An index Theorem for gerbes.

Abstract.

In this paper, we define the notion of vectorial gerbe, an example of a vectorial gerbe is the Clifford-gerbe defined on a riemannian manifold. We show an index theorem for these objects. This enables us to study the topology of riemannian manifolds.

Introduction.

In differential topology, the curvature tensors of riemannian metrics are very useful tools, for example, a compact manifold endowed with a metric whose sectional curvature is constant is the quotient of one of the following space by a discrete subgroup of isometries: a vector space endowed with the flat riemannian metric, the simply connected hyperbolic space, or the sphere $S^n$. If the second Stiefel-Whitney class of a manifold $N$ vanishes, there exists a vector bundle called the bundle of spinors which has many applications in differential topology. Suppose that the second Stiefel-Whitney class $w_2(N)$ of $N$ is not zero, we deduce from the Giraud classification theorem, the existence of a gerbe whose classifying cocycle is $w_2(N)$ which is according to Brylinski, and McLaughlin[2], an illuminating example of gerbe. It is natural to study the relations between this gerbe and the topology of $N$. For example one may expect to generalize the Lichnerowicz theorem for spinors. On this purpose, we need first to prove an index theorem for gerbes, which is our purpose.

In their preprint [6], Murray and Singer have shown an index theorem for bundle gerbes which is an example of the index formula of this paper.

1. On the notion of vectorial gerbes.

The aim of this section is to develop the notion of vectorial gerbes. First, we recall the definition of the notion of sheaf of categories on manifolds.

Definition 1.
Let $N$ be a manifold, a sheaf of categories $C$ on $N$, is a map $U \to C(U)$, where $U$ is an open subset of $N$, and $C(U)$ a category which satisfies the following properties:

- For each inclusion $U \to V$, there exists a functor $r_{U,V} : C(V) \to C(U)$ such that $r_{U,V} \circ r_{V,W} = r_{U,W}$.
- Glaubing conditions for objects,

Consider an open covering family $(U_i)_{i \in I}$ of an open subset $U$ of $N$, and for each $i$, an object $x_i$ of $C(U_i)$, suppose that there exists a map $g_{ij} : r_{U_i \cap U_j} \to r_{U_i \cap U_j}(x_i)$. Denote by $\varphi_{ij}^{ij}$, the restriction of the map $g_{ij}$ to $U_i \cap U_j \cap U_k$, and suppose that $\varphi_{ij}^{ij} \varphi_{ji}^{ij} = \varphi_{ii}^{ij}$, there then exists an object $x$ of $C(U)$ such that $r_{U_i,U}(x) = x_i$.

- Glaubing conditions for arrows,

Consider two objects $P$ and $Q$ of $C(N)$, the map $U \to Hom(r_{U,N}(P), r_{U,N}(Q))$ defined on the category of open subsets of $N$ is a sheaf.

Moreover, if the following conditions are satisfied, the sheaf of categories $C$ is called a gerbe.

G1
There exists an open covering family $(U_i)_{i \in I}$ of $N$, such that for each $i$, the category $C(U_i)$ is not empty.

G2
Let $U$ be an open subset of $N$, for each objects $x$ and $y$ of $C(U)$, there exists an open covering family $(U_i)_{i \in I}$ of $U$, such that $r_{U_i,U}(x)$ and $r_{U_i,U}(y)$ are isomorphic.

G3
Every arrow of $C(U)$ is invertible. There exists a sheaf $L$ of sections of a principal bundle over $N$ such that for each object $x$ of $C(U)$, $Hom(x,x)$ is isomorphic to $L(U)$, by an isomorphism which commutes with restriction maps.

The sheaf $L$ is called the band of the gerbe $C$, in the sequel, we will consider only gerbes with commutative band.

Notations.

For a covering family $(U_i)_{i \in I}$ of $N$, and an object $x_{i_1}$ of $C(U_{i_1})$, we denote by $x_{i_1}^{i_2 \cdot \cdot \cdot i_n}$ the element $r_{U_{i_1} \cap \cdot \cdot \cdot \cap U_{i_n}}(x_{i_1})$, by $U_{i_1 \cdot \cdot \cdot i_n}$ the intersection $U_{i_1} \cap \cdot \cdot \cdot \cap U_{i_n}$, and for a map $h_{i_1} : e_{i_1} \to e'_{i_1}$ between two objects, $e_{i_1}$ and $e'_{i_1}$ of $C(U_{i_1})$, we denote by $h_{i_1}^{i_2 \cdot \cdot \cdot i_n}$, its restriction to $U_{i_1 \cdot \cdot \cdot i_n}$. Suppose that the objects of the category $C(U)$ are vectors bundles, and consider $s_{i_1} : U_{i_1} \to e_{i_1}$, a section of the bundle $e_{i_1}$ defined over $U_{i_1}$, we denote by $s_{i_1}^{i_2 \cdot \cdot \cdot i_n}$ its restriction to $U_{i_1 \cdot \cdot \cdot i_n}$.

