Model Predictive Control for Linear Uncertain Systems Using Integral Quadratic Constraints

Lukas Schwenkel, Johannes Köhler, Matthias A. Müller, and Frank Allgöwer

Abstract—In this work, we propose a tube-based model predictive control (MPC) scheme for state- and input-constrained linear systems subject to dynamic uncertainties characterized by dynamic integral quadratic constraints (IQCs). In particular, we extend the framework of $\rho$-hard IQCs for exponential stability analysis to external inputs. This result yields that the error between the true uncertain system and the nominal prediction model is bounded by an exponentially stable scalar system. In the proposed tube-based MPC scheme, the state of this error bounding system is predicted along with the nominal model and used as a scaling parameter for the tube size. We prove that this method achieves robust constraint satisfaction and input-to-state stability despite dynamic uncertainties and additive bounded disturbances. A numerical example demonstrates the reduced conservatism of this IQC approach compared to state-of-the-art robust MPC approaches for dynamic uncertainties.

Index Terms—Integral quadratic constraints (IQCs), predictive control for linear systems, robust control, uncertain systems.

I. INTRODUCTION

When facing a control problem with hard state or input constraints, a popular approach that can guarantee stability and constraint satisfaction is to design a model predictive controller (MPC) (see, e.g., [1] or [2]). Throughout the past decades, the question how to adjust an MPC scheme to maintain these guarantees in the presence of disturbances or model uncertainties has been studied frequently for different kinds of uncertainties [3]. This led to several robust MPC schemes reaching from bounded disturbances (e.g., [4]) over stochastic disturbances (e.g., [5]), state and input dependent disturbances (e.g., [6]), and parametric uncertainties (e.g., [7]) to dynamic uncertainties (e.g., [8]). The reason for this wealth of approaches is not only the different nature of various uncertainties but also that there is a tradeoff between conservatism and complexity of the underlying uncertainty descriptions. While some control tasks require a fast and simple MPC scheme, there are other scenarios where a larger online computational complexity can be tolerated to gain tighter uncertainty descriptions and less cautious controllers, which can lead to significant performance improvements and much larger operating ranges. Interestingly, there is a lot of MPC literature on the rather simple case of additive bounded disturbances, whereas, on the other end of the scale, the handling of unmodeled dynamics, delays, errors from using reduced order models, or other dynamic uncertainties in MPC remains an open research area [2]. This is in contrast to classical robust control literature (e.g., [9]) where stability and performance of feedback interconnections of a known linear system $G$ and a dynamic uncertainty $\Delta$, as shown in Fig. 1, are studied comprehensively. We make use of analysis tools from the robust control literature for such interconnections and base our proposed MPC scheme on the powerful and efficient framework of integral quadratic constraints (IQCs, see [10] for the original article, or [11] for a tutorial overview). When the input–output behavior of an uncertainty is described by an IQC, stability and performance of the feedback interconnection can be verified with linear matrix inequalities (LMIs). In this article, we bridge this gap between classical robust control methods and robust MPC by providing an MPC design method for linear constrained systems subject to dynamic uncertainties characterized using $\rho$-hard IQCs as defined in [12]. Furthermore, the use of...
IQC schemes in robust MPC is a general and unifying approach since a multitude of uncertainties can be described with IQCs, such as \(\ell_2\)-gain bounds, uncertain time-delays, polytopic parameter uncertainties, or sector- and slope-restricted nonlinearities.

**Related Work:** A widespread approach to robustify MPC schemes is tube-based MPC, where a nominal MPC scheme is implemented with tighter constraints and the amount of the constraint tightening is determined from the size of a tube confining all possible trajectories of the true system. The main advantage of tube-based MPC compared to other robust MPC approaches like min-max MPC or multistage MPC is that the online computational complexity of tube-based MPC schemes is, if at all, only slightly larger than a nominal MPC. Tube-based MPC was first introduced by [4] and [13] for linear systems subject to additive bounded disturbances and later improved in [14]. Instead of using a static tube, in [15] the tube size is scaled with a parameter that is optimized online, thereby offering more flexibility. This idea is also used in [16] to develop an MPC scheme for systems subject to parametric uncertainties, which is in [7] extended to a mix of parametric uncertainties and bounded additive disturbances. Recently, in [17] the tube-based approach to parametric uncertainties is combined with a less conservative multistage MPC allowing the user to tradeoff between complexity and conservatism of the MPC scheme. In reality, however, uncertainties might not be parametric but are often more complex and dynamic. In [8] and [18], dynamic uncertainties are captured with a finite \(\ell_\infty\)-gain and conservatively overapproximated using a constant additive bound in order to use the MPC schemes designed for additive bounded disturbance. In [19], a dynamic bound in form of a stable scalar system is used to ensure robust constraint satisfaction and stability when applying MPC with a reduced order model, despite the dynamic uncertainty arising from the model order reduction. Similarly, in [20] and [21] such an error bounding system is used to describe the dynamic uncertainty and a multistage MPC is employed. However, no guarantees regarding robust constraint satisfaction or stability are provided and the computational demand increases exponentially compared to a nominal MPC. Instead of designing a new MPC scheme, existing MPC schemes have been tested in [22]–[24] for robust stability against dynamic uncertainties satisfying an IQC, however, without guarantees for robust state constraint satisfaction. Summing up, there is a need for a robust MPC scheme that can guarantee stability and constraint satisfaction for a general class of dynamic uncertainties. IQCs offer this generality and can describe a wide variety of uncertainty classes. In this article, we design a robust MPC scheme for systems subject to dynamic uncertainties that are bounded by IQCs and to the best knowledge of the authors, there exist no such MPC schemes ensuring robust stability and constraint satisfaction.

**Contribution and Outline:** We propose a tube-based MPC scheme for state and input constrained linear systems subject to dynamic uncertainties that are described by \(\rho\)-hard IQCs. In Section II, we start by describing the problem setup, providing a brief introduction into tube-based MPC and \(\rho\)-hard IQCs, as well as connecting the time-domain \(\rho\)-hard IQCs to frequency domain \(\rho\)-IQCs via positive–negative multipliers. In Section III, we extend the framework of \(\rho\)-hard IQCs to interconnections with external inputs by using a scalar exponentially stable system to bound the state of the extended system. In our tube-based MPC setup, we show that this scalar system provides an upper bound on the error between the true uncertain system and a nominal prediction model. In Section IV, we develop a tube-based MPC scheme that predicts the state of this dynamic error bound along with the nominal model and utilizes it as a scaling parameter for the tube size. This results in a tube dynamics which adjusts its size online according to the excitation of the dynamic uncertainty. As our key contribution, we prove that the proposed MPC scheme guarantees input-to-state stability (ISS) against bounded external disturbances as well as robust constraint satisfaction despite the dynamic uncertainty in the feedback loop. Further, in Section V, we demonstrate the flexibility and the reduced conservatism of the IQC approach in a numerical example and discuss some implementation aspects.

Preliminary results regarding the incorporation of IQCs in MPC can be found in the conference proceedings [25]. Compared to [25], the present article provides a more comprehensive analysis including connections to frequency domain IQCs, a more elaborate example, and a less conservative controller resulting from an improved scheme and a better proof technique. In particular, the initial MPC design in [25] considers a fixed nominal system, and hence the set of nominally feasible control actions is independent of the measured state, thus reducing to a robust trajectory planning with a linear stabilizing feedback. As one of the main technical contributions of the present work, we extend the tube dynamics to allow for an optimization of the initial state of the nominal system, thus, significantly increasing the flexibility of the proposed approach.

**Notation:** We denote the unit circle in the complex plane by \(T = \{z \in \mathbb{C} \mid |z| = 1\}\), the set of real rational and proper transfer matrices of dimension \(n \times m\) with \(\mathbb{RL}_{\infty}^{n \times m}\) and its subset of functions analytic outside the closed unit disk with \(\mathbb{RH}_{\infty}^{n \times n}\). Whenever the dimensions are obvious from the context, we write \(\mathbb{RL}_{\infty}\) and \(\mathbb{RH}_{\infty}\). The set of sequences in \(\mathbb{R}^n\) is denoted by \(\ell_2^n = \{(q_k)_{k \in \mathbb{N}} \mid q_k \in \mathbb{R}^n\}\), the subset of square summable sequences is denoted by \(\ell_2^n = \{(q_k)_{k \in \mathbb{N}} \mid \sum_{k=0}^{\infty} |q_k|^2 < \infty\}\), and for \(\rho \in (0,1)\) the subspace of exponentially square summable sequences is denoted by \(\ell_{2,\rho}^n = \{q \in \ell_2^n \mid \sum_{k=0}^{\infty} \rho^{-2k} ||q_k||^2 < \infty\}\). The \(z\)-transformation of a sequence \(q \in \ell_2^n\) is denoted by \(\tilde{q}(z) = \sum_{k=0}^{\infty} q_k z^{-k}\). For symmetric forms \(X^T P X\) with \(P \in \mathbb{R}^{n \times n}\) and \(X \in \mathbb{R}^{n \times m}\), we write \(\|x\|_P^2 = x^T P x\) for convenience. For matrices \(A, B, C, D\) with suitable dimensions we define \(\rho_{\mathcal{R}}\):

\[
\rho_{\mathcal{R}}[D] = D + C(zI - A)^{-1} B.
\]

For \(\rho \in (0,1)\) and \(\Pi \in \mathbb{RL}_{\infty}^{n \times m}\) we define the notation \(\Pi_{\rho}\) as the multiplier \(\Pi_{\rho} : \mathbb{C} \rightarrow \mathbb{C}^{n \times m}, z \rightarrow \Pi(z)\) and further, we denote the para-Hermitian conjugate with \(\Pi^\dagger(z^{-1})\). For positive definite matrices \(P > 0\) we define the norm \(\|x\|_{\rho}^2 = x^T P x\).

**II. Setup**

We consider the following linear discrete-time system:

\[
x_{t+1} = Ax_t + B_G^y w_t + B_G^d d_t + B_G^u u_t
\]

\[
y_t = Cx_t + D_G^y w_t + D_G^d d_t + D_G^u u_t
\]
with state vector \( x_t \in \mathbb{R}^{n_x} \), control input \( u_t \in \mathbb{R}^{n_u} \), external disturbance \( d_t \in \mathbb{D} = \{ d \in \mathbb{R}^{n_d} \mid ||d||_\infty \leq d_{\text{max}} \} \), \( d_{\text{max}} \geq 0 \), \( \Xi > 0 \), uncertainty signal \( w_t \in \mathbb{R}^{n_w} \), output \( y_t \in \mathbb{R}^{n_y} \), and the real matrices \( A, B_G, B^d_G, B^u_G, C, D^w_G, D^d_G, D^u_G \) with suitable dimensions. The system is interconnected in feedback with a bounded and causal uncertainty \( \Delta : \ell^2_{\mathbb{R}} \to \ell^2_{\mathbb{R}} \)

\[
w_t = \Delta(y)_t
\] (2)
which is dynamic and depends on the past measurements. Hence, the uncertainty may, for example, contain unmodeled dynamics, model mismatch, or delays. Note that the output \( y \) does not denote the vector of measured signals but the vector of signals that enter the uncertainty \( \Delta \).

**Assumption 1 (Well posedness):** The operator \( \Delta \) is bounded and causal and the interconnection of (1) and (2) is well-posed, i.e., for each \( d \in \ell^2_{\mathbb{R}} \) and \( u \in \ell^2_{\mathbb{R}} \) there exists a unique response \( y \in \ell^2_{\mathbb{R}} \), \( w \in \ell^2_{\mathbb{R}} \), \( x \in \ell^2_{\mathbb{R}} \).

