Properties of the five dimensions for the truncated overlap fermions

Yuko Murakami¹, Motoo Sekiguchi², Hiroaki Wada²,₃,⁴ and Masayuki Wakayama¹,²,₄

¹ Information Media Center, Hiroshima University, Hiroshima 739-8521, Japan
² School of Science and Engineering, Kokushikan University, Tokyo 154-8515, Japan
³ Department of General Education, Chiba Institute of Technology, Chiba 273-0023, Japan
⁴ Research Center for Nuclear Physics (RCNP), Osaka University, Ibaraki, Osaka 567-0047, Japan

Abstract

Lattice quantum chromodynamics (QCD) simulation with chiral fermions is an attractive tool for studies on the non-perturbative field theory of strong interaction. However, it requires a large number of computer assets and is computationally very expensive. We describe the propagators of the π meson and the ρ meson constructed from the truncated overlap fermions. These propagators are calculated using lattice QCD simulations with a quench approximation. We indicate the dependence of the five dimensions for the mass of the π and ρ mesons. The residual mass at the chiral limit is also investigated.

1. Introduction

Chiral symmetry is a feature for massless fermions under chiral transformation and plays a very important role in hadron physics. Quantum chromodynamics (QCD) is expected to reproduce the spontaneous breaking of chiral symmetry; however, the dynamics are a non-perturbative process and cannot be explained by perturbation theory. Lattice QCD (LQCD) is a numerical analysis based on lattice regularized gauge field theory to solve QCD for a non-perturbative process, and it has been particularly successful in studying phenomena that do not affect chiral symmetry. However, due to the Nielsen-Ninomiya theorem, it is not possible to describe a fermionic action with lattice gauge theory, which does not have the fermion doubling problem and satisfies chiral symmetry [1].

The realization of chiral symmetry on a lattice represents a breakthrough in this field. The Ginsparg-Wilson (GW) relation shows chiral symmetry on a lattice field by the block-spin formalism, which avoids the doubling problem [2],

$$\gamma_5 D + D \gamma_5 = aD \gamma_5 D.$$ (1)

In the continuum limit of a lattice spacing \(a \to 0\), the GW relation is equivalent to chiral symmetry [3]. The lattice fermion that satisfies the GW relation is called the lattice chiral fermion. Lattice fermion operators that satisfy lattice chiral symmetry are termed lattice chiral fermions. Several formulations have been developed for lattice chiral fermions, including the domain wall fermion (DWF) [4, 5] and the overlap fermion [6–8].

The DWF is defined in five dimensions by extending the four-dimensional spacetime operator to an extra flavour space, and left- and right-handed fermions appear on both boundaries of the five-dimensional coordinates by imposing Dirichlet boundary conditions on the five-dimensional finite space coordinates. The DWF realizes the GW relation approximately, whereas the overlap fermion strictly satisfies the GW relation.

Calculations of lattice chiral fermions are very expensive. The overlap fermion has a higher simulation cost than DWF, but DWF has a higher \(O(N_c)\) cost than the Wilson fermion. Therefore, we employ the truncated overlap fermion (TOF) formalism by Boriçi based on the DWF formalism [9]. A TOF is a lattice chiral fermion based on the DWF formalism, and simulations can be faster than with other lattice chiral fermions.

In this paper, we present the dependence of an extra fifth dimension size for several physical quantities. These are the key results in lattice simulations with chiral fermions.
2. Truncated overlap fermions

The TOFs [9] are defined by

$$S_{\text{TOF}} = \bar{\psi}D_{\text{TOF}}\psi$$

$$= \bar{\psi}_1(D\| - 1)\psi_1 + (D\| + 1)P_R\psi_2 - m_f(D\| + 1)P_L\psi_{N_5}$$

$$+ \sum_{i=2}^{N_5-1} \bar{\psi}_i(D\| - 1)\psi_i + (D\| + 1)P_R\psi_{i+1} + (D\| + 1)P_L\psi_{i-1}$$

$$+ \bar{\psi}_{N_5}(D\| - 1)\psi_{N_5} - m_f(D\| + 1)P_R\psi_1 + (D\| + 1)P_L\psi_{N_5-1},$$

(2)

$$D\| = D_{\text{WF}} - M_5,$$

(3)

where $D_{\text{WF}}$ is the Wilson fermion operator,

$$D_{\text{WF}}(x, y) = (4 - M_5)\delta_{x,y} - \frac{1}{2} \sum_{\mu=\pm 1}^{4} (1 - \gamma_\mu) U_\mu(x) \delta_{x,\hat{\mu} y}.$$

