Approximate $\ell$-State Solutions of a Spin-0 Particle for Woods-Saxon Potential

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Abstract

The radial part of Klein-Gordon equation is solved for the Woods-Saxon potential within the framework of an approximation to the centrifugal barrier. The bound states and the corresponding normalized eigenfunctions of the Woods-Saxon potential are computed by using the Nikiforov-Uvarov method. The results are consistent with the ones obtained in the case of generalized Woods-Saxon potential. The solutions of the Schrödinger equation by using the same approximation are also studied as a special case, and obtained the consistent results with the ones obtained before.

Keywords: Nikiforov-Uvarov Method, Klein-Gordon Equation, Woods-Saxon Potential

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I. INTRODUCTION

In recent years, the exact or approximate solutions of wave equations have received a great attention. The Woods-Saxon (WS) potential given by [1]

\[ V(r) = - \frac{V_0}{1 + e^{(r-r_0)/a}}, \]  

is one of the much studied potential in quantum mechanical problems [2]. \( V_0 \) is the potential depth, the parameter \( a \) is the thickness of surface, and we denote the width of the potential by \( r_0 \), which is proportional with target mass number \( A \). The coupled-channels approach is a powerful tool to reproduce in heavy-ion physics, and Woods-Saxon potential, as a internuclear potential, has an important role in the coupled-channels calculations [2]. Further, the nuclear optical-model potential is widely used to study elastic scattering processes of nucleons, and heavy particles, and generate distorted waves in nuclear reactions. The Woods-Saxon potential is one of the three parts of nucleon-nucleus potential in the view of optical-model [3].

The wave equations with the Woods-Saxon potential can be solved analytically for \( s \)-waves due to the centrifugal potential barrier, and these solutions including the wave functions have been obtained by using different methods [4, 9]. In this work, we give the energy eigenvalues and the corresponding eigenfunctions of the radial Klein-Gordon (KG) equation for usual Woods-Saxon potential for any \( \ell \) values by using an approximate term instead of the centrifugal potential barrier. We solve the radial part of KG equation by using Nikiforov-Uvarov (NU) method, which is a powerful method to solve the second-order, linear differential equations [10]. Further, we also study the energy spectrum, and the corresponding wave functions of the Schrödinger equation for any \( \ell \) value by using the same approximation. So, we check out the consistency of our new approximation scheme in the non-relativistic case.

The work is organized as follows. In Section II, we solve the radial part of KG equation for usual Woods-Saxon potential for any \( \ell \) state by using an approximate potential term replaced by centrifugal potential barrier. We find out the eigenvalues and corresponding normalized eigenfunctions, and also give the results for \( s \)-waves. We also apply our approximation to the case of the Schrödinger equation, and it makes possible to control the accuracy of our approximation in non-relativistic region. We summarize our concluding remarks in Section III.
II. BOUND STATES AND NIKIFOROV-UVAROV METHOD

In spherical coordinates, the radial part of KG equation can be written as [11]

\[
\left\{ -\frac{\hbar^2}{2m_0} \frac{d^2}{dr^2} + \frac{\hbar^2 \ell (\ell + 1)}{2m_0 r^2} + \frac{1}{2m_0 c^2} [m_0 c^4 - (E - V(r))^2] \right\} \phi(r) = 0, \tag{2}
\]

where \( E \) is the energy of the particle, \( m_0 \) is the rest mass of particle, and \( \ell \) is the angular momentum quantum number.

Let us write the potential as

\[
V(x) = -\frac{V_0}{1 + e^{\beta x}}, \tag{3}
\]

where \( x = r - r_0 \), and \( \beta \) is a short notation, i.e., \( \beta = 1/a \). Eq. (2) can not be solved exactly because of the centrifugal potential, but the term can be expand about \( x = 0 \) as the following

\[
V_1(r) = \frac{\hbar^2 \ell (\ell + 1)}{2m_0 r^2} = \frac{D}{(1 + \frac{x}{r_0})^2} = D(1 - 2 \frac{x}{r_0} + 3 \left( \frac{x}{r_0} \right)^2 + \ldots), \tag{4}
\]

because the nuclear distance \( r \) can not fluctuate very far from the equilibrium for rather high vibrational levels [12, 13, 14], which means that the series expansion in Eq. (4) is valid for small \( x \) values. The parameter \( D \) in the above equation is a short notation, i.e.,

\[
D = \frac{\hbar^2 \ell (\ell + 1)}{2m_0 r_0^2}.
\]

