BARTNIK HILBERT MANIFOLD STRUCTURE ON FIBERS OF THE
SCALAR CURVATURE AND THE CONSTRAINT OPERATOR

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ABSTRACT. We adapt the Bartnik method to provide a Hilbert manifold structure for
the space of solutions, without KID’s, to the vacuum constraint equations on compact
manifold of any dimension \( \geq 3 \). In the course, we prove that some fibers of the scalar
curvature or the constraint operator are Hilbert submanifolds. We also study some oper-
ators and inequalities related to the KID’s operator. Finally we comment the adaptation
to some non compact manifolds.

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weak regularity.

1. INTRODUCTION

The linearisation stability studies of Fischer, Marsden and Moncrief, started in 1975
(see [12,13,15]) implies the existence of a Fréchet manifold structure (modelled on \( C^\infty \))
for the set of solutions of the vacuum constraint equations on a compact manifold. A
similar structure has been proven by Andersson in the asymptotically flat setting in 1987,
[1]. With P. Chruściel, we obtained a Banach manifold structure (modelled on \( C^{k,\alpha} \))
of such set of solutions in three classical context of compact, or asymptotically flat ,or
asymptotically hyperbolic manifolds in 2004, [10]. At the same time R. Bartnik, provided
a Hilbert manifold structure (modelled on \( H^2 \times H^1 \)) on three dimensional asymptotically
flat manifolds ( [4], see also [22], [23], [26] for some adaptation to coupled equations in
dimension 3). The weak regularity assumptions concerning the metric involved (curvature

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only in $L^2$ in dimension 3) can be related to the context of the bounded $L^2$ curvature of S. Klainerman, I. Rodnianski and J. Szeftel [21]. Note also that R. Bartnik showed in [3] that these assumptions on the regularity are the weakest possible to define the ADM mass of the manifold.

With J. Fougeirol, we obtained an asymptotically hyperbolic version of the Bartnik Hilbert structure [11] on 3-manifolds. In the course we had to introduce and study two natural operators of second order $T$ and $U$ in order to overcome special difficulties to this asymptotic.

We could also mention here the weak local (around a more regular solution) Banach manifold structure modelled on $W^{2,p} \times W^{1,p}$ ($p > n$) for AF manifold, which follows from lemma 2.10 in [19] (which corrected a global structure affirmation of the papers cited as [8] and [11] there, see remark 2.11 of [19]).

The goal of the present paper is to obtain a Hilbert manifold structure, modelled on $H^{k+2} \times H^{k+1}$, on an $n$-dimensional compact manifold for $k + 2 > n/2$.

Thus, in the compact setting, we obtain a non trivial generalization of the Bartnik-Hilbert structure either for the dimension but also for the regularity.

Before going to the main theorem, let us introduce some notations. The constraint operator $\Phi$ we are studying act on couples of the form $(g, \pi)$ where $g$ is Riemannian metric and the field $\pi$ is a contravariant symmetric two tensor field valued in the n-forms. Its thorough definition is given in section 4, but let us write here $\Phi(g, \pi)$ as follows:

$$
\Phi_0(g, \pi) := (R(g) - 2\Lambda) \sqrt{g} - \left( |\pi|^2_g - \frac{1}{n-1} (\text{tr}_g \pi)^2 \right) / \sqrt{g},
$$
$$
\Phi_i(g, \pi) := g_{ij} \nabla_k \pi^{jk},
$$

where $R(g)$, $\sqrt{g}$, $\nabla$, are respectively the scalar curvature, the volume form, and the connexion of $g$, and $\Lambda$ is the cosmological constant. We start with the scalar curvature operator.

**Theorem 1.** Let $M$ be a smooth $n$-dimensional compact manifold with $n \geq 3$. Let $k \in \mathbb{N}$ such that $k + 2 > n/2$. Let $R : H^{k+2}_+ \to H^k$ be the scalar curvature operator. Then for every $\varepsilon \in H^k$, the non-static fiber of the scalar curvature map:

$$
S_0(\varepsilon) := \{ g \in H^{k+2}_+ : \ker DR(g)^* = \{0\}, R(g) = \varepsilon \}
$$

is a submanifold of $H^{k+2}$. In particular, the space of non static solutions of the constant scalar curvature equation has a Hilbert submanifold structure.

It is easy to prove that if $\varepsilon$ is negative, then any solution is non static, but in more general situation the non staticity may be proved (see [8, 9, 12] for smooth metrics).

For the constraint map, we work with a phase space $F$ consisting of pairs $(g, \pi)$ of $H^{k+2} \times H^{k+1}$ regularity.

We now state a formal version of our main result (see Theorem 3 for a precise statement).

**Theorem 2.** Let $n \geq 3$ and let $k \in \mathbb{N}$ such that $k + 2 > n/2$. Let $\Phi : F \to L^*$ be the constraint operator associated to the cosmological constant $\Lambda$. For every $\varepsilon \in L^*$, the no KID’s fiber of the constraint map

$$
C_0(\varepsilon) := \{ (g, \pi) \in F : \ker D\Phi(g, \pi)^* = \{0\}, \Phi(g, \pi) = \varepsilon \}
$$
is a submanifold of $\mathcal{F}$. In particular, the space of solutions of the vacuum constraint equations without KID's $C = C_0(0)$ has a Hilbert submanifold structure.

The low regularity of the metrics involved (curvature may be unbounded), and the non linear characteristic of the constraint operator, forces us to a very precise analysis of the different steps of the proof. All the usual instant thoughts with more regularity have to be reconsidered, including for instance, boundedness of operators, elliptic estimates or Fredholm properties (the operators also have weak regular coefficients here).

The kernel of the adjoint operator can be computed in a $H^k$ space and not in the all dual space $H^{-k}$, this regularity result take an important part of the paper and has his own interest.

The paper is written in the spirit of an easy adaptation to the non compact setting, where a reference metric $\hat{g}$ with an asymptotic constant curvature on each end is usually used. For this future applications, we recall the definitions of a Hessian-type operator $\hat{T}$ and a differential operator of order two, called $\hat{U}$, Combined from the first derivatives of the Killing operator $\hat{S}$. As in [11] they can be useful for non compact setting, once obtained new Poincaré and Korn-type estimates of second order on the ends. These estimates are very important to prove there is no KID's (with appropriate behaviour).

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2. Notations and conventions

Let $(\mathcal{M}, g)$ be a Riemannian manifold. We define $T^r_m(\mathcal{M})$ to be the bundle of tensor covariant of rank $m$ and contravariant of rank $r$. For all $u \in T^r_m(\mathcal{M})$, $|u|_g$ will denote the norm of $u$ with respect to the metric $g$. $d\mu(g)$ is the Riemannian measure determined by $g$. $\text{Riem } g, \text{Ric } g$ and $R(g)$ are respectively the Riemann tensor, the Ricci tensor and the scalar curvature of the metric $g$. For a Riemannian metric $g$ with connection $\nabla$, we set the following notations concerning the Hessian and Laplacian of a function $u$:

$$\nabla^2_{ij} u = \nabla_i \nabla_j u,$$
$$\Delta u = \text{tr}_g \nabla^2 u = g^{ij} \nabla^2_{ij} u.$$

We fix a smooth Riemannian metric $\hat{g}$. When working on non compact manifold such as asymptotically Euclidian or asymptotically hyperbolic, it is convenient to choose for $\hat{g}$ a special metric, for instance having Ricci curvature asymptotic to a constant at infinity like the model spaces.

We define the following norms for the usual Lebesgue spaces $L^p$ of functions or tensor fields.

$$\forall 1 \leq p < \infty \quad ||u||_p = \left( \int_{\mathcal{M}} |u|_g^p \, d\mu(\hat{g}) \right)^{1/p}$$

For $p = \infty$ \quad $||u||_\infty = \sup_{\mathcal{M}} (|u|_\hat{g})$

The norm on the Sobolev space $W^{k,p}$ is defined as
$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} ||\nabla^\alpha u||_p$$

where $\alpha$ is a multi-index of size $n$ and $\nabla^\alpha u = \nabla_{i_1}^{\alpha_1} \cdots \nabla_{i_n}^{\alpha_n} u$

$\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \sum_{i=1}^n \alpha_i$

If needed, we specify $W^{k,p}(T^r_m \mathcal{M})$ for the corresponding Sobolev space on tensors of type $(r, m)$ on $\mathcal{M}$.

For the special case where $p = 2$ we could use the notation $H^k$ for $W^{k,2}$. In that context we also need to define $H^k$ for negative integers $k$. Let $k \in \mathbb{N}$, for any $v \in L^2(M, T)$, we define the linear continuous map $L_v : H^k(M, T^* \otimes \Lambda^n \mathcal{M}) \to \mathbb{R}$ by

$$L_v(u) = \langle v, u \rangle = \int_M v u.$$  

$H^{-k}$ is the completion of $L^2$ for the norm

$$||v||_{-k} = ||L_v|| = \sup_{u \in H^k} \frac{\langle v, u \rangle}{||u||_k}.$$  

$H^{-k}$ is the dual of $H^k$. The $L^2$-product $L_v(u) = \langle v, u \rangle$ defined for $v \in L^2$ and $u \in H^k$ can be extended to $v \in H^{-k}$ by

$$\langle v, u \rangle = \lim_{n \to \infty} \langle v_n, u \rangle,$$

where $v_n \in L^2$ tends to $v$ in $H^{-k}$.

By the Riesz representation theorem, for $v \in H^{-k}$ there exist a unique $w \in H^k$ such that $v(u) = \langle w, u \rangle_{H^k}$. So $H^{-k}$ correspond to distributions on $\mathcal{M}$ who can be written in the form

$$v = \sum_{0 \leq l \leq k, \ 0 \leq i_1, \ldots, i_l \leq n} (-1)^l \nabla_{i_1} \cdots \nabla_{i_l} \nabla_{i_1} \cdots \nabla_{i_l} w$$

for some $w$ in $H^k$, or the more usual form

$$v = \sum_{0 \leq l \leq k, \ 0 \leq i_1, \ldots, i_l \leq n} \nabla_{i_1} \cdots \nabla_{i_l} l W_{i_1, \ldots, i_l},$$

where the $lW$’s are covariant tensors of rank $l$ in $L^2$. Similarly we may define Sobolev spaces of negative order $W^{-k,p}$ and more generally $W^{s,p}$ for any $s \in \mathbb{R}$ (see definitions 2 and 3 in [18]).

We also recall the usual H"older spaces , $C^{s,\alpha}(\mathcal{M}, g)$ with $0 < \alpha < 1$ and the following norm

$$||u||_{C^{s,\alpha}} = \max_{|k| \leq s} ||\nabla^k u||_{C^{0,\alpha}}$$

with

$$||u||_{C^{0,\alpha}} = \sup_{x \in \mathcal{M}} |u|_\tilde{g} + \sup_{x \in \mathcal{M}} \left( \sup_{d_{\tilde{g}}(x,y) \leq 1} \frac{|\tilde{u}(x) - \tilde{u}(y)|_{\tilde{g}}}{d_{\tilde{g}}(x,y)^{\alpha}} \right)$$

where $\tilde{u}$ et $\tilde{g}$ correspond to the tensors $u$ and $g$ in an orthonormal basis.
3. Analysis tools

We first describe a listing of some more or less classic inequalities (see eg. [2,5,6,18]).

Hölder inequalities:
• Let $p, q, r \in \mathbb{N}$ be such that
  
  \[ 1 \leq p \leq q \leq r \leq \infty \quad \text{and} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \]

  then
  \[ ||uv||_p \leq ||u||_q ||v||_r. \tag{1} \]

• Set $\lambda \in [0,1]$ and let $p, q, r \in \mathbb{N}$ be such that
  \[ 1 \leq p \leq q \leq r \leq \infty \quad \text{and} \quad \frac{1}{p} = \frac{\lambda}{q} + \frac{1-\lambda}{r}, \]

  then
  \[ ||u||_p \leq ||u||_q^{\lambda} ||u||_r^{1-\lambda}. \tag{2} \]

Sobolev embedding:
For all $1 \leq p \leq q < \infty$, for all $k \geq k'$, we have $W^{k,q} \subset W^{k',p}$, and there exists a positive constant $c = c (\hat{g}', k, k', n, p, q)$ such that

\[ ||u||_{k',p} \leq c ||u||_{k,q}. \]

If $q = \infty$, for all $1 \leq p \leq \infty$; for all $k \geq k'$, we get $W^{k,\infty} \subset W^{k',p}$, and there exists a positive constant $c$ such that

\[ ||u||_{k',p} \leq c ||u||_{k,\infty}. \]

Hölder embedding, Morrey’s inequality:
For all $0 < l + \alpha \leq k + 2 - n/2$, there exists a positive constant $c$ such that

\[ \forall u \in W^{k+2,2}(\mathcal{M}), \quad ||u||_{C^{l,\alpha}} \leq c ||u||_{k+2,2}. \tag{3} \]

Sobolev inequalities:
Set $1 \leq p < \infty$ and let $k, j$ be integers. In each of the following cases, there exists a positive constant $c$ such that for all $u \in W^{j+k,p}(\mathcal{M})$,

• If $pk < n$, $||u||_{j,q} \leq c ||u||_{j+k,p}, \quad \forall p \leq q \leq \frac{np}{n-kp}$.

