On the Local Solvability of Darboux’s Equation

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Abstract

We reduce the question of local nonsolvability of the Darboux equation, and hence of the isometric embedding problem for surfaces, to the local nonsolvability of a simple linear equation whose type is explicitly determined by the Gaussian curvature.

Let \((M^2, g)\) be a two-dimensional Riemannian manifold. A well-known problem is to ask, when can one realize this locally as a small piece of a surface in \(\mathbb{R}^3\)? That is, if the metric \(g = g_{ij}dx^i dx^j\) is given in the neighborhood of a point, say \((x^1, x^2) = 0\), when do there exist functions \(z_\alpha(x^1, x^2), \alpha = 1, 2, 3\), defined in a possibly smaller domain such that \(g = dz_1^2 + dz_2^2 + dz_3^2\)? This equation may be written in local coordinates as the following determined system

\[
\sum_{\alpha=1}^{3} \frac{\partial z_\alpha}{\partial x^i} \frac{\partial z_\alpha}{\partial x^j} = g_{ij}.
\]

Due to its severe degeneracy, in the sense that every direction happens to be a characteristic direction, little information has been obtained by studying this system directly. However a more successful approach has been to reduce this system to the following single equation of Monge-Ampère type, known as the Darboux equation:

\[
\det \nabla_{ij} z = K|g|(1 - |\nabla_g z|^2)
\]  

(1)

where \(\nabla_{ij}\) are second covariant derivatives, \(K\) is the Gaussian curvature, \(\nabla_g\) is the gradient with respect to \(g\), and \(|g| = \det g\). In fact, the local isometric embedding problem is equivalent to the local solvability of this equation (see the appendix).

Let us first recall the known results. Since equation (1) is elliptic if \(K > 0\), hyperbolic if \(K < 0\), and of mixed type if \(K\) changes sign, the manner in which \(K\) vanishes will play the primary role in the hypotheses of any result. The classical results state that a solution always exists in the case that \(g\) is analytic or \(K(0) \neq 0\); these results may be found in [4]. C.-S. Lin provides an affirmative answer in [10] and [11] when \(g\) is sufficiently smooth and satisfies \(K \geq 0\), or \(K(0) = 0\) and \(\nabla K(0) \neq 0\). When \(K \leq 0\) and \(\nabla K\) possesses a certain nondegeneracy, Han, Hong, and Lin [5] show that a smooth solution always exists if \(g\) is smooth. Lastly if the Gaussian curvature vanishes to finite order and the zero set \(K^{-1}(0)\) consists of Lipschitz curves

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intersecting transversely, then Han and the author [6] have proven the existence of smooth solutions if \( g \) is smooth. Related results may be found in [1], [2], [3], [7], [8].

A negative result has been obtained by Pogorelov [13] (see also [12]), who found a \( C^{2,1} \) metric with no local \( C^2 \) isometric embedding in \( \mathbb{R}^3 \). More recently, the author [9] has constructed \( C^\infty \) examples of degenerate hyperbolic and mixed type Monge-Ampère equations of the form

\[
\det(\partial_{ij} z + a_{ij}(p, z, \nabla z)) = k(p, z, \nabla z) \tag{2}
\]

which do not admit a local solution, where \( p = (x^1, x^2) \) and \( \partial_{ij} \) denote second partial derivatives. A fundamental part of the strategy in [9] is to reduce the local nonsolvability of (2), to the local nonsolvability of a quasilinear equation whose type is explicitly determined by the function \( k \). It is the purpose of this article to show that the Darboux equation possesses a similar property for a large class of Gaussian curvatures.

We begin by partially constructing the Gaussian curvature. Here we will denote the coordinates \( x^1 \) and \( x^2 \) by \( x \) and \( y \) respectively. Define sequences of disjoint open squares \( \{X^m\}_{m=1}^\infty \) and \( \{X^m_1\}_{m=1}^\infty \) whose sides are aligned with the \( x \) and \( y \)-axes, and such that \( X^m \), and \( X^m_1 \) are centered at \( q_m = (\frac{1}{m}, 0) \), \( X^n \subset X^n_1 \), and \( X^n, X^n_1 \) have widths \( \frac{1}{2n(n+1)}, \frac{1}{n(n+1)} \), respectively. Set \( K \equiv 0 \) in \( \mathbb{R}^2 - \bigcup_{n=1}^\infty X^n_1 \). Define

\[
X = \{(x, y) \mid |x| < 1, |y| < 1\}
\]

