Abstract

In this paper we show that the compactness of the central subgroup $G^0$ associated with the drift of a linear system $\Sigma_G$ on a connected Lie group $G$ is a necessary and sufficient condition for the boundedness of the $G^0$-periodic points of $\Sigma_G$. As a consequence, the control set containing the identity element of $G$ is bounded if and only if $G^0$ is a compact subgroup.

Key words: linear systems, periodic points, control sets

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1 Introduction

Essentially, a linear control system on a connected Lie group is an affine system whose drift is linear and the control vectors right-invariant ones. Its importance is highlighted by the Equivalence Theorem in \cite{6} and by the fact that it appears as the natural generalization of linear Euclidean system. One of the properties that this generalization inherit from the Euclidean case is the possibility to associated subgroups which are closely connected with its dynamics (see \cite{1,3,5}), called stable, central and unstable subgroups.

On the other hand, like singularities, periodic orbits are important to understand the dynamics of vector fields. In fact, dynamical systems may have stable limit sets determined by fixed points or periodic orbits, defining domain of attraction on the manifold, i.e. points from which the trajectories will converge to the corresponding limit set as the time goes to infinity. A perfect example of that is the Selgrade Theorem which give explicitly the Morse decomposition of a linear differential equation on $\mathbb{R}^n$ when projected to the projective space $\mathbb{R}P^{n-1}$. In fact, the Morse components are obtained from the projection of subspaces of $\mathbb{R}^n$ to the sphere $S^{n-1}$.

In order to understand the dynamic behavior of a linear system we introduce the notion of $F$-periodic point of the system as follows: Given a compact subset $F$ of $G$ we say that a point is $F$-periodic if it belongs to a trajectory of the system starting and finishing in $F$. Let us denote by $G^0$ the central subgroup associated with a linear system. Our main result shows that the assumption the compactness of $G^0$ is a necessary and sufficient condition for the boundedness of the $G^0$-periodic points. As a consequence, the control set containing the identity element of $G$ is bounded if and only if $G^0$ is a compact subgroup.
The paper is structured as follows: In Section 2 we introduce the concept of linear vector fields and the decompositions induced by them on the group and algebra level. We also introduce the concept of linear system and its \( F \)-periodic points, and prove some complementary results. In Section 3 we analyze a particular case of a linear system on a semi-direct product of a connected Lie group and a nilpotent, simply connected, connected Lie group which we identify with its Lie algebra by the exponential map. This particular results is not only the key to prove our main result but is also important by itself. Actually, the results in Section 3 gives a way to decompose linear system on simply connected nilpotent Lie group in coordinates in order that each coordinate depends only on the previous ones. Section 4 is used to prove the main result of the paper. In this section we introduce also the concept of control sets and show that the \( G^0 \)-periodic points coincides with the interior of control set containing the identity. We finish the section with some examples.

**Notations**

Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \). For any element \( g \in G \) we denote by \( L_g \) and \( R_g \) the left and right translations of \( G \) by \( g \), respectively. The conjugation \( C_g \) is by definition \( C_g = R_{g^{-1}} \circ L_g \). We denote by \( \text{Aut}(G) \) and by \( \text{Aut}(\mathfrak{g}) \) the set of automorphisms of \( G \) and \( \mathfrak{g} \), respectively. The adjoint map \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \) is the map defined by \( \text{Ad}(g) := (dC_g)_e \), where \( e \in G \) stands for the identity element of \( G \). If \( H \) is a connected Lie group and \( \rho : G \to \text{Aut}(H) \) is an homomorphism, the semi-direct product of \( G \) and \( H \) is the Lie group \( G \times \rho H \) whose subjacent manifold is \( G \times H \) and the product is given by

\[
(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 \rho(g_1) h_2).
\]

Its Lie algebra coincides, as a vector space, to the Cartesian product \( \mathfrak{g} \times \mathfrak{h} \).

**2 Preliminaries**

This section is devoted to present the main background needed in order to establish the main theorem. We also prove some new results that will be useful ahead.

**2.1 Decompositions at the algebra level**

Let \( \mathcal{D} \) be a derivation of \( \mathfrak{g} \) and \( \alpha \in \mathbb{C} \) an eigenvalue of \( \mathcal{D} \). The real generalized eigenspaces of \( \mathcal{D} \) are given by

\[
\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : (\mathcal{D} - \alpha \mathcal{I})^n X = 0 \text{ for some } n \geq 1 \}, \quad \text{if} \quad \alpha \in \mathbb{R}
\]

\[
\mathfrak{g}_\alpha = \text{span}\{\text{Re}(v), \text{Im}(v) : v \in \mathfrak{g}_\alpha\}, \quad \text{if} \quad \alpha \in \mathbb{C}
\]

where \( \mathfrak{g} = \mathfrak{g} + i\mathfrak{g} \) is the complexification of \( \mathfrak{g} \) and \( \mathfrak{g}_\alpha \) the generalized eigenspace of \( \mathcal{D} = \mathcal{D} + i\mathcal{D} \), the extension of \( \mathcal{D} \) to \( \mathfrak{g} \).

Following [1 Proposition 3.1] we have that \([\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta}\) when \( \alpha + \beta \) is an eigenvalue of \( \mathcal{D} \) and zero otherwise.

By considering in \( \mathfrak{g} \) the subspaces \( \mathfrak{g}_\lambda := \bigoplus_{\alpha; \text{Re}(\alpha) = \lambda} \mathfrak{g}_\alpha \), where \( \mathfrak{g}_\lambda = \{0\} \) if \( \lambda \in \mathbb{R} \) is not the real part of any eigenvalue of \( \mathcal{D} \), we get

\[
[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}] \subset \mathfrak{g}_{\lambda_1 + \lambda_2} \quad \text{when} \quad \lambda_1 + \lambda_2 = \text{Re}(\alpha) \quad \text{for some eigenvalue} \ \alpha \ \text{of} \ \mathcal{D} \ \text{and zero otherwise.}
\]

We define the *unstable*, *central* and *stable* subalgebras of \( \mathfrak{g} \), respectively, by

\[
\mathfrak{g}^+ = \bigoplus_{\alpha; \text{Re}(\alpha) > 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 = \bigoplus_{\alpha; \text{Re}(\alpha) = 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}^- = \bigoplus_{\alpha; \text{Re}(\alpha) < 0} \mathfrak{g}_\alpha.
\]

It holds that \( \mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^- \) are in fact \( \mathcal{D} \)-invariant Lie subalgebras with \( \mathfrak{g}^+ \), \( \mathfrak{g}^- \) nilpotent ones. Moreover, it turns out that \( \mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^- \).
2.2 Decompositions at the group level

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. A vector field $\mathcal{X}$ on $G$ is said to be linear if its flow $(\varphi_t)_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $\text{Aut}(G)$, the group of automorphisms of $G$. Associate to any linear vector field $\mathcal{X}$ there is a derivation $D$ of $\mathfrak{g}$ whose relation with the flow $\varphi_t$ is given by the formula

$$(d\varphi_t)_e = e^{tD} \quad \text{for all} \quad t \in \mathbb{R}.$$

Let us denote by $G^+, G^-, G^0, G^{+,0}$, and $G^{-,0}$ the connected Lie subgroups of $G$ with Lie algebras induced by $D$ given by $\mathfrak{g}^+, \mathfrak{g}^-, \mathfrak{g}^0, \mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively. By Proposition 2.9 of [3], all the above subgroups are $\varphi$-invariant, closed and have trivial intersection, that is,

$$G^+ \cap G^- = G^+ \cap G^{-,0} = \ldots = \{e\}.$$

The subgroups $G^+, G^0$ and $G^-$ are called the unstable, central and stable subgroups of $G$, respectively. We also use the notation $G^{+,\cdot}$ for the product of $G^+$ and $G^-$, that is, $G^{+,\cdot} = G^+ G^-.$ Following [5], we say that $G$ is decomposable if

$$G = G^{+,0} G^- = G^{-,0} G^+ = G^{+,\cdot} G^0.$$

By [3] Proposition 3.3] a sufficient condition for a group $G$ to be decomposable is that the central subgroup $G^0$ is compact. The next result explores some more properties coming from the assumption on the compactness of the central subgroup.

