IMPROVED LIEB–THIRRING TYPE INEQUALITIES FOR NON-SELFADJOINT SCHRÖDINGER OPERATORS

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Abstract. We improve the Lieb–Thirring type inequalities by Demuth, Hansmann and Katriel (J. Funct. Anal. 2009) for Schrödinger operators with complex-valued potentials. Our result involves a positive, integrable function. We show that in the one-dimensional case the result is sharp in the sense that if we take a non-integrable function, then an analogous inequality cannot hold.

1. Introduction

Lieb–Thirring inequalities appeared first in the work of Lieb and Thirring in the proof of stability of matter, see [11, 12]. Since then, the involved constants have been improved, and various attempts have been made to generalise the inequalities to allow for non-selfadjoint Schrödinger operators.

The classical Lieb–Thirring inequality for a Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$ reads

$$\sum_{\lambda \in \sigma_d(-\Delta + V)} |\lambda|^{p-d/2} \leq C_{d,p} \|V\|_{L^p}^p,$$

for real-valued potentials $V \in L^p(\mathbb{R}^d)$, where the range for $p$ depends on the dimension $d$ as follows:

- $p \geq 1$, if $d = 1$,
- $p > 1$, if $d = 2$,
- $p \geq \frac{d}{2}$, if $d \geq 3$.

Here $\sigma_d(-\Delta + V)$ denotes the set of discrete eigenvalues, outside the essential spectrum $\sigma_e(-\Delta + V) = [0, \infty)$. The inequality (1) cannot be true for complex-valued $V \in L^p(\mathbb{R}^d)$ with $p > (d + 1)/2$ since then $\sigma_d(-\Delta + V)$ can accumulate anywhere in the essential spectrum, see [2, 4].

It was proved by Demuth, Hansmann and Katriel in [4] that if $p \geq d/2 + 1$, then for any $\tau \in (0, 1)$ we have the inequality

$$\sum_{\lambda \in \sigma_d(-\Delta + V)} \frac{(\text{dist}(\lambda, [0, \infty]))^{p+\tau}}{|\lambda|^{d/2+\tau}} \leq C_{d,p,\tau} \|V\|_{L^p}^p,$$

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for any (complex-valued) \( V \in L^p(\mathbb{R}^d) \). Note that in the selfadjoint case \( (V \text{ real-valued}) \), the inequality reduces to (1) since all discrete eigenvalues are in \((-\infty, 0)\) and hence \( \text{dist}(\lambda, [0, \infty)) = |\lambda| \). In [5], it was published as an open problem to prove whether (2) remains true for \( \tau = 0 \), i.e. whether

\[
\sum_{\lambda \in \sigma_d(\mathcal{H})} \frac{(\text{dist}(\lambda, [0, \infty))|^p}{|\lambda|^{d/2}} \leq C_{p,d}||V||_{L^p}^p.
\]

For \( d = 1 \), a counterexample was found in [3]; it is still an open problem whether the inequality can hold in higher dimensions \( d \geq 2 \).

The main ingredient in the proof of (2) is the following Lieb–Thirring type inequality by Frank, Laptev, Lieb and Seiringer in [7], which sums only over all eigenvalues outside a sector: If \( p \geq d/2 + 1 \) then, for any \( t > 0 \),

\[
\sum_{\lambda \in \sigma_{d}(\Delta + V), \Re \lambda \geq t} |\lambda|^{p-d/2} \leq C_{d,p} \left( 1 + \frac{2}{t} \right)^p ||V||_{L^p}^p.
\]

Now (2) follows by taking a weighted integral over the parameter \( t \) of the sector. However, the weight of the integral wasn’t chosen to optimise the inequality. In this paper we improve the result by taking an optimal weight function.

In Theorem 1 we prove a Lieb-Thirring type inequality where the left hand side of (3) is multiplied by \( f \left( -\log \left( \text{dist}(\lambda, [0, \infty))/|\lambda| \right) \right) \) where \( f \) is a positive function that can decay quite slowly (slower than in the result (2) which corresponds to \( f(s) = e^{-\tau s} \)) but is still integrable, \( \int_0^\infty f(s) \, ds < \infty \).

In Theorem 2 we show that in dimension \( d = 1 \) the inequality is sharp in the sense that if we take a non-integrable function, \( \int_0^\infty f(s) \, ds = \infty \), then such an inequality cannot hold for all \( V \in L^p(\mathbb{R}) \). This suggests that the \( t \)-dependence (asymptotically \( t^{-p} \) as \( t \to 0 \)) on the right hand side of (4) is optimal. We prove this sharpness in Theorem 3. In Section 3 we discuss a few classes of integrable functions \( f \). Each class can be combined with Theorem 1 to give inequalities that are better than (2).

