Some results on intuitionistic L-fuzzy metric spaces
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Abstract
In this paper, some known results of ordinary metric spaces including Baire’s theorem for intuitionistic L-fuzzy metric spaces are stated and proved.

Keywords
Intuitionistic L-Fuzzy Metric Spaces, F-bounded, Dense.

AMS Subject Classification
03E72.

1 Introduction
In 1986, Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets which is introduced by Zadeh [16]. In 2004, Park [10] defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-co-norms. Recently, in 2006, Alaca et al. [1] using the notion of intuitionistic fuzzy sets and defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space which is introduced by Kramosil and Michalek [7]. Turkgul et al. [11] gave generalization of Jungck’s [5] common fixed point theorem in intuitionistic fuzzy metric spaces. S.Y.Mohamed and E.N.Begam [13] introduced intuitionistic L-fuzzy metric spaces and they studied connectedness properties, coupled fixed points theorem in intuitionistic L-fuzzy metric space.

In this paper, the complete intuitionistic L-fuzzy metric space is introduced. Baire’s theorem for intuitionistic L-fuzzy metric spaces are stated and proved.

2 Preliminaries

Definition 2.1. A 5-tuple $(X, M, N, *, ∗)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm, $*$ is a continuous-conorm and $M, N$ are fuzzy sets on $X^2 \times \mathbb{R}_+$ satisfying the conditions:
1. $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$
2. $M(x,y,0) = 0$ for all $x, y \in X$,
3. $M(x,y,t) = 1$ for all $x, y \in X$, and $t > 0$ if and only if $x = y$;
4. $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$;
5. $M(x,y,t) + M(y,z,s) \leq M(x,z,t+s)$ for all $x, y, z \in X$ and $t, s > 0$;
6. $M(x,y,.) : [0,\infty) \to [0,\infty]$ is left continuous, for all $x, y \in X$,
7. $\lim_{t \to 0^+} M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$;
8. $N(x,y,0) = 1$ for all $x, y \in X$,
9. $N(x,y,t) = 0$, for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
10. $N(x,y,t) = N(y,x,t)$ for all $x, y \in X$, and $t > 0$;
11. $N(x,y,t) + N(y,z,s) \leq N(x,z,t+s)$ for all $x, y, z \in X$ and $t, s > 0$;
12. $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous, for all $x, y \in X$;
13. $\lim_{t \to 0^+} N(x,y,t) = 0$ for all $x, y \in X$. 

References
The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. $t$ respectively.

**Definition 2.2.** A 5-tuple $(X,M,N,*,\circ)$ is said to be an intuitionistic $L$-fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuoust-norm, $\circ$ is a continuoust-conorm and $M,N$ are fuzzy sets on $X^2 \times [0,\infty)$ satisfying the conditions:

1. $M(x,y,t) + N(x,y,t) \leq 1$,
2. $M(x,y,0) = 0$
3. $M(x,y,t) = 1$ if and only if $x = y$;
4. $M(x,y,t) = M(y,x,t)$
5. $M(x,y,t) \ast M(y,z,s) \leq M(x,z,t+s)$,
6. $M(x,y,): [0,\infty) \to L$ is left continuous
7. $\lim_{t \to \infty} M(x,y,t) = 1$,
8. $N(x,y,0) = 1$,
9. $N(x,y,t) = 0$ if and only if $x = y$;
10. $N(x,y,t) = N(y,x,t)$,
11. $N(x,y,t) \circ N(y,z,s) \geq N(z,x,t+s)$.
12. $N(x,y,): [0,\infty) \to L$ is right continuous,
13. $\lim_{t \to \infty} N(x,y,t) = 0$.

The functions $M(x,y,t) : X^2 \times [0,\infty) \to L$ and $N(x,y,t) : X^2 \times [0,\infty) \to L$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. respectively, satisfying $M(x,y,t) \leq K(N(x,y,t))$, $K : L \to L$ is an unary involute order preserving operation.

**Example 2.3.** Define t-norm $a \ast b = \min\{a,b\}$ and t- co-norm $a \circ b = \max\{a,b\}$ and for all $x,y \in X$ and $t > 0$, $M(x,y,t) = \frac{t}{t+\frac{1}{x}}$, $N(x,y,t) = \frac{\frac{1}{x}}{t+\frac{1}{x}}$.

