A NOTE ON THE CATEGORY OF EQUIVALENCE RELATIONS

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Abstract. We make some beginning observations about the category $\mathbb{E}_q$ of equivalence relations on the set of natural numbers, where a morphism between two equivalence relations $R, S$ is a mapping from the set of $R$-equivalence classes to that of $S$-equivalence classes, which is induced by a computable function. We also consider some full subcategories of $\mathbb{E}_q$, such as the category $\mathbb{E}_q^{\Sigma_0^1}$ of computably enumerable equivalence relations (called ceers), the category $\mathbb{E}_q^{\Pi_0^1}$ of co-computably enumerable equivalence relations, and the category $\mathbb{E}_q(\text{Dark}^*)$ whose objects are the so-called dark ceers plus the ceers with finitely many equivalence classes. Although in all these categories the monomorphisms coincide with the injective morphisms, we show that in $\mathbb{E}_q^{\Sigma_0^1}$ the epimorphisms coincide with the onto morphisms, but in $\mathbb{E}_q^{\Pi_0^1}$ there are epimorphisms that are not onto. Moreover, $\mathbb{E}_q(\Sigma_0^1)$, and $\mathbb{E}_q(\text{Dark}^*)$ are closed under finite products, binary coproducts, and coequalizers, but we give an example of two morphisms in $\mathbb{E}_q(\Pi_0^1)$ whose coequalizer in $\mathbb{E}_q$ is not an object of $\mathbb{E}_q^{\Pi_0^1}$.

1. Introduction

In his monograph [6] Ershov introduces and thoroughly investigates the category of numberings. We recall that a numbering is a pair $N = \langle \nu, S \rangle$, where $\nu : \omega \to S$ is an onto function. Numberings are the objects of a category $\text{Num}$, called the category of numberings; the morphisms from a numbering $N_1 = \langle \nu_1, S_1 \rangle$ to a numbering $N_2 = \langle \nu_2, S_2 \rangle$ are the functions $\mu : S_1 \to S_2$ for which there is a computable function $f$ so that the diagram

$$
\begin{array}{ccc}
\omega & \xrightarrow{f} & \omega \\
\downarrow{\nu_1} & & \downarrow{\nu_2} \\
S_1 & \xrightarrow{\mu} & S_2
\end{array}
$$

commutes. We say in this case that the computable function $f$ induces the morphism $\mu$, and we write $\mu = \mu^{N_1,N_2}(f)$.

Now, numberings are equivalence relations in disguise, see our Theorem 2.3 below, where we show that the equivalence relations on the set $\omega$ of natural numbers can be structured into a category $\mathbb{E}_q$ which is equivalent to $\text{Num}$. In this paper, we rephrase in $\mathbb{E}_q$ some of the observations noticed by Ershov about $\text{Num}$, and we hopefully point out some useful, although simple, new facts about $\mathbb{E}_q$, and some of its full subcategories, such as the...
category $\text{Eq}(\Sigma^0_1)$ of computably enumerable equivalence relations (these relations are called ceers), the category $\text{Eq}(\Pi^0_1)$ of co-computably enumerable equivalence relations (called coceers), and the category $\text{Eq}(\text{Dark}^*)$ whose objects are the dark ceers and the finite ceers. Although in all these categories the monomorphisms trivially coincide with the injective morphisms, we see that in $\text{Eq}(\Sigma^0_1)$ the epimorphisms coincide with the onto morphisms, but in $\text{Eq}(\Pi^0_1)$ there are epimorphisms that are not onto. We also observe that $\text{Eq}$, $\text{Eq}(\Sigma^0_1)$, and $\text{Eq}(\text{Dark}^*)$ are closed under finite products, binary coproducts, and coequalizers. Although we were not able to show that $\text{Eq}(\Pi^0_1)$ is not closed under coequalizers, we give an example of two morphisms in $\text{Eq}(\Pi^0_1)$ whose coequalizer in $\text{Eq}$ provides an object which is not in $\text{Eq}(\Pi^0_1)$, in fact it is properly $\Sigma^0_2$.

The reader is referred to MacLane’s textbook for the basics about category theory. A category $\mathbb{C}$ is given by specifying $\text{ob}(\mathbb{C})$, i.e. the objects of $\mathbb{C}$, and for any pair $a, b \in \text{ob}(\mathbb{C})$ one must specify $\mathbb{C}(a, b)$, i.e. the morphisms from $a$ to $b$. We recall that there is a partial binary operation $\circ$ on morphisms: if $f \in \mathbb{C}(b, c)$ and $g \in \mathbb{C}(a, b)$ then $g \circ f \in \mathbb{C}(a, c)$, and for every object $a$ there is a special morphism $1_a \in \mathbb{C}(a, a)$: the operation $\circ$ is associative, and $f \circ 1_a = f$ for every $f \in \mathbb{C}(a, b)$, and $1_a \circ f = f$ for every $f \in \mathbb{C}(b, a)$. All other relevant notions of category theory which are used in this paper will be duly introduced when they are needed. Our basic reference for computability theory is Rogers’ textbook [7]. We use the notations $\langle \_ \_ \rangle$ for the Cantor pairing function, and $(\_)_0, (\_)_1$ for its projections. The the Cantor pairing function and its projections are computable (even primitive recursive) functions.

Let $1_\omega$ denote the identity function on $\omega$, i.e. $1_\omega(x) = x$. For every natural number $n \geq 1$, let $\text{Id}_n$ be the equivalence relation $x \equiv_n y$ if $x - y = nq$ for some integer $q$. Thus $\text{Id}_n$ has exactly $n$ equivalence classes. Let $\text{Id}$ denote equality, i.e. $x \equiv y$ if and only if $x = y$. For any equivalence relation $T$ on $\omega$, let $\omega_T$ be the set of equivalence classes into which $T$ partitions $\omega$, and let $\nu_T : \omega \rightarrow \omega_T$ be given by $\nu_T(x) = [x]_T$, where $[x]_T$ denotes the $T$-equivalence class of $x$.

