Genuine Secret-Sharing States

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Quantum secret sharing allows each player to have classical information for secret sharing in quantum mechanical ways. In this work, we construct a class of quantum states on which players can quantunmly perform secret sharing secure against dishonest players as well as eavesdropper. We here call them the genuine secret-sharing states. In addition, we show that if $N$ players share an $N$-party genuine secret-sharing state, then arbitrary $M$ players out of the total players can share an $M$-party genuine secret-sharing state by means of local operations and classical communication on the state. We also define the distillable rate with respect to the genuine secret-sharing state, and explain the connection between the distillable rate and the relative entropy of entanglement.

I. INTRODUCTION

Secret sharing [1, 2] is a way of allocating a secret among players, and a sufficient number of players must cooperate to restore the secret. To be more specific, in a $(k, n)$-threshold secret sharing scheme, a dealer distributes a secret to $n$ players, and $k$ or more players can reconstruct the secret if they collaborate, but fewer than $k$ players cannot do so, where $k \leq n$. Because of this feature, secret sharing can be used to deal with important and sensitive information.

We remark that quantum mechanics can provide us with unconditionally secure secret sharing. For example, there is a quantum protocol [3] based on the Greenberger-Horne-Zeilinger (GHZ) state [4] in which each player can obtain a classical bit for $(n, n)$-threshold secret sharing. More precisely, if $N$ players including a dealer share an $N$-qubit GHZ state, then they can carry out an $(N - 1, N - 1)$-threshold secret sharing through the protocol. As a matter of fact, this secret-sharing protocol can be considered as a natural generalization of the Bell-state based quantum key distribution (QKD) protocol [5].

Even though two players can securely share a secret key through the Bell state, there is a specific class of states on which perfectly secure QKD can be performed. The states in the class are called the private states [6, 7]. In a similar way to the private states in QKD, we can naturally ask the following question, what kinds of quantum states are required to accomplish a secure $(n, n)$-threshold secret sharing? In Ref. [8], this question has already been considered, and the quantum states, called the secret-sharing states, have been suggested as an answer to the question. It has been shown that if players share a secret-sharing state, then each player can have a classical bit for secret sharing, which is secure against any external eavesdropper. However, dishonest players, who have a fatal impact on the security of secret sharing, were not sufficiently considered in Ref. [8]. In particular, we can find secret-sharing states that provide each legitimate player with a secret bit, which is insecure against dishonest players, as we will see below. In other words, secret sharing is not in general guaranteed on the secret-sharing states.

To see this, we first look at the secret-sharing conditions presented in Ref. [8]: (i) The probability distributions of the players’ secret bits must be unbiased, and perfectly correlated, that is, if we let $p_I$ be the probability that the $N$ players get the random bit string $I \in \mathbb{Z}_2^N$, then $p_I = \frac{1}{2^{N-1}}$ for $I$ with even parity and $p_I = 0$ for $I$ with odd parity. (ii) Any eavesdropper cannot obtain any information about the players’ secret bits. For $1 \leq i \leq N$, let $A_i$ be the player $A_i$’s qubit system, and $A'_i$ be the $A_i$’s another system with arbitrary dimension. Then it has been shown [8] that $|\rho_{AA'}\rangle$ is a quantum state for secret sharing, that is, a secret-sharing state, where $A = A_1 \cdots A_N$ and $A' = A'_1 \cdots A'_N$ if and only if it is of the form

$$\frac{1}{2^{N-1}} \sum_{I,J \in \mathbb{Z}_2^N \text{ even parity}} |I\rangle_A \langle J| \otimes U_I \sigma_{A'} U'_I,$$

where $\sigma_{A'}$ is an arbitrary state, and the $U_I$’s are unitary operators on the system $A'$. Here, $A$ and $A'$ are called the secret part and the shield part of the state, respectively.

Suppose that three players share the following secret-sharing state,

$$|\Upsilon_1\rangle_{AA'} = \frac{1}{2}(|000\rangle_A |000\rangle_{A'} + |011\rangle_A |000\rangle_{A'} + |101\rangle_A |001\rangle_{A'} + |110\rangle_A |010\rangle_{A'}).$$

Then we can readily see that if $A_3$ is a dishonest player, then he/she can perfectly know the other players’ secret information by measuring his/her own secret part $A_3$ and shield part $A'_3$ in the computational basis.

There also exists a secret-sharing state on which each legitimate player has an insecure secret bit against dishonest players, even when dishonest players do not handle their shield parts. It can be easily seen from the following secret-sharing state,

$$|\Upsilon_2\rangle_{AA'} = \frac{1}{2\sqrt{2}}(|000\rangle_A |000\rangle_{A'} + |011\rangle_A |000\rangle_{A'} + |101\rangle_A |000\rangle_{A'} + |110\rangle_A |000\rangle_{A'} - |101\rangle_A |000\rangle_{A'} - |110\rangle_A |000\rangle_{A'} - |111\rangle_A |000\rangle_{A'}).$$
In this case, \( A_3 \) and \( A_4 \) can totally know the other players’ measurement outcomes by measuring their secret parts \( A_3 A_4 \) in the Bell basis, and they can also deceive legitimate players as if they measure their secret parts in the computational basis.

