Quantum isometry group of dual of finitely generated discrete groups- II
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Abstract
As a contribution of the programme of [18], we carry out explicit computations of \( Q(\Gamma, S) \), the quantum isometry group of the canonical spectral triple on \( C^*_r(\Gamma) \) coming from the word length function corresponding to a finite generating set \( S \), for several interesting examples of \( \Gamma \) not covered by the previous work [18]. These include the braid group of 3 generators, \( \mathbb{Z}_4^n \) etc. Moreover, we give another description of the quantum groups \( H^+_s(n, 0) \) and \( K^+_n \) (studied in [6], [4]) in terms of free wreath product. In the last section we give several new examples of groups for which \( Q(\Gamma) \) turn out to be doubling of \( C^*(\Gamma) \).

1 Introduction

It is a very important and interesting problem in the theory of quantum groups and noncommutative geometry to study ‘quantum symmetries’ of various classical and quantum structures. S.Wang initiated by studying quantum permutation group of finite sets and quantum automorphism groups of finite dimensional algebras. Later on, a number of mathematicians including Wang, Banica, Bichon and others ([22], [1], [12]) developed a theory of quantum automorphism groups of finite dimensional \( C^* \) algebras as well as quantum isometry groups of finite metric spaces and finite graphs. In [17] Goswami extended such constructions to the set up of possibly infinite dimensional \( C^* \) algebras, and more interestingly, that of spectral triples a la Connes [14], by defining and studying quantum isometry groups of spectral triples. This led to the study of such quantum isometry groups by many authors including Goswami, Bhowmick, Skalski, Banica, Bichon, Soltan, Das, Joardar and others. In the present paper, we are focusing on a particular class of spectral triples, namely those coming from the word-length metric of finitely generated discrete groups with respect to some given symmetric generating set. There have been several articles already on computations and study of the quantum isometry groups of such spectral triples, e.g [11], [21], [10], [6], [4] and references therein. In [18] together with Goswami we also studied the quantum isometry groups of such spectral triples in a systematic and unified way. Here we compute \( Q(\Gamma, S) \) for more examples of groups including braid groups, \( \mathbb{Z}_4 \ast \mathbb{Z}_4 \ast \cdots \ast \mathbb{Z}_4 \) etc.

The paper is organized as follows. In Section 2 we recall some definitions and facts related to compact quantum groups, free wreath product with quantum permutation group and quantum isometry group of spectral triples defined by
Bhowmick and Goswami in [9]. This section also contains the doubling procedure of a compact quantum group say $\mathcal{Q}$, which is denoted by $D(\mathcal{Q})$. In Section 3 we compute $\mathcal{Q}(\Gamma, S)$ for braid group with 3 generators, as a $C^*$ algebra it turns out to be four direct copies of the group $C^*$ algebra (in fact, it is precisely doubling of doubling of the group $C^*$ algebra as a Hopf algebra). Section 4 contains an interesting description of the quantum groups $H_n^+(n, 0)$ and $K_n^+(n)$ (studied in [6], [4]) in terms of free wreath product. Moreover, $\mathcal{Q}(\Gamma, S)$ is computed for $\Gamma = \mathbb{Z}_4 \ast \mathbb{Z}_4 \ast \cdots \ast \mathbb{Z}_4$ in $n$ copies. In the last section we present more examples of groups as in ([16], [21], Section 5 of [18]) where $\mathcal{Q}(\Gamma, S)$ turn out to be the doubling of $C^*(\Gamma)$.

2 Preliminaries

First of all, we fix some notational conventions which will be useful for the rest of the paper. Throughout the paper, the algebraic tensor product and the spatial (minimal) $C^*$ tensor product will be denoted by $\otimes$ and $\hat{\otimes}$ respectively. We’ll use the leg-numbering notation, consider the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$, it has two natural embeddings into $M(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q} \hat{\otimes} \mathcal{Q})$. The first one is obtained by extending the map $x \mapsto x \otimes 1$. The second one is obtained by composing this map with the flip on the last two factors. We will write $\omega^{12}$ and $\omega^{13}$ for the images of an element $\omega \in M(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ under these two maps respectively.

2.1 Compact quantum groups and free wreath product

Let us recall some definitions about compact quantum groups, its action on a $C^*$ algebra and free wreath product with quantum permutation group.

Definition 2.1 A Compact quantum group (CQG for short) is a pair $(\mathcal{Q}, \Delta)$ where $\mathcal{Q}$ is a unital $C^*$ algebra where $\Delta : \mathcal{Q} \to \mathcal{Q} \hat{\otimes} \mathcal{Q}$ is a unital $C^*$ homomorphism satisfying two conditions:
1. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ (co-associativity).
2. Each of the linear spans of $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$ and that of $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)$ is norm dense in $\mathcal{Q} \hat{\otimes} \mathcal{Q}$.

Definition 2.2 (Unitary representation): Let $(\mathcal{Q}, \Delta)$ be a CQG. A unitary representation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ is a $C^*$ linear map from $\mathcal{H}$ to the Hilbert module $\mathcal{H} \hat{\otimes} \mathcal{Q}$ such that 1. $<U(\xi), U(\eta)> = <\xi, \eta>$ for $\xi, \eta \in \mathcal{H}$. 2. $(U \otimes id)U = (id \otimes \Delta)U$.

Here $<., .>$ is the $C^*$ valued inner product and $\mathcal{H} \hat{\otimes} \mathcal{Q}$ denotes the completion of $\mathcal{H} \otimes \mathcal{Q}$ with respect to the natural $\mathcal{Q}$ valued inner product. Given such a unitary representation we have a unitary element $\hat{U}$ belonging to $M(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ given by $\hat{U}(\xi \otimes b) = U(\xi)b$, $(\xi \in \mathcal{H}, b \in \mathcal{Q})$ satisfying $(id \otimes \Delta)(\hat{U}) = \hat{U}^{12}\hat{U}^{13}$. 

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Definition 2.3 We say that CQG \((\mathcal{Q}, \Delta)\) acts on a unital C\(^*\) algebra \(\mathcal{B}\) if there is a unital C\(^*\) homomorphism (called action) \(\alpha : \mathcal{B} \to \mathcal{B} \hat{\otimes} \mathcal{Q}\) satisfying the following:

1. \((\alpha \otimes \text{id}) \alpha = (\text{id} \otimes \Delta) \alpha\).
2. Linear span of \(\alpha(B) (1 \otimes \mathcal{Q})\) is norm dense in \(\mathcal{B} \hat{\otimes} \mathcal{Q}\).

Given two CQG’s \(\mathcal{Q}_1, \mathcal{Q}_2\) the free product \(\mathcal{Q}_1 \ast \mathcal{Q}_2\) admits the natural CQG structure equipped with the following universal property (for more details see in [23]).