We prefer the following form instead of the centrifugal potential barrier

\[
V'_1(x) = \left[ D_0 + \frac{D_1}{1 + e^{\beta x}} + \frac{D_2}{(1 + e^{\beta x})^2} \right] D, \tag{5}
\]

where we use three new parameters \( D_0, D_1, \) and \( D_2 \). The parameter \( D_0 \) corresponds to the constant term in series expansion in Eq. (4), and the remaining two terms \( D_1, D_2 \) correspond to term proportional with \( 1/(1 + e^{\beta x}) \), and proportional with \( 1/(1 + e^{\beta x})^2 \) in Eq. (2), respectively.

Expanding the potential \( V'_1(x) \) around \( x = 0 \) under the same condition, and than combining equal powers with Eq. (4), one can find the arbitrary constants \( D_i (i = 0, 1, 2) \) in the new form of the potential as following
\begin{align*}
D_0 &= \frac{12}{\beta^2 r_0^2} - \frac{4}{\beta r_0} + 1, \\
D_1 &= -\frac{48}{\beta^2 r_0^2} + \frac{8}{\beta r_0}, \\
D_2 &= \frac{48}{\beta^2 r_0^2}. \quad (6)
\end{align*}

So, we get two different 'effective' potentials of the form

\begin{align*}
V_{\text{eff}}(x) &= -\frac{V_0}{1 + e^{\beta x}} + \frac{\hbar^2 \ell(\ell + 1)}{2m_0 r^2}, \\
V'_{\text{eff}}(x) &= -\frac{V_0}{1 + e^{\beta x}} + DD_0 + \frac{DD_1}{1 + e^{\beta x}} + \frac{DD_2}{(1 + e^{\beta x})^2}. \quad (8)
\end{align*}

In Figs. 1, and 2, we plot the variation of the 'effective' potentials \(V_{\text{eff}}(x)\), and \(V'_{\text{eff}}(x)\) with respect to \(\beta x\) for three different \(\ell\) values. The Figs. 1, and 2 show that there is a well consistency between of the potential \(V_{\text{eff}}(x)\), and our approximation in Eq. (5), where we use the numerical values \(V_0 = 43.1\) MeV, \(a_p = 0.67\) fm, \(a_a = 0.55\) fm, \(r_0 = 3.44731\) fm, \(m_p = 1.007825\) amu, \(m_a = 1.00866\) amu for particle, and antiparticle, respectively [19]. We see that the deviation from the effective potential \(V_{\text{eff}}(x)\) appears for higher \(\ell\) values \((\ell \geq 5)\) for the range starting from \(\beta(r - r_0) \approx 2.5\). So, if we use the geometric average value \(A = 56\) [20], we get an average as \(r_0 = 4.91623\), and taking \(a = 0.654\) fm [20], we could set an upper limit such as \(r < \sim 1.70974\) fm for a good consistency of our approximation.

Substituting Eq. (5) into Eq. (2), we get

\begin{align*}
\left\{ \frac{d^2}{dx^2} + [\delta^2(E^2 - m_0^2 c^4) - \frac{2m_0}{\hbar^2} DD_0] \\
+ [2\delta^2 E V_0 - \frac{2m_0}{\hbar^2} DD_1] \frac{1}{1 + e^{\beta x}} + [\delta^2 V_0^2 - \frac{2m_0}{\hbar^2} DD_2] \frac{1}{(1 + e^{\beta x})^2} \right\} \phi(x) = 0, \quad (9)
\end{align*}

By using the transformation \(z = 2(1 + e^{\beta x})^{-1} \quad (0 \leq x \leq \infty \rightarrow 0 \leq z \leq 1)\), we have

\begin{align*}
\frac{d^2 \phi(z)}{dz^2} + \frac{2(1 - z)}{z(2 - z)} \frac{d\phi(z)}{dz} + \frac{1}{z(2 - z)^2} \left[-a_1^2 z^2 - a_2^2 - a_3^2 \right] \phi(z) = 0. \quad (10)
\end{align*}

where
\[ -a_1^2 = \frac{1}{\beta^2} (\delta^2 V_0^2 - 2m_0 DD_2 / h^2), \]
\[ -a_2^2 = \frac{2}{\beta^2} (2\delta^2 EV_0 - 2m_0 DD_1 / h^2), \]
\[ -a_3^2 = \frac{4}{\beta^2} (\delta^2 (E^2 - m_0^2 c^4) - 2m_0 DD_0 / h^2). \] (11)

