On Topological Properties of Third type of Hex Derived Networks

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Abstract. In chemical graph theory, a topological index is a numerical representation of a chemical network while a topological descriptor correlates certain physico-chemical characteristics of underlying chemical compounds besides its chemical representation. Graph plays an vital role in modeling and designing any chemical network. F. Simonraj et al. derived new third type of Hex derived networks [27]. In our work, we discuss the third type of hex derived networks HDN3(r), THDN3(r) and RHDN3(r) and computed exact results for topological indices which are based on degrees of end vertices.

Keywords: General Randić index, Harmonic index, Augmented Zagreb index, Atom-bond connectivity (ABC) index, Geometric-arithmetic (GA) index, Third type of Hex Derived Networks, HDN3(r), THDN3(n), RHDN3(n).

1 Introduction and preliminary results

Topology indices application is now a standard procedure in studying chemical information, structure properties like QSAR and QSPR. Biological indicators such as the Randić Index, Zagreb Index, the Weiner Index and the Balaban index are used and predict and study the physical and chemical properties. There is too much research has been published in this field in the last few decades. The topological index is a numeric quantity associated with chemical constitutions purporting the correlation of chemical structures with many physicochemical properties, chemical reactivity or biological activity. Topological indices are made on the grounds of the transformation of a chemical network into a number that characterizes the topology of the chemical network. Some of the major types of topological indices of graphs are distance-based topological indices, degree-based topological indices, and counting-related topological indices. For any Graph G = (V,E) where V is be the vertex set and E to be the edge set of G. The degree $\kappa(x)$ of vertex x is the amount of edges of G episode with x. A graph can be spoken by a polynomial, a numerical esteem or by network shape.

Hexagonal mesh was derived by Chen et al. [8]. A set of triangles, made up a hexagonal mesh as shown in Fig. 1. Hexagonal mesh with dimension 1 does
not exist. Composition of six triangles made a 2-dimensional hexagonal mesh $HX(2)$ (see Fig. 1(1)). By adding a new layer of triangles around the boundary of $HX(2)$, we have a 3-dimensional hexagonal mesh $HX(3)$ (see Fig. 1(2)). Similarly, we form $HX(n)$ by adding $n$ layers around the boundary of each proceeding hexagonal mesh.

**Drawing algorithm of $HDN_3$ networks**

Step-1: First we draw a Hexagonal network of dimension $r$.

Step-2: Replace all $K_3$ subgraph in to Planar octahedron $POH$ once. The resulting graph is called $HDN_3$ (see Fig. 4) networks.

Step-3: From $HDN_3$ network, we can easily form $THDN_3$ (see Fig. ??) and $RHDN_3$ (see Fig. ??).

![Fig. 1. Hexagonal meshes: (1) HX(2), (2) HX(3), and (3) all faces in HX(2).](image)

In this paper, we consider $G$ as a network with $V(G)$ is the set of vertices and edge set $E(G)$, the degree of any vertex $\hat{p} \in V(G)$ is denoted by $\kappa(\hat{p})$. The $S_\rho = \sum_{\hat{p} \in N_G(\hat{p})} \deg(\hat{q})$ where $N_G(\hat{p}) = \{\hat{q} \in V(G) | \hat{p}\hat{q} \in E(G)\}$.

The beginning of degree based topological indices starts from Randić index [24] denoted by $R_{-\frac{1}{2}}(G)$ and acquainted by Milan Randić and written as

$$R_{-\frac{1}{2}}(G) = \sum_{\hat{p} \hat{q} \in E(G)} \frac{1}{\sqrt{\kappa(\hat{p})\kappa(\hat{q})}}. \quad (1)$$

B. Furtula and Ivan Gutman [12] introduced forgotten topological index (also called F-Index) defined as

$$F(G) = \sum_{\hat{p}\hat{q} \in E(G)} ((\kappa(\hat{p}))^2 + (\kappa(\hat{q}))^2). \quad (2)$$

Another topological index in view of the level of the vertex is the Balaban index [4,5]. This index for a graph $G$ of order ‘$n$’, size ‘$m$’ is characterized as

$$J(G) = \left(\frac{m}{m - n + 2}\right) \sum_{\hat{p}\hat{q} \in E(G)} \frac{1}{\sqrt{\kappa(\hat{p}) \times \kappa(\hat{q})}}. \quad (3)$$
Ranjini et al. [25] reclassified the Zagreb indices is to be specific the re-imagined in the first place, second and third Zagreb indices for graph G as

\[
ReZG_1(G) = \sum_{\hat{p} \hat{q} \in E(G)} \left( \frac{\kappa(\hat{p}) \times \kappa(\hat{q})}{\kappa(\hat{p}) + \kappa(\hat{q})} \right),
\]

(4)

\[
ReZG_2(G) = \sum_{\hat{p} \hat{q} \in E(G)} \left( \frac{\kappa(\hat{p}) + \kappa(\hat{q})}{\kappa(\hat{p}) \times \kappa(\hat{q})} \right),
\]

(5)

\[
ReZG_3(G) = \sum_{\hat{p} \hat{q} \in E(G)} (\kappa(\hat{p}) \times \kappa(\hat{q}))(\kappa(\hat{p}) + \kappa(\hat{q})).
\]

(6)

Only \( ABC_4 \) and \( GA_5 \) indices can be computed if we are able to find the edge partition of these networks, based on sum of the degrees of end vertices of each edge. The fourth version of \( ABC \) index is introduced by Ghorbani et al. [13] and defined as

\[
ABC_4(G) = \sum_{\hat{p} \hat{q} \in E(G)} \sqrt{S_{\hat{p}} + S_{\hat{q}} - 2 S_{\hat{p}} S_{\hat{q}}}. \]

(7)

The fifth version of \( GA \) index is proposed by Graovac et al. [14] and defined as

\[
GA_5(G) = \sum_{\hat{p} \hat{q} \in E(G)} \frac{2 \sqrt{S_{\hat{p}} S_{\hat{q}}}}{(S_{\hat{p}} + S_{\hat{q}})}. \]

(8)

2 Main Results for third type of Hex Derived networks

F. Simonraj et al. [27] derived new third type of Hex derived networks and found metric dimension of \( HDN_3 \) and \( PHDN_3 \). In this work, we discuss the newly derived third type of hex derived networks and compute exact results for degree based topological indices. In these days, there is an extensive research activity on these topological indices and their variants see, [12-18,20-21,22-23,26]. For basic definitions, notations, see [6,9,15,16,29].

2.1 Results for Hex Derived network of type 3, \( HDN_3(r) \)

In this section, we discuss the newly derived third type of hex derived network and compute numerical and exact results for forgotten index, Balaban index, reclassified the Zagreb indices, forth version of \( ABC \) index and fifth version of \( GA \) index for the very first time.

**Theorem 2.1.1.** Consider the Hex Derived network of type 3 \( HDN_3(n) \), then its forgotten index is equal to

\[
F(HDN_3(n)) = 6(5339 - 8132n + 3108n^2)
\]
Fig. 2. Hex Derived network of type 3 \( \text{HDN}_3(4) \)

| \( \kappa_x, \kappa_y \) where \( pq \in E(G_1) \) | Number of edges \( 18r^2 - 36r + 18 \) | \( \kappa_x, \kappa_y \) where \( pq \in E(G_1) \) | Number of edges \( 6r - 18 \) |
|-----------------|-----------------|-----------------|-----------------|
| (4, 4)          |                 | (7, 18)         |                 |
| (4, 7)          | 24              | (10, 10)        | 6               |
| (4, 10)         | 36r - 72        | (10, 18)        | 12r - 24        |
| (4, 18)         | 36r^2 - 108r + 84 | (18, 18)      | 9r^2 - 33r + 30 |
| (7, 10)         | 12              |                 |                 |

Table 1. Edge partition of Hex Derived network of type 3 \( \text{HDN}_3(r) \) based on degrees of end vertices of each edge.
Proof. Let $G_1$ be the Hex Derived network of type 3, $HDN3(r)$ shown in Fig. 4 where $r \geq 4$. The Hex Derived network $G_1$ has $21r^2 - 39r + 19$ vertices and the edge set of $G_1$ is divided into nine partitions based on the degree of end vertices. The first edge partition $E_1(G_1)$ contains $18r^2 - 36r + 18$ edges $\hat{pq}$, where $\kappa(\hat{p}) = \kappa(\hat{q}) = 4$. The second edge partition $E_2(G_1)$ contains 24 edges $\hat{pq}$, where $\kappa(\hat{p}) = 4$ and $\kappa(\hat{q}) = 7$. The third edge partition $E_3(G_1)$ contains $36r - 72$ edges $\hat{pq}$, where $\kappa(\hat{p}) = 4$ and $\kappa(\hat{q}) = 10$. The fourth edge partition $E_4(G_1)$ contains $36r^2 - 108r + 84$ edges $\hat{pq}$, where $\kappa(\hat{p}) = 4$ and $\kappa(\hat{q}) = 18$. The fifth edge partition $E_5(G_1)$ contains 12 edges $\hat{pq}$, where $\kappa(\hat{p}) = 7$ and $\kappa(\hat{q}) = 10$. The sixth edge partition $E_6(G_1)$ contains 6 edges $\hat{pq}$, where $\kappa(\hat{p}) = 7$ and $\kappa(\hat{q}) = 18$. The seventh edge partition $E_7(G_1)$ contains $6r - 18$ edges $\hat{pq}$, where $\kappa(\hat{p}) = \kappa(\hat{q}) = 10$ and the eighth edge partition $E_8(G_1)$ contains $12r - 24$ edges $\hat{pq}$, where $\kappa(\hat{p}) = 10$ and $\kappa(\hat{q}) = 18$ and the ninth edge partition $E_9(G_1)$ contains $9r^2 - 33r + 30$ edges $\hat{pq}$, where $\kappa(\hat{p}) = \kappa(\hat{q}) = 18$. Table 1 shows such an edge partition of $G_1$. Thus from (3) this follows that

\[
F(G) = \sum_{\hat{pq} \in E(G)} ((\kappa(\hat{p}))^2 + (\kappa(\hat{q}))^2)
\]

