1 Introduction

In previous papers, we have shown that the standard description of spin can be
generalized[1-7]. We have shown that more generalized probability amplitudes
than the standard forms in the literature can be derived for the cases of spin
$1/2$, spin $1$, and the singlet and triplet systems resulting from the addition of
the spins of two spin $1/2$ systems. There is no reason to suppose that we could
not obtain similar generalized results for any other spin system, though at this
stage we have only been able to treat the systems mentioned above.

The reasoning used in deriving the generalized results naturally leads us
to ask: in view of the similarities between intrinsic spin and orbital angular
momentum, is it possible to carry out a programme of generalization for the
ordinary spherical harmonics? If generalized spherical harmonics do indeed
exist, are they solutions of some differential eigenvalue equation? We show in
this paper that the answers to these questions are affirmative.

2 General Theory

2.1 Probability Amplitudes

Our considerations are based on the interpretation of quantum mechanics due to
Landé[8-11]. In this interpretation, a wave function is fundamentally a probabil-
ity amplitude connecting an initial and a final state of a system. In consequence,
it is characterized by two sets of quantum numbers - a set corresponding to the
initial state, and a set corresponding to the final state. To give an example, the
eigenfunctions $\psi_n(r)$ arising from solution of the time-independent Schrödinger
equation are probability amplitudes connecting two states. For each eigenfunc-
tion, the initial state corresponds to the energy eigenvalue $E_n$, and the final
state to the position $r$. Similarly, the spherical harmonic $Y_{l}^{m}(\theta, \varphi)$ is a prob-
ability amplitude referring to an initial state in which the angular momen-
tum projection is $m\hbar$ along the $z$ axis, while the final state is defined by the angular
position $(\theta, \varphi)$.

For a spin system, we denote the probability amplitudes by $\chi(m_{i}^{(a)}; m_{f}^{(c)})$.
The probability amplitude $\chi(m_{i}^{(a)}; m_{f}^{(c)})$ yields the probability that if the spin
projection of the system in the direction $\hat{a}$ (whose polar angles are $(\theta', \varphi')$) is
$m_{i}\hbar$, then a measurement of the spin projection in another direction $\hat{c}$ (whose
polar angles are $(\theta, \varphi)$) yields the projection $m_{f}\hbar$, where $m_{i}$ and $m_{f}$ are the
projection quantum numbers. For any spin system, the generalized probability
amplitudes take many forms because different choices of phase are possible[4].
For a spin-1 system, one form of these probability amplitudes is [3]:

\[
\chi((+1)^{(a)}; (+1)^{(c)}) = \cos^{2} \frac{\theta'}{2} \cos^{2} \frac{\theta}{2} e^{-i(\varphi'-\varphi)} + \sin^{2} \frac{\theta'}{2} \sin^{2} \frac{\theta}{2} e^{i(\varphi'-\varphi)}
+ \frac{1}{2} \sin \theta' \sin \theta,
\]

(1)

\[
\chi((+1)^{(a)}; 0^{(c)}) = \frac{1}{\sqrt{2}} \left[ \sin^{2} \frac{\theta'}{2} \sin \theta e^{i(\varphi'-\varphi)} - \cos^{2} \frac{\theta'}{2} \sin \theta e^{-i(\varphi'-\varphi)} \right]
\]
the three sets of probability amplitudes are connected by the law 

\[ \chi((+1)^{(a)}; (-1)^{(c)}) = \cos^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{-i(\varphi'-\varphi)} + \sin^2 \frac{\theta'}{2} e^{i(\varphi'-\varphi)} + \frac{1}{2} \sin \theta' \sin \theta, \]

\[ \chi((0)^{(a)}; (1)^{(c)}) = \frac{1}{\sqrt{2}} \left[ -\sin \theta' \sin^2 \frac{\theta}{2} e^{-i(\varphi'-\varphi)} + \sin \theta' \sin \frac{\theta}{2} e^{i(\varphi'-\varphi)} + \cos \theta' \sin \theta, \right] \]