To apply the NU-method, we rewrite Eq. (10) in the following form

\[ \phi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} \phi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \phi(z) = 0, \] (12)

where \( \sigma(z) \) and \( \tilde{\sigma}(z) \) are polynomials with second-degree, at most, and \( \tilde{\tau}(z) \) is a polynomial with first-degree. By using the following transformation for the total wave function

\[ \phi(z) = \xi(z)\psi(z) \] (13)

we get a hypergeometric type equation

\[ \sigma(z)\psi''(z) + \tau(z)\psi'(z) + \lambda\psi(z) = 0, \] (14)

where \( \xi(z) \) satisfies the equation

\[ \xi'(z)/\xi(z) = \pi(z)/\sigma(z). \] (15)

and the other part, \( \psi(z) \), is the hypergeometric type function whose polynomial solutions are given by

\[ \psi_n(z) = \frac{b_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \] (16)

where \( b_n \) is a normalization constant, and the weight function \( \rho(z) \) must satisfy the condition

\[ \frac{d}{dz} [\sigma(z)\rho(z)] = \tau(z)\rho(z). \] (17)
The function $\pi(z)$ and the parameter $\lambda$ required for this method are defined as follows

$$
\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)},
$$

(18)

$$
\lambda = k + \pi'(z)
$$

(19)

The constant $k$ is determined by imposing a condition such that the discriminant under the square root should be zero. Thus one gets a new eigenvalue equation

$$
\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2} \sigma'', (n = 0, 1, 2, \ldots)
$$

(20)

where

$$
\tau(z) = \tilde{\tau}(z) + 2\pi(z).
$$

(21)

and the derivative of $\tau(z)$ must be negative.

Comparing Eq. (10) with Eq. (12), we have

$$
\tilde{\tau}(z) = 2(1 - z), \quad \sigma(z) = z(2 - z), \quad \tilde{\sigma}(z) = -a_1^2 z^2 - a_2^2 z - a_3^2
$$

(22)

Substituting this into Eq. (18), we get

$$
\pi(z) = \pm \sqrt{(a_1^2 - k)z^2 + (a_2^2 + 2k)z + a_3^2}.
$$

(23)

The constant $k$ can be determined by the condition that the discriminant of the expression under the square root has to be zero

$$
(a_2^2 + 2k)^2 - 4a_3^2(a_1^2 - k) = 0.
$$

(24)

The roots of $k$ are $k_{1,2} = -\frac{1}{2} a_2^2 - \frac{1}{2} a_3^2 \mp \frac{1}{2} a_3 A$, where $A = \sqrt{a_3^2 + 2a_2^2 + 4a_1^2}$. Substituting these values into Eq. (23), we get for $\pi(z)$ for $k_1$...
\[ \pi(z) = \mp \left[ \left( \frac{A}{2} - \frac{a_3}{2} \right) z + a_3 \right], \tag{25} \]

and for \( k_2 \)

\[ \pi(z) = \mp \left[ \left( \frac{A}{2} + \frac{a_3}{2} \right) z - a_3 \right], \tag{26} \]

Now we find the polynomial \( \tau(z) \) from \( \pi(z) \) for \( k_2 \)

\[ \tau(z) = 2 + 2a_3 - 2\left( \frac{A}{2} + \frac{a_3}{2} + 1 \right)z. \tag{27} \]

so its derivative \(-2\left( \frac{A}{2} + \frac{a_3}{2} + 1 \right)\) is negative. We have from Eq. (19)

\[ \lambda = -\frac{1}{2} \left( a_2^2 + a_3^2 + Aa_3 + A + a_3 \right), \tag{28} \]

and Eq. (20) gives us

\[ \lambda_n = 2n\left( \frac{A}{2} + \frac{a_3}{2} + 1 \right) + n^2 - n. \tag{29} \]

