Scaling and Hierarchy in Urban Economies

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In several recent publications, Bettencourt, West and collaborators claim that properties of cities such as gross economic production, personal income, numbers of patents filed, number of crimes committed, etc., show super-linear power-scaling with total population, while measures of resource use show sub-linear power-law scaling. Re-analysis of the gross economic production and personal income for cities in the United States, however, shows that the data cannot distinguish between power laws and other functional forms, including logarithmic growth, and that size predicts relatively little of the variation between cities. The striking appearance of scaling in previous work is largely artifact of using extensive quantities (city-wide totals) rather than intensive ones (per capita rates). The remaining dependence of productivity on city size is explained by concentration of specialist service industries, with high value-added per worker, in larger cities, in accordance with the long-standing economic notion of the “hierarchy of central places”.

Abbreviations: MSA, metropolitan statistical area; GMP, gross metropolitan product; BEA, Bureau of Economic Analysis; RMS, root mean square

Introduction

Recent dramatic advances in explaining metabolic scaling relations in biology by the properties of optimal transport networks \(^1\) suggest the possibility of examining similar assemblages, especially cities, in similar terms. In a well-known series of papers, Bettencourt, West and collaborators \(^2\) claim that many social and economic properties of cities—gross economic production, total personal income, number of patents filed, number of people employed in “supercreative” occupations, number of crimes committed, etc.—grow as super-linear powers of population size, while measures of total resource use grow as sub-linear powers. These two claims imply that per capita output grows as a positive power of population, while per capita resource use shrinks as a negative power. If reliable and precise scaling laws of this type exist, they would be of considerable importance for both science and policy \(^2\).

Reasonable arguments from long-standing principles of economic geography would lead one to expect that larger cities would have higher economic output per capita, through a combination of the benefits to firms in related industries clustering together (“agglomeration economies”), and the tendency of firms and specialists with large increasing returns to scale to be located high in the “hierarchy of central places”. (For reviews of these concepts, including historical notes, mathematical models and empirical evidence, see Refs. \(^2\).) These arguments would carry over to producing technologically useful knowledge and to “supercreative” services as well. However, these economic considerations do not point to either a particular functional form for the growth of per-capita output with population, or suggest that it should be very strong. Moreover, these theories do not look at individual cities as isolated monads, as scaling arguments do, but rather rely on there being assemblages of multiple cities (and rural areas), coupled by common economic processes, and assuming distinct roles in those processes through a history of mutual interaction and combined and uneven development.

The purpose of this note is to argue that, at least for the United States, while there is indeed a tendency for per-capita economic output to rise with population, power-law scaling predicts the data no better than many other functional forms, and worse than some others. Furthermore, the impressive appearance of scaling displayed in Refs. \(^3\) is largely an aggregation artifact, arising from looking at extensive (city-wide) variables rather than intensive (per-capita) ones. The actual ability of city size to predict economic output, no matter what functional form is used, is quite modest. These conclusions hold whether economic output is measured by gross metropolitan product or by total personal income. If we control for metropolitan areas’ varying concentration of industrial sectors, we find that the remaining scaling with population is negligible, and much of the variance across cities is predicted by the extent to which they host specialist service providers with strongly increasing returns, as predicted by the idea of the hierarchy of central places.

I begin by re-analyzing the gross metropolitan product data, showing that scaling is far weaker than it seemed in Refs. \(^3\). I then re-analyzes the data on walking speed, originating in Ref. \(^10\) and presented in Ref. \(^3\), which makes the problems with the scaling analysis very clear. Per-capita productivity is better predicted by how much a city depends on industrial sectors which indicate a high position in the hierarchy of specialist service provision. This actually eliminates any significant role for scaling with size. The conclusions summarize the scientific import of the data analyses.

The supplemental information shows that (i) scaling also fails for personal income, (ii) the hypothesis of power-law scaling cannot be saved by positing a mixture of distinct scaling relations, and that (iii) contra Ref. \(^3\), neither a Gaussian nor a Laplace distribution is a good fit to the deviations from the power-law scaling relations.

