Muckenhoupt’s ($A_p$) condition and the existence of the optimal martingale measure

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Abstract

In the problem of optimal investment with utility function defined on $(0, \infty)$, we formulate sufficient conditions for the dual optimizer to be a uniformly integrable martingale. Our key requirement consists of the existence of a martingale measure whose density process satisfies the probabilistic Muckenhoupt ($A_p$) condition for the power $p = 1/(1 - a)$, where $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. We construct a counterexample showing that this ($A_p$) condition is sharp.

Keywords: utility maximization, optimal martingale measure, BMO martingales, ($A_p$) condition.

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1 Introduction

An unpleasant qualitative feature of the general theory of optimal investment with a utility function defined on $(0, \infty)$ is that the dual optimizer $\hat{Y}$ may not be a uniformly integrable martingale. In the presence of jumps, it may even fail to be a local martingale. The corresponding counterexamples can be found in [12]. In this paper, we seek to provide conditions under

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which the uniform martingale property for \( \hat{Y} \) holds and thus, \( \hat{Y} / \hat{Y}_0 \) defines the density process of the optimal martingale measure \( \hat{Q} \).

The question of whether \( \hat{Y} \) is a uniformly integrable martingale is of longstanding interest in mathematical finance and can be traced back to [8] and [10]. This problem naturally arises in situations involving utility-based arguments. For instance, it is relevant for pricing in incomplete markets, where according to [9] the existence of \( \hat{Q} \) is equivalent to the uniqueness of marginal utility-based prices for every bounded contingent claim.

Our key requirement consists of the existence of a dual supermartingale \( Z \), which satisfies the probabilistic Muckenhoupt (\( A_p \)) condition for the power \( p > 1 \) such that

\[
(1.1) \quad p = \frac{1}{1 - a}.
\]

Here \( a \in (0, 1) \) is a lower bound on the relative risk-aversion of the utility function. As we prove in Theorem 5.1, this condition, along with the existence of an upper bound for the relative risk-aversion, yields (\( A_{p'} \)) for \( \hat{Y} \) for some \( p' > 1 \). This property in turn implies that the dual minimizer \( \hat{Y} \) is of class (\( D \)), that is, the family of its values evaluated at all stopping times is uniformly integrable. In Proposition 6.1, we construct a counterexample showing that the bound (1.1) is the best possible for \( \hat{Y} \) to be of class (\( D \)) even in the case of power utilities and continuous stock prices.

A similar idea of passing regularity from some dual element to the optimal one has been employed in [6], [7] and [2] for respectively, quadratic, power and exponential utility functions defined on the whole real line. These papers use appropriate versions of the Reverse Hölder (\( R_q \)) inequality which is dual to (\( A_p \)). Note that contrary to (\( A_p \)), the uniform integrability property is not implied but rather required by (\( R_q \)). While this requirement is not a problem for real-line utilities, where the optimal martingale measures always exist, it is clearly an issue for utility functions defined on \((0, \infty)\).

Even if the dual minimizer \( \hat{Y} \) is of class (\( D \)), it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [12, Example 5.1']. In Proposition 4.2 we prove that every maximal dual supermartingale (in particular, \( \hat{Y} \)) is a local martingale if the ratio of any two positive wealth processes is \( \sigma \)-bounded.

Our main results, Theorems 5.1 and 5.3, are stated in Section 5. They are accompanied by Corollaries 5.5 and 5.6, which exploit well known connections between the (\( A_p \)) condition and BMO martingales.
2 Setup

We use the same framework as in [12, 13] and refer to these papers for more details. There is a financial market with a bank account paying zero interest and \(d\) stocks. The process of stocks’ prices \(S = (S^t)\) is a semimartingale with values in \(\mathbb{R}^d\) on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})\). Here \(T\) is a finite maturity and \(\mathcal{F} = \mathcal{F}_T\), but we remark that our results also hold for the case of infinite maturity.

A (self-financing) portfolio is defined by an initial capital \(x \in \mathbb{R}\) and a predictable \(S\)-integrable process \(H = (H^i)\) with values in \(\mathbb{R}^d\) of the number of stocks. Its corresponding wealth process \(X\) evolves as

\[
X_t = x + \int_0^t H_u dS_u, \quad t \in [0, T].
\]

We denote by \(\mathcal{X}\) the family of non-negative wealth processes:

\[
\mathcal{X} \triangleq \{X \geq 0 : X \text{ is a wealth process}\}
\]

and by \(\mathcal{Q}\) the family of equivalent local martingale measures for \(\mathcal{X}\):

\[
\mathcal{Q} \triangleq \{Q \sim \mathbb{P} : \text{every } X \in \mathcal{X} \text{ is a local martingale under } Q\}.
\]

We assume that

\[(2.1) \quad \mathcal{Q} \neq \emptyset,\]

which is equivalent to the absence of arbitrage; see [3, 5].

There is an economic agent whose preferences over terminal wealth are modeled by a utility function \(U\) defined on \((0, \infty)\). We assume that \(U\) is of Inada type, that is, it is strictly concave, strictly increasing, continuously differentiable on \((0, \infty)\), and

\[
U'(0) = \lim_{x \to 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0.
\]

For a given initial capital \(x > 0\), the goal of the agent is to maximize the expected utility of terminal wealth. The value function of this problem is denoted by

\[(2.2) \quad u(x) = \sup_{X \in \mathcal{X}, X_0 = x} \mathbb{E}[U(X_T)].\]

Following [12], we define the dual optimization problem to (2.2) as

\[(2.3) \quad v(y) = \inf_{Y \in \mathcal{Y}, Y_0 = y} \mathbb{E}[V(Y_T)], \quad y > 0,\]

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where $V$ is the convex conjugate to $U$:

$$V(y) = \sup_{x > 0} \{ U(x) - xy \}, \quad y > 0,$$

and $\mathcal{Y}$ is the family of “dual” supermartingales to $\mathcal{X}$:

$$\mathcal{Y} = \{ Y \geq 0 : XY \text{ is a supermartingale for every } X \in \mathcal{X} \}.$$

Note that the set $\mathcal{Y}$ contains the density processes of all $Q \in \mathcal{Q}$ and that, as $1 \in \mathcal{X}$, every element of $\mathcal{Y}$ is a supermartingale.

It is known, see [13, Theorem 2], that under (2.1) and (2.4)

$$v(y) < \infty, \quad y > 0,$$

the value functions $u$ and $-v$ are of Inada type, $v$ is the convex conjugate to $u$, and

$$v(y) = \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ V \left( y \frac{dQ}{dP} \right) \right], \quad y > 0.$$

The solutions $X(x)$ to (2.2) and $Y(y)$ to (2.3) exist. If $y = u'(x)$ or, equivalently, $x = -v'(y)$, then

$$U'(X_T(x)) = Y_T(y),$$

and the product $X(x)Y(y)$ is a uniformly integrable martingale.

