Multicomponent Bright Solitons in $F = 2$ Spinor Bose–Einstein Condensates

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We study soliton solutions for the Gross–Pitaevskii equation of the spinor Bose–Einstein condensates with hyperfine spin $F = 2$ in one-dimension. Analyses are made in two ways: by assuming single-mode amplitudes and by generalizing Hirota’s direct method for multi-components. We obtain one-solitons of single-peak type in the ferromagnetic, polar and cyclic states, respectively. Moreover, twin-peak type solitons both in the ferromagnetic and the polar state are found.

KEYWORDS: $F = 2$ Bose–Einstein condensate, hyperfine spin, multi-component solitons, integrability, direct method, multi-component Gross–Pitaevskii equation

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1. Introduction

Since the realization in experiments, Bose–Einstein condensate (BEC) of ultra-cold atoms has attracted much interest of theoretical physicists. Under optical dipole traps, the hyperfine spin of atoms remains to be active and leads to the spinor BEC. So far, spinor BECs have been found to show a variety of phases. For the hyperfine spin \( F = 1 \) state, the ground-state phase can be either polar (antiferromagnetic) with \(^{23}\text{Na}\) or ferromagnetic with \(^{87}\text{Rb}\). \(^{23}\text{Na}\) and \(^{87}\text{Rb}\) condensates have been realized with both \(^{23}\text{Na}\) and \(^{87}\text{Rb}\) atomic species. The \( F = 2 \) BECs are classified into three distinct phases referred to as ferromagnetic, polar, and “cyclic”. However, the ground-state phase of the \( F = 2 \) state at zero magnetic field is under discussion due to its very short lifetime (a few milliseconds) for which the equilibrium state cannot be reached. Compared to the \( F = 1 \) cases, the spin dynamics of the \( F = 2 \) BEC is less well-understood, especially in the cyclic phase, giving rise to experimental and theoretical challenges.

Recently, solitons of spinor BECs in one-dimension have been studied analytically and numerically. In experiments, matter-wave dark and bright solitons are produced only for single-component BEC. For a generic hyperfine spin \( F \), the dynamics is described by the \((2F + 1)\)-component Gross–Pitaevskii (GP) equation. The one-component GP equation is called the nonlinear Schrödinger equation (NLSE). If all the spin-dependent interactions vanish and only the intensity interaction exists, the multi-component GP equation is equivalent to the vector NLSE, which is also called the Manakov equation. The soliton solutions of these systems are well-known. For \( F = 1 \), at special sets of coupling constants one-soliton solutions and two-soliton collisions were explicitly shown for the bright soliton under the vanishing boundary conditions, and for the dark soliton and the bright soliton under the nonvanishing boundary conditions, by finding the map from the GP equation to the \( 2 \times 2 \) matrix NLSE, which can be solved by the inverse scattering method. However, for higher spins, such a map to a known integrable equation has not been found and we need to look for alternative methods.

In this paper, we aim at seeking higher-spinor BEC solitons in one-dimension. For this purpose, we employ two different methods for demonstrating \( F = 2 \) BEC bright one-solitons. One facile method, the single-mode analysis, effectively utilizes the reduction of the multi-component GP equation to the one-component one. Another new method is a generalization of Hirota’s direct method to multi-components. Hirota’s direct method has been successfully applied to get solitons in one-component equation systems, and we prove its applicability and strength even for the multi-component systems. Indeed, this method gives solitons beyond the single-mode analysis. Both methods are regarded to be effective even for nonintegrable equations, and suitable for the first step of investigation. The results include not only ordinary single-peak solitons but also twin-peak solitons which cannot be expressed as the superposition
of two single-peak solitons.

The paper is organized as follows. In Sec. 2 the GP equation for the $F = 2$ spinor BEC is introduced. In Sec. 3 we study the single-mode analysis. In Sec. 4 Hirota’s direct method is generalized for the $F = 2$ GP equation. In Sec. 5 we present one-soliton solutions obtained by Hirota’s method and discuss their properties. The last section is devoted to discussion and conclusion.

