FULLY HISTORY-DEPENDENT EVOLUTION
HEMIVARIATIONAL INEQUALITIES WITH CONSTRAINTS

STANISLAW MIGÓRSKI
College of Sciences, Beibu Gulf University
Qinzhou, Guangxi 535000, China
and
Jagiellonian University in Krakow
Chair of Optimization and Control
ul. Łojasiewicza 6, 30348 Krakow, Poland

YI-BIN XIAO
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, China

JING ZHAO
College of Sciences, Beibu Gulf University
Qinzhou, Guangxi 535000, China

Dedicated to Professor Meir Shillor on the occasion of his 70th birthday

Abstract. In this paper we study a new class of abstract evolution first order hemivariational inequalities which involves constraints and history-dependent operators. First, we prove the existence and uniqueness of solution by using a mixed equilibrium formulation with suitable selected bifunctions combined with a fixed-point principle for history-dependent operators. Next, we deduce existence, uniqueness and regularity results for some special subclasses of problems which include a constrained history-dependent variational–hemivariational inequality, an evolution quasi-variational inequality with constraints, and an evolution second order hemivariational inequality with constraints. Then, we provide an application of the results to a dynamic unilateral viscoelastic frictional contact problem and show its unique weak solvability.

1. Introduction. The mathematical theory of hemivariational and variational–hemivariational inequalities has been initiated in early 1980s with the pioneering works of Panagiotopoulos, see [35, 36, 37]. Hemivariational inequalities are variational descriptions of various physical phenomena that include nonconvex, nondifferentiable and locally Lipschitz functions. In the past decades, the theory of hemivariational inequalities has emerged as one of the most promising branches of pure, applied and industrial mathematics. This theory gives a powerful mathematical apparatus for studying a wide range of problems arising in diverse areas such as theoretical and applied mechanics, elasticity, economics, optimization, contact problems in

2010 Mathematics Subject Classification. Primary: 47J20, 47J22, 49J40, 49J45; Secondary: 74G25, 74G30, 74M15.

Key words and phrases. Hemivariational inequality; variational–hemivariational inequality, quasi-variational inequality, history-dependent operator, unilateral constraint, frictional contact.
solid and fluid mechanics, and others, see [7, 11, 19, 27, 38, 42, 39, 43, 44]. The representative literature on hemivariational inequalities and their applications is today very extensive and includes the following monographs [3, 14, 15, 20, 27, 33, 34, 40].

In this paper we study the Cauchy problem for a new class of evolution constrained history-dependent hemivariational inequalities of the following form: find \( w \in L^2(0, T; V) \) with \( w' \in L^2(0, T; V^*) \) such that \( w(t) \in K \) for a.e. \( t \in (0, T) \) and

\[
\begin{align*}
\langle w'(t) + A(t, (R_1 w)(t), w(t)) - f(t), v - w(t) \rangle_{V^* \times V} \\
+ j^0(t, (Rw)(t), (Sw)(t), w(t); v - w(t)) & \geq 0 \\
& \quad \text{for all } v \in K, \text{ a.e. } t \in (0, T), \\
w(0) &= w_0.
\end{align*}
\]

The problem (1) is investigated in the framework of evolution triple of spaces \( V \subset H \subset V^* \) which is used to define the standard Bochner space \( L^2(0, T; V) \), see Section 3. Here \( A: (0, T) \times X \times V \rightarrow V^* \) is a given nonlinear operator, \( j: (0, T) \times Y \times Z \rightarrow \mathbb{R}, j = j(t, y, z, v) \) is a given, in general nondifferentiable and nonconvex, function, called superpotential, which is assumed to be locally Lipschitz in \( v \), \( j^0(t, y, z, v; w) \) denotes the generalized Clarke subgradient of \( j(t, y, z, \cdot) \) at a point \( v \) in the direction \( w \), \( f: (0, T) \rightarrow V^* \) and \( w_0 \in \mathbb{V} \), \( (0, T) \) is a finite time interval, and \( X, Y \) and \( Z \) are Banach spaces. Moreover, throughout the paper, the prime denotes the distributional time derivative. The problem involves also a set of constraints \( K \subset V \), and three operators \( R_1, R \) and \( S \) which are so-called history-dependent operators.

The first order evolution hemivariational inequality (1) represents an extension of several abstract inequality problems extensively used in the study of a large number of dynamic models in nonsmooth contact mechanics. The main novelties of the paper are following. First, we prove a new existence, uniqueness and regularity result to the inequality (1). We will use the approach, different from those in \([4, 18, 24, 25, 27, 29, 30]\), and based on the mixed equilibrium problem involving bifunctions, see \([5, 6]\), combined with a fixed point principle for history-dependent operators, see \([40]\). Second, we obtain existence and uniqueness results for some particular cases of (1) which improve and generalize the results available in the literature. Third, we will illustrate the applicability of our results by a dynamic unilateral frictional problem in contact mechanics. Further, in the study of Problem (1), we will remove the following additional and stronger hypotheses and conditions met in the literature: the set of constraints is supposed to be a general nonempty, closed and convex without any additional restrictions as in [4], no compactness hypothesis on the embedding \( V \subset H \) in the evolution triple is assumed as in [24] and [27, Chapter 5], no additional fractional Sobolev spaces are needed as e.g. in [24, 25, 27], less restrictive smallness condition is required in comparison with e.g. [27, 29, 30, 31], and no compact operator composed with the superpotential \( j \) is needed, as in most of the aforementioned works.

The present paper is motivated by several recent contribution devoted to history-dependent variational and variational–hemivariational inequalities, see e.g. \([18, 26, 28, 46]\). For instance, the paper [46] contains results for a wide class of history-dependent quasi-variational inequalities which are applied to study quasistatic models in contact mechanics. The extension of these results to dynamic unilateral contact problems with viscoelastic materials is formulated in [46] as an open problem.
In this paper we provide results on a fully history-dependent evolution hemivariational inequality with constraints. To stress the generality of the problem (1), we formulate below its important three particular classes to which our results can be applied.

The first particular case reads as follows.

**Problem 1.** Find $w \in \mathcal{W}$ such that $w(t) \in K$ for a.e. $t \in (0, T)$, $w(0) = w_0$ and

$$
\begin{aligned}
\langle w'(t) + A(t, w(t)) - f(t), v - w(t) \rangle_{\mathcal{V}' \times \mathcal{V}} + g^0(t, Rw(t), w(t); v - w(t)) \\
+ \varphi(t, (Sw)(t), v) - \varphi(t, (Sw)(t), w(t)) \geq 0 \quad \text{for all} \quad v \in K, \text{ a.e. } t \in (0, T).
\end{aligned}
$$

Problem 1 is a constrained history-dependent variational–hemivariational inequality. It involves a nonconvex superpotential $g$, a convex potential $\varphi$, and the set of constraints $K$. Both potentials may depend on a history-dependent operator. It was treated in [31] with an additional compact operator $M$ under restrictive smallness condition and hypotheses on $A$, with the sign condition for $g$, and no constraints. Also, Problem 1 without constraints $K = \mathcal{V}$ has been recently studied in [18].

The second particular case is the dynamic quasi-variational inequality with constraints of the following form.

**Problem 2.** Find $u \in C(0, T; \mathcal{V})$ with $u' \in \mathcal{W}$ such that $u'(t) \in K$ for a.e. $t \in (0, T)$, $u(0) = u_0$, $u'(0) = w_0$ and

$$
\begin{aligned}
\langle u''(t) + A(t, u(t), u'(t)) - f(t), v - u'(t) \rangle_{\mathcal{V}' \times \mathcal{V}} \\
+ \varphi(t, u(t), v) - \varphi(t, u(t), u'(t)) \geq 0 \quad \text{for all} \quad v \in K, \text{ a.e. } t \in (0, T).
\end{aligned}
$$

This problem is a second order hyperbolic inequality where the governing operator depends on the unknown and its derivative, and the convex potential depends on the unknown function. Problem 2 is the dynamic counterpart of a quasistatic problem studied earlier in [46]. Moreover, Problem 2 was studied in [30] in a particular case $K = \mathcal{V}$, and with much stronger conditions on the growth of the subdifferential of $\varphi$ and smallness hypothesis.

The third example is the dynamic hyperbolic hemivariational inequality with constraints of the following form.

**Problem 3.** Find $u \in C(0, T; \mathcal{V})$ with $u' \in \mathcal{W}$ such that $u'(t) \in K$ for a.e. $t \in (0, T)$, $u(0) = u_0$, $u'(0) = w_0$ and

$$
\begin{aligned}
\langle u''(t) + A(t, u(t)) + B(t, u'(t)) - f(t), v - u'(t) \rangle_{\mathcal{V}' \times \mathcal{V}} \\
+ g^0(t, u(t), u'(t); v - u'(t)) \geq 0 \quad \text{for all} \quad v \in K, \text{ a.e. } t \in (0, T).
\end{aligned}
$$

This problem involves the nondifferentiable and nonconvex superpotential $g$, two operators depending separately on the unknown and its time derivative, respectively, and a set of constraints $K$. The results on existence of solution to Problem 3 can be found in the literature only in the case $K = \mathcal{V}$. We refer e.g. to [24] where additional fractional Sobolev spaces were used and $M$ was a trace operator, and to [29] and [27], Section 5.3 where the problem was studied with an additional compact operator $M$ involved in the superpotential $g$ which was supposed to be Clarke regular, and with a more complex smallness condition.

The main result of this paper on unique weak solvability, Theorem 3.1, is illustrated by an example of a dynamic frictional contact problem for a viscoelastic material with long memory and unilateral constraints in velocity. The model of this contact problem is new and, to our best knowledge, it has not been solved in the literature. Moreover, similar existence and uniqueness results can be demonstrated.
for other contact problems with unilateral boundary condition and a nonmonotone frictional law governed by a nonconvex superpotential. Results on existence of a weak solution to dynamic viscoelastic unilateral contact problems with constraints in displacement can be found in [1, 8, 12, 13, 23] and with constraints in velocity in [17, 32].

The outline of this paper is the following. In Section 2 we recall the notation and a preliminary material. The main result of the paper on constrained history-dependent variational–hemivariational inequality is stated and proved in Section 3. Section 4 discusses three consequences of the main theorem. Finally, in Section 5, we examine a dynamic nonsmooth contact problem of viscoelasticity with multivalued contact and friction conditions for which we prove existence and uniqueness of a weak solution.

2. Preliminary material. In this section we introduce the notation and recall some preliminary results.

Let \((X, \| \cdot \|_X)\) be a Banach space and \(X^*\) be its dual. We denote by \(J : X \rightarrow 2^{X^*}\) the duality mapping defined by
\[
J u = \{ u^* \in X^* | (u^*, u)_{X^* \times X} = \|u\|_X^2 = \|u^*\|_{X^*}^2 \} \quad \text{for all} \quad u \in X,
\]
where \((\cdot, \cdot)_{X^* \times X}\) is the duality pairing between \(X^*\) and \(X\). If \(X\) is a reflexive Banach space, then an equivalent norm can be introduced so that \(X\) is a strictly convex Banach space and, therefore, the duality map is a single-valued, monotone, continuous operator, and a mapping of class \((S_+)^\circ\), see e.g. [9, 47]. We use the standard notation “\(\rightarrow\)” and “\(\rightharpoonup\)” to denote the strong and the weak convergence, respectively. The space of linear continuous operators from a Banach space \(X_1\) to a Banach space \(X_2\) endowed with the usual norm \(\| \cdot \|_{L(X_1, X_2)}\) is denoted by \(L(X_1, X_2)\). For a set \(D \subset X\), \(\text{conv}(D)\) stands for the convex hull of \(D\).

In the literature one can find several notions of monotonicity of operators and functions. The monotone and maximal monotone bifunctions have been introduced in [2], another maximal monotone bifunctions were treated in [16], and other concepts of pseudomonotonicity and quasimonotonicity have been studied in [21]. To avoid confusion and to distinguished these concepts with the notion of monotonicity of operators (sometimes called in Brézis sense, see e.g. [47]), we recall some definitions for bifunctions and single-valued operators which can be found in [3, 5, 6, 10, 27, 45].

