Hydrodynamic chains and a classification of their Poisson brackets

Maxim V. Pavlov

Abstract

Necessary and sufficient conditions for an existence of the Poisson brackets significantly simplify in the Liouville coordinates. The corresponding equations can be integrated. Thus, a description of local Hamiltonian structures is a first step in a description of integrable hydrodynamic chains. The concept of $M$ Poisson bracket is introduced. Several new Poisson brackets are presented.

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1 Introduction

The theory of the hydrodynamic type systems
\[ u^i_t = v^i_j(u)u^j_x, \quad i, j = 1, 2, ..., N \]  
integrable by the generalized hodograph method (see [24]) starts from the hydrodynamic type systems equipped with the local Hamiltonian structure
\[ u^i_t = \{u^i, \tilde{h}\} = [g^{ij}\partial_x - g^{is}\Gamma^j_{sk}u^s_x]\frac{\delta\tilde{h}}{\delta u^i}, \]
determined by the Hamiltonian \( \tilde{h} = \int h(u)dx \) and the Dubrovin–Novikov bracket (a differential-geometric Poisson bracket of the first order, see [5])
\[ \{u^i(x), u^j(x')\} = [g^{ij}\partial_x - g^{is}\Gamma^j_{sk}u^s_x]\delta(x - x'), \quad i, j = 1, 2, ..., N, \]  
where the flat metric \( g^{ij}(u) \) is symmetric and non-degenerate, \( \Gamma^j_{sk} = \frac{1}{2}g^{jm}(\partial_x g_{mk} + \partial_k g_{ms} - \partial_m g_{sk}) \) are the Christoffel symbols. Such a local Hamiltonian structure can be written via so-called the Liouville coordinates \( A^k(u) \), so that the corresponding Poisson bracket is
\[ \{A^k(x), A^n(x')\} = [\mathcal{W}^{kn}(A)\partial_x + \partial_x \mathcal{W}^{nk}(A)]\delta(x - x'), \quad k, n = 1, 2, ..., N. \]

This paper deals with the hydrodynamic chains (cf. (1))
\[ U^k_t = \sum_{n=0}^{k+1} V^k_n(U)U^n_x, \quad k = 0, 1, 2, ..., \]  
where all components \( V^k_n(U) \) are functions of the field variables \( U^m \) and the index \( m \) runs from 0 up to \( k + 1 \) (except for possibly several first equations, see the end of the next section). This is a very important class of hydrodynamic chains. Plenty known hydrodynamic chains (see [2], [3], [8], [16], [20], [21], [23]) belong to this class. So, the hydrodynamic chains (4) are an extension of hydrodynamic type systems (1) to an infinite component case. Also, these hydrodynamic chains allow the invertible transformations
\[ \vec{U}^0 = \vec{U}^0(U^0), \quad \vec{U}^1 = \vec{U}^1(U^0, U^1), \quad \vec{U}^2 = \vec{U}^2(U^0, U^1, U^2), \ldots \]

The Poisson brackets (see (2))
\[ \{U^i, U^j\} = [G^{ij}(u)\partial_x + \Gamma^i_{kj}(u)U^k_x]\delta(x - x'), \quad i, j, k = 1, 2, 3, ... \]
for an infinite component case were considered by I. Dorfman in [4] (\( N \) component case was completely investigated by B.A. Dubrovin and S.P. Novikov in [5] for the non-degenerate matrix \( G^{ij} \); its degenerate case was considered by N. Grinberg in [11]). The Jacobi identity yields the set of restrictions (see [4], Theorem 5.14)
\[ G^{ij} = G^{ji}, \quad \partial_k G^{ij} = \Gamma^i_{kj} + \Gamma^j_{ki}, \quad G^{ik}\Gamma^j_k = G^{jk}\Gamma^i_k, \]
\[ 0 = \Gamma^{nm}_n\Gamma^m_k - \Gamma^{im}_n\Gamma^m_k + G^{in}(\partial_n \Gamma^m_k - \partial_k \Gamma^m_n), \]
\[ 0 = (\partial_n \Gamma^i_k - \partial_k \Gamma^i_n)\Gamma^m_p + (\partial_n \Gamma^m_i - \partial_k \Gamma^m_i)\Gamma^j_p + (\partial_n \Gamma^j_m - \partial_k \Gamma^j_n)\Gamma^i_p \]
\[ + (\partial_n \Gamma^i_j - \partial_p \Gamma^i_n)\Gamma^m_k + (\partial_n \Gamma^m_i - \partial_p \Gamma^m_n)\Gamma^j_k + (\partial_n \Gamma^j_m - \partial_p \Gamma^j_n)\Gamma^i_k, \]
which in the Liouville coordinates $A^i = A^i(U)$ (see (3) and (5); also [5], [15]) simplify to

$$(W^{ik} + W^{kj}) \partial_k W^{mj} = (W^{ij} + W^{kj}) \partial_k W^{mi},$$

$$\partial_n W^{ij} \partial_m W^{kn} = \partial_n W^{kj} \partial_m W^{in}.$$ (7)

In comparison with a finite-component case (hydrodynamic type systems (1)), where elements $W^{ik}$ depend on all field variables $u^i$ $(n = 1, 2, ..., N)$, the elements $W^{ik}$ in an infinite component case (hydrodynamic chains) depend on a finite number of moments $A^k$ only. Thus, the Liouville coordinates $A^k$ in the infinite component case is a natural generalization of the flat coordinates in the finite component case (see [5]).

The main problem in a classification of integrable hydrodynamic chains is the description of the Poisson brackets (cf. (3))

$$\{A^k(x), A^n(x')\} = [W^{kn}(A) \partial_x + \partial_x W^{nk}(A)] \delta(x - x'), \quad k, n = 1, 2, ...$$

**Definition:** The hydrodynamic chain (4) is said to be Hamiltonian, if it can be written in the form

$$A^i = [W^{ij}(A) \partial_x + \partial_x W^{ij}(A)] \frac{\delta \bar{H}}{\delta A^j}, \quad i, j = 0, 1, 2, ...$$

where the coefficients $W^{ij}(A)$ satisfy the Jacobi identity (7).

The Hamiltonian $\bar{H} = \int \bar{H} dx$ depends on several first moments $A^k$.

**Remark:** The above local Hamiltonian structures (8) and corresponding Poisson brackets still are not investigated properly. However, a lot of publications are devoted to their particular cases (see, for instance, [4], [12], [13]), some of them will be described below. In this paper I present the program of an investigation of these Poisson brackets.

