On Complex Conjugate Pair Sums and Complex Conjugate Subspaces

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Abstract—In this letter, we study a few properties of Complex Conjugate Pair Sums (CCPSs) and Complex Conjugate Subspaces (CCSs). Initially, we consider an LTI system whose impulse response is one period data of CCPS. For a given input \(x(n)\), we prove that the output of this system is equivalent to computing the first order derivative of \(x(n)\). Further, with some constraints on the impulse response, the system output is also equivalent to the second order derivative. With this, we show that a fine edge detection in an image can be achieved using CCPSs as impulse response over Ramanujan Sums (RSs). Later computation of projection for CCS is studied. Here the projection matrix has a circulant structure, which makes the computation of projections easier. Finally, we prove that CCS is shift-invariant and closed under the operation of circular cross-correlation.

Index Terms—Complex Conjugate Pair, CCPS, Derivative, CCS, Shift-Invariant, Projections.

I. INTRODUCTION

For a given \(q \in \mathbb{N}\), define a set of complex exponential sequences as \(H_q = \{S_{q,k}(n) = e^{j2\pi kn}/q | 0 \leq k \leq q - 1, (k, q) = 1\} \), where \(0 \leq n \leq q - 1\) and \((k, q)\) denotes the greatest common divisor (gcd) between \(k\) and \(q\). By adding all the elements of \(H_q\), mathematician Srinivasa Ramanujan introduced a trigonometric summation called as Ramanujan Sum (RS) in 1918, denoted as \(c_q(n)\) [1]. Later in 2014, P. P. Vaidyanathan introduced a finite length signal representation using \(c_q(n)\) and its circular shifts [2], [3].

Motivated by this, we introduced a trigonometric summation known as Complex Conjugate Pair Sum (CCPS) in one of our previous works [4]. As \((k, q) = (q - k, q)\), for every \(S_{q,k}(n) \in H_q\), there exists a complex conjugate sequence \(S_{q,q-k}(n) \in H_q\), both together form a complex conjugate pair. CCPS \((c_{q,k}(n))\) is defined by adding each complex conjugate pair, i.e.,

\[
c_{q,k}(n) = 2M \cos \left( \frac{2\pi kn}{q} \right),
\]

where \(M = \begin{cases} \frac{1}{2} & \text{if } q = 1 \text{ (or) } 2, \ k \in \hat{U}_q \text{ if } k > 1 \text{ and } k = 1 \text{ if } q = 1; \end{cases}\)

Notations: \((a, b)\) indicates the gcd between \(a\) and \(b\). A least common multiple is denoted as \(\text{lcm}\). Rounding the value \(a\) to the greatest integer less than or equal to \(a\) is denoted as \([a]\).

For a given \(n \in \mathbb{N}\), Euler’s totient function \(\varphi(n)\) is defined as \(\varphi(n) = \sum_{i=1}^{n} \frac{1}{(i, n)}\). As \((k, n) = (n - k, n)\), \(\varphi(n)\) is even for \(n \geq 3\). Symbol \(d/|N|\) denotes that \(d\) is a divisor of \(N\). Define a set \(\hat{U}_n = \{a \in \mathbb{N} | 1 \leq a \leq \frac{n}{2}, (a, n) = 1\}\), hence \#\(\hat{U}_n = \frac{\varphi(n)}{2}\).

II. CCPS AS DERIVATIVES AND APPLICATION

Here we perform an operation using CCPSs, which is equivalent to the derivative. In particular, we prove the following:

Theorem 1. Consider an LTI system whose impulse response is \(\tilde{c}_{q,k}(n)\), \(q > 1\), then for a given input \(x(n)\) the output \(y(n)\)
of the system, i.e., \(x(n) * \bar{c}_{q,k}(n)\) is equivalent to computing the first order derivative of \(x(n)\).

**Proof:** Let \(x(n) = C\), where \(C\) is a constant value, then

\[
y(n) = C \sum_{l=0}^{q-1} \bar{c}_{q,k}(l) = 0, \quad \text{as} \quad \sum_{l=0}^{q-1} \bar{c}_{q,k}(l) = 0, \quad \text{for} \quad q > 1.
\]