**Definition 2.1.** A function \(f : X \rightarrow \mathbb{R}\) is said to be

(i) lower (upper) semicontinuous \((\text{lsc}) \,(\text{usc}, \text{respectively})\) at \(x_0 \in X\), if for any sequence \(\{x_n\} \subset X\) with \(x_n \rightharpoonup x_0\), we have \(f(x_0) \leq \liminf f(x_n)\) \((\limsup f(x_n) \leq f(x_0))\), respectively,

(ii) \(f\) is said to be lsc \((\text{usc})\) on \(X\), if \(f\) is lsc \((\text{usc})\) at \(x\), for all \(x \in X\).

**Definition 2.2.** Let \(K\) be a nonempty, closed and convex subset of \(X\). A real-valued bifunction \(F : K \times K \rightarrow \mathbb{R}\) is said to be

(i) monotone, if \(F(u, v) + F(v, u) \leq 0\) for all \(u, v \in K\),

(ii) quasimonotone, if for any sequence \(\{u_n\} \subset K\) with \(u_n \rightharpoonup u\) in \(X\), we have \(\liminf F(u_n, u) \leq 0\),

(iii) pseudomonotone, if for any \(\{u_n\} \subset K\) with \(u_n \rightharpoonup u\) in \(X\) and \(\liminf F(u_n, u) \geq 0\), we have \(\limsup F(u_n, v) \leq F(u, v)\) for all \(v \in K\).
Remark 1. (i) If the operator \( F : K \times K \to \mathbb{R} \) is pseudomonotone, if for any sequence \( \{u_n\} \subset K \) with \( u_n \to u \) in \( X \) and for every convex function \( \psi : K \to \mathbb{R} \) with \( \psi(u) = 0 \), we have
\[
\psi(v) \geq F(v, u) \quad \text{for all } v \in K \implies \psi(v) \geq -F(u, v) \quad \text{for all } v \in K.
\]

Definition 2.3. Let \( K \) be a nonempty, closed and convex subset of \( X \). Let \( F : K \times K \to \mathbb{R} \) be a real-valued bifunction with \( F(u, u) = 0 \) for all \( u \in K \). The bifunction \( F \) is said to be maximal monotone if for every \( u \in K \) and for every convex function \( \psi : K \to \mathbb{R} \) with \( \psi(u) = 0 \), we have
\[
\psi(v) \geq F(v, u) \quad \text{for all } v \in K \implies \psi(v) \geq -F(u, v) \quad \text{for all } v \in K.
\]

Definition 2.4. A single-valued operator \( A : X \to X^\ast \) is said to be

(i) demicontinuous, if \( u_n \to u \) in \( X \) implies \( Au_n \to Au \) in \( X^\ast \),

(ii) monotone, if \( \langle Au - Av, u - v \rangle_{X^\ast \times X} \geq 0 \) for all \( u, v \in X \),

(iii) maximal monotone, if it is monotone and the conditions \( (u, u^\ast) \in X \times X^\ast \) and \( \langle u^\ast - Av, u - v \rangle_{X^\ast \times X} \geq 0 \) for all \( v \in X \) imply \( u^\ast = Au \).

(iv) quasimonotone, if \( \limsup \langle Au_n, u_n - u \rangle_{X^\ast \times X} \geq 0 \) for any sequence \( \{u_n\} \subset X \) with \( u_n \to u \) in \( X \),

(v) pseudomonotone, if for any sequence \( \{u_n\} \subset X \) such that \( u_n \to u \) in \( X \) and \( \limsup \langle Au_n, u_n - u \rangle_{X^\ast \times X} \leq 0 \), we have \( \liminf \langle Au_n, u_n - v \rangle_{X^\ast \times X} \geq \langle Au, u - v \rangle_{X^\ast \times X} \) for all \( v \in X \),

(vi) bounded, if it maps bounded subsets of \( X \) into bounded subsets of \( X^\ast \).

The following properties collected from [5, 6] will be exploited in next sections. Below, \( K \) denotes a nonempty, closed and convex subset of \( X \).

Remark 1. (i) If the operator \( A : X \to X^\ast \) is pseudomonotone (respectively, quasimonotone), then the bifunction \( F : K \times K \to \mathbb{R} \) defined by
\[
F(u, v) = \langle Au, v - u \rangle_{X^\ast \times X}
\]
for \( u, v \in K \), is pseudomonotone (respectively, quasimonotone).

(ii) If the bifunction \( F \) is usc with respect to the first argument for the weak topology, then it is pseudomonotone. Further, if the condition \( F(u, u) \geq 0 \) for all \( u \in K \) is satisfied, then \( F \) is quasimonotone.

(iii) If \( F, G : K \times K \to \mathbb{R} \) are pseudomonotone bifunctions such that \( F(u, u) \leq 0 \) and \( G(u, u) \leq 0 \) for all \( u \in K \), then \( F + G \) is also pseudomonotone.

(iv) If the operator \( A : X \to X^\ast \) is maximal monotone, then the bifunction \( F : K \times K \to \mathbb{R} \) defined by \( F(u, v) = \langle Au, v - u \rangle_{X^\ast \times X} \) for \( u, v \in K \), is monotone and maximal monotone.

(v) If the bifunction \( F : K \times K \to \mathbb{R} \) is such that \( F(u, v) + F(v, u) = 0 \) for all \( u, v \in K \), then \( F \) is monotone and maximal monotone.

We recall a crucial abstract result on existence of solution to the mixed equilibrium problem involving bifunctions. Let \( U \) be a subset of a reflexive Banach space \( X \).

Problem 4. Find \( u \in U \) such that
\[
F(u, v) + G(u, v) \geq 0 \quad \text{for all } v \in U.
\]

We need the following hypotheses on the data of Problem 4.

\( H(U) : \) \( U \) is a nonempty, closed and convex subset of \( X \).

\( H(F) : \) \( F : U \times U \to \mathbb{R} \) is such that
\begin{enumerate}
  \item \( F \) is monotone and maximal monotone.
  \item \( F(u, \cdot) \) is convex and lsc for all \( u \in U \).
\end{enumerate}
(3) \( F(u, u) = 0 \) for all \( u \in U \).

**H(G):** \( G: U \times U \to \mathbb{R} \) is such that

1. \( G \) is pseudomonotone.
2. For each finite subset \( D \) of \( U \), \( G(\cdot, v) \) is usc on \( \text{conv}(D) \), for all \( v \in U \).
3. \( G(u, \cdot) \) is convex for all \( u \in U \).
4. \( G(u, u) = 0 \) for all \( u \in U \).

\( (H_{\text{coer}}) \): There exists a nonempty weakly compact subset \( W \) such that for each \( \lambda > 0 \) small enough, there exists a weakly compact and convex subset \( B_\lambda \) of \( U \) satisfying the following condition:

\[ \forall u \in K \setminus W, \exists v \in B_\lambda \quad \text{such that} \quad G(u, v) + \lambda \langle J u, v - u \rangle_{X^* \times X} < F(v, u). \]

We have the following existence result which proof can be found in [5, Theorem 3.4], [6, Theorem 2.4].

**Theorem 2.5.** Assume the hypotheses \( H(U) \), \( H(F) \), \( H(G) \), and \( (H_{\text{coer}}) \). Then, Problem 4 has at least one solution \( u \in U \).

**Remark 2.** (i) If, additionally, \( U \) is a weakly compact set, then the condition \( (H_{\text{coer}}) \) in Theorem 2.5 can be omitted.

(ii) If \( F(u, \cdot) \) is convex and lsc for all \( u \in U \), and \( F(u, u) = 0 \) for all \( u \in U \), then the condition \( (H_{\text{coer}}) \) is satisfied, if for some \( v_0 \in U \), we have

\[ \frac{G(u, v_0) + \lambda \langle J u, v_0 - u \rangle}{\| u - v_0 \|_X} \to -\infty \quad \text{uniformly in } \lambda, \quad \text{as } \| u - v_0 \|_X \to +\infty. \]

Next, we recall the notions of generalized directional derivative and the generalized gradient for a locally Lipschitz function, see [7, 9, 27].

**Definition 2.6.** Let \( X \) be a Banach space and \( j: X \to \mathbb{R} \) be a locally Lipschitz function. The generalized directional derivative of \( j \) at \( x \in X \) in the direction \( v \in X \) is defined by

\[ j^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}. \]

The generalized gradient of \( j \) at \( x \) is a set defined by

\[ \partial j(x) = \{ x^* \in X^* \mid \langle x^*, v \rangle \leq j^0(x; v) \quad \text{for all} \quad v \in X \}. \]

Finally, we state the fixed-point result which is a consequence of the Banach contraction principle, see [22, Lemma 7] or [40, Corollary 27].

**Lemma 2.7.** Let \( X \) be a Banach space, \( 0 < T < \infty \). Let \( \Lambda: L^2(0, T; X) \to L^2(0, T; X) \) be an operator such that

\[ \|(\Lambda \eta_1)(t) - (\Lambda \eta_2)(t)\|_X^2 \leq c \int_0^t \| \eta_1(s) - \eta_2(s) \|_X^2 \, ds \]

for all \( \eta_1, \eta_2 \in L^2(0, T; X) \), a.e. \( t \in (0, T) \) with a constant \( c > 0 \). Then \( \Lambda \) has a unique fixed point in \( L^2(0, T; X) \), i.e., there exists a unique \( \eta^* \in L^2(0, T; X) \) such that \( \Lambda \eta^* = \eta^* \).
3. **Existence and uniqueness result.** In this section we study a class of evolution hemivariational inequalities with constraints and history-dependent operators. We treat this class of problems in the framework of evolution triple of spaces. Our main goal is to establish a result on the unique weak solvability.

For the convenience of the reader, we recall that $(V, H, V^*)$ is an evolution triple of spaces if $V$ is a separable, reflexive Banach space, $H$ is a separable Hilbert space, the embedding $V \subset H$ is dense and continuous, see e.g. [10, Section 8.4]. We need the following Bochner spaces defined on the finite time interval $(0, T)$ with values in Banach spaces

$$V = L^2(0, T; V), \quad V^* = L^2(0, T; V^*), \quad W = \{ v \in V \mid \frac{d}{dt} v \in V^* \},$$

where $\frac{d}{dt} v$ stands for the generalized time derivative of $v$, denoted also $v'$ in the sequel, i.e.,

$$\int_0^T v'(s) \phi(s) \, ds = - \int_0^T v(s) \phi'(s) \, ds \quad \text{for all } \phi \in C_0^\infty(0, T).$$

Note that the space $W$ is a separable and reflexive Banach space with the norm $\|v\|_W = \|v\|_V + \|v'\|_{V^*}$. It is well also known that the embeddings $V \subset L^2(0, T; H) \subset V^*$ and $W \subset C(0, T; H)$ are continuous, where $C(0, T; H)$ denotes the space of continuous functions on $[0, T]$ with values in $H$. Define $L v := v'$ with

$$D(L) = \{ v \in W \mid v(0) = 0 \}.$$ 

The generalized derivative restricted to $D(L)$ defines an operator $L \colon D(L) \subset V \to V^*$ which is a linear, densely defined and maximal monotone, see, e.g., [47, Proposition 32.10, p. 855]. Further, we note that the duality pairing between $V^*$ and $V$ is denoted by

$$\langle w, v \rangle_{V^* \times V} = \int_0^T \langle w(t), v(t) \rangle_{V^* \times V} \, dt \quad \text{for } w \in V^*, v \in V,$$

where $\langle \cdot, \cdot \rangle_{V^* \times V}$ stands for the duality pairing between $V^*$ and $V$.

Let $X$, $Y$ and $Z$ be Banach spaces. Consider the following evolution constrained history-dependent hemivariational inequality.

**Problem 5.** Find $w \in W$ such that $w(t) \in K$ a.e. $t \in (0, T)$ and

$$\begin{cases}
\langle w'(t) + A(t, (R_1 w)(t), w(t)), v(t) - f(t) \rangle_{V^* \times V} \\
\quad + \langle j^0(t, (R_1 w)(t), (Sw)(t), w(t) - v(t)), v - w(t) \rangle_{V^* \times V} \geq 0
\end{cases}
$$

for all $v \in K$, a.e. $t \in (0, T)$,

$$w(0) = w_0.$$

We need the following hypotheses on the data.

**H**($A$) : $A \colon (0, T) \times X \times V \to V^*$ is such that

(1) $A(\cdot, x, v)$ is measurable on $(0, T)$ for all $x \in X$, $v \in V$.

(2) $A(t, \cdot, v)$ is continuous on $X$ for all $v \in V$, a.e. $t \in (0, T)$.

