DIFFERENTIAL PROPERTIES OF
MATRIX ORTHOGONAL POLYNOMIALS

M.J. Cantero, L. Moral, L. Velázquez

Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain

December 2001

Abstract

In this paper a general theory of semi-classical matrix orthogonal polynomials is developed. We define the semi-classical linear functionals by means of a distributional equation $D(uA) = uB$, where $A$ and $B$ are matrix polynomials. Several characterizations for these semi-classical functionals are given in terms of the corresponding (left) matrix orthogonal polynomials sequence. They involve a quasi-orthogonality property for their derivatives, a structure relation and a second order differo-differential equation. Finally we illustrate the preceding results with some non-trivial examples.

Keywords and phrases: Orthogonal matrix polynomials, Semi-classical functionals, Differential equation, Structure relation.

(2000) AMS Mathematics Subject Classification : 42C05

Suggest running head: Differential properties of matrix O P.

Corresponding author: lmoral@posta.unizar.es
Tel: +34 976 76 11 41
Fax: +34 976 76 11 25

This work was supported by Dirección General de Enseñanza Superior (DGES) of Spain under grant PB 98-1615.
§ 1 - Introduction.

The study of semi-classical orthogonal polynomials in the scalar case (i.e., the study of orthogonal polynomials whose associated functional satisfies a distributional equation $D(u\Phi) = u\Psi$, where $\Phi, \Psi$ are polynomials with $\deg \Psi \geq 1$) was started by Shohat ([15]) in order to generalize the properties of classical orthogonal polynomials.

Among others, in ([2, 9]), an approach to such polynomials taking into account the quasi-orthogonality of the derivatives of the polynomials sequence is given. We also mention the results of Maroni ([11, 12]) where an algebraic theory of semi-classical orthogonal polynomials is presented. The purpose of this theory is the characterization of semi-classical orthogonal polynomials by means of the quasi-orthogonality of their derivatives, the structure relation and the differo-differential equation of second order.

In the last years, the study of matrix orthogonal polynomials attracted a great interest of the researchers, (see [4, 6, 10]). As for their differential properties it is known the result of Durán ([5]) who characterizes those matrix orthogonal polynomials (O P) satisfying a symmetric second order differential equation with polynomial coefficients. He proves that they are diagonal (up to a factor) with classical scalar O P in the diagonal. In spite of the variety of applications of matrix polynomials ([4, 5, 6]), there are not too many known families of semi-classical matrix O P out of the diagonal case.

One way to study many families of matrix orthogonal polynomials is to extend the analysis started by Durán of differential properties of matrix orthogonal polynomials. A natural way to do this is to generalize the theory of semi-classical scalar O P to the matricial case. In order to do that, we start with a Pearson type equation for matrix quasi-definite functionals. We obtain their characterization in terms of a (in general, non-symmetric) differo-differential equation for the related matrix orthogonal polynomials. In the way of the proof, we find another equivalences that generalize the scalar case.

This paper has been organized as follows. In Section 3 we introduce and study the concept of quasi-orthogonality for matrix polynomials.

The mean result of Section 4 is the characterization of semi-classical functionals in terms of the quasi-orthogonality of the corresponding matrix polynomials and their derivatives.

In Section 5 we prove for matrix O P with the above quasi-orthogonality properties, the structure relation and the differo-differential equation, showing that they are equivalent to a Pearson-type equation.

Finally, in Section 6, we illustrate the preceding with some examples and we find a way to construct non-diagonalizable semi-classical matrix functionals.

§ 2 - Basic tools.

We shall denote by $\mathbb{P}^{(m)}$ the $\mathbb{C}^{(m,m)}$-left-module

$$\mathbb{P}^{(m)} = \left\{ \sum_{k=0}^{n} \alpha_{k} x^{k} | \alpha_{k} \in \mathbb{C}^{(m,m)}; n \in \mathbb{N} \right\}$$
and by means of $\mathbb{P}^{(m)'}$ the $\mathbb{C}^{(m,m)}$-right-module $\text{Hom}(\mathbb{P}^{(m)}, \mathbb{C}^{(m,m)})$.

**2.1 Definition.**
(i) The duality bracket is defined by
\[
\langle \cdot, \cdot \rangle : \mathbb{P}^{(m)} \times \mathbb{P}^{(m)'} \to \mathbb{C}^{(m,m)}
\]
\[
(P, u) \to \langle P, u \rangle := u(P)
\]
(ii) For $k \in \mathbb{N}$ and $u \in \mathbb{P}^{(m)'}$ the linear functional $ux^k I \in \mathbb{P}^{(m)'}$ is defined by
\[
\langle P, ux^k I \rangle := \langle x^k P, u \rangle,
\]
where $I$ denotes the $m \times m$ identity matrix.
A linear extension gives the right-product $u \cdot A \in \mathbb{P}^{(m)'}$ for $u \in \mathbb{P}^{(m)'}$, $A \in \mathbb{P}^{(m)}$, with $A(x) = \sum_{k=0}^{n} \alpha_k x^k$, in the following way:
\[
\langle P, uA \rangle = \sum_{k=0}^{n} \langle x^k P, u \rangle \alpha_k.
\]
(iii) The inner product
\[
\langle \cdot, \cdot \rangle : \mathbb{P}^{(m)} \times \mathbb{P}^{(m)} \to \mathbb{C}^{(m,m)}
\]
\[
(P, Q) \to \langle P, Q \rangle_u := \langle P, u Q^* \rangle
\]
is defined for every $u \in \mathbb{P}^{(m)'}$, where $Q^*$ denotes the trasposed conjugated of $Q$.

**Remark.**
The duality bracket verifies the following linear properties:
\[
\langle \alpha_1 P_1 + \alpha_2 P_2, \beta_1 u_1 + \beta_2 u_2 \rangle = \langle \alpha_1 P_1, \beta_1 u_1 \rangle + \langle \alpha_1 P_1, \beta_2 u_2 \rangle + \langle \alpha_2 P_2, \beta_1 u_1 \rangle + \langle \alpha_2 P_2, \beta_2 u_2 \rangle,
\]
for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}^{(m,m)}$, $P_1, P_2 \in \mathbb{P}^{(m)}$ and $u_1, u_2 \in \mathbb{P}^{(m)'}$, while the inner product is sesquilinear:
(i) $\langle \alpha_1 P_1 + \alpha_2 P_2, \beta_1 Q_1 + \beta_2 Q_2 \rangle_u = \langle \alpha_1 P_1, \beta_1^* Q_1 \rangle_u + \langle \alpha_1 P_1, \beta_2^* Q_2 \rangle_u + \langle \alpha_2 P_2, \beta_1^* Q_1 \rangle_u + \langle \alpha_2 P_2, \beta_2^* Q_2 \rangle_u$,
for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}^{(m,m)}$, $P_1, P_2, Q_1, Q_2 \in \mathbb{P}^{(m)}$;
(ii) $\langle P, Q \rangle_u = \langle Q, P \rangle_u$
for all $P, Q \in \mathbb{P}^{(m)}$.

