GUAN-LI TYPE MEAN CURVATURE FLOW FOR FREE BOUNDARY HYPERSURFACES IN A BALL

GUOFANG WANG AND CHAO XIA

Abstract. In this paper we introduce a Guan-Li type volume preserving mean curvature flow for free boundary hypersurfaces in a ball. We give a concept of star-shaped free boundary hypersurfaces in a ball and show that the Guan-Li type mean curvature flow has long time existence and converges to a free boundary spherical cap, provided the initial data is star-shaped.

1. Introduction

Let $B^{n+1} \subset \mathbb{R}^{n+1}$ be the open unit Euclidean ball centered at the origin and $S^n = \partial B^{n+1} \subset \mathbb{R}^{n+1}$ the unit sphere. In this paper, we shall consider a mean curvature type flow for compact hypersurfaces in $B^{n+1}$ with free boundary on $S^n$.

Let $\Sigma \subset B^{n+1}$ be a properly embedded compact hypersurface with boundary, which is given by $x: M \to \overline{B}^{n+1}$, where $M$ is a compact Riemannian manifold with boundary $\partial M$. Here properly embedded means that $\text{int}(\Sigma) = x(\text{int}(M)) \subset B^{n+1}$ and $\partial \Sigma = x(\partial M) \subset \partial B^{n+1}$.

We further assume that $\Sigma$ has free boundary, in the sense that $\Sigma$ intersects $\partial B^{n+1} = S^n$ orthogonally, that is, $\langle \nu, \mu \circ x \rangle = 0$ on $\partial M$, where $\nu$ is a unit normal vector filed of $x$, which will be specified later, and $\mu$ is the outward unit normal vector field of $S^n$, i.e., $\mu \circ x = x$ along $\partial M$.

Let $e \in S^n \subset \mathbb{R}^{n+1}$ be a fixed unit vector field. Consider a family of properly embedded compact hypersurfaces $\{\Sigma_t\}_{t \in [0,T)}$ with free boundary, given by embeddings $x: M \times [0, T) \to \overline{B}^{n+1}$, satisfying

$$
\begin{align*}
\partial_t x &= (n \langle x, e \rangle - H \langle X_e, \nu \rangle) \nu \quad \text{in} \ M \times [0, T), \\
\langle \nu, \mu \circ x \rangle &= 0 \quad \text{on} \ \partial M \times [0, T).
\end{align*}
$$

2010 Mathematics Subject Classification. 53C21, 53C44.

GW is partly supported by Priority Programme “Geometry at Infinity” (SPP 2026) of DFG.

CX is supported by NSFC (Grant No. 11871406), the Natural Science Foundation of Fujian Province of China (Grant No. 2017J06003) and the Fundamental Research Funds for the Central Universities (Grant No. 20720180009).
with an initial surface $x(\cdot, 0) = x_0$. Here $\nu$ and $H$ are a unit normal vector filed and the mean curvature of $x(\cdot, t)$ respectively, $X_e$ is a fixed vector field in $\mathbb{R}^{n+1}$ given by

$$X_e = X_e(x) = \langle x, e \rangle x - \frac{1}{2}(|x|^2 + 1)e,$$

for a fixed unit vector $e$. This vector filed plays an important role in our recent paper [11]. We choose $\nu$ in the following way. Let $\Omega_t$ be the component of the enclosed domain by $\Sigma_t$ and $\mathbb{S}^n$ which containing $e$ in its interior. Then $\nu$ is chosen to be the outward normal of $\Sigma_t$ with respect to $\Omega_t$. Also, throughout this paper, we make the convention that the enclosed domain $\Omega_t$ of $\Sigma_t$ and $\mathbb{S}^n$ is the one containing $e$ in its interior. The volume of the enclosed domain $\Omega_t$ of $\Sigma$ is called the enclosed volume of $\Sigma_t$.

The flow is designed in this way so that the enclosed volume of $\Sigma_t$ is preserved along the flow $\Sigma_t$. We will discuss it later. Such kinds of flow was first considered by Guan-Li [6] in the setting of closed hypersurfaces in space forms and by Guan-Li-Wang [7] in the setting of closed hypersurfaces in warped product spaces.

The main objective of this paper is to study the existence and the convergence of the flow $\Sigma_t$. For this aim we introduce a concept of star-shaped hypersurfaces with free boundary in $\mathbb{B}^{n+1}$. To arrive at this, we should first make some comments on the vector field $X_e$ above. $X_e$ is a conformal Killing vector field with

$$\langle X_e(x), x \rangle = 0, \forall x \in \partial\mathbb{B}^{n+1}.$$

More precisely, denoting the Euclidean metric by $\delta$, we have

$$L_{X_e} \delta = \langle x, e \rangle \delta.$$

Let $\phi_t : \mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$ be the one-parameter family of conformal transformations generated by $X_e$. Let $\Pi_e$ be the hyperplane which passes through the origin and is orthogonal to $e$. For each point $p \in \Pi_e$, there exists a unique planar circle passing through $p$ and $\pm e$. One can check that the integral curves of $X_e$ are given by the intersection of all such planar circles with $\mathbb{B}^{n+1}$. We say a proper embedded hypersurface $\Sigma \subset \mathbb{B}^{n+1}$ is star-shaped (with respect to $e$) if $\Sigma$ intersects each integral curve of $X_e$ exactly once. It is easy to see that star-shapedness of $\Sigma$ is equivalent to $\langle X_e, \nu \rangle > 0$. For our purpose we consider strictly star-shaped hypersurfaces in $\mathbb{B}^{n+1}$, that is hypersurfaces with

$$\langle X_e, \nu \rangle > 0.$$

This condition is slightly stronger than the condition of star-shapedness. For simplicity we call hypersurfaces satisfying (2) star-shaped hypersurfaces.