Substituting the values of the parameters given by Eq. (11), and setting \( \lambda = \lambda_n \), one can find the energy eigenvalues for any \( \ell \)-state

\[ E_{n,\ell} = -\frac{[\beta^2 n_1^2 + 4\delta^2 V_0^2 - \frac{8m_0 D}{\hbar^2} (D_1 + D_2)] V_0}{2(\beta^2 n_1^2 + 4\delta^2 V_0^2)} \]

\[ \pm \frac{\beta n_1}{\delta} \sqrt{\frac{2m_0^2 c^4 \delta^2 + 2m_0 \delta^2 (2D_0 + D_1 + D_2)}{2(\beta^2 n_1^2 + 4\delta^2 V_0^2)}} - \left[ \frac{2m_0 \delta (D_1 + D_2)}{\beta^2 n_1^2 + 4\delta^2 V_0^2} \right]^2 - \frac{1}{16}, \tag{30} \]

where \( n \) is the principal quantum number, and

\[ n_1 = -(2n + 1) + \sqrt{1 + 4a_1^2}. \tag{31} \]

From this result, we can easily get the energy spectra for \( s \)-waves by setting \( D = D_0 = D_1 = D_2 = 0 \)
\[ E_{n,\ell=0} = -\frac{V_0}{2} \pm \frac{\beta n_1'}{\delta} \sqrt{\frac{m_0^2 c^4 \delta^2}{\beta^2 n_1'^2 + 4 \delta^2 V_0^2} - \frac{1}{16}}, \] (32)

where

\[ n_1' = -(2n + 1) + \sqrt{1 + 4a_1'^2}. \] (33)

It can be seen that the eigenvalues are real under the condition that

\[ \frac{2m_0^2 c^4 \delta^2 + 2m_0 D_0 (2D_0 + D_1 + D_2)}{2(\beta^2 n_1'^2 + 4 \delta^2 V_0^2)} > \left[ \frac{2m_0 D_1 (D_0 + D_2)}{\beta^2 n_1'^2 + 4 \delta^2 V_0^2} \right]^2 + \frac{1}{16}. \] Further, the parameters \( D_0, D_1, D_2 \) used to describe the approximate potential form in Eq. (5) are real in the case.

We give the variation of the bound states energy of particle + a nucleon with \( A \), and antiparticle + a nucleon with \( A \) with respect to \( n \) for different \( \ell \) values for in Figs. 3, and 4, respectively. We choose the target mass number as \( A = 20 \), and use the numerical values given above.

In order to find the eigenfunctions, we first compute the weight function from Eq. (17)

\[ \rho(z) = z^{a_3} (2 - z)^A, \] (34)

and the wave function becomes

\[ \psi_{n\ell} (z) = \frac{b_n}{z^{a_3} (2 - z)^A} \frac{d^n}{dz^n} \left[ z^{n+a_3} (2 - z)^{n+A} \right]. \] (35)

where \( b_n \) is a normalization constant. The polynomial solutions can be written in terms of the Jacobi polynomials [15-17]

\[ \psi_{n\ell} (z) = b_n P_n^{(a_3, A)} (1 - z), \quad A > -1, \quad a_3 > -1. \] (36)

On the other hand, the other part of the wave function is obtained from Eq. (15) as

\[ \xi(z) = z^{a_3/2} (2 - z)^{A/2}. \] (37)
Thus, the total eigenfunctions take

$$\phi_{n\ell}(z) = b'_n (2 - z)^{A/2} z^{a_3/2} P_n^{(a_3, A)}(1 - z).$$  \(38\)

where $b'_n$ is the new normalization constant. It is obtained from

$$\frac{2}{\beta} \int_0^1 \left( \frac{\left| \phi_{n\ell}(z) \right|}{\sqrt{z^{-1}(2 - z)}} \right)^2 dz = 1. \quad (39)$$

To evaluate the integral, we use the following representation of the Jacobi polynomials [16, 17]

$$P_n^{(\sigma, \varsigma)}(z) = \frac{\Gamma(n + \sigma + 1)}{n! \Gamma(n + \sigma + \varsigma + 1)} \sum_{m=0}^{n} \frac{\Gamma(n + 1)}{\Gamma(m + 1) \Gamma(n - m + 1)} \frac{\Gamma(n + \sigma + \varsigma + m + 1)}{\Gamma(m + \sigma + 1)} (\frac{z - 1}{2})^m, \quad (40)$$