2.1 Remark: It is important to notice that on decomposable groups, any element can be uniquely decomposed into the product of elements in the stable, central and unstable subgroups.

2.2 Lemma: Let $N$ to be the nilradical of $G$. If $G^0$ is a compact subgroup, then

1. $G^+, G^- \subset N$;
2. $N \cap G^0$ is a compact connected ideal of $G$;

Proof: 1. Let us denote by $R$ the solvable radical of $G$. Under the assumption that $G^0$ is compact, we get that $(G/R)^0 = \pi(G^0)$ is also compact, where $\pi : G \to G/R$ is the canonical projection. Since $G/R$ is semi-simple, Proposition 3.3 of [3] implies that $G/R = (G/R)^0$ and consequently that $G = G^0 R$. In particular, $G^+, G^- \subset R$. On the other hand, Lemma 2.1 of [2] assures that the nilradical $\mathfrak{r}$ of $\mathfrak{g}$ contains $\mathfrak{g}_\alpha$ for any nonzero eigenvalue $\alpha$ of $D$. Therefore, $\mathfrak{g}^+, \mathfrak{g}^- \subset \mathfrak{n}$ and $G^+, G^- \subset N$.

2. To prove the second claim let us notice that $N^0 \subset N \cap G^0$ implying in particular that $N^0$ is a compact subgroup and therefore $N$ is decomposable. By the uniqueness of the decomposition of each element of $N$ we must have necessarily that $G^0 \cap N \subset N^0$ and hence $N^0 = N \cap G^0$, showing that $N \cap G^0$ is a compact connected Lie subgroup of $N$. However, is a standard fact that compact subgroups of nilpotent Lie groups are always central and hence $N^0 \subset Z(N)$ (see for instance [8] Theorem 1.6]). By the previous item, $G^+, G^- \subset N$, in order to show that $N \cap G^0$ is an ideal of $G$ we only need to prove that $G^0$ normalizes $N \cap G^0$. However, the fact that the nilradical is invariant by automorphisms, implies that $C_g(N) = N$ for any $g \in G$. In particular, if $g \in G^0$ and $h \in N \cap G^0$ we get that

$$G^0 \ni ghg^{-1} = C_g(h) \in N \implies ghg^{-1} \in N \cap G^0,$$

concluding the proof. \hfill \Box

The next lemma shows that, in the decomposable case, if $G^{+,\cdot}$ is a subgroup, then $G$ can be seeing as a semi-direct product. It will be important to reduce our general case to a particular one.

2.3 Lemma: If $G$ is decomposable and $G^{+,\cdot}$ is a subgroup then $G$ is isomorphic to the semi-direct product $G^0 \times_{\text{Ad} \mathfrak{g}^{+,\cdot}}$. 
**Proof:** Define the map
\[ \psi : G^0 \times _{Ad} g^{+,-} \rightarrow G, \quad (g, x) \in G^0 \times g^{+,-} \mapsto e^x g \in G. \]
If \( p : G \times G \rightarrow G \) stands for the product in \( G \) we have that
\[ \psi = p \circ (\exp \times \text{id}_G) |_{g^{+,-} \times G^0}, \]
therefore \( \psi \) is a continuous map. Furthermore, since \( G \) is decomposable and \( G^{+,-} \) is a nilpotent group, we have \( G = G^{+,-}G^0 = \exp(g^{+,-})G^0 \) and so \( \psi \) is surjective. Moreover, for any \((g_1, x_1), (g_2, x_2) \in G^0 \times g^{+,-} \) we obtain
\[ \psi((g_1, x_1)(g_2, x_2)) = \psi(g_1g_2, C(x_1, \text{Ad}(g_1)x_2)) = e^{C(x_1, \text{Ad}(g_1)x_2)} g_1g_2 = e^{x_1} e^{\text{Ad}(g_1)x_2} g_1g_2 = e^{x_1} g_1 e^{x_2} g_2 = \psi(g_1, x_1) \psi(g_2, x_2), \]
showing that \( \psi \) is in fact a homomorphism.
On the other hand, since \( G^{+,-} \cap G^0 = \{e\} \) it follows that
\[ (g, x) \in \ker \psi \iff e^x g = e \iff G^{+,-} \ni e^x = g^{-1} \in G^0 \iff e^x = g = e, \]
and consequently
\[ \psi \text{ is injective} \iff \ker \psi = \{(e, 0)\} \iff \exp : g^{+,-} \rightarrow G^{+,-} \text{ is injective}. \]
However, since \( g^{+,-} \) is a nilpotent Lie algebra \( \exp : g^{+,-} \rightarrow G^{+,-} \) is a covering map and hence \( \exp^{-1}(e) \subset g^{+,-} \) is a discrete subset. On the other hand,
\[ e^{tD(\exp^{-1}(e))} \subset \exp^{-1}(e), \quad \text{for all} \quad t \in \mathbb{R}, \]
thus \( \exp^{-1}(e) \subset \ker D \cap g^{+,-} = \{0\} \) showing that \( \exp \) is injective and therefore \( \psi \) is in fact an isomorphism. \( \square \)

### 2.3 Linear systems

Let \( G \) be a connected Lie group with Lie algebra \( g \) identified with the right-invariant vector fields and \( \Omega \subset \mathbb{R}^m \) a compact and convex subset containing the origin in its interior. The **set of the control functions** is by definition
\[ \mathcal{U} := \{ u : \mathbb{R} \rightarrow \mathbb{R}^m ; \text{ u is a piecewise constant function with } u(\mathbb{R}) \subset \Omega \}. \]
A linear system on \( G \) is given a family of ordinary differential equations
\[ \dot{y}(t) = \mathcal{X}(y(t)) + \sum_{j=1}^{m} u_j(t) Y_j(y(t)), \quad (\Sigma_G) \]
where the drift \( \mathcal{X} \) is a linear vector field, \( Y_1, \ldots, Y_m \in g \) and \( u = (u_1, \ldots, u_m) \in \mathcal{U} \). For any \( g \in G \) and \( u \in \mathcal{U} \), the solution \( t \mapsto \phi(t, g, u) \) of \( \Sigma_G \) is complete and satisfies
\[ \phi_{\tau, u} \circ R_g = R_{\phi_{\tau}(g)} \circ \phi_{\tau, u}, \quad \text{for any} \quad \tau \in \mathbb{R}, g \in G. \quad (1) \]

