PROPORTионаL аLГEBRаS, HОМОМОрPHISMоS, СОНГРУЕНCES, 
阿ND FUnCToRоS

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АBSTRACT. This paper introduces proportional algebras as algebras endowed with the 4-ary 
analogical proportion relation where the fundamental concepts of subalgebras, homomorphisms, congruences, and functors are constructed.

1. ІNTRODUCTION

The purpose of this paper is to introduce proportional algebras (or p-algebras; see Section 2) as algebras endowed with the 4-ary analogical proportion relation \( a : b :: c : d \) and to lift the fundamental concepts of subalgebras, homomorphisms, and congruences from ordinary to p-algebras. Fortunately, it turns out that this can be achieved without major complications. In particular, this yields the notion of a proportional homomorphism (or p-homomorphism; see Section 3) preserving the analogical proportion relation across different domains by satisfying a stronger version of the analogical inference principle (AIP) of Couceiro, Hug, Prade, and Richard (2017) corresponding to the analogical jump in Davies and Russell (1987) (cf. Couceiro & Lehtonen, 2022). We show in Theorem 10 that every isomorphism is a p-homomorphism and that there are homomorphisms which are not p-homomorphisms (Examples 7, 8). Since there are non-bijective p-homomorphisms (Example 5) this shows that isomorphisms, homomorphisms, and p-homomorphisms are different concepts (Remark 11). In Section 4 we then lift the fundamental concept of a congruence from ordinary to p-algebras and show in Theorem 16 that the kernel of every p-homomorphism is a p-congruence. In a similar vein, we then introduce the notion of a proportional functor (or p-functor; see Section 5) preserving the relationship between elements of the source domain. We show in Fact 21 that p-homomorphisms and p-functors are different but related concepts as p-functors satisfy the AIP given that the underlying algebra is p-transitive (see Definition 23). We show in Theorem 25 that the space of all p-functors on a p-transitive algebra forms a monoid and we show in Fact 27 that p-functors are p-idempotent (see Definition 26). Finally, in order to be able to compare functions between algebras with respect to the analogical proportion relation, we define in Section 6 the notion of functional proportionality and show in Theorem 34 that all p-functors on p-transitive algebras are functionally proportional.

It is important to emphasize that all constructions in this paper are kept very general and do not refer to a specific formalization of analogical proportions. However, in examples we do refer to the author’s model introduced in Antić (2022a) and we therefore expect the reader to be familiar with the basics of his framework. Moreover, we use here the same notation as in Antić (2022a, §2.1). For further references on analogical reasoning we refer the interested reader to Hall (1989) and Prade and Richard (2014).

In a broader sense, this paper is a further step towards a theory of analogical reasoning and learning systems with potential applications to fundamental AI-problems like commonsense reasoning and computational learning and creativity.
2. Proportional algebras and subalgebras

In the rest of the paper, $\mathbf{A}$ and $\mathbf{B}$ denote $L$-algebras over some joint language of algebras $L$ (cf. Antić, 2022a, §2.1).

**Definition 1.** A proportional algebra (or $p$-algebra) is an $L$-algebra $\mathbf{A} = (A,F)$ endowed with the 4-ary analogical proportion relation :: on $\mathbf{A}$ denoted by $\mathbf{A}^:: = (A,F,::)$.

With a notion of algebra there is always an associated notion of subalgebra.

**Definition 2.** An $L$-algebra $\mathbf{A}'$ is a proportional subalgebra (or $p$-subalgebra) of $\mathbf{A}$ iff $(\mathbf{A}')^::$ is a subalgebra of $\mathbf{A}^::$ in the ordinary sense, which means that in addition of $\mathbf{A}'$ being a subalgebra of $\mathbf{A}$, we have that

$$\mathbf{A}' \models a:b::c:d \iff \mathbf{A} \models a:b::c:d \text{ for all } a,b,c,d \in \mathbf{A}'.$$

Intuitively, a p-subalgebra preserves the analogical proportion relation between the elements which in general may not be the case. For example, it may be the case that $a:b::c:d$ holds in the algebra $\mathbf{A}'$, whereas if we consider the larger algebra $\mathbf{A}$ we can find some $d'$ such that the relation between $a$ and $b$ and between $c$ and $d'$ is more similar than between $c$ and $d$.

3. Proportional homomorphisms

This section introduces the notion of a proportional homomorphism preserving analogical proportions.

