Renormalization of the vacuum angle for a particle on a ring

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(Dated: March 23, 2022)

We analyze the vacuum (topological) angle $\theta$ renormalization for the quantum mechanical model of a particle moving around a ring, where $\theta$ is the magnetic flux through the ring. We construct a renormalization group (RG) transformation for this model and derive exact RG equations which lead to the flow diagram similar to that of the Quantum Hall effect. Renormalization of $\theta$ is seen to follow from the loss of information about the initial topological charge in the course of the RG procedure.

PACS numbers: 03.65.Vf, 05.10.Cc, 73.23.Hk

Topologically nontrivial field theories are of considerable interest both in high energy and condensed matter physics. Different non-linear sigma models with topological terms describe, e.g., the Quantum Hall effect (QHE)\cite{1,2}, antiferromagnetic spin chains\cite{3}, tunneling effects in metallic nanostructures\cite{4} (see also recent work\cite{5}). Common feature of these models is the renormalization of the vacuum angle $\theta$\cite{6,7}, which is the coefficient in front of the topological charge in the action. While generally accepted by now, this $\theta$ renormalization still may seem somewhat obscure. For this reason we study here how this renormalization occurs in a simple quantum mechanical model (a particle on a ring). After constructing a certain kind of the renormalization group (RG) transformation it appears possible to obtain analytically almost the same RG flow diagram as in the QHE both for weak and strong coupling.

Consider a particle of mass $m$ moving around a ring of unit radius threaded by a magnetic flux $\theta$ (in units $\hbar = c = 1$). The corresponding (euclidian) action at finite temperature may be written in terms of a planar unit vector $\mathbf{n}(\tau)$ ($n^2 = 1$) which depends on a one-dimensional coordinate (imaginary time)$S[\mathbf{n}] = \int_0^\beta \dot{\mathbf{n}}^2(\tau) d\tau - \frac{i}{2\pi} \int_0^\beta \epsilon_{ab} n_a(\tau) n^b(\tau) d\tau, \quad (1)$

where $\epsilon_{ab}$ is the two dimensional antisymmetric tensor and $\beta$ is the inverse temperature. Since $\mathbf{n}(0) = \mathbf{n}(\beta)$ the model is actually defined on a circle. The last term in\cite{1} has the form $i\theta Q$ where $Q$ is the topological charge which distinguishes inequivalent mappings $S^1 \rightarrow S^1$ and takes integer values (equal to a number of rotations the particle make in time $\beta$), making the theory periodic in $\theta$. This $(0+1)$-dimensional field theory may seem trivial, since in terms of the polar angle the action\cite{1} is quadratic, but this is not so due to topological effects (like in compact electrodynamics).\cite{1}

This model may be used also to describe a single electron box (SEB)\cite{8}, which is essentially a metallic island coupled to the outside circuit by a tunnel junction of capacitance $C$ and resistance $R$. If $R \rightarrow \infty$ the action for the SEB reduces to Eq.\cite{1} where the first term counts for the charging energy and $1/m = e^2/C$ while $\theta$ is an external charge. Certainly, the nonlocal model for finite $R$\cite{4,9} (which also corresponds to the particle on the ring with Ohmic friction)\cite{10} and at $C \rightarrow 0$ has more in common with two-dimensional sigma models (e.g. dimensionless coupling and asymptotic freedom\cite{11}, instantons of all sizes\cite{12}), but the model of Eq.\cite{1} is much simpler and seems to be most suitable for the analytical study of the vacuum angle renormalization.

Since for the particle on the ring $\theta$ is an external magnetic flux, one would normally expect its renormalization due to the screening of the flux by the magnetic field, produced by the rotating particle, but this mechanism does not work here, because such back reaction is not included in Eq.\cite{1}. We shall see below, that quite a different (‘informational’) mechanism is relevant here, related to the way the topological charge changes under the RG transformation.

Now we shall construct the RG transformation for the action\cite{1}. For this purpose let us introduce a lattice dividing the whole $\tau$ axis into intervals of the length $a$ and then fix the values of the field $\mathbf{n}(\tau)$ at the sites $a_i$ of the lattice. This results in a discrete configuration $\mathbf{n}_i$ which will later play the role of a slowly varying background field.

