REGULARITY CRITERION FOR SOLUTIONS OF
THREE-DIMENSIONAL TURBULENT CHANNEL FLOWS

CHONGSHENG CAO, JUNLIN QIN, AND EDRISS S. TITI

ABSTRACT. In this paper we consider the three-dimensional Navier–Stokes equations in
infinite channel. We provide a regularity criterion for solutions of the three-dimensional
Navier–Stokes equations in terms of the vertical component of the velocity field.

AMS Subject Classifications: 35Q35, 65M70
Key words: Three-dimensional Navier–Stokes equations, regularity criterion, turbulent
channel flows.

1. Introduction

Turbulence stands out as a prototype of multi-scale phenomenon that occurs in nature. It
involves wide ranges of spatial and temporal scales which makes it very difficult to study an-
alytically and prohibitively expensive to simulate computationally. Turbulent channel flows
are considered to be the simplest flows confined within physical boundaries that can be simu-
lated numerically and that demonstrates many of the common features of turbulence. In this
paper we consider three-dimensional finite energy turbulent flows of viscous incompressible
homogeneous fluids in the infinite channel \( \Omega = \mathbb{R}^2 \times [-L, L] \subset \mathbb{R}^3 \), subject to the no-slip
Dirichlet boundary conditions. These flows are governed by the three-dimensional Navier–Stokes
system of equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f \quad \text{in } \Omega \quad (1) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \quad (2) \\
u &= 0 \quad \text{on } \partial \Omega \quad (3) \\
\lim_{x \to \infty} u(t, x) &= 0 \quad (4) \\
 u(0, x) &= u_0(x) \quad \text{in } \Omega \quad (5)
\end{align*}
\]

Here, \( u = (u_1, u_2, u_3) \) represents the unknown velocity vector field, and \( p \) is the unknown
pressure scalar; where \( \nu > 0 \), the constant kinematic viscosity, \( f \), the body forcing term, and
\( u_0 \), the initial velocity, are given.

Mathematically, it is well-known that the three-dimensional system (1)–(5) has global (for
all time and all initial data) weak solutions (see, e.g., [8], [9], [13], [19], [20], [21] and referen-
ces therein). The question of well-posedness, in the sense of Hadamard, and in particular the
question of uniqueness, of these weak solutions is still an open problem. On the other hand,
it is also well-established (see, e.g., [8], [9], [13], [19], [20], [21] and references therein) that
the system (1)–(4) possesses a unique strong (regular) solution, which depends continuously
on the initial data, for a short interval of time \([0, T_*)\), where \( T_* \) depends on the size of initial
datum, \( u_0 \), on \( f, \nu \) and \( L \). Moreover, it is also well-known that the existence (for all time) and
uniqueness of strong (regular) solutions is guaranteed under suitable additional assumptions.

Date: September 23, 2006.
(see, e.g., [2, 3, 7, 10, 11, 12, 15, 16, 17, 22] and references therein). In particular, some of these recent results involve conditions on only one component of the velocity field of the 3D NSE in the whole space $\mathbb{R}^3$ or under periodic boundary conditions (see, e.g., [10, 12, 15, 22]). In this paper, we study this type of sufficient conditions for the global regularity of the 3D NSE in the infinite channel $\Omega$, subject to no-slip Dirichlet boundary condition on the physical boundary of the channel. Using the geophysical terminology, our condition is formulated in terms of the third component of the baroclinic mode $\tilde{u}_3$ (see (16), below, for the definition of the barotropic mode (vertically averaged mode), $\bar{u}$, and the baroclinic mode (the fluctuation about the barotropic mode)). Specifically, our results states that if $\tilde{u}_3$ satisfies

$$\nabla \tilde{u}_3 \in L^\infty([0, \infty), L^2(\Omega)), \quad (6)$$

then the strong (regular) unique solution of the 3D Navier-Stokes equation (1)–(5) exists for all time.

Let us observe that our condition (6) seems to be slightly tighter than the former ones (cf. e.g., [10, 12, 15, 22]). However, unlike the previous works we study here the 3D Navier-Stokes in a domain with physical boundaries under the no-slip Dirichlet boundary conditions. Furthermore, we emphasize that the techniques developed here, which are inspired by ideas presented in [5], are totally different than the previous ones.