If \(x(n) = u(n - n_0)\), where \(u(n)\) is an unit step sequence, then

\[
y(n) = \sum_{l=0}^{q-1} \bar{c}_{q,k}(l) u(n - n_0 - l),
\]

here \(y(n) \neq 0, \quad \forall \quad n_0 \leq n \leq n_0 + q - 2\). If \(x(n) = n\), then

\[
y(n) = n \sum_{l=0}^{q-1} \bar{c}_{q,k}(l) - n \sum_{l=0}^{q-1} l \bar{c}_{q,k}(l)
\]

\[
= -M \left[ q + \sum_{l=0}^{q-1} le^{i(2\pi k l)/q} + \sum_{l=0}^{q-1} le^{i(2\pi k l)/q} \right] + Mq,
\]

\[
= M \left[ \frac{q}{1 - e^{i2\pi k/q}} + \frac{q}{1 - e^{-i2\pi k/q}} \right] = Mq.
\]

From the above analysis, we draw the following conclusions regarding the system output. That is, the system output is:

1) Zero for constant input.
2) Non-zero at the on transient of the unit step sequence.
3) Non-zero constant along the ramps.

In the context of image processing, any function/operation satisfying the above three properties is equivalent to first order derivative [6]–[8]. Therefore, \(x(n) * \bar{c}_{q,k}(n)\) is equivalent to computing the first order derivative of \(x(n)\).

Further, with some modifications in the impulse response, the above operation is equivalent to the second order derivative. To be a second order derivative, the system output should satisfy the first two conclusions mentioned in Theorem 1 and it should be zero for \(x(n) = n\) [6]. That is, the term \(P\) in (4) should be equal to zero, but the assumption of \(\bar{c}_{q,k}(n)\) as impulse response leads to \(P = Mq\). So, instead of \(\bar{c}_{q,k}(n)\), we try by considering its circular shifts as an impulse response. Therefore,

\[
\sum_{l=0}^{q-1} l \bar{c}_{q,k}(l - m) = q \frac{\cos v}{1 - \cos v} [\cos (u + v) - \cos (uv)], \quad 1 \leq m \leq q - 1,
\]

where \(u = \frac{2\pi kl}{q}\) and \(v = \frac{2\pi k}{q}\). Now for what value of \(m\), \(\cos (u + v) = \cos (u)\)? Figure 1 depicts \(\cos (u + v)\) vs \(\cos (u)\) for different \(q\) and \(k\) values. It is observed that independent of \(k\) both the values are equal whenever \(q\) is an odd number and \(m = \frac{q - 1}{2}\). Hence, we can summarize the above discussion as:

**Theorem 2.** For a given odd number \(q\), the operation of linear convolution between a given signal \(x(n)\) and \(\bar{c}_{q,k}(n - \frac{q - 1}{2})\) is equivalent to computing the second order derivative of \(x(n)\).

**A. Application**

Edge detection of an image is crucial in many applications, where the derivative functions are used [7], [8]. In this work, we address this problem using CCPSs and compare the results with the results obtained using RSs [6]. Consider the Lena image and convert it into two one-dimensional signals, namely \(x_1(n)\) and \(x_2(n)\), by column-wise appending and row-wise appending respectively. Now compute \(x_1(n) * \bar{c}_{5,1}(n)\), \(x_2(n) * \bar{c}_{5,1}(n)\), \(x_1(n) * \bar{c}_{5,2}(n)\) and \(x_2(n) * \bar{c}_{5,2}(n)\), the results are depicted in figure 2 (a)-(d) respectively. From the results, it is clear that we can find the edges using CCPSs. Note that, performing convolution on \(x_1(n)\) and \(x_2(n)\) gives better detection of horizontal (Figure 2 (a) and (c)) and vertical (Figure 2 (b) and (d)) edges respectively. Let \(U_q = \{k_1, k_2, \ldots, k_{q-1}\}\), \(q \in \mathbb{N}\), then we can write the relationship between RSs and CCPSs as

\[
\bar{c}_q(n) = \bar{c}_{q,k_1}(n) + \bar{c}_{q,k_2}(n) + \cdots + \bar{c}_{q,k_{q-1}}(n).
\]

Using this and linearity property of convolution sum, we can write \(x_1(n) * \bar{c}_{5,1}(n) = x_1(n) * \bar{c}_{5,1}(n) + x_1(n) * \bar{c}_{5,2}(n)\). Figure 2 (e)-(f) validates the same. So, fine edge detection can be achieved using CCPSs over RSs, where RSs are integer-valued sequences and CCPSs are real-valued sequences. A similar kind of analysis can be done using CCPSs as the second derivative.

