A HYPERKÄHLER SUBMANIFOLD OF THE MONOPOLE MODULI SPACE

ROGER BIELAWSKI

Abstract. We discuss a $4[k/2]$-dimensional complete hyperkähler submanifold of the $(4k - 4)$-dimensional moduli space of strongly centred $SU(2)$-monopoles of charge $k$.

The isometry group of $\mathbb{R}^3$ acts isometrically on the moduli space $M_k$ of Euclidean $SU(2)$-monopoles of charge $k$ and, consequently, a fixed point set of any subgroup of this group is a totally geodesic submanifold. Similar statement holds for any subgroup of the orthogonal group $O(3)$ acting on the submanifold $M_0^k$ consisting of strongly centred monopoles (these are monopoles with the centre at the origin and total phase equal to 1). Houghton and Sutcliffe [8] have shown that the submanifold of $M_0^3$ consisting of monopoles symmetric about the origin is isometric to the Atiyah-Hitchin manifold (i.e. the moduli space of centred monopoles of charge 2). Surprisingly, monopoles of higher charges invariant under the reflection $x \mapsto -x$ seem not to have been considered in the literature. This reflection is particularly interesting since it extends to a reflection $\tau : (x, t) \mapsto (-x, t^{-1})$ on $\mathbb{R}^3 \times S^1$ which preserves the hyperkähler structure of $\mathbb{R}^3 \times S^1$. It is then easy to deduce that the submanifold $N_k$ of strongly centred monopoles symmetric about the origin is a (complete) hyperkähler submanifold of $M_0^k$ for any charge $k$. We have stumbled upon this hyperkähler manifold (for even $k$) in a completely different context in [3] and realised only a posteriori, by identifying the twistor space, that it must be a submanifold of $M_0^k$.

In the present paper we describe the submanifold $N_k$ in terms of Nahm’s equations. Since $N_k$ is $SO(3)$-invariant, all of its complex structures are equivalent and can be identified with a complex submanifold of based rational maps of degree $k$. This is straightforward, given that the involution $\tau$ acts on rational maps via $p(z)/q(z) \mapsto \tilde{p}(z)/q(-z)$, where $\tilde{p}(-z)p(z) - 1 = 0 \mod q(z)$, but we also show this directly using Nahm’s equations. We then show that $N_k$ is biholomorphic to the transverse Hilbert scheme of $n$ points [3] §5 on the $D_1$-surface if $k = 2n$, and on the $D_0$-surface if $k = 2n + 1$. This allows us to conclude that $N_{2n}$ is simply connected, while $N_{2n+1}$ has fundamental group of order 2. In Section 2 we present an alternative construction, which also gives a description of hyperkähler deformations of $N_{2n}$.

1. Description in terms of Nahm’s equations

The moduli space $M_0^k$ of strongly centred $SU(2)$-monopoles of charge $k$ is isomorphic to the moduli space of $su(k)$-valued solutions to Nahm’s equations on $(0, 2)$ with $T_1(t), T_2(t), T_3(t)$ having simple poles at $t = 0, 2$, the residues of which define
the standard $k$-dimensional irreducible representation of $\mathfrak{su}(2)$, i.e.
\[
\text{Res}T_1(t) = i \text{diag}(k - 1, k - 3, \ldots, 3 - k, 1 - k),
\]
\[
\text{Res}(T_2 + iT_3)(t)_{ij} = \begin{cases} \sqrt{J(i - j)} & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}
\]
The corresponding representation of $\mathfrak{sl}_2(\mathbb{C})$ is given by the action $y\partial_x, x\partial_y, x\partial_x - y\partial_y$ on binary forms of degree $k - 1$ with a basis
\[
(1.1) \quad v_i = \left(\frac{k - 1}{i - 1}\right)^{1/2} x^{k-i}y^{i-1}, \quad i = 1, \ldots, k.
\]
Write $V$ for the standard 2-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, so that the symmetric product $S^{k-1}V$ is the standard $k$-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. The equivariant isomorphisms
\[
\Lambda^2(S^{2n+1}V^*) \cong \bigoplus_{i=0}^{n} S^{4i}V^*, \quad S^2(S^{2n}V^*) \cong \bigoplus_{i=0}^{n} S^{4i}V^*,
\]

