THE WALL-CHAMBER STRUCTURES OF THE REAL GROTHENDIECK GROUPS

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ABSTRACT. For a finite-dimensional algebra $A$ over a field $K$, the real Grothendieck group $K_0(\text{proj } A) \otimes \mathbb{R}$ gives stability conditions of King. We study the associated wall-chamber structure of $K_0(\text{proj } A) \otimes \mathbb{R}$ by using the Koenig-Yang correspondences in silting theory. First, we introduce an equivalence relation on $K_0(\text{proj } A) \otimes \mathbb{R}$ called the TF equivalence by using numerical torsion pairs of Baumann–Kamnitzer–Tingley. Second, we show that the TF equivalence classes with nonempty interiors correspond bijectively with 2-term silting objects. Finally, we determine the wall-chamber structure of $K_0(\text{proj } A) \otimes \mathbb{R}$ in the case that $A$ is a path algebra of an acyclic quiver.

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1. Introduction

It is well-known that projective modules and simple modules are fundamental and important objects in the representation theory of a ring $A$. This paper is devoted to study mutual relationship between these modules in the case that $A$ is a finite-dimensional algebra over a field $K$. In this setting, there are only finitely many isomorphism classes $S_1, \ldots, S_n$ of simple $A$-modules in the category $\text{mod } A$ of finite-dimensional $A$-modules. They bijectively correspond to the isomorphism classes $P_1, \ldots, P_n$ of indecomposable projective $A$-modules in the category $\text{proj } A$ of finitely generated projective $A$-modules via taking the projective covers $P_i \to S_i$. Moreover, $\text{Hom}_A(P_i, S_j) \neq 0$ holds if and only if $i = j$.

Such relationship between projective modules and simple modules has been extended to derived categories. As a generalization of progenerators and classical tilting modules, Keller–Vossieck [KV]...
introduced silting objects (Definition 3.1) of the perfect derived category $K^b(\text{proj} A)$. Then, Koenig–Yang [KY] found that silting objects have one-to-one correspondences with many important notions, including bounded $t$-structures with length heart (Definition 3.4) and simple-minded collections (Definition 3.5) in the bounded derived category $D^b(\text{mod} A)$. These bijections are collectively called the Koenig–Yang correspondences, and have been developed by many authors such as [BY] [AIR] [Asai] [MS].

The Koenig–Yang correspondences can be studied from the point of view of the Grothendieck groups $K_0(\text{proj} A)$ and $K_0(\text{mod} A)$ and the Euler form. The Euler form is a $\mathbb{Z}$-bilinear form

$$\langle ?,! \rangle : K_0(\text{proj} A) \times K_0(\text{mod} A) \to \mathbb{Z}$$

defined by

$$\langle T, X \rangle := \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \text{Hom}_{D^b(\text{mod} A)}(T, X[k]).$$

for every $T \in K^b(\text{proj} A)$ and $X \in D^b(\text{mod} A)$. With respect to the Euler form, the families $(P_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ are dual bases of $K_0(\text{proj} A)$ and $K_0(\text{mod} A)$ in the following sense:

$$\langle P_i, S_j \rangle = \begin{cases} \dim_K \text{End}_{D^b(\text{mod} A)}(S_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$  

For example, [KR] [DF] [Dlj] studied the $g$-vector $[U] = [U^0] - [U^{-1}] \in K_0(\text{proj} A)$ of a $2$-term presilting object $U = (U^{-1} \to U^0)$ in $K^b(\text{proj} A)$ by using the presentation space $\text{Hom}_A(U^{-1}, U^0)$. Moreover, Aihara–Iyama [AI] showed that the $g$-vectors of the indecomposable direct summands of every silting object give a $\mathbb{Z}$-basis of the Grothendieck group $K_0(\text{proj} A)$, and Koenig–Yang [KY] showed that this basis is dual to the $\mathbb{Z}$-basis of $K_0(\text{mod} A)$ given by the corresponding simple-minded collection.

Each $\theta \in K_0(\text{proj} A)$ gives a stability condition for modules in $\text{mod} A$ in the sense of King [Kin] via the Euler form. Stability conditions play an important role in many aspects, including the construction of moduli spaces of modules in geometric invariant theory [Kin], the detailed study of crystal bases of quantum groups from preprojective algebras [BKT], and the investigation of scattering diagrams and quivers with potentials in cluster theory [Bri].

In this paper, we consider the real Grothendieck groups

$$K_0(\text{proj} A)_\mathbb{R} := K_0(\text{proj} A) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad K_0(\text{mod} A)_\mathbb{R} := K_0(\text{mod} A) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The stability condition given by each $\theta \in K_0(\text{proj} A)_\mathbb{R}$ is nothing but a collection of linear inequalities; namely, a module $M \in \text{mod} A$ is said to be $\theta$-semistable if $\theta(M) = 0$ and $\theta(X) \geq 0$ for all quotient modules $X$ of $M$. The subcategory $\mathcal{W}_\theta \subset \text{mod} A$ of $\theta$-semistable modules is a wide subcategory of $\text{mod} A$, that is, a subcategory closed under taking kernels, cokernels and extensions. In particular, $\mathcal{W}_\theta$ is an abelian length category, and the simple objects in $\mathcal{W}_\theta$ are precisely the $\theta$-stable modules. Every simple object $S$ in $\mathcal{W}_\theta$ is a brick, that is, the endomorphism ring $\text{End}_A(S)$ is a division $K$-algebra.

Each nonzero module $M$ gives the rational polyhedral cone $\Theta_M \subset K_0(\text{proj} A)_\mathbb{R}$ called the wall consisting of $\theta \in K_0(\text{proj} A)_\mathbb{R}$ such that $M$ is $\theta$-semistable. The subsets $\Theta_M$ for all $M$ give a wall-chamber structure in $K_0(\text{proj} A)_\mathbb{R}$ studied in [BST] [Bri].

In this paper, we study the wall-chamber structure of $K_0(\text{proj} A)_\mathbb{R}$ by using the two numerical torsion pairs $(T_\theta, F_\theta)$ and $(T_\theta, F_\theta)$ for each $\theta \in K_0(\text{proj} A)_\mathbb{R}$ introduced by [BKT], which are defined by linear inequalities in a similar way to stability conditions.

Our first aim in this paper is to investigate the wall-chamber structure of $K_0(\text{proj} A)_\mathbb{R}$ via the numerical torsion pairs. For this purpose, we define an equivalence relation on $K_0(\text{proj} A)_\mathbb{R}$ as
follows: we say that \( \theta \) and \( \theta' \) are TF equivalent if \((T_\theta, F_\theta) = (T_{\theta'}, F_{\theta'})\) and \((T_\theta, F_{\theta}) = (T_{\theta'}, F_{\theta'})\).
The following first main result of this paper characterizes the TF equivalence classes in terms of the walls \( \Theta_M \).

**Theorem 1.1** (Theorem 2.16). Let \( \theta, \theta' \in K_0(\text{proj} A)_\mathbb{R} \) be distinct elements. Then the following conditions are equivalent.

(a) The elements \( \theta \) and \( \theta' \) are TF equivalent.

(b) Any \( \theta'' \) in the linear segment \([\theta, \theta']\) is TF equivalent to \( \theta \).

(c) For any \( \theta'' \in [\theta, \theta'] \), the \( \theta'' \)-semistable subcategory \( \mathcal{W}_{\theta''} \) is constant.

(d) For any module \( M \), we have \( \{\theta, \theta'\} \cap \Theta_M = \emptyset \) or \( \{\theta, \theta'\} \subset \Theta_M \).

(e) There does not exist a brick \( S \) such that \( \{\theta, \theta'\} \cap \Theta_S \) has exactly one element.

Next, we study TF equivalence classes by using the Koenig–Yang correspondences. For each 2-term presilting object \( U \) in \( K^b(\text{proj} A)_{\mathbb{R}} \), we define

\[
C(U) := \{a_1[U_1] + \cdots + a_m[U_m] \mid a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}\},
\]

\[
C^+(U) := \{a_1[U_1] + \cdots + a_m[U_m] \mid a_1, \ldots, a_m \in \mathbb{R}_{>0}\},
\]

following Demonet–Iyama–Jasso \([DIJ]\). The following second main result of this paper, based on Yurikusa’s work \([Yur]\), shows that each 2-term presilting object \( U \) gives a TF equivalence class \( C^+(U) \).

**Theorem 1.2** (Theorem 3.17 (3)). Let \( U \in 2\text{-presilt} A \). Then, the subset \( C^+(U) \subset K_0(\text{proj} A)_{\mathbb{R}} \) is a TF equivalence class satisfying

\[
(T_\theta, F_\theta) = (\mathcal{H}^{-1}(\nu U), \text{Sub} \mathcal{H}^{-1}(\nu U)), \quad (T_{\theta'}, F_{\theta'}) = (\mathcal{F} \mathcal{H}^0(U), \mathcal{H}^0(U)^\perp).
\]

In particular, the correspondence \( U \mapsto C^+(U) \) gives an injection from the set \( 2\text{-presilt} A \) of basic 2-term presilting objects in \( K^b(\text{proj} A) \) to the set of TF equivalence classes. By restricting this map to the set \( 2\text{-silt} A \) of basic 2-term silting objects in \( K^b(\text{proj} A) \), we obtain the following result.

**Theorem 1.3** (Theorem 3.17 (3)). There exists a bijection

\[
2\text{-silt} A \to \{\text{TF equivalence classes whose interiors are nonempty}\}
\]
given by \( T \mapsto C^+(T) \).

It follows from Theorem 1.1 that the TF equivalence classes whose interiors are nonempty coincide with the chambers in the wall-chamber structure of \( K_0(\text{proj} A)_{\mathbb{R}} \). Therefore, Theorem 1.3 shows that all chambers come from 2-term silting objects.

In section 4, we describe how TF equivalence classes change under \( \tau \)-tilting reduction introduced in \([Jas]\). For a fixed 2-term presilting object \( U \), we consider the wide subcategory \( \mathcal{W}_U := \mathcal{H}^{-1}(\nu U) \cap \mathcal{H}^0(U) ^\perp \) and the subset \( 2\text{-presilt}_U A \subset 2\text{-presilt} A \) of basic 2-term presilting objects containing \( U \) as a direct summand. In this setting, there exist a category equivalence

\[
\varphi := \text{Hom}_A(T, ?) : \mathcal{W}_U \to \text{mod} B
\]

and a bijection

\[
\text{red} := \text{Hom}_{K^b(\text{proj} A)}(T, ?)[/U] : 2\text{-presilt}_U A \to 2\text{-presilt} B,
\]

where \( T \) is the Bongartz completion of \( U \) and \( B := \text{End}_{K^b(\text{proj} A)}(T)[/U] \).

We would like to know the wall-chamber structure of \( K_0(\text{proj} B)_{\mathbb{R}} \) in this situation. For this purpose, we define an open neighborhood \( N_U \) of \([U] \in K_0(\text{proj} A)_{\mathbb{R}} \) by

\[
N_U := \{\theta \in K_0(\text{proj} A)_{\mathbb{R}} \mid \mathcal{H}^0(U) \in T_\theta, \mathcal{H}^{-1}(\nu U) \in F_\theta\}.
\]
Clearly, \( N_U \) is a union of some TF equivalence classes in \( K_0(\text{proj}\ A)_{\mathbb{R}} \). We prove that the local wall-chamber structure of \( N_U \subset K_0(\text{proj}\ A)_{\mathbb{R}} \) around \([U]\) recovers the whole wall-chamber structure of \( K_0(\text{proj}\ B)_{\mathbb{R}} \) via the linear projection \( \pi : K_0(\text{proj}\ A)_{\mathbb{R}} \to K_0(\text{proj}\ B)_{\mathbb{R}} \) given by
\[
\pi(\theta) := \sum_{i=1}^{m} \frac{\theta(X_i)}{d_i} [P_i^B],
\]
where \( X_1, X_2, \ldots, X_m \) are the simple objects of \( \mathcal{W}_U \), \( d_i := \dim_K \text{End}_A(X_i) \), and \( P_i^B \) is the projective cover of the simple \( B \)-module \( \varphi(X_i) \) for each \( i \).

**Theorem 1.4** (Theorem 4.5). Let \( U \in 2\text{-presilt} A \). Then, we have the following properties.

1. For any \( \theta \in N_U \) and \( M \in \mathcal{W}_U \), the wall \( \Theta_{\varphi(M)} \) coincides with \( \pi(\Theta_M \cap N_U) \).
2. The linear map \( \pi \) induces a bijection
\[
\{ \text{TF equivalence classes in } N_U \} \to \{ \text{TF equivalence classes in } K_0(\text{proj}\ B)_{\mathbb{R}} \},
\]
\( [\theta] \mapsto \pi([\theta]) \).
3. We have the following commutative diagram:
\[
\begin{array}{ccc}
2\text{-presilt}_U A & \xrightarrow{\text{red}} & 2\text{-presilt} B \\
\downarrow \cong & & \downarrow \cong \\
\{ \text{TF equivalence classes in } N_U \} & \xrightarrow{\pi} & \{ \text{TF equivalence classes in } K_0(\text{proj}\ B)_{\mathbb{R}} \}
\end{array}
\]

As an application of this theorem, we give a simple proof of the following characterization of \( \tau \)-tilting finiteness by the cones \( C(T) \) for 2-term silting objects. Recall that \( A \) is said to be \( \tau \)-tilting finite if \( \#(2\text{-silt} A) < \infty \) [DL, AIR].

**Theorem 1.5** (Theorem 4.7). The algebra \( A \) is \( \tau \)-tilting finite if and only if \( K_0(\text{proj}\ A)_{\mathbb{R}} = \bigcup_{T \in 2\text{-silt} A} C(T) \).

Note that the “only if” part follows from [DL]. The “if” part was conjectured by Demonet [Dem] and a different proof was given by Zimmermann–Zvonareva [ZZ].

Finally, we give a combinatorial method to obtain the wall-chamber structure of \( K_0(\text{proj}\ A)_{\mathbb{R}} \) in the case that \( A \) is the path algebra of an acyclic quiver \( Q \) over an algebraically closed field \( K \). For each nonzero dimension vector \( d \), there exists a module \( M \) which gives the largest wall \( \Theta_M \) with respect to inclusion among all modules whose dimension vectors are \( d \) [Sch]. We write this largest wall \( \Theta_d \). Then, the wall \( \Theta_d \) can be determined inductively in the following way.

**Theorem 1.6** (Theorem 5.6). Let \( Q \) be an acyclic quiver and \( d = (d_i)_{i \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0} \) be a nonzero dimension vector, and set \( \text{supp } d := \{ i \in Q_0 \mid d_i \neq 0 \} \). Then, \( \Theta_d \) is given as follows.

1. If \( \# \text{supp } d = 1 \) and \( k \in \text{supp } d \), then \( \Theta_d = \bigoplus_{i \neq k} \mathbb{R}[P_i] \).
2. Assume that \( \# \text{supp } d = 2 \) and that the full subquiver of \( Q \) corresponding to \( \text{supp } d \subset Q \) is
\[
\begin{array}{c}
k \\
\vdots \\
\rightarrow l \quad (m \text{ arrows})
\end{array}
\]
with \( k, l \in \text{supp } d \) and \( m \in \mathbb{Z}_{\geq 0} \). We define \( a, b \in \mathbb{Z}_{\geq 0} \) by \( a := d_k / \gcd(d_k, d_l) \) and \( b := d_l / \gcd(d_k, d_l) \). Then,
\[
\Theta_d = \begin{cases} 
(\bigoplus_{i \neq k,l} \mathbb{R}[P_i]) \oplus \mathbb{R}_{\geq 0}(b[P_k] - a[P_l]) & (a^2 + b^2 - mab \leq 1) \\
(\bigoplus_{i \neq k,l} \mathbb{R}[P_i]) & \text{(otherwise)}
\end{cases}
\]
(3) If \( \text{supp} \, d \geq 3 \), then \( \Theta_d \) is the smallest polyhedral cone of \( K_0(\text{proj} \, A)_\mathbb{R} \) containing 
\[
\bigcup_{0 < c < d} (\Theta_c \cap \Theta_{d-c}).
\]

As an example of the theorem above, we give the wall-chamber structure of \( K_0(\text{proj} \, A)_\mathbb{R} \) in the case that \( Q \) is the wild quiver \( 1 \to 2 \to 3 \) in Example \ref{example:wall-chamber}.