This assumption guarantees that system (1) admits a unique solution, i.e., there is no algebraic loop, which trivially holds in the case \( D_G^d = 0 \). Considering well-posed interconnections of a known system and an unknown system is a classical robust control setup, e.g., similar to [26].

The control objective is ISS from \( d \) to \( x \), while satisfying the polytopic state and input constraints

\[
H \begin{bmatrix} x_t \\ u_t \end{bmatrix} \leq h
\] (3)
for all times \( t \geq 0 \). The rows of \( H \in \mathbb{R}^{n_h \times (n_x + n_u)} \) and \( h \in \mathbb{R}^{n_h} \) are denoted by \( H_i \) and \( h_i \) for each \( i = 1, \ldots, n_h \), respectively. To keep the theoretical derivations concise and clear, we assume that full state measurement is available. We base our approach to solve this problem on tube-based MPC, which is introduced in the following.

### A. Tube-Based Model Predictive Control

In this subsection we briefly sketch the idea of tube-based MPC, which was introduced almost simultaneously by [4] and [13] for the case of additive bounded disturbances. MPC in general is based on predicting the state trajectories with a model and as common in MPC (e.g., [7]), we denote the predictions at time \( t \) that predict \( k \) steps into the future with the index \( k|t \). In the presence of disturbances and model mismatches, however, precise predictions are impossible, and, thus, in tube-based MPC a set confining all possible uncertain trajectories is predicted—the so-called tube. This tube is centered around a nominal trajectory that follows the true system dynamic (1) for \( k \geq 0 \)

\[
x_{k+1|t} = Ax_{k|t} + B^w_Gw_{k|t} + B^d_Gd_{k+t} + B^u_Gu_{k|t}
\] (5a)
\[
y_{k|t} = Cx_{k|t} + D^w_Gw_{k|t} + D^d_Gd_{k+t} + D^u_Gu_{k|t}
\] (5b)
\[
w_{k|t} = \Delta(y)_{k|t}
\] (5c)
starting at the current state \( x_0|t = x_t \) and having the past \( y_{-k|t} = y_{t-k} \) for \( k \in [1, t] \). While we assume full state measurement of \( x_t \), the disturbances \( d_{k+t} \), the uncertainty \( \Delta \), and, thus, \( w_{k|t} \) are unknown, and, thus, \( x_{k|t} \) for \( k \geq 1 \) is unknown as well. Hence, the possible future state \( x_{k|t} \) is neither a prediction (unknown at time \( t \)) nor a realization (we might choose other inputs), it is a what-if state meaning where would the state \( x_{t+k} \) be if from now (time \( t \)) on we apply the inputs \( u_{0|t}, \ldots, u_{k-1|t} \) and the external disturbances \( d_{t}, \ldots, d_{t+k-1} \).

The error between the possible future state \( x_{k|t} \) and the nominal prediction \( \xi_{k|t} \) is denoted by \( e_{k|t} = x_{k|t} - \xi_{k|t} \). To ensure that the error \( e_{k|t} \) does not diverge, the MPC control action \( u_{k|t} \) is augmented with a feedback of the error

\[
u_{k|t} = u_{k|t} + Ke_{k|t}
\] (6)
where the feedback gain \( K \) is static. From a robust control point of view it might seem unusual and limiting to consider a static feedback \( K \), however, to keep the derivations concise and clear and to be consistent with tube-based MPC literature we use a static \( K \) in this work, although it might be possible to extend the framework to dynamic controllers \( K \). Hence, \( u_{k|t} \) is a what-if input that includes knowledge of the possible future error \( e_{k|t} \). Thereby, the feedback \( K \) regulates \( x_{k|t} \) toward the nominal trajectory \( \xi_{k|t} \), while the control action \( v_{k|t} \) steers the nominal trajectory. This key feature of tube-based MPC significantly reduces the conservatism as the feedback \( K \) can keep the tube confining all possible trajectories small by stabilizing the error dynamics

\[
e_{k+1|t} = A_Ge_{k|t} + B^w_Gw_{k|t} + B^d_Gd_{k+t} + B^u_Gu_{k|t}
\] (7a)
\[
y_{k|t} = C_Ge_{k|t} + D^w_Gw_{k|t} + D^d_Gd_{k+t} + D^u_Gu_{k|t} + r_{k|t}
\] (7b)
\[
w_{k|t} = \Delta(y)_{k|t}
\] (7c)
where \( A_G = A + B^w_GK \) and \( C_G = C + D^w_GK \). The feedback interconnection (7) of the error dynamics and the uncertainty \( \Delta \) is well posed\(^2\) and in the form shown in Fig. 1 with

\[
G = \begin{bmatrix} A_G & B^w_G & B^d_G \\ C_G & D^w_G & D^d_G \end{bmatrix}.
\] (8)

For now, we have considered predictions at a fixed time \( t \). In closed loop, the MPC controller solves an open-loop finite-horizon optimal control problem to decide on the new nominal initial condition \( \xi_{0|t} \) and a new nominal input sequence \( v_{0|t} \). When determining \( \xi_{0|t} \) and \( v_{0|t} \) it must be ensured that the constraints (3) are not only satisfied for \( \xi_{t} \) and \( v_{t} \) but for the whole tube around this nominal trajectory to ensure robust constraint satisfaction, i.e., these constraints must be tightened according to the size of the tube. Then, the first input of the control sequence

\(^{1}\)Note that this setup includes the special case of two different disturbances \( d^w \) on the state and \( d^d \) the output. In this special case often separate bounds \( d^w_{\text{max}} \) and \( d^d_{\text{max}} \) are known and can be considered to reduce the conservatism.

\(^{2}\)Well posedness follows from Assumption 1 and the fact that (1) and (7) have the same feedthrough matrix \( D_G^d \).
is applied to the system, i.e., \( u_t = u_{0|t} = v_{0|t} + K(x_t - \xi_{0|t}) \), which recursively renders \( w_t = w_{0|t} \) and \( x_{t+1} = x_{1|t} \) since \( \Delta \) is causal. Fixing the initial state \( \xi_{0|t} = \xi_{1|t-1} \) to follow the nominal dynamics (4a), as proposed in the early work [13], simplifies the analysis significantly. This case has been considered in the preliminary conference article [25]. Nevertheless, at each time \( t \), we obtain a new measurement \( x_t \) and we want to make use of this new information when initializing the nominal trajectory \( \xi_{0|t} \). In [4], it was suggested to initialize \( \xi_{0|t} = x_t \), but it is not guaranteed that this choice is actually better. Thus, in [14] \( \xi_{0|t} \) is treated as free decision variable and is optimized over all \( \xi_{0|t} \) that contain the current measurement in the tube centered around \( x_t \). In the present work, we want to use this additional degree of freedom since it leads to faster convergence as discussed in [1] and larger regions of attraction as we will see in our numerical example. This, however, implies that we need to specify how the error evolves in closed loop, i.e., if the time \( t \) increments and a new nominal initial state \( \xi_{0|0} \) is chosen

\[
\epsilon_{0|t+1} = \epsilon_{1|t} - \xi_{0|t+1} + \xi_{1|t}.
\]

Although tube-based MPC schemes have different definitions of the tube, they are always based on a bound on the error \( \epsilon_k \). A key step in this article is to derive such an error bound based on the assumption that the input–output behavior of \( \Delta \) can be described with a \( \rho \)-hard IQCs and that the disturbance \( d_t \) is bounded. Therefore, in the rest of this section, we give a short introduction into \( \rho \)-hard IQCs, which provide a general framework to analyze interconnections of the form in Fig. 1 for dynamic and static uncertainties \( \Delta \).

### B. \( \rho \)-Hard Integral Quadratic Constraints

IQC originate from the seminal work [10] and are a powerful tool to analyze feedback interconnections as in Fig. 1 of a known linear system \( G \) and an unknown, possibly nonlinear, operator \( \Delta \). Originally, the framework was developed from a continuous-time frequency-domain point of view, however, the IQC framework has been extended to time-domain formulations via dissipation inequalities [27] and to discrete-time systems [26]. We build our analysis on the framework of \( \rho \)-hard IQCs, which were developed in [12], to analyze exponential stability of discrete-time systems with time-domain IQCs. This enables us to construct a bound on \( \epsilon_k \) in form of an exponentially stable error bounding system. Let us start by defining a time-domain \( \rho \)-hard IQC.

**Definition 1 (\( \rho \)-hard IQC, [12, Definition 3]):** Let \( \rho \in (0, 1], M \in \mathbb{R}^{n_M \times n_M} \) and \( \Psi \in \mathbb{R}^{n_Y \times (n_Y + n_w)} \). A bounded operator \( \Delta : \ell^{\infty}_{2c} \rightarrow \ell^{\infty}_{2c} \) is said to satisfy the \( \rho \)-hard IQC defined by \( (\Psi, M) \) if for all \( y \in \ell_{2c}^{\infty} \) and for all \( T \geq 1 \) the following inequality holds:

\[
\sum_{t=0}^{T-1} \rho^{-2t} p_t \delta y T_t p_t \geq 0, \text{ where } p = \Psi \Delta(y).
\]

The key idea when analyzing interconnections as in Fig. 1, where \( \Delta \) satisfies the \( \rho \)-hard IQC defined by \( (\Psi, M) \) is to replace the uncertain component \( \Delta \) with the filter \( \Psi \) and to consider \( w \) as an input that obeys the output constraint (10). This is sketched in Fig. 2. With a state space realization of the filter

\[
\Psi = \begin{bmatrix} A \Psi & B_{w}^y \cr C \Psi & D_{w}^y \end{bmatrix}
\]

we can write the transfer function \( w \rightarrow p \) (\( d = 0, r = 0 \)) as

\[
\begin{bmatrix} A & \rho^2 P \cr C & D \end{bmatrix} = \begin{bmatrix} A_G & 0 \\ B_{w}^y C_G & A_w \end{bmatrix} \begin{bmatrix} B_{w}^y D_{w}^y + B_{w}^y D_{w}^y + B_{w}^y \\ D_{w}^y C_G \end{bmatrix} = 0.
\]

We will denote the state vector of \( \Psi \) at time \( t \) with \( \psi_t \).

The following assumption is made on the uncertainty.

**Assumption 2 (Uncertainty):** The operator \( \Delta \) satisfies the \( \rho \)-hard IQC defined by \( (\Psi, M) \) and \( \rho \in (0, 1] \). The filter \( \Psi \) is initialized with \( \psi_0 = 0 \). There exists \( P > 0 \) such that the following matrix inequality holds:

\[
\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} = \begin{bmatrix} A_G & 0 \\ B_{w}^y C_G & A_w \end{bmatrix} \begin{bmatrix} B_{w}^y D_{w}^y + B_{w}^y \\ D_{w}^y C_G \end{bmatrix} < 0.
\]

Note that in the tube-based MPC setting from Section II-A, the matrices \( A_G \) and \( C_G \), and, thus, also \( A \) and \( C \) depend on the feedback gain \( K \). Throughout this work, we assume that a suitable \( K \) satisfying Assumption 2 is given. For a fixed gain \( K \) and a fixed constant \( \rho \), (13) reduces to a linear matrix inequality (LMI), which can thus be embedded in a suitable offline computation of the matrix \( P \) (cf. Remark 6 and 9). Based on Assumption 2, the following exponential stability bound was shown in [12].

