Privacy Analysis of Online Learning Algorithms via Contraction Coefficients

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Abstract

We propose an information-theoretic technique for analyzing privacy guarantees of online algorithms. Specifically, we demonstrate that differential privacy guarantees of iterative algorithms can be determined by a direct application of contraction coefficients derived from strong data processing inequalities for $f$-divergences. Our technique relies on generalizing the Dobrushin’s contraction coefficient for total variation distance to an $f$-divergence known as $E_{\gamma}$-divergence. $E_{\gamma}$-divergence, in turn, is equivalent to approximate differential privacy. As an example, we apply our technique to derive the differential privacy parameters of gradient descent. Moreover, we also show that this framework can be tailored to batch learning algorithms that can be implemented with one pass over the training dataset.

I. INTRODUCTION

Differential privacy (DP) [2] has become a standard definition for designing large-scale privacy-preserving algorithms in both industrial [3–5] and academic [6–16] settings. Intuitively, a randomized algorithm is said to be differentially private if its output does not vary significantly with small changes in its input parameters. DP is usually cast in terms of the maximum change of the probability distribution of the output of an algorithm given small perturbation in its input values.

Online learning [17, 18] is a framework where a sequence of predictions are made given the knowledge of past actions. This framework is particularly well-suited for dynamic and adversarial environments where learning from data must be done in real-time. Online learning is ubiquitous in practical applications such as recommender systems [19], spam detection [20], portfolio optimization [21–23], and convex optimization [24, 25], to name a few. The main goal of online learning is to sequentially approximate a minimizer of

$$\min_{w \in \mathcal{W}} \sum_{t=1}^{n} f_t(w),$$

where $\{f_1, \ldots, f_n\}$ are convex cost functions and $\mathcal{W}$ is a parameter space. One popular example of this formulation is the one-pass empirical risk minimization (ERM) problem. In this problem, data points $x_1, \ldots, x_n$ are made available to a learner one at a time. The goal of the learner is to find a minimizer of

$$\min_{w \in \mathcal{W}} \sum_{t=1}^{n} \ell(w, x_t),$$

where $\ell(\cdot, \cdot)$ is a loss function. Observe that here $f_t(w) = \ell(w, x_t)$.

In general, an online learning algorithm proceeds as follows: the learner first selects a random point $W_1$ from a convex set $\mathcal{W} \subset \mathbb{R}^d$. After committing to $W_1$, the cost function $f_1$ is revealed to her by nature, specifying the cost $f_1(W_1)$ incurred by the choice $W_1$. Upon observing the cost function $f_t$ at time $t$,

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the learner constructs $W_{t+1} \in \mathcal{W}$ at time $t + 1$ according to some update rules based on all previous choices $W_1, \ldots, W_t$ and their associated costs. Denoting this update rule at time $t$ by $\Psi_t : \mathcal{W} \rightarrow \mathbb{R}^d$, this iterative algorithm can be expressed by

$$W_{t+1} = \Pi_{\mathcal{W}}[\Psi_t(W_t)],$$

where $\Pi_{\mathcal{W}}$ is the projection operator onto $\mathcal{W}$. A prominent example of such update function is stochastic gradient descent, where

$$\Psi_t(w) = w - \eta_t \nabla f_t(w)$$

and $\eta_t$ is the learning rate at time $t$.

Like other machine learning techniques, online algorithms may compromise privacy when applied to sensitive or individual-level data. One standard approach to address this is to resort to DP. Considering \{\{f_1, \ldots, f_n\} as the input of online algorithms, DP ensures that the output remains nearly the same if a single function $f_i$, for some $i \in [n] := \{1, \ldots, n\}$, changes to a different $f'_i$. For example, in the one-pass ERM problem, DP would ensure that the distribution over the minimizer $w$ would be approximately invariant to changes in one single data point $x_i$. DP provides a strong privacy guarantee against an adversary with significant side information about all cost functions except one.

The analysis of DP mechanisms in online learning is particularly challenging: a single change in the algorithm’s cost function at any time instance may have an accumulative impact on all future parameter updates. One standard way to address this issue is to add calibrated noise to $\Psi_t$ at each iteration [13–16]. Thus, most differentially private online learning algorithms can be expressed as

$$W_{t+1} = \Pi_{\mathcal{W}} (\Psi_t(W_t) + \sigma_t Z_t).$$

where \{\{Z_t\} is the collection of i.i.d. noise variables sampled from a known density with covariance matrix $I_d$ and $\sigma_t$ specifies the magnitude of noise at time $t$. In fact, this simple additive-noise mechanism subsumes many popular differentially private algorithms [6–16, 26–33].

In this paper, we provide a new framework for deriving DP guarantees for online algorithms based on the information-theoretic tool of contraction coefficients [34, 35]. Our main result establishes a bound on the "distance" between the outputs $W_{n+1}$ and $W'_{n+1}$ of process (1) after $n$ iterations when starting from two different random initial points $W_1$ and $W'_1$, respectively. As we delineate later, DP guarantees are fully characterized by a certain $f$-divergence called $E_\gamma$-divergence (a connection also observed in [12, 36, 37]). Given $\gamma \geq 1$, the $E_\gamma$-divergence between two probability distribution $\mu$ and $\nu$ is defined as $E_\gamma(\mu || \nu) := \int d(\mu - \gamma \nu)^+ + \gamma \nu) \pi u\mu$ where $\{0, a\}^+ := \max(0, a)$. We discuss this divergence and its connection with privacy in Section II-C.

The following theorem establishes a multiplicative upper bound on the $E_\gamma$-divergence of the output distributions of the iterative process (1) when initiating on two different starting points and noise variables $Z_i$ drawn from a Gaussian distribution. Due to the inherent connection between DP constraints and $E_\gamma$-divergence (to be discussed in Section II), Theorem 1 serves as the main tool for deriving bounds on the DP parameters of online algorithms of the form (1).

**Theorem 1.** Let $W_1 \sim \mu_1$ and $W'_1 \sim \mu'_1$ be two random initial points of the iterative process (1) with $\mu_1, \mu'_1$ being probability measures on $\mathcal{W}$. Let also $W_{n+1} \sim \mu_{n+1}$ and $W'_{n+1} \sim \mu'_{n+1}$ denote the output of the iteration $n$ starting from $W_1$ and $W'_1$, respectively. If $\{Z_t\}_{t=1}^n$ is sampled i.i.d. from $\mathcal{N}(0, I_d)$ and there exists $D > 0$ such that $\{\Psi_t(w) : w \in \mathcal{W}\}$ has diameter $D$ for all $t \in [n]$, then we have

$$E_\gamma(\mu_{n+1} || \mu'_{n+1}) \leq E_\gamma(\mu_1 || \mu'_1) \prod_{t=1}^{n} \theta_\gamma\left(\frac{D}{\sigma_t}\right),$$

where $\theta_\gamma(r) := Q\left(\frac{\log \gamma - r}{\frac{r}{2}}\right) - \gamma Q\left(\frac{\log \gamma + r}{\frac{r}{2}}\right)$ and $Q(a) := \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-u^2/2} du$. 


In information-theoretic parlance, this result establishes a strong data processing inequality [34] for the Gaussian channels under Eγ-divergence with the contraction coefficient specified by the mapping \( \theta_\gamma : (0, \infty) \to (0, 1) \) (see Section II-B for more details). The proof of this theorem relies on a result (Theorem 3) that extends the celebrated Dobrushin’s theorem [38] from total variation distance to Eγ-divergence.

We make the following assumptions in order to apply Theorem 1 for deriving DP guarantees of online learning algorithms:

**Assumption 1:** The parameter \( W_{t+1} \) in the process (1) only depends on \( f_t \) (as opposed to the entire \( \{f_1, \ldots, f_t\} \)). Equivalently, the update function \( \Psi_t \) only depends on \( f_t \). In order to emphasize this assumption, we henceforth write \( \Psi_{f_t} \) for \( \Psi_t \). Thus, we write the process (1) as

\[
W_{t+1} = \Pi_{TV}(\Psi_{f_t}(W_t) + \sigma_t Z_t), \tag{2}
\]

where \( \Psi_{f_t} \) is the update function corresponding to \( f_t \) at iteration \( t \).

**Assumption 2:** We assume that the online algorithm hides all intermediate updates \( W_1, \ldots, W_n \) and only releases the last parameter \( W_{n+1} \). Thus, our goal is to ensure that \( W_{n+1} \) is differentially private.

The Markovity condition in Assumption 1 is equivalent to a memory constraint on the online algorithm. Moreover, it is trivially satisfied by several popular online algorithms such as online (stochastic) gradient descent [39], online mirror descent [17], implicit gradient descent [40], the passive-aggressive algorithm [41], composite mirror descent [42], and the Frank-Wolfe algorithm [43]. Nevertheless, this assumption rules out algorithms such as the follow-the-leader (FTL) algorithm and its variants. We do note, however, that a particular popular variant of the FTL algorithm, namely Regularized FTL [44, 45] is equivalent to online mirror descent (see [46, Lemma 1] and [47]), rendering our framework applicable in this specific case.

Assumption 2 is also present in several recent works on last-iterate convergence of learning algorithms [48–52], the privacy amplification by iteration framework [12, 53] (see Section V for more details), and privacy-preserving generative model for data inspection [54]. Another practical scenario where Assumption 2 naturally arises is as follows: A learner intends to fit a model privately and publicly releases the model parameters after a target level of accuracy is met (for instance, after a certain number of iterations). Hiding intermediate updates not only leads to privacy amplification [12, 53], it may also drive the final parameters to be sparse [52]—a property which is often crucial for many applications [52].

The iterative process (2) can also correspond to batch (i.e., offline) algorithms where a dataset is fixed and is available to the learner in advance. For instance, consider the empirical risk minimization (ERM) problem: Given a dataset \( \{x_1, \ldots, x_n\} \in \mathcal{X}^n \), a learner seeks to solve \( \min_{w \in \mathcal{W}} \frac{1}{n} \sum_{i=1}^{n} \ell(w, x_i) \) for some convex loss function \( \ell : \mathcal{W} \times \mathcal{X} \to \mathbb{R}_+ \). The problem of ERM with DP constraints has been widely-studied with known asymptotically tight upper and lower bounds for the excess loss (i.e., the difference between achieved loss and the true minimum), e.g., [6–10, 26–29, 31–33]. This problem can be viewed as an instance of the online learning problem with cost functions \( f_t(w) = \ell(w, x_t) \) [55]. This observation enables us to translate the privacy analysis of the process (2) to the privacy guarantee of batch algorithms. The caveat here is that each data point \( x_i, i \in [n] \) must be involved in the training process exactly once. Examples of such algorithms include SGD with sub-sampling without replacement.

