Chiral Symmetry Restoration in the Instanton Liquid at Finite Density

Ralf Rapp *
Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794-3800, U.S.A.

The properties of the QCD partition function at finite chemical potential are studied within the instanton liquid model. It is shown that the density dependence of the quark-induced instanton-antiinstanton ($I-A$) interaction leads to the formation of topologically neutral $I-A$ pairs ('molecules'), resulting in a first order chiral phase transition at a critical chemical potential $\mu_q^c \simeq 310$ MeV. At somewhat higher densities ($\mu_q \geq 360$ MeV), the quark Fermi surface becomes unstable with respect to diquark condensation (Cooper pairs) generating BCS-type energy gaps of order 50 MeV.

1. INTRODUCTION

The investigation of the phase diagram of QCD is one of the central issues in understanding the properties of strong interactions. While the finite temperature axis has been theoretically explored in quite some detail, much less is known about the finite density ($\mu_q$-) axis, which is partly due to the fact that first principle QCD lattice calculations at finite $\mu_q$ encounter the problem of a complex fermionic determinant when integrating the QCD partition function. The phase structure of QCD at finite density, however, might be very rich, including new forms of condensates (other than the ordinary chiral condensate present in the QCD vacuum), different orders of phase transitions in the $\mu_q-T$-plane (entailing a tricritical point [1]), etc.. In this contribution, however, we will focus on the $T=0$, $\mu_q>0$- axis and try to examine the nature of the chiral phase transition in this regime. We will do so within the framework of the instanton liquid model (ILM) of QCD, which, although lacking explicit confinement, yields a very successful phenomenology of the QCD vacuum structure, and the low-lying hadron spectrum. Recently it also provided an interesting mechanism for chiral symmetry restoration at finite temperature, based on the rearrangement of the (anti-) instantons within the liquid, rather than on a mere disappearance of them as suggested earlier. Our objective here is to investigate whether a similar mechanism could be responsible for the restoration of chiral symmetry at finite density as well.

We start by briefly recalling some features of the instanton liquid model at zero density (sect. 2) and then turn to the finite density case (sect. 3), where we first calculate the $\mu_q$-dependence of the instanton-antiinstanton interaction and then assess its impact on the chiral phase transition. Under certain approximations we will be able to avoid a complex

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partition function and estimate the critical chemical potential within a mean-field type approach. We furthermore include some correlations between quarks at the Fermi surface and show how the recently discussed mechanism of color superconductivity \[2,3\] figures in our mean-field description.

2. THE INSTANTON LIQUID MODEL AT ZERO DENSITY

The instanton model in QCD is based on the assumption that the vacuum is dominated by non-perturbative gauge field configurations which constitute semiclassical ('instanton') solutions of the Yang-Mills equations. The QCD partition function in the instanton approximation then becomes

$$Z_{\text{QCD}}^{\text{inst}} = \sum_{N_+,N_-} \frac{1}{N_+!N_-!} \prod_{I=1}^{N_+} \int d\Omega_I n(\rho_I) e^{-S_{\text{int}}} \rho_I^{N_I} \prod_{f=1}^{N_f} \det(i \not{D} + im_f),$$

where the path-integral over all possible field configurations has been converted into an integration over the so-called collective coordinates $\Omega_I = \{z_I, \rho_I, u_I\}$ (position, size and color orientation) of $N_+$ instantons and $N_-$ antiinstantons. The single-instanton amplitude $n(\rho_I)$ (including quantum corrections) and the gluonic part of the instanton interaction $S_{\text{int}}$ determine the total instanton density in the absence of fermions to be $N/V = 2 \int d\rho n(\rho) e^{-S_{\text{int}}}$. The determinant of the Dirac operator, arising from the integration over the quark fields, is approximately calculated by keeping only the lowest lying modes, which should be reasonable for assessing the bulk properties of the system. The Dirac equation in the (anti-) instanton field possesses a (right-) left-handed zero energy solution. In the basis spanned by these zero modes, $\Psi^\dagger_0, I$ and $\Psi^\dagger_0, A$, and neglecting small current quark masses ($m_f \to 0$), the fermionic determinant reads

$$\det(i \not{D}) \simeq \det \begin{pmatrix} 0 & T_{IA} \\ T_{AI} & 0 \end{pmatrix} = |T_{IA}|^2,$$

where the overlap matrix element

$$T_{IA}(z, u) = \int d^4x \, \Psi^\dagger_{0,I}(x - z_I, u_I) (i \not{D}) \Psi^\dagger_{0,A}(x - z_A, u_A) \equiv i u \cdot \hat{z} f(z)$$

is linear in the relative $SU(3)$-color orientation characterized by a complex four vector $u_\mu$ ($z = z_A - z_I$ denotes the relative distance between $I$ and $A$). Lorentz invariance then implies that $T_{IA}$ is determined by a single scalar function $f(\sqrt{z^2})$.