(3) $A(t, x, \cdot)$ is demicontinuous on $V$ for all $x \in X$, a.e. $t \in (0, T)$.

(4) $\|A(t, x, v)\|_{V^*} \leq a_0(t) + a_1 \|x\|_X + a_2 \|v\|_V$ for all $x \in X$, $v \in V$, a.e. $t \in (0, T)$ with $a_0 \in L^2(0, T)$, $a_0 \geq 0$ and $a_1, a_2 \geq 0$. 

(5) there are constants $m_A > 0$ and $m_0 \geq 0$ such that
\[
\langle A(t, x_1, v_1) - A(t, x_2, v_2), v_1 - v_2 \rangle_{V' \times V} \geq m_A \|v_1 - v_2\|_V^2 - m_0 \|x_1 - x_2\|_X \|v_1 - v_2\|_V
\]
for all $x_1, x_2 \in X$, $v_1, v_2 \in V$, a.e. $t \in (0, T)$. 

$H(j)$: $j: (0, T) \times Y \times Z \times V \to \mathbb{R}$ is such that

1. $j(\cdot, y, z, v)$ is measurable on $(0, T)$ for all $y \in Y$, $z \in Z$, $v \in V$.
2. $j(\cdot, \cdot, \cdot, v)$ is continuous on $Y \times Z$ for all $v \in V$, a.e. $t \in (0, T)$.
3. $j(t, y, z, \cdot)$ is locally Lipschitz on $V$ for all $y \in Y$, $z \in Z$, a.e. $t \in (0, T)$.
4. $\|\partial j(t, y, z, v)\|_V \leq c_0(t) + c_1(t) \|y\|_Y + c_2(t) \|z\|_Z + c_3(t) \|v\|_V$ for all $y \in Y$, $z \in Z$, $v \in V$, a.e. $t \in (0, T)$ with $c_0(t) \in L^2(0, T)$, $c_1(t), c_2(t), c_3(t) \geq 0$.

5. there are constants $m_j, m_1, m_2 \geq 0$ such that
\[
j(t, y_1, z_1, v_1; v_2 - v_1) + j(t, y_2, z_2, v_2; v_1 - v_2)
\leq m_j \|v_1 - v_2\|_V + m_1 \|y_1 - y_2\|_Y + m_2 \|z_1 - z_2\|_Z \|v_1 - v_2\|_V
\]
for all $y_i, z_i \in Z$, $v_i \in V$, $i = 1, 2$, a.e. $t \in (0, T)$.

$H(R, S)$: $R: V \to L^2(0, T; X)$, $R: V \to L^2(0, T; Y)$, and $S: V \to L^2(0, T; Z)$ are such that

1. $\|(Rv_1(t) - (Rv_2(t))_X \leq c_R \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in V$, a.e. $t \in (0, T)$ with $c_R > 0$.

2. $\|(Rv_1(t) - (Rv_2(t))_Y \leq c_R \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in V$, a.e. $t \in (0, T)$ with $c_R > 0$.

3. $\|(Sv_1(t) - (Sv_2(t))_Z \leq c_S \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in V$, a.e. $t \in (0, T)$ with $c_S > 0$.

$H(K)$: $K$ is a nonempty, closed and convex subset of $V$.

$H(f)$: $f \in V'$, $w_0 \in V$.

$(H_1)$: $m_A > m_j$.

We have the following main existence and uniqueness result.

**Theorem 3.1.** Under hypotheses $H(A)$, $H(j)$, $H(R, S)$, $H(K)$, $H(f)$, and $(H_1)$, Problem 5 has a unique solution.

**Proof.** It is established in several steps. First, we suppose that $w_0 = 0$.

Let $\xi \in L^2(0, T; X)$, $\eta \in L^2(0, T; Y)$ and $\zeta \in L^2(0, T; Z)$ be fixed. Consider the following auxiliary problem.

\[
\begin{cases}
\text{Find } w = w_{\xi_0\eta_0\zeta} \in W \text{ with } w(t) \in K \text{ for a.e. } t \in (0, T) \text{ such that } \\
\langle w'(t) + A(t, \xi(t), w(t)) - f(t), v - w(t) \rangle_{V' \times V} \\
+ j^0(t, \eta(t), \zeta(t), w(t); v - w(t)) \geq 0 \text{ for all } v \in K, \text{ a.e. } t, \\
w(0) = 0.
\end{cases}
\]

**Step 1.** We show that the solution to the problem (2) is unique. We begin with the proof of uniqueness. For simplicity, we skip the subscripts $\xi, \eta$ and $\zeta$ for this part.
of the proof. Let \( w_i \in W, i = 1, 2 \) be solutions to the problem (2), i.e., \( w_i(t) \in K \) for a.e. \( t \in (0, T) \), \( w_i(0) = 0 \) and
\[
\langle w'_i(t) + A(t, \xi(t), w_i(t)) - f(t), v - w_i(t) \rangle_{V^* \times V} + j^0(t, \eta(t), \zeta(t), w_i(t); v - w_i(t)) \geq 0
\]
for all \( v \in K \), a.e. \( t \in (0, T) \). We choose \( v = w_2(t) \) in the inequality for \( w_1 \), \( v = w_1(t) \) in the inequality satisfied by \( w_2 \), and add the resulting relations to get
\[
\langle w'_1(t) - w'_2(t), w_1(t) - w_2(t) \rangle_{V^* \times V} + \langle A(t, \xi(t), w_1(t)) - A(t, \xi(t), w_2(t)), w_1(t) - w_2(t) \rangle_{V^* \times V} + j^0(t, \eta(t), \zeta(t), w_1(t); w_2(t) - w_1(t)) + j^0(t, \eta(t), \zeta(t), w_2(t); w_1(t) - w_2(t)) \leq 0
\]
for a.e. \( t \in (0, T) \). Integrating the above inequality on \( (0, t) \), using the integration by parts formula, see [10, Proposition 8.4.14], and the hypothesis \( H(A)(5) \) on the left hand side, and applying the hypothesis \( H(j)(5) \) to the right hand side, we obtain
\[
\frac{1}{2} \| w_1(t) - w_2(t) \|^2_H + m_A \int_0^t \| w_1(s) - w_2(s) \|^2_V ds \leq m_j \int_0^t \| w_1(s) - w_2(s) \|^2_V ds
\]
for all \( t \in [0, T] \). Hence
\[
(m_A - m_j) \| w_1 - w_2 \|^2_{L^2(0,T;V)} \leq 0,
\]
for all \( t \in [0, T] \). From the condition \( (H_1) \), it follows \( w_1 = w_2 \). This completes the proof of uniqueness of the solution.

**Step 2.** We prove that the problem (2) has a solution. To this end, we formulate it in an equivalent way. Let
\[
\mathcal{K} = \{ v \in V \mid v(t) \in K \text{ for a.e. } t \in (0, T) \}, \tag{3}
\]
and \( \mathcal{K}_1 = D(L) \cap \mathcal{K} \), where, recall, \( D(L) = \{ w \in W \mid w(0) = 0 \} \). Consider the following problem.

\[
\text{Find } w \in \mathcal{K}_1 \text{ such that}
\begin{align*}
\int_0^T \langle w'(t) + A(t, \xi(t), w(t)) - f(t), \varpi(t) - w(t) \rangle_{V^* \times V} dt \\
+ \int_0^T j^0(t, \eta(t), \zeta(t), w(t); \varpi(t) - w(t)) dt \geq 0 & \quad \text{for all } \varpi \in \mathcal{K}_1. \tag{4}
\end{align*}
\]

**Lemma 3.2.** Problems (2) and (4) are equivalent.

**Proof.** Let \( w \in W \) be a solution to the problem (2). This means that \( w \in \mathcal{K} \) and \( w(0) = 0 \) which entails \( w \in D(L) \cap \mathcal{K} \). Let \( \varpi \in \mathcal{K}_1 \). Then, we have
\[
\langle w'(t) + A(t, \xi(t), w(t)) - f(t), \varpi(t) - w(t) \rangle_{V^* \times V} + j^0(t, \eta(t), \zeta(t), w(t); \varpi(t) - w(t)) \geq 0 \quad \text{for a.e. } t \in (0, T).
\]
Integrating the latter on \( (0, T) \), since \( \varpi \in \mathcal{K}_1 \) is arbitrary, we deduce that \( w \in D(L) \cap \mathcal{K} \) is a solution to the problem (4).

Conversely, assume that \( w \in D(L) \cap \mathcal{K} \) solves the problem (4). We suppose, arguing by contradiction, that there exist a measurable subset \( S \subset (0, T) \) with \( \text{meas}(S) > 0 \) and \( \varpi^* \in K \) such that
\[
\langle w'(t) + A(t, \xi(t), w(t)) - f(t), \varpi^* - w(t) \rangle_{V^* \times V} + j^0(t, \eta(t), \zeta(t), w(t); \varpi^* - w(t)) < 0 \quad \text{for all } t \in S.
\]
We define the function
\[
\varpi(t) = \begin{cases} 
v^*, & \text{if } t \in S, \\
w(t), & \text{otherwise}.\end{cases}
\]
Since \( w \in D(L) \cap \mathcal{K} \), we have \( \varpi \in D(L) \cap \mathcal{K} \). Then, we take \( \varpi \) in the problem (4) to obtain
\[
0 \leq \int_0^T \langle w'(t) + A(t, \xi(t), w(t)) - f(t), \varpi(t) - w(t) \rangle_{V^\ast \times V} \, dt \\
+ \int_0^T j^0(t, \eta(t), \zeta(t), w(t); \varpi(t) - w(t)) \, dt \\
= \int_S \langle w'(t) + A(t, \xi(t), w(t)) - f(t), v^* - w(t) \rangle_{V^\ast \times V} \, dt \\
+ \int_S j^0(t, \eta(t), \zeta(t), w(t); v^* - w(t)) \, dt < 0,
\]
which is a contradiction. Therefore, we conclude that \( w \in D(L) \cap \mathcal{K} \) solves the problem (2), which completes the proof of the lemma.

From Step 1 and Lemma 3.2, it is clear that the problems (2) and (4) have the same unique solution. Therefore, to show the existence of a solution to the problem (2), it is enough to prove that there exists a solution to the problem (4).

In what follows, we prove that the problem (4) has a solution. To this end, we need some preparation. We introduce the operator \( A : V \to V^\ast \) and the function \( J : V \to \mathbb{R} \) defined by
\[
(Aw)(t) = A(t, \xi(t), w(t)) \quad \text{for all } w \in V, \ a.e. \ t \in (0, T),
\]
\[
J(w) = \int_0^T j(t, \eta(t), \zeta(t), w(t)) \, dt \quad \text{for } w \in V.
\]
From the measurability of \( t \mapsto \xi(t), t \mapsto \eta(t) \) and \( t \mapsto \zeta(t) \) on \((0, T)\), and hypotheses \( H(A)(2) \) and \( H(j)(2) \), it is clear that \( A \) and \( J \) are well defined. They depend on \( \xi, \eta \) and \( \zeta \), respectively, but, as before, we skip this dependence here. We have the following properties of \( A \) and \( J \).

**Claim 1.** The operator \( A : V \to V^\ast \) is demicontinuous. Indeed, let \( \{z_n\} \subset V, z_n \to z \) in \( V \). Using [27, Theorem 2.39], we may pass to a subsequence, if necessary and assume that \( z_n(t) \to z(t) \) in \( V \) for a.e. \( t \in (0, T) \) and there exists \( g \in L^2(0, T) \) such that \( \|z_n(t)\|_V \leq g(t) \) for a.e. \( t \in (0, T) \). From \( H(A)(3) \), we obtain \( A(t, \xi(t), z_n(t)) \to A(t, \xi(t), z(t)) \) in \( V^\ast \) for a.e. \( t \in (0, T) \) which means that
\[
\langle A(t, \xi(t), z_n(t)), v(t) \rangle_{V^\ast \times V} \to \langle A(t, \xi(t), z(t)), v(t) \rangle_{V^\ast \times V}
\]
for all \( v \in V \). Taking into account \( H(A)(4) \), we apply the Lebesgue dominated convergence theorem, see, e.g., [27, Theorem 2.38], to get
\[
\int_0^T \langle A(t, \xi(t), z_n(t)), v(t) \rangle_{V^\ast \times V} \, dt \to \int_0^T \langle A(t, \xi(t), z(t)), v(t) \rangle_{V^\ast \times V} \, dt.
\]
Hence, since \( v \in V \) is arbitrary, we have \( A z_n \to A z \) in \( V \). Finally, we can observe that the whole sequence \( \{A z_n\} \) weakly converges to \( A z \), which proves the claim.

