TRAVELING WAVE SOLUTIONS FOR LOTKA-VOLTERRA SYSTEM RE-VISITED

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Abstract. Using a new method of monotone iteration of a pair of smooth lower- and upper-solutions, the traveling wave solutions of the classical Lotka-Volterra system are shown to exist for a family of wave speeds. Such constructed upper and lower solution pair enables us to derive the explicit value of the minimal (critical) wave speed as well as the asymptotic rates of the wave solutions at infinities. Furthermore, the traveling wave corresponding to each wave speed is unique modulo a translation of the origin. The stability of the traveling wave solutions with non-critical wave speed is also studied by spectral analysis of the linearized operator in exponentially weighted Banach spaces.

1. INTRODUCTION

We re-visit the classical Lotka-Volterra competition system:

\begin{align*}
\begin{cases}
    u_t &= d_1 u_{xx} + u(a_1 - b_1 u - c_1 v), \\
    v_t &= d_2 v_{xx} + v(a_2 - b_2 u - c_2 v)
\end{cases} 
\end{align*}

(x, t) \in \mathbb{R} \times \mathbb{R}^+,

where \( u(x, t), v(x, t) \) are the population densities of two competing species, the constants \( d_i, a_i, b_i, c_i, i = 1, 2 \) are assumed to be positive. In this paper, we are trying to accomplish the following goals: providing a new and easy construction of upper- and lower-solutions to derive the traveling wave solutions of (1.1); obtaining an accurate description of the minimal wave speed and asymptotic behaviors (up to the first order) of the wave solutions; investigating the stability of the traveling wave solutions in various Banach spaces.

System (1.1) has been extensively studied. In [16-Leung] and [18-Pao], there are many applications and treatments of solutions of (1.1) in bounded spatial domain under various initial and boundary conditions. As is well known, system (1.1) and its cooperative counter-parts admit traveling wave solutions. In [22-Tang], Tang and Fife showed the existence of the traveling wave solutions connecting the extinction state with the co-existent state, In [09-Kanel], Kanel and Zhou studied the existence

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\begin{itemize}
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\end{itemize}

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of the traveling wave solutions connecting the coexistent state to a semi-extinction state. In [05-Fei], the traveling wave solution connecting two semi-extinction states were studied, they also estimated the minimal wave speed. For the other treatment of the traveling wave solutions of system (1.1) and its generalizations, please see [25-Wu], [01-Alexander], [07-Hosono], [24-Volpert].

Throughout this paper, we make the following assumptions:

- [H1]. \( \frac{a_2}{b_2} < \frac{a_1}{b_1} \).
- [H2]. \( \frac{a_2}{c_2} < \frac{a_1}{c_1} \).
- [H3]. \( \frac{b_2}{b_1} + \frac{c_1 a_2}{c_2 a_1} \leq 1 + \frac{a_2}{a_1} \).

Under these conditions system (1.1) has three equilibria \((0, 0)\), \((\frac{a_1}{b_1}, 0)\) and \((0, \frac{a_2}{b_2})\).

We will use a new monotone iteration method to investigate the traveling wave solutions of (1.1). The traveling wave solution connects two of the above equilibria. The monotone iteration method has been widely used in the study of the traveling wave solutions of reaction diffusion system such as (1.1), but most constructed upper- and lower-solutions in literature are non-smooth. The relatively larger ‘gap’ between the non-smooth upper and lower-solutions creates certain difficulties in deriving the accurate asymptotic estimates of the traveling wave solutions at infinities. Such estimates are valuable in applications, and enable one fully exploit the cooperative or competitive structure of the Lotka Volterra system.

The smooth upper-solution in the monotone iteration as in section 3 is derived from the traveling wave solutions of the KPP equation. Observing that if we take function \( v \) to be a constant, then the first equation of (1.1) is a generalized KPP equation, the same consideration is also true for the second equation. The existence, uniqueness, asymptotics as well as the stability of the traveling wave solutions of the KPP system are well known, so are the properties of the upper-solution. The most difficult part in section 3 is to construct the lower solution for system (1.1). Since such constructed upper solution is ‘nearly’ a solution, a compromise is made to derive the smooth lower-solution. In fact, the lower-solution does not satisfy the boundary condition at \( \infty \). Thanks to the realxed condition in [25-Wu], we can still apply the monotone iteration scheme as specified in [25-Wu], [03-Boumenir]. The trade off of such ‘shorter’ lower solution is that we can have some freedom to choose the lower solution with desired asymptotic rate at negative infinity. This leads to an accurate (up to first order) asymptotic estimates of the traveling wave solution at \(-\infty\), and an exact value of the minimal wave speed.

The asymptotics of the traveling wave solutions at infinities are obtained by comparison principle. Once the asymptotic behaviors of the traveling wave solutions are known, we can use the Maximum principle and Sliding domain method [15-Leung] to derive the uniqueness, strict monotonicity as well as the local stability of the traveling wave solutions.

The local stability of the traveling wave solutions is studied by means of spectral analysis in some weighted Banach spaces. We proceed to show that the linearized operator about the traveling wave solution has essential spectrum lying completely in the left complex plane, and that 0 is not an eigenvalue of the linearized operator in the weighted Banach spaces, whereas all the other eigenvalues of the linearized operator have negative real parts. This means the traveling wave solution is linearly
exponentially stable. Since such linear operator is sectorial, the linear stability implies the local nonlinear stability of the traveling wave solutions [01-Alexander], [24-Volpert], [06-Henry]. Though general theories on the stability of the traveling wave solutions are known [24-Volpert], [12-Kapitula], [01-Alexander], the verification of the conditions there is in fact a case by case study. The methods used in the stability study of the traveling wave solutions of (1.1) are similar in the spirit to those in [28-Xu], [21-Sattinger], [12-Kapitula], [01-Alexander], [27-Wu], [26-Wu], [02-Bates], but in comparing to the above methods, we need to further overcome the 'unstable' component of the system. This is done by studying an equivalent form of the linearized operator in a smaller weighted Banach spaces.

We remark that the stability is only for the traveling waves with non-critical wave speed. The stability of the traveling waves with critical wave speed is currently under investigation.

The paper is arranged as follows: in section 2, we study the steady states of the system (1.1) and obtain the attraction region of the stable steady state; in section 3, we show there are traveling wave solutions connecting the one of the unstable steady states with a stable one, corresponding to each wave speed, the traveling wave solution is unique and strictly monotone. The analysis is done by utilizing an accurate description of asymptotic behavior of the traveling wave solutions. Furthermore, we have obtained the estimate of the critical wave speed. In the last section of the paper, we show the traveling wave solutions are locally, nonlinearly exponentially stable.

2. THE EQUILIBRIA AND THEIR STABILITY

In this section, we analyze the constant steady states of system (1.1) under conditions H1-H3.

Consider system (1.1) with the initial conditions
\[
(2.1) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad x \in \mathbb{R}.
\]
where \(u_0\) and \(v_0\) are non-negative bounded smooth functions on \(\mathbb{R}\).

**Definition 2.1.** A pair of nonnegative bounded smooth functions \(\tilde{U} = (\tilde{u}, \tilde{v})\) and \(\hat{U} = (\hat{u}, \hat{v})\) are called coupled upper- and lower-solutions of the Cauchy problem (1.1)-(2.1) if \(\tilde{U} \geq \hat{U}\) on \(\mathbb{R} \times [0, \infty)\) and the following inequalities are satisfied
\[
(2.2) \quad \begin{align*}
\tilde{u}_t & \geq d_1 \tilde{u}_{xx} + \tilde{u}(a_1 - b_1 \tilde{u} - c_1 \tilde{v}), \\
\tilde{v}_t & \geq d_2 \tilde{v}_{xx} + \tilde{v}(a_2 - b_2 \tilde{u} - c_2 \tilde{v}); \\
\hat{u}_t & \leq d_1 \hat{u}_{xx} + \hat{u}(a_1 - b_1 \hat{u} - c_1 \hat{v}), \\
\hat{v}_t & \leq d_2 \hat{v}_{xx} + \hat{v}(a_2 - b_2 \hat{u} - c_2 \hat{v}),
\end{align*}
\]
in \(\mathbb{R} \times (0, \infty)\), as well as the initial conditions
\[
(2.3) \quad \tilde{u}(x, 0) \geq u_0(x) \geq \hat{u}(x, 0), \quad \tilde{v}(x, 0) \geq v_0(x) \geq \hat{v}(x, 0) \quad \text{in} \ \mathbb{R}.
\]

It is known from [16-Leung], [18-Pao] that if there exist coupled upper and lower solutions \(\tilde{U}\) and \(\hat{U}\) on \(\mathbb{R} \times [0, \infty)\), then the Cauchy problem (1.1)-(2.1) has a unique solution \(U(x, t) = (u(x, t), v(x, t))\) with \(\tilde{u}(x, t) \geq u(x, t) \geq \hat{u}(x, t)\) and
\( \tilde{v}(x,t) \geq v(x,t) \geq \hat{v}(x,t) \) on \( \mathbb{R} \times [0, \infty) \). The next theorem gives the asymptotic stability and attraction region of the equilibrium \((\hat{u},\hat{v}), 0\).

**Theorem 2.2.** Let \( \alpha = b_2 \left( \frac{a_1}{b_1} - \frac{a_2}{b_2} \right) \), \( B = \frac{b_1 b_2}{a_1 c_1} \left( \frac{a_1 + a_2}{b_2} - \frac{a_1}{b_1} \right) \), and \( \beta = \frac{b_1 b_2}{a_1 c_1} \left( \frac{a_1 + a_2}{b_2} - \frac{a_2}{b_2} \right) \). Assuming that the hypotheses (H1) and (H3) hold, we know that \( \alpha > 0 \), \( B \geq 0 \), and \( \beta > 0 \). If for some \( A > 0 \) and \( 0 < \rho(0) < a_1/b_1 \), the initial functions satisfy

\[
\left( \frac{a_1}{b_1} - \rho(0), 0 \right) \leq (u_0(x), v_0(x)) \leq \left( \frac{a_1}{b_1} + A \rho(0), B \rho(0) \right)
\]

on \( \mathbb{R} \), then the solution for \((1.1) - (2.1)\) satisfies

\[
(2.4) \quad \left( \frac{a_1}{b_1} - \rho(t), 0 \right) \leq (u(t,x), v(t,x)) \leq \left( \frac{a_1}{b_1} + A \rho(t), B \rho(t) \right)
\]

on \([0, \infty) \times \mathbb{R}\) where

\[
(2.5) \quad \rho(t) = [\beta/\alpha + (\rho(0)^{-1} - \beta/\alpha) e^{\alpha t}]^{-1}.
\]

**Proof.** We will show that \((\hat{u}, \hat{v}) = \left( \frac{a_1}{b_1} + A \rho(t), B \rho(t) \right)\) and \((\hat{u}, \hat{v}) = \left( \frac{a_1}{b_1} - \rho(t), 0 \right)\) are a pair of coupled upper-lower solutions defined in Definition \((2.2)\). One can easily see that \( \hat{v} = 0 \) satisfies the required inequalities in \((2.2)\).

