Pseudo-multipliers and Smooth Molecules on Hermite Besov and Hermite Triebel–Lizorkin Spaces

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Abstract
We obtain new molecular decompositions and molecular synthesis estimates for Hermite Besov and Hermite Triebel–Lizorkin spaces and use such tools to prove boundedness properties of Hermite pseudo-multipliers on those spaces. The notion of molecule we develop leads to boundedness of pseudo-multipliers associated to symbols of Hörmander-type adapted to the Hermite setting on spaces for which the smoothness allowed includes non-positive values; in particular, we obtain continuity results for such operators on Lebesgue and Hermite local Hardy spaces. As a byproduct of our results on boundedness properties of pseudo-multipliers, we show that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities.

Keywords Hermite operator · Besov and Triebel–Lizorkin spaces · Molecules · Pseudo-multipliers

Mathematics Subject Classification 42B35 · 42C15 · 35S05 · 33C45

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1 Introduction

This article contributes new results to the theory of function spaces and corresponding boundedness properties of pseudo-multipiers associated to the Hermite operator. This operator is defined as

$$\mathcal{L} = -\Delta + |x|^2, \quad x \in \mathbb{R}^n, \quad n \geq 1,$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. For a symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$, the Hermite pseudo-multiplier $T_\sigma$ is given by

$$T_\sigma f(x) = \sum_{k \in \mathbb{N}_0} \sigma(x, \lambda_k) \mathbb{P}_k f(x), \quad x \in \mathbb{R}^n,$$

where $\lambda_k = 2k + n$ and $\mathbb{P}_k$ are orthogonal projectors onto spaces spanned by Hermite functions (see Sect. 2). When $\sigma$ is independent of $x$, $T_\sigma$ is called a Hermite multiplier and can be expressed in the form

$$T_\sigma(f) = \mathcal{F}_\mathcal{L}^{-1} (\sigma \mathcal{F}_\mathcal{L}(f)),$$

where $\mathcal{F}_\mathcal{L}$ is the Fourier–Hermite transform. These operators are counterparts of the well-known pseudo-differential operators and Fourier multiplier operators defined in the Euclidean setting in terms of the Fourier transform.

In this paper, we obtain new molecular decompositions (Theorem 3.5) and molecular synthesis estimates (Theorem 3.6) for the Hermite Besov spaces, denoted $B^{p,q}_\alpha(\mathcal{L})$, and the Hermite Triebel–Lizorkin spaces, denoted $F^{p,q}_\alpha(\mathcal{L})$, and use such tools to prove new results on boundedness of Hermite pseudo-multipiers on those spaces (Theorems 4.5 and 4.6 and Corollary 4.7).

The works [26,33,34] pioneered the study of boundedness properties of Hermite multipliers on Lebesgue spaces. For instance, a Hörmander-Mikhlin type multiplier theorem for Hermite expansions was proved in [33]; more precisely, if $\sigma$ satisfies

$$|\Delta^\kappa \sigma(k)| \lesssim k^{-\kappa} \quad \forall k \in \mathbb{N}_0$$

for $\kappa = 0, \ldots, \lfloor \frac{n}{2} \rfloor + 1$, then $T_\sigma$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. The symbol $\Delta$ denotes the forward difference operator, that is, for a function $f$ defined over the integers, $\Delta f(k) = f(k + 1) - f(k)$ and $\Delta^\kappa f(k) = \Delta(\Delta^{\kappa-1} f(k))$ for $\kappa \geq 2$.

The first results on boundedness properties of Hermite pseudo-multipiers in Lebesgue spaces appeared in [14] and are restricted to dimension one. This work was followed by investigations in [1] where it is proved that if $T_\sigma$ is bounded on $L^2(\mathbb{R}^n)$ and

$$|\partial_\nu^\gamma \Delta^\kappa \sigma(x,k)| \lesssim (1 + k)^{-\kappa} \quad \forall (x,k) \in \mathbb{R}^n \times \mathbb{N}_0 \quad (1.1)$$
for κ = 0, 1, ..., n + 1 and ν ∈ \( \mathbb{N}_0^n \) such that 0 ≤ |ν| ≤ 1, then \( T_\sigma \) is bounded on \( L^p(\mathbb{R}^n) \) for 1 < p < ∞. The theorems in [1,14] on boundedness of Hermite pseudo-multipliers on \( L^p(\mathbb{R}^n) \) for some range of 1 < p < ∞ assume the boundedness of the operator on \( L^2(\mathbb{R}^n) \). For Hermite multipliers, boundedness on \( L^2(\mathbb{R}^n) \) is equivalent to the symbol being a bounded function on \( \mathbb{N} \) by Parseval’s identity for Hermite expansions; however, the situation is more complicated in the case of Hermite pseudo-multipliers. This prompted the authors in [1] to pose the question about what conditions on the symbol of a Hermite pseudo-multiplier imply its boundedness on \( L^2(\mathbb{R}^n) \).

Corollaries 5.4 and 5.6 of our results give an answer to this question; for instance, we obtain that \( T_\sigma \) is bounded on \( L^p(\mathbb{R}^n) \) for 1 < p < ∞ if σ satisfies (1.1) for κ = 0, ..., n + 1 and ν ∈ \( \mathbb{N}_0^n \) such that 0 ≤ |ν| ≤ 2[\( \frac{n+1}{2} \)] or if σ satisfies a certain cancellation condition along with the weaker estimate

\[
|\partial_\chi^\kappa \Delta_\nu^\kappa \sigma(x, k)| \lesssim (1 + \sqrt{k})^{-2\kappa+|\nu|} \quad \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}_0
\]

for κ = 0, 1, ..., n + 1 and ν ∈ \( \mathbb{N}_0^n \) such that 0 ≤ |ν| ≤ 1. Other sufficient conditions for boundedness on \( L^p(\mathbb{R}^n) \) for 1 < p < ∞ of Hermite pseudo-multipliers, as well as a comparison of our corresponding results with those in the literature, are presented in Sect. 5.1. For the range 0 < p ≤ 1, we obtain results in the context of Hermite local Hardy spaces, which are better suited than Lebesgue or Hardy spaces for the study of boundedness properties of pseudo-multipliers (see Sect. 5.2).

Continuity properties of pseudo-differential operators have been extensively studied in various function spaces that include the scale of the classical Besov and Triebel–Lizorkin spaces associated to the Laplacian operator (see for instance, [4,22,27,29,31,32,35]). While there are some recent works on the study of boundedness properties of pseudo-differential operators in spaces associated to other non-negative self-adjoint operators (see [6,19] addressed below), such investigations lead to results only in the context of spaces with positive smoothness. Our article seems to be the first one treating boundedness properties of Hermite pseudo-multipliers in the scales of the Hermite Besov spaces \( B_{p,q}^\alpha(\mathcal{L}) \) and the Hermite Triebel–Lizorkin spaces \( F_{p,q}^\alpha(\mathcal{L}) \) that allow for non-positive values of the smoothness parameter \( \alpha \) and for the whole range \((0, \infty)\) for the parameters \( p, q \) (in particular, \( \alpha = 0 \) and \( q = 2 \) give the scale of Lebesgue and Hermite local Hardy spaces). The spaces \( B_{p,q}^\alpha(\mathcal{L}) \) and \( F_{p,q}^\alpha(\mathcal{L}) \) are defined in terms of Littlewood-Paley operators associated to the Hermite operator. A \( \varphi \)-transform characterization for them, in the spirit of the fundamental works [15,16] for the Euclidean setting, was obtained in [30]. Among the main contributions in this article, we introduce new concepts of molecules for Hermite Besov spaces and Hermite Triebel–Lizorkin spaces; we then prove new corresponding almost orthogonality estimates, molecular decompositions and molecular synthesis estimates (see Sect. 3). The latter constitutes a crucial tool for the proofs of the boundedness properties of the Hermite pseudo-multipliers here studied, which require the use of both a \( \varphi \)-transform characterization as well as molecular synthesis estimates for the spaces.

The classical molecules in [15,16] satisfy the following cancellation condition

\[
\int_{\mathbb{R}^n} y^\gamma m(y) \ dy = 0 \quad \forall \gamma \in \mathbb{N}_0 \text{ such that } |\gamma| \leq M,
\]
for a suitable integer $M$. On the other hand, the (almost) cancellation condition imposed to the molecules we define in the Hermite context is expressed in terms of estimates of the type
\[
\left| \int_{\mathbb{R}^n} (y - x)^\gamma m(y) \, dy \right| \leq |B(x, r)|^{1/2} \varrho(x)^{|\gamma|} \left( \frac{r}{\varrho(x)} \right)^{M+1} \quad \forall \gamma \in \mathbb{N}_0 \text{ such that } |\gamma| \leq M,
\]
where $\varrho(x) = \frac{1}{1 + |x|}$, $x \in \mathbb{R}^n$ and $r \leq \varrho(x)$ (see Definition 3.1). This condition is essential in our proofs of boundedness results for Hermite pseudo-multipliers that allow for the smoothness index to be zero or negative.

Our results on Hermite Besov and Hermite Triebel–Lizorkin spaces complement several articles that have extended the work in [15,16] for the classical Besov and Triebel–Lizorkin spaces to a variety of other settings associated to non-negative self-adjoint operators. For instance, the authors of [7] develop molecular decompositions and molecular synthesis estimates for scales of Besov and Triebel–Lizorkin spaces associated to the Hermite operator that are defined in terms of heat kernels via square functions. We note that the spaces $B^{p,q}_\alpha (\mathcal{L})$ and $F^{p,q}_\alpha (\mathcal{L})$ and the concept of molecules introduced in this article are different to those treated in [7]; in particular, the notion of molecules we present is new in this context and closer in spirit to the molecules in [15,16] as described above. The authors of [25] present frame decompositions for Besov and Triebel–Lizorkin spaces in the context of a Dirichlet space with a doubling measure and local scale-invariant Poincaré inequality; the authors of [17,18] study discrete frame decompositions and atomic and molecular decompositions for homogeneous Besov and Triebel–Lizorkin spaces in the setting of a doubling metric measure space in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property; the authors of [5] obtain atomic decompositions for weighted Besov and Triebel–Lizorkin spaces in the context of spaces of homogeneous type with a non-negative self-adjoint operator satisfying Gaussian upper bounds on its heat kernels.

The classes of symbols we consider are reminiscent of the well-known Hörmander classes corresponding to the Euclidean setting; they include symbols that satisfy estimates of the type
\[
|\partial^\nu x \Delta_k^\sigma (x, k)| \lesssim g(x, k)(1 + \sqrt{k})^{m - 2\kappa + \delta |\nu|} \quad \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}_0 \quad (1.2)
\]
for $0 \leq |\nu| \leq \mathcal{N}$ and $0 \leq \kappa \leq \mathcal{K}$, where $\mathcal{N}, \mathcal{K} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $0 \leq \delta \leq 1$, and $g$ is a function that admits some exponential growth. Symbols in the spirit of (1.2) have been considered (albeit in the absence of the growth function $g$) in [6,19] for pseudo-multipliers associated with the Hermite operator and for pseudo-differential operators associated with non-negative self-adjoint operators, respectively. These works prove boundedness of such operators in corresponding Besov and Triebel–Lizorkin spaces, but only with positive smoothness and indices $1 < p < \infty$ and $1 < q < \infty$. They also do not address the endpoint $\delta = 1$ (which is analogous to the so-called “forbidden class” in the classical setting). In contrast, we are able to show that condition (1.2) alone with $0 \leq \delta < 1$, or condition (1.2) with $\delta = 1$ along with newly introduced cancellation conditions on $\sigma$ (see Definition 4.2) lead to boundedness properties of
From $B_{\alpha+m}^{p,q}(\mathcal{L})$ to $B_{\alpha}^{p,q}(\mathcal{L})$ and from $F_{\alpha+m}^{p,q}(\mathcal{L})$ to $F_{\alpha}^{p,q}(\mathcal{L})$ for $0 < p < \infty$, $0 < q < \infty$ and ranges of the smoothness parameter $\alpha$ that allow for negative values (see Sect. 4 for more results and details). It is also worth mentioning that while the smoothness and decay conditions that we impose on the symbols in (1.2) have some similarities with those in [6,19], our symbols admit some further exponential growth as described above. Moreover, the techniques used in [6,19] are different from ours and rely on decompositions of the symbols in elementary pieces.

Inspired by the works [3,27], our results on boundedness properties of pseudo-multipliers also imply that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities. More precisely, we prove that if $f$ is a real-valued function in $B_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ (or, $F_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$), then $H(f) \in B_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ (respectively, $H(f) \in F_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$), where $H$ is an infinitely differentiable function defined on $\mathbb{R}$ that satisfies $H(0) = 0$. In particular, the spaces $B_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ and $F_{p,q}^{\alpha}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ turn out to be closed under pointwise multiplication. We refer the reader to Sect. 5.4 for more details and conditions on the parameters.

The organization of the article is as follows. In Sect. 2, we introduce notation and present background material related to Hermite functions, Hermite tiles and Besov and Triebel–Lizorkin spaces in the Hermite setting. In Sect. 3, we introduce a novel notion of molecules associated to such spaces and prove new results concerning almost orthogonality (Lemma 3.4), molecular decomposition and synthesis (Theorems 3.5 and 3.6). In Sect. 4, we pursue the study of boundedness properties in Hermite Besov and Hermite Triebel–Lizorkin spaces for pseudo-multipliers with symbols in Hörmander-type classes adapted to the Hermite setting (Theorems 4.3, 4.4, 4.5 and 4.6 and Corollary 4.7). In Sect. 5, we present examples and applications of the theorems proved in Sect. 4; in particular, we discuss boundedness results for Hermite pseudo-multipliers in Lebesgue and Hermite local Hardy spaces (Corollaries 5.4, 5.6, 5.7 and 5.9) and show that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities (Theorem 5.12 and Corollary 5.13). Finally, the appendices contain the proofs of some technical lemmas.

**Notation:** We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $|x|_\infty = \max_{1 \leq i \leq n} |x_i|$. The notation $Q(x_Q, r_Q) = \{x \in \mathbb{R}^n : |x - x_Q|_\infty < r_Q\}$ denotes the cube centered at $x_Q \in \mathbb{R}^n$ of side-length $2r_Q$. The smallest integer greater than $\alpha \in \mathbb{R}$ is denoted by $\lceil \alpha \rceil$, while the largest integer not exceeding $\alpha$ is denoted $\lfloor \alpha \rfloor$. We also set $\alpha_+ = \max\{0, \alpha\}$ and $\alpha^* = \alpha - \lfloor \alpha \rfloor$. Given real numbers $a$ and $b$, we set $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

We make use of the following multi-index conventions. If $\alpha, \beta \in \mathbb{N}_0^n$ are multi-indices, then $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for each $i \in \{1, \ldots, n\}$, $D_\alpha = (\partial_1)^{\alpha_1} (\partial_2)^{\alpha_2} \cdots (\partial_n)^{\alpha_n}$, $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$.

For $i \in \{1, \ldots, n\}$ we set $A_i^{(k)} = -\frac{\partial}{\partial x_i} + x_i$. When the variable under consideration is clear we just write $A_i$. If $\alpha \in \mathbb{N}_0^n$, we define $A^{\alpha} = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}$.

We denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. The letter $n$ will always mean Euclidean dimension.
For a locally integrable function $f$ and measurable set $E \subset \mathbb{R}^n$ also use the notation $\int_E f = \frac{1}{|E|} \int f$ to denote the average of $f$ over $E$.

2 Preliminaries

In this section we introduce notation and present background material related to Hermite functions, Hermite tiles, and Besov and Triebel–Lizorkin spaces in the Hermite setting.

For each $k \in \mathbb{N}_0$, the Hermite function of degree $k$ is

$$h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2} \quad \forall \ t \in \mathbb{R},$$

where

$$H_k(t) = (-1)^k e^{t^2} \partial_t^k (e^{-t^2})$$

is the $k$-th Hermite polynomial.

The $n$-dimensional Hermite functions $h_\xi$ are defined over the multi-indices $\xi \in \mathbb{N}_0^n$ by

$$h_\xi(x) = \prod_{j=1}^n h_{\xi_j}(x_j) \quad \forall x \in \mathbb{R}^n.$$ 

The Hermite functions are eigenfunctions of $\mathcal{L}$ in the sense that

$$\mathcal{L}(h_\xi) = \lambda_{|\xi|} h_\xi,$$

where $\lambda_k = 2k + n$. Furthermore, they form an orthonormal basis for $L^2(\mathbb{R}^n)$.

Let $W_k = \text{span}\{h_\xi : |\xi| = k\}$ and $V_N = \bigoplus_{k=0}^N W_k$. We define the orthogonal projection of $f \in L^2(\mathbb{R}^n)$ onto $W_k$ by

$$\mathbb{P}_k f = \sum_{|\xi| = k} \langle f, h_\xi \rangle h_\xi \quad \text{with kernel} \quad \mathbb{P}_k(x, y) = \sum_{|\xi| = k} h_\xi(x) h_\xi(y).$$

We also define the orthogonal projection of $f$ onto $V_N$ by

$$\mathbb{Q}_N f = \sum_{k=0}^N \mathbb{P}_k f \quad \text{with kernel} \quad \mathbb{Q}_N(x, y) = \sum_{k=0}^N \sum_{|\xi| = k} h_\xi(x) h_\xi(y).$$

The following bounds are known (see [30, p.376]): there exists $\vartheta > 0$ such that for any $N \in \mathbb{N}$
We will use the following function throughout:

\[
e_N(x) = \begin{cases} 
1 & \text{if } |x| < \sqrt{N}, \\
e^{-\vartheta|x|^2} & \text{if } |x| \geq \sqrt{N}.
\end{cases}
\] (2.2)

It follows from (2.1) that for any \( \varepsilon > 4 \) and \( N \in \mathbb{N} \), we have

\[
\mathbb{Q}_{4j+N}(x, x) \lesssim 2^{jn}(e_{\varepsilon 4^j}(x))^2 \quad \forall \ j \in \mathbb{N},
\] (2.3)

where the implicit constant depends only on \( N, n, \varepsilon \) and \( \vartheta \).

### 2.1 Hermite Tiles

In our setting, the notion of Hermite tiles replaces that of dyadic cubes. We briefly present such a concept in this section; further details can be found in [9,30].

Fix a positive structural constant \( \delta^* \) small enough (\( 0 < \delta^* < \frac{1}{37} \) suffices) and, for each \( j \in \mathbb{N}_0 \), consider the collection \( \mathcal{X}_j \) of nodes defined as the set of \( n \)-tuples of zeros of the Hermite polynomial \( H_{2N_j} \) with

\[
N_j = \lfloor (1 + 11\delta^*)(\frac{4}{\pi})^24^j \rfloor + 3.
\]

To each node \( \zeta \in \mathcal{X}_j \), we associate a tile \( R_\zeta \) with sides parallel to the axes, so that each such tile contains precisely one node and any two different tiles with nodes in \( \mathcal{X}_j \) have disjoint interiors. The tiles are approximately cubes along the diagonals of \( \mathbb{R}^n \), and are rectangular boxes off the diagonal (see Lemma 2.1 below). Given any tile \( R \) we denote its node by \( x_R \).

We set \( \mathcal{E}_j = \{ R_\zeta \}_{\zeta \in \mathcal{X}_j} \) (i.e. \( \mathcal{E}_j \) is the collection of all \( j \)th level tiles) and we define

\[
\mathcal{E} = \bigcup_{j \geq 0} \mathcal{E}_j.
\]

Note that, by construction, \( \mathcal{E}_j \) contains approximately \( 4^{jn} \) tiles. Our article relies on the following properties of these tiles (see [30]), which holds if \( \delta^* \) is chosen small enough:

**Lemma 2.1** There exist constants \( c_0, c_1, c_2, c_3 \) and \( c_4 \) depending only on \( \delta^* \) and \( n \), such that for each \( j \in \mathbb{N}_0 \) and each tile \( R \in \mathcal{E}_j \) the following properties hold.

(a) If \( |x_R| \leq (1 + 4\delta^*)2^{j+1} \), it holds that

\[
R \subset Q(x_R, c_0 2^{-j}).
\]
(b) In general, it holds that
\[ Q(x_R, c_1 2^{-j}) \subset R \subset Q(x_R, c_2 2^{-j/3}). \]

(c) Set \( Q_j = \bigcup_{P \in \mathcal{E}_j} P = Q(0, \xi_{N_j} + 2^{-j/6}); \) it holds that
\[ Q(0, 2^j) \subset Q_j \subset Q(0, c_3 2^j). \]

(d) \( R \) can be subdivided into a disjoint union of subcubes with sidelength roughly equal to \( 2^{-j}; \) more precisely, each such subcube \( Q \) satisfies
\[ Q(x_Q, c_4 2^{-j-1}) \subset Q \subset Q(x_Q, c_4 2^{-j}). \]

Denoting by \( \widehat{\mathcal{E}}_j \) the collection of all subcubes corresponding to tiles in \( \mathcal{E}_j, \) it holds that
\[ \bigcup_{Q \in \widehat{\mathcal{E}}_j} Q = Q_j. \tag{2.4} \]

By considering the two types of tiles listed in Lemma 2.1 and using (2.2), the following estimate is obtained with \( \varepsilon = 4(1 + 4\delta_*)^2 \) and \( s > 0: \)
\[ |R|^{1/2} 2^{jn} (e_{\varepsilon 4^j}(x_R))^s \lesssim |R|^{-1/2} \quad \forall R \in \mathcal{E}_j, j \in \mathbb{N}_0. \tag{2.5} \]

Indeed, if \( |x_R| \leq (1 + 4\delta_*) 2^{j+1} \) then \( |R| \sim 2^{-jn} \) and (2.5) follows. On the other hand, if \( |x_R| > (1 + 4\delta_*) 2^{j+1} \) then (2.5) follows from the fact that \( 2^{-jn} \lesssim |R| \lesssim 2^{-jn/3} \) and the bound \( e_{\varepsilon 4^j}(x_R)^s \lesssim 2^{-j\beta} \) with \( \beta > 0. \)

We next observe, for later use, that if \( \beta \geq 0 \) then it holds that
\[ e_{\varepsilon 4^j}(x) \lesssim \left( 1 + \frac{|x|}{2^j} \right)^{-\beta} \quad \forall x \in \mathbb{R}^n, j \in \mathbb{N}_0, \tag{2.6} \]

where the implicit constant depends on \( \varepsilon \) and \( \beta. \) Indeed, if \( |x| \leq \sqrt{\varepsilon} 2^j \) then, for any \( \beta \geq 0, \) we have
\[ e_{\varepsilon 4^j}(x) = 1 \leq (1 + \sqrt{\varepsilon})^\beta \left( 1 + \frac{|x|}{2^j} \right)^{-\beta} \quad \forall x \in \mathbb{R}^n, \forall j \in \mathbb{N}_0. \]

On the other hand, if \( |x| > \sqrt{\varepsilon} 2^j \) then \( 2(1 + |x|) > 2^j + |x| \) since \( \varepsilon > 4; \) hence, for any \( \beta \geq 0, \) we obtain
\[ e_{\varepsilon 4^j}(x) = e^{-\beta |x|^2} \leq C_\beta (1 + |x|)^{-\beta} \leq C_\beta 2^\beta (2^j + |x|)^{-\beta} \quad \forall x \in \mathbb{R}^n, \forall j \in \mathbb{N}_0, \]

which gives (2.6) since \( 2^j \beta \geq 1. \)

The following estimates will be employed throughout the paper.
Lemma 2.2  Let \( \phi \) be a smooth function defined in \([0, \infty)\), set \( \phi_j(x) = \phi(2^{-j}x) \) for \( j \in \mathbb{N}_0 \) and consider \( \ell, N \in \mathbb{N}, N > \ell \).

(a) If \( \phi^{(m)}(0) = 0 \) for all \( m \in \mathbb{N} \), it holds that
\[
|\Delta^\ell_k(\phi_j(\sqrt{\lambda_k}))| \lesssim \|\phi^{(N)}\|_{L^\infty} 2^{-jN} \lambda_k^{-N/2-\ell} \quad \forall j, k \in \mathbb{N}_0,
\]
where the implicit constant depends on \( N \) and \( \ell \).

(b) If \( \phi^{(m)}(0) \neq 0 \) for some \( m \in \mathbb{N} \), it holds that
\[
|\Delta^\ell_k(\phi_j(\sqrt{\lambda_k}))| \lesssim \max_{1 \leq s \leq N} \{\|\phi^{(s)}\|_{L^\infty}\} \lambda_k^{-\ell/2} \quad \forall j, k \in \mathbb{N}_0,
\]
where the implicit constant depends on \( N \) and \( \ell \).