**2.2 Definition.** We denote by $C_k := \langle x^k I, u \rangle$ the $k$-th moment with respect to $u \in \mathbb{P}^{(m)'}$. Given $u \in \mathbb{P}^{(m)'}$ with moments $(C_k)_{k \in \mathbb{N}}$, we say that $u$ is quasi-definite (non-singular) if $\det \left( [C_k]_{k,j=0}^n \right) \neq 0$, $\forall n \geq 0$, where $(C_k)_{k,j=0}^n$ is the Hankel-block matrix
\[
\begin{pmatrix}
C_0 & C_1 & \cdots & C_n \\
C_1 & C_2 & \cdots & C_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
C_n & C_{n+1} & \cdots & C_{2n}
\end{pmatrix}
\]
4 DIFFERENTIAL PROPERTIES OF MATRIX ORTHOGONAL POLYNOMIALS

Remark. Given the sequence \((C_k)_{k=0}^{\infty} \subset \mathbb{C}^{(m,m)}\), there exists a unique \(u \in \mathbb{P}(m)'\) such that \(\langle x^k I, u \rangle = C_k\). This establishes an isomorphism between \(\mathbb{P}(m)'\) and the formal series with coefficients in \(\mathbb{C}^{(m,m)}\), \(\sum_{k=0}^{\infty} C_k x^k\).

2.3 Definition. Let \(u \in \mathbb{P}(m)\). We say that \(u\) is hermitian if \(C_k^* = C_k\), for all \(k \geq 0\).

2.4 Theorem. Let \(u \in \mathbb{P}(m)\) be quasi-definite. Then, there exists a unique (up to left non-singular matrix factors) sequence of left orthogonal polynomials \((P_n)_{n \geq 0}\) with respect to \(u\), that is

1. \(P_n \in \mathbb{P}(m)\), \(dg P_n = n\).
2. The leading coefficient of \(P_n\) is non-singular.
3. \(\langle P_n, P_m \rangle_u = K_n \delta_{nm}\), where \(K_n\) is non-singular.

This sequence verifies a recurrence relation

\[
x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0,
\]

\[
P_{-1}(x) = \theta,
\]

with \(\alpha_n, \beta_n, \gamma_n \in \mathbb{C}^{(m,m)}, \alpha_n, \gamma_n\) non-singular and we denote by \(\theta\), the zero matrix.

Proof. See [4, 14]. ∗∗

Remarks.

(i) If we consider the structure of right-modulus on \(\mathbb{P}(m)\) (so, left-modulus on \(\mathbb{P}(m)'\)), we define in a similar way a sequence of right orthogonal polynomials.

(ii) The inner product \(\langle P_n, x^n I \rangle_u\) is non-singular for every quasi-definite \(u \in \mathbb{P}(m)\).

§ 3 - Quasi-orthogonality.

Like in the scalar case, the semi-classical character will be closely related to the idea of quasi-orthogonality. But the scalar quasi-orthogonality, when generalized to the matricial case, splits in two different concepts. Both of them will play an important role in the characterization of semi-classical matrix functionals.

3.1 Definition. Let \((Q_n)_{n \geq 0}\) be a sequence of matrix polynomials such that the leading coefficient of \(Q_n\) is non-singular and \(dg Q_n = n\).

1. We will say that \((Q_n)_{n \geq 0}\) is a sequence of left quasi-orthogonal matrix polynomials of order \(r\) with respect to \(v \in \mathbb{P}(m)' \setminus \{0\}\) if

(a) \(\langle x^k Q_n, v \rangle = \theta, \quad 0 \leq k \leq n - r - 1, \quad n \geq r + 1\).

(b1) There exists \(n_0 \geq r\) such that \(\langle x^{n_0-r} Q_{n_0}, v \rangle \neq \theta\).

2. We will say that \((Q_n)_{n \geq 0}\) is a sequence of regularly left quasi-orthogonal matrix polynomials of order \(r\) with respect to \(v \in \mathbb{P}(m)' \setminus \{0\}\) if

(a) \(\langle x^k Q_n, v \rangle = \theta, \quad 0 \leq k \leq n - r - 1, \quad n \geq r + 1\).

(b2) There exists \(n_0 \geq r\) such that \(\langle x^{n_0-r} Q_{n_0}, v \rangle\) is non-singular.

Notice that if \(r = 0\) the previous definition is the classical orthogonality. Furthermore, for \(m = 0\), both definitions yield the scalar quasi-orthogonality.
Remember that given $w \in \mathbb{P}^{(m)'}$ we will say that $w = 0$ if $\langle P, w \rangle = \theta$, $\forall P \in \mathbb{P}^{(m)}$.

As we will see in the next proposition, when a sequence of quasi-orthogonal matrix polynomials is already orthogonal with respect to another functional, the conditions (b1), (b2) become stronger.

### 3.2 Proposition

Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite and let $(P_n)_{n \geq 0}$ be the corresponding sequence of left orthogonal matrix polynomials. Given $v \in \mathbb{P}^{(m)'}$, the sequence $(P_n)_{n \geq 0}$ is quasi-orthogonal of order $r$ with respect to $v$ if and only if

(a) $\langle x^k P_n, v \rangle = \theta$, $0 \leq k \leq n - r - 1$, $n \geq r + 1$.

(b1) $\langle x^{n-r} P_n, v \rangle \neq \theta$, $\forall n \geq r$.

The sequence $(P_n)_{n \geq 0}$ is regularly quasi-orthogonal of order $r$ with respect to $v$ if and only if

(a) $\langle x^k P_n, v \rangle = \theta$, $0 \leq k \leq n - r - 1$, $n \geq r + 1$.

(b1') $\langle x^{n-r} P_n, v \rangle$ is non singular $\forall n \geq r$.

**Proof.** Let $n \geq r + 1$ be verifying (b1) or (b2). Taking into account the recurrence relation (1.1), we obtain

$$
\gamma_{n+1} \langle x^{n-r} P_n, v \rangle = -\beta_{n+1} \langle x^{n-r} P_{n+1}, v \rangle - \alpha_{n+1} \langle x^{n-r} P_{n+2}, v \rangle + \langle x^{n-r+1} P_{n+1}, v \rangle = \langle x^{n+1-r} P_{n+1}, v \rangle
$$

and (b1) or (b2) holds for $n + 1$. In a similar way,

$$
\gamma_n \langle x^{n-1-r} P_{n-1}, v \rangle = \langle x^{n-r} P_n, v \rangle
$$

and (b1) or (b2) is satisfied for $n - 1$. $\diamond$

In what follows, the following result will be useful.