From now on we consider star-shaped hypersurfaces. Being such a hypersurface, it is necessary that $M$ is of ball type. Therefore we use $M = \mathbb{S}^n_+$, the closed hemisphere.

Our main result is the following

**Theorem 1.1.** Let $\Sigma \subset \mathbb{B}^{n+1}(n \geq 2)$ be a properly embedded compact hypersurface with free boundary, given by $x_0 : \mathbb{S}^n_+ \rightarrow \mathbb{B}^{n+1}$, which is star-shaped with respect to $e$. Then there exists a unique solution $x : \mathbb{S}^n_+ \times [0, \infty) \rightarrow \mathbb{B}^{n+1}$ to (1). Moreover, $x(\cdot, t)$ converges smoothly to a spherical cap, whose enclosed domain has the same volume as $\Sigma$. When $n \geq 3$, $x(\cdot, t)$ converges exponentially fast.

The family of spherical caps is given by

$$C_r(e) = \{x \in \mathbb{B}^{n+1} : |x - \sqrt{r^2 + 1}e| \leq r\}, r > 0.$$
It is clear that each spherical cap $C_r(e)$ has free boundary, that is, it intersects the support $\mathbb{S}^n$ orthogonally.

As a direct consequence, we give a flow proof of the isoperimetric problem for free boundary hypersurfaces in $\mathbb{B}^{n+1}$.

**Corollary 1.1.** *Among star-shaped free boundary hypersurfaces with fixed enclosed volume, the spherical caps have minimal area.*

For general hypersurfaces it is a classical result proved by Burago-Mazaya [3], Bokowsky-Sperner [2] and Almgren [1], by using the method of symmetrization.

The introduction of flow (1) is motivated by the paper of Guan-Li [5], in which they used at the first time the Minkowski formula for closed hypersurfaces to define a geometric flow for isoperimetric problems. In the same spirit, the flow (1) is based on the following two Minkowski formulas obtained in [11] for free boundary hypersurfaces

\begin{align}
  n \int_\Sigma \langle x, e \rangle &= \int_\Sigma \langle X_e, \nu \rangle H, \\
  \int_\Sigma \langle x, e \rangle H &= \frac{2}{n-1} \int \langle X_e, \nu \rangle \sigma_2(\kappa).
\end{align}

Here $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ are principal curvatures of $\Sigma$ and $\sigma_2(\kappa)$ is the 2nd order mean curvature. From these formulas, one can show that flow (1) preserves the volume of $\Omega_t$ and decreases the area of $\Sigma_t$. See Proposition 4.1. These are crucial properties of this flow.

To prove Theorem 1.1, we first transform the flow equation to a scalar flow (19) on $\mathbb{S}^n_+$ by using star-shapedness. By using the Möbius transformation between the half space $\mathbb{R}^{n+1}_+$ and the unit ball $\mathbb{B}^{n+1}$, a star-shaped hypersurface in $\mathbb{B}^{n+1}$ is equivalent to a classical star-shaped hypersurface in $\mathbb{R}^{n+1}$ with a conformal flat metric. We remark that a different reparametrization based on Möbius transformation between round cylinder and $\mathbb{B}^{n+1}$ was used by Lambert-Scheuer [7]. For the scalar flow (19), the $C^0$ estimate follows directly from the barrier argument. We then show the gradient estimate for (19).

Finally we mention some previous results on curvature flows with free boundary in $\mathbb{B}^{n+1}$. The classical mean curvature flow was considered by Stahl [9, 10], where it was shown that strictly convex initial data are driven to a round point in a finite time. The classical inverse mean curvature flow was treated by Lambert-Scheuer [7], where it was shown that strictly convex initial data are driven to a flat perpendicular $n$-ball in a finite time. Following a similar idea of this paper, a fully nonlinear inverse curvature type flow was considered by Scheuer and the authors [8] to show a class of new Alexandrov-Fenchel’s inequalities for convex free boundary hypersurfaces in $\mathbb{B}^{n+1}$.

The rest of this paper is organized as follows. In Section 2 we introduce the Möbius transformation between $\mathbb{R}^{n+1}_+$ and $\mathbb{B}^n$, and reduce flow (1) to a scalar flow (19), provided that all evolving hypersurfaces are star-shaped. In Section 3, we show that $C^0$ and $C^1$ estimates of (1). As consequence, we prove in Section 4 that the global convergence of (1), Theorem 1.1 and its consequence, Corollary 1.1.
2. A scalar flow

In this section we reduce (1) to a scalar flow, provided that all evolving hypersurfaces are star-shaped.

Without loss of generality, from now on, we assume \( e = E_{n+1} \), the \((n+1)\)-coordinate vector. Let

\[
\mathbb{R}^{n+1}_+ = \{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} : z_{n+1} > 0 \}
\]

be the half space. Define

\[
(5) \quad f : \mathbb{R}^{n+1}_+ \to \mathbb{B}^{n+1},
\]

\[
(6) \quad (z', z_{n+1}) \mapsto \left( \frac{2z'}{|z'|^2 + (1 + z_{n+1})^2}, \frac{|z|^2 - 1}{|z'|^2 + (1 + z_{n+1})^2} \right).
\]

Here \( z' = (z_1, \ldots, z_n) \in \mathbb{R}^n \). \( f \) is bijective and

\[
(7) \quad f(\mathbb{R}^{n+1}_+) = f(\mathbb{B}^{n+1}),
\]

\[
(8) \quad f(\partial \mathbb{R}^{n+1}_+) = \partial \mathbb{B}^{n+1},
\]

\[
(9) \quad f(\{|z| = 1\}) = \{ x_{n+1} = 0 \}.
\]