Next we evaluate the ‘probability’ $P[\mathbf{n}_i]$ of this configuration integrating out all remaining degrees of freedom, i.e.

$$P[\mathbf{n}_i] \sim \prod_i \int_{\mathbf{n}(a_i) = n_i} D\mathbf{n}(\tau) \delta(n^2(\tau) - 1) \exp(-S[\mathbf{n}]) \quad (2)$$

with the action $S$ from Eq.\cite{1}. Obviously, this is just the product of the Green functions $G(\mathbf{n}_i, \mathbf{n}_{i+1}; a)$ at the inverse temperature $a$, which are known exactly for the simple model in question. If we introduce polar angle $\phi$ instead of the planar vector then

$$G(\phi_i, \phi_{i+1}; a) = \left( \frac{m}{2\pi a} \right)^{\frac{1}{2}} \sum_q \exp \left[ \frac{m}{2a} (\phi_{i+1} - \phi_i + 2\pi q)^2 + i \frac{\theta}{2\pi} (\phi_{i+1} - \phi_i + 2\pi q) \right]$$
where the sum is over different winding numbers \( q = 0 \pm 1 \ldots \). Then

\[
P[\phi_i] \sim \prod_i G(\phi_i, \phi_{i+1}; a) \sim \exp(-S_{\text{eff}})
\]

with

\[
S_{\text{eff}} = \frac{m}{2a} \sum_i (\phi_{i+1} - \phi_i)^2 - \frac{i}{2\pi} \sum_i (\phi_{i+1} - \phi_i) - \sum_i \ln \sum_q \exp \left[ - \frac{2\pi^2 m}{a} q^2 + i\theta q - \frac{2\pi m}{a} q(\phi_{i+1} - \phi_i) \right]
\]

Up to now we did not specify the values of \( \phi_i \) and these were some arbitrary numbers. Now let us assume that adjacent \( \phi_i \) are very close to each other, i.e. they represent some smooth continuous field \( \phi(\tau) \). Then we may write \( \phi_{i+1} - \phi_i \approx \phi' a \) with \( \phi \to 0 \) and expand the effective action \( \phi \) in terms of the derivative. The result of straightforward calculations is given by the same formula Eq. (4).

\[
S_{\text{eff}}[\phi] = \frac{m'(a)}{2} \int_0^\beta \left( \phi'^2(\tau)d\tau - i\frac{\theta'(a)}{2\pi} \int_0^\beta \phi(\tau)d\tau \right) + \ldots
\]

but with renormalized coupling constants, which now depend on \( a \)

\[
m'(a) = m - 4\pi^2 m^2 \frac{\partial^2 f(\theta, a)}{\partial \theta^2},
\]

\[
\theta'(a) = \theta - 4\pi^2 m \frac{\partial f(\theta, a)}{\partial \theta},
\]

where

\[
f(\theta, a) = -\ln Z(\theta, a),
\]

\[
Z(\theta, a) = \left( \frac{m}{2\pi a} \right)^{\frac{\beta}{2}} \sum_q \exp \left[ - \frac{2\pi^2 m}{a} q^2 + i\theta q \right].
\]

Here \( Z(\theta, a) \) is the partition function for the particle on the ring (at inverse temperature \( a \)) represented as a sum over winding numbers.

If we introduce a dimensionless coupling constant

\[
g = \frac{a}{m}
\]

we can easily see from Eq. (5) and Eq. (6) that both \( \theta'(a) \) and \( g'(a) = a/m'(a) \) depend on the scale \( a \) only through \( g \). This means that derivatives of \( g' \) and \( \theta' \) with respect to \( \ln a \) can be expressed solely through the running couplings (after inverting Eqs. (5)) to obtain RG equations in more familiar form

\[
\frac{dg'}{d\ln a} = \beta_1(g', \theta'), \quad \frac{d\theta'}{d\ln a} = \beta_2(g', \theta')
\]

However, we do not actually need \( \beta \)-functions here, since we already have the solutions of these RG equations (in terms of initial values \( m \) and \( \theta \)) given by Eq. (7).

Let us now comment on the physical meaning of the transformation constructed. While this is close in spirit to the real space RG approach here we do not eliminate fast variables step by step. It is difficult to find a suitable decomposition of fields into fast and slow components in theories with constraints. Usually, in sigma models with \( n^2 = 1 \) one adopts the RG scheme due to Polyakov [1], but this describes only small fast vibrations of a slowly rotating unit vector, i.e. fast variables are topologically trivial. But as we know (and will see later) fast rotations are also to be taken into account, if one wishes to obtain \( \theta \) renormalization. These fast rotations are usually represented by instantons of small size \( \mathcal{Q} \), but this makes sense only in the weak coupling region.

What is done here is in fact the same as the preliminary step for the Wilson’s RG, when he introduces a smooth average order parameter \( M(x) \), with Fourier components lower than some ultraviolet cutoff and defines an effective action for \( M(x) \) performing a statistical averaging holding \( M(x) \) fixed for all \( x \). The only (unimportant) difference is that here we take a discrete set \( \{n_i\} \) as an order parameter, and only later take the continuous limit. It is not so obvious, however, how to use the effective action \( \phi \), since as we shall see later, at \( \theta \neq \pi \) the effective mass \( m'(a) \) goes to zero as \( a \to \infty \) and fluctuations become large. Therefore, if we have in mind further functional integration over the slow field \( \phi(\tau) \) we should take account of the higher powers of \( \phi \) (and higher derivatives).

