Mending the Big-Data Missing Information

Extended Abstract

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Abstract. Consider a high-dimensional data set, such that for every data-point there is incomplete information. Each object in the data set represents a real entity, which models as a point in high-dimensional space. We assume that all real entities are embedded in the same space, which means they have the same dimension. We model the lack of information for a given object as affine subspace in $\mathbb{R}^d$ with dimension $k$.

Our goal in this paper is to find clusters of objects. The main problem is to cope with the partial information. We studied a simple algorithm we call Data clustering using flats minimum distances, using the following assumptions: 1) There are $m$ clusters. 2) Each cluster is modeled as a ball in $\mathbb{R}^d$. 3) Each cluster contains a $\frac{n}{m}$, $k$-dimensional affine subspaces. 4) All $k$-dimensional affine subspaces, which belong in the same cluster, are intersected with the ball of the cluster. 5) Each $k$-dimensional affine subspace, that belong to a cluster, is selected uniformly among all $k$-dimensional affine subspaces that intersect the ball’s cluster. A data set that satisfy these assumptions will be called separable data.

Our suggested algorithm calculates pair-wise projection of the data. We use probabilistic considerations to prove the algorithm correctness. These probabilistic results are of independent interest, as can serve to better understand the geometry of high dimensional objects.

Keywords: big-data clustering, high-dimensional data, affine subspace distance,

1 Introduction

One of the main challenges that arise while handling Big-Data is not only the large volume, but also the high-dimensions of the data. Moreover, part of the information at the different dimensions may be missing. Assuming that the true (unknown) data is

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$d$-dimensional points, we suggest to represent the given data point (which may lack information at different dimensions) by $k$-affine space embedded in the Euclidean $d$ dimensional space $\mathbb{R}^d$. Denote the affine-Grassmannian set of all $k$-affine space, embedded in the Euclidean $d$ dimensional space, by $A(d,k)$. This means that a point in our data set is a point in the affine-Grassmannian $A(d,k)$.

A data object, that is incomplete in one or more features, corresponds to an affine subspace (called flat, for short) in $\mathbb{R}^d$, whose dimension is the number of missing features. This representation yields algebraic objects, which help us to better understand the data, as well as studying its properties. A central property of the data is clustering. Clustering refers to the process of partitioning a set of objects into subsets, consisting of similar objects. Finding a good cluster is a challenging problem. Due to its wide range of applications, the clustering problem has been investigated for decades, and continues to be actively studied not only in theoretical computer science, but in other disciplines, such as statistics, data mining and machine learning. A motivation for cluster analysis of high-dimensional data, as well as an overview on some applications where high-dimensional data occur, is given in [?].

Our underlying assumption is that the original data-points, the real entities, can be divided into different groups according to their distance in the $\mathbb{R}^d$. We assume that every group of points lie in the same $d$ dimensional ball (a.k.a. a solid sphere), since the distance between a flat and a point (the center of the ball) is well-defined. The classical clustering problems, such as $k$-means or $k$-centers (see [?] Chapter 8), can be defined on a set of flats. The clustering problem when the data is $k$-flats is to find the centers of the balls, that minimizes the sum of the distance between the $k$-flats and the center of their groups, which is the nearest center among all centers.

However, Lee & Schulman [?] argues that the running time of an approximation algorithm, with any approximation ratio, cannot be polynomial in even one of $m$ (the number of clusters) and $k$ (the dimension of the flats), unless $P = NP$. We overcome this obstacle by approaching the problem differently. Using a probabilistic assumption, based on the distribution of the data, we achieve a polynomial algorithm, which we use to identify the flats’ groups. Moreover, the presented probability arguments can help us in better understanding the geometric distribution of high dimensional data objects, which is of major interest and importance in the scope of Big Data research.

**Related works.**

The probability of flats intersections is appear at different setting in [?] and [?]. Using polar representation [?] measure the probability that $d$ $k$-flats going through a ball, will intersect each other inside the ball. E.g., for $d = 2$ and $k = 1$, random lines intersecting a disk will intersect each other inside the disk with probability $1/2$ and for $d = 3$ and $k = 2$, three planes that intersecting a convex figure $K$ will have their common point inside $K$ with probability $\pi^2/48$. This results are generalized in [?] for $n$ randomly chosen subspaces $f_k$ ($i = 1, 2, ..., n$) in $\mathbb{R}^d$ such that $k_1 + k_2 + ... + k_n \geq (n-1)d$, that intersect a convex body $K$. Formalized the probability that $f_{k_1} \cap f_{k_2} \cap ... \cap L_{k_n} \cap K \neq \emptyset$ using the integral: 

$$\int_{f_{k_1} \cap f_{k_2} \cap ... \cap L_{k_n}} d f_{k_1} \cap f_{k_2} \cap ... \cap d f_{k_n}$$