Proposition 2.4 (i) The canonical injections, say \(i_1, i_2\) from \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\) to \(\mathcal{Q}_1 \ast \mathcal{Q}_2\) is CQG morphism.

(ii) Given any CQG \(\mathcal{C}\) and morphisms \(\pi_1 : \mathcal{Q}_1 \to \mathcal{C}\) and \(\pi_2 : \mathcal{Q}_2 \to \mathcal{C}\) there always exists a unique morphism denoted by \(\pi := \pi_1 \ast \pi_2\) from \(\mathcal{Q}_1 \ast \mathcal{Q}_2\) to \(\mathcal{C}\) satisfying \(\pi \circ i_k = \pi_k\) for \(k = 1, 2\).

Now we recall the definition of free wreath product by the quantum permutation group (see in [13]). We denote by \(\nu_i\) the canonical homomorphism \(\nu_i : \mathcal{Q} \to \mathcal{Q} \ast N\).

Definition 2.5 The quantum permutation group denoted by \((C(S_N^+), \Delta)\) is the universal C\(^*\) algebra generated by \(N^2\) elements \(t_{ij}\) such that the matrix \((t_{ij})\) is unitary and \(t_{ij} = t_{ij}^* = t_{ji}^2\) \(\forall i, j\) (i.e. \((t_{ij})\) is magic unitary). The coproduct \(\Delta\) is given by \(\Delta(t_{ij}) = \sum_{k=1}^N t_{ik} \otimes t_{kj}\) and it admits the CQG structure.

For further details see in [22].

Definition 2.6 Let \(\mathcal{Q}\) be a compact quantum group and \(N > 1\). The free wreath product of \(\mathcal{Q}\) by the quantum permutation group \(C(S_N^+)^\ast\) is the quotient of the C\(^*\) algebra \(\mathcal{Q} \ast N \ast C(S_N^+)^\ast\) by the two sided ideal generated by the elements

\[\nu_k(a) t_{ki} - t_{ki} \nu_k(a), \ 1 \leq i, k \leq N, \ a \in \mathcal{Q},\]

where \((t_{ij})\) is the matrix coefficients of the quantum permutation group \(C(S_N^+)^\ast\) (see in [22]). This is denoted by \(\mathcal{Q} \ast_w C(S_N^+)^\ast\).

Furthermore, it admits the CQG structure. The comultiplication satisfies

\[\Delta(\nu_i(a)) = \sum_{k=1}^N \nu_i(a_{(1)}) t_{ik} \otimes \nu_k(a_{(2)}).\]

Here we use Sweedler notation \(\Delta(a) = a_{(1)} \otimes a_{(2)}\).

2.2 The definition of \(\mathcal{Q}(\Gamma, S)\)

First of all, we are defining the quantum isometry group of spectral triples defined by Bhowmick and Goswami in [9].
\textbf{Definition 2.7} Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type (a la Connes). Consider the category $Q(\mathcal{D}) \equiv Q(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ whose objects are triples $(Q, \Delta, U)$ where $(Q, \Delta)$ is a CQG having a unitary representation $U$ on the Hilbert space $\mathcal{H}$ and $U$ commutes with $(\mathcal{D} \otimes 1_Q)$. Morphism between two such objects $(Q, \Delta, U)$ and $(Q', \Delta', U')$ is a CQG morphism $\psi : Q \to Q'$ such that $U' = (id \otimes \psi)U$. If universal object exists in $Q(\mathcal{D})$ then we denote it by $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ and the corresponding largest Woronowicz subalgebra for which $ad_U$ is faithful (where $U$ is the unitary representation of $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$) is called the quantum group of orientation preserving isometries and denoted by $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Here we state the Theorem 2.23 of (11) for existing $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

\textbf{Theorem 2.8} Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type. Assume that $\mathcal{D}$ has one dimensional kernel spanned by a vector $\xi \in \mathcal{H}$ which is cyclic and separating for $\mathcal{A}^\infty$ and each eigenvector of $\mathcal{D}$ belongs to $\mathcal{A}^\infty\xi$. Then $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ exists.

Now we discuss the special case of our purpose. Let $\Gamma$ be a finitely generated discrete group with generating set $\mathcal{S} = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}\}$. We make the convention of choosing the generating set to be symmetric, i.e. $a_i \in \mathcal{S}$ implies $a_i^{-1} \in \mathcal{S}$ for all $i$. In case some $a_i$ has order 2, we include only $a_i$, i.e. not count it twice. Define length function on the group as $l(g) = \min \{ r \in \mathbb{N}, g = h_1h_2\cdots h_r \}$ where $h_i \in \mathcal{S}$ i.e. for each $h_i = a_j$ or $a_j^{-1}$ for some $j$. Notice that $\mathcal{S} = \{ g \in \Gamma, l(g) = 1 \}$, using this length function we can define a metric on $\Gamma$ by $d(a, b) = l(a^{-1}b) \forall a, b \in \Gamma$. This is called the word metric. Now consider the algebra $C_r^*(\Gamma)$, which is the $C^*$ completion of the group ring $\mathbb{C}\Gamma$ viewed as a subalgebra of $B(l^2(\Gamma))$ in the natural way via left regular representation. We define the Dirac operator $D_T(\delta_g) = l(\delta_g)$. In general, $D_T$ is an unbounded operator.

$$\text{Dom}(D_T) = \{ \xi \in l^2(\Gamma) : \sum_{g \in \Gamma} l(g)^2 |\xi(g)|^2 < \infty \}.$$  

Here, $\delta_g$ is the vector in $l^2(\Gamma)$ which takes value 1 at the point $g$ and 0 at all other points. Natural generators of the algebra $\mathbb{C}\Gamma$ (images in the left regular representation) will be denoted by $\lambda_\gamma$, i.e. $\lambda_\gamma(\delta_h) = \delta_{gh}$. It is easy to check that $(\mathbb{C}\Gamma, l^2(\Gamma), D_T)$ is a spectral triple. Now take $\mathcal{A} = C_r^*(\Gamma), \mathcal{A}^\infty = \mathbb{C}\Gamma, \mathcal{H} = l^2(\Gamma)$ and $\mathcal{D} = D_T$ as before, $\delta_\epsilon$ is cyclic separating vector for $\mathbb{C}\Gamma$ then $QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D_T)$ exists by Theorem 2.8. As the object depends on the generating set of $\Gamma$ it is denoted by $Q(\Gamma, S)$. Most of the time we denote it by $Q(\Gamma)$ if $S$ is understood from the context. Now as in (11) Its action $\alpha$ (say) on $C_r^*(\Gamma)$ is determined by