Definition 2.
A gerbe is a vectorial gerbe, if and only if for each open subset $U$ of $N$, the category $C(U)$ is a category whose objects are vector bundles over $U$, which typical fiber is the vector space $V$, and such that the maps between objects of $C(U)$ are isomorphisms of vector bundles. The vector space $V$ is called the typical fiber of the vectorial gerbe. The classifying cocycle of the gerbe is defined.
as follows: there exists an open covering family \((U_i)_{i \in I}\) of \(N\), a commutative subgroup \(H\) of \(Gl(V)\), such that there exist maps \(g'_{ij} : U_i \cap U_j \rightarrow Gl(V)\), which define isomorphisms

\[
g_{ij} : U_i \cap U_j \times V \rightarrow U_i \cap U_j \times V
\]

\[
(x, y) \rightarrow (x, g_{ij}(x)y)
\]

and such that \(c_{i_1i_2i_3} = g_{i_1i_2}^{-1}g_{i_2i_3}^{-1}g_{i_3i_1}\) is the classifying Cech 2-cocycle which takes its values in the sheaf of \(H\)-valued functions.

**Example.**

**The Clifford gerbe associated to a riemannian structure.**

Let \(N\) be a riemannian \(n\)-manifold, and \(O(N)\) the reduction of the bundle of linear frames which defines the riemannian structure of the manifold \(N\). The bundle \(O(N)\) is a locally trivial principal bundle over \(N\) which typical fiber is \(O(n)\), the orthogonal group of \(n \times n\) matrices. We have the exact sequence \(1 \rightarrow \mathbb{Z}/2 \rightarrow Spin(n) \rightarrow O(n) \rightarrow 1\), where \(Spin(n)\) is the universal cover of \(O(n)\). We can associate to this problem a gerbe whose band is \(\mathbb{Z}/2\), defined as follows: for each open subset \(U\) of \(N\), \(Spin(U)\) is the category of \(Spin(n)\)-bundles over \(U\) such that the quotient of each object of \(Spin(U)\) by \(\mathbb{Z}/2\) is the restriction \(O(U)\), of \(O(N)\) to \(U\). The classifying 2-cocycle of this gerbe is the second Stiefel-Whitney class.

One can associate to this gerbe, a vectorial gerbe named the Clifford gerbe \(Cl(N)\) which is defined as follows: for each open subset \(U\) of \(N\), \(Cl(U)\) is the category whose objects are Clifford bundles associated to the objects of \(Spin(U)\). The gerbe \(Cl(N)\) is a vectorial gerbe.

Let \(g'_{ij}\) be the transitions functions of the bundle \(O(N)\), for each map \(g_{ij}\), consider an element \(g_{ij}\) over \(g'_{ij}\) which take its values in \(Spin(n)\). Then the element \(g_{ij}(x)\) acts on \(Cl(\mathbb{R}^n)\) by left multiplication, we will denote by \(h_{ij}(x)\) the resulting automorphism of \(Cl(\mathbb{R}^n)\). The classifying cocycle of the Clifford gerbe is defined by the maps \(h_{i_1i_2}^{-1}h_{i_2i_3}^{-1}h_{i_3i_1}\).

**The gerbe defined by the lifting problem associated to a vectorial bundle.**

Consider a vector bundle \(E\) over \(N\), whose typical fiber is the vector space \(V\). One classically associate to \(E\), a principal \(Gl(V)\)-bundle. We suppose that this bundle has a reduction \(E_L\), where \(L\) is a subgroup of \(Gl(V)\). Consider a central extension \(1 \rightarrow H \rightarrow G \rightarrow L \rightarrow 1\). We denote by \(\pi : G \rightarrow L\), the projection and suppose it has local sections. This central extension defines a gerbe \(C_H\) on \(N\), such that for each open subset \(U\) of \(N\), the objects of \(C_H(U)\) are \(G\)-principal bundles over \(U\), whose quotient by \(H\) is the restriction of \(E_L\) to \(U\).

Suppose moreover defined a representation \(r : G \rightarrow Gl(W)\), and a surjection \(f : W \rightarrow V\) such that the following square is commutative:

\[
\begin{array}{ccc}
W & \xrightarrow{r(h)} & W \\
\downarrow f & & \downarrow f \\
V & \xrightarrow{\pi(h)} & V
\end{array}
\]
then one can define the vectorial gerbe $C_{H,W}$ on $N$ such that the objects of $C_{H,W}(U)$ are fiber products $e_U \propto r$, where $e_U$ is an object of $C_H(U)$. Let $(U_i)_{i \in I}$ be a trivialization of $E$ defined by the transitions functions $g_{ij}$, we consider a map $g_{ij} : U_i \cap U_j \to G$ over $g'_{ij}$. The classifying cocycle of the gerbe $C_{H,W}$ is defined by $r(g_{i_1j_2}^{-1})r(g_{i_2j_3}^{-1})r(g_{i_3j_1}^{-1})$.

Definition 3.

