Intelligent sampling for multiple change-points in exceedingly long time series with rate guarantees

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Summary. Change point estimation in its offline version is traditionally performed by optimizing over the data set of interest, by considering each data point as the true location parameter and computing a data fit criterion. Subsequently, the data point that minimizes the criterion is declared as the change point estimate. For estimating multiple change points, the procedures are analogous in spirit, but significantly more involved in execution. Since change-points are local discontinuities, only data points close to the actual change point provide useful information for estimation, while data points far away are superfluous, to the point where using only a few points close to the true parameter is just as precise as using the full data set. Leveraging this "locality principle", we introduce a two-stage procedure for the problem at hand, which in the 1st stage uses a sparse subsample to obtain pilot estimates of the underlying change points, and in the 2nd stage refines these estimates by sampling densely in appropriately defined neighborhoods around them. We establish that this method achieves the same rate of convergence and even virtually the same asymptotic distribution as the analysis of the full data, while reducing computational complexity to $\sqrt{N}$ time ($N$ being the length of data set), as opposed to at least $O(N)$ time for all current procedures, making it promising for the analysis on exceedingly long data sets with adequately spaced out change points. The main results are established under a signal plus noise model with independent and identically distributed error terms, but extensions to dependent data settings, as well as multiple stage ($>2$) procedures (which deliver even bigger computational gains without losing precision in truly humongous time series) are also provided. The performance of our procedure – which is coined “intelligent sampling” – is illustrated on both synthetic and real Internet data streams.

1. Introduction

Change point analysis has been extensively studied in both the statistics and econometrics literature [2], due to its wide applicability in many application areas, including economics and finance [8], quality control [22], neuroscience [15], etc. A change point represents a discontinuity in the parameters of the data generating process. The literature has investigated both the offline and online versions of the problem [2, 4]. In the former case, one is given a sequence of observations and questions of interest include: (i) whether there exists a change point and (ii) if there exists one (or multiple) change point(s), identify their location, as well as estimate the parameters of the data generating process to the left and right of it (them). In the latter case, one is obtaining new observations in a sequential manner and the main interest is in quickest detection of the change point.

In the era of big data, researchers are faced with large numbers of observations whose processing and storage pose a number of computational challenges. For example, monitoring traffic in high-speed computer and communication networks and trying to identify shifting user patterns, as well as malicious activities is of great interest to network engineers (see [12] and references therein). However, the number of observations produced corresponding to number of packets or their corresponding payload in bytes at a granular temporal scale is in the thousands per minute. Nevertheless, it is of value to network operators to examine these traces and if something of interest is identified, to store the appropriate segment for further analysis. Analogous problems come up in other systems including manufacturing processes [24], or other cyber-physical systems.
equipped with sensors [13]. This exercise requires temporary storage of the data, possibly in a distributed manner, as well as doing the necessary calculations on them.

In this paper, we address the offline problem for extremely long time-series data, the analysis of which by conventional modes of analysis turns out to be prohibitive owing to the massive size involved. The identification of change-points in a sequence, which constitute local discontinuities, requires some sort of a search procedure. A list of such methods, along with summaries of their results, can be found in [20]. Exhaustive search is typically infeasible with multiple change-points, since the search-time grows exponentially in the number of change-points. A variety of intelligent methods have been proposed in the literature, e.g. binary segmentation (see [25] and [7]), wild binary segmentation ([7]), multi-scale methods ([5] and [21]), \( \ell_1 \) penalization ([11] and [9]): however, the best feasible computation time is of the order \( N \) or \( N \log N \), which, while being reasonable, is still a huge ask when one is considering a sequence of observations of length in the tens or hundreds of millions or even larger.

Hence, our key objective is to propose an effective solution to this problem via a strategy called “intelligent sampling”, which proceeds by making two (or more) passes over the time-series at different levels of resolution. The first pass is carried out at a coarse time resolution and enables the analyst to obtain pilot estimates of the change-points, while the second investigates stretches of the time-series in (relatively) small neighborhoods of the initial estimates to produce updated estimates enjoying a high level of precision; in fact, essentially the same level of precision as would be achieved by an analysis of the entire massive time-series. The core advantage of our proposed method is that it reduces computational time from linear to sub-linear under appropriate conditions, and, in fact close to square-root order, if the number of change-points is small relative to the length of the time-series. It is established that the computational (and analogously other processing gains from input-output operations) can be achieved without compromising any statistical efficiency.

The point of view we adopt in the paper is the following: we are given a time-series of massive length and are interested in identifying the major changes in this series. While changes of various intensities in mean shifts may occur frequently, their duration is short lived and hence they do not affect the overall performance of most engineered or physical systems. Hence, it is reasonable to assume that number of truly significant changes that persist over time, is not particularly large. However, sifting through the entire mass of data to detect those changes is computationally expensive and on many occasions even prohibitive. As discussed above, intelligent sampling provides a clever way of overcoming this problem. The modeling strategy uses a very long piece-wise constant time-series model with multiple jumps – where the number of jumps increases at a rate much slower than the length of the series — that are not too small relative to the fluctuations induced by noise. The ‘piece-wise constant with jumps’ strategy has been considered by a variety of authors (see [20] and references therein) for the problem at hand and is convenient for developing the methodology. In addition, and consistent with our point of view, even if the signal between the big change-points is not completely flat, but shows some variability or small magnitude discontinuities that are inconsequential compared to noise, the posited piecewise constant model is an attractive working model.

Most of our results will be rigorously developed in the signal plus noise model context of independent and identically distributed (iid) errors, as is the case with a significant portion of the literature; however, we also provide results/indicate extensions when the errors exhibit short or long-range dependence and in the presence of non-stationarity, since both these features are likely to be present with very long time series. Some empirical evidence is provided to this effect. Further, the focus of the presentation is on 2-stage procedures that provide all the key insights into the workings of the intelligent sampling procedure. However, for settings where the size of the data exceeds \( 10^{10} \) multiple stages are required to bring down the analyzed subsample to a manageable size. We therefore cover extensions to multi-stage intelligent sampling procedures as well as address how samples should be allocated at these different stages. Furthermore, such massive time series, often, cannot be effectively stored in a single location. This does not pose a problem for intelligent sampling as it adapts well to distributed computing: it can be applied on the reduced size subsamples at the various locations where the original data are stored, followed by a single subsequent back and forth communication between the various locations and the central server, and subsequent calculations essentially carried out on the local servers. This is elaborated on in Section 5.

The remainder of the paper is organized as follows: Section 2 addresses intelligent sampling for the simpler single change point problem, which nevertheless provides much insight into the nature of the procedure and its theoretical and computational performance. Section 3 addresses the main topic of this study, namely intelligent sampling for the multiple change point problem. Extensions to the non-iid setting are presented in Section 4 and computational guidelines for practical implementation are discussed at length in Section 5.
2. Intelligent Sampling for the Single Change Point Problem

The key advantages of, as well as the intuition behind the proposed intelligent sampling strategy become abundantly clear in the simpler single change point problem, and thus this constitutes the starting point of our investigation.

The simplest possible setting for the change point problem is the stump model, where data \((X_1, Y_1), \ldots, (X_N, Y_N)\) are available with \(X_i = i/N, Y_i = f(X_i) + \varepsilon_i\) for \(i = 1, \ldots, N\), and where the error term \(\varepsilon_i\) is independent and identically distributed (iid) following a \(N(0, \sigma^2)\) distribution, while the function \(f\) takes the form

\[
f(x) = \alpha \cdot 1(x \leq \tau) + \beta \cdot 1(x > \tau), \quad x \in (0, 1),
\]

for some constants \(\alpha, \beta \in \mathbb{R}, \alpha \neq \beta,\) and \(\tau \in (0, 1)\): the so-called ‘stump’ model. For estimating the change point \(\tau\) we employ a least squares criterion, given by

\[
(\hat{\alpha}, \hat{\beta}, \hat{\tau}) := \arg \min_{(a,b,t) \in \mathbb{R}^2 \times (0,1)} \sum_{i=1}^{N} (Y_i - a \cdot 1(X_i \leq t) - b \cdot 1(X_i > t))^2.
\]

Using techniques similar to those in Section 14.5 of [17] that analyzes a stump model where the \(X_i\)'s are random variables generated from some density over \((0,1)\), we can establish that the estimator \(\hat{\tau}\) is consistent for \(\tau_N := [N\tau]/N\), which acts as the change point among the grid-points \(X_i\)'s.

Proposition 1. For the stump model with normal errors the following hold:
(i) Both \((\hat{\alpha} - \alpha)\) and \((\hat{\beta} - \beta)\) converge to 0 with rate \(O_p(N^{-1/2})\).
(ii) The change point estimate \(\hat{\tau}\) satisfies

\[
\mathbb{P}[N(\hat{\tau} - \tau_N) = k] \rightarrow \mathbb{P}[L = k] \text{ for all } k \in \mathbb{Z}
\]

where \(L := \arg\min_{i \in \mathbb{Z}} Z_i\), and the random walk \(\{Z_i\}_{i \in \mathbb{Z}}\) is defined as

\[
Z_i = \begin{cases} 
\Delta(\varepsilon_1 + \ldots + \varepsilon_{i-1} + \varepsilon_i) + i\Delta^2/2, & i > 0 \\
0, & i = 0 \\
-\Delta(\varepsilon_{i+1} + \ldots + \varepsilon_{-1} + \varepsilon_0) + |i|\Delta^2/2, & i < 0,
\end{cases}
\]

with \(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\) and \(\varepsilon_{-1}, \varepsilon_{-2}, \ldots\) being iid \(N(0, \sigma^2)\) random variables and \(\Delta := \beta - \alpha\).

Next, we make several notes on the random variable \(L\) introduced in the above proposition, as it appears multiple times throughout the remainder of this paper. Although, at a glance, the distribution of \(L\) depends on two parameters, \(\Delta\) and \(\sigma\), in actuality \(L\) is completely determined by the signal-to-noise ratio \(\Delta/\sigma\). To see this, note that we can re-write \(L = \arg\min_{i \in \mathbb{Z}} (Z_i/|\Delta\sigma|)\) where

\[
\frac{Z_i}{|\Delta\sigma|} = \begin{cases} 
\text{sgn}(\Delta)(\varepsilon_1/\sigma + \cdots + \varepsilon_i/\sigma) + i|\Delta/\sigma|/2, & i > 0 \\
0, & i = 0 \\
-\text{sgn}(\Delta)(\varepsilon_{i+1}/\sigma + \cdots + \varepsilon_0/\sigma) + |i|\Delta/\sigma|/2, & i < 0
\end{cases}
\]

Since \(\{\text{sgn}(\Delta)\varepsilon_i/\sigma\}_{i \in \mathbb{Z}}\) are iid \(N(0,1)\) random variables, invariant under \(\Delta\) and \(\sigma\), it follows that \(L\) only depends on the single parameter \(\Delta/\sigma\). Hence, from here on, we will denote the argmin of the random walks described in (4) as \(L(\Delta/\sigma)\).

Further, note that the \(L(\cdot)\) distribution is stochastically ordered. Specifically:

Proposition 2. Suppose we have constants \(\Delta_1, \Delta_2 \in \mathbb{R}\) such that \(0 < |\Delta_1| < |\Delta_2|\), then for any positive integer \(k\)

\[
\mathbb{P}[|L(\Delta_1)| \leq k] \leq \mathbb{P}[|L(\Delta_2)| \leq k]
\]
Practically, this stochastic ordering implies that if the $1 - \alpha$ quantile $Q_\alpha(\Delta_1)$ of $|L(\Delta_1)|$ is known, then $Q_\alpha(\Delta_1)$ can also serve as a conservative $1 - \alpha$ quantile of $|L(\Delta_2)|$ for any $\Delta_2 \geq \Delta_1$. This can be useful in settings where given $J > 0$ random variables $L_i \sim L(\Delta_1)$ for $i = 1, \ldots, J$, we desire positive integers $\ell_i$ for $i = 1, \ldots, J$ such that $\mathbb{P}[|L_i| \leq \ell_i] \geq 1 - \alpha$ for $i = 1, \ldots, J$. This scenario will appear in a multiple change point setting considered in Theorems 5 and 6. In such a situation, a simple solution is to take $\ell_i = Q_\alpha(\Delta_m)$ for all $i$ where $m = \arg \min_i |\Delta_i|$, or in other words letting each $\ell_i$ be the $1 - \alpha$ quantile of the $|L(\Delta_i)|$ with the smallest parameter. Alternatively we can generate a table of quantiles for distributions $L(\Delta_1), L(\Delta_2), \ldots$ for a mesh of positive constants $\delta_1 < \delta_2 < \ldots$ (e.g. we can let the $\delta_j = 0.1j$ for $j = 5, \ldots, 1000$), and let $\ell_i = Q_\alpha(\delta_j)$ where $\delta_j = \max\{\delta_k: \delta_k \leq \Delta_i\}$, for $i = 1, \ldots, J$.

2.1. The Intelligent Sampling Procedure and its Properties

Our proposed procedure leverages the following locality principle: in a stump model, only the points near the actual change point $\tau$ are relevant to its estimation. Put another way, consider the interval $U_N = [\tau - w(N), \tau + w(N)]$ for some sequence $w(N)$ such that $w(N) \to 0$ but $Nw(N) \to \infty$, then the least squares estimate using the dataset $\{(X_i, Y_i)\}_{X_i \in U_N}$ is asymptotically as accurate as the least squares estimate from the full dataset $\{(X_i, Y_i)\}_{1 \leq i \leq N}$. Intuitively, this is true because the interval $U_N$ contains $N^{-1}$-resolution data points surrounding the actual jump (change point), while the two intervals $[0, \tau - w(N))$ and $(\tau + w(N), 1]$ both contain data that are irrelevant to identifying the change-point. We also refer to [18], which demonstrated the validity of this locality principle in a different setting. The locality principle hints that if one identifies such an interval $U_N$ and only uses the data-points within it, then an estimate can be obtained that is much faster to compute than the conventional estimator using all data points, while at the same time without suffering any loss of accuracy. The following two steps will therefore be the main strategy for intelligent sampling: first aim to find a neighborhood containing $\tau$, and then estimate it using only points in that neighborhood. To execute the first step, we propose using a sparse, evenly spaced subsample to construct an initial confidence region for the change-point:

(\text{ISS1}): From the full data set of $\{(\frac{1}{N}, Y_1), (\frac{2}{N}, Y_2), \ldots, (1, Y_N)\}$, take an evenly spaced subsample of approximately size $N_1 = K_1 N^\gamma$ for some $\gamma \in (0, 1)$, $K_1 > 0$: thus, the data points are $\left(\frac{|N/N_1|}{N}, Y_{|N/N_1|}\right), \left(\frac{2|N/N_1|}{N}, Y_{2|N/N_1|}\right), \ldots$.

(\text{ISS2}): On this subsample apply least squares to obtain estimates $\left(\hat{\alpha}(1), \hat{\beta}(1), \hat{\tau}(1)_N\right)$ for parameters $(\alpha, \beta, \tau_N)$.

By the results for the single change-point problem presented above, $\hat{\tau}(1)_N - \tau_N$ is $O_p(\sqrt{N})$. Therefore, if we take $w(N) = K_2 N^{-\gamma + \delta}$ for some small $\delta > 0$ (much smaller than $\gamma$) and any constant $K_2 > 0$, with probability increasing to 1, $\tau_N \in [\hat{\tau}(1)_N - w(N), \hat{\tau}(1)_N + w(N)]$. In other words, this provides a neighborhood around the true change point as desired; hence, in the next stage only points within this interval will be used.

(\text{ISS3}): Fix a small constant $\delta > 0$. Consider all $i/N$ such that $i/N \in \left[\hat{\tau}(1)_N - K_2 N^{-\gamma + \delta}, \hat{\tau}(1)_N + K_2 N^{-\gamma + \delta}\right]$ and $(i/N, Y_i)$ was not used in the first subsample. Denote the set of all such points as $S(2)$.

(\text{ISS4}): Fix a step function on this second subsample by minimizing

$$\sum_{i/N \in S(2)} \left(Y_i - \hat{\alpha}(1)1(i/N \leq d) - \hat{\beta}(1)1(i/N > d)\right)^2$$

with respect to $d$, and take the minimizing $d$ to be the second stage change point estimate $\hat{\tau}(2)_N$.

**Remark 1.** We note that in this procedure, no first stage points are employed in the second stage. This is mainly for simplifying the mathematical mechanics, as conditional on $\left(\hat{\alpha}(1), \hat{\beta}(1), \hat{\tau}(1)_N\right)$, the $Y_i$’s used in the second stage are independent normal random variables with common variance. However, we still obtain the desired precision by leaving these points out, since out of the the approximately $2K_2 N^{1-\gamma + \delta}$ points in the interval $\left[\hat{\tau}(1)_N - K_2 N^{-\gamma + \delta}, \hat{\tau}(1)_N + K_2 N^{-\gamma + \delta}\right]$, only about $2K_2 N^\delta$ points, a vanishingly small fraction of the $2K_2 N^{1-\gamma + \delta}$ points in the interval, are omitted.
The next theorem establishes that the intelligent sampling estimator $\hat{\tau}_N^{(2)}$ is consistent with the same rate of convergence as the estimator based on the full data.

**Theorem 1.** For the stump single change point model, the estimator obtained based on intelligent sampling satisfies

$$|\hat{\tau}_N^{(2)} - \tau_N| = O_p(1/N).$$

**Proof.** Special case of Theorem 7 which is in turn proved in section 9.2.1. □

In addition, one expects the estimator $\hat{\tau}_N^{(2)}$ and the estimator from the full data $\tilde{\tau}_N$ to converge to the same distribution after re-centering and re-scaling. This turns out to be almost the case, with a slight deviation owing to not using the first stage points at stage two. The problem arises from the fact that $\hat{\tau}_N^{(2)}$ cannot equal any integer multiple of $\lfloor \frac{|N_1/N|}{N} \rfloor$. Depending on the value of $\tau$, this can even create situations where $N\left|\hat{\tau}_N^{(2)} - \tau_N\right|$ does not converge to any distribution, as explained below in Remark 2.

To address this technical issue, we introduce a slight modification to the definition of the true change point and define a new type of distance function $\lambda_2 : [0, 1]^2 \rightarrow \mathbb{Z}$, as follows. First, for convenience, denote the set of $X_i$’s of the first stage subsample as

$$S^{(1)} := \left\{ \frac{i}{N} : i \in \mathbb{N}, i < N, i \text{ is divisible by } \lfloor N/N_1 \rfloor \right\},$$

then for any $a, b \in (0, 1)$

$$\lambda_2(a, b) := \begin{cases} \sum_{i=1}^{N} 1 \left( \frac{i}{N} < a/N \leq \frac{i}{N} \notin S^{(1)} \right) & \text{if } a \leq b, \\ -\sum_{i=1}^{N} 1 \left( \frac{i}{N} < a/N \leq \frac{i}{N} \notin S^{(1)} \right) & \text{otherwise}. \end{cases}$$

The modified distance $\tilde{\tau}_N$ is $\lambda_2(\tau_N, \hat{\tau}_N^{(2)})$, instead of $N(\hat{\tau}_N^{(2)} - \tau_N)$, does converge weakly to a distribution.

**Theorem 2.** Let $L(\Delta/\sigma)$ be as defined above; then, for any integer $\ell$,

$$\mathbb{P} \left[ \lambda_2(\tau_N, \hat{\tau}_N^{(2)}) = \ell \right] \rightarrow \mathbb{P}[L(\Delta/\sigma) = \ell].$$

**Proof.** Special case of Theorem 8 which in turn is proved in Section 9.2.2. □

Hence, in terms of this $\lambda_2$ distance, the deviation between the true change point and the intelligent sampling estimator actually matches up with the full data estimator in terms of the asymptotic distribution. A natural question one could ask at this point is: how different is the $\lambda_2$ distance from the ”natural” distance? To answer this, we explain it in terms of confidence intervals. For any fixed $\alpha \in (0, 1)$, first let $L_\alpha$ be smallest non-negative integer such that

$$\mathbb{P} \left[ \left| L(\Delta/\sigma) \right| \leq L_\alpha \right] \geq 1 - \alpha. \tag{10}$$

The $1 - \alpha$ honest confidence interval for $\tau_N$ constructed using $\hat{\tau}_N^{(2)}$ would be $[A_N, B_N]$, where

$$A_N = \min \left\{ \frac{i}{N} : i \in \mathbb{N}, \lambda_2 \left( \frac{i}{N}, \hat{\tau}_N^{(2)} \right) = L_\alpha \right\},$$

$$B_N = \max \left\{ \frac{i}{N} : i \in \mathbb{N}, \lambda_2 \left( \frac{i}{N}, \hat{\tau}_N^{(2)} \right) = L_\alpha \right\}. \tag{11}$$

Meanwhile, the $1 - \alpha$ honest confidence interval constructed using the full data estimator $\tilde{\tau}_N$ would be $[\tilde{\tau}_N - L_\alpha/N, \tilde{\tau}_N + L_\alpha/N]$. The widths of the two confidence intervals are respectively $B_N - A_N$ and $2L_\alpha/N$. Because $\lambda_2(A_N, B_N) = 2L_\alpha$, for sufficiently large $N$, $B_N - A_N$ could only equal $2L_\alpha/N$ or $(2L_\alpha + 1)/N$. This is an increase of at most $1/N$ in width compared to its counterpart and therefore asymptotically negligible.

**Remark 2.** A ramification of Theorem 8 is that we cannot derive asymptotic distribution results for $N(\hat{\tau}_N^{(2)} - \tau_N)$. Consider the case where $\gamma = 0.5$, $N_1 = \sqrt{N}$ (or $\gamma = 0.5$), and the two subsequences $N = 2^j$ or $N = 3^j$ for some large integer $j$. In such cases the first stage subsample would choose points that have integer multiples of $1/2^j$ or $1/3^j$ as their x-coordinate.
• If $N = 2^j$, $\tau_N = \frac{[2^j \cdot 0.5]}{2^j} = 0.5$ is an integer multiple of $1/2^j$, and hence $\tau_N$ is an $x$-coordinate used in the first stage.

• If $N = 3^j$, then $\tau_N = \frac{[3^j \cdot 0.5]}{3^j}$, and it can be checked that $[3^j \cdot 0.5]$ is an even integer not divisible by 3. Since the $x$-coordinate of every first stage data point takes the form $\frac{3k}{3^j}$ for some integer $k$, this means $\tau_N$ is not used in the first stage.

Hence, in the former case we cannot ever have $\hat{\tau}_N^{(2)} = \tau_N$, while in the latter case, we have $\tau_N^{(2)} = \tau_N$ and our theorem tells us that $\mathbb{P}[\hat{\tau}_N^{(2)} = \tau_N]$ converges to the nonzero $\mathbb{P}[L(\Delta/\sigma) = 0]$ as $j$ increases. Clearly, we have two subsequences for which $\mathbb{P}[\hat{\tau}_N^{(2)} = \tau_N]$ converges to different values.

2.2. Computational Complexity and Numerical Illustration of the Procedure

The results above establish that the two stage procedure can, using a subset of the full data, be asymptotically almost as precise as employing the full dataset. In practice this allows for quicker estimation of big datasets without losing precision. The first stage uses about $N_1 \sim N^\gamma$ points to perform least squares fitting of a stump model, and this step takes $O(N^\gamma)$ computational time. The second stage applies a least-squares fit of a step function on the set $S^{(2)}$, which contains $O(N^{1-\gamma+\delta})$ points and therefore uses $O(N^{1-\gamma+\delta})$ time.

Hence, the two stage procedure requires order $N_1 \vee N^{1-\gamma+\delta}$ computation time, which is minimized by setting $\gamma = 1 - \gamma + \delta$, or $\gamma = \frac{1+\delta}{2}$. As $\delta$ tends to 0 (any small positive value of $\delta$ yields the above asymptotic results), the optimal $\gamma$ tends to $1/2$. Therefore, one should employ $N_1 = \sqrt{N}$ at the first stage and the second stage sample should be all points in the interval $[\hat{\tau}_N^{(1)} - K_2\sqrt{N}, \hat{\tau}_N^{(1)} + K_2\sqrt{N}]$, minus those at the first stage, where $K_2$ ensures that this interval contains $\tau$ with an acceptable high probability $1 - \alpha$. If one knows the jump size $\Delta$, $K_2$ can be determined as the $1 - \frac{\alpha}{2}$ quantile of the random variable $L(\Delta/\sigma)$; in the realistic unknown $\Delta$ case, a lower estimate of $\Delta$ can yield a corresponding conservative value of $K_2$.

To evaluate the performance of the intelligent sampling procedure, we conducted a simple set of simulations to demonstrate the accuracy of Theorems 1 and 2 as well as our analysis on the total overall computing time. The data generating mechanism used is as follows: $f(x) = 2 \cdot 1(x > 0.5)$ and the error term follows a standard normal distribution. For each of 15 values of $N$ between $10^4$ and $10^6.5$, chosen evenly on the log scale, we applied intelligent sampling to 1,000 replicates to evaluate the distribution of the deviation of $\hat{\tau}_N^{(2)}$ from $\tau_N$, and 2,000 replicates to determine the average running time. At the first stage, we subsampled $N_1 = \sqrt{N}$ points, and at the second stage we took an interval around $\hat{\tau}_N^{(1)}$ with half-width $8/\sqrt{N}$. Figure 1 shows that the rate of convergence and the computational time are of the predicted order of $O_p(1/N)$ and $O(\sqrt{N})$, respectively. Also included are the average times of fitting the stump model across the full data, across 100 replicates for each $N$, which predictably scales as $O(N)$. Finally, in terms of asymptotic distributions, Figure 2 demonstrates that the distribution of $\lambda_2(\tau_N, \hat{\tau}_N^{(2)})$ does closely match the predicted asymptotic distribution for large $N$.

![Fig. 2. distribution of $\lambda_2(\tau_N, \hat{\tau}_N^{(2)})$ values (blue) compared with the distribution of $L(\Delta/\sigma)$ from Theorem 2 (red).](image-url)
3. The Case of Multiple Change Points

Suppose one has access to a data set \( Y_1, Y_2, \ldots, Y_N \) generated according to the following model:

\[
Y_i = \theta_i + \varepsilon_i, \quad i = 1, 2, 3, \ldots, N
\]

where the \( \theta_i \)'s form a piecewise constant sequence for any fixed \( N \) and the \( \varepsilon_i \)'s are zero-mean error terms. The signal is flat apart from jumps at some unknown change points \( 1 = \tau_0 < \tau_1 < \ldots < \tau_J < \tau_{J+1} = N \): i.e. \( \theta_{i_1} = \theta_{i_2} \) whenever \( i_1, i_2 \in (\tau_j, \tau_{j+1}] \) for some \( j \in \{0, \ldots, J\} \). We let \( \nu_0 \) denote the common value of the signal on \((\tau_0, \tau_1]\), \( \nu_1 \) that on \((\tau_1, \tau_2]\) and so on. The number of change points \( J = J(N) \) is also unknown and needs to be estimated from the data. We impose the following basic restrictions on this model:

(M1): there exists a constant \( \bar{\theta} \in (0, \infty) \) not dependent on \( N \), such that \( \max_{i=1, \ldots, N} |\theta_i| \leq \bar{\theta} \);

(M2): there exists a constant \( \Delta \) not dependent on \( N \), such that \( \min_{i=0, \ldots, J} |\theta_{\tau_{i+1}} - \theta_{\tau_i}| \geq \Delta \);

(M3): there exists a \( \Xi \in [0, 1) \) and some \( K > 0 \), such that \( \delta_N := \min_{i=0, \ldots, J} (\tau_{i+1} - \tau_i) \geq K N^{1-\Xi} \) for all large \( N \);

(M4): \( \varepsilon_i \) for \( i = 1, \ldots, N \) are i.i.d. \( N(0, \sigma^2) \).

Assumption (M1) implies that the mean function is bounded, while (M2) bounds the minimum “signal”, which corresponds to the gap between successive levels away from 0. We note that estimation and inference on change-points can also be performed when \( \Delta \) is allowed to go to 0 slowly enough; however, as indicated in the introduction, the goal of this study is to identify substantial changes in the signal over long time stretches. Bounding the minimal signal away from 0 reflects that aspect of the problem.

Assumption (M3) places restrictions on the distance between consecutive change points and therefore limits how fast the number of change-points can grow as a function of \( N \). Regarding (M4), while our general theoretical results will be developed under this assumption, some results and discussions on dependent and possibly non-stationary errors (which would be expected for very long time series) will be presented in Section [4]. We start by describing how intelligent sampling works in this setting. Analogously to the single change point case, in the first stage, a sparse grid of point is used together with an established change point detection procedure to obtain: (i) a set of pilot estimates for the change points, together with their confidence intervals and (ii) a set of estimates for the true signal levels (the \( \mathbb{E}[Y_j] \)'s). The second stage will use these confidence intervals and level estimates to obtain finer estimates of the location of the change points. This strategy implicitly imposes another condition, namely, that there is some estimation method

\[\text{To be more precise we consider the triangular array of sequences } \theta_{i,N}, \text{ which are piecewise constant in } i. \text{ The error terms } \varepsilon_i = \varepsilon_{i,N} \text{ also form a triangular array, but we suppress the notation for brevity.}\]
at stage 1 which can provide confidence intervals for the true change points and consistent estimates for the true signal levels.

3.1. Details of the First Stage Procedure
To obtain confidence intervals for the $\hat{\tau}_j$'s for $j = 1, \ldots, J$, we desire a set of estimators $\hat{J}, \hat{\tau} := (\hat{\tau}_1, \cdots, \hat{\tau}_J)$ on model (12) such that

$$\mathbb{P} \left[ \hat{J} = J; \max_{j=1, \ldots, J} |\hat{\tau}_j - \tau_j| \leq w(N) \right] \geq 1 - B_N$$

(13)

for sequences $w(N)$ and $B_N$ with $w(N) = o(\delta_N)$ and $B_N \to 0$, respectively. This would allow us to construct non-overlapping confidence intervals of width $2w(N)$ that contain all true change points with probability approaching 1. As for the second ingredient, consistent estimators for the signal levels $\nu_j := \mathbb{E}[Y_{\tau_j+1}]$ for $j = 0, \ldots, J$, the natural candidates are

$$\hat{\nu}_i = \frac{1}{\hat{\tau}_{i+1} - \hat{\tau}_i} \left( \sum_{\hat{\tau}_i < j \leq \hat{\tau}_{i+1}} Y_j \right) \quad \text{for } i = 0, \ldots, \hat{J}$$

(14)

with the convention of $\hat{\tau}_0 := 0$ and $\hat{\tau}_{J+1} := N$. We also desire the property that

$$\mathbb{P} \left[ \hat{J} = J; \max_{j=0, \ldots, J} |\hat{\nu}_j - \nu_j| \leq \rho_N \right] \to 1$$

(15)

for some sequence $\rho_N \to 0$.

We note that the consistency result (15) follows directly from (13) under further conditions on the rates of growth/decrease of $w(N)$, $\delta_N$, and $B_N$. The next result builds in this direction.

Lemma 1. (i) Suppose that $Y_1, \cdots, Y_N$ are generated according to model (12) under conditions (M1) to (M4) and that we have an estimation scheme which gives estimates $\hat{J}$ for $J$ and $(\hat{\tau}_1, \cdots, \hat{\tau}_J)$ for $(\tau_1, \cdots, \tau_J)$ such that

$$\mathbb{P} \left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}_i - \tau_i| \leq w(N) \right] \geq 1 - B_N$$

(16)

for some sequences $w(N)$ and $B_N$, with $w(N) = o(\delta_N)$ and $B_N \to 0$. Then, for any positive sequence $\{\rho_N\}$ such that $\frac{w(N)}{\delta_N} = o(\rho_N)$, there exist constants $C_1$ and $C_2$, where

$$\mathbb{P} \left[ \hat{J} = J; |\hat{\nu}_i - \nu_i| \geq \rho_N \right] \leq B_N + C_1 w(N) \frac{\exp \left[ -C_2 \delta_N \rho_N^2 \right]}{\sqrt{\delta_N \rho_N}} \quad \text{for all } i = 1, \cdots, J, \text{ when } N \text{ is sufficiently large.}$$

(ii) Moreover,

$$\mathbb{P} \left[ \hat{J} = J; \max_{i=0, \cdots, J} |\hat{\nu}_i - \nu_i| \leq \rho_N \right] \geq 1 - \left( \frac{N}{\delta_N} + 2 \right) B_N - C_1 \left( \frac{N}{\delta_N} + 1 \right) w(N) \frac{\exp \left[ -C_2 \delta_N \rho_N^2 \right]}{\sqrt{\delta_N \rho_N}}. \quad \text{(18)}$$

So, in addition to the conditions in (i), if, furthermore, $NB_N/\delta_N \to 0$ and $(Nw(N)/\delta_N^{3/2} \rho_N) = o(\exp[C_2\delta_N \rho_N^2])$, the probability in (18) goes to 1. The $\hat{\nu}_i$’s are simultaneously consistent if $\rho_N$ also converges to 0.

