An inequality for tensor product of positive operators and its applications

Haixia Chang\textsuperscript{a}, Vehbi E. Paksoy\textsuperscript{b}, Fuzhen Zhang\textsuperscript{b,\ast}

\textsuperscript{a} Department of Applied Mathematics, Shanghai Finance University, Shanghai 201209, PR China
\textsuperscript{b} Farquhar College of Arts and Sciences, Nova Southeastern University, 3301 College Ave., Fort Lauderdale, FL 33314, USA

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\textbf{ABSTRACT}

We present an inequality for tensor product of positive operators on Hilbert spaces by considering the tensor products of operators as words on certain alphabets (i.e., a set of letters). As applications of the operator inequality and by a multilinear approach, we show some matrix inequalities concerning induced operators and generalized matrix functions (including determinants and permanents as special cases).

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\ast{} Corresponding author.
E-mail addresses: hcychang@163.com (H. Chang), vp80@nova.edu (V.E. Paksoy), zhang@nova.edu (F. Zhang).

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1. Introduction

Let \( \mathcal{H} \) be a Hilbert space over the complex number field \( \mathbb{C} \) with an inner product \( \langle \cdot, \cdot \rangle \). Denote by \( \mathcal{B}(\mathcal{H}) \) the C*-algebra of all bounded linear operators on \( \mathcal{H} \). We write \( A \geq 0 \) if \( A \) is a positive semidefinite operator on \( \mathcal{H} \) (we simply call it a positive operator), that is, \( A \) is self-adjoint and \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). For self-adjoint \( A, B \in \mathcal{B}(\mathcal{H}) \), we write \( A \geq B \) if \( A - B \geq 0 \). It is well known that if \( A \geq 0 \) and \( B \geq 0 \) then the sum \( A + B \geq 0 \) (on \( \mathcal{H} \)) and the tensor product \( A \otimes B \geq 0 \) (on \( \otimes^2 \mathcal{H} = \mathcal{H} \otimes \mathcal{H} \)). Moreover, if \( A \) is positive then the tensor product \( \otimes^m A = A \otimes \cdots \otimes A \) (\( m \) copies of \( A \)) is positive (on \( \otimes^m \mathcal{H} \)) for any positive integer \( m \). For the finite-dimensional case of \( \mathcal{H} \), we denote by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) the smallest and largest eigenvalues of the positive linear operator (matrix) \( A \) on \( \mathcal{H} \), respectively.

In Section 2, we present an inequality for positive operators with tensor product. In Section 3, as applications of our main result, we deduce inequalities for generalized matrix functions, including determinants and permanents. Our results may be regarded as additions to the recent ones in the research development of positivity (see, e.g., [1–7,14]).

2. Main results

We present our main result in this section. The proof is accomplished by using the idea of words (which, for instance, is used to show normality of matrices in [12]). Let \( \{A_1, \ldots, A_j\} \) be a (multi-)set of operators from \( \mathcal{B}(\mathcal{H}) \) in which \( t \) operators are distinct. For example, \( t = 2 \) for \( \{A_1, A_1, A_2\} \). A tensor word or word of operators \( A_1, \ldots, A_j \) on \( \mathcal{H} \) of length \( m \) and of \( t \) (distinct) representatives with respect to the tensor product \( \otimes \), symbolized by \( w^m_{t_1}(A_1, \ldots, A_j) \), or \( w^t(A_i, \ldots, A_j) \), or even simply \( w^t \) (if no confusion is caused), is a tensor product

\[
A_{s_1} \otimes \cdots \otimes A_{s_m},
\]

in which \( A_{s_1}, \ldots, A_{s_m} \) are taken from \( \{A_1, \ldots, A_j\} \), and among \( A_{s_1}, \ldots, A_{s_m} \), there are \( t \) distinct operators. For instance, \( A_1 \otimes A_1 = \otimes^2 A_1 \) is a \( w^1 \) word; \( A_1 \otimes A_2 \) is a \( w^2 \) word, and \( A_1 \otimes A_2 \) is also a \( w^2 \) word. Note that \( w^2_2(A_1, A_2) \) may represent any of \( A_1 \otimes A_1 \otimes A_2, A_1 \otimes A_2 \otimes A_1, A_2 \otimes A_1 \otimes A_1, A_1 \otimes A_2 \otimes A_2, A_2 \otimes A_1 \otimes A_2, A_2 \otimes A_2 \otimes A_1, A_2 \otimes A_2 \otimes A_1, A_2 \otimes A_2 \otimes A_1. \) So when we say a \( w^2(A_1, A_2) \) word, we mean one of those tensor words (with the given length \( m = 3 \)).

