HERMITE-HADAMARD TYPE INEQUALITIES FOR
FUNCTIONS WHOSE DERIVATIVES ARE STRONGLY
$\varphi$-CONVEX

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Abstract. In this paper several inequalities of the right-hand side of Hermite-Hadamard’s inequality are obtained for the class of functions whose derivatives in absolutely value at certain powers are strongly $\varphi$-convex with modulus $c > 0$.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. This inequality plays an important role in convex analysis and it has a huge literature dealing with its applications, various generalizations and refinements (see [1, 2, 3, 4, 5, 8, 11]).

Let us consider a function $\varphi : [a, b] \to [a, b]$, where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [12]:

**Definition 1.** A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$-convex on $[a, b]$ if for every two points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f \left( t\varphi(x) + (1 - t)\varphi(y) \right) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).$$

Obviously, if function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition.

Recall also that a function $f : I \to \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$f \left( tx + (1 - t)y \right) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2$$

for all $x, y \in I$ and $t \in (0, 1)$. Strongly convex functions have been introduced by Polyak in [9] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature see [6, 7, 9].

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1
Let \((X, \|\cdot\|)\) be a real normed space, \(D\) stands for a convex subset of \(X\), \(\varphi : D \rightarrow D\) is a given function and \(c\) is a positive constant. In [10] Sarikaya have introduced the notion of the strongly \(\varphi\)-convex functions with modulus \(c\) and some properties of them. Moreover in his paper, Sarikaya have presented a version Hermite-Hadamard-type inequalities for strongly \(\varphi\)-convex functions as follows:

**Definition 2.** A function \(f : D \rightarrow \mathbb{R}\) is said to be strongly \(\varphi\)-convex with modulus \(c\) if

\[
f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2
\]

for all \(x, y \in D\) and \(t \in [0,1]\).

The notion of \(\varphi\)-convex function corresponds to the case \(c = 0\). If function \(\varphi\) is the identity, then the strongly convexity with modulus \(c > 0\) is obtained from the previous definition.

**Theorem 1.** If \(f : [a, b] \rightarrow \mathbb{R}\) is strongly \(\varphi\)-convex with modulus \(c > 0\) for the continuous function \(\varphi : [a, b] \rightarrow [a, b]\), then

\[
f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) + \frac{c}{12}(\varphi(b) - \varphi(a))^2 \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{c}{6}(\varphi(b) - \varphi(a))^2
\]

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives at certain powers are strongly \(\varphi\)-convex with modulus \(c\).

### 2. Inequalities for functions whose derivatives are strongly \(\varphi\)-convex with modulus \(c\)

In order to prove our main results we need the following lemma:

**Lemma 1.** Let \(f : I \subset \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(I^o\), \(a, b \in I\) with \(a < b\). If \(f' \in L[a, b]\) and \(\varphi : [a, b] \rightarrow [a, b]\) a continuous function with \(\varphi(a) < \varphi(b)\), then the following equality holds:

\[
(2.1) \quad f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx = \frac{\varphi(b) - \varphi(a)}{2} \int_0^1 (2t - 1)f'(t\varphi(b) + (1-t)\varphi(a))dt
\]

**Proof.** By using partial integration in right hand of (2.1) equality, the proof is obvious. \(\square\)

The next theorem gives a new the upper Hermite-Hadamard inequality for strongly \(\varphi\)-convex functions with modulus \(c\) as follows:

**Theorem 2.** Let \(f : I \subset \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(I^o\), \(a, b \in I\) with \(a < b\) such that \(f' \in L[a, b]\). If \(|f'|^q\) is strongly \(\varphi\)-convex functions with modulus \(c\)
on \([a, b]\) for the continuous function \(\varphi : [a, b] \rightarrow [a, b]\) with \(\varphi(a) < \varphi(b)\) and \(q \geq 1\) then

\[
(2.2) \quad \left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \varphi^{(b)} \int f(x)dx\right| \leq \frac{\varphi(b) - \varphi(a)}{4} \left(\frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{2} - \frac{c}{8} \frac{(\varphi(b) - \varphi(a))^2}{\varphi^{(b)}}\right)^{\frac{1}{q}}
\]

**Proof.** Suppose that \(q = 1\). From Lemma 1 and using the strongly \(\varphi\)–convexity functions with modulus \(c\) of \(|f'|\), we have

\[
\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \varphi^{(b)} \int f(x)dx\right| \leq \frac{\varphi(b) - \varphi(a)}{2} \int_0^1 |2t - 1| |f'(t\varphi(b) + (1 - t)\varphi(a))| dt
\]

\[
\leq \frac{\varphi(b) - \varphi(a)}{2} \int_0^1 |2t - 1| \left[t |f'(\varphi(b))| + (1 - t) |f'(\varphi(a))| - ct (1 - t) (\varphi(b) - \varphi(a))^2\right] dt
\]