All calculations were done using R \(^11\), version 2.12. Code for reproducing figures and analyses is included in the supplemental information.

Reserved for Publication Footnotes

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1 Bettencourt and West summarize their claims regarding this “unified theory of urban living” \(^2\) thus: “We have recently shown that these general trends [to cities] can be expressed as simple mathematical laws”; “Our work shows that, despite appearances, cities are approximately scaled versions of one another …: New York and Tokyo are, to a surprising and predictable degree, non-linearly scaled-up versions of San Francisco in California or Nagoya in Japan. These extraordinary regularities open a window on underlying mechanism, dynamics and structure common to all cities”; “Surprisingly, size is the major determinant of most characteristics of a city: history, geography and design have secondary roles”. 

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Models

Scaling Models Ref. [3] reported a power-law scaling between the population of cities in the United States and their economic output. To be precise, the units of analysis are “metropolitan statistical areas” (MSAs) as defined by the official statistical agencies. The measure of economic output is the gross domestic product for each metropolitan area (“gross metropolitan product” or GMP), as calculated by the U.S. Bureau of Economic Analysis [http://www.bea.gov/regional/gdpmetro/], which is supposed to be the sum of all “incomes earned by labor and capital and the costs incurred in the production of goods and services” in the metropolitan area [13]. Ref. [3] analyzed data for 2006, deflated to constant 2001 dollars, and I will do likewise; the 2008 and 2004 data are not much different.

Ref. [3] propose that output scales as a power of population, $Y \propto N^b$. This is connected to the data via the linear regression model

$$\ln Y = \ln c + b \ln N + \epsilon,$$

with $c$ being a mean-zero noise term. For later comparisons, it will be convenient to denote this by $Y \sim cN^b$.

There is a simple test of the model which has not, so far as I know, been applied before. If production does scale as some power of population, $Y \sim cN^b$, then per-capita production $Y/N \equiv y$ should also scale,

$$y \sim cN^{b-1},$$

and vice versa. As shown below, this transformation drastically changes the apparent fit of the power-law scaling model.

It is worth noting that there is no theoretical reason to expect a power-law scaling relation of the form of Eq. 1 for urban economies (while there are such reasons for biological and vice versa. As shown below, this transformation drastically changes the apparent fit of the power-law scaling model.

An alternative to size scaling is hierarchy. An alternative to size scaling is hierarchical structure. The “hierarchy of central places”, introduced by Lösch and Christaller in the 1930s, has become a cornerstone of urban economic geography. In outline, the idea is that developed economies contain many specialized goods, and especially services, that the mass of consumers need only rarely (such as the services of a surgeon), or indirectly (such as the services of a professor of surgery, or a maker of surgical instruments). The provision of such services has comparatively high fixed costs (the time needed to train a surgeon) but low marginal costs (the time needed to perform an operation), leading to increasing returns to scale. It thus becomes economically efficient for these specialists to locate in central places, where their fixed costs can be distributed over large consumer bases, and the more specialized they are, the more centrally located they need to be, and the larger the customer base they require. This logic leads to the formation of a hierarchy of market centers and cities, in which increasingly specialized skills, with (as it were) increasingly increasing returns, can be had, and so predicts positive associations between the population of urban centers, the concentration of specialist skills within them, and (owing to increasing returns) their per-capita economic output. Good reviews of the theory, including historical citations and connections to modern economic models of increasing returns, may be found in Refs. [7, 9].

