ATOMIC AND ANTIMATTER SEMIGROUP ALGEBRAS
WITH RATIONAL EXPONENTS

FELIX GOTTI

ABSTRACT. In this paper, a semigroup algebra consisting of polynomial expressions
with coefficients in a field \( F \) and exponents in an additive submonoid \( M \) of \( \mathbb{Q}_{\geq 0} \) is
called a Puiseux algebra and denoted by \( F[M] \). Here we study the atomic structure
of Puiseux algebras. To begin with, we answer the Isomorphism Problem for the class
of Puiseux algebras, that is, we show that for a field \( F \) if two Puiseux algebras \( F[M_1] \)
and \( F[M_2] \) are isomorphic, then the monoids \( M_1 \) and \( M_2 \) are also isomorphic. Then
we construct three classes of Puiseux algebras satisfying the following well-known
atomic properties: the ACCP property, the bounded factorization property, and the
finite factorization property. Finally, we give a full description of the seminormal
closure, root closure, and complete integral closure of a Puiseux algebra, and use such
description to provide a class of antimatter Puiseux algebras (i.e., Puiseux algebras
containing no irreducibles).

1. Introduction

The study of group rings and algebras dates back to the mid-nineteen century. Most
of the initial research in this area focused mainly on the structure of the groups of
units of group rings and the Isomorphism Problem for groups over a given coefficient
ring; see G. Highman [27] and S. K. Sehgal [30], respectively. However, most of the
early ring-theoretical study of group rings and algebras was mainly carried out on a
non-commutative setting. It was not until the seventies with the work of R. Gilmer,
R. Matsuda, and other authors that the study of commutative group rings as well as
commutative semigroup rings started earning substantial attention (see [17, 19, 29] and
references therein). Much of the work done on commutative semigroup rings during
this decade focused on the following abstract problem: given a commutative ring \( R \) and
a commutative semigroup \( S \), establish conditions under which the semigroup ring of \( S \)
over \( R \) satisfies certain algebraic property. Answering instances of this problem often
requires a fair understanding of the algebraic properties of \( S \) and \( R \). As the structure
of commutative semigroups most of the time cannot be derived from that of abelian
groups, a new research direction in commutative algebra had emerged.

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zation domain, UFD, seminormal domain, isomorphism problem.
Much of the work on commutative semigroup rings carried out in the seventies was compiled by Gilmer in his celebrated book Commutative Semigroup Rings [16], which in turn has motivated a lot of research in the field. More recently, many authors including P. A. Grillet [25], J. Gubeladze [26], and H. Kim [28] have investigated algebraic and factorization properties of semigroup rings. Semigroup algebras, in particular, have permeated through various fields under active research, including algebraic combinatorics [6], discrete geometry [7], and functional analysis [1].

Additive semigroups of rationals have a complex atomic structure (see [14] and references therein). As a result, the semigroup rings they determine, here called Puiseux algebras, have played some important roles in commutative algebra. For instance, A. Grams in [24] localizes a Puiseux algebra to construct an integral domain to disprove P. M. Cohn’s assumption that every atomic domain satisfies the ACCP. In addition, J. Coykendall and the author recently appeal to Puiseux algebras [11] to partially answer a question on atomicity stated by Gilmer in 1984 [16, page 189]. There are further appearances of Puiseux algebras in recent literature (for instance, [3] and [20]); however, no systematic study of their atomic structure seems to have been carried out. Although this paper offers by no means a systematic study, it aims to provide further insight on the algebraic and atomic structure of Puiseux algebras. Here we address two algebraic problems: the Isomorphism Problem and the computation of the seminormal, root, and complete integral closures. Then we use both results to construct various infinite classes of Puiseux algebras with distinct atomic properties.

The first problem we shall address here is the Isomorphism Problem for Puiseux algebras. The Isomorphism Problem for a field $F$ and a class of monoids $\mathcal{C}$ is the question of whether two monoids in $\mathcal{C}$ are isomorphic provided they have isomorphic semigroup algebra over $F$. Versions of the Isomorphism Problem on classes of finitely generated monoids have been investigated before; see, for instance, [25] and [26]. However, the Isomorphism Problem on classes of non-finitely generated monoids seems to be rather unexplored. In Section 3 we give a positive answer to the Isomorphism Problem for Puiseux algebras.

An integral domain $R$ is a bounded factorization domain (BFD) if for every nonzero nonunit $x \in R$ there exists $N \in \mathbb{N}$ such that $x = a_1 \cdots a_n$ for irreducibles $a_1, \ldots, a_n \in R$ implies that $n \leq N$. In addition, $R$ is a finite factorization domain (FFD) if every nonzero element of $R$ has only finitely many non-associate divisors. The notions of BFDs and FFDs were introduced in [2] by D. D. Anderson, D. F. Anderson, and M. Zafrullah, where the authors studied an implication diagram of atomic classes of domains containing the chain UFD $\Rightarrow$ FFD $\Rightarrow$ BFD $\Rightarrow$ ACCP. These three implications are not reversible in general, and examples of atomic domains witnessing this observation are provided in [2]. To illustrate the complexity of the class of Puiseux algebras, for each of the three implications, we construct in Section 4 an infinite class
of Puiseux algebras witnessing the failure of its reverse implication. Our positive answer to the Isomorphism Theorem is key to guarantee that the classes we construct are infinite (up to isomorphism).

Following [10], we say that an integral domain is antimatter if it contains no irreducibles. Classes of antimatter Puiseux algebras have been constructed in [3]. In Section 5 we prove that the seminormal closure, root closure, and complete integral closure of a Puiseux algebra all coincide, and we provide a full description in terms of its monoid of exponents. We use our description to determine whether the seminormal closure of a Puiseux algebra is atomic/antimatter. To contrast the fact that the algebras we shall obtain by taking seminormal closures are seminormal, we conclude the paper providing a non-seminormal class of antimatter Puiseux algebras.

2. Background

Throughout this paper, we let \( \mathbb{N} \) and \( \mathbb{P} \) denote the set of positive integers and the set of primes, respectively. In addition, we set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). If \( j, k \in \mathbb{Z} \), then we let \( [j, k] \) denote the discrete interval from \( j \) to \( k \), i.e., \( [j, k] := \{ z \in \mathbb{Z} \mid j \leq z \leq k \} \). For \( X \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we define \( X \geq r := \{ x \in X \mid x \geq r \} \) and, in a similar manner, we define \( X < r \) and \( X > r \). If \( q \in \mathbb{Q} > 0 \), then we denote the unique \( n, d \in \mathbb{N} \) such that \( q = n/d \) and \( \gcd(n, d) = 1 \) by \( n(q) \) and \( d(q) \), respectively. Finally, for \( S \subseteq \mathbb{Q} > 0 \), we set \( n(S) := \{ n(s) \mid s \in S \} \) and \( d(S) := \{ d(s) \mid s \in S \} \).