Proof  By the mean value theorem for finite differences and Hoppe's chain rule we have
\[
|\Delta^\ell_k(\phi_j(\sqrt{\lambda_k}))| = |\partial^\ell(\phi_j(\sqrt{\lambda_k}))| = \left|\sum_{r=1}^{\ell} c_r \phi^{(r)}(2^{-j}\sqrt{\lambda_k}) \lambda_k^{r/2-\ell} 2^{-jr}\right| \quad (2.7)
\]
for some \( k \leq \nu \leq k + \ell \) and appropriate constants \( c_r \).

To prove (a), let \( \bar{x} \) be between 0 and \( x \) and such that
\[
\phi^{(r)}(x) = \sum_{s=0}^{N-r-1} \frac{\phi^{(r+k)}(0)}{s!} x^s + \frac{\phi^{(N)}(\bar{x})}{(N-r)!} x^{N-r} = \frac{\phi^{(N)}(\bar{x})}{(N-r)!} x^{N-r}, \quad r \in \mathbb{N}.
\]
This leads to \( |\phi^{(r)}(x)| \lesssim \|\phi^{(N)}\|_{L^\infty} |x|^{N-r} \) and therefore, by (2.7), we obtain
\[
|\Delta^\ell_k(\phi_j(\sqrt{\lambda_k}))| \lesssim \|\phi^{(N)}\|_{L^\infty} \sum_{r=1}^{\ell} |c_r| 2^{-j} \lambda_k^{r/2-\ell} 2^{-jr}
\]
\[
\lesssim \|\phi^{(N)}\|_{L^\infty} 2^{-jN} \lambda_k^{-N/2-\ell}.
\]
For part (b), (2.7) gives
\[
|\Delta^\ell_k(\phi_j(\sqrt{\lambda_k}))| \lesssim \max_{1 \leq s \leq \ell} \{\|\phi^{(s)}\|_{L^\infty}\} \lambda_k^{-\ell/2} \leq \max_{1 \leq s \leq N} \{\|\phi^{(s)}\|_{L^\infty}\} \lambda_k^{-\ell/2}.
\]

2.2 Hermite Besov and Hermite Triebel–Lizorkin Spaces

We begin this section by introducing some notation that will be used in the definition of the Hermite Besov and Hermite Triebel–Lizorkin spaces.
Definition 2.3 (Admissible functions) We say that \((\varphi_0, \varphi)\) is an admissible pair if 
\(\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)\) and for some constants \(b_0 > 0, 0 < b_1 < 1\) and \(1/4 < b_2 < b_3 < 1\),
\[
\text{supp } \varphi_0 \subset [0, 1], \quad |\varphi_0| > b_0 \quad \text{on } [0, b_1], \quad \varphi_0^{(m)}(0) = 0 \quad \forall m \in \mathbb{N},
\]
\[
\text{supp } \varphi \subset \left[\frac{1}{4}, 1\right], \quad |\varphi| > b_0 \quad \text{on } [b_2, b_3].
\]

Given an admissible pair \((\varphi_0, \varphi)\), we set \(\varphi_j(\lambda) = \varphi(2^{-j}\lambda)\) if \(j \in \mathbb{N}, \lambda \in \mathbb{R}_+\) and call the resulting collection \(\{\varphi_j\}_{j \in \mathbb{N}_0}\) an admissible system.

Recall that the Hermite functions \(h_\xi\) with \(\xi \in \mathbb{N}_0^n\) are members of \(\mathcal{S}(\mathbb{R}^n)\). Then for an admissible system \(\{\varphi_j\}_{j \in \mathbb{N}_0}\) we may define the operators \(\varphi_j(\sqrt{\lambda})\) on \(\mathcal{S}'(\mathbb{R}^n)\) by
\[
\varphi_j(\sqrt{\lambda}) f(x) = \sum_{k \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_k}) \mathbb{P}_k(f)(x) \quad \forall f \in \mathcal{S}'(\mathbb{R}^n), x \in \mathbb{R}^n,
\]
where \((f, \phi) = f(\phi)\) for \(f \in \mathcal{S}'(\mathbb{R}^n)\) in the expression for \(\mathbb{P}_k(f)\). The kernels of the operators \(\varphi_j(\sqrt{\lambda})\) are given by
\[
\varphi_j(\sqrt{\lambda}) (x, y) = \sum_{k \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_k}) \mathbb{P}_k(x, y) = \sum_{k \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_k}) \sum_{|\xi| = k} h_\xi(x) h_\xi(y).
\]

Denote by \(\{I_j\}_{j \in \mathbb{N}_0}\) the following subsets of \(\mathbb{N}_0\): \(I_j = \left[\frac{1}{2}4^{j-2} - \frac{3}{2}, \frac{1}{2}4^{j-2} - \frac{1}{2}\right]\cap \mathbb{N}_0\) for \(j \in \mathbb{N}, I_0 = \{0\}\) if \(n = 1\) and \(I_0 = \emptyset\) if \(n \geq 2\). In view of the support of \(\varphi_j\), it follows that
\[
\varphi_j(\sqrt{\lambda}) f(x) = \sum_{k \in I_j} \varphi_j(\sqrt{\lambda_k}) \mathbb{P}_k(f)(x).
\]

We note that the kernels \(\varphi_j(\sqrt{\lambda})(x, y)\) satisfy some useful smoothness and cancellation estimates that will play an important throughout this paper. These estimates can be found in Lemma A.1 in the Appendix.

We are now ready to define the Hermite Besov and Hermite Triebel–Lizorkin spaces.

Definition 2.4 (Hermite distribution spaces) Let \(\alpha \in \mathbb{R}\) and \(0 < q \leq \infty\). For \(0 < p \leq \infty\), we define the Hermite Besov space \(B^{p,q}_\alpha = B^{p,q}_\alpha(\mathcal{L})\) as the class of tempered distributions \(f \in \mathcal{S}'(\mathbb{R}^n)\) such that
\[
\|f\|_{B^{p,q}_\alpha} = \left( \sum_{j \in \mathbb{N}_0} (2^{j\alpha} \|\varphi_j(\sqrt{\lambda}) f\|_{L^p})^q \right)^{1/q} < \infty;
\]
for $0 < p < \infty$, we define the Hermite Triebel–Lizorkin space $F^p_q = F^p_q(L)$ as the class of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
\[
\|f\|_{F^p_q} = \left\| \left( \sum_{j \in \mathbb{N}_0} \left( 2^{jaq} |\varphi_j(\sqrt{L})f| \right)^q \right)^{1/q} \right\|_{L^p} < \infty.
\]

**Definition 2.5 (Hermite sequence spaces)** Let $\alpha \in \mathbb{R}$ and $0 < q \leq \infty$. For $0 < p \leq \infty$, we define the Hermite Besov sequence space $b^p_q = b^p_q(L)$ as the set of all sequences of complex numbers $s = \{s_R\}_{R \in \mathcal{E}}$ such that
\[
|s|_{b^p_q} = \left\{ \sum_{j \in \mathbb{N}_0} 2^{jaq} \left( \sum_{R \in \mathcal{E}_j} (|R|^{1/p-1/2} |s_R|)^p \right)^{q/p} \right\}^{1/q} < \infty;
\]
for $0 < p < \infty$, we define the Hermite Triebel–Lizorkin sequence space $f^p_q = f^p_q(L)$ as the set of all sequences of complex numbers $s = \{s_R\}_{R \in \mathcal{E}}$ such that
\[
|s|_{f^p_q} = \left\| \left( \sum_{j \in \mathbb{N}_0} 2^{jaq} \sum_{R \in \mathcal{E}_j} (1_R(\cdot)|R|^{-1/2} |s_R|)^q \right)^{1/q} \right\|_{L^p} < \infty.
\]

We will use the notation $A^p_q(L)$ (or $A^p_q$) to refer to $B^p_q(L)$ or $F^p_q(L)$, with the understanding that $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq \infty$ if $A = B$ and $0 < p < \infty$ if $A = F$. An analogous comment applies to the sequence spaces, denoted by $a^p_q(L)$ (or $a^p_q$).

The spaces $A^p_q(L)$ are independent of the choice of $(\varphi_0, \varphi)$ (see [30, Theorems 3 and 5] and also the earlier works [11,13] for Triebel–Lizorkin spaces). Moreover, $A^p_q(L)$ are quasi-Banach spaces continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$ (see [30, Proposition 4 and p.392]) and have $\mathcal{S}(\mathbb{R}^n)$ as a dense subspace for finite values of $p$ and $q$. In addition, as shown in [30], the spaces $A^p_q(L)$ are in general different from the classical Triebel–Lizorkin and Besov spaces associated to the Laplacian operator. On the other hand, it holds that $F^p_0(L) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with equivalent norms.

We adopt the notation
\[
n_{p,q} = \begin{cases} 
\min\{n, \frac{1}{p}\} \quad &\text{for } F^p_q(L), \\
\min\{n, \frac{1}{p}\} \quad &\text{for } B^p_q(L). 
\end{cases}
\]

### 2.2.1 Frame Decompositions

The construction of frames for Hermite Besov and Hermite Triebel–Lizorkin spaces given in [30] relies on a certain cubature formula for functions in $V_N$. Before presenting the cubature formula we introduce the function
\[
\tau(N, x) = \frac{1}{Q_N(x, x)} \quad \forall x \in \mathbb{R}, \; N \in \mathbb{N}_0,
\]
which is the well known Christoffel function. It has certain useful asymptotic properties which are listed in [30, p.376].

The following cubature formula [30, Corollary 2]

\[
\int_{\mathbb{R}^n} f(x) g(x) \, dx \sim \sum_{\zeta \in X_j} \tau_\zeta f(\zeta) g(\zeta), \quad \zeta = (\zeta_{a_1}, \ldots, \zeta_{a_n}),
\]

\[\tau_\zeta = \prod_{k=1}^n \tau(2N_j, \zeta_{a_k}), \tag{2.8}\]

is exact for all \( f \in V_k \) and \( g \in V_\ell \) with \( k + \ell \leq 4N_j - 1 \).

If \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) is an admissible system, for each tile \( R \in \mathcal{E}_j \) we set

\[
\varphi_R(x) = \tau_R^{1/2} \varphi_j(\sqrt{L})(x, x_R),
\]

where \( x_R \) is the node of \( R \) and \( \tau_R = \tau_{x_R} \) is the coefficient in the cubature formula (2.8). It holds that \( \tau_R \sim |R| \) for any tile \( R \) (see [30, (2.33)]). Following the convention introduced in [30], we refer to the functions \( \varphi_R \) as needlets.

Given admissible systems \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) and \( \{\psi_j\}_{j \in \mathbb{N}_0} \) we define the analysis operator \( S_\psi \) and synthesis operator \( T_\psi \) by

\[
S_\psi : f \mapsto \{\langle f, \varphi_R \rangle\}_{R \in \mathcal{E}} \quad \text{and} \quad T_\psi : \{s_R\}_{R \in \mathcal{E}} \mapsto \sum_{R \in \mathcal{E}} s_R \psi_R.
\]

The following frame decompositions were proved in [30] (see also [9]).

**Theorem 2.6** (Frame decomposition) Let \( \alpha \in \mathbb{R}, 0 < q \leq \infty, \) and \( 0 < p < \infty \) if \( A^{p,q}_a(\mathcal{L}) = F^{p,q}_a(\mathcal{L}) \) or \( 0 < p \leq \infty \) if \( A^{p,q}_a(\mathcal{L}) = B^{p,q}_a(\mathcal{L}) \). Then,

(a) the operator \( T_\psi : a^{p,q}_a(\mathcal{L}) \to A^{p,q}_a(\mathcal{L}) \) is bounded;

(b) the operator \( S_\psi : A^{p,q}_a(\mathcal{L}) \to a^{p,q}_a(\mathcal{L}) \) is bounded;

(c) if \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) and \( \{\psi_j\}_{j \in \mathbb{N}_0} \) satisfy

\[
\sum_{j \geq 0} \psi_j(\lambda)\varphi_j(\lambda) = 1 \quad \forall \lambda \geq 0, \tag{2.9}\]

then \( T_\psi \circ S_\psi = I \) on \( A^{p,q}_a(\mathcal{L}) \) (with convergence in \( \mathcal{S}'(\mathbb{R}^n) \) – [30, Proposition 3].)

### 3 Molecules in the Hermite Setting

In this section, we introduce a novel notion of molecules associated to the Hermite Besov and Hermite Triebel–Lizorkin spaces and prove new results concerning almost orthogonality (Lemma 3.4 in Sect. 3.1), molecular decomposition and synthesis (Theorem 3.5 in Sect. 3.2 and Theorem 3.6 in Sect. 3.3). Lemma 3.4 is used in the proof of Theorem 3.6, which is a crucial tool in the proof of the results of Sect. 4.
3.1 Molecules and Almost Orthogonality Results

We start this section with our new definition of molecule associated to the Hermite setting.

**Definition 3.1 (Smooth molecules)** Let $(M, \theta) \in \{N_0 \times (0, 1) \cup \{(-1, 1)\}, N \in N_0, 0 \leq \delta \leq 1$ and $\mu \geq 1$. A function $m \in C^N(\mathbb{R}^n)$ is said to be an $(M, \theta, N, \delta, \mu)$-molecule for $L$ associated with a tile $R \in \mathcal{E}_j$ for some $j \in N_0$ if

(i) for each multi-index $\gamma$ with $0 \leq |\gamma| \leq N$ we have

$$|\partial^\gamma m(x)| \leq \frac{|R|^{-1/2} \gamma_j}{(1 + 2^j |x - x_R|)^{\mu}} \frac{1}{\left(1 + \frac{|x|}{2^j}\right)^{N+\delta}} \quad \forall x \in \mathbb{R}^n,$$

(ii) for each multi-index $\gamma$ with $|\gamma| = N$ we have

$$|\partial^\gamma m(x) - \partial^\gamma m(y)| \leq |R|^{-1/2} \gamma_j \delta \frac{1}{(1 + 2^j |x - x_R|)^{\mu}}$$

for every $x, y \in \mathbb{R}^n$ with $|x - y| \leq 2^{-j}$.

(iii) for each multi-index $\gamma$ with $0 \leq |\gamma| \leq M$ we have

$$\left| \int_{\mathbb{R}^n} (y - x_R)^\gamma m(y) dy \right| \leq |R|^{-1/2} 2^{-j(n + |\gamma|)} \left(\frac{1 + |x_R|}{2^j}\right)^{M + \theta - |\gamma|}.$$

If $(M, \theta) = (-1, 1)$, part (iii) is taken to be void.

**Remark 3.2** The following calculations show that if $m$ satisfies (i) for some $N \in \mathbb{N}$, then $m$ also satisfies (ii) (modulo a constant) for $N - 1$, any $0 \leq \delta \leq 1$ and the same value of $\mu$.

If $|x - y| \leq 2^{-j}$ then $(2^j |x - y|) \leq (2^j |x - y|)^{\delta}$ for any $\delta \in [0, 1]$. By the mean value theorem and part (i) we have, for some $\tilde{x}$ between $x$ and $y$,

$$|\partial^\gamma m(x) - \partial^\gamma m(y)| \leq \sum_{|\beta|=N} |\partial^\beta m(\tilde{x})||x - y|$$

$$\leq |R|^{-1/2} 2^j |x - y| \left(1 + 2^j |	ilde{x} - x_R|\right)^{-\mu}$$

$$= |R|^{-1/2} 2^j |x - y| \left(\frac{|x - y|}{2^{-j}}\right) \left(1 + 2^j |	ilde{x} - x_R|\right)^{\delta} \left(1 + 2^j |	ilde{x} - x_R|\right)^{-\mu}$$

$$\leq |R|^{-1/2} 2^j |x - y| \left(\frac{|x - y|}{2^{-j}}\right)^{\delta} \left(1 + 2^j |	ilde{x} - x_R|\right)^{-\mu}.$$

We may then conclude our estimate after observing that the triangle inequality, along with the fact that $|x - \tilde{x}| \leq |x - y| \leq 2^{-j}$, yields

$$\frac{1}{(1 + 2^j |	ilde{x} - x_R|)^{\mu}} \leq \left(\frac{1 + 2^j |x - \tilde{x}|}{1 + 2^j |x - x_R|}\right)^{\mu} \leq \frac{2^\mu}{(1 + 2^j |x - x_R|)^{\mu}}.$$
The following lemma presents a first example of molecules.

**Lemma 3.3** (Needlets are molecules) If \( \{\varphi_j\}_{j \geq 0} \) is an admissible system, the functions \( \{\varphi_R\}_{R \in \mathcal{E}} \) are multiples of \((M, \theta, N, \delta, \mu)\)-molecules for any \((M, \theta) \in \{N_0 \times (0, 1)\} \cup \{(-1, 1)\}, N \in \mathbb{N}_0, 0 \leq \delta \leq 1 \) and \( \mu \geq 1 \).

**Proof of Lemma 3.3** We will use Lemma A.1 to show that \( \varphi_R \) is a constant multiple of a molecule. The estimate in part (i) of Definition 3.1, with a uniform constant in \( R \) and \( j \), follows for all \( \gamma \in \mathbb{N}_0 \) from (5.11) and estimates (2.5) and (2.6). The estimate in part (ii) of Definition 3.1 follows, with a uniform constant in \( R \) and \( j \), from Remark 3.2. Part (iii) of Definition 3.1 follows, with a uniform constant in \( R \) and \( j \), from (5.12) and (2.5) by choosing \( K = M + \theta \).

The following almost orthogonality result will be useful in the proof of Theorem 3.6 and is the main result of this subsection.

**Lemma 3.4** (Almost Orthogonality) Let \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) be an admissible system and \( \{m_R\}_{R \in \mathcal{E}} \) be a collection of \((M, \theta, N, \delta, \mu)\)-molecules for some \((M, \theta) \in \{N_0 \times (0, 1)\} \cup \{(-1, 1)\}, N \in \mathbb{N}_0, 0 \leq \delta \leq 1 \) and \( \mu > \max\{\eta, n + M + \theta\} \) for some \( \eta > 0 \). Then it holds that

\[
|\varphi_j(\sqrt{L})m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^{\eta}} 2^{-(n+M+\theta)[(k-j)\vee 0]-(N+\delta)[(j-k)\vee 0]} \tag{3.1}
\]

for all \( x \in \mathbb{R}^n, j, k \in \mathbb{N}_0 \) and \( R \in \mathcal{E}_k \).

**Proof of Lemma 3.4** We will use the estimates for \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) from Lemma A.1 with decay \( \eta_0 > \eta + n + N + \delta \) and \( |\gamma| \leq M + 1 \) (in estimate (5.11)), and \( K = N + \delta \) (in estimate (5.12)). We will prove each of the following four cases with constants independent of \( x, j, k \) and \( R \):

1a) \( j \leq k, (M, \theta) = (-1, 1), (N, \delta) \in \mathbb{N}_0 \times \{0, 1\}, \mu > \max\{\eta, n\} \)

\[
|\varphi_j(\sqrt{L})m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^{\eta}} 2^{-(k-j)n} \tag{3.2}
\]

1b) \( j \leq k, (M, \theta) \in \mathbb{N}_0 \times (0, 1), (N, \delta) \in \mathbb{N}_0 \times \{0, 1\}, \mu > \max\{\eta, n + M + \theta\} \)

\[
|\varphi_j(\sqrt{L})m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^{\eta}} 2^{-(k-j)[n+M+\theta]} \tag{3.3}
\]

2a) \( j > k, (M, \theta) \in \{N_0 \times (0, 1)\} \cup \{(-1, 1)\}, (N, \delta) = (0, 0), \mu \geq \eta \)

\[
|\varphi_j(\sqrt{L})m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^k |x - x_R|)^{\eta}} \tag{3.4}
\]

2b) \( j > k, (M, \theta) \in \{N_0 \times (0, 1)\} \cup \{(-1, 1)\}, (N, \delta) \in \mathbb{N}_0 \times \{0, 1\}, (N, \delta) \neq (0, 0), \mu \geq \eta \)

\[
|\varphi_j(\sqrt{L})m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^k |x - x_R|)^{\eta}} 2^{-(j-k)(N+\delta)} \tag{3.5}
\]
Estimate (3.1) will then follow by combining each of the above cases appropriately. Fix \( x \) and \( R \); to handle Cases 1a and 1b, it will be useful to divide \( \mathbb{R}^n \) into the following regions:

\[
\begin{align*}
\Omega_1 &= \{ y \in \mathbb{R}^n : |y - x_R| \leq 2^{-j} \}, \\
\Omega_2 &= \{ y \in \mathbb{R}^n : |y - x_R| > 2^{-j} \text{ and } |y - x| \leq \frac{1}{2} |x - x_R| \}, \\
\Omega_3 &= \{ y \in \mathbb{R}^n : |y - x_R| > 2^{-j} \text{ and } |y - x| > \frac{1}{2} |x - x_R| \}.
\end{align*}
\]

The following two observations will be useful in the sequel:

\[
1 + 2^j |x - x_R| \leq 2(1 + 2^j |x - y|) \quad \forall y \in \Omega_1 \cup \Omega_3, \tag{3.6}
\]

and

\[
2^{(k-j)}(1 + 2^j |x - x_R|) \leq 3 \cdot 2^k |y - x_R| \quad \forall y \in \Omega_2. \tag{3.7}
\]

Case 1a: We have

\[
|\varphi_j(\sqrt{L})m_R(x)| \leq \int_{\Omega_1 \cup \Omega_3} |\varphi_j(\sqrt{L})(x, y)| |m_R(y)| \, dy \\
+ \int_{\Omega_2} |\varphi_j(\sqrt{L})(x, y)| |m_R(y)| \, dy.
\]

From (5.11) with \(|\gamma| = 0\) and \(\eta_0 \geq \eta + n\), Definition 3.1 (i) with \(|\gamma| = 0\), and the inequality (3.6) we obtain

\[
\int_{\Omega_1 \cup \Omega_3} |\varphi_j(\sqrt{L})(x, y)| |m_R(y)| \, dy \\
\lesssim \int_{1+2^j|x-x_R| \leq 1+2^j|y-x|} \frac{2^j n}{(1 + 2^j |x - y|)^\eta_0 (1 + 2^k |y - x_R|)^\mu} \, dy \\
\lesssim \frac{|R|^{-1/2} 2^{-(k-j)n}}{(1 + 2^j |x - x_R|)^\eta}
\]

since \(\mu > n\) and \(\eta_0 \geq \eta\). Proceeding similarly and applying (3.7), we have

\[
\int_{\Omega_2} |\varphi_j(\sqrt{L})(x, y)| |m_R(y)| \, dy \lesssim \int_{\Omega_2} \frac{2^j n}{(1 + 2^j |x - y|)^\eta_0 (1 + 2^k |y - x_R|)^\mu} \, dy \\
\lesssim \frac{2^{-(k-j)n} |R|^{-1/2}}{(1 + 2^j |x - x_R|)^\eta}
\]

since \(j \leq k, \mu \geq \max\{n, \eta\}\) and \(\eta_0 > n\).
Case 1b: We write

\[
\varphi_j(\sqrt{\ell})m_R(x) = \int_{\mathbb{R}^n} \varphi_j(\sqrt{\ell})(x, y) m_R(y) \, dy
\]

\[
= \int_{\mathbb{R}^n} \left[ \varphi_j(\sqrt{\ell})(x, y) - \sum_{|\gamma| \leq M} \frac{1}{\gamma!} \partial_y^\gamma \varphi_j(\sqrt{\ell})(x, x_R) (y - x_R)^\gamma \right] m_R(y) \, dy
\]

\[
+ \sum_{|\gamma| \leq M} \frac{1}{\gamma!} \partial_y^\gamma \varphi_j(\sqrt{\ell})(x, x_R) \int_{\mathbb{R}^n} (y - x_R)^\gamma m_R(y) \, dy
\]

\[=: I + II.\]

To study term \(I\) we further subdivide \(I = I_1 + I_2 + I_3\) where

\[
I_i = \int_{\Omega_i} \left[ \varphi_j(\sqrt{\ell})(x, y) - \sum_{|\gamma| \leq M} \frac{1}{\gamma!} \partial_y^\gamma \varphi_j(\sqrt{\ell})(x, x_R) (y - x_R)^\gamma \right] m_R(y) \, dy.
\]

By Taylor’s theorem we have

\[
I_1 = \sum_{|\gamma| = M+1} \frac{1}{\gamma!} \int_{\Omega_1} \partial_y^\gamma \varphi_j(\sqrt{\ell})(x, \gamma y) (y - x_R)^\gamma m_R(y) \, dy
\]

where \(\gamma y\) lies on the line segment connecting \(y\) and \(x_R\). Then, by (5.11) with \(|\gamma| = M+1\) and Definition 3.1 (i) with \(|\gamma| = 0\), we have

\[
[I_1] \lesssim \int_{\Omega_1} \frac{2^{j(n+M+1)}}{(1 + 2^j |x - \gamma y|)^{\eta_0}} |y - x_R|^{M+1} \frac{|R|^{-1/2}}{(1 + 2^k |y - x_R|)^{\mu}} \, dy
\]

\[
\lesssim \frac{|R|^{-1/2} 2^{j(n+M+1)}}{(1 + 2^j |x - x_R|)^{\eta_0 - 1 - \theta}} \int_{\Omega_1} \frac{1}{(1 + 2^j |y - x_R|)^{1 - \theta}} |y - x_R|^{M+1} \frac{|y - x_R|^{M+\theta}}{(1 + 2^k |y - x_R|)^{\mu}} \, dy
\]

\[
\lesssim \frac{|R|^{-1/2} 2^{j(n+M+\theta)}}{(1 + 2^j |x - x_R|)^{\eta_0 - 1 + \theta}} \int_{\mathbb{R}^n} \frac{1}{(1 + 2^j |y - x_R|)^{1 - \theta}} |y - x_R|^{M+\theta} \frac{|y - x_R|^{M+\theta}}{(1 + 2^k |y - x_R|)^{\mu}} \, dy
\]

\[
\lesssim \frac{|R|^{-1/2} 2^{j(n+M+\theta)}}{(1 + 2^j |x - x_R|)^{\eta}}
\]

since \(\mu > n + M + \theta\) and \(\eta_0 > \eta + n > \eta + 1 - \theta\). In the second step we applied the triangle inequality along with the facts \(|y - x_R| \leq 2^{-j}\) and \(|\gamma y - x_R| \leq 2^{-j}\).