With the key parameters taken as $N/V \simeq (1--1.4) \text{ fm}^{-4}$ and $\rho_I = \rho_A \simeq 1/3 \text{ fm}$ (which have also been confirmed in lattice calculations) a successful phenomenology of the QCD vacuum and the low-lying hadronic spectrum can be obtained \[4\].

3. THE INSTANTON LIQUID MODEL AT NON-ZERO DENSITY

3.1. $I$-$A$ Interaction at Finite $\mu_q$: Quark Zero Modes and $T_{IA}$

To study medium modifications of the instanton ensemble we first have to construct their interactions at finite quark chemical potential. We start from the finite density Dirac equation (in euclidean space), which still has zero mode solutions satisfying

$$(i \not{D}_I - i \mu_q \gamma_4) \Psi_{0,I} = 0.$$
Constraining ourselves to zero temperature, the gluonic (instanton-) fields entering the covariant derivative are not affected by $\mu_q$. The explicit form of the quark zero modes has been determined in ref. [5],

$$
\Psi_{0,I}(|\vec{x}|, x_4; \mu_q) = \frac{1 + \gamma_5}{2} \rho \frac{e^{\mu_q x_4}}{\Pi^I(x)} \left( \partial - \frac{\partial \Pi(x)}{\Pi(x)} \right) \cos(\mu_q |\vec{x}|) + \frac{x_4}{x^2} \sin(\mu_q |\vec{x}|) e^{-\mu_q x_4} \chi
$$

with the function $\Pi(x) = (1 + \rho^2/x^2)$ and a spin-color spinor $\chi$. Note that the solution of the adjoint Dirac equation,

$$
\Psi^\dagger_{0,I}(x; -\mu_q) \left( i \not{D} - i \mu_4 \gamma_4 \right) = 0 \ ,
$$

(6)

carries the chemical potential argument with opposite sign. This is necessary for a consistent definition of expectation values at finite $\mu_q$, and in particular renders a finite norm

$$
\int d^4x \, \Psi^\dagger_{0,I}(x; -\mu_q) \Psi_{0,I}(x; \mu_q) = 1 \ .
$$

(7)

With the properly constructed quark wave functions we can now evaluate the fermionic overlap matrix element representing the zero mode part of the full Dirac operator. Choosing for simplicity the sum ansatz for the gauge-field configurations, $A_\mu = A^I_\mu + A^A_\mu$, entering the covariant derivative, allows us to replace the latter by an ordinary one, yielding

$$
T_{IA}(z, u; \mu_q) = -\int d^4x \, \Psi^\dagger_{0,I}(x - z_I; -\mu_q) \left( i \not{\partial} - i \mu_4 \gamma_4 \right) \Psi_{0,A}(x - z_A; \mu_q)
$$

$$
\equiv i \, u_4 \, f_1(\tau, r; \mu_q) + i \, \left( \vec{u} \cdot \vec{r} \right) f_2(\tau, r; \mu_q) \ .
$$

(8)

The breaking of Lorentz invariance in the medium implies the existence of two independent (real-valued) functions $f_1, f_2$ which are shown in fig. 1. Similar to what has been found at finite temperature [3], $T_{IA}$ is strongly enhanced in temporal direction with increasing $\mu_q$, but oscillates as $\sim \sin(2\mu_q r)$ in spatial direction [7]. The latter effectively suppresses the interaction once integrating over $r$.

### 3.2. Thermodynamics of the ILM at Finite $\mu_q$

To investigate the finite density properties of the thermodynamic potential (or free energy) we here resort to the so-called cocktail model introduced in ref. [8]. It amounts to a mean-field type description including three major components in the system: essentially random (anti-) instantons (the 'atomic' component), strongly correlated $I$-$A$-pairs (the 'molecular' component) and a Fermi sphere of constituent quarks ('quasiparticles'). Thus

$$
\Omega(\mu_q) = \Omega_{\text{inst}}(\mu_q) + \Omega_{\text{quark}}^{QP}(\mu_q) \ ,
$$

(9)

where the constituent quark contribution

$$
\Omega_{\text{quark}}^{QP}(m_q; \mu_q) = \epsilon_q(m_q; \mu_q) - \mu_q \, n_q(m_q; \mu_q)
$$