2.1. Commutative and Atomic Monoids. Within the scope of our exposition, each monoid is tacitly assumed to be cancellative and commutative. Unless we specify otherwise, monoids here are written additively. Let \( M \) be a monoid. We set \( M^* := M \setminus \{0\} \), and we let \( \mathcal{U}(M) \) denote the set of units (i.e., invertible elements) of \( M \). If \( \mathcal{U}(M) = \{0\} \), then \( M \) is called reduced. The monoid \( M/\mathcal{U}(M) \), denoted by \( M_{\text{red}} \), is clearly reduced. If \( x, y \in M \), then \( y \) divides \( x \) in \( M \), in symbols, \( y \mid_M x \), if there exists \( z \in M \) such that \( x = y + z \).

The difference group of \( M \), denoted here by \( \text{gp}(M) \), is the abelian group (unique up to isomorphism) satisfying that any abelian group containing a homomorphic image of \( M \) also contains a homomorphic image of \( \text{gp}(M) \). The monoids

- \( M' := \{ x \in \text{gp}(M) \mid \text{there exists } N \in \mathbb{N} \text{ such that } nx \in M \text{ for all } n \geq N \} \),
- \( \widehat{M} := \{ x \in \text{gp}(M) \mid nx \in M \text{ for some } n \in \mathbb{N} \} \), and
- \( \widehat{M} := \{ x \in \text{gp}(M) \mid \text{there exists } c \in M \text{ such that } c + nx \in M \text{ for all } n \in \mathbb{N} \} \)

are called the seminormal closure, root closure, and complete integral closure of \( M \), respectively. It is not hard to verify that \( M \subseteq M' \subseteq \widehat{M} \subseteq \widehat{M} \subseteq \text{gp}(M) \). The monoid \( M \) is called seminormal (resp., root closed or completely integrally closed) if \( M' = M \) (resp., \( M = \widehat{M} = M \)).
An element \( a \in M \setminus \mathcal{U}(M) \) is called an atom if for all \( x, y \in M \) with \( a = x + y \) either \( x \in \mathcal{U}(M) \) or \( y \in \mathcal{U}(M) \). The set of atoms of \( M \) is denoted by \( \mathcal{A}(M) \). Atomicity and antimatterness play a fundamental role in this paper.

**Definition 2.1.** Let \( M \) be a monoid.

1. If each nonunit of \( M \) can be written as a sum of atoms, then \( M \) is atomic.
2. If \( M \) contains no atoms, then \( M \) is antimatter.

A subset \( I \) of \( M \) is an ideal of \( M \) provided that \( I + M \subseteq I \). The ideal \( I \) is principal if \( I = x + M \) for some \( x \in M \). The monoid \( M \) satisfies the ascending chain condition on principal ideals (or ACCP) if each increasing sequence of principal ideals of \( M \) eventually stabilizes. It is well known that each monoid satisfying the ACCP is atomic [15, Proposition 1.1.4]. For \( S \subseteq M \), we let \( \langle S \rangle \) denote the submonoid of \( M \) generated by \( S \) (i.e., the smallest (under inclusion) submonoid of \( M \) containing \( S \)). The monoid \( M \) is called finitely generated if it can be generated by a finite set, while \( M \) is called cyclic if it can be generated by a singleton. Each finitely generated monoid satisfies the ACCP [15, Proposition 2.7.8].

In this paper we study monoid algebras whose exponents lie in \( \mathbb{Q}_{\geq 0} \). Additive submonoids of \( \mathbb{Q}_{\geq 0} \) are known as Puiseux monoids. Additive submonoids of \( \mathbb{Q} \) account, up to isomorphism, for all rank-one torsion-free monoids [13, Section 24], and every additive submonoid of \( \mathbb{Q} \) that is not a group is isomorphic to a Puiseux monoid [16, Theorem 2.9]. The atomic spectrum of the class of Puiseux monoids is broad, whose members ranging from antimatter monoids (e.g., \( \langle 1/2^n \mid n \in \mathbb{N} \rangle \)) to non-finitely generated atomic monoids (e.g., \( \langle 1/p \mid p \in \mathbb{P} \rangle \)). The atomic structure of Puiseux monoids has been systematically studied during the last three years, and the most relevant achieved results can be found in the survey [9]. Puiseux monoids have also been studied in connection with factorizations of matrices [4] and commutative rings [11].

### 2.2. Factorizations

Recall that a multiplicative monoid \( F \) is the free commutative monoid on \( P \subseteq F \) provided that every element \( x \in F \) can be written uniquely in the form \( x = \prod_{p \in P} p^{v_p(x)} \), where \( v_p(x) \in \mathbb{N}_0 \) and \( v_p(x) > 0 \) only for finitely many \( p \in P \). For each set \( P \), there exists a unique (up to canonical isomorphism) monoid \( F \) that is free commutative on \( P \). We let \( \mathbb{Z}(F) \) denote the (multiplicative) free commutative monoid on \( \mathcal{A}(M_{\text{red}}) \). The elements of \( \mathbb{Z}(F) \) are called factorizations. If \( z = a_1 \cdots a_{\ell} \in \mathbb{Z}(F) \) for some \( \ell \in \mathbb{N}_0 \) and \( a_1, \ldots, a_{\ell} \in \mathcal{A}(M_{\text{red}}) \), then \( \ell \) is called the length of \( z \) and is denoted by \( |z| \). Because \( \mathbb{Z}(F) \) is free, there exists a unique monoid homomorphism \( \pi: \mathbb{Z}(F) \to M_{\text{red}} \) satisfying that \( \pi(a) = a \) for all \( a \in \mathcal{A}(M_{\text{red}}) \). For each \( x \in M \), we set

\[
\mathbb{Z}(x) := \mathbb{Z}_M(x) := \pi^{-1}(x + \mathcal{U}(M)) \subseteq \mathbb{Z}(F).
\]

Clearly, \( M \) is an atomic monoid if and only if \( \mathbb{Z}(x) \) is nonempty for all \( x \in M \). The monoid \( M \) is called a unique factorization monoid (or a UFM) if \( |\mathbb{Z}(x)| = 1 \) for all
$x \in M$. On the other hand, $M$ is called a finite factorization monoid (or an FFM) if $1 \leq |Z(x)| < \infty$ for all $x \in M$. Clearly, each UFM is an FFM. For each $x \in M$, we set

$$L(x) := L_M(x) := \{|z| : z \in Z(x)\}.$$ 

The monoid $M$ is called a bounded factorization monoid (or a BFM) if $1 \leq |L(b)| < \infty$ for all $b \in M$. It is clear that each FFM is a BFM. In addition, it is well known and easy to prove that each BFM satisfies the ACCP (see the chain 4.1).