For the second term \(I_2\) we have

\[
|I_2| \leq \int_{\Omega_2} |\varphi_j(\sqrt{\ell})(x, y)||m_R(y)| \, dy
\]

\[
+ \sum_{|\gamma| \leq M} |\partial_y^\gamma \varphi_j(\sqrt{\ell})(x, x_R)| \int_{\Omega_2} |y - x_R|^{\gamma} |m_R(y)| \, dy.
\]
We next apply (5.11) for $|\gamma| \leq M$ and Definition 3.1 (i) with $|\gamma| = 0$ to each integral above. The first integral in $I_2$ can be estimated in a similar way to the integral over $\Omega_2$ in Case 1a to obtain

$$\int_{\Omega_2} |\varphi_j(\sqrt{\sum})(x, y)||m(R(y))| dy \lesssim \frac{2^{-(k-j)\mu}|R|^{-1/2}}{(1 + 2^j|x - x_R|)^\mu} \leq \frac{2^{-(k-j)(n + M + \theta)}|R|^{-1/2}}{(1 + 2^j|x - x_R|)^\eta},$$

since $j \leq k$ and $\mu > \max\{\eta, n + M + \theta\}$. For the second term in $I_2$ we apply (3.7) to obtain, for each $|\gamma| \leq M$,

$$|\partial_r^\gamma \varphi_j(\sqrt{\sum})(x, x_R)| \int_{\Omega_2} |y - x_R|^{\gamma}|m(R(y))| dy \lesssim \frac{2^{j(n + |\gamma|)}|R|^{-1/2}}{(1 + 2^j|x - x_R|)^\eta_0} \int_{\Omega_2} |y - x_R|^{\gamma} dy \lesssim \frac{|R|^{-1/2}2^{j(n + |\gamma|)2^{-(k-j)\mu}}}{(1 + 2^j|x - x_R|)^{\eta_0 + \mu}} \int_{\Omega_2} |y - x_R|^{\gamma} dy \lesssim |R|^{-1/2}2^{-(k-j)\mu} \frac{(2^j|x - x_R|)^{n + M}}{(1 + 2^j|x - x_R|)^{\eta_0 + \mu}} \frac{1}{(1 + 2^j|x - x_R|)^\eta},$$

since $\mu \geq n + M + \theta$ and $\eta_0 \geq \eta$.

For the third term, we apply (5.11) with $|\gamma| \leq M$, Definition 3.1 (i) with $|\gamma| = 0$, and use that $|x - y| \gtrsim |x - x_R|$, $\mu \geq n + M + \theta$ and $\eta_0 \geq \eta$, to obtain

$$|I_3| \lesssim \frac{|R|^{-1/2}2^{jn}}{(1 + 2^j|x - x_R|)^{\eta_0}} \int_{\Omega_3} \left[1 + \sum_{|\gamma| \leq M} (2^j|y - x_R|)^{\gamma} \right] \frac{1}{(1 + 2^k|y - x_R|)^\mu} dy \lesssim \frac{|R|^{-1/2}2^{j(n + M)}}{(1 + 2^j|x - x_R|)^{\eta_0}} \int_{|y - x_R| \geq 2^{-j}} \frac{|y - x_R|^M}{(1 + 2^k|y - x_R|)^\mu} dy \lesssim \frac{|R|^{-1/2}2^{-(k-j)(n + M + \theta)}}{(1 + 2^j|x - x_R|)^\eta}.$$
since $\eta_0 \geq \eta$. By considering separately the cases when $|x_R| \leq \sqrt{\varepsilon}2^j$ and $|x_R| > \sqrt{\varepsilon}2^j$ along with (2.2), it follows that

$$2^{-j(M+\theta-|\gamma|)}(1 + |x_R|)^{M+\theta-|\gamma|}e_{\varepsilon 4^j}(x_R) \leq C_{M,\theta,\theta,\varepsilon}. \quad (3.8)$$

Inserting this bound into the preceding estimate gives

$$|II| \lesssim \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^\eta}.$$ 

Combing the estimates for both terms $I$ and $II$ yields (3.3).

Case 2a: Using (5.11) and Definition 3.1 (i) with $|\gamma| = 0$, $\mu \geq \eta$, the triangle inequality taking into account the $k < j$, and that $\eta_0 > \eta + n$, we obtain

$$\varphi_j(\sqrt{L})m_R(x) \leq \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^\eta} \int \frac{2^{jn}}{(1 + 2^j |y - x|)^{n_0 - n}} dy \lesssim \frac{|R|^{-1/2}}{(1 + 2^j |x - x_R|)^\eta}.$$ 

Case 2b: We argue as in Case 1b, but reverse the roles of $\varphi_j(\sqrt{L})$ and $m_R$. We have

$$\varphi_j(\sqrt{L})m_R(x) = I + II$$ 

where

$$I = \int_{\mathbb{R}^n} \varphi_j(\sqrt{L})(x, y) \left[ m_R(y) - \sum_{|\gamma| < N} \frac{1}{\gamma!} \partial^\gamma m_R(x) (y - x)^\gamma \right] dy$$

$$- \sum_{|\gamma| = N} \frac{1}{\gamma!} \partial^\gamma m_R(x) \int_{\mathbb{R}^n} (y - x)^\gamma \varphi_j(\sqrt{L})(x, y) dy,$$

$$II = \sum_{|\gamma| \leq N} \frac{1}{\gamma!} \partial^\gamma m_R(x) \int_{\mathbb{R}^n} (y - x)^\gamma \varphi_j(\sqrt{L})(x, y) dy.$$ 

By Taylor’s remainder theorem we write, for some $\tilde{y}$ on the line segment between $x$ and $y$,

$$I = \sum_{|\gamma| = N} \frac{1}{\gamma!} \int_{\mathbb{R}^n} \left[ \partial^\gamma m_R(\tilde{y}) - \partial^\gamma m_R(x) \right] (y - x)^\gamma \varphi_j(\sqrt{L})(x, y) dy =: I_1 + I_2$$ 

where

$$I_1 = \sum_{|\gamma| = N} \frac{1}{\gamma!} \int_{|x - y| \leq 2^{-j}} \left[ \partial^\gamma m_R(\tilde{y}) - \partial^\gamma m_R(x) \right] (y - x)^\gamma \varphi_j(\sqrt{L})(x, y) dy$$ 

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and
\[
I_2 = \sum_{|\gamma|=N} \frac{1}{\gamma!} \int_{|x-y|>2^{-k}} \left[ \partial^\gamma m_R(\tilde{y}) - \partial^\gamma m_R(x) \right] (y-x)^\nu \varphi_j(\sqrt{L})(x,y) \, dy.
\]

By setting \(\eta_0 > \eta + N + n + \delta\), we have by (5.11) with \(|\gamma| = 0\) and Definition 3.1 (ii) with \(|\gamma| = N\),
\[
|I_1| \lesssim \int_{|x-y| \leq 2^{-k}} \frac{2^{jn}|R|^{-1/2}2^{k(N+\delta)} |x-\tilde{y}|^\delta |y-x|^N}{(1 + 2^k |x - x_R|)\mu} \frac{1}{(1 + 2^j |x - y|)\eta_0} \, dy.
\]

Taking into account \(\mu \geq \eta\) and \(|\tilde{y} - x| \leq |y - x|\) we have
\[
|I_1| \lesssim \frac{|R|^{-1/2}2^{k(N+\delta)}}{(1 + 2^k |x - x_R|)\eta} \int_{\mathbb{R}^n} \frac{2^{jn}|y-x|^{N+\delta}}{(1 + 2^j |y - x|)\eta_0} \, dy \lesssim \frac{|R|^{-1/2}2^{-(j-k)(N+\delta)}}{(1 + 2^k |x - x_R|)^{\eta}}.
\]

For \(I_2\) we apply instead Definition 3.1 (i) to obtain
\[
|I_2| \leq \sum_{|\gamma|=N} \frac{1}{\gamma!} \int_{|x-y|>2^{-k}} \left[ |\partial^\gamma m_R(\tilde{y})| + |\partial^\gamma m_R(x)| \right] |y-x|^N |\varphi_j(\sqrt{L})(x,y)| \, dy
\]
\[
\lesssim \int_{|x-y|>2^{-k}} \left[ \frac{1}{(1 + 2^k |\tilde{y} - x_R|)\mu} + \frac{1}{(1 + 2^k |x - y|)\eta_0} \right] 
\times \frac{2^{jn}|R|^{-1/2}2^{kN} |y-x|^N}{(1 + 2^k |x - y|)^{\eta_0}} \, dy.
\]

Taking into account that \(j > k\), \(\mu \geq \eta\) and \(|\tilde{y} - x| \leq |y - x|\) and using the triangle inequality we have
\[
\frac{1}{(1 + 2^k |\tilde{y} - x_R|)\mu} \leq \left( \frac{1 + 2^k |\tilde{y} - x|}{1 + 2^k |x - x_R|} \right)^\eta \leq \left( \frac{1 + 2^j |y-x|}{1 + 2^k |x - x_R|} \right)^\eta.
\]

Inserting this inequality into the previous estimate, and using the fact that \((2^k |x - y|)^\delta > 1\), we arrive at
\[
|I_2| \lesssim \frac{|R|^{-1/2}2^{k(N+\delta)}}{(1 + 2^k |x - x_R|)\eta} \int_{|x-y|>2^{-k}} \frac{2^{jn}|y-x|^{N+\delta}}{(1 + 2^j |y - x|)\eta_0 - \eta} \, dy
\]
\[
\lesssim \frac{|R|^{-1/2}2^{-(j-k)(N+\delta)}}{(1 + 2^k |x - x_R|)^{\eta}}.
\]

Let us turn to term \(II\). Here we apply (5.12) with \(K = N + \delta\) and Definition 3.1 (i) with \(0 \leq |\gamma| \leq N\), and use that \(\mu \geq \eta\), to obtain
\[ |II| \lesssim \sum_{|\gamma| \leq N} |R|^{-1/2} 2^k |\gamma| \left( \frac{1 + |x|}{2^k} \right)^{N-\delta} 2^{-j|\gamma|} \left( \frac{1 + |x|}{2^j} \right)^{N+\delta-|\gamma|} \]

\[ = \sum_{|\gamma| \leq N} |R|^{-1/2} 2^{-(j-k)(N+\delta)} 2^{-k(N+\delta-|\gamma|)} (1 + |x|)^{N+\delta-|\gamma|} \left( \frac{1 + |x|}{2^k} \right)^{N-\delta}. \]

Considering \(|x| \leq 2^k\) and \(|x| \geq 2^k\), we see that

\[ 2^{-k(N+\delta-|\gamma|)} (1 + |x|)^{N+\delta-|\gamma|} \left( \frac{1 + |x|}{2^k} \right)^{N-\delta} \lesssim 1, \]

which completes the estimate for term \(II\). In conjunction with the estimate for \(I\) (as encapsulated in the estimates for \(I_1\) and \(I_2\)) we arrive at (3.5).

Thus, we have obtained (3.2)–(3.5), concluding the proof of Lemma 3.4. □

### 3.2 Molecular Decomposition

The molecular decomposition of Besov and Triebel spaces follows from their frame decompositions.

**Theorem 3.5** \((\text{Molecular decomposition})\) Let \(\alpha \in \mathbb{R}, 0 < q \leq \infty, \) and \(0 < p < \infty\) if \(A^{p,q}_\alpha(L) = F^{p,q}_\alpha(L)\) or \(0 < p \leq \infty\) if \(A^{p,q}_\alpha(L) = B^{p,q}_\alpha(L).\) Let \(\mu \geq 1, (M, \theta) \in \{\mathbb{N}_0 \times (0, 1)\} \cup \{(-1, 1)\}, N \in \mathbb{N}_0\) and \(0 \leq \delta \leq 1.\) Then there exists a family of \((M, \theta, N, \delta, \mu)-\)molecules \(\{m_R\}_{R \in \mathcal{E}}\) such that for any \(f \in A^{p,q}_\alpha(L)\) there is a sequence of scalars \(\{s_R\}_{R \in \mathcal{E}}\) satisfying

\[ f = \sum_R s_R m_R \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad (3.9) \]

and

\[ \|s\|_{a^{p,q}_\alpha} \lesssim \|f\|_{A^{p,q}_\alpha}. \quad (3.10) \]

**Proof of Theorem 3.5** Let \(\{\varphi_j\}_{j \in \mathbb{N}_0}\) and \(\{\psi_j\}_{j \in \mathbb{N}_0}\) be admissible systems satisfying (2.9). By the frame decompositions given in Theorem 2.6, we have

\[ f = \sum_{R \in \mathcal{E}} (f, \varphi_R) \psi_R = \sum_{R \in \mathcal{E}} s_R m_R, \]

where \(s_R = c^{-1}(f, \varphi_R)\) and \(m_R = c \psi_R.\) By Lemma 3.3, \(m_R\) is a \((M, \theta, N, \delta, \mu)\)-molecule for some appropriate uniform constant \(c;\) this gives (3.9). Secondly, by part (b) of Theorem 2.6, we have

\[ \|s\|_{a^{p,q}_\alpha} = c^{-1} \| \{(f, \varphi_R)\} \|_{a^{p,q}_\alpha} = c^{-1} \| S\varphi f \|_{a^{p,q}_\alpha} \lesssim \|f\|_{A^{p,q}_\alpha}, \]

which gives (3.10) and concludes our proof. □
3.3 Molecular Synthesis

We next state and prove molecular synthesis estimates. Recall that the notation \( n_{p,q} \) has been defined in Sect. 2.2.

**Theorem 3.6** (Molecular synthesis) Let \( \alpha \in \mathbb{R}, 0 < q \leq \infty, \) and \( 0 < p < \infty \) if \( A_{\alpha}^{p,q}(\mathcal{L}) = F_{\alpha}^{p,q}(\mathcal{L}) \) or \( 0 < p \leq \infty \) if \( A_{\alpha}^{p,q}(\mathcal{L}) = B_{\alpha}^{p,q}(\mathcal{L}) \). Suppose \( \{m_R\}_{R \in \mathcal{E}} \) is a collection of \((M, \theta, N, \delta, \mu)\)-molecules satisfying

(i) \( M \geq \max([n_{p,q} - n - \alpha], -1) \),

(ii) \( \theta = \begin{cases} \max[n_{p,q}^*, (n_{p,q} - \alpha)^*] & \text{if } n_{p,q} - n - \alpha \geq 0, \\ 0 & \text{if } n_{p,q} - n - \alpha < 0, \end{cases} \)

(iii) \( N \geq \max([\alpha], 0) \),

(iv) \( \delta > \alpha^* \) if \( \alpha \geq 0 \), and \( \delta \geq 0 \) if \( \alpha < 0 \),

(v) \( \mu > \max(n_{p,q}, n + M + \theta) \).

Then for any sequence of numbers \( s = \{s_R\}_{R \in \mathcal{E}} \in a_{\alpha}^{p,q}(\mathcal{L}) \), we have

\[
\left\| \sum_{R \in \mathcal{E}} s_R m_R \right\|_{A_{\alpha}^{p,q}} \lesssim \|s\|_{a_{\alpha}^{p,q}}.
\]

**Remark 3.7** (i) Note that our hypotheses on \( M, \theta, N \) and \( \delta \) ensure that

\[
N + \delta > \alpha \quad \text{and} \quad n + M + \theta + \alpha > n_{p,q}. \tag{3.11}
\]

In fact, it can be seen from its proof that Theorem 3.6 holds if conditions (i)-(iv) are replaced by the weaker inequalities in (3.11). This fact will be used in the proofs of our results on Hermite pseudo-multipliers in Sect. 4.

(ii) The following are examples of minimal conditions on \((M, \theta, N, \delta, \mu)\) in Theorem 3.6 for some special cases of \( \alpha, p \) and \( q \).

1. \( \alpha > 0, \min\{p, q\} \geq 1: n_{p,q} = n, M = -1, \theta = 1, N = [\alpha], \delta > \alpha^* \) and \( \mu > n \).
2. \( \alpha = 0, \min\{p, q\} \geq 1: n_{p,q} = n, M = 0, \theta \in (0, 1), N = 0, \delta > 0 \) and \( \mu > n + \theta \).
3. \( \alpha < 0, p, q > 0: M = [n_{p,q} - n - \alpha], \theta > \max\{n_{p,q}^*, (n_{p,q} - \alpha)^*\}, N = 0, \delta = 0, \mu > [n_{p,q} - \alpha] + \theta \).
4. \( \alpha > n_{p,q} - n, p, q > 0: M = -1, \theta = 1, N = [\alpha], \delta > \alpha^* \), \( \mu > n_{p,q} \).
5. \( 0 \leq \alpha < n_{p,q} - n, p, q > 0: M = [n_{p,q} - n - \alpha], \theta > \max\{n_{p,q}^*, (n_{p,q} - \alpha)^*\}, N = [\alpha], \delta > \alpha^* \), \( \mu > [n_{p,q} - n - \alpha] + \theta \).

Examples of spaces corresponding to the cases described include Sobolev type spaces for case 1, \( L^p \) spaces for case 2, and Hardy type spaces for case 5.

The proof of Theorem 3.6 requires certain inequalities involving a maximal operator, which we next present. For each \( s > 0 \) and for a locally integrable function \( f \) on \( \mathbb{R}^n \), we define

\[
\mathcal{M}_s f(x) = \sup_{s \in \mathbb{Q}} \left( \int_{Q} |f(y)|^s \, dy \right)^{1/s} \quad \forall x \in \mathbb{R}^n, \tag{3.12}
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes that contain $x$. The reader may observe that $M_s$ coincides with the usual Hardy–Littlewood maximal operator for $s = 1$. The maximal operator $M_s$ satisfies the following well-known inequality. For the case $s = 1$ one may consult [32], from which the general case follows readily.

**Lemma 3.8** *(Fefferman–Stein inequality)* If $0 < p < \infty$, $0 < q < \infty$ and $0 < s < \min\{p, q\}$, it holds that

$$\left\| \left( \sum_{j \in \mathbb{N}} |M_s(f_j)|^q \right)^{1/q} \right\|_{L^p} \leq C \left\| \left( \sum_{j \in \mathbb{N}} |f_j|^q \right)^{1/q} \right\|_{L^p}$$

for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of locally integrable functions defined on $\mathbb{R}^n$.

We next state and prove a lemma involving sequences of numbers and the maximal operator that is an extension of [30, Lemma 4].

**Lemma 3.9** Let $r > 0$, $\eta > \frac{n}{\min\{1, r\}}$ and $j, k \in \mathbb{N}_0$. Given a sequence of numbers $\{a_R\}_{R \in \mathcal{E}_k}$, set

$$a_k^n(x) = \sum_{R \in \mathcal{E}_k} \frac{|a_R|}{(1 + 2^{j+k}|x-x_R|)^\eta} \quad \forall x \in \mathbb{R}^n.$$ 

Then it holds that

$$a_k^n(x) \lesssim 2^{\frac{n}{1+r}(k-j)^+0} M_r \left( \sum_{R \in \mathcal{E}_k} |a_R| \mathbf{1}_R \right)(x) \quad \forall x \in \mathbb{R}^n,$$

where the implicit constant is independent of $k$ and $j$.

**Proof** For $k \leq j$ one may apply [30, Lemma 4] directly to obtain

$$a_k^n(x) \lesssim M_r \left( \sum_{R \in \mathcal{E}_k} |a_R| \mathbf{1}_R \right)(x).$$

For the case $k > j$, we proceed as in the proof of [30, Lemma 4] with

$$\tilde{a}_k(x) = \sum_{R \in \mathcal{E}_k} \frac{|a_R|}{(1 + 2^j d(x, R))^\eta},$$

where $d(x, R) = \inf_{y \in R} \|x - y\|_\infty$; note that $a_k^n(x) \lesssim \tilde{a}_k(x)$ for all $x \in \mathbb{R}^n$. Let $c_3$ and $c_4$ be the constants from Lemma 2.1. We consider two cases.

Case 1: $|x|_\infty > 2(c_3 + c_4)2^k$.

For each $R \in \mathcal{E}_k$ we have $d(x, R) > |x|_\infty/2$ by part (c) of Lemma 2.1 and the assumption on $|x|_\infty$. Set $v = 1 - \min(1, 1/r)$. Recalling that $\eta > n/\min\{1, r\}$, and
using the fact that \( \# \mathcal{E}_k \sim 4^{kn} \) along with Hölder’s inequality if \( r > 1 \) or the triangle inequality if \( r \leq 1 \), we have

\[
\tilde{a}_k(x) \leq \sum_{R \in \mathcal{E}_k} \frac{|a_R|}{(1 + 2^j d(x, R))^\eta} \lesssim 2^{(k-j)\eta} \left( \frac{2^{-k}}{|x|_\infty} \right)^\eta \sum_{R \in \mathcal{E}_k} |a_R| \lesssim 2^{(k-j)\eta} \left( \frac{2^{-k}}{|x|_\infty} \right)^\eta 4^{kn} \left( \sum_{R \in \mathcal{E}_k} |a_R|^r \right)^{1/r}.
\]

Let \( Q_x = Q(0, 2|x|_\infty) \); then \( x \in Q_x \) and \( \bigcup_{R \in \mathcal{E}_k} R \subseteq Q_x \) by part (c) of Lemma 2.1 and the fact that \( |x|_\infty > c_3 2^k \). Invoking Hölder’s inequality with \( 1/r \) if \( r < 1 \) or the triangle inequality if \( r \geq 1 \), and using that \( |R| \gtrsim 2^{-kn} \) for every tile \( R \in \mathcal{E}_k \) by part (b) of Lemma 2.1, we obtain

\[
\left( \sum_{R \in \mathcal{E}_k} |a_R|^r \right)^{1/r} \lesssim |Q_x|^{1/r} \left\{ \int_{Q_x} \left( \sum_{R \in \mathcal{E}_k} |a_R||R|^{-1/r} \mathbf{1}_R(y) \right)^r \, dy \right\}^{1/r} \lesssim (2^k |x|_\infty)^{n/r} \mathcal{M}_r \left( \sum_{R \in \mathcal{E}_k} |a_R| \mathbf{1}_R \right)(x).
\]

Inserting this estimate into the previous calculation and applying the assumption \( |x|_\infty \gtrsim 2^k \) we obtain

\[
\tilde{a}_k(x) \lesssim 2^{(k-j)\eta} 2^{-2k(\frac{\eta}{1+\eta} - \frac{n}{\gamma})} \mathcal{M}_r \left( \sum_{R \in \mathcal{E}_k} |a_R| \mathbf{1}_R \right)(x).
\]

By considering \( r \leq 1 \) and \( r > 1 \) separately, we see that \( 2^{-2k(\frac{\eta}{1+\eta} - \frac{n}{\gamma})} = 1 \), completing the proof of case 1.

Case 2: \( |x|_\infty \leq 2(c_3 + c_4)2^k \).