1.1. Notation. Throughout this paper, \( K \) is a field and \( A \) is a finite-dimensional \( K \)-algebra. Unless otherwise stated, all algebras and modules are finite-dimensional. We set \( \text{proj} \, A \) as the category of finitely generated projective right \( A \)-modules, and let \( P_1, \ldots, P_n \) be all the non-isomorphic indecomposable projective modules in \( \text{proj} \, A \). Similarly, we write \( \text{mod} \, A \) for the category of finite-dimensional right \( A \)-modules, and let \( S_1, \ldots, S_n \) be all the non-isomorphic simple modules in \( \text{mod} \, A \). We may additionally assume that \( S_i \) is the top of \( P_i \) for each \( i \in \{1, \ldots, n\} \).

The symbol \( K^b(\text{proj} \, A) \) denotes the homotopy category of the bounded complex category of \( \text{proj} \, A \), and \( D^b(\text{mod} \, A) \) stands for the derived category of the bounded complex category of \( \text{mod} \, A \). Both categories are triangulated categories, and their shifts are denoted by \([1]\).

Any subcategory appearing in this paper is a full subcategory, and is assumed to be closed under isomorphism classes.

2. Stability conditions and TF equivalence

We start by recalling the definition of Grothendieck groups. Let \( \mathcal{C} \) be an exact category or a triangulated category, then the Grothendieck group \( K_0(\mathcal{C}) \) is the quotient group of the free abelian group on the set of isomorphism classes \([X]\) of \( \mathcal{C} \) by the relations \([X] - [Y] + [Z] = 0\) for all admissible short exact sequences \( 0 \to X \to Y \to Z \to 0 \) or all admissible triangles \( X \to Y \to Z \to X[1] \).

It is well-known that the Grothendieck group \( K_0(\text{proj} \, A) \) has a \( \mathbb{Z} \)-basis \( (P_i)_{i=1}^n \) given by all the non-isomorphic indecomposable projective modules, and that it is canonically isomorphic to \( K_0(K^b(\text{proj} \, A)) \). Similarly, \( K_0(\text{mod} \, A) \) is also a free abelian group of rank \( n \), and the family \( (S_i)_{i=1}^n \) of all the non-isomorphic simple modules is a \( \mathbb{Z} \)-basis of \( K_0(\text{mod} \, A) \). The Grothendieck group \( K_0(\text{mod} \, A) \) can be canonically identified with \( K_0(D^b(\text{mod} \, A)) \); see \[Hap\] for details.

For these Grothendieck groups \( K_0(\text{proj} \, A) \) and \( K_0(\text{mod} \, A) \), we consider a non-degenerate \( \mathbb{Z} \)-bilinear form \( \langle \cdot, ! \rangle : K_0(\text{proj} \, A) \times K_0(\text{mod} \, A) \to \mathbb{Z} \) called the Euler form defined by

\[
\langle T, X \rangle := \sum_{k \in \mathbb{Z}} (-1)^k \dim K \text{Hom}_{D^b(\text{mod} \, A)}(T, X[k])
\]

for \( T \in K^b(\text{proj} \, A) \) and \( X \in D^b(\text{mod} \, A) \). The families \( (P_i)_{i=1}^n \) and \( (S_i)_{i=1}^n \) give dual bases of \( K_0(\text{proj} \, A) \) and \( K_0(\text{mod} \, A) \) with respect to the Euler form in the following sense:

\[
\langle P_i, S_j \rangle = \begin{cases} \dim K \text{End}_{D^b(\text{mod} \, A)}(S_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.
\]

In this paper, we consider the real Grothendieck groups

\[
K_0(\text{proj} \, A)_\mathbb{R} := K_0(\text{proj} \, A) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad K_0(\text{mod} \, A)_\mathbb{R} := K_0(\text{mod} \, A) \otimes_{\mathbb{Z}} \mathbb{R}.
\]

Then, they are identified with the Euclidean space \( \mathbb{R}^n \) as topological spaces and vector spaces. The Euler form is naturally extended to an \( \mathbb{R} \)-bilinear form \( \langle \cdot, ! \rangle : K_0(\text{proj} \, A)_\mathbb{R} \times K_0(\text{mod} \, A)_\mathbb{R} \to \mathbb{R} \). We regard each \( \theta \in K_0(\text{proj} \, A)_\mathbb{R} \) as an \( \mathbb{R} \)-linear form \( \theta, ? : K_0(\text{mod} \, A)_\mathbb{R} \to \mathbb{R} \); in other words, we set \( \theta(M) := \langle \theta, M \rangle \). For each \( \theta \in K_0(\text{proj} \, A)_\mathbb{R} \), King \[Kin\] associated a stability condition as follows.

**Definition 2.1.** \[Kin\] Definition 1.1] Let \( \theta \in K_0(\text{proj} \, A)_\mathbb{R} \).

1. A module \( M \in \text{mod} \, A \) is said to be \( \theta \)-semistable if
We define the $\theta$-semistable subcategory $\mathcal{W}_\theta$ as the full subcategory consisting of all the $\theta$-semistable modules in $\text{mod} \ A$.

(2) A module $M \in \text{mod} \ A$ is said to be $\theta$-stable if

- $M \neq 0$,
- $\theta(M) = 0$, and
- for any nonzero proper quotient module $X$ of $M$, we have $\theta(X) > 0$.

The $\theta$-semistable subcategory $\mathcal{W}_\theta$ is a wide subcategory of $\text{mod} \ A$, that is, a full subcategory closed under kernels, cokernels, and extensions in $\text{mod} \ A$. In particular, $\mathcal{W}_\theta$ is an abelian category, so all its simple objects are bricks. Here, a module $S \in \text{mod} \ A$ is called a brick if its endomorphism ring $\text{End}_A(S)$ is a division ring. We write $\text{brick} \ A$ for the set of isomorphism classes of bricks in $\text{mod} \ A$.

By definition, we obtain the following property.

**Lemma 2.2.** Let $\theta \in K_0(\text{proj} \ A)_{\mathbb{R}}$ and $M \in \mathcal{W}_\theta$, then $M$ is a simple object in $\mathcal{W}_\theta$ if and only if $M$ is $\theta$-stable.

To investigate semistable subcategories, we associate a wall for each nonzero module in $\text{mod} \ A$ as in Brüstle–Smith–Treffinger [BST, Definition 3.2] and Bridgeland [Bri, Definition 6.1].

**Definition 2.3.** For any nonzero module $M \in \text{mod} \ A \setminus \{0\}$, we set

$$\Theta_M := \{ \theta \in K_0(\text{proj} \ A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta \},$$

and call $\Theta_M$ the wall associated to the module $M$.

These walls define a wall-chamber structure of $K_0(\text{proj} \ A)_{\mathbb{R}}$. Clearly, we have $\Theta_{M_1 \oplus M_2} = \Theta_{M_1} \cap \Theta_{M_2}$ for any $M_1, M_2 \in \text{mod} \ A \setminus \{0\}$, so we sometimes consider the walls only for indecomposable modules. We here give an easy example.

**Example 2.4.** Let $A$ be the path algebra $K(1 \to 2)$. The indecomposable $A$-modules are $S_1, S_2, P_1$, and the corresponding walls are $\Theta_{S_1} = \mathbb{R}[P_2]$, $\Theta_{S_2} = \mathbb{R}[P_1]$ and $\Theta_{P_1} = \mathbb{R}_{\geq 0}[P_1 - [P_2]]$, since there exists a short exact sequence $0 \to S_2 \to P_1 \to S_1 \to 0$. These walls are depicted as follows.

To investigate the walls $\Theta_M$ more geometrically, we here cite some basic notions and properties on rational polyhedral cones in a Euclidean space from [Ful, Section 1.2].

Let $F$ be a free abelian group of finite rank, and set $F^* := \text{Hom}_{\mathbb{Z}}(F, \mathbb{Z})$. Then, we have two finite-dimensional $\mathbb{R}$-vector spaces $V := F \otimes_{\mathbb{Z}} \mathbb{R}$ and $V^* := F^* \otimes_{\mathbb{Z}} \mathbb{R} \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$. A subset $D \subset V$ is called a polyhedral cone if there exist finitely many elements $v_1, \ldots, v_m \in V$ such that

$$D = \left\{ \sum_{i=1}^m r_i v_i \mid r_i \in \mathbb{R}_{\geq 0} \right\}.$$
A polyhedral cone $D$ is said to be rational if we can take $v_1, \ldots, v_m$ above so that $v_1, \ldots, v_m \in F \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a polyhedral cone $D$ in $V$, we define the dual $D^\vee \subset V^*$ of $D$ by

$$D^\vee := \{ u \in V^* \mid \text{for all } v \in D, \langle u, v \rangle \geq 0 \},$$

where $\langle u, v \rangle := u(v)$. Then, $D^\vee$ is a polyhedral cone in $V^*$, that is, there exist finitely many elements $u_1, \ldots, u_m \in V^*$ such that

$$D^\vee = \left\{ \sum_{i=1}^m r_i u_i \mid r_i \in \mathbb{R}_{\geq 0} \right\}.$$

Moreover, if $D$ is rational, then $D^\vee$ is rational.

We can consider the dual polyhedral cone $C^\vee$ in $V$ of a polyhedral cone $C$ in $V^*$ in a similar way, and then, $(C^\vee)^\vee$ coincides with $C$.

Let $C$ be a polyhedral cone in $V^*$. A subset $C' \subset C$ is called a face if there exists some $v \in C^\vee$ such that $C' = C \cap \text{Ker} (?)_v$, or equivalently, if $C'$ admits finitely many elements $v_1, v_2, \ldots, v_m \in C^\vee$ which satisfy $C' = C \cap (\bigcap_{i=1}^m \text{Ker} (?)_{v_i})$. Any face of a (rational) polyhedral cone is a (rational) polyhedral cone again. We define the dimension $\dim C$ of the polyhedral cone $C$ as the dimension $\dim_{\mathbb{R}} (\mathbb{R} \cdot C)$ of the $\mathbb{R}$-vector subspace $\mathbb{R} \cdot C \subset V^*$ spanned by $C$. We say that a polyhedral cone $C$ is strongly convex if the vector space $C \cap (-C)$ is $\{0\}$.

By setting $F := K_0(\text{mod } A)$, we get $F^* \cong K_0(\text{proj } A)$ via the Euler form. For each $M \in \text{mod } A \setminus \{0\}$, consider the rational polyhedral cone $D_M$ in $K_0(\text{mod } A)_\mathbb{R}$ generated by the set

$$\{ [X] \mid X \text{ is a quotient module of } M \} \cup \{-[M]\},$$

then the wall $\Theta_M$ coincides with the dual $(D_M)^\vee$, so $\Theta_M$ is a rational polyhedral cone in $K_0(\text{proj } A)_\mathbb{R}$. A finite set $\{X_1, X_2, \ldots, X_m\}$ of quotient modules of $M$ gives a face $\Theta_M \cap (\bigcap_{i=1}^m \text{Ker} (?)_{X_i})$, and we can check that all faces of $\Theta_M$ are obtained in this way. Since $\Theta_M \subset \text{Ker} (?)_M$, we have $\dim \Theta_M \leq n - 1$.

From now on, we will characterize some conditions on $\Theta_M$ as a polyhedral cone in terms of representation theoretic properties of $M$.

We first consider the question when $\Theta_M$ is strongly convex. The answer is given by the sincerity of the module $M$. We say that $M \in \text{mod } A$ is sincere if $\text{supp } M = \{1, 2, \ldots, n\}$, where we set

$$\text{supp } M := \{ i \in \{1, \ldots, n\} \mid S_i \text{ appears in a composition series of } M \text{ in } \text{mod } A \}.$$

**Lemma 2.5.** Let $M \in \text{mod } A \setminus \{0\}$, and set $H_1 := \bigoplus_{i \in \text{supp } M} \mathbb{R}[S_i]$ and $H_2 := \bigoplus_{j \notin \text{supp } M} \mathbb{R}[S_j]$. Then, we have the following assertions.

1. We have $\Theta_M \cap (-\Theta_M) = H_2$ as vector subspaces of $K_0(\text{proj } A)_\mathbb{R}$.
2. The wall $\Theta_M$ is strongly convex if and only if $M$ is sincere.
3. The wall $\Theta_M$ coincides with $(\Theta_M \cap H_1) \oplus H_2$, and $\Theta_M \cap H_1$ is a strongly convex polyhedral cone in $H_1$.

**Proof.**

(1) We first show $\Theta_M \cap (-\Theta_M) \subset H_2$. Assume $\theta \in \Theta_M \cap (-\Theta_M)$, and take a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ in $\text{mod } A$. Since $\theta \in \Theta_M \cap (-\Theta_M)$, we have $\theta(M_k) \leq 0$ and $-\theta(M_k) \leq 0$ for all $k$, so $\theta(M_k) = 0$ for any $k$. Therefore, $\theta(M_k/M_{k-1}) = 0$ holds for all $k \in \{1, \ldots, l\}$, which clearly implies $\theta \in H_2$. The converse inclusion is obvious.

(2) This straightly follows from (1).

(3) The first statement is clear. We take the idempotent $e \in A$ such that $S_i e = 0$ if and only if $i \in \text{supp } M$, then $M$ is a sincere $A/\langle e \rangle$-module. We define the wall $\Theta_M^{\langle e \rangle}$ associated to $M \in \text{mod } A/\langle e \rangle$ in $K_0(\text{mod } A/\langle e \rangle)_\mathbb{R}$, which is strongly convex by (2). Under the canonical inclusion
$K_0(\text{mod } A/\langle e \rangle) \to K_0(\text{mod } A)_R$, the image of $\Theta_{M,\langle e \rangle}$ is $K_0(\text{mod } A/\langle e \rangle)_R$ is $\Theta_{M,H}$ is a strongly convex polyhedral cone in $H_1$.

We next focus on the faces of $\Theta_M$ as a polyhedral cone in $K_0(\text{proj } A)_R$. For $M \in \text{mod } A \setminus \{0\}$ and $\theta \in \Theta_M$, we define

$$\text{supp}_\theta \ M := \{S \in W_\theta \mid S \text{ is a simple object appearing in a composition series of } M \text{ in } W_\theta\}.$$

Lemma 2.6. Let $M \in \text{mod } A \setminus \{0\}$ and $\theta \in \Theta_M$, and set $H := \text{Ker}(?,\text{supp}_\theta \ M) \subset K_0(\text{proj } A)_R$. Then, we have the following properties.

1. The element $\theta$ belongs to the interior of $\Theta_M \cap H$ in $H$, and $\Theta_M \cap H$ is the smallest face of $\Theta_M$ containing $\theta$.
2. Let $\theta' \in \Theta_M$. Then, $\theta' \in \Theta_M \cap H$ holds if and only if $\text{supp}_\theta \ M \subset W_{\theta'}$.
3. Let $\theta' \in \Theta_M$ and set $H' := \text{Ker}(?,\text{supp}_{\theta'} \ M)$. Then, $H = H'$ holds if and only if $\text{supp}_\theta \ M = \text{supp}_{\theta'} \ M$.

Proof. (1) Clearly, we have $\theta \in \Theta_M \cap H$. For each $X \in \text{supp}_\theta \ M$, there exists an open subset $N_X \subset K_0(\text{proj } A)_R$ such that $\theta \in N_X$ by Lemma 2.2 [2]. Then, $\theta \in (\bigcap_{X \in \text{supp}_\theta \ M} N_X) \cap \Theta_M \cap H$ holds. Since $\text{supp}_\theta \ M$ is a finite set, $\bigcap_{X \in \text{supp}_\theta \ M} N_X$ is an open subset of $K_0(\text{proj } A)_R$. Therefore, $\theta$ belongs to the interior of $\Theta_M \cap H$ in $H$.