**2.4 Definition:** Let \( F \subset G \) be a nonempty compact subset. We say that \( g \in G \) is a **F-periodic point** of \( \Sigma_G \) if there exist \( f_1, f_2 \in F, \tau_1, \tau_2 > 0 \) and \( u_1, u_2 \in \mathcal{U} \) such that
\[ \phi(\tau_1, f_1, u_1) = g \quad \text{and} \quad \phi(\tau_2, g, u_2) = f_2. \quad (2) \]
We denote by \( \text{Per}(F; \Sigma_G) \) the set of the F-periodic points of \( \Sigma_G \).
2.5 Remark: If $F$ is $\varphi$-invariant, then $F \subset \text{Per}(F; \Sigma_G)$. In fact, if $g \in F$ we have by the $\varphi$-invariance of $F$ that

$$f_1 = \varphi_\tau(g) \in F \quad \text{and} \quad f_2 = \varphi_{\tau}(g) \in F.$$  

By considering $u_1 = u_2$ to constant equal to zero, we have that

$$\phi(\tau, f_1, u_1) = \varphi(\tau_1) = g \quad \text{and} \quad \phi(\tau, g, u) = \varphi(\tau) = f_2$$

showing that $g$ is $F$-periodic.

Next we show that the whole curve connecting a point $x \in \text{Per}(G^0, \Sigma_G)$ to $G^0$ is in fact contained in $\text{Per}(G^0, \Sigma_G)$.

2.6 Lemma: $g \in \text{Per}(F; \Sigma_G)$ if and only if there exists $f \in F$, $\tau > 0$ and $u \in \mathcal{U}$ such that

$$g \in \{ \phi(t, f, u), \; t \in (0, \tau) \} \quad \text{and} \quad \phi(\tau, f, u) \in F. \quad (3)$$

Hence, if $\phi(\tau, f, u) \in F$ for some $f \in F$, $\tau > 0$ and $u \in \mathcal{U}$, then $\phi(t, f, u) \in \text{Per}(F; \Sigma_G)$ for any $t \in (0, \tau)$.

Proof: Let $g \in \text{Per}(F; \Sigma_G)$ and consider $f_1, f_2 \in F$, $\tau_1, \tau_2 > 0$ and $u_1, u_2 \in \mathcal{U}$ satisfying $[2]$. The function

$$u(t) := \begin{cases} u_1(t), & t \in (-\infty, \tau_1), \\ u_2(t - \tau_1), & t \in [\tau_1, +\infty) \end{cases} \quad \text{belongs to} \quad \mathcal{U},$$

and it holds that

$$\phi(\tau_1, f_1, u) = \phi(\tau_1, f_1, u_1) = g, \quad \text{and} \quad \phi(\tau_1 + \tau_2, f_1, u) = \phi(\tau_2, \phi(\tau_1, f_1, u_1), \theta_{\tau_1} u) = \phi(\tau_2, g, u_2) = f_2 \in F,$n

showing that $[3]$ is satisfied. Reciprocally, let $f \in F$, $\tau > 0$ and $u \in \mathcal{U}$ such that $[3]$ is satisfied. Let $\tau_1 \in (0, \tau)$ and $f_2 \in F$ such that

$$g = \phi(\tau_1, f, u) \quad \text{and} \quad f_2 = \phi(\tau, f, u)$$

and set $f_1 = f$, $\tau_2 = \tau - \tau_1 > 0$, $u_1 = u$ and $u_2 = \theta_{\tau_1} u$. Then

$$\phi(\tau_1, f_1, u_1) = \phi(\tau_1, f, u) = g$$

and

$$\phi(\tau_2, g, u_2) = \phi(\tau_2, \phi(\tau_1, f, u), \theta_{\tau_1} u) = \phi(\tau_2 + \tau_1, f, u) = \phi(\tau, f, u) = f_2,$n

which implies that $g$ in $F$-periodic and concluding the proof. \hfill \square

2.7 Remark: In the particular case where $F = \{ g \}$ the previous lemma shows that the set $\text{Per}(g; \Sigma_G)$ consists of closer periodic orbits passing by $g \in G$.

Let $G, H$ be connected Lie groups and $\psi : G \to H$ a surjective homomorphism. We say that two linear system $\Sigma_G$ and $\Sigma_H$, on $G$ and $H$, respectively, are $\psi$-conjugated if

$$\psi \left( \phi^G(t, g, u) \right) = \phi^H(t, \psi(g), u), \quad \text{for any} \quad g \in G, t \in \mathbb{R} \quad \text{and} \quad u \in \mathcal{U}. \quad (4)$$

The next result relates the set of $F$-periodic points with its image by a conjugation.

2.8 Proposition: Let $\psi$ be as previously and assume that $F \cdot \ker \psi = F$. Then

$$\psi(\text{Per}(F; \Sigma_G)) = \text{Per}(\psi(F); \Sigma_H) \quad \text{and} \quad \psi^{-1}(\text{Per}(\psi(F); \Sigma_H)) = \text{Per}(F; \Sigma_G).$$

In particular, if $\ker \psi$ is a compact subgroup, then $\text{Per}(F; \Sigma_G)$ is a bounded subset if and only if $\text{Per}(\psi(F); \Sigma_H)$ is a bounded subset.
Proof: By the relation between the sets it is enough to prove that
\[ \psi(\text{Per}(F; \Sigma_G)) \subset \text{Per}(\psi(F); \Sigma_H) \quad \text{and} \quad \psi^{-1}(\text{Per}(\psi(F); \Sigma_H)) \subset \text{Per}(F; \Sigma_G). \]
If \( g \in \text{Per}(F; \Sigma_G) \) it follows from (3) that
\[ g \in \{ \phi^G(t, f, u), \ t \in (0, \tau) \} \quad \text{and} \quad \phi^G(\tau, f, u) \in F, \]
for some \( f \in F, \ \tau > 0 \) and \( u \in U \). By equation (4) we get that
\[ \psi(g) \in \psi(\{ \phi^G(t, f, u), \ t \in (0, \tau) \}) = \{ \phi^G(t, g, u), \ t \in (0, \tau) \} \]
and
\[ \psi(\phi^G(\tau, f, u)) \in \psi(F) \iff \phi^H(\tau, \psi(f), u) \in \psi(F), \]
showing that
\[ \psi(\text{Per}(F; \Sigma_G)) \subset \text{Per}(\psi(F); \Sigma_H). \]
Reciprocally, if \( g \in \psi^{-1}(\text{Per}(\delta, \psi(F); \Sigma_H)) \) there exist \( f_1, f_2 \in F, \ \tau_1, \tau_2 > 0 \) and \( u_1, u_2 \in U \) satisfying
\[ \phi^H(\tau_1, \psi(f_1), u_1) = \psi(g) \quad \text{and} \quad \phi^H(\tau_2, \psi(g), u_2) = \psi(f_2), \]
and hence,
\[ \phi^G(\tau_1, f_1, u_1)h_1 = g \quad \text{and} \quad \phi^G(\tau_2, g, u_2) = f_2h_2, \quad \text{for some} \ h_1, h_2 \in \ker \psi. \]
Using equation (3) for the control \( u \equiv 0 \) gives us that
\[ \psi \circ \varphi^G_t = \varphi^H_t \circ \psi, \ \forall t \in \mathbb{R} \quad \text{and hence} \quad \varphi^G_t(\ker \psi) \subset \ker \psi. \]
By defining \( \tilde{f}_1 := f_1 \varphi^G_{-\tau_1}(h_1) \) and \( \tilde{f}_2 := f_2h_2 \) the assumption that \( F \cdot \ker \psi = F \) implies that \( \tilde{f}_1, \tilde{f}_2 \in F \). Moreover, by equation (2.16)
\[ \phi^G(\tau_1, \tilde{f}_1, u_1) = \phi^G(\tau_1, f_1 \varphi^G_{-\tau_1}(h_1), u_1) = \phi^G(\tau_1, f_1, u_1)h_1 = g \]
and
\[ \phi^G(\tau_2, g, u_2) = f_2h_2 = \tilde{f}_2, \]
hence
\[ \psi^{-1}(\text{Per}(\psi(F); \Sigma_H)) \subset \text{Per}(F; \Sigma_G), \]
as stated.
For the compactness assumption, it certainly holds that \( \text{Per}(\psi(F); \Sigma_H) \) is a bounded set as soon as \( \text{Per}(F; \Sigma_G) \) is a bounded set. Reciprocally, if \( \text{Per}(\psi(F); \Sigma_H) \) is bounded, there exists a compact set \( K \subset G \) such that \( \text{Per}(\psi(F); \Sigma_H) \subset \psi(K) \) which by the previous equalities implies that
\[ \text{Per}(F; \Sigma_G) = \psi^{-1}(\text{Per}(\psi(F); \Sigma_H)) \subset \psi^{-1}(\psi(K)) = K \ker \psi \]
and consequently that \( \text{Per}(F; \Sigma_G) \) is a bounded subset if \( \ker \psi \) is a compact subgroup of \( G \). \( \square \)