2. $F = 2$ Spinor BEC in One-Dimension

In the mean-field theory, the $F = 2$ spinor BEC is characterized by the local order parameter (or, the macroscopic wavefunction) with five components, $\Phi = (\Phi_2, \Phi_1, \Phi_0, \Phi_{-1}, \Phi_{-2})$, reflecting the five spin degrees of freedom. For the magnetic quantum number $j = -2, \cdots, 2$ with respect to the quantization axis chosen in the $z$-direction, $\Phi_j = \Phi_j(x, t) = \langle \hat{\Psi}_j(x, t) \rangle$. In words, $\Phi_j$ are given by the ground state expectation value of the boson operators $\hat{\Psi}_j(x, t)$, which satisfy the equal-time commutation relation $[\hat{\Psi}_\alpha(x, t), \hat{\Psi}_\beta^\dagger(x', t)] = \delta_{\alpha\beta}\delta(x - x')$ for $\alpha, \beta = -2, \cdots, 2$.

We consider the dynamics of the $F = 2$ spinor BEC in one-dimension. The evolution equation for the local order parameters is described by the multi-component Gross–Pitaevskii (GP) equation,

$$
i\hbar \frac{\partial \Phi}{\partial t} = \frac{\delta E_{\text{GP}}[\Phi]}{\delta \Phi^*}.$$  \tag{1}

Here the energy functional is defined by $^{9-12}$

$$E_{\text{GP}}[\Phi] = \int_{-\infty}^{\infty} dx \left( \frac{\hbar^2}{2m} |\partial_x \Phi|^2 + \frac{c_0}{2} n^2 + \frac{c_2}{2} f^2 + \frac{c_4}{2} |\Theta|^2 \right).$$ \tag{2}

The coupling constants $c_i$ are real and can be expressed in terms of a transverse confinement radius and a linear combination of the $s$-wave scattering lengths of atoms. \cite{16} The interaction energy is derived from the short-range interactions of atoms in the scattering channel with total spin $0, 2, 4$, and is given in terms of the number density

$$n = \sum_{\alpha = -2, \cdots, 2} \Phi_\alpha^* \Phi_\alpha,$$ \tag{3}

the spin densities $f = (f^x, f^y, f^z)$, where for $i = x, y, z$,

$$f_i = \sum_{\alpha, \beta = -2, \cdots, 2} \Phi_\alpha^* \Phi_\beta^* \Phi_{\beta},$$ \tag{4}

and the singlet-pair amplitude $^{11, 12}$

$$\Theta = 2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2.$$ \tag{5}

The meaning of $\Theta$ is clear if we write with the Clebsch-Gordan coefficient as $\Theta = \sqrt{5} \sum_{j'j} \langle 00|2j;2j'\rangle \Phi_j \Phi_{j'}$, i.e. it measures the formation of spin-singlet “pairs” of bosons. The prefactor $\sqrt{5}$ is introduced just for convenience. The $c_4$-term in (2) includes the scattering process $2 + (-2) \leftrightarrow 0 + 0$, which changes the $z$-component of the spin states of bosons.
by two and is absent for \( F = 1 \). We also write the spin densities as \( f^\pm = f^x \pm if^y \). The spin matrices \( f^i \) in \( F = 2 \) are explicitly represented as

\[
f^x = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{6}/2 & 0 & 0 \\
0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\
0 & 0 & \sqrt{6}/2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
f^y = \begin{bmatrix}
0 & i\sqrt{6}/2 & 0 & -i\sqrt{6}/2 & 0 \\
i0 & -i\sqrt{6}/2 & 0 & 0 & 0 \\
0 & i\sqrt{6}/2 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 \\
\end{bmatrix},
f^z = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -2 \\
\end{bmatrix}.
\]

(6)

Without the magnetic field, the energy (2) is invariant under an SU(2) rotation and the system has an SU(2) symmetry. In particular, an obvious symmetry is the one under \( \exp[i\pi f^x] : \Phi_j \mapsto -\Phi_j \).