2. The category of equivalence relations

If $R, S$ are equivalence relations on the set $\omega$ of natural numbers, we say that a function $f$ is $(R, S)$-equivalence preserving if

$$(\forall x, y)[x \equiv_R y \Rightarrow f(x) \equiv_S f(y)].$$

If $f$ is $(R, S)$-equivalence preserving, then $f$ induces a well defined mapping $\alpha^{R, S}(f) : \omega_R \rightarrow \omega_S$ given by $\alpha^{R, S}(f)([x]_R) = [f(x)]_S$, i.e. $\alpha^{R, S}(f)$ is the
unique mapping $\alpha : \omega/R \to \omega/S$ such that the diagram

$$
\begin{array}{ccc}
\omega & \xrightarrow{f} & \omega \\
\nu_R \downarrow & & \downarrow \nu_S \\
\omega/R & \xrightarrow{\alpha} & \omega/S
\end{array}
$$

commutes.

In the following we will consider only $(R, S)$-equivalence preserving functions that are computable. As $\alpha^{R,T}(g \circ f) = \alpha^{S,T}(g) \circ \alpha^{R,S}(f)$, and for every equivalence relation $T$ the identity on $\omega/T$ is $\alpha^{T,T}(1_\omega)$ where $1_\omega$ is the identity function on $\omega$, we are led to the following definition:

**Definition 2.1.** The category of equivalence relations $\mathbb{E}q$ is defined as follows:

- the objects of $\mathbb{E}q$ are the equivalence relations on $\omega$;
- if $R, S \in \text{ob}(\mathbb{E}q)$ then the morphisms from $R$ to $S$ are the elements of the set $\mathbb{E}q(R, S)$ consisting of all $\alpha : \omega/R \to \omega/S$ such that $\alpha = \alpha^{R,S}(f)$ for some $(R, S)$-equivalence preserving computable $f$.

**Remark 2.2.** In the following we will generally use small Greek letters as variables on morphisms of $\mathbb{E}q$. Given equivalence relations $R, S$, the morphism $\alpha^{R,S}(f) : R \to S$ induced by some computable function $f$ will be written simply as $\alpha(f)$ when the pair of equivalence relations will be clear from the context.

We observe:

**Theorem 2.3.** The categories $\mathbb{E}q$ and $\mathbb{N}um$ are equivalent. In fact there exist functors $F : \mathbb{N}um \to \mathbb{E}q$ and $G : \mathbb{E}q \to \mathbb{N}um$ such that $1_{\mathbb{E}q} = F \circ G$, and $1_{\mathbb{N}um} \simeq G \circ F$, where the relation $\simeq$ on functors denotes natural equivalence.

**Proof.** If $N = \langle \nu, S \rangle$ is a numbering then define $F(N)$ to be the equivalence relation

$$
x F(N) y \iff \nu(x) = \nu(y),
$$

and if $\mu : N_1 \to N_2$ is a morphism in $\mathbb{N}um$, with $\mu = \mu^{N_1,N_2}(f)$ for some computable $f$, then define $F(\mu) = \alpha^{F(N_1),F(N_2)}(f)$. The definition is well given, as it does not depend on the choice of $f$.

Conversely, if $R$ is an equivalence relation on $\omega$, then define $G(R) = \langle \nu_R, \omega/R \rangle$, where $\nu_R(x) = [x]_R$, and if $\alpha : R_1 \to R_2$ is a morphism in $\mathbb{E}q$, with $\alpha = \alpha^{R_1,R_2}(f)$ for some computable $f$, then let $G(\alpha(f)) = \mu^{G(R_1),G(R_2)}(f)$. Again, the definition is well given.

It is now trivial to check that the claims hold: in particular, the set of $\mathbb{N}um$-morphisms

$$\{ N \overset{\mu_N}{\to} G(F(N)) : N \in \text{ob}(\mathbb{N}um) \},$$

where $\mu_N(s) = [\nu(s)]_F(N)$ (with $N = \langle \nu, S \rangle$ and $s \in S$) provides a natural equivalence $1_{\mathbb{N}um} \simeq G \circ F$. \qed
3. Monomorphisms and epimorphisms

In a category $\mathbb{C}$, a morphism $\gamma : b \to c$ is a **monomorphism** if $\alpha = \beta$ for every commutative $\mathbb{C}$-diagram of the form

$$
\begin{array}{ccc}
  a & \xrightarrow{\alpha} & b \\
  \beta & \searrow & \gamma \\
  & \downarrow & \\
  & c 
\end{array}
$$

where, “commutative” simply means: $\gamma \circ \alpha = \gamma \circ \beta$.

Dually, a morphism $\gamma : c \to a$ is an **epimorphism** if $\alpha = \beta$ for every commutative $\mathbb{C}$-diagram of the form

$$
\begin{array}{ccc}
  c & \xrightarrow{\gamma} & a \\
  \alpha & \nearrow & \beta \\
  & \downarrow & \\
  & b 
\end{array}
$$

i.e., when $\alpha$ and $\beta$ are morphisms such that $\alpha \circ \gamma = \beta \circ \gamma$.

**Lemma 3.1.** In the category $\text{Eq}$ the monomorphisms coincide with the 1-1 morphisms.

**Proof.** Consider any morphism $\gamma : R \to S$, with $\gamma = \alpha(f)$ for some computable function $f$. Assume for a contradiction that $\gamma$ is not 1-1, and let $[a_1]_R, [a_2]_R$ be distinct equivalence classes such that $\gamma([a_1]_R) = \gamma([a_2]_R)$. For $i = 1, 2$, define the computable function $g_i(x) = a_i$. Then, for every equivalence relation $E$, the functions $g_1$ and $g_2$ induce distinct morphisms $\alpha_1 = \alpha(g_1), \alpha_2 = \alpha(g_2) : E \to R$, such that $\gamma \circ \alpha_1 = \gamma \circ \alpha_2$, showing that $\gamma$ is not mono.