Moreover, \( f \) is a conformal diffeomorphism between \((\mathbb{R}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+})\) and \((\mathbb{B}^{n+1}, \delta_{\mathbb{B}^{n+1}})\). Here \( \delta_{\mathbb{R}^{n+1}_+} \) and \( \delta_{\mathbb{B}} \) denote the restriction of the Euclidean metric to \( \mathbb{R}^{n+1}_+ \) and \( \mathbb{B}^{n+1} \) respectively. Precisely,

\[
f^* \delta_{\mathbb{B}} = e^{2w} \delta_{\mathbb{R}^{n+1}_+} = \frac{4}{(|z'|^2 + (1 + z_{n+1})^2)^2} \delta_{\mathbb{R}^{n+1}_+}.
\]

In other words, \((\mathbb{R}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+})\) and \((\mathbb{B}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}_+})\) are isometric.

In \( \mathbb{R}^{n+1}_+ \), we use the polar coordinates \((\rho, \varphi, \theta) \in [0, \infty) \times [0, \pi/2] \times \mathbb{S}^{n-1}, \)

\[
\rho^2 = |z'|^2 + z_{n+1}^2, \quad z_{n+1} = \rho \cos \varphi
\]

and \( \theta \in \mathbb{S}^{n-1} \) is the spherical coordinate.

By using \((\rho, \varphi, \theta)\) in \( \mathbb{R}^{n+1}_+ \), the mapping \( f \) can be rewritten as

\[
(10) \quad f(\rho, \varphi, \theta) = \left( \frac{2\rho \sin \varphi \tilde{\theta}}{1 + \rho^2 + 2\rho \cos \varphi}, \frac{\rho^2 - 1}{1 + \rho^2 + 2\rho \cos \varphi} \right).
\]

Here \( \tilde{\theta} \) denotes the position vector of the point \( \frac{z'}{\|z'\|} \in \mathbb{S}^{n-1} \). We also have

\[
f^* \delta_{\mathbb{B}} = e^{2w} \delta_{\mathbb{R}^{n+1}_+} = \frac{4}{(1 + \rho^2 + 2\rho \cos \varphi)^2} (d\rho^2 + \rho^2 d\varphi^2 + \rho^2 \sin^2 \varphi g_{\mathbb{S}^{n-1}}),
\]

where

\[
w = w(\rho, \varphi, \theta) = \log 2 - \log(1 + \rho^2 + 2\rho \cos \varphi).
\]

One may also check that the conformal Killing vector field \( X_{n+1} \) on \( \mathbb{B}_+ \) is transformed to

\[
(11) \quad \tilde{X} = (f^{-1})_*(X_{n+1}) = -\rho \partial_{\rho} \text{ on } \mathbb{B}_+.
\]

The integral curves of \( \tilde{X} \) are clearly the rays in \( \mathbb{R}^{n+1}_+ \) initiating from the origin.
Let $\Sigma \subset \bar{\mathbb{R}}^{n+1}$ be a properly embedded compact hypersurface with boundary, given by an embedding \( x : \bar{\mathbb{S}}_+ \to \bar{\mathbb{R}}^{n+1} \). We associate $\Sigma$ with a corresponding hypersurface $\tilde{\Sigma} \subset \mathbb{R}^{n+1}$ given by the embedding
\[
\tilde{x} = f^{-1} \circ x : \tilde{\mathbb{S}}_+ \to \mathbb{R}^{n+1}.
\]

In view of (11), $\Sigma$ is star-shaped with respect to $E_{n+1}$ if and only if $\tilde{\Sigma}$ is star-shaped (with respect to the origin) in $\mathbb{R}^{n+1}$, that is, $\tilde{\Sigma}$ intersects each of the rays in $\mathbb{R}^{n+1}$ initiating from the origin exactly once, or in other words, $\tilde{\Sigma}$ is a graph over $\bar{\mathbb{S}}_+$.

Since $(\bar{\mathbb{R}}^{n+1}, \delta_{\bar{\mathbb{R}}^{n+1}})$ and $(\mathbb{R}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}})$ are isometric, a proper embedding $x : \bar{\mathbb{S}}_+ \to \bar{\mathbb{R}}^{n+1}$ can be identified as an embedding $\tilde{x} : \mathbb{S}_+ \to (\bar{\mathbb{R}}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}})$. In the following, we use $\bar{\cdot}$ to denote the corresponding quantity for $\tilde{x} : \mathbb{S}_+ \to (\bar{\mathbb{R}}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}})$.

Given a star-shaped hypersurface $\tilde{\Sigma}$ in $(\bar{\mathbb{R}}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}})$, by using the polar coordinate $(\rho, \varphi, \theta) \in \bar{\mathbb{R}}^{n+1}$, we may write
\[
\tilde{x} = \rho(y) = \rho(\varphi, \theta)y, y = (\varphi, \theta) \in \bar{\mathbb{S}}_+.
\]

We use $\sigma = d\varphi^2 + \sin^2 \varphi d\theta^2$ and $\nabla^\sigma$ to denote the round metric and the covariant derivative on $\bar{\mathbb{S}}_+$. Set
\[
\gamma = \log \rho, \quad v = \sqrt{1 + |\nabla^\sigma \gamma|^2}.
\]

We have the following correspondence for several geometric quantities.