We set \( \hbar = 1, 2m = 1 \) to simplify the expressions. With an operator \( \mathcal{L} = i\partial_t + \partial_x^2 \), the explicit form of GP equation (1) is

\[
\mathcal{L}\Phi_{\pm 2} = c_0 n \Phi_{\pm 2} + c_2 \left( \pm 2 f^z \Phi_{\pm 2} + f^\mp \Phi_{\pm 1} \right) + c_4 \Theta \Phi_{\mp 2}^*,
\]

(7a)

\[
\mathcal{L}\Phi_{\pm 1} = c_0 n \Phi_{\pm 1} + c_2 \left( f^\pm \Phi_{\pm 2} \pm f^\mp \Phi_{\pm 1} + \frac{\sqrt{6}}{2} f^z \Phi_0 \right) - c_4 \Theta \Phi_{\mp 1}^*,
\]

(7b)

\[
\mathcal{L}\Phi_0 = c_0 n \Phi_0 + c_2 \left( \frac{\sqrt{6}}{2} f^+ \Phi_1 + \frac{\sqrt{6}}{2} f^- \Phi_{-1} \right) + c_4 \Theta \Phi_0^*.
\]

(7c)

The right-hand sides include cubic terms with respect to \( \Phi_j \). If the spin-dependent interactions are absent, \( i.e. c_2 = c_4 = 0 \), the GP equation is reduced to the Manakov equation with five components and solutions for the initial problem as well as multi-solitons are known in the formalism of the inverse scattering method.\textsuperscript{30,31} However, this is a trivial reduction for spinor condensates. In the presence of the spin-dependent interactions where rich phenomena are expected, the GP equation becomes highly correlated and hard to be solved explicitly.

We concentrate on soliton solutions for the \( F = 2 \) GP equation. A soliton propagates keeping its own wave properties. Through its free translational motion, physical quantities given by the integral of densities characterize the soliton, such as the particle number \( N = \int n dx \), the spin \( F = \int f dx \) and the volume of the singlet-pair \( S = \int |\Theta| dx \).

In the subsequent sections, we attempt to derive one-soliton solutions with non-trivial spin degrees of freedom and clarify their physical properties. We apply two methods and obtain several one-soliton solutions for the GP equation (7).
3. Single-Mode Analysis

The single-mode analysis assumes the following amplitude for the order parameters:\(^{18}\)

\[
\Phi(x, t) = A \phi(x, t),
\]

where \( A = (A_2, A_1, A_0, A_{-1}, A_{-2}) \). That is, the order parameters have the same spatial profile but can have different magnitude. Normalization is such that \( \sum_j |A_j|^2 = 1 \). We require the GP equation to lead to the one-component nonlinear Schrödinger equation for \( \phi \),

\[
i \partial_t \phi + \partial_x^2 \phi - C |\phi|^2 \phi = 0
\]

with \( C \) being a real constant. This imposes the consistency conditions on the nonlinear terms of the GP equation. By the freedom of SU(2) rotation, we can fix the spin in the \( z \)-direction to have \( f_+ + f_- = 0 \). Then the conditions read

\[
E_{j_1} = \cdots = E_{j_k},
\]

for \( j_1, \cdots, j_k \in \{-2, \cdots, 2\} \) with \( A_{j_1}, \cdots, A_{j_k} \neq 0 \), where

\[
\begin{align*}
E_2 &= 2c_2 \bar{f}^2 + c_4 \bar{\Theta} A_{-2}^*/A_2, \\
E_1 &= c_2 \bar{f}^2 - c_4 \bar{\Theta} A_{-1}^*/A_1, \\
E_0 &= c_4 \bar{\Theta} A_0^*/A_0, \\
E_{-1} &= -c_2 \bar{f}^2 - c_4 \bar{\Theta} A_1^*/A_{-1}, \\
E_{-2} &= -2c_2 \bar{f}^2 + c_4 \bar{\Theta} A_{-2}^*/A_{-2}.
\end{align*}
\]

Here, \( \bar{f} = \sum_{\alpha, \beta} A^*_\alpha f_{\alpha \beta} A_{\beta} \) and \( \bar{\Theta} = 2A_2 A_{-2} - 2A_1 A_{-1} + A_0^2 \) for simplicity.

We summarize the result of the examination of the consistency conditions. It is sufficient to specify amplitudes by their representatives by virtue of the SU(2) symmetry.