The converse, i.e. if $\gamma$ is 1-1 then $\gamma$ is mono, is trivial.

Given equivalence relations $R, S$ on $\omega$ we recall that $R$ is **reducible to** $S$ (notation $R \preceq S$) if there exists a computable function $f$ such that

$$(\forall x, y)[x \ R \ y \iff f(x) \ S \ f(y)].$$

In other words, $R \preceq S$ if and only if there exists a 1-1-morphism $\mu : R \to S$. Thus we have the following:

**Corollary 3.2.** If $R$ and $S$ are equivalence relations on $\omega$, then $R \preceq S$ if and only if there is a monomorphism $\mu : R \to S$.

**Proof.** It immediately follows from the coincidence of monomorphisms with injective morphisms.

From the point of view of category theory, $R \preceq S$ may also be expressed by saying that $R$ is a **subobject** of $S$, see MacLane [1, p. 122] and Ershov [2, 3].

We now move on to consider epimorphisms and their relations with the onto morphisms.

**Lemma 3.3.** In $\text{Eq}$ every onto morphism is an epimorphism.

**Proof.** Trivial.

However, we now show that the converse is not always true:
Theorem 3.4. There are epimorphisms which are not onto.

Proof. Let $A, B$ be two disjoint undecidable $\Pi_1^0$ sets such that their union is undecidable. For instance take $A = 2^K$ and $B = 2^K + 1$, where $K$ denotes any undecidable co-c.e. set. Thus $A \cup B = K \oplus K$ is an undecidable $\Pi_1^0$ set. Consider the coceer $R$ whose equivalence classes are $A, B$ and then all singletons. Since $C = \overline{A} \cup \overline{B}$ is an infinite c.e. set, we can fix a computable bijection $f$ of $\omega$ with $C$. Clearly this function provides a reduction

$$\alpha(f) : \text{Id} \rightarrow R$$

such that the range of $f$ is $C$. The monomorphism $\alpha = \alpha(f)$ induced by this $f$ is not onto, as it leaves out the two equivalence classes $A, B$. We claim that $\alpha$ is epi. Suppose that $\alpha(f_1), \alpha(f_2) : R \rightarrow S$, for some coceer $S$ and computable functions $f_1, f_2$, are distinct morphisms such that $\alpha(f_1) \circ \alpha = \alpha(f_2) \circ \alpha$. As these morphisms may be distinct only because of the values they take on $A$ and $B$. We distinguish the following cases:

1. $\alpha(f_1)(A) \neq \alpha(f_2)(A)$ and $\alpha(f_1)(B) = \alpha(f_2)(B)$. Then

$$(\forall x)[x \in \overline{A} \Rightarrow f_1(x) = f_2(x)],$$



giving that $\overline{A}$ is co-c.e., hence $A$ is decidable, contradiction;

2. $\alpha(f_1)(A) = \alpha(f_2)(A)$ and $\alpha(f_1)(B) \neq \alpha(f_2)(B)$. A similar argument as in the previous item shows that $B$ is decidable, contradiction;

3. $\alpha(f_1)(A) \neq \alpha(f_2)(A)$ and $\alpha(f_1)(B) \neq \alpha(f_2)(B)$. In this case

$$(\forall x)[x \in \overline{A} \cup \overline{B} \Rightarrow f_1(x) \neq f_2(x)],$$



showing that $A \cup B$ is decidable, which is again a contradiction.

We recall that $\mathbb{D}$ is a full subcategory of $\mathcal{C}$ if $\text{ob}(\mathbb{D}) \subseteq \text{ob}(\mathcal{C})$, and, for all $a, b \in \text{ob}(\mathbb{D})$, we have that $\mathbb{D}(a, b) = \mathcal{C}(a, b)$.

Definition 3.5. Let $\mathcal{C}$ be a class of equivalence relations on $\omega$. Then by $\text{Eq}(\mathcal{C})$ we denote the full subcategory of $\text{Eq}$ whose objects are exactly the equivalence relations in $\mathcal{C}$.

Corollary 3.6. In $\text{Eq}(\Pi_1^0)$ there are epimorphisms which are not onto.

Proof. Immediate by the proof of Theorem 3.4, as $\text{Id}$ and $R$ are coceers.

On the other hand,

Theorem 3.7. In $\text{Eq}(\Sigma_1^0)$ the epimorphisms coincide with the onto morphisms.

Proof. Suppose that $R, S$ are ceers, and $\alpha : R \rightarrow S$ is a morphism which is not onto. Let $h$ be a computable function such that $\alpha = \alpha(h)$; let $A = \{x : (\exists y)[y \in \text{ran}(h) \& y S x]\}$, and let $a$ be such that $[a]_S \notin \text{ran}(\alpha)$. Consider any nontrivial precomplete coeer $T \neq \text{Id}_1$ (see \[\framebox[9pt]{4}], \[\framebox[9pt]{3}], or \[\framebox[9pt]{1}]), and by definition of
precompleteness, let \( f(e, x) \) be a totalizer for \( T \), i.e., a computable function such that

\[
(\forall e, x)[\varphi_e(x) \downarrow \Rightarrow \varphi_e(x) T f(e, s)].
\]