We mention that in [6, 8, 9, 10], different Lieb-Thirring type inequalities were proved. They don’t reduce to (1) in the selfadjoint case, and are therefore difficult to compare with the equalities proved here.

2. New Lieb–Thirring type inequalities

In this section we prove new Lieb-Thirring type inequalities and discuss their sharpness.

**Theorem 1.** Let \( d \in \mathbb{N} \) and \( p \geq d/2 + 1 \). Let \( f : [0, \infty) \to (0, \infty) \) be a continuous, non-increasing function. If \( \int_0^\infty f(s) \, ds < \infty \), then there exists \( C_{d,p,f} > 0 \) such that, for any \( V \in L^p(\mathbb{R}^d) \),

\[
\sum_{\lambda \in \sigma_{d}(\Delta + V)} \frac{\text{dist}(\lambda, [0, \infty))|\lambda|^{p}{|\lambda|^{d/2}} f \left( -\log \left( \frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|} \right) \right) \leq C_{d,p,f} ||V||_{L^p}^p.
\]
Proof. First we show that it suffices to prove the theorem for a continuous, non-increasing, integrable, piecewise $C^1$-function $f$ for which there exists $c > 0$ with $(\log f)' \geq -c$ almost everywhere. To this end, first note that since $f$ is continuous and non-increasing, we find a non-increasing $C^1$-function $f_1 : [0, \infty) \to (0, \infty)$ with $f(s) \leq f_1(s) \leq 2f(s)$ for all $s \in [0, \infty)$. Let $c > 0$. Assume that there exist $0 \leq a_1 < b_1 \leq \infty$ such that $(\log f_1)' \geq -c$ on $[0, a_1]$ and $(\log f_1)' < -c$ on $(a_1, b_1)$. Then

$$f_1(s) = \exp \left( \log(f_1(a_1)) + \int_{a_1}^{s} (\log f_1)'(t) \, dt \right) < f_1(a_1)e^{-c(s-a_1)}, \quad s \in (a_1, b_1).$$

Let $b_1' \in (b_1, \infty)$ be the smallest point where the function $f_1$ intersects $s \mapsto f_1(a_1)e^{-c(s-a_1)}$; set $b_1' = \infty$ if they don’t intersect. Let $f_2 : [0, \infty) \to (0, \infty)$ be the continuous, non-increasing, piecewise $C^1$-function defined by

$$f_2(s) = \begin{cases} f_1(s), & s \notin (a_1, b_1'), \\ f_1(a_1)e^{-c(s-a_1)}, & s \in (a_1, b_1'). \end{cases}$$

If there exists one or more intervals in $[b_1', \infty)$ on which $(\log f_1)' < -c$, we repeat the procedure (inductively) and change the function analogously. We then have finitely or infinitely many intervals $(a_n, b_n')$ (sorted by increasing $a_n$) such that the piecewise $C^1$-function $f_\infty : [0, \infty) \to (0, \infty)$ defined by

$$f_\infty(s) = \begin{cases} f_1(s), & s \notin (a_n, b_n') \text{ for any } n, \\ f_1(a_n)e^{-c(s-a_n)}, & s \in (a_n, b_n'), \end{cases}$$

is continuous, non-increasing, and satisfies $(\log f_\infty)' \geq -c$ almost everywhere. Note that, using $f_1 \leq 2f$ and $f_1(a_n)e^{-c(b_n'-a_n)} = f_1(b_n')$ by construction of $b_n'$,

$$\int_0^{b_n'} f_\infty(s) \, ds \leq \int_0^{b_n'} f_1(s) \, ds + \sum_n \int_{a_n}^{b_n'} f_1(a_n)e^{-c(s-a_n)} \, ds \leq 2 \int_0^{b_n'} f(s) \, ds + \frac{1}{c} \sum_n f_1(a_n)(1 - e^{-c(b_n'-a_n)})$$

$$= 2 \int_0^{b_n'} f(s) \, ds + \frac{1}{c} \sum_n (f_1(a_n) - f_1(b_n'))$$

$$\leq 2 \int_0^{b_n'} f(s) \, ds + \frac{1}{c} f_1(a_1),$$

where in the last inequality we used that $-f_1(b_n') + f_1(a_{n+1}) \leq 0$ since $b_n' \leq a_{n+1}$ and $f_1$ is non-increasing. This implies that since $f$ is integrable, so is $f_\infty$. Now if we can show the theorem for $f_\infty$, then it also follows for $f$ since $f \leq f_1 \leq f_\infty$.

