Power fluctuations in a driven damped chaotic pendulum

Yadin Y. Goldschmidt
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh PA 15260, U.S.A.
(Received: November 8, 2018)

In this paper we investigate the power fluctuations in a driven, damped pendulum. When the motion of the pendulum is chaotic, the average power supplied by the driving force is equal to the average dissipated power only for an infinite long time period. We measure the fluctuations of the supplied power during a time equal to the period of the driving force. Negative power fluctuations occur and we estimate their probability. In a chaotic state the histogram of the power distribution is broad and continuous although bounded. For a value of the power not too close to the edge of the distribution the Fluctuation Theorem of Gallavotti and Cohen is approximately satisfied.

PACS numbers: 05.45.Pq, 05.40-a, 05.45.Ac, 05.70.Ln

Recently Gallavotti and Cohen (GC) [1] derived a fluctuation theorem (FT) for chaotic systems. This theorem is concerned with the rate of entropy production \( \sigma_{\tau} \) averaged over an observation time \( \tau \). It states that the ratio of probabilities of having a given entropy production \( \sigma_{\tau} \) to that of having \( -\sigma_{\tau} \) is given by \( \exp(\tau \sigma_{\tau}) \).

The theorem was proven for thermostatted Hamiltonian systems which are driven by external forces, under the condition that the system is sufficiently chaotic (of the Anosov type [2]) and the number of degrees of freedom is very large. All the systems considered had a certain form of time reversibility even though they were dissipative. This was the result of the way the thermostat was imposed. The fluctuation theorem is actually a generalization of the fluctuation dissipation theorem (FDT) to systems away from equilibrium i.e. driven systems in a steady state. Gallavotti [3] has shown that in the limit of vanishing driving force the FT reduces to the FDT.

More recently Kurchan [4] and subsequently Lebowitz and Spohn [5] extended the FT to stochastic dynamics. Kurchan showed that the FT holds for finite systems undergoing Langevin dynamics. The role of chaos is replaced by the stochastic fluctuations. Lebowitz and Spohn generalized the FT to general Markov processes. Particular role is played by systems displaying a form of local detailed balance.

Our goal in this work was to test the FT for the case of a simple driven dissipative chaotic system that does not satisfy the conditions set by GC. For example, this system is most likely not of the Anosov type [2], and it is also not strictly thermostatted (energy is conserved only over a long period of time). Thus, it is deterministic but it is not time reversible because of the role of dissipation. This example may serve as a representative that is more closely related to real life realizations. Surprisingly, (or not) we find that approximately, the FT is satisfied, taking into account that we have only three degrees of freedom, whereas the FT derived by GC should apply for the case of a large number of degrees of freedom.

The system that we consider is that of a driven damped pendulum [6]. The driving force is periodic. Varying the strength of the driving force the pendulum can display simple periodic behavior, a motion characterized by period doubling, tripling etc., or a fully chaotic motion. There is a lot of literature on such a system. What we had in mind was to consider the power bestowed on the system by the driving force during a complete period of the latter (or during an integer multiple of the period). When the motion is periodic, the value of the power bestowed during any full period of the driving force is constant. But when the system is truly chaotic, there is a broad distribution and occasionally the power bestowed is negative. We are going to investigate numerically the power distribution, and discuss the similarity to the GC Fluctuation Theorem. We are going to find that the power distribution is actually a good fingerprint to determine if the system is truly chaotic.

We consider a planar pendulum subject to the force of gravity, a damping force resulting for instance from a viscous medium in which the pendulum is immersed, and a sinusoidal driving force acting around the pivot point. The equation of motion for the angle \( \theta \) is [6]:

\[
\ddot{\theta} = -\frac{b}{m\ell^2} \dot{\theta} - \frac{g}{\ell} \sin \theta + \frac{N_d}{m\ell^2} \cos \omega_d t.
\]

(1)

Here \( b \) is the damping coefficient and \( N_d \) is the driving torque of angular frequency \( \omega_d \). We are going to switch to dimensionless parameters. Denoting by \( \omega_0^2 = g/\ell \) and \( t_0 = 1/\omega_0 \), we set \( t' = t/t_0 \) and \( \omega = \omega_d/\omega_0 \). We also define

\[
x = \theta, \quad c = \frac{b}{m\ell^2 \omega_0}, \quad F = \frac{N_d}{mg\ell}.
\]

