Michael–Simon inequalities for $k$-th mean curvatures

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Abstract This paper continues the study of Alexandrov–Fenchel inequalities for quermass-integrals for $k$-convex domains. It focuses on the application to the Michael–Simon type inequalities for $k$-curvature operators. The proof uses optimal transport maps as a tool to relate curvature quantities defined on the boundary of a domain.

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1 Introduction

The classical Michael–Simon inequality is the Sobolev inequality on immersed submanifolds.

Theorem 1.1 ([14]) Let $i : M^n \to \mathbb{R}^N$ be an isometric immersion ($N > n$). Let $U$ be an open subset of $M$. For a function $\varphi \in C^\infty_c(U)$, there exists a constant $C$, such that

$$\left( \int_M |\varphi|^{\frac{n}{n-k}} d\mu_M \right)^{\frac{n-k}{n}} \leq C \int_M (|\vec{H}| \cdot |\varphi| + |\nabla \varphi|) d\nu_M. \quad (1)$$

In the special case when we take $\varphi \equiv 1$, the Michael–Simon inequality gives an inequality between the area of the boundary and the integral of its mean curvature. In this note, we derive a natural generalization of (1) in which we establish inequalities between fully nonlinear curvature quantities $\sigma_{k-1}(L)$ and $\sigma_k(L)$ if the hypersurface $M$ is $(k + 1)$-convex, where $\sigma_k(L)$ denotes the $k$-th elementary symmetric function of the second fundamental form $L$. 
Theorem 1.2 Let $i : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. Let $U$ be an open subset of $M$ and $\varphi$ be a $C^\infty_c(U)$ function. For $k = 2, \ldots, n-1$, if $M$ is $(k + 1)$-convex, then for any $0 \leq l \leq k - 1$, there exists a constant $C$ depending on $n$, $k$ and $l$, such that

$$\left( \int_M \sigma_l(L)|\varphi|^\frac{n-l}{n-k} d\mu_M \right)^\frac{n-k}{n-l} \leq C \int_M (\sigma_k(L)|\varphi| + \sigma_{k-1}(L)|\nabla \varphi| + \cdots + |\nabla^k \varphi|) d\mu_M.$$ 

If $k = n$, then the inequality holds when $M$ is $n$-convex. If $k = 1$, then the inequality holds when $M$ is $1$-convex. (When $k = 1$ and $l = 0$, the inequality is the Michael–Simon inequality for hypersurfaces.)

Theorem 1.2 generalizes previous works [6,7] on the Alexandrov–Fenchel inequalities for quermassintegrals of $k + 1$-convex domains. See [10–12,15], etc. for previous works on the Alexandrov–Fenchel inequalities. The proof of Theorem 1.2 follows from the same outline as that of Theorem 1.3 in [7], using optimal transport map. For interested readers, see Cordero-Erausquin, Nazaret, Villani [8] and McCann [13], etc. for other works using optimal transport method to derive geometric inequalities. The added complication is due to the present of the weight $\varphi$ and its higher order derivatives. The main technical part lies in the proof of Proposition 3.1. We reduce the proof of Proposition 3.1 into four types of estimates, which are defined to be the $I$-type, the $J$-type, the $K$-type and the $N$-type estimate (in Sect. 5). Among them, $K$-type estimate is quite different from the one in [7], and $N$-type estimate is new. In the proof we will briefly go through the $I$-type estimate and the $J$-type estimate which are similar to those in [7]; we then focus on the $K$-type estimate and the $N$-type estimate, especially the interplay between them.

The organization of this paper is as follows. In Sect. 2, we will recall some preliminary facts on elementary symmetric functions and curvature properties of embedded hypersurfaces. In Sect. 3, we will demonstrate the method of optimal transport and reduce the proof of Theorem 1.2 to the technical proposition (Proposition 3.1). In Sect. 4, we will present the proof of Proposition 3.1 for the special case $k = 2$. In Sect. 5, we will prove Proposition 3.1 for general $k$ by a delicate induction argument.

We remark that it is an open question to prove the Michael–Simon inequality (1) with sharp constant $C$. The Michael–Simon inequality for higher order curvatures we derive in this paper does not yield any sharp constants either.

2 Preliminaries

2.1 $\Gamma^+_k$ cone

In this subsection, we will describe some properties of the $k$-th elementary symmetric function $\sigma_k$ and its associated convex cone.

2.1.1 Definitions and basic properties

**Definition 2.1** The $k$-th elementary symmetric function for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is

$$\sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$
The elementary symmetric functions are special cases of hyperbolic polynomials introduced by Garding [9], which enjoy the following properties in their associated positive cones.

**Definition 2.2**

\[ \Gamma^+_k := \{ \lambda \in \mathbb{R}^n | \text{the connected component of } \sigma_k(\lambda) > 0 \text{ which contains the identity} \]
\[ = (1, \ldots, 1) \in \mathbb{R}^n \]

is called the positive \( k \)-cone. Equivalently,

\[ \Gamma^+_k = \{ \lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}. \]

In particular, \( \Gamma^+_n \) is the positive cone

\[ \{ \lambda \in \mathbb{R}^n | \lambda_1 > 0, \ldots, \lambda_n > 0 \}, \]

and \( \Gamma^+_1 \) is the half space \( \{ \lambda \in \mathbb{R}^n | \lambda_1 + \cdots + \lambda_n > 0 \} \). It is also obvious from Definition 2.2 that \( \Gamma^+_k \) is an open convex cone and that

\[ \Gamma^+_n \subset \Gamma^+_{n-1} \subset \cdots \subset \Gamma^+_1. \]

By Garding’s theory of hyperbolic polynomials [9], one knows that

\[ \sigma_1(\cdot) \text{ and } (\frac{\sigma_k(\cdot)}{\sigma_l(\cdot)})^{1-k-l} (k > l) \]

are concave functions in \( \Gamma^+_k \).

**Definition 2.3** A symmetric matrix \( A \) is in \( \tilde{\Gamma}^+_k \) cone, if its eigenvalues

\[ \lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)) \in \Gamma^+_k. \]

When there is no confusion, we will denote \( \tilde{\Gamma}^+_k \) by \( \Gamma^+_k \) and \( \sigma_k(\lambda(A)) \) by \( \sigma_k(A) \) for simplicity. An equivalent definition of \( \sigma_k(A) \) is

\[ \sigma_k(A) := \frac{1}{k!} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} A_{i_1 j_1} \cdots A_{i_k j_k}. \]

**Definition 2.4** The Newton transformation tensor is defined as

\[ [T_k]_{ij}(A_1, \ldots, A_k) := \frac{1}{k!} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \]

(2)

**Definition 2.5** With the notion of Newton transformation tensor \([T_k]_{ij} \), one may define the polarization of \( \sigma_k \) by

\[ \Sigma_k(A_1, \ldots, A_k) := (A_1)_{ij} \cdot [T_k]_{ij}(A_2, \ldots, A_k) = \frac{1}{(k-1)!} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \]

(3)

We remark here that \( \Sigma_k(A, \ldots, A) \) and \( \sigma_k(A) \) only differs by a multiplicative constant:

\[ \sigma_k(A) = \frac{1}{k} \Sigma_k(A, \ldots, A). \]

Therefore it is called the polarization of \( \sigma_k \).

**Notation 2.6** When some components are the same, we adopt the notational convention that

\[ \Sigma_k(B, \ldots, B, C, \ldots, C) := \Sigma_k(B, \ldots, B, C, \ldots, C), \]
and
\[ [T_k]_{ij}(B, \ldots, B, C, \ldots, C) := [T_k]_{ij}(B, \ldots, B, C, \ldots, C). \]

Also for simplicity, we denote
\[ [T_k]_{ij}(A) := [T_k]_{ij}(A, \ldots, A). \]

Some relations between the Newton transformation tensor \([T_k]_{ij}\) and \(\sigma_k\) are listed below. For any symmetric matrix \(A\), we denote the trace of \(A\) by \(\text{Tr}(A)\). Then
\[ \sigma_k(A) = \frac{1}{n-k} \text{Tr}([T_k]_{ij}(A)), \quad (4) \]
and
\[ \sigma_{k+1}(A) = \frac{1}{k+1} \text{Tr}([T_k]_{im}(A) \cdot A_{mj}). \quad (5) \]

On the other hand, one can write \([T_k]_{ij}\) in terms of \(\sigma_k\) by the formula
\[ [T_{k-1}]_{ij}(A) = \frac{\partial \sigma_k(A)}{\partial A_{ij}}, \]
and
\[ [T_k]_{ij}(A) = \sigma_k(A) \delta_{ij} - [T_{k-1}]_{im}(A) A_{mj}. \quad (6) \]

This last formula implies the following fact which we will repeatedly use later in our proof.