The last two properties actually characterize optimal $X(x)$ and $Y(y)$. For convenience of future references, we recall this “verification” result.

**Lemma 2.1.** Let $\hat{X} \in \mathcal{X}$ and $\hat{Y} \in \mathcal{Y}$ be such that

$$U'(\hat{X}_T) = \hat{Y}_T, \quad \mathbb{E} \left[ V(\hat{Y}_T) \right] < \infty, \quad \mathbb{E} \left[ \hat{X}_T\hat{Y}_T \right] = \hat{X}_0\hat{Y}_0.$$

Then $\hat{X}$ solves (2.2) for $x = \hat{X}_0$ and $\hat{Y}$ solves (2.5) for $y = \hat{Y}_0$.

**Proof.** The result follows immediately from the identity

$$U(\hat{X}_T) = V(\hat{Y}_T) + \hat{X}_T\hat{Y}_T$$

and the inequalities

$$U(X_T) \leq V(\hat{Y}_T) + X_T\hat{Y}_T, \quad X \in \mathcal{X},$$

$$U(\hat{X}_T) \leq V(Y_T) + \hat{X}_TY_T, \quad Y \in \mathcal{Y},$$

after we recall that $XY$ is a supermartingale for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. \qed
The goal of the paper is to find sufficient conditions for the lower bound in (2.5) to be attained at some \( Q(y) \in Q \) called the \textit{optimal martingale measure} or, equivalently, for the dual minimizer \( Y(y) \) to be a \textit{uniformly integrable martingale}; in this case,

\[
Y_T(y) = y \frac{dQ(y)}{dP}.
\]

Our criteria are stated in Theorem 5.1 below, where a key role is played by the probabilistic version of the classical Muckenhoupt \((A_p)\) condition.

### 3 \((A_p)\) condition for the dual minimizer

Following [11, Section 2.3], we recall the probabilistic \((A_p)\) condition.

\textbf{Definition 3.1.} Let \( p > 1 \). An optional process \( R \geq 0 \) satisfies \((A_p)\) if \( R_T > 0 \) and there is a constant \( C > 0 \) such that for every stopping time \( \tau \)

\[
\mathbb{E} \left[ \left( \frac{R_\tau}{R_T} \right)^{\frac{1}{p-1}} \mid \mathcal{F}_\tau \right] \leq C.
\]

Observe that if \( R \) satisfies \((A_p)\), then \( R \) satisfies \((A_{p'})\) for every \( p' \geq p \).

An important consequence of the \((A_p)\) condition is a uniform integrability property. For continuous local martingales this fact is well known and can be found e.g., in [11, Section 2.3].

\textbf{Lemma 3.2.} If an optional process \( R \geq 0 \) satisfies \((A_p)\) for some \( p > 1 \) and \( \mathbb{E} [R_T] < \infty \), then \( R \) is of class \((D)\):

\[
\{ R_\tau : \tau \text{ is a stopping time} \} \text{ is uniformly integrable.}
\]

\textit{Proof.} Let \( \tau \) be a stopping time. As \( p > 1 \), the function \( x \mapsto x^{-\frac{1}{p-1}} \) is convex. Hence, by Jensen’s inequality,

\[
\mathbb{E} \left[ \left( \frac{R_\tau}{R_T} \right)^{\frac{1}{p-1}} \mid \mathcal{F}_\tau \right] = R_\tau^{\frac{1}{p-1}} \mathbb{E} \left[ \left( \frac{1}{R_T^{\frac{1}{p-1}}} \mid \mathcal{F}_\tau \right) \right] \geq R_\tau^{\frac{1}{p-1}} \left( \mathbb{E} [R_T \mid \mathcal{F}_\tau] \right)^{-\frac{1}{p-1}}.
\]

Using the constant \( C > 0 \) from \((A_p)\), we obtain that

\[
R_\tau \leq C^{p-1} \mathbb{E} [R_T \mid \mathcal{F}_\tau],
\]

and the result follows. \( \square \)
To motivate the use of the \((A_p)\) condition in the study of the dual mini-
mimizers \(Y(y), y > 0\), we first consider the case of power utility with a positive
power.

**Proposition 3.3.** Let \((2.1)\) hold. Assume that

\[
U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,
\]

with the relative risk-aversion \(a \in (0, 1)\) and denote \(p \triangleq \frac{1}{1-a} > 1\). Then for
\(y > 0\), the solution \(Y(y)\) to the dual problem \((2.3)\) exists if and only if

\[
E \left[ Y_T^{-\frac{1}{p-1}} \right] < \infty \quad \text{for some} \quad Y \in \mathcal{Y}
\]

and, in this case, for every \(Y \in \mathcal{Y}, Y > 0\) and every stopping time \(\tau\),

\[
E \left[ \left( \frac{Y_{\tau}(y)}{Y_T(y)} \right)^{\frac{1}{p-1}} \right| \mathcal{F}_\tau] \leq E \left[ \left( \frac{Y_T}{Y} \right)^{\frac{1}{p-1}} \right| \mathcal{F}_\tau].
\]

In particular, \(Y(y)\) satisfies \((A_p)\) if and only if there is \(Y \in \mathcal{Y}\) satisfying
\((A_p)\).

**Proof.** Observe that the convex conjugate to \(U\) is given by

\[
V(y) = \frac{a}{1-a} y^{-\frac{1-a}{1-a}} = (p-1)y^{-\frac{1}{p-1}}, \quad y > 0.
\]

Then \((3.1)\) is equivalent to \((2.4)\), which, in turn, is equivalent to the existence
of the optimal \(Y(y), y > 0\). Denote \(\tilde{Y} \triangleq Y(1)\). Clearly, \(Y(y) = y\tilde{Y}\).