Fortunately, the BEA also provides estimates of the shares of gross metropolitan products attributable to different industrial sectors, some of which correspond to the specializations emphasized in central place theory. I specifically consider “Information, Communication, and Technology (ICT)”, “Financial activities”, “Professional and technical services” and “Management of companies and enterprises” (industry codes 106, 102, 58 and 62, respectively). Writing the proportions of gross metropolitan product deriving from each of these sectors as $x_1$ through $x_4$, the level of per-capita production can be predicted by a log-additive model [13] which incorporates power-law scaling with city size:

$$\ln y = \ln c + b \ln N + \sum_{j=1}^{4} f_j(x_j) + \epsilon,$$

where each of the “partial response” functions $f_j$ summarizes the contribution of the $j$th industrial sector. For comparison with the power-law scaling model (Eq. 1), I have constrained the partial response function for size to be logarithmic; the other partial response functions can be nonlinear, though they must be smooth.

Statistical Methods

Power-law scaling relations, like Eq. 4, were estimated through ordinary least squares, i.e., minimizing $n^{-1} \sum_{i=1}^{n} (\ln Y_i - \ln c - b \ln N_i)^2$, where the index $i$ runs over metropolitan areas, of which there are $n = 366$. As is well known [10], this is a consistent estimator of regression parameters for transformed regressions, even when Gaussian noise assumptions are violated, though the nominal values of standard errors and confidence intervals cannot be trusted. The nonlinear but parametric models (Eq. 3 and 4) were fit by nonlinear least squares.

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The non-parametric size scaling relation, Eq. 5 was fit by means of a smoothing spline [15][16] on the loged data. This is, the estimated spline is the function s minimizing

\[ n^{-1} \sum_i (\ln y_i - s(\ln N_i))^2 + \lambda \int (s''(x))^2 dx \]  

with the smoothness penalty \( \lambda > 0 \) chosen by cross-validation. Smoothing splines of this type are universal approximating functions, and picking the penalty by cross-validation controls the risk of over-fitting non-generalizing aspects of the data — see Ref. [15] for details.[4]

Finally, the additive model (Eq. 6) was estimated by combining spline smoothing for the non-parametric partial response functions \( f_j \), and an iterative “back-fitting” procedure [19]. (I used the mgcv library [19].) This adjusts for the correlations between industrial sectors, and between city size and industrial sectors, so that each estimated partial response function is, as far as possible, the unique additive contribution of that variable to economic output.

**Results**

**Weakness of Scaling in Gross Metropolitan Products.** Fitting Eq. 1 by least squares, I estimate \( b \) to be 1.12, in agreement with Ref. [3], with a 95% bootstrap confidence interval [16] of (1.10, 1.15). Figure 1 shows the data and the fitted trend, with both axis plotted on a logarithmic scale, so that a power law relationship appears as a straight line. The root-mean-squared (RMS) error for predicting \( \ln Y \) is 0.23, and the “coefficient of determination” \( R^2 \) is 0.96, i.e., the fitted values retain 96% of the variance in the actual data.

Visually, this looks like reasonable data collapse. Plotting the per-capita values \( y \) however, as in Figure 2 reveals a very different picture, though the two should be logically equivalent under the power-law model.

Figure 2 shows a trend curve for the the power-law scaling implied by Ref. [3]. (The exponent estimated for \( y \) is 0.12, matching that estimated for \( Y \), as it must.) The figure also shows the fitted logarithmic scaling relationship (Eq. 3), which is extremely close to the power law over the range of the data, the logistic scaling relationship (Eq. 4), and the non-parametric smoothing spline, corresponding to the relationship \( y \sim e^{\lambda \ln N} \). Note that the latter curve is not even monotonically increasing in \( N \).

While the curves in Figure 2 correspond to very different modeling assumptions — the differences between the implications of power-law and logistic scaling are perhaps especially striking — they all account for the data about equally well, or rather, equally poorly, because most of the variation in per-capita production is, in fact, unrelated to population. (Note that the vertical axis is linear, not logarithmic.) The RMS errors for \( \ln y \) of the power law, of logarithmic scaling and of logistic scaling are, respectively, 0.232, 0.234 and 0.229, while that of the spline is 0.225. They would predict, for a randomly chosen city, to within ±26.1, ±26.3, ±25.7 and ±25.3 percent, respectively. Predicting the same value of \( y \) for all cities, however, has an RMS error of 0.27, a margin of ±30%. Thus the \( R^2 \) values are, respectively, 0.24, 0.23, 0.26 and 0.29.

