IRREducible FRACTal STRUCTURES FOR MORAN’S TYPE THEOREMS

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Abstract. In this paper, we characterize a novel separation property for IFS-attractors on complete metric spaces. Such a separation property is weaker than the strong open set condition (SOSC) and becomes necessary to reach the equality between the similarity and the Hausdorff dimensions of strict self-similar sets. We also investigate the size of the overlaps from the viewpoint of that separation property. In addition, we contribute some equivalent conditions to reach the equality between the similarity dimension and a new Hausdorff type dimension for IFS-attractors introduced by the authors in terms of finite coverings.

1. Introduction

A classical problem in Fractal Geometry consists in determining under what conditions on the pieces of a strict self-similar set $K$, the equality between the similarity and the Hausdorff dimensions of $K$ holds. A classical result contributed by Australian mathematician P.A.P. Moran in the forties (c.f. [18, Theorem III]) states that under the open set condition (OSC), which is a property required to the pieces of $K$ to guarantee that their overlaps are thin enough, the desired equality stands. Afterwards, Lalley introduced the strong open set condition (SOSC) by further requiring that the (feasible) open set provided by the OSC must intersect the attractor $K$. It is worth pointing out that the next chain of implications and equivalences stands in the case of Euclidean self-similar sets and is best possible (c.f. [21]):

$$(1.1) \quad \text{SOSC} \iff \text{OSC} \iff \mathcal{H}_{\alpha}^0(K) > 0 \Rightarrow \dim_{\mathcal{H}}(K) = \alpha,$$

where $\mathcal{H}_{\alpha}^0$ is the $\alpha$–dimensional Hausdorff measure, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension, and $\alpha$ is the similarity dimension of the attractor. Interestingly, Schief proved that $\mathcal{H}_{\alpha}^0(K) > 0 \Rightarrow \text{SOSC}$ (c.f. [21] Theorem 2.1)) which implies that both the SOSC and the OSC are equivalent for Euclidean IFS-attractors. A counterexample due to Mattila (c.f. Section 4) guarantees that the last implication in Eq. (1.1) does not hold, in general. Accordingly, the OSC becomes only sufficient to reach the equality between those dimensions. A further extension of the problem above takes place in the more general context of attractors on complete metric spaces. Schief also explored such a problem and justified the following chain of implications (c.f. [22]):

$$(1.2) \quad \mathcal{H}_{\alpha}^0(K) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_{\mathcal{H}}(K) = \alpha,$$

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i.e., the SOSC is necessary for \( \mathcal{H}_H^K(\mathcal{K}) > 0 \) and only sufficient for \( \dim_H(\mathcal{K}) = \alpha \). Once again, the above-mentioned result of Mattila implies that Eq. (1.2) is best possible. From both Eqs. (1.1) and (1.2), it holds that the SOSC is a sufficient condition on the pre-fractals of \( \mathcal{K} \) leading to \( \dim_H(\mathcal{K}) = \alpha \).

In this paper, we make use of the concept of a fractal structure (first contributed in [7]) to explore and characterize a novel separation property in both contexts: Euclidean attractors and self-similar sets in complete metric spaces. Such a separation property, weaker than the OSC, becomes necessary to reach the equality between the similarity dimension of the attractor and its Hausdorff dimension. Accordingly, we shall conclude (in the general case) that

\[
\mathcal{H}_H^K(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(\mathcal{K}) = \alpha \Rightarrow \text{WSC},
\]

where WSC refers to the weak separation condition for attractors we shall introduce in upcoming Section [7].

Moreover, we will prove that the WSC holds if and only if \( \dim_4^f(\mathcal{K}) = \alpha \), where \( \dim_4^f \) is fractal dimension IV introduced in [11] (c.f. Section 3).

2. Preliminaries

2.1. General notation. Along the sequel, we shall use the following notation from the domain of words [5, 13, 17]. Let \( \Sigma = \{1, \ldots, k\} \) be a finite (nonempty) set (also called an alphabet). For each natural number \( n \), let \( \Sigma^n = \{i_1 \cdots i_n : i_j \in \Sigma, j = 1, \ldots, n\} \) be the set consisting of all the words of length \( n \) from \( \Sigma \). In addition, let \( \Sigma^\infty \) denote the collection of either all finite (\( \cup_{n \in \mathbb{N}} \Sigma^n \)) or infinite (\( \Sigma^\infty \)) words from \( \Sigma \), i.e., \( \Sigma^\infty = \cup_{n \in \mathbb{N}} \Sigma^n \cup \Sigma^\infty \). Thus, the prefix order \( \sqsubseteq \) is defined on \( \Sigma^\infty \) as follows: \( x \sqsubseteq y \), if and only if, \( x \) is a prefix of \( y \).

2.2. IFS-attractors. Let \( k \geq 2 \). By an iterated function system (IFS), we shall understand a finite collection of similitudes on a complete metric space \( (X, \rho) \), say \( \mathcal{F} = \{f_1, \ldots, f_k\} \), where each self-map \( f_i : X \to X \) satisfies the following identity:

\[
\rho(f_i(x), f_i(y)) = c_i \cdot \rho(x, y), \quad \text{for all} \ x, y \in X,
\]

with \( c_i \in (0, 1) \) being the similarity ratio associated with \( f_i \). In particular, if \( X = \mathbb{R}^d \), then \( \mathcal{F} \) is said to be an Euclidean IFS (EIFS, hereafter). Under the previous assumptions, there always exists a unique (nonempty) compact subset \( \mathcal{K} \subseteq X \) such that

\[
(2.1) \quad \mathcal{K} = \cup \{f_i(\mathcal{K}) : i = 1, \ldots, k\}.
\]

The previous equality is usually known as Hutchinson’s equation [12] and \( \mathcal{K} \) is said to be the IFS-attractor (also the self-similar set) generated by \( \mathcal{F} \). It is worth pointing out that \( \mathcal{K} \) consists of smaller self-similar copies of itself, \( \mathcal{K}_i \), named as pre-fractals of \( \mathcal{K} \). Thus, \( \mathcal{K}_i = f_i(\mathcal{K}) \) for all \( i = 1, \ldots, k \). In addition, we shall write \( \mathcal{K}_{ij} = f_i(f_j(\mathcal{K})) \), and so on. Hence, if \( f_1 = f_{i_1} \circ \cdots \circ f_{i_n}, c_1 = c_{i_1} \cdots c_{i_n} \), and \( \mathcal{K}_i = f_i(\mathcal{K}) \), then Eq. (2.1) can be rewritten in the following terms:

\[
\mathcal{K} = \cup \{\mathcal{K}_i : i \in \Sigma^n\}.
\]

In addition, the address map \( \pi : \Sigma^\infty \to \mathcal{K} \) stands as a continuous map from the collection \( \Sigma^\infty \) of all the words of infinite length (sequences) onto the IFS-attractor \( \mathcal{K} \). It is worth pointing out that if the similarity ratios \( c_i \) are small, then the pre-fractals \( \mathcal{K}_i \) are disjoint, \( \pi \) is a homemomorphism, and \( \mathcal{K} \) becomes a Cantor set.
2.3. The open set condition. In the Euclidean case, there are, at least, three equivalent descriptions regarding the open set condition (OSC in the sequel), which controls the overlaps among the pre-fractals of $K$.

