Analysis of the $p$-adic $q$-Volkenborn integrals: An approach to generalized Apostol-type special numbers and polynomials and their applications

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Abstract: The goal of this paper is to construct some families of generalized Apostol-type special numbers and polynomials attached to the Dirichlet character. Using the $p$-adic $q$-Volkenborn integrals including the bosonic and the fermionic $p$-adic integrals on $p$-adic integers, we give generating functions for these numbers and polynomials. These numbers and polynomials are associated with some well-known special numbers and polynomials such as the Peters polynomials, the Boole polynomials, the generalized Apostol–Bernoulli numbers and polynomials, the generalized Apostol–Euler numbers and polynomials, the generalized Apostol–Dahee numbers and polynomials, the Stirling numbers and the Bernoulli numbers of the first kind. We investigate some properties of these numbers and polynomials with their generating functions. Using these generating functions and their functional equation, we derive some identities and relations including some special numbers and polynomials. Finally, we give $p$-adic $q$-Volkenborn integral representations for these numbers and polynomials with combinatorial sums.

ABOUT THE AUTHOR

Yilmaz Simsek received the PhD degree in (Generalized Dedekind and Hardy sums) at Cukurova University Adana of Turkey. His research interests are in the areas: Modular Forms, $p$-adic Analysis, $q$-series, Special numbers, Special Polynomials, Special Functions, Generating Functions, Dedekind and Hardy Sums, Umbral Algebra, and Umbral Analysis. He has published research articles related above areas in distinguished international journals of mathematical and engineering sciences. He has been invited to many international conference, seminars, visiting professor, and other scientific activities. He is referee and editor of many mathematical journals. Professor Simsek’s carrier has been started from Adana, Cukurova University, Mersin University, and now continue at Antalya, Akdeniz University.

PUBLIC INTEREST STATEMENT

Recently, the special numbers and polynomials have been studied by many authors because these numbers and polynomials have been used in many branches of pure and applied mathematics, physics, engineering and computer science. Polynomials are used in elementary word problems to complicated problems in the sciences, approximate or curve fit experimental data, calculate beam deflection under loading, represent some properties of gases, and perform computer-aided geometric design in engineering; moreover, polynomials are used as solutions of differential equations. Using the $p$-adic $q$-Volkenborn integrals as a novel method, we construct new families of generalized Apostol-type special numbers and polynomials attached to the Dirichlet character. We give relations between these polynomials and some well-known special numbers and polynomials such as the Bernoulli and Euler polynomials, Bernstein polynomials, Peters polynomials, the Boole polynomials, the generalized Apostol-type numbers and polynomials, and the Stirling numbers.
1. Introduction

The main motivation of this paper is to construct generating functions for some certain new families of numbers and polynomials including generalized Apostol–Dahee numbers attached to Dirichlet character $\chi$ and the others. In order to construct generating functions for the generalized Apostol-type numbers and polynomials, we can use the $p$-adic $q$-Volkenborn integrals techniques including integral equations of the bosonic and the fermionic $p$-adic integrals on $p$-adic integers. The generalized Apostol-type numbers and polynomials are related to some well-known numbers and polynomials such as the generalized Bernoulli numbers and polynomials and the generalized Euler numbers and polynomials, the Apostol–Bernoulli numbers and polynomials and the Apostol–Euler numbers and polynomials, the Stirling numbers of the first and of the second kind, the Daehee numbers and polynomials and also the Changhee numbers and polynomials. The special functions and numbers have many applications in almost all branches of mathematics and the other science.

In this paper, we use the following notations, definitions and some families of the special numbers and polynomials.

Throughout this paper $\lambda$ is a complex number. $\chi$ is a non-trivial Dirichlet character with conductor $d$.

Firstly, we start with very important family of special numbers that are the Apostol–Bernoulli polynomials, $B_n(x;\lambda)$, defined by means of the following generating function:

$$F_A(t, x; \lambda) = \frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!},$$

where $|t| < 2\pi$ when $\lambda = 1$ and $|t| < |\log \lambda|$ when $\lambda \neq 1$) with

$$B_n(\lambda) = B_n(0; \lambda)$$

denotes so-called $\lambda$-Bernoulli numbers (cf. Kim, 2006; Luo & Srivastava, 2011; Ozden, Simsek, & Srivastava, 2010; Srivastava, 2011; Srivastava & Choi, 2001; Srivastava, Kim, & Simsek, 2005; see also the references cited in each of these earlier works). We easily see that

$$B_n = B_n(0; 1)$$

is a Bernoulli polynomials of the first kind (cf. Apostol, apostol; Cakic & Milovanovic, 2004; Charalambides, 2002; Dere, Simsek, & Srivastava, 2013; Djordjevic & Milovanovic, 2014; Hu & Kim, 2017; Jang & Pak, 2002; Jordan, 1950; Kim, 2002, 2003, 2006, 2007, 2008, 2009; Kim & Kim, 2013, 2012, 2014; Kim, Kim & Seo, 2013; Kim & Son, 2007; Kim, Kim, Dolgy, & Seo, 2016; Kim, Rim, Simsek, & Kim, 2008; Lu & Srivastava, 2011; Luo & Srivastava, 2011; Ozden et al., 2010; Qi, 2014; Roman, 2005; Schikhof, 1984; Simsek, 2005, 2013, 2016, in press; Srivastava, 2000, 2011; Srivastava & Choi, 2012; Srivastava et al., 2005; Srivastava, Ozarslan, & Kaanoglu, 2013; see also the references cited in each of these earlier works).
Srivastava (2000) gave many novel identities, relations and formulas related to the classical Bernoulli and Euler polynomials. He also gave finite sums involving the interpolation functions of the Bernoulli and Euler polynomials.