3 An important particular case

Our aim in this section is to analyze the solutions of a linear system on the special semi-direct product of a connected Lie group \( H \) by a simply connected nilpotent Lie group.

Let \( u \) be a nilpotent Lie algebra. For any \( X, Y \in u \) we denote by \( c(X, Y) \) the Campbell-Baker-Hausdorff (BCH) formula whose first terms are
\[ c(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12} \left( [X, [X, Y]] + \frac{1}{12} [Y, [Y, X]] \right) - \frac{1}{24} [Y, [X, [X, Y]]] + \cdots. \]
The simply connected nilpotent Lie group associated with \( u \) is given by \((u,\ast)\), where the product reads as

\[ X \ast Y = c(X,Y) , \quad X,Y \in g. \]

The identification between algebra and group allows us to work indistinctly with their elements. However, we will use capital letters \( X,Y,Z, \ldots \) for the elements in \( u \) seeing as Lie algebra and small letters \( x,y,z, \ldots \) for the elements in \( u \) seeing as Lie group. Since the group of automorphisms of \( u \) as Lie group coincides with \( \text{Aut}(u) \), any linear vector field \( \mathcal{X} \) on \( u \) coincides with its associated derivation. In fact, since \( \{ \varphi_t \}_{t \in \mathbb{R}} \subset \text{Aut}(u) \) is a one-parameter group of automorphism, there exists a derivation \( D \in \text{Lie}(\text{Aut}(u)) = \text{Der}(u) \) such that \( \varphi_t = e^{tD} \) for any \( t \in \mathbb{R} \). Consequently,

\[ \mathcal{X}(x) = \frac{d}{ds}_{s=0} \varphi_s(x) = \frac{d}{ds}_{s=0} e^{sD} x = Dx. \]

Let \( H \) be a connected Lie group with Lie algebra \( h \) and assume the existence of a continuous homomorphism \( \rho : H \to \text{Aut}(u) \). Our interest in this section is to analyze a linear system on the semi-direct product \( H \times_\rho u \). In order to do that, let us first obtain an expression for the right-invariant vector fields on \( H \times_\rho u \).

In order to avoid cumbersome notation, we denote only by \( \varphi_t \) the solution of 5 associated with \( (e^s\rho_Z)h, x \)

\[ \mathcal{X}(x) = \frac{d}{ds}_{s=0} \varphi_s(x) = \frac{d}{ds}_{s=0} e^{sD} x = Dx. \]

Let \( H \) be a connected Lie group with Lie algebra \( h \) and assume the existence of a continuous homomorphism \( \rho : H \to \text{Aut}(u) \). Our interest in this section is to analyze a linear system on the semi-direct product \( H \times_\rho u \). In order to do that, let us first obtain an expression for the right-invariant vector fields on \( H \times_\rho u \). Since \( T_{(e,0)}(H \times_\rho u) = h \times u \), for any given \((Y,Z) \in h \times u \) the associated right-invariant vector field \((Y,Z) \) on \( H \times_\rho u \) is, by definition, \((Y,Z)(g,x) = (dR_{(g,x)}) (0) \alpha(0), \) where \( \alpha : (-\varepsilon,\varepsilon) \to H \times_\rho u \) is any differentiable curve satisfying \( \alpha(0) = (e,0) \) and \( \alpha'(0) = (Y,Z) \).

In particular, the curve \( \alpha(s) := (e^sY, sZ) \) satisfies the above and so

\[ R_{(h,x)}(\alpha(s)) = (e^sY, s\rho(h)Z)(h,x) = (e^sY g, s\rho(h)Z \ast x), \]  

Since,

\[ s\rho(g)Z \ast x = s\rho(g)Z + x + \frac{1}{2}(s\rho(g)Z,x) + \frac{1}{12} \left( [s\rho(g)Z, [s\rho(g)Z,x]] + [x, [x,s\rho(g)Z]] \right) + \cdots, \]

by differentiating at \( s = 0 \), we get

\[ (Y,Z)(g,x) = (Y(g), (\rho(g)Z)(x)), \quad \text{where} \ (\rho(g)Z)(x) := \sum_{p=0}^{k-1} c_p \text{ad}(x)^p \rho(g)Z \]

where \( k \in \mathbb{N} \) is the smallest natural number that satisfies \( u^{k+1} = \{0\} \) and the coefficients \( c_j \) are the ones given by the BCH formula. For instance, \( c_0 = 1, \ c_1 = -1/2, \ c_2 = 1/12, \ldots \)

If \( \mathcal{X} \) is a linear vector field on \( G \) and \( D \) a derivation of \( u \), a linear vector field on \( H \times_\rho u \) is given by

\[ (\mathcal{X} \times D)(g,x) = (\mathcal{X}(g), Dx), \quad (h,x) \in H \times_\rho u. \]

Consider then the following linear system

\[
\begin{cases}
\dot{h} = \mathcal{X}(h) + \sum_{j=1}^{m} u_j Y_j(h) \\
\dot{x} = Dx + \sum_{j=1}^{m} u_j (\rho(h)Z_j)(x).
\end{cases}
\]

Let us fix \( (h,x) \in H \times_\rho u \) and \( u \in \mathcal{U} \). In order to avoid cumbersome notation, we denote only by \( t \mapsto (h_t, x_t) \) the solution of \( (5) \) associated with \( u \in \mathcal{U} \) and satisfying \((h_0,x_0) = (h,x)\). Define

\[ t \in \mathbb{R} \mapsto Z_t \in u, \quad \text{where} \ Z_t := \rho(h_t) \left( \sum_{j=1}^{m} u_j(t)Z_j \right). \]