Let \( \mathcal{E}_k \) be the collection of cubes defined in (2.4). For each \( Q \in \mathcal{E}_k \) we set \( a_Q = a_R \) whenever \( Q \subset R \). We have

\[
\tilde{a}_k(x) \lesssim \sum_{Q \in \mathcal{E}_k} \frac{|a_Q|}{(1 + 2^j d(x, Q))^\eta} \quad \text{and} \quad \sum_{R \in \mathcal{E}_k} |a_R| \mathbf{1}_R = \sum_{Q \in \mathcal{E}_k} |a_Q| \mathbf{1}_Q. \tag{3.13}
\]

For \( m \geq 1 \), define

\[
A_0 = A_0(x, k, j) = \{ Q \in \mathcal{E}_k : |x - x_Q|_\infty \leq c_4 2^{-j} \},
\]

\[
A_m = A_m(x, k, j) = \{ Q \in \mathcal{E}_k : c_4 2^{m-j-1} < |x - x_Q|_\infty \leq c_4 2^{m-j} \}.
\]
For \( m \geq 0 \), set

\[
B_m = B_m(x, j) = Q(x, c_4 2^{m+1-j}).
\]

Note that these sets satisfy the following properties:

\[
\#A_m \lesssim 2^{(m-j+k)n}, \quad \widehat{E}_k = \bigcup_{m \geq 0} A_m, \quad \bigcup_{Q \in A_m} Q \subseteq B_m. \tag{3.14}
\]

The first inequality in (3.14) holds because

\[
\#A_m \sim \frac{|\bigcup_{Q \in A_m} Q|}{2^{-kn}} \leq \frac{|B_m|}{2^{-kn}} \sim \frac{2^{(m-j)n}}{2^{-kn}},
\]

where we have used that the cubes in \( A_m \) are disjoint and have measure comparable to \( 2^{-kn} \). Using (3.13), the fact that \( d(x, Q) \sim 2^{m-j} \) whenever \( m \geq 2 \) (recall that \( j < k \) and part (d) of Lemma 2.1), and either the \( r \)-Hölder inequality along with the first property in (3.14) if \( r > 1 \) or the triangle inequality otherwise, we have

\[
\tilde{a}_k(x) \leq \sum_{m \geq 0} \sum_{Q \in A_m} \frac{|a_Q|}{(1 + 2^j d(x, Q))^{\eta}} \lesssim \sum_{m \geq 0} 2^{-m(\eta+(m-j+k)n)} \left( \sum_{Q \in A_m} |a_Q|^r \right)^{1/r}.
\]

From the last property in (3.14) and the \( 1/r \)-Hölder or triangle inequality as appropriate, for each \( m \geq 0 \), we have

\[
\left( \sum_{Q \in A_m} |a_Q|^r \right)^{1/r} \leq \left( \int_{B_m} \left( \sum_{Q \in A_m} |a_Q|^r |Q|^{-1/2} 1_Q(y) \, dy \right)^{1/r}
\]

\[
\leq |B_m|^{1/r} \left( \int_{B_m} \left( \sum_{Q \in A_m} |a_Q||Q|^{-1/2} 1_Q(y) \, dy \right)^r \right)^{1/r}
\]

\[
\lesssim 2^{(m+k-j)\frac{n}{r}} \mathcal{M}_r \left( \sum_{R \in \mathcal{E}_k} |a_R| |1_R| \right)(x),
\]

where in the last inequality we applied the estimates \( |B_m| \sim 2^{(m-j)n} \) and \( |Q| \sim 2^{-kn} \), the fact that \( B_m \) contains \( x \) and (3.13). Combining the previous two calculations gives

\[
\tilde{a}_k(x) \lesssim 2^{(k-j)\frac{n}{1/r}} \mathcal{M}_r \left( \sum_{R \in \mathcal{E}_k} |a_R| |1_R| \right)(x) \sum_{m \geq 0} 2^{-m(\eta-n\nu-\frac{n}{r})}.
\]

Since the assumption \( \eta > \frac{n}{1/r} \) ensures that the sum is finite, the proof of the Lemma is finished. \( \square \)
We turn to the proof of Theorem 3.6. In the rest of this section, for a given sequence of numbers \( \{s_R\}_{R \in \mathcal{E}} \), \( \alpha \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) we set
\[
s_k(\alpha, x) = 2^{k\alpha} \sum_{R \in \mathcal{E}_k} |s_R| |R|^{-1/2} 1_R(x).
\]

**Proof of Theorem 3.6** We separate the proof in two cases, one for the Triebel–Lizorkin spaces and another for the Besov spaces.

Case \((A_{\alpha}^{p,q}(\mathcal{L}), a_{\alpha}^{p,q}(\mathcal{L}))=(F_{\alpha}^{p,q}(\mathcal{L}), f_{\alpha}^{p,q}(\mathcal{L}))\).

Let \( \eta > 0 \) and \( 0 < r < 1 \) be such that
\[
\min\{n + M + \theta + \alpha, \mu\} > \eta > \frac{n}{r} > n_{p,q}.
\]
This is possible because of our hypotheses on \( M, \theta, \mu \) and \( \alpha \).

Set \( n_r = n/r \). Using the hypothesis on \( \mu \), (3.15) and Lemma 3.4 in the second inequality, and using (3.15) and Lemma 3.9 in the third inequality, we obtain
\[
\left\| \sum_R s_R m_R \right\|_{F_{\alpha}^{p,q}} \\
\leq \left\| \sum_{j \geq 0} \left( 2^{j\alpha} \sum_{k \geq 0} \sum_{R \in \mathcal{E}_k} |s_R| |\varphi_j(\sqrt{\mathcal{L}} m_R)| \right)^q \right\|_{L^p}^{1/q} \\
\leq \left\| \sum_{j \geq 0} \left( 2^{j\alpha} \sum_{k \geq 0} \sum_{R \in \mathcal{E}_k} |s_R| |R|^{-1/2} \frac{2^{-(n+M+\theta)(|k-j|\vee 0)-(N+\delta)(|j-k|\vee 0)}}{(1 + 2^{j\wedge k} |\cdot - x_R|)^{\eta}} \right)^q \right\|_{L^p}^{1/q} \\
\leq \left\| \sum_{j \geq 0} \left( 2^{j\alpha} \sum_{k \geq 0} \frac{2^{-(n+M+\theta-n_r)(|k-j|\vee 0)-(N+\delta)(|j-k|\vee 0)}}{\mathcal{M}_r \left( \sum_{R \in \mathcal{E}_k} |s_R| |R|^{-1/2} 1_R \right)} \mathcal{M}_r \left( s_k(\alpha, \cdot) \right) \right)^q \right\|_{L^p}^{1/q} \\
= \left\| \sum_{j \geq 0} \left( \sum_{k \geq 0} a_{j-k} b_{k}(\cdot) \right)^q \right\|_{L^p}^{1/q},
\]
where, for \( j \in \mathbb{Z} \),
\[
a_j = 2^{j\alpha+(j\wedge 0)(n+M+\theta-n_r)-(j\vee 0)(N+\delta)} \quad \text{and} \quad b_j(x) = \mathcal{M}_r \left( s_j(\alpha, \cdot) \right)(x) 1_{j \geq 0}(j).
\]

Define \( a = \{a_j\}_{j \in \mathbb{Z}} \) and \( b(x) = \{b_j(x)\}_{j \in \mathbb{Z}} \); note that for any \( t > 0 \), it holds that
\[
\|a\|_{l^t} < \infty.
\]

(3.16)
Indeed, we have
\[ \|a\|_{\ell_t} = \sum_{j \in \mathbb{Z}} 2^{j[\alpha+(j\wedge 0)(n+M+\theta-n_r)-(j\vee 0)(N+\delta)t]}
= \sum_{j \geq 0} 2^{-j(N+\delta-\alpha)t} + \sum_{j < 0} 2^{j(n+M+\theta-n_r+\alpha)t}. \]

Then first sum converges because \(N+\delta > \alpha\) by our hypotheses on \(N\) and \(\delta\), and the second sum converges because of (3.15).

We next use estimate (3.16) with \(t = 1 \wedge q\) and Young’s inequality with exponent \(q\) if \(q \geq 1\) or the \(q\)-triangle inequality with Young’s inequality with exponent 1 if \(q < 1\). This gives
\[ \left\| \sum_{R \in \mathcal{E}} s_R m_R \right\|_{F_{\ell_{1\wedge q}}^p} \lesssim \|a\|_{\ell_{1\wedge q}} \left\| b(\cdot) \right\|_{L^q} \lesssim \left\| b(\cdot) \right\|_{L^q} \]
\[ = \left\| \left( \sum_{j \geq 0} (\mathcal{M}_\ell(s_j(\alpha, \cdot)))^q \right)^{1/q} \right\|_{L^p}. \]

Inequality (3.15) and Lemma 3.8 lead to
\[ \left\| \sum_{R \in \mathcal{E}} s_R m_R \right\|_{F_{\ell_{1\wedge q}}^p} \lesssim \left\| \left( \sum_{j \geq 0} s_j(\alpha, \cdot)^q \right)^{1/q} \right\|_{L^p} = \|s\|_{\ell_{p,q}}. \]

This concludes the proof for the Triebel–Lizorkin spaces.

Case \((A_{\ell_{p,q}}^p(\mathcal{L}), a^p(\mathcal{L})) = (B_{\ell_{p,q}}^p(\mathcal{L}), b^p(\mathcal{L})).\)

Let \(\eta > 0\) and \(0 < r < 1\) be such that
\[ \min\{n + M + \theta + \alpha, \mu\} > \eta > \frac{n}{r} > n_{p,q}, \quad (3.17) \]

which is possible because of our hypotheses on \(M, \theta\) and \(\mu\).

As in the previous case we set \(n_r = n/r\). Using the hypothesis on \(\mu\), (3.17) and Lemma 3.4 in the first inequality, and using (3.17) and Lemma 3.9 in the second inequality, leads to
\[ \left\| \sum_{R \in \mathcal{E}_k} s_R m_R \right\|_{B_{\ell_{1\wedge q}}^p} \]
\[ \lesssim \left\{ \sum_{j \geq 0} \left(2^{j\alpha} \sum_{k \geq 0} \sum_{R \in \mathcal{E}_k} |s_R||R|^{-1/2} \frac{2^{-[(n+M+\theta)](k-j\vee 0)-(N+\delta)(j-k\vee 0)}}{(1 + 2^{j\wedge k}|\cdot - x_R|)\eta} \right)^q \right\}^{1/q} \]
\[ \lesssim \left\{ \sum_{j \geq 0} \left(2^{j\alpha} \sum_{k \geq 0} 2^{-[(n+M+\theta-n_r)](k-j\vee 0)-(N+\delta)(j-k\vee 0)} \right)^q \right\}^{1/q} \]
\[ \mathcal{M}_\ell \left( \sum_{R \in \mathcal{E}_k} |s_R| |R|^{-1/2} 1_R \right)^q \]
Lemma 3.8 gives

\[ \left\{ \sum_{j \geq 0} \left( \sum_{k \geq 0} 2^{j \alpha} \right) \left( \sum_{R \in \mathcal{E}_j} \right) \left| s_R \right| \left| R \right|^{-1/2} \left| R \right|^{1/2} \right\}^{q/p} \leq \| s \|_{b_{q,p}}. \]

d and, since for each \( j \) the tiles in \( \mathcal{E}_j \) are disjoint, we obtain

\[ \left\| \sum_{R} s_{R} m_{R} \right\|_{b_{q,p}} \leq \left\{ \sum_{j \geq 0} 2^{j \alpha} \left( \sum_{R \in \mathcal{E}_j} \right) \left| s_R \right| \left| R \right|^{-1/2} \left| R \right|^{1/2} \right\}^{q/p} = \| s \|_{b_{q,p}}. \]

This concludes the proof of the theorem for the Besov spaces. \( \square \)
4 Hermite Pseudo-multipliers

In this section, we pursue the study of boundedness properties in Hermite Besov and Hermite Triebel–Lizorkin spaces for pseudo-multipliers with symbols in Hörmander-type classes adapted to the Hermite setting. In Sect. 4.1, we define the classes of symbols and study the action of the corresponding pseudo-multipliers on needlets by proving smoothness and cancellation estimates (Theorems 4.3 and 4.4, respectively). In Sect. 4.2, we state the theorems regarding boundedness results and present their proofs: Theorem 4.5 gives results for spaces with positive smoothness; by assuming extra cancellation conditions on the symbols or more regularity conditions on the symbols, Theorem 4.6 gives results for spaces with zero or negative smoothness as well. The proofs of Theorems 4.5 and 4.6 use as tools the results of Theorems 2.6, 3.6, 4.3 and 4.4.

4.1 Classes of Symbols and the Action of Pseudo-multipliers on Needlets

Given a symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ we define the operator $T_\sigma$ by

$$T_\sigma f(x) = \sum_{k \in \mathbb{N}_0} \sigma(x, \lambda_k) \mathbb{P}_k f(x) = \sum_{k \in \mathbb{N}_0} \sigma(x, \lambda_k) \sum_{|\xi| = k} \langle f, h_\xi \rangle h_\xi(x). \quad (4.1)$$

We introduce the function

$$\varrho(x) = \frac{1}{1 + |x|}, \quad x \in \mathbb{R}^n; \quad (4.2)$$

observe that

$$\varrho(y) \sim \varrho(x) \quad \forall \ y \in B(x, \varrho(x)). \quad (4.3)$$

We call a non-negative function $g : \mathbb{R}^n \times \mathbb{N}_0 \to [0, \infty)$ an *admissible growth function* if for some $0 \leq \varkappa < 1$ and $\varepsilon > 16$

$$g(x, k) \lesssim e_k(x)^{-\varkappa} \quad (4.4)$$

and

$$g(x, k) \sim g(y, k) \quad \forall \ y \in B(x, \varrho(x)), \quad (4.5)$$

where $e_N(x)$ was defined in (2.2) for $N \geq 0$.

**Definition 4.1 (Symbols with growth)** Let $m \in \mathbb{R}$, $\rho, \delta \geq 0$, and $\mathcal{N}, \mathcal{K} \in \mathbb{N}_0 \cup \{\infty\}$. We say the symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ satisfies $\sigma \in S^{m, \mathcal{K}, \mathcal{N}}_{\rho, \delta}$ if $\sigma(\cdot, k) \in C^\mathcal{N}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ and there exists an admissible growth function $g$ such that

$$|\partial^\nu_x \triangle^\kappa_k \sigma(x, k)| \lesssim g(x, k)(1 + \sqrt{k})^{m - 2\rho \kappa + \delta |\nu|} \quad \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}_0 \quad (4.6)$$
for \( v \in \mathbb{N}_0^n \) satisfying \( 0 \leq |v| \leq N \) and \( 0 \leq \kappa \leq K \). If \( N = \infty \) (respectively \( K = \infty \)) then we mean that (4.6) holds for every multi-index \( v \in \mathbb{N}_0^n \) (respectively every \( \kappa \in \mathbb{N}_0 \)) with the implicit constant depending on \( v \) (respectively \( \kappa \)).

We note that \( S_{\rho', \delta}' \subseteq \mathcal{S}_{\rho, \delta} \) if \( \rho \leq \rho' \) and \( \mathcal{S}_{\rho, \delta} \subseteq S_{\rho', \delta}' \) if \( \delta' \leq \delta \). In particular, all results stated below for \( \mathcal{S}_{1, \delta} \) hold true for \( \mathcal{S}_{1, \delta} \) with \( 0 \leq \delta \leq 1 \).

For situations that require some degree of cancellation or orthogonality, we introduce the following condition on the symbols. A related condition, sufficient for the boundedness of a pseudo-differential operator on \( L^2(\mathbb{R}^n) \), has been considered in [24]; see in particular [24, Corollary 2.2].

**Definition 4.2 (Cancellation class)** Let \( m \in \mathbb{R} \) and \( M \in \mathbb{N}_0 \cup \{\infty\} \). We say the symbol \( \sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C} \) belongs to \( \mathcal{C}^m,M \) if

\[
\left( \int_{B(x, \rho(x))} |\varphi(y)|^2 \partial_{\gamma}^\sigma(y, k)^2 \, dy \right)^{1/2} \lesssim (1 + \sqrt{k})^m \quad \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}_0 \quad (4.7)
\]

for \( \gamma \in \mathbb{N}_0^n \) satisfying \( 0 \leq |\gamma| \leq 2[(n + M)/2] + 2 \) and where the implicit constant may depend on \( \gamma \).

In view of (4.3), the condition (4.7) is equivalent to

\[
\left( \int_{B(x, \rho(x))} |\partial_{\gamma}^\sigma(y, k)|^2 \, dy \right)^{1/2} \lesssim (1 + \sqrt{k})^m \varphi(x)^{-|\gamma|} \quad \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}_0.
\]

We next consider the action of \( T_\sigma \) on needlets.

**Theorem 4.3 (Smoothness estimates for \( T_{\sigma \varphi_R} \))** Let \( m \in \mathbb{R}, N \in \mathbb{N}_0 \) and \( K \in \mathbb{N} \), and suppose that \( \sigma \in \mathcal{S}^{m, K, N}_{1, \delta} \). Let \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) be an admissible system. Then there exists \( 0 \leq \varepsilon < 1 \) and \( \varepsilon > 4 \) such that for each \( \gamma \in \mathbb{N}_0^n \) satisfying \( 0 \leq |\gamma| \leq N \) and \( 1 \leq N \leq K \), it holds that

\[
|\partial_{x}^\gamma T_\sigma \varphi_R(x)| \lesssim \frac{|R|^{-1/2} 2^{j(m+|\gamma|)}}{(1 + 2^{j}|x - x_R|)^N} e_{x, \varepsilon} \varphi_{j}(x)^{1-\varepsilon} \quad \forall j \in \mathbb{N}_0, R \in \mathcal{E}_j, x \in \mathbb{R}^n. \tag{4.8}
\]

Before giving the proof of Theorem 4.3, note that if \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) is an admissible system, by (4.1) and the orthogonality of the Hermite functions \( \{h_{\xi}\}_{\xi \in \mathbb{R}_0} \), we have the expression

\[
T_\sigma \varphi_R(x) = \tau R^{1/2} \sum_{k \in I_j} \sigma(x, \lambda_k) \varphi_j(\sqrt{\lambda_k}) P_k(x, x_R), \tag{4.9}
\]

where the sets \( I_j \) are defined in Sect. 2.2.

In the sequel, the notation \( \Delta_k \sigma(x, \lambda_k) \) means that the finite difference is being applied to \( \sigma(x, \lambda_k) \) as a function of \( k \); that is, \( \Delta_k \sigma(x, \lambda_k) = \sigma(x, \lambda_{k+1}) - \sigma(x, \lambda_k) \).
Proof of Theorem 4.3 Let $j \in \mathbb{N}_0$ and $R \in \mathcal{E}_j$.

We first consider the case $|\gamma| = 0$.

Subcase 1 for $|\gamma| = 0$: $|x - x_R| \geq 2^{-j}$.

Let $N \in \mathbb{N}$ be such that $1 \leq N \leq K$. We apply the identity (5.13) to (4.9) to get, for $i = 1, \ldots, n$,

$$2^N (x_i - x_{R,i})^N \sum_{\ell} \sum_{k \in I_j} \Delta_\ell^k \left[ \sigma(x, \lambda_k) \phi_j(\sqrt{\lambda_k}) \right] (A_i^{(x_R)} - A_i^{(x)}) 2^{\ell} \pi_k (x, x_R).$$

Since $|\tau_R| \sim |R|$, we have

$$\left| (x_i - x_{R,i})^N \sum_{\ell} \sum_{k \in I_j} \Delta_\ell^k \left[ \sigma(x, \lambda_k) \phi_j(\sqrt{\lambda_k}) \right] (A_i^{(x_R)} - A_i^{(x)}) 2^{\ell} \pi_k (x, x_R) \right| \lesssim |R|^{1/2} \sum_{\ell \leq N} \sum_{k \in I_j} \Delta_\ell^k \left[ \sigma(x, \lambda_k) \phi_j(\sqrt{\lambda_k}) \right] \left| (A_i^{(x_R)} - A_i^{(x)}) 2^{\ell} \pi_k (x, x_R) \right|.$$ (4.10)

We next estimate each factor in the summation over $k$. Firstly, from the Leibniz formula for finite differences (5.19) we have

$$\left| \Delta_\ell^k \left[ \sigma(x, \lambda_k) \phi_j(\sqrt{\lambda_k}) \right] \right| \lesssim \sum_{r=0}^{\ell} \left| \Delta_\ell^r \phi_j(\sqrt{\lambda_k}) \right| \left| \Delta_{\ell-r} \sigma(x, \lambda_{k+r}) \right|.$$ (4.11)

Lemma 2.2 gives

$$\left| \Delta_\ell^r \phi_j(\sqrt{\lambda_k}) \right| \lesssim \lambda_k^{N/2-r} 2^{-jN}.$$ (4.11)

By the assumption on $\sigma$, there exists an admissible growth function $g$, $0 \leq \varepsilon < 1$ and $\varepsilon > 4$ such that

$$\left| \Delta_{\ell-r} \sigma(x, \lambda_{k+r}) \right| \lesssim \lambda_k^{m/2-\ell+r} g(x, k) \lesssim \lambda_k^{m/2-\ell+r} e_{4k} (x)^{-\varepsilon}.$$ (4.12)

These last three facts give

$$\left| \Delta_\ell^k \left[ \sigma(x, \lambda_k) \phi_j(\sqrt{\lambda_k}) \right] \right| \lesssim 2^{-jN} \lambda_k^{N/2+m/2-\ell} e_{4k} (x)^{-\varepsilon}.$$ (4.12)

It may be worth observing at this point that the implicit constant in (4.12) depends on $N, \|\phi(N)\|_{L^\infty}$ and $\|\phi_0(N)\|_{L^\infty}$. Secondly, since $0 \leq \ell - N/2 \leq N/2$, then observe that

$$(2(k + 2\ell - N) + 2)^{\ell-N/2} \leq (2k + 2N + 2)^{\ell-N/2} \leq (2N + 2)^{N/2} \lambda_k^{\ell-N/2}.$$
With this observation in mind, the binomial theorem and an application of (5.17) gives
\[
\left| \left( A_i^{(x_R)} - A_i^{(x)} \right)^{2\ell - N} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right|
\leq \sum_{|\xi|=k} \sum_{l=0}^{2\ell - N} \left( 2^{\ell - N} \right) \left| \left( A_i^{(x)} \right)^{2\ell - N - I} h_\xi(x) \right| \left| \left( A_i^{(x_R)} \right)^I h_\xi(x_R) \right|
\lesssim \lambda_k^{\ell - N/2} \sum_{|\xi|=k} \sum_{l=0}^{2\ell - N} \left( 2^{\ell - N} \right) \left| h_\xi + (2\ell - N - I)e_i(x) \right| \left| h_\xi + Ie_i(x_R) \right|,
\]
with an implicit constant that depends on \( N \). Applying the Cauchy–Schwarz inequality we obtain
\[
\left| \left( A_i^{(x_R)} - A_i^{(x)} \right)^{2\ell - N} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right|
\lesssim \lambda_k^{\ell - N/2} \left( \sum_{l=0}^{2\ell - N} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right)^{1/2} \left( \sum_{l=0}^{2\ell - N} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right)^{1/2}.
\] (4.13)

Recall that \( \lambda_k^{m/2} \sim 2^{jm} \). Then inserting the estimates (4.12) and (4.13) into (4.10) and using the Cauchy–Schwarz inequality, we get
\[
\left| (x_i - x_{R,i})^N T_\sigma \psi_R(x) \right|
\lesssim |R|^{1/2} 2^{j(m-N)e_{4j}(x)^{-\alpha}} \sum_{|\xi| \leq N k \in I_j} \sum_{l=0}^{2\ell - N} \sum_{k=0}^{2\ell + 2\ell - N - I} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right)^{1/2}
\times \left( \sum_{l=0}^{2\ell - N} \begin{bmatrix} k(x, x_R) \end{bmatrix} \right)^{1/2}
\lesssim |R|^{1/2} 2^{j(m-N)e_{4j}(x)^{-\alpha}} Q_{4j+N}(x, x)^{1/2} Q_{4j+N}(x, x)^{1/2}.
\]

Since this estimate holds for \( i = 1, \ldots, n \), it follows that
\[
|x - x_R|^N |T_\sigma \psi_R(x)| \lesssim |R|^{1/2} 2^{j(m-N)e_{4j}(x)^{-\alpha}} Q_{4j+N}(x, x)^{1/2} Q_{4j+N}(x, x)^{1/2}
\]
with constants depending on \( N, m, \varphi, \sigma \).