**Claim 2.** The operator \( A + \partial J : V \to 2^V \) is strongly monotone with constant \( m_A - m_J > 0 \). We use [27, Corollary 4.15] to infer that \( J \) is Lipschitz continuous.
on bounded subsets of \( V \), hence also locally Lipschitz on \( V \), and

\[
J^0(w; z) \leq \int_0^T j^0(\eta(t), \zeta(t), w(t); z(t)) \, dt \quad \text{for } z \in V. \tag{5}
\]

Using (5), we obtain

\[
J^0(w; z - w) + J^0(z; w - z) \leq m_j \| w - z \|^2_V \quad \text{for all } w, z \in V.
\]

The latter is equivalent, see [18, Remark 8], to the relaxed monotonicity of the operator \( w \mapsto \partial J(w) \), i.e.,

\[
\langle \partial J(w) - \partial J(z), w - z \rangle_{V^* \times V} \geq -m_j \| w - z \|^2_V \quad \text{for all } w, z \in V. \tag{6}
\]

On the other hand, by \( H(A)(5) \), we easily get

\[
\langle A(w) - Az, w - z \rangle_{V^* \times V} \geq m_A \| w - z \|^2_V \quad \text{for all } w, z \in V. \tag{7}
\]

Combining (6), (7) and (H1), we deduce that

\[
\langle (A + \partial J)(w) - (A + \partial J)(z), w - z \rangle_{V^* \times V} \geq (m_A - m_j) \| w - z \|^2_V \quad \text{for all } w, z \in V
\]

which proves the claim. \( \square \)

Next, we need the following functions \( F, G : K_1 \times K_1 \to \mathbb{R} \) given by

\[
F(w, z) = \langle Lw, z - w \rangle_{V^* \times V}, \tag{8}
\]

\[
G(w, z) = \langle Aw - f, z - w \rangle_{V^* \times V} + J^0(w; z - w) \tag{9}
\]

for \( w, z \in K_1 \). With this notation, we consider another auxiliary problem.

Find \( w \in K_1 \) such that \( F(w, z) + G(w, z) \geq 0 \) for all \( z \in K_1 \). \tag{10}

From (5), it follows that every solution to the problem (10) is a solution to the problem (4). Combining this fact with Lemma 3.2 it is clear that to establish the existence of solution to the problem (2) it will be sufficient to show that the problem (10) has a solution.

**Lemma 3.3.** Problem (10) has a solution.

**Proof.** We prove the existence of a solution to the problem (10) by using Theorem 2.5. We will verify the hypotheses of this theorem for \( F \) and \( G \) given by (8) and (9), and \( U = K_1 \).

First, we show that the function \( F \) defined by (8) satisfies the condition \( H(F) \). We begin with \( H(F)(1) \). Since \( F(w, z) + F(z, w) = -\langle Lw - Lz, w - z \rangle_{V^* \times V} \leq 0 \) for all \( w, z \in K_1 \), it is obvious that \( F \) is monotone. We show that \( F \) is maximal monotone. Assume that for every \( w \in U \) and every convex function \( \psi : U \to \mathbb{R} \) with \( \psi(w) = 0 \), we have

\[
\psi(z) \geq F(z, w) \quad \text{for all } z \in U.
\]

By the maximal monotonicity of the operator \( L : V \to V^* \), it follows that

\[
\psi(z) \geq -F(w, z) \quad \text{for all } z \in U,
\]

which means that the function \( F \) is maximal monotone.

Next, we will show that \( F(w, \cdot) \) is convex and lsc for all \( w \in U \). Let \( w, z_1, z_2 \in U \) and \( \lambda \in (0, 1) \). The following inequality

\[
F(w, \lambda z_1 + (1 - \lambda)z_2) = \langle Lw, \lambda z_1 + (1 - \lambda)z_2 - w \rangle_{V^* \times V}
\]

\[
\leq \lambda \langle Lw, z_1 - w \rangle_{V^* \times V} + (1 - \lambda)\langle Lw, z_2 - w \rangle_{V^* \times V} = \lambda F(w, z_1) + (1 - \lambda)F(w, z_2),
\]

\[
\lambda \| z_1 - z_2 \|^2_V.
\]
implies that \( F(w, \cdot) \) is convex for all \( w \in U \). Next, for \( \{z_n\} \subset U, z_n \to z \) in \( U \) and \( w \in U \), we have
\[
\liminf F(w, z_n) = \liminf (Lw, z_n - w)_{V^\ast \times V} \geq (Lw, z - w)_{V^\ast \times V} = F(w, z),
\]
which entails that \( F(w, \cdot) \) is lsc for all \( w \in U \), i.e., \( H(F)(2) \) is satisfied. The condition \( H(F)(3) \) is obvious.

Now we prove that the function \( G \) defined by (9) satisfies the condition \( H(G) \). We show that \( G \) is pseudomonotone on \( U \). Let \( \{w_n\} \subset U, w_n \to w \in U \) and \( \liminf G(w_n, w) \geq 0 \). We will prove that
\[
\limsup G(w_n, v) \leq G(w, v) \quad \text{for all} \quad v \in U. \tag{11}
\]
Let \( v \in U \). It follows from [27, Proposition 3.23] that there exists \( \eta_n \in \partial J(w_n) \) such that
\[
J^0(w_n; v - w_n) = \langle \eta_n, v - w_n \rangle_{V^\ast \times V}.
\]
Since the operator \( w \mapsto \partial J(w) \) is bounded by hypothesis \( H(j)(4) \), we know that \( \{\eta_n\} \) lies in a bounded subset of \( V^\ast \) independently of \( n \). Thus, by passing to a subsequence if necessary, we may assume
\[
\eta_n \to \eta \quad \text{in} \quad V^\ast \quad \text{with} \quad \eta \in V^\ast. \tag{12}
\]
Using the assumption \( \liminf G(w_n, w) \geq 0 \), we have
\[
0 \leq \liminf G(w_n, w) = \liminf (\langle Aw_n - f, w - w_n \rangle_{V^\ast \times V} + J^0(w_n; w - w_n))
= \liminf (\langle Aw_n + \eta_n, w - w_n \rangle_{V^\ast \times V} - \langle f, w - w_n \rangle_{V^\ast \times V})
= \liminf \langle Aw_n + \eta_n, w - w_n \rangle_{V^\ast \times V} = - \limsup \langle Aw_n + \eta_n, w_n - w \rangle_{V^\ast \times V},\tag{13}
\]
where \( \bar{\eta}_n \in \partial J(w_n) \) which implies
\[
\limsup \langle Aw_n + \bar{\eta}_n, w_n - w \rangle_{V^\ast \times V} \leq 0. \tag{13}
\]
By the strong monotonicity of the operator \( A + \partial J \), we have
\[
(m_A - m_f)\|w_n - w\|^2_V \leq \langle Aw_n + \bar{\eta}_n, w_n - w \rangle_{V^\ast \times V} - \langle (A + \partial J)w, w_n - w \rangle_{V^\ast \times V}
\]
which from (13) implies
\[
w_n \rightharpoonup w \quad \text{in} \quad V. \tag{14}
\]
Since \( A: V \to V^\ast \) is demicontinuous by Claim 1, from (14), we obtain
\[
Aw_n \rightharpoonup Aw \quad \text{in} \quad V^\ast. \tag{15}
\]
Next, we use the fact that the graph of the generalized gradient is strongly-weakly closed, see [27, Proposition 3.23(v)], and from (12) and (14), we get
\[
\eta \in \partial J(w). \tag{16}
\]
Exploiting (12), (14), (15) and (16), we have
\[
\limsup G(w_n, v) = \limsup (\langle Aw_n - f, v - w_n \rangle_{V^\ast \times V} + J^0(w_n; v - w_n))
= \limsup \langle Aw_n + \eta_n - f, v - w_n \rangle_{V^\ast \times V} = \langle Aw + \eta - f, v - w \rangle_{V^\ast \times V}
\leq \langle Aw, v - w \rangle_{V^\ast \times V} + J^0(w; v - w) = G(w, v).\tag{16}
\]
Since \( v \in U \) is arbitrary, we deduce (11). Hence, \( G \) is pseudomonotone.

Subsequently, we prove that \( H(G)(2) \). The proof is similar to the proof of condition \( H(G)(1) \). Let \( D \) be a finite subset of \( U, v \in U \) and \( \{w_n\} \subset \text{conv}(D) \) be such
that \( w_n \to w \) in \( U \). We show that \( \limsup G(w_n, v) \leq G(w, v) \). Since \( A : V \to V^* \) is demicontinuous by Claim 1, it follows that
\[
A\!w_n \to A\!w \text{ in } V^*.
\]
Further, from [27, Proposition 3.23] there exists \( \eta_n \in \partial J(w_n) \) such that
\[
J^0(w_n; v - w_n) = \langle \eta_n, v - w_n \rangle_{V^* \times V}.
\]
By the boundedness of the operator \( \sup \) suppose that \( \partial J \)
\[
is positively homogeneous and subadditive, see [27, Proposition 3.23(i)]. Hence
\[
\text{condition (19) holds with } \lambda = 0 \text{ uniformly in } \lambda, \text{ as } \|w - v_0\|_V \to +\infty, \tag{19}
\]
where \( w \in U \). Let \( v_0 = 0 \in U \). Then, we have
\[
G(w, 0) + \lambda\langle jw, v_0 - w \rangle_{V^* \times V} = \langle Aw - f, -w \rangle_{V^* \times V} + J^0(w; -w) + \lambda\langle jw, -w \rangle_{V^* \times V}
\]
\[
= \langle Aw - A0, 0 - w \rangle_{V^* \times V} + \langle A0, 0 - w \rangle_{V^* \times V} + \langle f, w \rangle_{V^* \times V}
\]
\[
+ J^0(0; w - 0) - J^0(0; w - 0) - J^0(0; w - 0) + \lambda\langle jw, -w \rangle_{V^* \times V}. \tag{20}
\]
From hypotheses \( H(j)(3) \), Proposition 3.23(iii) in [27] and (5), we obtain
\[
J^0(0; w - 0) \leq (\|c_{ij}\|_{L^2(0, T)} + c_{1j}\|\eta\|_{L^2(0, T; Y)} + c_{2j}\|\zeta\|_{L^2(0, T; Z)}) \|w\|_V \tag{21}
\]
for all \( w \in U \) while \( H(A)(4) \) entails
\[
\|A0\|_{V^*} \leq \sqrt{2} (\|a_0\|_{L^2(0, T)} + a_1\|\xi\|_{L^2(0, T; X)}). \tag{23}
\]
Recalling that \( \langle jw, w \rangle_{V^* \times V} = \|w\|_V^2 \) for all \( w \in V \) and using (21), (22) and (23), we obtain
\[
G(w, 0) + \lambda\langle jw, -w \rangle_{V^* \times V}
\]
\[
\leq -(m_A - m_j + \lambda)\|w\|_V + \sqrt{2} (\|a_0\|_{L^2(0, T)} + a_1\|\xi\|_{L^2(0, T; X)})
\]
\[
+ \|f\|_V + \|c_{ij}\|_{L^2(0, T)} + c_{1j}\|\eta\|_{L^2(0, T; Y)} + c_{2j}\|\zeta\|_{L^2(0, T; Z)}.
\]
Hence, we deduce that the condition (19) holds with \( v_0 = 0 \).

Having verified all hypotheses of Theorem 2.5, from this result, we deduce that the problem (10) has a solution \( w \in K_1 \). This completes the proof of the lemma. \( \square \)
We conclude, by Lemma 3.3, that the problem (2) has a solution \( w \in \mathcal{W} \). This completes Step 2.