Since \( \rho(h_t) \in \text{Aut}(u) \), by linearity we obtain

\[ \sum_{j=1}^{m} u_j(t)(\rho(h_t))Z_j(x_t) = \sum_{j=1}^{m} u_j(t) \sum_{p=0}^{n} c_p \text{ad}(x_t)^p \rho(h_t)Z_j = \sum_{p=0}^{n} c_p \text{ad}(x_t)^p Z_t = Z_t(x_t) \]

and the second equation in \( (5) \) can be rewritten as

\[ \dot{x}_t = Dx_t + Z_t(x_t). \]
Consider \( u^1 \supset u^2 \supset \cdots \supset u^k \supset u^{k+1} = \{0\} \) the central series of \( u \) defined as

\[
    u^i = u \quad \text{and} \quad u^{i+1} = [u^i, u], \quad \text{for } i \in \{1, \ldots, k\}.
\]

Let \( i \in \{1, \ldots, k\} \) and choose \( V_i \subset u^i \) to be a complementary space of \( u^{i+1} \) in \( u^i \), that is,

\[
    V_i \oplus u^{i+1} = u^i. \quad \text{In particular, } u^i = \bigoplus_{l=1}^k V_i, \quad \text{for } i \in \{1, \ldots, k\}. \tag{7}
\]

We denote \( x = (x^1, \ldots, x^k) \) to emphasize the decomposition of \( x \in u \) in the components \( x^i \in V_i \). Our aim is to write the solutions of the system(4) using the decomposition(7) in the \( V_i \)-components.

For any derivation \( D \), the fact that \( Du^i \subset u^i \) gives us a block-triangular decomposition form

\[
    D = \begin{pmatrix}
    D_{11} & 0 & 0 & \cdots & 0 \\
    D_{21} & D_{22} & 0 & \cdots & 0 \\
    D_{31} & D_{32} & D_{33} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    D_{k1} & D_{k2} & D_{k3} & \cdots & D_{kk}
\end{pmatrix}, \quad \text{where } D_{ij} : V_j \to V_i \text{ is a linear map.} \tag{8}
\]

Let \( x \in N \) and \( p \in \{1, \ldots, k-1\} \) and consider \( B^p_{ij}(x) : V_j \to V_i \) to be the linear maps obtained from the block-matrix of \( \text{ad}(x)^p \) as previously. We have the following result.

**3.1 Lemma:** For any \( p \in \{1, \ldots, k-1\} \) it holds that

\[
    B^p_{ij}(x) = \begin{cases} 
        0 & \text{for } i < p+j \\
        B^p_{ij}(x^1, \ldots, x^{j-p+1}) & \text{for } i \geq p+j
    \end{cases},
\]

where \( x = (x^1, \ldots, x^k) \).

**Proof:** The proof will proceed by induction. Since \( V_i \subset u^i \) we have that \( \text{ad}(x_i)V_j \subset u^{i+l} = \bigoplus_{q=j+i}^k V_q \) for any \( x_i \in V_i \). Hence, \( B_{ij}(x^l) = 0 \) for any \( x_i \in V_i \) if \( i < l + j \), implying that

\[
    \text{ad}(x) = \sum_{l=1}^{k-1} \text{ad}(x^l) \implies B_{ij}(x) = \begin{cases} 
        0 & \text{for } i < j+1 \\
        B_{ij}(x^1, \ldots, x^{j}) & \text{for } i \geq j+1
    \end{cases},
\]

and showing the result for \( p = 1 \). If the result is true for \( p \), a simple calculation shows that

\[
    B^{p+1}_{ij}(x) = \sum_{l=1}^k B_{il}(x)B^p_{lj}(x).
\]

By inductive hypothesis, it holds that

\[
    B_{il}(x) = \begin{cases} 
        0 & \text{for } i < 1 + l \\
        B_{il}(x^1, \ldots, x^{i-l}) & \text{for } i \geq 1 + l
    \end{cases} \quad \text{and} \quad B^p_{lj}(x) = \begin{cases} 
        0 & \text{for } l < p+j \\
        B^p_{lj}(x^1, \ldots, x^{j-p+1}) & \text{for } l \geq p+j
    \end{cases}.
\]

Therefore, \( B^{p+1}_{ij}(x) = 0 \) for \( i < (p + 1) + j \) and

\[
    B^{p+1}_{ij}(x) = \sum_{p+j \leq l \leq i-1} B_{il}(x^1, \ldots, x^{i-l}) B^p_{lj}(x^1, \ldots, x^{j-p+1}),
\]

that certainly only depends on \( x^1, \ldots, x^{i-j-(p+1)+1} \) and so

\[
    B^{p+1}_{ij}(x) = B^{p+1}_{ij} \left( x^1, \ldots, x^{i-j-(p+1)+1} \right) \quad \text{if } i \geq (p + 1) + j,
\]
concluding the proof.

If $Z = (Z^1, \ldots, Z^k) \in u$, then

$$(\text{ad}(x)^p Z)^i = \sum_{j=1}^{k} B_{ij}^p(x) Z^j = \sum_{j=1}^{i-p} B_{ij}^p(x^1, \ldots, x^{i-j-p+1}) Z^j, \quad \text{if } p < i$$

and $(\text{ad}(x)^p Z)^i = 0$ if $p \geq i$. It follows that,

$$(Z(x))^i = \left( \sum_{p=0}^{k-1} c_p \, (\text{ad}(x)^p Z)^i \right) = \sum_{p=0}^{k-1} c_p \, (\text{ad}(x)^p Z)^i = Z^i + \sum_{p=1}^{i-1} c_p \sum_{j=1}^{i-p} B_{ij}^p(x^1, \ldots, x^{i-j-p+1}) Z^j,$$

and the $V_i$-component of $Z(x)$ only depends on $x^1, \ldots, x^{i-1}$. For any $i \in \{1, \ldots, k\}$ we can define the continuous maps $G^i : V_i \times \cdots \times V_{i-1} \times u \to V_i$ by

$$G^i(Z) := Z^i \quad \text{and} \quad G^i(x^1, \ldots, x^{i-1}; Z) := \sum_{j=1}^{i-1} D_{ij} x^j + (Z(x))^i, \quad \text{for } i \geq 2.$$ 

The next result gives us a decomposition of $G^i$ in terms of the maps $G^j$ above.