Subcase 2 for \( |\gamma| = 0 \): \( |x - x_R| < 2^{-j} \).

By the assumption on \( \sigma \) and the Cauchy–Schwarz inequality, we get
\[
|T_\sigma \psi_R(x)| \leq \left| \tau_{1/2} \right| \sum_{k \in I_j} \left| \sigma(x, \lambda_k) \right| \left| \varphi_j(\sqrt{\lambda_k}) \right| \left| \begin{bmatrix} k(x, x_R) \end{bmatrix} \right|
\lesssim |R|^{1/2} 2^{jm} e_{4j}(x)^{-\alpha} \sum_{k \in I_j} \sum_{|\xi| = k} \left| h_\xi(x) \right| \left| h_\xi(x_R) \right|
\lesssim |R|^{1/2} 2^{jm} e_{4j}(x)^{-\alpha} Q_{4j+N}(x, x)^{1/2} Q_{4j+N}(x, x)^{1/2}
\]

\[\text{Birkhäuser}\]
Combining the estimates for both subcases along with (2.3) and (2.5) we have

\[ |T_\sigma \varphi_R(x)| \lesssim \frac{|N|^{1/2} 2^{jm}e_\sigma(x) - \infty}{(1 + 2^j|x - x_R|)^N} \Omega_{j+1}^{\sigma}(x, x)^{1/2} \Omega_{j+1}^{\sigma}(x_R, x_R)^{1/2} \]

\[ \lesssim |N|^{1/2} 2^{jm}e_\sigma(x_R) \frac{2^{jm}}{(1 + 2^j|x - x_R|)^N} e_\sigma(x)^{1 - \infty} \]

\[ \lesssim |N|^{-1/2} \frac{2^{jm}}{(1 + 2^j|x - x_R|)^N} e_\sigma(x)^{1 - \infty}, \]

with constants independent of \( j \in \mathbb{N}_0, R \in \mathcal{E}_j \) and \( x \in \mathbb{R}^n \).

We turn to the case \(|y| > 0\). Note first that we can represent \( \partial^y \) by

\[ \partial^y = \sum_{\alpha + \beta \leq y} C_{\alpha, \beta} A^\alpha x^\beta; \]

see [30, (6.15)]. This means that in order to prove (4.8) it suffices to show

\[ |A^\alpha x^\beta T_\sigma \varphi_R(x)| \lesssim \frac{|N|^{-1/2} 2^{jm + |\alpha| + |\beta|}}{(1 + 2^j|x - x_R|)^N} e_\sigma(x)^{1 - \infty} \]

(4.14)

for any \( \alpha, \beta \in \mathbb{N}_0^n \) such that \( 0 \leq |\alpha| + |\beta| \leq N \) and \( 1 \leq N \leq \mathcal{K} \).

Subcase 1 for \(|y| > 0\): \(|x - x_R| \geq 2^{-j}\).

We first prove bounds for each component \( i = 1, \ldots, n \) by expressing the operator \( A^\alpha x^\beta \) in terms of two commuting operators:

\[ (A^{(x)})^\alpha x^\beta = (A^{(x)})^{\alpha - \alpha_ie_i} x^{\beta - \beta_ie_i} (A_i^{(x)})^{\alpha_i} x_i^{\beta_i}, \]

where \( \{e_i\}_{1 \leq i \leq n} \) is the canonical basis for \( \mathbb{R}^n \). Using identity (5.15), we have

(5.15)

By (5.13) and the Leibniz rule for finite differences (5.19),

\[ (x_i - x_{R,i})^N (A^{(x)})^\alpha x^\beta T_\sigma \varphi_R(x) \]

\[ = \sum_{s=0}^{\alpha_i} (a_i^s) (N!)^{1/(N-s)!} (A^{(x)})^{\alpha - s \epsilon_i} x^{\beta - \beta_i e_i} (x_i - x_{R,i})^{N-s} T_\sigma \varphi_R(x). \]

(5.15)
Applying \((A(x))^{\alpha - se_i}x^\beta\) to this expression and using (5.20), we get

\[
(A(x))^{\alpha - se_i}x^\beta (x_i - x_{R,i})^N T_\sigma \varphi_R (x) \n
= \tau_R^{1/2} 2^{-(N-s)} \sum_{N-s \leq \ell \leq N-s} c_{\ell,N-s} \sum_{k \ell M_j} \sum_{r=0}^\ell (\ell \Delta_k^j (\sqrt{\lambda_k})) 
\times \sum_{\nu \leq \alpha - se_i} (\alpha - se_i)_\nu (-1)^\nu \partial_x^\nu \Delta_k^r \sigma (x, \lambda_{k+r}) (A(x))^{\alpha - se_i - \nu} 
\times [x^\beta (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s} \mathbb{P}_k (x, x_R)].
\] (4.16)

Writing \(x^\beta = \alpha - \beta e_i \alpha^\beta_i\), applying (5.14) to \(x_i^\beta (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s}\), and commuting \(x^\beta - \beta e_i\) with powers of \(A_i^{(x_R)} - A_i^{(x)}\), we obtain

\[
x^\beta (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s} \mathbb{P}_k (x, x_R) 
= x^\beta - \beta e_i \sum_{t=0}^{\beta_i} \left(\beta_i \right) \frac{(2\ell - N+s)!}{(2\ell - N+s-t)!} (-1)^t (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s-t} x_i^{\beta_i - t} \mathbb{P}_k (x, x_R) 
= \sum_{t=0}^{\beta_i} \left(\beta_i \right) \frac{(2\ell - N+s)!}{(2\ell - N+s-t)!} (-1)^t (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s-t} x_i^{\beta_i - t} \mathbb{P}_k (x, x_R).
\]

Applying \((A(x))^{\alpha - se_i - \nu}\) to this expression, expanding powers of \((A_{i}(x_R) - A_{i}(x))\) by the binomial theorem, and then absorbing \(x^\beta - \beta e_i\) into \(\mathbb{P}_k (x, x_R)\) via (5.16), we get

\[
(A(x))^{\alpha - se_i - \nu} [x^\beta (A_{i}(x_R) - A_{i}(x))^{2\ell - N+s} \mathbb{P}_k (x, x_R)] 
= \sum_{t=0}^{\beta_i} \left(\beta_i \right) \frac{(2\ell - N+s)!}{(2\ell - N+s-t)!} (-1)^t \sum_{l=0}^{2\ell - N+s-t} (-1)^l (A_{i}(x_R))^{2\ell - N+s-t-l} (A(x))^{\alpha - se_i - \nu + l e_i} 
\times \sum_{|\xi| = k} \sum_{\omega \leq \beta - te_i} b_{\omega, \beta - te_i} (\xi) h_{\xi + \beta - te_i - 2\omega} (x) h_{\xi} (x_R).
\]

Inserting this last expression into (4.16), and its result into (4.15), we arrive at

\[
(x_i - x_{R,i})^N \left(\sum_{s=0}^{\alpha_i} \sum_{k \ell M_j} \sum_{r=0}^\ell \sum_{t=0}^\ell \sum_{l=0}^C q^\ell (\varphi_j (\sqrt{\lambda_k})) \partial_x^\nu \Delta_k^r \sigma (x, \lambda_{k+r}) \sum_{|\xi| = k} \sum_{\omega \leq \beta - te_i} b_{\omega, \beta - te_i} (\xi) \right) 
\times (A_{i}(x_R))^{2\ell - N+s-t-l} (A(x))^{\alpha - se_i - \nu + l e_i} h_{\xi + \beta - te_i - 2\omega} (x) h_{\xi} (x_R),
\] (4.17)
where
\[ C' = (-1)^{t+l+v} s^{-N} \binom{\alpha_i}{s} \binom{\beta_j}{t} \binom{2\ell-N+s-t}{l} \frac{N!}{(N-s)!} \frac{(2\ell-N+s)!}{(2\ell-N+s-t)!} C_{\ell,N-s}. \]

We now estimate the above expression. By (5.17) and (5.18), we have
\[
|A(x_R)|^{2\ell-N+s-t-l} (A(x))^{\alpha-se_i-v+le_i} h_{\xi+\beta-te_i-2\omega} (x) h_{\xi} (x_R) \lesssim \lambda_k^{\ell-N/2+|a-v|/2-\ell/2} |h_{\xi+2\ell-N+s-t-l} e_i (x_R)| \longfrac{1}{2} |h_{\xi+\alpha+\beta-v-2\omega+(l-t-s)e_i} (x)|,
\]
and by part (c) of Lemma B.1, we have
\[
|b_{\omega,\beta-te_i} (\xi)| \lesssim \lambda_k^{(|\beta|-1)/2}.
\]

Furthermore, our assumption on \( \sigma \) gives
\[
|\partial_x^v \Delta_k^{\ell-r} \sigma (x, \lambda_{k+r})| \lesssim \lambda_k^{m/2+|v|/2-\ell+r} g(x, k) \lesssim \lambda_k^{m/2+|v|/2-\ell+r} e_{e_k} (x)^{-\infty}.
\]

Inserting these three estimates along with (4.11) into (4.17), and noting the binomial bounds \( \binom{a}{b} \leq 2^a \) and \( \binom{a}{a-b} \leq d_b \), we obtain
\[
|A(x)\xi^{\alpha} x^\beta T_\sigma \varphi_R (x)| \lesssim 2^{-jN} |R|^{1/2} e_{e_4 / 4} (x)^{-\infty} 2^{j(m+|\alpha|+|\beta|)} \times \sum_{k \in I_j} \sum_{s=0}^{N-s} |h_{\xi+2\ell-N+s-t-l} e_i (x_R)| \longfrac{1}{2} |h_{\xi+\alpha+\beta-v-2\omega+(l-t-s)e_i} (x)|.
\]

By two applications of the Cauchy–Schwarz inequality, it follows that
\[
\sum_{k \in I_j} \sum_{l=0}^{2\ell-N+s-t} \sum_{|\xi|=k} |h_{\xi+2\ell-N+s-t-l} e_i (x_R)|\longfrac{1}{2} |h_{\xi+\alpha+\beta-v-2\omega+(l-t-s)e_i} (x)| \leq \sum_{k \in I_j} \sum_{l=0}^{2\ell-N+s-t} \mathbb{P}_{k+|\alpha-v|+|\beta-2\omega|+l-t-s} (x, x, x)^{1/2} \mathbb{P}_{k+2\ell-N+s-t-l} (x_R, x_R)^{1/2} \lesssim \mathcal{Q}_{4j+N+|\alpha|+|\beta|} (x, x)^{1/2} \mathcal{Q}_{4j+N+|\alpha|+|\beta|} (x_R, x_R)^{1/2}.
\]

We then conclude that
\[
|A(x)\xi^{\alpha} x^\beta T_\sigma \varphi_R (x)| \lesssim 2^{-jN} |R|^{1/2} e_{e_4 / 4} (x)^{-\infty} 2^{j(m+|\alpha|+|\beta|)} \mathcal{Q}_{4j+N+|\alpha|+|\beta|} (x, x)^{1/2} \mathcal{Q}_{4j+N} (x_R, x_R)^{1/2}.
\]
with constants depending only on $N$, $\alpha, \beta, \varphi, \sigma$.

Subcase 2 for $|\gamma| > 0$: $|x - x_R| < 2^{-j}$.

By (5.20) and (5.16), we get

$$
(A^{(x)})^\alpha x^\beta T_\sigma \varphi_R (x)
\quad
= \tau_R \frac{1}{k} \sum_{k \in I_j} \varphi_j (\lambda_k) (A^{(x)})^\alpha \sigma (x, \lambda_k) x^\beta \mathbb{P}_k (x, x_R)
\quad
= \tau_R \frac{1}{k} \sum_{k \in I_j} \varphi_j (\lambda_k) \sum_{v \leq \alpha} (\alpha_v) (-1)^v \partial_x^v \sigma (x, \lambda_k) (A^{(x)})^{\alpha - v} x^\beta \mathbb{P}_k (x, x_R)
\quad
= \tau_R \frac{1}{k} \sum_{k \in I_j} \varphi_j (\lambda_k) \sum_{v \leq \alpha} (\alpha_v) (-1)^v \partial_x^v \sigma (x, \lambda_k)
\quad
\times \sum_{\omega \leq \beta} \sum_{|\xi| = k} b_{\omega, \beta} (\xi) (A^{(x)})^{\alpha - v} h_{\xi + \beta - 2\omega} (x) h_\xi (x_R).
$$

From (5.18), part (c) of Lemma B.1 and our assumption on $\sigma$, we have the following three estimates:

$$
\left| (A^{(x)})^\alpha x^\beta T_\sigma \varphi_R (x) \right| \lesssim \lambda_k^{m + \frac{|\beta|}{\alpha}} \left| h_{\xi + \beta - 2\omega - \alpha - v} (x) \right|,
\left| b_{\omega, \beta} (\xi) \right| \lesssim \lambda_k^{\frac{|\beta|}{\alpha}},
\left| \partial_x^v \sigma (x, \lambda_k) \right| \lesssim \lambda_k^{\frac{m + |\beta|}{\alpha}} e_{4k} (x)^{-\infty}.
$$

Applying these estimates and making use of the Cauchy–Schwarz inequality, we obtain

$$
\left| (A^{(x)})^\alpha x^\beta T_\sigma \varphi_R (x) \right| \lesssim |R| \frac{1}{k} \sum_{k \in I_j} \lambda_k^{m + |\alpha| + |\beta|} e_{4k} (x)^{-\infty} \sum_{\omega \leq \beta} \sum_{|\xi| = k} \left| h_{\xi + \beta - 2\omega - \alpha - v} (x) \right| \left| h_\xi (x_R) \right|
\lesssim |R| \frac{1}{k} e_{4i} (x)^{-\infty} 2^{j(m + |\alpha| + |\beta|)} \mathbb{Q}_{4i + N + |\alpha| + |\beta|} (x, x) \frac{1}{2} \mathbb{Q}_{4i + N} (x_R, x_R)^{\frac{1}{2}}.
$$

Finally, combining the estimates for both subcases lead to

$$
\left| (A^{(x)})^\alpha x^\beta T_\sigma \varphi_R (x) \right| \lesssim |R| \frac{1}{k} 2^{j(m + |\alpha| + |\beta|)} e_{4i} (x)^{-\infty} \frac{\mathbb{Q}_{4i + N + |\alpha| + |\beta|} (x, x) \frac{1}{2} \mathbb{Q}_{4i + N} (x_R, x_R)^{\frac{1}{2}}}{(1 + 2^j |x - x_R|)^N}.
$$

By making use of (2.3) and (2.5) we obtain (4.14), completing the proof of Theorem 4.3.

\begin{theorem}
(Cancellation estimates for $T_\sigma \varphi_R$) Let $m \in \mathbb{R}$, $M, N, K \in \mathbb{N}_0$ and $K \geq n + M + 1$. Assume that $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ satisfies one of the following conditions:
\end{theorem}
(a) \( \sigma \in S^{m,K,N}_{1,1} \cap C^{m,M} \),
(b) \( \sigma \in S^{m,K,N}_{1,\delta} \) for some \( 0 \leq \delta < 1 \) and \( N \geq 2 \left[ \frac{n+M+1}{2(1-\delta)} \right] \).

Let \( \{ \varphi_j \}_{j \geq 0} \) be an admissible system and \( 0 < \theta \leq 1 \). Then for each \( \gamma \in \mathbb{N}^n_0 \) satisfying \( 0 \leq |\gamma| \leq M \), it holds that

\[
\left| \int_{\mathbb{R}^n} (x - x_R) \gamma T_\sigma \varphi_R(x) \, dx \right| \lesssim |R|^{\theta-\frac{1}{2}2^{j(m-|\gamma|)}} \left( \frac{1 + |x_R|}{2^j} \right)^{M+\theta-|\gamma|} \tag{4.18}
\]

for every \( j \in \mathbb{N}_0 \) and each \( R \in E_j \).

**Proof of Theorem 4.4** Let \( j \in \mathbb{N}_0 \), \( R \in E_j \) and \( \gamma \in \mathbb{N}^n_0 \) such that \( |\gamma| \leq M \).

We first prove the theorem assuming (a). Set \( B = B(x_R, \varphi(x_R)) \); fix a function \( \chi_R \in C^\infty(\mathbb{R}^n) \) supported in \( B \) that satisfies \( \chi_R = 1 \) on \( \frac{1}{2}B \), \( 0 \leq \chi_R \leq 1 \) and

\[
\|\chi_R^{(\eta)}\|_\infty \lesssim \frac{1}{\varrho(x_R)^{|\eta|}} \quad \forall \eta \in \mathbb{N}^n_0.
\]

We split the integral into two terms:

\[
\int_{\mathbb{R}^n} (x - x_R) \gamma T_\sigma \varphi_R(x) \, dx = \int_{\mathbb{R}^n} (1 - \chi_R(x))(x - x_R) \gamma T_\sigma \varphi_R(x) \, dx \\
+ \int_{\mathbb{R}^n} \chi_R(x)(x - x_R) \gamma T_\sigma \varphi_R(x) \, dx
\]

=: \( I + II \).

To estimate \( I \) we use the bounds on \( T_\sigma \varphi_R \) in Theorem 4.3, which hold because \( \sigma \in S^{m,K,N}_{1,1} \). We have

\[
|I| \lesssim |R|^{-\frac{1}{2}2^{j(m-|\gamma|)}} \int_{\mathbb{R}^n \setminus \frac{1}{2}B} \frac{(2^j |x - x_R|)^{|\gamma|}}{(1 + 2^j |x - x_R|)^K} \, dx \\
\leq |R|^{-\frac{1}{2}2^{j(m-|\gamma|)}} \left( \frac{1 + |x_R|}{2^j} \right)^{M+\theta-|\gamma|} \int_{\mathbb{R}^n} \frac{1}{(1 + 2^j |x - x_R|)^{K-M-\theta}} \, dx,
\]

which yields (4.18) since \( K \geq n + M + 1 > n + M + \theta \).

For the second term, we apply the Cauchy-Schwarz inequality and obtain

\[
|II| = \left| \tau^{1/2} \sum_{k \in I_j} \varphi_j(\sqrt{\lambda_k}) \sum_{|\xi|=k} h_\xi(x_R) \int_{\mathbb{R}^n} \chi_R(x)(x - x_R) \gamma T_\sigma (x, \lambda_k) h_\xi(x) \, dx \right| \\
\lesssim \|\varphi\|_{L^\infty} |R|^{1/2} \mathcal{Q}_{4j}(x_R, \varphi(x_R))^{1/2} \\
\times \left( \sum_{k \in I_j} \sum_{|\xi|=k} \int_{\mathbb{R}^n} \chi_R(x)(x - x_R) \gamma T_\sigma (x, \lambda_k) h_\xi(x) \, dx \right)^2 \right\}^{1/2},
\]

\( \mathcal{Q}_{4j}(x_R, \varphi(x_R)) \) is the quadratic form associated with \( \mathcal{F}\varphi \).
We next estimate the second factor. For $|\xi| = k$ and $N \in \mathbb{N}_0$, we have

$$
\left| \int_{\mathbb{R}^n} \chi_R(x)(x - x_R)^{\gamma} \sigma(x, \lambda_k) h_\xi(x) \, dx \right|
= \lambda_k^{-N} \left| \int_{\mathbb{R}^n} \mathcal{L}_x^N \left[ \chi_R(x)(x - x_R)^{\gamma} \sigma(x, \lambda_k) \right] h_\xi(x) \, dx \right|
\leq \lambda_k^{-N} \left\| \mathcal{L}_x^N \left[ \chi_R(\cdot)(x - x_R)^{\gamma} \sigma(\cdot, \lambda_k) \right] \right\|_{L^2(B)} \|h_\xi\|_{L^2(B)}.
$$

Repeated application of the Leibniz’ rule gives

$$
\mathcal{L}_x^N \left[ \chi_R(\cdot)(x - x_R)^{\gamma} \sigma(\cdot, \lambda_k) \right](x)
= \sum_{a,b,\beta,\eta,\nu} C_{a,b,\beta,\eta,\nu} \chi^a(x - x_R)^{\gamma - \beta} X_R^{(\eta)}(x) \partial_\nu^\beta \sigma(x, \lambda_k),
$$

where the sum runs over indices such that $|a| + |b| \leq 2N$, $\beta + \eta + \nu = b$ and $|\beta| \leq |\gamma|$. Set $N = [(n + M)/2] + 1$; by applying the condition $\sigma \in \mathcal{C}^m, M$, we have

$$
\left\| \mathcal{L}_x^N \left[ \chi_R(\cdot)(x - x_R)^{\gamma} \sigma(\cdot, \lambda_k) \right] \right\|_{L^2(B)}
\lesssim \sum_{a,b,\beta,\eta,\nu} Q(x_R)^{|\gamma| - |\beta| - |\eta|} \sup_{x \in B} |x|^{|a|} \left( \int_B |\partial_\nu^\beta \sigma(x, \lambda_k)|^2 \, dx \right)^{1/2}
\lesssim \sum_{a,b,\beta,\eta,\nu} Q(x_R)^{|\gamma| - |\beta| - |\eta| - |\nu| + n/2} (1 + |x_R|)^{|a|} \lambda_k^{m/2}
\lesssim \lambda_k^{m/2} (1 + |x_R|)^{2N - |\gamma| - n/2},
$$

where the sums run over indices such that $|a| + |b| \leq 2N$, $\beta + \eta + \nu = b$ and $|\beta| \leq |\gamma|$. Thus we obtain

$$
\left| \int_{\mathbb{R}^n} \chi_R(x)(x - x_R)^{\gamma} \sigma(x, \lambda_k) h_\xi(x) \, dx \right| \lesssim \lambda_k^{-N + m/2} (1 + |x_R|)^{2N - |\gamma| - n/2} \|h_\xi\|_{L^2(B)}.
$$

Inserting this into the estimate for $II$, we get

$$
|II| \lesssim |R|^{1/2} \mathbb{Q}_{4^j}(x_R, x_R)^{1/2} \left( \sum_{k \in I_j, |\xi| = k} \lambda_k^{-N + m/2} (1 + |x_R|)^{2N - |\gamma| - n/2} \right)^{1/2} \|h_\xi\|_{L^2(B)}^{1/2}
\lesssim |R|^{1/2} \left( \frac{1 + |x_R|}{2^j} \right)^{2N - n/2 - |\gamma|} \left( \sum_{k \in I_j, |\xi| = k} \|h_\xi\|_{L^2(B)}^2 \right)^{1/2} \mathbb{Q}_{4^j}(x_R, x_R)^{1/2}.
$$

Observe that

$$
\sum_{k \leq 4^j} \sum_{|\xi| = k} \|h_\xi\|_{L^2(B)}^2 = \int_{B} \sum_{k \leq 4^j} \sum_{|\xi| = k} h_\xi(y)^2 \, dy = \int_{B} \mathbb{Q}_{4^j}(y, y) \, dy. \quad (4.19)
$$
This, the bounds in (2.3) and the estimate (2.5) give

$$
|II| \lesssim |R|^{1/2} \left( \frac{1 + |x_R|}{2^j} \right)^{2N-n/2-|\gamma|} 2^j \left( m - |\gamma| - n/2 \right) \left( 2^j n e_{4^j} (x_R) \right)^{1/2} \left( 2^j n |B| \right)^{1/2} \\
\sim \left( \frac{1 + |x_R|}{2^j} \right)^{2N-n-|\gamma|} 2^j \left( m - |\gamma| - n \right) |R|^{1/2} 2^j n e_{4^j} (x_R) \\
\sim \left( \frac{1 + |x_R|}{2^j} \right)^{2N-n-|\gamma|} 2^j \left( m - |\gamma| - n \right) |R|^{-1/2}.
$$

Finally, since $|x_R| \lesssim 2^j$ and $2N = 2 \left( (n + M)/2 \right) + 2 \geq n + M + 1 > n + M + \theta$, we conclude that $|II|$ can be controlled by the right hand side of (4.18).

We next prove the theorem assuming (b). We proceed along similar lines to the proof for (a). Since $\sigma \in S^{m,K,N}_{1,\delta}$, we may utilize Theorem 4.3 and estimate term $I$ as above.