**Definition 3.** A proportional homomorphism (or $p$-homomorphism) is any homomorphism $F : \mathbf{A}^:: \to \mathbf{B}^::$, which means that in addition of $F$ being a homomorphism $\mathbf{A} \to \mathbf{B}$, we have that for all $a,b,c,d \in \mathbf{A}$,

$$\mathbf{A} \models a:b::c:d \iff \mathbf{B} \models F(a):F(b)::F(c):F(d).$$

A proportional isomorphism (or $p$-isomorphism) is a bijective $p$-homomorphism.

**Remark 4.** The only if part “$\Rightarrow$” of the equivalence in Definition 3 is called the analogical inference principle by Couceiro et al. (2017) and it can be viewed as a particular case of the so-called analogical jump by Davies and Russell (1987).

**Example 5.** Let $S : \mathbb{N} \to \mathbb{N}$ be the unary successor function $S(a) := a + 1$. The translation function $S^k : (\mathbb{N},S) \to (\mathbb{N},S)$ given by $S^k(a) := a + k$, $k \in \mathbb{N}$, is a $p$-homomorphism as a direct consequence of Antić’s (2022b, Difference proportion theorem) which says that $(\mathbb{N},S) \models a:b::c:d$ iff $a - b = c - d$.

**Example 6.** The negation function $\neg$ is a $p$-isomorphism on $(\{0,1\},\lor,\neg)$ by the following argument. By Antić’s (2022c, Theorem 14), we have

$$(\{0,1\},\lor,\neg) \models a:b::c:d \iff (a = b \text{ and } c = d) \text{ or } (a \neq b \text{ and } c \neq d)$$

$$\iff (-a = -b \text{ and } -c = -d) \text{ or } (-a \neq -b \text{ and } -c \neq -d)$$

$$\iff (\{0,1\},\lor,\neg) \models \neg a :: \neg b :: \neg c :: \neg d,$$

where the second equivalence follows from the fact that

$$(1) \quad a = b \text{ iff } \neg a = \neg b \text{ and } a \neq b \text{ iff } \neg a \neq \neg b.$$
Example 7. Let $F : (\mathbb{Z}, +, 0, 1) \to (\{0,1\}, +, 0, 1)$ be the homomorphism sending every integer $a$ to $a \mod 2$. We claim that $F$ is not a p-homomorphism by showing

$$(\mathbb{Z}, +, 0, 1) \models F \colon a : b :: c : d \not\models (\{0,1\}, +, 0, 1) \models (a \mod 2) : (b \mod 2) :: (c \mod 2) : (d \mod 2).$$

We can interpret addition mod 2 as the logical XOR-operation and it is not hard to show by proofs similar to those in Antić (2022c) that

$$(\{0,1\}, +, 0, 1) \models a :: b :: c :: d \iff (a = b \text{ and } c = d) \text{ or } (a \neq b \text{ and } c \neq d).$$

On the other hand, we have

$$(\mathbb{Z}, +, 0, 1) \models 0 :: 0 :: 1 :: 2$$

characteristically justified via Antić’s (2022a, Functional proportion theorem) by $z \to z + z$ since $z + z$ is injective in $(\mathbb{Z}, +, 0, 1)$.

The following example shows that there are homomorphisms which are not p-homomorphisms.

Example 8. Let $A := ([1,4])$ and $B := ([5,6])$ be algebras consisting only of universes. Define $F : A \to B$ by $F(1) := F(2) := F(3) := 5$ and $F(4) := 6$. That the mapping $F$ is a homomorphism holds trivially as there are no fundamental operations involved. On the other hand, we have

$$A \models 1 :: 2 :: 3 :: 4 \text{ whereas } B \not\models 5 :: 5 :: 5 :: 6,$$

where $A \models 1 :: 2 :: 3 :: 4$ holds by Antić’s (2022a, Theorem 33) and $B \not\models 5 :: 5 :: 5 :: 6$ follows from the determinism of the analogical proportion relation (cf. Antić, 2022a, Theorem 28).

Example 9. Define the algebras $A := ([1,4], S)$ and $B := ([5,6], S)$ by

One can verify that $F$ as defined by the dashed arrows in the figure above is a p-homomorphism from $A$ to $B$. 

Theorem 10. Every isomorphism is a p-homomorphism.

