$T^3$-fibrations on compact six-manifolds

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Abstract

We describe a simple way of constructing torus fibrations $T^3 \to X \to S^3$ which degenerate canonically over a knot or link $\mathcal{t} \subset S^3$. We show that the topological invariants of $X$ can be computed algebraically from the monodromy representation of $\pi_1(S^3 \setminus \mathcal{t})$ on $H_1(T^3, \mathbb{Z})$. We use this to obtain some previously unknown $T^3$-fibrations $S^3 \times S^3 \to S^3$ and $(S^3 \times S^3) \# (S^3 \times S^3) \# (S^4 \times S^2) \to S^3$ whose discriminant locus is a torus knot $t(p, q) \subset S^3$.

1 Introduction

Our study of $T^3$-fibrations in dimension six is strongly motivated by recent advances surrounding the SYZ conjecture in mirror symmetry, especially Mark Gross’ work on special Lagrangian fibrations [3, 4]. Gross defines a class of $T^3$-fibrations $X \to S^3$ which he calls “well-behaved and admissible” and shows that, at a topological level, they display the properties the SYZ-conjecture leads us to expect of special Lagrangian fibrations on Calabi-Yau manifolds. In this paper we investigate a special case of Gross’ well-behaved, admissible fibrations. They develop canonical singularities over a smoothly embedded knot or link $\mathcal{t} \subset S^3$. However, it turns out that the total spaces of our $T^3$-fibrations are often rather far from being Calabi-Yau which places this work outside the immediate context of mirror symmetry, as an independent piece of differential geometry.

An affine $T^n$-bundle is a fibre bundle $f : X \to B$ whose fibres are $n$-tori, equipped with an affine structure. More precisely, if we write $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ for the standard
linear torus, an affine structure on the $T^n$-bundle $X$ can be specified by giving local trivialisations

$$\varphi_i : X|_{U_i} \to U_i \times \mathbb{T}^n$$

for some cover $\{U_i\}$ of $B$ so that $\varphi_j \circ \varphi_i^{-1}$ restricts to an affine transformation on each torus fibre $\{x\} \times \mathbb{T}^n$ with $x \in U_i \cap U_j$. We always assume that the total space $X$ and the base $B$ are orientable and connected. An affine $T^n$-bundle $f : X \to B$ is determined, up to equivalence, by its monodromy representation

$$\varrho : \pi_1(B, b) \to \text{SL}(H_1(F_b, \mathbb{Z})) \cong \text{SL}(n, \mathbb{Z})$$

where $b \in B$ is any basepoint and $F_b = f^{-1}(b)$, together with its Chern class

$$c(f) \in H^2(B, \mathcal{L})$$

where $\mathcal{L}$ is a bundle of lattices with $\mathcal{L}_b = H_1(F_b, \mathbb{Z}) \cong \mathbb{Z}^n$ and monodromy $\varrho$. The affine $T^n$-bundle $f : X \to B$ admits a section if and only if $c(f) = 0$. In this case it is called a linear $T^n$-bundle, where we think of the section as determining an origin in each fibre. Such a linear $T^n$-bundle $X$ with section can be recovered from its monodromy representation $\varrho$ as the bundle

$$X = \hat{B} \times_\varrho \mathbb{T}^n$$

where $\hat{B}$ is the universal cover of $B$ and $\pi_1(B, b)$ acts on $\hat{B}$ by deck transformations and on $\mathbb{T}^n$ via $\varrho$, preserving the linear structure on $\mathbb{T}^n$.

Not every $T^n$-bundle with section can be equipped with such a linear structure on the fibres. For instance, there are no non-trivial monodromy representations and Chern-classes on $S^m$ for $m > 2$, so every affine $T^n$-bundle $X \to S^m$ is trivial, $X \cong S^m \times \mathbb{T}^n$. General $T^n$-bundles over $S^m$ on the other side are classified by $\pi_{m-1}$(Diff($T^n$)) which is known to be non-trivial for certain values of $m, n$.

Few manifolds support the structure of a linear or affine $T^n$-bundle. We have a better chance of finding, on a given manifold $X$, an affine $T^n$-fibration. This is roughly speaking an affine $T^n$-bundle with some singular fibres. We define

**Definition 1.1** An affine $T^n$-fibration is a proper smooth map $f : X \to B$ between smooth manifolds so that for some proper closed submanifold $\mathfrak{k} \subset B$ the restriction $f|_{B\setminus\mathfrak{k}}$ is an affine $T^n$-bundle.

The assumption that the discriminant locus $\mathfrak{k}$ (that is the set of critical values of $f$) be a submanifold makes this a somewhat unorthodox definition of a fibration, but this paper is only concerned with fibrations in the sense of Definition 1.1, and we do not want to invent a new name for them.
In Section 2 we show that a linear $T^3$-bundle $f_0: X_0 \to B_0 = S^3 \setminus \mathfrak{k}$ (where $\mathfrak{k}$ is a compact one-dimensional submanifold, in other words, a knot or link) whose monodromy representation satisfies some algebraic condition can be canonically compactified to a $T^3$-fibration $f: X \to B$.

In Section 3 and Appendix A we study the relevant aspects of the representation theory of knot (link) groups, in particular, for torus knots $t(p, q)$ we determine all representations $\varphi: G(p, q) \to \text{SL}(n, \mathbb{Z})$ of the corresponding knot groups $G(p, q) = \pi_1(S^3 \setminus t(p, q))$ whose associated $T^3$-fibrations can be compactified in this way.

In Section 4 we show how to compute the topological invariants of the total space $X$ of such a fibration by purely algebraic means from the monodromy representation $\varphi$.

In Section 5 we apply the techniques developed in Section 4 to work out some concrete examples. We construct an infinite number of non-isomorphic affine $T^3$-fibrations on the manifolds $S^3 \times S^3$ and $(S^3 \times S^3)\#(S^3 \times S^3)\#(S^4 \times S^2)$. The critical locus is a torus knot $t(2p', 3q') \subset S^3$ with $\gcd(2p', 3q') = 1$, where $p' \in \mathbb{N}$ is odd in the former case and is even in the latter case.

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2 The construction

We describe a construction of $T^3$-fibrations using the ideas of [4]. Let the matrix

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be given and define an action of the infinite cyclic group $\langle \tilde{A} \rangle \cong \mathbb{Z}$ generated by $\tilde{A}$ on $\mathbb{C} \times T^2$ by the formula

$$\tilde{A}^k \cdot (z, \xi) = (z + 2\pi ik, \tilde{A}^k \xi)$$

where $\tilde{A} \in \text{SL}(2, \mathbb{Z})$ acts on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ in the usual way. The quotient

$$S_0 = (\mathbb{C} \times T^2)/\langle \tilde{A} \rangle$$

is a linear $T^2$-bundle over $\mathbb{C}^*$ with section. Its monodromy with respect to the obvious basis of $H_1(T^2, \mathbb{Z})$ is given by (2.4). It is well-known that there is a unique compactification

$$S_0 \subset S \xrightarrow{p} \mathbb{C}$$
which is topologically equivalent to an $I_1$ singularity in Kodaira’s classification of singularities in elliptic fibrations. The singular fibre $F_0$ over $0 \in \mathbb{C}$ is a $T^2$ with one circle collapsed to a point. We are going to construct $T^n$-fibrations $f: X^{n+k} \to B^k$ with the property that every $b \in B^k$ for which $F_b = f^{-1}(b)$ is singular has a neighbourhood $b \in U \subset B^k$ such that there is a commutative diagram

$$
\begin{array}{ccc}
f^{-1}(U) & \xrightarrow{\Phi} & S \times (\mathbb{T}^{n-2} \times \mathbb{R}^{k-2}) \\
\downarrow f & & \downarrow (p,pr_2) \\
U & \xrightarrow{\phi} & \mathbb{C} \times \mathbb{R}^{k-2}.
\end{array}
$$

(2.4)

The fibration (2.4) has a section. With view towards our application on six-manifolds we now restrict to $T^3$-fibrations over $S^3$, thus $n = k = 3$. However, much of this has obvious generalizations to arbitrary base manifolds and higher dimensions. If a fibration has singularities of type (2.4), then the discriminant locus $k = \phi^{-1}(\{0\} \times \mathbb{R})$ (the set of critical values of $f$) is an embedded one-dimensional submanifold. Because $k \subset S^3$ must be closed and $S^3$ is compact, $k$ is compact as well. This motivates the following definition.

**Definition 2.1** A link is a smoothly embedded compact one-dimensional submanifold $k \subset S^3$. A link with just one connected component is a knot.

Knot theorists often prefer to work in the category of piecewise linear spaces and maps. Also we have chosen to define a knot as a submanifold rather than an equivalence class of oriented submanifolds under the action of $\text{Diff}^+(S^3)$, however, all constructions in what follows will only depend on the equivalence class of the knot or link. The idea is to compactify a $T^3$-bundle $T^3 \to X_0 \to S^3 \setminus k$ over the complement of a knot or link by “gluing in” singular fibres of type (2.4) along $k$ so that the section and the linear structure are preserved. We need to explain under which circumstances this is possible.

**Definition 2.2** Suppose $k \subset S^3$ is a knot or link and $b \in S^3 \setminus k$. An element $m \in \pi_1(S^3 \setminus k, b)$ is called a meridian if there is a small open ball $U \subset S^3$ for which $U \cap k$ is a connected arc and if we can choose an $x \in U \setminus k$, a loop $\gamma_1: [0, 1] \to U \setminus k$ based at $x$ representing a generator of $\pi_1(U \setminus k, x) \cong \mathbb{Z}$ and a path $\gamma_2: [0, 1] \to S^3 \setminus k$ with $\gamma_2(0) = b$, $\gamma_2(1) = x$ so that $m$ is represented by $\gamma_2^{-1} \ast \gamma_1 \ast \gamma_2$. An element $\ell \in \pi_1(S^3 \setminus k, b)$ is called a longitude if there is a (“Seifert”-) surface $\Sigma \subset S^3$ with $\partial \Sigma$ a connected component of $k$ and a loop $\lambda_1: [0, 1] \to \Sigma$ which is based at some $x \in \Sigma$ and which is homotopic inside $\Sigma$ to $\partial \Sigma = k$ so that $\ell$ is represented by $\lambda_2^{-1} \ast \lambda_1 \ast \lambda_2$ for some path $\lambda_2$ from $b$ to $x$. 

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Using this language, we can see that if \( f : X \to S^3 \) is a \( T^3 \)-fibration whose singularities are locally equivalent to (2.4), the monodromy around any meridian must necessarily be conjugate to
\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}(3, \mathbb{Z}). \tag{2.5}
\]

**Theorem 2.3** Suppose \( X_0 \to S^3 \setminus \mathfrak{f} \) is a linear \( T^3 \)-bundle with section whose monodromy \( \rho \) sends all meridians into the conjugacy class of the matrix \( A \) given in (2.5). Then there exists a compactification \( X \to S^3 \) with singularities of the form (2.4).

**Proof.** Suppose we have chosen \( b \in S^3 \setminus \mathfrak{f} \), a loop \( \mu \) which is based at \( b \) and represents a meridian \( m = [\mu] \in \pi_1(S^3 \setminus \mathfrak{f}, b) \), and an isomorphism \( \text{SL}(H_1(F_b, \mathbb{Z})) \cong \text{SL}(3, \mathbb{Z}) \) which identifies the monodromy representation with a homomorphism
\[
\rho : \pi_1(S^3 \setminus \mathfrak{f}, b) \to \text{SL}(3, \mathbb{Z})
\]
such that \( \rho([\mu]) = A \). Then let \( B = \rho([\lambda]) \) be the image of the longitude \( \ell = [\lambda] \) under \( \rho \). Obviously \( [\mu][\lambda] = [\lambda][\mu] \), because if \( \mathfrak{K} \) is an open “thickening” of the knot then \( \partial(S^3 \setminus \mathfrak{K}) \cong T^2 \) and the inclusion of the boundary \( \iota : T^2 \hookrightarrow S^3 \setminus \mathfrak{K} \) satisfies \( \iota_*(\pi_1(T^2)) = ([\lambda], [\mu]) \), and hence \([\lambda]\) and \([\mu]\) commute because \( \pi_1(T^2) \) is Abelian. Consequently
\[
[A, B] = [\rho([\lambda]), \rho([\mu])] = \rho([[\lambda], [\mu]]) = 0.
\]
Then from definition (2.2) we can see that the standard action of \( B \) on \( T^3 = T^2 \times T^1 \) descends to the quotient (2.3) and induces an isomorphism \( \varphi_B \) of the linear \( T^3 \)-bundle on \( S_0 \times T^1 \times \mathbb{R} \to \mathbb{C}^* \times \mathbb{R} \).

