - \nabla_c \xi_d - \nabla_d \xi_c) \]

and the primary constraint that holds at each space point is

\[ g\mathcal{H} \equiv gR - p_{ij} p_{ij} + \frac{1}{2} p^2 = 0, \]

where \( p \) denotes the trace of \( p_{ij} \). In addition, variation of the BSW action with respect to \( \xi_i \) leads to the secondary momentum constraint

\[ \frac{1}{2} \sqrt{g} \mathcal{H} \equiv p_{ij;i} = 0. \]

(4) and (5) are respectively the Hamiltonian and momentum constraints of GR. The corresponding Euler–Lagrange equations ensure that these propagate, so the constraint algebra is closed. At first glance, one would expect the BSW action to be invariant only with respect to the global reparametrization \( \lambda \rightarrow \lambda'(\lambda) \), for \( \lambda' \) a monotonic arbitrary function of \( \lambda \). But in fact the action is invariant under the far more general local transformation

\[ \lambda \rightarrow \lambda'(\lambda), \quad g_{ij}(x, \lambda) \rightarrow g_{ij}(x, \lambda'), \quad \xi_i(x, \lambda) \rightarrow d\lambda' \frac{d\lambda}{d\lambda} \xi_i(x, \lambda). \]

This remarkable invariance does not hold for the generalization of the BSW action that BFÓ started with:

\[ S_{\text{BFO}} = \int d\lambda \int d^3x \sqrt{g} \sqrt{\Lambda + P(x, \lambda)} \sqrt{T_W}, \]

where \( \Lambda \) is an arbitrary constant, the potential \( P \) is some arbitrary scalar formed from \( g_{ij} \) and its spatial derivatives up to a given order, and \( T_W \) is the same as \( T_g \) except that it contains a generalized supermetric, \( G_{abcd}^{\text{fg}} = g^{ac} g^{bd} - W g^{ab} g^{cd} \). Their first result is as follows. The action (10) is defined solely in terms of 3-dimensional concepts, and associates an action with curves on the space \{Riem\} × \Xi, where \( \Xi \) is the vector space to which \( \xi_i \) belongs. Then, the presence of the local square root in (11) gives the primary constraint

\[ g\mathcal{H} \equiv g(P + \Lambda) - p_{ij} p_{ij} + \frac{2W}{2(3W - 1)} p^2 = 0 \]

and variation with respect to \( \xi_i \) leads to an unchanged secondary constraint, (8). The latter can be regarded as a differential equation for \( \xi_i \) (which is contained in \( p_{ij} \)). If this can be solved (for the
issues, as yet not fully resolved, that are involved, see the papers [23], the action will depend only on the curve in superspace. This follows from the constraints being free of $\xi_i$, and the momentum constraint reducing the number of degrees of freedom to 3, which is the number of degrees of freedom per space point in a 3-geometry.

The question posed in [1] – and answered in the affirmative – is whether GR can be derived solely from 3-dimensional arguments, that is, without any recourse to arguments related to 4-dimensional general covariance. The approach succeeds because of the need to propagate the quadratic constraint $H$ acts as a powerful filter of viable theories, which are already strongly restricted by the universal linear 3-diffeomorphism constraint 3. Up to second-order spatial derivatives of $g_{ij}$, this works for

$$S_{BFO} = \int d\lambda \int d^3x \sqrt{g} \sqrt{\Lambda + \epsilon R \sqrt{T(W=1)}},$$  

where $\epsilon \in \{-1, 0, 1\}$, and the subscript of $T$ indicates that the a priori free parameter $W$ must take the DeWitt value, 1. The cases $\epsilon = 1$ and $\epsilon = -1$ correspond to Lorentzian and Euclidean GR respectively. The case $\epsilon = 0$ is called strong gravity [19], because it is the limit as Newton’s gravitational constant, $\kappa$, goes to infinity of the other two cases. This is a theory which is 4-dimensionally generally covariant in the sense of having four constraints per 3-space point but cannot, unlike the other two cases, be represented by tensorial equations on a 4-dimensional spacetime manifold. This signature freedom $\epsilon$ and the freedom to have a $\Lambda$ (which we identify as a cosmological constant) is what we mean in the introduction by GR being essentially singled out. Furthermore, all the higher-derivative corrections considered in [1] were found not to give a propagating $H$. However, a conformal generalization of the above work (which has not yet be fully elaborated) is also possible, giving another theory which has no spacetime manifold interpretation [18]. In fact, we anticipate that the full significance of the 3-space approach will not become apparent until the full generalizations to conformal superspace, which will result in a fully scale-invariant theory, and the fermionic sector have been completed.

