The concept of quantization consists in replacing commutative quantities by noncommutative ones. In mathematical language an algebra of continuous functions on a locally compact topological space is replaced with a noncommutative $C^*$-algebra. Some classical topological notions have noncommutative generalizations. This article is concerned with a generalization of coverings.

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3 Noncommutative infinite coverings
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1 Motivation. Preliminaries

Gelfand-Naïmark theorem \cite{2} states the correspondence between locally compact Hausdorff topological spaces and commutative $C^*$-algebras.

**Theorem 1.1.** \cite{2} (Gelfand-Naïmark). Let $A$ be a commutative $C^*$-algebra and let $\mathcal{X}$ be the spectrum of $A$. There is the natural $\ast$-isomorphism $\gamma : A \rightarrow C_0(\mathcal{X})$.

So any (noncommutative) $C^*$-algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space. Following theorem yields a pure algebraic description of finite-fold coverings of compact spaces.

**Theorem 1.2.** \cite{12} Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \rightarrow \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

$$(f \xi)(y) = f(y)\xi(p(y)), \quad f \in C(\mathcal{Y}), \quad \xi \in C(\mathcal{X}),$$

then $p$ is a finite-fold covering.

This article contains pure algebraic generalizations of following topological objects:

- Coverings of noncompact spaces,
- Infinite coverings.
This article assumes elementary knowledge of following subjects:

1. Set theory [9],
2. Category theory [15],
3. Algebraic topology [15],
4. C*-algebras, C*-Hilbert modules [3][13].

The words "set", "family" and "collection" are synonyms.
Following table contains special symbols.

| Symbol | Meaning |
|--------|---------|
| $\hat{A}$ | Spectrum of a $C^*$- algebra $A$ with the hull-kernel topology (or Jacobson topology) |
| $A_+$ | Cone of positive elements of $C^*$- algebra, i.e. $A_+ = \{ a \in A \mid a \geq 0 \}$ |
| $A^G$ | Algebra of $G$ - invariants, i.e. $A^G = \{ a \in A \mid ga = a, \forall g \in G \}$ |
| $\text{Aut}(A)$ | Group of * - automorphisms of $C^*$- algebra $A$ |
| $A''$ | Enveloping von Neumann algebra of $A$ |
| $B(\mathcal{H})$ | Algebra of bounded operators on a Hilbert space $\mathcal{H}$ |
| $C$ (resp. $\mathbb{R}$) | Field of complex (resp. real) numbers |
| $C(\mathcal{X})$ | $C^*$- algebra of continuous complex valued functions on a compact space $\mathcal{X}$ |
| $C_0(\mathcal{X})$ | $C^*$- algebra of continuous complex valued functions on a locally compact topological space $\mathcal{X}$ equal to 0 at infinity |
| $C_c(\mathcal{X})$ | Algebra of continuous complex valued functions on a topological space $\mathcal{X}$ with compact support |
| $C_b(\mathcal{X})$ | $C^*$- algebra of bounded continuous complex valued functions on a locally compact topological space $\mathcal{X}$ |
| $G \left( \tilde{\mathcal{X}} \mid \mathcal{X} \right)$ | Group of covering transformations of covering $\tilde{\mathcal{X}} \to \mathcal{X}$ [15] |
| $\mathcal{H}$ | Hilbert space |
| $\mathcal{K} = \mathcal{K}(\mathcal{H})$ | $C^*$- algebra of compact operators on the separable Hilbert space $\mathcal{H}$ |
| $\mathcal{K}(A)$ | Pedersen ideal of $C^*$-algebra $A$ |
| $\lim \sup\limits_{\rightarrow}$ | Direct limit |
| $\lim \inf\limits_{\leftarrow}$ | Inverse limit |
| $M(A)$ | A multiplier algebra of $C^*$-algebra $A$ |
| $M_n(A)$ | The $n \times n$ matrix algebra over $C^*$-algebra $A$ |
| $\mathbb{N}$ | A set of positive integer numbers |
| $\mathbb{N}^0$ | A set of nonnegative integer numbers |
| $\mathcal{U}(A) \subset A$ | Group of unitary operators of algebra $A$ |
| $\mathbb{Z}$ | Ring of integers |
| $\mathbb{Z}_n$ | Ring of integers modulo $n$ |
| $k \in \mathbb{Z}_n$ | An element in $\mathbb{Z}_n$ represented by $k \in \mathbb{Z}$ |
| $X \setminus A$ | Difference of sets $X \setminus A = \{ x \in X \mid x \notin A \}$ |
| $|X|$ | Cardinal number of a finite set $X$ |
| $[x]$ | The range projection of element $x$ of a von Neumann algebra. |
| $f|_{A'}$ | Restriction of a map $f : A \to B$ to $A' \subset A$, i.e. $f|_{A'} : A' \to B$ |
1.1 Prototype. Inverse limits of coverings in topology

1.1.1 Topological construction

This subsection is concerned with a topological construction of the inverse limit in the category of coverings.

Definition 1.3. [15] Let \( \tilde{\pi} : \tilde{X} \to X \) be a continuous map. An open subset \( U \subset X \) is said to be evenly covered by \( \tilde{\pi} \) if \( \tilde{\pi}^{-1}(U) \) is the disconnected union of open subsets of \( \tilde{X} \) each of which is mapped homeomorphically onto \( U \) by \( \tilde{\pi} \). A continuous map \( \tilde{\pi} : \tilde{X} \to X \) is called a covering projection if each point \( x \in X \) has an open neighborhood evenly covered by \( \tilde{\pi} \). \( \tilde{X} \) is called the covering space and \( X \) the base space of the covering.

Definition 1.4. [15] A fibration \( p : \tilde{X} \to X \) with unique path lifting is said to be regular if, given any closed path \( \omega \) in \( X \), either every lifting of \( \omega \) is closed or none is closed.

Definition 1.5. [15] A topological space \( X \) is said to be locally path-connected if the path components of open sets are open.

Denote by \( \pi_1 \) the functor of fundamental group [15].

Theorem 1.6. [15] Let \( p : \tilde{X} \to X \) be a fibration with unique path lifting and assume that a nonempty \( \tilde{X} \) is a locally path-connected space. Then \( p \) is regular if and only if for some \( \tilde{x}_0 \in \tilde{X} \), \( \pi_1 (p) \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \) is a normal subgroup of \( \pi_1 (X, p(\tilde{x}_0)) \).

Definition 1.7. [15] Let \( p : \tilde{X} \to X \) be a covering projection. A self-equivalence is a homeomorphism \( f : \tilde{X} \to \tilde{X} \) such that \( p \circ f = p \). This group of such homeomorphisms is said to be the group of covering transformations of \( p \) or the covering group. Denote by \( G \left( \tilde{X} \mid X \right) \) this group.

Proposition 1.8. [15] If \( p : \tilde{X} \to X \) is a regular covering projection and \( \tilde{X} \) is connected and locally path connected, then \( X \) is homeomorphic to space of orbits of \( G \left( \tilde{X} \mid X \right) \), i.e. \( X \approx \tilde{X} / G \left( \tilde{X} \mid X \right) \). So \( p \) is a principal bundle.

Corollary 1.9. [15] Let \( p : \tilde{X} \to X \) be a fibration with a unique path lifting. If \( \tilde{X} \) is connected and locally path-connected and \( \tilde{x}_0 \in \tilde{X} \) then \( p \) is regular if and only if \( G \left( \tilde{X} \mid X \right) \) transitively acts on each fiber of \( p \), in which case

\[
\psi : G \left( \tilde{X} \mid X \right) \approx \pi_1 (X, p(\tilde{x}_0)) / \pi_1 (p) \pi_1 \left( \tilde{X}, \tilde{x}_0 \right).
\]

Remark 1.10. Above results are copied from [15]. Below the covering projection word is replaced with covering.

Definition 1.11. [10] A compactification of a space \( X \) is a compact Hausdorff space \( Y \) containing \( X \) as a subspace and the closure \( \overline{X} \) of \( X \) is \( Y \), i.e \( \overline{X} = Y \).
The algebraic construction requires following definition

**Definition 1.12.** A covering \( \pi : \tilde{X} \to X \) is said to be a **covering with compactification** if there are compactifications \( X \hookrightarrow Y \) and \( \tilde{X} \hookrightarrow \tilde{Y} \) such that:

- There is a covering \( \tilde{\pi} : \tilde{Y} \to Y \),
- The covering \( \pi \) is the restriction of \( \tilde{\pi} \), i.e. \( \pi = \tilde{\pi}|_\tilde{X} \).

**Example 1.13.** Let \( g : S^1 \to S^1 \) be an \( n \)-fold covering of a circle. Let \( X = \tilde{X} = S^1 \times [0,1) \). The map

\[
\pi : \tilde{X} \to X, \\
\pi = g \times \text{Id}_{[0,1)}
\]

is an \( n \)-fold covering. If \( Y = \tilde{Y} = S^1 \times [0,1] \) then a compactification \( [0,1) \hookrightarrow [0,1] \) induces compactifications \( X \hookrightarrow Y, \tilde{X} \hookrightarrow \tilde{Y} \). The map

\[
\pi : \tilde{Y} \to Y, \\
\pi = g \times \text{Id}_{[0,1]}
\]

is a covering such that \( \tilde{\pi}|_\tilde{X} = \pi \). So if \( n > 1 \) then \( \pi \) is a nontrivial covering with compactification.

**Example 1.14.** Let \( \mathcal{X} = \mathbb{C} \setminus \{0\} \) be a complex plane with punctured 0, which is parametrized by the complex variable \( z \). Let \( X \hookrightarrow Y \) be any compactification. If both \( \{z'_n \in \mathcal{X} \}_{n \in \mathbb{N}}, \{z''_n \in \mathcal{X} \}_{n \in \mathbb{N}} \) are Cauchy sequences such that \( \lim_{n \to \infty} |z'_n| = \lim_{n \to \infty} |z''_n| = 0 \) then form

\[
x_0 = \lim_{n \to \infty} z'_n = \lim_{n \to \infty} z''_n \in Y.
\]

If \( \tilde{X} = \mathcal{X} \) then for any \( n \in \mathbb{N} \) there is a finite-fold covering

\[
\pi : \tilde{X} \to \mathcal{X}, \\
z \mapsto z^n.
\]

If both \( \mathcal{X} \hookrightarrow Y, \tilde{X} \hookrightarrow \tilde{Y} \) are compactifications, and \( \tilde{\pi} : \tilde{Y} \to Y \) is a covering such that \( \tilde{\pi}|_\tilde{X} = \pi \) then from (1.1) it turns out \( \tilde{\pi}^{-1}(x_0) = \{\tilde{x}_0\} \) where \( \tilde{x}_0 \) is the unique point such that following conditions hold:

\[
\tilde{x}_0 = \lim_{n \to \infty} \tilde{z}_n \in \tilde{Y}, \\
\lim_{n \to \infty} |\tilde{z}_n| = 0.
\]

It turns out \( |\tilde{\pi}^{-1}(x_0)| = 1 \). However \( \tilde{\pi} \) is an \( n \)-fold covering and if \( n > 1 \) then \( |\tilde{\pi}^{-1}(x_0)| = n > 1 \). It contradicts with \( |\tilde{\pi}^{-1}(x_0)| = 1 \), and from the contradiction it turns out that for any \( n > 1 \) the map \( \pi \) is not a covering with compactification.
Definition 1.15. The sequence of regular finite-fold coverings

\[ \mathcal{X} = \mathcal{X}_0 \leftarrow \cdots \leftarrow \mathcal{X}_n \leftarrow \cdots \]

is said to be a (topological) finite covering sequence if following conditions hold:

- The space \( \mathcal{X}_n \) is a second-countable \([10]\) locally compact connected Hausdorff space for any \( n \in \mathbb{N}_0 \),
- If \( k < l < m \) are any nonnegative integer numbers then there is the natural exact sequence
  \[ \{e\} \to G(\mathcal{X}_m | \mathcal{X}_l) \to G(\mathcal{X}_m | \mathcal{X}_k) \to G(\mathcal{X}_l | \mathcal{X}_k) \to \{e\}. \]

For any finite covering sequence we will use a following notation

\[ \mathcal{S} = \{\mathcal{X} = \mathcal{X}_0 \leftarrow \cdots \leftarrow \mathcal{X}_n \leftarrow \cdots\} = \{\mathcal{X}_0 \leftarrow \cdots \leftarrow \mathcal{X}_n \leftarrow \cdots\}, \quad \mathcal{S} \in \mathcal{F}_{\text{inTop}}. \]

Example 1.16. Let \( \mathcal{S} = \{\mathcal{X} = \mathcal{X}_0 \leftarrow \cdots \leftarrow \mathcal{X}_n \leftarrow \cdots\} \) be a sequence of locally compact connected Hausdorff spaces and finite-fold regular coverings such that \( \mathcal{X}_n \) is locally path-connected for any \( n \in \mathbb{N} \). It follows from Lemma \([16]\) that if \( p > q \) and \( f_{pq} : \mathcal{X}_p \to \mathcal{X}_q \) then \( \pi_1(f_{pq}) \pi_1(\mathcal{X}_p, x_0) \) is a normal subgroup of \( \pi_1(\mathcal{X}_q, f_{pq}(x_0)) \). Otherwise from the Corollary \([15]\) it turns out

\[ G(\mathcal{X}_p | \mathcal{X}_q) \cong \pi_1(\mathcal{X}_q, f_{pq}(x_0)) / \pi_1(f_{pq}) \pi_1(\mathcal{X}_p, x_0). \]

If \( k < l < m \) then a following sequence

\[ \{e\} \to \pi_1(\mathcal{X}_l, f_{ml}(x_0)) / \pi_1(f_{ml}) \pi_1(\mathcal{X}_m, x_0) \to \pi_1(\mathcal{X}_k, f_{mk}(x_0)) / \pi_1(f_{mk}) \pi_1(\mathcal{X}_m, x_0) \to \pi_1(\mathcal{X}_k, f_{mk}(x_0)) / \pi_1(f_{lk}) \pi_1(\mathcal{X}_l, f_{ml}(x_0)) \to \{e\} \]

is exact. Above sequence is equivalent to the following sequence

\[ \{e\} \to G(\mathcal{X}_m | \mathcal{X}_l) \to G(\mathcal{X}_m | \mathcal{X}_k) \to G(\mathcal{X}_l | \mathcal{X}_k) \to \{e\} \]

which is also exact. Thus \( \mathcal{S} \in \mathcal{F}_{\text{inTop}}. \)

Definition 1.17. Let \( \{\mathcal{X} = \mathcal{X}_0 \leftarrow \cdots \leftarrow \mathcal{X}_n \leftarrow \cdots\} \in \mathcal{F}_{\text{inTop}} \), and let \( \hat{\mathcal{X}} = \varprojlim \mathcal{X}_n \) be the inverse limit in the category of topological spaces and continuous maps (cf. \([13]\)). If \( \hat{\pi}_0 : \hat{\mathcal{X}} \to \mathcal{X}_0 \) is the natural continuous map then a homeomorphism \( g \) of the space \( \hat{\mathcal{X}} \) is said to be a covering transformation if a following condition holds

\[ \hat{\pi}_0 = \hat{\pi}_0 \circ g. \]

The group \( \hat{\mathcal{G}} \) of such homeomorphisms is said to be the group of covering transformations of \( \mathcal{S} \). Denote by \( G(\hat{\mathcal{X}} | \mathcal{X}) \quad \text{def} \quad \hat{\mathcal{G}}. \)
Lemma 1.18. Let \( \{ \mathcal{X} = X_0 \leftarrow ... \leftarrow X_n \leftarrow ... \} \in \mathfrak{InTop} \), and let \( \hat{\mathcal{X}} = \lim^{-} \mathcal{X}_n \) be the inverse limit in the category of topological spaces and continuous maps. There is the natural group isomorphism \( G(\hat{\mathcal{X}} \mid \mathcal{X}) \cong \lim G(\mathcal{X}_n \mid \mathcal{X}) \). For any \( n \in \mathbb{N} \) there is the natural surjective homomorphism \( h_n : G(\hat{\mathcal{X}} \mid \mathcal{X}) \to G(\mathcal{X}_n \mid \mathcal{X}) \) and \( \bigcap_{n \in \mathbb{N}} \ker h_n \) is a trivial group.

Proof. For any \( n \in \mathbb{N} \) there is the natural continuous map \( \hat{\pi}_n : \hat{\mathcal{X}} \to X_n \). Let \( x_0 \in X_0 \) and \( \hat{x}_0 \in \hat{\mathcal{X}} \) be such that \( \hat{\pi}_0(\hat{x}_0) = x_0 \). Let \( \hat{x}' \in \hat{\mathcal{X}} \) be such that \( \hat{\pi}_0(\hat{x}') = x_0 \). If \( x'_n = \hat{\pi}_n(\hat{x}') \) and \( x_n = \hat{\pi}_n(\hat{x}_0) \) then \( \pi_n(x_n) = \pi_n(x'_n) \), where \( \pi_n : \mathcal{X}_n \to \mathcal{X} \) is the natural covering. Since \( \pi_n \) is regular for any \( n \in \mathbb{N} \) there is the unique \( g_n \in G(\mathcal{X}_n \mid \mathcal{X}) \) such that \( x'_n = g_n x_n \). In result there is a sequence \( \{ g_n \in G(\mathcal{X}_n \mid \mathcal{X}) \}_{n \in \mathbb{N}} \) which satisfies to the following condition

\[
g_m \circ \pi^n_m = \pi^n_m \circ g_n
\]

where \( n > m \) and \( \pi^n_m : \mathcal{X}_n \to \mathcal{X}_m \) is the natural covering. The sequence \( \{ g_n \} \) naturally defines an element \( \hat{g} \in \lim G(\mathcal{X}_n \mid \mathcal{X}) \). Let us define an homeomorphism \( \varphi_{\hat{g}} : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \) by a following construction. If \( \hat{x}'' \in \hat{\mathcal{X}} \) is any point then there is a sequence \( \{ x''_n \in \mathcal{X}_n \}_{n \in \mathbb{N}} \) such that

\[
x''_n = \hat{\pi}_n(\hat{x}'').
\]

On the other hand there is the sequence \( \{ x''_n \in \mathcal{X}_n \}_{n \in \mathbb{N}} \)

\[
x''_n = g_n x''
\]

which for any \( n > m \) satisfies to the following condition

\[
\pi^n_m(x''_n) = x''_m.
\]

From the above equation and properties of inverse limits it follows that there is the unique \( \hat{x}''_{\hat{g}} \in \hat{\mathcal{X}} \) such that

\[
\hat{\pi}_n(\hat{x}''_{\hat{g}}) = x''_n, \quad \forall n \in \mathbb{N}.
\]

The required homeomorphism \( \varphi_{\hat{g}} \) is given by

\[
\varphi_{\hat{g}}(\hat{x}') = \hat{x}''_{\hat{g}}.
\]

From \( \hat{\pi} \circ \varphi_{\hat{g}} = \hat{\pi} \) it follows that \( \varphi_{\hat{g}} \) corresponds to an element in \( G(\hat{\mathcal{X}} \mid \mathcal{X}) \) which mapped onto \( g_n \) for any \( n \in \mathbb{N} \). Otherwise \( \varphi_{\hat{g}} \) naturally corresponds to the element \( \hat{g} \in \lim G(\mathcal{X}_n \mid \mathcal{X}) \), so one has the natural group isomorphism \( G(\hat{\mathcal{X}} \mid \mathcal{X}) \cong \lim G(\mathcal{X}_n \mid \mathcal{X}) \).

From the above construction it turns out that any homeomorphism \( \hat{g} \in G(\hat{\mathcal{X}} \mid \mathcal{X}) \) uniquely depends on \( \hat{x}' = \hat{g}\hat{x}_0 \in \hat{\pi}_0^{-1}(x_0) \). It follows that there is the 1-1 map \( \varphi : \hat{\pi}_0^{-1}(x_0) \overset{\approx}{\to} G(\hat{\mathcal{X}} \mid \mathcal{X}) \). Since the covering \( \pi_n : \mathcal{X}_n \to \mathcal{X} \) is regular there is the 1-1 map \( \varphi_n : \pi_n^{-1}(x_0) \overset{\approx}{\to} G(\mathcal{X}_n \mid \mathcal{X}) \). The natural surjective map

\[
\pi_0^{-1}(x_0) \to \pi_n^{-1}(x_0)
\]
induces the surjective homomorphism \( G \left( \hat{X} \mid \mathcal{X} \right) \to G \left( \mathcal{X}_n \mid \mathcal{X} \right) \). If \( \hat{g} \in \bigcap_{n \in \mathbb{N}} \ker h_n \) is not trivial then \( \hat{g} \hat{x_0} \neq \hat{x_0} \) and there is \( n \in \mathbb{N} \) such that \( \hat{\pi}_n (\hat{x}_0) \neq \hat{\pi}_n (\hat{g} \hat{x}_0) = h_n (\hat{g}) \hat{\pi}_n (\hat{x}_0) \), so \( h_n (\hat{g}) \in G \left( \mathcal{X}_n \mid \mathcal{X} \right) \) is not trivial and \( \hat{g} \notin \ker h_n \). From this contradiction it follows that \( \bigcap_{n \in \mathbb{N}} \ker h_n \) is a trivial group.

**Definition 1.19.** Let \( \mathcal{G} = \{ \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \} \) be a finite covering sequence. The pair \( (\mathcal{Y}, \{ \pi_n^\mathcal{Y} \}_{n \in \mathbb{N}}) \) of a (discrete) set \( \mathcal{Y} \) with and surjective maps \( \pi_n^\mathcal{Y} : \mathcal{Y} \to \mathcal{X}_n \) is said to be a coherent system if for any \( n \in \mathbb{N}^0 \) a following diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\pi_n^\mathcal{Y}} & \mathcal{Y}_{n-1} \\
\downarrow & & \downarrow \\
\mathcal{X}_n & \xrightarrow{\pi_n} & \mathcal{X}_{n-1}
\end{array}
\]

is commutative.

**Definition 1.20.** Let \( \mathcal{G} = \{ \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \} \) be a topological finite covering sequence. A coherent system \( (\mathcal{Y}, \{ \pi_n^\mathcal{Y} \}_{n \in \mathbb{N}}) \) is said to be a connected covering of \( \mathcal{G} \) if \( \mathcal{Y} \) is a connected topological space and \( \pi_n^\mathcal{Y} \) is a regular covering for any \( n \in \mathbb{N} \). We will use following notation \( (\mathcal{Y}, \{ \pi_n^\mathcal{Y} \}_{n \in \mathbb{N}}) \downarrow \mathcal{G} \) or simply \( \mathcal{Y} \downarrow \mathcal{G} \).

**Definition 1.21.** Let \( (\mathcal{Y}, \{ \pi_n^\mathcal{Y} \}_{n \in \mathbb{N}}) \) be a coherent system of \( \mathcal{G} \) and \( y \in \mathcal{Y} \). A subset \( \mathcal{V} \subset \mathcal{Y} \) is said to be special if \( \pi_n^\mathcal{Y} (\mathcal{V}) \) is evenly covered by \( \mathcal{X}_1 \to \mathcal{X}_0 \) and for any \( n \in \mathbb{N}^0 \) following conditions hold:

- \( \pi_n^\mathcal{Y} (\mathcal{V}) \subset \mathcal{X}_n \) is an open connected set,

- The restriction \( \pi_n^\mathcal{Y} |_\mathcal{V} : \mathcal{V} \to \pi_n^\mathcal{Y} (\mathcal{V}) \) is a bijective map.

**Remark 1.22.** For any \( n \in \mathbb{N}^0 \) the space \( \mathcal{X}_n \) is second-countable, so from the Theorem 1.32 for any point \( x \in \mathcal{X}_n \) there is an open connected neighborhood \( \mathcal{U} \subset \mathcal{X}_n \).

**Remark 1.23.** If \( (\mathcal{Y}, \{ \pi_n^\mathcal{Y} \}_{n \in \mathbb{N}}) \) is a covering of \( \mathcal{G} \) then the set of special sets is a base of the topology of \( \mathcal{Y} \).

**Lemma 1.24.** Let \( \hat{X} = \varprojlim \mathcal{X}_n \) be the inverse limit of the sequence \( \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \) in the category of topological spaces and continuous maps. Any special set of \( \hat{X} \) is a Borel subset of \( \hat{X} \).

**Proof.** If \( \mathcal{U}_n \subset \mathcal{X}_n \) is an open set then \( \hat{\pi}_n^{-1} (\mathcal{U}_n) \subset \hat{X} \) is open. If \( \hat{\mathcal{U}} \) is a special set then \( \hat{\mathcal{U}} = \bigcap_{n \in \mathbb{N}} \hat{\pi}_n^{-1} \circ \hat{\pi}_n (\hat{\mathcal{U}}) \), i.e. \( \hat{\mathcal{U}} \) is a countable intersection of open sets. So \( \hat{\mathcal{U}} \) is a Borel subset.

**Definition 1.25.** Let us consider the situation of the Definition 1.20. A morphism from \( (\mathcal{Y}', \{ \pi_n^{\mathcal{Y}'} \}_{n \in \mathbb{N}}) \downarrow \mathcal{G} \) to \( (\mathcal{Y}'', \{ \pi_n^{\mathcal{Y}''} \}_{n \in \mathbb{N}}) \downarrow \mathcal{G} \) is a covering \( f : \mathcal{Y}' \to \mathcal{Y}'' \) such that

\[
\pi_n^{\mathcal{Y}''} \circ f = \pi_n^{\mathcal{Y}'}
\]

for any \( n \in \mathbb{N} \).
1.26. There is a category with objects and morphisms described by Definitions 1.20, 1.25. Denote by $\downarrow S$ this category.

**Lemma 1.27.** There is the final object of the category $\downarrow S$ described in 1.26.

**Proof.** Let $\hat{X} = \lim X_n$ be the inverse limit of the sequence $X_0 \leftarrow ... \leftarrow X_n \leftarrow ...$ in the category of topological spaces and continuous maps. Denote by $\hat{X}$ a topological space such that

- $\hat{X}$ coincides with $\hat{X}$ as a set,
- A set of special sets of $\hat{X}$ is a base of the topology of $\hat{X}$.

If $x_n \in X_n$ is a point then there is $\bar{x} \in \hat{X}$ such that $x_n = \hat{\pi}(\bar{x})$ and there is a special subset $\mathcal{U}$ such that $\bar{x} \in \mathcal{U}$. From the construction of special subsets it follows that:

- $U_n = \hat{\pi}(\mathcal{U})$ is an open neighborhood of $x_n$;
- $\hat{\pi}^{-1}(U_n) = \bigsqcup_{g \in \ker(G(X_n | X) \to G(X_n | X))} g\mathcal{U}$;

- For any $g \in \ker(G(X_n | X) \to G(X_n | X))$ the set $g\mathcal{U}$ mapped homeomorphically onto $U_n$.

So the natural map $\hat{\pi}_n : \hat{X} \to X_n$ is a covering. If $\tilde{X} \subset \hat{X}$ is a nontrivial connected component then the map $\tilde{X} \to X_n$ is a covering, hence $\tilde{X}$ is an object of $\downarrow \mathcal{S}$. Let $G \subset \hat{G}$ be a maximal subgroup such that $G\tilde{X} = \tilde{X}$. The subgroup $G \subset \hat{G}$ is normal. If $g \in \hat{G}\setminus G$ then $g\tilde{X} \cap \tilde{X} = \emptyset$, however $g$ is a homeomorphism, i.e. $g : \hat{X} \xrightarrow{\sim} g\tilde{X}$. If $\bar{x} \in \tilde{X}$ then there is $\bar{x} \in \hat{X}$ such that $\hat{\pi}_0(\bar{x}) = \hat{\pi}_0(\bar{x})$, hence there is $g \in \hat{G}$ such that $\bar{x} = g\bar{x}$ and $\bar{x} \in g\tilde{X}$. It follows that

$$\hat{X} = \bigsqcup_{g \in J} g\tilde{X}$$

(1.2)

where $J \subset \hat{G}$ is a set of representatives of $\hat{G}/G$. If $(\mathcal{V}, \{\pi_0^V\})$ is a connected covering of $\mathcal{S}$ then there is the natural continuous map $\mathcal{V} \to \hat{X}$, because $\hat{X}$ is the inverse limit. Since the continuous map $\hat{X} \to \hat{X}$ is bijective there is the natural map $\pi : \mathcal{V} \to \hat{X}$. Let $\bar{x} \in \hat{X}$ be such that $\bar{x} \in \pi(\mathcal{V})$, i.e. $\exists y \in \mathcal{V}$ which satisfies to a condition $\bar{x} = \pi(y)$. Let $G^y \subset G(\mathcal{V} | X)$ be such that $\pi(G^y) = \{\bar{x}\}$. If $\mathcal{U}$ is a special neighborhood of $\bar{x}$ then there is a connected neighborhood $\mathcal{V}$ of $y$ which is mapped homeomorphically onto $\hat{\pi}_0(\mathcal{U}) \subset \mathcal{Y}_0$. It follows that

$$\pi^{-1}(\mathcal{U}) = \bigsqcup_{g \in G^y} g\mathcal{V}$$

(1.3)

i.e. $\mathcal{U}$ is evenly covered by $\pi$. It turns out the map $\pi : \mathcal{V} \to \hat{X}$ is continuous. From (1.2) it turns out that there is $g \in \hat{G}$, such that $\bar{x} \in g\tilde{X}$. The space $\mathcal{V}$ is connected so it is mapped
into $g\mathcal{X}$, hence there is a continuous map $\tilde{\pi} = g^{-1} \circ \mathcal{Y} : \mathcal{Y} \to \mathcal{X}$. The set $\tilde{\pi}(\mathcal{Y}) \subset \mathcal{X}$ contains a nontrivial open subset. Denote by $\mathcal{X}_\tilde{\pi} \subset \mathcal{X}$ (resp. $\mathcal{X}_\tilde{\pi} \subset \mathcal{X}$) maximal open subset of $\tilde{\pi}(\mathcal{Y})$ (resp. minimal closed superset of $\tilde{\pi}(\mathcal{Y})$). The space $\mathcal{X}$ is connected (i.e. open and closed), hence from $\tilde{\pi}(\mathcal{Y}) \neq \mathcal{X}$ it turns out $\mathcal{X}_\tilde{\pi} \mathcal{X}_\tilde{\pi} \neq \emptyset$. Let $\bar{x} \in \mathcal{X}_\tilde{\pi} \mathcal{X}_\tilde{\pi}$, and let $\mathcal{U}$ be a special neighborhood of $\bar{x}$. There is $y \in \mathcal{Y}$ such that $y \in \tilde{\pi}^{-1}(\mathcal{U})$ and there is a special connected neighborhood $\mathcal{V} \subset \mathcal{Y}$ of $y$ such that $\tilde{\pi}(\mathcal{V})$ is mapped homeomorphically onto $\pi_0^-\mathcal{V}(\mathcal{U}) \subset X_0$. Otherwise $\mathcal{U} \subset \mathcal{X}$ is mapped homeomorphically onto $\pi_0^-\mathcal{U}$, it follows that $\mathcal{V}$ is mapped onto $\mathcal{U}$. The open set $\mathcal{U}$ is such that:

- $\mathcal{U}$ is a neighborhood of $\bar{x}$,
- $\mathcal{U} \subset \tilde{\pi}(\mathcal{Y})$.

From the above conditions it follows that $\bar{x}$ lies in open subset $\mathcal{U} \subset \tilde{\pi}(\mathcal{Y})$, hence $\bar{x} \in \mathcal{X}_\tilde{\pi}$.

This fact contradicts with $\bar{x} \notin \mathcal{X}_\tilde{\pi} \mathcal{X}_\tilde{\pi}$ and from the contradiction it turns out $\tilde{\pi}(\mathcal{Y}) = \mathcal{X}$, i.e. $\tilde{\pi}$ is surjective. From \ref{1.3} it follows that $\tilde{\pi} : \mathcal{Y} \to \mathcal{X}$ is a covering. Thus $\mathcal{X}$ is the final object of the category $\downarrow \mathcal{S}$.

**Definition 1.28.** The final object $(\tilde{\mathcal{X}}, \{\pi_n\})$ of the category $\downarrow \mathcal{S}$ is said to be the (topological) inverse limit of $\downarrow \mathcal{S}$. The notation $(\tilde{\mathcal{X}}, \{\pi_n\}) = \lim_{\downarrow \mathcal{S}}$ or simply $\tilde{\mathcal{X}} = \lim_{\downarrow \mathcal{S}}$ will be used. The space $\mathcal{X}$ from the proof of the Lemma \ref{1.27} is said to be the disconnected inverse limit of $\mathcal{S}$.