**Proposition 2.4** The map \( \varphi_B \) extends to an isomorphism of the \( T^3 \)-fibration \( S \times T^1 \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \) which preserves the section.

This can be checked explicitly by writing down the action of \( \varphi_B \) in our local model. We can now define a \( \mathbb{Z} \)-action on the linear \( T^3 \)-fibration \( S \times (T^1 \times \mathbb{R}) \) by the formula
\[
k \cdot (x, \vartheta, t) = \varphi_B^n(x, \vartheta, t + k).
\]
By dividing out this \( \mathbb{Z} \)-action we obtain a \( T^3 \)-fibration \( X_1 \to \mathbb{C} \times S^1 \) with section whose smooth part is a \( T^3 \)-bundle over \( \mathbb{C}^* \times S^1 \) with monodromy given, in a suitable basis, by \( A, B \). By construction, the singularities are locally equivalent to (2.4). Since a \( T^n \)-bundle with section is defined by its monodromy representation we can now compactify \( X_0 \) by gluing in (some open subset of) \( X_1 \) using an isomorphism of affine \( T^3 \)-bundles which preserves the linear structure and section. \( \square \)
Definition 2.5 We call fibrations of this kind good $T^3$-fibrations.

Thus a good $T^3$-fibration always has a section and an affine structure on its fibres and hence is completely determined by its monodromy representation.

3 Representations of knot groups

In the previous section we described a construction of $T^3$-fibrations with section degenerating over a knot or link $\mathfrak{k} \subset S^3$, using a gluing construction to compactify a (linear) $T^3$-bundle over the link complement whose monodromy around each meridian was given by (2.3). We define

Definition 3.1 Let $\mathfrak{k} \subset S^3$ be a knot or link, $b \in S^3 \setminus \mathfrak{k}$. By an $M$-homomorphism we mean a homomorphism $\varrho: G(\mathfrak{k}) \to SL(3, \mathbb{Z})$ of the group $G(\mathfrak{k}) = \pi_1(S^3 \setminus \mathfrak{k}, b)$ of $\mathfrak{k}$ which has the property that each meridian is mapped to the conjugacy class $C(A)$ of the matrix $A$ in (2.3). More generally, if $\Lambda$ is a free $\mathbb{Z}$-module of rank $\text{rk}(\Lambda) = 3$, we call $\varrho: G(\mathfrak{t}) \to SL(\Lambda)$ an $M$-homomorphism if $\psi \varrho \psi^{-1}$ is an $M$-homomorphism for some (hence any) trivialisation $\psi: \Lambda \to \mathbb{Z}^3$. Two homomorphisms $\varrho, \varrho': G(\mathfrak{t}) \to SL(\Lambda)$ are conjugate if there is a fixed $M \in SL(\Lambda)$ with $\varrho' = M^{-1} \varrho M$. An $M$-representation is a conjugacy class of $M$-homomorphisms.

The only input data for our construction of $T^3$-fibrations is an $M$-representation $\varrho: G \to SL(3, \mathbb{Z})$. We will now study these $M$-representations of knot groups.

Definition 3.2 We call an element $\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{Z}^3$ primitive if

$$\gcd(\xi_1, \xi_2, \xi_3) = 1,$$

that is, if it is not a non-trivial multiple of any other element in the lattice.

Define the usual bilinear form on $\mathbb{Z}^3$ by

$$\langle \eta, \xi \rangle = \eta_1 \xi_1 + \eta_2 \xi_2 + \eta_3 \xi_3.$$ 

The following proposition is an easy to prove characterization of the conjugacy class $C(A) \subset SL(3, \mathbb{Z})$.

Proposition 3.3 A matrix $\tilde{A} \in SL(3, \mathbb{Z})$ is conjugate to $A \in SL(3, \mathbb{Z})$ if and only if it is of the form

$$\tilde{A} = \text{Id} + \eta \xi^T$$

where $\eta, \xi \in \mathbb{Z}^3$ are primitive column vectors and $\langle \eta, \xi \rangle = 0$. The pair $(\xi, \eta)$ is uniquely determined by $\tilde{A}$ up to a common sign.
Proof. If \( \tilde{A} \) and \( A \) are conjugate then
\[
\operatorname{rk}(\tilde{A} - \text{Id}) = \operatorname{rk}(A - \text{Id}) = 1
\]
and thus \( \tilde{A} = \text{Id} + \eta \xi^T \) for some \( \eta, \xi \in \mathbb{Z}^3 \). Moreover
\[
3 + \langle \eta, \xi \rangle = \text{Trace}(\tilde{A}) = \text{Trace}(A) = 3
\]
and so \( \langle \eta, \xi \rangle = 0 \) as desired. Finally since \( \text{Im}(A - \text{Id}) \cong \mathbb{Z} e_1 \) is generated by a primitive element so is \( \text{Im}(\tilde{A} - \text{Id}) \) which implies that \( \eta, \xi \) must be primitive.

Conversely let \( \tilde{A} = \text{Id} + \eta \xi^T \) be given with these properties. We have to find \( M \in \text{SL}(3, \mathbb{Z}) \) such that
\[
M^{-1} \eta \xi^T M = A - \text{Id} = e_1 e_2^T.
\]
This amounts to solving the equations
\[
M^{-1} \eta = e_1, \quad M^T \xi = e_2.
\]
The first of these equations simply says that \( \eta \) has to be the first column in \( M \).
Since \( \xi \) is primitive, we can find \( \rho = (\rho_1, \rho_2, \rho_3)^T \in \mathbb{Z}^3 \) such that
\[
\langle \rho, \xi \rangle = \rho_1 \xi_1 + \rho_2 \xi_2 + \rho_3 \xi_3 = 1.
\]
Let \( \kappa = \eta \times \rho \) where the cross product is as in \( \mathbb{R}^3 \). Then \( \kappa \) is primitive since if \( \kappa \) was divisible, then so would be \( \kappa \times \xi \), however,
\[
\kappa \times \xi = (\eta \times \rho) \times \xi = \langle \eta, \xi \rangle \rho - \langle \rho, \xi \rangle \eta = -\eta
\]
is primitive where we have used that \( \langle \eta, \xi \rangle = 0 \). Hence we can solve the equation \( \langle \tilde{\sigma}, \kappa \rangle = 1 \) by some \( \tilde{\sigma} \in \mathbb{Z}^3 \). But the latter sum is just \( \det(\eta, \rho, \tilde{\sigma}) = 1 \). Now set \( m = \langle \tilde{\sigma}, \xi \rangle \). Then if we replace \( \sigma = \tilde{\sigma} - m \rho \) and set
\[
M = (\eta, \rho, \sigma) = \begin{pmatrix}
\eta_1 & \rho_1 & \sigma_1 \\
\eta_2 & \rho_2 & \sigma_2 \\
\eta_3 & \rho_3 & \sigma_3
\end{pmatrix}
\]
we will have
\[
\det(M) = \det(\eta, \rho, \sigma) = \det(\eta, \rho, \tilde{\sigma} - m \rho) = \det(\eta, \rho, \tilde{\sigma}) = 1
\]
and the desired equations \( M e_1 = \eta \) and \( M^T \xi = e_2 \) follow. It is obvious that \( \xi, \eta \) are uniquely determined up to a common sign. \( \square \)
Proposition 3.4 Take $\eta, \xi, \tilde{\eta}, \tilde{\xi}$ with the properties given in Proposition 3.3.
Let $\bar{A} = \text{Id} + \tilde{\eta} \tilde{\xi}^T$ and $M = \text{Id} + \eta \xi^T$. Then
\[
M^{-1} \bar{A} M = \text{Id} + (\eta - \langle \xi, \tilde{\eta} \rangle \eta) (\tilde{\xi} + \langle \tilde{\xi}, \eta \rangle \xi)^T.
\]

Proof. The proof is straightforward. Note that $M^{-1} = \text{Id} - \eta \xi^T$.
\[
M^{-1} \bar{A} M = (\text{Id} - \eta \xi^T)(\text{Id} + \tilde{\eta} \tilde{\xi}^T)(\text{Id} + \eta \xi^T)
\]
\[
= \text{Id} + \tilde{\eta} \tilde{\xi}^T - (\eta \xi^T)(\tilde{\eta} \tilde{\xi}^T) + (\tilde{\eta} \tilde{\xi}^T)(\eta \xi^T) - (\eta \xi^T)(\tilde{\eta} \tilde{\xi}^T)(\eta \xi^T)
\]
\[
= \text{Id} + \tilde{\eta} \tilde{\xi}^T - \langle \xi, \tilde{\eta} \rangle \eta \xi^T + \langle \tilde{\xi}, \eta \rangle \tilde{\eta} \xi^T - \langle \xi, \tilde{\eta} \rangle \langle \tilde{\xi}, \eta \rangle \eta \xi^T
\]
\[
= \text{Id} + (\eta - \langle \xi, \tilde{\eta} \rangle \eta) (\tilde{\xi} + \langle \tilde{\xi}, \eta \rangle \xi)^T.
\]
The expression is invariant under the substitutions $(\xi, \eta) \rightarrow (-\xi, -\eta)$ and $(\tilde{\xi}, \tilde{\eta}) \rightarrow (-\tilde{\xi}, -\tilde{\eta})$ and hence well-defined. $\blacksquare$

We can use this formalism to find the $M$-representations of knot groups, starting from any Wirtinger presentation. A Wirtinger presentation describes a knot or link group $G$ in terms of a set of meridians $g_1, \ldots, g_n$, subject to relations which can be read off a link diagram. The meridians $g_1, \ldots, g_n$ all have to be mapped into the conjugacy class of the matrix $A$ in (2.5) under an $M$-representation $\varrho$. So we can make an Ansatz
\[
\varrho(g_i) = \text{Id} + \xi_i \eta_i^T, \quad i = 1, \ldots, n,
\]
where the $(\xi_i, \eta_i)$ are required to be pairs of primitive orthogonal elements of $\mathbb{Z}^3$. We translate the relations given by a Wirtinger presentation of $G$ into a set of equations for the pairs $(\eta_i, \xi_i)$. The procedure is best explained by an example.

Theorem 3.5 Let $\mathfrak{k} = t(2, 3) \subset S^3$ be the trefoil knot, $b \in S^3 \setminus \mathfrak{k}$ and $G(\mathfrak{k}) = \pi_1(S^3 \setminus \mathfrak{k}, b)$. Then the $M$-representations of $G(\mathfrak{k})$ can be classified in the following way: There is one trivial $M$-representation which factorizes through the Abelianisation $G(\mathfrak{k})^{ab} = G(\mathfrak{k})/[G(\mathfrak{k}), G(\mathfrak{k})] \cong \mathbb{Z}$, and one non-trivial $M$-representation whose image in $\text{SL}(3, \mathbb{Z})$ is isomorphic to $\text{SL}(2, \mathbb{Z})$ (see (3.14)).

Proof. As we can read from Figure 1, $G(\mathfrak{k})$ has a Wirtinger presentation
\[
G(\mathfrak{k}) = \langle g_1, g_2, g_3 \mid g_2 g_1 = g_1 g_3 = g_3 g_2 \rangle.
\]
We rewrite these relations as
\[
g_1^{-1} g_2 g_1 = g_3 \tag{3.2}
\]
\[
g_3^{-1} g_1 g_3 = g_2 \tag{3.3}
\]
\[
g_2^{-1} g_3 g_2 = g_1. \tag{3.4}
\]
Note that $G(t)$ is in fact generated by $g_1$ and $g_2$ alone due to (3.2). Also the relation (3.4) is a consequence of (3.2) and (3.3) so that we only need to satisfy the first two. We use the convention that for a manifold with basepoint $(B, b)$ and elements $\gamma_1, \gamma_2$ of its loop-group the composition $\gamma_1 * \gamma_2$ is the loop going first through $\gamma_2$ and then through $\gamma_1$. This way our $M$-representations $\pi_1(S^3 \setminus t, b) \rightarrow SL(3, \mathbb{Z})$ will be homomorphisms rather than anti-homomorphisms if we let $SL(3, \mathbb{Z})$ act on $\mathbb{Z}^3$ from the left, as usual.