We first start with \(\hat{u} = \frac{a_1}{b_1} + A \rho(t)\). From \((2.2)\), it needs to satisfy the differential inequality

\[
A \rho'(t) \geq \left( \frac{a_1}{b_1} + A \rho \right)[a_1 - b_1 \left( \frac{a_1}{b_1} + A \rho \right)] = -A(a_1 \rho + b_1 A \rho^2)
\]

For \( \rho(t) \geq 0 \), it suffices to show that

\[
(2.6) \quad \rho'(t) + a_1 \rho(t) \geq -b_1 A \rho^2(t).
\]

Also for \( \hat{v} = B \rho(t) \), we see from \((2.2)\) that it needs to satisfy the differential inequality

\[
B \rho'(t) \geq B \rho[a_2 - b_2 \left( \frac{a_1}{b_1} - \rho \right) - c_2 B \rho]
\]

and it suffices to show that

\[
(2.7) \quad \rho'(t) + b_2 \left( \frac{a_1}{b_1} - \frac{a_2}{b_2} \right) \rho(t) \geq (b_2 - c_2 B) \rho^2(t).
\]

Finally we look at \(\hat{u} = \frac{a_1}{b_1} - \rho(t)\). Again from \((2.2)\), it needs to satisfy the differential inequality

\[
-\rho'(t) \leq \left( \frac{a_1}{b_1} - \rho \right)[a_1 - b_1 \left( \frac{a_1}{b_1} - \rho \right) - c_1 B \rho],
\]

or equivalently,

\[
\rho'(t) \geq \left( \frac{a_1}{b_1} - \rho \right)(c_1 B - b_1) \rho.
\]

For \( \rho(t) \geq 0 \) it suffices to show that

\[
(2.8) \quad \rho'(t) + \frac{a_1}{b_1} (b_1 - c_1 B) \rho(t) \geq (b_1 - c_1 B) \rho^2(t).
\]

From hypotheses (H1) and (H3) we observe the fact that

\[
a_1 > \frac{a_1}{b_1} - a_2 = b_2 \frac{a_1}{b_1} - \frac{a_2}{b_2} = \alpha > 0
\]

Setting \( \frac{a_1}{b_1}(b_1 - c_1 B) = \alpha \) leads to the choice of

\[
(2.9) \quad B = \frac{b_1 b_2}{a_1 c_1} \left( \frac{a_1 + a_2}{b_2} - \frac{a_1}{b_1} \right) > 0.
\]
From the hypotheses (H3), one can obtain that

$$(b_1 - c_1B) - (b_2 - c_2B) = (b_2 - \frac{a_2b_1}{a_1}) - (b_2 - c_2B)$$

$$= \frac{b_1c_2}{a_1c_1}(a_1 + a_2) - \frac{b_2c_2}{c_1} - \frac{a_2b_1}{a_1}$$

$$= \frac{b_1c_2}{c_1}[1 + \frac{a_2}{a_1} - \frac{b_2}{b_1} - \frac{a_2c_1}{a_1c_2}] \geq 0.$$ 

Noting that $b_1 - c_1B = \frac{b_1b_2(a_1 - a_2)}{a_1} = \beta$, we can now conclude that the three differential inequalities $(2.6)-(2.8)$ will all be satisfied if the function $\rho(t)$ is a positive solution of the differential equation

$$(2.10) \quad \rho'(t) + \alpha\rho(t) = \beta\rho^2(t).$$

This leads to the function $\rho(t)$ given in $(2.4)$ with $\rho(0) < \alpha/\beta = a_1/b_1$ and $\lim_{t \to \infty} \rho(t) = 0$. \hfill \Box

From the arbitrariness of constant $A$ in Theorem $(2.2)$, we then have the attraction region for the equilibrium $(a_1/b_1, 0)$. When the hypotheses (H1) and (H3) hold, for all the initial density functions $(u_0, v_0)$ in the rectangular area

$$(0, \infty) \times [0, \frac{b_2}{c_1}(\frac{a_1 + a_2}{b_2} - \frac{a_1}{b_1})],$$

the solution $(u, v)$ of the system $(1.1)-(2.1)$ converges to the equilibrium $(a_1/b_1, 0)$ uniformly on $\mathbb{R}$ as $t \to \infty$ with the rate $e^{-\alpha t}$.

In the meantime, by adding hypothesis (H2), we can quickly find that the equilibriums $(0, a_2/c_2)$ and $(0, 0)$ are both unstable. For this purpose we construct a pair of upper-lower solutions

$$(\hat{u}, \hat{v}) = (M, a_2/c_2 - \rho(t)) \quad \text{and} \quad (\check{u}, \check{v}) = (A\rho(t), 0),$$

where $M \geq a_1/b_1$ is a constant. Constant $A$ and function $\rho(t)$ will be determined later. The differential inequalities in $(2.2)$ are automatically satisfied by $\hat{u}$ and $\check{v}$. For $\hat{u}$ and $\check{v}$, the following relations need to hold:

$$\rho'(t) \leq (a_1 - \frac{a_2c_1}{c_2})\rho + (c_1 - Ab_1)\rho^2,$$

and

$$\rho'(t) \leq \frac{a_2}{c_2} - \rho)(c_2 - Ab_2)\rho.$$ 

The above inequalities are equivalent to

$$(2.11) \quad \rho'(t) - (a_1 - \frac{a_2c_1}{c_2})\rho \leq -(b_1A - c_1)\rho^2,$$

and

$$\rho'(t) - \frac{a_2}{c_2}(b_2A - c_2)\rho \leq -(b_2A - c_2)\rho^2.$$ 

Setting

$$a_1 - \frac{a_2c_1}{c_2} = \frac{a_2}{c_2}(b_2A - c_2),$$

from hypothesis (H2) we have

$$(2.12) \quad A = \frac{1}{b_2}(c_2 - c_1 + \frac{a_1c_2}{a_2}) > \frac{c_2}{b_2}.$$
From the hypotheses (H3), one can obtain that
\[
(b_1A - c_1) - (b_2A - c_2) = \frac{b_1}{b_2}(c_2 - c_1 + \frac{a_1c_2}{a_2}) - \frac{a_1c_2}{a_2} \\
= \frac{b_1c_2}{a_1c_1}(a_1 + a_2) - \frac{b_2c_2}{c_1} - \frac{a_2b_1}{a_1} \\
= \frac{a_1b_1c_2}{a_2b_2}(1 + \frac{a_2}{a_1} - \frac{b_2}{b_1} - \frac{a_2c_1}{a_1c_2}) \geq 0.
\]

Both the inequalities in (2.11) can be satisfied by choosing \(\rho(t)\) as the solution of the differential equation
\[
(2.13)\quad \rho'(t) - \gamma \rho = -\delta \rho^2
\]
where
\[
\gamma = a_1 - \frac{a_2c_1}{c_2} > 0 \quad \text{and} \quad \delta = b_1A - c_1 \geq b_2A - c_2 = \frac{a_1c_2}{a_2} - c_1 > 0.
\]

This results in the function
\[
(2.14)\quad \rho(t) = \frac{\gamma}{\delta + C e^{-\gamma t}}
\]
with an arbitrary constant \(C > 0\). For arbitrarily small \(\epsilon \) \(> 0\), one can always find a constnt \(C\) large enough such that \(\rho(0) = \gamma/(\delta + C) < \epsilon\). The fact that \(\lim_{t \to -\infty} \rho(t) = \gamma/\delta\) leads to the following theorem indicating that \((0, a_2/c_2)\) and \((0, 0)\) are both unstable.

**Theorem 2.3.** Let
\[
(2.15)\quad \gamma = a_1 - \frac{a_2c_1}{c_2}, \quad A = \frac{1}{b_2}(c_2 - c_1 + \frac{a_1c_2}{a_2}) \quad \text{and} \quad \delta = \frac{b_1}{b_2}(c_2 - c_1 + \frac{a_1c_2}{a_2}) - c_1.
\]
Assuming that the hypotheses (H1), (H2) and (H3) hold, we know that \(\gamma > 0\), \(A > c_2/b_2\) and \(\delta \geq \frac{a_1c_2}{a_2} - c_1 > 0\). For any arbitrarily small \(\epsilon\) with \(0 < \epsilon < \min\{A\gamma/\delta, \gamma/\delta\}\), if the initial functions \((u_0, v_0)\) satisfies \(u_0(x) \geq \epsilon\) and \(0 \leq v_0(x) \leq a_2/c_2 - \epsilon\) on \(\mathbb{R}\), then the solution \((u(x, t), v(x, t))\) of (1.1) and (2.1) satisfies
\[
(2.16)\quad \liminf_{t \to -\infty} u(x, t) \geq \frac{A\gamma}{\delta} \quad \text{and} \quad \limsup_{t \to -\infty} v(x, t) \leq \frac{a_2}{c_2} - \frac{\gamma}{\delta}.
\]

### 3. THE TRAVELING WAVES

In section 2 we showed system (1.1) has two unstable constant steady states: \((0, 0)\), \((0, a_2/c_2)\) and one asymptotically stable constant steady state \((\frac{\gamma}{\delta}, 0)\). We will show that there are traveling wave solutions of (1.1) having the form
\[
(3.1)\quad (u(x, t), v(x, t)) = (kw(\sqrt{\frac{a_1}{d}} x + ca_1t), qz(\sqrt{\frac{a_1}{d}} x + ca_1t)),
\]
and connecting the unstable state \((0, \frac{\gamma}{\delta})\) with \((\frac{\gamma}{\delta}, 0)\) as the variable \(\sqrt{\frac{a_1}{d}} x + ca_1t\) runs from \(-\infty\) to \(+\infty\). The constant \(c\) in (3.2) is the wave speed and the minimal speed is also called the critical wave speed. Throughout the rest of the paper, we assume \(d_1 = d_2 = d\).

To simplify notions, we introduce the following transformations to (1.1):


\[ r = a_1^{-1} c_1 q, \quad \epsilon_1 = a_1^{-1} a_2, \]

\[ b = b_2 b_1^{-1}, \quad \epsilon_2 = a_2^{-1} c_2 q - 1, \]

\[ k = a_1 b_1^{-1}, \quad \text{and} \quad q \text{ is a constant satisfying } a_2 c_2^{-1} < q < a_1 c_1^{-1}. \]

Under transformations (3.1) and (3.2), system (1.1) is changed into

\[
\begin{cases}
-w_\xi + cw_\xi = w(1 - w - rz) \\
-z_\xi + cz_\xi = z(\epsilon_1 - bw - \epsilon_1(1 + \epsilon_2)z)
\end{cases}
\]

with the corresponding boundary conditions

\[
\begin{align*}
\lim_{\xi \to -\infty} (w(\xi), z(\xi)) &= (0, \frac{1}{1 + \epsilon_2}), \\
\lim_{\xi \to \infty} (w(\xi), z(\xi)) &= (1, 0).
\end{align*}
\]

where \( \xi = \sqrt{\frac{a}{2}} x + ca_1 t \) in (3.3), (3.4) for \( x \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \).

