Dynamical Analysis of Infected Predator-Prey Model with Saturated Incidence Rate

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Abstract. A predator-prey model with disease in both populations is proposed to illustrate the possibility of disease transmission between prey and predator through contact and predation. We used saturated incidence rate which takes behavioural changes of healthy population into consideration when disease spreads around them. The existence of eight non-negative equilibrium points is analysed and their local stability has been investigated. Numerical simulations are given to illustrate analytic results.

1. Introduction

In 1927, Lotka and Volterra proposed a model representing the growth of predator and prey which interact in such a closed habitat in order to find out the evolution and existence of both populations in the future. Later, similar research has conducted widely and many researchers are interested in developing more realistic models. On the other hand, Kermack McKendrick [1] pioneered research in epidemiology by constructing a model of infectious diseases transmission which consists of three classes of population, namely S (Susceptible), I (Infected), and R (Recovered). One component that influences the dynamics of disease transmission is the incidence rate, which is defined as the number of new infected individual resulted by direct contact between susceptible and infected one. The incidence rate in Kermack McKendrick epidemic model is bilinear, $\beta SI$ with $\beta$ indicates the infection rate. When the number of infected or susceptible population is increasing, then the disease transmission is also increasing, vice versa.

The study that combines ecology and epidemiology has received special attention from researchers and is referred as eco-epidemiology. This model studies the spread of disease in a habitat where predator and prey are interacting. Some studies proposed eco-epidemiology models by using bilinear incidence rates ([2] - [4]). Johri, et al. [2] analyzed local and global stability of the eco-epidemiology model when only prey population is infected. Kant and Kumar [3] and Bera et al. [4] analyzed the local stability of the eco-epidemiology model with disease spreads in both populations. Bera and Kumar [3] considered that prey is able to migrate, while Bera et al. [4] assumed that infected predator is not able to catch susceptible prey and ignored prey’s ability to migrate.

In some cases, when the disease spreads and the number of infected population increases significantly, then susceptible population tends to reduce contact with infected population as a sign of psychological changes. Hence, disease transmission decreases when the number of infected population increases rapidly. Therefore, Capasso and Serio [5] introduced saturated incidence rate, $\beta SI/(\alpha + I)$ with $\alpha$ expresses inhibition effect of transmission, i.e. when susceptible individuals restricts the contact with infected individuals. Several studies analyzed eco-epidemiology model with disease spreads only...
in prey population by applying saturated incidence rates ([6] and [7]). Naji and Mustafa [6] assumed predation rate follows Holling type II response function, while Wang, et al. [7] used generalized functional responses and considering prey ability to refuge.

This paper proposed a model which is a modification of Kant and Kumar's model [3] by changing the bilinear incidence rate into saturated incidence rate in prey population. The local stability analysis of the constructed model is carried out to find out the dynamics of the system. Moreover, some numerical simulations are performed to illustrate the analytical results.

2. Model Formulation

The model we formulate consists of two populations, the prey populations \(X(t)\) and the predator population \(Y(t)\) and is following some assumptions stated below:

a. In the absence of disease, prey grows logistically with growth rate \(r\) and carrying capacity \(k\).

b. Prey is able to be infected due to external source of infection, while predator is infected because of predation of infected prey.

c. Disease transmission in prey follows saturated incidence rate \(\frac{\beta X_1 X_2}{\alpha + X_2}\) with \(\beta\) indicates infection rate in prey and \(\alpha\) expressed inhibition rate. Meanwhile the disease spread in predator population follows bilinear incidence rate \(\gamma Y_1 Y_2\) with \(\gamma\) expressed infection rate in predator.

d. Infected prey is easier to catch by predator and infected predator cannot eat susceptible prey.

e. Infected prey and infected predator are neither recovered nor get immune.

f. Predation rate follows Lotka Volterra functional response.