imply that, if $k$ is even (resp. if $k$ is odd), then there exists a unique (up to scaling) $\mathfrak{sl}_2(\mathbb{C})$-invariant skew-symmetric (resp. symmetric) bilinear form on $S^{k-1}V$. This invariant form is a classical object and is called transvectant. In the basis \( (1.1) \) the transvectant $T(f, g)$ of $f = \sum_{i=1}^{k} a_i v_i$, $g = \sum_{i=1}^{k} b_i v_i$ is given \cite{10} Ex.2.7
\[
T(f, g) = \sum_{i=0}^{k-1} (-1)^i a_{i+1} b_{k-i}.
\]
We conclude that the residues of $T_1, T_2, T_3$ belong to a symplectic subalgebra of $\mathfrak{su}(k)$ if $k$ is even, and to an orthogonal subalgebra if $k$ is odd. These subalgebras are defined as
\[
(1.2) \quad \{ A \in \mathfrak{su}(k); \ AJ + JAT = 0 \},
\]
where $J$ is an antidiagonal matrix with $J_{i, k+1-i} = (-1)^{i-1}$. In other words, they are the fixed point sets of the involution
\[
(1.3) \quad \sigma(A) = -JA^TJ^{-1}.
\]
We denote by $\mathfrak{su}(k)^\sigma$ the subalgebra \cite{12} and by $SU(k)^\sigma$ the corresponding subgroup of $SU(k)$ ($SU(k)^\sigma \cong Sp(n)$ if $k = 2n$ and $SU(k)^\sigma \cong SO(2n+1)$ if $k = 2n+1$).

We consider the space $\mathcal{A}^\sigma$ of $\mathfrak{su}(k)^\sigma$-valued solutions to Nahm’s equations on $[0, 2]$. Let $G$ denote the group of $SU(k)$-valued gauge transformations which are identity at $t = 0, 2$, and let $G^\sigma$ be its subgroup of $SU(k)^\sigma$-valued gauge transformations. It is easy to verify that two $G$-equivalent elements of $\mathcal{A}^\sigma$ are also $G^\sigma$-equivalent. Thus the natural map $\mathcal{A}^\sigma/G^\sigma \to M^0_k$ is an embedding and we view $N_k = \mathcal{A}^\sigma/G^\sigma$ as a submanifold of $M^0_k$. $N_k$ is the fixed point set of an involution $\sigma$ which sends each $T_i(t)$ to $\sigma(T_i(t))$ (and acts the same way on gauge transformations) and therefore a complete hyperkähler submanifold of $M^0_k$.

**Proposition 1.1.** With respect to any complex structure $N_k$ is biholomorphic to the space of based rational maps $\tilde{q}(z)/\tilde{q}(z^2)$ of degree $k$ such that

(i) if $k = 2n$, then $q(z) = \tilde{q}(z^2)$ for a monic polynomial $\tilde{q}$ of degree $n$ and $p(z)p(-z) \equiv 1 \pmod{q(z)}$;
(ii) if \( k = 2n + 1 \), then \( q(z) = zq(z^2) \) for a monic polynomial \( q \) of degree \( n \), \( p(0) = 1 \) and \( p(z)p(-z) \equiv 1 \bmod q(z) \).

In particular \( \dim_{\mathbb{R}} N_{2n} = \dim_{\mathbb{R}} N_{2n+1} = 4n \).

Remark 1.2. If the roots of \( q(z) \) are distinct, then the above condition means that \( p(w)p(-w) = 1 \) for each root \( w \) of \( q(z) \) (and \( p(0) = 1 \) if \( k \) is odd). The full \( N_k \) is then the closure of this set inside the space of all rational maps.

