COMPUTABLE COPIES OF $\ell^p$

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Abstract. Suppose $p$ is a computable real so that $p \geq 1$. It is shown that the halting set can compute a surjective linear isometry between any two computable copies of $\ell^p$. It is also shown that this result is optimal in that when $p \neq 2$ there are two computable copies of $\ell^p$ with the property that any oracle that computes a linear isometry of one onto the other must also compute the halting set. These results hold in both the real and complex case.

1. Introduction

We start by considering the very general question “Given two computable and linearly isometric Banach spaces, how hard is it to compute a linear isometry from one onto the other?” (Roughly speaking, a Banach space is computable if there are algorithms that compute its scalar multiplication, vector addition, and norm.) We specialize this question to the case of spaces that are linearly isometric to spaces of the form $\ell^p$ where $p$ is a computable real (i.e. a real whose decimal expansion is computable). Our first result is that this is no harder than computing membership in the halting set (i.e. the set of all computer programs that halt on at least one input). Namely, we show that when $p$ is a computable real so that $p \geq 1$, the halting set is capable of computing a surjective linear isometry between any two computable copies of $\ell^p$. Our second result is that this problem is not easier than the halting set. Namely, when $p$ is a computable real so that $p \geq 1$ and $p \neq 2$, there are two computable copies of $\ell^p$ so that any oracle that computes a surjective linear isometry from one onto the other must also compute the halting set. It is already known that any two computable copies of $\ell^2$ are computably linearly isometric [12]. This is essentially due to the fact that $\ell^2$ is a Hilbert space and mirrors the classical fact that any two infinite-dimensional separable Hilbert spaces are linearly isometric [8].

The first of our two results is based on a sharpening of an inequality due to J. Lamperti which we prove in Section 4. In the main, our second result was previously shown for $p = 1$ by Pour-El and Richards [13]. Their proof rests on an observation about the extreme points of the closed unit ball of $\ell^1$ that does not generalize to $\ell^p$ when $p > 1$. The proof presented here uses the characterization of the linear isometries of $L^p$ spaces due to S. Banach and J. Lamperti [2], [6], [10].

Our results can be recast in the setting of computable categoricity. A mathematical structure is computably categorical if any two of its computable copies are isomorphic via a computable map. A structure is $\Delta^0_2$-categorical if the halting set can compute an isomorphism between any two of its computable copies [1]. Our

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results can be interpreted in the setting of computable categoricity as follows: when \( p \) is a computable real so that \( p \geq 1 \), \( \ell^p \) is \( \Delta^0_2 \)-categorical, and \( \ell^p \) is computably categorical if and only if \( p = 2 \).

While all these results are proven for the complex version of \( \ell^p \), they also hold for the real version of \( \ell^p \).

The paper is organized as follows. Background and preliminaries from functional analysis and the theory of computation are covered in Section 2. In Section 3 we give an overview of the strategy behind the proofs of the main results. The remainder of the work is then divided into three parts each corresponding to a different mathematical universe: the classical world (Section 4), wherein we have full access to all the concepts, results, and methods of classical mathematics, the approximate world (Section 5) wherein we can only see approximations of our classical objects, and the computable world (Section 6) wherein we can only see approximations of classical objects and can only access computable operations on these objects. Section 7 presents concluding remarks and questions for further investigation.

2. Background and preliminaries

Only a small amount of functional analysis and computability theory is necessary to prove the results herein. We therefore hope in this section to cover enough from these two areas to make the following sections accessible to scholars from both of these disciplines.

2.1. Background and preliminaries from functional analysis. Throughout this paper, it is assumed that all Banach spaces are Banach spaces over the field of complex numbers. Let \( \mathbb{C} \) denote the field of complex numbers.

We begin with some notation and terminology. Let \( \mathcal{B} = (V, +, \| \cdot \|) \) be a Banach space. By a subspace of \( \mathcal{B} \) we will always mean a linear subspace of \( \mathcal{B} \) that is topologically closed. When \( S \subseteq V \) and \( F \subseteq \mathbb{C} \), we let

\[
\mathcal{L}_F(S) = \{ \sum_{j=0}^{M} \alpha_j v_j : \alpha_0, \ldots, \alpha_M \in F \land v_0, \ldots, v_M \in S \}.
\]

We then let \( \mathcal{L}(S) = \mathcal{L}_\mathbb{C}(S) \). That is, \( \mathcal{L}(S) \) is the linear span of \( S \). The subspace generated by \( S \) is the closure of the linear span of \( S \). We say that \( G \subseteq V \) is a generating set for \( \mathcal{B} \) if it generates all of \( \mathcal{B} \); i.e. \( V = \overline{\mathcal{L}(G)} \). A set of vectors \( \mathcal{B} = \{ u_0, u_1, \ldots \} \) is a Schauder basis for \( \mathcal{B} \) if for each \( v \in \mathcal{B} \) there is a unique sequence of complex numbers \( a_0, a_1, \ldots \) so that \( v = \sum_{n=0}^{\infty} a_n u_n \).

A map between two Banach spaces is linear if it preserves scalar multiplication and vector addition; it is an isometry (or is isometric) if it preserves the metric induced by the norm; i.e. \( \| T(x) - T(y) \| = \| x - y \| \). Thus, every isometry is injective. An endomorphism of a Banach space is a linear map of the space onto itself.

When \( p \) is a positive number, \( \ell^p \) denotes the space of all sequences of complex numbers \( \{ a_n \}_{n=0}^{\infty} \) so that

\[
\sum_{n=0}^{\infty} |a_n|^p < \infty.
\]
$\ell^p$ is a vector space over $\mathbb{C}$ with the usual scalar multiplication and vector addition. When $p \geq 1$ it is a Banach space under the norm defined by

$$\|\{a_n\}_n\| = \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{1/p}.$$  

We regard the vectors in $\ell^p$ as functions from $\mathbb{N}$ into $\mathbb{C}$. It will be convenient on a few occasions to view $\ell^p$ as $L^p(\mu)$ where $\mu$ is the counting measure on $\mathbb{N}$.

Throughout the rest of this paper, we assume that $p \geq 1$.

When $f \in \ell^p$, the support of $f$ is the set of all $t \in \mathbb{N}$ so that $f(t) \neq 0$; we denote this set by supp$(f)$. If $f_0, f_1, \ldots$ are vectors so that supp$(f_m) \cap$ supp$(f_n) = \emptyset$ whenever $m \neq n$, then we say that $f_0, f_1, \ldots$ are disjointly supported.

When $S \subseteq \mathbb{N}$, let $\chi_S$ denote the characteristic function of $S$.

Let $e_j = \chi_{\{j\}}$. Thus, $E := \{e_0, e_1, \ldots\}$ is a generating set for $\ell^p$ which we refer to as the standard generating set for $\ell^p$.

Throughout the rest of this paper it is assumed that $F = \{f_0, f_1, \ldots\}$ denotes a generating set for $\ell^p$.

The proof of our second main result is based on the following.

**Theorem 2.1** (Banach-Lamperti). Suppose $p \neq 2$. If $T : \ell^p \to \ell^p$ is a linear isometry, then $T$ preserves the ‘disjoint support’ relation; that is supp$(T(f)) \cap$ supp$(T(g)) = \emptyset$ if supp$(f) \cap$ supp$(g) = \emptyset$. Furthermore, $T$ is an isometric endomorphism of $\ell^p$ if and only if there are unimodular constants $\lambda_0, \lambda_1, \ldots$ and a permutation of $\mathbb{N}$, $\phi$, so that $T(e_n) = \lambda_n e_{\phi(n)}$ for all $n$.

In his seminal text on linear operators, S. Banach stated Theorem 2.1 for the case of $\ell^p$ spaces over the reals $[2]$. He also stated a classification of the linear isometries of $L^p[0,1]$ in the real case. Banach’s proofs of these results were sketchy and did not easily generalize to the complex case. In 1958, J. Lamperti rigorously proved a generalization of Banach’s claims to real and complex $L^p$-spaces of sigma-finite measures $[10]$. Theorem 2.1 follows from J. Lamperti’s work as it appears in Theorem 3.2.5 of $[6]$. Note that Theorem 2.1 fails when $p = 2$. For, $\ell^2$ is a Hilbert space. So, if $\{f_0, f_1, \ldots\}$ is any orthonormal basis for $\ell^2$, then there is a unique surjective linear isometry of $\ell^2$, $T$, so that $T(e_n) = f_n$ for all $n$.

### 2.2. Background and preliminaries from computability and computable analysis

Computability theory is the mathematical theory of the limits and potentialities of discrete computing devices such as the digital computer. Computable analysis is the branch of this theory that investigates computation with continuous data such as the reals and therefore provides a rigorous foundation for scientific computation. Computable analysis is related to the field of constructive analysis in that a constructive proof of a theorem yields an algorithm for computing its solution operator; so classical theorems with incomputable solution operators can not be proven constructively.

We cover here the basic notions from computability theory and computable analysis necessary to understand the results herein. A more expansive treatment of these subjects can be found in $[4, 14, 16]$. Constructive analysis is explicated in $[3]$.

We start with classical computability theory over the natural numbers. Let $f : \subseteq A \to B$ denote that $f$ is a function whose domain is included in $A$ and whose range is included in $B$.
A function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ is **computable** if there is an algorithm that, given any $n \in \mathbb{N}$ as input, produces $f(n)$ as output if $n \in \text{dom}(f)$ and does not halt if $n \notin \text{dom}(f)$. It is well-known that there are only countably many computable functions. (Informally, since a mathematical algorithm can always be represented in a programming language, there are only countably many algorithms.) Thus, most functions on the natural numbers are incomputable.

A set $A \subseteq \mathbb{N}$ is **computable** if its characteristic function is computable.

A set of nonnegative integers is **computably enumerable** (abbreviated ‘c.e.’) if it is the domain of a computable function. When $A \subseteq \mathbb{N}$, an **effective enumeration** of $A$ is a computable sequence $\{a_n\}_{n \in \mathbb{N}}$ so that $A = \{a_0, a_1, \ldots\}$. It is well-known that every infinite c.e. set has a one-to-one effective enumeration.

A family $\{A_e\}_{e \in \mathbb{N}}$ of sets is **uniformly computably enumerable** if there is a computable function $\phi : \subseteq \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ so that $A_e = \{(e, n) : (e, n) \in \text{dom}(\phi)\}$ for all $e$.

A fundamental result of computability theory is that there is a computable function $U : \subseteq \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ with the property that for every computable $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$, there is a number $e \in \mathbb{N}$ such that $f(n) = U(e, n)$ for all $n \in \mathbb{N}$. We fix such a function $U$, and let $\phi_e$ denote $U(e, \cdot)$. If $\phi_e = f$, then we say that $e$ is an index of $f$. Set $W_e = \text{dom}(\phi_e)$. Then, a set $A$ is c.e. if and only if $A = W_e$ for some $e$ in which case we call $e$ an index of $A$.

The set

$$K := \{e : e \in W_e\}$$

is called the **halting set**. It is an example of a computably enumerable set that is not computable.

Fix $A, B \subseteq \mathbb{N}$. We say that $A$ is **Turing reducible** to $B$ if there is a so-called oracle algorithm that given any $n \in \mathbb{N}$ as input determines if $n \in A$ after asking finitely many questions of the form ‘$x \in B$?’ and receiving correct answers. In this case, we also say that $B$ **computes** $A$. When this happens, we view $B$ as having at least as much information as $A$. If $B$ computes $A$ but not vice-versa, then we view $B$ as having more information than $A$. Such an algorithm is also called a **Turing reduction**.

These notions can be extended to other countable domains by means of codings. A **coding** of a set $X$ is a surjection $c : \subseteq \mathbb{N} \rightarrow X$. If $c$ is a coding of $X$, and if $Y \subseteq X$, then $Y$ is computable (computably enumerable) with respect to $c$ if $c^{-1}[Y]$ is computable (computably enumerable). When $X_1$ and $X_2$ are sets for which we have established codings $c_1$ and $c_2$ respectively, and when $f : \subseteq X_1 \rightarrow X_2$, then we say that $f$ is computable with respect to $c_1$ and $c_2$ if there is a computable $F : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ such that $c_2(F(n)) = f(c_1(n))$ for all $n$. In practice, all natural codings of countable sets yield the same class of computable subsets and functions and so their mention is usually omitted.

A natural example of an incomputable set is given by Hilbert’s 10th problem: the set of all diophantine equations with at least one solution over the integers is incomputable [5].

We now turn to computability notions in the complex plane. We say that a point $z \in \mathbb{C}$ is **computable** if there is an algorithm that, given any $k \in \mathbb{N}$ as input, produces a rational point $q$ so that $|q - z| < 2^{-k}$. Just as with computable functions on the natural numbers, there are only countably many computable points, so most points in the plane are incomputable. A sequence of complex numbers $\{z_n\}_{n \in \mathbb{N}}$ is
computable} if there is an algorithm that, given any \( n,k \in \mathbb{N} \) as input, produces a rational point \( q \) so that \( |q - z_n| < 2^{-k} \).

We now describe computability over \( \ell^p \). Suppose \( p \) is a computable real. We say that \( F \) is an effective generating set for \( \ell^p \) if there is an algorithm that given any \( f \in \mathcal{L}_{Q(i)}(F) \) and a nonnegative integer \( k \) as input computes a rational number \( q \) so that \( \|f\| - q < 2^{-k} \); less formally, the map \( f \in \mathcal{L}_{Q(i)} \mapsto \|f\| \) is computable. Thus, the standard generating set is an effective generating set for \( \ell^p \).

Suppose \( F \) is an effective generating set for \( \ell^p \). We say that a vector \( g \in \ell^p \) is computable with respect to the standard generating set if and only if it is computable as a sequence in the sense just defined.

When \( f \in \ell^p \) and \( r > 0 \), let \( B(f;r) \) denote the open ball with center \( f \) and radius \( r \). When \( f \in \mathcal{L}_{Q(i)}(F) \) and \( r \) is a positive rational number, we call \( B(f;r) \) a rational ball (with respect to \( F \)).