(2)

The equation of motion can now be written as a set of two first order differential equations (omitting the prime over the rescaled time variable):

\[
x'(t) = y,
\]

(3)

\[
y'(t) = -cy - \sin(x) + F \cos(\omega t).
\]

(4)
We have solved these equations numerically using Mathematica. We have used the values
\[ c = 0.05, \ \omega = 0.7 \]  
and considered different values of the driving strength \( F \) in the range 0-1.0. Defining
\[ \tau_d = \frac{2\pi}{\omega}, \]  
to be the period of the driving force, we will be interested in evaluating the quantity
\[ \langle P \rangle_{\tau,m} = \frac{F}{\tau} \int_{t_m}^{t_m + \tau} dt \ y(t) \ \cos(\omega t), \]  
with
\[ \tau = n\tau_d, \ \ t_m = m\tau \]  
and \( n \) is a fixed integer. We will be particularly interested in the case \( n = 1 \). \( \langle P \rangle_{\tau} \) is the average power supplied to the system by the driving force during \( n \) complete cycles of the latter. In the following we will calculate \( \langle P \rangle_{\tau,m} \) along a phase-space trajectory of the system, starting with \( m = 100 \) to allow the transient to decay. We calculate consecutive averages over time periods of duration \( \tau \) and observe the fluctuations of the measured quantities.

If we multiply Eq.(4) by \( y \) we get
\[ \frac{d}{dt} \left( \frac{1}{2} y^2 - \cos x \right) = F y \cos(\omega t) - c y^2. \]  
Thus averaging over an arbitrary time period \( \tau \)
\[ \langle \frac{d}{dt} E \rangle_{\tau} = F \langle y \cos(\omega t) \rangle_{\tau} - c \langle y^2 \rangle_{\tau}, \]  
where \( E \) is the total energy of the pendulum (kinetic plus gravitational). The first term on the right hand side of the equation represents the power input from the driving force and the second term represents the power output through dissipation with e.g. a viscous medium. The second term is always positive, whereas there is no restriction on the sign of the first term. We expect that in the limit of large \( \tau \), the left hand side \( (\langle E(\tau) - E(0) \rangle / \tau) \) will approach zero, if the system is in a steady state, and thus in that limit the two terms on the right should cancel each other. For a simple harmonic oscillator, i.e. if the sin \( x \) in Eq.(4) is replaced by \( x \), then after the transient term dies out the left hand side of Eq. (10) is zero for any \( \tau = n \tau_d \) and
\[ \langle P \rangle_{\tau} = \frac{c}{2} \frac{F^2 \omega^2}{(1 - \omega^2)^2 + \omega^2 c^2}. \]  
It has a maximum value of \( F^2/(2c) \) when \( \omega = 1 \) (at resonance). This result is of course not applicable for the pendulum where the non-linear potential is present. We will see below that for the pendulum with fixed \( c \) and \( \omega \), \( \langle P \rangle_{\tau} \) behaves very differently as a function of \( F \) than this simple quadratic form.

For small \( n \), and in particular for \( n = 1 \), we will be interested in the fluctuations in \( \langle P \rangle_{\tau} \), and in particular in its histogram. We will see that \( \langle P \rangle_{\tau} \) has a non-vanishing probability to be negative in the chaotic regime, and we will compare the probability of having \( \langle P \rangle_{\tau} = a \), to the probability of \( \langle P \rangle_{\tau} = -a \), within the valid range of \( a \).

We should also emphasize here the difference between our system and a thermostatted system considered in Ref. (1). For a thermostatted system the dissipation is adjusted in such a way, by adjusting its velocity dependence, that the energy of the system remain identically constant, not only on the average over a long period of time, as is the case here.