\[ \chi((0)^{(a)}; (0)^{(c)}) = \frac{1}{\sqrt{2}} \left[ -\sin \theta' \sin \theta e^{-i(\varphi'-\varphi)} + \sin \theta' \sin \theta e^{i(\varphi'-\varphi)} + \cos \theta' \sin \theta, \right] \]

\[ \chi((-1)^{(a)}; (1)^{(c)}) = \sin^2 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{-i(\varphi'-\varphi)} + \cos^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{i(\varphi'-\varphi)} + \frac{1}{2} \sin \theta' \sin \theta, \]

\[ \chi((-1)^{(a)}; (0)^{(c)}) = \frac{1}{\sqrt{2}} \left[ -\sin^2 \frac{\theta'}{2} \sin \theta e^{-i(\varphi'-\varphi)} + \cos^2 \frac{\theta'}{2} \sin \theta e^{i(\varphi'-\varphi)} - \cos \theta' \cos \theta \right] \]

and

\[ \chi((-1)^{(a)}; (-1)^{(c)}) = \sin^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{-i(\varphi'-\varphi)} + \cos^2 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{i(\varphi'-\varphi)} + \frac{1}{2} \sin \theta' \sin \theta. \]

These probability amplitudes reduce to the standard forms if \( \theta = \varphi = 0 \), which happens if the direction \( \mathbf{c} \) is taken to be the \( z \) direction[3].

### 2.2 Law of Probability Addition

Let \( A, B \) and \( C \) be observables of a quantum system with respective eigenvalue spectra \( A_1, A_2, ..., B_1, B_2, ..., \) and \( C_1, C_2, ..., \) respectively. Then there are three sets of probability amplitudes belonging to this system. Let \( \psi(A, C_n) \) be the probability amplitudes for measurements of \( C \) when the system is in an eigenstate \( A \). Let \( \xi(A_1, B_j) \) be the probability amplitudes for measurements of \( B \) when the system is in an eigenstate \( A \). Let \( \phi(B_2, C_n) \) be the probability amplitudes for measurements of \( C \) when the system is in an eigenstate of \( B \). Then the three sets of probability amplitudes are connected by the law...
The probability amplitudes obey a Hermiticity condition because the corresponding probabilities $P$ are symmetrical with regard to the measurement direction\cite{8}. Thus, because $P(A_i, C_n) = P(C_n, A_i)$ for example, the condition

$$\psi(A_i, C_n) = \psi^*(C_n, A_i)$$

is satisfied.

### 3 Application to Orbital Angular Momentum

#### 3.1 Spherical Harmonics

With these preliminaries, we can consider the spherical harmonics. As observed above, they are fundamentally probability amplitudes. The spherical harmonic $Y_{lm}(\theta, \varphi)$ is the probability amplitude that if the system is in the state of angular momentum projection $m\hbar$ along the $z$ direction, then a measurement of the angular position of the system gives the value $(\theta, \varphi)$ in the solid element $d\Omega$\cite{6}.

It is evident that the spherical harmonics are generalizable. If the initial quantization direction is not the $z$ direction but is defined by the unit vector $\hat{a}$, then the probability amplitude corresponding to measurement of the angular position becomes a generalized spherical harmonic.

Let us denote the generalized spherical harmonics by $Y(l, m; \theta, \varphi)$. In this notation, the ordinary spherical harmonics are denoted by $Y(l, m; \theta, \varphi)$, since $\hat{k}$ is the unit vector defining the $z$ direction. The generalized spherical harmonics can be constructed with the aid of Eq. (10). In order to do this, we make the identification $\psi(A_i, C_n) = Y(l, m; \theta, \varphi)$, so that $A$ is the angular momentum projection in the direction $\hat{a}$, while $C$ is the angular position of the system. We also need an appropriate third observable $B$ to facilitate the expansion Eq. (10). The choice of $B$ must result in the probability amplitudes $\xi(A_i, B_j)$ and $\phi(B_j, C_n)$ being known or easy to deduce. We choose $B$ to be the angular momentum projection in the $z$ direction. This choice indeed results in the probability amplitudes $\xi(A_i, B_j)$ and $\phi(B_j, C_n)$ in the expansion being known.