2.3. Integral Domains and Semigroup Rings. Let $R$ be an integral domain, and let $R^* := R \setminus \{0\}$ denote the multiplicative monoid of $R$. As usual, $R^\times$ denote the group of units of $R$ (clearly, $R^\times = \mathcal{U}(R^*)$). We say that $R$ is atomic (resp., antimatter, a BFD, an FFD) if $R^*$ is atomic (resp., antimatter, a BFM, an FFM). Also, we let $\mathcal{A}(R)$ and $Z(R)$ denote $\mathcal{A}(R^*)$ and $Z(R^*)$, respectively, and for a nonzero nonunit $r \in R$, we let $Z(r) := Z_R(r)$ and $L(r) := L_R(r)$ denote $Z_R^*(r)$ and $L_R^*(r)$, respectively.

For a ring $R$ and a semigroup $S$, consider the set $R[X;S]$ comprising all functions $f: S \to R$ satisfying that $\{s \in S \mid f(s) \neq 0\}$ is finite. We shall conveniently represent an element $f \in R[X;S]$ by $f = \sum_{i=1}^{n} f(s_i)X^{s_i}$, where $s_1, \ldots, s_n$ are precisely those $s \in S$ satisfying that $f(s) \neq 0$. With addition and multiplication defined as for polynomials, $R[X;S]$ is a ring, which is called the semigroup ring of $S$ over $R$. Following Gilmer [16], we shall write $R[S]$ instead of $R[X;S]$. As we are mainly concerned with semigroup rings of Puiseux monoids over a given field, the following terminology seems appropriate.

**Definition 2.2.** Let $F$ be a field, and let $M$ be a Puiseux monoid. Then we call $F[M]$ a Puiseux algebra.

Let $F[M]$ be a Puiseux algebra. It follows from [16, Theorem 8.1 and Theorem 11.1] that $F[M]$ is an integral domain satisfying that $F[M]^\times = F$. We say that the element $f = \alpha_1X^{q_1} + \cdots + \alpha_kX^{q_k} \in F[M]^*$ is represented in canonical form if $\alpha_i \neq 0$ for each $i \in \{1, k\}$ and $q_1 > \cdots > q_k$. Observe that each element of $F[M]^*$ has a unique representation in canonical form. In this case, $\text{supp}(f) := \{q_1, \ldots, q_k\}$ and $\text{deg}(f) := q_1$ are called the support and the degree of $f$, respectively. As for polynomials, the degree identity $\deg fg = \deg f + \deg g$ holds for all $f, g \in F[M]^*$.

3. The Isomorphism Problem

For each integral domain $R$, a monoid isomorphism $M_1 \to M_2$ always induces a ring isomorphism $R[M_1] \to R[M_2]$. More generally, we have the following proposition, which is an immediate consequence of [16, Theorem 7.2(2)].

**Proposition 3.1.** Let $R$ be an integral domain, and let $\varphi: M_1 \to M_2$ be a monoid homomorphism. If $\varphi$ is injective (resp., surjective), then the ring homomorphism $\bar{\varphi}: R[M_1] \to R[M_2]$ determined by $X^s \mapsto X^{\varphi(s)}$ is also injective (resp., surjective).
In the context of monoid algebras, the Isomorphism Problem refers to the veracity of the reverse implication of Proposition 3.1.

**Isomorphism Problem.** Let $F$ be a field, and let $\mathcal{C}$ be a nonempty class of monoids. For $M_1, M_2 \in \mathcal{C}$, does $F[M_1] \cong F[M_2]$ as $F$-algebras guarantee that $M_1 \cong M_2$?

A brief early survey on this problem is offered by Gilmer in [16, Section 25]. Although the cases when $\mathcal{C}$ is a class consisting of groups have been studied since the 1960s [5], it was not until the 1980s that some attention was given to the more general case of monoids. In 1982, A. S. Demushkin proposed in [12] a positive answer to the Isomorphism Problem on the class of finitely generated torsion-free monoids [16, Section 11] for definitions). However, his proof involved various invalid arguments. In 1998, Gubeladze provided a final positive answer to the Isomorphism Problem on the class of finitely generated torsion-free monoids [26]. We proceed to offer a positive answer to the Isomorphism Problem on the class of Puiseux monoids.

**Theorem 3.2.** Let $F$ be a field, and let $M_1$ and $M_2$ be Puiseux monoids. Then $M_1 \cong M_2$ as monoids if and only if $F[M_1] \cong F[M_2]$ as $F$-algebras.

**Proof.** The direct implication is an immediate consequence of Proposition 3.1. To prove the reverse implication, suppose that $\bar{\varphi} : F[M_1] \to F[M_2]$ is an $F$-algebra isomorphism. By virtue of Proposition 3.1 one can replace $M_1$ by an isomorphic copy $rM_1$ (for a suitable $r \in \mathbb{Q}_{>0}$), and therefore, assume that both conditions $F[M_1] \cong F[M_2]$ and $X \in F[M_1]$ hold. Now define $\varphi : M_1 \to M_2$ by $\varphi(q) = \deg \bar{\varphi}(X^q)$, and notice that for each $q \in M_1^*$,

\[
\varphi(q) = \deg \bar{\varphi}(X^q) = \frac{1}{d(q)} \deg \bar{\varphi}(X^q)^{d(q)} = \frac{1}{d(q)} \deg \bar{\varphi}(X)^{n(q)} = q \deg \varphi(X).
\]

Hence after setting $q_0 := \deg \bar{\varphi}(X)$, we see that $\varphi$ is the monoid homomorphism consisting in multiplying by $q_0$. Since $\bar{\varphi}$ is an $F$-algebra isomorphism, $q_0 > 0$. Therefore $\varphi$ is not only injective, but also strictly increasing. Finally, we show that $\varphi$ is surjective. Since $\varphi$ is strictly increasing, for each element $f := \sum_{k=1}^{n} c_k X^{q_k} \in F[M_1]^*$ represented in canonical form, one obtains that $\deg \bar{\varphi}(X^{q_0}) > \deg \bar{\varphi}(X^{q_k})$ for every $k \in [1, n - 1]$, and as a consequence,

\[
(3.1) \quad \deg \varphi(f) = \deg \sum_{k=1}^{n} c_k \bar{\varphi}(X^{q_k}) = \deg \bar{\varphi}(X^{q_0}) = \varphi(q_0) = q_0 \deg f.
\]

Clearly, $M_1^* = \{ \deg f \mid f \in F[M_1] \setminus F \}$ and $M_2^* = \{ \deg g \mid g \in F[M_2] \setminus F \}$. As $\varphi$ is a surjective function satisfying that $\varphi(F) = F$, it follows from (3.1) that

\[
M_2^* = \{ \deg \bar{\varphi}(f) \mid f \in F[M_1] \setminus F \} = q_0 \{ \deg f \mid f \in F[M_1] \setminus F \} = q_0 M_1^*.
\]

As $q_0 M_1 = M_2$, the homomorphism $\varphi$ is surjective. Hence $M_1 \cong M_2$ as monoids. \qed
4. Classes of Atomic Puiseux Algebras

The chain (4.1) of refined classes of atomic domains was introduced by Anderson, Anderson, and Zafrullah in [2]. Since then this chain has received a significant amount of consideration in the literature of both commutative algebra and semigroup theory.

\[
\text{UFD} \implies \text{FFD} \implies \text{BFD} \implies \text{ACCP} \implies \text{atomic domain}
\]

As illustrated in [2], none of the implications in (4.1) is in general reversible. This section is devoted to study the potential failure of each of the reverse implications in (4.1) when we restrict to the class of Puiseux algebras. We will construct atomic Puiseux algebras witnessing such failure for the three leftmost implications, illustrating, as a byproduct, the diversity and complexity of the atomic structure of Puiseux algebras. We still do not know whether there exists an atomic Puiseux algebra failing to satisfy the ACCP. However, we suspect that this is the case, and we propose a potential witness at the end of this section.