Based on assumptions said above, we formulate mathematical model of predator-prey with disease spreads in both populations:

\[
\frac{dx_1}{dt} = r x_1 \left( 1 - \frac{x_1 + x_2}{k} \right) - \frac{\beta x_1 x_2}{\alpha + x_2} - p_1 x_1 y_1 - a_1 x_1 \\
\frac{dx_2}{dt} = \frac{\beta x_1 x_2}{\alpha + x_2} - p_2 x_2 y_1 - p_4 x_2 y_2 - a_2 x_2 \\
\frac{dy_1}{dt} = a_3 x_1 y_1 + a_4 x_2 y_1 - \gamma y_1 y_2 - d_3 y_1 \\
\frac{dy_2}{dt} = a_5 x_2 y_2 + \gamma y_1 y_2 - a_6 y_2
\]

(2.1)

where \(a_1 = m_1 + d_1, a_3 = m_2 + c + d_2, a_4 = q_1 p_1, a_5 = q_2 p_2, a_6 = d_4 + d_5\), and \(X_1(t), X_2(t), Y_1(t),\) and \(Y_2(t)\) respectively denote susceptible prey, infected prey, susceptible predator and infected predator. \(m_1(m_2)\) is migrating rate of susceptible prey (or infected prey). \(p_1(p_2)\) is predation coefficient of susceptible predator to susceptible prey (to infected prey), meanwhile \(p_4\) is predation coefficient of infected predator to infected prey. \(q_1(q_2)\) expresses conversion rate from susceptible prey (or infected prey) to susceptible predator and \(q_4\) is conversion rate from infected prey to infected predator. Each of \(X_1, Y_1, X_2,\) and \(Y_2\) have their own natural death rates \(d_1, d_2, d_3,\) and \(d_4\). Death rate of infected prey and infected predator due to infection are \(c\) and \(d_5\), respectively.

3. Dynamics Analysis of Model

This section provides the analysis of the model, including equilibrium points, their existence, and also their stability conditions.

3.1. Equilibrium Points and Existence Conditions

By solving \(\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dy_1}{dt} = \frac{dy_2}{dt} = 0\), we obtain the following equilibrium points:

(i) The trivial equilibrium point \(E_0(0,0,0)\).

(ii) The axial equilibrium point \(E_1 \left( \frac{kr - ka_1}{r}, 0,0 \right) \) exists provided \(r > a_1\).
(iii) The disease-free point $E_2\left(\frac{d_3}{a_3}, 0, \frac{k(r-a_1)x_{2}^{(2)}r}{k_p_1}, 0\right)$ exists when $r > a_1 \left(\frac{k a_3}{a_3 - d_3}\right)$.

(iv) The predator-free point $E_3\left(\frac{a_2(\alpha + x_2^{(3)})}{\beta}, x_2^{(3)}, 0, 0\right)$ where $x_2^{(3)}$ is positive root of quadratic equation $A\left(x_2^{(3)}\right)^2 + Bx_2^{(3)} + C = 0$. Furthermore, it exists when $r \geq a_1 \left(\frac{\beta k}{\beta k - a_2\alpha}\right)$.

(v) The infected-prey-free point $E_4\left(\frac{k r y - k a_1 y - k p_1 a_6}{y r}, a_3, a_3 x_1^{(4)} - d_3\right)$ which exists when $r > \frac{a_2 k (y a_1 + p_1 a_6)}{y (a_3 - d_3)}$.

(vi) The susceptible-predator-free point $E_5\left(\frac{(a_6 + a a_2)(p_4 y_2^{(5)} + a_2)}{a_5 \beta}, a_6, y_2^{(5)}, Y_2^{(5)}\right)$ where

\[ y_2^{(5)} = \frac{(a_6 + a a_2)(r - a_3 + k r a - a_1 - k p_2) - r(a_6 - a_2 r(a_6 + a a_2))}{p_4 a a_6 (a_6 + a a_2)} \]

exists if $y_2^{(5)}$ is positive.

