Associated Spaces of Generalized Classical Lorentz Spaces $G\Lambda_{p,\psi;\varphi}$

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Abstract. In this paper we have calculated the associate norms of the $G\Lambda_{p,\psi;\varphi}$ generalized classical Lorentz spaces.

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1. Introduction

This paper aims at calculating the associate norm of the generalized classical Lorentz spaces. We have used the characterization of the weighted reverse Hardy inequality to calculate the associate norm of the spaces $G\Lambda_{p,\psi;\varphi}$.

2. Definitions and Preliminary Tools

Let $E$ be a measurable subset of $\mathbb{R}^n$. We denote by $L_{p,E}$ the class of all measurable functions $f$ defined on $E$ for which

$$\|f\|_{L_{p,E}} := \left(\int_E |f(y)|^p dy \right)^{\frac{1}{p}} < \infty, 0 < p < \infty,$$

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∥f∥_{L_\infty,E} := \sup\{\alpha: \{|y \in E: |f(y)| \geq \alpha| > 0\},
and denote by WL_p the weak L_p space such that
∥f∥_{WL_p,E} := \{f: \sup_{\alpha>0} \alpha \mu_f(\alpha)^{1/p} < \infty\},
where \mu_f(\alpha) denotes the distribution function of f given by
\mu_f(\alpha) = \mu\{x \in E: |f(x)| > \alpha\}.

Let for 0 < p, q \leq \infty and r > 0
∥f∥_{p,q;0,\infty} := \|t^{\frac{1}{p} - \frac{1}{q}} f^*(t)\|_{q,0,\infty},
where f^* is the decreasing rearrangement of f defined by
f^*(t) = \inf \{\lambda > 0: \mu_f(\lambda) \leq t\}, \ \forall t \in (0, \infty).

We denote by \mathcal{M}(\mathbb{R}^n, \mu) be the set of all extended real valued \mu-measurable functions on \mathbb{R}^n and \mathcal{M}^+(0, \infty) the set of all non-negative measurable functions on (0, \infty), \mathcal{M}^+(0, \infty; \uparrow) the set of all non-decreasing functions from \mathcal{M}^+(0, \infty).

Now we recall definitions of Lorentz, classical Lorentz and generalized classical Lorentz spaces.

**Definition 2.1.** The Lorentz space \( L_{p,q} \equiv L_{p,q}(\mathbb{R}^n) \), 0 < p, q \leq \infty is the collection of all measurable functions \( f \) on \( \mathbb{R}^n \) such the quantity
\[ \|f\|_{p,q} := \|f\|_{p,q;0,\infty} = \|t^{\frac{1}{p} - \frac{1}{q}} f^*(t)\|_{q,0,\infty} \] (2.2)
is finite.

Note that \( L_{p,\infty}(\mathbb{R}^n) = WL_p(\mathbb{R}^n) \) (see, for example, [16]).

Note that, \( L_{p,p} = L_p \) for 0 < p \leq \infty.

If 1 \leq q \leq p or p = q = \infty, then the functional \( \|f\|_{p,q} \) is a norm.

For 0 < q \leq r \leq \infty we have, with continuous embeddings, that
\[ L_{p,q} \subset L_{p,r}. \]

The function \( f^{**}: (0, \infty) \to [0, \infty] \) is defined as
\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds. \]

In the case 0 < p, q \leq \infty, we give a functional \( \|\cdot\|^*_{p,q} \) by
\[ \|f\|^*_{p,q} := \|f\|^*_{p,q;0,\infty} = \|t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t)\|_{q,0,\infty} \]
by
\[ \|f\|_{p,q} := \|f\|_{p,q;0,\infty} = \|t^{\frac{1}{p} - \frac{1}{q}} f^*(t)\|_{q,0,\infty} \]
(with the usual modification if \(0 < p \leq \infty, q = \infty\)) which is a norm on \(L_{p,q}(\mathbb{R}^n)\) for \(1 < p < \infty, 1 \leq q \leq \infty\) or \(p = q = \infty\).