**Step 3.** Let \((\xi_i, \eta_i, \zeta_i) \in L^2(0, T; X \times Y \times Z), i = 1, 2\) and \(w_1 = w_{\xi_1, \eta_1, \zeta_1}, w_2 = w_{\xi_2, \eta_2, \zeta_2} \in \mathcal{W}\) with \(w_1(t), w_2(t) \in K\) for a.e. \(t \in (0, T)\), be the unique solutions to the problem (2) corresponding to \((\xi_1, \eta_1, \zeta_1)\) and \((\xi_2, \eta_2, \zeta_2)\), respectively. We will prove

\[
\|w_1 - w_2\|_{L^2(0, t; V)} \leq c \left(\|\xi_1 - \xi_2\|_{L^2(0, t; X)} + \|\eta_1 - \eta_2\|_{L^2(0, t; Y)} + \|\zeta_1 - \zeta_2\|_{L^2(0, t; Z)} \right) \tag{24}
\]

for all \(t \in [0, T]\), where \(c\) is a positive constant. By the definition of solution to the problem (2), it follows that

\[
\begin{align*}
\langle w_1'(t) + A(t, \xi_1(t), w_1(t)) - f(t), w_2(t) - w_1(t) \rangle_{V^* \times V} + j^0(t, \eta_1(t), \zeta_1(t), w_1(t); w_2(t) - w_1(t)) & \geq 0 \\
\langle w_2'(t) + A(t, \xi_2(t), w_2(t)) - f(t), w_1(t) - w_2(t) \rangle_{V^* \times V} + j^0(t, \eta_2(t), \zeta_2(t), w_2(t); w_1(t) - w_2(t)) & \geq 0
\end{align*}
\]

for a.e. \(t \in (0, T)\) and \(w_1(0) = w_2(0) = 0\). We add these two inequalities to obtain

\[
\begin{align*}
\langle w_1'(t) - w_2'(t), w_1(t) - w_2(t) \rangle_{V^* \times V} & + \langle A(t, \xi_1(t), w_1(t)) - A(t, \xi_2(t), w_2(t)), w_1(t) - 2_1(t) \rangle_{V^* \times V} \\
& \leq j^0(t, \eta_1(t), \zeta_1(t), w_1(t); w_2(t) - w_1(t)) + j^0(t, \eta_2(t), \zeta_2(t), w_2(t); w_1(t) - w_2(t))
\end{align*}
\]

for a.e. \(t \in (0, T)\). Next, we integrate the above inequality on \((0, t)\), and use hypotheses \(H(A)(5)\) and \(H(j)(5)\) to get

\[
\begin{align*}
\frac{1}{2} \left\| w_1(t) - w_2(t) \right\|_{H^1}^2 & - \frac{1}{2} \left\| w_1(0) - w_2(0) \right\|_{H^1}^2 \leq m_A \int_0^t \left\| w_1(s) - w_2(s) \right\|_V^2 ds \\
& - m_0 \int_0^t \|\xi_1(s) - \xi_2(s)\|_X \left\| w_1(s) - w_2(s) \right\|_V ds \leq m_j \int_0^t \left\| w_1(s) - w_2(s) \right\|_V^2 ds \\
& + m_1 \int_0^t \|\eta_1(s) - \eta_2(s)\|_Y \left\| w_1(s) - w_2(s) \right\|_V ds \\
& + m_2 \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_Z \left\| w_1(s) - w_2(s) \right\|_V ds
\end{align*}
\]

for all \(t \in [0, T]\). Next, using the Hölder inequality, we have

\[
(m_A - m_j) \left\| w_1 - w_2 \right\|_{L^2(0, t; V)} \leq m_0 \left\| \xi_1 - \xi_2 \right\|_{L^2(0, t; X)} \left\| w_1 - w_2 \right\|_{L^2(0, t; V)} + m_1 \left\| \eta_1 - \eta_2 \right\|_{L^2(0, t; Y)} \left\| w_1 - w_2 \right\|_{L^2(0, t; V)} + m_2 \left\| \zeta_1 - \zeta_2 \right\|_{L^2(0, t; Z)} \left\| w_1 - w_2 \right\|_{L^2(0, t; V)}
\]

for all \(t \in [0, T]\). Hence, by hypothesis \((H_1)\), the inequality (24) follows.

**Step 4.** We use a fixed point argument. We define the operator \(\Lambda: L^2(0, T; X \times Y \times Z) \rightarrow L^2(0, T; X \times Y \times Z)\) by

\[
\Lambda(\xi, \eta, \zeta) = (R_1 w_{\xi_1}, R_2 w_{\xi_2}, S_1 w_{\xi_1}), \quad \text{for all } (\xi, \eta, \zeta) \in L^2(0, T; X \times Y \times Z),
\]

where \(w_{\xi_1} \in \mathcal{W}\) denotes the unique solution to the problem (2) corresponding to \((\xi, \eta, \zeta)\).
From hypothesis $H(R, S)$, inequality (24) and by the Jensen inequality, we find a constant $c > 0$ such that

$$\|\Lambda(\xi_1, \eta_1, \zeta_1)(t) - \Lambda(\xi_2, \eta_2, \zeta_2)(t)\|_{X \times Y \times Z}^2 \leq \left( c_R \int_0^t \|w_1(s) - w_2(s)\|_V \, ds \right)^2 + \left( c_R \int_0^t \|w_1(s) - w_2(s)\|_V \, ds \right)^2 \leq c \int_0^t \|w_1(t) - w_2(t)\|_{L^2(0,t,V)}^2 \, ds$$

which entails

$$\|\Lambda(\xi_1, \eta_1, \zeta_1)(t) - \Lambda(\xi_2, \eta_2, \zeta_2)(t)\|_{X \times Y \times Z} \leq c \int_0^t \|\xi_1 - \xi_2\|_{L^2(0,t,X)}^2 + \|\eta_1 - \eta_2\|_{L^2(0,t,Y)}^2 + \|\zeta_1 - \zeta_2\|_{L^2(0,t,Z)}^2 \, ds$$

for a.e. $t \in (0, T)$. By Lemma 2.7, we deduce that there exists a unique fixed point $(\xi^*, \eta^*, \zeta^*)$ of $\Lambda$, i.e.,

$$(\xi^*, \eta^*, \zeta^*) \in L^2(0,T; X \times Y \times Z) \quad \text{and} \quad \Lambda(\xi^*, \eta^*, \zeta^*) = (\xi^*, \eta^*, \zeta^*).$$

Next, let $(\xi^*, \eta^*, \zeta^*) \in L^2(0,T; X \times Y \times Z)$ be the unique fixed point of the operator $\Lambda$. We define $w_{\xi^*, \eta^*, \zeta^*} \in W$ to be the unique solution to the problem (2) corresponding to $(\xi^*, \eta^*, \zeta^*)$. From the definition of the operator $\Lambda$, we have

$$\xi^* = R_1(w_{\xi^*, \eta^*, \zeta^*}), \quad \eta^* = R(w_{\xi^*, \eta^*, \zeta^*}) \quad \text{and} \quad \zeta^* = S(w_{\xi^*, \eta^*, \zeta^*}).$$

Finally, we use these relations in the problem (2), and conclude that $w_{\xi^*, \eta^*, \zeta^*}$ is the unique solution to Problem 5. This completes the proof of the theorem in the case $w_0 = 0$.

**Step 5.** Let $w_0 \in V$ be arbitrary. We prove existence of solution to Problem 5, i.e., we will find $w \in W$ such that $w(t) \in K$ for a.e. $t \in (0, T)$ and

$$\begin{cases}
\langle w'(t) + A(t, R_1 w)(t), w(t) \rangle - f(t, v - w(t))_{V^* \times V} \\
+j^0(t, (Rw)(t), (Sw)(t), w(t); v - w(t)) \geq 0
\end{cases} \quad \text{for all } v \in K, \text{ a.e. } t \in (0, T),$$

$$w(0) = w_0.$$

To this end, we introduce a new variable in (26) to reduce the initial condition to the homogeneous one. We observe that $w \in W$ is a solution to (26) if and only if $\overline{w}(t) = w(t) - w_0$ satisfies: $\overline{w} \in W$ such that $\overline{w}(t) \in K_{\theta}$ for a.e. $t \in (0, T)$ and

$$\begin{cases}
\langle \overline{w}'(t) + \overline{A}(t, \overline{R}_1 \overline{w})(t), \overline{w}(t) + w_0 \rangle - f(t, \overline{v} - \overline{w}(t))_{V^* \times V} \\
+j^0(t, \overline{(Rw)}(t), \overline{(Sw)}(t), \overline{w}(t) + w_0; \overline{v} - \overline{w}(t)) \geq 0
\end{cases} \quad \text{for all } \overline{v} \in K_{\theta}, \text{ a.e. } t \in (0, T),$$

$$\overline{w}(0) = 0,$$

where the operator $\overline{A} : (0, T) \times X \times V \to V^*$ is defined by

$$\overline{A}(t, x, v) = A(t, x, v + w_0) \quad \text{for } x \in X, v \in V, \text{ a.e. } t \in (0, T),$$
the function $\bar{f} : (0,T) \times Y \times Z \times V \to \mathbb{R}$ is given by
\[
\bar{f}(t, y, z, v) = f(t, y, z, v) + w(t) \quad \text{for } y \in Y, z \in Z, v \in V, \text{ a.e. } t \in (0,T),
\]
the operators $\bar{R}_1 : V \to L^2(0,T;X)$, $\bar{R} : V \to L^2(0,T;Y)$, $\bar{S} : V \to L^2(0,T;Z)$ are defined by
\[
\bar{R}_1 \bar{w} = R_1(\bar{w} + w_0), \quad \bar{R} \bar{w} = \bar{R} (\bar{w} + w_0) \quad \text{and} \quad \bar{S} \bar{w} = \bar{S} (\bar{w} + w_0) \quad \text{for } \bar{w} \in V,
\]
the set $\mathcal{K} = K - w_0 \subset V$, and $v = v - w_0$.

Next, we can check that the data $\bar{A}, \bar{f}, \bar{R}_1, \bar{R}, \bar{S}$ and $\mathcal{K}$ satisfy the hypotheses $H(A), H(f), H(R,S)$, and $H(K)$, respectively. Thus, the problem (27) with homogeneous initial condition has the unique solution $\bar{w} \in \mathcal{W}$ with $\bar{w}(t) \in \mathcal{K}$ for a.e. $t \in (0,T)$. Finally, $w(t) = \bar{w}(t) + w_0$ in the unique solution to the problem (26). This completes the proof of the theorem.

\[\Box\]

4. Particular cases. In this section we formulate three particular cases of Problem 5, already formulated in Section 1, for which existence and uniqueness results are obtained as consequences of Theorem 3.1.

We begin with Problem 1. Let $Y$ and $Z$ be Banach spaces.

\[H(A)_1 : \quad A : (0,T) \times V \to V^* \text{ is such that}\]
\begin{enumerate}
\item[(1)] $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$.
\item[(2)] $A(t, \cdot)$ is demicontinuous on $V$ for a.e. $t \in (0,T)$.
\item[(3)] $\|A(t, v)\|_V \leq a_0(t) + a_2\|v\|_V$ for all $v \in V$, a.e. $t \in (0,T)$ with $a_0 \in L^2(0,T)$, $a_0 \geq 0$ and $a_2 \geq 0$.
\item[(4)] there is constant $m_A > 0$ such that
\[
\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{V \times V} \geq m_A\|v_1 - v_2\|_V^2
\]
for all $v_1, v_2 \in V$, a.e. $t \in (0,T)$.
\end{enumerate}

\[H(g) : \quad g : (0,T) \times Y \times Z \to \mathbb{R} \text{ is such that}\]
\begin{enumerate}
\item[(1)] $g(\cdot, y, v)$ is measurable on $(0,T)$ for all $y \in Y$, $v \in V$.
\item[(2)] $g(t, \cdot, v)$ is continuous on $Y$ for all $v \in V$, a.e. $t \in (0,T)$.
\item[(3)] $g(t, y, \cdot)$ is locally Lipschitz on $V$ for all $y \in Y$, a.e. $t \in (0,T)$.
\item[(4)] $\|\partial g(t, y, v)\|_V \leq c_{0g}(t) + c_{1g}y\|y\|_Y + c_{2g}\|v\|_V$ for all $y \in Y$, $v \in V$, a.e. $t \in (0,T)$ with $c_{0g} \in L^2(0,T)$, $c_{0g}, c_{1g}, c_{2g} \geq 0$.
\item[(5)] there are constants $m_g, m_1 \geq 0$ such that
\[
g^0(t, y_1, v_1; v_2 - v_1) + g^0(t, y_2, v_2; v_1 - v_2) \leq m_g\|v_1 - v_2\|_V^2 + m_1\|y_1 - y_2\|_Y\|v_1 - v_2\|_V
\]
for all $y_i \in Y$, $v_i \in V$, $i = 1, 2$, a.e. $t \in (0,T)$.
\end{enumerate}

