Algebraic properties of chromatic roots

Peter J. Cameron

Co-authors
The problem was suggested by Sir David Wallace, director of the Isaac Newton Institute, during the programme on “Combinatorics and Statistical Mechanics” during the first half of 2008. Apart from him, others who have contributed include Vladimir Dokchitser, F. M. Dong, Graham Farr, Bill Jackson, Kerri Morgan, James Sellers, Alan Sokal, and Dave Wagner.

Chromatic roots
A proper colouring of a graph $G$ is a function from the vertices of $G$ to a set of $q$ colours with the property that adjacent vertices receive different colours.

The chromatic polynomial $P_G(q)$ of $G$ is the function whose value at the positive integer $q$ is the number of proper colourings of $G$ with $q$ colours. It is a monic polynomial in $q$ with integer coefficients, whose degree is the number of vertices of $G$.

A chromatic root is a complex number $\alpha$ which is a root of some chromatic polynomial.

Integer chromatic roots
An integer $m$ is a root of $P_G(q) = 0$ if and only if the chromatic number of $G$ (the smallest number of colours required for a proper colouring of $G$) is greater than $m$.

Hence every non-negative integer is a chromatic root. (For example, the complete graph $K_{m+1}$ cannot be coloured with $m$ colours.)

On the other hand, no negative integer is a chromatic root.

Real chromatic roots

Theorem 1. • There are no negative chromatic roots, none in the interval $(0, 1)$, and none in the interval $(1, \frac{32}{27}]$.

• Chromatic roots are dense in the interval $[\frac{32}{27}, \infty)$.

The non-trivial parts of this theorem are due to Bill Jackson and Carsten Thomassen.

Complex chromatic roots
For some time it was thought that chromatic roots must have non-negative real part. This is true for graphs with fewer than ten vertices. But Alan Sokal showed:

Theorem 2. Complex chromatic roots are dense in the complex plane.

This is connected with the Yang–Lee theory of phase transitions.

Algebraic properties, I
We first observe that any chromatic root is an algebraic integer. The main question is, which algebraic integers are chromatic roots?

Let $G + K_n$ denote the graph obtained by adding $n$ new vertices to $G$, joined to one another and to all existing vertices. Then

$$P_{G+K_n}(q) = q(q-1) \cdots (q-n+1)P_G(q-n).$$

We conclude that if $\alpha$ is a chromatic root, then so is $\alpha + n$, for any natural number $n$.

However, the set of chromatic roots is far from being a semiring; it is not closed under either addition or multiplication. (Consider $\alpha + \bar{\alpha}$ and $\alpha \bar{\alpha}$, where $\alpha$ is non-real and close to the origin.)
Algebraic properties, II

We were led to make two conjectures, as follows.

**Conjecture 1 (The \(\alpha + n\) conjecture).** Let \(\alpha\) be an algebraic integer. Then there exists a natural number \(n\) such that \(\alpha + n\) is a chromatic root.

**Conjecture 2 (The \(na\) conjecture).** Let \(\alpha\) be a chromatic root. Then \(na\) is a chromatic root for any natural number \(n\).

If the \(\alpha + n\) conjecture is true, we can ask, for given \(\alpha\), what is the smallest \(n\) for which \(\alpha + n\) is a chromatic root?

**An example**

The golden ratio \(\alpha = (\sqrt{5} - 1)/2\) is not a chromatic root, as it lies in \((0, 1)\).

Also, \(\alpha + 1\) and \(\alpha + 2\) are not chromatic roots since their algebraic conjugates are negative or in \((0, 1)\). However, there are graphs (e.g., the truncated icosahedron) which have chromatic roots very close to \(\alpha + 2\), the so-called “golden root”.

We do not know whether \(\alpha + 3\) is a chromatic root or not.

However, \(\alpha + 4\) is a chromatic root (the smallest such graph has eight vertices), and hence so is \(\alpha + n\) for any natural number \(n \geq 4\).

**Quadratic roots**

**Theorem 3.** Let \(\alpha\) be an integer in a quadratic number field. Then there is a natural number \(n\) such that \(\alpha + n\) is a chromatic root.

If \(\alpha\) is irrational, then the set \(\{\alpha + n : n \in \mathbb{Z}\}\) is the set of all quadratic integers with given discriminant. So it is enough to show that, for any non-square \(d\) congruent to 0 or 1 mod 4, there is a quadratic integer with discriminant \(d\) which is a chromatic root.

I will sketch the ideas behind the proof of this and partial results for higher-degree algebraic integers.

Rings of cliques

A ring of cliques is the graph \(R(a_1, \ldots, a_n)\) whose vertex set is the union of \(n + 1\) complete subgraphs of sizes \(1, a_1, \ldots, a_n\), where the vertices of each clique are joined to those of the cliques immediately preceding or following it mod \(n + 1\).

**Theorem 4 (Read).** The chromatic polynomial of \(R(a_1, \ldots, a_n)\) is a product of linear factors and the polynomial

\[
\frac{1}{q} \left( \prod_{i=1}^{n} (q - a_i) - \prod_{i=1}^{n} (-a_i) \right).
\]

We call this the interesting factor.

**Examples**

- If \(a_i = 1\) for all \(i\) (so that the graph is an \((n + 1)\)-cycle), the interesting factor is \(\frac{1}{q} (q - 1)^n - \frac{1}{q} (-1)^n \) (which is a cycle of order \(n + 1\)).
- If \(n = 3\), the interesting factor of \(R(1, 1, 4)\) is \(q^2 - 7q + 11\), with roots \((7 \pm \sqrt{5})/2\). This is the eight-vertex graph promised earlier.