Hence, from Eq. (39), and with the help of Eq. (40), we get

$$\frac{1}{\beta} |g(n, m) \times g(r, s)| |b'_n|^2 \int_0^1 z^{m+s+a_3+1} (2 - z)^{A-1} dz = 1, \quad (41)$$

where $g(n, m)$, and $g(r, s)$ are two arbitrary functions of the parameters $A$, and $a_3$, and given by

$$g(n, m) = (-1)^{m+1/2} (-2)^{-m+1/2} \frac{\Gamma(a_3 + n + 1)}{n! \Gamma(A + a_3 + n + 1)} \times \sum_{m=0}^{n} \frac{\Gamma(n + 1)}{\Gamma(m + 1) \Gamma(n - m + 1)} \frac{\Gamma(A + a_3 + n + m + 1)}{\Gamma(a_3 + A + 1)}, \quad (42)$$

and

$$g(r, s) = g(n, m) \left( n \to r; m \to s \right). \quad (43)$$

The integral in Eq. (41) can be evaluated by using the following integral representation of hypergeometric type function $\,\,_{2}F_1(a, b; c; z)$ [13]
\[ 2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} \, dt, \]  

(44)

by setting the variable \( z \rightarrow \frac{z}{2} \), and taking \( c = 1 + b, z = 1 \), one gets

\[ \int_0^1 t^{b-1} (2 - t)^{-a} \, dt = \frac{\Gamma(b)\Gamma(1)}{2\Gamma(1 + b)} 2F_1(a, b; 1 + b; \frac{1}{2}), \]  

(45)

From last equation

\[ \int_0^1 z^{m+s+a_3+1} (2 - z)^{A-1} \, dz = \frac{\Gamma(m + s + a_3 + 2)\Gamma(1)}{2\Gamma(m + s + a_3 + 3)} \]  

\[ \times 2F_1(1 - A, m + s + a_3 + 2; m + s + a_3 + 3; \frac{1}{2}), \]  

(46)

where we set \( b - 1 = m + s + a_3 + 1 \), and \( a = 1 - A \).

By using the following identities of hypergeometric type functions [18]

\[ 2F_1(a, b; c; -1) = \frac{\Gamma\left(\frac{1}{2}b + 1\right)\Gamma(b - a + 1)}{\Gamma(b + 1)\Gamma\left(\frac{1}{2}b - a + 1\right)}, \quad (a - b + c = 1, \ b > 0) \]  

(47)

\[ 2F_1(a, b; c; \frac{1}{2}) = 2^a 2F_1(a, c - b; c; -1), \]  

(48)

the hypergeometric type function of \( 2F_1(a, b; c; z) \) in Eq. (46) can be evaluated as

\[ 2F_1(1 - A, b; 1 + b; \frac{1}{2}) = \frac{\sqrt{\pi}}{2^A} \frac{\Gamma(1 + A)}{\Gamma\left(\frac{1}{2} + A\right)}, \quad (A = b - 1), \]  

(49)

Finally, we get the normalization constant as

\[ |b'_n|^2 = \frac{2^{1+A}}{\beta \sqrt{\pi}} \frac{\Gamma(m + s + a_3 + 3)\Gamma\left(\frac{1}{2} + A\right)}{\Gamma(m + s + a_3 + 2)\Gamma(1 + A)[g(n, m) \times g(r, s)]}. \]  

(50)

It is interested to study the approximation scheme of potential given by Eq. (5) in the case of Schrödinger equation (SE). It also makes to possible to check out the results obtained by using the approximation Eq. (5) in the non-relativistic region.
The radial part of SE is given by

\[-\frac{\hbar^2}{2m_0} \frac{d^2 \phi(r)}{dr^2} + \left\{ \frac{\hbar^2 \ell(\ell + 1)}{2m_0 r^2} + V(r) - E \right\} \phi(r) = 0, \tag{51}\]

By using the approximate expression of the centrifugal potential barrier, and the transformation \(z = (1 + e^{\beta x})^{-1}\), we have

\[\frac{d^2 \phi(z)}{dz^2} + \frac{2z - 1}{z(z - 1)} \frac{d\phi(z)}{dz} + \frac{1}{[z(z - 1)]^2} \left[ -\epsilon^2 - \gamma^2 z - \kappa^2 z^2 \right] \phi(z) = 0. \tag{52}\]

where

\[\epsilon^2 = \frac{2m_0}{\beta^2 \hbar^2} (DD_0 - E), \]
\[\gamma^2 = \frac{2m_0}{\beta^2 \hbar^2} (DD_1 - V_0), \]
\[\kappa^2 = \frac{2m_0}{\beta^2 \hbar^2} DD_2. \tag{53}\]