We first show that for $\Lambda \cong \mathbb{Z}^3$, every non-trivial $M$-homomorphism on $\Lambda$ belongs to a family $\varrho_{abc} : G(t) \rightarrow SL(\Lambda)$ which is parametrised bijectively by all $(a, b, c) \in \mathbb{Z}^3$. Given any $M$-homomorphism $\varrho_\Lambda : G(t) \rightarrow SL(\Lambda)$ we can find a trivialisation $\Lambda \rightarrow \mathbb{Z}^3$ so that $\varrho_\Lambda$ becomes an $M$-homomorphism $\varrho : G(t) \rightarrow SL(3, \mathbb{Z})$ of the form (see (3.1))

$$\varrho(g_1) = \text{Id} + e_1 e_2^T = A \quad (3.5)$$
$$\varrho(g_2) = \text{Id} + \eta \xi^T \quad (3.6)$$

for some $(\eta, \xi)$ primitive and orthogonal. Then $\varrho(g_3)$ is determined by (3.2). Namely, using Proposition 3.4 and (3.3), (3.4) we can express $\varrho(g_3)$ as

$$\varrho(g_3) = \varrho(g_1)^{-1}\varrho(g_2)\varrho(g_1)$$
$$= \text{Id} + (\eta - \langle e_2, \eta e_1 \rangle) (\xi + \langle \xi, e_1 \rangle e_2)^T$$
$$= \text{Id} + (\eta - \eta e_1) (\xi + \xi e_1)^T. \quad (3.7)$$

If we apply $\varrho$ to (3.3) and use (3.5) and (3.7), we obtain the equation

$$\varrho(g_2) = \varrho(g_3)^{-1}\varrho(g_1)\varrho(g_2)$$
$$= \text{Id} + (e_1 - \langle \xi + \xi e_1, e_1 \rangle (\eta - \eta e_1)) (e_2 + \langle e_2, \eta - \eta e_1 \rangle (\xi + \xi e_2))^T$$
$$= \text{Id} + (e_1 - \xi (\eta - \eta e_1)) (e_2 + \eta (\xi + \xi e_2))^T. \quad (3.8)$$

Substituting this into (3.6) yields the equation

$$\pm(\eta, \pm \xi) = (e_1 - \xi (\eta - \eta e_1), e_2 + \eta (\xi + \xi e_2)) \quad (3.9)$$
which is a necessary and sufficient condition for (3.5) - (3.6) to define an $M$-homomorphism. Now observe that the right hand side of (3.8) remains unchanged if we replace $(\xi, \eta)$ by $(-\xi, -\eta)$, hence if we choose the sign appropriately (3.9) is equivalent to the following two equations

\begin{align*}
(1 + \xi_1)\eta &= (1 + \xi_1\eta_2)e_1 \\
(1 - \eta_2)\xi &= (1 + \xi_1\eta_2)e_2.
\end{align*}

(3.10) (3.11)

We first want to find all solutions to these equations with $\langle \eta, \xi \rangle = 0$ and $\eta, \xi$ primitive. Let us look at (3.10) to begin with. We distinguish the two cases whether the sides of the equation are or are not zero. If they are not we can argue as follows: $\eta$ is a multiple of $e_1$, hence being primitive it must equal $\eta = \pm e_1$. But then $\eta_2 = 0$ and (3.11) says $\xi = e_2$. Therefore, $\xi_1 = 0$ and going back to (3.10) this determines the sign of $\eta = e_1$. So $\varrho(g_2) = \text{Id} + e_1 e_2^T = A$ which gives us the trivial $M$-representation $\varrho_{\text{trivial}}$ (trivial in the sense that it factors through $G(\mathfrak{g})_{\text{ab}}$). So now we assume that both sides in (3.10) are zero. This means $\xi_1 = -1$ and $\eta_2 = 1$. Then because both $\eta$ and $\xi$ have a coefficient $\pm 1$ the primitivity condition places no further restriction on what the other $\eta_j, \xi_j \in \mathbb{Z}$ can be, and the only other equation to be satisfied is

$$0 = \langle \eta, \xi \rangle = -\eta_1 + \xi_2 + \eta_3 \xi_3.$$ 

So let $a, b, c \in \mathbb{Z}$ be any integers and let us make an Ansatz $\eta_1 = a$, $\eta_3 = b$, $\xi_3 = c$. Having fixed those coefficients, $\xi_2 = a - bc$ is uniquely determined. This yields

$$\varrho(g_2) = \text{Id} + \begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \begin{pmatrix} -1, a - bc, c \end{pmatrix} = \begin{pmatrix} 1 - a & a^2 - abc & ac \\ -1 & a - bc + 1 & c \\ -b & ab - b^2c & bc + 1 \end{pmatrix}. \quad (3.12)$$

Finally it is easy to determine the image of $g_3$ from (3.2) or (3.7); it is given by

$$\varrho(g_3) = \begin{pmatrix} 2 - a & 1 - 2a + a^2 - abc + bc & ac - c \\ -1 & a - bc & c \\ -b & ab - b^2c & bc + 1 \end{pmatrix}. \quad (3.13)$$

Now conversely, using the Wirtinger relations (3.2), (3.3) and (3.4) one can easily check that for all $a, b, c \in \mathbb{Z}$ (3.12), (3.13) extends to a homomorphism $\varrho_{abc}: G(\mathfrak{g}) \to \text{SL}(3, \mathbb{Z})$ that sends all meridians into the conjugacy class of $A$.

Are any of these $M$-homomorphisms equivalent? We check that the matrix

$$C = \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$$

10
commutes with \( A \), and \( C^{-1}g_{abc}C = \varrho_{000} \), and all the \( M \)-homomorphisms \( \varrho_{abc} \) are equivalent via conjugation to the \( M \)-homomorphism \( \varrho_{000} \) sending
\[
g_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 \mapsto \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.14)
\]

On the other side, \( \varrho_{000} \) is certainly not equivalent to the trivial (cyclic) \( M \)-homomorphism \( \varrho_{\text{trivial}} \) with \( \varrho_{\text{trivial}}(g_j) = A \) for \( j = 1, 2, 3 \) since \( \varrho_{\text{trivial}}(G(k)) \cong \mathbb{Z} \) while the matrices \( A = \varrho_{000}(g_1) \) and \( A' = \varrho_{000}(g_2) \) are well-known to generate a subgroup of \( \text{SL}(3, \mathbb{Z}) \) isomorphic to \( \text{SL}(2, \mathbb{Z}) \). \( \square \)

This formalism can be used to determine the \( M \)-representations of more complicated knots (see also Appendix A).

4 Topology of \( T^3 \)-fibrations

4.1 Some general facts about knots and knot groups

The following facts are well-known and only listed for reference. See for instance [1] and [2] for proofs and more details.

**Proposition 4.1** Let \( k \subset S^3 \) be a link with \( \mu \) components and \( B_0 = S^3 \setminus k \). Then the integral (co-) homology of \( B_0 \) is given by

\[
\begin{array}{c|c|c|c|c}
 p & 0 & 1 & 2 & 3 \\
\hline
 H^p(B_0, \mathbb{Z}) & \mathbb{Z} & \mathbb{Z}^\mu & \mathbb{Z}^\mu & 0 \\
 H_p(B_0, \mathbb{Z}) & \mathbb{Z} & \mathbb{Z}^\mu & \mathbb{Z}^\mu & 0 \\
\end{array}
\]

In order to compute the (co-) homology of a \( T^3 \)-bundle \( f_0 : X_0 \to S^3 \setminus k \) we will often use the Leray spectral sequence of a fibre bundle, whose \( E_2 \)-term (in the case of cohomology) is a double complex \( (E_2^{p,q})_{p,q} \) of modules
\[
E_2^{p,q} = H^p(S^3 \setminus k, R^q f_{0*} \mathbb{Z}). \quad (4.1)
\]

These modules can, under certain circumstances, be identified with the group cohomology modules of \( G(k) \) twisted by \( \varrho \) and can thus be determined algebraically from the abstract representation \( \varrho \). We explain this more precisely.

**Definition 4.2** We call a link \( k \subset S^3 \) irreducible if every embedding \( \iota : S^2 \hookrightarrow S^3 \setminus k \) is the restriction to \( \partial B^3 = S^2 \) of an embedding \( \iota' : B^3 \hookrightarrow S^3 \setminus k \) of a closed three-ball.

With this terminology, we can quote the following well-known theorem.
Theorem 4.3 (Papakyriakopoulos [8]) If \( \mathfrak{t} \subset S^3 \) is an irreducible link, then \( B_0 = S^3 \setminus \mathfrak{t} \) is an Eilenberg-MacLane space \( K(\pi_1(B_0), 1) = K(G(\mathfrak{t}), 1) \).

Now \( H^q(T^n, \mathbb{Z}) = \wedge^q H^1(T^n, \mathbb{Z}) \) and therefore \( R^q f_* \mathbb{Z} \) is the local system with monodromy \( \wedge^q \varrho \). So if \( \mathfrak{t} \) is irreducible, the \( E_2 \)-term of the Leray spectral sequence (4.1) can be interpreted as the double complex whose \((p, q)\)-term is the group cohomology module

\[
E^{p,q}_2 \cong H^p(G(\mathfrak{t}), \wedge^q \varrho).
\] (4.2)

We can use the powerful machinery of group cohomology to compute these modules. It will be particularly helpful that we do not have to mess about with any explicit representing cochains. The case of homology is similar. If \( G \) operates on a \( \mathbb{Z} \)-module \( \Lambda \) via \( \varrho \) we also write \( H_*(G, \Lambda) \) for \( H_*(G, \varrho) \). If we use the “bar resolution” to define the (co-)homology of a group \( G \) with coefficients in \( \Lambda \), the following formulas for the low-dimensional (co-)homology modules are immediate.

Proposition 4.4 Let \( G \) be a group and \( \Lambda \) a \( G \)-module. Then

\[
\begin{align*}
H_0(G, \Lambda) & = \frac{\Lambda}{\langle g\xi - \xi \mid g \in G, \xi \in \Lambda \rangle} = \Lambda^G \\
H^0(G, \Lambda) & = \{ \xi \in \Lambda \mid \forall g \in G: g \cdot \xi = \xi \} = \Lambda^G \\
H^1(G, \Lambda) & = \frac{\text{Der}(G, \Lambda)}{\text{Der}_0(G, \Lambda)} = \Lambda^G
\end{align*}
\]

Here \( \Lambda^G \) is the space of \( G \)-invariants, the largest submodule of \( \Lambda \) on which \( G \) acts trivially, while \( \Lambda_G \) is the space of \( G \)-coinvariants, the largest quotient of \( \Lambda \) on which \( G \) acts trivially.

Computing the higher degree homology or cohomology of a group can be rather difficult, even for everyday groups. However, the computation of \( H_*(G, \Lambda) \) and \( H^*(G, \Lambda) \) is quite easy if \( G \) is infinite cyclic.