In this paper we consider the very important case

$$W^{kn} = W^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}).$$

Corresponding Poisson brackets we call $M$–brackets. The simplest sub-cases $W^{kn} = W^{kn}_{(0)}(B^0, B^1, ..., B^{k+n})$ and $W^{kn} = W^{kn}_{(1)}(A^0, A^1, ..., A^{k+n-1})$ are considered in details below. The first example of such Poisson bracket

$$\{A^k, A^n\} = (kA^{k+n-1} \partial_x + n \partial_x A^{k+n-1}) \delta(x - x')$$

was found in [13], the second example is (see [12])

$$\{B^k, B^n\} = [(k + \beta)B^{k+n} \partial_x + (n + \beta) \partial_x B^{k+n}] \delta(x - x').$$

Infinitely many local Hamiltonian structures for the Benney hydrodynamic chain (see [22]) and for the Kupershmidt hydrodynamic chains (see [21]) are examples of these $M$–brackets. The third example

$$\{A^k, A^n\}_2 = [(k + 1)A^{k+n} \partial_x + (n + 1) \partial_x A^{k+n} + n(k + 1)A^{k-1} \partial_x A^{n-1}$$

$$+ \sum_{m=0}^{n-1} (mA^{n-1-m} \partial_x A^{k-1+m} + (k - n + m)A^{k-1+m} \partial_x A^{n-1-m})] \delta(x - x')$$

$$+ \sum_{m=0}^{n-1} (mA^{n-1-m} \partial_x A^{k-1+m} + (k - n + m)A^{k-1+m} \partial_x A^{n-1-m})] \delta(x - x')$$
The main observation successfully utilized in this paper is that the Jacobi identity (7)

\[ \sum_{k=0}^{n+j-M} (W_{(M)}^{nk} + W_{(M)}^{ki}) \partial_k W_{(M)}^{nj} = \sum_{k=0}^{n+i-M} (W_{(M)}^{jk} + W_{(M)}^{kj}) \partial_k W_{(M)}^{ni}, \]  

(11)

is a system of nonlinear PDE’s which can be solved completely, because all coefficients \( W_{(M)}^{ij} \) depend on different number of the moments \( A^k \) (cf. a similar problem in [20]). Corresponding illustrative examples are given in this paper for \( M = 0 \) and \( M = 1 \).

Remark: Of course, the \( M \)-bracket is just one among many other examples. For instance, another new Poisson bracket is

\[ \{ A^k, A^n \} = (A^{k-n} \partial_x + \partial_x A^{k-n}) \delta(x - x'). \]

All other Poisson brackets (8) and corresponding integrable hydrodynamic chains will be discussed elsewhere.

At the end of this paper local Poisson brackets are generalized on the simplest nonlocal case (in \( N \) component non-degenerate case known as the nonlocal Hamiltonian structure associated with a metric of constant curvature [9]). As example, a one parametric family of such nonlocal Poisson brackets is presented.

2 The general case

The Poisson brackets determining the hydrodynamic chains (4) written in the Liouville coordinates

\[ \{ A^k, A^n \} = \left[ W_{(M)}^{kn}(A^0, A^1, ..., A^{k+n-M}) \partial_x + \partial_x W_{(M)}^{mk}(A^0, A^1, ..., A^{k+n-M}) \right] \delta(x - x') \]  

(12)

satisfy the Jacobi identity (11).

The corresponding hydrodynamic chain (4) is

\[ A_t^k = \left\{ A^k, H_{M+1} \right\}_M = \left[ W_{(M)}^{kn}(A^0, A^1, ..., A^{k+n-M}) \partial_x + \partial_x W_{(M)}^{mk}(A^0, A^1, ..., A^{k+n-M}) \right] \frac{\delta H_{M+1}}{\delta A^n}, \]

where the Hamiltonian is \( H_{M+1} = \int H_{M+1}(A^0, A^1, A^2, ..., A^{M+1})dx \). The Hamiltonian structure is determined by the Hamiltonian, the momentum and the Casimirs. If a hydrodynamic chain has one more conservation law density, then this hydrodynamic chain

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is integrable, because every extra conservation law density can be used as an extra Hamiltonian density determining commuting flow (see \cite{12}). Thus, we have several interesting sub-cases:

1. $M = 1$. The famous example of such a Poisson bracket is the Kupershmidt–Manin bracket (9). This Poisson bracket has the momentum $A^1$ and the Casimir (annihilator) $A^0$.

If $M \geq 1$, then corresponding Poisson brackets have $M$ Casimirs. Since the Hamiltonian is $\bar{H}_{M+1} = \int H_{M+1}(A^0, A^1, A^2, ..., A^{M+1})dx$, then the momentum can be chosen as $\bar{H}_M = \int A^M dx$. The Casimirs can be chosen as $\bar{H}_k = \int A^k dx$ ($k = 0, 1, 2, ..., M - 1$). Then the auxiliary (natural) restrictions (“normalization”) are

\[ A^k = \mathcal{W}^M_k(A^0, A^1, ..., A^k), \quad k = 0, 1, 2, ... \]

\[ 0 = \mathcal{W}^k(A^0, A^1, ..., A^{k-s-M}), \quad 0 \leq s < M, \quad k \geq M - s. \]

\[ \mathcal{W}^k_n = \mathcal{W}^k_n = \text{const}, \quad k = 0, 1, 2, ..., M - 1, \quad 0 \leq n \leq M - 1 - k. \] (13)

Thus, such a hydrodynamic chain has at least $M + 2$ conservation laws (for an arbitrary Hamiltonian density $H_{M+1}$), where the first $N$ conservation laws of the Casimirs are

\[ A^k_t = \partial_x \left( \sum_{n=0}^{M-k-1} (\mathcal{W}^k_n + \mathcal{W}^n_k) \frac{\partial H_{M+1}}{\partial A^n} + \sum_{n=M-k}^{M+1} \mathcal{W}^n_m \frac{\partial H_{M+1}}{\partial A^n} \right), \quad k = 0, 1, 2, ..., M-1. \]

The conservation law of the momentum is

\[ A^M_t = \partial_x \left( \sum_{n=0}^{M+1} (\mathcal{W}^n(M) + A^n) \frac{\partial H_{M+1}}{\partial A^n} - H_{M+1} \right). \]

The conservation law of the energy is

\[ \partial_t H_{M+1} = \partial_x \left[ \sum_{k=0}^{M+1} \sum_{n=0}^{M+1} \mathcal{W}_{kn}(M) \frac{\partial H_{M+1}}{\partial A^k} \frac{\partial H_{M+1}}{\partial A^n} \right]. \]

2. $M = 0$. The important example of such a Poisson bracket is the Kupershmidt brackets (10). These Poisson brackets have the momentum $A^0$ only.