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S} \lambda_{\gamma'} \otimes q_{\gamma', \gamma},$$

where the matrix $[q_{\gamma', \gamma}]_{\gamma, \gamma' \in S}$ is called the fundamental unitary in $M_{\text{card}(S)}(Q(\Gamma, S))$. Now we fix some notational conventions which will be useful in later
sections. Note that the action $\alpha$ is of the form

\[
\alpha(\lambda_{a_1}) = \lambda_{a_1} \otimes A_{11} + \lambda_{a_1}^{-1} \otimes A_{12} + \lambda_{a_2} \otimes A_{13} + \lambda_{a_2}^{-1} \otimes A_{14} + \cdots + \\
\alpha(\lambda_{a_k}) = \lambda_{a_k} \otimes A_{11}^{(2k-1)} + \lambda_{a_k}^{-1} \otimes A_{1(2k)},
\]

\[
\alpha(\lambda_{a_1}^{-1}) = \lambda_{a_1} \otimes A_{12}^{*} + \lambda_{a_1}^{-1} \otimes A_{11}^{*} + \lambda_{a_2} \otimes A_{14}^{*} + \lambda_{a_2}^{-1} \otimes A_{13}^{*} + \cdots + \\
\alpha(\lambda_{a_k}^{-1}) = \lambda_{a_k} \otimes A_{12}^{*^{(2k-1)}} + \lambda_{a_k}^{-1} \otimes A_{1(2k)},
\]

\[
\alpha(\lambda_{a_2}) = \lambda_{a_2} \otimes A_{22} + \lambda_{a_2}^{-1} \otimes A_{21} + \lambda_{a_3} \otimes A_{24} + \lambda_{a_3}^{-1} \otimes A_{23} + \cdots + \\
\alpha(\lambda_{a_k}) = \lambda_{a_k} \otimes A_{22}^{(2k)} + \lambda_{a_k}^{-1} \otimes A_{21}^{(2k)},
\]

\[
\vdots \\
\vdots \\
\alpha(\lambda_{a_k}) = \lambda_{a_k} \otimes A_{1k} + \lambda_{a_k}^{-1} \otimes A_{1k}^{(2k-1)}, \\
\alpha(\lambda_{a_k}^{-1}) = \lambda_{a_k} \otimes A_{1k}^{*} + \lambda_{a_k}^{-1} \otimes A_{1k}^{*^{(2k-1)}}.
\]

From this we get the unitary corepresentation

\[
U \equiv ((u_{ij})) = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2k-1)} & A_{1(2k)} \\
A_{12}^* & A_{11}^* & A_{13}^* & A_{14}^* & \cdots & A_{1(2k-1)}^* & A_{1(2k)}^* \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2k-1)} & A_{2(2k)} \\
A_{22}^* & A_{21}^* & A_{23}^* & A_{24}^* & \cdots & A_{2(2k-1)}^* & A_{2(2k)}^* \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{k1} & A_{k2} & A_{k3} & A_{k4} & \cdots & A_{k(2k-1)} & A_{k(2k)} \\
A_{k2}^* & A_{k1}^* & A_{k3}^* & A_{k4}^* & \cdots & A_{k(2k-1)}^* & A_{k(2k)}^* \\
\end{pmatrix}.
\]

The coefficients $A_{ij}$ and $A_{ij}^*$ ’s generate a norm dense subalgebra of $Q(\Gamma, S)$. In fact, it is easy to see that $Q(\Gamma, S)$ is the CQG generated by $A_{ij}$ as above subject to the relation $U$ is a unitary and $\alpha$ given above * homomorphism on $C^*_r(\Gamma)$. We also note that the antipode of $Q(\Gamma, S)$ maps $u_{ij}$ to $u_{ij}^*$.  

### 2.3 $Q(\Gamma)$ as the doubling of certain quantum groups

In this subsection we briefly recall the doubling procedure of a compact quantum group from [16, 20]. Let $(Q, \Delta)$ be a CQG with a CQG-automorphism $\theta$ such that $\theta^2 = id$. The doubling of this CQG, say $(D(Q), \tilde{\Delta})$ is given by $D(Q) := Q \oplus Q$ (direct sum as a C$^*$ algebra), and the coproduct is defined by the following, where we have denoted the injections of $Q$ onto the first and second coordinate
Below we give some sufficient conditions for the quantum isometry group to be the doubling of the certain algebras. For this, it is convenient to use a slightly different notational convention: let $U_1, U_2$ be an automorphism of order 2 of the group algebra which gives a permutation $\sigma$ on the set \{1,2,...,2k-1,2k\}. Now assume the following:

1. $B_i := U_{i,\sigma(i)} \neq 0 \forall i,$ and $U_{i,j} = 0 \forall j \notin \{\sigma(i),i\},$
2. $A_iB_j = B_jA_i = 0, \forall j$ such that $\sigma(j) \neq j$, where $A_i = U_{i,i},$
3. All $U_{i,j}U_{i,j}^*$ are central projections,
4. There are well defined $C^*$ isomorphisms $\pi_1, \pi_2$ from $C^*(\Gamma)$ to $C^*\{A_i, i = 1,2,\cdots,2k\}$ and $C^*\{B_i, i = 1,2,\cdots,2k\}$ respectively such that $\pi_1(\lambda_{a_i}) = A_i, \pi_2(\lambda_{a_i}) = B_i \forall i.$

Then $\mathbb{Q}(\Gamma)$ is doubling of the group algebra (i.e. $\mathbb{Q}(\Gamma) \cong D(C^*(\Gamma))$) corresponding to the given automorphism $\theta$. Moreover the fundamental unitary takes the following form

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 & \cdots & 0 & B_1 \\
0 & A_2 & 0 & 0 & \cdots & B_2 & 0 \\
0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & A_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & B_{2k-1} & 0 & 0 & \cdots & A_{2k-1} & 0 \\
B_{2k} & 0 & 0 & 0 & \cdots & 0 & A_{2k}
\end{pmatrix}.
\]

The proof is given in [15]. Now we give a sufficient condition for $\mathbb{Q}(\Gamma)$ to be $D(D(C^*(\Gamma))).$

**Proposition 2.10** Let $\Gamma$ be a group with $k$ generators \{a_1, a_2, \cdots a_k\} (say) and $\theta_1, \theta_2, \theta_3$ be the automorphisms of order 2 of the group algebra which gives the permutations $\sigma_1, \sigma_2, \sigma_3$ respectively on the set \{1,2,\cdots,2k-1,2k\}. Now assume the following:

1. $B_i^{(s)} := U_{i,\sigma_i(i)} \neq 0 \forall i,$ and $s = 1,2,3$ also $U_{i,j} = 0 \forall j \notin \{\sigma_i(i),i\},$
2. $A_iB_j^{(s)} = B_j^{(s)}A_i = 0, \forall j$ such that $\sigma_s(j) \neq j,$ where $A_i = U_{i,i},$
3. All $U_{i,j}U_{i,j}^*$ are central projections,
There are well defined $C^*$ isomorphisms $\pi_1, \pi_2^{(s)}$ from $C^*(\Gamma)$ to $C^*\{A_i, i = 1, 2, \cdots, 2k\}$ and $C^*\{B_i^{(s)}, i = 1, 2, \cdots, 2k\}$ respectively where $s=1, 2, 3$ such that

$$\pi_1(\lambda_{a_i}) = A_i, \pi_2^{(s)}(\lambda_{a_i}) = B_i^{(s)} \forall i.$$ 

Then $Q(\Gamma)$ is doubling of $D(C^*(\Gamma))$ corresponding to the given automorphisms. Moreover the fundamental unitary takes the following form

$$\begin{pmatrix}
A_1 & B_1^{(1)} & 0 & 0 & \cdots & B_1^{(2)} & B_1^{(3)} \\
B_2^{(1)} & A_2 & 0 & 0 & \cdots & B_2^{(2)} & B_2^{(3)} \\
0 & 0 & A_3 & B_3^{(1)} & \cdots & 0 & 0 \\
0 & 0 & B_4^{(1)} & A_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_{2k-1}^{(2)} & B_{2k-1}^{(3)} & 0 & 0 & \cdots & A_{2k-1} & B_{2k-1}^{(3)} \\
B_{2k}^{(2)} & B_{2k}^{(3)} & 0 & 0 & \cdots & B_{2k} & A_{2k} 
\end{pmatrix}.$$

The proof is very similar to the Proposition 2.9 thus omitted. We end the discussion of Section 2 with the following easy observation.

**Proposition 2.11** If $UV = 0$ for two normal elements in a $C^*$ algebra then

$$U^*V = VU^* = 0,$$

$$V^*U = UV^* = VU = 0.$$ 

The proof is very straightforward, hence omitted. This will be helpful in later sections.

## 3 QISO computation for the braid group

In this section we will compute the quantum isometry group of the braid group with 3 generators. The group has the presentation

$$\Gamma = \langle a, b, c| ac = ca, aba = bab, cbc = bcb >.$$ 

Here $S = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$.

**Theorem 3.1** Let $\Gamma$ be the braid group with above presentation. Then $Q(\Gamma, S)$ will be $D(D(C^*(\Gamma)))$, this means as a $C^*$ algebra it is same as four direct copies of group $C^*$ algebra.

**Proof:**

The action is defined as,

$$\alpha(\lambda_a) = \lambda_a \otimes A + \lambda_{a^{-1}} \otimes B + \lambda_b \otimes C + \lambda_{b^{-1}} \otimes D + \lambda_c \otimes E + \lambda_{c^{-1}} \otimes F,$$
$\alpha(\lambda_a^{-1}) = \lambda_a \otimes B^* + \lambda_{a-1} \otimes A^* + \lambda_b \otimes D^* + \lambda_{b-1} \otimes C^* + \lambda_c \otimes F^* + \lambda_{c-1} \otimes E^*$,
$\alpha(\lambda_b) = \lambda_a \otimes G + \lambda_{a-1} \otimes H + \lambda_b \otimes I + \lambda_{b-1} \otimes J + \lambda_c \otimes K + \lambda_{c-1} \otimes L$,
$\alpha(\lambda_{b-1}) = \lambda_a \otimes H^* + \lambda_{a-1} \otimes G^* + \lambda_b \otimes J^* + \lambda_{b-1} \otimes I^* + \lambda_c \otimes L^* + \lambda_{c-1} \otimes K^*$,
$\alpha(\lambda_c) = \lambda_a \otimes M + \lambda_{a-1} \otimes N + \lambda_b \otimes O + \lambda_{b-1} \otimes P + \lambda_c \otimes Q + \lambda_{c-1} \otimes R$,
$\alpha(\lambda_{c-1}) = \lambda_a \otimes N^* + \lambda_{a-1} \otimes M^* + \lambda_b \otimes P^* + \lambda_{b-1} \otimes O^* + \lambda_c \otimes R^* + \lambda_{c-1} \otimes Q^*$.

Fundamental unitary is of the form
\[
\begin{pmatrix}
A & B & C & D & E & F \\
B^* & A^* & D^* & C^* & F^* & E^* \\
G & H & I & J & K & L \\
H^* & G^* & J^* & I^* & L^* & K^* \\
M & N & O & P & Q & R \\
N^* & M^* & P^* & Q^* & R^* & Q^*
\end{pmatrix}
\]

First we need some lemmas to prove the theorem.

**Lemma 3.2** All the entries of the above matrix are normal.

**Proof:**
First, using the condition $\alpha(\lambda_{ac}) = \alpha(\lambda_{ca})$ comparing the coefficients of $\lambda_a^2, \lambda_{a-2}$, $\lambda_b^2, \lambda_{b-2}, \lambda_c^2, \lambda_{c-2}$ on both sides we have,

$AM = MA, BN = NB, CO = OC, DP = PD, EQ = QE, FR = RF.$

Applying the antipode we get,

$AE = EA, BF = FB, GK = KG, HL = LH, MQ = QM, NR = RN.$

Similarly from the relation $\alpha(\lambda_{ac^{-1}}) = \alpha(\lambda_{c^{-1}a})$ following the same argument as above one can deduce,

$AF = FA, BE = EB, GL = LG, HK = KH, NQ = QN, MR = RM.$

Again we know, $AE^* + FB^* = 0$ by comparing the coefficient of $\lambda_{ac^{-1}}$ in the expression of $\alpha(\lambda_a)\alpha(\lambda_{a-1})$. This shows that $AE^*A^* = 0$ as $B^*A^* = 0$. Thus, $(AE)(AE)^* = AEE^*A^* = E(AE^*A^*) = 0$. Similarly, all the terms of the above are zero.