**Lemma 2.7** Suppose \(B\) and \(C\) are two symmetric matrices, then
\[ [T_{k-1}]_{im}(B, \ldots, B, C, \ldots, C) C_{mj} \]
\[ = \frac{c_i^l}{k c_{k-1}^l} \cdot \Sigma_k(B, \ldots, B, C, \ldots, C) \delta_{ij} - \frac{c_i^l}{c_{k-1}^l} \cdot [T_k]_{ij}(B, \ldots, B, C, \ldots, C) \]
\[ - \frac{c_i^{l-1}}{c_{k-1}^{l-1}} \cdot [T_{k-1}]_{im}(B, \ldots, B, C, \ldots, C) B_{mj}. \quad (7) \]

We omit the proof here since it is quite straightforward by formula (6) and the multilinearity of \([T_k]_{ij}(\cdot)\) and \(\sigma_k(\cdot)\). One can also refer to Lemma 2.7 in [7] for a complete proof.

We finish this section by listing some basic inequalities based on the Garding’s theory of hyperbolic polynomials, which we will use in the present paper.

(i) if \(\lambda \in \Gamma_k^+\), then
\[ \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} > 0, \quad \text{for} \quad i = 1, \ldots, n; \]
(ii) if \(A_1, \ldots, A_k \in \Gamma_{k+1}^+\), then \([(T_k)_{ij}]\) is a positive definite matrix, i.e.
\[ [T_k]_{ij}(A_1, \ldots, A_k) > 0; \]
(iii) if \(A_1, \ldots, A_k \in \Gamma_k^+\), then
\[ \Sigma_k(A_1, \ldots, A_k) > 0; \]
iv) if $A - B \in \Gamma^+_k$ and $A_2, \ldots, A_k \in \Gamma^+_k$, then

$$
\Sigma_k(B, A_1, \ldots, A_k) < \Sigma_k(A, A_2, \ldots, A_k).
$$

Finally, we recall two technical lemmas regarding the derivative of the Newton transformation tensor $[T_k]_{ij}$.

**Lemma 2.8** Let $L$ denote the second fundamental form of the hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$. Let $[T_k]_{ij}(L)$ be the Newton transform tensor of $L$. Then the divergence of $[T_k]_{ij}(L)$ is equal to 0, i.e.

$$
([T_k]_{ij}(L))_i = 0 \quad \text{for each } j.
$$

The proof uses the Codazzi equation

$$
L_{ij,k} = L_{ik,j}
$$

and properties of $[T_k]_{ij}$. We refer interested readers to Lemma 5.1 in [7].

**Lemma 2.9** Suppose $v$ is a smooth function defined on the hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$. Denote the Hessian of $v$ on $M$ by $D^2v$, the second fundamental form of $M$ by $L$. Consider the polarized Newton transformation tensor $[T_k]_{ij}(D^2v, \ldots, D^2v, L, \ldots, L)$ introduced in Definition. The divergence of it satisfies

$$
([T_k]_{ij}(D^2v, \ldots, D^2v, L, \ldots, L))_i = -l \cdot [T_k]_{ij}(D^2v, \ldots, D^2v, L, \ldots, L)L_{mi}v_m \quad \text{for each } j.
$$

The proof of this lemma uses the above Codazzi equation and the Gauss equation

$$
0 = \tilde{R}_{ijkl} = R_{ijkl} - L_{ik}L_{jl} + L_{il}L_{jk}, \quad \text{(Gauss equation)}
$$

where the curvature tensor of $M$ and the curvature tensor of the ambient space $\mathbb{R}^{n+1}$ are denoted by $R_{ijkl}$ and by $\tilde{R}_{ijkl}$, respectively. See the detailed proof of (122) in [7].

2.2 Restriction of a convex function to a submanifold

Consider an isometric immersion $i : M^n \hookrightarrow \mathbb{R}^{n+1}$. Let $\nabla$ and $D^2$ (resp. $\tilde{\nabla}$ and $\tilde{D}^2$) be the gradient and the Hessian on $M$ (resp. on $\mathbb{R}^{n+1}$). We also denote the second fundamental form of $M$ by $L_{ij}$ and the inner unit normal by $\tilde{n}$. Suppose $\tilde{V} : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function and $v = \tilde{V}|_M$ is its restriction to $M$. Then the Hessian of $v$ with respect to the metric on $M$ relates to the Hessian of $\tilde{V}$ on the ambient space $\mathbb{R}^{n+1}$ by

$$
D^2_{ij}v = \tilde{D}^2_{ij}\tilde{V} + \langle (\tilde{\nabla}\tilde{V}, \tilde{n})\rangle L_{ij}
$$

$$
= \tilde{D}^2_{ij}\tilde{V} + b(x) \cdot L_{ij},
$$

where $b(x) := \langle (\tilde{\nabla}\tilde{V}, \tilde{n})\rangle(x)$. We remark in general $b(x)$ changes sign on $M$ and $|b(x)| \leq |\tilde{\nabla}\tilde{V}|$. 

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3 Proof of the main theorem

To prove Theorem 1.2, the main technical part of this paper is the following proposition.

**Proposition 3.1** Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional linear subspace, and $p$ be the orthogonal projection from $\mathbb{R}^{n+1}$ to $E$. Suppose $V : E \to \mathbb{R}$ is a $C^3$ convex function that satisfies $|\nabla V| \leq 1$. Define its extension to $\mathbb{R}^{n+1}$ by $\bar{V} := V \circ p$, and define the restriction of $\bar{V}$ to the immersed hypersurface $M$ by $v$. Suppose also that $M$ is $(k + 1)$-convex if $2 \leq k \leq n - 1$, i.e., the second fundamental form $L_{ij} \in \Gamma_{k+1}^+$. And suppose that $M$ is $n$-convex if $k = n$. Then for each $k$, constant $a > 1$ and function $\varphi \in C_c^\infty(U)$, there exists a constant $C$, which depends only on $k$, $n$ and $a$, such that

$$\int_M \sigma_k(D^2 v + aL)|\varphi|d\mu_M \leq C \int_M \sigma_k(L)|\varphi| + \sigma_{k-1}(L)|\nabla \varphi| + \cdots + |\nabla^k \varphi|d\mu_M. \quad (13)$$

Note that $C$ does not depend on $v$.

Our proof of Proposition 3.1 uses a multi-layer induction process and is quite complicated. We will first illustrate the idea of the proof of the proposition for the (easy) case $k = 2$ in Sect. 4, and we will finish the proof for all integers $k$ in Sect. 5. In the rest of this section, we will prove the main theorem assuming Proposition 3.1. The proof follows the outline similar to that of the main theorem in [7] which is inspired by the work of Castillon [5]. Since such an argument is standard and has appeared with minor difference in [7] already, we will only describe the difference of its proof from the one in [7] without repeating the whole paragraph.