Proof. See section 10.1. \hfill □

Remark 3. It follows that under the conditions imposed in (i) and (ii) of Lemma 1,

$$\mathbb{P} \left[ \hat{J} = J; \max_{j=1, \ldots, J} |\hat{\tau}_j - \tau_j| \leq w(N); \max_{j=0, \cdots, J} |\hat{\nu}_j - \nu_j| \leq \rho_N \right] \to 1,$$

(19)

by Bonferroni’s inequality, and the estimates of change point locations and signal levels are both consistent. Binary segmentation (BinSeg), a popular method for multiple change-point detection, does satisfy these conditions as we will show in the next section. Further, in Section 11.0.1 of the Supplement we show that wild binary segmentation (introduced in [2]) can also provide consistent estimates of change-point locations and signal levels.
3.1.1. Binary Segmentation

This section is developed to an exposition of Binary segmentation (BinSeg), a popular method for multiple change-point detection, which satisfies the properties desired of the first stage procedure outlined above. We first, briefly, describe the BinSeg algorithm (for a comprehensive exposition see [7]) and some additional properties that relate to our procedure. Consider the model given in [12]. For any positive integers $1 \leq s \leq b < e \leq N$, let $n = e - s + 1$ and define the Cumulative Sum (CUSUM) statistic at $b$ with endpoints $(s, e)$ as

$$
\hat{Y}_{s,e}^b = \sqrt{\frac{e - b}{n(b - s + 1)}} \sum_{t=s}^{b} X_t - \sqrt{\frac{b - s + 1}{n(e - b)}} \sum_{t=b+1}^{e} X_t.
$$

Binary segmentation is performed by iteratively maximizing the CUSUM statistics over the segment between change point estimates, accepting a new change point if the maximum passes a threshold parameter $\zeta_N$. Specifically,

(a) Fix a threshold value $\zeta_N$ and initialize the segment set $SS = \{(1, N)\}$ and the change point estimate set $\hat{\tau} = \emptyset$.

(b) Pick any ordered pair $(s, e) \in SS$, remove it from $SS$ (update $SS$ by $SS \leftarrow SS - \{(s, e)\}$). If $s \geq e$ then skip to step 5, otherwise continue to step 3.

(c) Find the argmax and max of the CUSUM statistic over the chosen $(s, e)$ from the previous step:

$$
b_0 = \arg\max_{b \in \{s, \ldots, e - 1\}} |\hat{Y}_{s,e}^b|, \quad |\hat{Y}_{s,e}^{b_0}|.$$

(d) If $|\hat{Y}_{s,e}^{b_0}| \geq \zeta_N$, then add $b_0$ to the list of change point estimates (add $b_0$ to $\hat{\tau}$), and add ordered pairs $(s, b_0)$ and $(b_0 + 1, e)$ to $SS$, otherwise skip to step 5.

(e) Repeat steps 2-4 until $SS$ contains no elements.

For the model under consideration, this algorithm provides consistent estimates of both the location of the change points and the corresponding levels, given further restrictions on the minimal separation. Specifically, imposing the following condition in addition to (M1) through (M4),

(M5 (BinSeg)): $\Xi$ (from condition (M3)) is further restricted by $\Xi \in [0, 1/4)$,

yields the following result that can be obtained from Theorem 3.1 of [7].

**Theorem 3.** Suppose that conditions (M1) to (M4) and (M5 (BinSeg)) are satisfied, and the tuning parameter $\zeta_N$ is chosen appropriately so that

- if $\Xi > 0$ then $\zeta_N = c_1 N^{\xi}$ where $\xi \in (1/2 - \Xi)$ and $c_1 > 0$
- if $\Xi = 0$ then $c_2 (\log(N))^p \leq \zeta_N \leq c_3 N^{\xi}$ where $p > 1/2, \xi < 1/2$, and $c_2, c_3 > 0$.

Define $E_N = \left(\frac{N}{\delta N}\right)^2 \log(N)$. Then, there exist positive constants $C, C_1$ such that

$$
P \left[ \hat{J} = J; \quad \max_{i=1, \ldots, J} |\hat{\tau}_i - \tau_i| \leq C E_N \right] \geq 1 - C_1 N^{-1}.
$$

**Remark 4.** The theorem in [7] applies to the more general setting where $\Delta$, the minimum signal jump, can decrease to 0 as $N \to \infty$. The above theorem is a corollary of Theorem 3.1 in Frylewicz (2014) to the setting where $\Delta$ is not dependent on $N$.

We can then apply the conclusions of Lemma [4] to the results of binary segmentation given in Theorem 3. Note that $\delta_N \geq C_1 N^{1-\Xi}$ and $\Xi < 1/4$. For some positive constant $C_2$, set $w(N) = C_2 E_N = C_2 \left(\frac{N}{\delta N}\right)^2 \log(N)$. Then, there is a positive constant $C_4$ such that $w(N) \leq C_4 N^{2\Xi} \log(N)$, and let $B_N = C_1 N^{-1}$. Now,

- since $N^{2\Xi} \log(N) = o(N^{1-\Xi})$ this does allow $w(N) = o(\delta_N)$ to be satisfied;

†Further, in Section 11.0.1 of the Supplement we show that wild binary segmentation (introduced in [11]) can also provide consistent estimates of change-point locations and signal levels.
Specifically, under conditions (M1) to (M5 (BinSeg)), the following holds:

\[ \frac{N}{\delta^{3/2}} \leq C \log(N) \]

which is \( o(\delta_N) \):

\[ Nw(N) / \delta^{3/2} \rho_N \leq N((3\Xi - 1)/2) \theta - \exp[C_2\delta_N N^{1/4}] \geq \exp[C_2 N^{1-\Xi + 2\theta}] \]

and as \( 1 - \Xi + 2\theta > 0 \), it follows that:

\[ Nw(N) / \delta^{3/2} \rho_N = o(\exp[C_2\delta_N N^{1/4}]) \]

Therefore, all conditions of Lemma 1 for a sequence \( \rho_N \) tending to 0 are satisfied. Next, combining the results of Theorem 3 and Lemma 1, we establish the simultaneous consistency of \( \hat{J} \), the \( \tilde{\tau}_i \)'s, and the \( \tilde{\nu}_i \)'s. Specifically, under conditions (M1) to (M5 (BinSeg)), the following holds:

\[
\mathbb{P} \left[ \hat{J} = J; \max_{i=1, \ldots, J} |\tilde{\tau}_i - \tau_i| \leq CE_N; \max_{i=0, \ldots, J} |\tilde{\nu}_i - \nu_i| \leq \rho_N \right] \rightarrow 1
\]

for any \( \rho_N = N^\theta \), where \( \theta \in (3\Xi - 1) \cup (-3/8) \)

(21)

as \( N \rightarrow \infty \).

Remark 5. Constructing confidence intervals based on Theorem 3 would require putting a value on \( CE_N = C(N/\delta_N)^2 \log(N) \) from (20). An estimate of \( \delta_N \) can be obtained from the minimum difference of consecutive \( \tilde{\tau}_j \)'s, but an explicit expression for \( C \) is also unavailable, and the existing literature on binary segmentation does not appear to provide such an explicit expression. In the context of using BinSeg as a key component of intelligent sampling, this issue can be resolved in practical settings via a calibration method to be introduced in Section 3.4.

### 3.2. Intelligent Sampling on Multiple Change Points

The intelligent sampling procedure in the multiple change-points case works in two (or more) stages: in the two-stage version, as in Section 2, the first stage aims to find rough estimates of the change points using a uniform subsample (Steps ISM1-ISM4) and the second stage produces the final estimates (Steps ISM5 and ISM6).

**ISM1:** Start with a data set \( Y_1, \ldots, Y_N \) described in (12).

**ISM2:** Take \( N_1 = K_1 N^\gamma \) for some \( K_1 \) and \( \gamma \in (2, 1) \) such that \( N/N_1 = o(\delta_N) \); for \( j = 1, \ldots, \left\lfloor \frac{N}{N/N_1} \right\rfloor \), consider the subsample \( \{Z_j\} = \{Y_j|N/N_1\} \).

The subsample \( Z_1, Z_2, \ldots \) can also be considered a time series structured as in (12), and since \( \delta_N >> N/N_1 \), there are jumps in the signal at \( \tau_j^* := \left\lfloor \frac{\tilde{\tau}_j}{N/N_1} \right\rfloor \) for \( j = 1, \ldots, J \), with corresponding minimum spacing

\[
\delta_{N_1} := \min_{i=1, \ldots, J+1} |\tau_i^* - \tau_{i-1}^*| = \frac{1}{N/N_1} \min_{i=1, \ldots, J+1} |\tau_i - \tau_{i-1} + O(1)| = \left( \frac{N_1}{N} \delta_N \right)^{1 + o(1)}.
\]

(22)

**ISM3:** Apply some multiple change point estimation procedure (such as binary segmentation) to the set of \( Z_j \)'s to obtain estimates \( \hat{\tau}_1^*, \ldots, \hat{\tau}_J^* \) for the \( \tau_j^* \)'s and \( \hat{\nu}_1^*(1), \ldots, \hat{\nu}_J^* \) for the levels \( \nu_0, \nu_1, \ldots, \nu_J \).

- the choice of the procedure does not matter so long as the estimates satisfy

\[
\mathbb{P} \left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}_i^* - \tau_i| \leq w^*(N), \max_{i=0, \ldots, J} |\hat{\nu}_i^*(1) - \nu_i| \leq \rho_N \right] \rightarrow 1
\]

for some sequence \( w^*(N) \) such that \( w^*(N) \rightarrow \infty, w^*(N) = o(\delta^*_N) \) and \( \rho_N \rightarrow 0 \).

**ISM4:** Convert these into estimates for the \( \tau_j \)'s by letting \( \hat{\tau}_j^* := \frac{\hat{\tau}_j}{N/N_1} \) for \( j = 1, \ldots, \hat{J} \).

- taking \( w(N) := (w^*(N) + 1)|N/N_1| \), expression (23) gives

\[
\mathbb{P} \left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}_i - \tau_i| \leq w(N), \max_{i=0, \ldots, J} |\hat{\nu}_i^*(1) - \nu_i| \leq \rho_N \right] \rightarrow 1.
\]

(24)
(ISM5): Fix any integer $K > 1$, and consider the intervals $[\hat{\tau}_i^{(1)} - Kw(N), \hat{\tau}_i^{(1)} + Kw(N)]$ for $i = 1, ..., J$. Denote by $S^{(2)}(\hat{\tau}_i^{(1)})$ all integers in this interval not divisible by $[N/N_1]$.

(ISM6): For each $i = 1, ..., J$, let

$$\hat{\tau}_i^{(2)} = \min_{d \in S^{(2)}(\hat{\tau}_i^{(1)})} \left( \sum_{j \in S^{(2)}(\hat{\tau}_i^{(1)})} \left[ Y_j - (\hat{\rho}_{i-1}^{(1)} 1(j < d) + \hat{\rho}_i^{(1)} 1(j \geq d)) \right]^2 \right) \tag{25}$$

The motivation for fitting a stump model in ISM6 comes from the nature of the data in each of the second stage intervals. By (24), the intervals $[\hat{\tau}_1^{(1)} - Kw(N), \hat{\tau}_1^{(1)} + Kw(N)]$, for $j = 1, \ldots, J$, satisfy the following properties with probability growing to 1:

- Each contains a change point, and because the width of the interval $2Kw(N)$ is $o(\delta_N)$, it contains only a single change point.
- All $\tau_j$’s are contained in one of these intervals.

Hence, with probability $\to 1$ the multiple change point problem has simplified to $\hat{J}$ single change point problems. As to whether we can satisfy (24), under certain conditions BinSeg does guarantee (23) on the subsample $\{Z_j\}$, and therefore satisfies (24). See Section 4.3 for more details.

The next Theorem establishes the properties of intelligent sampling for the multiple change points problem.

**Theorem 4.** Suppose conditions (M1) to (M4) are satisfied and the first stage estimates satisfy the consistency result (27). Then, for any $\varepsilon > 0$, we can find some constant $C$ such that

$$\mathbb{P}[\hat{J} = J, \max_{k=1, ..., J} |\hat{\tau}_k^{(2)} - \tau_k| \leq CJ] \geq 1 - \varepsilon \tag{26}$$

for all sufficiently large $N$.

**Proof.** See section 10.1.1. \qed

**Remark 6.** The rate of convergence given in Theorem 4 can be thought of as an extension of Theorem 7 to a setting with more than one change point, and indeed in the case where $J(N) = 1$, for any $\varepsilon > 0$ there exists a constant $C$ where

$$\mathbb{P}[\hat{J} = 1; \ |\hat{\tau}_1^{(2)} - \tau_1| \leq C] \geq 1 - \varepsilon. \tag{27}$$

This can be restated as $\hat{\tau}^{(2)} - \frac{\tau_1}{N} = O_p(N^{-1})$, if we define

$$\hat{\tau}^{(2)} := \begin{cases} \frac{\tau_1^{(2)}}{N} & \text{whenever } \hat{J} = 1 \\ \text{anything} & \text{otherwise} \end{cases} \tag{28}$$

which is exactly the result of Theorem 7.

**Asymptotic distributions of the intelligent sampling based estimators:** Next, we derive the joint asymptotic distribution of the intelligent sampling estimates in the case where the number of change points remains constant, as captured by the following assumption:

(M6): The number of change points $J$ is a finite constant not dependent on $N$, and the jump sizes $\Delta_j := \nu_j - \nu_{j-1}$ for $j = 1, \ldots, J$ are also constants not dependent on $N$.

Due to the omission of the first stage subsampling points we define the familiar distance function $\lambda_2 = \lambda_{2,N}$ as

$$\lambda_2(a, b) := \begin{cases} \sum_{i=1}^N 1(a < i \leq b) \cdot 1(i \neq k\lfloor N/N_1 \rfloor \text{ for any integer } k) & \text{if } a \leq b \\ -\sum_{i=1}^N 1(b < i \leq a) \cdot 1(i \neq k\lfloor N/N_1 \rfloor \text{ for any integer } k) & \text{otherwise} \end{cases} \tag{29}$$

Using this notation, we establish
Theorem 5. Suppose conditions (M1) to (M4), (M6), and the consistency condition \([24]\) are satisfied. Then, the deviations \(\left\{\lambda_2 \left(\tau_j, \hat{\tau}_j^{(2)}\right)\right\}_{j=1}^J\) jointly converge to the distribution of \(\left(L \left(\frac{\Delta}{\Theta}\right), \ldots, L \left(\frac{\Delta}{\Theta}\right)\right)\), where the \(L(\Delta)\)'s are mutually independent. That is, for any integers \(k_1, \ldots, k_J\),

\[
P \left[\hat{J} = J, \lambda_2 \left(\tau_j, \hat{\tau}_j^{(2)}\right) = k_j \right. \text{ for } 1 \leq j \leq J \left.\right] = \prod_{j=1}^J P \left[L \left(\frac{\Delta_j}{\Theta}\right) = k_j\right].
\]

Proof. See section 10.2.

The \(L(\cdot)\) limit distribution of Theorem 2 appears again here. This is no coincidence, since the second stage of the procedure works with disjoint intervals containing a single change point each. As mentioned before, this makes the problem behave like \(J\) independent single change point problems.

Discussion of Condition (M6): The requirement for \(J\) to be finite is rather stringent and was not imposed for previously established results for the multiple change point problem. While it may be possible to derive asymptotic distribution results for a setting where \(J(N)\) grows at a slow rate, there are major technical difficulties which we describe in detail in the Supplement (see Section 13.1). To summarize, denote \(Q_j(\alpha)\) as the \(1 - \alpha\) quantile of \(L(\Delta_j/\Theta)\) for any \(\alpha \in (0, 1]\), and any \(j \in \mathbb{N}\). Extending the result of Theorem 2 requires convergence results on how quickly the probabilities \(P \left[\hat{J} = J, \lambda_2 \left(\tau_j, \hat{\tau}_j^{(2)}\right) \leq Q_j(\alpha_N)\right]\) converge to \(1 - \alpha_N\), for a sequence \(\alpha_N \to 0\). Unfortunately, since they are the argmins of random walks, the speed at which the cdf of \(\lambda_2 \left(\tau_j, \hat{\tau}_j^{(2)}\right)\) converges to the cdf of \(L(\Delta_j/\Theta)\) is not readily tractable, neither do there appear to be any results in the probability literature along these directions.

3.3. Computational Considerations

Suppose that binary segmentation is used in ISM3. From the spacing condition (M5), we require \(\delta_{N_i} \geq KN_i^\Theta\) for some \(K > 0\) and \(\Theta > 3/4\). In \([22]\), we can explicitly write out \(w^*(N) = CE_{N_i}^*\) where \(C\) is a positive constant and

\[
E_{N_i}^* = \left(\frac{N_i}{\delta_{N_i}}\right)^2 \log(N_i) = \gamma \left(\frac{N}{\delta_{N}}\right)^2 \log(N)(1 + o(1))
\]

Clearly \(w^*(N) \to \infty\). The condition \(w^*(N) = o(\delta_{N})\) where \(\delta_{N_i} \sim (N_i/N)\delta_N\) by \([22]\), implies \((N_i/N)\delta_N \gg (N^2/\delta_{N})\log N\), which by simple algebra and noting that \(\delta_N \sim N^{1-\Xi}\) translates to \(N^\gamma \gg N^{3\Xi}\log N\), an inequality satisfied whenever \(\gamma > 3\Xi\). The condition \((N_i/\delta_{N_i})B_{N_i} \to 0\) is easily satisfied since \(B_{N_i} = C_1/N_i\) and \(\delta_{N_i} \to \infty\). Next, starting from the \(\delta_N \geq KN_i^{1-\Xi}\) we can obtain (by expressing \(N_i\) in terms of \(N\) and \(\gamma\)) \(\delta_{N_i} \geq K_1N_i^\Theta\) for some \(K_1 > 0\) and \(\Theta = 1 - \Xi/\gamma\). A choice of \(\Theta > 3/4\) will be satisfied when \(\gamma > 4\Xi\).

By choosing \(\gamma > 4\Xi\) and \(\rho_{N_i} = N_i^{-\theta}\) for some \(\theta \in \left(0, \left(\frac{2}{3}\right) \wedge \left(1 - 3\Xi\right)\right)\), it is not difficult to verify that the conditions

\[
\frac{w^*(N)}{\delta_{N_i}} = o(\rho_{N_i}) \quad \text{and} \quad \frac{N_i w^*_N}{(\delta_{N_i})^{3/2} \rho_{N_i}} = o(\exp[C_2\delta_{N_i}^2 \rho_{N_i}])
\]

hold true. Hence, by an appeal to Lemma 1, it follows that binary segmentation used in the first stage satisfies \([23]\). The above discussion leads to the following condition for using binary segmentation at stage 1:

(M7 (BinSeg)): \(N_i\), the size of the first stage subsample is chosen so that \(N_i = K_1N_i^\gamma\) for some \(K_1 > 0\) and \(\gamma > 4\Xi\).

For an analysis of the computational run time we make the simplifying assumptions that \(\delta_N/N^{1-\Xi} \to C_1\) and \(J(N)/N^\Lambda \to C_2\) for some \(\Lambda \in [0, \Xi]\) and some positive constants \(C_1, C_2\). To summarize (for the full details of the analysis see Section 11.1 in the Supplement), BinSeg on the first stage subsample takes \(O(N^\gamma \log(N))\) time while the second stage takes on average \(O(N^{1-\gamma + 2\Xi + \Lambda} \log(N))\) time. Minimizing the total computational would be accomplished by setting \(\gamma\) to equal \(\gamma_{\min} = \frac{1 + 2\Xi + \Lambda}{2}\), but due to condition (M7), this is not quite the case, and the optimal \(\gamma\) formula needs some modification depending on the value of \(\Xi\).

We list the different values of \(\gamma_{\min}\) and the associated order of computational time, including the extreme cases \(\Lambda = 0\) and \(\Lambda = \Xi\) (which correspond to the least and highest number of change points allowable for
Table 1. Table of $\gamma_{min}$ and computational times for various values of $\Xi$. Also shown are their values for extreme value of $\Lambda$ ($\Lambda = 0$ and $\Lambda = \Xi$). For $\Xi \geq 1/4$ no values of $\gamma$ can theoretically assure binary segmentation will be consistent.

| $\Xi$ | $[0, 1/6)$ | $[1/6, 1/5)$ | $[1/5, 1/4)$ | $[1/4, 1]$ |
|-------|------------|--------------|--------------|------------|
| $\gamma_{min}$ | $\frac{1+2\Xi+\Lambda}{2}$ | $\max\{\frac{1+2\Xi+\Lambda}{4}, \frac{1+2\Xi+\Lambda}{2}\}$ | $4\Xi + \delta$ | N/A |
| Order of Time | $N^{(1+2\Xi+\Lambda)/2} \log(N)$ | $\max\{N^{(1+2\Xi+\Lambda)/2}, N^{4\Xi+\delta}\} \cdot \log(N)$ | $N^{4\Xi+\delta} \log(N)$ | N/A |
| $\gamma_{min}$ ($\Lambda = 0$) | $\frac{1+2\Xi}{2}$ | $4\Xi + \delta$ | $4\Xi + \delta$ | N/A |
| Time ($\Lambda = 0$) | $N^{(1+2\Xi)/2} \log(N)$ | $N^{4\Xi+\delta} \log(N)$ | $N^{4\Xi+\delta} \log(N)$ | N/A |
| $\gamma_{min}$ ($\Lambda = \Xi$) | $\frac{1+3\Xi}{2}$ | $\frac{1+3\Xi}{2}$ | $4\Xi + \delta$ | N/A |
| Time ($\Lambda = \Xi$) | $N^{(1+3\Xi)/2} \log(N)$ | $N^{(1+3\Xi)/2} \log(N)$ | $N^{4\Xi+\delta} \log(N)$ | N/A |

$\delta_N \sim N^{1-\Xi}$, in Table 1 and depicted in Figure 3 below. In the below table, $\delta$ is a small positive number that can be chosen to be as close to 0 as one wants.

$\gamma$ vs $\Xi$ for BinSeg

![Graph showing $\gamma$ vs $\Xi$ for BinSeg](image)

Fig. 3. Blue triangle encompasses all valid values of $\gamma$ vs $\Xi$ as set by (M7 (BinSeg)). Pink region, solid red lines, and dotted red lines denotes $\gamma_{min}$ for each $\Xi$ ($\gamma_{min}$ can vary for different values of $\Lambda$ even when $\Xi$ is fixed, hence the red region).

It can be seen that the biggest decrease in order of average computational time is for small values of $\Xi$, and in fact for $\Xi = 0$ it is $O(\sqrt{N} \log(N))$ and marginally slower than intelligent sampling on a single change point. For larger values of $\Xi$, there is less than a square root drop in $N \log(N)$ (order of using BinSeg on the whole data) to $N^{\gamma_{min}} \log(N)$ (intelligent sampling), to the point where as $\Xi \to 1/4$, both procedures take near the same order of time.

The main directions of this analysis could be extended to methods beyond BinSeg at stage one, with the consistency regimes of ($\Lambda, \Xi, \gamma$) and computational time order varying from method to method. As an example, consider wild binary segmentation, a procedure first introduced in [7]. The consistency results for WBinSeg require less stringent conditions than the consistency results for BinSeg, and hence WBinSeg can serve as an alternative to BinSeg. In Section 11.0.1 in the Supplement we perform similar analysis on intelligent sampling with WBinSeg, which requires a less stringent condition of $\Xi < 1$ (compared to $\Xi < 1/4$ for BinSeg). However the optimal order of computational time for BinSeg and WBinSeg is the same for $\Xi < 1/4$, and the time scales rather expensively using WBinSeg for larger values of $\Xi$, even with intelligent sampling. Due to the lack of improvement on the sparse change point regime (i.e. small $\Xi$) and the greater popularity of BinSeg, we present our analysis with BinSeg in the main text.
Remark 7. Note that when implementing the intelligent sampling strategy knowledge of \( \Xi \) is desirable, but in practice, its value is unknown. If one is willing to impose an upper bound on \( \Xi \), intelligent sampling can be implemented with this (conservative) upper-bound.

3.4. Calibration of intervals used in Stage 2

It is not difficult to see that for the single change point problem the asymptotic distribution of \( \hat{\tau}^{(1)} \) after centering and re-scaling is the minimizer of a random walk with absolute value drift, denoted by \( L(\Delta/\sigma) \) in Section 2. Therefore once \( \Delta \) and \( \sigma \) are estimated, for any \( \alpha > 0 \) we can calculate a number \( K_\alpha \) and work with the interval \([\hat{\tau}^{(1)} - K_\alpha/N_1, \hat{\tau}^{(1)} + K_\alpha/N_1]\) that contains the true value of \( \tau_N \) with probability at least \( 1 - \alpha \).

For the multiple change point problem coupled with intelligent sampling and the BinSeg algorithm, there is no analogous explicit expression for the asymptotic distribution in order to calibrate the second stage interval widths. Theorem 3 shows that there exist simultaneous confidence intervals (with confidence level approaching 1) of width \( KE_N \) for some \( K > 0 \). The question of how to explicitly derive such a \( K \) was not a focus of prior work e.g. \[7\] or \[25\]). Since there are no available theory on the asymptotic distribution of the BinSeg estimators, this leaves us with no principled way of adjusting the width of the intervals which we take the second stage subsample from.

To rectify this, we propose a calibration method where we use the initial estimators from steps (ISM3) and (ISM4): fit stump models on data with indices \([\hat{\tau}_{j-1}^{(1)} + 1, \hat{\tau}_{j+1}^{(1)}]\), as each of these stretches forms almost a stump model with probability going to 1. Consider starting from after step (ISM4) (e.g., Figure 4) where we have rough estimates \( \hat{\tau}_{j}^* \)'s of the change points (with respect to the \( \{Z_i\} \) time series) and \( \hat{\nu}_i^{(1)} \)'s of the signals, obtained from the roughly \( N_1 \) sized subsample \( \{Z_i\} \).

![Fig. 4. Green points are \( Z_i \)'s, solid green line is the BinSeg estimate.](image)

Fig. 4. Green points are \( Z_i \)'s, solid green line is the BinSeg estimate.

We then pick a different subsample \( \{V_i\} \) of equal size to the \( \{Z_i\} \) subsample and consider the \( \hat{\tau}_i^* \)'s and \( \hat{\nu}_i^{(1)} \)'s as estimates for the parameters of this time series (e.g. Figure 5).

![Fig. 5. \( Z_i \)'s are light green points, BinSeg estimates as dashed green line, \( V_i \)'s as red points.](image)

Fig. 5. \( Z_i \)'s are light green points, BinSeg estimates as dashed green line, \( V_i \)'s as red points.

Next, for each \( j \), fit a one-parameter stump model \( f_j^{(d)}(k) = \hat{\nu}_{j-1}^{(1)} 1(k \leq d) + \hat{\nu}_{j+1}^{(1)} 1(k > d) \) (here \( d \) is the discontinuity parameter) to the subset of \( \{V_k : \hat{\tau}_j^* - 1 \leq k \leq \hat{\tau}_j^* + 1\} \) given by \( \{V_k : \hat{\tau}_j^* - D_j \leq k \leq \hat{\tau}_j^* + D_j\} \).

\[\text{Specifically } N_1(\hat{\tau}^{(1)} - \tau_N) \text{ converges, where } N_1 := N/\lfloor N/N_1 \rfloor \text{ and } \hat{\tau}_N = \left\lfloor \frac{N \hat{\tau}^{(1)}}{\lfloor N/N_1 \rfloor} \right\rfloor \frac{\lfloor N/N_1 \rfloor}{N}.\]
where \( D_j = \min\{(\hat{\tau}^*_j - \hat{\tau}^*_{j-1}), (\hat{\tau}^*_{j+1} - \hat{\tau}^*_j)\} \), to get an updated least squares estimate of \( \tau_j \) (e.g. Figure 6).

Fig. 6. Solid green and solid red lines denote stump estimates using \( V_i \)'s from \( \{V_k : \hat{\tau}^*_j - D_j \leq k \leq \hat{\tau}^*_j + D_j\} \) intervals.

In specific detail, the calibration steps are:

(ISM4-1): pick a positive integer \( k_N \) less than \( \lfloor N/N_1 \rfloor \), take a subsample \( \{V_i\} \) from the dataset of \( \{Y_i\} \) which is the same size as the \( \{Z_i\} \) subsample, by letting \( V_i = Y_i\lfloor N/N_1 \rfloor - k_N \) for all \( i \).

The \( \{V_i\} \) subsample also conforms to the model given in (12), with change points \( \tau^*_{i*} = \max\{j \in \mathbb{N} : j\lfloor N/N_1 \rfloor - k_N \leq \tau_i\} \) and minimum spacing \( \delta^*_{N_1} := \min_k(\tau^*_{k+1} - \tau^*_k) \) which satisfies \( |\delta^*_{N_1} - \delta^*_{N_1}| \leq 1 \).

(ISM4-2): For each \( i = 1, ..., J \), consider the estimates \( \hat{\tau}^*_i \) (obtained from the \( \{Z_i\} \) subsample at step (ISM3)) as estimators for \( \tau^*_{i*} \). From (23) it is possible to derive that

\[
\mathbb{P}\left[ \hat{J} = J, \max_{i=1,...,J} |\hat{\tau}^*_i - \tau^*_{i*}| \leq w^*(N) + 1, \max_{i=0,...,J} |\hat{\nu}_i^{(1)} - \nu_i| \leq \rho_N \right] \to 1
\]

(33)

where \( w^*(N) + 1 \to \infty \) and \( (w^*(N) + 1)/\delta^*_{N_1} \to 0 \).

\( \bullet \) We use this subset instead of the full interval to avoid situations where \( \frac{\tau_j - \tau_{j-1}}{\tau_{j+1} - \tau_j} \to 0 \) or \( \infty \), which makes matters easier for theoretical derivations.

\( \parallel \) A good pick is \( k_N = \left\lfloor \frac{N/N_1}{2} \right\rfloor \)
Remark 8. with probability approaching 1 − \( \alpha \), data set, we can obtain with high probability, simultaneous confidence intervals for the change-points in

The proposed intelligent sampling procedure for multiple change point problems has so far been presented in the setting of i.i.d. data for a signal-plus-noise model. However, time series data usually exhibit temporal correlation. Hence, it is of interest to examine the properties of the procedure under a non-i.i.d. data generating mechanism. Specifically, the \( \varepsilon_j \)'s are marginally \( N(0, \sigma^2) \) and the time series has an autocorrelation function \( r : Z \rightarrow \mathbb{R} \) [defined as \( r(k) = \text{cov}(X_j, X_{j+k}) \) for all \( j \) and \( k \)] such that \( r(k) = 0 \) whenever \( |k| > M \), for some positive integer \( M \). We also assume conditions (M1) to (M3), and (M6). Suppose we apply intelligent sampling with binary segmentation [further assuming (M5) and (M7 (BinSeg))] on the stage 1, samples \( \{Z_j\} \) described in step (ISM2). This subsample is taken from an increasingly sparse sub-grid from **Consistent estimates \( \hat{\Delta}_j \) and \( \hat{\sigma} \) are trivially computable.

††To be able to use Theorem 6 rigorously we need to split the data into two independent parts, however in appears, from exploratory simulations, that it is not required to split the data and both the initial estimates and the re-fitting could both be applied to the full data.
the full data, and hence for \( N \) large (specifically \( N/N_1 > M \)), the error terms from this subsample are i.i.d, and the consistency result found in (24) holds.

As in the i.i.d. case, the intervals \( [\hat{\tau}^{(1)}_j - Kw(N), \hat{\tau}^{(1)}_j + Kw(N)] \) for \( j = 1, \ldots, J \) [obtained at step (ISM5)] would each contain only one change point with probability approaching one. We are therefore still justified in fitting stump models on each interval, although with a slight modification. Unlike the i.i.d. error terms scenario, the joint distribution of the error terms at the second stage does change when we condition on \( \tau \). We conclude the discussion on M-dependent error terms with the following:

**Proposition 3.** Assume (M1) to (M3) are true, (M6) holds, and the error terms \( \varepsilon_i \)'s form an M-dependent Gaussian stationary time series. Next, suppose that the first stage estimators satisfy (23).

Further, assume that the second stage estimates are \( O_p(1) \): for any \( \epsilon > 0 \) there exists a constant \( C \) such that for all sufficiently large \( N \),

\[
\mathbb{P} \left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}^{(2)}_i - \tau_i| \leq CJ \right] \geq 1 - \epsilon.
\]

For \( j = 1, \ldots, J \) let \( \{\varepsilon_{i,j}\}_{i \in \mathbb{Z}} \) be a time series identical in distribution to the error terms \( \{\varepsilon_i\}_{i \in \mathbb{Z}} \). Define

\[
Z_{i,j} := \begin{cases} 
\Delta_j(\varepsilon_1 + \cdots + \varepsilon_j) + i\Delta_j^2/2, & i > 0 \\
0, & i = 0 \\
-\Delta_j(\varepsilon_{i+1} + \cdots + \varepsilon_0) + |i|\Delta_j^2/2, & i < 0
\end{cases}
\]

for \( j = 1, \ldots, J \), and denote the minimizer of the \( j \)’th process by \( L_j := \arg \min_{i \in \mathbb{Z}} Z_{i,j} \).