Proof. Recall [6] that \( M_k \) is biholomorphic to the space of based (i.e. \( f(\infty) = 0 \)) rational maps of degree \( k \), and \( M^0_k \) to the submanifold consisting of rational maps \( p(z)/q(z) \) such that the sum of the poles \( z_i \) is equal to 0 and \( \prod_{i=1}^{k} p(z_i) = 1 \). From the point of view of Nahm’s equations, the rational map is obtained by applying a (singular) complex gauge transformation \( g \) to the complex Nahm equation \( \dot{\beta} = [\beta, \alpha] \) in order to make \( \beta \) a constant matrix of the form

\[
S = \begin{pmatrix}
0 & \ldots & 0 & s_n \\
1 & \ddots & 0 & s_{n-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & s_2 \\
0 & \ldots & 1 & s_1 \\
\end{pmatrix}
\]

The value of the complex gauge transformation \( g(t) \) at \( t = 2 \) (modulo a fixed singular gauge transformation) is then an element \( u \) of the centraliser of \( S \) in \( GL(k, \mathbb{C}) \). The pair \((S, u)\) corresponds to the rational map \( u(z) = z^{-1} \) (this is the description given in [2]). The complex structures of \( M^0_k \) and of \( N_k \) are obtained by assuming that \( S \) and \( u \) belong to the appropriate subalgebra and subgroup (i.e. to \( \mathfrak{sl}(k, \mathbb{C}), SL(n, \mathbb{C}) \) for \( M^0_k \) and to \( \mathfrak{sl}(k, \mathbb{C})^\sigma, SL(n, \mathbb{C})^\sigma \) for \( N_k \)). It is enough to consider the subset where the poles of the rational map, i.e. the eigenvalues of \( S \), are distinct (since \( N_k \) is the closure of this set in the space of all rational maps).

A Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{sl}(k, \mathbb{C})^\sigma \) is given by the diagonal matrices \( h \) satisfying \( h_{ii} + h_{k+1-i,k+1-i} = 0 \), \( i = 1, \ldots, k \). It is then immediate that if \((S, u)\) is conjugate to an element of \( \mathfrak{h} \times H \) (\( H = \exp \mathfrak{h} \)), then the corresponding rational map satisfies the conditions in the statement. \( \square \)

In [3] Ex. 5.4 we have identified the complex manifold described in the above proposition for \( k = 2n \) as the Hilbert scheme of \( n \) points on the \( D_1 \)-surface \( x^2 - zy^2 = 1 \) transverse to the projection \((x, y, z) \mapsto z\) (similarly, the space of all rational maps of degree \( k \) is the Hilbert scheme of \( k \) points on \( \mathbb{C}^* \times \mathbb{C} \) transverse to the projection onto the second factor [1 Ch. 6]). It turns out that for odd \( k \) the complex structure of \( N_k \) is that of the transverse Hilbert scheme of points on the \( D_0 \)-surface \( x^2 - zy^2 - y = 0 \):

Proposition 1.3. (i) With respect to any complex structure \( N_{2n} \) is biholomorphic to the Hilbert scheme of \( n \) points on the \( D_1 \)-surface \( x^2 - zy^2 = 1 \) transverse to the projection \((x, y, z) \mapsto z\).

(ii) With respect to any complex structure \( N_{2n+1} \) is biholomorphic to the Hilbert scheme of \( n \) points on the \( D_0 \)-surface \( x^2 - zy^2 - y = 0 \) transverse to the projection \((x, y, z) \mapsto z\).

Remark 1.4. The fact that \( N_3 \) is biholomorphic to the \( D_0 \)-surface has been observed by Houghton and Sutcliffe [9].
subvarieties, namely the transverse Hilbert scheme of points on the 
subvariety inside the affine variety of all rational maps. Since the two closed affine
the condition
fundamental group of the full Hilbert scheme
X
p
p
to the condition
easy to see that if the roots of
q
f
Let us write
π
X
X
π
Puppe [5, Thm. 12.15] which says that