Next, we show that $\Theta_M \cap H$ is a face of $\Theta_M$. Take a composition series $0 \subset M_0 \subset M_1 \subset \cdots \subset M_l = M$ in $W_\theta$, then $\text{supp}_\theta \ M = \{M_i/M_{i-1} \mid i \in \{1,2,\ldots,l\}\}$. Thus, for each $\theta' \in \Theta_M$, the condition $\theta(\text{supp}_\theta \ M) = 0$ holds if and only if $\theta(M/M_i) = 0$ for all $i \in \{0,1,\ldots,l\}$, so we get

$\Theta_M \cap H = \bigcap_{i=1}^l (\Theta_M \cap \text{Ker}(?,M/M_i))$.

Since $M/M_i$ is a quotient module of $M$, the subset $\Theta_M \cap H$ is a face of $\Theta_M$.

(2) The “if” part is obvious, so we consider the “only if” part. Let $\theta' \in \Theta_M \cap H$ and $X \in \text{supp}_\theta \ M$. It suffices to show $X \in W_{\theta'}$. First, $\theta'(X) = 0$ follows from the definition of $H$. Next, assume that $Y$ is a quotient module of $X$. Consider the composition series in the proof of (1), then we can take $i \in \{1,2,\ldots,l\}$ such that $X \cong M_i/M_{i-1}$. It follows that there exists a quotient module $N$ of $M$ admitting a short exact sequence $0 \to Y \to N \to M/M_i \to 0$. Since $\theta' \in \Theta_M$, we have $\theta'(N) \geq 0$. Moreover, $\theta'(M/M_i) = 0$ because $\theta' \in H$. Thus, $\theta'(Y) = \theta'(N) \geq 0$. Consequently, $X \in W_{\theta'}$.

(3) We get the “if” part straightforwardly. Conversely, assume $H = H'$. For any $X \in \text{supp}_\theta \ M$, we get $X \in W_{\theta'}$ by (2). Take a nonzero quotient module $Y$ of $X$ which is simple in $W_{\theta'}$. Then, by (2) again, $Y \in W_\theta$. Since $X$ is a simple object in $W_\theta$ and $Y$ is a quotient module of $X$, we have $Y = X$. Therefore, $X$ is a simple object of $W_{\theta'}$. This property holds for all $X \in \text{supp}_\theta \ M$, so a composition series of $M$ in $W_\theta$ is a composition series of $M$ in $W_{\theta'}$. Thus, $\text{supp}_\theta \ M = \text{supp}_{\theta'} \ M$.

The property above yields that $\dim \Theta_M = n - 1$ if and only if there exists $\theta \in \Theta_M$ such that any $X \in \text{supp}_\theta \ M$ satisfies $\text{[X]} \in \mathbb{Q}[M]$. We remark that these conditions are not equivalent to that $M$ admits $\theta \in \Theta_M$ such that $M$ is a simple object in $W_\theta$, because $\Theta_M = \Theta_M \cap M$ holds for any $M \in \text{mod } A \setminus \{0\}$.

Moreover, the following property tells us that the dimension of every maximal wall with respect to inclusion is $n - 1$ and that such a wall is always realized by a brick.

Proposition 2.7. Let $M \in \text{mod } A \setminus \{0\}$. Then, there exists $S \in \text{brick } A$ such that $\Theta_S \supset \Theta_M$ and that $\dim \Theta_S = n - 1$.

Proof. Take $\theta \in \Theta_M$ such that $\theta$ does not belong to any proper subspace of $\Theta_M$, and set $H := \text{Ker}(?,\text{supp}_\theta \ M)$. By Lemma 2.6 (1), $\Theta_M \cap H$ is the smallest face containing $\theta$, but it must be $\Theta_M$ itself. Thus, we get $\Theta_M \cap H = \Theta_M$. This and Lemma 2.6 (2) imply that every $\theta' \in \Theta_M$ satisfies

$$\text{supp}_{\theta'} \ M = \text{supp}_\theta \ M.$$
\( \text{supp}_\theta M \subset W_\theta. \) Now, we take \( S \in \text{supp}_\theta M \), then \( S \in W_\theta \) holds for all \( \theta' \in \Theta_M \), and this means \( \Theta_S \supset \Theta_M. \)

Next, we show that the wall \( \Theta_M \) can be given in terms of the walls \( \Theta_{M'} \) for other modules \( M' \) such that \( \dim_K M' < \dim_K M. \) For \( M', M'' \in \text{mod} A \), set \( M' \ast M'' \) as the collection of modules \( M \in \text{mod} A \) admitting a short exact sequence \( 0 \to M' \to M \to M'' \to 0. \)

**Lemma 2.8.** Suppose that \( M \in \text{mod} A \) and that \( \# \text{supp} M \geq 3. \) Then, \( \Theta_M \) is the smallest polyhedral cone of \( K_0(\text{proj} A)_R \) containing

\[
\bigcup_{M', M'' \in \text{mod} A \setminus \{0\}, M \in M' \ast M''} (\Theta_{M'} \cap \Theta_{M''}).
\]

To prove this, we need the following geometrical property.

**Lemma 2.9.** Let \( C \subset \mathbb{R}^m \) be a strongly convex polyhedral cone, and assume \( \dim C \geq 2. \) We define \( \partial C \) as the boundary of \( C \) in \( \mathbb{R}^m. \) Then, \( C \) coincides with the smallest polyhedral cone in \( \mathbb{R}^m \) containing \( \partial C. \)

**Proof.** First, consider the canonical sphere

\[ \Sigma := \{(a_1, \ldots, a_m) \in \mathbb{R}^m \mid a_1^2 + \cdots + a_m^2 = 1\}. \]

By the assumption \( \dim C \geq 2, \) we have \( m \geq 2, \) so \( \Sigma \) is connected. On the other hand, \( (C \cap \Sigma) \cup (-C \cap \Sigma) \) is a disjoint union of two proper closed subsets of \( \Sigma, \) since \( C \) is strongly convex. Therefore, the disjoint union \( (C \cap \Sigma) \cup (-C \cap \Sigma) \) cannot be equal to the connected space \( \Sigma. \) Thus, we can take \( v_0 \in \Sigma \) satisfying \( v_0 \notin C \cup (-C). \)

Now, let \( v \) be in the interior of \( C \) in \( \mathbb{R}^m. \) It is enough to find \( u_1, u_2 \in \partial C \) such that \( v = u_1 + u_2. \) We define a subset \( \Gamma := \{r \in \mathbb{R} \mid v + rv_0 \in C\} \) of \( \mathbb{R}. \) Clearly, \( \Gamma \) is convex in \( \mathbb{R}, \) and \( 0 \in \Gamma \) is in the interior of \( \Gamma \) in \( \mathbb{R}, \) since \( C \) is a neighborhood of \( v \) in \( \mathbb{R}^m. \) Also, \( \Gamma \) is bounded from above; otherwise, \( (1/r)v + v_0 \in C \) holds for all \( r > 0, \) so we get \( v_0 \in C \) because \( C \) is closed in \( \mathbb{R}^m, \) but it contradicts the choice of \( v_0. \) Similarly, \( \Gamma \) is bounded from below. Thus, \( \Gamma \) is actually a bounded closed interval \([r_1, r_2]\) with \( r_1 < r_2, \) and \( v + r_1 v_0, v + r_2 v_0 \in \partial C. \) Then, the equation

\[
v = \frac{r_2}{r_2 - r_1} (v + r_1 v_0) + \frac{-r_1}{r_2 - r_1} (v + r_2 v_0)
\]

implies that \( v \) is a sum of two points in \( \partial C. \)

We apply Lemma 2.9 by setting \( m := n - 1 \) to prove Lemma 2.8.

**Proof of Lemma 2.8.** By Lemma 2.5 we may assume that \( M \) is sincere, and in this case, \( \Theta_M \) is strongly convex. Then, \( n \geq 3 \) follows by assumption. We write \( C \) for the smallest polyhedral cone of \( K_0(\text{proj} A)_R \) containing

\[ C^* := \bigcup_{M', M'' \in \text{mod} A \setminus \{0\}, M \in M' \ast M''} (\Theta_{M'} \cap \Theta_{M''}). \]

We first show \( C \subset \Theta_M. \) Since \( \Theta_M \) is a polyhedral cone, it suffices to check \( C^* \subset \Theta_M. \) Assume that \( \theta \in C^* \), then we can take \( M', M'' \in \text{mod} A \setminus \{0\} \) satisfying \( \theta \in \Theta_{M'} \cap \Theta_{M''} \) and \( M \in M' \ast M''. \) By definition, \( M', M'' \in W_\theta, \) which implies \( M \in M' \ast M'' \subset W_\theta. \) Thus, \( \theta \in \Theta_M, \) and we have \( C^* \subset \Theta_M \); hence \( C \subset \Theta_M. \)

Therefore, it remains to show the converse \( \Theta_M \subset C. \)

We first consider the case that \( \dim \Theta_M \leq n - 2. \) Then, any \( \theta \in \Theta_M \) belongs to the boundary of \( \Theta_M \) in the vector subspace \( \text{Ker}(?, M) \) of dimension \( n - 1. \) By Lemma 2.6, \( M \) is not a simple object
in \( \mathcal{W}_\theta \), so there exists an object \( M' \in \mathcal{W}_\theta \) such that \( 0 \subset M' \subset M \). Then, \( \theta \in \Theta_{M'} \cap \Theta_{M''} \subset C' \subset C \) with \( M' := M/M' \neq 0 \). Thus, \( \Theta_M \subset C \).

The other case is that \( \dim \Theta_M = n - 1 \). We set \( \partial \Theta_M \) as the boundary of \( \Theta_M \) in the hyperplane \( \ker(\cdot, M) \). Since \( \dim \Theta_M = n - 1 \geq 2 \) and \( \Theta_M \) is strongly convex, Lemma 2.9 implies that \( \Theta_M \) is the smallest polyhedral cone containing \( \partial \Theta_M \). Thus, it suffices to check \( \partial \Theta_M \subset C \). Let \( \theta \in \partial \Theta_M \). Then, \( M \) is not a simple object in \( \mathcal{W}_\theta \) by Lemma 2.6. As in the previous case, we can show \( \partial \Theta_M \subset C \), and \( \Theta_M \subset C \).

In both cases, we have obtained the assertion \( \Theta_M = C \) as desired.

To investigate the walls \( \Theta_M \) more, we will use numerical torsion pairs, so we here shortly recall the definition of torsion pairs. Let \( \mathcal{T}, \mathcal{F} \) be two full subcategories of \( \text{mod} \ A \), then the pair \((\mathcal{T}, \mathcal{F})\) is called a torsion pair in \( \text{mod} \ A \) if \( \text{Hom}_A(\mathcal{T}, \mathcal{F}) = 0 \) holds and every \( M \in \text{mod} \ A \) admits a short exact sequence \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) with \( M' \in \mathcal{T} \) and \( M'' \in \mathcal{F} \). A full subcategory \( \mathcal{T} \subset \text{mod} \ A \) is called a torsion class if \( \mathcal{T} \) admits \( \mathcal{F} \subset \text{mod} \ A \) such that \((\mathcal{T}, \mathcal{F})\) is a torsion pair in \( \text{mod} \ A \), and this condition is equivalent to that \( \mathcal{T} \) is closed under taking extensions and quotient modules. Dually, torsion-free classes in \( \text{mod} \ A \) are defined, and they are precisely the full subcategories closed under taking extensions and submodules.

Now, we associate two torsion classes and two torsion-free classes to each \( \theta \in K_0(\text{proj} \ A)_\mathbb{R} \) as in Baumann–Kamnitzer–Tingley [BKT]. See also [Bri] Lemma 6.6.

**Definition 2.10.** [BKT Subsection 3.1] Let \( \theta \in K_0(\text{proj} \ A)_\mathbb{R} \). Then, we define numerical torsion classes \( \mathcal{T}_\theta \) and \( \mathcal{T}_\theta \) as

\[
\mathcal{T}_\theta := \{ M \in \text{mod} \ A \mid \text{for any quotient module } X \text{ of } M, \theta(X) \geq 0 \},
\]

\[
\mathcal{T}_\theta := \{ M \in \text{mod} \ A \mid \text{for any quotient module } X \neq 0 \text{ of } M, \theta(X) > 0 \}.
\]

Dually, we define numerical torsion-free classes \( \mathcal{F}_\theta \) and \( \mathcal{F}_\theta \) as

\[
\mathcal{F}_\theta := \{ M \in \text{mod} \ A \mid \text{for any submodule } X \text{ of } M, \theta(X) \leq 0 \},
\]

\[
\mathcal{F}_\theta := \{ M \in \text{mod} \ A \mid \text{for any submodule } X \neq 0 \text{ of } M, \theta(X) < 0 \}.
\]

Clearly, \( \mathcal{T}_\theta \subset \mathcal{T}_\theta \) and \( \mathcal{F}_\theta \subset \mathcal{F}_\theta \) hold, and their “differences” are expressed by the \( \theta \)-semistable subcategory \( \mathcal{W}_\theta = \mathcal{T}_\theta \cap \mathcal{F}_\theta \). Thus, the three conditions \( \mathcal{T}_\theta = \mathcal{T}_\theta, \mathcal{F}_\theta = \mathcal{F}_\theta \), and \( \mathcal{W}_\theta = \{0\} \) are all equivalent. The numerical torsion(-free) classes form torsion pairs in \( \text{mod} \ A \) as follows.

**Proposition 2.11.** [BKT Proposition 3.1] For \( \theta \in K_0(\text{proj} \ A)_\mathbb{R} \), the pairs \((\mathcal{T}_\theta, \mathcal{F}_\theta)\) and \((\mathcal{T}_\theta, \mathcal{F}_\theta)\) are torsion pairs in \( \text{mod} \ A \).

Therefore, \( \mathcal{T}_\theta \) and \( \mathcal{F}_\theta \) determine each other, and so do \( \mathcal{T}_\theta \) and \( \mathcal{F}_\theta \). By using numerical torsion(-free) classes, we introduce an equivalence relation on \( K_0(\text{proj} \ A)_\mathbb{R} \) as follows.

**Definition 2.12.** Let \( \theta \) and \( \theta' \) be elements in \( K_0(\text{proj} \ A)_\mathbb{R} \). We say that \( \theta \) and \( \theta' \) are TF equivalent if both \( \mathcal{T}_\theta = \mathcal{T}_{\theta'} \) and \( \mathcal{F}_\theta = \mathcal{F}_{\theta'} \) hold. We define \( [\theta] \subset K_0(\text{proj} \ A)_\mathbb{R} \) as the TF equivalence class of \( \theta \).

Now, we give an example on TF equivalence classes.

**Example 2.13.** Let \( A \) be the path algebra \( K(1 \rightarrow 2) \). Since the Auslander–Reiten quiver of \( \text{mod} \ A \) is

\[
\begin{array}{cc}
P_1 \swarrow & \searrow \ \\
S_2 & S_1
\end{array}
\]

we can express an additive full subcategory \( C \) of \( \text{mod} \, A \) by writing \( \bullet \) or \( \circ \) instead of each \( * \) in the diagram: each \( * \) corresponds to the module in the same place in the Auslander–Reiten quiver, and \( \bullet \) means that the module belongs to \( C \) and \( \circ \) means not. For example, \( \circ \bullet \bullet \) denotes \( \text{add} \{ P_1, S_1 \} \).

Under this notation, \( T_\theta \) and \( F_\theta \) for \( \theta \in K_0(\text{proj} \, A)_\mathbb{R} \) are as in the following pictures, respectively.

Each domain contains a line or a point in its boundary if it is described by a solid line or a black point, and does not contain if it is denoted by a dotted line or a white point.

Therefore, \( K_0(\text{proj} \, A)_\mathbb{R} \) is divided to eleven TF equivalence classes, which are the origin, the five half-lines without the origin, and the five colored open domains in the following picture:

The following property is easily deduced, but important. We write
\[
[\theta, \theta'] := \{(1 - r)\theta + r\theta' \mid r \in [0, 1]\}
\]
for the line segment between \( \theta, \theta' \in K_0(\text{proj} \, A)_\mathbb{R} \).