Let $G_1$ be the Hex Derived network of type 3, $HDN3(r)$. By using edge partition from Table 1, the result follows. The forgotten index can be calculated by using (2) as follows.

\[
F(G_1) = \sum_{\hat{pq} \in E(G)} ((\kappa(\hat{p}))^2 + (\kappa(\hat{q}))^2) = \sum_{\hat{pq} \in E(G)} \sum_{j=1}^{9} ((\kappa(\hat{p}))^2 + (\kappa(\hat{q}))^2)
\]

\[
F(G_1) = 32|E_1(G_1)| + 65|E_2(G_1)| + 116|E_3(G_1)| + 340|E_4(G_1)| + 149|E_5(G_1)| + 373|E_6(G_1)| + 200|E_7(G_1)| + 424|E_8(G_1)| + 648|E_9(G_1)|
\]

By doing some calculations, we get

\[
\Rightarrow F(G_1) = 6(5339 - 8132n + 3108n^2).
\]

\[
\square
\]

In the following theorem, we compute Balaban index of Hex Derived network of type 3, $G_1$.

**Theorem 2.1.2.** For Hex Derived network $G_1$, the Balaban index is equal to

\[
J(G_1) = \frac{(20 - 41n + 21n^2)(1595.47 + 7(-307 - 270\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})n)}{70(43 - 84n + 42n^2)} + \frac{210(5 + 3\sqrt{2})n^2}{70(43 - 84n + 42n^2)}
\]

Proof. Let $G_1$ be the Hex Derived network $HDN3(r)$. By using edge partition from Table 1, the result follows. The Balaban index can be calculated by using
as follows.

\[ J(G_1) = \left( \frac{m}{m - n + 2} \right) \sum_{\rho \notin E(G)} \frac{1}{\sqrt{\kappa(\rho) \times \kappa(\bar{\rho})}} = \left( \frac{m}{m - n + 2} \right) \sum_{\rho \notin E_i(G)} \sum_{j=1}^9 \frac{1}{\sqrt{\kappa(\rho) \times \kappa(\bar{\rho})}} \]

\[ J(G_1) = \left( \frac{63n^2 - 123n + 60}{43 - 84n + 42n^2} \right) \frac{1}{4} |E_1(G_1)| + \frac{1}{2 \sqrt{7}} |E_2(G_1)| + \frac{1}{2 \sqrt{10}} |E_3(G_1)| + \frac{1}{6 \sqrt{2}} |E_4(G_1)| + \frac{1}{2 \sqrt{10}} |E_5(G_1)| + \frac{1}{3 \sqrt{14}} |E_6(G_1)| + \frac{1}{10} |E_7(G_1)| + \frac{1}{18} |E_8(G_1)| \]

By doing some calculation, we get

\[ \Rightarrow J(G_1) = \frac{(20 - 41n + 21n^2)(1595.47 + 7(-307 - 270\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})n)}{70(43 - 84n + 42n^2)} + \frac{210(5 + 3\sqrt{2})n^2}{70(43 - 84n + 42n^2)}. \]

Now, we compute \( ReZG_1, ReZG_2 \) and \( ReZG_3 \) indices of Hex Derived network \( G_1 \).

**Theorem 2.1.3.** Let \( G_1 \) be the Hex Derived network, then

- \( ReZG_1(G_1) = 19 - 39n + 21n^2; \)
- \( ReZG_2(G_1) = \frac{115452}{425} - \frac{5637n}{11} + \frac{2583n^2}{11}; \)
- \( ReZG_3(G_1) = 12(27381 - 38996n + 13692n^2). \)

**Proof.** By using edge partition given in Table 1, the \( ReZG_1(G_1) \) can be calculated by using (4) as follows.

\[ ReZG_1(G) = \sum_{\rho \notin E(G)} \left( \frac{\kappa(\rho) \times \kappa(\bar{\rho})}{\kappa(\rho) + \kappa(\bar{\rho})} \right) = \sum_{j=1}^9 \sum_{\rho \notin E_j(G)} \left( \frac{\kappa(\rho) \times \kappa(\bar{\rho})}{\kappa(\rho) + \kappa(\bar{\rho})} \right) \]

\[ ReZG_1(G_1) = 2|E_1(G_1)| + \frac{28}{11} |E_2(G_1)| + \frac{20}{7} |E_3(G_1)| + \frac{36}{11} |E_4(G_1)| + \frac{70}{17} |E_5(G_1)| + \frac{126}{25} |E_6(G_1)| + 5|E_7(G_1)| + \frac{45}{7} |E_8(G_1)| + 9|E_9(G_1)| \]
By doing some calculation, we get

\[ \implies ReZG1(G_1) = 19 - 39n + 21n^2 \]

The ReZG2(G_1) can be calculated by using (5) as follows.

\[ ReZG2(G) = \sum_{\rho q \in E(G)} \left( \frac{\kappa(\hat{p}) + \kappa(\hat{q})}{\kappa(\hat{p}) \times \kappa(\hat{q})} \right) = \sum_{j=1}^{9} \sum_{\rho q \in E_j(G_1)} \left( \frac{\kappa(\hat{p}) + \kappa(\hat{q})}{\kappa(\hat{p}) \times \kappa(\hat{q})} \right) \]

\[ ReZG2(G_1) = \frac{1}{2} |E_1(G_1)| + \frac{11}{28} |E_2(G_1)| + \frac{7}{20} |E_3(G_1)| + \frac{11}{36} |E_4(G_1)| + \frac{17}{70} |E_5(G_1)| + \frac{25}{126} |E_6(G_1)| + \frac{1}{5} |E_7(G_1)| + \frac{7}{45} |E_8(G_1)| + \frac{1}{9} |E_9(G_1)| \]

By doing some calculation, we get

\[ \implies ReZG2(G_1) = \frac{115452}{425} - \frac{5637n}{11} + \frac{2583n^2}{11} \]

The ReZG3(G_1) index can be calculated from (6) as follows.

\[ ReZG3(G) = \sum_{\rho q \in E(G)} (\kappa(\hat{p}) \times \kappa(\hat{q}))(\kappa(\hat{p}) + \kappa(\hat{q})) \]

\[ = \sum_{j=1}^{9} \sum_{\rho q \in E_j(G_1)} (\kappa(\hat{p}) \times \kappa(\hat{q}))(\kappa(\hat{p}) + \kappa(\hat{q})) \]

\[ ReZG3(G_1) = 128|E_1(G_1)| + 308|E_2(G_1)| + 560|E_3(G_1)| + 1584|E_4(G_1)| + 1190|E_5(G_1)| + 3150|E_6(G_1)| + 2000|E_7(G_1)| + 5040|E_8(G_1)| + 11664|E_9(G_1)| \]

By doing some calculation, we get

\[ \implies ReZG3(G_1) = 12(27381 - 38996n + 13692n^2) \]

Now, we compute \( ABC_4 \) and \( GA_5 \) indices of Hex Derived network \( G_1 \).

**Theorem 2.1.4.** Let \( G_1 \) be the Hex Derived network, then

- \( ABC_4(G_1) = 51.766 + \frac{3}{25} \sqrt{79} (-5 + n) + 3 \sqrt{23} (-4 + n) + \frac{3}{5} \sqrt{109} (-4 + n) + \sqrt{\frac{111}{11}} (-4 + n) + \frac{3}{35} \sqrt{109} (-4 + n) + 3 \sqrt{23} (-3 + n) + 3 \sqrt{23} (-3 + n) + 12 \sqrt{23} (-3 + n) + 2 \sqrt{\frac{111}{11}} (-3 + n) + \sqrt{\frac{52}{7}} (-3 + n) + \sqrt{\frac{78}{11}} (-2 + n) + \frac{9}{11} \sqrt{\frac{52}{7}} (-2 + n) + \frac{1}{3} \sqrt{\frac{35}{7}} (-5 + 2n) + \cdots \)
\[ \left( \kappa_x, \kappa_y \right) \text{ where } \{p, q\} \in E(G_1) \text{ Number of edges} \]

\begin{array}{ccc}
(25, 34) & 12 & (44, 44) & 12r - 72 + 72 \\
(25, 36) & 12 & (44, 129) & 36 \\
(25, 54) & 12 & (44, 140) & 48r - 144 \\
(25, 77) & 12 & (44, 156) & 36r^2 - 180r + 228 \\
(28, 36) & 12r - 36 & (54, 77) & 12 \\
(28, 57) & 12 & (54, 129) & 6 \\
(28, 80) & 12r - 48 & (77, 80) & 12 \\
(33, 34) & 12 & (77, 129) & 12 \\
(33, 54) & 12 & (77, 140) & 12 \\
(33, 129) & 12 & (80, 80) & 6r - 30 \\
(36, 36) & 12r - 30 & (80, 140) & 12r - 48 \\
(36, 44) & 12r - 24 & (129, 140) & 12 \\
(36, 77) & 48 & (129, 156) & 6 \\
(36, 80) & 24r - 96 & (140, 140) & 6r - 24 \\
(36, 129) & 24 & (140, 156) & 12r - 36 \\
(36, 140) & 24r - 72 & (156, 156) & 9r^2 - 51r + 72 \\
\end{array}

