Instantons in the Maximally Abelian Gauge

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We investigate the Maximally Abelian (MA) Projection for a single SU(2) instanton in continuum gauge theory. We find that there is a class of solutions to the differential MA gauge condition with circular monopole loops of radius $R$ centered on the instanton of width $\rho$. However, the MA gauge fixing functional $G$ decreases monotonically as $R/\rho \to 0$. Its global minimum is the instanton in the singular gauge. We point out that interactions with nearby anti-instantons are likely to excite these monopole loops.

1. Introduction

The Abelian projection reduces a non-Abelian gauge theory to a theory of monopoles and Abelian gauge fields interacting with charged gluons. It has been claimed both by numerical simulations[1–3] and analytical calculations[3–5] that there is a correlation between magnetic currents of the Abelian projected fields and instantons of the non-Abelian theory. Here we report on a thorough analytical and numerical study of the MA Abelian projection of the instanton.

In our investigation we use the widely accepted definition for the Maximally Abelian (MA) gauge[6], the minimization of the functional

$$G = \frac{1}{4} \int d^4x \{ A^1_\mu(x)^2 + A^2_\mu(x)^2 \},$$

or in differential form $(\partial_\mu \pm ieA^3_\mu(x))A^\pm_\mu(x) = 0$.

There are three key points to our results:

(a) Instantons of width $\rho$ do contain magnetic monopole loops of radius $R$. (b) The functional $G$ decreases monotonically as $(R/\rho)^4 \log(R/\rho)$ as $R/\rho \to 0$, thus small monopole loops are favored. (c) Interactions with nearby anti-instantons stabilize monopole loop formation.

2. Instanton in the MA gauge

We would like to find the gauge transformation $\Omega$ that rotates an instanton to the MA gauge. $\Omega$ satisfies

$$D^2_\mu(A)\Phi + \sigma\Phi = 0, \quad |\Phi| = 1,$$

where $\Phi(x) \cdot \tau \equiv \Phi(x) = \Omega_\tau \Omega^\dagger$, and $\sigma$ is a Lagrange multiplier. The SU(2) instanton field in the singular gauge is given by

$$A^s_\mu = -\tau^\alpha \eta^\alpha_{\mu\nu} x^\nu \frac{1}{x^2} \frac{\rho^2}{x^2 + \rho^2}.$$

It can be readily shown that this configuration satisfies the MA gauge condition. The functional $G$ is finite and takes the value $G = 4\pi^2 \rho^2$. Similarly the instanton in the non-singular gauge,

$$A^{ns}_\mu = -\tau^\alpha \eta^\alpha_{\mu\nu} x^\nu \frac{1}{x^2} \frac{\rho^2}{x^2 + \rho^2},$$

also satisfies the differential MA condition, but $G$ diverges. We observe that, in the singular gauge, the largest contribution to $G$ comes from the region near the instanton center and it is well-behaved at infinity. On the other hand, for the non-singular gauge, the contribution to $G$ from the origin is suppressed and the divergence comes from the infinity. It is possible to consider an intermediate gauge where the gauge potential approaches the behavior of the singular gauge at infinity and the behavior of the non-singular gauge at the origin. This is our key idea in searching

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for solutions to (2). Since the instanton in the singular gauge already yields a finite value for $G$, we will use it as the starting point of our investigation.

We parameterize $\Phi$, which is an iso-vector of unit length, by its spherical coordinates: the polar angle $\beta$ and the azimuthal angle $\alpha$. In this parameterization, equation (3) becomes

$$\partial_\mu^2 \beta - \frac{1}{2} \sin(2\beta)(\partial_\mu \alpha)^2 = 2 \sin \beta \partial_\mu \alpha \bar{A}_\mu \beta$$  \hspace{1cm} (5)
$$\partial_\mu^2 \alpha + 2 \cot \beta \partial_\mu \alpha \partial_\mu \beta = -2 \partial_\mu \beta \bar{A}_\mu \beta$$  \hspace{1cm} (6)

where $\bar{A}_\alpha^3 = \bar{A}_\mu \cdot \Phi$.

One solution to (4) corresponds to that discussed by Chernodub and Gubarev [4]: $\beta = \theta$ and $\alpha = \varphi$, where $\theta$ and $\varphi$ are the polar and azimuthal angles for the spatial three-vector $\vec{x}$. This static solution leads to a divergent $G$ and certainly is not preferred in the single instanton case.

Another obvious solution to (2) corresponds to the one that rotates the singular gauge to the non-singular gauge. In our parameterization that solution is $\beta = 2\theta$ and $\alpha = \phi - \psi$, where $\theta \equiv \tan^{-1}(u/v)$, $\phi \equiv \tan^{-1}(y/x)$, $\psi \equiv \tan^{-1}(t/z)$ and $v^2 \equiv t^2 + z^2$, $u^2 \equiv x^2 + y^2$.