3. $M = -1$. These brackets do not have a momentum and annihilators.

4. If $M < -1$, then an investigation of the Poisson brackets and corresponding hydrodynamic chains should start from commuting flows determined by a local Hamiltonian density. For instance, if $M = -2$, then the first nontrivial Hamiltonian hydrodynamic chain (cf. (4))

\[ A^k_t = \sum_{n=0}^{k+2} V^k_n(A)A^n_x, \quad k = 0, 1, 2, ... \]

is determined by the first local Hamiltonian $\bar{H}_0 = \int H_0(A^0)dx$. 

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**Examples:** The Kupershmidt hydrodynamic chains (see \[12\]) have an infinite set of local Hamiltonian structures determined by \( M \)-brackets \((12), \) where the first of them are (for the indexes \( M = 0, 1, 2 \), respectively; see \[21\])

\[
\{C^k, C^n\} = [(\beta k + 1)C^{k+n}\partial_x + (\beta n + 1)\partial_x C^{k+n}]\delta(x - x'),
\]

\[
\{B^0, B^0\} = \beta\delta'(x - x'), \quad \{B^k, B^n\} = [(\beta k + 1 - \beta)B^{k+n-1}\partial_x + (\beta n + 1 - \beta)\partial_x B^{k+n-1}]\delta(x - x'),
\]

\[
\{A^0, A^1\} = \{A^1, A^0\} = \beta\delta'(x - x'), \quad \{A^1, A^1\} = (\beta - 1)(A^0\partial_x + \partial_x A^0)\delta(x - x'),
\]

\[
\{A^k, A^n\} = [(\beta k + 1 - 2\beta)A^{k+n-2}\partial_x + (\beta n + 1 - 2\beta)\partial_x A^{k+n-2}]\delta(x - x').
\]

If the first Poisson bracket is the well-known Kupershmidt bracket \((10), M = 0\), two others \((M = 1 \text{ and } M = 2)\) are new.

The most interesting class of these Poisson brackets is provided by the brackets, whose coefficients \( \mathcal{W}^{kn}_{(M)} \) are polynomials with respect to the moments \( A^k \). The simplest case is given by the linear Poisson brackets determined by

\[
\mathcal{W}^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}) = e^{kn}_{(M)}A^{k+n-M},
\]

where \( e^{kn}_{(M)} \) are some constants satisfying the algebraic system (see \(7\))

\[
(e^{p,k+n-M}_{(M)} + e^{k+n-M,p}_{(M)})e^{nk}_{(M)} = (e^{k,n+p-M}_{(M)} + e^{n+p-M,k}_{(M)})e^{np}_{(M)}, \quad e^{sp}_{(M)}e^{k,s+p-M}_{(M)} = e^{kp}_{(M)}e^{s,k+p-M}_{(M)};
\]

possibly connected with infinite dimensional analogue of the Frobenius algebras (cf. \[1\]).

Let us look for particular solution in the form

\[
e^{kn}_{(M)} = (Ak + Bn + C),
\]

where \( A \) and \( B \) are some constants. The substitution of this ansatz in the above algebraic system yields the Kupershmidt brackets (see \[12\]) determined by an arbitrary value \( A \), but \( B = 0 \). More general linear Poisson brackets

\[
\mathcal{W}^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}) = \sum_{s=0}^{k+n-M} e^{kn}_{(M)s}A^s,
\]

were considered by I. Dorfman in \[4\]. For instance,

\[
\{A^0, A^0\} = \varepsilon\delta'(x - x'), \quad \{A^k, A^n\} = \sum_{m=0}^{M} \gamma_m[kA^{m+k+n-1}\partial_x + n\partial_x A^{m+k+n-1}]\delta(x - x'),
\]

where \( \varepsilon \) and \( \gamma_k \) are arbitrary constants.
Higher order homogeneous polynomials create more complicated Poisson brackets, which can be described by algebraic tools. For instance, the second nontrivial case is the quadratic Poisson brackets determined by

\[ \mathcal{W}_{(M)}^{kn}(A^0, A^1, ..., A^{k+n-M}) = \frac{1}{2} \sum_{m=0}^{k+n-M} e_{(M)m}^{kn} A^m A^{k+n-m-M}, \]

where \( e_{(M)m}^{kn} \equiv e_{(M)k+n-m-M}^{kn} \) are some constants, which can be found by direct substitution in the Jacobi identity (see (11)). A corresponding system of algebraic relations is very complicated and could be investigated in details elsewhere.

**Remark:** For every Poisson bracket with respect to highest moment \( A^{M+1} \) the Hamiltonian density \( H_{M+1} \) can be linear \( H_{M+1} = A^{M+1} + F(A^0, ..., A^M) \), quasilinear \( H_{M+1} = G(A^0, ..., A^M) A^{M+1} + F(A^0, ..., A^M) \) and fully nonlinear. If \( M > 0 \) just in the linear case \( H_{M+1} = A^{M+1} + F(A^0, A^1) \) the hydrodynamic chain has the form (4). In general case the corresponding hydrodynamic chain is

\[ A^k_t = \sum_{n=0}^{M+1} V^n_A A^n_x, \quad k = 0, 1, ..., M - 1, \quad A^k_t = \sum_{n=0}^{k+1} V^n_A A^n_x, \quad k = M, M + 1, ... \] (14)

### 3 \( M = 0 \)

We omit the sub-index in all components \( W_{(0)}^{mn} \) of the Poisson bracket (see (12), \( M = 0 \))

\[ \{B^k, B^n\} = [\mathcal{W}^{kn}(B^0, B^1, ..., B^{k+n}) \partial_x + \partial_x \mathcal{W}^{mn}(B^0, B^1, ..., B^{k+n})] \delta(x - x'). \]

The Jacobi identity (11)

\[ \sum_{m=0}^{n+k} (\mathcal{W}^{pm} + \mathcal{W}^{mp}) \partial_m \mathcal{W}^{nk} = \sum_{m=0}^{n+p} (\mathcal{W}^{km} + \mathcal{W}^{mk}) \partial_m \mathcal{W}^{np}, \]

\[ \sum_{m=0}^{s+p} \partial_m \mathcal{W}^{sp} \partial_n \mathcal{W}^{km} = \sum_{m=0}^{p+k} \partial_m \mathcal{W}^{kp} \partial_n \mathcal{W}^{sm}. \]

is a nonlinear PDE system describing a family of the local Poisson brackets connected with the Hamiltonian hydrodynamic chains.