• If $pk = n$, $||u||_{j,q} \leq c ||u||_{j+k,p}, \quad \forall p \leq q < \infty$.

• If $pk > n$, $||u||_{j,q} \leq c ||u||_{j+k,p}, \quad \forall p \leq q \leq \infty$.

Ehrling inequality:
Let $j$ and $k$ be two integers such that $0 < j < k$. For all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

\[ \forall u \in W^{k,p}, \quad ||u||_{j,p} \leq \varepsilon ||u||_{k,p} + C(\varepsilon) ||u||_{p}. \tag{4} \]

Rellich - Kondrachov Theorem:
For all $k > k'$, the inclusion $W^{k,2} \subset W^{k',2}$ is compact.

We will also need to control the norm of some products (see eg. [5,6,18]).
Lemma 1. Let $0 \leq m \leq l \leq k$, $q, p \in [1, +\infty]$, $kp > n$. Suppose that $(l, q)$ is such that the Sobolev embedding $W^{k,p} \subset W^{l,q}$ holds. Then the product map $W^{k-m,p} \times W^{l,q} \ni (f, g) \mapsto fg \in W^{l-m,q}$ is continuous.

Remark 1. For our applications we usually use that when $k + 2 > n/2$, 
\[\|uv\|_{k+2,2} + \|u\nabla v\|_{k+1,2} + \|u \nabla^2 v\|_{k,2} + \|\nabla u \nabla v\|_{k,2} \leq c \|u\|_{k+2,2} \|v\|_{k+2,2}. \tag{5}\]

The following estimates will also be useful.

Lemma 2. Let $p \in [2, +\infty)$ such that $p < \frac{2n}{n-4}$ if $n > 4$. Let $q \in [2, \frac{2n}{n-4})$. For all $k \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a positive constant $c = c(\varepsilon)$ such that
\[\forall u \in W^{k+2,2}(\mathcal{M}) \rightarrow ||u||_{k,p} \leq \varepsilon ||u||_{k+2,2} + c(\varepsilon)||u||_{0,2}. \tag{6}\]
\[\forall u \in W^{k+1,2}(\mathcal{M}) \rightarrow ||u||_{k,q} \leq \varepsilon ||u||_{k+1,2} + c(\varepsilon)||u||_{0,2}. \tag{7}\]

Proof. Let $r > p$ such that $r \leq \frac{2n}{n-1}$ if $n > 4$. There exist $\lambda \in (0, 1]$ such that
\[\frac{1}{p} = \frac{\lambda}{2} + \frac{1-\lambda}{r}. \]

By the Sobolev inequality (2) used for $u$ and its derivatives up to order $k$, we find 
\[\|u\|_{k,p} \leq C_k \|u\|_{k,2}^{1-\lambda}, \]
thus \[\|u\|_{k,p} \leq C \|u\|_{k,2} \|u\|_{k+2,2}^{1-\lambda}. \]
We can now use 
\[a^\lambda b^{1-\lambda} \leq \lambda^\lambda (a + b) \leq (a + b), \]
with $b = \varepsilon ||u||_{k+2,2}$ and $a = C_1^{1/\lambda} \varepsilon^{1-\lambda} ||u||_{k,2}$. We obtain the desirable estimate by applying the Ehrling inequality (4) in order to control $a$.

Let $s > q$ such that $s \leq \frac{2n}{n-1}$ then there exist $\lambda \in (0, 1]$ such that
\[\frac{1}{q} = \frac{\lambda}{2} + \frac{1-\lambda}{s}. \]
As before, we have 
\[\|u\|_{k,q} \leq C_k \|u\|_{k,2} \|u\|_{k,s}^{1-\lambda}, \]
so 
\[\|u\|_{k,q} \leq C \|u\|_{k,2} \|u\|_{k+1,2}^{1-\lambda}, \]
and we conclude in the same way $\square$

Remark 2. For our applications, we will often use the lemma 2 combined with the inequality 
\[||uv||_{k,2} \leq ||u||_{k,p} ||v||_{k,2}, \]
true when $kp > n$ (see lemma 1) so we need $\frac{2n}{k} < \frac{2n}{n-4}$ if $n > 4$ for $p$ to exist, it is possible precisely when $k + 2 > \frac{n}{2}$. We will also frequently use the lemma 2 together with 
\[||uv||_{k,2} \leq ||u||_{k,q} ||v||_{k+1,2}, \]
true when $(k + 1)q > n$ (see lemma 1) so we need $\frac{n}{k+1} < \frac{2n}{n-2}$ for $q$ to exist, which is true again exactly when $k + 2 > \frac{n}{2}$. 

Elliptic operators.
Here we recall a classical results about elliptic operators (see eg. [2]). Let $B_1$ and $B_2$ be
two tensor bundles over a compact manifold $(M, g)$ and $A : \mathcal{C}^\infty(B_1) \to \mathcal{C}^\infty(B_2)$ be a
partial differential linear operator of order $m$ defined by

$$A = \sum_{|\alpha| \leq m} a_\alpha \nabla^\alpha.$$  \hfill (8)

We say that $A$ is an elliptic operator if

- For all $\alpha$ such that $|\alpha| = m$, for all $\xi^\alpha = \xi^\alpha_1 \ldots \xi^\alpha_n \neq 0$,
  $a_\alpha \xi^\alpha : B_1 \to B_2$ is a tensor bundles isomorphism.
- There exists two constants $c_1$ and $c_2$ such that, for all $\xi$,

$$||a_\alpha \xi^\alpha||_g < c_1 |\xi^\alpha|_g \quad \text{and} \quad ||(a_\alpha \xi^\alpha)^{-1}||_g < c_2 |\xi^\alpha|_g.$$

Let us recall the following classical elliptic estimate.

**Lemma 3.** For every elliptic operator $A$ of order $m$, there exists a positive constant
c $c = c(g, k)$ such that for all $u \in L^1$ such that $Au \in W^{k,2}$ then $u \in W^{k+m,2}$ with :

$$||u||_{k+m,2} \leq c (||Au||_{k,2} + ||u||_1).$$  \hfill (9)

Moreover $A : W^{k+m,2} \to W^{k,2}$ is semi-Fredholm, i.e. $A$ has finite dimensional kernel and
closed range.

The Killing operator
We study the Killing operator $\hat{S}$ defined on 1-forms by

$$\hat{S}(Y)_{ij} = \frac{1}{2} (\nabla_i Y_j + \nabla_j Y_i) = \nabla_{(i} Y_{j)}.$$  \hfill (10)

This operator plays an important role when studying the formal adjoint to the constraint
operator (also called the KID’s operator). The goal of this section is to recall a Korn
inequality.

**Lemma 4.** Let $k \geq -1$. There exist a constants $C > 0$, such that for any one form
$X \in L^2$ such that $\hat{S}(X) \in H^{k+1}$ then $X \in H^{k+2}$ and

$$||X||_{k+2,2} \leq C (||\hat{S}(X)||_{k+1,2} + ||X||_{0,2}).$$

**Proof.** For $k = -1$, we compute

$$2 \int |\hat{S}(X)|^2 = 2 \int \hat{S}(X)_{ij} \hat{\nabla}^i X^j = \int |\hat{\nabla} X|^2 + \int \hat{\nabla} j X_i \hat{\nabla}^i X^j = \int |\hat{\nabla} X|^2 - \int X_j \hat{\nabla} j \hat{\nabla}^i X^j,$$
we deduce

$$2 \int |\hat{S}(X)|^2 = \int |\hat{\nabla} X|^2 + \int \text{div}(X)^2 + \text{Ric}(\dot{g})(X, X),$$
so we easily obtain

$$||X||_{1,2} \leq C (||\hat{S}(X)||_{0,2} + ||X||_{0,2}).$$  \hfill (11)

If $k \in \mathbb{N}$ we also use the following identity (see e.g. equation (29) of [4] for example)

$$\hat{\nabla}_{kj} X_i := \hat{\nabla}_k \hat{\nabla}_j X_i = Riem \dot{g}_{ijkl} X^l + \hat{\nabla}_k \hat{S}(X)_{ij} + \hat{\nabla}_j \hat{S}(X)_{ik} - \hat{\nabla}_i \hat{S}(X)_{jk}.$$  \hfill (12)
This leads to
\[ ||\hat{\nabla}^2 X||_{k,2} \leq ||\text{Riem} \hat{g} X||_{k,2} + c ||\hat{\nabla}\hat{S}(X)||_{k,2} \leq c ||X||_{k,2} + c ||\hat{\nabla}\hat{S}(X)||_{k,2}. \]  
(13)

The equations (11), (13) and the Ehrling inequality (4) close the proof. □

**Remark 3.** During the proof the following interesting operator appear
\[ \hat{U}(X)_{kji} := \hat{\nabla}_k \hat{\nabla}_j X_i - \text{Riem} \hat{g}_{kji} X^l = \hat{\nabla}_k \hat{S}(X)_{ij} + \hat{\nabla}_j \hat{S}(X)_{ik} - \hat{\nabla}_i \hat{S}(X)_{jk}. \]

**A shifted Hessian operator**

When studying the KID’s, a natural operator acting on function appear:
\[ L(N)_{ij} := \nabla_i \nabla_j N - g_{ij} \Delta g N - [R_{ij} - \frac{1}{2}(R(g) - 2\Lambda)g_{ij}]N. \]

We thus define the operator
\[ T(N) = \nabla\nabla N - \frac{1}{2(n-1)}(R(g) + 2\Lambda)g_{ij}N, \]
so \( L(N) = T(N) - \text{tr}_g(T(N))g. \) Using the Ehrling inequality (4) we immediately obtain:

**Lemma 5.** Let \( k \in \mathbb{N} \). There exist a constant \( C > 0 \) such that If \( N \in L^2 \) is such that \( \hat{T}(N) \in H^k \) then \( N \in H^{k+2} \) and
\[ ||N||_{k+2,2} \leq C(||\hat{T}(N)||_{k,2} + ||N||_{0,2}). \]

4. **The constraint operator**

Let \( \mathcal{M} \) be a \( n \)-dimensional connected smooth compact oriented manifold. We fix a smooth Riemannian metric \( \hat{g} \) on \( \mathcal{M} \). Let \( \tau \in \mathbb{R} \) and let
\[ \hat{K} = \tau \hat{g}. \]

We consider \( \mathcal{M} \) as a spacelike hypersurface of a \((n + 1)\)-dimensional Lorentzian manifold \((\mathcal{N}, \gamma)\), from now on called spacetime. We will identify the two manifolds by different indices: Latin indices will take values from 1 to \( n \) and are spatial indices whereas Greek indices will take values from 0 to \( n \) and are spacetime indices. \( K \) is the second fundamental form of \( \mathcal{M} \) in \( \mathcal{N} \) defined by
\[ K(X, Y) = \gamma(X, (\gamma)\nabla_Y \hat{n}), \]  
(14)
where \((\gamma)\nabla \) is the spacetime connection on \( T\mathcal{N}, X, Y \in T\mathcal{M} \) and \( \hat{n} \) is the future-directed unit normal to \( \mathcal{M} \) in \( \mathcal{N} \). It is convenient to consider the conjugate momentum \( \pi \) as a reparametrisation of \( K \)
\[ \pi^{ij} = \hat{\pi}^{ij} \sqrt{\hat{g}} \quad \text{with} \quad \hat{\pi}^{ij} = K^{ij} - \text{tr}_g Kg^{ij}, \]  
(15)
where \( \sqrt{\hat{g}} \) is the (relative) volume measure of the metric \( g \) :
\[ \sqrt{\hat{g}} = \frac{\sqrt{\text{det}(g)}}{\sqrt{\text{det}(\hat{g})}}, \]
identified with the volume form
\[ d\mu(g) = \sqrt{\hat{g}} d\mu(\hat{g}). \]
The field $\pi$ is a section of the bundle $S^2TM$ of symmetric bilinear forms on $\mathcal{M}$, whereas $\pi$ is a section of the bundle $S^2TM \otimes \Lambda^nT^*\mathcal{M}$ of symmetric 2-tensors-valued densities ($n$-forms) on $\mathcal{M}$.