Then \( |\hat{\tau}^{(2)}_j - \tau_j|'s \) for \( j = 1, \ldots, J \) jointly converge to the distribution of \( (L_1, \ldots, L_J) \): for any integers \( k_1, \ldots, k_J \),

\[
\mathbb{P} \left[ \hat{J} = J; \hat{\tau}^{(2)}_j = k_j \text{ for } 1 \leq j \leq J \right] \rightarrow \prod_{j=1}^J \mathbb{P}[L_j = k_j]
\]

Next, we aim to generalize the result of Proposition 3 to settings imposing even less restrictive conditions on the error terms \( \varepsilon_i \)'s. Again, suppose that the time series is in the form of (12) and satisfies conditions (M1) to (M3) and (M6). Upon the error terms, we impose the assumption that they have an autocorrelation structure which dies out at a polynomial rate or faster, and locally around the change points assume that the joint distributions of the errors are fixed [i.e. invariant to \( N \)]:

**M4-alt1:** \( \varepsilon_i \)'s are each marginally \( N(0, \sigma_j^2) \), and there exist positive constants \( \sigma_{\max} \), \( B \) and \( \alpha \), independent of \( N \), such that \( \sigma_j \leq \sigma_{\max} \) and \( \text{cor}(\varepsilon_j, \varepsilon_{j+k}) \leq Bk^{-\alpha} \) for any \( j \) and \( j + k \) from 1 to \( N \).

**M4-alt2:** there exists a sequence \( w_c(N) \rightarrow \infty \) and Gaussian time series \( \{\varepsilon_{i,j}\}_{i \in \mathbb{Z}} \) (not required to be stationary) for \( j = 1, \ldots, J \), such that for all \( j = 1, \ldots, J \) and all sufficiently large \( N \), \( \{\varepsilon_{\tau_j - w_c(N)}, \ldots, \varepsilon_{\tau_j + w_c(N)}\} \) has the same joint distribution as \( \{\varepsilon_{-w_c(N)}, \ldots, \varepsilon_{w_c(N)}\}. \)

**Remark 9.** The upper bound for the autocovariance function in condition (M4-alt1) arises from the work in [10] which established desirable properties for stump estimation when the autocorrelation is bounded as \( \text{cor}(\varepsilon_j, \varepsilon_{j+k}) \leq Bk^{-\alpha} \). Further, \( \alpha < 1 \) also gives us the flexibility of considering long-range dependence. Condition (M4-alt2) allows us to consider non-stationary errors in the vicinity of the change point. To give
an example, suppose the system outputting the data experiences a shock at a change point $\tau$, that changes not only the mean signal but also the error distribution to the right of $\tau$. In other words, the error terms $\varepsilon_i$’s have different joint distributions (locally) to the left and right of $\tau$. Condition (M4-alt2) would allow us to derive asymptotics under this scenario, and as will be seen shortly, the $\{\varepsilon_{i,j}\}_{i\in\mathbb{Z}}$’s will figure in the limit.

On a set of data where (M4-alt1) and (M4-alt2) hold (along with assumptions (M1) to (M3), and (M6)), we still want steps (ISM1) and (ISM4) to go through with some procedure that ensures

$$
\Pr \left( \hat{j} = J, \max_{i=1,\ldots,J} |\tau_i^{(1)} - \tau_i| \leq w(N), \max_{i=0,\ldots,J} |\hat{\nu}_i^{(1)} - \nu_i| \leq \rho_N \right) \to 1. \tag{39}
$$

for some sequence $w(N) \to \infty$ and $w(N) = o(\delta N)$. This is not unreasonable: the procedure used at stage 1 will work on a sparse subsample instead, and therefore the correlation between spaced $k$ indices apart on the subsample is actually of the same order as $(kN^{1-\alpha})^{-\alpha}$, which is vanishingly small even if $k$ is fixed. In other words, as $N$ grows, the error terms of the first stage points will increasingly resemble an independent sequence, so we expect binary segmentation to work.

We could also consider other methods consistent under non-i.i.d. errors that have been explored in multiple papers, such as [1] and [2].

Next, we desire for the final estimators $\hat{\tau}_i^{(2)}$ to be $O_p(1)$ consistent and have the property where for each $\epsilon > 0$ there exists a constant $C$ such that

$$
\Pr \left( \hat{j} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i| \leq CJ \right) \geq 1 - \epsilon. \tag{40}
$$

for all sufficiently large $N$. This, again, is expected under the setting considered. As noted earlier, [10] demonstrated that in a single change point problem with stationary errors and auto-correlation function bounded as in condition (M4-alt1), the change point estimator has the same rate of convergence as when the errors are i.i.d. Even in a multiple change point problem with stationary errors, the second stage will reduce to $J$ asymptotically independent single change point problems, and (40) will follow from the finiteness of $J$. Although this does not extend immediately to a heteroscedastic/non-stationary setting, it points to perhaps a similar result for the framework we’ve considered, possibly after placing heavier restrictions on $\alpha$.

**Proposition 4.** Suppose conditions (M1) to (M3), (M4-alt1), (M4-alt2), and (M6) are satisfied. Next, suppose the first stage estimators satisfy (39) and the second stage estimators satisfy (40). Define the random walks

$$
Z_{i,j} = \begin{cases} 
\Delta_j (\varepsilon_{1,j} + \cdots + \varepsilon_{i,j}) - i\Delta^2_j / 2, & i > 0 \\
0, & i = 0 \\
\Delta_j (\varepsilon_{i+1,j} + \cdots + \varepsilon_{0,j}) - i\Delta^2_j / 2, & i < 0,
\end{cases} \tag{41}
$$

with the $\varepsilon_{i,j}$’s from condition (M4-alt2), for $j = 1, 2, \ldots, J$, and denote $\tilde{L}_j := \arg \min_{i \in \mathbb{Z}} Z_{i,j}$. Then $|\hat{\tau}_i^{(2)} - \tau_j|$’s for $j = 1, \ldots, J$ jointly converge to the distribution of $(\tilde{L}_1, \ldots, \tilde{L}_J)$: for any integers $k_1, \ldots, k_J$,

$$
\Pr \left[ \hat{j} = J; |\hat{\tau}_i^{(2)} - \tau_j| = k_j \text{ for } 1 \leq j \leq J \right] \to \prod_{j=1}^{J} \Pr[\tilde{L}_j = k_j]. \tag{42}
$$

**Remark 10.** We note that the asymptotic distribution given above and in Theorem 2 have the same form, since as before, conditional on (39) being true, intelligent sampling simplifies the problem into multiple single change point problems. Because of condition (M4-alt1), these single change point problems are essentially independent (again after conditioning on (39)) as the separation between change points grows without bound. A rigorous demonstration of (39) and (40) for a variety of dependent error-structures under BinSeg (or WBinSeg) will provide interesting opportunities for future research on this topic.

††Additionally, result (40) was obtained for wild binary segmentation for $\alpha > 1$ in [16], albeit in a change points in variance rather than change points in means model. Nevertheless, this points to perhaps a similar result for the framework we’ve considered, possibly after placing heavier restrictions on $\alpha$. 

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Remark 11. Using Proposition 4 to construct confidence intervals in a practical setting requires an idea of the joint distribution of the $\{\epsilon_{ij}\}$’s. Although we have presented Proposition 4 under less restrictive assumptions, in practice it would be difficult to estimate the distribution structure of a time series without placing any assumption on its variance structure besides bounding inequalities. In practice one would impose some structural conditions, e.g. assuming an ARMA or ARIMA structure on long stretches of the error at the left and right of the change points.

5. Computational Analysis for a Methodological Setting

In the asymptotic setting of Section 3.3 we were concerned about minimizing the order of computational time required for locating the change point through intelligent sampling, assuming that certain important quantities were known. The focus in this section is on obtaining explicit expressions for the minimum sample size that the procedure requires to correctly identify the underlying change points. Obviously, the minimum sample size is the key driver in the computational time formulas provided, albeit not the single one, and also addresses computer memory usage issues. In order to develop explicit expressions for the total computational time, one would need to know exactly how fast BinSeg runs versus data size, in terms of its model parameters, and this is unavailable as an exact expression. Therefore, we look at minimizing the subsample utilized as a proxy, with the added benefit of deriving the least amount of data that must be held in memory at a single time.

We have already investigated the optimal order of the first stage subsample, denoted $N_1$, and demonstrated in Section 3.3 that in the best cases the size of both the first and second stage subsamples scales (up to log factors) as $\sqrt{N}$. Additionally, in Sections 12.1.1 and 12.2.1 we argue that intelligent sampling with more than two stages can have computational time scaling even slower than $\sqrt{N}$. Although valid, these previous analyses only apply to an abstract asymptotic setting. In practice, given a data set with fixed (large) $N$, a different approach is needed to determine the optimal number of points to use at each different stage. Next, we establish, given the number of change points and their associated SNR’s, how to optimally allocate samples in order to minimize the total number used, for intelligent sampling.

We start with the two-stage intelligent sampling procedure and assume that in stage 1, roughly $N_1$ points are used for BinSeg and another $N_1$ points, for the calibration steps described in Section 3.4. At stage 2, we work with $\hat{J}$ (which is $\approx J$) intervals. Using Theorem 6, setting the width of the second stage intervals to be $(Q_j(\alpha, J) + 1) \frac{\sqrt{N}}{N_1}$ will ensure that they cover the true change points with probability around $1 - \alpha$, where $Q_j(\alpha, J)$’s are any integers satisfying

$$P[|L(\Delta_j/\sigma)| \leq Q_j(\alpha, J) \text{ for all } j = 1, \ldots, J] \geq 1 - \alpha.$$  \hfill (43)

A good choice for the $Q_j^{(\alpha)}(J)$’s are the $\sqrt{1 - \alpha}$’s quantiles of $L(\Delta_j/\sigma)$’s. Assuming $N_1$ is large enough so that the first stage is accurate (ie $\hat{J} = J$ and $\max_j |\hat{\tau}_j^{(1)} - \tau_j|$ is small with high probability), the number of points used in the two stages, combined, is approximately

$$2N_1 + \frac{2}{N} \left( \sum_{j=1}^J \left( Q_j^{(\alpha)}(J) + 1 \right) \right).$$ \hfill (44)

This presents a trade-off, e.g. if we decrease $N_1$ by a factor of 2, the second term in (44) increases by a factor of 2. To use a minimal number of points in both stages, we need to set $N_1 = \sqrt{N} \sum (Q_j(\alpha, J) + 1)$. In turn this yields a minimum of $4 \sqrt{N} \sum (Q_j(\alpha, J) + 1)$ when plugged into (44). For any given values of $N$, $J$, and SNR, this provides a lower bound on the minimum number of points that intelligent sampling must utilize, and Tables 7 and 8 depict some of these lower bounds for a select number of these parameters.

§§For example, BinSeg will generally terminate in fewer steps on a dataset with fewer change points than on another dataset of the same length but more change points.
For $N = 1.5 \times 10^7$, the minimal percentage of data that must be used for various values of $J$ and SNR, assuming all jumps have equal SNR, $Q_j(\alpha, J)$’s are the $\sqrt{0.99}$ quantiles of the $|L(SNR)|$ distribution to ensure 99% probability of simultaneous coverage of change-points by the confidence intervals on which the analysis at Stage 2 will be based.

| $\Delta/\sigma$ | 40 | 60 | 80 | 100 |
|----------------|----|----|----|-----|
| 1              | 4.33 | 5.43 | 6.4 | 7.3 |
| 1.25           | 3.46 | 4.38 | 5.14 | 5.84 |
| 1.5            | 2.92 | 3.07 | 3.33 | 4.95 |
| 1.75           | 2.53 | 3.28 | 3.81 | 4.26 |
| 2              | 2.26 | 2.77 | 3.33 | 3.72 |
| 2.25           | 1.96 | 2.53 | 3.92 | 4.34 |
| 2.5            | 1.85 | 2.26 | 2.61 | 3.1 |
| 2.75           | 1.73 | 2.12 | 2.44 | 2.73 |
| 3              | 1.65 | 1.90 | 2.26 | 2.53 |

For $N = 1.5 \times 10^{10}$, the minimal percentage of data that must be used for various values of $J$ and SNR, assuming all jumps have equal SNR, $Q_j(\alpha, J)$’s are the $\sqrt{0.99}$ quantiles of the $|L(SNR)|$ distribution.

It can be seen that for larger values of $N$, intelligent sampling uses a smaller fraction of the data (Table 8), which generally has lower values than Table 7 despite looking at higher values of $J$. However, in absolute terms, the method requires a very large subsample for higher values of $N$: even just 0.57% of $1.5 \times 10^{10}$ is a very large dataset of $8.6 \times 10^7$, which requires server type computer capabilities. The situation becomes more tenuous for larger values of $N$. This suggests that a larger number of stages is in order for sample sizes of $N$ exceeding $10^{10}$.

The multi-stage intelligent sampling procedure is described in Section 12.2.1. For a three-stage implementation, suppose $\approx N_1$ points are utilized at stage 1, letting us form simultaneous confidence intervals that are (approximately) of the form

$$\hat{\tau}_j^{(1)} - (Q_j(\alpha, J) + 1) \frac{N}{N_1}, \hat{\tau}_j^{(1)} + (Q_j(\alpha, J) + 1) \frac{N}{N_1}$$

for $j = 1, \ldots, J$ (assuming $J = J$ for simplification). At stage 2, suppose at the $j$'th confidence interval we subsample roughly $N_2(j)$ points, giving us a subsample which skips approximately every $2Q_j(\alpha, J) \frac{N}{N_1N_2(j)}$ points. Hence, at stage 3 we work with confidence intervals that are (approximately) of the form

$$\hat{\tau}_j^{(2)} - (Q_j(\alpha, J) + 1) \left(2 \left(Q_j(\alpha, J) + 1\right) \frac{N}{N_1N_2(j)}\right), \hat{\tau}_j^{(2)} + (Q_j(\alpha, J) + 1) \left(2 \left(Q_j(\alpha, J) + 1\right) \frac{N}{N_1N_2(j)}\right)$$

for $j = 1, \ldots, J$. In total all three stages use around a total of

$$2N_1 + \sum_{j=1}^{J} N_2(j) + \frac{4N}{N_1} \left(\sum_{j=1}^{J} \frac{(Q_j(\alpha, J) + 1)^2}{N_2(j)}\right)$$

(47)

points. This expression is minimized by setting

$$N_1 = N^{1/3} \left(\sum_{k=1}^{J} (Q_k(\alpha, J) + 1)^3\right)^{2/3} \text{ and } N_2^{(j)} = 2N_1^{1/3} \frac{Q_j(\alpha, J) + 1}{\left(\sum_{k=1}^{J} (Q_k(\alpha, J) + 1)^3\right)^{1/3}}$$

for $j = 1, \ldots, J$, which in turn gives a minimum of $6N^{1/3} \left(\sum_{j=1}^{J} (Q_j(\alpha, J) + 1)^3\right)^{2/3}$ for (47). A similar analysis on a four-stage procedure would have the optimal subsample allocation as $N_1 = N^{1/4} \left(\sum_{k=1}^{J} (Q_k(\alpha, J) + 1)^4\right)^{3/4}$ and $N_2^{(j)} = N_3^{(j)} = 2(Q_j(\alpha, J) + 1) N^{1/4} \left(\sum_{k=1}^{J} (Q_k(\alpha, J) + 1)^3\right)^{-1/4}$ for $j = 1, \ldots, J$, which yields a total
of $8N^{1/4} \left( \sum_{k=1}^{J} (Q_k(\alpha, J) + 1) \right)^{3/4}$ points utilized.

Fig. 9. For $N = 1.5 \times 10^{10}$, minimal percentage of the data that must be used for a three stage procedure, assuming all jumps have equal SNR. $Q_j(\alpha, J)$'s were the $\sqrt{0.99}$ quantiles of the $|L\text{SNR}|$ distribution to ensure 99% probability of simultaneous coverage of change-points by the confidence intervals on which the analysis at Stage 3 will be based.

| $\Delta/\sigma$ | $J$  | 100  | 250  | 500  | 1000 |
|----------------|-----|------|------|------|------|
| 1              | 0.03| 0.061| 0.101| 0.160|
| 1.25           | 0.023| 0.045| 0.075| 0.125|
| 1.5            | 0.018| 0.038| 0.066| 0.099|
| 1.75           | 0.013| 0.029| 0.049| 0.082|
| 2              | 0.012| 0.025| 0.041| 0.068|
| 2.25           | 0.011| 0.022| 0.036| 0.066|
| 2.5            | 0.009| 0.016| 0.031| 0.052|
| 2.75           | 0.009| 0.017| 0.027| 0.046|
| 3              | 0.008| 0.014| 0.025| 0.043|

Comparing Figures 8 and 9, we focus on the case of 1000 change points with SNR 1.5: using three stages allows us to decrease the minimal required points by a factor of around five. The ease on computations is greater when looking at the largest amount of data the computer must handle at a time: are $N_1 \approx 2.1 \times 10^7$ for two stages and $N_1 \approx 2.5 \times 10^6$ for three stages, a decrease by a factor of 9. Meanwhile for a dataset of size 1.5 trillion, using four stages allows us to work with subsamples of size at most $N_1 \approx 4.7 \times 10^6$ for the more demanding scenario of SNR 1.5 and 2000 change points, a very manageable dataset for most computers.

We note here that these optimal allocations are valid assuming that BinSeg is able to pin down $\hat{\tau}_j$ and the change points with the initial subsample. In general, this will be the case provided the SNR is reasonable, and the initial subsample is large enough so that the change-points are adequately spaced apart. For example, in the context of the above tables, one can ask whether BinSeg will accurately estimate the parameters on a 2.4 million length dataset with 1000 evenly spaced change points, or 2000 change points on a 4.7 million length data with 2000 evenly spaced change points, under a constant SNR of 1.5 (which is of modest intensity). To this end, we ran a set of simulations and concluded that if there are over 1000 data points between consecutive change points of SNR 1.5, based on these two settings and for appropriate tuning parameters, BinSeg’s estimators satisfy $\hat{J} = J$ and max $|\hat{\tau}_j - \tau_j| \leq 150$ with probability over 99%.

Observe also that the formulas provided depend on the values of the SNRs at the change points and the actual number of change points ($J$). In practice, neither will be known, and the practitioner will not be able to determine the derived allocations exactly. In such situations, conservative lower bounds on the SNRs and a conservative higher bound on $J$, can yield valid (but conservative) sampling allocations when plugged into the expressions derived through this section. Such bounds can be obtained if background information about the problem and the data are available, or via rough pilot estimates on an appropriately sparse subsample.

It is also worth pointing out that the intelligent sampling procedure is readily adaptable to a distributed computing environment, which can come into play, especially with data sets of length exceeding $10^{12}$ that are stored sequentially across several storage disks. In such cases, the two sparse subsamples at the first stage, which are of much smaller order, can be transferred over to a central server (a much easier exercise than transferring all the data on to one server), where the first is analyzed via binary segmentation to determine the initial change-points, and the other used for the re-estimation procedure and associated confidence intervals as described in Section 3.4. As the number of disks on which the data are stored is of a much smaller order than the length of the data, each re-estimated change-point and its associated confidence interval will typically belong to a stretch of data completely contained within one storage disk, and the

†For intelligent sampling the largest data subset the computer has to work with and hold in memory at any moment, under these optimal allocations and when all change points have equal SNR, is the roughly $N_1$ sized data set used at the initial step for BinSeg. All subsequent steps can work with sub-intervals of data less than $N_1$ in size.
subsequent resampling and estimation steps can be performed on the local processor, after the information on the confidence interval has been transferred back from the central server. An occasional communication between two machines may be necessary.

6. Performance Evaluation of Intelligent Sampling Simulation Results

We next illustrate, through a set of simulation studies, the theoretical properties of the intelligent sampling procedure: the rate of convergence and the lower than $O(N \log(N))$ computational running time, along with the validity of the asymptotic distribution. All simulations in this section were performed on a server with 24 Intel Xeon ES-2620 2.00 GHz CPUs, with a total RAM of 64 GB.

Implementation of Intelligent Sampling via BinSeg: There are numerous parameters associated with the multiple change point problem. Of importance are not only just the minimal separation $\delta_N$, minimum jump size $\Delta$, and the number of change points $J$, which are the main parameters that appear in the theory, but also how the change points are distributed across the time series (which can vary wildly if $(J + 1)\delta_N < N$), the actual values of the jumps $\nu_i - \nu_{i-1}$, for $i = 0, \ldots, J$, and the first-stage subsample size $N_1$. All of these can affect the accuracy of the procedure, particularly in the first stage rather than the zoom in estimation at the second stage (which is usually accurate if the first stage was accurate to begin with).

In addition to the re-estimation procedure described in Section 2.4, we also included some ad-hoc methods to practically improve the accuracy of binary segmentation for the sparse subsample. For the initial subsample $Z_1, Z_2, \ldots$ with binseg estimates $\hat{\tau}_i^*, i = 1, \ldots, J$ as was described in step (ISM2), consider the two drop steps: fix positive constants $\delta_D$ and $\Delta_D$, and,

(D1): if for some $1 \leq i \leq J$ we have $|\hat{\tau}_i^* - \hat{\tau}_{i-1}^*| \leq \delta_D$, then remove $\hat{\tau}_i^*$ from the list of estimates;

(D2): continuing to denote the remaining estimates as $\hat{\tau}_i^*, \ldots, \hat{\tau}_J^*$ for convenience, let each $\hat{\nu}_i^*$ be the mean of the $Z_i$’s from $\hat{\tau}_i^* + 1$ to $\hat{\tau}_{i+1}^*$; if for some $i$ we have $|\hat{\nu}_i^* - \hat{\nu}_{i-1}^*| \leq \Delta_D$ then drop $\hat{\tau}_i^*$ from the list of estimates.

The intuition behind the first step is to set $\delta_D$ as a reasonably small integer, then for any dataset where $\delta_N$ is large, no two change points should be $\delta_D$ apart. Similarly, when $\Delta_D$ is set to be a number lower than $\Delta$ (or some estimate thereof), step (D2) drops any estimate which does not exhibit a large enough signal change before and after. These two steps, when executed after binary segmentation and after the refitting method described from Section 2.4, ensure that the first stage estimates are more robust to overestimating $J$.

To illustrate the rate of convergence, intelligent sampling was applied to time series of length $N$ varying from $10^5$ to $10^7.5$, evenly on the log scale, with the number of change points being $J \approx \log_{10}(N)^2$. The change point location and the signal levels were randomly generated:

- The spacings $(\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots, N - \tau_J)$ were generated as the sum of the constant $\frac{N}{1+J}$ and the Multinom $\left(N - \frac{(J+1)N}{1+J}, (p_0, \ldots, p_J)\right)$ distribution

  $$ - (p_0, \ldots, p_J) \text{ is generated as the consecutive differences of the order statistics of } J+1 \text{ Unif}(0,1) \text{ random variables}$$

- The signals were generated as a Markov chain with $\nu_0$ initialized as 0, and iteratively, given $\nu_i, \nu_{i+1}$ is generated from a density proportional to $f(x) = \exp(-0.3(x - \nu_i - \Delta))1(\nu_i + \Delta \leq x \leq M) + \exp(0.3(x + \nu_i - \Delta))1(\nu_i - \Delta \geq x \geq -M)$, where $M$ was taken to be 10 and $\Delta$ was taken to be 1.

For each of 10 values of $N$, 50 configurations of change points and signals were generated, and on each of those configurations 40 datasets with iid $N(0,1)$ error terms simulated, and intelligent sampling was performed on each simulated dataset with binary segmentation taken at stage 1 on a subsample of size roughly $N_1 = 50\sqrt{N}$ and thresholding parameter $\zeta_N = N^{0.2}$, a valid choice according to Theorem 3. Additionally the drop steps (D1) and (D2) were applied right after binary segmentation (with $\delta_D = 15$ and $\Delta_D = 0.5$), the re-estimation procedures (ISM4-1) to (ISM4-3) were run, and the drop steps (D1) and (D2) were applied again. With this setup, the stage 1 binary segmentation was accurate in determining the correct value of $J$ over 99% of times for all $N$. In the second stage we let the width of the sampling interval around $\hat{\tau}_j^*(1)$ be the $\sqrt{0.99}$
quantile of the $\left| L \left( \frac{\hat{\nu}(j) - \nu(j)}{\sigma} \right) \right|$ distribution, for $j = 1, \ldots, \hat{J}$. After the second stage of intelligent sampling, the maximum deviations $\max_{j=1,\ldots,\hat{J}} |\lambda_2(\tau_j, \hat{\tau}_j^{(2)})|$ were recorded. The running time of intelligent sampling was also recorded, and we compared it to the running time of binary segmentation on the full data (only the BinSeg procedure itself, without steps (D1) and (D2)). For the latter, we ran 100 iterations for each $N$, 2 runs per configuration of parameters. The results are depicted in Figure 11 and are in accordance with the theoretical investigations.

Fig. 11. Left: Quantiles of the max deviations versus $N$. Over the observed regime of parameters, the maximal deviation scales linearly or sub-linearly with $\hat{J}$, as was predicted by Theorem 4. Right: Log-log plot of mean computational time when using intelligent sampling to obtain the final change-point estimates at stage two, and using BinSeg on the full data to construct change-point estimates, with reference lines of slope 1 (black) and 0.5 (red) respectively. To give some sense of the actual values, for $N = 10^7$, the average time for intelligent sampling vs full data were, respectively, 1.70 and 83.557 seconds.

A second set of simulation experiments is used to illustrate the asymptotic distribution of the change point deviations. We considered four scenarios, each with $N = 10^7$ and 55 change points:

(Setup 1): one set of signal and change point locations generated as in the previous set of simulations, with i.i.d. $N(0, 1)$ error terms;

(Setup 2): change points evenly spaced apart with signals 0,1,0,1,..., repeating, and i.i.d. $N(0, 1)$ error terms

(Setup 3): change points evenly spaced part with 0,1,0,1,..., repeating signals, and error terms generated as $\varepsilon_i = \frac{\varepsilon_i^* + 0.5\varepsilon_{i+1} + 0.25\varepsilon_{i+2}}{\sqrt{1+0.5+0.25}}$ for all $i = 1,\ldots, N$, where the $\varepsilon_i^*$’s are generated as i.i.d. $N(0, 1)$;

(Setup 4): change points evenly spaced with signals 0,1,0,1,..., and error terms generated from an AR(1) time series with parameter 0.2, and each marginally $N(0, 1)$.

For all 4 cases the first stage of intelligent sampling was performed identically as for the previous set of simulations, and with the same tuning parameters. At the second stage, first stage subsample points were omitted for data with setups 1 and 2, but not for setups 3 and 4. At stage 2, the subsampling intervals were 1.1 times the $\sqrt{0.99}$ of the estimated SNR. From 2500 iterations on each of the 4 simulation setups, the distributions of the maximum deviations (maximum of $|\lambda_2(\tau_j, \hat{\tau}_j^{(2)})|$ for the first two setups and $|\hat{\tau}_j^{(2)} - \tau_j|$ for the other two setups) are seen to match well with their predicted asymptotic distributions. To illustrate the convergence of the individual change point estimates, we also show that the distribution of the 27th change point matches with the asymptotic distribution from Theorem 5 (for the first two scenarios) and Proposition 4 (for the third and fourth scenarios).
Fig. 12. Distributions of $\max_{1 \leq j \leq 55} \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right)$ and $\lambda_2 \left( \tau_{27}, \hat{\tau}_{27}^{(2)} \right)$ from simulations of setup 1.

Fig. 13. Distributions of $\max_{1 \leq j \leq 55} \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right)$ and $\lambda_2 \left( \tau_{27}, \hat{\tau}_{27}^{(2)} \right)$ from setup 2.

Fig. 14. Distributions of $\max_{1 \leq j \leq 55} \left| \tau_j - \hat{\tau}_j^{(2)} \right|$ and $\left| \tau_{27} - \hat{\tau}_{27}^{(2)} \right|$ from setup 3.
The distribution of the deviations for setup 1 is the least spread out, with the primary reason that the jump between signals is randomly generated but lower bounded by 1 while the other 3 setups have signal jumps all fixed at 1. Setups 3 and 4 have the most spread out distributions, as the dependence among the error terms cause the estimation to be less accurate but only up to a constant and not an order of magnitude. Nevertheless, in all 4 setups, the change point estimates behave very closely to what was predicted by Theorem 5 (for the first two scenarios) and Proposition 4 (for the third and fourth scenarios).

These results clearly show that the distributions of the estimates match up well with the true asymptotic distributions $L(\Delta_i/\sigma)$ for $i = 1, \ldots, J$, but in a practical setting the true values of $\Delta_i/\sigma$ are unknown and we would be unable to generate from the true asymptotic distribution. A solution to this is to use estimates $\hat{\Delta}_i/\hat{\sigma}$ and to perform inference based on on the distributions $L(\hat{\Delta}_i/\hat{\sigma})$, with a possible set of estimates given by calculating the sample moments based on the data-points used in the first and second subsample while assuming $\hat{\tau}_j^{(2)}$'s are the true change points for $j = 1, \ldots, J$. We conducted set of simulations to investigate the effect of using the estimated SNRs, the $\hat{\Delta}_i/\hat{\sigma}$’s, on our inference. Using the parameters of setup 1, and the same estimation procedure and tuning parameters, for each $j$ from 1 to 55 we recorded the proportion of 2000 iterations for which $\tau_j$ was covered by the 0.95 level confidence interval based on $\hat{\tau}_j^{(2)}$ and $L(\hat{\Delta}_j, \hat{\sigma})$. The coverage proportions were all very close to 95%, as shown in Figure 16, demonstrating the robustness of using an estimated SNR for inference.

**Implementation in a heteroscedastic error model:** Finally, we explored the validity of Proposition 4 by looking at a case with heteroscedastic errors. We again generated a time series of length $N = 10^7$ with
55 evenly spaced changed points and signals of $0, 1, 0, 1, 0, 1, \ldots, 1$. Instead of generating error terms from a stationary series, we, instead, generated them as independent segments of Gaussian processes as follows. For $j = 1, 2, 3, \ldots$,

(a) from $\tau_{4j}$ to $\frac{\tau_{4j+3}+\tau_{4j+1}}{2}$ the errors are iid $N(0, 1)$;

(b) from $\frac{\tau_{4j+3}+\tau_{4j+1}}{2}$ to $\frac{\tau_{4j+2}+\tau_{4j+3}}{2}$ the errors are $\varepsilon_i = \frac{\varepsilon^*_i+0.5\varepsilon^*_{i+1}+0.25\varepsilon^*_{i+2}}{\sqrt{4+0.5^2+0.25^2}}$ where the $\varepsilon^*_i$'s are iid $N(0, 1)$ (and will be treated as a generic iid $N(0, 1)$ sequence from here on);

(c) from $\frac{\tau_{4j+3}+\tau_{4j+2}}{2}$ to $\tau_{4j+3}$, error terms are $\varepsilon_i = 0.5 \cdot \frac{\varepsilon^*_{i-1}+\varepsilon^*_{i}+\varepsilon^*_{i+1}}{\sqrt{2}}$;

(d) from $\tau_{4j+3}$ to $\tau_{4j+4}$ the error terms are $\varepsilon_i = 0.7 \cdot \frac{\varepsilon^*_{i-1}+\varepsilon^*_{i}}{\sqrt{2}}$;

and the error terms generated in each stretch are independent of those in any other stretch. This creates a situation where around $\tau_{4j+1}$ the error terms are iid $N(0, 1)$, around $\tau_{4j+2}$ the error terms are stationary, and around $\tau_{4j+3}$ and $\tau_{4j+4}$ the error terms are stationary to the left and to the right, but their autocorrelation and marginal variances change at the change points. With the same intelligent sampling procedure as setups 3 and 4, and the same tuning parameters, we ran 2000 replicates of this setup and recorded the $\hat{\tau}^{(2)}_j - \tau_j$ values for $j = 1, \ldots, 55$.

![Distribution of Maximal Deviation](image1.png)

![Deviations for Change Point #25](image2.png)

![Deviations for Change Point #30](image3.png)

![Deviations for Change Point #31](image4.png)

![Deviations for Change Point #32](image5.png)

**Fig. 17.** Above: Predicted and actual distribution of the maximal deviation $\max_{1 \leq j \leq 55} |\hat{\tau}^{(2)}_j - \tau_j|$. Below: Predicted and actual distributions of the individual deviations $|\hat{\tau}^{(2)}_j - \tau_j|$ for $j = 29, 30, 31, \text{and } 32$.