If \(1 < p \leq \infty, 1 \leq q \leq \infty\), then
\[
\|f\|_{p,q} \leq \|f\|_{p,q}^* \leq \frac{p}{p - 1} \|f\|_{p,q}.
\] (2.3)

About \(L_{p,q}(\mathbb{R}^n)\) Lorentz spaces see [11, 16].

**Definition 2.2.** Let \(0 < p, q \leq \infty\) and \(\psi \in \mathcal{M}^+(0, \infty)\). We denote by \(\Lambda_{p,\psi}(\mathbb{R}^n)\) the classical Lorentz spaces, the spaces of all measurable functions with finite quasinorm
\[
\Lambda_{p,\psi}(\mathbb{R}^n) := \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{\Lambda_{p,\psi}} := \|\psi f^*\|_{p,(0,\infty)}\}.
\]

Therefore the following statement
\[
\Lambda_{p,\psi}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)
\]
is valid.

The spaces \(\Lambda^p(w) \equiv \Lambda_{p,w,1/p}\) were introduced by Lorentz in 1951 in [14]. Spaces whose norms involve \(f^{**}\) appeared explicitly for the first time in Calderon’s paper [4]. In [15] Sawyer give description of the dual of \(\Lambda^p(w)\).

Lorentz [14] proved that, for \(p \geq 1\), \(\|f\|_{\Lambda^p(w)}\) is a norm if and only if \(w\) is nonincreasing. The class of weights for which \(\|f\|_{\Lambda^p(w)}\) is merely equivalent to a Banach norm is however considerably larger. In fact it consists of all those weights \(w\) which, for some \(C\) and all \(t > 0\), satisfy
\[
i^p \int_t^\infty x^{-p} w(x) dx \leq C \int_0^t w(x) dx \quad \text{when } p \in (1, \infty)
\]
([15], Theorem 4], see also [1]), or
\[
\frac{1}{t} \int_0^t w(x) dx \leq \frac{C}{s} \int_0^s w(x) dx \quad \text{for } 0 < s \leq t \quad \text{when } p = 1
\]
([5], Theorem 2.3).

In [10], Theorem 1.1 (see also [7], Corollary 2.2, [12], p. 6) it was observed that the functional \(\|f\|_{\Lambda^p(w)}; 0 < p \leq \infty, does not have to be a quasinorm. It was shown that it is a quasinorm if and only if the function \(W(t) = \int_0^t w(s) ds\) satisfies the \(\Delta_2\)-condition, i.e.,
\[
W(2t) \leq CW(t) \quad \text{for some } C > 1 \quad t \in (0, \infty).
\]

In [8] were given necessary and sufficient conditions for \(\Lambda^p(w)\) to be a linear space. About historical developments classical Lorentz spaces see [6].

**Definition 2.3.** Let \(0 < p, q \leq \infty\) and let \(\varphi, \psi \in \mathcal{M}^+(0, \infty)\). We denote by \(GL_{p,q;\varphi}(\mathbb{R}^n)\) the generalized Lorentz spaces, the spaces of all measurable functions with finite quasinorm
\[
GL_{p,q;\varphi}(\mathbb{R}^n) := \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{GL_{p,q;\varphi}} := \sup_{r > 0} \varphi(r) \|t^{\frac{1}{q} - \frac{1}{p}} f^*(t)\|_{q,(0,r)}\}.
\]
Definition 2.4. Let $0 < p \leq \infty$ and let $\varphi, \psi \in \mathcal{M}^+(0, \infty)$. We denote by $G\Lambda_{p,\psi;\varphi}(\mathbb{R}^n)$ the generalized classical Lorentz spaces, the spaces of all measurable functions with finite quasinorm
\[
G\Lambda_{p,\psi;\varphi}(\mathbb{R}^n) := \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{G\Lambda_{p,\psi;\varphi}} := \sup_{r > 0} \varphi(r) \| \psi(\cdot) f^*(\cdot) \|_{p;(0,r)} \}.
\]