Suppose \( F \) and \( G \) are effective generating sets for \( \ell^p \). We say that a map \( T : \ell^p \rightarrow \ell^p \) is computable with respect to \((F,G)\) if there is an algorithm \( P \) that meets the following three criteria.

- **Approximation**: Given as input a ball that is rational with respect to \( F \), \( P \) either does not halt or produces a ball that is rational with respect to \( G \).
- **Correctness**: If \( B_2 \) is the output of of \( P \) on input \( B_1 \), then \( T(f) \in B_2 \) whenever \( f \in B_1 \).
- **Convergence**: If \( U \) is a neighborhood of \( T(f) \), then \( f \) belongs to a rational ball \( B_1 \) with the property that \( P \) halts on \( B_1 \) and produces a rational ball that is included in \( U \).

When we speak of an algorithm accepting a rational ball \( B(\sum_{j=0}^{M} \alpha_j f_j;r) \) as input, we of course mean that it accepts some representation of the ball such as a code of the sequence \( (r,M,\alpha_0,\ldots,\alpha_M) \).

All of these definitions have natural relativizations. For example, if \( F \) is an effective generating set, then we say that \( X \) computes a vector \( g \in \ell^p \) with respect to \( F \) if there is a Turing reduction that given the oracle \( X \) and an input \( k \) computes \( f \in \mathcal{L}_{Q(i)}(F) \) so that \( \|g - f\| < 2^{-k} \).

By default, when we speak of computable vectors, sequences of vectors, or functions \( T : \ell^p \rightarrow \ell^p \), the computability is with respect to the standard generating set. Other generating sets can yield very different classes of computable vectors. For example, let \( \zeta \) be an incomputable unimodular point (i.e. a point of modulus 1). For each \( n \), let \( g_n = \zeta e_n \). Let \( G = \{g_0,g_1,\ldots\} \). Thus, \( G \) is an effective generating set. However, the vector \( \zeta e_0 \) is computable with respect to \( G \) even though it is not computable with respect to the standard generating set \( E \). On the other hand, note that there is an isometric endomorphism of \( \ell^p \) that is computable with respect to \((E,G)\); namely multiplication by \( \zeta \). Thus, \( E \) and \( G \) give the same computability theory on \( \ell^p \) even though they yield very different classes of computable vectors.

The definitions just given for \( \ell^p \) can easily be adapted to any separable Banach space. Suppose \( G = \{g_0,g_1,\ldots\} \) is an effective generating set for a Banach space \( \mathcal{B} \). The pair \((\mathcal{B},G)\) is called a computable Banach space. Suppose \( T \) is a linear
isometric mapping of \( B \) onto \( \ell^p \). Then, \( T[G] \) is an effective generating set for \( \ell^p \), and \( T \) is computable with respect to \( (G,T[G]) \). Thus, the study of computable Banach spaces that are linearly isometric to \( \ell^p \) spaces can be reduced to the study of computability notions on \( \ell^p \) spaces with respect to different generating sets. Accordingly, we formally state our two main results as follows.

**Theorem 2.2.** Suppose \( p \) is a computable real so that \( p \geq 1 \), and suppose \( F \) is an effective generating set for \( \ell^p \). Then, with respect to \( (E,F) \), the halting set computes an isometric endomorphism of \( \ell^p \).

**Theorem 2.3.** Suppose \( p \) is a computable real so that \( p \geq 1 \) and \( p \neq 2 \). Suppose \( C \) is a c.e. set. Then, there is an effective generating set for \( \ell^p \), \( F \), so that with respect to \( (E,F) \), \( C \) computes an isometric endomorphism of \( \ell^p \) and so that any oracle that computes an isometric endomorphism of \( \ell^p \) with respect to \( (E,F) \) must also compute \( C \).

So if we take \( C \) to be the halting set in Theorem 2.3, then it follows that the problem of computing isometric endomorphisms of \( \ell^p \) with respect to \( (E,F) \) is at least as hard as computing membership in the halting set.

We close this section by mentioning some related work. A.G. Melnikov and K.M. Ng investigated computable categoricity questions with regards to the space \( C[0,1] \) of continuous functions on the unit interval with the supremum norm \( [12,13] \). The study of computable categoricity for countable structures goes back at least as far as the work of S. Goncharov and has been a very fruitful and important direction in mathematical logic \( [7] \). The text of Ash and Knight has a thorough discussion of other directions in the countable computable structures program \( [9] \).

### 2.3. Preliminaries from combinatorics.

When \( A \) is a finite set, let \( |A| \) denote the cardinality of \( A \).

Let \( \mathbb{N}^* \) denote the set of all finite sequences of natural numbers. Such a sequence is regarded as a function of the form \( f : \{j : j < n\} \to \mathbb{N} \) where \( n \) is the length of the sequence. We regard \( \emptyset \) as such a sequence; i.e. the sequence of length 0. When \( \alpha \in \mathbb{N}^* \) let \( |\alpha| \) denote the length of \( \alpha \).

When \( \alpha, \beta \in \mathbb{N}^* \), write \( \alpha \subseteq \beta \) if \( \alpha \) is a prefix of \( \beta \). (In this case it follows that \( \alpha \) is a subset of \( \beta \) when both functions are viewed as sets of ordered pairs.) Note that \( \emptyset \subseteq \alpha \) for every \( \alpha \in \mathbb{N}^* \). Note also that \( \subseteq \) is a partial order on \( \mathbb{N}^* \).

If \( \alpha \in \mathbb{N}^* \), and if \( t \leq |\alpha| \), let \( \alpha \uparrow t \) denote the prefix of \( \alpha \) whose length is \( t \).

We will refer to the sequences in \( \mathbb{N}^* \) as nodes. When \( \alpha, \beta \in \mathbb{N}^* \) have the property that \( \alpha \subseteq \beta \) and there is no \( \gamma \) so that \( \alpha \subseteq \gamma \subseteq \beta \), we say that \( \beta \) is a child of \( \alpha \) and the \( \alpha \) is the parent of \( \beta \). Let \( \beta^- \) denote the parent of \( \beta \) when \( \beta \neq \emptyset \).

A tree is a nonempty subset of \( \mathbb{N}^* \) that is closed under prefixes; i.e. \( T \subseteq \mathbb{N}^* \) is a tree if \( \alpha \in T \) whenever there exists a \( \beta \in T \) so that \( \alpha \subseteq \beta \). Thus, the empty sequence belongs to every tree and is referred to as the root of the tree. The \( s \)-th level of \( T \) consists of all nodes in \( T \) whose length is \( s \).

### 3. Overview

Herein, we will cover the main theorems and definitions used in the proofs of Theorems 2.2 and 2.3 and how they come together to prove these two results. The proofs of these intermediate conclusions are given in later sections.
3.1. Overview of the proof of Theorem 2.2. Most of the paper is spent proving Theorem 2.2. We divide our overview of this proof into three parts: the classical, the approximate, and the computable.

3.1.1. Classical world. The following observation is the foundation of our approach.

**Theorem 3.1.** Suppose \( g_0, g_1, \ldots \) are disjointly supported nonzero vectors in \( \ell^p \). Then, there is a unique linear isometry \( T : \ell^p \to \ell^p \) so that \( T(e_n) = \|g_n\|^{-1} g_n \) for all \( n \). If \( \{g_0, g_1, \ldots\} \) is a generating set, then this isometry is an endomorphism.

**Definition 3.2.** Suppose \( \lambda e \) is a linear isometry. Note that the atoms in this ordering are the vectors of the form \( g \) where \( \lambda \) is a non-atomic vector in \( \ell^p \). We say that \( \lambda e \) of all vectors with respect to \((E, F)\) is a disintegration and that \( \lambda e \) is a disintegration if it meets the following conditions.

(1) The partial order \((G, \preceq)\) has no infinite ascending chains and no infinite intervals; i.e. if \( g, g' \in G \), then there are only finitely many \( h \in G \) so that \( g \preceq h \preceq g' \).
(2) If \( g, g' \in G \) are incomparable, then their supports are disjoint.
(3) \( G \) is a generating set.
(4) If \( g \in G \) is not an atom, then \( g \) is the sum of its predecessors in \((G, \preceq)\).

To make complete sense of Condition (4), we define the sum of the empty set to be the zero vector. So Condition (4) entails that each non-atomic vector in \( G \) has a least one predecessor in \((G, \preceq)\); hence, it must have at least 2 predecessors. Note that by Condition (2), it has countably many predecessors.

Suppose \( G \) is a disintegration and that \( g \in G \). It follows from Conditions (1) and (2) that there is a unique maximal chain in \((G, \preceq)\) whose least element is \( g \); let \( C \) denote this chain. It also follows that this chain is finite, and we call the number \(#C - 1\) the level of \( g \) in \( G \). We then let \( G^{(s)} \) denote the \( s \)-th level of \( G \); i.e. the set of all vectors \( g \in G \) so that the level of \( g \) is \( s \).

Again, suppose \( G \) is a disintegration. We use \( G \) to induce an isometric endomorphism of \( \ell^p \) as follows. First, we will show (Proposition 4.3) that if \( g \) is a non-atomic vector in \( G \), then

\[
\max\{\|g'\| : g' \text{ is a predecessor of } g \text{ in } (G, \preceq)\}
\]
exists. We then say that $h$ is an almost norm-maximizing predecessor of $g$ in $(G, \leq)$ if $h$ is a predecessor of $g$ in $(G, \leq)$ and if
\[
\max\{\|g^\prime\|^p : g^\prime \text{ is a predecessor of } g \text{ in } (G, \leq)\} \leq \|h\|^p + 2^{-s}
\]
where $s$ is the level of $g$ in $G$. Suppose $C$ is a chain in $(G, \leq)$. We say that $C$ is almost norm maximizing if for each non-atomic $g \in C$ an almost norm-maximizing predecessor of $g$ in $(G, \leq)$ belongs to $C$. We show (Proposition 3.4) that every chain in $(G, \leq)$ has an infimum. We say that a chain in $(G, \leq)$ avoids zero if its infimum is nonzero. In Section 4 we prove the following.

**Theorem 3.3.** Let $G$ be a disintegration. Let $C_0, C_1, \ldots$ be a partition of $G$ into almost norm-maximizing chains. Then, the infima of the zero-avoiding chains in $\{C_0, C_1, \ldots\}$ induce an isometric endomorphism of $\ell^p$.

A transitional step to producing a disintegration is the concept of a shattering which we define now.

**Definition 3.4.** A shattering is a pair $(G, \kappa)$ so that the following hold.

1. $G$ is a finite set of nonzero vectors in $\ell^p$, and $\kappa \in \mathbb{N}$.
2. If $g, g^\prime \in G^{(s)}$ are incomparable, then their supports are disjoint.
3. When $0 \leq j < \kappa, d(f_j, L(G)) < 2^{-\kappa}$.
4. If $g \in G$ and if there is a predecessor of $g$ in $(G, \leq)$, then $g$ has at least two predecessors in $(G, \leq)$ and
\[
\left\| g - \sum_{g^\prime \in \mathcal{P}} g^\prime \right\| < 2^{-\kappa}
\]
where $\mathcal{P}$ is the set of all predecessors of $g$ in $(G, \leq)$.

Suppose $(G, \kappa)$ is a shattering. Thus, $G$ is finite, and it again follows that for each $g \in G$ there is a unique maximal chain $C$ in $(G, \leq)$ so that $g = \min(C)$. We again refer to $\#C - 1$ as the level of $g$ in $G$ and let $G^{(s)}$ denote the $s$-th level of $G$.

Suppose $(G, \kappa)$ and $(G^\prime, \kappa^\prime)$ are shatterings. We say that $(G^\prime, \kappa^\prime)$ extends $(G, \kappa)$ if $\kappa^\prime > \kappa$ and the $s$-th level of $(G^\prime, \leq)$ includes the $s$-th level of $(G, \leq)$ for all $s$.

In Section 4 we prove the following.

**Theorem 3.5.**

1. If $(G_0, \kappa_0), (G_1, \kappa_1), \ldots$ are shatterings so that $(G_{n+1}, \kappa_{n+1})$ extends $(G_n, \kappa_n)$ for all $n$, then $\bigcup_n G_n$ is a disintegration.
2. Suppose $(G, \kappa)$ is a shattering. Then, for each $\kappa^\prime > \kappa$, there is a set of vectors $G^\prime$ so that $(G^\prime, \kappa^\prime)$ is an extension of $(G, \kappa)$. Furthermore, $G^\prime$ may be chosen so that $\max\{\|g^\prime\| : g^\prime \in G^\prime\} \leq \max\{\|g\| : g \in G\}$.

If $G = \emptyset$ (in which case $\kappa$ must be 0), we interpret $\max\{\|g\| : g \in G\}$ to be infinite.

As an intermediate step towards translating shatterings into the world of approximations, we introduce an upper bound on the distance from $(f, g)$ to the nearest pair of disjointly supported vectors as follows. Suppose $p \neq 2$. We let $c_p = |4 - 2\sqrt{2}|^{-1}$. When $z, w \in \mathbb{C}$ set:
\[
\sigma_1(z, w) = |2|z|^p + 2|w|^p - |z - w|^p - |z + w|^p
\]
\[
\sigma_2(z, w) = c_p \sigma_1(z, w)
\]
J. Lamperti proved that $\sigma_1(z, w) = 0$ if and only if $zw = 0$. This is the fundamental fact behind his proof that linear isometries of $\ell^p$ preserve the disjoint support relation. We sharpen this result as follows.
Theorem 3.6. Suppose \( p \geq 1 \) and \( p \neq 2 \). Then,
\[
\min\{|z|^p, |w|^p\} \leq \sigma(z, w)
\]
for all \( z, w \in \mathbb{C} \). Furthermore, if \( 1 \leq p < 2 \), then
\[
2|z|^p + 2|w|^p - |z + w|^p - |z - w|^p > 0
\]
and if \( 2 < p \) then
\[
2|z|^p + 2|w|^p - |z + w|^p - |z - w|^p < 0.
\]

The ‘furthermore’ part of Theorem 3.6 was shown by Lamperti, but the inequality (3.1) appears to be new.