Results from the simulation are very consistent with Proposition 4. Even for change points which have different distributions of error terms to the left and right, the deviations match up very closely with the stated asymptotic distributions.

7. **Real Data Application**

The effectiveness of the proposed intelligent sampling procedure is illustrated on an Internet traffic data set, obtained from the public CAIDA repository [http://data.caida.org/datasets/passive/passive-oc48/20020814](http://data.caida.org/datasets/passive/passive-oc48/20020814).
that contains traffic traces from an OC48 (2.5 Gbits/sec) capacity link. The trace under consideration contains traffic for a two hour period from large west coast Internet service provider back in 2002. The original trace contains all packets that went through the link in an approximately 2 hour interval, but after some aggregation into bins of length 300 microseconds, the resulting time series comprises of \( N = 1.5 \times 10^7 \) observations. After applying a square-root transformation, a snapshot from this time series is depicted in Figure 18 and some of its statistical characteristics in Figure 19, respectively.

Fig. 18. First 5000 time points of the data after a square root transformation.

Fig. 19. QQ plot and estimated ACF of first 5000 points of data set, after square root transformation, note the normality of the data after a square root transformation.

It can be seen that the data are close to marginally normally distributed, while their autocorrelation decays rapidly and essentially disappears after a lag of 10. Similar exploratory analyses performed for multiple stretches of the data leads to similar conclusions. Hence, for the remainder of the analysis, we work with the square-root transformed data and model them as a short range dependent sequence.

To illustrate the methodology, we used an emulation setting, where we injected various mean shifts to the mean level of the data of random durations, as described next. This allows us to test the proposed intelligent sampling procedure, while at the same time retaining all features in the original data.

In our emulation experiments, we posit that there are two types of disruptions, short term spikes that may be the result of specific events (release of a software upgrade, a new product or a highly anticipated broadcast) and longer duration disruptions that may be the result of malicious activity [3, 12]. To emulate these scenarios, at random intervals \([V_i, W_i]\) we increased the signal in the time series by a randomly generated constant \(\Delta_i\), changing the data as \(Y_j \leftarrow Y_j + \Delta_i \cdot 1_{[V_i, W_i]}(j)\), as follows:

(sig-1): A set of 31 stretches \((V_{1,i}, W_{1,i})\) for \(i = 1, \ldots, 31\) were created by first generating \((W_{1,1}, W_{1,2} - W_{1,1}, W_{1,3} - W_{1,2}, \ldots, W_{1,30} - W_{1,29}, N - W_{1,30})\) from the \(1.5 \times 10^5 + \text{Multinom}(N-31 \times 1.5 \times 10^5, (p_0, \ldots, p_{30}))\) distribution, conditional on \((p_0, \ldots, p_{120})\) which is generated as the order statistics of Unif(0, 1). Then each \(V_{1,i}\) are taken by generating each \(W_{1,i} - V_{1,i}\) as \(75000 + \text{Binom}(W_{1,i} - W_{1,i-1} - 75000, 0.5)\), and the
increase in signals $\Delta_{1,i}$’s are generated from a $\text{Unif}(1.3\hat{\sigma}, 2\hat{\sigma})$ where $\hat{\sigma}$ is the standard deviation of the data.

(sig-2): Stretches $(V_{2,i}, W_{2,i})$ for $i = 1, \ldots, 201$ were independently generated by setting $(W_{2,1}, W_{2,2} - W_{2,1}, \ldots, N-W_{2,200})$ from the 50050+$\text{Multinom}(N-201\times50050, (p_0, \ldots, p_{201}))$ conditioned on $(p_0, \ldots, p_{120})$ which is generated as the order statistics of $\text{Unif}(0, 1)$. Each $W_{2,i} - V_{2,i}$ is generated as 50+$\text{Binom}(W_{2,i} - W_{2,i-1} - 50000, 0.0001)$, and the increases in signals as $\Delta_{2,i} \sim \text{Unif}(10\hat{\sigma}, 15\hat{\sigma})$

This scheme randomly places a fixed number of stretches of traffic increases (a combined value of 232 in fact), without placing the stretches too close together. Stretches from (sig-1) emulate longer, milder increases of a bump in the time series, as each $W_{1,i} - V_{1,i} \geq 75,000$, while stretches from (sig-2) emulate short but more dramatic increases, as each $W_{i} - V_{i}$ is guaranteed to be higher than 50 but not likely to be much higher. Both types of traffic increases can occur when looking for increase in user traffic or attacks by third parties. A depiction of a segment of the data with the emulated signal is given in Figure 20.

![Fig. 20. Example of emulated data. The intervals $[V_{1,i}, W_{1,i}]$ emulate persistent stretches of mild increase in traffic, while the intervals $[V_{2,i}, W_{2,i}]$ emulate very short stretches of high traffic increase.](image)

As mentioned in the introduction, the main objective of the proposed methodology is to identify long duration, persistent shifts in the time series using a limited number data points; in the emulation scenario used, this corresponds to change points induced by sig-1, while we remain indifferent to those induced by sig-2.$^\dagger$

The two-stage intelligent sampling procedure was implemented as follows: (i) the first stage subsample comprised observations 100 time points apart in the original time series. BinSeg with thresholding parameter $\zeta_{N_i} = N_i^{0.2} = 150,000^{0.2}$ was employed, followed by steps (D1) and (D2) from Section 8 with $\delta_D = 15$ and $\Delta_D = 0.5$, and calibration from Section 3.4 applied with a different subsample, and again an application of steps (D1) and (D2). (ii) For each $j$, the second stage interval surrounding $\hat{\tau}_j^{(1)}$ was chosen to have half width $Q_j(\sqrt{0.99}) \times 100$, where $Q_j(\sqrt{0.99})$ is the $\sqrt{0.99}$ quantile of $\left\lfloor L \left( \frac{\hat{\phi}_j^{(1)} - \hat{\phi}_j^{(1)}}{\hat{\sigma}} \right) \right\rfloor$. A stump model was then fitted to the data in each second stage interval to obtain the final estimates of the change-points and the final (2nd stage) CIs were constructed.

To assess the accuracy, we calculated the coverage proportion of the 90%, 95%, and 98% level confidence intervals over different emulation settings. To construct these confidence intervals we had to randomly generate time series with identical distribution structure as the data (which would give us a random sample of $L$-type distributions and their quantiles). We generated these time series as marginally normal random variables, with marginal standard deviation the same as the sample sd of the first 50,000 points of the $1.5 \times 10^7$ length data. Finally, the ACF of the generated series was matched with the sample ACF of the first 50,000 points up to a lag of 20: we first generated vectors of iid normal variables, then multiplied them

$^\dagger$We note that the theoretical development does not include spiky signals. Nonetheless, we included spiky signals in our emulation to mimic the pattern of internet traffic data. As will be seen later, our method is quite robust to the presence of this added feature.
with the Cholesky square root of the band matrix created with the sample ACF (bandwidth of this matrix is 20, and non-zero entries taken from the first 20 values of the sample ACF).

Intelligent sampling exhibits satisfactory performance: among all 61 change points corresponding to sig-1, the lowest coverage probability for the 90%, 95%, and 98% nominal confidence intervals were 0.886, 0.926, and 0.954 respectively, while average coverage probabilities were around 0.908, 0.946, and 0.970, respectively. On the other hand, for change points induced by sig-2, the average coverage probability was lower than 0.002 even for the 98th confidence interval. However, since the focus of intelligent sampling is on long duration persistent signals, missing the spiky signals is of no great consequence. In terms of computational burden, the average emulation setting utilized 3.46% of the full dataset, requiring an average time of 1.43 seconds to perform the estimation. We note that Table 7 corresponds to the length of this dataset in Section 7. As we are concerned with SNR between 1.3 and 2, the percentage of data used in our emulation experiment is quite consistent with the numbers presented in that table.

**Fig. 21.** Coverage proportions, the proportion of time when the change point was covered by some confidence interval, for the 90% level (green bars), 95% level (blue bars), and 98% level (red bars) within the 500 iterations, for a select number of 20 change points (change point # 2 is always the second one in order, # 3 is the third in order, etc). Horizontal reference lines are at 0.9 (green), 0.95 (blue), and 0.98 (red).

8. Concluding Remarks and Discussion

This paper introduced sampling methodology that reduces significantly the computational requirements in multi-change point problems, while not compromising on the statistical accuracy of the resulting estimates. It leverages the locality principle, which is obviously at work in the context of the classical signal-plus-noise model employed throughout this study. A natural extension, further enhancing the scope of the methodology, would deal with a piece-wise linear signal plus noise model with discontinuities between the linear stretches. We expect that the locality principle naturally extends to such a setting, based on prior work together with numerical results in a related problem [18] in a design setting provides. Extensions to problems involving multiple (potentially high-dimensional) time series produced by cyber-physical systems equipped with a multitude of sensors monitoring physical or man-made phenomena are of obvious interest.

The focus in this paper has primarily been on a two-stage procedure, which is easiest to implement in practice and suitable for many applications. Nevertheless, as discussed in Section 12.2.1 and illustrated in Section 5 in specific settings involving data sets of length exceeding $10^{10}$ points, a multi-stage procedure may be advantageous, as discussed in Section 12.2.1. Furthermore, in Section 12.2.2 we also discuss how the procedure developed in this paper could be extended to scenarios that also involve weak signals (i.e. situations where we do not bound the minimum jump size away from 0) and short-lived spiky signals.

A key technical requirement for intelligent sampling is that the procedure used to obtain the 1st stage estimates needs to exhibit consistency properties, e.g. [21]. The choice of binary segmentation in our exposition (or its wild binary segmentation variant presented in detail in the Appendix) is due to its computational
attractiveness and the fact that it provides consistent estimates of the number of change points and their locations. Nevertheless, there are other methods that fit the bill, as discussed next.

Two popular methods used for models defined as in \((12)\) are the estimation of multiple structural breakpoints introduced in [1] and PELT as described in [14]. The method described in [1] does give consistent estimates, but only under the much stricter condition that \(J\) is a constant and there exists values \(\beta_1, \ldots, \beta_J \in (0, 1)\) such that \(\tau_j = [\beta_j N]\) for all \(j = 1, \ldots, J\) and \(N\). Further, to run the actual procedure would require the use of dynamic programming which is computationally expensive \(O(N^2)\) time. With the PELT procedure, the implementation itself runs in a more manageable \(O(N)\) time; however, this works under the very different Bayesian setting where the spacings \(\tau_{j+1} - \tau_j\) are iid generated from some distribution. Further, PELT was built upon a procedure described in [14], which examines another Bayesian model where every point \(\{1, \ldots, N\}\) has a probability \(p\) of being a change point, and the development did not go into details regarding rates of convergence of the change point estimates. Due to the theoretical and computational restrictions of the multiple structural breakpoints method and the differing framework under which PELT works, we focused our analysis on binary segmentation.

We also mention the SMUCE procedure, introduced in [2], where lower probability bounds for the events \(P[J \neq J]\) and \(P[\max_{j=1,\ldots,J} \min_{i=1,\ldots,J} |\tilde{\tau}_i - \tau_j| \leq c_N]\), for any sequence \(c_N\), were derived. These results can be combined to yield \(P[J = J; \max_{j=1,\ldots,J} |\tilde{\tau}_j - \tau_j| \leq c_N] \rightarrow 1\) under certain restrictions and for some sequences \(c_N\) that are \(o(\delta_N)\), and therefore could be used in the first stage of intelligent sampling. SMUCE has the flexibility of working for a broader class of error terms but as was stated in [2], the procedure involves dynamic programming which runs in \(O(N^2)\) time. This last point is less of an issue for a modified version of SMUCE designed for iid Gaussian errors with heterogeneous variances. H-SMUCE, in [21], could run the procedure in as low as \(O(N)\) time in some cases. Overall, SMUCE could be used as the first part of intelligent sampling, and the regimes of \(\delta_N\) and restrictions on the subsample size \(N_1\) needed for intelligent sampling to be consistent could be fleshed out in a similar manner as in this paper. However, as BinSeg and WBinSeg are somewhat easier to implement computationally, we chose to perform our analysis with them instead.

Another modeling assumption permeating our presentation is that of Gaussian error terms. However, such an assumption is not strictly needed. The results from Theorems 4, 5, and 6 of the random walk described in [14] but with the \(\epsilon_j\) terms having the same distributions as the error terms. Extensions to a wider class of errors than sub-Gaussian would require the proofs to be modified, but since much of the mathematical details depend on concentration inequalities, these feasibly could be generalized further.

The only other issue with working under non-Gaussian error terms would be to ensure the estimates at the first stage are consistent. Although results from BinSeg found in [25] and [2] are for a Gaussian framework, other methods found in [1] and [3] do work under a more general context. The method presented in [1] works when the errors form a mixingale/martingale following certain assumptions, while SMUCE as presented in [2] can work when the data are generated from a class of exponential families, and therefore can handle discrete data (e.g. Binomial, Poisson) in particular. Both could be used in the first stage of intelligent sampling on data with non-Gaussian error terms.

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9. Appendix A (Single Change Point Problem)

9.1. Single Change Point Problem

Instead of proving Theorems 1 and 2 directly, we shall consider a more general nonparametric result from which the two theorems will follow as a special case. As before suppose the time series data is \((X_1, Y_1),..., (X_N, Y_N)\), where \(X_i = i/n\) and \(Y_i = f(X_i) + \epsilon_i\) for \(i = 1,..., N\). We will make the weaker assumptions that

- \(f\) is a right continuous function in \([0, 1]\) with a single left discontinuity at some point \(\tau_0 \in (0, 1)\), with jump size \(f(\tau_0^+) - f(\tau_0^-) = \Delta\)
- there exists a \(\beta_f > 0\) where \(|f(x) - f(y)| \leq \beta_f |x - y|\) whenever \((x - \tau_0)(y - \tau_0) > 0\)
- the errors \(\epsilon_i\)'s are iid \(N(0, \sigma^2)\) error terms

The main difference between this model and the model presented in section ?? is the looser restriction on the signal \(f\): here \(f\) could be any Lipschitz continuous function with a single discontinuity and not be constrained to the family of piecewise constant functions.

We will first remark on some background regarding this this model before moving on to proving some results. Estimation procedures for such a dataset can be found in [19], where one-sided polynomial fitting was used to obtain an estimate \(\hat{\tau}_N\) for \(\tau_N := [N \tau_0]/N\). In summary, fix a sequence of bandwidth \(h = h_N\), a non-negative integer \(p\), and a kernel function \(K\) with support on \([-1, 1]\). Next, for all \(X_m \in (h, 1-h)\), consider the signal estimates

\[
\begin{align*}
\hat{f}_-(X_m) := & \pi_1 \left( \arg \min_{(a_0, ..., a_p) \in \mathbb{R}^{p+1}} \left( \sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right) \left( Y_{m-j} - a_0 - a_1 j - ... - a_p j^p \right)^2 \right) \right) \\
\hat{f}_+(X_m) := & \pi_1 \left( \arg \min_{(a_0, ..., a_p) \in \mathbb{R}^{p+1}} \left( \sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right) \left( Y_{m+j} - a_0 - a_1 j - ... - a_p j^p \right)^2 \right) \right),
\end{align*}
\]

where \(\pi_1\) is the projection functions such that \(\pi_1(a_0, ..., a_p) = a_0\). The change point estimate is

\[
\hat{\tau}_N := \arg \max_{X_i \in (h, 1-h)} |\hat{f}_+(X_i) - \hat{f}_-(X_i)|.
\]

This estimator is consistent under a few regularity conditions on the kernel \(K\) and conditions on how fast \(h\) converges to 0. For the sake of brevity we will not mention all those conditions here, but we will note that under said conditions, \(\mathbb{P}[\hat{\tau}_N - \tau_N = k] \rightarrow \mathbb{P}[L(\Delta/\sigma) = k]\) for all \(k \in \mathbb{Z}\), which is the exactly asymptotic result for \(\hat{\tau}_N\) obtained by least squares in a stump model setting. Finally, for our purposes we propose estimators \(\hat{\alpha}\) and \(\hat{\beta}\) for \(f(\tau_0^-)\) and \(f(\tau_0^+)\), respectively, defining them as

\[
\begin{align*}
\hat{\alpha} := & \frac{\sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right) Y_{N \hat{\tau}_N - j - 1}}{\sum_{j=1}^{Nh} K \left( \frac{j}{Nh} \right)} \\
\hat{\beta} := & \frac{\sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right) Y_{N \hat{\tau}_N + j}}{\sum_{j=1}^{Nh} K \left( \frac{j}{Nh} \right)}.
\end{align*}
\]

These two estimators are consistent:

**Lemma 2.** \(|\hat{\alpha} - f(\tau_0^-)| \text{ and } |\hat{\beta} - f(\tau_0^-)|\) are \(O_p(h \vee (Nh)^{-1/2})\).

It is possible to perform intelligent sampling to this nonparametric setting as in steps (ISS1)-(ISS4), though with a slight adjustment. Instead of fitting a stump function at step (ISS2), use one-sided local
polynomial fitting with bandwidth $h$ on the first stage subsample to obtain estimates $(\hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N)$ for the parameters $(f(\tau_0^-), f(\tau_0^+), \tau_N)$. These first stage estimators satisfy the following consistency result:

$$\mathbb{P} \left[ |\hat{\tau}_N^{(1)} - \tau_N| \leq w(N); \ |\hat{\alpha}^{(1)} - f(\tau_0^-)| \vee |\hat{\beta}^{(1)} - f(\tau_0^+)| \leq \rho_N \right] \rightarrow 1 \quad (51)$$

for the sequence $w(N) = CN^{1-\gamma-\delta}$ where $\delta$ and $C$ can be any positive constants, and some sequence $\rho_N \rightarrow 0$ (an explicit sequence can be derived by Lemma 2). The consistency condition in (51) is sufficient for a generalized versions of Theorems 1 and 2.

Theorem 7.

$$\hat{\tau}^{(2)} - \tau_N = O_p(N^{-1}) \quad (52)$$

Theorem 8. Suppose the conditions of Theorem 8 are satisfied, then for all integers $k \in \mathbb{Z}$ we have

$$\mathbb{P} \left[ \lambda_2(\tau_N, \hat{\tau}_N^{(2)}) = k \right] \rightarrow \mathbb{P} \left[ L \left( \frac{\Delta}{\sigma} \right) = k \right] \quad (53)$$

These results can hold under a general nonparametric setting, but they still hold for the stump model from Section 2. Since consistency condition (51) is if $f$ is a stump function and least square fitting was used at step (ISS2) as it was written in Section 2.1, Theorems 7 and 8 do imply Theorems 1 and 2. The proof of Theorem 7 will be covered in Section 9.2.1, while the proof of Theorem 8 will be covered in Appendix B at Section 9.2.2.

Remark 12. We note that not only do the consistency results of intelligent sampling for stump models generalize to this nonparametric setting, the computational time aspects translates also does not change. Local polynomial fitting on $n$ data points takes $O(n)$ computational time, see e.g. [23]. Therefore the analysis at the beginning of Section 2.2 still holds for this nonparametric case.

To further validate the extension to this nonparametric setting, we also ran a set of simulations for when $Y_i = 2 \sin(4\pi X_i) + 2 \cdot 1(X_i > 0.5) + \epsilon_i$, where $\epsilon_i$ are iid $N(0, 1)$. We took 15 values of $N$ between 2500 and $10^6$, chosen evenly on the log scale, and applied intelligent sampling on 1000 replicates. For each of these values of $N$. First stage used roughly $N_1 = \sqrt{N}$ points, which were subjected to one sided local polynomial fitting with a parabolic kernel and bandwidth $h = N_1^{-0.3}$, while the second stage interval had half-width $8/\sqrt{N}$. Figures 22 and 9.1 show results consistent with Theorems 7 and 8.

![Fig. 22.](image)  

**Fig. 22.** Left graph shows log-log plot of the quantiles of $|\hat{\tau}_N^{(2)} - \tau_N|$ versus $N$, with the solid black line having a slope of exactly -1. Some datapoints for the quantiles of the 50th quantiles do not appear since for some $N$, the median of $|\hat{\tau}_N^{(2)} - \tau_N|$ was 0. Right graph is a log-log plot of the mean computational time of using all datapoints (black) and intelligent sampling (red), with the solid black line having a slope of exactly 1 and the solid red a slope of exactly 0.5.
9.2. Proof of Corollary

Proof. We will show that for any $\epsilon > 0$, 

$$
P\left[|\hat{\beta} - f(\tau +)| > C_0(h \vee (Nh)^{-1/2})\right] \leq \epsilon. \tag{54}
$$

We start off by utilizing the rate of convergence of the change point estimator: there is a constant $C_1 > 0$ such that 

$$
P\left[|\hat{\tau}_N - \tau| > \frac{C_1}{N}\right] < \frac{\epsilon}{2}
$$

for all sufficiently large $N$. Hence, for any $C > 0$ we have, 

$$
P\left[|\hat{\beta}_N - f(\tau +)| > C\right] \leq 
P\left[|\hat{\beta}_N - f(\tau +)| > C \text{ and } |\hat{\tau}_N - \tau| \leq \frac{C_1}{N}\right] + P\left[|\hat{\tau}_N - \tau| > \frac{C_1}{N}\right] \leq 
P\left[|\hat{\beta}_N(t) - f(\tau +)| > C \text{ for some } |t - \tau| \leq \frac{C_1}{N}\right] + \frac{\epsilon}{2} \tag{55}
$$

where 

$$
\hat{\beta}_N(t) := \frac{\sum_{j=0}^{Nh} K\left(\frac{j}{Nh}\right) Y_{Nt+j}}{\sum_{j=1}^{Nh} K\left(\frac{j}{Nh}\right)} \tag{56}
$$

Next, we bound the first term above, so consider only the case where $|t - \tau| \leq \frac{C_1}{N}$. By expanding we have 

$$
|\hat{\beta}(t) - f(\tau +)| = \left| \frac{\sum_{j=0}^{Nh} K\left(\frac{j}{Nh}\right) [f(t + j/N) + \varepsilon_{Nt+j}] - f(\tau +)}{\sum_{j=1}^{Nh} K\left(\frac{j}{Nh}\right)} \right|
\leq \frac{\sum_{j=0}^{Nh} K\left(\frac{j}{Nh}\right) |f(t + j/N) - f(\tau +)|}{\sum_{j=1}^{Nh} K\left(\frac{j}{Nh}\right)} + \left| \frac{\sum_{j=0}^{Nh} K\left(\frac{j}{Nh}\right) \varepsilon_{Nt+j}}{\sum_{j=1}^{Nh} K\left(\frac{j}{Nh}\right)} \right|
:= A(t) + |B(t)| \tag{57}
$$
First, we derive a bound for \( A(t) \). If \( t \geq \tau \) then we have

\[
A(t) = \frac{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) |f(t + j/N) - f(\tau +)|}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
\leq \frac{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \beta_j |t + j/N - \tau|}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
\leq \frac{\beta_j \sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \left( \frac{C_1 + j}{N} \right)}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
= \frac{\beta_j N_h \sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \left( \frac{N-Nh}{Nh} \right)}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} + \frac{C_1}{N} \tag{58}
\]

Note that since \( Nh \to \infty \) as \( N \to \infty \), we have \( \frac{1}{Nh} \sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \to \int_0^1 xK(x) \, dx \) (which exists) and \( \sum_{j=1}^{N_h} K \left( \frac{j}{Nh} \right) \to \int_0^1 K(x) \, dx = 1 \), as \( N \to \infty \), hence we can find a constant \( M > 0 \) such that

\[
A(t) \leq \beta_j M h + \frac{C_1}{N} \tag{59}
\]

for all sufficiently large \( N \). On the other hand, suppose \( t < \tau \). For sufficiently large \( N \) we would have \( N(\tau - t) \leq C_1 < Nh \) and so

\[
A(t) = \frac{\sum_{j=0}^{N(\tau-t)-1} K \left( \frac{j}{Nh} \right) |f(t + j/N) - f(\tau +)|}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} + \frac{\sum_{j=N(\tau-t)}^{N_h} K \left( \frac{j}{Nh} \right) |f(t + j/N) - f(\tau +)|}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
\leq \frac{\sum_{j=0}^{N(\tau-t)-1} K \left( \frac{j}{Nh} \right) (\Delta + \beta_j (\tau - t - j/N))}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} + \frac{\sum_{j=N(\tau-t)}^{N_h} K \left( \frac{j}{Nh} \right) \beta_j (t + j/N - \tau)}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
\leq \frac{K^\dagger N(\tau - t) \Delta}{\sum_{j=1}^{N_h} K \left( \frac{j}{Nh} \right)} + \frac{\beta_j \sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \left( \frac{C_1 + j}{N} \right)}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)}
\]

\[
\leq (K^\dagger \Delta) \frac{1}{Nh} \frac{C_1}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} + \beta_j \frac{1}{Nh} \frac{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \left( \frac{N-Nh}{Nh} \right)}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} + \frac{C_1}{N} \tag{60}
\]

which, for sufficiently large \( N \), can be bounded by \( \frac{M}{Nh} + \beta_j M h + \frac{C_1}{N} \) for some constants \( M, M_1 > 0 \). Hence, this shows that \( A(t) \) itself is \( O(h \lor (Nh)^{-1}) \) for all \( t \) where \( |t - \tau| \leq \frac{C_1}{N} \).

Next, we consider the random term

\[
B(t) = \frac{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right) \varepsilon_{Nt+j}}{\sum_{j=0}^{N_h} K \left( \frac{j}{Nh} \right)} \tag{61}
\]
which satisfies

\[
\mathbb{E}[B(t)] = 0
\]

\[
\text{var}(B(t)) = \frac{\sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right)^2}{\left( \sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right)^2 \right)^2} = (Nh)^{-1} \frac{\frac{1}{Nh} \sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right)^2}{\left( \frac{1}{Nh} \sum_{j=0}^{Nh} K \left( \frac{j}{Nh} \right)^2 \right)^2}
\]

\[
\leq (Nh)^{-1} 2 \int_0^1 K(x)^2 \, dx, \quad \text{for all sufficiently large } N. \quad (62)
\]

Thus \( B(t) = O_p((Nh)^{-1/2}) \) by Chebychev’s inequality.

Combining these results on \( A(t) \) and \( B(t) \) derived above, one can find constants \( C_2, C_3 > 0 \) such that for all \( N > N_2 \) for some integer \( N_2 \) we have

\[
A(t) \leq C_2 [h \lor (Nh)^{-1}]
\]

\[
\mathbb{P}[|B(t)| > C_3(h \lor (Nh)^{-1/2})] \leq \frac{\epsilon}{2(2C_1 + 3)} \quad (63)
\]

for all \( |t - \tau| \leq \frac{C_1}{N} \), to get from (55):

\[
\mathbb{P} \left[ |\hat{\beta}_N - f(\tau^+)| > (C_1 + C_2)(h \lor (Nh)^{-1/2}) \right] \leq \\
\mathbb{P} \left[ |\hat{\beta}_N(t) - f(\tau^+)| > (C_1 + C_2)(h \lor (Nh)^{-1/2}) \text{ for some } |t - \tau| \leq \frac{C_1}{N} \right] + \frac{\epsilon}{2} \leq \\
\mathbb{P} \left[ A(t) + |B(t)| > (C_1 + C_2)(h \lor (Nh)^{-1/2}) \text{ for some } |t - \tau| \leq \frac{C_1}{N} \right] + \frac{\epsilon}{2} \leq \\
\sum_{t:|t-\tau| \leq C_1/N} \mathbb{P} \left[ A(t) + |B(t)| > (C_1 + C_2)(h \lor (Nh)^{-1/2}) \right] + \frac{\epsilon}{2} \leq \\
\sum_{t:|t-\tau| \leq C_1/N} \left[ |B(t)| > C_2(h \lor (Nh)^{-1/2}) \right] + \frac{\epsilon}{2} \leq \\
\sum_{t:|t-\tau| \leq C_1/N} \frac{\epsilon}{2(2C_1 + 3)} + \frac{\epsilon}{2} \leq \epsilon \quad (64)
\]

for all \( N \geq N_1 \lor N_2 \). This establishes that \( |\hat{\beta}_N - f(\tau^+)| \) is \( O_p(h \lor (Nh)^{-1/2}) \), and the proof for \( |\hat{\alpha}_N - f(\tau^-)| \) proceeds similarly. \( \square \)

9.2.1. Proof of Theorem

The structure of this proof will be similar to the rate of convergence proof found in Lan et al (2007). We will initially set some notations: let \( \tau_N := \lfloor N \tau \rfloor / N \), and define

\[
\tau_N^{(2)} := \begin{cases} 
\tau_N & \text{if } \tau_N \text{ is not in first subsample} \\
\tau_N - 1/N & \text{if } \tau_N \text{ is in first subsample}
\end{cases} \quad (65)
\]

We will show that \( \left( \hat{\tau}_N^{(2)} - \tau_N^{(2)} \right) = O_p(1/N) \), which will also demonstrate the same rate of convergence for \( \left( \hat{\tau}_N - \tau_N \right) \). An additional property of \( \tau_N^{(2)} \), used later on, is the fact that \( \lambda_2 \left( \tau_N^{(2)}, \hat{\tau}_N^{(2)} \right) = \lambda_2 \left( \tau_N, \hat{\tau}_N \right) \). This will be utilized in the proof of Theorems 2 and 3.
Proof. Denote $G_N$ as the joint distribution of $(\hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N^{(1)})$. Given any constant $\epsilon > 0$, there is a positive constant $C_\epsilon$ such that for all sufficiently large $N$ we have
\[
(\hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N^{(1)}) \in \left[ f(\tau-) - \rho N, f(\tau-) + \rho N \right] \\
\times \left[ f(\tau+) - \rho N, f(\tau+) + \rho N \right] \times \left[ \tau - C_\epsilon/N^\gamma, \tau + C_\epsilon/N^\gamma \right]
\]
with probability at least $1 - \epsilon$. Denote this event as $R_N$. It follows that for any sequence $\{a_N\}$,
\[
P\left[ N | \hat{\tau}_N^{(2)} - \tau_N^{(2)} > a_N \right] \leq \\
\int_{R_N} \mathbb{P}\left[ N | \hat{\tau}_N^{(2)} - \tau_N^{(2)} > a_N \left( \hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N^{(1)} \right) = (\alpha, \beta, t) \right] dG_N(\alpha, \beta, t) + \epsilon \leq \\
\sup_{(\alpha, \beta, t) \in R_N} \mathbb{P}\left[ N | \hat{\tau}_N^{(2)} - \tau_N^{(2)} > a_N \left( \hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N^{(1)} \right) = (\alpha, \beta, t) \right] + \epsilon
\]
Next, we show that this first term is smaller than any $\epsilon > 0$ for a sequence $a_N = O(1/N)$ and all sufficiently large $N$, by bounding the probability that $\mathbb{P}_{\alpha, \beta, t} \left[ N | \hat{\tau}_N^{(2)} - \tau > a_N \right] := \mathbb{P} \left[ N | \hat{\tau}_N^{(2)} - \tau > a_N \left( \hat{\alpha}^{(1)}, \hat{\beta}^{(1)}, \hat{\tau}_N^{(1)} \right) = (\alpha, \beta, t) \right]$ for any given $(\alpha, \beta, t) \in R_N$.