Let us denote by $L^q(\Omega), L^q(\mathbb{R}^2)$, and $H^m(\Omega), H^m(\mathbb{R}^2)$ the usual $L^q$–Lebesgue and Sobolev spaces, respectively ([1]). We denote by

$$\|\phi\|_q = \begin{cases} \left(\int_{\Omega} |\phi|^q \, dx \right)^{\frac{1}{q}} & \text{for every } \phi \in L^q(\Omega) \\ \left(\int_{\mathbb{R}^2} |\phi|^q \, dx \right)^{\frac{1}{q}} & \text{for every } \phi \in L^q(\mathbb{R}^2). \end{cases} \quad (7)$$

Let

$$\mathcal{V} = \{ v \in C^\infty_0(\Omega) : \nabla \cdot v = 0 \}.$$

Since we are interested in flows of finite energy in the infinite channel $\Omega$, we consider the spaces $H$ and $V$, defined to be the closures of the set $\mathcal{V}$ in $L^2(\Omega)$ under $L^2$–topology, and in $H^1(\Omega)$ under $H^1$–topology, respectively. Denote by $P : L^2 \to H$, the orthogonal projection, and let $A = -P\Delta$ be the Stokes operator subject to the homogeneous Dirichlet boundary condition ([3]). It is well known that the Navier–Stokes equations (NSE) (14)–(15) are equivalent to the functional differential equation (see, e.g., [8, 19, 20, 21])

$$\frac{du}{dt} + \nu Au + B(u, u) = f, \quad (8)$$
$$u(0) = u_0, \quad (9)$$

where $B(u, u) = P((u \cdot \nabla)u)$, the nonlinear (bilinear) term. We say $u$ is a Leray–Hopf weak solution to the system ([3]–[5]) if $u$ satisfies (see, e.g., [8, 20, 21])

1. $u \in C([0, T], H)$–weak $\cap L^2([0, T], V)$, and $\partial_t u \in L^2([0, T], V)'$, where $V'$ is the dual space of $V$,

2. the weak formulation:

$$\int_{\Omega} u(t, x) \cdot \phi(x) \, dx - \int_{\Omega} u(t_0, x) \cdot \phi(x) \, dx$$
$$= -\int_{t_0}^t \int_{\Omega} \left( \nu(\nabla u(s, x) : \nabla \phi(x)) + (u(s, x) \cdot \nabla)u(s, x) \cdot \phi(x) \right) \, dx \, ds + \int_{t_0}^t \int_{\Omega} \left( f(s, x) \cdot \phi(x) \right) \, dx \, ds,$$

for every $\phi \in \mathcal{V}$, and almost every $t$, $t_0 \in [0, T]$. 
(3) the energy inequality:
\[
\|u(t)\|_2^2 - \|u(0)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 \, ds \leq 2 \int_0^t \int_\Omega f(s, x) \cdot u(s, x) \, dx \, ds,
\]
for all \( t \in [0, T] \), and for almost every \( t_0 \) in the interval \([0, t]\).

Moreover, a weak solution is called strong solution of (8)–(9) on \([0, T]\) if, in addition, it satisfies
\[
u \in C([0, T], V) \cap L^2([0, T], H^2(\Omega)).
\]

