Abstract. We establish a duality between the category of involutive bisemilattices and the category of semilattice inverse systems of Stone spaces, using Stone duality from one side and the representation of involutive bisemilattices as Płonka sum of Boolean algebras, from the other. Furthermore, we show that the dual space of an involutive bisemilattice can be viewed as a GR space with involution, a generalization of the spaces introduced by Gierz and Romanowska equipped with an involution as additional operation.

Keywords: Duality, Involutive bisemilattice, Stone space, Płonka sum, Paraconsistent weak Kleene.

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1. Introduction

It is a common trend in mathematics to study natural dualities for general algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [41]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [34]. The two above mentioned are the prototypical examples of dualities obtained via dualizing objects and will be both recalled and constructively used in the present work.

These kind of dualities have an intrinsic value: they are indeed a way of describing the very same mathematical object from two different perspectives, the target category and its dual. More generally, dualities between algebraic structures and corresponding topological spaces may open the way to applications as algebraic problems can possibly be translated into topological ones, or new insights can be obtained via the representation of a particular algebra as an algebra of continuos functions over a certain space (for a more detailed exposition of applications see [10,11]).
The starting point of our analysis is the duality established by Gierz and Romanowska [14] between distributive bisemilattices and compact totally disconnected partially ordered left normal bands with constants, which we refer to as GR spaces. The relevance of the result lies mainly in using the technique of Plonka sums as an essential tool for proving the duality.

Our aim is to provide a duality between the categories of involutive bisemilattices and those topological spaces, here christened as GR spaces with involution. The former consists of a class of algebras introduced and extensively studied in [4] as algebraic counterpart for paraconsistent weak Kleene logic, introduced in [16] from an idea by Kleene [21] and extensively studied in [1, 8, 27, 39] as the logic of Non-sense. The logical interests around these structures is relatively recent; on the other hand, it is easily checked that involutive bisemilattices, as introduced in [4], are equivalent to the regularization of the variety of Boolean algebras, axiomatized by Plonka [31, 32]. For this reason, involutive bisemilattices are strictly connected to Boolean algebras, as they are representable as Plonka sums over a direct system of Boolean algebras. The connection between logics and Plonka sums is currently under investigation by one of the authors of this paper. In particular, the technique of Plonka sums can be extended to logical matrices so to provide algebraic counterparts to the variable inclusion companion of a given logic [5]. These logics features the presence of a non-sensical, infectious truth-value [9, 42]. For this reason, they are applied in modeling reasonings with non-existing objects [36], computer-programs affected by errors [12] (see also [26]) as well as recent developments in the theory of truth [43]. An algebraic approach to these kind of logics can also be found in [13]. It has also been argued, for instance by Williamson [44], that three-valued logics can be applied to the problem of vagueness.

The present work consists of two main results. On one hand, taking advantage of the Plonka sums representation in terms of Boolean algebras and Stone duality, we are able to describe the dual spaces of an involutive bisemilattice as semilattice inverse systems of Stone spaces (Theorem 4.6). On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with involution as an additional operation (Theorem 4.19). As a byproduct of our analysis we get a topological description of *semilattice inverse systems* of Stone spaces (Corollary 4.20).

The paper is structured as follows. In Section 2 we summarize all the necessary notions and known results about bisemilattices, Gierz and Romanowska duality and involutive bisemilattices. In Section 3 we introduce the categories of *semilattice* direct and inverse systems, proving that they are dually equivalent, when constructed out of dually equivalent categories.
In Section 4, we introduce GR spaces with involution and prove the main results. Finally, in Section 5 we make some considerations about categories admitting both topological duals and a representation in terms of Plonka sums. By using Priestley duality, we then extend our results to the category of distributive bisemilattices.

2. Preliminaries

A **distributive bisemilattice** is an algebra $A = \langle A, +, \cdot \rangle$ of type $\langle 2, 2 \rangle$ such that both $+$ and $\cdot$ are idempotent, associative and commutative operations and, moreover, $+$ ($\cdot$ respectively) distributes over $\cdot$ ($+$ respectively). Distributive bisemilattices have been introduced by Plonka [29], who called them “quasi lattices”; nowadays these structures are studied in a more general setting under the name of Birkhoff systems (see [17, 18]). Throughout the paper we will refer to these algebras simply as **bisemilattices**. Observe that every distributive lattice is an example of bisemilattice and every semilattice is a bisemilattice, where the two operations have a common result. Any bisemilattice induces two different partial orders, namely $x \leq y$ iff $x \cdot y = x$ and $x \leq_+ y$ iff $x + y = y$.

**Examples 2.1.** The 3-element algebra $3 = \langle \{0, 1, \alpha\}, +, \cdot \rangle$, whose operations are defined by the so-called **weak Kleene tables** (see below), is the main example of bisemilattice:

\[
\begin{array}{ccc}
\cdot & 0 & \alpha & 1 \\
0 & 0 & \alpha & 0 \\
\alpha & \alpha & \alpha & \alpha \\
1 & 0 & \alpha & 1 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 0 & \alpha & 1 \\
0 & 0 & \alpha & 1 \\
\alpha & \alpha & \alpha & \alpha \\
1 & 1 & \alpha & 1 \\
\end{array}
\]

The two partial orders may be represented by the following Hasse diagrams:

\[
\begin{array}{c}
\alpha \\
1 \\
\leq_+ \\
0
\end{array}
\quad
\begin{array}{c}
1 \\
\leq . \\
0 \\
\alpha
\end{array}
\]
The algebra 3 generates the variety of bisemilattices (see [20]).

A duality for bisemilattices has been established in [14], by using 3 as dualizing object. We recall here all the notions needed to state the main result.

A left normal band is an idempotent semigroup \( \langle A, * \rangle \) satisfying the additional identity \( x * (y * z) = x * (z * y) \), which is a weak form of commutativity. A left normal band can equipped with a partial order.

**Definition 2.2.** A partially ordered left normal band is an algebra \( \langle A, *, \leq \rangle \) such that

(i) \( \langle A, * \rangle \) is a left normal band
(ii) \( \langle A, \leq \rangle \) is a partially ordered set
(iii) if \( x \leq y \) then \( x * z \leq y * z \) and \( z * x \leq z * y \)
(iv) \( x * y \leq x \)

In any partially ordered left normal band it is possible to define a second partial order via \(*\) and \(\leq\): \(a \sqsubseteq b \) iff \(a * b \leq b \) and \(b * a = b\). A partially ordered left normal band may be equipped with constants.