**III. Properties of CCS**

Construct a circulant matrix \(D_{q,k} \in \mathbb{M}_q(\mathbb{R})\) as given below

\[
D_{q,k} = \begin{bmatrix}
\bar{c}_{q,k}^0 & \cdots & \bar{c}_{q,k}^1 & \cdots & \bar{c}_{q,k}^{q-1}
\end{bmatrix},
\]
where \( \hat{c}_{q,k} \) indicates \( l \) times circular downshift of the sequence \( c_{q,k} \). Using the factorization property of \( D_{q,k} \), we can write \( D_{q,k} = B_{q,k}B_{k}^{H} \), where

\[
B_{k}^{H} = \begin{bmatrix}
S_{q,k}(0) & S_{q,k}(1) & \ldots & S_{q,k}(q-1) \\
S_{q,k}(0) & S_{q,k}(1) & \ldots & S_{q,k}(q-1)
\end{bmatrix}_{2 \times q}.
\]

(8)

From (8), the column space of \( D_{q,k} \) is equal to \( v_{q,k} \). Moreover, the first two columns of \( D_{q,k} \) are linearly independent [4]. So, any signal \( x \in v_{q,k} \) can be written as,

\[
[x]_{q \times 1} = \begin{bmatrix}
\hat{c}_{q,k}^T \\
\hat{c}_{q,k}^{l+1}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_{q,k}^T \\
\end{bmatrix}_{2 \times 1}. \tag{9}
\]

Let \( q \in \mathbb{N} \) and \( N = lcm(q_1, q_2) \), then the orthogonality property of CCPSs is

\[
\|e_{q_1,kq_1} \|_{N \times 1} = \|e_{q_2,kq_2} \|_{N \times 1} = 2N \cos \left( \frac{2 \pi k_1(q_1 - q_2)}{q_1} \right) \delta(k_1 - k_2), \tag{10}
\]

where \( \delta(k_1 - k_2) \) is obtained by repeating \( \hat{c}_{q_1,kq_1} \) periodically \( \frac{N}{q_1} \) times. Using this, we can conclude the following: The subspaces \( v_{q_1,k}, v_{q_2,k} \) are orthogonal to each other. With this basic introduction, now we discuss a few of CCS properties in the following subsections.

A. shift-invariant Subspace

A natural question for a signal belongs to CCS is: What is the effect if we consider a shifted version of the input signal? We answer this question by proving the following theorem:

**Theorem 3.** CCS is a circular shift-invariant subspace, i.e., if \( x \in v_{q,k} \), then \( x' \in v_{q,k} \), here the shift is interpreted as circular shift (modulo \( q \)).

**Proof:** From (9), circular shifting \( x \) by an amount \( l \) gives

\[
x' = \begin{bmatrix}
\hat{c}_{q,k} \\
\hat{c}_{q,k}^{l+1}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_{q,k} \\
\end{bmatrix}. \tag{11}
\]

\( \hat{c}_{q,k} \) for any \( l \in \mathbb{Z} \) is still a column in \( D_{q,k} \). It implies both \( \hat{c}_{q,k} \) and \( \hat{c}_{q,k}^{l+1} \) belong to \( v_{q,k} \). This results in \( x' \in v_{q,k} \). In fact, using the row-Vandermonde structure of \( B_{k}^{H} \) one can prove that any two consecutive columns of \( D_{q,k} \) act as the basis for \( v_{q,k} \), hence \( x' \in v_{q,k} \).

Since CCS is shift-invariant, it may be useful in applications like wireless communication [9], sub-space tracking: which play a vital role in video surveillance, source localization in radar and sonar, etc., [10].

B. Computing the Projections

In CCPT, a signal \( x \in M_{N,1}(\mathbb{C}) \) is represented as a linear combination of signals belongs to CCSs [4],

\[
x = \sum_{q_1 \in N} \sum_{k=1}^{N} \frac{1}{N} E_{q_1,k} \beta_{q_1,k} = [T_{N}]_{N \times N} [\beta]_{N \times 1}. \tag{12}
\]

Here, \( T_{N} \) is the transformation matrix, \( \beta \) is the transform coefficient vector, \( x_{q_1,k} \) denotes the projection of \( x \) onto \( v_{q_1,k} \), \( E_{q_1,k} \) is the projection matrix of \( x_{q_1,k} \) onto \( v_{q_1,k} \).