We define the constraint operator $\Phi = (\Phi_0, \Phi_I)$ as follows:

$$\Phi_0(g, \pi) := (R(g) - 2\Lambda - |K|^2_g + (\text{tr}_g K)^2) \sqrt{g}$$

$$= (R(g) - 2\Lambda) \sqrt{g} - \left( |\pi|^2_g - \frac{1}{n-1}(\text{tr}_g \pi)^2 \right) / \sqrt{g}. \quad (16)$$

$$\Phi_I(g, \pi) := 2(\nabla^j K_{ij} - \nabla_i (\text{tr}_g K)) \sqrt{g}$$

$$= 2g_{ij} \nabla_k \pi^{jk} = 2g_{ij} \nabla_k \pi^{jk} \sqrt{g}. \quad (17)$$

If the spacetime satisfies Einstein’s equations, the normalisation chosen insures that the constraint operator and the energy-momentum tensor are related by

$$\Phi_\alpha = 16\pi G T_{\alpha \mu} \sqrt{g},$$

where $G$ is Newton’s gravitational constant. $\xi = (N, X^i)$ is the lapse-shift associated to the spacetime foliation.

We denote by $T := TN_{T\mathcal{M}}$ the spacetime tangent bundle restricted to $\mathcal{M}$. The following spaces will be used along the paper (recall we would like an easy adaptation to non compact setting)

$$\mathcal{G} := W^{k+2}(S).$$

$$\mathcal{K} := \{ \pi : \pi - \pi \in W^{k+1,2}(\mathfrak{S}) \}. \quad \mathcal{G}^+ := \{ g : g - \tilde{g} \in \mathcal{G}, g > 0 \}. \quad \mathcal{G}^+ := \{ g \in \mathcal{G}^+ : \lambda \hat{g} < g < \lambda^2 \hat{g} \}, 0 < \lambda < 1. \quad \mathcal{L}^* := W^{k,2}(T^* \otimes \Lambda^n T^* \mathcal{M})$$

is the dual space of $\mathcal{L} := W^{-k,2}(T)$. From (3), tensors in $\mathcal{G}$ are Hölder-continuous and thus, matrices inequalities in $\mathcal{G}^+$ are satisfied pointwise. In particular, for all metric $g \in \mathcal{G}^+_+$, metrics $g$ and $\hat{g}$ are equivalent in the following sense:

$$\forall x \in \mathcal{M}, \forall v \in T_x \mathcal{M}, \lambda \hat{g}_{ij}(x) v^iv^j < g_{ij}(x) v^iv^j < \lambda^{-1} \hat{g}_{ij}(x) v^iv^j. \quad (18)$$

$\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}$ will be the phase space of the constraint operator $\Phi$. We will use $(g, \pi)$ as well as $(g, K)$ to denote coordinates on $\mathcal{F}$.

Let $\hat{\Gamma}$ and $\hat{\nabla}$ (resp. $\Gamma$ and $\nabla$) be respectively the Christoffel symbols and the Levi-Civita connection of $\hat{g}$ (resp. $g$). We define

$$A^{\ell}_{ij} = \Gamma^{\ell}_{ij} - \hat{\Gamma}^{\ell}_{ij}. \quad (19)$$

It is well known that

$$A^{\ell}_{ij} = \frac{1}{2} g^{kl} (\nabla_i g_{jl} + \nabla_j g_{il} - \nabla_l g_{ij}). \quad (20)$$

The scalar curvature of $g$ can be formulate using $\nabla$ and $A_{ik}^j$ (see eq. (21) of [4]):

$$R(g) = g^{jk} Ric  \hat{g} \hat{g}_{jk} + g^{jk} (\nabla_i A^j_{ik} - \nabla_j A^i_{ik} + A^i_{jk} A^j_{il} - A^j_{ik} A^i_{jl})$$

$$= g^{jk} Ric  \hat{g} \hat{g}_{jk} + Q(g^{-1}, \nabla g) + g^{jk} g^{il} (\nabla^2_{ij} g_{kl} - \nabla^2_{ik} g_{jl}) \quad (21)$$

where $Q$ is a sum of quadratic terms in $g^{-1}, \nabla g$. 

This result relies on the following fact:

\[ \text{Ric}_{jk} - \text{Ric}_{\hat{g}jk} = \tilde{\nabla}_i A^i_{jk} - \hat{\nabla}_j A^i_{ik} + A^i_{jk} A^i_{\mu} - A^i_{j\mu} A^i_{k\nu} \]  

(22)

Here we show \( \Phi \) is a well-defined mapping between the Hilbert spaces \( \mathcal{F} \) and \( \mathcal{L}^* \).

**Proposition 1.** Let \( 0 < \lambda < 1 \). There exists a positive constant \( c = c(\lambda, \hat{g}) \) such that for all \( (g, \pi) \in G^+_{\lambda} \times \mathcal{K} \),

\[
\|\Phi_0(g, \pi)\|_{k,2} \leq c \left( 1 + \|g - \hat{g}\|_{k+2,2}^2 + \|\pi - \hat{\pi}\|_{k+1,2}^2 \right) \\
\|\Phi_1(g, \pi)\|_{k,2} \leq c \left( \|\hat{\nabla}(\pi - \hat{\pi})\|_{k,2} + \|\hat{\nabla}g\|_{k+1,2}(1 + \|\pi - \hat{\pi}\|_{k+1,2}) \right)
\]

**Proof:** From \( R(g) \) expression (21), we get

\[
\Phi_0(g, \pi) = (R(g) - 2\Lambda) \sqrt{g} - (\|\pi\|_g^2 - \frac{1}{\pi - 1} (\text{tr}_g \pi)^2) / \sqrt{g} \\
= \left[ R(g) - R(\hat{g}) + R(\hat{g}) - 2\Lambda + n(n-1)\tau^2 \right] \sqrt{g} - \|\pi - \hat{\pi}\|_g^2 + 2(\pi - \hat{\pi})_{ij} \hat{\pi}^{ij} \right] / \sqrt{g} \\
+ \frac{1}{n-1} \left( [\text{tr}_g(\pi - \hat{\pi})]^2 + ((g - \hat{g})_{ij} \hat{\pi}^{ij})^2 + 2(g - \hat{g})_{ij} \hat{\pi}^{ij} \text{tr}_g \hat{\pi} + 2 \text{tr}_g(\pi - \hat{\pi}) \text{tr}_g \hat{\pi} \right] / \sqrt{g}.
\]

Since \( g \in G^+_{\lambda} \), we can use (18) and from Cauchy-Schwarz inequality and \( 2ab \leq a^2 + b^2 \)

\[
\|\Phi_0(g, \pi)\|_{g} \leq \left[ \|R(g) - R(\hat{g})\|_g + \|R(\hat{g}) - 2\Lambda + n(n-1)\tau^2 \|_g \right] \sqrt{g} \\
+ c \left[ 1 + \|\pi - \hat{\pi}\|_g^2 + \|g - \hat{g}\|_g^2 \right] \sqrt{g}.
\]

From (22), \( \text{Ric}_g - \text{Ric}_{\hat{g}} \simeq \tilde{\nabla} A + A^2 \simeq (\tilde{\nabla} g)^2 + g \tilde{\nabla}^2 g + g^{-2}(\tilde{\nabla} g)^2 \).

Using lemma 1 and remark 2,

\[
\|\text{Ric}_g - \text{Ric}_{\hat{g}}\|_{k,2} \leq c \|g - \hat{g}\|_{k+2,2},
\]

and

\[
\|R(g) - R(\hat{g})\|_{k,2} \leq c \|g - \hat{g}\|_{k+2,2}.
\]

In particular, we have

\[ \text{Ric}_g - \text{Ric}_{\hat{g}} \in W^{k,2}, \]

(25)

\[ R(g) - R(\hat{g}) \in W^{k,2}. \]

(26)

Thanks to (26) and lemma 1, we obtain the estimate

\[
\|\Phi_0(g, \pi)\|_{k,2} \leq c \left( 1 + \|\pi - \hat{\pi}\|^2_{k+2,2} + \|g - \hat{g}\|^2_{k+2,2} \right) \\
\leq c \left( 1 + \|\pi - \hat{\pi}\|^2_{k+1,2} + \|g - \hat{g}\|^2_{k+2,2} \right),
\]

hence \( \Phi_0(g, \pi) \in \mathcal{L}^* \).

For \( \Phi_1(g, \pi) \), using (19), we have

\[
\Phi_1(g, \pi) = 2g_{ij}(\tilde{\nabla}_k(\pi - \hat{\pi})^{jk} + A^j_{k\mu} (\pi - \hat{\pi})^{k\mu}) + A^j_{k\mu} \hat{\pi}^{k\mu}.
\]

Considering (20), \( \Phi_1(g, \pi) \) is of the form

\[
\Phi_1(g, \pi) \simeq g(\tilde{\nabla}(\pi - \hat{\pi}) + g^{-1} \tilde{\nabla} g (\pi - \hat{\pi}) + g^{-1} \tilde{\nabla} g \hat{\pi}),
\]

(27)
thus again by lemma 1 and remark 2,
\[ ||\Phi_1(g, \pi)||_{k,2} \leq c (||\nabla_\pi (\pi - \tilde{\pi})||_{k,2} + ||\nabla_\pi (\pi - \tilde{\pi})||_{k,2} + ||\nabla_\pi (\pi - \tilde{\pi})||_{k,2}) \]
\[ \leq c (||\nabla_\pi (\pi - \tilde{\pi})||_{k,2} + ||\nabla_\pi (\pi - \tilde{\pi})||_{k+1,2} + ||\nabla_\pi (\pi - \tilde{\pi})||_{k+1,2}) \]
\[ \leq c (||\nabla_\pi (\pi - \tilde{\pi})||_{k,2} + ||\nabla_\pi (\pi - \tilde{\pi})||_{k+1,2}) (1 + ||\pi - \tilde{\pi}||_{k+1,2}) \].

We now bring in the principal result of the section.

**Proposition 2.** The map \( \Phi : \mathcal{F} \to \mathcal{L}^* \) is a smooth map between Hilbert spaces.

**Proof:** We recall the proof of [4] for completeness. From Proposition 1,
\[ ||\Phi(g, \pi)||_{L^\infty} \leq c(1 + ||g - \hat{g}||_G^2 + ||\pi - \tilde{\pi}||_G^2), \]
i.e. \( \Phi \) is locally bounded on \( \mathcal{F} \). The polynomial structure of the constraint operator allows us to show \( \Phi \) is smooth, i.e. indefinitely differentiable in a Fréchet sense. From the expression (21) of scalar curvature and given (27), \( \Phi \) can be expressed as
\[ \Phi(g, \pi) = F(g, g^{-1}, \sqrt{|g|}, \nabla g, \nabla^2 g, \pi, \nabla \pi), \]
where \( F = F(a_1, \ldots, a_8) \) is a polynomial function quadratic in \( a_5 \) and \( a_7 \) and linear in the remaining parameters. The map \( g \mapsto (g, g^{-1}, \sqrt{|g|}, \nabla g, \nabla^2 g, \pi, \nabla \pi) \) is analytic on the space of positive definite matrices and the maps \( g \mapsto \nabla g, g \mapsto \nabla^2 g \) and \( \pi \mapsto \nabla \pi \) are bounded linear, thus smooth, from \( \mathcal{F} \) to \( \mathcal{L}^* \), which are Hilbert spaces. A result from Hille [17] on locally bounded polynomial functional shows \( \Phi \) admit continuous Fréchet-derivatives of all orders. \( \Box \)

5. Linearised constraint and KID’s operator

The set \( \mathcal{C} = \{(g, \pi) \in \mathcal{G}^+ \times \mathcal{K} : \Phi(g, \pi) = 0\} := \Phi^{-1}(\{0\}) \subset \mathcal{F} \) is the set of initial data for the vacuum Einstein’s equations. To prove that \( \mathcal{C} \) is a submanifold of \( \mathcal{F} \), we show that 0 is a regular value of \( \Phi \), so we are interested in the surjectivity of the differential of \( \Phi \), also related to the injectivity of its adjoint. We recall the expression of the linearization of the constraint operator \( \Phi \) and its formal adjoint, the KID’s operator (see [4] or [14] for example).