The original full diagram containing the chain (4.1) also involves the class of half-factorial domains. An integral domain \( R \) is called half-factorial (or an HFD) provided that \( |L(x)| = 1 \) for every \( x \in R^* \). We shall not explicitly consider half-factoriality here because a Puiseux algebra is an HFD if and only if it is a UFD [22, Theorem 4.4].

If a Puiseux algebra \( F[M] \) is atomic, then so is the monoid \( M \) [28, Proposition 1.4]. The converse statement was posed by Gilmer in [16, page 189] and has been answered negatively by Coykendall and the author [11, Theorem 5.4]. In the main result of this section (Theorem 4.3), we identify three infinite classes of atomic Puiseux monoids whose corresponding Puiseux algebras are also atomic but play different roles in the chain of atomic classes (4.1).

For \( S \subseteq \mathbb{N} \), we let \( M_S \) denote the Puiseux monoid \( \langle 1/s \mid s \in S \rangle \). The next lemmas will be useful in the proof of Theorem 4.3.

**Lemma 4.1.** The following statements hold.

1. For each \( P \subseteq \mathbb{P} \), the monoid \( M_P \) satisfies the ACCP, and therefore, it is atomic. In addition, \( \mathcal{A}(M_P) = \{1/p \mid p \in P \} \).

2. If \( P, P' \subseteq \mathbb{P} \), then \( M_P \cong M_{P'} \) if and only if either \( P = P' \) or \( |P| = |P'| = 1 \).

**Proof.** (1) It is easy to check that for each \( P \subseteq \mathbb{P} \) the monoid \( M_P \) is atomic with \( \mathcal{A}(M_P) = \{1/p \mid p \in P \} \). Since \( M_P \) is reduced, proving the first statement amounts to arguing that \( M := M_P \) satisfies the ACCP. Let \((p_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence with underlying set \( \mathbb{P} \). It is not hard to verify that for all \( b \in M \) there exist a unique \( N \in \mathbb{N}_0 \) and a unique sequence of nonnegative integers \((c_n)_{n \in \mathbb{N}}\) such that

\[
b = N + \sum_{n \in \mathbb{N}} c_n \frac{1}{p_n},
\]

where \( c_n \in [0, p_n - 1] \) and \( c_n = 0 \) for all but finitely many \( n \in \mathbb{N} \). With notation as in (4.2), we set \( N(b) := N \) and \( s(b) := \sum_{n \in \mathbb{N}} c_n \). It is clear that if \( b' \mid_M b \) for some
b' ∈ M, then N(b') ≤ N(b). In addition, observe that if b' is a proper divisor of b in M, then N(b') = N(b) implies that s(b') < s(b). As a consequence of these two observations, one deduces that each sequence (qa)n∈N in M satisfying that qa+1 |M qa for every n ∈ N must stabilize. Hence M satisfies the ACCP.

(2) To prove the direct implication, take P, P' ⊆ P such that M_P ≅ M_{P'}. Clearly, if |P| = 0 (resp., |P| = 1), then both M_P and M_{P'} are trivial (resp., cyclic) and so P = P' = ∅ (resp., |P| = |P'| = 1). Assume therefore that |P| ≥ 2. It follows from [23, Proposition 3.2] that M_{P'} = qM_P for some q ∈ Q_{>0}. If p ∈ P, then q/p ∈ S(F_{P'}) and so q = p/p' for some p' ∈ P'. Similarly, if p_i ∈ P \ {p}, then q = p_i/p' for some p'_i ∈ P'. The equality pp'_i = p_i p' now implies that p = p' ∈ P'. Thus, P ⊆ P'. In a similar manner one can argue that P' ⊆ P. The reverse implication is obvious. □

The following lemma is well known and has a straightforward proof.

Lemma 4.2. Let F be a field, and let M be a reduced monoid. If M satisfies the ACCP, then F[M] also satisfies the ACCP.

We are now in a position to prove the main result of this section.

Theorem 4.3. Let F be a field.

(1) If P ⊆ F is an infinite set, then F[M_P] satisfies the ACCP, but it is not a BFD. In particular, F[M_P] is atomic. In addition, F[M_P] ≅ F[M_{P'}] when P' ⊆ F and P' ≠ P.

(2) If M_p := {0} ∪ (Z_p ∩ Q_{>1}), where p ∈ P and Z_p is the localization of the ring Z at p, then F[M_p] is a BFD, but it is not an FFD. In particular, F[M_p] is atomic. In addition, F[M_p] ≅ F[M_{P'}] when P' ⊆ F \ {p}.

(3) If S_r := {p^n | n ∈ N_0} for some r ∈ Q_{>1} \ N, then F[S_r] is an FFD, but it is not a UFD. In particular, F[S_r] is atomic. In addition, F[S_r] ≅ F[S_t] when t ∈ Q_{>1} \ {r}.

Proof. (1) Let P be an infinite subset of F. Lemma 4.1 guarantees that the monoid M_P satisfies the ACCP, and Lemma 4.2 guarantees that the domain F[M_P] also satisfies the ACCP. To verify that F[M_P] is not a BFD, first notice that a ∈ S(F[M_P]) if and only if X^a ∈ S(F[M_P]). Hence X^{1/p} ∈ S(F[M_P]) for every p ∈ P. Because X = (X^{1/p})^P, it follows that L_{F[M_P]}(X) = P. Since |P| = ∞, the domain F[M_P] is not a BFD.

To argue the last statement of (1), assume that P' ⊆ F such that F[M_P] ≅ F[M_{P'}]. Then Theorem 3.2 ensures that M_P ≅ M_{P'}. Since |P| = ∞, the equality P = P' follows from part (2) of Lemma 4.1.