### 3.3 Lemma

Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite and let $(P_n)_{n \geq 0}$ be the corresponding sequence of left orthogonal matrix polynomials. Then, given $w \in \mathbb{P}^{(m)'}$, $w = 0$ if and only if there exists $n_0$ such that

$$
\langle x^k P_n, w \rangle = \theta, \quad 0 \leq k \leq n, \quad n \geq n_0.
$$

**Proof.** As a consequence of the recurrence relation (1.1),

$$
< x^k P_{n_0 - 1}, w > = \gamma_{n_0}^{-1} < x^{k+1} P_{n_0}, w > - \alpha_{n_0} < x^k P_{n_0 + 1}, w > - \beta_{n_0} < x^k P_{n_0}, w > = \theta
$$

for $0 \leq k \leq n_0 - 1$. So, the hypothesis is true for $n \geq n_0 - 1$, $0 \leq k \leq n$ too, and as a consequence, for all $n \geq 0$, $k \geq 0$. That means $< x^k, w > = \theta$ for all $k \geq 0$ and then, $w = 0$. $\diamond$

Given a sequence of orthogonal matrix polynomials with respect to a functional $u$, its quasi-orthogonality with respect to another functional $v$ can be characterized by a simple relation between $u$ and $v$.

### 3.4 Theorem

Let $u$, $v \in \mathbb{P}^{(m)'}$ such that $u$ is quasi-definite and let $(P_n)_{n \geq 0}$ be the corresponding sequence of left orthogonal matrix polynomials. Then,
(i) \((P_n)_{n \geq 0}\) is quasi-orthogonal of order \(r\) with respect to \(v\) if and only if there exists \(A \in \mathbb{P}^{(m)}\), with \(dgA = r\), and such that \(v = uA\).

(ii) \((P_n)_{n \geq 0}\) is regularly quasi-orthogonal of order \(r\) with respect to \(v\) if and only if there exists \(A \in \mathbb{P}^{(m)}\), with \(dgA = r\), and non-singular leading coefficient such that \(v = uA\).

Moreover, in any case, the matrix polynomial \(A\) is unique.

Proof.
We will do the proof just for (i) because for (ii) the arguments are analogous.

\(\Leftarrow\) We consider \(A(x) = \sum_{j=0}^{r} a_j x^j\), with \(a_r \neq \theta\). Then, for \(v = uA\)

\[
\langle x^k P_n, v \rangle = \sum_{j=0}^{r} \langle x^{k+j} P_n, u \rangle a_j.
\]

For \(0 \leq k \leq n - r - 1\), we have

\[
\langle x^{k+j} P_n, u \rangle = \theta, \quad (0 \leq j \leq r)
\]

and, for \(k = n - r\),

\[
\langle x^k P_n, v \rangle = \langle x^n P_n, u \rangle a_r \neq \theta.
\]

So, \((P_n)_{n \geq 0}\) is regularly quasi-orthogonal of order \(r\) with respect to \(v\).

\(\Rightarrow\) We will prove that \(0 = v - \sum_{j=0}^{r} x^j u a_j\) has an unique solution in the unknowns \((a_j)_{j=0}^{r}\).

For all \(n \geq r\) fixed, the system of \(r + 1\) equations

\[
\begin{align*}
\langle x^{n-r} P_n, v \rangle & = \langle x^n P_n, u \rangle a_r^{(n)}, \\
\cdots & \\
\langle x^n P_n, v \rangle & = \langle x^n P_n, u \rangle a_0^{(n)} + \cdots + \langle x^{n+r} P_n, u \rangle a_r^{(n)},
\end{align*}
\]

has a unique solution in the unknowns \((a_j^{(n)})_{j=0}^{r}\). Notice that the first equation leads to \(a_r^{(n)} \neq \theta\).

We will prove that \((a_j^{(n)})_{j=0}^{r}\) are independent of \(n\). Keeping in mind the recurrence relation (1.1), we have that the relation

\[
\gamma_{n+1} \langle x^k P_n, \hat{v} \rangle + \beta_{n+1} \langle x^k P_{n+1}, \hat{v} \rangle + \alpha_{n+1} \langle x^k P_{n+2}, \hat{v} \rangle = \langle x^{k+1} P_{n+1}, \hat{v} \rangle
\]

(2.2)

is verified \(\forall k \geq 0\) and \(\forall \hat{v} \in \mathbb{P}^{(m)'}\). The first equation of (2.1) leads to

\[
\begin{align*}
\begin{cases}
\langle x^{n-r} P_n, v \rangle = \langle x^n P_n, u \rangle a_r^{(n)} \\
\langle x^{n+1-r} P_{n+1}, v \rangle = \langle x^{n+1} P_{n+1}, u \rangle a_r^{(n+1)},
\end{cases}
\end{align*}
\]

(2.3)

and, from (2.2)

\[
\begin{align*}
\begin{cases}
\gamma_{n+1} \langle x^{n-r} P_n, v \rangle - \langle x^{n+1-r} P_{n+1}, v \rangle = \theta, \\
\gamma_{n+1} \langle x^n P_n, u \rangle - \langle x^{n+1} P_{n+1}, u \rangle = \theta,
\end{cases}
\end{align*}
\]

(2.4)
Equations (2.3) and (2.4) imply \( a_r^{(n)} = a_r^{(n+1)} \).

We suppose \( a_i^{(n)} = a_i^{(n+1)} \) \( (l = m+1, \ldots, r, \ m < r, \ \forall n \geq r) \), and we will prove that \( a_m^{(n)} = a_m^{(n+1)} \). Taking into account (2.1) we have

\[
\begin{align*}
\langle x^{n-m} P_n, v \rangle &= \sum_{j=m}^{r} \langle x^{n-m+j} P_n, u \rangle a_j^{(n)} \\
\langle x^{n-m+1} P_{n+1}, v \rangle &= \sum_{j=m}^{r} \langle x^{n+1-m+j} P_{n+1}, u \rangle a_j^{(n+1)}
\end{align*}
\]  \( (2.5) \)

and from (2.2)

\[
\begin{align*}
\gamma_{n+1} \langle x^{n-m} P_n, v \rangle + \beta_{n+1} \langle x^{n-m} P_{n+1}, v \rangle + \alpha_{n+1} \langle x^{n-m} P_{n+2}, v \rangle &= \langle x^{n-m+1} P_{n+1}, v \rangle, \\
\gamma_{n+1} \langle x^{n-m+j} P_n, u \rangle + \beta_{n+1} \langle x^{n-m+j} P_{n+1}, u \rangle + \alpha_{n+1} \langle x^{n-m+j} P_{n+2}, u \rangle &= \langle x^{n+1-m+j} P_{n+1}, u \rangle.
\end{align*}
\]  \( (2.6) \)

By substitution of (2.5) into (2.6) we obtain

\[
\begin{align*}
\gamma_{n+1} \sum_{j=m}^{r} \langle x^{n-m+j} P_n, u \rangle a_j^{(n)} + \beta_{n+1} \sum_{j=m+1}^{r} \langle x^{n-m+j} P_{n+1}, u \rangle a_j^{(n+1)} + \\
\alpha_{n+1} \sum_{j=m}^{r} \langle x^{n-m+j} P_{n+2}, u \rangle a_j^{(n+1)} &= \sum_{j=m+2}^{r} \langle x^{n+1-m+j} P_{n+1}, u \rangle a_j^{(n+2)}.
\end{align*}
\]  \( (2.7) \)

Now, keeping in mind the relation (2.7) and by application of the induction hypothesis, we have

\[
\gamma_{n+1} \langle x^n P_n, u \rangle a_m^{(n)} - \langle x^{n+1} P_{n+1}, u \rangle a_m^{(n+1)} = \theta.
\]

Thus, by (2.4), we conclude \( a_m^{(n)} = a_m^{(n+1)} \).