In the following we write again $f$ instead of $f_\infty$. To prove the theorem, we proceed in a similar way as in [4, Corollary 3]. The main difference is that we use a function $g$ such that $t \mapsto t^{-p}g(t)$ may go to infinity at $t = 0$ faster than with the function $g(t) = t^{p-1+\tau}$ ($\tau \in (0, 1)$) used in [4, Corollary 3] but
is still integrable on \((0, m)\) (for \(m > 0\) sufficiently small). We start with the estimate from [7 Theorem 1 part 2], see (4). We estimate the left hand side further by taking the sum only over \(\lambda\) with \(\text{Re} \lambda > 0\) and \(\text{dist}(\lambda, [0, \infty)) = |\text{Im} \lambda| \geq t|\lambda|\). Then, multiplying both sides by \(g(t) := t^{p-1}f(-\log t)\) and integrating over \(t \in (0, 1)\), we get

\[
\int_0^1 \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0, \text{dist}(\lambda, [0, \infty)) \geq t|\lambda|} |\lambda|^{p-d/2} g(t) \, dt \leq C_{d,p} \|V\|_{L^p}^p \int_0^1 \left(1 + \frac{2}{t}\right)^p g(t) \, dt.
\]

Using that \(1 \leq 1/t\) for \(t \in (0, 1)\) and substituting \(s = -\log t\), the integral on the right hand side of (6) side becomes

\[
\int_0^1 \left(1 + \frac{2}{t}\right)^p g(t) \, dt \leq \int_0^1 3^p t^{-1} f(-\log t) \, dt = 3^p \int_0^\infty f(s) \, ds < \infty.
\]

The left hand side of (6) becomes, using the substitution \(s = -\log t\),

\[
\int_0^1 \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0} |\lambda|^{p-d/2} g(t) \, dt = \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0} |\lambda|^{p-d/2} \int_0^{\text{dist}(\lambda, [0, \infty))/|\lambda|} g(t) \, dt
\]

\[
= \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0} |\lambda|^{p-d/2} \int_0^{\text{dist}(\lambda, [0, \infty))/|\lambda|} e^{-ps} f(s) \, ds.
\]

Recall that by assumption or construction, \(f = f_\infty\) satisfies \((\log f)' \geq -c\) almost everywhere. This implies \(f'(s) \geq -cf(s)\) and thus integration by parts yields

\[
\int_a^\infty e^{-ps} f(s) \, ds \geq \frac{1}{p} \left(e^{-pa} f(a) - c \int_a^\infty e^{-ps} f(s) \, ds\right),
\]

whence

\[
\int_a^{\infty} e^{-ps} f(s) \, ds \geq \frac{1}{p + c} e^{-pa} f(a).
\]

Thus the last line in (7) can be estimated from below by

\[
\frac{1}{p + c} \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0} |\lambda|^{p-d/2} e^{-pa} f(a) \bigg|_{a = -\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)}
\]

\[
= \frac{1}{p + c} \sum_{\lambda \in \sigma_d(-\Delta + V), \text{Re} \lambda > 0} |\lambda|^{p-d/2} \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)^p f \left(-\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)\right).
\]

Combining the estimates of the left and right hand sides of (6), we obtain (5) with the left hand side restricted to \(\text{Re} \lambda > 0\). For the sum over the eigenvalues with \(\text{Re} \lambda \leq 0\), we use (4) with \(t = 1\) (in fact, any \(t > 0\)
would work here). Note that then all \( \lambda \) satisfying \( \text{Re}\lambda \leq 0 \) also satisfy \( |\text{Im}\lambda| \geq t\text{Re}\lambda \). Thus we get
\[
\sum_{\lambda \in \sigma_d(-\Delta+V), \text{Re}\lambda \leq 0} |\lambda|^{p-d/2} \leq C_{d,p}3^p\|V\|_{L^p}^p.
\]
This implies, with \( \text{dist}(\lambda, [0, \infty)) = |\lambda| \) if \( \text{Re}\lambda \leq 0 \),
\[
\sum_{\lambda \in \sigma_d(-\Delta+V), \text{Re}\lambda \leq 0} \frac{\text{dist}(\lambda, [0, \infty))^p}{|\lambda|^{d/2}} f\left(-\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)\right)
= \sum_{\lambda \in \sigma_d(-\Delta+V), \text{Re}\lambda \leq 0} |\lambda|^{p-d/2}f(0) \leq C_{d,p}3^pf(0)\|V\|_{L^p}^p,
\]
which concludes the proof. \( \square \)

In dimension \( d = 1 \), Theorem 1 is optimal in the sense that if \( f \) is no longer integrable, then the corresponding Lieb–Thirring inequality is false.