For convenience, we recall the following Gagliardo-Nirenberg, Ladyzhenskaya, and Sobolev inequalities (cf. e.g., [1], [8], [9], [14], and [19]) in \(\mathbb{R}^2\):
\[
\|\phi\|_{L^r(\mathbb{R}^2)} \leq C_r \|\phi\|_{L^2(\mathbb{R}^2)}^{2/r} \|\phi\|_{H^1(\mathbb{R}^2)}^{2/r}, \quad r < \infty,
\]
for every \( \phi \in H^1(\mathbb{R}^2) \), and in \(\mathbb{R}^3\):
\[
\|\psi\|_{L^\alpha(\Omega)} \leq C_\alpha \|\psi\|_{L^2(\Omega)}^{\frac{6-\alpha}{2}} \|\psi\|_{H^1(\Omega)}^{\frac{3(\alpha-2)}{2}},
\]
for every \( u \in H^1(\Omega), 2 \leq \alpha \leq 6 \). Here \( C_r \) and \( C_\alpha \) are scale invariant constants. We also recall the Poincaré inequality:
\[
\|\nabla v\|_2 \geq \frac{C_0}{L} \|v\|_2 \quad \forall v \in V
\]
\[
\|Av\|_2 \geq \frac{C_0}{L} \|\nabla v\|_2 \quad \forall v \in D(A),
\]
where \( C_0 \) is a scale invariant constant. Also, we recall the integral version of Minkowsky inequality for the \(L^r\) spaces, \( r \geq 1 \). Let \( \Omega_1 \subset \mathbb{R}^{m_1} \) and \( \Omega_2 \subset \mathbb{R}^{m_2} \) be two measurable sets, where \( m_1 \) and \( m_2 \) are two positive integers. Suppose that \( \phi(\xi, \eta) \) is measurable over \( \Omega_1 \times \Omega_2 \). Then,
\[
\left( \int_{\Omega_1} \left( \int_{\Omega_2} |\phi(\xi, \eta)| \, d\eta \right)^r \, d\xi \right)^{1/r} \leq \int_{\Omega_2} \left( \int_{\Omega_1} |\phi(\xi, \eta)|^r \, d\xi \right)^{1/r} \, d\eta.
\]

\section{Global Existence of the Strong Solution}

In this section we will show the global existence of the strong solutions to the three-dimensional Navier–Stokes system \([11]–[15]\) under assumption \([13]\).

We will denote by
\[
\theta(x_1, x_2) = \frac{1}{2L} \int_{-L}^L \theta(x_1, x_2, x_3) \, dx_3 \quad \text{and} \quad \tilde{\theta} = \theta - \theta.
\]

Following the geophysical fluid dynamics terminology we will call \( \theta \) the barotropic mode and \( \tilde{\theta} \) the baroclinic mode.

From now on, we will denote by \( \nabla_h = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) \) and \( \Delta_h = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \). First, let us prove the following Lemma.

\textbf{Lemma 1.} Suppose that \( \xi(x_1, x_2) \in H^1(\mathbb{R}^2), \phi \in H^1(\Omega) \) and \( \psi \in L^2(\Omega) \). Then,
\[
\int_{\Omega} |\xi| |\phi| |\psi| \, dx_1 \, dx_2 \, dx_3 \leq C \|\xi\|_2^{1/2} (||\xi||_2 + ||\nabla_h \xi||_2)^{1/2} \|\phi\|_2^{1/2} (||\phi||_2 + ||\nabla_h \phi||_2)^{1/2} \|\psi\|_2.
\]
Proof. Notice that
\[
\int_{\Omega} |\xi| |\phi| |\psi| \, dx_1 dx_2 dx_3 = \int_{\mathbb{R}^2} \left[ |\xi| \left( \int_{-L}^{L} |\phi| \, dx_3 \right) \right] \, dx_1 dx_2.
\]
We will estimate the above term by applying the same method used to establish Proposition 2.2 in [4]. First, by Cauchy–Schwarz inequality, we obtain
\[
\int_{-L}^{L} |\phi| |\psi| \, dx_3 \leq \left( \int_{-L}^{L} |\phi|^2 \, dx_3 \right)^{\frac{1}{2}} \left( \int_{-L}^{L} |\psi|^2 \, dx_3 \right)^{\frac{1}{2}}.
\]
Thus, by the above and Hölder inequality, we reach
\[
\int_{\Omega} |\xi| |\phi| |\psi| \, dx_1 dx_2 dx_3 \leq \int_{\mathbb{R}^2} \left[ |\xi| \left( \int_{-L}^{L} |\phi|^2 \, dx_3 \right)^{\frac{1}{2}} \left( \int_{-L}^{L} |\psi|^2 \, dx_3 \right)^{\frac{1}{2}} \right] \, dx_1 dx_2
\]
By using Minkowsky inequality (15), we get
\[
\left[ \int_{\mathbb{R}^2} \left( \int_{-L}^{L} |\phi|^2 \, dx_3 \right)^{\frac{1}{2}} \, dx_1 dx_2 \right]^{\frac{1}{2}} \leq \int_{-L}^{L} \left( \int_{\mathbb{R}^2} \left( \int_{-L}^{L} |\phi|^2 \, dx_3 \right)^2 \, dx_1 dx_2 \right)^{\frac{1}{2}} \, dx_3.
\]
Thanks to (11) with \( r = 4 \), for every fixed \( x_3 \) we have
\[
\left( \int_{\mathbb{R}^2} |\phi|^4 \, dx_1 dx_2 \right)^{\frac{1}{4}} \leq C \|\phi\|_{L^4(\mathbb{R}^2)} \|\phi\|_{H^1(\mathbb{R}^2)}.
\]
As a result of the above and the Cauchy–Schwarz inequality, we obtain
\[
\int_{-L}^{L} \left( \int_{\mathbb{R}^2} |\phi|^4 \, dx_1 dx_2 \right)^{\frac{1}{4}} \, dx_3
\]
Therefore,
\[
\left[ \int_{\mathbb{R}^2} \left( \int_{-L}^{L} |\phi|^2 \, dx_3 \right)^2 \, dx_1 dx_2 \right]^{\frac{1}{2}} \leq C \|\phi\|_{L^2(\mathbb{R}^2)} \|\phi\|_{H^1(\mathbb{R}^2)}
\]
By using (11) with \( r = 4 \), we have
\[
\left[ \int_{\mathbb{R}^2} |\xi|^4 \, dx_1 dx_2 \right]^{\frac{1}{4}} \leq C \|\xi\|^{1/2}_{L^4(\mathbb{R}^2)} \|\xi\|^{1/2}_{H^1(\mathbb{R}^2)}.
\]
Thus, by the above and \( \| \psi \|_2 \leq C \| \xi \|_2^{1/2} \| \nu \|_2^{1/2} \| \psi \|_2 \), we get

