A PARABOLIC MINIMAL SURFACE IN $\mathbb{R}^3$ INTERSECTS EVERY NONFLAT PROPERLY EMBEDDED MINIMAL SURFACE OF BOUNDED CURVATURE

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Abstract. We show that the image of a nonconstant conformal harmonic map $\mathbb{C} \to \mathbb{R}^3$, not necessarily proper and possibly with branch points, intersects every properly embedded nonflat minimal surface of bounded curvature in $\mathbb{R}^3$. The same holds if $\mathbb{C}$ is replaced by any open conformal surface without nonconstant bounded subharmonic functions.

1. Introduction

The Strong Halfspace Theorem of Hoffman and Meeks [20, Theorem 2] says that any two proper minimal surfaces in $\mathbb{R}^3$ intersect, unless they are parallel planes. There are other known separation theorems for minimal surfaces in which properness is replaced by geometric conditions. In particular, it was shown by Bessa et al. [6] that a pair of complete immersed minimal surfaces in $\mathbb{R}^3$ of bounded curvature intersect, unless they are parallel planes.

The following related question was motivated by the recently introduced hyperbolicity theory for minimal surfaces; see [11, Problem 1.16].

Problem 1.1. Let $M$ be a nonflat properly embedded minimal surface in $\mathbb{R}^3$. Is there a nonconstant conformal harmonic map $\mathbb{C} \to \mathbb{R}^3$ whose image does not intersect $M$?

If $M$ is a plane in $\mathbb{R}^3$ then the image of every conformal harmonic map $\mathbb{C} \to \mathbb{R}^3 \setminus M$ lies in a plane parallel to $M$, so this case need not be considered.

We give the following negative answer to Problem 1.1 in several cases of interest.

Theorem 1.2. The image of a nonconstant conformal harmonic map $\mathbb{C} \to \mathbb{R}^3$ intersects every properly embedded nonflat minimal surface in $\mathbb{R}^3$ of bounded Gaussian curvature. The same holds for nonconstant conformal harmonic maps $\mathbb{R} \to \mathbb{R}^3$ from any open conformal surface without nonconstant bounded subharmonic functions.

The map $R \to \mathbb{R}^3$ in the above theorem is not assumed to be proper, it may have double points and branch points, and there is no assumption on the curvature of the image surface. Note that every compact conformal surface punctured at finitely many points satisfies the assumption on $R$ in the theorem. The result is close to optimal with respect to the conformal type of the source surface, since open conformal surfaces of hyperbolic type tend to admit nonconstant bounded conformal harmonic maps to $\mathbb{R}^3$. This holds in particular for the disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and more generally for any bordered conformal surface.

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Recall that a map \( f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n \) from an open set \( U \subset \mathbb{C} \) is harmonic if every component \( f_i \) is a harmonic function. The map is conformal if it preserves angles at every immersion point; equivalently, if \( z = x + iy \) is a complex coordinate on \( U \) then the partial derivatives of \( f \) satisfy \( f_x \cdot f_y = 0 \) and \( |f_x| = |f_y| \). Here, the dot denotes the Euclidean inner product and \( | \cdot | \) the Euclidean length. These conditions imply that the rank of \( f \) at any point is either two (an immersion point) or zero (a branch point), and the branch points form a closed discrete set unless \( f \) is constant. The analogous definition applies to maps from any conformal surface in local isothermal coordinates. The image of a conformal harmonic map from an open conformal surface is a minimal surface, and every minimal surface is of this form. For background see, e.g., the monographs \([2,8,37]\) and the surveys \([1,29,30]\).

Our proof of Theorem 1.2, given in Section 2, relies on two main ingredients. The regular neighbourhood theorem of Meeks and Rosenberg \([36, \text{Theorem 5.1}]\) shows that for an embedded minimal surface \( M \subset \mathbb{R}^3 \) of bounded curvature there is an \( \epsilon > 0 \) such that every point in the \( \epsilon \)-neighbourhood \( V_\epsilon(M) \) of \( M \) has a unique closest point in \( M \), and the signed distance function \( \delta_M \) to \( M \) (2.2) is real analytic on \( V_\epsilon(M) \) (see Theorem 2.1). Next, by suitably convexifying \( \delta_M \) we construct a minimal plurisubharmonic function \( \rho : \Omega \to (-c,0) \) for some \( c > 0 \) whose level sets \( \{ \rho = t \} \) for \( t \in (-c,0) \) coincide with the level sets of \( \delta_M \). Assuming that \( M \) is nonflat, none of these level sets contains any minimal surfaces. If \( f : \mathbb{C} \to \Omega \) is a nonconstant conformal harmonic map then \( \rho \circ f \) is a bounded subharmonic function on \( \mathbb{C} \), hence constant. It follows that \( f(\mathbb{C}) \) lies in the domain \( \{ x \in \Omega : \rho(x) \leq -c \} \), so it has positive distance to \( M \) depending only on \( M \), and by translating \( M \) we infer that such \( f \) cannot exist. The same argument holds if \( \mathbb{C} \) is replaced by any open conformal surface without nonconstant bounded subharmonic functions.

Our proof breaks down at several points for embedded minimal surfaces with unbounded curvature in \( \mathbb{R}^3 \), and hence Problem 1.1 remains open for such surfaces.