**Theorem 2.** Let \( d = 1 \) and \( p \geq 1 \). Let \( f : [0, \infty) \to (0, \infty) \) be a continuous, non-increasing function with \( \int_0^\infty f(s) \, ds = \infty \). Then
\[
\sup_{V \in L^p(\mathbb{R})} \frac{\sum_{\lambda \in \sigma_d(-\Delta+V)} \text{dist}(\lambda, [0, \infty))^p}{|\lambda|^{d/2}} f\left(-\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)\right)\|V\|_{L^p}^p = \infty.
\]

**Remark 3.** Note that Theorem 2 holds for \( p \geq 1 \) but Theorem 1 is proved for \( p \geq 3/2 \) in dimension \( d = 1 \). It is still an open question how to optimise the Lieb-Thirring type inequalities in \( d = 1 \) for \( p \in (1, 3/2) \), and in higher dimensions. In particular, it is not known whether Theorem 1 is optimal for \( d \geq 2 \).

**Proof of Theorem 2.** We use a slight modification of the example in \[3\] Proposition 10 with potential \( V_h(x) = ih\chi_{[-1,1]}(x) \) where \( h > 0 \) with \( h \to \infty \). In the proof of \[3\] Proposition 10], asymptotic formulas were derived for all eigenvalues \( \lambda_j = \mu_j^2 + ih \) with \( \alpha \log h \leq \text{Im}\mu_j \leq \alpha \log h \), \( \text{Re}\mu_j \leq -h^\gamma|\text{Im}\mu| \) where \( \alpha, \beta, \gamma > 0 \) are chosen to be \( h \)-independent constants with \( \gamma < 2\alpha < 2\beta < 1 \). We now want to allow \( \alpha, \gamma \) to be \( h \)-dependent \( (\beta \in (0, 1/2) \) still constant), with \( 0 < \gamma(h) < 2\alpha(h) \to 0 \) as \( h \to \infty \), but sufficiently slowly so that the asymptotics still hold. It turns out that if we still have \( h^{-\gamma(h)} \to 0 \), then
\[
\mu_j = \frac{\pi}{4}(7 - 8j) + i\log \left(\frac{\pi(8j - 7)}{2\sqrt{h}}\right) + O(h^{-\gamma(h)})
\]
uniformly for all integers \( j \) with \( h^{\alpha(h)+1/2} j \leq h^{\beta+1/2} \). Hence we set \( \varepsilon(h) = h^{-\gamma(h)} \) (converging to zero arbitrarily slowly) and \( \alpha(h) = \gamma(h) \). The latter implies \( h^{\alpha(h)+1/2} = \frac{h^{1/2}}{\varepsilon(h)} \). Analogously we then obtain that there exists
\( h_0 > 0 \) such that for all \( h > h_0 \) and \( \frac{h^{1/2}}{\varepsilon(h)} \leq j \leq h^{\beta + 1/2} \), we have the estimates

\[
\text{dist}(\lambda_j, [0, \infty)) = \text{Im} \lambda_j > \frac{h}{2}, \quad \pi j \leq |\lambda_j|^{1/2} \leq 2\pi j.
\]

(8) This implies, for all \( h > h_0 \),

\[
\sum_{\lambda \in \sigma_d(-\Delta + V_h)} \frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|^{1/2}} f\left(-\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)\right) \geq \sum_{h^{1/2}/\varepsilon(h) \leq j \leq h^{\beta + 1/2} \frac{1}{\varepsilon(h)}} \frac{1}{j} f\left(-\log \left(\frac{h}{2(2\pi j)^2}\right)\right).
\]