**3.2 Theorem:** With the previous notations the system (6) in coordinates reads as

$$\begin{aligned}
\begin{cases}
\dot{x}_1^1 &= D_{11} x_1^1 + G^1 (Z_t) \\
\dot{x}_1^2 &= D_{22} x_1^2 + G^2 (x_1^1; Z_t) \\
\dot{x}_1^3 &= D_{33} x_1^3 + G^3 (x_1^1, x_1^2; Z_t) \\
& \vdots \\
\dot{x}_1^k &= D_{kk} x_1^k + G^k (x_1^1, \ldots, x_1^{k-1}; Z_t)
\end{cases}
\end{aligned}$$

In particular, its solution $x_t = (x_1^1, \ldots, x_1^k)$ satisfies

$$x_1^i = \int_0^t e^{(t-s)D_{ii}} G^i (x_1^1, \ldots, x_1^{i-1}; Z_s) ds + e^{tD_{ii}} x_1^i, \quad \text{for } i = 1, \ldots, k. \quad (10)$$

**Proof:** Since

$$(\mathcal{D} x)^i = \sum_{j=1}^{i} D_{ij} x^j = D_{ii} x^i + \sum_{j=1}^{i-1} D_{ij} x^j,$$

we get

$$\dot{x}_i^i = (\mathcal{D} x_t + Z_t(x_t))^i = (\mathcal{D} x_t)^i + (Z_t(x))^i = D_{ii} x^i + \sum_{j=1}^{i-1} D_{ij} x^j + (Z_t(x))^i = D_{ii} x^i + G^i (x_1^1, \ldots, x_t^{i-1}; Z_t),$$

which proves equations (9). Equation (10) follows directly from integration. In fact,

$$\dot{x}_i^i = D_{ii} x_1^1 + G^i (x_1^1, \ldots, x_t^{i-1}; Z_t) \iff \frac{d}{dt} e^{-tD_{ii}} x_1^i = e^{-tD_{ii}} G(x_1^1, \ldots, x_t^{i-1}; Z_t)$$

$$\iff e^{-tD_{ii}} x_1^i - x_1^i = \int_0^t e^{-sD_{ii}} G(x_1^1, \ldots, x_t^{i-1}; Z_s) ds \iff x_i^i = \int_0^t e^{(t-s)D_{ii}} G(x_1^1, \ldots, x_t^{i-1}; Z_s) ds + e^{tD_{ii}} x_1^i. \quad \square$$
3.3 Remark: It is important to notice the previous calculations where made for a choice of \((h, x) \in H \times \rho u\) and \(u \in \mathcal{U}\). We will use the notations

\[ h_{t,u}, \ x_{t,u,h}, \text{ and } Z_{t,u,h}, \]

if we want to emphasize the dependence on them.

The next lemma will be central in the proof of our main result.

3.4 Lemma: If \(H\) is a compact group and \(\mathcal{D}\) has only eigenvalues with nonzero real part then

\[ \text{Per} \left( H \times \{0\}; \Sigma_{H \times \rho u} \right), \]

is a bounded set.

**Proof:** For simplicity let us use the notation \(P := \text{Per} \left( H \times \{0\}; \Sigma_{H \times \rho u} \right)\). Since \(H\) is a compact group we only have to show that \(\pi_2(P)\) is bounded in \(u\), where \(\pi_2\) is the projection onto the second coordinate. By defining

\[ \pi_{2,i} : H \times \rho u \to V_i, \ (h, (x^1, \ldots, x^k)) \to x^i, \]

we obtain \(\pi_2(P)\) is bounded in \(u\) if and only if \(\pi_{2,i}(P)\) is bounded in \(V_i\) for any \(i = 1, \ldots, k\), which we will prove recurrently after some preliminaries.

By the block-triangular decomposition form of \(\mathcal{D}\) given in (3) it follows directly that if \(\mathcal{D}\) has only eigenvalues with nonzero real part the same is true for \(\mathcal{D}_{ii}\) for any \(i \in \{1, \ldots, k\}\). Therefore, for any \(i \in \{1, \ldots, k\}\) we can consider the decomposition \(V_i = V_i^+ \oplus V_i^-\), where \(V_i^+\) (resp. \(V_i^-\)) are the sum of the real generalized eigenspaces of \(\mathcal{D}_{ii}\) associated with eigenvalues with positive (resp. negative) real parts. Therefore, if \(|\cdot|\) is a norm in \(u\) there exist constants \(\kappa_i, \mu_i > 0\) such that

\[ |e^{t\mathcal{D}_{ii}} \pi^+_i(x)| \leq \kappa_i e^{-t \mu_i |\pi^-_i(x^i)|} \quad \text{and} \quad |e^{-t\mathcal{D}_{ii}} \pi^-_i(x^i)| \leq \kappa_i e^{-t \mu_i |\pi^+_i(x^i)|}, \quad \text{for any } t > 0, \ x^i \in V_i, \]

where \(\pi^\pm_i : V_i \to V_i^\pm\) are the projections associated with the decomposition \(V_i = V_i^+ \oplus V_i^-\). Let us fix \(M_1 > 0\) such that

\[ |G^i(Z_{t,u,h})| \leq M_1, \quad \text{for all} \quad t \geq 0, h \in H, \ u \in \mathcal{U}, \]

which exists by the compactness of \(H \times \mathcal{U}\) and the continuity of \(G^i\).

Let then \(x^1 \in \pi_{2,1}(P)\) and consider \(h_1, h_2 \in H, \tau_1, \tau_2 > 0\) and \(u_1, u_2 \in \mathcal{U}\) such that

\[ x^1_{0,u_1,h_1} = x^1_{\tau_2,u_2,h_2} = 0 \quad \text{and} \quad x^1_{0,u_2,h_2} = x^1_{\tau_1,u_1,h_1} = x^1. \]

By Theorem 3.2 we obtain

\[ x^1 = \int_0^{\tau_1} e^{(\tau_1-s)\mathcal{D}_{ii}} G^i(z_{s,u_1,h_1}) \, ds \quad \text{and} \quad 0 = \int_0^{\tau_2} e^{(\tau_2-s)\mathcal{D}_{ii}} G^i(z_{s,u_2,h_2}) \, ds + e^{\tau_2\mathcal{D}_{ii}} x^1, \]

implying that

\[ |\pi^+_1(x)| \leq \int_0^{\tau_1} \left| e^{(\tau_1-s)\mathcal{D}_{ii}} \pi^+_i \left( G^i(z_{s,u_1,h_1}) \right) \right| \, ds \leq \int_0^{\tau_1} \kappa_i \left| e^{-(\tau_1-s)\mu_i} \pi^-_i \left( G^i(z_{s,u_1,h_1}) \right) \right| \, ds \leq \frac{\kappa_i}{\mu_i} M_1 (1 - e^{-(\tau_1 \mu_i)}) \leq \frac{\kappa_i}{\mu_i} M_1 \]

and

\[ |\pi^-_1(x)| \leq \int_0^{\tau_2} \left| e^{-s\mathcal{D}_{ii}} \pi^-_i \left( G^i(z_{s,u_2,h_2}) \right) \right| \, ds \leq \int_0^{\tau_2} \kappa_i \left| e^{-s\mu_i} \pi^+_i \left( G^i(z_{s,u_2,h_2}) \right) \right| \, ds \leq \frac{\kappa_i}{\mu_i} M_1 (1 - e^{-\tau_2 \mu_i}) \leq \frac{\kappa_i}{\mu_i} M_1. \]
Consequently

\[ |x^1| = |\pi_1^+(x^1) + \pi_1^-(x^1)| \leq |\pi_1^+(x^1)| + |\pi_1^-(x^1)| \leq 2 \frac{k_1}{\mu_1} M_1 \]

proving the boundedness of \( \pi_{2,1}(P) \).