The existence of commuting hydrodynamic chains

\[ B^k_t = [\mathcal{W}^{kn}(B^0, B^1, ..., B^{k+n}) \partial_x + \partial_x \mathcal{W}^{mn}(B^0, B^1, ..., B^{k+n})] \frac{\delta \bar{H}_1}{\delta B^n}. \] (15)

\[ B^k_y = [\mathcal{W}^{kn}(B^0, B^1, ..., B^{k+n}) \partial_x + \partial_x \mathcal{W}^{mn}(B^0, B^1, ..., B^{k+n})] \frac{\delta \bar{H}_2}{\delta B^n}. \] (16)

where the Hamiltonians are \( \bar{H}_1 = \int H_1(B^0, B^1) dx \) and \( \bar{H}_2 = \int H_2(B^0, B^1, B^2) dx \), implies the existence of a hierarchy of integrable hydrodynamic chains (see (4)).
The auxiliary restrictions ("normalization")

\[ W^{0k} \equiv B^k \]

can be obtained by virtue the existence of a conservation law of the momentum \( \bar{H}_0 = \int B^0 dx \). Thus, first two conservation laws are

\[ \begin{align*}
B_0^0 &= \partial_x \left( 2B^0 \left( \frac{\partial H_1}{\partial B^0} \right) + (W^{10} + B^1) \left( \frac{\partial H_1}{\partial B^1} \right) - H_1 \right), \\
\partial_t H_1 &= \partial_x \left[ B^0 \left( \frac{\partial H_1}{\partial B^0} \right)^2 + (W^{10} + B^1) \left( \frac{\partial H_1}{\partial B^0} \right) \left( \frac{\partial H_1}{\partial B^1} \right) + W^{11} \left( \frac{\partial H_1}{\partial B^0} \right)^2 \right].
\end{align*} \]

Let us write several first nonlinear PDE’s from the Jacobi identity (11)

\[ (W^{10} + B^1) \partial_0 W^{10} + 2W^{11} \partial_1 W^{10} = 2B^0 \partial_0 W^{11} + (B^1 + W^{10}) \partial_1 W^{11} + (B^2 + W^{20}) \partial_2 W^{11}, \]

\[ \partial_1 W^{10} \partial_0 W^{21} = \partial_1 W^{20} \partial_0 W^{11} + \partial_2 W^{20} \partial_0 W^{12}, \]

\[ \partial_0 W^{10} \partial_1 W^{20} + \partial_1 W^{10} \partial_1 W^{21} = \partial_0 W^{20} \partial_1 W^{10} + \partial_1 W^{20} \partial_1 W^{11} + \partial_2 W^{20} \partial_1 W^{12}, \]

\[ \partial_0 W^{10} \partial_2 W^{20} + \partial_1 W^{10} \partial_2 W^{21} = \partial_1 W^{20} \partial_2 W^{11} + \partial_2 W^{20} \partial_2 W^{12}, \]

\[ \partial_1 W^{10} \partial_3 W^{21} = \partial_2 W^{20} \partial_3 W^{12}. \]

This system involves just four moments \( B^0, B^1, B^2, B^3 \). It is enough to find coefficients \( W^{10}, W^{11}, W^{20} \). For instance, \( W^{10} \) is a solution of the Monge–Ampere equation

\[ (W^{10})_{00} (W^{10})_{11} - [(W^{10})_{01}]^2 + \varphi^2(B^0) [(W^{10})_{11}]^2 = 0, \]

where \( \varphi(B^0) \) is a some function (determined from the compatibility conditions \( [(W^{11})_{11}]_0 = [(W^{11})_{10}]_0 = [(W^{11})_{11}]_0 = [(W^{11})_{10}]_0 \), see expressions below).

\[ W^{20} = B^2 (W^{10})_1 + G, \]

where \( G(p, q) \) is a solution of the Euler–Darboux–Poisson equation

\[ G_{pq} = \frac{\varphi''(B^0)}{4\varphi^2(B^0)} (G_p - G_q), \quad \varphi(B^0) = \frac{1}{2} (p - q), \quad q = \ln[(W^{10})_1] - \varphi(B^0), \quad p = \ln [(W^{10})_1] + \varphi(B^0). \]

Explicit expressions for all other coefficients (depending on higher moments \( B^k \), \( k = 3, 4, 5, ... \)) can be found recursively in complete differentials. For instance, the equation

\[ (W^{11})_2 = \frac{2[(W^{10})_1]^2(B^0(W^{10})_{00} + (W^{10} + B^1)(W^{10})_{10} + W^{11}(W^{10})_{11})}{MB^2 + N}. \]
where

\[ M = [1 + (\mathcal{W}^{10})_1(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11} + 2B^0[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{111} - (\mathcal{W}^{10})_{10}(\mathcal{W}^{10})_{11}]
\]
\[ + (\mathcal{W}^{10} + B^1)[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{111} - ((\mathcal{W}^{10})_{11})^2], \]
\[ N = G(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11} + 2B^0[G_{10}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{10}]
\]
\[ + [(\mathcal{W}^{10})_1 + B^1][G_{11}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{11}], \]

can be solved up to a some function of \( B^0 \) and \( B^1 \), which can be found in complete differentials by a substitution into other derivatives

\[ (\mathcal{W}^{11})_1 = (\mathcal{W}^{11})_2 \frac{[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{111} - ((\mathcal{W}^{10})_{11})^2]B^2 + G_{11}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{11}}{2(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11}} + K, \]
\[ (\mathcal{W}^{11})_0 = (\mathcal{W}^{11})_2 \frac{[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{110} - (\mathcal{W}^{10})_{10}(\mathcal{W}^{10})_{11}]B^2 + G_{10}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{10}}{2(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11}} - L, \]

where

\[ K = \frac{(\mathcal{W}^{10})_0(\mathcal{W}^{10})_1}{(\mathcal{W}^{10})_{11}} - 2(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{10}, \]
\[ L = \frac{(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{10}}{(\mathcal{W}^{10})_{11}}. \]