We further introduce the transformations

\[ u_1(\xi) = w(\xi), \quad u_2(\xi) = \frac{1}{1 + \epsilon_2} - z(\xi) \]

to change system (3.3) into the following monotone (cooperative) system:

\[
\begin{cases}
-(u_1)_\xi + c(u_1)_\xi = u_1(\frac{1 + \epsilon_2 - r}{1 + \epsilon_2} - u_1 + ru_2), \\
-(u_2)_\xi + c(u_2)_\xi = (\frac{1}{1 + \epsilon_2} - u_2)(bu_1 - \epsilon_1(1 + \epsilon_2)u_2)
\end{cases}
\]

with boundary conditions

\[
\begin{align*}
\lim_{\xi \to -\infty} (w(\xi), z(\xi)) &= (0, 0), \\
\lim_{\xi \to \infty} (w(\xi), z(\xi)) &= (1, \frac{1}{1 + \epsilon_2}).
\end{align*}
\]

**Remark 3.1.** Note that from hypotheses [H1]-[H3] and relations (3.2), we have the following inequalities:

\[ 0 < \epsilon_1 < b, \quad 0 < r < 1, \quad \epsilon_2 > 0 \quad \text{and} \quad 1 - \frac{r}{1 + \epsilon_2} > b - \epsilon_1 > 0. \]

Before showing the existence of the traveling wave solutions for (3.6) with boundary conditions (3.7), we first recall the following well known fact: (please see \[ \text{[4-Kolmogorov, 21-Sattinger]} \] for the proof)

Let a function \( f \) be a \( C^2 \) function on the interval \([0, \beta], \beta > 0\), with \( f > 0 \) on \((0, \beta), \) and \( f(0) = f(\beta) = 0, f'(0) = \alpha_1 > 0, f'(\beta) = -\beta_1 < 0.\)
Lemma 3.1. Corresponding to every \( c \geq 2\sqrt{\alpha_1} \), the boundary value problem

\[
\begin{align*}
\omega''(\xi) - c\omega'(\xi) + f(\omega(\xi)) &= 0, \\
\omega(-\infty) &= 0, \quad \omega(+\infty) = b.
\end{align*}
\tag{3.8}
\]

has a unique monotonically increasing traveling wave solution \( \omega_c(\xi), \xi \in \mathbb{R} \), where the lower index denotes the dependence of the wave solution \( \omega \) on \( c \).

We next show the existence of the traveling wave solution for system (3.6)-(3.7).

Theorem 3.2. Let the parameters \( \epsilon_1, \epsilon_2, b \) and \( r \) satisfy conditions in Remark 3.1, then corresponding to every \( c \geq \frac{2\sqrt{1-r}}{\epsilon_1 + \epsilon_2} \), system (3.6) has a monotone traveling wave solution satisfying the boundary condition (3.7). (Recall hypotheses \([H1]-[H3]\) imply all the conditions in Remark 3.1 are valid.)

Proof. The proof will be done by monotone iterating a pair of smooth upper- and lower-solutions. We first construct a twice differentiable smooth upper-solutions. According to lemma 3.1, for every \( c \geq \frac{2\sqrt{1-r}}{\epsilon_1 + \epsilon_2} \), there is correspondingly a monotonically increasing \( C^2 \) function \( \bar{Y}(\xi), \xi \in \mathbb{R} \) satisfying

\[
\begin{align*}
\bar{Y}_{\xi\xi} - c\bar{Y}_\xi + (1 - r_{/1+\epsilon_2})\bar{Y}(1 - \bar{Y}) &= 0, \\
\bar{Y}(-\infty) &= 0, \quad \bar{Y}(\infty) = 1.
\end{align*}
\tag{3.9}
\]

Define

\[
\bar{u}_1(\xi) = \bar{Y}(\xi), \quad \bar{u}_2(\xi) = \frac{1}{1 + \epsilon_2}Y(\xi).
\tag{3.10}
\]

For \( 0 \leq \bar{u}_2(\xi) \leq \bar{u}_2(\xi) \), we readily verify that

\[
\begin{align*}
-\bar{u}_1''(\xi) + c\bar{u}_1'(\xi) - \bar{u}_1\left(\frac{1 + \epsilon_2 - r}{1 + \epsilon_2} - \bar{u}_1 + ru_2\right) &= (1 - \frac{r}{1 + \epsilon_2})\bar{u}_1(1 - \bar{u}_1) - \bar{u}_1(1 - \frac{r}{1 + \epsilon_2} - \bar{u}_1 + ru_2) \\
&= \bar{u}_1 \left\{ (1 - \frac{r}{1 + \epsilon_2})(1 - \bar{u}_1) - 1 + \frac{r}{1 + \epsilon_2} + \bar{u}_1 - ru_2 \right\} \\
&= \bar{u}_1 \left\{ \frac{r}{1 + \epsilon_2} \bar{u}_1 - ru_2 \right\} \\
&\geq \bar{u}_1 \left\{ \frac{r}{1 + \epsilon_2} \bar{u}_1 - r\bar{u}_2 \right\} \equiv 0
\end{align*}
\tag{3.11}
\]
for all \(-\infty < \xi < \infty\). For \(0 \leq u_1(\xi) \leq \bar{u}_1(\xi)\), one verifies

\[
-\bar{u}''_2(\xi) + c\bar{u}'_2(\xi) - \left(\frac{1}{1 + \epsilon_2} - \bar{u}_2\right)(bu_1 - \epsilon_1(1 + \epsilon_2)\bar{u}_2) = \frac{1}{1 + \epsilon_2} \left\{-Y'' + cY' - (1 - Y)(bu_1 - \epsilon_1 Y)\right\}
\]

(3.12)

\[
\geq \frac{1}{1 + \epsilon_2} (1 - Y) \left\{(1 - \frac{r}{1 + \epsilon_2})Y + (1 - Y)(\epsilon_1 Y - bu_1)\right\}
\]

\[
= \frac{1}{1 + \epsilon_2} (1 - Y) \left\{1 - \frac{r}{1 + \epsilon_2} + \epsilon_1 - b\right\} \geq 0.
\]

The last inequality is true provided \(b < 1 - \frac{r}{1 + \epsilon_2} + \epsilon_1\), which is valid due to hypothesis H. It is also straightforward to verify that \((\bar{u}_1, \bar{u}_2)\) satisfies the boundary conditions (3.7).

We next construct a twice continuously differentiable lower solution for the system (3.6)-(3.7). Let the function \(Z(\xi), \xi \in \mathbb{R}\) be the solution of

\[
\begin{cases}
Z_{\xi\xi} - cZ_\xi + \left(1 - \frac{r}{1 + \epsilon_2}\right)Z(1 - \frac{1 - rl}{1 + \epsilon_2} Z) = 0, \\
Z(-\infty) = 0, \quad Z(\infty) = \frac{1 - rl}{1 + \epsilon_2}.
\end{cases}
\]

(3.13)

Here \(l\) is some number in the interval \((0, 1)\) to be determined. One can readily verify that the solutions of (3.9) and (3.13) are related by the following

\[
Z(\xi) = \frac{1 - \frac{r}{1 + \epsilon_2}}{1 - \frac{lr}{1 + \epsilon_2}} Y(\xi), \quad \xi \in \mathbb{R}.
\]

(3.14)

Since \(0 < l < 1\), we have

\[
Z(\xi) < Y(\xi), \quad \xi \in \mathbb{R}.
\]

(3.15)

We define a lower solution of (3.6), (3.7) by setting

\[
\tilde{u}_1 = Z, \quad \tilde{u}_2 = \frac{l}{1 + \epsilon_2} Z,
\]

(3.16)

where \(l \in (0, 1)\) is to be determined. We readily verify that they satisfy

\[
-\tilde{u}''_1(\xi) + c\tilde{u}'_1(\xi) - \tilde{u}_1(\frac{1 + \epsilon_2 - r}{1 + \epsilon_2} - \tilde{u}_1 + r\tilde{u}_2) = Z \left\{(1 - \frac{r}{1 + \epsilon_2}) - (1 - \frac{lr}{1 + \epsilon_2})Z - \left(\frac{1 + \epsilon_2 - r}{1 + \epsilon_2} + Z - \frac{rl}{1 + \epsilon_2} Z\right)\right\} = 0.
\]

Moreover, we have
where

\[ \lim \]

Again, by the limit and comparison argument in the proof of Theorem 3.6 in [25-Wu], we obtain for \( i \)

\[ \text{From the comparison argument with } \bar{\rho} \]

\[ \text{can apply the monotone iteration methods provided in [25-Wu] or [03-Boumenir] to derive the conclusion of this Theorem. Here we only sketch the ideas.} \]

\[ \text{The last inequality is valid provided that} \]

\[ 0 < l \leq \min\{1, \frac{b}{1 + \epsilon_1 - \frac{r}{1 + \epsilon_2}}\}. \]

Also by the limiting boundary conditions of (3.13) we see \((\tilde{u}_1, \tilde{u}_2)(-\infty) = (0, 0)\) and \((\tilde{u}_1, \tilde{u}_2)(+\infty) = \left(\frac{l + \epsilon_2 + r}{1 + \epsilon_2}, \frac{l(1 + \epsilon_2 + r)}{1 + \epsilon_2}\right)\). Inequality (3.17) along with (3.18) show that \((\tilde{u}_1, \tilde{u}_2)\) consists of a pair of lower-solutions for system (3.16), (3.17).

Noting that such constructed upper- and lower-solution pairs are ordered. We can apply the monotone iteration methods provided in [25-Wu] or [03-Boumenir] to derive the conclusion of this Theorem. Here we only sketch the ideas.

Following the notions in [25-Wu], we write \( \beta = \text{diag.}(0, 0), K = (K_1, K_2) := (1, 1) \), lower-solution \( \tilde{\rho}(\xi) = (Z(\xi), \frac{1}{1 + \epsilon_2} Z(\xi)) \), and upper-solution \( \bar{\rho}(\xi) = (Y(\xi), \frac{1}{1 + \epsilon_2} Y(\xi)) \), \( \xi \in \mathbb{R} \), as described above. As explained below, a slight variant of Theorem 3.6 or Theorem 3.6' in [25-Wu] is needed, because \((F_1, F_2)\) defined in (3.24) below has an additional zero at \((0, \frac{1}{1 + \epsilon_2})\) between \(0 := (0, 0)\) and \(K\). By means of the iterative procedure in the proof of Theorem 3.6 in [25-Wu], we first obtain a solution of (3.16) \( \phi(\xi) := (u_1(\xi), u_2(\xi)) \), satisfying the inequality

\[ \tilde{\rho}(\xi) \leq \phi(\xi) \leq \bar{\rho}(\xi), \quad \xi \in \mathbb{R}. \]

From the comparison argument with \( \tilde{\rho}(\xi) \) in the proof of Theorem 3.6 in [25-Wu], we have

\[ \lim_{\xi \to -\infty} u_1(\xi) = \lim_{\xi \to -\infty} u_2(\xi) = 0. \]

Again, by the limit and comparison argument in the proof of Theorem 3.6 in [25-Wu], we obtain for \( i = 1, 2 \),

\[ \lim_{\xi \to -\infty} u_i(\xi) = Q_i, \quad F_i(Q_1, Q_2) = 0, \]

where

\[ F_1(\rho_1, \rho_2) = \rho_1(1 + \epsilon_2 - r - \rho_1 + r \rho_2), \]

\[ F_2(\rho_1, \rho_2) = \left( \frac{1}{1 + \epsilon_2} - \rho_2 \right) (b \rho_1 - \epsilon_1 (1 + \epsilon_2) \rho_2); \]
and
\[ (3.23) \]
\[ 0 < \lim_{\xi \to \infty} Z(\xi) \leq Q_1 \leq K_1 = 1, \quad 0 < \lim_{\xi \to \infty} \frac{\xi}{1 + \epsilon_2} Z(\xi) \leq Q_2 \leq K_2 = \frac{1}{1 + \epsilon_2}. \]

We then deduce from condition (H) and \((3.20), (3.21), (3.23)\) that we must have
\[ (3.24) \]
\[ Q_1 = K_1 = 1, \quad Q_2 = K_2 = \frac{1}{1 + \epsilon_2}. \]

**Remark 3.2.** One can translate Theorem 3.2 into the following: Assuming hypotheses \([H1], [H2], [H3]\), then for any \(c \geq 2 \sqrt{1 - \frac{a_2 a_1}{\epsilon_1}}\), system \((1.1)\) has a traveling wave solution connecting \((0, a_2)\) with \((a_1 b_1, 0)\) as the variable \(\sqrt{\frac{2}{a_1}} x + ct\) running from \(-\infty\) to \(+\infty\).