We are going to define two computable functions \( g_1, g_2 \) which are \((S, T)\)-equivalence preserving, and induce distinct morphisms \( \alpha_1 = \alpha(g_1), \alpha_2 = \alpha(g_2) \) that coincide (and are constant!) on \( \text{ran}(\alpha) \). Our construction is somewhat modelled on the proofs of [8, Theorem 2.6 and Corollary 2.8]. Let \( b, c_1, c_2 \) be such that the equivalence classes \([b]_T, [c_1]_T, [c_2]_T \) are pairwise distinct: we are using the fact that a nontrivial precomplete equivalence relation has infinitely many equivalence classes. Let \( \{A_s : s \in \omega\} \) be a computable approximation to \( A \) (i.e., \( A_0 = \emptyset, A_s \subseteq A_{s+1}, A = \bigcup_s A_s \), and each \( A_s \) is a finite set uniformly given by its strong index); let \( \{S_s : s \in \omega\} \) be a computable approximation to \( S \), meaning that for every \( s, S_s \) is a decidable equivalence relation, \( S_0 = \text{Id}, S_s \subseteq S_{s+1}, S = \bigcup_s S_s \), and there exists a computable \( r \) such that, for every \( s \) and \( i \geq r(s) \) we have that \([i]_{S_s} = [i]_s \); finally, assume that \( \{T_s : s \in \omega\} \) is a similar approximation to \( T \). Assume that \( S_{2s+2} = S_{2s+1}, \) and \( A_{2s+2} = A_{2s+1} \), for every \( s \). We may as well assume that the above approximations satisfy: If \( i \notin A_s \) and \( i \in S_s \) and \( j \in S_s \) then \( j \notin A_s \) (and of course \( j \in S_s \)). At even stages neither \( A \) nor \( S \) changes, so we only devote these stages to make sure that the construction has not placed \([c_1]_T \) in the range of the morphism induced by \( g_1 \), or has made \( g_1 \) not \((S, T)\)-equivalence preserving.

Define \( g_1(i) = f(e, i) \) where \( e \) is a fixed point that we control by the Recursion Theorem. We define \( \varphi_e \) in stages. At any stage we may call a special clause (named \((*)\)), which, if called, “freezes” the construction. At stage \( s + 1 \), if clause \((*)\) has not been called at any previous stage then the following inductive assumption (referred to as \((\dagger)\)) will be true: that if \( i \notin A_s \) and \( i \in S_s \) and \( \varphi_e(i) \downarrow \), then \( i \) is not least in its \( S_s\)-equivalence class.

Stage 0. Let \( \varphi_{e,0}(i) \) be undefined for all \( i \).

Stage \( s + 1 \), with \( s = 2t \). If we have called \((*)\) at any previous stage then let \( \varphi_{e,s+1} = \varphi_{e,s} \). Otherwise, if \( \varphi_{e,s}(i) \) is still undefined then:

1. if \( i \in A_s \) define \( \varphi_{e,s+1}(i) = b \); if \( i \in A_s \) define \( \varphi_{e,s+1}(i) = c_1 \);
2. if \( i \notin A_s \) and \( i \in S_s \) and there exists \( j < i \) with \( j \in S_s \) and \( \varphi_{e,s}(j) \downarrow \), then define \( \varphi_{e,s+1}(i) = f(e, j) \) for the least such \( j \).

Note that this preserves the inductive assumption \((\dagger)\).

Stage \( s + 1 \), with \( s = 2t + 1 \): If we have called \((*)\) at any previous stage then let \( \varphi_{e,s+1} = \varphi_{e,s} \). Otherwise, if \( i \notin A_s \) and \( i \in S_s \) and \( \varphi_{e,s}(i) \) is still undefined, then:

1. if \( f(e, i) \) \( T_s b \) or \( f(e, i) \) \( T_s c_1 \) then define \( \varphi_{e,s+1}(i) = c_2 \). After this, call clause \((*)\);
2. if \( f(e, i) \) \( T_s c_2 \) then define \( \varphi_{e,s+1}(i) = c_1 \). After this, call clause \((*)\).
Notice that by inductive assumption (†), for every \( i \) such that \( i \notin A \), and \( i \mathrel{\mathcal{S}} a \) there is always \( j \) such that \( j \notin A \) and \( j \mathrel{\mathcal{S}} a \), so that we certainly act on \( j \) as in (1) or (2).

It is now easy to check the following:

- We never call clause (\( \ast \)). Indeed, if we call it in (1) of an even stage then for some \( i \) we would have \( c_2 = \varphi_e(i) T f(e, i) \) with \( f(e, i) \mapsto b \) or \( f(e, i) \mapsto T c_1 \); if we call (\( \ast \)) in (2) of an even stage then for some \( i \) we would have \( c_1 = \varphi_e(i) T f(e, i) \mapsto c_2 \). Both cases give rise to \( c_1 T c_2 \), a contradiction.
- If \( \varphi_e(i) \) diverges then \( i \notin A \cup [a]_S \) and \( i \) is least in its \( S \)-equivalence class.
- If \( i \in A \) then \( f(e, i) \mapsto T b \) and of \( i \in [a]_S \) then \( f(e, i) \mapsto T c_1 \). To see this, as \( \varphi_e(i) \) is defined, let \( i_0 < i_1 < \ldots < i_n \) be such that \( i_h S i_k \) for all \( h, k \leq n \), \( i_n = i \), and for every \( 0 < k \leq n \), \( \varphi_e(i_k) \) has been defined \( \varphi_e(i_k) = f(e, i_{k-1}) \) through (2) of an odd stage, whereas \( \varphi_e(i_0) \) has been defined \( \varphi_e(i_0) = b \) if \( i_0 \in A \) through (1) of an odd stage, or \( \varphi_e(i_0) = c_1 \) if \( i \in [a]_S \) through (1) of an odd stage. Then as \( f(e, i) T f(e, i_0) \) we have that \( f(e, i) \mapsto T b \) if \( i \in A \), or \( f(e, i) \mapsto T c_1 \) if \( i \in [a]_S \).
- If \( i, j \notin A \cup [a]_S \) and \( i S j \) then \( f(e, i) \mapsto T f(e, j) \). To see this, assume \( [i]_S = [j]_S \) and let

\[
[i]_S = \{i_0 < i_1 < \ldots \}.
\]

Then \( \varphi_e(i_0) \) is undefined, and by induction on \( n \) it is easy to see (by an argument similar to the previous item, since if \( h > 0 \) then \( \varphi_e(i_h) \) is defined through (2) of an odd stage), that \( f(e, i_h) \mapsto T f(e, i_0) \).