Further using the condition $\alpha(\lambda_a)\alpha(\lambda_{a-1}) = \alpha(\lambda_{a-1})\alpha(\lambda_a) = \lambda_c \otimes I_\mathbb{Q}$ one can deduce,

$AC^* = AD^* = CA^* = C^*A = DA^* = D^*A = 0,$
$A^*C = A^*D = BD^* = D^*B = BC^* = B^*C = C^*B = 0.$

Applying the antipode we have,

$AG^* = G^*A = AH = HA = BG = BH^* = H^*B = GB = 0.$
Similarly from $\alpha(\lambda_b)\alpha(\lambda_{b-1}) = \alpha(\lambda_{b-1})\alpha(\lambda_b) = \lambda_c \otimes 1_Q$ one obtains
\[
CJ = JC = CI^* = I^*C = C^*I = IC^* = J^*C^* = C^*J^* = 0,
\]
\[
DI = ID = DJ^* = J^*D = 0.
\]
Again using $\alpha(\lambda_c)\alpha(\lambda_{c-1}) = \alpha(\lambda_{c-1})\alpha(\lambda_c) = \lambda_c \otimes 1_Q$ we must have,
\[
EL = LE = EK^* = K^*E = 0,
\]
\[
FK = KF = FL^* = L^*F = 0.
\]
Moreover, using the relation $\alpha(\lambda_{aba}) = \alpha(\lambda_{bab})$ we obtain $\alpha(\lambda_{ab}) = \alpha(\lambda_{baba}^{-1})$.
From $\alpha(\lambda_{ab}) = \alpha(\lambda_{baba}^{-1})$ comparing the coefficients of $\lambda_{ab}$ and $\lambda_{b-2}$ on both sides we obtain $CI = DJ = 0$. Now applying the antipode we get $I^*G^* = JH = 0$. This implies $GI = JH = 0$. Again from $\alpha(\lambda_{ab}^{-1}) = \alpha(\lambda_{b-1a^{-1}ba})$ and applying previous argument we can deduce $CJ^* = DI^* = 0$. Applying antipode we get $GJ = IH = 0$. Now from unitarity condition we know $GG^* + HH^* + JI^* + JJ^* + KK^* + LL^* = 1$. This shows that $G^2G^* = G$ as we already get $GH = GI = GJ = GK = GL = 0$. Also we have $G^*G^2 = G$ in a similar way. Thus we can conclude $G$ is normal. Using the same argument as before we can show that $H, I, J, K, L$ are normal, i.e. all elements of 3rd row become normal. Using the antipode the normality of $C, D, O, P$ follows.
Now we are going to show $A, B, E, F, M, N, Q, R$ are normal too. Using $AA^* + BB^* + CC^* + DD^* + EE^* + FF^* = 1$ we can write,
\[
A = A(AA^* + BB^* + CC^* + DD^* + EE^* + FF^*) = A^2A^* + ACC^* + ADD^* \quad (\text{as } AB = AE = AF = 0)
\]
\[
= A^2A^* + (AC^*)C + (AD^*)D \quad (\text{as } C, D \text{ are normal})
\]
\[
= A^2A^* \quad (\text{as } AC^* = AD^* = 0).
\]
Similarly $A^*A^2 = A$, thus $A$ is normal. Following exactly a similar argument one can show the normality of the remaining elements. □

**Lemma 3.3** $C = D = G = H = K = L = O = P = 0$.

**Proof:**
From the relation $\alpha(\lambda_{ac}) = \alpha(\lambda_{ca})$ equating the coefficients of $\lambda_{ba}, \lambda_{ab}, \lambda_{ab-1}, \lambda_{b-1a}$ on both sides we get $AO = MC, CM = OA, AP = MD, CN = OB$. This implies that $CMM^* = OAM^* = 0, CNN^* = OBN^* = 0$ as $AM^* = BN^* = 0$. Similarly one can obtain $CQQ^* = CRR^* = 0$. Now using $(AA^* + BB^* + GG^* + HH^* + MM^* + NN^*) = 1$ we have,
\[
C = C(AA^* + BB^* + GG^* + HH^* + MM^* + NN^*)
\]
\[
= C(AA^* + BB^* + GG^* + HH^*) \quad (\text{as } CMM^* = CNN^* = 0)
\]
\[
= C(GG^* + HH^*) \quad (\text{as } CAA^* = CA^*A = 0, CBB^* = CB^*B = 0).
\]

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Further one can write,

\[ C = C(EE^* + FF^* + KK^* + LL^* + QQ^* + RR^*) \]
\[ = C(KK^* + LL^* + QQ^* + RR^*) \text{ (as } CE^* = CF^* = 0) \]
\[ = C(KK^* + LL^*) \text{ (as } CQQ^* = CRR^* = 0). \]

Using the above equations we get that

\[ C(KK^* + LL^*)(GG^* + HH^*) = C(GG^* + HH^*) = C = 0 \text{ (as } KG = KH = LG = LH = 0). \]

Then we have \( G = H = 0 \) by using antipode. Moreover, \( AO = MC = 0, AP = OB = BO = 0. \) This gives us \( O = (A^*A + B^*B + M^*M + N^*N))O = 0. \) Similarly, we get \( P = 0, K = L = 0. \)

Applying the above lemma, the fundamental unitary is reduced to the form

\[
\begin{pmatrix}
A & B & 0 & 0 & E & F \\
B^* & A^* & 0 & 0 & F^* & E^* \\
0 & 0 & I & J & 0 & 0 \\
0 & 0 & J^* & I^* & 0 & 0 \\
M & N & 0 & 0 & Q & R \\
N^* & M^* & 0 & 0 & R^* & Q^*
\end{pmatrix}
\]

**Lemma 3.4**

\[ AIA = IAI, BJB = JBJ, AQ = QA, \]
\[ QIQ = IQI, RJR = JRJ, BR = RB, \]
\[ AJ = BI = AR = BQ = IR = JQ = 0, \]
\[ EIE = IEI, FJF = JFJ, EM = ME, \]
\[ MIM = IMI, NIN = JNJ, FN = NF, \]
\[ EJ = FI = EN = FM = IN = JM = 0. \]

**Proof:**
First of all we deduce the following relations among the generators,

\[ aba = bab, a^{-1}b^{-1}a^{-1} = b^{-1}a^{-1}b^{-1}, ab^{-1}a^{-1} = b^{-1}a^{-1}b, \]
\[ a^{-1}ba = bab^{-1}, ba^{-1}b^{-1} = a^{-1}b^{-1}a, b^{-1}ab = aba^{-1}. \]

We also get same relations replacing \( a \) by \( c \). Using the condition \( \alpha(\lambda_{aba}) = \alpha(\lambda_{bab}) \) and comparing on both sides the coefficients of \( \lambda_{aba}, \lambda_{a^{-1}b^{-1}a^{-1}}, \lambda_{ab^{-1}a^{-1}}, \lambda_{a^{-1}ba}, \lambda_{ba^{-1}b^{-1}}, \lambda_{b^{-1}ab} \) one can get

\[ AIA = IAI, BJB = JBJ, AIB = JBI, \]
\[ BIA = IAJ, IBJ = BJA, JAI = AIB. \]
Moreover, comparing the coefficients of $\lambda_{ab^{-1}a}, \lambda_{a^{-1}ba^{-1}}, \lambda_{ba^{-1}b}, \lambda_{b^{-1}ab^{-1}}$ on both sides we have

$$AJA = BIB = IBI = JAJ = 0.$$  

Similarly, equating the coefficients of $\lambda_{cbe}, \lambda_{c^{-1}b^{-1}c^{-1}}, \lambda_{cb^{-1}c^{-1}}, \lambda_{c^{-1}bc}, \lambda_{bc^{-1}b^{-1}}, \lambda_{b^{-1}cb}$ we also find