These problems are caused by the lack of sufficient consideration on dishonest players in the secret-sharing conditions. Hence, in this work, we modify the secret-sharing conditions to fully cover \((n, n)\)-threshold secret sharing scenarios, and introduce a class of quantum states on which each player can obtain a classical information for secret sharing secure against not only eavesdropper but also dishonest players. We call them the genuine secret-sharing (GSS) states. In addition, we show that if \( N \) players share an \( N \)-party GSS state, local operations and classical communication (LOCC) enables any \( M \) players out of the \( N \) players to share an \( M \)-party GSS state. It can be an important property in a quantum network connected by repeaters, since if the network consists of a GSS state then players can share their own GSS state with proper smaller size without providing any information to the repeaters.

Furthermore, we define the distillable rate with respect to the GSS state, and also show that the distillable rate is upper bounded by the relative entropy of entanglement [6, 7] between any bipartition of the total players. Using this property, we discuss the irreducible GSS states, which players cannot have additional information for secret sharing from the shield part of the state via LOCC.

Our paper is organized as follows. We introduce the GSS state, and investigate its properties in Sec. III. We define the GSS distillable rate in Sec. IIII and give examples of the GSS states in Sec. IV. Finally, we conclude our work with some discussion in Sec. V.

II. GENUINE SECRET-SHARING STATE AND ITS PROPERTIES

To perfectly deal with \((n, n)\)-threshold secret sharing scenarios, we should also regard dishonest players as internal eavesdroppers who may conspire with an external one. Thus we modify the secret-sharing conditions in Ref. [8] as follows.

(i) The probability distributions of all players’ secret information must be unbiased and perfectly correlated.

(ii) Any eavesdropper and dishonest players cannot get any information about the legitimate players’ secret information.

Since the modified conditions include the previous ones, the quantum states on which players can have secret information that satisfies the modified conditions also have the form of the secret-sharing states, but they must be different from the states. The difference can be seen in the following theorem. From now on, we let \( A_i \) be the qudit system for all \( i \) to handle more general situations.

**Theorem 1.** \( \Upsilon_{AA'} \) is a quantum state on which players can obtain secret information that obeys the modified secret-sharing conditions by measuring their secret parts in the computational basis if and only if for any bipartite split \( \{P_i, P'_i\} \) of the players with \( |P_i| \geq 2 \), the given state \( \Upsilon_{AA'} \) can be written as

\[
\frac{1}{d^{N-1}} \sum_{I_1 I_2 J_1 J_2 \in \mathcal{G}_N^t} |I_1 I_2|_{P_i P'_i} \langle J_1 J_2 | \otimes \left( U_{P'_1}^{I_1} V_{A'_1}^{I_2} \right) \sigma_{A'} \left( U_{P'_1}^{I_1} V_{A'_1}^{I_2} \right)^\dagger,
\]

where

\[
\mathcal{G}_N^t \equiv \left\{ I = i_1 i_2 \cdots i_N \in \mathbb{Z}_d^N : \sum_{j=1}^N i_j \equiv t \pmod{d} \right\},
\]

\( P_k \) and \( P'_k \) are the secret part and the shield part of \( P_k \), respectively, \( \sigma_{A'} \) is an arbitrary state, and the \( \{U_{P'_1}^{I_1}\} \) and \( \{V_{A'_1}^{I_2}\} \) are unitary operators on the system \( P'_1 \) and \( A'_1 \), respectively. We call the state the GSS state.

**Proof.** We first give a proof of the forward direction. Suppose that \( |\Psi\rangle_{AA'E} \) is a purification of \( \Upsilon_{AA'} \), that is,

\[
|\Psi\rangle_{AA'E} = \sum_{I \in \mathbb{Z}_d^N} \sqrt{p_I} |I\rangle_A |\Psi_I\rangle_{A'E},
\]

where \( E \) is the reference system for the purification, which can be considered as the system of the external eavesdropper. From the condition (i), we have \( p_I = 1/d^{N-1} \) for \( I \in \mathcal{G}_N^t \) and \( p_I = 0 \) for \( I \notin \mathcal{G}_N^t \).

For a dealer \( A_k \), the worst case is that the other players except one player \( A_i \) (\( i \neq k \)) are dishonest. In this case, by changing the order of the systems, we can rewrite the state \( |\Psi\rangle_{AA'E} \) as follows:

\[
|\Psi\rangle_{A_k A_i D A'E} = \frac{1}{\sqrt{d^2}} \sum_{i_k, i_l \in \mathbb{Z}_d} |i_k, i_l\rangle_{A_k A_i} |\eta_{i_k, i_l}\rangle_{D A'E},
\]

where \( D \) is the secret part of the dishonest players and

\[
|\eta_{i_k, i_l}\rangle_{D A'E} = \frac{1}{\sqrt{d^{N-3}}} \sum_{\xi \in \mathcal{G}_N^{i_k-i_l}} |\xi\rangle_D |\Psi_{i_k, i_l, \xi}\rangle_{A'E}.
\]