Let a stopping time \(\tau\) and a process \(Y \in \mathcal{Y}, Y > 0\), be such that

\[
E \left[ \left( \frac{Y_{\tau}}{Y_T} \right)^{\frac{1}{p-1}} \right| \mathcal{F}_\tau] < \infty.
\]

We have to show that

\[
\xi \triangleq E \left[ \left( \frac{\tilde{Y}_\tau}{Y_T} \right)^{\frac{1}{p-1}} \right| \mathcal{F}_\tau] - E \left[ \left( \frac{Y_T}{Y} \right)^{\frac{1}{p-1}} \right| \mathcal{F}_\tau] \leq 0.
\]

For a set \(A \in \mathcal{F}_\tau\), the process

\[
Z_t \triangleq \tilde{Y}_\tau 1_{\{t \leq \tau\}} + \tilde{Y}_\tau \left( \frac{Y_{t}}{Y_T} 1_A + \frac{\tilde{Y}_\tau}{Y_T}(1 - 1_A) \right) 1_{\{t > \tau\}}, \quad t \in [0, T],
\]

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belongs to $\mathcal{Y}$ and is such that $Z_0 = 1$ and $Z_\tau = \hat{Y}_\tau$. We obtain that

$$
\mathbb{E}\left[ \left( \frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \left| \mathcal{F}_\tau \right. \right] = \mathbb{E}\left[ \left( \frac{Y_\tau}{Y_T} \right)^{\frac{1}{p-1}} \left| \mathcal{F}_\tau \right. \right] \mathbf{1}_A + \mathbb{E}\left[ \left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{\frac{1}{p-1}} \left| \mathcal{F}_\tau \right. \right] (1 - \mathbf{1}_A)
$$

$$
= \mathbb{E}\left[ \left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{\frac{1}{p-1}} \left| \mathcal{F}_\tau \right. \right] - \mathbb{E}\left[ \left( \frac{1}{\hat{Y}_T} \right)^{\frac{1}{p-1}} \max(\xi, 0) \right].
$$

Dividing both sides by $Z_T^{\frac{1}{p-1}} = \hat{Y}_T^{\frac{1}{p-1}}$ and choosing $A = \{ \xi \geq 0 \}$, we deduce that

$$
\mathbb{E}\left[ \left( \frac{1}{Z_T} \right)^{\frac{1}{p-1}} \right] = \mathbb{E}\left[ \left( \frac{1}{Y_T} \right)^{\frac{1}{p-1}} \right] - \mathbb{E}\left[ \left( \frac{1}{\hat{Y}_T} \right)^{\frac{1}{p-1}} \max(\xi, 0) \right].
$$

However, the optimality of $\hat{Y} = Y(1)$ implies that

$$
\mathbb{E}\left[ \left( \frac{1}{Y_T} \right)^{\frac{1}{p-1}} \right] \leq \mathbb{E}\left[ \left( \frac{1}{Z_T} \right)^{\frac{1}{p-1}} \right].
$$

Hence $\xi \leq 0$. \hfill \square

We now state the main result of the section.

**Theorem 3.4.** Let (2.1) hold. Suppose that there are constants $0 < a < 1$, $b \geq a$ and $C > 0$ such that

$$
\frac{1}{C} \left( \frac{y}{x} \right)^a \leq \frac{U'(x)}{U'(y)} \leq C \left( \frac{y}{x} \right)^b, \quad x \leq y,
$$

and there is a supermartingale $Z \in \mathcal{Y}$ satisfying $(A_p)$ with

$$
p = \frac{1}{1-a}.
$$

Then for every $y > 0$, the solution $Y(y)$ to (2.3) exists and satisfies $(A_{p'})$ with

$$
p' = 1 + \frac{b}{1-a}.
$$

**Remark 3.5.** Notice that if the relative risk-aversion of $U$ is well-defined and bounded away from $0$ and $\infty$, then in (3.2) we can take $C = 1$ and choose $a$ and $b$ as lower and upper bounds:

$$
0 < a \leq -\frac{xU''(x)}{U'(x)} \leq b < \infty, \quad x > 0.
$$
In particular, if
\[ 1 \leq - \frac{xU''(x)}{U'(x)} \leq b, \quad x > 0, \]
then choosing \( a \in (0, 1) \) sufficiently close to 1 we fulfill the conditions of Theorem 3.4 if there exists a supermartingale \( Z \in \mathcal{Y} \) satisfying \((A_p)\) for some \( p > 1 \).

Observe also that for the positive power utility function \( U \) with relative risk-aversion \( a \in (0, 1) \) we can select \( b = a \) and then obtain same estimate as in Proposition 3.3:
\[ p' = 1 + \frac{a}{1 - a} = \frac{1}{1 - a} = p. \]

The proof of Theorem 3.4 relies on the following lemma.

**Lemma 3.6.** Assume (2.1) and suppose that there are constants \( 0 < a < 1 \) and \( C_1 > 0 \) such that
\[ \frac{1}{C_1} \left( \frac{y}{x} \right)^a \leq \frac{U'(x)}{U'(y)}, \quad x \leq y, \]
and there is a supermartingale \( Z \in \mathcal{Y} \) satisfying \((A_p)\) with
\[ p = \frac{1}{1 - a}. \]
Then for every \( y > 0 \) the solution \( Y(y) \) to (2.3) exists, and there is a constant \( C_2 > 0 \) such that for every stopping time \( \tau \) and every \( y > 0 \),
\[ \mathbb{E} [I(Y_T(y))Y_T(y)|\mathcal{F}_\tau] \leq C_2 I(Y_\tau(y))Y_\tau(y), \]
where \( I = -V' \).

**Remark 3.7.** Recall that for \( x = -v'(y) \) the optimal wealth process \( X(x) \) has the terminal value
\[ X_T(x) = -V'(Y_T(y)) = I(Y_T(y)) \]
and the product \( X(x)Y(y) \) is a uniformly integrable martingale. It follows that for every stopping time \( \tau \)
\[ X_\tau(x) = \frac{1}{Y_\tau(y)} \mathbb{E} [I(Y_T(y))Y_T(y)|\mathcal{F}_\tau] \]
and therefore, inequality (3.4) is equivalent to
\[ X_\tau(x) \leq C_2 I(Y_\tau(y)). \]
Proof of Lemma 3.6. To show the existence of $Y(y)$ we need to verify (2.4). As $I = -V'$ is the inverse function to $U$, condition (3.3) is equivalent to

$$(3.5) \quad \frac{I(x)}{I(y)} \leq C_3 \left( \frac{y}{x} \right)^{1/a}, \quad x \leq y,$$

where $C_3 = C_{1}^{1/a}$. From (3.5) we deduce that for $y \leq 1$

$$V(y) = V(1) + \int_0^1 I(t)dt \leq V(1) + C_3 I(1) \int_y^1 t^{-1/a}dt$$

$$= V(1) + C_3 I(1) \frac{a}{1-a} (y^{-1/a} - 1)$$

$$= V(1) + C_3 I(1) (p-1)(y^{-1/p} - 1).$$

Hence, there is a constant $C_4 > 0$ such that

$$V(y) \leq C_4(1 + y^{-1/p}), \quad y > 0.$$  

As $Z$ satisfies $(A_p)$, we have

$$\mathbb{E} \left[ Z_T^{-\frac{1}{p-1}} \right] < \infty.$$  

It follows that

$$v(y) \leq \mathbb{E} \left[ V(yZ_T/Z_0) \right] < \infty, \quad y > 0,$$

which completes the proof of the existence of $Y(y)$.