**Remark 3.3.** Theorem 3.2 does not insure the strict monotonicity of the resulting traveling wave solutions, as the iteration is only applied for the upper-solution, and the lower solution is served as a nonzero barrier so that the iteration limit does not converge to zero.

To further study the asymptotics of the traveling wave solutions as obtained in Theorem 3.2, we shall need the following Lemma concerning the scalar problem \((3.8)\).

**Lemma 3.3.** The solution \(w_c(\xi)\) to \((3.8)\), described in Lemma 3.1, has the following asymptotic behaviors:
1. Corresponding to the wave speed \(c > 2 \sqrt{\alpha_1}\),
\[ (3.25) \]
\[ \begin{align*}
\omega_c(\xi) &= a_c e^{c \sqrt{\alpha_1} \xi} + o(e^{c \sqrt{\alpha_1} \xi}), \quad \text{as} \ \xi \to -\infty \\
\omega_c(\xi) &= \beta - b_c e^{-c \sqrt{\alpha_1} \xi} + o(e^{-c \sqrt{\alpha_1} \xi}), \quad \text{as} \ \xi \to +\infty,
\end{align*} \]
where \(a_c\) and \(b_c\) are positive constants;
2. Corresponding to minimal wave speed \(c = 2 \sqrt{\alpha_1}\),
\[ (3.26) \]
\[ \begin{align*}
\omega_c(\xi) &= (a_c + d_c \xi)e^{\sqrt{\alpha_1} \xi} + o(\xi e^{\sqrt{\alpha_1} \xi}), \quad \text{as} \ \xi \to -\infty \\
\omega_c(\xi) &= \beta - b_c e^{-c \sqrt{\alpha_1} \xi} + o(e^{-c \sqrt{\alpha_1} \xi}), \quad \text{as} \ \xi \to +\infty;
\end{align*} \]
where the constant \(d_c < 0, b_c > 0\) and \(a_c \in \mathbb{R}\).

**Proof.** The conclusion follows [21-Sattinger], [23-Thiery] with slight changes. \(\square\)

Based on Lemma 3.3, we study the asymptotic behaviors of the traveling wave solutions of system \((3.6), (3.7)\) at infinities.

**Corollary 3.4.** Assume hypotheses \([H1] to [H3]\), and thus all the conditions in Remark 3.1 are satisfied. Let \(\alpha = 1 - \frac{\epsilon_1}{1 + \epsilon_2}\), then the traveling wave solutions \((u_1(\xi), u_2(\xi))\) of system \((3.6), (3.7)\) as obtained in Theorem 3.2 have the following asymptotic behaviors:
1. Corresponding to each wave speed $c > 2\sqrt{\alpha}$, the traveling wave solution $(u_1(\xi), u_2(\xi))$ satisfies

$$
(3.27) \quad \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}} + o(e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}})
$$

as $\xi \to -\infty$; and

$$
(3.28) \quad \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{1 + \epsilon_2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}} + o(e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}})
$$

as $\xi \to +\infty$, where $A_1$, $A_2$, $\tilde{A}_1$, $\tilde{A}_2$ are positive constants;

2. Corresponding to the wave speed $c_{\text{critical}} = 2\sqrt{\alpha}$, the traveling wave solution $(u_1(\xi), u_2(\xi))$ satisfies

$$
(3.29) \quad \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} A_{11c} + A_{12c} \xi \\ A_{21c} + A_{22c} \xi \end{pmatrix} e^{\sqrt{\alpha} \xi} + o(\xi e^{\sqrt{\alpha} \xi})
$$

as $\xi \to -\infty$; and

$$
(3.30) \quad \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{1 + \epsilon_2} \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{22} \end{pmatrix} e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}} + o(e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}})
$$

as $\xi \to +\infty$, where $A_{12c}, A_{22c} < 0$, $A_{11c}, A_{21c} \in \mathbb{R}$ and $\tilde{A}_{11}, \tilde{A}_{22} > 0$.

**Proof.** For $c = 2\sqrt{1 - \frac{p}{1 + \epsilon_2}}$; according to Lemma 3.3 the upper-solution $(\bar{u}_1, \bar{u}_2)$ and the lower-solution $(\tilde{u}_1, \tilde{u}_2)$ as defined in (3.10), (3.10) have the following respective asymptotic behaviors at $-\infty$,

$$
\begin{pmatrix} \bar{u}_1(\xi) \\ \bar{u}_2(\xi) \end{pmatrix} = \begin{pmatrix} A_{11c} + \tilde{A}_{12c} \xi \\ A_{21c} + \tilde{A}_{22c} \xi \end{pmatrix} e^{\sqrt{\alpha} \xi} + o(\xi e^{\sqrt{\alpha} \xi}),
$$

and

$$
\begin{pmatrix} \tilde{u}_1(\xi) \\ \tilde{u}_2(\xi) \end{pmatrix} = \begin{pmatrix} \tilde{B}_{11c} + \tilde{B}_{12c} \xi \\ \tilde{B}_{21c} + \tilde{B}_{22c} \xi \end{pmatrix} e^{\sqrt{\alpha} \xi} + o(\xi e^{\sqrt{\alpha} \xi});
$$

While with wave speed $c > 2\sqrt{1 - \frac{p}{1 + \epsilon_2}}$,

$$
\begin{pmatrix} \bar{u}_1(\xi) \\ \bar{u}_2(\xi) \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}} + o(e^{\frac{-c}{2} \sqrt{\alpha + 4\alpha \xi}}),
$$

and
\[
\begin{pmatrix}
\tilde{u}_1(\xi) \\
\tilde{u}_2(\xi)
\end{pmatrix} = \begin{pmatrix}
\tilde{A}_1 \\
\tilde{A}_2
\end{pmatrix} e^{\sqrt{\frac{c^2}{4} - 4\alpha^2} \xi} + o\left(e^{\sqrt{\frac{c^2}{4} - 4\alpha^2} \xi}\right).
\]

where \(\tilde{A}_i, \tilde{A}_i, \bar{A}_i, \tilde{B}_i c_i = 1, 2\) are positive constants and \(\bar{A}_{1c}, \tilde{B}_{1c}\) are constants.

Noting the upper- and lower-solutions have the same asymptotic growth rate at \(-\infty\), we immediately have (3.27) and (3.29) via comparison.

We next derive the asymptotic behaviors of the traveling wave solutions of (3.6), (3.7) at \(+\infty\). Since the traveling wave solution \((u_1(\xi), u_2(\xi))^T\) monotonically approaches the steady state \((1, 1 + \epsilon_2)\) as \(\xi \to +\infty\), by letting \((w_1(\xi), w_2(\xi))\) be the derivative of the traveling wave solution, we have \((w_1(+\infty), w_2(+\infty)) = (0, 0)\).

We linearize system (3.6) about the traveling wave solution \((u_1(\xi), u_2(\xi))\) to obtain

\[
\begin{align*}
\phi_{\xi\xi} - c\phi_{\xi} + (1 - r + 2u_1 + ru_2)\phi + ru_1\psi &= 0, \\
\psi_{\xi\xi} - c\psi_{\xi} + b(\frac{1}{1+\epsilon_2} - u_2)\phi + [-\epsilon_1 - bu_1 + 2\epsilon_1(1 + \epsilon_2)u_2]\psi &= 0.
\end{align*}
\]

It is easy to see that \((w_1, w_2)\) solves (3.31).

The limit system of (3.31) at \(+\infty\) is a constant coefficient system, and is given by

\[
\begin{align*}
(\phi^+)_{\xi\xi} - c(\phi^+)_{\xi} - \phi^+ + r\psi^+ &= 0, \\
(\psi^+)_{\xi\xi} - c(\psi^+)_{\xi} - (\epsilon_1 - b)\psi^+ &= 0.
\end{align*}
\]

The exponential growth rates of the traveling wave solutions of (3.6)-(3.7) are determined by those of the solutions of (3.32). The justification is as follows: first we note that (3.32) admits exponential dichotomy. By the roughness of exponential dichotomy, solutions of (3.31) grow/decay exponentially (possibly with a different exponential rate) \[04-Coddington, 20-Sandstede\]. Since the derivative \((w_1, w_2)\) of the traveling wave solution solves (3.31), the traveling wave solutions of (3.6) approach exponentially to the steady state \((1, \frac{1}{1+\epsilon_2})\).

To find out the exact asymptotic rates of the solutions of (3.31), we first change (3.31), (3.32) into the first order systems. Letting \(\phi_{\xi} = \phi_1, \psi_{\xi} = \psi_1, \phi_{\xi}^+ = \phi_1^+, \psi_{\xi}^+ = \psi_1^+\), we have

\[
\frac{d}{d\xi} \begin{pmatrix}
\phi \\
\phi_1 \\
\psi \\
\psi_1
\end{pmatrix} = \mathcal{R}^+ \begin{pmatrix}
\phi \\
\phi_1 \\
\psi \\
\psi_1
\end{pmatrix}
\]

and

\[
\frac{d}{d\xi} \begin{pmatrix}
\phi^+ \\
\phi_1^+ \\
\psi^+ \\
\psi_1^+
\end{pmatrix} = \mathcal{R}^\infty \begin{pmatrix}
\phi^+ \\
\phi_1^+ \\
\psi^+ \\
\psi_1^+
\end{pmatrix}.
\]
It is easy to check that the matrix \( R \) implies that the asymptotic exponential rates of the solutions of (3.31) are the same as those of the solutions of (3.32). We search for the solutions of (3.32) with zero limit at \(+\infty\). Owing to conditions in Remark 3.1, the second equation of (3.32) has general solution of the form:

\[
\psi^+(\xi) = c_{21} e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}} + c_{22} e^{\frac{\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}}
\]

with \( \frac{c - \sqrt{c^2 + 4(b - \epsilon)}}{2} < 0 \) and \( c_{21}, c_{22} \) two constants.

Substituting (3.33) into the first equation of (3.32), we then have

\[
(\phi^+)_{\xi} - c(\phi^+) = -r(c_{21} e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}} + c_{22} e^{\frac{\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}})
\]

with general solution of the form

\[
\phi^+(\xi) = c_{11} e^{\frac{c - \sqrt{c^2 + 4(b - \epsilon)}}{2} \xi} + c_{12} e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}} + c_{21} e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}} + c_{22} e^{\frac{\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}}
\]

Comparing (3.31), (3.32) and using the fact that \((w_1, w_2)\) is a solution of (3.31) with \((w_1(\infty), w_2(\infty)) = (0, 0)\), we deduce from (3.35) that as \( \xi \to +\infty \),

\[
w_1(\xi) = (\hat{c}_{12} + o(1)) e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}}{2} \xi} + (\hat{c}_{21} + o(1)) e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}}
\]

\[
w_2(\xi) = (\hat{c}_{21} + o(1)) e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}}{2} \xi}
\]

with \( \hat{c}_{21} \neq 0 \) and \( \hat{c}_{21} \neq 0 \). Since \( \frac{c - \sqrt{c^2 + 4(b - \epsilon)}}{2} < \frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2} < 0 \), by [H3], we may write

\[
w_1(\xi) = (\hat{c}_{21} + o(1)) e^{\frac{-\sqrt{c^2 + 4(b - \epsilon)}\xi}{2}}
\]

as \( \xi \to +\infty \).

We therefore conclude this proof by integrating \((w_1(\xi), w_2(\xi))^T\) from \( \xi \) to \(+\infty\) with \( \xi \) sufficiently large. \( \square \)
We next derive a corollary that is important to section [4].