Following the same procedure, the energy eigenvalues are written

\[E_{n\ell} = \frac{\hbar^2 \ell(\ell + 1)}{2m_0 r_0^2} D_0 \]
\[\quad \quad - \frac{\hbar^2}{2m_0 a^2} \left[ \frac{1}{4} \left( \frac{n + 1}{2n + 1} + \sqrt{1 + \frac{4\ell(\ell + 1)a^2}{r_0^2 D_2}} \right) - \frac{\ell(\ell + 1)a^2}{r_0^2 D_2} \right] - \frac{\ell(\ell + 1)a^2}{r_0^2 D_2} \left( D_1 + D_2 \right) - \frac{2m_0 V_0 a^2}{h^2} \right]^2. \tag{54}\]

We plot the variation of the energy eigenvalues obtained from Eq. (54) with respect to \(n\) for \(\ell = 0\) in Fig. 5. We give the dependence of the energy spectrum to \(n\) for the values \(\ell = 1, 2\) in Fig. 6 by using the numerical values given above.

We get easily the energy spectrum for s-waves

\[E_{n,\ell=0} = -\frac{\hbar^2}{2m_0 a^2} \left[ \left( \frac{n + 1}{2} \right)^2 + \left( \frac{m_0 V_0 a^2}{\hbar^2 (n + 1)} \right)^2 + \frac{m_0 V_0 a^2}{\hbar^2} \right]. \tag{55}\]

which is consistent with obtained in Ref. [9].

The corresponding eigenfunctions are written

11
\[ \phi_{n\ell}(z) = a_n (1 - z)^{-A} \ z^{-\epsilon} \ P_n^{(-2\epsilon, -2A)} (1 - 2z). \]  

(56)

where \( A = \sqrt{\epsilon^2 + \gamma^2 + \kappa^2} \), and \( a_n \) is normalization constant. Let us discuss the behavior of the approximate wave function at the origin. Taking into account that \( z = (1 + e^{\beta x})^{-1} \) and \( x = r - r_0 \), we get the following limits

\[
\begin{align*}
    z &= \begin{cases} 
        0 & \text{if } x \to \infty \\
        \frac{1}{2} & \text{if } x \to 0 
    \end{cases} \\
\end{align*}
\]

(57)

We should take the limit \( z \to \frac{1}{2} \) to discuss the behaviour of the wave function at origin. We obtain the wave function in this limit as

\[ \phi_{n\ell}(z) \sim \left( \frac{1}{2} \right)^{-A-\epsilon} P_n^{(-2\epsilon, -2A)} (1 - 2z \to 0), \]

(58)

The Jocobi polynomials has the following form in this limit [17,18]

\[
P_n^{(-2\epsilon, -2A)}(y) = \frac{1}{2^n} \sum_{k=0}^{n} \left( \frac{n - 2\epsilon}{k} \right) \left( \frac{n - 2A}{n - k} \right) (1 + y)^k (y - 1)^{n-k},
\]

(59)

where \( y \) identifies \( 1 - 2z \). So we write the approximate wave function in the first order of \( z \)

\[
\phi_{n\ell}(z) \sim \left\{ (-1)^{n-k} + \left[ (-1)^{n-k-1} (n - k) + (-1)^{n-k} k \right] z + \ldots \right\} f(n, k, \epsilon, A),
\]

(60)

where

\[
f(n, k, \epsilon, A) = \frac{1}{2^n} \sum_{k=0}^{n} \left( \frac{n - 2\epsilon}{k} \right) \left( \frac{n - 2A}{n - k} \right).
\]

(61)

It is seen from Eq. (60) that the approximate wave function is finite at the origin. It will have the same behavior with the approximate solution.
As a final test, we compare our results obtained from Eq. (54) with the ones given in Ref. [21], where the problem is solved numerically, and the results are obtained by using a MATLAB package called as the MATSLISE. So, we use the following form of the Woods-Saxon potential in Eq. (3)

\[ V(x) = -\frac{50}{1 + e^{(x-r)/a_s}}, \]  

(62)

where we should stress that the variable \( x \) in last equation denotes the radial coordinate \( r \) in the rest of the present work.