On the linear scale, i.e., in terms of dollars per person-year \( y \), the RMS errors of the power law, logarithmic, and logistic and spline curves are, respectively, 7.9 \times 10^3, 7.9 \times 10^3, 7.8 \times 10^3 and 7.7 \times 10^3, as compared to 9.2 \times 10^3 for predicting the mean for all cities.[1] In other words, even allowing for quite arbitrary functional forms, city size does not predict economic output very well.

The similarity of the RMS errors, and indeed of the curves, arises in part from the limited range of \( y \). The difference between the largest and smallest per-capita products (6.3 \times 10^4 dollars/person-year) is “only” a factor of 5.2, i.e., not even one order of magnitude. This is too small, with only 366 observations, to distinguish among competing functional forms for the trend, while still being quite consequential in human and economic terms. Per-capita production is simply not very strongly related to population.

Taking any per-capita (intensive) quantity which is statistically independent of population, and looking at the corresponding aggregate (extensive) quantities will yield a scaling exponent close to one. The overwhelming majority of the apparent fit of the scaling relationship in Figure 1 is just such an artifact of aggregation. This can be shown in three different ways: by algebra; by extrapolating the different per-capita functional forms back to city-wide totals; and by simulation.

Algebraically, suppose that \( y \) was statistically independent of \( N \). Then \( \ln Y = \ln y + \ln N \) would be the sum of two independent random variables, so its variance would be the sum of their variances. The \( R^2 \) of a linear regression of \( \ln y \) on \( \ln N \), with the slope constrained to be 1, would be 0.96,[4]

Figure 3 shows the same data and scaling curve as Figure 1, but three additional trend lines. These are the logarithmic, logistic and spline fits to the per-capita data (from Figure 2) extrapolated back to the implied aggregated values \( Y \). These are, visually, almost indistinguishable from the fitted power law; all have \( R^2 = 0.96 \).

Figure 4 demonstrates in a different way that the data do not support the idea of power-law scaling. The circles in the figure show the actual data values. The stars, by contrast, are surrogate data simulated from the fitted logistic scaling model, with the actual population sizes. The surrogate per-capita output values \( \tilde{y} \) were set equal to the fitted values under the model of Eq. 4 and then randomly perturbed according to the empirical distribution of deviations from that model. The figure plots the surrogate aggregate products \( \tilde{y}N \), which look very much like the data.

If a power-law scaling relation is fit to the surrogate data from the logistic-scaling regression, then, averaging over many simulations, the median scaling exponent is 1.12, with 95% of the estimates falling between 1.10 and 1.15, and the median \( R^2 \) of the power-law was 0.96. Recall that the estimate for the actual data was 1.12, with a 95% confidence interval of (1.10, 1.15), and \( R^2 = 0.96 \).

The RMS error for \( \ln y \) on the real data is very slightly lower for the logistic model (0.229) than for the power law (0.232). The difference is minute, but is, in fact, statistically significant: when repeating both fits on surrogate data simulated from the power law, gaps of this size or larger occur only ≈1% of the time. Not too much should be read into this, however, owing to the small magnitude of the difference, the large errors around both regression curves, and the comparatively small number of observations. Reliably discriminating between the two models simply requires more information (in the sense of [23]) than the data provides: either much smaller

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[1] A smoothing spline fit to the un-transformed data was similar, but visually somewhat more jagged.
[2] All these measures of error are calculated on the same data used to fit the models, exaggerating the models’ predictive powers. However, using six-fold cross-validation to approximate the out-of-sample risk gives RMS errors of 0.234 for the power law, 0.236 for logarithmic scaling, 0.232 for logistic scaling, and 0.231 for the spline. The differences are small, but bootstrapping shows they are significant at the 5% level (at least).
[3] Examples like this are why regression textbooks advise against using \( R^2 \) to check goodness of fit [23][25][26]
fluctuations of $y$ around the regression curve, or many more data points.