(i) The Moran's open set condition (due to P.A.P. Moran, c.f. [18]). We say that $\mathcal{F} = \{f_1, \ldots, f_k\}$ (or its attractor $K$) is under the OSC if there exists a nonempty open subset $\mathcal{V} \subseteq \mathbb{R}^d$ such that the images $f_i(\mathcal{V})$ are pairwise disjoint with all of them being contained in $\mathcal{V}$, which is called a feasible open set of $\mathcal{F}$ (resp., of $K$).

(ii) The finite clustering property (contributed by Schief, c.f. [21]). There exists an integer $N$ such that at most $N$ incomparable pieces $K_j$ of size $\geq \varepsilon$ can intersect the $\varepsilon$–neighborhood of a piece $K_i$ of diameter equal to $\varepsilon$. It has to be mentioned here that two pieces of $K, K_j$ and $K_k$, are said to be incomparable if $j \not\subseteq k$ and $k \not\subseteq j$.

(iii) Positive $\alpha$–dimensional Hausdorff measure (c.f. [18, 21]): $\mathcal{H}^\alpha_\varepsilon(\mathcal{K}) > 0$, where $\alpha$ is the similarity dimension of $K$, i.e., the (unique) solution of the equation $\sum_{i=1}^k \epsilon_i^\alpha = 1$ (c.f. Definition 4.2).

Lalley strengthened the OSC since the feasible open set $\mathcal{V}$ and the attractor $K$ may be disjoint. Thus, the classical OSC may be too weak in order to obtain results regarding the fractal dimension of $K$. In this way, the strong open set condition (SOSC) stands, if and only if, it holds, in addition to the OSC, that $K \cap \mathcal{V} \neq \emptyset$ (c.f. [15]). Schief proved that both the OSC and the SOSC are equivalent on Euclidean spaces (c.f. [21] Theorem 2.2). Such a result has been further extended to conformal IFSs [20], and self-conformal random fractals [19], as well.

Finally, we should mention here that Schief has already explored some conditions to guarantee the equality between the similarity dimension (c.f. Definition 4.2) and the Hausdorff dimension of IFS–attractors on complete metric spaces. In this case, though, the OSC no longer leads to the equality between such fractal dimensions (c.f. [22]).

2.4. Fractal structures. Fractal structures were first sketched by Bandt and Retta in [7] and introduced and applied afterwards by Arenas and Sánchez-Granero in [11] to characterize non-Archimedean quasi-metrization. By a covering of a nonempty set $X$, we shall understand a family $\Gamma$ of subsets such that $X = \bigcup \{A : A \in \Gamma\}$. Let $\Gamma_1$ and $\Gamma_2$ be two coverings of $X$. The notation $\Gamma_2 \prec \Gamma_1$ means that $\Gamma_2$ is a refinement of $\Gamma_1$, namely, for all $A \in \Gamma_2$, there exists $B \in \Gamma_1$ such that $A \subseteq B$. Moreover, $\Gamma_2 \ll \Gamma_1$ denotes that $\Gamma_2 \prec \Gamma_1$, and additionally, for all $B \in \Gamma_1$, it holds that $B = \bigcup \{A \in \Gamma_2 : A \subseteq B\}$. Thus, a fractal structure on $X$ is a countable family of coverings $\Gamma = \{\Gamma_n\}_{n \in \mathbb{N}}$ such that $\Gamma_{n+1} \ll \Gamma_n$, for all natural number $n$. The covering $\Gamma_n$ is called level $n$ of $\Gamma$. It is worth mentioning that a fractal structure induces a transitive base of quasi-uniformity (and hence, a topology) given by the transitive family of entourages $U_{\Gamma_n} = \{(x, y) \in X \times X : y \in X \setminus \bigcup_{A \in \Gamma_n, x \not\in A} A\}$. Along the sequel, we shall allow that a set could appear twice or more in any level of a fractal structure. Let $\Gamma$ be a fractal structure on $X$ and assume that $\text{St}(x, \Gamma) = \{\text{St}(x, \Gamma_n)\}_{n \in \mathbb{N}}$ is a neighborhood base for all $x \in X$, where $\text{St}(x, \Gamma_n) = \bigcup \{A \in \Gamma_n : x \in A\}$. Then $\Gamma$ is called a starbase fractal structure. A fractal structure is said to be finite if all its levels are finite coverings. An example of a finite fractal structure is the one which any IFS–attractor can be always endowed with naturally. Such a fractal structure plays a key role in this paper so we shall formally define it next.
Definition 2.1 (c.f. [4], Definition 4.4). Let \( \mathcal{F} \) be an IFS whose attractor is \( \mathcal{K} \). The natural fractal structure on \( \mathcal{K} \) as a self-similar set is given by the countable family of coverings \( \Gamma = \{ \Gamma_n \}_{n \in \mathbb{N}} \), where \( \Gamma_n = \{ f_i(\mathcal{K}) : i \in \Sigma^n \} \).

Alternatively, the levels of the natural fractal structure for any IFS-attractor \( \mathcal{K} \) can be described as \( \Gamma_1 = \{ f_i(\mathcal{K}) : i \in \Sigma \} \), and \( \Gamma_{n+1} = \{ f_i(A) : A \in \Gamma_n, i \in \Sigma \} \) for every \( n \in \mathbb{N} \). It is also worth mentioning that such a natural fractal structure is starbase (c.f. [4] Theorem 4.7]). In addition, the next remark will result useful for upcoming purposes.

Remark 2.2. All the elements in a same level \( \Gamma_n \) are incomparable.

3. Fractal dimensions for fractal structures

The fractal dimension models for a fractal structure involved along this paper, i.e., fractal dimensions III and IV, have been explored in previous works by the authors (c.f. [9, 11]) and can be considered as subsequent models from those studied in [10]. It is worth pointing out that they allowed to generalize both box dimension (c.f. [9, Definition 4.2] and [11, Definition 3.2]) and can be considered as subsequent models from those studied in [10].

Definition 3.1. (c.f. [9, Definition 4.2] and [11, Definition 3.2]) Assume that \( \text{diam} (\Gamma_n) \to 0 \) and consider the following expression, where \( k = 3, 4 \):

\[
\mathcal{H}^n_{k,k}(F) = \inf \left\{ \sum \text{diam} (A_i)^k : \{A_i\}_{i \in I} \in \mathcal{A}_n(F) \right\},
\]

(i) \( \mathcal{A}_{n,3}(F) = \{ A_i(F) : i \geq n \} \).