(13.39),(14.2) show that the measure of all $k$-flats $f_k$ that intersect a convex body $K$ in $\mathbb{R}^d$ is

$$\frac{O_{d-1} - O_{d-2}}{\sqrt{d}d_{d-2} - d_{d-1}}$$

where
\(O_d\) denotes the surface area of the \(d\)-dimensional unit sphere). Another related result one can extract from \([?]\) work is the probability of a hyperplane \(L_{d-1}\) and a line \(L_1\) that intersect a ball to have an intersection inside the ball, which equals \(1/d\).

However, the polar representation (i.e., the coordinates consist of a radial coordinate and \(d-1\) angular coordinates) gives high weights to the first coordinates where the weight of the follows decrease (since the coefficients are multiples of sine and cosine). Hence our assumption on normal distribution over the different coordinates is not fulfill.

Canas et al \([?]\) study the problem of estimating a manifold from random \(k\)-flats. Given collections of \(k\)-flats in \(\mathbb{R}^d\) their (Lloyd-type) algorithm, analogously to \(k\)-means, aims at finding the set of \(k\)-flats that minimizes an empirical reconstruction over the whole collection. Although they also deal with the input of \(k\)-flat, their framework and goals is different from ours, and specifically, impractical for the clustering task.

The distance between pairs of \(k\)-flats as well as measuring the geometry of the midpoints was studied in \([?]\) and generalized at \([?]\). Although these papers considering the probabilistic aspects of flats intersection as we do, they focus only on stationary processes (such as Poisson processes) that does not suffice the uniform and Gaussian distributions we assume here.

As mention above, Lee & Schulman \([?]\) presented algorithms and hardness results for clustering general \(k\)-flats in \(\mathbb{R}^d\). After proving that the exponential dependence on \(k\) (the internal dimension of the flat) and \(m\) (the number of clusters) is inevitable they suggest an algorithm which runs in time exponential in \(k\) and \(m\) but is linear in \(n\) and \(d\).

Their theoretical results is based on the assumption that the flats are axis-parallel. Our model overcomes their exponential bound due to the randomized assumption that does not allow the axis-parallel case.

**Our contributions.**

We face the challenge of mending the missing information at the different dimension, by representing the objects as affine subspaces. In particular, we work in the framework of flat in \(\mathbb{R}^d\), where the missing features correspond to the (intrinsic) dimension of the flat. This representation is accurate and flexible, in the sense that it saves all the features of the origin data, as well as allowing algebraic calculation over the objects. In this paper we study the pairwise distance between the flats, and based on our probabilistic and geometrical results, we developed polynomial algorithm that achieve clustering of the flats with high probability.

The main result of the paper summarized in the following theorem, while the precise definition and the detailed proof are presented in the sequel.

**Theorem 1** Given the separable data set \(\mathbf{P}\) of \(n\) affine subspaces in \(\mathbb{R}^d\), for any \(\epsilon > 0\) and for sufficiently large \(d\) (depending on \(\epsilon\)), with probability \(1 - \epsilon\), we can cluster \(\mathbf{P}\) according to \(\mathbb{F}^d\), using their pair-wise distance projection in \(\text{poly}(n,k,d)\) time.

Although in the scope of big-data the goal is to achieve good performance for high dimensions, as we proved that our algorithm does, we also show that the algorithm works well for low dimensions. In addition, we claim that we can relax the model assumption to any size of cluster, and even to achieve a poly-logarithmic running time.
To enhance the readability of our text, Section 2 contains the basic notions, from convex and stochastic geometry, which are needed in the following. In particular, we recall the notion of flats and provide the model assumptions. In Section 3 we prove our main result, summarized in Theorem 1 and Algorithm 1. We supplement our theoretical results with experimental data in Section 4. Finally, in Section 5 we discuss the geometric and algebraic representation, comparing our approach against others proposals. For the sake of readability, assurance and space limitation; some of the proofs are given only as a sketch.

2 Preliminaries

General notation.

Throughout the following, we work in $d$-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Hence, $\| x - y \|$ is the Euclidean distance of two points $x, y \in \mathbb{R}^d$, and $\text{dist}(X, Y) := \inf \{ \| x - y \| : x \in X, y \in Y \}$ is the distance of two sets $X, Y \subseteq \mathbb{R}^d$. We refer to any set $S \subseteq X$, which is closest to $Y$, i.e., satisfies $\| Y - S \| = \text{dist}(X, Y)$, as a projection of $Y$ on $X$. In general, there can be more than one projection of $Y$ on $X$, i.e., several subsets in $Y$ closest to $X$.

