Two forms of the action variable for the relativistic harmonic oscillator

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(Dated: March 28, 2022)

Abstract

The frequency of a classical periodic system can be obtained using action variables without solving the dynamical equations. We demonstrate the construction of two equivalent forms of the action variable for a one dimensional relativistic harmonic oscillator and obtain its energy dependent frequency. This analysis of oscillation is compared with the traditional solution of the problem which requires the use of hypergeometric series.
I. KEY WORDS

Hamilton-Jacobi theory, action variable, contour integration, relativity, simple harmonic oscillator, hypergeometric series.

II. INTRODUCTION

Classical periodic systems can be analyzed elegantly in terms of their action and angle variables which constitute a set of canonically conjugate momenta and coordinates. Action variables are proportional to $\oint p_j dq_j$, where $(q_j, p_j)$ are the system’s coordinates and canonical momenta. For conservative systems they are constants of motion in the manner of angular momentum and energy. The frequencies of periodic systems can be found using the functional relationship between the action variables and total mechanical energy without requiring a complete solution of the dynamical equations. We first summarize here that theory and situate action variables within its matrix. In Section II we apply this formalism to determine the frequency of a non-relativistic simple harmonic oscillator using two equivalent contour integral definitions of the action variable. In Section III we extend this formalism to a relativistic simple harmonic oscillator, obtain its frequency in two equivalent series representations and demonstrate that it has the correct non-relativistic limit. In Section IV we derive the expression for the period of this relativistic oscillator by direct integration and compare it with the one obtained using action variables.

The time evolution of a classical system is governed by its Hamiltonian $H$ which is a function of its coordinates $x_i$, the conjugate momenta $p_i$ and the time $t$. The dynamics of such a system is determined by Hamilton’s equations of motion

$$\dot{x}_i = \frac{\partial H(x_i, p_i, t)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(x_i, p_i, t)}{\partial x_i}. \tag{1}$$

In the case of a particle of mass $m$ moving in one dimension under the influence of a time independent potential energy function $V(x)$ the Hamiltonian is given by $H = \frac{p^2}{2m} + V(x)$. Such a Hamiltonian is a constant of the motion and is the total energy $E$ of the system. Thus,

$$\frac{p^2}{2m} + V(x) = E. \tag{2}$$
Canonical transformations are those transformations of one set of coordinate and momentum \((x, p)\) to another set \((X, P)\) that preserve the form of Hamilton’s equations. One such transformation is generated by the generating function \(W_C(x, P)\) (the suffix \(C\) in this and similar variables refers to the ”classical” rather than the ”quantum” nature of the mechanics considered), whose arguments are the ”old” coordinate, \(x\), and the ”new” momentum, \(P\):

\[
p = \frac{\partial W_C(x, P)}{\partial x}, \quad X = \frac{\partial W_C(x, P)}{\partial P}.
\]

If this transformation transforms the Hamiltonian into a function only of \(P\), then, using \((1)\),

\[
\dot{P} = -\frac{\partial H(P)}{\partial X} = 0 \Rightarrow P(t) = P, \text{ a constant},
\]

\[
\dot{X} = \frac{\partial H(P)}{\partial P} = V_0, \text{ a constant}, \Rightarrow X(t) = V_0 t + X_0.
\]

Thus \(X\) and \(P\) evolve very simply in time. The former progresses linearly in time and the latter is a constant. \(W_C(x, P)\), which generates a canonical transformation in which the transformed Hamiltonian is independent of the new coordinate \(X\), is the Hamilton’s characteristic function. It is related to Hamilton’s principle function \(S_C\) through \(S_C(x, P) = W_C(x, P) - Et\) (which generates another canonical transformation, not considered here) and, for the case of time independent Hamiltonians, satisfies the Hamilton-Jacobi equation obtained by using \((3)\) in \((2)\):

\[
\frac{1}{2m} \left( \frac{\partial W_C(x, P)}{\partial x} \right)^2 + V(x) = E(P).
\]

The use of this method to solve the dynamical problem involves the following steps: (i) Define a suitable new constant momentum \(P\), (ii) Integrate Eq. \((5)\) to obtain \(W_C(x, E(P))\), (iii) Obtain \(x(X, P)\) and \(p(X, P)\) using Eq. \((3)\), and (iv) Express \(X\) and \(P\) in terms of the initial values \(x_0, p_0\) and \(t\).