**Corollary 3.5.** Assume the hypotheses of Corollary 3.4. For every wave speed with \( c \geq 2 \sqrt{1 - \frac{1}{1 + \epsilon_2}} \), the corresponding traveling wave solution \((u_1(\xi), u_2(\xi))^T\) of system (3.6), (3.7) obtained in Corollary 3.4 is a strict monotonic function for \( \xi \in \mathbb{R} \).

**Proof.** According to the monotone iteration (25-Wu), the traveling wave solution \((u_1(\xi), u_2(\xi))^T\) to (3.6), (3.7) is monotone, which implies that \((u'_1(\xi), u'_2(\xi))^T \geq 0\) for \( \xi \in \mathbb{R} \). By Corollary 3.4 one sees that \((u'_1(\pm \infty), u'_2(\pm \infty))^T = 0\). The monotonicity of system (3.6) and the maximum principle lead to the conclusion that \((u'_1(\xi), u'_2(\xi))^T > 0\) for \( \xi \in \mathbb{R} \). \(\square\)

We next show the uniqueness of the traveling wave solutions.

**Theorem 3.6.** Assume hypotheses [H1] to [H3]. The traveling wave solution to system (3.6)-3.7), obtained for each wave speed \( c \geq 2 \sqrt{1 - \frac{1}{1 + \epsilon_2}} \), with properties described in Corollary 3.4 and 3.5, is unique up to a translation of the origin.

**Proof.** We only prove the conclusion for traveling wave solutions with asymptotic behaviors (3.27) and (3.28), since other case can be proved similarly. Let \( U_1(\xi) \) and \( U_2(\xi) \) be traveling wave solutions of system (3.6)-3.7), with speed \( c > 2 \sqrt{\alpha} \) and properties described. There exist positive constants \( A_i, B_i, i = 1, 2, 3, 4 \) and a large number \( N > 0 \) such that for \( \xi < -N \),

\[
U_1(\xi) = \begin{pmatrix} (A_1 + o(1))e^{c-\sqrt{\alpha-4}\theta_{\xi}} \
(A_2 + o(1))e^{-c+\sqrt{\alpha-4}\theta_{\xi}} \end{pmatrix}
\]

and for \( \xi > N \),

\[
U_1(\xi) = \begin{pmatrix} 1 - (B_1 + o(1))e^{c-\sqrt{\alpha-4}(1-\epsilon_1)\theta_{\xi}} \
1 + \epsilon_2 - (B_2 + o(1))e^{-c+\sqrt{\alpha-4}(1-\epsilon_1)\theta_{\xi}} \end{pmatrix}
\]

\[
U_2(\xi) = \begin{pmatrix} 1 - (B_3 + o(1))e^{c-\sqrt{\alpha-4}(1+\epsilon_1)\theta_{\xi}} \
1 + \epsilon_2 - (B_4 + o(1))e^{-c+\sqrt{\alpha-4}(1+\epsilon_1)\theta_{\xi}} \end{pmatrix}
\]

The traveling wave solutions of system (3.6) are translation invariant, thus for any \( \theta > 0 \), \( U^\theta_1(\xi) := U_1(\xi + \theta) \) is also a traveling wave solution of (3.6). By (3.36) and (3.38), the solution \( U_1(\xi + \theta) \) has the following asymptotic behaviors:

\[
U^\theta_1(\xi) = \begin{pmatrix} (A_1 + o(1))e^{c+\sqrt{\alpha-4}\theta_{\xi}} \theta_{\xi} e^{c+\sqrt{\alpha-4}\theta_{\xi}} \
(A_2 + o(1))e^{c-\sqrt{\alpha-4}\theta_{\xi}} \theta_{\xi} e^{c-\sqrt{\alpha-4}\theta_{\xi}} \end{pmatrix}
\]
for $\xi \leq -N - \theta$;

$$
\begin{aligned}
U_1^\theta(\xi) &= \\
\begin{cases}
1 - (B_1 + o(1)) e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \xi} \\
1 + e_2 - (B_2 + o(1)) e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \xi}
\end{cases}
\end{aligned}
$$

for $\xi \geq N$.

It is clear that for $\theta$ large enough, we have

$$
\begin{aligned}
A_1 e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} &> A_3, \\
A_2 e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} &> A_4, \\
B_1 e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} &< B_3, \\
B_2 e^{-\sqrt{\frac{c^2}{e} + 4(1-\varepsilon^2)} \theta} &< B_4.
\end{aligned}
$$

Thus for some $N > 0$, formulas (3.42) - (3.46) imply that for $\theta$ large enough,

$$
U_1^\theta(\xi) > U_2(\xi)
$$

for $\xi \in (-\infty, -N]\cup[N, \infty)$. Here, the inequality $"\; > \;"$ in (3.40) is component-wise. We now consider system (3.6) on $[-N, +N]$. First, suppose $U_1^\theta(\xi) \geq U_2(\xi)$ on $[-N, +N]$, then the function $W(\xi) := U_1^\theta(\xi) - U_2(\xi) \geq 0$ and satisfies for some $\zeta \in (0,1)$:

$$
W'' - cW' + \left[ \frac{\partial E}{\partial u_1}(U_2 + \zeta_1(U_1^\theta - U_2)), \frac{\partial E}{\partial u_2}(U_2 + \zeta_1(U_1^\theta - U_2)) \right] = 0,
$$

for $\xi \in (-N, N)$, and $W(-N) > 0$, $W(+N) > 0$.

Since the above system is monotone, we can readily deduce by maximum principle that $W > 0$, on $[-N, N]$. Consequently, we have $U_1^\theta(\xi) > U_2(\xi)$ on $\mathbb{R}$ in this case.

Second, suppose there are some points in $(-N, N)$ such that $U_1^\theta(\xi) < U_2(\xi)$. We then increase $\theta$, that is shift $U_2^\theta(\xi)$ further left so that $U_2^\theta(-N) > U_2(-N)$, $U_2^\theta(N) > U_2(N)$. By the monotonicity of $U_1^\theta$ and $U_2$, we can find a $\bar{\theta} \in (0,2N)$ such that in the interval $(-N, N)$, we have $U_1^\theta(\xi + \bar{\theta}) > U_2(\xi)$. Shifting $U_1^\theta(\xi + \bar{\theta})$ back until one component of $U_1^\theta(\xi + \bar{\theta})$ first touches one component of $U_2(\xi)$ at some point $\bar{\xi} \in (-N, N)$. Then by maximum principle for that component again, we find that component of $U_1^\theta$ and $U_2$ are identically equal for all $\xi \in [-N, N]$ for a larger $\theta$ than the original one such that (3.40) holds. This is a contradiction. Therefore, we must have

$$
U_1^\theta(\xi) > U_2(\xi)
$$

for all $\xi \in R$, where $\theta$ is the one chosen by means of (3.42) - (3.46) as described above.

Now, decrease $\theta$ until one of the following situations happens.

1. There exists $\bar{\theta} \geq 0$, such that $U_1^\theta(\xi) \equiv U_2(\xi)$. In this case we have finished our proof.
2. For \( \bar{\theta} \geq 0 \), there exists \( \xi_1 \in \mathbb{R} \), such that one of the components of \( U\bar{\theta} \) and \( \tilde{U} \) are equal at the point \( \xi_1 \); and for all \( \xi \in \mathbb{R} \), we have \( U_{\bar{\theta}}(\xi) \geq U_2(\xi) \). We then consider the system (3.47) on \((-N,N)\) and \( \theta = \bar{\theta} \) in the definition for \( W \). To fix ideas, we suppose that the first component of \( U_{\bar{\theta}} \) and \( U_2 \) is equal at the point \( \xi_1 \). The maximum principle for this component implies that the first component of \( U_{\bar{\theta}}(\xi) \) is identically equal to that of \( U_2(\xi) \). Also, we readily obtain that for large \(+\xi\), the limiting equation for (3.47) is the same as (3.32). Since the first component of \( \tilde{W} \) is identically zero and the off diagonal limit coefficient \( r \) in the first equation in (3.32) is not equal to zero, we conclude from the first equation in (3.47) that the second component of \( \tilde{W} \) must vanish for all large \( \xi \). By the maximum principle for the second equation, we conclude that the second component of \( \tilde{W} \) is also identically zero for all \( \xi \in \mathbb{R} \). Similarly, we next consider the case that the second component of \( U_{\bar{\theta}} \) and \( U_2 \) are equal at the point \( \xi_1 \). We first obtain the limiting equation of (3.47) for large \(-\xi\). The off diagonal limit coefficient of the second equation will be \( \frac{h}{1+\tau_2} \neq 0 \). We first deduce that the second component of \( \tilde{W} \) is identically zero, and then the first component must also be identically zero for \( \xi \in \mathbb{R} \).

Consequently, in either situation, there exists a \( \bar{\theta} \geq 0 \), such that

\[
U_{1,\bar{\theta}}(\xi) \equiv U_2(\xi).
\]

for all \( \xi \in \mathbb{R} \).

**Theorem 3.7.** Assume hypotheses [H1] to [H3]. System (3.6)–(3.7) does not have strict monotonic traveling wave solution tending to \((0,0)^T\) as \( \xi \to -\infty \) for \( c < 2\sqrt{\alpha} \). Here, \( \alpha = 1 - \frac{r}{1+\tau_2} \).

**Proof.** Suppose there is a constant \( c \) with \( 0 < c < 2\sqrt{\alpha} \) and a corresponding solution \( V(\xi) = (v_1(\xi), v_2(\xi))^T \) of (3.7) tending to \((0,0)^T\) as \( \xi \to -\infty \). Similar to the proof of Corollary 3.4, we can deduce by integrating the asymptotic approximation of its derivative that the asymptotic behaviors of \( V(\xi) \) at \(-\infty \) must be of the form:

\[
\begin{pmatrix}
v_1(\xi) \\
v_2(\xi)
\end{pmatrix} = \begin{pmatrix} A_s \\ B_s
\end{pmatrix} e^{-\sqrt{\alpha\alpha/c}} - \xi + \begin{pmatrix} A_s \\ B_s
\end{pmatrix} e^{\sqrt{\alpha\alpha/c}} - \xi + h.o.t.,
\]

where \((A_s, B_s)^T\) and \((A_s, B_s)^T\) can not be both zero, and h.o.t. is the short notation for the higher order terms. The condition \( 0 < c < 2\sqrt{\alpha} \) implies that \( V(\xi) \) is oscillating. This says that such solution of (3.6) with \( c < 2\sqrt{\alpha} \) is not monotone.

\[\square\]

4. **STABILITY OF THE TRAVELING WAVES WITH NON-CRITICAL SPEEDS**

In this section, we always hypotheses [H1] to [H3] for system (1.1); thus all the conditions in Remark 3.1 are satisfied for (3.6) and subsequent systems. We first show that the traveling wave solutions with the non-critical speed obtained in Theorem 3.2 is unstable in the space of continuous function \( C(\mathbb{R}) \times C(\mathbb{R}) \) (Please see definitions below). This motivates us to investigate the stability in the “smaller” exponentially weighted Banach spaces. We will concentrate on the stability of the traveling waves with non-critical wave speeds.