Note that \(-\log \left(\frac{h}{2(2\pi j)^2}\right) = \log \left(\frac{j^2}{h}\right) + \log(8\pi^2)\). Since \( f \) is non-increasing, with \( f(s) \leq f(0) \) for \( s \geq 0 \), we see that

\[
\sum_{h^{1/2}/\varepsilon(h) \leq j \leq h^{\beta + 1/2} \frac{1}{\varepsilon(h)}} \frac{1}{j} f\left(\log \left(\frac{j^2}{h}\right) + \log(8\pi^2)\right) \geq \int_{h^{\beta + 1/2} \frac{1}{\varepsilon(h)}} \frac{1}{x} f\left(\log \left(\frac{x^2}{h}\right) + \log(8\pi^2)\right) \, dx - 2f(0)
\]

where we have used the substitution \( s = \log \left(\frac{x^2}{h}\right) \). Note that \( 2\beta \log(h) \to \infty \). Since \( f \) is not integrable, we can choose \( \varepsilon(h) \to 0 \) so slowly (i.e. \(-2\log(\varepsilon(h)) \to \infty \) so slowly) that \( \int_{-2\log(\varepsilon(h))}^{2\beta \log(h)} f(s) \, ds \to \infty \). Now, with \( \|V_h\|_{L^p}^p = 2h^p \), we arrive at

\[
\sum_{\lambda \in \sigma_d(-\Delta + V_h)} \frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|^{1/2}} f\left(-\log \left(\frac{\text{dist}(\lambda, [0, \infty))}{|\lambda|}\right)\right) \geq \left(\frac{1}{2}\right)^p \frac{1}{4\pi} \left(\int_{-2\log(\varepsilon(h))}^{2\beta \log(h)} f(s) \, ds - 2f(0)\right) \to \infty, \quad h \to \infty.
\]

This completes the proof. \( \square \)

Finally we also prove that in one dimension, the \( t \)-dependence in (4) is optimal.
Theorem 4. Let $d = 1$ and $p \geq 1$. Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ a continuous function with $\varphi(t) = o(t^{-p})$ as $t \rightarrow 0$. Then
\[
\limsup_{t \rightarrow 0} \sup_{V \in L^p(\mathbb{R})} \frac{\sum_{\lambda \in \sigma_d(-\Delta + V), \text{Im } \lambda \geq t \text{Re } \lambda} |\lambda|^{p-1/2} \varphi(t) \|V\|_{L^p}^p}{\|V\|_{L^p}^p} = \infty.
\]

Proof. We use the same potentials $V_h$ as in the proof of Theorem 2. Recall that $\|V_h\|_{L^p}^p = 2h^p$. Again we use the index $j \in \mathbb{Z}$ with $\frac{h^{1/2}}{\varepsilon(h)} \leq j \leq h^{\beta+1/2}$ to enumerate the eigenvalues $\lambda_j$ with uniform estimates in (8); we take $\varepsilon(h) \rightarrow 0$ with $1/\varepsilon(h) = o(h^\beta)$. We take $t$ to be $h$-dependent as $t(h) = \frac{h^{-\beta}}{\varepsilon(h)}$. Then one can check that each of these $\lambda_j$ satisfies $|\text{Im } \lambda_j| \geq t(h)|\lambda_j| \geq t(h)\text{Re } \lambda_j$. Therefore,
\[
\sum_{\lambda \in \sigma_d(-\Delta + V_h), \text{Im } \lambda \geq t(h) \text{Re } \lambda} |\lambda|^{p-1/2} \geq \sum_{\frac{h^{1/2}}{\varepsilon(h)} \leq j \leq h^{\beta+1/2}} (\pi j)^2(p-1/2)
\]
In the limit $h \rightarrow \infty$, the sum on the right hand side is of the same order as
\[
\int_{\frac{h^{1/2}}{\varepsilon(h)}}^{h^{\beta+1/2}} f^{2p-1} \frac{h^{2\beta p} - \frac{h^p}{\varepsilon(h)^{2p}}}{2p}.
\]
Thus, up to a multiplicative constant, the right hand side of (9) is asymptotically
\[
\frac{h^{2\beta p}}{\varphi(t(h))} = \frac{1}{(8\pi^2)^p} \frac{t(h)^{-p}}{\varphi(t(h))} \rightarrow \infty,
\]
where we used the assumption $\varphi(t) = o(t^{-p})$ as $t \rightarrow 0$. This proves the claim. \qed

3. Examples

In this section we verify the assumptions of Theorem 1 for a few examples. If we combine the below part (i) with Theorem 1, we recover [4, Corollary 3], which was the hitherto best known result. Parts (ii)–(v) yield improvements of this result.