**Definition 2.3.** A partially ordered left normal band with constants is an algebra \( \langle A, *, \leq, c_0, c_1, c_\alpha \rangle \) such that \( \langle A, *, \leq \rangle \) is a partially ordered left normal band and \(c_0, c_1\) and \(c_\alpha\) are constants satisfying

(1) \( x * c_\alpha = c_\alpha * x = c_\alpha \)
(2) \( x * c_0 = x * c_1 = x \)
(3) \( c_0 \sqsubseteq x \leq c_1 \) and \( c_\alpha \leq x \sqsubseteq c_\alpha \)
(4) if \( c_0 * x = c_1 * x \) then \( x = c_\alpha \)

**Definition 2.4.** A GR space is a structure \( \langle A, *, \leq, c_0, c_1, c_\alpha, \tau \rangle \), such that \( \langle A, *, \leq, c_0, c_1, c_\alpha \rangle \) is a partially ordered left normal band with constants and \( \tau \) is a topology making \(*\): \( A \times A \to A \) a continuous map and \( \langle A, \leq, \tau \rangle \) is a compact totally order disconnected space.\(^1\)

**Examples 2.5.** The support set of 3, namely \( \{0, 1, \alpha\} \), equipped with the discrete topology, where \( \leq \equiv \leq \cdot \), \( c_0 = 0 \), \( c_1 = 1 \), \( c_\alpha = \alpha \) and \(*\) is defined as follows:

---

\(^1\)A topological space is totally order disconnected if (1) \( \{(a, b) \in A \times A: a \leq b\} \) is closed; (2) if \( a \not\leq b \) then there is an open and closed (clopen) lower set \( U \) such that \( b \in U \) and \( a \not\in U \).
A Duality for Involutive Bisemilattices

\[
a \ast b = \begin{cases} 
a & \text{if } b \neq \alpha \\
b & \text{otherwise,}
\end{cases}
\]

is a GR space (it is not difficult to check that operation \(a \ast b = a + a \cdot b = a \cdot (a + b)\) and that the induced order \(\sqsubseteq\) coincides with \(\leq_+\)).

We call \(\mathcal{DB}\) the concrete category of bisemilattices (whose morphisms are homomorphisms of bisemilattices) and \(\mathcal{GR}\) the category of GR spaces (whose morphisms are continuous maps preserving \(\ast\), constants and the order). The main result in [14] is the following:

**Theorem 2.6.** [14, Theorem 7.5] The categories \(\mathcal{DB}\) and \(\mathcal{GR}\) are dual to each other under the invertible functor \(\text{Hom}_b(-, 3) : \mathcal{DB} \to \mathcal{GR}\) and its inverse \(\text{Hom}_{GR}(-, 3) : \mathcal{GR} \to \mathcal{DB}\).

In detail, given a bisemilattice \(S\), its dual GR space is \(\hat{S} = \text{Hom}_b(S, 3)\), i.e. the space of the homomorphisms (of bisemilattices) from \(S\) to \(3\). Analogously, if \(A\) is a GR space, then the dual is given by \(\hat{A} = \text{Hom}_{GR}(A, 3)\), the bisemilattice of morphisms of \(\mathcal{GR}\) from \(A\) to \(3\).

The isomorphism between \(S\) and \(\hat{S}\) is given by:

\[
\varepsilon_S : S \to \hat{S}, x \mapsto \varepsilon_S(x), \varepsilon_S(x)(\varphi) = \varphi(x),
\]

for every \(x \in S\) and \(\varphi \in \hat{S}\).

Analogously, for \(A\) and \(\hat{A}\), the isomorphism is given by:

\[
\delta_A : A \to \hat{A}, x \mapsto \varepsilon_A(x), \varepsilon_A(x)(\varphi) = \varphi(x),
\]

for every \(x \in A\) and \(\varphi \in \hat{A}\).

The class of involutive bisemilattices has been introduced in [4] as the most suitable candidate to be the algebraic counterpart of PWK logic.

**Definition 2.7.** An involutive bisemilattice is an algebra \(B = \langle B, +, \cdot, ', 0, 1 \rangle\) of type \((2, 2, 1, 0, 0)\) satisfying:

11. \(x + x = x\);
12. \(x + y = y + x\);
13. \(x + (y + z) = (x + y) + z\);
14. \((x')' = x\);
15. \(x \cdot y = (x' + y')'\);
16. \(x \cdot (x' + y) = x \cdot y\);
17. \(0 + x = x\);
18. \(1 = 0'\).
We denote the variety of involutive bisemilattices by $\text{IBSL}$. Every involutive bisemilattice has, in particular, the structure of a join semilattice with zero, in virtue of axioms (I1)–(I3) and (I7). More than that, it is possible to prove [4, Proposition 20] that $\cdot$ distributes over $+$ and vice versa, therefore the reduct $\langle B, +, \cdot \rangle$ is a bisemilattice. Notice that, in virtue of axioms (I5) and (I8), the operations $\cdot$ and 1 are completely determined by $+, +',$ and 0. It is not difficult to check that every involutive bisemilattice has also the structure of a meet semilattice with 1, and that the equations $x + y = (x' \cdot y)'$, $x + y = x + (x' \cdot y)$ are satisfied. There are different equivalent ways to define involutive bisemilattices: it is not difficult to check that $\text{IBSL}$ corresponds to the regularization of the variety of Boolean algebras described in [32].