Kim et al. (2008) defined the $\lambda$-Bernoulli polynomials (Apostol-type Bernoulli polynomials), $\mathfrak{B}_n(x; \lambda)$ by means of the following generating function:

$$F_{\lambda}(t, x; \lambda) = \frac{\log \frac{\lambda + t}{\lambda} e^t - 1}{e^t - 1} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda) \frac{t^n}{n!}$$  \hspace{1cm} (1.2)

(||t|| < 2\pi when $\lambda = 1$ and ||t|| < ||log $\lambda$$|| when $\lambda \neq 1$) with
$$\mathfrak{B}_n(\lambda) = \mathfrak{B}_n(0; \lambda)$$

denotes the $\lambda$-Bernoulli numbers (Apostol-type Bernoulli numbers) (cf. Jang & Pak, 2002; Kim et al., 2008; Lu & Srivastava, 2011; Simsek, in press; Srivastava et al., 2005, 2013). A few of the $\lambda$-Bernoulli numbers are given as follows:

$$\mathfrak{B}_0(\lambda) = \frac{\log \lambda}{\lambda - 1},$$

and

$$\mathfrak{B}_1(\lambda) = \frac{\lambda - 1 - \lambda \log \lambda}{(\lambda - 1)^2}.$$  \hspace{1cm} (1.3)

If $n > 1$, than we have
$$\mathfrak{B}_n(\lambda) = \lambda (\mathfrak{B}_n(\lambda) + 1)^n.$$  

Kim et al. gave a relation between these numbers and the Frobenius-Euler numbers (Kim et al., 2008, Theorem 1, p. 439) as follows:

$$\mathfrak{B}_0(\lambda) = \frac{\log \lambda}{\lambda - 1} H_0 \left( \frac{1}{\lambda} \right)$$

and

$$\mathfrak{B}_n(\lambda) = \frac{\log \lambda}{\lambda - 1} H_n \left( \frac{1}{\lambda} \right) + \frac{n}{\lambda - 1} H_{n-1} \left( \frac{1}{\lambda} \right)$$

where $H_n \left( \frac{1}{\lambda} \right)$ denotes the Frobenius–Euler numbers, defined by means of the following generating function:

Let $u$ be a complex numbers with $u \neq 1$.

$$F_{\lambda}(t, u) = \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}$$  \hspace{1cm} (1.3)

(cf. Kim & Kim, 2012; Kim et al., 2008, Theorem 1, p. 439; Simsek, 2005; Srivastava et al., 2005; see also the references cited in each of these earlier works).

Let $r$ be a positive integer, and let $\lambda \neq 1$ be any nontrivial $r$-th root of 1. Observe that the $\lambda$-Bernoulli numbers are reduced to the Apostol–Bernoulli numbers and the twisted Bernoulli numbers (cf. Kim et al., 2008, Theorem 1, p. 439; see also the references cited in each of these earlier works).

We (Simsek, 2016) gave the following functional equation:

$$F_{\lambda}(t, u) = \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}$$  \hspace{1cm} (1.3)
Using the above functional equation, we have (cf. Simsek, 2016)

\[ F_{p}(t, 0; \lambda) = \frac{\log \lambda}{\lambda - 1} F_{p}\left( t, \frac{1}{\lambda} \right) + F_{p}(t, 0; \lambda). \]

The generalized Apostol–Bernoulli numbers attached to Dirichlet character, \( B_{n, \chi}(\lambda) \), are defined by means of the following generating function:

\[ \sum_{p=0}^{d-1} j! e^{jt(\chi(j))} t^{p} \lambda^{d} - 1 = \sum_{n=0}^{\infty} B_{n, \chi}(\lambda) \frac{t^{n}}{n!}, \tag{1.4} \]

where \(|t + \log \lambda| < \frac{\pi}{\lambda} \) (cf. Apostol, 1951; Kim, 2007, 2008; Kim & Son, 2007; Srivastava et al., 2005; see also the references cited in each of these earlier works).