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The Gaussian curvature function \( K \) of a minimal surface \( M \) in \( \mathbb{R}^n \) is nonpositive. We say that \( M \) has bounded curvature if \( |K| : M \to \mathbb{R}_+ \) is a bounded function, and it has finite total curvature if \( \int_M K \, dA > -\infty \), where \( dA \) is the surface area measure of \( M \). The surface is said to be complete if it is complete in the induced metric. Completeness is a strong hypothesis: a complete injectively immersed minimal surface of bounded curvature in \( \mathbb{R}^3 \) is proper (see Meeks and Rosenberg \([36, \text{Theorem 2.1}]\)).

A complete minimal surface in \( \mathbb{R}^n \) of finite total curvature has the conformal type of a compact conformal surface punctured at finitely many points (see Huber \([22]\)). Every such surface has bounded curvature function which tends to zero at the ends (see Jorge and Meeks \([23]\)), so Theorem 1.2 holds for every complete embedded minimal surface of finite total curvature in \( \mathbb{R}^3 \). It was classically known that the plane and the catenoid are such. In 1984, Costa \([9]\) discovered a new example of genus one with three embedded ends, which was proved to be embedded by Hoffman and Meeks in 1985 \([19]\). Kapouleas \([24]\) found complete embedded minimal surfaces of finite total curvature in \( \mathbb{R}^3 \) with an arbitrary number of ends. There are many works treating such surfaces; see \([18,34,32]\), among others.
There exist several properly embedded minimal surfaces in $\mathbb{R}^3$ of bounded curvature but infinite total curvature. An example is the helicoid. Meeks, Pérez, and Ros proved in [33, Theorem 1] that every properly embedded minimal surface in $\mathbb{R}^3$ of finite genus has bounded curvature, so Theorem 1.2 holds for it. Furthermore, such a surface is conformally diffeomorphic to a compact Riemann surface punctured in a finite or a countable closed subset, which has exactly two limit points if the subset is infinite. An interesting family of examples of genus zero are Riemann’s minimal surfaces which form a 1-parameter family; see Meeks and Pérez [31] for more information. On the other hand, Traizet [39] found properly embedded minimal surfaces of infinite genus with unbounded curvature function.

There are helicoids of positive genera constructed by Hoffman, Traizet, and White [21]. They have finitely many handles (every genus is possible) and a single end asymptotic to the standard helicoid, hence they are of infinite total curvature. However, by the above mentioned result of Meeks, Pérez, and Ros [33] the curvature function is bounded, so Theorem 1.2 holds for these generalized helicoids. Bernstein and Breiner [5] and Meeks and Perez [32] proved that every nonflat properly embedded minimal surface in $\mathbb{R}^3$ of finite genus and a single end is asymptotic to a helicoid at infinity, but so far the only known examples are those in [21].

Besides these, we only know periodic embedded minimal surfaces of infinite total curvature in $\mathbb{R}^3$. Meeks and Rosenberg [35] proved that a properly embedded minimal surface $M$ in a complete non-simply connected flat 3-manifold has finite total curvature if and only if it is of finite topological type, i.e., the first homology group $H_1(M, \mathbb{Z})$ is finitely generated. Since a periodic minimal surface in $\mathbb{R}^3$ is the same thing as a minimal surface in a quotient of $\mathbb{R}^3$, which is a flat non-simply connected three-manifold, it follows that a fundamental domain of an embedded periodic minimal surface is of finite total curvature if and only if it is of finite topology. Meeks and Rosenberg also proved that the curvature is then bounded and goes to zero at the ends. Hence, every periodic properly embedded minimal surface in $\mathbb{R}^3$ whose fundamental domain is of finite topological type satisfies Theorem 1.2. Examples include Scherk’s surfaces; see [2, Sect. 2.8.3] and the references therein.

We now place Theorem 1.2 in the context of another line of recent developments which provided the original motivation for Problem 1.1.

In 2021, Forstnerič and Kalaj [14] introduced on any domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) a Finsler pseudometric $g_\Omega : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$, the minimal pseudometric, and the associated pseudodistance $\text{dist}_\Omega : \Omega \times \Omega \rightarrow \mathbb{R}_+$, by using conformal harmonic discs $D \rightarrow \Omega$ in the same way as the Kobayashi–Royden pseudometric and pseudodistance are defined on complex manifolds by using holomorphic discs; see [25, 26, 38]. Let $\text{CH}(D, \Omega)$ denote the space of conformal harmonic discs $D \rightarrow \Omega$, possibly with branch points. Denote by $z = x + iy$ the complex coordinate on $D$. The minimal pseudometric at a point $p \in \Omega$ on a tangent vector $v \in \mathbb{R}^n$ is defined as follows:

$$g_\Omega(p, v) = \inf \left\{ \frac{1}{r} > 0 : \exists f \in \text{CH}(D, \Omega), \ f(0) = p, \ fx(0) = rv \right\} \geq 0.$$ 

The pseudodistance $\text{dist}_\Omega$ is the integrated form of $g_\Omega$. The same definitions apply in any Riemannian manifold $(X, g)$ of dimension at least three: the Riemannian metric $g$ determines the class of conformal harmonic discs in $X$, and these discs are used to define the minimal pseudometric $g_X$ and pseudodistance $\text{dist}_X$. The manifold $(X, g)$ is said to be hyperbolic if $\text{dist}_X$ is a distance function, and complete hyperbolic if $(X, \text{dist}_X)$ is a complete metric space.
In [10], Drinovec Drnovšek and Forstnerič developed the basic hyperbolicity theory for domains in Euclidean spaces endowed with the Euclidean metric, and they posed several problems. In particular, they asked the following; see [10, Problem 12.4].