Conditional on the first stage estimates equaling $(\alpha, \beta, t)$, we can rewrite $\hat{\tau}_N^{(2)}$ and $\tau$ as maximizers of:
\[
\hat{\tau}_N^{(2)} = \arg\min_{d \in S^{(2)}} \left( \frac{1}{\lambda_2(S^{(2)}(t))} \sum_{i : i/N \in S^{(2)}(t)} \left( Y_i - \frac{\alpha + \beta}{2} \right) \left( 1(i/N \leq d) - 1(i/N \leq \tau) \right) \right) \\
= \arg\min_{d \in S^{(2)}} M_n(d)
\]
\[
\tau_N^{(2)} = \arg\min_{d \in S^{(2)}} \left( \frac{1}{\lambda_2(S^{(2)}(t))} \sum_{i : i/N \in S^{(2)}(t)} \left( Y_i - \frac{\alpha + \beta}{2} \right) \left( 1(i/N \leq d) - 1(i/N \leq \tau) \right) \right) \\
= \arg\min_{d \in S^{(2)}} M_n(d)
\]
Since $\tau_N^{(2)} - C_\epsilon/N^\gamma \leq t \leq \tau_N^{(2)} + C_\epsilon/N^\gamma + 2/N$, for $N$ large enough so that $K N^{-\gamma} / 2 > (C_\epsilon + 2)$, we have $t - K N^{-\gamma + \delta} < \tau_N^{(2)} - K N^{-\gamma + \delta}/2 < \tau_N^{(2)} + K N^{-\gamma + \delta}/2 < t + K N^{-\gamma + \delta}$. This enables us to define the function $A(r)$ in the domain where $8N^{-1+\gamma-\delta} < r < K/2$, such that
\[
a(r) := \min \{ M_n(d) : |d - \tau_N^{(2)}| \geq r N^{-\gamma + \delta} \} \\
= \min_{|d - \tau_N^{(2)}| \geq r N^{-\gamma + \delta}} \sum_{i : i/N \in S^{(2)}} \left( f(i/N) - \frac{\alpha + \beta}{2} \right) \left( 1(i/N \leq d) - 1(i/N \leq \tau) \right) / \lambda_2(t - K N^{-\gamma + \delta}, t + K N^{-\gamma + \delta})
\]
To make $a(r)$ simpler to work with, we show that for sufficiently large $N$, there exists a constant $A > 0$ such that $a(r) \geq Ar$. First, because
\[
S^{(2)}(t) := \left\{ i/N : i \in \mathbb{N}, \ i/N \in [t \pm K N^{-\gamma + \delta}], \ [N/N_1] \text{ does not evenly divide } i \right\} \\
\subset [t - K N^{-\gamma + \delta}, t + K N^{-\gamma + \delta}] \\
\subset [\tau^{(2)} - 2 K N^{-\gamma + \delta}, \tau^{(2)} + 2 K N^{-\gamma + \delta}]
\]
this implies
\[
|f(i/N) - f(\tau+)| \leq 2 \beta f K N^{-\gamma + \delta} \quad \text{for all } i/N \in S^{(2)}, i/n > \tau \\
|f(i/n) - f(\tau-)| \leq 2 \beta f K N^{-\gamma + \delta} \quad \text{for all } i/n \in S^{(2)}, i/n \leq \tau
\]
Combine this with the fact that $|\alpha - f(\tau-)|$ and $|\beta - f(\tau+)|$ are $O(1)$, which implies for sufficiently large $N$, and for any $i/n \in S^{(2)}(t)$,
\[
f(i/n) - \frac{\alpha + \beta}{2} > \frac{\Delta}{4} \quad \text{if } i/n > \tau \\
f(i/n) - \frac{\alpha + \beta}{2} < -\frac{\Delta}{4} \quad \text{if } i/n \leq \tau
\]
37
The preceding fact implies that every term in the summand of (69) is positive, and therefore the minimizing $d$ for (69) would be either $\tau(2) \pm rN^{-\gamma+\delta}$:

$$a(r) = \left( \sum_{i/j/n \in S^{(2)}(t)} \left( f(i/n) - \frac{\alpha + \beta}{2} \right) \left( 1(i/n \leq \tau^{(2)} + rN^{-\gamma+\delta}) - 1(i/n \leq \tau) \right) \right) \wedge \left( \sum_{i/j/n \in S^{(2)}(t)} \left( f(i/n) - \frac{\alpha + \beta}{2} \right) \left( 1(i/n \leq \tau^{(2)} - rN^{-\gamma+\delta}) - 1(i/n \leq \tau) \right) \right) \geq \frac{\Delta}{4} \cdot \frac{\lambda_2(\tau^{(2)}, \tau^{(2)} + rN^{-\gamma+\delta}) \wedge \lambda_2(\tau^{(2)} - rN^{-\gamma+\delta}, \tau^{(2)})}{\lambda_2[t - KN^{-\gamma+\delta}, t + KN^{-\gamma+\delta}]}$$  \hspace{1cm} (71)

It can also be shown that for $N$ large enough (specifically $|N^{1-\gamma}| \geq 2$) and any $d_1, d_2 \in [t - KN^{-\gamma+\delta}, t + KN^{-\gamma+\delta}]$ such that $d_2 - d_1 \geq 8/N$, we have

$$\lambda_2(d_1, d_2) \geq [N(d_2 - d_1) - 2] - \left[ \frac{N(d_2 - d_1) + 1}{|N^{1-\gamma}|} \right] \geq \frac{N(d_2 - d_1)}{8}$$

In a slightly similar fashion, it can be argued that for all large $N$, $\lambda_2[t - KN^{-\gamma+\delta}, t + KN^{-\gamma+\delta}] \leq 3KN^{1-\gamma+\delta}$. Since we restricted $r$ to be greater than $8N^{-1+\gamma-\delta}$, this means

$$a(r) \geq \frac{\Delta}{96K^\gamma}$$ \hspace{1cm} (72)

Hence, this shows that $a(r)$ is greater than some linear function with 0 intercept.

Now define $b(r) = (a(r) - M_n(\tau_N^{(2)}))/3 = a(r)/3$, then we have the following relation:

$$\sup_{d \in S^{(2)}} |M_n(d) - M_n(d)| \leq b(r) \Rightarrow |\tau_N^{(2)} - \tau_N^{(2)}| \leq rN^{-\gamma+\delta}$$ \hspace{1cm} (73)

To show the above is true, suppose $d \in [t - KN^{-\gamma+\delta}, t + KN^{-\gamma+\delta}]$ and $|d - \tau_N^{(2)}| > rN^{-\gamma+\delta}$. If, in addition, the left expression above holds, then

$$M_n(d) \geq M_n(d) - b(r) \geq a(r) - b(r) \Rightarrow M_n(d) - M_n(\tau_N^{(2)}) \geq a(r) - b(r) - M_n(\tau_N^{(2)}) - b(r) = b(r) > 0$$ \hspace{1cm} (74)

Since $M_n(d) > M_n(\tau_N^{(2)})$ and $\tau_N^{(2)}$ minimizes $M_n$ among all points in $S^{(2)}(t)$, this implies $d$ could not equal $\tau_N^{(2)}$, showing that $|\tau_N^{(2)} - \tau_N^{(2)}| \leq rN^{-\gamma}$.

Next, we bound $P_{\alpha, \beta, t} \left[ |\tau_N^{(2)} - \tau_N^{(2)}| \leq rN^{-\gamma+\delta} \right]$. First, we split it into the two parts:

$$P_{\alpha, \beta, t} \left[ |\tau_N^{(2)} - \tau_N^{(2)}| > rN^{-\gamma+\delta} \right] \leq P_{\alpha, \beta, t} \left[ rN^{-\gamma+\delta} < |\tau_N^{(2)} - \tau_N^{(2)}| \leq \eta N^{-\gamma+\delta} \right] + P_{\alpha, \beta, t} \left[ |\tau_N^{(2)} - \tau_N^{(2)}| > \eta N^{-\gamma+\delta} \right] := P_N(\alpha, \beta, t) + Q_N(\alpha, \beta, t)$$ \hspace{1cm} (75)

where $\eta = K/3$. We first consider the term $P_n(\alpha, \beta, t)$. Because

$$rN^{-\gamma+\delta} < |\tau_N^{(2)} - \tau_N^{(2)}| \leq \eta N^{-\gamma+\delta} \Rightarrow \inf_{\tau_N^{(2)} + rN^{-\gamma+\delta} < d \leq \tau_N^{(2)} + \eta N^{-\gamma+\delta}} M_n(d) \leq M_n(\tau)$$ or

$$\inf_{\tau_N^{(2)} - \eta N^{-\gamma+\delta} < d \leq \tau_N^{(2)} - rN^{-\gamma+\delta}} M_n(d) \leq M_n(\tau),$$ \hspace{1cm} (76)
we can first split \( P_n(\alpha, \beta, t) \) into the two terms

\[
P_N(\alpha, \beta, t) \leq P_{N, 1}(\alpha, \beta, t) + P_{N, 2}(\alpha, \beta, t)
\]

\[
=: \mathbb{P}_{\alpha, \beta, t} \left[ \sup_{\tau_N^{(2)} + r N^{-\gamma + \delta} < d \leq \tau_N^{(2)} + \eta N^{-\gamma + \delta}} (M_n(\tau) - M_n(d)) \geq 0 \right] +
\]

\[
\mathbb{P}_{\alpha, \beta, t} \left[ \sup_{\tau_N^{(2)} - \eta N^{-\gamma + \delta} < d \leq \tau_N^{(2)} - r N^{-\gamma + \delta}} (M_n(\tau) - M_n(d)) \geq 0 \right]
\]

(77)

We first form an upper bound for \( P_{N, 1}(\alpha, \beta, t) \) for all \((\alpha, \beta, t) \in R_n\). Note that

\[
M_n(\tau_N^{(2)}) - M_n(d) = -(M_n(d) - M_n(d)) - M_n(d)
\]

\[
= -\sum_{i : i/N \in S(t)} \left( Y_i - \frac{\alpha + \beta}{2} \right) - \left( f(i/N) - \frac{\alpha + \beta}{2} \right) (1(i/N \leq d) - 1(i/N \leq \tau))
\]

\[
\leq \frac{\lambda_2[t - K N^{-\gamma + \delta}, t + K N^{-\gamma + \delta}]}{\lambda_2[t - K N^{-\gamma + \delta}, t + K N^{-\gamma + \delta}]} - M_n(d)
\]

(78)

As previously explained, the \( f(i/N) - \frac{\alpha + \beta}{2} \) term in the second summand can be bounded below by \( \Delta/4 \) for all sufficiently large \( N \), and hence this leads to:

\[
M_n(\tau) - M_n(d) \geq 0 \Rightarrow -\sum_{i : i/N \in S(t)} \varepsilon_i \geq \frac{\Delta}{4} \lambda_2(\tau^{(2)}, d)
\]

(79)

It thus follows that

\[
P_{N, 1}(\alpha, \beta, t) \leq \mathbb{P}_{\alpha, \beta, t} \left[ \sup_{\tau_N^{(2)} + r N^{-\gamma + \delta} < d \leq \tau_N^{(2)} + \eta N^{-\gamma + \delta}} \left( \frac{1}{\lambda_2(\tau^{(2)}, d)} \right) \sum_{i : i/N \in S(t) \cap (\tau^{(2)}, d]} \varepsilon_i \right] \geq \frac{\Delta}{4}
\]

(80)

and by the Hajek-Renyi inequality, we get

\[
\leq 16 \frac{\lambda_2(\tau^{(2)}, \tau^{(2)} + r N^{-\gamma + \delta})}{\lambda_2(\tau^{(2)}, \tau^{(2)} + r N^{-\gamma + \delta})} + \sum_{j = \lambda_2(\tau^{(2)}, \tau^{(2)} + r N^{-\gamma + \delta})} \frac{1}{j^2}
\]

\[
\leq 32 \frac{1}{\Delta^2} \lambda_2(\tau^{(2)}, \tau^{(2)} + r N^{-\gamma + \delta})
\]

(81)

We argued earlier that \( \lambda_2(\tau^{(2)}, \tau^{(2)} + r N^{-\gamma + \delta}) \geq r N^{1-\gamma + \delta}/8 \) for \( N \) sufficiently large enough, thus

\[
P_{N, 1}(\alpha, \beta, t) \leq \frac{8B}{r N^{1-\gamma + \delta}}
\]

(82)

where \( B = 32/\Delta^2 \). From this expression we arrive at \( P_{N, 1}(\alpha, \beta, t) \leq \epsilon \) (for any \( \epsilon > 0 \)) eventually, by setting \( r = CN^{-1+\gamma - \delta} \) where \( C \) is any constant satisfying \( C > 8 \) and \( 8B/C \leq \epsilon \).

To bound \( Q_N(\alpha, \beta, t) \), from (72) we’ve argued that \( a(r) \) is eventually greater than a multiple of \( r \) when
\[ r > 8N^{-1+\gamma-\delta}. \] Since we’ve defined \( b(r) = a(r)/3, \) we can find some positive constant \( B' \) where \( b(r) \geq B'r \) when \( r > 8N^{-1+\gamma-\delta} \) (and for all large \( N \)). Since \( \eta = K/3 > 8N^{-1+\gamma-\delta} \) eventually, this leads to

\[
\mathbb{P}_{\alpha,\beta,t} \left[ |\hat{\tau}^{(2)} - d| > \eta N^{-\gamma+\delta} \right] 
\leq \mathbb{P}_{\alpha,\beta,t} \left[ \sup_{d \in S^{(2)}(t)} \left| M_n(d) - M_n(d) \right| > b(\eta) \right] 
\leq \mathbb{P}_{\alpha,\beta,t} \left[ \sup_{d \in S^{(2)}(t)} \left| M_n(d) - M_n(d) \right| > B' \eta \right] 
= \mathbb{P}_{\alpha,\beta,t} \left[ \sup_{d \in S^{(2)}(t)} \frac{\sum_{i : i/N \in S^{(2)}(t)} \epsilon_i (1(i/N \leq d) - 1(i/N \leq \tau))}{\lambda_2(S^{(2)}(t))} > B' \eta \right] 
\tag{83}
\]

Using Corollary 8.8 from [8], the latter expression is bounded by \( C_1 \exp(-C_2 \eta^2 \lambda_2(S^{(2)}(t))) \) for some positive constants \( C_1, C_2, \) which converges to 0.

9.2.2. Proof of Theorem 8

Proof. Let \( \{X_1^{(2)}, X_2^{(2)}, \ldots\} \) be the x-coordinates of the data, not used in the first stage, with corresponding response variable \( \{Y_1^{(2)}, Y_2^{(2)}, \ldots\} \) and error terms \( \{\epsilon_1^{(2)}, \epsilon_2^{(2)}, \ldots\} \). As a set, \( \{X_1^{(2)}, X_2^{(2)}, \ldots\} \) equals \( \{X_1, \ldots, X_N\} - \left\{ \left\lfloor \frac{i}{N} \right\rfloor + \left\lfloor \frac{2N}{N} \right\rfloor \right\} \). Note that we do not have \( X_j^{(2)} = j/N \) for every integer \( j \), and additionally we can write \( \tau^{(2)} = X_m^{(2)} \) for some integer \( m \). Since our estimate will also be one of the \( X_i^{(2)} \)'s, we can then denote \( \hat{m} \) be the integer such that \( \hat{\tau}^{(2)} = X_{\hat{m}}^{(2)} \). Note that we have the following relation between \( \hat{m} - m \) and the \( \lambda_2 \) function on intervals:

\[
\hat{m} - m = \begin{cases} 
\lambda_2(\tau^{(2)}, \hat{\tau}^{(2)}) & \text{when } \hat{\tau}^{(2)} > \tau \\
-\lambda_2(\hat{\tau}^{(2)}, \tau^{(2)}) & \text{when } \hat{\tau}^{(2)} \leq \tau 
\end{cases}
\tag{84}
\]

Hence we can write results on \( \lambda_2(\tau^{(2)}, \hat{\tau}^{(2)}) \) in terms of \( \hat{m} - m \).

After taking a subset \( S^{(2)} \) of \( \{X_1^{(2)}, X_2^{(2)}, \ldots, X_N^{(2)}\} \) (specifically \( S^{(2)} \) are those within \( KN^{-\gamma+\delta} \) of the pilot estimate \( \hat{\tau}^{(1)} \)), we minimize

\[
\hat{\Delta}^{(2)}(t) := \sum_{i : X_i \in S^{(2)}} \left(Y_i - \hat{\alpha}^{(1)} N + \hat{\beta}^{(1)} \right) \left(1(X_i \leq t) - 1(X_i \leq \tau) \right) 
= \sum_{i : X_i \in S^{(2)}} \left(Y_i^{(2)} - \hat{\alpha}^{(1)} N + \hat{\beta}^{(1)} \right) \left(1(X_i^{(2)} \leq t) - 1(X_i^{(2)} \leq \tau^{(2)}) \right) 
\tag{85}
\]

over all points \( t \in S^{(2)} \) to obtain the estimate for the change point. Equivalently the domain of \( \hat{\Delta}^{(2)}(t) \) can be extended to all \( t \in \{X_1^{(2)}, X_2^{(2)}, \ldots\} \), letting

\[
\hat{\Delta}^{(2)}(t) = \max\left\{ \hat{\Delta}^{(2)}(r) : r \in S^{(2)} \right\} + 1 \quad \text{for } t \notin S^{(2)}
\]

The argmin of this extension is the argmin of the function restricted to \( S^{(2)} \). This extended definition will be used for the next result:

**Lemma 3.** For any fixed positive integer \( j_0 > 0 \),

\[
\hat{\Delta}^{(2)}(X_{m+j}^{(2)}) = \frac{j \Delta}{2} + \epsilon_{m+1}^{(2)} + \ldots + \epsilon_{m+j}^{(2)} + o_p(1) \quad \text{for } 1 \leq j \leq j_0 \\
\hat{\Delta}^{(2)}(X_m^{(2)}) = 0 + o_p(1) \\
\hat{\Delta}^{(2)}(X_{m-j}^{(2)}) = \frac{j \Delta}{2} - \epsilon_{m-j+1}^{(2)} - \ldots - \epsilon_m^{(2)} + o_p(1) \quad \text{for } 1 \leq j \leq j_0 
\tag{86}
\]
From this lemma it is straightforward to show the asymptotic distribution of \( \lambda_2 \left( \tau_N, \hat{\tau}_N^{(2)} \right) \) is the distribution of \( L(\Delta/\sigma) \), the argmax of the random process

\[
Z_j = \begin{cases} 
\frac{|j\Delta|}{2} - \varepsilon_{j-1} - \ldots - \varepsilon_j & , \text{for } j < 0 \\
0 & , \text{for } j = 0 \\
\frac{|j\Delta|}{2} + \varepsilon_j^* + \ldots + \varepsilon_j^* & , \text{for } j > 0
\end{cases}
\]

where the \( \{\varepsilon_j\}_{j \in \mathbb{Z}} \) are iid \( N(0, \sigma^2) \) random variables.

For any fixed \( \varepsilon > 0 \) and integer \( j \), we will show that \( |P[\hat{m} - m = j] - P[L(\Delta/\sigma) = j]| \leq \varepsilon \) for all sufficiently large \( N \). To do this we will first establish 3 probability bounds.

**First Bound:** First we will show that with high probability we can approximate the stochastic process \( L(\Delta/\sigma) \), which has support \( \mathbb{Z} \), with a stochastic process \( L_k(\Delta/\sigma) \), which has a finite support \( \mathbb{Z} \cap [-k,k] \).

We note that there exists an integer \( j_1 > |j| \), such that \( |L(\Delta/\sigma)| > j_1 \) with probability less than \( \varepsilon/3 \). For any integer \( k \) with \( k \geq j_1 \), define \( L_k(\Delta/\sigma) := \text{argmin}\{Z_i\} \). In the case that \( |L(\Delta/\sigma)| \leq k \), we have \( L_k(\Delta/\sigma) = L(\Delta/\sigma) \), and using this we can show that \( P[L(\Delta/\sigma) = j] \) is within \( \varepsilon/3 \) of \( P[L_k(\Delta/\sigma) = j] \):

\[
|P[L(\Delta/\sigma) = j] - P[L_k(\Delta/\sigma) = j]| \\
= \left| P\left[L(\Delta/\sigma) = j, |L(\Delta/\sigma)| \leq k \right] - P\left[L_k(\Delta/\sigma) = j, |L_k(\Delta/\sigma)| \leq k \right] - P\left[L_k(\Delta/\sigma) = j, |L(\Delta/\sigma)| > k \right]\right| \\
\leq \left| P\left[L(\Delta/\sigma) = j, |L(\Delta/\sigma)| \leq k \right] - P\left[L_k(\Delta/\sigma) = j, |L_k(\Delta/\sigma)| \leq k \right] \right| + P[|L(\Delta/\sigma)| > k] \\
= \left| P\left[L_k(\Delta/\sigma) = j, |L(\Delta/\sigma)| \leq k \right] - P\left[L_k(\Delta/\sigma) = j, |L(\Delta/\sigma)| \leq k \right] \right| + P[|L(\Delta/\sigma)| > k] \\
\leq 0 + \frac{\varepsilon}{3}
\]

**Second Bound:** We will show that there exists an integer \( j_0 > j_1 \) such that \( |\hat{m} - m| \leq j_0 \) with probability greater than \( 1 - \frac{\varepsilon}{4} \). From our theorem on the rate of convergence, we can find some integer \( j_0 > j_1 \) such that for all sufficiently large \( N \),

\[
P\left[|\hat{\tau}^{(2)} - \tau| \leq \frac{j_0 - 2}{N}\right] > 1 - \frac{\varepsilon}{3}.
\]

When \( |\hat{\tau}^{(2)} - \tau| \leq \frac{j_0 - 2}{N} \), we have \( |\hat{m} - m| \leq j_0 \); first we can show

\[
|X_m^{(2)} - X_m^{(2)}| \leq |\hat{\tau}^{(2)} - \tau| + |\tau - \tau^{(2)}| \\
\leq \frac{j_0 - 2}{N} + \frac{2}{N} \\
= \frac{j_0}{N}.
\]

and second, because the \( \{X^{(2)}_1, X^{(2)}_2, \ldots\} \) grid is just the equally spaced \( \{1/N, 2/N, \ldots, N/N\} \) with some points taken out, the result of (90) implies \( |\hat{m} - m| \leq j_0 \). Hence

\[
P[|\hat{m} - m| \leq j_0] \geq P\left[|\hat{\tau}^{(2)} - \tau| \leq \frac{j_0 - 2}{N}\right] \\
> 1 - \frac{\varepsilon}{3}
\]

**Third Inequality:** Define \( \hat{\tau}_{j_0}^{(2)} \) to be the minimizer of \( \hat{\Delta}^{(2)}(\cdot) \) on the set \( \{X^{(2)}_{m-j_0}, X^{(2)}_{m-j_0+1}, \ldots, X^{(2)}_{m+j_0}\} \), and let \( \hat{m}_{j_0} \) be its corresponding index such that \( \hat{\tau}_{j_0}^{(2)} = X^{(2)}_{\hat{m}_{j_0}} \). In the case when \( |\hat{m} - m| \leq j_0 \), then \( \hat{\tau}_{j_0}^{(2)} \) would
be equal to \( \hat{\tau}(2) \), and \( \hat{m} = \hat{m}_{j_0} \). Using this notation we can obtain the following bound:

\[
|P[\hat{m} - m = j] - P[\hat{m}_{j_0} - m = j]| \\
= |P[\hat{m} - m = j, |\hat{m} - m| \leq j_0] - P[\hat{m}_{j_0} - m = j, |\hat{m} - m| \leq j_0] - P[\hat{m}_{j_0} - m = j, |\hat{m} - m| > j_0]| \\
= |P[\hat{m}_{j_0} - m = j, |\hat{m} - m| \leq j_0] - P[\hat{m}_{j_0} - m = j, |\hat{m} - m| \leq j_0] - P[\hat{m}_{j_0} - m = j, |\hat{m} - m| > j_0]| \\
\leq P[|\hat{m} - m| > j_0] \\
\leq \frac{\epsilon}{3}
\]

(92)

Consider the stochastic process \( \hat{\Delta}^{(2)}(X^{(2)}_{m+i}) \) for \( i \in \{-j_0, \ldots, 0, \ldots, j_0\} \). The previous lemma showed that, as a random variable in \( \mathbb{R}^{2j_0+1} \), \( \left( \hat{\Delta}^{(2)}(X^{(2)}_{m-j_0}), \ldots, \hat{\Delta}^{(2)}(X^{(2)}_{m+j_0}) \right) \) converges in distribution to \( (Z_{-j_0}, \ldots, Z_{j_0}) \). Also consider the function \( \text{Ind}_{\min} : \mathbb{R}^{2j_0+1} \to \mathbb{Z} \), defined as

\[
\text{Ind}_{\min}(a_1, \ldots, a_{2j_0+1}) = \left( \arg \min_{i=1, \ldots, 2j_0+1} (a_i) \right) - (j_0 + 1).
\]

(93)

It can be easily checked that \( \text{Ind}_{\min} \) is a continuous function, and by definition, we also have

\[
L_{j_0}(\Delta/\sigma) = \text{Ind}_{\min}(Z_{-j_0}, \ldots, Z_{j_0}) \\
\hat{m}_{j_0} - m = \text{Ind}_{\min} \left( \hat{\Delta}^{(2)}(X^{(2)}_{m-j_0}), \ldots, \hat{\Delta}^{(2)}(X^{(2)}_{m+j_0}) \right)
\]

(94)

Hence, by the continuous mapping theorem we have \( \hat{m}_{j_0} - m \) converging to \( L_{j_0}(\Delta/\sigma) \) in distribution. For sufficiently large \( N \), the absolute difference between \( P[L_{j_0}(\Delta/\sigma) = j] \) and \( P[\hat{m}_{j_0} - m = j] \) will be less than \( \epsilon/3 \).

Combining what we have just shown, for sufficiently large \( N \) we have

\[
|P[\hat{m} - m = j] - P[L(\Delta/\sigma) = j]| \\
\leq |P[\hat{m} - m = j] - P[\hat{m}_{j_0} - m = j]| + |P[\hat{m}_{j_0} - m = j] - P[L_{j_0}(\Delta/\sigma) = j]| \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
\]

(95)

Proof of Lemma \[3\]

Proof. First note that with probability increasing to 1, \( X^{(2)}_{m-j_0}, X^{(2)}_{m-j_0+1}, \ldots, X^{(2)}_{m+j_0} \) are all contained inside \( S^{(2)} \), and this fact will be shown first. Since \( \hat{\tau}(1) - \tau = O_p(N^{-\gamma}) \), for any \( \epsilon > 0 \) it is possible to find a constant \( C > 0 \) such that

\[
P \left[ \hat{\tau}(1) - CN^{-\gamma} \leq \tau \leq \hat{\tau}(1) + CN^{-\gamma} \right] > 1 - \epsilon
\]

(96)

for all sufficiently large \( N \). Additionally, for all sufficiently large \( N \) we have \( \frac{4 + 2j_0}{N} \leq (KN^{\delta} - C)N^{-\gamma} \), and which means that if \( |\tau(1) - \tau| \leq CN^{-\gamma} \) then

\[
\hat{\tau}(1) - KN^{-\gamma+\delta} = \hat{\tau}(1) - (KN^{-\gamma} - C)N^{-\gamma} - CN^{-\gamma} \\
\leq \tau - \frac{4 + 2j_0}{N}
\]

\[
\hat{\tau}(1) + KN^{-\gamma+\delta} = \hat{\tau}(1) + (KN^{-\gamma} - C)N^{-\gamma} + CN^{-\gamma} \\
\geq \tau + \frac{4 + 2j_0}{N}
\]

(97)
Finally, for all sufficiently large \( N \), we have \([N^{1-\gamma}] > 2\), i.e. the first stage subsample chooses points which are spaced more than \( 2/N \) points apart. Hence,

\[
X_{m-j_0}^{(2)} \geq X_m^{(2)} - 2 \left( \frac{j_0 + 2}{N} \right) = \tau^{(2)} - 2 \left( \frac{j_0 + 2}{N} \right) \\
\geq \tau - 2j_0 + \frac{4}{N}
\]

\[
X_{m+j_0}^{(2)} \leq X_m^{(2)} + 2 \left( \frac{j_0 + 2}{N} \right) \\
\leq \tau + 2j_0 + \frac{4}{N},
\]

which leads to the conclusion that for all \( N \) large enough, we have

\[
1 - \epsilon < \mathbb{P} \left[ \hat{\tau}^{(1)} - CN^{-\gamma} \leq \tau \leq \hat{\tau}^{(1)} + CN^{-\gamma} \right] \\
\leq \mathbb{P} \left[ \hat{\tau}^{(1)} - KN^{-\gamma+\delta} \leq \tau - \frac{2j_0 + 4}{N} < \tau + \frac{2j_0 + 4}{N} \leq \hat{\tau}^{(1)} + KN^{-\gamma+\delta} \right] \\
\leq \mathbb{P} \left[ X_{m-j_0}^{(2)} \geq \hat{\tau}^{(1)} - KN^{-\gamma+\delta} \text{ and } X_{m+j_0}^{(2)} \leq \hat{\tau}^{(1)} + KN^{-\gamma+\delta} \right] \\
= \mathbb{P} \left[ X_{m-j_0}^{(2)} \text{ and } X_{m+j_0}^{(2)} \text{ are in } S^{(2)} \right] \tag{98}
\]

Therefore, consider the case for which \( X_{m-j_0}^{(2)} \) through \( X_{m+j_0}^{(2)} \) are contained in \( S^{(2)} \). Under this condition we have \( \hat{\Delta}^{(2)} \left( X_{m}^{(2)} \right) = 0 \) by simple calculation, and for any \( 0 < j \leq j_0 \),

\[
\left| \hat{\Delta}^{(2)}(X_{m+j}^{(2)}) - \left( \frac{j\Delta}{2} + \sum_{i=1}^{j} \epsilon_{m+i}^{(2)} \right) \right| \\
= \left| \sum_{i.X_{m+i}^{(2)} \in S^{(2)} \cap \{X_{m}^{(2)}, X_{m+j}^{(2)}\}} (Y_i^{(2)} - \hat{\alpha}_N + \hat{\beta}_N) - \left( \frac{j\Delta}{2} + \sum_{i=1}^{j} \epsilon_{m+i}^{(2)} \right) \right| \\
= \left| \sum_{i=1}^{j} \left( f(X_{m+i}^{(2)}) - \frac{\hat{\alpha}_N^{(1)} + \hat{\beta}_N^{(1)}}{2} \right) - \frac{j\Delta}{2} \right| \\
= \left| \sum_{i=1}^{j} \left[ \left( f(X_{m+i}^{(2)}) - f(\tau^+) \right) + \left( \frac{f(\tau^+) + f(\tau^-)}{2} - \hat{\alpha}_N^{(1)} + \hat{\beta}_N^{(1)} \right) + \left( \frac{f(\tau^+) - f(\tau^-)}{2} - \frac{\Delta_0}{2} \right) \right] \right| \\
\leq \left| \sum_{i=1}^{j} \left( f(X_{m+i}^{(2)}) - f(\tau^+) \right) \right| + \frac{j_0}{2} \left( \left| \hat{\alpha}_N^{(1)} - f(\tau^-) \right| + \left| \hat{\beta}_N^{(1)} - f(\tau^+) \right| \right) \tag{100}
\]

For the first term above, using an earlier argument we make the case that for sufficiently large \( N \) we have \([N/N_1] > 2\) and \( X_{m+i}^{(2)} \leq \tau + \frac{2i+4}{N} \), hence

\[
\left| \sum_{i=1}^{j} \left( f(X_{m+i}^{(2)}) - f(\tau^+) \right) \right| \leq \sum_{i=1}^{j} \beta_f \left| X_{m+i}^{(2)} - \tau \right| \\
\leq \beta_f \sum_{i=1}^{j} \frac{2i+4}{N} \\
\leq \frac{\beta_f}{N} (2j_0^2 + 4j_0) \tag{101}
\]
For the second term, it was shown earlier that both \( |\hat{\alpha}_N^{(1)} - f(\tau^-)| \) and \( |\hat{\beta}_N^{(1)} - f(\tau^+)| \) are \( O_p \left( h \sqrt{\frac{1}{N_L}} \right) \), and hence, so is their sum. Overall, this shows that for sufficiently large \( N \), \( \hat{\Delta}^{(2)}(X_{m+j}^{(2)}) - \left( \frac{j \Delta}{2} + \sum_{i=1}^{j} \epsilon_{m-i+1}^{(2)} \right) \) is (uniformly for all \( 0 \leq j \leq j_0 \)) bounded above by the random variable
\[
\frac{\beta f}{N} (2j_0^2 + 4j_0) + \frac{j_0}{2} \left( |\hat{\alpha}_N^{(1)} - f(\tau^-)| + |\hat{\beta}_N^{(1)} - f(\tau^+)| \right),
\]
which is \( o_p(1) \). Similarly, again for any \( 0 < j \leq j_0 \),
\[
\left| \hat{\Delta}^{(2)}(X_{m-j}^{(2)}) - \left( \frac{j \Delta}{2} - \sum_{i=1}^{j} \epsilon_{m-i+1}^{(2)} \right) \right| \leq \sum_{i=0}^{j-1} \left( f(X_{m-i}^{(2)}) - f(\tau) \right) + \frac{j_0}{2} \left( |\hat{\alpha}_N^{(1)} - f(\tau^-)| + |\hat{\beta}_N^{(1)} - f(\tau^+)| \right)
\]
which is again uniformly bounded by the expression in (102).