Let us write the first two equations from the (15) and the first equation from (16)

\[ B^0_t = (B^0_0\partial_x + \partial_x B^0) \frac{\delta \mathcal{H}_1}{\delta B^0} + [B^1 \partial_x + \partial_x \mathcal{W}^{10}(B^0, B^1)] \frac{\delta \mathcal{H}_1}{\delta B^1}, \]
\[ B^1_t = [\mathcal{W}^{10}(B^0, B^1) \partial_x + \partial_x B^1] \frac{\delta \mathcal{H}_1}{\delta B^0} + [\mathcal{W}^{11}(B^0, B^1, B^2) \partial_x + \partial_x \mathcal{W}^{11}(B^0, B^1, B^2)] \frac{\delta \mathcal{H}_1}{\delta B^1}, \]
\[ B^0_y = (B^0 \partial_x + \partial_x B^0) \frac{\delta \mathcal{H}_2}{\delta B^0} + [B^1 \partial_x + \partial_x \mathcal{W}^{10}(B^0, B^1)] \frac{\delta \mathcal{H}_2}{\delta B^1} + [B^2 \partial_x + \partial_x \mathcal{W}^{10}(B^0, B^1, B^2)] \frac{\delta \mathcal{H}_2}{\delta B^2}. \]

These three equations can be written in the conservative form

\[ \partial_t H_0 = \partial_x F_{00}(H_0, H_1), \quad \partial_t H_1 = \partial_x F_{01}(H_0, H_1, H_2), \quad \partial_y H_0 = \partial_x F_{10}(H_0, H_1, H_2). \]

Let us introduce the potential function \( z \), where \( H_0 = z_x, \ F_{00}(H_0, H_1) = z_t, \ F_{10}(H_0, H_1, H_2) = z_y \). Then one can substitute \( H_1(z_x, z_t) \) and \( H_2(z_x, z_t, z_y) \) into the second equation

\[ \partial_t H_1(z_x, z_t) = \partial_x F_{01}(H_0, H_1(z_x, z_t), H_2(z_x, z_t, z_y)). \]

This 2+1 quasilinear equation of the second order is integrable by the method of hydrodynamic reductions [7] (or by introducing pseudopotentials; see again [7] and [25]). The Hamiltonian structure of this equation is unknown, while the Hamiltonian structure of corresponding hydrodynamic chains (15) and (16) is defined.
4 $M = 1$

We omit the sub-index in all components $\mathcal{W}^{nk}_{(1)}$ of the Poisson bracket (see (12), $M = 1$)

$$\{A^k, A^n\} = [\mathcal{W}^{kn}(A^0, A^1, ..., A^{k-1})w + \partial_x \mathcal{W}^{mk}(A^0, A^1, ..., A^{k-1})] \delta(x-x').$$

The Jacobi identity (11) is

$$\sum_{m=0}^{n+k-1} (\mathcal{W}^{pm} + \mathcal{W}^{mp}) \partial_m \mathcal{W}^{nk} = \sum_{m=0}^{n+p-1} (\mathcal{W}^{km} + \mathcal{W}^{mk}) \partial_m \mathcal{W}^{np},$$

$$\sum_{m=0}^{s+p-1} \partial_m \mathcal{W}^{sp} \partial_n \mathcal{W}^{km} = \sum_{m=0}^{p+k-1} \partial_m \mathcal{W}^{kp} \partial_n \mathcal{W}^{sm}.$$

Thus, the above system of nonlinear PDE’s describes a family of the local Poisson brackets connected with the Hamiltonian hydrodynamic chains.

The existence of commuting hydrodynamic chains

$$A^k_t = [\mathcal{W}^{kn}(A^0, A^1, ..., A^{k-1})w + \partial_x \mathcal{W}^{mk}(A^0, A^1, ..., A^{k-1})] \frac{\delta \bar{H}_2}{\delta A^n},$$

$$A^k_y = [\mathcal{W}^{kn}(A^0, A^1, ..., A^{k-1})w + \partial_x \mathcal{W}^{mk}(A^0, A^1, ..., A^{k-1})] \frac{\delta \bar{H}_3}{\delta A^n},$$

where the Hamiltonians are $\bar{H}_2 = \int H_2(A^0, A^1, A^2) dx$ and $\bar{H}_3 = \int H_3(A^0, A^1, A^2, A^3) dx,$ implies the existence of a hierarchy of integrable hydrodynamic chains.

The auxiliary restrictions (“normalization”)

$$\mathcal{W}^{0k}(A^0, A^1, ..., A^{k-1}) \equiv \bar{W}^{00} \delta^{0k}, \quad \mathcal{W}^{1k}(A^0, A^1, ..., A^k) \equiv A^k,$$

where $\bar{W}^{00} = \text{const}$ (and $\delta^{ik}$ is the Kronecker symbol), can be obtained by virtue of an existence of conservation laws of the Casimir $\bar{H}_0 = \int A^0 dx$ and the momentum $\bar{H}_1 = \int A^1 dx.$ Then the first three conservation laws are

$$A_0^t = \partial_x \left( 2\mathcal{W}^{00} \frac{\partial \mathcal{H}_2}{\partial A^0} + A^0 \frac{\partial \mathcal{H}_2}{\partial A^1} + \mathcal{W}^{20} \frac{\partial \mathcal{H}_2}{\partial A^2} \right),$$

$$A_1^t = \partial_x \left( A^0 \frac{\partial \mathcal{H}_2}{\partial A^0} + 2A^1 \frac{\partial \mathcal{H}_2}{\partial A^1} + (\mathcal{W}^{21} + A^2) \frac{\partial \mathcal{H}_2}{\partial A^2} - \mathcal{H}_2 \right),$$

$$\partial_t \mathcal{H}_2 = \partial_x [\varepsilon \left( \frac{\partial \mathcal{H}_2}{\partial A^0} \right)^2 + A^0 \frac{\partial \mathcal{H}_2}{\partial A^0} \frac{\partial \mathcal{H}_2}{\partial A^1} + \mathcal{W}^{20} \frac{\partial \mathcal{H}_2}{\partial A^0} \frac{\partial \mathcal{H}_2}{\partial A^2} + \mathcal{W}^{21} \frac{\partial \mathcal{H}_2}{\partial A^1} \frac{\partial \mathcal{H}_2}{\partial A^2} + \mathcal{W}^{22} \left( \frac{\partial \mathcal{H}_2}{\partial A^2} \right)^2],$$

where $\varepsilon \equiv \bar{W}^{00}.$
Let us write several first nonlinear PDE’s from the Jacobi identity (11)

\[ [2\varepsilon \partial_0 + A^0 \partial_1 + W^{20} \partial_2] W^{21} = [A^0 \partial_0 + 2A^1 \partial_1] W^{20} \]

\[ \partial_1 W^{20} \partial_0 W^{31} = \partial_1 W^{30} \partial_0 W^{21} + \partial_2 W^{30} \partial_0 W^{22}, \]

\[ \partial_0 W^{30} \partial_1 W^{20} + \partial_1 W^{30} \partial_1 W^{31} + \partial_2 W^{30} \partial_1 W^{32} = \partial_0 W^{20} \partial_1 W^{30} + \partial_1 W^{20} \partial_1 W^{31}, \]

\[ \partial_0 W^{20} \partial_2 W^{30} + \partial_1 W^{20} \partial_2 W^{31} = \partial_1 W^{30} \partial_2 W^{21} + \partial_2 W^{30} \partial_2 W^{22}, \]

\[ \partial_1 W^{20} \partial_2 W^{31} = \partial_2 W^{30} \partial_3 W^{22}. \]