Let $\tau$ be a stopping time and let $y > 0$. We set $\hat{Y} \triangleq Y(y)$ and define the process

$$Y_t \triangleq \hat{Y}_t 1_{\{t \leq \tau\}} + \hat{Y}_\tau Z_t 1_{\{t > \tau\}}, \quad t \in [0, T].$$

Clearly, $Y \in \mathcal{Y}$ and $Y_0 = \hat{Y}_0 = y$. We represent

$$I(\hat{Y}_T)\hat{Y}_T = \xi_1 + \xi_2 + \xi_3,$$

by multiplying the left-side on the elements of the unity decomposition:

$$1 = 1_{\{\hat{Y}_\tau \leq \hat{Y}_T\}} + 1_{\{\hat{Y}_\tau \leq \hat{Y}_T < \hat{Y}_r\}} + 1_{\{\hat{Y}_\tau < \hat{Y}_T, \hat{Y}_T < \hat{Y}_r\}}.$$

For the first term, since $I = -V'$ is a decreasing function, we have that

$$\xi_1 = I(\hat{Y}_T)\hat{Y}_T 1_{\{\hat{Y}_\tau \leq \hat{Y}_T\}} \leq I(\hat{Y}_\tau)\hat{Y}_T.$$

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Using the supermartingale property of $\hat{Y}$, we obtain that
\[
\mathbb{E} \left[ \xi_1 | \mathcal{F}_\tau \right] \leq I(\hat{Y}_\tau)\hat{Y}_\tau.
\]

For the second term, we deduce from (3.5) that
\[
\xi_2 = I(\hat{Y}_T)\hat{Y}_T 1\{Y_T \leq \hat{Y}_T < \bar{Y}_T\} = I(\hat{Y}_T)\hat{Y}_T^{1-a} 1\{Y_T \leq \hat{Y}_T < \bar{Y}_T\}
\]
\[
\leq C_3 I(\hat{Y}_T)\hat{Y}_T^{1-a} Y_T^{1-a} = C_3 I(\hat{Y}_T)\hat{Y}_T \left( \frac{Z_\tau}{Z_T} \right)^{1-a} = C_3 I(\hat{Y}_T)\hat{Y}_T \left( \frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}}
\]
and the $(A_p)$ condition for $Z$ yields the existence of a constant $C_5 > 0$ such that
\[
\mathbb{E} \left[ \xi_2 | \mathcal{F}_\tau \right] \leq C_5 I(\hat{Y}_T)\hat{Y}_T.
\]

For the third term, we deduce from (3.5) that
\[
\xi_3 = I(\hat{Y}_T)\hat{Y}_T 1\{\hat{Y}_T < Y_T, \hat{Y}_T < \bar{Y}_T\} \leq I(\hat{Y}_T)\hat{Y}_T 1\{\hat{Y}_T < Y_T\}
\]
\[
= I(\hat{Y}_T)^a \hat{Y}_T I(\hat{Y}_T)^{1-a} 1\{\hat{Y}_T < Y_T\} \leq C_1 I(Y_T)^a Y_T I(\hat{Y}_T)^{1-a}
\]
\[
= C_1 (I(Y_T)Y_T)^a (I(\hat{Y}_T)Y_T)^{1-a}
\]
and then from Hölder’s inequality that
\[
\mathbb{E} \left[ \xi_3 | \mathcal{F}_\tau \right] \leq C_1 (\mathbb{E} \left[ I(Y_T)Y_T | \mathcal{F}_\tau \right])^a \left( \mathbb{E} \left[ I(\hat{Y}_T)Y_T | \mathcal{F}_\tau \right] \right)^{1-a}.
\]

We recall that the terminal wealth of the optimal investment strategy with $\hat{X}_0 = -\nu'(y)$ is given by
\[
I(\hat{Y}_T) = \hat{X}_T.
\]

It follows that
\[
\mathbb{E} \left[ I(\hat{Y}_T)Y_T | \mathcal{F}_\tau \right] = \mathbb{E} \left[ \hat{X}_TY_T | \mathcal{F}_\tau \right] \leq \hat{X}_T Y_T = \hat{X}_T \hat{Y}_T
\]
\[
= \mathbb{E} \left[ \hat{X}_T \hat{Y}_T | \mathcal{F}_\tau \right] = \mathbb{E} \left[ I(\hat{Y}_T)\hat{Y}_T | \mathcal{F}_\tau \right].
\]

To estimate $\mathbb{E} \left[ I(Y_T)Y_T | \mathcal{F}_\tau \right]$ we decompose
\[
I(Y_T)Y_T = I(Y_T)Y_T 1\{\hat{Y}_T \leq Y_T\} + I(Y_T)Y_T 1\{\hat{Y}_T > Y_T\}.
\]

Since $I$ is decreasing, we have that
\[
I(Y_T)Y_T 1\{\hat{Y}_T \leq Y_T\} \leq I(\hat{Y}_T)Y_T.
\]
As $Y$ is a supermartingale and $Y_\tau = \hat{Y}_\tau$, we obtain that
\[
\mathbb{E}\left[ I(Y_T)Y_T 1\{\hat{Y}_\tau \leq Y_\tau \} \bigg| \mathcal{F}_\tau \right] \leq I(\hat{Y}_\tau)Y_\tau = I(\hat{Y}_\tau)\hat{Y}_\tau.
\]

For the second term, using (3.5) we deduce that
\[
I(Y_T)Y_T 1\{\hat{Y}_\tau > Y_\tau \} = I(Y_T)\frac{1}{T} Y_T^{-\frac{1-a}{a}} 1\{\hat{Y}_\tau > Y_\tau \} \leq C_3 I(\hat{Y}_\tau)\hat{Y}_\tau^{-\frac{1-a}{a}}
\]
\[
= C_3 I(\hat{Y}_\tau)\hat{Y}_\tau \left( \frac{\hat{Y}_\tau}{Y_T} \right)^{\frac{1-a}{a}} = C_3 I(\hat{Y}_\tau)\hat{Y}_\tau \left( \frac{Z_\tau}{Z_T} \right)^{\frac{1}{1-a}}
\]

and the $(A_p)$ condition for $Z$ implies that
\[
\mathbb{E}\left[ I(Y_T)Y_T 1\{\hat{Y}_\tau > Y_\tau \} \bigg| \mathcal{F}_\tau \right] \leq C_5 I(\hat{Y}_\tau)\hat{Y}_\tau.
\]