**Lemma 2.14.** In \( K_0(\text{proj} \, A)_\mathbb{R} \), any TF equivalence class is convex.

**Proof.** Assume that \( \theta \) and \( \theta' \) in \( K_0(\text{proj} \, A)_\mathbb{R} \) are TF equivalent and that \( \theta'' \in [\theta, \theta'] \). By definition, \( T_{\theta''} \) contains \( T_{\theta'} \cap T_{\theta''} \), which is equal to \( T_{\theta} \) by assumption. Similarly, \( F_{\theta''} \) contains \( F_{\theta'} \cap F_{\theta''} = F_{\theta} \). Thus, the torsion pair \((T_{\theta''}, F_{\theta''})\) must coincide with \((T_{\theta'}, F_{\theta'})\). We can also prove that \((T_{\theta''}, F_{\theta''}) = (T_{\theta}, F_{\theta})\) in the same way. Therefore, \( \theta'' \) is also TF equivalent to \( \theta \). \( \square \)

Each TF equivalence class is not a closed subset in general, but the closure enjoys the following nice property.

**Lemma 2.15.** Let \( \theta, \theta' \in K_0(\text{proj} \, A)_\mathbb{R} \). Then, \( \theta' \) belongs to the closure \([\theta]\) if and only if \( F_\theta \subset F_{\theta'} \) and \( T_\theta \subset T_{\theta'} \).

**Proof.** The “only if” part follows from the definition.
Conversely, assume \( \mathcal{T}_\theta \subset \mathcal{T}_{\theta'} \) and \( \mathcal{T}_\theta \subset \mathcal{T}_{\theta''} \). In this case, we can show that any \( \theta'' \in [\theta, \theta'] \) satisfies \( \mathcal{T}_{\theta''} \supset \mathcal{T}_\theta \cap \mathcal{T}_{\theta''} = \mathcal{T}_\theta \) and \( \mathcal{T}_{\theta''} \supset \mathcal{T}_\theta \cap \mathcal{T}_{\theta''} = \mathcal{T}_\theta \). Thus, we have \( (\mathcal{T}_{\theta''}, \mathcal{F}_{\theta''}) = (\mathcal{T}_\theta, \mathcal{F}_\theta) \). Similarly, \( (\mathcal{T}_{\theta'}, \mathcal{F}_{\theta'}) = (\mathcal{T}_\theta, \mathcal{F}_\theta) \) holds. Therefore, every \( \theta'' \in [\theta, \theta'] \) belongs to the TF equivalence class \([\theta]\), so \( \theta' \) is in the closure \([\theta]\).

Moreover, we are able to characterize TF equivalence in terms of the walls \( \Theta_M \) as follows.

**Theorem 2.16.** Let \( \theta, \theta' \in K_0(\text{proj} A)_{\mathbb{R}} \) be distinct elements. Then, the following conditions are equivalent.

(a) The elements \( \theta \) and \( \theta' \) are TF equivalent.
(b) Any \( \theta'' \in [\theta, \theta'] \) is TF equivalent to \( \theta \).
(c) For any \( \theta'' \in [\theta, \theta'] \), the \( \theta'' \)-semistable subcategory \( \mathcal{W}_{\theta''} \) is constant.
(d) For any module \( M \), we have \([\theta, \theta'] \cap \Theta_M = \emptyset \) or \([\theta, \theta'] \subset \Theta_M \).
(e) There does not exist \( S \in \text{brick} A \) such that \([\theta, \theta'] \cap \Theta_S \) has exactly one element.

To prove this, we also prepare a fact on coincidence of torsion pairs.

**Lemma 2.17.** Let \( (\mathcal{T}, \mathcal{F}) \) and \( (\mathcal{T}', \mathcal{F}') \) be torsion pairs in \( \text{mod} A \). Then, \( (\mathcal{T}, \mathcal{F}) \) and \( (\mathcal{T}', \mathcal{F}') \) coincide if and only if \( \mathcal{T} \cap \mathcal{F} = \mathcal{T}' \cap \mathcal{F} = \emptyset \).

**Proof.** If \( (\mathcal{T}, \mathcal{F}) = (\mathcal{T}', \mathcal{F}') \), then we clearly have \( \mathcal{T} \cap \mathcal{F}' = \emptyset \cap \mathcal{F} \) = \( (\mathcal{T}', \mathcal{F}') \cap \mathcal{F} = \emptyset \). Conversely, we assume \( \mathcal{T} \cap \mathcal{F}' = \emptyset \) and \( \mathcal{T}' \cap \mathcal{F} = \emptyset \). Then, \( \text{Hom}(\mathcal{T}, \mathcal{F}') = 0 \) and \( \text{Hom}(\mathcal{T}', \mathcal{F}) = 0 \) hold, and they imply \( \mathcal{F}' \subset \mathcal{F} \) and \( \mathcal{F} \subset \mathcal{F}' \), respectively. Therefore, we get \( \mathcal{F} = \mathcal{F}' \) and \( \mathcal{T} = \mathcal{T}' \).

The following criterion is obtained by simply applying Lemma 2.17 to the numerical torsion pairs \( (\mathcal{T}_\theta, \mathcal{F}_\theta) \) and \( (\mathcal{T}_\theta, \mathcal{F}_\theta) \).

**Lemma 2.18.** Let \( \theta, \theta' \in K_0(\text{proj} A)_{\mathbb{R}} \). Then the following assertions hold.

(a) The torsion classes \( \mathcal{T}_\theta \) and \( \mathcal{T}_{\theta'} \) coincide if and only if \( \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} = \mathcal{T}_{\theta'} \cap \mathcal{F}_\theta = \emptyset \).
(b) The torsion-free classes \( \mathcal{F}_\theta \) and \( \mathcal{F}_{\theta'} \) coincide if and only if \( \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} = \mathcal{T}_{\theta'} \cap \mathcal{F}_\theta = \emptyset \).

Now, we can prove Theorem 2.16.

**Proof of Theorem 2.16.** (a) \( \Rightarrow \) (b) follows from Lemma 2.14.
(b) \( \Rightarrow \) (c) and (c) \( \Rightarrow \) (d) are clear by definition.
(d) \( \Rightarrow \) (e) is also obvious, since \( \theta \neq \theta' \).
(e) \( \Rightarrow \) (a): We assume that \( \theta \) and \( \theta' \) are not TF equivalent and will find some \( S \in \text{brick} A \) such that \([\theta, \theta'] \cap \Theta_S \) has exactly one element.

Then, since \( \theta \) and \( \theta' \) are not TF equivalent, \( \mathcal{T}_\theta \neq \mathcal{T}_{\theta'} \) or \( \mathcal{F}_\theta \neq \mathcal{F}_{\theta'} \) holds. We only consider the case \( \mathcal{T}_\theta \neq \mathcal{T}_{\theta'} \), because a similar proof works in the other case.

In this case, we have \( \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \neq \emptyset \) or \( \mathcal{T}_{\theta'} \cap \mathcal{F}_\theta \neq \emptyset \) from Lemma 2.18. By exchanging \( \theta \) and \( \theta' \), we may assume \( \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \neq \emptyset \). We can take a nonzero module \( S \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \) such that \( \dim K S \leq \dim K M \) holds for all nonzero modules \( M \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \). Then, \( S \) is a brick by \([\text{DIRRT}]\) Lemma 3.8.

Since \( S \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \), we have \( \theta(S) \geq 0 \) and \( \theta'(S) < 0 \). Thus, there uniquely exists \( \theta'' \in [\theta, \theta'] \) such that \( \theta''(S) = 0 \), and it suffices to show \( \theta'' \in \Theta_S \).

By minimality, any proper nonzero quotient module of \( S \) must belong to \( \mathcal{T}_{\theta'} \). We have \( S \in \mathcal{T}_\theta \cap \mathcal{T}_{\theta'} \), and this clearly implies \( S \in \mathcal{T}_{\theta''} \). We can similarly prove that \( S \in \mathcal{F}_{\theta''} \). Therefore, \( S \in \mathcal{W}_{\theta''} \), which means that \( \theta'' \in \Theta_S \).

In general, there may exist infinitely many TF equivalence classes.
We define the following notions.

Definition 3.1. Let $U$ be a direct summand of $U$ isomorphic to a direct summand of $U$ closed under taking direct summands.

Definition 3.2. $K^b(proj A)$ for the set of isomorphism classes of basic 2-term presilting objects (resp. 2-term silting objects) in $\mathcal{K}$.

We write $\text{silt}_b A$ for the set of isomorphism classes of basic silting objects in $\mathcal{K}$.

Example 2.19. Let $A$ be the path algebra of the $m$-Kronecker quiver:

\[
\begin{array}{c}
1 \\
\vdots \\
2 \quad (m \text{ arrows})
\end{array}
\]

In Example 5.1 later, we give the wall-chamber structure of $K_0(proj A)_{\mathbb{R}}$, and this and Theorem 2.16 tell us the cardinality of the set of TF equivalence classes as follows:

- if $m = 0, 1$, then only finitely many TF equivalence classes exist;
- if $m = 2$, then the set of TF equivalence classes is infinite and countable;
- if $m \geq 3$, then there exist uncountably many TF equivalence classes.

3. The wall-chamber structures and the Koenig–Yang correspondences

3.1. Preparations on the Koenig–Yang correspondences. In this section, we study the relationship between stability conditions and the Koenig–Yang correspondences established in [KY, BY]. The Koenig–Yang correspondences are a collection of bijections between many important notions in the perfect derived category $K^b(proj A)$ and the bounded derived category $D^b(mod A)$, such as silting objects in $K^b(proj A)$, bounded t-structures with length heart in $D^b(mod A)$, and 2-term simple-minded collections in $D^b(mod A)$.

Before explaining the detail, we recall some notions here.

Let $\mathcal{C}$ be a triangulated category. We say that a triangulated subcategory $\mathcal{C}' \subset \mathcal{C}$ is thick if $\mathcal{C}'$ is closed under taking direct summands.

For every $U \in K^b(proj A)$, we define the full subcategory $\text{add} U \subset K^b(proj A)$ as the additive closure of $U$, that is, $U' \in \text{add} U$ holds if and only if there exists some $s \in \mathbb{Z}_{\geq 0}$ such that $U'$ is isomorphic to a direct summand of $U^s$.

Every $U \in K^b(proj A)$ admits a decomposition $U \cong \bigoplus_{i=1}^m U_i^{s_i}$ with all $U_i$ indecomposable, $s_i \geq 1$, and $U_i \not\cong U_j$ for any $i \neq j$. We set $|U| := m$, and say that $U$ is basic if $s_i = 1$ for all $i$.

Now, we recall the definition of silting objects in $K^b(proj A)$.

Definition 3.1. We define the following notions.

1. An object $U \in K^b(proj A)$ is said to be presilting if $\text{Hom}_{K^b(proj A)}(U, U[>0]) = 0$.
2. An object $T \in K^b(proj A)$ is said to be silting in $K^b(proj A)$ if $T$ is a presilting object and the smallest thick subcategory of $K^b(proj A)$ containing $U$ is $K^b(proj A)$ itself.

We write $\text{silt}_b A$ for the set of isomorphism classes of basic silting objects in $K^b(proj A)$.

In this paper, we mainly focus on the 2-term versions of these notions.

Definition 3.2. An object $U$ in $K^b(proj A)$ is said to be 2-term if $U$ is isomorphic to some complex $(P^{-1} \rightarrow P^0)$ whose terms except $-1$st and $0$th ones vanish. We write $2\text{-presilt}_b A$ (resp. 2-silt$_b A$) for the set of isomorphism classes of basic 2-term presilting objects (resp. 2-term silting objects) in $K^b(proj A)$.

2-term presilting and silting objects satisfy many nice properties as below; see also [KY Corollary 5.1] for (4).

Proposition 3.3. Let $U \in 2\text{-presilt}_b A$. Then the following assertions hold.

1. [AIR Proposition 2.16] There exists some $T \in 2\text{-silt}_b A$ such that $U \in \text{add} T$.
2. [AIR Proposition 3.3] The condition $|U| = n$ is equivalent to $U \in 2\text{-silt}_b A$.
3. [AIR Corollary 3.8] If $|U| = n - 1$, then $\#\{T \in 2\text{-silt}_b A \mid U \in \text{add} T\} = 2$.
4. [AI Theorem 2.27, Corollary 2.28] The indecomposable direct summands of each $T \in 2\text{-silt}_b A$ gives a $\mathbb{Z}$-basis of $K_0(proj A)$. 

Definition 3.4. Let $\mathcal{U}, \mathcal{V}$ be full subcategories of $D^b(\text{mod} A)$. Then, the pair $(\mathcal{U}, \mathcal{V})$ is called a t-structure if the following conditions hold:

(a) $\mathcal{U}[1] \subset \mathcal{U}$ and $\mathcal{V}[-1] \subset \mathcal{V}$;
(b) $\text{Hom}_{D^b(\text{mod} A)}(\mathcal{U}, \mathcal{V}[-1]) = 0$;
(c) for any $X \in D^b(\text{mod} A)$, there exists a triangle $X' \to X \to X'' \to X'[1]$ with $X' \in \mathcal{U}$ and $X'' \in \mathcal{V}[-1]$.

A t-structure $(\mathcal{U}, \mathcal{V})$ is said to be bounded if

$$\bigcup_{i \in \mathbb{Z}} \mathcal{U}[i] = D^b(\text{mod} A) = \bigcup_{i \in \mathbb{Z}} \mathcal{V}[i].$$

For a t-structure $(\mathcal{U}, \mathcal{V})$, the intersection $\mathcal{U} \cap \mathcal{V}$ is called the heart, which is an abelian category $\text{BBD}$. If the heart $\mathcal{U} \cap \mathcal{V}$ is an abelian length category, then the t-structure $(\mathcal{U}, \mathcal{V})$ is said to be with length heart. We define $t\text{-str} A$ as the set of bounded t-structures in $D^b(\text{mod} A)$ with length heart.

Moreover, a t-structure $(\mathcal{U}, \mathcal{V})$ is said to be intermediate if $D^{\leq -1} \subset \mathcal{U} \subset D^{\leq 0}$, where

$$D^{\leq k} := \{X \in D^b(\text{mod} A) \mid H^i(X) = 0 \text{ for } i > k\}.$$  

An intermediate t-structure in $D^b(\text{mod} A)$ is always bounded, and we write $\text{int-t-str} A$ for the set of intermediate t-structures in $D^b(\text{mod} A)$ with length heart.

We also use 2-term simple-minded collections, which are defined as follows.

Definition 3.5. A set $\mathcal{X}$ of isomorphism classes of objects in $D^b(\text{mod} A)$ is called a simple-minded collection in $D^b(\text{mod} A)$ if the following conditions are satisfied:

(a) for any $X \in \mathcal{X}$, the endomorphism ring $\text{End}_{D^b(\text{mod} A)}(X)$ is a division ring;
(b) for any $X_1, X_2 \in \mathcal{X}$ with $X_1 \neq X_2$, we have $\text{Hom}_{D^b(\text{mod} A)}(X_1, X_2) = 0$;
(c) for any $X_1, X_2 \in \mathcal{X}$ and $k \in \mathbb{Z}_<$, we have $\text{Hom}_{D^b(\text{mod} A)}(X_1, X_2[k]) = 0$;
(d) the smallest thick subcategory of $D^b(\text{mod} A)$ containing $\mathcal{X}$ is $D^b(\text{mod} A)$ itself.

We write $\text{smc} A$ for the set of simple-minded collections in $D^b(\text{mod} A)$.

Moreover, a simple-minded collection $\mathcal{X}$ in $D^b(\text{mod} A)$ is called a 2-term simple-minded collection in $D^b(\text{mod} A)$ if the $i$th cohomology $H^i(X)$ vanishes for any $X \in \mathcal{X}$ and $i \in \mathbb{Z} \setminus \{-1, 0\}$. We write $\text{2-smc} A$ for the set of 2-term simple-minded collections in $D^b(\text{mod} A)$.

Each simple-minded collection $\mathcal{X}$ in $D^b(\text{mod} A)$ has exactly $n$ elements [KY] Corollary 5.5, which is equal to the number of indecomposable direct summands of a silting object in $K^b(\text{proj} A)$.

The following bijections between $\text{silt} A$, $t\text{-str} A$ and $\text{smc} A$ are included in the Koenig–Yang correspondences.