**Problem 1.3.** Let \( M \) be an embedded minimal surface in a bounded domain \( \Omega \subset \mathbb{R}^3 \). Is the minimal distance in \( \Omega \setminus M \) to any point \( p \in M \) infinite? In other words, is the complement of an embedded minimal surface in \( \mathbb{R}^3 \) locally complete hyperbolic at every point of the surface?

Schwarz lemma for positive harmonic functions gives an affirmative answer to the above problem if \( M \) is a plane [10, Lemma 5.2]. It is shown in [10, Theorem 9.2] that if \( \Omega \) is a domain in \( \mathbb{R}^3 \) whose boundary \( M = b\Omega \) is of class \( C^2 \) and has positive mean curvature at \( p \), then \( p \) is at infinite minimal distance from \( \Omega \). While we do not know the answer to Problem 1.3 for nonflat minimal surfaces, Theorem 1.2 shows that every connected component of the complement of a properly embedded minimal surface of bounded curvature in \( \mathbb{R}^3 \) has the following weaker hyperbolicity property.

**Definition 1.4.** A domain \( \Omega \) in \( \mathbb{R}^n, n \geq 3 \), is weakly hyperbolic if every conformal harmonic map \( \mathbb{C} \to \Omega \) is constant.

Every hyperbolic domain is also weakly hyperbolic. Indeed, given a nonconstant conformal harmonic map \( f : \mathbb{C} \to \Omega \subset \mathbb{R}^n \) and a point \( a \in \mathbb{C} \) where \( df_a \neq 0 \), the maps \( f_r : \mathbb{D} \to \Omega \) \((r > 0)\) given by \( f_r(z) = f(a + rz) \) are conformal harmonic discs centred at the point \( f(a) \), with the differential at \( 0 \) given by \( df_r(0) = rdf_a \). Letting \( r \to +\infty \) we see that the minimal pseudometric \( g_\Omega \) vanishes at the point \( p = f(a) \) on all vectors in the tangent plane \( df_a(\mathbb{C}) \). The converse is false as shown by the following example.

**Example 1.5.** We show that the domain

\[
\Omega = \{(x, y, z) \in \mathbb{R}^3 : |x| < 1, x^2(y^2 + z^2) < 1, |y| < 1 \text{ if } x = 0\}
\]

is weakly hyperbolic but not hyperbolic. If \( f = (f_1, f_2, f_3) : \mathbb{C} \to \Omega \) is a conformal harmonic map then \( |f_1| < 1 \), so \( f_1 = c \) is constant. If \( c \neq 0 \) then \( f_2^2 + f_3^2 < 1/c^2 < +\infty \), so \( f_2 \) and \( f_3 \) are constant as well. If \( c = 0 \) then \( |f_2| < 1 \), so \( f_2 \) is constant. By conformality, \( f_3 \) is also constant. Thus, \( \Omega \) is weakly hyperbolic. However, the minimal distance in \( \Omega \) between any two points in the fibre of \( \Omega \) over \( x = 0 \) (the strip \( \{(y, z) \in \mathbb{R}^2 : |y| < 1\} \)) vanishes. For example, taking \( A = (0, 0, 0) \) and \( B = (0, 0, 1) \), we set \( A_k = (1/k, 0, 0) \) and \( B_k = (1/k, 0, 1) \) for \( k = 2, 3, \ldots \). Note that \( A_k \) and \( B_k \) belong to the disc \( \{y = 0\} \setminus \Omega \), so \( \lim_{k \to \infty} \operatorname{dist}_\Omega(A, A_k) = 0 \) and \( \lim_{k \to \infty} \operatorname{dist}_\Omega(B, B_k) = 0 \), while the sequence of discs \( \{(1/k, y, z) : y^2 + z^2 < k^2\} \subset \Omega \) shows that \( \lim_{k \to \infty} \operatorname{dist}_\Omega(A, B_k) = 0 \). Hence, \( \operatorname{dist}_\Omega(A, B) = 0 \).

Weak hyperbolicity is an analogue of Brody hyperbolicity in complex analysis: a complex manifold \( X \) is said to be **Brody hyperbolic** if every holomorphic map \( \mathbb{C} \to X \) is constant. Every Kobayashi hyperbolic manifold is also Brody hyperbolic, and Brody [7] showed that the converse holds on compact complex manifolds. Examples similar to our Example 1.5 show that the converse fails for some unbounded domains in \( \mathbb{C}^2 \) (see [7, p. 219]).