(4)

$P_{R/L}$ are the chiral projection operators, $P_{R/L} = (1 \pm \gamma_5)/2$, and $m_f$ and $M_5$ are the bare quark mass and the height of the domain wall, respectively. From the low effective theory, the Dirac operator of the TOFs is derived with

$$D = (P^\dagger D^0 P) D_{\text{TOF}} P_{1,1},$$

(5)

$$P_{x,y} = P_L\delta_{x,y} + P_R\delta_{x+1,y} + P_R\delta_{x,y+1}.$$  

(6)

$D_{\text{PV}}$ is the Pauli–Villars matrix given by $D_{\text{PV}} = D_{\text{TOF}}(m_f = 1)$. $D$ in (5) is a four-dimensional operator subtracted from the fifth-dimensional matrix, $N_5 \times N_5$ and described with

$$D = \frac{1 + m_f}{2} + \frac{1 - m_f}{2}\gamma_5 \frac{1 - T_{N_5}}{1 + T_{N_5}},$$

(7)

where $T$ is a transfer matrix defined using the Hamiltonian $H = \gamma_5 D\|$, $T = \frac{1 + H}{1 - H}$.  

(8)

In the $N_5 \rightarrow \infty$ limit, the Dirac operator $D$ becomes the overlap fermion and the lattice chiral symmetry is satisfied exactly.

3. Simulations and analysis

We construct $\pi$ and $\rho$ meson operators as color-singlet bilinear operators of iso-doublet quarks, $u$ and $d$, and anti-quarks, $\bar{u}$ and $\bar{d}$, where the quarks are represented by the TOFs and calculate the propagator of the $\pi$ and $\rho$ mesons using LQCD simulations with a quench approximation. The parameters of the TOF action are the quark mass $m_f$, the length of the five-dimensional direction $N_5$, and the mass in the five-dimensional direction $m_5 (= M_5/a)$, although $m_5$ has almost no effect on the mass of the meson. We find the effective masses of mesons from their propagators and investigate how the effective masses of these mesons are dependent on $m_f$ and $N_5$.

We apply the restriction that the $\pi$ and $\rho$ mesons have degenerate quark mass with $m_f = m_\pi = m_\rho$. The gauge configurations were generated with the plaquette gauge action (the standard Wilson gauge action) [10] using a heat bath algorithm with a lattice size of $8^3 \times 24$ and $\beta = 5.7$. After the gauge configurations were updated from a cold start to 20,000 times to maintain the thermal equilibrium, the gauge configurations were saved every 1,000 updates and used for calculation of the meson propagator functions. In this simulation, three quark masses were selected at $m_f = 0.040, 0.060$ and 0.080, and the five-dimensional lengths used were $N_5 = 3, 8, 16, 24, 32$ and 48.

The quark mass in the five-dimensional direction was $m_5 a = 1.650$ with lattice spacing $a$. Table 1 shows the number of gauge configurations used in the simulation of the meson propagators. The typical CPU times per configuration with $m_f = 0.040$ are 34 minutes, 2.6 hours, 11.5 hours and 30 hours for $N_5 = 8, 16, 32$ and 48, respectively on the SX-ACE at RCNP/CMC. Also, even with the same $N_5$, the CPU time becomes shorter as $m_f$ increases. For $N_5 = 48$, CPU times are 20.8 hours and 15.8 hours with $m_f = 0.060$ and 0.080, respectively. We interrupted the calculation for $N_5 = 48$ and $m_f = 0.040$ due to needing about 300 thousand CPU hours to get the meaningful results. Simulation was conducted under periodic boundary conditions; therefore, the effective masses $m_{\text{eff}}$ of each meson from the propagators of the $\pi$ and $\rho$ mesons were derived by fitting with (9), where $G(t)$ is a meson propagation function at time $t$.

$$G(t) = e^{-m_{\text{eff}}(t)a}\frac{e^{-m_{\text{eff}}(t)a(T-1)} + e^{-m_{\text{eff}}(t)a(T-t)}}{e^{-m_{\text{eff}}(t)a(t+1)} + e^{-m_{\text{eff}}(t)a(F-(t+1))}}.$$

(9)
The chiral limit is realized when the quark mass $m_f$ is zero. The mass of $\pi$ decreases as $N_5$ increases until $N_5$ becomes 32; however, the rate of decrease of the $\pi$ mass is extremely low when $N_5$ exceeds 32. Since the amount of calculation required increases as the value of $N_5$ increases, it can be said that an $N_5$ value larger than the value given in this simulation is not necessary. Alternatively, we would like to derive the physical result at the limit where $N_5 \to \infty$ from the results of the simulations at reasonable finite values of $N_5$. However, the relational expression of the mass of $\pi$ with respect to $N_5$ has not been clarified. Therefore, the result of the