Varying the strength of the applied force the pendulum can display a simple periodic motion, a motion displaying periodic doubling, tripling, etc., or truly chaotic motion. In Fig. 1 we show the average power supplied by the driving force to the pendulum by using a time period of \( \tau = 400 \tau_d \) for the averaging (after the transient component has practically died). The same results were obtained for the average power dissipated i.e. \( c \langle y^2 \rangle_{\tau} \). It is evident that beyond the initial segment \( F = 0-0.4 \) which corresponds to periodic motion, the power curve is not smooth but vary significantly from point to point. The big jump in the power that occurs for \( F \sim 0.41 \) is associated with the threshold for the pendulum to go over the top. The error bars are associated with chaotic motion when we do not have enough statistics to pinpoint the average power exactly.

We now consider the histogram of the values of the supplied power during a time period \( \tau = \tau_d \). When the mo-
FIG. 2. Histogram of the supplied power during a period $\tau_d$. For example, for $F=0.4$ we get a single peak at $=0.0134$. For the case of $F=0.5$ there is period tripling, i.e. the Poincaré plot shows three points. In that case, the histogram consists of three peaks at $P=-0.030$, $0.128$ and $0.181$. Thus the number of peaks in the power histogram coincides with the multiplicity of the period. For the case of fully chaotic motion the histogram becomes broad and continuous. We see that the histograms can be used as a good signature for chaotic motion. Only in the case of chaotic motion we obtain a broad, continuous distribution of the power. Since we consider a system with a few degrees of freedom the distribution of power is bounded from below and from above.

To plot the distribution of power we accumulated the data into bins. For $F=0.6$ we have chosen the bin size to be $\Delta = 0.036$, and there are 21 bins spanning the range $(-0.378,0.378)$. The bin counts for a run of 11300 periods of $\tau_d$ were found to be:

$$\{0, 0, 0, 0, 83, 305, 394, 677, 728, 648, 659, 795, 1091, 1440, 1108, 1081, 729, 825, 429, 228, 80\}.$$  

This is depicted in Figure 2.

Using the histogram we calculate the ratio of

$$ratio(i) = \frac{freq(11+i)}{freq(11-i)}, \quad i = 0, \cdots, 6. \quad (12)$$

The center of the 11'th bin corresponds to $\langle P \rangle_\tau = 0$. Let us define the effective temperature of the pendulum by

$$\frac{1}{2}kT = \frac{1}{2} \langle y^2 \rangle_\infty = \frac{1}{2c} \langle P \rangle_\infty, \quad (13)$$

where the average is over an infinite time period. We can then define the rate of entropy production as

$$\sigma_\tau = \frac{\langle P \rangle_\tau}{kT} = \frac{c \langle P \rangle_\tau}{\langle P \rangle_\infty}. \quad (14)$$

Since $\langle P \rangle_\tau = \Delta i$, we can express $ratio$ as a function of $\sigma_\tau$ instead of $i$. In Fig. 3 we plot the probability of having an entropy production value $\sigma_\tau$ during a time interval $\tau_d$ vs. entropy production $-\sigma_\tau$. We fit this to an exponential of the form

$$\exp(\lambda \tau_d \sigma_\tau), \quad (15)$$

with $\lambda$ a single fitting parameter. $\lambda = 1$ provides a perfect fit for the first three points (solid line). A good fit to the first six points is obtained with $\lambda = 1.25$ (dashed line). This is quite close to the value of $\lambda = 1$ predicted by the FT for a system with a large number of degrees of freedom. Of course unlike the case for a system of many degrees of freedom the fit has to be cut off since the power distribution is bounded (the next data point is at infinity). Thus if one is considering the properties of the histogram not too close to the edge, it approximately satisfies the FT. The deviation from $\lambda = 1$ seems to be associated with the fact that the distribution is bounded due to the small number of degrees of freedom (finite size effect).

We should mention that the dissipated power given by $c \langle y^2 \rangle_\tau$ also has a broad distribution for the case of chaotic motion, but of course it is never negative. Thus unlike the thermostatted case the two histograms of the supplied power and dissipated power are different.

We also calculated the histogram of supplied power for a time period $\tau = 2\tau_d$, see Fig. 4. The sum of frequencies is 10,000 and the bin size is 0.027. In that case the histogram becomes broad and continuous. We see that the histograms can be used as a good signature for chaotic motion. Only in the case of chaotic motion we obtain a broad, continuous distribution of the power.
case the distribution is narrower as expected, although there are still periods of negative power. Of course in the limit of large $\tau$ the histogram will be peaked at the average value.