\textbf{Table 2.} Edge partition of Hex Derived network of type 3 \( HDN3(r) \) based on degrees of end vertices of each edge.

\[ \frac{1}{26} \sqrt{\frac{155}{2}(24 - 17n + 3n^2)} + 3 \sqrt{\frac{6}{11}(19 - 15n + 3n^2)}; \]

- \( ABC_4(G_1) = 110.66 + \frac{6}{37} \sqrt{1365(-6 + n)} + \frac{24}{11} \sqrt{7(-5 + n)} + \frac{18}{11} \sqrt{35(-5 + n)} + \frac{24}{37} \sqrt{385(-5 + n)} + \frac{64}{37} \sqrt{5(-4 + n)} + \frac{9}{11} \sqrt{21(-4 + n)} + \frac{8}{5} \sqrt{35(-4 + n)} + \frac{36}{37} \sqrt{22(-2 + n) - 12n + 3n^2 + \frac{2}{11}(42 - 13n + n^2) + \frac{6}{37} \sqrt{429(30 - 11n + n^2)}}. \]

\textbf{Proof.} By using edge partition given in Table 2, the \( ABC_4(G_1) \) can be calculated by using (7) as follows.

\[ ABC_4(G) = \sum_{\{p, q\} \in E(G)} \sqrt{S_p + S_q - 2} S_p S_q \]

\[ = \sum_{j=10}^{41} \sum_{\{p, q\} \in E_j(G_1)} \sqrt{S_p + S_q - 2} S_p S_q \]
On Topological Properties of Third type of Hex Derived Networks

\[ ABC_4(G_1) = \frac{2}{5} \sqrt{\frac{14}{33}} |E_{10}(G_1)| + \frac{\sqrt{59}}{30} |E_{11}(G_1)| + \frac{1}{15} \sqrt{\frac{77}{6}} |E_{12}(G_1)| + \]

\[ \frac{36}{11} \sqrt{\frac{7}{14}} |E_{13}(G_1)| + \frac{1}{6} \sqrt{\frac{31}{14}} |E_{14}(G_1)| + \frac{1}{14} \sqrt{\frac{103}{11}} |E_{15}(G_1)| + \]

\[ \frac{1}{4} \sqrt{\frac{53}{70}} |E_{16}(G_1)| + \frac{1}{5} \sqrt{\frac{67}{33}} |E_{17}(G_1)| + \frac{1}{9} \sqrt{\frac{85}{22}} |E_{18}(G_1)| + \]

\[ \frac{4}{3} \sqrt{\frac{10}{473}} |E_{19}(G_1)| + \frac{1}{3} \sqrt{\frac{32}{2}} |E_{20}(G_1)| + \frac{1}{2} \sqrt{\frac{13}{66}} |E_{21}(G_1)| + \]

\[ \frac{1}{2} \sqrt{\frac{37}{231}} |E_{22}(G_1)| + \frac{1}{2} \sqrt{\frac{19}{30}} |E_{23}(G_1)| + \frac{1}{6} \sqrt{\frac{163}{129}} |E_{24}(G_1)| + \]

\[ \frac{1}{2} \sqrt{\frac{29}{210}} |E_{25}(G_1)| + \frac{1}{2} \sqrt{\frac{43}{2}} |E_{26}(G_1)| + \frac{1}{2} \sqrt{\frac{57}{473}} |E_{27}(G_1)| + \]

\[ \frac{1}{2} \sqrt{\frac{13}{110}} |E_{28}(G_1)| + \frac{1}{2} \sqrt{\frac{3}{26}} |E_{29}(G_1)| + \frac{1}{3} \sqrt{\frac{43}{134}} |E_{30}(G_1)| + \]

\[ \frac{1}{4} \sqrt{\frac{181}{86}} |E_{31}(G_1)| + \frac{1}{4} \sqrt{\frac{31}{77}} |E_{32}(G_1)| + \frac{1}{3} \sqrt{\frac{17}{3311}} |E_{33}(G_1)| + \]

\[ \frac{1}{14} \sqrt{\frac{33}{11}} |E_{34}(G_1)| + \frac{1}{40} \sqrt{\frac{79}{2}} |E_{35}(G_1)| + \frac{1}{20} \sqrt{\frac{109}{14}} |E_{36}(G_1)| + \]

\[ \frac{1}{2} \sqrt{\frac{99}{1505}} |E_{37}(G_1)| + \frac{1}{6} \sqrt{\frac{283}{559}} |E_{38}(G_1)| + \frac{1}{70} \sqrt{\frac{139}{2}} |E_{39}(G_1)| + \]

\[ \frac{1}{2} \sqrt{\frac{7}{130}} |E_{40}(G_1)| + \frac{1}{78} \sqrt{\frac{155}{2}} |E_{41}(G_1)| \]

By doing some calculation, we get

\[ \Rightarrow ABC_4(G_1) = 51.706 + \frac{3}{20} \sqrt{\frac{79}{2}} (-5 + n) + 3 \sqrt{\frac{53}{70}} (-4 + n) + \frac{3}{5} \sqrt{\frac{109}{14}} (-4 + n) + \]

\[ \sqrt{\frac{114}{5}} (-4 + n) + \frac{3}{35} \sqrt{\frac{139}{2}} (-4 + n) + 3 \sqrt{\frac{14}{65}} (-3 + n) + \]

\[ 12 \sqrt{\frac{26}{50}} (-3 + n) + 2 \sqrt{\frac{174}{35}} (-3 + n) + \frac{62}{7} (-3 + n) + \]

\[ \sqrt{\frac{78}{11}} (-2 + n) + \frac{9}{11} \sqrt{\frac{43}{2}} (-2 + n)^2 + \frac{1}{3} \sqrt{\frac{35}{2}} (-5 + 2n) + \frac{1}{26} \sqrt{\frac{155}{2}} \]

\[ (24 - 17n + 3n^2) + 3 \sqrt{\frac{6}{13}} (19 - 15n + 3n^2) \]

The \( GA_5(G_1) \) index can be calculated from (8) as follows.

\[ GA_5(G) = \sum_{\rho \neq \emptyset \in E(G)} \frac{2 \sqrt{\sum_{\rho \neq \emptyset \in E(G)} S_{\rho}}}{S_{\rho} + S_{\emptyset}} = \sum_{j=1}^{41} \sum_{\rho \neq \emptyset \in E(G)} \frac{2 \sqrt{\sum_{\rho \neq \emptyset \in E(G)} S_{\rho}}}{S_{\rho} + S_{\emptyset}} \]
GA_5(G_1) = \frac{5}{29}\sqrt{33}|E_{10}(G_1)| + \frac{60}{11}|E_{11}(G_1)| + \frac{30}{79}\sqrt{6}|E_{12}(G_1)| + \\
\frac{5}{51}\sqrt{77}|E_{13}(G_1)| + \frac{3}{8}\sqrt{7}|E_{14}(G_1)| + \frac{4}{11}\sqrt{11}|E_{15}(G_1)| + \\
\frac{4}{27}\sqrt{35}|E_{16}(G_1)| + \frac{4}{23}\sqrt{33}|E_{17}(G_1)| + \frac{6}{29}\sqrt{22}|E_{18}(G_1)| + \\
\frac{1}{31}\sqrt{395}|E_{19}(G_1)| + |E_{20}(G_1)| + \frac{3}{10}\sqrt{119}|E_{21}(G_1)| + \\
\frac{12}{113}\sqrt{77}|E_{22}(G_1)| + \frac{12}{29}\sqrt{5}|E_{23}(G_1)| + \frac{4}{55}\sqrt{129}|E_{24}(G_1)| + \\
\frac{3}{22}\sqrt{35}|E_{25}(G_1)| + |E_{26}(G_1)| + \frac{4}{173}\sqrt{1419}|E_{27}(G_1)| + \\
\frac{1}{23}\sqrt{385}|E_{28}(G_1)| + \frac{1}{25}\sqrt{429}|E_{29}(G_1)| + \frac{6}{131}\sqrt{462}|E_{30}(G_1)| + \\
\frac{6}{61}\sqrt{86}|E_{31}(G_1)| + \frac{8}{157}\sqrt{385}|E_{32}(G_1)| + \frac{1}{103}\sqrt{9933}|E_{33}(G_1)| + \\
\frac{4}{31}\sqrt{55}|E_{34}(G_1)| + |E_{35}(G_1)| + \frac{4}{11}\sqrt{7}|E_{36}(G_1)| + \\
\frac{4}{269}\sqrt{4515}|E_{37}(G_1)| + \frac{4}{95}\sqrt{559}|E_{38}(G_1)| + |E_{39}(G_1)| + \\
\frac{1}{37}\sqrt{1365}|E_{40}(G_1)| + |E_{41}(G_1)|

By doing some calculation, we get

\[ GA_5(G_1) = 315.338 + \frac{288}{29}\sqrt{5}(-4 + r) + \frac{48}{11}\sqrt{7}(-4 + r) + \frac{16}{9}\sqrt{35}(-4 + r) + \frac{9}{2}
\]
\[ \sqrt{7}(-3 + r) + \frac{36}{11}\sqrt{35}(-3 + r) + \frac{48}{23}\sqrt{385}(-3 + r) + \frac{12}{37}\sqrt{1365}(-3 + r) + \\
\frac{18}{5}\sqrt{11}(-2 + r) - 99r + 27r^2 + \frac{12}{25}\sqrt{429}(19 - 15r + 3r^2) \]

2.2 Results for Third type of Rectangular Hex Derived network THDN3(r)

Now, we discuss the newly derived third type of rectangular hex derived network and compute numerical and exact results for forgotten index, Balaban index, reclassified the Zagreb indices, forth version of ABC index and fifth version of GA index for THDN3(r).