If one assumes $\alpha = \phi - \psi$ and $\beta = \beta(u,v)$, Eq. (1) is automatically satisfied. This ansatz allows solutions with a monopole loop in the 3-4 and/or 1-2 planes. All other allowed orientations of the monopole loop can be generated by one of the chiral SU(2)/U(1) cosets of the Lorentz group.

Having in mind a solution with $A_\mu(x)$ behaving like the non-singular gauge at the origin, and like the singular gauge at infinity, we can consider a variational ansatz for $\beta$,

$$\beta_0(x, \theta) \equiv 2\theta - (\theta_+ + \theta_-) + \pi,$$  \hspace{1cm} (7)

with $\theta_{\pm} = \tan^{-1}[u/(v \pm R)]$ and $x^2 = u^2 + v^2$, which leads to a single monopole loop of radius $R$ in the $v$-plane (3-4 plane). Note that $\beta_0$ has a jump from zero to $\pi$ at $v = R$ in the $v$-plane, $(u = 0)$. This is where the magnetic monopole loop is located.

The above variational ansatz is in fact an exact solution to (3) in the limit $R/\rho \to 0$. Furthermore, in the limit $R/\rho \to \infty$, it corresponds to the gauge rotation that takes you from the singular gauge to the non-singular gauge. Thus one can say that the instanton in the non-singular gauge contains a monopole loop of infinite radius, while the instanton in the singular gauge contains a zero-size loop.

We have found numerically solutions that contain a monopole loop of arbitrary radius $R$ in the $v$-plane by setting $\alpha = \phi - \psi$ and solving the remaining differential equation (6). Finiteness of $G$ requires that $\beta(0,v)$ and $\beta(u,0)$ can only take on integer multiples of $\pi$. The loop of radius $R$ is introduced by enforcing the boundary conditions; at $\theta = 0$ ($v$-plane) $\beta$ has a jump (0 to $\pi$) at $v = R$ and at $\theta = \pi/2$ ($u$-plane) $\beta = \pi$. The value of $G$ as a function of $R$ is plotted in Fig.1.

![Figure 1](image-url)  

Note that $G$ is monotonically increasing with $R$. At small $R/\rho$ it goes like $(R/\rho)^4 \log(R/\rho)$. Although $R = 0$ corresponds to the true minimum, the absence of a $R^2$-behavior corresponds to the presence of a "zero-mode" so that infinitesimal-size loops are easily produced. Thus interactions or quantum fluctuations may excite these loop modes. In fact we have done numerical experiments which demonstrate that small deformations in the instanton potential make finite loops favorable.
3. 4D Euclidean Lattice

In order to see if there are other configurations that might give lower values to the MA functional we did a lattice minimization of $G$, approximated by

$$G = \frac{a^2}{2} \sum_{\mu,x} \{1 - \frac{1}{2} Tr(\Phi(x)U_\mu(x)\Phi(x+\mu)U^\dagger_\mu(x))\}.$$  

Since we expect at infinity that the functional drops like $1/x^4$ and that the gauge rotation is $1$, we can use a large enough volume with open boundary conditions to approximate $G$. That will also help us put on the lattice a “better” instanton, free of defects at the boundaries.

Our minimization show that there is magnetic current loop formation on the lattice, the gauge rotation satisfies our ansatz; i.e. $\alpha = \phi - \psi$ and $\beta$ only depends on $u$ and $v$, but the radius $R$ does not scale with the instanton size $\rho$. This is in contradiction with what was observed by Hart and Teper\(^\text{[7]}\). Perhaps their use of periodic boundary conditions causes the loop radius to scale via interactions with the images of the instanton.

![Figure 2. Extrapolation to the continuum limit gives $G_{\text{ma}}/G_i = (1.000 \pm 0.001)$. $G_i \equiv G[A^i_\mu]$.](image)

If the loop is a lattice artifact, in the continuum limit it should shrink to zero in physical units, and the MA functional should achieve the value $4\pi^2\rho^2$. We did finite size and volume analysis and we were able to extrapolate to the continuum (infinite volume and zero lattice spacing). We found that indeed in the continuum limit $G$ is $4\pi^2\rho^2$ within errors (Fig. 2).

We also studied the interacting case. Although this work is still in progress, we find clear evidence that the instanton anti-instanton (I-A) system does possess a monopole loop that survives the continuum limit. There is a critical distance for which the individual loops fuse into a single loop orbiting the I-A pair. On the other hand, the instanton-instanton system does not give rise to loops that survive the continuum limit.

4. Conclusions

The bottom line of our study is that the single instanton in the maximally Abelian gauge possesses a monopole loop. We find that the global minimum of $G$ for the isolated instanton is the singular gauge which is equivalent to a zero sized monopole loop. The behavior of $G$ at small $R$ indicates that quantum fluctuations or interactions with (anti)instantons probably cause large loop formation. Our numerical study shows that interactions with anti-instantons cause large loop formation.

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