and let \( \phi \in C^\infty(\overline{X}) \) be such that \( \phi \) vanishes to infinite order on \( \partial X \), and either \( \phi(q) > 0 \) or \( \phi(q) < 0 \) for all \( q \in X \) (here \( \overline{X} \) denotes the closure of \( X \)). We now define \( K \) in \( X^n \) by

\[
K(q) = \gamma_n \phi(4n(n+1)(q - q_n)), \quad q \in \overline{X}^n,
\]

where \( \{\gamma\}_{n=1}^\infty \) is a sequence of positive numbers that are to be chosen with the property that \( \lim_{n \to \infty} \gamma_n = 0 \) in order to insure that \( K \in C^\infty(\mathbb{R}^2) \). A description of how \( K \) should be prescribed in the remaining region \( \bigcup_{n=1}^\infty (X^n_1 - X^n) \) shall be given below.

**Theorem 1.** Suppose that \( K \) adheres to the description given above, and that a local \( C^5 \) solution \( z \) of the Darboux equation exists in a domain containing the origin. Then in a neighborhood of a point on \( \partial X^n \) for some \( n \) sufficiently large, there exists a \( C^2 \) function \( u \) constructed from \( z \) which after an appropriate change of coordinates satisfies the equation

\[
\partial_{tt} u + K \partial_{ss} u = K f, \tag{3}
\]

where \( f \in C^0 \) also depends on \( z \) and is strictly positive.

This theorem suggests a strategy for constructing smooth counterexamples to the local solvability of the Darboux equation, or equivalently the local isometric embedding problem. Namely, complete the construction of a smooth Gaussian curvature
function in the region $\bigcup_{n=1}^{\infty}(X_1^n - X^n)$, in such a way that the linear equation (3) can have no local solution. Whether this is possible is still an open question, however as pointed out above, a similar strategy was successfully employed for the related Monge-Ampère equation (2). Note that in order for this strategy to be utilized for the Darboux equation, it must be shown that given a smooth function $K$ there always exists a locally defined smooth metric $g$ having Gaussian curvature $K$. This may be accomplished in the following way. Let $\Omega$ be a neighborhood of the origin, and let $G \in C^\infty(\Omega)$ be the unique solution of the equation

$$\partial_{xx}G + KG = 0, \quad G(0, y) = 1, \quad \partial_x G(0, y) = 0.$$  

By choosing $\Omega$ sufficiently small we have that $G > 0$. Then

$$g = dx^2 + G^2dy^2$$

is a smooth Riemannian metric and has Gaussian curvature $K$ in the domain $\Omega$.

The first step in verifying Theorem 1, will be to show that certain second covariant derivatives of any solution of (1) cannot vanish on $\partial X^n$ for $n$ sufficiently large. Suppose that a local solution $z \in C^2$ of (1) exists, so that upon rewriting the equation we have

$$b^{ij} \nabla_{ij}z = 2K(1 - |\nabla_g z|^2), \quad (4)$$

where the Einstein summation convention concerning raised and lowered indices has been used (this convention will also be utilized in what follows) and

$$(b^{ij}) = |g|^{-1} \begin{pmatrix} \nabla_{22}z & -\nabla_{12}z \\ -\nabla_{12}z & \nabla_{11}z \end{pmatrix}.$$

Then integrating by parts yields

$$\int_{X^n} b^{ij} \nabla_{ij}zd\omega_g = -\int_{X^n} \nabla_j z \nabla_i b^{ij}d\omega_g + \int_{\partial X^n} b^{ij} n_i \nabla_j zd\sigma_g, \quad (5)$$

where $d\omega_g$ and $d\sigma_g$ are the elements of area and length with respect to $g$, and $(n_1, n_2)$ is the unit outer normal to $\partial X^n$ also with respect to $g$. In order to calculate the interior term on the right-hand side we note that $b^{ij}$ is a contravariant 2-tensor, so that

$$\nabla_i b^{ij} = \partial_i b^{ij} + \Gamma^i_{il} b^{lj} + \Gamma^j_{il} b^{il}$$

where $\Gamma^i_{ij}$ are Christoffel symbols. Therefore

$$\nabla_i b^{ij} = |g|^{-1}(\partial_1 \nabla_{22}z - \partial_2 \nabla_{12}z) + |g|^{-2}(-\partial_1 |g| \nabla_{22}z + \partial_2 |g| \nabla_{12}z)$$

$$+ |g|^{-3/2}(\partial_1 |g|^{1/2} \nabla_{22}z - \partial_2 |g|^{1/2} \nabla_{12}z) + \Gamma^1_{il} b^{il}$$

$$= |g|^{-1}(\partial_1 \nabla_{22}z - \partial_2 \nabla_{12}z + |g| \Gamma^1_{ij} b^{ij}) - \Gamma^i_{ij} b^{ij},$$