One particular form of Hamilton-Jacobi theory is particularly suited for the study of periodic motion. If an inspection of the Hamiltonian indicates that the motion is periodic, then by a particular choice of the new momentum \(P\) we can evaluate the period of motion without obtaining a complete solution of the dynamical problem. The new canonically conjugate coordinate and momentum are chosen to be \(X = w, P = J_C\) with

\[
J_C = \frac{1}{2\pi} \oint p_C(x, E)dx,
\]
where \( p_C(x, E) \), from \(^2\), is \( \sqrt{2m[E - V(x)]} \) and the integral in phase space is performed over one cycle of the periodic motion. \( J_C \) is the classical action variable and \( w \) the angle variable. Since \( J_C = J_C(E) \) we can invert it to obtain \( E = E(J_C) \). From Eq. \(^4\) the time evolution of the new coordinate is \( w(t) = \omega t + w_0 \) where the constant "velocity" is

\[
\omega = \frac{\partial H(J_c)}{\partial J_C} = \frac{\partial E(J_c)}{\partial J_C}.
\]

(7)

It can be shown that \( \omega \) is the angular frequency of this periodic motion\(^1\). Thus the mathematical problem of finding the frequency of classical periodic motion for a conservative system is reduced to that of performing the integral \(^6\), solving for \( E \) to get \( E(J_C) \), and evaluating \( \partial E/\partial J_C \). This is a simple and elegant method for evaluating the frequency of a system known to be periodic. Charles-Eugène Delaunay (1816-1872) invented action and angle variables in the course of his study of periodicity of lunar motion\(^2\). In the early days of quantum mechanics, Sommerfeld, in his treatment of the motion of an electron in the hydrogen atom, made use of the action variable, and evaluated it using a contour integral in the complex coordinate plane\(^3\). We follow his example and obtain the classical relativistic harmonic oscillator’s frequency by evaluating the action variable using two different suitably defined contour integrals. There is an equivalent formalism in contemporary quantum mechanics where a quantum version of the action variable, also evaluated by contour integrals, is employed to determine the energy eigenvalues of a bound quantum system. This has received attention in the last two decades\(^4,5,6\).

An equivalent definition of \( J_C \) is that it is the contour integral

\[
J_C = \frac{1}{2\pi} \oint_C p_C(x, E) dx,
\]

(8)
in the complex \( x \) plane over a contour \( C \) (specified next), where \( p_c(x, E) \) is a complex valued function of the complex argument \( x \), and is defined as a suitable branch of

\[
p_c(x, E) = \sqrt{2m[E - V(x)]}.
\]

(9)
The turning points \( x_1 \) and \( x_2 \), both real, are defined by \( p_c(x_1, E) = p_c(x_2, E) = 0 \). These are also the branch points of \( p_c(x, E) \) in the complex-\( x \) plane. We choose a branch cut connecting \( x_1 \) and \( x_2 \) along the real axis. \( p_c(x, E) \) is chosen as that branch of the square root which is positive along the bottom of the cut. The counterclockwise contour \( C \) surrounds the two turning points \( x_1 \) and \( x_2 \) and the branch cut connecting them. The integral in \(^8\) is
performed by enlarging $C$ outward to another contour $\gamma$, expanding $p_c(x, E)$ in a Laurent series in an annulus that contains $\gamma$ and using Cauchy’s residue theorem.

An equivalent construction of the action variable has the $-\frac{1}{2\pi} \oint_{C'} x_c(p, E) dp$ form, where the clockwise contour $C'$ in the complex-$p$ plane encloses the two turning momenta, $p_1$ and $p_2$, and the branch cut connecting them along the real axis. This second form of the action variable is insufficiently explored in the literature, and there is no account of the frequency determination of the relativistic oscillator using the action variable. We will demonstrate the use of this alternate form of the action variable for the harmonic oscillator in the relativistic case, and for comparison and completeness, in the non-relativistic case also.

III. ACTION VARIABLE FOR NON-RELATIVISTIC HARMONIC OSCILLATOR

The Hamiltonian for the non-relativistic harmonic oscillator with spring constant $k$ is $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. We will refer to the angular frequency of this oscillator, $\sqrt{\frac{k}{m}}$, as $\omega_0$. We first demonstrate the technique of constructing the classical action variable for this simple case using the two methods outlined earlier. We will extend it to the case of the relativistic harmonic oscillator in a very similar manner.