Grassmannians.

For $k \in \{1, \ldots, d\}$, we denote by $G(d, k)$ and $A(d, k)$ the spaces of $k$-dimensional linear and affine subspaces of $\mathbb{R}^d$, respectively, both supplied with their natural topologies (see e.g., [?]). The elements of $A(d, k)$ are also called $k$-flats (for $k = 0$, points; for $k = 1$, lines; for $k = 2$, planes; and for $k = d - 1$, hyperplanes). Recall that two subspaces $L \in G(d, k_1)$ and $M \in G(d, k_2)$ are said to be in general position if the linear hull of $L \cup M$ has dimension $k_1 + k_2$ if $k_1 + k_2 < d$ or if $L \cap M$ has dimension $k_1 + k_2 - d$ if $k_1 + k_2 \geq n$. We also say that two flats $E \in A(d, k_1)$ and $F \in A(d, k_2)$ are in general position, if this is the case for $L(E)$ and $L(F)$, where $L(E)$ is the linear subspace parallel to $E$.

Geometric and Probabilistic definitions.

Let $P = \{P_1, P_2, \ldots, P_n\}$ be the set of $n$ random flats that we want to cluster. For simplicity, we consider the situation where all of them are of dimension $k$, where $k$ is taken to be the greatest dimension of any flat at $P$. Hence, every flat $P$ is represented by a set of $d - k$ linear equations, each with $d$ variables. Alternatively, we can represent any $k$-flat using parametric notation, such that $P$ is given by a set of $d$ linear equations, each with $d - k$ variables.

Call the $j$th coordinate trivial when there is no flat with fixed $j$th coordinate. We can assume that no coordinate is trivial, since, otherwise, simply removing this coordinate from all flats will decrease $k$ and $d$ by 1, while not affecting the clustering cost.

For $c \in \mathbb{R}^d$, let $\mathbb{B}_c^d$ be the unit ball of dimension $d$, centered at $c$ and $\mathbb{B}_0^d$ denote the unit ball centered at the origin. Two balls, $\mathbb{B}_{c_i}$ and $\mathbb{B}_{c_j}$, are $\Delta$-distinct if $\text{dist}(c_i, c_j) \geq \Delta$. The ball $\mathbb{B}_c^d$ intersects the subset of flats $P = \{P_1, \ldots, P_n\}$ if it intersects each flat in $P$. 
Denote by \( P^i \in \mathbb{P} \) a \( k \)-flat intersecting the unit ball \( \mathbb{B}^d \). Denote by \( P_i(r) \in \mathbb{P} \) a \( k \)-flat in \( \mathbb{R}^d \) passing through the point \((r, 0, \ldots, 0)\). E.g., for the \( k = 1 \) (line in \( \mathbb{R}^d \)) \( P_i(r) = (\alpha_1 t_1 + r, \alpha_2 t_2, \ldots, \alpha_d t_d) \) where \( \alpha_j \sim \mathcal{N}(0, \sigma) \).

Let \( \ast \) be an equivalence relation such that for a point \( u \in \mathbb{B}^d \), \( u^\ast \) is the antipodal point of \( u \) (i.e., \( u \) and \( u^\ast \) are opposite through the center \( c \)). For a \( k \)-flat \( P^c \) intersecting the unit ball \( \mathbb{B}^d \) in one point only (i.e., tangent to the balls surface), \( P^c \) denotes its antipodal \( k \)-flat.

If \( E \) and \( F \) are in general position, there are unique points \( x_E \in E \) and \( x_F \in F \), so that \( \text{dist}(E, F) = ||x_E - x_F|| \). We call the point \( p = \text{midpoint}(E, F) := (x_E + x_F) / 2 \) the midpoint of \( E \) and \( F \).

The probability, expectation and variance; will be denoted by the common notations \( \Pr(\cdot), E(\cdot) \) and \( V(\cdot) \) respectively. For a random variable \( A \) dependent on \( d \), we denote by \( A \to p \) the converges in probability, namely, \( \forall \varepsilon, \lim_{d \to \infty} \Pr(||A - c|| \leq \varepsilon) = 1. \)

### Model assumptions.

Throughout the paper we assume that the data is separable, namely, satisfy the following assumptions:

- Two independent random flats \( E, F \in A(d, k) \), with distribution \( \mathbb{Q} \), are in general position with probability one.
- \( 1 \leq k \leq \lfloor d/2 \rfloor \) which ensures that the flats do not intersect each other with probability one.
- The (unknown) balls \( \mathbb{B}^d, \ldots, \mathbb{B}^d \) are \( \Delta \)-distinct with probability one.
- The given flats set \( \mathbb{P} \) is a superset of \( m \) groups \( \mathbb{P} = \{P_1, \ldots, P_m\} \), such that every group \( P_i \in \mathbb{P} \) contains \( n/m \) flats that intersect the ball \( \mathbb{B}^d \). Moreover, each flat \( P \in P_i \) have normally distributed location and direction at the ball \( \mathbb{B}^d \). We model this assumption by normally distributed coefficients. The parametric representation of a \( k \)-flat \( P \) is:

\[
P = (\alpha_{0,1} t_1 + \alpha_{1,1} t_1 + \alpha_{2,1} t_2 + \ldots + \alpha_{k,1} t_k, \ldots, \alpha_{0,d} + \alpha_{1,d} t_1 + \alpha_{2,d} t_2 + \ldots + \alpha_{k,d} t_k)
\]

where \( \alpha_j \sim \mathcal{N}(\mu, \sigma) \).

### 3 \( k \)-flats Clustering

Given the set \( \mathbb{P} \) of \( n \) \( k \)-flats in \( \mathbb{R}^d \), we start our procedure by calculating pair-wise projection of \( \mathbb{P} \), namely, find the distance and the midpoint between every pair in \( \mathbb{P} \). Let \( P_1 = \{x \in \mathbb{R}^d : Ex = c\} \) and \( P_2 = \{y \in \mathbb{R}^d : Fy = f\} \) be a pair of \( k \)-flats in \( \mathbb{P} \). Note that the matrices dimensions \( \text{Dim}(E) = \text{Dim}(F) = (d - k) \times d \) since each flat \( P \in \mathbb{P} \) is represented by \( d - k \) equations with \( d \) variables. The suggested algorithm calculate the minimum distance points (i.e., midpoint) between the pair using Euclidean norm minimization:

\[
\text{minimize } ||Ax - b||
\]
where \( A = \begin{pmatrix} E \\ F \end{pmatrix}, x = (x_1, \ldots, x_d), b = \begin{pmatrix} e \\ f \end{pmatrix} \).

Since the norm is always nonnegative, we can just as well solve the least squares problem

\[
\text{minimize} \|Ax - b\|^2
\]

(2)

The problems are clearly equivalent, while the objective in the first one is not differentiable at any \( x \) with \( Ax - b = 0 \), whereas the objective in the second is differentiable for all \( x \).

**Proposition 1** The least squares minimization (Eq. 2) gives unique solution \( p \) such that 

\[ p = \text{mid point} (P_i, P_j). \]

**Proof.** Using the equation

\[
\text{minimize} \|Ax - b\|^2 = (Ax - b)^T (Ax - b) = A^T x A x - 2b^T A x + b^T b
\]

this problem is simple enough to have a well known analytical solution - a point \( p \) minimizes the function \( f = A^T x A x - 2b^T A x + b^T b \) if and only if

\[
\nabla f = 2A^T A p - 2A^T b = 0
\]

i.e., if and only if \( p \) satisfies normal equations

\[
A^T A p = A^T b
\]

which always have a solution (note that the system is square or over-determiment since \( 2(d - k) \geq d \) for \( 1 \leq k \leq d/2 \)). The columns of \( A \) are the different coordinates of the two flats, hence they are independent, and has the unique solution:

\[ p = (A^T A)^{-1} A^T b. \]

\( \square \)

**Proposition 2** Using the midpoint \( p = \text{mid point} (P_i, P_j) \) one can find the distance between the two flats \( \text{dist} (P_i, P_j) \).

**Proof.** Theorem 1 at \( \square \) calculates the Euclidean distance between the two affine subspaces using the matrices range and null space. Alternatively, since we already have the midpoint \( p \) between the flats we can find the distance between them by projecting \( p \) onto the flats and then calculating the distance between the projected points. This projection can be made by a least squares method with constraints, more precisely, to solve the following two optimization problems:

\[
\text{min} \left\{ \|p - x\|^2 : Ex = e \right\}
\]

and

\[
\text{min} \left\{ \|p - x\|^2 : Fx = f \right\}
\]

or any other efficient orthogonal projection method (e.g. \( \square \)).