- The ferromagnetic states, \(|\bar{f}| > 0\) and \(|\bar{\Theta}| \geq 0\).

\[
\begin{align*}
A &= (p_2, 0, 0, 0, p_{-2}), & 4c_2 = c_4. \\
A &= (0, p_1, 0, p_{-1}, 0), & c_2 = c_4. \\
A &= (p_2, p_1, p_0, -p_{-1}, p_{-2}), & c_2 = 0, \bar{\Theta} = 0.
\end{align*}
\]

- The polar states, \( \bar{f} = 0 \) and \( |\bar{\Theta}| > 0 \).

\[
\begin{align*}
A &= (p_2, 0, p_0, 0, p_{-2}), & |p_2| = |p_{-2}|, c_4 = 0. \\
A &= (p, q, r, -q^*, p^*), & p, q \in \mathbb{C}, r \in \mathbb{R}.
\end{align*}
\]

- The cyclic state, \( \bar{f} = 0 \) and \( |\bar{\Theta}| = 0 \).

\[
A = (p, 0, 0, \sqrt{2}q, 0), \quad p, q \in \mathbb{C}, |p| = |q|.
\]
Here we can take \( p_i \in \mathbb{C} \). These three spin states for \( F = 2 \) spinor BEC are specified in Ref. 11. The cyclic state, which is absent in the \( F = 1 \) case and is available for spin \( F \geq 2 \) boson systems, exhibits unusual features such as phase-locking phenomena and kink excitations\(^{15}\) owing to a unique nature that the condensate energy depends on the relative value among the phase factors of \( \Phi_j \). It is remarkable that any one-solitons in the three states are obtained.

The coefficient of the nonlinear term in (9) turns out to be

\[
C = \begin{cases} 
  c_0, & \text{for } \Theta = 0, \\
  c_0 + c_4, & \text{otherwise}.
\end{cases}
\] (18)

When the effective coupling is the attractive one, \( C < 0 \), we have the bright one-soliton

\[
\phi(x, t) = \sqrt{\frac{2k_i}{|C|}} e^{i\chi_i} \text{sech} \chi_r.
\] (20)

The position function \( \chi_r \) and the phase function \( \chi_i \) of the soliton are given by

\[
\chi_r = 2k_r k_i t - k_i x + \delta_r,
\]

\[
\chi_i = - (k_r^2 - k_i^2) t + k_r x + \delta_i,
\] (21)

respectively, where \( \chi \equiv \chi_r + i\chi_i = kx - k^2 t + \delta \) with \( k = k_r + ik_i \). One can also see the plane-wave solution \( \phi(x, t) = \exp[i(Kx - \Omega t)] \), where \( K \) and \( \Omega \) are real with \( \Omega = K^2 + C \). Under \( C < 0 \), this plane-wave is unstable against the modulation and is decomposed into bright solitons during time-evolution. In the case of the repulsive coupling \( C > 0 \), a dark-soliton is formed under the nonvanishing boundary conditions \( |\phi| \to \text{const.} \) as \( x \to \pm \infty \).

We remark on the reduction to \( F = 1 \). If \( c_4 = 0 \) and \( \Phi = (0, \Phi_0', \Phi_{-1}'/\sqrt{3}, \Phi_{-1}' ) \) such that \( \Phi^\prime_0 \Phi_0' + \Phi^\prime_{-1} \Phi_{-1}' = 0 \), the GP equation for \( \Phi_j \) is equivalent to the \( F = 1 \) GP equation. The polar single-mode soliton with (16) and \( p = 0 \) is reduced to the \( F = 1 \) polar one-soliton in Refs. 16–18.

4. Hirota’s Direct Method

In this section, we introduce Hirota’s direct method.\(^ {34}\) Hirota’s direct method is powerful for getting solitons in both integrable and nonintegrable one-component partial differential equation (PDE) systems. We generalize this method for the \( F = 2 \) GP equation.

By putting \( \Phi_j = G_j/H \) for \( j = -2, \cdots, 2 \), the GP equation is transformed into the form,

\[
(iD_t + D_x^2)G_j \cdot H - \left( \frac{c_2}{2} \frac{\delta^2}{\delta \Phi_j^*} \right) H^2 = 0,
\] (23a)

\[
2D_x^2 H \cdot H + c_0 \sum_{\alpha=-2,\cdots,2} |G_\alpha|^2 = 0.
\] (23b)

Here the Hirota derivative is defined as

\[
D^m_t D^n_x a \cdot b = \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right)^m \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n a(x_1, t_1) b(x_2, t_2) \bigg|_{x_1=x_2=x, t_1=t_2=t}.
\] (24)
In the Manakov and the one-component systems, the $c_2$- and $c_4$-terms in eqs. (23a) are absent and the transformation is called the bilinear transformation.