\[
\int_\Omega |\xi| \| \phi \| \| \psi \|_{dx_1 dx_2 dx_3} \leq C \| \xi \|_2^{1/2} \| \phi \|_2 \| \psi \|_2.
\]

\( \square \)

**Theorem 2.** Let \( f \in L^\infty([0, \infty), L^2(\Omega)) \), \( u_0 \in V \). Let \( u = (u_1, u_2, u_3) \) be a weak solution of the system \([8]–[9]\) in \([0, \infty)\). Suppose that for \( T > 0 \), \( \nabla \tilde{u}_3 \in L^\infty([0, T], L^2(\Omega)) \); that is, \( \tilde{u}_3 \) satisfies

\[
\sup_{0 \leq t \leq T} \| \nabla \tilde{u}_3(t) \|_2 < \infty. \tag{18}
\]

Then \( u \) is the strong solution of the system \([8]–[9]\) on \([0, T]\).

**Proof.** Let \( u_0 \in V \). Following, for instance, the Galerkin method one can show that there exists a unique strong solution \( u \) for the system \([8]–[9]\), with the initial datum \( u_0 \), for a short interval of time (see, e.g., \([8], [13], [19], [20]\) and \([21]\). Suppose that \([0, T^*] \) is the maximal interval of existence of this strong solution \( u \). It is also well known (see, e.g., the above references) that there exists a Leray-Hopf weak solution for the system \([8]–[9]\), with the same initial datum \( u_0 \), which exists globally in time, i.e. for all time \( t \geq 0 \). Most importantly, following the work of J. Sather and J. Serrin in \([13]\) on can show that all the Leray-Hopf weak solutions coincide with the unique strong solution, \( u \), on the interval \([0, T^*] \).

To conclude our proof we need to show that \( T < T^* \). Suppose, arguing by contradiction, that \( T^* \leq T \), and that \([13]\) holds. If we show that the \( \limsup_{t \to T^-} \| u(t) \|_{H^1} < \infty \) then \([0, T^*] \) is not a maximal interval of existence, which leads to a contradiction.