\( \square \)

Having the midpoint and the distance between all the pairs we filter the irrelevant midpoints using their corresponding distance as shown in the following Lemmas. First we argue that the flats’ pairwise projection emphasizes the origin balls, namely, the midpoints that arise from the same ball are centered around that ball:
Lemma 1 Let $P = \{P_1, P_2, \ldots, P_j\} \subseteq P$ be a set of $k$-flats in $\mathbb{R}^d$ intersecting the ball $\mathbb{B}^d_c$. Let $p = \{p_{12}, p_{13}, \ldots, p_{ij}, \ldots, p_{(j-1)j}\}$ be the set of the midpoints of all $\binom{j}{2}$ pairs of $P$. The mean of this set $E[p]$ equals to $c$ (the center of $\mathbb{B}^d_c$), and the variance $V[p]$ is bounded.

Proof. Let $P_i, P_j \in P$ be two flats intersecting the ball $\mathbb{B}^d_c$ where their distance midpoint is $p_{ij}$. Denote by $p_{ij}^*$ the antipodal point of $p_{ij}$. Since the directions and the location of flats at $P$ are normally distribution around $c$ (see the model assumptions at Section 2), we get that the probability that $p_{ij} \in p$ equals to the probability that $p_{ij}^* \in p$, which implies that their expected value $E[p_{ij}, p_{ij}^*]$ equals to $c$. This geometric-probabilistic consideration holds to the whole set $p$, hence, we get that $E[p] = c$.

For proofing that the variance is bounded we argue in propositions $\Box$ and $\Box$ that the distance $r_{ij}$ between $p_{ij}$ and the center of the ball $c$ is bounded, hence $p$'s variance is bounded around $c$. \hfill $\Box$

At this point, for every pair of flats $(P_i, P_j)$ we have the corresponding midpoint and the distance $(p_{ij}, d_{ij})$. We would like to show that if we eliminate all the midpoints $p_{ij}$ that their distance $d_{ij}$ is greater than 2, we are left with those that arise from the same cluster. The following Lemma argue that this is the case when $d$ is big enough:

Lemma 2 Let $P_i, P_j \in P$ be a pair of $k$-flats in $\mathbb{R}^d$.

1. If $P_i$ and $P_j$ intersecting the same ball $\mathbb{B}^d_c$ then the probability that the distance between them is less then 2 is $P(\text{dist}(P_i, P_j) \leq 2) = 1$.
2. Otherwise, for any $\varepsilon > 0$, $\lim_{d \to \infty} \Pr(\text{dist}(P_i, P_j) \geq 2(\Delta - \varepsilon)) = 1$.

Proof. When both flats are intersecting the same unit ball, the minimum distance between them is $\leq 2 \cdot \text{radius} (\mathbb{B}^d_c) = 2$ which implies the first part of the lemma. Applying Proposition $\Box$ with $\text{dist}(P_i, Q_i) \leq 2$ (by the first part of the Lemma), we get that for any $\varepsilon$ the distance between the two flats approach $2(\Delta - \varepsilon)$. \hfill $\Box$

Proposition 3 Let $P_i, Q_i$ and $R_i$ be flats intersecting the $\Delta$ – distinct balls $\mathbb{B}_{c_i}$ and $\mathbb{B}_{c_j}$ (respectively). Then, for any $\varepsilon > 0$, $\lim_{d \to \infty} \Pr(\text{dist}(R_i, Q_i) \geq (\Delta - \varepsilon) \text{dist}(P_i, Q_i)) = 1$.

Note: This proposition appears at [?] for random points. Here we reproduce a proof for the distance between the flats.

Proof. Let $\mu = E(\text{dist}(P_i, Q_i))$, $V = \frac{\text{dist}(P_i, Q_i)}{\mu}$ and $W = \frac{\text{dist}(R_i, Q_i)}{\mu}$. Using Lemma 1 and the weak law of large numbers we get that $V \to_p 1$. Proposition $\Box$ implies that $W \to_p \Delta$. Thus, $\frac{\text{dist}(R_i, Q_i)}{\text{dist}(P_i, Q_i)} = \frac{\mu \text{dist}(R_i, Q_i)}{\mu \text{dist}(P_i, Q_i)} = \frac{W}{V} \to_p \Delta$ (see Corollary 1 at [?]). By definition of convergence in probability for any $\varepsilon > 0$, $\lim_{d \to \infty} \Pr \left( \left| \frac{\text{dist}(R_i, Q_i)}{\text{dist}(P_i, Q_i)} - \Delta \right| \leq \varepsilon \right) = 1$. So

$\lim_{d \to \infty} \Pr \left( \Delta - \varepsilon \leq \frac{\text{dist}(R_i, Q_i)}{\text{dist}(P_i, Q_i)} \leq \Delta + \varepsilon \right) = 1$ which implies $\lim_{d \to \infty} \Pr(\text{dist}(R_i, Q_i) \geq (\Delta - \varepsilon) \text{dist}(P_i, Q_i)) = 1$. \hfill $\Box$
Proposition 4 Let $P_i$ and $P_j$ be flats intersecting the $\Delta$-distinct balls $B_{c_i}$ and $B_{c_j}$ (respectively), then $E(dist(P_i, P_j))$ is linear function of $\Delta$.

