Spacetime Deformations and Electromagnetism in Material Media

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Abstract

This paper is intended to investigate the relation between electrodynamics in anisotropic material media and its analogous formulation in an spacetime, with non-null Riemann curvature tensor. After discussing the electromagnetism via chiral differential forms, we point out the optical activity of a given material medium, closely related to topological spin, and the Faraday rotation, associated to topological torsion. Both quantities are defined in terms of the magnetic potential and the electric and magnetic fields and excitations. We revisit some properties of material media and the associated Green dyadics. Some related features of ferrite are also investigated. It is well-known that the constitutive tensor is essentially equivalent to the Riemann curvature tensor. In order to investigate the propagation of electromagnetic waves in material media, we prove that it is analogous to consider the electromagnetic wave propagation in the vacuum, but this time in a curved spacetime, which is obtained by a deformation of the Lorentzian metric of Minkowski spacetime. Spacetime deformations leave invariant the form of Maxwell equations. Also, there exists a close relation between Maxwell equations in curved spacetime and in an anisotropic material medium, indicating that electromagnetism and spacetime properties are deeply related. For instance, the equations of holomorphy in Minkowski spacetime are essentially Maxwell equations in vacuum. Besides, the geometrical aspects of wave propagation can be described by an effective geometry which represents a modification of the Lorentzian metric of Minkowski spacetime, i.e., a kind of spacetime deformation.

Key words: electromagnetism, constitutive tensor, Green dyadic, optical activity, Faraday rotation, spacetime deformations.

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Introduction

The metric-free formulation of electromagnetism is an old concept \[ 1, 6, 9, 22, 28, 31, 33, 35, 10 \], and it is more natural, correct, precise and geometrically sensible if differential forms, intrinsically endowed with chirality \[ 16, 28 \], are used \[ 13, 33 \]. The excitations \( \mathbf{D}(x), \mathbf{H}(x) \) (respectively the electric displacement and the magnetic field) and the fields \( \mathbf{E}(x), \mathbf{B}(x) \) (respectively the electric field and the magnetic induction) are naturally described in a spacetime destituted of a metric, which is introduced only when the constitutive relations are to be defined. The metric-free (and affine-free) formulation of electrodynamics brings a geometric character and a clear physical interpretation. But if we want to relate the Faraday excitation 2-form field \( \mathbf{G}(x) \) and the electromagnetic field strength 2-form field \( \mathbf{F}(x) \), we need to consider a constitutive tensor \( \chi \), that gives the relation \( \mathbf{G}(x) = \chi \mathbf{F}(x) \). Equivalently, spacetime must be endowed with a metric, since the constitutive law depends on the spacetime metric \[ 9 \].

The constitutive tensor (CT) is more than a relation between \( \mathbf{F}(x) \) and \( \mathbf{G}(x) \), describing physical intrinsic properties of matter or spacetime. It is essentially the Hodge star operator \[ 15 \] that, besides the well-known duality between \( k \)-forms and \( (n-k) \)-forms in a \( n \)-dimensional vector space (endowed with a metric), changes the parity of differential forms, but with an additional information about the medium structure. In this sense, CT describes the properties of spacetime (magnetic) permeability and (electric) permittivity, since a general medium can be arbitrarily anisotropic. Under this viewpoint, a CT immediately brings a light-cone conformal structure to spacetime \[ 15 \]. Formally, the link between CT and spacetime structure is expected, since CT and the Riemann curvature tensor have the same mathematical properties. A CT can reveal precious informations about spacetime, for example, the CT scalar curvature is identically null in any medium possessing central symmetry \[ 33 \].

In this paper the CT \( \chi \) that describes any linear (in particular, crystalline) media, in the general case presenting optical activity, is expressed by a conformal transformation of the vacuum CT \( \chi_0 \). The metric associated to the medium is derived from the CT (up to a conformal factor) and vice-versa \[ 20, 22, 23, 25 \]. In this sense, the Lorentzian metric of Minkowski spacetime, associated with \( \chi_0 \), is deformed into a general metric of a riemannian spacetime, related to \( \chi \).

We prove that in order to completely describe the CT of any linear medium presenting natural optical activity, we only need the matrix \( \gamma \), that describes the optical activity of such a medium, and \( \chi_0 \). In the particular investigation of crystalline media, we describe the constitutive tensor associated to the 32 crystal classes presenting natural optical activity uniquely from \( \chi_0 \), i.e., from the spacetime metric, since \( \chi_0 \) can be written as second order metric tensor combination \[ 22, 20 \]. This paper is organized as follows: in Sec. 1 we review the geometric description of fields and excitations

\[ ^1 \text{We follow the nomenclature given in 22.} \]
in electromagnetic theory, well-established originally in the papers by Maxwell [21], Schönberg [10], Hehl [22], Kiehn [31], Post [33] and Jancewicz [27, 28], among others. The Maxwell equations carry information about the nature of the different fields and excitations, and the theory arises with an essential geometric character, if the fields and excitations are correctly interpreted, as in [27, 28, 22, 33], as even and odd differential form fields. In Sec. 2 we revisit the main features of electromagnetism in anisotropic material media. The Green dyadic is obtained and we treat the particular case of an electrically anisotropic material medium. Plane waves are investigated, illustrating the present approach. Also, optically active media are treated. For instance, ferrite is explicitly studied. In Sec. 3 Maxwell equations in curved spacetime and in an anisotropic material medium are proved to be equivalent. It sheds some new light on the differential geometric aspects of electromagnetic wave propagation, that can be described by a geometry which represents a modification of the metric in Minkowski spacetime. We prove that all the information contained in the constitutive tensor associated to any linear media (arbitrarily describing optical activity), is precisely given by the vacuum constitutive tensor and by the matrix that describes the optical activity associated with the medium. Such matrix is essentially viewed as a perturbation of the original constitutive tensor that does not describe optically active media. Finally, in the Appendix, the main results concerning differential forms are reviewed.

1 Electromagnetism in the exterior algebra

Heretofore $\Omega^k(M)$ denotes the space of $k$-form fields defined on a manifold $M$. Given the map $E : M \to \mathbb{R}^3$, the electric field $E(x), x \in M$, is an even 1-form field ($E(x) \in \Omega^1(\mathbb{R}^3)$), since $E(x)$ is a linear map from the infinitesimal vector $dr(x)$ to the infinitesimal scalar potential $dV(x)$, given by $dV(x) = -E(x) \cdot dr(x)$. The physical dimension of $E(x)$ in the SI, $[E(x)] = \text{Vm}^{-1}$, agrees with this interpretation. Analogously the magnetic induction $B(x)$ is an even 2-form field ($B(x) \in \Omega^2(\mathbb{R}^3)$), since $B(x)$ is a linear map from the infinitesimal bivector $dS(x)$ to the infinitesimal scalar $d\phi(x)$. Explicitly we have $d\phi(x) = -B(x) \cdot dS(x)$, where $\phi(x)$ is the magnetic flux. The physical dimension $B(x)$ in the SI, $[B(x)] = \text{Wbm}^{-2} = \text{T} = \text{Tesla}$, again agrees with such an interpretation [21, 28].

From now on we call an even (odd) differential form field the one that doesn’t (does) change sign under parity transformations\(^2\) [16, 28]. Even form fields are elements of $\Omega_+(M)$, hereon simply denoted by $\Omega(M)$, while odd form fields are elements of $\Omega_-(M)$. Such forms are called chiral differential forms.

\(^2\)A parity transformation is defined in $\mathbb{R}^n$ as the inversion of an odd number of basis vectors of $\mathbb{R}^n$. 
1.1 Homogeneous Maxwell equations and potentials

The differential operator \(d: \Omega^p(M) \to \Omega^{p+1}(M)\) does not change the differential forms parity. From now on we denote \(\partial_\alpha = \partial / \partial \alpha\). We also adopt natural units, in particular the speed of light \(c = 1\), in what follows.