**Lemma 1.29.** Suppose $\mathcal{S} = \{\mathcal{X} = X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \} \in \mathfrak{Fin} \mathfrak{Top}$, and $\tilde{\mathcal{X}} = \lim_{\downarrow \mathcal{S}} \mathcal{X}_n$. If $\mathcal{X}$ a topological space which coincides with $\tilde{\mathcal{X}}$ as a set and the topology on $\mathcal{X}$ is generated by special sets then there is the natural isomorphism $G(\mathcal{X} | \mathcal{X}) \cong G(\tilde{\mathcal{X}} | \mathcal{X})$ induced by the map $\mathcal{X} \to \tilde{\mathcal{X}}$.

**Proof.** Since $\mathcal{X}$ coincides with $\tilde{\mathcal{X}}$ as a set, and the topology of $\mathcal{X}$ is finer than the topology of $\tilde{\mathcal{X}}$ there is the natural injective map $G(\mathcal{X} | \mathcal{X}) \hookrightarrow G(\tilde{\mathcal{X}} | \mathcal{X})$. If $\tilde{g} \in G(\tilde{\mathcal{X}} | \mathcal{X})$ and $\mathcal{U}$ is a special set, then for any $n \in \mathbb{N}$ following condition holds

$$\tilde{\pi}_n(\tilde{g})(\mathcal{U}) = h_n(\tilde{g}) \circ \tilde{\pi}_n(\mathcal{U}) \quad \text{(1.4)}$$

where $\tilde{\pi}_n : \tilde{\mathcal{X}} \to \mathcal{X}_n$ is the natural map, and $h_n : G(\tilde{\mathcal{X}} | \mathcal{X}) \to G(\mathcal{X}_n | \mathcal{X})$ is given by the Lemma \ref{1.18}. Clearly $h_n(\tilde{g})$ is a homeomorphism of $\mathcal{X}_n$, so from \ref{1.18} it follows that $\tilde{\pi}_n(\tilde{g})(\mathcal{U})$ is an open subset of $\mathcal{X}_n$. Hence $\tilde{g}(\mathcal{U})$ is special. So $\tilde{g}$ maps special sets onto special sets. Since topology of $\mathcal{X}$ is generated by special sets the map $\tilde{g}$ is a homeomorphism of $\mathcal{X}$, i.e. $\tilde{g} \in G(\mathcal{X} | \mathcal{X})$. 

\[\Box\]
1.1.2 Algebraic construction in brief

The inverse limit of coverings \( \tilde{X} \) is obtained from inverse limit of topological spaces \( \hat{X} \) by a change of a topology. The topology of \( \tilde{X} \) is finer than topology of \( \hat{X} \), it means that \( C_0 \left( \tilde{X} \right) \) is a subalgebra of \( C_b \left( \hat{X} \right) \). The topology of \( \tilde{X} \) is obtained from topology of \( \hat{X} \) by addition of special subsets. Addition of new sets to a topology is equivalent to addition of new elements to \( C_0 \left( \hat{X} \right) \). To obtain \( C_b \left( \tilde{X} \right) \) we will add to \( C_0 \left( \hat{X} \right) \) special elements (cf. Definition 3.5). If \( \tilde{U} \subset \tilde{X} \) is a special set and \( \tilde{a} \in C_c \left( \tilde{X} \right) \) is positive element such that \( \tilde{a} \mid_{\tilde{X} \setminus \tilde{U}} = \{0\} \), and \( a \in C_c \left( \hat{X}_0 \right) \) is given by \( a = \sum_{\hat{g} \in \hat{G}} \hat{g} \tilde{a} \), then following condition holds

\[
\tilde{a} \left( \tilde{\pi}_n \left( \tilde{x} \right) \right) = \left( \sum_{\hat{g} \in \hat{G}} \hat{g} \tilde{a} \right) \left( \tilde{\pi}_n \left( \tilde{x} \right) \right) = \begin{cases} a \left( \tilde{x} \right) & \tilde{x} \in \tilde{U} \\ 0 & \tilde{x} \notin \tilde{U} \end{cases}
\]

From above equation it follows that

\[
\left( \sum_{\hat{g} \in \hat{G}} \hat{g} \tilde{a} \right)^2 = \sum_{\hat{g} \in \hat{G}} \hat{g} \tilde{a}^2. \tag{1.5}
\]

The equation (1.5) is purely algebraic and related to special subsets. From the Theorem 4.13 it follows that the algebraic condition (1.5) is sufficient for construction of \( C_0 \left( \tilde{X} \right) \). Thus noncommutative inverse limits of coverings can be constructed by purely algebraic methods.

1.2 Locally compact spaces

There are two equivalent definitions of \( C_0 \left( \mathcal{X} \right) \) and both of them are used in this article.

**Definition 1.30.** An algebra \( C_0 \left( \mathcal{X} \right) \) is the \( C^\ast \)-norm closure of the algebra \( C_c \left( \mathcal{X} \right) \) of compactly supported continuous functions.

**Definition 1.31.** A \( C^\ast \)-algebra \( C_0 \left( \mathcal{X} \right) \) is given by the following equation

\[
C_0 \left( \mathcal{X} \right) = \{ \varphi \in C_b \left( \mathcal{X} \right) \mid \forall \varepsilon > 0 \ \exists K \subset \mathcal{X} \ (K \text{ is compact}) \ & \forall x \in \mathcal{X} \setminus K \ | \varphi \left( x \right) | < \varepsilon \},
\]

i.e.

\[
\| \varphi \mid_{\mathcal{X} \setminus K} \| < \varepsilon.
\]

**Theorem 1.32.** [4] For a locally compact Hausdorff space \( \mathcal{X} \), the following are equivalent:

(a) The Abelian \( C^\ast \)-algebra \( C_0 \left( \mathcal{X} \right) \) is separable;

(b) \( \mathcal{X} \) is \( \sigma \)-compact and metrizable;

(c) \( \mathcal{X} \) is second-countable.
Corollary 1.33. If $X$ is a locally compact second-countable Hausdorff space then for any $x \in X$ and any open neighborhood $U \subset X$ there is a bounded positive continuous function $a : X \to \mathbb{R}$ such that $a(x) \neq 0$ and $a(X \setminus U) = \{0\}$.

Definition 1.34. If $\phi : X \to \mathbb{C}$ is continuous then the support of $\phi$ is defined to be the closure of the set $\phi^{-1}(\mathbb{C}\setminus\{0\})$. Thus if $x$ lies outside the support, there is some neighborhood of $x$ on which $\phi$ vanishes. Denote by $\text{supp} \phi$ the support of $\phi$.

1.3 Hilbert modules

We refer to [3] for definition of Hilbert $C^*$-modules, or simply Hilbert modules. Let $A$ be a $C^*$-algebra, and let $X_A$ be an $A$-Hilbert module. Let $\langle \cdot, \cdot \rangle_{X_A}$ be the $A$-valued product on $X_A$. For any $\xi, \zeta \in X_A$ let us define an $A$-endomorphism $\theta_{\xi, \zeta}$ given by $\theta_{\xi, \zeta}(\eta) = \xi \langle \zeta, \eta \rangle_{X_A}$ where $\eta \in X_A$. The operator $\theta_{\xi, \zeta}$ shall be denoted by $\langle \xi, \zeta \rangle$. The norm completion of a generated by operators $\theta_{\xi, \zeta}$ algebra is said to be an algebra of compact operators $K(X_A)$. We suppose that there is a left action of $K(X_A)$ on $X_A$ which is $A$-linear, i.e. action of $K(X_A)$ commutes with action of $A$.

1.4 $C^*$-algebras and von Neumann algebras

In this section I follow to [13].

Definition 1.35. Let $A$ be a $C^*$-algebra. The strict topology on the multiplier algebra $M(A)$ is the topology generated by seminorms $\|x\|_a = \|ax\| + \|xa\|, (a \in A)$.

Definition 1.36. Let $\mathcal{H}$ be a Hilbert space. The strong topology on $B(\mathcal{H})$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto \|x\xi\|, x \in B(\mathcal{H}), \xi \in \mathcal{H}$.

Definition 1.37. Let $\mathcal{H}$ be a Hilbert space. The weak topology on $B(\mathcal{H})$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto |\langle x\xi, \eta \rangle|, x \in B(\mathcal{H}), \xi, \eta \in \mathcal{H}$.

Theorem 1.38. Let $M$ be a $C^*$-subalgebra of $B(\mathcal{H})$, containing the identity operator. The following conditions are equivalent:

- $M = M''$ where $M''$ is the bicommutant of $M$;
- $M$ is weakly closed;
- $M$ is strongly closed.

Definition 1.39. Any $C^*$-algebra $M$ is said to be a von Neumann algebra or a $W^*$-algebra if $M$ satisfies to the conditions of the Theorem 1.38.

Definition 1.40. Let $A$ be a $C^*$-algebra, and let $S$ be the state space of $A$. For any $s \in S$ there is an associated representation $\pi_s : A \to B(\mathcal{H}_s)$. The representation $\bigoplus_{s \in S} \pi_s : A \to \bigoplus_{s \in S} B(\mathcal{H}_s)$ is said to be the universal representation. The universal representation can be regarded as $A \to B(\bigoplus_{s \in S} \mathcal{H}_s)$. 
Proposition 1.44. Let $A$ be a $C^*$-algebra, and let $A \to B(H)$ be the universal representation $A \to B(H)$. The strong closure of $\pi(A)$ is said to be the enveloping von Neumann algebra or the enveloping $W^*$-algebra of $A$. The enveloping von Neumann algebra will be denoted by $A''$.

Proposition 1.42. The enveloping von Neumann algebra $A''$ of a $C^*$-algebra $A$ is isomorphic, as a Banach space, to the second dual of $A$, i.e. $A'' \approx A^{**}$.

Lemma 1.43. Let $\Lambda$ be an increasing net in the partial ordering. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be an increasing net of self-adjoint operators in $B(H)$, i.e. $\lambda \leq \mu$ implies $x_\lambda \leq x_\mu$. If $\|x_\lambda\| \leq \gamma$ for some $\gamma \in \mathbb{R}$ and all $\lambda$ then $\{x_\lambda\}$ is strongly convergent to a self-adjoint element $x \in B(H)$ with $\|x_\lambda\| \leq \gamma$.

For each $x \in B(H)$ we define the range projection of $x$ (denoted by $[x]$) as projection on the closure of $xH$. If $M$ is a von Neumann algebra and $x \in M$ then $[x] \in M$.

Proposition 1.44. For each element $x$ in a von Neumann algebra $M$ there is a unique partial isometry $u \in M$ and positive $|x| \in M_+$ with $uu^* = [x]$ and $x = |x|u$.

Definition 1.45. The formula $x = |x|u$ in the Proposition 1.44 is said to be the polar decomposition.

1.46. Any separable $C^*$-algebra $A$ has a state $\tau$ which induces a faithful GNS representation [11]. There is a $C$-valued product on $A$ given by

$$\langle a, b \rangle = \tau(a^*b).$$

This product induces a product on $A/I_\tau$ where $I_\tau = \{a \in A \mid \tau(a^*a) = 0\}$. So $A/I_\tau$ is a pre-Hilbert space. Let denote by $L^2(A, \tau)$ the Hilbert completion of $A/I_\tau$. The Hilbert space $L^2(A, \tau)$ is a space of a GNS representation of $A$.

## 2 Noncommutative finite-fold coverings

### 2.1 Basic construction

Definition 2.1. If $A$ is a $C^*$-algebra then an action of a group $G$ is said to be involutive if $g(a^*) = (ga)^*$ for any $a \in A$ and $g \in G$. The action is said to be non-degenerated if for any nontrivial $g \in G$ there is $a \in A$ such that $ga \neq a$.

Definition 2.2. Let $A \hookrightarrow \tilde{A}$ be an injective $*$-homomorphism of unital $C^*$-algebras. Suppose that there is a non-degenerated involutive action $G \times \tilde{A} \to \tilde{A}$ of a finite group $G$, such that $A = \tilde{A}^G \overset{\text{def}}{=} \{a \in \tilde{A} \mid a = ga; \forall g \in G\}$. There is an $A$-valued product on $\tilde{A}$ given by

$$\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g(a^*b) \quad (2.1)$$

and $\tilde{A}$ is an $A$-Hilbert module. We say that a triple $(A, \tilde{A}, G)$ is an unital noncommutative finite-fold covering if $\tilde{A}$ is a finitely generated projective $A$-Hilbert module.
Remark 2.3. Above definition is motivated by the Theorem 1.2.

Definition 2.4. Let $A, \tilde{A}$ be C*-algebras and let $A \hookrightarrow \tilde{A}$ be an inclusion such that following conditions hold:

(a) There are unital C*-algebras $B, \tilde{B}$ and inclusions $A \subset B, \tilde{A} \subset \tilde{B}$ such that $A$ (resp. $B$) is an essential ideal of $\tilde{A}$ (resp. $\tilde{B}$) and $A = B \cap \tilde{A}$,

(b) There is a unital noncommutative finite-fold covering $(B, \tilde{B}, G)$,

(c) $G\tilde{A} = \tilde{A}$.

The triple $(A, \tilde{A}, G)$ is said to be a noncommutative finite-fold covering with compactification. The group $G$ is said to be the covering transformation group (of $(A, \tilde{A}, G)$) and we use the following notation

$$G \left( \tilde{A} | A \right) \overset{\text{def}}{=} G.$$ (2.2)

Remark 2.5. The Definition 2.4 is motivated by the Lemma 4.1.

Remark 2.6. Any unital noncommutative finite-fold covering is a noncommutative finite-fold covering with compactification.

Definition 2.7. Let $A, \tilde{A}$ be C*-algebras, $A \hookrightarrow \tilde{A}$ an injective *-homomorphism and $G \times \tilde{A} \rightarrow \tilde{A}$ an involutive non-degenerated action of a finite group $G$ such that following conditions hold:

(a) $A \cong \tilde{A}^G \overset{\text{def}}{=} \left\{ a \in \tilde{A} \mid Ga = a \right\}$,

(b) There is a family $\left\{ \tilde{I}_\lambda \subset \tilde{A} \right\}_{\lambda \in \Lambda}$ of closed ideals of $\tilde{A}$ such that

$$G\tilde{I}_\lambda = \tilde{I}_\lambda.$$ (2.3)

Moreover $\bigcup_{\lambda \in \Lambda} \tilde{I}_\lambda$ (resp. $\bigcup_{\lambda \in \Lambda} \left( A \cap \tilde{I}_\lambda \right)$) is a dense subset of $\tilde{A}$ (resp. $A$), and for any $\lambda \in \Lambda$ there is a natural noncommutative finite-fold covering with compactification $(\tilde{I}_\lambda \cap A, \tilde{I}_\lambda, G)$.

We say that the triple $(A, \tilde{A}, G)$ is a noncommutative finite-fold covering.

Remark 2.8. The Definition 2.7 is motivated by the Theorem 4.3.

Remark 2.9. Any noncommutative finite-fold covering with compactification is a noncommutative finite-fold covering.

Definition 2.10. The injective *-homomorphism $A \hookrightarrow \tilde{A}$ from the Definition 2.7 is said to be a noncommutative finite-fold covering.
Definition 2.11. Let \((A, \tilde{A}, G)\) be a noncommutative finite-fold covering. The algebra \(\tilde{A}\) is a Hilbert \(A\)-module with an \(A\)-valued product given by
\[
\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g(a^* b); \ a, b \in \tilde{A}.
\] (2.4)
We say that this structure of Hilbert \(A\)-module is induced by the covering \((A, \tilde{A}, G)\). Henceforth we shall consider \(\tilde{A}\) as a right \(A\)-module, so we will write \(\tilde{A}_A\).

2.2 Induced representation

2.12. Let \((A, \tilde{A}, G)\) be a noncommutative finite-fold covering, and let \(\rho : A \to B(\mathcal{H})\) be a representation. If \(X = \tilde{A} \otimes_A \mathcal{H}\) is the algebraic tensor product then there is a sesquilinear \(\mathbb{C}\)-valued product \((\cdot, \cdot)_X\) on \(X\) given by
\[
(a \otimes \xi, b \otimes \eta)_X = (\xi, \langle a, b \rangle_{\tilde{A}} \eta)_\mathcal{H}
\] (2.5)
where \((\cdot, \cdot)_\mathcal{H}\) means the Hilbert space product on \(\mathcal{H}\), and \(\langle \cdot, \cdot \rangle_{\tilde{A}}\) is given by (2.4). So \(X\) is a pre-Hilbert space. There is a natural map \(p : \tilde{A} \times (\tilde{A} \otimes_A \mathcal{H}) \to \tilde{A} \otimes_A \mathcal{H}\) given by
\[
(a, b \otimes \xi) \mapsto ab \otimes \xi.
\]

Definition 2.13. Use notation of the Definition 2.11, and 2.12. If \(\tilde{\mathcal{H}}\) is the Hilbert completion of \(X = \tilde{A} \otimes_A \mathcal{H}\) then the map \(p : \tilde{A} \times (\tilde{A} \otimes_A \mathcal{H}) \to \tilde{A} \otimes_A \mathcal{H}\) induces the representation \(\tilde{\rho} : \tilde{A} \to B(\tilde{\mathcal{H}})\). We say that \(\tilde{\rho}\) is induced by the pair \((\rho, (A, \tilde{A}, G))\).

Remark 2.14. Below any \(\tilde{a} \otimes \xi \in \tilde{A} \otimes_A \mathcal{H}\) will be regarded as element in \(\tilde{\mathcal{H}}\).

Lemma 2.15. If \(A \to B(\mathcal{H})\) is faithful then \(\tilde{\rho} : \tilde{A} \to B(\tilde{\mathcal{H}})\) is faithful.

Proof. If \(\tilde{a} \in \tilde{A}\) is a nonzero element then
\[
a = \langle \tilde{a} \tilde{a}^*, \tilde{a} \tilde{a}^* \rangle_{\tilde{A}} = \sum_{g \in G} g(\tilde{a}^* \tilde{a} \tilde{a}^*) \in A
\]
is a nonzero positive element. There is \(\xi \in \mathcal{H}\) such that \(\langle \xi, \tilde{a} \xi \rangle_{\mathcal{H}} > 0\). However
\[
\langle \xi, \tilde{a} \xi \rangle_{\mathcal{H}} = \langle \tilde{a} \xi, \tilde{a} \xi \rangle_{\tilde{\mathcal{H}}}
\]
where \(\tilde{\xi} = \tilde{a}^* \otimes \xi \in \tilde{A} \otimes_A \mathcal{H} \subset \tilde{\mathcal{H}}\). Hence \(\tilde{a} \xi \neq 0\). \(\square\)
2.16. Let \((A, \tilde{A}, G)\) be a noncommutative finite-fold covering, let \(\rho : A \to B(\mathcal{H})\) be a faithful representation, and let \(\bar{\rho} : \tilde{A} \to B(\tilde{\mathcal{H}})\) is induced by the pair \((\rho, (A, \tilde{A}, G))\).

There is the natural action of \(G\) on \(\tilde{\mathcal{H}}\) induced by the map

\[ g(\tilde{a} \otimes \xi) = (g\tilde{a}) \otimes \xi; \quad \tilde{a} \in \tilde{A}, \ g \in G, \ \xi \in \mathcal{H}. \]

There is the natural orthogonal inclusion \(\mathcal{H} \subset \tilde{\mathcal{H}}\) induced by inclusions

\[ A \subset \tilde{A}; \quad A \otimes_A \mathcal{H} \subset \tilde{A} \otimes_A \mathcal{H}. \]

Action of \(g\) on \(\tilde{A}\) can be defined by representation as \(g\tilde{a} = g\tilde{a}g^{-1}\), i.e.

\[ (g\tilde{a})\xi = g\left(\tilde{a}\left(g^{-1}\xi\right)\right); \quad \forall \xi \in \tilde{\mathcal{H}}. \]

**Definition 2.17.** If \(M(\tilde{A})\) is the multiplier algebra of \(\tilde{A}\) then there is the natural action of \(G\) on \(M(\tilde{A})\) such that for any \(\tilde{a} \in M(\tilde{A}), \ \tilde{b} \in \tilde{A}\) and \(g \in G\) a following condition holds

\[ (g\tilde{a})\tilde{b} = g\left(\tilde{a}\left(g^{-1}\tilde{b}\right)\right). \]

We say that action of \(G\) on \(M(\tilde{A})\) is induced by the action of \(G\) on \(\tilde{A}\).

**Lemma 2.18.** If an action of \(G\) on \(M(\tilde{A})\) is induced by the action of \(G\) on \(\tilde{A}\) then

\[ M(\tilde{A})^G \subset M(\tilde{A}^G). \quad (2.6) \]

**Proof.** If \(a \in M(\tilde{A})^G\) and \(b \in \tilde{A}^G\) then \(ab \in \tilde{A}\) is such that \(g(ab) = (ga)(gb) = ab \in \tilde{A}^G\). \(\square\)

## 3 Noncommutative infinite coverings

### 3.1 Basic construction

This section contains a noncommutative generalization of infinite coverings.

**Definition 3.1.** Let

\[ \mathcal{G} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi^{n+1}} \ldots \right\} \]

be a sequence of \(C^*\)-algebras and noncommutative finite-fold coverings such that:

(a) Any composition \(\pi_{n_1} \circ \ldots \circ \pi_{n_0+1} \circ \pi_{n_0} : A_{n_0} \to A_{n_1}\) corresponds to the noncommutative covering \((A_{n_0}, A_{n_1}, G(A_{n_1} | A_{n_0})�);\)
(b) If \( k < l < m \) then \( G(A_m \mid A_k)A_l = A_l \). (Action of \( G(A_m \mid A_k) \) on \( A_l \) means that 
\( G(A_m \mid A_k) \) acts on \( A_m \), so \( G(A_m \mid A_k) \) acts on \( A_l \) since \( A_l \) a subalgebra of \( A_m \)).

(c) If \( k < l < m \) are nonegative integers then there is the natural exact sequence of covering transformation groups

\[
\{e\} \to G(A_m \mid A_l) \xrightarrow{i} G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k) \to \{e\}
\]

where the existence of the homomorphism \( G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k) \) follows from (b).

The sequence \( S \) is said to be an (algebraical) finite covering sequence. For any finite covering sequence we will use the notation \( S \in \text{FinAlg} \).

**Definition 3.2.** Let \( \hat{A} = \lim_{\to} A_n \) be the \( C^* \)-inductive limit \([11]\), and suppose that \( \hat{G} = \lim_{\leftarrow} G(A_n \mid A) \) is the projective limit of groups \([15]\). There is the natural action of \( \hat{G} \) on \( \hat{A} \).

A non-degenerate faithful representation \( \hat{A} \to B(\mathcal{H}) \) is said to be equivariant if there is an action of \( \hat{G} \) on \( \mathcal{H} \) such that for any \( \xi \in \mathcal{H} \) and \( g \in \hat{G} \) the following condition holds

\[
(ga)\xi = g\left(a\left(g^{-1}\xi\right)\right).
\]

**Example 3.3.** Let \( S \) be the state space of \( \hat{A} \), and let \( \hat{A} \to B(\bigoplus_{s \in S} \mathcal{H}_s) \) be the universal representation. There is the natural action of \( \hat{G} \) on \( S \) given by

\[
(gs)\left((a)\right) = s(ga); \ s \in S, \ a \in \hat{A}, \ g \in \hat{G}.
\]

The action of \( \hat{G} \) on \( S \) induces the action of \( \hat{G} \) on \( \bigoplus_{s \in S} \mathcal{H}_s \). It follows that the universal representation is equivariant.

**Example 3.4.** Let \( s \) be a faithful state which corresponds to the representation \( \hat{A} \to B(\mathcal{H}_s) \) and \( \{\xi_n \in \hat{G}\}_{n \in \mathbb{N}} = \hat{G} \) is a bijection. The state

\[
\sum_{n \in \mathbb{N}} \xi_n^{g_n} \xi^n
\]

corresponds to an equivariant representation \( \hat{A} \to B\left(\bigoplus_{g \in \hat{G}} \mathcal{H}_{gs}\right) \).

**Definition 3.5.** Let \( \pi : \hat{A} \to B(\mathcal{H}) \) be an equivariant representation. A positive element \( \pi \in B(\mathcal{H})_+ \) is said to be special (with respect to \( \pi \)) if following conditions hold:

(a) For any \( n \in \mathbb{N}^0 \) the following series

\[
a_n = \sum_{g \in \ker(\hat{G} \to G(A_n \mid A))} g\pi
\]

is strongly convergent and the sum lies in \( A_n \), i.e. \( a_n \in A_n \).
If \( f : \mathbb{R} \to \mathbb{R} \) is given by
\[
f(x) = \begin{cases} 
0 & x \leq \varepsilon \\
x - \varepsilon & x > \varepsilon 
\end{cases}
\]
(3.2)
then for any \( n \in \mathbb{N}^0 \) and for any \( z \in A \) following series
\[
b_n = \sum_{g \in \ker \hat{G} \to G (A_n | A)} g(z\bar{a} z^*) , \quad c_n = \sum_{g \in \ker \hat{G} \to G (A_n | A)} g(z\bar{a} z^*)^2 , \quad d_n = \sum_{g \in \ker \hat{G} \to G (A_n | A)} g f(x)(z\bar{a} z^*)
\]
are strongly convergent and the sums lie in \( A_n \), i.e. \( b_n, c_n, d_n \in A_n \);

For any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) (which depends on \( a \) and \( z \)) such that for any \( n \geq N \) a following condition holds
\[
\| b_n^2 - c_n \| < \varepsilon.
\]
(3.3)

An element \( \bar{a} \in B(\mathcal{H}) \) is said to be weakly special if
\[
\bar{a} = x\bar{a} y, \text{ where } x, y \in \hat{A}, \text{ and } \bar{a} \in B(\mathcal{H}) \text{ is special.}
\]

**Lemma 3.6.** If \( \bar{a} \in B(\mathcal{H})_+ \) is a special element and \( \overline{\mathcal{C}}_n = \ker \hat{G} \to G (A_n | A) \) then from
\[
a_n = \sum_{g \in \mathcal{C}_n} g\bar{a}_n
\]
it follows that \( \bar{a} = \lim_{n \to \infty} a_n \) in the sense of the strong convergence. Moreover one has \( \bar{a} = \inf_{n \in \mathbb{N}} a_n \).

**Proof.** From the Lemma 3.1 it follows that the decreasing lower-bounded sequence \( \{ a_n \} \) is strongly convergent and \( \lim_{n \to \infty} a_n = \inf_{n \in \mathbb{N}} a_n \). From \( a_n > \bar{a} \) it follows that \( \inf_{n \in \mathbb{N}} a_n \geq \bar{a} \).

If \( \inf_{n \in \mathbb{N}} a_n > \bar{a} \) then there is \( \xi \in \mathcal{H} \) such that
\[
\left( \xi, \inf_{n \in \mathbb{N}} a_n \right) > \left( \xi, \bar{a} \right),
\]
however one has
\[
\left( \xi, \inf_{n \in \mathbb{N}} a_n \right) = \inf_{n \in \mathbb{N}} \left( \xi, a_n \bar{\xi} \right) = \inf_{n \in \mathbb{N}} \left( \xi, \left( \sum_{g \in \mathcal{C}_n} g\bar{a}_n \right) \bar{\xi} \right) = \inf_{n \in \mathbb{N}} \sum_{g \in \mathcal{C}_n} \left( \xi, g\bar{a}_n \bar{\xi} \right) = \left( \xi, \bar{a} \bar{\xi} \right).
\]
It follows that \( \bar{a} = \inf_{n \in \mathbb{N}} a_n \).

\[\square\]
Corollary 3.7. Any weakly special element lies in the enveloping von Neumann algebra $\hat{A}''$ of $A = \lim\downarrow A_n$. If $\overline{A}_\pi \subset B(\mathcal{H})$ is the C*-norm completion of an algebra generated by weakly special elements then $\overline{A}_\pi \subset \hat{A}''$.

Lemma 3.8. If $\mathbf{\pi} \in B(\mathcal{H})$ is special, (resp. $\mathbf{\pi}' \in B(\mathcal{H})$ weakly special) then for any $g \in \hat{G}$ the element $g\mathbf{\pi}$ is special, (resp. $g\mathbf{\pi}'$ is weakly special).

Proof. If $\mathbf{\pi} \in B(\mathcal{H})$ is special then $g\mathbf{\pi}$ satisfies to (a)-(c) of the Definition 3.5, i.e. $g\mathbf{\pi}$ is special. If $\mathbf{\pi}'$ is weakly special then form

$$\mathbf{\pi}' = x\mathbf{\pi};$$

where $x, y \in \hat{A}$, and $g \pi \in B(\mathcal{H})$ is special,

it turns out that

$$g\mathbf{\pi}' = (gx)(g\mathbf{\pi})(gy),$$

i.e. $g\mathbf{\pi}'$ is weakly special. \hfill \Box

Corollary 3.9. If $\overline{A}_\pi \subset B(\mathcal{H})$ is the C*-norm completion of algebra generated by weakly special elements, then there is a natural action of $\hat{G}$ on $\overline{A}_\pi$.

Definition 3.10. Let $\mathfrak{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \cdots \right\}$ be an algebraical finite covering sequence. Let $\mathfrak{S} : \hat{A} \rightarrow B(\mathcal{H})$ be an equivariant representation. Let $\overline{A}_\pi \subset B(\mathcal{H})$ be the C*-norm completion of algebra generated by weakly special elements. We say that $\overline{A}_\pi$ is the disconnected inverse noncommutative limit of $\downarrow \mathfrak{S}$ (with respect to $\pi$). The triple $(A, \overline{A}_\pi, G(\overline{A}_\pi \mid A) \overset{\text{def}}{=} \hat{G})$ is said to be the disconnected infinite noncommutative covering of $\mathfrak{S}$ (with respect to $\pi$). If $\pi$ is the universal representation then “with respect to $\pi$” is dropped and we will write $(A, \overline{A}_\pi, G(\overline{A}_\pi \mid A))$.

Definition 3.11. Any maximal irreducible subalgebra $\overline{A}_\pi \subset \overline{A}_\pi$ is said to be a connected component of $\mathfrak{S}$ (with respect to $\pi$). The maximal subgroup $G_\pi \subset G(\overline{A}_\pi \mid A)$ among subgroups $G \subset G(\overline{A}_\pi \mid A)$ such that $G\overline{A}_\pi = \overline{A}_\pi$ is said to be the $\overline{A}_\pi$-invariant group of $\mathfrak{S}$. If $\pi$ is the universal representation then “with respect to $\pi$” is dropped.

Remark 3.12. From the Definition 3.11 it follows that $G_\pi \subset G(\overline{A}_\pi \mid A)$ is a normal subgroup.

Definition 3.13. Let

$$\mathfrak{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \cdots \right\} \in \mathfrak{SinAlg},$$

and let $(A, \overline{A}_\pi, G(\overline{A}_\pi \mid A))$ be a disconnected infinite noncommutative covering of $\mathfrak{S}$ with respect to an equivariant representation $\pi : \lim A_n \rightarrow B(\mathcal{H})$. Let $\overline{A}_\pi \subset \overline{A}_\pi$ be a connected component of $\mathfrak{S}$ with respect to $\pi$, and let $G_\pi \subset G(\overline{A}_\pi \mid A)$ be the $\overline{A}_\pi$-invariant group of $\mathfrak{S}$. Let $h_n : G(\overline{A}_\pi \mid A) \rightarrow G(A_n \mid A)$ be the natural surjective homomorphism. The representation $\pi : \lim A_n \rightarrow B(\mathcal{H})$ is said to be good if it satisfies to following conditions:
(a) The natural *-homomorphism \( \lim_{\to} A_n \to M (\tilde{A}_\pi) \) is injective,

(b) If \( J \subset G (\tilde{A}_\pi \mid A) \) is a set of representatives of \( G (\tilde{A}_\pi \mid A) / G_\pi \), then the algebraic direct sum

\[
\bigoplus_{g \in J} \tilde{A}_\pi
\]

is a dense subalgebra of \( \tilde{A}_\pi \),

(c) For any \( n \in \mathbb{N} \) the restriction \( h_n \mid G_\pi \) is an epimorphism, i.e. \( h_n (G_\pi) = G (A_n \mid A) \).