(ii) \( \mathcal{A}_{n,4}(F) = \{ \{ A_i \}_{i \in I} : A_i \in \bigcup_{i \geq n} \Gamma_i, F \subseteq \bigcup_{i \in I} A_i, \text{Card} (I) < \infty \} \). Here, \( \text{Card} (I) \) denotes the cardinal number of \( I \).

In addition, let \( \mathcal{H}^n_k(F) = \lim_{n \to \infty} \mathcal{H}^n_{k,k}(F) \). By the fractal dimension III (resp., IV) of \( F \), we shall understand the (unique) critical point satisfying the identity

\[
\dim_k^F(F) = \sup \{ s \geq 0 : \mathcal{H}^k_s(F) = \infty \} = \inf \{ s \geq 0 : \mathcal{H}^k_s(F) = 0 \}.
\]

It is worth pointing out that fractal dimension III always exists since the sequence \( \{ \mathcal{H}^n_{3,k}(F) \}_{n \in \mathbb{N}} \) is monotonic in \( n \in \mathbb{N} \). Likewise, the hypothesis \( \text{diam} (\Gamma_n, \Gamma_n) \to 0 \), though necessary in such a definition, is not too restrictive as the following remark points out.

Remark 3.2. Let \( \mathcal{K} \) be an IFS-attractor (with \( \Gamma \) the natural fractal structure as a self-similar set). Then it holds that \( \text{diam} (\mathcal{K}, \Gamma_n) \to 0 \), since the sequence of diameters \( \{ \text{diam} (\Gamma_n) \}_{n \in \mathbb{N}} \) decreases geometrically.

4. Some Moran’s type theorems under the OSC

One of the main goals in this paper is to explore some separation conditions for IFS-attractors in the context of fractal structures. To deal with, we shall endow each attractor with its natural fractal structure as self-similar set (c.f. either Definition 2.1).
or Remark 3.2. Next, we collect several assumptions regarding the IFS-attractors involved along this paper. They are stated in the general context of complete metric spaces.

**IFS conditions 4.1.** Let \((X, \mathcal{F})\) be an IFS, where \(X\) is a complete metric space, \(\mathcal{F} = \{f_1, \ldots, f_k\}\) is a finite collection of similitudes on \(X\), and \(K\) is the IFS-attractor of \(\mathcal{F}\). In addition, let \(\Gamma\) be the natural fractal structure on \(K\) as a self-similar set (c.f. Definition 2.1), and \(c_i\) be the similarity ratio of each \(f_i \in \mathcal{F}\).

If \((X, \mathcal{F})\) satisfies IFS conditions 4.1 then we shall say, for short, that \(\mathcal{F}\) is under IFS conditions 4.1. It is worth mentioning that all the results provided in this paper stand under IFS conditions 4.1.

Next, we recall the concept of similarity dimension for IFS-attractors.

**Definition 4.2.** Let \(\mathcal{F}\) be an IFS and \(K\) its attractor. By the similarity dimension of \(K\), we shall understand the unique solution \(\alpha > 0\) of the equation \(\sum_{i=1}^{k} c_i^\alpha = 1\). In other words, the similarity dimension of \(K\) is the unique value \(\alpha > 0\) such that \(P(\alpha) = 0\), where \(P(s) = \sum_{i=1}^{k} c_i^s - 1\).

Along the sequel, \(\alpha\) will denote the similarity dimension of an IFS-attractor. It is worth noting that (without any additional assumption) \(\mathcal{H}^3_H(K) < \infty\) for any IFS-attractor \(K\) (c.f. [12, Proposition 4 (i)])

The two results that follow are especially useful for upcoming purposes. The first one states that fractal dimension III (c.f. Definition 3.1 (3)) equals the similarity dimension of IFS-attractors without requiring the IFS \(\mathcal{F}\) (resp., the attractor \(K\)) any additional separation property. On the other hand, we also recall the classical Moran's Theorem, a standard result that gives the equality between both the Hausdorff and the similarity dimensions of IFS-attractors lying under the OSC.

**A note to readers.** Each theoretical result provided along this paper has been assigned one of the two following labels: IFS or EIFS. In the first case, it helps the reader to keep in mind that the corresponding result stands for attractors on complete metric spaces, whereas the label EIFS means that the result holds for Euclidean IFS-attractors.

**Theorem 4.3 (IFS).** (c.f. [9] Theorem 4.20) \(\dim_{\Gamma}^4(K) = \alpha\) and \(0 < \mathcal{H}^3_H(K) < \infty\).

**Moran’s Theorem (1946) (EIFS).** OSC \(\Rightarrow \dim_H(K) = \alpha\) and \(0 < \mathcal{H}^3_H(K) < \infty\).

By a Moran’s type theorem, we shall understand a result that yields the equality between a fractal dimension \(\dim\) of an IFS-attractor \(K\) and its similarity dimension, i.e., \(\dim(K) = \alpha\).

Theorem 4.3 and Moran’s Theorem give the following result involving the fractal dimension III of \(K\) in the Euclidean case.

**Corollary 4.4 (EIFS).** (c.f. [9] Corollary 4.22) OSC \(\Rightarrow \dim_{\Gamma}^3(K) = \dim_{\Gamma}^4(K) = \alpha\).

The following result we recall is quite general and stands for finite fractal structures.

**Lemma 4.5.** (c.f. [11] Proposition 3.5 (3)) Let \(\Gamma\) be a finite fractal structure on a metric space \((X, \rho)\), \(F\) be a subset of \(X\), and assume that \(\text{diam}(F, \Gamma_n) \to 0\). Then \(\dim_{\Gamma}(F) \leq \dim^4_{\Gamma}(F) \leq \dim^3_{\Gamma}(F)\).

**Corollary 4.6 (IFS).** \(\dim_{\Gamma}(K) \leq \dim^4_{\Gamma}(K) \leq \dim^3_{\Gamma}(K) = \alpha\).
Proof. It follows as a consequence of Lemma 4.5. In fact, recall that the natural fractal structure which any IFS−attractor can be endowed with is finite. Further, such a fractal structure also satisfies that diam \((K, \Gamma_n) = \text{diam}(\Gamma_n) \to 0\), since the sequence of diameters \(\{\text{diam}(\Gamma_n)\}_{n \in \mathbb{N}}\) decreases geometrically in the case of self-similar sets (c.f. Remark 3.2). Finally, Theorem 4.3 gives dim_3(K) = \(\alpha\).

□

The following result provides a Moran’s type theorem (under the OSC) involving fractal dimension IV as a consequence of previous corollaries. It can be understood as an extension of Corollary 4.4.

**Theorem 4.7 (EIFS).** OSC \(\Rightarrow\) dim_3(H(K)) = dim_4(\(\Gamma(K)\)) = dim_3(\(\Gamma(K)\)) = \(\alpha\).

**Proof.** Notice that dim_3(H(K)) \(\leq\) dim_4(\(\Gamma(K)\)) \(\leq\) dim_3(\(\Gamma(K)\)) = \(\alpha\) by Corollary 4.6. Corollary 4.4 gives the result. □

To conclude this section, we recall two key results explored by Schief (c.f. [21, 22]). They provide sufficient conditions to reach Moran’s type theorems in both contexts: complete metric spaces and Euclidean IFS−attractors. Such conditions consist of appropriate separation properties for IFS−attractors.

