Kepler Equation for the Compact Binaries under the Spin-spin Interaction

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Abstract. In this work we consider compact binaries on eccentric orbit under the spin-spin interaction. Using the post-Newtonian formalism, the binaries undergo a perturbed Keplerian motion. Here we investigate only the radial motion and derive the contribution of the second post-Newtonian order spin-spin effect to the solution.

1. Introduction

The solution to the relativistically perturbed two body problem (Damour & Deruelle 1985) has received important applications since the discoveries of compact binary pulsars (Hulse & Taylor 1975). The predictions of the general theory of relativity for the motion of such systems were verified to a high degree. Nowadays compact binary systems composed of neutron stars and/or black holes are considered among the most promising sources for the Earth-based gravitational wave observatories. Before the final coalescence there is a regime where a post-Newtonian (PN) description of the motion and of the gravitational radiation is suitable.

The dynamics of a two-body system without spins to 2PN orders (Damour & Schafer 1987) and then the perturbation induced by the spin-orbit interaction (Schafer & Wex 1993; Wex 1995) was solved using suitable quasi-Keplerian parametrizations. A systematic treatment (Gergely et al. 2000) has yielded such parametrizations (the true and eccentric anomaly parametrizations) for the radial motion of a wide class of perturbed Keplerian motions, including the generic perturbing force of Brumberg (Brumberg 1991). These parametrizations have convenient features over those introduced by (Ryan 1996; Rieth & Schafer 1997).

The first imprint of the presence of the spins in a compact binary are the spin-orbit effects (Rieth & Schafer 1997; Gergely et al. 1998). At the 2PN order, also spin-spin effects appear (Gergely 2000). Therefore it would be desirable to have the corresponding parametrization of the orbit available. The main impediment in the parametrization of the perturbed Keplerian motion with the inclusion of the spin-spin effects is that the magnitude of the orbital angular momentum is not conserved, even in the absence of gravitational radiation. This feature of the spin-spin type perturbation is outside the framework settled in (Gergely et al. 2000). The radial motion in terms of true anomaly parametrization was solved (Gergely 2000). However the complete dynamics with the inclusion of angular degrees of freedom was not described yet.
2. The Radial Motion

The motion of the binary under the influence of the spin-spin interaction is governed by the Lagrangian (Kidder 1995): \[ \mathcal{L} = \mathcal{L}_N + \mathcal{L}_{SS}, \]

where
\[ \mathcal{L}_N = \frac{\mu v^2}{2} + \frac{G\mu}{r}, \]
\[ \mathcal{L}_{SS} = \frac{G}{c^2r^3} \left[ (S_1S_2) - \frac{3}{r^2} (rS_1) (rS_2) \right]. \]

The perturbed motion is characterized by the conservation of the total energy
\[ E = E_N + E_{SS}, \quad E_N = \frac{\mu v^2}{2} - \frac{G\mu}{r}, \quad E_{SS} = -\mathcal{L}_{SS}. \]

and of the total angular momentum vector
\[ J = L + S, \]

where \( L = L_N = \mu (r \times v) \) is the orbital angular momentum and \( S = S_1 + S_2 \) is the total spin. Using the equation of the motion the magnitude of the orbital angular momentum is not constant: it changes according to (2.14) of (Gergely 2000). However the secular evolution of the orbital angular momentum vector (Barker & O’Connell 1970) is a precessional motion. Consequently, there is no change in the magnitude of the orbital angular momentum over one radial period due to the spin-spin interaction. Using the procedure given in (Gergely 2000) the magnitude of the orbital angular momentum can be written in the following form:
\[ L = \mathcal{T} + \delta L (\chi), \]

where the first term is the angular average of the magnitude of the orbital angular momentum over one radial period and the second appears due to the spin-spin interaction. \( \chi \) is the true anomaly parameter in the Newtonian limit. We can characterize the radial motion by the total energy and \( \mathcal{T} \) instead of \( L \).

The expression of the energy and orbital angular momentum in terms of spherical coordinates give the radial equation, which for the accuracy needed here is:
\[ r^2 = \frac{2E}{\mu} + \frac{2Gm}{r} - \frac{\mathcal{T}^2}{\mu^2r^2} - \frac{2\mathcal{T}\delta L}{\mu^2r^2} + \frac{GS_1S_2\alpha}{c^2\mu r^3}, \]

where \( \delta L \) and \( \alpha \) is given in (Gergely 2000). To solve the radial equation we introduce the generalized true and eccentric anomaly parametrization (Gergely 2000). The relation between the two parametrizations:
\[ \xi = 2 \arctan \left( \sqrt{\frac{1 - e_r}{1 + e_r}} \tan \frac{\chi}{2} \right). \]
Here the eccentric anomaly parameter is denoted by $\xi$ and the true anomaly parameter by $\chi$. Using these parametrizations the integration of the radial equation emerges after a long computation. The solution of the radial equation is

$$r = a \left(1 - e_r \cos \xi\right),$$

$$n (t - t_0) = \xi - e_t \sin \xi + \frac{f_t}{c^2} \sin \left[\chi + 2 \left(\psi_0 - \overline{\psi}\right)\right],$$

where

$$a_r = \frac{Gm\mu}{-2E} \left[1 + \frac{ES_1 S_2}{c^2 m L^2} (\alpha_{SS} + \beta_{SS})\right],$$

$$e_r = \frac{\overline{A}}{Gm\mu} \left\{1 - \frac{ES_1 S_2}{c^2 m L^2 \overline{A}^2} \left[\left(G^2 m^2 \mu^2 + \overline{A}^2\right) \alpha_{SS} + \overline{A}^2 \beta_{SS}\right]\right\},$$

$$n = \frac{2\pi}{T} = \frac{1}{Gm\mu} \left(\frac{-2E}{\mu}\right)^{3/2},$$

$$e_t = \frac{\overline{A}}{Gm\mu} \left\{1 - \frac{ES_1 S_2 G^2 m \mu^2 \alpha_{SS}}{c^2 L^2 \overline{A}^2}\right\},$$

$$f_t = - \left(\frac{-2E}{\mu}\right)^{3/2} \frac{\mu S_1 S_2}{c^2 m AL} \sin \kappa_1 \sin \kappa_2,$$

with

$$\alpha_{SS} = 3 \cos \kappa_1 \cos \kappa_2 - \cos \gamma,$$

$$\beta_{SS} = \sin \kappa_1 \sin \kappa_2 \cos 2 \left(\psi_0 - \overline{\psi}\right).$$

The magnitude of the Laplace-Runge-Lenz vector $\overline{A}$ belongs to a Keplerian motion characterized by the energy $E$ and $L$. The $\kappa$ and $\gamma$ are relative angles of the orbital angular momentum vector and the spins of the bodies. The angles $\psi_0$ and $\psi_1$ are subtended by the intersection line of the planes perpendicular to $L_N$ and $J$ with the position of periastron line and the projections of the spins in the plane of the orbit. All angles are shown in Fig. 1 in (Gergely et al. 1998).

### 3. Concluding Remarks

We have presented the solution of the radial motion of a Newtonian problem perturbed by spin-spin interaction. In part, the orbital elements are characterized by the energy $E$ and the angular average of the magnitude of the orbital angular momentum $\overline{T}$. The magnitude of the spins, angles between the spins and angles between the spins and the orbital angular momentum also appear in the characterization of the motion. During one radial period the angles $\kappa$, $\gamma$, $\psi_1$ and $\psi_0$ can be considered constants (Gergely et al. 1998), thus the orbital elements are constants as well.

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