If \( \pi \) is the universal representation we say that \( \mathcal{S} \) is good.

**Definition 3.14.** Let \( \mathcal{S} = \{ A = A_0 \to A_1 \to ... \to A_n \to ... \} \in \mathfrak{FinAlg} \) be an algebraical finite covering sequence. Let \( \pi : \tilde{A} \to B (\mathcal{H}) \) be a good representation. A connected component \( \tilde{A}_\pi \subset \tilde{A}_\pi \) is said to be the inverse noncommutative limit of \( \downarrow \mathcal{S} \) (with respect to \( \pi \)). The \( \tilde{A}_\pi \)-invariant group \( G_\pi \) is said to be the covering transformation group of \( \mathcal{S} \) (with respect to \( \pi \)). The triple \( (A, \tilde{A}_\pi, G_\pi) \) is said to be the infinite noncommutative covering of \( \mathcal{S} \) (with respect to \( \pi \)). We will use the following notation

\[
\lim_{\to} \downarrow \mathcal{S} \overset{\text{def}}{=} \tilde{A}_\pi,
G \left( \tilde{A}_\pi \mid A \right) \overset{\text{def}}{=} G_\pi.
\]

If \( \pi \) is the universal representation then "with respect to \( \pi \)" is dropped and we will write \( (A, \tilde{A}, G) \), \( \lim_{\to} \downarrow \mathcal{S} \overset{\text{def}}{=} \tilde{A} \) and \( G \left( \tilde{A} \mid A \right) \overset{\text{def}}{=} G \).

**Definition 3.15.** Let \( \mathcal{S} = \{ A = A_0 \to A_1 \to ... \to A_n \to ... \} \in \mathfrak{FinAlg} \) be an algebraical finite covering sequence. Let \( \pi : \tilde{A} \to B (\mathcal{H}) \) be a good representation. Let \( (A, \tilde{A}, G_\pi) \) be the infinite noncommutative covering of \( \mathcal{S} \) (with respect to \( \pi \)). Let \( K (\tilde{A}_\pi) \) be the Pedersen ideal of \( \tilde{A}_\pi \). We say that \( \mathcal{S} \) allows inner product (with respect to \( \pi \)) if following conditions hold:

(a) Any \( \tilde{a} \in K (\tilde{A}_\pi) \) is weakly special,

(b) For any \( n \in \mathbb{N} \), and \( \tilde{a}, \tilde{b} \in K (\tilde{A}_\pi) \) the series

\[
a_n = \sum_{g \in \ker (G \to G (A_n \mid A))} g \left( \tilde{a} \tilde{b} \right)
\]

is strongly convergent and \( a_n \in A_n \).
Remark 3.16. If $\mathcal{S}$ allows inner product (with respect to $\pi$) then $K\left(\tilde{A}_\pi\right)$ is a pre-Hilbert $A$ module such that the inner product is given by

$$\langle \tilde{a}, \tilde{b} \rangle = \sum_{g \in \hat{G}} g\left(\tilde{a}^* \tilde{b}\right) \in A$$

where the above series is strongly convergent. The completion of $K\left(\tilde{A}_\pi\right)$ with respect to a norm

$$\|\tilde{a}\| = \sqrt{\|\langle \tilde{a}, \tilde{a} \rangle\|}$$

is an $A$-Hilbert module. Denote by $X_A$ this completion. The ideal $K\left(\tilde{A}_\pi\right)$ is a left $\tilde{A}_\pi$-module, so $X_A$ is also $\tilde{A}_\pi$-module. Sometimes we will write $\tilde{A}_\pi X_A$ instead $X_A$.

Definition 3.17. Let $\mathcal{S} = \{A = A_0 \rightarrow A_1 \rightarrow ... \rightarrow A_n \rightarrow ...\} \in \text{FinAlg}$ and $\mathcal{S}$ allows inner product (with respect to $\pi$) then then we say that given by the Remark 3.16 $A$-Hilbert module $\tilde{A}_\pi X_A$ corresponds to the pair $(\mathcal{S}, \pi)$. If $\pi$ is the universal representation then we say that $\tilde{A}_\pi X_A$ corresponds to $\mathcal{S}$.

3.2 Induced representation

Let $\pi: \hat{A} \rightarrow B\left(\overline{H}_\pi\right)$ be a good representation. Let $(A, \tilde{A}_\pi, G_\pi)$ be an infinite noncommutative covering with respect to $\pi$ of $\mathcal{S}$. Denote by $\tilde{W}_\pi \subset B\left(\overline{H}_\pi\right)$ the $\hat{A}$-bimodule of weakly special elements, and denote by

$$\tilde{W}_\pi = \tilde{W}_\pi \cap \tilde{A}_\pi.$$ (3.4)

If $\pi$ is the universal representation then we write $\tilde{W}$ instead $\tilde{W}_\pi$.

Lemma 3.18. If $\tilde{a}, \tilde{b} \in \tilde{W}_\pi$ are weakly special elements then a series

$$\sum_{g \in G_\pi} g\left(\tilde{a}^* \tilde{b}\right)$$

is strongly convergent.

Proof. From the definition of weakly special element one has

$$\tilde{a}^* = x\tilde{c}y$$

where $\tilde{c}$ is a (positive) special element and $x, y \in \hat{A}$. A series

$$\sum_{g \in G_\pi} g\tilde{c}$$

is strongly convergent.
is strongly convergent. For any \( \xi \in \overline{\mathcal{H}}_{\pi} \) and \( \varepsilon > 0 \) there is a finite subset \( G' \subset G_{\pi} \) such that for any finite \( G'' \) which satisfies to \( G' \subset G'' \subset G_{\pi} \) following condition holds

\[
\left\| \sum_{g \in G'' \setminus G'} (g \tilde{b}) \xi \right\| < \frac{\varepsilon}{\| x \| \| \sum_{g \in G_{\pi}} g \tilde{c} \| \| y \|}.
\]

Hence one has

\[
\left\| \sum_{g \in G'' \setminus G'} \left( g \left( \tilde{a}^* \tilde{b} \right) \right) \xi \right\| < \varepsilon,
\]

i.e. the series

\[
\sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{b} \right)
\]

is strongly convergent and \( \sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{b} \right) \in \tilde{A}'' \).

**Definition 3.19.** Element \( \tilde{a} \in \tilde{A}_{\pi} \) is said to be square-summable if the series

\[
\sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{a} \right)
\]

is strongly convergent to a bounded operator. Denote by \( L^2 \left( \tilde{A}_{\pi} \right) \) (or \( L^2 \left( \tilde{A} \right) \)) the \( \mathbb{C} \)-space of square-summable operators.

**Remark 3.20.** If \( \tilde{b} \in \tilde{A} \), and \( \tilde{a} \in L^2 \left( \tilde{A} \right) \) then

\[
\left\| \sum_{g \in G_{\pi}} g \left( \tilde{b} \tilde{a}^* \right) \left( \tilde{b} \tilde{a} \right) \right\| \leq \| \tilde{b} \|^2 \left\| \sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{a} \right) \right\|,
\]

it turns out

\[
\tilde{A} L^2 \left( \tilde{A}_{\pi} \right) \subset L^2 \left( \tilde{A}_{\pi} \right), \quad L^2 \left( \tilde{A}_{\pi} \right) \tilde{A} \subset L^2 \left( \tilde{A}_{\pi} \right),
\]

i.e. there is the left and right action of \( \tilde{A} \) on \( L^2 \left( \tilde{A} \right) \).

**Remark 3.21.** If \( a, b \in L^2 \left( \tilde{A}_{\pi} \right) \) then sum \( \sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{b} \right) \in \tilde{A}'' \) is bounded and \( G_{\pi} \)-invariant, hence \( \sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{b} \right) \in A'' \).

**Remark 3.22.** From the Lemma 3.18 it turns out \( \tilde{W}_{\pi} \subset L^2 \left( \tilde{A}_{\pi} \right) \)

3.23. Let \( A \rightarrow B \left( \mathcal{H} \right) \) be a representation. Denote by \( \tilde{\mathcal{H}} \) a Hilbert completion of a pre-Hilbert space

\[
L^2 \left( \tilde{A}_{\pi} \right) \otimes_A \mathcal{H},
\]

with a scalar product \( \left( \tilde{a} \otimes \xi, \tilde{b} \otimes \eta \right)_{\tilde{\mathcal{H}}} = \left( \tilde{x}, \left( \sum_{g \in G_{\pi}} g \left( \tilde{a}^* \tilde{b} \right) \right) \eta \right)_{\mathcal{H}} \).

(3.7)
There is the left action of \( \hat{A} \) on \( L^2 \left( \hat{\mathcal{A}}_\pi \right) \otimes_\mathcal{A} \mathcal{H} \) given by

\[
\tilde{b} (\tilde{a} \otimes \xi) = \tilde{b} \tilde{a} \otimes \xi
\]

where \( \tilde{a} \in L^2 \left( \hat{\mathcal{A}}_\pi \right) \), \( \tilde{b} \in \hat{A} \), \( \xi \in \mathcal{H} \). The left action of \( \hat{A} \) on \( L^2 \left( \hat{\mathcal{A}}_\pi \right) \otimes_\mathcal{A} \mathcal{H} \) induces following representations

\[
\hat{\rho} : \hat{A} \to B \left( \mathcal{H} \right), \\
\hat{\rho} : \hat{\mathcal{A}}_\pi \to B \left( \mathcal{H} \right).
\]

**Definition 3.24.** The constructed in 3.23 representation \( \hat{\rho} : \hat{\mathcal{A}}_\pi \to B \left( \mathcal{H} \right) \) is said to be induced by \((\rho, \mathcal{G}, \pi)\). We also say that \( \hat{\rho} \) is induced by \((\rho, \hat{\mathcal{A}}_\pi, \mathcal{G}, \hat{\mathcal{A}}_\pi | A)\). If \( \pi \) is an universal representation we say that \( \hat{\rho} \) is induced by \((\rho, \mathcal{G})\) and/or \((\rho, \hat{\mathcal{A}}_\pi, \mathcal{G}, \hat{\mathcal{A}}_\pi | A)\).

**Remark 3.25.** If \( \rho \) is faithful, then \( \rho \) is faithful.

**Remark 3.26.** There is an action of \( G_\pi \) on \( \tilde{\mathcal{H}} \) induced by the natural action of \( G_\pi \) on the \( \tilde{\mathcal{A}}_\pi \)-bimodule \( L^2 \left( \tilde{\mathcal{A}}_\pi \right) \). If the representation \( \tilde{\mathcal{A}}_\pi \to B \left( \tilde{\mathcal{H}} \right) \) is faithful then an action of \( \tilde{\mathcal{A}}_\pi \) on \( \tilde{\mathcal{A}}_\pi \) is given by

\[
(g \tilde{a}) \tilde{\pi} = g \left( \tilde{a} \left( g^{-1} \tilde{\pi} \right) \right); \forall g \in G, \forall \tilde{a} \in \tilde{\mathcal{A}}_\pi, \forall \tilde{\pi} \in \tilde{\mathcal{H}}.
\]

3.27. If \( \mathcal{G} \) allows inner product with respect to \( \pi \) then for any representation \( A \to B \left( \mathcal{H} \right) \) an algebraic tensor product \( _{\tilde{\mathcal{A}}_\pi}X_A \otimes_{\mathcal{A}} \mathcal{H} \) is a pre-Hilbert space with the product given by

\[
(a \otimes \xi, b \otimes \eta) = (\xi, (a, b) \eta)
\]

(cf. Definitions 3.17 and 3.18)

**Lemma 3.28.** Suppose \( \mathcal{G} \) allows inner product with respect to \( \pi \) and any \( \tilde{a} \in K \left( \tilde{\mathcal{A}}_\pi \right) \) is weakly special. If \( \tilde{\mathcal{H}} \) (resp. \( \tilde{\mathcal{H}}' \)) is a Hilbert norm completion of \( W_\pi \otimes \mathcal{A} \mathcal{H} \) (resp. \( \tilde{\mathcal{A}}_\pi X_A \otimes_{\mathcal{A}} \mathcal{H} \)) then there is the natural isomorphism \( \tilde{\mathcal{H}} \cong \tilde{\mathcal{H}}' \).

**Proof.** From \( K \left( \tilde{\mathcal{A}}_\pi \right) \subset W_\pi \) and taking into account that \( K \left( \tilde{\mathcal{A}}_\pi \right) \) is dense in \( \tilde{\mathcal{A}}_\pi X_A \) it turns out \( \tilde{\mathcal{H}}' \subset \tilde{\mathcal{H}} \). If \( \tilde{a} \in W_\pi \) is a positive element and \( f_\xi \) is given by (3.2) then

1. \( f_\xi \left( \tilde{a} \right) \in K \left( \tilde{\mathcal{A}}_\pi \right) \),
2. \( \lim_{\xi \to 0} f_\xi \left( \tilde{a} \right) = \tilde{a} \).

From (a) it follows that \( f_\xi \left( \tilde{a} \right) \otimes \xi \in \tilde{\mathcal{A}}_\pi X_A \otimes_{\mathcal{A}} \mathcal{H} \) for any \( \xi \in \mathcal{H} \). From (b) it turns out \( \tilde{a} \otimes \xi \in \tilde{\mathcal{H}}' \). From this fact it follows the natural inclusion \( \tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}' \). Mutually inverse inclusions \( \mathcal{H} \subset \mathcal{H}' \) and \( \mathcal{H}' \subset \mathcal{H} \) yield the isomorphism \( \mathcal{H} \cong \mathcal{H}' \).

\[\square\]
3.29. Let \( \mathcal{H}_n \) be a Hilbert completion of \( A_n \otimes_A \mathcal{H} \) which is constructed in the section 2.2. Clearly

\[
L^2 \left( \tilde{\mathcal{A}}_{\pi} \right) \otimes_{A_n} \mathcal{H}_n = L^2 \left( \tilde{\mathcal{A}}_{\pi} \right) \otimes_{A_n} (A_n \otimes_A \mathcal{H}) = L^2 \left( \tilde{\mathcal{A}}_{\pi} \right) \otimes_A \mathcal{H}.
\]

(3.8)

4 Quantization of topological coverings

4.1 Finite-fold coverings

The following lemma supplies the quantization of coverings with compactification.

**Lemma 4.1.** If \( \mathcal{X}, \tilde{\mathcal{X}} \) are locally compact spaces, and \( \pi : \tilde{\mathcal{X}} \to \mathcal{X} \) is a surjective continuous map, then following conditions are equivalent:

(i) The map \( \pi : \tilde{\mathcal{X}} \to \mathcal{X} \) is a finite-fold covering with compactification,

(ii) There is a natural noncommutative finite-fold covering with compactification \((C_0(\mathcal{X}), C_0(\tilde{\mathcal{X}}), G)\).

**Proof.** (i)=>(ii) Denote by \( \mathcal{X} \subseteq \mathcal{Y}, \tilde{\mathcal{X}} \subseteq \tilde{\mathcal{Y}} \) compactifications such that \( \pi : \tilde{\mathcal{Y}} \to \mathcal{Y} \) is a finite-fold (topological) covering. Let \( G = G \left( \tilde{\mathcal{Y}} | \mathcal{Y} \right) \) be a group of covering transformations. If \( B = C(\mathcal{Y}) \) and \( \tilde{B} = C \left( \tilde{\mathcal{Y}} \right) \) then \( A = C_0(\mathcal{X}) \) (resp. \( \tilde{A} = C_0 \left( \tilde{\mathcal{X}} \right) \)) is an essential ideal of \( B \) (resp. \( \tilde{B} \)). Taking into account \( A = C_0(\mathcal{X}) = C_0 \left( \tilde{\mathcal{X}} \right) \cap C(\mathcal{Y}) = B \cap \tilde{A} \) one concludes that these algebras satisfy to the condition (a) of the Definition 2.4. From the Theorem 1.2 it turns out that the triple \((B, \tilde{B}, G)\) is an unital noncommutative finite-fold covering. So the condition (b) of the Definition 2.4 holds. From \( G_{\tilde{X}} = \tilde{X} \) it turns out that \( GC_0 \left( \tilde{\mathcal{X}} \right) = C_0 \left( \tilde{\mathcal{X}} \right) \), i.e. the condition (c) of the Definition 2.4 holds. 

(ii)=>(i) If \( A = C_0(\mathcal{X}), \tilde{A} = C_0 \left( \tilde{\mathcal{X}} \right) \) and inclusions \( A \subseteq B, \tilde{A} \subseteq \tilde{B} \) are such that \( A \) (resp. \( B \)) is an essential ideal of \( \tilde{A} \) (resp. \( \tilde{B} \)) then there are compactifications \( \mathcal{X} \hookrightarrow \mathcal{Y} \) and \( \tilde{\mathcal{X}} \hookrightarrow \tilde{\mathcal{Y}} \) such that \( B = C(\mathcal{Y}), \tilde{B} = C \left( \tilde{\mathcal{Y}} \right) \). From the condition (b) of the Definition 2.7 it turns out that the triple \((B, \tilde{B}, G) = (C(\mathcal{Y}), C \left( \tilde{\mathcal{Y}} \right), G)\) is an unital noncommutative finite-fold covering. From the Theorem 1.2 it follows that the \( * \)-homomorphism \( C(\mathcal{Y}) \hookrightarrow C \left( \tilde{\mathcal{Y}} \right) \) induces a finite-fold (topological) covering \( \pi : \tilde{\mathcal{Y}} \to \mathcal{Y} \). From condition (c) of of the Definition 2.4 it turns out \( GC_0 \left( \tilde{\mathcal{X}} \right) = C_0 \left( \tilde{\mathcal{X}} \right) \) or, equivalently

\[
G_{\tilde{X}} = \tilde{X}.
\]

(4.1)

From \( A = B \cap \tilde{A} \) or, equivalently \( C_0(\mathcal{X}) = C_0 \left( \tilde{\mathcal{X}} \right) \cap C(\mathcal{Y}) \) and (4.1) it turns out that \( \pi \) is the restriction of finite-fold covering \( \pi \), i.e. \( \pi = \pi |_{\tilde{\mathcal{X}}} \). So \( \pi \) is a finite-fold covering. \( \square \)
Lemma 4.2. Let \( \pi: \widetilde{X} \to X \) be a surjective map of topological spaces such that there is a family of open subsets \( \{ U_\lambda \subset X \} _{\lambda \in \Lambda} \) such that

(a) \( X = \bigcup _{\lambda \in \Lambda} U_\lambda \),

(b) For any \( \lambda \in \Lambda \) the natural map \( \pi ^{-1} (U_\lambda ) \to U_\lambda \) is a covering.

Then the map \( \pi: \widetilde{X} \to X \) is a covering.

Proof. For any point \( x_0 \in X \) there is \( \lambda \in \Lambda \) such that \( x_0 \in U_\lambda \). The map \( \pi ^{-1} (U_\lambda ) \to U_\lambda \) is a covering, it follows that there is an open neighborhood \( V \) of \( x_0 \) such that \( V \subset U_\lambda \) and \( V \) is evenly covered by \( \pi \).

Theorem 4.3. If \( X, \widetilde{X} \) are locally compact spaces, and \( \pi: \widetilde{X} \to X \) is a surjective continuous map, then following conditions are equivalent:

(i) The map \( \pi: \widetilde{X} \to X \) is a finite-fold regular covering,

(ii) There is the natural noncommutative finite-fold covering \( \left( C_0 (X), C_0 (\widetilde{X}), G \right) \).

Proof. (i)\( \Rightarrow \) (ii) We need check that \( \left( C_0 (X), C_0 (\widetilde{X}), G \right) \) satisfies to condition (a), (b) of the Definition.

(a) Covering \( \pi \) is regular, so from the Proposition it turns out \( X = \widetilde{X} / G \) where \( G = G \left( \widetilde{X} \mid X \right) \) is a covering group. From \( X = \widetilde{X} / G \) it follows that \( C_0 (X) = C_0 \left( \widetilde{X} \right) ^G \).

(b) The space \( X \) is locally compact, so for any \( x \in X \) there is a compact neighborhood \( U \). The maximal open subset \( U \subset \overline{U} \) is an open neighborhood of \( x \). So there is family of open subsets \( \{ U_\lambda \subset X \} _{\lambda \in \Lambda} \) such that

- \( X = \bigcup _{\lambda \in \Lambda} U_\lambda \),

- For any \( \lambda \in \Lambda \) the closure \( \overline{U_\lambda} \) of \( U_\lambda \) in \( X \) is compact.

Since \( \pi \) is a finite-fold covering the set \( \pi ^{-1} (\overline{U_\lambda}) \) is compact for any \( \lambda \in \Lambda \). If \( \overline{U_\lambda} \subset C_0 \left( \widetilde{X} \right) \) is a closed ideal given by

\[
\overline{U_\lambda} \overset{\text{def}}{=} C_0 \left( \pi ^{-1} (U_\lambda ) \right) \cong \left\{ \tilde{a} \in C_0 \left( \widetilde{X} \right) \mid \tilde{a} \left( \widetilde{X} \setminus \pi ^{-1} (U_\lambda ) \right) = \{ 0 \} \right\}
\]

then \( \overline{U_\lambda} \subset C \left( \overline{U_\lambda} \right) \) is an essential ideal of the unital algebra \( C \left( \pi ^{-1} (U_\lambda ) \right) \). From \( G \pi ^{-1} (U_\lambda ) = \pi ^{-1} (U_\lambda ) \) it follows that \( G \overline{U_\lambda} = \overline{U_\lambda} \). If \( I_\lambda = C_0 (X) \cap \overline{U_\lambda} \) then one has

\[
I_\lambda = C_0 (U_\lambda ) \cong \{ a \in C_0 (X) \mid a \left( X \setminus U_\lambda \right) = \{ 0 \} \}
\]

hence \( I_\lambda \) is an essential ideal of an unital algebra \( C \left( \overline{U_\lambda} \right) \). The restriction map

\[
\pi _{\pi ^{-1} (\overline{U_\lambda })}: \pi ^{-1} (\overline{U_\lambda }) \to \overline{U_\lambda}
\]
is a finite-fold covering of compact spaces, so from the Theorem 1.2 it follows that
\[
\left( C \left( U_\lambda \right), C \left( \pi^{-1} \left( U_\lambda \right) \right), G \right)
\]
is an unital noncommutative finite-fold covering. It turns out that
\[
\left( I_\lambda, \tilde{I}_\lambda, G \right) = \left( C_0 \left( U_\lambda \right), C_0 \left( \pi^{-1} \left( U_\lambda \right) \right), G \right)
\]
is a noncommutative finite-fold covering with compactification. From \( X = \bigcup_{\lambda \in \Lambda} U_\lambda \) (resp. \( \tilde{X} = \bigcup_{\lambda \in \Lambda} \pi^{-1} \left( U_\lambda \right) \)) it turns out that \( \bigcup_{\lambda \in \Lambda} I_\lambda \) (resp. \( \bigcup_{\lambda \in \Lambda} \tilde{I}_\lambda \)) is a dense subset of \( C_0 \left( X \right) \) (resp. \( C_0 \left( \tilde{X} \right) \)).

(ii)\( \Rightarrow \) (i) Let \( \left\{ \tilde{I}_\lambda \subset C_0 \left( \tilde{X} \right) \right\}_{\lambda \in \Lambda} \) be a family of closed ideals from the condition (b) of the Definition 2.7, and let \( I_\lambda = \tilde{I}_\lambda \cap C_0 \left( X \right) \). If \( \tilde{U}_\lambda \subset \tilde{X} \) is a given by
\[
\tilde{U}_\lambda \overset{\text{def}}{=} \left\{ \tilde{x} \in \tilde{X} \mid \exists \tilde{a} \in \tilde{I}_\lambda; \tilde{a} \left( \tilde{x} \right) \neq 0 \right\}
\]
then from \( G\tilde{I}_\lambda = \tilde{I}_\lambda \) it turns out \( G\tilde{U}_\lambda = \tilde{U}_\lambda \). If \( U_\lambda \subset X \) is given by
\[
U_\lambda = \left\{ x \in X \mid \exists a \in I_\lambda; a \left( x \right) \neq 0 \right\}
\]
then \( U_\lambda = \pi \left( \tilde{U}_\lambda \right) \), and \( \tilde{U}_\lambda = \pi^{-1} \left( U_\lambda \right) \), hence there is the natural *-isomorphism
\[
\tilde{I}_\lambda \cong C_0 \left( \pi^{-1} \left( U_\lambda \right) \right).
\]

Any covering is an open map, so if \( \tilde{U}_\lambda \) is the closure of \( U_\lambda \) in \( X \) then \( \pi^{-1} \left( \tilde{U}_\lambda \right) \) is the closure of \( \tilde{U}_\lambda \) in \( \tilde{X} \). Following conditions hold:

- \( \tilde{U}_\lambda \) (resp. \( \pi^{-1} \left( \tilde{U}_\lambda \right) \)) is a compactification of \( U_\lambda \), (resp. \( \pi^{-1} \left( U_\lambda \right) \)),
- \( I_\lambda = C_0 \left( U_\lambda \right) \), (resp. \( \tilde{I}_\lambda = C_0 \left( \pi^{-1} \left( U_\lambda \right) \right) \)) is an essential ideal of \( C \left( \tilde{U}_\lambda \right) \) (resp. \( C \left( \pi^{-1} \left( \tilde{U}_\lambda \right) \right) \)),
- The triple \( \left( C \left( U_\lambda \right), C \left( \pi^{-1} \left( U_\lambda \right) \right), G \right) \) is an unital noncommutative finite-fold covering.

It follows that the triple \( \left( I_\lambda, \tilde{I}_\lambda, G \right) = \left( C_0 \left( U_\lambda \right), C_0 \left( \pi^{-1} \left( U_\lambda \right) \right), G \right) \) is a noncommutative finite-fold covering with compactification, hence from the Lemma 4.1 it follows that the natural map \( \pi^{-1} \left( U_\lambda \right) \to U_\lambda \) is a covering. From (b) of the Definition 2.7 it follows that \( \bigcup_{\lambda \in \Lambda} I_\lambda \) is dense subset of \( C_0 \left( X \right) \) it turns out \( X = \bigcup U_\lambda \), hence from the Lemma 4.2 it follows that \( \pi : \tilde{X} \to X \) is a finite-fold covering. \( \square \)
4.2 Infinite coverings

This section supplies a purely algebraic analog of the topological construction given by the Subsection 1.1. Suppose that

\[ S = \{ X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \} \]

is a topological finite covering sequence. From the Theorem 4.3 it turns out that \( S_{\mathcal{C}^0} \left( \mathcal{X} \right) \) is an algebraical finite covering sequence. The following theorem and the corollary give the construction of \( \hat{\mathcal{C}}^0(\mathcal{X}) = \lim_{\leftarrow} \mathcal{C}^0(\mathcal{X}_n) \).

**Theorem 4.4.** [16] If a \( \mathcal{C}^\ast \)-algebra \( A \) is a \( \mathcal{C}^\ast \)-inductive limit of \( A_\gamma (\gamma \in \Gamma) \), the state space \( \Omega \) of \( A \) is homeomorphic to the projective limit of the state spaces \( \Omega_\gamma \) of \( A_\gamma \) (\( \gamma \in \Gamma \)).

**Corollary 4.5.** [16] If a commutative \( \mathcal{C}^\ast \)-algebra \( A \) is a \( \mathcal{C}^\ast \)-inductive limit of the commutative \( \mathcal{C}^\ast \)-algebras \( A_\gamma \) (\( \gamma \in \Gamma \)), the spectrum \( \mathcal{X} \) of \( A \) is the projective limit of spectrums \( \mathcal{X}_\gamma \) of \( A_\gamma \) (\( \gamma \in \Gamma \)).

4.6. From the Corollary 4.5 it turns out \( \hat{\mathcal{C}}^0(\mathcal{X}) = \mathcal{C}^0(\hat{\mathcal{X}}) \) where \( \hat{\mathcal{X}} = \lim_{\leftarrow} \mathcal{X}_n \). If \( \mathcal{X} \) is the disconnected inverse limit of \( \mathcal{G}X \) then there is the natural bicontinuous map \( f : \mathcal{X} \rightarrow \hat{\mathcal{X}} \).

The map induces the injective *-homomorphism \( \mathcal{C}^0(\hat{\mathcal{X}}) \rightarrow \mathcal{C}^b(\mathcal{X}) \). It follows that there is the natural inclusion of enveloping von Neumann algebras \( \mathcal{C}^0(\hat{\mathcal{X}})^\prime \rightarrow \mathcal{C}^0(\mathcal{X})^\prime \). Denote by \( G_n = G(\mathcal{X}_n | \mathcal{X}) \) groups of covering transformations and \( \hat{G} = \lim_{\leftarrow} G_n \). Denote by \( \overline{\pi} : \overline{\mathcal{X}} \rightarrow \mathcal{X}, \overline{\pi}_n : \overline{\mathcal{X}} \rightarrow \mathcal{X}_n, \pi^n : \mathcal{X}_n \rightarrow \mathcal{X}, \pi^m_n : \mathcal{X}_m \rightarrow \mathcal{X}_n (m > n) \) the natural covering projections.

**Lemma 4.7.** Following conditions hold:

(i) If \( \overline{\mathcal{U}} \subset \mathcal{X} \) is a compact set then there is \( N \in \mathbb{N} \) such that for any \( n \geq N \) the restriction \( \overline{\pi}_n|_{\overline{\mathcal{U}}} : \overline{\mathcal{U}} \rightarrow \mathcal{X}_n \) is a homeomorphism,

(ii) If \( \overline{\pi} \in \mathcal{C}_c(\overline{\mathcal{X}}) \) is a positive element then there is \( N \in \mathbb{N} \) such that for any \( n \geq N \) following condition holds

\[ a_n(\overline{\pi}_n(\overline{\mathcal{X}})) = \begin{cases} \overline{\pi}(\overline{\mathcal{X}}) & \overline{\mathcal{X}} \in \text{supp} \overline{\pi} \text{ & } \overline{\pi}_n(\overline{\mathcal{X}}) \in \text{supp} a_n \\ 0 & \overline{\pi}_n(\overline{\mathcal{X}}) \notin \text{supp} a_n \end{cases} \quad \text{(4.2)} \]

where

\[ a_n = \sum_{g \in \ker(\hat{G} \rightarrow G_n)} g\overline{\pi}. \]

**Proof.** (i) The set \( \overline{\mathcal{U}} \) is compact, hence \( \text{supp} \overline{\pi} \) is a finite disconnected union of connected compact sets, i.e.

\[ \overline{\mathcal{U}} = \bigsqcup_{j=1}^M \overline{\mathcal{U}}_j. \]
It is known \[15\] that any covering is an open map, and any open map maps any closed set onto a closed set, so for any \(n \in \mathbb{N}\) the set \(\pi_n(U)\) is compact. For any \(n \in \mathbb{N}\) denote by \(c_n \in \mathbb{N}\) the number of connected components of \(\pi_n(U)\). If \(n > m\) then any connected component of \(\pi_n(U)\) is mapped into a connected component of \(\pi_m(U)\), it turns out \(c_n \geq c_m\). Clearly \(c_n \leq M\). The sequence \(\{c_n\}_{n \in \mathbb{N}}\) is non-decreasing and \(c_n \leq M\). It follows that there is \(N \in \mathbb{N}\) such that \(c_N = M\). For any \(n > N\) the set \(\pi_n(U)\) is mapped homeomorphically onto \(\pi_N(U)\), hence from the sequence of homeomorphisms it follows

\[
\ldots \cong \pi_n(U) \cong \ldots \cong U
\]

it follows that \(\pi_n|U : \overline{U} \cong \pi_n(U)\) is a homeomorphism.