It is now clear that chaos is characterized by a wide, continuous distribution of entropy production rates, whereas in the non chaotic regime the histogram is characterized by discrete peaks whose number is equal to the multiplicity of the period in terms of the period of the driving force. Thus the power distribution is a good signature for the chaotic state. The supplied power can be negative for time periods equal to the period of the driving force, but the the distribution is centered about a positive average value. For values of the power not too close to the edge the ratio of probabilities

$$\frac{\pi_{\tau_d}(P)}{\pi_{\tau_d}(-P)} \approx \exp \left( \frac{\tau_d P}{kT} \right) = \exp \left( \frac{\tau_d c}{\langle P \rangle_\infty} P \right).$$  \hspace{1cm} (16)

This suggests that at least near the center one may approximate the power distribution by a gaussian of the form

$$\pi_\star(P) \propto \exp \left( -\frac{\tau_c}{4\langle P \rangle_\infty^2} (P - \langle P \rangle_\infty)^2 \right).$$ \hspace{1cm} (17)

It should be emphasized that a gaussian is only one possible solution satisfying the FT, and other solutions deviating from a gaussian are possible.

If we consider $N$ identical (uncoupled) driven dissipative pendulum, then the total power supplied which is the sum of individual powers will have a gaussian distribution in the limit $N \to \infty$ as predicted by the central limit theorem. One can consider $N$ coupled pendulums, and in that case it will be very interesting to check for the validity of the fluctuation theorem when the motion of the system is chaotic and $N \to \infty$. In that case the power distribution is expected to be unbounded. It need not necessarily be gaussian. Deviations from gaussian behavior were observed for turbulent flows [12].

Gallavotti and Cohen used the time reversal invariance of the particular thermostatted systems that they considered in order to prove the FT. Such a symmetry is not present in our system, although energy is still conserved when averaged over an infinitely long time period. Of course it will be very interesting to find a proof of the theorem for a steady state of a dissipative chaotic system for which the energy is conserved only over long time periods if indeed the theorem is still true.

Another ingredient of the proof of GC is the validity of the SRB measure on the attractor. This measure implies that the relative time the system spends at a certain region of the attractor is inversely proportional to the positive expansion rate at that region as given by the local positive Lyapunov exponents (for a long trajectory centered at that local). We carried out [11] a preliminary investigation of the attractor of the chaotic pendulum and showed that there is a positive correlation (although not perfect) between the time spent by the system at a region of the attractor (divided into cells) and the inverse local expansion rate as given by the local positive Lyapunov exponent.

This research is supported by the US Department of Energy (DOE), grant No. DE-FG02-98ER45686. I thank Michael Widom and Itamar Procaccia for some useful discussions.

---

1. G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. 74, 2694 (1995); J. Stat Phys. 80, 931 (1995).
2. D. V. Anosov, Proc. Steklov Inst. Math. 90, 1 (1967); G. Gallavotti, Chaos 8, 384 (1998).
3. G. Gallavotti, Phys. Rev. Lett, 77, 4334 (1996).
4. J. Kurchan, J. Phys. A 31, 3719 (1998).
5. J. L. Lebowitz and H. Spohn, J. Stat Phys. 95, 333 (1999).
6. Y-C Lai, C. Gribogi, J. A. York and I. Kan, Nonlinearity 6, 779 (1993).
7. J. B. Marion and S. T. Thornton in Classical dynamics of particles and systems, forth edition, Harcourt Brace & Company, 1995.
8. D. D’Humieres, M.R. Beasley, B.A.Huberman and A. Libchaber, Phys. Rev. A 26, 3483 (1982).
9. E.G. Gwinn and R.M. Westervelt, Phys. Rev. Lett. 54, 1613 (1985).
10. G.L. Baker and J.P. Gollub in Chaotic dynamics: an introduction Cambridge University Press, 1990.
11. Y. Y. Goldschmidt, unpublished.
12. S. T. Bramwell et al., Nature 396, 552 (1998).