**Theorem 4.8.**

(EIFS) \(\text{OSCC }\Leftrightarrow\text{ OSC }\Leftrightarrow H_0^\alpha(K) > 0 \Rightarrow \text{dim}_H(K) = \alpha\).

(IFS) \(\text{H}_0^\alpha(K) > 0 \Rightarrow \text{SOSC }\Rightarrow \text{dim}_H(K) = \alpha\).

Interestingly, Mattila provided the following counterexample which allows to justify that Theorem 4.8 is best possible.

**Mattila’s Counterexample** (c.f. [21, 22]). Let \(\mathcal{F} = \{f_1, f_2, f_3\}\) be an EIFS on \(\mathbb{R}^2\), where \(f_i(x) = x_i + \frac{1}{3}(x - x_i)\) with \(x_1 = (0, 0), x_2 = (1, 0),\) and \(x_3 = (\frac{1}{2}, \frac{1}{2})\). The attractor of \(\mathcal{F}, K,\) is a nonconnected Sierpiński gasket in the plane which satisfies the SOSC and has similarity dimension \(\alpha = 1\). Thus, almost all Lebesgue projections of \(K\) on 1−dimensional subspaces of \(\mathbb{R}^2\) (that are self-similar set themselves), have Hausdorff dimension 1 (due to Marstrand’s Projection Theorem, c.f. [8] Projection theorem 6.1) and \(H_1^\alpha\) but zero \(H_1\) measure.

From Theorem 4.8 it holds that the OSC provides a sufficient (though not necessary) condition to get the equality between the similarity and the Hausdorff dimensions of Euclidean IFS−attractors. It is worth mentioning that the implication \(\text{H}_0^\alpha(K) > 0 \Rightarrow \text{SOSC}\) was contributed by Schief (c.f. [21 Theorem 2.1]). This guarantees the equivalence among OSC, SOSC, and the \(\alpha\)−dimensional Hausdorff measure of \(K\). However, in the case of complete metric spaces, the OSC no longer leads to a Moran’s type theorem (c.f. [22 Example 3.1]). Thus, it must be replaced by the SOSC for that purpose (c.f. [22 Theorem 2.6]). In addition, [22 Example 3.2] highlights that the SOSC does not suffice to guarantee a positive Hausdorff measure in the general case. Also, from Mattila’s Counterexample it holds that the implication dim_H(K) = \(\alpha\) \(\Rightarrow\) SOSC does not stand, in general.

5. **Irreducible Fractal Structures**

In this section, we introduce the concept of an irreducible fractal structure and characterize it in terms of the similarity dimension of an IFS−attractor and its fractal dimensions III and IV, as well.
Definition 5.1. Let $\Gamma$ be a covering of $X$. We say that $\Gamma$ is irreducible provided that it has no proper subcoverings (c.f. [23, Problem 20D]). By an irreducible fractal structure, we shall understand a fractal structure whose levels are irreducible coverings.

First, we provide a sufficient condition leading to irreducible fractal structures. It consists of the equality between both fractal dimensions III and IV.

Proposition 5.2 (IFS). $\dim^4_1(\mathcal{K}) = \dim^3_1(\mathcal{K}) = \alpha \Rightarrow \Gamma$ irreducible.

Proof. Assume, by the contrary, that $\Gamma$ is not irreducible. Then there exist $m \in \mathbb{N}$ and $i \in \Sigma^m$ such that $\Gamma_m \setminus \{f_i(\mathcal{K})\}$ is a covering of $\mathcal{K}$. Let $J = \Sigma^m \setminus \{i\}$. Since $\sum_{j \in J} c_j^\alpha = 1$ with $\alpha = \dim^4_1(\mathcal{K})$, then $\sum_{j \in J} c_j^\alpha < 1$. Thus, if $t$ is the unique solution of the equation $\sum_{j \in J} c_j^t = 1$, then $t < \alpha$ since the function $s \mapsto \sum_{i=1}^k c_i^t$ is strictly decreasing in $s$ (c.f. [23, Convention (2)]). On the other hand, $\mathcal{H}^t_{n,\text{dim}}(\mathcal{K}) = \mathcal{H}^t_{1,\text{dim}}(\mathcal{K})$ for each $n \in \mathbb{N}$, since any element $f_i(\mathcal{K}) \in \Gamma_k$ with $i \in \Sigma^k$ for some $k \in \mathbb{N}$, can be replaced by $\{f_j(\mathcal{K}) : j \in J\}$, where $\text{diam}(f_j(\mathcal{K})) = \sum_{j \in J} \text{diam}(f_i(\mathcal{K}))^t$. Letting $n \to \infty$, it holds that $\mathcal{H}^t_{1}(\mathcal{K}) = \mathcal{H}^t_{1,\text{dim}}(\mathcal{K}) < \infty$. Accordingly, $\dim^4_1(\mathcal{K}) \leq t < \alpha$, a contradiction. $\square$

The following result stands from Proposition 5.2. It provides another sufficient condition to get irreducible fractal structures involving both fractal dimension III and Hausdorff dimension.

Corollary 5.3 (IFS). $\dim_H(\mathcal{K}) = \dim^3_1(\mathcal{K}) = \alpha \Rightarrow \Gamma$ irreducible.

Proof. Corollary 4.6 gives $\dim_H(\mathcal{K}) \leq \dim^4_1(\mathcal{K}) \leq \dim^3_1(\mathcal{K}) = \alpha$. Then $\dim_H(\mathcal{K}) = \alpha$ implies $\dim^4_1(\mathcal{K}) = \dim^3_1(\mathcal{K}) = \alpha$. Hence, $\Gamma$ is irreducible by Proposition 5.2. $\square$

Interestingly, the reciprocal of Proposition 5.2 can also be stated.

Proposition 5.4 (IFS). $\Gamma$ irreducible $\Rightarrow \mathcal{H}^\alpha_{1}(\mathcal{K}) > 0$.