**Theorem 1 (Exponential stability, [12, Th. 4]):** Let Assumption 1, 2, and \( (d, r) = 0 \) hold. Then system (7) is \( \rho \)-exponentially stable, i.e., \( \|e_t\| \leq \sqrt{\text{cond}(P)} \rho^t \|e_0\| \) for all \( t \geq 0 \).

Notice that the authors of [12] analyzed this interconnection with \( d = 0 \) and \( r = 0 \). Hence, this result cannot immediately be applied to our setup, as we have \( d \neq 0 \) and \( r \neq 0 \). However, Lemma 1 in Section III below extends this result under the same Assumption 2 to \( d \neq 0 \) and \( r \neq 0 \) and additionally includes an estimate to account for the nominal initial state optimization (9).

In the rest of this section, we provide more insights into \( \rho \)-hard IQCs by bridging the gap from the time-domain perspective to the frequency-domain framework of \( \rho \)-IQC [28], whereas the error bounds relevant for the MPC are developed in Section III.

\( \text{To be precise, the theorem is slightly altered compared to [12]: First, [12] requires only semi definiteness of the LMI (13), i.e., } \leq 0. \text{ And second, in [12] the theorem is stated for } D_{w}^y = 0. \) \text{ However, the result still holds due to the well-posedness assumption; the proof is analogous.}
particular, we will see that a general class of frequency-domain $\rho$-IQCs can be equivalently formulated as time-domain $\rho$-hard IQCs.

**Definition 2** ($\rho$-IQC, [28, Definition 6]): Let $\rho \in (0, 1)$ and $\Pi = \Pi^{*} \in \mathbb{R}^{(n_y+n_u) \times (n_y+n_u)}$. A bounded operator $\Delta : \ell_{2e}^{\rho} \to \ell_{2e}^{\rho}$ is said to satisfy the $\rho$-IQC defined by the multiplier $\Pi$ if for all $y \in \ell_{2,\rho}^{\rho}$ and $w = \Delta(y)$ the following inequality holds:

$$
\int_{T} \begin{bmatrix} \hat{y}(\rho z) \\ \hat{w}(\rho z) \end{bmatrix}^{*} \Pi(\rho z) \begin{bmatrix} \hat{y}(\rho z) \\ \hat{w}(\rho z) \end{bmatrix} \, dz \geq 0. 
$$

(14)

There is also an exponential stability result with $\rho$-IQCs formulated in the frequency domain.

**Theorem 2** (Exponential stability, [28, Th. 8]): Let $\rho \in (0, 1)$, $G_{\rho} \in \mathbb{R}^{\infty \times \infty}$ and $\Delta$ be a bounded causal operator. Suppose that for all $\tau \in [0, 1]$:

1. the interconnection of $G$ and $\tau \Delta$ is well posed;
2. $\tau \Delta$ satisfies the $\rho$-IQC defined by $\Pi$;
3. there exists $\varepsilon > 0$ such that

$$
\begin{bmatrix} G(\rho z) \\ I \end{bmatrix}^{*} \Pi(\rho z) \begin{bmatrix} G(\rho z) \\ I \end{bmatrix} \leq -\varepsilon I \forall z \in \mathbb{T}.
$$

(15)

Then the interconnection of $G$ and $\Delta$, as shown in Fig. 1, is exponentially stable with rate $\rho$ for $r = 0$, $d = 0$.

In order to compare Theorems 1 and 2, we note that each frequency-domain $\rho$-IQC can be related to time domain by a factorization of the multiplier $\Pi$.

**Definition 3** ($\rho$-factorization): Let $\Psi \in \mathbb{R}^{\infty, \infty}$, $\Pi \in \mathbb{R}^{\infty, \infty}$, and $M \in \mathbb{R}^{n_y \times n_p}$. We call $(\Psi, M)$ a $\rho$-factorization of $\Pi$ if $\Pi = \Psi^{*} M \Psi_{\rho}$ and $\Psi_{\rho} \in \mathbb{R}^{\infty, \infty}$.

If $(\Psi, M)$ is a $\rho$-factorization of $\Pi$, then, as shown in [28, Remark 10] by applying Parseval’s theorem, the $\rho$-IQC defined by $\Pi$ is satisfied if and only if (10) holds for $T = \infty$. Hence, $\rho$-hard IQCs imply $\rho$-IQCs, while the opposite is in general not true. Further, it was shown in [28, Corollary 12] that (15) is equivalent to the existence of $P = P^{*}$ satisfying (13). Note that again, the time-domain requirement $P > 0$ is strict. This leads to the interesting question for which multipliers the frequency-domain $\rho$-IQC also implies a $\rho$-hard IQC in the time domain and for which multipliers $P$ is guaranteed to be positive definite. If $\rho = 1$, it is known that so-called strict positive negative (PN) multipliers have both properties [26].

As defined in [26, Definition 4], strict PN-multipliers are multipliers $\Pi = \Pi^{*} \in \mathbb{R}^{\infty, \infty}$, where the first block diagonal entry with dimension $n_y \times n_y$ is positive definite and the second block diagonal entry with dimension $n_u \times n_u$ is negative definite for all frequencies $\varepsilon \in \mathbb{T}$. The following theorem extends the results of [26] to $\rho$-IQCs with general $\rho \in (0, 1]$ and show that strict PN multipliers admit $\rho$-hard factorizations and that (15) leads to (13) with $P > 0$.

**Theorem 3 (From $\rho$-IQCs to $\rho$-hard IQCs):** Let $\rho \in (0, 1)$, $G_{\rho} \in \mathbb{R}^{\infty, \infty}$, $\Pi \in \mathbb{R}^{\infty, \infty}$, and $\Pi_{\rho}$ be a strict PN multiplier. Then there exists a $\rho$-factorization $(\Psi, M)$ of $\Pi$ such that all $\Delta$ satisfying the $\rho$-IQC defined by $\Pi$ also satisfy the $\rho$-hard IQC defined by $(\Psi, M)$. Further, if (15) holds, then there exists a $P > 0$ such that (13) holds.

A proof of this result can be found in Appendix A. The theorem shows that a large class of frequency-domain $\rho$-IQC multipliers have a $\rho$-hard time-domain factorization. This is especially helpful since many IQCs are more conveniently derived in the frequency domain.

### III. Exponentially Stable Error Bounding System

In this section, we derive a bound on the error $e_{k|t}$ based on Assumption 2 by making use of the $\rho$-hard IQC that bounds $\Delta$. Instead of a static error bound, as in [8], we reduce conservatism with a dynamic error bound in form of a scalar exponentially stable system with the inputs $d$ and $r$, i.e., that depends on the disturbance and the nominal excitation of the uncertainty. This idea is inspired by the work [19], where an error bounding system was used to describe the deviation of model order reductions.

First, let us introduce the notation $\psi_{k|t}$ for the possible future trajectory of the state of filter $\Psi$ from (11)

$$
\psi_{k+1|t} = A_{\Psi} \psi_{k|t} + B_{\Psi} y_{k|t} + B_{\Psi}^{e} e_{k|t}
$$

(16a)

$$
p_{k|t} = C_{\Psi} \psi_{k|t} + D_{\Psi} y_{k|t} + D_{\Psi}^{e} e_{k|t}
$$

(16b)

with $\psi_{0|t+1} = \psi_{|t}$ and $\psi_{0|0} = \psi_{0}$. Note that this guarantees $\psi_{0|t} = \psi_{t}$ and $p_{0|t} = p_{t}$ since $w_{0|t} = w_{t}$ and $y_{0|t} = y_{t}$ as discussed above. Second, with the help of this notation and based on Assumption 2, we can bound the discrepancy $e_{k|t}$ between the nominal state $\xi_{k|t}$ and the possible future state $x_{k|t}$, as shown in the following lemma.

**Lemma 1 (Exponentially stable error bounding system):** Consider the interconnection (7) and let Assumption 2 hold. Then, there exist $\Gamma \in \mathbb{R}^{n_y \times n_p}$, $\Gamma > 0$ and $\gamma > 0$ satisfying the following LMI:

$$
\begin{bmatrix}
\begin{bmatrix} I \quad 0 \\
0 \quad A \end{bmatrix} \\
\begin{bmatrix} B_{G}^{e} \\
0 \end{bmatrix}
\end{bmatrix} < 0

$$

(17)

with $\Lambda = \text{diag}(\gamma \Xi, \Gamma)$. Further, for any sequences $d \in \ell_{2e}^{\rho}$, $(r_{k|t})_{t \in \mathbb{N}}$, $r_{k|t} \in \ell_{2e}^{\rho}$, and $e_{0|t} \in \ell_{2e}^{\rho}$ the following inequality holds for all times $t \geq 0$ and all predictions $k \geq 0$:

$$
\| \begin{bmatrix} e_{k|t} \\
\psi_{k|t} \end{bmatrix} \|_{P}^{2} \leq c_{k|t}
$$

(18)

with $c_{0|0} = \| e_{0|0} \|_{P}^{2}$ and $c_{k|t}$ recursively defined by

$$
k_{k+1|t} = \rho^{2} c_{k|t} + \gamma \| d_{t+k} \|_{I}^{2} + \| r_{k|t} \|_{I}^{2}
$$

(19a)

$$
c_{0|t+1} = C_{1|t} + \| \begin{bmatrix} e_{0|t+1} \\
\psi_{0|t+1} \end{bmatrix} \|_{P}^{2} - \| \begin{bmatrix} e_{1|t} \\
\psi_{1|t} \end{bmatrix} \|_{P}^{2}.
$$

(19b)

**Proof:** First, we derive (17) from (13) by using Finsler’s lemma; second, we use (17) to show that a dissipation inequality holds; and third, summing up this inequality from 0 to $t$ yields (18) and (19).
The LMI (13) guarantees that (17) holds whenever multiplied from left with \( [\epsilon_k] \), \( \psi_k \), \( w_k \), \( d_k \), \( r_k \) \( ] \) and from right with its transpose for \( d_k = 0 \), \( r_k = 0 \). Thus, Finsler’s lemma [29] guarantees the existence of \( \gamma > 0 \) large enough such that (17) holds with \( \Lambda = \gamma I \) also for nonzero \( r_k \), \( d_k \). Therefore, we know that any \( \Gamma \geq \gamma I \) and \( \Xi \geq \gamma I \) satisfy (17) with \( \Lambda = \text{diag}(\Gamma, \Xi) \geq \gamma I \) as well.