For the rest of this proof we consider the strong solution, \( u \), in the the interval \([0, T^*] \). From the energy inequality of \([10]\), which is satisfied by all the Leray-Hopf weak solutions (see, for example, \([8], [13], [19], [20]\) or \([21]\) for details), we have

\[
\| u(t) \|_2^2 \leq C \frac{\| f \|_2 L^4 F^2 \nu^2}{\nu^2} + e^{-\frac{2\nu}{F}} \| u_0 \|_2^2, \tag{19}
\]

\[
\nu \int_0^t \| \nabla u(s) \|_2^2 ds \leq C \frac{L^2 F^2 \nu^2}{\nu^2} + \| u_0 \|_2^2, \tag{20}
\]

for all \( t \in [0, T^*] \), where

\[
F = \| f \|_{L^\infty([0, \infty), L^2(\Omega))}. \tag{21}
\]

In particular, since \( t < T^* \leq T \) we have

\[
\| u(t) \|_2^2 + \nu \int_0^t \| \nabla u(s) \|_2^2 ds \leq K_1, \tag{22}
\]

where

\[
K_1 = C \frac{\| f \|_2 L^4 \nu^2 + \nu T}{\nu^2} + 2 \| u_0 \|_2^2. \tag{23}
\]

Taking the inner product of the equation \([8]\) with \( -\Delta_h u \) in \( H \), and notice that \( P \partial_{x_i} \partial_{x_i} P \) for \( i = 1, 2 \), we get

\[
\frac{1}{2} \frac{d \| \nabla_h u \|_2^2}{dt} + \nu \| \Delta_h u \|_2^2 + \nu \| \nabla_h u \|_2^2 = - \int_{\Omega} \langle f - B(u, u) \rangle \cdot \Delta_h u dx_1 dx_2 dx_3.
\]
By integration by parts we get

\[
- \int_{\Omega} B(u, u) \cdot \Delta_h u \, dx_1 dx_2 dx_3 = \int_{\Omega} \left\{ \sum_{k=1}^{3} \left( \frac{\partial u_1}{\partial x_k} \right)^2 + \left( \frac{\partial u_2}{\partial x_k} \right)^2 \right\} dx_1 dx_2 dx_3
\]

\[
= \int_{\Omega} \left\{ \sum_{k=1}^{3} \left( \frac{\partial u_1}{\partial x_k} \right)^2 + \left( \frac{\partial u_2}{\partial x_k} \right)^2 \right\} dx_1 dx_2 dx_3
\]

Then, from the above and the Cauchy–Schwarz inequality we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \nabla_h u \|^2 + \nu \| \Delta_h u \|^2 + \nu \| \nabla_h u_2 \|^2 \leq F \| \Delta_h u \|_2 + C \int_{\Omega} |u| \| \nabla_h u| \, dx_1 dx_2 dx_3
\]

where \( F \) is given in (21). By applying Lemma \textbf{1} we obtain

\[
\int_{\Omega} \left[ \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_j} \right)^2 + \left( \frac{\partial u_j}{\partial x_j} \right)^2 \right] dx_1 dx_2 dx_3
\]

\[
\leq C \| u \|_{2}^{1/2} \| \nabla_h u_2 \|_2^{1/2} \| \nabla_h u \|_2^{1/2} \| \nabla_h u_2 \|_2^{1/2} \| \nabla_h u \|_2^{1/2} \| \nabla_h u_2 \|_2^{1/2} \| \nabla_h u \|_2^{1/2} \| \nabla_h u_2 \|_2^{1/2} \| \nabla_h u \|_2^{1/2} .
\]
By Hölder inequality, we reach
\[
\int_{\Omega} |\nabla \tilde{u}_3| |\nabla_h u|^2 \, dx_1 dx_2 dx_3 \\
\leq C \|\nabla \tilde{u}_3\|_2 \|\nabla_h u\|_2^2 \leq C \|\nabla \tilde{u}_3\|_2 \|\nabla_h u\|_2^{1/2} (\|\nabla_h u\|_2 + \|\nabla_h \nabla u\|_2)^{3/2}.
\] (25)