Since $A$ is an angular momentum projection and $B$ is an angular momentum projection, the $\xi(A_i, B_j)$ are probability amplitudes for angular momentum projection measurements from one direction to another. The initial projection direction is $\hat{a}$, while the final projection direction is the $z$ direction. For a given value of $l$ the probability amplitudes for angular momentum projection measurements from one direction to another should be identical to the probability amplitudes $\chi(m_i; m_f)$ for spin projection measurements for these directions for $s = l$.

Since $B$ corresponds to angular momentum projection along the $z$ direction and $C$ is the angular position of the system, it follows that the $\phi(B_j, C_n)$ are just the ordinary spherical harmonics. Thus, the expansion for the generalized spherical harmonics becomes
3.2 Generalized Spherical Harmonics for \( l = 1 \)

The case \( l = 1 \) is the simplest to which we can apply these arguments. Then the probability amplitudes \( \chi(l,m_1,\hat{a};l,m_j,\hat{k}) \) are specialized forms of the expressions for \( s = 1 \). They result if we set \( \hat{c} = \hat{k} \) so that \( \theta = \varphi = 0 \) in Eqs. (1)-(9). We obtain

\[
\chi((+1,\hat{a});(+1,\hat{k})) = \cos^2 \frac{\theta'}{2} e^{-i\varphi'},
\]

\[
\chi((+1,\hat{a});0,\hat{k}) = \frac{1}{\sqrt{2}} \sin \theta',
\]

\[
\chi((+1,\hat{a});(-1,\hat{k})) = \sin^2 \frac{\theta'}{2} e^{i\varphi'},
\]

\[
\chi(0,\hat{a};(+1,\hat{k})) = -\frac{1}{\sqrt{2}} \sin \theta' e^{-i\varphi'},
\]

\[
\chi(0,\hat{a};0,\hat{k}) = \cos \theta',
\]

\[
\chi(0,\hat{a};(-1,\hat{k})) = \frac{1}{\sqrt{2}} \sin \theta' e^{i\varphi'},
\]

\[
\chi((-1,\hat{a});(+1,\hat{k})) = -\sin^2 \frac{\theta'}{2} e^{-i\varphi'},
\]

\[
\chi((-1,\hat{a});0,\hat{k}) = \frac{1}{\sqrt{2}} \sin \theta' e^{i\varphi'},
\]

and

\[
\chi((-1,\hat{a});(-1,\hat{k})) = -\cos^2 \frac{\theta'}{2} e^{-i\varphi'}.
\]

The ordinary spherical harmonics \( Y_{m}^{l}(\theta,\varphi) = Y(l,m,\hat{k};\theta,\varphi) \) for \( l = 1 \) are

\[
Y(1,1,\hat{k};\theta,\varphi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi},
\]

\[
Y(1,0,\hat{k};\theta,\varphi) = \sqrt{\frac{3}{4\pi}} \cos \theta
\]

and

\[
Y(1,-1,\hat{k};\theta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi}.
\]

The summation over \( j \) in Eq. (12) produces the values \( m_j = -1,0,1 \). Thus, the generalized spherical harmonics are
\[ Y(1,1(\mathbf{a}); \theta, \varphi) = \sqrt{\frac{3}{8\pi}} [\cos \theta' \sin \theta \cos (\varphi' - \varphi) + \sin \theta' \cos \theta + i \sin \theta \sin (\varphi' - \varphi)], \quad (25) \]

\[ Y(1,0(\mathbf{a}); \theta, \varphi) = \sqrt{\frac{3}{4\pi}} [\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi' - \varphi)] \quad (26) \]

and

\[ Y(1,-1(\mathbf{a}); \theta, \varphi) = \sqrt{\frac{3}{8\pi}} [-\sin \theta \cos \theta' \cos (\varphi' - \varphi) + \sin \theta' \cos \theta - i \sin \theta \sin (\varphi' - \varphi)]. \quad (27) \]

These functions satisfy the orthogonality condition

\[ \int \int \ Y^* (l, m(\mathbf{a}); \theta, \varphi) Y(l, m'(\mathbf{a}); \theta, \varphi) \sin \theta d\theta d\varphi = \delta_{mm'}. \quad (28) \]