To sum up these results, the appearance of a strong, super-linear relationship between gross production $Y$ and population $N$ is mostly driven by production growing in proportion to population — that is, linearly. Per-capita production $y$ does not have a strong scaling relationship of any form with $N$, and the data are unable to distinguish between different functional forms for such trends as there are. The same is true of personal income (SI, section 1). Lacking ready access to the data sets on patents, crime, infrastructure and resource consumption which Ref. 6 analyzed in the same way as economic output and personal income, I cannot say whether the reported scaling relations for those aggregate variables suffer from the same problem.

**Superiority of the Urban-Hierarchy Model.** To address the question of why there is a weak and noisy tendency for per-capita output to rise with population, I turn to the log-additive model, Eq. 5. Fitting to the data yields the partial response functions shown in Figure 6. As expected from the urban-hierarchy argument, all four of the partial response functions are monotonically increasing, so that rising shares of those industries predict increasing per capita production. Very notably, however, the estimated power-law scaling exponent is actually negative, $-2.6 \times 10^{-3}$, but statistically indistinguishable from zero (standard error $2.8 \times 10^{-3}$). That is, in the log-additive model, controlling for these four industrial sectors makes population effectively irrelevant for predicting urban productivity. Indeed, dropping population from the model altogether produces no appreciable difference in the fit. At least at the level of expectation values, controlling for these four industrial sectors “screens off” the effects of city size on per-capita production.

Statistically, there is no question that the log-additive model predicts better than the simple scaling model. The RMS error of the former, on the log scale, is 0.218, corresponding to an $R^2$ of 38.8%, and an accuracy of $\pm 24\%$ or $\pm 6.8 \times 10^3$, better than any model based on size alone. The log-additive model is a more flexible specification, and so over-fitting to the data is an issue, but this can be addressed by cross-validation, which directly measures the ability of a model to extrapolate from one part of the data to another 17. The cross-validated mean squared error of the log-additive model is 0.053, while that of the power-law is 0.067, clearly showing that the extra complexity of the former is being used to capture genuinely predictive patterns, and not merely to memorize the training data.

The simple log-additive model is unlikely to be a fully adequate predictor of systematic differences in urban productivity. If nothing else, these four coarse-grained industrial sectors were selected merely for convenience, as approximate indicators of position in the urban hierarchy, and presumably could be improved. Moreover, the model does not even try to represent the interactive processes which lead cities to have the industrial mixes that they do. In reality, these industries can be so concentrated towards the largest cities, at the top of the hierarchy (e.g., New York), and away from lower-rank cities (e.g., San Francisco, Peoria), only because all these cities are part of a single national, and even international, division of labor 6.

“The Pace of Life”. A further claim of Ref. 3 is that the speed at which people walk grows as a positive power of the number of people in a city. The source given for this is Ref. 10, a two-page letter to *Nature* in 1976. The authors of Ref. 10 went to 15 cities, towns and villages, picked locations and individuals which seemed to them to be comparable, and timed how long it took them to walk fifty feet (15.2 meters). Such unsystematic data, however intriguing, is too weak to support substantial scientific conclusions. Nonetheless, it is instructive to examine it, as in Figure 5.

The original plot (Figure 1 in Ref. 10) showed population on a log scale, and speed on a linear scale, as in Figure 6. The linear regression, for this transformation of the data, corresponds to assuming that speed grows logarithmically with population, $v \sim r \ln N/k$. Figure 2a in Ref. 3 re-plots the same data, but with the vertical axis on a logarithmic scale, so the linear regression assumes speed grows as a power of population, $v \sim c N^b$. (Neither figure included error bars, though Bornstein and Bornstein give the standard deviations in their caption.) As can be seen from Figure 6, both of these regressions, along with logistic scaling, are very similar in this data, while they embody very different assumptions, and at most one can be right.