(2) Take p ∈ F. Since 0 is not a limit point of M^*_p, it follows from [21, Proposition 4.5] that M_p is a BFM, and therefore, it must satisfy the ACCP. In light of Lemma 4.2, the integral domain F[M_p] also satisfies the ACCP. In particular, F[M_p] is atomic. To check that F[M_p] is a BFD, fix f ∈ F[M_p]\F and notice that if f = g_1 · · · g_ℓ for some g_1, . . . , g_ℓ ∈ S(F[M_p]), then ℓ ≤ ∑_{i=1}^ℓ deg g_i = deg f. Thus, L_{F[M_p]}(f) is bounded by
deg \ f. Hence F[\mathcal{M}_p] is a BFD. Let us verify that F[\mathcal{M}_p] is not an FFD. One can readily check that \mathcal{A}(\mathcal{M}_p) = [1, 2] \cap \mathbb{Z}_p, and as a consequence, the set of irreducibles in F[\mathcal{M}_p] dividing X^3 is A = \{X^q \mid q \in [1, 2] \cap \mathbb{Z}_p\}. Clearly, the irreducibles in A are pairwise non-associate. So the equalities X^3 = X^{1+1/p}\mathbb{Z}^2 - 1/p\mathbb{Z} \quad \text{(for every } n \in \mathbb{N}) \quad \text{yield infinitely many factorizations of } X^3 \text{ in } F[\mathcal{M}_p], \text{ which implies that } F[\mathcal{M}_p] \text{ is not an FFD.}

For the last statement of (2), take p' \in \mathbb{P} such that F[\mathcal{M}_p] \cong F[\mathcal{M}_{p'}]. Using Theorem 3.2 we obtain that \mathcal{M}_p \cong \mathcal{M}_{p'}, \text{ and therefore, [23, Proposition 3.2] guarantees that } \mathcal{M}_p' = q \mathcal{M}_p \text{ for some } q \in \mathbb{Q}_{>0}. \text{ As inf } \mathcal{M}_p' = \inf \mathcal{M}_{p'} = 1, \text{ we see that } q = 1, \text{ whence } \mathcal{M}_p = \mathcal{M}_{p'}.

(3) Take } r \in \mathbb{Q}_{>1} \setminus \mathbb{N}. \text{ Because } S_r \text{ is generated by the increasing sequence } (r^n)_{n \in \mathbb{N}_0}, \text{ it follows from [21, Theorem 5.6] that } S_r \text{ is an FFM and, in particular, atomic. It is not hard to verify that } \mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}. \text{ As in part (2), we can argue that } F[S_r] \text{ satisfies the ACCP, and so it is an atomic domain. To verify that } F[S_r] \text{ is indeed an FFD, suppose for the sake of a contradiction that there exists } f \in F[S_r] \setminus F \text{ such that } D_f := \{g \in F[S_r]^*: g |_{F[S_r]} f\} \text{ contains infinitely many non-associate divisors of } f. \text{ Since } S_r \text{ is increasingly generated, the set } S_r \cap (0, \text{deg } f) \text{ is finite. Clearly, for each } g \in D_f \text{ the inclusion } \text{supp}(g) \subseteq S_r \cap (0, \text{deg } f) \text{ holds. Hence there exists } S \subseteq S_r \cap (0, \text{deg } f) \text{ such that the set } \{g \in D_f: \text{supp}(g) = S\} \text{ contains infinitely many non-associate divisors of } f. \text{ Let } m \text{ be the least common multiple of } \text{d}(S_r \cap (0, \text{deg } f)). \text{ Observe that } \{g(X^m) \mid g \in D_f \text{ and supp}(g) = S\} \text{ is a subset of } F[X] \text{ consisting of infinitely many non-associate divisors of } f(X^m) \text{ in } F[X]. \text{ However, this contradicts that } F[X] \text{ is a UFD. Thus, each element of } F[S_r] \text{ has only finitely many non-associate divisors and, because } F[S_r] \text{ is atomic, it follows from [2, Theorem 5.1] that } F[S_r] \text{ is an FFD. To see that } F[S_r] \text{ is not a UFD, note that } \mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\} \text{ implies that } X \text{ and } X^r \text{ are non-associate irreducibles in } F[S_r], \text{ and therefore, } X^{n(r)} \text{ and } (X)^{d(r)} \text{ (as elements of } Z(F[S_r])) \text{ are two distinct factorizations of } X^{n(r)}. \text{ Finally, let us argue the last statement of (3). To do so take } t \in \mathbb{Q}_{>1} \text{ such that } F[S_r] \cong F[S_t]. \text{ As in part (2), there exists } q \in \mathbb{Q}_{>0} \text{ such that } S_t = qS_r. \text{ Since } 1 \in \mathcal{A}(S_t), \text{ it follows that } q \in \mathcal{A}(S_t), \text{ that is, } q = t^k \text{ for some } k \in \mathbb{N}_0. \text{ Now for each } n \in \mathbb{N}_0, \text{ the fact that } t^{k+n} \in S_t = t^kS_r \text{ implies that } t^n \in S_r. \text{ Thus, } S_t \subseteq S_r. \text{ Along the same lines, we can prove the reverse inclusion.} \]

\begin{remark}
\text{The monoids } S_r \text{ (for all } r \in \mathbb{Q}_{>0}) \text{ have been recently studied in [8] under the term \textit{rational cyclic semirings}.}
\end{remark}

We have just constructed Puiseux algebras witnessing the failure of all the reverse implications of the chain (4.1) except the last one. Although we still do not know whether the last implication is reversible, we suspect it is not. We would like to finish this section proposing a Puiseux algebra as a potential counterexample. Let } r \text{ be a rational number in } (0, 1) \text{ with } n(r) \neq 1, \text{ and consider the Puiseux monoid } S_r. \text{ Since } n(r)r^n = d(r)r^{n+1} = (d(r) - n(r))r^{n+1} + n(r)r^{n+1} \text{,}

\begin{align*}
\end{align*}
for every $n \in \mathbb{N}$, the sequence $(n(r)r^n + S_r)_{n \in \mathbb{N}}$ is an ascending chain of principal ideals of $S_r$. Clearly, this sequence does not stabilize, and so $S_r$ does not satisfy the ACCP. Now consider a field $F$. Lemma 4.2 guarantees that the integral domain $F[S_r]$ does not satisfy the ACCP. However, we believe that $F[S_r]$ is an atomic domain. The case when $r \geq 1$ in the following conjecture follows from part (3) of Theorem 4.3.

**Conjecture 4.5.** Let $F$ be a field, and take $r \in \mathbb{Q}_{>0}$. If $S_r$ is an atomic monoid, then $F[S_r]$ is an atomic domain.

5. Classes of Antimatter Puiseux Algebras

In this section we prove that the seminormal closure, root closure, and complete integral closure of a Puiseux algebra are equal, and we describe such closures in terms of the exponent Puiseux monoid. Our description will yield a class of antimatter and seminormal Puiseux algebras. We will also offer another class of antimatter Puiseux algebras that are not seminormal. Before proceeding, we would like to emphasize that antimatter domains were first investigated in [10] and classes of antimatter Puiseux algebras were first constructed in [3].