Figures 1–3 show graphs of the time dependence of the effective masses of the $\pi$ and $\rho$ mesons obtained for $N_5 = 3, 8, 16, 24, 32$ and 48. The effective masses of $\pi$ and $\rho$ with $N_5 = 24$ have large error bars because the number of gauge configurations is much smaller than the others. These results indicate that the curve of the effective masses of the mesons becomes almost constant in the time range of $t \geq 7$. Therefore, we find the effective masses of the mesons with $N_5 = 16, 24, 32$ and 48 averaged in the time region of $7 \leq t \leq 10$, and that with $N_5 = 3$ and 8 averaged in the time region of $8 \leq t \leq 10$. The results for averaging of the effective masses are listed in Table 1. Here, the statistical errors of the effective masses were estimated by the jack-knife method with 10 clusters.

The chiral limit is realized when the five-dimensional length $N_5$ is infinite and the $\pi$ mass is zero. The dependence of $\pi$ mass on $N_5$ and the quark mass $m_f$ is shown in Figure 4. The mass of $\pi$ decreases as $N_5$ increases until $N_5$ becomes 32; however, the rate of decrease of the $\pi$ mass is extremely low when $N_5$ exceeds 32. Since the amount of calculation required increases as the value of $N_5$ increases, it can be said that an $N_5$ value larger than the value given in this simulation is not necessary. Alternatively, we would like to derive the physical result at the limit where $N_5 \to \infty$ from the results of the simulations at reasonable finite values of $N_5$. However, the relational expression of the mass of $\pi$ with respect to $N_5$ has not been clarified. Therefore, the result of the

Figure 1. Dependences of the effective masses of $\pi$ and $\rho$ mesons on $N_5$ where $m_f = 0.040$ and lattice size is $8^3 \times 24 \times N_5$.

Table 1. Effective masses of $\pi$ and $\rho$ mesons and the number of gauge configurations that correspond to the $N_5$. All simulations were performed with a lattice size of $8^3 \times 24 \times N_5$, except for the last line results of $N_5 = 24 (16^3 \times 24 \times 24)$.

| $m_f$ a = 0.040 | $m_f$ a = 0.060 | $m_f$ a = 0.080 |
|-----------------|-----------------|-----------------|
| $N_5 = 3$       | $N_5 = 8$       | $N_5 = 16$      |
| $N_5 = 24$      | $N_5 = 32$      | $N_5 = 48$      |
| $N_5 = 48$      | $N_5 = 24$      |                  |
| $m_{\pi,0}$     | 0.9276(47)      | 0.9771(13)      | 1.0262(20)    |
| $m_{\rho,0}$    | 1.0585(46)      | 1.0992(14)      | 1.1385(23)    |
| $m_{\pi,0}$     | 0.6611(15)      | 0.7270(14)      | 0.7898(12)    |
| $m_{\rho,0}$    | 0.8938(44)      | 0.9381(31)      | 0.9831(24)    |
| $m_{\pi,0}$     | 0.5505(15)      | 0.6263(32)      | 0.6992(21)    |
| $m_{\rho,0}$    | 0.8616(11)      | 0.8984(47)      | 0.9451(38)    |
| $m_{\pi,0}$     | 0.804(74)       | 0.900(43)       | 0.966(42)     |
| $m_{\rho,0}$    | 0.5143(95)      | 0.5968(73)      | 0.6722(60)    |
| $m_{\pi,0}$     | 0.861(12)       | 0.8913(83)      | 0.9368(49)    |
| $m_{\rho,0}$    | 0.7860          | 0.8000          | 0.9660        |
| $m_{\pi,0}$     | 0.500(91)       | 0.5876(60)      | 0.6647(59)    |
| $m_{\rho,0}$    | 0.861(12)       | 0.8913(83)      | 0.9368(49)    |
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| $m_{\pi,0}$     | 0.500(91)       | 0.5876(60)      | 0.6647(59)    |
| $m_{\rho,0}$    | 0.861(12)       | 0.8913(83)      | 0.9368(49)    |
simulation of $N_5 \to \infty$ at the chiral limit has been estimated so far by finding $m_\pi \to 0$ for $N_5$ as large as possible such as [11]. The parameter $N_5$ is introduced to achieve the lattice chiral symmetry during lattice QCD simulation. Therefore, when we take the chiral limit, it should realize first the lattice chiral symmetry by taking the limit of $N_5 \to \infty$, in principle, and then take the limit of $m_\pi \to 0$ as in method 1 described later. In this

Figure 2. Dependences of the effective masses of $\pi$ and $\rho$ mesons on $N_5$ where $m_f = 0.060$ where lattice size is $8^3 \times 24 \times N_5$.