This system involves just four moments $A^0, A^1, A^2, A^3$. It is enough to find coefficients $W^{20}, W^{21}, W^{30}$. For instance, $W^{20}$ is a solution of the Monge–Ampere equation

\[ (W^{20})_{00}(W^{20})_{11} - [(W^{20})_{01}]^2 + \varphi^2(A^0)(W^{20})_{11} = 0, \quad (18) \]

where $\varphi(A^0)$ is some function (determined from the compatibility conditions $[(W^{21})_{11}]_1 = [(W^{21})_{11}]_0, \quad [(W^{21})_{10}]_1 = [(W^{21})_{10}]_0, \quad [(W^{21})_{02}]_1 = [(W^{21})_{02}]_0$, see these expressions below).

\[ W^{30} = A^2(W^{20})_1 + G, \]

where $G(p, q)$ is a solution of the Euler–Darboux–Poisson equation

\[ G_{pq} = \frac{\varphi''(A^0)}{4\varphi^2(A^0)}(G_p - G_q), \quad \varphi(A^0) = \frac{1}{2}(p - q), \]

\[ q = \ln[(W^{20})_1] - \varphi(A^0), \quad p = \ln[(W^{20})_1] + \varphi(A^0). \]

Explicit expressions for all other coefficients (depending on higher moments $A^k, k = 3, 4, 5, ...$) can be found recursively in complete differentials. For instance,

\[ dW^{21} = (W^{21})_0 dA^0 + (W^{21})_1 dA^1 + (W^{21})_2 dA^2, \]

where

\[ (W^{21})_2 = \frac{2[A^1(W^{20})_{11} + A^0(W^{20})_{10}] + \varepsilon(W^{20})_{00}}{2\varepsilon \left( \frac{(W^{20})_{11}A^2 + G_1}{(W^{20})_1} \right)_0 + A^0 \left( \frac{(W^{20})_{11}A^2 + G_1}{(W^{20})_1} \right)_1 + \frac{(W^{20})(W^{20})_{11}}{(W^{20})_1}}, \]

\[ (W^{21})_1 = \frac{(W^{20})_1}{(W^{20})_{11}} \left( \frac{(W^{20})_{11}A^2 + G_1}{(W^{20})_1} \right)_1(W^{21})_2 + (W^{20})_0 - 2 \frac{(W^{20})_1(W^{20})_{10}}{(W^{20})_{11}}, \]

\[ (W^{21})_0 = \frac{(W^{20})_1}{(W^{20})_{11}} \left( \frac{(W^{20})_{11}A^2 + G_1}{(W^{20})_1} \right)_0(W^{21})_2 - \frac{(W^{20})_1(W^{20})_{00}}{(W^{20})_{11}}. \]

**Remark:** The coincidence of the coefficients $W^{20}$ from this section and $W^{10}$ from the previous section is easy to understand if one takes into account that if the Hamiltonian
density $H_2$ is a function of the moments $A^1$ and $A^2$ only, then the corresponding hydrodynamic chain has the Hamiltonian structure coinciding (by the “shift” $A^k \to B^{k-1}$, $k = 1, 2, ...$) with the Hamiltonian structure presented in the previous section.

**Remark:** The Monge–Ampere equation (18) was derived in [8]. In this paper solutions of this equation classify the integrable hydrodynamic chains

$$\partial_t H_0 = \partial_x F_1(H_0, H_1), \quad \partial_t H_1 = \partial_x F_2(H_0, H_1, H_2), \quad \partial_t H_2 = \partial_x F_3(H_0, H_1, H_2, H_3), ...$$

These hydrodynamic chains can coincide with the integrable hydrodynamic chains (17) determined by the Hamiltonian density $H_2 = A^2 + Q(A^0, A^1)$.

All other Hamiltonian hydrodynamic chains (8) can be investigated in the same way using the Jacobi identity (7).

## 5 The Miura type and reciprocal transformations

Suppose two Hamiltonian hydrodynamic chains

$$B^k_t = \sum_{n=0}^{N_k} F^k_n(B) B^n_x, \quad A^k_t = \sum_{m=0}^{M_k} G^k_n(A) A^n_x, \quad k = 1, 2, ...,$$

where $N_k$ and $M_k$ are some integers, are related by the Miura type transformations

$$A^k = A^k(B^0, B^1, ..., B^{k+1}), \quad k = 0, 1, 2, ...$$

**Theorem:** The corresponding Poisson brackets (see (6) and (12))

$$\{B^k, B^n\} = [G^{kn}_{(L)}(B^0, B^1, ..., B^{k+n-L}) \partial_x + \Gamma^{kn}_{(L)m}(B) B^m_x] \delta(x - x'),$$

$$\{A^k, A^n\} = [G^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}) \partial_x + \Gamma^{kn}_{(M)m}(A) A^m_x] \delta(x - x')$$

are related by the above Miura type transformations if $L = M + 1$.

**Proof:** It is easy to check by taking into account the highest order dependence on the moments $B^0$ only.

$$\{A^k, A^n\} = \sum_{i=0}^{k+1-n} \sum_{j=0}^{k+1-n} \frac{\partial A^k}{\partial B^i} \{B^i, B^j\} \frac{\partial A^n}{\partial B^j} \delta(x - x') \sim \frac{\partial A^k}{\partial B^{k+1}} \{B^{k+1}, B^{n+1}\} \frac{\partial A^n}{\partial B^{n+1}} \delta(x - x').$$

Thus, $G^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}) \sim G^{k+1,n+1}_{(L)}(B^0, B^1, ..., B^{k+n+2-L})$. So, indeed, $L = M + 1$.