**Brief outline of the Proof of Theorem 1.2** The difference of the proof from that in [7] is to first take a different function $f$ on $M$. Namely, instead of taking

$$f := \frac{\sigma_{k-1}(L)J_{E}^{\frac{1}{n-k}}}{\int_M \sigma_{k-1}(L)J_{E}^{\frac{1}{n-k}}d\mu_M} \quad (14)$$

we define

$$f := \frac{\sigma_l(L)|\varphi|^{\frac{n-l}{n-k}}J_{E}^{\frac{k-l}{n-k}}}{\int_M \sigma_l(L)|\varphi|^{\frac{n-l}{n-k}}J_{E}^{\frac{k-l}{n-k}}d\mu_M}. \quad (15)$$

$f(x)d\mu_M$ is again a probability measure on $M$. Thus we follow the same argument to derive inequality (37) in [7]:

$$\left(\omega_n f(x)J_E(x)\right)^{\frac{k-l}{n-l}} \cdot \frac{\sigma_l(\bar{D}^2 \bar{V} + (a - 1)L)}{(\det(\bar{D}^2 \bar{V})(x))^{\frac{k-l}{n-l}}} \leq \frac{\sigma_l(\bar{D}^2 \bar{V} + (a - 1)L)}{(\det(\bar{D}^2 \bar{V})(x))^{\frac{k-l}{n-l}}} \cdot \frac{\sigma_l(\bar{D}^2 \bar{V} + (a - 1)L)}{\sigma_l(\bar{D}^2 \bar{V})^{\frac{k-l}{n-l}}} \quad (16)$$

Denote the left hand side (resp. right hand side) of this inequality by $LHS$ (resp. $RHS$). By exactly the same argument, we construct the optimal transport map as in [7]. The existence and uniqueness of the map is proved by Brenier [1]. For more details on optimal transport maps, interested readers are referred to Villani’s books [16,17].

$$RHS \leq C_{n,k}^{\frac{1}{n-k+1}} \sigma_k(D^2 v + aL); \quad (17)$$
while on the other hand, by taking the newly defined function \( f \), we obtain

\[
LHS \geq \left( a - 1 \right)^{\frac{1}{2}} \frac{1}{2} \int_M \sum_2(D^2v + aL)\phi d\mu_M \geq \int_M \sum_2(D^2v + aL)\phi d\mu_M .
\]

Now we multiply \(|\phi|\) on both LHS and RHS, and integrate both of them over \( M \). This gives rise to

\[
(a - 1)^{\frac{1}{2}} \int_M \sum_1 L_k(D^2v + aL)\phi d\mu_M \leq C_n^2 \int_M \sum_2(D^2v + aL)\phi d\mu_M .
\]

This inequality plays the same role as inequality (47) in [7]. By applying Proposition 3.1, the argument after this inequality follows exactly in the same way as that in [7]. This finishes the brief description of the differences of the proof from the one in [7].

We remark here that regularity issue for optimal transport of non-convex domains will appear as it does in our previous paper [7]. Again, we can handle this problem by using the approximation argument together with L. Caffarelli’s regularity result [2–4] for strictly convex domains. Such a method has been demonstrated in [7] already, so we will not repeat it here.

4 \( k = 2 \) case of Proposition 3.1

In this section, we are going to prove

\[
\int_M \sum_2(D^2v + aL)\phi d\mu_M \leq C \int_M \sum_1 L_2(D^2v + aL)\phi d\mu_M.
\]

Proof First of all, we can write

\[
\int_M \sum_2(D^2v + aL)\phi d\mu_M = \int_M \frac{1}{2} \sum_2(D^2v + aL, D^2v + aL)\phi d\mu_M
\]

\[
= \int_M \frac{1}{2} \sum_2(D^2v, D^2v)\phi + a \sum_2(D^2v, L)\phi + \frac{a^2}{2} \sum_2(L, L)\phi d\mu_M
\]

\[
:= \frac{1}{2} I + a \cdot II + \frac{a^2}{2} III .
\]

To bound the term I, by Definition 2.4 and the integration by parts formula

\[
I := \int_M \sum_2(D^2v, D^2v)\phi d\mu_M
\]

\[
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\]
\[ = \int_M v_{ij} [T_1]_{ij} (D^2 v) \varphi d\mu_M \]

\[ = \int_M -v_j ([T_1]_{ij} (D^2 v))_i \varphi - v_j [T_1]_{ij} (D^2 v) \varphi_i d\mu_M. \tag{22} \]

For the first term, we apply the Riemannian curvature equation,

\[((T_1)_{ij} (D^2 v))_i = v_{ii,j} - v_{j,i} = R_{miij} v_m = (L_{mi} L_{ij} - L_{mj} L_{ii}) v_m = -[T_1]_{ij} (L) L_{mi} v_m.\]

Thus

\[\int_M -v_j ([T_1]_{ij} (D^2 v))_i \varphi d\mu_M = \int_M [T_1]_{ij} (L) L_{mi} v_j v_m \varphi d\mu_M. \tag{23} \]

By the assumption \( L_{ij} \in \Gamma_3^+\), \([T_1]_{ij} (L) L_{mi} \leq \sigma_2(L) g_{ij}. \) In fact, one can diagonalize \( L_{ij} \sim diag(\lambda_1, \ldots, \lambda_n)\); thus \([T_1]_{ij} (L) L_{mi}\) is also diagonalized,

\[ [T_1]_{ij} (L) L_{mi} \sim diag(\lambda_1 (\sigma_1 (L) - \lambda_1), \ldots, \lambda_n (\sigma_1 (L) - \lambda_n)). \]

We remark here that \([T_1]_{ij} (L) L_{mi} = L_{mj} L_{ii} - L_{mi} L_{ij}\) is a symmetric matrix. Note that

\[ \lambda_i (\sigma_1 (L) - \lambda_i) + \frac{\partial \sigma_3 (L)}{\partial \lambda_i} = \sigma_2 (L), \quad \text{for each } i. \]

Also \( L_{ij} \in \Gamma_3^+\) implies \( \frac{\partial \sigma_3 (L)}{\partial \lambda_i} \geq 0.\) Thus \( \lambda_i (\sigma_1 (L) - \lambda_i) \leq \sigma_2 (L)\) for each \( i.\) Therefore

\[ [T_1]_{ij} (L) L_{mi} \leq \sigma_2 (L) g_{ij}. \]

Applying this to (23), we obtain

\[\int_M -v_j ([T_1]_{ij} (D^2 v))_i \varphi d\mu_M \leq \int_M \sigma_2 (L) |\nabla v|^2 \cdot |\varphi| d\mu_M \leq \int_M \sigma_2 (L) |\varphi| d\mu_M. \tag{24} \]

with the last inequality following from \( |\nabla v| \leq 1.\)

For the second term \( \int_M -v_j [T_1]_{ij} (D^2 v) \varphi_i d\mu_M\) in (22), we use the relation \( D^2 v = \bar{D}^2 \bar{V} + b(x) L.\)

\[\int_M -v_j [T_1]_{ij} (D^2 v) \varphi_i d\mu_M \]

\[= \int_M -v_j [T_1]_{ij} (\bar{D}^2 \bar{V} + b(x) L) \varphi_i d\mu_M \]

\[= \int_M -v_j [T_1]_{ij} (\bar{D}^2 \bar{V}) \varphi_i d\mu_M + \int_M -v_j [T_1]_{ij} (b(x) L) \varphi_i d\mu_M. \tag{25} \]

Since \([T_1]_{ij} (\bar{D}^2 \bar{V}) \geq 0, [T_1]_{ij} (L) \geq 0, |b(x)| \leq 1,\) we have

\[ -[T_1]_{ij} (\bar{D}^2 \bar{V}) v_j \varphi_i \leq Tr([T_1]_{ij} (\bar{D}^2 \bar{V})) |\nabla \varphi| \cdot |\nabla v|, \]

and

\[ -b(x)[T_1]_{ij} (L) v_j \varphi_i \leq Tr([T_1]_{ij} (L)) |\nabla \varphi| \cdot |\nabla v|, \]
where \( Tr([T_1]_{ij}(\bar{D}^2 \bar{V})) \) denotes the trace of \([T_1]_{ij}(\bar{D}^2 \bar{V})\) and \( Tr([T_1]_{ij}(L)) \) denotes the trace of \([T_1]_{ij}(L)\). Thus

\[
\int_M -v_j [T_1]_{ij}(\bar{D}^2 \bar{V} + b(x)L)\phi_i d\mu_M \\
\leq \int_M Tr([T_1]_{ij}(\bar{D}^2 \bar{V}))|\nabla \phi| \cdot |\nabla v| + Tr([T_1]_{ij}(L))|\nabla \phi| \cdot |\nabla v| d\mu_M \\
= \int_M (n - 1)\sigma_1(\bar{D}^2 \bar{V})|\nabla \phi| \cdot |\nabla v| + (n - 1)\sigma_1(L)|\nabla \phi| \cdot |\nabla v| d\mu_M. \quad (26)
\]