**Quadratic and cubic integers**

For \(n = 3\), the interesting factor of \(R(a,b,c)\) is \(x^2 - (a + b + c)x + (ab + bc + ca)\). The discriminant of this quadratic is \((a + b + c)^2 - 4(ab + bc + ca)\).

It takes but a little ingenuity to show that this discriminant takes all possible values congruent to 0 or 1 mod 4.

For \(n = 4\), we have a four-parameter family of cubics for the interesting factors. Are these enough to prove the \(\alpha + n\) conjecture for cubic integers? (We have a long list of cubics obtained from this construction but don’t seem to have hit everything!)

**A higher-dimensional family**

Let \(G\) be a graph whose vertex set is the union of two cliques, of sizes \(n\) and \(m\). For \(i = 1, \ldots, m\), let \(F_i\) be the set of neighbours in the first clique of the \(i\)th vertex of the second. We may assume without loss of generality that the union of all the sets \(F_i\)
is the whole $n$-clique, and that their intersection is empty.

The chromatic polynomial can be computed by inclusion-exclusion in terms of the sizes of the $F_i$ and their intersections.

If $m = 2$, $|F_1| = a$ and $|F_2| = b$, we have a ring of cliques $R(1,a,b)$.

For $m = 3$, we get a six-parameter family of cubics as the “interesting factors”. We have not been able to find suitable specialisations to prove the $a + n$ conjecture using this family.

A remark on the $na$ conjecture

The only small piece of evidence is the following. If $a$ is a root of the interesting factor of $R(a_1, \ldots, a_m)$, then for any natural number $n$, $na$ is a root of the interesting factor of $R(na_1, \ldots, na_m)$.

Wild speculation 3. Let $a$ be a root of the chromatic polynomial of $G$. For any natural number $n$, let $G[n]$ be the graph obtained from $G$ by blowing up all but one of the vertices of $G$ up into a clique of size $n$. Then $na$ is a root of the chromatic polynomial of $G[n]$.

Galois groups

A weaker form of our conjecture (modulo the Inverse Galois Problem(!)) would assert:

Conjecture 4. Every finite permutation group of degree $n$ is the Galois group of an extension of $\mathbb{Q}$ generated by a chromatic root.

This conjecture is amenable to computation. We computed the Galois groups of many of the interesting factors of rings of cliques $R(a_1, \ldots, a_n)$. Note that we can assume without loss that $\gcd(a_1, \ldots, a_n) = 1$.

Note also that, if $n$ is prime, then the interesting factor is $n$th cyclotomic polynomial in $x = q - 1$, so that the cyclic groups of prime order all occur as Galois groups.

The next table shows what happens for small values.

Small rings of cliques

For given $n$, we test all non-decreasing $n$-tuples $(a_1, \ldots, a_n)$ of positive integers with $\gcd 1$ and $a_n \leq l$. $G$ is the Galois group, in case the polynomial is irreducible. $S_n$ and $A_n$ are the symmetric and alternating groups of degree $n$, $C_n$ the cyclic group of order $n$, $V_4$ the Klein group of order $4$, $D_n$ the dihedral group of order $2n$, and $\lambda$ denotes the wreath product of permutation groups.

- $n = 4, l = 20$: 774 reducible, 3 with $G = A_3$, 7215 with $G = S_3$.
- $n = 5, l = 20$: 586 reducible, 6 with $C_4$, 5 with $V_4, 360$ with $D_4$, 6 with $A_4$, and 39250 times $S_4$. So every transitive permutation group of degree up to 4 occurs as a Galois group.
- $n = 6, l = 20$: 4902 reducible, one Frobenius group of order 20, one $A_5$, 166671 times $S_5$. In this case, we are missing $C_5$ and $D_5$.

More small rings

- $n = 7, l = 15$: 734 reducible, one $C_6$, six of $S_2 \wr S_3$, 52 times $S_3 \wr S_2$, five times $\text{PGL}(2,5)$, and 113401 times $S_6$. There are sixteen transitive groups of degree 6, of which eleven don’t occur in this range.
- $n = 8, l = 10$: 1132 reducible, 22630 times $S_7$. No other transitive group occurs in this range.
- $n = 9, l = 8$: 152 reducible, three $S_4 \wr S_2$, 11054 of $S_8$.
- $n = 10, l = 8$: 1061 reducible, 18089 of $S_9$.
- $n = 11, l = 6$: 29 reducible, one $C_{10}$, 4248 times $S_{10}$.
- $n = 12, l = 6$: 592 reducible, 5492 times $S_{11}$.
- $n = 13, l = 6$: 33 reducible, one $C_{12}$, 8415 times $S_{12}$.
- $n = 14, l = 6$: 884 reducible, 10609 times $S_{13}$.
- $n = 15, l = 6$: 307 reducible, 15045 times $S_{14}$.
- $n = 16, l = 6$: 1366 reducible, 18813 times $S_{15}$.

Not overwhelming support for our conjecture!
Other families of graphs
We have done similar analysis on other families of graphs, including

- complete bipartite graphs;
- "theta-graphs" (one of these consists of $p$ paths of length $s$ with the endpoints identified) – these were the graphs used by Sokal to show that chromatic roots are dense in the complex plane;
- small graphs.

The results are similar but there is no time to present them here.

Further speculation
The Galois group of a “random” polynomial is typically the symmetric group of its degree.

The chromatic polynomial of a random graph cannot be irreducible, since it will have many linear factors $q - m$, for $m$ up to the chromatic number. Bollobás showed that the chromatic number is almost surely close to $n / (2 \log_2 n)$.

Wild speculation 5. The chromatic polynomial of a random graph is almost surely a product of linear factors and one irreducible factor whose Galois group is the symmetric group of its degree.