5.1. Algebraic Closures. Let $R$ be an integral domain with quotient field denoted by $qf(R)$. The seminormal closure, root closure, and complete integral closure of $R$, respectively denoted by $R'$, $\overline{R}$, and $\widehat{R}$, are the overrings of $R$ whose multiplicative monoids are $R'^*$, $\overline{R}^*$, and $\widehat{R}^*$, respectively. Thus,

(5.1) \[ R \subseteq R' \subseteq \overline{R} \subseteq \widehat{R} \subseteq qf(R). \]

The integral domain $R$ is called seminormal (resp., root closed or completely integrally closed) if $R' = R$ (resp., $\overline{R} = R$ or $\widehat{R} = R$). In general, $R' \neq R$ and $\overline{R} \neq R$ even in the context of monoid algebras.

**Example 5.1.**

(1) In [7, Example 2.56], W. Bruns and J. Gubeladze exhibit an additive submonoid $M$ of $\mathbb{N}_0^2$ that is seminormal but not root closed. Since $\mathbb{Q}[M]' = \mathbb{Q}[M']$ by [7, Corollary 4.77] and $\overline{\mathbb{Q}[M]} = \mathbb{Q}[^{\overline{M}}]$ by [16, Corollary 12.11], one obtains that $\mathbb{Q}[M]' \neq \mathbb{Q}[M]$.

(2) Consider the additive submonoid $M := \{(0,0)\} \cup \mathbb{N}^2$ of $\mathbb{N}_0^2$, which satisfies that $\text{gp}(M) = \mathbb{Z}^2$. It follows immediately that $M$ is root closed, and therefore, [16, Corollary 12.11] guarantees that the monoid algebra $\mathbb{Q}[M]$ is also root closed. Notice, on the other hand, that $M$ is not completely integrally closed because $(1,1) + n(0,1) \in M$ for every $n \in \mathbb{N}$ even though $(0,1) \notin M$. So it follows from [16, Corollary 12.7] that $\mathbb{Q}[M]$ is not completely integrally closed. As a result, $\mathbb{Q}[M] \neq \mathbb{Q}[M]$.
However, as we shall prove in the next theorem, in the class consisting of Puiseux algebras the three algebraic closures above coincide. First, let us argue the following lemma.

**Lemma 5.2.** Let $F$ be a field, and let $M$ be a Puiseux monoid. Then the equality $F[M] \cap F(X) = F[X]$ holds.

**Proof.** It suffices to argue that $F[M] \cap F(X) \subseteq F[X]$, as the reverse inclusion follows immediately. To do this, take $f = \sum_{i=1}^{k} \alpha_i X^{q_i} \in F[M] \cap F(X)$ represented in canonical form as an element of $F[M]$. Let $\ell$ be the least common multiple of $d(q_1), \ldots, d(q_k)$. Then take $g = \sum_{i=1}^{m} \beta_i X^{m_i}$ and $h = \sum_{i=1}^{n} \theta_i X^{n_i}$ in $F[X]$, both of them represented in canonical form, such that $f = g/h$. Then

$$
\sum_{i=1}^{n} \theta_i X^{\ell n_i} \sum_{i=1}^{k} \alpha_i X^{k_i} = h(X^{\ell}) f(X^{\ell}) = g(X^{\ell}) = \sum_{i=1}^{m} \beta_i X^{\ell m_i},
$$

in $F[X]$, where $k_i := \ell q_i \in \mathbb{N}$ for every $i \in [1, k]$. Let us argue inductively that $q_i \in \mathbb{N}$ for every $i \in [1, k]$. As $k_1 = \ell n_1 - \ell n_1$, we see that $q_1 \in \mathbb{N}$. Suppose that $q_1, \ldots, q_j \in \mathbb{N}$ for some $j \in [1, k - 1]$. Consider the monomial $\theta_1 \alpha_j X^{\ell n_1 + k_j}$ that shows when one multiplies out the leftmost part of (5.2). If $\ell n_1 + k_j \in \text{supp} g(X^{\ell})$, then $\ell$ must divide $\ell n_1 + k_j$, and therefore, $q_j \in \mathbb{N}$. If $\ell n_1 + k_j \notin \text{supp} g(X^{\ell})$, then the monomial $\theta_1 \alpha_j X^{\ell n_1 + k_j}$ should cancel with monomials of the form $\theta_i \alpha_t X^{\ell n_t + k_t}$ with $t < j$, in which case $\ell$ must divide $k_j - k_t$. As $\ell$ divides $k_t$, it follows that $q_j \in \mathbb{N}$. Then we conclude that $f \in F[X]$. Thus, $F[M] \cap F(X) \subseteq F[X]$. \hfill $\square$

**Theorem 5.3.** Let $F$ be a field, and let $M$ be a Puiseux monoid. Then the following statement hold.

1. $F[M]' = \overline{F[M]} = \overline{F[M]} = F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}]$.
2. $F[M]'$ is atomic if $M$ is finitely generated.
3. $F[M]'$ is antimatter if $M$ is not finitely generated and $F$ is algebraically closed.

**Proof.** (1) By virtue of (5.1), it suffices to argue that $\overline{F[M]} \subseteq F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}]$ and $F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}] \subseteq F[M]'$. To verify the latter inclusion, it is enough to observe that the equality $\text{gp}(M) \cap \mathbb{Q}_{\geq 0} = M'$ holds by [14, Proposition 3.1] while the equality $F[M'] = F[M]'$ holds by [7, Corollary 4.77].

To prove the former inclusion, take $f$ in the complete integral closure of $F[M]$, and then take $g \in F[M]$ such that $g f^n \in F[M]$ for every $n \in \mathbb{N}$. Write $f = f_1/f_2$ for $f_1, f_2 \in F[M]$ with $f_2 \neq 0$. Assume, by way of contradiction, that $f_2$ does not divide $f_1$ in $F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}]$. Now let $\ell$ be the least common multiple of the set $\bigcup_{s \in S} d(\text{supp}(s))$, where $S := \{g, f_1, f_2\}$. It is clear that $n/\ell \in \text{gp}(M) \cap \mathbb{Q}_{\geq 0}$ for every $n \in \mathbb{N}$. Therefore setting $q(X) := f_1(X^{\ell})/f_2(X^{\ell}) \in F(X)$, one can see that $q \notin F[X]$ as otherwise
\[ f_1/f_2 = q(X^{1/\ell}) \in F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}], \] which is not possible. Then \( f_2(X^\ell) \) does not divide \( f_1(X^\ell) \) in \( F[X] \). As \( F[X] \) is a UFD, there exist \( a \in \mathcal{O}(F[X]) \) and \( m \in \mathbb{N} \) such that
\[ a(X)^m \mid_{F[X]} f_2(X^\ell) \quad \text{but} \quad a(X)^m \nmid_{F[X]} f_1(X^\ell). \]