For attractive spin-independent interaction, we can set $c_0 = -2$ by scaling the order parameters. Equations (23) give solitons with the vanishing boundary conditions $|\Phi| \to 0$ as $x \to \pm \infty$, that is, bright solitons. For the nonvanishing boundary conditions, extra terms are needed.

Within Hirota’s method, a one-soliton solution may be obtained by a finite-order perturbation,

$$H = 1 + \varepsilon^2 h_2 + \varepsilon^4 h_4, \quad (25a)$$
$$G_j = \varepsilon g_{1,j} + \varepsilon^3 g_{3,j}. \quad (25b)$$

Substituting (25) into (23), we solve them order by order. At the first order, we have

$$g_{1,j} = \Pi_j e^{i(kx - k^2 t)} \quad (26)$$

with $k = k_r + ik_i \in \mathbb{C}$ and $\Pi_j (j = -2, \cdots , 2)$ being free parameters for one-solitons. A success of terminating the perturbation expansion at a finite order leads to soliton solutions. We cannot expect soliton solutions for generic values of parameters. Our strategy is that by utilizing the freedom of 7 parameters, i.e. 5 for one-soliton $\Pi_j$ and 2 for interaction couplings $c_2$ and $c_4$, we look for both one-soliton solutions and the valid parameters in order to have those solutions.

It is known that for one-component PDEs, Hirota’s direct method is applicable for up to two-solitons even in nonintegrable systems. In fact, the expansion (25) is usually taken for a two-soliton. In our case, as we will show later, we have a twin-peak one-soliton for the $F = 2$ GP equation. In a sense, a twin-peak one-soliton may be considered as a degenerate two-soliton. This kind of observation is also seen in Refs. 19,33. For a general two-soliton, we take (26) as a combination of two plane waves. In principle, those calculations which include more higher orders and multi-solitons are possible in integrable systems, but we do not reach the argument of integrability and postpone them to future works.

5. Results for One-Solitons

We present one-soliton solutions through Hirota’s direct method. A one-soliton is determined by the following parameters; $k_r$: (half of) the phase velocity of the envelope soliton, $k_i$: the amplitude of the soliton, $\Pi = (\Pi_2, \Pi_1, \Pi_0, \Pi_{-1}, \Pi_{-2})$: distribution among spin components. The position function and the phase function of the soliton are the same as the one-component one’s, (21) and (22), respectively. We normalize $\Pi$ by $\sum_j |\Pi_j|^2 = 1$ since its factor can be absorbed in the shift of $\chi$.

5.1 Single-mode (Single-peak) soliton

Single-mode one-solitons are reproduced with specific values of coupling constants.
a) A single-mode soliton in the ferromagnetic state, with $|f| > 0$ and $\Theta = 0$.

\[
\Pi = (p_2, p_1, p_0, p_{-1}, p_{-2}), \quad p_i \in \mathbb{C}, \quad 2p_2p_{-2} - 2p_1p_{-1} + p_0^2 = 0. \tag{27}
\]

\[c_2 = 0, \quad c_4 = \text{arbitrary}.\]

b) A single-mode soliton in the polar state, with $f = 0$ and $|\Theta| > 0$.

\[
\Pi = (p_2, 0, p_0, 0, p_{-2}), \quad p_i \in \mathbb{C}, \quad |p_2| = |p_{-2}|. \tag{28}
\]

\[c_2 = \text{arbitrary}, \quad c_4 = 0.\]

c) A single-mode soliton in the cyclic state, with $f = 0$ and $\Theta = 0$.

\[
\Pi = (p, q, r, -q^*, p^*), \quad p, q \in \mathbb{C}, \quad r \in \mathbb{R}. \tag{29}
\]

\[c_2 = \text{arbitrary}, \quad c_4 = \text{arbitrary}.\]

For the cases $a) \sim c)$, the one-soliton solution is

\[
\Phi = k_i \mathrm{e}^{\chi_i \mathrm{sech} \chi_i} \Pi. \tag{30}
\]

It should be remarked that the cyclic soliton in $c)$ exists for all interaction couplings.

d) A single-mode soliton in the polar state, with $f = 0$, $|\Theta| > 0$ and a doubled particle number compared to the cases $a) \sim c)$ for the same $k_r$ and $k_i$.