**Theorem 2.2.1.** Consider the Third type of Triangular Hex Derived network of THDN3(n), then its forgotten index is equal to

\[ F(HD\text{N}3(n)) = 12(990 - 997r + 259r^2) \]
Fig. 3. Hex Derived network of type 3 (HDN3(4))

| (κ_u, κ_v) where p̅q̅ ∈ E(G_1) | Number of edges | (κ_u, κ_v) where p̅q̅ ∈ E(G_1) | Number of edges |
|--------------------------------|----------------|--------------------------------|----------------|
| (4, 4)                         | 3r^2 - 6r + 9   | (10, 10)                       | 3r - 6         |
| (4, 10)                        | 18r - 30        | (10, 18)                       | 6r - 18        |
| (4, 18)                        | 6r^2 - 30r + 36 | (18, 18)                       | (3r^2 - 21r + 36) |

Table 3. Edge partition Third type of Triangular Hex Derived network THDN3(r) based on degrees of end vertices of each edge.

Proof. Let G_2 be the Hex Derived network of type 3, HDN3(r) shown in Fig. 4 where r ≥ 4. The Hex Derived network G_2 has \(\frac{7r^2 - 11r + 6}{2}\) vertices and the edge set of G_2 is divided into six partitions based on the degree of end vertices. The first edge partition \(E_1(G_2)\) contains \(3r^2 - 6r + 9\) edges \(p̅q̅\), where \(κ(p̅) = κ(q̅) = 4\). The second edge partition \(E_2(G_2)\) contains \(18r - 30\) edges \(p̅q̅\), where \(κ(p̅) = 4\) and \(κ(q̅) = 10\). The third edge partition \(E_3(G_2)\) contains \(6r^2 - 30r + 36\) edges \(p̅q̅\), where \(κ(p̅) = 4\) and \(κ(q̅) = 18\). The fourth edge partition \(E_4(G_2)\) contains \(3r - 6\) edges \(p̅q̅\), where \(κ(p̅) = κ(q̅) = 10\). The fifth edge partition \(E_5(G_2)\) contains \(6r - 18\) edges \(p̅q̅\), where \(κ(p̅) = 10\) and \(κ(q̅) = 18\) and the sixth edge partition \(E_6(G_2)\) contains \(\frac{3r^2 - 21r + 36}{2}\) edges \(p̅q̅\), where \(κ(p̅) = κ(q̅) = 18\). Table 3, shows such an edge partition of G_2. Thus from (3) this follows that

\[
F(G) = \sum_{p̅q̅ ∈ E(G)} ((κ(p̅))^2 + (κ(q̅))^2)
\]

Let G_2 be the third type of triangular hex derived network, THDN3(r). By using edge partition from Table 3, the result follows. The forgotten index can be calculated by using (2) as follows.

\[
F(G_2) = \sum_{p̅q̅ ∈ E(G)} ((κ(p̅))^2 + (κ(q̅))^2) = \sum_{p̅q̅ ∈ E_j(G)} \sum_{j=1}^{6} ((κ(p̅))^2 + (κ(q̅))^2)
\]
By doing some calculations, we get
\[ F(G_2) = 12(990 - 997r + 259r^2). \]

In the following theorem, we compute Balaban index of third type of triangular hex Derived network, \( G_2 \).

**Theorem 2.2.2.** For triangular hex derived network \( G_2 \), the Balaban index is equal to
\[ J(G_2) = \left( \frac{1}{40(8 - 14r + 7r^2)} \right) \left( 6 - 13r + 7r^2 \right) \left( 159 + 1802\sqrt{2} - 36\sqrt{5} - 90\sqrt{10} + (-107 - 150\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})r + 10(5 + 3\sqrt{2})r^2 \right) \]

**Proof.** Let \( G_2 \) be the triangular hex derived network \( THDN3(r) \). By using edge partition from Table 3, the result follows. The Balaban index can be calculated by using (3) as follows.

\[ J(G_2) = \left( \frac{m}{m - n + 2} \right) \sum_{\rho \in E(G)} \frac{1}{\sqrt{\kappa(p) \times \kappa(q)}} = \left( \frac{m}{m - n + 2} \right) \sum_{\rho \in E(G)} \sum_{j=1}^{6} \frac{1}{\sqrt{\kappa(p) \times \kappa(q)}} \]

\[ J(G_2) = \frac{3}{2} \left( \frac{6 - 13r + 7r^2}{8 - 14r + 7r^2} \right) \frac{1}{4} |E_1(G_2)| + \frac{1}{2\sqrt{10}} |E_2(G_2)| + \frac{1}{6\sqrt{2}} |E_3(G_2)| + \frac{1}{10} |E_4(G_2)| + \frac{1}{6\sqrt{5}} |E_5(G_2)| + \frac{1}{18} |E_6(G_2)| \]

By doing some calculation, we get
\[ \Rightarrow J(G_2) = \left( \frac{1}{40(8 - 14r + 7r^2)} \right) \left( 6 - 13r + 7r^2 \right) \left( 159 + 1802\sqrt{2} - 36\sqrt{5} - 90\sqrt{10} + (-107 - 150\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})r + 10(5 + 3\sqrt{2})r^2 \right). \]

\[ \square \]

Now, we compute \( ReZG_1 \), \( ReZG_2 \) and \( ReZG_3 \) indices of triangular hex derived network \( G_2 \).

**Theorem 2.2.3.** Let \( G_2 \) be the triangular hex derived network, then

- \( ReZG_1(G_2) = \frac{3}{181}(3408 - 5117r + 2009r^2); \)
• $ReZG_2(G_2) = \frac{1}{2}(6 - 11r + 7r^2)$;
• $ReZG_3(G_2) = 24(6192 - 5185r + 1141r^2)$.

Proof. By using edge partition given in Table 3, the $ReZG_1(G_2)$ can be calculated by using (4) as follows.

\[
ReZG_1(G) = \sum_{\hat{p} \in E(G)} \left( \frac{\kappa(\hat{p}) \times \kappa(\hat{q})}{\kappa(\hat{p}) + \kappa(\hat{q})} \right) = \sum_{j=1}^{6} \sum_{\hat{p} \in E_j(G_2)} \left( \frac{\kappa(\hat{p}) \times \kappa(\hat{q})}{\kappa(\hat{p}) + \kappa(\hat{q})} \right)
\]

\[
ReZG_1(G_2) = 2|E_1(G_2)| + \frac{20}{7}|E_2(G_2)| + \frac{36}{11}|E_3(G_2)| + 5|E_4(G_2)| + \frac{45}{7}|E_5(G_2)| + 9|E_6(G_2)|
\]

By doing some calculation, we get

\[
\Rightarrow ReZG_1(G_2) = \frac{3}{154}(3408 - 5117r + 2009r^2)
\]

The $ReZG_2(G_2)$ can be calculated by using (5) as follows.

\[
ReZG_2(G) = \sum_{\hat{p} \in E(G)} \left( \frac{\kappa(\hat{p}) + \kappa(\hat{q})}{\kappa(\hat{p}) \times \kappa(\hat{q})} \right) = \sum_{j=1}^{9} \sum_{\hat{p} \in E_j(G_2)} \left( \frac{\kappa(\hat{p}) + \kappa(\hat{q})}{\kappa(\hat{p}) \times \kappa(\hat{q})} \right)
\]

\[
ReZG_2(G_2) = \frac{1}{2}|E_1(G_2)| + \frac{7}{20}|E_2(G_2)| + \frac{11}{36}|E_3(G_2)| + \frac{1}{5}|E_4(G_2)| + \frac{7}{45}|E_5(G_2)| + \frac{1}{9}|E_6(G_2)|
\]

By doing some calculation, we get

\[
\Rightarrow ReZG_2(G_2) = \frac{1}{2}(6 - 11r + 7r^2)
\]

The $ReZG_3(G_2)$ index can be calculated from (6) as follows.

\[
ReZG_3(G) = \sum_{\hat{p} \in E(G)} (\kappa(\hat{p}) \times \kappa(\hat{q}))(\kappa(\hat{p}) + \kappa(\hat{q}))
\]

\[
= \sum_{j=1}^{6} \sum_{\hat{p} \in E_j(G_2)} (\kappa(\hat{p}) \times \kappa(\hat{q}))(\kappa(\hat{p}) + \kappa(\hat{q}))
\]

\[
ReZG_3(G_2) = 128|E_1(G_2)| + 560|E_2(G_2)| + 1584|E_3(G_2)| + 2000|E_4(G_2)| + 5040|E_5(G_2)| + 11664|E_6(G_2)|
\]
By doing some calculation, we get

\[ \Rightarrow ReZG3(G_2) = 24(6192 - 5185r + 1141r^2) \]

Now, we compute \( ABC_4 \) and \( GA_5 \) indices of triangular hex derived network \( G_2 \).