A property directly opposite to hyperbolicity is **flexibility** (for minimal surfaces), introduced in [11, Definition 1.1]. A domain \( \Omega \subset \mathbb{R}^n, n \geq 3 \), is said to be flexible if, given an open conformal surface \( M \), a compact set \( K \subset M \) whose complement has no relatively
compact connected components, and a conformal harmonic immersion \( U \to \Omega \) from an open neighbourhood \( U \subset M \) of \( K \), we can approximate \( f \) uniformly on \( K \) and interpolate it at finitely many given points of \( K \) by conformal harmonic immersions \( M \to \Omega \). Clearly, a flexible domain admits many conformal harmonic images of \( \mathbb{C} \), so it is not weakly hyperbolic. Conversely, a weakly hyperbolic domain is not flexible. A halfspace in \( \mathbb{R}^n \) is neither (weakly) hyperbolic nor flexible. On the other hand, Theorem 1.2 implies the following.

**Corollary 1.6.** If \( M \) is a properly embedded nonflat minimal surface in \( \mathbb{R}^3 \) of bounded curvature, then every connected component of \( \mathbb{R}^3 \setminus M \) is weakly hyperbolic and is not flexible.

The situation is quite different in \( \mathbb{R}^4 \). It was shown in [11, Example 1.9] that every domain in \( \mathbb{R}^4 \) with coordinates \((x_1, x_2, x_3, x_4)\), given by

\[
\Omega = \{ x_4 > -a|x_2| + b|x_3| \quad \text{for some} \ a > 0 \text{ and } b \in \mathbb{R} \},
\]

is flexible. Taking \( b > 0 \), the complementary domain \( \Omega' = \mathbb{R}^4 \setminus \overline{\Omega} \) is of the same type with the reversed roles of \( x_2 \) and \( x_3 \), so it is also flexible. It was shown by Alarcón and López [3] that each of these domains contains a properly immersed conformal minimal surface parameterised by an arbitrary open Riemann surface. (In fact, their result holds for any concave wedge in \( \mathbb{R}^3 \) obtained by intersecting \( \Omega \) with the hyperplane \( x_3 = 0 \).) This gives many pairs of disjoint properly immersed minimal surfaces in \( \mathbb{R}^4 \) of any given conformal type. There also exist pairs of disjoint catenoids in \( \mathbb{R}^4 \) whose ends are asymptotic to a pair of orthogonal 2-planes in \( \mathbb{R}^4 \), so their closures in \( \mathbb{R}P^4 \) are disjoint as well. On the other hand, a pair of complex curves in \( \mathbb{C}^2 \) intersect, or their closures in \( \mathbb{C}P^2 \) intersect at infinity. (Note that every complex curve is also a minimal surface.)

2. **Proof of Theorem 1.2**

Given a subset \( M \) of \( \mathbb{R}^n \), we denote by \( d_M \) the Euclidean distance function to \( M \):

\[
d_M(x) = \inf \{|x - p| : p \in M\}, \quad x \in \mathbb{R}^n.
\]

For \( a \in \mathbb{R}^n \) and \( \epsilon > 0 \) we let \( \mathbb{B}(a, \epsilon) = \{ x \in \mathbb{R}^n : |x - a| < \epsilon \} \). By

\[
(2.1) \quad V_\epsilon(M) = \bigcup_{a \in M} \mathbb{B}(a, \epsilon) = \{ x \in \mathbb{R}^n : d_M(x) < \epsilon \}
\]

we denote the open \( \epsilon \)-neighbourhood of \( M \).

Let \( M \) be a properly embedded connected hypersurface in \( \mathbb{R}^n \). Such \( M \) is necessarily oriented, and its complement \( \mathbb{R}^n \setminus M = M^+ \cup M^- \) consists of a pair of connected domains. The *signed distance function* to \( M \) is defined by

\[
(2.2) \quad \delta_M(x) = \begin{cases} 
  d_M(x), & \text{if } x \in M^+ \cup M; \\
  -d_M(x), & \text{if } x \in M^-.
\end{cases}
\]

The following result of Meeks and Rosenberg [36, Theorem 5.3] will be used in the proof of Theorem 1.2. (See also [30, Corollary 2.6.6].)

**Theorem 2.1.** If \( M \subset \mathbb{R}^3 \) is a properly embedded minimal surface of bounded curvature, then there is an \( \epsilon > 0 \) such that every point in the \( \epsilon \)-neighbourhood \( V_\epsilon(M) \) (see (2.1)) has a unique nearest point to \( M \), and the signed distance function \( \delta_M \) is real analytic on \( V_\epsilon(M) \).
Since $M$ is proper and hence unbounded, the existence of a number $\epsilon > 0$ satisfying the first part of the theorem essentially depends on the bounded curvature hypothesis. On the other hand, the regularity statement holds for any smooth embedded hypersurface $M \subset \mathbb{R}^n$: if every point in a tubular neighbourhood $V$ of $M$ has a unique nearest point to $M$ and $M$ is of class $\mathcal{C}^r$ for some $r \in \{2, 3, \ldots, \infty, \omega\}$, then the signed distance function $\delta_M$ is also of class $\mathcal{C}^r$ on $V$ (see Gilbarg and Trudinger [15, Lemma 14.16] or Krantz and Park [27]). In our case, $M$ is a minimal surface, hence real analytic, so $\delta_M$ is also real analytic.