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after making use of the identity
\[ \Gamma^i_{ij} = |g|^{-1/2} \partial_j |g|^{1/2}. \]

Moreover direct computation shows that
\[
\begin{align*}
\partial_1 \nabla_{22} z - \partial_2 \nabla_{12} z + |g| \Gamma^i_{ij} b^{ij} \\
= -\Gamma^i_{j2} \partial_1 z + \Gamma^i_{j1} \partial_2 z \\
+ (\partial_2 \Gamma^i_{12} - \partial_1 \Gamma^i_{22} - \Gamma^i_{11} \Gamma^i_{22} + 2 \Gamma^i_{12} \Gamma^i_{12} - \Gamma^i_{22} \Gamma^i_{11}) \partial_1 z \\
= |g|((\Gamma^i_{j2} b^{ij} + \Gamma^i_{j1} b^{ij}) \\
+ (\partial_2 \Gamma^i_{12} - \partial_1 \Gamma^i_{22} - \Gamma^i_{11} \Gamma^i_{22} + 2 \Gamma^i_{12} \Gamma^i_{12} - \Gamma^i_{22} \Gamma^i_{11} - \Gamma^i_{j2} \Gamma^i_{12} + \Gamma^i_{j1} \Gamma^i_{22}) \partial_1 z, \\
\end{align*}
\]
and we observe that the coefficient of \( \partial_1 z \) is in fact a curvature term. More precisely, if it is denoted by \( \chi^i \) then
\[
\chi^i = \partial_2 \Gamma^i_{12} - \partial_1 \Gamma^i_{22} + \Gamma^i_{12} \Gamma^i_{12} - \Gamma^i_{22} \Gamma^i_{11} = -R^i_{212} = -g^{i1} |g| K
\]
where \( R^i_{jkl} \) is the Riemann tensor. We now have
\[
\begin{align*}
\partial_1 \nabla_{22} z - \partial_2 \nabla_{12} z + |g| \Gamma^i_{ij} b^{ij} \\
= |g|((\Gamma^i_{j2} b^{ij} + \Gamma^i_{j1} b^{ij}) \\
+ (\partial_2 \Gamma^i_{12} - \partial_1 \Gamma^i_{22} - \Gamma^i_{11} \Gamma^i_{22} + 2 \Gamma^i_{12} \Gamma^i_{12} - \Gamma^i_{22} \Gamma^i_{11} - \Gamma^i_{j2} \Gamma^i_{12} + \Gamma^i_{j1} \Gamma^i_{22}) \partial_1 z, \\
\end{align*}
\]
so that
\[
\nabla_i b^{i1} = -K z^1. \quad (6)
\]
Similarly
\[
\nabla_i b^{i2} = -K z^2. \quad (7)
\]
With the help of (4), (6), and (7) it follows that (5) becomes
\[
\int_{X^n} K (2 - 3|\nabla_g z|^2) d\omega_g \\
= \int_{\partial X^n} |g|^{-1/2}[(\nabla_1 z \nabla_{22} z - \nabla_2 z \nabla_{12} z) \vec{m}_1 + (\nabla_2 z \nabla_{11} z - \nabla_1 z \nabla_{12} z) \vec{m}_2]d\sigma,
\]
where \((\vec{m}_1, \vec{m}_2)\) is the Euclidean unit outer normal to \( \partial X^n \) and \( d\sigma \) is the Euclidean element of length.

The integral equality (8) will now be used to show that certain second covariant derivatives of any solution of the Darboux equation cannot vanish on \( \partial X^n \) for \( n \) sufficiently large. Let \(-v_n, +v_n\) represent the left and right vertical portions of \( \partial X^n \), respectively, and let \(+h_n, -h_n\) represent the top and bottom horizontal portions of \( \partial X^n \), respectively.

**Lemma 2.** Suppose that \( K \) satisfies the hypotheses of Theorem 1. Then it is not possible for a \( C^2 \) solution \( z \) of (1) to satisfy the following property for any \( n \) sufficiently large:
\[
\nabla_{22} z|_{\pm v_n} = 0, \quad \nabla_{11} z|_{\pm h_n} = 0. \quad (9)
\]
Proof. We proceed by contradiction and assume that property (9) holds. Then since $K |_{\partial X^n} = 0$, the Darboux equation implies that $\nabla_{12} z |_{\partial X^n} = 0$. Therefore the right-hand side of (8) vanishes. However this yields a contradiction, as the left-hand side is nonzero for large $n$. To see this last fact observe that according to the appendix, any solution of the Darboux equation yields an isometric embedding $F = (z_1, z_2, z)$ of the metric $g$. So that by performing an appropriate rigid body motion of this embedding, to obtain $\overline{F} = AF$ where $A$ is an orthogonal matrix, we can ensure that the new third component $\overline{z}$ of $\overline{F}$ satisfies $|\nabla z|_{(0,0)} = 0$. Furthermore the appendix also shows that $\overline{z}$ must satisfy the Darboux equation, and so we have $2 - 3 |\nabla g z|^2 > 1$ inside $X^n$ if $n$ is chosen sufficiently large. Therefore since $K$ never vanishes on $X^n$, integral equality (8) yields a contradiction. □