Proposition 4.5 Suppose \( G \) is a free cyclic group and \( \Lambda \) is a \( G \)-module. Then

\[
\begin{align*}
H^0(G, \Lambda) & \cong H_1(G, \Lambda) \cong \Lambda^G \\
H^1(G, \Lambda) & \cong H_0(G, \Lambda) \cong \Lambda_G \\
H^i(G, \Lambda) & \cong H_i(G, \Lambda) \cong \{0\} \quad \text{for } i > 1.
\end{align*}
\]

In case a group \( G \) is an amalgamated sum of groups with known (co-)homology the following Mayer-Vietoris sequence can be used to compute the (co-)homology of \( G \) from that of its factors.
Theorem 4.6 (Lyndon [3], Swan [10]) Let \( G = G_1 \ast_A G_2 \) be a free product with amalgamated subgroup \( A \), and let \( \varrho: G \to \text{Aut}_Z(\Lambda) \) be a representation of \( G \) on a \( Z \)-module \( \Lambda \), making \( \Lambda \) into a \( ZG \)-module. Then the following naturally defined Mayer-Vietoris sequences are exact:

\[
\cdots \to H_{n+1}(G, \Lambda) \to H_n(A, \Lambda) \to H_n(G_1, \Lambda) \oplus H_n(G_2, \Lambda) \to H_n(G, \Lambda) \to \cdots
\]

\[
\cdots \to H^{n-1}(A, \Lambda) \to H^n(G, \Lambda) \to H^n(G_1, \Lambda) \oplus H^n(G_2, \Lambda) \to H^n(A, \Lambda) \to \cdots
\]

The fact that if \( G(k) \) is the group of a knot or irreducible link \( k \subset S^3 \) then \( S^3 \setminus k \) is a \( K(G(k), 1) \) space by Theorem 4.3, so the group (co-)homology of \( G(k) \) is identified with the singular (co-)homology of \( S^3 \setminus k \) which is a smooth open three-manifold of cohomological dimension \( \text{cd}(S^3 \setminus k) \leq 2 \).

Proposition 4.7 The group of a knot or irreducible link has no cohomology with coefficients in degree \( > 2 \) for any representation.

Proof. If \( G(k) \) is the group of a knot or irreducible link \( k \subset S^3 \) then \( S^3 \setminus k \) is a \( K(G(k), 1) \) space by Theorem 4.3, so the group (co-) homology of \( G(k) \) is identified with the singular (co-) homology of \( S^3 \setminus k \) which is a smooth open three-manifold of cohomological dimension \( \text{cd}(S^3 \setminus k) \leq 2 \). \( \Box \)

4.2 Fundamental group and characteristic classes

We now show how to compute the topological invariants of an affine \( T^3 \)-fibration of the type considered in this paper from its monodromy representation.

Theorem 4.8 Suppose \( k \) is an irreducible link and \( b \in S^3 \setminus k \). Let \( G(k) = \pi_1(S^3 \setminus k, b) \) and \( \varrho: G(k) \to \text{SL}(3, Z) \) an \( M \)-representation. If

\[
H_0(G, \varrho) = \frac{H_1(F_b, Z)}{\{ \varrho(g)\xi - \xi \mid \xi \in H_1(F_b, Z), g \in G(k) \}} = \{0\}, \quad (4.3)
\]

the total space \( X \) of the good \( T^3 \)-fibration (with section) \( f: X \to S^3 \) associated to \( \varrho \) is simply connected.

Proof. Let \( b \in S^3 \setminus k \) and \( \iota: F_b \to X \) the inclusion of the smooth fibre over \( b \) in \( X \). We claim that (4.3) implies that for every loop \( \gamma \) in \( F_b \) with base point \( \sigma(b) \), the push-forward \( \iota_*\gamma \) is null-homotopic in \( X \). The reason is, geometrically speaking, that “\( H_1(F_b, Z) \) is generated by vanishing cycles”. We will explain this more precisely now. Note that \( G(k) \) is generated by finitely many meridians \( g_1, \ldots, g_l \) and

\[
\varrho(g_i,g_j)\xi - \xi = [\varrho(g_i)(\varrho(g_j)\xi - \varrho(g_j)\xi)] + [\varrho(g_j)\xi - \xi],
\]

13
thus by induction on the length of words in $g_1, \ldots, g_l$ we conclude from (4.3) that every $\xi \in H_1(F_b, \mathbb{Z})$ can be given in terms of the $g_i$ by an expression

$$\xi = \sum_{i=1}^{r} \rho(g_{k_i})\xi_i - \xi_i, \quad k_i \in \{1, \ldots, l\}, \quad \xi_i \in H_1(F_b, \mathbb{Z}).$$  (4.4)

The image of the map

$$(\rho(g_i) - \text{Id}): H_1(F_b, \mathbb{Z}) \to H_1(F_b, \mathbb{Z})$$

is a $\mathbb{Z}$-module $E_i \subset H_1(F_b, \mathbb{Z})$ with $\text{rk}(E_i) = 1$, and by (4.3) and (4.4),

$$E_1 + \cdots + E_l = H_1(F_b, \mathbb{Z}).$$

In our local model (2.4), there is a fibre-preserving $S^1$-action in a neighbourhood of every singular fibre which is free outside the singular points, and whose fixed points are precisely the critical points. If $x \in F_b$, the orbit of $x$ under the $S^1$-action represents a non-trivial homology class $[\varepsilon_i] \in H_1(F_b, \mathbb{Z})$ which generates $E_i$ ($[\varepsilon_i]$ is the “vanishing cycle” for $g_i$). Now let $[\gamma] \in \pi_1(F_b, \sigma(b))$, then since $H_1(F_b, \mathbb{Z}) \cong \pi_1(F_b, \sigma(b))$ we can write the representing loop

$$\gamma = \sum_{i=1}^{l} r_i \varepsilon_i, \quad r_i \in \mathbb{Z},$$

thus realizing $[\gamma]$ as a “linear combination” of vanishing cycles. But now the collapsing $S^1$-action yields a homotopy from $\gamma$ to a constant loop, and

$$\iota_*(\pi_1(F_b, \sigma(b))) = \{1\} \subset \pi_1(X, \sigma(b)).$$

Now let $\gamma: S^1 \to X$ be an arbitrary (say, piecewise linear or smooth) loop. Since $f^{-1}(\mathfrak{t})$ is contained in a finite union of codimension 2 submanifolds of $X$, we can move $\gamma$ slightly inside its homotopy class to make it disjoint from $C = f^{-1}(\mathfrak{t})$. It then represents an element of $\pi_1(X \setminus C, \sigma(b))$. Now we have the following homotopy exact sequence (omitting basepoints):

$$\pi_2(S^3 \setminus \mathfrak{t}) \longrightarrow \pi_1(F_b) \longrightarrow \pi_1(X \setminus C) \longrightarrow \pi_1(S^3 \setminus \mathfrak{t}) \longrightarrow \pi_0(F_b)$$

$$\cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow$$

$$\{0\} \longrightarrow H_1(F_b, \mathbb{Z}) \longrightarrow H_1(F_b, \mathbb{Z}) \times_1 G(\mathfrak{t}) \longrightarrow G(\mathfrak{t}) \longrightarrow \{*\}$$

Because $X \to S^3$ has a section $\sigma$ this sequence splits and we can write

$$\gamma = (\iota_*\gamma_1)(\sigma_*\gamma_2)$$
with \([\gamma_1] \in \pi_1(F_b, \sigma(b))\) and \([\gamma_2] \in \pi_1(S^3 \setminus \mathfrak{k}, b)\). However, we just said that under the inclusion \(\pi_1(X \setminus C, \sigma(b)) \to \pi_1(X, \sigma(b))\) the element \([\iota_* \gamma_1] \mapsto 1 \in \pi_1(X, \sigma(b))\) and \([\sigma_* \gamma_2] \mapsto 1 \in \pi_1(X, \sigma(b))\) because \(S^3\) is simply connected. Thus
\[
\pi_1(X, \sigma(b)) = \{1\}
\]
and \(X\) is simply connected. This proof works more generally for \(T^n\)-fibrations \((n \geq 2)\) with singularities of the form \((2.4)\) as long as \((4.3)\) holds. \(\Box\)

**Proposition 4.9** Every good \(T^3\)-fibration \(X \to S^3\) degenerating over a link \(\mathfrak{k}\) with monodromy representation \(\varrho: G(\mathfrak{k}) \to \text{SL}(3, \mathbb{Z})\) is orientable.

**Proof.** Let \(X_0 = X \setminus f^{-1}(\mathfrak{k})\) be the smooth part of the fibration and \(B_0 = S^3 \setminus \mathfrak{k}\). Then
\[
TX_0 = \mathcal{V} \oplus f^*TB_0
\]
where \(\mathcal{V}\) is the vertical bundle of the \(T^3\)-bundle \(f_0: X_0 \to B_0\). Now
\[
\mathcal{V} = B_0 \times_\varrho \mathbb{R}^n,
\]
hence we get
\[
\Lambda^6TX_0 = (B_0 \times_\wedge^3 \varrho \mathbb{R}) \otimes f^*(\Lambda^3TS^3)|_{B_0}.
\]
This is trivial because \(\varrho(G(\mathfrak{k})) \subset \text{SL}(3, \mathbb{Z})\) and so \(\wedge^3 \varrho\) is trivial, as is \(TS^3\), hence \(X_0\) is orientable. But \(\dim(X \setminus X_0) = 4 = \dim(X) - 2\) and so \(X\) itself is also orientable. \(\Box\)

**Theorem 4.10** Let \(f: X \to S^3\) be a good \(T^3\)-fibration with section \(\sigma\) and \(X^\# = X \setminus \text{Crit}(f)\). Then the vertical bundle \(\mathcal{V} \subset TX^\#\) and \(TX^\#\) itself are trivial.

**Proof.** The linear structure on the \(T^3\)-bundle \(X_0 \to S^3 \setminus \mathfrak{k}\) extends to the smooth part of the singular fibres, hence there is a short exact sequence of sheaves
\[
0 \to R^{n-1}(f^\#)_* \mathbb{Z} \to \sigma^* \mathcal{V} \to X^\# \to 0
\]
and hence
\[
\mathcal{V} \cong f^*(\sigma^* \mathcal{V})
\]
trivial as a pull-back of a bundle on the base \(S^3\). (Every vector bundle on \(S^3\) is trivial since \(\pi_2(\text{GL}(n, \mathbb{R})) = \{0\}\)). Moreover,
\[
TX^\# = \mathcal{V} \oplus f^*TS^3|_{X^\#}
\]
and \(TS^3\) is trivial, hence \(TX^\#\) is trivial, too. \(\Box\)
Theorem 4.11  The underlying six-manifold $X$ of a good $T^3$-fibration $f: X \to S^3$ has vanishing second Stiefel-Whitney class, $w_2(X) = 0$.

Proof. (see [1]) By Proposition 4.9, a good $T^3$-fibration is orientable. Since $\text{Crit}(f) \subset X$ is of codimension $\geq 4$, we have $w_2(X) = w_2(X^\#)$. But $TX^\#$ is trivial by Theorem 4.10, thus $w_2(X^\#) = 0$. 

Theorem 4.12  In a six-manifold $X$ carrying a good $T^3$-fibration, the critical locus represents the first Pontryagin class:

$$c \left[\text{Crit}(f)\right] = p_1(X) \in H^4(X, \mathbb{Z})$$

for some constant $c \in \mathbb{Z}$ under Poincaré-duality.

Proof. This theorem is also contained in [1], but since we are in the simpler situation of good $T^3$-fibrations, we can avoid the use of $K$-theory.

Let $f: X \to S^3$ be a good $T^3$-fibration. Then the bundle $TX^\#$ is trivial. Choose a trivialisation of $TS^3$, this gives us a preferred trivialisation of $TX^\#$ which in turn defines a relative Pontryagin class $p_{\text{rel}}(X, X^\#) \in H^4(X, X^\#, \mathbb{Z})$. There is a short exact sequence

$$0 \to H^3(X, \mathbb{Z}) \xrightarrow{\phi} H^3(X^\#, \mathbb{Z}) \xrightarrow{\delta} H^4(X, X^\#, \mathbb{Z}) \xrightarrow{\psi} H^4(X, \mathbb{Z}).$$

The image $p_1(X) = \psi(p_{\text{rel}}(X, X^\#))$ is the first Pontryagin class of $X$. (It is independent of the chosen trivialisation of $TS^3$ because any other choice is given by a map $t: S^3 \to SO(3)$ and changes $p_{\text{rel}}(X, X^\#)$ by $\delta((f^\#)^*t^*\omega)$ where $\omega \in H^3(SO(3))$ is the canonical degree-3 class. But $(f^\#)^*(t^*\omega) = \phi(f^*(t^*\omega))$ and hence $\delta((f^\#)^*(t^*\omega) = 0$.)

But now $H^4(X, X^\#, \mathbb{Z}) \cong H_2(X \setminus X^\#, \mathbb{Z}) \cong \mathbb{Z}$ and $p_1(X)$ is represented by some multiple of the generator $[\text{Crit}(f)] \in H_2(X \setminus X^\#, \mathbb{Z})$. 