Let us denote by \( (a_j)_{j=0}^{r} \) the solutions of (2.1) for \( n \geq r \), that we have proved are independent \( n \). If \( A(x) = \sum_{j=0}^{r} a_j x^j \) and \( w = v - uA \), we have

\[
\langle x^k P_n, w \rangle = \langle x^k P_n, v \rangle - \sum_{j=0}^{r} \langle x^{k+j} P_n, u \rangle a_j, \ 0 \leq k \leq n, \ n \geq r.
\]

When \( 0 \leq k \leq n-r-1 \), taking into account the orthogonality with respect to the functional \( u \) as well as the quasi-orthogonality with respect to the functional \( v \), we get \( \langle x^k P_n, w \rangle = \theta \). If \( n-r \leq k \leq n \), then \( \langle x^k P_n, w \rangle = \theta \), because \( (a_j)_{j=0}^{r} \) are solutions of the equations (2.1). So, \( w = 0 \), according to Proposition 2.2.
To prove the uniqueness of the matrix polynomial $A$, let us suppose $A, B \in \mathbb{P}(m)$ such that $v = uA = uB$. Then, from (i), it must be $r = dgA = dgB$ since $(P_n)_{n \geq 0}$ can not be quasi-orthogonal of different order with respect to $v$. Hence, $A(x) = \sum_{j=0}^{r} a_j x^j$, $B(x) = \sum_{j=0}^{r} b_j x^j$ with $a_r, b_r \neq \theta$. From

$$\langle x^k P_n, v \rangle = \sum_{j=0}^{p} \langle x^{k+j} P_n, u \rangle a_j = \sum_{j=0}^{q} \langle x^{k+j} P_n, u \rangle b_j,$$

and taking $k = n - r, \ldots, n$, we have

$$\begin{cases} 
\langle x^n P_n, u \rangle (a_r - b_r) = \theta, \\
\cdots \cdots \\
\langle x^{n+r} P_n, u \rangle (a_r - b_r) + \cdots + \langle x^n P_n, u \rangle (a_0 - b_0) = \theta.
\end{cases}$$

Therefore, $a_j = b_j$, $(j = 0, \cdots, r)$. ☐

§ 4 - Semi-classical functionals.

We consider the derivative operator on the space $\mathbb{P}(m)'$ as the linear operator $D : \mathbb{P}(m)' \to \mathbb{P}(m)'$ such that $\langle P, Du \rangle = - \langle P', u \rangle$. From this and the definition of the right-product it is straightforward to prove that $D(uA) = (Du)A + uA'$, for all $u \in \mathbb{P}(m)'$ and $A \in \mathbb{P}(m)$.

4.1 Definition. Let $u \in \mathbb{P}(m)'$ be quasi-definite. We will say that $u$ is semi-classical if there exist $A, B \in \mathbb{P}(m)$, with det $A \neq 0$, such that it is verified the distributional equation $D(uA) = uB$. We will also say that the corresponding sequence of left orthogonal matrix polynomials $(P_n)_{n \geq 0}$ is semi-classical.

Remark. Given $u \in \mathbb{P}(m)'$, the linear functionals $u_{i,j} : \mathbb{P}(m) \to \mathbb{C}$, $(i, j = 1, \cdots, m)$ defined by $u_{i,j}(P) = u(P)_{i,j}$, $\forall P \in \mathbb{P}(m)$ are called the components of $u$. Then, $(uA)_{i,j} = \sum_{k=1}^{m} u_{i,k} A_{k,j}$ with the obvious definition for the multiplication of $u_{i,j}$ by a scalar polynomial. Therefore, det $A \neq 0$ is the minimal requirement to ensure that the previous definition will involve to all the components of the functional $u$.

4.2 Lemma. Let $u \in \mathbb{P}(m)'$ such that $D(uA) = uB$. Then, for every $C \in \mathbb{P}(m)$,

$$D(uAC) = u(AC' + BC)$$

Proof. It is just a consequence of the rule for the derivation of the right-product. ☐

The previous result implies that, for every $u \in \mathbb{P}(m)'$, the set

$$A_u = \{ A \in \mathbb{P}(m) / D(uA) = uB, B \in \mathbb{P}(m) \}$$
is a right-ideal of $\mathbb{P}^{(m)}$.

When dealing with scalar semi-classical functionals we arrive in this way to an ideal of $\mathbb{P}$, which is necessarily generated by a unique (up to non-trivial factors) polynomial. This generator is used to classify the scalar semi-classical functionals, ([11, 12]).

Since in the matricial case, a right-ideal of $\mathbb{P}^{(m)}$ is not necessarily principal, we cannot use $\mathcal{A}_u$ for the classification of semi-classical functionals. However, if $u \in \mathbb{P}^{(m)\prime}$ is semi-classical then there exists $\alpha \in \mathbb{P} \setminus \{0\}$ (where $\mathbb{P} \equiv \mathbb{P}^{(1)}$) such that $\alpha I \in \mathcal{A}_u$. To see this just notice that if $A \in \mathcal{A}_u$ with $\det A \neq 0$, then $(\det A)I \in \mathcal{A}_u$ since $(\det A)I = AA^+ \text{ with } A^+ = \text{adj} A \in \mathbb{P}^{(m)}$. Therefore, for every semi-classical matrix functional $u \in \mathbb{P}^{(m)\prime}$, the set

$$\vartheta_u = \{ \alpha \in \mathbb{P} / \langle u \alpha I \rangle = uB, B \in \mathbb{P}^{(m)} \}$$

is a non-trivial ideal of $\mathbb{P}$. We can use the “essentially” unique generator of this ideal to classify the semi-classical matrix functionals similarly to the scalar case.

4.3 Definition. Let $u \in \mathbb{P}^{(m)\prime}$ be semi-classical and let $\alpha \in \mathbb{P} \setminus \{0\}$ be a polynomial with smallest degree such that $D(u \alpha I) = uB, B \in \mathbb{P}^{(m)}$. Then, we say that $u$ is of class $s = \max\{p - 2, q - 1\}$, where $p = \text{deg} \alpha$ and $q = \text{deg} B$.

Remarks.

(i) The preceding discussion shows that this definition is well done: it always exists such a polynomial $\alpha$ and the definition of class does not depend on the choice of $\alpha$.

(ii) When $u \in \mathbb{P}^{(m)\prime}$ is semi-classical there exists $A \in \mathcal{A}_u$ with $\det A \neq 0$, but it is possible for the leading coefficient of $A$ to be singular. However, there always exists $A \in \mathcal{A}_u$ with non-singular leading coefficient. To see this just take $A = (\det A)I$.

The following theorem gives the first characterization of semi-classical matrix functionals.