- By the previous two items, we get that \( g_1 \) is \( (S, T) \)-equivalence preserving, and \( \alpha(g_1) \) is a morphism from \( S \) to \( T \).
- \( [c_2]_T \notin \operatorname{ran}(\alpha_1) \). This follows from the fact that we never use clause (1) at even stage. Moreover \( [c_1]_T \in \operatorname{ran}(\alpha_1) \) as \( \alpha_1([a]_S) = [c_1]_T \).

In a similar way, but interchanging \( c_1 \) and \( c_2 \) at each stage, we define a computable function \( g_2 \) such that, letting \( \alpha_2 = \alpha(g_2) \), we eventually have

- \( (\forall [x]_S \in \operatorname{ran}(\alpha))[\alpha_1([x]_S) = \alpha_2([x]_S) = [b]_T] \); thus \( \alpha_1 \circ \alpha = \alpha_2 \circ \alpha \);
- \( \alpha_1 \neq \alpha_2 \) as \( [c_i]_T \in \operatorname{ran}(\alpha_i) \setminus \operatorname{ran}(\alpha_{i-1}) \).

\[\square\]

4. Products and coproducts in \( \mathbb{E}q \)

We recall the definitions of products and coproducts in a category \( C \). If \( a, b \in \text{ob}(C) \), then a product of the pair \( (a, b) \) is, when it exists, a triple \((a \times b, \pi_a, \pi_b)\) with \( a \times b \mathrel{\sim^\Delta} a, a \times b \mathrel{\sim^\Delta} b \), such that for all pairs of morphisms \( c \mathrel{\rightarrow^\rho} a, c \mathrel{\rightarrow^\rho} b \), there exists a unique morphism \( \rho_a \times \rho_b \) which makes the
following diagram

\[
\begin{array}{ccc}
\rho_a \times \rho_b & \rightarrow & c \\
p_a \downarrow & & \downarrow p_b \\
a \times b & \rightarrow & b
\end{array}
\]

commute. It is a well known fact of category theory that products are unique up to isomorphisms (we recall that a pair of objects \(a, b\) are isomorphic in a category if there is an isomorphism \(\alpha : a \rightarrow b\), i.e. a morphism for which there is \(\beta \in \text{ob}(C)\) such that \(\beta \circ \alpha = 1_a\) and \(\alpha \circ \beta = 1_b\)), so we will talk about the product of two objects, when the two objects have a product.

A coproduct of the pair \((a, b)\) is, when it exists, a triple \(a \sqcup b, i_a, i_b\) with \(i_a : a \sqcup b, i_b : a \sqcup b\), such that for all pairs of morphisms \(\rho_a : a \rightarrow c, \rho_b : b \rightarrow c\), there exists a unique morphism \(\rho_{a \sqcup b}\) which makes the following diagram commute:

\[
\begin{array}{ccc}
\rho_a \times \rho_b & \rightarrow & c \\
p_a \downarrow & & \downarrow p_b \\
a \times b & \rightarrow & b
\end{array}
\]

Again, coproducts are unique up to isomorphisms, so we will talk about the coproduct of two objects, when the two objects have a coproduct.

The following is a simple observation essentially from [5].

**Theorem 4.1.** The category \(\text{Eq}\) has all nonempty finite products and nonempty finite coproducts.

**Proof.** The product of \(R, S\) is the triple \((R \times S, \pi_R, \pi_S)\) so that

\[
\langle x, y \rangle R \times S \langle u, v \rangle \Leftrightarrow x R u \& y S v,
\]

with \(\pi_R = \alpha^{R \times S, R}(p_0)\), and \(\pi_S = \alpha^{R \times S, S}(p_1)\), where \(p_0(x) = (x)_0\) and \(p_1(x) = (x)_1\) are the projections of the Cantor pairing function. If \(T\) is another equivalence relation and \(T \overset{\rho_T}{\rightarrow} R, T \overset{T_S}{\rightarrow} S\) are two morphisms, with say \(\rho_R = \alpha^{T, R}(f_R)\) and \(\rho_S = \alpha^{T, S}(f_S)\) where \(f_R\) and \(f_S\) are computable functions then take \(\rho_R \times \rho_S = \alpha^{T, R \times S}(f_R \times f_S)\) where \(f_R \times f_S(x) = \langle f_R(x), f_S(x) \rangle\). This makes the defining diagram commute. To show uniqueness, suppose that \(\beta : T \rightarrow R \times S\) makes the defining diagram commute. Then if \(\beta([x]_T) = ([u,v]_{R \times S}\) we have that \(\pi_R([u,v]_{R \times S}) = [u]_R = \rho_R([x]_T) = [f_R(x)]_R\), and thus \(u R f_R(x)\), and similarly \(v S f_S(x)\), giving \(\langle u, v \rangle R \times S \langle f_R(x), f_S(x) \rangle\). This yields

\[
\rho_R \times \rho_S([x]_T) = [f_R(x), f_S(x)]_{R \times S} = [\langle u, v \rangle]_{R \times S} = \beta([x]_T),
\]

i.e., \(\beta = \rho_R \times \rho_S\).
The coproduct of \( R, S \) is the triple \( (R \oplus S, i_R, i_S) \), often called the uniform join of \( R, S \), see e.g. [2], i.e.,

\[
R \oplus S = \{(2x, 2y) : x \leq y \} \cup \{(2x + 1, 2y + 1) : x \leq y \},
\]

with \( i_R = \alpha(\text{ev}) \) and \( \text{ev}(x) = 2x \), and \( i_S = \alpha(\text{odd}) \) and \( \text{odd}(y) = 2y + 1 \).