Let \( i_k \) be \( A_k \)'s measurement outcome, then the quantum state of dishonest players and eavesdropper after the measurement becomes

\[
\Upsilon_{DD'E}^{i_k} = \frac{1}{d} \sum_{i_l \in \mathbb{Z}_d} \text{tr}_{A_k A_l} |\eta_{i_k, i_l}\rangle_{D A'E} \langle \eta_{i_k, i_l} |,
\]

where \( D' \) is the shield part of the dishonest players. Since the eavesdropper and dishonest players cannot get any information about the \( A_k \)'s outcome, \( \Upsilon_{DD'E}^{i_k} = \Upsilon_{DD'E}^i \) for
any \( i_k, i_k' \in \mathbb{Z}_d \). We note that \( \text{tr}_{A'_k A'_i} |\eta_{i_k, i'i} \rangle_{DA'E} \langle \eta_{i_k, i'i} | \)

is written as
\[
\frac{1}{d^{N-3}} \sum_{\xi, \zeta \in \mathbb{S}_{N-2}^{-i_k}} |\zeta \rangle_{\mathcal{D}} \langle \xi | \otimes \text{tr}_{A'_k A'_i} |\Psi_{i_k, i'i, \xi} \rangle_{A'E} \langle \Psi_{i_k, i'i, \zeta} |.
\]

Hence, \( \Upsilon_{DD'E}^k = \Upsilon_{DD'E}^{i_k} \) implies that if \( i_l \) and \( i'_l \) satisfy \( i_k + i_l = i_k' + i'_l \) (mod \( d \)), then
\[
\text{tr}_{A'_k A'_i} |\eta_{i_k, i'i} \rangle_{DA'E} \langle \eta_{i_k, i'i} | = \text{tr}_{A'_k A'_i} |\eta_{i_k, i'i} \rangle_{DA'E} \langle \eta_{i_k, i'i} |.
\]

It follows from Hughston-Jozsa-Wooters theorem [11] that for \( i_k, i'_k, i, i'_i \in \mathbb{Z}_d \), if \( i_k + i_l = i_k' + i'_l \) (mod \( d \)), there is a unitary operator \( \tilde{U}_{A'_k A'_i}^{i_k, i'i, i} \) on the system \( A'_k A'_i \) such that
\[
\tilde{U}_{A'_k A'_i}^{i_k, i'i, i} |\Psi_{i_k, i'i, \xi} \rangle_{A'E} = |\Psi_{i_k, i'i, \xi} \rangle_{A'E}.
\]

for all \( \xi \in \mathbb{S}_{N-2}^{-i_k} \).

Let us now divide the players into two parties, \( P_1 \) and \( P_2 \), where \( |P_1| = M \geq 2 \). Then Eq. (3) can be rewritten as
\[
|\Psi \rangle_{P_1 P_2 A'E} = \sum_{i_1, i_2} |I_1 i_2 \rangle_{P_1 P_2} |\Psi_{I_1, I_2} \rangle_{A'E}.
\]

Here, all possible cases of secret sharing, including the case where \( P_2 \) is the party of the dishonest players, should be considered. Thus, by symmetry and the Eq. (4), it can be shown that there are unitary operators \( U_{P_1}^{I_1} \) and \( V_{A'}^{I_1} \) such that
\[
|\Psi_{I_1, I_2} \rangle_{A'E} = U_{P_1}^{I_1} |\Psi_{I_1, I_2} \rangle_{A'E} = U_{P_1}^{I_1} V_{A'}^{I_1} |\Psi_{00\ldots0} \rangle_{A'E}.
\]

for \( I_1 \in \mathbb{S}_M^\alpha \) and \( I_2 \in \mathbb{S}_N^{d-\alpha} \), where \( I_1^\alpha = 0 \ldots 0 \alpha \in \mathbb{S}_M^\alpha \).

For instance, if \( I = i_1 i_2 \ldots i_N \in \mathbb{S}_N^\alpha \), then
\[
|\Psi_{I} \rangle_{A'E} = U_{A'_1 A'_2}^{i_1 i_2} U_{A'_3 A'_4}^{i_3 i_4} \ldots U_{A'_{N-1} A'_N}^{i_{N-1} i_N} |\Psi_{00\ldots0} \rangle_{A'E},
\]

where \( U_{A'_k A'_i}^{i_k i_i} = \left( \tilde{U}_{A'_k A'_i}^{i_k, i'i, i} \right)^\dagger \) and \( j_l \equiv i_1 + \ldots + i_l \) (mod \( d \)). Therefore, if we let \( \text{tr}_{A'} (|\Psi_{00\ldots0} \rangle \langle \Psi_{00\ldots0}|) = \sum_x \lambda_x |\phi_x \rangle \langle \phi_x | \) be its spectral decomposition, we have
\[
|\Psi_{I_1, I_2} \rangle_{A'E} = \sum_x \sqrt{\lambda_x} U_{P_1}^{I_1} V_{A'}^{I_1} |\psi_x \rangle_{A'E} |\phi_x \rangle_{A'E},
\]

where \( \{ |\psi_x \rangle \} \) forms an orthonormal set for the system \( A' \), and thus \( \Upsilon_{AA'} \) has the form in Eq. (2).

Conversely, we now assume that for any bipartite split \( \{ P_1, P_2 \} \) of the players with \( |P_1| = M \geq 2 \), the given state \( \Upsilon_{AA'} \) is of the form in Eq. (2). Then it can be readily checked that players have secret information that satisfies the condition (i) by measuring their secret parts in the computational basis. It remains to show that the secret information satisfies the condition (ii').