Therefore the following statement
\[
L_{p,q} = GL_{p,q;1}, \quad L_{p;q} = GL_{p,\psi;\varphi}, \quad L_{p;p} = GL_{p,p;1}, \quad L_{p;\psi} = GL_{p,\psi;\varphi}, \quad L_{p;1} = GL_{p,1;1}
\]
is valid. Note that, the space
\[
L_{p;\varphi}(\mathbb{R}^n) = \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{L_{p;\varphi}} := \sup_{r > 0} \varphi(r) \| f^* \|_{p;(0,r)} < \infty \}
\]
we called generalized Lebesgue spaces.

Recall the definition of generalized Marcinkiewicz space
\[
M_{p;\varphi}(\mathbb{R}^n) = \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{M_{p;\varphi}} := \sup_{E} \varphi(|E|) \| f \|_{p,E} < \infty \},
\]
where the supremum is taken for all measurable subset of $\mathbb{R}^n$. It easy to see that
\[
\sup_{E} \varphi(|E|) \| f \|_{p,E} = \sup_{t > 0} \varphi(t) \| f \|_{p,(0,t)}
\]
Therefore, we have $G\Lambda_{p,1;\varphi}(\mathbb{R}^n) = M_{p;\varphi}(\mathbb{R}^n)$ and $G\Lambda_{1,1;\varphi}(\mathbb{R}^n) = M_{\varphi}(\mathbb{R}^n)$.

Remark 2.5. Note that, the classical Lorentz space $\Lambda_{p,\psi}(\mathbb{R}^n)$ is not a linear space, therefore the generalized classical Lorentz spaces $G\Lambda_{p,\psi;\varphi}(\mathbb{R}^n)$ are also not linear spaces in general. It is easy to see that the condition
\[
\int_0^{2t} \psi^p(x)dx \leq C \int_0^t \psi^p(x)dx \quad \text{for some} \quad C > 1 \quad t \in (0, \infty)
\]
is sufficient for the generalized classical Lorentz spaces $G\Lambda_{p,\psi;\varphi}(\mathbb{R}^n)$ to be a quasinorm space.

Also the condition
\[
t^p \int_t^\infty x^{-p}\psi^p(x)dx \leq C \int_0^t \psi^p(x)dx \quad \text{when} \quad p \in (1, \infty)
\]
or
\[
\frac{1}{t} \int_0^t \psi(x)dx \leq \frac{C}{s} \int_0^s \psi(x)dx \quad \text{for} \quad 0 < s \leq t \quad \text{when} \quad p = 1
\]
is sufficient for the generalized classical Lorentz spaces $G\Lambda_{p,\psi;\varphi}(\mathbb{R}^n)$ to be a norm space.

Remark 2.6. The problems mentioned in the Remark 2.5 are not as easy as the problems in the case of the classical Lorentz spaces. At this point it’s required to have the characterizations of the embedding between these generalized classical Lorentz spaces and the boundedness of the maximal operator in these spaces. We haven’t got the solution of these problems yet.
To solve these problems it will be useful to find the characterizations of the associate spaces of the generalized classical Lorentz spaces. In this paper we give the characterizations of the associate spaces of these spaces. Note that, in the special case $\varphi(t) = t^{-\frac{1}{\lambda}}$ the generalized Lorentz space $GL_{p,q,\varphi}(\mathbb{R}^n)$ was introduced and investigated in [2].

**Definition 2.7.** Let $X$ be a set of functions from $\mathcal{M}(\mathcal{R}, \mu)$ endowed with a positively homogenous functional $\| \cdot \|_X$ defined for every $f \in \mathcal{M}(\mathcal{R}, \mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$, we define the associate space $X'$ of $X$ as the set of all functions $f \in \mathcal{M}(\mathcal{R}, \mu)$ such that $\|f\|_{X'} < \infty$, where

$$\|f\|_{X'} = \sup\{ \int_X fg d\mu : \|g\|_X \leq 1 \}$$

In what follows we assume $\mathcal{R} = \mathbb{R}^n$ and $d\mu = dx$.