The special case of Theorem 2.1 when $M$ is a complete embedded minimal surface of finite total curvature follows from the description of such surfaces given by Jorge and Meeks [23], combined with the results of Federer [12, Sect. 4]. In Federer's terminology, the first statement of the theorem says that a properly embedded minimal surface $M \subset \mathbb{R}^3$ of bounded curvature has positive reach. Every compact piece of a smooth embedded surface has positive reach, so the problem lies in the ends of the surface. If $M$ is embedded and of finite total curvature, then every end of $M$ can be represented as a graph over the complement of a disc in $\mathbb{R}^2$ whose graphing function has bounded second order partial derivatives, and the conclusion then follows from [12, Lemma 4.11]. In the general case the proof in [36] is more involved.

Given a $\mathcal{C}^2$ function $\rho : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{R}^n$, we denote by $\text{Hess}_\rho(x)$ the Hessian of $\rho$ at the point $x \in \Omega$, i.e., the quadratic form on the tangent space $T_x \mathbb{R}^n = \mathbb{R}^n$ represented by the matrix $(\frac{\partial^2 \rho}{\partial x_i \partial x_j}(x))$ of second order partial derivatives of $\rho$ at $x$.

We recall the notion of a minimal plurisubharmonic function on a domain in $\mathbb{R}^n$, referring to [2, Sect. 8.1] or [17] for the details.

Let $G_2(\mathbb{R}^n)$ denote the Grassman manifold of 2-planes in $\mathbb{R}^n$. Given $\Lambda \in G_2(\mathbb{R}^n)$, we denote by $\text{tr}_\Lambda \text{Hess}_\rho(x) \in \mathbb{R}$ the trace of the restriction of $\text{Hess}_\rho(x)$ to $\Lambda$. We say that $\rho$ is minimal plurisubharmonic on $\Omega$ if

$$\text{tr}_\Lambda \text{Hess}_\rho(x) \geq 0 \quad \text{for all } x \in \Omega \text{ and } \Lambda \in G_2(\mathbb{R}^n).$$

The function $\rho$ is said to be strongly minimal plurisubharmonic if $\text{tr}_\Lambda \text{Hess}_\rho(x) > 0$ holds for all pairs $(x, \Lambda) \in \Omega \times G_2(\mathbb{R}^n)$. If $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of $\text{Hess}_\rho(x)$ then these conditions are equivalent to

$$\lambda_1(x) + \lambda_2(x) \geq 0, \quad \text{resp. } \lambda_1(x) + \lambda_2(x) > 0 \quad \text{for every } x \in \Omega.$$

A function $\rho : \Omega \to \mathbb{R}$ is minimal plurisubharmonic if and only if for every conformal harmonic map $f : \mathbb{D} \to \Omega$ from the disc the composition $h \circ f$ is a subharmonic function on $\mathbb{D}$ (see [2, Proposition 8.1.2]). This characterisation allows one to extend the notion of being minimal plurisubharmonic to upper semicontinuous functions.

**Proof of Theorem 1.2** Let $\Omega$ be a domain in $\mathbb{R}^3$ bounded by a properly embedded nonflat minimal surface $M = b\Omega$ of finite total curvature. Let $\epsilon > 0$ be such that Theorem 2.1 holds. Hence, the signed distance function $\delta = \delta_M$ (see (2.2)) is well-defined and real analytic on the $\epsilon$-neighbourhood $V_\epsilon(M)$ of $M$, and we choose the sign so that $\delta < 0$ on the collar

$$(2.3) \quad C_\epsilon := \Omega \cap V_\epsilon(M) = \{x \in \Omega : \text{dist}(x, M) < \epsilon\}.$$

We shall recall some further properties of $\delta$, referring to Bellettini [4, Theorem 1.18, p. 14] and Gilbarg and Trudinger [15, Section 14.6]. (This description was also used in [13].)
There is a real analytic projection $\xi : V_\varepsilon(M) \to M$ such that for every $x \in V_\varepsilon(M)$ the point $p = \xi(x) \in M$ is the unique nearest point to $x$ on $M$. The gradient $\nabla \delta$ has constant norm $|\nabla \delta| = 1$ on $V_\varepsilon$, and it has constant value on the intersection of $V_\varepsilon(M)$ with the normal line $N_p = p + \mathbb{R} \cdot \nabla \delta(p)$ at $p \in b\Omega = M$. There is an orthonormal basis $(v_1, v_2, v_3 = \nabla \delta(p))$ of $\mathbb{R}^3$ such that $v_1, v_2$ are a basis of $T_pM$ and these vectors diagonalize the matrix $A(p)$ of the Hessian $\text{Hess}_\delta(p)$. Let $\nu : M \to \mathbb{R}_+$ denote the principal normal curvature function of $M$. Up to switching $v_1$ and $v_2$ if necessary we have that

$$A(p)v_1 = -\nu(p)v_1, \quad A(p)v_2 = \nu(p)v_2, \quad A(p)v_3 = 0.$$ 

For any point $x \in N_p \cap V_\varepsilon(M)$ the same basis $v_1, v_2, v_3$ then diagonalizes $\text{Hess}_\delta(x)$, and the corresponding eigenvalues equal

$$(2.4) \quad \nu_1(x) = \frac{-\nu(p)}{1 - \delta(x)\nu(p)}, \quad \nu_2(x) = \frac{\nu(p)}{1 + \delta(x)\nu(p)}, \quad \nu_3(x) = 0.$$ 

Note that

$$(2.5) \quad -\nu(p) \leq \nu_1(x) \leq 0 \quad \text{and} \quad \nu_2(x) \geq \nu(p) \geq 0 \quad \text{for} \ x \in N_p \cap C_\varepsilon,$$

where $C_\varepsilon$ is the interior $\varepsilon$-collar of $M$ in $\Omega$ (see (2.3)).