Therefore, given any \( \epsilon > 0 \), we have
\[
\mathbb{P} \left[ \left| \hat{\Delta}^{(2)}(X_{m+j}^{(2)}) - \left( \frac{j \Delta}{2} + \sum_{i=1}^{j} \epsilon_{m-i+1}^{(2)} \right) \right| \geq \epsilon \right] \\
\leq \mathbb{P} \left[ X_{m-j_0}^{(2)} \notin S^{(2)} \text{ and/or } X_{m+j_0}^{(2)} \notin S^{(2)} \right] + \mathbb{P} \left[ X_{m-j_0}^{(2)}, X_{m+j_0}^{(2)} \in S^{(2)}, \text{ and } \left| \hat{\Delta}^{(2)}(X_{m+j}^{(2)}) - \left( \frac{j \Delta}{2} + \sum_{i=1}^{j} \epsilon_{m-i+1}^{(2)} \right) \right| \geq \epsilon \right] \\
\leq \mathbb{P} \left[ X_{m-j_0}^{(2)} \notin S^{(2)} \text{ and/or } X_{m+j_0}^{(2)} \notin S^{(2)} \right] + \mathbb{P} \left[ \frac{\beta f}{N} (2j_0^2 + 4j_0) + \frac{j_0}{2} \left( |\hat{\alpha}_N^{(1)} - f(\tau^-)| + |\hat{\beta}_N^{(1)} - f(\tau^+)| \right) \geq \epsilon \right] \\
\rightarrow 0 + 0 \quad \text{for all } 0 < j \leq j_0
\]
and similarly,
\[
\mathbb{P} \left[ \left| \hat{\Delta}^{(2)}(X_{m}^{(2)}) \right| \geq \epsilon \right] \rightarrow 0 \\
\mathbb{P} \left[ \left| \hat{\Delta}^{(2)}(X_{m-j}^{(2)}) - \left( \frac{j \Delta}{2} + \sum_{i=1}^{j} \epsilon_{m-i+1}^{(2)} \right) \right| \geq \epsilon \right] \rightarrow 0
\]

9.3. Proof of Proposition

Proof. Because \( L(\Delta) \) has a symmetric distribution for any \( |\Delta| > 0 \), without loss of generality assume \( 0 < \Delta_1 < \Delta_2 \). Let \( \epsilon_i \) be iid \( N(0,1) \) random variables and denote the random processes
\[
Z_i^{(1)} = \begin{cases} 
\epsilon_1 + \ldots + \epsilon_i + i \Delta_1/2, & i > 0 \\
0, & i = 0 \\
-(\epsilon_{i+1} + \ldots + \epsilon_0) + |i| \Delta_1/2, & i < 0,
\end{cases} \\
Z_i^{(2)} = \begin{cases} 
\epsilon_1 + \ldots + \epsilon_i + i \Delta_2/2, & i > 0 \\
0, & i = 0 \\
-(\epsilon_{i+1} + \ldots + \epsilon_0) + |i| \Delta_2/2, & i < 0,
\end{cases}
\]
(106)

Consider the event \( \left| \arg \min_{i \in \mathbb{Z}} Z_i^{(1)} \right| \leq k \), i.e., there exists an integer \( m \) such that \( Z_m^{(1)} \geq Z_i^{(1)} \) for all \( i \in \mathbb{Z} \) and \( |m| \leq k \). Now note that for any \( j \in \mathbb{Z} \) we have
\[
Z_j^{(2)} = Z_j^{(1)} + \frac{|j| (\Delta_2 - \Delta)}{2},
\]
(107)
which means for all integers $i$ with $|i| > k$ we have

\[
Z_i^{(2)} = Z_i^{(1)} + \frac{|i|(\Delta_2 - \Delta_1)}{2} \\
\geq Z_m^{(1)} + \frac{|m|(\Delta_2 - \Delta_1)}{2} \\
= Z_m^{(2)}.
\] (108)

This shows that the event \( \left\{ \arg \min_{i \in \mathbb{Z}} Z_i^{(1)} \leq k \right\} \) implies the event \( \left\{ \arg \min_{i \in \mathbb{Z}} Z_i^{(2)} \leq k \right\} \), and hence

\[
P[|L(\Delta_2)| \leq k] = P \left[ \left\{ \arg \min_{i \in \mathbb{Z}} Z_i^{(2)} \leq k \right\} \right] \leq P \left[ \left\{ \arg \min_{i \in \mathbb{Z}} Z_i^{(1)} \leq k \right\} \right] = P[|L(\Delta_1)| \leq k] \] (109)

10. Appendix B (Proofs for Multiple Change Point Setting)

10.1. Proofs For Lemma 1

First we focus on part (i):

\[\text{Proof.}\] Consider the separate cases of $i = 0$ or $i = J$ (case 1), and $1 \leq i < J$ (case 2). We know

\[
P \left[ \hat{j} = J; \ \max_{i=1,\ldots,J} |\hat{\tau}_i - \tau_i| \leq w(N) \right] \geq 1 - B_N
\] (110)

for some sequences $w(N)$ and $B_N$ which are $o(\delta_N)$, and $o(1)$, respectively. Also as in the statement of the theorem, assume there is some sequence $\rho_N$ such that $w(N)/\delta_N = o(\rho_N)$. For the rest of the proof assume $N$ is large enough so that

- $1 < w(N) < \frac{\delta_N}{\delta}$
- $6\frac{w(N)}{\delta_N} \hat{\theta} < \frac{\rho_N}{\delta}$
- $\delta_N > 3$

\textbf{Case 1}: For $i = 0$, \( \hat{\nu}_0 \) is the average of all $Y_t$'s where $t$ lies between 1 and $\hat{\tau}_1$, inclusive. We have

\[
P \left[ \hat{j} = J; \ |\hat{\nu}_0 - \nu_0| \geq \rho_N \right] \\
\leq P[\hat{j} = J; \ |\hat{\tau}_1 - \tau_1| > w(N)] + P[\hat{j} = J; \ |\hat{\nu}_0 - \nu_0| \geq \rho_N; \ |\hat{\tau}_1 - \tau_1| \leq w(N)] \\
\leq B_N + \sum_{\tau: |\tau - \tau_1| \leq w(N)} \mathbb{P} \left[ \hat{j} = J; \ |\hat{\nu}_0 - \nu_0| \geq \rho_N; \ \hat{\tau}_1 = \tau \right] \\
\leq B_N + \sum_{\tau: |\tau - \tau_1| \leq w(N)} \mathbb{P} \left[ \hat{j} = J; \ \hat{\tau}_1 = \tau; \ \frac{1}{\tau} \sum_{j=1}^{\tau} Y_j - \nu_0 \geq \rho_N \right] \\
\leq B_N + \sum_{\tau: |\tau - \tau_1| \leq w(N)} \mathbb{P} \left[ \left| \frac{1}{\tau} \sum_{j=1}^{\tau} (Y_j - \nu_0) \right| \geq \rho_N \right] \] (111)
For all $\tau_1 - w(N) \leq \tau \leq \tau_1$, we have $\frac{1}{\tau} \sum_{j=1}^{\tau} (Y_j - \tau_0) \sim N(0, \sigma^2/\tau)$, and hence

$$
\Pr \left[ \left| \frac{1}{\tau} \sum_{j=1}^{\tau} (Y_j - \tau_0) \right| \geq \rho_N \right] = 2 \left( 1 - \Phi \left( \frac{\rho_N \sqrt{\tau}}{2} \right) \right)
$$

\begin{align*}
&\leq 2 \frac{\phi \left( \frac{\rho_N \sqrt{\tau}}{2} \right)}{\sqrt{\pi} \rho_N/\sigma} \\
&\leq \sqrt{\frac{2}{\pi}} \exp \left( -\frac{(\tau_1 - w(N))\rho_N^2/(2\sigma^2)}{\sqrt{\tau_1 - w(N)} \rho_N/\sigma} \right) \\
&\leq 2 \frac{\sigma}{\sqrt{\pi}} \exp \left( -\frac{\delta_N \rho_N^2/(4\sigma^2)}{\sqrt{\delta_N \rho_N}} \right) \\
&\quad (\text{by } \tau_1 w(N) > \delta_N/2)
\end{align*}

(112)

For all $\tau_1 < \tau \leq \tau_1 + w(N)$ we have $\frac{1}{\tau} \sum_{j=1}^{\tau} (Y_j - \nu_0) \sim N \left( \frac{\tau - \tau_1}{\tau} (\nu_1 - \nu_0), \frac{\sigma^2}{\tau} \right)$. Because

$$
\left| \frac{\tau - \tau_1}{\tau} (\nu_1 - \nu_0) \right| \leq \frac{w(N)}{\delta_N} (2\theta) \leq \frac{\rho_N}{2},
$$

the magnitude of the z-score of both $\pm \rho_N$ for the $N \left( \frac{\tau - \tau_1}{\tau} (\nu_1 - \nu_0), \frac{\sigma^2}{\tau} \right)$ distribution is at least $\frac{\rho_N \sqrt{\tau}}{2\sigma}$, and hence

\begin{align*}
\Pr \left[ \left| \frac{1}{\tau} \sum_{j=1}^{\tau} (Y_j - \tau_0) \right| \geq \rho_N \right] &\leq 2 \left( 1 - \Phi \left( \frac{\rho_N \sqrt{\tau}}{2} \right) \right) \\
&\leq 2 \frac{\phi \left( \frac{\rho_N \sqrt{\tau}}{2\sigma} \right)}{\frac{\rho_N}{\sigma}} \\
&\leq 2 \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\sigma}{\sqrt{\pi}} \cdot \frac{\exp \left( -\frac{\delta_N \rho_N^2/(4\sigma^2)}{\sqrt{\delta_N \rho_N}} \right)}{\sqrt{\delta_N \rho_N}}
\end{align*}

(113)

Therefore, the expression (112) can be bounded from above by

$$
B_N + (w(N) + 1) \frac{2\sigma}{\sqrt{\pi}} \cdot \frac{\exp \left( -\frac{\delta_N \rho_N^2/(4\sigma^2)}{\sqrt{\delta_N \rho_N}} \right)}{\sqrt{\delta_N \rho_N}} + w(N) \frac{2\sigma \sqrt{2}}{\sqrt{\pi}} \cdot \frac{\exp \left( -\frac{\delta_N \rho_N^2/(8\sigma^2)}{\sqrt{\delta_N \rho_N}} \right)}{\sqrt{\delta_N \rho_N}}
$$

(114)

For $i = J$, a very similar argument will bound $\Pr \left[ \hat{J} = J; \ |\hat{\nu}_N - \nu_N| \geq \rho_N \right]$ by the same expression in (115).

**Case 2:** The procedure for this case will be similar to the steps for Case 1, but there are a few modifications. For $0 < i < N$, $\hat{\nu}_i$ is the average of all $X_j$ for $\hat{\tau}_i < t \leq \hat{\tau}_{i+1}$. For the following part we re-write this average by considering the midpoint $\hat{\tau}_i^{(m)} := \frac{\hat{\tau}_i + \hat{\tau}_{i+1}}{2}$ where $1 < i < J$.

In the case where $\hat{\tau}_i$ and $\hat{\tau}_{i+1}$ are within $\delta_N/3$ (which is less than $|\tau_{i+1} - \tau_i|/3$) of $\tau_i$ and $\tau_{i+1}$ respectively,
we have \( \hat{\tau}_i < \tau_i^{(m)} < \hat{\tau}_{i+1} \), and hence we can bound \( |\hat{\nu}_i - \nu_i| \) by

\[
\left| \frac{1}{\hat{\tau}_{i+1} - \hat{\tau}_i} \sum_{j=\hat{\tau}_i+1}^{\hat{\tau}_{i+1}} (Y_j - \nu_i) \right|
= \left| \frac{\tau_i^{(m)} - \hat{\tau}_i}{\hat{\tau}_{i+1} - \hat{\tau}_i} \left( \frac{1}{\tau_i^{(m)} - \hat{\tau}_i} \sum_{j=\hat{\tau}_i+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right) + \frac{\hat{\tau}_{i+1} - \tau_i^{(m)}}{\hat{\tau}_{i+1} - \hat{\tau}_i} \left( \frac{1}{\hat{\tau}_{i+1} - \tau_i^{(m)}} \sum_{j=\tau_i^{(m)} + 1}^{\hat{\tau}_{i+1}} (X_j - \nu_i) \right) \right|
\leq \frac{\tau_i^{(m)} - \hat{\tau}_i}{\hat{\tau}_{i+1} - \hat{\tau}_i} \left| \frac{1}{\tau_i^{(m)} - \hat{\tau}_i} \sum_{j=\hat{\tau}_i+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right| + \frac{\hat{\tau}_{i+1} - \tau_i^{(m)}}{\hat{\tau}_{i+1} - \hat{\tau}_i} \left| \frac{1}{\hat{\tau}_{i+1} - \tau_i^{(m)}} \sum_{j=\tau_i^{(m)} + 1}^{\hat{\tau}_{i+1}} (X_j - \nu_i) \right|
\]

(116)

In order for \( |\hat{\nu}_i - \nu_i| \) to exceed \( \rho_N \), at least one of \( \frac{1}{\tau_i^{(m)} - \hat{\tau}_i} \sum_{j=\hat{\tau}_i+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \) or \( \frac{1}{\hat{\tau}_{i+1} - \tau_i^{(m)}} \sum_{j=\tau_i^{(m)} + 1}^{\hat{\tau}_{i+1}} (X_j - \nu_i) \) must exceed \( \rho_N \), or in other words,

\[
P \left[ \hat{j} = J; \ |\hat{\nu}_i - \nu_i| \geq \rho_N \right]
\leq P \left[ \hat{j} = J; \ |\hat{\tau}_i - \tau_i| > w(N) \right. \left. \text{ or } |\hat{\tau}_{i+1} - \tau_i| > w(N) \right] +
\]

\[
P \left[ \hat{j} = J; \ |\hat{\tau}_i - \tau_i| \leq w(N); \ |\hat{\tau}_{i+1} - \tau_i| \leq w(N); \ |\hat{\nu}_i - \nu_i| \geq \rho_N \right]
\leq B_N + P \left[ \hat{j} = J; |\hat{\tau}_i - \tau_i| \leq w(N); |\hat{\tau}_{i+1} - \tau_i| \leq w(N); \ \left| \frac{1}{\tau_i^{(m)} - \hat{\tau}_i} \sum_{j=\hat{\tau}_i+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right| \geq \rho_N \right]
\]

\[
+ P \left[ \hat{j} = J; |\hat{\tau}_i - \tau_i| \leq w(N); |\hat{\tau}_{i+1} - \tau_i| \leq w(N); \ \left| \frac{1}{\hat{\tau}_{i+1} - \tau_i^{(m)}} \sum_{j=\tau_i^{(m)} + 1}^{\hat{\tau}_{i+1}} (X_j - \nu_i) \right| \geq \rho_N \right]
\leq B_N + \sum_{\tau: |\tau - \tau_i| \leq w(N)} P \left[ \left| \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right| \geq \rho_N \right]
\]

\[
+ \sum_{\tau: |\tau - \tau_{i+1}| \leq w(N)} P \left[ \left| \frac{1}{\tau - \tau_i^{(m)}} \sum_{j=\tau_i^{(m)} + 1}^{\tau} (Y_j - \nu_i) \right| \geq \rho_N \right]
\]

(117)

Next, we will bound \( P \left[ \left| \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right| \geq \rho_N \right] \) for each \( \tau \) such that \( |\tau - \tau_i| \leq w(N) \). For \( \tau_i \leq \tau \leq \tau_i + w(N) \) we have \( \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \sim N \left(0, \frac{\sigma^2}{\tau_i^{(m)} - \tau} \right) \), and hence

\[
P \left[ \left| \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \right| \geq \rho_N \right] \leq 2 \left( 1 - \Phi \left( \rho_N \sigma^{-1} \sqrt{\tau_i^{(m)} - \tau} \right) \right)
\leq \frac{2\sigma\sqrt{3}}{\sqrt{\pi}} \exp(-\delta_N \rho_N^2/(12\sigma^2)) \rho_N \sqrt{\delta_N},
\]

(118)

where we used the fact that \( \tau_i^{(m)} - \tau > \tau_i^{(m)} - \tau_i - w(N) > \delta_N/3 - \delta_N/6 \).
For $\tau_i - w(N) \leq \tau < \tau_i$, we have
$$\frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (Y_j - \nu_i) \sim N \left( \frac{\tau_i - \tau}{\tau_i^{(m)} - \tau} (\nu_i - \nu_{i-1}), \frac{\sigma^2}{\tau_i^{(m)} - \tau} \right).$$

The z-scores of $\pm \rho_N$ would have magnitudes greater than
$$\sigma^{-1} \sqrt{\tau_i^{(m)} - \tau} \left( \rho_N - \left| \frac{\tau_i - \tau}{\tau_i^{(m)} - \tau} (\nu_i - \nu_{i-1}) \right| \right) \geq \sigma^{-1} \sqrt{\frac{\delta N}{3}} \left( \rho_N - \frac{w(N)}{\delta N/3} \right) \geq \sqrt{\frac{\delta N}{3}} \cdot \frac{\rho_N}{2\sigma}$$

(119)

Hence, this gives the probability bound
$$\Pr \left[ \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (X_j - \nu_i) \geq \rho_N \right] \leq 2 \left( 1 - \Phi \left( \sqrt{\frac{\delta N}{3}} \cdot \frac{\rho_N}{2\sigma} \right) \right) \leq \frac{2\sigma \sqrt{6}}{\sqrt{\pi}} \cdot \frac{\exp \left( -\delta^2 N/(24\sigma^2) \right)}{\rho_N \sqrt{\delta N}}$$

(120)

Putting together the bounds in (118) and (120) will give
$$\sum_{\tau:|\tau - \tau_i| \leq w(N)} \Pr \left[ \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau+1}^{\tau_i^{(m)}} (X_j - \nu_i) \geq \rho_N \right] \leq \sum_{\tau:|\tau - \tau_i| \leq w(N)} \Pr \left[ \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau_i^{(m)}+1}^{\tau_i^{(m)}} (X_j - \nu_i) \geq \rho_N \right] \leq 3w(N) \cdot \frac{2\sigma \sqrt{6}}{\sqrt{\pi}} \cdot \frac{\exp \left( -\delta^2 N/(24\sigma^2) \right)}{\rho_N \sqrt{\delta N}}$$

(121)

In an extremely similar manner, it can be argued that
$$\sum_{\tau:|\tau - \tau_i| \leq w(N)} \Pr \left[ \frac{1}{\tau_i^{(m)} - \tau} \sum_{j=\tau_i^{(m)}+1}^{\tau_i^{(m)}} (X_j - \nu_i) \geq \rho_N \right] \leq 3w(N) \cdot \frac{2\sigma \sqrt{6}}{\sqrt{\pi}} \cdot \frac{\exp \left( -\delta^2 N/(24\sigma^2) \right)}{\rho_N \sqrt{\delta N}}$$

(122)

Therefore, (117) can be bounded by
$$B_N + \frac{12\sigma \sqrt{6}}{\sqrt{\pi}} \cdot \frac{w(N) \exp \left( -\delta^2 N/(24\sigma^2) \right)}{\rho_N \sqrt{\delta N}}$$

(123)

By taking constants $C_1$ and $C_2$ to be the "worse" of the coefficients in (115) and (123), which are $\frac{12\sigma \sqrt{6}}{\sqrt{\pi}}$ and $1/(24\sigma^2)$ respectively, we can combine the result of both cases and establish
$$\Pr \left[ \hat{J} = J; \ |\hat{\nu}_i - \nu_i| \geq \rho_N \right] \leq B_N + C_1 w(N) \frac{\exp \left( -C_2 \delta^2 N \rho^2_N \right)}{\rho_N \sqrt{\delta N}}$$

(124)

for all $i = 1, \ldots, J$

Using part (i), previously shown, it is straightforward to show part (ii):

**Proof.** The complement of the event $\{J = J; \ \max_{i=0, \ldots, J} |\hat{\nu}_i - \nu_i| < \rho_N\}$ is the event where either $\hat{J} \neq J$ or $\hat{J} = J$ and $|\hat{\nu}_i - \nu_i| \geq \rho_N$ for some $i$. For all sufficiently large $N$ and some positive constants $C_1$ and $C_2$, we have

$$1 - \Pr \left[ \hat{J} = J; \ \max_{i=0, \ldots, J} |\hat{\nu}_i - \nu_i| < \rho_N \right] \leq \Pr[\hat{J} \neq J] + \sum_{i=0}^{J} \Pr \left[ \hat{J} = J; \ |\hat{\nu}_i - \nu_i| \geq \rho_N \right] \leq B_N + (J + 1) \left( B_N + C_1 w(N) \frac{\exp \left[ -C_2 \delta^2 N \rho^2_N \right]}{\sqrt{\delta N \rho_N}} \right) \leq B_N + \left( \frac{N}{\delta N} + 1 \right) \left( B_N + C_1 w(N) \frac{\exp \left[ -C_2 \delta^2 N \rho^2_N \right]}{\sqrt{\delta N \rho_N}} \right) \rightarrow 0$$

(125)
10.1.1. Proof of Theorem 4

Before we proceed with the proof, we define

$$
\tau_j^{(2)} := \begin{cases} \tau_j - 1/N & \text{if } \tau_j \text{ was a first stage subsample point} \\ \tau_j & \text{otherwise} \end{cases}
$$

for \( j = 1, \ldots, J \). We will also define the sets

$$
S^{(2)}(t) := \{ i \in \mathbb{N} : |i - t| \leq Kw(N), \ Y_i \text{ not used in 1st stage subsample} \}.
$$

Essentially \( S^{(2)}(t) \) is the second stage subsample around \( t \), if \( t \) were a change point estimate from the first stage of intelligent sampling.

**Proof.** Define the event

$$
R_N := \left\{ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N; \max_{i=0,\ldots,J} |\nu_i^{(1)} - \nu_i| \leq \rho_N \right\},
$$

Denote \( G_N \) as the joint distribution of \( J, \hat{\tau}_1^{(1)}, \ldots, \hat{\tau}_J^{(1)}, \nu_0^{(1)}, \ldots, \nu_J^{(1)} \); the domain of \( G_N \) would be \( \mathbb{N}^{k+1} \times \mathbb{R} \).

Then, for any sequence \( \{a_N\} \), we can bound \( \mathbb{P} \left[ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N \right] \) from below by:

$$
\begin{align*}
\mathbb{P} & \left[ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N \right] \\
& \geq \sum_{k=0}^{N-1} \int_{0<t_1<\ldots<t_k<\ldots<t_{k+1}<N} \int_{\alpha_1,\ldots,\alpha_k \in \mathbb{R}} dG_N(k, t_1, \ldots, t_k, \alpha_0, \ldots, \alpha_k) \\
& \quad \times \mathbb{P} \left[ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N \left| \hat{J} = J, \hat{\tau}_j^{(1)} = t_j, \nu_j^{(1)} = \alpha_j \text{ for } j \leq k \right. \right] \\
& \quad \times \mathbb{P} \left[ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(1)} - \tau_i^{(1)}| \leq Kw(N); \max_{i=0,\ldots,J} |\nu_i^{(1)} - \nu_i| \leq \rho_N \right] \\
& \quad \times \mathbb{P} \left[ R_N \text{ is false} \right]
\end{align*}
$$

We wish to show that for all \( \varepsilon > 0 \), there exists a sequence \( a_N = O(J(N)) \) such that

$$
\mathbb{P} \left[ \hat{J} = J; \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N \right] > 1 - \varepsilon
$$

for all large \( N \). It is sufficient to show this is satisfied by the second to last line of (129), as \( R_N \) is true with probability increasing to 1. Henceforth, we will work with the probability

$$
\mathbb{P} \left[ \max_{i=1,\ldots,J} |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq a_N \left| \hat{J} = J, \hat{\tau}_j^{(1)} = t_j, \nu_j^{(1)} = \alpha_j \text{ for } j \leq J \right. \right]
$$
and, in the domain $|t_i - \tau_i| \leq Kw(N)$ and $|\alpha_i - \nu_i| \leq \gamma N$ for all $i$, we show that it is greater than $1 - \varepsilon$ for all sufficiently large $N$ and $a_N = CJ$ for some $C > 0$.

For sufficiently large $N$, we have $Kw(N) \leq \delta_N / 4$, and therefore no two of the second stage intervals $[t_i - Kw(N), t_i + Kw(N)]$ for $i = 1, \ldots, J$ intersect. Because each $\hat{\tau}_j^{(2)}$ is a function of all $Y_i$'s in the disjoint index sets $S^{(2)}(t_j) \subset [t_j - Kw(N), t_j + Kw(N)]$ and the two level estimates $\hat{\nu}_j^{(1)}$ and $\hat{\nu}_j^{(1)}$, conditional independence holds:

$$|\tau_j^{(2)} - \tau_j^{(2)}| \leq a_N \hat{\tau}_j = J, \hat{\nu}_j^{(1)} = \alpha_j \text{ for } j \leq J$$

$$\prod_{i=1}^{J} \mathbb{P} \left[ \left| \hat{\tau}_i^{(2)} - \tau_i^{(2)} \right| \leq a_N \hat{\tau}_i = J, \hat{\nu}_i^{(1)} = \alpha_j \text{ for all } j \right]$$

To show the above product is eventually greater than some $1 - \varepsilon$, it would suffice to show that, for all $1 \leq k \leq J$ and sufficiently large $N$,

$$\mathbb{P}_{J, \alpha, t} \left[ \left| \hat{\tau}_k^{(2)} - \tau_k^{(2)} \right| > a_N \right] = \mathbb{P} \left[ \left| \hat{\tau}_k^{(2)} - \tau_k^{(2)} \right| > a_N \right] \leq \frac{C_{\varepsilon}/J}{(30)}$$

for some $C_{\varepsilon} < -\log(1 - \varepsilon)$. Henceforth, we will fix some $k$ between 1 to $J$ inclusive. Conditioning on the first stage values, we can write explicit expressions for $\tau_k^{(2)}$ and $\hat{\tau}_k^{(2)}$. To simplify, we will assume that $\nu_k > \nu_{k-1}$, as the situation for the reversed inequality can be handled analogously. For sufficiently large $N$ we have $\alpha_{k-1} \leq \nu_{k-1} + \rho_N < \nu_k - \rho_N < \alpha_k$, and hence

$$\hat{\tau}_k^{(2)} = \arg\min_{d \in S^{(2)}(t_k)} \left( \sum_{i \in S^{(2)}(t_k)} \left( Y_i - \frac{\alpha_i + \gamma}{2} \right) \left[ 1(i \leq d) - 1\left(i \leq \tau_k^{(2)}\right) \right] \right)$$

$$= \arg\min_{d \in S^{(2)}(t_k)} M_k(d)$$

$$\tau_k^{(2)} = \arg\min_{d \in S^{(2)}(t_k)} \left( \sum_{i \in S^{(2)}(t_k)} \left( \theta_i - \frac{\alpha_i + \gamma}{2} \right) \left[ 1(i \leq d) - 1\left(i \leq \tau_k^{(2)}\right) \right] \right)$$

$$= \arg\min_{d \in S^{(2)}(t_k)} M_k(d)$$

(131)

Since $t_k \in [\tau_k - w(N), \tau_k + w(N)]$, for $N$ large enough so that $K - 1 \geq 1$ we have

$$t_k - Kw(N) \leq \tau_k - (K - 1)w(N) \leq \tau_k^{(2)} - \frac{K - 1}{2}w(N) <$$

$$\tau_k^{(2)} + \frac{K - 1}{2}w(N) \leq \tau_k + (k - 1)w(N) \leq t_k + Kw(N)$$

(132)

Therefore, we can define, for $\frac{6}{w(N)} < r < \frac{K - 1}{2}$, the functions

$$A_k(r) := \min \left\{ M(d) : |d - \tau_k^{(2)}| \geq r w(N) \right\}$$

$$B_k(r) := A(r) / 3$$

(133)

For sufficiently large $N$, the first stage subsample is from a grid obtained by skipping more than 1 point, and we have $|\alpha_j - \nu_j| \wedge |\alpha_{j-1} - \nu_{j-1}| \leq \tilde{f} / 4$, which implies $A(r)$ is bounded below for some positive multiple
of \( r \) within this domain:

\[
A_k(r) = \left| \frac{\alpha_k - \alpha_{k-1}}{2} \right| \frac{\lambda_2 \left( \left( \tau_k^{(2)} - \tau_k^{(2)} + rw(N) \right) \right) \wedge \lambda_2 \left( \left( \tau_k^{(2)} - rw(N), \tau_k^{(2)} \right) \right)}{\lambda_2(S^{(2)}(t_k))} \\
\geq \frac{\hat{\theta}}{4} \frac{rw(N)}{2} (3Kw(N))^{-1} \\
\geq \frac{\hat{\theta}}{4} \frac{rw(N)}{6} (3Kw(N))^{-1} = \frac{\hat{\theta}}{72K} r
\]

(134)

Additionally,

\[
\left\{ \sup_{d \in S^{(2)}(t_k)} |M_k(d) - M_k(d)| \leq B_k(r) \right\} \rightarrow \left\{ |\hat{\tau}_k^{(2)} - \tau_k^{(2)}| \leq rw(N) \right\}
\]

(135)

For suppose \( d \in S^{(2)}(t_k) \) with \( |d - \tau_k^{(2)}| > rw(N) \) and

\[
\sup_{d \in S^{(2)}(t_k)} |M_k(d) - M_k(d)| \leq B_k(r),
\]

(136)

this implies

\[
M_k(d) \geq M_k(d) - B(r) \geq A_k(r) - B_k(r) \rightarrow \\
M_k(d) - \hat{M}_k(\tau_k^{(2)}) \geq A_k(r) - B_k(r) - \hat{M}_k(\tau_k^{(2)}) - B_k(r) > 0.
\]

(137)

Since \( \hat{M}_k(\cdot) \) cannot achieve a minimum at \( d \), \( d \) cannot equal to \( \hat{\tau}_k^{(2)} \), meaning \( |\hat{\tau}_k^{(2)} - \tau_k^{(2)}| \leq B(r) \).

We can now show the inequality in (139). First, divide the probabilities as

\[
\mathbb{P} \left[ |\hat{\tau}_k^{(2)} - \tau_k^{(2)}| > rw(N) \mid J = J, \hat{\nu}_j^{(1)}, \hat{\nu}_j^{(1)} = \alpha_j \text{ for all } j \right] \\
= \mathbb{P}_{J, \alpha, \tau} \left[ rw(N) \leq |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| \leq \frac{K - 1}{2} w(N) \right] + \mathbb{P}_{J, \alpha, \tau} \left[ |\hat{\tau}_i^{(2)} - \tau_i^{(2)}| > \frac{K - 1}{2} w(N) \right] \\
:= P_{J, \alpha, \tau}^{(k)} + Q_{J, \alpha, \tau}^{(k)}
\]

(138)

We first bound the term:

\[
P_{J, \alpha, \tau}^{(i)} = \mathbb{P}_{J, \alpha, \tau} \left[ \sup_{\tau_i^{(2)} + rw(N) \leq d \leq \tau_i^{(2)} + \frac{K - 1}{2} w(N)} \left( \hat{M}_k(\tau_i^{(2)}) - M_k(d) \right) \geq 0 \right] \\
\geq \mathbb{P}_{J, \alpha, \tau} \left[ \sup_{\tau_i^{(2)} - \frac{K - 1}{2} w(N) \leq d \leq \tau_i^{(2)} - rw(N)} \left( \hat{M}_k(\tau_i^{(2)}) - M_k(d) \right) \geq 0 \right]
\]

(139)

The steps and expressions used to bound both terms are extremely similar, so we work only with the first term. For \( d \in \left[ \tau_i^{(2)} + rw(N), \tau_i^{(2)} + \frac{K - 1}{2} w(N) \right] \), we have

\[
\hat{M}_k(\tau_k^{(2)}) - M_k(d) = -\frac{1}{\lambda_2(S^{(2)}(t_k))} \sum_{i \in S^{(2)}(t_k)} \left( \theta_i + \varepsilon_i - \frac{\alpha_k - \alpha_{k-1}}{2} \right) \left( 1(i \leq d) - 1(i \leq \tau_k^{(2)}) \right)
\]

(140)

and so \( \hat{M}_k(\tau_i^{(2)}) - M_k(d) > 0 \) if and only if

\[
- \sum_{i \in S^{(2)}(t_k)} \varepsilon_i \left( 1(i \leq d) - 1(i \leq \tau_k^{(2)}) \right) \geq \sum_{i \in S^{(2)}(t_k)} \left( \theta_i - \frac{\alpha_k + \alpha_{k+1}}{2} \right) \left( 1(i \leq d) - 1(i \leq \tau_k^{(2)}) \right)
\]

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For sufficiently large $N$, we have \( \theta_j - \frac{\alpha_j + \bar{\alpha}_j}{2} \geq \frac{\tilde{\theta}}{4} \) for all $i \in S^{(2)}(t_i)$ and $i > \tau_i^{(2)}$; hence, the right side above is actually bounded below by $\frac{\tilde{\theta}}{4} \geq \lambda_2 \left( \tau_k^{(2)} + d \right)$, and hence,

\[
P_{j,\alpha, t} \left[ \sup_{\tau_k^{(2)} + rw(N) \leq d \leq \tau_k^{(2)} + K - \frac{1}{2} N} \left( M_k(\tau_k^{(2)}) - M_k(d) \right) \geq 0 \right]
\]

\[
\leq P_{j,\alpha, t} \left[ \sup_{\tau_k^{(2)} + rw(N) \leq d \leq \tau_k^{(2)} + K - \frac{1}{2} N} \left( \frac{1}{\lambda_2 \left( \tau_k^{(2)} + d \right)} \sum_{i \in \{ \tau_k^{(2)} + d \}} (-\varepsilon_i) \right) > \frac{\tilde{\theta}}{4} \right]
\]

\[
\leq \left( \frac{\tilde{\theta}}{4} \right)^{-2} \left( \frac{\lambda_2(\tau_k^{(2)} + rw(N))E[\varepsilon_1^2]}{\lambda_2(\tau_k^{(2)} + K - \frac{1}{2} N)} \right) + \frac{\lambda_2(\tau_k^{(2)} + K - \frac{1}{2} N)}{\lambda_2(\tau_k^{(2)} + rw(N)) + 2} \left( \frac{1}{r^2} \right)
\]

\[
\leq 2 \left( \frac{\tilde{\theta}}{4} \right)^{-2} \frac{1}{\lambda_2(\tau_k^{(2)} + rw(N))}
\]

\[
\leq 4 \left( \frac{\tilde{\theta}}{4} \right)^{-2} \frac{4}{rw(N)}
\]

(141)

We now bound $Q^{(k)}_{N, \alpha, t}$:

\[
P_{N, \alpha, t} \left[ \left| \bar{z}_k^{(2)} - \tau_k^{(2)} \right| \geq \frac{K - 1}{2} w(N) \right]
\]

\[
\leq P_{N, \alpha, t} \left[ \sup_{d \in S^{(2)}(t_i)} \left| M_k(d) - M(d) \right| > B_k \left( \frac{K - 1}{2} \right) \right]
\]

\[
\leq P_{N, \alpha, t} \left[ \sup_{d \in S^{(2)}(t_i)} \sum_{i \in S^{(2)}(t_i)} \varepsilon_i \left( 1(i \leq d) - 1(i \leq \tau_k^{(2)}) \right) \right] > B_k \left( \frac{K - 1}{2} \right)
\]

\[
\leq C_5 \exp \left( -C_6 \lambda_2(S^{(2)}(t_k)) \right)
\]

\[
\leq C_5 \exp \left( -C_6 N^{1 - \gamma} \right),
\]

(142)

for some constants $C_5$ and $C_6$ not dependent on $T$. Hence, we have

\[
P_{N, \alpha, t}^{(k)} + Q_{N, \alpha, t}^{(k)} \leq \frac{C_4}{rw(N)} + C_5 \exp \left( -C_6 N^{5} \right)
\]

(143)

for some constant $C_4$ not changing with $T$. This expression can be made smaller than any $C/N$ for sufficiently large $N$ by setting

\[
r = 0.9 \cdot \frac{J}{Cw(N)}
\]

(144)

10.2. Proof of Theorem

First we define some notations. The entire data $(Y_1, Y_2, Y_3, ..., Y_N)$ is indexed from 1 to $N$, but at the second stage all indices that are multiples of $[N/N_1]$ are taken out. Hence let $\{1^*, 2^*, 3^*, ...\}$ be the set $\{1, ..., N\} - \{[N/N_1], 2[N/N_1], ...\}$ where $1^* < 2^* < 3^* < ...$. This notation also allows for the following convenience: for positive integers $a$ and $b$ in $\{1, ..., N\}$ we have $\lambda_2(a^*, b^*) = b - a$. We also have $\tau_i^{(2)} = n_i^*$ for some integers $n_1, ..., n_J$. 