The following set up of the problem is standard: Let \( U = (u,v)^T \), \( F(U) = (u_1(\frac{1+\tau_2}{1+\tau_2} - u_1 + ru_2), (\frac{1}{1+\tau_2} - u_2)(bu_1 - c_1(1+\tau_2)u_2))^T \) and write system (3.6)
in the moving coordinates $\xi = x + ct$, $c > 0$. In terms of $(\xi, t)$ variable, (3.6) is changed into
\begin{equation}
(4.1) \begin{cases}
U_t &= U_{\xi\xi} - cU_\xi + F(U), \\
U(\xi, 0) &= \bar{U},
\end{cases}
\end{equation}
where $\bar{U}$ the initial function. Let $U^*(\xi) = (u^*(\xi), v^*(\xi))^T$, $\xi = x + c^* t$ be the traveling wave solution of (3.6) with wave speed $c^* > 2\sqrt{\alpha}$. It is easy to see that $U^*(\xi)$ is a non-constant steady state of the system (4.1). We consider the perturbation of this steady state solution. By letting $U(\xi, t) = U^*(\xi) + V(\xi, t)$, we obtain the system for the perturbation function $V$:
\begin{equation}
(4.2) \begin{cases}
V_t &= LV + N(V, U^*), \\
V(\xi, 0) &= \bar{U} - U^*(\xi),
\end{cases}
\end{equation}
where
\begin{equation}
(4.3) LV = V_{\xi\xi} - c^* V_\xi + \frac{\partial F}{\partial U}(U^*)V
\end{equation}
is a linear operator, and
\begin{equation}
(4.4) N(V, U^*) = F(U^* + V) - F(U^*) - \frac{\partial F}{\partial U}(U^*)V
\end{equation}
is a nonlinear operator.

The stability of the traveling wave solution $U^*(\xi)$ in certain Banach space is determined by the location of the spectrum, $\sigma(L)$, of $L$.

Let
\begin{equation}
\sigma_p(L) = \{ \lambda \in \sigma(L) \mid \lambda \text{ is an eigenvalue of } L \},
\end{equation}
and $\sigma_e(L)$ be the essential spectrum of $L$, which are points in $\sigma(L)$ outside $\sigma_p(L) \cap \{ \text{isolated eigenvalues of } L \text{ with finite multiplicity} \}$. Note that $\sigma_e(L)$ includes the continuous spectrum of $L$. [01-Alexander, 06-Henry, 24-Volpert]. Let $C(\mathbb{R})$ be the space of all continuous functions on the real line and $C_0(\mathbb{R})$ be its subspace
\begin{equation}
C_0(\mathbb{R}) := \{ U(\xi) \in C(\mathbb{R}) \times C(\mathbb{R}) \mid \lim_{|\xi| \to +\infty} U(\xi) = 0 \}
\end{equation}
along with norm
\begin{equation}
\| U \|_{C_0(\mathbb{R})} := \sup_{\xi \in \mathbb{R}} \| U(\xi) \| .
\end{equation}
We also need the following weighted Banach spaces: for non-negative numbers $\sigma_1$, $\sigma_2$, the space $C_{\sigma_1, \sigma_2}$ is defined as:
\begin{equation}
C_{\sigma_1, \sigma_2} = \{ U(\xi) \in C_0(\mathbb{R}) \mid U(\xi)(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}) \in C_0(\mathbb{R}) \},
\end{equation}
on which we define the norm
\begin{equation}
\| U \|_{C_{\sigma_1, \sigma_2}} = \sup_{\xi \in \mathbb{R}} \| U(\xi)(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}) \| .
\end{equation}
Similarly, we can define $C_{\sigma_1, \sigma_2}^{(i)}$, $i = 1, 2, ...$ as well, for example:
\begin{equation}
C_{\sigma_1, \sigma_2}^{(2)} = \{ U \mid U(\xi), U''(\xi), U''(\xi) \in C_{\sigma_1, \sigma_2} \text{ and } \xi \in \mathbb{R} \}
\end{equation}
with norm
\[ \|U\|_{C^{2,2}} = \sup_{\xi \in \mathbb{R}} \sum_{i=0}^{2} \left\| \left( e^{\sigma_1 \xi} + e^{-\sigma_2 \xi} \right) \frac{d^i U(\xi)}{d\xi^i} \right\|. \]

It can be readily verified that these spaces are Banach spaces.

**Theorem 4.1.** Assume \([H1]\) to \([H3]\), and let \(U^*(\xi) = (u_1(\xi), u_2(\xi))^T\) be the traveling wave solution of (3.6) or (4.1) with wave speed \(c^* > 2\sqrt{\alpha}\), \(\alpha = 1 - \frac{r}{1+\epsilon_2}\) as described in Corollary 3.4. Then \(U^*(\xi)\) is unstable with initial conditions in \(C_0\).

**Proof.** This theorem holds for the traveling wave solutions with critical and non-critical wave speeds. We need to prove the trivial solution of (4.2) is unstable. Thus, it suffices to show that in the space \(C_0\) the operator \(L\) in (4.3) has essential spectrum with positive real part. As is well known ([06-Henry], [24-Volpert]) the location of the continuous spectrum of the operator \(L\) is bounded by the spectrum of \(L\) at \(\pm \infty\), which we denote by \(L^+\) and \(L^-\) respectively. More precisely, we let

\[
L^+ V = V_{\xi\xi} - c^* V_{\xi} + \frac{\partial F}{\partial U}(U^*_+)^T V
\]

(4.5)

\[
= V_{\xi\xi} - c^* V_{\xi} + \begin{bmatrix}
-1 & r \\
0 & \epsilon_1 - b
\end{bmatrix} V,
\]

\[
L^- V = V_{\xi\xi} - c^* V_{\xi} + \frac{\partial F}{\partial U}(U^*_-)^T V
\]

(4.6)

\[
= V_{\xi\xi} - c^* V_{\xi} + \begin{bmatrix}
1 - \frac{r}{1+\epsilon_2} & 0 \\
\frac{b}{1+\epsilon_2} & -\epsilon_1
\end{bmatrix} V.
\]

Here, \(U^*_\pm\) respectively denote the limit of \(U^*(\xi)\) as \(\xi \to \pm \infty\).

Now consider the equation

\[
\frac{\partial V}{\partial t} = L^+ V.
\]

Following [24-Volpert] and [06-Henry], we replace \(V\) by \(e^{(\lambda t + i\zeta \xi)} I\), where \(I\) is an identity matrix and \(\lambda\) is a complex number and \(\zeta\) is real. We then have

\[
e^{(\lambda t + i\zeta \xi)} (-\zeta^2 I - c^* \zeta i I + \frac{\partial F}{\partial U}(U^*_+)^T - \lambda I) = 0.
\]

(4.7)

The spectrum of the operator \(L^+\) consists of curves given by:

\[
\det(-\zeta^2 I - c^* \zeta i I + \frac{\partial F}{\partial U}(U^*_+)^T - \lambda I) = 0.
\]

(4.8)

Solving (4.8), we have

\[
-\zeta^2 - c^* \zeta i - 1 - \lambda = 0,
\]

(4.9)

or

\[
-\zeta^2 - c^* \zeta i + \epsilon_1 - b - \lambda = 0.
\]

(4.10)

Letting \(\lambda = x + yi\) for \(x, y \in \mathbb{R}\), then by (4.9) we have

\[
x = -\frac{y^2}{(c^*)^2} - 1,
\]

(4.11)
or by (4.10),
\[
\frac{y^2}{(c^*)^2} + 1 = \epsilon_1 - b. 
\]
Similarly, the spectrum of \(L^-\) consists of curves:
\[
\frac{y^2}{(c^*)^2} + 1 - \frac{r}{1 + \epsilon_2}, 
\]
or
\[
\frac{y^2}{(c^*)^2} - \epsilon_1 
\]
in the complex plane. Consequently, by theory described in [06-Henry], we have
\[
\max \Re \sigma_e(L) \geq \max \{-1, \epsilon_1 - b, 1 - \frac{r}{1 + \epsilon_2}, -\epsilon_1\} = 1 - \frac{r}{1 + \epsilon_2} > 0. 
\]
Hence, by [06-Henry] again, the traveling wave solution \(U^*(\xi)\) of (4.1) is essentially unstable in \(C_0(\mathbb{R})\).

In order to obtain stability for the traveling solution \(U^*\), we will restrict the initial conditions and the operator \(L\) to a "smaller" Banach space \(C_{\sigma_1,\sigma_2}\) with \(\sigma_1 \geq 0, \sigma_2 \geq 0\) and \(\sigma_1^2 + \sigma_2^2 \neq 0\). To relate the operator \(L\) in \(C_0(\mathbb{R})\) to an equivalent operator in \(C_{\sigma_1,\sigma_2}\), we introduce the mapping \(T: C_{\sigma_1,\sigma_2} \rightarrow C_0(\mathbb{R})\) as follows:
\[
TV := (e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})V. 
\]
\(T\) is thus linear, bounded and has a bounded inverse \(T^{-1} : C_0 \rightarrow C_{\sigma_1,\sigma_2}\) with \(T^{-1}V = (e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})^{-1}V\). Consider operator
\[
\tilde{L} : V = TL^{-1}V. 
\]
One readily sees that \(\tilde{L}\) is a linear operator with domain \(C^{(2)}(\mathbb{R}) \times C^{(2)}(\mathbb{R})\). By relation (4.16), considering \(L\) in \(C_{\sigma_1,\sigma_2}\) is equivalent to considering \(\tilde{L}\) in \(C_0(\mathbb{R})\), which is:
\[
\tilde{L}V = V_{\xi\xi} - (2g_1 + c^*)V_{\xi} + (2g_1^2 - g_2 + c^* g_1 + \frac{\partial F}{\partial U}(U^*))V, 
\]
where \((2g_1 + c^*), 2g_1^2 - g_2 + c^* g_1\) in (4.17) are short notions for the matrices \((2g_1 + c^*)I\) and \(M(\xi) \equiv (2g_1^2 - g_2 + c^* g_1)I\), where
\[
\begin{align*}
g_1(\xi) &= \frac{\sigma_1 e^{\sigma_1 \xi} - \sigma_2 e^{-\sigma_2 \xi}}{e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}}, \\
g_2(\xi) &= \frac{\sigma_1^2 e^{\sigma_1 \xi} + \sigma_2^2 e^{-\sigma_2 \xi}}{e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}},
\end{align*}
\]
with
\[
\begin{align*}
\lim_{\xi \to \infty} g_1(\xi) &= \sigma_1, & \lim_{\xi \to -\infty} g_1(\xi) &= -\sigma_2; \\
\lim_{\xi \to \infty} g_2(\xi) &= \sigma_1^2, & \lim_{\xi \to -\infty} g_2(\xi) &= \sigma_2^2.
\end{align*}
\]
We now locate the essential spectrum of the operator \(\tilde{L}\) in the space \(C_0(\mathbb{R})\).
Lemma 4.2. Suppose \( \sigma_1 \) and \( \sigma_2 \) satisfying

\[
0 \leq \sigma_1 < \frac{c^* + \sqrt{c^* + 4(b - \epsilon_1)}}{2},
\]

\[
0 < \frac{c^* - \sqrt{c^* - 4(1 - \epsilon_2)}}{2} < \sigma_2 < \frac{c^* + \sqrt{c^* - 4(1 - \epsilon_2)}}{2},
\]

then the essential spectrum of the operator \( \tilde{L} \) in the space \( C_0(\mathbb{R}) \) is contained in some closed sector in the left half complex plane with vertex on the horizontal axis left of the origin. Outside this sector, there are only a finite number of eigenvalues of \( \tilde{L} \).