Proposition 3.6. The following assertions hold.

(1) [KY] Theorem 6.1 There exist the following bijections:

(a) $\text{silt} A \to t\text{-str} A$ sending $T \in \text{silt} A$ to the t-structure $(T[<0], T[>0])$, where

$$T[<0] := \{U \in K^b(\text{proj} A) \mid \text{Hom}_{D^b(\text{mod} A)}(T[k], U) = 0 \text{ holds for any } k < 0\},$$

$$T[>0] := \{U \in K^b(\text{proj} A) \mid \text{Hom}_{D^b(\text{mod} A)}(T[k], U) = 0 \text{ holds for any } k > 0\};$$ 

(b) $t\text{-str} A \to t\text{-str} A$ sending $T \in t\text{-str} A$ to the t-structure $(T[<0], T[>0])$, where

$$T[<0] := \{U \in K^b(\text{proj} A) \mid \text{Hom}_{D^b(\text{mod} A)}(T[k], U) = 0 \text{ holds for any } k < 0\},$$

$$T[>0] := \{U \in K^b(\text{proj} A) \mid \text{Hom}_{D^b(\text{mod} A)}(T[k], U) = 0 \text{ holds for any } k > 0\};$$ 

(c) $t\text{-str} A \to \text{smc} A$ sending $T \in t\text{-str} A$ to the simple-minded collection $\mathcal{X}$ consisting of all objects in $T[<0] \cup T[>0]$.

(d) $\text{smc} A \to \text{smc} A$ sending $\mathcal{X} \in \text{smc} A$ to the simple-minded collection $\mathcal{Y}$ consisting of all objects in $\mathcal{X}$.
Definition 3.8. Let families \((X_i)_{i=1}^n\) be (2-Proposition 3.6) and \((Y_i)_{i=1}^n\) satisfying (Corollary 4.3)

(b) \(t\)-str \(A \to \text{scl} \ A \) sending \((U, V) \in t\)-str \(A \) to the set of isomorphism classes of simple objects of the heart \(U \cap V \).

(2) \([BY]\) Lemma 5.3] Let \(X \in \text{scl} \ A \) correspond to \(T \in \text{silt} \ A \) under the bijections in (1). Then, there exist families \((T_i)_{i=1}^n\) and \((X_i)_{i=1}^n\) satisfying

(a) \(T = \bigoplus_{i=1}^n T_i\);

(b) \(X = \{X_i\}_{i=1}^n\);

(c) set \(R_j := \text{End}_{D^b_{\text{mod} A}}(X_j)\), then

\[
\text{Hom}_{D^b_{\text{mod} A}}(P_i, X_j) \cong \begin{cases} \text{R}_j \text{ as left } \text{R}_j\text{-modules} & (i = j) \\ 0 & (i \neq j) \end{cases}
\]

We also need the 2-term restrictions of the Koenig–Yang correspondences.

Proposition 3.7. \([BY\) Corollary 4.3] The bijections \(\text{silt} \ A \to t\)-str \(A \) and \(t\)-str \(A \to \text{scl} \ A \) given in Proposition 3.6 are restricted to bijections \(2\text{-silt} \ A \to \text{int}\t\)-str \(A \) and \(\text{int}\t\)-str \(A \to 2\text{-scl} \ A \).

Thus, we shall use the following notation in the rest of this paper.

Definition 3.8. Let \(X \in 2\text{-scl} \ A \) correspond to \(T \in 2\text{-silt} \ A \) in the bijections above. We take families \((T_i)_{i=1}^n\) and \((X_i)_{i=1}^n\) satisfying

(a) \(T = \bigoplus_{i=1}^n T_i\);

(b) \(X = \{X_i\}_{i=1}^n\);

(c) \((T_i)_{i=1}^n\) and \((X_i)_{i=1}^n\) give dual bases of \(K_0(\text{proj} \ A)\) and \(K_0(\text{mod} \ A)\); more precisely,

\[
\langle T_i, X_j \rangle = \begin{cases} \dim_K \text{End}_{D^b_{\text{mod} A}}(X_j) & (i = j) \\ 0 & (i \neq j) \end{cases}
\]

3.2. Cones of presilting objects. Now, we define cones \(C(U), C^+(U) \subset K_0(\text{proj} \ A) \) for each object \(U \in K^b(\text{proj} \ A)\). Decompose \(U\) as \(\bigoplus_{i=1}^n U_i \in K^b(\text{proj} \ A)\) with \(U_i\) indecomposable, and then set

\[
C(U) := \{a_1[U_1] + \cdots + a_m[U_m] | a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}\},
\]

\[
C^+(U) := \{a_1[U_1] + \cdots + a_m[U_m] | a_1, \ldots, a_m \in \mathbb{R}_{> 0}\}.
\]

In particular, \(C(0) = C^+(0) = \{0\}\) for \(0 \in K^b(\text{proj} \ A)\).

We mainly deal with \(C(U)\) and \(C^+(U)\) for \(U \in 2\text{-presilt} \ A\). In this case, \(C^+(U)\) is a relative interior of the cone \(C(U)\), since the indecomposable direct summands of \(U\) are linearly independent by Proposition 3.3. If \(T \in 2\text{-silt} \ A\) and its indecomposable direct summands are \(T_1, \ldots, T_n\), then the cone \(C(T)\) has exactly \(n\) walls \(C(T/T_i)\) with \(i \in \{1, \ldots, n\}\), and each wall \(C(T/T_i)\) corresponds to the mutation of \(T\) at \(T_i\).

When we consider the intersection of cones for 2-term presilting objects, the following properties on uniqueness of presilting objects by \([DLJ]\) is crucial. We remark that (1) is an analogue of \([DK\) 2.3, Theorem].

Proposition 3.9. Let \(U, U'\) be (not necessarily basic) 2-term presilting objects in \(K^b(\text{proj} \ A)\). Then, we have the following assertions.

(1) \([DLJ]\) Theorem 6.5] If \([U] = [U'] \in K_0(\text{proj} \ A)\), then \(U \cong U'\) in \(K^b(\text{proj} \ A)\).

(2) The following conditions are equivalent:

(a) \(\text{add} U = \text{add} U'\),

(b) \(C(U) = C(U')\),

(c) \(C^+(U) = C^+(U')\).
(3) If \( U'' \in K^b(\text{proj } A) \) satisfies \( \text{add } U \cap \text{add } U' = \text{add } U'' \), then \( C(U) \cap C(U') = C(U'') \).

Proof. Parts (2) and (3) immediately follow from (1) as in [DHI, Corollary 6.7].

In particular, if \( T' \in 2\text{-silt } A \) is not isomorphic to \( T \in 2\text{-silt } A \), then \( C^+(T) \cap C(T') = \emptyset \).

We here prepare some symbols. For each \( M \in \mod A \), we define the following subcategories of \( \mod A \):

- \( M^\perp := \{ X \in \mod A \mid \text{Hom}_A(M, X) = 0 \} \),
- \( ^\perp M := \{ X \in \mod A \mid \text{Hom}_A(X, M) = 0 \} \),
- \( \text{Fac } M := \{ X \in \mod A \mid \text{there exists a surjection } M^{\oplus s} \to X \} \),
- \( \text{Sub } M := \{ X \in \mod A \mid \text{there exists an injection } X \to M^{\oplus s} \} \).

We write \( \text{inj } A \) for the category of finite-dimensional injective \( A \)-modules, and let \( \nu \) denote the Nakayama functor \( K^b(\text{proj } A) \to K^b(\text{inj } A) \). Now, we can state one of the main results of this section.

**Theorem 3.10.** Let \( U \in 2\text{-presilt } A \). Then, the subset \( C^+(U) \subset K_0(\text{proj } A) \) is a \( TF \) equivalence class satisfying

\[
(\mathcal{T}_U, \mathcal{F}_U) = (\perp H^{-1}(\nu U), \text{Sub } H^{-1}(\nu U)), \quad (\mathcal{T}_U, \mathcal{F}_U) = (\text{Fac } H^0(U), \text{Sub } H^0(U))
\]

To prove the theorem above, we apply some results on \( \tau \)-tilting theory to 2-term presilting objects. First, \( H^0(U) \) is \( \tau \)-rigid and \( H^{-1}(\nu) \) is \( \tau^{-1} \)-rigid by [AIR, Lemma 3.4]. Thus, the torsion classes and torsion-free classes

\[
\mathcal{T}_U := \perp H^{-1}(\nu U), \quad \mathcal{F}_U := \text{Sub } H^{-1}(\nu U), \quad \mathcal{T}_U := \text{Fac } H^0(U), \quad \mathcal{F}_U := \text{Sub } H^0(U)
\]

are functorially finite by [AS, Theorem 5.10] (see [MS, Section 2] for the definition of functorially finite subcategories in \( \mod A \)). It is easy to see that \( \mathcal{T}_U \subset \mathcal{T}_U \) and that \( \mathcal{F}_U \subset \mathcal{F}_U \).

If \( T \in 2\text{-silt } A \), then \( H^0(T) \) is a support \( \tau \)-tilting module in \( \mod A \) by [AIR, Proposition 3.6], so we have \( \mathcal{T}_T = \mathcal{T}_T \) and \( \mathcal{F}_T = \mathcal{F}_T \) from [AIR, Proposition 2.16]. Conversely, if \( \mathcal{T}_U = \mathcal{T}_U \) and \( \mathcal{F}_U = \mathcal{F}_U \) hold, then \( U \) is 2-term silting by [AIR, Theorem 2.12].

The torsion classes \( \mathcal{T}_U \subset \mathcal{T}_U \) satisfy the following property.

**Lemma 3.11.** [AIR, Proposition 2.9] Let \( U \in 2\text{-presilt } A \) and \( T \in 2\text{-silt } A \). Then, \( U \in \text{add } T \) if and only if \( \mathcal{T}_U \subset \mathcal{T}_U \subset \mathcal{T}_U \).

One of the main results of [AIR] is that the set \( 2\text{-silt } A \) has bijections to the set \( \text{f-tors } A \) of functorially finite torsion classes and the set \( \text{f-torf } A \) of functorially finite torsion-free classes in \( \mod A \).

**Proposition 3.12.** [AIR, Theorems 2.7, 3.2] There exist bijections

\[
\begin{align*}
2\text{-silt } A & \to \text{f-tors } A, & T & \mapsto \mathcal{T}_T = \mathcal{T}_T; \\
2\text{-silt } A & \to \text{f-torf } A, & T & \mapsto \mathcal{F}_T = \mathcal{F}_T.
\end{align*}
\]

Therefore, for \( U \in 2\text{-presilt } A \), there uniquely exists \( T \in 2\text{-silt } A \) satisfying \( U \in \text{add } T \) and \( \mathcal{T}_T = \mathcal{T}_U \). We call this \( T \) the Bongartz completion of \( U \). Similarly, we can uniquely take \( T' \in 2\text{-silt } A \) satisfying \( U \in \text{add } T \) and \( \mathcal{T}_{T'} = \mathcal{T}_U \), and such \( T' \) is called the co-Bongartz completion of \( U \). For these two completions, we have the following properties.

**Lemma 3.13.** Let \( U \in 2\text{-presilt } A \) and \( T, T' \) be the Bongartz completion and the co-Bongartz completion of \( U \), respectively.

1. We have \( \text{add } T \cap \text{add } T' = \text{add } U \).
2. If \( U' \in 2\text{-presilt } A \) satisfies \( \mathcal{T}_{U'} = \mathcal{T}_U \) and \( \mathcal{T}_{U'} = \mathcal{T}_U \), then \( U \cong U' \).
(3) Let $V \in \text{2-presilt } A$, then $V \in \text{add } U$ if and only if $T_V \subset T_U \subset T_V$.

Proof. (1) Clearly, $add T \cap add T' \supset add U$, so we let $V \in (add T \cap add T') \setminus add U$ be indecomposable and deduce a contradiction. We consider the mutation $T''$ of $T$ at $V$, then $U$ is a direct summand of $T''$, since $V \notin \text{add } U$. Thus, $T_U \subset T_{T''} \subset T_V$ by Lemma 3.11. On the other hand, due to $V \in \text{add } T \cap \text{add } T'$ and the choice of $T$ and $T'$, we have $T_V \subset T_U = T_V$ and $T_U = T_T \subset T_V$.
Therefore, $T_V \subset T_{T''} \subset T_V$, so Lemma 3.11 implies that $V$ is a direct summand of $T''$. This contradicts that $T''$ is the mutation of $T$ at $V$. Therefore, $add T \cap add T' = add U$.

(2) By assumption, $T$ and $T'$ are the Bongartz completion and the co-Bongartz completion also of $U'$. Then, (1) implies $\text{add } U = \text{add } U'$. Since $U$ and $U'$ are basic, we get $U \cong U'$.

(3) The “only if” part is easy. For the “if” part, assume $T_V \subset T_U \subset T_T \subset T_V$. This implies $T_V \subset T_{T''} \subset T_T \subset T_V$, so we obtain that $V \in \text{add } T \cap \text{add } T'$ from Lemma 3.11. Then, (1) tells us that $V \in \text{add } U$.

In order to connect numerical torsion pairs and functorially finite torsion pairs, the following result by Yurikusa [Yur] is important.

**Proposition 3.14.** [Yur, Proposition 3.3] Let $U \in \text{2-presilt } A$ and $\theta \in C^+(U)$. Then,

$$\langle \Theta_\theta, F_\theta \rangle = \langle T_U, F_U \rangle, \quad \langle \Theta_\theta, F_\theta \rangle = \langle T_U, F_U \rangle.$$  

This implies that $C^+(U)$ is contained in a TF equivalence class. Thus, to prove Theorem 3.10, it remains to show the converse. For this purpose, we recall the following result on 2-term simple-minded collections by Brüstle–Yang [BY].

**Lemma 3.15.** [BY, Remark 4.11] Let $X \in 2$-smc $A$, then every $X \in X$ satisfies $X \in \text{brick } A$ or $X \in \text{(brick } A)[1]$ up to isomorphisms in $D^b(\text{mod } A)$.

Therefore, it is natural to consider the intersections $X \cap \text{mod } A$ and $X[-1] \cap \text{mod } A$ for $X \in 2$-smc $A$. We call a subset $S$ of brick $A$ a semibrick in $\text{mod } A$ if $\text{Hom}_A(S, S') = 0$ holds for any two different (hence, non-isomorphic) elements $S, S' \in S$, and write $\text{brick } A$ for the set of semibricks in $\text{mod } A$. By definition, the sets $X \cap \text{mod } A$ and $X[-1] \cap \text{mod } A$ are semibricks.

In [Asa], we introduced the notions of left-finiteness and right-finiteness of semibricks: a semibrick $S$ is said to be left finite (resp. right finite) if the smallest torsion (resp. torsion-free) class $T(S)$ (resp. $F(S)$) in $\text{mod } A$ containing $S$ is functorially finite in $\text{mod } A$. We write $f_l$-$\text{brick } A$ (resp. $f_r$-$\text{brick } A$) for the set of left finite (resp. right finite) semibricks. In that paper, we obtained the following bijections.

**Proposition 3.16.** [Asa, Theorem 2.3] There exist bijections

$$2\text{-smc } A \rightarrow f_l\text{-brick } A, \quad X \mapsto X \cap \text{mod } A,$$

$$2\text{-smc } A \rightarrow f_r\text{-brick } A, \quad X \mapsto X[-1] \cap \text{mod } A.$$  

Moreover, if $X \in 2\text{-smc } A$ corresponds to $T \in 2\text{-silt } A$ in the bijections in Proposition 3.7, then $(T(X \cap \text{mod } A), F(X[-1] \cap \text{mod } A)) = (T_T, F_T) = (T_T, F_T).$

Now, we can prove Theorem 3.10.

**Proof of Theorem 3.10.** It suffices to prove that $\theta \in C^+(U)$ holds if and only if $(\Theta_\theta, F_\theta) = (T_U, F_U)$ and $(\Theta_\theta, F_\theta) = (T_U, F_U)$.

The “only if” part is nothing but Proposition 3.14.