**Proposition 2.1.**

(i) \[ x_{n+1} = \langle f(\tilde{x}), E_{n+1} \rangle = \frac{1}{2} (\rho^2 - 1) e^w. \]

(ii) \[ |X_{n+1}| = e^w | - \rho \partial_\rho | = e \rho_e^w. \]

(iii) \[ (X_{n+1}, \nu) = e^{2w} ( - \rho \partial_\rho, \bar{\nu} ) = \frac{\rho e^w}{v} \]

(iv) The Weingarten transformation $h_i^j = g^{jk} h_{ik}$ satisfies
\[
h_i^j = \bar{h}_i^j = \frac{1}{\rho v e^w} (\sigma^{kj} - \frac{\gamma^k \gamma^j}{v^2}) \gamma_{ik} + \left[ \frac{\sin \varphi \gamma_{\varphi}}{v} + \frac{(\rho^2 - 1)}{2 \rho v} \right] \delta_i^j.
\]

(v) \[ H = \bar{H} = \frac{1}{\rho v e^w} (\sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2}) \gamma_{ij} + \frac{n \sin \varphi \gamma_{\varphi}}{v} + \frac{n (\rho^2 - 1)}{2 \rho v}.
\]

**Remark 2.1.** We see from (iii) that in case we have $C^0$ estimate, a positive lower bound for $\langle X_{n+1}, \nu \rangle$ is equivalent to the gradient estimate for $\gamma$.

**Proof.** (i) follows from (10) and (ii) follows from (11).

It is clear that the unit outward normal is given by
\[
\tilde{\nu} = e^{-w} \nu_\delta = e^{-w} \rho^{-1} \nabla^\sigma \gamma - \partial_\rho,
\]
where $\nu_\delta$ is the unit outward normal of $\tilde{\Sigma} \subset (\bar{\mathbb{R}}^{n+1}, \delta_{\mathbb{R}^{n+1}})$. Then (iii) follows from (11) and (12).
By a well-known transformation law for the Weingarten transformation under a conformal change, we know that \( \tilde{h}^j_i \) of \( \Sigma \subset (\mathbb{R}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}}) \) with respect to \(-\tilde{\nu}\) is given by

\[
(13) \quad \tilde{h}^j_i = e^{-w}((h_\delta)^j_i + \nabla^\delta_{\nu_\delta} w \delta^j_i),
\]

where \((h_\delta)^j_i \) is the Weingarten transformation with respect to \(-\nu_\delta \) of \( \tilde{\Sigma} \subset (\mathbb{R}^{n+1}, \delta_{\mathbb{R}^{n+1}}) \) and \( \nabla^\delta \) is the Euclidean derivative.

It is known that

\[
(14) \quad (h_\delta)^j_i = -\frac{1}{\rho v} \delta^j_i + \frac{1}{\rho v} (\sigma^{kj} - \frac{\gamma^k \gamma^j}{v^2}) \gamma_{ik},
\]

On the other hand, using \( e^{-w} = \frac{1}{2} (1 + \rho^2 + 2\rho \cos \varphi) \), we have

\[
(15) \quad \nabla_{\nu_\delta}^\delta (e^{-w}) = \left( (\rho + \cos \varphi) \partial_\rho - \rho^{-1} \sin \varphi \partial_\varphi, \rho^{-1} \nabla^\sigma \gamma - \partial_\rho \right) = -\frac{1}{v} (\rho + \cos \varphi + \sin \varphi \gamma_{\varphi}).
\]

(iv) follows from (13), (14) and (15). (v) follows from (iv) by taking trace. \( \square \)

We return to the flow problem (11) in \((\mathbb{R}^{n+1}, \delta_{\mathbb{R}}) \). By the identification using \( f \), the corresponding family of embeddings \( \tilde{x} : S^n_+ \to (\mathbb{R}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}}) \) satisfies

\[
(16) \quad \left\{ \begin{array}{l}
\partial_t \tilde{x} = (n(f(\tilde{x}), E_{n+1}) - \tilde{H} e^{2w} (-\rho \partial_\rho, \tilde{\nu})) \tilde{\nu} \quad \text{in} \ S^n_+ \times (0, T), \\
\langle \tilde{\nu}, \hat{\nu} \circ \tilde{x} \rangle = 0, \quad \text{on} \ \partial S^n_+ \times (0, T),
\end{array} \right.
\]

with an initial surface \( \tilde{x}(\cdot, 0) = \tilde{x}_0 \). Here \( \hat{\nu} \) is the downward unit normal of \((\mathbb{R}^{n+1}, e^{2w} \delta_{\mathbb{R}^{n+1}}) \).

As long as \( \tilde{x}(\cdot, t) \) is star-shaped in \( \mathbb{R}^{n+1} \), we may reduce (16) to a scalar flow.

Using a standard argument (see [4], Eq. (2.4.21)) and Proposition 2.1 we see that

\[
(17) \quad \partial_t \gamma = -\frac{v}{\rho e^w} \left( \frac{n}{2} (\rho^2 - 1) e^w - \tilde{H} \rho e^w \right) \]

\[
= \frac{1}{\rho v e^w} \left( \sigma_{ij} - \frac{\gamma^i \gamma^j}{v^2} \right) \gamma_{ij} + \frac{n \sin \varphi \gamma_{\varphi}}{v} - \frac{n (\rho^2 - 1) |\nabla^\sigma \gamma|^2}{2 \rho v} \]

\[
= \text{div}_\sigma \left( \frac{\nabla^\sigma \gamma}{\rho v e^w} \right) - \frac{n + 1}{v} \sigma \left( \nabla^\sigma \gamma, \nabla^\sigma \left( \frac{1}{\rho v e^w} \right) \right).
\]