(vii) The infected-predator-free point $E_6\left(\frac{d_3 - a_4 x_2^{(6)}}{a_3}, x_2^{(6)}, Y_1^{(6)}, Y_1^{(6)}, 0\right)$, where

\[ Y_1^{(6)} = \frac{d_3 \beta - a_4 x_2^{(6)} - a_2 a_3 - a_3 a_2 x_2^{(6)}}{a_3 p_2 (a + x_2^{(6)})}\]

and $x_2^{(6)}$ is positive root of quadratic equation

\[ K\left(x_2^{(6)}\right)^2 + Lx_2^{(6)} + M = 0, \]

where

\[ K = (a_3 - a_4) p_2 r\]

\[ L = p_2 a a_3 (k + a_2 - r) + p_2 (d_3 r + r a (a_3 - a_4)) - p_1 k (a_4 + a_2 a_3)\]

\[ M = r a (d_3 p_2 + a_3 p_2 k) + k a a_3 (a_3 p_2 - a_2 p_1) + d_3 p_1 k \beta.\]

This equilibrium point exists if $0 < x_2^{(6)} < \frac{d_3 \beta - a_2 a_3}{a_4 \beta + a_3 a_2}$.

(viii) The interior equilibrium point $E_7\left(x_1^{(7)}, x_2^{(7)}, a_2 x_1^{(7)} + a_4 x_2^{(7)} - d_3, a_6 - a x_2^{(7)}\right)$, where

\[ x_1^{(7)} = \frac{(\alpha + x_2^{(7)})(p_2 a_6 - a_2 x_2^{(7)}) + a_4 (a_6 x_2^{(7)} - d_3) + a_2 y}{\beta y - (\alpha + x_2^{(7)})p_4 a_3}\]

and $x_2^{(7)}$ is positive root of cubic equation, where

\[ P\left(x_2^{(7)}\right)^3 + Q\left(x_2^{(7)}\right)^2 + R x_2^{(7)} + S = 0 \quad (3.1)\]

\[ P = p_4 (a y (a_4 - a_3) + a_3 a_5 k p_1) - a_5 p_2 y r\]

\[ Q = 2 p_4 y a (a_4 - a_3) + y r (a_6 p_2 + a_6 + \beta) r + a_3 p_4 k - 2 a_5 p_2 a - d_3 p_4 a + a_3 k p_1 p_4 (2 a_5 a - a_6) - \gamma \beta k (a_5 p_1 + a_3 p_4) - a_5 a_3 p_4 k y\]

\[ R = p_4 a^2 r y (a_4 - a_3) + \beta y^2 (a r + k (\beta + a_1 - r)) + k p_2 (a_6 p_1 - a (a_5 p_1 + a_3 p_4)) + a_3 p_4 k a (a_6 - 2 a_6) + 2 a_3 k k p_1 y a r - a_1 + a y p_2 (2 a_6 - a_5 a) + 2 a_2 y - 2 d_3 p_4 a\]

\[ S = a^2 y r (a_6 p_2 + a_2 y + p_4 (a_3 k - d_3)) + k a a_3 y (a_1 - r) + a_6 p_1) - a_3 p_4 a^2 (a_1 y - a_3 p_1 k)\]

The existence of positive root of equation (3.1) can be determined using Cardan's method [8].

3.2. Stability Analysis of Model
To analyze local stability of system (2.1), we need to linearized system (2.1) around equilibrium points $E_i, i = 0, 1, \ldots, 7$ with Jacobian matrix is given by:

\[
J(E_i) = \begin{bmatrix}
A_{11} & A_{12} & -p_1 X_1^{(i)} & 0 \\
A_{21} & A_{22} & -p_2 X_2^{(i)} & -p_4 X_2^{(i)} \\
a_3 Y_1^{(i)} & a_4 Y_1^{(i)} & A_{33} & -y Y_1^{(i)} \\
0 & a_5 Y_2^{(i)} & y Y_2^{(i)} & A_{44}
\end{bmatrix}
\]

where
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\[ A_{11} = r \left( 1 - \frac{2X_1(i) + X_2(i)}{k} \right) - \frac{\beta X_1(i)}{a + X_2(i)} - p_1Y_1(i) - a_4, \quad A_{21} = \frac{\beta X_2(i)}{a + X_2(i)}. \]
\[ A_{22} = \frac{\beta X_1(i)}{a + X_2(i)} - \frac{\beta X_1(i)X_2(i)}{(a + X_2(i))^2} - p_2Y_1(i) - p_4Y_2(i) - a_2, \quad A_{33} = a_3X_1(i) + a_4X_2(i) - \gamma Y_2(i) - d_3, \]
\[ A_{12} = - \frac{rX_1(i)}{k} - \frac{\beta X_1(i)}{a + X_2(i)} + \frac{\beta X_1(i)}{(a + X_2(i))^2}, \quad A_{44} = a_5X_2(i) + \gamma Y_2(i) - a_6. \]