Since \(|\nabla v| \leq 1\),

\[
\int_M \sigma_1(L)|\nabla \phi| \cdot |\nabla v| d\mu_M \leq \int_M \sigma_1(L)|\nabla \phi| d\mu_M. \quad (27)
\]

On the other hand,

\[
\int_M \sigma_1(\bar{D}^2 \bar{V})|\nabla \phi| \cdot |\nabla v| d\mu_M \\
\leq \int_M \sigma_1(\bar{D}^2 \bar{V})|\nabla \phi| d\mu_M \\
= \int_M \sigma_1(D^2 v - b(x)L)|\nabla \phi| d\mu_M \\
\leq \int_M \sigma_1(L)|\nabla \phi| d\mu_M + \int_M \sigma_1(D^2 v)|\nabla \phi| d\mu_M. \quad (28)
\]

By integration by parts, the last line is equal to

\[
\int_M \sigma_1(L)|\nabla \phi| d\mu_M - \int_M v_j ((\nabla \phi)_i d\mu_M \\
\leq \int_M \sigma_1(L)|\nabla \phi| d\mu_M + \int_M |\nabla v| \cdot |\nabla^2 \phi| d\mu_M \\
\leq \int_M \sigma_1(L)|\nabla \phi| + |\nabla^2 \phi| d\mu_M. \quad (29)
\]

Here we have used \(|\nabla|\nabla \phi|| \leq |\nabla^2 \phi|\). Plugging (27)-(29) into (26) and then (26), we obtain

\[
\int_M -v_j [T_1]_{ij}(\bar{D}^2 \bar{V} + b(x)L)\phi_i d\mu_M \\
\leq \int_M 2(n - 1)\sigma_1(L)|\nabla \phi| + (n - 1)|\nabla^2 \phi| d\mu_M. \quad (30)
\]
Thus the second term in (22) is bounded by $\int_M (n-1)|\nabla^2 \varphi| + 2(n-1)\sigma_1(L)|\nabla \varphi| d\mu_M$. Therefore, we conclude from (24) and (30) that

$$I := \int_M \Sigma_2(D^2v, D^2v)\varphi d\mu_M$$

$$= \int_M -v_j([T_1]_{ij}(D^2v))_i\varphi - v_j[T_1]_{ij}(D^2v)\varphi_i d\mu_M$$

$$\leq \int_M \sigma_2(L)|\varphi| + 2(n-1)\sigma_1(L)|\nabla \varphi| + (n-1)|\nabla^2 \varphi| d\mu_M$$

$$\leq C \int_M \sigma_2(L)|\varphi| + \sigma_1(L)|\nabla \varphi| + |\nabla^2 \varphi| d\mu_M,$$  \hspace{1cm} (31)

where $C$ depends only on $k$ and $n$. (Note that in this section, $k$ is equal to 2.) This finishes the estimate of $I$.

To bound the term $II$ in (21),

$$II := \int_M \Sigma_2(D^2v, L)\varphi d\mu_M$$

$$= \int_M v_{ij}[T_1]_{ij}(L)\varphi d\mu_M$$

$$= \int_M -v_j([T_1]_{ij}(L))_i\varphi - v_j[T_1]_{ij}(L)\varphi_i d\mu_M.$$ \hspace{1cm} (32)

Recall that $([T_1]_{ij}(L))_i = 0$ by Lemma 2.8. Further, since $L \in \Gamma_3^+ \subseteq \Gamma_2^+$, $[T_1]_{ij}(L) \geq 0$. This, with $|\nabla v| \leq 1$, implies that

$$II = \int_M -v_j[T_1]_{ij}(L)\varphi_i d\mu_M$$

$$\leq (n-1) \int_M \sigma_1(L) |\nabla v| \cdot |\nabla \varphi| d\mu_M$$

$$\leq (n-1) \int_M \sigma_1(L) |\nabla \varphi| d\mu_M.$$ \hspace{1cm} (33)

Finally, the estimate of term $III$ in (21) is straightforward, since $\int_M \sigma_2(L)\varphi d\mu_M \leq \int_M \sigma_2(L)|\varphi| d\mu_M$. In conclusion,

$$\int_M \sigma_2(D^2v + aL)\varphi d\mu_M = \frac{1}{2} I + a \cdot II + \frac{a^2}{2} \cdot III$$

$$\leq C \int_M (\sigma_2(L)|\varphi| + \sigma_1(L)|\nabla \varphi| + |\nabla^2 \varphi|) d\mu_M.$$ \hspace{1cm} (34)
This implies
\[
\int_M \sigma_2(D^2v + aL)|\varphi|d\mu_M = \frac{1}{2}I + a \cdot II + \frac{a^2}{2} \cdot III \\
\leq C \int_M (\sigma_2(L)|\varphi| + \sigma_1(L)|\nabla \varphi| + |\nabla^2 \varphi|)d\mu_M.
\] (35)

This completes the proof of Proposition 3.1 when \( k = 2 \).

\[ \square \]

5 General \( k \) case of Proposition 3.1

By the multilinearity of \( \Sigma_k(\cdot, \ldots, \cdot) \), it is sufficient to prove
\[
\int_M \Sigma_k(D^2v, D^2v, L, \ldots, L)|\varphi|d\mu_M \\
\leq C \int_M (\sigma_k(L)|\varphi| + \sigma_{k-1}(L)|\nabla \varphi| + \cdots + |\nabla^k \varphi|)d\mu_M
\] (36)

for each \( 0 \leq i_0 \leq k \). In the following, we first prove (36) for two initial values \( i_0 = 1 \) and \( i_0 = 2 \). We need two initial cases to start the induction argument since the index \( i_0 \) decreases by 2 in each induction step.

For \( i_0 = 1 \),
\[
\int_M \Sigma_k(D^2v, L, \ldots, L)|\varphi|d\mu_M \\
= \int_M v_{ij}[T_{k-1}]_{ij}(L)|\varphi|d\mu_M \\
= \int_M -v_j([T_{k-1}]_{ij}(L)) \varphi - v_j [T_{k-1}]_{ij}(L)|\varphi|d\mu_M. 
\] (37)

By Lemma 2.8, \( ([T_{k-1}]_{ij}(L))_i = 0 \). Thus
\[
\int_M \Sigma_k(D^2v, L, \ldots, L)|\varphi|d\mu_M \\
= \int_M -v_j [T_{k-1}]_{ij}(L)|\varphi|d\mu_M. 
\] (38)

Now \( L \in \Gamma^+_{k+1} \subseteq \Gamma^+_k \) implies \( [T_{k-1}]_{ij}(L) \geq 0 \). Thus
\[
\int_M -v_j [T_{k-1}]_{ij}(L)|\varphi|d\mu_M \leq \int_M Tr([T_{k-1}]_{ij}(L))|\nabla \varphi| \cdot |\nabla v|d\mu_M, 
\] (39)
where $Tr([T_{k-1}]_{ij}(L))$ denotes the trace of $[T_{k-1}]_{ij}(L)$, which is, by (4), equal to $(n - k + 1)\sigma_{k-1}(L)$. Hence

$$
\int_{M} -v_j [T_{k-1}]_{ij}(L) \varphi_i d\mu_M \leq C \int_{M} \sigma_{k-1}(L) |\nabla \varphi| d\mu_M. \quad (40)
$$

where $C$ depends only on $n$ and $k$.