Proof. Denote the mean distance integral between two $k$-flats in $\mathbb{R}^d$ by $S = E(dist(P_i, P_j))$. Given that the probability density function of the flats is $\rho$, the expected value of the distance function, is given by the inner product of the functions $dist$ and $\rho$. E.g., for the $d$ dimensional lines $P(1) = (\alpha_1 t_1 + 1, \alpha_2 t_1, \ldots, \alpha_d t_1)$ and $P(-1) = (\beta_1 t_2 - 1, \beta_2 t_2, \ldots, \beta_d t_2)$ such that $\alpha_i, \beta_i \sim N(\mu, \sigma)$, the mean distance integral is

$$S = \int_{-\infty}^{\infty} dist(P(1), P(-1)) \rho(\alpha_1, \beta_1) d\alpha_1 d\alpha_2 \cdots d\alpha_d d\beta_1 \cdots d\beta_d$$

Let $S_0$ be the solution of the integral $S$ for two $k$-flats intersecting the unit ball $\mathbb{B}_{c_0}^d$ and $S_1$ be the solution of $S$ for two antipodals $k$-flats tangents the surface of $\mathbb{B}_{c_0}^d$. The following proposition holds:

Proposition 5 Let $S_0$ and $S_1$ be the mean distance integral solution as defined above, then $0 < S_0 < S_1 \leq 2$.

Proof. Since the degree of the flats is $\leq d/2$ the probability that the flats intersect is $= 0$ which implies that $0 < S_0$. The mean distance integral $S$ contains a density function $\rho(\mu, \sigma)$ and a geometric distance $dist(\cdot, \cdot)$. The density is dependent only on the mean and the variance of the coefficients which are invariant. The distance function get its maximum value for antipodal pair, which implies that $S_0 < S_1$. Finally, since the two flats intersecting the same unit ball, the minimum distance between them is $\leq 2$(the ball radius) $= 2$ which implies $S_1 \leq 2$ as needed.

Observing that $S_1$ is equals to any antipodal pair of flats that tangents to the surface of $\mathbb{B}_{c_0}^d$, w.l.o.g. we use the pair of flats $(P(-1), P(1))$. Denote by $S_1$ and $S_\Delta$ the solutions of the integral $S$ for the pairs $(P(-1), P(1))$ and $(P(-\Delta), P(\Delta))$, respectively, the following proposition argue that shifting the flats by $\Delta$ cause scaling the mean distance with $\Delta$:

Proposition 6 Let $S_1$ and $S_\Delta$ be the integral solutions as defined above, then $S_\Delta = S_1 \Delta$.

Proof. The mean distance of pair of $k$-flats act symmetrically on the two pairs $(P(-1), P(1))$ and $(P(-\Delta), P(\Delta))$. Namely, the density function is invariant while the distance scaling only in one direction, which implies a linear change in $\Delta$ in the solution of $S$, i.e., $S_\Delta = \Delta S_1$.

Integrating the above proposition complete the proof of Proposition 4

Lemma 2 implies the correctness of our algorithms when $d \to \infty$. The following Proposition argue that for any dimension $d$, when we dropped the flats that their distance is $\leq 2$ we eliminate linear fraction of the whole set:

Proposition 7 Given $n$ flats such that $\delta$-fraction of them arise intersecting $\Delta$-distinct balls. Dropping all the pairs with distance $\leq 2$, guaranty that we eliminate linear fraction of the flats.
Proof. Let $X$ denote a random variable of $\text{dist}(P_{1}(\Delta), P_{1}(\Delta))$. From all the non-negative random variables $Y$ that their mean is equal to $S_{1}\Delta$ and $\text{Pr}(Y \leq 2\Delta) = 1$, we would like to find the one that maximize the probability of $\text{Pr}(Y \leq 2)$. We get this by defined $Y$ to get 2 if $\text{dist}(P_{1}(\Delta), P_{1}(\Delta)) \leq 2$ and $2\Delta$ otherwise. Applying Proposition 6 and the expectation definition we get that $E(Y_{i}) = 2q + 2\Delta q = S_{1}\Delta$. Solving the equation and generates a power series expansion for $q$ we got $(1-S_{1}/2)+(1-S_{1}/2)^{3}/3 + o(1/\Delta^{2})$. Since $S_{1} < 2$ (see Proposition 5 below) we get that $0 < q < 1$. Since $q$ is a bound on the probability to accept the flats $P(\Delta)$ and $P_{1}(\Delta)$, it holds that the probability $p = \text{Pr}(X > 2)$ to drop $P_{1}(\Delta)$ and $P_{1}(\Delta)$ is $p \geq 1 - q > 0$. I.e., we dropped the $p$ fraction of the $(\binom{n}{2})$ pairs we have got.