The explanation for this apparent paradox is that the range of reported walking speeds is small, from 0.7 m/s to 1.8 m/s, and if $|x| \ll 1$, then $\ln 1 + x \approx x$. Observed over a narrow range, then, logarithmic and power law scaling simply are very similar, and hard to distinguish. This is also why the the power-law and logarithmic fits to per-capita production in Figure 6 were so close.

**Discussion**

Neither gross metropolitan product nor personal income scales with population size for U.S. metropolitan areas. The appearance of scaling in Refs. 3, 4 is an artifact of inappropriately looking at extensive variables (city-wide totals) rather than intensive ones (per-capita values). Scaling is also unpersuasive for walking speed. I was not able to examine the other variables claimed to show scaling in Refs. 3, 4, but, as they were all extensive variables, the analyses reported there would be subject to the same aggregation artifacts. It is also possible that cities in the contemporary United States are anomalous, and that scaling of income and economic output holds elsewhere.

It is evident from Figures 2 (and Supplemental Figure S1) that there is a weak tendency for per-capita output and income to rise with population, though the relationship is simply too loose to qualify as a scaling law. (Arguably, the real trend in those figures is for the minimum per-capita output to rise with population, though I would press this point.) Qualitatively, this is what one would expect from well-established findings of economic geography. The data do not really support any stronger quantitative statement. In particular, asserting any specific functional form, such as a power law, goes far beyond the what the data can support. Nor is there any theory, supported on independent grounds, which predicts a specific functional form. Accordingly, extrapolations based on such claims (e.g., the finite-time singularity in the model for city growth in 2) are speculative at best. The amplitude of fluctuations around the trend lines are, in any case, so large that predictions based on size alone can have very little utility.

By taking account of the shares of just a few industries in the gross metropolitan product, we can obtain much better predictions of the level of per-capita production. In this statistical model, summarized in Eq. 3 population plays no significant direct role in predicting per capita economic output, and could in fact be profitably ignored. Rather, the industrial sectors used are chosen as signs of where metropolitan areas

8 Dropping population size $N$ from the log-additive model altogether actually improves the cross-validation score, very slightly, to 0.052.
stand in the urban hierarchy, which is also related, of course, to size. One could interpret this as the mechanism by which size scaling happens (to the extent that it does), but this would imply that an exogenous increase in a city’s population would automatically shift its industrial pattern, which is implausible. Indeed, the whole scaling picture for cities seems to rest on an oddly monadic, interaction-free view of metropolitan areas. The logic of central place theory, in contrast, relies on cities being part of an interactive assemblage, coupled by processes of production, distribution and exchange. This not only seems more plausible, but also better matches the evidence at hand.

As Refs. 2, 3, 4 have stressed, developing a sound scientific understanding of cities should be a priority for an increasingly urban species. In seeking such understanding, it is a sound strategy to begin with simple hypotheses, and to reject them in favor of more complicated ones only as they prove unable to explain the data. This is not because the truth is more likely to be simple, in some metaphysical sense, but because this strategy leads us to the truth faster and more reliably than ones which invoke needless complexities 22. The elegant hypothesis of power-law scaling marked a step forward in our understanding of cities, but it is now time to leave it behind.

Appendix

Personal Income
The BEA also makes available estimates of personal income by metropolitan area, a variable closely related to, but not quite the same as, the gross metropolitan product. (See http://www.bea.gov/regional/reis// for definitions, estimation techniques, and data.) Ref. 3 reports that total personal income L also scales as a power of population, implying per capita personal income $L/N \equiv l$ should scale likewise. Figure 7 plots $l$ versus $N$, with the best-fitting power law, logarithmic relationship, and spline.