Proof. Part (i) has already been shown in [3, Ex. 5.4]. For part (ii) recall from [3] that the Hilbert scheme of \( n \) points on the \( D_0 \)-surface \( x^2 - zy^2 + y = 0 \) transverse to the projection \( (x, y, z) \to z \) is an affine variety in \( \mathbb{C}^n \) given by the same equation, but for polynomials. More precisely its points are polynomials \( x(z), y(z), r(z) \) with degrees of \( x \) and \( y \) at most \( n - 1 \) and \( r(z) \) a monic polynomial of degree \( n \), satisfying the condition
\[
x(z)^2 - zy(z)^2 - y(z) = 0 \mod r(z).
\]
As in [3] write \( z = u^2 \) and rewrite the above equation as
\[
(x(u^2) + uy(u^2))(x(u^2) - uy(u^2)) - \frac{(x(u^2) + uy(u^2)) - (x(u^2) - uy(u^2))}{u} = 0 \mod r(u^2).
\]
Let us write \( f(u) = x(u^2) + uy(u^2), \ p(u) = 1 + uf(u), \ q(u) = ur(u^2) \). Then it is easy to see that if the roots of \( q \) are distinct, then the last equation is equivalent to the condition \( p(w)p(-w) = 1 \) for any nonzero root \( w \) of \( q(u) \). Since this last equation is polynomial in the coefficients of \( p \) and \( q \), it describes a closed affine subvariety inside the affine variety of all rational maps. Since the two closed affine subvarieties, namely the transverse Hilbert scheme of points on the \( D_0 \)-surface and the variety described in Proposition [14] have a common open dense subset, they must coincide.

We can now compute the fundamental group of \( N_k \):

**Proposition 1.5.**

\[
\pi_1(N_k) = \begin{cases} 
1 & \text{if } k \text{ is even}, \\
\mathbb{Z}_2 & \text{if } k \text{ is odd}.
\end{cases}
\]

**Proof.** The fundamental groups of the \( D_0 \)- and \( D_1 \)-surface (i.e. the Atiyah-Hitchin manifold and its double cover) are well-known [11] and equal to \( \mathbb{Z}_2 \) and to 1 respectively. The result follows from

**Lemma 1.6.** Let \( X \) be a smooth complex surface and \( \pi : X \to C \) a holomorphic submersion onto a connected Riemann surface \( C \). Suppose further that, over an open dense subset of \( C \), \( \pi \) is a locally trivial fibration with connected fibres. Then the fundamental group of the transverse Hilbert scheme \( X^{[n]} \) of \( n \) points, \( n \geq 2 \), is equal to \( H_1(X, \mathbb{Z}) \).

We first observe that if \( X \) is a smooth complex surface and \( n \geq 2 \), then the fundamental group of the full Hilbert scheme \( X^{[n]} \) of \( n \) points, \( n \geq 2 \), is equal to \( H_1(X, \mathbb{Z}) \). This follows by combining two facts: 1) a classical result of Dold and Puppe [5, Thm. 12.15] which says that \( \pi_1(S^n X) \simeq H_1(X, \mathbb{Z}) \) for \( n \geq 2 \), and 2) a theorem of Kollár [11, Thm. 7.8] which implies that \( \pi_1(X^{[n]}) \simeq \pi_1(S^n X) \).

We now aim to show that \( \pi_1(X^{[n]}_+ \simeq \pi_1(X^{[n]}) \). Let \( Y \) denote a submanifold of \( X^{[n]} \) consisting of \( D \in X^{[n]} \) with \( \pi(D) = z_0 + E \), where \( E \) has length \( n - 2 \) in \( C \), \( z_0 \notin \text{supp } E \), and \( D \cap \pi^{-1}(z_0) \) consists of two distinct points. Clearly \( Y \cap X^{[n]}_+ = \emptyset \). Denote by \( U \) a tubular neighbourhood of \( Y \) and let \( Z = U \cup X^{[n]}_+ \), \( W = U \cap X^{[n]}_+ \).