When \( f, g \in \ell^p \), let \( \sigma_1(f, g) = |2 \|f\|^p + 2 \|g\|^p - \|f - g\|^p - \|f + g\|^p| \) and let \( \sigma(f, g) = c_p \sigma_1(f, g) \). So, if \( f, g \in \ell^p \), by the ‘furthermore’ part of Theorem 3.6
\[
\sigma_1(f, g) = \sum_{t=0}^\infty \sigma_1(f(t), g(t)).
\]

Thus, when \( f, g \in \ell^p \), \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \) if and only if \( \sigma(f, g) = 0 \), and \( g \preceq f \) if and only if \( \sigma(f - g, g) = 0 \). By means of (3.1), we show (Lemma 4.4) that \( \sigma(f, g) \) is an upper bound on the distance from \( (f, g) \) to the nearest pair of disjointly supported vectors; furthermore, \( \sigma(f, g) \) approaches 0 as \( (f, g) \) approaches a pair of disjointly supported vectors. In addition, \( \sigma(f - g, g) \) is an upper bound on the distance from \( g \) to the nearest subvector of \( f \) (Lemma 4.5) and \( \sigma(f - g, g) \) approaches 0 as \( g \) approaches a subvector of \( f \).

We use \( \sigma \) to address the following problem: given a finite tree \( T \subseteq \mathbb{N}^* \) and a map \( \lambda : T - \{\emptyset\} \rightarrow \ell^p \) that maps incomparable nodes to disjointly supported vectors, i.e. \( \text{supp}(\lambda(\nu)) \cap \text{supp}(\lambda(\nu')) = \emptyset \) whenever \( \nu, \nu' \in T - \{\emptyset\} \) are incomparable. (Herein, when we say that \( \psi : T - \{\emptyset\} \rightarrow \ell^p \) is an order homomorphism we mean \( \psi(\nu') \preceq \psi(\nu) \) if \( \nu' \succ \nu \).) We give an upper estimate on this distance as follows. Suppose \( T \) is a finite subtree of \( \mathbb{N}^* \) and \( \lambda : T - \{\emptyset\} \rightarrow \ell^p \). Set:
\[
\epsilon_1(T, \lambda) = \max\{\sigma(\lambda(\nu), \lambda(\nu')) : \nu, \nu' \in T - \{\emptyset\} \land \nu, \nu' \text{ are incomparable}\}
\]
\[
\epsilon_2(T, \lambda) = \max\{\sigma(\lambda(\nu) - \lambda(\nu'), \lambda(\nu')) : \emptyset \neq \nu \subset \nu' \in T\}
\]
\[
\epsilon_0(T, \lambda) = \max\{\epsilon_1(T, \lambda), \epsilon_2(T, \lambda)\}
\]
Let \( N_p = \lfloor c_p \rfloor \). Set
\[
\Delta(T, \lambda) = (\#T(160)N_p^2)^{\#T-1}(\#T)^{1/p}\epsilon_0(T, \lambda)^{-2(#T-1)}.\]

In Subsection 4.3 we prove the following.

Theorem 3.7. Suppose \( p \neq 2 \). Suppose \( T \) is a finite subtree of \( \mathbb{N}^* \), and suppose \( \lambda : T - \{\emptyset\} \rightarrow \ell^p \) is such that \( \max_{\nu} p(2 \|\lambda(\nu)\|)^{p-1} \leq 1 \) and \( \epsilon_0(T, \lambda) < 1 \). Then, there is an order homomorphism \( \Lambda : T - \{\emptyset\} \rightarrow \ell^p \) that maps incomparable nodes to disjointly supported vectors and for which \( \|\Lambda(\nu) - \lambda(\nu)\| < \Delta(T, \lambda) \) for all non-root \( \nu \in T \).

In other words, \( \Delta(T, \lambda) \) is an upper bound on the distance from \( \lambda \) to the nearest order homomorphism \( \Lambda : T - \{\emptyset\} \rightarrow \ell^p \) that maps incomparable nodes to disjointly supported vectors. In addition, \( \Delta(T, \lambda) \) approaches 0 as \( \lambda \) approaches such a map.
We conclude the section on the classical world (Section 3.1) by giving a short proof of the Banach-Lamperti characterization of the surjective linear isometries of \( \ell^p \) (Theorem 2.4); this result will be used in the proof of Theorem 2.3.

3.1.2. Approximate world. In Section 3.1, we translate the material on shatterings and disintegrations into the world of discrete approximations as follows. Recall that \( F = \{ f_0, f_1, \ldots \} \) is a fixed generating set for \( \ell^p \). Suppose \( T \) is a finite subtree of \( \mathbb{N}^* \) and that \( \lambda : T - \{ \emptyset \} \to \mathcal{L}_{Q(i)}(F) \). Suppose that in addition to \( T \), \( \lambda \) we have nonnegative integers \( p, \kappa \) and a map \( \beta : T - \{ \emptyset \} \to \mathbb{Q}(i) - \{ 0 \} \) for each \( j < \kappa \).

Define \( v_1(T, \lambda, \kappa, \{ \beta_j \}) \) to be the minimum of the following quantities.

\[
\begin{align*}
(1) \quad & 2^{-p} \\
(2) \quad & \frac{1}{2} \min \{ \| \lambda(\nu) \| : \nu \in T - \{ \emptyset \} \} \\
(3) \quad & \min_{0 \leq j < \kappa} \{ (2^{-\kappa} - \| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \| (\sum_{\nu} | \beta_j(\nu) |)^{-1} \} \\
(4) \quad & (1 + \#T)^{-1}(2^{-\kappa} - \| \lambda(\nu) - \sum_{\nu} | \beta_j(\nu) | \|)^{-1} \text{ whenever } \nu \in T - \{ \emptyset \} \text{ is non-terminal.}
\end{align*}
\]

If \( p = 1 \), then define \( v(T, \lambda,\kappa,\{ \beta_j \}) \) to be \( v_1(T, \lambda, \kappa, \{ \beta_j \}) \). If \( p > 1 \), then we define \( v(T, \lambda, \kappa, \{ \beta_j \}) \) to be the minimum of \( v_1(T, \lambda, \kappa, \{ \beta_j \}) \) and \( \min \{ 2^{-p} - \| \lambda(\nu) \| : \nu \in T - \{ \emptyset \} \} \). We then define an approximate shattering as follows.

**Definition 3.8.** An approximate shattering of \( F \) is a quintuple \( \mathcal{A} = (T, \lambda, \kappa, \{ \beta_j \}_{j < \kappa}, p) \) that meets the following conditions.

\[
\begin{align*}
(1) \quad & T \text{ is a finite subtree of } \mathbb{N}^*. \text{ Each non-terminal non-root node of } T \text{ has at least two children in } T. \\
(2) \quad & \lambda : T - \{ \emptyset \} \to \mathcal{L}_{Q(i)}(F) \text{ and } \beta_j : T - \{ \emptyset \} \to \mathbb{Q}(i) - \{ 0 \} \text{ when } j < \kappa. \\
(3) \quad & \varepsilon_0(T, \lambda) < 1. \\
(4) \quad & \Delta(T, \lambda) < v(\mathcal{A}).
\end{align*}
\]

The last inequality ensures that \( (\text{ran} (\phi), \kappa) \) is a shattering if \( \phi : T - \{ \emptyset \} \to \ell^p \) is an order homomorphism that maps incomparable nodes to disjointly supported vectors and so that \( \| \lambda(\nu) - \phi^{-1}(\nu) \| < \Delta(T, \lambda) \) for all \( \nu \). This is proven rigorously in Lemma 5.2. If \( \mathcal{A} = (T, \lambda, \kappa, \{ \beta_j \}_{j < \kappa}, p) \) is an approximate shattering we set \( \Delta(\mathcal{A}) = \Delta(T, \lambda) \).

We now define what it means for one approximate shattering to extend another.

**Definition 3.9.** Suppose \( \mathcal{A} = (T, \lambda, \kappa, \{ \beta_j \}_{j < \kappa}, p) \) and \( \mathcal{A}' = (T', \lambda', \kappa', \{ \beta'_j \}_{j < \kappa'}, p') \) are approximate shatterings of \( F \). We say that \( \mathcal{A}' \) extends \( \mathcal{A} \) if it satisfies the following conditions.

\[
\begin{align*}
(1) \quad & T \subseteq T'. \\
(2) \quad & \kappa' > \kappa. \\
(3) \quad & p' > p. \\
(4) \quad & \| \lambda'(\nu) - \lambda(\nu) \| < \min \{ 2^{-p} - 2^{-p'}, \frac{1}{2} \| \lambda(\nu) \| \} \text{ whenever } \nu \in T - \{ \emptyset \}. \\
(5) \quad & \Delta(\mathcal{A}') < v(\mathcal{A}).
\end{align*}
\]

Our key results on approximate shatterings are the following two theorems.

**Theorem 3.10.** Suppose \( p \neq 2 \). Suppose \( \mathcal{A}_0, \ldots, \mathcal{A}_r \) are approximate shatterings so that \( \mathcal{A}_t \) extends \( \mathcal{A}_t \) when \( t < t' \). Then, there is an approximate shattering \( \mathcal{A}_{r+1} \) that extends \( \mathcal{A}_0, \ldots, \mathcal{A}_r \).
Theorem 3.11. Suppose \( p \neq 2 \). Suppose that for each \( t \in \mathbb{N} \), \( A_t = (T_t, \lambda_t, \kappa_t, \{\beta_j^{(t)}\}_{j < \kappa_t}, p_t) \) is an approximate shattering. Suppose also that \( A_{t'} \) extends \( A_t \) whenever \( t < t' \). Then:

1. \( \lim_{t \to \infty} \lambda_t(\nu) \) exists and is nonzero for all \( \nu \in \bigcup_t T_t \); let \( \lambda(\nu) \) denote this limit and let \( T = \bigcup_t T_t \).
2. \( \lambda \) is an order monomorphism of \( T \) into \( (\ell^p, \preceq) \) and maps incomparable nodes to disjointly supported vectors.
3. \( G := \lambda[T] \) is a disintegration.

The proof of Theorem 3.10 is based on Theorem 3.5.2.

3.1.3. Computable world. Moving into the computable world (Section 6), let us suppose that \( p \) is a computable real so that \( p \neq 2 \) and that \( F \) is an effective generating set. It then follows that the set of approximate shatterings is c.e. as is the ‘extends’ relation on approximate shatterings. It follows from Theorem 3.10 that via a search procedure we can compute a sequence of approximate shatterings \( A_0, A_1, \ldots \) so that \( A_{t'} \) extends \( A_t \) whenever \( t' > t \). Set \( A_t = (T_t, \lambda_t, \kappa_t, \{\beta_j^{(t)}\}_{j < \kappa_t}, p_t) \). Let \( \lambda, T, \) and \( G \) be as in Theorem 3.11. Hence, \( T \) is c.e. and \( \lambda \) is computable. We then prove the following.

Theorem 3.12. There is a c.e. \( S \subseteq T \) and a uniformly c.e. family \( \{N_{\nu}\}_{\nu \in S} \) so that \( \{\lambda[N_{\nu}]\}_{\nu \in S} \) is a partition of \( G \) into almost norm-maximizing chains.

Theorem 3.13. Let \( S \) be as in Theorem 3.12. Then, the halting set computes the following.

1. An enumeration of the set of all \( \nu \in S \) so that \( \lambda[N_{\nu}] \) is zero-avoiding.
2. For each \( \nu \in S \), the infimum of \( \lambda[N_{\nu}] \).

It follows from Theorem 3.3 that with respect to \( F \), the halting set computes a sequence of vectors that induces an isometric endomorphism of \( \ell^p \). It then follows (as shown in detail in Lemma 6.1) that the halting set computes this isometric endomorphism. This proves Theorem 2.2.

3.2. Overview of the proof of Theorem 2.3. Let \( C \) be an incomputable c.e. set. Without loss of generality, we assume \( 0 \notin C \). Let \( \{e_n\}_{n \in \mathbb{N}} \) be a one-to-one effective enumeration of \( C \). Set

\[
\gamma = \sum_{k \in C} 2^{-k}.
\]

Thus, \( 0 < \gamma < 1 \), and \( \gamma \) is an incomputable real. Set:

\[
\begin{align*}
f_0 &= (1 - \gamma)^{1/p} e_0 + \sum_{n=0}^{\infty} 2^{-e_n/p} e_{n+1} \\
f_{n+1} &= e_{n+1} \\
F &= \{f_0, f_1, f_2, \ldots\}
\end{align*}
\]

Since \( 1 - \gamma > 0 \), we can use the standard branch of \( \sqrt[p]{\cdot} \). In Subsection 6.2, we show that \( F \) is an effective generating set and that any set that computes an isometric endomorphism of \( \ell^p \) with respect to \((E, F)\) must also compute \( C \). The proofs of these claims are very similar to the proofs of corresponding results for \( \ell^1 \) in [14] and the reader familiar with this material could skip Subsection 6.2.

This concludes the overview of the paper.
4. Classical World

4.1. Proof of Theorem 3.1. The proof of Theorem 3.1 is based on the following observations and Proposition 4.1. Suppose \( f, g \in \ell^p \), and suppose the supports of \( f \) and \( g \) are disjoint. It follows that the supports of \( zf \) and \( wg \) are disjoint for all \( z, w \in \mathbb{C} \) and that \( \|f + g\|^p = \|f\|^p + \|g\|^p \).