Theorem 4.13  For any good $T^3$-fibration $f: X \to B$ the Euler-characteristic vanishes, $\chi(X) = 0$.

Proof. For a fibre bundle $f: X \to B$ with fibre $F$ we have the formula $\chi(X) = \chi(F)\chi(B)$. Write the total space of our $T^3$-fibration as $X = X_0 \cup X_1$ where $X_0 \to B \setminus \mathfrak{t}$ is the smooth part of the fibration and $X_1 \to \mathfrak{r}$ is the singular fibration over an open solid torus. Then

$$\chi(X) = \chi(X_0) + \chi(X_1) - \chi(X_0 \cap X_1).$$

Now $\chi(X_0) = \chi(T^3)\chi(B \setminus \mathfrak{t}) = 0$, and $X_1$ is homotopy equivalent to a fibre bundle $X_1' \to S^1$ with fibre $F' \cong I_1 \times S^1$, hence $\chi(X_1) = \chi(F')\chi(S^1) = 0$. Finally $\chi(X_0 \cap X_1) = 0$ because the intersection retracts onto a compact smooth five-manifold (a $T^3$-bundle over $T^2$). Thus $\chi(X) = 0$. 

16
5 Examples

We study the $T^3$-fibrations associated to the simplest non-trivial $M$-representations of torus knot groups discussed in Section 4, namely those $M$-representations whose image is a subgroup of $\text{SL}(3,\mathbb{Z})$ isomorphic to $\text{SL}(2,\mathbb{Z})$. Geometrically this means that we first construct a $T^2$-fibred five-manifold $Y \to S^3$ and then consider $X = T^1 \times Y$ or, more generally, (if we do not insist on the existence of a section) principal-$T^1$-bundles $X \to Y$.

5.1 The trefoil $t(2, 3)$ and $S^3 \times S^3$

Let $\mathfrak{t} = t(2, 3) \subset S^3$ be the trefoil knot. We now show that the non-trivial $M$-representation of the trefoil group $\varphi: G(\mathfrak{t}) \to \text{SL}(2,\mathbb{Z}) \subset \text{SL}(3,\mathbb{Z})$ which was given in Theorem 3.5 (for an explicit formula see (3.14)) gives us $T^3$-fibrations on spaces such as $S^1 \times S^2 \times S^3$ and $S^3 \times S^3$.

We begin with a short detour which offers a geometric description of the $T^3$-fibration associated to $\varphi$. Recall the classical fact (see [6]) that

$$S^3 \setminus \mathfrak{t} \simeq \frac{\text{SL}(2,\mathbb{R})}{\text{SL}(2,\mathbb{Z})}. \tag{5.1}$$

The right hand side is naturally the space of lattices of unit volume in $\mathbb{C}$. To any lattice $\Lambda \subset \mathbb{C}$ we can associate the unique pair $(g_2(\Lambda), g_3(\Lambda))$ of complex numbers

$$g_2(\Lambda) = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3(\Lambda) = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6} \tag{5.2}$$

so that the elliptic curve $\mathbb{C}/\Lambda$ is equivalent to an elliptic curve in $\mathbb{C}P^2$ defined by the Weierstraß equation

$$4x^3 - g_2(\Lambda)xz^2 - g_3(\Lambda)z^3 - y^2z = 0, \tag{5.3}$$

where $[x : y : z]$ are homogeneous coordinates on $\mathbb{C}P^2$. If we rescale the lattice $\Lambda \to t\Lambda$ by some $t > 0$, its invariants transform by

$$g_2(t\Lambda) = t^{-4}g_2(\Lambda), \quad g_3(t\Lambda) = t^{-6}g_3(\Lambda)$$

as (5.2) shows. So if $\Lambda_0 = \mathbb{Z} \oplus i\mathbb{Z} \subset \mathbb{C}$ is the standard lattice (of Gauß integers) there exists a unique function $t: \text{SL}(2,\mathbb{R}) \to \mathbb{R}^+$ so that $\Lambda_M = t(M)\Lambda_0$ satisfies

$$|g_2(\Lambda_M)|^2 + |g_3(\Lambda_M)|^2 = 1.$$
for all $M \in \text{SL}(2, \mathbb{R})$, where the action of $\text{SL}(2, \mathbb{R})$ on the complex plane comes from the standard isomorphism $\mathbb{R}^2 \cong \mathbb{C}$. Obviously $g_i(\Lambda_M) = g_i(\Lambda)$ for $i = 2, 3$ if and only if $M \in \text{SL}(2, \mathbb{Z})$. Given $M \in \text{SL}(2, \mathbb{R})$, the isomorphism (5.1) sends the $\text{SL}(2, \mathbb{Z})$ orbit of $M$ to $(g_2(\Lambda_M), g_3(\Lambda_M)) \in S^3 \subset \mathbb{C}^2$. A pair $(g_2, g_3) \in S^3$ can arise in this manner if and only if the discriminant satisfies

$$\Delta = 27g_2^2 - g_3^3 \neq 0. \quad (5.4)$$

It is well-known $[7]$ that (5.4) describes the complement of a trefoil in $S^3$.

Using this isomorphism let us now define a “tautological” $\mathbb{Z}^2$-bundle $L \to S^3 \setminus \mathfrak{t}$ whose fibre at $\Lambda = (g_2, g_3) \in S^3 \setminus \mathfrak{t}$ is $\Lambda$ itself. In other words, if we embed

$$\varphi: \text{SL}(2, \mathbb{R}) \times \Lambda_0 \hookrightarrow \text{SL}(2, \mathbb{R}) \times \mathbb{C}$$

$$(M, \xi) \mapsto (M, t(M)M\xi),$$

then by the isomorphism (5.1) $L$ is the quotient

$$L = \frac{\text{Im}(\varphi)}{\text{SL}(2, \mathbb{Z})}$$

where $\text{SL}(2, \mathbb{Z})$ acts on $\text{SL}(2, \mathbb{R})$ by right-multiplication and on $\mathbb{C}$ by its real standard representation.

Now from (3.14) there is a short exact sequence

$$\{1\} \to 2\mathbb{Z}(G) \to G \to \text{SL}(2, \mathbb{Z}) \to \{1\}.$$

The centre $Z(G) \cong \mathbb{Z}$ is infinite cyclic and $2\mathbb{Z}(G) = \ker(\tilde{\rho})$ is the subgroup of index two in $\hat{Z}(G)$. Thus the $M$-representation $\tilde{\rho}$ descends to a representation $\overline{\rho}$ of the quotient $G(\mathfrak{t})/2\mathbb{Z}(G) \cong \text{SL}(2, \mathbb{Z})$, and it is easy to check explicitly that $\overline{\rho}$ is the standard representation of $\text{SL}(2, \mathbb{Z})$. Let $\hat{\text{SL}}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R})$ be the universal cover. Then

$$Y_0 = \frac{\hat{S}^3 \setminus \mathfrak{t} \times_\phi \mathbb{T}^2}{\text{SL}(2, \mathbb{R}) \times_\phi \mathbb{T}^2}$$

$$= \frac{\hat{\text{SL}}(2, \mathbb{R}) \times_\phi \mathbb{T}^2}{\text{SL}(2, \mathbb{R}) \times_\phi \mathbb{C}}$$

This is a $T^2$-bundle whose fibre at $\Lambda = (g_2, g_3) \in S^3 \setminus \mathfrak{t}$ is $\mathbb{C}/\Lambda$. Hence we can see that there is a natural embedding $Y_0 \subset \mathbb{C}P^2 \times (S^3 \setminus \mathfrak{t})$ given by the equation

$$4x^3 - g_2xz^2 - g_3z^3 - y^2z = 0 \quad (5.5)$$

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with \([x : y : z] \in \mathbb{CP}^2\) and \((g_2, g_3) \in S^3\). Note that the \(T^2\)-bundle \(f: Y_0 \to S^3 \setminus \mathfrak{k}\) also has a section \(\sigma_0: S^3 \to X\) mapping

\[(g_2, g_3) \in S^3 \mapsto ([0 : 1 : 0], (g_2, g_3)) \in X,\]

where \(\sigma(g_2, g_3)\) is the point at infinity of \(f^{-1}(g_2, g_3)\). By construction its monodromy is given by the unique non-trivial \(M\)-representation \(\varphi: G(t(2, 3)) \to \text{SL}(2, \mathbb{Z})\). However, equation (5.5) is well-defined on \(\mathbb{CP}^2 \times S^3\) and thus defines a natural compactification \(Y_0 \subset Y \subset \mathbb{CP}^2 \times S^3\). The section extends to a section \(\sigma: S^3 \to Y\).

**Proposition 5.1** \(Y\) is a smooth compact \(T^2\)-fibred five-manifold.

*Proof.* Use inhomogeneous coordinates to write \(Y\) as the vanishing locus of a smooth function \(f\) with 0 as a regular value. The unitary condition \(g_2\bar{g}_2 + g_3\bar{g}_3 = 1\) ensures that there are no critical points in \(f^{-1}(0)\). The implicit function theorem then shows that \(Y\) is smooth, and it is obviously compact. \(\square\)

From Theorem \[theoremin textbooks\] we know that \(Y\) is spin, but we can also see this directly in this situation.

**Theorem 5.2** \(Y\) is a spin manifold, that is the second Stiefel-Whitney class vanishes, \(w_2(Y) = 0\).

*Proof.* Let \(\iota: Y \hookrightarrow \mathbb{CP}^2 \times S^3\) be the inclusion and let

\[\varpi \in H^2(\mathbb{CP}^2 \times S^3, \mathbb{Z}/2) \cong H^2(\mathbb{CP}^2, \mathbb{Z}/2) \cong \mathbb{Z}/2\]

be the non-trivial degree 2 cohomology class. Let \(N \subset T(\mathbb{CP}^2 \times S^3)|_Y\) be the normal bundle of the submanifold \(Y\), then

\[w_2(Y) + w_2(N) = \iota^*w_2(\mathbb{CP}^2 \times S^3) = \iota^*(\text{pr}_1^*w_2(\mathbb{CP}^2)) = \iota^*\varpi.\]

But since the equation (5.3) is of odd degree 3 we get \(N = \iota^*\text{pr}_1^*\mathcal{O}(-3)\) and

\[w_2(N) = -3\iota^*\varpi = \iota^*\varpi,\]

and thus \(w_2(Y) = 0\) and \(Y\) is spin. \(\square\)

We would now like to identify this manifold. We first compute its integral homology. We define:

\[
\begin{align*}
B_0 &= S^3 \setminus \mathfrak{k}, & Y_0 &= f^{-1}(B_0), \\
B_1 &= N(\mathfrak{k}), & Y_1 &= f^{-1}(B_1).
\end{align*}
\]

(5.6)

As usual, \(N(\mathfrak{k})\) denotes a tubular neighbourhood of the knot.
Theorem 5.3 The integral homology of the five-manifolds \( Y_0, Y_1 \) and \( Y_0 \cap Y_1 \) is as follows:

| \( p \) | \( H_p(Y_0) \) | \( H_p(Y_1) \) | \( H_p(Y_0 \cap Y_1) \) |
|-------|----------------|----------------|----------------|
| 0     | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |
| 1     | \( \mathbb{Z} \) | \( \mathbb{Z} \oplus \mathbb{Z}/2 \) | \( \mathbb{Z}^2 \oplus \mathbb{Z}/2 \) |
| 2     | \( \mathbb{Z} \oplus \mathbb{Z}/2 \) | \( \mathbb{Z} \) | \( \mathbb{Z}^2 \oplus \mathbb{Z}/2 \) |
| 3     | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z}^2 \) |
| 4     | 0               | 0               | \( \mathbb{Z} \) |
| 5     | 0               | 0               | 0               |

**Proof.** Homology of \( Y_0 \). We can compute \( H_*(Y_0, \mathbb{Z}) \) from the Leray spectral sequence of the fibre bundle whose \( E^2 \)-term we now show is the following:

\[
\begin{array}{ccccc}
\mathbb{Z} & \mathbb{Z} & 0 & 0 \\
0 & \mathbb{Z}/2 & 0 & 0 \\
\mathbb{Z} & \mathbb{Z} & 0 & 0 \\
\end{array}
\]  \hspace{1cm} (5.7)