Figure 3. Dependences of the effective masses of $\pi$ and $\rho$ mesons on $N_5$ where $m_f = 0.080$ where lattice size is $8^3 \times 24 \times N_5$.

Figure 4. Dependence of $m_\pi a$ and $(m_\pi a)^2$ on $N_5$ with $m_f = 0.040, 0.060$ and 0.080.
paper, we examine the chiral limit for the masses of the \( \pi \) and \( \rho \) mesons using an ideal procedure (Method 1) and a conventional procedure (Method 2), in which we take the limit \( m_\pi \to 0 \) in the beginning. To show the validity of the conventional procedure (Method 2), we demonstrate our simulation results agree with the previous results by method 2 and compare the results of the residual mass \( m_{\text{res}} \) in methods 1 and 2. In method 1, we can present the explicit relational expression between the masses of \( \pi \) and \( \rho \) mesons and \( N_5 \). We take the chiral limit with two methods as follows.

Method 1: We take the limit of \( m_\pi \to 0 \) after \( N_5 \). The \( 1/N_5 \) dependence of the mass for the \( \rho \) meson makes it possible to fit with a linear function in figure 5. The results of our simulations tend to be linear for \( m_\pi \) and \( (m_\rho a)^2 \) with \( 1/N_5 \) and each \( m_f \) in figure 6 when the analysis is performed excluding the data of \( N_5 = 3 \).

\begin{table}
| \( m_f \) | \( m_\pi a \) | \( (m_\rho a)^2 \) |
|--------|-------------|-------------|
| 0.040  | 0.405(92)   | 0.532(72)   |
| 0.060  | 0.505(57)   | 0.576(74)   |
| 0.080  | 0.605(57)   | 0.732(74)   |
\end{table}

Figure 5. Dependence of \( \rho \) mass on \( 1/N_5 \) with \( m_f = 0.040, 0.060 \) and 0.080. The parameters of the fitted linear functions are summarized in table 2. (Method 1).

Figure 6. Dependence of \( m_\pi a \) and \( (m_\rho a)^2 \) on \( 1/N_5 \) with \( m_f \), where \( N_5 = 3, 8, 16, 32 \) and 48. The linear functions in the figure were fitted by the data for \( N_5 = 8, 16, 32 \) and 48. (Method 1).
After obtaining the \( p_{ma}^2 \) and \( m_{\rho a} \) at \( 1/N_5 \to 0 \) as table 3, we fit the dependency of the \( m_{\rho a} \) on \( p_{ma}^2 \) with a linear function and extrapolate \( m_{\rho a} \) to the chiral limit, i.e. \( p_{ma}^2 \to 0 \), as shown in figure 7. The mass of the \( \rho \) meson at the chiral limit was obtained as \( m_{\rho a} = 0.7595(215) \). Using the mass of the \( \rho \) meson as \( m_{\rho} = 775.26 \text{ MeV} \) [12] as an input revealed that the lattice spacing \( a \) was \( 0.1930(55) \text{ fm} \) and \( 1/a = 1021(29) \text{ MeV} \). Finally, figure 8 shows the linear function that fits the relationship between \( p_{ma}^2 \) and the quark mass. According to low energy effective field theories based on chiral symmetry, \( p_{ma}^2 \) and \( m_{\rho a} \) are described as a linear function of \( m_{fa} \). It indicates the quark mass at the chiral limit when \( \rho_{ma}^2 = 0 \). The value obtained by reversal of the sign of the quark mass in the chiral limit is defined as the residual mass; therefore, the residual mass \( m_{res} \) is \(-1.77(47) \text{ MeV} \).

Method 2: We take the limit of \( N_5 \to \infty \) after \( (m_{\rho a})^2 \to 0 \). We first seek the residual masses where the \( (m_{\rho a})^2 = 0 \) from figure 9. In the limit of \( (m_{\rho a})^2 \to 0 \), the residual masses \( m_{res a} \) and \( \rho \) meson masses \( m_{\rho a} \) for each \( N_5 \) are listed in table 4. According to the DWF case by Blum et al [11], the residual mass at \( N_5 = 48 \) was \( m_{res} \sim 8 \text{ MeV} \). Our value of \( m_{res} = 8.29(42) \text{ MeV} \) at \( N_5 = 48 \) agrees with the result of Blum et al. The inverse of the lattice spacing \( 1/a \) is calculated by the input \( m_{\rho} = 775.26 \text{ MeV} \) [12]. The residual masses \( m_{res} \) are obtained from \( m_{res a} \) and \( 1/a \) in table 4. The linearity as shown in figure 10 allows the residual mass \( m_{res} = -2.37(51) \text{ MeV} \) to be extrapolated when \( 1/N_5 \to 0 \). This value is in agreement with the result of method 1.