Barbour, Foster and O’Murchadha included a scalar field $\phi$ by considering the action

$$S_{BSW} = \int d\lambda \int d^3x \sqrt{g} \sqrt{R + U_\phi \sqrt{T_g + T_\phi}}$$  

(13)

with the gravitationally best-matched scalar kinetic term $T_\phi = (\dot{\phi} - \xi \phi)^2$ and the potential ansatz $U_\phi = - (C/4) g^{ab} \phi, a \phi, b + \sum (n) A(n) \phi^n$. Then the square local root gives as an identity the primary constraint

$$g^{\phi} H \equiv g(R + U_\phi) - p^i p_i + \frac{1}{2} p^2 - \pi^2 = 0,$$  

(14)

where $\pi$ is the momentum conjugate to $\phi$. Variation with respect to $\xi_i$ gives the momentum constraint

$$\frac{1}{2} \sqrt{g} \phi^H^i = p^i - \frac{1}{2} \pi \phi^i = 0.$$  

(15)

The constraint $\phi H$ contains the canonical propagation speed $C$ of the scalar field. A priori, $C \neq 1$, which means there is no reason for the scalar field to obey the same light-cone as gravity. However, imposing $\phi^H \approx 0$ gives a putative secondary constraint

$$\frac{(1 - C)}{N} (N^2 \pi \phi, i)^i = 0$$  

(16)

3Throughout this paper, the momentum constraints are automatically propagated because the action has been deliberately constructed to be invariant under ($\lambda$-dependent) 3-diffeomorphisms.
and the theory has just one scalar degree of freedom, so if the cofactor of \((1 - C)\) were zero, the scalar dynamics would be trivial. Thus one has derived that \(C = 1\): scalar fields must obey the gravitational light-cone. Notice also that this scheme gives minimal coupling of the scalar field to gravity. However, there is one other possibility because the most general kinetic term includes also a metric–scalar cross-term. This gives Brans–Dicke theory \([1]\).

For our use in Sec. 4, we now sharpen up the procedure used by BFÓ on inclusion of a single vector field. They considered the action

\[
S_{BSW} = \int d\lambda \int d^3x \sqrt{g} \sqrt{R + U_{A} \sqrt{T_A + T_L}},
\]

for \(T_A = g^{ab}(\dot{A}_a - \xi A_a)(\dot{A}_b - \xi A_b)\) the quadratic gravitationally best-matched kinetic term of \(A_a\) \(^4\) and the potential ansatz \(U_{A} = C_1 A_{a:b} A^{b:a} + C_2 A_{a:b} A^{a:b} + C_3 A_{a} A_{b} A^{b:a} + \sum_{(k)} B_{(k)} (A_{a} A^{a})^{k}\). Then, the local square root gives as an identity the primary constraint

\[
g^{A \xi}H \equiv g(R + U_{A}) - p^{ij} p_{ij} + \frac{1}{2} p^2 - \pi^a \pi_a = 0,
\]

where \(\pi^a\) is the momentum conjugate to \(A_a\). Variation with respect to \(\xi_i\) gives the momentum constraint

\[
\frac{1}{2} \sqrt{g} A^{ij} \dot{h} = p^{ij} - \frac{1}{2} (\pi^c (A_c : j - A_j ; c) - \pi^c ; c A^j) = 0.
\]

Then, imposition of \(A^j \dot{h} \approx 0\) gives rise to

\[
\frac{4C_1 + 1}{N} (N^2 \pi^a A_a)_{,b} + \frac{4C_2 - 1}{N} (N^2 \pi^a A^a)_{,b} + \frac{4C_3}{N} (N^2 \pi^b A^{a})_{,b} - \frac{1}{N} (N^2 \pi^a_a A^b)_{,b} = 0.
\]

This gives a nontrivial theory only for \(C_1 = -C_2 = -1/4, C_3 = 0\) and if there is a secondary constraint \(\bar{G} = \pi_a^{,a}\). The conditions on the \(C\)’s mean that the gravitational light-cone is obeyed, and furthermore that the derivative terms in \(U_{A}\) are \(-(\text{curl} A)^2/4\). One then requires that \(\bar{G} \approx 0\), which gives rise to \(2\sqrt{g}(N \Sigma_{(k)} B_{(k)} (A_{a} A^{a})^{(k-1)} A^j)_{,i} = 0\). This forces the \(B_{(k)}\) to be zero. In particular, \(B_{(1)} = 0\) means that the vector field must be massless. The working for these two steps is a subcase of that in Sec. 4, so the details of the calculation are implicitly contained there.