Here \( E^2_{p,q} \cong H_p(G, \wedge^q \varrho) \) by (4.2) for \( 0 \leq p \leq 3 \) and \( 0 \leq q \leq 2 \). Since \( \wedge^0 \varrho = \wedge^2 \varrho \) is trivial we have \( E^2_{0,0} \cong E^2_{0,2} \cong H_p(B_0, \mathbb{Z}) \) and so the top and bottom rows are given by Proposition 4.1. For the middle row we have from Proposition 4.4

\[
E^2_{0,1} = H_0(G, \varrho) = \frac{\Lambda_0}{\{g\xi - \xi \mid \xi \in \Lambda_0, \ g \in G\}} \cong \frac{\mathbb{Z}^2}{\{\varrho(g)\xi - \xi \mid \xi \in \mathbb{Z}^2, \ g \in G\}}.
\]

However, with respect to the standard basis \( \Lambda_0 \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \) and meridians \( g_1, g_2 \) as before we have \( \varrho(g_2)e_1 - e_1 = -e_2 \) and \( \varrho(g_1)e_2 - e_2 = e_1 \), thus the right hand side is trivial and \( H_0(G, \varrho) \cong E^2_{0,1} \cong \{0\} \). Also \( H_3(G, \varrho) \cong E^2_{3,1} \cong \{0\} \). To compute \( E^2_{2,1} \) and \( E^2_{1,1} \) we use the Mayer-Vietoris sequence. We can write \( G = \langle x \rangle \ast \langle x^2 = y^3 \rangle \langle y \rangle \) as the amalgamated sum of two cyclic groups \( G_1 = \langle x \rangle \) and \( G_2 = \langle y \rangle \) along the cyclic subgroup \( A \cong \langle x^2 \rangle \cong \langle y^3 \rangle \). Cyclic groups have no homology in degree \( > 0 \) and the first term which has a chance of being non-zero is \( H_2(G, \varrho) \cong E^2_{2,1} \), so the tail of this sequence looks like

\[
0 \longrightarrow E^2_{2,1} \longrightarrow H_1(A, \varrho) \xrightarrow{\psi_1} H_1(G_1, \varrho) \oplus H_1(G_2, \varrho) \longrightarrow \cdots.
\]  \hspace{1cm} (5.8)

By Proposition 4.5 and Proposition 4.4

\[
\begin{align*}
H_1(A, \varrho) & \cong H^0(A, \varrho) \cong \{\xi \in \mathbb{Z}^2 \mid \varrho(x^2)\xi = \xi\} \\
H_1(G_1, \varrho) & \cong H^0(G_1, \varrho) \cong \{\xi \in \mathbb{Z}^2 \mid \varrho(x)\xi = \xi\} \\
H_1(G_2, \varrho) & \cong H^0(G_2, \varrho) \cong \{\xi \in \mathbb{Z}^2 \mid \varrho(y)\xi = \xi\} \\
\end{align*}
\]
We check from the matrices given in Theorem 3.5 that
\[ \varrho(x) = \varrho(g_3g_2g_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
\[ \varrho(y) = \varrho(g_3g_2) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \]
\[ \varrho(x^2) = \varrho(y^3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \]

None of the representations induced from \( \varrho \) on \( A, G_1 \) and \( G_2 \) has invariants, so
\[ H_1(G_1, \varrho) \cong H_1(G_2, \varrho) \cong H_1(A, \varrho) \cong \{0\}. \]

Hence \( E^2_{2,1} \cong \{0\} \). But also \( \text{coker}(\psi_1) = \{0\} \) in (5.8) and so \( E^2_{1,1} \cong H_1(G, \varrho) \) is the kernel of the map \( \psi_0 \) in the short exact sequence
\[ 0 \to E^2_{1,1} \to H_0(A, \varrho) \xrightarrow{\psi_0} H_0(G_1, \varrho) \oplus H_0(G_2, \varrho) \to H_0(G, \varrho) \cong \{0\}. \] (5.9)

We compute these 0-th homology groups from Proposition 4.4. Note that
\[ B = \varrho(x) - \text{Id} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \]
\[ B' = \varrho(y) - \text{Id} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \]
\[ B'' = \varrho(x^2) - \text{Id} = \varrho(y^3) - \text{Id} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \]

Since \( \det(B) = 2 \) and \( \det(B') = 1 \) we deduce that
\[ H_0(G_1, \varrho) \cong \frac{\mathbb{Z}^2}{B\mathbb{Z}^2} \cong \mathbb{Z}/2, \]
\[ H_0(G_2, \varrho) \cong \frac{\mathbb{Z}^2}{B'\mathbb{Z}^2} \cong \{0\}, \]
\[ H_0(A, \varrho) \cong \frac{\mathbb{Z}^2}{B''\mathbb{Z}^2} \cong \mathbb{Z}^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2. \]

Thus (5.3) becomes a short exact sequence of groups
\[ 0 \longrightarrow E^2_{1,1} \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \]

The only group fitting into this sequence is \( E^2_{1,1} \cong \mathbb{Z}/2 \). It is obvious that the boundary operator \( \partial_2 : E^2_{p,q} \to E^2_{p-2,q+1} \) of the \( E^2 \)-term must be zero since all non-trivial homology is concentrated in degree 0 and 1. Hence the homology \( Y_0 \) is the total
homology of $E^{2}_{p,q}$ which is the homology given in the proposition.

**Homology of $Y_{1}$**. The singular fibration $Y_{1} \to B_{1}$ is homotopy equivalent to a fibre bundle of singular elliptic curves of type $I_{1}$ over the circle. Note that a longitude $\ell$ acts on $\Lambda_{0} \cong \mathbb{Z}^{2}$ via

$$\varrho(\ell) = \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix}. \quad (5.10)$$

Let $F_{0}$ be a singular fibre. Then $H_{p}(F_{0}, \mathbb{Z}) \cong \mathbb{Z}$ for $p = 0, 1, 2$ and $H_{p}(F_{0}, \mathbb{Z}) = \{0\}$ for $i > 2$, and (5.10) implies that the monodromy around $\ell$ acts as $(-1)$ on $H_{1}(F_{0}, \mathbb{Z}) \cong \mathbb{Z}$. Cover $S^{1}$ with two open intervals $U', V'$ and let $U = Y_{1}|_{U'}$ and $V = Y_{1}|_{V'}$ so that

$$H_{p}(U, \mathbb{Z}) \oplus H_{p}(V, \mathbb{Z}) \cong H_{p}(U \cap V, \mathbb{Z}) \cong H_{p}(F_{0}, \mathbb{Z}) \cong \mathbb{Z}$$

for $p = 0, 1, 2$. Choosing appropriate trivialisations the map $\psi_{p}$ in the Mayer-Vietoris sequence

$$\cdots \to H_{p+1}(Y_{1}, \mathbb{Z}) \to H_{p}(U \cap V, \mathbb{Z}) \xrightarrow{\psi_{p}} H_{p}(U, \mathbb{Z}) \oplus H_{p}(V, \mathbb{Z}) \to H_{p}(Y_{1}, \mathbb{Z}) \to \cdots \quad (5.11)$$

takes the form $(a, b) \in \mathbb{Z}^{2} \mapsto (a - b, a - \mu_{p}b) \in \mathbb{Z}^{2}$ where $\mu_{p}$ is the monodromy action on $H_{p}(F_{0}, \mathbb{Z})$. We have $\mu_{0} = \mu_{2} = 1$ and $\mu_{1} = -1$. Because $H_{*}(F_{0}, \mathbb{Z})$ is torsion free and all modules are Abelian, the Mayer-Vietoris sequence splits at all of its boundary operators, and

$$H_{p}(Y_{1}, \mathbb{Z}) = \ker(\psi_{p-1}) \oplus \text{coker}(\psi_{p}) \quad \text{ (5.12)}$$

for all $p$. Let $\psi^{\pm} : \mathbb{Z}^{2} \to \mathbb{Z}^{2}$ be the map $(a, b) \mapsto (a - b, a \mp b)$, then

$$\ker(\psi^{+}) = \text{diag}(\mathbb{Z}^{2}) \cong \mathbb{Z},$$

$$\text{coker}(\psi^{+}) = \frac{\mathbb{Z}^{2}}{\text{diag}(\mathbb{Z}^{2})} \cong \mathbb{Z},$$

$$\ker(\psi^{-}) = \{(a, b) \in \mathbb{Z}^{2} \mid a + b = a - b = 0\} \cong \{0\}, \quad \text{(no 2 torsion)}$$

$$\text{coker}(\psi^{-}) = \frac{\mathbb{Z}^{2}}{\{(a - b, a + b) \mid a, b \in \mathbb{Z}\}} \cong \mathbb{Z}/2.$$  

Now $\psi_{0} = \psi_{2} = \psi^{+}, \psi_{1} = \psi^{-}$ and $\psi_{p} = 0$ for $p > 2$. Together with (5.12) this implies the claim.
Homology of $Y_0 \cap Y_1$. We compute again the Leray spectral sequence for the $T^2$-bundle over $B_0 \cap B_1 \simeq T^2$. We claim that the $E^2$-term has the following form:

$$
\begin{array}{ccc}
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \\
\mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\
\mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}
\end{array}
$$

(5.13)

Again, the top and bottom rows are just the integral homology of $B_0 \cap B_1 \simeq T^2$. For the middle row note that the inclusion $B_0 \cap B_1 \hookrightarrow B_0$ is an inclusion of $K(\pi_1,1)$-complexes corresponding to the embedding $G' = \langle m, \ell \rangle \to G$ of the subgroup of $G$ generated by a longitude $\ell$ and a meridian $m$ into $G$. Hence let $\varrho' = \varrho|G'$. Then $E^2_{p,1} \cong H_p(G', \varrho')$ for $p = 0, 1, 2$. As before we compute $H_0(G', \varrho')$ from Proposition 4.4 and equations (2.5) and (5.10) which give explicitly the actions of $m$ and $\ell$ on $\Lambda_0$ with respect to the standard trivialisation of $\Lambda_0 \cong \mathbb{Z}^2$. We find

$$E_{0,1}^2 \cong H_0(G', \varrho') \cong \mathbb{Z}/2.$$

Note that $\varrho \sim \varrho^*$ because the representation takes values in $SL(2, \mathbb{Z})$ and

$$E_{2,1}^2 \cong H_2(G', \varrho') \cong H^0(G', (\varrho')^*) \cong H^0(G', \varrho') \cong \{0\}$$

by Poincaré-duality ((5.10) shows that $\varrho'$ has no invariants). Finally in order to compute $E_{1,1}^2$ we again use Poincaré-duality and find

$$E_{1,1}^2 \cong H_1(G', \varrho') \cong H^1(G', \varrho').$$

So we need to determine the derivations of $\varrho'$. We make an Ansatz

$$d(m) = \mu = \begin{pmatrix} r \\ s \end{pmatrix}, \quad d(\ell) = \lambda = \begin{pmatrix} t \\ u \end{pmatrix}.$$ 

This defines a derivation if and only if $0 = d(1) = d(m\ell m^{-1}\ell^{-1})$ which by virtue of the relations $d(gh) = dg + g \cdot dh$ and $d(g^{-1}) = -g^{-1} \cdot dg$ transforms by expansion into the equation

$$(\varrho'(m) - \text{Id})\lambda = (\varrho'(\ell) - \text{Id})\mu$$

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
r \\
s
\end{pmatrix}
=
\begin{pmatrix}
-2 & 6 \\
0 & -2
\end{pmatrix}
\begin{pmatrix}
t \\
u
\end{pmatrix}.
$$

The most general solution is given by choosing $r, t \in \mathbb{Z}$ arbitrary and setting $s = 0$, $u = -2t$. Therefore $\text{Der}(\varrho') \cong \mathbb{Z}^2$. The corresponding derivation $d_{r,t}$ given by $\mu = (r, 0)$ and $\lambda = (t, -2r)$ is principal if and only if there is $\kappa \in \mathbb{Z}^2$ so that

$$\mu = (\varrho'(m) - \text{Id})\kappa \quad \text{and} \quad \lambda = (\varrho'(\ell) - \text{Id})\kappa.$$
The unique solution to these equations is

$$\kappa = \left( -\frac{t}{2} - 3r \right)$$

which is integral only if $t \equiv 0 \mod 2$, hence the principal derivations correspond to the submodule $\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z}^2$ and so

$$H_1(G', \hat{g}') \cong H^1(G', \hat{g}') \cong \left( \frac{\mathbb{Z}^2}{2\mathbb{Z} \oplus \mathbb{Z}} \right) \cong \mathbb{Z}/2.$$