**Figure 7.** Dependence of the \( \rho \) meson mass on the \( \pi \) mass squared. (Method 1).

**Figure 8.** Dependence of \( m_{\rho a}^2 \) on \( m_{fa} \). (Method 1).

**Table 3.** \( (m_{\rho a})^2 \) and \( m_{\rho a} \) at \( 1/N_5 \to 0 \). (Method 1).

| \( m_{fa} \) | 0.040 | 0.060 | 0.080 |
|------------|-------|-------|-------|
| \( (m_{\rho a})^2 \) | 0.1884(67) | 0.2884(44) | 0.3862(35) |
| \( m_{\rho a} \) | 0.8427(103) | 0.8706(74) | 0.9192(56) |
As an additional consideration, we investigate the dependence of the spatial length of the lattice size on \( \rho \) by a simulation performed with \( N_5 = 24 \) and two lattice sizes of \( 8^3 \times 24 \) and \( 12^3 \times 24 \). As shown in table 1, the mass of \( \pi \) tends to be light when the finite volume is large.

### 4. Conclusion

We have presented the results of studies on quenched LQCD using TOF action. Simulations for the mass of \( \pi \) and \( \rho \) mesons suggest that the large size \( N_5 > 32 \) case has a small effect on lowering the mass of mesons in figures 1–4, whereas the numerical computation cost increases with size \( N_5 \), so that the simulations of large size

| \( N_5 \) | \( m_{\rho}a \) | \( m_{\pi}a \) | \( 1/a [\text{GeV}] \) | \( m_{\text{res}} [\text{MeV}] \) |
|---|---|---|---|---|
| 8 | 0.053 47(95) | 0.6843(24) | 1.1329(39) | 60.6 ± 1.1 |
| 16 | 0.025 35(87) | 0.713(12) | 1.088(18) | 27.6 ± 1.0 |
| 32 | 0.012 58(37) | 0.751(16) | 1.032(22) | 12.98 ± 0.47 |
| 48 | 0.007 92(32) | 0.743(22) | 1.047(32) | 8.29 ± 0.42 |

Figure 9. Dependence of \((m_{\pi}, a)^2\) and \(m_{\rho}a\) on \( m_{\pi} \) with \( N_5 = 8, 16, 32 \) and 48. (Method 2).

Figure 10. Relationship between the residual mass \( m_{\text{res}} [\text{MeV}] \), and \( 1/N_5 \). (Method 2).
$N_5$ are not realistic, even with the current high-performance computer technology. Our result of method 2, $N_5 \rightarrow \infty$ after $(m_\pi a)^2 \rightarrow 0$, is very consistent with simulations of DWFs [11, 13]. We have explored the dependence of the $\pi$ and $\rho$ meson masses for five dimensions, the analysis of which indicates $(m_\pi)^2$ and $m_\rho$, tend toward a linear function of the quark mass for each value of $1/N_5$ in method 1. These results are consistent with the chiral effective theory. Moreover, $m_{\pi a}$ is determined to be a linear function of $1/N_5$. LQCD simulation with chiral symmetry is attractive; however, it is difficult due to the high computational costs. These are key results in simulations on LQCD with lattice chiral symmetry. Our next step is to study the details of the relationships for low energy effective field theories based on chiral symmetry, such as the chiral condensate with a chiral limit by TOFs.

Acknowledgments

We would like to thank S Date (Osaka University) for his excellent advice for the lattice code and A Nakamura for valuable discussions. The simulations were performed on the SX-ACE supercomputer system at Research Center for Nuclear Physics (RCNP) and the Cybermedia Center (CMC), Osaka University, and on the SX-Aurora TSUBASA system at Kokushikan University. This work was supported by ‘Joint Usage/Research Center for Interdisciplinary Large-scale Information Infrastructures’ in Japan (Project ID: jh200030-NAH and jh210032-NAH).

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Hiroaki Wada © https://orcid.org/0000-0001-5449-1819
Masayuki Wakayama © https://orcid.org/0000-0002-4269-953X

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