**Examples 2.8.** Every Boolean algebra, in particular the 2-element Boolean algebra $B_2$, is an involutive bisemilattice. Also every semilattice with zero (we call $S_2$ the two element member of the class), endowed with the identity as its unary operation is an example of involutive bisemilattice. In this case, notice that the two constants realize the same element. The most prominent example of involutive bisemilattice is the 3-element algebra $\text{WK}$, which is obtained by expanding the language of $\mathbf{3}$ with an involution behaving as follows:

$$
\begin{array}{c|c|c}
\prime & 1 & 0 \\
\hline
1 & \alpha & \alpha \\
0 & \alpha & 1 \\
\end{array}
$$

Upon considering the partial order $\leq$ induced by the product in its bisemilattice reduct, it becomes a 3-element chain with $\alpha$ as its bottom element. $B_2$, $S_2$ and $\text{WK}$ can be represented by means of the following Hasse diagrams (the dashes represent the order, while the arrows represent the negation):

$B_2 = \begin{array}{c|c}
1 & \alpha \\
\hline
0 & \alpha \\
\end{array}$

$S_2 = \begin{array}{c|c}
0 & 1 \\
\hline
\alpha & \alpha \\
\end{array}$

$\text{WK} = \begin{array}{c|c}
1 & \alpha \\
\hline
0 & \alpha \\
\end{array}$

It is not difficult to verify that $B_2$ is a subalgebra of $\text{WK}$, while $S_2$ is a quotient.
The subdirectly irreducible members of the variety \( \mathcal{IBSL} \) coincide with the above mentioned ones, namely \( \mathbf{WK}, \mathbf{S}_2 \) and \( \mathbf{B}_2 \) [4] (this follows also from general result in [22]). Therefore the algebra \( \mathbf{WK} \) generates the variety.

### 3. The Categories of Semilattice Inverse and Direct Systems

The concepts that we are going to introduce in this section have been already treated before, though in a different setting and using a different language (see [37, 38]). For our purposes we need to strengthen the concepts of inverse and direct system of a category, introducing the notions of semilattice inverse and semilattice direct systems. For sake of simplicity, we opt for presenting these topics following the trend in algebraic topology (see [24] for details).

**Definition 3.1.** Let \( \mathcal{C} \) be an arbitrary category, a semilattice inverse system in the category \( \mathcal{C} \) is a triple \( \mathcal{X} = \langle X_i, p_{ii'}, I \rangle \) such that

(i) \( I \) is a semilattice with zero;
(ii) for each \( i \in I \), \( X_i \) is an object in \( \mathcal{C} \);
(iii) \( p_{ii'} : X_{i'} \to X_i \) is a morphism of \( \mathcal{C} \), for each pair \( i \leq i' \), satisfying that \( p_{ii} \)

is the identity in \( X_i \) and such that \( i \leq i' \leq i'' \) implies \( p_{ii'} \circ p_{ii''} = p_{ii''} \).

\( I \) is called the **index set** of the system \( \mathcal{X} \), \( X_i \) are the **terms** and \( p_{ii'} \) are referred to as **bonding morphisms** of \( \mathcal{X} \). For convention, we indicate with \( \vee \) the semilattice operation on \( I \) and \( \leq \) the induced order.

The only difference making an inverse system a semilattice inverse system is the requirement on the index set to be a semilattice with lower bound instead of a directed preorder.

**Definition 3.2.** Given two semilattice inverse systems \( \mathcal{X} = \langle X_i, p_{ii'}, I \rangle \) and \( \mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle \), a morphism between \( \mathcal{X} \) and \( \mathcal{Y} \) is a pair \( (\varphi, f_j) \) such that

(i) \( \varphi : J \to I \) is a semilattice homomorphism;
(ii) for each \( j \in J \), \( f_j : X_{\varphi(j)} \to Y_j \) is a morphism in \( \mathcal{C} \), such that whenever \( j \leq j' \), then the diagram in Figure 1 commutes.

Notice that, for morphisms of semilattice inverse systems, the assumption that \( \varphi : J \to I \) is a (semilattice) homomorphism implies that whenever \( j \leq j' \) then \( \varphi(j) \leq \varphi(j') \). Given three semilattice inverse systems \( \mathcal{X} = \langle X_i, p_{ii'}, I \rangle \), \( \mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle \), \( Z = \langle Z_k, r_{kk'}, K \rangle \), the composition of morphisms is defined in the same way as for inverse systems.

\(^2\)The constants zero stands here for the least element of the induced order.
Lemma 3.3. The composition of morphisms between semilattice inverse system is a morphism.

Proof. Let \((\varphi, f_j): \mathcal{X} \to \mathcal{Y}, (\psi, g_k): \mathcal{Y} \to \mathcal{Z}\), then \((\chi, h_k) = (\psi, g_k)(\varphi, f_j): \mathcal{X} \to \mathcal{Z}\) is \(\chi = \varphi \psi, h_k = g_k f_{\chi(k)}\). \(\chi\) is the composition of two (semilattice) homomorphisms, hence it is a semilattice homomorphism. The claim follows from the commutativity of the following diagram (we omitted the indexes for the maps \(p, q, r, f, g\) to make the notation less cumbersome)

Proposition 3.4. Let \(\mathcal{C}\) an arbitrary category. Then Sem-inv-\(\mathcal{C}\) is the category whose objects are semilattice inverse systems in \(\mathcal{C}\) with morphisms as defined above.
Proof. The composition of morphisms between systems is associative and the identity morphism is \((1_I, 1_i)\), where \(1_I : I \to I\) is the identity homomorphism on \(I\) and \(1_i : X_i \to X_i\) is the identity morphism in the category \(\mathcal{C}\).

The category of semilattice direct systems of a given category \(\mathcal{C}\) is obtained by reversing morphisms of Sem-inv-\(\mathcal{C}\) as follows:

Definition 3.5. Let \(\mathcal{C}\) be an arbitrary category. A semilattice direct system in \(\mathcal{C}\) is a triple \(X = \langle X_i, p_{ii'}, I \rangle\) such that

1. \(I\) is a semilattice with zero;
2. \(X_i\) is an object in \(\mathcal{C}\), for each \(i \in I\);
3. \(p_{ii'} : X_i \to X_{i'}\) is a morphism of \(\mathcal{C}\), for each pair \(i \leq i'\), satisfying that \(p_{ii}\) is the identity in \(X_i\) and such that \(i \leq i' \leq i''\) implies \(p_{i'i''} \circ p_{ii'} = p_{ii''}\).

We call \(I\), \(X_i\), the index set and the terms of the direct system, respectively, while we refer to \(p_{ii'}\) as transition morphisms to stress the crucial difference with respect to inverse systems.

A morphism between two semilattice direct systems \(X\) and \(Y\) is a pair \((\varphi, f_i) : X \to Y\) s. t.