Proof. Let us assume, by the contrary, that $\mathcal{H}^\alpha_{1}(\mathcal{K}) = 0$. Observe that any element $f_i(\mathcal{K}) \in \Gamma_k$ with $i \in \Sigma^k$ for some $k \in \mathbb{N}$, can be replaced by $\{f_j(\mathcal{K}) : j \in \Sigma\}$, where $\text{diam}(f_i(\mathcal{K}))^\alpha = \sum_{j \in \Sigma} \text{diam}(f_j(\mathcal{K}))^\alpha$, as many times as needed. This leads to $\mathcal{H}^\alpha_{1,\text{dim}}(\mathcal{K}) = \mathcal{H}^\alpha_{1}(\mathcal{K})$, and letting $n \to \infty$, we have $0 = \mathcal{H}^\alpha_{1}(\mathcal{K}) = \mathcal{H}^\alpha_{1,\text{dim}}(\mathcal{K})$. On the other hand, let $\varepsilon = \frac{1}{4} \sum_{i \in \Sigma} \text{diam}(f_i(\mathcal{K}))^\alpha$ and $\mathcal{A}$ be a finite covering of $\mathcal{K}$ by elements of $\bigcup_{\mathcal{N} \in \mathcal{N}\Gamma_n} (\mathcal{A})^\alpha < \varepsilon$. In addition, let $n$ be the greatest level containing (at least) one element of $\mathcal{A}$. Thus, for all $m \leq n$ and each $f_i(\mathcal{K}) \in \mathcal{A} \cap \Gamma_m$, we can replace $f_i(\mathcal{K})$ by $\{f_j(\mathcal{K}) : j \in \Sigma^m, i \subseteq j\}$, where $\text{diam}(f_j(\mathcal{K}))^\alpha = \sum_{J \in \Sigma^m} \text{diam}(f_j(\mathcal{K}))^\alpha$. Hence, we can construct a covering $\mathcal{A}' \subseteq \Gamma_n$ from $\mathcal{A}$ such that $\sum_{A \in \mathcal{A}'} \text{diam}(A)^\alpha = \sum_{A \in \mathcal{A}} \text{diam}(A)^\alpha < \varepsilon$. Since $\Gamma_n$ is an irreducible covering and $\mathcal{A}'$ is a subcovering of $\Gamma_n$, then $\mathcal{A}' = \Gamma_n$ but $\sum_{A \in \Gamma_n} \text{diam}(A)^\alpha = \sum_{i \in \Sigma} \text{diam}(f_i(\mathcal{K}))^\alpha = 2\varepsilon$, a contradiction. $\square$

Corollary 5.5 (IFS). $\Gamma$ irreducible $\Rightarrow \dim^4_1(\mathcal{K}) = \dim^3_1(\mathcal{K}) = \alpha$.

Proof. First, notice that $\dim^4_1(\mathcal{K}) \leq \dim^3_1(\mathcal{K}) = \alpha$ due to Corollary 4.6. In addition, Proposition 5.4 gives $\mathcal{H}^\alpha_{1}(\mathcal{K}) > 0$, so $\alpha \leq \dim^4_1(\mathcal{K})$. $\square$

It is worth pointing out that Corollary 5.5 can be understood as a Moran’s type theorem for fractal dimensions III and IV. Interestingly, both Proposition 5.2 and Corollary 5.4 lead to one of the key results we state in this paper.
Theorem 5.6 (IFS). \( \Gamma \) irreducible \( \iff \dim_1^4(\mathcal{K}) = \dim_3^1(\mathcal{K}) = \alpha. \)

In this way, if \( \Gamma \) is the natural fractal structure on \( \mathcal{K} \) as a self-similar set (c.f. Definition 2.1), then the condition \( \dim_1^4(\mathcal{K}) = \dim_3^1(\mathcal{K}) = \alpha \) is equivalent to \( \Gamma \) being irreducible. Thus, it allows to characterize irreducible fractal structures in terms of the equality among the similarity dimension of an IFS-attractor and its fractal dimensions III and IV.

6. The level separation property for complete metric spaces

In this section, we introduce a novel separation condition for each level of the natural fractal structure \( \Gamma \) that any IFS-attractor can be endowed with (c.f. Definition 2.1). We shall prove that such a separation property is equivalent to \( \Gamma \) being irreducible. Moreover, we show additional equivalences between that separation property and other conditions to describe the structure of self-similar sets, e.g., \( \Gamma \) being a tiling.

**Definition 6.1.** We shall understand that \( \mathcal{F} \) satisfies the so-called level separation property (LSP) if the two following conditions hold for each level of \( \Gamma \):

\[
\text{(LSP1)} \quad \hat{A} \cap \hat{B} = \emptyset, \text{ for all } A, B \in \Gamma_n : A \neq B.
\]

\[
\text{(LSP2)} \quad \hat{A} \neq \emptyset, \text{ for each } A \in \Gamma_n,
\]

where the interiors have been considered in \( \mathcal{K} \).

Thus, \( \Gamma \) is under the LSP if for each level of the natural fractal structure on \( \mathcal{K} \), it holds that the interiors of any two (different) elements are pairwise disjoint with their interiors being nonempty. It is worth pointing out that the LSP does not depend on an external open set, unlike the OSC. First, we prove that the LSP implies \( \Gamma \) being irreducible.

**Proposition 6.2** (IFS). LSP \( \Rightarrow \) \( \Gamma \) irreducible.

*Proof.* Assume that \( \Gamma \) is not irreducible. Then there exist \( n \in \mathbb{N} \) and \( A \in \Gamma_n \) such that \( A \subseteq \bigcup\{B \in \Gamma_n : B \neq A\} \). Let us denote \( \beta = \{B \in \Gamma_n : B \neq A\} \). By LSP2, there exists \( x \in A^\circ \). Since \( \Gamma \) is starbase, there exists \( m \geq n \) with \( \text{St}(x, \Gamma_m) \subseteq A \).

Since \( x \in A \subseteq \bigcup_{B \in \beta} B \), there exists \( B \in \Gamma_n \) with \( B \neq A \) and \( x \in B \). By LSP1, we have \( \hat{A} \cap \hat{B} = \emptyset \). Then there exists \( C \in \Gamma_m \) such that \( x \in C \subseteq B \), and hence \( \hat{C} \subseteq \hat{B} \). On the other hand, \( C \subseteq \text{St}(x, \Gamma_m) \subseteq A \). Therefore \( \hat{C} \subseteq \hat{B} \cap \hat{A} = \emptyset \), which is a contradiction with LSP2. We conclude that \( \Gamma \) is irreducible.

The reciprocal of Proposition 6.2 is also true.

**Proposition 6.3** (IFS). \( \Gamma \) irreducible \( \Rightarrow \) LSP.

*Proof.* Let \( \Gamma \) be an irreducible fractal structure.

- First, we check that LSP2 holds. To deal with, let \( n \in \mathbb{N} \) and \( A \in \Gamma_n \). Since \( \Gamma_n \) is an irreducible covering, then it follows that \( \bigcup_{B \in \Gamma_n, B \neq A} B \neq \mathcal{K} \). Thus, \( \mathcal{K} \setminus \bigcup_{B \in \Gamma_n, B \neq A} B \) is a nonempty open set contained in \( A \) and hence, \( \hat{A} \neq \emptyset \).
Next step is to prove LSP1. Assume, by the contrary, that $F$ does not satisfy LSP1. Then there exist a level $n \in \mathbb{N}$, $x \in X$ and elements $A, B \in \Gamma_n : A \neq B$ such that $x \in \bar{A} \cap \bar{B}$. Moreover, since $\Gamma$ is starbase, then there exists $m > n$ such that $\text{St}(x, \Gamma_m) \subseteq \bar{A} \cap \bar{B}$. We can write $A = f_i(\mathcal{K}), B = f_j(\mathcal{K}) : i, j \in \Sigma^n, i \neq j$. Let $C \in \Gamma_m$ such that $C = f_I(\mathcal{K})$, where $I \subseteq \Gamma$ and $x \in C$, so $j \not\subseteq 1$, then $C \subseteq \text{St}(x, \Gamma_m) \subseteq A \cap \bar{B} \subseteq A \cap B$. Since $C \subseteq B = \bigcup_{k \in \Sigma_{m-n}} f_{jk}(\mathcal{K})$, then $\Gamma_m \setminus \{C\}$ is a cover of $\mathcal{K}$, so $\Gamma_n$ is not irreducible, a contradiction.