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Proof. Denote \( S_k^{(2)} := S^{(2)} \left( \tilde{r}_k^{(1)} \right) \) for \( k = 1, \ldots, J \) as shorthand for the second stage subsamples. We also define the following random functions on integers: if \( J = J \) then for \( k = 1, \ldots, J \), let

\[
\hat{\Delta}_k^{(2)}(d) := \begin{cases} \frac{\sum_{j \in S_k^{(2)}(\tilde{r}_k^{(1)})} \left( Y_j - \frac{\nu_k - \nu_{k+1}}{2} \right) \left( 1(j \leq (n_k + d)^*) - 1(j \leq \tau_k^{(2)}) \right)}{\text{sgn}(\hat{\nu}_k^{(1)} - \hat{\nu}_{k+1}^{(1)})} & \text{for } (n_k + d)^* \in S_k^{(2)} \\
\infty & \text{otherwise} \end{cases}
\]

and if \( J \neq J \) then

\[
\hat{\Delta}_i^{(2)}(d) := d.
\]

so that the arg min of \( \hat{\Delta}_k^{(2)} \) is \( -\infty \) whenever \( J \neq J \). Using this definition, for all sufficiently large \( N \), the event \( \{ \hat{J} = J, \lambda_2(\tau_k^{(2)}, \tilde{r}_k^{(2)}) = j_k \text{ for } k = 1, \ldots, J \} \) is equivalent to the event \( \{ \text{arg min } \hat{\Delta}_k^{(2)}(d) = j_k \text{ for } k = 1, \ldots, J \} \).

Further, for any integer \( M > 0 \), we can easily obtain convergence properties by restricting the function \( \hat{\Delta}_k^{(2)} \) to the set \( \{ -M, -(M-1), \ldots, M \} \). For sufficiently large \( N \), when event \( A_N \) (defined in (128)) occurs, we have for any \( d \in \{ -M, \ldots, M \} \), and for all \( k = 1, \ldots, J \),

\[
\hat{\Delta}_k^{(2)}(d) = \text{sgn}(\Delta_k) \sum_{j \in S_k^{(2)}(\tilde{r}_k^{(1)})} \left( Y_j - \frac{\nu_k - \nu_{k+1}}{2} \right) \left( 1(j \leq (n_k + d)^*) - 1(j \leq \tau_k^{(2)}) \right)
\]

\[
= \begin{cases} \text{sgn}(\Delta_k) \sum_{j=n_k+1}^{n_k+d} \left( Y_j - \nu_k + \frac{\nu_k - \nu_{k+1}}{2} + \frac{1}{2} \left( \nu_k - \hat{\nu}_k^{(1)} + \nu_{k-1} - \hat{\nu}_{k-1}^{(1)} \right) \right) & \text{for } d > 0 \\
0 & \text{for } d = 0 \\
-\text{sgn}(\Delta_k) \sum_{j=n_k+d+1}^{n_k+d} \left( Y_j - \nu_{k-1} + \frac{\nu_k - \nu_{k+1}}{2} + \frac{1}{2} \left( \nu_k - \hat{\nu}_k^{(1)} + \nu_{k-1} - \hat{\nu}_{k-1}^{(1)} \right) \right) & \text{for } d < 0 \end{cases}
\]

\[
= \begin{cases} \frac{d|\Delta_k|}{2} + \text{sgn}(\Delta_k) \sum_{j=n_k+1}^{n_k+d} \epsilon_j + \frac{d}{2} \left( \nu_k - \hat{\nu}_k^{(1)} + \nu_{k-1} - \hat{\nu}_{k-1}^{(1)} \right) & \text{for } d > 0 \\
0 & \text{for } d = 0 \\
-\frac{d|\Delta_k|}{2} - \text{sgn}(\Delta_k) \sum_{j=n_k+d+1}^{n_k+d} \epsilon_j + \frac{d}{2} \left( \nu_k - \hat{\nu}_k^{(1)} + \nu_{k-1} - \hat{\nu}_{k-1}^{(1)} \right) & \text{for } d < 0 \end{cases}
\]

Because \( \max_{i=1,\ldots,J} |\hat{\nu}_i^{(1)} - \nu_0| \leq \gamma_N \) under \( A_N \), this gives the uniform bound

\[
\left| \frac{d}{2} \left( \nu_k - \hat{\nu}_k^{(1)} + \nu_{k-1} - \hat{\nu}_{k-1}^{(1)} \right) \right| \leq \frac{1}{2} M \gamma_N.
\]

for all \( k \) and \( d \). The right side of the above inequality converges to 0 since \( \gamma_N \to 0 \), and because all of this occurs with probability \( P[A_N] \to 1 \), this shows that the \( \Delta_k(d) \)'s all jointly converge. Specifically, let \( \epsilon_{i,j} \) for \( i = 1, \ldots, J \) and \( j \in \mathbb{Z} \) be iid copies of the error terms, and define the random variables

\[
\Lambda_{k,d} := \begin{cases} \frac{d|\Delta_k|}{2} + \text{sgn}(\Delta_k) \sum_{j=1}^{d} \epsilon_{k,j} & \text{for } d > 0 \\
0 & \text{for } d = 0 \\
-\frac{d|\Delta_k|}{2} - \text{sgn}(\Delta_k) \sum_{j=d+1}^{0} \epsilon_{k,j} & \text{for } d < 0 \end{cases}
\]

Then, we have the joint weak convergence

\[
\left( \hat{\Delta}_1^{(2)}(-M), \hat{\Delta}_1^{(2)}(-M+1), \ldots, \hat{\Delta}_1^{(2)}(M), \hat{\Delta}_2^{(2)}(-M), \hat{\Delta}_2^{(2)}(-M+1), \ldots, \hat{\Delta}_2^{(2)}(M), \ldots, \hat{\Delta}_k^{(2)}(-M), \hat{\Delta}_k^{(2)}(-M+1), \ldots, \hat{\Delta}_k^{(2)}(M) \right) \Rightarrow \left( \Lambda_{1,-M}, \Lambda_{1,-M+1}, \ldots, \Lambda_{1,M}, \Lambda_{2,-M}, \Lambda_{2,-M+1}, \ldots, \Lambda_{2,M}, \ldots, \Lambda_{k,-M}, \Lambda_{k,-M+1}, \ldots, \Lambda_{k,M} \right)
\]
Hence, define \( L_k := \arg \min_{j \in \mathbb{Z}} \Lambda_{k,j} \) and \( L_k^{(M)} := \arg \min_{|j| \leq M} \Lambda_{k,j} \). We have the joint weak convergence

\[
\left( \arg \max_{|j| \leq M} \hat{\Delta}_1^{(2)}(d), \ldots, \arg \max_{|j| \leq M} \hat{\Delta}_K^{(2)}(d) \right) \Rightarrow \left( L_1^{(M)}, \ldots, L_K^{(M)} \right)
\]  

(151)

by the continuous mapping theorem, because \( \arg \max \) is a continuous function on \( \mathbb{R}^{2M+1} \) (except when at least two of the coordinates are equal, which has probability 0 here since the \( \Lambda_{i,j} \)'s have densities).

Next, we establish that \( \mathbb{P} \left[ \hat{J} = J, \lambda_2(\tau_k^{(2)}, \hat{\tau}_k^{(2)}) = j_k \text{ for } k = 1, \ldots, J \right] \) converges to the product of \( \mathbb{P}[L_k = j_k] \) for \( k = 1, \ldots, J \). We will do this by showing for any fixed \( \epsilon > 0 \), the absolute difference between the two is smaller than \( \epsilon \) for all large \( N \). As in the proof of the single change point problem, this is accomplished through three main inequalities.

**First Inequality:** From the previous theorem we can find an integer \( K_0 \) greater than \( \max_k |j_k| \), such that for any \( K_1 \geq K_0 \), we have for sufficiently large \( N \)

\[
\mathbb{P} \left[ \hat{J} = J, \max_{k=1, \ldots, J} |\hat{\tau}_k^{(2)} - \tau_k| \leq (2K_1 + 2) \right] \geq 1 - \frac{\epsilon}{3}
\]  

(152)

For all sufficiently large \( N \), \( \left\{ \hat{J} = J, \max_{k=1, \ldots, J} |\hat{\tau}_k^{(2)} - \tau_k| \leq (2K_1 + 2) \right\} \) would mean \( \hat{J} = J, \max_{k=1, \ldots, J} |\lambda_2(\tau_k^{(2)}, \hat{\tau}_k^{(2)})| \leq K_1 \) and hence

\[
1 - \frac{\epsilon}{3} \leq \mathbb{P} \left[ \hat{J} = J, \max_{k=1, \ldots, J} \lambda_2(\tau_k^{(2)}, \hat{\tau}_k^{(2)}) \leq K_1 \right] = \mathbb{P} \left[ \max_{k=1, \ldots, J} \arg \min_{d \in \mathbb{Z}} \Delta_k^{(2)}(d) \leq K_1 \right]
\]  

(153)

Now

\[
\arg \min_{d \in \mathbb{Z}} \Delta_k^{(2)}(d) = j_k \text{ for } k = 1, \ldots, J
\]

\[
\iff \arg \min_{|d| \leq K_1} \Delta_k^{(2)}(d) = j_k \text{ for } k = 1, \ldots, J, \text{ and } \max_{k=1, \ldots, J} \arg \min_{d \in \mathbb{Z}} \Delta_k^{(2)}(d) \leq K_1
\]  

(154)

With steps very similar to those used in (92), it can be shown that

\[
\mathbb{P} \left[ \arg \min_{d \in \mathbb{Z}} \Delta_k^{(2)}(d) = j_k \text{ for } k = 1, \ldots, J \right] - \mathbb{P} \left[ \arg \min_{|d| \leq K_1} \Delta_k^{(2)}(d) = j_k \text{ for } k = 1, \ldots, J \right] \leq \mathbb{P} \left[ \max_{k=1, \ldots, J} \arg \min_{d \in \mathbb{Z}} \Delta_k^{(2)}(d) > K_1 \right] \leq \epsilon/3
\]  

(155)

**Second Inequality** We can find some integer \( K_2 > K_0 \) such that

\[
\mathbb{P} \left[ \max_{k=1, \ldots, J} |L_k| \leq K_1 \right] \geq 1 - \frac{\epsilon}{3}
\]  

(156)

Now \( L_k = j_k \) for \( k = 1, \ldots, J \) if and only if both \( L_k^{(K_2)} = j_k \) for \( k = 1, \ldots, J \) and \( \max_k |L_k| \leq K_2 \). With steps very similar to those in (88), we have

\[
\mathbb{P} \left[ L_k = j_k \text{ for } k = 1, \ldots, J \right] - \mathbb{P} \left[ L_k^{(K_2)} = j_k \text{ for } k = 1, \ldots, J \right] \leq \mathbb{P} \left[ \max_{k=1, \ldots, J} |L_k| > K_1 \right] \leq \epsilon/3
\]  

(157)
Thus, by following steps similar to the proof for Theorems 7 and 4, we are able to arrive at (161).

**Proof.** We know that for some sequence $N$ for all large $\nu$ occurs with probability approaching 1. Importantly, conditioned on the event there fore also independent from the estimates $\hat{\tau}^{re}_j$'s and $\hat{\epsilon}^{*}$’s which are functions of the $\{Z_i\}$ dataset.

Combining the inequalities in (155), (157), and (159) will give

$$\begin{align*}
\left| \mathbb{P}[L_k = j_k \text{ for } k = 1, \ldots, J] - \mathbb{P}[\hat{\tau}^{re}_j = j_k \text{ for } k = 1, \ldots, J] \right| &\leq \frac{\epsilon}{3} \\
&\leq \epsilon \quad (160)
\end{align*}$$

### 10.3. Proof of Theorem

We will first state some results on the asymptotic distribution of the first stage re-estimates $\hat{\tau}^{re}_j$ for $j = 1, \ldots, J$, as defined in step (ISM4-3). In performing the re-estimation at stage 1 we have two time series $Z_1, \ldots, Z_n$ and $V_1, \ldots, V_n$, where $n = N(1 + o(1))$. They satisfy

- $Z_i = E[Z_i] + \epsilon_i^*$ and $V_i = E[V_i] + \epsilon_i^{**}$ where $\epsilon_i^*$’s and $\epsilon_i^{**}$’s are all iid $N(0, \sigma^2)$.
- The sequences $\{E[Z_i]\}$ and $\{E[V_i]\}$ are both piecewise constant in $i$ and with change points $\{\tau_1^*, \ldots, \tau_J^*\}$ and $\{\tau_1^{**}, \ldots, \tau_J^{**}\}$.
- $E[Z_i] = E[V_i]$ whenever $i \in (\tau_j^*, \tau_{j+1}^*)$ and $i \in (\tau_j^{**}, \tau_{j+1}^{**})$ for some $j$.
- $\max_j |\tau_j^* - \tau_j^{**}| \leq 1$

**Lemma 4.** Under the same assumptions as Theorem, for any $\epsilon > 0$ and $j \in \{1, \ldots, J\}$ there exists a constant $C$ for which

$$\mathbb{P}[\hat{\tau}^{re}_j = j; |\hat{\tau}^{re}_j - \tau_j^*| \leq C] \geq 1 - \epsilon \quad (161)$$

for all large $N$.

**Proof.** We know that for some sequence $w^*(N) = o(\delta_{N, i})$ and $\rho_N \to 0$, the event

$$R^{(j)}_N := \left\{ \hat{\tau}^{(j)}_j = j; |\hat{\tau}^{(j)}_j - \tau_j^*| \leq w^*(N) \max \left\{ |\hat{\nu}_j^{(1)} - \nu_j - 1|, |\hat{\nu}_j^{(1)} - \nu_j - 1| \right\} \leq \rho_N \right\} \quad (162)$$

occurs with probability approaching 1. Importantly, conditioned on the event $R^{(j)}_N$, the joint distribution of the $\{V_j\}$ dataset does not change, as this time series is independent of the $\{Z_i\}$ dataset from step (ISM2) and therefore also independent from the estimates $\hat{\tau}^{(j)}_j$’s, and $\hat{\nu}_j^{(1)}$’s which are functions of the $\{Z_i\}$ dataset. Thus, by following steps similar to the proof for Theorems 7 and 4 we are able to arrive at (161).

Next, we provide the proof of Theorem

Namely if (M1)-(M4), (M8), and (28) are satisfied.
Proof. First, define the event

\[ A_N := \{ \hat{J} = J; \quad \max_{j=1,\ldots,J} |\hat{\tau}_j - \tau_j^*| \leq w^*(N); \quad \max_{j=0,\ldots,J} |\hat{\nu}_j^{(1)} - \nu_j| \leq \rho_N \}. \]  

(163)

For any positive integer \( C \) define the event

\[ B_N(C) := \{ \hat{J} = J; \quad \max_{j=1,\ldots,J} |\hat{\tau}_j^C - \tau_j^*| \leq C \}. \]

(164)

We know that \( \mathbb{P}[A_N] \to 1 \), and by Lemma 4 we know that for any \( \epsilon > 0 \) there exists a \( C_\epsilon \) for which \( \mathbb{P}[B_N(C_\epsilon)] \geq 1 - \epsilon \) for all large \( N \).

For \( j = 1, \ldots, J \) and all \( c \in \mathbb{N} \) define

\[ \hat{\Delta}_j^C(d) := \text{sgn}(\nu_j - \nu_{j-1}) \left( \sum_{k=\tau_j^*-C}^{\tau_j^*+C} \left(V_k - \hat{\nu}_j^{(1)} + \hat{\nu}_j^{(1)}(k - 1\leq \tau_j^* + d) - 1(k \leq \tau_j^* + d) \right) \right). \]

(165)

For sufficiently large \( N \), on the event \( A_N \cap B_N(C) \) we have \( \hat{\tau}_j^C = \arg\min_{-C \leq d \leq C} \Delta_j^C(d) \). We also have

\[ \left( \arg\min_{-C \leq d \leq C} \Delta_1^C(d), \ldots, \arg\min_{-C \leq d \leq C} \Delta_J^C(d) \right) \Rightarrow (L_1^C, \ldots, L_J^C) \]

(166)

where \( L_1^C, \ldots, L_J^C \) are independent variables with distributions defined as they were after expression (150). As noted previously, if we have independent random variables \( L(\Delta_1/\sigma), \ldots, L(\Delta_J/\sigma) \), on the event \( \{ \max_{j=1,\ldots,J} |L(\Delta_j/\sigma)| \leq C \} \) we have \( L(\Delta_j/\sigma) = L_j^C \) for \( j = 1, \ldots, J \).

Since \( \mathbb{P}[A_N \cap B_N(C)] \) and \( \mathbb{P}[\max_{j=1,\ldots,J} |L(\Delta_j/\sigma)| < C] \) can be made arbitrarily close to 1 by raising the value of \( C \), the proof can be completed as in Theorem 4 or by using a slightly modified version of the Converging Together Lemma (see [18], Lemma 3.3).

\[ \square \]

11. Appendix C (Wild Binary Segmentation)

11.0.1. Wild Binary Segmentation

We next discuss the WBinSeg algorithm, introduced in [7]. The steps of this algorithm are:

(a) Fix a threshold value \( \zeta_N \) and initialize the segment set \( SS = \{(1, N)\} \), the change point estimate set \( \hat{\mathbf{c}} = \emptyset \), and \( M_N \) intervals \([s_1, e_1], \ldots, [s_{M_N}, e_{M_N}]\) where each \( s_j \) and \( e_j \) are uniformly picked from \([1, N]\).

(b) Pick any ordered pair \((s, e) \in SS\), remove it from \( SS \) (update \( SS \) by \( SS \leftarrow SS - \{(s, e)\}\)). If \( s \geq e \) then skip to step 6, otherwise continue to step 3.

(c) Define \( \mathcal{M}_{s,e} := \{ [s_i, e_i] : [s_i, e_i] \subseteq [s, e] \} \).

- As an optional step, also take \( \mathcal{M}_{s,e} \leftarrow \mathcal{M}_{s,e} \cup \{(s, e)\} \).

(d) Find a \([s^*, e^*] \in \mathcal{M}_{s,e}\) such that

\[ \max_{b \in [s^*, e^*-1]} |\hat{Y}_{s^*, e^*}^b| = \max_{|s', e'] \in \mathcal{M}_{s,e}} \left( \max_{b \in [s', e'-1]} |\hat{Y}_{s', e'}^b| \right) \]

and let \( b_0 = \arg\max_{b \in [s^*, e^*-1]} |\hat{Y}_{s^*, e^*}^b| \).

(e) If \( |\hat{Y}_{s^*, e^*}^{b_0}| \geq \zeta_N \), then add \( b_0 \) to the list of change point estimates (add \( b_0 \) to \( \hat{\mathbf{c}} \)), and add ordered pairs \((s, b_0)\) and \((b_0 + 1, e)\) to \( SS \), otherwise skip to step 5.

\[ \text{See the explanation after Second Inequality of Section 10.1.1} \]

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(f) Repeat steps 2-4 until $SS$ contains no elements.

Roughly speaking, WBinSeg performs very much like binary segmentation but with steps that maximize change point estimates over $M_N$ randomly chosen intervals. The consistency results in [7] imply that in our setting, the following holds:

**Theorem 9.** Suppose conditions (M1) to (M4) are satisfied and the tuning parameter $\zeta_N$ is chosen appropriately such that there exists positive constants $C_1$ and $C_2$ with $C_1\sqrt{\log(N)} \leq \zeta_N \leq C_2\sqrt{N}$. Denote $\hat{J}, \hat{\tau}_1, \ldots, \hat{\tau}_J$ as the estimates obtained from wild binary segmentation. Then, there exists positive constants $C_3, C_4$ where

$$\mathbb{P}\left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}_i - \tau_i| \leq C_3 \log(N) \right] \geq 1 - C_4N^{-1} - \left( \frac{N}{\delta_N} \right) \left( 1 - \left( \frac{\delta_N}{3N} \right)^2 \right)^{M_N}$$

(167)

We remark that the right side of (167) does not necessarily converge to 1 unless $M_N \to \infty$ fast enough. Using some simple algebra, it was shown in the original paper, that for sufficiently large $N$, this expression can be bounded from below by $1 - CN^{-1}$ for some $C > 0$ if $M_N \geq \left( \frac{3N}{\delta_N} \right)^2 \log(2N/\delta_N)$, a condition on $M_N$ which we assume from here on in order to simplify some later analysis.

Compared with the consistency result for binary segmentation given in Theorem 3, $\max_{j=1, \ldots, M_N} |\hat{\tau}_j - \tau_j|$ can be bounded by some constant times $\log(N)$, which grows much slower than $E_N = (N/\delta N)^3 \log(N)$ whenever $N/\delta_N \to \infty$. However, this comes at the cost of computational time. Suppose we perform wild binary segmentation with $M_N$ random intervals $[s_1, e_1], \ldots, [s_{M_N}, e_{M_N}]$, then for large $N$ the most time-consuming part of the operation is to maximize the CUSUM statistic over each interval, with the other tasks in the WBinSeg procedure taking much less time. This takes an order of $\sum_{j=1}^{M_N} (e_j - s_j)$ time, and since the interval endpoints are drawn from $\{1, \ldots, N\}$ with equal probability, we have $\mathbb{E}[e_j - s_j] = \frac{N}{3}(1 + o(1))$ for all $j = 1, \ldots, M_N$. Hence the scaling of the average computational time for maximizing the CUSUM statistic in all random intervals, and WBinSeg as a whole, is $O(NM_N) = O\left( \frac{N}{\delta_N} \log(N^2/\delta_N) \right)$ time, which is greater than $O(N \log(N))$ time for binary segmentation whenever $N/\delta_N \to \infty$. The trade-off between the increased accuracy of the estimates and the bigger computational time, as they pertain to intelligent sampling, will be analyzed later.

Finally, we note that under conditions (M1) through (M4), wild binary segmentation does satisfy the requirements of Lemma 1. Specifically with $w(N) = C_1 \log(N)$, $B_N = C_2N^{-1}$, and $\rho_N = N^\theta$ for some positive constants $C_1$ and $C_2$, we have

$$\mathbb{P}\left[ \hat{J} = J; \max_{i=1, \ldots, J} |\hat{\tau}_i - \tau_i| \leq w(N); \max_{i=0, \ldots, J} |\hat{\nu}_i - \nu_i| \leq \rho_N \right] \to 1$$

for any $\rho_N = N^\theta$, where $\theta \in \left( \frac{\Xi - 1}{2}, 0 \right)$. (168)

**Remark 13.** As with the BinSeg algorithm, the WBinSeg procedure is asymptotically consistent but faces the same issues as BinSeg in a practical setting where the goal is to obtain confidence bounds $[\hat{\tau}_i \pm C_3 \log(N)]$ for the change point $\tau_i$. Namely, as in Remark 9, there are unspecified constants associated with the tuning parameter $\zeta_N$ and the confidence interval width $C_3 \log(N)$ in (167). The issue of choosing a confidence interval width will be resolved in Section 3.4, in the context of the intelligent sampling scheme and finitely many change-points, and the discussion of the $\zeta_N$ parameter for the numerical work in this paper, provided later.

**Wild Binary Segmentation at Stage 1:** Suppose now that WBinSeg is used at step ISM3. This would involve different requirements on $\Xi$ and $\gamma$ for consistency. Namely, condition (M7 (BinSeg)) is no longer required to satisfy (23). Since WBinSeg on a dataset of size $N$ was shown to be consistent whenever the minimal spacing satisfies $\delta_N > KN^\varepsilon$ for any positive constants $K$ and $\varepsilon$, we only need the weaker assumption:

(M8 (WBinSeg)): $N_1$, the size of the first stage subsample is chosen so that $N_1 = K_1N^\gamma$ for some $K_1 > 0$ and $\gamma > \Xi$.
Table 2. Table of $\gamma_{\min}$ and computational times for various values of $\Xi$, using WBinSeg at stage 1.

| $\Xi$       | $[0, 1/4)$ | $[1/4, 1/3)$ | $[1/3, 1)$ |
|-------------|------------|--------------|------------|
| $\gamma_{\min}$ | $\max\{\frac{1-2\Xi+\Lambda}{2}, \Xi + \delta\}$ | $\Xi + \delta$ | $\Xi + \delta$ |
| Order of Time | $N^{(1+2\Xi+\Lambda)/2} \log(N)$ | $N^{(1+2\Xi+\Lambda)/2}, N^{3\Xi+\delta} \log(N)$ | $N^{3\Xi+\delta} \log(N)$ |
| $\gamma_{\min}$ (\(\Lambda = 0\)) | $\Xi + \delta$ | $\Xi + \delta$ | $\Xi + \delta$ |
| Time (\(\Lambda = 0\)) | $N^{(1+2\Xi)/2} \log(N)$ | $N^{(1+2\Xi+\Lambda)/2} \log(N)$ | $N^{3\Xi+\delta} \log(N)$ |
| $\gamma_{\min}$ (\(\Lambda = \Xi\)) | $\Xi + \delta$ | $\Xi + \delta$ | $\Xi + \delta$ |
| Time (\(\Lambda = \Xi\)) | $N^{(1+3\Xi)/2} \log(N)$ | $N^{(1+3\Xi)/2} \log(N)$ | $N^{3\Xi+\delta} \log(N)$ |

With a subsampling of $N_1 \sim N^\gamma$ points at stage 1, the minimum spacing for stage 1 subsample is

$$\delta_{N_1}^* \geq K_1 N_1^{\gamma - \Xi} \geq K_2 N_1^{1-\Xi/\gamma}$$

for some constants $K_1$ and $K_2$. From the discussion around (168), we see that $w^*(N) = C \log(N)$ for some $C > 0$, which satisfies $w^*(N) = o(\delta_{N_1}^*)$, and $\rho_N \equiv \rho_{N_1} = N_1^{-\theta}$ for any $\theta \in (0, \Xi/(2\gamma))$. In summary, if WBinSeg is used at stage 1 it is sufficient to assume conditions (M1) to (M5) and (M8 (WBinSeg)) in order for intelligent sampling to be consistent.

To provide some insight on the computational time when using WBinSeg at stage 1, we again assume that $\delta_{N_1}/N^\Xi \rightarrow K_1$ and $J(N)/N^\Lambda \rightarrow K_2$ for some constant $\Lambda \leq \Xi$ and positive constants $K_1, K_2$. To summarize the details (relegated to Section 11.1 in the supplement), for $N_1 \sim N^\gamma$, the average time for the first stage is $O(N^\gamma + 2\Xi \log(N))$, while the second stage takes, on average, $O(N^\gamma + \Lambda \log(N))$ time. Together with condition (M8 (WBinSeg)) and setting, $\gamma > \Xi$, the order of average time for both stages combined is minimized by setting $\gamma_{\min} = \max\{\frac{1-2\Xi+\Lambda}{2}, \Xi + \delta\}$ for any small constant $\delta$, with the average total computational time being $O(N^{\gamma_{\min} + 2\Xi} \log(N))$.

![Fig. 24. Blue triangle encompasses all valid values of $\gamma$ vs $\Xi$ as set by (M8 (WBinSeg)). Pink region, solid red lines, and dotted red lines denotes $\gamma_{\min}$ for each $\Xi$.](image)

Although the $\gamma_{\min}$ values are, across the board, smaller than those given in Table 1, the order of the actual computational time is actually the same up to $\Xi \leq 1/6$ (but could also be the same up to $\Xi \leq 1/5$ depending on $\Lambda$). There is an advantage of using WBinSeg at stage 1 for higher values of $\Xi$, e.g. for $\Xi > 1/5$ using WBinSeg is on average faster for large $N$ (as $N^{4\Xi+\delta} \log(N) < N^{(1+3\Xi)/2} \log(N)$), and WBinSeg is theoretically guaranteed consistency for $\Xi \in [1/4, 1]$ whereas BinSeg is not. However, despite the theoretical flexibilities of using WBinSeg in the regime where $\Xi$ is high, we comment on why intelligent sampling is better suited for situations where the change points are sparse and widely spread out, specifically when $\Xi$...
is low and the order of time does not change whether using BinSeg or WBinSeg.

We note that computation-wise, the closer the change points are placed together (i.e. the higher the value of $\Xi$), the less we see a drop in computational time using intelligent sampling as opposed to using the full data. Using BinSeg, it takes order $N^{\gamma_{\min}} \log(N) \left( N^{\gamma_{\min} + 2\Xi} \log(N) \right)$ for WBinSeg) time for using intelligent sampling as compared to order $N \log(N) \left( N^{1+2\Xi} \log(N) \right)$ for WBinSeg time for full data analysis. The gaps between these orders shrink as $\Xi$ increases, as one can easily check. In other words, intelligent sampling gives better gains when $\Xi$ is low. We note that on the theoretical side, Theorems 5 and 6 (a result useful for practical implementation, explained later) operate under the assumption that the number of change points is finite, a situation that is more compatible with a high $\delta_N$ (low $\Xi$).

Intelligent sampling is therefore well-suited to the needles in haystack problem where an extremely long stretch of sequential data is available with relatively few jumps in between. In the extreme case where $\Xi = 0$, the optimal $\gamma_{\min}$ is $1/2$ (whether using BinSeg or WBinSeg at stage 1) and the average computational time is the same order as $\sqrt{N} \log(N)$. These gains also carry over to a logarithmic regime: suppose instead $\delta_N/(N/\log^\beta(N)) \rightarrow K$ and $J(N)/\log^\beta(N) \rightarrow K_2$ for some positive constants $K_1$, $K_2$, and $\beta$. Using BinSeg (or WBinSeg) at stage 1, we would take order $N^{\gamma} \log(N)$ (or $N^{\gamma} \log^{2\beta+1}(N)$ with WBinSeg) time, and the second stage takes $N^{1-\gamma} \left( \log(N) \right)^{3\beta+1}$ (or $N^{1-\gamma} \log^{\beta+1}(N)$ with WBinSeg). With binary segmentation, setting $\gamma \equiv \gamma_N = \frac{1}{2} + \frac{3\beta}{2} \log_2 N \left( \log(N) \right)$ places the computational minimum time at $O \left( N^{1/2} \log(N)^{2\beta+2} \right)$, while with WBinSeg setting $\gamma \equiv \gamma_N = \frac{1}{2} - \frac{\beta}{2} \log_2 N \left( \log(N) \right)$ gives the minimum computational time of $O \left( N^{1/2} \log(N)^{3\beta+2} \right)$. Either way, they are close to the $O(\sqrt{N} \log(N))$ time which is achieved for $\Xi = 0$.