The privacy properties of SGD algorithms with Gaussian perturbations was initiated in [53] where the privacy guarantee was given in terms of a variant of DP, namely Rényi differential privacy (RDP) [56]. However, this notion of privacy is known to suffer from lack of operational interpretation [37]. In Section V, we revisit the model studied in [53] and compute DP guarantees directly. We demonstrate through numerical examples that our results can be substantially tighter than what would be obtained by converting the results of [53] on RDP to DP via the best known conversion formula (recently given in [57]).
A. Additional Related Work

The problem of analyzing DP guarantees of online algorithms was introduced in [58] for the simple setting where $\mathcal{W}$ is a probability simplex and the cost functions are linear with binary coefficients. Inspired by this work, [13–15] developed techniques for making a large class of online learning algorithms differentially private. The settings studied in these works violate both of our assumptions; the output of the algorithm are the entire sequence $W_1, \ldots, W_{n+1}$ (sometimes called the continual observation [58] setting), and their construction relies on the tree-based aggregation [58] which selects $W_{t+1}$ depending on $f_1, \ldots, f_t$.

Strong data processing inequalities (SDPI) for KL divergence and total variation distance are ubiquitous in information theory and statistics. They appear, for example, in the study of ergodicity of Markov processes [38], the uniqueness of Gibbs measures [38], contraction of mutual information (and generalized mutual information) in Markov chains [59–63] and in Bayesian networks [64], comparison of channels [65], distributed estimation [66], communication complexity of statistical estimation [67], distributed function computation [68], and private estimation problems [69, 70]. Incidentally, the contraction coefficients of general Markov kernels under total variation distance and KL divergence have simple expressions, as outlined in [38] and [34, 59], respectively. In particular, Dobrushin [38] derived a remarkably simple expression for the case of total variation distance. A formula for the contraction coefficients under general $f$-divergences is derived in [35, Theorem 5.2] for differentiable $f$. This formula, however, is not applicable for the specific case of $E_\gamma$-divergence as $f(t) = (t - \gamma)^+$ is not differentiable. More recently, Kamalaruban [71, Theorem 3.10] derived a closed-form expression for the contraction coefficient under another $f$-divergence, namely DeGroot’s statistical information [72]. We initiate the study of the contraction coefficient of general Markov kernels under $E_\gamma$-divergence and demonstrate that it has a simple closed-form expression (Theorem 3) of which Dobrushin’s characterization is a special case.

B. Paper Organization

The rest of the paper is organized as follows. Section II presents preliminary definitions and results. In Section III, we study the contraction coefficient of Markov kernels under $E_\gamma$-divergence. In Section IV, we present our main result on the connection between contraction coefficient and the DP parameters of the iterative process (2). We then specialize this result to distributed SGD in Section V and online gradient descent in Section VI.

C. Notation

We use upper-case letters (e.g., $W$) to denote random variables and calligraphic letters to represent their alphabets (e.g., $\mathcal{W}$). The set of all distributions on $\mathcal{W}$, is denoted by $\mathcal{P}(\mathcal{W})$. For a signed measure $\phi$ over $\mathcal{W}$, its total variation is defined as

$$\|\phi\| := \phi^+ (\mathcal{W}) + \phi^- (\mathcal{W}),$$

where $(\phi^+, \phi^-)$ is the Hahn-Jordan decomposition of $\phi$ (see, e.g., [73, Page 421]). Observe that if $\mu$ and $\nu$ are probability measures,

$$\|\mu - \nu\| = 2 \sup_{A \subseteq \mathcal{W}} [\mu(A) - \nu(A)].$$

The total variation distance between two distributions $\mu$ and $\nu$ is

$$\text{TV}(\mu, \nu) := \frac{1}{2} \|\mu - \nu\|.$$
\( \mu \), and is given by \( \mu K := \int \mu(w)K(w) \). We use \( \mathbb{E}_\mu[\cdot] \) to write the expectation with respect to \( \mu \) and \([n]\) for an integer \( n \geq 1 \) to denote \( \{1, \ldots, n\} \). For \( a \in [0, 1] \), we define \( \bar{a} = 1 - a \). For a set \( D \subset \mathbb{R}^d \), we let \( \text{dia}(D) \) be its diameter, i.e.,
\[
\text{dia}(D) = \sup_{w_1, w_2 \in D} \|w_2 - w_1\|.
\]

II. BACKGROUND

In this section, we recall the framework for online learning and the definition of differential privacy in this context. Also, we briefly review definitions of \( f \)-divergence and contraction coefficient. Finally, we present a relation between differential privacy and a particular type of \( f \)-divergence known as \( \mathbb{E}_\gamma \)-divergence.

A. Online Learning Algorithms and Differential Privacy

Let \( \mathcal{W} \) denote a parameter space, e.g., the coefficients of a linear regression model. We describe typical online learning algorithms under Assumptions 1 and 2 in the next definition.

**Definition 1** (Online Algorithms). An online learning algorithm \( \mathcal{M} \) proceeds as follows. A learner initiates the algorithm by taking a random point \( W_1 \) from \( \mathcal{W} \). Once \( W_1 \) is chosen, a convex real-valued cost function \( f_1 : \mathcal{W} \rightarrow \mathbb{R} \) is revealed to her, implying the cost associated with \( W_1 \) is \( f_1(W_1) \). Upon observing \( f_t \) and \( W_t \), the learner at time \( t + 1 \) chooses \( W_{t+1} \) according to the update rule \( \Psi_t \), i.e., \( W_{t+1} = \Pi_{\mathcal{M}}(\Psi_t(W_t)) \). After \( n \) iterations, the algorithm outputs \( W_{n+1} \). Hence, letting \( F \) be the collection of all possible convex cost functions, the algorithm can be viewed as the mapping
\[
\mathcal{M} : F^n \rightarrow \mathcal{W},
\]
given by
\[
\mathcal{M}(\{f_1, \ldots, f_n\}) = W_{n+1}.
\]

For brevity, we denote \( \{f_1, \ldots, f_n\} \) by \( \{f_t\} \). It is worth noting that in this setting we assume the collection of cost functions \( \{f_1, \ldots, f_n\} \) are fixed ahead of time before the algorithm is started—often referred to as the oblivious setting in literature [17, Section 5.5]. We say that \( \mathcal{M} \) is a randomized online algorithm if \( W_{t+1} \) is a random variable on \( \mathcal{W} \) given \( \{f_t\} \) and \( W_t \).

The goal of the learner is to minimize regret, i.e., the difference between the cumulative cost and that of the best fixed (offline) solution in hindsight. Specifically, the regret of a randomized algorithm \( \mathcal{M} \) after \( n \) iterations is defined as
\[
R_{\mathcal{M}}(n) := \mathbb{E} \left[ \sum_{t=1}^{n} f_t(W_t) - \min_{w \in \mathcal{W}} \sum_{t=1}^{n} f_t(w) \right],
\]
where the expectation is taken over the algorithm’s randomness. Two examples of such algorithms are (1) one-pass empirical risk minimization (ERM) where \( f_t(w) = \ell(w, x_t) \) with \( \ell(\cdot, \cdot) \) being a loss function and \( x_t \) being the ground truth observation (e.g., stock price) at time \( t \), and (2) online gradient descent where \( f_t(w) = w - \eta_t \nabla f_t(w) \) with \( \eta_t \) being the learning rate at time \( t \). These two examples are studied in details in Sections V and VI, respectively. In all online learning algorithms a sublinear regret is sought, as it implies that asymptotically the online algorithm performs almost as well as the optimizer in hindsight.

In many practical scenarios, the cost functions \( \{f_t\} \) might leak private information about the learner. For example, linear cost in ERM problem may reveal significant information about an observation \( x_t \). Other examples of privacy leakage in online algorithms are listed in [74, 75]. As such, we focus on privacy attacks aimed to infer information about the cost function \( f_t \) for some \( i \in [n] \) upon observing the output of the algorithm, i.e., \( \mathcal{M}(\{f_i\}) = W_{n+1} \). Thus, our goal is to design online algorithms such that \( W_{n+1} \) does not reveal significant information about any single cost function in \( \{f_t\} \). The following
definition formalizes this goal. We say that two collection of cost functions \( \{f_t\} \) and \( \{f'_t\} \) are neighboring at index \( i \in [n] \), if \( f_t = f'_t \) for all \( t \in [n] \setminus i \) and \( f_i \neq f'_i \). We denote this by \( \{f_t\} \sim \{f'_t\} \).

**Definition 2** ([2]). A randomized online algorithm \( \mathcal{M} \) is said to be \((\varepsilon, \delta)\)-differentially private, for \( \varepsilon \geq 0 \) and \( \delta \in [0, 1] \), if we have

\[
\sup_{i \in [n]} \sup_{\{f_t\} \sim \{f'_t\}} \sup_{A \subseteq \mathcal{W}} \left[ \Pr(\mathcal{M}(\{f_t\}) \in A) - e^{\varepsilon} \Pr(\mathcal{M}(\{f'_t\}) \in A) \right] \leq \delta.
\]

**Remark 1.** The above definition of DP is specific to online learning. Here, a pair of neighboring collections of cost functions is considered rather than the "neighboring datasets" used in batch learning scenarios. Definition 2 is also the standard notion of DP for online learning algorithm considered in [13–16, 76]. Nevertheless, Definition 2 differs from previous works on differentially private online algorithms in that privacy is ensured with respect to only the last parameter \( W_{n+1} \) rather than the entire set of parameters \( \{W_1, \ldots, W_{n+1}\} \).

In order to quantify the DP parameters \( \varepsilon \) and \( \delta \) of an online algorithm, we model each of its iterations as a Markov kernel (i.e., a channel) and apply the relation between the DP constraint and the contraction coefficient of a Markov kernel, defined next.

**B. f-Divergence and Contraction Coefficients**

Given a convex function \( f : (0, \infty) \rightarrow \mathbb{R} \) such that \( f(1) = 0 \), the \( f \)-divergence between two probability measures \( \mu \) and \( \nu \) with \( \mu \ll \nu \) is defined in [77] and [78] as

\[
D_f(\mu \parallel \nu) := \mathbb{E}_\nu \left[ f \left( \frac{d\mu}{d\nu} \right) \right].
\]

Let \( \mathcal{D}, \mathcal{W} \subset \mathbb{R}^d \) and \( K : \mathcal{D} \rightarrow \mathcal{P}(\mathcal{W}) \) be a Markov kernel. Following the information-theoretic approach initiated by Ahlswede and Gács [34], we define the contraction coefficient (or strong data processing coefficient) of \( K \) under \( D_f(\cdot \parallel \cdot) \) as

\[
\eta_f(K) := \sup_{\mu, \nu \in \mathcal{P}(\mathcal{D}) : D_f(\mu \parallel \nu) \neq 0} \frac{D_f(\mu \parallel \nu K)}{D_f(\mu \parallel \nu)}. \tag{4}
\]

This quantity has been long studied for several \( f \)-divergences, e.g., KL-divergence for which \( f(t) = t \log(t) \), \( \chi^2 \)-divergence for which \( f(t) = (t - 1)^2 \), and total variation distance for which \( f(t) = \frac{1}{2} |t - 1| \). In particular, Dobrushin [38] showed that the contraction coefficient \( \eta_{TV} \) under total variation distance TV has the remarkably simple expression

\[
\eta_{TV}(K) = \sup_{w_1, w_2 \in \mathcal{W}} TV(K(w_1), K(w_2)). \tag{5}
\]

It is worth noting that (5) has been extensively used as a main tool for studying ergodicity of Markov processes as well as Gibbs measures, e.g., see [35, 38, 60].