Proof. A direct consequence of Antić’s (2022a, Second isomorphism theorem). □

Remark 11. Notice that the p-homomorphism $S^k$ of Example 5 is not bijective for $k \neq 0$, which means that it cannot be an isomorphism. In combination with Example 8 and Theorem 10 this shows that p-homomorphisms, isomorphisms, and homomorphisms are different concepts.

Theorem 12. The space of all p-homomorphisms of the form $A \rightarrow A$ forms a monoid with respect to function composition with the neutral element given by the identity function.

Proof. First, it follows from the definition that p-homomorphisms are closed under composition, that is, if $F$ and $G$ are p-homomorphisms, then $F \circ G$ is a p-homomorphism as well by the following computation:

\[
\begin{align*}
A &\models a : b :: c : d \\
A &\models F(a) : F(b) :: F(c) : F(d) \\
A &\models G(F(a)) : G(F(b)) :: G(F(c)) : G(F(d)).
\end{align*}
\]

The identity function on $A$ is clearly a p-homomorphism. □

4. PROPORTIONAL CONGRUENCES

In universal algebra, congruences provide a mechanism for factorizing algebras into equivalence classes compatible with the algebraic operations. Here, we require in addition that the classes preserve the analogical proportion relation giving rise to the notion of a proportional congruence. We will show in Theorem 16 that proportional congruences and proportional homomorphisms are connected in the same way as congruences and homomorphism via kernels. The concept of a proportional congruence is motivated by the fact that analogical proportions and congruences are, in general, not compatible as has been observed in Antić (2022b, Theorem 6).

Definition 13. A proportional congruence (or p-congruence) on $A$ is any congruence $\theta$ on $A$ which means that in addition of being a congruence, $\theta$ satisfies for all elements $a, a', b, b', c, c', d, d' \in A$:

\[
\begin{align*}
a \theta a' &\quad b \theta b' &\quad c \theta c' &\quad d \theta d' \\
A &\models a : b :: c : d &\theta.
\end{align*}
\]

The following definition is essentially the well-known definition of factor algebras in universal algebra with the only difference that we use here p-congruences instead of congruences.

Definition 14. Given a p-congruence $\theta$ on a p-algebra $A$ and an element $a \in A$, we denote the equivalence class of $a$ in $A$ by $[a]_\theta$ (or simply by $[a]$ in case $\theta$ is understood). The factor algebra $A/\theta$ of $A$ with respect to $\theta$ is defined as usual, that is, the universe of $A/\theta$ is given by the set of all equivalence classes $\{[a]_\theta \mid a \in A\}$ and the fundamental operations are given by

\[
f^{A/\theta}([a_1]_\theta, \ldots, [a_{rk(f)}]_\theta) := [f^A(a_1, \ldots, a_{rk(f)})]_\theta.
\]

A standard construction in universal algebra is given by the kernel of a homomorphism (cf. Burris & Sankappanavar, 2000, Definition 6.7) which we directly adapt here to p-homomorphisms.
**Definition 15.** The kernel of a p-homomorphism $F : A \rightarrow B$ is given by
$$\ker(F) := \{(a, b) \in A \times A \mid F(a) = F(b)\}.$$ 

**Theorem 16.** The kernel of every p-homomorphism $F : A \rightarrow B$ is a p-congruence on $A$, that is, in addition of being a congruence, we have that for all elements $a, a', b, b', c, c', d, d' \in A$:
$$F(a) = F(a') \quad \ldots \quad F(d) = F(d') \quad A \models a : b :: c : d$$

**Proof.** It is well-known that the kernel of every homomorphism is a congruence (cf. Burris & Sankappanavar, 2000, Theorem 6.8) and since every p-homomorphism is a homomorphism, we know that the kernel of $F$ is a congruence. It remains to prove the above derivation rule:

\[
\begin{array}{c}
A \models a : b :: c : d \\
B \models F(a) : F(b) :: F(c) : F(d)
\end{array}
\quad F \text{ is p-hom.}
\quad
\begin{array}{c}
F(a) = F(a') \quad \ldots \quad F(d) = F(d')
\end{array}
\quad
\begin{array}{c}
B \models F(a') : F(b') :: F(c') : F(d')
\end{array}
\quad F \text{ is p-hom.}
\]

$\square$

**5. Proportional Functors**

In Section 3, we have defined proportional homomorphisms satisfying the equivalence in Definition 3 and in particular the analogical inference principle (see Remark 4). In this section, we are interested in a related but different notion of analogical proportion preserving function defined as follows.