**Proposition 3.** The differential of the constraint operator \( \Phi \) at \((g, \pi)\) in the direction \((h, p)\) is
\[ D\Phi_0(g, \pi)(h, p) = \nabla^i \nabla^j h_{ij} - \Delta_g \text{tr}_g h) \sqrt{|g|} - h_{ij}[R^{ij} - \frac{1}{2} R(g - 2\Lambda) g^{ij}] \sqrt{|g|} + p h_{ij} (\frac{2}{n-1} \text{tr}_g \pi \eta_{ij} - 2 \pi^{i} \pi^{k}ij + \frac{1}{2} |\pi|^2 g^{ij} - \frac{1}{2(n-1)} (\text{tr}_g \pi)^2 g^{ij}) \sqrt{|g|} + p^{ij} (\frac{2}{n-1} \text{tr}_g \pi \eta_{ij} - 2 \pi^{i} \pi^{k}) \sqrt{|g|}. \]

Using notations of [4], we define
\[ \delta g \delta \pi h = \nabla^i \nabla^j h_{ij}, \]
\[ E^{ij} = R^{ij} - \frac{1}{2} R(g - 2\Lambda) g^{ij}, \]
\[ \Pi^{ij} = \left( \frac{2}{n-1} \text{tr}_g \pi \eta_{ij} - 2 \pi^{i} \pi^{k}ij + \frac{1}{2} |\pi|^2 g^{ij} - \frac{1}{2(n-1)} (\text{tr}_g \pi)^2 g^{ij} \right) / (\sqrt{|g|})^2. \]
We can express $D\Phi$ in the following form

$$D\Phi(g, \pi)(h, p) = \begin{bmatrix} \sqrt{\gamma} (\delta_g \delta_p - \Delta_g \text{tr}_g + \Pi - E) & -2K \\ 2\delta_g \end{bmatrix} \begin{bmatrix} h \\ p \end{bmatrix},$$

(30)

with $\hat{\pi} \nabla h = \hat{\pi}^{ij} \hat{\nabla}_j h_{kl} = (\pi^i k \delta^j_i + \pi^j i \delta^k_i - \pi^k l \delta^j_l) \nabla_j h_{kl}$.

To prove surjectivity of the differential of $\Phi$, we study the injectivity of the adjoint operator. Integrating by parts and ignoring boundary terms leads (cf. [14] for example) to the expression of the formal $L^2(d\mu(\hat{g}))$-adjoint of $D\Phi(g, \pi)$, also called the KID’s operator:

$$\int_M D\Phi(g, \pi)(h, p)(N, X) = \int_M (h, p) \bullet D\Phi(g, \pi)^*(N, X).$$

The detail of the product is given by the following equalities

$$(h, p) \bullet D\Phi_0(g, \pi)^* N = h_{ij} \nabla^i N \nabla^j N - g^{ij} \Delta_g N - [R^{ij} - \frac{1}{2}(R(g) - 2\Lambda)g^{ij}]N\sqrt{g} + Nh_{ij} \left(\frac{2}{2n-1}\text{tr}_g \pi^i \pi^j - 2\pi^{ik} \pi^{kj} + \frac{1}{2} |\pi^i_2 g^{ij}| - \frac{1}{2(n-1)} (\text{tr}_g \pi)^2 g^{ij} \right) \sqrt{g} + N \pi^{ij} \left(\frac{2}{2n-1}\text{tr}_g g_{ij} - 2\pi_{ij}\right) \sqrt{g}.$$

$$(h, p) \bullet D\Phi_1(g, \pi)^* X^i = h_{ij} (X^k \nabla_k \pi^{ij} + \nabla_k X^k \pi^{ij} - 2\nabla_k (\pi_{i}^{j} k)) - 2p^{ij} \nabla_i X_j.$$

Then we can put the KID’s operator $D\Phi^*$ in the matrix form

$$D\Phi(g, \pi)^*(N, X) = \begin{bmatrix} \sqrt{\gamma} (\nabla^2 - \Delta_g + \Pi - E) & \nabla \pi - \hat{\pi} \nabla \\ -2K & -L_g \end{bmatrix} \begin{bmatrix} N \\ X \end{bmatrix},$$

(31)

with

$$(\nabla \pi - \hat{\pi} \nabla) X = L_X \pi = \nabla X^i \pi^{ij} - \hat{\pi}^i_j \nabla_k X^l, \quad L_g (X) = L_g g = 2 \nabla_i X_j = 2 S(X).$$

$D\Phi(g, \pi)^1_1, \xi$ and $D\Phi(g, \pi)^2_2, \xi$ will denote the two components of $D\Phi(g, \pi)^*$ in (31).

We will denote by $W^{k, 2}_\xi$ any terms of the form $u \xi$ such that $||u||_{k, 2} \leq C$, where $C$ is a constant depending on $\hat{g}$ and $||((g, \pi))||_F$. We have

$$D\Phi(g, \pi)^1_1, \xi = [\nabla_i \nabla_j N - g_{ij} \Delta_g N + (\Pi_{ij} - E_{ij})]N\sqrt{g} + (\nabla \pi - \hat{\pi} \nabla) X,$$

$$D\Phi(g, 0)^* (N, 0) + \Pi_{ij} N \sqrt{g} + (\nabla \pi - \hat{\pi} \nabla) X,$$

(32)

$$(\nabla \pi - \hat{\pi} \nabla) X = X^k \nabla_k \pi^{ij} - (\pi^i_k \delta^j_i + \pi^j_k \delta^i_l - \pi^k l \delta^j_l) \nabla_k X^l,$$

$$W^{k, 2} X + W^{k+1, 2} \nabla X + (n - 1) r(2 \hat{S}(X) - \hat{g} \text{tr}_g \hat{S}(X)),$$

$$\Pi(g, \pi) N = W^{k, 2} N + \Pi(\hat{g}, \hat{\pi}) N,$$

$$W^{k, 2} N - \frac{1}{2} (n-1)(n-4) r^2 \hat{g} N.$$

If $(g, \pi) \in F$ we have

$$\Pi(g, \pi) - \Pi(\hat{g}, \hat{\pi}) = \Pi(g, \pi) + \frac{1}{2} (n-1)(n-4) r^2 \hat{g} \in W^{k, 2},$$

(33)

and

$$E - k(n-1) \hat{g} - \frac{1}{2} n(n-1) r^2 \hat{g} \in W^{k, 2},$$

(34)

(this last quantity being equal to $E - \hat{E}$ if $\text{Ric}(\hat{g}) = k(n-1) \hat{g}$).
On one hand, we find
\[
D \Phi(g, \pi)_{1,2}^* \xi / \sqrt{g} = \nabla^2 N - g \Delta g N - k(n-1)\hat{g}N + [\Pi + \frac{1}{2}(n-1)(n-4)\tau^2 \hat{g}]N \\
+ (n-1)\tau (2\hat{S}(X) - \hat{g} \mathrm{tr}_g \hat{S}(X)) - [E - k(n-1)\hat{g} - \frac{1}{2}n(n-1)\tau^2 \hat{g}]N \\
- (n-1)(n-2)\tau^2 \hat{g}N + W^{k,2} \xi + W^{k+1,2} \nabla X
\]
\[
= \nabla^2 N - g \Delta g N - k(n-1)N + (n-1)\tau (2\hat{S}(X) - \hat{g} \mathrm{tr}_g \hat{S}(X)) \\
- (n-1)(n-2)\tau^2 \hat{g}N + W^{k,2} \xi + W^{k+1,2} \nabla X.
\]

On the other hand, we can write
\[
D \Phi(g, \pi)_{2,2}^* \xi = -2KN - 2\hat{S}(X) \\
= -2(\hat{S}(X) + \tau \hat{g}N) + W^{k+1,2} \xi.
\]

From the definition of the operator \( T = \nabla^2 N - Ng \) and the expression of \( \hat{S} \) related to \( D \Phi(g, \pi)_{2,2}^* \xi \), we obtain
\[
D \Phi(g, \pi)_{1,1}^* \xi / \sqrt{g} = T - g \mathrm{tr}_g T + (n-1)\tau (2\hat{S}(X) - \hat{g} \mathrm{tr}_g \hat{S}(X)) \\
- (n-1)(n-2)\tau^2 \hat{g}N + W^{k,2} \xi + W^{k+1,2} \nabla X.
\]
\[
D \Phi(g, \pi)_{2,2}^* \xi = -2(\hat{S}(X) + \tau \hat{g}N) + W^{k+1,2} \xi.
\]

It is useful to restructure \( D \Phi^* \) into the operator \( P^* \) defined by
\[
P^*(\xi) = P^*_{(g, \pi)}(\xi) = \left[ g^{1/4} \left( \nabla^i \nabla_j N - \delta^i_j \Delta g N + (\Pi^i_j - E^i_j)N \right) + g^{-1/4} \nabla_l (K^i_j N + S(X)_{ij}) \right] \\
= \zeta \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & \nabla \end{array} \right] \circ D \Phi(g, \pi)^* \xi,
\]
where \( g^{1/4} = (\det(g)/\det(\hat{g}))^{1/4} d\mu(\hat{g}) \) is a density of weight \( \frac{1}{2} \) and
\[
\zeta = \zeta(g) = \left[ \begin{array}{cc} g^{-1/4} g_{jk} & 0 \\ 0 & g^{1/4} g^{jk} \end{array} \right].
\]

Finally, we can put \( P^*_{(g, \pi)}(\xi) \) into the form
\[
P^*_{(g, \pi)}(\xi) = \left( \begin{array}{c} g^{-1/4} D \Phi(g, \pi)^* \xi \\ g^{1/4} \nabla D \Phi(g, \pi)_{2,2}^* \xi \end{array} \right).
\]

Expression (38) of \( P^* \) allows us to rewrite the \( L^2(d\mu(\hat{g})) \)-adjoint of \( P^* \) as follows
\[
P_{(g, \pi)} = D \Phi(g, \pi) \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & -\delta_g \end{array} \right] \circ \zeta,
\]
with \( \delta_g \eta = \nabla^l (q^i_l) \) so that \( P(f^j_l, q^j_l) = D \Phi(f^i_j, q^i_l) \) and so the composition \( PP^* \) is well defined.
6. Elliptic estimates for the KID’s operator

In this section, we gather elliptic estimates satisfied by the adjoint operator $D\Phi^*$. We start with:

**Proposition 4.** If $k + 2 > n/2$, there exists a positive constant $C = C(\bar{g}, \lambda, ||g||_X)$ such that the following elliptic estimate hold: $\forall \xi \in W^{k+2,2}(T)$,

$$||\xi||_{k+2,2} \leq c \left( ||D\Phi(g, \pi)^*\xi||_{k+2,} + ||D\Phi(g, \pi)_{2}\xi||_{k+1,2} \right) + C ||\xi||_{0,2},$$  

(42)

**Proof:** Considering expression (37) of $\hat{S}$ as a function of $D\Phi(g, \pi)_{2}\xi$,

$$T - g \text{tr}_g T = D\Phi(g, \pi)^*\xi / \sqrt{g} + \tau(D\Phi(g, \pi)_{2}\xi - \frac{1}{2}g \text{tr}_g D\Phi(g, \pi)_{2}\xi)$$

$$+ W^{k,2}\xi + W^{k+1,2} \nabla X.$$  

(43)

From lemma 5, equation (1), and the remark 2 we have

$$||N||_{k+2,2} \leq c \left( ||D\Phi(g, \pi)^*\xi||_{k+2,} + (n-1)\tau \right) \left( 1 + \frac{\beta^2}{4} \right) ||D\Phi(g, \pi)_{2}\xi||_{k+2,}$$

$$+ C \left( ||\xi||_{k,1} + ||\nabla \xi||_{k,2} + ||N||_{0,2} \right).$$

(44)

Using (6), (7) and Sobolev embedding, there exists a positive constant $C = C(\bar{g}, \lambda, ||g||_X)$ such that

$$||\xi||_{k+2,2} \leq c \left( ||D\Phi(g, \pi)^*\xi||_{k+2,} + (n-1)\tau \right) \left( 1 + \frac{\beta^2}{4} \right) ||D\Phi(g, \pi)_{2}\xi||_{k+2,}$$

$$+ \varepsilon ||\xi||_{k+2,2} + C ||\xi||_{0,2}. $$

(45)

Now from (37) and (6), we get the estimate

$$||\hat{S}(X)||_{k+2,2} \leq \frac{1}{4} ||D\Phi(g, \pi)_{2}\xi||_{k+2,} + n|\tau| ||N||_{k+1,2} + \varepsilon ||\xi||_{k+2,2} + C ||\xi||_{0,2}.$$  

(46)

Consequently, using the lemma 4 there exists a constant $C$ depending on $\bar{g}, \lambda, \varepsilon$ and $||\xi||_X$ such that

$$||X||_{k+2,2} - n\epsilon \leq \frac{1}{4} ||D\Phi(g, \pi)_{2}\xi||_{k+2,} + \varepsilon ||\xi||_{k+2,2} + C ||\xi||_{0,2}. $$

(47)

We can choose a small positive constant $\epsilon_0$ so that (44) + $\epsilon_0$ implies (42).  