So, the orthogonal projection \( x_{q_1,k} \) is computed as follows:

\[
x_{q_1,k} = E_{q_1,k} = \frac{\hat{P}}{E_{q_1,k} = \frac{q_1}{N} \frac{\hat{P}}{E_{q_1,k}}}, \tag{13}
\]

Now consider a matrix \( \hat{P}_{q_1,k} = \frac{q_1}{N} \frac{\hat{P}}{E_{q_1,k}} \), then \( \hat{P}_{q_1,k} = \frac{B_{q_1,k}B_{q_1,k}^{H}}{q_1^2} \), and \( \hat{P}_{q_1,k} = \frac{B_{q_1,k}B_{q_1,k}^{H}}{q_1^2} \), since \( B_{q_1,k}B_{q_1,k} = q_1I \). Moreover, the even symmetry property of CCS makes \( \hat{P}_{q_1,k} \). So, we can say that \( \hat{P}_{q_1,k} \) is an orthogonal projection matrix. Since \( v_{q_1,k} \) is the column space of \( D_{q,k} \), we can write \( \hat{P}_{q_1,k} = \frac{D_{q_1,k}}{q_1} \). With this, (16) can be modified as,

\[
y_{q_1,k} = \frac{1}{N} D_{q_1,k} x_{iN} = \frac{1}{N} D_{q_1,k} x. \tag{17}
\]

Here, \( D_{q_1,k} \) is the transformation matrix, \( \beta \) is the transform coefficient vector, \( x_{q_1,k} \) denotes the projection of \( x \) onto \( v_{q_1,k} \), \( E_{q_1,k} \) is the projection matrix of \( x_{q_1,k} \) onto \( v_{q_1,k} \).

The number of multiplications (computational complexity) involved in computing \( x_{q_1,k} \) is \( q_1^2 + q_1 \). From (11), there are total of \( \frac{1}{N} \sum_{q_1 \in N} \phi(q_1) = \frac{N}{2} \) number of CCSs in representing \( x \). So, the total number of multiplications (\( M_{total} \)) required for computing projections for all \( \frac{N}{2} \) CCSs are

\[
M_{total} = \begin{cases}
2 + \frac{1}{N} \sum_{q_1 \in N} \phi(q_1)(q_1^2 + q_1), & \text{if } 2 \mid N \\
8 + \frac{1}{N} \sum_{q_1 \in N} \phi(q_1)(q_1^2 + q_1), & \text{if } 2 \not\mid N.
\end{cases} \tag{18}
\]
For few $N$ values both $M_{total}$ and $\lfloor N^{2.81} \rfloor$ are tabulated

| $N$ | 3  | 4  | 5  | 6  | 8  | 12 | 16  | 32  | 64  |
|-----|----|----|----|----|----|----|-----|-----|-----|
| $M_{total}$ | 21 | 151| 344| 16961| 238680 |
| $\lfloor N^{2.81} \rfloor$ | 21 | 151| 344| 16961| 238680 |

in Table I. From the table, we can conclude that $M_{total} \ll O( N^{2.81})$ and there is an approximate of 40% reduction in the computational complexity for the values given in the table. Therefore, estimating the period and frequency information through projection computation requires less computational complexity over the direct inverse computation method. Apart from this computational advantage, in some applications, it is required to know the existence of certain periods in an observed signal. In such scenarios, computing projections for those CCSs are sufficient.

C. Correlation of Sequences in CCS

DFT of CCSs: For a given $q\in\mathbb{N}$ and $k\in\tilde{U}_q$,

$$DFT[\tilde{c}_{q,k}(n)] = \tilde{C}_{q,k}(K) = \begin{cases} q, & \text{if } K = k \text{ (or) } q - k, \\ 0, & \text{Otherwise.} \end{cases}$$

(19)

Let $r_c(l)$ denotes the circular autocorrelation of $\tilde{c}_{q,k}(l)$, then

$$DFT[r_c(l)] = R_c(K) = \tilde{C}_{q,k}(K)\tilde{C}_{q,k}(K) = q\tilde{C}_{q,k}(K).$$

Taking IDFT of above equation leads to $r_c(l) = q\hat{c}_{q,k}(l)$. Using this relation, we prove the following Theorem.

**Theorem 4.** CCS is closed under circular cross-correlation operation, i.e., if $x(n)\in v_{q,k}$ and $y(n)\in v_{q,k}$ then the circular cross-correlation $r_{xy}(l)$ also belongs to $v_{q,k}$.