11.1. Computational Time Analysis for Multiple Change Points

11.1.1. Stage 1 with BinSeg

**Computational time:** We deduce this as follows. Given $n$ points from a multiple change point signal, it takes $O(n \log(n))$ computational time to apply binary segmentation, and given $n$ points, $O(n)$ time to fit a step function signal. The first stage utilizes $O(N^\gamma)$ points and hence takes time $O(N^\gamma \log(N))$, while the expected amount of time used in the second stage is of the same order as $\mathbb{E} \left[ \hat{J} \right] \cdot N^{1-\gamma} \left( \frac{N}{\delta_N} \right)^2 \log(N)$ points, where the factor multiplying $\mathbb{E} \left[ \hat{J} \right]$ is simply the order of $w(N)$ [as defined in (ISM4)]. From the fact that there exists some constant $c > 0$ such that $P[\hat{J} \neq J] \leq cN^{-1}$ for all large $N$ (by the result of Theorem 3), we can further deduce that

$$\mathbb{E} \left[ \hat{J} \right] \leq (1 - cN^{-1}) \cdot J + cN^{-1} \cdot N_1 \leq J + c \tag{169}$$

and hence the average time for the second stage is $O \left( J \cdot N^{1-\gamma} \left( \frac{N}{\delta_N} \right)^2 \log(N) \right)$. The minimal order of computation is achieved by setting $\gamma$ such that these two stages take around the same order of time.

To simplify this analysis, suppose that for some $0 \leq \Lambda \leq \Xi$, $J/N^\Lambda$ converges to a positive constant $c$. Then

$$J \cdot N^{1-\gamma} \left( \frac{N}{\delta_N} \right)^2 \log(N) = N^{1-\gamma + 2\Xi + \Lambda} \log(N)(1 + o(1)),$$

meaning the second stage will take $O(N^{1-\gamma + 2\Xi + \Lambda} \log(N))$ time and the whole procedure, $O(N^{\gamma \vee (1-\gamma + 2\Xi + \Lambda)} \log(N))$.

Ignoring the logarithmic term, the problem reduces to the following:

Minimize $\gamma \vee (1 - \gamma + 2\Xi + \Lambda)$ subject to the constraints (a) $\Lambda \leq \Xi$, and (b) $\gamma > 4\Xi$, where $0 < \gamma < 1$.

Denote this constrained minimizer by $\gamma_0$. Consider first the unconstrained solution without enforcing (a) and (b). This is seen to be achieved by setting $\gamma = 1 - \gamma + 2\Xi + \Lambda$, i.e. $\gamma_{\min} = (1 + 2\Xi + \Lambda)/2$. For this to be also the solution under constraints (a) and (b) we require that $\gamma_{\min} > 4\Xi$, which translates to $\Lambda > 6\Xi - 1$, and $\Lambda \leq \Xi$, which therefore means that $\Xi < 6\Xi - 1$, i.e. $\Xi < 1/5$. 

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Now, since \( \Lambda \geq 0 \), the inequality \( \Lambda > 6\Xi - 1 \) is automatically satisfied for \( \Xi < 1/6 \). This means that for \( 0 \leq \Xi < 1/6 \), the unconstrained solution \( \gamma_{\text{min}} \) is also the constrained solution. The worst case scenario is when \( \Lambda = \Xi \) which gives an optimum \( \gamma_0 \) of \( (1 + 3\Xi)/2 \) whereas the best case scenario corresponds to \( \Lambda = 0 \) which gives \( \gamma_0 = (1/2 + \Xi) \).

Next, consider \( \Xi \in [1/6, 1/5] \) in which range \( 6\Xi - 1 \leq \Xi \) with equality at \( \Xi = 1/5 \). When \( \Lambda \in (6\Xi - 1, \Xi] \), the optimal \( \gamma_0 \) is still given by the expression \( \gamma_{\text{min}} \). However, when \( \Lambda \leq 6\Xi - 1 \), \( \gamma_{\text{min}} \) is no longer an admissible choice for \( \gamma_0 \). Our goal is now to minimize \( \gamma \vee (1 - \gamma + 2\Xi + \Lambda) \) over \( \gamma > 4\Xi \), or equivalently minimize \( (4\Xi + \eta) \vee (1 - 2\Xi + \Lambda - \eta) \) when \( \eta > 0 \). Now, since \( \Lambda \leq 6\Xi - 1 \), \( -2\eta \leq 6\Xi - 1 \) for all \( \eta > 0 \), which translates to \( 4\Xi + \eta \geq 1 - 2\Xi + \Lambda - \eta \). It follows that \( \gamma_0 \) should be taken just above \( 4\Xi \) [an exact minimizer is not achievable, one has to work with a very small \( \eta \)]. So, the range of possible \( \gamma_0 \)'s (depending on \( \Lambda \)) ranges in \((4\Xi, (1 + 3\Xi)/2)\).

Finally consider \( \Xi \geq 1/5 \). Now, we have \( 6\Xi - 1 \geq \Xi \) with equality only at \( \Xi = 1/5 \), so \( \Lambda \leq 6\Xi - 1 \) necessarily. Essentially the argument above shows that, in this range again \( \gamma_0 \) should be chosen above \( 4\Xi \) but close to it.

**Remark 14.** We note here that for the best implementation of the intelligent sampling strategy, knowledge of \((\Lambda, \Xi)\) is desirable. However, in practice, these values are unknown. If one is willing to impose an upper bound on the \( \Xi \) parameter, intelligent sampling can be implemented with this (conservative) upper-bound, and letting \( \Lambda = \Xi \) (again a conservative assumption). The sparser the number of change-points, the better the gains in computational time, as the below figure demonstrates. Intelligent sampling is therefore well-suited to the needles in haystack problem where an extremely long stretch of sequential data is available with relatively few jumps in between. As \( \Xi \) approaches 0, note that the optimal \( \gamma_0 \) converges to 1/2, its value in the fixed number of change-points problem. If we model a sparse number of change-points by writing \( J \) as some constant times \((\log N)^\beta \) for some \( \beta > 0 \) (instead of a polynomial regime as assumed above), then we would have the first stage taking order \( N^{1-\gamma} \log(N) \) time, and the second stage taking \( N^{1-\gamma} (\log(N))^{3\beta+1} \). By setting \( \gamma = \gamma_N = 1/2 + 3\beta+1 \log(N) (\log(N)) \) this places the computational minimum time at \( O(N^{1/2}(\log N)^{3\beta+1}) \), which is a significant drop from \( O(N \log N) \) used by the conventional binary segmentation procedure.

### 11.1.2. Stage 1 with WBinSeg

In terms of computational time, we again consider the two stages. Given \( n \) data points from (12) with minimum separation \( \delta_n \) between change points, it takes an order of \( 3\delta_n \log(n/\delta_n) \) to perform the procedure. The first stage works with a time series data of length approximately \( N \) with minimal separation \( \delta_N \), and hence has an average computational time that is of the same order as \((N_1/\delta_{N_1}) \cdot N_1 \log(N_2/\delta_{N_2})\), which is the same order as \( N^{\gamma+2\Xi} \log(N) \). The second stage works with \( J \) intervals, each of width \( CN^{1-\gamma} \log(N) \) for some constant \( C \). Because we have

\[
P[\hat{J} = J] \geq 1 - CN^{-1} \quad \text{for some } C > 0 \tag{170}
\]

by our earlier condition on \( M_N \), we arrive at \( E[\hat{J}] = O(J) \) by the same arguments used in (169). This in turn shows the expected computational time of the second stage is \( O(JN^{1-\gamma} \log(N)) \) which simplifies to \( O(N^{1-\gamma+\Lambda} \log(N)) \) if we assume \( J/N^\Lambda \rightarrow C \) for some \( \Lambda \in [0, \Xi] \) and constant \( C > 0 \).

Both stages combined are expected to take \( O(N^{(\gamma+2\Xi)/(1-\gamma+\Lambda)} \log(N)) \) time. This fact combined with the requirement \( \gamma > 3 \Xi \geq \Lambda \) lead to an optimal way to choose \( \gamma \) to minimize the amount of computational time:

- On the region \( \Xi < 1/4 \) we can solve the equation \( \gamma_{\text{min}} + 2\Xi = 1 - \gamma_{\text{min}} + \Lambda \) to get the minimizing \( \gamma \) as \( \gamma_{\text{min}} = \frac{1-2\Xi+\Lambda}{2} \), which satisfies \( \gamma_{\text{min}} > \Xi \). This results in \( O(N^{1-\gamma+\Lambda} \log(N)) \) computational time.

- On the region \( \Xi \in [1/4, 1/3] \):
  - If \( \frac{1+\Lambda}{2} > \Xi \) we would still have \( \frac{1-2\Xi+\Lambda}{2} > \Xi \) and hence set \( \gamma_{\text{min}} = \frac{1-2\Xi+\Lambda}{2} \) resulting in \( O(N^{1-\gamma+\Lambda} \log(N)) \) computational time.

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– Otherwise if $\frac{1-2\Xi+\Lambda}{2} \leq \Xi$, set $\gamma_{\min} = \Xi + \eta$, where $\eta > 0$ is small, for $O(N^{3\Xi+\eta} \log(N))$ computational time.

• For $\Xi \in [1/3, 1)$ also set $\gamma_{\min} = \Xi + \eta$, where $\eta > 0$ is small, for $O(N^{3\Xi+\eta} \log(N))$ computational time.

We will note that in the regime where $\Xi$ is low enough so that $\frac{1+\Lambda-2\Xi}{2} > \Xi$, this comes out to $O(N^{\frac{1+\Lambda+2\Xi}{2}} \log(N))$ computational time. We will also note that the $\gamma > \Xi$ requirement for WBinSeg is much more flexible than the $\gamma > 4\Xi$ condition derived in the previous section, which allows us to extend Theorem 4 to values $\Xi$ greater than $1/4$. However computational time also scales up in this regime, and for $\Xi \geq 1/3$, intelligent sampling with WBinSeg is expected to take $O(N^\beta \log(N))$ time where $1 \leq \beta < 3$, which can be very time intensive for large datasets.

11.2. Simulation Results for WBinSeg

We next looked at how effective intelligent sampling with WBinSeg would work in practice, by running a set of simulations with the same set of model parameters as in Setup 2 of Section 6, and used the exact same method of estimation except with WBinSeg used in place of the BinSeg algorithm. For the tuning parameters of WBinSeg, the same $\zeta_N$ was retained and the number of random intervals was taken as $M_N = 20000$. Although we could have used the theoretically prescribed value of $M_N = 9(\frac{\log(N)}{\delta_1^*})^{1/2}$, this turns out to be over 400,000 and is excessive as setting $M_N = 20,000$ gave accurate estimates.

Fig. 25. Distributions of $\max_{1 \leq j \leq 55} \lambda_2(\tau_j, \hat{\tau}^{(2)}_j)$ and $\lambda_2(\tau_27, \hat{\tau}^{(2)}_{27})$ from 1000 trials using the same parameters as setup 2 but employing WBinSeg instead of BinSeg.

Using WBinSeg along with steps (D1) and (D2), the event $\{\hat{J} = J\}$ also occurred over 99% of the time during simulations. One can also see from Figure 25 that the distribution of the $\hat{\tau}^{(2)}_j$’s again match with Theorem 5. Because the performance of intelligent sampling is near identical for this setup, regardless of whether BinSeg or WBinSeg was used, the reader may wonder why the latter isn’t used for the previous few simulations. The reason is the following: when the re-fitting method from Section 3.4 is implemented, it results in the second stage intervals being of width $Q_{1-\alpha/J}(1)N^{1-\gamma}$, irrespective of which of BinSeg or WBinSeg was used at stage one [where $Q_{1-\alpha/J}(1)$ is the $1 - \alpha/J$ quantile of $|L(N)|$]. Hence, WBinSeg loses any possible advantage from the tighter confidence bound of width $O(\log(N))$ rather than $O(E_N)$ for BinSeg from stage one. So, in a sparse change point setting and with stage 1 refitting, WBinSeg provides no accuracy advantages but adds to the computational time, e.g., the 1000 iterations used to create Figure 25 averaged $\approx 293$ seconds, while the iterations used to create Figure 13 averaged $\approx 7$ seconds.

12. Appendix D (Discussion of Extensions)

12.1. Extensions of Single Change Point Model
12.1.1. Intelligent Sampling with > 2 Steps

We can extend our 2-stage procedure by adding in more stages. In the 2-stage version, we first use a subsample of size $N^\gamma$ to find some interval $[\hat{\tau}^{(1)} - K_1 N^{-\gamma+\delta_1}, \hat{\tau}^{(1)} + K_1 N^{-\gamma+\delta_1}]$ which contains the true value of $\tau$ with probability going to 1. At the second stage, we do not necessarily have to use all the data
contained within this interval. We can, instead, apply a two stage procedure on this interval as well, that is, take a subset of $N^\zeta$ (for some $0 < \zeta < 1 - \gamma + \delta_1$) points from the second stage interval for estimation. From this, we obtain a second stage estimate $\hat{\tau}^{(2)}$ and an interval $[\hat{\tau}^{(2)} - K_2 N^{-\gamma - \zeta + \delta_1 + \delta_2}, \hat{\tau}^{(2)} + K_2 N^{-\gamma - \zeta + \delta_1 + \delta_2}]$ (note that $\delta_1$ and $\delta_2$ can be as small as one pleases) which contains $\tau$ with probability going to 1. In the third stage, we finally take all points in the aforementioned interval (leaving aside those used in previous stages) to obtain an estimate $\hat{\tau}^{(3)}$.

Such a procedure will have the same rate of convergence as the one using the full data: $(\hat{\tau}^{(3)} - \tau) = O_p(1/N)$, and the same asymptotic distribution (in terms of a "third stage distance" similar to how $\lambda_2$ was defined) as the one and two stage procedures. In terms of computational time, the first stage takes $O(N^\tau)$ time, the second stage $O(N^\zeta)$ time, and the final stage $O(N^{1-\gamma - \zeta + \delta_1 + \delta_2})$ time, for a total of $O((N^{\gamma} \vee N^{\zeta} \vee N^{1-\gamma - \zeta + \delta_1 + \delta_2})$ time, which can reach almost $O(N^{1/3})$ time.

In general, we can use a $k$ stage procedure which can operate in almost as low as $O(N^{1/k})$ time. A possibly interesting theoretical question is letting $k = k(N)$ increase with $N$ and at each stage taking approximately $O(N^{1/k(N)})$ points. Such a scheme may let the computational time dip as low as the $\log(N)$ regime. Heuristically, consider the very extreme example where at each stage of intelligent sampling we take only a constant number of points at each stage (i.e. taking the order of $N^{1/k(N)} \rightarrow 1$ subsample at each stage), say 100 points. At each stage we evenly take a subsample of 100 points, fit a stump model, and construct a confidence interval for the true change point. Then we take 100 points from the confidence interval and repeat. At around the $\log_{100}(N)$’s step, the resolution of the subsample would be the same as the resolution of the data and intelligent sampling terminates. Each stage involved estimating with only 100 points, an operation bounded above in running time by a constant, and hence the whole process takes $O(\log_{100}(N))$ time. Although this example would need to be modified to make the final estimator consistent, it motivates $O(\log(N))$ running time as the lower limit to what we can achieve with intelligent sampling.

In summary, the use of an increasing number of stages for detecting a single change point is an interesting extension of intelligent sampling. However in this work we are mainly concerned with the problem of detecting multiple change points, and as we shall elaborate in Section [12.2.1] there are practical issues with using more than 2 stages of intelligent sampling to the more general model concerning multiple change points. Hence, we conclude this topic as it is more fitting to study this extension in future work.

### 12.2. Multiple Change Point Problem

#### 12.2.1. Multiple Stage Intelligent Sampling

We can also consider intelligent sampling with multiple (> 2) stages of estimation for model (4). An $m$-stage intelligent sampling procedure would entail:

(a) Take a uniform subsample $Y_{[N/N_1]}, Y_{[2N/N_1]}, Y_{[3N/N_1]}, \ldots$, where $N_1 = KN^\gamma$ for some $K > 1, \gamma \in (0, 1)$, to obtain estimates $\hat{J}, \hat{\tau}_1^{(1)}, \ldots, \hat{\tau}_J^{(1)}$, and confidence intervals $[\hat{\tau}_j^{(1)} - w(N), \hat{\tau}_j^{(1)} + w(N)]$, $1 \leq j \leq \hat{J}$, for the change points.

(b) On each interval $[\hat{\tau}_j^{(1)} - w(N), \hat{\tau}_j^{(1)} + w(N)]$ for $1 \leq j \leq \hat{J}$ perform the $(m - 1)$ stage intelligent sampling procedure for the single change point.

The final estimators $\hat{\tau}_j^{(m)}$ for $j = 1, \ldots, \hat{J}$ will display similar asymptotic behavior by the two-stage estimators in Theorems as recorded in [4] and [5]. This is, again, under the assumption that the first stage estimators satisfy condition (12).

Computationally, an $m$ stage procedure would be faster than a two stage procedure, which can be immediately seen by the fact that using intelligent sampling to fit a stump model on each $[\hat{\tau}_j^{(1)} - w(N), \hat{\tau}_j^{(1)} + w(N)]$ is faster than using virtually every point inside this interval for the same purpose. In fact, this second step would require an average time that is almost $O(J w(N)^{1/(m-1)})$, which goes down with $m$. As a specific example, suppose we have $\delta_N/N^{1-\Xi}$ converges to a constant for some $\Xi \in (3/4, 1)$, then the first step would take $O(N^\gamma \log(N))$ time if BinSeg is used, and if $\gamma$ was chosen to be greater than $4\Xi$ per condition (M7

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(BinSeg)), then we can set the confidence interval widths as \( w(N) = N^{1-\gamma+2\Xi} \log(N) \). The second step, involving stages 2 to \( m \), has an average time which can be as low as \( O(Jw(N)^{1/(m-1)}) \leq O(N^{\Xi+1/(4m^{\Xi})} \log(N)) \) time. Thus for \( m \) large enough such that \( \frac{1-\gamma+2\Xi}{m} < 3\Xi \), the first stage will always dominate the other stages, combined, in terms of computational time, and the whole procedure can be accomplished in almost as low as \( O(N^{\Xi} \log(N)) \) time by setting \( \gamma \) close to \( 4\Xi \). This is uniformly less or equal to the optimal computational time derived in Section 11.1 for BinSeg, and a similar argument applied to WBinSeg would result in an average total computational time almost as low as \( O(N^{3\Xi} \log(N)) \) which also uniformly less than equal to the optimal computational time for a two stage procedure.

Although, asymptotically, more stages imply a faster procedure, whether to use more stages in practice depends on how sparsely one can afford to subsample at stage 1. For smaller \( N \), and configurations where change-points are closer, or are associated with smaller SNR, larger \( N_1 \) relative to \( N \) is preferred. Consequently, \( N/N_1 \) is small and the confidence intervals produced from the first stage data are too narrow to warrant a multi-stage procedure on each. For example, when we used data of length on the order of \( 10^7 \) in Section 11.1 we had second stage intervals that contained a few thousand points each. Fitting a stump model to all points in such an interval would take a small fraction of a second and there is little reason to use more stages. On larger data sets with sparse change points, \( N/N_1 \) can be larger and using more than two stages will start to show significant gains. In Section 11.1 we deduced that for \( N = 1.5 \times 10^{10} \) and around 1000 change points, we can work with subsamples around 5 times smaller by moving from a 2 stage procedure to a 3 stage procedure. As a second example, for the even larger values of \( N = 1.5 \times 10^{12} \) with 2000 change points, using a four stage procedure allows us to work with data on the order of \( 10^6 \), whereas using few stages would force us to work with substantially larger datasets that may not be easy to handle on many computational platforms.

12.2.2. Modeling Weak Jumps

Condition (M2) in the multiple change point model (the condition that jumps are bounded from below by a constant \( \Delta \)) is motivated from the angle that only those change-points that show substantially large jumps count, while (M3) ensures substantial separation between consecutive change-points. However, a more thorough treatment would be to consider a model where in addition to change-points of the above types, some jumps can decrease to 0 and others can be short-lived and spiky. As before, consider (12) satisfying (M1) and (M4), with change points \( \tau_1, \ldots, \tau_M \) (this time not necessarily in order), where \( M = M(N) \) can possibly increase to \( \infty \). Suppose for some \( J = J(N) \leq M(N) \) and \( K = K(N) \leq M(N) - J(N) \), there are \( J(N) \) change points \( \tau_1, \ldots, \tau_J \) which satisfy

\[
\min_{j=1,\ldots,J} \left( \min_{i=1,\ldots,K} |\tau_j - \tau_i| \right) \geq \delta_N
\]

\[
\min_{j=1,\ldots,J} |\theta_{\tau_{j+1}} - \theta_{\tau_j}| \geq \Delta
\]

for some sequence \( \delta_N \to \infty \) and constant \( \Delta \); there are \( K(N) \) change points \( \tau_{J+1}, \ldots, \tau_{J+K} \) which satisfy

\[
\min_{j=J+1,\ldots,J+K} \left( \min_{i=1,\ldots,J+K} |\tau_j - \tau_i| \right) \geq \delta_N^J
\]

\[
\max_{j=J+1,\ldots,J+K} |\theta_{\tau_{j+1}} - \theta_{\tau_j}| \leq \Delta_N \text{ where } \Delta_N \to 0;
\]

for some sequence \( \delta_N^J \to \infty \); and finally, the remaining \( M - J - K \) change points, \( \tau_{J+K+1}, \ldots, \tau_M \), are short-lasting in terms of how they affect the signal: there is a constant \( \delta_{\max} \) not dependent on \( N \) such that

\[
\text{for any } J + K < i < j \leq M, \text{ either } |\tau_j - \tau_i| \leq \delta_{\max} \text{ or } |\tau_j - \tau_i| \geq \delta_N^J
\]

\[
\theta_{\tau_j+\delta_{\max}} = \theta_{\tau_j-\delta_{\max}} \text{ for } j = J + K + 1, \ldots, M.
\]

for some \( \delta_{\max}^J \to \infty \). Regarding inferential goals, we are certainly interested in estimating for \( \hat{\tau}_1, \ldots, \hat{\tau}_J \) to start with. We may also want to estimate the locations \( \tau_{J+1}, \ldots, \tau_{J+K} \), but we will not be able to estimate \( \tau_{J+K}, \ldots, \tau_M \). As the change points \( \tau_j \) for \( J + K < j \leq M \) are located within discrete intervals of widths at most \( 2\delta_{\max} + 1 \) each, the sparse subsampling at stage 1 will eventually pick up at most 1 point from each of these intervals, precluding estimation. The only way we can pick up these change points is at stage 2,
when some of them happen to fall inside one of the second stage subsampling intervals. Even then, fitting a stump model to these data intervals of growing size would eventually ignore these fluctuations of finite length so long as $\delta_N^j$ does not grow too slowly (and hence there not too many spikes in this interval). Therefore, we can only hope to detect change points $\tau_j$ for $1 \leq j \leq J + K$, and whether we estimate $\tau_{J+1}, \ldots, \tau_{J+K}$ will determine our strategy.

Suppose we are only interested in the $\tau_1, \ldots, \tau_J$ and do not care for the other change points. Then, once we have captured $\tau_1, \ldots, \tau_j$’s with confidence intervals from a subsample, we can update these estimates at stage two in the usual manner. Specifically, if from a subsample $Y_{[N/N_1]}, Y_{2[N/N_1]}, \ldots$ we obtain change point estimates $\hat{\tau}_1^{(1)}, \ldots, \hat{\tau}_R^{(1)}$ such that

$$\mathbb{P} \left[ \bigcup_{j=1}^J \left\{ |\hat{\tau}_k^{(1)} - \tau_j| \leq w(N) \text{ for some } k \in \{1, \ldots, R\} \right\} \right] \rightarrow 1, \tag{171}$$

for some sequence $w(N)$ that does not increase too rapidly, then a stump model fit on each confidence interval would recover the location of $\tau_1, \ldots, \tau_J$ with rates and asymptotic distributions similar to those found in Theorems 4 and 5. This, of course, assumes sufficient conditions on the minimum separation $\delta_N = \min_{1 \leq j, k \leq K} |\tau_j - \tau_k|$, the maximum of the decreasing jump sizes $\Delta_N$, and potentially other parameter values, which would need to be fleshed out for a rigorous development. In finding such sufficient conditions, one needs to take into consideration the accuracy of fitting a stump model within a confidence interval of the form $[\hat{\tau}_j^{(1)} - w(N), \hat{\tau}_j^{(1)} + w(N)]$ when it contains not only a single big change point, but potentially also smaller change points from $\{\tau_j+1, \ldots, \tau_{J+K}\}$ if we consider the regimes where $\delta_N^j \ll \delta_N$. Secondly, we have to consider the set of sufficient conditions under which the first stage procedure recovers only the significant change points $\tau_1, \ldots, \tau_J$ with probability going to 1, but this needs further study into BinSeg or WBinSeg (or some other multiple change point estimating procedure), rather than intelligent sampling.

The second approach deals with the case where we want to estimate every change point $\tau_1, \ldots, \tau_J, \ldots, \tau_{J+K}$. In this case, we require confidence intervals from a stage 1 subsample for all $\tau_j$ from $1 \leq j \leq J + K$, and results from [7] can be brought into play, supposing either BinSeg or WBinSeg were applied to the full dataset,

$$\mathbb{P} \left[ R = J + K, \max_{j=1, \ldots, J+K} \left( \min_{i=1, \ldots, R} |\hat{\tau}_i - \tau_j| \right) \leq w(N) \right] \rightarrow 1 \tag{172}$$

under appropriate conditions on $\delta_N$ and $\Delta_N$, for a sequence $w(N) = o(\delta_N)$. Firstly, this provides us with assurance that on a first stage subsample we can obtain consistent estimates for all change points, and secondly, guarantees that the second stage intervals will each contain at most one change point. What remains now is to show that the second stage estimates of $\tau_{J+1}, \ldots, \tau_{J+K}$ are consistent as well. Overall, this approach is easier than the previous one, but requires more stringent conditions. We point out that in order to obtain (172), $\delta_N$ and $\delta_N'$ would need to be larger than $\delta_N$ for the case when none of the jumps tends to 0. This would entail stronger conditions than (M5 (BinSeg)), (M7 (BinSeg)), and (M8 (WBinSeg)) in order for intelligent sampling to work as a whole. This is where the previous approach has an advantage: we may not require stronger conditions if we are simply interested in estimating $\tau_1, \ldots, \tau_J$. In conclusion, the first approach requires more mathematical verification and model building, but could be more flexible in terms of the conditions required, while the second approach already has much of its foundation worked out, but appears to require more stringent conditions than what we have now.

13. **Appendix E (Miscellaneous)**

13.1. **Finite $J$ Assumption**

First we will need to re-frame the result given in Theorem 5 since even if ever $\lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right)$ marginally converge to the previously described distribution $L_j$, for a growing $J(N)$ we would have to re-interpret convergence to $(L_1, \ldots, L_J)$, itself changing with $N$. Instead we consider joint quantiles which would come into play in a confidence interval setting: for any $\alpha \in (0,1)$ let $q_{J(N)}(1, \alpha), \ldots, q_{J(N)}(J, \alpha)$ be the $\sqrt{\alpha}/\sqrt{1 - \alpha}$
quantiles of $|L_1|, \ldots, |L_J|$, so that
\[
\mathbb{P} \left[ \bigcap_{j=1}^J \{ |L_j| \leq q_{J(N)}(j, \alpha) \} \right] = 1 - \alpha, \tag{173}
\]
then we wish to show that
\[
\mathbb{P} \left[ \hat{j} = J \text{ and } \bigcap_{j=1}^J \left\{ \left| \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) \right| \leq q_{J(N)}(j, \alpha) \right\} \right] \to 1 - \alpha \tag{174}
\]
for all $\alpha$. To motivate the form of this desired result, we note that if for every $N$, every $L_j$ has the same distribution and each $q_{J(N)}(j, \alpha)$ are equal to a common value $q_{J(N)}(\alpha)$, then expression (174) reduces to
\[
\mathbb{P} \left[ \hat{j} = J \text{ and } \max_{j=1, \ldots, J} \left| \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) \right| \leq q_{J(N)}(\alpha) \right] \to 1 - \alpha \tag{175}
\]
an expression not only ties in to extreme value theory as a convergence of maximums, but also bears similarity to Theorem 3.

For simplicity let’s assume that for all sufficiently large $N$ we have
\[
\mathbb{P} \left[ \hat{j} = J \text{ and } \bigcap_{j=1}^J \left\{ \left| \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) \right| \leq q_{J(N)}(j, \alpha) \right\} \right] = \left( \prod_{j=1}^J \mathbb{P} \left[ \hat{j} = J \text{ and } \left| \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) \right| \leq q_{J(N)}(j, \alpha) \right] \right) + o(1). \tag{176}
\]
Although this requires effort to show, this is a reasonable assumption given the results of Theorem 5. Now denote, for $j = 1, \ldots, J(N)$,
\[
\Delta_{j,N}^{(P)} := \mathbb{P} \left[ \hat{j} = J \text{ and } \left| \lambda_d \left( \tau_j, \hat{\tau}_j^{(2)} \right) \right| \leq q_{J(N)}(j, \alpha) \right] - \frac{j(N)}{\sqrt{1 - \alpha}}. \tag{177}
\]
Then in order for (174) to be satisfied, this requires
\[
o(1) = 1 - \alpha - \prod_{j=1}^J \left( \sqrt{1 - \alpha} + \Delta_{j,N}^{(P)} \right) = (1 - \alpha) \left( 1 - \prod_{j=1}^J \left( 1 + \frac{\Delta_{j,N}^{(P)}}{\sqrt{1 - \alpha}} \right) \right) = (1 - \alpha) \left( 1 - \exp \left( \sum_{j=1}^J \log \left( 1 + \frac{\Delta_{j,N}^{(P)}}{\sqrt{1 - \alpha}} \right) \right) \right) \tag{178} \]
which is equivalent to the requirement that
\[
\sum_{j=1}^{J(N)} \log \left( 1 + \frac{\Delta_{j,N}^{(P)}}{\sqrt{1 - \alpha}} \right) \to 0. \tag{179}
\]
Hence we must show that the probability deviations $\Delta_{1,N}^{(P)}, \ldots, \Delta_{J,N}^{(P)}$ not only does all have to converge to 0, but in such a way such that the above summation also converge to 0.

Now we apply this informal result. As one can see in the proofs for Theorem 2 and 5 to show the convergence in distribution of the $\lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right)$’s we re-write it as
\[
\lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) = \arg \min_{k \in \mathbb{Z}} W_{j,K}(k) \tag{180}
\]
where for \( j = 1, ..., J \), \( W_{\sum N}(\cdot) \) are each a stochastic process that can be expressed in terms of the dataset \( Y = (Y_1, ..., Y_N) \). From here it can be demonstrated that the random walks \( W_{\sum N}(\cdot) \) for \( j = 1, ..., J \) in a sense converges in distribution to the random walks \( Z_{\sum 1}, ..., Z_{\sum J} \) as denoted from Theorem 5. By continuous mapping, the random variables \( \lambda_2 \left( \tau_j, \hat{\tau}_j^{(2)} \right) \) converge to the argmax of the \( Z_{\sum j} \)'s. Hence to derive results for a \( J(N) \to \infty \) setting, we need to show the probability differences

\[
P \left[ \hat{J} = J \quad \text{and} \quad \arg \min_{k \in \mathbb{Z}} W_{\sum N}(k) \leq q_{J(N)}(j, \alpha) \right] - P \left[ \arg \min_{i \in \mathbb{Z}} Z_{i,j} \leq q_{J(N)}(j, \alpha) \right]
\]  

(181)

all converge to 0 fast enough to satisfy (174). The difficulty comes in here: although the speed of convergence of the \( W_{\sum N}(\cdot) \)'s to \( Z_{\sum j} \)'s can be analyzed, the speed of convergence of their argmins is more difficult to handle.

In summary, while it may be possible to extend the results derived in Theorem 5 to a setting where \( J(N) \) grows without bound, to do so would require deriving results for the rate of convergence of probabilities associated with the argmins of random walks. Because of such difficulties we only derived results assuming condition (M8), but we surmise such a condition is not strictly necessary.