**Proposition 8** Let $S_{1}$ be the integral solutions as defined above, then $S_{1} < 2$.

Proof. By its definition, $S_{1}$ is the mean distance between two flats passing through the points $(-1,0,...,0)$ and $(1,0,...,0)$. Fixing the flat $P(-1)$, we can observe that if $P(1)$ is intersecting the ball $B(-1,0,...,0)$ than $\text{dist}(P(-1), P(1)) \leq 1$. Let $\alpha$ denote the probability of this event, i.e. $\alpha = \text{Pr}(P(1) \cap B(-1,0,...,0) \neq \emptyset)$. To complete the proof it is enough to prove that $\alpha > 0$ (since $S_{1} \leq E(\text{dist}(P(-1), P(1))) = 1 * \alpha + 2 * (1 - \alpha)$).

Observing that $P(1) \cap B(-1,0,...,0)$ is a spherical cap with nonzero volume (relatively to the measure of all the flats), one can show that the probability that two random flats passing through $(-1,0,...,0)$ and $(1,0,...,0)$ has distance $1$ is greater than zero, i.e., $\theta > 0$.

3.1 Algorithm

We summarize the clustering procedure of a set of $n$ $k$-flats in $\mathbb{R}^{d}$ in Algorithm 1.

We note that:

1. The midpoint of the flats (Line 3) is calculated using Proposition 1.
2. The distance between the flats (Line 4) is calculated as described in Proposition 2.
3. Line 9 – 16 is a simple ‘union find’ algorithm that clusters flats with distance $\leq 2$.
4. The final step (lines 18-21) eliminate all the clusters that their density is low.

4 Experimental Studies of $k$-Flat Clustering

As part of the main theorem proof, Lemma 2 tells us what happens when we take the dimensionality to infinity. In practice, it is interesting to know at what dimensionality do we anticipate that the flat pairwise projection to midpoints, implies good separation to different clusters. In other words, Lemma 2 describes some convergence, but does not indicate the convergence rate. We addressed this issue through empirical studies.

We ran the following experiments using synthetic data set, producing the flats inputs with normally distributed location and direction, as described in the model assumption. W.l.o.g. we choose the balls’ center to be $c_{1} = (-100, ..., 0)$ and $c_{2} = (100, 0, ..., 0)$ and $k$ (the flats dimension) equals to $d/3$.

Each cluster contain 10 random flats, all together we have 20 random flats. Our algorithm computes the center point for all pairs of flats; all together we have 190 center
Algorithm 1 Data clustering using flats minimum distances

**Input:** a set $P$ of $n$ $k$ dimensions flats in $\mathbb{R}^d$, a threshold $M = n/m$ for the minimal size of every cluster

**Output:** a set $C$ of $m$ clusters

1: $p \leftarrow \emptyset$
2: for each $(P_i, P_j) \in P$ do
3:     $p_{ij} \leftarrow \text{mid point} (P_i, P_j)$
4:     $d_{ij} \leftarrow \text{dist} (P_i, P_j)$
5:     if $d_{ij} \leq 2$ then
6:         $p \leftarrow p \cup p_{ij}$
7:     end if
8: end for
9: for each $p_i \in p$ do
10:    $c_i \leftarrow \text{makeSet} (p_i)$
11: end for
12: for each $c_i, c_j$ do
13:    if $\text{dist} (c_i, c_j) \leq 2$ then
14:        $\text{union} (c_i, c_j)$
15:    end if
16: end for
17: $C \leftarrow \emptyset$
18: for each $c_i$ do
19:    if $\text{size} (c_i) > M$ then
20:        $C \leftarrow C \cup c_i$
21:    end if
22: end for
23: return $C$
points. see Figure 1 which shows four different experiments, each done for different dimension. Those center points are divided into three groups: the first 45 are shown as a red dot close to the center $c_1$. Furthermore, they are close to one another that the eye cannot distinguish between them. The second group is also comprised of 45 points, shown as a red dot to the right, close to $c_2$. The third group has 100 points, centered around 0 point. Those points are shown in black, with a distance of $>2$. This means that the algorithm reject all the points in the third group as was anticipated. The four images illustrate how the variance is decreasing, while increasing the dimension. This illustrate that our algorithm preforms better for higher dimension.