We have

\[
\int_0^1 |2t - 1| t dt = \frac{1}{4}, \quad \int_0^1 |2t - 1| (1 - t) dt = \frac{1}{4}
\]

and

\[
\int_0^1 |2t - 1| t (1 - t) dt = \frac{1}{16}
\]

hence we obtain

\[
\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \varphi^{(b)} \int f(x)dx\right| \leq \frac{\varphi(b) - \varphi(a)}{4} \left(\frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{2} - \frac{c}{8} \frac{(\varphi(b) - \varphi(a))^2}{\varphi^{(b)}}\right)
\]

which completes the proof for this case.
Suppose now that $q \in (1, \infty)$. From Lemma 1 and using the Hölder’s integral inequality, we have

$$\int_0^1 |2t - 1| |f'(t \varphi(b) + (1 - t) \varphi(a))| dt$$

(2.3)

$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left( \int_0^1 |2t - 1| dt \right)^{\frac{q-1}{q}} \left( \int_0^1 |2t - 1| |f'(t \varphi(b) + (1 - t) \varphi(a))|^q dt \right)^{\frac{1}{q}}$$

Since $|f'|^q$ is strongly $\varphi-$convex functions with modulus $c$ on $[a, b]$, we know that for every $t \in [0, 1]$

(2.4)

$$|f'(t \varphi(b) + (1 - t) \varphi(a))|^q \leq t |f'(\varphi(b))|^q + (1-t) |f'(\varphi(a))|^q - ct(1-t)(\varphi(b) - \varphi(a))^2$$

From (2.1), (2.3) and (2.4) we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right|$$

$$\leq \frac{\varphi(b) - \varphi(a)}{4} \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} - \frac{c}{8} (\varphi(b) - \varphi(a))^2 \right)^{\frac{1}{q}}$$

which completes the proof. \(\square\)

**Corollary 1.** Suppose that all the assumptions of Theorem 2 are satisfied, in this case:

1. For $c = 0$, i.e. if $|f'|^q$ is $\varphi-$convex functions, we have

(2.5)

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right|$$

$$\leq \frac{\varphi(b) - \varphi(a)}{4} \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} \right)^{\frac{1}{q}}$$

2. For $\varphi(t) = t$ in (2.3) inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{4} \left( \frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}}$$

we get the same result in [8, Theorem 1].
(3) If we take $\varphi(t) = t$ in (2.3) for strongly convex functions with modulus $c > 0$ on $[a, b]$ inequality, then we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( |f'(b)|^q + |f'(a)|^q - \frac{c}{8} (b-a)^2 \right)^{\frac{1}{q}}.$$

**Theorem 3.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^*$, $a, b \in I$ with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is strongly $\varphi$-convex functions with modulus $c$ on $[a, b]$ for the continuous function $\varphi : [a, b] \to [a, b]$ with $\varphi(a) < \frac{\varphi(a) + \varphi(b)}{2} < \varphi(b)$ and $q > 1$ then

$$\left( \frac{1}{2} \right)^{\frac{1}{p}} \left[ \left( \int_0^1 (1 - 2t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(t\varphi(b) + (1-t)\varphi(a))|^q dt \right)^{\frac{1}{q}} \right].$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** From Lemma [1] and using the Hölder inequality, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \right| \leq \frac{\varphi(b) - \varphi(a)}{2} \left( \int_0^1 (1 - 2t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(t\varphi(b) + (1-t)\varphi(a))|^q dt \right)^{\frac{1}{q}} + \frac{\varphi(b) - \varphi(a)}{2} \left( \int_0^1 (2t - 1)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(t\varphi(b) + (1-t)\varphi(a))|^q dt \right)^{\frac{1}{q}} \leq \frac{\varphi(b) - \varphi(a)}{2} \left( \frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left[ \left( \int_0^1 (1 - 2t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(t\varphi(b) + (1-t)\varphi(a))|^q dt \right)^{\frac{1}{q}} \right].$$

where we use the fact that

$$\int_0^1 (1 - 2t)^p dt = \int_0^1 (2t - 1)^p dt = \frac{1}{2(p+1)}.$$
and by Theorem 1 we get
\[
\int_0^{\frac{1}{\lambda}} |f'(t, \varphi(b) + (1-t)\varphi(a))|^q \, dt = \frac{1}{2} \frac{2}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} |f'(x)|^q \, dx
\]
\[
\leq \left( \frac{1}{4} \right) \left( \left| f' \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|^q + \left| f'(\varphi(a)) \right|^q - \frac{c}{3} (\varphi(b) - \varphi(a))^2 \right) .
\]
\[
\frac{\lambda + \mu}{\mu} \int_0^{1} |f'(tb + (1-t)a)|^q \, dt = \frac{1}{2} \frac{2}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} |f'(x)|^q \, dx
\]
\[
\leq \left( \frac{1}{4} \right) \left( \left| f' \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|^q + \left| f'(\varphi(b)) \right|^q - \frac{c}{3} (\varphi(b) - \varphi(a))^2 \right) .
\]