**Proposition 2.8.** ([3], p. 58) Let $\| \cdot \|$ be a rearrangement-invariant function norm over a resonant measure space $(X, d\mu)$. Then the associate norm $\| \cdot \|_{X'}$ is also rearrangement invariant. Furthermore,

$$\|f\|_{X'} = \sup\{ \int_X fg d\mu : \|g\|_X \leq 1 \} = \sup\{ \int_0^{\infty} f^*(t)g^*(t)dt : \|g\|_X \leq 1 \}$$

holds.

Throughout the paper, we write $A \lesssim B$ if there exists a positive constant $C$, independent of appropriate quantities such as functions, satisfying $A \leq CB$. We write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

### 3. Reverse Hardy Inequality

Let us recall some results from [9].

Let $u, v$ and $w$ will denote weights, that is, locally integrable non-negative functions on $(0, \infty)$. We set, once and for all

$$U(t) = \int_0^t u(s)ds, V(t) = \int_0^t v(s)ds, W(t) = \int_0^t w(s)ds.$$ 

We assume that $U(t) > 0$ for every $t \in (0, \infty)$. We then denote

$$f^{**}_u(t) = \frac{1}{U(t)} \int_0^t f^*(s)u(s)ds, t \in (0, \infty).$$

When $u \equiv 1$ (hence $U(t) = t$), we will omit the subscript $u$.

Furthermore, for given $q \in (0, \infty)$ and every $f \in \mathcal{M}(\mathcal{R}, \mu)$, the necessary and sufficient conditions for the inequality

$$\left( \int_0^{\infty} f^*(t)^q w(t)dt \right)^{\frac{1}{q}} \lesssim \text{ess sup} \ f^{**}_u(t)v(t) \quad (3.1)$$
were established in [9].

**Definition 3.1.** Let \( \theta \) be a continuous strictly increasing function on \([0, \infty)\) such that \( \theta(0) = 0 \) and \( \lim_{t \to \infty} \theta(t) = \infty \). Then we say \( \theta \) is admissible.

Let \( \theta \) be an admissible function. We say that a function \( h \) is \( \theta \)-quasiconcave if \( h \) is equivalent to a non-decreasing function on \([0, \infty)\) and \( \frac{h}{\theta} \) is equivalent to a non-increasing function on \((0, \infty)\). We say that a \( \theta \)-quasiconcave function \( h \) is non-degenerate if

\[
\lim_{t \to 0^+} h(t) = \lim_{t \to \infty} \frac{1}{h(t)} = \lim_{t \to \infty} \frac{h(t)}{\theta(t)} = \lim_{t \to 0^+} \frac{\theta(t)}{h(t)} = 0.
\]

The family of non-degenerate \( \theta \)-quasiconcave functions will be denoted by \( \Omega_\theta \).

**Lemma 3.2.** Let \( u, v \) be weights above and let \( \sigma \) defined by

\[
\sigma(t) := \text{ess sup}_{s \in (0,t]} U(s) \text{ess sup}_{\tau \in (s,\infty]} \frac{v(\tau)}{U(\tau)}, t \in (0, \infty) \tag{3.2}
\]

then,

\[
\sigma(t) \approx \text{ess sup}_{s \in (0,\infty)} v(s) \frac{U(t)}{U(s) + U(t)}.
\]

**Definition 3.3.** Let \( \sigma \) be an admissible function and let \( \nu \) be a non-negative Borel measure on \([0, \infty)\). We say that the function \( h \) defined as

\[
h(t) := \sigma(t) \int_{[0,\infty)} \frac{d\nu(s)}{\sigma(s) + \sigma(t)}, t \in (0, \infty),
\]

fundamental function of the measure \( \nu \) with respect to \( \sigma \). We will also say that the function \( \nu \) is a representation measure of \( h \) with respect to \( \sigma \).