**Proof:** Given $x(n)\in v_{q,k}$ and $y(n)\in v_{q,k}$ then

$$r_{xy}(l) = \sum_{m=0}^{q-1} \beta_{m1}\tilde{c}_{q,k}(l - m1) \sum_{n=0}^{q-1} \tilde{c}_{q,k}(n - l - m2)$$

$$= \sum_{m=0}^{q-1} \sum_{n=0}^{q-1} \beta_{m1}\tilde{c}_{q,k}(n - m1)\tilde{c}_{q,k}(n - l - m2)$$

$$= \sum_{m=0}^{q-1} \sum_{n=0}^{q-1} \beta_{m1}\tilde{c}_{q,k}(l - m2 - m1)$$

From above, $r_{xy}(l)$ is a weighted linear combination of signals belongs to $v_{q,k}$, hence $r_{xy}(l)\in v_{q,k}$.

Further, the autocorrelation of a $N$-length signal $x(n)$ is

$$r_x(l) = \sum_{n=0}^{N-1} x(n)x^*(n - l)$$

(20)

Using (11), the above equation can be modified as

$$r_x(l) = \sum_{q_i|N} \sum_{k_i=1}^{N/q_i} \sum_{q_j|N} \sum_{k_j=1}^{N/q_j} \sum_{N-1} \sum_{n=0}^{N-1} x_{q_1,k_1}(n)x_{q_2,k_2}(n - l)$$

(21)

where $Q = \left\lfloor \frac{N}{q_i} x_{q_1,k_1}(l) \right\rfloor$, if $q_i = q_j$ & $k_1 = k_2$, since CCS is shift-invariant and both $v_{q_1,k_1}$, $v_{q_2,k_2}$ are orthogonal to each other for $k_1 \neq k_2$. Now substituting $Q$ in (21) leads to

$$\frac{1}{N} r_x(l) = \sum_{q_i|N} \sum_{k_i=1}^{N/q_i} \sum_{N-1} \sum_{n=0}^{N-1} x_{q_1,k_1}(n)x_{q_2,k_2}(n - l)$$

(22)

The above result is summarized as follows:

**Theorem 5.** The circular autocorrelation of a signal is equal to the linear combination of its projections (onto CCSs) autocorrelation, with a proper normalization.

This property is useful if the given signal is contaminated with additive noise. In such scenarios projecting $r_x(l)$ onto CCSs gives an accurate period and frequency estimation over projecting $x(n)$.

IV. CONCLUSION

In this work, we have shown how to use CCSs as first and second order derivatives. Here, the first order derivative is used to find the edges in an image. It is shown that fine edge detection can be achieved using CCSs over RSs. Later we discussed few properties of CCSs, which may find their applications in signal processing. In particular, the signal information can be estimated with lesser computational complexity using CCS projections.

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**References**

[1] S. Ramanujan, “On certain trigonometrical sums and their applications in the theory of numbers,” Trans. Cambridge Philos. Soc., vol. 22, no. 13, pp. 259–276, 1918.

[2] P. P. Vaidyanathan, “Ramanujan Sums in the Context of Signal Processing-Part I: Fundamentals,” IEEE Transactions on Signal Processing, vol. 62, pp. 4145–4157, Aug 2014.

[3] P. P. Vaidyanathan, “Ramanujan Sums in the Context of Signal Processing-Part II: FIR Representations and Applications,” IEEE Transactions on Signal Processing, vol. 62, pp. 4158–4172, Aug 2014.

[4] B. S. Shaik, V. K. Chakka, and A. S. Reddy, “A new signal representation using complex conjugate pair sums,” IEEE Signal Processing Letters, vol. 26, pp. 252–256, Feb 2019.

[5] S. w. Deng and J. q. Han, “Signal Periodic Decomposition With Conjugate Subspaces,” IEEE Transactions on Signal Processing, vol. 64, pp. 5981–5992, Nov 2016.

[6] D. K. Yadav, G. Kuldeep, and S. D. Joshi, “Ramanujan sums as derivatives and applications,” IEEE Signal Processing Letters, vol. 25, pp. 413–416, March 2018.

[7] A. K. Jain, Fundamentals of digital image processing. Prentice-Hall, Inc., 1989.

[8] C. Rafael Gonzalez and R. Woods, “Digital image processing.” Pearson Education, 2002.

[9] K. W. Forsythe, “Capacity of flat-fading channels associated with a subspace-invariant detector,” in Conference Record of the Thirty-Fourth Asilomar Conference on Signals, Systems and Computers (Cat. No.00CH37154), vol. 1, pp. 411–416 vol.1, Oct 2000.

[10] M. Cho and Y. Chi, “Shift-invariant subspace tracking with missing data,” in 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 8222–8225, May 2019.

[11] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to algorithms. MIT press, 2009.