Since $M$ is assumed to have bounded curvature, we have that

$$(2.6) \quad 0 \leq \nu_0 := \sup_{p \in M} \nu(p) < \infty.$$ 

The formulas (2.4) make sense when $|\delta(x)| < 1/\nu_0$, which shows that $\varepsilon \leq 1/\nu_0$. For $|t| < \varepsilon$ the mean curvature of the surface $M_t = \{ \delta = t \}$ at a point $x \in M_t$ equals

$$H(x) = \nu_1(x) + \nu_2(x) = \frac{-\nu(p)}{1 + t\nu(p)} + \frac{\nu(p)}{1 - t\nu(p)} = \frac{-2t\nu(p)^2}{1 - t^2\nu(p)^2},$$

where $p = \xi(x) \in M$. Note that

$$(2.7) \quad H(x) \geq 0 \quad \text{for} \ x \in C_\varepsilon,$$

the function $H$ increases as we move away from $M = b\Omega$ into $\Omega$, and it vanishes on a normal line $N_p$ at $p \in M$ if and only if the Gaussian curvature $K(p) = -\nu(p)^2$ of $M$ at $p$ vanishes.

We now show that a suitable convexification of the signed distance function $\delta$ on the collar $C_\varepsilon \subset \Omega$ in (2.3) extends to a bounded minimal plurisubharmonic function $\rho : \Omega \to (-\infty, 0)$ whose level sets $\{ \rho = t \}$ for $t < 0$ close to zero coincide with the level sets of $\delta$. The argument is similar to the one in [13, proof of Theorem 1.1], except that we are now dealing with an unbounded domain $\Omega$, and we obtain uniform estimates of the quantities involved.

Recall that $\nu_0 \in [0, +\infty)$ is defined by (2.6). Choose a number $\alpha > \nu_0$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth, convex, increasing function with $h(0) = 0$, $\dot{h}(0) = 1$, and $\ddot{h}(0) = \alpha$. Then,

$$(2.8) \quad h(t) < 0 \quad \text{and} \quad 0 \leq \ddot{h}(t) < 1 \quad \text{for} \quad -\infty < t < 0.$$ 

Choose a number $t_0 \in (-\varepsilon, 0)$ such that

$$(2.9) \quad \ddot{h}(t) > \nu_0 \quad \text{for} \ t_0 \leq t \leq 0.$$ 

Consider the function $h \circ \delta : C_\varepsilon \to (-\infty, 0)$. We have that

$$\text{Hess}_{h \circ \delta} = (\ddot{h} \circ \delta)\text{Hess}_\delta + (\ddot{h} \circ \delta) \nabla \delta \cdot (\nabla \delta)^T.$$
Here, \((\nabla \delta)^T\) is the transpose of \(\nabla \delta\) and \(\nabla \delta \cdot (\nabla \delta)^T\) is a 3 \(\times\) 3 matrix. Assume that for \(p \in M = b\Omega\) the orthonormal vectors \(v_1, v_2, v_3 = \nabla \delta(p)\) diagonalize \(\text{Hess}_\delta(p)\), where \(T_p M = \text{span}\{v_1, v_2\}\). Then the vectors \(v_1, v_2\) lie in the kernel of the matrix \(\nabla \delta(p) \cdot (\nabla \delta)^T\), while \(v_3\) is an eigenvector with eigenvalue 1. The same basis \(v_1, v_2, v_3\) then diagonalizes \(\text{Hess}_{h_0 \delta}\) at every point \(x = p + \delta(x) v_3 \in N_p \cap C_{\tau}\), the eigenvalues corresponding to \(v_1\) and \(v_2\) get multiplied by the number \(\hat{h}(\delta(x))\), and the eigenvalue in the normal direction \(v_3 = \nabla \delta(x)\) equals \(\tilde{h}(\delta(x))\). Summarizing, the eigenvalues of \(\text{Hess}_{h_0 \delta}\) at any point \(x \in C_{\varepsilon}\) equal

\[
\begin{align*}
\hat{h}(\delta(x)) & \nu_1(x), \\
\hat{h}(\delta(x)) & \nu_2(x), \\
\tilde{h}(\delta(x)).
\end{align*}
\]