(ii) The set \(\text{supp} a = U\) is compact, it follows that from (i) and \(a > 0\) that \(\text{supp} a\) is mapped homeomorphically onto \(\text{supp} a\). It turns out that if \(a_n = \sum_{g \in \ker(\hat{G} \to G_n)} g \overline{a}\) and \(n \geq N\) then \(a_n\) is given by (4.2).

\[\text{Lemma 4.8.}\] If \(X\) is a locally compact Hausdorff space then any positive element \(\overline{a} \in C_c(\overline{X})_+\) is special.

\[\text{Proof.}\] From the Lemma \[4.7\] it follows that there is \(N \in \mathbb{N}\) such that the equation (4.2) holds. It turns out that for any \(z \in C_0(X), n \geq N\) and \(f \) given by (3.2) the series

\[
\begin{align*}
    b_n &= \sum_{g \in \ker(\hat{G} \to G_n)} g (z \overline{z}^*), \\
    c_n &= \sum_{g \in \ker(\hat{G} \to G_n)} g (z \overline{z}^*)^2, \\
    d_n &= \sum_{g \in \ker(\hat{G} \to G_n)} f (z \overline{z}^*)
\end{align*}
\]

are given by

\[
\begin{align*}
    b_n (\pi_n(x)) &= 0, & \pi \in \text{supp} \overline{a} & \& \pi_n(x) \in \text{supp} \ a_n, \\
    c_n (\pi_n(x)) &= \begin{cases} (z (\pi_n(x)) \overline{\pi(x)} z^* (\pi_n(x)))^2 & \pi \in \text{supp} \overline{a} & \& \pi_n(x) \in \text{supp} \ a_n, \\
        0 & \pi_n(x) \notin \text{supp} \ a_n. \end{cases}
\end{align*}
\]

From (4.3) it turns out \(b_n^2 = c_n\), i.e. \(\overline{a}\) satisfies to the condition (c) of the Definition 3.5.
Otherwise (4.2), (4.3) from that \(a_n, b_n, c_n, d_n \in C_0(\mathcal{X}_n)\) for any \(n \geq N\). If \(n < N\) then

\[
\begin{align*}
a_n &= \sum_{g \in G(\mathcal{X}_n \mid X_n)} g\bar{a}_N, \\
b_n &= \sum_{g \in G(\mathcal{X}_n \mid X_n)} g\bar{b}_N, \\
c_n &= \sum_{g \in G(\mathcal{X}_n \mid X_n)} g\bar{c}_N, \\
d_n &= \sum_{g \in G(\mathcal{X}_n \mid X_n)} g\bar{d}_N.
\end{align*}
\]

Above sums are finite, it turns out \(a_n, b_n, c_n, d_n \in C_0(\mathcal{X}_n)\) for any \(n \in \mathbb{N}^0\), i.e. \(\bar{a}\) satisfies to conditions (a), (b) of the Definition 3.5.

\[\square\]

**Corollary 4.9.** If \(\overline{A}\) is a disconnected inverse noncommutative limit of

\[
\mathcal{G}_{C_0(\mathcal{X})} = \{C_0(\mathcal{X}_0) \to \cdots \to C_0(\mathcal{X}_n) \to \cdots\}
\]

then \(C_0(\overline{\mathcal{X}}) \subset \overline{A}\).

**Proof.** From the Lemma 4.8 it follows that \(C_c(\overline{\mathcal{X}}) \subset \overline{A}\), and taking into account the Definition 1.30 one has \(C_0(\overline{\mathcal{X}}) \subset \overline{A}\). \(\square\)

**Lemma 4.10.** Suppose that \(\mathcal{X}\) is a locally compact Hausdorff space. Let \(\bar{a} \in C_0(\overline{\mathcal{X}})^{\prime\prime}\) be such that following conditions hold:

(a) If \(f_\varepsilon\) is given by (3.2) then following series

\[
\begin{align*}
a_n &= \sum_{g \in \ker(G \to G_n)} g\bar{a}, \\
b_n &= \sum_{g \in \ker(G \to G_n)} g\bar{b}, \\
c_n &= \sum_{g \in \ker(G \to G_n)} g f_\varepsilon(\bar{a}),
\end{align*}
\]

are strongly convergent and \(a_n, b_n, c_n \in C_0(\mathcal{X}_n)\).

(b) For any \(\varepsilon > 0\) there is \(N \in \mathbb{N}\) such that

\[
\|a_n^2 - b_n\| < \varepsilon, \quad \forall n \geq N.
\]

Then \(\bar{a} \in C_0(\overline{\mathcal{X}})^{\prime\prime}\).
Proof. The dual space $C_0(\overline{X})^*$ of $C_0(\overline{X})$ is a space of Radon measures on $\overline{X}$. If $\overline{f} : \overline{X} \to \mathbb{R}$ is given by

$$\overline{f}(\overline{x}) = \lim_{n \to \infty} a_n(\pi_n(\overline{x})) = \inf_{n \in \mathbb{N}} a_n(\pi_n(\overline{x}))$$

then from the Proposition 1.42 and the Lemma 3.6 it follows that $\overline{f}$ represents $\pi$, i.e. the following conditions hold:

- The function $\overline{f}$ defines a following functional

$$C_0(\overline{X})^* \to \mathbb{C},$$

$$\mu \mapsto \int_{\overline{X}} \overline{f} \, d\mu$$

where $\mu$ is a Radon measure on $\overline{X}$,

- The functional corresponds to $\overline{\pi} \in C_0(\overline{X})^{**} = C_0(\overline{X})''$.

If $m > n$ then

$$a_n = \sum_{g \in G(\mathcal{X}_m | \mathcal{X}_n)} g a_m,$$

$$b_n = \sum_{g \in G(\mathcal{X}_m | \mathcal{X}_n)} g b_m. \tag{4.4}$$

Let $M \in \mathbb{N}$ be such that for any $n \geq M$ following condition holds

$$\|a_n^2 - b_n\| < 2\varepsilon^2. \tag{4.5}$$

Let $n > M$, $p_n = \pi^n_M : \mathcal{X}_m \to \mathcal{X}_M$, and let $\overline{x}_1, \overline{x}_2 \in \mathcal{X}_n$ be such that

$$\overline{x}_1 \neq \overline{x}_2,$$

$$p_n(\overline{x}_1) = p_n(\overline{x}_2) = x, \tag{4.6}$$

$$a_n(\overline{x}_1) \geq \varepsilon; \quad a_n(\overline{x}_2) \geq \varepsilon.$$

From (4.4) it turns out

$$a_M(x) = \sum_{\overline{x} \in p_n^{-1}(x)} a_n(\overline{x}),$$

$$b_M(x) = \sum_{\overline{x} \in p_n^{-1}(x)} b_n(\overline{x}).$$

From the above equation and $a_n^2 \geq b_n$ it turns out

$$a_M^2(x) = \sum_{\overline{x} \in p_n^{-1}(x)} a_n^2(\overline{x}) + \sum_{(\overline{x}',\overline{x}'') \in p_n^{-1}(x) \times p_n^{-1}(x)} a_n(\overline{x}') a_n(\overline{x}'') \geq$$

$$\geq \sum_{\overline{x} \in p_n^{-1}(x)} b_n(\overline{x}) + a_n(\overline{x}_1) a_n(\overline{x}_2) + a_n(\overline{x}_2) a_n(\overline{x}_1) =$$

$$= b_M(x) + 2a_n(\overline{x}_1) a_n(\overline{x}_2).$$

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Taking into account \( a_n (\bar{x}_1) \geq \epsilon, a_n (\bar{x}_2) \geq \epsilon \) one has
\[
a^2_M (x) - b_M (x) \geq 2 \epsilon^2,
\]
\[
\left\| a^2_M - b_M \right\| \geq 2 \epsilon^2.
\]
So (4.6) contradicts with (4.5), it follows that
\[
p_n (\bar{x}_1) = p_n (\bar{x}_2) = x \quad \& \quad a_n (\bar{x}_1) \geq \epsilon \quad \& \quad a_n (\bar{x}_2) \geq \epsilon \Rightarrow \bar{x}_1 = \bar{x}_2.
\]
If \( f_\epsilon \) is given by (3.2) and
\[
c_n = \sum g \in \ker (\hat{G} \to G_n) g f_\epsilon (a)
\]
then
\[
\text{supp} \, c_n = \left\{ x \in X_n \mid \inf_{m > n} \max_{\bar{x} \in \pi_n^M (x)} a_m (\bar{x}) \geq \epsilon \right\}
\]
\[
= \left\{ x \in X_n \mid \exists \bar{x} \in X; \pi_n (\bar{x}) = x \quad \& \quad f (\bar{x}) \geq \epsilon \right\}.
\]
Indeed \( f_\epsilon (a) \) as a functional on \( C_0 (X) \) is represented by the following function
\[
f_\epsilon : X \to \mathbb{R},
\]
\[
\bar{x} \mapsto f_\epsilon (f (\bar{x})).
\]
From \( \bar{x} \in \text{supp} \, c_n \) it turns out \( a_n (\bar{x}) \geq \epsilon \) and taking into account (4.7) one concludes that the restriction \( \pi^M_n |_{\text{supp} \, c_n} \) is an injective map. Clearly \( \pi^M_n (\text{supp} \, c_n) = \text{supp} \, c_M \), so there is a bijection \( \text{supp} \, c_n \xrightarrow{\approx} \text{supp} \, c_M \). The map \( \pi^M_n \) is a covering and it is known [15] that any covering is an open map. Any bijective open map is a homeomorphism, hence one has a sequence of homeomorphisms
\[
\text{supp} \, c_M \leftarrow \ldots \leftarrow \text{supp} \, c_n \leftarrow \ldots
\]
If \( \mathcal{U} \subset X \) is given by
\[
\mathcal{U} = \bigcap_{n=M}^{\infty} \pi_n^{-1} (\text{supp} \, c_n)
\]
then from (4.8) it turns out that \( \pi_M \) homeomorphically maps \( \mathcal{U} \) onto \( \text{supp} \, c_M \). Moreover following condition holds
\[
f_\epsilon (\bar{a}) (\bar{x}) = \begin{cases} c_M (\pi_M (\bar{x})) & \bar{x} \in \mathcal{U} \\ 0 & \bar{x} \notin \mathcal{U} \end{cases}.
\]
From the above equation it follows that \( f_\epsilon (a) \) is a continuous function, i.e. \( f_\epsilon (a) \in C_b (X) \)
From the Definition [1.31] it turns out \( D = \{ x \in X_M \mid a_M (x) \geq \epsilon \} \) is compact, therefore the closed subset \( \text{supp} \, c_M \subset D \) is compact, hence \( \overline{\mathcal{U}} = \text{supp} \, f_\epsilon (a) \approx \text{supp} \, c_M \) is also compact. It turns out \( f_\epsilon (a) \in C_c (X) \). From \( \| f_\epsilon (a) - \bar{a} \| \leq \epsilon \) it follows that \( \bar{a} = \lim_{\epsilon \to 0} f_\epsilon (a) \) and from the Definition [1.30] it turns out \( \bar{a} \in C_0 (X) \).
Corollary 4.11. If $\mathcal{S}_{C_0(\mathcal{X})} = \{C_0(\mathcal{X}_0) \to ... \to C_0(\mathcal{X}_n) \to ...\}$ and $\bar{\mathcal{A}}$ is a disconnected inverse noncommutative limit of $\downarrow \mathcal{S}_{C_0(\mathcal{X})}$ then following conditions hold:

(i) Any special element $\bar{a} \in C_0(\mathcal{X})''$ of $\mathcal{S}_{C_0(\mathcal{X})}$ lies in $C_0(\mathcal{X})$, i.e. $\bar{a} \in C_0(\mathcal{X})$.

(ii) $C_0(\mathcal{X}) \subset \bar{\mathcal{A}}$.

Proof. (i) Let $\{e_\lambda \in C_0(\mathcal{X})\}_{\lambda \in \Lambda}$ be an approximate unit of $C_0(\mathcal{X})$. From the Definition 3.5 it follows that $\overline{b_\lambda} = e_\lambda \bar{a} e_\lambda$ satisfies to conditions of the Lemma 4.10. Otherwise from the Lemma 4.10 it turns out $\overline{b_\lambda} \in C_0(\mathcal{X})$. From the $C^*$-norm limit $\lim_{\lambda \in \Lambda} \overline{b_\lambda} = \bar{a}$ it follows that $\bar{a} \in C_0(\mathcal{X})$.

(ii) Follows from (i) and the Definitions 3.5, 3.10.

4.12. Let $\bar{\mathcal{X}} \subset \mathcal{X}$ be a connected component of $\mathcal{X}$ and suppose that $G \subset G \left( \lim_{\leftarrow} C_0(\mathcal{X}_n) \mid C_0(\mathcal{X}) \right)$ is the maximal among subgroups $G'$ such that $G' \bar{\mathcal{X}} = \bar{\mathcal{X}}$. If $J \subset \hat{G}$ is a set of representatives of $\hat{G} / G$ then from the (1.2) it follows that

$$\bar{\mathcal{X}} = \bigsqcup_{g \in J} g \bar{\mathcal{X}}$$

and $C_0(\mathcal{X})$ is a $C^*$-norm completion of the direct sum

$$\bigoplus_{g \in J} C_0(g \bar{\mathcal{X}}). \quad (4.9)$$

Theorem 4.13. If $\mathcal{S}_\mathcal{X} = \{\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\} \in \mathfrak{S}\text{InTop}$ and $\mathcal{S}_{C_0(\mathcal{X})} = \{C_0(\mathcal{X}) = C_0(\mathcal{X}_0) \to ... \to C_0(\mathcal{X}_n) \to ...\} \in \mathfrak{S}\text{InAlg}$ is an algebraical finite covering sequence then following conditions hold:

(i) $\mathcal{S}_{C_0(\mathcal{X})}$ is good,

(ii) There are isomorphisms:

- $\lim \downarrow \mathcal{S}_{C_0(\mathcal{X})} \approx C_0 \left( \lim \downarrow \mathcal{S}_\mathcal{X} \right)$;
- $G \left( \lim \downarrow \mathcal{S}_{C_0(\mathcal{X})} \mid C_0(\mathcal{X}) \right) \approx G \left( \lim \downarrow \mathcal{S}_\mathcal{X} \mid \mathcal{X} \right)$.

Proof. The proof of this theorem uses a following notation:

- The topological inverse limit $\bar{\mathcal{X}} = \lim \downarrow \mathcal{S}_\mathcal{X}$;
- The limit in the category of spaces and continuous maps $\bar{\mathcal{X}} = \lim_{\rightarrow} \mathcal{X}_n$.
• The disconnected covering space \( \overline{X} \) of \( \mathcal{G}_X \);
• The disconnected covering algebra \( \mathcal{A} \) of \( \mathcal{G}_{C_0(X)} \);
• A connected component \( \tilde{A} \subset \mathcal{A} \);
• The disconnected \( G_X = \lim_{\leftarrow} (X_n | X) \) and the connected \( \mathcal{G}_X = \mathcal{G} \left( \overline{X} | X \right) = \mathcal{G} \left( \lim_{\leftarrow} \mathcal{G}_X | X \right) \) covering groups of \( \mathcal{G}_X \);
• The disconnected \( G_{C_0(X)} = \lim_{\leftarrow} (C_0(X_n) | C_0(X)) \) and \( \tilde{A} \)-invariant group \( G_A \).

From the Corollary 4.9 it follows that \( \mathcal{A} \subset C_0(X) \). From the Corollary 4.11 it turns out \( C_0(X) \subset A \), hence \( A = C_0(X) \). If \( J \subset G_X \) is a set of representatives of \( G_X / G(\tilde{X} | X) \) then \( \overline{X} = \bigsqcup_{g \in J} g\tilde{X} \) is the disconnected union of connected homeomorphic spaces, i.e. \( \overline{X} \approx \tilde{X} \).

(i) We need check (a) - (c) of the Definition 3.13. \( A = C_0(X) \) is the \( C^* \)-norm completion of the algebraical the direct sum (4.9). Any maximal irreducible subalgebra of \( A \) is isomorphic to \( C_0(\tilde{X}) \). The map \( \tilde{X} \to X_n \) is a covering for any \( n \in \mathbb{N} \), it turns out \( C_0(X_n) \hookrightarrow C_b(\tilde{X}) = M(C_0(\tilde{X})) \) is injective \(*\)-homomorphism. It follows that the natural \(*\)-homomorphism \( C_0(\tilde{X}) = \lim_{n \to \infty} C_0(X_n) \hookrightarrow C_b(\tilde{X}) = M(C_0(\tilde{X})) \) is injective, i.e. the condition (a) holds. The algebraic direct sum \( \bigoplus_{g \in J} gC_0(X) \) is is a dense subalgebra of \( \mathcal{A} \), i.e. condition (b) holds. The homomorphism \( G(\tilde{X} | X) \to G(X_n | X) \) is surjective for any \( n \in \mathbb{N} \). From the following isomorphisms

\[
G_A \cong G(\tilde{X} | X), \\
G(C_0(X_n) | C_0(X)) \cong G(X_n | X),
\]

it turns out that \( G_A \to G(C_0(X_n) | C_0(X)) \) is surjective, i.e. condition (c) holds.

(ii) From the proof of (i) it turns out

\[
\lim_{\leftarrow} \mathcal{G}_X = \tilde{X}; \quad \tilde{A} = C_0(\tilde{X}), \\
\lim_{\leftarrow} \mathcal{G}_{C_0(X)} = \tilde{A} = C_0(\tilde{X}) = C_0(\lim_{\leftarrow} \mathcal{G}_X), \\
G(\lim_{\leftarrow} \mathcal{G}_X | X) = G_X = G_A = G(\lim_{\leftarrow} \mathcal{G}_{C_0(X)} | C_0(X)).
\]

\( \square \)
5 Continuous trace $C^*$-algebras and their coverings

Let $A$ be a $C^*$-algebra. For each positive $x \in A_+$ and irreducible representation $\pi : A \to B(\mathcal{H})$ the (canonical) trace of $\pi(x)$ depends only on the equivalence class of $\pi$, so that we may define a function $\hat{x} : \hat{A} \to [0,\infty]$ by $\hat{x}(t) = \text{Tr}(\pi(x))$ where $\hat{A}$ is the space of equivalence classes of irreducible representations. From Proposition 4.4.9 [13] it follows that $\hat{x}$ is lower semicontinuous function in the Jacobson topology.

**Definition 5.1.** [13] We say that element $x \in A$ has continuous trace if $\hat{x} \in C_b(\hat{A})$. We say that $C^*$-algebra has continuous trace if a set of elements with continuous trace is dense in $A$.

**Definition 5.2.** [13] A positive element in $C^*$-algebra $A$ is Abelian if subalgebra $xAx \subset A$ is commutative.

**Definition 5.3.** [13] We say that a $C^*$-algebra $A$ is of type I if each non-zero quotient of $A$ contains a non-zero Abelian element. If $A$ is even generated (as $C^*$-algebra) by its Abelian elements we say that it is of type $I_0$.

**Proposition 5.4.** [13] A positive element $x$ in $C^*$-algebra $A$ is Abelian if $\dim \pi(x) \leq 1$ for every irreducible representation $\pi : A \to B(\mathcal{H})$.

**Theorem 5.5.** [13] For each $C^*$-algebra $A$ there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.

**Definition 5.6.** The ideal $K(A)$ from the theorem 5.5 is said to be the Pedersen ideal of $A$. Henceforth Pedersen ideal shall be denoted by $K(A)$.

**Proposition 5.7.** [13] Let $A$ be a $C^*$-algebra with continuous trace. Then

(i) $A$ is of type $I_0$;

(ii) $\hat{A}$ is a locally compact Hausdorff space;

(iii) For each $t \in \hat{A}$ there is an Abelian element $x \in A$ such that $\hat{x} \in K(\hat{A})$ and $\hat{x}(t) = 1$.

The last condition is sufficient for $A$ to have continuous trace.

**Remark 5.8.** From [6], Proposition 10, II.9 it follows that a continuous trace $C^*$-algebra $A$ is always a CCR-algebra, i.e. for every irreducible representation $\rho : A \to B(H)$ following condition hold

$$\rho(A) \approx K$$ (5.1)

**Theorem 5.9.** (Dixmier–Douady). Any stable separable algebra $A$ of continuous trace over a second-countable locally compact Hausdorff space $\mathcal{X}$ is isomorphic to $\Gamma_0(\mathcal{X})$, the sections vanishing at infinity of a locally trivial bundle of algebras over $\mathcal{X}$, with fibres $K$ and structure group $\text{Aut}(K) = PU = U/T$. Classes of such bundles are in natural bijection with the Čech cohomology group $H_3(\mathcal{X},\mathbb{Z})$. The 3-cohomology class $\delta(A)$ attached to (the stabilisation of) a continuous-trace algebra $A$ is called its Dixmier–Douady class.
Remark 5.10. Any commutative $C^*$-algebra has continuous trace. So described in the Section 4 case is a special case of described in the Section 5 construction.

5.11. For any $x \in \hat{A}$ denote by $\rho_x : A \to B(H)$ a representation which corresponds to $x$. For any $a \in A$ denote by $\text{supp } a \subset \hat{A}$ the closure of the set $\text{supp } a \overset{\text{def}}{=} \{ x \in \hat{A} | \rho_x(a) \neq 0 \}$.

5.1 Basic construction

Let $A$ be a continuous trace $C^*$-algebra such that the spectrum $\hat{A} = X$ of is a second-countable locally compact Hausdorff space. For any open subset $U \subset X$ denote by $A(U) = \{ a \in A | \rho_x(a) = 0; \forall x \in X \setminus U \}$

where $\rho_x$ is an irreducible representation which corresponds to $x \in X$. If $V \subset U$ then there is a natural inclusion $A(V) \hookrightarrow A(U)$. Let $\pi : \tilde{X} \to X$ be a topological covering. Let $\tilde{U} \subset \tilde{X}$ be a connected open subset homeomorphically mapped onto $U = \pi(\tilde{U})$, and suppose that the closure of $\tilde{U}$ is compact. Denote by $\tilde{A}(\tilde{U})$ the algebra such that $\tilde{A}(\tilde{U}) \cong A(U)$. If $\tilde{V} \subset \tilde{U}$ and $V = \pi(\tilde{V})$ then the inclusion $A(\tilde{V}) \hookrightarrow A(\tilde{U})$ naturally induces an inclusion $i_{\tilde{V}} : \tilde{A}(\tilde{V}) \hookrightarrow \tilde{A}(\tilde{U})$. Let us consider $\tilde{U}$ as indexes, and let

$$A' = \bigoplus_{\tilde{U}} \tilde{A}(\tilde{U}) / I,$$

where $\oplus$ means the algebraic direct sum of $C^*$-algebras and $I$ is the two sided ideal generated by elements $\tilde{1}_{\tilde{U}_1 \cap \tilde{U}_2}(a) - \tilde{1}_{\tilde{U}_1 \cap \tilde{U}_2}(a)$, for any $a \in A(\tilde{U}_1 \cap \tilde{U}_2)$. There is the natural $C^*$-norm of the direct sum on $\bigoplus_{\tilde{U}} \tilde{A}(\tilde{U})$ and let us define the norm on $A' = \bigoplus_{\tilde{U}} \tilde{A}(\tilde{U}) / I$ given by

$$\|a + I\| = \inf_{a' \in I} \|a + a'\|; \forall (a + I) \in \bigoplus_{\tilde{U}} \tilde{A}(\tilde{U}) / I \quad (5.2)$$

Definition 5.12. If $A(\tilde{X})$ is completion of $A'$ with respect to the given by (5.2) then we say that $A(\tilde{X})$ is an induced by $\pi : \tilde{X} \to X$ covering of $A$.

The action of $G(\tilde{X} | X)$ on $\tilde{X}$ induces an action of $G(\tilde{X} | X)$ on $A'$, so there is a natural action of $G(\tilde{X} | X)$ on $A(\tilde{X})$.

Definition 5.13. We say that the action of $G(\tilde{X} | X)$ on $\tilde{X}$ induces the action of $G(\tilde{X} | X)$ on $A(\tilde{X})$.
From the Proposition 5.7 it follows that $A \left( \tilde{X} \right)$ is a continuous trace $C^*$-algebra, and the spectrum of $A \left( \tilde{X} \right)$ coincides with $\tilde{X}$. If $G = G \left( \tilde{X} \mid X \right)$ is a finite group then

$$A = A \left( \tilde{X} \right)^G$$

and the above equation induces an injective *-homomorphism $A \hookrightarrow A \left( \tilde{X} \right)$.

**Lemma 5.14.** Let $A$ be a continuous trace $C^*$-algebra, and let $\mathcal{X} = \hat{A}$ be a spectrum of $A$. Suppose that $\mathcal{X}$ is a second-countable locally compact Hausdorff space and $B$ is a $C^*$-algebra such that

- $A \subset B \subset A''$,
- For any $b \in B_+$ and $x_0 \in \mathcal{X}$ such that $\rho_{x_0}(b) \neq 0$ there is an open neighborhood $W \subset \mathcal{X}$ of $x_0$ and an Abelian $z \in A$ such that

$$\text{supp } z \subset W,$$

$$\text{tr } (zbz) \in C_0(\mathcal{X}),$$

$$\text{tr } (zbz)(x_0) \neq 0.$$  

Then $B = A$.

**Proof.** The spectrum $\hat{B}$ of $B$ coincides with the spectrum of $A$ as a set. Let $V \subset \mathcal{X}$ be a closed subset with respect to topology of $\hat{B}$. There is a closed ideal $I \subset B$ which corresponds to $V$. Denote by $I_+$ the positive part of $I$. For any $x_0 \in \mathcal{X}\setminus V$ there is $b \in I_+$ such that $\rho_{x_0}(b) \neq 0$. There is an Abelian element $z \in A$ such that $\text{tr}(\rho_{x_0}(zbz)) \neq 0$. If $W \subset \mathcal{X}$ is an open neighborhood of $x_0$ then from the Corollary 1.33 it follows that there is a bounded positive continuous function $a : \mathcal{X} \to \mathbb{R}$ such that $a(x_0) \neq 0$ and $a(\mathcal{X}\setminus W) = \{0\}$. If $z = az$ then $z$ is an Abelian document, $\text{tr}(zbz)(x_0) = (\text{tr}(a^2)(x_0))a^2(x_0) \neq 0$ and $\text{supp } z \subset W$. From $\text{tr}(zbz) \in C_0(\mathcal{X})$ it turns out that there is an open (with respect to topology of $\hat{A}$) neighborhood $U$ of $x_0$ such that $\text{tr}(\rho_{x}(zbz)) \neq 0$ for any $x \in U$, i.e. $V \cap U = \emptyset$. It follows that $V$ is a closed subset with respect to the topology of $\hat{A}$. Hence there is a homeomorphism $\hat{A} \approx \hat{B}$. Below we apply the method of proof of the Theorem 6.1.11 [13]. Let us consider the set $M$ of elements in $B_+$ with continuous trace, $M$ is hereditary and the closure of $M$ is the positive part of an ideal $J$ of $B$. However for any $x \in \mathcal{X} = \hat{B}$ there is an Abelian $a \in K(A)$ such that $\text{tr}(\rho_{x}(a)) \neq 0$. It turns out $J$ is not contained in any primitive ideal of $B$, hence $J = B$. It turns out $B$ has continuous trace. From this fact it turns out $\rho_{x}(A) = \rho_{x}(B) \approx \mathcal{K}$ for any $x \in \mathcal{X}$. Taking into account $\rho_{x}(A) = \rho_{x}(B)$, homeomorphism $\mathcal{A} \approx \hat{B}$ and the Theorem 5.9 one has $B = A$.  

### 5.2 Finite-fold coverings

If $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is a finite-fold covering, such that $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are compact Hausdorff spaces, then there is a finite family $\{\mathcal{U}_i \subset \mathcal{X}\}_{i \in I_0}$ of connected open subsets of $\mathcal{X}$ evenly covered
by \( \pi \) such that \( \mathcal{X} = \bigcup_{i \in I_0} U_i \). There is a partition of unity subordinated to \( \{U_i\} \), i.e.

\[
1_{C(\mathcal{X})} = \sum_{i \in I_0} a_i
\]

where \( a_i \in C(\mathcal{X})_+ \) is such that \( \text{supp } a_i \subset U_i \). Denote by \( e_i = \sqrt{a_i} \in C(\mathcal{X})_+ \). For any \( i \in I_0 \) we select \( \tilde{U}_i \subset \tilde{\mathcal{X}} \) such that \( \tilde{U}_i \) is homeomorphically mapped onto \( U_i \). If \( \tilde{e}_i \in C(\tilde{\mathcal{X}}) \) is given by

\[
\tilde{e}_i(\tilde{x}) = \begin{cases} 
  e_i(\pi(\tilde{x})) & \tilde{x} \in \tilde{U}_i \\
  0 & \tilde{x} \notin \tilde{U}_i 
\end{cases}
\]

and \( G = G\left(\tilde{\mathcal{X}} \mid \mathcal{X}\right) \) then

\[
1_{C(\tilde{\mathcal{X}})} = \sum_{(g,i) \in G \times I_0} g\tilde{e}_i^2,
\]

and \( G = G\left(\tilde{\mathcal{X}} \mid \mathcal{X}\right) \) then

If \( I = G \times I_0 \) and \( \tilde{e}_{(g,i)} = g\tilde{e}_i \) the from the above equation it turns out

\[
1_{C(\tilde{\mathcal{X}})} = \sum_{i \in I} \tilde{e}_i(\tilde{e}_i)
\]

where \( \tilde{e}_i(\tilde{e}_i) \) means a compact operator induced by the C*-Hilbert module structure given by (2.1).

**Proposition 5.15.** If \( B \) is a C*-subalgebra of \( A \) containing an approximate unit for \( A \), then \( M(B) \subset M(A) \) (regarding \( B'' \) as a subalgebra of \( A'' \)).

**Lemma 5.16.** Let \( A \) be a continuous trace algebra, and let \( \hat{\mathcal{X}} = \mathcal{X} \) be the spectrum of \( A \). Suppose that \( \mathcal{X} \) is a locally compact second-countable Hausdorff space, and let \( \pi : \tilde{\mathcal{X}} \to \mathcal{X} \) be a finite-fold covering. There is the natural *-isomorphism \( M(A) \cong M\left(\tilde{\mathcal{X}} \mid \mathcal{X}\right) \) \( G \) of multiplier algebras.

**Proof.** For any \( x \in \mathcal{X} \) there is an open neighborhood \( U \) such that \( A(U) \cong C_0(U) \otimes \mathcal{K} \). Since \( \mathcal{X} \) is second-countable there is an enumerable family \( \{U_k\}_{k \in \mathbb{N}} \) such that \( A(U_k) \cong C_0(U_k) \otimes \mathcal{K} \) for any \( k \in \mathbb{N} \). There is a family \( \{a_k \in C_0(\mathcal{X})_+\}_{k \in \mathbb{N}} \) such that

- \( \text{supp } a_k \subset U_k \),
- \( 1_{C_b(\mathcal{X})} = \sum_{k=0}^{\infty} a_k \)

where sum of the series means the strict convergence (cf. Definition 1.35).