By combining (1.4) with (1.1), we have

\[ B_{n, \chi}(\lambda) = d^{n-1} \sum_{j=0}^{d-1} j! e^{jt(\chi(j))} B_{j}(\lambda) \]

for the trivial character \( \chi \equiv 1 \), we have (cf. Apostol, 1951; Kim, 2007, 2008; Kim & Son, 2007; Srivastava et al., 2005).

\[ B_{n}(\lambda) = B_{n, 1}(\lambda) \]

The Apostol-Euler polynomials of first kind, \( E_{n}(x, \lambda) \), are defined by means of the following generating function:

\[ F_{p1}(t, x; k, \lambda) = \frac{2}{\lambda e^{t} + 1} e^{tx} = \sum_{n=0}^{\infty} E_{n}(x, \lambda) \frac{t^{n}}{n!}, \tag{1.5} \]

\(|t| < \pi\) when \( \lambda = 1 \) and \(|t| < |\ln(\lambda)|\) when \( \lambda \neq 1 \), \( \lambda \in \mathbb{C} \). (cf. Dere et al., 2013; Kim, 2006, Ozden et al., 2010; Srivastava, 2011; Srivastava & Choi, 2001; Srivastava et al., 2005; see also the references cited in each of these earlier works). Substituting \( x = 0 \) into (1.5), we have The first kind Apostol-Euler numbers of order \( k \):

\[ E_{n}(\lambda) = E_{n}^{(1)}(0, \lambda) \]

Setting \( \lambda = 1 \) into (1.5), one has the first kind Euler numbers (cf. Djordjevic & Milovanovic, 2014; Hu & Kim, 2017; Jang & Pak, 2002; Jordan, 1950; Kim, 2002, 2003, 2006, 2007, 2008, 2009; Kim & Kim, 2012, 2013, 2014; Kim, Kim & Seo, 2013; Kim & Son, 2007; Kim, Kim, Dolgy, & Seo, 2016; Kim, Rim, Simsek, & Kim, 2008; Lu & Srivastava, 2011; Luo & Srivastava, 2011; Ozden et al., 2010; Qi, 2014; Roman, 2005; Schikhof, 1984; Simsek, 2005, 2013, 2016; in press; Srivastava, 2000, 2011; Srivastava & Choi, 2012; Srivastava et al., 2005; Srivastava, Özarslan, & Kaanoglu, 2013; see also the references cited in each of these earlier works).

\[ E_{n} = E_{n}^{(1)}(1) \]

The generalized Apostol-Euler numbers attached to Dirichlet character, \( E_{n, \chi}(x, \lambda) \), are defined by means of the following generating function:

\[ 2 \sum_{j=0}^{d-1} j! e^{jt(\chi(j))} t^{p} \lambda^{d} - 1 = \sum_{n=0}^{\infty} E_{n, \chi}(\lambda) \frac{t^{n}}{n!} \tag{1.6} \]
where $|t + \log \lambda| < \frac{C}{\lambda}$ (cf. Kim, 2007, 2008; Srivastava et al., 2005; see also the references cited in each of these earlier works).

By combining (1.6) with (1.5), we have

$$E_{n,t}(\lambda) = d^n \sum_{j=0}^{d-1} \chi'(j) E_{n,j} \left( \frac{j}{\lambda} \right)$$

for the trivial character $\chi \equiv 1$, we have (cf. Kim, 2007, 2008; Srivastava et al., 2005)

$$E_{n,1}(\lambda) = E_{n,1}(\lambda)$$

The Stirling numbers of the first kind, $S_1(n,k)$ are defined by means of the following generating function:

$$F_{S_1}(t, k) = \frac{(\log(1 + t))^k}{k!} \sum_{n=0}^{\infty} S_1(n,k) \frac{t^n}{n!}.$$ (1.7)

Some properties of the Stirling numbers of the first kind are given as follows:

- $S_1(0,0) = 1, S_1(0,k) = 0$ if $k > 0, S_1(n,0) = 0$ if $n > 0, S_1(n,k) = 0$ if $k > n$. For these numbers, recurrence relation is given by
- $S_1(n+1,k) = -nS_1(n,k) + S_1(n,k-1)$

(cf. Charalambides, 2002; Roman, 2005; Simsek, 2013; and see also the references cited in each of these earlier works).

The Bernoulli polynomials of the second kind, $b_n(x)$ are defined by means of the following generating function (cf. Roman, 2005, pp. 113–117; see also the references cited in each of these earlier works):

$$F_{b_2}(t, x) = \frac{t}{\log(1 + t)} (1 + t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}.$$ (1.8)

The Bernoulli numbers of the second kind $b_n(0)$ are defined by means of the following generating function:

$$F_{b_2}(t) = \frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n(0) \frac{t^n}{n!}.$$ (1.9)

These numbers are computed by the following formula:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} b_k(0) = n! \delta_{n,1},$$

where $\delta_{n,1}$ denotes the Kronecker delta (cf. Roman, 2005, p. 116). The Bernoulli polynomials of the second kind are defined by

$$b_n(x) = \int_x^{x+1} (u)_n du.$$
The numbers \( b_n(0) \) are also so-called the Cauchy numbers (cf. Charalambides, 2002; Qi, 2014; Roman, 2005, p. 116; see also the references cited in each of these earlier works). Kim et al. (2016) gave a computation method for the Bernoulli polynomials of the second kind are defined as follows:

\[
b_n(x) = \sum_{l=0}^{n} \frac{S_1(n, l)}{l+1} \left( (x+1)^{l+1} - x^{l+1} \right)
\]

and also Roman (2005, p. 115) gave

\[
b_n(x) = b_n(0) + \sum_{l=1}^{n} \frac{nS_1(n-l, l-1)}{l} x^l.
\]