1. \(\varphi : I \to J\) is a semilattice homomorphism
2. \(f_i : X_i \to Y_{\varphi(i)}\) is a morphism of \(\mathcal{C}\), making the diagram in Figure 2 commutative, for each \(i, i' \in I\), \(i \leq i'\):

![Figure 2](image-url)

The composition of two morphisms is defined as \((f_i, \varphi)(g_j, \psi) = (h_i, \chi)\),

\[
\chi = \psi \varphi, \quad h_i = g_{\varphi(i)} f_i : X_i \to Z_{\chi(i)}.
\]
It is easily verified that the composition \((h, \chi)\) is a morphism and it is associative and that the element \((1_I, 1_i)\), where \(1_I : I \to I\) is the identity map on \(I\) and \(1_i : X_i \to X_i\) is the identity morphism in \(C\), is the identity morphism between semilattice direct systems. Therefore semilattice direct systems form a category which we will call \(\text{Sem-dir-} C\).

**Remark 3.6.** It is easily proven that, if \(C\) and \(D\) be dually equivalent categories, then \(\text{Sem-dir-} C\) and \(\text{Sem-inv-} D\) are dually equivalent. In detail, the contravariant functors \(\mathcal{F}\) and \(\mathcal{G}\)

![Diagram](image)

yielding the duality between \(C\) and \(D\) can be lifted to semilattice systems,

![Diagram](image)

upon defining

\[
\tilde{\mathcal{F}}(X) := \langle \mathcal{F}(X_i), \mathcal{F}(p_{ii'}), I \rangle \quad \tilde{\mathcal{F}}(\varphi, f_i) := (\varphi, \mathcal{F}(f_i)),
\]

where \(X = \langle X_i, p_{ii'}, I \rangle\) is an object and \((\varphi, f_i)\) a morphism in the category \(\text{Sem-dir-} C\). \(\tilde{\mathcal{G}}\) is defined analogously.

The above Remark somehow resembles *semilattice-based dualities* established by Romanowska and Smith in [37,38], where the authors essentially show how to lift a duality between two categories, in particular an algebraic category and its dual representation spaces, to a duality involving the correspondent semilattice representations. The substantial difference is that Romanowska and Smith indeed consider, from one side, the semilattice sum of an algebraic category but, on the other, the semilattice representation of the dual spaces, and thus the duality, is obtained by *dualizing* the semilattice of the index sets (relying on the duality due to Hofmann, Mislove and Stralka [19]). Our approach uses the very same index set on both categories making the duality, on one hand, easy to lift and, on the other hand, useful for applications (see e.g. [6]).
4. The Category of Involutive Bisemilattices and its Dual

Plonka introduced [28,30] a construction to build algebras out of semilattice direct systems of algebras.

**Definition 4.1.** If $A$ is a semilattice direct system of algebras of a fixed type $\nu$, then the *Plonka sum* over $A$ is the algebra $P(A) = \bigcup A_i$, whose universe is the disjoint union of algebras in $A$ and the operations $g^P$ are defined as follows: for every $n$-ary $g \in \nu$, and $a_1, \ldots, a_n \in \bigcup A_i$, where $n \geq 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \lor \cdots \lor i_n$ and define

$$g^P(a_1, \ldots, a_n) = g^A_j(\varphi_{i_1j}(a_1), \ldots, \varphi_{i_nj}(a_n)).$$

In case $\nu$ contains constants, then, for every constant $g \in \nu$, we define $g^A = g^{A_{i_0}}$.

Involutive bisemilattices, as well as bisemilattices, admits a representation in terms of Plonka sums.

**Theorem 4.2.** ([4, Theorem 46])

1. If $A$ is a semilattice direct system of Boolean algebras, then the $P(A)$ is an involutive bisemilattice.
2. If $B$ is an involutive bisemilattice, then $B$ is isomorphic to the Plonka sum over a semilattice direct system of Boolean algebras.

The above result states that every involutive bisemilattice admits a unique representation as Plonka sum of Boolean algebras. We summarize here the categories we are dealing with:

| Category      | Objects                      | Morphisms                   |
|---------------|------------------------------|-----------------------------|
| $\mathcal{B}A$ | Boolean Algebras             | Homomorphisms of B.A.       |
| $\mathcal{IBS}L$ | Involutive bisemilattices    | Homomorphisms of I.B.       |
| Sem-dir-$\mathcal{B}A$ | semilattice direct systems of B.A. | Homomorphisms of s.d.s.    |
| $\mathcal{S}A$    | Stone spaces                 | continuous maps             |
| Sem-inv-$\mathcal{S}A$ | semilattice inverse s. of Stone sp. | Homomorphisms of s.i.s.    |

Theorem 4.2 states that the objects of the category $\mathcal{IBS}L$ are isomorphic to the objects of the category Sem-dir-$\mathcal{B}A$. Actually, the two categories are also equivalent (see Theorem 4.5).

In order to establish this, we need the following lemmata.

**Lemma 4.3.** Let $A = \langle A_i, p_{ii'}, I \rangle$ and $B = \langle B_j, q_{jj'}, J \rangle$ be semilattice direct systems of Boolean algebras and $h \in \text{Hom}(P(A), P(B))$, then for any $i \in I$ there exists a $j \in J$ such that...
\[ (1) \ h(A_i) \subseteq B_j; \]
\[ (2) \ h|_{A_i} \text{ is a Boolean homomorphism from } A_i \text{ into } B_j. \]

**Proof.** (1) As first notice that, from the construction of Plonka sums, we have that for any \( x \in A_i \), also \( x' \in A_i \). Consequently, for any \( h(x) \in B_j \), for a certain \( j \in J \), then also \( h(x)' \in B_j \). Let \( a \in A_i \) for some \( i \in I \), then there exists a \( j \in J \) such that \( h(a) \in B_j \). Therefore \( h(0_{A_i}) = h(a \land a') = h(a) \land h(a') = h(a) \land h(a)' = 0_{B_j} \), where the last equality holds since \( h(a) \) and \( h(a)' \) belong to the same Boolean algebra \( B_j \). Similarly, \( h(1_{A_i}) = h(a \lor a') = h(a) \lor h(a)' = 1_{B_j} \).