To prove (36) for $i_0 = 2$,

$$
\int_{M} \Sigma_k(D^2 v, D^2 v, L, \ldots, L) \varphi d\mu_M = \int_{M} v_{ij} [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \varphi d\mu_M = \int_{M} -v_{ij} ([T_{k-1}]_{ij}(D^2 v, L, \ldots, L)) \varphi - v_j [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \varphi_i d\mu_M. \quad (41)
$$

By Lemma 2.9, $([T_{k-1}]_{ij}(D^2 v, L, \ldots, L))_i = -[T_{k-1}]_{ij}(L) L_m v_m$,

$$
\int_{M} \Sigma_k(D^2 v, D^2 v, L, \ldots, L) \varphi d\mu_M = \int_{M} [T_{k-1}]_{ij}(L, \ldots, L) L_m v_j v_m \varphi - [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) v_j \varphi_i d\mu_M. \quad (42)
$$

For the first term on the last line of (42), by (6)

$$
[T_{k-1}]_{ij}(L) L_m v_i = \sigma_k(L) \delta_{mj} - [T_k]_{mj}(L).
$$

Thus we have

$$
\int_{M} [T_{k-1}]_{ij}(L, \ldots, L) L_m v_j v_m \varphi d\mu_M = \int_{M} \sigma_k(L) |\nabla v|^2 \varphi d\mu_M - \int_{M} [T_k]_{mj}(L) v_j v_m \varphi d\mu_M. \quad (43)
$$

Note that $|\nabla v| \leq 1$, so

$$
\int_{M} \sigma_k(L) |\nabla v|^2 \varphi d\mu_M \leq \int_{M} \sigma_k(L) |\varphi| d\mu_M.
$$

Also, due to the fact that $L \in \Gamma^+_{k+1}$, $[T_k]_{mj}(L) \geq 0$. Thus

$$
- \int_{M} [T_k]_{mj}(L) v_j v_m \varphi d\mu_M \leq C \int_{M} \sigma_k(L) |\nabla v|^2 |\varphi| d\mu_M \leq C \int_{M} \sigma_k(L) |\varphi| d\mu_M. \quad (44)
$$
For the second term in (42), we first use $D^2 v = \tilde{D}^2 \tilde{V} + b(x)L$. Using the facts $|b(x)| \leq 1$, $\tilde{D}^2_{ij} \tilde{V} \geq 0$, $[T_{k-1}]_{ij}(L) \geq 0$ and $|\nabla v| \leq 1$, we obtain

$$\int [T_{k-1}]_{ij}(D^2 v, L, \ldots, L)v_j \varphi_i d\mu_M$$

$$= \int -b(x)[T_{k-1}]_{ij}(L, \ldots, L)v_j \varphi_i - [T_{k-1}]_{ij}(\tilde{D}^2 \tilde{V}, L, \ldots, L)v_j \varphi_i d\mu_M$$

$$\leq \int Tr([T_{k-1}]_{ij}(L))|\nabla \varphi| + Tr([T_{k-1}]_{ij}(\tilde{D}^2 \tilde{V}, L, \ldots, L))|\nabla \varphi| \cdot |\nabla v| d\mu_M$$

$$\leq C \int \sigma_{k-1}(L)|\nabla \varphi| + \Sigma_{k-1}(\tilde{D}^2 \tilde{V}, L, \ldots, L)|\nabla \varphi| d\mu_M.$$  

(45)

We now apply $D^2 v = \tilde{D}^2 \tilde{V} + b(x)L$ again. Then

$$C \int \sigma_{k-1}(L)|\nabla \varphi| + \Sigma_{k-1}(\tilde{D}^2 \tilde{V}, L, \ldots, L)|\nabla \varphi| d\mu_M$$

$$= C \int (1 - b(x))\sigma_{k-1}(L)|\nabla \varphi| + \Sigma_{k-1}(D^2 v, L, \ldots, L)|\nabla \varphi| d\mu_M$$

$$\leq C \int \sigma_{k-1}(L)|\nabla \varphi| + \Sigma_{k-1}(D^2 v, L, \ldots, L)|\nabla \varphi| d\mu_M.$$  

(46)

Now by our earlier proof of 36 for $i_0 = 1$,

$$\int \Sigma_k(D^2 v, L, \ldots, L)|\varphi| d\mu_M \leq C \int \sigma_{k-1}(L)|\nabla \varphi| d\mu_M$$

for any integer $1 \leq k \leq n$ and function $\varphi$. So in particular, this inequality holds for $k - 1 \leq n$ and function $|\nabla \varphi|$. Namely

$$\int \Sigma_{k-1}(D^2 v, L, \ldots, L)|\nabla \varphi| d\mu_M \leq C \int \sigma_{k-2}(L)|\nabla^2 \varphi| d\mu_M.$$  

(47)

Here we have used the fact $|\nabla|\nabla \varphi|| \leq |\nabla^2 \varphi|$. To conclude, by (45)-(47), we obtain

$$\int [T_{k-1}]_{ij}(D^2 v, L, \ldots, L)v_j \varphi_i d\mu_M$$

$$\leq C \int \sigma_{k-1}(L)|\nabla \varphi| + \Sigma_{k-1}(\tilde{D}^2 \tilde{V}, L, \ldots, L)|\nabla \varphi| d\mu_M$$

$$\leq C \int \sigma_{k-1}(L)|\nabla \varphi| + \sigma_{k-2}(L)|\nabla^2 \varphi| d\mu_M.$$  

(48)
This finishes the estimate of the second term in (42). Therefore
\[
\int_M \Sigma_k(D^2 v, D^2 v, L, \ldots, L) \varphi d\mu_M \leq C \int_M \sigma_k(L) |\varphi| + \sigma_{k-1}(L) |\nabla \varphi| + \sigma_{k-2}(L) |\nabla^2 \varphi| d\mu_M.
\]
(49)
This finishes the proof of (36) for \( i_0 = 2 \).
Now we aim to prove (36) for \( i_0 = 3, \ldots, k \); i.e.
\[
I_{k,m}(\varphi) := \int_M \Sigma_k(D^2 v, D^2 v, L, \ldots, L) \varphi d\mu_M
\]
\[
\leq C \int_M \sigma_k(L) |\varphi| + \sigma_{k-1}(L) |\nabla \varphi| + \cdots + |\nabla^k \varphi| d\mu_M,
\]
(50)
for some \( C \) depending only on \( n \) and \( k \). To begin the inductive argument, we assume (50) holds for \( m = 1, \ldots, i_0 - 1 \) where \( i_0 \geq 3 \), which we call the inductive assumption in the following; with this we will show (50) for \( m = i_0 \). To simplify \( I_{k,i_0}(\varphi) \), we apply a similar integration by parts argument as the one to show formula (128) in [7]. Such an argument splits the estimate of \( I_{k,i_0} \) into four parts.
\[
I_{k,i_0}(\varphi) = (i_0 - 1) \frac{C_{i_0-2}}{k} I_{k,i_0-2}(\varphi)
\]
\[
+ \frac{C_{i_0-2}}{C_{i_0-2}} \cdot J_{k,i_0-2}(\varphi) + (i_0 - 1) \frac{C_{i_0-3}}{C_{i_0-2}} \cdot K_{k,i_0-3}(\varphi)
\]
\[
+ N_{k,i_0-1}(\varphi),
\]
(51)
where
\[
I_{k,l}^{(u)}(\varphi) := \int_M \Sigma_k(D^2 v, \ldots, D^2 v, L, \ldots, L) u(x) \varphi(x) d\mu_M,
\]
(52)
\[
J_{k,l}^{(u)}(\varphi) := \int_M [T_k]_{mj}(D^2 v, \ldots, D^2 v, L, \ldots, L) v_j v_m u(x) \varphi(x) d\mu_M,
\]
(53)
\[
K_{k,l}^{(u)}(\varphi) := \int_M [T_{k-1}]_{ij}(D^2 v, \ldots, D^2 v, L, \ldots, L) v_{mj} v_j v_m u(x) \varphi(x) d\mu_M,
\]
(54)
and
\[
N_{k,l}^{(u)}(\varphi) := \int_M [T_{k-1}]_{ij}(D^2 v, \ldots, D^2 v, L, \ldots, L) v_j \varphi_i u(x) d\mu_M.
\]
(55)
We remark that in the above definitions, \( \varphi(x) \) is the test function that has appeared in the statement of the main theorem, while \( u(x) \) is a bounded coefficient function which may vary from line to line in our later argument.
In the following we will call any term that takes the form \( I_{k,l}^{(u)}(\varphi) \), \( J_{k,l}^{(u)}(\varphi) \), \( K_{k,l}^{(u)}(\varphi) \), \( N_{k,l}^{(u)}(\varphi) \) the \( I \)-type term, the \( J \)-type term, the \( K \)-type and the \( N \)-type term, respectively. In the special case when \( u = 1 \), we will denote \( I_{k,l}^{(1)}(\varphi) \), \( J_{k,l}^{(1)}(\varphi) \), \( K_{k,l}^{(1)}(\varphi) \), \( N_{k,l}^{(1)}(\varphi) \) by \( I_{k,l}(\varphi) \), \( J_{k,l}(\varphi) \), \( K_{k,l}(\varphi) \), \( N_{k,l}(\varphi) \) for simplicity.