Arguing as in the case of products, it is easy to see that our definition turns \( R \oplus S, i_R, i_S \) into a coproduct. If \( \rho_R : R \to T \) and \( \rho_S : S \to T \), then \( \rho_R \sqcup \rho_S = \alpha_{R \oplus S, T}(f_R \oplus f_S) \), where \( f_R \) and \( f_S \) induce \( \rho_R \) and \( \rho_S \), respectively, and

\[
f_R \oplus f_S(x) = \begin{cases}
    f_R(y), & \text{if } x = 2y, \\
    f_S(y), & \text{if } x = 2y + 1.
\end{cases}
\]

To show uniqueness, suppose that \( \beta : R \oplus S \to T \) makes the defining diagram commute: then

\[
\beta(\{[2x]_{R \oplus S}\}) = \beta(i_R([x]_R)) = \rho_R([x]_R) = (\rho_R \sqcup \rho_S)(\{2x\}_{R \oplus S}),
\]

and similarly \( \beta(\{[2x + 1]_{R \oplus S}\}) = (\rho_R \sqcup \rho_S)(\{2x + 1\}_{R \oplus S}) \).

\[\square\]

**Corollary 4.2.** For every \( n \), \( \text{Eq}(\Sigma_n^0) \) and \( \text{Eq}(\Pi_n^0) \) have nonempty finite products and coproducts.

**Proof.** Trivial, since \( R \times S \) and \( R \sqcup S \) are \( \Sigma_n^0 \) (\( \Pi_n^0 \)) if both \( R, S \) are \( \Sigma_n^0 \) (\( \Pi_n^0 \)). \[\square\]

We recall that a *terminal object* in a category is an object \( a \) such that for every object \( b \) there exists a unique morphism \( b \to a \). Terminal objects are unique up to isomorphisms. A terminal object can be described as an empty product.

**Theorem 4.3.** \( \text{Eq} \) has a terminal object, thus \( \text{Eq} \) has all finite products.

**Proof.** It is easy to see that \( \text{Id}_1 \) is a terminal object. \[\square\]

Dually, an *initial object* in a category is an object \( a \) such that for every object \( b \) there exists a unique morphism \( a \to b \). Initial objects are unique up to isomorphisms. An initial object can be described as an empty coproduct.

**Theorem 4.4.** \( \text{Eq} \) has no initial object, thus \( \text{Eq} \) does not have empty coproducts.

**Proof.** No equivalence relation \( X \) can be initial, as there are two distinct morphisms from \( X \) to \( \text{Id}_2 \). \[\square\]

**Corollary 4.5.** For every \( n \geq 1 \), \( \text{Eq}(\Sigma_n^0) \) and \( \text{Eq}(\Pi_n^0) \) have terminal objects (and thus they have all finite products), but no initial objects.

**Proof.** Immediate. \[\square\]
5. Equalizers and coequalizers

From category theory we recall the following definition. Given two morphisms \( a \xrightarrow{\alpha} b \) and \( a \xrightarrow{\beta} b \), a coequalizer of \( \alpha, \beta \), when it exists, is a pair \( (c, \gamma) \) with \( \gamma : b \rightarrow c \) such that \( \gamma \circ \alpha = \gamma \circ \beta \) and for every morphism \( \gamma' : b \rightarrow c' \) such that \( \gamma' \circ \alpha = \gamma' \circ \beta \) there exists a unique morphism \( \gamma'' : c \rightarrow c' \) so that the following diagram commutes:

\[
\begin{array}{ccc}
    a & \xrightarrow{\alpha} & b \\
    \downarrow{\gamma} & & \downarrow{\gamma''} \\
    c' & & c \\
\end{array}
\]

It is known from category theory that coequalizers are unique up to isomorphisms.

**Theorem 5.1.** The category \( \mathbb{E}_Q \) has coequalizers.

**Proof.** Suppose that \( X, Y \) are equivalence relations, and \( \alpha, \beta : X \rightarrow Y \) are morphisms, with \( \alpha = \alpha(f_1) \) and \( \beta = \alpha(f_2) \). Consider the equivalence relation \( Z \) generated by the set of pairs \( Y \cup \{(f_1(x), f_2(x)) : x \in \omega\} \). Then \( \gamma = \alpha^{Y,Z}(1_\omega) \) is an onto morphism \( \gamma : Y \rightarrow Z \) which is the coequalizer of \( \alpha \) and \( \beta \). The following diagram verifies the defining property of coproducts for every \((Y, U)\)-equivalence preserving \( g \):

\[
\begin{array}{ccc}
    X & \xrightarrow{\alpha} & Y \\
    \downarrow{\alpha^{Y,U}(g)} & & \downarrow{\alpha^{Z,U}(g)} \\
    U & & U. \\
\end{array}
\]

We use here that \( g \) is \((Z, U)\)-equivalence preserving if it is \((Y, U)\)-equivalence preserving, and thus \( \alpha^{Z,U}(g) \) is a morphism from \( Z \) to \( U \) as well. \( \square \)

**Corollary 5.2.** For every \( n \geq 1 \), \( \mathbb{E}_Q(\Sigma^0_n) \) has coequalizers.

**Proof.** Immediate by the previous proof, since if \( Y \) is \( \Sigma^0_n \), with \( n \geq 1 \), then \( Z \) is \( \Sigma^0_n \) as well. \( \square \)

**Corollary 5.3.** Every object of \( \mathbb{E}_Q(\Sigma^0_1) \) is a coequalizer of a pair of morphisms \( \text{Id} \xrightarrow{\alpha} \text{Id} \).