□

As a preliminary result, we shall establish a lemma corresponding to the Time-symmetric version of proposition 5 given later.

**Lemma 6.** Let $k \in \mathbb{N}$ such that $k + 2 > \frac{n}{2}$. The operator

$$D\Phi(g, 0)^* : W^{k+2,2}(\mathcal{M}) \rightarrow W^{k,2}(\tilde{S})$$

is bounded and depends on $g$ in a Lipschitz way,

$$||D\Phi(g, 0)^* - D\Phi(\bar{g}, 0)^* ||_{k+2,2} \leq C ||g - \bar{g}||_X ||N||_{k+2,2},$$  

(47)

where the constant $C$ depends on $\bar{g}, ||g||_X$ and $||\bar{g}||_X$.  

**Proof:** Let us recall the statement of $D\Phi(g, 0)^* :$

$$D\Phi(g, 0)^*(N, 0) = [\nabla, \nabla g]N - g_{ij} \Delta_g N - [R_{ij} - \frac{1}{2}(Rg - 2\Lambda)g_{ij}]N \sqrt{g}. $$  

(48)

We begin by showing $D\Phi(g, 0)^*$ is bounded. Let us define the operator acting on functions

$$O(N) = \nabla^2 N - g \Delta_g N$$  

(49)
and note that \( O(N) = L(\nabla^2 N) \) where \( L \) is a linear invertible operator. Thus
\[
||O(N)||_{k,2} \leq c ||\nabla^2 N||_{k,2} \leq c \left(||\nabla^2 N||_{k,2} + ||A dN||_{k,2}\right)
\]
\[
\leq C ||N||_{k+2,2},
\]
indeed, \( A dN \simeq g^{-1} \nabla g \ dN \) and using Hölder inequality (1), (5) and Sobolev inclusion,
\[
||A dN||_{k,2} \leq ||g^{-1}||_{k+2,2} ||\nabla g \ dN||_{k,2}
\]
\[
\leq c ||\nabla g||_{k+1,2} ||dN||_{k+1,2}
\]
\[
\leq C ||N||_{k+2,2}. \tag{50}
\]
We go on with
\[
||D \Phi(g, 0)^* (N, 0)/\sqrt{g}||_{k,2} \leq ||O(N)||_{k,2} + ||(Ric \ g - \ g \ \tilde{\Phi}) \ N||_{k,2}
\]
\[
+ ||(\tilde{\Phi} \ g + (n - 1) \ g \ N)||_{k,2} + ||(R(g) - 2\Lambda + n(1 - \tau^2) \ g \ N)||_{k,2}
\]
\[
+ \frac{2}{3} ||(R(g) - R(\tilde{g})) \ N||_{k,2}.
\]
Considering (25), (80), we have
\[
||\tilde{\Phi} \ g + (n - 1) \ g \ N||_{k,2} \leq c ||N||_{k+2,2}.
\]
For the scalar curvature term, using the lemma 1 together with (81) and (26),
\[
||R(g) - (2\Lambda + n(1 - \tau^2) \ g \ N)||_{k,2} \leq C ||N||_{k+2,2},
\]
Now we have of course \( ||(n - 1) \ g \ N||_{k,2} \leq C ||N||_{k+2,2} \), and similarly, \( ||\frac{2}{3} n(1 - \tau^2) \ g \ N||_{k,2} \leq c ||N||_{k+2,2} \). We end up with
\[
||D \Phi(g, 0)^* (N, 0)/\sqrt{g}||_{k,2} \leq C ||N||_{k+2,2},
\]
and finally using again the lemma 1
\[
||D \Phi(g, 0)^* (N, 0)||_{k,2} \leq C ||\sqrt{g}||_{k+2,2} ||N||_{k+2,2} \leq C ||N||_{2,2}, \tag{51}
\]
where \( C \) is a constant depending upon \( \tilde{g} \) and \( ||g||_\mathcal{X} \). The boundedness of the map is then proved.

We now proceed to the proof of equation (47). Let us denote respectively by \( \tilde{\nabla}, \tilde{\Delta}, Ric(\tilde{g}) \) and \( R(\tilde{g}) \) the Levi-Civita connection, the Laplacian, the Ricci tensor and the scalar curvature of the Riemannian metric \( \tilde{g} \). In order to lighten notations, we also set
\[
D \Phi_0(\tilde{g}) N := D \Phi(g, 0)^* (N, 0) \quad \text{and} \quad D \Phi_0(\tilde{g})^* N := D \Phi(\tilde{g}, 0)^* (N, 0).
\]
We split
\[
[D \Phi_0(g)^* - D \Phi_0(\tilde{g})^*] N = (\sqrt{g} - \sqrt{\tilde{g}}) \frac{D \Phi_0(g)^* N}{\sqrt{g}} + \sqrt{g} \left[ \frac{D \Phi_0(g)^* N}{\sqrt{g}} - \frac{D \Phi_0(\tilde{g})^* N}{\sqrt{\tilde{g}}} \right].
\]
It directly implies
\[
\left\| [D\Phi_0(g)^* - D\Phi_0(\hat{g})^*]N \right\|_{k,2} \leq \left\| g - \hat{g} \right\|_X \left\| \frac{D\Phi_0(g)^*N}{\sqrt{g}} \right\|_{k,2} + c \left\| \frac{D\Phi_0(g)^*N}{\sqrt{g}} - \frac{D\Phi_0(\hat{g})^*N}{\sqrt{\hat{g}}} \right\|_{k,2}.
\]

Now, because
\[
\left( \frac{D\Phi_0(g)^*N}{\sqrt{g}} - \frac{D\Phi_0(\hat{g})^*N}{\sqrt{\hat{g}}} \right) = (\nabla - \tilde{\nabla}) dN + g \Delta_g N - \hat{g} \Delta \hat{N} - [Ric(g) - Ric(\hat{g})]N
\]
\[
+ \frac{1}{2} [(R(g) - 2\Lambda)g - (R(\hat{g}) - 2\Lambda)\hat{g}] N,
\]
we have
\[
\left\| \frac{D\Phi_0(g)^*N}{\sqrt{g}} - \frac{D\Phi_0(\hat{g})^*N}{\sqrt{\hat{g}}} \right\|_{k,2} \leq \left\| (\nabla - \tilde{\nabla}) dN \right\|_{k,2} + ||g \Delta_N - \hat{g} \Delta \hat{N}||_{k,2}
\]
\[
+ \frac{1}{2} \left\| [(R(g) - 2\Lambda)g - (R(\hat{g}) - 2\Lambda)\hat{g}] N \right\|_{k,2}
\]
\[
- \left\| [Ric(g) - Ric(\hat{g})]N \right\|_{k,2}.
\]
We will estimate each terms of the right hand side of the above inequality.

- For the Hessians term, we write
\[
\nabla - \tilde{\nabla} = (g^{-1} - \hat{g}^{-1})\tilde{\nabla} g + \hat{g}^{-1}\nabla (g - \hat{g}).
\]

Using (5), we obtain
\[
\left\| (\nabla - \tilde{\nabla}) dN \right\|_{k,2} \leq C \left\| g - \hat{g} \right\|_{k+2,2} \left\| N \right\|_{k+2,2}.
\]

- For the Laplacian terms, we decompose
\[
g \Delta_N - \hat{g} \Delta \hat{N} = g \Delta_N - \hat{g} \Delta \hat{N} + (g - \hat{g}) \Delta g N - \hat{g} \Delta \hat{N}
\]
\[
= (g - \hat{g}) \Delta g N + \hat{g} (\Delta g N - \Delta \hat{N})
\]
\[
= (g - \hat{g}) g^{-1} \nabla dN + \hat{g} (g^{-1} - \hat{g}^{-1}) \nabla dN + \hat{g} \hat{g}^{-1} (\nabla - \tilde{\nabla}) dN.
\]

Using (5), we deduce
\[
\left\| g \Delta_N - \hat{g} \Delta \hat{N} \right\|_{k,2} \leq c \left\| g - \hat{g} \right\|_{k+2,2} \left\| \nabla dN \right\|_{k,2} + c \left\| (\nabla - \tilde{\nabla}) dN \right\|_{k,2}.
\]

Now, considering (50) and given that \( \nabla \simeq A + \tilde{\nabla} \), we have
\[
\left\| \nabla dN \right\|_{k,2} \leq C \left\| N \right\|_{k+2,2}.
\]

Using (54) and (55), we finally get
\[
\left\| g \Delta_N - \hat{g} \Delta \hat{N} \right\|_{k,2} \leq C \left\| g - \hat{g} \right\|_{k+2,2} \left\| N \right\|_{k+2,2}.
\]

- For the Ricci tensors term, we define \( \tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k - \hat{\Gamma}_{ij}^k \) and we set
\[
\tilde{T} := \tilde{\nabla} - \nabla = \hat{g}^{-1} \tilde{\nabla} \hat{g} - g^{-1} \nabla g = (g^{-1} - \hat{g}^{-1}) \nabla g + \hat{g} \hat{g}^{-1} (\nabla - \tilde{\nabla}) g.
\]

Using (5),
\[
\left\| \tilde{T} \right\|_{k+1,2} \leq C \left\| g - \hat{g} \right\|_{k+2,2}.
\]
We can show, adding and subtracting $Ric(\hat{g})$ and using (22), that
\[
[Ric(g) - Ric(\hat{g})]N \simeq (\nabla \hat{T} + \tilde{A}T + \tilde{T}^2)N,
\]
which leads to
\[
||(Ric(g) - Ric(\hat{g}))N||_{k,2} \leq ||\nabla \hat{T}N||_{k,2} + ||\tilde{A}TN||_{k,2} + ||\tilde{T}^2N||_{k,2}. \tag{57}
\]
Using (5) and (56)
\[
||\nabla \hat{T}N||_{k,2} \leq C ||g - \hat{g}||_{k+2,2} ||N||_{k+2,2}
\]
The same method for the term $\tilde{A}TN$ gives, considering (5)
\[
||\tilde{A}TN||_{k,2} \leq C ||g - \hat{g}||_{k+2,2} ||N||_{k+2,2}.
\]
In the same way with (56),
\[
||\tilde{T}^2N||_{k,2} \leq C ||g - \hat{g}||^2_{k+2,2} ||N||_{k+2,2}.
\]
Replacing in (57), we obtain
\[
||(Ric(g) - Ric(\hat{g}))N||_{k,2} \leq C ||g - \hat{g}||_{k+2,2} ||N||_{k+2,2}. \tag{58}
\]
- For the scalar curvatures term, we write
\[
(R(g) - 2\Lambda)g - (R(\hat{g}) - 2\Lambda)\hat{g} = (g - \hat{g})(R(g) - 2\Lambda) + \hat{g}\hat{g}^{-1}(Ric g - Ric \hat{g})
\]
\[
+ \hat{g}(g^{-1} - \hat{g}^{-1})Ric g
\]
\[
= (g - \hat{g}) \left[ R(g) - 2\Lambda + n(n - 1)\tau^2 \right]
\]
\[
- n(n - 1)\tau^2(g - \hat{g}) + \hat{g}\hat{g}^{-1}(Ric g - Ric \hat{g})
\]
\[
+ \hat{g}(g^{-1} - \hat{g}^{-1}) \{ Ric g - Ric \hat{g} \}.
\]
The inequality (5) and (58) will yield
\[
||(R(g) - 2\Lambda)g - (R(\hat{g}) - 2\Lambda)\hat{g}||_{k,2} \leq C ||g - \hat{g}||_{k+2,2} ||N||_{k+2,2},
\]
because for instance $\forall u \in W^{k+2,2}, \forall v \in W^{k,2}$,
\[
||(g - \hat{g}) u v N||_{k,2} \leq ||g - \hat{g}||_{k+2,2} ||u||_{k+2,2} ||v||_{k,2} ||N||_{k+2,2}
\]
\[
\leq C ||g - \hat{g}||_{k+2,2} ||u||_{k+2,2} ||v||_{k,2} ||N||_{k+2,2},
\]
where $C$ is a positive constant depending on $\hat{g}$ and $||g||_F$.
Putting the pieces all together in (52) and taking (51) into account lead to
\[
||(D\Phi_0(g)^* - D\Phi_0(\hat{g})^*)N||_{k,2} \leq C ||g - \hat{g}||_{k+2,2} ||N||_{k+2,2},
\]
and close the proof.
\[
\square
\]
The dependence in $(g, \pi)$ of $P^*$ is controlled as follows:

**Proposition 5.** When $k + 2 > n/2$, the operator $P^* : W^{k+2,2}(\mathcal{T}) \rightarrow W^{k,2}$ is bounded and satisfies
\[
||\xi||_{k+2,2} \leq C ||P^*\xi||_{k,2} + C ||\xi||_{0,2}, \tag{59}
\]
where $C$ depends on $\hat{g}$ and $||(g, \pi)||_F$.
Moreover, $P_{(g, \pi)}^*$ depends on $(g, \pi) \in \mathcal{F}$ in a Lipschitz way,
\[
||(P_{(g, \pi)}^* - P_{(\hat{g}, \hat{\pi})}^*)\xi||_{k,2} \leq C_1 ||(g - \hat{g}, \pi - \hat{\pi})||_F ||\xi||_{k+2,2}, \tag{60}
\]
where constant $C_1$ depends on $\hat{g}, \| (g, \pi) \|_F$ and $\| (\tilde{g}, \tilde{\pi}) \|_F$.

**Proof:** Let us begin by showing $P^*$ is bounded, i.e.

$$||P^* \xi||_{k,2} \leq C ||\xi||_{k+2,2}.$$

We set

$$\begin{cases}
P^* = P^*_{(g, \pi)}, \\
D\Phi^*_1 = D\Phi(g, \pi)^*_1, \\
D\Phi^*_2 = D\Phi(g, \pi)^*_2.
\end{cases}$$

From (40), we have

$$||P^* \xi||_{k,2} \leq c (||D\Phi^*_1 \xi||_2 + ||\nabla D\Phi^*_2 \xi||_2) \leq c (||D\Phi^*_1 \xi||_{k,2} + ||\nabla D\Phi^*_2 \xi||_{k,2} + ||AD\Phi^*_2 \xi||_{k,2}).$$

From (36), (5), we first estimate

$$||D\Phi^*_1 \xi||_{k,2} \leq c (||T||_{k,2} + ||\hat{S}(X)||_{k,2} + ||N||_{k,2}) + C (||\xi||_{k+2,2} + ||\nabla X||_{k+1,2} + ||\nabla N||_{k+1,2}) \leq c (||T||_{k,2} + ||N||_{k,2}) + C (||\xi||_{k+2,2} + ||\nabla X||_{k+1,2} + ||\nabla N||_{k+1,2}) \leq C ||\xi||_{k+2,2}.$$  

From (37) along with (1), (5), we can also control

$$||D\Phi^*_2 \xi||_{k,2} \leq c (||\hat{S}(X)||_{k,2} + ||N||_{k,2}) + C ||\xi||_{k+1,2},$$

$$||AD\Phi^*_2 \xi||_{k,2} \leq c (||A\hat{S}(X)||_{k,2} + ||AN||_{k,2}) + ||\xi||_{k+2,2} \leq C (||\hat{S}(X)||_{k+1,2} + ||N||_{k+1,2}) + ||\xi||_{k+2,2},$$

$$||\nabla D\Phi^*_2 \xi||_{k,2} \leq c (||\nabla \hat{S}(X)||_{k,2} + ||\nabla N||_{k,2}) + ||\xi||_{k+2,2} \leq C ||\xi||_{k+2,2}.$$  

Consequently,

$$||D\Phi^*_2 \xi||_{k,2} \leq ||D\Phi^*_1 \xi||_{k+1,2} \leq C ||\xi||_{k+1,2}.$$  

Every term of the right hand side of (62) is then dominated by $||\xi||_{k+2,2}$ leading to (61). The estimate (59) satisfied by $P^*$ directly comes from (42). We now look into the Lipschitz behaviour of $P^*$. We set

$$\begin{cases}
P^{\ast} = P^{\ast}_{(\hat{g}, \tilde{\pi})}, \\
D\tilde{\Phi}^*_1 = D\Phi(\hat{g}, \tilde{\pi})^*_1, \\
D\tilde{\Phi}^*_2 = D\Phi(\hat{g}, \tilde{\pi})^*_2.
\end{cases}$$

Let us write

$$(P^* - P^\ast) \xi = \begin{pmatrix}
g^{-1/4} D\Phi^*_1 \xi - \hat{g}^{-1/4} D\tilde{\Phi}^*_1 \xi \\
g^{1/4} \nabla D\Phi^*_2 \xi - \hat{g}^{1/4} \nabla D\tilde{\Phi}^*_2 \xi
\end{pmatrix} = \begin{pmatrix}
E \\
F
\end{pmatrix},$$

so

$$|| (P^* - P^\ast) \xi ||_{k,2} \leq || E ||_{k,2} + || F ||_{k,2}.$$  

We start to estimate

$$E = g^{-1/4} D\Phi^*_1 \xi - \hat{g}^{-1/4} D\tilde{\Phi}^*_1 \xi = (g^{-1/4} - \hat{g}^{-1/4}) D\Phi^*_1 \xi + \hat{g}^{-1/4} (D\Phi^*_1 \xi - D\tilde{\Phi}^*_1 \xi).$$
Using (5),
\[ ||E||_{k,2} \leq ||(g^{-1/4} - \hat{g}^{-1/4}) D\Phi_1^* \xi||_{k,2} + ||\hat{g}^{-1/4} (D\Phi_1^* \xi - D\tilde{\Phi}_1^* \xi)||_{k,2} \leq c ||g - \hat{g}||_{\mathcal{F}} ||D\Phi_1^* \xi||_{k,2} + c ||D\Phi_1^* \xi - D\tilde{\Phi}_1^* \xi||_{k,2}.\]

From (32), we expand
\[ D\Phi_1^* \xi - D\tilde{\Phi}_1^* \xi = [D\Phi(g, 0)^* - D\Phi(\tilde{g}, 0)^*] (N, 0) + (\Pi \sqrt{g} - \tilde{\Pi} \sqrt{\hat{g}}) N + X \nabla (\pi - \tilde{\pi}) + \hat{\pi} \nabla X + AX (\pi - \tilde{\pi}).\]

Using (1), (5), we already have
\[ ||(\pi - \tilde{\pi}) \nabla X||_{k,2} \leq ||\pi - \tilde{\pi}||_{k+1,2} ||\nabla X||_{k+1,2} + ||\tilde{\nabla} (\pi - \tilde{\pi})||_{k,2} ||X||_{k+2,2}, \]
and
\[ ||AX (\pi - \tilde{\pi})||_{k,2} \leq ||A (\pi - \tilde{\pi})||_{k,2} ||X||_{k+2,2}, \]

Now, we write formally
\[ \Pi \sqrt{g} - \tilde{\Pi} \sqrt{\hat{g}} \sim \frac{1}{\sqrt{g}} g^{-1} \pi^2 - \frac{1}{\sqrt{\hat{g}}} \hat{\pi}^2 \]
\[ \sim \frac{1}{\sqrt{g}} (g^{-1} - \hat{g}^{-1}) \pi^2 + \frac{1}{\sqrt{\hat{g}}} \hat{\pi}^{-1} (\pi^2 - \hat{\pi}^2) + \left( \frac{1}{\sqrt{g}} - \frac{1}{\sqrt{\hat{g}}} \right) \hat{\pi}^{-1}, \]
leading to
\[ ||(\Pi \sqrt{g} - \tilde{\Pi} \sqrt{\hat{g}}) N||_{k,2} \leq C ||(g - \hat{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||N||_{k+2,2}. \]

Given also (47), we obtain
\[ ||D\Phi_1^* \xi - D\tilde{\Phi}_1^* \xi||_{k,2} \leq C ||(g - \hat{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||\xi||_{k+2,2}, \]
and taking (63) into account,
\[ ||E||_{k,2} \leq C ||(g - \hat{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||\xi||_{k+2,2}. \]

We will now estimate the term
\[ F' = g^{1/4} \nabla D\Phi_2^* \xi - \hat{g}^{1/4} \hat{\nabla} D\tilde{\Phi}_2^* \xi \]
\[ = g^{1/4} (\nabla - \hat{\nabla}) D\Phi_2^* \xi + (g^{1/4} - \hat{g}^{1/4}) \hat{\nabla} D\tilde{\Phi}_2^* \xi + g^{1/4} \hat{\nabla} (D\Phi_2^* \xi - D\tilde{\Phi}_2^* \xi). \]

Using (53),(1), (5),
\[ ||F||_{k,2} \leq c ||\nabla - \hat{\nabla}||_{k+1,2} ||D\Phi_2^* \xi||_{k+1,2} + ||g^{1/4} - \hat{g}^{1/4}||_{k+2,2} ||\nabla D\tilde{\Phi}_2^* \xi||_{k,2} + c ||\hat{\nabla} (D\Phi_2^* \xi - D\tilde{\Phi}_2^* \xi)||_{k,2} + c ||\nabla (D\Phi_2^* \xi - D\tilde{\Phi}_2^* \xi)||_{k,2} + c ||A (D\Phi_2^* \xi - D\tilde{\Phi}_2^* \xi)||_{k,2}. \]
Considering (64) and (65), one has
\[ ||F||_{k,2} \leq C ||g - \tilde{g}||_{\mathcal{F}} ||\xi||_{k+2,2} + c ||\nabla(D\Phi^{*}_2 \xi - D\tilde{\Phi}^{*}_2 \xi)||_{k,2} \]
\[ + c ||A(D\Phi^{*}_2 \xi - D\tilde{\Phi}^{*}_2 \xi)||_{k,2}, \]
with, formally,
\[ D\Phi^{*}_2 \xi - D\tilde{\Phi}^{*}_2 \xi \sim (K - \tilde{K})N + (A - \tilde{A})X \]
\[ \sim (\pi - \tilde{\pi})N + (\nabla - \tilde{\nabla})X. \]
Using (53), (1), (5), we deduce
\[ \nabla(D\Phi^{*}_2 \xi - D\tilde{\Phi}^{*}_2 \xi)||_{k,2} \leq c ||\nabla(\pi - \tilde{\pi})||_{k,2} ||N||_{k+2,2} + c ||\nabla - \tilde{\nabla}||_{k+1,2} ||\nabla N||_{k+1,2} \]
\[ + c ||\nabla - \tilde{\nabla}||_{k,2} ||X||_{k+2,2} + c ||\nabla - \tilde{\nabla}||_{k+1,2} ||\nabla X||_{k+1,2} \]
\[ \leq C ||(g - \tilde{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||\xi||_{k+2,2}. \]
In the same way,
\[ ||A(D\Phi^{*}_2 \xi - D\tilde{\Phi}^{*}_2 \xi)||_{k,2} \leq c ||A(\pi - \tilde{\pi})||_{k,2} ||N||_{k+2,2} + ||A(\nabla - \tilde{\nabla})||_{k,2} ||X||_{k+2,2} \]
\[ \leq C ||(g - \tilde{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||\xi||_{k+2,2}. \]
We deduce from (68)
\[ ||F||_{k,2} \leq C ||(g - \tilde{g}, \pi - \tilde{\pi})||_{\mathcal{F}} ||\xi||_{k+2,2}. \]
The desired Lipschitz estimate (60) arises from (66), considering (67) and (69). □

We claim that the estimate (42) of Corollary 3 is also satisfied by weak solutions \( \xi \) only in \( W^{-k,2}(\mathcal{T}) \). More precisely, we say that \( \xi \in \mathcal{L} \) is a weak solution of \( D\Phi(g, \pi)^* \xi = (f_1, f_2) \), with \( (f_1, f_2) \in W^{k,2}(\tilde{S}) \times W^{k+1,2}(\mathcal{S}) \) when
\[ \langle \xi, D\Phi(g, \pi)(h, p) \rangle = \int_{\mathcal{M}} \langle (f_1, f_2), (h, p) \rangle_{\tilde{g}}, \forall (h, p) \in W^{k+2,2}(\mathcal{S}) \times W^{k+1,2}(\tilde{S}). \]
Note that it suffices to verify the equality for any \( (h, p) \in C^\infty_c(S) \times C^\infty_c(\tilde{S}) \) by density.