**Theorem 2.2.4.** Let \( G_2 \) be the triangular hex derived network, then

- \( ABC_4(G_2) = 24.131 + 3\sqrt{\frac{7}{10}}(-6 + r) + 6\sqrt{\frac{25}{10}}(-5 + r) + \frac{3}{10}\sqrt{\frac{109}{14}}(-5 + r) + \frac{3}{8}\sqrt{\frac{75}{2}}(-5 + r) + \frac{3}{8}\sqrt{\frac{139}{2}}(-5 + r) + \frac{3}{8}\sqrt{\frac{33}{2}}(-4 + r) + \frac{3}{8}\sqrt{\frac{39}{2}}(-4 + r) + \frac{3}{8}\sqrt{\frac{57}{10}}(-4 + r) + \frac{3}{8}\sqrt{\frac{42}{2}}(-4 + r)^2 + \frac{1}{3}\sqrt{\frac{35}{2}}(-3 + r) + 2\sqrt{\frac{11}{2}}(-2 + r) + \frac{1}{3}\sqrt{\frac{155}{2}}(42 - 13r + r^2) + 3\sqrt{\frac{11}{20}}(30 - 11r + r^2) \);

- \( GA_5(G_2) = 110.66 + \frac{9}{10}\sqrt{1365}(-6 + r) + \frac{9}{10}\sqrt{7}(-5 + r) + \frac{18}{11}\sqrt{35}(-5 + r) + \frac{24}{23}\sqrt{385}(-5 + r) + \frac{44}{29}\sqrt{5}(-4 + r) + \frac{4}{7}\sqrt{11}(-4 + r) + \frac{4}{9}\sqrt{35}(-4 + r) + \frac{36}{29}\sqrt{27}(-2 + r) - 12r + 3r^2 + \frac{3}{2}(42 - 13r + r^2) + \frac{9}{25}\sqrt{129}(30 - 11r + r^2) \).

**Proof.** By using edge partition given in Table 4, the \( ABC_4(G_2) \) can be calculated by using (7) as follows.

\[
ABC_4(G) = \sum_{p\notin E(G)} \sqrt{S_p + S_q - 2} - \frac{32}{j=7} \sum_{p\notin E_j(G_2)} \sqrt{S_p + S_q - 2}
\]
By doing some calculation, we get

\[ ABC_4(G_2) = \frac{1}{11} \sqrt{\frac{21}{2} \left| E_7(G_2) \right| + \frac{6}{17} \left| E_8(G_2) \right| + \frac{7}{11} \left| E_9(G_2) \right| + \frac{43}{6} \left| E_{10}(G_2) \right| + \frac{23}{462} \left| E_{11}(G_2) \right| + \frac{53}{70} \left| E_{12}(G_2) \right| + \frac{32}{2} \left| E_{13}(G_2) \right| + \frac{13}{66} \left| E_{14}(G_2) \right| + \frac{5}{3} \sqrt{\frac{4}{18} \left| E_{15}(G_2) \right| + \frac{19}{30} \left| E_{16}(G_2) \right| + \frac{7}{6} \left| E_{17}(G_2) \right| + \frac{1}{2} \sqrt{\frac{29}{210} \left| E_{18}(G_2) \right| + \frac{43}{2} \left| E_{19}(G_2) \right| + \frac{83}{2} \left| E_{20}(G_2) \right| + \frac{13}{110} \left| E_{21}(G_2) \right| + \frac{3}{20} \left| E_{22}(G_2) \right| + \frac{65}{2} \left| E_{23}(G_2) \right| + \frac{3}{110} \left| E_{24}(G_2) \right| + \frac{47}{2046} \left| E_{25}(G_2) \right| + \frac{131}{2170} \left| E_{26}(G_2) \right| + \frac{1}{20} \left| E_{27}(G_2) \right| + \frac{109}{14} \left| E_{28}(G_2) \right| + \frac{131}{2170} \left| E_{29}(G_2) \right| + \frac{1}{2} \sqrt{\frac{139}{2} \left| E_{30}(G_2) \right| + \frac{7}{130} \left| E_{31}(G_2) \right| + \frac{155}{2} \left| E_{32}(G_2) \right|} \]

By doing some calculation, we get

\[ \Rightarrow ABC_4(G_2) = 24.131 + 3 \sqrt{\frac{7}{130} (-6 + r) + 6 \sqrt{\frac{26}{55} (-5 + r) + \sqrt{\frac{174}{35} (-5 + r) + \frac{3}{10} \sqrt{\frac{109}{14} (-5 + r) + \frac{3}{40} \sqrt{\frac{79}{2} (-5 + r) + \frac{3}{70} \sqrt{\frac{139}{2} (-5 + r) + \frac{3}{2} \sqrt{\frac{53}{70} (-4 + r) + \frac{39}{22} (-4 + r) + \sqrt{\frac{37}{10} (-4 + r) + \frac{3}{22} \sqrt{\frac{43}{2} (-4 + r)^2 + \frac{3}{1} \sqrt{\frac{35}{2} (-3 + r) + 2 \sqrt{\frac{7}{11} (-2 + r) + \frac{1}{52} \sqrt{\frac{155}{2} (42 - 13r + r^2) + 3 \sqrt{\frac{3}{26} (30 - 11r + r^2)}}} \right) \right) \right) \right) \right) \right) \right) \right)} \]

The \( GA_5(G) \) index can be calculated from (8) as follows.

\[ GA_5(G) = \sum_{\rho q \in E(G)} \frac{2 \sqrt[3]{S_p S_q}}{(S_p + S_q)} = \sum_{j=7}^{32} \sum_{\rho q \in E(G)} \frac{2 \sqrt[3]{S_p S_q}}{(S_p + S_q)} \]
By doing some calculation, we get

\[
GA_5(G_2) = |E_7(G_2)| + \frac{2}{25}\sqrt{154}|E_8(G_2)| + \frac{6}{29}\sqrt{22}|E_9(G_2)| + \frac{1}{2}\sqrt{3}|E_{10}(G_2)| + \\
\frac{2}{47}\sqrt{462}|E_{11}(G_2)| + \frac{4}{27}\sqrt{35}|E_{12}(G_2)| + |E_{13}(G_2)| + \\
\frac{3}{10}\sqrt{11}|E_{14}(G_2)| + \frac{2}{17}\sqrt{66}|E_{15}(G_2)| + \frac{12}{29}\sqrt{5}|E_{16}(G_2)| + \\
\frac{3}{20}\sqrt{31}|E_{17}(G_2)| + \frac{3}{22}\sqrt{35}|E_{18}(G_2)| + \frac{1}{25}\sqrt{129}|E_{22}(G_2)| + \\
\frac{1}{21}\sqrt{34}|E_{20}(G_2)| + \frac{1}{23}\sqrt{385}|E_{21}(G_2)| + \frac{1}{25}\sqrt{229}|E_{22}(G_2)| + \\
\frac{4}{73}\sqrt{330}|E_{23}(G_2)| + \frac{2}{95}\sqrt{2046}|E_{24}(G_2)| + \frac{1}{E_{26}(G_2)} + \\
\frac{4}{51}\sqrt{155}|E_{27}(G_2)| + \frac{1}{11}\sqrt{7}|E_{28}(G_2)| + \frac{1}{33}\sqrt{1085}|E_{29}(G_2)| + \\
\frac{1}{37}\sqrt{1365}|E_{31}(G_2)| + \frac{1}{37}|E_{32}(G_2)|
\]

By doing some calculation, we get

\[
\Rightarrow GA_5(G_2) = 315.338 + \frac{288}{29}\sqrt{5}(-4 + r) + \frac{48}{11}\sqrt{7}(-4 + r) + \frac{16}{9}\sqrt{35}(-4 + r) + \frac{9}{2}
\]

\[
\sqrt{7}(-3 + r) + \frac{36}{11}\sqrt{35}(-3 + r) + \frac{48}{23}\sqrt{385}(-3 + r) + \frac{12}{37}\sqrt{1365}(-3 + r) + \\
\frac{18}{5}\sqrt{11}(-2 + r) - 99r + 27r^2 + \frac{12}{25}\sqrt{129}(19 - 15r + 3r^2)
\]

### 2.3 Results for Third type of Rectangular Hex Derived network \(RHDN3(r)\)

Now, we calculate certain degree based topological indices of Rectangular Hex Derived network of type 3, \(RHDN3(r, s)\) of dimension \(r = s\). We compute forgotten index, Balaban index, reclassified the Zagreb indices, forth version of \(ABC\) index and fifth version of \(GA\) index in the coming theorems of \(RHDN3(r, s)\).