There is an enumerable family \( \{e_k \in \mathcal{K}\}_{k \in \mathbb{N}} \) of rank-one positive mutually orthogonal operators such that

\[
1_{M(\mathcal{K})} = \sum_{k=0}^{\infty} e_k
\]
where above sum assumes strict topology (cf. Definition 1.35). The family of products \( \{ u_{jk} = a_j \otimes e_k \}_{j,k \in \mathbb{N}} \) is enumerable and let us introduce an enumeration of \( \{ u_{jk} \}_{j,k \in \mathbb{N}} \), i.e. \( \{ u_{jk} \}_{j,k \in \mathbb{N}} = \{ u_p \}_{p \in \mathbb{N}} \). From (5.5) and (5.6) it follows that

\[
1_{M(A)} = \sum_{j=1}^{\infty} u_{jk} = \sum_{p=0}^{\infty} u_p. \tag{5.7}
\]

If \( h \in A \) is given by

\[
h = \sum_{p=0}^{\infty} \frac{1}{2^p} u_p
\]

and \( \tau : A \to \mathbb{C} \) is a state such that \( \tau (h) = 0 \) then from \( u_p > 0 \) for any \( p \in \mathbb{N} \) it follows that \( \tau (u_p) = 0 \) for any \( p \in \mathbb{N} \). However from (5.7) it turns out

\[
1 = \tau \left( 1_{M(A)} \right) = \tau \left( \sum_{p=0}^{\infty} u_p \right) = \sum_{p=0}^{\infty} \tau (u_p),
\]

and above equation contradicts with \( \tau (u_p) = 0 \) for any \( p \in \mathbb{N} \). It follows that \( \tau (h) \neq 0 \) for any state \( \tau \), i.e. \( h \) is strictly positive element of \( A \). Similarly one can prove that \( h \) is strictly positive element of \( \tilde{A} (\tilde{X}) \) because

\[
1_{M(A(\tilde{X}))} = \sum_{p=0}^{\infty} u_p.
\]

From the Proposition 5.15 it follows that there is the natural injective *-homomorphism \( f : M(A) \hookrightarrow M \left( \tilde{A} \left( \tilde{X} \right) \right) \). Clearly \( gf(a) = f(ga) = f(a) \) for any \( a \in A \) and \( g \in G \), it follows that \( f \left( M(A) \right) \subset M \left( \tilde{A} \right)^G \), or equivalently \( M(A) \subset M \left( \tilde{A} \right)^G \). Otherwise from the Lemma 2.18 one has \( M \left( \tilde{A} \right)^G \subset M(A) \). Taking into account mutually inverse inclusions \( M \left( \tilde{A} \right)^G \subset M(A) \) and \( M(A) \subset M \left( \tilde{A} \right)^G \) we conclude that

\[
M(A) \cong M \left( A \left( \tilde{X} \right) \right)^G.
\]

Lemma 5.17. Let \( A \) be a continuous trace algebra, and let \( \hat{A} = \mathfrak{X} \) be the spectrum of \( A \). Suppose that \( \mathfrak{X} \) is a locally compact second-countable Hausdorff space, and let \( \pi : \tilde{X} \to \mathfrak{X} \) be a finite-fold covering with compactification. Then the triple \( \left( A, A \left( \tilde{X} \right), G = G \left( \tilde{X} \mid \mathfrak{X} \right) \right) \) is a finite-fold noncommutative covering with compactification.
Proof. We need check conditions (a) - (c) of the Definition \[2.4\]

(a) There is the action of \(G\) on \(A \left(\tilde{X}\right)\) induced by the action of \(G\) on \(\tilde{X}\). From \[5.3\] it turns out that \(A = A \left(\tilde{X}\right)^G\) and there is an injective *-homomorphism \(A \hookrightarrow A \left(\tilde{X}\right)^G\). Denote by \(M\left(A\right)\) and \(M\left(A \left(\tilde{X}\right)\right)\) multiplier algebras of \(A\) and \(A \left(\tilde{X}\right)\). Denote by \(\mathcal{X} \rightarrow \mathcal{Y}\), \(\tilde{X} \rightarrow \tilde{Y}\) compactifications such that \(\tilde{\pi} : \tilde{Y} \rightarrow \mathcal{Y}\) is a (topological) finite covering and \(\pi = \tilde{\pi}|_{\tilde{X}}\). From \(C_b\left(\mathcal{X}\right) \subset M\left(A\right)\), \(C_b\left(\tilde{X}\right) \subset M\left(A \left(\tilde{X}\right)\right)\) and \(C\left(\mathcal{Y}\right) \subset C_b\left(\mathcal{X}\right)\), \(C\left(\tilde{Y}\right) \subset C_b\left(\tilde{X}\right)\) it follows that \(C\left(\mathcal{Y}\right) \subset M\left(A\right)\), \(C\left(\tilde{Y}\right) \subset M\left(A \left(\tilde{X}\right)\right)\). If \(B = C\left(\mathcal{Y}\right)M\left(A\right)\) and \(\tilde{B} = C\left(\tilde{Y}\right)M\left(A \left(\tilde{X}\right)\right)\) then \(\tilde{A}\) (resp. \(A \left(\tilde{X}\right)\)) is an essential ideal of \(\tilde{B}\) (resp. \(B\)). Clearly \(A = \tilde{B} \cap A \left(\tilde{X}\right)\).

(b) Since \(G\tilde{Y} = \tilde{Y}\) the action \(G \times M\left(A \left(\tilde{X}\right)\right) \rightarrow M\left(A \left(\tilde{X}\right)^G\right)\) induces an action \(G \times \tilde{B} \rightarrow \tilde{B}\). From the Lemma \[5.16\] on has the natural *-isomorphism \(M\left(A \left(\tilde{X}\right)^G\right) \cong M\left(A\right)\). It follows that \(B = C\left(\mathcal{Y}\right)M\left(A\right) \cong C\left(\mathcal{Y}\right)M\left(A \left(\tilde{X}\right)\right)^G = \tilde{B}^G\). From \[5.4\] it turns out that there is a finite family \(\{e_i \in C\left(\mathcal{Y}\right)\}_{i \in I}\) such that

\[
1_{C\left(\mathcal{Y}\right)} = 1_{\tilde{B}} = \sum_{i \in I} e_i b_i,
\]

It turns out that any \(\bar{b} \in \tilde{B}\) is given by

\[
\bar{b} = \sum_{i \in I} \bar{e}_i b_i,
\]

i.e. \(\tilde{B}\) is a finitely generated (by \(\{e_i\}_{i \in I}\)) right \(B\)-module. From the Kasparov Stabilization Theorem \[3\] it turns out that \(\tilde{B}\) is a projective \(B\)-module. So \(\left(\tilde{B}, \tilde{B}, G\right)\) is an unital finite-fold noncommutative covering.

(c) Follows from \(G\tilde{X} = \tilde{X}\). \(\square\)

**Theorem 5.18.** Let \(A\) be a continuous trace algebra, and let \(\tilde{A} = \tilde{X}\) be the spectrum of \(A\). Suppose that \(\tilde{X}\) is a locally compact second-countable Hausdorff space, and let \(\pi : \tilde{X} \rightarrow X\) be a finite-fold covering. Then the triple \((A, A\left(\tilde{X}\right), G = G \left(\tilde{X} \mid X\right))\) is a finite-fold noncommutative covering.

Proof. We need check (a), (b) of the Definition \[2.7\]

(a) Follows from \(\mathcal{X} = \tilde{X}/G\).

(b) Let us consider a family \(\{U_\lambda \subset \mathcal{X}\}_{\lambda \in \Lambda}\) of open sets such that

- \(\mathcal{X} = \bigcup_{\lambda \in \Lambda} U_\lambda\),
- The closure \(\overline{U_\lambda}\) of \(U_\lambda\) in \(\mathcal{X}\) is compact \(\forall \lambda \in \Lambda\).
Clearly \( \pi^{-1} (\mathcal{U}_\lambda) \to \mathcal{U}_\lambda \) is a covering, so \( \pi^{-1} (\mathcal{U}_\lambda) \to \mathcal{U}_\lambda \) is a covering with compactification. If \( \bar{I}_\lambda \equiv A (\pi^{-1} (\mathcal{U}_\lambda)) \subset A (\bar{X}) \) and \( I_\lambda = \bar{I}_\lambda \cap A \) then from \( G\pi^{-1} (\mathcal{U}_\lambda) = \pi^{-1} (\mathcal{U}_\lambda) \) it follows that

\[
G\bar{I}_\lambda = \bar{I}_\lambda, \quad I_\lambda = A (\mathcal{U}_\lambda),
\]

i.e. \( \bar{I}_\lambda \) satisfies to \( (2.3) \). From the Lemma \( 5.17 \) it follows that there is a finite-fold non-commutative covering with compactification \( (I_\lambda, \bar{I}_\lambda, G) = (A (\mathcal{U}_\lambda), A (\pi^{-1} (\mathcal{U}_\lambda)), G) \). From the Definition \( 5.12 \) and \( X = \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda \) it follows that \( \bigcup_{\lambda \in \Lambda} I_\lambda = \bigcup_{\lambda \in \Lambda} A (\mathcal{U}_\lambda) \) (resp. \( \bigcup_{\lambda \in \Lambda} \bar{I}_\lambda = \bigcup_{\lambda \in \Lambda} A (\pi^{-1} (\mathcal{U}_\lambda)) \)) is dense in \( A \) (resp. \( A (\bar{X}) \)).

**5.3 Infinite coverings**

Let \( A \) be a continuous trace \( C^* \)-algebra such that the spectrum \( \hat{A} = X \) of is a second-countable locally compact Hausdorff space. Suppose that

\[
\mathcal{S}_X = \{ X = X_0 \leftarrow ... \leftarrow X_n \leftarrow ... \}
\]

is a topological finite covering sequence. From the Theorem \( 5.18 \) it turns out that

\[
\mathcal{S}_{A(X)} = \{ A = A (X_0) \to ... \to A (X_n) \to ... \}
\]

is an algebraical finite covering sequence. If \( \hat{A} = \varinjlim A (X_n) \) then from the Theorem \( 4.4 \) it follows that there is the spectrum of \( \hat{A} \) is homeomorphic to \( \hat{X} = \varprojlim X_n \).

**Lemma 5.19.** If \( \mathcal{S}_X = \{ X = X_0 \leftarrow ... \leftarrow X_n \leftarrow ... \} \in \mathcal{F}\mathcal{Top} \) and \( \bar{X} \) is disconnected inverse limit of \( \mathcal{S}_X \), then there is the natural inclusion of \( \hat{A}'' \to A (\bar{X})'' \) of von Neumann enveloping algebras.

**Proof.** Surjective maps \( \bar{X} \to X_n \) give injective \(*\)-homomorphisms \( A (X_n) \hookrightarrow M (A (\bar{X})) \), which induce the injective \(*\)-homomorphism \( \hat{A} \hookrightarrow M (A (\bar{X})) \). It turns out the injective \(*\)-homomorphism of von Neumann enveloping algebras \( \hat{A}'' \to A (\bar{X})'' \). 

**5.20.** Denote by \( G_n = G (X_n \mid X) \) groups of covering transformations and \( \hat{G} = \varinjlim G_n \). Denote by \( \mathcal{P}_n : \bar{X} \to X_n, \pi^n : X_n \to X, \pi^m_n : X_m \to X_n (m > n) \) the natural covering projections.

**Lemma 5.21.** If \( \mathcal{U} \subset \bar{X} \) is an open subset mapped homeomorphically onto \( U \subset X \) then any positive element in \( \pi \in A (\mathcal{U})_+ \subset A (\bar{X})_+ \) is special.
Proof. If \( U_n = \pi_n (\overline{U}) \) then there is a *-isomorphism \( \overline{\varphi}_n : A (\overline{U}) \xrightarrow{\cong} A (U_n) \). For any \( n \in \mathbb{N}_0 \) and \( z \in A \) and \( f_\varepsilon \) given by (3.2)

\[
\begin{align*}
  a_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g \overline{a} = \overline{\varphi}_n (\overline{a}) , \\
  b_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g (z \overline{a} z^*) = z \overline{\varphi}_n (\overline{a}) z^* , \\
  c_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g (z \overline{a} z^*)^2 = (z \overline{\varphi}_n (\overline{a}) z^*)^2 , \\
  d_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g f_\varepsilon (z \overline{a} z^*) = f_\varepsilon (z \overline{\varphi}_n (\overline{a}) z^*) .
\end{align*}
\]

From the above equations it follows that \( a_n, b_n, c_n, d_n \in A (X_n) \) and \( b_n^2 = c_n \), i.e. \( \overline{a} \) satisfies to the Definition 3.5.

\[ \square \]

**Corollary 5.22.** If \( \overline{A} \) is the disconnected inverse noncommutative limit of \( \mathcal{S}_A (X^\prime) \), then \( A (\overline{X}) \subset \overline{A} \).

Proof. From the Lemma 5.21 it turns out \( A (\overline{U}) \subset \overline{A} \). However \( A (\overline{X}) \) is the \( C^* \)-norm completion of its subalgebras \( A (\overline{U}) \subset A (\overline{X}) \).

\[ \square \]

**Lemma 5.23.** If \( \overline{a} \in A (\overline{X})'' \) is a special element and \( z \in K (A) \) is an Abelian element then \( \overline{b} = \text{tr} (z \overline{a} z) \in C_0 (\overline{X}) \).

Proof. Any Abelian element is positive, hence \( z = z^* \). If \( f_\varepsilon \) is given by (3.2) and \( \overline{b} = z \overline{a} z \) then from (b) of the Definition 3.5 it turns out

\[
\begin{align*}
  b'_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g \overline{b} \in A (X_n) , \\
  c'_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g \overline{b}^2 \in A (X_n) , \\
  d'_n &= \sum_{g \in \ker (\hat{G} \to G_n)} g f_\varepsilon (\overline{b}) \in A (X_n) .
\end{align*}
\]

From \( z \in K (A) \) it turns out that \( \text{supp} \text{tr} (z) \) is compact. The map \( \pi_n : X_n \to X \) is a finite-fold covering, it turns out \( \pi_n^{-1} (\text{supp} \text{tr} (z)) \) is compact. If \( \text{supp} b'_n, \text{supp} c'_n, \text{supp} d'_n \subset \pi_n^{-1} (\text{supp} \text{tr} (z)) \) it turns out that all sets \( \text{supp} b'_n, \text{supp} c'_n, \text{supp} d'_n \) are compact. Taking into account that all \( b'_n, c'_n, d'_n \) are Abelian one has \( b'_n, c'_n, d'_n \in K (A (X_n)) \) where \( K (A (X_n)) \)
means the Pedersen ideal of $A (\mathcal{X}_n)$. It follows that
\[
b_n = \text{tr} (b'_n) = \sum_{g \in \ker (G \to G_n)} \text{tr} (g \overline{b'}) \in C_c (\mathcal{X}_n),
\]
\[
b_n^2 = \text{tr} \left( b_n^2 \right) = \text{tr} \left( (b'_n)^2 \right) \in C_c (\mathcal{X}_n),
\]
\[
c_n = \text{tr} (c'_n) = \sum_{g \in \ker (G \to G_n)} \text{tr} (g \overline{b'}^2) = \sum_{g \in \ker (G \to G_n)} \text{tr} \left( g \overline{b'}^2 \right) \in C_c (\mathcal{X}_n),
\]
\[
d_n = \text{tr} (d'_n) = \sum_{g \in \ker (G \to G_n)} \text{tr} \left( \left( g f_\varepsilon \left( \overline{b'} \right) \right) \right) = \sum_{g \in \ker (G \to G_n)} f_\varepsilon \left( \text{tr} \left( g \overline{b'} \right) \right) \in C_c (\mathcal{X}_n).
\]

From the above equations it follows that $b_n, c_n, d_n$ satisfy to the condition (a) of the Lemma 4.10. From the condition (c) the Definition 3.5 it follows that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for any $n \geq N$ following condition holds
\[
\left\| b_n^2 - c'_n \right\| < \varepsilon. \tag{5.8}
\]

Both $b'_n$ and $c'_n$ are Abelian and the range projection of $b'_n$ equals to the range projection of $c'_n$, i.e. $[b'_n] = [c'_n]$, it turns out
\[
\left\| b_n^2 - c_n \right\| = \left\| \text{tr} (b')^2 - \text{tr} (c') \right\| = \left\| b_n^2 - c'_n \right\|.
\]

From (5.8) it follows that $\left\| b_n^2 - c_n \right\| < \varepsilon$ for any $n \geq N$. It means that $b_n$ and $c_n$ satisfy to condition (b) of the Lemma 4.10. From the Lemma 4.10 it turns out that $\overline{b} = \text{tr} (z \overline{a} \overline{z}) \in C_0 (\overline{\mathcal{X}})$.

**Lemma 5.24.** If $\overline{A}$ is the disconnected inverse noncommutative limit of $\downarrow \mathcal{S}_A (\mathcal{X})$, then $\overline{A} = A (\overline{\mathcal{X}})$.

**Proof.** From the Corollary 5.22 it follows that $A (\overline{\mathcal{X}}) \subset \overline{A}$. From the Corollary 3.7 it follows that
\[
A (\overline{\mathcal{X}}) \subset \overline{A} \subset A (\overline{\mathcal{X}})''. \tag{5.9}
\]

Let $\overline{\pi} : \overline{\mathcal{X}} \to \mathcal{X}$ and let $\overline{\pi} \in A (\overline{\mathcal{X}})''$. Let $\overline{x} \in \overline{\mathcal{X}}$ be such that $\rho_\pi (\overline{\pi}) \neq 0$ and let $\overline{W}$ be an open neighborhood of $x$ such that $\overline{\pi}$ homeomorphically maps $\overline{W}$ onto $\mathcal{W} = \overline{\pi} (\overline{W})$. If $z \in K (A (\overline{\mathcal{X}}))$ is an Abelian element such that $\text{supp} \overline{z} \subset \overline{W}$ and $\rho_\pi (z \overline{\pi} \overline{z}) \neq 0$ then the element $\overline{z} = \sum_{g \in G} g \overline{z} \in A$ is Abelian and $\text{supp} \overline{z} \subset \mathcal{W}$. If $\overline{\pi}$ is special, then from the Lemma 5.23 it turns out that
\[
\text{tr} (z \overline{\pi} \overline{z}) \in C_0 (\overline{\mathcal{X}}).
\]

However from
\[
\rho_\pi (z \overline{\pi} \overline{z}) (\overline{\pi}) = \begin{cases} 
\rho_\pi (z \overline{a} \overline{z}) & \overline{x} \in \overline{W} \\
0 & \overline{x} \notin \overline{W}
\end{cases}
\]

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it turns out
\[
\begin{align*}
\text{tr} (\pi \varpi) &\in C_0 (\varpi), \\
\text{tr} (\pi \varpi) (x) &\neq 0.
\end{align*}
\] (5.10)

The set of special elements is dense in \( A \), it turns out that any \( a \in A \) satisfies to (5.10). Taking into account this fact and (5.9) it turns out

• \( A (\mathcal{X}) \subset A \subset A (\mathcal{X})'' \),

• For any \( \pi \in \overline{\mathcal{X}} \) and \( x \in \overline{\mathcal{X}} \) such that \( \rho (\pi) \neq 0 \) there is an open neighborhood \( \mathcal{W} \subset \overline{\mathcal{X}} \) of \( x \) and an Abelian \( \varpi \in A (\overline{\mathcal{X}}) \) such that

\[
\text{supp} \varpi \subset \mathcal{W},
\]

\[
\text{tr} (\pi \varpi) \in C_0 (\overline{\mathcal{X}}),
\]

\[
\text{tr} (\pi \varpi) (x) \neq 0.
\]

From the Lemma 5.14 it follows that \( A = A (\overline{\mathcal{X}}) \).

5.25. Let \( \overline{\mathcal{X}} \subset \overline{\mathcal{X}} \) be a connected component and let \( G \subset G \left( \lim \right. C_0 (\mathcal{X}_n) | C_0 (\mathcal{X}) \left. \right) \) be maximal subgroup among subgroups \( G' \subset G \left( \lim \right. C_0 (\mathcal{X}_n) | C_0 (\mathcal{X}) \left. \right) \) such that \( G' \overline{\mathcal{X}} = \overline{\mathcal{X}} \). If \( J \subset \hat{G} \) is a set of representatives of \( \hat{G} / G \) then from the (1.2) it follows that

\[
\overline{\mathcal{X}} = \bigcup_{g \in J} g \overline{\mathcal{X}}.
\]

and the algebraic direct sum
\[
\bigoplus_{g \in J} A (g \overline{\mathcal{X}}) \subset A (\overline{\mathcal{X}}).
\] (5.11)

is a dense subalgebra of \( A (\overline{\mathcal{X}}) \).

**Theorem 5.26.** Let \( A \) be \( C^* \)-algebra of continuous trace, and let \( \mathcal{X} \) be the spectrum of \( A \). Let

\[
\mathcal{S}_X = \{ \mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \} \in \text{FinTop}
\]

be a topological finite covering sequence, and let

\[
\mathcal{S}_{A(\mathcal{X})} = \{ A = A (\mathcal{X}_0) \rightarrow ... \rightarrow A (\mathcal{X}_n) \rightarrow ... \} \in \text{FinAlg}
\]

be an algebraical finite covering sequence. Following conditions hold:

(i) \( \mathcal{S}_{A(\mathcal{X})} \) is good,

(ii) There are isomorphisms:

\[
\begin{align*}
\lim \downarrow \mathcal{S}_{A(\mathcal{X})} &\approx A \left( \lim \downarrow \mathcal{S}_X \right), \\
G \left( \lim \downarrow \mathcal{S}_{A(\mathcal{X})} \mid A \right) &\approx G \left( \lim \downarrow \mathcal{S}_X \mid X \right).
\end{align*}
\]

**Proof.** Similar to the proof of the Theorem 4.13.
6 Noncommutative tori and their coverings

6.1 Fourier transformation

There is a norm on $\mathbb{Z}^n$ given by

$$\|(k_1, \ldots, k_n)\| = \sqrt{k_1^2 + \ldots + k_n^2}. \quad (6.1)$$

The space of complex-valued Schwartz functions on $\mathbb{Z}^n$ is given by

$$S(\mathbb{Z}^n) = \left\{ a = \{a_k\}_{k \in \mathbb{Z}^n} \in C^{\infty}(\mathbb{Z}^n) \mid \sup_{k \in \mathbb{Z}^n} (1 + \|k\|^s |a_k| < \infty, \forall s \in \mathbb{N}\right\}. \quad (6.3)$$

Let $T^n$ be an ordinary $n$-torus. We will often use real coordinates for $T^n$, that is, view $T^n$ as $\mathbb{R}^n/\mathbb{Z}^n$. Let $C^\infty(T^n)$ be an algebra of infinitely differentiable complex-valued functions on $T^n$. There is the bijective Fourier transformations $\mathcal{F}_T : C^\infty(T^n) \xrightarrow{\approx} S(\mathbb{Z}^n); f \mapsto \hat{f}$ given by

$$\hat{f}(p) = \mathcal{F}_T(f)(p) = \int_{T^n} e^{-2\pi i x \cdot p} f(x) \, dx \quad (6.2)$$

where $dx$ is induced by the Lebesgue measure on $\mathbb{R}^n$ and $\cdot$ is the scalar product on the Euclidean space $\mathbb{R}^n$. The Fourier transformation carries multiplication to convolution, i.e.

$$\hat{f} \hat{g}(p) = \sum_{r+s=p} \hat{f}(r) \hat{g}(s).$$

The inverse Fourier transformation $\mathcal{F}_T^{-1} : S(\mathbb{Z}^n) \xrightarrow{\approx} C^\infty(T^n); \hat{f} \mapsto f$ is given by

$$f(x) = \mathcal{F}_T^{-1} \hat{f}(x) = \sum_{p \in \mathbb{Z}^n} \hat{f}(p) e^{2\pi i x \cdot p}. \quad (6.2)$$

There is the $\mathbb{C}$-valued scalar product on $C^\infty(T^n)$ given by

$$(f, g) = \int_{T^n} f g \, dx = \sum_{p \in \mathbb{Z}^n} \hat{f}(-p) \hat{g}(p).$$

Denote by $S(\mathbb{R}^n)$ be the space of complex Schwartz (smooth, rapidly decreasing) functions on $\mathbb{R}^n$.

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{a,\beta} < \infty \quad \forall a = (a_1, \ldots, a_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n \right\}, \quad \|f\|_{a,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^a D^\beta f(x) \right| \quad (6.3)$$

where

$$x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad D^\beta = \frac{\partial}{\partial x_1^{\beta_1}} \cdots \frac{\partial}{\partial x_n^{\beta_n}}.$$

The topology on $S(\mathbb{R}^n)$ is given by seminorms $\| \cdot \|_{a,\beta}$. 

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**Definition 6.1.** Denote by $S' (\mathbb{R}^n)$ the vector space dual to $S (\mathbb{R}^n)$, i.e. the space of continuous functionals on $S (\mathbb{R}^n)$. Denote by $\langle \cdot, \cdot \rangle : S' (\mathbb{R}^n) \times S (\mathbb{R}^n) \to \mathbb{C}$ the natural pairing. We say that $\{a_n \in S' (\mathbb{R}^n)\}_{n \in \mathbb{N}}$ is weakly-* convergent to $a \in S' (\mathbb{R}^n)$ if for any $b \in S (\mathbb{R}^n)$

$$\lim_{n \to \infty} \langle a_n, b \rangle = \langle a, b \rangle .$$

We say that

$$a = \lim_{n \to \infty} a_n$$

in the sense of weak-* convergence.

Let $F$ and $F^{-1}$ be the ordinary and inverse Fourier transformations given by

$$\left( Ff \right) (u) = \int_{\mathbb{R}^n} f(t) e^{-2\pi it \cdot u} dt, \quad \left( F^{-1} f \right) (u) = \int_{\mathbb{R}^n} f(t) e^{2\pi it \cdot u} dt \quad (6.4)$$

which satisfy following conditions

$$F \circ F^{-1} |_{S (\mathbb{R}^n)} = F^{-1} \circ F |_{S (\mathbb{R}^n)} = \text{Id}_{S (\mathbb{R}^n)}.$$

There is the $\mathbb{C}$-valued scalar product on $S (\mathbb{R}^n)$ given by

$$\langle f, g \rangle_{L^2 (\mathbb{R}^n)} = \int_{\mathbb{R}^n} f g dx = \int_{\mathbb{R}^n} Ff Fg dx. \quad (6.5)$$

which if $F$-invariant, i.e.

$$\langle f, g \rangle_{L^2 (\mathbb{R}^n)} = \langle Ff, Fg \rangle_{L^2 (\mathbb{R}^n)} . \quad (6.6)$$

There is the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ such that

$$g x = x + g; \ x \in \mathbb{R}^n, \ g \in \mathbb{Z}^n$$

and $\mathbb{T}^n \approx \mathbb{R}^n / \mathbb{Z}^n$. For any $x \in \mathbb{R}^n$ and $C \in \mathbb{R}$ the series

$$\sum_{k \in \mathbb{Z}^n} \frac{C}{1 + |x + k|^{n+1}}$$

is convergent, and taking into account (6.3) one concludes that for $f \in S (\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the series

$$\sum_{g \in \mathbb{Z}^n} D^\beta f (x + g) (x) = \sum_{g \in \mathbb{Z}^n} \left( g D^\beta f \right) (x)$$

is absolutely convergent. It follows that the series

$$\tilde{h} = \sum_{g \in \mathbb{Z}^n} g f$$
is point-wise convergent and $\tilde{h}$ is a smooth $\mathbb{Z}^n$-invariant function. The periodic smooth function $\tilde{h}$ corresponds to an element of $\tilde{h} \in C^\infty (T^n)$. This construction provides a map

$$S(\mathbb{R}^n) \to C^\infty (T^n), \quad f \mapsto \sum_{g \in \mathbb{Z}^n} g f.$$  

(6.7)

If $\mathcal{U} = (0, 1)^n \subset \mathbb{R}^n$ is a fundamental domain of the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ then $\tilde{h}_\mathcal{U}$ can be represented by the Fourier series

$$\tilde{h}_\mathcal{U} (x) = \sum_{p \in \mathbb{Z}^n} c_p e^{2\pi i px}, \quad c_p = \int_{\mathcal{U}} \tilde{h} (x) e^{-2\pi i px} \, dx = \sum_{g \in \mathbb{Z}^n} \int_{\mathcal{U}} f (x + g) e^{-2\pi i px} \, dx = \int_{\mathbb{R}^n} f (x) e^{-2\pi i px} \, dx = \hat{f} (p)$$

where $\hat{f} = \mathcal{F} f$ is the Fourier transformation of $f$. So if $\hat{h} = \mathcal{F}^T h$ is the Fourier transformation of $h$ then for any $p \in \mathbb{Z}^n$ a following condition holds

$$\hat{h} (p) = \hat{f} (p).$$  

(6.8)

### 6.2 Noncommutative torus $T^n_\Theta$

Denote by $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the scalar product on the Euclidean vector space $\mathbb{R}^n$. Let $\Theta$ be a real skew-symmetric $n \times n$ matrix, we will define a new noncommutative product $\ast_\Theta$ on $S (\mathbb{Z}^n)$ given by

$$\left( \hat{f} \ast_\Theta \hat{g} \right) (p) = \sum_{r + s = p} \hat{f} (r) \hat{g} (s) e^{-\pi i r \cdot \Theta s}.$$  

(6.9)

and an involution

$$\hat{f}^* (p) = \hat{f} (-p).$$

In result there is an involutive algebra $C^\infty (T^n_\Theta) = (S (\mathbb{Z}^n), +, \ast_\Theta, \ast)$. There is a tracial state on $C^\infty (T^n_\Theta)$ given by

$$\tau (f) = \hat{f} (0).$$  

(6.10)

From $C^\infty (T^n_\Theta) \approx S (\mathbb{Z}^n)$ it follows that there is a $C$-linear isomorphism

$$\varphi_\infty : C^\infty (T^n_\Theta) \xrightarrow{\approx} C^\infty (T^n).$$  

(6.11)

such that following condition holds

$$\tau (f) = \frac{1}{(2\pi)^n} \int_{T^n} \varphi_\infty (f) \, dx.$$  

(6.12)
Similarly to 1.46 there is the Hilbert space $L^2 \left( C^\infty \left( T^n_\Theta \right), \tau \right)$ and the natural representation $C^\infty \left( T^n_\Theta \right) \to B \left( L^2 \left( C^\infty \left( T^n_\Theta \right), \tau \right) \right)$ which induces the $C^*$-norm. The $C^*$-norm completion $C \left( T^n_\Theta \right)$ of $C^\infty \left( T^n_\Theta \right)$ is a $C^*$-algebra and there is a faithful representation

$$C \left( T^n_\Theta \right) \to B \left( L^2 \left( C^\infty \left( T^n_\Theta \right), \tau \right) \right). \quad (6.13)$$

We will write $L^2 \left( C \left( T^n_\Theta \right), \tau \right)$ instead of $L^2 \left( C^\infty \left( T^n_\Theta \right), \tau \right)$. There is the natural $C$-linear map $C^\infty \left( T^n_\Theta \right) \to L^2 \left( C \left( T^n_\Theta \right), \tau \right)$ and since $C^\infty \left( T^n_\Theta \right) \approx S \left( Z^n \right)$ there is a linear map $\Psi : S \left( Z^n \right) \to L^2 \left( C \left( T^n_\Theta \right), \tau \right)$. If $k \in Z^n$ and $U_k \in S \left( Z^n \right) = C^\infty \left( T^n_\Theta \right)$ is such that

$$U_k \left( p \right) = \delta_{kp} : \forall p \in Z^n \quad (6.14)$$

then

$$U_k U_p = e^{-\pi ik \cdot \Theta p} U_{k+p}, \quad U_k U_p = e^{-2\pi ik \cdot \Theta p} U_p U_k. \quad (6.15)$$

If $\xi_k = \Psi \left( U_k \right)$ then from (6.9), (6.10) it turns out

$$\tau \left( U_k^* + \Theta U_l \right) = \left( \xi_k, \xi_l \right) = \delta_{kl}, \quad (6.16)$$

i.e. the subset $\{ \xi_k \}_{k \in Z^n} \subset L^2 \left( C \left( T^n_\Theta \right), \tau \right)$ is an orthogonal basis of $L^2 \left( C \left( T^n_\Theta \right), \tau \right)$. Hence the Hilbert space $L^2 \left( C \left( T^n_\Theta \right), \tau \right)$ is naturally isomorphic to the Hilbert space $l^2 \left( Z^n \right)$ given by

$$l^2 \left( Z^n \right) = \left\{ \xi = \{ \xi_k \in C \}_{k \in Z^n} \in C^{Z^n} \mid \sum_{k \in Z^n} |\xi_k|^2 < \infty \right\}$$

and the $C$-valued scalar product on $l^2 \left( Z^n \right)$ is given by

$$\left( \xi, \eta \right)_{l^2 \left( Z^n \right)} = \sum_{k \in Z^n} \overline{\xi_k} \eta_k.$$ 

An alternative description of $C \left( T^n_\Theta \right)$ is such that if

$$\Theta = \begin{pmatrix} 0 & \theta_{12} & \ldots & \theta_{1n} \\ \theta_{21} & 0 & \ldots & \theta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \ldots & 0 \end{pmatrix} \quad (6.17)$$

then $C \left( T^n_\Theta \right)$ is the universal $C^*$-algebra generated by unitary elements $u_1, \ldots, u_n \in U \left( C \left( T^n_\Theta \right) \right)$ such that following condition holds

$$u_j u_k = e^{-2\pi i \delta_{jk}} u_k u_j. \quad (6.18)$$

Elements $u_j$ are given by

$$u_j = U_j, \quad j = 0, \ldots, 1, \ldots, 0.$$
**Definition 6.2.** Unitary elements \( u_1, \ldots, u_n \in U \left( C \left( T^n_{\Theta} \right) \right) \) which satisfy the relation (6.18) are said to be generators of \( C \left( T^n_{\Theta} \right) \). The set \( \{ U_l \}_{l \in \mathbb{Z}^n} \) is said to be the basis of \( C \left( T^n_{\Theta} \right) \).