We use the notation $g^{0n}$ for the $n$-th iterated function $g$, i.e. $g^{00}(s) = s$, $g^{01}(s) = g(s)$, $g^{02}(s) = g(g(s))$ etc. We also use the super-logarithm (to the basis $e$) defined for an $s \in \mathbb{R}$ by $\text{log}(s) := \min\{n \in \mathbb{N}_0 : \log^{0n}(s) \leq 1\}$.

Proposition 5. The following continuous, non-increasing functions $f : [0, \infty) \rightarrow (0, \infty)$ satisfy $\int_0^\infty f(s) \, ds < \infty$:

(i) $f(s) = e^{-\varepsilon s}$;
(ii) $f(s) = \frac{1}{s^{1+\varepsilon}}$ for $\varepsilon > 0$;
(iii) $f(s) = \begin{cases} 1/e & s \leq e, \text{ for } \varepsilon > 0; \\ \frac{1}{s \log(s)^{1+\varepsilon}} & s > e, \end{cases}$
Remark 6. The functions in (i)–(iv) are the piecewise derivatives of the following respective functions:

\[ f(s) = \begin{cases} \frac{1}{\exp^{en}(1)}, & \text{slog}(s) \leq n \text{ (i.e. } s \leq \exp^{en}(1)), \\ \left(\prod_{j=0}^{n-1} \frac{1}{\log^{en}(s)^{\epsilon j+1}}\right) \frac{1}{\log^{en}(s)^{\epsilon s}}, & \text{slog}(s) \geq n + 1 \text{ (i.e. } s > \exp^{en}(1)), \end{cases} \]

for \( n \in \mathbb{N} \) and \( \epsilon > 0 \) (where \( n = 1 \) corresponds to the case (iii));

\[ f(s) = \left(\prod_{j=0}^{\text{slog}(s)-1} \frac{1}{\log^{en}(s)}\right) \frac{1}{(\text{slog}(s)-1+\log\text{clog}(s)/(s))^{1+\epsilon}}, \]

\( s \leq e, \) \( s > e, \) for \( \epsilon > 0. \)

In each case (i)–(v), if we set \( \epsilon = 0 \) then \( \int_{0}^{\infty} f(s) \, ds = \infty. \)

Remark 6. The functions in (i)–(iv) are the piecewise derivatives of the following respective functions:

(i) \( F(s) = -\frac{1}{s} \exp^{-\epsilon s}, \)
(ii) \( F(s) = -\frac{1}{s} \exp^{-\epsilon s}, \)
(iii) \( F(s) = -\frac{1}{\epsilon \log(s)^{\epsilon}} \) for \( s > e; \)
(iv) \( F(s) = -\frac{1}{\epsilon \log^{en}(s)^{\epsilon}} \) for \( s > \exp^{en}(1); \)
(v) \( F(s) = -\frac{1}{\epsilon} \frac{1}{(\text{slog}(s)-1+\log\text{clog}(s)/(s))^{\epsilon}} \) for \( s > e. \)

In (v), note that \( \log\text{clog}(s) \in (0,1], \) so asymptotically we have \( F(s) \sim -\frac{1}{\epsilon (\text{slog}(s))^{\epsilon}} \) as \( s \to \infty. \) More examples can be generated by taking \( F \) to be a piecewise continuous function with \( F(s) \nearrow 0 \) very slowly as \( s \to \infty, \) and then take \( f = F' \) piecewise. There exists no function with the slowest convergence, as if \( F(s) = -1/g(s) \) with \( g(s) \nearrow \infty \) slowly, then \( g(g(s)) \nearrow \infty \) even slower and hence \( -1/g(g(s)) \nearrow 0 \) even slower than \( F(s). \)

Proof of Proposition 5. By Remark 6 we can write \( \int_{0}^{\infty} f(s) \, ds = \int_{0}^{a} f(s) \, ds + F(a) \) for any \( a \in (0, \infty). \) This proves that \( f \) is integrable. It is left to show that if \( \epsilon = 0, \) then \( \int_{0}^{\infty} f(s) \, ds = \infty. \) The corresponding antiderivatives (modulo additive constants) are the following functions:

(i) \( F(s) = s; \)
(ii) \( F(s) = \log(s); \)
(iii) \( F(s) = \log^{o2}(s) \) for \( s > e; \)
(iv) \( F(s) = \log^{o(n+1)}(s) \) for \( s > \exp^{en}(1); \)
(v) \( F(s) = \log(\log(s) - 1 + \log\text{clog}(s)/(s)) \) for \( s > e. \)

Since each of these functions satisfies \( \lim_{s \to \infty} F(s) = \infty, \) the corresponding \( f = F' \) is not integrable.

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