It follows from (2.7), (2.8), and (2.9) that the sum of any two of these eigenvalues is nonnegative at every point \(x \in C_{t_0}\) (see (2.3)). In other words, the function \(h \circ \delta\) is minimal plurisubharmonic on \(C_{t_0}\). Hence, the continuous function \(\rho : \Omega \to [h(t_0), 0)\) given by

\[
\rho(x) = \begin{cases} 
\hat{h}(\delta(x)), & \text{if } x \in C_{t_0}; \\
\tilde{h}(t_0), & \text{if } x \in \Omega_{t_0} := \{x \in \Omega : \delta(x) \leq t_0\}
\end{cases}
\]

is minimal plurisubharmonic. (Note that near \(bC_{t_0} = \{x \in \Omega : \delta(x) = t_0\} = b\Omega_{t_0}\) we have \(\rho = \max\{h \circ \delta, h(t_0)\}\), and the maximum of two minimal plurisubharmonic functions is also minimal plurisubharmonic; see [16, Sect. 6].) We can approximate \(\rho\) from above as closely as desired, uniformly on \(\Omega\), by a smooth minimal plurisubharmonic function which agrees with \(\rho\) on a smaller collar \(C_{t_1}\) around \(M\) for some \(t_0 < t_1 < 0\). To do this, one uses a regularized maximum instead of maximum in the definition of \(\rho\); see [13] proof of Theorem 1.1, p. 7. However, this is inessential for the argument to follow.

We can now conclude the proof of Theorem 1.2. Assume that \(f : \mathbb{C} \to \Omega\) is a nonconstant conformal harmonic map. Then, \(\rho \circ f\) is a bounded subharmonic function on \(\mathbb{C}\), hence constant. This means that the image \(f(\mathbb{C})\) either lies in the domain \(\Omega_{t_0} := \{x \in \Omega : \delta(x) \leq t_0\}\) or in the surface \(M_t = \{x \in \Omega : \delta(x) = t\}\) for some \(t \in (t_0, 0)\).

Let us first discard the second possibility. Since the minimal surface \(M = b\Omega\) is assumed to be nonflat, the zero set of the principal curvature function \(\nu : M \to \mathbb{R}_+\) has empty interior. By (2.7) and the sentence following it, every surface \(M_t\) for \(t_0 < t < 0\) has nonnegative mean curvature, and the zero set of the mean curvature function has empty interior. Hence, \(M_t\) does not contain any minimal surface. Since the local image of \(f\) at every immersion point is a minimal surface, the assumption \(f(\mathbb{C}) \subset M_t\) for \(t_0 < t < 0\) implies that \(f\) has rank zero at each point, so it is constant, contradicting our assumption.

This shows that \(f(\mathbb{C}) \subset \Omega_{t_0}\), so the distance between \(f(\mathbb{C})\) and the hypersurface \(M\) is at least \(r := |t_0| > 0\). If we now translate \(M\) for the distance \(r/2\) in any direction, the same argument (with the same constants) applies to the translated surface \(M'\), so \(f(\mathbb{C}) \cap M' = \emptyset\) implies that \(\text{dist}(f(\mathbb{C}), M') \geq r\). Since this holds for every translate of \(M\) for \(r/2\), we infer that \(\text{dist}(f(\mathbb{C}), M) \geq 3r/2\). Repeating this argument shows that the distance from \(f(\mathbb{C})\) to \(M\) must be infinite, hence no such map \(f\) exists.

Since the function \(\rho\) constructed above is negative and bounded from below, the same argument applies if \(\mathbb{C}\) is replaced by any open conformal surface \(R\) which does not carry any nonconstant bounded subharmonic functions. This holds in particular if \(R\) is obtained by removing finitely many points from a compact conformal surface. \(\square\)
Remark 2.2 (Concerning the paper [13]). By [13, Theorem 1.1], every bounded domain $\Omega$ in $\mathbb{R}^n$ ($n \geq 3$) whose boundary is of class $C^{r,\alpha}$ for some $r \geq 2$ and $0 < \alpha \leq 1$ and is $p$-convex for some $p \in \{1, \ldots, n-1\}$ admits a $p$-plurisubharmonic defining function of class $C^{r,\alpha}$. The proof uses the result of Li and Nirenberg [28] that the signed distance function to a hypersurface of class $C^{r,\alpha}$ for such $(r, \alpha)$ is of the same class $C^{r,\alpha}$ near the hypersurface. By using the result of Gilbarg and Trudinger [15, Lemma 14.16] and Krantz and Parks [27], we see that [13, Theorem 1.1] also holds in smoothness classes $C^r$ for $r = 2, 3, \ldots, \infty$.