4.4 Theorem. Let $u \in \mathbb{P}^{(m)\prime}$ be quasi-definite and $(P_n)_{n \geq 0}$ be the corresponding sequence of left orthogonal matrix polynomials. Then, the following statements are equivalent:

(i) $u$ is semi-classical.

(ii) There exists $v \in \mathbb{P}^{(m)\prime} \setminus \{0\}$ such that $(P'_n)_{n \geq 0}$ is quasi-orthogonal with respect to $v$, and $(P_n)_{n \geq 0}$ is regularly quasi-orthogonal with respect to $v$.

Proof.

(i) $\Rightarrow$ (ii) Notice that if $u$ semi-classical then $D(u \alpha I) = uB, \alpha \in \mathbb{P} \setminus \{0\}$ according to previous comments. Therefore,

$$\left< x^k P_n, u \alpha I \right> + \left< x^k P_n, uB \right> = \theta.$$

Let $\alpha(x) = \sum_{j=0}^{p} a_j x^j, B(x) = \sum_{j=0}^{q} b_j x^j$ with $a_p \neq 0$ and $b_q \neq 0$ whenever $B \neq 0$.

With this notation the above equation becomes

$$\sum_{j=0}^{p} \left< kx^{k-1+j} P_n, u \right> a_j + \sum_{j=0}^{q} \left< x^{k+j} P_n, u \right> b_j = - \left< x^k P_n, u \alpha I \right>. \quad (3.1)$$
The left hand side is equal to zero matrix if $0 \leq k \leq n - p$ and $0 \leq k \leq n - (q + 1)$. Let $r = \max\{p - 1, q\}$. Then, $\langle x^k P_n', u\alpha I \rangle = 0$ if $0 \leq k \leq n - r - 1$. For $k = n - r$, (3.1) becomes

$$(n - r) \langle x^n P_n, u \rangle a_{r+1} + \langle x^n P_n, u \rangle b_r = - \langle x^{n-r} P_n', u\alpha I \rangle,$$

and thus

$$\langle x^{n-r} P_n', u\alpha I \rangle = - \langle x^n P_n, u \rangle [(n - r)a_{r+1} + b_r],$$

which is equal to zero matrix for at most one $n$. So, the sequence $(Q_n)_{n \geq 0}$ with $Q_n = P_{n+1}'$ is quasi-orthogonal with respect to $u\alpha I$, of order $r - 1$.

Obviously, since $\alpha I$ has non-singular leading coefficient, $(P_n)_{n \geq 0}$ is regularly quasi-orthogonal with respect to $u\alpha I$, of order $p$.

$(ii) \Rightarrow (i)$ If $(P_n)_{n \geq 0}$ is regularly quasi-orthogonal of order $p$ with respect to $v$, there exists $A \in P^{(m)}$ with $dg\alpha = p$ and non-singular leading coefficient of $A$, such that $v = uA$. Notice that if the leading coefficient of $A$ is non-singular, then $det A \neq 0$.

Let $w = D(uA)$. It is, $\langle P, w \rangle = - \langle P', uA \rangle$ and for $P = x^k P_n$ we have

$$\langle x^k P_n, w \rangle = -k \langle x^{k-1} P_n, uA \rangle - \langle x^k P_n', uA \rangle.$$

From the quasi-orthogonality, the right hand side vanishes if $0 \leq k \leq n - p$ and $0 \leq k \leq n - s - 2$, where $s$ is the order of quasi-orthogonality of $(P_{n+1}')_{n \geq 0}$. So, $(P_n)_{n \geq 0}$ is quasi-orthogonal with respect to the functional $w$ with order at most $\max\{p - 1, s + 1\}$. Thus, there exists $B \in P^{(m)}$ such that $dgB \leq \max\{p - 1, s + 1\}$ and $w = uB$. ∞

Remark. Notice that if $(P_{n+1}')_{n \geq 0}$ is quasi-orthogonal of order $s$ with respect to $u\alpha I$, then $D(u\alpha I) = uB$ where $dg\alpha = p$, $dg\beta = q$ and this implies that $s = \max\{p - 2, q - 1\}$.

§ 5 - Structure relation and differo-differential equation.

5.1 Theorem. (Structure relation)

Let $u \in P^{(m)}$ be quasi-definite and let $(P_n)_{n \geq 0}$ be the associated sequence of left orthogonal matrix polynomials. Then, the following statements are equivalent:

$(i)$ $u$ is semi-classical.

$(ii)$ There exist a polynomial $\alpha \in P \setminus \{0\}$ with $dg\alpha = p$, $s \in \mathbb{N} \cup \{0\}$, $s \geq p - 2$ and $\Theta^{(n)}_j \in \mathbb{C}^{(m,m)}$ $(n \geq 0, -s \leq j \leq p)$, such that

$$\alpha(x)P_{n+1}'(x) = \sum_{j=-s}^{p} \Theta^{(n)}_j P_{n+j}(x),$$

where $\Theta^{(n)}_j \neq \Theta$ for some $n \geq s$ (we use the convention $P_{k} = \Theta$ for $k < 0$).

Proof.

$(i) \Rightarrow (ii)$ If $u$ is semi-classical then there exist $\alpha \in P \setminus \{0\}$ and $B \in P^{(m)}$ such that $D(u\alpha I) = uB$. Hence, $(P_{n+1}')_{n \geq 0}$ is quasi-orthogonal with respect to $u\alpha I$ of order $s = \max\{p - 2, q - 1\}$, where $p = dg \alpha$ and $q = dg B$. This implies

$$\langle x^k P_{n+1}, u\alpha I \rangle = \Theta, \quad 0 \leq k \leq n - s - 1,$$
\[<x^{n-s}P'_{n+1}, u\alpha I> \neq \theta, \text{ for some } n \geq s.\]

Since \(\alpha P'_{n+1} \in \mathbb{P}_n^{(m)}\) there exist \(\Theta_j^{(n)}\) verifying (i).

\((ii) \Rightarrow (i)\) Let \(v = u\alpha I\). Then

\[\langle x^k P'_{n+1}, v \rangle = \sum_{j=-s}^{p} \Theta_j^{(n)} \langle x^k P_{n+j}, u \rangle = \theta, \quad 0 \leq k \leq n-s-1, \quad n \geq s,
\]

and, since \(x^{n-s}P_{n-s}, u \rangle \) is non-singular,

\[\langle x^{n-s}P'_{n+1}, v \rangle = \Theta_j^{(n)} \langle x^{n-s}P_{n-s}, u \rangle \neq \theta, \text{ for some } n \geq s.\]

Thus, \((P'_{n})_{n \geq 0}\) is quasi-orthogonal of order \(s\) with respect to \(v\). Obviously \((P_{n})_{n \geq 0}\) is regularly quasi-orthogonal of order \(p\) with respect to \(v\) and, so, \(u\) is semi-classical. \(\diamond\)

5.2 Theorem. (Differo-differential equation)

Let \(u \in \mathbb{P}_n^{(m)}\) be quasi-definite and let \((P_{n})_{n \geq 0}\) be the corresponding sequence of left orthogonal matrix polynomials. Then, the following statements are equivalent:

(i) \(u\) is semi-classical.