Suppose that \( P_1 \) and \( P_2 \) are parties of legitimate players and dishonest players, respectively. Let \( \sigma_{A'} = \sum_x \lambda_x |\mu_x \rangle \langle \mu_x | \) be a spectral decomposition of \( \sigma_{A'} \), and
\[
|\Psi_{I_1, I_2} \rangle_{A'E} = \sum_x \sqrt{\lambda_x} U_{P_1}^{I_1} V_{A'}^{I_1} |\mu_x \rangle_{A'} |\mu_x \rangle_{E},
\]

where \( \{ |\mu_x \rangle_{E} \} \) is an orthonormal set for the subsystem \( E \). Then we can see that the state of the form in Eq. (3) is a purification of \( \Upsilon_{AA'} \).

Let \( I_1 \in \mathbb{S}_M^\alpha \) be the legitimate players’ measurement outcome after measuring their secret parts, then the eavesdropper and dishonest players’ state after the measurement becomes
\[
\Upsilon_{I_1 P_2 P_2 E} = \frac{1}{d^{N-1}} \sum_{I_1, I_2} |I_1 i_2 \rangle_{P_1 P_2} |I_1 \rangle_{E} \otimes \text{tr}_{P_1} |\Psi_{I_1, I_2} \rangle_{A'E} |\Psi_{I_1, I_2} \rangle_{A'E}.
\]

for \( 2 \leq M \leq N - 1 \) and \( \Upsilon_{I_1 P_1 P_2 P_2 E} = \Upsilon_{I_1 P_2 P_2 P_2 E} \) for any \( I_1, I_1 \in \mathbb{S}_M^\alpha \). This means that the legitimate players’ secret information is secure against dishonest players and eavesdropper.

As seen in Eq. (3), unitary operators on the shield part of the GSS state can be expressed as the product of unitary operators acting on two players’ shield parts.

Corollary 2. For any rearranged order \( i_1 i_2 \ldots i_N \), the GSS state can be written as
\[
\frac{1}{d^{N-1}} \sum_{I, J} |I \rangle_{A_1 A_2 A_N} \langle J | \otimes U_I \sigma_A U_J^\dagger,
\]

where \( U_I = U_{A_1}^{i_1 i_2} U_{A_2}^{i_3 i_4} \ldots U_{A_{N-1}}^{i_N-1 i_N} \) for some unitary operators \( U_{A_1}^{i_1 i_2} U_{A_2}^{i_3 i_4} \ldots U_{A_{N-1}}^{i_N-1 i_N} \), with \( j_l \equiv i_1 + \ldots + i_l \) (mod \( d \)).

Let us now investigate the properties of the GSS state. We remark that if \( N \) players share the \( N \)-party GHZ state, then any \( M \) players among the total players can share the \( M \)-party GHZ state by all players’ LOCC. We can see from the following theorem that the GSS state has the similar property.

Theorem 3. Suppose that players share a GSS state \( \Upsilon_{AA'} \). Then for any bipartite split \( \{ P_1, P_2 \} \) of the players with \( |P_1| = M \geq 2 \), \( P_1 \) can share a GSS state by means of LOCC.

Proof. Let us divide \( P_1 \) into legitimate players’ party \( P_L \) and dishonest players’ party \( P_D \) with \( |P_L| = K \geq 2 \). By rearranging the order, let \( P_L = \{ A_{i_1}, \ldots, A_{i_K} \}, P_D = \ldots \)
\{A_{k+1}, \ldots, A_M\}$, and $P_2 = \{A_{m+1}, \ldots, A_N\}$. Since $\Upsilon_{\text{AA}}$ is a GSS state, it follows from the Corollary 2 that $\Upsilon_{\text{AA}}$ is written as in Eq. (1).

Assume that players in $P_2$ measure their secret parts in the computational basis, and have the measurement outcome $j_2 = m_{m+1} \cdots m_N \in \mathcal{B}_{N-M}$. Then $j_{1k} = d - \beta - (i_{k+1} + \cdots + i_M)$, $j_{1M} = d - \beta$, and the post-measurement state becomes as follows:

$$\frac{1}{d^{M-1}} \sum_{I_L I_D, J_L J_D \in \mathcal{B}_{M}} |I_L I_D J_L J_D\rangle \langle I_L I_D J_L J_D| \otimes \left(U_{P_L}^I V_{P_1}^I W_{A'} \sigma_A \left(U_{P_L}^J V_{P_1}^J W_{A'} \right) \right)',$$

where

$$U_{P_L}^I = U_{A_1 'A_2 'A_3 '} \cdots U_{A_{K-1} 'A_K '}$$

$$V_{P_1}^I = U_{A_K 'A_{K+1} 'A_{K+2} '} \cdots U_{A_{M-1} 'A_M '}$$

$$W_{A'} = U_{A_1 'A_2 'A_3 '} \cdots U_{A_{N-1} 'A_N '}.$$

By tracing out the $P_2$’s system, we can see that $P_1$’s system is written as

$$\frac{1}{d^{M-1}} \sum_{I_L I_D, J_L J_D \in \mathcal{B}_{M}} |I_L I_D\rangle \langle I_L I_D| \otimes \left(U_{P_L}^I V_{P_1}^I \right) \bar{\sigma}_{P_1} \left(U_{P_L}^J V_{P_1}^J \right)$$

for some state $\bar{\sigma}_{P_1}$. Hence, after one of players $A_k$ in $P_1$ applies unitary operator $T_\beta = \sum_i |i + \beta\rangle \langle i|$ on his/her secret part, their state becomes a GSS state. \hfill $\Box$

Since quantum networks in general require repeaters connecting players because of the distance limitation of quantum communication, and players want to get a secure information without providing their information to repeaters, this property can play an important role in quantum networks. Hence, in quantum networks, sharing a GSS state can be a goal for secure multipartite quantum communication.