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References

[1] A. Alarcón and F. Forstnerič. New complex analytic methods in the theory of minimal surfaces: a survey. J. Aust. Math. Soc., 106(3):287–341, 2019.
[2] A. Alarcón, F. Forstnerič, and F. J. López. Minimal surfaces from a complex analytic viewpoint. Springer Monographs in Mathematics. Springer, Cham, 2021.
[3] A. Alarcón and F. J. López. Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$. J. Differential Geom., 90(3):351–381, 2012.
[4] G. Bellettini. Lecture notes on mean curvature flow, barriers and singular perturbations, volume 12 of Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2013.
[5] J. Bernstein and C. Breiner. Symmetry of embedded genus 1 helicoids. Duke Math. J., 159(1):83–97, 2011.
[6] G. P. Bessa, L. P. Jorge, and G. Oliveira-Filho. Half-space theorems for minimal surfaces with bounded curvature. J. Differ. Geom., 57(3):493–508, 2001.
[7] R. Brody. Compact manifolds and hyperbolicity. Trans. Amer. Math. Soc., 235:213–219, 1978.
[8] T. H. Colding and W. P. Minicozzi, II. Minimal surfaces, volume 4 of Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York, 1999.
[9] C. J. Costa. Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. Bol. Soc. Brasil. Mat., 15(1-2):47–54, 1984.
[10] B. D. Drnovšek and F. Forstnerič. Hyperbolic domains in real euclidean spaces. Pure Appl. Math. Q., to appear. [https://arxiv.org/abs/2109.06943](https://arxiv.org/abs/2109.06943)
[11] B. D. Drnovšek and F. Forstnerič. Flexible domains for minimal surfaces in euclidean spaces. arXiv e-prints, May 2022. [https://arxiv.org/abs/2204.14254](https://arxiv.org/abs/2204.14254)
[12] H. Federer. Curvature measures. Trans. Am. Math. Soc., 93:418–491, 1959.
[13] F. Forstnerič. Every smoothly bounded $p$-convex domain in $\mathbb{R}^n$ admits a $p$-plurisubharmonic defining function. Bull. Sci. Math., 175:10, 2022. Id/No 103100.
[14] F. Forstnerič and D. Kalaj. Hyperbolicity theory for minimal surfaces in Euclidean spaces. arXiv e-prints, March 2021. [https://arxiv.org/abs/2102.12403](https://arxiv.org/abs/2102.12403)
[15] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order, volume 224 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 1983.
[16] F. R. Harvey and H. B. Lawson, Jr. Plurisubharmonicity in a general geometric context. In Geometry and analysis. No. 1, volume 17 of Adv. Lect. Math. (ALM), pages 363–402. Int. Press, Somerville, MA, 2011.
[17] F. R. Harvey and H. B. Lawson, Jr. $p$-convexity, $p$-plurisubharmonicity and the Levi problem. Indiana Univ. Math. J., 62(1):149–169, 2013.
[18] D. Hoffman and H. Karcher. Complete embedded minimal surfaces of finite total curvature. In Geometry V: Minimal surfaces. Transl. from the Russian, pages 5–93. Berlin: Springer, 1997.
[19] D. Hoffman and W. H. Meeks, III. A complete embedded minimal surface in $\mathbb{R}^3$ with genus one and three ends. J. Differential Geom., 21(1):109–127, 1985.
[20] D. Hoffman and W. H. Meeks, III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101(2):373–377, 1990.

[21] D. Hoffman, M. Traizet, and B. White. Helicoidal minimal surfaces of prescribed genus. *Acta Math.*, 216(2):217–323, 2016.

[22] A. Huber. On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.*, 32:13–72, 1957.

[23] L. P. d. M. Jorge and W. H. Meeks, III. The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, 22(2):203–221, 1983.

[24] N. Kapouleas. Complete embedded minimal surfaces of finite total curvature. *J. Differ. Geom.*, 47(1):95–169, 1997.

[25] S. Kobayashi. Invariant distances on complex manifolds and holomorphic mappings. *J. Math. Soc. Japan*, 19:460–480, 1967.

[26] S. Kobayashi. *Hyperbolic manifolds and holomorphic mappings. An introduction*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2005.

[27] S. G. Krantz and H. R. Parks. Distance to $C^k$ hypersurfaces. *J. Differ. Equations*, 40:116–120, 1981.

[28] Y. Li and L. Nirenberg. Regularity of the distance function to the boundary. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 29:257–264, 2005.

[29] W. H. Meeks, III and J. Pérez. The classical theory of minimal surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 48(3):325–407, 2011.

[30] W. H. Meeks, III and J. Pérez. *A survey on classical minimal surface theory*, volume 60 of *University Lecture Series*. Amer. Math. Soc., Providence, RI, 2012.

[31] W. H. Meeks, III and J. Pérez. The Riemann minimal examples. In *The legacy of Bernhard Riemann after one hundred and fifty years. Vol. II*, volume 35 of *Adv. Lect. Math. (ALM)*, pages 417–457. Int. Press, Somerville, MA, 2016.

[32] W. H. Meeks, III and J. Pérez. Embedded minimal surfaces of finite topology. *J. Reine Angew. Math.*, 753:159–191, 2019.

[33] W. H. Meeks, III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus. II: Nonexistence of one limit end examples. *Invent. Math.*, 158(2):323–341, 2004.

[34] W. H. Meeks, III, J. Pérez, and A. Ros. Properly embedded minimal planar domains. *Ann. of Math.* (2), 181(2):473–546, 2015.

[35] W. H. Meeks, III and H. Rosenberg. The geometry of periodic minimal surfaces. *Comment. Math. Helv.*, 68(4):538–578, 1993.

[36] W. H. Meeks, III and H. Rosenberg. Maximum principles at infinity. *J. Differential Geom.*, 79(1):141–165, 2008.

[37] R. Osserman. *A survey of minimal surfaces*. Dover Publications, Inc., New York, second edition, 1986.

[38] H. L. Royden. Remarks on the Kobayashi metric. In *Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970)*, pages 125–137. Lecture Notes in Math., Vol. 185, 1971.

[39] M. Traizet. A minimal surface with one limit end and unbounded curvature. *Indiana Univ. Math. J.*, 61(3):1325–1350, 2012.

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