The last line above follows from the fact

\[
\sigma \left( \nabla^\sigma \gamma, \nabla^\sigma \left( \frac{1}{\rho v e^w} \right) \right) = \frac{\rho^2 - 1}{2 \rho} |\nabla^\sigma \gamma|^2 - \sin \varphi \gamma_{\varphi}.
\]

Next we examine the boundary condition. Note that \( \mu \perp \partial \mathbb{R}^{n+1} \). Since the conformal change \( f \) preserves angles, we have \( \hat{\mu} \perp \partial \mathbb{R}^{n+1} \) and in turn

\[
\hat{\mu} = -e^{-w} \partial_\varphi.
\]

In view of (12), the boundary condition in (16) reduces to

\[
(18) \quad \nabla_{\partial_\varphi}^\sigma \gamma = 0 \text{ on } \partial S^n_+.
\]
In summary, the flow problem (16) reduces to solve the scalar PDE
\[ \partial_t \gamma = \frac{1}{\rho v v} \left( \sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2} \right) \gamma_{ij} + \frac{n \sin \varphi \gamma \varphi}{v} - \frac{n(\rho^2 - 1)|\nabla^\sigma \gamma|^2}{2\rho v}, \]
in \( S^n_+ \times [0,T) \),
with the initial and the boundary conditions
\[ \gamma(\cdot, 0) = \gamma_0, \quad \text{in } S^n_+, \]
\[ \nabla^p_{\partial_+} \gamma = 0, \quad \text{on } \partial S^n_+ \times [0,T). \]
where \( \gamma_0 \) is the corresponding function for \( x_0 \).

3. A priori estimates

The short time existence of the scalar flow (19) follows by the standard parabolic PDE theory. Next we show the \( C^0 \) and \( C^1 \) estimates for (19). The a priori \( C^0 \) estimate follows directly from the maximum principle.

**Proposition 3.1.** Let \( \gamma : S^n_+ \times [0,T) \rightarrow \mathbb{R} \) solve (19). Then
\[ \min_{S^n_+} \gamma_0 \leq \gamma \leq \max_{S^n_+} \gamma_0. \]

The key point is the following gradient estimate for \( \gamma \).

**Proposition 3.2.** Let \( \gamma : S^n_+ \times [0,T) \rightarrow \mathbb{R} \) solve (19). Then there exists a constant \( C \), depending on \( \|\gamma_0\|_{C^1} \) and \( \min_{S^n_+} \gamma_0 \), such that
\[ |\nabla^\sigma \gamma|^2 \leq C. \]
Moreover, if \( n \geq 3 \), we have
\[ |\nabla^\sigma \gamma|^2 \leq C_1 e^{-C_2 t}. \]

**Proof.** For notation simplicity, we use \( \nabla = \nabla^\sigma \) in the proof. Denote
\[ F(\nabla^2 \gamma, \nabla \gamma, \rho, \varphi) = \frac{1}{\rho v v} \left( \sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2} \right) \gamma_{ij} + \frac{n \sin \varphi \gamma \varphi}{v} - \frac{n(\rho^2 - 1)|\nabla \gamma|^2}{2\rho v}, \]
and
\[ F^{ij} = \frac{\partial F}{\partial \gamma_{ij}}, \quad F^p = \frac{\partial F}{\partial \gamma_p}, \quad F^\rho = \frac{\partial F}{\partial \rho}, \quad F^\varphi = \frac{\partial F}{\partial \varphi}. \]
Then
\[ \partial_t |\nabla \gamma|^2 = 2\gamma_k (\gamma_k)_k = 2F^{ij} \gamma_{ij} + F^p \nabla_p |\nabla \gamma|^2 + 2F^\rho \rho |\nabla \gamma|^2 + 2F^\varphi \gamma \varphi. \]

By a direct computation, we have
\[ F^{ij} = \frac{1}{\rho v v} \left( \sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2} \right), \]
\[ F^p = \frac{\rho^2 - 1}{2\rho^2 v} \left( \sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2} \right) \gamma_{ij} - \frac{n(\rho^2 + 1)}{2\rho^2 v} |\nabla \gamma|^2, \]
\[ F^\varphi = -\sin \varphi \left\{ \frac{1}{v} \left( \sigma^{ij} - \frac{\gamma^i \gamma^j}{v^2} \right) \gamma_{ij} + \frac{n \cos \varphi}{v} \gamma \varphi. \right. \]
Using the Ricci identity
\[ \gamma_{ijk} = \gamma_{kij} + \gamma_j \sigma_{ki} - \gamma_k \sigma_{ij}, \]
and (21), we have

\[
2F^{ij} \gamma_k \gamma_{ijk} = F^{ij} \nabla^2_{ij} |\nabla \gamma|^2 - 2 \frac{1}{\rho v \varepsilon} \left( \sigma^{ij} - \gamma^{ij} \frac{\gamma}{v^2} \right) \partial_{jk} \gamma - \frac{2(n-1)}{\rho v \varepsilon} |\nabla \gamma|^2
\]

(24)

Replacing (22), (23) and (24) into (20), we get

\[
\partial_t |\nabla \gamma|^2 = F^{ij} \nabla^2_{ij} |\nabla \gamma|^2 + F^p_{ij} \nabla_p |\nabla \gamma|^2
- \frac{2}{\rho v \varepsilon} |\nabla^2 \gamma|^2 + \frac{1}{2 \rho v \varepsilon} |\nabla |\nabla \gamma|^2|^2 - \frac{2(n-1)}{\rho v \varepsilon} |\nabla \gamma|^2
+ 2 \left[ \frac{\rho^2 - 1}{2 \rho v^2} \left( \sigma^{ij} - \gamma^{ij} \frac{\gamma}{v^2} \right) \gamma_{ij} - \frac{n(\rho^2 + 1)}{2 \rho v^2} |\nabla \gamma|^2 \right] \rho |\nabla \gamma|^2
+ 2 \left[ - \sin \varphi \frac{1}{v} \left( \sigma^{ij} - \gamma^{ij} \frac{\gamma}{v^2} \right) \gamma_{ij} + \frac{n \cos \varphi}{v} \gamma \right] \rho |\nabla \gamma|^2
\]