**Remark:** Suppose that both coordinate systems are the Liouville coordinates. Then, one obtains the linear system describing such Miura type transformations

$$\partial_m \left[ \frac{\partial A^k}{\partial B^i} (W^{ij}_{(L)} + W^{ij}_{(L)}) \frac{\partial^2 A^n}{\partial B^j \partial B^s} + W^{ij}_{(L)} \frac{\partial A^n}{\partial B^j} \right] = \partial_s \left[ \frac{\partial A^k}{\partial B^i} (W^{ij}_{(L)} + W^{ij}_{(L)}) \frac{\partial^2 A^n}{\partial B^j \partial B^m} + W^{ij}_{(L)} \frac{\partial A^n}{\partial B^j} \right].$$

If the first Hamiltonian structure written in the Liouville coordinates (19) and the Miura type transformations (12) are given, then the second Hamiltonian structure (20) can be reconstructed in complete differentials, where

$$\frac{\partial W^{nk}_{(M)}(A)}{\partial B^s} = \frac{\partial A^k}{\partial B^i} (W^{ij}_{(L)} + W^{ij}_{(L)}) \frac{\partial^2 A^n}{\partial B^j \partial B^s} + W^{ij}_{(L)} \frac{\partial A^n}{\partial B^j}. $$
**Example:** The Benney hydrodynamic chain (see (9), [2] and [13])

\[ A^k_t = A^{k+1}_x + kA^{k-1}A_x^0 = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}]\frac{\partial H_2}{\partial A^n}, \quad k = 0, 1, 2, \ldots, \]

where the Hamiltonian density \( H_2 = A^2 + (A^0)^2 \), is connected with the modified Benney hydrodynamic chain (see [23], the particular case of the Kupershmidt hydrodynamic chains [12])

\[ B^k_t = B^{k+1}_x + B^0B_x^0 + (k + 2)B^kB_x^0 = [(k + 1)B^{k+n}\partial_x + (n + 1)\partial_x B^{k+n}]\frac{\partial H_0}{\partial B^n}, \]

where the Hamiltonian density \( H_0 = B^1 + (B^0)^2 \), by the Miura type transformations (see [22])

\[ A^0 = B^1 + (B^0)^2, \quad A^1 = B^2 + 3B^0B^1 + 2(B^0)^3, \ldots \]

In this case \( L = 0 \) and \( M = -1 \). Thus, the Benney moment chain has the second local Hamiltonian structure determined by the Poisson bracket (20), whose coefficients can be found by using the above Miura type transformations.

**Remark:** In the theory of dispersive integrable systems the Miura transformation is a tool to reduce the Hamiltonian structure to canonical form \( \frac{d}{dx} \) (the infinitely many component analogue of the Darboux theorem, see [19]). In the theory of the Poisson brackets associated with hydrodynamic chains an application of the Miura type transformation reduces the dependence on a number of the moments \( A^k \). It means that the following diagram exists (see (6) and (12))

\[
\begin{align*}
\{A^k, A^n\} &= [G_{(-M)}^{kn}(A^0, \ldots, A^{k+n+M}) + \ldots]\delta(x - x'), \\
\downarrow \\
\{B^k, B^n\} &= [G_{(1-M)}^{kn}(B^0, \ldots, B^{k+n+M-1}) + \ldots]\delta(x - x'), \\
\downarrow \\
\{C^k, C^n\} &= [G_{(2-M)}^{kn}(C^0, \ldots, C^{k+n+M-2}) + \ldots]\delta(x - x'), \\
\downarrow \\
\text{.....}
\end{align*}
\]

**Conjecture:** Possibly, the number of such Miura type transformations is **infinite**, and any local Hamiltonian structure with the index \(-M\) can be reduced to local Hamiltonian structure with an arbitrary index \(N\), where the triangular block \( W_{(N)}^{kn} = \text{const} \) increases (see (13), if \( N \) increases) in complete accordance with the Darboux theorem.

Let us apply the reciprocal transformation

\[ dz = F(A^0, A^1, \ldots)dx + G(A^0, A^1, \ldots)dt, \quad dy = dt \]

to the hydrodynamic chain (14)

\[ A^k_t = \sum_{n=0}^{M+1} V_n^k(A)A^n_x, \quad k = 0, 1, \ldots, M - 1, \quad A^k_t = \sum_{n=0}^{k+1} V^n_k(A)A^n_x, \quad k = M, M + 1, \ldots \]
If the conservation law density \( F(A^0, A^1, \ldots) \) is a linear function (with special choice of constants, see [6]) of \( M \) Casimirs \( A^k \) and the momentum \( A^M \), then new hydrodynamic chain

\[
A^k_y = \sum_{n=0}^{M+1} (FV_n^k(A) - G_n^k) A^n_z, \quad k = 0, 1, ..., M-1, \quad A^k_y = \sum_{n=0}^{k+1} (FV_n^k(A) - G_n^k) A^n_z, \quad k = M, M+1, ...
\]

preserves the local Hamiltonian structure (see (6) and (12)), i.e.

\[
\{A^k(x), A^n(x')\} = [G^k_n(A^0, ..., A^{k+n-M}) + ...] \delta(x - x'),
\]

\[
\downarrow
\]

\[
\{A^k(z), A^n(z')\} = [G^k_n(A^0, ..., A^{k+n-M}) + ...] \delta(z - z').
\]

Possibly, more general (generalized) reciprocal transformations (see [6] and [10]) preserving local Hamiltonian structures for integrable hydrodynamic chains can be found.

### 6 Nonlocal Hamiltonian structures

The nonlocal Poisson brackets (cf. (2))

\[
\{U^i, U^j\} = [G^ij(U) \partial_x + \Gamma^ij_k(U) U^k_x + \varepsilon U^i_k \partial_x^{-1} U^j_k \partial_x \delta(x - x'), \quad i, j, k = 1, 2, 3, ... \quad (21)
\]

for \( N \) component case were completely investigated by E.V. Ferapontov and O.I. Mokhov in [9] for the non-degenerate matrix \( G^{ij} \); its degenerate case was considered by O.I. Mokhov in [17]. The Jacobi identity yields the set of restrictions

\[
G^{ij} = G^{ji}, \quad \partial_k G^{ij} = \Gamma^ij_k + \Gamma^ji_k, \quad G^{ik} \Gamma^{jn} = G^{jk} \Gamma^{in},
\]

\[
\varepsilon (G^{ij} \delta_p^k - G^{ij} \delta_p^m) = [G^{ij} \Gamma^{pm}_k - \Gamma^{ij} \Gamma^{pm}_k + G^{mn} (\partial_n \Gamma^{mj}_k - \partial_k \Gamma^{mj}_n)],
\]

\[
- \varepsilon (\Gamma^{ij}_k - \Gamma^{ij} \Gamma^{ij}_k) \delta_p^m + (\Gamma^{mj}_k - \Gamma^{mj} \Gamma^{ij}_k) \delta_p^j + (\Gamma^{jm}_k - \Gamma^{jm} \Gamma^{ij}_k) \delta_j^k
\]

\[
+ (\Gamma^{ij}_p - \Gamma^{ij} \Gamma^{ij}_p) \delta_p^m + (\Gamma^{mj}_p - \Gamma^{mj} \Gamma^{ij}_p) \delta_p^j + (\Gamma^{jm}_p - \Gamma^{jm} \Gamma^{ij}_p) \delta_j^k
\]