There exists however a different interpretation which seems to be more instructive. One can view the procedure described as a kind of a continuous measurement. In fact, \( P[n_i] \) is an amplitude for a process when successive measurements of the unit vector directions at times \( a_i \) give the values \( n_i \). One may think that we look at our system at discrete time moments and make a series of snapshots which are then combined into a ‘movie’. This cinematic sequence (similar to the ‘coarse-grained history’ of C. von M. von J. Hartle) looks like a smooth trajectory and the corresponding amplitude is determined by the effective action \( \phi \). This interpretation provides us with additional physical meaning of parameters \( m'(a) \) and \( \theta'(a) \) — they characterize the slow motion of a particle being continuously measured by some external observer (or environment) with time resolution \( a \).

Between the measurements the particle moves freely and can even perform many revolutions around the ring which we are unaware of. To state this more rigorously, we are always mistaken when we ascribe a certain topological charge \( Q \) to a coarse grained field. The true topological charge \( Q \) should include also a number of fast rotations (topologically nontrivial short wavelength fluctuations, e.g. instantons and anti-instantons) which we are unable to resolve. Then, to reproduce the true phase factor of a given field configuration \( \exp(i\theta Q) \) we have to change the value of the vacuum angle \( \theta \) and to make it
a perturbative contribution is in fact due to the explicit
\( Z \) may use the dual representation of the partition function
arise from the fast instanton-like rotations of the vector
coming from the terms with winding numbers
\( n \) is trivial. The first term in
\( \beta \) with \( \beta \) functions (here we omit the primes)
\( \beta_1(g, \theta) = g - g^2 D(g) e^{-2 a^2 \pi \cos \theta}, \)
\( \beta_2(g, \theta) = D(g) e^{-2 a^2 \sin \theta}, \) (11)
where \( D(g) = 8 \pi^2 (1 - 2 \pi^2/g)/g^2. \)
These formulas strongly resembles the instanton induced renormalization in the Yang-Mills theory \( \theta' \) in or in
the QHE \( \theta' \). It should be stressed however that in our model \( \theta' \) there are no instantons (as solutions of classical equations) of small size. Nevertheless the physics is just the same. Exponentially small contributions in \( \theta' \), coming from the terms with winding numbers \( q = \pm 1 \), arise from the fast instanton-like rotations of the vector \( n \) between the time moments \( a_i \) and \( a_{i+1} \). Note that the same fluctuations are responsible also for the mass renormalization since without the topological effects the model \( \theta' \) is trivial. The first term in \( \beta_1(g, \theta) \) which looks like a perturbative contribution is in fact due to the explicit scale dependence of the coupling constant \( g \).

In the opposite limit of the strong coupling \( g \gg 1 \) we may use the dual representation of the partition function \( Z(\theta, a) \) as a sum over the energy levels
\[ Z(\theta, a) = \sum_n \exp \left[ -\frac{a}{2m} \left( n - \frac{\theta}{2\pi} \right)^2 \right], \] (12)
where \( n = 0, \pm 1, \ldots \) is the angular momentum of the particle measured in units of \( \hbar \). Then, at \( a \to \infty \) only the ground state contributes to \( Z(\theta, a) \), i.e.
\[ Z(\theta, a) \simeq \exp(-a E_0(\theta)), \] (13)
where \( E_0 \) is the ground state energy for the particle on the ring
\[ E_0(\theta) = \frac{1}{2m} \left( \frac{\theta}{2\pi} \right)^2, \quad 0 < \theta \leq \pi, \]
\[ E_0(\theta) = \frac{1}{2m} \left( 1 - \frac{\theta}{2\pi} \right)^2, \quad \pi < \theta < 2\pi \] (14)
Recall that there is level crossing at \( \theta = \pi \) and the ground state is degenerate at this point. Substituting these formulas into Eq. (6) we immediately obtain
\[ m'(a) \to 0, \quad \theta \neq \pi, \]
\[ \theta'(a) \to \begin{cases} 0, & 0 < \theta < \pi \vspace{1pt} \\ 2\pi, & \pi < \theta < 2\pi \end{cases} \] (15)
Hence, in the long wavelength limit \( \theta'(a) \) tends to a step-wise function, just like the Hall conductivity in the QHE, while \( m'(a) \) vanishes like the diagonal conductivity. The calculation at \( \theta = \pi \) is slightly more involved (the degeneracy of the ground level should be taken into account) and results in a linear growth of the effective mass with \( a \) according to \( m'(a) \sim a/4 \). Therefore, when expressed in terms of \( g \) the RG flow has a fixed point \( \theta' = \pi, g^* = 4 \).