We say that \( \nu \) is non-degenerate if the following conditions are satisfied for every \( t \in (0, \infty) \):

\[
\int_{[0,\infty)} \frac{d\nu(s)}{\sigma(s) + \sigma(t)} < \infty, \int_{[0,1]} \frac{d\nu(s)}{\sigma(s)} = \int_{[1,\infty)} d\nu(s) = \infty.
\]

**Remark 3.4.** Let \( \sigma \) be an admissible function and let \( \nu \) be a non-negative non-degenerate Borel measure on \([0, \infty)\). Let \( h \) be the fundamental function of \( \nu \) with respect to \( \sigma \). Then

\[
h(t) \approx \int_0^t \int_{[s,\infty)} \frac{d\nu(y)}{\sigma(y)} d\sigma(s), t \in (0, \infty),
\]

and also

\[
h(t) \approx \int_{[0,t]} d\nu(s) + \sigma(t) \int_{[t,\infty]} \frac{d\nu(s)}{\sigma(s)}, t \in (0, \infty).
\]
Theorem 3.5. \cite{9} Let \( q \in (0, \infty) \) and let \( u, v, w \) be weights. Assume that \( u \) is such that \( U \) is admissible. Let \( \sigma \) defined by \cite{3,2}, be non-degenerate with respect to \( U \). Let \( \nu \) be the representation measure of \( U^q/\sigma^q \) with respect to \( U^q \).

(i) If \( 1 \leq q < \infty \), then (3.1) holds for all \( f \) if and only if

\[
A(1) = \left( \int_0^\infty \sup_{s \in (t, \infty)} \frac{W(s) d\nu(t)}{U(s)^q} \right)^{\frac{1}{q}} < \infty.
\]

Moreover, the optimal constant \( C \) in (3.1) satisfies \( C \approx A(1) \).

(ii) If \( 0 < q < 1 \), then (3.1) holds for all \( f \) if and only if

\[
A(2) = \left( \int_0^\infty \frac{\zeta(t) d\nu(t)}{U(t)^q} \right)^{\frac{1}{q}} < \infty,
\]

where

\[
\zeta(t) = W(t) + U(t)^q \left( \int_t^\infty \left( \frac{W(s)}{U(s)} \right)^{\frac{q}{1-q}} w(s) ds \right)^{1-q}, \quad t \in (0, \infty).
\]

Moreover, the optimal constant in (3.1) satisfies \( C \approx A(2) \).

4. Associated spaces of generalized classical Lorentz spaces

\( G\Lambda_{p,\psi;\varphi} \)

The associated spaces of classical Lorentz spaces \( \Lambda_{p,\psi} \) was calculated in \cite{13}.

Theorem 4.1. Let \( 0 < p < \infty, \ \psi \in M^+(0, \infty) \). Then the associate spaces of \( \Lambda_{p,\psi} \) are described as follows:

(i) If \( 0 < p \leq 1 \), then

\[
\|f\|_{(\Lambda_{p,\psi})'} = \sup_{t>0} \frac{tf^{**}(t)}{\Psi_p(t)},
\]

where \( \Psi_p(t) = \|\Psi\|_{p,(0,t)} \).

(ii) If \( 1 < p < \infty \), then

\[
\|f\|_{(\Lambda_{p,\psi})'} = \int_0^\infty \left( \frac{tf^{**}(t)}{\Psi_p(t)^p} \right)^{\frac{1}{p'}} \psi^p(t) dt.
\]

In this section by using results of previous section we calculate the associated spaces of generalized classical Lorentz spaces.

Theorem 4.2. Let \( 0 < p < \infty, \ \psi \in M^+(0, \infty), \ \varphi \in M^+(0, \infty, \downarrow) \) and \( \varphi(r)^{\frac{1}{r^p}} \in M^+(0, \infty, \uparrow) \). Then the associate spaces of \( G\Lambda_{p,\psi;\varphi} \) are described as follows:

(i) If \( 0 < p \leq 1 \), then

\[
\|f\|_{(G\Lambda_{p,\psi;\varphi})'} = \int_0^\infty \sup_{s \in (t, \infty)} \frac{sf^{**}(s)}{\Psi_p(s)} d\nu(t),
\]
where $ν$ is the representation measure of $\frac{1}{φ(t)}$ with respect to $∥ψ∥_{p,(0,t)}$.