Now, the form of \(U_{A}\) allowed is invariant under the gauge transformation \(A_a \rightarrow A_a + \nabla_a \Lambda\), so we are dealing with a gauge theory. Thus, if we introduce an auxiliary variable \(\Phi\) into \(T_A\) such that variation with respect to it encodes \(\bar{G}\), then we should do so according to the best matching corresponding to this gauge symmetry. This uniquely fixes the form of \(T_A(A, \Phi)\) to be

\[
T_A = (\dot{A}_a - \xi A_a - \Phi, a)(\dot{A}^a - \xi A^a - \Phi^a).
\]

Thus, if one identifies \(\Phi\) as \(A_0\), this derivation forces \(A_0 = [A_0, A_a]\) to obey Maxwell’s equations minimally-coupled to gravity \([13]\). Moreover, as noted by Giulini \([24]\) and reported in \([1]\), the massive vector field does not fit in the conceptual scheme of the 3-space approach although it is a generally covariant theory. We see that the 3-space approach does not yield all generally covariant theories. But \([1]\) and the present paper show that it does at least yield the bosonic fields hitherto observed in nature.

\(^4\)The Lie derivative of \(A_a\) with respect to \(\xi\) is \(\mathcal{L}_\xi A_a = A_a^{\,\xi, c} \xi_c + A_c \xi_a^{\, c}\).
Finally, on attempting to couple $A_a$ to scalar fields by the inclusion of interaction terms, BFÓ have shown similarly that demanding the propagation of $A^a\phi H$, and of any secondary constraints arising from it, leads to $U(1)$ gauge theory. We have thus a chain of successively more sophisticated theories, each arising from its predecessor by iteration of constraint propagation consistency. This not only prises open the door to classical physics. It shows one a way to derive it. Thus, Dirac’s work, applied to best matching theories with local square roots in their actions, leads to a striking alternative to Einstein and Minkowski’s 4-dimensional foundation of physics. The above outline of ‘Relativity without Relativity’ will make the remainder of this paper into an almost algorithmic formality, from which Yang–Mills gauge theory will emerge from allowing a general collection of 3-vector fields to interact with each other.

4 $K$ Interacting Vector Fields

We consider a BSW-type action containing the a priori unrestricted vector fields $A^I_a$ ($I = 1$ to $K$),

$$S_{BSW_{A^I}} = \int d\lambda \int d^3x\sqrt{g}(g_{ij} \dot{g}^{ij} + A^I_i \dot{A}^I_i - \mathcal{L}, N, N') = \int d\lambda \int d^3x\sqrt{g}(R + U_{A^I}\sqrt{T_k + T_{A^I}}).$$

We use the most general homogeneous quadratic best-matched kinetic term $T_{A^I}$, and a general ansatz for the potential term $U_{A^I}$. We could have constructed these using the inverse 3-metric $g^{ab}$ as the only possible means of contracting spatial indices. But, for greater generality, we have also introduced the totally antisymmetric tensor density $\epsilon^{abc}$ for this purpose.

We note that no kinetic cross-term $\dot{g}_{ab} \dot{A}^I_a$ is possible. This is because the only way to contract 3 spatial indices is to use $\epsilon^{abc}$, and $\dot{g}_{ab}$ is symmetric. Then $T_{A^I}$ is unambiguously

$$T_{A^I} = P_{IJ} g^{ad}(\dot{A}^I_a - \mathcal{L}_{\xi} A^I_a)(\dot{A}^J_d - \mathcal{L}_{\xi} A^J_d),$$

for $P_{IJ}$ without loss of generality a symmetric constant matrix. We will assume that $P_{IJ}$ is positive-definite so that the quantum theory of $A^I_a$ has a well-behaved inner product. In this case, we can take $P_{IJ} = \delta_{IJ}$ by rescaling the vector fields.