Proof. As in the proof of Theorem 4.1, we first study the operator \( \tilde{L} \) at infinity. We have

\[
\tilde{L}_+ V = V_{\xi \xi} - (2\sigma_1 + c^*)V_\xi + (\sigma_1^2 + c^*\sigma_1 + \frac{\partial F}{\partial U}(U^*))V,
\]

\[
\tilde{L}_- V = V_{\xi \xi} - (2\sigma_2 + c^*)V_\xi + (\sigma_2^2 + c^*\sigma_2 + \frac{\partial F}{\partial U}(U^*))V,
\]

where \( \sigma_1^2 + c^*\sigma_1 + \frac{\partial F}{\partial U}(U^*) \) and \( \sigma_2^2 - c^*\sigma_2 + \frac{\partial F}{\partial U}(U^*) \) correspond respectively to the matrices:

\[
M^+ = \begin{bmatrix}
\sigma_1^2 + c^*\sigma_1 - 1 & r \\
0 & \sigma_1^2 + c^*\sigma_1 + \epsilon_1 - b
\end{bmatrix}
\]

and

\[
M^- = \begin{bmatrix}
\sigma_2^2 - c^*\sigma_2 + 1 - \frac{r}{1 + \epsilon_2} & 0 \\
\frac{b}{1 + \epsilon_2} & \sigma_2^2 - c^*\sigma_2 - \epsilon_1
\end{bmatrix}.
\]

Similar to the proof of Theorem 4.1, we find the right most points of the corresponding parabolas are on the horizontal axis given by

\[
\max \{ \sigma_1^2 + c^*\sigma_1 - 1, \sigma_1^2 + c^*\sigma_1 + \epsilon_1 - b, \sigma_2^2 - c^*\sigma_2 + 1 - \frac{r}{1 + \epsilon_2}, \sigma_2^2 - c^*\sigma_2 - \epsilon_1 \}.
\]

A simple calculation shows the number above is negative by the choice of \( \sigma_1 \) and \( \sigma_2 \) in (4.18). Thus by the theory in \cite{Volpert}, the essential spectrum of \( \tilde{L} \) is contained in a closed sector in the left complex plane with vertex on the horizontal axis left of the origin. Moreover, we may choose this sector with the further property that outside it there is a finite number of eigenvalues of \( \tilde{L} \).

Corollary 4.3. Assuming all the hypotheses of Lemma 4.2 and \( \sigma_1, \sigma_2 \) satisfying \( 4.1 \), the essential spectrum of the operator \( L \) in the space \( C_{\sigma_1, \sigma_2} \) is contained in some closed sector in the left half complex plane with vertex on the horizontal axis left of the origin. Outside this sector, there are only a finite number of eigenvalues of \( L \).

Proof. The conclusion follows immediately from Lemma 4.2 and relation (4.16). \( \square \)
Having established the location of the essential spectrum of the operator $L$ in the space $C_{\sigma_1, \sigma_2}$, we next study the location of its eigenvalues. We first note that from Corollary 3.4, for $c > 2 \sqrt{\alpha}$, we have $(U^*(\xi))(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})$ is unbounded as $\xi \to -\infty$, which is different from the situations met in [02-Bates], [28-Xu], therefore their methods cannot be carried over to our case.

**Lemma 4.4.** Let $\sigma_1$ and $\sigma_2$ satisfy (4.18). Then 0 is not an eigenvalue of the operator $L$ in the space $C_{\sigma_1, \sigma_2}(\mathbb{R})$.

**Proof.** Let $U^*$ be a traveling wave solution of (4.11) as obtained in Theorem 3.2. It is easy to see that $(U^*)' \in C_0$ and satisfies the equation

$$LV = 0.$$  

This shows that 0 is an eigenvalue of the operator $L$ in $C_0$. Suppose that there exists a nonzero function $\bar{V} \in C_{\sigma_1, \sigma_2}$ satisfying equation (4.24), we then claim that the inequality $|r\bar{V}(\xi)| \leq (U^*)'(\xi)$ is consequently true for all $r \in \mathbb{R}$ and all $\xi \in \mathbb{R}$. Writing $S := \{r \in \mathbb{R}||r\bar{V}(\xi)| \leq (U^*)'(\xi), \xi \in \mathbb{R}\}$, we verify the following properties:

1. $S$ is non-empty, since $0 \in S$.
2. $S$ is closed. Let $r_i \in S$, $i = 1, 2, ...$ and $r_i \to r$ as $i \to +\infty$, then we will have $|r_i\bar{V}(\xi)| \leq (U^*)'(\xi)$ which implies that $|r\bar{V}(\xi)| \leq (U^*)'(\xi)$, we therefore have $r \in S$.
3. $S$ is open. Let $r \in S$, we will show that there exists a $\delta > 0$ such that $(r - \delta, r + \delta) \subset S$. We claim that $|r\bar{V}(\xi)| \leq (U^*)'(\xi)$ implies $|r\bar{V}(\xi)| < (U^*)'(\xi)$. In fact, let $W(\xi) = (U^*)'(\xi) - r\bar{V}(\xi)$ then $W(\xi) \geq 0$, $\xi \in \mathbb{R}$ and satisfies the following equation:

$$w''_1 - cw'_1 + A_{11}w_1 + A_{12}w_2 = 0,$$

$$w''_2 - cw'_2 + A_{21}w_1 + A_{22}w_2 = 0,$$

$$(w_1, w_2) (-\infty) = (w_1, w_2) (+\infty) = 0,$$  

where $A_{ij}$, $i, j = 1, 2$ are the entries of the Jacobian $\frac{\partial F}{\partial \bar{r}}(U^*)$. Since $A_{12} \geq 0$ and $A_{21} \geq 0$, the Maximum Principle implies that $W(\xi) = (w_1(\xi), w_2(\xi))^T > 0$ for $\xi \in \mathbb{R}$, unless $W$ is identically 0 and the Lemma is proved. We thus have $(U^*)'(\xi) - r\bar{V}(\xi) > 0$, $\xi \in \mathbb{R}$. Similarly we can show that $(U^*)'(\xi) + r\bar{V}(\xi) > 0$ for $\xi \in \mathbb{R}$. The claim then follows.

We next show that the claim further implies $|r\bar{V}(\xi)| < (U^*)'(\xi)$ as long as $r$ is sufficiently close to $r$. According to condition (4.18) and the assumption that $\bar{V} \in C_{\sigma_1, \sigma_2}$, for any fixed $\tilde{r} \in \mathbb{R}$, there exists $N > 0$ sufficiently large such that $(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})(U^*)'(\xi) - \tilde{r}\bar{V}(\xi) > 0$ for all $\xi \in (-\infty, N)$. This implies that $\tilde{r}\bar{V}(\xi) < (U^*)'(\xi)$ also holds there. Furthermore, due to the claim, and the boundedness of the functions $(U^*)'$ and $\bar{V}$, we can find $\delta > 0$ such that for any $\tilde{r} \in (-\delta + r, \delta + r)$, we have $(U^*)'(\xi) > \tilde{r}\bar{V}(\xi)$ on the finite interval $[-N, N]$.

Now we fix $\tilde{r} = \bar{r}$ and show $|(U^*)'(\xi) - \bar{r}\bar{V}(\xi)| > 0$ for $\xi \in [N, +\infty)$. Noting the diagonal entries of the matrix $\frac{\partial F}{\partial \bar{r}}(U^*(+\infty))$ are both negative, we can choose column vector $P_+ > 0$ such that (increasing $N$ if necessary) $\frac{\partial F}{\partial \bar{r}}(U^*(+\infty))P_+ < 0$ for $\xi \in [N, +\infty)$.

We have to consider the following two cases:

Case A. If we already have $|(U^*)'(\xi) - \bar{r}\bar{V}(\xi)| \geq 0$ for $\xi \geq N$, then the Maximum Principle implies that $|(U^*)'(\xi) - \bar{r}\bar{V}(\xi)| > 0$ on $[N, +\infty)$. Analogously, we have $|(U^*)'(\xi) + \bar{r}\bar{V}(\xi)| > 0$ is also true for $\xi \in \mathbb{R}$. Consequently, $S$ is open.
Case B. If there is a point in the interval \((N, +\infty)\) such that one of the components of vector \((U^*)'(\xi) - r \tilde{V}(\xi)\) takes negative local minimum at this point, we consider function \(\tilde{W}(\xi) := (U^*)'(\xi) - r \tilde{V}(\xi) + \tau P_+\). The asymptotic rates of \((U^*)'(\xi) - r \tilde{V}(\xi)\) and \(\tilde{V}\) at \(+\infty\) imply that there is a sufficiently large \(\tau > 0\) such that \(\tilde{W}(\xi) = (U^*)'(\xi) - r \tilde{V}(\xi) + \tau P_+ \geq 0\) for \(\xi \in (N, +\infty)\). We further assume that one of the components of \(\tilde{W}(\xi)\), say \(\tilde{w}_1\) for example, takes minimum at a finite point \(\xi_2\) in \((N, +\infty)\). It is not hard to verify that there is a \(\tau = \tau_2\) such that the corresponding \(\tilde{W}(\xi)\) satisfying \(\tilde{w}_1(\xi_2) = 0\) and \(\tilde{W}(\xi) \geq 0\) for \(\xi \in (N, +\infty)\). For such \(\tau_2\) on the one hand, we have

\[
L \tilde{W} = \hat{W}_{\xi\xi} - \hat{c} \hat{W}_\xi + \frac{\partial F}{\partial U}(U^*) \hat{W} = \tau_2 \frac{\partial F}{\partial U}(U^*) P_+ < 0;
\]

on the other hand at \(\xi = \xi_2\), the first component on the left hand side of (4.26) is larger than or equal to zero. We then have a contradiction, and consequently \((U^*)'(\xi) - r \tilde{V}(\xi) \geq 0\) for \(\xi \in (N, +\infty)\). We are again in the situation described by case A. By a similar argument, we can show that \((U^*)'(\xi) + r \tilde{V}(\xi) \geq 0\) for \(\xi \in [N, +\infty)\).

In summary, both case A and Case B show that for any \(\bar{r} \in (-\bar{\delta} + r, \bar{\delta} + r)\), \(|r \tilde{V}(\xi)| < (U^*)'(\xi), \xi \in \mathbb{R}\), i.e., \(S\) is open.

Now the set \(S\) is a non-empty, open and closed subset of \(\mathbb{R}\), hence \(S \equiv \mathbb{R}\). However, this is impossible by the definition of \(S\), since \((U^*)'(\xi)\) is bounded. Therefore the equation \(LV = 0\) cannot have a nontrivial solution in \(C_{\sigma_1, \sigma_2}\).

\(\square\)

The next lemma shows that there is no eigenvalue of the operator \(L\) in \(C_{\sigma_1, \sigma_2}\) with positive real part.

**Lemma 4.5.** Let \(C^0_0\) be the complexified space of \(C_0(\mathbb{R})\) and \(\lambda\) be an eigenvalue of the operator \(\hat{L}\), given by (4.16), with corresponding eigenfunction \(\hat{U} \in C^0_0\), then \(\text{Re} \lambda < 0\).

**Proof.** Let the eigenvalue \(\lambda = \lambda_1 + \lambda_2 i\) and eigenfunction \(\hat{U}(\xi) = U^1(\xi) + iU^2(\xi)\) for \(\xi \in \mathbb{R}\), where \(\lambda_1 \in \mathbb{R}\) and \(U^1(\xi) \in C_0(\mathbb{R})\).

Consider the Cauchy problem (02-Bates, 28-Xu):

\[
V_t = \hat{L}V - \lambda_1 V, \quad V(\xi, 0) = U^1(\xi).
\]

It is easy to verify that \(V(\xi, t) = U^1(\xi) \cos(\lambda_2 t) - U^2(\xi) \sin(\lambda_2 t)\) solves (4.27) for \(\xi \in \mathbb{R}\) and \(t \geq 0\) and is bounded. We suppose that at least one of the components of \(V\) assumes positive value at some \(\xi\) and \(t\) (we can consider \(-V\) if otherwise).

Suppose \(\lambda_1 \geq 0\), then the following claim is true.