$$EIE = IEI, FJF = JFJ, EJF = JFI, FIE = IEJ, IFJ = FJE, JEI = EIF.$$  

Furthermore, comparing the coefficients of $\lambda_{cb^{-1}c}, \lambda_{c^{-1}bc^{-1}}, \lambda_{bc^{-1}b}, \lambda_{b^{-1}cb^{-1}}$ on both sides we have

$$EJE = FIF = IFI = JEJ = 0.$$  

Now our aim is to show $JA = IB = 0$. We have $JAI^2 = AIB$ as $JAI = AIB$, this implies $JAI^2 = 0$ because of $IBI = 0$. This shows that $JAI = 0$ as we proved before $I^2I^* = I$. Thus we can deduce

$$JA = JA(II^* + JJ^*)$$  

$$= (JAI)I^* + (JAJ)J^* \text{ (as } JAI = JAJ = 0)$$  

$$= 0.$$  

Similarly, it follows that $IB = 0$. Now using Proposition 2.11 we get $JA = AJ = IB = BI = 0$. In a similar way one can prove that $EJ = JE = IF = FI = 0$, and $MJ = IN = IR = JQ = 0$ as well. Now

$$AR = A(IJ^* + JJ^*)R$$  

$$= (AI^*)(IR) + (AJ)(J^*R)$$  

$$= 0 \text{ (as } IR = AJ = 0).$$  

We get $BQ = EN = FM = 0$ applying similar arguments as above. The only remaining part of the lemma is to prove $AQ = QA, BR = RB, EM = ME, FN = NF$. Using Lemma 4.5 of [18] we can get the desired equality. □

Combining the above lemmas the desired algebra $C^*\{A, B, E, F, I, J, M, N, Q, R\}$ is isomorphic to $D(D(C^*(\Gamma))$ by Proposition 2.10. In this case the automorphisms are as follows:

$$\theta_1(a) = a^{-1}, \theta_1(b) = b^{-1}, \theta_1(c) = c^{-1},$$  

$$\theta_2(a) = c, \theta_2(b) = b, \theta_2(c) = a,$$  

$$\theta_3(a) = c^{-1}, \theta_3(b) = b^{-1}, \theta_3(c) = a^{-1}.$$  

This completes the proof. □

Remark 3.5 We can prove the obvious analogue of Theorem 3.1 for the braid group of 2 generators in a very similar way.
4 Alternative description of the Quantum groups $H_+^+(n, 0), K_+^n$ and computing the QISO of free copies of $\mathbb{Z}_4$

We recall the quantum groups $H_+^+(n, 0), K_+^n$ which are discussed in [6,7] and [8]. $K_+^n$ is the universal $C^*$ algebra generated by the unitary matrix $((u_{ij}))$ which is described in Subsection 2.2 subject to the conditions given below.

1. Each $u_{ij}$ is normal, partial isometry.
2. $u_{ij}u_{ik} = 0, u_{ij}^*u_{kj} = 0 \forall i, j, k.$

$H_+^+(n, 0)$ is the universal $C^*$ algebra satisfying the above conditions and moreover, $u_{ij}^* = u_{ij}^{-1}$. In this section we are giving another description of these objects in terms of free wreath product motivated from the fact $H_+^+(n, 0) \cong C^*(\mathbb{Z}_2) \ast_w C(S_n^+)$ (see in [7]). First of all, we compute the quantum isometry group of $n$ free copies of $\mathbb{Z}_4$. The group is presented as follows:

$$\Gamma = \langle a_1, a_2, \cdots a_n | a_i = 4 \forall i \rangle$$

Now the fundamental unitary is of the form

$$U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & 0 & \cdots & 0 & A_{21} & A_{22} & A_{23} & 0 & \cdots & 0 \\
A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{10}^* & A_{21}^* & A_{22}^* & A_{24}^* & A_{23}^* & \cdots & A_{20}^* \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
A_{21}^* & A_{22}^* & A_{23}^* & A_{24}^* & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
A_{n1}^* & A_{n2}^* & A_{n3}^* & A_{n4}^* & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.$$

First of all, our aim is to show

$$A_{i(2j-1)}A_{i(2j)} + A_{i(2j-1)}A_{i(2j)} = 0 \forall i, j.$$ 

Using the condition $a(\lambda_n) = (\lambda_n)^{-1}$ we obtain

$$A_{k(2j-1)} = (A_{k(2j-1)}^2 + A_{k(2j)}^2)A_{k(2j-1)} + A_{k(2j)}(A_{k1}A_{k2} + A_{k2}A_{k1} + \cdots + A_{k(n-1)}A_{k(n)} + A_{k(n-1)}A_{k(n)}),$$

Moreover, considering the condition $a(\lambda_n^k) = a(\lambda_n^k)^{-1}$ we can get

$$A_{k(2j-1)}A_{k(2j)}A_{k(2j)} = 0 \forall i \neq j.$$ 

$$A_{k(2j-1)}A_{k(2j)} = 0 \forall i \neq j.$$ 

$$A_{k(2j-1)}A_{k(2j)} = 0 \forall j.$$ 

Thus, using the above relations we can find

$$A_{k(2j-1)}A_{k(2j)} = (A_{k(2j-1)}^2 + A_{k(2j)}^2)A_{k(2j-1)}A_{k(2j)}.$$

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\[ A^*_k(2j) A^*_k(2j-1) = (A^2_k(2j-1) + A^2_k(2j)) A_k(2j) A^*_k(2j-1). \]

Now adding these two equations we get
\[ A^*_k(2j-1) A^*_k(2j) + A^*_k(2j) A^*_k(2j-1) = (A^2_k(2j-1) + A^2_k(2j))(A_k(2j-1) A^*_k(2j)) + A_k(2j) A^*_k(2j-1) = 0. \]
Taking the adjoint we have \( A_k(2j-1) A_k(2j) + A_k(2j) A_k(2j-1) = 0 \). For any \( k \) we can prove this result, which means \( A_i(2j-1) A_i(2j) + A_i(2j) A_i(2j-1) = 0 \) \( \forall i, j \).

Using this we have
\[ A^*_i(2j-1) = (A^2_i(2j-1) + A^2_i(2j)) A_i(2j-1), \]
\[ A^*_i(2j) = (A^2_i(2j-1) + A^2_i(2j)) A_i(2j-1). \]
From \( A_i(2j-1) A_i(2j) + A_i(2j) A_i(2j-1) = 0 \) \( \forall i, j \), one can easily get \( A^2_i(2j-1) A_i(2j) = A^2_i(2j) A_i(2j-1) \). Thus, we can also conclude all \( A_{ij} \)'s are normal using the above equations.