### 3.3 Probabilities

The probabilities corresponding to the generalized spherical harmonics are given by

\[ P(1, m(\mathbf{a}); \theta, \varphi) = |Y(1, m(\mathbf{a}); \theta, \varphi)|^2. \]

They are

\[ P(1, 1(\mathbf{a}); \theta, \varphi) = \frac{3}{8\pi} [\cos^2 \theta' \sin^2 \theta \cos^2 (\varphi' - \varphi) \]
\[ + \sin^2 \theta' \cos^2 \theta + \sin^2 \theta \sin^2 (\varphi' - \varphi) \]
\[ + \frac{1}{2} \sin 2\theta \sin 2\theta' \cos (\varphi' - \varphi)], \quad (29) \]

\[ P(1, 0(\mathbf{a}); \theta, \varphi) = \frac{3}{4\pi} [\cos^2 \theta' \cos^2 \theta + \sin^2 \theta \sin^2 \theta' \cos^2 (\varphi' - \varphi) \]
\[ + \frac{1}{2} \sin 2\theta \sin 2\theta' \cos (\varphi' - \varphi)] \quad (30) \]

and

\[ P(1, -1(\mathbf{a}); \theta, \varphi) = \frac{3}{8\pi} [\cos^2 \theta' \sin^2 \theta \cos^2 (\varphi' - \varphi) \]
\[ + \sin^2 \theta' \cos^2 \theta + \sin^2 \theta \sin^2 (\varphi' - \varphi) \]
\[ - \frac{1}{2} \sin 2\theta \sin 2\theta' \cos (\varphi' - \varphi)]. \quad (31) \]

If \( P(1, m(\mathbf{a}); \theta, \varphi) \) is integrated over \((\theta, \varphi)\), the result is unity:

\[ \int \int P(1, m(\mathbf{a}); \theta, \varphi) d\Omega = 1. \quad (32) \]

We observe that if we set \( \theta' = \varphi' = 0 \), so that \( \mathbf{a} = \mathbf{k} \), we obtain the standard results

\[ P(1, 1(\mathbf{k}); \theta, \varphi) = |Y_1^1(\theta, \varphi)|^2 = \frac{3}{8\pi} \sin^2 \theta, \quad (33) \]
\[ P(1, 0; \theta, \varphi) = |Y_0^1(\theta, \varphi)|^2 = \frac{3}{4\pi} \cos^2 \theta \]  
and
\[ P(1, -1; \theta, \varphi) = |Y_{-1}^1(\theta, \varphi)|^2 = \frac{3}{8\pi} \sin^2 \theta. \]

### 3.4 Generalized Operators

We expect that the generalized spherical harmonics are solutions of an eigenvalue equation. In order to deduce this equation, we look more carefully at the general features of differential eigenvalue equations. Consider the time-independent Schrödinger equation; this is obtained by forming an eigenvalue equation for the Hamiltonian operator \( H(r) \). Now, the Hamiltonian is expressed in terms of the position coordinate \( r \), which also acts as the eigenfunction characterising the final state. Similarly, in the Legendre equation, the operator for the square of the total angular momentum, \( L^2 \), is expressed in terms of the angles \((\theta, \varphi)\), which also define the eigenvalue corresponding to the final state. In both these cases, the variables which form the arguments of the eigenfunction define the final eigenvalue pertaining to the measurement which this eigenfunction is the probability amplitude for. Furthermore, we observe that the variables corresponding to the final eigenvalue are the ones in terms of which the derivatives in the differential operator are defined. We therefore conclude that this is a general rule, and will use it in the construction of the operator for the generalized spherical harmonics.