**Proposition 4.1.** Suppose \( f, f_1, \ldots \in \ell^p \) are disjointly supported. Then, \( \sum_{n=0}^{\infty} f_n \in \ell^p \) if and only if \( \sum_{n=0}^{\infty} \|f_n\|^p < \infty \) in which case
\[
\left\| \sum_{n=0}^{\infty} f_n \right\|^p = \sum_{n=0}^{\infty} \|f_n\|^p.
\]

**Proof.** By Lemma 28 of Chapter 11 Section 7 of [15]. \( \square \)

**Proof of Theorem 3.1.** For the moment, fix \( f \in \ell^p \). Since \( g_0, g_1, \ldots \) are disjointly supported, \( f(0) \|g_0\|^{-1} g_0, f(1) \|g_1\|^{-1} g_1, \ldots \) are disjointly supported. So by Proposition 4.1,
\[
\sum_{n=0}^{\infty} \frac{f(n)}{\|g_n\|} g_n \in \ell^p
\]
and
\[
\left\| \sum_{n=0}^{\infty} \frac{f(n)}{\|g_n\|} g_n \right\|^p = \sum_{n=0}^{\infty} |f(n)|^p = \|f\|^p.
\]

So, for all \( f \in \ell^p \), set
\[
T(f) = \sum_{n=0}^{\infty} \frac{f(n)}{\|g_n\|} g_n.
\]

Thus, \( T : \ell^p \to \ell^p \), and \( T \) is an isometry. By definition, \( T \) is linear.

Since \( E \) is a basis for \( \ell^p \), the uniqueness of \( T \) is automatic. \( \square \)

4.2. Proof of Theorem 3.3. The proof of Theorem 3.3 is based on the following two propositions.

**Proposition 4.2.** Every nonempty chain in \( (\ell^p, \preceq) \) has an infimum. If \( C = \{g_0 \succ g_1 \succ g_2 \ldots\} \) is a chain in \( (\ell^p, \preceq) \), and if \( g \) is the infimum of \( C \), then \( \lim_{n \to \infty} \|g - g_n\| = 0 \).

**Proof.** Let \( C \) be a nonempty chain in \( (\ell^p, \preceq) \). Fix \( f_0 \in C \). Let
\[
S = \bigcap_{g \in C} \text{supp}(g).
\]
Set \( f_1 = f_0|_S \).

We claim that \( f_1 \) is a lower bound on \( C \). For, let \( f \in C \). Then, \( f \preceq f_0 \) or \( f_0 \preceq f \). By definition, \( S \subseteq \text{supp}(f) \). Let \( t \in S \). Then, \( f_1(t) = f_0(t) = f(t) \). Thus, \( f_1 \preceq f \).

We now claim that \( f_1 \) is the infimum of \( C \). For, suppose \( g \preceq f \) for all \( f \in C \). Thus, \( \text{supp}(g) \subseteq \text{supp}(f) \) for all \( f \in C \) and so \( \text{supp}(g) \subseteq S = \text{supp}(f_1) \). Let \( t \in \text{supp}(g) \). Thus, since \( g \preceq f_0 \), \( g(t) = f_0(t) = f_1(t) \). Thus, \( g \preceq f_1 \).

Now, suppose \( C = \{g_0 \succ g_1 \succ g_2 \ldots\} \), and set \( g = \inf(C) \). It follows from the above argument that \( g \) is the pointwise limit of \( \{g_n\}_n \).

We claim that \( |g_n(t) - g(t)| \leq |g_0(t)| \). For if \( t \notin \text{supp}(g_n) \), then \( t \notin \text{supp}(g) \) and so \( |g_n(t) - g(t)| = 0 \). Suppose \( t \in \text{supp}(g_n) \). Then, \( t \in \text{supp}(g_0) \). So, \( |g_n(t) - g(t)| =
\begin{align*}
|g_0(t) - g(t)| &. \text{ If } t \notin \text{supp}(g), \text{ then } |g_0(t) - g(t)| = |g_0(t)|. \text{ Suppose } t \in \text{supp}(g). \\
\text{Then, } g_0(t) = g(t) \text{ and so } |g_0(t) - g(t)| = 0.
\end{align*}

It now follows from the Dominated Convergence Theorem that \( \lim_{n \to \infty} \| g_n - g \| = 0. \)

**Proposition 4.3.** If \( \mathcal{G} \) is a disintegration, and if \( g \in \mathcal{G} \) is non-atomic, then

\[
\max \{ \| g' \|^p : g' \text{ is a predecessor of } g \text{ in } (\mathcal{G}, \preceq) \}
\]

exists.

**Proof.** Let \( \mathcal{S} \) be the set of all predecessors of \( g \) in \( (\mathcal{G}, \preceq) \). Since \( \mathcal{G} \) is a disintegration, by Definition 3.2, \( g = \sum_{g' \in \mathcal{S}} g' \) and the supports of distinct vectors in \( \mathcal{S} \) are disjoint. Thus, by Proposition 4.1

\[
\sum_{g' \in \mathcal{S}} \| g' \|^p = \| g \|^p < \infty.
\]

Therefore, there is a finite set \( \{ g'_0, \ldots, g'_1 \} \subseteq \mathcal{S} \) so that \( \| g' \|^p \leq \max \{ \| g'_0 \|^p, \ldots, \| g'_1 \|^p \} \) whenever \( g' \in \mathcal{S} - \{ g'_0, \ldots, g'_1 \} \). Thus, \( \sup \{ \| g' \|^p : g' \in \mathcal{S} \} = \max \{ \| g'_0 \|^p, \ldots, \| g'_1 \|^p \}. \)

**Proof of Theorem 3.3.** We first claim that for every \( j \), there is a \( k \) so that \( j \) belongs to the support of inf\((\mathcal{C}_k)\). If there is an atom in \( \mathcal{G} \) whose support contains \( j \), then there is nothing left to prove. So, suppose \( j \) does not belong to the support of any atom in \( \mathcal{G} \).

We claim that there is a \( g \in \mathcal{G}(0) \) so that \( j \in \text{supp}(g) \). For otherwise, \( j \notin \text{supp}(g) \) for all \( g \in \mathcal{G} \). But, since \( \mathcal{G} \) is a disintegration, \( \mathcal{G} \) is a generating set. This is a contradiction.

Since \( \mathcal{G} \) is a disintegration, each non-atomic vector in \( \mathcal{G} \) is the sum of its predecessors in \( (\mathcal{G}, \preceq) \). So, it follows by induction that for each \( s \), \( j \) belongs to the support of at least one vector in \( \mathcal{G}^{(s)} \). Since distinct vectors in \( \mathcal{G}^{(s)} \) are disjointly supported, this vector is unique. So, for each \( s \), let \( g_s \) denote the vector in \( \mathcal{G}^{(s)} \) whose support contains \( j \). Since distinct vectors in \( \mathcal{G}^{(s)} \) are disjointly supported, \( g_{s+1} \leq g_s \) for all \( s \). Thus, \( g_s(j) = g_0(j) \neq 0 \) for all \( s \).

Now, for each \( s \), let \( k_s \) denote the \( k \) so that \( g_s \in \mathcal{C}_k \). We claim that \( \lim_s k_s \) exists. By way of contradiction suppose otherwise. Let \( s_0 < s_1 < \ldots \) be the increasing enumeration of all values of \( s \) for which \( k_s \neq k_{s+1} \). Since \( g_{s_m+1} \leq g_{s_m} \), \( g_{s_m} \) has at least one predecessor in \( (\mathcal{G}, \preceq) \). Therefore, since \( \mathcal{C}_{k_{m+1}} \) is almost norm-maximizing, it must contain a predecessor of \( g_{s_m} \) in \( (\mathcal{G}, \preceq) \); let \( \lambda_m \) denote this vector. Thus, \( \lambda_m < g_{s_m} \) and the supports of \( \lambda_m \) and \( g_{s_m+1} \) are disjoint (since they are distinct vectors at the same level of \( \mathcal{G} \)). In addition, since \( \lambda_m \) is an almost norm-maximizing predecessor of \( g_{s_m} \), \( |g_0(j)|^p = |g_{s_m+1}(j)|^p \leq \| \lambda_m \|^p + 2^{-s_m+1} \). Since \( \lambda_{m+r} \leq g_{s_m+r} \), the supports of \( \lambda_m \) and \( \lambda_{m+r} \) are disjoint if \( r > 0 \). That is to say, \( \text{supp}(\lambda_m) \cap \text{supp}(\lambda_{m'}) = \emptyset \) whenever \( m \neq m' \). Thus, \( \infty = \sum_m \| \lambda_m \|^p \leq \| g_0 \|^p \) - a contradiction. Thus, \( k := \lim_s k_s \) exists, and so \( j \) belongs to the support of inf\((\mathcal{C}_k)\).

We now claim that for all \( k \), the support of inf\((\mathcal{C}_k)\) contains at most one number. For, let \( g = \text{inf}(\mathcal{C}_k) \). By way of contradiction, suppose \( j_0, j_1 \in \text{supp}(g) \) are distinct. Since the supports of distinct vectors at each level of \( \mathcal{G} \) are disjoint, it follows that for every \( g' \in \mathcal{G} \), \( j_0 \in \text{supp}(g') \) if and only if \( j_1 \in \text{supp}(g') \). Thus,

\[
\text{supp}(g) = \{ f \in \ell^p : f(j_0) = f(j_1) \}.
\]

But, \( \mathcal{G} \) is a generating set; this is a contradiction.
This entails that there are infinitely many zero-avoiding chains in \( \{C_k\}_k \). We now claim that the supports of the infima of these chains are pairwise disjoint. For, suppose \( k \neq k' \). It suffices to show that there are incomparable vectors \( g, g' \) so that \( g \in C_k \) and \( g' \in C_{k'} \). By way of contradiction, suppose this is not the case. Therefore, there is no \( s \) so that \( C_k \) and \( C_{k'} \) both contain a vector of level \( s \). Let \( h \) be the \( \preceq \)-maximal vector in \( C_k \) and let \( h' \) be the \( \preceq \)-maximal vector in \( C_{k'} \). Let \( s \) denote the level of \( h \), and let \( s' \) denote the level of \( h' \). Thus, \( s \neq s' \). Without loss of generality, assume \( s < s' \). Therefore, \( C_k \cap C_{s'} = \emptyset \). Thus, \( C_k \) contains an atom \( g \), and \( g = \inf(C_k) \). Let \( t \) denote the level of \( g \). Therefore, \( t < s' \). However, \( g \) and \( h' \) are comparable since \( g \in C_k \) and \( h' \in C_{k'} \). Thus, since \( g \) is an atom, \( g \preceq h' \), a contradiction. Thus, there are incomparable vectors \( g, g' \) so that \( g \in C_k \) and \( g' \in C_{k'} \). Hence, the supports of \( \inf(C_k) \) and \( \inf(C_{k'}) \) are disjoint.

So, by Theorem 3.1, the infima of the zero-avoiding chains in \( C_k \) induce a linear isometry of \( \ell^p \). However, by what has just been shown, these infima also constitute a generating set. Thus, by Theorem 3.1 again, this isometry is also an endomorphism. 

4.3. Proof of Theorem 3.5

(1): Suppose \( (G_0, \kappa_0), (G_1, \kappa_1), \ldots \) are shatterings and that \( (G_{n+1}, \kappa_{n+1}) \) extends \( (G_n, \kappa_n) \) for all \( n \). Set \( G = \bigcup_n G_n \). We show that \( G \) is a disintegration. Since \( G_{n+1}^{(s)} \supseteq G_n^{(s)} \) for all \( n \), it follows that Condition (1) of Definition 3.2 is satisfied. In particular, \( g \) belongs to the \( s \)-th level of \( G \) if and only if \( g \in G_n^{(s)} \) for some \( n \).

Since each \( G_n \) is a shattering, it follows that incomparable vectors in \( G \) are disjointly supported.

Since \( G_{n+1} \) extends \( G_n \) for all \( n \), \( \lim_n \kappa_n = \infty \). So it follows from the definition of ‘shattering’ (Definition 3.3), that \( G \) is a generating set and that each non-atomic vector in \( G \) is the sum of its predecessors in \( (G, \preceq) \).

(2): Suppose \( (G, \kappa) \) is a shattering and that \( \kappa' > \kappa \). We show that there is a set of vectors \( G' \) so that \( (G', \kappa') \) is a shattering and extends \( (G, \kappa) \). We first observe that if \( f \in \ell^p \), then \( \lim_{k \to \infty} f \|_{[k, \infty)} = 0 \). So, choose \( k \) so that for all \( g \in G^{(0)} \cup \{f_0, \ldots, f_{k-1}\} \), \( \|g\|_{[k, \infty)} < 2^{-\kappa} \). For each \( t \leq k \), let \( n_t \) be the smallest natural number so that \( t \in f_{n_t} \). Choose a positive number \( \epsilon \) so that \( \epsilon |f_{n_t}(t)| \leq \max\{\|g\| : g \in G\} \) for all \( t \leq k \). Set

\[
U = G^{(0)} \cup \{\epsilon f_{n_t}|_{\{t\}} : t \leq k \land t \notin \bigcup_{g \in G^{(0)}} \text{supp}(g)\}.
\]

Note that if \( g, g' \in U \) are distinct, then they are incomparable with respect to \( \preceq \) and their supports are disjoint.

For each \( g \in U \), choose a set of atoms \( A_g \) such that \( g' \preceq g \) for all \( g' \in A_g \) and so that \( \left\|g - \sum_{g' \in A_g} g'\right\| < 2^{-\kappa'} \). If \( g \) is not atomic, ensure that \( \#A_g \geq 2 \). Set \( G' = G \cup \bigcup_{g \in U} A_g \).

We first claim that \( (G', \kappa') \) is a shattering. Since every vector in \( G' - G \) is an atom, every vector in \( G' \) is nonzero. Suppose \( g, g' \) are distinct vectors on the same level of \( G \). Without loss of generality, suppose \( g' \notin G' - G \).