(ii) If $1 < p < ∞$, then

$$∥f∥(Γ_{p,ψ,ν})′ = \int_0^∞ \left( \int_t^∞ \left( \frac{sf^{**}(s)}{Ψ_p(s)} \right)^p ψ^p(s)ds \right)^{1/p'} dv(t),$$

where $ν$ is the representation measure of $\frac{1}{φ(t)}$ with respect to $Ψ_p(t)$.

**Proof.** From Proposition 2.8 we have

$$∥f∥(Γ_{p,ψ,ν})′ = \sup_{g≥0} \frac{∫_0^∞ f^*(t)g^*(t)dt}{∫_0^∞ sup_{r>0} φ(r) \left( ∫_0^∞ g^*(t)^p ψ^p(t)dt \right)^{1/p'}}.$$  \hspace{1cm} (4.1)

If we take $g^*(t) = h^*(t)^{1/r}$, then we can write

$$∥f∥(Γ_{p,ψ,ν})′ = \left[ sup_{h≥0} \frac{∫_0^∞ f^*(t)h^*(t)^{1/r} dt}{∫_0^∞ sup_{r>0} φ(r)^p ∫_0^r h^*(t)ψ^p(t)dt} \right]^{1/p}.$$  \hspace{1cm} (4.2)

If we define the function $u(t) = ψ^p(t)$, then we get

$$h_{u^*}^*(t) = \frac{1}{Ψ_p^p(t)} f^*(t)∞ h^*(s)ψ^p(s)ds.$$  \hspace{1cm} (4.2)

Therefore

$$\left[ sup_{h≥0} \frac{∫_0^∞ f^*(t)h^*(t)^{1/r} dt}{∫_0^∞ sup_{r>0} φ(r)^p ∫_0^r h^*(t)ψ^p(t)dt} \right]^{1/p} \approx \left[ sup_{h≥0} \frac{∫_0^∞ f^*(t)h^*(t)^{1/r} dt}{∫_0^∞ sup_{r>0} φ(r)^p h_{u^*}^*(r)} \right]^{1/p}.$$  \hspace{1cm} (4.2)

(i) Let $1 < p < ∞$.

In Theorem 3.5 if we take $q = \frac{1}{p}$, $w(t) = f^*(t)$, $U(t) = Ψ_p^p(t)$, $v(t) = φ(t)^pΨ_p^p(t)$ and $ν$ be the representation measure of $\frac{1}{φ(t)}$ with respect to $Ψ_p(t)$, which means

$$\frac{1}{φ(t)} \approx ∫_0^t dv(s) + Ψ_p^p(t) ∫_t^∞ dv(s) / Ψ_p^p(s).$$

Then we get

$$RHS[4.2] ≃ ∫_0^∞ \frac{ζ(t)}{Ψ_p^p(t)} dv(t),$$

where

$$ζ(t) = tf^{**}(t) + Ψ_p^p(t) \left( ∫_t^∞ \left( \frac{sf^{**}(s)}{Ψ_p^p(s)} \right)^{1/p} f^*(s)ds \right)^{1/p'} , t ∈ (0, ∞).$$
Furthermore, we have

\[ \zeta(t) \approx \Psi_p(t) \left( \int_t^{\infty} \left( \frac{sf^{**}(s)}{\Psi_p(s)} \right)^{p'} \psi^p(s) ds \right)^{1/p'} =: \zeta_1(t). \]  

(4.3)