The first homogeneous Maxwell equation is an expression relating even 2-form fields:

\[ dE(x) + \partial_t B(x) = 0. \]

(1)

The absence of magnetic monopoles can be described by the equation

\[ dB(x) = 0. \]

(2)

Eqs. (1) and (2) are the homogeneous Maxwell equations. From eq. (2), using the Poincaré lemma, there exists an even 1-form field \(A(x)\) (the magnetic potential) satisfying the relation

\[ B(x) = dA(x). \]

(3)

Substituting in eq. (1), one obtains the expression \(dE(x) + \partial_t dA(x) = 0\), or \(d(E(x) + \partial_t A(x)) = 0\). Using the Poincaré lemma, there exists a scalar field potential \(\Phi(x) \in \Omega^0(\mathbb{R}^3)\) such that \(E(x) + \partial_t A(x) = -d\Phi(x)\), implying that

\[ E(x) = -\partial_t A(x) - d\Phi(x). \]

(4)

1.2 Non-homogeneous Maxwell equations

The electric current density \(j(x)\) is an element of \(\Omega^2(\mathbb{R}^3)\), an odd 2-form field, which changes sign under parity transformations. It is clear that \(dj \in \Omega^2(\mathbb{R}^3)\), and then the continuity equation (local form of electric charge conservation) can be written as

\[ dj(x) + \partial_t \rho(x) = 0, \]

(5)

where \(\rho(x)\), the electric charge density, is an odd 3-form field. Obviously \(d\rho(x) = 0\), and the Poincaré lemma again asserts that there exists \(D(x) \in \Omega^2(\mathbb{R}^3)\) such that

\[ \rho(x) = dD(x). \]

(6)

The 1-form field \(D(x)\) is called electric displacement. The unity of \(D(x)\) in the SI is \([D(x)] = \text{Cm}^{-2}\) (C = Coulomb). Substituting in eq. (5), we have \(dj(x) + \partial_t (dD(x)) = 0\), and \(d(j(x) + \partial_t D(x)) = 0\). There exists an odd 1-form field \(H(x) \in \Omega^1(\mathbb{R}^3)\) such that

\[ j(x) + \partial_t D(x) = dH(x). \]

(7)
It describes the Ampère-Oersted law. The SI unit of $H(x)$ is $[H(x)] = \text{Am}^{-1}$ (Ampère). The odd form fields $D(x)$ and $H(x)$ are potentials with sources $\rho(x)$ and $j(x)$, respectively.

The Poynting vector $S(x)$ describes the electromagnetic strength energy flux density. It is possible to write

$$S(x) = E(x) \wedge H^*(x).$$

From the algebraic viewpoint, the product above is the unique possibility, since quantities representing flux densities are described by odd differential 3-form fields, elements of $\Omega^3_3(R^3)$. The electric ($w_e(x)$) and magnetic ($w_m(x)$) energy densities are elements of $\Omega^3_3(R^3)$, expressed by

$$w_e(x) = \frac{1}{2}E(x) \wedge D(x) = \frac{1}{2}D(x) \wedge E(x), \quad w_m = \frac{1}{2}B(x) \wedge H(x) = \frac{1}{2}H(x) \wedge B(x).$$

The electromagnetic field energy density is written as

$$w = w_e + w_m = \frac{1}{2}(E(x) \wedge D(x) + B(x) \wedge H(x)).$$

### 1.3 The Hodge star operator

It is well-known that the vector spaces $\Omega^k(R^3)$ and $\Omega^{3-k}(R^3)$ have the same dimension, since $\dim \Omega^k(R^3) = \binom{3}{k} = \binom{3}{3-k} = \dim \Omega^{3-k}(R^3)$. The same result is valid to any $n$-dimensional space (see Appendix). Meanwhile, it does not exist any canonical isomorphism between $\Omega^k(R^3)$ and $\Omega^{3-k}(R^3)$. The isomorphism given by the Hodge star operator $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$ always satisfies $\star^2 = \pm id$. For more details, see the Appendix. The contraction is a generalization of the interior product, and it can be written in terms of the Hodge star operator and the exterior product, as

$$\psi \llcorner \phi = \star((\star \phi) \wedge \psi),$$

where $\psi, \phi \in \Omega(R^3)$. For more details see, e.g., [36, 37].

### 1.4 The Poynting theorem

If we take the $\mathbb{C}$-conjugation of eqs. (11) and (17) and respectively multiply by $E^*(x)$ and $H^*(x)$ we obtain

$$E(x) \wedge dH^*(x) - H^*(x) \wedge dE(x) = \frac{1}{\epsilon}[E(x) \wedge (\partial_t \star \varepsilon E^*(x)) + \star \mu^{-1}B(x) \wedge (\partial_t B^*(x))]$$

$$= \star [(E(x) \llcorner (\partial_t \varepsilon E^*(x)) + (\mu^{-1}B(x) \llcorner (\partial_t B^*(x)))].$$

Eq. (12) can be written in a coordinate system as

$$\frac{1}{2} \partial_t \left(E(x) \llcorner (\varepsilon E(x)) + B(x) \llcorner (\mu^{-1}B(x))\right) dx \wedge dy \wedge dz.$$

and from the expression

$$-dS(x) = H^*(x) \wedge dE(x) - E(x) \wedge dH^*(x),$$

(13)
it follows that

$$dS(x) = \partial_t w(x),$$

the so-called \textit{Poynting theorem} [2, 13, 14].

### 1.5 Electromagnetic Intensity and Excitation

The electromagnetic field strength $F(x) \in \Omega^2(\mathbb{R}^{1,3})$ is an even 2-form in $\mathbb{R}^{1,3}$, also called the \textit{Faraday 2-form field} [13]. If an arbitrary, but fixed, time vector is chosen in $\mathbb{R}^{1,3}$, we can split spacetime in space plus time. Then it is possible to use $E(x)$ and $B(x)$ to describe $F(x)$ as

$$F(x) = B(x) + E(x) \wedge dt.$$  \hspace{1cm} (15)

The electromagnetic excitation $G(x) \in \Omega^2(\mathbb{R}^{1,3})$ can also be considered as an odd 2-form field given by [28, 23]

$$G(x) = D(x) - H(x) \wedge dt.$$  \hspace{1cm} (16)

Eqs. (1,2) can be summarized as

$$dF(x) = 0,$$  \hspace{1cm} (17)

and eqs. (6,7) are synthetically written as

$$dG(x) = J(x),$$  \hspace{1cm} (18)

when the odd 3-form current density field $J(x) = \rho(x) - j(x) \wedge dt$ is defined [28]. If we admit primarily eq. (17), the electric and magnetic fields are only defined after a spacetime splitting.

### 1.6 Vacuum constitutive relations

Hereon it is assumed implicitly that the Hodge star operator \textit{changes} the parity of the differential forms$^4$.