We consider the most general $U_{A^I}$ up to first derivatives of $A_{Ia}$, and up to four spatial index contractions. We note that the lattice is equivalent to the necessary naive power-counting requirement for the renormalizability of any emergent four-dimensional quantum field theory for $A_{Ia}$. Then $U_{A^I}$ has the form

$$U_{A^I} = O_{IK} C^{abcd} A^I_{a} A^K_{c} + B^I_{JK} C^{abcd} A^J_{a} A^J_{b} A^K_{c} + I_{IKLM} C^{abcd} A^I_{a} A^J_{b} A^J_{c} A^K_{d} + \frac{1}{\sqrt{g}} \epsilon^{abc} (D_{IK} A^I_{a} A^K_{c} + E_{IK} A^I_{a} A^K_{b} + F_{IJK} A^I_{a} A^J_{b} A^K_{c} + M_{IK} g^{ab} A^I_{a} A^I_{b} A^K_{c},$$

where $C^{abcd} = C_1 g^{ac} g^{bd} + C_2 g^{ad} g^{bc} + C_3 g^{ab} g^{cd}$ is a generalized supermetric, and similarly for $\tilde{C}$ with distinct coefficients. $O_{IK}, B_{IJK}, I_{IKLM}, D_{IK}, E_{IJK}, F_I, M_{IK}$ are constant arbitrary arrays. Without loss of generality $O_{IK}, M_{IK}$ are symmetric and $E_{IJK}$ is totally antisymmetric.

Defining $2N = (T_k + T_{A^I})/(R + U_{A^I})$, the conjugate momenta are given by

$$p^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} = \frac{\sqrt{g}}{2N} (g^{ij} g^{cd} - g^{ij} g^{cd}) (\dot{g}_{cd} - \nabla_c \xi_d - \nabla_d \xi_c),$$

$$\pi^i_j = \frac{\partial \mathcal{L}}{\partial \dot{A}^I_i} = \frac{\sqrt{g}}{2N} (\dot{A}^I_i - \mathcal{L}_{\xi} A^I_i),$$

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which can be inverted to give expressions for $\dot{g}_{ij}$ and $\dot{A}_i^I$. The local square root gives as an identity the primary Hamiltonian constraint,

$$g^{Ij} \mathcal{H} = g(R + U_{A_I}) - p^{ij} p_{ij} + \frac{1}{2} p^2 - \pi^i_1 \pi^i_1 = 0. \quad (27)$$

We get the secondary momentum constraint by varying with respect to $\xi$:

$$\frac{1}{2} \sqrt{g} g^{Ij} \mathcal{H}^j = p^{ij} \pi_1 + \frac{1}{2} (\pi^I_1(A_{Ic}^j - A_{Ic}^{j:c}) - \pi^c_1 A^I_1) = 0. \quad (28)$$

The Euler–Lagrange equations for $g_{ij}$ and $A_i^I$ are

$$\frac{\partial p_{ij}}{\partial \xi} = \frac{\partial}{\partial g_{ij}} = \sqrt{g} N (g^{ij} R - R^{ij}) - \frac{2N}{\sqrt{g}} \left( p^{im} p_{mj} - \frac{1}{2} p^{ij} p \right) + \sqrt{g} (Ng^{ij} U_{A_I} + N;ij - g^{ij} \nabla^2 N)$$

$$- \sqrt{g} N \pi^I_1 \pi^I_1^{ij} + \xi \pi^{ij} + \sqrt{g} O^{IK} (N(2A_{I(b)d}A_{K}(j)C^{bd}(i)) - A^I_{Kbd}C^{bd}(ij))$$

$$- \sqrt{g} O^{IJK} \left( C_1 (A_{Ia}^{Ia} A_{K}^{Ia} + A_{Ia}^{Ia} A_{K}^{Ia}(j)) + C_2 (A_{Ia}^{Ia} A_{K}^{Ia} + A_{Ia}^{Ia} A_{Ia}^{Ia}(j)) \right)$$

$$+ B^I_{JK} \sqrt{g} \frac{\pi^I_1}{2} \left( \left( C_1 + C_2 \right) (N(A_{I}^{(i)} A_{Ib}^{(j)}) A_{Kb}^{Kb} + A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb}) - A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb}) \right)$$