We now have to prove that for any \( a \in A_i \), with \( a \neq 0_{A_i} \) we have that \( h(a) \in B_j \). Suppose, by contradiction, that \( a \in A_i \), and \( h(a) \in B_k \), with \( j \neq k \). Then \( 0_{B_j} = h(0_{A_i}) = h(a \land a') = h(a) \land h(a') = h(a) \land h(a)' = 0_{B_k} \), which is impossible, as, by construction \( B_j \cap B_k = \emptyset \), hence, necessarily \( h(A_i) \subseteq B_j \).

(2) follows from the fact that \( h \) preserves joins, meets and complements by definition and we already proved that \( h(0_{A_i}) = 0_{B_j} \) and \( h(1_{A_i}) = 1_{B_j} \).

---

Theorem 4.2 together with Lemma 4.3 state that \( \mathcal{IBS\ell} \)–homomorphisms are nothing but homomorphisms between the correspondent (unique) Plonka sum representations. The statement of Lemma 4.3 can be exposed more precisely saying that there exists a map \( \varphi : I \rightarrow J \) such that for every homomorphism \( h : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \), \( h(A_i) \subseteq B_{\varphi(i)} \). It is not difficult to prove that such map is actually a semilattice homomorphism.

**Lemma 4.4.** Let \( A = \langle A_i, p_{ii'}, I \rangle \) and \( B = \langle B_j, q_{jj'}, J \rangle \) be semilattice direct systems of Boolean algebras, \( h \in \text{Hom}(\mathcal{P}(A), \mathcal{P}(B)) \) and \( \varphi : I \rightarrow J \) such that \( h(A_i) \subseteq B_{\varphi(i)} \). Then \( \varphi_h \) is a semilattice homomorphism.

**Proof.** Let \( a_1 \in A_i \) and \( a_2 \in A_i' \), with \( i, i' \in I \); by definition of \( \mathcal{P}(A) \), \( a_1 \land a_2 \in A_{i \lor i'} \) and \( h(a_1) \in B_{\varphi_h(i)} \), \( h(a_2) \in B_{\varphi_h(i')} \), then \( h(a_1 \land a_2) = h(a_1) \land h(a_2) \in B_{\varphi_h(i \lor i')} \). But since \( h(a_1 \land a_2) \in B_{\varphi_h(i \lor i')} \), then necessarily \( \varphi_h(i \lor i') = \varphi_h(i) \lor \varphi_h(i') \), i.e. \( \varphi_h \) is a semilattice homomorphism.

**Theorem 4.5.** The categories \( \mathcal{IBS\ell} \) and \( \text{Sem-dir-BA} \) are equivalent.
Proof. The equivalence is given upon defining the following functors:

\[ \mathcal{F} \]

\[ \mathcal{I} \mathcal{B} \mathcal{S} \mathcal{L} \]

\[ \text{Sem-dir-BA} \]

\[ \mathcal{G} \]

\( \mathcal{F} \) associates to an involutive bisemilattices \( A \cong \mathcal{P}_l(A) \) (due to Theorem 4.2), the semilattice direct system of Boolean algebras \( A \). On the other hand, for a semilattice direct system of Boolean algebras \( A \), we define \( \mathcal{G}(A) := \mathcal{P}_l(A) \).

Recall that a Stone space is topological space which is compact, Hausdorff and totally disconnected. Stone spaces can be viewed as a category, which we refer to as \( \mathcal{G} \mathcal{A} \) with continuous maps as morphisms.

It is well known that the category of Stone spaces is the dual of the category of Boolean algebras [41]. Theorem 4.5 and Remark 3.6 yields the following first characterization of the dual category of \( \mathcal{I} \mathcal{B} \mathcal{S} \mathcal{L} \).

**Theorem 4.6.** The category \( \text{Sem-inv-}\mathcal{G} \mathcal{A} \) is the dual of \( \mathcal{I} \mathcal{B} \mathcal{S} \mathcal{L} \).

Theorem 4.6 gives a description of the dual category of involutive bisemilattices in terms of Stone spaces, i.e. the natural dual category of Boolean algebras, objects coming into play due to Theorem 4.2.

The above theorem together with Theorem 4.2 should be compared with the following theorem due to Haimo [15], where direct limits are considered instead of Plonka sums. In the following statement, \( \text{lim} \rightarrow \), \( \text{lim} \leftarrow \) denote the direct and inverse limit, respectively.

**Theorem 4.7.** ([15], Theorem 9) Let \( \{ A_i \} \) be a direct system of Boolean algebras and \( \{ A_i^* \} \) the corresponding family of Stone spaces. Then

\[
(\text{lim} A_i)^* \cong \text{lim} A_i^*.
\]

In Theorem 4.19 (see below) we will give a concrete topological description of the dual space of an involutive bisemilattice based on Gierz and Romanowska duality (see Theorem 2.6), where, instead of GR spaces, we use GR spaces with involution.

**Definition 4.8.** A GR space with involution is a GR space \( \mathcal{G} \) with a continuous map \( \neg : \mathcal{G} \to \mathcal{G} \) such that, for any \( a \in \mathcal{G} \):

**G1.** \( \neg(\neg a) = a \)
G2. \( \neg(a \ast b) = \neg a \ast \neg b \)

G3. if \( a \leq b \) then \( \neg b \sqsubseteq \neg a \)

G4. \( \neg c_0 = c_1, \neg c_1 = c_0 \) and \( \neg c_\alpha = c_\alpha \)

G5. the space \( \text{Hom}_\text{GR}(A, 3) \) (see Section 2) equipped with natural involution \( \neg \), i.e. \( \neg \varphi(a) = (\varphi(\neg a))' \) satisfies \( \varphi \cdot (\neg \varphi + \psi) = \psi \cdot \varphi \), where operations are defined pointwise;

G6. there exist \( \varphi_0, \varphi_1 \in \text{Hom}_\text{GR}(A, 3) \) such that \( \neg \varphi_0 = \varphi_1 \) and \( \varphi + \varphi_0 = \varphi \), for each \( \varphi \in \text{Hom}_\text{GR}(A, 3) \).