A. Action variable in the $\oint p dx$ form

The two turning points of the oscillator, $x_1$ and $x_2$ for energy $E$, are obtained, using (2), by setting $p_C(x_1, E) = p_C(x_2, E) = 0$. As functions of energy, they are

$$-x_1 = x_2 = \sqrt{\frac{2E}{k}}. \quad (10)$$

We write the momentum in the form

$$p_C(x, E) = \sqrt{2m \left[ E - \frac{kx^2}{2} \right]} = i\sqrt{mk} x \left[ 1 - \left( \frac{x_2}{x} \right)^2 \right]^\frac{1}{2} \quad (11)$$

and extend it analytically in the complex-$x$ plane, with a branch cut connecting the turning points $x_1$ and $x_2$ along the real axis. The Laurent series for $p_C(x, E)$ in the region $|x| > x_2$ in powers of $\frac{x_2}{x}$ is obtained by using the binomial series for the square root:

$$p_C(x, E) = \sum_{j=1}^{\infty} a_j x^{3-2j} = i\sqrt{mk} x \left[ 1 - \frac{1}{2} \left( \frac{x_2}{x} \right)^2 - \frac{1}{8} \left( \frac{x_2}{x} \right)^4 + \ldots \right] \quad (12)$$
Deforming the contour $C$ outward into a circular contour $\gamma$ centered at the origin with radius greater than $x_2$, and evaluating the contour integral using Cauchy’s residue theorem, we get

$$J_C = \frac{1}{2\pi i} (2\pi i) i \sqrt{mk} \left( a_2 = -\frac{1}{2} x_2^2 \right) = \frac{1}{\omega_0} E. \quad (13)$$

The angular frequency, from Eq. (7), is $\frac{\partial E}{\partial J_C}$, or $\omega = \omega_0$. For other kinds of oscillators where the relation $J_C = J_C(E)$ cannot be inverted to obtain $E = E(J_C)$ in a closed form, we can use $\frac{1}{\omega} = \frac{\partial J_C}{\partial E}$. The non-dependence of $\omega$ on the simple harmonic oscillator’s energy, and thus on its amplitude of oscillation, arises from the purely linear relation between $E$ and $J_C$, which is characteristic of this simple classical periodic system, with its quadratic potential energy function. As we will see in the next section there is a richer structure to the relationship between $J_C$ and $E$ for a relativistic harmonic oscillator.

B. Action variable’s $- \oint xdp$ form

We demonstrate an alternate, but equivalent, form of the action variable for the simple harmonic oscillator. The turning momenta, $p_1$ and $p_2$, of the oscillator are defined, using (2), by $x_C(p_1, E) = x_C(p_2, E) = 0$, or

$$-p_1 = p_2 = \sqrt{2mE}. \quad (14)$$

We write the coordinate in powers of $\frac{p_2}{p}$ as

$$x_C(p, E) = \sqrt{\frac{2}{k}} \sqrt{E - \frac{p_2^2}{2m}} = \frac{-i}{\sqrt{mk}} p \left[ 1 - \left( \frac{p_2}{p} \right)^2 \right]^\frac{1}{2}. \quad (15)$$

Using (15) we extend $x_C(p, E)$ into the complex-$p$ plane with a branch cut connecting $p_1$ and $p_2$ along the real axis. We choose that branch of $x_C(p, E)$ that is positive just above the branch cut. The alternate form of $J_C(E)$ is

$$J_C = -\frac{1}{2\pi} \oint_{C'} x_C(p, E) dp, \quad (16)$$

where the clockwise contour $C'$ encloses the branch cut. For $|p| > p_2$, we expand $x_C(p, E)$ in the Laurent series

$$x_C(p, E) = \sum_{j=1}^{\infty} a_j p^{3-2j} = \frac{-i}{\sqrt{mk}} p \left[ 1 - \frac{1}{2} \left( \frac{p_2}{p} \right)^2 - \frac{1}{8} \left( \frac{p_2}{p} \right)^4 \ldots \right]. \quad (17)$$
Deforming the contour $C'$ outward into a circular contour $\gamma'$ centered at the origin with radius greater than $p_2$ and evaluating the integral in (16), we get

$$J_C = -\frac{1}{2\pi} \left(2\pi i\right) \left(i = -\frac{1}{2}p_2^2\right) = \frac{1}{\omega_0} E$$  \hspace{1cm} (18)

The expected result is that this $J_C(E)$ yields the angular frequency $\omega = \partial E/\partial J_C = \omega_0$.