Let \( \mu = \max\{n \in \mathbb{N} \mid a(X)^n \mid_{F[X]} f_1(X^\ell)\} \). Take \( N \in \mathbb{N} \) such that the inequality \( N \deg a > \ell \deg g \) holds, and then take \( h_N \in F[M] \) such that \( g f_{1,N}^N = f_2^N h_N \). Observe that \( h_N(X^\ell) = g(X^\ell) f_1(X^\ell)^N f_2(X^\ell)^{-N} \in F(X) \). Then it follows from Lemma 5.2 that \( h_N(X^\ell) \in F[X] \). As a result, the factors in \( g(X^\ell) f_1(X^\ell)^N = f_2(X^\ell)^N h_N(X^\ell) \) are polynomials in \( F[X] \). This, together with the fact that \( a(X)^m \mid_{F[X]} f_2(X^\ell) \), implies that \( a(X)^{mN} \mid_{F[X]} g(X^\ell) f_1(X^\ell)^N \). So there exists \( b(X) \in F[X] \) such that
\[ b(X) a(X)^{N(m-\mu)} = g(X^\ell) \left( \frac{f_1(X^\ell)}{a(X)^\mu} \right)^N. \]

Since \( m > \mu \) and \( F[X] \) is a UFD, \( a(X)^N \mid_{F[X]} g(X^\ell) \). However, this contradicts the inequality \( N \deg a > \ell \deg g \). Therefore \( f \in F[\text{gp}(M) \cap \mathbb{Q}] \). We conclude that \( \overline{F[M]} \subseteq F[\text{gp}(M) \cap \mathbb{Q}_{\geq 0}] \).

(2) Suppose that \( M \) is finitely generated, namely, \( M = \langle q_1, \ldots, q_n \rangle \) for some \( n \in \mathbb{N} \) and \( q_1, \ldots, q_n \in \mathbb{Q}_{>0} \). Letting \( \ell \) be the least common multiple of \( d(q_1), \ldots, d(q_n) \) and \( g \) be the greatest common divisor of \( n(q_1), \ldots, n(q_n) \), one can check that \( N := \ell g^{-1} M \) is a numerical monoid. Therefore \( N' = N0 \). It is clear that \( M \cong N \). Then it follows from [7, Corollary 4.77] and Theorem 3.2 that \( F[M]' = F[M'] \cong F[N'] \cong F[X] \), and so \( F[M]' \) is a UFD and, in particular, an atomic domain.

(3) Suppose now that \( M \) is not finitely generated. In light of Theorem 3.2, one can replace \( M \) by \((\gcd n(M^*))^{-1} M \) and assume that \( \gcd n(M^*) = 1 \). Then [14, Proposition 3.1] ensures that \( M' = \langle d(q)^{-1} \mid q \in M^* \rangle \). From the fact that \( M \) is not finitely generated, one can deduce that \( |d(M^*)| = |d(M^*)| = \infty \). We check that \( M' \) is pure (i.e., for each \( b \in M' \) there exists \( n \in \mathbb{N}_{\geq 2} \) such that \( b/n \in M' \)). To do so, take \( b \in M^* \). As \( |d(M^*)| = \infty \), we can take \( d \in d(M^*) \) such that \( d \not\mid d(b) \). It is clear that the least common multiple \( \ell \) of \( d(b) \) and \( d \) belongs to \( d(M^*) \). Setting \( n := \ell \langle d(b) \rangle \), we obtain that \( n \geq 2 \) and \( b/n \in (n(b)/\ell) \in M' \). Hence \( M' \) is pure. Since \( F \) is algebraically closed, it follows from [7, Corollary 4.77] and [3, Theorem 1] that \( F[M]' = F[M'] \) is an antimatter domain. \( \square \)

Recall that for each \( r \in \mathbb{Q}_{>0} \), the Puiseux monoid \( \langle r^n \mid n \in \mathbb{N}_0 \rangle \) is denoted by \( S_r \).

**Corollary 5.4.** Let \( F \) be an algebraically closed field. For each \( p \in \mathbb{P} \), the Puiseux algebra \( F[S_{1/p}] \) is antimatter. In addition, \( F[S_{1/p}] \not\cong F[S_{1/q}] \) if \( q \in \mathbb{P} \setminus \{p\} \).

**Proof.** As \( -S_{1/p} \cup S_{1/p} \) is an additive subgroup of \( \mathbb{Q} \), the equality \( S_{1/p} = \text{gp}(S_{1/p}) \cap \mathbb{Q}_{\geq 0} \) holds, and so \( F[S_{1/p}] = F[\text{gp}(S_{1/p}) \cap \mathbb{Q}_{\geq 0}] = F[S_{1/p}]' \) by part (1) of Theorem 5.3. Since \( M \) is not finitely generated and \( F \) is algebraically closed, it follows from part (3) of Theorem 5.3 that \( F[S_{1/p}] \) is antimatter.
To argue the second statement, suppose for the sake of a contradiction that there exist \( p, q \in \mathbb{P} \) with \( p \neq q \) such that \( F[S_{1/p}] \cong F[S_{1/q}] \). It follows from Theorem 3.2 that \( S_{1/p} \cong S_{1/q} \), and therefore, [23, Proposition 3.2] guarantees that \( S_{1/q} = rS_{1/p} \) for some \( r \in \mathbb{Q}_{>0} \). Taking \( n \in \mathbb{N} \) such that \( p^n \not| \, n(r) \), one obtains that \( r/p^n \in rS_{1/p} = S_{1/q} \), which contradicts that \( d(S_{1/q}) = \{ q^n \mid n \in \mathbb{N}_0 \} \). \( \square \)

In the direction of part (3) of Theorem 5.3 we have the following question.

**Question 5.5.** Is there an antimatter Puiseux monoid that is not root closed such that the algebra \( F[M] \) is antimatter over any (or some) algebraically closed field \( F \)?

The antimatter Puiseux algebras we have seen so far come from part (3) of Theorem 5.3 and are, therefore, seminormal. By contrast, we would like to construct a class of antimatter Puiseux algebras that are not seminormal. For distinct \( p, q \in \mathbb{P} \), let \( M_{p,q} \) denote the Puiseux monoid \( \langle p^{-m}q^{-n} \mid m, n \in \mathbb{N}_0 \rangle \).

**Proposition 5.6.** Let \( F \) be a perfect field of finite characteristic \( p \). For each \( q \in \mathbb{P} \setminus \{ p \} \), the Puiseux algebra \( F[M_{p,q}] \) is antimatter but fails to be seminormal. In addition, \( F[M_{p,q}] \not\cong F[M_{p,q}'] \) for any \( q' \in \mathbb{P} \setminus \{ p, q \} \).

**Proof.** Fix \( q \in \mathbb{P} \setminus \{ p \} \). For each \( x \in M_{p,q} \) it is clear that \( x/p \in M_{p,q} \), and therefore, \( M_{p,q} \) is antimatter. To argue that \( F[M_{p,q}] \) is an antimatter domain, consider the element

\[
\alpha_1 X^{q_1} + \cdots + \alpha_n X^{q_n} \in F[M_{p,q}] \setminus F.
\]

As \( F \) is a perfect field of characteristic \( p \), the Frobenius homomorphism \( x \mapsto x^p \) is surjective, and so for each \( i \in [1, n] \), there exists \( \beta_i \in F \) with \( \alpha_i = \beta_i^p \) for some \( \beta_i \in F \). In addition, it follows from our initial observation that \( q_i/p \in M_{p,q} \) for every \( i \in [1, n] \). Therefore the element \( f = (\beta_1 X^{q_1/p} + \cdots + \beta_n X^{q_n/p})^p \) is not irreducible in \( F[M_{p,q}] \). Hence \( F[M_{p,q}] \) is an antimatter Puiseux algebra.