Since the complement of \( Z \) in \( X^{[n]} \) has (complex) codimension 2, the fundamental groups of \( Z \) and of \( X^{[n]} \) coincide. Since \( W \) is a punctured disc bundle over \( Y \), the long exact sequence of homotopy groups implies that the map \( \pi_1(W) \to \pi_1(U) \) is surjective and its kernel consists of at most the “meridian loop” around \( Y \). Owing
to the assumption, we can choose a point \( y \) of \( Y \) so that a neighbourhood of the fibre containing two points is isomorphic to \( B \times F \), where \( B \) is a disc \( \{ z \in \mathbb{C} : |z| < 1 + \varepsilon \} \) and \( F \) is the generic fibre of \( \pi \). The intersection of a neighbourhood of \( y \) in \( Z \) with \( W \) is then of the form \( S^2_2(B \times F) \times V \), where the subscript \( \pi \) means that the pair of points \( \{ x_1, x_2 \} \subset B \times F \) satisfies \( \pi(x_1) \neq \pi(x_2) \) and \( V \) is an open subset in \( X^{[n-2]}_\pi \).

Let \( \rho \) be the projection \( (B \times F) \times (B \times F) \to S^2_2(B \times F) \). A “meridian loop” in \( W \) around \( Y \) can then be chosen to be

\[
t \mapsto (\rho(e^{\pi i t}, f, e^{-\pi i t}, f), v), \quad t \in [0, 1]
\]

for constant \( f \) and \( v \). This loop is contractible in \( X^{[n]}_\pi \): the homotopy

\[
H(r, t) = (\rho(re^{\pi i t}, f, re^{-\pi i t}, f), v), \quad t, r \in [0, 1]
\]

contracts it to \((D, v)\) where \( D \) is the double point \( \{(z^2 = 0, f) \} \subset B \times F \). To recapitulate: we have shown that the map \( \pi_1(W) \to \pi_1(U) \) is surjective and its kernel has trivial image in \( \pi_1(X^{[n]}_\pi) \). It follows that the amalgamated free product \( \pi_1(U)*_{\pi_1(W)} \pi_1(X^{[n]}_\pi) \) is isomorphic to \( \pi_1(X^{[n]}_\pi) \) and, hence, van Kampen’s theorem implies that \( \pi_1(Z) \cong \pi_1(X^{[n]}_\pi) \).

Remark 1.7. The \( D_0 \)-surface \( X \) is the quotient of the \( D_1 \)-surface \( \tilde{X} \) by a free action of \( \mathbb{Z}_2 \) given by \((x, y, z) \mapsto (-x, -y, z)\). This induces a free \( \mathbb{Z}_2 \)-action on the transverse Hilbert scheme \( \tilde{X}^{[n]}_\pi \) of \( n \) points for any \( n \), but \( \tilde{X}^{[n]}_\pi / \mathbb{Z}_2 \not\cong X^{[n]}_\pi \), unless \( n = 1 \). Certainly, there is a surjective holomorphic map \( \tilde{X}^{[n]}_\pi \to X^{[n]}_\pi \) for any \( n \), which, in the description of these spaces provided in Proposition 1.1, sends a rational function \( p(z)/q(z) \in \tilde{X}^{[n]}_\pi \) to \( \tilde{p}(z)/zq(z) \), where \( \tilde{p}(z) = p(z)^2 \mod zq(z) \). This map is constant on \( \mathbb{Z}_2 \)-orbits, but it is generically \( 2^n \)-to-1 and not a covering (the preimage of a point consists of \( 2^m \) points, where \( 2m \) is the number of distinct roots of \( q(z) \)).