$$\left. + C_3 (2N A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb}) - g^{ij} (N A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb}) \right)$$

$$+ N \sqrt{g} I^{KLM} \left( C_1 A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb} A_{Ib}^{Kb} + A_{I}^{(i)} A_{Ib}^{(j)} A_{Kb}^{Kb}) \right)$$

$$- \sqrt{g} F^{I} \left( N A_{I}^{(i)} - (N A_{I}^{(i)})^j + \frac{1}{2} g^{ij} (N A_{I}^{(i)}) \right)$$

$$- N \sqrt{g} M^{JK} A_{I}^{(i)} A_{I}^{(j)} - N \sqrt{g} A_{I}^{(i)} A_{I}^{(j)} a_{abcd} (D_{I} A_{I}^{(i)} A_{I}^{(j)} A_{I}^{(i)} A_{I}^{(j)})$$

$$\frac{\partial p_{ij}}{\partial \xi} = \frac{\partial}{\partial A_I} = -2 \sqrt{g} O^{IJK} (C_1 (N A_{I}^{(i)} A_{Ib}^{(j)}) + C_2 (N A_{I}^{(i)} A_{Ib}^{(j)}))$$

$$+ \sqrt{g} (N A_{I}^{(i)} A_{Ib}^{(j)} A_{I}^{(i)} A_{Ib}^{(j)} + C^{Iibc} B_{I}^{JM} + C^{Iibc} B_{I}^{JM}) + (N A_{I}^{(i)} A_{Ib}^{(j)} - C^{Iibc} B_{I}^{JM})$$

$$+ \sqrt{g} N (C^{abcd} I^{JKLM} + C^{abcd} I^{JKLM} + C^{abcd} I^{JKLM}) A_{Kb} A_{Lc} A_{M}$$

$$+ C^{abcd} (D_{I} A_{I}^{(i)} A_{I}^{(j)} A_{I}^{(i)} A_{I}^{(j)} A_{I}^{(i)} A_{I}^{(j)}) + 3 \epsilon^{Iabc} E^{JNK} (N A_{I}^{(i)} A_{I}^{(j)} A_{I}^{(i)} A_{I}^{(j)} A_{I}^{(i)} A_{I}^{(j)})$$

$$- \sqrt{g} F^{I} N^{I} + 2 \sqrt{g} M^{JK} A_{I}^{(i)} A_{I}^{(j)} + \xi \pi^{I} A_{I}^{(i)} A_{I}^{(j)}.$$
The evolution of the Hamiltonian constraint is then

\[
\frac{\partial}{\partial \lambda} \left( \sqrt{g}(R + U_{A_1}) - \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{1}{2}p^2 + \pi^I \pi_I \right) \right) =
\]

\[
2N_{ij} \left( 2p^{ij} - (\pi^{Ic}(A_{Ic}^{ij} - A_I^{ij};c) - \pi_{Ic;c}A^{IJ}) \right) + N \left( 2p^{ij} - (\pi^{Ic}(A_{Ic}^{ij} - A_I^{ij};c) - \pi_{Ic;c}A^{IJ}) \right)_{ij}
\]

\[
+ \frac{N_N}{\sqrt{g}} \left( \sqrt{g}(R + U_{A_1}) - \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{1}{2}p^2 + \pi^I \pi_I \right) \right) + \zeta \left( \sqrt{g}(R + U_{A_1}) - \frac{1}{\sqrt{g}} \left( p^{ij} p_{ij} - \frac{1}{2}p^2 + \pi^I \pi_I \right) \right)
\]

\[
+ \frac{1}{N} \left( (4C_1O^{IK} + \delta^{IK})(N^2\pi^a_{Ia}A_{K(ab)};b) + (4C_2O^{IK} - \delta^{IK})(N^2\pi^a_{Ia}A_{K(ab)};b) + 4C_3O^{IK}(N^2\pi^a_{Ia}A_{K(ab)};a) \right)
\]

\[-\frac{1}{N} \left( N^2\pi^a_{Ia}A_{K(ab)};b \right) + \frac{2}{N} C^{abcd} B^K_{JK} (N^2\pi^a_{Ia}A_{K(ab)};b) + \frac{2}{N} \epsilon^{abc} D^{IK}(N^2\pi^a_{Ia}A_{K(ab)};a)
\]

\[-\frac{2}{N} F^I (N^2\pi^a_{Ia}A_{K(ab)};a)
\]

\[
-\frac{1}{N} B^I_{JK} \left( N^2 (p_{ij} - \frac{1}{2}g_{ij}) A^a_{Ia} A^K_{Ia} (2A^i_c C^{ajbd} - A^i_c C^{ajbd})_{;a} \right) + \frac{2}{N} F^I (N^2\pi^a_{Ia}A_{K(ab)};a).
\]