Suppose \( g, g' \) have an upper bound in \( (\ell^p, \preceq) \). Then, since \( g' \) is an atom, \( g' \preceq g \) if \( \text{supp}(g') \cap \text{supp}(g) \neq \emptyset \). So, in this case, the supports of \( g \) and \( g' \) are disjoint.
So, suppose \( g, g' \) do not have an upper bound in \((\ell^p, \preceq)\). By construction of \( \mathcal{G}' \), there exist \( h, h' \in U \) so that \( g \preceq h \) and \( g' \preceq h' \). Thus, \( h, h' \) are incomparable. Therefore, \( \text{supp}(h) \cap \text{supp}(h') = \emptyset \).

By construction, \([0, k] \subseteq \bigcup_{g \in \mathcal{G}''} \text{supp}(g)\). Since each vector in \( \mathcal{G}' - \mathcal{G} \) is an atom, by the choice of \( k \), \( d(f_j, \mathcal{L}(\mathcal{G}'')) < 2^{-\kappa'} \) if \( j < \kappa' \).

Finally, suppose \( g \) has a predecessor in \((\mathcal{G}', \preceq)\). Let \( S \) denote the set of all predecessors of \( g \) in \((\mathcal{G}', \preceq)\). Since \( g \) is not an atom, by definition of \( \mathcal{G}' \), \( g \in \mathcal{G} \). There is a \( g_1 \in \mathcal{G}^{(0)} \) so that \( g \preceq g_1 \). Let \( S_1 = \{ g' \in \mathcal{G}' - \mathcal{G} : g' \preceq g_1 \} \). Let \( S = \{ g' \in \mathcal{G}' - \mathcal{G} : g' \preceq g \} \). By construction of \( \mathcal{G}' \), \( \| g_1 - \sum_{g' \in S_1} g' \| < 2^{-\kappa'} \). At the same time,

\[
\| g_1 - \sum_{g' \in S_1} g' \| = \| (g_1 - g) - \sum_{g' \in S_1 - S} g' + g - \sum_{g' \in S} g' \|.
\]

By construction, every vector in \( \mathcal{G}' - \mathcal{G} \) is an atom. So, if \( g' \in S_1 - S \), then \( g' \preceq g_1 - g \). Thus,

\[
(g_1 - g) - \sum_{g' \in S_1 - S} g' \preceq g_1 - g,
\]

and

\[
g - \sum_{g' \in S} g' \preceq g.
\]

Since \( g \preceq g_1 \), the supports of \( g \) and \( g_1 - g \) are disjoint. So, we have

\[
\| (g_1 - g) - \sum_{g' \in S_1 - S} g' + g - \sum_{g' \in S} g' \| = \| (g_1 - g) - \sum_{g' \in S_1 - S} g' \| + \| g - \sum_{g' \in S} g' \|^p \geq \| g - \sum_{g' \in S} g' \|^p.
\]

Thus,

\[
\| g - \sum_{g' \in S} g' \| < 2^{-\kappa'}.
\]

Since every vector in \( \mathcal{G}' - \mathcal{G} \) is an atom, it follows that the \( s \)-th level of \( \mathcal{G}' \) includes the \( s \)-th level of \( \mathcal{G} \) for all \( s \). Thus, \((\mathcal{G}', \kappa')\) extends \((\mathcal{G}, \kappa)\).

By the choice of \( \epsilon \), \( \| g' \| \leq \max\{ \| g \| : g \in \mathcal{G} \} \) whenever \( g' \in \mathcal{G}' \).

4.4. Proof of Theorem 3.6 and its consequences.

Proof of Theorem 3.6. Without loss of generality, assume \( 0 < |z| \leq |w| \). Set \( w/z = te^{i\theta} \) where \( t \geq 1 \). Then, (3.1) reduces to

\[
1 \leq \frac{|2 + 2t^p - |1 + te^{i\theta}|^p - |1 - te^{i\theta}|^p|}{|4 - 2\sqrt{2}^p|}.
\]

This leads to consideration of the function

\[
f_p(\theta, t) := \begin{cases} 
2 + 2t^p - |1 + te^{i\theta}|^p - |1 - te^{i\theta}|^p & 1 \leq p < 2 \\
|1 + te^{i\theta}|^p + |1 - te^{i\theta}|^p - 2t^p - 2 & p > 2
\end{cases}
\]
We show that
\[
\min_{t \geq 1} f_p(\theta, t) = |4 - 2p|.
\]
We note that \(f_p(\theta + \pi, t) = f_p(\theta, t)\). So, we restrict attention to values of \(\theta\) between 0 and \(\pi\). We use basic multivariable calculus to minimize \(f_p(\theta, t)\) in the region \([0, \pi] \times [1, \infty)\). To this end, we first note that
\[
\frac{\partial}{\partial t}|1 + te^{i\theta}| = \frac{t \pm \cos(\theta)}{|1 + te^{i\theta}|}
\]
and that
\[
\frac{\partial}{\partial \theta}|1 + te^{i\theta}| = \frac{\mp t \sin(\theta)}{|1 + te^{i\theta}|}.
\]
It follows that when \(1 \leq p < 2\):
\[
\frac{\partial f_p}{\partial t}(\theta, t) = 2pt^{p-1} - p(t - \cos(\theta))|1 - te^{i\theta}|^{p-2} + (t + \cos(\theta))|1 + te^{i\theta}|^{p-2}
\]
\[
\frac{\partial f_p}{\partial \theta}(\theta, t) = pt \sin(\theta)|1 + te^{i\theta}|^{p-2} - |1 - te^{i\theta}|^{p-2}.
\]
The signs are reversed when \(p > 2\).
So, when \(0 < \theta_0 < \pi\) and \(t_0 \geq 1\),
\[
\frac{\partial f_p}{\partial \theta}(\theta_0, t_0) = 0 \iff |1 + t_0e^{i\theta_0}| = |1 - te^{i\theta_0}|
\]
\[
\iff \theta_0 = \pi/2.
\]
We now claim that \(\frac{\partial f_p}{\partial t}(\pi/2, t_0) > 0\) when \(t_0 \geq 1\). We first consider the case \(1 \leq p < 2\). We have
\[
\frac{\partial f_p}{\partial t}(\pi/2, t) = 2pt^{p-1} - 2pt|1 + ti|^{p-2}.
\]
Since \(t < |1 + ti|\) and \(p - 2 < 0\), \(t^{p-2} > |1 + ti|^{p-2}\). Thus, \(\frac{\partial f_p}{\partial t}(\pi/2, t_0) > 0\). The case \(2 < p\) is symmetric; the signs are merely reversed and \(p - 2 > 0\).

We next claim that \(\frac{\partial f_p}{\partial t}(0, t) \geq 0\) if \(t \geq 1\). We first consider the case \(1 \leq p < 2\). In this case the claim reduces to
\[
2 \geq \left(\frac{t}{t+1}\right)^{p-1} + \left(\frac{t+1}{t}\right)^{p-1}.
\]
Since \(0 \leq p - 1 < 1\), \(x \mapsto x^{p-1}\) is concave. Thus,
\[
1 = \left(\frac{t-1 + t+1}{2}\right)^{p-1} \geq \frac{1}{2} \left[ \left(\frac{t-1}{t}\right)^{p-1} + \left(\frac{t+1}{t}\right)^{p-1} \right].
\]
This verifies the claim when \(1 \leq p < 2\). The case \(2 < p\) is symmetric: signs are reversed and the function \(x \mapsto x^{p-1}\) is convex.

Thus, \(\frac{\partial f_p}{\partial t}(\pi, t) \geq 0\) if \(1 \leq t\).

So, let \(t_0 > 1\), and let \(R\) denote the rectangle \([0, \pi] \times [1, t_0]\). It follows from what has just been shown that the minimum of \(f_p\) on \(R\) is achieved on the lower line segment \([0, \pi] \times \{1\}\). Moreover, it is achieved at one of the points \((0, 1), (\pi/2, 1), (\pi, 1)\). \(f_p(0, 1) = f_p(0, \pi) = |2 - 2^p|\) and \(f_p(0, \pi/2) = |4 - 2\sqrt{2}^p|\). Since \(p \neq 2\), it follows that \(|4 - 2^p| > |4 - 2\sqrt{2}^p|\). Thus, the minimum of \(f_p\) on \(R\) is \(|4 - 2\sqrt{2}^p|\). Since \(t_0\) can be any number larger than 1, the minimum of \(f_p\) on \([0, \pi] \times [1, \infty)\) is \(|4 - 2\sqrt{2}^p|\).
The theorem now follows. □

The proof of Theorem 3.7 is based on the following lemmas.

**Lemma 4.4.** Suppose \( p \neq 2 \), and suppose \( g_1, \ldots, g_n \in \ell^p \). There exist disjointly supported \( h_1, \ldots, h_n \in \ell^p \) so that \( h_j \leq g_j \) for all \( j \) and \( \|h_j - g_j\|^p \leq \max_{m \neq k} \sigma(g_m, g_k) \times n \).

**Proof.** Let \( j \in \{1, \ldots, n\} \). When \( j \neq k \) set

\[
S_{j,k} = \{ t : 0 < |g_j(t)| \leq |g_k(t)| \}.
\]

Set \( S_j = \bigcup_k S_{j,k} \). By Theorem 3.6 for all \( t \in S_{j,k} \), \( |g_j(t)|^p \leq \sigma(g_j(t), g_k(t)) \). Set \( h_j = g_j - g_j |S_j \). Then, \( \|h_j - g_j\|^p = \|g_j|S_j\|^p \leq \max_k \sigma_{j,k} \sigma(\sigma_j, \sigma_k) \times n \).

Suppose \( t \in \text{supp}(h_j) \) and \( k \neq j \). Then, \( t \notin S_j \). Hence, \( |g_j(t)| > \max_k \|g_k\| \).

Thus, \( S_{j,k} \) gives an upper bound on the distance from \( (g_1, \ldots, g_n) \) to the nearest \( n \)-tuple of disjointly supported vectors. Again, by the continuity of \( \sigma \), this upper bound tends to 0 as \( g_1, \ldots, g_n \) approach disjointly supported vectors.

**Lemma 4.5.** Suppose \( p \neq 2 \), and suppose \( f, g \in \ell^p \). Then, there exists \( g_1 \leq f \) so that \( \|g_1 - g\|^p \leq \sigma(f - g, g) \).

**Proof.** When \( t \in \mathbb{N} \), let

\[
g_1(t) = \begin{cases} 0 & \text{if } |g(t)| \leq |f(t) - g(t)| \\ f(t) & \text{otherwise} \end{cases}
\]

Thus, \( g_1 \leq f \). Also, \( |g_1(t) - g(t)|^p = \min\{|g(t)|^p, |f(t) - g(t)|^p\} \). So, by Theorem 3.6 \( |g_1(t) - g(t)|^p \leq \sigma(f(t) - g(t), g(t)) \) for all \( t \). Thus, \( \|g_1 - g\|^p \leq \sigma(f - g, g) \). □

So, Lemma 4.5 gives an upper bound on the distance from \( g \) to the nearest subvector of \( f \). By the continuity of \( \sigma \), this upper bound tends to 0 as \( g \) approaches a subvector of \( f \).

The next result gives an upper bound on the distortion of \( \sigma(f, g) \) as \( f, g \) are varied.

**Lemma 4.6.** Suppose \( f_1, f_2, g_1, g_2 \in \ell^p \), and suppose \( \|f_1\|, \|f_2\|, \|g_1\|, \|g_2\| \leq M \). Then,

\[
\|\sigma(f_1, g_1) - \sigma(f_2, g_2)\| \leq 4pcp(2M)^{p-1}(\|f_1 - f_2\| + \|g_1 - g_2\|).
\]

**Proof.** By the Mean Value Theorem, when \( 0 \leq a, b \leq 2M \), \( |a^p - b^p| \leq p(2M)^{p-1}|a - b| \). By assumption, \( \|f_j\|, \|g_j\|, \|f_j + g_j\| \leq 2M \). Thus,

\[
\|f_1 - f_2\|^p \leq p(2M)^{p-1} \|f_1\| - \|f_2\| \leq p(2M)^{p-1} \|f_1 - f_2\|,
\]

\[
\|g_1 - g_2\|^p \leq p(2M)^{p-1} \|g_1\| - \|g_2\| \leq p(2M)^{p-1} \|g_1 - g_2\|,
\]

\[
\|f_1 + g_1 - f_2 - g_2\|^p \leq p(2M)^{p-1} \|f_1 + g_1\| - \|f_2 + g_2\| \leq p(2M)^{p-1} \|f_1 - f_2\| + \|g_1 - g_2\|.
\]

The lemma follows. □
Proof of Theorem 3.7: We first note that since \( p \geq 1 \), \( a \leq a^{1/p} < 1 \) when \( 0 \leq a < 1 \) and \( a^{1/p} < a \) when \( a > 1 \). By assumption, \( \epsilon_0(T, \lambda) < 1 \).

Set
\[
\delta(s) = (\#T(160)N_p^2)^{s-1}(\#T)^{1/p}\epsilon_0(T, \lambda)^{1/p^{2s-1}}.
\]

Note that \( \delta \) is increasing since \( \epsilon_0(T, \lambda) < 1 \). Note also that \( \delta(\#T) = \Delta(T, \lambda) \).

By Definition of \( \epsilon_0(T, \lambda) \), \( \sigma(\lambda(\nu), \lambda(\nu')) \leq \epsilon_0(T, \lambda) \) whenever \( \nu, \nu' \) are distinct nodes in the same level of \( T \). By Lemma 4.5, we can choose the values of \( \Lambda \) on level 1 of \( T \) so that they are disjointly supported, so that \( \Lambda(\nu) \cap \Lambda(\nu') = \emptyset \) when \( \nu \neq \nu' \) and so that \( ||\Lambda(\nu) - \Lambda(\nu)||^p \leq \#T\epsilon_0(T, \lambda) \). So,
\[
||\Lambda(\nu) - \Lambda(\nu)|| \leq (\#T)^{1/p}\epsilon_0(T, \lambda)^{1/p} = \delta(1).
\]

Now, suppose \( \nu \) belongs to the \((s + 1)\)st level of \( T \) where \( s \geq 1 \). By way of induction suppose \( ||\Lambda(\nu^s) - \Lambda(\nu^-)|| \leq \delta(s) \). Set \( C = \delta(s)\epsilon_0(T, \lambda)^{-1/p^{2s-1}} \). Thus, by definition of \( \delta, C \geq 1 \).