Using the above formula for the Bernoulli polynomials and numbers of the second kind, few of these numbers are computed as follows, respectively:

\[
b_0(x) = 1,
\]
\[
b_1(x) = \frac{1}{2}(2x + 1),
\]
\[
b_2(x) = \frac{1}{6}(6x^2 - 1),
\]
\[
b_3(x) = \frac{1}{4}(4x^3 - 6x^2 + 1),
\]
\[
b_4(x) = \frac{1}{30}(30x^4 - 120x^3 + 120x^2 - 19),
\]

and

\[
b_0(0) = 1, \ b_1(0) = \frac{1}{2}, \ b_2(0) = -\frac{1}{12}, \ b_3(0) = \frac{1}{24}, \ b_4(0) = -\frac{19}{720}, \ldots.
\]

The generating function for the Stirling numbers of the second kind, \( S_2(n, k) \) is given as follows:

\[
F_2(t, k) = \frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!},
\]

where \( k \in \mathbb{N}_0 \). Some properties of these numbers are given as follows:

\[
S_2(0, 0) = 1, \ S_2(n, k) = 0 \text{ if } k > n, \ S_2(n, 0) = 0 \text{ if } n > 0 \text{ and also recurrence relation is given by (cf. Charalambides, 2002; Roman, 2005; Srivastava & Choi, 2012; see also the references cited in each of these earlier works):}
\]

\[
S_2(n+1, k) = S_2(n, k-1) + kS_2(n, k)
\]

The Peters polynomials \( S_k(x; \lambda, \mu) \), which are Sheffer polynomials, are defined by means of the following generating functions (cf. Jordan, 1950; Roman, 2005):

\[
\frac{1}{(1+(1+t)^{-\mu})(1+t)^\lambda} = \sum_{n=0}^{\infty} S_k(x; \lambda, \mu) \frac{t^n}{n!}
\]

If \( \mu = 1 \) then the polynomials \( S_k(x; \lambda, \mu) \) are reduced to the Boole polynomials. If \( \lambda = 1 \) and \( \mu = 1 \), then these polynomials are also reduced to the Changhee polynomials, which are given in Section 3 (cf. Kim & Kim, 2014; Roman, 2005).
In this section, we need the following definitions and notations.

Let \( \mathbb{Z}_p \) be a set of \( p \)-adic integers. Let \( \mathbb{K} \) be a field with a complete valuation and \( C^1(\mathbb{Z}_p \to \mathbb{K}) \) be a set of continuous derivative functions. That is \( C^1(\mathbb{Z}_p \to \mathbb{K}) \) is contained in the following set

\[
\left\{ f: \mathbb{K} \to \mathbb{K}; f(x) \text{ is differentiable and } \frac{d}{dx} f(x) \text{ is continuous} \right\}.
\]

**Definition 1** (Schikhof, 1984, p. 167, Definition 55.1) The Volkenborn integral (\( p \)-adic bosonic integral) of the function \( f \in C^1(\mathbb{Z}_p \to \mathbb{K}) \)

is given by

\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),
\]

(1.11)

We observe that \( \mu_1(x) = \mu_1 \left( x + p^N \mathbb{Z}_p \right) \) is the Haar distribution, defined by (cf. Hu & Kim, 2017; Kim, 2009, 2002, Schikhof, 1984; see also the references cited in each of these earlier works):

\[
\mu_1 \left( x + p^N \mathbb{Z}_p \right) = \frac{1}{p^N}.
\]

In work of Kim (2002), the Volkenborn integral is also so-called the bosonic \( p \)-adic Volkenborn integral on \( \mathbb{Z}_p \).

The Volkenborn integral in terms of the Mahler coefficients is given by the following formula:

\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n,
\]

where (cf. Schikhof, 1984, p. 168, Proposition 55.3):

\[
f(x) = \sum_{n=0}^{\infty} a_n \left( \begin{array}{c} x \\ j \end{array} \right) \in C^1(\mathbb{Z}_p \to \mathbb{K})
\]

From the above observation, we have the theorem:

**THEOREM 1**

\[
\int_{\mathbb{Z}_p} \left( \begin{array}{c} x \\ j \end{array} \right) d\mu_1(x) = \frac{(-1)^j}{j+1}.
\]

(1.12)

Theorem 1 was proved by Schikhof (1984).