In order to prove (50) we need to estimate the \( I \)-type term, the \( J \)-type term, the \( K \)-type and the \( N \)-type term individually. The main idea of the proof is that each of the four terms in (51) is of a decreased index \( i_0 - 1 \), \( i_0 - 2 \) or \( i_0 - 3 \); if we can bound them by the \( I \)-type terms with indices strictly less than \( i_0 \), then we can apply the inductive assumption to derive the estimate. We will show both the \( I \)-type term and the \( J \)-type term are bounded by \( \sum_{l=1} I_{k,s}(\varphi) \); the \( N \)-type term is bounded by \( \sum_{s=1} I_{k,s}(\varphi) \); and the \( K \)-type term is inductively bounded by the \( K \)-type term \( \sum_{s=1} K_{k,s}(\varphi) \) and the \( N \)-type term \( \sum_{s=1} N_{k,s}(\varphi) \), thus bounded by

\[
\sum_{s=1} I_{k,s}(\varphi) + \sum_{s=1} I_{k,s}(\nabla \varphi).
\]

We begin by looking at the \( I \)-type term and the \( J \)-type term. They can be estimated using similar arguments as the ones to prove Lemma 6.3 and Claim 2 in [7]. We present the results here without proof.

**Proposition I** For any bounded function \( u(x) \), let us denote \( \max_{x \in M} |u(x)| \) by \( U_0 \). Then for any \( l \geq 0 \) and function \( \varphi \), there exist positive constants \( A_0, \ldots, A_l \) depending on \( U_0 \), \( k \), and \( n \), such that

\[
I_{k,l}^{(u)}(\varphi) \leq \sum_{s=0}^{l} A_s I_{k,s}(|\varphi|).
\]

(56)

In particular, one can choose \( A_l = U_0 \).

**Proposition J** For any bounded function \( u(x) \), let us denote \( \max_{x \in M} |u(x)| \) by \( U_0 \). Then for any \( l \geq 0 \) and function \( \varphi \), there exist positive constants \( A_0, \ldots, A_l \) depending on \( U_0 \), \( k \), and \( n \), such that

\[
J_{k,l}^{(u)}(\varphi) \leq \sum_{s=0}^{l} A_s I_{k,s}(|\varphi|).
\]

(57)

On the other hand, the \( K \)-type and the \( N \)-type estimates are quite different from those in [7]. They will be the focus of the arguments below. We begin by proving the \( N \)-type estimate.

**Proposition N** For any bounded function \( u(x) \), let us denote \( \max_{x \in M} |u(x)| \) by \( U_0 \). Then for any \( l \geq 0 \) and function \( \varphi \), there exist positive constants \( A_0, \ldots, A_l \) depending on \( U_0 \), \( k \), and \( n \), such that

\[
N_{k,l}^{(u)}(\varphi) \leq \sum_{s=0}^{l} \tilde{A}_s I_{k-l,s}(|\nabla \varphi|).
\]

(58)

**Proof** Recall that

\[
N_{k,l}^{(u)}(\varphi) := \int_M [T_{k-1}]_{ij} \left( D^2 v, \ldots, D^2 v, L, \ldots, L \right) v_j \varphi_i u(x) d\mu_M.
\]

(59)
By $D^2 v = \bar{D}^2 \bar{V} + b(x)L$ with $|b(x)| \leq 1$, we have

$$N_{k,l}^{(u)} (\varphi) = \sum_{s=0}^{l} \int_{\mathcal{M}} b_s(x)[T_{k-1}]_{ij} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) v_j \varphi_i u(x) d\mu_M, \quad (60)$$

where $b_s(x)$ are some bounded functions with bounds only depending on $n$ and $k$. Note that

$$[T_{k-1}]_{ij} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) \geq 0,$$

and that $|\nabla v| \leq 1$. Thus

$$[T_{k-1}]_{ij} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) v_j \varphi_i u \leq U_0 \cdot Tr ([T_{k-1}]_{ij} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L)) \cdot |\nabla \varphi|$$

$$= U_0 \cdot \frac{n - (k - 1)}{k - 1} \Sigma_{k-1} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) \cdot |\nabla \varphi|. \quad (61)$$

So

$$N_{k,l}^{(u)} (\varphi) \leq \sum_{s=0}^{l} \tilde{A}_s \cdot \int_{\mathcal{M}} \Sigma_{k-1} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) \cdot |\nabla \varphi| d\mu_M. \quad (62)$$

Here $\tilde{A}_s$ are constants only depending on $U_0, k, n$. We then apply $D^2 v = \bar{D}^2 \bar{V} + b(x)L$ again. By the multilinearity of $\Sigma_{k-1} (\cdot, \cdot, \cdot, \cdot)$,

$$\sum_{s=0}^{l} \tilde{A}_s \cdot \int_{\mathcal{M}} \Sigma_{k-1} (\bar{D}^2 \bar{V}, \ldots, \bar{D}^2 \bar{V}, L, \ldots, L) \cdot |\nabla \varphi| d\mu_M$$

$$= \sum_{s=0}^{l} \tilde{b}_s(x) \cdot \int_{\mathcal{M}} \Sigma_{k-1} (D^2 v, \ldots, D^2 v, L, \ldots, L) \cdot |\nabla \varphi| d\mu_M$$

$$= \sum_{s=0}^{l} I_{k-1,s} (|\nabla \varphi|), \quad (63)$$

where $\tilde{b}_s(x)$ are bounded functions with bounds depending on $U_0, k, n$.

By Proposition I,

$$\sum_{s=0}^{l} I_{k-1,s} (|\nabla \varphi|) \leq \sum_{s=0}^{l} \bar{A}_s I_{k-1,s} (|\nabla \varphi|). \quad (64)$$

Here $\bar{A}_s$ are positive constants which are different from the ones in (62). But again they only depend on the bounds of $\tilde{b}_s(x), n$ and $k$; thus they only depend on $U_0, k, n$. In conclusion,

$$N_{k,l}^{(u)} (\varphi) \leq \sum_{s=0}^{l} \bar{A}_s I_{k-1,s} (|\nabla \varphi|), \quad (65)$$
for some $\tilde{A}_i$ depending on $U_0$, $k$, and $n$. This ends the proof of Proposition N.

**Proposition K** For any bounded function $u(x)$, let us denote $\max_{x \in M} |u(x)|$ by $U_0$. Then for any $i_0 \geq 3$ and function $\varphi$, there exist positive constants $A_0, \ldots, A_{i_0-3}$, and $\tilde{A}_0, \ldots, \tilde{A}_{i_0-3}$ depending on $U_0$, $k$, and $n$ such that

$$K_{k, i_0-3}^{(-1)}(\varphi) \leq \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0-3} \tilde{A}_s I_{k-1,s}(|\nabla\varphi|).$$

(66)

Before proving Proposition K, we first show the following two inequalities.