And also by Hölder inequality, we obtain
\[
\int_{\Omega} |u| |\nabla \tilde{u}_3| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3 \\
\leq C \int_{\mathbb{R}^2} \left\{ \|u(x_1, x_2, \cdot)\|_{\infty} \|\nabla \tilde{u}_3(x_1, x_2, \cdot)\|_2 \|\nabla_h \nabla u(x_1, x_2, \cdot)\|_2 \right\} \, dx_1 dx_2 \\
\leq C \int_{\mathbb{R}^2} \left\{ \|u(x_1, x_2, \cdot)\|_2^{1/2} \|\frac{\partial u}{\partial x_3}(x_1, x_2, \cdot)\|_2^{1/2} \|\nabla \tilde{u}_3(x_1, x_2, \cdot)\|_2 \|\nabla_h \nabla u(x_1, x_2, \cdot)\|_2 \right\} \, dx_1 dx_2 \\
\leq C \left\{ \int_{\mathbb{R}^2} \|u(x_1, x_2, \cdot)\|_2 \, dx_1 dx_2 \right\}^{1/8} \left\{ \int_{\mathbb{R}^2} \|\frac{\partial u}{\partial x_3}(x_1, x_2, \cdot)\|_2 \, dx_1 dx_2 \right\}^{1/8} \\
\times \left\{ \int_{\mathbb{R}^2} \|\nabla \tilde{u}_3(x_1, x_2, \cdot)\|_2 \, dx_1 dx_2 \right\}^{1/4} \|\nabla_h \nabla u\|_2 \\
\leq C \|u\|_2^{1/4} \|\nabla_h u\|_2^{1/2} \left\| \frac{\partial u}{\partial x_3} \right\|_2^{1/4} \left\| \frac{\partial \nabla_h u}{\partial x_3} \right\|_2^{1/4} \|\nabla \tilde{u}_3\|_2^{1/2} \|\nabla_h \nabla \tilde{u}_3\|_2^{1/2} \|\nabla_h \nabla u\|_2.
\] (26)

By (21), (23) and Young’s inequality we get
\[
\frac{d}{dt} \|\nabla_h u\|_2^2 + \nu \|\nabla_h \nabla u\|_2^2 \leq CF^2 + C \|u\|_2^2 \|\nabla u\|_2^2 \\
+C \left( \|u\|_2^{1/2} \left\| \nabla \tilde{u}_3\right\|_2 + \|\nabla \tilde{u}_3\|_2^{1/2} \|u\|_2 \right) \left\| \frac{\partial u}{\partial x_3} \right\|_2 \|\nabla \tilde{u}_3\|_2 \|\nabla_h u\|_2^2,
\]
where \( F \) is given in (21). Thanks to Gronwall inequality, we obtain, for all \( t \in [0,T_\ast) \),
\[
\|\nabla_h u(t)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 \, ds \leq K_2,
\]
where
\[
K_2 = e^{CK_1^2 + C(T + K_1^2)} \max_{0 \leq s \leq T} \|\nabla \tilde{u}_3(s)\|_2 \left[ \|u_0\|_{H^1(\Omega)}^2 + F^2 + CK_1^2 \right].
\] (27)

Recall that \( \|u\|_V^2 = \int u \cdot Au \, dx_1 dx_2 dx_3 \). It is well know that \( \|u\|_V^2 \) is equivalent to \( \|\nabla u\|_2^2 \) (see, e.g., [3]). Taking the inner product of the equation (2) with \( Au \) in \( H \), we get
\[
\frac{1}{2} \frac{d}{dt} \|u\|_V^2 + \nu \|Au\|_2^2 = \int_{\Omega} \left( f - B(u, u) \right) \cdot Au \, dx_1 dx_2 dx_3 \\
\leq F \|Au\|_2 + C \|u\|_6 \|\nabla u\|_3 \|Au\| \\
\leq F \|Au\|_2 + C \|\nabla_h u\|_2^{2/3} \left\| \frac{\partial u}{\partial x_3} \right\|_2^{1/3} \|\nabla u\|_2^{1/2} \|Au\|^{3/2}.
\]

Here, we used (cf., e.g., [3], p. 33)
\[
\|u\|_6 \leq C \left\| \frac{\partial u}{\partial x_1} \right\|_2^{1/3} \left\| \frac{\partial u}{\partial x_2} \right\|_2^{1/3} \left\| \frac{\partial u}{\partial x_3} \right\|_2^{1/3}.
\]
By Young’s inequality we obtain
\[ \frac{d}{dt}\|u\|^2_{V} + \nu \|Au\|^2_{V} \leq CF^2 + C\|\nabla u\|^s_{2} \left\| \frac{\partial u}{\partial x_3} \right\|^t_{2} \|u\|^2_{V}. \]