Jacobian matrix at \( E_0 \) is

\[
\begin{bmatrix}
 r - a_4 & 0 & 0 & 0 \\
 0 & -a_2 & 0 & 0 \\
 0 & 0 & -d_3 & 0 \\
 0 & 0 & 0 & -a_6
\end{bmatrix},
\]

which has eigen values \( \lambda_1 = r - a_4, \lambda_2 = -a_2 < 0, \lambda_3 = -d_3 < 0, \lambda_4 = -a_6 < 0. \) Thus, \( E_0 \) is locally asymptotically stable if only if \( r < a_4. \) At point \( E_1, \) Jacobian matrix (3.2) becomes

\[
\begin{bmatrix}
 a_1 - r & (r - a_1)(-1 - \frac{\beta k}{ar}) & -p_1k(r - a_1) & 0 \\
 0 & \frac{\beta k}{ar} (r - a_1) - a_2 & 0 & 0 \\
 0 & 0 & a_3k(r - a_3) & -d_3 \\
 0 & 0 & 0 & -a_6
\end{bmatrix}.
\]

We obtain eigen values \( \lambda_1 = a_1 - r, \lambda_2 = \frac{\beta k}{ar} (r - a_1) - a_2, \lambda_3 = \frac{a_3k(r-a_1)}{r} - d_3, \) and \( \lambda_4 = -a_6. \)

Note that \( \lambda_4 < 0. \) Hence, \( E_1 \) is locally asymptotically stable if only if all eigen values are negative that is when \( a_1 < r < \min \left\{ a_1 + \frac{a_3ar}{\beta k}, a_1 + \frac{d_3r}{a_3k} \right\}. \) Characteristic equation of Jacobian matrix (3.2) at \( E_2 \) is

\[
\left( \gamma Y_1(i) - a_6 - \lambda \right) \left( \frac{\beta X_1(i)}{\alpha} - p_2 Y_1(i) - a_2 - \lambda \right) \left( \lambda^2 + \frac{r^2 Y_1(i)}{k} \lambda + a_3p_1X_1(i)Y_1(i)^2 \right) = 0
\]

The corresponding eigen values are \( \lambda_1 = \gamma Y_1(i) - a_6, \lambda_2 = \frac{\beta X_1(i)}{\alpha} - p_2 Y_1(i) - a_2, \) and \( \lambda_{3,4} = -\frac{r^2 Y_1(i)}{k} \pm \frac{\sqrt{r^4 Y_1(i)^2}}{k} - 4a_3p_1X_1(i)Y_1(i)^2. \)

Note that \( \frac{r^2 Y_1(i)^2}{k} - 4a_3p_1X_1(i)Y_1(i)^2 \geq \frac{r^2 Y_1(i)^2}{k} \) guarantees \( \lambda_{3,4} < 0. \) Thus, \( E_2 \) is locally asymptotically stable if \( \gamma Y_1(i) - a_6 < 0 \) and \( \frac{\beta x_1(i)}{\alpha} - p_2 Y_1(i) - a_2 < 0. \)

At \( E_3, \) characteristic equation from (3.2) is

\[
\left( a_5X_2(i) - a_6 - \lambda \right) \left( a_3X_1(i) + a_4X_2(i) - d_3 - \lambda \right) \left( (B_{11} - \lambda)(B_{22} - \lambda) - (B_{21})(B_{12}) \right) = 0
\]

where

\[
B_{11} = r \left( 1 - \frac{2X_1(i) + X_2(i)}{k} \right) - \frac{\beta X_1(i)}{a + X_2(i)} - a_1, \quad B_{21} = \frac{\beta X_2(i)}{a + X_2(i)}, \]
\[
B_{12} = X_1(i) \left( -\frac{r}{k} - \frac{\beta X_1(i)}{a + X_2(i)} + \frac{\beta X_1(i)}{(a + X_2(i))^2} \right), \quad B_{22} = X_1(i) \left( \frac{\beta}{a + X_2(i)} - \frac{\beta X_1(i)}{a + X_2(i)^2} \right) - a_2.
\]