\[
\Pi = (p, q, r, -q^*, p^*), \quad p, q \in \mathbb{C}, \quad r \in \mathbb{R}, \tag{31}
\]

\[c_2 = \text{arbitrary}, \quad c_4 = 1.\]

e) A single-mode soliton in the ferromagnetic state, with $|f| > 0$, $|\Theta| \geq 0$ and a doubled particle number compared to the cases $a) \sim c)$ for the same $k_r$ and $k_i$.

\[
\Pi = (p_2, 0, 0, 0, p_{-2}), \quad p_i \in \mathbb{C}. \tag{32}
\]

\[c_2 = 1/4, \quad c_4 = 1.\]

\[
\Pi = (0, p_1, 0, p_{-1}, 0), \quad p_i \in \mathbb{C}. \tag{34}
\]

\[c_2 = 1, \quad c_4 = 1.\]
For the cases \( d \) and \( e \), the one-soliton solution is
\[
\Phi = \sqrt{2} k_i e^{i \chi_i} \text{sech} \chi_r \Pi.
\] (36)

Compared to (30) in the previous cases, the amplitude is larger by a factor \( \sqrt{2} \). From the single-mode analysis in Sec. 3, it is shown that the effective coupling \( C \) for the one-component equation has the value \( c_0 + c_4 = -1 \), not \( c_0 = -2 \). Accordingly, the amplitude gets the factor \( \sqrt{2} \), and hence the particle number becomes doubled.

The single-mode solitons in this section are included in the result in Sec. 3.

One can see that for single-mode one-solitons, zero local spin is allowed for arbitrary \( c_2 \), and zero singlet-pair amplitude is allowed for arbitrary \( c_4 \), since the corresponding interaction energy in (2) is ineffective, respectively. Moreover, the solitons in the cases \( a \)~\( c \) turn to have vanishing interaction terms with \( c_2 \) and \( c_4 \) in the energy (2), and they are regarded as the solitons in the Manakov system.

5.2 Twin-peak soliton

We further investigate one-solitons which cannot be expressed within a single-mode form. They have a wave-form with twin peaks. The distance of the twin peaks is freely adjusted by changing the parameters of the one-soliton. One may be tempted to say that the twin-peak soliton is the superposition of two identical single-peak solitons with shift of their positions, but it is not true because physical densities of the twin-peak soliton are not always just the sum of those of two single-peak solitons. Such a twin-peak one-soliton was already discovered in the \( F = 1 \) spinor BEC,\(^{16,17} \) but is allowed only for the polar state, i.e. with zero total spin.

In our result for \( F = 2 \), we find that twin-peak one-soliton occurs both in the polar state and in the ferromagnetic state.

f) A twin-peak soliton in the polar state, with \( F = 0 \) (but locally \( f \neq 0 \)) and \( |\Theta| > 0 \).

\[
\Pi = (p_2, p_1, p_0, p_{-1}, p_{-2}), \quad p_i \in \mathbb{C},
\]
\[
c_2 = 0, \quad c_4 = 1.
\] (37)

The wavefunctions have the form
\[
\Phi_j = \frac{\sqrt{2} k_i (-1)^j \sigma p_j^* e^{i \chi_r} + p_j e^{-i \chi_r}}{\sqrt{|\mathcal{F}|} \cosh 2 \chi_r + \cosh \omega} e^{i \chi_i},
\] (38)

for \( j = -2, \cdots, 2 \), where \( \mathcal{F} = 2p_2 p_{-2} - 2p_1 p_{-1} + p_0^2 \), \( N = \sum_j |p_j|^2 \), \( \sigma = \mathcal{F}/|\mathcal{F}| \) and \( \cosh \omega = N/|\mathcal{F}| \). Physical densities are calculated as follows;
\[
n = k_i^2 \left[ \text{sech}^2 \left( \chi_r - \frac{\omega}{2} \right) + \text{sech}^2 \left( \chi_r + \frac{\omega}{2} \right) \right],
\] (39)
\[
f^i = -4k_i^2 \frac{f^i}{|\mathcal{F}|} \frac{\sinh 2 \chi_r}{(\cosh 2 \chi_r + \cosh \omega)^2},
\] (40)
\[
|\Theta| = \frac{4k_i^2}{\cosh 2 \chi_r + \cosh \omega};
\] (41)
Fig. 1. Density plots for $f$) (twin-peak polar soliton) with $\Pi = (1, 3, 1, 3), k_r = k_i = 1$. (a) Densities for each component ($|\Phi_2|^2$: red, $|\Phi_1|^2$: purple, $|\Phi_0|^2$: blue, $|\Phi_{-1}|^2$: yellow, $|\Phi_{-2}|^2$: green). (b) The number density. (c) The spin densities ($f^x$: green, $f^y$: blue, $f^z$: pink). (d) The absolute value of the singlet-pair amplitude.