(ii) There exist two polynomials \(\alpha, \beta \in \mathbb{P}\) with \(p = d\alpha \geq 0, q = d\beta \) and matrices \(\Lambda_k^{(n)} \in \mathbb{C}^{(m,m)} (n \geq 0, -s \leq k \leq s)\), such that

\[\alpha(x)P''_{n}(x) + \beta(x)P'_{n}(x) = \sum_{k=-s}^{s} \Lambda_k^{(n)} P_{n+k}(x),\]

where \(s \geq \max\{p - 2, q - 1\}\) (we use the convention \(P_k = \theta\) for \(k < 0\)).

Proof.

\((i) \Rightarrow (ii)\) Let \(u\) be semiclassical. By Theorem 5.1, there exist a polynomial \(a \in \mathbb{P}\), with \(d\alpha = p_1 \geq 0\), a non-negative integer \(s_1 \geq p_1 - 2\), and matrices \(\Theta_j^{(n)} \in \mathbb{C}^{(m,m)} (n \geq 0, -s_1 \leq j \leq p_1)\), such that

\[a(x)P''_{n}(x) = \sum_{j=-s_1}^{p_1} \Theta_j^{(n-1)} P_{n-1+j}(x),\quad (5.1)\]

Taking derivatives in (5.1), we obtain that

\[a^2 P''_{n} + a\alpha P'_{n} = \sum_{j=-s_1}^{p_1} \Theta_j^{(n-1)} aP'_{n-1+j},\]

If we use (5.1) in the right hand side, it follows that

\[a^2 P''_{n} + a\alpha P'_{n} = \sum_{j=-s_1}^{p_1} \Theta_j^{(n-1)} \sum_{k=-s_1}^{p_1} \Theta_k^{(n-2+j)} P_{n-2+j+k},\]
Let us denote $\alpha(x) = a^2(x)$, $\beta(x) = a(x) a'(x)$, $A^{(n)}_{j+k} = \Theta_j^{(n-1)} \Theta_k^{(n-2+j)}$, and $p = 2p_1 = \text{deg } a$, $q = 2p_1 - 1 = \text{deg } \beta$, $s = \max \{2p_1 - 2, 2s_1 + 2\}$. Then,

$$\alpha P'' + \beta P' = \sum_{j=-s}^s \Lambda_j^{(n)} P_{n+j},$$

with $s \geq \max \{p - 2, q - 1\}$.

(ii) $\Rightarrow$ (i) Let $(\pi_n)_{n \geq 0}$ be the dual basis of $(P_n)_{n \geq 0}$, that is, $\pi_n = u P_n^* E_n^{-1}$, where $E_n := < P_n, P_n >_u$. Let $(Q_n)_{n \geq 0}$ be the basis of $\mathbb{P}^m$ given by $Q_n(x) = \frac{1}{n+1} P'_{n+1}$, and let $(\rho_n)_{n \geq 0}$ be the corresponding dual basis. Thus,

$$D \rho_n = -(n+1) \pi_{n+1}, \quad n \geq 0. \quad (5.2)$$

We consider for every $n \geq 0$ the linear functional $-D(\pi_n \alpha) + \pi_n \beta$. There exists $(\lambda_k^{(n)})_{k \geq 0} \subseteq \mathbb{C}^{m,n}$ such that

$$-D(\pi_n \alpha) + \pi_n \beta = \sum_{k=0}^{\infty} \rho_k \lambda_k^{(n)},$$

where from

$$< Q_j, -D(\pi_n \alpha) + \pi_n \beta > = \sum_{k=0}^{\infty} < Q_j, \rho_k > = \lambda_k^{(n)}$$

holds for $j \geq 0$. Hence, our hypothesis implies that

$$\lambda_k^{(n)} = \frac{1}{n+1} \sum_{j=-s}^s \Lambda_j^{(n+1)} \delta_{j+1+k,n}. \quad (5.3)$$

For $n = 0$, (5.3) leads to $\lambda_j^{(0)} = 0$ if $j \geq s$, and

$$-D(\pi_0 \alpha) + \pi_0 \beta = \sum_{k=0}^{s-1} \rho_k \lambda_k^{(0)},$$

or

$$-D(u \alpha) P_0^* E_0^{-1} + u \beta P_0^* E_0^{-1} = \sum_{k=0}^{s-1} \rho_k \lambda_k^{(0)}.$$

If we denote $\tilde{\lambda}_k^{(0)} = \lambda_k^{(0)} E_0 (P_0^*)^{-1}$, then

$$-D(u \alpha) + u \beta = \sum_{k=0}^{s-1} \rho_k \tilde{\lambda}_k^{(0)} \quad (5.4)$$

In a similar way, for $n = 1$ (5.3) gives

$$-D(u \alpha) P_1^* E_1^{-1} + u \beta P_1^* E_1^{-1} = \sum_{k=0}^{s} \rho_k \lambda_k^{(1)}.$$
because $\lambda_j^{(1)} = 0$ for $j \geq s + 1$. Thus, 

$$-D(u\alpha P_1^*) + u\beta P_1^* = \sum_{k=0}^{s} \rho_k \tilde{\lambda}_k^{(1)}$$

(5.5)

with $\tilde{\lambda}_k^{(1)} = \lambda_k^{(1)} E_1$.

Keeping in mind that $P_1(x) = M_1 x + M_2$, with $M_1$ non-singular, (5.5) remains

$$-D(u\alpha P_1^*) - u\beta P_1^* + u\beta P_1^* = \sum_{k=0}^{s} \rho_k \tilde{\lambda}_k^{(1)},$$

and, using (5.4),

$$-u\alpha M_1^* = \sum_{k=0}^{s} \rho_k \tilde{\lambda}_k^{(1)} - \left(\sum_{k=0}^{s-1} \rho_k \tilde{\lambda}_k^{(0)}\right) P_1^*.$$

Taking derivatives and applying (5.2) we obtain

$$-D(u\alpha) M_1^* = - \sum_{k=0}^{s} (k+1) \pi_{k+1} \tilde{\lambda}_k^{(1)} + \left(\sum_{k=0}^{s-1} (k+1) \pi_{k+1} \tilde{\lambda}_k^{(0)}\right) P_1^* - \left(\sum_{k=0}^{s-1} \rho_k \tilde{\lambda}_k^{(0)}\right) M_1^* =$$

$$= - \sum_{k=0}^{s} (k+1) u P_{k+1}^* E_{k+1}^{-1} \tilde{\lambda}_k^{(1)} + \left(\sum_{k=0}^{s-1} (k+1) u P_{k+1}^* E_{k+1}^{-1} \tilde{\lambda}_k^{(0)}\right) P_1^* - \left(\sum_{k=0}^{s-1} \rho_k \tilde{\lambda}_k^{(0)}\right) M_1^*.$$

As a consequence, the polynomial $\Psi \in \mathbb{P}^{(m)}$ given by

$$\Psi(x) := \left[\sum_{k=0}^{s} (k+1) P_{k+1}^* E_{k+1}^{-1} \tilde{\lambda}_k^{(1)} - \left(\sum_{k=0}^{s-1} (k+1) P_{k+1}^* E_{k+1}^{-1} \tilde{\lambda}_k^{(0)}\right) P_1^*(x)\right] (M_1^*)^{-1},$$

whose degree is not greater than $s + 1$, verifies that

$$D(u\alpha) = u\Psi + \sum_{k=0}^{s-1} \rho_k \tilde{\lambda}_k^{(0)}.$$

From this and (5.4) we get

$$D(u\alpha) = uB$$

with $B = \frac{\Psi + \beta I}{2}$, that is, $u$ is semi-classical.