In light of Lemma 2, there must exist a point $p \in \partial X^n$ at which one of the given second covariant derivatives is nonzero. As arguments similar to those presented below may be applied if $p \in -v_n$ or $p \in \pm h_n$, we assume without loss of generality that $p \in +v_n$ so that $\nabla_{22} z(p) \neq 0$. It follows that after a change of coordinates near $p$, a solution $u$ of equation (3) may be constructed. The following lemma will complete the proof of Theorem 1.

**Lemma 3.** Suppose that there exists a $C^5$ solution $z$ of the Darboux equation satisfying $\nabla_{22} z(p) \neq 0$. Then there exists a $C^3$ local change of coordinates near $p = (p^1, p^2)$ given by

$$t = x - p^1, \quad s = s(x, y),$$

and a $C^2$ solution $u$ of the equation

$$\partial_{tt} u + K \partial_{ss} u = K f,$$

where $f \in C^0$ and is strictly positive if $n$ is sufficiently large.

Proof. The desired coordinates $(t, s)$ will be chosen to eliminate the mixed second covariant derivative appearing in (4). Since $b^{ij}$ is a contravariant 2-tensor, under a coordinate change $\overline{\xi} = \overline{\xi}(x^1, x^2)$ it transforms by

$$\overline{b}^{ij} = b^{lm} \frac{\partial \overline{\xi}^i}{\partial x^l} \frac{\partial \overline{\xi}^j}{\partial x^m}.$$ 

Therefore by setting $t = \overline{\xi}^1 = x - p^1$, we seek $s = \overline{\xi}^2$ such that

$$\overline{b}^{12} = b^{11} \partial_x s + b^{12} \partial_y s = 0, \quad s(p^1, y) = cy,$$

where $c$ is a nonzero constant to be determined. Observe that since $b^{11} = |g|^{-1} \nabla_{22} z \neq 0$ near $p$, the line $x = p^1$ is noncharacteristic for (10). Thus the theory of first order partial differential equations guarantees the existence of a unique local solution $s \in C^3$, in light of the fact that $b^{11}, b^{12} \in C^3$. 

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We now calculate each of the new coefficients for the Darboux equation. First note that $b_{11}^{11} = b_{11}^{1}$, and with the help of (10)

$$b_{22}^{22} = b_{11}^{11} (\partial_z s)^2 + 2b_{12}^{12} \partial_z s \partial_y s + b_{22}^{22} (\partial_y s)^2 = (b_{11}^{11})^{-1} (\partial_y s)^2 \det b_{ij}^{ij} = (|g|b_{11}^{11})^{-1} (\partial_y s)^2 K(1 - |\nabla g z|^2).$$

Therefore in the new coordinates Darboux’s equation (4) is given by

$$b_{11}^{11} \nabla_{11} z + K f \nabla_{22} z = 2K(1 - |\nabla g z|^2),$$

(11)

where $\nabla_{ij}$ denote covariant derivatives with respect to the new coordinates $(t, s)$ and $f = (|g|b_{11}^{11})^{-1} (\partial_y s)^2 (1 - |\nabla g z|^2)$.

Notice that if we choose

$$c = b_{11}^{11} |g|^{1/2}(1 - |\nabla g z|^2)^{-1/2}(p),$$

then $(b_{11}^{11})^{-1} f(p) = 1$. Moreover by setting

$$u(t, s) = z(t, s) - \int_0^t \left( \int_0^{t'} (\Gamma_{11}^1 \partial_t z + \Gamma_{11}^2 \partial_s z)(t'', s) dt'' \right) dt'$$

we have $\partial_{tt} u = \nabla_{11} z$, so that (11) becomes

$$\partial_{tt} u + K \partial_{ss} u = K f$$

with

$$f = (b_{11}^{11})^{-1} [2(1 - |\nabla g z|^2) + (f(p) - \overline{f}) \nabla_{22} z] + \Gamma_{22}^1 \partial_t z + \Gamma_{22}^2 \partial_s z + \partial_s a(u - z).$$