Thus, it remains to show the “if” part. We assume $(\Theta_\theta, F_\theta) = (T_U, F_U)$ and $(\Theta_\theta, F_\theta) = (T_U, F_U)$. We take the Bongartz completion $T = \bigoplus_{i=1}^n T_i$ of $U$ with $T_i$ indecomposable.
If $\theta \in C(T)$, then there exists some direct summand $U'$ of $T$ such that $\theta \in C^+(U')$. Then, $\mathcal{T}_U' = \mathcal{T}_\theta = \mathcal{T}_U$ and $\mathcal{T}_U'' = \mathcal{T}_0 = \mathcal{T}_U$ follow from Proposition 3.14 and the assumption, and we get $U' \cong U$ by Lemma 3.13(2), thus, it is sufficient to prove $\theta \in C(T)$. Since $[T_1], \ldots, [T_n]$ is a basis of $K_0(\text{proj} A)_{\mathbb{R}}$, there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $\theta = \sum_{i=1}^n a_i [T_i]$. Thus, we prove $a_i \geq 0$ for all $i$. We take $X = \{X_i\}^n_{i=1} \in 2\text{-smt} A$ corresponding to $T \in 2\text{-silt} A$ as in Definition 3.8. For each $i$, we have $\theta(X_i) = a_i \dim_K \text{End}_{D^b(\text{mod} A)}(X_i)$.

By Proposition 3.16, if $X_i \in \text{mod} A$, then $X_i$ belongs to $\mathcal{T}_T = \mathcal{T}_\theta$, so $\theta(X_i) \geq 0$ holds; and otherwise, $X_i$ belongs to $\text{mod} A[1]$ by Lemma 3.15 and we get $X_i[-1] \in \mathcal{F}_T = \mathcal{F}_U = \mathcal{F}_\theta$ and $\theta(X_i[-1]) < 0$, which implies $\theta(X_i) > 0$. Therefore, $a_i \geq 0$ holds for all $i$, and we obtain $\theta \in C(T)$. 

### 3.3. All chambers come from silting objects.

We conclude this section by results on the chambers of the wall-chamber structure of $K_0(\text{proj} A)_{\mathbb{R}}$. The chambers are nothing but the connected components of the open subset $K_0(\text{proj} A)_{\mathbb{R}} \setminus \bigcup_{M \in \text{mod} A} \Theta_M$, and by Theorem 2.16, they are precisely the TF equivalent classes whose interiors are nonempty. We have a chamber $C^+(T)$ for each $T \in 2\text{-silt} A$, and actually, all chambers are obtained in this way. More precisely, we have the following properties.

**Theorem 3.17.** We set $$K_0(\text{proj} A)_Q := \{a_1[P_1] + \cdots + a_n[P_n] \mid a_1, \ldots, a_n \in \mathbb{Q}\} \subset K_0(\text{proj} A)_{\mathbb{R}}.$$ Then, the following statements hold.

1. In $K_0(\text{proj} A)_Q$, we have an equation
$$\prod_{T \in 2\text{-silt} A} (C^+(T) \cap K_0(\text{proj} A)_Q) = K_0(\text{proj} A)_Q \setminus \bigcup_{M \in \text{mod} A \setminus \{0\}} \Theta_M.$$ 

2. In $K_0(\text{proj} A)_{\mathbb{R}}$, we have an equation
$$\prod_{T \in 2\text{-silt} A} C^+(T) = K_0(\text{proj} A)_{\mathbb{R}} \setminus \bigcup_{M \in \text{mod} A \setminus \{0\}} \Theta_M,$$
where the left-hand side is the decomposition into the connected components.

3. There exists a bijection
$$2\text{-silt} A \rightarrow \{\text{all the TF equivalence classes whose interiors are nonempty}\}$$
given by $T \mapsto C^+(T)$.

To show this, we use the bijection entitled the *Happel–Reiten–Smalø tilt* $[\text{HRS}]$ between the set of intermediate $t$-structures in $D^b(\text{mod} A)$ and the set of torsion pairs in $\text{mod} A$. This bijection sends a torsion pair $(\mathcal{T}, \mathcal{F})$ to the intermediate $t$-structure $(\mathcal{U}, \mathcal{V})$ in $D^b(\text{mod} A)$, where
$$\mathcal{U} := \{X \in D^b(\text{mod} A) \mid H^0(X) \in \mathcal{T} \text{ and } H^i(X) = 0 \text{ for } i > 0\},$$
$$\mathcal{V} := \{X \in D^b(\text{mod} A) \mid H^0(X) \in \mathcal{F} \text{ and } H^i(X) = 0 \text{ for } i < 0\},$$
and the inverse map sends an intermediate $t$-structure $(\mathcal{U}, \mathcal{V})$ to the torsion pair $(\mathcal{U} \cap \text{mod} A, \mathcal{V} \cap \text{mod} A)$ in $\text{mod} A$.

For our purpose, it is important to know which torsion pairs in $\text{mod} A$ correspond to intermediate $t$-structures with length heart. The answer is given by the following result of $[\text{BY}]$.

**Proposition 3.18.** $[\text{BY}, \text{Theorem 4.9}]$ There exist bijections $\text{int-t-str} A \rightarrow f\text{-tors} A$ and $\text{int-t-str} A \rightarrow f\text{-torf} A$ given by $(\mathcal{U}, \mathcal{V}) \mapsto \mathcal{U} \cap \text{mod} A$ and $(\mathcal{U}, \mathcal{V}) \mapsto \mathcal{V} \cap \text{mod} A$, respectively. Moreover, for $T \in 2\text{-silt} A$, we have $T[<0]^+ \cap \text{mod} A = \mathcal{T}_T$ and $T[>0]^+ \cap \text{mod} A = \mathcal{F}_T$. 

Now, we can show Theorem 3.17 as follows. We remark that some important part of our proof has been already done by Bridgeland [Bri].

Proof of Theorem 3.17 (1) It suffices to show that there exists some $T \in 2$-silt $A$ such that $\theta \in C^+(T)$ if and only if $W_0 = \{0\}$ in the case that $\theta \in K_0(\text{proj} A)_R$.

The “only if” part follows from Propositions 3.12 and 3.14.

Conversely, we assume that $W_0 = \{0\}$. Clearly, $(\mathcal{T}_\theta, \mathcal{F}_\theta) = (\mathcal{T}_0, \mathcal{F}_0)$. Since $\theta \in K_0(\text{proj} A)_R$, the proof of [Bri, Lemma 7.1] tells us that the heart $\mathcal{H}$ of the corresponding $t$-structure $(\mathcal{U}, \mathcal{V})$ for the torsion pair $(\mathcal{T}_\theta, \mathcal{F}_\theta)$ is a length category. Thus, there exists some $T \in 2$-silt $A$ such that $(T[<0]^+, T[>0]^+) = (\mathcal{U}, \mathcal{V})$ by Proposition 3.7 and then, $(\mathcal{T}_\theta, \mathcal{F}_\theta) = (\mathcal{T}_T, \mathcal{F}_T)$ by Proposition 3.18. Applying Theorem 3.10, we get $\theta \in C^+(T)$. The “if” part has been proved.

(2) We set $C := \bigsqcup_{T \in 2\text{-silt} A} C^+(T)$ and $W := \bigcup_{M \in \text{mod} A \setminus \{0\}} \Theta_M$.

As in the proof of (1), $C \subset K_0(\text{proj} A)_R \setminus W$ follows, and since the left-hand side is an open subset of $K_0(\text{proj} A)_R$, we have $C \subset K_0(\text{proj} A)_R \setminus W$.

On the other hand, if $\theta \in K_0(\text{proj} A)_R \setminus W$, then there exists a convex neighborhood $N \subset K_0(\text{proj} A)_R \setminus W$ of $\theta$, and we can take $\theta' \in N \cap K_0(\text{proj} A)_R$. Then, there exists $T \in 2$-silt $A$ such that $\theta' \in C^+(T)$ by (1). Because $N$ is convex, the line segment $[\theta, \theta']$ is contained in $N \subset K_0(\text{proj} A)_R \setminus W$, thus $\theta$ and $\theta'$ are TF equivalence by Theorem 2.16. Since $C^+(T)$ is a TF equivalence class by Theorem 3.10, $\theta$ also belongs to $C^+(T)$. Thus, $\theta \in C$ as desired.

Clearly, $C = \bigsqcup_{T \in 2\text{-silt} A} C^+(T)$ is the decomposition into the connect components.

(3) This map is well-defined by Theorem 3.10 and injective by Proposition 3.12. Now, we begin the proof of the surjectivity. Let $E$ be a TF equivalence class in $K_0(\text{proj} A)_R$ whose interior is nonempty, and take $\theta \in E \cap K_0(\text{proj} A)_R$. Then, $W_0 = \{0\}$ follows; indeed, for any $M \in W_0$, we have $\theta'(M) = \theta(M) = 0$ for all $\theta' \in E$, and then $M$ must be zero, because the interior of $E$ is nonempty. Thus, $\theta$ does not belong to $\bigcup_{M \in \text{mod} A \setminus \{0\}} \Theta_M$, and (1) implies that there exists $T \in 2$-silt $A$ such that $\theta \in C^+(T)$. Since $C^+(T)$ is a TF equivalence class, $E = [\theta] = C^+(T)$. Consequently, the surjectivity has been proved. 

$\square$

4. Reduction of the wall-chamber structures

4.1. $\tau$-tilting reduction and the local wall-chamber structures. Recall that we obtained an injection from $2$-presilt $A$ to the set of TF equivalence classes sending $U$ to $C^+(U)$ in Theorem 3.10. For $\theta \in C^+(U)$, the $\theta$-semistable subcategory $W_\theta$ is a wide subcategory $W_U := \mathcal{T}_U \cap \mathcal{F}_U$. The wide subcategory $W_U$ was investigated by [Asa, DIRRT] as $\tau$-tilting reduction in the context of $\tau$-rigid pairs, and they found that $W_U$ is equivalent to the module category $\text{mod} B$ for an algebra $B$ constructed from $U$; see [Asa, Theorem 3.8] and [DIRRT, Theorem 4.12].

The corresponding result for 2-term presilting objects is given as follows, and we write a direct proof by using [Asa, LY] for the convenience of the readers. We can check that our $\varphi$ is compatible with their original equivalence $\text{Hom}_A(H^0(T), ?) : W_U \rightarrow \text{mod} B$ as in the proof of [Asa, Theorem 3.16].

Proposition 4.1. Let $U \in 2$-presilt $A$. Define $T \in 2$-silt $A$ as its Bongartz completion, and set an algebra $B$ as $\text{End}_{K^0(\text{proj} A)}(T)/(e)$, where $e$ is the idempotent $(T \rightarrow U \rightarrow T) \in \text{End}_{K^0(\text{proj} A)}(T)$. Then, we have an equivalence

$$\varphi := \text{Hom}_A(T, ?) : W_U \rightarrow \text{mod} B.$$ 

Moreover, let $T = \bigoplus_{i=1}^n T_i$ and $U = \bigoplus_{i=m+1}^n T_i$ with $T_i$ indecomposable, take $X \in 2$-smc $A$ corresponding to $T$, and define $X_1, X_2, \ldots, X_n \in X$ as in Definition 3.8. Then, $\{X_1, X_2, \ldots, X_m\}$ is the set of simple objects in $W_U$, and sent to the set of simple $B$-modules.
Proof. First, we check that $X_i \in \mathcal{W}_U$ for $i \in \{1, \ldots, m\}$. Let $T'$ be the mutation of $T$ at $T_i$, then since $U$ is a direct summand of $T/T_i$, we have $\mathcal{T}_{T'} \subset \mathcal{T}_{T'/T_i} \subset \mathcal{T}_T$ from Lemma 3.13. Then, from the proof of [Asa Theorem 3.12], $X_i \in \text{mod } A$ follows, and moreover, $X_i \in \mathcal{T}_{T} \cap \mathcal{T}_{T'}$. By Lemma 3.13 again, we get $X_i \in \mathcal{T}_T \cap \mathcal{T}_{T'} \in \mathcal{T}_U \cap \mathcal{T}_{U'} = \mathcal{W}_U$.

Now, we recall the equivalence $\text{Hom}_{\text{mod}(\text{mod } A)}(T, ?) : T[\neq 0] \rightarrow \text{mod } \text{End}_{\text{K}^b(\text{proj } A)}(T)$ [IX Proposition 4.8], which sends $\mathcal{X}$ to the set of simple $\text{End}_{\text{K}^b(\text{proj } A)}(T)$-modules. Then, the same strategy as the proof of [Asa Theorem 3.15] yields that the equivalence above is restricted to an equivalence $\varphi = \text{Hom}_A(T, ?) : \mathcal{W}_U \rightarrow \text{mod } B$ between their Serre subcategories, sending $\{X_1, X_2, \ldots, X_m\}$ to the simple objects in $\text{mod } B$. \hfill \Box

Moreover, Jasso proved that the set $2\text{-silt } B$ has a bijection from the subset $2\text{-silt}_U A := \{V \in 2\text{-silt } A \mid U \in \text{add } V\}$ of $2\text{-silt } A$ compatible with Proposition 3.12. Actually, this bijection can be extended to $2\text{-term presilting objects}.

**Proposition 4.2.** Let $U \in 2\text{-presilt } A$, and consider the functor

$$\text{red} := \text{Hom}_{\text{K}^b(\text{proj } A)}(T, ?)/[U] : \text{K}^b(\text{proj } A) \rightarrow \text{K}^b(\text{proj } B),$$

where $[U]$ is the ideal in $\text{K}^b(\text{proj } A)$ generated by $U$.

1. [Jas Theorems 3.14, 3.16, 4.12] The functor red induces a bijection $\text{red} : 2\text{-silt}_U A \rightarrow 2\text{-silt } B$ satisfying $\mathcal{T}_{\text{red}(V)} = \varphi(\mathcal{T}_V \cap \mathcal{W}_U) \in \text{f-tors } B$ and $\mathcal{F}_{\text{red}(V)} = \varphi(\mathcal{F}_V \cap \mathcal{W}_U) \in \text{f-tors } B$.

2. Set $2\text{-presilt}_U A := \{V \in 2\text{-presilt } A \mid U \in \text{add } V\}$. Then, the functor red gives a bijection $\text{red} : 2\text{-presilt}_U A \rightarrow 2\text{-presilt } B$.

**Proof.** The part (2) is verified by considering the Bongartz completion and the co-Bongartz completion of each $V$ and using Lemma 3.13. \hfill \Box

Next, we will investigate the relationship between the Grothendieck groups $K_0(\text{proj } A)_{\mathbb{R}}$ and $K_0(\text{proj } B)_{\mathbb{R}}$ in terms of the functor $\varphi_U$. For this purpose, we define a subset $N_U \subset K_0(\text{proj } A)_{\mathbb{R}}$ by

$$N_U := \{\theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid T_U \subset \mathcal{T}_\theta \subset \mathcal{T}_\theta \subset \mathcal{T}_U\}$$

for each $U \in 2\text{-presilt } A$. If $\theta \in N_U$, then $\mathcal{F}_\theta \subset \mathcal{T}_U$; hence, $\mathcal{W}_\theta \subset \mathcal{W}_U$. By definition, $N_U$ is a union of some TF equivalent classes in $K_0(\text{mod } A)_{\mathbb{R}}$.

The following property is easy to deduce, but crucial.

**Lemma 4.3.** Let $U \in 2\text{-presilt } A$. Then, $N_U \subset K_0(\text{proj } A)_{\mathbb{R}}$ is an open neighborhood of $C^+(U)$.

**Proof.** We can check that $T_U \subset \mathcal{T}_\theta \subset \mathcal{T}_\theta \subset \mathcal{T}_U$ if and only if both $H^0(U) \subset \mathcal{T}_\theta$ and $H^{-1}(\nu U) \subset \mathcal{T}_\theta$ hold. The latter conditions can be written as a collection of finitely many strict linear inequalities on $\theta$, and only if both $H^0(U) \subset \mathcal{T}_\theta$ and $H^{-1}(\nu U) \subset \mathcal{T}_\theta$ hold, then $U$ is an open subset of $K_0(\text{proj } A)_{\mathbb{R}}$. Moreover, Proposition 3.14 tells us that $C^+(U)$ is contained in $N_U$. \hfill \Box

In the setting of Proposition 4.1, $\varphi(X_i)$ is a simple $B$-module, so set $P_i^B \in \text{proj } B$ as the projective cover of $S_i^B := \varphi(X_i)$. We define a linear map $\pi : K_0(\text{proj } A)_{\mathbb{R}} \rightarrow K_0(\text{proj } B)_{\mathbb{R}}$ by

$$\pi(\theta) := \sum_{i=1}^m \frac{\theta(X_i)}{d_i} [P_i^B],$$

where $d_i := \dim_K \text{End}_A(X_i) = \dim_K \text{End}_B(\varphi(X_i))$. Then, $\pi$ satisfies the following nice properties in the subset $N_U$.