Once again, the appearance of power-law scaling in the aggregate variable is not supported by examination of the per-capita values. The RMS error, on the log scale, of predicting a constant per capita income over all cities is 0.179, while the RMS errors of the power-law, logarithmic and logistic scaling relations are 0.157, 0.158 and 0.156, and that of the spline 0.154. Even the spline thus has an $R^2$ of only 0.26. Repeating the procedures of Figures 3 and 4 from the main text yields similar results. Thus, personal income also fails to display non-trivial scaling with population.

Mixtures of Scaling Relations
Recall that the posited scaling relation is $y \sim cN^b$. As shown above, this does not fit the data, at least not assuming, following Ref. 3, that both parameters, the scaling exponent $b$ and the pre-factor $c$, are the same for all cities. A natural way to try to reconcile the data with the model would be to modify the latter, allowing $c$ to depend on the type of the city. The rationale for such a regression would be that there are several different kinds of cities, and that city type shifts the over-all level of production up or down, but, once that is factored out, all cities scale with size in the same way. This common scaling exponent would not, naturally, be the same as the one estimated from the pooled data.

Formally, we introduce a latent variable $Z$ for each city, treated as a discrete random variable independent of $N$, and consider the statistical model $y \sim cZN^b$. This leads to a “mixture-of-regressions” or “latent-class regression” model, which can be fit by the expectation-maximization algorithm 23. Such fitting would lead not only to estimates of $b$ and the pre-factors $c$, but also to the probability that each city belonged to each of the different city types or mixture components, categorizing cities inductively from the data.

To investigate this, I fit mixture-of-regression models to the data from Figure 2 in the main text, varying the number of mixture components from 1 to 10, using the software of Ref. 24 25. To determine the correct number of mixture components, I used both Schwarz’s “Bayesian” information criterion and cross-validation, which are both known to be consistent for such mixture problems, unlike the Akaike information criterion, which over-fits 22. Both BIC and cross-validation strongly favored one mixture component, meaning that the fit to the data is not actually improved by allowing for multiple scaling curves.

This does not completely rule out the $y \sim cZN^b$ model, as only 366 observations may not have enough information to simultaneously induce appropriate categories and fit scaling relations. An alternative would be to expand the information available, by defining the categorical variable $Z$ in terms of measurable attributes of cities other than $N$ and $y$, such as geographic location or the mix of industries. (See Ref. 27 on such variable-intercept, constant-slope regressions with known categories.) Success with such models hinges on selecting categories to represent important features of the data-generating process, a task I must leave to other inquirers.

Assuming that such a statistical model works, there would still be the question of its interpretation. Whether one would judge such a model to really show scaling in urban assemblages would depend on how much importance one gives, on the one hand, to a common scaling exponent, and on the other to most of the fit coming from the un-modeled differences across city types.

Residuals
Ref. 3 proposes ranking cities not by their per capita values of quantities like economic production or patents or crime, but by the deviation, positive or negative, from the scaling relationship, i.e., by the residuals of the trend lines. (It does not compare this to ranking by per capita values. The Spearman rank correlation between the two variables is 0.87 for GMP and 0.83 for personal income.) They consider both a Gaussian distribution for the residuals, i.e., a probability density function $f(x) \propto e^{-x^2/2a^2}$, and a Laplace distribution, $f(x) \propto e^{-|x|/a}$, and claim that both fit very well.

Figure 8 shows the situation for GMP. Visually, neither distribution matches the residuals well. Quantitatively, goodness-of-fit can be checked by “data-driven smooth tests” 28, which transform their inputs so that they will be uniform if and only if the postulated distribution holds, and then measure departures from uniformity (coefficients from expanding the transformed empirical distribution in a series of orthogonal polynomials). Such tests reject both the Gaussian and the Laplace distribution with high confidence ($p$-values of $1 \times 10^{-3}$ and $8 \times 10^{-3}$, respectively, calculated using code provided by Ref. 29).