Corresponding eigen values are \( \lambda_1 = a_5X_2(i) - a_6, \lambda_2 = a_3X_1(i) + a_4X_2(i) - d_3, \) and \( \lambda_{3,4} = \frac{(B_{11} + B_{22})^2 + 4(B_{11} + B_{22})(B_{11}B_{22} - B_{21}B_{12})}{2}. \)

So, \( E_3 \) is locally asymptotically stable if \( a_5X_2(i) - a_6, a_3X_1(i) + a_4X_2(i) < d_3, \) \( B_{11} + B_{22} < 0 \) and \( B_{11}B_{22} - B_{21}B_{12} > 0. \)
At $E_4$, characteristic equation of (3.2) can be written as

$$(C_{22} - \lambda)(C_{11} - \lambda) \left( (C_{33} - \lambda)(C_{44} - \lambda) + \gamma^2 Y_1^{(4)} Y_2^{(4)} \right) = 0,$$

where

$$C_{11} = r \left(1 - \frac{2x_1^{(4)} + x_2^{(4)}}{k} \right) - p_1 Y_1^{(4)} - a_1,$$

$$C_{12} = X_1^{(4)} \left(1 - \frac{r}{k} - \frac{\beta}{a}\right),$$

$$C_{22} = \frac{\beta x_1^{(4)}}{a} - p_2 Y_1^{(4)} - p_4 Y_2^{(4)} - a_2,$$

$$C_{33} = a_3 X_1^{(4)} - \gamma Y_2^{(4)} - d_3,$$

$$C_{44} = \gamma Y_1^{(4)} - a_6.$$

Corresponding eigen values are $\lambda_1 = C_{11}$, $\lambda_2 = C_{22}$, and $\lambda_{3,4} = \frac{(c_{33} + c_{44}) \pm \sqrt{(c_{33} + c_{44})^2 - 4 c_{33} c_{44} - \gamma^2 Y_1^{(4)} Y_2^{(4)}}}{2}.$

Hence we have conditions so that $E_3$ is locally asymptotically stable:

- $\frac{\beta x_1^{(4)}}{a} < p_2 Y_1^{(4)} + p_4 Y_2^{(4)} + a_2$ and $r < r \left(\frac{2x_1^{(4)} + x_2^{(4)}}{k} \right) + p_1 Y_1^{(4)} + a_1$,
- $\lambda_3 < 0$ and $\lambda_4 < 0$ are obtained if $C_{33} + C_{44} < 0$ and $C_{33} C_{44} - \gamma^2 Y_1^{(4)} Y_2^{(4)} > 0$.

At $E_5$, we have characteristic equation

$$(D_{33} - \lambda) \left( (D_{11} - \lambda)(D_{22} - \lambda)(D_{44} - \lambda) + a_5 p_4 X_2^{(5)} Y_2^{(5)} \right) - D_{21} \left( X_1^{(5)} D_{12} \right) (D_{44} - \lambda) = 0,$$

where

$$D_{11} = r \left(1 - \frac{2x_1^{(5)} + x_2^{(5)}}{k} \right) - \frac{\beta x_1^{(5)}}{a + x_2^{(5)}} - a_1,$$

$$D_{12} = X_1^{(5)} \left(1 - \frac{r}{k} - \frac{\beta}{a}\right),$$

$$D_{21} = \frac{\beta x_1^{(5)}}{a + x_2^{(5)}},$$

$$D_{22} = X_1^{(5)} \left(1 - \frac{r}{k} - \frac{\beta}{a}\right),$$

$$D_{33} = a_3 X_1^{(5)} + a_4 X_2^{(5)} - \gamma Y_2^{(5)} - d_3,$$

$$D_{44} = a_3 X_2^{(5)} - a_6.$$

We obtain an eigen value $\lambda_1 = D_{33}$. Other corresponding eigen values are roots of cubic equation

$$\lambda^3 + \varphi_1 \lambda^2 + \varphi_2 \lambda + \varphi_3 = 0,$$

where

$$\varphi_1 = D_{12} D_{21} X_1^{(5)} - D_{22} - D_{11} - D_{44},$$

$$\varphi_2 = D_{44} (D_{11} + D_{22} - D_{12} D_{21} + X_1^{(5)}) + p_4 a_5 X_2^{(5)} Y_2^{(5)} + D_{11} (D_{22} - D_{12} D_{21} X_1^{(5)}),$$

$$\varphi_3 = -D_{44} (D_{11} D_{12} D_{21} X_1^{(5)} - D_{11} D_{22}) + a_5 p_4 D_{11} X_2^{(5)} Y_2^{(5)}.$$