Thus we have
\[
\mathbb{E}\left[ I(Y_T)Y_T 1\{\hat{Y}_\tau \leq Y_\tau \} \bigg| \mathcal{F}_\tau \right] \leq C_5 I(\hat{Y}_\tau)\hat{Y}_\tau.
\]

Adding together the estimates for $\mathbb{E}\left[ \xi_i \big| \mathcal{F}_\tau \right]$ we obtain that
\[
\mathbb{E}\left[ I(\hat{Y}_T)\hat{Y}_T \bigg| \mathcal{F}_\tau \right] \leq \eta + C_1 \eta^a \left( \mathbb{E}\left[ I(\hat{Y}_T)\hat{Y}_T \bigg| \mathcal{F}_\tau \right] \right)^{1-a}.
\]

It follows that
\[
\mathbb{E}\left[ I(\hat{Y}_T)\hat{Y}_T \bigg| \mathcal{F}_\tau \right] \leq x^* \eta = x^* (1 + C_5) I(\hat{Y}_\tau)\hat{Y}_\tau,
\]

where $x^*$ is the root of
\[
x = 1 + C_1 x^{1-a}, \quad x > 0.
\]

We thus have proved inequality (3.4) with $C_2 = (1 + C_5)x^*$. \hfill \square

**Proof of Theorem 3.4.** Fix $y > 0$. In view of Lemma 3.6, we only have to verify that $\hat{Y} \equiv Y(y)$ satisfies $(A_{p'})$.

Denote $\tilde{X} \equiv X(-v'(y))$ and recall that by Lemma 3.6 and Remark 3.7, there is $C_2 > 0$ such that, for every stopping time $\tau$,
\[
\tilde{X}_\tau \leq C_2 I(\hat{Y}_\tau).
\]
Observe also that as $I = -V'$ is the inverse function to $U'$, the second inequality in (3.2) is equivalent to

$$\frac{y}{x} \leq C \left( \frac{I(x)}{I(y)} \right)^b, \quad x \leq y.$$ 

We fix a stopping time $\tau$. Since $I(\hat{Y}_T) = \hat{X}_T$, we deduce from the inequalities above that

$$\left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{1/b} \leq \max \left( 1, C^{1/b} \frac{I(\hat{Y}_T)}{I(\hat{Y}_\tau)} \right) \leq \max \left( 1, C_3 \frac{\hat{X}_T}{\hat{X}_\tau} \right),$$

where $C_3 = C^{1/b} C_2$. It follows that

$$\left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{\frac{1-a}{b}} = \left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{1-a} \leq \max \left( 1, C_3^{1-a} \left( \frac{\hat{X}_T}{\hat{X}_\tau} \right) \right) \leq 1 + C_3^{1-a} \left( \frac{\hat{X}_T \hat{Z}_T}{\hat{X}_\tau \hat{Z}_\tau} \right)^{1-a} \left( \frac{Z_\tau}{Z_T} \right)^{1-a}.$$

Denoting by $C_1 > 0$ the constant in the $(A_p)$ condition for $Z$, we deduce from Hölder’s inequality and the supermartingale property of $\hat{X}Z$ that

$$\mathbb{E} \left[ \left( \frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{\frac{1-a}{b}} \bigg| \mathcal{F}_\tau \right] \leq 1 + C_3^{1-a} \left( \mathbb{E} \left[ \frac{\hat{X}_T \hat{Z}_T}{\hat{X}_\tau \hat{Z}_\tau} \bigg| \mathcal{F}_\tau \right] \right)^{1-a} \left( \mathbb{E} \left[ \left( \frac{Z_\tau}{Z_T} \right)^{\frac{1-a}{b}} \bigg| \mathcal{F}_\tau \right] \right)^a \leq 1 + C_3^{1-a} C_1^{a}.$$ 

Hence, $\hat{Y}$ satisfies $(A_{p'})$. \hfill $\Box$

4 Local martingale property for maximal elements of $\mathcal{Y}$

Even if the dual minimizer $Y(y)$ is uniformly integrable, it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [12, Example 5.1]. Proposition 4.2 below yields sufficient conditions for every maximal element of $\mathcal{Y}$ (in particular, for $Y(y)$) to be a local martingale.

A semimartingale $R$ is called $\sigma$-bounded if there is a predictable process $h > 0$ such that the stochastic integral $\int h dR$ is bounded. Following [14], we make the following assumption.

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Assumption 4.1. For all $X$ and $X'$ in $\mathcal{X}$ such that $X > 0$, the process $X'/X$ is $\sigma$-bounded.

Assumption 4.1 holds easily if stock price $S$ is continuous. Theorem 3 in Appendix of [14] provides a sufficient condition in the presence of jumps. It states that every semimartingale $R$ is $\sigma$-bounded if there is a finite-dimensional local martingale $M$ such that every bounded purely discontinuous martingale $N$ is a stochastic integral with respect to $M$.

Proposition 4.2. Suppose that Assumption 4.1 holds. Let $Y \in \mathcal{Y}$ be such that $YX'$ is a local martingale for some $X' \in \mathcal{X}$, $X' > 0$. Then $YX$ is a local martingale for every $X \in \mathcal{X}$. In particular, $Y$ is a local martingale.

Proof. We assume first that $X' = Y = 1$. Let $X \in \mathcal{X}$. As $X$ is $\sigma$-bounded, there is a predictable $h > 0$ such that

$$\left| \int h dX \right| \leq 1.$$ 

Since the bounded non-negative processes $1 \pm \int h dX$ belong to $\mathcal{X}$, they are supermartingales, which is only possible if $\int h dX$ is a martingale. It follows that $X$ is a non-negative stochastic integral with respect to a martingale:

$$X = X_0 + \int \frac{1}{h} d(\int h dX) \geq 0.$$ 

Therefore, $X$ is a local martingale, see [1]. Under the condition $X' = Y = 1$, the proof is obtained.

We now consider the general case. Without loss of generality, we can assume that $X_0' = Y_0 = 1$. By localization, we can also assume that the local martingale $YX'$ is uniformly integrable and then define a probability measure $Q$ with the density

$$\frac{dQ}{dP} = X'_T Y_T.$$ 

Let $X \in \mathcal{X}$. We have that $XY'$ is a local martingale under $P$ if and only if $X/X'$ is a local martingale under $Q$.

By Assumption 4.1, the process $X/X'$ is $\sigma$-bounded. Elementary computations show that $X/X'$ is a wealth process in the financial market with stock price

$$S' = \left( \frac{1}{X'}, \frac{S}{X'} \right);$$

see [4]. The result now follows by applying the previous argument to the $S'$-market whose reference probability measure is given by $Q$. \hfill $\square$
5 Existence of the optimal martingale measure

Recall that $X(x)$ denotes the optimal wealth process for the primal problem (2.2), while $Y(y)$ stands for the minimizer to the dual problem (2.3). As usual, the density process of a probability measure $\mathbb{R} \ll \mathbb{P}$ is a uniformly integrable martingale (under $\mathbb{P}$) with the terminal value $\frac{d\mathbb{R}}{d\mathbb{P}}$.