The next property is about a relation between the GSS state and the Holevo information [12], which is one of the important quantities in security analysis. Suppose that Alice and Bob share a state $\rho_{AB}$, and Alice measures her system. For each Alice’s measurement outcome $x$, let $p_x$ and $\rho_B^x$ be its probability and Bob’s resulting state after her measurement, respectively. The Holevo information between Alice’s measurement outcomes and Bob’s state is given by

$$\chi_B(x : B) = S(\bar{\rho}) - \sum_x p_x S(\rho_B^x),$$

where $S$ is the von Neumann entropy and $\bar{\rho} = \sum_x p_x \rho_B^x$. Zero Holevo information between Alice’s measurement outcomes and Bob’s state, that is, $\chi_B(x : B) = 0$ means that Bob cannot have any information about Alice’s measurement outcomes. Thus, the modified condition (ii’) can be replaced as follows: For a given quantum state $\Upsilon$, $\chi_B(k_i : DE) = 0$ for any party $DE$ of dishonest players and eavesdropper, where $k_i$ is the legitimate player $A_i$’s secret information. In other words, the Holevo information is zero if and only if the state satisfying the condition (i) for secret sharing is a GSS state. For example, since $\chi(i_1 : A_3) = 1$ in Eq. (1), where $i_1$ is the $A_1$’s measurement outcome when he/she measures his/her secret part in the computational basis, the secret-sharing state in Eq. (1) is not a GSS state.

**Theorem 4.** Let $\Gamma_{\text{AA'}}$ be a quantum state and $p_I$ be the probability that players’ measurement outcome is $I$ when they measure the system $A$ in the computational basis. Suppose that $p_I > 0$ for $I \in \mathcal{B}_N$, and $p_I = 0$ for $I \notin \mathcal{B}_N$. Then $\Gamma_{\text{AA'}}$ is a GSS state if and only if for any $D \subset \{A_1, \ldots, A_N\}$ with $|D| = N - 2$, the Holevo information $\chi_B(i_k : DE)$ equals zero, where $i_k$ is the measurement outcome of the $A_k$’s secret part and $A_k \notin D$.

**Proof.** We show that $p_I$’s are identical for all $I \in \mathcal{B}_N$ if for any $D \subset \{A_1, \ldots, A_N\}$ with $|D| = N - 2$, $\chi_B(i_k : DE) = 0$, where $i_k$ is the measurement outcome of $A_k \notin D$. If it is true, then Theorem 1 completes the proof. Without loss of generality, we may assume that $D = \{A_3, \ldots, A_N\}$, and $\chi_B(i_k : DE) = 0$ for $k = 1, 2$. Note that the following state is a purification of the given state:

$$|\Psi\rangle_{A_1 A_2 D A' E} = \sum_{i_1,i_2=0}^{d-1} |i_1,i_2\rangle_{A_1 A_2} |\psi_{i_1,i_2}\rangle_{D A' E},$$

where $D$ is the secret part of $D$ and

$$|\psi_{i_1,i_2}\rangle_{D A' E} = \sum_{\xi \in \mathcal{B}_{N-2}} \sqrt{p_{i_1,i_2,\xi}} |\xi\rangle_D |\psi_{i_1,i_2,\xi}\rangle_{A' E}.$$

When $A_1$’s measurement outcome is $i$, the state of $D$ and eavesdropper can be described as

$$\rho_{DE}_{i} = \frac{1}{q_i} \sum_{i_{2z=0}^{d-1}} \text{tr}_{A'_2 |\psi_{i_1,i_2}\rangle_{D A' E} \langle \psi_{i_1,i_2}|}$$

where $q_i = \sum_{i_{2z=0}^{d-1}} \sum_{\xi \in \mathcal{B}_{N-2}} p_{i_1,i_2,\xi}$. Since the Holevo information $\chi_B(i_1 : DE)$ equals zero, $\rho_{DE}_{i_1} = \rho_{DE}_{i_2}$ for any $A_1$’s measurement outcomes $i$ and $i'$. Similarly, since $\chi_B(i_2 : DE) = 0$, we also have $\rho_{DE}_{i_2} = \rho_{DE}_{i_2'}$ for any $A_2$’s measurement outcomes $j$ and $j'$.
with $r_j = \sum_{i} \sum_{\xi \in \mathcal{S}_{N-2}^{i-j}} p_{i,j,\xi}$. Thus, if $i + j = i' + j'$ (mod $d$),