With these observations we are able to deduce the required operators. In the primary coordinate system \( S \) defined by the unit vectors \( \hat{i}, \hat{j} \) and \( \hat{k} \), the various angular momentum operators are

\[ L_x = -i\hbar \left[ -\sin \varphi \frac{\partial}{\partial \theta} - \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right], \]  
\[ L_y = -i\hbar \left[ \cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right], \]  
\[ L_z = -i\hbar \frac{\partial}{\partial \varphi} \]  
and
\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta \partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]. \]

In a new orthogonal coordinate system \( S' \) defined by the unit vectors \( \hat{u}, \hat{v}, \) and \( \hat{w} \), corresponding to the \( x', y' \), and \( z' \) directions, the operators are

\[ L'_x = \hat{u} \cdot \mathbf{L}, \quad L'_y = \hat{v} \cdot \mathbf{L} \quad \text{and} \quad L'_z = \hat{w} \cdot \mathbf{L}. \]

Let the polar angles of the unit vectors \( \hat{u}, \hat{v}, \) and \( \hat{w} \) be \((\theta_{\hat{u}}, \varphi_{\hat{u}})\), \((\theta_{\hat{v}}, \varphi_{\hat{v}})\) and \((\theta_{\hat{w}}, \varphi_{\hat{w}})\) respectively. These form an orthogonal set of axes, and so

\[ \hat{u} \cdot \hat{v} = \hat{u} \cdot \hat{w} = \hat{v} \cdot \hat{w} = 0. \]

In addition, they form a right-handed coordinate system, so that

\[ \hat{u} \times \hat{v} = \hat{w}, \quad \hat{v} \times \hat{w} = \hat{u} \quad \text{and} \quad \hat{w} \times \hat{u} = \hat{v}. \]
We can define all the unit vectors in terms of one set of polar angles \((\theta^\prime, \varphi^\prime)\) = \((\theta^\wedge_w, \varphi^\wedge_w)\), with the angles \((\theta^u, \varphi^u)\) and \((\theta^v, \varphi^v)\) defined in such a way as to ensure that the conditions Eqs. (41) and (42) are satisfied. This means that the unit vector \(\hat{w}\) is to be identified with the unit vector \(\hat{a}\). One choice for the vectors \(\hat{u}\) and \(\hat{v}\) is
\[
\theta_u = \theta^\prime - \pi/2, \quad \varphi_u = \varphi^\prime; \quad \theta_v = \pi/2, \quad \varphi_v = \varphi^\prime - \pi/2.
\]

Thus, while
\[
\hat{w} = (\sin \theta^\prime \cos \varphi^\prime, \sin \theta^\prime \sin \varphi^\prime, \cos \theta^\prime),
\]
we have
\[
\hat{u} = (-\cos \theta^\prime \cos \varphi^\prime, -\cos \theta^\prime \sin \varphi^\prime, \sin \theta^\prime)
\]
and
\[
\hat{v} = (\sin \varphi^\prime, -\cos \varphi^\prime, 0).
\]

Using Eqs. (36)- (38) and Eqs. (40), we find that the new operators are
\[
L_x^\prime = -i\hbar \left( \cos \theta^\prime \sin(\varphi - \varphi^\prime) \frac{\partial}{\partial \theta} + [\cos \theta^\prime \cot \theta \cos(\varphi - \varphi^\prime) + \sin \theta^\prime] \frac{\partial}{\partial \varphi} \right),
\]
\[
L_y^\prime = i\hbar \left( \cos(\varphi - \varphi^\prime) \frac{\partial}{\partial \theta} - \cot \theta \sin(\varphi - \varphi^\prime) \frac{\partial}{\partial \varphi} \right)
\]
and
\[
L_z^\prime = i\hbar \left( \sin \theta^\prime \sin(\varphi - \varphi^\prime) \frac{\partial}{\partial \theta} + [\sin \theta^\prime \cot \theta \cos(\varphi - \varphi^\prime) - \cos \theta^\prime] \frac{\partial}{\partial \varphi} \right).
\]

The square of the total angular momentum remains the same, of course. If our reasoning is correct, the generalized spherical harmonics should be eigenfunctions of \(L_z^\prime\) and \(L^2\), Eqs. (49) and Eq. (39), respectively. We find that this is indeed so:
\[
L^2 Y(l, m; \theta, \varphi) = l(l + 1)\hbar^2 Y(l, m; \theta, \varphi)
\]
and
\[
L_z^\prime Y(l, m; \theta, \varphi) = m \hbar Y(l, m; \theta, \varphi).
\]