(31)

We demand that this vanishes weakly. The first four terms vanish weakly by definition, leaving us with ten extra terms. Suppose that these do not to automatically vanish; then we would require new constraints. We shall deal with this possibility by implementing our interpretation of Dirac that we presented in Sec. 2. In this case we have at most 2 + 3K degrees of freedom, so if we had 3K or more new constraints, the vector field theory would be trivial. Furthermore, all constraints must be independent of N. Thus, terms in N^a must be of the form (N^aV_{Ja})S^J for the theory to be nontrivial (and we cannot have more than 3K independent scalar constraint factors {S} in total). Most of these scalars will vanish strongly, which means they will fix coefficients in the potential ansatz. Finally, (31) is such that all the non-automatically vanishing terms in N are partnered by terms in N^a. So the above big restriction on the terms in N^a affects all the terms. The above argument applied to (31) may be conveniently subdivided into the following three steps.

1) The first, second, third, sixth and seventh extra terms have no nontrivial scalar factors, so we are forced to have O^{IK} = \delta^{IK}, C_1 = -C_2 = -1/4, C_3 = 0, D_{IK} = 0 and F_I = 0. The conditions on the C’s correspond to the vector fields obeying the gravitational light-cone.

2) This automatically implies that the eighth and tenth terms also vanish. The only nontrivial possibilities for the vanishing of the ninth term are C_1 = 0 and either B_{IJK} = 0 or C_1 = -C_2 \equiv -g/4, say. In fact, because of the symmetry properties of this ansatz term, the second and third of these imply each other if the first holds.

3) This finally leaves us with K new scalar constraint factors from the fourth and fifth terms,

\[
\mathcal{G}_J \equiv \pi^{a}_{J,a} - g B_{IJK} \pi^I_a A^K a \approx 0.
\]

(32)
Next, we examine the evolution of this internal-index vector of new constraints

\[
\begin{align*}
\frac{\partial}{\partial x}(\pi^a_{J;\alpha} - g B_{IJK} \pi^I_{a;K}) = \\
\mathcal{L}_\xi (\pi^a_{J;\alpha} - g B_{IJK} \pi^I_{a;K}) & - \frac{2N}{\sqrt{g}} \pi^I_{\alpha} \pi^I_{\mu} B_{IJK} \\
+ \sqrt{g} A^{Ki;\beta}(A_{I;\beta} - A_{I;\alpha}) B^I_{JK} \\
+ \sqrt{g}(NA^dK A^{iL} A^M_{\alpha;L})_{;\alpha} \left( C_1(IJKLM + JKLM + IJKLM + IMLJK) \\
+ C_2(IJKML + IJMLK + ILMJK + IMLJK; \\
+2C_3(I_{JL}KLM + I_{KM}JL) - \frac{1}{2} g^2 B^I_{JK} B_{1IML} \\
- \frac{\sqrt{g}}{2} g^2 N A^L_i A^M b^K_{b_i} (B^I_{JK} B_{1LM} + B^I_{JM} B_{1KL} + B^I_{JL} B_{1MK}) \\
- B^Q_{JLP} g A^{iP} A^{Kd} A^L A^M_{d} \left( C_1(I_{1QKLM} + I_{KLQI} + I_{LMQI} + I_{MQIL}) \\
+ C_2(I_{1KQLM} + I_{KLQM} + I_{1QKML} + I_{1KQML}) \\
+ 2C_3(I_{1KMQI} + I_{1QMLK}) \\
+ 3 \epsilon^{i\beta c}(E_{JNK}(NA^K_{\alpha} A^\alpha_b); \alpha) + g E_{JNK} NA^K_{\alpha} A^\alpha_{i} B^Q_{JLP} \\
+ 2\sqrt{g}(M_{JK}(NA^K_{\alpha}); \alpha) - g M_{QK} B^Q_{JLP} A^K_{\alpha} A^P)
\end{align*}
\]  

(33)

and we demand that this vanishes weakly. The first term vanishes weakly by definition, leaving us with seven extra terms. Again, we first consider the $N^{ia}$ parts of the terms, which lead us to the following extra steps.