Since \( p(\#\lambda(\nu))\delta^{-1} \leq 1 \), it follows from Lemma 4.6 that
\[
|\sigma(\Lambda(\nu^-) - \Lambda(\nu), \lambda(\nu)) - \sigma(\Lambda(\nu^-) - \lambda(\nu), \lambda(\nu))| \leq 4N_p \|\Lambda(\nu^-) - \lambda(\nu^-)\|
\leq 4N_pC\epsilon_0(T, \lambda)^{1/p^{2s-1}}.
\]

By definition of \( \epsilon_0(T, \lambda) \), \( \sigma(\Lambda(\nu^-) - \lambda(\nu), \lambda(\nu)) \leq \epsilon_0(T, \lambda) \). Therefore,
\[
\sigma(\Lambda(\nu^-) - \lambda(\nu), \lambda(\nu)) \leq \sigma(\Lambda(\nu^-) - \lambda(\nu), \lambda(\nu)) + 4N_pC\epsilon_0(T, \lambda)^{1/p^{2s-1}}
\leq \epsilon_0(T, \lambda) + 4N_pC\epsilon_0(T, \lambda)^{1/p^{2s-1}}
\leq 5N_pC\epsilon_0(T, \lambda)^{1/p^{2s-1}}.
\]

By Lemma 4.5, there is a vector \( \Lambda_1(\nu) \leq \Lambda(\nu^-) \) so that
\[
||\Lambda_1(\nu) - \lambda(\nu)||^p \leq 5N_pC\epsilon_0(T, \lambda)^{1/p^{2s-1}}.
\]

So, since \( p \geq 1 \) and \( N_p, C \geq 1 \),
\[
||\Lambda_1(\nu) - \lambda(\nu)|| \leq 5N_pC\epsilon_0(T, \lambda)^{1/p^{2s}}.
\]

Now, by Lemma 4.6 again, when \( \nu, \nu' \in T \) and \( |\nu| = |\nu'| = s + 1 \),
\[
|\sigma(\Lambda_1(\nu), \Lambda_1(\nu')) - \sigma(\Lambda(\nu), \lambda(\nu'))| \leq 8N_p \left[ 5N_pC\epsilon_0(T, \lambda)^{1/p^{2s}} \right]
= 40N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s}}
\]

Thus,
\[
\sigma(\Lambda_1(\nu), \Lambda_1(\nu')) \leq \sigma(\Lambda(\nu), \lambda(\nu')) + 40N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s}}
\leq \epsilon_0(T, \lambda) + 40N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s}}
\leq 80N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s}}.
\]

Thus, by Lemma 4.4, we can choose the values of \( \Lambda \) on the \((s + 1)\)st level of \( T \) so that they are disjointly supported, so that \( \Lambda(\nu) \leq \Lambda_1(\nu) \), and so that
\[
||\Lambda(\nu) - \Lambda_1(\nu)||^p \leq \#T80N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s}}.
\]

Thus,
\[
||\Lambda(\nu) - \Lambda_1(\nu)|| \leq \#T80N_p^2C\epsilon_0(T, \lambda)^{1/p^{2s+1}}.
\]
So,
\[
\|\Lambda(\nu) - \lambda(\nu)\| \leq 5N_p C_{\ell_0} (T, \lambda)^{1/p^2} + \#T80N_p^2C_{\ell_0}(T, \lambda)^{1/p^2+1} \\
\leq \delta(s+1).
\]
This completes the proof of the theorem.

We now give a short proof of the Banach-Lamperti classification of the isometries of \(\ell^p\) that was mentioned in Section 2.

**Proof of Theorem 2.1.** The first part of the theorem is an immediate consequence of Theorem 3.6.

Suppose \(T\) is an isometric endomorphism of \(\ell^p\). From the first part of the theorem, it follows that the supports of \(T(e_0), T(e_1), \ldots\) are pairwise disjoint. In addition, these vectors have norm 1 and generate \(\ell^p\). It follows that each is an atom of norm 1. The existence of \(\phi, \lambda_0, \lambda_1, \ldots\) follows. The converse follows from Theorem 5.1.

\[\square\]

5. Approximate World

We now prove Theorems 3.10 and 3.11. The proofs are based on the following lemmas.

**Lemma 5.1.** Suppose \(T\) is a finite tree, \(\lambda : T - \{\emptyset\} \to \ell^p\), and \(\beta_j : T - \{\emptyset\} \to \mathbb{C}\) for each \(j < \kappa\). Suppose \(\Lambda : T - \{\emptyset\} \to \ell^p\) is such that \(\|\lambda(\nu) - \Lambda(\nu)\| < v(T, \lambda, \kappa, \{\beta_j\}_j, p)\) for all \(\nu \in T - \{\emptyset\}\). Then:

1. \(\|\Lambda(\nu)\| > \frac{1}{2} \|\lambda(\nu)\|\).
2. \(\|f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu)\| < 2^{-\kappa}\)
3. \(\|\Lambda(\nu) - \sum_{\nu \in T} \lambda(\nu \setminus (a))\| < 2^{-\kappa}\) whenever \(\nu\) is a non-root non-terminal node of \(T\).

**Proof.** 1. By definition of \(v\) (see Section 3), \(v(T, \lambda, \kappa, \{\beta_j\}_j, p) \leq \frac{1}{2} \|\lambda(\nu)\|\) for all \(\nu \in T - \{\emptyset\}\). Thus, \(\|\lambda(\nu) - \Lambda(\nu)\| < \frac{1}{2} \|\lambda(\nu)\|\), and so \(\|\Lambda(\nu)\| > \frac{1}{2} \|\lambda(\nu)\|\).

2. By definition of \(v\),
\[
v(T, \lambda, \kappa, \{\beta_j\}_j, p) \cdot \sum_{\nu} |\beta_j(\nu)| \leq 2^{-\kappa} - \left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\|.
\]
We have
\[
\left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\| \leq \left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\| + \left\| \sum_{\nu} \beta_j(\nu) (\Lambda(\nu) - \lambda(\nu)) \right\|
\leq \left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\| + \left\| \sum_{\nu} \beta_j(\nu) \right\| + \left\| \sum_{\nu} |\beta_j(\nu)| \right\|
\leq \left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\| + 2^{-\kappa} - \left\| f_j - \sum_{\nu} \beta_j(\nu) \lambda(\nu) \right\| = 2^{-\kappa}.
\]
Suppose \( \nu \) is a non-root non-terminal node of \( T \). By definition of \( \nu \),

\[
(1 + \#T)v(T, \lambda, \kappa, \{ \beta_j \}_{j<\kappa}, \nu) \leq 2^{-\kappa} - \left\| \lambda(\nu) - \sum_{\nu^-(a) \in T} \lambda(\nu^-(a)) \right\|
\]

We have

\[
\left\| \Lambda(\nu) - \sum_{\nu^-(a) \in T} \Lambda(\nu^-(a)) \right\| \leq \left\| \Lambda(\nu) - \lambda(\nu) \right\| + \left\| \lambda(\nu) - \sum_{\nu^-(a) \in T} \lambda(\nu^-(a)) \right\|
\]

\[
+ \left\| \sum_{\nu^-(a) \in T} (\Lambda(\nu^-) - \lambda(\nu^-)) \right\|
\]

\[
< v(T, \lambda, \kappa, \{ \beta_j \}_{j<\kappa}, t) + \left\| \lambda(\nu) - \sum_{\nu^-(a) \in T} \lambda(\nu^-(a)) \right\|
\]

\[
+ (#T)v(T, \lambda, \kappa, \{ \beta_j \}_{j<\kappa}, t)
\]

\[
\leq \left\| \lambda(\nu) - \sum_{\nu^-(a) \in T} \lambda(\nu^-) \right\| + 2^{-\kappa} - \left\| \lambda(\nu) - \sum_{\nu^-(a) \in T} \lambda(\nu^-) \right\| = 2^{-\kappa}.
\]

\[\square\]

**Lemma 5.2.** Suppose \( A = (T, \lambda, \kappa, \{ \beta_j \}_{j<\kappa}, \nu) \) is an approximate shattering. Suppose \( \Lambda : T - \emptyset \rightarrow \mathcal{P}^\nu \) is an order homomorphism so that \( \|\Lambda(\nu) - \lambda(\nu)\| < v(A) \) for all \( \nu \in T - \{\emptyset\} \) and so that \( \Lambda \) maps incomparable vectors to disjointly supported vectors. Then, \( (\text{ran}(\Lambda), \kappa) \) is a shattering, and \( \Lambda \) is an order monomorphism.

**Proof.** Set \( G = \text{ran}(\Lambda) \). By Lemma 5.1 each vector in \( G \) is nonzero.

We first show that \( \Lambda \) is an order monomorphism. Suppose \( \emptyset \neq \nu \subset \nu' \in T \). We show that \( \Lambda(\nu) \neq \Lambda(\nu') \). We can assume \( \nu' \) is a chid of \( \nu \). Since \( A \) is an approximate shattering, each non-terminal non-root node of \( T \) has at least two children (see Definition 5.4). So, let \( \nu'' \) be a child of \( \nu \) in \( T \) so that \( \nu'' \neq \nu' \). By assumption, the supports of \( \Lambda(\nu') \) and \( \Lambda(\nu'') \) are disjoint and \( \Lambda(\nu') \), \( \Lambda(\nu'') \) are nonzero. Since \( \Lambda(\nu'') \) and \( \Lambda(\nu') \) are nonzero, \( \Lambda(\nu') \neq \Lambda(\nu) \).

We now show that \( (G, \kappa) \) is a shattering. Suppose \( g, g' \in G \) are incomparable. Let \( \nu = \Lambda^{-1}(g) \), and let \( \nu' = \Lambda^{-1}(g') \). Thus, since \( \Lambda \) is an order homomorphism, \( \nu \) and \( \nu' \) are incomparable. So, by assumption, the supports of \( g \) and \( g' \) are disjoint. The remaining conditions of Definition 3.4 now follow from Lemma 5.1. \[\square\]

The statements of the next two lemmas require the following.

**Definition 5.3.** Let \( A = (T, \lambda, \kappa, \{ \beta_j \}_{j<\kappa}, \nu) \) be an approximate shattering. Let \( (G, \kappa) \) be a shattering. We say that \( A \) approximates \( (G, \kappa) \) if there is an order isomorphism \( \phi \) of \( G \) onto \( T - \{\emptyset\} \) so that \( \|\lambda(\nu) - \phi^{-1}(\nu)\| < v(A) \) for all \( \nu \in T - \{\emptyset\} \). In this case we also say that \( A \) approximates \( (G, \kappa) \) via \( \phi \).
Lemma 5.4 (Simultaneous Approximation Lemma). Suppose $A_0, \ldots, A_r$ are approximate shatterings so that $A_r$ extends $A_t$ whenever $t' > t$. Then, there exist shatterings $(G_0, \kappa_0), \ldots, (G_r, \kappa_r)$ and a map $\phi$ so that $A_r$ approximates $(G_t, \kappa_t)$ via $\phi$ for each $t$ and $(G_{t+1}, \kappa_{t+1})$ extends $(G_t, \kappa_t)$ for all $t$.

Proof. Let $A_r = (T_r, \lambda_r, \kappa_r, \{\beta^0_j\}_{j < \kappa_r, p_r})$. Since $A_r$ extends $A_t$, $\Delta(A_r) < v(A_t)$ (by Definition 3.9).

Since $A_r$ is a shattering, by Definition 3.8, $e_0(T_r, \lambda_r) < 1$. If $p > 1$, then since $\Delta(A_r) < v(A_t)$, $\frac{1}{p} - \frac{1}{(p-1)} < \|\lambda(\nu)\| > 0$ and so $p(\|\lambda(\nu)\|)^{p-1} < 1$; if $p = 1$ then $p(\|\lambda(\nu)\|)^{p-1} = 1$. By Theorem 3.7, there is an order isomorphism $\lambda : T_r - \{\emptyset\} \to \ell^p$ so that $\|\lambda(\nu) - \lambda_r(\nu)\| \leq \Delta(A_r)$ and so that $\text{supp}(\lambda(\nu)) \cap \text{supp}(\lambda(\nu')) = \emptyset$ whenever $\nu, \nu'$ are incomparable nodes of $T$.

Set $\phi = \lambda^{-1}$. When $0 < t < r$, set $G_t = \lambda[T_t]$. By Lemma 5.2 $(G_t, \kappa_t)$ is a shattering and the restriction of $\phi$ to $T_t$ is an order isomorphism of $(G_t, \geq)$ with $T - \{\emptyset\}$. By construction, $A_t$ approximates $(G_t, \kappa_t)$ via $\phi$.

Since $T_t \subseteq T_{t'}$ when $t' > t$, and since $\phi$ is an order isomorphism, $G_t^{(s)} \geq G_{t'}^{(s)}$ for all $s$. Thus, $(G_t, \kappa_t)$ extends $(G_t, \kappa_t)$.

Lemma 5.5 (Simultaneous Extension Lemma). Suppose $A_0, \ldots, A_r$ are approximate shatterings that approximate $(G_0, \kappa_0), \ldots, (G_r, \kappa_r)$ respectively. Suppose further that there is a single map $\phi$ so that $A_i$ approximates $(G_i, \kappa_i)$ via $\phi$ for each $i$. Finally, suppose $(G_{r+1}, \kappa_{r+1})$ extends $(G_r, \kappa_r)$ and $p(\|g\|)^{p-1} < 1$ for all $g \in G_{r+1}$ if $p > 1$.

Then, there is an approximate shattering $A_{r+1}$ that extends each of $A_0, \ldots, A_r$ and that approximates $(G_{r+1}, \kappa_{r+1})$ via a map $\psi$ that extends $\phi$.