Clearly

\[ tf^{**}(t) = (p - 1) \frac{1}{p'} \Psi_p(t) \left( \int_t^{\infty} \frac{\psi^p(s)}{\Psi_p(s)^{p'}} ds \right)^{1/p'} \]

\[ \lesssim \Psi_p(t) \left( \int_t^{\infty} \frac{sf^{**}(s)}{\Psi_p(s)} \psi^p(s) ds \right)^{1/p'} =: \zeta_1(t). \]  

(4.4)

Also by partial integration

\[ \int_t^{\infty} \frac{sf^{**}(s)}{\Psi_p(s)} \psi^p(s) ds \]

\[ = \frac{(sf^{**}(s))^{p'}}{\Psi_p(s)^{p'}} \Big|^{\infty}_{t} - \int_t^{\infty} (sf^{**}(s))^{p'} d\left( \Psi_p(s) \right)^{-\frac{1}{p'}} \]

\[ = - (tf^{**}(t))^{p'} (\Psi_p(t))^{-\frac{1}{p'}} + \frac{1}{p - 1} \int_t^{\infty} \left( \frac{sf^{**}(s)}{\Psi_p(s)} \right)^{p'} \psi^p(s) ds. \]  

(4.5)

Then from (4.4) and (4.5) we get

\[ \zeta(t) \lesssim \zeta_1(t). \]  

(4.6)

Furthermore, from (4.5) we also have

\[ \zeta_1(t) \leq \Psi_p(t) \left( \int_t^{\infty} \frac{sf^{**}(s)}{\Psi_p(s)} \psi^p(s) ds \right)^{1/p'} \]

Therefore together with (4.6) we get (4.3) and consequently we have

\[ \text{RHS}(4.2) \approx \int_0^{\infty} \left( \int_t^{\infty} \frac{sf^{**}(s)}{\Psi_p(s)} \psi^p(s) ds \right)^{1/p'} d\nu(t). \]

(ii) Let \( 0 < p \leq 1 \). If we take \( q = \frac{1}{p} \) in Theorem 3.5 then \( 1 \leq q < \infty \) and we obtain

\[ \text{RHS}(4.2) \approx \int_0^{\infty} \sup_{s \in [t, \infty)} \frac{sf^{**}(s)}{\Psi_p(s)} d\nu(t). \]

From the Theorem 4.2 we get the following theorem.

**Theorem 4.3.** Let \( 0 < p, q < \infty, \psi, w \in M^{+}(0, \infty), \varphi \in M^{+}(0, \infty, \downarrow) \) and \( \varphi(r)^{\frac{1}{p'}} \in M^{+}(0, \infty, \uparrow) \). Then the following embedding

\[ G\Lambda_{p, \psi ; \varphi} \hookrightarrow \Lambda_{q,w} \]  

is valid iff
i) \(0 < p \leq q < \infty\)

\[
\int_0^\infty \sup_{s \in (t, \infty)} \frac{\int_0^t w^q(s)ds}{\Psi^q(s)} d\nu(t) < \infty
\]

where \(\nu\) is the representation measure of \(\frac{1}{\psi'(t)}\) with respect to \(\Psi^q(t) = \|\psi^q\|_{\frac{q}{p}, (0,t)}\).

(ii) If \(0 < q < p < \infty\), then

\[
\int_0^\infty \left( \int_0^\infty \left( \frac{\int_0^t w^q(s)ds}{\Psi^q(s)} \right)^{\frac{q}{p}} \psi^p(s)ds \right)^{1/\left(\frac{q}{p}\right)'} d\nu(t),
\]

where \(\nu\) is the representation measure of \(\frac{1}{\psi'(t)}\) with respect to \(\Psi^q(t)\).

Proof. Using a simple observation function, it is obvious that \(f\) is decreasing if and only if, \(f^q\) is decreasing for all \(q > 0\). We see that the embedding (4.7) holds if and only if the following embedding holds

\[
G\Lambda_{\frac{q}{p}, \psi^q; \psi^p} \hookrightarrow \Lambda_{1, w^q}.
\]

One can get the required result by using Theorem 4.2. \(\square\)

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