Let \( f: \mathbb{Z}_p \to \mathbb{K} \) be an analytic function and \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) with \( x \in \mathbb{Z}_p \). The Volkenborn integral of this analytic function is given by (cf. Schikhof, 1984, Proposition 55.4, p. 168):

\[
\int_{\mathbb{Z}_p} \left( \sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n \left( \sum_{x=0}^{\infty} x^n d\mu_1(x) \right)
\]

The following property is very important to our new results on special numbers:
\begin{equation}
\int_{\mathbb{Z}_p} f(x + m) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + \sum_{j=0}^{m-1} f'(j),
\end{equation}

(1.13)

\begin{equation}
f'(j) = \left. \frac{d}{dx} f(x) \right|_{x=j}
\end{equation}

where

(cf. Kim, 2002, 2006; Schikhof, 1984; see also the references cited in each of these earlier works)

The \( p \)-adic \( q \)-Volkenborn integral was defined by Kim (2002). The distribution on \( \mathbb{Z}_p \) is given by

\[ \mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{|p^N|}, \]

where \( q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \) and (cf. Kim, 2002)

\[ [x] = [x q] = \left\{ \begin{array}{ll}
\frac{1-q}{1-q}, & q \neq 1 \\
1, & x, q = 1
\end{array} \right. \]

The \( p \)-adic \( q \)-integral of a function \( f \in C^1(\mathbb{Z}_p \to \mathbb{K}) \) is defined by Kim (2002) as follows:

\[ \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{|p^N|} \sum_{x=0}^{p^N-1} f(x) q^x. \]

Observe that

\[ \lim_{q \to 1} |x:q| = x. \]

The Witt’s formula for the Bernoulli numbers and polynomials are given as follows, respectively

\[ \int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_n \]

(1.14)

and (cf. Kim, 2002, 2006; Schikhof, 1984; see also the references cited in each of these earlier works)

\[ \int_{\mathbb{Z}_p} (z + x)^n d\mu_q(x) = B_n(z) \]

(1.15)

Let \( f \in C^1(\mathbb{Z}_p \to \mathbb{K}) \) and

\[ E^d f(x) = f(x + d). \]

Kim (2008, Theorem 1) defined the following functional equation for the \( q \)-bosonic \( p \)-adic Volkenborn integral on \( \mathbb{Z}_p \) as follows:

\[ q^n \int_{\mathbb{Z}_p} E^d f(x) d\mu_q(x) - \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \frac{q-1}{\log q} \left( \sum_{j=0}^{n-1} q^j f'(j) + \log q \sum_{j=0}^{n-1} q^j f(j) \right), \]

(1.16)

where \( n \) is a positive integer.
The fermionic $p$-adic integral on $\mathbb{Z}_p$ is given by
\[
\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}-1} (-1)^x f(x) \tag{1.17}
\]
where (cf. Kim, 2009, 2006).

\[
\mu_{-1}(x) = \mu_{-1}(x + p^N \mathbb{Z}_p) = \frac{(-1)^x}{p^N}
\]
Let $p$ be a fixed prime. For a fixed positive integer $d$ with $(p, d) = 1$, we set (see Kim, 2002)
\[
\mathbb{X} = \mathbb{X}_d = \lim_{N \to \infty} \mathbb{Z}/p^N \mathbb{Z}, \mathbb{X}_1 = \mathbb{Z}_p, \mathbb{X}^* = \bigcup_{0 < a < dp} a + dp \mathbb{Z}_p
\]
and $a + dp^N \mathbb{Z}_p = \{ x \in \mathbb{X} | x \equiv a (\text{mod} \ (dp^N)) \}$ where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

Let $f \in UD(\mathbb{Z}_p^*, \mathbb{C}_p)$. In work of Kim (2003), we see that
\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{\mathbb{X}} f(x) d\mu_1(x). \tag{1.18}
\]

The following integral equation is given by Kim (2008, Theorem 3):
\[
q^d \int_{\mathbb{Z}_p} E^d f(x) d\mu_{-q}(x) - (-1)^d \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^{d-1-l} q^l f(j). \tag{1.19}
\]
where $d$ is a positive integer.

When $d = 1$, the above integral equation is reduced to the fermionic integral equation (cf. Kim et al., 2013):
\[
\int_{\mathbb{Z}_p} \left( \frac{x}{j} \right) d\mu_{-1}(x) = \frac{(-1)^j}{2^j} \tag{1.20}
\]

We summarize our paper as follows: In Section 2, using the bosonic $p$-adic integral on $\mathbb{Z}_p$, we give generating functions for the generalized Apostol–Dahee numbers and polynomials attached to Dirichlet character $\chi$. These numbers are related to many well-known numbers and polynomials. We also give not only relations between these numbers, the $\lambda$-Bernoulli numbers, the Stirling numbers, the Bernoulli numbers of the second kind, the generalized Bernoulli numbers, the generalized Euler numbers and the Daehee numbers and polynomials, but also a bosonic integral representation of these numbers. In Section 3, we give $p$-adic $q$-Volkenborn integral representation for the generalized Apostol-type Daehee numbers and polynomials with combinatorial sums.

2. Generalized Apostol–Dahee numbers attached to Dirichlet character $\chi$ on the bosonic $p$-adic integral

In this section, using the bosonic $p$-adic integral on $\mathbb{Z}_p$, we construct generating functions for the generalized Apostol–Dahee numbers and polynomials attached to Dirichlet character $\chi$. We give relations between these numbers, the $\lambda$-Bernoulli numbers, the Stirling numbers, the Bernoulli numbers of the second kind, the generalized Bernoulli numbers, the generalized Euler numbers and the Daehee numbers and polynomials. We also give bosonic integral representation of these numbers. Firstly, we give some standard notations for the Volkenborn integral.