**Proof.** Let \( Z \) be a ceer, and let \( h \) be a computable function enumerating \( Z \), i.e. \( u Z v \) if and only if \( \langle u, v \rangle \in \text{ran}(h) \). Let \( f_1, f_2 \) be computable functions defined by \( f_1(x) = (h(x))_0 \) and \( f_2(x) = (h(x))_1 \). Then \( Z \) is the ceer generated by the pairs \( \{(f_1(x), f_2(x)) : x \in \omega\} \), and thus, by (the proof of) Theorem 5.1, is the coequalizer of the morphisms induced by \( f_1 \) and \( f_2 \). \( \square \)
Unfortunately, the construction in the proof of Theorem [5.1] does not always produce $\Pi^0_n$ equivalence relations $Z$ when starting from $\Pi^0_n$ equivalence relations $X,Y$. We show below that this is the case even for $n = 1$. We need the following preliminary lemma.

**Lemma 5.4.** There exist computable functions $f_1, f_2$ (in fact with $f_1(x) = 0$ for every $x$) and a c.o.e. $Y$ such that the equivalence relation $Z$ generated by the set of pairs $Y \cup \{(f_1(x), f_2(x)) : x \in \omega\}$ has exactly two equivalence classes, at least one of which is not $\Pi^0_1$, hence $Z$ is not $\Pi^0_1$.

**Proof.** We construct in stages a c.o.e. $Y$ and a c.e. set $U$. At stage $s$ we build an equivalence relation $Y_s$ and a finite set $U_s$ such that $\{Y_s : s \in \omega\}$ and $\{U_s : s \in \omega\}$ are a computable approximation to $Y$ (that is, $\{Y_s : s \in \omega\}$ is a computable sequence with $Y_s \supseteq Y_{s+1}$, and $Y = \bigcup_s Y_s$) and an approximation to $U$ (that is, in this case, $\{U_s : s \in \omega\}$ is a computable sequence of computable sets $U_s \subseteq U_{s+1}$, and $U = \bigcup_s U_s$). We work with computable approximations $\{V_{e,s} : e, s \in \omega\}$ to the $\Pi^0_0$ sets (meaning that the predicate “$x \in V_{e,s}$” is decidable in $e, x, s$, $V_{e,s} \supseteq V_{e,s+1}$ and $V_e = \bigcup_s V_{e,s}$, for all $e, s$).

**Stage 0.** Let $Y_0$ be the set of pairs corresponding to the equivalence relation consisting of the two equivalence classes $\{0\}$ and $\omega \setminus \{0\}$; let $U_0 = \emptyset$.

**Stage $s+1$.** Extract from $Y_s$ all pairs $\langle e + 2, y \rangle$ such that $y \neq e + 2$ and $e + 2 \in V_{e,s} \setminus V_{e,s+1}$; we say in this case that we $Y$-isolate $e + 2$ at $s + 1$. Let $Y_{s+1}$ be the remaining set of pairs: notice that, looking at equivalence classes, $Y_{s+1}$ looks like $Y_s$ but having a computable set of additional singletons, namely all those $\{e + 2\}$ which have been $Y$-isolated at $s + 1$. Add to $U_s$ all numbers $e + 2$ which have been $Y$-isolated at $s + 1$.

This ends the construction.

Define $f_1(x) = 0$, and let $f_2$ be any computable function such that $\text{ran}(f_2) = U$. Finally, let $Z$ be the equivalence relation generated by $Y$ and the set of pairs $\{(f_1(x), f_2(x)) : x \in \omega\}$, that is by $Y$ and the pairs $\{(0, e + 2)\}$ so that $e + 2$ has been $Y$-isolated at some stage. We now check that the construction works. The sequences $\{Y_s : s \in \omega\}$ and $\{U_s : s \in \omega\}$ are indeed computable sequences so that $Y = \bigcap_s Y_s$ is a c.o.e. $U$ and is a c.e. set. The singleton $\{e + 2\}$ is an equivalence class of $Y$ if and only if $e + 2 \notin V_e$. Thus $[e + 2]_Y \cap ([0]_Y \cup [1]_Y) = \emptyset$ if and only if $e + 2 \notin V_e$. Thus $0 \notin Z$ if and only if $e + 2 \notin V_e$, hence $[0]_Z \neq V_e$ for every $e$. All numbers $x$ different from 0 and from those $e + 2$ which have been $Y$-isolated at the stage, are eventually $Y$-equivalent to 1 as they were so at stage 0, and therefore $x \in Z$ as the construction does not ask to involve these numbers in any extraction or to merge their classes to other classes at any stage bigger than 0. Therefore $Z$ has only two equivalence classes, and the equivalence class $[0]_Z$ is not $\Pi^0_1$, hence $Z$ is not $\Pi^0_1$.\[\square\]
Corollary 5.5. There are morphisms $\text{Id} \xrightarrow{\alpha} Y$ in $\text{Eq}(\Pi_1^0)$ such that their coequalizer in $\text{Eq}$ is properly $\Sigma_2^0$.

Proof. Let $f_1, f_2, Y, Z$ be as in the previous lemma. Then, by the construction of Theorem 5.1, the identity $1_\omega$ induces a morphism $\gamma = \alpha(1_\omega)$, $\gamma : Y \to Z$ which is a coequalizer of $\alpha, \beta : \text{Id} \to Y$, where $\alpha = \alpha(f_1)$ and $\beta = \alpha(f_2)$ (notice that for every equivalence relation $R$ and any computable function $f$, we have that $f$ induces a morphism from $\text{Id}$ to $R$) in the category $\text{Eq}(\Sigma_2^0)$. Clearly $Z \in \Sigma_2^0$, but, as already observed, $Z \notin \Pi_1^0$. □

The above observation shows that $\text{Eq}(\Pi_1^0)$ is not closed under coequalizers. Unfortunately it cannot be used to conclude that $\text{Eq}(\Pi_1^0)$ does not have coequalizers. So we raise the following question.