The following is the main result of the paper.

**Theorem 5.1.** Let Assumption 4.1 hold. Suppose that there are constants $0 < a < 1$, $b \geq a$ and $C > 0$ such that

\[
\frac{1}{C} \left( \frac{y}{x} \right)^a \leq \frac{U'(x)}{U'(y)} \leq C \left( \frac{y}{x} \right)^b, \quad x \leq y,
\]

and there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ whose density process $Z$ satisfies $(A_p)$ with

\[
p = \frac{1}{1 - a}.
\]

Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process $Y(y)/y$ satisfies $(A_{p'})$ with

\[
p' = 1 + \frac{b}{1 - a}.
\]

**Proof.** From Theorem 3.4 we obtain that the dual minimizer $Y(y)$ exists and satisfies $(A_{p'})$ and then from Lemma 3.2 that it is of class $(D)$. The local martingale property of $Y(y)$ follows from Proposition 4.2, if we account for Assumption 4.1 and the martingale property of $X(-v'(y))Y(y)$. Thus, $Y(y)$ is a uniformly integrable martingale and hence, $Y(y)/y$ is the density process of the optimal martingale measure $\mathbb{Q}(y)$. \qed

We refer the reader to Remark 3.5 for a discussion of the conditions of Theorem 5.1.

**Example 5.2.** In a typical situation, the role of the “testing” martingale measure $\mathbb{Q}$ is played by the minimal martingale measure, that is, by the optimal martingale measure for logarithmic utility. For a model of stock prices driven by a Brownian motion, its density process $Z$ has the form:

\[
Z_t = \mathcal{E} (-\lambda \cdot B)_t := \exp \left( -\int_0^t \lambda dB - \frac{1}{2} \int_0^t |\lambda_s|^2 ds \right), \quad t \in [0, T],
\]
where $B$ is an $N$-dimensional Brownian motion and $\lambda$ is a predictable $N$-dimensional process of the market price of risk. We readily deduce that $Z$ satisfies $(A_p)$ for all $p > 1$ if both $\lambda$ and the maturity $T$ are bounded. This fact implies the assertions of Theorem 5.1, provided that inequalities (5.1) hold for some $a \in (0, 1)$, $b \geq a$ and $C > 0$ or, in particular, if the relative risk-aversion of $U$ is bounded away from 0 and $\infty$.

The following result shows that the key bound (5.2) is the best possible.

**Theorem 5.3.** Let constants $a$ and $p$ be such that

\[ 0 < a < 1 \quad \text{and} \quad p > \frac{1}{1-a}. \]

Then there exists a financial market with a continuous stock price $S$ such that

1. There is a $Q \in \mathcal{Q}$ whose density process $Z$ satisfies $(A_p)$.

2. In the optimal investment problem with the power utility function

\[ U(x) = x^{1-a} \quad \text{for} \quad x > 0, \]

the dual minimizers $Y(y) = y\hat{Y}$, $y > 0$, are well-defined, but are not uniformly integrable martingales. In particular, the optimal martingale measure $\hat{Q} = Q(y)$ does not exist.

The proof of Theorem 5.3 follows from Proposition 6.1 below, which contains an exact counterexample.

We conclude the section with a couple of corollaries of Theorem 5.1 which exploit connections between the $(A_p)$ condition and BMO martingales. Hereafter, we shall refer to [11] and therefore, restrict ourselves to the continuous case.

**Assumption 5.4.** All local martingales on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ are continuous.

From Assumption 5.4 we deduce that the density process of every $Q \in \mathcal{Q}$ is a continuous uniformly integrable martingale and that the dual minimizer $Y(y)$ is a continuous local martingale.

We recall that a continuous local martingale $M$ with $M_0 = 0$ belongs to BMO if there is a constant $C > 0$ such that

\[ \mathbb{E} [\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau] \leq C \]

for every stopping time $\tau$. 

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where $\langle M \rangle$ is the quadratic variation process for $M$. It is known that BMO is a Banach space with the norm
\[
\|M\|_{\text{BMO}} \triangleq \inf \left\{ \sqrt{C} > 0 : (5.3) \text{ holds for } C > 0 \right\}.
\]
We also recall that for a continuous local martingale $M$ with $M_0 = 0$,

(i) The stochastic exponential \( E(M) \triangleq e^{M - \langle M \rangle / 2} \) satisfies \((A_p)\) for some \( p > 1 \) if and only if \( M \in \text{BMO} \); see Theorem 2.4 in [11].

(ii) The stochastic exponentials \( E(M) \) and \( E(-M) \) satisfy \((A_p)\) for all \( p > 1 \) if and only the martingale
\[
q(M)_t \triangleq \mathbb{E} \left[ \langle M \rangle_T | \mathcal{F}_t \right] - \mathbb{E} \left[ \langle M \rangle_T \right], \quad t \in [0, T],
\]
is well-defined and belongs to the closure in \( \|\cdot\|_{\text{BMO}} \) of the space of bounded martingales; see Theorem 3.12 in [11].

**Corollary 5.5.** Let Assumption 5.4 hold. Suppose that there are constants \( b \geq 1 \) and \( C > 0 \) such that
\[
\frac{1}{C} \left( \frac{y}{x} \right)^{a} \leq \frac{U'(x)}{U'(y)} \leq C \left( \frac{y}{x} \right)^{b}, \quad x \leq y,
\]
and there is a martingale measure \( Q \in Q \) with density process \( Z = E(M) \) with \( M \in \text{BMO} \). Then for every \( y > 0 \) the optimal martingale measure \( Q(y) \) exists and its density process is given by \( Y(y)/y = E(M(y)) \) with \( M(y) \in \text{BMO} \).

**Proof.** From (i) we deduce that \( Z \) satisfies \((A_p)\) for some \( p > 1 \). Clearly, \((5.5)\) implies \((5.1)\) for every \( a \in (0, 1) \) and in particularly for \( a \) satisfying \((5.2)\). Theorem 5.1 then implies that \( Y(y)/y \) satisfies \((A_{p'})\) for some \( p' > 1 \) and another application of (i) yields the result.

We notice that by (i) and Theorem 5.3 the power 1 in the first inequality of \((5.5)\) cannot be replaced with any \( a \in (0, 1) \), in order to guarantee that the optimal martingale measure \( Q(y) \) exists.