Now consider the transpose of the above matrix, we denote the entries \((u_{ij})\), then from the co-associativity condition we can easily deduce the co-product given by \( \Delta(u_{ij}) = \sum_{k=1}^{2n} u_{ik} \otimes u_{kj}. \)

Let us define
\[ U_i = \sum_{j=1}^{n} A_{ji}, \]
\[ U_{i+1} = \sum_{j=1}^{n} A_{j(i+1)}, \]
for \( i = 1, 3, 5, \ldots, (2n - 1) \). Consider the algebras \( C^*\{U_i, U_{i+1}\} \) \( \forall i \), it can be easily verified that \( C^*\{U_i, U_{i+1}\} \equiv Q(\mathbb{Z}_4) \) by using the above relations. This means \( U_i \) and \( U_{i+1} \) satisfy the following relations:
\[ U_i U_i^* + U_{i+1} U_{i+1}^* = 1, \]
\[ U_i U_{i+1} + U_{i+1} U_i = 0, \forall i \]

\[ U_i^* = (U_i^2 + U_{i+1}^2) U_i, \quad U_{i+1}^* = (U_i^2 + U_{i+1}^2) U_{i+1}, \forall i \]
Moreover, the coproduct is given by
\[ \Delta(U_i) = U_i \otimes U_i + U_{i+1}^* \otimes U_i, \]
\[ \Delta(U_i) = U_i \otimes U_i + U_{i+1}^* \otimes U_i. \]

Our aim is to deduce
\[ C^*\{A_{ij} | i = 1, \cdots n, j = 1, \cdots (2n - 1), 2n\} \equiv Q(\mathbb{Z}_4) \ast_w C(S_n^+). \]

Let define
\[ t_{ij} = A_j(2i-1) A^*_j(2i-1) + A_j(2i) A^*_j(2i), \]
\( t_{ij} \) satisfy all the conditions of the quantum permutation groups. Furthermore, \( \Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}. \), Now it can be easily checked that
\[ Q(\mathbb{Z}_4 \ast \mathbb{Z}_4 \cdots \ast \mathbb{Z}_4) \equiv Q(\mathbb{Z}_4) \ast_w C(S_n^+). \]
Remark 4.1 The quantum groups \( H_s^+(n,0), K_n^+ \) can be described also such way. For finite \( s > 2 \)

\[
H_s^+(n,0) \cong [C^*(\mathbb{Z}_s) \oplus C^*(\mathbb{Z}_s)] *_{w} C(S_n^+),
\]

and

\[
K_n^+ \cong [C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z})] *_{w} C(S_n^+),
\]

where \([C^*(\mathbb{Z}_s) \oplus C^*(\mathbb{Z}_s)], [C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z})]\) admit the quantum group structure as in [11]. The above facts will follow by essentially the same arguments of the previous theorem.

Corollary 4.2 Using the above theorem, remark and the result of [7] we can conclude that for every finite \( s \)

\[
Q(\mathbb{Z}_s * \mathbb{Z}_s * \cdots * \mathbb{Z}_s) \cong Q(\mathbb{Z}_s) *_{w} C(S_n^+).
\]

Remark 4.3 If we consider \( \Gamma = \mathbb{Z}_n * \mathbb{Z}_n \) where \( n \) is finite, then \( Q(\Gamma) \) is doubling of the quantum group \( Q(\mathbb{Z}_n) * Q(\mathbb{Z}_n) \). This means

\[
Q(\Gamma) \cong (Q(\mathbb{Z}_n) * Q(\mathbb{Z}_n)) \oplus (Q(\mathbb{Z}_n) * Q(\mathbb{Z}_n))
\]

as a \( C^* \) algebra. In particular for \( n = 2 \), \( Q(\Gamma) \) becomes doubling of the group algebra as \( Q(\mathbb{Z}_n) \cong C^*(\mathbb{Z}_2) \) and \( C^*(\mathbb{Z}_2) * C^*(\mathbb{Z}_2) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \).

5 Examples of \((\Gamma, S)\) for which \( Q(\Gamma) \cong D(C^*(\Gamma)) \)

We already observed in [20] that, if there exists a non trivial automorphism of order 2 which preserves the generating set, then \( D(C^*(\Gamma)) \) ([20],[16]) will be always a quantum subgroup of \( Q(\Gamma) \). In ([11],[16],[21]) the authors could show that \( Q(\Gamma) \) coincides with doubled group algebra for some examples. In Section 5 of [18] together with Goswami we also present few examples of groups where this happens. Our aim in this section is to give few more examples of such groups.

5.1 \( \mathbb{Z}_9 * \mathbb{Z}_3 \)

The above group is presented by \( \Gamma = \langle h, g | o(g) = 9, o(h) = 3, h^{-1}gh = g^4 \rangle \).

Using Lemma 5.3 of [18] its fundamental unitary is of the form

\[
\begin{pmatrix}
A & B & 0 & 0 \\
B^* & A^* & 0 & 0 \\
0 & 0 & G & H \\
0 & 0 & H^* & G^*
\end{pmatrix}.
\]
Now the action is defined as,

\[ \alpha(\lambda h) = \lambda h \otimes A + \lambda_{h^{-1}} \otimes B, \]
\[ \alpha(\lambda_{h^{-1}}) = \lambda h \otimes B^* + \lambda_{h^{-1}} \otimes A^*, \]
\[ \alpha(\lambda g) = \lambda g \otimes G + \lambda_{g^{-1}} \otimes H, \]
\[ \alpha(\lambda_{g^{-1}}) = \lambda g \otimes H^* + \lambda_{g^{-1}} \otimes G^*. \]

First we are going to show that \( B = 0 \).

We have \( \alpha(\lambda h g) = \alpha(\lambda h g^4) \), \( \alpha(\lambda g^4) = \lambda g^4 \otimes G^4 + \lambda_{g^{-1}} \otimes H^4 \) as \( GH = HG = 0 \).

Equating all the terms of \( \alpha(\lambda h g) = \alpha(\lambda h g^4) \) on both sides we deduce,

\[ GA = AG^4, HA = AH^4, GB = HB = BG^4 = BH^4 = 0. \]

Thus, \( B = (G^*G + H^*H)B = 0 \) as \( (G^*G + H^*H) = 1, GB = HB = 0 \).