Constitutive relations are written as

$$G(x) = \star F(x).$$  \hspace{1cm} (19)

This relation can be expressed in the vacuum, after a spacetime splitting, as:

$$D(x) = \varepsilon_0 \star E(x), \hspace{1cm} B(x) = \mu_0 \star H(x),$$  \hspace{1cm} (20)

where $\varepsilon_0$ denotes the vacuum electric permittivity and $\mu_0$ denotes the vacuum magnetic permeability. From eq. (17) it is possible to find $A(x) \in \Omega^1(\mathbb{R}^{1,3})$ such that

$$F(x) = dA(x).$$  \hspace{1cm} (21)

$^4$This Hodge star operator is, \textit{de facto}, the composition of the Hodge star operator with a pseudoscalar [12, 13, 14]. This new operator is then able to lead odd (even) form fields to even (odd) ones. (By abuse of notation we also denote this new operator by $\star$.}
The even 1-form field \( A(x) \) denotes the well-known electromagnetic potential. In components, eq. (21) is written as

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. 
\]  

Eq. (21) is invariant under the maps \( A(x) \mapsto A(x) + \omega(x), \omega(x) \in \Omega^1(\mathbb{R}^{1,3}) \) such that \( d\omega(x) = 0 \). In particular eq. (21) is invariant when \( A(x) \mapsto A(x) + d\lambda(x), \lambda(x) \in \Omega^1(\mathbb{R}^{1,3}) \). The existence of form fields that are closed, but not exact, gives rise to the

The physical monopole and solitons in fluids, concerning paramount and striking applications such as superconductivity, topological defects and turbulent non-equilibrium thermodynamics of fluids, exhaustively investigated by Kiehn [31].

One [31] defines the odd 3-form field topological spin \( S(x) = A(x) \wedge G(x) \in \Omega^3(\mathbb{R}^{1,3}) \) and the even 3-form field topological torsion \( T(x) = A(x) \wedge F(x) \in \Omega^3(\mathbb{R}^{1,3}) \).

Under a spacetime splitting it can be seen that

\[
T(x) = A(x) \wedge F(x) = (A(x) - \phi(x)dt) \wedge (B(x) + E(x) \wedge dt) = A(x) \wedge B(x) + (A(x) \wedge E(x) - \phi(x)B(x))dt 
\]  

and

\[
S(x) = A(x) \wedge G(x) = (A(x) - \phi(x)dt) \wedge (D(x) - H(x) \wedge dt) = A(x) \wedge D(x) + (A(x) \wedge H(x) - \phi(x)D(x))dt
\]  

Kiehn [31] shows that \( T(x) \) is related to the helicity, while \( S(x) \) is associated to chirality of the electromagnetic fields. The 3-form field energy-momentum, is defined if an arbitrary direction \( e_i \) is chosen:

\[
U_i(x) = \frac{1}{2} [F(x) \wedge (e_i \wedge G(x))] - G(x) \wedge (e_i \wedge F(x)).
\]

The 3-form field energy-momentum is invariant under pseudodual maps \( F(x) \mapsto \varphi(x)G(x) \) and \( G(x) \mapsto -F(x)/\varphi(x) \), where \( \varphi(x) \) is an arbitrary scalar field non-null in all points of \( \mathbb{R}^{1,3} \).

2 Revisiting electromagnetism in non-homogeneous media

In the last decade, a lot of manuscripts have been concerning electrodynamics in material media via differential forms. For instance, see [3, 4, 5, 7, 27].

For permeability and permittivity tensors such that the product \( \varepsilon'_\mu \mu^{-1} \) is diagonalizable, the expression for the Green diadic [3, 4, 5] is given by

\[
g = \frac{\det \mu}{4\pi} \begin{pmatrix}
\exp(i m_1 \vec{r}) & 0 & 0 \\
0 & \exp(i m_2 \vec{r}) & 0 \\
0 & 0 & \exp(i m_3 \vec{r})
\end{pmatrix},
\]  

(26)
where \( \tilde{r} = \sqrt{\det \mu \left( \frac{x^2}{\mu_1} + \frac{y^2}{\mu_2} + \frac{z^2}{\mu_3} \right)^{\frac{1}{2}}} \), and \( m_1, m_2, m_3 \) denote eigenvalues associated with the matrix \( \varepsilon \mu^{-1} \), such that \( \text{Re} \, m_i > 0 \), \( i=1,2,3 \). \( \mu_1, \mu_2, \mu_3 \) denote eigenvalues associated with \( \mu \).

The diadic given by eq. (26) can be immediately written as the tensor product [4, 5]:

\[
g = \frac{\det \mu}{4\pi} \left( e^{im_1 \tilde{r}} dx_1 \otimes dx^2 + e^{im_2 \tilde{r}} dy_1 \otimes dy^2 + e^{im_3 \tilde{r}} dz_1 \otimes dz^2 \right). \tag{27}
\]

### 2.1 Material media

A medium is completely isotropic if the electric permittivity \( \varepsilon \) and the magnetic permeability \( \mu \) can be written, respectively, as \( \mu = \text{diag}(\mu_1, \mu_1, \mu_1) \) and \( \varepsilon = \text{diag}(\varepsilon_1, \varepsilon_1, \varepsilon_1) \). If the elements of \( \mu \) and \( \varepsilon \) in an anisotropic medium are hermitian, such medium is called electrically or magnetically girotropic. For instance, a plasma with static magnetic field in the \( z \) axis

\[
\varepsilon = \begin{pmatrix}
\varepsilon_1 & -i\varepsilon_p & 0 \\
-\varepsilon_p & \varepsilon_1 & 0 \\
0 & 0 & \varepsilon_z
\end{pmatrix} \tag{28}
\]
is electrically girotropic. A magnetic girotropic medium is given by

\[
\mu = \begin{pmatrix}
\mu_1 & -i\mu_2 & 0 \\
-\mu_2 & \mu_1 & 0 \\
0 & 0 & \mu_z
\end{pmatrix}. \tag{29}
\]

### 2.2 Electrically anisotropic media

Suppose that, in a given material medium, \( \varepsilon = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \) and \( \mu = I \), where \( I \) denotes the identity matrix. It is immediate that \( \tilde{r} = \mu_1 r \) and \( \det \mu = \mu_1^3 \), from where it can be seen that \( \varepsilon \mu^{-1} = \mu_1^{-1} \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \). Since the Green form \( g \) is given by [4]

\[
g = \left[ (\det \mu)^{-1} k^2 \mu kI - \omega^2 \varepsilon \mu^{-1} \right]^{-1}, \tag{30}
\]
then the diagonal components of \( g \) are given by

\[
g_{jj}(k) = \left[ \frac{k^2}{\mu_1^2} - \omega^2 \varepsilon_j \right]^{-1} \mu_1^2 \frac{k^2}{k^2 - \omega^2 \varepsilon_j \mu_1}. \tag{31}
\]
A solution of eq. (30) is given by

\[
g = \frac{\mu_1^2}{4\pi r} \begin{pmatrix}
\exp(i\omega \sqrt{\varepsilon_1 \mu_1} r) & 0 & 0 \\
0 & \exp(i\omega \sqrt{\varepsilon_2 \mu_1} r) & 0 \\
0 & 0 & \exp(i\omega \sqrt{\varepsilon_3 \mu_1} r)
\end{pmatrix} \tag{32}
\]
and then the Green diadic is given by

\[
g(\vec{r}_1, \vec{r}_2) = \frac{\mu_1^2}{4\pi r} \left( \exp(i\omega \sqrt{\varepsilon_1 \mu_1 r}) \, dx_1 \otimes dx_2 + \exp(i\omega \sqrt{\varepsilon_2 \mu_1 r}) \, dy_1 \otimes dy_2 \\
+ \exp(i\omega \sqrt{\varepsilon_3 \mu_1 r}) \, dz_1 \otimes dz_2 \right)
\]

(33)

where \( r = \|\vec{r}_1 - \vec{r}_2\| \). This equation is originally obtained by Warnick [4].