= \[
(\partial_n \Gamma^{ij}_k - \partial_k \Gamma^{ij}_n) \Gamma^{pm}_p + (\partial_n \Gamma^{mj}_p - \partial_k \Gamma^{mj}_n) \Gamma^{nj}_k + (\partial_n \Gamma^{jm}_p - \partial_k \Gamma^{jm}_n) \Gamma^{ni}_p
\]

\[
+ (\partial_p \Gamma^{ij}_k - \partial_k \Gamma^{ij}_n) \Gamma^{nm}_m + (\partial_p \Gamma^{mj}_k - \partial_k \Gamma^{mj}_n) \Gamma^{nj}_n + (\partial_p \Gamma^{jm}_k - \partial_k \Gamma^{jm}_n) \Gamma^{ni}_n
\]

which in the Liouville coordinates \( A^i = A^i(U) \) determined by the conditions (see [15], [17])

\[
G^{ij} = \mathcal{W}^{ij} + \varepsilon A^i A^j, \quad \Gamma^{ij} = \partial_k \mathcal{W}^{ij} - \varepsilon \delta_k^i A^j,
\]
can be classified completely by virtue of the Jacobi identity (because all integrable hydrodynamic chains (the solution of the several first nonlinear PDE’s allows one to compute all other coefficients determined system of infinitely many nonlinear PDE’s. The L symmetric and non-degenerated (coordinates is (see one can develop for the infinite component case. Then the Poisson bracket in the Liouville coordinates has the form (the numeration of the moments A runs from 0 up to infinity)

\[ \{A^i, A^j\} = [\mathcal{W}^{ij} \partial_x + \partial_x \mathcal{W}^{ji} - \mathcal{A}^j \partial_x A^i + \epsilon \mathcal{A}^j \partial_x^{-1} A^i] \delta(x - x'), \quad i, j = 0, 1, \ldots \]

Similar classification of these nonlocal Poisson brackets (see (22)) will be made elsewhere.

The first example of such a nonlocal Poisson bracket (21) is (see [21])

\[ \{U^k, U^n\} = [\beta(\beta k + \beta + 1)U^{k+n+1} \partial_x + \beta(\beta n + \beta + 1) \partial_x U^{k+n+1} + (\beta k + \beta + 2)(\beta n + \beta + 2)U^k U^n \partial_x + (\beta k + \beta + 2)(\beta n + \beta + 1)U^k(U^n)_x \]

\[ + (\beta n + \beta + 2)U^n(U^k)_x - (U^k)_x \partial_x^{-1}(U^n)_x] \delta(x - x'). \]

**Remark:** A more general Poisson brackets have been introduced by E.V. Ferapontov (see [6]) for N component non-degenerate case. A similar extension as in the above case one can develop for the infinite component case. Then the Poisson bracket in the Liouville coordinates is (see [18])

\[ \{U^k, U^n\} = [(\Phi^{kn} + \Phi^{nk} - \epsilon pq V^{(p)k} V^{(q)n}) \partial_x + (\partial_m \Phi^{nk} - \epsilon pq V^{(q)n} \partial_m V^{(p)k}) A^m_x \]

\[ + \epsilon pq \partial_m V^{(p)k} A^m_x \partial_x^{-1} \partial_s V^{(q)n} A^s_x] \delta(x - x'), \]

where \(\Phi^{kn}\) and \(V^{(p)k}\) are functions of the moments \(A^m\), the constant \(L \times L\) matrix \(\epsilon_{pq}\) is symmetric and non-degenerated (\(L\) is “co-dimension” of the pseudo-Riemannian space).

### 7 Conclusion and outlook

The **main claim** of this paper is that the Poisson brackets

\[ \{A^k, A^n\} = [\mathcal{W}^{kn}_{(M)}(A^0, ..., A^{k+n-M}) \partial_x + \partial_x \mathcal{W}^{nk}_{(M)}(A^0, ..., A^{k+n-M})] \delta(x - x'), \quad k, n = 0, 1, 2, \ldots \]

can be classified completely by virtue of the Jacobi identity (7), which is the overdetermined system of infinitely many nonlinear PDE’s. The **main observation** is that the solution of the several first nonlinear PDE’s allows one to compute all other coefficients recursively in complete differentials.

Nevertheless, the problem of complete description of these Poisson brackets is solvable, because all integrable hydrodynamic chains (4) are already found (see [8]). Thus, one
should just identify this classification and hydrodynamic chains generated by the Poisson brackets described above.

**Conjecture:** Moreover, we believe that any integrable hydrodynamic chain

\[ A^k_t = \sum_{n=0}^{N_k} V^k_n(A) A^n_x, \quad k = 0, 1, 2, \ldots, \]

where set of functions \( V^k_n(A) \) depends on the moments \( A^m \) \((m = 0, 1, \ldots, M_k; M_k \text{ and } N_k \text{ are arbitrary integers})\), possesses infinitely many local Hamiltonian structures \((8)\).

Let us consider a bi-Hamiltonian structure (see, for instance, \((9)\) and below) determined by the couple of \( M \)-brackets

\[ A^k_t = [\mathcal{W}_{(M)}(\partial_x) + \partial_x \mathcal{W}_{(M)}^{nk}] \frac{\delta H_{M+1}}{\delta A^n} = [G_{(M+1)}^{kn} + \Gamma_{(M+1)s}^{kn} A^s_x] \frac{\delta H_{M+2}}{\delta A^n} \]

or, for instance,

\[ A^k_t = [\mathcal{W}_{(M)}^{kn} \partial_x + \partial_x \mathcal{W}_{(M)}^{nk}] \frac{\delta H_{M+1}}{\delta A^n} = [\tilde{G}_{(M-1)}^{kn} + \tilde{\Gamma}_{(M-1)s}^{kn} A^s_x] \frac{\delta H_{M}}{\delta A^n}. \]

Suppose, that all \( M \)-brackets are classified already. If any pair of such Poisson brackets is compatible (see, for instance, \([14]\)), then this pair automatically creates an integrable hydrodynamic chain together with recursion (hereditary) operator.

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