\[H(\varphi) : \quad \varphi : (0,T) \times Z \times V \to \mathbb{R} \text{ is such that}\]
\begin{enumerate}
\item[(1)] $\varphi(\cdot, z, v)$ is measurable on $(0,T)$ for all $z \in Z$, $v \in V$.
\item[(2)] $\varphi(t, \cdot, v)$ is continuous on $Z$ for all $v \in V$, a.e. $t \in (0,T)$.
\item[(3)] $\varphi(t, z, \cdot)$ is convex and lsc on $V$ for all $z \in Z$, a.e. $t \in (0,T)$.
\item[(4)] $\|\partial \varphi(t, z, v)\|_V \leq c_{0\varphi}(t) + c_{1\varphi}\|z\|_Z + c_{2\varphi}\|v\|_V$ for all $z \in Z$, $v \in V$, a.e. $t \in (0,T)$ with $c_{0\varphi} \in L^2(0,T)$, $c_{0\varphi}, c_{1\varphi}, c_{2\varphi} \geq 0$. 
\end{enumerate}
Corollary 2. Assume the hypotheses $H(A)_1$, $H(g)$, $H(H)$, $H(K)$, $H(f)$ and $m_A > m_g$. Let the operators $R$ and $S$ satisfy $H(R, S)$. Then, there exists a unique $w \in \mathcal{W}$ such that $w(t) \in K$ for a.e. $t \in (0, T)$, $w(0) = w_0$ and

$$
\begin{cases}
(w'(t) + A(t, w(t)) - f(t), v - w(t))_{V^* \times V} + g^0(t, (Rw)(t), w(t); v - w(t)) \\
+ \varphi(t, (Sw)(t), v) - \varphi(t, (Sw)(t), w(t)) \geq 0
\end{cases}
$$

for all $v \in K$, a.e. $t \in (0, T)$.

**Proof.** It is obvious that the hypothesis $H(A)_1$ implies condition $H(A)$. We define

$$j(t, y, z, v) = g(t, y, v) + \varphi(t, z, v)$$

for $y \in Y$, $z \in Z$, $v \in V$, a.e. $t \in (0, T)$.

It is easy to observe that $H(j)(1)$, (2), and (4) hold from [9, Proposition 5.2.10] and $H(\varphi)(3)$, it follows that $\varphi(t, z, \cdot)$ is locally Lipschitz for all $z \in Z$, a.e. $t \in (0, T)$. Hence and by $H(g)(3)$, we infer that $H(j)(3)$ holds. Also, using [27, Proposition 3.35(ii)] together with $H(g)(5)$ and $H(\varphi)(5)$, we verify that the condition $H(j)(5)$ is satisfied with $m_j = m_g$. Then, using Theorem 3.1, we deduce the existence of a unique solution to Problem 1 with the desired regularity. This completes the proof of the theorem.

We now proceed with the study of Problem 2.

Corollary 2. Assume the hypotheses $H(A)$, $H(\varphi)$, $H(K)$, $H(f)$ and $u_0 \in V$. Then, there exists a unique $u \in C(0, T; V)$ with $u' \in \mathcal{W}$ such that $u'(t) \in K$ for a.e. $t \in (0, T)$, $u(0) = u_0$, $u'(0) = w_0$ and

$$
\begin{cases}
(u''(t) + A(t, u(t), u'(t)) - f(t), v - u'(t))_{V^* \times V} \\
+ \varphi(t, u(t), v) - \varphi(t, u(t), u'(t)) \geq 0
\end{cases}
$$

for all $v \in K$, a.e. $t \in (0, T)$.

**Proof.** We will apply Theorem 3.1. Let $w = u'$ and $X = Y = Z = V$. Consider the operators $R_1$, $R$, $S$: $V \to V$ and the function $j: (0, T) \times Y \times Z \times V \to \mathbb{R}$ defined by

$$j(t, y, z, v) = \varphi(t, z, v)$$

for $z \in Z$, $v \in V$, a.e. $t \in (0, T)$.

Note that under this notation the function $u$ is a solution to Problem 2 if and only if $w$ is a solution to the inequality:

$$
\begin{cases}
\text{find } w \in \mathcal{W} \text{ such that } w(t) \in K \text{ for a.e. } t \in (0, T), w(0) = w_0 \\
(w'(t) + A(t, (R_1w)(t), w(t)) - f(t), v - w(t))_{V^* \times V} + \varphi(t, (Sw)(t), v) \\
- \varphi(t, (Sw)(t), w(t)) \geq 0
\end{cases}
$$

(28)

for all $v \in K$, a.e. $t \in (0, T)$.

The operators $R_1$ and $S$ satisfy $H(R, S)$. It is easy to see that the function $j$ satisfies $H(j)$ with $m_j = 0$. Thus ($H_1$) holds and no smallness condition is needed. As a consequence of Theorem 3.1, we deduce that (28) has a unique solution $w \in \mathcal{W}$ with $w(t) \in K$ for a.e. $t \in (0, T)$. Hence, we have $u' \in \mathcal{W}$, $u'(t) \in K$ for a.e. $t \in (0, T)$ while the equality $u(t) = u_0 + \int_0^t w(s) \, ds$ and $u_0 \in V$ implies $u \in C(0, T; V)$ which concludes the proof.

\[\square\]
We end this section with a result on Problem 3. We need the following hypotheses.

\[ H(A)_2 : A : (0, T) \times V \to V^* \text{ is such that} \]

1. \(A(\cdot, v)\) is measurable on \((0, T)\) for all \(v \in V\).
2. \(A(t, \cdot)\) is continuous on \(V\) for a.e. \(t \in (0, T)\).
3. \(\|A(t, v)\|_{V^*} \leq a_0(t) + a_2\|v\|_{V}\) for all \(v \in V\), a.e. \(t \in (0, T)\) with \(a_0 \in L^2(0, T)\), \(a_0 \geq 0\) and \(a_2 \geq 0\).

\[ H(B) : B : (0, T) \times V \to V^* \text{ is such that} \]

1. \(B(\cdot, v)\) is measurable on \((0, T)\) for all \(v \in V\).
2. \(B(t, \cdot)\) is demicontinuous on \(V\) for a.e. \(t \in (0, T)\).
3. \(\|B(t, v)\|_{V^*} \leq b_0(t) + b_2\|v\|_{V}\) for all \(v \in V\), a.e. \(t \in (0, T)\) with \(b_0 \in L^2(0, T)\), \(b_0 \geq 0\) and \(b_2 \geq 0\).
4. there is constant \(m_B > 0\) such that

\[ \langle B(t, v_1) - B(t, v_2), v_1 - v_2 \rangle_{V^* \times V} \geq m_B\|v_1 - v_2\|_{V}^2 \]

for all \(v_1, v_2 \in V\), a.e. \(t \in (0, T)\).

**Corollary 3.** Assume the hypotheses \(H(A)_2\), \(H(B)\), \(H(g)\), \(H(K)\), \(H(f)\), \(m_B > m_g\) and \(u_0 \in V\). Then, there exists a unique \(u \in C(0, T; V)\) with \(u' \in \mathcal{W}\) such that \(u'(t) \in K\) for a.e. \(t \in (0, T)\), \(u(0) = u_0\), \(u'(0) = u_0\) and

\[
\begin{align*}
\langle u''(t) + A(t, u(t)) + B(t, u'(t)) - f(t), v - u'(t) \rangle_{V^* \times V} \\
+ g(t, u(t), u'(t); v - u'(t)) \geq 0 
\end{align*}
\]

for all \(v \in K\), a.e. \(t \in (0, T)\).

**Proof.** We will apply Theorem 3.1. Let \(w = u'\), \(X = Y = V\), \(\varphi = 0\), and \(R = R_1\), \(R_1 : V \to V\), \((R_1 w)(t) = u_0 + \int_0^t w(s) ds \) for \(w \in V\). Let the operator \(A : (0, T) \times V \times V \to V^* \) and the function \(j : (0, T) \times V \times V \to \mathbb{R}\) be defined by

\[
A(t, x, v) := A(t, x) + B(t, v) \text{ for } x, v \in V, \text{ a.e. } t \in (0, T)
\]

\[
j(t, y, z) := g(t, y, v) \text{ for } y, v \in V, \text{ a.e. } t \in (0, T).
\]

From \(H(A)_2(2)\) and \(H(B)(4)\), it follows that \(H(A)\) is satisfied with \(m_A = m_B\). Other conditions in \(H(A)\) hold easily. Using \(H(g)\) we can verify that \(H(j)\) is satisfied with \(m_j = m_g\). The application of Theorem 3.1 completes the proof.

5. **Dynamic unilateral frictional contact problem.** In this section, we study a new unilateral viscoelastic frictional contact problem to which our main results of Section 3 can be applied. We give the classical formulation of the contact problem, provide its variational formulation, and prove a result on its unique weak solvability.

The physical setting is as follows. A viscoelastic body occupies a bounded Lipschitz domain \(\Omega \subset \mathbb{R}^d\), \(d = 1, 2, 3\). The boundary \(\Gamma\) of \(\Omega\) consists of four mutually disjoint and measurable parts \(\Gamma_1, \Gamma_2, \Gamma_3, \text{ and } \Gamma_4\) such that \(\text{meas}(\Gamma_1) > 0\). The outward unit normal at the boundary is denoted by \(\nu\). Let \(Q = \Omega \times (0, T)\) with \(0 < T < \infty\). The symbol \(\mathbb{S}^d\) denotes the space of \(d \times d\) symmetric matrices with the standard inner product and norm. For a vector field \(\xi\) defined on the boundary, \(\xi_\nu\) and \(\xi_\tau\) stand for its normal and tangential components defined by \(\xi_\nu = \xi \cdot \nu\) and \(\xi_\tau = \xi - \xi_\nu\nu\). Moreover, given a tensor \(\sigma\), the symbols \(\sigma_\nu\) and \(\sigma_\tau\) denote its normal and tangential components on the boundary, i.e., \(\sigma_\nu = (\sigma\nu) \cdot \nu\) and \(\sigma_\tau = \sigma\nu - \sigma_\nu\nu\). We consider the classical model for the contact process.
Problem 6. Find a displacement field \( u : \Omega \times (0, T) \to \mathbb{R}^d \) and a stress field \( \sigma : \Omega \times (0, T) \to \mathbb{S}^d \) such that for all \( t \in (0, T) \),

\[
\begin{align*}
\sigma(t) & = A(t, \varepsilon(u'(t))) + B(t, \varepsilon(u(t))) + \int_0^t C(t-s)\varepsilon(u'(s)) \, ds \quad \text{in } \Omega, \\
u''(t) & = \text{Div} \sigma(t) + f_0(t) \quad \text{in } \Omega, \\
u(t) & = 0 \quad \text{on } \Gamma_1, \\
\sigma(t) \nu & = f_N(t) \quad \text{on } \Gamma_2, \\
u_\nu'(t) & \leq g_0, \quad \sigma_\nu(t) + \eta(t) \leq 0, \quad (u_\nu'(t) - g_0)(\sigma_\nu(t) + \eta(t)) = 0, \\
\eta(t) & \in k(u_\nu(t)) \partial_j \psi(u_\nu'(t)) \quad \text{on } \Gamma_3, \\
\sigma_\nu'(t) & = 0 \quad \text{on } \Gamma_4,
\end{align*}
\]

and

\[
\begin{align*}
u(0) & = u_0, \quad u'(0) = w_0 \quad \text{in } \Omega.
\end{align*}
\]

The equation (29) is the constitutive law for viscoelastic materials with long memory. Here, \( A \) represents the viscosity operator, \( B \) is the elasticity operator and \( C \) denotes the relaxation tensor, while \( \varepsilon(u) \) is the linearized strain tensor given by

\[
\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega.
\]

Equation (30) stands for the equation of motion in which \( \text{Div} \sigma = (\sigma_{ij,j}) \) is the divergence operator and \( f_0 \) denotes the time dependent density of the body forces. The displacement boundary condition (31) means that the body is fixed on \( \Gamma_D \) and (32) represents the traction boundary condition with surface tractions of density \( f_N \) acting on the part \( \Gamma_N \). Condition (33) is the frictionless Signorini unilateral contact boundary condition for the normal velocity in which \( g_0 > 0 \), and \( \partial j_\nu \) denotes the Clarke subgradient of a given nonsmooth function \( j_\nu \). Condition \( \eta(t) \in k(u_\nu(t)) \partial j_\nu(u_\nu'(t)) \) on \( \Gamma_3 \) represents a multivalued extension of the normal damped response condition where \( k \) is a given damper coefficient depending on the normal displacement. The first condition in (34) is a multivalued version of the normal compliance contact condition in which \( m \) is a positive stiffness coefficient of the obstacle and \( p \) represents the current value of the normal displacement. The friction law is given in (34), where \( j_\nu \) is a prescribed function and its subdifferential in the second variable is taken in the sense of Clarke. Several examples including the nonmonotone friction law, the Tresca friction law, the power-law friction can be formulated in this way. Finally, conditions (35) are the initial conditions in which \( u_0 \) and \( w_0 \) represent the initial displacement and the initial velocity, respectively.