**Remark 1.** The distributional equation $D(u\alpha) = u\beta$ for a scalar functionals $u$ implies that

$$< x^k (\alpha p_n''(x) + \beta p_n'(x)), u > =$$

$$= < (x^k p_n'(x))', u\alpha > - k < x^{k-1} p_n'(x), u\alpha > + < x^k p_n'(x), u\beta > =$$

$$= - k < x^{k-1} p_n'(x), u\alpha >,$$
wich vanishes for \( k = 0, \ldots, n - s - 1 \). So,

\[
\alpha p_n'(x) + \beta p_n(x) = \sum_{k=-s}^{s} \lambda_{nk} p_{n+k}(x)
\]

In the matricial case, the equality \( D(uaI) = uB \) does not imply \( < x^k P_n'(x), uB(x) >= < x^k B(x)P_n'(x), u > \) for the non-commutativity, and more generally, there not exists a polynomial \( B(x; n) \) such that \( < x^k P_n'(x), uB(x) > = < x^k B(x; n)P_n'(x), u > \), neither.

In this situation it is not possible to obtain a differo-differential equation with the polynomial \( B \) explicitly.

5.2 Remark 2. Obviously, the differo-differential equation given by the Theorem 5.2 is not unique because we can modify it by adding any structure relation.

6 - Some examples.

We will illustrate the preceding results with some examples.

1. In the first one, we consider the scalar Laguerre weight \( w = xe^{-x} \) that verifies the Pearson type equation

\[
(wx)' = w(2 - x).
\]

Now, we define the matrix weight function

\[ u := w \begin{pmatrix} 1 & 1 \\ 1 & 1 + x^2 \end{pmatrix} \]

that also verifies a Pearson type equation. In fact

\[ D(uxI) = uB(x) \]

where \( B(x) = \begin{pmatrix} 2 - x & -2 \\ 0 & 4 - x \end{pmatrix} \).

Moreover, there exists \( T \in GL(\mathbb{R}^2) \) such that

\[ \hat{u} = TuT^t = \begin{pmatrix} xe^{-x} & 0 \\ 0 & x^3e^{-x} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \]

i.e., \( u \) is congruent with a diagonal weight with classical scalar weights in its diagonal and then, the functional \( u \) is positive definite.

Let \( l_n^{(a)} \) be the monic Laguerre polynomials asociated to the weight \( x^a e^{-x}, a > -1 \). Then,

\[ L_n(x) = \begin{pmatrix} l_n^{(1)}(x) & 0 \\ 0 & l_n^{(3)}(x) \end{pmatrix} \]

constitutes a sequence of matrix O P related to the diagonal weight \( \hat{u} \). Obviously \( \hat{u} \) satisfies a Pearson equation,

\[ D(\hat{u}xI) = \hat{u}\hat{B}(x), \quad \hat{B}(x) = (T^t)^{-1}B(x)T^t = \begin{pmatrix} 2 - x & 0 \\ 0 & 4 - x \end{pmatrix}, \]
and a sequence of matrix $O P$ associated to $u$ is given by

$$P_n(x) = L_n(x)T, \quad n \geq 0$$

It is easy to obtain a structure relation

$$xP'_{n+1}(x) = (n+1)P_{n+1} + (n+1) \left( \begin{array}{cc} n+2 & 0 \\ 0 & n+4 \end{array} \right) P_n(x),$$

as well as, for all $n \geq 2$, a differential equation,

$$x^2 P''_n(x) + xP'_n(x) = n(n+1)P_n(x) +$$

$$+ n^2 \left( \begin{array}{cc} n+1 & 0 \\ 0 & n+3 \end{array} \right) P_{n-1}(x) + n(n-1) \left( \begin{array}{cc} n(n+1) & 0 \\ 0 & (n+2)(n+3) \end{array} \right) P_{n-2}(x).$$

Moreover, the polynomials $l_n^{(a)}$ satisfy the differential equation,

$$x l''_n(x) + (x-a-1)l'_n(x) = n l_n^{(a)}(x),$$

and so, we can obtain for the polynomials $P_n$ the following differential equation:

$$xP''_n(x) + \left( \begin{array}{cc} x-2 & 0 \\ 0 & x-4 \end{array} \right) P'_n(x) = nP_n(x).$$

For the monic polynomial $\tilde{P}_n = T^{-1} P_n$ we have

$$x\tilde{P}''_n(x) + T^{-1} \left( \begin{array}{cc} x-2 & 0 \\ 0 & x-4 \end{array} \right) T\tilde{P}'_n(x) = n\tilde{P}_n(x),$$

and then

$$x\tilde{P}''_n(x) + B'(x)\tilde{P}'_n(x) = n\tilde{P}_n(x)$$

The polynomials $\tilde{P}_n$ can be obtained by means of a Rodrigues type formula

$$\tilde{P}_n(x) = \frac{1}{n!} (T^{-1} T)^{-1} u^{-1} \frac{d^n}{dx^n} (T^{-1} T).$$

2. Now, we consider the the matrix weight $u := e^{-x^2} R(x)$ where $R(x) = \left( \begin{array}{cc} 0 & 1 \\ 1 & x \end{array} \right)$. So, $R(x)$ is not positive definite. Then, $u$ verifies the distributional equation

$$u' = e^{-x^2} R'(x) - 2xe^{-x^2} R(x) = u(-2xI + R^{-1}(x)R'(x)) = uB(x),$$

where $B(x) = \left( \begin{array}{cc} -2x & 1 \\ 0 & -2x \end{array} \right)$. Moreover, $u$ has not a congruent diagonal.

Now, let

$$P_n(x) = (-2)^{(-n)} e^{x^2} S^{-1}(x) \frac{d^n}{dx^n} \left( e^{-x^2} S(x) \right), \quad n \geq 0,$$
where \( S(x) = \begin{pmatrix} 1 + x & 1 \\ x & 1 \end{pmatrix} \).

By derivation in (6.2)

\[
P_{n+1}(x) = -\frac{1}{2} [-2xI + S^{-1}(x)S'(x)]P_n(x) - \frac{1}{2} P'_n(x).
\]

Then,

\[
P_{n+1}(x) = -\frac{1}{2} [B'(x)P_n(x) + P'_n(x)],
\]

and the leading coefficient of \( P_n \) is \( I \) because \( P_0(x) = I \).