$$p_{i,j,\xi} = \frac{q_{i}}{q_{i'}} = \frac{r_{j}}{r_{j'}}$$

for $\xi \in \mathcal{S}_{N-2}^{i-j}$. In particular, since

$$q_{\beta} = \frac{p_{\alpha,\beta,\alpha,\xi}}{p_{\alpha,\beta,\alpha,\xi}} = \frac{r_{\beta,\alpha}}{r_{\alpha}}$$

for $\xi \in \mathcal{S}_{N-2}^{i-j}$ and $\xi \in \mathcal{S}_{N-2}^{\beta}$, we obtain

$$q_{\beta} = q_{\beta} \sum_{\alpha=0}^{d-1} q_{\alpha} = q_{\beta} = q_{0}$$

for any $\beta \in \mathbb{Z}_d$, that is, $q_i$'s are all equal. Hence, if $i + j = i' + j'$ (mod $d$), then $p_{i,j,\xi} = p_{i',j',\xi}$ for all $\xi \in \mathcal{S}_{N-2}^{i-j}$. By symmetry, it follows that $p_{i,j}$'s are identical for all $i \in \mathcal{S}_0$. \hfill $\Box$

Theorem 4 shows that players’ secret information must be unbiased in order to be secure against dishonest players and eavesdropper. Therefore, the condition (ii') of the modified secret-sharing conditions includes unbiasedness of the players’ secret information, and so the unbiasedness can be omitted in the condition (i) of the modified conditions.

### III. GSS DISTILLABLE RATE

In this section, we discuss the distillable rate with respect to the GSS state. Before defining the distillable rate, we need to consider one issue arising from the presence of dishonest players. For instance, let us look at the secret-sharing state in Eq. (1) once more. For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, if we let $|\mu_0\rangle = |\alpha|0 + |\beta|1$ and $|\mu_1\rangle = |\beta^*|0 - |\alpha^*|1$; the state can be written as

$$\frac{1}{2} |\mu_0\rangle_{A_1} |01A_2\rangle = |\alpha^*|00A_3A_4^* + |\beta^*|11A_3A_4^* \rangle$$

$$+ \frac{1}{2} |\mu_0\rangle_{A_1} |10A_2\rangle = |\alpha^*|10A_3A_4^* + |\beta^*|01A_3A_4^* \rangle$$

$$+ \frac{1}{2} |\mu_1\rangle_{A_1} |00A_2\rangle = |\beta|00A_3A_4^* - |\alpha|11A_3A_4^* \rangle$$

$$+ \frac{1}{2} |\mu_1\rangle_{A_1} |11A_2\rangle = |\beta|10A_3A_4^* - |\alpha|01A_3A_4^* \rangle.$$.

Hence, for any $A_1$'s measurement basis $\{|\mu_0\rangle, |\mu_1\rangle\}$, player $A_3$ can get the other players' secret information by properly measuring his/her system. However, since this is a secret-sharing state, it is GHZ distillable \[5\], that is, the GHZ state can be asymptotically distilled from the state by LOCC. Thus, if we define the distillable rate as how many copies of the given state are required to asymptotically distill a GSS state through LOCC, the completely insecure quantum state may have a strictly positive rate.

In order to avoid such an issue, we employ the Devetak-Winter rate $[13, 14]$. Let $\rho_{A_1}\ldots A_N$ be a given state. We say that $\rho_{A_1}\ldots A_N$ has positive Devetak-Winter rate if there is a set of measurement operations $\{\mathcal{M}_k\}_{1 \leq k \leq N}$ such that for any $\mathcal{D} H_{\mathcal{D}} \leq N - 2$, $I (m_i : \tilde{m}_i) \geq \chi (m_i : \mathcal{D}E) > 0$, where $I (X : Y)$ is the mutual information between $X$ and $Y$, $A_i \notin \mathcal{D}$, $m_i$ is the $A_i$'s measurement outcome, and $\tilde{m}_i$ is the sum of the measurement outcomes of players except $A_i$. If we define the distillable rate only for quantum states with positive Devetak-Winter rate, then we can rule out quantum states that are completely insecure.

**Definition 1.** For given state $\rho_{A_1}\ldots A_N$ with positive Devetak-Winter rate, let $P_n$ be a sequence of LOCC operations such that $P_n (\rho^\otimes n) = \sigma_n$. We call $P \equiv \bigcup_{n=1}^{\infty} \{P_n\}$ a GSS distillation protocol of the state $\rho$ if $\lim_{n \rightarrow \infty} \|\sigma_n - Y_{\mathcal{D}_n}\| = 0$, where $Y_{\mathcal{D}_n}$ is a GSS state that has a secret part with dimension $d_n^\mathcal{D}$. For given protocol $P$, its rate is defined as

$$R (P) = \lim_{n \rightarrow \infty} \frac{\log d_n}{n},$$

and the GSS distillable rate of the state $\rho$ is given by

$$D_G (\rho) = \sup_{P} \log d_n / n.$$

We note that when players are divided into two parties, $P_1$ and $P_2$, they can have a private state between two parties if they share a GSS state. Thus, we have

$$D_G (\rho_{A_1}\ldots A_N) \leq K_{P_1}^{P_2} (\rho_{A_1}\ldots A_N),$$

where $K_{P_1}^{P_2}$ is the distillable key rate between $P_1$ and $P_2$, which is defined by protocols to distill private states between $P_1$ and $P_2$. In addition, since the relative entropy of entanglement (REE) is an upper bound of the distillable key rate, we obtain

$$D_G (\rho_{A_1}\ldots A_N) \leq E_{P_1}^{P_2} (\rho_{A_1}\ldots A_N),$$

where $E_{P_1}^{P_2}$ is REE between $P_1$ and $P_2$, that is,

$$E_{P_1}^{P_2} (\rho) = \inf_{\sigma \in \mathcal{S}} S (\rho || \sigma).$$

Here, $S (\rho || \sigma) = -S (\rho) - T_\rho \log \sigma$ is the relative entropy, and $S (\rho || \sigma)$ is the set of bipartite separable states of the system $P_1P_2$.