where $\bar{f}^i = \sum_{\alpha, \beta} \Pi_\alpha^i f^{i}_{\alpha\beta} \Pi_\beta$. The total amounts of the quantities are obtained by integrating these densities as

$$N = 4k_i, \quad (42)$$
$$F = (0, 0, 0), \quad (43)$$
$$S = 4k_i \omega \cosech \omega. \quad (44)$$

Figure 1 shows an example of the twin-peak polar soliton. We observe that the spins contained in the two peaks have the same amount with the opposite sign, and form a polarization. Therefore, in total the soliton has zero spin. The reduction to a single-peak soliton is achieved by sending $\mathcal{T} \to 0$. In this limit, the two peaks get infinitely far apart and eventually, the remained single-peak soliton contains nonzero total spin and no singlet-pair amplitude, and coincides with the ferromagnetic one (14) in $a$). We also observe that the singlet-pair amplitude is localized around the center of the twin peaks. This indicates that the twin-peak soliton is not just the superposition of two identical single-peak solitons. The same argument
also holds for the next case \(g\).

For the special case with zero local spin, where we make \(f^i = 0\) for \(i = x, y, z\), the following set of parameters is allowed:

\[
\begin{align*}
\Pi &= (p_2, 0, p_0, 0, p_{-2}), \quad p_i \in \mathbb{C}, \quad |p_2| = |p_{-2}|. \\
c_2 &= \text{arbitrary}, \quad c_4 = 1. 
\end{align*}
\]  

(45)

The wavefunctions are

\[
\begin{align*}
\Phi_{\pm 1} &= \frac{\sqrt{2}k_i p_{\pm}}{(2p_2p_{-2} + p_0^2)^2} e^{\chi r + \xi} + e^{-\chi r} e^{i\chi_i}, \\
\Phi_{0} &= \frac{\sqrt{2}k_i p_0}{(2p_2p_{-2} + p_0^2)^2} e^{\chi r + \xi'} + e^{-\chi r} e^{i\chi_i},
\end{align*}
\]  

(46a) (46b) (46c)

where \(\cosh \omega = \frac{2|p_2p_{-2}| + |p_0|^2}{|2p_2p_{-2} + p_0^2|} \), \(\xi = \arg \left(1 + \frac{p_0^2}{2p_2p_{-2}}\right)\) and \(\xi' = \arg \left(1 + \frac{2p_2p_{-2}}{p_0^2}\right)\). In particular, \(\xi = \xi' = 0\) (cosh \(\omega = 1\)) gives the single-mode soliton of \(d\).

\(g)\) A twin-peak soliton in the ferromagnetic state, with \(|f| > 0\) and \(|\Theta| > 0\).

\[
\begin{align*}
\Pi &= (p_+, 0, 0, 0, p_-), \quad p_i \in \mathbb{C}.
\end{align*}
\]  

(47)

\[
\begin{align*}
\Pi &= (0, p_+, 0, p_-, 0), \quad p_i \in \mathbb{C}.
\end{align*}
\]  

(48)

We use \(s = 2\) for the case (47) and \(s = 1\) for the case (48). Then, the wavefunctions are

\[
\Phi_{\pm s} = \frac{\sqrt{2}k_i p_{\pm}}{\sqrt{|p_+|^2 - |p_-|^2}} e^{\pm \sigma \chi r + \chi_i} + e^{-\chi r} e^{i\chi_i},
\]  

(49)

and the others are constantly zero, where \(\sigma = \text{sign} \left(|p_+|^2 - |p_-|^2\right)\) and \(\cosh \omega = \frac{|p_+|^2 - |p_-|^2}{|p_+|^2 + |p_-|^2}\).