Remark 1.8. It is instructive to compare spectral curves of monopoles in \( N_{2n+1} \) to those in \( N_{2n} \). It follows from Proposition 1.1 that the spectral curve \( S \) of a monopole in \( N_{2n+1} \) is always singular and given by an equation of the form \( \eta P(\zeta, \eta) = 0 \), where \( \zeta \) is the affine coordinate of \( \mathbb{P}^1 \), \( \eta \) is the induced fibre coordinate in \( TP^1 \), and \( P \) a polynomial of the form \( \eta^{2n} + \sum_{i=1}^n a_i(\zeta)^{2n-2i}, \deg a_i(\zeta) = 4i \). A spectral curve of a monopole satisfies the condition \( L^2|S \simeq O \), where \( L^2 \) is a line bundle on \( TP^1 \) with transition function \( \exp(2\eta/\zeta) \). It follows that the line bundle \( L^2 \) is also trivial on the curve \( \tilde{S} \) defined by the equation \( P(\zeta, \eta) = 0 \). Conversely, the spectral curve \( \tilde{S} \) of a monopole in \( N_{2n} \) admits a section \( s(\zeta, \eta) \) of \( L^2 \) which satisfies \( s(\zeta, \eta)s(\zeta, -\eta) \equiv 1 \mod P(\zeta, \eta) \), where \( P(\zeta, \eta) = 0 \) is the equation of \( \tilde{S} \). It follows that \( s(\zeta, 0) = \pm 1 \) and if we set \( \tilde{s}(\zeta, \eta) = s^2(\zeta, \eta) \) on \( \tilde{S} \) and \( s(\zeta, \eta) \equiv 1 \) on \( \eta = 0 \), we obtain a nonvanishing section of \( L^4 \) on the curve \( \eta P(\eta, \zeta) = 0 \), i.e. a section of \( L^2 \) on the curve \( S \) given by \( \tilde{\eta} P(\tilde{\eta}/2, \zeta) = 0 \), where \( \tilde{\eta} = 2\eta \). In the case \( n = 1 \), Houghton and Sutcliffe [6] have shown that for \( n = 1 \) these maps \( S \mapsto \tilde{S} \) and \( \tilde{S} \mapsto S \) send spectral curves of monopoles in \( N_3 \) to spectral curves of monopoles in \( N_2 \) and vice versa [4], but for higher \( n \) this is not the case. The reason is that Hitchin’s nonsingularity condition \( H^0(S, L^4(k - 2)) = 0, t \in (0, 2) \), is not necessarily satisfied for the resulting curves.

\footnote{Strictly speaking, the curve \( \tilde{S} \) obtained from \( S \) must be rescaled via \( \eta = 2\tilde{\eta} \) in order to be the spectral curve of a monopole in \( N_2 \).}
2. Deformations and coverings

Dancer [4] has shown that the $D_1$-surface admits a 1-parameter family of deformations carrying complete hyperkähler metrics. As we observed in [3], the transverse Hilbert schemes of points on these deformations also admit natural complete hyperkähler metrics. We wish to describe these metrics as deformations of manifolds $N_{2n}$. We begin by describing $N_k$ without reference to an embedding into $M_k$.

Let $G_k$ (resp. $g_k$) denote $Sp(n)$ (resp. $sp(n)$) if $k = 2n$ and $SO(2n + 1)$ (resp. $so(k)$) if $k = 2n + 1$. The construction of the previous section shows that $N_k$ is the moduli space of $g_k$-valued solutions to Nahm’s equations on $(0, 2)$ with simple poles at $t = 0, 2$ and residues defining the principal homomorphism $su(2) \to g_k$, modulo $G_k$-valued gauge transformations which are identity at $t = 0, 2$. This moduli space is, in turn, a finite-dimensional hyperkähler quotient of a simpler hyperkähler manifold. Let $W_k^-$ (resp. $W_k^+$) be the moduli space of $g_k$-valued solutions to Nahm’s equations on $(0, 1]$ (resp. $[1, 2]$) with the above boundary behaviour at $t = 0$ (resp. at $t = 2$) and regular at $t = 1$, modulo $G_k$-valued gauge transformations which are identity at $t = 0, 1$ (resp. at $t = 1, 2$). $W_k^\pm$ are hyperkähler manifolds (biholomorphic to $G_k^\pm \times \mathbb{C}^n$ [2]) with an isometric and triholomorphic action of $G_k$ obtained by allowing gauge transformations with an arbitrary value at $t = 1$. Then $N_k$ is the hyperkähler quotient of $W_k^- \times W_k^+$ by the diagonal $G_k$.