Thus we find the term

$$E^2_{1,1} \cong \mathbb{Z}/2$$

which completes the computation of the last missing term in the spectral sequence. Again the sequence degenerates because $E^2_{2,1} = \{0\}$, and the other arrow $\partial_2: E^2_{2,0} \to E^2_{0,1}$ vanishes because

$$H_2(B_0 \cap B_1, \mathbb{Z}) \xrightarrow{\sigma_*} E^2_{2,0} \xrightarrow{\partial_2} E^2_{0,1}$$

is an exact sequence. Here $\sigma$ is a section with $\sigma_* = (f_*|E^2_{2,0})^{-1}$, hence $\sigma_*$ is surjective and $\partial_2 = 0$. The homology is thus given by the total homology of the $E^2$-term which is as described in the proposition. \hfill \Box

**Theorem 5.4** The integral homology of $Y$ is as follows:

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| $H_p(Y, \mathbb{Z})$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |

**Proof.** We apply the Mayer-Vietoris sequence

$$\cdots \to H_{p+1}(Y, \mathbb{Z}) \to H_p(Y_0 \cap Y_1, \mathbb{Z}) \xrightarrow{\Phi_p} H_p(Y_0, \mathbb{Z}) \oplus H_p(Y_1, \mathbb{Z}) \to H_p(Y, \mathbb{Z}) \to \cdots$$

(5.14)

to $Y = Y_0 \cup Y_1$. Let

$$[\mu], [\lambda] \in \pi_1(B_0 \cap B_1) \cong H_1(B_0 \cap B_1, \mathbb{Z})$$

be the classes of a meridian and a longitude, respectively, let $F$ be a fibre and $\sigma$ a section of $Y_0 \to B_0$. Now obviously $H_0(Y, \mathbb{Z}) = \mathbb{Z}$. We also know already that $H_1(Y, \mathbb{Z}) = \{0\}$ because Theorem 4.8 states that $Y$ is simply connected.
But observe that $H_1(Y_0 \cap Y_1, \mathbb{Z})\text{free} \cong \mathbb{Z}^2$ is generated by $\sigma_*[\mu]$ and $\sigma_*[\lambda]$, while $H_1(Y_0, \mathbb{Z}) \cong \mathbb{Z}(\sigma_*[\mu])$ and $H_1(Y_1, \mathbb{Z})\text{free} \cong \mathbb{Z}(\sigma_*[\lambda])$. Then

$$\psi_1(\sigma_*[\mu]) = (\sigma_*[\mu], 0) \quad \text{and} \quad \psi_1(\sigma_*[\lambda]) = (0, -\sigma_*[\lambda]) \in H_1(Y_0, \mathbb{Z}) \oplus H_1(Y_1, \mathbb{Z}).$$

This implies that $\psi_1$ induces an isomorphism of the free parts

$$H_1(Y_0 \cap Y_1, \mathbb{Z})\text{free} \xrightarrow{\cong} H_1(Y_0, \mathbb{Z}) \oplus H_1(Y_1, \mathbb{Z})\text{free}.$$ (5.15)

Let $[\tau_1] \in H_1(Y_1, \mathbb{Z})$ and $[\tau_2] \in H_1(Y_0 \cap Y_1, \mathbb{Z})$ be the unique torsion elements. Since $\psi_1$ has to be onto in order for $H_1(Y, \mathbb{Z})$ to vanish, we conclude $[\tau_1] \in \text{im}(\psi_1)$. Given the isomorphism (5.15) this is only possible if $\psi_1([\tau_2]) = [\tau_1]$ for the torsion class, and thus $\psi_1$ is an isomorphism of $\mathbb{Z}$-modules. So $\text{coker}(\psi_1) = \ker(\psi_1) = \{0\}$. Therefore $H_2(Y, \mathbb{Z}) \cong \text{coker}(\psi_2)$. Note that

$$H_2(Y_0, \mathbb{Z})\text{free} \cong H_2(Y_1, \mathbb{Z}) \cong \mathbb{Z}[F]$$

is generated by the class of a fibre, while for a generator $T \in H_2(B_0 \cap B_1, \mathbb{Z})$

$$H_2(Y_0 \cap Y_1, \mathbb{Z})\text{free} \cong \mathbb{Z}[F] \oplus \mathbb{Z}(\sigma_*[T]).$$

Thus $\psi_2([F]) = ([F], -[F])$ while $\psi_2(\sigma_*[T]) = 0$. Now denote the degree-2 torsion classes by $[\tau_3] \in H_2(Y_0, \mathbb{Z})$ and $[\tau_4] \in H_2(Y_0 \cap Y_1, \mathbb{Z})$. Suppose $\psi_2([\tau_4]) = [\tau_3]$. In this case we get

$$\text{ker}(\psi_2) = \mathbb{Z}([\sigma_*T]) \cong \mathbb{Z},$$
$$\text{coker}(\psi_2) = \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[F] \oplus \mathbb{Z}/2[\tau_3]}{\{a[F], -a[F] \mid a \in \mathbb{Z}\} \oplus \mathbb{Z}/2[\tau_3]} \cong \frac{\mathbb{Z}^2}{\text{diag}(\mathbb{Z}^2)} \cong \mathbb{Z}$$ (5.17)

and therefore

$$H_2(Y, \mathbb{Z}) \cong \text{coker}(\psi_2) \cong \mathbb{Z}[T] \cong \mathbb{Z}.$$

Otherwise if $\psi_2([\tau_4]) \neq \tau_3$ then $\psi_2([\tau_4]) = 0$ since it is a torsion class. But then we obtain $H_2(Y, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. This is a contradiction since the torsion subgroup $H_2(Y, \mathbb{Z})_{\text{tors}} \cong \frac{1}{2}T \oplus \frac{1}{2}T$ of a simply connected spin five-manifold is always decomposable as a direct sum of two isomorphic torsion modules (see [4]). So we can rule out the case $[\tau_4] \in \text{ker}(\psi_2)$ and $H_2(Y, \mathbb{Z}) \cong \mathbb{Z}$. For the remaining homology groups $H_3(Y_0, \mathbb{Z})$ with $p > 2$ we only give a sketch since Smale’s Theorem ([3]) implies that they are already uniquely determined by $H_2(Y, \mathbb{Z})$. For $Y = Y_0, Y_1, Y_0 \cap Y_1$ there is an isomorphism

$$\tau_F: H_1(Y, \mathbb{Z})\text{free} \longrightarrow H_3(Y, \mathbb{Z}).$$
This isomorphism sends the classes of \( m \) and \( \ell \) to \([f^{-1}(m)]\) and \([f^{-1}(\ell)]\). Then
\[
\psi_3 \circ \tau[F] = (\tau[F], \tau[F]) \circ \psi_1,
\]
hence \( \psi_3 \) is an isomorphism as is \( \psi_1 \). Thus
\[
H_3(Y, \mathbb{Z}) = \ker(\psi_2) \cong \mathbb{Z}
\]
by (5.17). For \( i > 3 \), we have \( \psi_i = 0 \) trivially, and
\[
H_4(Y, \mathbb{Z}) = \coker(\psi_4) = \frac{\{0\}}{\{0\}} = \{0\}
\]
since \( H_4(Y_i, \mathbb{Z}) = \{0\} \) for \( i = 0, 1 \). Finally
\[
H_5(Y, \mathbb{Z}) = \ker(\psi_4) = H_4(Y_0 \cap Y_1, \mathbb{Z}) \cong \mathbb{Z},
\]
which is also clear otherwise since \( Y \) is compact, orientable and \( \dim(Y) = 5 \). \( \square \)

**Corollary 5.5** \( Y \) is an (integral) homology \( S^2 \times S^3 \).

We are now in a position to show that \( Y \) is diffeomorphic to \( S^2 \times S^3 \). This uses the following theorem of Smale:

**Theorem 5.6** (Smale [9]) There is a bijective correspondence between the category of smooth, closed, orientable, simply connected, 5-dimensional spin manifolds and the category of finitely generated Abelian groups. The correspondence is realized by sending a manifold \( Y \) in this category to the group \( F \oplus \frac{1}{2}T \), where \( F \oplus T = H_2(Y, \mathbb{Z}) \) is a decomposition of the second integral homology of \( Y \) into its free and its torsion subgroups, and \( T = \frac{1}{2}T \oplus \frac{1}{2}T \) is a direct sum decomposition of the torsion subgroup.

This is an analogue for five-manifolds of Wall’s classification theorem in dimension six, however, it is easier to apply since we do not need to know the ring-structure on homology, and it is also more general in that it includes manifolds with torsion in homology.

**Theorem 5.7** \( Y \) is diffeomorphic to \( S^2 \times S^3 \).

**Proof.** The five-manifold \( Y \) is smooth and compact by Proposition 5.1 and simply connected by Theorem 4.8, it is orientable by Proposition 4.9, \( Y \) is torsion free and has the right Betti-numbers by Theorem 5.3. \( Y \) is spin from Theorem 5.2. Hence the claim follows by applying Theorem 5.6. \( \square \)

This gives us torus fibrations on well-known manifolds. We formulate this in the following theorem.
Theorem 5.8 There are $T^3$-fibrations on $S^1 \times S^2 \times S^3$ and $S^3 \times S^3$ whose singularities are locally equivalent to (2.4).

Proof. For the first we simply take the product $S^1 \times Y$ and extend the fibration map $f: Y \to S^3$ trivially, for the second we take the Hopf map $H: S^3 \to S^2$ and let $\tilde{f}: S^3 \times S^3 \to S^3$ be given by $\tilde{f} = f \circ (H, \text{id})$. Note that this latter fibration has no section, while the first one obviously has one.  

5.2 General torus knots

We give a brief account of the general torus knot $t(p, q)$ for arbitrary $p, q \in \mathbb{N}$. First observe that we can easily modify the argument given in Paragraph 5.1 to obtain a smooth compactification of a $T^2$-bundle over $S^3 \setminus t(p, q)$ for any pair of coprime integers $(p, q) \in \mathbb{N}^2$ such that $2|p$ and $3|q$. As in (5.3), we define $Y_{p, q} \subset \mathbb{C}P^2 \times S^3$ by the (single-valued) equation

$$4x^3 - g_{2}^{q/3}xz^2 - g_{3}^{p/2}z^3 - y^2z = 0$$

and let again the fibration $f: Y_{p, q} \to S^3$ be given by $f = \text{pr}_2|Y_{p, q}$ where $\text{pr}_2: \mathbb{C}P^2 \times S^3 \to S^3$ is projection onto the second factor. As before, a local calculation shows $Y_{p, q}$ to be a smooth compact submanifold of $\mathbb{C}P^2 \times S^3$. For generic $(g_2, g_3) \in S^3$ the fibre $f^{-1}(g_2, g_3)$ is an elliptic curve in $\mathbb{C}P^2$ determined by the Weierstraß equation with coefficients $g_2^{q/3}$ and $g_3^{p/2}$. The fibre is singular if and only if the discriminant vanishes, that is

$$\Delta = 27 \left( g_3^{p/2} \right)^2 - \left( g_2^{q/3} \right)^3 = 27g_3^p - g_2^q = 0.$$ 

From [7] the discriminant locus is a $(p, q)$-torus knot. The argument for the vanishing of the second Stiefel-Whitney class in Paragraph 5.1 carries over directly to $Y_{p, q}$. From Theorem 4.8 we deduce that $Y_{p, q}$ is simply connected. To summarize:

Theorem 5.9 $Y_{p, q} \subset \mathbb{C}P^2 \times S^3$ is a smooth, compact, orientable submanifold with $\pi_1(Y_{p, q}) = \{1\}$ and $w_2(Y_{p, q}) = 0$. There is a $T^2$-fibration $f: Y_{p, q} \to S^3$ with singularities of type (2.4) which degenerates over a $(p, q)$-torus knot $t(p, q) \subset S^3$.