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10 For computational reasons, it is easier to fit the more general specification in which the scaling exponent is allowed to vary, $y \sim cZN^{bZ}$. (Sharing a parameter across the regressions complicates the maximization step of the expectation-maximization algorithm.) If the constant-exponent model is right, the estimated exponents for each mixture component should agree to within statistical precision.
Results for personal income are similar (Figure 3). The Gaussian distribution can be rejected with high confidence ($p < 10^{-4}$). While the data do not rule out the Laplace distribution in the same way ($p = 0.27$), the limited power of the test at the comparatively small sample size means that there is not strong evidence in its favor either. (See Ref. 20 on the evidential interpretation of significance tests.)

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Fig. 1. Horizontal axis: population of the 366 US metropolitan statistical areas in 2006, log scale; vertical axis, 2006 gross product of each MSA, in constant 2001 dollars, log scale. (In all figures, grey inner ticks on axes mark observed values.) Solid line: ordinary least squares regression of log gross metropolitan product on log population, i.e., the regression $Y \sim cN^b$, with estimated exponent $\hat{b} = 1.12$. 

Gross product (2001 dollars/year)

MSA population
Fig. 2. Horizontal axis: population, as in Figure 1 log scale. Vertical axis: gross product per capita, but on a linear and not a logarithmic scale. The two largest values are $7.8 \times 10^4$ dollars/person-year (in Bridgeport-Stamford-Norwalk, CT, a center for hedge funds and other financial firms) and $7.7 \times 10^4$ dollars/person-year (in San Jose-Sunnyvale-Santa Clara, CA, i.e., Silicon Valley), and the smallest are $1.5 \times 10^4$ dollars/person-year (in McAllen-Edinburg-Mission, TX and Palm Coast, FL). Black line: fitted power-law scaling relation. Blue line: fitted logarithmic scaling relationship. Grey line: logistic scaling. Red line: smoothing spline fitted to the logged data.
Fig. 3. As in Figure 1, but with the addition of curves showing the scaling relations from Figure 2, extrapolated to aggregate rather than per-capita values. These are visually all but indistinguishable.
Fig. 4. Axes: as in Figure 1 and 3. Circles: Actual values. Stars: simulated values, with per-capita production figures drawn from the logistic (not power-law) scaling model.
Fig. 5. Horizontal axis: city population, logarithmic scale. Vertical axis: estimated pedestrian speed in meters/second, plus or minus one standard deviation, linear scale. Blue line: the regression $v \sim r \ln(N/k)$, as proposed by Ref. [10]. Black line: the regression $v \sim cN^{b}$, as proposed by Ref. [3]. Grey line: logistic scaling. (Data from Ref. [10], who report the mean and standard deviation of the time taken to walk 50 feet = 15.2 meters; I calculated standard deviations by propagation of error.)
Fig. 6. Partial response functions for the log-additive model (Eq 6). Horizontal axes indicate the fraction of each metropolitan area’s gross product derived from each industry, while the vertical axis shows the predicted logarithmic increase, or decrease, to per capita output, relative to the baseline of the mean over all cities. Solid curves are the main estimate, with dashed curves at ±2 standard errors in the partial response function. Dots show “partial residuals”, the difference between actual In y values and those predicted by the model including all the other variables.
Fig. 7. Personal income per capita versus population, 2006. Horizontal axis: population of MSAs (log scale). Vertical axis: personal income per capita, in nominal 2006 dollars (linear scale). Black line: power-law scaling curve (estimated exponent 0.082). Blue line: logarithmic scaling curve. Grey line: logistic scaling curve. Red line: spline fit to logged data.
Fig. 8. Horizontal axis: magnitude of residuals from power-law scaling of gross metropolitan product on population, i.e., from regressing $\ln Y$ on $\ln N$. Vertical axis: probability density of the residual distribution. Solid line: Nonparametric kernel density estimate (Gaussian kernel, default bandwidth choice — see Ref. [31]). Dashed line: maximum likelihood Gaussian fit to residuals. Dotted line: maximum likelihood Laplace (double-exponential) fit to residuals.
Fig. 9. As in Figure 8, but showing the deviations of personal income from power-law scaling.