(10)

is zero for Fermi energies $\mu_q < m_q$, with $m_q$ denoting the constituent quark mass. The instanton part of the free energy,

$$
\Omega_{\text{inst}}(n_a, n_m; \mu_q) = -\ln\left[ Z_{\text{inst}}(n_a, n_m; \mu_q) \right] / V_4 = -n_a \ln \left[ \frac{\epsilon z_a}{n_a} \right] - n_m \ln \left[ \frac{\epsilon z_m}{n_m} \right] \ ,
$$

(11)
Figure 1. Quark-induced $I-A$-interaction at finite density for the most attractive color orientation $u_4=1$, $\vec{u}=0$ as well as $r=0$ (left panel) and for $u_4=0$, $|\vec{u}|=1$ and $\tau=0$ (right panel).

is related to the atomic and molecular ’activities’

$$z_a = 2C\,\rho^{b-4}\,e^{-S_{int}}\,\langle T_{IA}(\mu_q)T_{AI}(\mu_q)\rangle^{N_f/2}$$

$$z_m = C^2\,\rho^{2(b-4)}\,e^{-2S_{int}}\,\langle [T_{IA}(\mu_q)T_{AI}(\mu_q)]^{2N_f}\rangle.$$  \hspace{1cm} (12)

The minimization of $\Omega$ with respect to the corresponding (4-dimensional) densities $n_a$ and $n_m$ determines the equilibrium state of the system at fixed $\mu_q$. In particular, $\Omega_{inst}$ encodes all the features of the $T=\mu_q=0$ instanton ensemble, as e.g. the quark condensate and the constituent quark mass, which in mean-field approximation are given by $\langle \bar{q}q \rangle = -1/(\pi \rho)(3/2n_a)^{1/2}$ and $m_q \propto -\rho^2 \langle \bar{q}q \rangle$. Therefore the normalization constant $C \propto (\Lambda_{QCD})^b$ can be fixed to give $N/V=1.4$ fm$^{-4}$, being realized for $n_a=1.34$ fm$^{-4}$ and $n_m=0.03$ fm$^{-4}$, which is not unreasonable. The gluonic interaction has been approximated by an average repulsion $S_{int} = -\kappa \rho^4 (n_a + 2n_m)$.

At finite $\mu_q$ the Dirac operator is not hermitian any more, i.e. $T_{AI}(\mu_q) \neq T_{AI}^\dagger(\mu_q)$, resulting, in general, in a complex fermionic determinant, entailing the well-known ’sign’ problem. However, assuming an average gluonic interaction that does not depend on density allows us to perform the color averages implied in eqs. (12) analytically, i.e.

$$z_a \propto \int du T_{IA}(\mu_q)T_{IA}^\dagger(-\mu_q) = \frac{1}{2N_c}[f_1^+ f_1^- + f_2^+ f_2^-]$$

$$z_m \propto \int du [T_{IA}(\mu_q)T_{IA}^\dagger(-\mu_q)]^{N_f} = \frac{[(2N_c-1)(f_1^+ f_1^- + f_2^+ f_2^-)^2 + \{f_1^+ f_2^- f_2^- f_1^+\}^2]}{4N_c(N_c^2-1)}$$  \hspace{1cm} (13)

$$(N_f = 2, f_i^\pm \equiv f_i(\pm \mu_q)), \text{which ensures the pressure to remain real. Fig. 2 shows our results as function of the quark chemical potential (left panel) for $N_c = 3, N_f = 2$. At small $\mu_q$ essentially nothing happens until, at a critical value $\mu_q^c \simeq 310$ MeV, the system...}$$
Figure 2. Our results for the instanton (‘atomic’) and molecule densities (upper left panel), the pressure \( p = -\Omega \) (middle left panel) and the constituent quark mass (lower left panel) after minimizing the free energy, eq. (9), with respect to \( n_a \) and \( n_m \); additionally including scalar diquark correlations (as discussed in sect. 3.3) adds the dashed-dotted line to the lower left panel, representing the BCS energy gap at the quark Fermi surface. The right panel shows the free energy as a function of constituent quark mass \( m_q \propto n_a^{1/2} \), indicating a first order transition from the minimum at finite \( m_q \) to the one at \( m_q = 0 \).