Then $(X,M,N,*,\circ)$ is an intuitionistic $L$-fuzzy metric space.

**Theorem 3.1.** Every open ball in an intuitionistic $L$-fuzzy metric space $(X,M,N,*,\circ)$ is an open set.

**Proof.** Consider an open ball $B(x,r,t)$. Now, $y \in B(x,r,t)$ implies that $M(x,y,t) > 1 - r$ and $N(x,y,t) < r$.

Since $M(x,y,t) > 1 - r$ and $N(x,y,t) < r, 0 < t_0 < t$, such that $M(x,y,t_0) > 1 - r$ and $N(x,y,t_0) < r$.

Let $r_0 = M(x,y,t_0) > 1 - r$ and $N(x,y,t_0) < r$.

Since $r_0 > 1 - r$ and $r_0 < r$, we can find $s, 0 < s < 1$, such that $r_0 > 1 - s > 1 - r, r_0 < 1 - s < r$.

Now for a given $r_0$ and $s$, such that $r_0 > 1 - s$.

We can find $r_1, 0 < r_1 < 1$, such that $r_0 \ast r_1 \geq 1 - s$.

Now consider the ball $B(y,1 - r_1,t - t_0)$.

We claim $B(y,1 - r_1,t - t_0) \subset B(x,r,t)$.

Now $z \in B(y,1 - r_1,t - t_0)$ implies that $M(y,z,t - t_0) > 1 - r_1, N(y,z,t - t_0) < r_1$.

Therefore $M(x,z,t) \geq M(x,y,t) \ast M(y,z,t - t_0)$

$\geq r_0 \ast r_1$

$\geq 1 - s > 1 - r$.

Similarly,$N(x,z,t) \leq N(x,y,t_0) \circ N(y,z,t - t_0)$

$\leq r_0 \circ r_1$

$\leq 1 - s < r$.

Therefore $z \in B(x,r,t)$ and hence $B(y,1 - r_1,t - t_0) \subset B(x,r,t)$.

**Theorem 3.2.** Every intuitionistic $L$-fuzzy metric space is Hausdorff.

**Proof.** Let $(X,M,*,\circ)$ be an intuitionistic $L$-fuzzy metric space.

Let $x, y$ be two distinct points of $X$.

Then $0 < M(x,y,t) < 1, 0 < N(x,y,t) < 1$.

Let $M(x,y,t) = r, N(x,y,t) = 1 - r$ for some $r, 0 < r < 1$.

For each $r_0, 0 < r_0 < 1$, we can find an $r_1$ such that $r_1 \ast r_1 \geq r_0$.

Now consider the open balls $B(x,1 - r_1,\frac{1}{2})$ and $B(y,1 - r_1,\frac{1}{2})$.  

3. **Topology Induced By an Intuitionistic L-Fuzzy Metric Space**

**Theorem 3.1.** Every open ball in an intuitionistic $L$-fuzzy metric space $(X,M,N,*,\circ)$ is an open set.
Theorem 3.3. Let $A$ be a compact subset of $X$.

Proof. Let $A$ be a compact subset of $X$.

Fix $t > 0$ and $0 < r < 1$.

Consider an open cover $\{B(x,r,t) : x \in A\}$ of $A$.

Since $A$ is compact, there exists $x_1, x_2, x_3, ..., x_n \in A$ such that $A \subseteq \bigcup B(x_i, r, t)$.

Let $x, y \in A$. Then $x \in B(x_1, r, t)$ and $y \in B(x_j, r, t)$ for some $i, j$.

Therefore $M(x, x_i, t) > 1 - r$ and $N(x, x_j, t) < r$.

Since $A$ is compact, there exists $x_1, x_2, x_3, ..., x_n \in A$ such that $A \subseteq \bigcup B(x_i, r, t)$.

Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some $i, j$.

Therefore $M(x, x_i, t) > 1 - r$ and $N(x, x_j, t) < r$.

Now, let $\alpha = \min \{M(x, x_i, t) : 1 \leq i \leq n, 1 \leq j \leq n\}$.

Then $\alpha > 0$.

Now

$M(x, y, 3t) \geq M(x, x_i, t) * M(x_i, y, t) * M(y, y, t) \geq (1 - r) * (1 - r) * \alpha$.