**C. Differential Privacy and \( E_\gamma \)-Divergence**

The definition of \((\varepsilon, \delta)\)-DP given in Definition 2 can be reformulated in terms of a certain \( f \)-divergence, namely \( E_\gamma \)-divergence (aka hockey-stick divergence) [79–81]. Given \( \gamma \geq 1 \), the \( E_\gamma \)-divergence between two probability measures \( \mu \) and \( \nu \) is defined as

\[
E_\gamma(\mu \parallel \nu) := \int_{\mathcal{W}} d(\mu - \gamma \nu)^+ = \sup_{A \subseteq \mathcal{W}} [\mu(A) - \gamma \nu(A)] \tag{6}
\]
Theorem 3. Let \( \eta \) to derive the contraction coefficient of the projected additive Gaussian kernel. For ease of notation, we under analysis of popular additive-noise mechanism in DP literature, e.g., the Gaussian mechanism.

Lemma 1. A Markov kernel. A similar approach has been exploited in [12] to quantify the improvement of DP shall make this relationship precise in Section IV by casting each iteration of an iterative algorithm as a Markov kernel. A similar approach has been exploited in [12] to quantify the improvement of DP parameters due to post-processing (aka privacy amplification).

By relating DP to \( E_\gamma \)-divergence, this theorem enables us to quantify the DP parameters of randomized algorithms via the contraction coefficient of Markov kernels under \( E_\gamma \)-divergence (see Theorem 1). We shall make this relationship precise in Section IV by casting each iteration of an iterative algorithm as a Markov kernel. A similar approach has been exploited in [12] to quantify the improvement of DP parameters due to post-processing (aka privacy amplification).

Before formalizing this relationship, we first compute in the next section the contraction coefficient of Gaussian Markov kernels under \( E_\gamma \)-divergence. This contraction coefficient is instrumental for the privacy analysis of popular additive-noise mechanism in DP literature, e.g., the Gaussian mechanism.

III. CONTRACTION COEFFICIENT OF MARKOV KERNELS UNDER \( E_\gamma \)-DIVERGENCE

In this section we establish a closed-form expression for the contraction coefficient of Markov kernels under \( E_\gamma \)-divergence which generalizes Dobrushin’s formula in (5). We then instantiate this expression to derive the contraction coefficient of the projected additive Gaussian kernel. For ease of notation, we let \( \eta_\gamma(K) := \eta_{E_\gamma}(K) \).

Theorem 3. Let \( D, W \subset \mathbb{R}^d \) and \( K : D \to \mathcal{P}(W) \) be a Markov kernel. For any \( \gamma \geq 1 \), we have

\[
\eta_\gamma(K) = \sup_{w_1, w_2 \in D} \frac{E_\gamma(K(w_1))}{E_\gamma(K(w_2))}. \tag{8}
\]

The proof of this theorem is given in Appendix A. Similar to Dobrushin’s result (5) in characterizing \( \eta_{TV} \), this theorem enables us to reduce the computation of \( \eta_\gamma(K) \) to evaluating the \( E_\gamma \)-divergence between \( K(w_1) \) and \( K(w_2) \) for any two inputs \( w_1, w_2 \in D \). The following lemma (proved in Appendix B) is useful for this purpose.

Lemma 1. For \( m \in \mathbb{R}^d \) and \( \sigma > 0 \), let \( \mathcal{N}(m, \sigma^2 I_d) \) denote the multivariate Gaussian distribution with mean \( m \) and covariance matrix \( \sigma^2 I_d \). If \( m_1, m_2 \in \mathbb{R}^d \) and \( \sigma > 0 \), then

\[
E_\gamma(\mathcal{N}(m_1, \sigma^2 I_d)||\mathcal{N}(m_2, \sigma^2 I_d)) = Q \left( \frac{\log \gamma}{\beta} - \frac{\beta}{2} \right) - \gamma Q \left( \frac{\log \gamma}{\beta} + \frac{\beta}{2} \right),
\]

where \( \beta = \frac{\|m_2 - m_1\|}{\sigma} \).

This lemma motivates the following definition.

Definition 3. For \( \gamma \geq 1 \), we define \( \theta_\gamma \) at \( [0, \infty) \rightarrow [0, 1] \) by

\[
\theta_\gamma(r) := E_\gamma(\mathcal{N}(ru, I_d)||\mathcal{N}(0, I_d)) = Q \left( \frac{\log \gamma}{r} - \frac{r}{2} \right) - \gamma Q \left( \frac{\log \gamma}{r} + \frac{r}{2} \right),
\]

where \( u \) is a vector of length \( d \).
where \( u \in \mathbb{R}^d \) is any vector of unit norm.

**Remark 2.** Notice that the \( E_\gamma \)-divergence between Gaussian distributions \( \mathcal{N}(m_1, \sigma^2 I_d) \) and \( \mathcal{N}(m_2, \sigma^2 I_d) \) can be concisely expressed as

\[
E_\gamma(\mathcal{N}(m_1, \sigma^2 I_d) \| \mathcal{N}(m_2, \sigma^2 I_d)) = \theta_\gamma \left( \frac{\|m_2 - m_1\|}{\sigma} \right).
\]

(9)

The mapping \( r \mapsto \theta_\gamma(r) \) is closely related to the functions \( R_\alpha \), introduced by Feldman et al. in [53, Definition 10], and \( \theta(r) \), introduced by Polyanskiy and Wu in [60, Section 2.2].

The additive Gaussian kernel \( K : \mathbb{R}^d \rightarrow \mathcal{P}(\mathbb{R}^d) \) is the kernel determined by \( K(w) = \mathcal{N}(w, \sigma^2 I_d) \) for some \( \sigma > 0 \). This kernel can represent the privacy mechanism which maps \( w \mapsto w + \sigma Z \) with \( Z \sim \mathcal{N}(0, I_d) \). An application of Theorem 3 and Lemma 1 shows that, under \( E_\gamma \)-divergence, the contraction coefficient of the additive Gaussian kernel is trivial, i.e., \( \eta_\gamma(K) = 1 \). Fortunately, the input and output of kernels appearing in applications tend to be bounded. Consider, for example, the kernel which models the online update of parameters of a machine learning model during training (e.g., weights of a neural network). In this case the parameters are bounded, either explicitly by design, implicitly by some regularization mechanisms, or simply due to finite precision of floating-point operations in computers. Motivated by this observation, we define the projected additive Gaussian kernel as follows.

**Definition 4.** Let \( D \subset \mathbb{R}^d \) be a bounded set, \( W \subset \mathbb{R}^d \) a closed convex set, and \( \sigma > 0 \). The projected additive Gaussian kernel associated with \((D, W, \sigma)\) is the kernel \( K : D \rightarrow \mathcal{P}(W) \) which models the map \( w \mapsto \Pi_W(w + \sigma Z) \) where \( \Pi_W(\cdot) \) is the projection operator onto \( W \) and \( Z \sim \mathcal{N}(0, I_d) \).

The following proposition establishes \( \eta_\gamma \) for projected Gaussian kernels.

**Proposition 1.** Let \( \gamma \geq 1 \), \( D \subset \mathbb{R}^d \) a bounded set, \( W \subset \mathbb{R}^d \) a closed convex set, and \( \sigma > 0 \). If \( K : D \rightarrow \mathcal{P}(W) \) is the projected additive Gaussian kernel associated with \((D, W, \sigma)\), then

\[
\eta_\gamma(K) = \theta_\gamma \left( \frac{\text{dia}(D)}{\sigma} \right).
\]

The proof of this proposition is deferred to Appendix C. Having identified \( \eta_\gamma \) for projected additive Gaussian kernels, we invoke the SDPI relationship and Theorem 2 to derive the DP parameters of iterative process (2) in the next section.

**A. Digression: An upper bound for output \( f \)-divergences**

If \( f \) is twice differentiable and convex, then \( D_f(\mu \| \nu) \) can be expressed in terms of \( E_\gamma \)-divergence [84, Proposition 3] as

\[
D_f(\mu \| \nu) = \int_1^\infty f''(\gamma)E_\gamma(\mu \| \nu) + \gamma^{-3} f''(\gamma^{-1})E_\gamma(\nu \| \mu) d\gamma
\]

for any pair of probability measures \( \mu \) and \( \nu \). Combined with Theorem 3, this leads to the following general result.

**Proposition 2.** Let \( f : (0, \infty) \rightarrow \mathbb{R} \) be a convex and twice differentiable function with \( f(1) = 0 \) and \( K : W \rightarrow \mathcal{P}(W) \) be a Markov kernel. Then, for any pair of probability measures \( \mu, \nu \in \mathcal{P}(W) \) we have

\[
D_f(\mu \| \nu K) \leq \int_1^\infty \eta_\gamma(K) \left[ f''(\gamma)E_\gamma(\mu \| \nu) + \gamma^{-3} f''(\gamma^{-1})E_\gamma(\nu \| \mu) \right] d\gamma.
\]

\( ^1 \)This is not surprising given the facts that \( \eta_{TV}(K) = 1 \) for any Gaussian channels \( K \) without input constraints [60] and \( \eta_{TV}(K) = 1 \) if and only if \( \eta_f(K) = 1 \) for any non-linear functions \( f \) [83, Proposition II.4.12].
The previous proposition can be contrasted with the celebrated result

$$\eta_f(K) \leq \eta_{TV}(K),$$

for any convex functions $f$ satisfying $f(1) = 0$ and Markov kernels $K$, originally proved in [83] for discrete case and subsequently extended to the general case in [85]. The inequality (11) implies that

$$D_f(\mu K \| \nu K) \leq \eta_{TV}(K) D_f(\mu \| \nu),$$

for any convex functions $f$ satisfying $f(1) = 0$, Markov kernels $K$, and any pair of probability measures $\mu$ and $\nu$. To see the improvement of Proposition 2 over (12), we consider $\chi^2$-divergence (i.e., $f(t) = (t-1)^2$) in the following two examples.