4) For the theory to be nontrivial, the third, sixth and seventh non-automatically vanishing terms force us to have, without loss of generality, $I_{JKLM} = B^I_{JK} B_{1LM}$, $\tilde{C}_2 = -\tilde{C}_1 = g^2/16$, $\tilde{C}_3 = 0$, $E_{JNK} = 0$ and $M_{IK} = 0$. This last condition means that the interacting vector fields must be fundamentally massless.

5) We are then left with the first, second, fourth and fifth terms. The fourth term forces upon us the Jacobi identity

\[
B^I_{JK} B_{1LM} + B^I_{JM} B_{1KL} + B^I_{JL} B_{1MK} = 0.
\]  

(34)

Thus, the $B_{IJK}$ are axiomatically the structure constants of some Lie algebra, $A$. Furthermore, the vanishing of the first term forces us to have $B_{IJK} = B_{[IJK]}$, which means that the $B_{IJK}$ are totally antisymmetric. Finally, the second and fifth terms are then automatically zero. So the potential term must be

\[
U_{A_{J}} = -\frac{1}{8} (A^i_{a;b} - A_b^i_{a;\alpha} + g B_{IJK} A^I_a A^K_b)(A^{i;b}_{\alpha} - A^{b;i}_{\alpha} + g B_{1LM} A^1_a A^{M;b}),
\]  

(35)

We will now investigate what the total antisymmetry of $B_{IJK}$ means. In the standard approach to Yang–Mills theory in flat spacetime, one starts with Lorentz and parity invariance, which restricts the Lagrangian to be $L^{(4)}_{A_{J}} = -Q_{AB} F^A_{\mu\nu} F^{B\rho\sigma}$, where

\[
F^A_{\mu\nu} = (A^A_{\mu;\nu} - A^A_{\nu;\mu} + g_c f^A_{JK} A^J_{\mu} A^K_{\nu})
\]  

(36)

is the field strength tensor, $f^I_{JK}$ are structure constants and $g_c$ is a coupling constant. Furthermore, one demands gauge invariance, $\delta L^{(4)}_{A_{J}} = 0$, under the gauge transformation

\[
A^I_{\alpha} \rightarrow A^I_{\alpha} + g_c f^I_{JK} A^J_{\alpha} A^K_{\alpha}
\]  

(37)

which is equivalent to

\[
Q_{A_{J|B} F^A_{C;D}} = 0.
\]  

(38)
For $Q_{AB}$ positive-definite, Gell-Mann and Glashow \[20, 12\] have shown that this and the following two statements are equivalent:

\[ \exists \text{ basis in which } B_{ABC} = B_{[ABC]} \]  
(39)

$A$ is a direct sum of compact simple and U(1) Lie subalgebras.
(40)

Also, (38) $\iff$ (39) $\iff \dot{G}_J \approx 0$ in the usual flat spacetime canonical working.

In contrast, we have started with 3-dimensional vector fields on 3-geometries, obtained $H$ as an identity and demanded that $\dot{H} \approx 0$, which has forced us to have the secondary constraints $\mathcal{G}_J$. But once we have the $\mathcal{G}_J$, we can use them to do much the same as above. $\dot{G}_J \approx 0 \iff (39) \iff (38)$, so our scheme allows the usual restriction (40) on the type of Lie algebra. We can moreover take (38) to be equivalent to the gauge invariance of $U A_I$ under

\[ A_I^a \rightarrow A_I^a + g B_{JK} A_J^a A^K_a. \]  
(41)

Thus, if we introduce $K$ auxiliary variables $\Phi^K$ such that variation with respect to them encodes $\mathcal{G}_K$ then we should do so according to the best matching corresponding to this gauge symmetry. This uniquely fixes the form of $T_{A_I}(A_{Ia}, \Phi_I)$ to be

\[ T_{A_I} = g^{ad}(\dot{A}_a^I - L_\xi A_a^I - \Phi_a^I + g B_{JK} A_a^J A^K_a)(\dot{A}_{Id} - L_\xi A_{Id} - \Phi_{Id} + g B_{LM} A^L_{Id} \Phi^M). \]  
(42)

Finally, if we identify $\Phi^K$ with $A^K_0$, we arrive at Yang–Mills theory \[8, 11, 12\] for $A^K_0 = [A_0, A_a]$, with coupling constant $g$ and gauge group $A$ (corresponding to the structure constants $B_{IJK}$). So this work constitutes a unique derivation, from 3-dimensional principles alone, of Yang–Mills theory minimally-coupled to general relativity.