**Theorem 2.3.1.** Consider the third type of rectangular hex derived network \(RHDN3(n)\), then its forgotten index is equal to

\[
F(HDN3(n)) = 19726 - 20096r + 6216r^2
\]

**Proof.** Let \(G_3\) be the Rectangular Hex Derived network of type 3, \(RHDN3(r)\) shown in Fig. ??, where \(r = s \geq 4\). The Rectangular Hex Derived network \(G_3\) has \(7r^2 - 12r + 6\) vertices and the edge set of \(G_3\) is divided into nine partitions based on the degree of end vertices. The first edge partition \(E_1(G_3)\) contains \(9r^2 - 12r + 10\) edges \(\hat{p}\), where \(\kappa(\hat{p}) = \kappa(\hat{q}) = 4\). The second edge partition \(E_2(G_1)\) contains \(8\) edges \(\hat{p}\), where \(\kappa(\hat{p}) = 4\) and \(\kappa(\hat{q}) = 7\). The third edge partition \(E_3(G_3)\) contains \(24r - 44\) edges \(\hat{p}\), where \(\kappa(\hat{p}) = 4\) and \(\kappa(\hat{q}) = 10\).
Fig. 4. Hex Derived network of type 3 (HDN3(4))

| (κx, κy) where ´pq´ ∈ E(G1) | Number of edges (κx, κy) where ´pq´ ∈ E(G1) | Number of edges |
|-----------------------------|---------------------------------|--------------------|
| (4, 4)                      | 6r² - 12r + 10                 | (7, 18)            |
| (4, 7)                      | 8                              | (10, 10)           | 4r - 10            |
| (4, 10)                     | 24r - 44                       | (10, 18)           | 8r - 20            |
| (4, 18)                     | 12r² - 48r + 48                | (18, 18)           | 3r² - 16r + 21     |
| (7, 10)                     | 4                              |                     |                    |

Table 5. Edge partition of Rectangular Hex Derived network of type 3, RHDN3(r) based on degrees of end vertices of each edge.

The fourth edge partition $E_4(G_3)$ contains $12r^2 - 48r + 48$ edges ´pq´, where $\kappa(p) = 4$ and $\kappa(q) = 18$. The fifth edge partition $E_5(G_3)$ contains 4 edges ´pq´, where $\kappa(p) = 7$ and $\kappa(q) = 10$. The sixth edge partition $E_6(G_3)$ contains 2 edges ´pq´, where $\kappa(p) = 7$ and $\kappa(q) = 18$. The seventh edge partition $E_7(G_3)$ contains 4 edges ´pq´, where $\kappa(p) = 7$ and $\kappa(q) = 10$ and the eighth edge partition $E_8(G_3)$ contains 8 edges ´pq´, where $\kappa(p) = 10$ and $\kappa(q) = 18$ and the ninth edge partition $E_9(G_3)$ contains 3 edges ´pq´, where $\kappa(p) = \kappa(q) = 18$.

Table 3 shows such an edge partition of $G_3$. Thus from (3) this follows that

$$F(G) = \sum_{\rho q \in E(G)} ((\kappa(p))^2 + (\kappa(q))^2)$$

Let $G_3$ be the third type of triangular hex derived network, THDN3(r). By using edge partition from Table 3, the result follows. The forgotten index can be calculated by using (2) as follows.

$$F(G_3) = \sum_{\rho q \in E(G)} ((\kappa(p))^2 + (\kappa(q))^2) = \sum_{\rho q \in E(G)} \sum_{j=1}^{9} ((\kappa(p))^2 + (\kappa(q))^2)$$

$$F(G_3) = 32|E_1(G_3)| + 65|E_2(G_3)| + 116|E_3(G_3)| + 340|E_4(G_3)| + 149|E_5(G_3)| + 373|E_6(G_3)| + 200|E_7(G_3)| + 424|E_8(G_3)| + 648|E_9(G_3)|$$
By doing some calculations, we get
\[ F(G_3) = 19726 - 20096r + 6216r^2. \]

In the following theorem, we compute Balaban index of third type of triangular hex Derived network, \( G_3 \).

**Theorem 2.3.2.** For triangular hex derived network \( G_3 \), the Balaban index is equal to
\[
J(G_3) = \left( \frac{1}{315(15 - 28r + 14r^2)} \right) 7(-157 - 180\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})r +
105(5 + 3\sqrt{2})r^2(19 - 40r + 21r^2)(3(280 + 420\sqrt{2} - 70\sqrt{5} +
60\sqrt{7} - 231\sqrt{10} + 5\sqrt{14} + 6\sqrt{70})) \]

**Proof.** Let \( G_3 \) be the rectangular hex derived network \( RHDRN3(r) \). By using edge partition from Table 5, the result follows. The Balaban index can be calculated by using (3) as follows.

\[
J(G_3) = \left( \frac{1}{m - n + 2} \right) \sum_{\hat{p} \in E(G)} \frac{1}{\sqrt{\kappa(p) \times \kappa(q)}} = \left( \frac{m}{m - n + 2} \right) \sum_{\hat{p} \in E(G)} \sum_{j=1}^{9} \frac{1}{\sqrt{\kappa(p) \times \kappa(q)}}
\]

\[
J(G_3) = \left( \frac{19 - 40r + 21r^2}{15 - 28r + 14r^2} \right) \left( \frac{1}{4} |E_1(G_3)| + \frac{1}{2\sqrt{7}} |E_2(G_3)| + \frac{1}{2\sqrt{10}} |E_3(G_3)| +
\frac{1}{6\sqrt{2}} |E_4(G_3)| + \frac{1}{\sqrt{70}} |E_5(G_3)| + \frac{1}{3\sqrt{14}} |E_6(G_3)| + \frac{1}{10} |E_7(G_3)| +
\frac{1}{6\sqrt{5}} |E_8(G_3)| + \frac{1}{18} |E_9(G_3)| \right)
\]

By doing some calculation, we get
\[ J(G_3) = \left( \frac{1}{315(15 - 28r + 14r^2)} \right) 7(-157 - 180\sqrt{2} + 12\sqrt{5} + 54\sqrt{10})r +
105(5 + 3\sqrt{2})r^2(19 - 40r + 21r^2)(3(280 + 420\sqrt{2} - 70\sqrt{5} +
60\sqrt{7} - 231\sqrt{10} + 5\sqrt{14} + 6\sqrt{70})). \]
Theorem 2.3.3. Let $G_3$ be the rectangular hex derived network, then

- $\text{Re}ZG_1(G_3) = \frac{10102843}{32725} - \frac{2036r}{11} + \frac{861n^2}{11}$;

- $\text{Re}ZG_2(G_3) = 56 - 12r + 7r^2$;

- $\text{Re}ZG_3(G_3) = 4(50785 - 50608r + 13692r^2)$.

Proof. By using edge partition given in Table 3, the $\text{Re}ZG_1(G_3)$ can be calculated by using (4) as follows.

\[
\text{Re}ZG_1(G) = \sum_{\{p, q\} \in E(G)} \left( \frac{\kappa(p) \times \kappa(q)}{\kappa(p) + \kappa(q)} \right) = \sum_{j=1}^{9} \sum_{\{p, q\} \in E_j(G_3)} \left( \frac{\kappa(p) \times \kappa(q)}{\kappa(p) + \kappa(q)} \right)
\]

\[
\text{Re}ZG_1(G_3) = 2|E_1(G_3)| + \frac{28}{11}|E_2(G_3)| + \frac{20}{7}|E_3(G_3)| + \frac{36}{11}|E_4(G_3)| + \frac{70}{17}|E_5(G_3)| + \frac{126}{25}|E_6(G_3)| + 5|E_7(G_3)| + \frac{45}{7}|E_8(G_3)| + \frac{9}{9}|E_9(G_3)|
\]

By doing some calculation, we get

\[
\Rightarrow \text{Re}ZG_1(G_3) = \frac{10102843}{32725} - \frac{2036r}{11} + \frac{861n^2}{11}
\]

The $\text{Re}ZG_2(G_3)$ can be calculated by using (5) as follows.

\[
\text{Re}ZG_2(G) = \sum_{\{p, q\} \in E(G)} \left( \frac{\kappa(p) + \kappa(q)}{\kappa(p) \times \kappa(q)} \right) = \sum_{j=1}^{9} \sum_{\{p, q\} \in E_j(G_3)} \left( \frac{\kappa(p) + \kappa(q)}{\kappa(p) \times \kappa(q)} \right)
\]

\[
\text{Re}ZG_2(G_3) = \frac{1}{2}|E_1(G_3)| + \frac{11}{28}|E_2(G_3)| + \frac{7}{20}|E_3(G_3)| + \frac{11}{36}|E_4(G_3)| + \frac{17}{70}|E_5(G_3)| + \frac{25}{126}|E_6(G_3)| + \frac{1}{5}|E_7(G_3)| + \frac{7}{45}|E_8(G_3)| + \frac{1}{9}|E_9(G_3)|
\]

By doing some calculation, we get

\[
\Rightarrow \text{Re}ZG_2(G_3) = 56 - 12r + 7r^2
\]

The $\text{Re}ZG_3(G_3)$ index can be calculated from (6) as follows.