5.2.1 $M$-representations

Using the techniques developed in Section 3 one can show that a torus knot $t(p, q) \subset S^3$ admits a non-trivial $M$-representation (that is, one whose image is $\not\subset \mathbb{Z}$) if and only if $2|p$ and $3|q$ (after exchanging $p$ and $q$ if necessary). In this case, there is precisely one $M$-representation whose image in $\text{SL}(3, \mathbb{Z})$ is isomorphic to
SL(2, \mathbb{Z}). This is the monodromy of the $T^2$-fibration $Y_{p,q} \to S^3$ described above. The case $p = 2, q = 3$ was dealt with in Theorem 3.5. We give a short description of the other SL(2, \mathbb{Z})-valued $M$-representations which occur as monodromy representations in the $T^2$-fibrations $Y_{p,q} \to S^3$. We refer to Appendix A for more details and a complete classification of all $M$-representations.

Theorem A.1 and its proof show that the group $G = G(t(4,3))$ of the $(4,3)$-torus knot $t(4,3)$ has a unique non-trivial $M$-representation $\varrho = \varrho_0: G \to \text{SL}(2,\mathbb{Z}) \subset \text{SL}(3,\mathbb{Z})$ which can be given, with respect to a presentation $G = \langle x,y \mid x^4 = y^3 \rangle$, by

$$
\varrho(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varrho(y) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

(5.18)

see Appendix A. Note that $\varrho(x)^4 = \varrho(y)^3 = \text{Id}$ and the meridian $m = x^{-1}y$ is mapped to the standard monodromy matrix $A$ given in (2.5).

Now suppose we have $p,q \in \mathbb{N}$ with $2 \nmid p$, $3 \nmid q$, and $\gcd(p,q) = 1$. Let $G(t) = \langle x, y \mid x^p = y^q \rangle$ and $G(2,3) = \langle x, y \mid x^2 = y^3 \rangle$. Let $\varrho: G(2,3) \to \text{SL}(2,\mathbb{Z})$ the unique non-trivial $M$-representation of $G(2,3)$. Then

$$
\varrho(x) = \varrho(\bar{x})\pm^1 \quad \text{and} \quad \varrho(y) = \varrho(\bar{y})\pm^1.
$$

Also note that the image of a longitude $\ell$ under our representation is

$$
\varrho(\ell) = \begin{pmatrix} -1 & pq & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \varrho(\bar{\ell})^{pq/6}.
$$

If $p,q \in \mathbb{N}$ with $4 \nmid p$ and $3 \nmid q$ the unique $M$-representation with image $\text{SL}(2,\mathbb{Z})$ can be derived similarly from the SL(2, \mathbb{Z})-valued $M$-representation (5.18) of $G(t(4,3))$.

5.2.2 $t(4,3)$ and $(S^3 \times S^3)\#(S^3 \times S^3)\#(S^4 \times S^2)$

We now study the $T^2$-bundle $f: Y = Y_{4,3} \to S^3$ associated to the SL(2, \mathbb{Z})-valued $M$-representation of $G(t(4,3))$. As before, let

$$
B_0 = S^3 \setminus t(4,3), \quad Y_0 = f^{-1}(B_0),
$$

$$
B_1 = N(t(4,3)), \quad Y_1 = f^{-1}(B_1).
$$

and denote by $M$ the $\mathbb{Z}G$-module given by the monodromy representation $\varrho$. Since the computations are somewhat lengthy and all the necessary techniques were already illustrated in the last example, we will only give the results.

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Theorem 5.10 The integral homology of $Y_0$, $Y_1$ and $Y_0 \cap Y_1$ is given by the following table:

| $p$ | $H_p(Y_0)$ | $H_p(Y_1)$ | $H_p(Y_0 \cap Y_1)$ |
|-----|------------|------------|----------------------|
| 0   | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$         |
| 1   | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}^3$       |
| 2   | $\mathbb{Z}^3$ | $\mathbb{Z}^2$ | $\mathbb{Z}^4$       |
| 3   | $\mathbb{Z}^3$ | $\mathbb{Z}$  | $\mathbb{Z}^3$       |
| 4   | 0           | 0          | $\mathbb{Z}$         |
| 5   | 0           | 0          | 0                    |

By applying the Mayer-Vietoris sequence one finds the homology of $Y = Y_0 \cup Y_1$.

Theorem 5.11 The integral homology of $Y = Y_0 \cup Y_1$ is as follows:

| $p$ | $H_p(Y, \mathbb{Z})$ |
|-----|----------------------|
| 0   | $\mathbb{Z}$         |
| 1   | $\mathbb{Z}^2$       |
| 2   | $\mathbb{Z}^2$       |
| 3   | 0                     |
| 4   | 0                     |
| 5   | $\mathbb{Z}$         |

The following corollary follows from Smale’s Theorem 5.6 in the same way as the corresponding statements in Paragraph 5.1.

Corollary 5.12 $Y$ is diffeomorphic to $(S^2 \times S^3) \# (S^2 \times S^3)$.

Proof. $Y$ is closed, spin, and simply connected by Theorem 5.9. The description of the integral homology in Theorem 5.11 together with Smale’s Theorem 5.6 yields the result. \qed

As before we can construct $\mathbb{T}^3$-fibrations $X \to Y \to S^3$ where $X \to Y$ is a $\mathbb{T}^1$-bundle. If we chose the trivial $\mathbb{T}^1$-bundle then

$$X \cong S^1 \times (S^2 \times S^3) \# (S^2 \times S^3),$$

and if $X \to Y$ is a $\mathbb{T}^1$-bundle with primitive Chern-class in $H^2((S^2 \times S^3) \# (S^2 \times S^3), \mathbb{Z}) \cong \mathbb{Z}^2$ then we obtain the space

$$X \cong (S^3 \times S^3) \# (S^3 \times S^3) \# (S^4 \times S^2).$$

This uses again the Leray spectral sequence.
5.2.3 The \((p, q)\)-torus knot

We argue that we do not get any new manifolds by using other torus knots.

**Theorem 5.13** Let \(p, q \in \mathbb{N}\) be coprime positive integers with \(2 \mid p\) and \(3 \mid q\), let \(t = t(p,q) \subset S^3\) be the \((p, q)\)-torus knot and let \(f : Y_{p,q} \to S^3\) be the \(T^2\)-bundle with section associated to the unique \(M\)-representation \(\rho = \rho_0 : G(t) \to \text{SL}(2, \mathbb{Z}) \subset \text{SL}(3, \mathbb{Z})\). Then either \(4 \nmid p\) and \(Y_{p,q} \cong S^2 \times S^3\) or \(4 \mid p\) and \(Y_{p,q} \cong (S^2 \times S^3) \# (S^2 \times S^3)\).

**Sketch of proof.** One can then check that all calculations for \(t(2, 3)\) and \(t(4, 3)\) remain valid. \(\square\)

**Corollary 5.14** There are infinitely many pairwise inequivalent \(T^3\)-fibrations with singularities of type \((2.4)\).

\[
\begin{align*}
S^3 \times S^3 & \to S^3 \quad (5.19) \\
S^1 \times S^2 \times S^3 & \to S^3 \quad (5.20) \\
S^1 \times ((S^2 \times S^3) \# (S^2 \times S^3)) & \to S^3 \quad (5.21) \\
(S^3 \times S^3) \# (S^3 \times S^3) \# (S^4 \times S^2) & \to S^3 \quad (5.22)
\end{align*}
\]

The discriminant locus can be any torus knot \(t(2p', 3q')\), with \(p'\) odd in \((5.19), (5.21)\) and \(p'\) even in \((5.21), (5.22)\). More generally such fibrations exist on every \(T^1\)-bundle over \(S^2 \times S^3\) or \((S^2 \times S^3) \# (S^2 \times S^3)\).

### A Computations with knot groups

**Theorem A.1** Let \(\mathfrak{t} \subset S^3\) be the \((4, 3)\)-torus knot and \(G = \pi_1(S^3 \setminus \mathfrak{t})\) its group. Then the \(M\)-representations of \(G\) are given by the trivial \(M\)-representation \(\rho_{\text{trivial}}\) and an infinite family \(\rho_k\) of non-Abelian \(M\)-representations parametrised by \(k \in \mathbb{Z}\), with \(\rho_{-k} = (\rho_k^{-1})^* = \rho_k^*\).

![](image)

**Figure 2:** The \((4, 3)\)-torus knot \(t(4, 3)\) (or \(S_{19}\)).
We do not give a full proof which is very similar to that of Theorem 3.5, but we do describe explicitly these $M$-representations. From the above knot-diagram we obtain the following Wirtinger-presentation of $G$:

\begin{align*}
&g_1^{-1}g_3g_1 = g_2 \quad \text{(A.1)} \quad g_2^{-1}g_8g_2 = g_5 \quad \text{(A.3)} \\
g_3^{-1}g_6g_3 = g_1 \quad \text{(A.2)} \quad g_4^{-1}g_1g_4 = g_8 \quad \text{(A.4)} \\
g_5^{-1}g_2g_5 = g_6 \quad \text{(A.5)} \quad g_5^{-1}g_5g_1 = g_4 \quad \text{(A.7)} \\
g_2^{-1}g_4g_2 = g_7 \quad \text{(A.6)} \quad g_5^{-1}g_7g_5 = g_3 \quad \text{(A.8)}
\end{align*}

Making an Ansatz as in the case of a trefoil we get the family $(\varrho_k)_{k \in \mathbb{Z}}$ of $M$-representations, determined by sending 

\begin{align*}
g_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}

and sending the remaining generator

\begin{align*}
g_3 \mapsto \begin{pmatrix} 0 & 1 & k \\ -1 & 2 & k \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k < 0, \quad \text{and} \quad g_3 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ k & -k & 1 \end{pmatrix} \text{ if } k \geq 0.
\end{align*}

For $k = 0$ we get an $M$-representation with image $\text{SL}(2, \mathbb{Z}) \subset \text{SL}(3, \mathbb{Z})$. This is the unique non-trivial $M$-representation of $G(t(4, 3))$ with this property and was used in Paragraph 5.2.

Similar computations show that the possible $M$-representations of a general torus knot can be obtained from those of the $(2, 3)$ and $(4, 3)$ torus knots. We do not carry this out here but only state the result.

**Theorem A.2** The group $G(p, q)$ of the $(p, q)$-torus knot $t(p, q)$ has non-trivial $M$-representations if and only if (after perhaps exchanging $p$ and $q$) $p = 2^mp' \text{ and } q = 3^kq'$ with $\gcd(p', 6) = \gcd(q', 6) = \gcd(p', q') = 1$ and $m, k > 0$. In case $m = 1$ there is a bijection $M\text{-Rep}(G(p, q)) \to M\text{-Rep}(G(2, 3))$ between the $M$-representations of $G(p, q)$ and $G(2, 3)$. In case $m > 1$ there is a bijection $M\text{-Rep}(G(p, q)) \to M\text{-Rep}(G(4, 3))$.

If $G = \langle x, y \mid x^2 = y^3 \rangle$ is the trefoil group and $G = \langle \bar{x}, \bar{y} \mid \bar{x}^{2p'} = \bar{y}^{3q'} \rangle$ the group of the $(2p', 3q')$ torus knot, with $p', q'$ odd and relative prime, this correspondence is
given by sending an $M$-representation $\varrho$ of $G$ to the $M$-representation $\bar{\varrho}$ of $\bar{G}$ which is defined by

\[ \bar{\varrho}(\bar{x}) = \varrho(x)^{\pm 1}, \quad \bar{\varrho}(\bar{y}) = \varrho(y)^{\pm 1}, \]  

(A.9)

for a unique choice of exponent $\pm 1$ which is determined by $p'$ and $q'$. This is well-defined because $\varrho(x)^2 = \varrho(y)^3 = -\text{Id}$ and $p', q'$ are both odd, hence (A.9) does define a homomorphism, and there is a unique choice of exponents $\pm 1$ in (A.9) to make it an $M$-homomorphism. The case of $t(4p', 3q')$ with $\gcd(2p', q') = 1$ is analogous.

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