By Gronwall inequality and (27), we obtain, for all \( t \in [0,T^*] \),
\[ \|u(t)\|^2_{V} + \nu \int_{0}^{t} \|Au(s)\|^2_{2} ds \leq K, \]
where
\[ K = e^{CK_2^2K_1^2} \left[ \|u_0\|^2_{H^1(\Omega)} + F^2 \right]. \]

Therefore,
\[ \limsup_{t \to T^*} \|u(t)\|^2_{V} \leq K, \]
which leads to a contradiction that \([0,T^*)\) is the maximal interval of existence, and this completes the proof.

Acknowledgements

E.S.T. would like to thank the Bernoulli Center of the École Polytechnique Fédéral de Lausanne for the kind hospitality where this work was completed. This work was supported in part by the NSF grant no. DMS-0504619, the BSF grant no. 2004 271, the ISF grant no. 120/06, and by the MAOF Fellowship of the Israeli Council of Higher Education.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] L.C. Berselli, *On a regularity criterion for the solutions to the 3D Navier–Stokes equations*, Differential Integral Equations 15 (2002), 1129–1137.
[3] L.C. Berselli and G.P. Galdi, *Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations*, Proc. Amer. Math. Soc, 130 (2002), 3585–3595.
[4] C. Cao and E.S. Titi, *Global well-posedness and finite dimensional global attractor for a 3–D planetary geostrophic viscous model*, Comm. Pure Appl. Math. 56 (2003), 198-233.
[5] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Annals of Mathematics, 165 (2007), (to appear).
[6] C. Cao and E.S. Titi, *Pressure regularity criterion for the three-dimensional Navier–Stokes equations in infinite channel*, Preprint.
[7] D. Chae and J. Lee, *Regularity criterion in terms of pressure for the Navier–Stokes equations*, Nonlinear Anal, Ser. A: Theory Methods, 46 (2001), 727–735.
[8] P. Constantin and C. Foias, *Navier–Stokes Equations*, The University of Chicago Press, 1988.
[9] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. I & II, Springer-Verlag, 1994.
[10] C. He, *New sufficient conditions for regularity of solutions to the Navier–Stokes equations*, Adv. Math. Sci. Appl. 12 (2002), 535–548.
[11] T. Kato, *Strong \( L^p \) solutions of the Navier–Stokes equation in \( R^n \), with applications to weak solutions*, Math. Z. 187 (1984), 471–480.
[12] I. Kukavica and M. Ziane, *One component regularity for the Navier–Stokes equation*, Nonlinearity 19 (2006), 453–470.
[13] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, English transl., Gordon and Breach, New York, London, 1969.
[14] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, 1985.
[15] M. Pokorný, *On the result of He concerning the smoothness of solutions to the Navier–Stokes equations*, Electron. J. Differential Equations 11 (2003), 1–8.
[16] G. Prodi, *Un teorema di unicità per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl. 48 (1959), 173–182.
[17] J. Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Rational Mech. Anal. 9 (1962), 187–191.
[18] J. Serrin, *The initial value problem for the Navier–Stokes equations*, Nonlinear Problems (R.E. Langer, ed.), Univ. of Wisconsin Press, Madison, 1963, 69–98.
[19] H. Sohr, *The Navier–Stokes Equations: An Elementary Functional Analytical Approach*. Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2001.
[20] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS Regional Conference series, No. 41, SIAM, Philadelphia, 1983.
[21] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd revised edition, North-Holland, 2001.
[22] Y. Zhou, *A new regularity criterion for the Navier–Stokes equations in terms of the gradient of one velocity component*, Methods Appl. Anal. 9 (2002), 563–578.

(C. Cao) Department of Mathematics, Florida International University, University Park, Miami, FL 33199, USA
E-mail address: caoc@fiu.edu

(J. Qin) Department of Mathematics, Xian Jiaotong University, Xian, Shaanxi, China
E-mail address: jlqin@mail.xjtu.edu.cn

(E.S. Titi) Department of Mathematics, and Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697-3875, USA, also, Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: etiti@math.uci.edu and edriss.titi@weizmann.ac.il