We obtain energy eigenvalues of the Woods-Saxon potential given in Eq. (62) by using the approach in Eq. (5) numerically, and summarize the results in Table I. We give the eigenfunctions obtained without, and with using the approximation given in Eq. (5) for the Woods-Saxon potential given in Eq. (62) in Fig. 7 for \( n = 1 \), and \( \ell = 1 \), and Fig. 8 for \( n = 6 \), and \( \ell = 5 \), respectively. The deviations between the results obtained with and without the approximation increase as the values of \( n \), and \( \ell \) increase, and there is a shifting to the left between the approximate and exact eigenfunctions. These are expected results, because the validity of the approximation is specified in the Figs. 1, and 2 for the range as \( (r < \sim 1.70794 \text{ fm}) \), while we set the molecular distance \( r \) as \( r_0 = 7 \) in Eq. (62).

### III. CONCLUSION

We have solved analytically the radial part of the KG equation for the usual Woods-Saxon potential in the framework of an approximation to the centrifugal potential term for any \( \ell \) values. The energy spectra and the corresponding wave functions are obtained by applying the NU-method. We give in Figs. 1, and 2 the variations of \( V_{eff}(r) \), and \( V'_{eff}(r) \) with respect to \( \beta(r - r_0) \) for \( \ell = 1, 2, 5 \). We have pointed out that the consistency between \( V_{eff} \), and our new approximation is very well, and reliable results can be obtained by using the approximation scheme used in the present work. To check our results, we have also calculated analytically the energy eigenvalues of the particle and antiparticles for the \( s \)-waves. We have found that the analytical results are consistent with those in Ref. [4] if \( q \to 1 \). Further, we have also studied independently the energy spectrum of the Schrödinger equation by using the same approximation to check out our results in the non-relativistic
region. We have seen that the results obtained analytically for the case of the Schrödinger equation are the same for s-waves with Ref. [9]. Further, we give some numerical results for the eigenvalues of the Schrödinger equation with different values of the quantum numbers $n$, and $\ell$ in Table I. We compare the eigenfunctions for two different $\ell$ values obtained with, and without using the new approximation in Figs. 7-8.

IV. ACKNOWLEDGMENTS

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TABLE I: The energy eigenvalues of Woods-Saxon potential in Eq. (57) derived in MATSLISE.

|  | Our Work | Numerical Solution |
|---|----------|--------------------|
| 1s | -49.57   | -49.57             |
| 2s | -48.50   | -48.50             |
| 2p | -49.52   | -49.17             |
| 3s | -46.96   | -46.96             |
| 3p | -48.45   | -47.84             |
| 3d | -49.40   | -48.68             |
| 4s | -45.02   | -45.02             |
| 4p | -46.91   | -46.09             |
| 4d | -48.33   | -47.11             |
| 4f | -49.22   | -48.12             |
| 5s | -40.11   | -40.11             |
| 5p | -44.96   | -43.96             |
| 5d | -46.79   | -45.15             |
| 5f | -48.16   | -46.32             |
| 5g | -48.99   | -47.49             |
| 6s | -37.21   | -37.21             |
| 6p | -42.67   | -41.50             |
| 6d | -44.85   | -42.85             |
| 6f | -46.62   | -44.17             |
| 6g | -47.93   | -45.48             |
| 6h | -48.70   | -46.79             |
FIG. 1: The variations of $V_{\text{eff}}(r)$, and $V'_{\text{eff}}(r)$ with respect to $\beta(r - r_0)$ for $\ell = 1$.

FIG. 2: The variations of $V_{\text{eff}}(r)$, and $V'_{\text{eff}}(r)$ with respect to $\beta(r - r_0)$ for $\ell = 2, 5$. 
FIG. 3: The dependence of a particle energy levels in the case of the Klein-Gordon to $n$ for different $\ell$ values 1, 2, 3, 4.

FIG. 4: The dependence of a antiparticle energy levels in the case of the Klein-Gordon to $n$ for different $\ell$ values 1, 2, 3, 4.
FIG. 5: The variation of the Schrödinger energy level with respect to $n$ for $\ell = 0$.

FIG. 6: The variation of the Schrödinger energy levels with respect to $n$ for $\ell = 1, 2$. 
FIG. 7: The Schrödinger eigenfunctions without (full line), and with using the approximation in Eq. (5) for Woods-Saxon potential given in Eq. (57) for $n = 1$, and $\ell = 1$.

FIG. 8: The same as Fig. 7 but for $n = 6$, and $\ell = 5$. 