**Example 1.** Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(2, \sigma^2)$. Furthermore, let $X_B$ and $Y_B$ be obtained by projecting $X$ and $Y$ onto $B = \{x \in \mathbb{R} : |x| \leq \frac{1}{2}\}$, respectively. Denote their probability distributions by $\mu$ and $\nu$, respectively. Suppose $K$ is a projected Gaussian kernel associated with $(B, B, \sigma)$. In this case, $\eta_{TV}(K) = \theta_1\left(\frac{1}{b}\right)$ and $\eta_2(K) = \theta_2\left(\frac{1}{b}\right)$. Thus, according to (12) we have for $\sigma = 1$

$$\chi^2(\mu K \| \nu K) \leq \eta_{TV}(K) \chi^2(\mu \| \nu)$$

$$= 2\theta_1(1) \int_1^{\infty} \left[ E_\gamma(\mu \| \nu) + \gamma^{-3} E_\gamma(\nu \| \mu) \right] d\gamma$$

$$\leq 2\theta_1(1) \int_1^{\infty} \left[ E_\gamma(\mathcal{N}(0, 1) \| \mathcal{N}(2, 1)) + \gamma^{-3} E_\gamma(\mathcal{N}(2, 1) \| \mathcal{N}(0, 1)) \right] d\gamma$$

$$= 2\theta_1(1) \int_1^{\infty} \left[ (1 + \gamma^{-3}) \theta_2(2) \right] d\gamma$$

$$= 0.49,$$

where the first equality comes from (10) and the second inequality is due to the data processing inequality. It, however, follows from Proposition 2 that

$$\chi^2(\mu K \| \nu K) \leq 2 \int_1^{\infty} \theta_1(1) \left[ E_\gamma(\mu \| \nu) + \gamma^{-3} E_\gamma(\nu \| \mu) \right] d\gamma$$

$$\leq 2 \int_1^{\infty} \theta_1(1) \left[ E_\gamma(\mathcal{N}(0, 1) \| \mathcal{N}(2, 1)) + \gamma^{-3} E_\gamma(\mathcal{N}(2, 1) \| \mathcal{N}(0, 1)) \right] d\gamma$$

$$= 2 \int_1^{\infty} \theta_1(1) \left[ (1 + \gamma^{-3}) \theta_2(2) \right] d\gamma$$

$$= 0.26.$$

**Example 2.** Suppose $K$ is the binary input binary output channel with crossover probabilities $a, b \in [0, \frac{1}{2}]$ defined by

$$K^{a,b} = \begin{bmatrix} \bar{a} & a \\ b & \bar{b} \end{bmatrix}.$$  

When $a = b = \tau$, then $K^{\tau,\tau}$ corresponds to the binary symmetric channel with crossover probability $\tau \in [0, \frac{1}{2}]$. It can be easily verified that $\eta_2(K^{a,b}) = \max\{(\bar{a} - \gamma b)^+, (\bar{b} - \gamma a)^+\}$ for all $\gamma \geq 1$ and in particular $\eta_{TV}(K^{a,b}) = 1 - a - b$. Applying (12), we obtain for any binary probability measures $\mu$ and $\nu$

$$\chi^2(\mu K^{a,b} \| \nu K^{a,b}) \leq (1 - a - b) \chi^2(\mu \| \nu)$$

$$= 2(1 - a - b) \int_1^{\infty} \left[ E_\gamma(\mu \| \nu) + \gamma^{-3} E_\gamma(\nu \| \mu) \right] d\gamma,$$

$$= \frac{2(1 - a - b)}{\theta_2(2)}.$$
where the equality comes from (10). Thus, for \( a = 0.1, b = 0.4, \mu = \text{Bernoulli}(0.1), \) and \( \nu = \text{Bernoulli}(0.4) \) the bound (12) yields
\[
\chi^2(\mu K^{a,b}||\nu K^{a,b}) \leq 0.19.
\]

On the other hand, Proposition 2 renders
\[
\chi^2(\mu K^{a,b}||\nu K^{a,b}) \leq 2 \int_{1}^{\max\{\frac{a}{b}, \frac{b}{a}\}} \max\{(a - \gamma b)^+, (b - \gamma a)^+\} \left[ E_\gamma(\mu||\nu) + \gamma^{-2} E_\gamma(\nu||\mu) \right] d\gamma,
\]
implying for the same values \( a \) and \( b \) and probability measures \( \mu, \nu \)
\[
\chi^2(\mu K^{a,b}||\nu K^{a,b}) \leq 0.17.
\]

IV. PRIVACY ANALYSIS OF NOISY ITERATIVE ALGORITHMS

In this section, we develop a technique for computing the DP parameters of general iterative process (2) via the contraction coefficient introduced in the previous section. The main goal is to establish a bound for the DP parameters \( \varepsilon \) and \( \delta \) achievable by adding Gaussian noise to the update rules \( \{\Psi_{f_t}\} \).

The subsequent analysis is based on two key observations.

- **Iterative processes as composition of Markov kernels:** Assume \( \{Z_t\} \) are i.i.d. samples drawn from a Gaussian distribution. As illustrated in Fig. 1, the iterative process (2) can be viewed as a collection of Markov kernels \( \{K_{t_i}\} \) for \( t \in [n] \) where each \( K_{t_i} \) is a concatenation of the update rule \( \Psi_{f_t} \), an additive Gaussian channel with variance \( \sigma_f^2 \), and the projection operator \( \Pi_{\mathcal{W}}(\cdot) \). Consequently, we can express \( K_{t_i} \) as \( K_{t_i} \circ \Psi_{f_t} : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{W}) \) where \( K_t \) is the projected additive Gaussian Markov kernel associated with \( (\Psi_{f_t}(\mathcal{W}), \mathcal{W}, \sigma_f) \). Consequently, the distribution of the final parameter \( W_{n+1} \) is given by \( \mu_1 K_{t_1} \cdots K_{t_n} \), where \( \mu_1 \) is the distribution from which the initial point \( W_1 \) is sampled.

- **Per-Iteration DP:** The overall premise of the DP framework is that \( \mu_{n+1} \) (the distribution of \( W_{n+1} \) the final output of the process) does not change significantly when the family of input cost functions \( \{f_t\} \) are replaced by \( \{f'_t\} \) satisfying \( \{f_t\} \overset{\text{d}}{\sim} \{f'_t\} \) for some \( i \in [n] \). According to Theorem 2, DP parameter \( \delta \) is given by the \( E_\gamma \)-divergence between \( \mu_{n+1} \) and \( \mu'_{n+1} \). Due to the additive nature of the process, this divergence might, however, change drastically depending on \( i \). To see this, notice that if \( i = n \), then \( E_\gamma(\mu_{n+1}||\mu'_{n+1}) \) depends only the Markov kernel \( K_{t_n} \), whereas if \( i = 1 \), then \( E_\gamma(\mu_{n+1}||\mu'_{n+1}) \) is the output divergence of a concatenation of \( n \) Markov kernels \( \{K_{t_i}\}_{i=1}^n \). To account for this (rather unpleasant) phenomenon, Feldman et al. [53] adopted a modified definition of DP (see also [86]) given below.

**Definition 5** ([53]). A randomized online algorithm \( \mathcal{M} \) satisfying Assumption 2 is said to be \((\varepsilon, \delta)\)-DP at index \( i \in [n] \) if
\[
\sup_{\{f_t\}, \{f'_t\}} \sup_{A \subseteq \mathcal{W}} \left[ \Pr(\mathcal{M}(\{f_t\}) \in A) - e^{\varepsilon} \Pr(\mathcal{M}(\{f'_t\}) \in A) \right] \leq \delta, \tag{14}
\]
or equivalently,
\[
\sup_{\{f_t\}, \{f'_t\}} E_{\nu'}(\mu_{n+1}||\mu'_{n+1}) \leq \delta.
\]

In our privacy analysis, we first exploit the contraction coefficient under \( E_\gamma \)-divergence (see (4)) to compute the DP parameters at a given index \( i \in [n] \) and then use the joint convexity of \( E_\gamma \)-divergence to "homogenized" it over all \( i \). The following lemma is key for our privacy analysis (see Appendix D for a proof).
Assume that \( \text{Theorem 5.} \)

\[
\delta \quad \text{per-iteration DP parameter}
\]

result is the linchpin for our privacy analysis in the sequel. \( \delta \) per-iteration privacy guarantee given in this theorem to privacy guarantee independent of index \( i \). To this end, we follow \[53\] to consider the randomly-stopped variant of process (2): Instead of terminating after pre-determined \( n \) iterations, the algorithm stops at a random time \( T \) uniformly chosen in \([n]\).

The detailed proof of this theorem is given in Appendix E. We will use this theorem to quantify the DP guarantee of popular algorithms. Before we delve into instantiating this result, we first convert the per-iteration privacy guarantee given in this theorem to privacy guarantee independent of index \( i \).

While the above lemma follows from a routine application of SDPI, it provides a natural framework to study privacy guarantees of general iterative processes. In fact, inequality (15) allows us to obtain per-iteration DP parameter \( \delta \) of iterative process (2) by bounding \( E_{\epsilon^*}(\mu_i K_{f_{i}} \| \mu_i K_{f_{i}'} \) and \( \eta_{\epsilon^*}(K_{f_{i}}) \). This result is the linchpin for our privacy analysis in the sequel.

**Theorem 4.** Assume that \( \mathcal{W} \subset \mathbb{R}^d \) is a closed convex set and \( P_Z = N(0, I_d) \). The iterative process (2) is \((\epsilon, \delta)\)-DP at index \( i \in [n] \) for \( \epsilon \geq 0 \) and \( \delta \) given by

\[
\delta = \epsilon^* \left( \frac{\psi}{\sigma_i} \right) \sum_{t = i+1}^{n} \theta_{\epsilon^*} \left( \frac{\text{dia}(\Psi_{f_i}(\mathcal{W}))}{\sigma_t} \right),
\]

where \( \psi := \sup_{f_i, f \in F, w \in \mathcal{W}} \| \Psi_{f_i}(w) - \Psi_f(w) \| \).

The detailed proof of this theorem is given in Appendix E. We will use this theorem to quantify the DP guarantee of popular algorithms. Before we delve into instantiating this result, we first convert the per-iteration privacy guarantee given in this theorem to privacy guarantee independent of index \( i \). To this end, we follow \[53\] to consider the randomly-stopped variant of process (2): Instead of terminating after pre-determined \( n \) iterations, the algorithm stops at a random time \( T \) uniformly chosen in \([n]\).