**Corollary 5.6.** Let Assumption 5.4 hold and let inequality \((5.1)\) be satisfied for some constants \( 0 < a < 1, b \geq a \) and \( C > 0 \). Suppose also that there is a martingale measure \( Q \in Q \) whose density process \( Z = E(M) \) is such that the martingale \( q(M) \) in \((5.4)\) is well-defined and belongs to the closure in \( \|\cdot\|_{\text{BMO}} \) of the space of bounded martingales. Then for every \( y > 0 \) the optimal martingale measure \( Q(y) \) exists and its density process is given by \( Y(y)/y = E(M(y)) \) with \( M(y) \in \text{BMO} \).

**Proof.** The result follows directly from (ii) and Theorem 5.1.
6 Counterexample

In this section we construct an example of financial market satisfying the conditions of Theorem 5.3. For a semimartingale $R$, we denote by $\mathcal{E}(R)$ its stochastic exponential, that is, the solution of the linear equation:

$$d\mathcal{E}(R) = \mathcal{E}(R) - dR, \quad \mathcal{E}(R)_0 = 1.$$  

We start with an auxiliary filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$, which supports a Brownian motion $B = (B_t)$ and a counting process $N = (N_t)$ with the stochastic intensity $\lambda = (\lambda_t)$ given in (6.3) below; $B_0 = N_0 = 0$. We define the process

$$S_t \triangleq \mathcal{E}(B)_t = e^{B_t - t/2}, \quad t \geq 0,$$

and the stopping times

$$T_1 \triangleq \inf \{t \geq 0 : S_t = 2\},$$
$$T_2 \triangleq \inf \{t \geq 0 : N_t = 1\},$$
$$T \triangleq T_1 \wedge T_2 = \min(T_1, T_2).$$

We fix constants $a$ and $p$ such that

$$0 < a < 1 \quad \text{and} \quad p > \frac{1}{1 - a}$$

and choose a constant $b$ such that

$$a < b < \frac{1}{q} \quad \text{and} \quad \gamma \leq \frac{1}{2} \delta (1 - \delta),$$

where

$$q \triangleq \frac{p}{p - 1} < \frac{1}{a},$$
$$\delta \triangleq b - a > 0,$$
$$\gamma \triangleq \frac{b}{2}(1 - qb) > 0.$$  

With this notation, we define the stochastic intensity $\lambda = (\lambda_t)$ as

$$\lambda_t \triangleq \frac{\gamma}{1 - (S_t/2)^q} 1_{\{t < T_1\}} + \gamma 1_{\{t \geq T_1\}}, \quad t \geq 0.$$  

Recall that $N - \int \lambda dt$ is a local martingale under $\mathbb{Q}$.  

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Finally, we introduce a probability measure $\mathbb{P} \ll \mathbb{Q}$ with the density
\[
\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{\mathbb{E}^\mathbb{Q}[S_T^b]} S_T^b.
\]
Notice that
\[
\{ \frac{d\mathbb{P}}{d\mathbb{Q}} = 0 \} = \{ S_T = 0 \} = \{ \mathcal{E}(B)_T = 0 \} = \{ T = \infty \}
\]
and therefore, the stopping time $T$ is finite under $\mathbb{P}$:
\[
\mathbb{P}(T < \infty) = 1.
\]

**Proposition 6.1.** Assume (6.1) and (6.2) and consider the financial market with the price process $S$ and the maturity $T$ defined on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Then

1. The probability measure $\mathbb{Q}$ belongs to $\mathcal{Q}$ and the density process $Z$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies (A_p).

2. In the optimal investment problem with the power utility function
\[
U(x) = x^{1-a} \quad x > 0,
\]
the dual minimizers $Y(y) = y\hat{Y}$, $y > 0$, are well-defined but are not uniformly integrable martingales. In particular, the optimal martingale measure $\hat{\mathbb{Q}} = \mathbb{Q}(y)$ does not exist.

The proof is divided into a series of lemmas.

**Lemma 6.2.** The stopping time $T$ is finite under $\mathbb{Q}$ and the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

**Proof.** In view of (6.4), we only have to show that
\[
\mathbb{Q}(T < \infty) = 1.
\]
Indeed, by (6.3), the intensity $\lambda$ is bounded below by $\gamma > 0$ and hence,
\[
\mathbb{Q}(T > t) \leq \mathbb{Q}(T_2 > t) \leq e^{-\gamma t} \to 0, \quad t \to \infty.
\]
\[\square\]
From the construction of the model and Lemma 6.2 we deduce that \( \mathbb{Q} \in \mathcal{Q} \). To show that the density process \( Z \) of \( \mathbb{Q} \) with respect to \( \mathbb{P} \) satisfies \((A_p)\) we need the following estimate.

**Lemma 6.3.** Let \( 0 < \epsilon < 1 \) be a constant and \( \tau \) be a stopping time. Then

\[
\mathbb{E}^\mathbb{Q}[S^\epsilon_T | \mathcal{F}_\tau] \leq S^\epsilon_\tau \leq \left( 1 + \frac{\epsilon(1 - \epsilon)}{2\gamma} \right) \mathbb{E}^\mathbb{Q}[S^\epsilon_T | \mathcal{F}_\tau].
\]

**Proof.** We denote

\[
\theta = \frac{1}{2}\epsilon(1 - \epsilon)
\]

and deduce that

\[
S^\epsilon_t = \mathcal{E}(B)^\epsilon_t = \mathcal{E}(\epsilon B)_t e^{-\theta t}, \quad t \in [0, T].
\]

In particular, \( S^\epsilon \) is a \( \mathbb{Q} \)-supermartingale, and the first inequality in the statement of the lemma follows.

To verify the second inequality, we define local martingales \( L \) and \( M \) under \( \mathbb{Q} \) as

\[
L_t = \int_0^t \frac{\theta}{\lambda r} (dN_r - \lambda_r d\tau),
\]

\[
M_t = \mathcal{E}(\epsilon B)_t \mathcal{E}(L)_t,
\]

and observe that

\[
M_t = S^\epsilon_t, \quad t \leq T, \quad t < T_2,
\]

\[
M_T = \left( 1 + \frac{\theta}{\lambda_T} \right) S^\epsilon_T, \quad T = T_2.
\]

Since \( \lambda \geq \gamma \), we obtain that

\[
S^\epsilon_t \leq M_t \leq \left( 1 + \frac{\theta}{\gamma} \right) S^\epsilon_T, \quad t \in [0, T].
\]

As \( S \leq 2 \), we deduce that \( M \) is a bounded \( \mathbb{Q} \)-martingale and the result readily follows.