5 Discussion

The analysis of incomplete data is one of the major challenges in the scope of big data. Typically, data objects are represented by points in $\mathbb{R}^d$, here we suggest that the incomplete data is corresponding to affine subspaces. With this motivation we study the problem of clustering $k$-flats, where two objects are similar when the Euclidean distance between them is small. The paper presented simple clustering algorithm for $k$-flats in $\mathbb{R}^d$, as well as studying the probability of pair-wise intersection of these objects.

The key idea of our algorithm is to formulate the pairs of flats as midpoints, which preserves the distance features. This way, the geometric location of midpoints that arise from the same cluster, identify the center of the cluster with high probability (as shown in Lemma 1). Moreover, we also show (Lemma 2) that when the dimension $d$ is big enough, the corresponding distance of flats, that arise from different clusters, approach the mean distance of the cluster’s center. Using this we can eliminate the irrelevant midpoints with high probability.

For low dimensions, we did not identify the exact probability that we dropped all the irrelevant flats (i.e., those that arise from different clusters), however, we do show that we eliminate a linear fraction of those irrelevant flats. In addition, using experimental results, we support our claim that the algorithm works well also in low dimensions.

Given a set of size $n$ with $k$-flats in $\mathbb{R}^d$, we find the distance and the midpoints of every pair, by solving least squares linear problem in $O((kd)^{\omega})$ time (where $\omega$ is the matrix multiplication complexity). By doing this for $\binom{n}{2}$ pairs, hence we got a $\text{poly}(n,k,d)$ running time algorithm. Observing that when sampling poly-logarithmic number of flats (say $\log^2 n$) we get the right results with high probability, therefor, we can improve the running time to $\text{polylog}(n,k,d)$.

as a last remark, note that the algorithm we presented above works fine for a set of $m$ clusters, for which each one of them contains the same number of flats. We can relax this assumption by using a recursive version of the algorithm, where in every step the algorithm identify the biggest clusters, eliminates their corresponding flats, and continue to find the smaller ones.
Proposition 9  Let \( \ell' \) and \( \ell'' \) be two random 2D lines that intersect the unit disk and \( p \) be their intersecting point. With probability \( 1 - O(1) \) the distance \( r \) from \( p \) to the origin is bounded.

Proof. Observing that the maximum distance between the intersecting point \( p \) and the origin occur when \( \ell' \) and \( \ell'' \) are tangents to the disk, we will consider only this case. Let \( \phi \in [0, \pi] \) be the intersection angle between the two lines. When \( \phi = \pi \) the two lines are joined together and \( r \) equals to 1 (the disk radius). While reducing \( \phi \) toward the zero angle, \( r \) is increased toward infinity (i.e., when \( \phi \to 0 \) the lines are parallel and \( r \to \infty \)). For example, when \( \phi = \pi/2 \) then \( r = \sqrt{2} \), for \( \phi = \pi/3 \) we get \( r = 2 \) and generally, \( r = 1/\sin \phi \). Since \( \phi \) is uniformly distributed over \([0, \pi]\), we get that with probability \( 1 - \varepsilon \), the distance \( r \) is \( \leq r_0 \), where \( \varepsilon = 2 \arcsin(1/r_0) \). E.g., with probability \( \geq 2/3 \) we have \( r \leq 2 \). \( \square \)

Proposition 10  Let \( P_{c_0}^{i} \) and \( P_{c_0}^{j} \) be two \( k \)-flats that intersect the unit ball \( B_{c_0}^{d} \) and \( p \) be their midpoint point. With probability \( 1 - O(1) \) the distance \( r \) from \( p \) to the origin is bounded.

Proof. Starting with three dimensional space, the relation between the two flats can be expressed using the distance between them, their relative direction (azimuthal angle) and its relative orientation (polar angle). Fixing the orientation, the variation of the direction is described in the 2D case (see proposition 9). When the two flats’ directions cause a small distance between \( p \) and the origin, changing the orientation will not increase this distance (but may decrease it). Generally, changing the flat orientation will increase the probability that the distance from \( p \) to the origin is bound.

For general \( d \), using the same idea, the flats can be represented by a spherical coordinate (i.e., the coordinates consist of a radial coordinate and \( d - 1 \) angular coordinates), implies that the distance between the midpoint and the ball’s center is bounded by a probability that increases as \( d \) increases. \( \square \)
Fig. 1. Given two sets of flats from two clusters located at $B_{(-100,0,...,0)}$ and $B_{(100,0,...,0)}$, the black points are the midpoints of all the pairs and the red points indicate those who left after eliminate flats that their corresponding distance is greater than 2.