Using the GSS distillable rate, one can define irreducible GSS state, as irreducible private state is defined in Ref. \[2\]. For any GSS state $\mathcal{Y}_{\mathcal{A}A'}$, whose secret part is of dimension $d^\mathcal{A}$, the state is said to be irreducible if $D_G (\mathcal{Y}_{\mathcal{A}A'}) = \log d$. Then the following theorem provides a way to check that a given GSS state is irreducible.

**Theorem 5.** Let $\mathcal{Y}_{\mathcal{A}A'}$ be a GSS state and $A_k$ denote the party of players except $A_k$. If $\mathcal{Y}_{\mathcal{A}A'}$ is written as

$$\mathcal{Y}_{\mathcal{A}A'} = \frac{1}{d^{N-1}} \sum_{i_k,j_k \in \mathcal{A}} |i_k\rangle |j_k\rangle_{A_k A_k'} \langle i_k| \otimes U_{A_k}^j V_{A_k'}^j \langle \sigma | U_{A_k}^j V_{A_k'}^j \rangle,$$

then $\mathcal{Y}_{\mathcal{A}A'}$ is irreducible.
where $\sigma_{A'}$ is a state on the system $A'$, and $\tilde{A}_k$ and $\tilde{A}_k'$ are the secret part and the shield part of $\tilde{A}_k$, respectively, then

$$\left. E_{r_{A_k:k}}^{A_k:A_k} (\Upsilon) \leq \log d + \frac{1}{d} \sum_{i_k} E_{r_{A_k:k}}^{A_k:A_k} \left( V_{i_k}^{A', \sigma_{A'} (V_{i_k}^{A'})^\dagger} \right), \right. \tag{9}$$

Proof. Let us consider the unitary operator

$$W_{\tilde{A}_k \tilde{A}_k'} = \sum_{I \in \mathbb{E}_d^{N-1}} \left| I \right\rangle_{\tilde{A}_k} \langle I | \otimes U_{I_{\tilde{A}_k}}^\dagger.$$ 

Since the REE is invariant under local unitary operation, $E_{r_{A_k:k}}^{A_k:A_k} (\Upsilon) = E_{r_{A_k:k}}^{A_k:A_k} (W \Upsilon W^\dagger)$. Note that

$$W \Upsilon W^\dagger = \frac{1}{d} \sum_{i_k,j_k} \langle i_k \psi_{i_k} | \tilde{A}_k \tilde{A}_k' \rangle_{\tilde{A}_k} \langle j_k \psi_{j_k} | \otimes V_{i_k}^{A', \sigma_{A'} (V_{i_k}^{A'})^\dagger},$$

where

$$| \psi_{i_k} \rangle_{\tilde{A}_k} = \frac{1}{\sqrt{dN-1}} \sum_{I \in \mathbb{E}_d^{N-1}} \left| I \right\rangle_{\tilde{A}_k}.$$

Hence, we can regard $W \Upsilon W^\dagger$ as a private state. By Theorem 3 in Ref. [7], we can prove this theorem. \qed

Combining inequalities (5) and (6), we have

$$D_G (\Upsilon_{AA'}) \leq \log d + \frac{1}{d} \sum_{i_k} E_{r_{A_k:k}}^{A_k:A_k} \left( V_{i_k}^{A', \sigma_{A'} (V_{i_k}^{A'})^\dagger} \right).$$

Hence, if there is a $k$ such that

$$E_{r_{A_k:k}}^{A_k:A_k} \left( V_{i_k}^{A', \sigma_{A'} (V_{i_k}^{A'})^\dagger} \right) = 0$$

for all $i_k$, the given GSS state is irreducible, since $D_G (\Upsilon_{AA'}) \geq \log d$.

IV. EXAMPLES

Example 1. Let us consider a quantum network consisting of only private states. In order for each player to get a secret bit for secret sharing in the network, all players should be connected via private states. More precisely, assume that Alice and Bob, Bob and Charlie, and Charlie and Alice share the following private states:

$$\gamma_{A_1B_2A'_1B'_2} = \frac{1}{2} \sum_{i,i'=0}^{1} \langle i' | A_1B_2 \langle i' | \otimes U_i \sigma_{A_1B_2} U_i^\dagger,$$

$$\tilde{\gamma}_{B_2C_2B'_2C'_2} = \frac{1}{2} \sum_{j,j'=0}^{1} \langle j' | B_2C_2 \langle j' | \otimes \tilde{U}_j \tilde{\sigma}_{B_2C_2} \tilde{U}_j^\dagger,$$