If \( a \in C \left( T^n_{\Theta} \right) \) is presented by a series
\[
a = \sum_{l \in \mathbb{Z}^n} c_l U_l; \quad c_l \in \mathbb{C}
\]
and the series \( \sum_{l \in \mathbb{Z}^n} |c_l| \) is convergent then from the triangle inequality it follows that
\[
\|a\| \leq \sum_{l \in \mathbb{Z}^n} |c_l|.
\]

**Definition 6.3.** If \( \Theta \) is non-degenerated, that is to say, \( \sigma(s, t) \overset{\text{def}}{=} s \cdot \Theta t \) to be symplectic. This implies even dimension, \( n = 2N \). Then one selects
\[
\Theta = \theta J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}
\]
where \( \theta > 0 \) is defined by \( \theta^{2N} \overset{\text{def}}{=} \det \Theta \). Denote by \( C^\infty \left( T^n_{\Theta} \right) \overset{\text{def}}{=} C^\infty \left( T_{\Theta}^{2N} \right) \) and \( C \left( T^n_{\Theta} \right) \overset{\text{def}}{=} C \left( T_{\Theta}^{2N} \right) \).

### 6.3 Finite-fold coverings

In this section we write \( ab \) instead \( a \ast \Theta b \). Let \( \Theta \) be given by (6.17), and let \( C \left( T^n_{\Theta} \right) \) be a noncommutative torus. If \( (k_1, \ldots, k_n) \in \mathbb{N}^n \) and
\[
\tilde{\Theta} = \begin{pmatrix} 0 & \tilde{\theta}_{12} & \cdots & \tilde{\theta}_{1n} \\ \tilde{\theta}_{21} & 0 & \cdots & \tilde{\theta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\theta}_{n1} & \tilde{\theta}_{n2} & \cdots & 0 \end{pmatrix}
\]
is a skew-symmetric matrix such that
\[
e^{-2\pi i \tilde{\Theta}_{rs}} = e^{-2\pi i \tilde{\theta}_{rs}},
\]
then there is a *-homomorphism \( C \left( T^n_{\Theta} \right) \rightarrow C \left( T^n_{\Theta} \right) \) given by
\[
u_j \mapsto v_j^{k_j}; \quad j = 1, \ldots, n
\]
where \( u_1, \ldots, u_n \in C \left( T^n_{\Theta} \right) \) (resp. \( v_1, \ldots, v_n \in C \left( T^n_{\Theta} \right) \)) are unitary generators of \( C \left( T^n_{\Theta} \right) \) (resp. \( C \left( T^n_{\Theta} \right) \)). There is an involutive action of \( G = \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n} \) on \( C \left( T^n_{\Theta} \right) \) given by
\[
\left( \overline{\nu}_1, \ldots, \overline{\nu}_n \right) v_j = e^{-2\pi i \nu_j^{k_j}} v_j,
\]
and a following condition holds \( C\left(T^n_\Theta\right) = C\left(T^n_\tilde{\Theta}\right)^G \). Otherwise there is a following \( C\left(T^n_\Theta\right) \)-module isomorphism

\[
C\left(T^n_\Theta\right) = \bigoplus_{\{\nu_i\} \in \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n}} v^{p_1}_1 \cdot \ldots \cdot v^{p_n}_n C\left(T^n_\Theta\right) \approx C\left(T^n_\Theta\right)^{k_1 \ldots k_n}
\]

i.e. \( C\left(T^n_\Theta\right) \) is a finitely generated projective Hilbert \( C\left(T^n_\Theta\right) \)-module. It turns out the following theorem.

**Theorem 6.4.** The triple \( \left(C\left(T^n_\Theta\right), C\left(T^n_\tilde{\Theta}\right), \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n}\right) \) is an unital noncommutative finite-fold covering.

### 6.4 Moyal plane and a representation of the noncommutative torus

**Definition 6.5.** Denote the Moyal plane product \( \star_\theta \) on \( \mathcal{S}\left(\mathbb{R}^{2N}\right) \) given by

\[
(f \star_\theta h)(u) = \int_{y \in \mathbb{R}^{2N}} f\left(u - \frac{1}{2} \Theta y\right) g\left(u + v\right) e^{2\pi i y \cdot v} dy dv
\]

where \( \Theta \) is given by (6.20).

**Definition 6.6.** Denote by \( \mathcal{S}'\left(\mathbb{R}^n\right) \) the vector space dual to \( \mathcal{S}\left(\mathbb{R}^n\right) \), i.e. the space of continuous functionals on \( \mathcal{S}\left(\mathbb{R}^n\right) \). The Moyal product can be defined, by duality, on larger sets than \( \mathcal{S}\left(\mathbb{R}^{2N}\right) \). For \( T \in \mathcal{S}'\left(\mathbb{R}^{2N}\right) \), write the evaluation on \( g \in \mathcal{S}\left(\mathbb{R}^{2N}\right) \) as \( \langle T, g \rangle \in \mathbb{C} \); then, for \( f \in \mathcal{S} \) we may define \( T \star_\theta f \) and \( f \star_\theta T \) as elements of \( \mathcal{S}'\left(\mathbb{R}^{2N}\right) \) by

\[
\langle T \star_\theta f, g \rangle \overset{\text{def}}{=} \langle T, f \star_\theta g \rangle \\
\langle f \star_\theta T, g \rangle \overset{\text{def}}{=} \langle T, g \star_\theta f \rangle \\
\]

(6.22)

using the continuity of the star product on \( \mathcal{S}\left(\mathbb{R}^{2N}\right) \). Also, the involution is extended to by \( \langle T^*, g \rangle \overset{\text{def}}{=} \langle T, g^* \rangle \).

**Remark 6.7.** It is proven in [7] that the domain of the Moyal plane product can be extended up to \( L^2\left(\mathbb{R}^{2N}\right) \).

**Lemma 6.8.** [7] If \( f, g \in L^2\left(\mathbb{R}^{2N}\right) \), then \( f \star_\theta g \in L^2\left(\mathbb{R}^{2N}\right) \) and \( \| f \|_\text{op} < (2\pi \theta)^{-\frac{N}{2}} \| f \|_2 \).

where \( \| \cdot \|_2 \) is the \( L^2 \)-norm given by

\[
\| f \|_2 \overset{\text{def}}{=} \left| \int_{\mathbb{R}^{2N}} |f|^2 dx \right|^{\frac{1}{2}}.
\]

(6.23)

and the operator norm \( \| T \|_\text{op} \overset{\text{def}}{=} \sup \{ \| T \ast g \|_2 / \| g \|_2 : 0 \neq g \in L^2\left(\mathbb{R}^{2N}\right) \} \).
**Definition 6.9.** Denote by $\mathcal{S}(\mathbb{R}^2)$ (resp. $L^2(\mathbb{R})^2$) the operator algebra which is $C$-linearly isomorphic to $\mathcal{S}(\mathbb{R}^2)$ (resp. $L^2(\mathbb{R})^2$) and product coincides with $\ast_g$. Both $\mathcal{S}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ act on the Hilbert space $\mathcal{L}(\mathbb{R}^2)$. Denote by

$$\Psi : \mathcal{S}(\mathbb{R}^2) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}^2)$$

(6.24)

the natural $C$-linear isomorphism.

6.10. There is the tracial property [7] of the Moyal product

$$\int_{\mathbb{R}^2} (f \ast_g g) (x) \, dx = \int_{\mathbb{R}^2} f (x) g (x) \, dx.$$  

(6.25)

The Fourier transformation of the star product satisfies to the following condition.

$$\mathcal{F}(f \ast_g g) (x) = \int_{\mathbb{R}^2} \mathcal{F}f (x - y) \mathcal{F}g (y) e^{\pi iy \cdot \Theta x} \, dy.$$  

(6.26)

**Definition 6.11.** [7] Let $\mathcal{S}'(\mathbb{R}^2)$ be a vector space dual to $\mathcal{S}(\mathbb{R}^2)$. Denote by $C_b(\mathbb{R}^2) \overset{\text{def}}{=} \{ T \in \mathcal{S}'(\mathbb{R}^2) : T \ast_g g \in L^2(\mathbb{R}^2) \text{ for all } g \in L^2(\mathbb{R}^2) \}$, provided with the operator norm

$$\|T\|_{\text{op}} \overset{\text{def}}{=} \sup \{ \|T \ast_g g\|_2 / \|g\|_2 : 0 \neq g \in L^2(\mathbb{R}^2) \}.$$  

(6.27)

Denote by $C_0(\mathbb{R}^2)$ the operator norm completion of $\mathcal{S}(\mathbb{R}^2)$.

**Remark 6.12.** Obviously $\mathcal{S}(\mathbb{R}^2) \hookrightarrow C_b(\mathbb{R}^2)$. But $\mathcal{S}(\mathbb{R}^2)$ is not dense in $C_b(\mathbb{R}^2)$, i.e. $C_0(\mathbb{R}^2) \not\subseteq C_b(\mathbb{R}^2)$ (cf. [7]).

**Remark 6.13.** $L^2(\mathbb{R}^2)$ is the $\| \cdot \|_2$ norm completion of $\mathcal{S}(\mathbb{R}^2)$ hence from the Lemma

6.8 it follows that

$$L^2(\mathbb{R}^2) \subseteq C_0(\mathbb{R}^2).$$  

(6.28)

**Remark 6.14.** Notation of the Definition 6.11 differs from [7]. Here symbols $A_\Theta, A_\theta, A_0^\Theta$ are replaced with $C_b(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2), C_0(\mathbb{R}^2)$ respectively.

**Remark 6.15.** The $C_0(\mathbb{R}^2)$ does not isomorphic to $C_0(\mathbb{R}^2)$ (cf. [7]).

6.16. [7] By plane waves we understand all functions of the form

$$x \mapsto \exp(ik \cdot x)$$

for $k \in \mathbb{R}^2$. One obtains for the Moyal product of plane waves:

$$\exp(ik \cdot) \ast_\Theta \exp(ik \cdot) = \exp(ik \cdot) \ast_\Phi \exp(ik \cdot) = \exp(i(k + l) \cdot) e^{-\pi i k \cdot \Theta l}$$

(6.29)

**Remark 6.17.** [7] The algebra $C_b(\mathbb{R}^2)$ contains all plane waves.

**Remark 6.18.** If $\{c_k \in \mathbb{C}\}_{k \in \mathbb{N}_0}$ is such that $\sum_{k=0}^{\infty} |c_k| < \infty$ then from $\| \exp(ik \cdot) \|_{\text{op}} = 1$ it turns out $\| \sum_{k=0}^{\infty} \exp(ik \cdot) \|_{\text{op}} < \sum_{k=0}^{\infty} |c_k| < \infty$, i.e. $\sum_{k=0}^{\infty} c_k \exp(ik \cdot) \in C_b(\mathbb{R}^2)$. 

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6.19. The equation (6.29) is similar to the equation (6.15) which defines $C(T^2_{\theta})$. From this fact and from the Remark 6.18 it follows that there is an injective *-homomorphism $C^\infty (T^2_{\theta}) \hookrightarrow C_b (R^2_{\theta})$. An algebra $C^\infty (T^2_{\theta})$ is dense in $C(T^2_{\theta})$ so there is an injective *-homomorphism $C(T^2_{\theta}) \hookrightarrow C_b (R^2_{\theta})$. The faithful representation $C_b (R^2_{\theta}) \to B (L^2 (R^2))$ gives a representation $\pi : C(T^2_{\theta}) \to B (L^2 (R^2))$

$$\pi : C(T^2_{\theta}) \to B \left( L^2 \left( R^2 \right) \right),$$

(6.30)

where $U_k \in C(T^2_{\theta})$ is given by the Definition 6.2.

6.20. Let us consider the unitary dilation operators $E_a$ given by

$$E_a f (x) \overset{\text{def}}{=} a^{N/2} f \left( a^{1/2} x \right).$$

It is proven in [7] that

$$f \circ g = (\theta/2)^{-N/2} E_{\theta/2} f \circ E_{\theta/2} g.$$

(6.31)

We can simplify our construction by setting $\theta = 2$. Thanks to the scaling relation (6.31) any qualitative result can be true if it is true in case of $\theta = 2$. We use the following notation

$$f \times g \overset{\text{def}}{=} f \circ g$$

(6.32)

Lemma 6.21. Let $a, b \in S (R^2)$. For any $\Delta \in R^2$ let $a_\Delta \in S (R^2)$ be such that $a_\Delta (x) = a (x + \Delta)$. For any $m \in N$ there is a constant $C_{a,b}^m$ such that

$$\| a_\Delta \times b \|_2 \leq \frac{C_{a,b}^m}{\left( 1 + \| \Delta \| \right)^m}$$

where $\| \cdot \|_2$ is given by (6.23).

Proof. From the definition of Schwartz functions it follows that for any $f \in S (R^2)$ and any $m \in N$ there is $C_{m}^f > 0$ such that

$$\| f (u) \| < \frac{C_{m}^f}{\left( 1 + \| u \| \right)^m}.$$  

(6.33)

From (6.26) it follows that

$$\mathcal{F} (a_\Delta \times b) (x) = \int_{R^2} F_{a_\Delta} (x - y) F_{b} (y) e^{\pi i y \cdot \Theta x} dy = \int_{R^2} c (y - \Delta - x) d (y) e^{\pi i y \cdot \Theta x} dy$$

where $c (x) = \mathcal{F} a (-x)$, $d (x) = \mathcal{F} b (x)$ If $\xi = \mathcal{F} (a_\Delta \times b)$ then $\xi \in L^2 \left( R^2 \right)$. Let $\xi = \xi_1 + \xi_2$ where $\xi_1, \xi_2 \in L^2 \left( R^2 \right)$ are given by

$$\xi_1 (x) = \begin{cases} \mathcal{F} (a_\Delta b) (x) & \| x \| \leq \frac{\| \Delta \|}{2} \\ 0 & \| x \| > \frac{\| \Delta \|}{2} \end{cases}$$

(6.34)

$$\xi_2 (x) = \begin{cases} 0 & \| x \| \leq \frac{\| \Delta \|}{2} \\ \mathcal{F} (a_\Delta b) (x) & \| x \| > \frac{\| \Delta \|}{2} \end{cases}.$$
From (6.33) it turns out

\[ |\xi_1(x)| \leq \int \left| c(t-\Delta-x) d(t) e^{-\frac{\|t\|}{M}} \right| dt \leq \int_{\mathbb{R}^{2N}} \frac{C_M}{(1+\|t-\Delta-x\|)^M} \frac{C_{2M}^d}{(1+\|t\|)^M} dt = \int_{\mathbb{R}^{2N}} \frac{C_M^*}{(1+\|t-\Delta-x\|)^M} \frac{C_{2M}^{d*}}{1+\|t\|^M} dt \leq \sup_{x \in \mathbb{R}^{2N}, \|x\| \leq \frac{\|\Delta\|}{2}, s \in \mathbb{R}^{2N}} \frac{C_M^* C_{2M}^{d*}}{(1+\|s-\Delta-x\|)^M} \frac{C_M^*}{(1+\|s\|)^M} \int_{\mathbb{R}^{2N}} \frac{1}{(1+\|t\|)^M} dt. \]  

(6.35)

If \( x, y \in \mathbb{R}^{2N} \) then from the triangle inequality it follows that \( \|x+y\| > \|y\| - \|x\| \), hence

\[ (1+\|x\|)^M (1+\|x+y\|)^M \geq (1+\|x\|)^M (1+\max(0,\|y\| - \|x\|))^M. \]

If \( \|x\| \leq \frac{\|y\|}{2} \) then \( \|y\| - \|x\| \geq \frac{\|y\|}{2} \) and

\[ (1+\|x\|)^M (1+\|x+y\|)^M > \left( \frac{\|y\|}{2} \right)^M. \]  

(6.36)

Clearly if \( \|x\| > \frac{\|y\|}{2} \) then condition (6.36) also holds, hence (6.35) is always true. Clearly if \( \|x\| > \frac{\|y\|}{2} \) then condition 6.36 also holds, hence (6.36) is always true. It turns out from \( \|\Delta-x\| > \frac{\|\Delta\|}{2} \) and (6.36) that

\[ \inf_{x \in \mathbb{R}^{2N}, \|x\| \leq \frac{\|\Delta\|}{2}, s \in \mathbb{R}^{2N}} \frac{C_M^*}{(1+\|s-\Delta-x\|)^M} \frac{C_{2M}^{d*}}{(1+\|s\|)^M} > \left( \frac{\|\Delta\|}{4} \right)^M, \]

hence from (6.35) it turns out

\[ |\xi_1(x)| \leq \frac{4MC_M^* C_{2M}^{d*}}{\|\Delta\|^M} \frac{1}{(1+\|t\|)^M} dt \]

There is the well known integral

\[ \int_{x \in \mathbb{R}^{2N}, \|x\| \leq \frac{\|\Delta\|}{2}} \frac{1}{\Gamma(N+1)} \left( \frac{\|\Delta\|}{2} \right)^{2N} dx, \]

where \( \Gamma \) is the Euler gamma function. If \( M > 2N \) then the integral \( C' = \int_{\mathbb{R}^{2N}} \frac{1}{(1+\|t\|)^M} dt \) is convergent, it turns out

\[ |\xi_1|^2 \leq \left( \frac{4MC_M^* C_{2M}^{d*}}{\|\Delta\|^M} \right)^2 \int_{x \in \mathbb{R}^{2N}, \|x\| \leq \frac{\|\Delta\|}{2}} \frac{1}{\Gamma(N+1)} \left( \frac{\|\Delta\|}{2} \right)^{2N} dx = \frac{4MC_M^* C_{2M}^{d*}}{\|\Delta\|^M} \frac{1}{\Gamma(N+1)} \left( \frac{\|\Delta\|}{2} \right)^{2N}. \]

If \( M = 2N + m \) then from the above equation it turns out that there is \( C_1 > 0 \) such that

\[ |\xi_1|^2 \leq \frac{C_1}{\|\Delta\|^m}. \]  

(6.37)
If \((\cdot, \cdot)_{L^2(\mathbb{R}^N)}\) is the given by (6.5) scalar product then from (6.6) it turns out

\[
|\xi_2(x)| \leq \left| \int c(t - \Delta - x) d(t) e^{\pi ix \Theta t} dt \right| \leq \left| \left( c(\Theta \cdot - \Delta - x) , d(\Theta \cdot) e^{\pi ix \Theta \cdot} \right)_{L^2(\mathbb{R}^N)} \right| = \\
= \left| \left( \mathcal{F}(c(\Theta \cdot - \Delta - x)) , \mathcal{F}(d(\Theta \cdot)) e^{\pi i x \Theta \cdot} \right)_{L^2(\mathbb{R}^N)} \right| = \\
= \left| \int_{\mathbb{R}^N} \mathcal{F}(c)(\Theta \cdot - \Delta - x) (u) \mathcal{F}(d(\Theta \cdot)) (u + \pi \Theta x) du \right| \leq \\
\leq \int_{\mathbb{R}^N} \frac{C^{\mathcal{F}(c)}_{\cdot M}}{(1 + ||u||)^{3M}} \frac{C^{\mathcal{F}(d)}_{\cdot M}}{(1 + ||u - \pi \Theta x||)^{2M}} |\Delta|^M \frac{1}{M} du.
\]

Since we consider the asymptotic dependence \(\|\Delta\| \to \infty\) only large values of \(\|\Delta\|\) are interesting, so we can suppose that \(\|\Delta\| > 2\). If \(\|\Delta\| > 2\) then from \(||x|| > \frac{\|\Delta\|}{2}\) it follows that \(\|\pi \Theta x\| > 1\), and from (6.36) it follows that

\[
\inf_{x \in \mathbb{R}^N, ||x|| > \frac{\|\Delta\|}{2}} \frac{1}{(1 + ||x||)^{2M}} > \frac{\|\pi \Theta x\|}{4},
\]

hence

\[
|\xi_2(x)| \leq \frac{C^{\mathcal{F}(c)}_{\cdot M}}{\|\pi \Theta x\|^M} \int_{\mathbb{R}^N} \frac{1}{(1 + ||u||)^M} du.
\]

If \(m \geq 1\) and \(M = 2N + m\) then the integral \(C' \int_{\mathbb{R}^N} \frac{1}{(1 + ||u||)^M} du\) is convergent and

\[
|\xi_2(x)| \leq \frac{C^{\mathcal{F}(c)}_{\cdot M}}{\|\pi \Theta x\|^M}.
\]

Taking into account (6.20) and \(\theta = 2\) one has

\[
\|\Theta z\| = \|2z\|; \forall z \in \mathbb{R}^{2N}.
\]
It follows that
\[ |\xi_1|^2 \leq \int_{x \in \mathbb{R}^{2N}, \|x\| > \|x\|_{\Delta}} \left( \frac{C_{2M}(c) C_{2M}(d)}{2\pi \|\Delta\|^M \|2\pi x\|^M} \right)^2 \, dx. \]

Since above integral is convergent one has there is a constant \( C_2 \) such that
\[ |\xi_2|^2 \leq C_2 \left( \frac{\pi^{\frac{1}{2}}}{\pi} \right)^{2M} \]
(6.39)

Since \( \xi_1 \perp \xi_2 \) one has \( |\xi|^2 = |\xi_1|^2 + |\xi_2|^2 \) and taking into account (6.37), (6.39) it follows that for any \( m \in \mathbb{N} \) there is \( C_m > 0 \) such that
\[ \|\xi\|^2 = \|F(a_\Delta \times b)\|^2 \leq \frac{C_m^{a,b}}{(1 + \|\Delta\|)^m}. \]

From (6.6) it turns out
\[ \|a_\Delta \times b\|^2 = \|F(a_\Delta \times b)\|^2 \leq \frac{C_m^{a,b}}{(1 + \|\Delta\|)^m}. \]

\[ \square \]

**Proposition 6.22.** [7] The algebra \( \mathcal{S}(\mathbb{R}^{2N}, \star_\theta) \) has the (nonunique) factorization property: for all \( h \in \mathcal{S}(\mathbb{R}^{2N}) \) there exist \( f, g \in \mathcal{S}(\mathbb{R}^{2N}) \) that \( h = f \star_\theta g \).

**Lemma 6.23.** Following conditions hold:

(i) Let \( \{a_n \in C_b(\mathbb{R}^{2N}_\theta)\}_{n \in \mathbb{N}} \) be a sequence such that

- \( \{a_n\} \) is weakly-* convergent (cf. Definition 6.1),
- If \( a = \lim_{n \to \infty} a_n \) in the sense of weak-* convergence then \( a \in C_b(\mathbb{R}^{2N}_\theta) \).

Then the sequence \( \{a_n\} \) is convergent in sense of weak topology \( \{a_n\} \) (cf. Definition 1.37) and \( a \) is limit of \( \{a_n\} \) with respect to the weak topology. Moreover if \( \{a_n\} \) is increasing or decreasing sequence of self-adjoint elements then \( \{a_n\} \) is convergent in sense of strong topology (cf. Definition 1.36) and \( a \) is limit of \( \{a_n\} \) with respect to the strong topology.

(ii) If \( \{a_n\} \) is strongly and/or weakly convergent (cf. Definitions 1.36, 1.37) and \( a = \lim_{n \to \infty} a_n \) is strong and/or weak limit then \( \{a_n\} \) is weakly-* convergent and \( a \) is the limit of \( \{a_n\} \) in the sense of weakly-* convergence.

**Proof.** (i) If \( \langle \cdot, \cdot \rangle : \mathcal{S}'(\mathbb{R}^{2N}) \times \mathcal{S}(\mathbb{R}^{2N}) \to \mathbb{C} \) is the natural pairing then one has
\[ \lim_{n \to \infty} \langle a_n, b \rangle = \langle a, b \rangle \quad \forall b \in \mathcal{S}(\mathbb{R}^{2N}). \] (6.40)
Let $\xi, \eta \in L^2(\mathbb{R}^{2N})$ and let $\{x_j \in \mathcal{S}(\mathbb{R}^{2N})\}_{j \in \mathbb{N}}$, $\{y_j \in \mathcal{S}(\mathbb{R}^{2N})\}_{j \in \mathbb{N}}$ be sequences such that there are following limits

$$
\lim_{j \to \infty} x_j = \xi, \quad \lim_{j \to \infty} y_j = \eta
$$

(6.41)
in the topology of the Hilbert space $L^2(\mathbb{R}^{2N})$. If $(\cdot, \cdot) : L^2(\mathbb{R}^{2N}) \times L^2(\mathbb{R}^{2N}) \to \mathbb{C}$ is the Hilbert pairing then from (6.22), (6.41) it follows that

$$(a_n \xi, \eta) = \lim_{j \to \infty} \langle a_n x_j, y_j \rangle = \lim_{j \to \infty} \langle a_n x_j, y_j \rangle = \lim_{j \to \infty} \langle a_n, x_j * \theta y_j \rangle = (a_n \xi, \eta),$$

(6.42)
hence, taking into account (6.40) one has

$$
\lim_{n \to \infty} (a_n \xi, \eta) = \lim_{n \to \infty} \lim_{j \to \infty} (a_n x_j, y_j) = \lim_{n \to \infty} \lim_{j \to \infty} (a_n, x_j * \theta y_j) = \lim_{j \to \infty} \langle a_n, x_j * \theta y_j \rangle = (a_n, b)
$$

(6.43)
i.e. $\{a_n\}$ is weakly convergent to $a$. If $\{a_n\}$ is an increasing sequence then $a_n \leq a$ for any $n \in \mathbb{N}$ and from the Lemma [1.43] it turns out that $\{a_n\}$ is strongly convergent. Clearly the strong limit coincides with the weak one. Similarly one can prove that $\{a_n\}$ is an decreasing then $\{a_n\}$ is strongly convergent.

(ii) If $b \in \mathcal{S}(\mathbb{R}^{2N})$ then from the Proposition 6.22 it follows that $b = x * \theta y$ where $x, y \in \mathcal{S}(\mathbb{R}^{2N})$. The sequence $\{a_n\}$ is strongly and/or weakly convergent it turns out that

$$(x, a_n * \theta y) = \langle a_n, x * \theta y \rangle = \langle a_n, b \rangle$$

is convergent. Hence $\{a_n\}$ is weakly-* convergent.

There are elements $f_{mn} \in \mathcal{S}(\mathbb{R}^2)$ which have very useful properties. To present $f_{mn}$ explicitly, we use polar coordinates $q + ip = re^{i\alpha}$, where $p, q \in \mathbb{R}^2$ and $\rho = (p, q) \in \mathbb{R}^2$

Note that $\|\rho\|^2 = |p|^2 + |q|^2$.

$$
f_{mn} = 2 (-1)^n \sqrt{\frac{1}{m!}} \epsilon^{i\alpha(m-n)} \|\rho\|^{|m-n|} L_n^{m-n} \left(\|\rho\|^2\right) e^{-\|\rho\|^2/2},
$$

$$
f_{mn} (\rho, \alpha) = 2 (-1)^n L_n \left(\|\rho\|^2\right) e^{-\|\rho\|^2/2}
$$

where $L_n^{m-n}, L_n$ are Laguerre functions. From this properties it follows that $C_0(\mathbb{R}^2)$ is the $C^*$-norm completion of linear span of $f_{mn}$ (cf. [3]).

**Lemma 6.24.** [8] Let $m, n, k, l \in \mathbb{N}$. Then $f_{mn} * \theta f_{kl} = \delta_{mk} f_{ml}$ and $f_{mn} = f_{nm}$. Thus $f_{mn}$ is an orthogonal projection and $f_{mn}$ is nilpotent for $m \neq n$. Moreover, $\langle f_{mn}, f_{kl} \rangle = 2^N \delta_{mk} \delta_{nl}$. The family $\{f_{mn} : m, n \in \mathbb{N} \} \subset \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ is an orthogonal basis.

**Proposition 6.25.** [78] Let $N = 1$. Then $\mathcal{S}(\mathbb{R}^{2N}) = \mathcal{S}(\mathbb{R}^2)$ has a Fréchet algebra isomorphism with the matrix algebra of rapidly decreasing double sequences $c = (c_{mn})$ such that, for each $k \in \mathbb{N}$,

$$
r_k(c) = \left( \sum_{m, n=0}^\infty \theta_{2k} \left(m + \frac{1}{2}\right)^k \left(n + \frac{1}{2}\right)^k |c_{mn}|^2 \right)^{1/2}
$$

(6.43)
is finite, topologized by all the seminorms \((r_k)\); via the decomposition \(f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}\) of \(S(\mathbb{R}^2)\) in the \(\{f_{mn}\}\) basis. The twisted product \(f \ast \theta g\) is the matrix product \(ab\), where

\[
(ab)_{mn} \overset{\text{def}}{=} \sum_{k=0}^{\infty} a_{mk} b_{kn}.
\] (6.44)

For \(N > 1\), \(C^\infty(\mathbb{R}_\theta^{2N})\) is isomorphic to the (projective) tensor product of \(N\) matrix algebras of this kind, i.e.

\[
S(\mathbb{R}_\theta^{2N}) \cong S(\mathbb{R}_\theta^2) \otimes \cdots \otimes S(\mathbb{R}_\theta^2),
\] (6.45)

with the projective topology induced by seminorms \(r_k\) given by (6.43).

**Remark 6.26.** If \(A\) is \(C^*\)-norm completion of the matrix algebra with the norm (6.43) then \(A \cong \mathcal{K}\), i.e.

\[
C_0(\mathbb{R}_\theta^2) \cong \mathcal{K}.
\] (6.46)

Form (6.45) and (6.46) it follows that

\[
\underbrace{C_0(\mathbb{R}_\theta^2)}_{N\text{-times}} \cong \underbrace{\mathcal{K} \otimes \cdots \otimes \mathcal{K}}_{N\text{-times}} \approx \mathcal{K} \otimes \cdots \otimes \mathcal{K} \approx \mathcal{K}
\] (6.47)

where \(\otimes\) means minimal or maximal tensor product (\(\mathcal{K}\) is nuclear hence both products coincide).

### 6.5 Infinite coverings

Let us consider a sequence

\[
\mathfrak{G}_{C(T_\theta^0)} = \left\{ C\left( T_\theta^0 \right) = C\left( T_\theta^0 \right) \overset{\pi^1}{\rightarrow} \ldots \overset{\pi^j}{\rightarrow} C\left( T_\theta^0 \right) \overset{\pi^{j+1}}{\rightarrow} \ldots \right\}
\] (6.48)

of finite coverings of noncommutative tori. The sequence (6.48) satisfies to the Definition 3.1 i.e. \(\mathfrak{G}_{C(T_\theta^0)} \in \mathfrak{FinAlg}\).

**6.27.** Let \(\Theta = J\theta\) where \(\theta \in \mathbb{R}\setminus \mathbb{Q}\) and

\[
J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.
\]

Denote by \(C(T_\theta^{2N}) \overset{\text{def}}{=} C(T_\Theta^{2N})\). Let \(\{p_k \in \mathbb{N}\}_k \in \mathbb{N}\) be an infinite sequence of natural numbers such that \(p_k > 1\) for any \(k\), and let \(m_j = \prod_{k=1}^{j} p_k\). From the (6.3) it follows that there is a sequence of *-homomorphisms

\[
\mathfrak{G}_\theta = \left\{ C\left( T_\theta^{2N} \right) \rightarrow C\left( T_\theta^{2N} / m_j^2 \right) \rightarrow C\left( T_\theta^{2N} / m_j^2 \right) \rightarrow \ldots \right\}
\] (6.49)

such that...
(a) For any $j \in \mathbb{N}$ there are generators $u_{j-1,1}, \ldots, u_{j-1,2N} \in U \left( C \left( T^{2N}_{\theta/m_j^2} \right) \right)$ and generators $u_{j,1}, \ldots, u_{j,2N} \in U \left( C \left( T^{2N}_{\theta/m_j} \right) \right)$ such that the $*$-homomorphism $C \left( T^{2N}_{\theta/m_j^2} \right) \to C \left( T^{2N}_{\theta/m_j} \right)$ is given by $u_{j-1,k} \mapsto u_{j,k'}^{p_j} \quad \forall k = 1, \ldots, 2N.$ There are generators $u_1, \ldots, u_{2N} \in U \left( C \left( T^{2N}_{\theta} \right) \right)$ such that $*$-homomorphism $C \left( T^{2N}_{\theta} \right) \to C \left( T^{2N}_{\theta/m_j^2} \right)$ is given by $u_j \mapsto u_{1,j}^{p_j} \quad \forall j = 1, \ldots, 2N,$

(b) For any $j \in \mathbb{N}$ the triple $\left( C \left( T^{2N}_{\theta/m_j^2} \right), C \left( T^{2N}_{\theta/m_j} \right), Z_{p_j} \right)$ is a noncommutative finite-fold covering.