$E_5$ is locally asymptotically stable if $a_3 X_1^{(5)} + a_4 X_2^{(5)} < \gamma Y_2^{(5)} + d_3$, $\varphi_1 > 0, \varphi_2 > 0, \text{ and } \varphi_3 > 0$.

Characteristic equation (3.2) at $E_6$ is

$$(G_{44} - \lambda) \left( (G_{11} - \lambda)(G_{22} - \lambda)(G_{33} - \lambda) + a_4 p_2 X_2^{(6)} Y_2^{(6)} \right) - G_{21} \left( X_1^{(6)} G_{12} (G_{33} - \lambda) + a_4 p_1 X_1^{(6)} Y_1^{(6)} \right) + a_3 Y_1^{(6)} \left( p_2 Y_1^{(6)} G_{12} + p_1 X_1^{(6)} (G_{22} - \lambda) \right) = 0,$$

where

$$G_{11} = r \left(1 - \frac{2x_1^{(6)} + x_2^{(6)}}{k} \right) - \frac{\beta x_1^{(6)}}{a + x_2^{(6)}} - p_1 Y_1^{(6)} - a_1,$$

$$G_{22} = X_1^{(6)} \left(1 - \frac{r}{k} - \frac{\beta}{a}\right),$$

$$G_{21} = \frac{\beta x_1^{(6)}}{a + x_2^{(6)}},$$

$$G_{33} = a_3 X_1^{(6)} + a_4 X_2^{(6)} - d_3.$$
\[ G_{12} = \left( -\frac{r}{\kappa} - \frac{\beta}{a + x_2^0} + \frac{\beta x_2^{(6)}}{(a + x_2^{(6)})^2} \right), \quad G_{44} = a_5 x_2^{(6)} + \gamma y_1^{(6)} - a_6. \]

One of eigen values is obtained \( \lambda_1 = G_{44} \). Other corresponding eigen values are roots of cubic
\[ \lambda^3 + \theta_1 \lambda^2 + \theta_2 \lambda + \theta_3 = 0 \]
where
\[ \theta_1 = -G_{11} + G_{22} - G_{33}, \]
\[ \theta_2 = G_{11}(G_{22} + G_{33}) + G_{22} G_{33} + a_4 p_2 x_2^{(6)} y_1^{(6)} - G_{12} G_{21} x_1^{(6)} + a_3 p_1 x_1^{(6)} y_1^{(6)}, \]
\[ \theta_3 = -G_{11} G_{22} G_{33} - G_{11} a_4 p_2 x_2^{(6)} y_1^{(6)} + G_{21} (G_{11} G_{33} x_1^{(6)} + a_4 p_1 x_1^{(6)} y_1^{(6)}) - a_3 x_1^{(6)} (G_{22} p_1 x_1^{(6)} - p_2 x_1^{(6)} x_2^{(6)} G_{12}). \]

\( E_0 \) is locally asymptotically stable when \( a_5 x_2^{(6)} + \gamma y_1^{(6)} < a_6, \theta_1 > 0, \theta_3 > 0, \) and \( \theta_1 \theta_2 - \theta_3 > 0. \)