**Lemma 4.4.** Let $U \in 2\text{-presilt } A$. Then, the following assertions hold.

1. The restriction $\pi|_{N_U} : N_U \rightarrow K_0(\text{proj } B)_{\mathbb{R}}$ is surjective.
(2) For any $\theta \in K_0(\text{proj} A)_R$ and $M \in \mathcal{W}_U$, we have $\pi(\theta)(\varphi(M)) = \theta(M)$.

(3) For any $\theta \in N_U$, we have the following equations in $\text{mod } B$:

\[
\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}, \\
\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}, \\
\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}, \\
\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}, \\
\varphi(W_\theta) = W_{\pi(\theta)}.
\]

(4) Let $\theta, \theta' \in N_U$. The elements $\theta$ and $\theta'$ are TF equivalent in $K_0(\text{proj} A)_R$ if and only if $\pi(\theta)$ and $\pi(\theta')$ are TF equivalent in $K_0(\text{proj} B)_R$. In particular, if $\pi(\theta) = \pi(\theta')$, then $\theta$ and $\theta'$ are TF equivalent.

**Proof.** (1) Since $N_U$ is an open neighborhood of $[U]$, the image $\pi(N_U)$ is an open neighborhood of 0 in $K_0(\text{proj} B)_R$. Moreover, $N_U = \mathbb{R}_{>0} \cdot N_U$ follows from the definition, so $\pi(N_U) = \mathbb{R}_{>0} \cdot \pi(N_U)$. Thus, $\pi(N_U)$ must be $K_0(\text{proj} B)_R$.

(2) Since $\theta$ and $\pi(\theta)$ are linear maps, we may assume that $M$ is simple in $\mathcal{W}_U$; in other words, $M = X_j$ for some $j \in \{1, 2, \ldots, m\}$. Then, $\varphi(M) = S_j^B$. Because $P_i^B$ is the projective cover of $S_i^B$ in $\text{mod } B$ for $i \in \{1, 2, \ldots, m\}$, we have $\langle P_j^B, S_j^B \rangle = d_j$ and $\langle P_i^B, S_j^B \rangle = 0$ for $i \neq j$. Therefore, $\pi(\theta)(\varphi(M)) = \pi(\theta)(S_j^B) = \theta(X_j)$ as desired.

(3) From (1), we have $\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) \subset \mathcal{T}_{\pi(\theta)}$ and $\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) \subset \mathcal{F}_{\pi(\theta)}$. Thus, if $\langle \varphi(\mathcal{T}_\theta \cap \mathcal{W}_U), \varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) \rangle$ is a torsion pair in $\text{mod } B$, then we get $\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$ and $\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$.

To show that $\langle \varphi(\mathcal{T}_\theta \cap \mathcal{W}_U), \varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) \rangle$ is a torsion pair in $\text{mod } B$, it suffices to check that $\langle \mathcal{T}_\theta \cap \mathcal{W}_U, \mathcal{F}_\theta \cap \mathcal{W}_U \rangle$ is a torsion pair in $\mathcal{W}_U$. Clearly, $\text{Hom}_A(\mathcal{T}_\theta \cap \mathcal{W}_U, \mathcal{F}_\theta \cap \mathcal{W}_U) = 0$. Moreover, for all $M \in \mathcal{W}_U$, there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$ with $M' \in \mathcal{T}_\theta$ and $M'' \in \mathcal{F}_\theta$ in $\text{mod } A$. Since $\theta \in N_U$, we get $M' \in \mathcal{T}_\theta \subset \mathcal{T}_U$. On the other hand, since $M \in \mathcal{W}_U$, we get $M' \in \mathcal{F}_U$. Thus, $M'$ also belongs to $\mathcal{W}_U$, and so does $M''$. Therefore, $0 \to M' \to M \to M'' \to 0$ is a short exact sequence with $M' \in \mathcal{T}_\theta \cap \mathcal{W}_U$ and $M'' \in \mathcal{F}_\theta \cap \mathcal{W}_U$ in $\mathcal{W}_U$. We have proved that $\langle \mathcal{T}_\theta \cap \mathcal{W}_U, \mathcal{F}_\theta \cap \mathcal{W}_U \rangle$ is a torsion pair in $\mathcal{W}_U$.

From the argument in the first paragraph, $\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$ and $\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$ hold. Similarly, we obtain $\varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$ and $\varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$.

Finally, since $\mathcal{W}_\theta \subset \mathcal{W}_U$ by $\theta \in N_U$,

\[
\varphi(W_\theta) = \varphi(\mathcal{T}_\theta \cap \mathcal{F}_\theta \cap \mathcal{W}_U) = \varphi(\mathcal{T}_\theta \cap \mathcal{W}_U) \cap \varphi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)} \cap \mathcal{F}_{\pi(\theta)} = W_{\pi(\theta)}.
\]

(4) [Jas] Theorem 3.12] tells us the following property: let $\mathcal{T}, \mathcal{T}'$ be two torsion classes in $\text{mod } A$ satisfying $\mathcal{T}_U \subset \mathcal{T}, \mathcal{T}' \subset \mathcal{T}_U$, then $\mathcal{T} \cap \mathcal{W}_U = \mathcal{T}' \cap \mathcal{W}_U$ holds if and only if $\mathcal{T} = \mathcal{T}'$. This fact and (3) imply the assertion. 

Therefore, the wall-chamber structure of $N_U \subset K_0(\text{proj} A)_R$ recovers that of $K_0(\text{proj} B)_R$ via $\pi$ as follows.

**Theorem 4.5.** Let $U \in 2$-presilt $A$. Then, we have the following properties.

(1) For any $\theta \in N_U$ and $M \in \mathcal{W}_U$, the wall $\varphi(M)$ coincides with $\pi(\Theta_M \cap N_U)$.

(2) The linear map $\pi$ induces a bijection

\[
\{\text{TF equivalence classes in } N_U\} \to \{\text{TF equivalence classes in } K_0(\text{proj} B)_R\}, \\
[\theta] \mapsto \pi([\theta]).
\]
(3) We have the following commutative diagram:

\[
\begin{array}{ccc}
2 \text{-presilt}_U A & \xrightarrow{\sim} & 2 \text{-presilt} B \\
\{\text{TF equivalence classes in } N_U\} & \xrightarrow{\pi} & \{\text{TF equivalence classes in } K_0(\text{proj } B)_R\}
\end{array}
\]

Proof. (1) This follows from Lemma 4.4 (3).

(2) Note that \(\pi|_{N_U} : N_U \to K_0(\text{proj } B)_R\) is surjective by Lemma 4.4 (1). Then, Lemma 4.4 (4) yields that the linear map \(p\) sends each TF equivalence class \([\theta]\) in \(N_U\) to a TF equivalence class \(\pi(\theta)\) in \(K_0(\text{proj } B)_R\), and that this correspondence is bijective.

(3) Let \(V \in 2 \text{-presilt}_U A\). By Theorem 3.10, \(C^+(V)\) is a TF equivalence class in \(K_0(\text{proj } A)_R\), and it is contained in \(N_U\) by Lemma 3.13. Then, the bijection in (2) sends \(C^+(V)\) to the TF equivalence class \(\pi(C^+(V))\) in \(K_0(\text{proj } B)_R\), which must be \(C^+(\text{red}(V))\) by Proposition 4.2 (2) and Lemma 4.4 (3).

We remark that \(N_U\) itself is not very important to investigate the wall-chamber structure of \(K_0(\text{proj } B)_R\). By using Theorem 4.5, we can obtain all information on walls, chambers, and TF equivalence classes in \(K_0(\text{proj } B)_R\) from a subset \(N' \subset N_U\) and the restriction \(\pi|_{N'} : N' \to p(N')\) as long as \(0 \in K_0(\text{proj } B)_R\) is in the interior of \(p(N')\) in \(K_0(\text{proj } B)_R\).

We conclude this subsection by giving an example.

Example 4.6. Let \(A\) be the path algebra \(K(1 \to 2 \to 3)\), and take injections \(f : P_2 \to P_1\) and \(g : P_3 \to P_1\). We can check that \(U := (P_2 \xrightarrow{f} P_1)\) and \(V := (P_3 \xrightarrow{g} P_1)\) belong to \(2 \text{-presilt} A\).

Consider the \(\tau\)-tilting reduction at \(U\). We here use the setting of Proposition 4.1 for the Bongartz completion \(T = T_1 \oplus T_2 \oplus T_3\) of \(U\) as \(T_1 = P_1, T_2 = P_3,\) and \(T_3 = U\).

Then, the algebra \(B\) is isomorphic to \(K(1 \to 2)\), and the simple objects of \(W_U\) are \(X_1 = \text{Coker } g \cong P_1/P_3\) and \(X_2 = S_3\). The indecomposable objects of \(W_U\) are \(X_1, X_2\) and \(P_1\), which correspond to \(S^B_1, S^B_2\) and \(P^B_1\) in \(\text{mod } B\), respectively. The walls for the indecomposable modules in \(W_U\) are

\[\Theta_{X_1} = \mathbb{R}_{>0}([P_1] - [P_2]) \oplus \mathbb{R}_{>0}[P_3], \quad \Theta_{X_2} = \mathbb{R}[P_1] \oplus \mathbb{R}[P_2],\]

\[\Theta_{P_1} = \mathbb{R}_{>0}([P_1] - [P_2]) \oplus \mathbb{R}_{>0}([P_2] - [P_3]).\]

Since \(H^0(U) = S_1\) and \(H^{-1}(\nu U) = S_2\), we get \(N_U = \mathbb{R}_{>0}[P_1] \oplus \mathbb{R}_{>0}(-[P_2]) \oplus \mathbb{R}[P_3]\), so

\[\Theta_{X_1} \cap N_U = \mathbb{R}_{>0}([P_1] - [P_2]) \oplus \mathbb{R}[P_3], \quad \Theta_{X_2} \cap N_U = \mathbb{R}_{>0}[P_1] \oplus \mathbb{R}_{>0}(-[P_2]),\]

\[\Theta_{P_1} \cap N_U = \mathbb{R}_{>0}([P_1] - [P_2]) \oplus \mathbb{R}_{>0}([P_2] - [P_3]).\]

Since \([X_1] = [S_1] + [S_2]\) and \([X_2] = [S_3]\) in \(K_0(\text{mod } A)\), the linear map \(\pi\) sends \([P_1], [P_2], [P_3]\) to \([P^B_1], [P^B_2], [P^B_1] \oplus [P^B_2]\), respectively. Thus,

\[\pi(\Theta_{X_1} \cap N_U) = \mathbb{R}[P^B_1], \quad \pi(\Theta_{X_2} \cap N_U) = \mathbb{R}[P^B_1], \quad \pi(\Theta_{P_1} \cap N_U) = \mathbb{R}_{>0}([P^B_1] - [P^B_2]).\]

On the other hand, the set \(2 \text{-silt}_U A\) has exactly five elements:

\[T^{(1)} := P_1 \oplus P_3 \oplus U, \quad T^{(2)} := P_1 \oplus V \oplus U, \quad T^{(3)} := V \oplus P_3[1] \oplus U,\]

\[T^{(4)} := P_2[1] \oplus P_3 \oplus U, \quad T^{(5)} := P_2[1] \oplus P_3[1] \oplus U,\]

and the functor \(\text{red} = \text{Hom}_{K^b(\text{proj } A)}(T, ?)/[U]\) acts to their indecomposable direct summands as

\[P_1 \mapsto P^B_1, \quad P_3 \mapsto P^B_2, \quad V \mapsto (P^B_2 \to P^B_1), \quad P_2[1] \mapsto P^B_1[1], \quad P_3[1] \mapsto P^B_2[1].\]
THE WALL-CHAMBER STRUCTURES OF THE REAL GROTHENDIECK GROUPS

Figure 1. The local wall-chamber structures around \( [U] \in K_0(\text{proj} \ A)_\mathbb{R} \) and \( 0 \in K_0(\text{proj} \ B)_\mathbb{R} \)

The corresponding elements in \( K_0(\text{proj} \ A)_\mathbb{R} \) are sent by \( \pi \) as

\[
[P_1] \mapsto [P_1^B], \quad [P_3] \mapsto [P_2^B], \quad [V] \mapsto [P_1^B] - [P_2^B], \quad -[P_2] \mapsto -[P_2^B], \quad -[P_3] \mapsto -[P_3^B].
\]

Thus, \( p \) is compatible with the bijection \( \text{red}: 2\text{-silt}_U \ A \to 2\text{-silt} \ B \).

Consequently, the local wall-chamber structures around \( [U] \in K_0(\text{proj} \ A)_\mathbb{R} \) and \( 0 \in K_0(\text{proj} \ B)_\mathbb{R} \) are depicted as in Figure 1 above.

4.2. Application to \( \tau \)-tilting finiteness. By [DIJ, Theorem 3.8] and Proposition 3.12, an algebra \( A \) is said to be \( \tau \)-tilting finite if \( 2\text{-silt} \ A \) is a finite set, or equivalently, if the set \( 2\text{-ipresilt} \ A \) of 2-term indecomposable presilting objects in \( K^b(\text{proj} \ A) \) is a finite set. In this section, we give a proof of the following characterization of \( \tau \)-tilting finiteness in terms of the cones for 2-term silting objects by using the subset \( N_U \) for each \( U \in 2\text{-presilt} \ A \).

**Theorem 4.7.** The algebra \( A \) is \( \tau \)-tilting finite if and only if \( K_0(\text{proj} \ A)_\mathbb{R} = \bigcup_{T \in 2\text{-silt} \ A} C(T) \).

The “if” part follows from [DIJ, Theorem 5.4, Corollary 6.7]. We give a simple proof for the convenience of the readers.

**Proposition 4.8.** If \( A \) is \( \tau \)-tilting finite, then \( K_0(\text{proj} \ A)_\mathbb{R} = \bigcup_{T \in 2\text{-silt} \ A} C(T) \).

**Proof.** Set subsets \( F_1, F_2 \subset K_0(\text{proj} \ A)_\mathbb{R} \) by

\[
F_1 := \bigcup_{T \in 2\text{-silt} \ A} C(T) \quad \text{and} \quad F_2 := \bigcup_{U \in 2\text{-presilt} \ A} C(U).
\]

Since \( F_2 \subset F_1 \), it suffices to show \( F_1 \setminus F_2 = K_0(\text{proj} \ A)_\mathbb{R} \setminus F_2 \).
For $U \in 2\text{-presilt}_A$ with $|U| = n - 1$, we can take the two distinct elements $T, T'$ in $2\text{-presilt}_U A$ by Proposition 3.3 (3), and then, $C'(U) := C^+(T) \cup C^+(U) \cup C^+(T')$ is open in $K_0(\text{proj } A)_\mathbb{R} \setminus F_2$. Thus,

$$F_1 \setminus F_2 = \bigcup_{U \in 2\text{-presilt}_A \mid |U| = n - 1} C^+(U) = \bigcup_{U \in 2\text{-presilt}_A \mid |U| = n - 1} C'(U)$$

is an open subset of $K_0(\text{proj } A)_\mathbb{R} \setminus F_2$. On the other hand, $F_1 \subset K_0(\text{proj } A)_\mathbb{R}$ is closed since $\#(2\text{-silt}) < \infty$, so $F_1 \setminus F_2$ is a closed subset of $K_0(\text{proj } A)_\mathbb{R} \setminus F_2$. Clearly, $F_1 \setminus F_2$ is nonempty.