Turning to term $II$, the assumption $\sigma \in S^{m,K,N}_{1,\delta}$ and (4.5) give, for every $|v| \leq N$,

$$
\left( \int_B |\partial_x^\nu \sigma (x, \lambda_k) |^2 dx \right)^{1/2} \lesssim \lambda_k^{(m+|\delta|)|v|/2} \left( \int_B |g(x,k) |^2 dx \right)^{1/2} \\
\sim \lambda_k^{(m+|\delta|)|v|/2} g(x,R) q(x,R)^{n/2}.
$$

Set $N = \lceil \frac{n+M+1}{2(1-\delta)} \rceil$; then $N \geq 2N$ and by the above estimate, the fact that $\delta \leq 1$, and inequality (4.4) with some $\varepsilon > 4$ and $0 \leq \varepsilon < 1$, we have

$$
\| L^N \left[ \chi_R (\cdot - x_R)^\nu \sigma (\cdot, \lambda_k) \right] \|_{L^2(B)} \\
\lesssim \sum_{a,b,\beta,\eta,v,\gamma} q(x,R)^{|\gamma| - |\beta| - |\eta| + n/2} (1 + |x_R|)^{|\alpha|} \lambda_k^{(m+|\delta|)|v|/2} g(x,R,k) \\
\leq \sum_{a,b,\beta,\eta,v,\gamma} q(x,R)^{|\gamma| - |\beta| - |\eta| - |v| + n/2} (1 + |x_R|)^{|\alpha|} \lambda_k^{m/2} \left( \sqrt{\lambda_k} \right)^{\delta|v|} g(x,R,k) \\
\lesssim \lambda_k^{m/2} \max \left\{ 1, \frac{\sqrt{\lambda_k}}{1 + |x_R|} \right\}^{2N\delta} (1 + |x_R|)^{2N - |\gamma| - n/2} g(x,R,k) \\
\lesssim \lambda_k^{m/2} \max \left\{ 1, \frac{\sqrt{\lambda_k}}{1 + |x_R|} \right\}^{2N\delta} (1 + |x_R|)^{2N - |\gamma| - n/2} e_{4^j k} (x_R)^{-\varepsilon},
$$

where the sums run over indices such that $|\alpha| + |b| \leq 2N$, $\beta + \eta + v = b$ and $|\beta| \leq |\gamma|$. Therefore, we obtain

$$
\left| \int_{\mathbb{R}^n} \chi_R (x - x_R)^\nu \sigma (x, \lambda_k) h_x (x) dx \right| \\
\lesssim \lambda_k^{-N+m/2} \max \left\{ 1, \frac{\sqrt{\lambda_k}}{1 + |x_R|} \right\}^{2N\delta} (1 + |x_R|)^{2N - |\gamma| - n/2} e_{4^j k} (x_R)^{-\varepsilon} \| h_x \|_{L^2(B)}.
$$
Note that since $|x_R| \lesssim 2^j$, we have
\[
\max \left\{ 1, \frac{\sqrt{\lambda_k}}{1 + |x_R|} \right\}^{2Nδ} \lesssim \left( \frac{2^j}{1 + |x_R|} \right)^{2Nδ} \quad \forall k \in I_j.
\]
Inserting the above two estimates into term $II$, gives
\[
|II| \lesssim |R|^{1/2} Q_{4j} (x_R, x_R)^{1/2} \left( \sum_{k \in I_j} \sum_{|ξ| = k} |λ_k|^{-N+m/2} \max \left\{ 1, \frac{\sqrt{\lambda_k}}{1 + |x_R|} \right\}^{2Nδ} \right)^{1/2} \\
\times (1 + |x_R|)^{2N-|γ|-n/2} e_{4j}(x_R)^{-2} \left\| h_ξ \right\|_{L^2(B)}^{2} \right)^{1/2} \\
\lesssim |R|^{1/2} \left( \frac{1 + |x_R|}{2^j} \right)^{2N(1−δ)−n/2−|γ|} \\
\times 2^j(m−|γ|−n/2) e_{4j}(x_R)^{-2} Q_{4j} (x_R, x_R)^{1/2} \left( \sum_{k \in I_j} \sum_{|ξ| = k} \left\| h_ξ \right\|_{L^2(B)}^{2} \right)^{1/2}.
\]
By (4.19), (2.3) and (2.5), we have
\[
|II| \lesssim \left( \frac{1 + |x_R|}{2^j} \right)^{2N(1−δ)−n−|γ|} \left\| R \right\|^{1/2} 2^j n e_{4j}(x_R)^{1−2ξ} \\
\lesssim \left( \frac{1 + |x_R|}{2^j} \right)^{2N(1−δ)−n−|γ|} \left\| R \right\|^{-1/2}.\]

Our proof is finished once we observe that our choice of $N$ ensures $2N(1−δ) \geq n + M + 1 > n + M + θ$. □

### 4.2 Pseudo-multipliers on Distribution Spaces

In this section, we state and prove our results on boundedness properties of pseudo-multipliers in Hermite Besov and Hermite Triebel–Lizorkin spaces.

For smoothness index $α > 0$ we have the following result.

**Theorem 4.5** (No cancellation of molecules required) Let $m ∈ \mathbb{R}$ and $N, K ∈ \mathbb{N}$, and suppose that $σ ∈ S^m_{1,1}.K$. Let $α > 0$, $0 < q ≤ \infty$ and $0 < p < \infty$ for Triebel–Lizorkin spaces or $0 < p ≤ \infty$ for Besov spaces. If $α, p, q$ satisfy
\[
n_{p,q} - n < α < N \quad \text{and} \quad n_{p,q} < K, \tag{4.20}
\]
then the operator $T_σ$ extends to a bounded operator from $A^p_{α+m}(\mathcal{L})$ to $A^{p,q}_{α} (\mathcal{L})$.

The next result allows for negative values of $α$ by taking $M$ large enough.

**Theorem 4.6** (Cancellation of molecules required) Let $m ∈ \mathbb{R}$, $M ∈ \mathbb{N}_0$ and $N, K ∈ \mathbb{N}$. Assume that $σ : \mathbb{R}^a × \mathbb{N}_0 \to \mathbb{C}$ satisfies one of the following conditions:
\( \sigma \in S^{m,K,N}_{1,1} \cap C^{m,M} \),

(b) \( \sigma \in S^{m,K,N}_{1,\delta} \) for some \( 0 \leq \delta < 1 \) and \( N \geq 2\left\lceil \frac{n+M+1}{2(1-\delta)} \right\rceil \).

Let \( \alpha \in \mathbb{R} \), \( 0 < q \leq \infty \) and \( 0 < p < \infty \) for Triebel–Lizorkin spaces or \( 0 < p \leq \infty \) for Besov spaces. If \( \alpha \), \( p \), \( q \) satisfy

\[
n_{p,q} - n - M - 1 < \alpha < N \quad \text{and} \quad \max\{n_{p,q}, n + M\} < K, \quad (4.21)
\]

then the operator \( T_{\sigma} \) extends to a bounded operator from \( A^{p,q}_{\alpha+m}(\mathcal{L}) \) to \( A^{p,q}_{\alpha}(\mathcal{L}) \).

Before proving Theorems 4.5 and 4.6, we state a direct corollary regarding the classes \( S^{m,\infty,\infty}_{1,\delta} \) for \( 0 \leq \delta \leq 1 \).

**Corollary 4.7** Let \( m \in \mathbb{R} \), \( \alpha \in \mathbb{R} \), \( 0 < q \leq \infty \), \( 0 < p < \infty \) for Triebel–Lizorkin spaces or \( 0 < p \leq \infty \) for Besov spaces. Assume that \( \sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C} \) satisfies one of the following conditions:

(a) \( \sigma \in S^{m,\infty,\infty}_{1,1} \cap C^{m,\infty} \),

(b) \( \sigma \in S^{m,\infty,\infty}_{1,\delta} \) for some \( 0 \leq \delta < 1 \),

(c) \( \sigma \in S^{m,\infty,\infty}_{1,1} \) and \( \alpha > n_{p,q} - n \).

Then the operator \( T_{\sigma} \) extends to a bounded operator from \( A^{p,q}_{\alpha+m}(\mathcal{L}) \) to \( A^{p,q}_{\alpha}(\mathcal{L}) \).

We first prove Theorem 4.6 and then briefly sketch the proof of Theorem 4.5, which follows similarly.

**Proof of Theorem 4.6** Fix a \( \theta \in (0, 1) \) such that \( \alpha > n_{p,q} - n - M - \theta \) and \( K > \max\{n_{p,q}, n + M + \theta\} \), which is possible from our assumption on \( \sigma \) in (4.21).

Let \( \{\psi_j\}_{j \in \mathbb{N}_0} \) be an admissible system; we will first show that if \( j \in \mathbb{N}_0 \) and \( R \in \mathcal{E}_j \), then \( 2^{-jm}T_{\sigma}\psi_R \) is a constant multiple of a \((M, \theta, N - 1, 1, K)\)-molecule for \( A^{p,q}_{\alpha}(\mathcal{L}) \) under assumptions (a) or (b).

Firstly, note that the smoothness estimates (parts (i) and (ii) in Definition 3.1) follow from Theorem 4.3 and (2.6). Indeed, since \( \sigma \in S^{m,K,N}_{1,\delta} \), Theorem 4.3 implies that there exists \( 0 \leq \varepsilon < 1 \) and \( \varepsilon > 4 \) such that for \( \gamma \in \mathbb{N}_0^n \) satisfying \( 0 \leq |\gamma| \leq N \), it holds that

\[
|2^{-jm} \partial_x^\gamma T_{\sigma}\psi_R(x)| \lesssim \frac{|R|^{-1/2} 2^{j|\gamma|}}{(1 + 2^j|x - x_R|)^K} e_{4j}(x)^{1-\varepsilon} \quad \forall j \in \mathbb{N}_0, R \in \mathcal{E}_j, x \in \mathbb{R}^n.
\]

Then, by (2.6), and for \( \beta \geq 0 \),

\[
|2^{-jm} \partial_x^\gamma T_{\sigma}\psi_R(x)| \lesssim \frac{|R|^{-1/2} 2^{j|\gamma|}}{(1 + 2^j|x - x_R|)^K} \left(1 + \frac{|x|}{2^j}\right)^{-\beta} \quad \forall j \in \mathbb{N}_0, R \in \mathcal{E}_j, x \in \mathbb{R}^n,
\]

where \( 0 \leq |\gamma| \leq N \). Taking \( \beta = N \) in (4.22) gives the smoothness estimates in part (i) of Definition 3.1 for \( 0 \leq |\gamma| \leq N - 1 \); taking \( \beta = N \) and \( |\gamma| = N \) in (4.22) and

\[\text{Birkhäuser}\]
using Remark 3.2 lead to the smoothness estimates in part (ii) of Definition 3.1 for \(|\gamma| = N - 1\).

Secondly, from Theorem 4.4 we have the cancellation estimates (part (iii) of Definition 3.1)
\[
\left| \int_{\mathbb{R}^n} (x - x_R)^\gamma 2^{-jm} T_\sigma \psi_R(x) \, dx \right| \lesssim |R|^{-1/2} 2^{-j|\gamma|} \left( \frac{1 + |x_R|}{2j} \right)^{M + \theta - |\gamma|},
\]
(4.23)
for \(0 \leq |\gamma| \leq M\), \(j \in \mathbb{N}_0\), and \(R \in \mathcal{E}_j\).

Notice that (4.21) and the facts that \(\alpha > n_{p,q} - n - M - \theta\) and \(0 < \theta < 1\) ensure that
\[
\mathcal{N} > \alpha, \quad n + M + \theta + \alpha > n_{p,q}, \quad \mathcal{K} > \max\{n_{p,q}, n + M + \theta\}.
\]

We may then apply Theorem 3.6 (see Remark 3.7 (i)) to yield
\[
\left\| \sum_{R \in \mathcal{E}} s_R 2^{-jm} T_\sigma \psi_R \right\|_{A^{p,q}_\alpha} \lesssim \left\| \{s_R\}_{R \in \mathcal{E}} \right\|_{a^{p,q}_\alpha},
\]
(4.24)
for any sequence of numbers \(\{s_R\}_{R \in \mathcal{E}}\).

Using Theorem 2.6 (c) and the linearity of \(T_\sigma\), we have
\[
\|T_\sigma f\|_{A^{p,q}_\alpha} = \left\| \sum_{R \in \mathcal{E}} 2^{jm} \langle f, \varphi_R \rangle 2^{-jm} T_\sigma (\psi_R) \right\|_{A^{p,q}_\alpha},
\]
where the admissible systems \(\{\varphi_j\}_{j \in \mathbb{N}_0}\) and \(\{\psi_j\}_{j \in \mathbb{N}_0}\) satisfy (2.9). By (4.24), the definition of sequence spaces and Theorem 2.6 (b), we finally have
\[
\|T_\sigma f\|_{A^{p,q}_\alpha} \lesssim \left\| \{2^{jm} \langle f, \varphi_R \rangle\}_{R \in \mathcal{E}} \right\|_{a^{p,q}_\alpha} = \left\| \{(f, \varphi_R)\}_{R \in \mathcal{E}} \right\|_{a^{p,q}_{\alpha+m}} \lesssim \|f\|_{A^{p,q}_{\alpha+m}}.
\]
This completes the proof of Theorem 4.6.

Proof of Theorem 4.5 The proof of Theorem 4.5 is similar to the proof of Theorem 4.6 except that only the estimates in parts (i) and (ii) of Definition 3.1 hold. Thus \(2^{-jm} T_\sigma \psi_R\) are \((-1, 1, \mathcal{N} - 1, 1, \mathcal{K})\)-molecules. The conditions (4.20) and the fact that \(n_{p,q} \geq n\) ensure
\[
\mathcal{N} > \alpha, \quad \alpha + n > n_{p,q}, \quad \mathcal{K} > \max\{n_{p,q}, n\}.
\]
Therefore, (4.24) holds in view of Remark 3.7 (i). The rest of the proof follows as in the proof of Theorem 4.6.

\[\square\]
5 Examples and Applications

In this section, we present examples and applications of the results obtained in Sect. 4. We start by giving examples of symbols in the classes $S^{m,K,N}_{\rho,\delta}$. We next consider implications of Theorem 4.6 for boundedness properties of pseudo-multipliers on Lebesgue spaces $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ (Corollaries 5.4 and 5.6) and we compare them with existing results in the literature; as a byproduct, we obtain weighted estimates for pseudo-multipliers with symbols of order zero (Corollary 5.7). For $0 < p \leq 1$, we show that Theorem 4.6 leads to boundedness of pseudo-multipliers in the setting of Hermite local Hardy spaces (Corollary 5.9). We also comment on boundedness properties of Hermite multipliers. Finally, we present an example of a linearization process of a non-linear problem inspired by the works [3,27] (Theorem 5.12) that along with Theorem 4.5 implies that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities (Corollary 5.13).

**Example 5.1** (A symbol in $S^{m,\infty,\infty}_{\rho,\delta}$) Consider $\sigma(x, k) = \Phi(x)\Psi(k)$ where $\Phi \in C_0^\infty(\mathbb{R}^d)$ and $\Psi \in \mathcal{S}$. From the mean-value property $|\Delta_k^\alpha \Psi(k)| = |\Psi(\kappa)(\nu)|$ for some $\nu \in (k, k + \kappa)$; it then follows that $\sigma \in S^{m,\infty,\infty}_{\rho,\delta}$ for any $m, \rho, \delta$.

**Example 5.2** (A symbol in $S^{0,\infty,\infty}_{1,\delta}$) For $0 \leq \delta \leq 1$, let

$$
\sigma(x, k) = \sum_{j \in \mathbb{N}} \sigma_j(x) \varphi_j(\sqrt{\lambda_k})
$$

where $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is an admissible system and $\sigma_j \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$
|\partial^\alpha \sigma_j(x)| \leq C_{\nu} 2^{j\delta|\nu|} \left(1 + \frac{|x|}{2^j}\right)^\beta, \quad \forall \ \nu \in \mathbb{N}_0^n, \ j \in \mathbb{N}, \ x \in \mathbb{R}^n,
$$

for some $\beta \geq 0$. We next show that $\sigma \in S^{0,\infty,\infty}_{1,\delta}$.

For $j \in \mathbb{N}$, $\varphi_j(\sqrt{\lambda_k})$ is supported, as a function of $k \in \mathbb{N}_0$, in $I_j = [\frac{1}{2}4^{-j} - \frac{n}{2}, \frac{1}{2}4^{-j} + \frac{n}{2}] \cap \mathbb{N}_0$; for each fixed $k$, the sum contains at most five nonzero terms, and these occur for those $j$ such that $\lambda_k \sim 4^j$ (more precisely, $4^{j-2} \leq \lambda_k \leq 4^j$). For a given $\nu \in \mathbb{N}_0^n$, we have

$$
|\partial^\alpha \sigma_j(x)| \lesssim \lambda_k^{\delta|\nu|/2} \left(1 + \frac{|x|}{\sqrt{\lambda_k}}\right)^\beta, \quad \lambda_k \sim 4^j.
$$

From Lemma 2.2 (a), given $\kappa \in \mathbb{N}_0$ and $N > \kappa$,

$$
|\Delta_k^\alpha \varphi_j(\sqrt{\lambda_k})| \lesssim \lambda_k^{N/2-\kappa} 2^{-jN} \|\varphi(N)\|_\infty \sim \lambda_k^{N-\kappa}, \quad \lambda_k \sim 4^j.
$$

Therefore, we obtain

$$
|\partial^\alpha \Delta_k^\alpha \sigma(x, k)| \lesssim \lambda_k^{\delta|\nu|/2-\kappa} \left(1 + \frac{|x|}{\sqrt{\lambda_k}}\right)^\beta \sim (1 + \sqrt{k})^{\delta|\nu|-2\kappa} g(x, k)
$$
where\[ g(x, k) = \left(1 + \frac{|x|}{1 + \sqrt{k}}\right)^{\beta}. \]

It follows immediately from (2.6) that \( g \) satisfies (4.4) and (4.5).

**Example 5.3** (A symbol in \( S^0_{1,1} \cap C^{0,M} \)) Consider Example 5.2 with \( \delta = 1 \) and assume further that \( \sigma_j \) is supported in the set \( \{ x : 2^j \leq |x| < 2^{j+1} \} \). Then \( \sigma \in S^0_{1,1} \cap C^{0,M} \). Indeed, from Example 5.2 we have \( \sigma \in S^0_{0,1} \); thus it remains to be checked that \( \sigma \in C^{0,M} \). We first note if \( |x| \geq 2 \) then there is a unique \( j \in \mathbb{N} \) such that \( x \in \text{supp} \sigma_j \) and we have \[ 2^{j-1} \leq 1 + |x| \leq 2^{j+2}; \]

therefore, \( \varrho(x) \sim 2^{-j} \). For each such \( x \) and any \( \gamma \in \mathbb{N}^n_0 \), we obtain

\[ |\partial^\gamma x \sigma(x, k)| \leq \sum_{\nu \in \mathbb{N}} |\partial^{\nu} \sigma(x)| |\varphi_\nu(\sqrt{k}x)| \lesssim 2^{j|\gamma|} \sim (1 + |x|)^{|\gamma|} = \varrho(x)^{-|\gamma|}. \]

If \( |x| \leq 2 \) then \( \sigma(x, k) = 0 \) and the above estimate holds trivially. Thus for any \( x \in \mathbb{R}^n \) we get

\[ \left( \int_{B(x, \varrho(x))} |\partial^\gamma y \sigma(y, k)|^2 \, dy \right)^{1/2} \lesssim \left( \int_{B(x, \varrho(x))} \varrho(y)^{-2|\gamma|} \, dy \right)^{1/2} \sim \varrho(x)^{-|\gamma|}, \]

where in the last step we used property (4.3). This gives that \( \sigma \in C^{0,M} \).

### 5.1 Results for the \( L^p \) scale, \( 1 < p < \infty \)

In this section we present consequences of Theorem 4.6 for boundedness properties of pseudo-multipliers on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \). We first recall the relation

\[ F_0^{p,2}(\mathcal{L}) = L^p(\mathbb{R}^n), \quad 1 < p < \infty, \]

with equivalent norms; see [30, Proposition 5] or [13, Theorem 1.2] for dimension 1. Combining (5.1) with Theorem 4.6 we obtain the following result.

**Corollary 5.4** (\( L^p \) boundedness for symbols with growth) Assume that \( \sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C} \) satisfies one of the following conditions:

(a) \( \sigma \in S^0_{1,1} \cap C^{0,0} \) for some \( K \geq n + 1 \),

(b) \( \sigma \in S^0_{1,1} \cap C^{0,1} \) for some \( 0 \leq \delta < 1 \), \( N \geq 2 \left\lceil \frac{n+1}{2(1-\delta)} \right\rceil \) and \( K \geq n + 1 \).

Then \( T_\sigma \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \).
Proof For $1 < p < \infty$ and $q = 2$, we have $n_{p,q} = n$. The conditions in (4.21) are satisfied with $n_{p,q} = n$, $M = 0$, $\alpha = 0$, $N \in \mathbb{N}$ and $K \geq n + 1$. Applying Theorem 4.6 with those values and with $m = 0$ yield the desired result through the use of (5.1).

\[ \square \]

5.1.1 Comparisons with Other Results on $L^p$

In this section we compare our results in Corollary 5.4 with existing results in the literature and give some further consequences. The works [1,10,14] address symbols that satisfy (4.6) but without admissible growth (i.e. $g \equiv 1$). In order to continue the discussion we define the following class.

**Definition 5.5 (Symbols without admissible growth)** Let $m \in \mathbb{R}$, $\rho, \delta \geq 0$, and $N, K \in \mathbb{N}_0 \cup \{\infty\}$. We say the symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ satisfies $\sigma \in S^{m,K,N}_{\rho,\delta}$ if

\[ |\partial_x^\nu \Delta_k^\sigma(x,k)| \lesssim (1 + \sqrt{k})^{m-2\rho x + \delta|\nu|} \quad \forall (x,k) \in \mathbb{R}^n \times \mathbb{N}_0 \tag{5.2} \]

for $\nu \in \mathbb{N}_0^m$ such that $0 \leq |\nu| \leq N$ and $0 \leq \kappa \leq K$. If $N = \infty$ or $K = \infty$, the implicit constant in (5.2) may depend on $\nu$ or $\kappa$ respectively.

Since $S^{m,K,N}_{\rho,\delta} \subset S^{m,K,N}_{\rho,\delta}$ we have the following immediate consequence of Corollary 5.4.

**Corollary 5.6 (L$^p$ boundedness for symbols without growth)** Assume that $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ satisfies one of the following conditions:

(a) $\sigma \in S^{0,K,1}_{1,0} \cap C^{0,0}$ for some $K \geq n + 1$,

(b) $\sigma \in S^{0,K,N}_{1,\delta}$ for some $0 \leq \delta < 1$, $N \geq 2\lceil \frac{n+1}{2(1-\delta)} \rceil$ and $K \geq n + 1$.

Then $T_\sigma$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

In addition, by invoking [1, Theorem 1.4], we obtain the following weighted estimates, where $A_p$ denotes the Muckenhoupt class of weights.

**Corollary 5.7 (Weighted L$^p$ boundedness)** If $\sigma \in S^{0,K,1}_{1,0} \cap C^{0,0}$ with $K \geq n + 1$ (which implies assumption (a) of Corollary 5.6) or $\sigma$ satisfies assumption (b) of Corollary 5.6 with $\delta = 0$, the operator $T_\sigma$ is bounded on $L^p_w(\mathbb{R}^n)$ for every $w \in A_p$ and $1 < p < \infty$.

Let us compare Corollary 5.6 with existing results. First, assuming that $T_\sigma$ is a priori bounded on $L^2(\mathbb{R}^n)$, the authors in [1, Theorem 1.4] prove $L^p(\mathbb{R}^n)$ boundedness for all $1 < p < \infty$ (actually $L^p_w(\mathbb{R}^n)$ with $w \in A_p$) provided that $\sigma \in S^{0,n+1,1}_{0,0}$. On the other hand, [10, Corollary 2.12] does not assume $L^2(\mathbb{R}^n)$ boundedness but assumes $N = 0$ and $K = 2n + 1$, requiring no regularity in $x$ but more regularity in $k$. By contrast, say for $\delta = 0$, in place of $L^2(\mathbb{R}^n)$ boundedness we assume more regularity in the $x$ variable with $N = 2\lceil (n+1)/2 \rceil = 2\lceil n/2 \rceil + 2$, but retain the same level of $k$-regularity as [1] with $K = n + 1$. It is worth noting that [10, Corollary 2.12] is a consequence of [10, Theorem 1.1], which addresses symbols satisfying estimates in terms of Sobolev norms (Hörmander-type conditions). Such conditions do not imply those assumed in Corollary 5.6.
5.2 Results for $0 < p \leq 1$

For $0 < p \leq 1$, the local Hardy spaces $h^p(\mathbb{R}^n)$ (as defined in [20]) are better suited than $L^p(\mathbb{R}^n)$ or the Hardy spaces $H^p(\mathbb{R}^n)$ for boundedness of pseudo-differential operators in the Euclidean setting; see for instance [20,23,28]. In this context, it holds that $h^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n)$ for $0 < p \leq 1$ and $h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n)$ for $1 < p < \infty$, where $F_0^{p,2}(\mathbb{R}^n)$ are the classical Triebel–Lizorkin spaces in Euclidean space.