This last step is not immediate. Picking $Q_{AB} = \delta_{AB}$, the $(3 + 1)$ decomposition of $L_{A_I}^{(4)}$ yields

\[ T_{A_I} = g^{ad}(\dot{A}_a^I - A^I_{0,a} + g c f_{JK} A_a^J A^K_a - \xi^m F^I_{am})(\dot{A}_{Id} - A_{0,d} + g c f_{LM} A^L_{Id} A^M_0 - \xi^n F_{0bn}). \]  
(43)

One must then integrate by parts and discard $\xi^m A^I_m$ $G_I$ to show that this is equivalent to (2).

5 Discussion

This work shows that the ‘Relativity without Relativity’ formalism can accommodate many examples of physical theories. We can immediately write down a gravity-coupled formalism with the $SU(3)$ gauge group of the strong force, or with larger groups such as $SU(5)$ or $O(10)$, used in grand unified theories. However, the work does not restrict attention to a single simple gauge group, since it also holds for the direct sum (40). One can then rescale the structure constants of each U(1) or compact simple subalgebra separately, which is equivalent to each subalgebra having a distinct coupling constant \[12\]. The simplest example of this is to have $B_{ijk} = 0$, which corresponds to K non-interacting copies of electromagnetism. Other examples include the $SU(2) \times U(1)$ electroweak theory and the $SU(3) \times SU(2) \times U(1)$ Standard Model. We stress that our formalism does not have the power to single out what the gauge groups of nature are.

BFÓ showed that a scalar field, a 3-vector field, and a 3-vector field coupled to scalar fields all obey the same light-cone as gravity. In this paper we have shown that this is also true for $K$
interacting vector fields, so there is a universal light-cone for bosonic fields, derived entirely from 3-dimensional principles. Investigation of the fermionic sector would tell us whether this light-cone is indeed universal for all the known fields of nature. We also note that our formalism reveals that the universality of the light-cone and gauge theory have a common origin resulting from the universal application of best matching to implement 3-diffeomorphisms in conjunction with the need to propagate the quadratic Hamiltonian constraint.

In our 3-dimensional formalism, fundamental vector fields are not allowed to have mass. The only bosonic fields allowed to have mass are scalar fields. This would make spontaneous symmetry breaking a necessity if we are to describe the real world, since the $W^+, W^-$ and $Z$ bosons, believed to be responsible for the weak force, are massive. Moreover, if the study of fermions in our formalism were to reveal these to be also fundamentally massless, one would have a 3-dimensional derivation that mass necessarily arises from Higgs scalar fields.

It would be interesting to consider whether our formalism can accommodate topological terms. Although it is not free of controversy, 't Hooft’s standard explanation of the low energy QCD spectrum makes use of an extra topological term

$$
\frac{\Theta^2 g^2_{\text{strong}}}{32\pi^2} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta}
$$

in the classical Lagrangian. The parameter $\Theta$ is constrained to be small by the non-observation of the neutron dipole moment. The inclusion of this term corresponds to dropping the parity-invariance of the Lagrangian. The new term is a total derivative. Nevertheless it makes a contribution to the action when the QCD vacuum is nontrivial.

Baierlein, Sharp and Wheeler discarded their gravitational total derivatives. If kept, these would simply give a further additive term:

$$
S_{\text{BSW}} = \int d\lambda \int d^3 x \sqrt{g} \sqrt{T_\gamma} \sqrt{R} + \text{surface integral.}
$$

One should thus treat (44) as another such additive term. So in our formalism, just as in any other, we can, and should, allow for further surface contributions to the action. We argue also that the accommodation of topological terms is not yet a problem for us, because so far we are only describing a classical, unbroken, fermion-free world. But the need for the new term arises from quantum-mechanical considerations when quarks (which are spin-1/2 fermions) are present. Furthermore, the quarks must be massive in order for the four-dimensional theory to predict $\Theta$-dependent physics.

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