**Claim:** There exists a \(r > 0\) such that \(V(\xi, t) \leq r T(U^*)(\xi)\) for \(\xi \in \mathbb{R}\) and \(t \geq 0\). (Recall the operator \(T\) is defined in (4.15).)

In fact since the vector \(T(U^*)(\xi) \to +\infty\) as \(\xi \to -\infty\) we can choose a sufficiently large \(\xi_0 > 0\) such that

\[
V(\xi, t) < T(U^*)(\xi) \text{ for } \xi = -\xi_0 \text{ and } t \geq 0.
\]

Furthermore the positivity of \(T(U^*)(\xi)\) for \(\xi \in \mathbb{R}\) implies that there is a \(r > 0\) such that
(4.29)  \[ V(\xi, t) \leq rT(U^*)(\xi) \quad \text{for} \quad \xi \in [-\xi_0, \xi_0] \quad \text{and} \quad t \geq 0. \]

Let \( \bar{r} = \max\{1, r\} \), we then have

(4.30)  \[ V(\xi, t) \leq \bar{r}T(U^*)'(\xi) \]

for \( \xi \leq \xi_0 \) and \( t \geq 0 \). We next adjust \( \bar{r} \) suitably such that an equality in (4.30) holds on at least one component at a point \((\xi_1, t_1)\) with \( \xi_1 \in (-\infty, \xi_0] \) and \( t_1 \geq 0 \).

We proceed to show that the assumption \( \lambda_1 \geq 0 \) implies that (4.30) is also true for \( \xi \geq \xi_0 \) and \( t \geq 0 \).

From the limits of \( g_1, g_2 \), the choice of \( \sigma_1, \sigma_2 \), and (4.21), we can find a vector \( \hat{P}^+ > 0 \) and increasing \( \xi_0 \) if necessary such that

(4.31)  \[ M(\xi) \hat{P}^+ < 0 \quad \text{for} \quad \xi \geq \xi_0, \]

where

\[ M(\xi) := (2g_1(\xi) - g_2(\xi) + c^* g_1(\xi))I + \frac{\partial F}{\partial U}(U^*(\xi)). \]

We can also choose a small \( \bar{\epsilon} > 0 \) such that

(4.32)  \[ (\bar{\epsilon}^2 \left( \begin{array}{cc} 1 \\ 1 \end{array} \right) - \bar{\epsilon}(2g_1 + c^*)I + M(\xi))\hat{P}^+ < 0 \quad \text{for} \quad \xi \geq \xi_0, \]

Now suppose that we can find a \( \xi_1 > \xi_0 \) and a \( t_1 \geq 0 \) such that \( V(\xi_1, t_1) > \bar{r}T(U^*)'(\xi_1) \). Let

\[ Q^+(\xi) := e^{\bar{\epsilon} \xi} \hat{P}^+. \]

Since \( Q^+(\xi) \to +\infty \) as \( \xi \to +\infty \), there is a \( \hat{r} > 0 \) such that \( V(\xi, t) \leq \hat{r}T(U^*)'(\xi) + \hat{r}Q^+(\xi) \) for all \( \xi \geq \xi_0 \) and \( t \geq 0 \), and for at least one index \( j \) and a \( \xi_2 \geq \xi_0 \) and a \( t_2 \geq 0 \), we have the equality for the \( j \)-th component:

\[ V_j(\xi_2, t_2) = \hat{r}T(U^*_j)'(\xi_2) + \hat{r}Q^+_j(\xi_2). \]

Let \( Y_j(\xi, t) = \hat{r}T(U^*)'(\xi_j) + \hat{r}Q^+(\xi) - V(\xi, t) \), then \( Y_j \) has the following properties:

\[ Y_j(\xi_2, t_2) = 0, \quad Y_j(\xi_0, t) > 0, \quad Y_j(\xi, t) \geq 0 \quad \text{for} \quad \xi \geq \xi_0, \quad t \geq 0 \quad \text{and} \quad \text{it then follows} \]

that \( Y_{j,t}(\xi_2, t_2) = 0, \quad Y_{j,\xi}(\xi_2, t_2) = 0 \) and \( Y_{j,\xi}(\xi_2, t_2) \geq 0 \) and that \( Y_j(\xi, t) \) satisfies

(4.33)

\[
\begin{align*}
Y_{j,t} &= -V_{j,t} \\
&= -(\mathcal{L}V - \lambda_1 V)_j \\
> &\ (\mathcal{L}V + \lambda_1 V + \hat{L}^{\tau}T(U^*)' + \hat{L}^{\tau}Q^+ - \lambda_1 (\hat{r}T(U^*)' + \hat{r}Q^+))_j \\
= &\ (\mathcal{L}Y - \lambda_1 Y)_j \\
= &\ Y_{j,\xi} - (2g_1 + c^*)Y_{j,\xi} + M_{ij}(U^*)Y_1 + M_{2j}(U^*)Y_2 - \lambda_1 Y_j.
\end{align*}
\]

Note that in the third line above we have \( \hat{L}^{\tau}T(U^*)' = 0, \hat{L}^{\tau}Q^+ < 0 \) by (4.32), \( \lambda_1 \geq 0, \hat{r}T(U^*)' > 0 \) and \( \hat{r}Q^+ > 0 \). In the last line of (4.33), \( M_{ij} \) denotes the \( ij \)-th entry of the matrix \( M \) in (4.31). However, at \((\xi_2, t_2)\) the left hand side of (4.33) is equal to zero, while the right hand side is greater than or equal to zero because the off diagonal entries of \( M \) is nonnegative. We have a contradiction and consequently the claim is proved.

We have thus established the fact that the set

\[ S = \{ r \geq 0 | V(\xi, t) \leq rT(U^*)'(\xi) \quad \text{for} \quad \xi \in \mathbb{R} \} \]
is non-empty. Let \( r_0 \) denotes the greatest lower bound of \( S \). In what follows we will show \( r_0 = 0 \). Suppose \( r_0 > 0 \), we have

\[
V(\xi, t) \leq r_0 \mathcal{T}(U^*)(\xi) \quad \text{for} \quad \xi \in \mathbb{R}. \tag{4.34}
\]

We first assume that an equality occurs at a point in \([1, \xi, \tilde{t}]\) at the \( i-th \) component at a point \((\tilde{\xi}, \tilde{t})\) with \( \tilde{\xi} \in \mathbb{R} \) and \( \tilde{t} > 0 \). Let

\[
\lambda X(\xi, t) := r_0 \mathcal{T}(U^*)(\xi) - V(\xi, t).
\]

From (4.27), we obtain the following inequality

\[
X_{i,t} \geq (\lambda X - \lambda_1 X) \tag{4.35}
\]

The positivity of \( M_{ij}(U^*) \geq 0 \) if \( i \neq j \). From the positivity theorem for parabolic equations we deduce that \( X_i(\xi, t) > 0 \) for \( \xi \in \mathbb{R} \) and \( t > \tilde{t} \) (cf p.14 in [16-Leung]). However by the \( t \)-periodicity of \( V \), we have that \( X_i(\xi, t) > 0 \) for all \( \xi \in \mathbb{R} \) and \( t > 0 \). Contradiction with the existence of \( \tilde{\xi} \). This shows that

\[
V(\xi, t) < r_0 \mathcal{T}(U^*)(\xi) \quad \text{for} \quad \xi \in \mathbb{R}, \ t \geq 0. \tag{4.36}
\]

Again since \( \mathcal{T}(U^*)(\xi) \to +\infty \) monotonically as \( \xi \to -\infty \), there exist a sufficiently small \( \delta_1 > 0 \) and a large \( \bar{M} > 0 \) such that

\[
V(\xi, t) < (r_0 - \delta_1) \mathcal{T}(U^*)(\xi) \quad \text{for} \quad \xi \leq -\bar{M} \quad \text{and} \quad t \geq 0. \tag{4.36}
\]

The positivity of \( \mathcal{T}(U^*)(\xi) \) implies that we can extend inequality (4.36) to the interval \((-\infty, \bar{M})\) with an even smaller \( \delta > 0 \). We have

\[
V(\xi, t) < (r_0 - \delta) \mathcal{T}(U^*)(\xi) \quad \text{for} \quad \xi \in (-\infty, \bar{M}) \quad \text{and} \quad t \geq 0. \tag{4.36}
\]

We are then in the same situation as in the proof of the claim at the beginning of this lemma. Using similar arguments as in the proof of the claim, we extend the above inequality to:

\[
V(\xi, t) \geq (r_0 - \delta) \mathcal{T}(U^*)(\xi) \quad \text{for} \quad \xi \in \mathbb{R}, \ t \geq 0. \tag{4.37}
\]

It then follows that \( r_0 - \delta \in S \). Contradiction with the definition of \( r_0 \). Hence \( r_0 = 0 \). However, this contradicts the assumption that at least one component of \( V \) assume positive value. Thus we must have \( \lambda_1 < 0 \). This concludes the proof of the lemma. \( \square \)

**Theorem 4.6.** Assume \([H1]\) to \([H3]\) and that \( \sigma_1 \) and \( \sigma_2 \) satisfy \([4.18]\), the operator \( L \) in \( C_{\sigma_1, \sigma_2} \) has a dense domain of definition. For any complex number with \( \text{Re} \lambda > 0 \) large enough, \((\lambda - L)^{-1}\) exists and is defined on all of \( C_{\sigma_1, \sigma_2} \), and satisfies the following estimate

\[
\| (\lambda - L)^{-1} \|_{C_{\sigma_1, \sigma_2}} \leq \frac{\bar{c}}{1 + |\lambda|}, \tag{4.37}
\]

where \( \bar{c} > 0 \) is a constant.
Proof. The proof follows the same idea as in [24-Volpert] but with resolvent estimates in $C_{0,\tau}$ replaced by in the space $C_{\sigma_1,\sigma_2}$. We skip the proof. □

Theorem 4.7. Under the hypotheses of Theorem 4.7 the operator $L$ generates an analytical semigroup in $C_{\sigma_1,\sigma_2}$, where $\sigma_1$ and $\sigma_2$ satisfy (4.18).

Proof. The conclusion follows from Theorem (4.7) and Hille-Yoshida Theorem. □

We now state the stability theorem,

Theorem 4.8. Assume hypotheses [H1] to [H3], and $\sigma_1, \sigma_2$ satisfy (4.18). The traveling wave solution $U^*$ of (3.6) - (3.7), with wave speed $c^* > 2\sqrt{\alpha}$, is asymptotically stable according to norm $|| \cdot || := || \cdot ||_{C_{\sigma_1,\sigma_2}}$. That is, there exists $\epsilon > 0$ such that if the initial condition $U(\xi, 0) = \bar{U}(\xi) \in C$ with $(\bar{U}(\xi) - U^*(\xi)) \in C_{\sigma_1,\sigma_2}$ and $||\bar{U} - U^*|| < \epsilon$, then the solution $U(\xi, t)$ exists uniquely for all $t > 0$ and

$$ ||U(\xi, t) - U^*(x + ct)|| \leq Me^{-bt}. $$

Here, the constants $M > 0, b > 0$ are independent of $t$ and $\bar{U}$.

Proof. The stability of $U^*$ leads to the consideration of the stability of the trivial solution for system (4.2), and the analysis of the spectrum of the operator $L$ in (4.3). Corollary 4.3, Lemma 4.4 and Lemma 4.5 show that the spectrum of the operator $L$ in the space $C_{\sigma_1,\sigma_2}$ is contained in a closed angular region in the left open complex plane. Thus, following the methods in Theorem 2.1 on p.227 in [24-Volpert], we obtain the conclusion of this theorem. □

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