**IV. RELATIVISTIC HARMONIC OSCILLATOR**

The relativistic motion of the harmonic oscillator is governed by the Hamiltonian $H(x, p_{CR}) = \sqrt{p_{CR}^2 c^2 + m^2 c^4} + \frac{1}{2}kx^2$. The additional suffix $R$ indicates that the canonical momentum is relativistic. The total mechanical energy of the relativistic oscillator will be referred to as $E$, and $\tilde{E} = E - mc^2$ is its mechanical energy in excess of its rest mass energy. The dimensionless energy related parameter we will use is $\epsilon = \frac{\tilde{E}}{mc^2}$. The non-relativistic case is characterized by $\epsilon << 1$.

**A. $\oint p\, dx$ form of action variable**

The relativistic orbit equation, obtained from $H(x, p_{CR}) = \tilde{E} + mc^2$, is

$$p_{CR} = \sqrt{2m \left[\tilde{E} - \frac{1}{2}kx^2\right] \left[1 + \frac{\tilde{E} - \frac{1}{2}kx^2}{2mc^2}\right]}.$$  \hspace{1cm} (19)

There are four branch points, $x_jR, \ j = 1, 2, 3, 4$, in the complex-$x$ plane where $p_{CR}(x, \tilde{E})$ vanishes. Two of these are the physical turning points $x_{1R}$ and $x_{2R}$ given by

$$-x_{1R} = x_{2R} = \sqrt{\frac{2\tilde{E}}{k}}.$$  \hspace{1cm} (20)

Their locations in the complex-$x$ plane are similar to those of the turning points $x_1$ and $x_2$ in the non-relativistic case, given by Eq. (10). The other two branch points of $p_{CR}(x, \tilde{E})$ are on the real axis at

$$-x_{3R} = x_{4R} = \sqrt{\frac{2\tilde{E}}{k} \sqrt{1 + \frac{2}{\epsilon}}}.$$  \hspace{1cm} (21)

Their form indicates that these two branch points, unlike $x_{1R}$ and $x_{2R}$, are entirely relativistic in character. It is clear that $x_{2R} < x_{4R}$ for all energies. We choose a branch cut of $p_{CR}(x, \tilde{E})$ connecting $x_{1R}$ and $x_{2R}$ along the real axis. Two other branch cuts connect $x_{3R}$ and $x_{4R}$.
with \( x = \infty \) along the real axis. The branch of \( p_{CR}(x, \tilde{E}) \) that we choose for complex values of \( x \) is positive just below the cut connecting \( x_{1R} \) and \( x_{2R} \).

Using Eq. (19) we rewrite \( p_{CR} \) in the form

\[
p_{CR}(x, \tilde{E}) = \sum_{j=-\infty}^{\infty} A_j x^{3-2j} = i \sqrt{mk} \ x \left[ 1 - \left( \frac{x_{2R}}{x} \right)^2 \right]^{\frac{1}{4}} (1 + \epsilon)^{\frac{1}{2}} \left[ 1 - \left( \frac{x}{x_{4R}} \right)^2 \right]^{\frac{1}{2}}. \tag{22}
\]

Comparing this with Eq. (11) we see that the multiplicative factor \((1 + \epsilon)^{\frac{1}{2}}[1 - (\frac{x}{x_{4R}})^2]^{\frac{1}{2}}, \) which is very nearly 1 for low energies and for \(|x| < |x_{4R}|\), modifies the non-relativistic \( p_C(x, E) \) into the relativistic \( p_{CR}(x, \tilde{E}) \).