In light of [7, Corollary 4.77], proving that the Puiseux algebra \( F[M_{p,q}] \) is not seminormal amounts to verifying that \( M_{p,q} \) is not a seminormal monoid. Assume, by way of contradiction, that \( M \) is seminormal. Then \( \frac{1}{pq} \in M \) by [14, Proposition 3.1]. So we can write

\[
\frac{1}{pq} = \sum_{i=1}^t \frac{\alpha_i}{p^i} + \sum_{i=1}^s \frac{\beta_i}{q^i}
\]

for some coefficients \( \alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_s \in \mathbb{N}_0 \) with \( \alpha_i \neq 0 \) and \( \beta_s \neq 0 \). After simplifying if necessary, we can assume that \( p \not| \, \alpha_i \) and \( q \not| \, \beta_j \) for any \( i \in [1, t] \) and \( j \in [1, s] \). Multiplying (5.3) by \( p^jq^s \) one obtains that \( t = s = 1 \). However, \( \frac{\alpha_1}{p} + \frac{\beta_1}{q} \geq \frac{1}{p} + \frac{1}{q} > \frac{1}{pq} \), which contradicts (5.3). Hence \( F[M_{p,q}] \) is not a seminormal domain.

To argue that \( F[M_{p,q}] \not\cong F[M_{p,q}'] \) for any \( q' \in \mathbb{P} \setminus \{ p, q \} \) one can merely mimic the lines of the second paragraph of the proof of Corollary 5.4. \( \square \)
Remark 5.7. Proposition 5.6 is a version of [3, Theorem 5(2)], which states that if $R$ is an antimatter GCD-domain whose quotient field is perfect of finite characteristic, then $R[Q_{\geq 0}]$ is also an antimatter GCD-domain.

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References

[1] M. Amini: Module amenability for semigroup algebras, Semigroup Forum 69 (2004) 243–254.
[2] D. D. Anderson, D. F. Anderson, and M. Zafrullah: Factorizations in integral domains, J. Pure Appl. Algebra 69 (1990) 1–19.
[3] D. D. Anderson, J. Coykendall, L. Hill and M. Zafrullah: Monoid domain constructions of antimatter domains, Comm. Alg. 35 (2007) 3236–3241.
[4] N. R. Baeth and F. Gotti: Factorization of upper triangular matrices over information algebras, J. Algebra (to appear). Preprint on arXiv: https://arxiv.org/pdf/2002.09828.pdf
[5] S. D. Berman: Group algebras of countable abelian $p$-groups, Publ. Math. Debrecen 14 (1967), 365–405.
[6] E. Briales, A. Campillo, C. Marijuan, and P. Pison: Combinatorics and syzygies for semigroup algebras, Coll. Math. 49 (1998) 239–256.
[7] W. Bruns and J. Gubeladze: Polytopes, Rings and K-theory, Springer Monographs in Mathematics, Springer, Dordrecht, 2009.
[8] S. T. Chapman, F. Gotti, and M. Gotti, Factorization invariants of Puiseux monoids generated by geometric sequences, Comm. Algebra 48 (2020) 380–396.
[9] S. T. Chapman, F. Gotti, and M. Gotti: When is a Puiseux monoid atomic?, Amer. Math. Monthly (to appear). Preprint on arXiv: https://arxiv.org/pdf/1908.09227.pdf
[10] J. Coykendall, D. E. Dobbs, and B. Mullins: On integral domains with no atoms, Comm. Alg. 27 (1999) 5813–5831.
[11] J. Coykendall and F. Gotti: On the atomicity of monoid algebras, J. Algebra 539 (2019) 138–151.
[12] A. S. Denushkina: Combinatorial invariance of toric singularities, Moscow Univ. Math. Bull. 37 (1982) 104–111.
[13] L. Fuchs, Infinite Abelian Groups I, Academic Press, 1970.
[14] A. Geroldinger, F. Gotti, and S. Tringali: On strongly primary monoids, with a focus on Puiseux monoids. Preprint on arXiv: https://arxiv.org/pdf/1910.10270.pdf
[15] A. Geroldinger and F. Halter-Koch: Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics Vol. 278, Chapman & Hall/CRC, Boca Raton, 2006.
[16] R. Gilmer: Commutative Semigroup Rings, Chicago Lectures in Mathematics, The University of Chicago Press, London, 1984.
[17] R. Gilmer: Idempotents of commutative semigroup rings, Houston J. Math. 3 (1977) 369–385.
[18] R. Gilmer and T. Parker: Divisibility properties of semigroup rings, Mich. Math. J. 21 (1974) 65–86.
[19] R. Gilmer and R. Heitmann: The group of units of a commutative semigroup ring, Pacific J. Math. 85 (1979) 49–64.
[20] R. Gipson and H. Kulosman: *For which Puiseux monoids are their monoid rings over fields AP?* Int. Electron. J. Algebra 7 (2020) 43–60.

[21] F. Gotti: *Increasing positive monoids of ordered fields are FF-monoids*, J. Algebra 518 (2019) 40–56.

[22] F. Gotti: *Irreducibility and factorizations in monoid rings*. In: Numerical Semigroups (Eds: V. Barucci, S. T. Chapman, M. D’Anna, and R. Fröberg) pp. 129–139. Springer INdAM Series, vol. 40. Springer Cham, 2020.

[23] F. Gotti: *Puiseux monoids and transfer homomorphisms*, J. Algebra 516 (2018) 95–114.

[24] A. Grams: *Atomic rings and the ascending chain condition for principal ideals*. Math. Proc. Cambridge Philos. Soc. 75 (1974) 321–329.

[25] P. A. Grillet, *Isomorphisms of one-relator semigroup algebras*, Comm. Alg. 23 (1995) 4757–4779.

[26] J. Gubeladze: *The isomorphism problem for commutative monoid rings*, J. Pure Appl. Algebra 129 (1998) 35–65.

[27] G. Higman: *The units of group rings*, Proc. London Math. Soc. 46 (1940) 231–248.

[28] H. Kim: *Factorization in monoid domains*, Comm. Algebra 29 (2001) 1853–1869.

[29] R. Matsuda: *Torsion-free abelian semigroup rings* V. Bull. Fac. Sci. Ibaraki Univ. 11 (1979) 1–37.

[30] S. K. Sehgal: *On the isomorphism of integral group rings. I*, Canad. J. Math. 21 (1969) 410–413.

Department of Mathematics, MIT, Cambridge, MA 02139

*E-mail address: fgotti@mit.edu*