(25)

Now we examine the boundary normal derivative of $|\nabla \gamma|^2$ and have

\[
\nabla_{\partial_s} |\nabla \gamma|^2 = 2(\gamma_{\theta_a} \gamma_{\theta_v} + \gamma_{\gamma} \gamma_{\theta_v}) = \gamma_{\theta_a} \left[ \nabla_{\theta_a} (\gamma_{\theta_v}) - (\nabla_{\theta_b} \partial_{\theta_v}) \gamma \right] = 0.
\]

Here we used $\gamma_{\theta_v} = 0$ along $\partial S^n_+$ and the fact that $\nabla_{\theta_b} \partial_{\theta_v} = 0$.

Assume for $t \in [0, T)$, $\max_{S^n_+} |\nabla \gamma|^2(\cdot, t) = |\nabla \gamma|^2(x_t, t)$. If $x_t \in S^n_+$, it follows from the maximum point condition that

\[
\nabla |\nabla \gamma|^2 = 0, \quad \nabla^2 |\nabla \gamma|^2 \leq 0.
\]

If $x_t \in \partial S^n_+$, we see from (26) that $\nabla_{\partial_s} |\nabla \gamma|^2 = 0$, and in turn we also have (27). Thus, for each $t \in [0, T)$, at $x_t$, we have (27). We choose at $x_t$ local coordinates $x^1, \cdots, x^n$ such that $\gamma_1 = |\nabla \gamma|$. One has $\gamma_{1i} = 0$ for all $i$ by (27). By further rotating the $\{x^2, \cdots, x^n\}$ coordinate, we can assume $\nabla^2 \gamma$ is diagonal. Then

\[
|\nabla^2 \gamma|^2 \geq \frac{1}{n-1} (\Delta \gamma)^2.
\]
It follows from (25) that at \( x_t \),
\[
0 \leq \partial_t |\nabla \gamma|^2 (x_t, t) \leq -\frac{2}{\rho v e^w} |\nabla^2 \gamma|^2 - \frac{2(n-1)}{\rho v e^w} |\nabla \gamma|^2
\]
\[
+ \frac{\rho^2 - 1}{\rho v} \Delta |\nabla \gamma|^2 - \frac{n(\rho^2 + 1)}{\rho v} |\nabla \gamma|^4 + \frac{2n \cos \varphi}{v} \gamma^2 - \frac{2 \sin \varphi}{v} \Delta \gamma \varphi
\]
\[
\leq - \frac{2(1 - \epsilon)}{(n-1) \rho v e^w} \left( \Delta \gamma - \frac{(n-1)(\rho^2 - 1)e^w}{4(1 - \epsilon)} |\nabla \gamma|^2 \right)^2
\]
\[
- \frac{2\epsilon}{(n-1) \rho v e^w} \left( \Delta \gamma + \frac{(n-1) \rho e^w \sin \varphi}{2\epsilon} \gamma \varphi \right)^2
\]
\[
+ \frac{1}{\rho v} \left( \frac{(n-1)(\rho^2 - 1)^2 e^w}{8(1 - \epsilon)} - n(\rho^2 + 1) \right) |\nabla \gamma|^4
\]
\[
+ \frac{1}{v} \left( \frac{2(n-1)}{\rho e^w} |\nabla \gamma|^2 + 2n \cos \varphi \gamma \varphi + \frac{(n-1) \rho e^w \sin^2 \varphi}{2\epsilon} \gamma^2 \right).
\]

Choosing \( \epsilon = \frac{3}{4} \), we have
\[
\frac{(n-1)(\rho^2 - 1)^2 e^w}{8(1 - \epsilon)} - n(\rho^2 + 1)
\]
\[
< \frac{ne^w}{2} \left[ (\rho^2 - 1)^2 - (\rho^2 + 1)(1 + \rho^2 + 2\rho \cos \varphi) \right] \leq -n\rho^2 e^w
\]
and
\[
- \frac{2(n-1)}{\rho e^w} |\nabla \gamma|^2 + 2n \cos \varphi \gamma \varphi + \frac{(n-1) \rho e^w \sin^2 \varphi}{2\epsilon} \gamma^2
\]
\[
\leq \left( \frac{(n-1)(1 + \rho^2 + 2\rho \cos \varphi)}{\rho} + 2n \cos \varphi + \frac{4(n-1)}{3} \frac{\rho}{1 + \rho^2 + 2\rho \cos \varphi} \right) |\nabla \gamma|^2
\]
\[
\leq (-2(n-1) + 2 \cos \varphi + \frac{2(n-1)}{3}) |\nabla \gamma|^2
\]
\[
\leq (-\frac{4}{3} n + \frac{10}{3}) |\nabla \gamma|^2.
\]
Thus
\[
0 \leq \partial_t |\nabla \gamma|^2 \leq -\frac{n e^w}{v} |\nabla \gamma|^4 + \frac{4(n-1)}{3} \frac{1}{\rho v} |\nabla \gamma|^2.
\]

It follows from (28) that \( |\nabla \gamma|^2 \leq C \). Moreover, when \( n \geq 3 \), one sees from (28) that \( |\nabla \gamma|^2 \leq C_1 e^{-C_2 t} \).