Let $\chi$ be a non-trivial Dirichlet character with conductor $d$. Let $\lambda \in \mathbb{Z}_p$. We set
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\[ f(x, t; \lambda) = x^t (1 + \lambda t)^x \chi(x). \]  

(2.1)

Substituting (2.1) into (1.16), we get

\[
\int x^t (1 + \lambda t)^x \chi(x) d\mu_q(x) = \frac{(q - 1) \log (\lambda + \lambda^2 t)}{(\lambda q)^d (1 + \lambda t)^d - 1} \sum_{j=0}^{d-1} (\lambda q)^j (1 + \lambda t)^j \chi(j)
\]

\[
+ \frac{q - 1}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j (1 + \lambda t)^j \chi(j).
\]

(2.2)

Using the above integral equation, we define the following generating function for generalized Apostol–Daehee numbers attached to Dirichlet character \( \chi \) with conductor \( d \) as follows:

\[
F_{q,t}(t; q, \lambda, \chi) = \frac{(q - 1)(\log q + \log \lambda + \log (1 + \lambda t))}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j (1 + \lambda t)^j \chi(j)
\]

and

\[
F_{q,t}(t; q, \lambda, \chi) = \sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(\lambda, q) \frac{t^n}{n!}.
\]

(2.3)

The generalized Apostol–Daehee polynomials attached to Dirichlet character \( \chi \) are defined by means of the following generating function:

\[
F_{q,t}(z, t; q, \lambda, \chi) = F_{q,t}(t; q, \lambda, \chi) (1 + \lambda t)^z = \sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(z; \lambda, q) \frac{t^n}{n!}
\]

(2.4)

so that, obviously,

\[
\mathfrak{D}_{n,\chi}(\lambda, q) = \mathfrak{D}_{n,\chi}(0; \lambda, q).
\]

Combining the above function with (2.3), we have

\[
\sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(z; \lambda, q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{z}{n} x^n \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(\lambda, q) \frac{t^n}{n!}.
\]

Therefore

\[
\sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(z; \lambda, q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathfrak{D}_{j,\chi}(\lambda, q) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we arrive at the following theorem:

**THEOREM 2** Let \( n \in \mathbb{N}_q \). Then we have

\[
\mathfrak{D}_{n,\chi}(z; \lambda, q) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathfrak{D}_{j,\chi}(\lambda, q).
\]

(2.5)
Comparing the coefficients of \( t^n \) on both sides of the above equation, we arrive at the following theorem:

**Theorem 3** Let \( m \in \mathbb{N} \). Then we have

\[
\Phi_{m-1, \lambda}(\lambda, q) = \left( \frac{(q - 1) \log \lambda}{\log q} + q - 1 \right) \sum_{j=0}^{d-1} q^j x(j) \sum_{l=0}^{m} \binom{m}{l} \lambda^{m+j-l-1} b_{m-l}(0) \\
\times \sum_{n=0}^{1} d^{n-1} B_n \left( \frac{j}{d} (\lambda q)^{-1} \right) S_l(l, n) \frac{t^n}{m!} \\
+ \frac{q - 1}{\log q} \sum_{j=0}^{d-1} q^j x(j) \sum_{n=0}^{m-1} d^{n-1} B_n \left( \frac{j}{d} (\lambda q)^{-1} \right) S_l(m - 1, n) \frac{t^n}{m!}.
\]

If \( q \to 1 \) in (2.6), we get the following corollary:
COROLLARY 1  Let $m \in \mathbb{N}$. Then we have

$$
D_{m-1,\lambda}(\lambda) = \frac{\log \lambda}{m} \sum_{j=0}^{d-1} \chi(j) \sum_{i=0}^{m} \binom{m}{i} (\lambda^{m+i-1} b_{m-i}(0))
\times \sum_{n=0}^{m-1} d^{n-1} B_n \left( \frac{i}{d} \right) S_1(n, m) + \sum_{j=0}^{d-1} \lambda^j (i)
\times \sum_{n=0}^{m-1} d^{n-1} B_n \left( \frac{i}{d} \right) S_1(m, n - 1).
$$

(2.7)

If $\lambda = 1$, we get the following corollary:

COROLLARY 2  Let $m \in \mathbb{N}_0$. Then we have

$$
D_{m,\lambda} = \sum_{j=0}^{d-1} \chi(j) \sum_{m=0}^{d-1} d^{n-1} B_n \left( \frac{j}{d} \right) S_1(m, n).
$$

(2.8)

By combining (2.8) with the following well-known formula for the generalized Bernoulli numbers

$$
B_{n,\lambda} = d^{n-1} \sum_{j=0}^{d-1} \chi(j) B_n \left( \frac{j}{d} \right),
$$

we arrive at the following result.