$N(x, y, 3t) \leq N(x, x_i, t) * N(x_i, y, t) * N(y, y, t) < r * r * \alpha$.

$t 

Taking $t \geq 3t$ and $(1 - r) * (1 - r) * \alpha > 1 - s, 0 < s < 1$, we have

$M(x, y) > s, N(x, y) < 1 - s, \forall x, y \in A$.

Hence $A$ is $F$-bounded. 

Corollary 3.4. An intuitionistic $L$-fuzzy metric space in which every Cauchy sequence is convergent is called a complete intuitionistic $L$-fuzzy metric space.

Theorem 3.5. Let $(X, M, N, *, \phi)$ be a complete intuitionistic $L$-fuzzy metric space. Then the intersection of a countable number of dense open sets is dense.

Proof. Let $A$ be the given complete intuitionistic $L$-fuzzy metric space.

Let $B_0$ be a nonempty open set.

Let $D_1, D_2, D_3, ..., D_n, ..., \text{be dense open sets in } X$.

Since $D_1$ is dense in $X, B_0 \cap D_1 \neq \emptyset$.

Let $x_1 \in B_0 \cap D_1$. Since $B_0 \cap D_1$ is open, there exists $0 < r_1 < 1, t > 0$ such that $B(x_1, r_1, t) \subseteq B_0 \cap D_1$.

Choose $r_1' < r_1$ and $t_0 = \min\{t_1, 1\}$ such that $B(x_1, r_1', t_1') \subseteq B_0 \cap D_1$.

Let $B_1 = B(x_1, r_1', t_1')$. Since $D_2$ is dense in $X, B_0 \cap D_2 \neq \emptyset$.

Let $x_2 \in B_0 \cap D_2$.

Since $B_0 \cap D_2$ is open, there exists $0 < r_2 < \frac{t_1}{2}$ and $t_2 > 0$ such that $B(x_2, r_2, t_2) \subseteq B_1 \cap D_2$.

Choose $r_2' < r_2$ and $t_2' = \min\{t_2, r_1'\}$ such that $B(x_2, r_2', t_2') \subseteq B_1 \cap D_2$.

Let $B_2 = B(x_2, r_2', t_2')$. Similarly proceeding by induction we can find an $x_n \in B_{n-1} \cap D_n$.

Since $B_{n-1} \cap D_n$ is open, there exists $0 < r_n < \frac{1}{n}$ and $t_n > 0$ such that $B(x_n, r_n, t_n) \subseteq B_{n-1} \cap D_n$.

Choose $r_n' < r_n, t_n' = \min\{t_n, \frac{1}{n}\}$ such that $B(x_n, r_n', t_n') \subseteq B_{n-1} \cap D_n$.

Let $B_n = B(x_n, r_n', t_n')$.

Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0, \varepsilon > 0$ choose $n_0$ such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} < \varepsilon$. Then for $n \geq n_0, m \geq n$, we have

$M(x_n, x_m) < \varepsilon, N(x_n, x_m) < \varepsilon, \forall x_n, x_m \in A$.
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\[ M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n}) \]
\[ \geq 1 - \frac{1}{n} \]
\[ \geq 1 - \varepsilon. \]
\[ N(x_n, x_m, t) \leq N(x_n, x_m, \frac{1}{n}) \]
\[ \leq \frac{1}{n} \]
\[ \leq \varepsilon. \]

Therefore \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, the sequence \( \{x_n\} \) converges to \( x \) in \( X \). But \( x_k \in B[x_n, r'_n, t'_n] \) for all \( k \geq n \) and \( B[x_n, r'_n, t'_n] \) is a closed set.

Hence \( x \in B[x_n, r'_n, t'_n] \subset B_{n-1} \cap D_n \) for all \( n \).

Therefore \( B_0 \cap (\cap_{n=1}^m D_n) \neq \phi \).

Hence \( \cap_{n=1}^m D_n \) is dense in \( X \).

4. Conclusion

In this paper, we have proved every open ball is an open set and Baire’s theorem for intuitionistic L-fuzzy metric spaces. Also, any complete intuitionistic L-fuzzy metric space cannot be represented as the union of a sequence of nowhere dense sets and hence it is not of first category.

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