In the particular case of an uniaxial medium, with \( \varepsilon_1 = \varepsilon_2 = \varepsilon \) and \( \varepsilon_3 = \varepsilon_z \), eq.(33) is led to

\[
g(\vec{r}_1, \vec{r}_2) = \frac{\mu_1^2}{4\pi r} \left( \exp(i\omega \sqrt{\varepsilon \mu_1 r}) \, dx_1 \otimes dx_2 + \exp(i\omega \sqrt{\varepsilon \mu_1 r}) \, dy_1 \otimes dy_2 \\
+ \exp(i\omega \sqrt{\varepsilon_z \mu_1 r}) \, dz_1 \otimes dz_2 \right)
\]

(34)

In any material isotropic medium we have \( \varepsilon_z = \varepsilon \), and the vacuum is obtained when \( \varepsilon \mapsto \varepsilon_0 \) and \( \mu \mapsto \mu_0 \). In this case,

\[
g = \frac{\mu_0^2}{4\pi r} f
\]

(35)

### 2.3 Plane waves

Heretofore we denote \( \mathbf{E} = \mathbf{E}(x), \mathbf{B} = \mathbf{B}(x), \ldots \), in order to simplify the notation to be used. Suppose that a plane wave propagates in the \( s \) direction and let the electric field be expressed by

\[
\mathbf{E} = E_0 \exp(i(k\hat{s} \cdot \vec{r} - \omega t)).
\]

(36)

Denoting \( n = k/\omega = 1/v \), eqs.\([11]\) and \([17]\) gives

\[
\mathbf{D} = -n \mathbf{H} \wedge s, \quad \mathbf{B} = n \mathbf{E} \wedge s.
\]

(37)

Using the constitutive relation \( \mathbf{B} = \star_\mathbf{s} \mathbf{H} \), eqs.\([17]\) give

\[
\mathbf{D} = \frac{n^2}{\mu} \star (s \wedge \mathbf{E}) \wedge s
\]

\[
= \frac{n^2}{\mu} \star \left[ \mathbf{E} - s(s \cdot \mathbf{E}) \right].
\]

(38)

By abuse of notation, here \( \mu \) denotes the magnetic permeability, a real number, instead of the tensor \( \mu \). Define the component \( \mathbf{E}_\perp \) of \( \mathbf{E} \) such that \( \mathbf{E}_\perp \cdot s = 0 \). Then, since \( \mathbf{E}_\perp \) is in the plaquette defined by \( \mathbf{E} \wedge s \), eq.\([38]\) is written as

\[
\mathbf{D} = \frac{n^2}{\mu} \mathbf{E}_\perp.
\]

(39)

### 2.4 Fresnel equations

Now let \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be the eigenvalues of the matrix representation of \( \varepsilon \). From eq.\([38]\) we have

\[
E_i = \frac{n^2 s_i}{n^2 - \mu \varepsilon_i} \mathbf{E} \cdot s,
\]

(40)
from where we obtain
\[
\frac{s_1^2}{n_1^2 - \mu \varepsilon_1} \mathbf{E} \cdot s + \frac{s_2^2}{n_2^2 - \mu \varepsilon_2} \mathbf{E} \cdot s + \frac{s_3^2}{n_3^2 - \mu \varepsilon_3} \mathbf{E} \cdot s = \frac{1}{n^2} \mathbf{E} \cdot s
\] (41)

\[
\sum_{i=1}^{3} \frac{s_i^2}{n_i^2 - \mu \varepsilon_i} = \frac{1}{n^2}
\] (42)

Since \(s \cdot s = 1\), then \(\sum_{i=1}^{3} s_i^2 = 1\), and from eq. (42) it follows that
\[
\sum_{i=1}^{3} \frac{s_i^2}{n_i^2 - \mu \varepsilon_i} = 0.
\] (43)

If we define the so-called principal propagation velocity \(v_i := (\mu \varepsilon_i)^{-1/2}, 0, 0)\), eq. (43) is lead to
\[
\frac{s_1^2}{v_1^2 - v_1^2} + \frac{s_2^2}{v_2^2 - v_2^2} + \frac{s_3^2}{v_3^2 - v_3^2} = 0.
\] (44)

Eqs. (42), (43) and (44) are called Fresnel wave equations [14].

2.5 Ferrite

Ferrite is a material medium defined by
\[
\mu = \mu_0 \begin{pmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} =: \mu_0 \mu_r
\] (45)

From now on we consider \(\beta < \alpha\), and it is easy to see that \(\det \mu = \mu_0^3 (\alpha^2 - \beta^2) \gamma\) and that the eigenvalues of \(\mu_r\) are \(\alpha + \beta, \alpha - \beta\) and \(\gamma\). From the expression
\[
\mathbf{r} := (\alpha^2 - \beta^2)^{1/2} \gamma^{1/2} \mu_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]

it follows that
\[
\mathbf{r} := \|\mathbf{r}\| = \mu_0 (\alpha^2 - \beta^2)^{1/2} \gamma^{1/2} \left( \frac{x^2}{\alpha + \beta} + \frac{y^2}{\alpha - \beta} + \frac{z^2}{\gamma} \right)^{1/2}.
\] (46)

The square roots of the eigenvalues of \(\omega^2 \varepsilon \mu^{-1}\) are given by
\[
m_1 = \frac{\varepsilon}{(\alpha + \beta) \mu_0}, \quad m_2 = \omega \sqrt{\frac{\varepsilon}{(\alpha - \beta) \mu_0}}, \quad m_3 = \omega \sqrt{\frac{\varepsilon}{\gamma \mu_0}},
\] (47)

where \(\varepsilon\) is the diagonal element of the ferrite permittivity tensor. \(g\) is given by
\[
g = \frac{\mu_0^2 (\alpha^2 - \beta^2)^{3/2} \gamma^{1/2}}{4 \pi r^3} \text{ diag}(e^{i \omega \sqrt{\varepsilon \mu_0 (\alpha - \beta) \gamma r}}, e^{i \omega \sqrt{\varepsilon \mu_0 (\alpha + \beta) \gamma r}}, e^{i \omega \sqrt{\varepsilon \mu_0 (\alpha^2 - \beta^2) r}}),
\] (48)

where \(\mathbf{r} = (\alpha^2 - \beta^2) \gamma \mu_0 r\) and \(\mathbf{r}' = \frac{x}{\sqrt{\alpha + \beta}}, \frac{y}{\sqrt{\alpha - \beta}}, \frac{z}{\sqrt{\gamma}}\). Eq. (48) is equivalent to the expression
\[
g(\mathbf{r}_1, \mathbf{r}_2) = g_0 \left( e^{i \omega \sqrt{(\alpha - \beta) \gamma \mu_0 r}} dx_1 \otimes dx_2 + e^{i \omega \sqrt{(\alpha + \beta) \gamma \mu_0 r}} dy_1 \otimes dy_2 + e^{i \omega \sqrt{(\alpha^2 - \beta^2) \mu_0 r}} dz_1 \otimes dz_2 \right)
\] (49)
where \( g_0 = \mu_0^2 (\alpha^2 - \beta^2) \xi / 4\pi r' \), \( r' = ||\vec{r}_1' - \vec{r}_2'|| \) and \( \vec{r}_i' \) are analogously defined as \( \vec{r}' \). When \( \beta = 0 \) and \( \alpha = \gamma \) in eq.(48) it follows that

\[
g = \frac{\mu_0^2 \alpha^2}{4\pi r} \exp(i\omega \sqrt{\varepsilon \mu_0} \alpha r) I,
\]

where \( r_i = \sqrt{x_i^2 + y_i^2 + z_i^2} \). Denoting \( \mu = \alpha \mu_0 \) eq.(50) can be written as

\[
g = \frac{\mu^2}{4\pi r} \exp(i\omega \sqrt{\varepsilon \mu} r) I,
\]

which is the well-known expression for an isotropic medium.