**Definition 17.** A proportional functor (or p-functor) is any mapping $F : A \rightarrow B$ satisfying

$$(A, B) \models a : b :: F(a) : F(b), \text{ for all } a, b \in A.$$ 

Roughly, a p-functor preserves the relationship between source elements. Notice that we do not require a p-functor to be a homomorphism.

**Example 18.** The translation function $S^k : (N, S) \rightarrow (N, S)$ given by $S^k(a) := a + k$, $k \in N$, is a p-functor as a direct consequence of Antić’s (2022b, Difference proportion theorem) which says that $(N, S) \models a : b :: c : d$ iff $a - b = c - d$.

**Example 19.** By Antić’s (2022c, Theorems 13 and 14), we have for all $B \subseteq \{0, 1\}$:

$$(\{0, 1\}, \neg, B) \models a : b :: c : d \iff (a = b \text{ and } c = d) \text{ or } (a \neq b \text{ and } c \neq d)$$

and

$$(\{0, 1\}, \lor, \neg, B) \models a : b :: c : d \iff ((\{0, 1\}, \neg, B) \models a : b :: c : d).$$

Together with (1), this immediately shows that the negation operator is a p-functor on $(\{0, 1\}, \neg, B)$ and $(\{0, 1\}, \lor, \neg, B)$.

**Example 20.** Recall the situation in Example 19, where we have seen that $F$ is a p-homomorphism from $A$ to $B$. It is not hard to show, however, that $F$ is not a p-functor as we have (cf. Antić, 2022a, Example 39)

$$(A, B) \not\models 1 : 3 :: F(1) : F(3).$$
5.1. **Isomorphisms and homomorphisms.** We now want to compare the notion of a p-functor with the well-known notions of a homomorphism and isomorphism.

**Fact 21.** Every isomorphism is a p-functor and there is a p-homomorphism which is not a p-functor.

**Proof.** A direct consequence of Antić’s (2022a, First isomorphism theorem) and Example 20. □

**Remark 22.** Notice that the p-functor \( S^k \) of Example 18 is not bijective for \( k \neq 0 \), which means that it cannot be an isomorphism. In combination with Fact 21, this shows that p-functors, isomorphisms, homomorphisms, and p-homomorphisms are different concepts.

5.2. **Analogical inference principle.** Recall from Remark 4 that the analogical inference principle (Couceiro et al., 2017) is the inference rule

\[
\begin{align*}
A \models a \:: b \:: c \:: d \quad & \quad B \models F(a) \:: F(b) \:: F(c) \:: F(d) \quad \text{AIP.}
\end{align*}
\]

In Theorem 24 we will see that p-functors satisfy the analogical inference principle given that the underlying algebra is p-transitive which is defined as follows.

**Definition 23.** An algebra \( A \) is called proportional transitive (or p-transitive) iff it satisfies the transitivity axiom of analogical proportions for all elements in \( A \) given by (cf. Antić, 2022a, §4.3):

\[
\begin{align*}
A \models a \:: b \:: c \:: d \quad & \quad A \models c \:: d \:: e \:: f \\
A \models a \:: b \:: e \:: f
\end{align*}
\]

(transitivity).

By proportional symmetry (or p-symmetry) we mean the axiom \((A, B) \models a \:: b \:: c \:: d \iff (B, A) \models c \:: d \:: a \:: b\) valid in all pairs of algebras by Antić’s (2022a, Theorem 28).

**Theorem 24.** Let \( F : A \to A \) be a p-functor, and let \( A \) be a p-transitive algebra. Then \( F \) satisfies the analogical inference principle.

**Proof.** We have the following derivation:

\[
\begin{align*}
A \models a \:: b \:: c \:: d \quad & \quad A \models c \:: d \:: F(c) \:: F(d) \\
A \models a \:: b \:: F(c) \:: F(d) \quad & \quad \text{p-transitivity} \\
A \models a \:: b \:: F(c) \:: F(d) \quad & \quad \text{p-symmetry} \\
A \models F(c) \:: F(d) \:: a \:: b \\
A \models F(c) \:: F(d) \:: F(a) \:: F(b) \quad & \quad \text{p-trans.} \\
A \models F(a) \:: F(b) \:: F(c) \:: F(d) \quad & \quad \text{p-symmetry}
\end{align*}
\]

5.3. **Closedness under composition.** The composition of two p-functors yields another p-functor given that the underlying algebra is p-transitive (cf. Definition 23) which is shown in the next result.