**Theorem 5.** Assume that \( \mathcal{W} \subset \mathbb{R}^d \) is a closed convex set and \( P_Z = N(0, I_d) \). The randomly-stopped iterative process described in Algorithm 1, is \((\epsilon, \delta)\)-DP for \( \epsilon \geq 0 \) and

\[
\delta = \min_{i \in [n]} \left\{ \frac{1}{n} \sum_{t = i}^{n} \theta_{\epsilon^*} \left( \frac{\psi}{\sigma_t} \right) \prod_{j = i+1}^{t} \theta_{\epsilon^*} \left( \frac{D}{\sigma_j} \right) \right\},
\]

where \( \psi := \sup_{f_i, f \in F, w \in \mathcal{W}} \| \Psi_{f_i}(w) - \Psi_f(w) \| \).

\[
\delta \leq \frac{1}{n} \theta_{\epsilon^*} \left( \frac{\psi}{\sigma} \right) \left[ 1 - \theta_{\epsilon^*} \left( \frac{D}{\sigma} \right) \right]^{-1},
\]

\( \text{Here, we use the convention that } \prod_{t = i+1}^{n} \eta_f(K_{f_{i}}) = 1 \text{ for } i = n. \)
In this setup each cost function $f_t$ depends only on data point $x_t$. Thus, the neighboring relationship between two collections of cost functions $\{f_t\}$ and $\{f'_t\}$ reduces to that of the datasets $D = \{x_1, \ldots, x_n\}$ and $D' = \{x'_1, \ldots, x'_n\}$. That is, assuming $x \leftrightarrow \ell(\cdot, x)$ is injective, we have $\{f_t\} \sim \{f'_t\}$ if and only if the datasets $D \sim D'$, meaning $x_i \neq x'_i$ and $x_j = x'_j$ for all $j \in [n]\{i\}$. Consequently, the definition of DP in Definition 2 can be equivalently given in terms of neighboring datasets and thus reduces to the original definition of DP in [2].

The randomly-stopped PNSGD algorithm is described in Algorithm 2. The privacy guarantee of this algorithm has been recently studied by Feldman et al. [53] under the name of privacy amplification by}
Algorithm 2 Randomly stopped Projected Noisy SGD (PNSGD)

Require: Dataset $\mathcal{D} = \{x_1, \ldots, x_n\}$, learning rate $\eta > 0$, convex set $\mathcal{W} \subset \mathbb{R}^d$, and noise parameter $\sigma$
1: Pick $W_1 \sim \mu_1 \in \mathcal{P}(\mathcal{W})$
2: Take $T$ uniformly on $[n]$ 
3: for $t \in \{1, \ldots, T\}$ do 
4: \hspace{1em} $W_{t+1} = \Pi_{\mathcal{W}}(W_t - \eta[\nabla_w \ell(W_t, x_t) + \sigma Z_t]), \quad Z_t \sim \mathcal{N}(0, \mathbf{I}_d)$ 
5: end for 
6: return $W_{T+1}$

iteration. However, the notion of privacy used in their work is Rényi differential privacy that is known to suffer from lack of operational interpretation [37]. We can now apply the machinery developed in Section IV to derive the privacy guarantee of this algorithm directly in terms of DP. Specifically, the following corollary specializes Theorem 5 to the PNSGD algorithm; with the proof given in Appendix G.

Corollary 1. Let $\mathcal{W} \subset \mathbb{R}^d$ be a convex set with diameter 1 and $\{\ell(\cdot, x)\}_{x \in \mathcal{X}}$ be a family of convex $L$-Lipschitz functions over $\mathcal{W}$. Then the randomly-stopped PNSGD algorithm is $(\varepsilon, \delta)$ with $\varepsilon \geq 0$ and

$$
\delta = \frac{1}{n} \theta e \left( \frac{2L}{\sigma} \right) \left[ 1 - \theta e \left( \frac{1 + 2\eta L}{\eta \sigma} \right) \right]^{-1}.
$$

(19)

If we further assume that $w \mapsto \ell(w, x)$ is $3\beta$-smooth for any $x \in \mathcal{X}$, then a standard calculation in convex optimization shows that $w \mapsto \Psi^{\text{SGD}}(w)$ is 1-Lipschitz for $\eta \leq \frac{\beta}{3}$ (see Appendix H for a detailed proof). A similar argument as in the proof of Corollary 1 reveals that with this extra assumption (19) can be improved as (cf. Appendix H)

$$
\delta = \frac{1}{n} \theta e \left( \frac{2L}{\sigma} \right) \left[ 1 - \theta e \left( \frac{1}{\eta \sigma} \right) \right]^{-1},
$$

(20)

if $\eta \leq \frac{2}{\beta}$. This enables us to formally compare our result with [53]. To do so, we need the following definition. Given $\alpha > 1$, a mechanism $\mathcal{M}$ is called $(\alpha, \zeta)$-Rényi differentially-private (RDP) if

$$
\sup_{\mathbb{D} \sim \mathbb{D}'} D_{\alpha}(\mu_{n+1}||\mu'_{n+1}) \leq \zeta,
$$

where $D_{\alpha}(\cdot||\cdot)$ denotes the Rényi divergence of order $\alpha$ and $\mu_{n+1}$ and $\mu'_{n+1}$ are the distributions of $W_{n+1}$ and $W'_{n+1}$ the outputs of $\mathcal{M}$ when running on neighboring $\mathbb{D}$ and $\mathbb{D}'$, respectively.

Theorem 6 ([53, Theorem 26]). Let $\mathcal{W} \subset \mathbb{R}^d$ be a convex set and $\{\ell(\cdot, x)\}_{x \in \mathcal{X}}$ be a family of convex, $L$-Lipschitz and $\beta$-smooth loss functions over $\mathcal{W}$. Then, for any $\eta \leq \frac{3}{\beta}$ and $\alpha > 1$, and $\sigma \geq L \sqrt{2(\alpha - 1)\alpha}$, the randomly-stopped PNSGD algorithm is $(\alpha, \zeta)$-RDP for

$$
\zeta = \frac{4\alpha L^2 \log n}{n \sigma^2}.
$$

While Corollary 1 provides the privacy guarantee for any $\sigma > 0$, this theorem is restricted to $\sigma$ greater than $L \sqrt{2(\alpha - 1)\alpha}$. This discrepancy stems from the fact that, unlike $\epsilon, \delta$-divergence, $(\mu||\nu) \rightarrow D_{\alpha}(\mu||\nu)$ is not convex for $\alpha > 1$. To get around this issue, Feldman et al. [53, Lemma 25] presented a "weak" form of joint convexity for the Rényi divergence.

In order to compare Corollary 1 with Theorem 6, we need to convert $(\alpha, \zeta)$-RDP guarantee to $(\varepsilon, \delta)$-DP. To do so, we invoke two existing RDP-to-DP conversion formulae, making both analytical and numerical comparison possible.

A function $f : \mathcal{W} \rightarrow \mathbb{R}$ is $\beta$-smooth if $\|\nabla f(w_1) - \nabla f(w_2)\| \leq \beta \|w_1 - w_2\|$ for all $w_1, w_2 \in \mathcal{W}$.
Fig. 2. The privacy parameters of PNSGD Algorithm 2 obtained from Corollary 1 and converting [53, Theorem 26] according to (22) and (23), respectively. Here, we vary the learning rate $\eta$ and $\sigma$. Other parameters are as follows: $L = 1, \beta = 1$, and $n = 100$.

Lemma 3 ([56, Proposition 3]). If a mechanism $\mathcal{M}$ is $(\alpha, \zeta)$-RDP for $\alpha > 1$, then it is $(\varepsilon, e^{-(\alpha-1)(\varepsilon-\zeta)})$-DP.

Due to the efficiency of RDP (especially in the context of composition), this lemma has been extensively used in many recent private ML algorithms, see e.g., [5, 11, 89, 90] and has been implemented in the Google’s open-source TensorFlow Privacy\textsuperscript{4} [4]. In light of this result, Theorem 6 implies that the randomly stopped PNSGD is $(\varepsilon, \hat{\delta})$-DP for any $\varepsilon \geq 0$ and

$$\hat{\delta} := \inf_{\alpha \in (1, \alpha^*)} e^{-(\alpha-1)(\varepsilon-\zeta)},$$

(21)

where $\alpha^* := \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{2\varepsilon^2}{L^2}} \right]$ and $\zeta = \frac{4\alpha L^2 \log n}{n \alpha^2}$. The restriction on the range of $\alpha$ was caused by the constraint on $\sigma$ in Theorem 6. Since $\zeta$ is linear in $\alpha$, the minimization in (21) can be solved analytically:

$$\hat{\delta} = e^{-(\alpha^* - 1)(\varepsilon - \rho \alpha^*)},$$

(22)

where $\rho := \frac{4L^2 \log n}{n \sigma^2}$ and $\alpha^* := \begin{cases} \alpha^*, & \text{if } \varepsilon^2 \geq \rho^2 \left( 1 + \frac{2\varepsilon^2}{L^2} \right), \\ \frac{1}{2} + \frac{\varepsilon^2}{2\rho}, & \text{otherwise}. \end{cases}$

Hence, for sufficiently large $\varepsilon$ (i.e., $\varepsilon \geq O\left( \frac{\log n}{n} \right)$), $\hat{\delta}$ behaves approximately like $e^{-O(\varepsilon)}$ while our privacy analysis yields $\delta$ in (20) that decays at the rate $e^{-O(\varepsilon^2)}$.

Despite its prevalence in practice, Lemma 3 is not tight in general. This issue has recently prompted Asoodeh et al. [57] to derive the optimal translation of $(\alpha, \zeta)$-RDP into $(\varepsilon, \delta)$-DP. Invoking [57, Lemma 2], we deduce that the randomly stopped PNSGD is $(\varepsilon, \tilde{\delta})$-DP for $\varepsilon \geq 0$ and

$$\tilde{\delta} := \inf_{\alpha \in (1, \alpha^*)} \min \left\{ \kappa e^{-(\alpha-1)(\varepsilon-\zeta)}, \frac{e^{(\alpha-1)\zeta} - 1}{\alpha (e^{(\alpha-1)\varepsilon} - 1)} \right\},$$

(23)

where $\kappa := \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right)^{\alpha-1}$ and $\zeta = \frac{4\alpha L^2 \log n}{n \sigma^2}$. Although this result is tighter than Lemma 3, it is not possible to analytically express $\tilde{\delta}$. In Fig. 2, we compare the privacy parameters of the randomly-stopped PNSGD given in Corollary 1 with $\hat{\delta}$ and $\tilde{\delta}$ given in (22) and (23), respectively. As illustrated in this

\textsuperscript{4}See the function “compute_eps” in the analysis directory of TensorFlow Privacy in [91]. At the time of submission of this paper, Lemma 3 is particularly implemented in the line 216 of [91].
where \( Z \) it is implicitly assumed in this definition that the algorithm stops at time \( \mu \). However, these assumptions are not applicable in our setting: the algorithm stops at \( T \) where the expectation is taken with respect to the randomness in stopping time \( F \) over \( (0, \infty) \).

Note that \( nSR \) is a standard quantity for measuring the utility of such algorithm is regret (defined in (3)). However, Corollary 1 over [53] is more pronounced for smaller values of \( \eta \), for sufficiently large \( \varepsilon \), compared to the technique developed in [53]. Inspired by Chen et al. [92], we define stochastic regret of the PNSGD algorithm. Corollary 1 formalizes this intuition by establishing a precise rate at which \( \delta \) decreases as \( \eta \) increases for any given \( \varepsilon \).