### 2.6 Faraday rotations

From eq.(17) and eq.(36), it follows that

\[
\ast d\mathbf{H} = -i\omega \varepsilon \mathbf{E}.
\]

Taking the differential of the last equation we obtain

\[
d \ast d\mathbf{H} = \omega^2 \varepsilon \ast_h \mathbf{H} = \omega^2 \varepsilon \mu \mathbf{H}.
\]

Now, if we solve eqs.(37) for \( \mathbf{B} \), we obtain

\[
\mathbf{B} = \frac{n^2}{\varepsilon} \ast_h \mathbf{H}_\perp,
\]

where \( \mathbf{H}_\perp = \mathbf{H} - (s \cdot \mathbf{H}) s \). In components, the field given by eq.(54) is written as

\[
B_i = \frac{n_i^2 \mu_i}{n^2 - c^2 \varepsilon \mu_i} (s \cdot H)_i s_i.
\]

From eq.(55) the component of \( \mathbf{H} \) in the \( z \)-direction is zero, and if we make the assumption that \( \mathbf{H} = (H_1 dx + H_2 dy)e^{i(kz-\omega t)} \), eq.(55) gives

\[
\omega^2 \varepsilon \mu_0 \begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha \end{pmatrix} = k^2 \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}
\]

which solution is given by

\[
k^2_+ = \omega^2 \varepsilon \mu_0 (\alpha + \beta), \quad k^2_- = \omega^2 \varepsilon \mu_0 (\alpha - \beta),
\]

describing two (left- and right-handed) circularly polarized plane waves. Now, substituting eqs.(57) in eq.(56) it follows that \( H_1 = \pm iA \), if \( H_2 = A, A \in \mathbb{C} \). The general solution of the system is

\[
\mathbf{H} = \begin{bmatrix} -ic_1 A e^{ikx} + ic_2 A e^{ikz} \\ c_1 A e^{ikx} + c_2 A e^{ikz} \end{bmatrix} e^{-i\omega t} dx + \begin{bmatrix} -c_1 A e^{ikx} + c_2 A e^{ikz} \\ c_1 A e^{ikx} + c_2 A e^{ikz} \end{bmatrix} e^{-i\omega t} dy.
\]

Choosing \( c_1 = -c_2 = \frac{1}{2} \), we obtain

\[
H_1(z) = \frac{A}{2} e^{ikz} + \frac{A}{2} e^{ikx}, \quad H_2(z) = \frac{iA}{2} e^{ikz} - \frac{iA}{2} e^{ikx},
\]

(59)
which can be written as

\[
H_1(z) = A \cos \left( \frac{k_+ - k_-}{2} z \right) \exp(-i(k_+ + k_-)z/2),
\]

\[
H_2(z) = A \sin \left( \frac{k_+ - k_-}{2} z \right) \exp(-i(k_+ + k_-)z/2).
\]

(60)

Let \( \theta \in \mathbb{R} \) such that

\[
\tan \theta = \frac{H_2(z)}{H_1(z)} = \tan \left( \frac{k_+ - k_-}{2} z \right).
\]

(61)

It is immediate that

\[
\theta_k = \left( \frac{k_+ - k_-}{2} z \right) + 2k\pi, \quad k \text{ is an integer.}
\]

(62)

Restricting \( \theta \in [0, 2\pi) \) it is clear that the phase difference between the left- and right-handed components is \( 2\theta_0 \), where

\[
\theta_0 = \frac{1}{2} \varepsilon \omega \sqrt{\varepsilon \mu_0 \alpha} \left( \sqrt{1 + \beta^2} - \sqrt{1 - \beta^2} \right).
\]

(63)

Consider \( k_\pm \) in eq. (57) given by a second-order approximation, i.e.,

\[
k_\pm = \varepsilon \mu_0 \left( \frac{1}{2} \alpha + \frac{1}{8} \beta^2 + \mathcal{O} \left( \frac{\alpha^3}{\beta^3} \right) \right).
\]

(64)

Substituting in eq. (63) we have

\[
\theta_0 = \frac{\beta}{2\alpha} \varepsilon \omega \sqrt{\alpha \mu_0 \varepsilon}.
\]

(65)

It shows the well-known result asserting that ferrite is indeed a non-reciprocal medium.

3 Material media viewed as spacetime deformations in vacuum

In the formalism in, e.g. [23, 34] that describes the electromagnetism in linear media, the dual Hodge operator action is equivalent to the constitutive \( \chi \) tensor action on 2-form fields:

\[
\ast \Delta = \chi \Delta, \quad \Delta \in \Omega^2_{\pm}(\mathbb{R}^{1,\beta}).
\]

(66)

If cartesian coordinates are introduced, eq. (66) is equivalent to

\[
\ast \Delta = \frac{1}{4} \epsilon_{\mu\nu\sigma\kappa} \chi_{\sigma\kappa\tau\rho} \Delta_{\tau\rho} dx^\mu \wedge dx^\nu.
\]

(67)

3.1 The constitutive tensor

In this subsection we present and discuss the main results in, e.g., [33, 34], concerning the relation between the constitutive and the Riemann curvature tensors. In
linear media, the 2-form electromagnetic intensity $F(x) \in \Omega^2(\mathbb{R}^{1,3})$ is related to the electromagnetic excitation $G \in \Omega^2(\mathbb{R}^{1,3})$ by the equation

$$F(x) = \chi G(x).$$

(68)

Using cartesian coordinates, $F(x)$ and $G(x)$ are expressed as

$$G(x) = \frac{1}{2} G_{\mu\nu}(x) dx^\mu \wedge dx^\nu, \quad F(x) = \frac{1}{2} F_{\mu\nu}(x) dx^\mu \wedge dx^\nu$$

(69)

and $F_{\mu\nu}(x)$ and $G_{\mu\nu}(x)$ are related by

$$G_{\mu\nu}(x) = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \chi^\alpha\beta\sigma\lambda F_{\sigma\lambda}(x),$$

(70)

where $\epsilon$ is the Levi-Civita tensor. The symmetry

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\kappa\nu\sigma}, \quad \chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa},$$

(71)

arises, since $F_{\mu\nu}(x)$ and $G_{\mu\nu}(x)$ are antisymmetric. Besides, the lagrangian density $L(x) = G(x) \wedge F(x)$ is written as \[33, 34\]

$$L(x) = \frac{1}{4} \chi^{\lambda\nu\sigma\kappa} F_{\lambda\nu}(x) F_{\sigma\kappa}(x).$$

(72)

From the relation

$$G^{\lambda\nu}(x) = 2 \frac{\partial L(x)}{\partial F_{\lambda\nu}(x)} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}(x).$$

(73)

there exists the relation

$$\chi^{\lambda\nu\sigma\kappa} = 2 \frac{\partial^2 L(x)}{\partial F_{\lambda\nu}(x) \partial F_{\sigma\kappa}(x)} = 2 \frac{\partial^2 L(x)}{\partial F_{\sigma\kappa}(x) \partial F_{\lambda\nu}(x)}$$

$$= \chi^{\sigma\kappa\lambda\nu}$$

(74)

For more details, see \[33, 34\]. The number of independent coordinates is 21, (using eqs.\(71, 74\)), which comes from the analogy to the Riemann curvature tensor. Only the antisymmetric combinations are non-trivial. An order two antisymmetric tensor has exactly six components and then there would exist 36 components. Expressing $\chi$ as a 6 $\times$ 6 matrix ($\chi \in \text{Hom}(\mathbb{R}^6, \mathbb{R}^6)$) and using the bivectorial notation, (where the indices $I, J, \ldots = 01, 02, 03, 23, 31, 12$ are defined) we can see from eq.\(74\) that the matrix $\chi^{IJ}$ is symmetric ($\chi^{IJ} = \chi^{JI}$), and there exists $(6 \times 7)/2$ independent components in $\chi$. Using physical arguments, one can show that in uniform media we have the relation \[33, 34\]

$$\chi^{[\lambda\nu\sigma\kappa]} = 0.$$  

(75)

In vacuum $\chi$ can be written as

$$\chi^{\lambda\nu\sigma\kappa} = Y_0 \sqrt{g} (g^{\lambda\sigma} g^{\nu\kappa} - g^{\lambda\kappa} g^{\nu\sigma}),$$

(76)

where $g$ is the determinant of $g^{\mu\nu}$ and $Y_0$ is the vacuum admittance.