We refer to [18, 27, 41] and references therein for the detailed discussion of boundary conditions in this contact problem.

We provide the weak formulation of Problem 6. We use the spaces

\[
\mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad V = \{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \}.
\]
The space $\mathcal{H}$ is a Hilbert space endowed with the inner product

$$\langle \sigma, \varepsilon \rangle_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}(x) \varepsilon_{ij}(x) \, dx \quad \text{for all } \sigma, \varepsilon \in \mathcal{H},$$

and the associated norm $\| \cdot \|_{\mathcal{H}}$. The inner product and norm on $V$ are given by

$$(\mathbf{u}, \mathbf{v})_{V} = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \| \mathbf{v} \|_{V} = \| \varepsilon(\mathbf{v}) \|_{\mathcal{H}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

The symbol $\| \gamma \|$ denotes the norm of the linear bounded trace operator $\gamma : V \to L^{2}(\Gamma; \mathbb{R}^{d})$. The space of fourth order tensor fields is given by

$$Q_{\infty} = \{ \mathbf{\sigma} = (\sigma_{ijkl}) | \sigma_{ijkl} = \sigma_{klji} \in L^{\infty}(\Omega), 1 \leq i, j, k, l \leq d \}$$

which is a real Banach space with the norm

$$\| \mathbf{\sigma} \|_{Q_{\infty}} = \sum_{1 \leq i, j, k, l \leq d} \| \sigma_{ijkl} \|_{L^{\infty}(\Omega)} \quad \text{for all } \mathbf{\sigma} \in Q_{\infty}.$$

We need the following hypotheses on Problem 6.

**\(H(A)\):** $\mathcal{A} : Q \times S^{d} \to S^{d}$ is such that

1. $\mathcal{A}(\cdot, \varepsilon)$ is measurable on $Q$ for all $\varepsilon \in S^{d}$.
2. there exists $L_{\mathcal{A}} > 0$ such that $\| \mathcal{A}(x, t, \varepsilon_{1}) - \mathcal{A}(x, t, \varepsilon_{2}) \| \leq L_{\mathcal{A}} \| \varepsilon_{1} - \varepsilon_{2} \|$ for all $\varepsilon_{1}, \varepsilon_{2} \in S^{d}$, a.e. $(x, t) \in Q$.
3. there exists $m_{\mathcal{A}} > 0$ such that $\mathcal{A}(x, t, \varepsilon_{1}) - \mathcal{A}(x, t, \varepsilon_{2}) \cdot (\varepsilon_{1} - \varepsilon_{2}) \geq m_{\mathcal{A}} \| \varepsilon_{1} - \varepsilon_{2} \|^{2}$ for all $\varepsilon_{1}, \varepsilon_{2} \in S^{d}$, a.e. $(x, t) \in Q$.
4. $\mathcal{A}(x, t, 0) = 0$ for a.e. $(x, t) \in Q$.

**\(H(\mathcal{B})\):** $\mathcal{B} : Q \times S^{d} \to S^{d}$ is such that

1. $\mathcal{B}(\cdot, \varepsilon)$ is measurable on $Q$ for all $\varepsilon \in S^{d}$.
2. there exists $L_{\mathcal{B}} > 0$ such that $\| \mathcal{B}(x, t, \varepsilon_{1}) - \mathcal{B}(x, t, \varepsilon_{2}) \| \leq L_{\mathcal{B}} \| \varepsilon_{1} - \varepsilon_{2} \|$ for all $\varepsilon_{1}, \varepsilon_{2} \in S^{d}$, a.e. $(x, t) \in Q$.
3. $\mathcal{B}(x, t, 0) = 0$ for a.e. $(x, t) \in Q$.

**\(H(\mathcal{C})\):** $\mathcal{C} \in L^{\infty}(0, T; Q_{\infty})$.

**\(H(k)\):** $k : \Gamma_{3} \times \mathbb{R} \to \mathbb{R}$ is such that

1. $k(\cdot, r)$ is measurable on $\Gamma_{3}$ for all $r \in \mathbb{R}$.
2. there exist $k_{1}, k_{2}$ such that $0 < k_{1} \leq k(x, r) \leq k_{2}$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_{3}$.
3. there exists $L_{k} > 0$ such that $|k(x, r_{1}) - k(x, r_{2})| \leq L_{k}|r_{1} - r_{2}|$ for all $r_{1}, r_{2} \in \mathbb{R}$, a.e. $x \in \Gamma_{3}$.

**\(H(\psi)\):** $\psi : \Gamma_{3} \times \mathbb{R} \to \mathbb{R}$ is such that

1. $\psi(\cdot, r)$ is measurable on $\Gamma_{3}$ for all $r \in \mathbb{R}$.
2. $\psi(x, \cdot)$ is convex and lsc, for a.e. $x \in \Gamma_{3}$.

**\(H(m)\):** $m : \Gamma_{4} \times \mathbb{R} \to \mathbb{R}^{+}$ is such that

1. $m(\cdot, r)$ is measurable on $\Gamma_{4}$ for all $r \in \mathbb{R}$.
2. there exists $m_{\Gamma}$ such that $|m(x, r)| \leq m_{\Gamma}$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_{4}$.
3. there exists $L_{m} > 0$ such that $|m(x, r_{1}) - m(x, r_{2})| \leq L_{m}|r_{1} - r_{2}|$ for all $r_{1}, r_{2} \in \mathbb{R}$, a.e. $x \in \Gamma_{4}$.

**\(H(p)\):** $p : \Gamma_{4} \times (0, T) \times \mathbb{R} \to \mathbb{R}^{+}$ is such that

1. $p(\cdot, \cdot, r)$ is measurable on $\Gamma_{4} \times (0, T)$ for all $r \in \mathbb{R}$.
(2) there exists \( \overline{p} \) such that \( |p(x, t, r)| \leq \overline{p} \) for all \( r \in \mathbb{R} \), a.e. \( (x, t) \in \Gamma_4 \times (0, T) \).

(3) there exists \( L_p > 0 \) such that \( |p(x, t, r_1) - p(x, t, r_2)| \leq L_p |r_1 - r_2| \) for all \( r_1, r_2 \in \mathbb{R} \), a.e. \( (x, t) \in \Gamma_4 \times (0, T) \).

\[ H(j_r) : j_r : \Gamma_4 \times (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R} \] is such that

(1) \( j_r(\cdot, \cdot, \xi) \) is measurable on \( \Gamma_4 \times (0, T) \) for all \( \xi \in \mathbb{R}^d \) and there exists \( e \in L^2(\Gamma_4; \mathbb{R}^d) \) such that \( j_r(\cdot, \cdot, e(\cdot)) \in L^1(\Gamma_4 \times (0, T)) \).

(2) \( j_r(x, t, \cdot) \) is locally Lipschitz on \( \mathbb{R}^d \) for a.e. \( (x, t) \in \Gamma_4 \times (0, T) \).

(3) \( \| j_r(x, t, \xi) \|_{\mathbb{R}^d} \leq c_0 + c_1 \| \xi \|_{\mathbb{R}^d} \) for all \( \xi \in \mathbb{R}^d \), a.e. \( (x, t) \in \Gamma_4 \times (0, T) \) with \( c_0, c_1 \geq 0 \).

(4) \( j^0_r(x, t, \xi_1; x_2 - x_1) + j_r^0(x, t, \xi_2; x_1 - x_2) \leq m_j \| \xi_1 - \xi_2 \|_{\mathbb{R}^d}^2 \) for all \( \xi_1, \xi_2 \in \mathbb{R}^d \), a.e. \( (x, t) \in \Gamma_4 \times (0, T) \) with \( m_j \geq 0 \).

\[ (H_3) : f \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)), f_N \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d)), u_0, w_0 \in V. \]

Further, we introduce the set \( U \) defined by

\[ U = \{ v \in V \mid v_\nu \leq g_0 \text{ on } \Gamma_C \}, \]

and an element \( f \in V^* \) given by

\[ (f, v)_{V^* \times V} = (f_0, v)_{L^2(\Omega; \mathbb{R}^d)} + (f_N, v)_{L^2(\Gamma_2; \mathbb{R}^d)} \quad \text{for} \quad v \in V. \]

Exploiting the standard procedure, see e.g. [19, 27, 38], we obtain the following weak formulation of Problem 6.

**Problem 7.** Find \( u : (0, T) \rightarrow V \) such that \( u(0) = u_0, u'(0) = w_0 \) and

\[
\int_{\Omega} u''(t) \cdot (v - u'(t)) \, dx + \langle A(t, e(u'(t))), \varepsilon(v) - \varepsilon(u'(t)) \rangle_H \\
+ \langle B(t, e(u(t))) + \int_0^t C(t,s) \varepsilon(u'(s)) \, ds, \varepsilon(v) - \varepsilon(u'(t)) \rangle_H \]

\[
+ \int_{\Gamma_3} k(u_\nu(t)) (\psi(v_\nu) - \psi(u'_\nu(t))) \, d\Gamma \\
+ \int_{\Gamma_4} m_2 (\int_0^t u_\nu(s) \, ds) p(t, u_\nu(t)) (v_\nu - u'_\nu(t)) \, d\Gamma \]

\[
+ \int_{\Gamma_4} j^0_r(t, u'_\nu(t); v_\nu - u'_\nu(t)) \, d\Gamma \geq \langle f(t), v - u'(t) \rangle_{V^* \times V}
\]

for all \( v \in U \), a.e. \( t \in (0, T) \).

The following result concerns the unique solvability solution to Problem 7.

**Theorem 5.1.** Assume hypotheses \( H(A), H(B), H(C), H(k), H(\psi), H(m), H(p), H(j_r), (H_3) \) and the following smallness condition

\[ m_A > m_j \| \gamma \|_2^2. \]

Then Problem 7 has a unique solution \( u \in C([0, T]; V), u' \in W \) with \( u'(t) \in U \) for a.e. \( t \in (0, T) \).

**Proof.** We will apply Theorem 3.1. To this end, we introduce the operator \( \zeta : V \rightarrow V \) defined by

\[ (\zeta w)(t) = \int_0^t w(s) \, ds + u_0 \]
for \( w \in V \) and \( t \in (0, T) \). We denote by \((\zeta_w)(t)\) its normal component, i.e.,

\[
(\zeta_w)(t) = (\zeta w(t))_\nu = \int_0^t w_\nu(s) \, ds + u_0. \tag{39}
\]

Then, Problem 7 in terms of velocity \( w = u' \) can be equivalently formulated as follows.

**Problem 8.** Find \( w \in W \) such that \( w(t) \in U \) for a.e. \( t \in (0, T) \), \( w(0) = w_0 \) and

\[
\langle w'(t), v - w(t) \rangle_{V^* \times V} + (A(t, \varepsilon(w(t))), \varepsilon(v) - \varepsilon(w(t)))_H
\]

\[
+ \langle B(t, \varepsilon((\zeta w(t)))) + \int_0^t \mathfrak{C}(t - s) \varepsilon(w(s)) \, ds, \varepsilon(v) - \varepsilon(w(t)) \rangle_H
\]

\[
+ \int_{\Gamma_3} k((\zeta_w(t))(\psi(v_\nu) - \psi(w_\nu(t))) \, d\Gamma
\]

\[
+ \int_{\Gamma_4} m(\int_0^t (\zeta w(s) \, ds) p(t, (\zeta_w(t))(v_\nu - w_\nu(t)) \, d\Gamma
\]

\[
+ \int_{\Gamma_4} \rho(t, w(t); v - w(t)) \, d\Gamma \geq \langle f(t), v - w(t) \rangle_{V^* \times V}
\]

for all \( v \in U \), a.e. \( t \in (0, T) \).