(c) There is the sequence of groups and epimorphisms

$$\mathbb{Z}_{m_1}^{2N} \twoheadrightarrow \mathbb{Z}_{m_2}^{2N} \twoheadrightarrow \cdots$$

which is equivalent to the sequence

$$G \left( C \left( T^{2N}_{\theta/m_1^2} \right) \mid C \left( T^{2N}_{\theta/m_1} \right) \right) \leftarrow G \left( C \left( T^{2N}_{\theta/m_2^2} \right) \mid C \left( T^{2N}_{\theta/m_2} \right) \right) \leftarrow \cdots$$

The sequence (6.49), is a specialization of (6.48), hence $\mathcal{G}_\theta \in \mathfrak{Nil}_{\mathfrak{Alg}}.$ Denote by $C \left( T^{2N}_{\theta/m_j^2} \right) \defeq \lim_{\longrightarrow} C \left( T^{2N}_{\theta/m_j} \right), \quad \mathcal{G} \defeq \lim_{\leftarrow} G \left( C \left( T^{2N}_{\theta/m_j^2} \right) \mid C \left( T^{2N}_{\theta/m_j} \right) \right).$ The group $\mathcal{G}$ is Abelian because it is the inverse limit of Abelian groups. Denote by $0_{\mathcal{G}}$ (resp. "$+"") the neutral element of $\mathcal{G}$ (resp. the product operation of $\mathcal{G}$).

6.28. For any $\tilde{a} \in \mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)$ from (6.7) it turns out that the series

$$a_j = \sum_{g \in \ker \left( \mathbb{Z}^{2N} \rightarrow \mathbb{Z}_{m_j}^{2N} \right)} \tilde{a}$$

is point-wise convergent and $a_j$ satisfies to following conditions:

- $a_j \in \mathcal{S}' \left( \mathbb{R}^{2N} \right),$
- $a_j$ is a smooth $m_j$ - periodic function.

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It follows that the above series is weakly-* convergent (cf. Definition 6.1) and from the Lemma 6.23 it turns out that the series is weakly convergent. From (6.8) it follows that

\[ a_j = \sum_{k \in \mathbb{Z}^{2N}} c_k \exp \left( 2\pi i \frac{k}{m_j} \cdot \right) \]

where \( \{c_k \in \mathbb{C} \}_{k \in \mathbb{Z}^{2N}} \) are rapidly decreasing coefficients given by

\[ c_k = \frac{1}{m_j^{2N}} \int_{\mathbb{R}^{2N}} \tilde{a}(x) \exp \left( 2\pi i \frac{k}{m_j} \cdot x \right) dx = \frac{1}{m_j^{2N}} \mathcal{F}\tilde{a} \left( \frac{k}{m_j} \right). \quad (6.50) \]

On the other hand

\[ \tilde{a} = \lim_{j \to \infty} a_j \quad (6.51) \]

in sense of weakly-* convergence, and from the Lemma 6.23 it follows that (6.51) is a limit in sense of the weak topology.

**Lemma 6.29.** Let \( G_j = \ker \left( \mathbb{Z}^{2N} \to \mathbb{Z}^{2N}_{m_j} \right) \). Let \( \tilde{a} \in S \left( \mathbb{R}^{2N}_b \right) \) and let

\[ a_j = \sum_{g \in G_j} g\tilde{a} \quad (6.52) \]

where the sum the series means weakly-* convergence. Following conditions hold:

(i) \( a_j \in C^\infty (\mathbb{R}^{2N}) \),

(ii) The series (6.52) is convergent with respect to the strong topology (cf. Definition 1.36),

(iii) There is a following strong limit

\[ \tilde{a} = \lim_{j \to \infty} a_j. \quad (6.53) \]

**Proof.** (i) From (6.50) it turns out that

\[ a_j = \sum_{k \in \mathbb{Z}^{2N}} c_k U_k \]

where \( \{c_k\} \) is a rapidly decreasing sequence, hence \( a_j \in C^\infty (\mathbb{R}^{2N}) \).

(ii) From the Lemma 6.23 it turns out that the series

\[ c = \sum_{g \in \mathbb{Z}^{2N}} g (\tilde{a}^* \tilde{a}) \]

is strongly convergent, and the series (6.52) is weakly convergent. If \( k = \max \left( 1, \sqrt{\|c\|} \right) \) then for any \( \eta \in L^2 (\mathbb{R}^{2N}) \) and any subset \( G \subset \mathbb{Z}^{2N} \) following condition holds

\[ \left\| \left( \sum_{g \in G} g\tilde{a} \right) \eta \right\|_2 \leq k \|\eta\|_2 \]
that

Proof. (iii) If \( \eta \in L^2(\mathbb{R}^{2N}) \) then for any \( \epsilon > 0 \) there is \( \bar{b} \in S(\mathbb{R}^{2N}) \) such that

\[
\| \eta - \bar{b} \|_2 < \frac{\epsilon}{2k} \quad (6.54)
\]

From the Lemma 6.21 it follows that for any \( m \in \mathbb{N} \) there is a constant \( C_m > 0 \) such that

\[
\| (g\bar{a}) \bar{b} \|_2 < \frac{C_m}{(1 + \| g \|)^m}; \quad \forall g \in \mathbb{Z}^{2N} \quad (6.55)
\]

where \( \| g \| \) is given by (6.1). If \( m > 2N \) then there is \( M \in \mathbb{N} \) such that if \( G_0 = \{-M, \ldots, M\}^{2N} \subset \mathbb{Z}^{2N} \) such that

\[
\sum_{g \in \mathbb{Z}^{2N} \setminus G_0} \frac{C_m}{(1 + \| g \|)^m} < \frac{\epsilon}{2} \quad (6.56)
\]

It follows that

\[
\left\| \left( \sum_{\gamma \in \mathcal{G}_j} \bar{a} - \sum_{\gamma \in \mathcal{G}_j \cap G_0} \bar{a} \right) \bar{b} \right\|_2 = \left\| \left( \sum_{\gamma \notin \mathcal{G}_j \setminus (\mathcal{G}_j \cap G_0)} \bar{a} \right) \bar{b} \right\|_2 < \sum_{g \notin \mathcal{G}_j \setminus (\mathcal{G}_j \cap G_0)} \frac{C_m}{(1 + \| g \|)^m} < \frac{\epsilon}{2}.
\]

Otherwise from (6.54)-(6.56) one has

\[
\left\| \left( \sum_{\gamma \in \mathcal{G}_j} \bar{a} - \sum_{\gamma \in \mathcal{G}_j \cap G_0} \bar{a} \right) \xi \right\|_2 < \left\| \left( \sum_{\gamma \notin \mathcal{G}_j \setminus (\mathcal{G}_j \cap G_0)} \bar{a} \right) \bar{b} \right\|_2 + \left\| \left( \sum_{\gamma \in \mathcal{G}_j \setminus G_0} \bar{a} \right) (\xi - \bar{b}) \right\|_2 < \frac{\epsilon}{2} + k \| \xi - \bar{b} \|_2 < \epsilon.
\]

Above equation means that the series (6.52) is strongly convergent.

(iii) If \( j \in \mathbb{N} \) is such that \( m_j > M \) then

\[
\| (a_j - \bar{a}) \xi \|_2 = \| \sum_{\gamma \in \mathcal{G}_j} g\bar{a} - \bar{a} \|_2 = \| \sum_{\gamma \notin \mathcal{G}_j \setminus \{0\}} g\bar{a} \|_2 \]

where 0 is the neutral element of \( \mathbb{Z}^{2N} \). However from \( m_j > M \) it turns out \( G_0 \cap (\mathcal{G}_j \setminus \{0\}) = \emptyset \), so from (6.54)-(6.56) one has

\[
\| (a_j - \bar{a}) \xi \|_2 < \| (a_j - \bar{a}) \bar{b} \|_2 + k \| \xi - \bar{b} \|_2 < \left\| \sum_{\gamma \in \mathcal{G}_j} g\bar{a} - \bar{a} \right\|_2 + \frac{\epsilon}{2} < \epsilon.
\]

Above equation means that there is the strong limit (6.53).

\[\square\]

**Corollary 6.30.** Any \( \bar{a} \in S(\mathbb{R}^{2N}) \) lies in \( C(\mathbb{T}^{2N}_\theta)^\prime \).

**Proof.** There is a strong limit (6.52), i.e. \( \bar{a} = \lim_{j \to \infty} a_j \). For any \( j \in \mathbb{N} \) one has \( a_j \in C(\mathbb{T}^{2N}_\theta) \) it turns out \( \bar{a} = \lim_{j \to \infty} a_j \in C(\mathbb{T}^{2N}_\theta)^\prime \).

\[\square\]
6.5.1 Equivariant representation

Denote by \( \{ U_{\theta}^{g/mj} \} \) the basis of \( C(T^{2N}_\theta) \). Similarly to (6.30) there is the representation \( \pi_j : C(T^{2N}_\theta) \rightarrow B(L^2(R^{2N})) \) given by

\[
\pi_j(U_{\theta}^{g/mj}) = \exp \left( 2\pi i \frac{k}{m_j} \cdot g \right).
\]

There is a following commutative diagram.

\[
\begin{array}{ccc}
C(T^{2N}_\theta) & \xrightarrow{\pi_j} & C(T^{2N}_\theta) \\
\downarrow & & \downarrow \\
B(L^2(R^{2N})) & \xrightarrow{\pi_j+1} & B(L^2(R^{2N}))
\end{array}
\]

This diagram defines a faithful representation \( \hat{\pi} : C(T^{2N}_\theta) \rightarrow B(L^2(R^{2N})) \). There is the action of \( \mathbb{Z}^{2N} \times R^{2N} \rightarrow R^{2N} \) given by

\[
(k, x) \mapsto k + x.
\]

The action naturally induces the action of \( \mathbb{Z}^{2N} \) on both \( L^2(R^{2N}) \) and \( B(L^2(R^{2N})) \). Otherwise the action of \( \mathbb{Z}^{2N} \) on \( B(L^2(R^{2N})) \) induces the action of \( \mathbb{Z}^{2N} \) on \( C(T^{2N}_\theta) \). There is the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}^{2N} & \rightarrow & G \left( C(T^{2N}_\theta) \mid C(T^{2N}_\theta) \right) \\
\downarrow & & \downarrow \\
G_j = G \left( C(T^{2N}_\theta) \mid C(T^{2N}_\theta) \right) \approx \mathbb{Z}^{2N}_{m_j}
\end{array}
\]

From the above diagram it follows that there is the natural homomorphism \( \mathbb{Z}^{2N} \rightarrow \hat{G} \), and \( \mathbb{Z}^{2N} \) is a normal subgroup. Let \( J \subset \hat{G} \) be a set of representatives of \( \hat{G}/\mathbb{Z}^{2N} \), and suppose that \( 0_{\hat{G}} \in J \). Any \( g \in \hat{G} \) can be uniquely represented as \( g = g_j + g_z \) where \( g \in J \), \( g_z \in \mathbb{Z}^{2N} \). For any \( g_1, g_2 \in \hat{G} \) denote by \( \Phi_j(g_1, g_2) \in J \), \( \Phi_z(g_1, g_2) \in \mathbb{Z}^{2N} \), such that

\[
g_1 + g_2 = \Phi_j(g_1, g_2) + \Phi_z(g_1, g_2).
\]

Let us define an action of \( \hat{G} \) on \( \bigoplus_{g \in J} L^2(R^{2N}) \) given by

\[
g_1 \left( 0, ..., \underbrace{\xi}_{\text{\( g_2 \)th place}}, ..., 0, ... \right) = \left( 0, ..., \underbrace{\Phi_z(g_1, g_2) \xi}_{\text{\( g_1 \)th place}}, ..., 0, ... \right).
\]
Let $X \subset \bigoplus_{g \in J} L^2(\mathbb{R}^{2N})$ be given by

$$X = \left\{ \eta \in \bigoplus_{g \in J} L^2(\mathbb{R}^{2N}) \mid \eta = \left( 0, \ldots, \zeta, \ldots, 0, \ldots \right) \right\}.$$

Taking into account that $X \approx L^2(\mathbb{R}^{2N})$, we will write $L^2(\mathbb{R}^{2N}) \subset \bigoplus_{g \in J} L^2(\mathbb{R}^{2N})$ instead of $X \subset \bigoplus_{g \in J} L^2(\mathbb{R}^{2N})$. This inclusion and the action of $\hat{G}$ on $\bigoplus_{g \in J} L^2(\mathbb{R}^{2N})$ enable us write $\bigoplus_{g \in J} g L^2(\mathbb{R}^{2N})$ instead of $\bigoplus_{g \in J} L^2(\mathbb{R}^{2N})$. If $\hat{\pi}^{\oplus} : C(\mathbb{T}_{\theta}^{2N}) \to B\left( \bigoplus_{g \in J} g L^2(\mathbb{R}^{2N}) \right)$ is given by

$$\hat{\pi}^{\oplus} (a) (g \xi) = g \left( \hat{\pi} \left( g^{-1}a \right) \xi \right); \quad \forall a \in \mathcal{C}(\mathbb{T}_{\theta}^{2N}), \quad \forall g \in J, \quad \forall \xi \in L^2(\mathbb{R}^{2N})$$

then $\hat{\pi}^{\oplus}$ is an equivariant representation.

### 6.5.2 Inverse noncommutative limit

If $\tilde{a} \in \mathcal{S}(\mathbb{R}^{2N})$ then from the Corollary 6.30 it turns out $\tilde{a} \in \mathcal{C}(\mathbb{T}_{\theta}^{2N})''$. Since $\hat{\pi}^{\oplus}$ is a faithful representation of $\mathcal{C}(\mathbb{T}_{\theta}^{2N})$, one has an injective homomorphism $\mathcal{S}(\mathbb{R}^{2N}) \hookrightarrow \hat{\pi}^{\oplus} \left( \mathcal{C}(\mathbb{T}_{\theta}^{2N}) \right)''$ of involutive algebras.

For any $\tilde{a} \in \mathcal{S}(\mathbb{R}^{2N})$ following condition holds

$$\sum_{\tilde{g} \in \ker(G \to G)} g \hat{\pi}^{\oplus} (\tilde{a}) = \sum_{g' \in J} g' \left( \sum_{g'' \in \ker(\mathbb{Z}^{2N} \to G)} g'' \hat{\pi} (\tilde{a}) \right) = \sum_{g \in J} g P.$$

where

$$P = \sum_{\tilde{g} \in \ker(\mathbb{Z}^{2N} \to G)} g \hat{\pi} (\tilde{a}).$$

If $J \subset G$ is a set of representatives of $G/\mathbb{Z}^{2N}$ and $g', g'' \in J$ are such that $g' \neq g''$ then operators $g' P, g'' P$ act on mutually orthogonal Hilbert subspaces $g' L^2(\mathbb{R}^{2N}), g'' L^2(\mathbb{R}^{2N})$ of the direct sum $\bigoplus_{g \in J} g L^2(\mathbb{R}^{2N})$, and taking into account $\| P \| = \| g P \|$ one has

$$\left\| \sum_{\tilde{g} \in \ker(G \to G)} \hat{\pi}^{\oplus} (\tilde{a}) \right\| = \left\| \sum_{\tilde{g} \in \ker(G \to G)} \hat{\pi} (\tilde{a}) \right\| = \left\| \sum_{g \in J} g P \right\| = \left\| \sum_{\tilde{g} \in \ker(\mathbb{Z}^{2N} \to G)} \hat{\pi} (\tilde{a}) \right\|. \quad (6.57)$$

**Lemma 6.31.** Let $a \in \mathcal{S}(\mathbb{R}^{2N})$, and let $a_{\Delta} \in \mathcal{S}(\mathbb{R}^{2N})$ be given by

$$a_{\Delta} (x) = a(x + \Delta); \quad \forall x \in \mathbb{R}^{2N} \quad (6.58)$$
where $\Delta \in \mathbb{R}^{2N}$. For any $m \in \mathbb{N}$ there is a dependent on a real constant $C_m > 0$ such that for any $j \in \mathbb{N}$ following condition holds

$$
\left\| \sum_{g \in \ker(G \to G_j)} \hat{\pi}^j (a_\Delta a) \right\| \leq \frac{C_m}{\|\Delta\|^m}.
$$

**Proof.** From (6.57) it follows that

$$
\left\| \sum_{g \in \ker(G \to G_j)} \hat{\pi}^j (a_\Delta a) \right\| = \left\| \sum_{g \in \ker(\mathbb{Z}^{2N} \to \mathbb{Z}_j^{2N})} \hat{\pi}^j (a_\Delta a) \right\|.
$$

From (6.53) it follows that for any $f \in \mathcal{S}(\mathbb{R}^{2N})$ and any $m \in \mathbb{N}$ there is $C_m$ such that

$$
|f(u)| < \frac{C_m}{(1 + |u|)^m}.
$$

Let $M = 2N + 1 + m$. From (6.19) and (6.50) it follows that

$$
\left\| \sum_{g \in \ker(\mathbb{Z}^{2N} \to \mathbb{Z}_j^{2N})} g \hat{\pi}^j (a_\Delta a) \right\| \leq \frac{1}{m_j^N} \sum_{l \in \mathbb{Z}^{2N}} \left| \mathcal{F} (a_\Delta a) \left( \frac{1}{m_j} \right) \right|.
$$

Otherwise from (6.26) it follows that

$$
\mathcal{F} (a_\Delta a) (x) = \int_{\mathbb{R}^{2N}} \mathcal{F} a_\Delta (x - y) \mathcal{F} a (y) e^{\pi i y \cdot \Theta x} dy.
$$

From the above equations it turns out

$$
\frac{1}{m_j^N} \sum_{l \in \mathbb{Z}^{2N}} \left| \mathcal{F} (a_\Delta a) \left( \frac{1}{m_j} \right) \right| = \\
= \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}} \left| \int \mathcal{F} a \left( \frac{1}{m_j} + \Delta - t \right) \mathcal{F} a (t) e^{\pi i \Theta t} dt \right| \leq \\
\leq \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}} \int \left| b \left( t - \Delta - \frac{l}{m_j} \right) c (t) e^{\pi i \Theta t} \right| dt = \\
= \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \left\| l \right\| \leq \frac{\|\Delta\|}{m_j}} \int \left| b \left( t - \Delta - \frac{l}{m_j} \right) c (t) e^{\pi i \Theta t} \right| dt + \\
+ \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \left\| l \right\| > \frac{\|\Delta\|}{m_j}} \int \left| b \left( t - \Delta - \frac{l}{m_j} \right) c (t) e^{\pi i \Theta t} \right| dt
$$

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where \( b(u) = F a(-u) \), \( c(u) = F a(u) \). From (6.33) it turns out

\[
\frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \int \left| \frac{b \left( t - \Delta - \frac{l}{m_j} \right) c(t) e^{\frac{\|\Theta\|}{2}}}{M} \right| dt \leq
\]

\[
\leq \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \int_{\mathbb{R}^{2N}} \frac{C_b^M}{\left( 1 + \|t - \Delta - \frac{l}{m_j}\| \right)^M (1 + \|t\|)^M} \frac{C_{cM}^2}{dt} \leq
\]

\[
= \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \int_{\mathbb{R}^{2N}} \frac{C_b^M}{\left( 1 + \|t - \Delta - \frac{l}{m_j}\| \right)^M (1 + \|t\|)^M} \frac{C_{cM}^2}{dt} \leq
\]

\[
\leq \frac{N_{m_j}^\Delta}{m_j^{2N}} \sup_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \frac{C_b^M C_{cM}^2}{\left( 1 + \|s - \Delta - \frac{l}{m_j}\| \right)^M (1 + \|s\|)^M} \times \int_{\mathbb{R}^{2N}} \frac{1}{(1 + \|t\|)^M} dt.
\]

where \( N_{m_j}^\Delta = \left\{ l \in \mathbb{Z}^{2N} \mid \frac{l}{m_j} < \frac{\|\Delta\|}{2} \right\} \). The number \( N_{m_j}^\Delta \) can be estimated as a number of points with integer coordinates inside \( 2N \)-dimensional cube

\[
N_{m_j}^\Delta < \|m_j\|^{2N}.
\]

From \( M > 2N \) it turns out the integral \( \int_{\mathbb{R}^{2N}} \frac{1}{(1 + \|t\|)^M} dt \) is convergent, hence

\[
\frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \int \left| \frac{b \left( t - \Delta - \frac{l}{m_j} \right) c(t) e^{\frac{\|\Theta\|}{2}}}{M} \right| dt \leq
\]

\[
\leq C_1^M \sup_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}} \frac{\|\Delta\|^{2N}}{\left( 1 + \|s - \Delta - \frac{l}{m_j}\| \right)^M (1 + \|s\|)^M} \int_{\mathbb{R}^{2N}} \frac{1}{(1 + \|t\|)^M} dt.
\]

where

\[
C_1^M = C_b^M C_{cM}^2 \int_{\mathbb{R}^{2N}} \frac{1}{(1 + \|t\|)^M} dt.
\]

It turns out from the (6.36) that

\[
\inf_{l \in \mathbb{Z}^{2N}, \frac{l}{m_j} < \frac{\|\Delta\|}{2}, s \in \mathbb{R}^{2N}} \left( 1 + \|s - \Delta - \frac{l}{m_j}\| \right)^M (1 + \|s\|)^M > \left\| \frac{\Delta}{4} \right\|^M.
\]
From $M = 2N + 1 + m$ it turns out

$$
\frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \left| \int b \left( t - \Delta - \frac{1}{m_j} \right) c(t) e^{\frac{\pi i}{m_j} \cdot \Theta t} dt \right| \leq C_1 \Delta^{-m}
$$

where $C_1 = C_1' / 4^m$. Clearly

$$
\frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \left| \int b \left( t - \Delta - \frac{1}{m_j} \right) c(t) e^{\frac{\pi i}{m_j} \cdot \Theta t} dt \right| =
$$

$$
= \frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \left| \left( b \left( \cdot - \Delta - \frac{1}{m_j} \right), c(\cdot) e^{\frac{\pi i}{m_j} \cdot \Theta \cdot} \right) \right|
$$

where $(\cdot, \cdot)$ means the given by (6.5) scalar product. From the $\mathcal{F}$-invariance of $(\cdot, \cdot)$ it follows that

$$
\frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \left| \int_{\mathbb{R}^{2N}} \mathcal{F}(b) \left( \cdot - \Delta - \frac{1}{m_j} \right)(u) \mathcal{F}(c(\cdot) e^{\frac{\pi i}{m_j} \cdot \Theta \cdot})(u) du \right| \leq
$$

$$
\leq \frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \int_{\mathbb{R}^{2N}} e^{-i \left( \Delta - \frac{1}{m_j} \right) u} \mathcal{F}(b)(u) \mathcal{F}(c(u + \Theta \frac{\pi i}{m_j}))(u) du \leq
$$

$$
\leq \frac{1}{m_j^2} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \int_{\mathbb{R}^{2N}} \frac{C_{3M}^{(b)}}{(1 + \|u\|)^M} \frac{C_{2M}^{(c)}}{(1 + \|\Theta \frac{\pi i}{m_j}\|)^M} \frac{1}{(1 + \|u\|)^M} du \leq
$$

$$
\leq \sup_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \sum_{s \in \mathbb{R}^{2N}} \frac{C_{3M}^{(b)}}{(1 + \|s\|)^M} \frac{C_{2M}^{(c)}}{(1 + \|s - \Theta \frac{\pi i}{m_j}\|)^M} \frac{1}{(1 + \|u - \Theta \frac{\pi i}{m_j}\|)^M} \frac{1}{(1 + \|u\|)^M} \times
$$

$$
\times \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{|A_1|}{2}} \frac{1}{(1 + \|u\|)^M} du.
$$
Since we consider the asymptotic dependence \( \| \Delta \| \to \infty \) only large values of \( \| \Delta \| \) are interesting, so we can suppose that \( \| \Delta \| > 2 \). If \( \| \Delta \| > 2 \) then from \( \| \frac{I_m}{m_j} \| > \frac{\| \Delta \|}{2} \) it follows that \( \| \Theta \frac{\Delta}{m_j} \| > 1 \), and from (6.36) it follows that

\[
(1 + \| u \|)^M \left( 1 + \left\| u - \Theta \frac{\pi l m_j}{m_j} \right\| \right)^M > \left\| \Theta \frac{\pi l m_j}{m_j} \right\|^M,
\]

\[
\inf_{l \in \mathbb{Z}^{2N}, \| u \| > \| \Delta \|} (1 + \| s \|)^M \left( 1 + \left\| s - \Theta \frac{\pi l m_j}{m_j} \right\| \right)^M > \left\| \Delta \right\|^M / 4,
\]

hence, taking into account (6.38), one has

\[
\frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \| u \| > \| \Delta \|} \left| \left( b \left( \bullet + \Delta - \frac{l}{m_j} \right), c \left( \bullet \right) e^{\pi \Theta l \bullet} \right) \right| \leq \frac{1}{m_j^{2N}} C'_2 \sum_{l \in \mathbb{Z}^{2N}, \| u \| > 1} \int_{\mathbb{R}^{2N}} \frac{1}{m_j^M} \left( 1 + \| u \| \right)^M = \frac{C'_2}{\| \Delta \|^M m_j^{2N}} \left( \sum_{l \in \mathbb{Z}^{2N}, \| u \| > 1} \left\| \frac{\pi \Theta l}{m_j} \right\|^M \left( \int_{\mathbb{R}^{2N}} \frac{1}{m_j^M} \left( 1 + \| u \| \right)^M d u \right) \right),
\]

where \( C'_2 = C_3^{(b)} C_2^{(c)} \). Since \( M \geq 2N + 1 \) the integral \( \int_{\mathbb{R}^{2N}} \frac{1}{m_j^M} \left( 1 + \| u \| \right)^M d u \) is convergent. The infinite sum in the above equation can be represented as an integral of step function, in particular following condition holds

\[
\frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \| u \| > 1} \left\| \frac{\pi \Theta l}{m_j} \right\|^M = \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \| x \| > 1 \}} f_{m_j}(x) \, dx
\]

where \( f_{m_j} \) is a multidimensional step function such that

\[
f_{m_j} \left( \frac{2\pi l m_j}{m_j} \right) = \frac{1}{\| \frac{2\pi l m_j}{m_j} \|^M}
\]

From

\[
f_{m_j}(x) < \frac{2}{\| \frac{2\pi x m_j}{m_j} \|^M}
\]
it follows that
\[ \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \|x\| > 1 \}} f_{m_j}(x) \, dx < \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \|x\| > 1 \}} \frac{2}{\|2\pi x\|^M} \, dx. \]

From \( m > 2N + 1 \) it turns out the integral
\[ \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \|x\| > 1 \}} \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \|x\| > 1 \}} \frac{2}{\|2\pi x\|^M} \, dx \]
is convergent, hence
\[ \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{1}{m_j}} \frac{1}{m_j^M} < C_2'' = \int_{\mathbb{R}^{2N} - \{ x \in \mathbb{R}^{2N} | \|x\| > 1 \}} \frac{2}{\|2\pi x\|^M} \, dx. \]

From above equations it follows that
\[ \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}, \|l\| > \frac{1}{m_j}} \left\| \int_{\mathbb{R}^{2N}} b \left( \frac{w}{m_j} - l \right) c(w + \Delta) e^{i \frac{1}{m_j} \frac{w}{m_j}} \right\| \leq \frac{C_2}{\|\Delta\|^M}, \]
where \( M = 2N + 1 + m \) and \( C_2 = C_2' C_2'' \int_{\mathbb{R}^{2N}} \frac{1}{(1 + \|u\|)^M} \, du \). In result for any \( m > 0 \) there is \( C_m \in \mathbb{R} \) such that
\[ \left\| \sum_{g \in \ker(G \to G_j)} \hat{\pi}^{\oplus}(a) \right\| < \frac{1}{m_j^{2N}} \sum_{l \in \mathbb{Z}^{2N}} \int_{\mathbb{R}^{2N}} b \left( \frac{t + \Delta - l}{m_j} \right) c(t) e^{i \frac{\pi}{m_j} \Theta t} \, dt < \frac{C_m}{\|\Delta\|^M}. \]

\[ \square \]

**Lemma 6.32.** If \( \pi \) in \( \mathcal{S}(\mathbb{R}_0^{2N}) \) is positive then following conditions hold:

(i) For any \( j \in \mathbb{N}^0 \) the following series
\[ a_j = \sum_{g \in \ker(G \to G_j)} g \bar{a}, \]
\[ b_j = \sum_{g \in \ker(G \to G_j)} g \bar{a}^2 \]
are strongly convergent and the sums lie in \( C^\infty \left( T_{\sigma/m_j}^{2N} \right) \), i.e. \( a_j, b_j \in C^\infty \left( T_{\sigma/m_j}^{2N} \right) \); (ii) For any \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that for any \( j \geq N \) the following condition holds
\[ \|a_j^2 - b_j\| < \epsilon. \]
Proof. (i) Follows from the Lemmas 6.23 and/or 6.29.
(ii) Denote by \( J_j = \ker (\mathbb{Z}^{2N} \to G_j) = m_j \mathbb{Z}^{2N} \). If
\[
a_j = \sum_{g \in I_j} g \bar{a},
b_j = \sum_{g \in I_j} g \bar{a}^2
\]
then
\[
a_j^2 - b_j = \sum_{g \in I_j} g \bar{a} \left( \sum_{g' \in I_j \setminus \{g\}} g' \bar{a} \right).
\]
From (6.58) it follows that \( g \bar{a} = \bar{a}_g \) where \( \bar{a}_g (x) = \bar{a} (x + g) \) for any \( x \in \mathbb{R}^{2N} \) and \( g \in \mathbb{Z}^{2N} \). Hence the equation (6.59) is equivalent to
\[
a_j^2 - b_j = \sum_{g \in I_j} \bar{a}_g \left( \sum_{g' \in I_j \setminus \{g\}} \bar{a}_g' \right).
\]
Let \( m > 1 \) and \( M = 2N + 1 + m \). From the Lemma 6.31 it follows that there is \( C \in \mathbb{R} \) such that
\[
\left\| \sum_{g \in I_j} g (aa_\Delta) \right\| < \frac{C}{\| \Delta \|^M}.
\]
From the triangle inequality it follows that
\[
\left\| a_j^2 - b_j \right\| \leq \sum_{g' \in \mathbb{Z}^{2N \setminus \{0\}}} \left\| \sum_{g \in I_j} g (aa_\Delta) \right\| \leq \sum_{g' \in \mathbb{Z}^{2N \setminus \{0\}}} C \frac{\| m_j g' \|^M}{\| m_j g' \|^M}.
\]
From \( M > 2N \) it turns out that the series
\[
C' = \sum_{g' \in \mathbb{Z}^{2N \setminus \{0\}}} \frac{C}{\| m_j g' \|^M}
\]
is convergent and
\[
\sum_{g \in \mathbb{Z}^{2N \setminus \{0\}}} \frac{C}{\| m_j g \|^M} = \frac{C'}{m_j^M}.
\]
If $\varepsilon > 0$ is a small number and $N \in \mathbb{N}$ is such $m_N > \frac{M}{\varepsilon}$ then from above equations it follows that for any $j \geq N$ the following condition holds

$$\|a_j^2 - b_j\| < \varepsilon.$$  

Lemma 6.33. Let us consider a dense inclusion

$$S\left(\mathbb{R}^2_\theta\right) \otimes \cdots \otimes S\left(\mathbb{R}^2_\theta\right) \subset S\left(\mathbb{R}^{2N}_\theta\right) \quad \text{N-times}$$

of algebraic tensor product which follows from (6.45). If $\overline{\sigma} \in S\left(\mathbb{R}^{2N}_\theta\right)$ is a positive such that

- $\overline{\sigma}$ is a rank-one operator.

then $\overline{\sigma}$ is special.