**Examples 4.9.** WK equipped with discrete topology is the canonical example of GR space with involution.

**Remark 4.10.** Although the Definition of GR-space is the correct one to provide an axiomatization for spaces dually equivalent to involutive bisemilattices (see Corollary 4.20), it does not sound very elegant, due to the presence of conditions G5 and G6. In order to avoid these conditions, a natural solution would be having, for every GR space \( A \), two maps \( \varphi_0, \varphi_1 \in \text{Hom}_\text{GR}(A, 3) \) such that:

\[
\varphi_0(a) = 0 \quad \varphi_1(a) = 1,
\]

for every \( a \in A \setminus \{c_0, c_1, c_\alpha\} \). However, it is not difficult to show that, in general, \( \varphi_0 \) and \( \varphi_1 \) are not morphisms of GR spaces, as witnessed by the following example. Consider the direct system formed by \( \mathbb{N}^\infty \), the Alexandroff compactification\(^3\) of \( \mathbb{N} \), a trivial GR space \( \{c\} \) and the unique map from \( \mathbb{N}^\infty \) into \( \{c\} \) (one should also, pedantically, add the identity maps). Upon defining \( n \ast m = \min(n, \max(n, m)) \), \( \mathbb{N}^\infty \) turns into a partially ordered left zero band\(^4\) (where \( \leq \) is the usual ordering over \( \mathbb{N} \) and \( n \leq \infty \), for every \( n \in \mathbb{N} \)). Since, by \([14, \text{Theorem 4.8}]\), every partially order left normal bands is the Plonka of partially ordered left zero bands, then the Plonka sum \( \mathbb{N}^\infty \oplus \{c\} \) is a partially ordered left normal band that can be turned into a GR space, topologised with the disjoint union topology and, moreover, upon setting \( c_0 = 0, c_1 = \infty \) and \( c_\alpha = c \). It is easy to see that \( \varphi_0 : \mathbb{N}^\infty \oplus \{c\} \rightarrow 3 \) is not continuous, since \( \varphi_0^{-1}(\{1\}) = \infty = \mathbb{N}^\infty \setminus \mathbb{N} \) is not an open set.

**Definition 4.11.** \( \mathfrak{IGR} \) is the category whose objects are GR spaces with involution and whose morphisms are GR-morphisms preserving involution.

\(^3\)See [40] for details.

\(^4\)See [14] for details.
Given a GR space with involution $G$, we can consider its GR space reduct (simply its involution free reduct), say $A$, which can be associated to the dual distributive bisemilattice $\hat{A} = \text{Hom}_{GR}(A, 3)$. Aiming at turning it into an involutive bisemilattice, we define an involution on $\hat{A}$ as follows:

$$\neg \Phi(a) = (\Phi(-a))',$$

for each $\Phi \in \hat{A}$ and $a \in G$, where $\neg$ and $'$ are the involutions of $G$ and $WK$, respectively.

**Lemma 4.12.** If $\Phi \in \hat{A}$ then $\neg \Phi \in \hat{A}$.

**Proof.** Assuming that $\Phi$ is a morphism of GR spaces, we have to verify that also $\neg \Phi$ is, i.e. that it is a continuous map, preserving operation $\ast$, constants and the order $\leq$. Observe that $\neg \Phi$ is continuous as it is the composition of continuous maps.

Concerning operations and constants, we have:

$$\neg (\Phi(a \ast b))' = (\Phi(-a \ast -b))' = (\Phi(-a) \ast \Phi(-b))' = (\Phi(-a))' \ast (\Phi(-b))' = -\Phi(a) \ast -\Phi(b).$$

$$\neg (\Phi(c_0))' = (\Phi(c_1))' = 1' = 0.$$

Similarly, $\neg (\Phi(c_1)) = (\Phi(-c_1))' = (\Phi(c_0))' = 0' = 1$ and $\neg (\Phi(c_\alpha)) = (\Phi(-c_\alpha))' = (\Phi(c_\alpha))' = \alpha' = \alpha$.

As for the order, let $a \leq b$, but then $-b \sqsubseteq -a$. Since $\Phi$ preserve both the orders, $\Phi(-b) \leq \Phi(-a)$, thus $(\Phi(-a))' \leq (\Phi(-a))'$, i.e. $\neg \Phi(a) \leq \neg \Phi(b)$. 

**Proposition 4.13.** $\hat{G} = \langle \hat{A}, \neg \rangle$ is an involutive bisemilattice.

**Proof.** We have to check that conditions $I_1$–$I_8$ of Definition 2.7 hold for $\hat{G}$. Clearly, $I_1$, $I_2$ and $I_3$ hold as $\hat{A}$ is a distributive bisemilattice, while $I_6$, $I_7$ and $I_8$ hold by definition. For the remaining ones, let $\varphi \in \hat{A}$ and $a \in A$.

**I4.** $\neg (-\varphi(a)) = -\varphi(-\varphi(a))' = \varphi(-\varphi(a))'' = \varphi(a)$.

$$\neg (\varphi + \psi)(a) = (\varphi + \psi(-a))' = (\varphi(-a) + \psi(-a))' = ('(\varphi(-a))' \cdot (\psi(-a))' =$$

$$\neg \varphi(a) \cdot \neg \psi(a).$$

**Proposition 4.14.** $G \cong \hat{G}$.

**Proof.** We make good use of the duality established in [14], from which it follows $A \cong \hat{A}$, where $A$ is the GR space reduct of $G$. To prove our claim
we only have to prove that the isomorphism, given by (2), \( \delta_A(x)(\varphi) = \varphi(x) \), for \( x \in A \) and \( \varphi \in \hat{A} \), preserve the involution. This is easily checked, indeed
\[
(-\delta_A(x))(\varphi) = (\delta_A(x)(-\varphi))' = (-\varphi(x))' = (\varphi(-x))'' = \varphi(-x).
\]

Given an arbitrary involutive bisemilattice \( I \), we consider its bisemilattice reduct \( S = \langle I, +, \cdot \rangle \), which is distributive [4, Proposition 20], and therefore can be associated to its natural dual GR space, \( \hat{S} = \text{Hom}_b(S, \mathbf{3}) \) (see Section 2). The bisemilattice \( \mathbf{3} \) turns into \( \mathbf{WK} \) just by adding the usual involution and the constants \( 0, 1 \), so it makes sense to define an involution on \( \hat{S} \) as:
\[
\neg\varphi(x) = (\varphi(x'))',
\]
for any \( \varphi \in \hat{S} \) and \( x \in S \). We prove that \( \hat{S} \) is closed with respect to the above defined involution.