**Lemma 5.1** Let $v$ be a function on $M$ with $|\nabla v| \leq 1$. For any integer $3 \leq i_0 \leq k$,

$$K_{k,0}^{(\pm)|\nabla v|^{i_0-3}}(\varphi) \leq \sum_{s=0}^{i_0} A_s I_{k,s}(|\varphi|) + \tilde{A}_0 I_{k-1,0}(|\nabla\varphi|)$$

$$= A_0 I_{k,0}(|\varphi|) + A_1 I_{k,1}(|\varphi|) + \tilde{A}_0 I_{k-1,0}(|\nabla\varphi|), \text{ when } i_0 \text{ is odd; (67)}$$

and

$$K_{k,1}^{(\pm)|\nabla v|^{i_0-4}}(\varphi) \leq \sum_{s=0}^{i_0} A_s I_{k,s}(|\varphi|) + \tilde{A}_s I_{k-1,s}(|\nabla\varphi|), \text{ when } i_0 \text{ is even. (68)}$$

**Proof** To prove (67) when $i_0$ is odd, we first write

$$K_{k,0}^{(\pm)|\nabla v|^{i_0-3}}(\varphi) : = \pm \int_M [T_{k-1}]_{ij}(L, \ldots, L) v_{mi} v_j v_m |\nabla v|^{i_0-3} \varphi d\mu_M$$

$$= \pm \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{i_0} v_j |\nabla v|^{i_0-1} \varphi d\mu_M$$

$$\mp \int_M ([T_{k-1}]_{ij}(L, \ldots, L))_i \frac{1}{i_0} v_j |\nabla v|^{i_0-1} \varphi d\mu_M$$

$$\mp \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{i_0} v_{ij} |\nabla v|^{i_0-1} \varphi d\mu_M$$

$$\mp \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{i_0} v_j \varphi |\nabla v|^{i_0-1} d\mu_M.$$  

(69)

Note that by Lemma 2.8,

$$([T_{k-1}]_{ij}(L, \ldots, L))_i = 0.$$  

(70)

So we only need to estimate the rest two terms. First of all,

$$\mp \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{i_0} v_{ij} |\nabla v|^{i_0-1} \varphi d\mu_M$$

$$= \mp \frac{1}{i_0} \int_M \Sigma_k (D^2 v, L, \ldots, L) |\nabla v|^{i_0-1} \varphi d\mu_M$$

$$= \mp \frac{1}{i_0} I_{k,1}^{(|\nabla v|^{i_0-1})}(\varphi),$$  

(71)
by the definition of $I_{k,l}^{(u)}(\varphi)$ in (52). Now by the $I$-type estimate proved in Proposition I,

$$\mp \frac{1}{l_0 - 1} I_{k,1}^{(|\nabla v|^{i_0 - 1})}(\varphi) \leq \sum_{s=0}^{1} A_s I_{k,s}(\varphi) = A_0 I_{k,0}(\varphi) + A_1 I_{k,1}(\varphi), \quad (72)$$

for some constants $A_0$ and $A_1$ depending only on $n$ and $k$.

Another term

$$\mp \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{l_0 - 1} v_j \varphi_i |\nabla v|^{i_0 - 1} d\mu_M$$

is an $N$-type term. In fact,

$$\mp \int_M [T_{k-1}]_{ij}(L, \ldots, L) \frac{1}{l_0 - 1} v_j \varphi_i |\nabla v|^{i_0 - 1} d\mu_M = \mp \frac{1}{l_0 - 1} N^{(|\nabla v|^{i_0 - 1})}(\varphi). \quad (73)$$

Therefore by Proposition N, this term is bounded by

$$\tilde{A}_0 I_{k-1,0}(\varphi) = \tilde{A}_0 \int_M \sigma_{k-1}(L)(\nabla \varphi) d\mu_M. \quad (74)$$

The estimates of these two terms lead to

$$K_{k,0}^{(|\nabla v|^{i_0 - 3})}(\varphi) \leq \sum_{s=0}^{1} A_s I_{k,s}(\varphi) + \tilde{A}_0 I_{k-1,0}(\nabla \varphi)$$

$$= A_0 I_{k,0}(\varphi) + A_1 I_{k,1}(\varphi) + \tilde{A}_0 I_{k-1,0}(\nabla \varphi). \quad (75)$$

This finishes the proof of (67).

To prove (68), since $i_0$ is even, we have

$$K_{k,1}^{(|\nabla v|^{i_0 - 4})}(\varphi) := \pm \int_M [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) v_{mi} v_j v_m |\nabla v|^{i_0 - 4} d\mu_M$$

$$= \pm \int_M [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \frac{1}{l_0 - 2} v_j |\nabla v|^{i_0 - 2} d\mu_M$$

$$= \mp \int_M [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \frac{1}{l_0 - 2} v_j |\nabla v|^{i_0 - 2} d\mu_M$$

$$= \mp \int_M [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \frac{1}{l_0 - 2} v_j |\nabla v|^{i_0 - 2} d\mu_M. \quad (76)$$

For the first term in the last equality of (76)

$$\mp \int_M [T_{k-1}]_{ij}(D^2 v, L, \ldots, L) \frac{1}{l_0 - 2} v_j |\nabla v|^{i_0 - 2} d\mu_M,$$

we recall Lemma 2.9

$$[T_{k-1}]_{ij}(D^2 v, L, \ldots, L) = -[T_{k-1}]_{ij}(L) L_{mi} v_m.$$
Thus
\[
\begin{align*}
\mp & \int_{M} ([T_{k-1}]_{ij} (D^{2}v, L, \ldots, L)) \frac{1}{i_{0} - 2} v_{j} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \pm \frac{1}{i_{0} - 2} \int_{M} [T_{k-1}]_{ij} (L) L_{m_{i}v_{j}} v_{m} |\nabla v|^{i_{0}-2} \varphi d\mu_{M}.
\end{align*}
\]
(77)

By formula (6), and the definition of \(I_{k,l}^{(u)}(\varphi), J_{k,l}^{(u)}(\varphi)\) in (52), (53)
\[
\begin{align*}
& \pm \frac{1}{i_{0} - 2} \int_{M} [T_{k-1}]_{ij} (L, \ldots, L) L_{mi} v_{mj} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \int_{M} \{ \pm C_{1} \Sigma_{k} (L, \ldots, L) \delta_{jm} \mp C_{2} [T_{k}]_{jm} (L, \ldots, L) \} v_{m} v_{j} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \int_{M} \pm C_{1} \Sigma_{k} (L, \ldots, L) |\nabla v|^{i_{0}} \varphi \mp C_{2} [T_{k}]_{jm} (L, \ldots, L) v_{m} v_{j} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \pm C_{1} I_{k,0}^{(|\nabla v|^{i_{0}})} (\varphi) \mp C_{2} J_{k,0}^{(|\nabla v|^{i_{0}-2})} (\varphi),
\end{align*}
\]
(78)

where \(C_{1}, C_{2}\) are positive constants depending only on \(n\) and \(k\). Note that \(|\nabla v| \leq 1\). Thus by Proposition I and Proposition J, the \(I\)-type term \(\pm C_{1} I_{k,0}^{(|\nabla v|^{i_{0}})} (\varphi)\) and the \(J\)-type term \(\mp C_{2} J_{k,0}^{(|\nabla v|^{i_{0}-2})} (\varphi)\) are both bounded by \(A_{0} I_{k,0}(|\varphi|)\) for some positive constants \(A_{0}\), namely
\[
\pm C_{1} I_{k,0}^{(|\nabla v|^{i_{0}})} (\varphi) \mp C_{2} J_{k,0}^{(|\nabla v|^{i_{0}-2})} (\varphi) \leq A_{0} I_{k,0}(|\varphi|).
\]
(79)

By (77)–(79), we obtain
\[
\pm \int_{M} ([T_{k-1}]_{ij} (D^{2}v, L, \ldots, L)) \frac{1}{i_{0} - 2} v_{j} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \leq A_{0} I_{k,0}(|\varphi|).
\]

This completes the estimate of the term \(\mp \int_{M} ([T_{k-1}]_{ij} (D^{2}v, L, \ldots, L)) \frac{1}{i_{0} - 2} v_{j} |\nabla v|^{i_{0}-2} \varphi d\mu_{M}\) in (76).