From the constitutive tensor $\chi$ Post defines two invariants:

$$\chi_1 = \chi_{\lambda\nu} \lambda\nu,$$

(77)
which is called scalar curvature of the medium described by $\chi$, and

$$\chi_2 = \varepsilon_{\lambda\nu\sigma\kappa} \chi^{\lambda\nu\rho\tau} \chi^{\sigma\kappa\mu\alpha} \epsilon_{\rho\tau\mu\alpha}. \quad (78)$$

Post [Po72] proves that $\chi_2$ is non-zero for any medium and $\chi_1 = \chi^{[\lambda\nu\sigma\kappa]}$ is identically null in any medium possessing central symmetry. The constitutive tensor is explicitly represented by:

$$\chi = \left( \begin{array}{ccc} -\varepsilon & \gamma & \mu^{-1} \\ \gamma^t & \varepsilon & \mu^{-1} \\ \mu & \mu & \mu \end{array} \right)$$

The matrix $\mu_{lk}$ is the magnetic permeability matrix, $\varepsilon_{lk}$ is the electric permittivity matrix and $\gamma_{lk}$ is a matrix that describes the electric and magnetic polarization effects. One can prove that in media possessing central symmetry, the matrix $\gamma_{lk}$ is null [33, 34]. In isotropic media the relations

$$\gamma_{lk} = 0, \quad \varepsilon_{lk} = \varepsilon_0 \delta_{lk}, \quad \zeta_{lk} = \mu_0^{-1} \delta_{lk}, \quad (79)$$

are satisfied. In this case, $\chi_1 = 0$ and $\chi_2 = -12\varepsilon_0 / \mu_0$ [33].

We shall study the light propagation in crystalline media presenting optical activity, which are characterized by 32 classes [35, 33]. Each class is represented by a symmetry represented in the table:

| 1 | C | 9 | C, z, x, z | 17 | C, z_4 | 25 | z_0 |
|---|---|---|------------|---|-------|---|-----|
| 2 | - | 10 | z_3, x_2 | 18 | z_4 | 26 | z_3, x_2, E_x |
| 3 | C, z_2 | 11 | z_3, E_x | 19 | S_z, x_2 | 27 | z_3, E_x |
| 4 | E_x | 12 | C, z_3 | 20 | S_z | 28 | C, x_4, y_4 |
| 5 | z_2 | 13 | z_3 | 21 | C, z_6, x_2 | 29 | x_4, y_4 |
| 6 | C, z_2, x_2 | 14 | C, z_4, x_2 | 22 | z_6, x_2 | 30 | S_x, S_y |
| 7 | z_2, x_2 | 15 | z_4, x_2 | 23 | z_6, E_x | 31 | C, x_2, y_2, S |
| 8 | z_2, E_x | 16 | z_4, E_x | 24 | C, z_6 | 32 | x_2, y_2, S |

$C$ denotes central symmetry, $S$ is the cyclic permutation of the indices, $E_x$ is a reflection with respect to the $yz$ plane (analogous definition for $E_y$ and $E_z$), $S_x$ is a rotation using the $x$ axis, followed by a reflection related to the $yz$ plane (And analogous definitions for $S_y$ and $S_z$).

All crystal classes described in the above table present natural optical activity, and the corresponding respective matrices $\gamma_{kl}$, composing the tensor $\chi$, are described
below (the number before the matrices indicates the class number above described):

\[
\begin{pmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{pmatrix}
\]

2 4 5

\[
\begin{pmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{22} & 0 \\
0 & 0 & \gamma_{33}
\end{pmatrix}
\]

7 8 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25

\[
\begin{pmatrix}
\gamma_{11} & \gamma_{12} & 0 \\
-\gamma_{12} & \gamma_{11} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

11, 16, 23 13, 18, 25 19

\[
\begin{pmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{11} & 0 \\
0 & 0 & \gamma_{11}
\end{pmatrix}
\]

20, 29, 32

The matrices corresponding to the classes 29, 32 describe the chiral vacuum [31].

4 Arbitrary constitutive tensors from the vacuum CT

From now on we adopt the notation \( F = F(x), G = G(x), \ldots \), omitting the argument \( x \).

**Spectral Theorem:** There always exists a conformal transformation that diagonalizes the constitutive tensor \( \chi \).

Considering the splitting \( R^{1,3} \simeq R^3 \times R \), we can write

\[
G = \chi F \iff \begin{pmatrix} D \\ H \end{pmatrix} = \begin{pmatrix} -\varepsilon & \gamma \\ \gamma & \mu^{-1} \end{pmatrix}_{6 \times 6} \begin{pmatrix} -E \\ B \end{pmatrix}
\]

(80)

where \( \varepsilon = \varepsilon_0 I \) and \( \mu^{-1} = \mu_0^{-1} I \). By the theorem above, there exists a matrix \( \Gamma \) composed by the eigenvectors of \( \chi \) such that \( \Gamma^{-1} \chi \Gamma = \Lambda \) is a diagonal matrix.

4.1 The chiral vacuum

In order to illustrate the general approach, we firstly consider the chiral vacuum, described by the matrix

\[
\begin{pmatrix}
-\varepsilon & \gamma_{\circ} \\
\gamma_{\circ} & \mu^{-1}
\end{pmatrix}_{6 \times 6}
\]

(81)

where \( \gamma_{\circ} = \gamma_{11} I \). The matrix [31] has eigenvalues \( \sigma_1, \sigma_2 \) and eigenvectors \( \{ (0, 0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0) \} \), where
\[ \sigma_{1,2} = \mu_0^{-1} - \varepsilon_0 \pm \frac{\sqrt{(\mu_0^{-1} + \varepsilon_0)^2 - 4\gamma^2_{11}}}{2\gamma_{11}}. \] 

(82)

Then

\[ \Gamma^{-1} \chi \Gamma = \Lambda = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}_{6 \times 6} \] 

(83)

where \( \Sigma_1 = -\sigma_1 J \) and \( \Sigma_2 = \sigma_2 J \). Denoting \( \hat{F} = \Gamma^{-1}F \) and \( \hat{G} = \Gamma^{-1}G \), we obtain

\[ \begin{pmatrix} \hat{D} \\ \hat{H} \end{pmatrix} = \chi \begin{pmatrix} -\hat{E} \\ B \end{pmatrix} \Rightarrow \Gamma \begin{pmatrix} \hat{D} \\ \hat{H} \end{pmatrix} = \chi \Gamma \begin{pmatrix} -\hat{E} \\ B \end{pmatrix} = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\hat{E} \\ B \end{pmatrix}. \] 

(84)

Define the odd form fields \( \mathbf{D} \) and \( \mathbf{H} \) as

\[ \mathbf{D} = \Sigma_1^{-1} \hat{\mathbf{D}}, \quad \mathbf{H} = \Sigma_2^{-1} \hat{\mathbf{H}}. \] 

(85)

It follows that

\[ \begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\hat{\mathbf{D}} \\ \hat{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\hat{\mathbf{E}} \\ B \end{pmatrix}, \] 

(86)

and we prove that

\[ G = \chi_0 \hat{F}, \] 

(87)

where \( \chi_0 \) vacuum constitutive tensor\(^5\). We obtain for the chiral vacuum, after doing the inverse maps, the constitutive relation

\[ G = \chi^\odot F \] 

(88)

where

\[ \chi^\odot = \Gamma \Lambda \chi_0 \Gamma^{-1} \] 

(89)

Then the constitutive tensor \( \chi^\odot \), related to the chiral vacuum, is completely described by the matrix \( \gamma \) and the vacuum constitutive tensor. We only used conformal maps in \( \mathbb{R}^{1,3} \), which are elements of the group. This kind of structure in electromagnetism was discovered by Bateman [39], who was the first to observe that the Maxwell equations are invariant under the conformal group [31, 32].