These three statements imply that $F_1 \setminus F_2 = K_0(\text{proj } A)_\mathbb{R} \setminus F_2$, since $K_0(\text{proj } A)_\mathbb{R} \setminus F_2$ is connected. Now, the assertion follows. \qed

We next proof the “if” part. We remark that this was conjectured by Demonet [Dem] Question 3.48, and that Zimmermann–Zvonareva [ZZ] have given a more geometrical proof.

**Proposition 4.9.** If $K_0(\text{proj } A)_\mathbb{R} = \bigcup_{T \in 2\text{-silt}_A} C(T)$, then $A$ is $\tau$-tilting finite.

**Proof.** Clearly, any $\theta \in C(T) \setminus \{0\}$ admits some nonzero $V \in 2\text{-presilt}_A$ such that $\theta \in C^+(V)$. Lemma 3.13 and Proposition 3.14 yield that if $U \in 2\text{-ipresilt}_A$ and $V \in 2\text{-presilt}_A$ satisfy $U \in \text{add } V$, then $C^+(V) \subset N_U$. Thus, the assumption $K_0(\text{proj } A)_\mathbb{R} = \bigcup_{T \in 2\text{-silt}_A} C(T)$ implies that $K_0(\text{proj } A)_\mathbb{R} \setminus \{0\} = \bigcup_{U \in 2\text{-ipresilt}_A} N_U$.

We consider the canonical sphere $\Sigma \subset K_0(\text{proj } A)_\mathbb{R}$; more precisely,

$$\Sigma := \left\{ \sum_{i=1}^{n} a_i [P_i] \in K_0(\text{proj } A)_\mathbb{R} \mid \sum_{i=1}^{n} a_i^2 = 1 \right\}.$$

Then, $\Sigma = \bigcup_{U \in 2\text{-ipresilt}_A} (N_U \cap \Sigma)$. Since $N_U \cap \Sigma$ is an open subset of the compact space $\Sigma$ for every $U \in 2\text{-ipresilt}_A$, there exists a finite set $I \subset 2\text{-ipresilt}_A$ such that $\Sigma = \bigcup_{U \in I} (N_U \cap \Sigma)$. Clearly, this implies that $K_0(\text{proj } A)_\mathbb{R} \setminus \{0\} = \bigcup_{I \in I} N_U$.

It is sufficient to show that $2\text{-ipresilt}_A = I$, so let $V \in 2\text{-ipresilt}_A$. Since $[V] \in K_0(\text{proj } A)_\mathbb{R} \setminus \{0\}$, there exists some $U \in 2\text{-ipresilt}_A$ such that $[V] \in N_U$. By Lemma 3.13 and Proposition 3.14, we have $V \in \text{add } U$, and since $U$ and $V$ are indecomposable, we get $V \cong U$. Thus, $V \in I$. Now, we get that $2\text{-ipresilt}_A$ coincides with the finite set $I$, and this means that $A$ is $\tau$-tilting finite. \qed

Now, we can show Theorem 4.7

**Proof of Theorem 4.7** It follows from Propositions 4.8 and 4.9. \qed

5. **The wall-chamber structures for path algebras**

In this section, we give a combinatorial algorithm to obtain the wall-chamber structure of $K_0(\text{proj } A)_\mathbb{R}$ in the case that $A$ is a path algebra. Throughout this section, $K$ is an algebraically closed field, $Q$ is an acyclic quiver with $\#Q_0 = n$, and $A := KQ$. We use the symbol $XQ_0$ for the set of maps from $Q_0$ to $X$.

We need some fundamental facts on module varieties. Let $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$ be a dimension vector, and write $d_i = d(i)$ for each $i \in Q_0$. Then, we set

$$\text{mod}(A, d) := \prod_{(\alpha : i \to j) \in Q_1} \text{Mat}_K(d_j, d_i).$$

This is exactly the set of representations of the quiver $Q$, and we can regard $\text{mod}(A, d)$ as the set of all $A$-modules $M$ with $\dim M = d$. By considering the Zariski topology, $\text{mod}(A, d)$ has a structure of an algebraic variety, so we call $\text{mod}(A, d)$ the *module variety* of $A$ associated to the dimension
vector $d$. The module variety $\mod(A, d)$ is clearly irreducible. In particular, any nonempty open subset is dense in $\mod(A, d)$. For our purpose, the following property is very crucial, where $c \leq d$ means that $c_i \leq d_i$ for all $i \in Q_0$.

**Proposition 5.1.** [Sch, Lemma 3.1] Let $c \leq d \in (\mathbb{Z}_{\geq 0})^{Q_0}$ be two dimension vectors. We define $F_c \subset \mod(A, d)$ consisting of all $M \in \mod(A, d)$ admitting a submodule $L \subset M$ with $\dim L = c$. Then, $F_c$ is closed in $\mod(A, d)$.

As in Theorem 3.17 the union of the walls is more important than each wall itself, so we here define

$$\Theta_d := \bigcup_{M \in \mod(A, d)} \Theta_M$$

for every nonzero dimension vector $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$. Actually, $\Theta_d$ is realized as the wall of some module in $\mod(A, d)$.

**Lemma 5.2.** Let $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$ be a nonzero dimension vector. Then, there exists $M \in \mod(A, d)$ such that $\Theta_M = \Theta_d$; hence $\Theta_d$ is a rational polyhedral cone in $K_0(\proj A)_\mathbb{R}$.

*Proof.* We define $G_c$ as the complement of $F_c \subset \mod(A, d)$ for each dimension vector $c \leq d$, and set

$$G := \bigcap_{c \leq d, \ G_c \neq \emptyset} G_c.$$

By Proposition 5.1, $G$ is the intersection of finitely many open dense subsets, so $G$ is also open and dense. In particular, $G$ is nonempty.

We take $M \in G$. Then, we have $\Theta_M \supset \Theta_{M'}$ for all $M' \in \mod(A, d)$ by definition, so $\Theta_M = \Theta_d$ follows. Since $\Theta_M$ is a rational polyhedral cone, so is $\Theta_d$. \hfill $\Box$

A dimension vector $d$ is called a Schur root if there exists an open dense subset $G$ of $\mod(A, d)$ such that every $M \in G$ is indecomposable.

Here, we associate a g-vector $g(d) \in K_0(\proj A)$ to each $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$ by $g(d) := f^{-1}(d)$, where $f$ is the linear isomorphism $K_0(\proj A) \to K_0(\mod A)$ sending $[P_1] \in K_0(\proj A)$ to $[P_1] \in K_0(\mod A)$. For $d, d' \in (\mathbb{Z}_{\geq 0})^{Q_0}$, we set $\langle d, d' \rangle := \langle g(d), d' \rangle$, which is the classical Euler form.

In this setting, a Schur root $d$ is said to be

- real if $\langle d, d \rangle = 1$,
- imaginary if $\langle d, d \rangle < 0$,
- isotropic if $\langle d, d \rangle = 0$.

It is well-known that every Schur root is real or imaginary [Kac, Propositions 1.1, 1.6].

We next show that we can determine whether $d$ is a Schur root from the dimension of the wall $\Theta_d$ and the value $\langle d, d \rangle$ of the Euler form.

**Proposition 5.3.** Let $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$ be a nonzero dimension vector.

1. Assume $\langle d, d \rangle \geq 0$.
   a. The dimension vector $d$ is a Schur root of $Q$ if and only if $d$ is indivisible and $\dim \Theta_d = n - 1$.
   b. There exist an integer $k \in \mathbb{Z}_{\geq 1}$ and a Schur root $d'$ of $Q$ such that $d = kd'$ if and only if $\dim \Theta_d = n - 1$.
2. Assume $\langle d, d \rangle < 0$. Then, $d$ is a Schur root of $Q$ if and only if $\dim \Theta_d = n - 1$. 
Example 5.4. Assume that $Q$ is the $m$-Kronecker quiver:

$$
\begin{array}{c}
1 \\
\vdots \\
2
\end{array}
\qquad m 	ext{ arrows}.
$$

By using [Kac], we know all Schur roots. Then, for each Schur root $d$, [Sch] Theorem 6.1 guarantees the existence of an open dense subset $G \subset \text{mod}(A, d)$ and a stability condition $\theta$ such that every module in $G$ is $\theta$-stable. Therefore, the wall-chamber structures of $K_0(\text{proj} A)_{\mathbb{R}}$ for $m = 0, 1, 2, 3$ are given as follows (see also [Bri] Figures 1–3):}

where there exists a wall $\mathbb{R}_{>0} \cdot \theta$ for each rational point $\theta$ in the gray domain in the picture for $m = 3$. We write the wall $\Theta_d$ for each $d = (a, b) \in (\mathbb{Z}_{>0})^2 \setminus \{0\}$ more explicitly below.

First, if $m = 0$, then the real roots of $Q$ are $(0, 1)$ and $(1, 0)$, and $Q$ admits no imaginary roots; hence,

$$
\Theta_d = \begin{cases} 
\mathbb{R}[P_2] & (a = 0) \\
\mathbb{R}[P_1] & (b = 0) \\
\{0\} & \text{(otherwise)}
\end{cases}
$$
Second, consider the case that \( m = 1 \). In this case, the real roots of \( Q \) are \((0, 1), (1, 1), \) and \((1, 0)\), and no imaginary roots of \( Q \) exist; hence,

\[
\Theta_d = \begin{cases} 
  \mathbb{R}[P_2] & (a = 0) \\
  \mathbb{R}_{\geq 0}(l[P_1] - [P_2]) & (a = b) \\
  \mathbb{R}[P_1] & (b = 0) \\
  \{0\} & (\text{otherwise})
\end{cases}.
\]

Next, assume that \( m = 2 \). Then, the real roots of \( Q \) are \((i, i + 1)\) and \((i + 1, i)\) for all \( i \in \mathbb{Z}_{\geq 0} \). The unique imaginary root of \( Q \) is \((1, 1)\). Thus,

\[
\Theta_d = \begin{cases} 
  \mathbb{R}[P_2] & (a = 0) \\
  \mathbb{R}_{\geq 0}((i + 1)[P_1] - i[P_2]) & ((a, b) \in \mathbb{Z}_{\geq 1}(i, i + 1), \ i \in \mathbb{Z}_{\geq 1}) \\
  \mathbb{R}_{\geq 0}(l[P_1] - [P_2]) & (a = b) \\
  \mathbb{R}_{\geq 0}(i[P_1] - (i + 1)[P_2]) & ((a, b) \in \mathbb{Z}_{\geq 1}(i + 1, i), \ i \in \mathbb{Z}_{\geq 1}) \\
  \mathbb{R}[P_1] & (b = 0) \\
  \{0\} & (\text{otherwise})
\end{cases}.
\]

We finally consider the case that \( m \geq 3 \). In this case, the real roots of \( Q \) are \((s_i, s_{i+1})\) and \((s_{i+1}, s_i)\) for all \( i \in \mathbb{Z}_{\geq 0} \), where the sequence \((s_i)_{i=0}^{\infty} \) is defined by \( s_0 := 0, s_1 := 1, \) and \( s_{i+2} := ms_{i+1} - s_i \). The imaginary roots of \( Q \) are all \((a, b)\) satisfying \( a^2 + b^2 - mab < 0 \). Thus,

\[
\Theta_d = \begin{cases} 
  \mathbb{R}[P_2] & (a = 0) \\
  \mathbb{R}_{\geq 0}(s_{i+1}[P_1] - s_i[P_2]) & ((a, b) \in \mathbb{Z}_{\geq 1}(s_i, s_{i+1}), \ i \in \mathbb{Z}_{\geq 1}) \\
  \mathbb{R}_{\geq 0}(b[P_1] - a[P_2]) & (a^2 + b^2 - mab < 0) \\
  \mathbb{R}_{\geq 0}(s_i[P_1] - s_{i+1}[P_2]) & ((a, b) \in \mathbb{Z}_{\geq 1}(s_{i+1}, s_i), \ i \in \mathbb{Z}_{\geq 1}) \\
  \mathbb{R}[P_1] & (b = 0) \\
  \{0\} & (\text{otherwise})
\end{cases}.
\]

We set \( \text{supp} \ d := \#\{i \in Q_0 \mid d_i > 0\} \) for each dimension vector \( d \in (\mathbb{Z}_{\geq 0})^{Q_0} \). Generalizing the example above by applying Lemma 2.5, we can determine \( \Theta_d \) in the case \( 1 \leq \# \text{supp} \ d \leq 2 \).

**Lemma 5.5.** Suppose \( d \in (\mathbb{Z}_{\geq 0})^{Q_0} \) is a dimension vector with \( \# \text{supp} \ d \in \{1, 2\} \).

1. If \( \# \text{supp} \ d = 1 \) and \( k \in \text{supp} \ d \), then \( \Theta_d = \bigoplus_{i \neq k} \mathbb{R}[P_i] \).
2. Assume that \( \# \text{supp} \ d = 2 \) and that the full subquiver of \( Q \) corresponding to \( \text{supp} \ d \subset Q \) is

   \[
   k \xrightarrow{a} l \quad (m \text{ arrows})
   \]

   with \( k, l \in \text{supp} \ d \) and \( m \in \mathbb{Z}_{\geq 0} \). We define \( a, b \in \mathbb{Z}_{\geq 0} \) by \( a := d_k / \gcd(d_k, d_l) \) and \( b := d_l / \gcd(d_k, d_l) \). Then,

   \[
   \Theta_d = \begin{cases} 
     \bigoplus_{i \neq k, l} \mathbb{R}[P_i] & (a = 0) \\
     \mathbb{R}_{\geq 0}(b[P_k] - a[P_l]) & (a^2 + b^2 - mab \leq 1) \\
     \bigoplus_{i \neq k, l} \mathbb{R}[P_i] & (\text{otherwise})
   \end{cases}.
   \]

On the other hand, if \( \# \text{supp} \ d \geq 3 \), then we can apply Lemma 2.8 to obtain a recurrence relation on \( \Theta_d \).
Proposition 5.6. Suppose \( d \in (\mathbb{Z}_{\geq 0})^Q_0 \) is a dimension vector with \( \# \text{supp } d \geq 3 \). Then, \( \Theta_d \) is the smallest polyhedral cone of \( K_0(\text{proj } A)_R \) containing 
\[
\bigcup_{0 < c < d} (\Theta_c \cap \Theta_{d-c}).
\]

Proof. By Lemma 5.2, \( \Theta_d \) itself is a polyhedral cone. Lemma 2.8 tells us that \( \Theta_d \) is the polyhedral cone of \( K_0(\text{proj } A)_R \) generated by
\[
\bigcup_{M', M'' \in \text{mod } A \setminus \{0\}, (M' \cdot M'') \setminus \text{mod } (A,d) \neq \emptyset} (\Theta_{M'} \cap \Theta_{M''}) = \bigcup_{0 < c < d, M' \in \text{mod } (A,c), M'' \in \text{mod } (A,d-c)} (\Theta_{M'} \cap \Theta_{M''}) = \bigcup_{0 < c < d} (\Theta_c \cap \Theta_d),
\]
where we use Lemma 5.2 again for the latter equality. \( \square \)

As a consequence, we can determine the wall-chamber structure of \( K_0(\text{proj } A)_R \) for any path algebra \( A \).

Theorem 5.7. Let \( d \in (\mathbb{Z}_{\geq 0})^Q_0 \) be a nonzero dimension vector. Then, \( \Theta_d \) is determined by Lemma 5.5 and Proposition 5.6.

We end this paper by giving an example of Theorem 5.7.

Example 5.8. Let \( Q \) be a quiver \( 1 \to 2 \to 3 \). The following picture is the wall-chamber structure of \( K_0(\text{proj } A)_R \) on the subset
\[
\{a_1[P_1] - a_2[P_2] - a_3[P_3] \mid a_1, a_2, a_3 \geq 0, \ a_1 + a_2 + a_3 = 1\}.
\]

Compare our figure in \( K_0(\text{proj } A)_R \) with the diagram in \( K_0(\text{mod } A)_R \) in [DW, Example 11.3.9], then we find that the chambers in our figure are sent to the triangles expressing tilting modules in their diagram under the linear map \( f : K_0(\text{proj } A) \to K_0(\text{mod } A) \) sending \([P_i] \in K_0(\text{proj } A)\) to \([P_i] \in K_0(\text{mod } A)\).

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