Proof. Set $A_r = (T_r, \lambda_r, \kappa_r, \{\beta^0_j\}_{j < \kappa_r, p_r})$. By Definition 5.3, $\phi$ is an order isomorphism. Since $(G_{r+1}, \kappa_{r+1})$ extends $(G_r, \kappa_r)$, $G_{r+1}^{(s)} \supseteq G_r^{(s)}$ for all $s$. Thus, there is a tree $T_{r+1}$ so that $\phi$ extends to an order isomorphism of $G_{r+1}$ with $T - \{\emptyset\}$; let $\psi$ denote such an extension. Since $G_{r+1}$ is a shattering, each vector in $G_{r+1}$ that has at least one predecessor in $(G_{r+1}, \leq)$ has at least two such predecessors. Therefore, each node that has a child in $T_{r+1}$ has at least two children in $T_{r+1}$.

For the moment, fix $j < \kappa_{r+1}$. Since $(G_{r+1}, \kappa_{r+1})$ is a shattering, there exists $\beta^{(r+1)}_j : T_{r+1} - \{\emptyset\} \to \ell^p(i) - \{0\}$ such that

$$\left\| f_j - \sum_{\nu} \beta^{(r+1)}_j(\nu) \psi^{-1}(\nu) \right\| < 2^{-\kappa_{r+1}}.$$
If \( p > 1 \), then by assumption \( p(2\|g\|)^{p-1} < 1 \) whenever \( g \in \mathcal{G}_{r+1} \). Since \( A_t \) approximates \((\mathcal{G}_t, \kappa_t)\) via \( \phi \), \( \|\phi^{-1}(\nu) - \lambda(\nu)\| < v(A_t) \) for all \( \nu \in T_t - \{\emptyset\} \). Thus, \( \|\phi^{-1}(\nu) - \lambda(\nu)\| < \frac{1}{2}\|\lambda(\nu)\| \). So, by the continuity of \( \sigma \) and the choice of \( p_{r+1} \), we can choose \( \lambda_{r+1} \) as required by choosing \( \lambda_{r+1} \) sufficiently close to \( \psi^{-1} \). Set \( A_{r+1} = (T_{r+1}, \lambda_{r+1}, \kappa_{r+1}, \{\beta_j\}^f_{j=r+1}) \).

By the choice of \( \lambda_{r+1} \), \( A_{r+1} \) is an approximate shattering that approximates \((\mathcal{G}_{r+1}, \kappa_{r+1})\) via \( \psi \). By the choice of \( p_{r+1} \) and \( \lambda_{r+1} \), \( A_{r+1} \) extends each of \( A_0, \ldots, A_r \).

**Proof of Theorem 3.10.** By the Simultaneous Approximation Lemma, there exist \((\mathcal{G}_0, \kappa_0), \ldots, (\mathcal{G}_r, \kappa_r) \) and \( \phi \) so that each \((\mathcal{G}_t, \kappa_t)\) is a shattering and \( A_t \) approximates \((\mathcal{G}_t, \kappa_t)\) via \( \phi \). Therefore, \( \|\phi^{-1}(\nu) - \lambda(\nu)\| < v(A_t) \) for all \( \nu, \tau \). By definition of \( \nu, \tau \), \( p(2\|\phi^{-1}(\nu)\|)^{p-1} < 1 \) if \( p > 1 \). Let \( \kappa_{r+1} > \kappa_r \). By Theorem 3.9, there is a set of vectors \( \mathcal{G}_{r+1} \) so that \((\mathcal{G}_{r+1}, \kappa_{r+1})\) is a shattering that extends \((\mathcal{G}_r, \kappa_r)\). In addition, if \( p > 1 \), then Theorem 3.9 guarantees that we can choose \( \mathcal{G}_{r+1} \) so that \( p(2\|g\|)^{p-1} < 1 \) for all \( g \in \mathcal{G}_{r+1} \). By the Simultaneous Extension Lemma, \( A_{r+1} \) exists.

**Proof of Theorem 3.11.** (1): Since \( A_{r+1} \) extends \( A_r \), \( p_{r+1} > p_r \) and so \( \lim p_r = \infty \). Since \( A_{r+1} \) extends \( A_r \), \( \|\lambda_{r+1}(\nu) - \lambda(\nu)\| < 2^{-p_r} \) (see Condition 3.9) of Definition 3.9). Therefore, \( \lim_{s \to \infty} \lambda_s(\nu) \) exists. By Definition 3.9, \( \|\lambda_s(\nu) - \lambda(\nu)\| < \frac{1}{2}\|\lambda(\nu)\| \) whenever \( s \geq t \). Thus, \( \lambda(\nu) \) is nonzero.

(2): We first claim that \( \epsilon_0(T_{t'}, \lambda_{t'}) < 2^{-p_{t'}} \) if \( t' > t \). For, let \( t \in \mathbb{N} \). Since \( \epsilon_0(T_t, \lambda_t) < 1 \), and since \( p \geq 1 \), it follows from the definition of \( \Delta \) that \( \Delta(A_t) > \epsilon_0(T_t, \lambda_t) \). Since \( A_t \) extends \( A_r \), by Definition 3.9 \( \Delta(A_t) < v(A_t) \). By Definition of \( v \), \( v(A_t) \leq 2^{-p_t} \).

So, suppose \( \nu, \nu' \in T_t \). Then, \( \nu, \nu' \in T_t \) for all sufficiently large \( t \). Suppose \( \nu < \nu' \). Then, \( \sigma(\lambda_t(\nu) - \lambda_t(\nu')) \leq \epsilon_0(T_t, \lambda_t) \) for all sufficiently large \( t \). Thus, by what has just been shown, \( \sigma(\lambda(\nu) - \lambda(\nu')) < 2^{-p_r} \) for all sufficiently large \( t \). Thus, \( \sigma(\lambda(\nu) - \lambda(\nu')) = 0 \), and \( \lambda(\nu') \neq \lambda(\nu) \). Similarly, if \( \nu, \nu' \) are incomparable, then \( \sigma(\lambda(\nu), \lambda(\nu')) = 0 \) and so the supports of \( \lambda(\nu) \) and \( \lambda(\nu') \) are disjoint. It then follows as in the proof of Lemma 5.2 that \( \lambda \) is an order monomorphism.

(3): Set \( \mathcal{G}_t = \lambda[T_{t+1}] \). We claim that \((\mathcal{G}_{t+1}, \kappa_t)\) is a shattering. We verify the conditions of Definition 3.4 one by one. Condition (1) holds by definition of \( \mathcal{G}_t \). Condition (2) holds by what has just been shown. Conditions (3) and (4) hold since \( A_t \) extends \( A_{t+1} \) when \( t' > t + 1 \) and since \( \kappa_t < \kappa_{t+1} \).

### 6. Computable world

#### 6.1. Proof of Theorems 3.12 and 3.13

We now prove Theorems 3.12 and 3.13 via the following sequence of lemmas.

**Throughout this subsection we assume \( p \) is computable and \( p \neq 2 \). We also assume \( F \) is an effective generating set for \( \ell^p \).**

**Lemma 6.1.** Suppose \( \{g_n\}_n \) induces a linear isometry of \( \ell^p \). If \( X \subseteq \mathbb{N} \) computes \( \{g_n\}_n \) with respect to \( F \), then \( X \) computes the linear isometry induced by \( \{g_n\}_n \) with respect to \( (E, F) \).
Lemma 6.2. There is a computable sequence of approximate shatterings \( \{A_t\}_{t \in \mathbb{N}} \) so that \( A_{t+1} \) extends \( A_t \) for all \( t \).

Proof. We first note that since \( F \) is an effective generating set, the set of approximate shatterings is computably enumerable as is the ‘extends’ relation on approximate shatterings.

Set \( A_0 = (\{\emptyset\}, \emptyset, 0, 2^{-0}, \emptyset) \). Suppose \( A_0, \ldots, A_n \) have been defined and that \( A_{t'} \) extends \( A_t \) when \( t' > t \). Then, by Theorem 3.10 there is an approximate shattering \( A_{t+1} \) that extends \( A_0, \ldots, A_n \) and so such a shattering can be found by a search procedure.

Throughout the rest of this section:
1. \( \{A_t\}_{t \in \mathbb{N}} \) is a computable sequence of approximate shatterings so that \( A_{t+1} \) extends \( A_t \) for all \( t \).
2. \( A_t = (T_t, \lambda_t, \kappa_t, \{\beta_j^t\}_j, p_t) \).
3. \( \lambda = \lim_t \lambda_t \).
4. \( T = \bigcup_t T_t \).
5. \( G = \lambda[T] \).

Thus, by Theorem 3.11 \( G \) is a disintegration.

Lemma 6.3. \( \lambda \) is computable (with respect to \( F \)).

Proof. Since \( A_{t+1} \) extends \( A_t \), \( \| \lambda_t(\nu) - \lambda_{t+1}(\nu) \| < 2^{-p_t} \) when \( \nu \in T_t \) (see Definition 3.9); it also follows that \( p_{t+1} > p_t \). Thus, \( \lambda \) is computable.

By definition, \( T \) is a c.e. subtree of \( \mathbb{N}^* \). When \( \nu \in T \), let \( \nu^+ \) denote the set of all children of \( \nu \) in \( T \). When \( \nu \in T_s \), let \( \nu^+_s \) denote the set of all children in \( T_s \).

Lemma 6.4. If \( \nu \in T - \{\emptyset\} \), and if \( \nu^+ \neq \emptyset \), then there is a number \( s \) so that

\[
(6.1) \quad \| \lambda(\nu) \|^p - \sum \| \lambda(\mu) \|^p < \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+_s \}
\]

When \( s \) is such a number,
\[
\max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+_s \} = \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \}.
\]

Proof. Suppose \( \nu^+ \neq \emptyset \). By Proposition 3.3 there is a \( \mu_0 \in \nu^+ \) so that
\[
\| \lambda(\mu_0) \|^p = \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \}.
\]

Since \( G \) is a disintegration,
\[
\lim_{s \to \infty} \| \lambda(\nu) \|^p - \sum_{\mu \in \nu^+_s} \| \lambda(\mu) \|^p = 0.
\]

Thus, there is a number \( s \) so that (6.1) holds.
Thus, \( \mu \in \nu^+ \). By Lemma 6.4, there is a stage \( M \). So, there is a number and bound on \( \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \} \). Therefore, 
\[
\| \lambda(\nu) \|^p < \sum_{\mu \in \nu^+} \| \lambda(\mu) \|^p + \| \lambda(\nu_0) \|^p.
\]
Since \( \nu_0 \in \nu^+ \) but \( \nu_0 \notin \nu^+_s \), it follows that \( \mu_0 \) is incomparable with every node in \( \nu^+_s \). So, by Proposition 4.1, 
\[
\sum_{\mu \in \nu^+} \| \lambda(\mu) \|^p + \| \lambda(\nu_0) \|^p \leq \sum_{\mu \in \nu^+} \| \lambda(\nu) \|^p = \| \lambda(\nu) \|^p.
\]
This is a contradiction. Therefore, \( \| \lambda(\nu_0) \|^p = \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \} \). □

By Lemma 6.6, there is a computable function \( q : T - \{ \emptyset \} \times \mathbb{N} \to \mathbb{Q} \) so that \( |q(\nu, s) - \| \lambda(\nu) \|^p| < 2^{-s} \). Set:
\[
m(\nu, s) = \min\{q(\nu, s) - 2^{-s}, 0\}
\]
\[
M(\nu, s) = q(\nu, s) + 2^{-s}
\]
\[
\Sigma^-(X, s) = \sum_{\nu \in X} m(\nu, s)
\]
\[
\overline{m}(X, s) = \max\{m(\mu, s) : \mu \in X\}
\]
Thus, \( m(\nu, s) \) is a lower bound on \( \| \lambda(\nu) \|^p \), and \( M(\nu, s) \) is an upper bound on \( \| \lambda(\nu) \|^p \). \( \Sigma^-(\nu^+_s, s) \) is a lower bound on \( \sum_{\mu \in \nu^+} \| \lambda(\mu) \|^p \), and \( \overline{m}(\nu^+_s, s) \) is a lower bound on \( \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \} \).

**Lemma 6.5.** Suppose \( \nu \in T - \{ \emptyset \} \) and \( \nu^+ \neq \emptyset \). Then, there is a stage \( s \) so that 
\[
M(\nu, s) - \Sigma^- (\nu^+_s, s) < \overline{m}(\nu^+_s, s).
\]
At such a stage \( s \),
\[
0 \leq \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \} - \overline{m}(\nu^+_s, s) \leq 2^{-s+1}
\]
**Proof.** By Lemma 6.4 there is a stage \( s_0 \) so that
\[
\| \lambda(\nu) \|^p - \sum_{\mu \in \nu^+_s} \| \lambda(\mu) \|^p < \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+_s \}.
\]
Set \( U = \nu^+_s \). Then,
\[
\lim_{s \to \infty} M(\nu, s) - \Sigma^- (U, s) = \| \lambda(\mu) \|^p - \sum_{\mu \in U} \| \lambda(\mu) \|^p
\]
and,
\[
\lim_{s \to \infty} \overline{m}(\nu^+_s, s) = \max \{ \mu \in \nu^+ : \| \lambda(\mu) \|^p \}.
\]
So, there is a number \( s_1 > s_0 \) so that 
\[
M(\nu, s_1) - \Sigma^- (U, s_1) < \overline{m}(\nu^+_s, s_1).
\]
\( m(\nu, s) \geq 0 \) by definition and \( U \subseteq \nu^+_s \) since \( s_1 > s_0 \), so 
\[
M(\nu, s_1) - \Sigma^- (\nu^+_s, s_1) < \overline{m}(\nu^+_s, s_1).
\]
Now, suppose \( M(\nu, s) - \Sigma^- (\nu^+_s, s) < \overline{m}(\nu^+_s, s) \). By definition of \( M, \Sigma^- \), 
\[
\| \lambda(\nu) \|^p - \sum_{\mu \in \nu^+_s} \| \lambda(\mu) \|^p \leq M(\nu, s) - \Sigma^- (\nu^+_s, s),
\]
and
\[
\overline{m}(\nu^+_s, s) \leq \max_{\mu \in \nu^+_s} \| \lambda(\mu) \|^p
\]
So, by Lemma 6.4
\[
\max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+ \} = \max \{ \| \lambda(\mu) \|^p : \mu \in \nu^+_s \}.
\]
Furthermore,
\[
\max \{ \| \lambda(\nu) \|^p : \mu \in \nu^+_s \} \leq m(\nu, s) + 2^{-s}.
\]
This proves the lemma. \qed

Lemma 6.6. For each \( \nu \in T - \{ \emptyset \} \), there is a c.e. set \( \Phi_\nu \subseteq T \), so that \( \lambda[\Phi_\nu] \) is an almost norm-maximizing chain and so that \( \lambda(\nu) \) is the \( \preceq \)-largest vector in \( \lambda[\Phi_\nu] \). Furthermore, an index of \( \Phi_\nu \) can be computed uniformly from \( \nu \).