Characteristic equation (3.2) at \( E_7 \) is
\[
\begin{pmatrix}
H_{11} - \lambda & X_1^{(7)} H_{12} & -p_3 X_1^{(7)} & 0 \\
H_{21} & H_{22} - \lambda & -p_2 X_2^{(7)} & -p_4 Y_2^{(7)} \\
a_3 Y_1^{(7)} & a_4 Y_1^{(7)} & H_{33} - \lambda & -Y_1^{(7)} \\
0 & a_5 Y_2^{(7)} & Y_2^{(7)} & H_{44} - \lambda
\end{pmatrix} = 0,
\]
where
\[ H_{11} = r \left( 1 - \frac{2 X_1^{(7)} x_1^{(7)}}{k} \right) - \frac{\beta x_2^{(7)}}{a + x_2^{(7)}} - p_1 y_1^{(7)} - a_1, \quad H_{21} = \frac{\beta x_2^{(7)}}{a + x_2^{(7)}}, \]
\[ H_{22} = X_1^{(7)} \left( \frac{\beta}{a + x_2^{(7)}} - \frac{\beta x_2^{(7)}}{(a + x_2^{(7)})^2} \right) - p_2 y_1^{(7)} - p_4 y_2^{(7)} - a_2, \quad H_{33} = a_3 X_1^{(7)} + a_4 X_2^{(7)} - d_3, \]
\[ H_{12} = \left( -\frac{r}{\kappa} - \frac{\beta}{a + x_2^{(7)}} + \frac{\beta x_2^{(7)}}{(a + x_2^{(7)})^2} \right), \quad H_{44} = a_5 X_2^{(7)} + \gamma Y_1^{(7)} - a_6. \]

Corresponding eigen values \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are roots of fourth order equation
\[ \lambda^4 + \eta_1 \lambda^3 + \eta_2 \lambda^2 + \eta_3 \lambda + \eta_4 = 0, \tag{3.3} \]
where
\[ \eta_1 = -H_{11} - H_{22} - H_{33} - H_{44}, \]
\[ \eta_2 = H_{11} H_{22} + H_{11} H_{33} + H_{11} H_{44} + H_{22} H_{33} + H_{22} H_{44} + H_{33} H_{44} + \gamma^2 Y_1^{(7)} Y_2^{(7)} + a_4 p_2 X_2^{(7)} Y_1^{(7)} - a_5 p_2 X_2^{(7)} Y_2^{(7)} - H_{12} H_{21} x_1^{(7)} - \lambda H_{12} H_{21} X_1^{(7)} - a_3 p_1 X_1^{(7)} Y_1^{(7)}, \]
\[ \eta_3 = -H_{11} H_{22} H_{33} - H_{11} H_{22} H_{44} - H_{11} H_{33} H_{44} - H_{11} a_4 p_2 X_2^{(7)} Y_1^{(7)} + H_{11} a_5 p_2 x_2^{(7)} Y_2^{(7)} - a_5 p_2 x_2^{(7)} X_2^{(7)} - H_{22} H_{33} X_4^{(7)} - H_{22} H_{33} Y_2^{(7)} Y_1^{(7)} - H_{22} H_{33} Y_2^{(7)} Y_1^{(7)} - H_{22} H_{33} X_2^{(7)} + H_{11} p_2 x_2^{(7)} Y_2^{(7)} + H_{11} p_2 x_2^{(7)} Y_1^{(7)} + a_4 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{11} p_2 x_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{11} p_2 x_2^{(7)} X_2^{(7)} + a_5 p_2 x_2^{(7)} X_2^{(7)} - a_5 p_2 x_2^{(7)} Y_2^{(7)} + H_{12} H_{21} X_1^{(7)} + a_3 p_1 Y_1^{(7)} Y_2^{(7)} Y_1^{(7)} + H_{12} H_{21} X_2^{(7)} - a_3 p_1 Y_1^{(7)} Y_2^{(7)} H_{22} H_{44}, \]
\[ \eta_4 = -H_{11} H_{22} H_{33} H_4^{(7)} + H_{11} H_{22} Y_1^{(7)} Y_2^{(7)} + H_{11} H_{44} a_4 p_2 X_2^{(7)} Y_1^{(7)} + H_{11} a_5 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{11} a_5 p_2 x_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{11} a_5 p_2 x_2^{(7)} X_2^{(7)} + a_5 p_2 x_2^{(7)} X_2^{(7)} + a_3 p_4 H_{12} X_1^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{12} H_{21} H_{33} H_4^{(7)} + a_3 p_4 H_{12} H_4^{(7)} Y_1^{(7)} Y_2^{(7)} + a_3 p_4 H_{12} H_4^{(7)} X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + a_3 p_4 H_{12} H_4^{(7)} X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{21} H_{44} a_4 p_1 X_1^{(7)} Y_1^{(7)} - a_5 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{12} H_{21} H_{33} H_4^{(7)} + a_3 p_4 H_{12} H_4^{(7)} Y_1^{(7)} Y_2^{(7)} + a_3 p_4 H_{12} H_4^{(7)} X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{21} H_{44} a_4 p_1 X_1^{(7)} Y_1^{(7)} - a_5 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{12} H_{21} H_{33} H_4^{(7)} + a_3 p_4 H_{12} H_4^{(7)} Y_1^{(7)} Y_2^{(7)} + a_3 p_4 H_{12} H_4^{(7)} X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{21} H_{44} a_4 p_1 X_1^{(7)} Y_1^{(7)} - a_5 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{12} H_{21} H_{33} H_4^{(7)} + a_3 p_4 H_{12} H_4^{(7)} Y_1^{(7)} Y_2^{(7)} + a_3 p_4 H_{12} H_4^{(7)} X_2^{(7)} Y_1^{(7)} Y_2^{(7)} + H_{21} H_{44} a_4 p_1 X_1^{(7)} Y_1^{(7)} - a_5 p_2 X_2^{(7)} Y_1^{(7)} Y_2^{(7)} \]
Using Routh-Hurwitz criterion, roots of equation (3.3) have negative real part if only if \( \eta_1 > 0, \eta_4 > 0, \eta_1 \eta_2 - \eta_3 > 0, \) and \( \eta_3(\eta_1 \eta_2 - \eta_3) - \eta_1^2 \eta_4 > 0. \)
4. Numerical Simulation