VI. PRIVACY-UTILITY TRADEOFF OF THE ONLINE GRADIENT DESCENT ALGORITHM

In this section, we apply the technique developed in Section IV to study the privacy guarantee of the online gradient descent (OGD) algorithm [39]. By drawing on standard results from online convex optimization, we also expound the tradeoff between privacy and utility in the OGD algorithm.

The OGD algorithm is an instance of the iterative process (2) with update function

\[
\Psi_{t}^{\text{OGD}}(w) := w - \eta_{t} \nabla f_{t}(w).
\]

Similar to Section V, we add Gaussian noise to the gradient term and thus the noisy OGD algorithm iterates as

\[
W_{t+1} = \Pi_{W}(W_{t} - \eta_{t} \nabla f_{t}(W_{t}) + \eta_{t} \sigma_{t} Z_{t}),
\]

where \( Z_{t} \sim \mathcal{N}(0, I_{d}) \).

A standard quantity for measuring the utility of such algorithm is regret (defined in (3)). However, it is implicitly assumed in this definition that the algorithm stops at time \( n \) and all \( W_{1}, \ldots, W_{n+1} \) are available for analysis. However, these assumptions are not applicable in our setting: the algorithm stops at a random time \( T \in [n] \) and only \( W_{n+1} \) is available for analysis. We therefore need to consider a variant of regret that takes these requirements into consideration.

Recall that \( F \) denotes the collection of all possible real-valued cost functions over \( W \). Here, we assume that \( F \) consists of all convex functions over \( W \) with uniformly bounded gradients and let \( \mu \) be a distribution over \( F \). We further assume that the cost functions \( \{f_{t}\} \) are independently sampled from \( F \) according to \( \mu \). Inspired by Chen et al. [92], we define stochastic regret for the randomly stopped OGD algorithm as

\[
SR(n) := \mathbb{E}[f(W_{T})] - \inf_{w \in W} f(w),
\]

where the expectation is taken with respect to the randomness in stopping time \( T \) and \( f(w) = \mathbb{E}_{\mu}[f_{t}(w)] \).

Note that \( nSR(n) \) can be thought of as the average regret of an algorithm when applied several times. The utility of the randomly stopped OGD algorithm is stated next.

**Proposition 3.** Assume that \( W \subset \mathbb{R}^{d} \) is a closed convex set, \( F \) is a family of convex functions over \( W \) with uniformly bounded gradients, \( \{f_{t}\} \) are \( n \) independent samples from \( F \), and \( P_{Z} = \mathcal{N}(0, I_{d}) \). If \( \eta_{t} = \frac{\text{dia}(W)}{M \sqrt{t}} \) with \( M = \sup_{f_{t} \in F} \sup_{w \in W} \| \nabla f_{t}(w) \| \), then the randomly stopped OGD algorithm satisfies

\[
SR(n) \leq \frac{3M\text{dia}(W)}{2 \sqrt{n}} + \frac{d}{2n} \sum_{t=1}^{n} \eta_{t} \sigma_{t}^{2}.
\]

The previous proposition follows from standard results in online convex optimization, see, e.g., [17]. For the reader’s convenience, we provide a full proof in Appendix I. Combined with Theorem 5, this

\footnote{Abadi et al. [11] empirically observed that the accuracy of PNSGD algorithm is stable for a learning rate in the range of \((0.01, 0.07)\) and peaks around 0.052.}
proposition can be applied to capture the balance between privacy and utility of OGD algorithm. Consider the following two examples. For ease of notation, we let $B$ denote the maximum of $\text{dia}(W)$ and $D$ defined in Theorem 5, i.e.,

$$B = \max \left\{ \text{dia}(W), \max_{t \in [n]} \text{dia}(\Psi^{\text{O}}_{t} (W)) \right\}.$$ 

Example 3. If $\sigma_t = \sigma$ for some $\sigma > 0$, then the randomly stopped OGD algorithm is $(\varepsilon, \delta)$-DP with

$$\delta = \frac{1}{n} \theta_{e^*} \left( \frac{\psi M \sqrt{n}}{B \sigma} \right) \left[ 1 - \theta_{e^*} \left( \frac{M \sqrt{n}}{\sigma} \right) \right]^{-1},$$

and it satisfies

$$SR(n) \leq \frac{B}{\sqrt{n}} \left( \frac{3M}{2} + \frac{d \sigma^2}{M} \right).$$

It is worth mentioning that Zinkevich [39] showed that the regret (3) of the OGD algorithm (without privacy constraint) scales as $O(\sqrt{n})$ for $\eta_t = \frac{1}{\sqrt{t}}$ and cannot be improved in general. One can use the above utility bound to show that $O(\sqrt{n})$ regret can be obtained on average for the differentially private OGD algorithm (24). Nevertheless, the privacy guarantee becomes trivial for sufficiently large $n$.

Example 4. If $\eta_t \sigma_t = \lambda$ for some $\lambda > 0$, then the randomly stopped OGD algorithm is $(\varepsilon, \delta)$-DP with

$$\delta = \frac{1}{n} \theta_{e^*} \left( \frac{\psi \lambda}{M} \right) \left[ 1 - \theta_{e^*} \left( \frac{B \lambda}{\sigma} \right) \right]^{-1},$$

and it satisfies

$$SR(n) \leq \frac{M}{2} \left( \frac{3D}{\sqrt{n}} + \frac{d \lambda^2}{B \sqrt{n}} \right).$$

Observe that in this case the utility upper bound becomes trivial for large $n$ as it diverges as $n \to \infty$. However, the privacy guarantee increases in $n$ at a rate $O(1/n)$.

The above two examples illustrate the intricate balance between privacy and utility. In particular, they elucidate that if $\eta_t = O(\frac{1}{\sqrt{t}})$, then a necessary condition for non-trivial privacy and utility is $\sigma_t \to \infty$ and $\frac{\sigma_t}{\sqrt{t}} \to 0$, i.e., $\{\sigma_t\}_{t}$ must be an increasing sequence but with a rate slower than $\sqrt{t}$.

VII. CONCLUDING REMARKS AND FUTURE WORKS

In this work, we developed a new information-theoretic framework for studying the privacy guarantee of online optimization algorithms obtained by adding independent noise at each iteration. This framework is applicable to algorithms that process only one input data at a time. Examples of such algorithms include one pass empirical risk minimization and online gradient descent algorithms. To characterize the differential privacy guarantee after $n$ iterations, we first demonstrated that these algorithms can be decomposed into a concatenation of $n$ Markov kernels each with bounded input and output. We then showed that the differential privacy parameters of an algorithm is fully determined by the contraction coefficients of these Markov kernels under a certain $f$-divergence, namely $E_{\gamma}$-divergence. This allowed us to translate privacy analysis of an iterative algorithm into the problem of computing contraction coefficients of its constituent Markov kernels. When the additive noise is Gaussian, the constituent Markov kernels were shown to be projected Gaussian Markov kernels whose contraction coefficient under $E_{\gamma}$-divergence can be computed in a closed form expression. Using this expression, we characterize a provably stronger privacy guarantee of one-pass empirical risk minimization algorithm compared to the existing

\footnote{In particular, we apply (18). Observe that, depending on the values of $\psi$, $D$ and $\varepsilon$, the bound in (17) might yield tighter results.}
results. It is worth emphasizing that while the technique developed in this work is applicable to algorithms that process data points one at a time, it can be adapted to multi-pass algorithms such as batch stochastic gradient descent algorithms. The caveat is that at each iteration a batch of data points must be selected without replacement. With this assumption in mind, the framework in this work can be adopted to distributed optimization problems (e.g., federated learning) where user-to-user communication channels are encrypted and thus the privacy leakage occurs only by revealing the last parameter update.

This work opens an avenue for the privacy analysis of iterative algorithms through the lens of information theory. The technique developed in this paper can be readily adopted to study privacy guarantees of algorithms where the privacy is measured in terms of more general $f$-divergences (as opposed to merely $\Gamma$-divergence). For instance, recently DP was defined in terms of total variation distance [93] and Pearson-Vajda divergence [89]. This can potentially lead to an interesting generalization of DP with different statistical and estimation-theoretic operational interpretations and is the subject of further research.

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A. Proof of Theorem 3

Observe that the case $\gamma = 1$ corresponds to Dobrushin’s formula (5), so we assume that $\gamma > 1$. We begin by showing that, for any two probability measures $\mu, \nu \in \mathcal{P}(D)$,

$$E_{\gamma}(\mu K \| \nu K) \leq E_{\gamma}(\mu \| \nu) \sup_{w_1, w_2 \in D} E_{\gamma}(K(w_1) \| K(w_2)),$$

or equivalently

$$\eta_{\gamma}(K) \leq \sup_{w_1, w_2 \in D} E_{\gamma}(K(w_1) \| K(w_2)).$$

(25)

Let $\phi := \mu - \gamma \nu$ and $(\phi^+, \phi^-)$ be its Hahn-Jordan decomposition. By the definition of $E_{\gamma}$ in (6), we have that $E_{\gamma}(\mu \| \nu) = \| \phi^+ \|$ and

$$E_{\gamma}(\mu \| \nu) = \frac{1}{2}\| \phi^+ \| + \frac{1}{2}\| \phi^- \| + \frac{1}{2}\| \phi^+ \| - \frac{1}{2}\| \phi^- \|$$

$$= \frac{1}{2}\| \phi \| + \frac{1}{2}(1 - \gamma),$$

(26)

where the last equality follows from the fact that $\phi^+$ and $\phi^-$ are positive measures and thus

$$\| \phi^+ \| - \| \phi^- \| = \phi^+(D) - \phi^-(D) = \phi(D).$$

Mutatis mutandis, we have that

$$E_{\gamma}(\mu K \| \nu K) = \frac{1}{2}\| \phi K \| + \frac{1}{2}(1 - \gamma).$$

(27)

It can be verified that if $\phi^+$ is the zero measure, then so is $(\phi K)^+$. In this case, (25) holds trivially as $E_{\gamma}(\mu K \| \nu K) = \| (\phi K)^+ \| = 0$. Assume that $\phi^+$ is not the zero measure. Observe that

$$\phi^+(D) - \phi^-(D) = \phi(D) = 1 - \gamma < 0,$$

which implies that $\phi^-(D) > 0$. In particular, the measures $\| \phi^+ \|^{-1} \phi^+$ and $\| \phi^- \|^{-1} \phi^-$ are probability measures. A straightforward manipulation shows that

$$\| \phi K \| = \left\| \int K(w_1) d\phi^+(w_1) - \int K(w_2) d\phi^-(w_2) \right\|$$

$$= \left\| \int \int \left[ \| \phi^+ \| K(w_1) - \| \phi^- \| K(w_2) \right] d\phi^+(w_1) d\phi^-(w_2) \right\|$$

$$\leq \sup_{w_1, w_2 \in D} \left\| \| \phi^+ \| K(w_1) - \| \phi^- \| K(w_2) \right\|.$$