Proof. We define a computable function \( \psi \) as follows. Set \( \psi(0) = \nu \). Given that \( \psi(t) \) has been defined, define \( \psi(t+1) \) as follows. Set \( \mu = \psi(t) \). Wait for a successor of \( \mu \) to appear in \( T \). Then, search for \( s > t + |\nu| \) so that
\[
M(\mu, s) - \Sigma^{-}(\mu^1_s, s, s) < \overline{m}(\mu^1_s, s).
\]
Then, find \( \tau \in \mu^1_s \) so that \( m(\tau, s) = \overline{m}(\mu^1_s, s) \) and set \( \psi(t+1) = \tau \). Therefore,
\[
\max \{ \| \lambda(\mu') \|^p : \mu' \in \mu^1_s \} = \max \{ \| \lambda(\mu') \|^p : \mu' \in \mu^1_s \}
\leq m(\psi(t+1), s) + 2^{-s}
\leq m(\psi(t+1), s) + 2^{-(t+|\nu|)}
\leq \| \lambda(\psi(t+1)) \|^p + 2^{-(t+|\nu|)}
\]
Thus, \( \psi(t+1) \) is an almost norm-maximizing successor of \( \psi(t) \) in \( (G, \preceq) \).

Set \( \Phi_\nu = \text{ran}(\psi) \). Therefore, \( \lambda[\Phi_\nu] \) is an almost norm-maximizing chain, is computably enumerable, and \( \lambda(\nu) \) is its \( \preceq \)-largest vector. \qed

Proof of Theorem 3.12. By simultaneous recursion, we define a uniformly c.e. family of sets \( \{ S_\nu \} \), and for each \( \nu \in S := \bigcup \nu S_\nu \) a c.e. subset of \( T, \mathcal{N}_\nu \). Set \( \Phi_\nu = \lambda[\mathcal{N}_\nu] \) for all \( \nu \in S \). We maintain the following invariants.

1. \( \Phi_\nu \) is an almost norm-maximizing chain and \( \lambda(\nu) \) is the \( \preceq \)-largest vector in \( \Phi_\nu \).
2. If \( \nu, \mu \in \bigcup \nu \leq S_\nu \) are distinct, then \( \Phi_\nu \) and \( \Phi_\mu \) are disjoint.
3. \( S_\nu \subseteq T \cap \mathbb{N}^{r+1} \).
4. If \( \nu \in T \cap \mathbb{N}^{r+1} \), then there is a unique number \( s \) so that \( 1 \leq s \leq r + 1 \) and so that \( \nu \in \mathcal{N}_{\nu|s} \).

We begin by defining \( S_0 = T \cap \mathbb{N}^1 \). By Lemma 6.6 for each \( \nu \in S_0 \) we can compute an index of a c.e. subset of \( T, \mathcal{N}_\nu \), so that \( \lambda[\mathcal{N}_\nu] \) is an almost norm-maximizing chain whose \( \preceq \)-largest element is \( \lambda(\nu) \). If \( \nu, \mu \in S_0 \) are distinct, then the supports of \( \lambda(\nu) \) and \( \lambda(\mu) \) are disjoint and so \( \Phi_\nu \) and \( \Phi_\mu \) are disjoint. By definition, \( \nu \in \Phi_\nu \) for all \( \nu \in T \cap \mathbb{N}^1 \).

We now define \( S_{r+1} \) and \( \mathcal{N}_\nu \) for each \( \nu \in S_{r+1} \). Suppose \( \nu \in T \cap \mathbb{N}^{r+2} \). Then, there exists a unique \( s \) so that \( 1 \leq s \leq r + 1 \) and \( \nu^s \in \mathcal{N}_{\nu|s} \). Since the \( \mathcal{N}_\nu \)'s are uniformly c.e., we can compute \( s \) in finitely many steps. Since \( \nu^- \) has at least one child in \( T \), it follows that \( \mathcal{N}_{\nu|s} \) contains exactly one child of \( \nu^- \) in \( T \); label this child node \( \mu \). Again, since we can compute an index of \( \mathcal{N}_{\nu|s} \), we can compute \( \mu \) in finitely many steps. If \( \mu \neq \nu \), then enumerate \( \nu \) into \( S_{r+1} \) and compute an index for a c.e. set \( \mathcal{N}_\nu \subseteq T \) so that \( \lambda[\mathcal{N}_\nu] \) is an almost norm-maximizing chain whose \( \preceq \)-largest element is \( \lambda(\nu) \). Note that if \( \nu \in S_{r+1} \), then \( \nu \in \Phi_\nu \). Note also that if \( \nu \in S_{r+1} \), then \( \Phi_\nu \) and \( \Phi_{\nu|s} \) are disjoint. It follows that all invariants are maintained.
Finally, it follows from the invariants that \( \{ \Phi_\nu \}_{\nu \in S} \) is a partition of \( G \).

\[ \square \]

**Proof of Theorem 3.13.** Let \( S, N_\nu \) be as in Theorem 3.12. Let \( \Phi_\nu = \lambda[N_\nu] \).

By Proposition 4.2, \( \inf \Phi_\nu \) exists for each \( \nu \). Set \( h_\nu = \inf(\Phi_\nu) \). By Theorem 6.3, the infima of the zero-avoiding chains in \( \{ \Phi_\nu : \nu \in S_1 \} \) induce an isometric endomorphism of \( \ell^p \). Let \( S_1 \) denote the set of all \( \nu \in S \) so that \( h_\nu \neq 0 \). Thus, the halting set computes a one-to-one enumeration of \( S_1 \); let \( \{ \nu_j \} \) denote such an enumeration. It follows that the halting set computes \( j \mapsto \| \lambda(h_{\nu_j}) \| \).

We now claim that the halting set computes \( j \mapsto \lambda(h_{\nu_j}) \) (with respect to \( F \)).

For, by Proposition 4.2, \( \lim_{\mu \in N_\nu} \| \lambda(\mu) - h_{\nu_j} \| = 0 \). So, given \( k \) as input, we can use the halting set to compute a \( \mu \in N_{\nu_j} \) so that \( \| \| \lambda(\mu) \| - \| h_{\nu_j} \| \| < 2^{-k} \). Since \( h_{\nu_j} \leq \lambda(\mu) \), \( \| \lambda(\mu) - h_{\nu_j} \| = \| \lambda(\mu) \| - \| h_{\nu_j} \| \). By Lemma 6.3, \( \lambda \) is computable (with respect to \( F \)). Thus, the halting set computes \( j \mapsto \lambda(h_{\nu_j}) \).

So, let \( T : \ell^p \to \ell^p \) be the linear isometry induced by \( \{ h_{\nu_j} \} \). By Theorem 3.1, \( T \) is surjective. It follows from Lemma 5.1 that the halting set computes \( T \) with respect to \( (E, F) \).

This completes the proof of Theorem 2.2.

### 6.2. Proof of Theorem 2.3

**Lemma 6.7.** \( F \) is an effective generating set.

**Proof.** Since

\[
(1 - \gamma)^{1/p} e_0 = f_0 - \sum_{n=1}^{\infty} 2^{-\gamma(n-1)/p} f_n
\]

the closed linear span of \( F \) includes \( E \). Thus, \( F \) is a generating set for \( \ell^p \). Note that \( \| f_0 \| = 1 \).

Suppose \( \alpha_0, \ldots, \alpha_M \) are rational points. When \( 1 \leq j \leq M \), set

\[
E_j = |\alpha_0 2^{-c_j -1/p} + \alpha_j|^p - |\alpha_0|^p 2^{-c_j -1}.
\]

It follows that

\[
\| \alpha_0 f_0 + \ldots + \alpha_M f_m \|^p = |\alpha_0|^p \| f_0 \|^p + E_1 + \ldots + E_M = |\alpha_0|^p + E_1 + \ldots + E_M.
\]

Since \( E_1, \ldots, E_M \) can be computed from \( \alpha_0, \ldots, \alpha_M \), \( \| \alpha_0 f_0 + \ldots + \alpha_M f_m \| \) can be computed from \( \alpha_0, \ldots, \alpha_M \). Thus, \( F \) is an effective generating set.

**Lemma 6.8.** Every oracle that with respect to \( F \) computes a scalar multiple of \( e_0 \) whose norm is 1 must also compute \( C \).

**Proof.** Suppose that with respect to \( F \), \( X \) computes a vector of the form \( \lambda e_0 \) where \( |\lambda| = 1 \). It suffices to show that \( X \) computes \( (1 - \gamma)^{-1/p} \).

Fix a rational number \( q_0 \) so that \( (1 - \gamma)^{-1/p} \leq q_0 \). Let \( k \in \mathbb{N} \) be given as input. Compute \( k' \) so that \( 2^{-k'} \leq q_0 2^{-k} \). Since \( X \) computes \( \lambda e_0 \) with respect to \( F \), we can use oracle \( X \) to compute rational points \( \alpha_0, \ldots, \alpha_M \) so that

\[
\| \lambda e_0 - \sum_{j=0}^{M} \alpha_j f_j \| < 2^{-k'}.
\]

(6.2)
We claim that $|(1 - \gamma)^{-1/p} - \alpha_0| < 2^{-k}$. For, it follows from 0.2 that $|\lambda - \alpha_0(1 - \gamma)^{1/p}| < 2^{-k}$. Thus, $|1 - |\alpha_0|(1 - \gamma)^{1/p}| < 2^{-k}$. Hence,

$$|(1 - \gamma)^{-1/p} - \alpha_0| < 2^{-k'} (1 - \gamma)^{-1/p} \leq 2^{-k'} q_0 \leq 2^{-k}.$$  

Since $X$ computes $\alpha_0$ from $k$, $X$ computes $(1 - \gamma)^{-1/p}$.

Lemma 6.9. If $X$ computes a surjective linear isometry of $\ell^p$ with respect to $(E, F)$, then $X$ must also compute $C$.

Proof. By Lemma 6.8 and Lemma 6.1.

Lemma 6.10. With respect to $F$, $C$ computes $e_0$.

Proof. Fix an integer $M$ so that $(1 - \gamma)^{-1/p} < M$.

Let $k \in \mathbb{N}$. Using oracle $C$, we can compute an integer $N_1$ so that $N_1 \geq 3$ and

$$\left\| \sum_{n=N_1}^{\infty} 2^{-c_n-1/p}e_n \right\| \leq \frac{2^{-(kp+1)/p} + M}{2^{-(kp+1)/p}}.$$  

We can use oracle $C$ to compute a rational number $q_1$ so that $|q_1 - (1 - \gamma)^{-1/p}| \leq 2^{-(kp+1)/p}$. Set

$$g = q_1 \left[ f_0 - \sum_{n=1}^{N_1-1} 2^{-c_n-1/p} f_n \right].$$  

It suffices to show that $\|e_0 - g\| < 2^{-k}$. Note that since $1 - \gamma < 1$,

$$|q_1 (1 - \gamma)^{1/p} - 1| \leq 2^{-(kp+1)/p}.$$  

Note also that $|q_1| < M + 2^{-(kp+1)/p}$. Thus,

$$\|e_0 - g\|^p = \left\| e_0 - q_1 (1 - \gamma)^{1/p} e_0 - q_1 \sum_{n=N_1}^{\infty} 2^{-c_n-1/p} e_n \right\|^p \leq |q_1 (1 - \gamma)^{1/p} - 1|^p + |q_1|^p \left\| \sum_{n=N_1}^{\infty} 2^{-c_n-1/p} e_n \right\|^p < 2^{-(kp+1)} + 2^{-(kp+1)} = 2^{-kp}.$$  

Thus, $\|e_0 - g\| < 2^{-k}$. This completes the proof of the lemma.

Lemma 6.11. With respect to $(E, F)$, $C$ computes a surjective linear isometry of $\ell^p$.

Proof. By Lemma 6.10 and the relativization of Lemma 6.1.  

7. Conclusion

To summarize, we have investigated the complexity of isometries between computable copies of $\ell^p$. We have shown that the halting set bounds the complexity of computing these isometries and that this bound is optimal. Along the way we have strengthened an important inequality due to J. Lamperti. These results suggest some questions for further inquiry. For example, what is the complexity of the injective case? In particular, is there a computable copy of $\ell^p$, $B$, with the property that there is no computable linear isometry $T : \ell^p \to B$? Let us say that a set $C$ represents a degree of categoricity for $\ell^p$ if it has the following property: there are computable copies of $\ell^p$ so that $C$ computes a surjective linear isometry of one onto the other and any set that computes such an isometry must also compute $C$. Thus,
we have shown that when $p \neq 2$, every c.e. set represents a degree of categoricity for $(\ell^p)$. Are there any other such sets (besides those that are Turing equivalent to c.e. sets)?

Finally, the results in Section 4 point to some questions about the ordering $(\ell^p, \preceq)$. For example, which partial orders can be embedded in $(\ell^p, \preceq)$ in such a way that incomparable elements are mapped to disjointly supported vectors?

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