Question 5.6. Does $\text{Eq}(\Pi_1^0)$ have coequalizers?

The dual notion of a coequalizers is that of an equalizer: Given two morphisms $a \xrightarrow{\alpha} b$ an equalizer of $\alpha, \beta$, when it exists, is a pair $(c, \gamma)$ with $\gamma : c \to a$ such that $\alpha \circ \gamma = \beta \circ \gamma$ and for every morphism $\gamma' : c' \to a$ such that $\alpha \circ \gamma' = \beta \circ \gamma'$ there exists a unique morphism $\gamma'' : c' \to c$ so that the following diagram commutes:

\[
\begin{array}{ccc}
c & \xrightarrow{\gamma} & a \\
\downarrow{\gamma''} & & \downarrow{\gamma'} \\
c'
\end{array}
\]

As to equalizers, the situation is much simpler than for coequalizers, as follows from the following observation.

Remark 5.7. The pair of morphisms $\alpha(f_1), \alpha(f_2) : \text{Id} \to \text{Id}$ induced by the computable functions $f_0, f_1 : \text{Id}_1 \to \text{Id}_2$, with $f_0(x) = 0$ and $f_1(x) = 1$ have no equalizer.

Proof. Trivial, since for no $x$ we have $f_0(x) \text{Id}_2 f_1(x)$. □

6. Subcategories of $\text{Eq}(\Sigma_1^0)$ and closure under binary coproducts and coequalizers

Binary coproducts and binary coequalizers can be used to build more complex objects in a category, at least those finite colimits that can be built without an initial object. Let us recall that a category has all finite colimits if and only if it has coequalizers (which are special colimits) and finite coproducts (including an initial object, which is not available in $\text{Eq}$).

Corollary 6.1. For every $n \geq 1$, the category $\text{Eq}(\Sigma_1^0)$ is closed under coequalizers and nonempty finite coproducts, although it does not have an initial object.
Proof. From Corollary 4.2, Corollary 5.2, and Corollary 4.4.

Andrews and Sorbi [2] have proposed a partition of the ceers into the three classes $F$, Light, Dark, where $F$ is comprised of the finite ceers, i.e. the ceers with only finitely many equivalence classes; Light is comprised of the light ceers, i.e. the ceers $R$ such that $\text{Id} \leq R$, where we recall that $\text{Id}$ denotes the identity ceer; Dark is comprised of the dark ceers, i.e. the ceers which are neither finite nor light. These classes have been extensively investigated in relation to the existence or non-existence of joins and meets in the poset of degrees of ceers under the reducibility mentioned in Corollary 3.2: for instance no pair of incomparable degrees of dark ceers has join or meet. It is easy to see that these classes are closed under isomorphisms (in the category-theoretic sense). The classes of degrees corresponding to the classes of the above partition are first order definable within the poset of degrees of ceers, under the already mentioned reducibility, in the language of partial orders.

It might be of some interest to know whether Dark or Light allow for the constructions corresponding to finite colimits in category theory. Corollary 5.3 excludes that the proof of Theorem 5.1 may yield that $\text{Eq}(\text{Light})$ (or even $\text{Eq}(\text{Light} \cup F)$) has coequalizers. However, let $\text{Dark}^* = \text{Dark} \cup \mathcal{F}$. Then

**Corollary 6.2.** The category $\text{Eq}(\text{Dark}^*)$ is closed under coequalizers and nonempty finite coproducts.

**Proof.** First of all, $\text{Eq}(\text{Dark}^*)$ is closed under binary coproducts, as it is easy to see (see [2]) that Dark is closed under uniform joins, $\mathcal{F}$ is closed under uniform joins, and the uniform join of a dark ceer and a finite ceer is still dark.

Moreover, every coequalizer $Z$ of a diagram $\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \beta & \downarrow & \\end{array}$ where $Y$ is dark or finite, is still dark or finite, since building $Z$ as in the proof of Theorem 5.1 makes $Y \subseteq Z$, and thus if $\text{Id} \not\leq Y$ then $\text{Id} \not\leq Z$.

**Remark 6.3.** It might be the case to observe that it is necessary to include the finite ceers in the previous corollary, since the coequalizer $Z$ built in the proof of Corollary 5.2 starting from two dark ceers might be finite: consider for instance the pair of morphisms

$\begin{array}{ccc} X & \xrightarrow{\alpha} & X \oplus \text{Id}_1 \\ \alpha(\text{ev}) & \alpha(\text{odd}) & \\end{array}$

where $X$ is dark (hence $x \oplus \text{Id}_1$ is dark too [2]), $\text{ev}(x) = 2x$ and odd is the function taking the constant value 1: then, as follows from the proof of Theorem 5.1 (or its “local” version Corollary 5.2) these two morphisms have coequalizer $\gamma : X \oplus \text{Id}_1 \rightarrow \text{Id}_1$. Thus we see that the coequalizer is the finite ceer consisting of only one class.

We see from the proof of Theorem 3.7 that in $\text{Eq}(\text{Light})$ epimorphisms coincide with the onto morphisms, as the proof makes use of precomplete
ceers which are known to be light (in fact, see [3], every ceer, hence Id as well, is reducible to any nontrivial precomplete ceer). It is therefore natural to ask the following question:

**Question 6.4.** Do epimorphisms coincide with the onto morphisms in the category $\mathbb{Eq}(\text{Dark})$?

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