**Theorem 1.** Let \( A_1, A_2, \ldots, A_k \in \mathcal{B}(\mathcal{H}) \) be positive operators. Then

\[
\otimes^m(A_1 + A_2 + \cdots + A_k) \geq \\
\sum_{1 \leq i_1 < \cdots < i_{k-1} \leq k} \otimes^m(A_{i_1} + A_{i_2} + \cdots + A_{i_{k-1}})
\]

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\[ + \sum_{1 \leq j_1 < \ldots < j_{k-2} \leq k} \bigotimes^m (A_{j_1} + A_{j_2} + \cdots + A_{j_{k-2}}) - \cdots \]
\[ + (-1)^{k-t} \sum_{1 \leq x_1 < \ldots < x_t \leq k} \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t}) + \cdots \tag{3} \]
\[ + (-1)^{k-2} \sum_{1 \leq p_1 < p_2 \leq k} \bigotimes^m (A_{p_1} + A_{p_2}) \tag{4} \]
\[ + (-1)^{k-1} \sum_{q=1}^{k} \bigotimes^m A_q \geq 0. \tag{5} \]

**Proof.** The idea of the proof is to show that all the tensor words \( w^t \) with \( t < k \) representatives are canceled out in the reduced form (after the additions and subtractions). That is, after the calculations, the only tensors that survive will be the ones of the form in the summation

\[ \sum \text{each of } A_1, A_2, \ldots, A_k \text{ appears at least once} \]

\[ (A_{i_1} \otimes \cdots \otimes A_{i_m}). \]

Consider the words \( w^t \), i.e., the tensors of one operator, \( \bigotimes^m A_i \), say \( \bigotimes^m A_1 \). The tensor \( \bigotimes^m A_1 \) appears in (1) once, in (2) \( (k-1)(k-2) \) times, \ldots, in (4) \( (k-1) \) times, and in (5) once. Therefore, after cancellation, the total number of \( \bigotimes^m A_1 \) left in the reduced form is

\[ 1 - \binom{k-1}{k-2} + \cdots + (-1)^{k-2} \binom{k-1}{1} + (-1)^{k-1} = (1-1)^{k-1} = 0. \]

Now we consider a general tensor word \( A_{i_1} \otimes \cdots \otimes A_{i_m} \). Let it be a \( w^t \) word with \( t < k \), and let \( A_{x_1}, A_{x_2}, \ldots, A_{x_t} \) be the distinct representatives of \( A_{i_1}, A_{i_2}, \ldots, A_{i_m} \). We show that \( A_{i_1} \otimes \cdots \otimes A_{i_m} \) vanishes in the reduced form. Since it is a \( w^t \) word, it cannot be a \( w^{t-1} \) word. Therefore, this \( w^t \) word does not appear in the expansion of any sum-tensor \( \bigotimes^m (A_{y_1} + A_{y_2} + \cdots + A_{y_s}), s < t \). That is, \( \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t}) \) in (3) is the only sum-tensor containing all \( A_{x_1}, A_{x_2}, \ldots, A_{x_t} \). It is important to observe that a \( w^l \) word contains all \( A_{x_1}, A_{x_2}, \ldots, A_{x_t} \) if and only if \( l \geq t \) and \( w^l \) is obtained from a sum-tensor \( \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t} + A_{y_1} + \cdots + A_{y}) \), in which, after expansion, each word \( w^l (A_{x_1}, A_{x_2}, \ldots, A_{x_t}) \) appears once and only once; so does \( A_{i_1} \otimes \cdots \otimes A_{i_m} \).

Note that for each fixed word \( w^l (A_{x_1}, A_{x_2}, \ldots, A_{x_t}) \) (say, \( A_{i_1} \otimes \cdots \otimes A_{i_m} \)) in the sum-tensor \( \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t}) \), there are \( k-t = \binom{k-1}{t} \) many terms \( \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t} + A_x) \) in the expression prior to (3); they are

\[ \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t} + A_x), \quad x \in \{1, 2, \ldots, k\} \setminus\{x_1, \ldots, x_t\}. \]

Similarly, considering one level above, there are \( \binom{k-2}{t} \) many terms in

\[ \sum \bigotimes^m (A_{x_1} + A_{x_2} + \cdots + A_{x_t} + A_x + A_y), \quad x < y, \quad x, y \in \{1, 2, \ldots, k\} \setminus\{x_1, \ldots, x_t\} \]