With this in mind, given an admissible system $\{\varphi_j\}_{j \in \mathbb{N}_0}$, we make the following definition.

**Definition 5.8 (Hermite local Hardy spaces)** For $0 < p \leq 1$, we define the Hermite local Hardy space $h^p(\mathcal{L})$ as the class of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{h^p(\mathcal{L})} = \left\| \left( \sum_{j \in \mathbb{N}_0} |\varphi_j(\sqrt{\mathcal{L}}) f|^2 \right)^{1/2} \right\|_{L^p} < \infty.
$$

Note that

$$
F_0^{p,2}(\mathcal{L}) = h^p(\mathcal{L}), \quad 0 < p \leq 1. \tag{5.3}
$$

In particular, as discussed in Sect. 2.2, the spaces $h^p(\mathcal{L})$ are independent of the choice of $\{\varphi_j\}_{j \in \mathbb{N}_0}$ and are quasi-Banach spaces. These spaces coincide with those introduced in [12]; see Remark 5.11 below.

From Theorem 4.6 and (5.3) we have the following result.

**Corollary 5.9** ($h^p$ boundedness) Let $0 < p \leq 1$, $M \in \mathbb{N}_0$ and $M \geq \lfloor n(\frac{1}{p} - 1) \rfloor$.
Assume that $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ satisfies one of the following conditions:

(a) $\sigma \in S_{1,\delta}^{0,K,1} \cap C^{0,M}$ for some $K \geq \lfloor n/p \rfloor + 1$,
(b) $\sigma \in S_{1,\delta}^{0,K,N'}$ for some $0 \leq \delta < 1$, $N' \geq 2\lceil \frac{n+M+1}{2(1-\delta)} \rceil$ and $K \geq \lfloor n/p \rfloor + 1$.

Then $T_\sigma$ extends to a bounded operator on $h^p(\mathcal{L})$.

**Proof** For $0 < p \leq 1$ and $q = 2$, we have $n_{p,q} = n/p$. The conditions in (4.21) are satisfied with $n_{p,q} = n/p$, $M > n/p - n - 1$, $\alpha = 0$, $N' \in \mathbb{N}$ and $K \geq \max\{n/p, n + M\}$. Thus, we can apply Theorem 4.6 with $0 < p \leq 1$, $q = 2$, $\alpha = 0$, $M \geq \lfloor n(\frac{1}{p} - 1) \rfloor$, $K \geq \lfloor n/p \rfloor + 1$, $N' \geq 1$ or $N' \geq 2\lceil \frac{n+M+1}{2(1-\delta)} \rceil$ to obtain the desired result through the use of (5.3). \qed

**Remark 5.10** The following consequences can be easily deduced from Corollary 5.9. If $\sigma \in S_{1,\delta}^{0,\infty,\infty} \cap C^{0,M}$ for some $M \in \mathbb{N}_0$ then $T_\sigma$ is bounded on $h^p(\mathcal{L})$ for $\frac{n}{n+M+1} < p \leq 1$. On the other hand, if $\sigma \in S_{1,\delta}^{0,\infty,\infty}$ for some $0 \leq \delta < 1$, then $T_\sigma$ is bounded on $h^p(\mathcal{L})$ for every $0 < p \leq 1$.

**Remark 5.11** We next observe that Corollary 5.9 also yields results on the classical local Hardy spaces $h^p(\mathbb{R}^n)$. Indeed, if $\sigma \in S_{1,\delta}^{0,\infty,\infty} \cap C^{0,M}$ for some $M \in \mathbb{N}_0$, then
$T_\sigma$ maps $h^p(\mathbb{R}^n)$ into $h^p(L)$ for $\frac{n}{n+M+1} < p \leq 1$, while if $\sigma \in S_{1,\delta}^{0,\infty,\infty}$ for some $0 \leq \delta < 1$, then $T_\sigma$ maps $h^p(\mathbb{R}^n)$ into $h^p(L)$ for every $0 < p \leq 1$.

In order to show this, we first recall the atomic Hardy space associated to the Hermite operator introduced in [12]. Given $0 < p \leq 1$, a function $a$ defined on $\mathbb{R}^n$ is a $p$-atom if there exists a ball $B = B(x, r)$, where $x \in \mathbb{R}^n$ and $r > 0$, such that

(i) $\text{supp } a \subset B$,
(ii) $\|a\|_\infty \leq |B|^{-1/p}$,
(iii) if $r \leq \frac{1}{2}\varrho(x)$ then $\int_B x^\gamma a(x) \, dx = 0$ for every $|\gamma| \leq \lfloor n\left(\frac{1}{p} - 1\right)\rfloor$.

The atomic Hardy space $h^p_\varrho(\mathbb{R}^n)$ associated to the Hermite operator is defined as the class of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{h^p_\varrho} = \inf \left\{ \left( \sum_{j \in \mathbb{N}} |c_j|^p \right)^{1/p} \right\} < \infty,$$

where the infimum is taken over all representations $f = \sum_{j \in \mathbb{N}} c_j a_j$ with scalars $c_j$ and $p$-atoms $a_j$. Note that $h^p(\mathbb{R}^n) \subset h^p_\varrho(\mathbb{R}^n)$ for $0 < p \leq 1$ since atoms in $h^p(\mathbb{R}^n)$, as defined in [20], satisfy conditions (i), (ii) and (iii).

It was recently proved that the atomic Hardy space $h^p_\varrho(\mathbb{R}^n)$ coincides with the Hermite Triebel–Lizorkin space $F^{p,2}_0(\mathcal{L})$ for every $0 < p \leq 1$ (see [21, Theorem 9] and [8, Remarks 2.20 and 2.7]). Thus, the relations

$$h^p(\mathbb{R}^n) \subset h^p_\varrho(\mathbb{R}^n) = h^p(\mathcal{L})$$

and Corollary 5.9 lead to the desired results.

### 5.3 Hermite Multipliers

Let $\sigma = \sigma(k)$ be a symbol in $S^{m,K,\infty}_{1,0}$ for some $m \in \mathbb{R}$ and $K \in \mathbb{N}_0$; thus, $\sigma$ satisfies

$$|\Delta^k \sigma(k)| \lesssim (1 + \sqrt{k})^{m-2k} \quad \forall k \in \mathbb{N}_0, \ 0 \leq k \leq K. \quad (5.4)$$

For the case $m = 0$, it was shown in [34, Theorem 4.2.1] that $T_\sigma$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ provided $K = \lfloor n/2 \rfloor + 1$. In dimension $n = 1$ with $m = 0$ and $K = 1$, the boundedness on $F^{p,q}_0(\mathcal{L})$ for $p, q > 1$ and $\alpha \in \mathbb{R}$ was obtained in [14, Theorem 1].

By applying Theorem 4.6, we obtain that if $M \in \mathbb{N}_0$ and $K > \max\{n_{p,q}, n + M\}$, then $T_\sigma$ is bounded from $A^{p,q}_{\alpha+m}(\mathcal{L})$ to $A^{p,q}_{\alpha}(\mathcal{L})$ provided

$$n_{p,q} - n - M - 1 < \alpha.$$

In particular, if $K = \infty$ then the result holds for all $\alpha \in \mathbb{R}$ and every $p, q > 0$. 

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5.4 On a Result of Meyer and Bony

In this section, we present an example of a linearization process of a non-linear problem that along with Theorem 4.5 implies that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities.

Theorem 5.12 (A linearization formula) Let \( H \in C^\infty(\mathbb{R}) \) be such that \( H(0) = 0 \). If \( f \in \mathcal{S}(\mathbb{R}^n) \) is real-valued, there exists \( \sigma_f \in S^{0,\infty,\infty}_{1,1} \) such that \( H(f) = T_{\sigma_f}(f) \). In particular, if \( \nu \in \mathbb{N}_0^n \) and \( \kappa \in \mathbb{N}_0 \), the symbol \( \sigma_f \) satisfies

\[
|\partial^{\nu}_{\xi} \Delta^\kappa_x \sigma_f(x, k)| \lesssim \left( \sup_{|\lambda| \leq \|f\|_{L^\infty}} \sum_{\ell=0}^{[\nu]} |H^{(\ell+1)}(\lambda)| \|f\|_{L^\infty}^\ell \right) \lambda_{\nu}/2^{\nu/2 - \kappa} \quad \forall x \in \mathbb{R}^n, k \in \mathbb{N}_0,
\]

where the implicit constant may depend on \( \nu \) and \( \kappa \) and is independent of \( f \).

As a consequence of Theorems 5.12 and 4.5 we obtain the following result.

Corollary 5.13 (Closure under non-linearities) Assume \( 0 < p < \infty, 0 < q < \infty, \alpha > n/p, q - n \) and \( H \in C^\infty(\mathbb{R}) \) is such that \( H(0) = 0 \). If \( f \in A^{p,q}_\alpha(L) \cap L^\infty(\mathbb{R}^n) \) is real-valued, then \( H(f) \in A^{p,q}_\alpha(L) \cap L^\infty(\mathbb{R}^n) \).

We next proceed with the proofs of the stated results.

Proof of Theorem 5.12 Let \( f \in \mathcal{S}(\mathbb{R}^n) \) be real-valued. Assume that \( (\varphi_0, \varphi) \) is an admissible pair such that \( \sum_{j=0}^{\infty} \varphi_j(\lambda) = 1 \) for all \( \lambda \geq 0 \) and \( \varphi_0^{(\ell)}(0) = 0 \) for all \( \ell \in \mathbb{N} \). We have

\[
\sum_{j \in \mathbb{N}_0} \varphi_j(\sqrt{L}) f = f,
\]

where the series converges absolutely and uniformly in \( \mathbb{R}^n \) since \( f \in \mathcal{S}(\mathbb{R}^n) \). Define \( f_j = \sum_{\ell=0}^{j} \varphi_\ell(\sqrt{L}) f \) for \( j \in \mathbb{N}_0 \) and \( f_{-1} = 0 \). Since \( f_j \to f \) uniformly on \( \mathbb{R}^n \), \( H \) is continuous and \( H(0) = 0 \), it follows that

\[
H(f) = \lim_{j \to \infty} H(f_j) = \sum_{j \in \mathbb{N}_0} H(f_j) - H(f_{j-1})
\]

pointwise in \( \mathbb{R}^n \) (even more, through the Mean Value Theorem, it follows that the convergence is uniform using that \( \sup_{j \in \mathbb{N}_0} \|f_j\|_{L^\infty} < \infty \), as shown below, and that \( H' \) is continuous). The Mean Value Theorem gives that

\[
H(f_j) - H(f_{j-1}) = \int_0^1 H'(tf_j + (1-t)f_{j-1}) \, dt \, (f_j - f_{j-1})
\]

\[
= \int_0^1 H'(f_{j-1} + t\varphi_j(\sqrt{L}) f) \, dt \varphi_j(\sqrt{L}) f.
\]
Setting \( m_j = \int_0^1 H'(f_{j-1} + t \varphi_j(\sqrt{\ell}) f) \, dt \), we then have

\[
H(f)(x) = \sum_{j \in \mathbb{N}_0} m_j(x) \varphi_j(\sqrt{\ell}) f(x) = \sum_{k \in \mathbb{N}_0} \left( \sum_{j \in \mathbb{N}_0} m_j(x) \varphi_j(\sqrt{\lambda_k}) \right) \Pi_k(f)(x),
\]

which means that \( H(f) \) can be realized as the action on \( f \) of the pseudo-multiplier with symbol

\[
\sigma_f(x, k) = \sum_{j \in \mathbb{N}_0} m_j(x) \varphi_j(\sqrt{\lambda_k}).
\]

We next prove that \( \sigma_f \in \dot{S}^{\infty, \infty}_{1,1} \) by showing (5.5). We have

\[
\partial_x^\nu \Delta_k^\kappa \sigma_f(x, k) = \sum_{j \in \mathbb{N}_0} \partial_x^\nu m_j(x) \Delta_k^\kappa (\varphi_j(\sqrt{\lambda_k}))
\]

\[
= \sum_{j = \lfloor \frac{1}{2} (\log_2 \lambda_k - 1) \rfloor}^{\lfloor \frac{1}{2} (\log_2 (\lambda_k + 2\kappa) + 1) \rfloor} \partial_x^\nu m_j(x) \Delta_k^\kappa (\varphi_j(\sqrt{\lambda_k})).
\]

Note that the number of terms in the last sum is bounded by a number independent of \( k \) and dependent on \( \kappa \). It is then enough to show that

\[
|\partial_x^\nu m_j(x) \Delta_k^\kappa (\varphi_j(\sqrt{\lambda_k}))| \lesssim \left( \sup_{|\lambda| \lesssim \|f\|_{L^\infty}} \sum_{\ell = 0}^{|\nu|} |H^{(\ell+1)}(\lambda)\|f\|_{L^\infty}^\ell \right) \lambda_k^{1/2 - \kappa}
\]

\[
\forall x \in \mathbb{R}^n, k \in \mathbb{N}_0,
\]

for \( j = \lfloor \frac{1}{2} (\log_2 \lambda_k - 1) \rfloor, \ldots, \lfloor \frac{1}{2} (\log_2 (\lambda_k + 2\kappa) + 1) \rfloor \) and where the implicit constant may depend on \( \nu \) and \( \kappa \) and is independent of \( f \).

Using Lemma 2.2 and taking \( N \) so that \( N > \kappa \), it holds that

\[
|\Delta_k^\kappa (\varphi_j(\sqrt{\lambda_k}))| \lesssim \lambda_k^{N/2 - \kappa} 2^{-jN} \lesssim \lambda_k^{-\kappa},
\]

where it was used that \( \sqrt{\lambda_k} \sim 2^j \) for \( j = \lfloor \frac{1}{2} (\log_2 \lambda_k - 1) \rfloor, \ldots, \lfloor \frac{1}{2} (\log_2 (\lambda_k + 2\kappa) + 1) \rfloor \).

We next observe that if \( \gamma \in \mathbb{N}_0^d \) then

\[
\max\{\|\partial_x^\gamma f_j\|_{L^\infty}, \|\partial_x^\gamma \varphi_j(\sqrt{\ell}) f\|_{L^\infty}\} \lesssim 2^{j|\gamma|} \|f\|_{L^\infty}.
\]

Indeed, the estimate for \( \|\partial_x^\gamma \varphi_j(\sqrt{\ell}) f\|_{L^\infty} \) follows from (5.11). Regarding \( \|\partial_x^\gamma f_j\|_{L^\infty} \), note that \( f_j = \varphi_0 j(\sqrt{\ell}) f \) with \( \varphi_0 j(\lambda) = \varphi_0(2^{-j} \lambda) \) since \( \sum_{l \in \mathbb{N}_0} \varphi_l \equiv 1 \) gives that \( \varphi_0(2^{-j} \lambda) = \sum_{l=0}^j \varphi_l(\lambda) \) for \( \lambda \geq 0 \). This and the fact that \( \varphi_0(l)(0) = 0 \) for \( l \in \mathbb{N} \) imply that the estimate for \( \|\partial_x^\gamma f_j\|_{L^\infty} \) is also a consequence of (5.11) (see Remark A.3).
For $\nu = 0$, (5.8) gives

$$|m_j(x)| \leq \sup_{|\lambda| \lesssim \|f\|_{L^\infty}} |H'(\lambda)|. \quad (5.9)$$

To estimate $\partial^\nu m_j$ for $\nu \neq 0$ we will use Faà di Bruno’s formula for the partial derivatives of a composition $G(g)$, where $g : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are smooth functions. Namely, for $\nu \in \mathbb{N}_0^n$, it holds that

$$\partial^\nu G(g) = \sum_{\nu = \nu_1 + \cdots + \nu_\ell} G^{(\ell)}(g) \prod_{r=1}^\ell \partial^{\nu_r} g,$$

where the sum is over all the multi-index decompositions $\nu = \nu_1 + \cdots + \nu_\ell$ with $\ell \geq 1$, $\nu_r \in \mathbb{N}_0^n$ and $\nu_r \neq 0$ for $r = 1, \ldots, \ell$; notice that $\ell \leq |\nu|$. We then have

$$\partial^\nu m_j = \int_0^1 \partial^\nu (H'(f_{j-1} + t \varphi_j(\sqrt{L})f)) \, dt$$

$$= \int_0^1 \sum_{\nu = \nu_1 + \cdots + \nu_\ell} H^{(\ell+1)}(f_{j-1} + t \varphi_j(\sqrt{L})f) \prod_{r=1}^\ell \partial^{\nu_r} (f_{j-1} + t \varphi_j(\sqrt{L})f) \, dt.$$

By (5.8) and since $\sqrt{\lambda_k} \sim 2^j$ for $j = \lfloor \frac{1}{2} (\log_2 \lambda_k - 1) \rfloor, \ldots, \lfloor \frac{1}{2} (\log_2 (\lambda_k + 2\kappa) + 1) \rfloor$, we obtain

$$\left| \prod_{r=1}^\ell \partial^{\nu_r} (f_{j-1} + t \varphi_j(\sqrt{L})f)(x) \right| \lesssim \prod_{r=1}^\ell 2^{j|\nu_r|} \|f\|_{L^\infty} = 2^{j|\nu|} \|f\|^{\ell}_{L^\infty} \lesssim \lambda_k^{\nu_1/2} \|f\|^{\ell}_{L^\infty}.$$

This implies that

$$|\partial^\nu m_j(x)| \lesssim \lambda_k^{\nu_1/2} \sup_{|\lambda| \lesssim \|f\|_{L^\infty}} \sum_{\ell=1}^{|\nu|} |H^{(\ell+1)}(\lambda)||f||_{L^\infty} \forall x \in \mathbb{R}^n. \quad (5.10)$$

The desired estimate (5.6) then follows from (5.7), (5.9) and (5.10).

We note that (5.8) is true for functions in $L^\infty(\mathbb{R}^n)$, not just in $\mathcal{S}(\mathbb{R}^n)$. Also, if $f \in A_{\alpha}^{p,q}(\mathcal{L})$ and $\{f_j\}_{j \in \mathbb{N}}$ is as in the proof of Theorem 5.12 then $f_j \in \mathcal{S}(\mathbb{R}^n)$, since it is a finite linear combination of Hermite functions, and $f_j \to f$ in $A_{\alpha}^{p,q}(\mathcal{L})$; the latter can be proved using the same ideas as in [36, Section 2.3.3]. These facts will be used in the proof of Corollary 5.13.

**Proof of Corollary 5.13** If $f \in \mathcal{S}(\mathbb{R}^n)$, then $H(f) \in A_{\alpha}^{p,q}(\mathcal{L})$ as a consequence of Theorem 5.12 and Theorem 4.5. Consider $f \in A_{\alpha}^{p,q}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ and note that $H(f) \in L^\infty(\mathbb{R}^n)$ since $f$ is bounded and $H$ is continuous.
Let \( \{f_j\}_{j \in \mathbb{N}} \) be as in the proof of Theorem 5.12; then \( f_j \in \mathcal{S}(\mathbb{R}^n) \), \( f_j \to f \) in \( A_{\alpha}^{p,q}(\mathcal{L}) \) and, by (5.8), \( \sup_{j \in \mathbb{N}} \|f_j\|_{L^\infty} \leq \|f\|_{L^\infty} \). We have
\[
|H(f)(x) - H(f_j)(x)| = |H'(c_{j,x})||f_j(x) - f(x)| \leq \sup_{|\lambda| \leq \|f\|_{L^\infty}} H'(\lambda)|f_j(x) - f(x)|,
\]
where \( c_{j,x} \) is a convex linear combination of \( f_j(x) \) and \( f(x) \) and the supremum is finite since \( H' \) is continuous. This estimate and the fact that \( f_j \to f \) in \( L^r(\mathbb{R}^n) \) for some \( 1 < r < \infty \) (see Corollary C.2 in Appendix C) imply that \( H(f) \to H(f_j) \) in \( L^r(\mathbb{R}^n) \) and, in particular, in \( \mathcal{S}'(\mathbb{R}^n) \). Let \( \sigma_{f_j} \) be as given in Theorem 5.12; applying Theorem 4.5 and taking into account (5.5), we obtain
\[
\|H(f_j)\|_{A_{\alpha}^{p,q}} = \|T_{\sigma_{f_j}}(f_j)\|_{A_{\alpha}^{p,q}} \lesssim \sup_{|\lambda| \leq \|f\|_{L^\infty}} \sum_{\ell=0}^{[\alpha]} |H^{(\ell+1)}(\lambda)||f_j|_{L^\infty} \|f_j\|_{A_{\alpha}^{p,q}}.
\]
(See conditions in Theorem 4.5, and Appendix D.) Since \( \sup_{j \in \mathbb{N}} \|f_j\|_{L^\infty} \leq \|f\|_{L^\infty} \) and \( \|f_j\|_{A_{\alpha}^{p,q}} \to \|f\|_{A_{\alpha}^{p,q}} \), we conclude that
\[
\liminf_{j \to \infty} \|H(f_j)\|_{A_{\alpha}^{p,q}} \lesssim \sup_{|\lambda| \leq \|f\|_{L^\infty}} \sum_{\ell=0}^{[\alpha]} |H^{(\ell+1)}(\lambda)||f|_{L^\infty} \|f\|_{A_{\alpha}^{p,q}}.
\]
By the Fatou property of \( A_{\alpha}^{p,q}(\mathcal{L}) \) (see Appendix C), the above implies that \( H(f) \in A_{\alpha}^{p,q}(\mathcal{L}) \) and
\[
\|H(f)\|_{A_{\alpha}^{p,q}} \lesssim \sup_{|\lambda| \leq \|f\|_{L^\infty}} \sum_{\ell=0}^{[\alpha]} |H^{(\ell+1)}(\lambda)||f|_{L^\infty} \|f\|_{A_{\alpha}^{p,q}}.
\]
\( \square \)

**Remark 5.14** If \( 0 < p < \infty, 0 < q < \infty, \alpha > n_{p,q} - n \) and \( f, g \in A_{\alpha}^{p,q}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n) \), then \( fg \in A_{\alpha}^{p,q}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n) \). Indeed, assuming without loss of generality that \( f \) and \( g \) are real-valued and using Corollary 5.13 with \( H(x) = x^2 \), it follows that \( f^2, (f + g)^2, g^2 \in A_{\alpha}^{p,q}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n) \); since \( 2fg = (f + g)^2 - f^2 - g^2 \), we conclude that \( fg \in A_{\alpha}^{p,q}(\mathcal{L}) \cap L^\infty(\mathbb{R}^n) \).

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### Appendix A. Estimates for \( \varphi_j(\sqrt{\mathcal{L}}) \)

In this appendix, we state and prove Lemma A.1, which is used in the proofs of Lemmas 3.3 and 3.4, and Theorem 5.12.
Lemma A.1 Let \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) be an admissible system. If \( \eta \geq 1, \varepsilon \geq 4, \gamma \in \mathbb{N}_0^n \) and \( K \geq 0 \), it holds that

\[
| \partial_{y}^\gamma \varphi_j(\sqrt{L})(x, y) | + | \partial_{x}^\gamma \varphi_j(\sqrt{L})(x, y) | \lesssim 2^j (n + |\gamma|) (1 + 2^j |x - y|) \eta e^{\varepsilon 4^j} e(x) \forall x, y \in \mathbb{R}^n, j \in \mathbb{N}_0,
\]

and, for \( |\gamma| \leq K \),

\[
\left| \int_{\mathbb{R}^n} (x - y)^\gamma \varphi_j(\sqrt{L})(x, y) \, dy \right| \lesssim 2^{-j |\gamma|} \left( \frac{1 + |x|}{2j} \right)^{K - |\gamma|} e(x) \forall x \in \mathbb{R}^n, j \in \mathbb{N}_0.
\]

Remark A.2 Note that by the symmetry of the kernels \( \varphi_j(\sqrt{L})(x, y) \), (5.12) also holds with \( dx \) in place of \( dy \) on the left hand side, and \( y \) in place of \( x \) on the right hand side.

Remark A.3 Lemma A.1 holds true, with the same proof, for a family \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) where \( \varphi_j(\lambda) = \varphi(2^{-j} \lambda) \) and \( \varphi \) is a smooth function supported in \([0, c] \) for some \( c > 0 \) that satisfies \( \varphi^{(k)}(0) = 0 \) for all \( k \in \mathbb{N} \).