The action variable \( J_{CR}(E) \) is defined as

\[
J_{CR} = \frac{1}{2\pi} \oint_{C_R} p_{CR}(x, E) dx, \tag{23}
\]

where the counterclockwise contour \( C_R \) hugs the branch cut between \( x_{1R} \) and \( x_{2R} \). Expanding \( p_{CR} \) in a Laurent series in the annulus \( x_{2R} < |x| < x_{4R} \) we obtain

\[
p_{CR}(x, E) = i \sqrt{mk} \sqrt[4]{1 + \epsilon \over 2} \ x \left[ 1 - \frac{x_{2R}^2}{2} \left\{ 1 - \frac{1}{8} \left( \frac{x_{2R}}{x_{4R}} \right)^2 - \frac{1}{64} \left( \frac{x_{2R}}{x_{4R}} \right)^4 - \cdots \right\} \frac{1}{x^2} \right] + \text{powers of } x \text{ other than } x^{-1}. \tag{24}
\]

The coefficient of \( x^{-1} \) in this series, required for evaluating \( J_{CR} \), is a series in powers of \( \left( \frac{x_{2R}}{x_{4R}} \right)^2 = \frac{\epsilon}{2+\epsilon} < 1 \), which is \(<1 \) for low energies. Deforming the contour \( C_R \) outward into the circular counterclockwise contour \( \gamma_R \) centered at the origin with its radius less than \( x_{4R} \), and evaluating the integral in Eq. (23), we get

\[
J_{CR} = \frac{\tilde{E}}{\omega_0} \sqrt{1 + \frac{\epsilon}{2}} \left[ 1 - \frac{1}{8} \left( \frac{\epsilon}{2+\epsilon} \right)^2 - \frac{1}{64} \left( \frac{\epsilon}{2+\epsilon} \right)^4 - \cdots \right] \tag{25}
\]

Comparing \( J_{CR}(\tilde{E}) \) for \( \epsilon \ll 1 \) with \( J_{C}(E) \) in Eq. (13), we see that the relativistic action variable has the correct non-relativistic limit. For low energies, up to order \( \epsilon \), we have

\[
J_{CR} \approx \frac{\tilde{E}}{\omega_0}(1 + \frac{\epsilon}{4})(1 - \frac{\epsilon}{16}) \approx \frac{\tilde{E}}{\omega_0} (1 + \frac{3}{16}\epsilon). \]  

Finally, we calculate the relativistic angular frequency from the equation

\[
\frac{1}{\omega_R} = \frac{dJ_{CR}}{d\tilde{E}} = \frac{1}{mc^2} \frac{dJ_{CR}}{d\epsilon} = \frac{1}{\omega_0} \eta(\epsilon), \tag{26}
\]

where the energy dependent relativistic factor \( \eta(\epsilon) \) is

\[
\eta(\epsilon) = \frac{d}{d\epsilon} \left( \epsilon \sqrt{1 + \frac{\epsilon}{2}} \left\{ 1 - \frac{1}{8} \left( \frac{\epsilon}{2+\epsilon} \right)^2 - \frac{1}{64} \left( \frac{\epsilon}{2+\epsilon} \right)^4 - \cdots \right\} \right). \tag{27}
\]
\[ \eta(\epsilon) \text{ is very nearly 1 for low energies. This expression for } \omega_R^{-1}, \text{ valid for all energies, shows} \]

\[ \text{a decrease in frequency from the non-relativistic case due to time dilation, for an observer in} \]

\[ \text{the laboratory reference frame. We also see here the explicit dependence of this frequency on} \]

\[ \text{the oscillator’s energy, and therefore, on its amplitude. This expression is equivalent to the} \]

\[ \text{series representation of the period of this relativistic oscillator obtained by direct integration} \]

\[ \text{and shown in Section IV (See Eq. \((37)\)). Up to order } \epsilon, \text{ it reduces to} \]

\[ \frac{1}{\omega_R} \approx \frac{1}{\omega_0}(1 + \frac{3}{8}\epsilon). \]

**B. \( \oint \) xdp form of action variable**

Starting with Eq. \((19)\) and extending it into the complex \(p\) plane we write the coordinate as

\[ x_{CR} = -\sqrt{\frac{2}{k}} \sqrt{E - \sqrt{p^2c^2 + m^2c^4}}. \]  

(28)

We see that there are two turning momenta \(p_{jR}, j = 1, 2\), given by \(E - \sqrt{p_{jR}^2c^2 + m^2c^4} = 0\), or

\[ -p_{1R} = p_{2R} = \sqrt{2m\bar{E}} \left(1 + \frac{\epsilon}{2}\right). \]  

(29)