We present numerical simulation of system (2.1) in order to visualize analytic results from previous section. Firstly, we set value parameters as follow:

\[ r = 0.26, k = 1000, \alpha = \frac{1}{3}, \beta = \frac{1}{2}, \gamma = \frac{1}{3}, p_1 = \frac{1}{2}, p_2 = \frac{1}{2}, p_4 = \frac{1}{16}, d_3 = \frac{1}{8}, a_1 = \frac{7}{24}, a_2 = \frac{19}{24}, a_3 = \frac{1}{14}, a_4 = \frac{1}{12}, a_5 = \frac{1}{256}, a_6 = \frac{2}{3} \]

\( E_0(0,0,0,0) \) is always exist and asymptotically stable because \( r < a_1 \) while other equilibrium points do not exist at this condition. Using various initial values, we can notice that all population are coming into their extinction (Fig. 1).

![Figure 1. Numerical simulation of system with \( r = .26 \).](image1)

The dynamics of system is investigated further by changing the parameter value of \( r \). Take \( r = 0.2917 > a_1 \), It satisfies existence criteria for \( E_0 \) and \( E_1 = (0.114,0,0,0) \). Result shows system is asymptotically stable to \( E_1 \) because \( a_1 = .2916 < r < \min(0.2918, 0.2921) \) is satisfied (Fig. 2).

![Figure 2. Numerical simulation of system with \( r = .2917 \).](image2)

Now we take \( r = 0.4583 \). Because \( r > a_1 \left( \frac{ka_3}{k_0-\gamma} \right) = 0.2921 \) and \( r > a_1 \left( \frac{\beta k}{\beta k-a_2 a_5} \right) = 0.2918 \), hence it satisfies existence condition of \( E_0, E_2 = (1.75, 0, 0.331, 0) \), and \( E_3 = (0.86, 0.21, 0, 0) \). For various initial values, it shows that system is stable to \( E_3 \) where only prey population exists (Figure 3).
5. Conclusion

In this paper, we studied about eco-epidemiology model where infectious disease is able to spread in predator population and prey population using saturated incidence rate for disease transmission in prey population. We observed eight non-negative equilibrium points which exist under certain conditions and local stability for those points had been pointed out. In numerical simulation section, we changed the value of certain parameter (in this case the value of $r$) which satisfies existence and local stability conditions of equilibrium points to illustrate analytic results discussed in dynamical analysis section.

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