**Lemma 4.15.** If \( \varphi \in \hat{S} \) then \( \neg\varphi \in \hat{S} \).

**Proof.** Suppose that \( \varphi \in \hat{S} \), i.e. it is a map preserving sum and multiplication. It suffices to verify that also \( \neg\varphi \) preserves the two operations.
\[
\neg(\varphi(x+y)) = (\varphi(x+y))' = (\varphi(x'y'))' = (\varphi(x')\cdot\varphi(y'))' = (\varphi(x'))' + (\varphi(y'))' = \neg\varphi(x) + \neg\varphi(y).
\]

For multiplication the proof runs analogously.

**Remark 4.16.** The role of the dual space can not be played by \( \text{Hom}_i(I, \mathbf{WK}) \), the space of homomorphisms of involutive bisemilattices (namely those maps preserving also involution) with \( \varphi'(x) = \varphi(x') \). Indeed \( \text{Hom}_i(I, \mathbf{WK}) \) is not closed under such involution: \( \varphi'(x + y) = (\varphi(x + y))' = (\varphi(x) + \varphi(y))' = \varphi'(x)\cdot\varphi'(y) \), which is in general different from \( \varphi'(x) + \varphi'(y) \).

The above Remark highlights the fact that the duality presented in this paper cannot be constructed following the usual prescription of dualizing objects and homo functors. For example, in the case of Stone duality, the two element Boolean algebra \( \mathbf{2} = \{0, 1\} \) is the dualizing object as the duality is constructed looking at it once as a Boolean algebra and once as a Stone space. The same idea is used also in the duality for distributive bisemilattices [14], with \( \mathbf{3} \) as schizophrenic object. In our case, the object \( \mathbf{WK} \) is still chosen to belong to the target category and its dual and in this sense the duality is natural.

**Proposition 4.17.** \( \hat{I} = \langle \hat{S}, \neg \rangle \) is a GR space with involution.
**Proof.** By [14], we have that \( \hat{S} \) is a GR space, thus we only have to check that \( \neg \) has the required properties. Let \( \varphi, \psi \in \hat{S} \) and \( x \in S \); properties G1 – G4 can be easily verified as follows:

\[
\neg(\neg \varphi(x)) = \neg(\varphi(x'))' = (\varphi(x''))'' = \varphi(x).
\]

\[
\neg(\varphi \ast \psi)(x) = (\varphi \ast \psi(x'))' = (\varphi(x') \ast \varphi(x'))' = (\varphi(x'))' \ast (\psi(x'))' = \neg \varphi(x) \ast \neg \psi(x).
\]

Let \( \varphi \leq \psi \), i.e. \( \varphi(x) \leq \psi(x) \) for each \( x \in S \). In particular \( \varphi(x') \leq \psi(x') \), thus \((\psi(x'))' \leq_+ (\varphi(x'))'\), i.e. \( \neg \psi \subseteq \neg \varphi \).

Let \( \varphi_0, \varphi_1 \) and \( \varphi_\alpha \) the constant homomorphisms (of bisemilattices) on 0, 1 and \( \alpha \), respectively. \( \neg \varphi_0(x) = (\varphi_0(x'))' = 0' = 1 = \varphi_1(x); \neg \varphi_1(x) = (\varphi_1(x'))' = 1' = 0 = \varphi_0(x); \neg \varphi_\alpha(x) = (\varphi_\alpha(x'))' = \alpha' = \alpha = \varphi_\alpha(x) \).

In order to prove G5 and G6, it is enough to show that \( I \cong \hat{I} \). Recall that the bisemilattice reduct \( S \) of \( I \) is isomorphic to \( \hat{S} \) under the isomorphism given by (1), namely \( \varepsilon_S(x)(\varphi) = \varphi(x) \), for every \( \varphi \in \hat{S} \) and \( x \in S \). The map \( \varepsilon_S \) is obviously a homomorphism of bisemilattices and a bijection from \( I \setminus \{0, 1\} \) to \( \hat{I} \setminus \{\Phi_0, \Phi_1\} \), where by \( \Phi_0, \Phi_1 \) we indicate the constants in \( \hat{I} \).

This map can be extended to a bijection from \( I \) to \( \hat{I} \), by setting \( \varepsilon_S(0) = \Phi_0 \) and \( \varepsilon_S(1) = \Phi_1 \). We have to prove that \( \Phi_0 \) and \( \Phi_1 \) indeed play the role of the constants in \( \hat{I} \) and that \( \varepsilon_S \) also preserves involution. We start with the latter task:

\[
(\neg \varepsilon_S(x)(\varphi)) = (\varepsilon_S(x)(\neg \varphi))' = (\neg \varphi(x))' = (\varphi(x'))'' = \varphi(x').
\]

Regarding the constants, we only need to prove that \( \neg \Phi_0 = \Phi_1 \) and \( \Psi + \Phi_0 = \Psi \), for each \( \Psi \in \hat{I} \). Indeed, for any \( \varphi \in \hat{I} \), one has:

\[
\neg \Phi_0(\varphi) = \neg \varepsilon_S(0)(\varphi) = \varphi(0') = \varphi(1) = \varepsilon_S(1)(\varphi) = \Phi_1(\varphi).
\]

Finally, due to the surjectivity of \( \varepsilon_S \), for any \( \Psi \in \hat{I} \), there exists \( x \in I \) such that \( \Psi = \varepsilon_S(x) \). Therefore \( \Psi(\varphi) = \varepsilon_S(x)(\varphi) = \varepsilon_S(x + 0)(\varphi) = \varphi(x + 0) = \varphi(x) + \varphi(0) = \varepsilon_S(x)(\varphi) + \varepsilon_S(0)(\varphi) = (\Psi + \Phi_0)(\varphi) \) and we are done.

In order to prove Theorem 4.19 we are only left with proving that the functors \( \text{Hom}_b(-, \text{WK}) : \mathcal{IBSL} \rightarrow \mathcal{ISL} \) and \( \text{Hom}_{\text{GR}}(-, \text{WK}) : \mathcal{ISL} \rightarrow \mathcal{IBSL} \) are co-contravariant (we consider just the first functor as for the other the proof runs analogously).