For the second part, we use.

\[
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
\begin{bmatrix}
B_G^T \\
B_g^T \\
D_G^T \\
D_g^T
\end{bmatrix}
\begin{bmatrix}
\kappa_{\tau} \\
\psi_{\tau} \\
w_{\tau} \\
\nu_{\tau}
\end{bmatrix}
\begin{bmatrix}
e_{\kappa_{\tau}} \\
\psi_{\kappa_{\tau}} \\
w_{\kappa_{\tau}} \\
\nu_{\kappa_{\tau}}
\end{bmatrix}
\]

(7a,12) \[ A_G \kappa_{\tau} + B_G^T w_{\kappa_{\tau}} + B_g^T d_{\kappa_{\tau} + \tau} \\
A \kappa_{\tau} + B_G^T w_{\kappa_{\tau} + \tau} + B_g^T w_{\kappa_{\tau}} \\
C \psi_{\kappa_{\tau} + \tau} + D_G^T w_{\kappa_{\tau}} \\
D_g^T w_{\kappa_{\tau}}
\]

when multiplying (17) from the left with \( [\epsilon_{\kappa_{\tau}} \psi_{\kappa_{\tau}} w_{\kappa_{\tau}} d_{\kappa_{\tau} + \tau} \nu_{\kappa_{\tau}}] \) and from the right with its transpose, which leads to the dissipation inequality.

\[
\left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 - \rho^2 \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 + p_{\kappa_{\tau}}^T M_{\kappa_{\tau}} - \gamma \left\| d_{\kappa_{\tau} + \tau} \right\|_2^2 - \left\| r_{\kappa_{\tau}} \right\|_2^2 \leq 0.
\]

Now we use a telescoping sum argument by multiplying (22) with \( \rho^{2(k-1)} \) and sum over \( \kappa \) from 1 to \( k - 1 \)

\[
0 \geq \rho^{(22)} \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 - \rho^2 \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 + \sum_{\kappa=1}^{k-1} \rho^{2(k-\kappa-1)} \left\| p_{\kappa_{\tau}}^T M_{\kappa_{\tau}} - \epsilon_{\kappa_{\tau}} \right\|_p =: \Sigma_3.
\]

Finally, we sum up \( \Sigma_1 \), \( \Sigma_2 \), and \( \Sigma_3 \) with suitable factors of \( \rho^2 \). Since the sequence \( \hat{p}_i := p_{\kappa_{\tau}} \) for \( i = \tau = 0, \ldots, t \) appended with \( p_{\kappa_{\tau}} = p_{\kappa_{\tau}} \) for \( i = t = \kappa = 0, \ldots, k = 1 \) is a feasible filter output trajectory, the \( \rho \)-hard IQC (10) holds for this sequence and we can conclude.

\[
0 \geq \rho^{2(t+k-1)} \Sigma_1 + \rho^{2(k-1)} \Sigma_2 + \Sigma_3
\]

\[
= \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 - \hat{c}_{\kappa_{\tau}} + \rho^{2(t+k-1)} \sum_{i=0}^{t+k-1} \rho^{-2 i} \hat{p}_i^T M \hat{p}_i
\]

\[
\Rightarrow \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 - \hat{c}_{\kappa_{\tau}}\leq 0.
\]

Remark 1: Lemma 1 is not only relevant for considering MPC schemes but also for general \( \rho \)-hard IQC theory. In particular the error bound (18) and the error bound dynamics (19a) for all \( k \geq 0 \) and fixed \( t = 0 \) might be of interest to other settings considering the interconnection in Fig. 1, for example, reachability analysis with IQCs (compare [30]). Then, these equations provide a bound on the state \( \epsilon_{\kappa_{\tau}} \) of system \( G \) for nonzero initial conditions \( \epsilon_{\kappa=0} \) and nonzero external inputs \( r_{\kappa=0} \) and \( d_k \).

Unfortunately, we cannot use the bound (18) of Lemma 1 for constraint tightening in an MPC scheme, since it depends on the generally unknown future disturbances \( d_{\kappa+1} \) in the recursion (19a) and the generally unknown filter state \( \psi_{\kappa=0} = \psi_{\kappa=0} \) in the recursion (19b). Therefore, as a third step, we introduce a known upper bound \( s_{\kappa_{\tau}} \) on \( c_{\kappa_{\tau}} \).

Theorem 4 (Tube dynamics): Consider the interconnection (7), let Assumption 2 be hold. Further, let \( P \) be decomposed into \( P = \left[ P_1 P_2 \right] \) with \( P_1 \in \mathbb{R}^{n_x \times n_x} \) and define \( \mathcal{P} = \mathcal{P}_1 - P_2^T P_2 > 0 \). Then, there exist \( \Gamma > 0 \) and \( \gamma > 0 \) satisfying (17) and the following inequality holds for any sequences \( d \in l_2, \xi_0 \in \ell_2, (v_\ell)_{\ell \in \mathbb{N}}, v_\ell \in l_2, \) and any initial condition \( x_0 \in \mathbb{R}^{n_x} \).

\[
\left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p^2 \leq s_{\kappa_{\tau}}
\]

(23)

where \( s_{\kappa_{\tau}} \) is recursively defined by.

\[
\left\| s_{\kappa_{\tau}} \right\|_p = \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p + \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p + \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p + \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p
\]

(24a)

\[
s_{\kappa_{\tau}} = \rho^{2 s_{\kappa_{\tau}}} + \gamma d_{\kappa_{\tau}}^2 + \left\| r_{\kappa_{\tau}} \right\|_2^2
\]

(24b)

with \( s_{\kappa_{\tau}} = \left\| \begin{bmatrix} \epsilon_{\kappa_{\tau}} \\ \psi_{\kappa_{\tau}} \\ w_{\kappa_{\tau}} \\ d_{\kappa_{\tau} + \tau} \\ \nu_{\kappa_{\tau}} \end{bmatrix} \right\|_p \).
Recall Lemma 1 which guarantees the existence of $P$ which is positive semidefinite, which can be seen by looking at its Schur complement $P_2^T P_1 P_2 = 0$ for $0 \geq 0$. In view of Lemma 1, we can infer
\[
\|e_k\|^2 \leq \|\psi_{k}\|^2 P \leq c_k r.
\]
In order to prove (23), we show $c_{k|t}$ using a proof of induction. The induction basis is trivial, since $s_{0|0} = c_{0|0}$. Now we have to do two induction steps, one from $k|t$ to $k+1|t$ (for $t \geq 0$) and one from $1|t-1$ to $0|t$ (for $t \geq 1$). Let us start with the fromer by using the induction hypothesis (IH) $c_{k|t} \leq s_{k|t}$ and $\|d_{t+k}\|^2 \leq d_{k+1,max}^2$ (34b)
\[
s_{k+1|t} \geq \rho^2 c_{k|t} + \gamma \|d_{t+k}\|^2 + \|e_k\|^2 P \leq c_k + 1.\]
In order to take the step from $1|t-1$ to $0|t$, we show that $s_{0|t} = s_{1|t-1} = c_{0|t} - c_{1|t-1}$, which then implies $c_{0|t} \leq s_{0|t}$ by (IH). As stated in (19b), $c_{0|t} = c_{1|t-1}$ depends on $\psi_{1|t-1}$, thus we maximize it over all possible $\psi_{1|t-1}$, i.e., all that satisfy
\[
\left\| \frac{e_{1|t-1}}{\psi_{1|t-1}} \right\|^2 \leq s_{1|t-1}.\]
Hence, we solve
\[
\max_{\psi_{1|t-1}} \left\| \frac{e_{0|t}}{\psi_{1|t-1}} \right\|^2 - \left\| \frac{e_{1|t-1}}{\psi_{1|t-1}} \right\|^2.
\]
To this end, we transform (27) to an easier form. First, we apply the coordinate shift $\tilde{\psi} = \psi_{1|t-1} + P_2^{-1} P_2 e_{1|t-1}$, which transforms the constraint (26) to
\[
s_{1|t-1} \geq \left\| \frac{e_{1|t-1}}{\psi_{1|t-1}} \right\|^2 P_1 + 2e_{1|t-1}^T P_2^T (\tilde{\psi} - P_2^{-1} P_2 e_{1|t-1}) + \left\| \tilde{\psi} - P_2^{-1} P_2 e_{1|t-1} \right\|^2 P_{22}.
\]
and the objective of (27) to
\[
\left\| \frac{e_{0|t}}{\psi_{1|t-1}} \right\|^2 - \left\| \frac{e_{1|t-1}}{\psi_{1|t-1}} \right\|^2 P_1 + 2(e_{1|t-1}^T P_2 (e_{0|t} - e_{1|t-1})) + 2 \left\| \tilde{\psi} - P_2^{-1} P_2 e_{1|t-1} \right\|^2 P_1 + 2e_{1|t-1}^T P_2 (e_{0|t} - e_{1|t-1}) \right\|^2 P_{22}.
\]
Second, we apply the linear transformation $\tilde{\psi} = P_2^{1/2} \tilde{\psi}$ and obtain that the optimization problem (27) is equivalent to
\[
\max_{\tilde{\psi}} \left\| e_{0|t} \right\|^2 P - \left\| e_{1|t-1} \right\|^2 + \left\| e_{0|t} - e_{1|t-1} \right\|^2 P_{diff} + \left\| e_{1|t-1} \right\|^2 P_{aff} + 2 \left\| e_{0|t} - e_{1|t-1} \right\|^2 F_{diff} + \left\| \tilde{\psi} \right\|^2 \leq s_{1|t-1} - \left\| e_{1|t-1} \right\|^2 P.\]
This optimization problem has an affine objective and the constraint is a scaled unit ball. Thus the analytical maximum is attained at $\tilde{\psi} = s_{1|t-1} - \left\| e_{1|t-1} \right\|^2 F_{diff}$ with $\tilde{\psi}$ pointing in the direction of $P_2^{-1} P_2 e_{0|t} - e_{1|t-1}$, i.e., the maximum of (27) is
\[
\left\| e_{0|t} \right\|^2 P - \left\| e_{1|t-1} \right\|^2 + \left\| e_{0|t} - e_{1|t-1} \right\|^2 P_{diff} + \left\| e_{0|t} - e_{1|t-1} \right\|^2 P_{aff} + 2 \left\| e_{0|t} - e_{1|t-1} \right\|^2 F_{diff} + \frac{1}{2} \left\| e_{0|t} - e_{1|t-1} \right\|^2 P_{aff} \leq s_{1|t-1} - \left\| e_{1|t-1} \right\|^2 P .
\]
By definition (24a), this is $s_{0|t} - \left\| e_{1|t-1} \right\|^2$. To conclude, we have shown that $s_{0|t} \leq s_{1|t-1}$, which implies by (IH) $s_{0|t} \leq s_{0|t}$, and, thus, completes the proof by induction.

IV. PROPOSED MPC SCHEME

In this section, we propose an MPC scheme that handles dynamic uncertainties by using the error bounding system from Theorem 4 as tube dynamics, which ensures that the tube confines all possible trajectories. In order to ensure constraint satisfaction of the true but unknown system, we have to choose the nominal inputs $u_{k|t}$ such that the whole tube around the nominal trajectory is feasible.

Lemma 2 (Constraint tightening): Consider the interconnection (4)–(7) and the tube dynamics (24). Let Assumption 2 hold. Then, the constraints $H \left[ \begin{array}{c} x_{k|t} \\ u_{k|t} \end{array} \right] \leq h$ hold whenever
\[
H \left[ \begin{array}{c} x_{k|t} \\ y_{k|t} \end{array} \right] \leq h - \sqrt{g_{k|t}}
\]
holds, where $g_k = \|P - \frac{1}{2} I K^T \| H_k^T$ for $i = 1, \ldots, n_h$.

Proof: Since Assumption 2 holds, the error bound (23) from Theorem 4 is valid. This implies
\[
H \left[ \begin{array}{c} x_{k|t} \\ u_{k|t} \end{array} \right] = H \left[ \begin{array}{c} x_{k|t} + e_{k|t} \\ u_{k|t} + K e_{k|t} \end{array} \right] \leq \max_{\|e\| \leq s_{k|t}} H \left[ \begin{array}{c} x_{k|t} + \tilde{e} \\ u_{k|t} + K \tilde{e} \end{array} \right]
\]
where we can solve the maximization problem with the transformation $\tilde{e} = \frac{1}{2} \tilde{e}$
\[
H \left[ \begin{array}{c} x_{k|t} \\ u_{k|t} \end{array} \right] \leq \max_{\|e\| \leq s_{k|t}} H \left[ \begin{array}{c} x_{k|t} + \frac{1}{2} \tilde{e} \\ u_{k|t} + K \tilde{e} \end{array} \right]
\]
Hence, we have shown $H \left[ \begin{array}{c} x_{k|t} \\ u_{k|t} \end{array} \right] \leq H \left[ \begin{array}{c} x_{k|t} \\ u_{k|t} \end{array} \right] + g\sqrt{s_{k|t}} \leq h$.  