These two turning momenta, with a dependence on the relativistic factor \(\sqrt{1 + \frac{\epsilon}{2}}\) invite comparison with their non-relativistic counterparts in Eq. \((14)\). They are also branch points of \(x_{CR}(p, \bar{E})\) and we choose one of its branch cuts from \(p_{1R}\) to \(p_{2R}\). The presence of \(\sqrt{p^2c^2 + m^2c^4}\) in Eq. \((30)\) produces two additional branch points of \(x_{CR}(p, \bar{E})\) of relativistic origin, given by \(-p_{3R} = p_{4R} = imc\). We choose the second set of branch cuts along the imaginary axis connecting each of \(p_{3R}\) and \(p_{4R}\) to \(p = \infty\). Further, we choose the branch of \(x_{CR}(p, \bar{E})\) that is positive just above the branch cut connecting \(p_{1R}\) and \(p_{2R}\).

The coordinate in Eq. \((28)\) can be rewritten, using the turning momenta, in a form similar to the non-relativistic coordinate Eq. \((15)\), and suitable for Laurent expansion, as

\[ x_{CR}(p, \bar{E}) = \sum_{j=-\infty}^{\infty} A_j' p^{3-2j} = \frac{-i}{[mk(1 + \frac{\epsilon}{2})]^\frac{1}{2}} p \sqrt{1 - \left(\frac{p_{2R}}{p}\right)^2} f(p^2, \epsilon), \]  

(30)

where

\[ f(p^2, \epsilon) = \left[ (1 + \epsilon) - \left\{ 1 - \left(\frac{p}{p_{4R}}\right)^2 \right\}^{\frac{1}{2}} \right]^\frac{1}{2} \left[ -\epsilon \left(\frac{p}{p_{2R}}\right)^2 \left\{ 1 - \left(\frac{p}{p_{2R}}\right)^2 \right\}^{\frac{1}{2}} \right]. \]  

(31)
\( f(p^2, \epsilon) \) is analytic for \(|p^2| < |p_{4R}^2|\), is 1 at \( p^2 = 0 \), and thus its Laurent series consists only of the non-negative powers of \( p^2 \). We notice that, in this form, \( x_{CR}(p, \tilde{E}) \), apart from \( p^2 \) and two varieties of binomial series, one in powers of \( \frac{p_{2R}}{p} \) and the other in \( \frac{p}{p_{4R}} \), which converge uniformly in the annulus \( p_{2R} < p < p_{4R} \). Expanding the square roots in Eq. (30) we get

\[
x_{CR} = -\sqrt{-2\tilde{E}} \frac{p}{p_{2R}} \left[ 1 - \frac{1}{2} \left( \frac{p_{2R}}{p} \right)^2 - \frac{1}{8} \left( \frac{p_{2R}}{p} \right)^4 - \frac{1}{16} \left( \frac{p_{2R}}{p} \right)^6 \ldots \right] \left[ 1 + \sum_{j=1}^{\infty} f_j(\epsilon) p^{2j} \right],
\]

where \( f_j(\epsilon) \), which are inversely proportional to \( p_{4R}^{2j} \), are the expansion coefficients in the Laurent series of \( f(p^2, \epsilon) \). The coefficient of \( \frac{1}{p} \) necessary for evaluating the residue is a power series in the parameter \( \left( \frac{p_{2R}}{p_{4R}} \right)^2 = 2 \epsilon \left(1 + \frac{3}{8} \epsilon \right) \), which, for low energies, is of order \( \epsilon \).

The alternate definition of \( J_{CR}(\tilde{E}) \) is

\[
J_{CR} = -\frac{1}{2\pi} \int_{C_R'} x_{CR}(p, \tilde{E}) dp,
\]

where the clockwise contour \( C_R' \) embraces the branch cut connecting \( p_{1R} \) and \( p_{2R} \). We expand this contour outward into the circular clockwise contour \( \gamma_R' \) centered at the origin with a radius less than \( p_{4R} \). Evaluating the integral in Eq. (33) on \( \gamma_R' \) using Eq. (30) we get

\[
J_{CR}(\tilde{E}) = \frac{\tilde{E}}{\omega_0} \sqrt{1 + \frac{\epsilon}{2}} \left[ 1 - \frac{1}{16} \epsilon + \frac{7}{256} \epsilon^2 + \frac{1}{128} \epsilon^3 \ldots \right]
\]