Proof. Clearly $\overline{\sigma}$ is a rank-one operator. If $\overline{\sigma} \in S\left(\mathbb{R}^{2N}_\theta\right)$ then from the Lemmas 6.23 and/or 6.29 it turns out that $\overline{\sigma}$ satisfies to (a) of the Definition 3.5. If $z \in C\left(\mathbb{T}^{2N}_\theta\right)$ then from the injective *-homomorphism $C\left(\mathbb{T}^{2N}_\theta\right) \hookrightarrow C\left(\mathbb{T}^{2N}_\theta \otimes \mathbb{M}_2\right)$ it follows that $z$ can be regarded as element of $C\left(\mathbb{T}^{2N}_\theta \otimes \mathbb{M}_2\right)$, i.e. $z \in C\left(\mathbb{T}^{2N}_\theta \otimes \mathbb{M}_2\right)$. Denote by

$$b_j = \sum_{g \in \text{ker}(G \rightarrow G_j)} g \left(z \overline{\sigma} z^*\right) = z \left(\sum_{g \in \text{ker}(G \rightarrow G_j)} g \overline{\sigma}\right) z^*,$$

$$c_j = \sum_{g \in \text{ker}(G \rightarrow G_j)} g \left(z \overline{\sigma} z^*\right)^2 = z \left(\sum_{g \in \text{ker}(G \rightarrow G_j)} g \left(z \overline{\sigma} z^*\right)\right) z^*,$$

$$d_j = \sum_{g \in \text{ker}(G \rightarrow G_j)} g f_\varepsilon(z \overline{\sigma} z^*)$$

where $f_\varepsilon$ is given by (3.2). From $\overline{\sigma}$ in $S\left(\mathbb{R}^{2N}_\theta\right)$ it turns out $a_j = \sum_{g \in \text{ker}(G \rightarrow G_j)} g \overline{\sigma} \in C\left(\mathbb{T}^{2N}_\theta \otimes \mathbb{M}_2\right)$, hence $b_j = z a_j z^* \in C\left(\mathbb{T}^{2N}_\theta \otimes \mathbb{M}_2\right)$. If $\xi \in \mathcal{H}$ is eigenvector of $\overline{\sigma}$ such that
that follows that \( \parallel \) then \( \eta = z_\xi \) is an is eigenvector of \( \eta = z\xi \) such that \( z\xi \eta = \parallel z\xi \parallel \eta \). It follows that \((z\xi)^2 = kz\xi \) where \( k \in \mathbb{R}_+ \) is given by

\[
k = \frac{\parallel z\xi \parallel^2}{\parallel z\xi \parallel}.
\]

Hence \( c_j = kb_j \) and \( c_j \in C^\infty (T_{\theta/m}^N) \). Similarly \( f_\varepsilon (z\xi) = k' (z\xi) \) where

\[
k' = \max \left( 0, \frac{\parallel z\xi \parallel - \varepsilon}{\parallel z\xi \parallel} \right).
\]

Hence \( d_j = k'b_j \) and \( d_j \in C^\infty (T_{\theta/m}^N) \), it follows that \( \pi \) satisfies to the condition (b) of the Definition 3.5. Let \( \varepsilon > 0 \), and let \( \delta > 0 \) be such that

\[
\delta^4 \left( \sum_{g \in \hat{G}} \parallel \pi \parallel^2 \right)^2 + 2\delta^2 \left( \sum_{g \in \hat{G}} \parallel \xi \parallel \right) \left( \sum_{g \in \hat{G}} z\xi \right) < \frac{\varepsilon}{4}.
\]

\[
\delta^2 \sum_{g \in \hat{G}} g (\xi z\xi) < \frac{\varepsilon}{4}.
\]

\[
\left( \parallel \zeta \parallel + \delta \right)^2 \left( \delta^2 + 2\delta \parallel \zeta \parallel \right) \sum_{g \in \hat{G}} \parallel \xi \parallel^2 < \frac{\varepsilon}{4}.
\]

The algebra \( C^\infty (T_{\theta}^N) \) is a dense subalgebra of \( C (T_{\theta}^N) \), so there is \( y \in C^\infty (T_{\theta}^N) \) such that \( \parallel \zeta - y \parallel < \delta \). From

\[
\left\| b_j - y \left( \sum_{g \in \ker (G \rightarrow G_j)} g \pi \right) y^* \right\| \leq \left\| (z - y) \left( \sum_{g \in \hat{G}} g \pi \right) (z - y)^* \right\| < \delta^2 \left\| \sum_{g \in \hat{G}} \pi \right\|
\]

and taking into account \( \delta^4 \left( \sum_{g \in \hat{G}} \parallel \pi \parallel^2 \right)^2 + 2\delta^2 \left( \sum_{g \in \hat{G}} \parallel \xi \parallel \right) \left( \sum_{g \in \hat{G}} z\xi \right) < \frac{\varepsilon}{4} \) one has

\[
\left\| b_j^2 - \left( y \left( \sum_{g \in \ker (G \rightarrow G_j)} g \pi \right) y^* \right) \right\|^2 < \frac{\varepsilon}{4}.
\]

From

\[
\left\| c_j - y \left( \sum_{g \in \ker (G \rightarrow G_j)} g (\xi z\xi) \right) y^* \right\| \leq \left\| (z - y) \left( \sum_{g \in \hat{G}} g (\xi z\xi) \right) (z - y)^* \right\| < \delta^2 \left\| \sum_{g \in \hat{G}} (\xi z\xi) \right\|
\]

\[
< \frac{\varepsilon}{4}.
\]

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and taking into account \( \| \sum_{g \in \hat{G}} g (\pi z^* \pi) \| \delta^2 < \frac{\varepsilon}{4} \) one has

\[
\left\| c_j - y \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi z^* \pi) \right) y^* \right\| < \frac{\varepsilon}{4} \quad (6.61)
\]

From \( \| y \| < \| z \| + \delta \) it turns out

\[
\left\| y \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi z^* \pi) \right) y^* - y \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi y^* \pi) \right) y^* \right\| \leq (\| z \| + \delta)^2 \left( \delta^2 + 2\delta \| \pi \| \right) \left\| \sum_{g \in \hat{G}} \delta^2 \right\|
\]

and taking into account \( (\| z \| + \delta)^2 (\delta^2 + 2\delta \| \pi \|) \left\| \sum_{g \in \hat{G}} \delta^2 \right\| < \frac{\varepsilon}{4} \) one has

\[
\left\| y \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi z^* \pi) \right) y^* - y \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi y^* \pi) \right) y^* \right\| < \frac{\varepsilon}{4} \quad (6.62)
\]

From \( y \in C^\infty (T^2N) \) and \( \alpha \in S (\mathbb{R}_\theta^2 \times \cdots \times \mathbb{R}_\theta^2) \) it follows that \( \pi y^* \in S (\mathbb{R}_\theta^2) \), hence from the Lemma 6.32 it turns out the existence of \( N \in \mathbb{N} \) such that for any \( j \geq N \) following condition holds

\[
\left\| \left( \sum_{g \in \text{ker}(G \to G_j)} g (\pi y^*) \right)^2 - \sum_{g \in \text{ker}(G \to G_j)} (g (\pi y^*))^2 \right\| < \frac{\varepsilon}{4} \quad (6.63)
\]

From (6.60)-(6.63) it follows than for any \( j \geq N \) following condition holds

\[
\left\| b_j^2 - c_j \right\| < \varepsilon,
\]

i.e. \( \pi \) satisfies to the condition (c) of the Definition 3.5.

\[\square\]

**Corollary 6.34.** If \( \overline{A_{\hat{\pi}^0}} \) is the disconnected inverse noncommutative limit of \( S_\theta \) with respect to \( \hat{\pi}^0 \) then

\[
\bigoplus_{g \in I} gC_0 \left( \mathbb{R}_\theta^2 \right) \subset \overline{A_{\hat{\pi}^0}}
\]

**Proof.** From the Lemma 6.33 it turns out that \( \overline{A_{\hat{\pi}^0}} \) contains all elements

\[
f_{j_1 k_1} \otimes \cdots \otimes f_{j_N k_N} \in S (\mathbb{R}_\theta^2) \otimes \cdots \otimes S (\mathbb{R}_\theta^2) \subset S (\mathbb{R}_\theta^2) \quad (6.64)
\]

\[\text{N-times}\]
where \( f_{j,k_l} \) \((l = 1, \ldots, N)\) are given by the Lemma 6.24. However the linear span of given by (6.64) elements is dense in \( C_0 \left( \mathbb{R}_\theta^{2N} \right) \), hence \( C_0 \left( \mathbb{R}_\theta^{2N} \right) \subset \mathcal{A}_{\pi^\oplus} \). From the Corollary 3.9 it turns out
\[
\bigoplus_{g \in J} g C_0 \left( \mathbb{R}_\theta^{2N} \right) \subset \mathcal{A}_{\pi^\oplus}.
\]

6.35. From the Lemma 6.8 it turns out that \( L^2 \left( \mathbb{R}_\theta^{2N} \right) \subset B \left( L^2 \left( \mathbb{R}^{2N} \right) \right) \) is a Hilbert space with the norm \( \| \cdot \|_2 \) given by (6.23). One can construct the Hilbert direct sum
\[
X = \bigoplus_{g \in J} g L^2 \left( \mathbb{R}_\theta^{2N} \right) \subset \prod_{g \in J} B \left( g L^2 \left( \mathbb{R}_\theta^{2N} \right) \right),
\]
\[
X = \left\{ \overline{\pi} \in \prod_{g \in J} B \left( g L^2 \left( \mathbb{R}_\theta^{2N} \right) \right) \mid \| (\ldots, x_{g_k}, \ldots) \|_2 = \sqrt{\sum_{g \in J} \| x_g \|_2^2} < \infty \right\}.
\]

If \( \overline{\pi} \in X \) is a special element and \( b = \sum_{g \in \hat{G}} g \overline{\pi} \in C \left( \mathbb{T}^{2N}_\theta \right) \) then
\[
\tau \left( b \right) = \int_{\mathbb{R}_\theta^{2N}} \overline{\pi}^2 dx = \| \overline{\pi} \|_2^2
\]
where \( \tau \) is given by (6.10), or (6.11). On the other hand \( | \tau \left( b \right) | < \infty \) for any \( b \in C \left( \mathbb{T}^{2N}_\theta \right) \) it follows that \( \| \overline{\pi} \|_2^2 < \infty \) for a special element \( \overline{\pi} \). In result we have the following lemma.

**Lemma 6.36.** The special element \( \overline{\pi} \in \lim_{\rightarrow} C \left( \mathbb{T}^{2N}_{\theta/m^n} \right) \) lies in \( X = \bigoplus_{g \in J} g L^2 \left( \mathbb{R}_\theta^{2N} \right) \). Moreover if \( b = \sum_{g \in \hat{G}} g \left( \overline{\pi}^2 \right) \in C \left( \mathbb{T}^{2N}_\theta \right) \) then
\[
\| \overline{\pi} \|_2^2 = \tau \left( b \right) < \infty
\]
where \( \tau \) is the tracial state on \( C \left( \mathbb{T}^{2N}_\theta \right) \) given by (6.10), 6.12 and \( \| \cdot \|_2 \) is given by (6.23).

**Remark 6.37.** From \( L^2 \left( \mathbb{R}_\theta^{2N} \right) \subset C_0 \left( \mathbb{R}_\theta^{2N} \right) \) it follows that any special element in \( B \left( L^2 \left( \mathbb{R}_\theta^{2N} \right) \right) \) lies in \( C_0 \left( \mathbb{R}_\theta^{2N} \right) \).

6.38. Let \( \mathcal{A}_{\pi^\oplus} \) be the disconnected inverse noncommutative limit of \( \mathcal{G}_\theta \) with respect to \( \pi^\oplus \) of \( \mathcal{G}_\theta \). From the Corollary 6.34 it follows that
\[
C_0 \left( \mathbb{R}_\theta^{2N} \right) \subset \mathcal{A}_{\pi^\oplus} \cap B \left( L^2 \left( \mathbb{R}^{2N} \right) \right).
\]
From the Remark 6.37 it follows that
\[
\mathcal{A}_{\pi^\oplus} \cap B \left( L^2 \left( \mathbb{R}^{2N} \right) \right) \subset C_0 \left( \mathbb{R}_\theta^{2N} \right).
\]
In result we have
\[
\mathcal{A}_{\pi^\oplus} \cap B \left( L^2 \left( \mathbb{R}^{2N} \right) \right) = C_0 \left( \mathbb{R}_\theta^{2N} \right).
\]
Similarly for any \( g \in J \) on has
\[
\mathcal{A}_{\mathcal{R}^\oplus} \cap B \left( gL^2 \left( \mathcal{R}^{2N} \right) \right) = gC_0 \left( \mathcal{R}_g^{2N} \right).
\]
The algebra \( C_0 \left( \mathcal{R}_g^{2N} \right) \) is irreducible. Clearly \( C_0 \left( \mathcal{R}_g^{2N} \right) \subset \mathcal{A}_{\mathcal{R}^\oplus} \) is a maximal irreducible subalgebra.

**Theorem 6.39.** Following conditions hold:

(i) The representation \( \mathcal{R}^\oplus \) is good,

(ii) \[
\lim_{\mathcal{R}^\oplus} \downarrow \mathcal{S}_\theta = C_0 \left( \mathcal{R}_g^{2N} \right);
\]

\[
G \left( \lim_{\mathcal{R}^\oplus} \downarrow \mathcal{S}_\theta \mid C \left( \mathcal{T}_g^{2N} \right) \right) = \mathbb{Z}^{2N},
\]

(iii) The triple \( \left( C \left( \mathcal{T}_g^{2N} \right), C_0 \left( \mathcal{R}_g^{2N} \right), \mathbb{Z}^{2N} \right) \) is an infinite noncommutative covering of \( \mathcal{S}_\theta \) with respect to \( \mathcal{R}^\oplus \).

**Proof.** (i) There is the natural inclusion \( \mathcal{A}_{\mathcal{R}^\oplus} \hookrightarrow \prod_{g \in J} B \left( gL^2 \left( \mathcal{R}^{2N} \right) \right) \) where \( \prod \) means the Cartesian product of algebras. This inclusion induces the decomposition
\[
\mathcal{A}_{\mathcal{R}^\oplus} \hookrightarrow \prod_{g \in J} \left( \mathcal{A}_{\mathcal{R}^\oplus} \cap B \left( gL^2 \left( \mathcal{R}^{2N} \right) \right) \right).
\]
From (6.65) it turns out \( \mathcal{A}_{\mathcal{R}^\oplus} \cap B \left( gL^2 \left( \mathcal{R}^{2N} \right) \right) = gC_0 \left( \mathcal{R}_g^{2N} \right) \), hence there is the inclusion
\[
\mathcal{A}_{\mathcal{R}^\oplus} \hookrightarrow \prod_{g \in J} gC_0 \left( \mathcal{R}_g^{2N} \right).
\]

From the above equation it follows that \( C_0 \left( \mathcal{R}_g^{2N} \right) \subset \mathcal{A}_{\mathcal{R}^\oplus} \) is a maximal irreducible subalgebra. From the Lemma 6.36 it turns out that algebraic direct sum \( \bigoplus_{g \in J} gC_0 \left( \mathcal{R}_g^{2N} \right) \) is a dense subalgebra of \( \mathcal{A}_{\mathcal{R}^\oplus} \), i.e. the condition (b) of the Definition 3.13 holds. Clearly the map \( C \left( \mathcal{T}_g^{2N} \right) \to M \left( C_0 \left( \mathcal{R}_g^{2N} \right) \right) \) is injective, i.e. the condition (a) of the Definition 3.13 holds. If \( G \subset \hat{G} \) is the maximal group such that \( GC_0 \left( \mathcal{R}_g^{2N} \right) = C_0 \left( \mathcal{R}_g^{2N} \right) \) then \( G = \mathbb{Z}^{2N} \). The homomorphism \( \mathbb{Z}^{2N} \to \mathbb{Z}^{2N}_{m_j} \) is surjective, it turns out that the condition (c) of the Definition 3.13 holds.

(ii) and (iii) Follows from the proof of (i).

7 Isospectral deformations and their coverings

A very general construction of isospectral deformations of noncommutative geometries is described in [3]. The construction implies in particular that any compact spin-manifold
\( M \) whose isometry group has rank \( \geq 2 \) admits a natural one-parameter isospectral deformation to noncommutative geometries \( M_\gamma \). We let \((C^\infty(M), \mathcal{H} = L^2(M, S), \mathcal{D})\) be the canonical spectral triple associated with a compact spin-manifold \( M \). We recall that \( \mathcal{A} = C^\infty(M) \) is the algebra of smooth functions on \( M \), \( S \) is the spinor bundle and \( \mathcal{D} \) is the Dirac operator. Let us assume that the group \( \text{Isom}(M) \) of isometries of \( M \) has rank \( r \geq 2 \).

Then, we have an inclusion
\[
\mathbb{T}^2 \subset \text{Isom}(M),
\]
with \( \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2 \) the usual torus, and we let \( U(s), s \in \mathbb{T}^2 \), be the corresponding unitary operators in \( \mathcal{H} = L^2(M, S) \) so that by construction
\[
U(s) \mathcal{D} = \mathcal{D} U(s).
\]

Also,
\[
U(s) a U(s)^{-1} = a_s(a), \quad \forall a \in \mathcal{A}, \tag{7.1}
\]
where \( a_s \in \text{Aut}(\mathcal{A}) \) is the action by isometries on the algebra of functions on \( M \).

We let \( p = (p_1, p_2) \) be the generator of the two-parameters group \( U(s) \) so that
\[
U(s) = \exp(i(s_1 p_1 + s_2 p_2)).
\]

The operators \( p_1 \) and \( p_2 \) commute with \( D \). Both \( p_1 \) and \( p_2 \) have integral spectrum,
\[
\text{Spec}(p_j) \subset \mathbb{Z}, \quad j = 1, 2.
\]

One defines a bigrading of the algebra of bounded operators in \( \mathcal{H} \) with the operator \( T \) declared to be of bidegree \((n_1, n_2)\) when,
\[
\alpha_s(T) = U(s) T U(s)^{-1} \quad \text{as in} \quad (7.1),
\]
where \( \alpha_s(T) = U(s) T U(s)^{-1} \) as in (7.1).

Any operator \( T \) of class \( C^\infty \) relative to \( \alpha_s \) (i.e. such that the map \( s \to \alpha_s(T) \) is of class \( C^\infty \) for the norm topology) can be uniquely written as a doubly infinite norm convergent sum of homogeneous elements,
\[
T = \sum_{n_1, n_2} \tilde{T}_{n_1, n_2},
\]
with \( \tilde{T}_{n_1, n_2} \) of bidegree \((n_1, n_2)\) and where the sequence of norms \( ||\tilde{T}_{n_1, n_2}|| \) is of rapid decay in \((n_1, n_2)\). Let \( \lambda = \exp(2\pi i \theta) \). For any operator \( T \) in \( \mathcal{H} \) of class \( C^\infty \) we define its left twist \( l(T) \) by
\[
l(T) = \sum_{n_1, n_2} \tilde{T}_{n_1, n_2} \lambda^{n_2 p_1}, \tag{7.2}
\]
and its right twist \( r(T) \) by
\[
r(T) = \sum_{n_1, n_2} \tilde{T}_{n_1, n_2} \lambda^{n_1 p_2},
\]
Since \(|\lambda| = 1\) and \( p_1, p_2 \) are self-adjoint, both series converge in norm. Denote by \( C^\infty(M)_{n_1, n_2} \subset C^\infty(M) \) the \( \mathbb{C} \)-linear subspace of elements of bidegree \((n_1, n_2)\).

One has,
Lemma 7.1. 

a) Let $x$ be a homogeneous operator of bidegree $(n_1, n_2)$ and $y$ be a homogeneous operator of bidegree $(n'_1, n'_2)$. Then,

$$l(x) r(y) - r(y) l(x) = (xy - yx) \lambda'^1 n_2 \lambda^{n_2 \mu_1 + n'_1 \mu_2}$$

(7.3)

In particular, $[l(x), r(y)] = 0$ if $[x, y] = 0$.

b) Let $x$ and $y$ be homogeneous operators as before and define

$$x * y = \lambda'^1 n_2 xy;$$

then $l(x)l(y) = l(x * y)$.

The product $*$ defined in (7.4) extends by linearity to an associative product on the linear space of smooth operators and could be called a $*$-product. One could also define a deformed ‘right product’. If $x$ is homogeneous of bidegree $(n_1, n_2)$ and $y$ is homogeneous of bidegree $(n'_1, n'_2)$ the product is defined by

$$x * r y = \lambda'^1 n_2 xy.$$

Then, along the lines of the previous lemma one shows that $r(x)r(y) = r(x * r y)$.

We can now define a new spectral triple where both $\mathcal{H}$ and the operator $D$ are unchanged while the algebra $C^\infty (M)$ is modified to $l(C^\infty (M))$. By Lemma 7.1b) one checks that $l(C^\infty (M))$ is still an algebra. Since $D$ is of bidegree $(0, 0)$ one has,

$$[D, l(a)] = l([D, a])$$

which is enough to check that $[D, x]$ is bounded for any $x \in l(A)$. There is a spectral triple $(l(C^\infty (M)), \mathcal{H}, D)$.

Denote by $C(M_\theta)$ the operator norm completion (equivalently $C^*$-norm completion) of $l(C^\infty (M))$, and denote by $\rho : C(M) \to L^2 (M, S)$ (resp. $\pi_\theta : C(M_\theta) \to B (L^2 (M, S))$) natural representations.

7.1 Finite-fold coverings

Let $M$ be a spin-manifold with the smooth action of $\mathbb{T}^2$. Let $\pi : \tilde{M} \to M$ be a finite-fold covering. Let $\tilde{x}_0 \in \tilde{M}$ and $x_0 = \pi (\tilde{x}_0)$. Denote by $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{T}^2$ the natural covering. There are two closed paths $\omega_1, \omega_2 : [0, 1] \to M$ given by

$$\omega_1 (t) = \varphi (t, 0) x_0, \omega_2 (t) = \varphi (0, t) x_0.$$

There are lifts of these paths, i.e. maps $\tilde{\omega}_1, \tilde{\omega}_2 : [0, 1] \to \tilde{M}$ such that

$$\tilde{\omega}_1 (0) = \tilde{\omega}_2 (0) = \tilde{x}_0,$$

$$\pi (\tilde{\omega}_1 (t)) = \omega_1 (t),$$

$$\pi (\tilde{\omega}_2 (t)) = \omega_2 (t).$$
Since $\pi$ is a finite-fold covering there are $N_1, N_2 \in \mathbb{N}$ such that if
\[
\gamma_1(t) = \varphi(N_1 t, 0) x_0, \quad \gamma_2(t) = \varphi(0, N_2 t) x_0.
\]
and $\tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$) is the lift of $\gamma_1$ (resp. $\gamma_2$) then both $\tilde{\gamma}_1, \tilde{\gamma}_2$ are closed. Let us select minimal positive values of $N_1, N_2$. If $pr_n : S^1 \to S^1$ is an $n$ listed covering and $pr_{N_1, N_2}$ the covering given by
\[
\tilde{T}^2 = S^1 \times S^1 \xrightarrow{pr_{N_1} \times pr_{N_2}} S^1 \times S^1 = T^2
\]
then there is the action $\tilde{T}^2 \times \tilde{M} \to \tilde{M}$ such that
\[
\tilde{T}^2 \times \tilde{M} \xrightarrow{pr_{N_1} \times \pi} \tilde{M}
\]
\[
\tilde{T}^2 \times M \xrightarrow{pr_{N_1, N_2} \times \pi} M
\]
where $\tilde{T}^2 \approx T^2$. Let $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$ be the generator of the associated with $\tilde{T}^2$ two-parameters group $\tilde{U}(s)$ so that
\[
\tilde{U}(s) = \exp(i(s_1 \tilde{p}_1 + s_2 \tilde{p}_2)).
\]
The covering $\tilde{M} \to M$ induces an involutive injective homomorphism
\[
\varphi : C^\infty(M) \hookrightarrow C^\infty(\tilde{M}).
\]
Since $\tilde{M} \to M$ is a covering $C^\infty(\tilde{M})$ is a finitely generated projective $C^\infty(M)$-module, i.e. there is the following direct sum of $C^\infty(\tilde{M})$-modules
\[
C^\infty(\tilde{M}) \bigoplus P = C^\infty(M)^n
\] (7.5)
such that
\[
\varphi(C^\infty(M))_{n_1, n_2} \subset C^\infty(\tilde{M})_{n_1 N_1, n_2 N_2}.
\]
Let $\theta, \tilde{\theta} \in \mathbb{R}$ be such that
\[
\tilde{\theta} = \theta + n \frac{\pi}{N_1 N_2}, \text{ where } n \in \mathbb{Z}.
\]
If $\lambda = e^{2\pi i \theta}$, $\tilde{\lambda} = e^{2\pi i \tilde{\theta}}$ then $\lambda = \tilde{\lambda}^{N_1 N_2}$. There are isospectral deformations $C^\infty(M_\theta), C^\infty(\tilde{M}_\tilde{\theta})$ and $C$-linear isomorphisms $I : C^\infty(M) \to C^\infty(M_\theta), \tilde{I} : C^\infty(\tilde{M}) \to C^\infty(\tilde{M}_\tilde{\theta})$. These isomorphisms and the inclusion $\varphi$ induce the inclusion
\[
\varphi_\theta : C^\infty(M_\theta) \to C^\infty(\tilde{M}_\tilde{\theta}),
\]
\[
\varphi_{\tilde{\theta}}(C^\infty(M_\theta))_{n_1, n_2} \subset C^\infty(\tilde{M}_\tilde{\theta})_{n_1 N_1, n_2 N_2}.
\]
From (7.5) it follows that
\[ \tilde{I} \left( C^\infty \left( \tilde{M} \right) \right) \oplus I \left( P \right) = I \left( C^\infty \left( M \right) \right)^n, \]
or equivalently \( C^\infty \left( \tilde{M}_\theta \right) \oplus I \left( P \right) = C^\infty \left( M_\theta \right)^n, \)
i.e. \( C^\infty \left( \tilde{M}_\theta \right) \) is a finitely generated projective \( C^\infty \left( M_\theta \right) \) module. There is a projection \( p \in M_n \left( C^\infty \left( M_\theta \right) \right) \) such that
\[ C^\infty \left( \tilde{M}_\theta \right) = pC^\infty \left( M_\theta \right)^n. \]
If \( C \left( \tilde{M}_\theta \right) \) (resp. \( C \left( M_\theta \right) \)) is the operator norm completion of \( C^\infty \left( \tilde{M}_\theta \right) \) (resp. \( C^\infty \left( M_\theta \right) \)) then
\[ C \left( \tilde{M}_\theta \right) = pC \left( M_\theta \right)^n, \]
i.e. \( C \left( \tilde{M}_\theta \right) \) is a finitely generated projective \( C \left( M_\theta \right) \) module. Denote by \( G = G \left( \tilde{M} \mid M \right) \) the group of covering transformations. Since \( \tilde{I} \) is a \( C \)-linear isomorphism the action of \( G \) on \( C^\infty \left( \tilde{M} \right) \) induces a \( C \)-linear action \( G \times C^\infty \left( \tilde{M}_\theta \right) \to C^\infty \left( M_\theta \right) \). According to the definition of the action of \( \tilde{T}^2 \) on \( \tilde{M} \) it follows that the action of \( G \) commutes with the action of \( \tilde{T}^2 \). It turns out
\[ gC^\infty \left( \tilde{M} \right)_{n_1,n_2} = C^\infty \left( \tilde{M} \right)_{n_1,n_2} \]
for any \( n_1, n_2 \in \mathbb{Z} \) and \( g \in G \). If \( \tilde{a} \in C^\infty \left( \tilde{M} \right)_{n_1,n_2}, \tilde{b} \in C^\infty \left( \tilde{M} \right)_{n'_1,n'_2} \) then \( g \left( \tilde{a} \tilde{b} \right) = \left( g\tilde{a} \right) \left( g\tilde{b} \right) \in C^\infty \left( \tilde{M} \right)_{n_1+n'_1,n_2+n'_2} \). One has
\[ \tilde{I} \left( \tilde{a} \right) \tilde{I} \left( \tilde{b} \right) = \tilde{\lambda}^{n_1+n_2} \tilde{I} \left( \tilde{a} \tilde{b} \right), \]
\[ \tilde{\lambda}^{n_2} \tilde{I} \left( \tilde{b} \right) = \tilde{\lambda}^{n_1+n_2} \tilde{I} \left( \tilde{b} \right) \tilde{\lambda}^{n_2}, \]
\[ \tilde{I} \left( g\tilde{a} \right) \tilde{I} \left( g\tilde{b} \right) = g\tilde{a} \tilde{\lambda}^{n_2} \tilde{I} \left( g\tilde{a} \tilde{b} \right) = \tilde{\lambda}^{n_1+n_2} g \left( \tilde{a} \tilde{b} \right) \tilde{\lambda} \left( n_2+n_2' \right) \tilde{I}. \]
On the other hand
\[ g \left( \tilde{I} \left( \tilde{a} \right) \tilde{I} \left( \tilde{b} \right) \right) = \tilde{I} \left( \tilde{a} \tilde{b} \right) = \tilde{\lambda}^{n_1+n_2} g \left( \tilde{a} \tilde{b} \right) \tilde{\lambda} \left( n_2+n_2' \right) \tilde{I}. \]
From above equations it turns out
\[ \tilde{I} \left( g\tilde{a} \right) \tilde{I} \left( g\tilde{b} \right) = g \left( \tilde{I} \left( \tilde{a} \right) \tilde{I} \left( \tilde{b} \right) \right), \]
i.e. \( g \) corresponds to automorphism of \( C^\infty \left( \tilde{M}_\theta \right) \). It turns out that \( G \) is the group of automorphisms of \( C^\infty \left( \tilde{M}_\theta \right) \). Clearly form \( \tilde{a} \in C^\infty \left( \tilde{M}_\theta \right)_{n_1,n_2} \) it follows that \( \tilde{a}^* \in \).
One has
\[ g \left( \tilde{I} \tilde{a} \right)^* = g \left( \tilde{\lambda}^{-n_2\tilde{p}_1} \tilde{a}^* \right) = g \left( \tilde{\lambda}^{n_1n_2} \tilde{a}^* \tilde{\lambda}^{-n_2\tilde{p}_1} \right) = \tilde{\lambda}^{n_1n_2} g \left( \tilde{I} \tilde{a}^* \right). \]

On the other hand
\[ \left( g \tilde{I} (\tilde{a}) \right)^* = \left( (ga^*) \tilde{\lambda}^{-n_2\tilde{p}_1} \right)^* = \tilde{\lambda}^{-n_2\tilde{p}_1} \left( ga^* \right) = \tilde{\lambda}^{n_1n_2} \left( ga^* \tilde{\lambda}^{-n_2\tilde{p}_1} \right) = \tilde{\lambda}^{n_1n_2} g \left( \tilde{I} \tilde{a}^* \right), \]
i.e. \( g \left( \tilde{I} (\tilde{a}) \right)^* = \left( g \tilde{I} (\tilde{a}) \right)^* \). It follows that \( g \) corresponds to the involutive automorphism of \( C^\infty(\tilde{M}_\theta) \). Since \( C^\infty(\tilde{M}_\theta) \) is dense in \( C(\tilde{M}_\theta) \) there is the unique involutive action \( G \times C(\tilde{M}_\theta) \to C(\tilde{M}_\theta) \). From the above construction it turns out the following theorem.

**Theorem 7.2.** The triple \( (C(M_\theta), C(\tilde{M}_\theta), G(\tilde{M} | M)) \) is an unital noncommutative finite-fold covering.

### 7.2 Infinite coverings

Let \( \mathcal{S}_M = \{ M = M^0 \leftarrow M^1 \leftarrow \ldots \leftarrow M^n \leftarrow \ldots \} \in \mathfrak{FinTop} \) be an infinite sequence of spin-manifolds and regular finite-fold covering. Suppose that there is the action \( \mathbb{T}^2 \times M \to M \) given by (7.1). From the Theorem 7.2 it follows that there is the algebraical finite covering sequence

\[ \mathcal{S}_{C(M_\theta)} = \{ C(M_\theta) \to \ldots \to C(M^n_\theta) \to \ldots \}. \]

So one can calculate a finite noncommutative limit of the above sequence. This article does not contain detailed properties of this noncommutative limit, because it is not known yet by the author of this article.

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