At each time step $t \geq 0$ we measure the current state $x_t = x_{0|t} = x_{1|t-1}$ and solve the following optimization problem based on this measurement and the previously predicted nominal
state $\xi_{1:T} = 0$ and tub size $s_{1:T}$

$$\min_{v_k \in \mathcal{C}_{0|t}} \sum_{k=0}^{T-1} \left( \|\xi_k(t)\|_Q^2 + \|v_k(t)\|_R^2 \right) + \|\xi_{T\mathcal{C}_{0|t}}\|_S^2$$

(30a)

s.t. initial constraint (24a)

terminal constraints (29) for $k = 0, \ldots, T - 1$

nominal dynamics (4) for $k = 0, \ldots, T - 1$

tube dynamics (24b) for $k = 0, \ldots, T - 1$

tightened constraints (29) for $k = 0, \ldots, T - 1$

where $\Omega \subseteq \mathbb{R}^{n_x+1}$ is the terminal constraint set and $Q, S \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite weighting matrices. Hence, the initial nominal state $\xi_{0|t}$ is a decision variable.

We denote the estimator of problem (30) by $\hat{\xi}_{0|t}^*$ and $\hat{s}_{0|t}$, with the corresponding trajectories $\hat{\xi}_{0|t}^*$ and $\hat{s}_{0|t}$. Then, as is standard in tube-based MPC and as defined in (6), the control input is given by $u_k = v_0^\top + K(x_k - \hat{\xi}_{0|t}^*) + v_0^\top$. Note that this implies for the closed loop system that $x_1 = x_0^\top + \psi_t = \psi_0^\top + y_t = y_0^\top$, $p_t = p_0^\top$, and $w_t = w_0^\top$, where the stars denote that these signals result from $v_t = v_0^\top$ and $\hat{\xi}_{0|t} = \hat{\xi}_{0|t}^*$. Note that the stage cost only acts on the nominal state and input, comparable to standard tube-based MPC designs (e.g., [4], [14], [15], [19], etc.). Hence, the asymptotic behavior is dominated by $u = Kx$. Even further, and similar to [4], [15], and [19], the MPC optimizer chooses $\hat{\xi}_{0|t} = 0$ and $v_0 = 0$ if the controller $K$ is guaranteed to steer the true system toward 0 without violating the constraints. In this sense, the MPC is restrained and interferes only if robust constraint satisfaction of the system controlled by $K$ cannot be guaranteed. To ensure that problem (30) is recursively feasible, we need to design suitable terminal conditions.

**Assumption 3 (Terminal conditions):** The matrices $Q, R,$ and $S$ are positive definite. The terminal set $\Omega$ contains the origin in its interior and there exist $K_{0|t} \in \mathbb{R}^{n_x \times n_x}$ such that for all $[\xi^\top \ s] = \Omega$ we have

1) positive invariance

$$\left( A + B_G^\top K_{0|t} \right) \xi + \rho \gamma \| \xi \|_R^2 + \| (A + D_G^\top K_{0|t}) \xi \|_T^2 + \gamma \rho \alpha_{\max}^2 \right) \subseteq \Omega ;$$

2) constraint satisfaction: $H [ \xi | K_{0|t} ] \leq - \sqrt{\rho \gamma}$

3) terminal cost decrease

$$\| (A + B_G^\top K_{0|t}) \xi \|_S^2 \leq - \| \xi \|_Q^2 - \| K_{0|t} \xi \|_R^2 ;$$

In Section IV-B below, we will discuss how to construct $K_{0|t}$, $S$ and $\Omega$ that satisfy Assumption 3. Now, we have all ingredients to show that the MPC controller indeed stabilizes the system and guarantees robust constraint satisfaction.

**Theorem 5 (Stability and recursive feasibility):** Let Assumption 2 and 3 hold with $\rho < 1$. Let $\psi_0 = 0$ and assume that the optimization problem (30) is feasible at time $t = 0$. Then (30) is feasible for all $t \geq 0$, the constraints (3) are satisfied for all times $t \geq 0$ and the closed loop satisfies the following ISS bound: there are a constant $a_d > 0$ and a class $\mathcal{K}$ function $\alpha$ such that for all $N \geq 0$ it holds

$$\sum_{t=0}^N \| x_t \|_S^2 \leq \alpha(f(x_0)) + a_d \sum_{t=0}^{N-1} \| d_t \|_2^2 .$$

(31)

**Proof:** The proof is divided into three parts.

**Recursive Feasibility:** We show recursive feasibility by induction. Therefore, assume (30) is feasible at time $t - 1$, then define the following candidate solution:

$$\hat{v}_{k|t} = \begin{cases} v_{k+1|t-1}^* & \text{for } k = 0, \ldots, T - 2 \\ K_{0|t} \xi_{k|t-1}^* & \text{for } k = T - 1 \end{cases} \quad \hat{s}_{k|t} = \begin{cases} s_{k+1|t-1}^* & \text{for } k = 0, \ldots, T - 1 \\ \rho^2 \hat{s}_{k|t-1}^\top + \| \hat{r}_{T-1|t} \|_2^2 + \gamma \rho^2 \alpha_{\max}^2 & \text{for } k = T \end{cases}$$

where $\hat{x}_{T-1|t} = (C + D_G K_{0|t}) \xi_{T-1|t}$. This candidate solution follows the nominal dynamics (4) and the tube dynamics (24). The induction hypothesis yields (29) for $k \in [0, T - 2]$ as well as $[\xi_{T-1|t}^\top s_{T-1|t}^\top] \subseteq \Omega$. With 2) in Assumption 3 it follows (29) for $k = T - 1$ and with 1) in Assumption 3 it follows (30). Hence, this candidate solution is feasible.

**Robust constraint satisfaction:** Follows immediately from Lemma 2, recursive feasibility, and $x_t = x_0^\top$, $u_t = u_0^\top$.

**Input-to-state stability:** To derive the ISS bound (31), we note that the LMI (13) from Assumption 2 implies

$$\begin{bmatrix} -\rho \gamma & I \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} B_G^0 & B_G^u \\ B_G^0 & B_G^u \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} x_t \\ \psi_t \\ w_t \\ d_t \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} v_t - K \xi_t \end{bmatrix} \end{bmatrix} \end{bmatrix} = 0$$

(10, 16, 12)

for some $\gamma > 0$. This can be seen by applying Finsler’s lemma analogous to the proof of Lemma 1. Let us recall that $x_t = x_0^\top$, and, thus, $x_t = x_0^\top + \psi_t = \psi_0^\top + y_t = y_0^\top$, $p_t = p_0^\top$, $w_t = w_0^\top$, and similarly denote the close-loop nominal input $v_t = v_0^\top$ and state $\xi_t = \xi_t^\top$. Then, by multiplying $[x_t^\top \psi_t^\top \ w_t^\top \ d_t^\top (v_t - K \xi_t)]^\top$ from left to the above LMI and its transpose from right, we obtain with

$$A \hat{x} + B_G^0 \hat{w} + B_G^u \hat{d} + B_G^u \hat{u} = A \hat{x} + B_G^0 \hat{w} + B_G^u \hat{d} + B_G^u \hat{u} = A \hat{x}$$

(10, 16, 12)

$\alpha(0) = 0$.
the dissipation inequality
\[
\left\| \begin{bmatrix} x_{\tau+1} \\ \psi_{\tau+1} \end{bmatrix} \right\|_p^2 - \rho^2 \left\| \begin{bmatrix} x_{\tau} \\ \psi_{\tau} \end{bmatrix} \right\|_p^2 + p_1^\top M_1 p_1 - \gamma \| d_{\tau} \|_2^2 - \gamma \| v_{\tau} - K \xi_{\tau} \|_2^2 \leq 0.
\]
Multiplying this inequality with \( \rho^{2(t-\tau-1)} \), summing it up from \( \tau = 0 \) to \( \tau = t-1 \), and using (10) yields
\[
\| x_t \|_P^{2(\tau)} \leq \left\| \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} \right\|_P^2 \leq \rho^{2t} \| x_0 \|_{F_1}^2 + \sum_{\tau=0}^{t-1} \rho^{2(t-\tau-1)} \gamma \left( \| d_\tau \|_2^2 + \| v_\tau - K \xi_\tau \|_2^2 \right).
\]
Summing this inequality once more from \( t = 0 \) to \( t = N \) and using the geometric series \( \sum_{i=0}^{N} \rho^{2i} \leq \frac{1}{1-\rho^2} \) yields
\[
\sum_{t=0}^{N} \| x_t \|_P^2 \leq \frac{1}{1-\rho^2} \| x_0 \|_{F_1}^2 + \sum_{t=0}^{N-1} \frac{\gamma}{1-\rho^2} \| d_t \|_2^2 + \| v_t - K \xi_t \|_2^2.
\]
(32)

To proceed, we need to bound the sum over \( \xi_t \) and \( v_t \). Therefore, let us introduce the notation \( J_T(\xi_t, v_t) \) for the objective function in problem (30) and use the suboptimality of the candidate solution to obtain
\[
J_T(\xi_t^*, v_t^*) - J_T(\xi_{t-1}^*, v_{t-1}^*) \leq J_T(\xi_t, v_t) - J_T(\xi_{t-1}, v_{t-1})
\]
\[
= \| \xi_t \|_Q^2 + \| K_0 \xi_t \|_R^2 + \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2
\]
\[
+ \| (A + B_0^\top K_1) \xi_t \|_Q^2 - \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2.
\]
and further with the third property of Assumption 3 it follows:
\[
J_T(\xi_t^*, v_t^*) - J_T(\xi_{t-1}^*, v_{t-1}^*) \leq -\| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2 - \| \xi_{t-1} \|_Q^2.
\]
If we sum this inequality from \( t = 1 \) to \( t = N \) we find that
\[
\sum_{t=0}^{N-1} \left( \| \xi_t \|_Q^2 + \| v_t \|_R^2 \right) \leq J_T(\xi_0^*, v_0^*) - J_T(\xi_{N-1}^*, v_{N-1}^*) \leq J_T(\xi_0^*, v_0^*) \leq \alpha_0(\| x_0 \|)
\]
(33)
where the second inequality holds due to nonnegativity of \( J_T(\xi_{N-1}^*, v_{N-1}^*) \) for all \( N \) and the third inequality is discussed in the following. For \( (x_0, 0) \in \Omega \) we know that problem (30) is feasible, since we can choose \( \xi_0 = x_0 \), which implies \( s_{0|0} = \| x_0 \|_{F_1}^2 = 0 \), and the local controller \( \hat{v}_{0|0} = K_0 \hat{\xi}_{0|0} \), which is feasible due to Assumption 3. Thus, we conclude \( J_T(\xi_0^*, v_0^*) \leq J_T(\xi_0, \hat{v}_{0|0}) \leq \| \xi_0 \|_Q^2 = \| x_0 \|_Q^2 \), where the second inequality follows from repeatedly applying condition 3 of Assumption 3. Since the origin is in the interior of \( \Omega \) this bound holds in a neighborhood of \( x_0 = 0 \). We can extend such a bound by a class \( K \) function \( \alpha_0 \) over the whole feasible set, i.e., \( J_T(\xi_0^*, v_0^*) \leq \alpha_0(\| x_0 \|) \) (see [1, Prop. B.25]) due to local boundedness of \( J_T(\xi_0^*, v_0^*) \) for feasible \( x_0 \). Hence, it follows (33). The positive definiteness of \( Q \) and \( R \) ensures existence of a constant \( a_2 > 0 \) such that
\[
a_2 \sum_{t=0}^{N-1} \| v_t - K \xi_t \|_2^2 \leq \sum_{t=0}^{N-1} \left( \| \xi_t \|_Q^2 + \| v_t \|_R^2 \right) \leq a_0(\| x_0 \|)
\]
(34)
for all \( N \). If we use (34) in (32) and do some basic algebra to estimate the positive definite weighting matrices, then we obtain that there exist constants \( a_3, a_4, a_d > 0 \) such that
\[
\sum_{t=0}^{N} \| x_t \|_P^2 \leq a_3 \| x_0 \|_P^2 + a_4 \| x_0 \|_P^2 + a_d \sum_{t=0}^{N-1} \| d_t \|_2^2.
\]
Defining \( \alpha(\| x_0 \|) = a_3 \| x_0 \|_P^2 + a_4 \| x_0 \|_P^2 \) concludes the proof.