**Corollary 2.** Suppose that all the assumptions of Theorem 4 are satisfied, in this case:

1. Since $|f'|^q$ is strongly $\varphi$–convex functions with modulus $c$, from (2.6) inequality we get

\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right|
\]
\[
\leq \frac{\varphi(b) - \varphi(a)}{4} \left( \frac{1}{(p+1)} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} \right)^{\frac{1}{q}} - \frac{7c}{12} (\varphi(b) - \varphi(a))^2 \right]^{\frac{1}{q}}
\]

2. For $c = 0$, i.e. if $|f'|^q$ is $\varphi$–convex functions, we have

(2.7) \[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right|
\]
\[
\leq \frac{\varphi(b) - \varphi(a)}{4} \left( \frac{1}{(p+1)} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} \right)^{\frac{1}{q}} + \frac{7c}{12} (\varphi(b) - \varphi(a))^2 \right]^{\frac{1}{q}}
\]

3. For $\varphi(t) = t$ in (2.7) inequality :

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\]
\[
\leq \frac{b-a}{4} \left( \frac{1}{(p+1)} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ \left( \frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \frac{7c}{12} (\varphi(b) - \varphi(a))^2 \right]^{\frac{1}{q}}
\]

we get the same result in (3) for $s = 1$. 

If we take ϕ(t) = t in (2.6) for strongly convex functions with modulus c > 0 on [a, b] inequality, then we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ \left( \left| f'(a) \right|^q + \left| f'(b) \right|^q - \frac{c}{2} (b - a)^2 \right)^{\frac{1}{q}} \right]
\]

\[
\phi(b) - \phi(a) \leq \frac{1}{p+1} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \left| f'(b) \right|^q + \left| f'(a) \right|^q - \frac{c}{2} (b - a)^2 \right)^{\frac{1}{q}}.
\]

Theorem 4. Let f : I ⊂ ℝ → ℝ be a differentiable mapping on I°, a, b ∈ I with a < b such that f′ ∈ L[0, b]. If |f′| is strongly ϕ−convex functions with modulus c on [a, b] for the continuous function ϕ : [a, b] → [a, b] with ϕ(a) < ϕ(b) and q > 1 then

\[
\left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(x)dx \right| \leq \frac{\phi(b) - \phi(a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \left( \int_0^1 |f'(t\phi(b) + (1 - t)\phi(a))|^q dt \right)^{\frac{1}{q}}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1. \)

Proof. From Lemma\[1\] and using the Hölder’s integral inequality, we have

\[
\left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(x)dx \right| \leq \frac{\phi(b) - \phi(a)}{2} \int_0^1 |2t - 1| \left| f'(t\phi(b) + (1 - t)\phi(a)) \right| dt
\]

\[
\leq \frac{\phi(b) - \phi(a)}{2} \left( \int_0^1 |2t - 1| \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(t\phi(b) + (1 - t)\phi(a)) \right|^q dt \right)^{\frac{1}{q}}
\]

\[
\leq \frac{\phi(b) - \phi(a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \left( \int_0^1 \left| f'(\phi(b)) \right|^q + \left| f'(\phi(a)) \right|^q - ct (1 - t) (\phi(b) - \phi(a))^2 dt \right)^{\frac{1}{q}}
\]

\[
= \frac{\phi(b) - \phi(a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \left( \left| f'(\phi(b)) \right|^q + \left| f'(\phi(a)) \right|^q - \frac{c}{6} (\phi(b) - \phi(a))^2 \right)^{\frac{1}{q}}.
\]

\[\square\]

Corollary 3. Suppose that all the assumptions of Theorem\[4\] are satisfied, in this case:
(1) For $c = 0$, i.e. if $|f'|^{q}$ is $\varphi-$convex functions, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b)) - 1}{\varphi(b) - \varphi(a)} \int^{b}_{a} f(x) \, dx \right|^{\frac{p}{q}} \leq \frac{\varphi(b) - \varphi(a)}{2} \left( \frac{1}{p+1} \right)^{\frac{p}{q}} \left( \frac{\left| f'(\varphi(b)) \right|^{q} + \left| f'(\varphi(a)) \right|^{q}}{2} \right)^{\frac{1}{q}}$$

(2.9)

(2) We obtained the same result in [3, Theorem 2.3] for $\varphi(t) = t$ in (2.9) inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int^{b}_{a} f(x) \, dx \right| \leq \frac{b-a}{2} \left( \frac{1}{p+1} \right)^{\frac{p}{q}} \left( \frac{\left| f'(a) \right|^{q} + \left| f'(b) \right|^{q}}{2} \right)^{\frac{1}{q}}$$

(3) If we take $\varphi(t) = t$ in (2.8) for strongly convex functions with modulus $c > 0$ on $[a, b]$ inequality, then we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int^{b}_{a} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{p}{q}} \left( \frac{\left| f'(a) \right|^{q} + \left| f'(b) \right|^{q}}{2} \right)^{\frac{1}{q}} - \frac{c}{6} \left( \varphi(b) - \varphi(a) \right)^{\frac{1}{q}}$$

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