Lastly we observe that $f(t, s) > 0$ in a sufficiently small neighborhood of $p$ if $n$ is large, since as in the proof of Lemma 2 we may assume that $|\nabla z|(0, 0) = 0$. □

**Appendix**

Here we show that the local isometric embedding problem is equivalent to the local solvability of the Darboux equation (1). Assume that there exists a local $C^2$ embedding $F = (z_1, z_2, z_3)$ for a given metric $g$. Then according to the Gauss equations

$$\nabla_{ij} F = h_{ij} \nu,$$
where $h_{ij}$ are the components of the second fundamental form with respect to a unit normal $\nu$. Then by taking the Euclidean inner product of this equation with the vector $\vec{k} = (0, 0, 1)$, we obtain

$$\det \nabla_{ij}z = K|g|(\nu \cdot \vec{k})^2$$

where for convenience we denote $z_3$ by $z$. Furthermore, if $\times$ represents the cross product operation between two vectors in $\mathbb{R}^3$ then

$$(\nu \cdot \vec{k})^2 = 1 - \frac{|(\partial_1 F \times \partial_2 F) \times \vec{k}|^2}{|\partial_1 F \times \partial_2 F|^2} = 1 - g^{ij} \partial_i z \partial_j z = 1 - |\nabla_g z|^2,$$

where $g^{ij}$ are components of the inverse matrix $(g_{ij})^{-1}$. Clearly the remaining two components of $F$ must also satisfy equation (1). Conversely, if a local solution of (1) exists for a given metric $g$ and $|\nabla_g z| < 1$, then a calculation shows that $g - dz^2$ is a Riemannian metric and is flat. It follows that there exists a local change of coordinates $z_1 = z_1(x^1, x^2)$, $z_2 = z_2(x^1, x^2)$ such that $g - dz^2 = dz_1^2 + dz_2^2$.

References

[1] Q. Han, On the isometric embedding of surfaces with Gauss curvature changing sign cleanly, Comm. Pure Appl. Math., 58 (2005), 285-295, MR 2094852.

[2] Q. Han, Local isometric embedding of surfaces with Gauss curvature changing sign stably across a curve, Calc. Var. & P.D.E., 25 (2006), 79-103, MR 2183856.

[3] Q. Han, Smooth local isometric embedding of surfaces with Gauss curvature changing sign cleanly, preprint.

[4] Q. Han, & J.-X. Hong, Isometric Embedding of Riemannian Manifolds in Euclidean Spaces, Mathematical Surveys and Monographs, Vol. 130, AMS, Providence, RI, 2006, MR 2261749.

[5] Q. Han, J.-X. Hong, & C.-S. Lin, Local isometric embedding of surfaces with nonpositive Gaussian curvature, J. Differential Geom., 63 (2003), 475-520, MR 2015470.

[6] Q. Han, & M. Khuri, On the local isometric embedding in $\mathbb{R}^3$ of surfaces with Gaussian curvature of mixed sign, preprint.

[7] M. Khuri, The local isometric embedding in $\mathbb{R}^3$ of two-dimensional Riemannian manifolds with Gaussian curvature changing sign to finite order on a curve,
J. Differential Geom., 76 (2007), 249-291, MR 2330415.

[8] M. Khuri, *Local solvability of degenerate Monge-Ampère equations and applications to geometry*, Electron. J. Diff. Eqns., 2007 (2007), No. 65, 1-37, MR 2308865.

[9] M. Khuri, *Counterexamples to the local solvability of Monge-Ampère equations in the plane*, Comm. PDE, 32 (2007), 665-674, MR 2334827.

[10] C.-S. Lin, *The local isometric embedding in \( \mathbb{R}^3 \) of 2-dimensional Riemannian manifolds with nonnegative curvature*, J. Differential Geom., 21 (1985), no. 2, 213-230, MR 0816670.

[11] C.-S. Lin, *The local isometric embedding in \( \mathbb{R}^3 \) of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly*, Comm. Pure Appl. Math., 39 (1986), no. 6, 867-887, MR 0859276.

[12] N. Nadirashvili, & Y. Yuan, *Improving Pogorelov’s isometric embedding counterexample*, Calc. Var. Partial Differential Equations, 32 (2008), no. 3, 319-323, MR 2393070.

[13] A. Pogorelov, *An example of a two-dimensional Riemannian metric not admitting a local realization in \( E_3 \)*, Dokl. Akad. Nauk. USSR, 198 (1971), 42-43, MR 0286034.

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