Let \( i \in \{2, \ldots, k\} \) and assume that \( \pi_{2,j}(P) \subset V_j \) is a bounded set for \( j < i \). If \( x^i \in \pi_{2,i}(P) \), it holds that

\[ x^i_{0,1i,1} = x^i_{12,12,2} = 0 \quad \text{and} \quad x^i_{0,12,2} = x^i_{12,11,1} = x^i \]

for some \( h_1, h_2 \in H, \tau_1, \tau_2 > 0 \) and \( u_1, u_2 \in U \). By Theorem 3.2 we get that

\[ x^i = \int_0^{\tau_1} e^{(\tau_1-s)D_{i1}}G \left( x^1_{s,1i,1}, \ldots, x^{i-1}_{s,1i,1}; Z_{s,1i,1} \right) ds \]

and

\[ 0 = \int_0^{\tau_2} e^{(\tau_2-s)D_{i1}}G \left( x^1_{s,12,2}, \ldots, x^{i-1}_{s,12,2}; Z_{s,12,2} \right) ds + e^{\tau_2 D_{ii}} x^i. \]

On the other hand, by inductive hypothesis, \( \pi_{2,j}(P) \) is a bounded set for \( j = 1, \ldots, i-1 \). Hence, the continuity of \( G^i \) assures the existence of \( M_i > 0 \) such that

\[ |G_i(x^1, \ldots, x^{i-1}; Z_{t,u})| \leq M_i, \quad \text{for any} \quad x^j \in \pi_{2,j}(P), \quad j = 1, \ldots, i-1, \quad t > 0, \quad u \in U. \]

Since

\[ x^j_{s,1i,1} \in \pi_{2,j}(P), \quad s \in [0, \tau_1] \quad \text{and} \quad x^j_{s,12,2} \in \pi_{2,j}(P), \quad s \in [0, \tau_2] \]

for any \( j = 1, \ldots, i-1 \), we get

\[ |\pi^{-}_i(x^i)| \leq \int_0^{\tau_1} \left| e^{(\tau_1-s)D_{i1}} \pi^{-}_i \left( G \left( x^1_{s,1i,1}, \ldots, x^{i-1}_{s,1i,1}; Z_{s,1i,1} \right) \right) \right| ds \leq \frac{k_i}{\mu_i} M_i \left( 1 - e^{-\tau_1 \mu_i} \right) \leq \frac{k_i}{\mu_i} M_i \]

and

\[ |\pi^{+}_i(x^i)| \leq \int_0^{\tau_2} \left| e^{-sD_{i1}} \pi^{+}_i \left( G \left( x^1_{s,12,2}, \ldots, x^{i-1}_{s,12,2}; Z_{s,12,2} \right) \right) \right| ds \leq \frac{k_i}{\mu_i} M_i \left( 1 - e^{-\tau_2 \mu_i} \right) \leq \frac{k_i}{\mu_i} M_i, \]

implying that

\[ |x^1| = |\pi^{+}_i(x^i) + \pi^{-}_i(x^i)| \leq |\pi^{+}_i(x^i)| + |\pi^{-}_i(x^i)| \leq 2 \frac{k_i}{\mu_i} M_i. \]

Since \( x^i \in \pi_{2,i}(P) \) is arbitrary, we get that \( \pi_{2,i}(P) \) is a bounded set, which finishes the proof.

\[ \square \]

4 The main result

We can now state and prove our main result.

4.1 Theorem: It holds that

\( G^0 \) is a compact subgroup \( \iff \) \( \text{Per}(G^0, \Sigma_G) \) is a bounded subset of \( G \).

**Proof:** Since \( G^0 \) is \( \varphi \)-invariant, it follows that \( G^0 \subset \text{Per}(G^0, \Sigma_G) \) and hence \( G^0 \) is a compact subgroup as soon as \( \text{Per}(G^0, \Sigma_G) \) is a bounded subset of \( G \).

Reciprocally, assume that \( G^0 \) is a compact subgroup and also that \( G^{+,-} \) is a subgroup of \( G \). Since the compactness of \( G^0 \) implies the decomposability of \( G \), Lemma 2.3 gives us that the map

\[ \psi: (g, x) \in G^0 \times_{Ad} g^{+,-} \mapsto e^{\varphi} g \in G, \]

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In particular, for any $(g,x) \in G^0 \times \text{Adj}^{+,-}$ we get that
\[
\psi \left( \varphi_t(g), e^{t \mathfrak{d}|_{g^{+,-}}} x \right) = \psi \left( \varphi_t(g), e^{t \mathfrak{d}} x \right) = \exp(e^{t \mathfrak{d}} x) \varphi_t(g) = \varphi_t(e^t g) = \varphi_t(g(x)),
\]
which, by derivation, implies that $\psi \circ \left( \mathcal{L}_{G^0} \times D|_{g^{+,-}} \right) = \mathcal{X} \circ \psi$. Thus, the linear vector field of $\Sigma_{G^0 \times \text{Adj}^{+,-}}$ is given by $\mathcal{X}_{G^0} \times D|_{g^{+,-}}$. In particular, the linear system $\Sigma_{G^0 \times \text{Adj}^{+,-}}$ is of the form $[5]$. Since $G^0$ is a compact group and $D|_{g^{+,-}}$ has only eigenvalues with nonzero real part, Lemma 6.3 implies that
\[
\text{Per} \left( G^0 \times \{0\}, \Sigma_{G^0 \times \text{Adj}^{+,-}} \right)
\]
is a bounded subset.

Since $\psi$ is an isomorphism, Proposition 2.8 shows that
\[
\psi \left( \text{Per} \left( G^0 \times \{0\}, \Sigma_{G^0 \times \text{Adj}^{+,-}} \right) \right) = \text{Per} \left( \psi(G^0 \times \{0\}), \Sigma_G \right) = \text{Per} \left( G^0, \Sigma_G \right)
\]
and hence $\text{Per} \left( G^0, \Sigma_G \right)$ is a bounded subset if $G^{+,-}$ is a subgroup.

For the general case, let us consider as previously the compact, connected normal subgroup of $G$ given by $N^0 = N \cap G^0$. If $\hat{G} = G/N^0$ we have that $\hat{G}$ is a Lie group and the induced system $\hat{\Sigma}_G$ a linear system. Moreover, by [5] Lemma 2.3 it holds that
\[
\hat{G}^+ = \pi(G^+), \quad \hat{G}^0 = \pi(G^0) \quad \text{and} \quad \hat{G}^- = \pi(G^-),
\]
where $\pi : G \rightarrow \hat{G}$ is the canonical projection. Also, the fact that $\pi^{-1}(\pi(G^0)) = G^0 N^0 = G^0$ implies, by Proposition 2.8 that
\[
\text{Per}(G^0, \Sigma_G) \quad \text{is a bounded set} \quad \iff \quad \text{Per}(\hat{G}^0, \hat{\Sigma}_G) \quad \text{is a bounded set},
\]
and our work is reduced to show the result for the linear system $\Sigma_{\hat{G}}$. However, by Lemma 2.2 it holds that $N = G^{+,-} N^0$ and so
\[
\hat{G}^{+,-} = \pi(G^{+,-}) = \pi(G^{+,-} N^0) = \pi(N),
\]
therefore, $\hat{G}^{+,-}$ is a subgroup of $\hat{G}$. By the first case, we get that $\text{Per}(\hat{G}^0, \hat{\Sigma}_G)$ is a bounded set and the result follows.

### 4.1 Control sets

In this section we define control sets and use the previous results to obtain a characterization for its boundedness. Roughly speaking a control set is a maximal region on which the system is controllable. Such concept appear mainly due to the lack of controllability of many systems.