**Lemma 6.4.** The density process \( Z \) of \( \mathbb{Q} \) with respect to \( \mathbb{P} \) satisfies \((A_p)\).
Proof. Fix a stopping time $\tau$. As $Q \sim P$, we have
\[
E \left[ \left( \frac{Z_\tau}{Z_T} \right)^{1-p} \big| F_\tau \right] = E^Q \left[ \left( \frac{Z_\tau}{Z_T} \right)^{1-p} \big| F_\tau \right] = E^Q \left[ \left( \frac{Z_\tau}{Z_T} \right)^{1-p} \big| F_\tau \right] = E^Q \left[ \left( \frac{Z_\tau}{Z_T} \right)^{1-p} \big| F_\tau \right],
\]
where $\tilde{Z} = 1/Z$ is the density process of $P$ with respect to $Q$.

Recall that
\[
\tilde{Z}_T = CS^b_t,
\]
for some constant $C > 0$. Since $0 < b < bq < 1$, Lemma 6.3 yields that
\[
E^Q \left[ \tilde{Z}_\tau \big| F_\tau \right] = C \int_0^\tau \gamma \lambda_r dr - dN_r, \quad t \in [0, T].
\]
which implies the result.

We now turn our attention to the second item of Proposition 6.1. Of course, our financial market has been specially constructed in such a way that the solutions $X(x)$ and $Y(y)$ to the primal and dual problems are quite explicit.

Lemma 6.5. In the optimal investment problem with the utility function $U$ from (6.5), it is optimal to buy and hold stocks:
\[
X(x) = xS, \quad x > 0.
\]
The dual minimizers have the form $Y(y) = y\tilde{Y}$, $y > 0$, with
\[
\tilde{Y} = \mathcal{E}(L)Z,
\]
where $Z$ is the density process of $Q$ with respect to $P$ and
\[
L_t = \int_0^t \gamma \lambda_r (\lambda_r dr - dN_r), \quad t \in [0, T].
\]
Proof. We verify the conditions of Lemma 2.1. For the stochastic exponential $\mathcal{E}(L)$ we obtain that
\[
\mathcal{E}(L)_t = e^{\gamma t}, \quad t < T,
\]
and, as $S_{T_1} = 2$, that
\[ \mathcal{E}(L)_T = e^{\gamma T} \left( 1_{\{T=T_1\}} + \left( 1 - \frac{\gamma}{\lambda_T} \right) 1_{\{T=T_2\}} \right) \]
\[ = e^{\gamma T} \left( 1_{\{T=T_1\}} + \left( \frac{S_T}{2} \right)^{\delta} 1_{\{T=T_2\}} \right) \]
\[ = e^{\gamma T} \left( \frac{S_T}{2} \right)^{\delta}. \]

Hence for $\hat{Y}$ defined by (6.6) we have
\[ \hat{Y}_T = \mathcal{E}(L)_T Z_T = CS_T^{-a} = CU'(S_T), \]
for some constant $C > 0$.

Let $X \in \mathcal{X}$. Under $Q$, the product $X \mathcal{E}(L)$ is a local martingale, because $X$ is a stochastic integral with respect to the Brownian motion $B$ and $\mathcal{E}(L)$ is a purely discontinuous local martingale. It follows that $X\hat{Y} = X \mathcal{E}(L)Z$ is a non-negative local martingale (hence, a supermartingale) under $\mathbb{P}$. Thus, $\hat{Y} \in \mathcal{Y}$.

Observe that the convex conjugate to $U$ is given by
\[ V(y) = \frac{a}{1 - a} y^{\frac{1-a}{a}}, \quad y > 0. \]

It follows that
\[ V(y\hat{Y}_T) = V(y)\hat{Y}_T^{\frac{1}{a}} = V(y)C^{-\frac{1}{a}}\hat{Y}_T S_T \]
and therefore,
\[ \mathbb{E} \left[ V(y\hat{Y}_T) \right] \leq V(y)C^{-\frac{1}{a}} < \infty, \quad y > 0. \]

To conclude the proof we only have to show that the local martingale $S\hat{Y} = SE(L)Z$ under $\mathbb{P}$ is of class (D) or, equivalently, that the local martingale $SE(L)$ under $Q$ is of class (D). Actually, we have a stronger property:
\[ \{ S_{\tau} \mathcal{E}(L)_{\tau} : \tau \text{ is a stopping time} \} \text{ is bounded in } L^p(Q). \]

Indeed,
\[ S_t \mathcal{E}(L)_t \leq S_t e^{\gamma t} \leq 2^{1-b} S_t^b e^{\gamma t}, \quad t \in [0, T], \]
and then for a stopping time $\tau$,
\[
\mathbb{E}_Q^\mathbb{Q} \left[ (S_{\tau} \mathcal{E}(L)_{\tau})^q \right] \leq 2^{q(1-b)} \mathbb{E}_Q^\mathbb{Q} \left[ (S_{\tau}^b e^{\gamma \tau})^q \right] = 2^{q(1-b)} \mathbb{E}_Q^\mathbb{Q} \left[ \mathcal{E}(B)^{qb} e^{q\gamma \tau} \right] \\
= 2^{q(1-b)} \mathbb{E}_Q^\mathbb{Q} \left[ \mathcal{E}(qbB)_{\tau} \right] \leq 2^{q(1-b)}.
\]

The following lemma completes the proof of the proposition.

**Lemma 6.6.** For the dual minimizer $\hat{Y}$ constructed in Lemma 6.5 we have
\[
\mathbb{E} \left[ \hat{Y}_T \right] < 1.
\]
Thus, $\hat{Y}$ is not a uniformly integrable martingale.

*Proof.* Recall from the proof of Lemma 6.5 that for the local martingale $L$ defined in (6.7),
\[
\mathcal{E}(L)_T = e^{\gamma T} \left( \frac{S_T}{2} \right)^{\delta}.
\]
Using (6.2), we deduce that
\[
\mathcal{E}(L)_T = \frac{1}{2^\delta} e^{\gamma T} (\mathcal{E}(B)_T)^{\delta} = \frac{1}{2^\delta} e^{\gamma T} \mathcal{E}(\delta B)_T e^{-\frac{1}{2} \delta (1-\delta) T} \leq \frac{1}{2^\delta} \mathcal{E}(\delta B)_T.
\]
It follows that
\[
\mathbb{E} \left[ \hat{Y}_T \right] = \mathbb{E} \left[ \mathcal{E}(L)_T Z_T \right] = \mathbb{E}_Q^\mathbb{Q} \left[ \mathcal{E}(L)_T \right] \leq \frac{1}{2^\delta} \mathbb{E}_Q^\mathbb{Q} \left[ \mathcal{E}(\delta B)_T \right] \leq \frac{1}{2^\delta}.
\]

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