Proof of Lemma A.1 Regarding (5.11), recall that

\[
\varphi_j(\sqrt{L})(x, y) = \sum_{k \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_k}) \mathcal{P}_k(x, y).
\]

Then (5.11) can be proved employing the same ideas in the proof of Theorem 4.3 through the use of Lemmas B.1 and B.2 presented in Appendix B.

We turn to the proof of (5.12). Assume \( |\gamma| \leq K \) and fix \( x \in \mathbb{R}^n \) and \( j \in \mathbb{N}_0 \). Set \( B = B(x, \rho(x)) \) where the function \( \rho(\cdot) \) is defined in (4.2). Let \( \chi \) be a function in \( C^\infty(\mathbb{R}^n) \) supported in \( 2B \) that satisfies \( \chi = 1 \) on \( B \), \( 0 \leq \chi \leq 1 \) and

\[
\| \chi^{(v)} \|_\infty \leq \frac{C}{\rho(x)^{|v|}} \forall v \in \mathbb{N}_0^n.
\]

We split the integral into two terms:

\[
\int_{\mathbb{R}^n} (x - y)^\gamma \varphi_j(\sqrt{L})(x, y) \, dy = \int_{\mathbb{R}^n} (1 - \chi(y))(x - y)^\gamma \varphi_j(\sqrt{L})(x, y) \, dy + \int_{\mathbb{R}^n} \chi(y)(x - y)^\gamma \varphi_j(\sqrt{L})(x, y) \, dy =: I + II.
\]

To estimate \( I \) we use the bounds from (5.11) with \( \eta > n + K \) and recall that \( |\gamma| \leq K \) to obtain
so that

\[ |I| \lesssim e^{|4j(x)|} \int_{B^c} \frac{(2^j |x - y|)^{|\gamma - K|} 2^{j(n - |\gamma|)}}{(1 + 2^j |x - y|)^{\eta - K}} dy \]

\[ \leq \left( \frac{1 + |x|}{2^j} \right)^{K - |\gamma|} 2^{-j|\gamma|} e^{|4j(x)|} \int_{B^c} \frac{2^{jn}}{(1 + 2^j |x - y|)^{\eta - K}} dy \]

\[ \lesssim \left( \frac{1 + |x|}{2^j} \right)^{K - |\gamma|} 2^{-j|\gamma|} e^{|4j(x)|} . \]

For the second term we have, by employing the Cauchy-Schwarz inequality,

\[ |II| = \left| \sum_{k \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_k}) \sum_{|\xi| = k} h_{\xi}(x) \int_{\mathbb{R}^n} \chi(y)(y - x)^{\gamma} h_{\xi}(y) dy \right| \]

\[ \leq \|\varphi\|_\infty \left( \sum_{k \in I_j} \sum_{|\xi| = k} h_{\xi}(x)^2 \right)^{1/2} \left( \sum_{k \in I_j} \sum_{|\xi| = k} \left| \int_{\mathbb{R}^n} (y - x)^{\gamma} \chi(y) h_{\xi}(y) dy \right|^2 \right)^{1/2} , \]

where we recall that \( k \in I_j \) means \( \frac{1}{2} 4^{j-2} - |n/2| \leq k \leq \frac{1}{2} 4^j - |n/2| \).

To estimate the second factor we note that for any \( N \in \mathbb{N}_0 \), it holds that

\[ \left| \int_{\mathbb{R}^n} (y - x)^{\gamma} \chi(y) h_{\xi}(y) dy \right| = \lambda_{|\xi|}^{-N} \int_{\mathbb{R}^n} \mathcal{L}_y^N [(y - x)^{\gamma} \chi(y)] h_{\xi}(y) dy \]

\[ \leq \lambda_{|\xi|}^{-N} \left\| \mathcal{L}_y^N [(\cdot - x)^{\gamma} \chi(\cdot)] \right\|_{L^2(2B)} \|h_{\xi}\|_{L^2(2B)} \]

\[ \sim (1 + |\xi|)^{-N} \left\| \mathcal{L}_y^N [(\cdot - x)^{\gamma} \chi(\cdot)] \right\|_{L^2(2B)} \|h_{\xi}\|_{L^2(2B)} . \]

Repeated application of the Leibniz’ rule gives, with the sum running over indices such that \( |a| + |b| \leq 2N, \beta + \nu = b \) and \( |\beta| \leq |\gamma| \),

\[ \mathcal{L}_y^N [(\cdot - x)^{\gamma} \chi(\cdot)](y) = \sum_{a,b,\beta,\nu} C_{a,b,\beta,\nu} y^a (y - x)^{\beta - \nu} \chi^{(\nu)}(y) \]

so that

\[ \left\| \mathcal{L}_y^N [(\cdot - x)^{\gamma} \chi(\cdot)] \right\|_{L^2(2B)} \sim \sum_{a,b,\beta,\nu} \left( \int_{2B} |y|^{|a|} |y - x|^{|\gamma| - |\beta|} |\chi^{(\nu)}(y)|^2 dy \right)^{1/2} \]

\[ \lesssim \sum_{a,b,\beta,\nu} Q(x)^{\gamma - |\beta| - |\nu| + n/2} \sup_{y \in 2B} |y|^{|a|} \]

\[ \lesssim \sum_{|a| + |b| \leq 2N} (1 + |x|)^{|a| + |b| - |\gamma| - n/2} \]

\[ \lesssim (1 + |x|)^{2N - |\gamma| - n/2} . \]
Inserting this into the estimate for $II$ leads to

$$|II| \lesssim \left( \sum_{k \in I_j} \sum_{|\xi| = k} h_{\xi}(x)^2 \right)^{1/2} \left( \sum_{k \in I_j} \sum_{|\xi| = k} \frac{(1 + |x|)^{2N-|\gamma|-n/2}}{(1 + |\xi|)^N} \left\| h_{\xi} \right\|_{L^2(2B)}^2 \right)^{1/2}$$

$$\lesssim \left( \frac{1 + |x|}{2^j} \right)^{2N-|\gamma|-n/2} 2^{-j(|\gamma|+n/2)} \left( \sum_{k \in I_j} \sum_{|\xi| = k} h_{\xi}(x)^2 \right)^{1/2} \left( \sum_{k \in I_j} \sum_{|\xi| = k} \left\| h_{\xi} \right\|_{L^2(2B)}^2 \right)^{1/2}$$

$$\lesssim \left( \frac{1 + |x|}{2^j} \right)^{2N-|\gamma|-n/2} 2^{-j(|\gamma|+n/2)} \left( \sum_{k \in I_j} \sum_{|\xi| = k} h_{\xi}(x)^2 \right)^{1/2} \left( \int_{2B} Q_{4j}(y, y) \ dy \right)^{1/2}$$

where in the last line we used that $\sum_{k \leq 4j} \sum_{|\xi| = k} h_{\xi}(y)^2 = Q_{4j}(y, y)$. We next apply the bounds (2.3) to get

$$|II| \lesssim \left( \frac{1 + |x|}{2^j} \right)^{2N-|\gamma|-n/2} 2^{-j(|\gamma|+n/2)} \left( 2^{jn} e_{4j}(x)^2 \right)^{1/2} 2^{jn/2} |2B|^{1/2}$$

$$\approx \left( \frac{1 + |x|}{2^j} \right)^{2N-|\gamma|-n} 2^{-j|\gamma|} e_{4j}(x)$$

$$\leq \left( \frac{1 + |x|}{2^j} \right)^{K-|\gamma|} 2^{-j|\gamma|} e_{4j}(x)$$

by choosing $N$ appropriately depending on whether $\frac{1+|x|}{2^j}$ is larger or smaller than 1. 

\[ \square \]

**Appendix B. Useful Identities and Estimates**

In this appendix, we present identities and estimates used in the proof of Theorem 4.3.

**Lemma B.1** (a) Suppose that

$$\mathcal{F}(x, y) = \sum_{k \in \mathbb{N}_0} f(x, y, k) \mathbb{P}_k(x, y).$$

If $N \in \mathbb{Z}_+$, it holds that

$$2^N(x_i - y_i)^N \mathcal{F}(x, y) = \sum_{N \leq \ell \leq N} c_{\ell, N} \sum_{k \in \mathbb{N}_0} \Delta_k^\ell f(x, y, k) \left(A_i^{(y)} - A_i^{(x)}\right)^{2\ell - N} \mathbb{P}_k(x, y),$$

(5.13)

where $c_{\ell, N} = (-4)^{N-\ell} (2N - 2\ell - 1)!! (\frac{N}{2\ell - N})$.

(b) If $N, M \in \mathbb{Z}_+$, it holds that

$$x_i^M \left(A_i^{(x)} - A_i^{(y)}\right)^N = \sum_{k=0}^M \binom{M}{k} \binom{N}{N-k} \left(A_i^{(x)} - A_i^{(y)}\right)^{N-k} x_i^{M-k}$$

(5.14)
and

\[(x_i - y_i)^N (A_i^{(x)})^M = \sum_{k=0}^{M} \binom{M}{k} \frac{N!}{(N-k)!} (A_i^{(x)})^{M-k} (x_i - y_i)^{N-k}, \] (5.15)

where \(\frac{N!}{(N-k)!}\) is defined to be 0 whenever \(N < k\).

(c) If \(\beta \in \mathbb{N}_0^n\) and \(k \in \mathbb{N}_0\), it holds that

\[x^\beta \mathbb{P}_k(x, y) = \sum_{\omega \leq \beta} \sum_{|\xi| = k} b_{\omega, \beta}(\xi) h_{\xi + \beta - 2\omega}(x) h_{\xi}(y), \] (5.16)

where \(b_{\omega, \beta}(\xi) = \prod_{i=1}^n b_{\omega_i, \beta_i}(\xi_i)\) with \(b_{\omega_i, \beta_i}(\xi_i) = 0\) if \(\xi_i + \beta_i - 2\omega_i < 0\) and \(b_{\omega_i, \beta_i}(\xi_i) \sim \xi_i^{\beta_i/2}\) otherwise.

(d) If \(\xi, \alpha \in \mathbb{N}_0^n\), \(m \in \mathbb{N}_0\) and \(i \in \{1, \ldots, n\}\), it holds that

\[\left| (A_i^{(x)})^m h_{\xi}(x) \right| \leq \left\lfloor 2(\xi_i + m) + 2 \right\rfloor^{\frac{m}{2}} |h_{\xi+me_i}(x)| \] (5.17)

and

\[\left| (A^{(x)})^\alpha h_{\xi}(x) \right| \leq \left\lfloor 2(|\xi| + |\alpha|) + 2 \right\rfloor^{\frac{|\alpha|}{2}} |h_{\xi+\alpha}(x)|. \] (5.18)

Proof of Lemma B.1 The identity in part (a) can be found in [30, Lemma 8] and [34, p.72] with \(k\) as a function of \(k\) only. However, it can be checked that the proof also works when \(k\) depends on both \(x\) and \(y\). Part (b) is from [30, Lemma 9]. Part (c) is [30, equation (6.14)] with \(\mu = 0\). In part (d), estimate (5.17) follows from [30, equation (6.5)].

\begin{align*}
(A_i^{(x)})^m h_{\xi}(x) &= \prod_{r=0}^{m-1} \sqrt{2(\xi_i + r) + 2} h_{\xi+me_i}(x).
\end{align*}

The inequality (5.18) follows from applying (5.17) repeatedly.

Lemma B.2 (a) If \(\ell \in \mathbb{N}_0\), it holds that

\[\Delta_{k}^{\ell}(f(k) g(k)) = \sum_{r=0}^{\ell} \binom{\ell}{r} \Delta_{k}^{r} f(k) \Delta_{k}^{\ell-r} g(k + r). \] (5.19)

(b) If \(\alpha \in \mathbb{N}_0^n\), it holds that

\[A^{\alpha}(fg) = \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} (-1)^{\nu} \partial^{\nu} f A^{\alpha-\nu} g. \] (5.20)
Proof of Lemma B.2} Part (a) is well known. For part (b), first note that the following representation for Hermite derivatives holds:

\[
A_i^m = (-1)^m e^{x_i^2/2} \partial_i^m e^{-x_i^2/2} \quad \forall m \in \mathbb{N}_0.
\] (5.21)

This identity can be obtained by direct calculation for \(m = 1\) and by induction for all \(m\). We next show that (5.21) gives

\[
A_i^{\alpha_i} (fg) = \sum_{\nu_i=0}^{\alpha_i} \binom{\alpha_i}{\nu_i} (-1)^{\nu_i} \partial_i^{\nu_i} f A_i^{\alpha_i-\nu_i} g.
\] (5.22)

Indeed, by (5.21) and the Leibniz rule for differentiation we obtain

\[
A_i^{\alpha_i} (fg) = (-1)^{\alpha_i} e^{x_i^2/2} \partial_i^{\alpha_i} (e^{-x_i^2/2} fg)
\]
\[= (-1)^{\alpha_i} e^{x_i^2/2} \sum_{\nu_i=0}^{\alpha_i} \binom{\alpha_i}{\nu_i} \partial_i^{\nu_i} f \cdot \partial_i^{\alpha_i-\nu_i} (e^{-x_i^2/2} g)
\]
\[= \sum_{\nu_i=0}^{\alpha_i} \binom{\alpha_i}{\nu_i} (-1)^{\nu_i} \partial_i^{\nu_i} f \cdot (-1)^{\alpha_i-\nu_i} e^{x_i^2/2} \partial_i^{\alpha_i-\nu_i} (e^{-x_i^2/2} g).
\]

Equality (5.22) follows by applying (5.21) again. The identity (5.20) then follows by applying (5.22) to each component \(1 \leq i \leq n\). □

Appendix C. Remarks About Hermite Besov and Triebel–Lizorkin Spaces

In this appendix, we present some embeddings of Hermite Besov and Hermite Triebel–Lizorkin spaces. The embeddings stated in Corollary C.2, a consequence of Theorem C.1, are used in the proof of Corollary 5.13. In addition, we comment on the Fatou property of Hermite Besov and Hermite Triebel–Lizorkin spaces, which is also used in the proof of Corollary 5.13.

**Theorem C.1** (a) If \(\alpha \in \mathbb{R}, \varepsilon > 0, 0 < q \leq \infty, 0 < q_1 \leq \infty, \) and \(0 < p \leq \infty\) for Besov spaces or \(0 < p < \infty\) for Triebel–Lizorkin spaces, it holds that

\[
A_{\alpha+\varepsilon}^{p,q} (\mathcal{L}) \hookrightarrow A_{\alpha}^{p,q_1} (\mathcal{L}).
\]

(b) If \(0 < q \leq \infty, 0 < p < \infty\) and \(\alpha \in \mathbb{R},\) it holds that

\[
B_{\alpha}^{p,\min(p,q)} (\mathcal{L}) \hookrightarrow F_{\alpha}^{p,q} (\mathcal{L}) \hookrightarrow B_{\alpha}^{p,\max(p,q)} (\mathcal{L}).
\]
(c) If \(0 < q \leq \infty\), \(0 < p < p_1 < \infty\) and \(\alpha, \alpha_1 \in \mathbb{R}\) are such that \(\alpha_1 < \alpha\), it holds that

\[
A_{\alpha_1}^{p,q}(L) \hookrightarrow A_{\alpha}^{p_1,q}(L) \quad \text{if} \quad \alpha - \frac{n}{p} = \alpha_1 - \frac{n}{p_1}.
\]

The proofs of the embeddings stated in Theorem C.1 are the same as those in the Euclidean setting; see \([36, \text{p.}47, \text{Proposition 2}]\) for (a) and (b) and \([30, \text{Propositions 6 and 7}]\) for (c).

**Corollary C.2** Let \(0 < q \leq \infty\). If \(1 < p < \infty\) and \(\varepsilon > 0\), then \(A_{\varepsilon}^{p,q}(L) \hookrightarrow L^p(\mathbb{R}^n)\); if \(0 < p \leq 1\) and \(\varepsilon > n(\frac{1}{p} - 1)\), there exists \(p_1 > 1\) such that \(A_{\varepsilon}^{p,q}(L) \hookrightarrow L^{p_1}(\mathbb{R}^n)\).

**Proof** Let \(0 < q \leq \infty\).

Case 1 \(1 < p < \infty\) and \(\varepsilon > 0\): Taking \(\alpha = 0\) in part (a) of Theorem C.1 we obtain that \(F_{\varepsilon}^{p,q}(L) \hookrightarrow F_0^{p,2}(L) = L^p(\mathbb{R}^n)\). Using parts (a) and (b) of Theorem C.1 we have \(B_{\alpha+\varepsilon}^{p,q}(L) \hookrightarrow B_0^{\alpha,\min(p,2)}(L) \hookrightarrow F_0^{p,2}(L)\); then \(\alpha = 0\) implies \(B_{\alpha}^{p,q}(L) \hookrightarrow F_0^{p,2}(L) = L^p(\mathbb{R}^n)\).

Case 0 \(\leq p \leq 1\) and \(\varepsilon > n(\frac{1}{p} - 1)\): Let \(p_1 > 1\) be such that \(p_1 > p\) and \(\varepsilon > n(\frac{1}{p} - 1)\); such \(p_1\) exists since \(\varepsilon > n(\frac{1}{p} - 1) \geq 0\). Setting \(\alpha_1 = \varepsilon - n(\frac{1}{p} - \frac{1}{p_1})\), part (c) of Theorem C.1 and the previous case imply that

\[
A_{\varepsilon}^{p,q}(L) \hookrightarrow A_{\alpha_1}^{p_1,q}(L) \hookrightarrow L^{p_1}(\mathbb{R}^n).
\]

\(\square\)

Next, we comment about the Fatou property for Hermite Besov and Hermite Triebel–Lizorkin spaces.

Let \(\mathcal{A}\) be a quasi-Banach space such that \(\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)\). The space \(\mathcal{A}\) is said to have the Fatou property if for every sequence \(\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{A}\) that converges in \(\mathcal{S}'(\mathbb{R}^n)\), as \(j \to \infty\), and that satisfies \(\lim_{j \to \infty} \inf_{j \to \infty} \|f_j\|_{\mathcal{A}} < \infty\), it follows that \(\lim_{j \to \infty} f_j \in \mathcal{A}\) and \(\|\lim_{j \to \infty} f_j\|_{\mathcal{A}} \leq \lim_{j \to \infty} \inf_{j \to \infty} \|f_j\|_{\mathcal{A}}\), where the implicit constant is independent of \(\{f_j\}_{j \in \mathbb{N}}\).

It can be shown, using standard proofs (see for instance \([37, \text{p.}48, \text{Proposition 2.8}]\)), that \(A_{\alpha}^{p,q}(L)\) possess the Fatou property for any \(\alpha \in \mathbb{R}\), \(0 < q \leq \infty\), \(0 < p \leq \infty\) for Besov spaces and \(0 < p < \infty\) for Triebel–Lizorkin spaces. This is due to the following facts: (1) if \(f, g \in L^p(\mathbb{R}^n)\) and \(|f| \leq |g|\) pointwise a.e., then \(\|f\|_{L^p} \leq \|g\|_{L^p}\); (2) if \(\{f_j\}_{j \in \mathbb{N}} \subset L^p(\mathbb{R}^n)\) and \(f_j \geq 0\) pointwise a.e., then \(\|\liminf_{j \to \infty} f_j\|_{L^p} \leq \liminf_{j \to \infty} \|f_j\|_{L^p}\); (3) if \(f_j \to f\) in \(\mathcal{S}'(\mathbb{R}^n)\) then, for any \(k \in \mathbb{N}_0\) and any admissible pair \((\varphi_0, \varphi_j), \varphi_k(\sqrt{L}) f_j \to \varphi_k(\sqrt{L}) f\) pointwise as \(j \to \infty\).

**Appendix D. Operator Norm**

The following result about the operator norm of pseudo-multipliers is used in the proof of Corollary 5.13.
Lemma D.1 Let $m \in \mathbb{R}$, $\rho \geq 0$, $\delta \geq 0$, $\mathcal{N}, K \in \mathbb{N}_0$, $\alpha, \lambda \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$, and $0 < \tilde{p} \leq \infty$ for Besov spaces or $0 < \tilde{p} < \infty$ for Triebel–Lizorkin spaces. If $T_\sigma$ is bounded from $A_{\alpha}^{p,q}(\mathcal{L})$ to $A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L})$ for all $\sigma \in S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}$, it holds that

$$
\|T_\sigma\|_{A_{\alpha}^{p,q}(\mathcal{L}) \rightarrow A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L})} \lesssim \max_{0 \leq |\nu| \leq \mathcal{N}} \|\partial_1^{\nu} \Delta_1^\kappa \sigma(x,k)(1 + \sqrt{k})^{-m+2\rho_k-\delta|\nu|}\|_{\mathcal{L}(\mathbb{R}^n)}
$$

for all $\sigma \in S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}$.

Proof We follow ideas from the proof of [2, Lemma 2.6]. Set

$$
\|\sigma\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}} = \max_{0 \leq |\nu| \leq \mathcal{N}} \|\partial_1^{\nu} \Delta_1^\kappa \sigma(x,k)(1 + \sqrt{k})^{-m+2\rho_k-\delta|\nu|}\|_{\mathcal{L}(\mathbb{R}^n)}
$$

then $S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}$ is a Banach space with the norm $\|\cdot\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}}$. Define the linear operator

$$
\mathcal{U} : S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}} \rightarrow L(A_{\alpha}^{p,q}(\mathcal{L}), A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L})), \quad \mathcal{U}(\sigma) = T_\sigma,
$$

where $L(A_{\alpha}^{p,q}(\mathcal{L}), A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L}))$ is the quasi-Banach space of all linear bounded operators from $A_{\alpha}^{p,q}(\mathcal{L})$ to $A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L})$ with the usual operator norm. We will show that the graph of $\mathcal{U}$ is closed; as a consequence of the Closed Graph Theorem, it follows that $\mathcal{U}$ is continuous and therefore the desired result follows.

Let $\{(\sigma_j, T_{\sigma_j})\}_{j \in \mathbb{N}}$ be a sequence in the graph of $\mathcal{U}$ that converges to $(\sigma, T) \in S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}} \times L(A_{\alpha}^{p,q}(\mathcal{L}), A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L}))$ in the product topology. We will show that $T(f) = T_\sigma(f)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$; assuming the latter, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $A_{\alpha}^{p,q}(\mathcal{L})$ and $T_\sigma, T \in L(A_{\alpha}^{p,q}(\mathcal{L}), A_{\lambda}^{\tilde{p},\tilde{q}}(\mathcal{L}))$, it follows that $T_\sigma = T$. As a consequence, the graph of $\mathcal{U}$ is closed.

Given $f \in \mathcal{S}(\mathbb{R}^n)$ and $N$ sufficiently large, using the definition of $\|\cdot\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}}$, [30, Lemma 3] and that $h_\xi$ are bounded uniformly in $\xi$ by [34, Lemma 1.5.2, p.27], we obtain

$$
|T_\sigma(f)(x) - T_{\sigma_j}(f)(x)| = \left| \sum_{k \in \mathbb{N}_0} (\sigma(x, \lambda_k) - \sigma_j(x, \lambda_k)) \mathbb{P}_k(f)(x) \right|
$$

$$
\leq \|\sigma - \sigma_j\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}} \sum_{k \in \mathbb{N}_0} (1 + \sqrt{\lambda_k})^m \sum_{|\xi| = k} |\langle f, h_\xi \rangle| |h_\xi(x)|
$$

$$
\leq \|\sigma - \sigma_j\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}} \sum_{k \in \mathbb{N}_0} (1 + \sqrt{|\xi|})^m \frac{1}{(1 + |\xi|)^N}
$$

$$
\leq \|\sigma - \sigma_j\|_{S_{\rho,\delta}^{m,\mathcal{K},\mathcal{N}}},
$$
which implies that $T_{\sigma_j}(f)$ converges to $T_{\sigma}(f)$ uniformly in $\mathbb{R}^n$. On the other hand, we have

$$\|T_{\sigma_j}(f) - T(f)\|_{A^p_q} \lesssim \|T_{\sigma_j} - T\|_{A^p_q} \|f\|_{A^p_q} \to 0.$$  

The above implies that $T_{\sigma_j}(f)$ converges to $T_{\sigma}(f)$ in $\mathcal{S}'(\mathbb{R}^n)$ and $T_{\sigma_j}(f)$ converges to $T(f)$ in $\mathcal{S}'(\mathbb{R}^n)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ (for the latter see [30, Proposition 4, p. 385 and Section 5, p. 392], which state that $A^p_q (\mathcal{L}) \hookrightarrow S'(\mathbb{R}^n)$). Therefore $T_{\sigma}(f) = T(f)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, as desired. □

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