**Proposition 6.** Let \( k + 2 > \frac{2}{d} \) and \( (g, \pi) \in G^+ \times \mathcal{K}, (f_1, f_2) \in W^{k,2}(\tilde{S}) \times W^{k+1,2}(\mathcal{S}). \) Assume that \( \xi \in \mathcal{L} \) is a weak solution of \( D\Phi(g, \pi)^* \xi = (f_1, f_2) \), then \( \xi \in W^{k+2,2}(\mathcal{T}) \) is a strong solution and satisfies (42).

**Proof:** We start with an adaptation to the dimension \( n \) of the beginning of the proof of the proposition 3.5 of [4] . More precisely, the equation \( P^* \xi = f \) can be rewritten in local coordinate
\[ A.\partial^2 \xi + B.\partial \xi + C.\xi = f, \]
with \( A \in W^{k+2,2} \) invertible, \( B \in W^{k+1,2}, C \in W^{k,2}, \xi \in W^{-k,2}. \) This is equivalent to an equation of the form (see equation (39) in [4] )
\[ \partial^2 \xi + \partial(h\xi) + c \xi = f, \]
with \( b \in W^{k+1,2} \) and \( c, f \in W^{k,2}. \)
From now on the proof is different from the Bartnik’s one, we do not take a trace of the equation. Let \( p \in \mathbb{N}, p > 0 \) such that \( k + 2 - \frac{2}{d} > \frac{1}{p} \), by multiplications in Sobolev spaces.
(see [18] lemma 28 for instance) we infer that \( c \xi \in W^{-k-2+\frac{1}{p}, 2} \) and \( b \xi \in W^{-k-1+\frac{1}{p}, 2} \). This proves that \( \partial^2 \xi \in W^{-k-2+\frac{1}{p}, 2} \) by equation (70), so combined with \( \partial \xi \in W^{-k-1, 2} \subset W^{-k-2+\frac{1}{p}, 2} \) implies \( \partial \xi \in W^{-k-1+\frac{1}{p}, 2} \). Again this last fact added to \( \xi \in W^{-k, 2} \subset W^{-k-1+\frac{1}{p}, 2} \) leads to \( \xi \in W^{-k+\frac{1}{p}, 2} \). We have then improved the regularity of \( \xi \). Bootstrapping in this way (we can add \( 1/p \) to the regularity at each step) we will end up with \( \xi \in W^{k+2, 2} \). □

In the compact manifold setting, we may assume that the (weak) kernel of \( D\Phi(g, \pi)^* \) is trivial and forget this section. But either for a practical verification and for an adaptation to non compact manifold, where one has to prove there is no kernel under appropriate behaviour, this regularity result is very important.

7. The submanifold structure

We are ready to provide the smooth Hilbert submanifold structure of the set of solutions to the vacuum constraint equations.

We start with a well known fact.

Lemma 7. Let \( X, Y \) be two Banach spaces and \( T \) a linear operator with closed range.

\[ T : X \to Y \]
\[ T^* : Y^* \to X^* \]
then \( (\text{Coker} T)^* \simeq \ker T^* \), where \( \text{Coker} T := \frac{Y}{\text{Im} T} \) is a Banach space.

Proof: This is a classical argument, see [11] lemma 22 for instance □

We can now state our main result.

Theorem 3. Let \( \Phi : \mathcal{F} \to \mathcal{L}^* \) be the constraint operator and assume that \( k + 2 > \frac{n}{2} \). For every \( \varepsilon \in \mathcal{L}^* \), the set of solutions of the constraint equations without KID’s 

\[ C(\varepsilon) := \{(g, \pi) \in \mathcal{F} : \ker D\Phi(g, \pi)^* = \{0\}, \Phi(g, \pi) = \varepsilon \} \]

is a submanifold of \( \mathcal{F} \). In particular, the space of solutions, without KID’s, of the vacuum constraint equations \( C = C(0) \) has a Hilbert submanifold structure.

In order to prove the Theorem 3, we need to show:

- \( \ker D\Phi(g, \pi) \) splits.
- \( D\Phi(g, \pi) \) is surjective.

\( D\Phi(g, \pi) \) being a bounded operator, its kernel is closed by continuity and \( T(g, \pi)\mathcal{F} = W^{k+2, 2}(\mathcal{S}) \times W^{k+1, 2}(\tilde{\mathcal{S}}) \) can be written as a direct sum of \( \ker D\Phi(g, \pi) \) and its orthogonal complement \( (\ker D\Phi(g, \pi))^\bot \), which is always closed. Hence \( \ker D\Phi(g, \pi) \) splits.

The assumption about triviality of \( \ker D\Phi(g, \pi)^* \), leads to 

\[ (\ker D\Phi(g, \pi)^*)^\bot = \mathcal{L}^*. \]

Using the classical relation 

\[ (\ker D\Phi(g, \pi)^*)^\bot = \overline{\text{Im} D\Phi(g, \pi)}, \]

we get 

\[ \overline{\text{Im} D\Phi(g, \pi)} = \mathcal{L}^*. \]
Thus, in order to obtain the surjectivity of $D\Phi(g, \pi)$, we will prove it has closed range. For that it suffice to prove the range is the direct sum of a closed space and a finite dimensional space. To do so, we consider particular variations $(h, p)$ of $(g, \pi)$ of the form

$$
\begin{align*}
    h_{ij} & = 2y g_{ij} \\
    p_{ij} & = (2S(Y)^{ij} - g^{ij} \text{tr}_g S(Y) - (n - 1)(n - 2)\tau y g^{ij}) \sqrt{g}
\end{align*}
$$

(71)

determined from fields $(y, Y)$. We define the operator

$$
F(y, Y) = [F_0(y, Y), F_1(y, Y)] = [D\Phi_0(g, \pi)(h, p), D\Phi_1(g, \pi)(h, p)].
$$

(72)

The equations (28) and (29) provide,

$$
\begin{align*}
    F_0(y, Y) & = 2(n - 1)\sqrt{g} [-\Delta y - \kappa ny] + (4 - n) \Phi_0(g, \pi) y + 2(n - 2)\tau \text{div}Y \sqrt{g} \\
    + W^{k,2}[y + Y] + W^{k+1,2} \nabla Y,
\end{align*}
$$

\[
\begin{align*}
    F_1(y, Y) & = -2\sqrt{g} [-\Delta Y - \kappa(n - 1)Y] + 2 \Phi_1(g, \pi) y + W^{k+1,2} \nabla y + W^{k,2}[y + Y].
\end{align*}
\]

In order to prove Fredholm properties of $F$, we compare to the corresponding one related to $\hat{g}$

**Definition 1.** Let $k \in \mathbb{N}$. We say an operator $P$ of the form

$$
P u = a^i(x) \nabla^2_{ij} u + b^i(x) \partial_i u + c(x) u
$$

is well related to $\hat{\Delta}$ if there exists $n < q(k + 1) < \infty$, and two positive constants $C_1, \lambda$ such that

$$
\begin{align*}
    & \lambda |\xi|_g^2 \leq a^i(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|_g^2, \forall x \in \mathcal{M}, \xi \in \mathcal{T}\mathcal{M}.
\end{align*}
$$

Then

$$
\|a^i - \check{a}^i\|_{k+1, q} + \|b^i\|_{k, q} + \|c\|_{k,q,k+\frac{1}{2}} \leq C_1.
$$

From the lemma 1, the operator $P$ is then bounded from $W^{k+2,2}$ to $W^{k,2}$. In our situation, we are interested in the Laplacian relative to the metric $g$.

**Proposition 7.** Let $k \in \mathbb{N}$ such that $k + 2 > \frac{n}{2}$. Let $c \in W^{k,2}$ and let $g \in G^+$. Then $\Delta + c$ is well related to $\hat{\Delta}$.

**Proof:** Recall that

$$
\begin{align*}
    \Delta = g^{ij} \nabla^2_{ij} & = g^{ij} \nabla^2_{ij} + g^{ij} (\nabla_i - \nabla_i) \nabla_j \\
    & = g^{ij} \nabla^2_{ij} - g^{ij} A_k A_{ij} \nabla_k.
\end{align*}
$$

(73)

The metrics $g$ and $\hat{g}$ being equivalent, equation (18) directly gives

$$
\begin{align*}
    \lambda |\xi|_g^2 \leq g^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|_\hat{g}^2, \forall x \in \mathcal{M}, \xi \in \mathcal{T}\mathcal{M}.
\end{align*}
$$

Setting

$$
b^k = g^{ij} A_{ij}^k,
$$

then $b \in W^{k+1,2}$ from (20). Let us choose $q = \frac{k+2}{k+1}$, so $(k + 1)q > n$ and $2 = q_{k+1}$. Given the Sobolev inequality, there exists a constant $C_1 > 0$ such that

$$
\begin{align*}
    \|g^{ij} - \hat{g}^{ij}\|_{k+1, q} + \|b^k\|_{k, q} + \|c\|_{k,q,k+\frac{1}{2}} \leq C \left( \|g^{ij} - \hat{g}^{ij}\|_{k+2, 2} + \|b^k\|_{k+1, 2} + \|c\|_{k, 2} \right) \leq C_1.
\end{align*}
$$

\[\square\]

1Here a fixed constant $\kappa$ can be chosen to be zero for a compact manifold but we want to make the calculation easily adaptable to open manifold asymptotic to Einstein models, see section B
The operator $A = -\Delta - \kappa n$, acting on functions, will be of great interest. It satisfies a classical elliptic estimate, valid for any weight $s$.

**Proposition 8.** Let $g \in G^+$ and $A = -\Delta - \kappa n$. There exists a constant $C = C(n, p, q, s, C_1, \lambda)$ such that if $u \in L^2$ and $Au \in W^{k, 2}$, then $u \in W^{k+2, 2}$ and
\[
\|u\|_{k+2, 2} \leq C (\|Au\|_{k, 2} + \|u\|_{0, 2}).
\]  

**Proof:** By elliptic regularity, $u \in W^{2, 2}_{loc}$ and the estimate arises from interior estimates (see [16], [18], [7] for example) and partition of unity. □

We need a similar result for an operator acting on 1-forms. Let us define $\hat{B} = -\hat{\Delta} - \kappa(n - 1)$, be a Laplacian acting on 1-forms.

**Theorem 4.** We assume $g \in G^*$. Setting $B = -\Delta - \kappa(n - 1)$. Then $B : W^{k+2, 2}(\Lambda^1 M) \to W^{k, 2}(\Lambda^1 M)$ is bounded. Furthermore, it satisfies
\[
\|Y\|_{k+2, 2} \leq C (\|BY\|_{k, 2} + \|Y\|_{0, 2}).
\]

In particular, $B$ is a semi-Fredholm operator.

We are now ready to prove Fredholm property of the operator $F$ associated to our special variations.

**Theorem 5.** When $k + 2 > \frac{n}{2}$, the operator

$F : W^{k+2, 2}(\Lambda M) \times W^{k+2, 2}(\Lambda^3 T^* M) \to L^2(\Lambda^3 T^* M) := L^*$

is bounded. Furthermore, it satisfies
\[
\|(y, Y)\|_{k+2, 2} \leq C (\|F(y, Y)\|_{k, 2} + \|(y, Y)\|_{0, 2}).
\]

In particular, $F$ is a semi-Fredholm operator.

**Proof:** Starting from the definition of $F$, we find like before
\[
\|F(y, Y)\|_{k, 2} \leq C \|(y, Y)\|_{k+2, 2},
\]

where $C$ is a constant depending on $\hat{g}$ and $\|g\|_{\infty}$.