By adding and subtracting the term $\gamma\| \phi^+ \| K(w_2)$, the triangle inequality implies that

$$\| \| \phi^+ \| K(w_1) - \| \phi^- \| K(w_2) \| \leq \| \phi^+ \| \| K(w_1) - \gamma K(w_2) \| + \| \phi^- \| - \gamma \| \phi^+ \|$$

$$= \| \phi^+ \| \left( \| K(w_1) - \gamma K(w_2) \| + 1 - \gamma \right) - (1 - \gamma),$$

where we used the inequality $\| \phi^- \| - \gamma \| \phi^+ \| \geq 0$ and the identity $1 - \gamma = \| \phi^+ \| - \| \phi^- \|$. Therefore,

$$\| \phi K \| \leq \| \phi^+ \| \sup_{w_1, w_2 \in D} \left( \| K(w_1) - \gamma K(w_2) \| + 1 - \gamma \right) - (1 - \gamma)$$

$$= 2\| \phi^+ \| \sup_{w_1, w_2 \in D} E_{\gamma}(K(w_1) \| K(w_2)) - (1 - \gamma),$$
where the equality follows from (26). By plugging the previous inequality in (27), we obtain that
\[ E_\gamma(\mu K \parallel \nu K) \leq E_\gamma(\mu \parallel \nu) \sup_{w_1, w_2 \in \mathcal{D}} E_\gamma(K(w_1) \parallel K(w_2)). \]

Now we show that the equality in (25) is attained. For \( w \in \mathcal{D} \), let \( \delta_w \) be the Dirac mass at \( w \). Observe that for \( w_1, w_2 \in \mathcal{D} \) such that \( w_1 \neq w_2 \), we have that
\[
E_\gamma(\delta_{w_1} \parallel \delta_{w_2}) = (\delta_{w_1} - \gamma \delta_{w_2})^+(\mathcal{D}) = 1.
\]
Since \( \delta_w K = K(w) \) for each \( w \in \mathcal{D} \), we have that
\[
E_\gamma(\delta_{w_1} K \parallel \delta_{w_2} K) = E_\gamma(K(w_1) \parallel K(w_2)).
\]
By the definition of \( \eta_\gamma(K) \) in (4), we conclude that
\[
\eta_\gamma(K) \geq \sup_{w_1 \neq w_2} \frac{E_\gamma(\delta_{w_1} K \parallel \delta_{w_2} K)}{E_\gamma(\delta_{w_1} \parallel \delta_{w_2})} = \sup_{w_1 \neq w_2} E_\gamma(K(w_1) \parallel K(w_2)),
\]
as we wanted to show.

**B. Proof of Lemma 1**

A direct computation shows that
\[
\iota_{N_1 \parallel N_2}(t) = 2\langle t, m_1 - m_2 \rangle + \|m_2\|^2 - \|m_1\|^2 \sigma^2,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the inner product and \( \iota_{P \parallel Q} \) is defined in Section II-C. As a consequence, we have that
\[
\iota_{N_1 \parallel N_2}(Y) \sim \begin{cases} N(\beta^2/2, \beta^2) & \text{whenever } Y \sim N_1, \\ N(-\beta^2/2, \beta^2) & \text{whenever } Y \sim N_2, \end{cases}
\]
(28)
where \( \beta := \frac{\|m_1 - m_2\|}{\sigma} \). Therefore, by plugging (28) in the expression for \( E_\gamma \) given in (7),
\[
E_\gamma(N_1 \parallel N_2) = Q \left( \frac{\log \gamma}{\beta} - \frac{\beta}{2} \right) - \gamma Q \left( \frac{\log \gamma}{\beta} + \frac{\beta}{2} \right),
\]
as required.

**C. Proof of Proposition 1**

For ease of notation, let \( K_0 \) denote the (unbounded) additive Gaussian kernel. Note that \( K = \Pi_{W} \circ K_0 \).

By Theorem 3, we have that
\[
\eta_\gamma(K) = \sup_{w_1, w_2 \in \mathcal{D}} E_\gamma(\Pi_{K_0}(w_1) \parallel \Pi_W(K_0(w_2)))
\leq \sup_{w_1, w_2 \in \mathcal{D}} E_\gamma(K_0(w_1) \parallel K_0(w_2))
= \sup_{w_1, w_2 \in \mathcal{D}} E_\gamma(N(w_1, \sigma^2 I_d) \parallel N(w_2, \sigma^2 I_d)),
\]
where the inequality follows from the data processing inequality. It could be verified that \( r \mapsto \theta_\gamma(r) \) is increasing. Hence, by (9), we conclude that
\[
\eta_\gamma(K) = \sup_{w_1, w_2 \in D} \theta_\gamma \left( \frac{\|w_2 - w_1\|}{\sigma} \right) = \theta_\gamma \left( \frac{\text{dia}(\mathcal{D})}{\sigma} \right),
\]
as desired.

D. Proof of Lemma 2
Recall that, by definition,
\[
\mu_{n+1} = \mu_1 K_{t_1} \cdots K_{t_n} \quad \text{and} \quad \mu'_{n+1} = \mu_1 K'_{t_1} \cdots K'_{t_n}.
\]
Since \( \{f_i\} \sim \{f'_i\} \), we have \( K_{f_t} = K_{f'_t} \) for all \( t \in \{n\} \setminus \{i\} \) and \( K_{f_i} \neq K_{f'_t} \). In particular,
\[
\mu_{n+1} = \mu_i K_{f_t} K_{t_{i+1}} \cdots K_{t_n} \quad \text{and} \quad \mu'_{n+1} = \mu_i K'_{f_t} K_{t_{i+1}} \cdots K_{t_n}.
\]
By repeatedly invoking (4), we obtain that
\[
D_f(\mu_{n+1} || \mu'_{n+1}) = D_f(\mu_i K_{f_t} K_{t_{i+1}} \cdots K_{t_n} || \mu_i K'_{f_t} K_{t_{i+1}} \cdots K_{t_n})
\leq D_f(\mu_i K_{f_t} || \mu_i K'_{f_t}) \prod_{t=1}^{n} \eta_f(K_{f_t}),
\]
as we wanted to prove.

E. Proof of Theorem 4
Let \( \mu_1 \) be the initial distribution of the iterative process and \( \{f_i\} \sim \{f'_i\} \) be a pair of neighboring collections of cost functions. In light of Theorem 2 and Lemma 2, we have
\[
\delta = \mathbb{E}_{\psi^*}(\mu_i K_{f_t} || \mu_i K'_{f_t}) \prod_{t=i+1}^{n} \eta_{\psi^*}(K_{f_t}). \tag{29}
\]
We begin by bounding \( \eta_{\psi^*}(K_{f_t}) \). Each kernel \( K_{f_t} \) can be decomposed as
\[
K_{f_t} = K_t \circ \Psi_{f_t},
\]
where \( K_t \) is the projected additive Gaussian kernel associated with \( (\Psi_{f_t}(\mathcal{W}), \mathcal{W}, \sigma_t) \), as defined in Section III. By Theorem 3, we have
\[
\eta_{\psi^*}(K_{f_t}) = \sup_{w_1, w_2 \in \mathcal{W}} \mathbb{E}_{\psi^*}(K_t(\Psi_{f_t}(w_1)) || K_t(\Psi_{f_t}(w_2)))
\leq \sup_{w_1, w_2 \in \Psi_{f_t}(\mathcal{W})} \mathbb{E}_{\psi^*}(K_t(w_1) || K_t(w_2))
\leq \eta_{\psi^*}(K_t)
\leq \theta_{\psi^*} \left( \frac{\text{dia}(\Psi_{f_t}(\mathcal{W}))}{\sigma_t} \right), \tag{30}
\]
where the last equality comes from Proposition 1.

Next, we look at the first term in the RHS of (29). By Jensen’s inequality, we can write
\[
\mathbb{E}_{\psi^*}(\mu_i K_{f_t} || \mu_i K'_{f_t}) \leq \int_{\mathcal{W}} \mathbb{E}_{\psi^*}(K_{f_t}(w) || K'_{f_t}(w)) d\mu_i(w).
\]
Observe that, for every \( w \in \mathcal{W} \),
\[
K_{f_t}(w) \sim \Pi_{\mathcal{W}}(\Psi_{f_t}(w) + \sigma_t Z),
\]
where \( \Pi_{\mathcal{W}} \) projects \( \Psi_{f_t}(w) + \sigma_t Z \) onto \( \mathcal{W} \).
Therefore, Lemma 1 implies that

\[ E^\mu(K_t(w)\|K_{t'}(w)) \leq E^\nu(\Psi_f(w) + \sigma Z \| \Psi_{f'}(w) + \sigma Z). \]

where \( Z \sim \mathcal{N}(0, I_d) \). A similar relation holds for \( K_{t'} \). By the data processing inequality, we obtain that

\[ E^\nu(\Psi_f(w) + \sigma Z \| \Psi_{f'}(w) + \sigma Z). \]

where we used the fact that \( r \) is increasing. Plugging (30) and (31) into (29), we obtain the desired result.

**F. Proof of Theorem 5**

Assume that \( \{f_t\} \) and \( \{f'_t\} \) are collections of cost functions such that \( \{f_t\} \stackrel{i.d.}{\sim} \{f'_t\} \) for some \( i \in [n] \). Let \( T \) be a uniform random variable over \([n]\). If \( \mu_{T+1} \) and \( \mu'_{T+1} \) are the distributions of the output of Algorithm 1 applied to \( \{f_t\} \) and \( \{f'_t\} \), respectively, then

\[ \mu_{T+1} = \frac{1}{n} \sum_{t=1}^{n} \mu_1 K_{t_1} \ldots K_{t_t} \quad \text{and} \quad \mu'_{T+1} = \frac{1}{n} \sum_{t=1}^{n} \mu_1 K'_{t_1} \ldots K'_{t_t}. \]

The convexity\(^7\) of \((\mu, \nu) \mapsto E_{\nu}(\mu \| \nu)\), and Jensen’s inequality imply that

\[ E^\nu(\mu_{T+1} \| \mu'_{T+1}) \leq \frac{1}{n} \sum_{t=1}^{n} E^\nu(\mu_1 K_{t_1} \ldots K_{t_t} \| \mu_1 K'_{t_1} \ldots K'_{t_t}). \]