**Proposition 4.18.** Let \( f : I \rightarrow L \) be a morphism of \( \mathcal{IBSL} \), then it induces a morphism of \( \mathcal{ISL} \) \( f^* : L \rightarrow \hat{I} \), where \( L, \hat{I} \) are the dual spaces of \( L \) and \( I \), respectively.
Proof. $f^*$ is defined in the usual way, i.e. $f^*(\hat{j})(i) = \hat{j}(f(i))$, for each $i \in I$ and $\hat{j} \in \hat{J}$. It suffices to prove that $f^*$ preserves involution, namely $f^*(\neg \hat{j}) = \neg f^*(\hat{j})$, for all $j \in J$:
\[
(\neg f^*(\hat{j}))(i) = \neg \hat{j}(f(i)) = f^*(\neg \hat{j})(i),
\]

Surprisingly enough, we have established that semilattice inverse systems of Stone spaces are nothing but GR spaces with involution.

**Theorem 4.19.** The category of GR spaces with involution is the dual of the category of involutive bisemilattices.

**Corollary 4.20.** The category $\text{Sem-inv-}\mathfrak{A}$ is equivalent to the category of GR spaces with involution.

Theorem 4.19 highlights an interesting as well as unexpected topological properties of Stone spaces. Indeed the category of (semilattice) inverse systems of Stone spaces which deals with a possibly infinite family of them can be described by a specific class of topological spaces, namely GR spaces with involution.

5. Final Comments and Remarks

It is natural to wonder whether the result in Theorem 4.19 may be extended to other algebraic categories admitting topological duals such as bisemilattices and GR spaces. Indeed, recall that bisemilattices are Plonka sums of distributive lattices, according to the following

**Theorem 5.1.** [29, Theorem 3] An algebra $B$ is a bisemilattice iff it is the Plonka sum over a semilattice direct system of distributive lattices.

A Priestley space is an ordered topological space, i.e. a set $X$ equipped with a partial order $\leq$ and a topology $\tau$, such that $(X, \tau)$ is compact and, for $x \not\leq y$ there exists a clopen up-set $U$ such that $x \in U$ and $y \not\in U$. The category of Priestley spaces, $\mathcal{P}S$, is the category whose objects are Priestley spaces and morphisms are continuous maps preserving the ordering.

The category of Priestley spaces is the dual of the category of distributive lattices [34,35].

Let us call $\mathcal{BSL}$ the category of bisemilattices (objects are bisemilattices, morphisms homomorphisms of bisemilattices). It follows from our analysis and Theorem 5.1 that the objects in $\mathcal{BSL}$ are the same as in $\text{Sem-dir-}\mathcal{D}L$. 

where \( DL \) stands for the category of distributive lattices. We claim that the two categories of \( BSL \) and \( \text{Sem-dir-}DL \) coincide. We show that by using the same strategy applied in Section 4.

**Lemma 5.2.** Let \( L \) and \( M \) be two bisemilattices, the Plonka sums over the semilattice direct systems of distributive lattices \( L = \langle L_i, \varphi_{i,i'}, I \rangle \) and \( M = \langle M_j, \varphi_{j,j'}, J \rangle \), and let \( h : L \to M \) be a homomorphism. Then, for any \( i \in I \), there exists a \( j \in J \) such that \( h(L_i) \subseteq M_j \). Moreover, there exists a semilattice homomorphism \( \varphi : I \to J \), for every homomorphism \( h : \mathcal{P}(L) \to \mathcal{P}(M) \), \( h(A_i) \subseteq B_{\varphi(i)} \).

**Proof.** Let \( a, b \in L_i \); we claim that \( h(a), h(b) \in M_j \), for some \( j \in J \). Two cases may arise: either \( a, b \) are comparable with respect to the order \( \leq \) of \( L_i \) or they are not. Suppose \( a \) and \( b \) are comparable: let \( a \leq b \) and suppose that \( h(a) \in M_j, h(b) \in M_{j'} \) with \( j \neq j' \). Then, \( h(a) = h(a \land b) = h(a) \land h(b) \in M_{j \lor j'} \) (by definition of operations in the Plonka sum), therefore \( j = j \lor j' \).

On the other hand, \( h(b) = h(a \lor b) = h(a) \lor h(b) \in M_{j \land j'} \). Thus \( j = j' \).

The case of \( b < a \) can be proved analogously.

Suppose now that \( a \) is not comparable with \( b \), namely \( a \not\leq b \) and \( b \not\leq a \). Clearly \( a \land b \leq a \lor b \), hence, reasoning as above, \( h(a \lor b) \) and \( h(a \land b) \) will belong to the same \( M_j \) for some \( j \in J \). Now, both \( a \) and \( b \) are comparable with \( a \land b \) and \( a \lor b \), hence necessarily \( h(a) \in M_j \) and \( h(b) \in M_j \). Therefore \( h(L_i) \subseteq M_j \).

The proof of the second statement runs analogously as for Lemma 4.4.

As consequence of Theorem 5.1 and Lemma 5.2 we get

**Proposition 5.3.** The category \( BSL \) is equivalent to \( \text{Sem-dir-}DL \).

Using Priestley duality and Remark 3.6 we have

**Theorem 5.4.** The category \( \text{Sem-inv-PS} \) is the dual of \( BSL \).

As the category of GR spaces is the dual category of \( BSL \) (see Theorem 2.6), this means that \( \text{Sem-inv-PS} \) are equivalent to a single class of spaces, namely

**Corollary 5.5.** The category \( \text{Sem-inv-PS} \) is equivalent to the category of GR spaces.

A different kind of duality, for varieties of bisemilattices (see [25]), including De Morgan bisemilattices [7] and involutive bisemilattices, has been recently investigated in [23]. While we aim at constructing a duality founded on the Plonka sum representation for the considered variety, the duality in
[23] moves from a different representation, an adaptation of Balbes represen-
tation theorem for bisemilattices [2]. We believe that finding a link, a part
from the obvious equivalence between the dual spaces involved, is not an
easy task, due to the diversity of the representation theorems involved. Our
motivation for insisting on the Plonka sum representation, relies on the fact
that Plonka sums allows to represent a very large class of regular varieties:
the regularizations of any strongly irregular variety (see [33] for details). In-
deed, our approach can be generalized, yielding to dualities for any regular
variety, representable as the Plonka sum of dualizable algebras, see [3].

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