\section{4. Global convergence}

We first prove the nice properties of (1), mentioned in the Introduction.

\textbf{Proposition 4.1.} Flow (1) satisfies
\[
\frac{d}{dt} \text{Vol}(\Omega_t) = 0
\]
and
\[
\frac{d}{dt} \text{Area}(\Sigma_t) = -\frac{1}{n-1} \int \sum_{i<j} (\kappa_i - \kappa_j)^2 \langle X_{n+1}, \nu \rangle dA_t \leq 0.
\]
Proof. From (3), we get
\[
\frac{d}{dt} \text{Vol}(\Omega_t) = \int_\Sigma (nx_{n+1} - H \langle X_{n+1}, \nu \rangle) dA_t = 0.
\]
The first variational formula gives
\[
\frac{d}{dt} \text{Area}(\Sigma_t) = \int_\Sigma H(nx_{n+1} - H \langle X_{n+1}, \nu \rangle) dA_t.
\]
Using the Minkowski formula (4)
\[
\int_\Sigma Hx_{n+1} - \frac{2}{n-1} \sigma_2(\kappa) \langle X_{n+1}, \nu \rangle dA_t = 0,
\]
we get
\[
\frac{d}{dt} \text{Area}(\Sigma_t) = -\int_\Sigma \left( H^2 - \frac{2n}{n-1} \sigma_2(\kappa) \right) \langle X_{n+1}, \nu \rangle dA_t
\]
\[
= -\frac{1}{n-1} \int_\Sigma \sum_{i<j} (\kappa_i - \kappa_j)^2 \langle X_{n+1}, \nu \rangle dA_t \leq 0.
\]
□

Now we prove the global convergence.

Proof of Theorem 1.1. In view of Proposition 2.1 (iii), the \(C^0\) and \(C^1\) estimates in Propositions 3.1 and 3.2 imply that \(\langle X_{n+1}, \nu \rangle \geq c > 0\), that is, the star-shapedness of \(\Sigma_t\) is preserved under the flow (1).

Now we are ready to prove the long time existence in Theorem 1.1. Since equation (19) is a quasilinear parabolic PDE of divergent form, the higher order a priori estimates follows from the standard parabolic PDE theory, once we have the \(C^0\) and \(C^1\) estimates in Propositions 3.1 and 3.2. Hence we prove that (19) has a smooth solution for all time. The exponential convergence for \(n \geq 3\) follows directly from Proposition 3.2.

For the convergence part in two dimension, we examine the monotonicity of the area functional along the flow. In the following we restrict to \(n = 2\). By integrating (4.1) over \(t \in [0, \infty)\) and using the uniform estimate, we get
\[
\int_0^\infty \int_{\mathbb{S}^2} |\kappa_1(y, t) - \kappa_2(y, t)|^2 \langle X_{n+1}, \nu \rangle dA_t dt \leq C.
\]
where \(\kappa_i(y, t), i = 1, 2\) are the principal curvatures of the radial graph at \((y, t)\). It follows from the uniform bound for \(\langle X_{n+1}, \nu \rangle\) and \(dA_t\) that
\[
\max_{y \in \mathbb{S}^2_t} |\kappa_1 - \kappa_2|(y, t) = o_t(1),
\]
where \(o_t(1)\) denotes a quantity which goes to zero as \(t \to \infty\). See the proof of Proposition 5.5 in [5]. With the help of the property (31), we can show the convergence of flow (1) when \(n = 2\). This idea was used first by Guan-Li in [5].

Let us go back to the estimate at \(x_t\), where \(\max_{\mathbb{S}^2_{x_t}} |\nabla \gamma|^2(\cdot, t) = |\nabla \gamma|^2(x_t, t)\). Again we choose the local coordinate around \(x_t\) such that at \(x_t\),
\[
\gamma_1 = |\nabla \gamma|, \quad \gamma_{11} = 0.
\]
In view of Proposition 2.1 (iv), the Weingarten transformation $h_j^i$ is diagonal in this coordinate which means the coordinate directions are the principal directions of $x(\cdot, t)$ at $x_t$. Thus the principal curvature $\kappa_i$ at $x_t$ is given by

$$
\kappa_i = \frac{\gamma_{ii}}{\rho v e^w} + \frac{\sin \varphi \gamma_{\varphi}}{v} + \frac{(\rho^2 - 1)}{2 \rho v}, \quad i = 1, 2.
$$

It follows that at $x_t$,

$$
|\Delta \gamma| = |\gamma_{22} + \gamma_{11}| = |\gamma_{22} - \gamma_{11}| = \rho v e^w |\kappa_2 - \kappa_1| = o_t(1). \quad (32)
$$

Using (32) and the $C^1$ estimate, we get at $(x_t, t)$,

$$
\partial_t |\nabla \gamma|^2 \leq -\frac{2}{\rho v e^w} |\nabla^2 \gamma|^2 - \frac{2(n - 1)}{\rho v e^w} |\nabla \gamma|^2 \\
+ \frac{\rho^2 - 1}{\rho v} \Delta \gamma |\nabla \gamma|^2 - \frac{n(\rho^2 + 1)}{\rho v} |\nabla \gamma|^4 + \frac{2n \cos \varphi}{v} \gamma_{\varphi}^2 - \frac{2 \sin \varphi}{v} \Delta \gamma \gamma_{\varphi} \\
\leq -\frac{n(\rho^2 + 1)}{\rho v} |\nabla \gamma|^4 + \frac{1}{v} \left(-\frac{2}{\rho v e^w} |\nabla \gamma|^2 + 4 \cos \varphi \gamma_{\varphi}^2\right) + o_t(1) \quad (33)
$$

Here we have used

$$
-\frac{2}{\rho v e^w} |\nabla \gamma|^2 + 4 \cos \varphi \gamma_{\varphi}^2 \leq \left(-1 + \frac{\rho^2 + 2 \rho \cos \varphi}{\rho} + 4 \cos \varphi\right) |\nabla \gamma|^2 \leq 0.
$$

Now we claim that

$$
|\nabla \gamma|^2 = o_t(1).
$$

The smooth convergence follows from this claim and the interpolation theorem. We show the claim in 2 steps.