Both Propositions 6.5 and 6.6 give the following equivalence between the LSP and $\Gamma$ being an irreducible fractal structure.

**Theorem 6.4** (IFS). LSP $\iff$ $\Gamma$ irreducible.

An additional equivalence regarding the LSP is shown next.

**Proposition 6.5** (IFS). $\Gamma$ irreducible $\Rightarrow$ $F$ satisfies LSP2 and $A_i \subseteq A_j$ implies $j \subseteq i$.

**Proof.** Suppose that $\Gamma$ is irreducible. Then Proposition 6.3 guarantees LSP2, so $\bar{A} \neq \emptyset$, for each $A \in \Gamma_n$ and all $n \in \mathbb{N}$. In addition, let $i \in \Sigma^n, j \in \Sigma^m$ be such that $A_i \subseteq A_j$. Then one of the two following cases occurs.

(i) Assume that $m > n$. Then there exists $k \in \Sigma^m$ such that $i \subseteq k$ and $A_k \subseteq A_i$. Thus, $A_k \subseteq A_j$. Further, since $\Gamma_m$ is irreducible, then $k = j$. Hence, $i \subseteq j$ and $A_j \subseteq A_i$. Therefore, $i \subseteq j$ and $A_j = A_i$. But this is only possible if $n = m$, a contradiction.

(ii) Suppose that $m \leq n$. Then $A_i \subseteq A_j = \bigcup_{k \in \Sigma^n, j \subseteq k} A_k$. Since $\Gamma_n$ is irreducible, then $i = k$ for some $k \in \Sigma^n$ with $j \subseteq k$, and hence, $j \subseteq i$.

**Proposition 6.6** (IFS). If $F$ is under LSP2 and the condition $A_i \subseteq A_j$ implies $j \subseteq i$, then $\Gamma$ is irreducible.

**Proof.** Suppose that $F$ is under LSP2 and the condition $A_i \subseteq A_j$ implies $j \subseteq i$. Assume, in addition, that $\Gamma$ is not irreducible. Then there exists $n \in \mathbb{N}$ and $i \in \Sigma^n$ such that $A_i \subseteq \bigcup_{j \in \Sigma^n, j \neq i} A_j$.

By LSP2, there exists $x \in A_i$. Since $\Gamma$ is starbase, there exists $m > n$ such that $\text{St}(x, \Gamma_m) \subseteq A_i$. On the other hand, since $x \in A_i \subseteq \bigcup_{j \in \Sigma^n, j \neq i} A_j$, there exists $j \in \Sigma^n$ with $j \neq i$ and $x \in A_j$. Let $k \in \Sigma^m$ be such that $j \subseteq k$ and $x \in A_k$.

It follows that $x \in A_k \subseteq \text{St}(x, \Gamma_m) \subseteq A_i$, so $A_k \subseteq A_i$ and by hypothesis $i \subseteq k$, which is a contradiction, since $j \subseteq k$ and $j, i \in \Sigma^n$ with $i \neq j$.

Both Propositions 6.5 and 6.6 give the following equivalence.

**Theorem 6.7** (IFS). $\Gamma$ irreducible $\iff$ $F$ is under LSP2 and $A_i \subseteq A_j$ implies $j \subseteq i$.

We also provide a further characterization for $\Gamma$ being irreducible. To deal with, first we recall when a covering is said to be a tiling.

**Definition 6.8.** (c.f. [2, Section 2]) Let $\Gamma$ be a covering of $X$. We shall understand that $\Gamma$ is a tiling provided that all the elements of $\Gamma$ have disjoint interiors and are regularly closed, i.e., $\overline{\nu C} = \overline{C}$ for each $A \in \Gamma$. In particular, a fractal structure $\Gamma$ is called a tiling if each level $\Gamma_n$ of $\Gamma$ is a tiling itself.
From Definition 6.8, it is clear that $\Gamma$ tiling $\Rightarrow$ LSP. The following result deals with its reciprocal.

**Theorem 6.9 (IFS).** LSP $\Leftrightarrow$ $\Gamma$ tiling.

**Proof.**

(\(\Leftarrow\)) Obvious.

(\(\Rightarrow\)) Suppose that there exist $n \in \mathbb{N}$, $A \in \Gamma_n$, and $x \notin A \setminus \text{Fr}(A)$. Since $\Gamma$ is starbase, then there exists $m \in \mathbb{N}$ with $m \geq n$ and such that $\text{St}(x, \Gamma_m) \cap A^o = \emptyset$. Let $B \in \Gamma_m$ be such that $x \in B \subseteq A$. Then $B \subseteq A \setminus A^o \supseteq \text{Fr}(A)$, and hence $B^o = \emptyset$, a contradiction with LSP. Therefore, $A = \text{Fr}(A)$, i.e., $A$ is regularly closed.

\[\square\]

**Corollary 6.10 (IFS).** $\Gamma$ irreducible $\Leftrightarrow$ $\Gamma$ tiling.

All the results obtained so far are collected in the next result.

**Theorem 6.11 (IFS).** The following statements are equivalent:

(i) $\Gamma$ irreducible.

(ii) $\dim_4^4(\mathcal{K}) = \dim_3^3(\mathcal{K}) = \alpha$.

(iii) LSP.

(iv) LSP2 and $A_i \subseteq A_j$ implies $j \subseteq i$.

(v) $\Gamma$ tiling.

(vi) $H_\alpha^4(\mathcal{K}) > 0$.

**Proof.** Just notice that (i) $\Leftrightarrow$ (ii) is due to Theorem 5.6, (ii) $\Leftrightarrow$ (iii) holds by Theorem 6.4, (iii) $\Rightarrow$ (iv) follows from Theorem 6.7, (iv) $\Rightarrow$ (vi) follows from Proposition 5.4, and (vii) $\Leftrightarrow$ (vi) is a consequence of Corollary 6.10. Thus, we shall prove that (vi) $\Rightarrow$ (ii) to conclude the proof. In fact, $H_\alpha^4(\mathcal{K}) > 0$ leads to $\alpha \leq \dim_4^4(\mathcal{K})$. Hence, $\dim_4^4(\mathcal{K}) = \dim_3^3(\mathcal{K}) = \alpha$ due to Corollary 4.6.

\[\square\]

7. The weak separation condition for IFS—attractors

Theorem 6.11 allows us to introduce the so-called weak open set condition for IFS—attractors.