### 4.2 Arbitrary linear media: crystalline media, optical activity, magnetic and dielectric Faraday effects

The method is fundamentally analogous to the chiral vacuum case. Consider an arbitrary linear media described by the matrix

\[ \chi = \begin{pmatrix} -\varepsilon & \gamma \\ \gamma^\dagger & \mu^{-1} \end{pmatrix}, \] 

(90)

\(^5\)modulo dilation of the axis \( \mathbf{e}_4, \mathbf{e}_5 \) and \( \mathbf{e}_6 \) by \( \mu_0 \) and contraction of \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \) by \( \varepsilon_0 \).
where

\[ \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}. \]  

(91)

The matrix \( \gamma \) has eigenvalues \( \sigma_A \) \((A = 1, 2, \ldots, 6)\). Then

\[ \Gamma^{-1} \chi \Gamma = \Lambda = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}_{6 \times 6}. \]  

(92)

where \( \Sigma_1 = -\text{diag}(\sigma_1, \sigma_2, \sigma_3) \) and \( \Sigma_2 = -\text{diag}(\sigma_4, \sigma_5, \sigma_6) \). Denoting \( \tilde{F} = \Gamma^{-1} F \) and \( \tilde{G} = \Gamma^{-1} G \), we obtain

\[
\begin{pmatrix} \tilde{D} \\ \tilde{H} \end{pmatrix} = \chi \begin{pmatrix} -\tilde{E} \\ \tilde{B} \end{pmatrix} \Rightarrow \Gamma \begin{pmatrix} \tilde{D} \\ \tilde{H} \end{pmatrix} = \chi \Gamma \begin{pmatrix} -\tilde{E} \\ \tilde{B} \end{pmatrix} = \Gamma^{-1} \chi \Gamma \begin{pmatrix} -\tilde{E} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\tilde{E} \\ \tilde{B} \end{pmatrix}.
\]  

(93)

Defining the vectors \( \tilde{D} = \Sigma_1^{-1} \tilde{D}, \tilde{H} = \Sigma_2^{-1} \tilde{H} \), it follows that

\[
\begin{pmatrix} \tilde{D} \\ \tilde{H} \end{pmatrix} = \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\tilde{D} \\ -\tilde{H} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\tilde{E} \\ \tilde{B} \end{pmatrix}.
\]  

(94)

This implies that

\[ G = \chi_0 \tilde{F}, \]  

(95)

where \( \chi_0 \) is the vacuum constitutive relations. Calculating the inverse maps, we obtain for any medium the constitutive relation

\[ G = \chi F \]  

(96)

where

\[ \chi = \Gamma \chi_0 \Gamma^{-1}. \]  

(97)

The constitutive tensor associated to the 32 crystal classes presenting natural optical activity is described uniquely from \( \chi_0 \), i.e., from the spacetime metric, since

\[ \chi_0^{\lambda_0 \nu \kappa} = Y_0 \sqrt{g} (g^{\lambda \sigma} g^{\mu \kappa} - g^{\lambda \kappa} g^{\mu \sigma}). \]  

(98)

Using coordinates we write

\[
G_{\mu \nu} = \frac{Y_0}{4} \sqrt{g} \epsilon_{\mu \nu \alpha \beta} (\Gamma^4)_{\alpha \beta} \Gamma_\rho^\gamma \Lambda_\sigma^\delta (g^{\delta \theta} g^{\rho \lambda} - g^{\delta \lambda} g^{\rho \theta}) F_{\sigma \lambda}
\]  

(99)

Note that the expression above is the constitutive relation for any crystalline material, and it depends only of the matrix \( \gamma \) (given at the end of Sec. 3, for all crystal classes), that describes optical natural activity. Then it can be seen as the deformation of the metric of Minkowski spacetime into a metric of curved riemannian spacetime, since in order to describe the constitutive relations of any crystalline medium we only need the metric of Minkowski spacetime.

In particular, it is also possible to express, from the Lorentzian metric of Minkowski spacetime, the constitutive tensor associated to the dielectric and magnetic Faraday
rotations, and the natural optical activity in arbitrary rotational symmetric media. It is respectively given by the following matrices:

\[
\begin{pmatrix}
-\varepsilon_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & -\varepsilon & -i\varepsilon_{23} & 0 & 0 & 0 \\
0 & i\varepsilon_{23} & -\varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 1/\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 1/\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 1/\mu \\
\end{pmatrix},
\begin{pmatrix}
-\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & -\varepsilon & 0 & 0 & 0 & 0 \\
0 & 0 & -\varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & \zeta_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & \mu^{-1} & i\zeta_{23} \\
0 & 0 & 0 & 0 & -i\zeta_{23} & \mu^{-1} \\
\end{pmatrix},
\begin{pmatrix}
-\varepsilon & 0 & 0 & 0 & i\gamma_{11} & 0 \\
0 & -\varepsilon & 0 & 0 & i\gamma_{11} & 0 \\
0 & 0 & -\varepsilon & 0 & 0 & i\gamma_{11} \\
i\gamma_{11} & 0 & 0 & 1/\mu & 0 & 0 \\
0 & -i\gamma_{11} & 0 & 0 & 1/\mu & 0 \\
0 & 0 & -i\gamma_{11} & 0 & 0 & 1/\mu \\
\end{pmatrix},
\]

(99)

where \( \zeta_{ij} = (\mu^{-1})_{ij} \). Post proves [Po97] that electromagnetic waves propagate with phase velocity \( u \) given by

\[
u = \pm \left( (\varepsilon \pm \varepsilon_{23})\mu \right)^{-1/2},
\]

(dielectric Faraday rotation),

\[
u = \pm \sqrt{\frac{\zeta \pm \zeta_{23}}{\varepsilon}},
\]

(magnetic Faraday rotation) and

\[
u = \pm \frac{\gamma_{11}}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon \mu} + \frac{\gamma_{11}^2}{\varepsilon^2}}.
\]

(natural optical activity).

In the whole process described in this subsection, we only have accomplished conformal transformations in \( \mathbb{R}^{1,3} \).

Concluding Remarks

We investigated the relation between electrodynamics in anisotropic material media and its analogous formulation in an spacetime, with non-null Riemann curvature tensor. The propagation of electromagnetic waves in material media is proved to be analogous to consider the electromagnetic wave propagation in the vacuum, now in a curved spacetime, which is obtained by a deformation of the Lorenztian metric of Minkowski spacetime. Such process of performing deformations of the metric of Minkowski spacetime can be rigorously described using extensors. Also, there exists a close relation between Maxwell equations in curved spacetime and in an anisotropic material medium, indicating that electromagnetism and spacetime properties are deeply related. Besides, the geometrical aspects of wave propagation can be
described by an effective geometry which represents a modification of the Lorentzian metric of Minkowski spacetime.

We discussed the optical activity of a given material medium, closely related to topological spin, and the Faraday rotation, associated to topological torsion. Both quantities are defined in terms of the magnetic potential and the electric and magnetic fields and excitations. The existence of form fields that are closed, but not exact, gives rise to the monopole and solitons in fluids, concerning topological defects and turbulent non-equilibrium thermodynamics, exhaustively investigated by Kiehn [31].

In a forthcoming paper, since the integral over $\mathbb{R}^3$ of the topological torsion spatial component, introduced by eq. (23), is the writhe of a framed oriented link, it is possible to investigate link invariants in gauge theory, from the knot theory viewpoint.