First, we show that there exists a sequence $\{t_i\}$ with $t_i \to \infty$ such that

$$
\max_{\bar{S}_n^t} |\nabla \gamma(\cdot, t_i)|^2 \to 0 \quad \text{as} \quad i \to \infty.
$$

Assume this is not true. Then there exists $\epsilon_0 > 0$ and $T_0 > 0$ such that

$$
\max_{\bar{S}_n^t} |\nabla \gamma(\cdot, t)|^2 \geq \epsilon_0, \quad \text{for} \quad t > T_0.
$$

From (33) we have that for a large $T_1 > 0$ and for any $t > T_1$, we have

$$
\frac{d}{dt} \max_{\bar{S}_n^t} |\nabla \gamma|^2 \leq -C \max_{\bar{S}_n^t} |\nabla \gamma|^4 + \frac{1}{2} \epsilon_0^4 = -\frac{1}{2} C \epsilon_0^4,
$$

which is impossible.

Second, we show that any sequence $\{s_i\}$ with $s_i \to \infty$ such that

$$
\max_{\bar{S}_n^s} |\nabla \gamma(\cdot, s_i)|^2 \to 0 \quad \text{as} \quad i \to \infty.
$$

If not, there exists a sequence $\{s_i\}$ with $s_i \to \infty$ such that

$$
\max_{\bar{S}_n^s} |\nabla \gamma(\cdot, s_i)|^2 \geq \epsilon_1
$$
for any $s_i$ and for some positive constant $\epsilon_1$. Without loss of generality, we may assume that $t_i < s_i$. We consider the interval $I_i := [t_i, s_i]$ for sufficiently large $i$, such that we have from (33) at a maximum point $x_t \in \overline{S_n^+}$

\[
\frac{d}{dt} \max_{\overline{S_n^+}} |\nabla \gamma|^2 \leq -C \max_{\overline{S_n^+}} |\nabla \gamma|^4 + \frac{1}{2} C \epsilon_1^4
\]

for any $t \geq t_i$. Let $y_i \in \overline{S_n^+}$ and $\tilde{t}_i \in [t_i, s_i]$ such that

\[
|\nabla \gamma(y_i, \tilde{t}_i)|^2 = \max_{t \in [t_i, s_i]} \max_{\overline{S_n^+}} |\nabla \gamma(\cdot, t)|^2 \geq \epsilon_1.
\]

By the first step, we may assume that $\tilde{t}_i \neq t_i$ for $i$ large. It follows that

\[
\frac{d}{dt} \max_{\overline{S_n^+}} |\nabla \gamma|^2(\tilde{t}_i) \geq 0.
\]

Together with (34), implies that

\[
|\nabla \gamma(y_i, \tilde{t}_i)|^2 < \epsilon_1,
\]

a contradiction. $\square$

**Proof of Corollary 1.1** It follows from Theorem 1.1 and Proposition 4.1 $\square$

**References**

[1] F. J. Almgren, *Spherical symmetrization*, Proc. International workshop on integral functions in the calculus of variations, Trieste, 1985, Red. Circ. Mat. Palermo 2 Supple. (1987), 11–25.

[2] J. Bokowsky and E. Sperner, *Zerlegung konvexer Körper durch minimale Trennflächen*, J. Reine Angew. Math. 311/312 (1979), 80–100.

[3] Yu. D. Burago and V. G. Maz’ya, *Certain Questions of Potential Theory and Function Theory for Regions with Irregular Boundaries*, (Russian) Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 3 1967 152 pp; English translation: Potential Theory and Function Theory for Irregular Regions. Seminars in Mathematics, V. A. Steklov Mathematical Institute, Leningrad, Vol. 3 Consultants Bureau, New York 1969 vii+68 pp.

[4] C. Gerhardt, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., Sommerville, 2006.

[5] P. Guan and J. Li, *A mean curvature type flow in space forms*, Intern. Math. Res. Not. 2015 (2015), no. 13, 4716–4740.

[6] P. Guan, J. Li, and M.-T. Wang, *A volume preserving flow and the isoperimetric problem in warped product spaces*, Trans. Am. Math. Soc. 372 (2019), 2777–2798.

[7] B. Lambert and J. Scheuer, *The inverse mean curvature flow perpendicular to the sphere*, Math. Ann. 364 (2016), no. 3, 1069–1093.

[8] J. Scheuer, G. Wang and C. Xia, *Alexandrov-Fenchel inequalities for convex hypersurfaces with free boundary in a ball*, J. Differ. Geom., to appear, 2019.

[9] A. Stahl, *Convergence of solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. 4 (1996), no. 5, 421–441.

[10] A. Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. 4 (1996), no. 4, 385–407.

[11] G. Wang and C. Xia, *Uniqueness of stable capillary hypersurfaces in a ball*, Math. Ann. 374 (2019), no. 3–4, 1845–1882.
Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany

E-mail address: guofang.wang@math.uni-freiburg.de

School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China

E-mail address: chaoxia@xmu.edu.cn