### 3.5 Eigenvalue Equations for \(L_x^\prime\) and \(L_y^\prime\)

There is a simple prescription for obtaining \(L_x^\prime\) and \(L_y^\prime\) when the generalized expression for \(L_z^\prime\) is known[2]. To obtain \(L_x^\prime\) from \(L_z^\prime\) we simply make the transformation \(\theta^\prime \rightarrow \theta^\prime - \pi/2\), leaving \(\varphi^\prime\) unchanged. To obtain \(L_y^\prime\) we set \(\theta^\prime = \pi/2\) and \(\varphi^\prime \rightarrow \varphi^\prime - \pi/2\). These transformations also convert the eigenfunctions of \(L_z^\prime\) into the eigenfunctions of \(L_x^\prime\) or \(L_y^\prime\), as the case may be. We consider first the case of \(L_x^\prime\); its eigenvalue equation is
\[
L_x^\prime Y(l, m; \theta, \varphi) = m \hbar Y(l, m; \theta, \varphi).
\]
The eigenfunctions of the operator Eq. (47) are obtained from the generalized spherical harmonics Eqs. (25)-(27) by applying to these functions the transformation \( \theta' \rightarrow \theta' - \pi/2 \). The eigenfunctions of \( L_x \) are found to be

\[
Y(1, 1(\hat{n}); \theta, \varphi) = -\sqrt{3/8\pi}[\cos \theta' \cos \theta + \sin \theta \sin \theta' \cos(\varphi' - \varphi) - i \sin \theta \sin(\varphi' - \varphi)],
\]

(53)

\[
Y(1, 0(\hat{n}); \theta, \varphi) = \sqrt{3/4\pi} [\sin \theta' \cos \theta' - \cos \theta \sin(\varphi' - \varphi)]
\]

and

\[
Y(1, (-1)(\hat{n}); \theta, \varphi) = \sqrt{3/8\pi} [\cos \theta' \cos \theta + \sin \theta \sin \theta' \cos(\varphi' - \varphi) + i \sin \theta \sin(\varphi' - \varphi)].
\]

(55)

The interpretation of these eigenfunctions is evident: they are probability amplitudes for measurements of the angular position \((\theta, \varphi)\) if the system is initially in the state of angular momentum projection \( m \hbar \) in the direction \( \hat{u} \).

We next shift attention to \( L'_y \), whose eigenvalue equation is

\[
L'_z Y(1, m(\hat{v}); \theta, \varphi) = m(\hat{v}) \hbar Y(1, m(\hat{v}); \theta, \varphi).
\]

(56)

The transformations \( \theta' = \pi/2 \) and \( \varphi' \rightarrow \varphi' - \pi/2 \) applied to the expression for \( L'_z \) transform this operator to \( L'_y \). The same transformations applied to the eigenfunctions of \( L'_z \) yield from them the eigenfunctions of \( L'_y \). The eigenvalue equation is then found to have the eigenfunctions

\[
Y(1, 1(\hat{v}); \theta, \varphi) = \sqrt{3/8\pi} [\cos \theta + i \sin \theta \cos(\varphi' - \varphi)],
\]

(57)

\[
Y(1, 0(\hat{v}); \theta, \varphi) = \sqrt{3/4\pi} \sin \theta \sin(\varphi' - \varphi)]
\]

and

\[
Y(1, (-1)(\hat{v}); \theta, \varphi) = -\sqrt{3/8\pi} [\cos \theta + i \sin \theta \cos(\varphi - \varphi)].
\]

(59)

The eigenfunctions \( Y(1, m(\hat{v}); \theta, \varphi) \) of \( L'_y \) are probability amplitudes for measurements of the angular position of the system if its initial state corresponds to the angular momentum projection \( m \hbar \) in the direction \( \hat{v} \).

4 Conclusion

In this paper we have deduced generalized spherical harmonics and the eigenvalue equation they satisfy. It is striking that after postulating the generalized spherical harmonics, we were able to deduce the eigenvalue equation by plausibility arguments. Furthermore, the generalized spherical harmonics for the various components of the angular momentum have a reasonable and obvious interpretation. The next step is to extend these ideas to the case \( l = 2 \), and indeed higher values of \( l \). This work is in progress.
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