COROLLARY 3  Let $m \in \mathbb{N}$. Then we have

$$
D_{m,\lambda} = \sum_{n=0}^{m} B_{n,\lambda} S_1(m, n).
$$

(2.9)

Remark 2  If $\chi \equiv 1$, then (2.8) reduces to the following result:

$$
D_m = \sum_{n=0}^{m} B_n S_1(m, n)
$$

(c.f. Kim & Kim, 2013, Equation (2.10).

Substituting $\lambda t = e^u - 1$ into (2.3), we get

$$
\frac{(q - 1)(\log \lambda + \log q + u)}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j e^{\lambda q} \chi(j)
= \sum_{n=0}^{\infty} \frac{\Phi_{n,\lambda}(\lambda, q) (e^{u} - 1)^n}{\lambda^n}.
$$

By substituting (1.10) into the above equation,

$$
\frac{(q - 1)(\log \lambda + \log q + u)}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j e^{\lambda q} \chi(j)
= \sum_{n=0}^{\infty} \frac{\Phi_{n,\lambda}(\lambda, q)}{\lambda^n} \sum_{m=0}^{\infty} S_2(m, n) \frac{u^m}{m!}
$$

(2.10)
Since $S_2(m,n) = 0$ if $n > m$, we get

$$
\sum_{m=0}^{\infty} \sum_{n=0}^{m-1} \frac{\Phi_{m,n}(\lambda, q)S_2(m-1, n)}{\lambda^n} \frac{u^m}{m!}
$$

$$
= \frac{q - 1}{\log q} \sum_{m=0}^{\infty} m d^{m-1} \sum_{j=0}^{d-1} (\lambda q)^j x(j) B_n \left( \frac{j}{d} j \lambda q \right) \frac{u^m}{m!}
$$

$$
+ \frac{q - 1}{m \log q} \sum_{m=0}^{\infty} d^{m-1} \sum_{j=0}^{d-1} (\lambda q)^j x(j) B_m \left( \frac{j}{d} j \lambda q \right) \frac{u^m}{m!}.
$$

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

**THEOREM 4**  Let $m \in \mathbb{N}$. Then we have

$$
\sum_{n=0}^{m-1} \Phi_{m,n}(\lambda, q)S_2(m-1, n) \frac{x^n}{\lambda^n} = \frac{d^{m-1}(q - 1) \log (\lambda q)}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j x(j) B_n \left( \frac{j}{d} j \lambda q \right) \frac{u^m}{m!}
$$

$$
+ \frac{(q - 1)d^{m-1}}{m \log q} \sum_{j=0}^{d-1} (\lambda q)^j x(j) B_m \left( \frac{j}{d} j \lambda q \right) \frac{u^m}{m!}.
$$

(2.11)

If $\lambda = 1$ and $q \rightarrow 1$ in (2.11), we get the following corollary:

**COROLLARY 4**  Let $m \in \mathbb{N}$. Then we have

$$
\sum_{n=0}^{m} \Phi_{m,n}S_2(m,n) = d^{m-1} \sum_{j=0}^{d-1} x(j) B_n \left( \frac{j}{d} \right)
$$

or

$$
B_{m,x} = \sum_{n=0}^{m} \Phi_{m,n}S_2(m,n).
$$

(2.12)

**Remark 3**  If $\chi \equiv 1$, then (2.12) reduces to the following result:

$$
B_m = \sum_{n=0}^{m} D_nS_2(m,n)
$$

(cf. Kim & Kim, 2013, Theorem 4).

Using (2.10) with $S_2(m,n) = 0$ if $n > m$, we get

$$
\frac{(q - 1)(\log \lambda q + u)}{\lambda^q q - 1} \log q \sum_{j=0}^{d-1} (\lambda q)^j e^{ij} x(j) \left( 1 - \frac{1}{e^m} \right)
$$

$$
= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \Phi_{m,n}(\lambda, q)S_2(m,n) \frac{t^m}{m!} \lambda^n.
$$

By substituting (1.3) into the above equation, since $S_2(m,n) = 0$ if $n > m$, we get
Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

**THEOREM 5**

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{\Phi_{n,d}(\lambda, q)S_2(m, n)}{\lambda^n} \frac{t^m}{m!} = \frac{(q - 1)(\log \lambda q)}{(\lambda^d q^d - 1) \log q} \sum_{m=0}^{\infty} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) H_m \left( \frac{j}{d^d q^d} \right) \frac{t^m}{m!} + \frac{q - 1}{\log q} \sum_{m=0}^{\infty} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) H_{m-1} \left( \frac{j}{d^d q^d} \right) \frac{t^m}{m!}.
\]

If $q \to 1$ in (2.13), we get the following corollary:

**COROLLARY 5**  Let $m$ be a positive integer. Then we have

\[
\sum_{n=0}^{m} \frac{\Phi_{n,d}(\lambda, q)S_2(m, n)}{\lambda^n} = \frac{d^m(q - 1)(\log \lambda q)}{(\lambda^d q^d - 1) \log q} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) H_m \left( \frac{j}{d^d q^d} \right) + \frac{d^m q - 1}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) H_{m-1} \left( \frac{j}{d^d q^d} \right).
\]