The densities are given by

\[
\begin{align*}
n &= k_i^2 \left[ \text{sech}^2 \left( \chi r - \frac{\omega}{2} \right) + \text{sech}^2 \left( \chi r + \frac{\omega}{2} \right) \right], \\
f^x &= f^y = 0, \\
f^z &= \frac{4k_i^2 s \sigma}{\cosh 2\chi r + \cosh \omega}, \\
|\Theta| &= \frac{4k_i^2 |\sinh \omega \sinh 2\chi r|}{(\cosh 2\chi r + \cosh \omega)^2}.
\end{align*}
\]  

(50) (51) (52) (53)

The total amounts of the quantities are

\[
\begin{align*}
N &= 4k_i, \\
\mathbf{F} &= (0, 0, 4k_i s \sigma \omega \cosech \omega), \\
S &= 4k_i \left| \frac{p_-}{p_+} \right|^{\sigma}.
\end{align*}
\]  

(54) (55) (56)
Fig. 2. Density plots for \( g \) (twin-peak ferromagnetic soliton) with \( \Pi = (0, \sqrt{5}/2, 0, 1, 0), k_r = k_i = 1 \).
(a) Densities for each component (\( |\Phi_1|^2 \): purple, \( |\Phi_{-1}|^2 \): yellow). (b) The number density. (c) The spin densities (\( f^x \): green, \( f^y \): blue, \( f^z \): pink). (d) The absolute value of the singlet-pair amplitude.

Figure 2 illustrates an example of the twin-peak ferromagnetic soliton. One can see that the spin densities are localized around the center of the twin peaks. As \( |p_-|/|p_+| \to 1 \), the two peaks of the soliton get infinitely far apart and a single-peak soliton in the polar state with \( (15) \) in \( b \) is left. It is interesting that the reduction of the twin-peak soliton to the single-peak one changes the state from ferromagnetic to polar in \( g \) and vice versa in \( f \).

Note that (49) is the solution of the two-component coupled NLSE,

\[
\mathcal{L}\Phi_s = -\left(\alpha|\Phi_s|^2 + \beta|\Phi_{-s}|^2\right)\Phi_s, \tag{57a}
\]

\[
\mathcal{L}\Phi_{-s} = -\left(\beta|\Phi_s|^2 + \alpha|\Phi_{-s}|^2\right)\Phi_{-s}, \tag{57b}
\]

with \( \alpha = 1 \) and \( \beta = 3 \). It was established that two-component coupled NLSE is integrable only for \( \alpha = \beta \), corresponding to the original Manakov equation.

6. Discussion and Conclusion

We have studied one-soliton solutions for the Gross–Pitaevski (GP) equation of the \( F = 2 \) spinor Bose–Einstein condensates (BEC) by means of two methods, the single-mode analysis...
and the multi-component generalization of Hirota's direct method. The latter method has been successfully applied to show twin-peak solitons both in the ferromagnetic and the polar states, which cannot be accessed by the single-mode analysis.

Hirota’s method is not restricted to the present analysis. One can also find soliton solutions for higher-spinor BECs or other types of multi-component systems by generalizing this method as presented in this work. For instance, applying to the $F = 1$ spinor BEC, we reproduce the bright solitons in Refs. 16–18, at the order $\varepsilon^2$ for the single-mode solitons and at the order $\varepsilon^4$ for the twin-peak polar soliton. Mathematically, incorporating such as $c_2$- and $c_4$-terms of eqs. (23a) in Hirota’s method suggests a new direction for further extension of the framework.

One of our next interests is the integrability of the GP equation. In the integrable systems with multi-components, multi-solitons should be constructed from any combinations of one-solitons. Their collisions are factorized into successive two-soliton collisions, but in contrast to one-component systems, it is not always the case that each soliton keeps its shape after collisions, deforming its parameters for internal degrees of freedom. The results in this paper pick up specific interactions, and we hope that they especially include integrable points. Whether those systems are integrable or equivalent to already known systems is an interesting future problem. The Painlevé analysis may give a clue for the problem.

From the physical point of view, the discovery of all one-solitons in the ferromagnetic, polar and cyclic states is of much importance. As higher-spinor BECs should exhibit richer physics, we expect that higher-spinor solitons will make wider possibilities in various applications.
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