\[
\text{Re}ZG_3(G) = \sum_{\{p, q\} \in E(G)} (\kappa(p) \times \kappa(q)) (\kappa(p) + \kappa(q))
\]

\[
= \sum_{j=1}^{6} \sum_{\{p, q\} \in E_j(G_3)} (\kappa(p) \times \kappa(q)) (\kappa(p) + \kappa(q))
\]
Theorem 2.3.4. Let $G_3$ be the triangular hex derived network, then

\[ ABC_4(G_3) = 22.459 + 8\sqrt{\frac{26}{55}(-4 + r)} + 4\sqrt{\frac{26}{105}(-4 + r)} + \frac{4}{9}\sqrt{\frac{27}{17}(-4 + r)} + 3\sqrt{\frac{6}{13}(-4 + r)^2} + 2\sqrt{\frac{28}{55}(-3 + r)} + \frac{3}{11}\sqrt{\frac{31}{2}(-3 + r)^2} + \sqrt{\frac{14}{165}(-9 + 2r)} + \frac{1}{\sqrt{2}}\sqrt{\frac{14}{2}(-9 + 2r)} + \frac{1}{9}\sqrt{\frac{92}{2}(-5 + 2r)} + \frac{4}{\sqrt{5}}\sqrt{31(-5 + 2r)} + \frac{4}{9}\sqrt{\frac{27}{17}(-3 + 2r)} + \frac{2}{\sqrt{89}}\sqrt{(-3 + 2r)^2} + \frac{1}{9}\sqrt{\frac{35}{2}(-11 + 4r)} + \frac{1}{\sqrt{8}}\sqrt{\frac{155}{2}(65 - 28r + 3r^2)};\]

Now, we compute $ABC_4$ and $GA_5$ indices of triangular hex derived network $G_3$. 

\[ \Rightarrow \text{ReZG3}(G_3) = 4(50785 - 50608r + 13692r^2) \]

Table 6. Edge partition of Hex Derived network of type 3 $HDN3(r)$ based on degrees of end vertices of each edge.

| $\{\kappa_x, \kappa_y\}$ where $\kappa' \in E(G_3)$ | Number of edges | $\{\kappa_x, \kappa_y\}$ where $\kappa' \in E(G_3)$ | Number of edges |
|-----------------------------------------------|-----------------|-----------------------------------------------|-----------------|
| (22, 22)                                      | 2               | (44, 44)                                      | 6$r^2 - 36r + 54$ |
| (22, 28)                                      | 8               | (44, 124)                                     | 8               |
| (22, 63)                                      | 4               | (44, 129)                                     | 12              |
| (25, 33)                                      | 4               | (44, 140)                                     | 32$r - 128$     |
| (25, 36)                                      | 4               | (44, 156)                                     | 12$r^2 - 96r + 192$ |
| (25, 54)                                      | 4               | (54, 63)                                      | 4               |
| (25, 63)                                      | 4               | (54, 129)                                     | 2               |
| (28, 36)                                      | 8$r - 20$       | (63, 63)                                      | 4$r - 10$       |
| (28, 63)                                      | 8$r - 12$       | (63, 124)                                     | 8               |
| (33, 36)                                      | 4               | (63, 129)                                     | 4               |
| (33, 54)                                      | 4               | (63, 140)                                     | 8$r - 32$       |
| (33, 129)                                     | 4               | (129, 140)                                    | 4               |
| (36, 36)                                      | 8$r - 22$       | (129, 140)                                    | 4               |
| (36, 44)                                      | 8$r - 24$       | (129, 156)                                    | 2               |
| (36, 63)                                      | 16$r - 40$      | (140, 140)                                    | 4$r - 18$       |
| (36, 124)                                     | 16$r - 60$      | (140, 156)                                    | 8$r - 36$       |
| (36, 129)                                     | 8               | (156, 156)                                    | 3$r^2 - 28r + 65$ |
| (36, 140)                                     | 16$r - 64$      |                                           |                 |
• \( GA_3(G_3) = 173.339 + \frac{96}{25} \sqrt{3}(-4 + r) + \frac{24}{11} \sqrt{35}(-4 + r) + \frac{32}{21} \sqrt{385}(-4 + r) + \frac{12}{27} \sqrt{429}(-4 + r)^2 + \frac{12}{27} \sqrt{111}(-3 + r) - 48r + 9r^2 + \frac{4}{17} \sqrt{1365}(-9 + 2r) + \frac{3}{2} \sqrt{7}(-5 + 2r) + \frac{48}{17}(-3 + 2r) + \frac{42}{17} \sqrt{7}(-3 + 2r). \)

Proof. By using edge partition given in Table 4, the \( ABC_4(G_3) \) can be calculated by using (7) as follows.

\[
ABC_4(G) = \sum_{\rho \in E(G)} \sqrt{S_{\rho} + S_{\rho} - 2} = \sum_{j = 10}^{44} \sum_{\rho \in E_j(G_3)} \sqrt{S_{\rho} + S_{\rho} - 2}
\]

\[
ABC_4(G_3) = \frac{1}{17} \sqrt{\frac{21}{2}} |E_{10}(G_3)| + \sqrt{\frac{6}{77}} |E_{11}(G_3)| + \frac{1}{3} \sqrt{\frac{83}{154}} |E_{12}(G_3)| + \frac{1}{5} \sqrt{\frac{46}{33}} |E_{13}(G_3)| + \frac{1}{30} \sqrt{59} |E_{14}(G_3)| + \frac{1}{15} \sqrt{\frac{77}{6}} |E_{15}(G_3)| + \frac{1}{15} \sqrt{\frac{86}{7}} |E_{16}(G_3)| + \frac{1}{6} \sqrt{\frac{31}{14}} |E_{17}(G_3)| + \frac{1}{42} \sqrt{89} |E_{18}(G_3)| + \frac{1}{6} \sqrt{\frac{67}{33}} |E_{19}(G_3)| + \frac{1}{5} \sqrt{\frac{85}{22}} |E_{20}(G_3)| + \frac{1}{3} \sqrt{\frac{31}{4}} |E_{21}(G_3)| + \frac{1}{6} \sqrt{\frac{35}{2}} |E_{22}(G_3)| + \frac{1}{2} \sqrt{\frac{79}{66}} |E_{23}(G_3)| + \frac{1}{15} \sqrt{\frac{30}{7}} |E_{24}(G_3)| + \frac{1}{6} \sqrt{\frac{79}{62}} |E_{25}(G_3)| + \frac{1}{2} \sqrt{\frac{163}{129}} |E_{26}(G_3)| + \frac{1}{2} \sqrt{\frac{29}{210}} |E_{27}(G_3)| + \frac{1}{22} \sqrt{\frac{43}{7}} |E_{28}(G_3)| + \frac{1}{2} \sqrt{\frac{83}{682}} |E_{29}(G_3)| + \frac{1}{2} \sqrt{\frac{57}{473}} |E_{30}(G_3)| + \frac{1}{2} \sqrt{\frac{13}{110}} |E_{31}(G_3)| + \frac{1}{2} \sqrt{\frac{3}{26}} |E_{32}(G_3)| + \frac{1}{9} \sqrt{\frac{115}{42}} |E_{33}(G_3)| + \frac{1}{3} \sqrt{\frac{181}{86}} |E_{34}(G_3)| + \frac{1}{2} \sqrt{\frac{31}{63}} |E_{35}(G_3)| + \frac{1}{6} \sqrt{\frac{185}{217}} |E_{36}(G_3)| + \frac{1}{3} \sqrt{\frac{190}{903}} |E_{37}(G_3)| + \frac{1}{4} \sqrt{\frac{67}{15}} |E_{38}(G_3)| + \frac{1}{2} \sqrt{\frac{131}{2170}} |E_{39}(G_3)| + \frac{1}{2} \sqrt{\frac{89}{1505}} |E_{40}(G_3)| + \frac{1}{70} \sqrt{\frac{283}{559}} |E_{41}(G_3)| + \frac{1}{70} \sqrt{\frac{139}{2}} |E_{42}(G_3)| + \frac{1}{2} \sqrt{\frac{7}{130}} |E_{43}(G_3)| + \frac{1}{78} \sqrt{\frac{155}{2}} |E_{44}(G_3)|}
By doing some calculation, we get

\[ ABC_2(G_3) = 22.459 + 8\sqrt{\frac{26}{55}(-4 + r)} + 4\sqrt{\frac{58}{105}(-4 + r)} + \frac{4}{7}\sqrt{\frac{67}{15}} \]

\[ (-4 + r) + 3\sqrt{\frac{6}{13}(-4 + r)^2} + 2\sqrt{\frac{26}{33}(-3 + r)} + \frac{3}{11}\sqrt{\frac{43}{2}} \]

\[ (-3 + r)^2 + \sqrt{\frac{14}{65}(-9 + 2r)} + \frac{1}{35}\sqrt{\frac{139}{2}(-9 + 2r)} + \frac{1}{3}\sqrt{\frac{62}{7}} \]

\[ (-5 + 2r) + \frac{4}{63}\sqrt{31(-5 + 2r)} + \frac{4}{9}\sqrt{\frac{97}{7}(-3 + 2r)} + \frac{2}{21}\sqrt{89} \]

\[ (-3 + 2r) + \frac{1}{9}\sqrt{\frac{35}{2}(-11 + 4r)} + \frac{1}{78}\sqrt{\frac{155}{2}(65 - 28r + 3r^2)} \]