**Definition 7.1.** We shall understand that $\mathcal{F}$ is under the weak separation condition (WSC) if any of the equivalent statements provided in Theorem 6.11 holds.

Unlike the OSC, the WSC does not depend on any external open set (c.f. Definition 7.1). It is also worth pointing out that the WSC becomes a separation property for IFS—attractors weaker than the OSC. Additionally, it allows the calculation of their similarity dimensions. Next, we shall justify these facts.

**Corollary 7.2 (EIFS).** OSC $\equiv$ SOSC $\Rightarrow$ $\Gamma$ irreducible.

**Proof.** By Moran’s Theorem, OSC implies $\dim_H(\mathcal{K}) = \alpha$. Hence, $\Gamma$ is irreducible just by applying both Theorems 4.7 and 5.6.

However, Corollary 7.2 cannot be inverted, in general.

**Remark 7.3 (EIFS).** $\Gamma$ irreducible $\not\Rightarrow$ OSC.
Proof. Consider Mattila’s Counterexample. In fact, the IFS-attractor provided therein has zero $H_1^1$ measure. Thus, it does not satisfy the OSC. In addition, since the Hausdorff dimension of such an IFS-attractor equals its similarity dimension, then Corollary 7.3 leads to $\Gamma$ irreducible. □

The next result stands in the general case as a consequence of both Corollary 7.2 and Remark 7.3. It states that the WSC becomes necessary to fulfill the SOSC in the general case.

**Corollary 7.4 (IFS).** SOSC $\Rightarrow$ WSC and the reciprocal is not true, in general.

Thus, the WSC remains necessary for the SOSC in the general case. In short, as well as the SOSC is sufficient to achieve the equality $\dim_H(K) = \alpha$ in the context of complete metric spaces, the following Moran’s type theorem holds for both fractal dimensions III and IV provided that $\mathcal{F}$ is under the WSC.

**Theorem 7.5 (IFS).** WSC $\iff$ $\dim_F^1(K) = \dim_F^3(K) = \alpha$.

It is worth noting that Theorem 7.5 does not guarantee the identity $\dim_H(K) = \alpha$, unlike Schief’s Theorem 4.2 (IFS case). In return, it provides an equivalence between the WSC, a separation property for IFS-attractors weaker than the OSC, and the identity $\dim_F^1(K) = \dim_F^3(K) = \alpha$. Both results allow the calculation of the self-similarity dimension of $K$.

8. **Sufficient conditions for irreducible fractal structures**

Along this section, we shall explore several properties on a fractal structure leading to the WSC. To tackle with, recall that the order of a fractal structure $\Gamma$ is defined as $\text{Ord} (\Gamma) = \sup \{\text{Ord} (\Gamma_n) : n \in \mathbb{N}\}$, where $\text{Ord} (\Gamma_n)$ is the order of level $n$ as a covering, i.e., $\text{Ord} (\Gamma_n) = \sup \{\text{Ord} (x, \Gamma_n) : x \in X\}$, and $\text{Ord} (x, \Gamma_n) = \text{Card} (\{A \in \Gamma_n : x \in A\}) - 1$ (c.f. [3]).

The first result we provide in that direction deals with the general case of IFS-attractors on complete metric spaces.

**Proposition 8.1 (IFS).** Let us consider the following list of statements:

(i) There exists $k \in \mathbb{N}$ such that each element in $\Gamma_n$ intersects at most to $k$ other elements in that level, for each $n \in \mathbb{N}$.

(ii) $\text{Ord} (\Gamma) < \infty$.

(iii) $\Gamma$ irreducible.

Then $\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}$.

**Proof.** The implication $\text{(i)} \Rightarrow \text{(ii)}$ is clear. Thus, we shall be focused on how to tackle with $\text{(ii)} \Rightarrow \text{(iii)}$. Let us assume, by the contrary, that $\text{Ord} (\Gamma) < \infty$ with $\Gamma$ not being irreducible. Then there exist $m \in \mathbb{N}$ and $i \in \Sigma^m$ such that $A_i \subseteq \cup_{j \in \Sigma^m, j \neq i} A_j$. Since $\text{Ord} (\Gamma)$ is finite, then there exist $k \in \mathbb{N}$, $x \in K$, and $n \in \mathbb{N}$, such that $x$ belongs exactly to $k$ elements of $\Gamma_n$, and any other point in $K$ belongs at most to $k$ elements of $\Gamma_o$, for any $o \in \mathbb{N}$. Hence, $f_i(x)$ belongs to $k$ elements of $\Gamma_{n+m}$ of the form $A_j$ with $j \in \Sigma^{n+m}$ and $i \subseteq j$, and since $f_i(x) \in A_i \subseteq \cup_{j \notin \Sigma^{n+m}, j \neq i} A_j$, then it holds that $f_i(x)$ belongs to some $A_j$ with $j \in \Sigma^{n+m}$ and $i \nsubseteq j$. Thus, $f_i(x)$ belongs at least to $k + 1$ elements of $\Gamma_{n+m}$, a contradiction. □

Similarly to Proposition 8.1, the following result stands in the Euclidean case.

**Proposition 8.2 (EIFS).** Consider the following list of statements:
(i) OSC.
(ii) $\text{Ord}(\Gamma) < \infty$.
(iii) $\Gamma$ irreducible.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof. The implication $(ii) \Rightarrow (iii)$ has been proved in Proposition 8.1. To deal with $(i) \Rightarrow (ii)$, observe that the OSC implies that there exists an integer $N$ such that $A$ intersects to $\leq N$ incomparable pieces $A_j \in \bigcup_{k \in \mathbb{N}} \Gamma_k$ with $\text{diam}(A) \leq \text{diam}(A_j)$, for all $A \in \Gamma_n$ and all $n \in \mathbb{N}$. Let $x$ be a point, $n \in \mathbb{N}$, and assume that $x$ belongs to $> N + 1$ pieces of $\Gamma_n$. Let $A$ be the lowest diameter piece of $\Gamma_n$ containing $x$. Thus, $A$ intersects to $> N$ elements of $\Gamma_n$ that are incomparable since all of them lie in the same level of $\Gamma$, a contradiction. □

The result above states that irreducible fractal structures becomes necessary to fulfill the OSC in the Euclidean case.

9. Regarding the size of the overlaps

In a previous work due to Bandt and Rao (c.f. [6]), they proved the following result regarding the size of the overlaps.

**Theorem 9.1** (c.f. [6], Theorem 2). Let $\mathcal{K}$ be a connected self-similar set in $\mathbb{R}^2$. If $\mathcal{K}_i \cap \mathcal{K}_j$ is a finite set for $i \neq j$, then the OSC holds.

Proposition 8.2 yields the following consequence from Bandt-Rao’s theorem.