**Theorem 25.** Let \( A \) be a p-transitive algebra. The space of all p-functors on \( A \) forms a monoid with respect to function composition with the neutral element given by the identity function.
Proof. First, it follows from the definition that p-functors are closed under composition in case $A$ is p-transitive, that is, if $F$ and $G$ are p-functors, then $F \circ G$ is a p-functor as well by the following computation:

$$
A \models a : b :: F(a) : F(b) \quad A \models a : b :: G(F(a)) : G(F(b)).
$$

The identity function on $A$ is a p-functor as an immediate consequence of the reflexivity of analogical proportions (cf. Antić, 2022a, Theorem 28). □

5.4. Proportional idempotency. Every function $F : A \to A$ can be applied repeatedly, which motivates the following definition.

**Definition 26.** We say that $F : A \to A$ is **propionally idempotent** (or **p-idempotent**) iff

$$
A \models F(a) : F(b) :: F(F(a)) : F(F(b)),
$$

holds for all $a, b \in A$.

**Fact 27.** Every p-functor $F : A \to A$ is p-idempotent.

**Remark 28.** Notice that by p-symmetry of analogical proportions (cf. Definition 23), every p-idempotent function $F : A \to A$ on a p-transitive algebra $A$ satisfies

$$
A \models F^m(a) : F^m(b) :: F^n(a) : F^n(b) \quad \text{for all } m, n \geq 0 \text{ and } a, b \in A.
$$

6. **Functional proportionality**

We wish to be able to compare two functions $F$ and $G$ with respect to the analogical proportion relation, which motivates the following definition.

**Definition 29.** Given two functions $F, G : A \to B$, define

$$
F : G \iff B \models F(a) : F(b) :: G(a) : G(b), \quad \text{for all } a, b \in A.
$$

In case $F : G$ and $G : F$, we say that $F$ and $G$ are **functionally proportional** written $F :: G$.

**Example 30.** All translation functions $S^k, S^\ell$, $k, \ell \in \mathbb{N}$, are functionally proportional, that is, $S^k :: S^\ell$ holds for all $k, \ell \in \mathbb{N}$.

**Proposition 31.** Functional proportionality is reflexive and symmetric. If $B$ is p-transitive, then functional proportionality is transitive and therefore an equivalence relation with respect to $(A, B)$, for all $A$.

**Proof.** Reflexivity and symmetry follow from the reflexivity and symmetry of analogical proportions (cf. Antić, 2022a, Theorem 28), and the assumed p-transitivity of $B$ induces transitivity. □

**Fact 32.** $F :: G$ iff $F \circ H :: G \circ H$ for all $H$.

**Theorem 33.** For any p-functor $F : A \to B$ and function $G : A \to B$, if $B$ is p-transitive and $F : G$ then $G$ is a p-functor.

**Proof.** We have the following derivation:

\[1\text{See Example [13]}\]
Theorem 34. In every p-transitive algebra \( A \), all p-functors on \( A \) are functionally proportional, that is, for all p-functors \( F,G : A \to A \), we have \( F \preceq G \).

Proof. We have the following derivation:

\[
\frac{a : b :: F(a) : F(b)}{A} \quad \frac{F(a) : F(b) :: G(a) : G(b)}{B} \quad F : G
\]

\( (A,B) \models a : b :: F(a) : F(b) \quad F : G \quad (A,B) \models a : b :: G(a) : G(b) \)

7. Future work

From an artificial intelligence perspective, it is interesting to transfer the concepts introduced in this paper to settings relevant in AI-research like, for example, to logic programming by building on the author’s recent work on logic program proportions in Antić (2022d). In that context, p-homomorphisms and p-functors correspond to syntactic logic program transformations preserving the analogical relationships between programs—this can be interpreted as a form of learning novel logic programs by analogy-making which appears to be a very promising approach to symbolic learning.

From an algebraic point of view, it is interesting to study the mathematical properties of proportional algebras, homomorphisms, congruences, and functors and to analyze which concepts and results in the literature on universal algebra can be lifted from ordinary to proportional algebras. For instance, one might ask whether the set of all p-congruences on an algebra forms a complete lattice or whether the natural map of a p-congruence is a p-homomorphism.

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\(^2\)See Definition 23.
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