References

[1] Tonti E, *On the geometrical structure of electromagnetism*, in Ferraeze G (editor), Gravitation, Electromagnetism and Geometrical Structures, 281-308, Pitagora, Bologna 1996.

[2] Heaviside O, *Electromagnetic Theory*, Ernest Benn, London 1925.

[3] Warnick K F, Selfridge R H and Arnold D V *Teaching electromagnetic field theory using differential forms*, IEEE Trans. Educ. 40, 53-68 (1997).

[4] Warnick K F, *A Differential Forms Approach to Electromagnetics in Anisotropic Media*, PhD. Thesis, Department of Electric Engineering and Computation, Brigham Young University 1997.

[5] Warnick K F and Arnold D V, *Green forms for anisotropic, inhomogeneous media*, J. Electromagnet Wave 11, 1145-1164 (1997).

[6] Baldomir D and Hammond P, *Geometry of Electromagnetic Systems*, Clarendon Press, Oxford 1996.

[7] Freire I L, *Applications of Differential Forms to the Anisotropic Media Electrodynamics* (in Portuguese), M.Sc. Thesis, Department of Applied Mathematics, Unicamp, Campinas 2004.

[8] Helszajn J, *Principles of Microwave Ferrite Engineering*, Wiley-Interscience, London 1969.

[9] Puntigam R A, Lämmerzahl C and Hehl F W, *Maxwell’s theory on a post-Riemannian spacetime and the equivalence principle*, Class. Quantum Grav. 14 1347-1356 (1997).

[10] Schönberg M, *Electromagnetism and gravitation*, Revista Brasileira de Física, 91-122 (1971).

[11] Salingaros N, *Electromagnetism and the holomorphic properties of spacetime*, J. Math. Phys. 22 (9) 1919-1925 (1981).

[12] de Lorenci V A and Souza M A, *Electromagnetic wave propagation inside a material medium: an effective geometry interpretation*, Phys. Lett. B 512 417-422 (2001).

[13] Baylis W, *Electrodynamics: A Modern Approach*, Birkhäuser, Boston (1999).
[14] Born M and Wolf E, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, Cambridge Unive. Press, Cambridge 1999.

[15] Jadczyk A Z, *Electromagnetic Permeability of the Vacuum and Light-Cone Structure*, Bulletin de l’Academie Polonaise des Sciences, Serie des Sciences Physiques et Astron. 18, 91-94 (1979).

[16] Schouten J A, *Tensor Analysis for Physicists*, Dover, New York 1989.

[17] Gronwald F, Muench U, Macías A and Hehl F, *Volume elements of spacetime and a quartet of scalar fields*, [gr-qc/9712063](https://arxiv.org/abs/gr-qc/9712063).

[18] Lindell I and Loumesto P, *Differentiaalimuodot Sähkömagnetikkassa*, Helsinki University of Technology, Electromagnetics Laboratory report, Espoo 1995.

[19] Misner C, Thorne K and Wheeler J, *Gravitation*, Freeman, San Francisco 1973.

[20] Gross A and Rubilar G, *On the derivation of the spacetime metric from linear electrodynamics*, [gr-qc/0103016](https://arxiv.org/abs/gr-qc/0103016).

[21] Maxwell J C, *A Treatise on Electricity and Magnetism*, vols.1 and 2, Dover, New York 1954.

[22] Hehl F and Obukhov Y, *Spacetime metric from linear electrodynamics I, II* [gr-qc/9904067](https://arxiv.org/abs/gr-qc/9904067), [gr-qc/9911096](https://arxiv.org/abs/gr-qc/9911096).

[23] Hehl F and Obukhov Y, *A gentle introduction to the foundations of classical electrodynamics: the meaning of the excitations and the field strengths*, [physics/0005084](https://arxiv.org/abs/physics/0005084).

[24] Hehl F and Obukhov Y, *On the energy-momentum current of the electromagnetic field in a pre-metric axiomatic approach: I* [gr-qc/0103020](https://arxiv.org/abs/gr-qc/0103020).

[25] Hehl F, Obukhov Y and Rubilar G, *Light propagation in generally covariant electrodynamics and the Fresnel equation*, [gr-qc/0203096](https://arxiv.org/abs/gr-qc/0203096).

[26] Hehl F, Obukhov Y and Rubilar G, *On a possible new type of a T odd skewon field linked to electromagnetism*, [gr-qc/0203105](https://arxiv.org/abs/gr-qc/0203105).

[27] Jancewicz B, *A variable metric electrodynamics. The Coulomb and Biot-Savart laws in anisotropic media*, Annals of Physics, 245, 227 (1996).

[28] Jancewicz B, *The extended Grassmann algebra in R³*, chap. 28 in Baylis W E (editor), *Clifford (Geometric) Algebras with applications in Physics, Mathematics and Engineering*, Birkhäuser, Berlin 1996.

[29] Hillion P, *Manifestly covariant formalism for electromagnetism in chiral media*, Phys. Rev, R47(2), 1365-1374 (1993).

[30] Hillion P, *Electromagnetism in anisotropic chiral media*, Phys. Rev, E47(4), 2868-2873 (1993).

[31] Kiehn R M, (a) *Chirality and helicity vs. spin and torsion or differential topology and electromagnetism*; (b) *Topological Torsion, Pfaff Dimension and Coherent structures*; (c) *Non-equilibrium and irreversible electromagnetism from a topological perspective*; (d) *A topological perspective of non-equilibrium electromagnetism*; (e) *The Photon Spin and other Topological Features of Classical Electromagnetism*; (f) *Topological Defects, Coherent Structures and Turbulence*; (g) *Spinors, Minimal Surfaces, Torsion, Helicity,*
Chirality, Spin, Twistors, Orientation, Continuity, Fractals, Point Particles, Polarization, the Light Cone and the Hopf Map; (h) Electromagnetic Waves in the Vacuum with Torsion and Spin; (i) Topological Torsion and Spin form Coherent Structures in Plasmas and electromagnetic media; (j) Chirality and Helicity vs Topological Spin and Topological Torsion; (k) Optical Vortices and Topological Torsion; (l) The chiral vacuum, [http://www.cartan.pair.com](http://www.cartan.pair.com).

[32] Whitaker E T, A History of the Theories of Aether, Dublin Univ. Press, Dublin (1910).

[33] Post E J, The constitutive map and some of its ramifications, Ann. Phys., 71, 497-518 (1972).

[34] Post E J, Formal Structure of Electromagnetics, Dover, New York 1997.

[35] Salingaros N, Electromagnetism and the holomorphic properties of spacetime, J. Math. Phys., 22(9) 1919-1925 (1981).

[36] Benn I and Tucker R, An Introduction to Spinors and Geometry with applications in Physics, Adam Hilger, Bristol 1987.

[37] Burke W, Applied Differential Geometry, Cambridge Univ. Press, Cambridge 1985.

[38] Voigt W, Lehrbuch der Kristallphysik, B. Teubner, Leipzig 1910.

[39] Bateman H, The transformation of the electrodynamical equations, Proc. London Math. Soc. 8, 223-264 (1910).

[40] Weyl H, Symmetry, Princeton University Press, Princeton, New Jersey 1989.

[41] Teixeira F L and Chew W C, Unified analysis of perfectly matched layers using differential forms, Microwave and Optical Technology Letters 20 (2), 124-126 (1999).

[42] Barnabei M, Brini A and Rota G-C, On the exterior calculus of invariant theory, J. Algebra, 96, 120-160 (1985).

[43] Conradt O, A treatise on quantum Clifford algebras, Ph.D. thesis, Konstanz University, Konstanz 1999 [math-ph/0202059](http://arxiv.org/abs/math-ph/0202059).

[44] Conradt O, Mechanics in Space and Counterspace, J. Math. Phys. 41, 6995-7028 (2000).