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containing $\otimes^m(A_{x_1} + A_{x_2} + \cdots + A_{x_t})$. Thus, going in this way up to (1), we have the
total number of the $w^i(A_{x_1}, A_{x_2}, \ldots, A_{x_t})$ word $A_{i_1} \otimes \cdots \otimes A_{i_m}$ appearing in the tensors
$\otimes^m(A_{x_1} + A_{x_2} + \cdots + A_{x_t} + \cdots)$ in the reduced form:

$$1 - \binom{k - t}{k - t - 1} + \cdots + (-1)^{k-t-1} \binom{k - t}{1} + (-1)^{k-t} = (1 - 1)^{k-t} = 0.$$ 

It follows that the only remaining tensors are $w^k$ words, i.e., $A_{i_1} \otimes \cdots \otimes A_{i_m}$ in which
every $A_1, A_2, \ldots, A_k$ has to appear at least once (so $m \geq k$). That is, in the reduced form, namely, after computing all expressions (1) through (5), we obtain

$$\sum_{\text{each of } A_1, A_2, \ldots, A_k \text{ appears at least once}} (A_{i_1} \otimes \cdots \otimes A_{i_m}) \geq 0.$$ 

If $m < k$, then the left hand side of the above display is 0. \( \square \)

The inequality below is a special case of Theorem 1. We single it out as it is interesting
in its own right and all our results in Section 3 rely on it.

**Theorem 2.** Let $A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})$ be positive operators. Then, for any positive integer $m$,

$$(\otimes^m(A_1 + A_2 + A_3) + \otimes^m A_1 + \otimes^m A_2 + \otimes^m A_3)$$

$$- (\otimes^m(A_1 + A_2) + \otimes^m(A_1 + A_3) + \otimes^m(A_2 + A_3)) \geq 0.$$

If $\mathcal{H}$ is finite-dimensional, then the eigenvalues of the above difference lie between

$$(3(3^{m-1} - 2^m + 1)) \left( \min_{1 \leq i \leq 3} \lambda_{\text{min}}(A_i) \right)^m$$

and

$$(3(3^{m-1} - 2^m + 1)) \left( \max_{1 \leq i \leq 3} \lambda_{\text{max}}(A_i) \right)^m.$$ 

**Proof.** If $m = 1$ or $m = 2$, it is straightforward to check that the difference is zero. Let $m \geq 3$. From Theorem 1, we have

$$\otimes^m(A_1 + A_2 + A_3) + \otimes^m A_1 + \otimes^m A_2 + \otimes^m A_3$$

$$- (\otimes^m(A_1 + A_2) + \otimes^m(A_1 + A_3) + \otimes^m(A_2 + A_3))$$

$$= \sum_{\text{each of } 1, 2, 3 \text{ appears at least once}} (A_{i_1} \otimes \cdots \otimes A_{i_m}) \geq 0.$$
When the indices are taken from the set \{1, 2, 3\} and not all the same, there are \(3^m - 3\) different terms. When the indices are taken from the 2-element sets, each summation will consist of \(2^m - 2\) distinct tensors. Clearly, the first sum of tensors with indices from \{1, 2, 3\} contains all possible configurations of tensors in the other three sums. On the other hand, the first summation has additional tensors which contain all the three elements from \{1, 2, 3\}. So, the last sum in the above computation has exactly \(3^m - 3 - 3(2^m - 2) = 3(3^{m-1} - 2^m + 1)\) distinct positive semidefinite entries.

For the smallest eigenvalue, since for every tensor product \(A_{i_1} \otimes \cdots \otimes A_{i_m}\),

\[
\lambda_{\min}(A_{i_1} \otimes \cdots \otimes A_{i_m}) = \prod_{t=1}^{m} \lambda_{\min}(A_{i_t}),
\]

we see that the smallest eigenvalue of the difference has a lower bound

\[
(3(3^{m-1} - 2^m + 1)) \cdot \left( \min_{1 \leq i \leq 3} \lambda_{\min}(A_i) \right)^m.
\]

Similarly, we arrive at an upper bound for the largest eigenvalue:

\[
(3(3^{m-1} - 2^m + 1)) \cdot \left( \max_{1 \leq i \leq 3} \lambda_{\max}(A_i) \right)^m. \quad \square
\]

3. Applications

Let \(\mathcal{H}\) be finite-dimensional and \(\otimes^m \mathcal{H}\) be the tensor product space of \(m\) copies of \(\mathcal{H}\). Let \(G\) be a subgroup of the \(m\)-symmetric group \(S_m\) on \(m\) letters and let \(\chi\) be an irreducible character on \(G\). Denote by \(V_\chi(G)\) the symmetry class of tensors associated with \(G\) and \(\chi\) (see, e.g., [9, p. 154]). For a linear operator \(A\) on \(\mathcal{H}\), the induced operator \(K(A)\) of \(A\) with respect to \(G\) and \(\chi\) is the restriction of \(\otimes^m A\) on \(V_\chi(G)\); that is, \(K(A) = (\otimes^m A)|_{V_\chi(G)}\) (see, e.g., [9, p. 185, p. 235]).