To evaluate corrections to Eqs. (15) one has to take into account also the first exited level and after some algebra we obtain at \( g \to \infty \)
\[ \frac{1}{g'} \approx \exp(-g\Delta), \quad \theta' \approx 2\pi \exp(-g\Delta), \] (16)
at \( 0 < \theta < \pi \) (for \( \theta \) not too close to zero) where \( \Delta = (1 - \theta/\pi)/2 \) is the dimensionless energy gap. Therefore deviations from the ideal quantization at \( \theta \neq \pi \) are exponentially small at large scales as it should be in a system with the finite correlation length \( \xi \sim m^{\Delta^{-1}} \). Moreover, at \( g \gg 1 \) running couplings are related by a simple linear relation
\[ \frac{1}{g''(a)} = \frac{1}{2\pi} \theta'(a). \] (17)
The whole RG flow diagram is shown in Fig. 4 in the \((1/g, \theta)\)-plane for \( 0 < \theta < 2\pi \). The region below the fixed point is not available in our model, since for a finite initial \( m \) RG trajectories in Fig. 4 always start at infinity.

This is essentially the same diagram as proposed initially for the integer Quantum Hall effect [2, 14]. Its main features are: renormalization of \( 1/g \) to zero except for \( \theta = \pi \) where the saddle point exists at some finite \( g^* \) and a set of infra-red fixed points at quantized values of \( \theta = 0, \pm 2\pi, \ldots \). Our example shows that one can obtain such a non-trivial RG flow even for a very simple system provided the RG transformation is complicated enough. At a finite temperature \( T \) the RG flow should be stopped at \( a = 1/T \) since this is the size of the system [1] along the \( \tau \) axis.

For the model considered the vanishing of the effective particle mass (or \( 1/g \)) as \( a \to \infty \) is almost trivial. When
the scale \( a \) becomes larger than the correlation length \( \xi \sim m^{-1} \), adjacent \( n_i \) are no longer correlated which results in the loss of the corresponding stiffness, i.e. \( m(a) \to 0 \). From the point of view of one-dimensional statistical mechanics this means that the RG flow leads us further into the disordered phase (leading to the entanglement loss along the RG trajectories \( \xi \)). In the vicinity of \( \theta = \pi \) the correlation length diverges \( \xi \sim 1/|\theta - \pi| \) and the fixed point \( g^* \) results from the degeneracy of the ground state at \( \theta = \pi \).

The physical meaning of the renormalized vacuum angle \( \theta'(a) \) may be easily seen from Eq. (14). If we substitute \( Z(\theta, a) \) from Eq. (14) into Eq. (3) then

\[
\theta'(a)/2\pi = \langle n \rangle = \left \langle -i \frac{d}{d\phi} \right \rangle,
\]

where for the particle on the ring \( n \) is the canonical angular momentum and the brackets denote averaging over the ensemble with the temperature \( T = 1/a \). For the SEB \( \langle n \rangle \) is the average number of excess electrons in the box and hence quantization of \( \theta' \) at \( a \to \infty \) is just the zero temperature Coulomb blockade \( \frac{1}{n} \) (for the SEB with \( 1/R \neq 0 \) this was recently pointed out in [5]). Thus the \( \theta \) renormalization may be directly observed here (as in the QHE) as the formation of the stepwise dependence of \( \langle n \rangle \) on the external charge \( \theta \) (‘Coulomb staircase’) when the temperature is lowered.

In summary, we have studied the \( \theta \) angle renormalization for the quantum mechanical particle moving around the ring, where \( \theta \) is the magnetic flux through the ring. The appropriate RG transformation was constructed which resulted in the RG flow shown in Fig. 1. This flow diagram is similar to that of the Quantum Hall effect and gives one more example of the \( \theta \) renormalization. We argue that this renormalization occurs due to the information loss, because the RG transformation which eliminates fast fluctuations from a given field configuration changes its topological charge \( Q \) (mixes different topological sectors). Then, looking at the resulting coarse-grained field we see different topological charge than the true one, so that to recover the phase factor \( \exp(i\theta Q) \) we have to change the value of \( \theta \) and to make it scale dependent. The absence of the phase factor in the long wavelength limit (due to \( \theta \) renormalization) may probably be viewed as a complete screening of the background topological charge by fast instanton-like fluctuations.

I am grateful to V. Losyakov, A. Marshakov, A. Morozov, D. Nurgaliev, I. Tipunin and A.D. Zaikin for valuable discussions. The work was supported by the RFBR grant # 06-02-17459.

FIG. 1: RG flow diagram of Eqs. (12).

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