The \(G_5(G_3)\) index can be calculated from (8) as follows.

\[ G_5(G) = \sum_{\rho \in E(G)} \frac{2\sqrt{S_\rho \cdot S_\rho}}{(S_\rho + S_\rho)} = \sum_{j=10}^{44} \sum_{\rho \in E(G)} \frac{2\sqrt{S_\rho \cdot S_\rho}}{(S_\rho + S_\rho)} \]

\[ G_5(G_3) = 1|E_{10}(G_3)| + \frac{2}{25}\sqrt{154}|E_{11}(G_3)| + \frac{6}{85}\sqrt{154}|E_{12}(G_3)| + \]

\[ \frac{5}{29}\sqrt{33}|E_{13}(G_3)| + \frac{60}{61}|E_{14}(G_3)| + \frac{30}{79}\sqrt{6}|E_{15}(G_3)| + \]

\[ \frac{15}{44}\sqrt{7}|E_{16}(G_3)| + \frac{3}{8}\sqrt{7}|E_{17}(G_3)| + \frac{12}{13}|E_{18}(G_3)| + \frac{4}{23}\sqrt{33}|E_{19}(G_3)| + \]

\[ \frac{6}{29}\sqrt{22}|E_{20}(G_3)| + \frac{1}{27}\sqrt{473}|E_{21}(G_3)| + 1|E_{22}(G_3)| + \]

\[ \frac{3}{30}\sqrt{11}|E_{23}(G_3)| + \frac{4}{11}\sqrt{7}|E_{24}(G_3)| + \frac{3}{20}\sqrt{31}|E_{25}(G_3)| + \]

\[ \frac{4}{55}\sqrt{129}|E_{26}(G_3)| + \frac{3}{22}\sqrt{35}|E_{27}(G_3)| + |E_{28}(G_3)| + \]

\[ \frac{1}{21}\sqrt{341}|E_{29}(G_3)| + \frac{4}{173}\sqrt{1419}|E_{30}(G_3)| + \frac{1}{23}\sqrt{1385}|E_{31}(G_3)| + \]

\[ \frac{1}{25}\sqrt{129}|E_{32}(G_3)| + \frac{2}{13}\sqrt{42}|E_{33}(G_3)| + \frac{6}{61}\sqrt{86}|E_{34}(G_3)| + 1|E_{35}(G_3)| + \]

\[ \frac{12}{187}\sqrt{217}|E_{36}(G_3)| + \frac{1}{32}\sqrt{903}|E_{37}(G_3)| + \frac{12}{29}\sqrt{5}|E_{38}(G_3)| + \]

\[ \frac{1}{33}\sqrt{1085}|E_{39}(G_3)| + \frac{4}{269}\sqrt{4515}|E_{40}(G_3)| + \frac{4}{95}\sqrt{559}|E_{41}(G_3)| + \]

\[ 1|E_{42}(G_3)| + \frac{1}{37}\sqrt{1365}|E_{43}(G_3)| + 1|E_{44}(G_3)| \]
By doing some calculation, we get

\[ GA_5(G_3) = 173.339 + \frac{96}{29} \sqrt{5}(-4 + r) + \frac{24}{11} \sqrt{35}(-4 + r) + \]
\[ \frac{12}{25} \sqrt{129}(-4 + r)^2 + \frac{12}{5} \sqrt{11}(-3 + r) - 48r + 9r^2 + \]
\[ \frac{4}{37} \sqrt{1365}(-9 + 2r) + \frac{3}{2} \sqrt{7}(-5 + 2r) + \frac{48}{13} (-3 + 2r) + \]
\[ \frac{32}{11} \sqrt{7}(-3 + 2r) \]

3 Conclusion

In this paper, we have studied newly formed third type of hex derived networks, \( HDN_3 \), \( THDN_3 \) and \( RHDN_3 \). The exact results have been computed of Randić, Zagreb, Harmonic, Augmented Zagreb, atom-bond connectivity and Geometric-Arithmetic indices for the very first time of third type of hex-derived networks also find the numerical computation for all the networks. As these important results are help in many chemical point of view as well as for pharmaceutical sciences. We are looking forward in future to derived and compute new networks and topological indices.

References

1. M. Bača, J. Horváthová, M. Mokrišová, A. Semaničová-Feňovčíková, A. Suhányiová, On topological indices of carbon nanotube network, Canadian J. Chem. 93 (2015), 1-4.
2. A. Q. Baig, M. Imran, H. Ali, Computing Omega, Sadhana and PI polynomials of benzoid carbon nanotubes, Optoelectron. Adv. Mater. Rapid Communin. 9(2015), 248 – 255.
3. A. Q. Baig, M. Imran, H. Ali, On Topological Indices of Poly Oxide, Poly Silicate, DOX and DSL Networks, Canad. J. Chem., DOI:10.1139/cjc-2014-0490
4. A. T. Balaban, Highly discriminating distance-based topological index, Chem. Phys. Lett. 89(1982), 399 – 404.
5. A. T. Balaban, L. V. Quintas, The smallest graphs, trees, and 4-trees with degenerate topological index J. Math. Chem. 14(1993), 213 – 233.
6. J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, Macmilan, New York, 1997.
7. G. Caporossi, I. Gutman, P. Hansen, L. Pavlović, Graphs with maximum connectivity index, Comput. Bio. Chem. 27(2003), 85 – 90.
8. M. S. Chen, K. G. Shin, D. D. Kandlur, Addressing, routing, and broadcasting in hexagonal mesh multiprocessors, IEEE Trans. Comput. 39(1990), 1018.
9. M. V. Diudea, I. Gutman, J. Lorentz, Molecular Topology, Nova, Huntington, 2001.
10. E. Estrada, L. Torres, L. Rodríguez, I. Gutman, An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, Indian J. Chem. 37A(1998), 849 – 855.
11. B. Furtula, I. Gutman, Relation between second and third geometric-arithmetic indices of trees, J. of Chemometrics 25(2011), 87 – 91.
12. B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem. 53(4)(2015), 1184 – 1190.
13. M. Ghorbani, M. A. Hosseinzadeh, Computing $ABC_3$ index of nanostar dendrimers, Optoelectron. Adv. Mater. Rapid Commun. 4(2010), 1419 – 1422.
14. A. Graovac, M. Ghorbani, M. A. Hosseinzadeh, Computing fifth geometric-arithmetic index for nanostar dendrimers, J. Math. Nanosci. 1(2011), 3342.
15. I. Gutman, O. E. Polansky, Mathematical concepts in organic chemistry, Springer-Verlag, New York, 1986.
16. I. Gutman, B. Ruscic, N. Trinajstić, CF Wilcox Jr, Graph theory and molecular orbitals. XII. Acyclic polyenes, The Journal of Chemical Physics, 62(9)(1975), 3399-3405.
17. B. Furtula, A. Graovac and D. Vukičević, Augmented Zagreb index, J. Math. Chem. 48(2010), 370 – 380.
18. M. Imran, A.Q. Baig, H. Ali, On topological properties of dominating David derived networks, Canadian Journal of Chemistry, 94(2015), 137-148.
19. M. Imran, A.Q. Baig, S.U. Rehman, H. Ali, R. Hasni, Computing topological polynomials of mesh-derived networks, Discrete Mathematics, Algorithms and Application, 10(6)(2018), 1850077.
20. M. Imran, A.Q. Baig, H.M.A. Siddiqui, R. Sarwar, On molecular topological properties of diamond like networks, Canadian Journal of Chemistry, 95(7)(2017), 758-770.
21. A. Iranmanesh, M. Zeraatkar, Computing GA index for some nanotubes, Optoelectron. Adv. Mater. Rapid Commun. 4(2010), 1852 – 1855.
22. J.B. Liu, H. Ali, M.K. Shafiq, U. Munir, On Degree-Based Topological Indices of Symmetric Chemical Structures, Symmetry, 10(2018), 619; doi:10.3390/sym10110619.
23. J.B. Liu, M.K. Shafiq, H. Ali, A. Naseem, N. Maryam, S.S. Asghar, Topological Indices of nth Chain Silicate Graphs, Mathematics, 7(2019), 42; doi:10.3390/math7010042.
24. M. Randić, On Characterization of molecular branching, J. Amer. Chem. Soc., 97(1975), 6609 – 6615.
25. P. S. Ranjini, V. Lokesh, A. Usha, Relation between phenylene and hexagonal squeeze using harmonic index, Int J Graph Theory, 1(2013), 116 – 121.
26. F. Simonraj, A. George, Embedding of poly honeycomb networks and the metric dimension of star of david network, GRAPH-HOC, 4(2012), 11 – 28.
27. F. Simonraj, A. George, on the Metric Dimension of HDN3 and PHDN3, IEEE International Conference on Power, Control, Signals and Instrumentation Engineering (ICPCSI), (2017), 1333 – 1336.
28. D. Vukičević B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges, J. Math. Chem., 46(2009), 1369 – 1376.
29. H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc., 69(1947), 17 – 20.
30. L. Zhong, The harmonic index on graphs, Appl. Math. Lett. 25(2012), 561 – 566.