This gives the following reduction:

\[
\begin{pmatrix}
A & 0 & 0 & 0 \\
0 & A^* & 0 & 0 \\
0 & 0 & G & H \\
0 & 0 & H^* & G^*
\end{pmatrix}.
\]

Moreover, using the relations between the generators one can find

\[ A^*GA = G^4, A^*HA = H^4, A^*G = G^4A^*, A^*H = H^4A^*. \]

Now using the above relations we can easily show that \( G^*G, H^*H \) are central projections of the desired algebra, hence \( \mathbb{Q}(\Gamma, S) \) is isomorphic to \( D(C^*(\Gamma)) \) by proposition 2.9 corresponding to the automorphism \( g \mapsto g^{-1}, h \mapsto h \).

\[ \blacksquare \]

### 5.2 \((\mathbb{Z}_2 \ast \mathbb{Z}_2) \times \mathbb{Z}_2\)

The group is presented as \( \Gamma = \langle a, b, c | ba = ab, bc = cb, a^2 = b^2 = c^2 = e \rangle \).

Here \( S = \{a, b, c\} \). The action is given by,

\[ \alpha(\lambda a) = \lambda a \otimes A + \lambda b \otimes B + \lambda c \otimes C, \]
\[ \alpha(\lambda b) = \lambda a \otimes D + \lambda b \otimes E + \lambda c \otimes F, \]
\[ \alpha(\lambda c) = \lambda a \otimes G + \lambda b \otimes H + \lambda c \otimes K. \]

Write the fundamental unitary as

\[
\begin{pmatrix}
A & B & C \\
D & E & F \\
G & H & K
\end{pmatrix}.
\]

Our aim is to show \( D = B = F = H = 0 \).
Applying $\alpha(\lambda_{a^2}) = \lambda_c \otimes 1_\mathbb{Q}$ comparing the coefficients of $\lambda_{ac}, \lambda_{ca}$ on both sides we have $AC = CA = 0$. Using the antipode one can get $AG = GA = 0$.

Applying the same process with $b, c$ we can deduce,

$$DF = FD = BH = HB = 0,$$

$$GK = KG = CK = KC = 0.$$

Further, using the condition $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ comparing the coefficients of $\lambda_{ac}, \lambda_{ca}$ on both sides we can get $AF = DC, CD = FA$. Applying $\kappa$ we have $HA = GB, AH = BG$. Proceeding the same argument with $\alpha(\lambda_{cb}) = \alpha(\lambda_{bc})$ one can find,

$$DK = GF, KD = FG,$$

$$KB = HC, BK = CH.$$

Again we have, $GH + HG = 0$ from $\alpha(\lambda_{a^2}) = \lambda_c \otimes 1_\mathbb{Q}$ comparing the coefficient of $\lambda_{ab}$ on both sides. Now $AHG = BG^2$ as we know $AH = BG$. Further we have $-AGH = BG^2$ as $GH = -HG$. Thus we get $BG^2 = 0$ as $AG = 0$. Similarly it can be shown that $BK^2 = 0$, obviously $BH^2 = 0$ as $BH = 0$.

Hence, $B = B(G^2 + H^2 + K^2) = 0$ as $(G^2 + H^2 + K^2) = 1$. This gives $D = 0$ by using antipode.

Now $HA = 0, HC = 0$ as we get before $HA = GB, HC = KB$. This tells us $H = H(A^2 + C^2) = 0, and applying the antipode $F = 0$.

Thus the fundamental unitary is reduced to the form

$$\begin{pmatrix}
A & 0 & C \\
0 & E & 0 \\
G & 0 & K
\end{pmatrix}.$$

It now follows from Proposition 2.9 that $\mathbb{Q}(\Gamma) \cong D(C^*(\Gamma))$ with respect to the automorphism $a \mapsto c, c \mapsto a, b \mapsto b$. □

**Remark 5.1** The above quantum group can be described also another way, it is precisely $C^*\{A, C, E, G, K\} \cong \mathbb{Q}(\mathbb{Z}_2 \ast \mathbb{Z}_2) \otimes \mathbb{Q}(\mathbb{Z}_2)$.

### 5.3 Lamplighter Group

The group is presented as $\Gamma = \langle a, t | a^2 = [t^m a t^{-m}, t^n a t^{-n}] = e \rangle$ where $m, n \in \mathbb{Z}$.

Fundamental unitary is of the form

$$\begin{pmatrix}
A & B & C \\
D & E & F \\
D^* & F^* & E^*
\end{pmatrix}.$$

Now the aim is to show $B = C = D = 0$.

Using the condition $\alpha(\lambda_{a^2}) = \alpha(\lambda_e) = \lambda_c \otimes 1_\mathbb{Q}$ we deduce $D^2 = 0$, this implies $B^2 = C^2 = 0$ applying antipode. Further, we know $DD^* + EE^* + FF^* = 1,$
this gives us \( DEE^* + DFF^* = D \) as \( D^2 = 0 \). If we can show \( DE = DF = 0 \) then we will be able to prove our first claim i.e., \( D = 0 \).

Using group relations we deduce \( t^{(m-n)}a t^{-(m-n)}a = a t^{(m-n)}at^{-(m-n)} \) [where \( m, n \in \mathbb{Z} \)]. In particular, \( t^{-1}at = at^{-1}, tat^{-1}a = atat^{-1} \), this gives us \( at = tat^{-1}ata \). Now using the condition \( \alpha(\lambda at) = \alpha(\lambda tat^{-1}ata) \) comparing the coefficient of \( \lambda^2 \) on both sides we have \( BE = 0 \) because there are no terms with coefficient \( \lambda^2 \) on the right hand side as \( D^2 = BF = BE^* = FB = E^*B = 0 \). Applying the antipode one can get \( DE = 0 \). Similarly using the relation \( \alpha(\lambda at^{-1}) = \alpha(\lambda^{-1}atat^{-1}a) \) following the same argument we deduce \( BF^* = 0, DF = 0 \). Hence, we find that \( D = 0 \). This gives us \( B = C = 0 \) using the antipode. Thus, the fundamental unitary is reduced to the form

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & E & F \\
0 & F^* & E^*
\end{pmatrix}.
\]

From the relation of the group one can easily get,

\[
AE = EAE^* AEA, E^* A = AE^* AEA^*, AE^* = E^* AEA^* A.
\]

Thus we have,

\[
AEE^* = EAE^* AEA^* = E(AE^* AEA^*) = EE^* A,
\]

hence \( EE^* \) is a central projection. Similarly, \( FF^* \) is a central projection. Now we can define the map from \( C^*\{A, E, F\} \) to \( C^*(\Gamma) \oplus C^*(\Gamma) \) such as \( A \mapsto (\lambda_a \oplus \lambda_a), E \mapsto (\lambda_t \oplus 0), F \mapsto (0 \oplus \lambda_t^{-1}) \). This gives the isomorphism between these two algebras, which is also a CQG isomorphism and by Proposition \ref{prop:iso} corresponding to the automorphism \( a \mapsto a, t \mapsto t^{-1} \) we can conclude that \( Q(\Gamma) \cong D(C^*(\Gamma)) \). \( \square \)

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