Recall that \( f_j = f'_j \) for all \( j \neq i \). In particular, \( \mu_1 K_{t_1} \ldots K_{t_t} = \mu_1 K'_{t_1} \ldots K'_{t_t} \) for all \( t < i \) and hence

\[ E^\nu(\mu_{T+1} \| \mu'_{T+1}) \leq \frac{1}{n} \sum_{t=i}^{n} E^\nu(\mu_1 K_{t_1} \ldots K_{t_t} \| \mu_1 K'_{t_1} \ldots K'_{t_t}), \]

where \( \mu_i = \mu_1 K_{t_1} \ldots K_{t_{i-1}} \). As in the proof of Theorem 4, each summand can be bounded via Lemma 2 to obtain

\[ E^\nu(\mu_{T+1} \| \mu'_{T+1}) \leq \frac{1}{n} \sum_{t=i}^{n} E^\nu(\mu_1 K_{t_1} K_{t_{i+1}} \ldots K_{t_t}) \prod_{j=i+1}^{t} \eta_{\nu^*}(K_{t_j}). \]

Furthermore, (30) and (31) imply that

\[ E^\nu(\mu_{T+1} \| \mu'_{T+1}) \leq \frac{1}{n} \sum_{t=i}^{n} \theta_{\nu^*} \left( \frac{\psi}{\sigma} \right) \prod_{j=i+1}^{t} \theta_{\nu^*} \left( \frac{D}{\sigma} \right), \]

where we used the fact that \( r \mapsto \theta_r(r) \) is increasing. By taking the minimum over \( i \), Theorem 2 leads to (17). By further exploiting the monotonicity of \( r \mapsto \theta_r(r) \), we obtain that

\[ \delta \leq \frac{1}{n} \sum_{t=i}^{n} \theta_{\nu^*} \left( \frac{\psi}{\sigma} \right) \prod_{j=i+1}^{t} \theta_{\nu^*} \left( \frac{D}{\sigma} \right). \]

A straightforward manipulation leads to

\[ \delta \leq \frac{1}{n} \theta_{\nu^*} \left( \frac{\psi}{\sigma} \right) \sum_{t=0}^{n-i} \left[ \theta_{\nu^*} \left( \frac{D}{\sigma} \right) \right]^t. \]

\(^7\)For any convex function \( f \) on \( \mathbb{R}_+ \), its perspective, i.e., \((p, q) \mapsto qf \left( \frac{p}{q} \right)\), is convex on \( \mathbb{R}_+^2 \). Since \( D_f(\mu \| \nu) = E_{\nu} \left[ f \left( \frac{\mu}{\nu} \right) \right] \), it follows that \((\mu, \nu) \mapsto D_f(\mu \| \nu)\) is convex.
\[
\leq \frac{1}{n} \theta_w \left( \frac{\psi}{\sigma} \right) \left[ 1 - \theta_w \left( \frac{D}{\sigma} \right) \right]^{-1},
\]

as claimed in (18).

\textbf{G. Proof of Corollary 1}

Define \( \Psi_{x}^{\text{SGD}}(w) = w - \eta \nabla \ell(w, x) \) for \( x \in X \). The PNSGD algorithm can then be expressed as

\[ W_{t+1} = \Pi_{W} \left( \Psi_{x_t}^{\text{SGD}}(W_t) + \eta \sigma Z_t \right). \]

Applying Theorem 5, we obtain that this algorithm is \((\varepsilon, \delta)\)-DP for \( \varepsilon \geq 0 \) and

\[
\delta = \frac{1}{n} \theta_w \left( \frac{\psi}{\eta \sigma} \right) \left[ 1 - \theta_w \left( \frac{D}{\eta \sigma} \right) \right]^{-1},
\]

where

\[
\psi = \sup_{x_1, x_2 \in X} \sup_{w \in W} \| \Psi_{x_1}^{\text{SGD}}(w) - \Psi_{x_2}^{\text{SGD}}(w) \|
\]

and \( D = \max_{t \in [n]} \text{dia}(\Psi_{x_t}^{\text{SGD}}(W)) \). Note that

\[
\| \Psi_{x_1}^{\text{SGD}}(w) - \Psi_{x_2}^{\text{SGD}}(w) \| = \eta \| \nabla \ell(w, x_1) - \nabla \ell(w, x_2) \|
\]

\[
\leq \eta (\| \nabla \ell(w, x_1) \| + \| \nabla \ell(w, x_2) \|)
\]

\[
\leq 2\eta L,
\]

and thus

\[
\psi \leq 2\eta L.
\]

We can also write for each \( t \in [n] \)

\[
\text{dia}(\Psi_{x_t}^{\text{SGD}}(W)) = \sup_{w_1, w_2 \in W} \| \Psi_{x_t}^{\text{SGD}}(w_1) - \Psi_{x_t}^{\text{SGD}}(w_2) \|
\]

\[
= \sup_{w_1, w_2 \in W} \| w_1 - \eta \nabla \ell(w_1, x_t) - w_2 + \eta \nabla \ell(w_2, x_t) \|
\]

\[
\leq \| w_1 - w_2 \| + \eta \| \nabla \ell(w_1, x_t) - \nabla \ell(w_2, x_t) \|
\]

\[
\leq 1 + 2\eta L,
\]

implying

\[
D \leq 1 + 2\eta L
\]

Plugging (37) and (39) into (35), we obtain the desired result.

\textbf{H. Missing proof of Section V}

We first show that if \( w \mapsto f(w) \) is convex and \( \beta \)-smooth on \( W \), then \( G(w) := w - \eta \nabla f(w) \) is 1-Lipschitz for \( \eta \leq \frac{2}{\beta} \). To do so, we write for any \( w_1, w_2 \in W \)

\[
\| G(w_1) - G(w_2) \|^2 = \| w_1 - w_2 + \eta \nabla f(w_2) - \eta \nabla f(w_1) \|^2
\]

\[
= \| w_1 - w_2 \|^2 + \eta^2 \| \nabla f(w_1) - \nabla f(w_2) \|^2 - 2\eta \langle \nabla f(w_1) - \nabla f(w_2), w_1 - w_2 \rangle
\]

\[
\leq \| w_1 - w_2 \|^2 + \eta \left( \eta - \frac{2}{\beta} \right) \| \nabla f(w_1) - \nabla f(w_2) \|^2
\]

\[
\leq \| w_1 - w_2 \|^2,
\]

where the inequality in (40) follows from [94, Lemma 3.11] that states \( \langle \nabla f(w_1) - \nabla f(w_2), w_1 - w_2 \rangle \geq \frac{1}{\beta} \| \nabla f(w_1) - \nabla f(w_2) \|^2 \).
Replacing $f(w)$ with $\ell(w, x)$, we obtain that $w \mapsto \Psi^{\text{SGD}}(w)$ is 1-Lipschitz for all $x \in \mathcal{X}$. To obtain (20), we modify (38) in the proof of Corollary 1 as follows:

$$D = \max_{t \in [n]} \text{dia}(\Psi^{\text{SGD}}(W)) = \sup_{w_1, w_2 \in W} \|\Psi^{\text{SGD}}(w_1) - \Psi^{\text{SGD}}(w_2)\| \leq 1. \quad (42)$$

Plugging this and (37) into (35), we obtain (20).

I. Proof of Proposition 3

For ease of notation, let $w^* \in \arg \min_{w \in W} f(w)$ and $\tilde{D} = \text{dia}(W)$. Since $T$ is uniformly distributed on $[n]$, we have

$$\mathbb{E}[f(W_T) \mid W_1, \ldots, W_n] = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[f(W_t) \mid W_1, \ldots, W_n].$$

Therefore, we have that

$$\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) = \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^{n} \{f(W_t) - f(w^*)\} \right].$$

Since $F$ is a family of convex functions, the function $f(w) = \mathbb{E}[f_t(w)]$ is convex. In particular, it follows that

$$\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) \leq \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^{n} \langle \nabla f(W_t), W_t - w^* \rangle \right]. \quad (43)$$

Define

$$G_t := \nabla f_t(W_t) - \sigma_t Z_t.$$

Recall that $f_t$ and $Z_t$ are independent of $W_t$. Hence, by differentiation under the integral sign,

$$\mathbb{E}[G_t \mid W_t] = \nabla f(W_t).$$

As a consequence, we obtain that

$$\mathbb{E}[\langle G_t, W_t - w^* \rangle] = \mathbb{E}[\langle \mathbb{E}[G_t \mid W_t], W_t - w^* \rangle] = \mathbb{E}[\langle \nabla f(W_t), W_t - w^* \rangle].$$

Therefore, (43) becomes

$$\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[\langle G_t, W_t - w^* \rangle]. \quad (44)$$

Recall that, for every $w \in \mathbb{R}^d$ and every $w_0 \in W$,

$$\|\Pi_W(w) - w_0\| \leq \|w - w_0\|.$$

Thus, the update rule in (24) and the definition of $G_t$ imply that

$$\|W_{t+1} - w^*\|^2 \leq \|W_t - \eta_t G_t - w^*\|^2 = \|W_t - w^*\|^2 + \eta_t^2 \|G_t\|^2 - 2\eta_t \langle G_t, W_t - w^* \rangle.$$

A straightforward manipulation shows that

$$\mathbb{E}[\langle G_t, W_t - w^* \rangle] \leq \mathbb{E} \left[ \frac{\|W_t - w^*\|^2 - \|W_{t+1} - w^*\|^2}{2\eta_t} \right] + \frac{\eta_t (L^2 + \sigma_t^2 d)}{2},$$

where $L$ is the Lipschitz constant of $f$. This completes the proof of Proposition 3.
where the last inequality uses the fact that
\[ \mathbb{E}[\|G_t\|^2] \leq L^2 + \sigma_t^2d. \]

Therefore, (44) becomes
\[
\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) \leq \frac{1}{n} \sum_{t=1}^{n} \left\{ \mathbb{E} \left[ \frac{\|W_t - w^*\|^2 - \|W_{t+1} - w^*\|^2}{2\eta_t} \right] + \frac{\eta_t(L^2 + \sigma_t^2d)}{2} \right\}
\]
\[ = \frac{1}{n} \sum_{t=1}^{n} \frac{\mathbb{E}[\|W_t - w^*\|^2]}{2} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{1}{n} \sum_{t=1}^{n} \frac{\eta_t(L^2 + \sigma_t^2d)}{2}, \]

where for ease of notation we define $1/\eta_0 = 0$. Since $\eta_t = \frac{\bar{D}}{L\sqrt{t}}$ and $\|W_t - w^*\| \leq \bar{D}$ for every $t$, we have that
\[
\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) \leq \frac{L\bar{D}}{2\sqrt{n}} + \frac{1}{n} \sum_{t=1}^{n} \frac{\eta_t(L^2 + \sigma_t^2d)}{2},
\]

Finally, by the inequality $\sum_{t=1}^{n} \frac{1}{\sqrt{t}} \leq 2\sqrt{n}$, we conclude that
\[
\mathbb{E}[f(W_T)] - \inf_{w \in W} f(w) \leq \frac{L\bar{D}}{2\sqrt{n}} + \frac{L\bar{D}}{\sqrt{n}} + \frac{d}{2n} \sum_{t=1}^{n} \eta_t \sigma_t^2,
\]
as required.