If $\chi \equiv 1$, then (2.14) reduces to the following result:

**COROLLARY 6**  Let $m$ be a positive integer. Then we have

\[
H_{m-1} \left( \frac{1}{x} \right) = \frac{1}{m} \sum_{n=0}^{m} D_{n,d}(\lambda)S_2(m, n) \chi(n).
\]

If $q \to 1$ in (2.3), then we give the following functional equation:

\[
dF_{\beta} \left( \frac{e^u - 1}{\lambda}; 1, \lambda, \chi \right) = \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) F_{\beta} \left( du, \frac{j}{d^d q^d} \right).
\]

Combining this equation with (2.3) and (1.2), we get

\[
\sum_{n=0}^{\infty} \frac{\Phi_{n,d}(\lambda) (e^u - 1)^n}{\lambda^n n!} = \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) \sum_{m=0}^{\infty} d^{m-1} \Psi_m \left( \frac{j}{d^d q^d} \right) \frac{u^m}{m!}.
\]

Therefore

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{\Phi_{n,d}(\lambda) S_2(m, n)}{\lambda^n} \frac{u^m}{m!} = \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) \sum_{m=0}^{\infty} d^{m-1} \Psi_m \left( \frac{j}{d^d q^d} \right) \frac{u^m}{m!}.
\]

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:
Theorem 6
\[
\sum_{n=0}^{m} \frac{\mathfrak{D}_{\lambda,n}(\lambda)S_s(m,n)}{\chi^n} = d^{m-1} \sum_{j=0}^{d-1} \left( \frac{\lambda^j q^j}{\chi(j)} \right) \mathfrak{B}_m \left( \frac{j}{d}, \lambda^j \right).
\] (2.15)

If \( \chi \equiv 1 \), then (2.15) reduces to the following result:

Corollary 7
\[
\mathfrak{B}_m(\lambda) = \sum_{n=0}^{m} D_{\lambda,n}(\lambda)S_s(m,n) \frac{1}{\chi^n}.
\]

3. \( p \)-adic \( q \)-Volkenborn integral applications of the generalized Apostol-type Daehee numbers and polynomials

Here, we give \( p \)-adic \( q \)-Volkenborn integral representation the generalized Apostol-type Daehee numbers attached to the Dirichlet character. By applying \( p \)-adic Volkenborn integrals to the generalized Apostol-type Daehee polynomials on \( \mathbb{Z}_p \) we give two different \( p \)-adic integral formulas for this polynomials including combinatorial sums.

We firstly give \( p \)-adic \( q \)-Volkenborn integral representation for the generalized Apostol-type Daehee numbers as follows:

Combining (2.2) with (2.3), we get
\[
\sum_{n=0}^{\infty} \left( (x)_n \lambda^{n+q} x(x)d\mu_0(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{D}_{\lambda,n}(\lambda) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( C \) on both sides of the above equation, we arrive at the bosonic integral representation of the generalized Daehee numbers by the following theorem:

Theorem 7  Let \( n \in \mathbb{N}_q \) Then we have
\[
\mathfrak{D}_{\lambda,n}(\lambda) = \int_{\mathbb{C}_p} (x)_n \lambda^{n+q} x(x)d\mu_0(x).
\] (3.1)

Secondly, by applying bosonic integral to Equation (2.5), we get
\[
\int_{\mathbb{Z}_p} \mathfrak{D}_{\lambda,n}(\lambda) d\mu_1(z) = \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \lambda^{n-j} \mathfrak{D}_{\lambda,j}(\lambda) \int_{\mathbb{Z}_p} (z)_{n-j} d\mu_1(z).
\]

Combining the above equation with Equation (1.12), we arrive at the following theorem:

Theorem 8  Let \( n \in \mathbb{N}_q \) Then we have
\[
\int_{\mathbb{Z}_p} \mathfrak{D}_{\lambda,n}(\lambda) d\mu_1(z) = \sum_{j=0}^{n} (-1)^{n-j} \left( \begin{array}{c} n \\ j \end{array} \right) \frac{\lambda^{n-j}}{n-j+1} \mathfrak{D}_{\lambda,j}(\lambda).
\]

By applying fermionic integral to Equation (2.5), we get
\[
\int_{\mathbb{Z}_p} \mathfrak{D}_{\lambda,n}(\lambda) d\mu_{-1}(z) = \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \lambda^{n-j} \mathfrak{D}_{\lambda,j}(\lambda) \int_{\mathbb{Z}_p} (z)_{n-j} d\mu_{-1}(z).
\]
Combining the above equation with Equation (1.20), we arrive at the following theorem:

**Theorem 9** Let $n \in \mathbb{N}_p$. Then we have

$$
\int_{\mathbb{Z}_p} \Phi_{n, \lambda}(z;\lambda, q) d\mu_{1}(z) = \sum_{j=0}^{n} \binom{n}{j} \left( \frac{\lambda}{2} \right)^{n-j} \Phi_{j, \lambda}(\lambda, q).
$$

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