This expression for the relativistic action variable is a different series representation than the one in Eq. (25), and reduces to the non-relativistic \( J_C \) in Eq. (16) for \( \epsilon << 1 \). The angular frequency, \( \omega_R \), is given by

\[
\frac{1}{\omega_R} = \frac{dJ_{CR}}{d\tilde{E}} = \frac{1}{mc^2} \frac{dJ_{CR}}{d\epsilon} = \frac{1}{\omega_0} \frac{d}{d\epsilon} \left[ \epsilon \sqrt{1 + \frac{\epsilon}{2}} \left\{ 1 - \frac{1}{16} \epsilon + \frac{7}{256} \epsilon^2 + \frac{1}{128} \epsilon^3 \ldots \right\} \right]
\]

Truncating this series to the first order in \( \epsilon \) for low energies, we recover the previous result,

\[
\frac{1}{\omega_R} \approx \frac{1}{\omega_0} (1 + \frac{3}{8} \epsilon).
\]

V. PERIOD OF RELATIVISTIC HARMONIC OSCILLATOR - TRADITIONAL TREATMENT

We evaluate here the period of the relativistic oscillator by direct integration. The period \( \tau \) of a relativistic harmonic oscillator, for all energies, can be obtained in a closed form by
integrating $\oint dx$ over one cycle of the motion. For a potential energy function $V(x)$ which has the form of a symmetric well and is even in $x$,

$$\tau = \frac{4}{c} \int_{0}^{x_{2R}} \frac{[E-V(x)]dx}{\sqrt{[E-V(x)]^2 - m^2c^4}}$$

(36)

$x_{2R}$ is the relativistic turning point on the right given by $E - mc^2 - V(x_{2R}) = 0$. For the harmonic oscillator, with $V(x) = \frac{1}{2}kx^2$, the period integrates to

$$\tau = \frac{2\pi}{c} \sqrt{\frac{2}{k}} \left[ \sqrt{E + 2mc^2} \left( 1 - \frac{1}{4}\kappa^2 - \frac{3}{64}\kappa^4 \ldots \right) - \frac{mc^2}{\sqrt{E + 2mc^2}} F_1^2 \left( \frac{1}{2}, \frac{1}{2} | \kappa^2 \right) \right],$$

(37)

with $\kappa = \sqrt{\frac{E}{E + 2mc^2}}$, and the hypergeometric series $F_1^2$ given by

$$F_1^2(a, b | c | z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}z^2 \ldots$$

For the weak relativistic case, where $\tilde{E} << mc^2$, we retain terms up to order $\kappa^2$, and obtain

$$\tau \approx 2\pi \sqrt{\frac{m}{k}} \left[ 2 \left( 1 + \frac{\epsilon}{2} \right) \left( 1 - \frac{1}{4}\kappa^2 \right) - \left( 1 + \frac{\epsilon}{2} \right)^{-\frac{1}{2}} \left( 1 + \frac{1}{4}\kappa^2 \right) \right]$$

(38)

Further, $\kappa^2 \approx \frac{\epsilon}{2}(1 - \frac{\epsilon}{2})$ and

$$\tau \approx 2\pi \sqrt{\frac{m}{k}} \left[ 1 + \frac{3}{8} \epsilon \right] \Rightarrow \omega \approx \omega_0 \left[ 1 - \frac{3}{8} \epsilon \right]$$

(39)

VI. CONCLUSION

We have shown the utility of the contour integral definition of the action variable in determining the frequency of the relativistic harmonic oscillator. The formalism is easily extended to periodic systems in two and three dimensions for separable systems. The non-relativistic frequency emerges naturally for the $\epsilon << 1$ case. A series representation of the frequency of any other relativistic periodic system can be similarly obtained. Further, other Hamiltonian models of periodic systems lend themselves to this analysis. The central problem in this development is the identification of branch points of the temporally varying quantity (e.g., $p(x, E)$ or $x(p, E)$) expressed as a function of its conjugate variable and other constants of motion. There are four such points each in the two versions of the relativistic oscillator considered here, with the expansion parameter for the frequency being the ratio of
a "near" and a "far" branch point. The case of a general periodic system in one dimension is characterized by its frequency depending on several such ratios of magnitude less than 1, with each ratio depending on the system’s energy.

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