Next we need to estimate the second term in the last equality of (76). Note that by the definition of \(I_{k,l}^{(u)}(\varphi)\) in (52),
\[
\begin{align*}
\mp & \int_{M} [T_{k-1}]_{ij} (D^{2}v, L, \ldots, L) \frac{1}{i_{0} - 2} v_{ij} |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \mp \frac{1}{i_{0} - 2} \int_{M} \Sigma_{k} (D^{2}v, D^{2}v, L, \ldots, L) |\nabla v|^{i_{0}-2} \varphi d\mu_{M} \\
& = \mp \frac{1}{i_{0} - 2} J_{k,2}^{(|\nabla v|^{i_{0}-2})} (\varphi).
\end{align*}
\]
(80)

Thus using Proposition I, we obtain
\[
\pm \frac{1}{i_{0} - 2} J_{k,2}^{(|\nabla v|^{i_{0}-2})} (\varphi) \leq \sum_{s=0}^{2} A_{s} I_{k,s}(|\varphi|).
\]
Finally we estimate the last term \( \mp \int_M |T_{k-1}|_{ij}(D^2 v, L, \ldots, L) \frac{1}{i_0 - 2} v_j \phi_i |\nabla v|^{i_0 - 2} d\mu_M \) in (76).

\[
\mp \int_M |T_{k-1}|_{ij}(D^2 v, L, \ldots, L) \frac{1}{i_0 - 2} v_j \phi_i |\nabla v|^{i_0 - 2} d\mu_M = \mp \frac{1}{i_0 - 2} N_{k,1}^{i(|\nabla v|^{i_0 - 2})}(\varphi). \quad (81)
\]

Thus by Proposition N, this is bounded by \( \sum_{s=0}^1 \tilde{A}_s I_{k-1,s}(|\nabla \varphi|) \). By the above estimates, we conclude that

\[
K_{k,1}^{(\pm |\nabla v|^{i_0 - 4})}(\varphi) \leq 2 \sum_{s=0}^1 A_{s,k,s}(|\varphi|) + \sum_{s=0}^1 \tilde{A}_s I_{k-1,s}(|\nabla \varphi|).
\]

\( \square \)

**Proof of Proposition K** If \( i_0 = 3 \) or \( 4 \), \( K_{k,i_0 - 3}^{(-1)}(\varphi) \) is equal to either \( K_{k,0}^{(-1)}(\varphi) \) or \( K_{k,1}^{(-1)}(\varphi) \). The estimates of these two terms have already been proved in Lemma 5.1. Consequently, we can assume \( i_0 \geq 5 \) from now on. To estimate the \( K \)-type term \( K_{k,i_0 - 3}^{(-1)}(\varphi) \) for \( i_0 \geq 5 \), we first apply a similar argument as the one to derive formula (154) in [7]. This implies

\[
K_{k,i_0 - 3}^{(-1)}(\varphi) = \frac{1}{2} I_{k,i_0 - 2}^{(|\nabla v|^4)}(\varphi) - C_1 I_{k,i_0 - 4}^{(|\nabla v|^4)}(\varphi) + C_2 J_{k,i_0 - 4}^{(|\nabla v|^2)}(\varphi)
\]

\[
+ C_3 K_{k,i_0 - 5}^{(|\nabla v|^2)}(\varphi) + \frac{1}{2} N_{k,i_0 - 3}^{(|\nabla v|^2)}(\varphi).
\]

Here \( C_1, C_2, C_3 \) are positive constants depending only on \( n \) and \( k \). For detailed steps, one can refer to the similar arguments (156)–(161) present in [7]. By Proposition I, Proposition J, and Proposition N, there exist positive constants \( A_s \) and \( \tilde{A}_s \) for \( s = 0, \ldots, i_0 - 3 \) depending only on \( k, n, C_1, C_2 \) and \( \max_{x \in M} |\nabla v(x)| \leq 1 \), thus depending only on \( n \) and \( k \), such that

\[
\frac{1}{2} I_{k,i_0 - 2}^{(|\nabla v|^2)}(\varphi) \leq \sum_{s=0}^{i_0 - 2} A_s I_{k,s}(|\varphi|).
\]

\( (83) \)

\[
-C_1 I_{k,i_0 - 4}^{(|\nabla v|^4)}(\varphi) \leq \sum_{s=0}^{i_0 - 4} A_s I_{k,s}(|\varphi|).
\]

\( (84) \)

\[
C_2 J_{k,i_0 - 4}^{(|\nabla v|^2)}(\varphi) \leq \sum_{s=0}^{i_0 - 4} A_s I_{k,s}(|\varphi|).
\]

\( (85) \)

\[
\frac{1}{2} N_{k,i_0 - 3}^{(|\nabla v|^2)}(\varphi) \leq \sum_{s=0}^{i_0 - 3} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|).
\]

\( (86) \)

Here \( A_s \) in each inequality may be different. By these inequalities, (82) is deduced to

\[
K_{k,i_0 - 3}^{(-1)}(\varphi) \leq \sum_{s=0}^{i_0 - 2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0 - 3} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|) + C_3 K_{k,i_0 - 5}^{(|\nabla v|^2)}(\varphi). \quad (87)
\]

The induction argument stops if either \( i_0 - 5 = 0 \) or \( i_0 - 5 = 1 \); otherwise we perform a similar argument to \( K_{k,i_0 - 5}^{(|\nabla v|^2)}(\varphi) \) to obtain

\[
K_{k,i_0 - 5}^{(|\nabla v|^2)}(\varphi) \leq \sum_{s=0}^{i_0 - 4} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0 - 5} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|) + C_3 K_{k,i_0 - 7}^{(-1)|\nabla v|^4}(\varphi). \quad (88)
\]
We remark here that the constant $C_3$ in (88) may be different from the one in (87). But they are both positive constants depending only on $n$ and $k$, so we use the same notation when it is not necessary to distinguish them.

Such an inductive argument will stop at the $q$-th step, where $q = \frac{i_0 - 3}{2}$. If $i_0$ is odd, then when the induction stops, we obtain

\[ K_{k,i_0-3}^{(-1)}(\varphi) \leq \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0-3} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|) + C_3 K_{k,0}^{(-1) - \frac{3}{2} |\nabla \varphi|^{i_0-3}}(\varphi). \]  

(89)

If $i_0$ is even, then when the induction stops, we obtain

\[ K_{k,i_0-3}^{(-1)}(\varphi) \leq \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0-3} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|) + C_3 K_{k,1}^{(-1) - \frac{3}{2} |\nabla \varphi|^{i_0-4}}(\varphi). \]  

(90)

By Lemma 5.1 and the inductive formula (88), we conclude that

\[ K_{k,i_0-3} \leq \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0-3} \tilde{A}_s I_{k-1,s}(|\nabla \varphi|). \]

This finishes the proof of Proposition K. \qed

We are now ready to apply these four types of estimates to show (3.1) for $m = i_0$. With Proposition I, J and K, and the inductive formula (51), we obtain

\[ I_{k,i_0}(\varphi) := \int_M \sum_{s=0}^{i_0} \Sigma_k (D^2 v, \ldots, D^2 v, L, \ldots, L) \varphi d \mu_M \]

\[ \leq \sum_{s=0}^{i_0-2} A_s \int_M \sum_{s=0}^{s} \Sigma_k (D^2 v, \ldots, D^2 v, L, \ldots, L) |\varphi| d \mu_M \]

\[ + \sum_{s=0}^{i_0-1} \tilde{A}_s \int_M \sum_{s=0}^{s} \Sigma_{k-1} (D^2 v, \ldots, D^2 v, L, \ldots, L) |\nabla \varphi| d \mu_M. \]  

(91)

The first sum above is equal to

\[ \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|); \]

and the second sum is equal to

\[ \sum_{s=0}^{i_0-1} \tilde{A}_s I_{k,s}(|\nabla \varphi|). \]

As the index $s$ has dropped below $i_0$, both sums can be bounded by the inductive assumption, i.e. (50) holds for $1 \leq m \leq i_0 - 1$ and any function $\varphi$. Therefore we have, for some $C$ depending only on $n$ and $k$,
\[
\int_{M} \sum_{k}(D^2v, \ldots, D^2v, L, \ldots, L)\varphi d\mu_M
\]
\[
\leq \sum_{s=0}^{i_0-2} A_s I_{k,s}(|\varphi|) + \sum_{s=0}^{i_0-1} \tilde{A}_s I_{k,s}(|\nabla \varphi|)
\]
\[
\leq C \int_{M} (\sigma_k(L)|\varphi| + \sigma_{k-1}(L)|\nabla \varphi| + \cdots + |\nabla^k \varphi|) d\mu_M. \quad (92)
\]

This completes the proof of (36).

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