Hence $F$ is a bounded (continuous) operator. Plugging the expression of $F_0(y, Y^i)$ in (74) and using Hölder inequality (1), lemma 2 (see remark 2), Ehrling inequality (4) along with and $\Phi_0(g, \pi) \in W^{k, 2}$,
\[
\|y\|_{k+2, 2} \leq C (\|\Delta y - \kappa ny\|_{k, 2} + \|y\|_{0, 2})
\leq C (\|\Phi_0(g, Y)\|_{k, 2} + \|(y, Y)\|_{0, 2}).
\]  

Plugging the expression of $F_1(y, Y^i)$ in (75) and using Hölder inequality (1), (6), Ehrling inequality (4) along with lemma 1 and $\Phi_1(g, \pi) \in W^{k, 2}$,
\[
\|Y\|_{k+2, 2} \leq C (\|\Delta Y - \kappa(n - 1)Y\|_{k, 2} + \|Y\|_{0, 2})
\leq C (\|\Phi_1(y, Y)\|_{k, 2} + \|(y, Y)\|_{0, 2} + \|Y\|_{0, 2}).
\]  

Finally, combination of (77) and (78) gives (76). It is now standard to deduce from the estimate (76) that $F$ is semi-Fredholm. □
We can now argue like in the proof of corollary 1 to prove that $F$ is a Fredholm operator. We approximate the metric $g$ by a smooth one $g_\varepsilon$ to produce an operator $F_\varepsilon$ close to $F$. Now $F_\varepsilon$ and its adjoint $F_\varepsilon^*$ have similar structure

$$F_\varepsilon^*: W^{-k, 2}(T) \rightarrow W^{-k-2, 2}(T^* \otimes \Lambda^3 T^* \mathcal{M}).$$

Let $\widetilde{F}_\varepsilon^*$ be the restriction of $F_\varepsilon^*$ defined as follows

$$\widetilde{F}_\varepsilon^*: W^{k+2, 2}(T) \rightarrow W^{k, 2}(T^* \otimes \Lambda^3 T^* \mathcal{M}).$$

Booth $F_\varepsilon$ and $\widetilde{F}_\varepsilon^*$ satisfies an estimate like (76) and because of elliptic regularity the kernel of $\widetilde{F}_\varepsilon^*$ is the same as the kernel of $F_\varepsilon^*$. We conclude that $F_\varepsilon$ is Fredholm. $F$ being semi-Fredholm and a limit of Fredholm operators, it is Fredholm. Thus $\text{Im} F$ is closed and its cokernel is finite dimensional.

We can now close the proof of Theorem 3 by the following argument.

The space $\text{Coker} F = L^*_\mathcal{M}/ \text{Im} F$ is finite dimensional. The operator $F$ satisfies

$$\text{Im} F \subset \text{Im} D\Phi(g, \pi) \subset L^*.$$

Let $\pi$ be the canonical projection:

$$\pi: L^* \rightarrow L^*_\mathcal{M}.$$

$\pi(\text{Im} D\Phi(g, \pi))$ is a subspace of a finite dimensional vector space, so is closed. Because it is preimage of a closed set by a continuous map, $\text{Im}(D\Phi(g, \pi))$ is closed. (an equivalent argument is to note that $L^* = \text{Im} F \oplus G$ for a finite dimensional subspace $G$ so $\text{Im} D\Phi(g, \pi) = \text{Im} F \oplus (G \cap \text{Im} D\Phi(g, \pi)).$

This ends the proof of the manifold structure of $\mathcal{C}$, as a smooth submanifold of $\mathcal{F}$. In fact, all no KID’s fibers of $\Phi$ are smooth submanifolds of $\mathcal{F}$. □

APPENDIX A. A NOTE FOR THE SCALAR CURVATURE CASE

The manifold structure on the fiber of the scalar curvature operator is not obtained directly using $\Phi_0(g, 0)$, because the constant $\Lambda$ need to be replaced by a function $f$. Instead, one may consider the map

$$\phi(g) = (R(g) - 2f)\sqrt{g},$$

whose linearisation is

$$D\phi(g)h = (\nabla^i \nabla^j h_{ij} - \Delta_g \text{tr}_g h)\sqrt{g} - h_{ij}[R^{ij} - \frac{1}{2}(R(g) - 2f)g^{ij}]\sqrt{g},$$

and the adjoint given by

$$D\phi(g)^*N = [\nabla^i \nabla^j N - g^{ij} \Delta_g N - [R^{ij} - \frac{1}{2}(R(g) - 2f)g^{ij}]N]\sqrt{g}.$$ 

Like for the operator $T$ defined before, the kernel of $D\phi(g)^*$ is the same than the one of

$$N \mapsto \nabla \nabla N - [\text{Ric}(g) - \frac{1}{2(n-1)}(R(g) + 2f)g]N.$$ 

If $R(g) = 2f$ then $D\phi(g)^* = DR(g)^*$ thus the theorem 1 is obtained with $\varepsilon = 2f$ and the preimage of 0 by $\phi$. 

APPENDIX B. ABOUT NON COMPACT MANIFOLDS WITH SPECIAL ENDS

As already explained in the introduction, the paper is written in the spirit to an easy adaptation to some non compact setting such as the asymptotically Euclidian one or the asymptotically hyperbolic context. In such a case, the constraint operator $\Phi$ is studied for Riemannian metrics of the form $g = \hat{g} + h$ with $g$ “asymptotic” to a smooth model metric $\hat{g}$, i.e. $|g - \hat{g}|_\hat{g} = |h|_\hat{g}$ is controlled in a suitable weighted space.

In this section, we explain the choices made before and mention two natural operators related to the no KID’s condition.

We consider a smooth metric $\hat{g}$ on $\mathcal{M}$ as model. One can think of $\hat{g}$ has a metric of constant sectional curvature $\kappa$ on any end but this is a particular case. One work in some weighted Sobolev spaces, say $W^{k,2}$ weight, the weight is a real describing the asymptotic behaviour, it can change from line to line from now on. We usually ask $\hat{g}$ to satisfy the following

$$Riem \hat{g} - \frac{\kappa}{2} \hat{g} \otimes \hat{g} \in W^{k,2}_{\text{weight}},$$

(79)

where $(\hat{g} \otimes \hat{g})_{ijkl} = -2(\hat{g}_{il}\hat{g}_{jk} - \hat{g}_{ik}\hat{g}_{jl})$, or the weaker one,

$$Ric \hat{g} - \kappa(n - 1)\hat{g} \in W^{k,2}_{\text{weight}},$$

(80)

or only

$$R(\hat{g}) - n(n - 1)\kappa \in W^{k,2}_{\text{weight}}.$$  

(81)

We fix a real parameter $\tau$ and we set

$$\hat{K} = \tau \hat{g}.$$  

(82)

The cosmological constant $\Lambda$ is normalized here in dimension $n$ by

$$2\Lambda = n(n - 1)(\tau^2 + \kappa),$$  

(83)

so that $\Phi(\hat{g}, \hat{K}) = 0$ if $R(\hat{g}) = \kappa n(n - 1)$.

From the choice of $\hat{K}$, the conjugate momentum $\hat{\pi}$ is then

$$\hat{\pi}^{ij} = (\hat{K}^{ij} - tr_\hat{g} \hat{K}) d\mu(\hat{g}) = \tau(1 - n)\hat{g}^{ij} d\mu(\hat{g}).$$  

(84)

Note that we have $\hat{\nabla} \hat{\pi} = \nabla \hat{K} = 0$. If $Ric(\hat{g}) = \kappa(n - 1)\hat{g}$ then

$$\hat{\Pi} = -\frac{1}{2}(n - 1)(n - 4)\tau^2 \hat{g}^{\text{g}}^{-1},$$

$$\hat{E} = \frac{1}{2}(n - 1)(2k + n\tau^2)\hat{g}^{\text{g}}^{-1}$$

$$\hat{\Pi} - \hat{E} = -(n - 1)[(n - 2)\tau^2 + \kappa]\hat{g}^{\text{g}}^{-1}.$$  

If $h$ and $p$ are like in (71) with $Y = 0$ we see that

$$F_0(y, 0) = 2(1 - n)(\hat{\Delta} y + \kappa ny),$$

and if $h$ and $p$ are like in (71) with $y = 0$ we see that

$$F_i(0, Y) = 2(\hat{\Delta} Y_i + \kappa(n - 1)Y_i).$$

This explain the choice of the operators $A$ and $B$ of section 7.
In the non compact setting, the no KID’s condition is usually not assumed but has to be proved for a certain range of weight. In the course, some asymptotic inequalities are useful, we introduce two operators who have to be studied.

The operator $\mathcal{U}$

The operator $\mathcal{U}$, acting on 1-forms, inspired by the formula (12) and so related to the covariant derivatives of Killing operator $S$. It may be used to estimate the $W^{k+2}_{\text{weight}}$-norm of a 1-form $X$ with the $W^{k,2}_{\text{weight}}$-norms of $\hat{S}(X)$ and $(X)$, so with the $W^{k+1,2}_{\text{weight}}$-norm of $\hat{S}(X)$. Because of the constant sectional curvature model, we introduce the operator $\mathcal{U}$ defined on 1-forms by

$$\mathcal{U}_{kji}(X) = \nabla^2_{kji}X_i + \kappa(\hat{g}_{jk}X_i - \hat{g}_{ik}X_j).$$

(85)

An important step will be to prove that for any smooth one form $X$ supported on an end,

$$||X||_{2,\text{weight}} \leq C(||\mathcal{U}(X)||_{0,\text{weight}} + ||\hat{S}(X)||_{0,\text{weight}})$$

The operator $\hat{T}$

Inspired by the case $Ric(\hat{g}) = \kappa(n - 1)$ and the operator $\hat{T}$, we define the Obata type operator

$$\hat{T}(N) = \hat{\nabla}^2 N + \kappa \hat{g} N$$

In the same way, it has to be proven that for any smooth function $N$ supported on an end,

$$||N||_{2,\text{weight}} \leq C||\hat{T}(N)||_{0,\text{weight}}$$

Appendix C. Fredholm properties for elliptic operators with rough coefficients

We start with a result who can be found in [7] or [18] for instance (see also [25] and [24])

**Theorem 6.** Let $k \in \mathcal{N}$ such that $k + 2 > \frac{n}{2}$ and $g \in H^{k+2}$. Let $P = \Delta + c$ with $c \in H^{k}$. For all $j \in (-k, k+2]$ there exist a constant $C$ such that all $u \in H^{j}$ satisfies the estimate

$$||u||_{j,2} \leq C(||Pu||_{j-2,2} + ||u||_{j-2,2}).$$

In particular $P : H^{j} \to H^{j-2}$ is semi-Fredholm (with finite dimensional kernel and closed range).

We are now ready to deduce (see also [25] with [24])

**Corollary 1.** Let $k \in \mathcal{N}$ such that $k + 2 > \frac{n}{2}$ and $g \in H^{k+2}$. Let $P = \Delta + c$ with $c \in H^{k}$. For all $j \in (-k, k+2]$ the operator $P : H^{j} \to H^{j-2}$ is Fredholm.

**Proof:** We first recall that for any $j \in [-k, k+2]$ the operator $P$ is well defined and bounded from $H^{j}$ to $H^{j-2}$ ([7], Theorem A.1 with $p = p_{1} = p_{2} = 2$, $s_{1} = k + 2$ and $s = s_{2} = j$). From the theorem 6, if moreover $j \neq -k$, $P$ is semi-Fredholm so its (eventually infinite) index is well defined.

The metric $g$ and the 0-order term $c$ can be approximated by smooth ones to produce a family of operators $P_{\varepsilon}$ with smooth coefficients and such that

$$||P - P_{\varepsilon}|| \leq \varepsilon,$$
where the norm is the norm of operators from $H^j$ to $H^{j-2}$. We have the usual elliptic regularity estimate

$$||v||_{j,2} \leq C(||P_\varepsilon v||_{j-2,2} + ||v||_{j-2,2}),$$

in particular $P_\varepsilon$ is semi-Fredholm with finite dimensional kernel. Its formal $L^2$ adjoint (for the measure $d\mu_\varepsilon$ for instance) $P_\varepsilon^*: H^{2-j} \to H^{-j}$ has a similar structure, verify the same kind of estimate so is also semi-Fredholm with finite dimensional kernel. We easily deduce that $P_\varepsilon$ is a Fredholm operator. For $\varepsilon$ small enough, the (finite) index of $P_\varepsilon$ is equal to that of $P$ (see [20]) so the index of $P$ is finite thus $P$ is Fredholm.

\[\Box\]

**Remark 4.** We have just used that a semi-Fredholm limit of Fredholm operators is Fredholm. It is clear that the proof can be transposed to more general operators like those in definition 1.

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