So, for the positive operators \(A_1, A_2, A_3\) in Theorem 2 with \(\dim \mathcal{H} < \infty\),

\[
(\otimes^m(A_1 + A_2 + A_3))|_{V_\chi(G)} + (\otimes^m A_1)|_{V_\chi(G)} + (\otimes^m A_2)|_{V_\chi(G)} + (\otimes^m A_3)|_{V_\chi(G)}
- ((\otimes^m(A_1 + A_2))|_{V_\chi(G)} + (\otimes^m(A_1 + A_3))|_{V_\chi(G)} + (\otimes^m(A_2 + A_3))|_{V_\chi(G)}) \geq 0,
\]

that is,

\[
K(A_1 + A_2 + A_3) + K(A_1) + K(A_2) + K(A_3)
- (K(A_1 + A_2) + K(A_1 + A_3) + K(A_2 + A_3)) \geq 0.
\]
The induced operators are closely related to generalized matrix functions (see, e.g., [9, p. 213]). For the above \( G \) and \( \chi \), the generalized matrix function with respect to \( G \) and \( \chi \) defined on the space of \( m \times m \) matrices is

\[
d^g_{\chi}(X) = \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{m} x_{t\sigma(t)},\quad \text{where } X = (x_{ij}).
\]

If \( G = S_m \) and \( \chi \) is the signum function with values \( \pm 1 \), then the generalized matrix function becomes the usual matrix determinant (det); setting \( \chi(\sigma) = 1 \) for each \( \sigma \in G = S_m \) defines the permanent (per) of the matrix.

Let \( \dim \mathcal{H} = m \) and let \( \{e_1, e_2, \ldots, e_m\} \) be an orthonormal basis of \( \mathcal{H} \). Let \( P' \) (the transpose of \( P \)) be a matrix representation of a linear operator \( T \) on \( \mathcal{H} \) with respect to the basis \( \{e_1, e_2, \ldots, e_m\} \). Then (see, e.g., [9, p. 227])

\[
d^g_{\chi}(P) = \frac{o(G)}{o(\chi)} \langle K(T)e^*, e^* \rangle,
\]

where \( o(G) \) is the order of \( G \), \( o(\chi) \) is the degree of \( \chi \), and \( e^* = e_1 \star e_2 \star \cdots \star e_m \) is the decomposable symmetrized tensor of \( e_1, e_2, \ldots, e_m \) [9, p. 155].

The following result for generalized matrix functions are immediate.

**Theorem 3.** Let \( A_1, A_2, A_3 \) be \( m \times m \) positive semidefinite matrices. Let \( G \) be a subgroup of \( S_m \) and \( \chi \) be an irreducible character of \( G \). Then

\[
d^g_{\chi}(A_1 + A_2 + A_3) + d^g_{\chi}(A_1) + d^g_{\chi}(A_2) + d^g_{\chi}(A_3)
\]

\[
- \left( d^g_{\chi}(A_1 + A_2) + d^g_{\chi}(A_1 + A_3) + d^g_{\chi}(A_2 + A_3) \right) \geq 0.
\]

(6)

The determinant and permanent inequalities follow at once.

**Corollary 4.** Let \( A_1, A_2, A_3 \) be \( m \times m \) positive semidefinite matrices. Then

\[
\det(A_1 + A_2 + A_3) + \det(A_1) + \det(A_2) + \det(A_3)
\]

\[
- \left( \det(A_1 + A_2) + \det(A_1 + A_3) + \det(A_2 + A_3) \right) \geq 0.
\]

(7)

**Corollary 5.** Let \( A_1, A_2, A_3 \) be \( m \times m \) positive semidefinite matrices. Then

\[
\per(A_1 + A_2 + A_3) + \per(A_1) + \per(A_2) + \per(A_3)
\]

\[
- \left( \per(A_1 + A_2) + \per(A_1 + A_3) + \per(A_2 + A_3) \right) \geq 0.
\]

Inequality (7) is obtained by Lin in [8, Theorem 1.1] by using a majorization approach, while inequality (6) in Theorem 3 confirms the strong superadditivity of the generalized matrix functions – a question raised by Lin in [8, Section 3]. Additionally, our result gives
a different proof for the inequality in [10] in which an embedding approach is employed. These results are generalizations of the classic inequalities \( \det(A + B) \geq \det A + \det B \) and \( \text{per}(A + B) \geq \text{per} A + \text{per} B \) for positive semidefinite matrices \( A \) and \( B \) of the same size (see, e.g., [11, p. 121] and [13, p. 215]).

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