$$\tilde{\gamma}_{C_1A_2C'_1A'_2} = \frac{1}{2} \sum_{k,k'=0}^{1} \langle kk' | C_1A_2 \langle k'k' | \otimes \tilde{U}_{k} \tilde{\sigma}_{C_1A_2} \tilde{U}_{k'}^\dagger.$$
Example 2. As in Refs. [7, 8], it can be an interesting task to find a GSS state with low distillable entanglement. To this end, let us first consider the following state:

\[
\Gamma = a_0 |\psi_0\rangle_{ABC} \langle \psi_0| \otimes (\rho_0 \otimes \sigma_0 \otimes \tau_0) \\
+ a_1 |\psi_1\rangle_{ABC} \langle \psi_1| \otimes (\rho_1 \otimes \sigma_0 \otimes \tau_1) \\
+ a_2 |\psi_2\rangle_{ABC} \langle \psi_2| \otimes (\rho_1 \otimes \sigma_1 \otimes \tau_0) \\
+ a_3 |\psi_3\rangle_{ABC} \langle \psi_3| \otimes (\rho_0 \otimes \sigma_1 \otimes \tau_1),
\]

where |\psi_0\rangle = \frac{1}{\sqrt{3}} (|000\rangle + |011\rangle + |101\rangle + |110\rangle), |\psi_1\rangle = Z_A |\psi_0\rangle, |\psi_2\rangle = Z_B |\psi_0\rangle, |\psi_3\rangle = Z_C |\psi_0\rangle, and \rho_i, \sigma_j, and \tau_k have orthogonal supports on the system \(A'_i B'_j C'_k\), respectively. Then we can find unitary operators \(U_{A'_i B'_j}, V_{B'_j C'_k},\) and \(W_{C'_k A'_i}\) that satisfy \(U_{A'_i B'_j} \rho_i = (-1)^i \rho_i, V_{B'_j C'_k} \sigma_j = (-1)^j \sigma_j,\) and \(W_{C'_k A'_i} \tau_k = (-1)^k \tau_k\). Using these operators, it can be shown that \(\Gamma\) is a GSS state.

We now consider the state

\[
\Gamma = p |\psi_0\rangle \langle \psi_0| \otimes (\rho_s \otimes \sigma_0 \otimes \rho_s) \\
+ p |\psi_1\rangle \langle \psi_1| \otimes (\rho_a \otimes \sigma_0 \otimes \rho_a) \\
+ \left(\frac{1}{2} - p\right) |\psi_2\rangle \langle \psi_2| \otimes (\rho_a \otimes \sigma_1 \otimes \rho_s) \\
+ \left(\frac{1}{2} - p\right) |\psi_3\rangle \langle \psi_3| \otimes (\rho_s \otimes \sigma_1 \otimes \rho_a),
\]

where \(\rho_s\) and \(\rho_a\) are symmetric and antisymmetric Werner states [13]

\[
\rho_s = \frac{I + F}{d^2 + d},
\]

\[
\rho_a = \frac{I - F}{d^2 - d},
\]

with identity operator \(I\) on the \(d \otimes d\) system and the flip operator \(F = \sum_{i,j=0}^{d-1} |ij\rangle \langle ji|\). It follows from tedious but straightforward calculations that

\[
\left\| \Gamma^{T_{A_i A'_i}} \right\|_1 = 1 + \left(\frac{2}{d^2} + \frac{2}{d}\right)(1 + 2p),
\]

where \(\| \cdot \|_1\) is the trace norm. Since the distillable entanglement is upper bounded by the log negativity [16]

\[
E_N(\Gamma) = \log_2 \left(\left\| \Gamma^{T_{A_i A'_i}} \right\|_1\right),
\]

we can construct the GSS state that has arbitrarily low bipartite distillable entanglement between Alice and the rest of players by increasing \(d\).

V. CONCLUSION

In this work, we have presented the GSS state, and explored its properties. The GSS state provides a classical information for \((n, n)\)-threshold secret sharing, which is secure against dishonest players and eavesdropper. Furthermore, if \(N\) players share an \(N\)-party GSS state, LOCC enables arbitrary \(M\) players out of the total players to share an \(M\)-party GSS state. We have also defined the GSS distillable rate, and it has been shown that the REE between any bipartition of the total players is an upper bound on the GSS distillable rate.

The GSS state can be regarded as a generalization of the private state with respect to secret sharing. In addition, when players share a GSS state, any two players among them can share a private state by all players’ LOCC. Thus, by applying various research results related to the private state, the results can also be used to investigate multipartite communication.

In a quantum network, if players share a GSS state, any arbitrary parties of players can perform QKD or quantum secret sharing. We can naturally have the following question: Is the GSS state the only quantum state on which players can do them? If we answer this question, we can have a more in-depth discussion of the secure quantum network.

There are interesting future works related to the GSS state. First, we can think about a bound entangled state with positive GSS distillable rate. If we find such a state, it can help us to study the relationship between distillable multipartite entanglement and the GSS distillable rate in detail. Second, it can be an intriguing task to find a way to share a GSS state between players and repeaters in quantum networks. If there exists such a way, we can construct a quantum network that ensures secure multipartite communication among players. Therefore the GSS state could be considered as a new resource in multipartite quantum communication.

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