We can also describe in a similar manner the universal (i.e. double) covering space of $N_{2n+1}$: it is given by the same construction, but with $G_{2n+1} = Spin(2n+1)$ instead of $SO(2n+1)$.

An alternative construction of the $N_k$ proceeds as follows. Let $G$ denote one of the groups $Sp(n)$, $SO(2n + 1)$, or $Spin(2n + 1)$ and let $\tau$ be an automorphic involution on $G$ with fixed point set $K$. Consider the hyperkähler quotient $Y^-$ of $W_k^-$ by $K$ (with zero-level set of the moment map). Let $(T_0, T_1, T_2, T_3)$ be a solution to Nahm’s equations corresponding to a point in $Y^-$. Modulo gauge transformations we can assume that $T_0(1) = 0$. We can then extend this solution to a solution to Nahm’s equations on $(0, 2)$ by setting

\[(2.1)\]

\[T_i(2 - t) = -\tau(T_i(t)), \quad i = 0, 1, 2, 3.\]

This solution has the boundary behaviour of a solution in $N_k$ and we can describe the moduli space $Y$ of such extended solutions as the space of solutions on $(0, 2)$ having the correct poles and residues at $t = 0, 2$ and satisfying (2.1), modulo $G$-valued gauge transformations $g(t)$ such that

\[(2.2)\]

\[g(0) = g(2) = 1, \quad g(2 - t) = \tau(g(t)), \quad t \in [0, 2].\]

The map $Y^- \to Y$ is a triholomorphic homothety with factor 2. For dimensional reasons $Y^-$ (and consequently $Y$) is empty unless $\mathfrak{k} = u(n)$ for $k = 2n$ or $\mathfrak{k} = so(n) \oplus so(n + 1)$ for $k = 2n + 1$. Thus there are the following three possibilities for the symmetric pair $(G, K)$:

(i) $k = 2n$, $G = Sp(n)$ and $K = U(n);
(ii) k = 2n + 1, G = SO(2n + 1)$ and $K = S(O(n) \times O(n + 1));$
(iii) $k = 2n + 1$, $G = Spin(2n + 1)$ and $K$ is the diagonal double cover of $SO(n) \times SO(n + 1)$ (i.e. $K = Spin(n) \times Spin(n + 1) \setminus \{(1, 1), (-1, -1)\}.$

An easy computation shows that in each case $\dim Y = \dim N_k$. Moreover, the natural map $Y \to N_k$ is an isometric (and triholomorphic) immersion, and since
both \( N_k \) and \( Y \) are complete ([3] Thm. A.1), this map must be a covering. Thus it follows from Proposition 1.5 that \( Y \) is isometric to \( N_k \) in cases (i) and (ii), while in case (iii) \( Y \) is the universal cover of \( N_{2n+1} \).

The above construction allows us easily to describe a family of deformations of \( N_{2n} \). Indeed, the Lie algebra \( \mathfrak{k} = \mathfrak{u}(n) \) has a nontrivial centre and, therefore, we can take hyperkähler quotients of \( W_{2n}^{-} \) by \( K \) at nonzero level sets of the hyperkähler moment map. This produces a 3-parameter family of hyperkähler deformations of \( N_{2n} \).

Arguments analogous to those in [3] show that these are the natural hyperkähler metrics on the transverse Hilbert schemes of points on Dancer’s deformations of the \( D_1 \)-surface.

**Remark 2.1.** As already mentioned in Remark 1.7, \( N_{2n} \) admits a free action of \( \mathbb{Z}_2 \) for any \( n \). This action is also isometric and triholomorphic and, hence, \( N_{2n}/\mathbb{Z}_2 \) is a hyperkähler manifold. This manifold can be described in the same way as \( N_{2n} \) but with \( G_{2n} = \mathbb{P}Sp(n) \) rather than \( Sp(n) \).

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**Institut für Differentialgeometrie, Universität Hannover, Welfengarten 1, D-30167 Hannover**