Remark 2: The bound (31) can be formulated with \( \alpha(\| x_0 \|) = a_0 \| x_0 \|_P^2 \) if only a compact set of initial conditions is considered, e.g., due to compact constraints. In this case, the function \( \alpha_0 \) in the proof can be constructed as quadratic function from the local quadratic bound and the maximum of \( J_T \) on this compact set. The difficulty in achieving a quadratic bound without considering compact sets stems from the nonlinear constraints. Such a quadratic bound is desirable, since it guarantees not only asymptotic but exponential stability in the absence of disturbances.

A. Extensions, Special Cases, and Discussion

In this subsection, we discuss some special cases of the scheme and further extensions.

Remark 3: In special cases, the recursion of the error bound \( s_{0|t} \) in (24a) can be simplified.

1) If the initialization of the nominal predictions is set to follow the nominal dynamic, i.e., \( \xi_{0|t} = \xi_{1|t} \), then (24a) simplifies to \( s_{0|t+1} = s_{1|t} \). This is the special case that has been addressed in the preliminary conference paper [25].

2) If the LMI (13) from Assumption 2 can be satisfied with a blockdiagonal \( P \) having \( P_{21} = 0 \), then (24a) simplifies to
\[
s_{0|t} = s_{1|t-1} + \| \epsilon_{0|t} \|_{F_{11}}^2 + \| \epsilon_{1|t-1} \|_{F_{11}}^2.
\]

3) If the filter state \( \psi_t = \psi_{0|t} = \psi_{1|t-1} \) is known, then the tighter recursion (19b) can be used instead of (24a) to propagate \( s_{1|t-1} \) to \( s_{0|t} \). The filter state can be computed if (i) \( B_0^\top = 0 \) or \( B_0^\top w_{it} \) can be measured; and (ii) \( B_0^\top = 0 \) or \( B_0^\top y_{it} \) can be measured. If the filter is static we can also use (19b).

Remark 4: The increase in the computational complexity is moderate compared to a nominal MPC scheme. The scalar error bounding system can be interpreted as an additional state such that the number of decision variables increases as if the state dimension would increase by 1. However, we introduced nonlinear constraints (24) and (29), which might render the problem more complicated. Nevertheless, we can reformulate the nondifferentiable square root in the constraint (29) as an
equivalent differentiable constraint

\[
| s_k(t) g_i^2 \leq \left( h_i - H_i \left[ \xi_k \right] \right)^2 \land h_i \geq H_i \left[ \xi_k \right] \]

\forall i = 1, \ldots, n_h.

An analogous transformation can be applied to the constraint (24a) as well to get rid of the square root therein. If we further combine this with one of the simplifications from Remark 3, where (24a) need not be included, then the optimization problem becomes a quadratically constrained quadratic program (QCQP).

Remark 5: We note that the proposed MPC scheme can be further simplified by using a fixed tube size \( s_{\text{max}} \) instead of the tube dynamics. Then, a constraint on the nominal output \( r_k(t) \leq (1 - \rho^2 s_{\text{max}} - \gamma d_{\text{max}}) \) can be used to make sure that the actual \( s_k(t) \) is always less than or equal to \( s_{\text{max}} \). This special case of using a constant tube in combination with an output constraint is conceptually similar to [8], where exactly this procedure is proposed with an \( \ell_\infty \)-gain bound on \( \Delta \) instead of an IQC describing it. If we further want to optimize over the initial nominal state, we need to add a constraint on \( \xi_{0|T} \) that (24a) is less than or equal to \( s_{\text{max}} \) if we substitute \( s_{i(T-1)} = s_{\text{max}} \) in (24a).

Remark 6: The design parameters for the tube dynamics and the constraint tightening are \( \rho, P, \Gamma, \gamma, \text{ and } K \). When the prestabilizing control law \( K \) and the exponential decay rate \( \rho \) are fixed, the parameters \( P, \Gamma, \text{ and } \gamma \) can be determined as solutions of the LMI (17). Assumption 2 guarantees the existence of not only one but infinitely many solutions of the LMI (due to the strict inequality), which we can use to optimize over the matrices \( P \) that define the shape of the tube. Finding the matrix \( P \) that minimizes the constraint tightening \( g \) is actually a convex problem since \( g^2 \leq \gamma_i \) with \( \gamma_i > 0 \) can be reformulated as an LMI by using \( P = P_{11} - P_{21} P_{22}^{-1} P_{21} \) and applying the Schur complement twice

\[
g^2 \leq \gamma_i \iff H_i \left[ I \ K \right]^\top P^{-1} \left[ I \ K \right] H_i^\top \leq \gamma_i \\
\iff \left[ H_i \left[ I \ K \right]^\top \gamma_i \gamma_i \right] \geq 0 \\
\iff \left[ P_{22} \ P_{21} \ 0 \right] \left[ P_{21}^\top \ P_{11}^\top \ 0 \right] \left[ I \ K \ H_i^\top \right] \geq 0.
\]

Choosing \( \sum_{i=0}^{n_h} \gamma_i \) as objective yields a semidefinite program whose solution is a matrix \( P \) that minimizes the sum of all constraint tightenings. Further, since we can always rescale a solution of the LMI, we need to fix or at least bound \( \Gamma \) and \( \gamma \) when performing this optimization, otherwise the solutions of \( P, \Gamma, \text{ and } \gamma \) tend to infinity.

Remark 7: The shape of the tube resulting from the proposed approach is an ellipsoid defined by the shape matrix \( P = P_{11} - P_{21} P_{22}^{-1} P_{21} \). Using a fixed shape for the tube is important to keep the online computational complexity low and is standard in most tube-based MPC schemes (e.g., [4], [14], [15], [19], etc.). Nevertheless, we can reduce conservatism by using several tubes with different shape matrices \( P_i \) and scaling parameters \( s_{k,i} \) at the same time leading to an intersection of ellipsoids and a vector valued tube scaling parameter \( s_k \). Note that the feedback \( K \) must be the same for all tubes. Then, we can use Remark 6 but choose only one \( \gamma_i \) as objective to obtain the shape matrix \( P_n \), which constitutes the tube to tighten constraint \( i \) (and only constraint \( i \)). By repeating this for all constraints, we obtain \( n_h \) ellipsoidal tubes and use the intersection of them for the constraint tightening. If we do not optimize over initial conditions (compare Remark 3) and use the same \( \Gamma \) and \( \gamma \), then (24a) is independent of \( P \), which implies that all tube scalings \( s_{k,i} \) are identical, thus we need only one \( s_{k,i} \) and in this case do not increase the computational complexity.

B. Terminal Ingredients

The purpose of this section is to give a constructive proof how a local controller \( K_{\Omega} \), a terminal set \( \Omega \) and a terminal cost weight \( \tilde{Q} \) can be found that satisfy Assumption 3.

Proposition 1: Let the matrices \( Q > 0, R > 0, A, B_{G,C}, C, D_{G,H} \), the vectors \( h, g \) and the scalars \( \rho \in (0, 1), d_{\text{max}} \geq 0, \gamma > 0 \) be given. If \( A, B_{G,C} \) is stabilizable and \( h > \frac{\sqrt{\gamma_{\text{max}}}}{\sqrt{1-\rho^2}} \), then there exists \( K_{\Omega}, S > 0, \xi_{\Omega} > 0, \) and \( s_{\Omega} > 0 \) such that Assumption 3 holds with \( \Omega = \left\{ \xi \in \mathbb{R} | \xi^\top s \leq \xi_{\Omega}, 0 \leq s \leq s_{\Omega} \right\} \).

Proof: Since \( A, B_{G,C} \) is stabilizable, we can find \( K_{\Omega} \) such that \( A + B_{G,C} K_{\Omega} \) is Schur stable. Thus, for each \( Q > 0 \) there is \( S > 0 \) such that

\[
(A + B_{G,C} K_{\Omega})^\top S (A + B_{G,C} K_{\Omega}) - S = -\tilde{Q}.
\]

Setting \( \tilde{Q} = Q + K_{\Omega}^\top R K_{\Omega} \) renders 3) of Assumption 3 true for all \( \xi \). Further, for each \( \xi_{\Omega} > 0, \) the set \( \Omega_{\xi} = \{ \xi | \xi^\top S \xi \leq \xi_{\Omega} \} \) is a positive invariant set of the nominal dynamics (4a) controlled by \( u = K_{\Omega} \xi \). To choose \( s_{\Omega} \) such that \( \Omega \) is a positive invariant set of the augmented dynamics of \( [\xi \ s] \) and hence 1) of Assumption 3 holds, we set

\[
s_{\Omega} := \sup_{\xi \in \Omega_{\xi}} \frac{1}{1-\rho^2} \left\| (C + D_{G,H} K_{\Omega}) \xi \right\|_2^2 + \gamma d_{\text{max}}^2.
\]

Finally, we can choose \( \xi_{\Omega} > 0 \) and \( s_{\Omega} > \frac{\gamma d_{\text{max}}^2}{1-\rho^2} \) small enough such that \( H \left[ \xi \ K_{\Omega} \xi \right] \leq h - \sqrt{\gamma_{\text{max}}} g \) holds for all \( \| \xi \|_2^2 \leq \xi_{\Omega}, \) since \( h > \sqrt{\gamma_{\text{max}}}/\sqrt{1-\rho^2} \). Then 2) of Assumption 3 holds as well.

Remark 8: If the requirement \( h > \frac{\sqrt{\gamma_{\text{max}}}}{\sqrt{1-\rho^2}} g \) is not satisfied, then the worst-case disturbance \( d_{\text{max}} \) is too large to meet the constraints, such that no suitable terminal region exists (for this choice of \( P \) and \( K \) in Assumption 2, other \( P \) and \( K \) could change \( g \)). Such a requirement is intuitive as constraint satisfaction cannot be achieved if the disturbances get arbitrarily large.

Remark 9: We want to briefly summarize the main steps and offline computations necessary to implement the scheme as follows.