For any $x \in G$ the set of reachable points from $x$ is by definition the set
\[
\mathcal{O}^+(x) := \{ \phi(t, x, u), \; t \geq 0, u \in \mathcal{U} \}.
\]

Following [3] 3.1.2 a nonempty set $\mathcal{C} \subset G$ is said to be a control set of $\Sigma_G$ if it is maximal (w.r.t. set inclusion) satisfying

(i) For any $g \in \mathcal{C}$ there exists $u \in \mathcal{U}$ such that $\phi(\mathbb{R}^+, g, u) \in \mathcal{C}$;

(ii) For any $x \in \mathcal{C}$ it holds that $\mathcal{C} \subset \text{cl}(\mathcal{O}^+(x))$.

As a consequence of Theorem 4.1 we have the following:

### 4.2 Theorem:

Let $\mathcal{C}$ be the control set of $\Sigma_G$ containing the identity element. If $e \in \text{int} \mathcal{C}$ then
\[
\text{int} \mathcal{C} = \text{Per}(e, \Sigma_G) = \text{Per}(G^0, \Sigma_G).
\]

In particular, $\mathcal{C}$ is a bounded control set if and only if $G^0$ is a compact subgroup.
**Proof:** Certainly $\text{Per}(e, \Sigma_G) \subseteq \text{Per}(G^0, \Sigma_G)$ always holds. Furthermore, control sets with nonempty interior have the no-return property, that is, if $x \in C$ and for some $\tau > 0$ and $u \in U$ it holds that $\phi(\tau, x, u) \in C$, then $\phi(t, x, u) \in C$ for any $t \in [0, \tau]$ (see [7, Corollary 1.1]). Since $e \in \text{int} C$, it turns out that $\Omega^e(e)$ is open and hence Proposition 3.7 of [1] implies $G^0 \subseteq \text{int} C$. Hence, the no-return property proves that $\text{Per}(G^0, \Sigma_G) \subseteq \text{int} C$.

On the other hand, by [3, Theorem 2.4], any two points in $\text{int} C$ can be joined by a trajectory of $\Sigma_G$ implying that $\text{int} C \subseteq \text{Per}(e, \Sigma_G)$ and hence $\text{int} C = \text{Per}(e, \Sigma_G) = \text{Per}(G^0, \Sigma_G)$.

By Theorem 4.4 $G^0$ is a compact subgroup if and only if $\text{Per}(G^0, \Sigma_G)$. Since $\text{int} C$ is dense in $C$ the result follows. \qed

### 4.2 Examples

We use this section to provide some examples.

#### 4.3 Example:

If $G = G^0$ then $\text{Per}(G^0, \Sigma_G) = G$. However, if $e \in \text{int} C$ also holds, [1, Proposition 3.7] assures that $G^0 \subseteq \text{int} C$ and hence $G = \text{Per}(e, \Sigma_G) = C$.

#### 4.4 Example:

Let us consider the linear system $\Sigma_{\mathbb{R}^2}$ on $\mathbb{R}^2$ given in coordinates by

$$
\begin{cases}
\dot{x}(t) = x(t) + u(t) \\
\dot{y}(t) = -y(t) + u(t)
\end{cases}
$$

where $u(t) \in [-1, 1]$. Following [4, Example 3.2.27] the control set of $\Sigma_{\mathbb{R}^2}$ is given by

$$C = (-1, 1) \times [-1, 1].$$

In particular, the fact that $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ implies that $(\mathbb{R}^2)^0 = \{0\}$ and hence $\text{Per}(0, \Sigma_{\mathbb{R}^2}) = (-1, 1)^2$.

#### 4.5 Example:

Let us consider $\mathbb{R}^3$ endowed with the bracket obtained by the relations

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. The *Heisenberg group* $\mathbb{H}$ is the Lie group whose subjacent manifold is $\mathbb{R}^3$ and the product is given by

$$v \cdot w = v + w + \frac{1}{2}[v, w].$$

By identifying $\{(0, 0, p), p \in \mathbb{Z}\} = \mathbb{Z}$ we have the connected Lie group $G = \mathbb{H}/\mathbb{Z} \sim \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. On $G$, the maps

$$[v] \mapsto e_1 + e_2 + \frac{1}{2}[e_1 + e_2, v] \quad \text{and} \quad [v] \mapsto Dv,$$

define, respectively, a right-invariant vector field and a linear vector, where $D = \text{diag}(1, -1, 0)$ and $[v] = v + \mathbb{Z}$. In particular the linear system $\Sigma_G$ given in coordinates reads as

$$
\begin{cases}
\dot{x}(t) = x(t) + u(t) \\
\dot{y}(t) = -y(t) + u(t) \\
\dot{z}(t) = \frac{u(t)}{2}(y(t) - x(t)),
\end{cases}
$$

where $u(t) \in [-1, 1]$. Note that in this case $G^0 = Z(G) = \mathbb{R}/\mathbb{Z}$ is a compact subgroup and also an ideal of $G$. Moreover, by the previous equations, we have that $\Sigma_G$ and the linear system $\Sigma_{\mathbb{R}^2}$ from Example 1.4 are $\pi$-conjugated, where $\pi: G \rightarrow G/G^0$ is the canonical projection. Since $\ker \pi = G^0$, Proposition 2.8 implies

$$\pi(\text{Per}(G^0, \Sigma_G)) = \text{Per}(0, \Sigma_{\mathbb{R}^2}) \quad \text{and} \quad \pi^{-1}(\text{Per}(0, \Sigma_{\mathbb{R}^2})) = \text{Per}(G^0, \Sigma_G).$$

Consequently,

$$\text{Per}(G^0, \Sigma_G) = ((-1, 1)^2 \times \{0\}) + \mathbb{R}/\mathbb{Z} = (-1, 1)^2 \times \mathbb{R}/\mathbb{Z}.$$
References

[1] V. Ayala and A. Da Silva, *Controllability of Linear Control Systems on Lie Groups with Semisimple Finite Center*, SIAM Journal on Control and Optimization 55 No 2 (2017), 1332-1343.

[2] V. Ayala and A. Da Silva, *On the characterization of the controllability property for linear control systems on nonnilpotent, solvable threedimensional Lie groups*, Journal of Differential Equations, 266 (2019), 1-25.

[3] V. Ayala, A. Da Silva and G. Zsigmond, *Control sets of linear systems on Lie groups*. Nonlinear Differential Equations and Applications - NoDEA 24 No 8 (2017), 1 - 15.

[4] F. Colonius and C. Kliemann, *The Dynamics of Control*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2000.

[5] A. Da Silva, *Controllability of linear systems on solvable Lie groups*. SIAM Journal on Control and Optimization 54 No 1 (2016), 372-390.

[6] P. Jouan, *Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces*. ESAIM: Control Optimization and Calculus of Variations, 16 (2010) 956-973.

[7] C. Kawan, *Invariance Entropy for Deterministic Control Systems. An Introduction*. Lecture Notes in Mathematics, 2089. Springer, 2013.

[8] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Lie Algebras III - Structure of Lie Groups and Lie Algebras*. Berlin: Springer (1990).

[9] L. A. B. San Martin, *Algebras de Lie*, Second Edition, Editora Unicamp, (2010).