Let \( X = Z = V^* \) and \( K = U \). Define the operator \( A : (0, T) \times X \times V \to V^* \) by

\[
(A(t, x, v), z)_{V^* \times V} = \langle A(t, \varepsilon(v)), \varepsilon(z) \rangle_H + \langle x, z \rangle_{V^* \times V}
\]

for \( x \in X, v, z \in V \), a.e. \( t \in (0, T) \). Let \( R_1 : V \to V^* \) be given by \( R_1 = R_2 + R_3 \), where

\[
\langle (R_2w)(t), v \rangle_{V^* \times V} = \langle B(t, \varepsilon((\zeta w(t)))) + \int_0^t \mathfrak{C}(t - s) \varepsilon(w(s)) \, ds, \varepsilon(v) \rangle_H,
\]

\[
\langle (R_3w)(t), v \rangle_{V^* \times V} = \int_{\Gamma_4} m(\int_0^t (\zeta w(s) \, ds) p(t, (\zeta_w(t))(v_\nu - w_\nu(t)) \, d\Gamma
\]

for \( w \in V, v \in V \), a.e. \( t \in (0, T) \), and let \( S : V \to V^* \) be defined by \( (Sw)(t) = (\zeta_w(t)) \) for \( w \in V \), a.e. \( t \in (0, T) \). Moreover, let \( j : (0, T) \times Y \times Z \times V \to R \) be defined by

\[
j(t, y, z, v) = g(t, v) + \varphi(t, z, v) \quad \text{for} \quad z \in Z, v \in V \), a.e. \( t \in (0, T) \),
\]

where \( g : (0, T) \times V \to R \) and \( \varphi : (0, T) \times Z \times V \to R \) are given by

\[
g(t, v) = \int_{\Gamma_3} j_{\tau}(t, v_\tau) \, d\Gamma, \quad \varphi(t, z, v) = \int_{\Gamma_3} k(z) \psi(v_\nu) \, d\Gamma
\]

for \( z \in Z, v \in V \), a.e. \( t \in (0, T) \), respectively. Note that \( j \) is independent of \( y \). Using the above notation, we consider the following auxiliary problem.

**Problem 9.** Find \( w \in W \) such that \( w(t) \in U \) for a.e. \( t \in (0, T) \), \( w(0) = w_0 \) and

\[
\langle w'(t) + A(t, R_1 w(t), w(t)), v - w(t) \rangle_{V^* \times V}
\]

\[
+ j^0(t, (Rw)(t), (Sw)(t), w(t); v - w(t)) \geq 0 \quad \text{for all} \quad v \in U, \text{ a.e.} \ t \in (0, T).
\]

To prove existence of solution to Problem 9, we will verify the hypotheses \( H(A), H(j), H(R, S), H(K), H(f) \), and \( (H_1) \) of Theorem 3.1. First, following the lines of [27, Theorem 7.3] we can show that the operator \( A \) satisfies hypothesis \( H(A) \) with \( m_A = m_A \) and \( m_0 = 1 \). Next, we recall, that a convex and lsc function defined on a Banach space, is locally Lipschitz, see [10, Proposition 5.2.10]. From this fact,
We obtain that unique weak solution. Further, this solution has the regularity solution to Problem 6. Under the assumptions of Theorem 5.1, Problem 6 has a (36) satisfies H respectively. Hence z for all w, Problem 9 has a unique solution condition (37). satisfies U j verify that Corollary 4.15(iii), the continuity of the trace operator, and the property ξ is verified. Having verified the hypotheses of Theorem 3.1, we infer from this result that j is history-dependent with L operators with constants follows from [41, Theorems 14.2 and 14.3] that also proved directly.

Finally, from the relation u(t) = \int_0^t w(s) ds + u_0 for all t ∈ (0, T), we conclude that u ∈ C([0, T]; V), u' ∈ W with u'(t) ∈ U for a.e. t ∈ (0, T). This completes the proof.

A couple of functions (u, σ) which satisfies Problem 7 and (29) is called a weak solution to Problem 6. Under the assumptions of Theorem 5.1, Problem 6 has a unique weak solution. Further, this solution has the regularity u ∈ C([0, T]; V), u' ∈ W with u'(t) ∈ U for a.e. t ∈ (0, T), σ ∈ L^2(0, T; H), and Div σ ∈ V^*.

Acknowledgments. The project was supported by the European Union’s Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie Grant Agreement No. 823731 CONMECH, the Natural Science Foundation of Guangxi under Project No. 2018GXNSFAA281353, the National Natural Science Foundation of China under Project Nos. 11661012 and 11771067, the Applied Basic Project of Sichuan Province No. 2019JY0204, the Fundamental Research Funds for the Central Universities No. ZYGX2019J095, and the Beibu Gulf University under Projects Nos. 2018KYQD03 and 2018KYQD06.
REFERENCES

[1] J. Ahn and D.E. Stewart, Dynamic frictionless contact in linear viscoelasticity, *IMA J. Numer. Anal.*, **29** (2009), 43–71.
[2] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123–145.
[3] S. Carl, V.K. Le and D. Motreanu, *Nonsmooth Variational Problems and their Inequalities: Comparison Principles and Applications*, Springer Monographs in Mathematics. Springer, New York, 2007.
[4] S. Carl, V.K. Le and D. Motreanu, Evolutionary variational-hemivariational inequalities: Existence and comparison results, *J. Math. Anal. Appl.*, **345** (2008), 545–558.
[5] O. Chadli, Q.H. Ansari and S. Al-Homidan, Existence of solutions for nonlinear implicit differential equations: An equilibrium problem approach, *Numer. Funct. Anal. Optim.*, **37** (2016), 1385–1419.
[6] O. Chadli, Q.H. Ansari and J.-C. Yao, Mixed equilibrium problems and anti-periodic solutions for nonlinear evolution equations, *J. Optim. Theory Appl.*, **168** (2016), 410–440.
[7] F.H. Clarke, *Optimization and Nonsmooth Analysis*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.
[8] M. Cocou, Existence of solutions of a dynamic Signorini’s problem with nonlocal friction in viscoelasticity, *Z. angew. Math. Phys.*, **53** (2002), 1099–1109.
[9] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[10] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic Publishers, Boston, MA, 2003.
[11] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics, 270. Chapman/CRC Press, New York, Boca Raton, FL, 2005.
[12] C. Eck, J. Jarušek and M. Sofonea, A dynamic elastic-visco-plastic unilateral contact problem with normal damped response and Coulomb friction, *European J. Appl. Math.*, **21** (2010), 229–251.
[13] E.-H. Essoufi and M. Kabbaj, Existence of solutions of a dynamic Signorini’s problem with nonlocal friction for viscoelastic piezoelectric materials, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.*), **48** (2005), 181–195.
[14] D. Goeleven, D. Motreanu, Y. Dumont and M. Rochdi, *Variational and Hemivariational Inequalities: Theory, Methods and Applications, Volume I. Unilateral Analysis and Unilateral Mechanics*, Nonconvex Optimization and its Applications, 69. Kluwer Academic Publishers, Boston, MA, 2003.
[15] D. Goeleven and D. Motreanu, *Variational and Hemivariational Inequalities, Theory, Methods and Applications. Vol. II: Unilateral Problems*, Nonconvex Optimization and its Applications, 70. Kluwer Academic Publishers, Boston, MA, 2003.
[16] N. Hadjisavvas and H. Khatibzadeh, Maximal monotonicity of bifunctions, *Optimization*, **59** (2010), 147–160.
[17] J.F. Han, S. Migórski and H.D. Zeng, Analysis of a dynamic viscoelastic unilateral contact problem with normal damped response, *Nonlinear Anal. Real World Appl.*, **18** (2016), 229–250.
[18] W.M. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, AMS/IP Studies in Advanced Mathematics, 30. American Mathematical Society, Providence, RI, International Press, Somerville, MA, 2002.
[19] J. Haslinger, M. Miettinen and P.D. Panagiotopoulos, *Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications*, Nonconvex Optimization and its Applications, 35. Kluwer Academic Publishers, Dordrecht, 1999.
[20] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps, *J. Optim. Theory Appl.*, **18** (1976), 445–454.
[21] A. Kulig and S. Migórski, Solvability and continuous dependence results for second order nonlinear inclusion with Volterra-type operator, *Nonlinear Anal.*, **75** (2012), 4729–4746.
[22] K. Kuttler and M. Shillor, Dynamic contact with Signorini’s condition and slip rate dependent friction, *Electron. J. Differ. Equ.*, **2004** (2004), 21 pp.
[24] S. Migórski, Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction, *Appl. Anal.*, **84** (2005), 669–699.

[25] S. Migórski and A. Ochal, A unified approach to dynamic contact problems in viscoelasticity, *J. Elasticity*, **83** (2006), 247–275.

[26] S. Migórski, A. Ochal and M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, *Nonlinear Anal. Real World Appl.*, **12** (2011), 3384–3396.

[27] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, 26. Springer, New York, 2013.

[28] S. Migórski, A. Ochal and M. Sofonea, History-dependent variational-hemivariational inequalities in contact mechanics, *Nonlinear Anal. Real World Appl.*, **22** (2015), 604–618.

[29] S. Migórski, A. Ochal and M. Sofonea, Evolutionary inclusions and hemivariational inequalities, *Advances in Variational and Hemivariational Inequalities*, Adv. Mech. Math., Springer, Cham, **33** (2015), 39–64.

[30] S. Migórski and J. Ogorzaly, A class of evolution variational inequalities with memory and its application to viscoelastic frictional contact problems, *J. Math. Anal. Appl.*, **442** (2016), 685–702.

[31] S. Migórski and J. Ogorzaly, Dynamic history-dependent variational-hemivariational inequalities with applications to contact mechanics, *Z. angew. Math. Phys.*, **68** (2017), Art. 15, 22 pp.

[32] S. Migórski, M. Sofonea and S.D. Zeng, Well-posedness of history-dependent sweeping processes, *SIAM J. Math. Anal.*, **51** (2019), 1082–1107.

[33] D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Nonconvex Optimization and its Applications, 29. Kluwer Academic Publishers, Dordrecht, 1999.

[34] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, 188. Marcel Dekker, Inc., New York, 1995.

[35] P.D. Panagiotopoulos, *Nonconvex energy functions, hemivariational inequalities and substationary principles*, Acta Mech., **42** (1983), 111–130.

[36] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications. Convex and Non-convex Energy Functions*, Birkhäuser, Boston, Inc., Boston, MA, 1985.

[37] P. D. Panagiotopoulos, *Hemivariational Inequalities. Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[38] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys., 655. Springer, Berlin, Heidelberg, 2004.

[39] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Notes Series, 398. Cambridge University Press, Cambridge, 2012.

[40] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2018.

[41] M. Sofonea, S. Migórski and A. Ochal, Two history-dependent contact problems, *Advances in Variational and Hemivariational Inequalities*, Adv. Mech. Math., Springer, Cham, **33** (2015), 355–380.

[42] M. Sofonea, Y.-B. Xiao and S.D. Zeng, Generalized penalty method for history-dependent variational–hemivariational inequalities, submitted.

[43] M. Sofonea and Y.-B. Xiao, Tykhonov well-posedness of split problems, submitted.

[44] M. Sofonea and Y.-B. Xiao, Tykhonov well-posedness of a viscoplastic contact problem, *Evolution Equations and Control Theory*, to appear.

[45] Y.-M. Wang, Y.-B. Xiao, X. Wang and Y.J. Cho, Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, *J. Nonlinear Sci. Appl.*, **9** (2016), 1178–1192.

[46] Y.-B. Xiao and M. Sofonea, Fully history-dependent quasivariational inequalities in contact mechanics, *Appl. Anal.*, **95** (2016), 2464–2484.
[47] E. Zeidler, *Nonlinear Functional Analysis and Applications, II A/B*, Springer, New York, 1990.

Received for publication October 2019.

E-mail address: stanislav.migorski@uj.edu.pl
E-mail address: xiaoyb9999@uestc.edu.cn
E-mail address: zjgxmzdx@163.com