1) Find a \( \rho \)-hard IQC description of the uncertainty \( \Delta \).
2) Find suitable \( K \) such that Assumption 2 holds.
3) Compute \( P, \Gamma, \text{ and } \gamma \) according to Remark 6.
4) Compute terminal ingredients according to Proposition 1. A systematic synthesis procedure of step 2) is subject of current research. In contrast to step 3), it cannot be expected to result in a semidefinite program (compare [32], where the IQC synthesis is solved iteratively similar to a $D$-$K$ iteration).

V. NUMERICAL EXAMPLE

The following example demonstrates the advantages of using the much more flexible IQC framework to describe dynamic uncertainties compared to the $\ell_{\infty}$-gain that was used in earlier tube-based MPC schemes [8]. To this end, consider the following system:

$$x_{t+1} = \begin{bmatrix} 1.05 & -0.3 \\ 0 & 0.95 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{t-\tau},$$

with an unknown, possibly time-varying delay $\tau \in [0, \tau_{\text{max}}]$, $\tau_{\text{max}} = 2$ on the input signal $u$ and an external disturbance $d$ that satisfies $|d_t| \leq 0.001$ and acts on the unstable mode. Note that the control input (even for $\tau = 0$) has a larger relative degree to the unstable mode than the disturbance and additionally must go through the time delay. Further, the state constraint $\xi_{\text{max}} = 0.2$ and the input constraint $|u_t| \leq 0.1$ must be satisfied at all times. In order to write the system in the form of (1), we define the nominal case as $\tau = 0$. Hence, we obtain

$$x_{t+1} = Ax_t + B_G w_t + B_G d_t + B_G u_{t-\tau},$$

$$y_t = u_t$$

$$w_t = \Delta(y)_t = y_{t-\tau} - y_t.$$  

(35a)

(35b)

(35c)

It is straightforward to see that $\Delta$ is a causal bounded operator with $\ell_{\infty}$-gain of 2. However, based on the only information of the $\ell_{\infty}$-gain of $\Delta$, the unstable system cannot be robustly stabilized as the $\ell_{\infty}$-gain bound of 2 includes the case $\Delta (y)_t = -y_t$ which cancels all inputs. Thus, the approach from [8] cannot be used for this problem and we need a less conservative description of the uncertainty $\Delta$ as for example via IQCs. As proposed in this article, we can design a tube-based MPC scheme based on IQCs. Hence, we first define the filter $\Psi = \begin{bmatrix} A_{\Psi} & B_{\Psi}^y \\ C_{\Psi} & D_{\Psi}^y \end{bmatrix}$ with

$$A_{\Psi} = \begin{bmatrix} I & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & I \\ 0 & \cdots & \cdots & I \end{bmatrix}, \quad B_{\Psi}^y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{\Psi}^w = 0$$

$$C_{\Psi} = \begin{bmatrix} I \cdots I \\ \vdots \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \quad D_{\Psi}^y = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad D_{\Psi}^w = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

which results in the filter state $\psi_t = [y_{t-\tau_{\text{max}}} \cdots y_{t-1} ]^T$ and the output

$$p_t = [y_{t-\tau_{\text{max}}} - y_{t-\tau_{\text{max}} + 1} \cdots y_{t-1} - y_t]^T.$$  

As the next step we show that the delay uncertainty satisfies an IQC described by this filter. Therefore, let $X \in \mathbb{R}^{n_x \times n_x}, X > 0$ be arbitrary and let us denote the $\tau \times \tau$ all ones (all zeros) matrix by $\mathbf{1}_\tau$ (by $0_\tau$) and the Kronecker product by $\otimes$. Further, for $\tau \in [0, \tau_{\text{max}}]$ let $X_t = \text{diag}(0_{\tau_{\text{max}} - \tau}, I_\tau) \otimes X \in \mathbb{R}^{\tau_{\text{max}} n_y \times \tau_{\text{max}} n_y}$ and $M_\tau = \text{diag}(X_t, -X)$. Then, we obtain

$$0 = \|y_{t-\tau} - y_t\|_X^2 - \|w_t\|_X^2$$

$$= \|\sum_{k=1}^{\tau} (y_{t-k} - y_{t-k+1})\|_X^2 - \|w_t\|_X^2 = \|p_t\|_{M_\tau}^2.$$  

Hence, $\Delta$ satisfies the $\rho$-hard IQC defined by $(\Psi, M)$ if $M$ satisfies for all $\tau = 0, \ldots, \tau_{\text{max}}$ the LMI $M \succeq M_\tau$, independent of $\rho$. We choose $\rho = 0.95$ and $K = [0.18 -0.35]$ and observe that the resulting semidefinite program consisting of the LMI constraints (17), $M \succeq M_\tau, \Gamma > 0, \gamma > 0$, and $X \succeq 0$, as well as the decision variables $P, M, X, \Gamma, \gamma$ is solved with the objective described in Remark 6, which yields

$$P \approx \begin{bmatrix} 5.9 & -8.1 \cdots \cdots \cdots & -4.2 & -11.7 \\ -8.1 & 15.7 \cdots \cdots \cdots & 6.0 & 22.2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4.2 & 6.0 \cdots \cdots \cdots & -17.0 \cdots \cdots \cdots & 81.7 \\ -11.7 & 22.2 \cdots \cdots \cdots & -17.0 \cdots \cdots \cdots & 81.7 \end{bmatrix}, \quad M \approx \begin{bmatrix} 29.0 & 14.5 & 25.4 & 0 \\ 0 & 14.5 & 25.4 & 0 \\ 0 & 0 & -20.7 \end{bmatrix}.$$  

$$\gamma = \Gamma \approx 244, \quad X \approx 20.7.$$  

The cost function (30a) is defined by $Q = I$ and $R = 1$ and the prediction horizon $T = 25$. With the help of Prop. 1 we find that the terminal ingredients $K_{\Omega} \approx [0.19 \cdots -0.28], S \approx [9.2 \cdots -5.6 \cdots 7.7], \xi_{\Omega} \approx 0.0039, s_{\Omega} = 0.1$ satisfy the requirements of Assumption 3. The MPC optimization problems\footnote{Here, $\rho$ and $K$ were manually chosen by an LQR design by varying $\rho$ and the LQR weights until Assumption 2 became feasible.} are solved using CasADI [33] with the solver IPOPT.

The simulation results for two different initial conditions are shown in Fig. 3. One of the initial conditions is on the boundary of the constraints and the other one starts close to the eigenspace of the unstable eigenvalue. We observe for the first initial condition (top, orange) that the tube size starts very small as the controller must be more cautious when close to the constraints. When moving away from the constraints, the MPC controller has more freedom and can excite the uncertainty stronger, resulting in a growing tube size until the end of the prediction horizon is approached and the terminal constraint of the tube size $s$ must be satisfied. For this initial condition, we can see that the error bound $\|e_{k\mid t}\|_P^2 \leq s_{k\mid t}$ is conservative and that the tube grows much faster than the actual error, which indicates some conservatism in the variables $\gamma$ and $\Gamma$ or in the IQC description itself. In closed loop, we observe that the MPC scheme places the nominal state $\xi_{\Omega\mid t}$ after a few steps directly into the origin. This behavior can be explained by the fact that the MPC controller is designed to interfere only when necessary. After these few steps, the MPC does not need to intervene since the system state is far enough from the constraints and close enough to the origin, such that robust constraint satisfaction is guaranteed when solely applying the prestabilizing controller $K$.

For the second initial condition (middle, blue), the controller must be more aggressive in the beginning to push the state from

\footnote{To overcome numerical problems with square roots in the constraints, we apply the equivalence transformation discussed in Remark 4.}
Simulation results for initial conditions $\hat{x}$ and $y$ (blue). Degree of freedom (solid line).

The operators $\hat{\Pi}$ and $\hat{\Psi}$, respectively, we denote the IQC, which holds for all $\hat{\rho}$.

Further, the operators $\rho_+$ and $\rho_-$ are defined via $\rho_+ \circ \rho_-$, which demonstrates the

Remark 10: Note that a nominal MPC scheme does not stabilize this example. In the neighborhood of the origin, where no constraints are active the nominal MPC with the standard LQR terminal cost reduces to an LQR controller. However, an LQR with $Q = I$ and $R = 1$ for the nominal system does not stabilize the true system. Hence, when facing dynamic uncertainties, the robust MPC design is not only needed to handle constraints but also for stability. This is in contrast to the case of additive bounded disturbances, where a nominal MPC scheme is already ISS stable and a tube-based MPC is only needed to ensure robust constraint satisfaction.

VI. CONCLUSION

We have proposed a tube-based MPC scheme for linear systems subject to dynamic uncertainties and disturbances. The use of $\rho$-hard IQCs to capture the behavior of the dynamic uncertainty offers a more detailed description than in previous MPC schemes based on $\ell_\infty$-gain bounds. By extending the $\rho$-hard IQC theory, we were able to derive a dynamic bound on the error between the nominal state and the true system state. When incorporating this scalar error bounding system to predict the tube size in the MPC scheme, we can ensure recursive feasibility and ISS stability. Finally, we have demonstrated in a numerical example that the proposed scheme can reduce conservatism and is applicable to a larger class of systems compared to existing MPC schemes for dynamic uncertainties. An open issue regards the extension to dynamic output feedback and a more detailed investigation of the corresponding offline IQC-based feedback synthesis.

APPENDIX A

Proof of Theorem 3: Let us introduce some notation: We conveniently write $\Delta \in \text{IQC}(\rho, \Pi)$ and $\Delta \in \text{hardIQC}(\rho, \Psi, M)$ as short for $\Delta$ satisfies the $\rho$-IQC defined by $\Pi$ and the $\rho$-hard IQC defined by $(\Psi, M)$, respectively. Similarly, we denote the set of matrices $P = P^\top$ that satisfy (13) with $\text{LMI}(\rho, \Psi, M, G)$.

Further, the operators $\rho_+$ and $\rho_-$ are defined via $\rho_\pm \circ \rho_\pm$, as in [28, Definition 3].

In the first part of the proof, we will show that $\rho$-IQC imply $\rho$-hard IQCs. Note that $\Pi$ is a $\rho$-PN multiplier iff $\Pi_\rho$ is a strict PN multiplier in the sense of [26, Definition 4]. Hence, we can apply [26, Lemma 1 and 6] to $\Pi_\rho$ and obtain that there exists a (J-spectral) factorization $(\hat{\Psi}, \hat{M})$, with $\hat{M} = \text{diag}(I_{n_y}, -I_{n_w})$, $\hat{\Pi}_\rho = \hat{\Psi}^\top \hat{M} \hat{\Psi}$, and $\hat{\Psi} \in \mathbb{R}^{n_y}_{\infty}$ that has the following properties.

1) $\Delta'$ satisfies the 1-hard IQC defined by $(\hat{\Psi}, \hat{M})$ for all $\Delta'$ that satisfy the 1-IQC defined by $\Pi_\rho$.

2) For any $Y \in \mathbb{R}^{n_y}_{\infty}$: If $\tilde{P} \in \text{LMI}(1, \hat{\Psi}, \hat{M}, Y)$ then $\tilde{P} \succeq 0$.