Applying the Leibnitz rule for the derivatives in (6.2), we have

\[
P_{n+2}(x) = (-2)^{-n-\frac{1}{2}} e^{x^2} S^{-1}(x) \frac{d^{n+1}(-2xe^{-x^2}S(x) + e^{-x^2}S'(x))}{dx^{n+1}} =
\]

\[
= (-2)^{-n-1} xe^{x^2} S^{-1}(x) \frac{d^{n+1}(e^{-x^2}S(x))}{dx^{n+1}} + (n+1)(-2)^{-n-1} e^{x^2} S^{-1}(x) \frac{d^n(e^{-x^2}S(x))}{dx^n} +
\]

\[
+ (-2)^{-n-2} e^{x^2} S^{-1}(x)S'(x)S^{-1}(x) \frac{d^{n+1}(e^{-x^2}S(x))}{dx^{n+1}}.
\]

So, we obtain the recurrence relation,

\[
P_{n+2}(x) = -\frac{1}{2} [B'(x)P_{n+1}(x) + (n+1)P_n(x)].
\] (6.4)

As a consequence, \( (P_n) \) is a sequence of matrix orthogonal polynomials with respect to certain non-singular functional (non-positive definite because \(-\frac{1}{4}(n+1)I\) is a non-positive definite matrix).

Moreover, from (6.3) and (6.4) we can obtain a structure relation

\[
P_n(x) = nP_{n-1}(x),
\] (6.5)

and by derivation, the differo-differential equation

\[
P'_n(x) = n(n-1)P_{n-2}(x).
\] (6.6)

So, \( (P_n) \) is a semiclassical matrix orthogonal polynomial sequence.

Now, from (6.3) and (6.5) it follows the differential equation,

\[
P'_n(x) + B'(x)P'_n(x) = -2nP_n(x).
\] (6.7)

Indeed, \( (P_n) \) is an sequence of matrix orthogonal polynomials with respect to the matricial weight \( u \). In fact,

\[
\int_{-\infty}^{\infty} P'_n(x)e^{-x^2} R(x)P_m(x)dx =
\]

\[
= \int_{-\infty}^{\infty} [P'_n(x) + (-2xI + R'(x)R^{-1}(x))P'_n(x)]e^{-x^2} R(x)P_m(x)dx =
\]
\[-2n \int_{-\infty}^{\infty} P_n(x) e^{-x^2} R(x) P_m^t(x) dx,\]

the last equality from equation (6.7). In the same way, we obtain for the above integral that

\[\int_{-\infty}^{\infty} P_n'(x) e^{-x^2} R(x) P_m^t(x) dx = -2m \int_{-\infty}^{\infty} P_n(x) e^{-x^2} R(x) P_m^t(x) dx.\]

So, when \( n \neq m \) we have the orthogonality for \((P_n)\) with respect to \( u \).

As a consequence we have a quasi definite functional such that:

(i) It satisfies the distributional equation \( D(u\alpha I) = uB \).

(ii) The corresponding sequence of matrix orthogonal polynomials satisfies a Rodrigues type formula.

(iii) It verifies the structure relation (6.5), the differo-differential equation (6.6) and a differential equation (6.7). Keeping in mind Remark 1 to Theorem 5.2, notice that we have (6.7) because \( uB = (uB)^t = B^t u \) and \( B^t P_n = P_n B^t \).

3. In the last example, we consider the Jacobi scalar weight \( w = 1 - x^2 \) that satisfies the distributional equation

\[(w(1 - x^2))' = -w \cdot 4x, \quad x \in [-1, 1].\]

We define the matricial weight function,

\[u := w \begin{pmatrix} 1 & x \\ x & 2 - x^2 \end{pmatrix}, \quad x \in [-1, 1],\]

that verifies the Pearson type equation

\[D \left( u(1 - x^2)I \right) = uB(x),\]

where

\[B(x) = \begin{pmatrix} -9x & 2 + x^2 \\ 1 & -11x \end{pmatrix}.\]

Notice that \( u \) is positive definite and then, there exists a sequence of matrix orthogonal polynomials \((P_n)\) related to \( u \). Moreover, \( u \) not is congruent with a diagonal weight. So, the sequence \((P_n)\) satisfies a structure relation with \( s = 1, p = 2 \) and a differo-differential equation of second order in the following way:

\[(1 - x^2)^2 P_n''(x) - x(1 - 2x) P_n'(x) = \sum_{k=-4}^{2} \Lambda_{nk} P_{n+k}.\]
Acknowledgements.- This research was supported by Dirección General de Enseñanza Superior of Spain (DGES). Project PB98-1615.

The authors are very grateful to Professor Francisco Marcellán for his remarks and useful suggestions.

References

[1] M. Alfaro, A. Branquinho, F. Marcellán, and J. Petronilho. A generalization of a theorem of S. Bochner. Publicaciones del Seminario Matemático García de Galdeano. Serie II. Sección 1, número 11. (1992).

[2] S. Bonan, D.S. Lubinski, and P. Nevai. Orthogonal polynomials and their derivatives. SIAM J. Math. Anal. 18 (1987) 1163-1175.

[3] T.S. Chihara. “An introduction to orthogonal polynomials” Gordon and Breach, New York, 1978.

[4] A.J. Durán. On orthogonal polynomials with respect to a positive definite matrix of measures. Can. J. Math. 47 (1995) 88-112.

[5] A.J. Durán. Matrix inner product having a matrix symmetric second order differential equation. Rocky Mountain Journal of Mathematics 27 (1997) 585-600.

[6] A.J. Durán, W. Van Assche. Orthogonal matrix polynomials and higher-order recurrence relations. Linear Alg. Appl. 219 (1995) 261-280.

[7] G. Freud. “Orthogonal polynomials” Pergamon Press, Oxford, 1971.

[8] Y.L. Geronimus. “Orthogonal polynomials” Consultants Bureau, New York, 1961.

[9] E. Hendriksen, H. van Rossum. Semi-classical orthogonal polynomials. C. Brezinski et al. Eds., Lecture Notes in Math.1171 (Springer, Berlin, 1985) 354-361.

[10] F. Marcellán, H.O. Yakhlef. Recent trends on analytic properties of matrix orthonormal polynomials. Electronic Transactions in Numerical Analysis. To appear.

[11] P. Maroni. Variations around classical orthogonal polynomials. Connected problems. J. Comput. Appl. Math. 48 (1-2) (1993) 133-155.

[12] P. Maroni. Une théorie algébrique des polynômes orthogonaux. Application aux polynomes orthogonaux semiclassiques. C. Brezinski et al. Eds. Orthogonal Polynomials and Their Applications, IMACS Ann.Comput.Appl. Math. 9 (1991) 95-130.

[13] G. Sansigre. Polinomios ortogonales matriciales y matrices bloques. Doctoral Dissertation Universidad de Zaragoza, (1992). In Spanish.

[14] J.A. Shohat. A differential equation for orthogonal polynomials. Duke. Math. Journal. 5 (1939) 401-407.