**Corollary 9.2** (EIFS). Let $\mathcal{K}$ be a connected self-similar set in $\mathbb{R}^2$ endowed with its natural fractal structure. If $\mathcal{K}_i \cap \mathcal{K}_j$ is a finite set for $i \neq j$, then the WSC holds.

Thus, similarly to the problem appeared in Section 1 of [6] regarding the OSC, next we pose the analogous concerning the WSC:

**Problem.** Will the WSC be true if all the overlaps are finite sets?

In this occasion, though, an affirmative result can be proved in the general case.

**Theorem 9.3** (IFS). If $A \cap B$ is a finite set for all $A, B \in \Gamma_n$ with $A \neq B$ and each $n \in \mathbb{N}$, then the LSP stands. Hence, the WSC holds.

Proof. Observe that $\mathcal{K} \setminus \bigcup \{ B : A \neq B \} = A \setminus \bigcup \{ B : A \neq B \} = A \setminus F$ with $F$ being a finite set. Thus, $\mathcal{K} \setminus \bigcup \{ B : A \neq B \}$ is a nonempty open set (on $\mathcal{K}$) contained in $A$, so LSP2 holds. On the other hand, since $A^c \cap B^c \subseteq (A \cap B)^c$ and $A \cap B$ is finite for all $A, B \in \Gamma_n$, then $A^c \cap B^c = \emptyset$. Hence, LSP1 stands. □

**Corollary 9.4** (IFS). If $A \cap B$ is a finite set for all $A, B \in \Gamma_n$ with $A \neq B$ and all $n \in \mathbb{N}$, then the WSC holds.

It is worth pointing out that Theorem 9.3 stands in the general case of IFS-attractors on a complete metric space, unlike Corollary 9.2 which only stands in the Euclidean plane. In addition, Theorem 9.4 does not require the self-similar set to be connected.
In this section, we summarize all the results contributed along this paper.

**Theorem 10.1.** Consider the following statements:

(i) \( \mathcal{H}_0^0(K) > 0 \).
(ii) SOSC.
(iii) OSC.
(iv) \( \dim_H(K) = \dim_4^F(K) = \dim_3^F(K) = \alpha \).
(v) \( \dim_4^F(K) = \dim_3^F(K) = \alpha \).
(vi) \( \Gamma \) irreducible.
(vii) \( \Gamma \) tiling.
(viii) \( \mathcal{H}_4^0(K) > 0 \).

The next chains of implications and equivalences hold in each case:

(EIFS) \( (i) \iff (ii) \iff (iii) \Rightarrow (iv) \Rightarrow (v) \iff (vi) \iff (vii) \iff (viii) \).

(IFS) \( (i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (v) \iff (vi) \iff (vii) \iff (viii) \).

In addition, the next theorem highlights the connections among separation conditions for IFS-attractors.

**Theorem 10.2.** Consider the following list of statements:

(i) SOSC.
(ii) OSC.
(iii) WSC.

The next chains of implications and equivalences hold and are best possible:

(EIFS) \( (i) \iff (ii) \iff (iii) \iff (iv) \iff (v) \iff (vi) \iff (vii) \iff (viii) \).

(IFS) \( (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \iff (v) \iff (vi) \iff (vii) \iff (viii) \).

A counterexample due to Schief (c.f. [22, Example 3.1]), guarantees the existence of (non-Euclidean) IFS-attractors under the OSC that do not satisfy the SOSC.

**Remark 10.3.** Regarding Theorem 10.2, it is worth noting that the following implications are not true, in general:

(EIFS) \( (iv) \Rightarrow (iii) \Rightarrow (i) \) (c.f. Mattila’s Counterexample).

(IFS) \( (ii) \Rightarrow (iii) \Rightarrow (i) \) due to Schief’s Counterexample. In fact, the WSC does not hold there since \( f_1(K) = f_2(K) \), \( (iv) \Rightarrow (iii) \Rightarrow (i) \) (c.f. Mattila’s Counterexample), \( (iii) \Rightarrow (iv) \) (c.f. either Corollary 7.4 or Mattila’s Counterexample), and \( (iv) \Rightarrow (i) \) (c.f. Schief’s Counterexample).

To conclude this section, we provide two comparative theorems (one for each context, EIFS or IFS) involving our results and those obtained by Schief.

**Theorem 10.4** (EIFS, comparative theorem in Euclidean case).

**Schief’s**: \( \mathcal{H}_0^0(K) > 0 \Leftrightarrow \text{OSC} \Leftrightarrow \text{SOSC} \Rightarrow \dim_H(K) = \alpha \).

**Our’s**: \( \text{WSC} \Leftrightarrow \mathcal{H}_0^0(K) > 0 \Leftrightarrow \dim_4^F(K) = \alpha \).

**Theorem 10.5** (IFS, comparative theorem for complete metric spaces).

**Schief’s**: \( \mathcal{H}_0^0(K) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(K) = \alpha \).

**Our’s**: \( \text{WSC} \Leftrightarrow \mathcal{H}_0^0(K) > 0 \Leftrightarrow \dim_4^F(K) = \alpha \).

Both Theorems 10.4 and 10.5 are best possible since no other implications are valid, in general. Thus, it holds that in the Euclidean case, the SOSC becomes sufficient but
not necessary (c.f. Mattila’s Counterexample) to achieve the equality \( \dim_H(K) = \alpha \). On the other hand, we have proved the equivalence among the WSC and the conditions \( \dim^4_r(K) = \alpha \) and \( H^\alpha_r(K) > 0 \) (that is calculated by finite coverings), as well. In the general case, though, the SOSC is only sufficient to get the equality between \( \dim_H(K) \) and \( \alpha \). Our chain of equivalences involving the WSC still remains valid for complete metric spaces. We have also proved that the SOSC is stronger than the WSC (once again, the Mattila’s Counterexample works), a separation property for IFS-attractors which does not depend on any external open set (unlike the SOSC). Anyway, both statements in Theorem 10.5 (Schief’s and our’s) can be combined into the following summary result standing in the general case:

**Corollary 10.6 (IFS).**

\[ H^\alpha_K > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(K) = \alpha \Rightarrow \text{WSC} \iff \dim^4_r(K) > 0 \iff \dim^4_f(K) = \alpha. \]

Interestingly, Corollary 10.6 highlights that the WSC becomes necessary to reach the equality between the Hausdorff and the similarity dimensions of IFS-attractors. In other words, if the natural fractal structure which any IFS-attractor can be endowed with is not irreducible, then a Moran’s type theorem cannot hold.

To conclude this paper, we shall pose some natural questions still remaining open.

**Open question 10.7 (EIFS/IFS).** *Is it true that* \( \text{WSC} \Rightarrow \dim_H(K) = \alpha \)?

**Open question 10.8 (EIFS/IFS).** *Is it true that* \( \dim_H(K) = \dim^4_f(K) \)?

It is worth noting that an affirmative response to Open question 10.8 would imply that Open question 10.7 is true.

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