Defining $\hat{\Psi} = \rho_\rho^{-1}$ and $\hat{M} = \hat{M}$, we see that $\Psi_\rho^\top \hat{M} \Psi_\rho = \Psi^\top \hat{M} \Psi = \rho_\rho$ and $\rho_\rho = \Psi \in \mathbb{R}^{n_y}_{\infty}$, i.e., $(\rho, M)$ is a $\rho$-factorization of $\Pi$. Further, we define $\Delta' = \rho_\rho \circ (\rho_\rho \circ \rho_\rho)$ and obtain by using [28, Prop. 7] that $\Delta \in \text{IQC}(\rho, \Pi) \Rightarrow \Delta' \in \text{IQC}(1, \Pi_\rho)$. Now we can use 1) to conclude $\Delta \in \text{IQC}(\rho, \Pi) \Rightarrow \Delta' \in \text{hardIQC}(1, \hat{\Psi}, \hat{M})$. If we take a detailed look at this hard IQC, which holds for all $y \in \ell_\infty^{2n_y}$, and, thus, as well for all $y' := \rho_\rho \circ y$, we observe in two steps as follows:
First
\[ p' = \Psi \Delta(y) = \Psi \circ \rho_+ \Delta(y) = \rho_+ \Psi \Delta(y) \]
and second, for \( p := \Psi \Delta(y) \),
\[ \sum_{t=1}^{T-1} \rho_2^{-2} p_t^T M p_t = \rho_1^T \sum_{t=1}^{T-1} p_t^T M p_t \geq 0. \]

Thus, we have just shown \( \Delta \in \text{hardIQC}(\rho, \Psi, M) \) and altogether \( \Delta \in \text{IQC}(\rho, \Pi) \Rightarrow \Delta \in \text{hardIQC}(\rho, \Psi, M) \).

In the second part of the proof, we will show that (15) implies the existence of \( P > 0 \) such that (13) holds. Due to [28, Corollary 12], (15) is equivalent to existence of \( P = \tilde{P} \omega \) with \( P \in \text{LMI}(\rho, \Psi, M, G) \). This leads to \( \rho^2 P \in \text{LMI}(1, \Psi, M, G) \rho \) and
\[ \Psi = \Psi \rho = \begin{bmatrix} \rho^{-1} A^P \rho & \rho^{-1} B^P \rho & M & G \rho \end{bmatrix}, \]
since \( \Psi = \Psi \rho \). This leads to \( \rho^2 P \in \text{LMI}(1, \Psi, M, G) \rho \).

In the second part of the proof, we will show that (15) implies the existence of \( P > 0 \) such that (13) holds. Due to [28, Corollary 12], (15) is equivalent to existence of \( P = \tilde{P} \omega \) with \( P \in \text{LMI}(\rho, \Psi, M) \). This leads to \( \rho^2 P \in \text{LMI}(1, \Psi, M, G) \rho \). Since \( G \in \mathbb{R}^{n \times m} \) we can conclude with (2) that \( \rho^2 P \geq 0 \). Since the LMI holds strict, we can perturb \( P \) slightly to obtain \( P > 0 \).

\[ \square \]

REFERENCES

[1] J. B. Rawlings, D. Q. Mayne, and M. M. Diehl, Model Predictive Control: Theory, Computation, and Design, 2nd ed. Santa Barbara, CA, USA: Nob Hill Publishing, LLC, 2017.

[2] D. Q. Mayne, “Model predictive control: Recent developments and future promise,” Automatica, vol. 50, no. 12, pp. 2967–2986, 2014.

[3] A. Bemporad and M. Morari, “Robust model predictive control: A survey,” in Robustness in Identification and Control. London, U.K.: Springer, 1999, pp. 207–226.

[4] L. Chisci, J. A. Rossiter, and G. Zappa, “Systems with persistent disturbances: Predictive control with restricted constraints,” Automatica, vol. 37, no. 7, pp. 1019–1028, 2001.

[5] A. Mesbah, “Stochastic model predictive control: An overview and perspectives for future research,” IEEE Control Syst. Mag., vol. 36, no. 6, pp. 30–44, Dec. 2016.

[6] J. Köhler, R. Saloperto, A. M. Müller, and F. Allgöwer, “A computationally efficient robust model predictive control framework for uncertain nonlinear systems,” IEEE Trans. Automat. Control, vol. 66, no. 2, pp. 794–801, Dec. 2021.

[7] B. Kouvaritakis and M. Cannon, Model Predictive Control. Berlin, Germany: Springer, 2016.

[8] F. Patrignani and D. Q. Mayne, “Getting robustness against unstructured uncertainty: A tube-based MPC approach,” IEEE Trans. Autom. Control, vol. 59, no. 5, pp. 1290–1295, May 2014.

[9] K. Zhou, J. C. Doyle, and K. Glover, Robust and Optimal Control. London, U.K.: Pearson, 1995.

[10] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” IEEE Trans. Automat. Control, vol. 42, no. 6, pp. 819–830, Jun. 1997.

[11] J. Veenman, C. W. Scherer, and H. Köröglü, “Robust stability and performance analysis based on integral quadratic constraints,” Eur. J. Control, vol. 31, pp. 1–32, 2016.

[12] L. Lessard, B. Recht, and A. Packard, “Analysis and design of optimization algorithms via integral quadratic constraints,” SIAM J. Optim., vol. 26, no. 1, pp. 57–95, 2016.

[13] D. Q. Mayne and W. Langson, “Robustifying model predictive control of constrained linear systems,” Electron. Lett., vol. 37, no. 23, 2001, Art. no. 1422.

[14] D. Q. Mayne, M. M. Seron, and S. V. Raković, “Robust model predictive control of constrained linear systems with bounded disturbances,” Automatica, vol. 41, no. 2, pp. 219–224, 2005.

[15] S. V. Raković, B. Kouvaritakis, R. Findeisen, and M. Cannon, “Homothetic tube model predictive control,” Automatica, vol. 48, no. 8, pp. 1631–1638, 2012.

[16] J. Fleming, B. Kouvaritakis, and M. Cannon, “Robust tube MPC for linear systems with multiplicative uncertainty,” IEEE Trans. Automat. Control, vol. 60, no. 4, pp. 1087–1092, Apr. 2015.

[17] S. Subramanian, S. Lucia, R. Paulen, and S. Engell, “Tube-enhanced multi-stage model predictive control for flexible robust control of constrained linear systems with additive and parametric uncertainties,” Int. J. Robust Nonlinear Control, vol. 31, no. 9, pp. 4458–4487, 2021.

[18] C. Løvaa, M. M. Seron, and G. C. Goodwin, “Robust output-feedback model predictive control for systems with unstructured uncertainty,” Automatica, vol. 44, no. 8, pp. 1933–1943, 2008.

[19] M. Löhning, M. Reble, J. Hasenauer, S. Yu, and F. Allgöwer, “Model predictive control using reduced order models: Guaranteed stability for constrained linear systems,” J. Process. Control, vol. 24, no. 11, pp. 1647–1659, 2014.

[20] S. Tangavel, S. Subramanian, S. Lucia, and S. Engell, “Handling structural plant-model mismatch using a model-error model in the multi-stage NMPC framework,” in Proc. 18th IFAC Symp. Syst. Identification, 2018, pp. 1074–1079.

[21] S. Tangavel, S. Subramanian, and S. Engell, “Robust NMPC using a model-error model with additive bounds to handle structural plant-model mismatch,” in Proc. 12th IFAC Symp. Dyn. Control Process. Syst., 2019, pp. 592–597.

[22] W. P. Heath, G. Li, A. G. Willis, and B. Lennox, “The robustness of input constrained model predictive control to infinity-norm bound model uncertainty,” in Proc. 5th IFAC Symp. Robust Control Des., 2006, pp. 495–500.

[23] P. Petsagkourakis, W. P. Heath, and C. Theodoropoulos, “Stability analysis of piecewise affine systems with multi-model predictive control,” Automatica, vol. 111, 2020, Art. no. 108539.

[24] P. Petsagkourakis, W. P. Heath, J. Carrasco, and C. Theodoropoulos, “Robust stability of barrier-based model predictive control,” IEEE Trans. Automat. Control, vol. 66, no. 4, pp. 1879–1886, Apr. 2021.

[25] L. Schwenkel, J. Köhler, M. A. Müller, and F. Allgöwer, “Dynamic uncertainties in model predictive control: Guaranteed stability for constrained linear systems,” in Proc. 59th IEEE Conf. Decis. Control, 2020, pp. 1235–1241.

[26] B. Hu, M. J. Lacerda, and P. Seiler, “Robustness analysis of uncertain discrete-time systems with dissipation inequalities and integral quadratic constraints,” Int. J. Robust Nonlinear Control, vol. 27, no. 11, pp. 1940–1962, 2017.

[27] P. Seiler, “Stability analysis with dissipation inequalities and integral quadratic constraints,” IEEE Trans. Automat. Control, vol. 60, no. 6, pp. 1704–1709, Jun. 2015.

[28] R. Boczar, L. Lessard, and B. Recht, “Exponential convergence bounds using integral quadratic constraints,” in Proc. 54th IEEE Conf. Decis. Control, 2015, pp. 7516–7521.

[29] P. Finsler, “Über das Vorkommen definiter und semidefiniter Formen in scharen quadratischer Formen,” Commentarii Mathematici Helvetici, vol. 9, no. 1, pp. 188–192, 1936.

[30] H. Yin, A. Packard, M. Arcak, and P. Seiler, “Reachability analysis using dissipation inequalities for uncertain nonlinear systems,” Syst. Control Lett., vol. 142, 2020, Art. no. 104736.

[31] E. D. Sontag, “Comments on integral variants of ISS,” Syst. Control Lett., vol. 34, no. 1-2, pp. 93–100, 1998.

[32] J. Veenman and C. W. Scherer, “IQC-synthesis with general dynamic multipliers,” Int. J. Robust Nonlinear Control, vol. 24, no. 17, pp. 3027–3056, 2014.

[33] J. A. E. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, “CasADi: A software framework for nonlinear optimization and optimal control,” Math. Program. Comput., vol. 11, no. 1, pp. 1–36, 2019.
Johannes Köhler received the master's degree in engineering cybernetics, in 2017, and the Ph.D. degree in mechanical engineering, in 2021, from the University of Stuttgart, Stuttgart, Germany.

He is currently a Postdoctoral Researcher with the Institute for Dynamic Systems and Control, ETH Zürich. His research interests include model predictive control and control and estimation for nonlinear uncertain systems.

Matthias A. Müller received the diploma degree in engineering cybernetics from the University of Stuttgart, Stuttgart, Germany, the M.S. degree in electrical and computer engineering from the University of Illinois at Urbana-Champaign, Champaign, IL, USA, both in 2009, and the Ph.D. degree in mechanical engineering, in 2014, from the University of Stuttgart, Stuttgart, Germany.

Since 2019, he is a Director of the Institute of Automatic Control and Full Professor with the Leibniz University, Hannover, Germany. His research interests include nonlinear control and estimation, model predictive control, and data-/learning-based control, with application in different fields including biomedical engineering.

Prof. Müller was the recipient of the 2015 European Ph.D. award on control for complex and heterogeneous systems. He was the recipient of the ERC Starting Grant in 2020 and the recipient of the inaugural Brockett–Willems Outstanding Paper Award for the best paper published in Systems & Control Letters, in the period 2014–2018.

Frank Allgöwer received the degree in engineering cybernetics from the University of Stuttgart, Stuttgart, Germany, and the degree in applied mathematics from the University of California, Los Angeles, CA, USA, respectively, and the Ph.D. degree in chemical engineering from the University of Stuttgart.

Since 1999, he has been the Director of the Institute for Systems Theory and Automatic Control and Professor with the University of Stuttgart. His research interests include predictive control, networked control, cooperative control, and nonlinear control with application to a wide range of fields including systems biology.

Prof. Allgöwer served as President of the International Federation of Automatic Control, from 2017 to 2020, and Vice President of the German Research Foundation DFG, from 2012 to 2020.