jumps into the chirally restored phase, the latter being characterized by \( n_a = 0 \). The transition is of first order, as can be seen by inspection of the \( m_q \)-dependence of the free energy (right panel of fig. 2). Below \( \mu_q^c \), the pressure actually decreases slightly with increasing \( \mu_q \) indicating a mixed phase-type instability, similar to what has been discussed in refs. \[2,9\]. A significant difference, however, is given by the fact that in our approach the total instanton density at the transition (residing in \( I-A \)-molecules) is still appreciable, \( N/V = 2n_m \simeq 1.1 \text{fm}^{-4} \), providing the major part of the pressure at this point; in other words: a substantial part of the nonperturbative vacuum pressure persists in the chirally restored phase (of course, eventually it will be suppressed due to the Debye screening of the instanton fields, which we have not included here).
3.3. Color Superconductivity

As has recently been pointed out in refs. \[2,3\], the quark Fermi surface in the plasma phase might be unstable with respect to the formation of quark-Cooper pairs, once an attractive $q$-$q$ interaction is present. Using the instanton-induced interaction in the scalar, color-antitriplet diquark channel (which is essentially the Fierzed-transformed interaction leading to a deeply bound pion and is also phenomenologically well-supported by baryonic spectroscopy), BCS-type energy gaps of $\Delta_0 \simeq 50-100$ MeV have been predicted.

In the mean-field approach employed here, this interaction modifies the quark quasiparticle contribution according to

$$\Omega_\Delta^\mu (\mu) = tr \log [D(m_q, \Delta_0)] - tr [D(m_q, \Delta_0)\Sigma(\Delta_0)] + Gtr [F(m_q, \Delta_0)] tr [\bar{F}(m_q, \Delta_0)]$$

with the quasiparticle quark-propagator $D$ and the anomalous (Gorkov) propagator $F$ corresponding to creation/annihilation of a bound Cooper pair. $G$ is the effective coupling constant derived from the instanton $q$-$q$-vertex \[3\]. From $\Omega_\Delta^\mu$ one can obtain the standard gap equation for $\Delta_0$. However, here we also have to account for the fact that a finite $\Delta_0$ will damp the quark zero-mode propagators in the instanton part of the free energy, which results in a suppression of the quark induced $I$-$A$ interaction $T_{IA}$ of eq. (8). Thus, $\Omega_{\text{inst}}$ disfavors finite $\Delta_0$’s. The resulting expression for the free energy is of the form

$$\Omega(\mu_q) = \Omega_{\text{inst}}(n_a, n_m, \Delta_0; \mu_q) + \frac{1}{N_c} \left[ 2\Omega_\Delta^\mu (m_q, \Delta_0; \mu_q) + (N_c - 2)\Omega_q (m_q; \mu_q) \right],$$

(15)

(the last term accounting for unpaired quarks), which now has to be minimized w.r.t. $n_a$, $n_m$ and $\Delta_0$. We find that color superconductivity does not appear before $\mu_q \simeq 360$ MeV (see dashed-dotted curve in the lower left panel of fig. 2): although the quark-part of the free energy by itself, eq. (14), always favors a finite value for $\Delta_0$, the suppression caused by $\Delta_0$ in $\Omega_{\text{inst}}$ (i.e. the damping of quark-propagation in $T_{IA}$) prevents the formation of a diquark condensate below $\mu_q \simeq 360$ MeV. This again reflects the fact that the instanton contribution to the pressure is still dominant in the region somewhat above $\mu^c_q$, so that the gain due to a finite $\Delta_0$ in $\Omega_\Delta^\mu$ cannot overcome the ‘penalty’ in $\Omega_{\text{inst}}$. Above $\mu_q \simeq 360$ MeV energy gaps of order $\sim 50$ MeV arise, in line with the findings of ref. \[3\].

As far as the thermodynamic properties of the system are concerned, no changes as compared to the previous section occur below $\mu_q = 360$ MeV, and even above the results for $n_a(\mu_q)$, $n_m(\mu_q)$, $p(\mu_q)$ and $m_q(\mu_q)$ would be hardly distinguishable from the curves displayed in fig. 2.

In summary, we have studied the QCD partition function at finite density using the instanton model. Employing a simple mean-field approach, chiral symmetry restoration emerges at $\mu^c_q = 310$ MeV as a first order transition from an essentially random (anti-/instanton liquid into a phase of strongly correlated $I$-$A$ molecules (and massless quarks), driven by the density dependent increase of the quark-induced $I$-$A$-interaction. Neglecting possible medium modifications in the gluonic interaction, which are expected to be small, a complex pressure could be avoided. Accounting for correlations in the scalar diquark channel, color superconductivity sets in for chemical potentials $\mu_q \geq 360$ MeV, associated with BCS gaps of up to $\sim 50$ MeV.
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