- A riemannian metric on a vectorial gerbe $C$ is defined by the following data:

  For each object $e_U$ of $C(U)$, a riemannian metric $<,>_e$ on the vector bundle $e_U$. We suppose that each morphism $h : e_U \to e'_U$, between the objects $e_U$ and $e'_U$ of $C(U)$, is an isomorphism between the riemannian manifolds $(e_U, <,>_e)$ and $(e'_U, <,>_e')$. We remark that the band need to be contained in a sheaf of sections of a principal bundle whose typical fiber is a compact group in this case, since its elements preserve the riemannian metric.

An example of a scalar product on a gerbe is the following: consider the Clifford gerbe $Cl(N)$, we know that the group $Spin(n)$ is a compact group, its action on $Cl(\mathbb{R}^n)$ preserves a scalar product. This scalar product defines on each fiber of an object $e_U$ of $Cl(U)$, a scalar product which defines a riemannian metric $<,>_e$ on $e_U$. The family of riemannian metrics $<,>_e$ is a riemannian metric defined on the gerbe $Cl(N)$.

Definition 4.

- A global section of a vectorial gerbe $C$ associated to a 1-Cech chain $(g_{ij})_{i,j}$, is defined by an open covering family $(U_i)_{i \in I}$ of $N$, for each element $i$ of $I$, an object $e_i$ of $C(U_i)$, a section $s_i$ of $e_i$, a family of morphisms $g_{ij} : e_j^i \to e_i^j$ such that on $U_{ij}$ we have $s_i = g_{ij}(s_j)$.

Let $(s_i)_{i \in I}$ be a global section, on $U_{i_1i_2i_3}$, we have: $s_{i_1}^{i_2i_3} = s_{i_2}^{i_3i_1} = s_{i_3}^{i_1i_2}$, $s_{i_1i_2}^{i_3} = g_{i_3i_2}^{i_3i_1}(s_{i_3}^{i_1i_2})$. This implies that the restriction $s_i^{ij} = g_{ij}^{ij}(s_i)$ of $s_i$ to $U_{ij}$ verifies:

$s_{i_1}^{i_2i_3} = g_{i_3}^{i_1i_2}g_{i_2}^{i_3i_1}(s_{i_3}^{i_1i_2}) = g_{i_3}^{i_1i_2}g_{i_3}^{i_2i_1}(s_{i_3}^{i_1i_2})$. Or equivalently $g_{i_3}^{i_2i_1}g_{i_3}^{i_2i_1}(s_{i_3}) = s_{i_3}$. Remark that the restriction of the element $s_{i_3}$ to $U_{i_1i_2i_3}$ is not necessarily preserved by all the band.

Suppose that $N$ is compact, and $I$ is finite. We can suppose that there exists $i_0$ such that $T = U_{i_0} - \bigcup_{j \neq i_0} U_j$ is not empty. Consider a section $s_{i_0}$ of $e_{i_0}$ whose support is contained in $T$, then we can define a global section $(u_i)_{i \in I}$ such that $u_{i_0} = s_{i_0}$, and if $i \neq i_0$, $u_i = 0$. This ensures that $S(g_{ij})$ is not empty. We will denote by $S(g_{ij})$ the family of global sections associated to $(g_{ij})_{i,j \in I}$. Remark that $S(g_{ij})$ is a vector space.

Proposition 5.

Suppose that the vectorial gerbe $C$ is the gerbe associated to the lifting problem defined by the extension $1 \to H \to G \to L \to 1$ and the vector bundle $E$. Let $r : G \to GL(W)$ be a representation, we suppose that the condition of the diagram (1) is satisfied. Then for each $G$-chain $g_{ij}$, each element $(s_i)_{i \in I}$ of the vector space of global sections $S(g_{ij})$, there exists a section $s$ of $E$, such that $s|_{U_i} = f \circ s_i$. 


Proof.
Let $(s_i)_{i \in I}$ be a global section associated to the chain $S(g_{ij})$, then on $U_{ij}$, we have $s'_i = g_{ij}(s'_j)$, (where $s'_i$ is the restriction of $s_i$ to $U_{ij}$) this implies that on $U_{ij}$, $f(s'_i) = g_{ij}(f(s'_j))$. Thus the family $(f(s_i))_{i \in I}$ of local sections of $E$ defines a global section $s$ of $E$.

Remark.
Let $s$ be a section of the bundle $E$, locally we can define a family of sections $s_i$ of $e_i$, such that $f(s_i) = s_{i|U_i}$. We can consider the chain $s_{ij} = s'_i - g_{ij}(s'_j)$. The family $s_{jk} - s_{ik} + s_{ij}$ is a 2-cocycle. Whenever there exists a chain $g_{ij}$, a global section $s = (s_i)_{i \in I}$ such that $s_i = g_{ij}(s_j)$, and $f(s_i) = s_{i|U_i}$, it is not sure that such a global section exists for another chain $h_{ij}$. This motivates the following definition:

Definition 6.
We define the vector space $L$ of formal global sections of the vector gerbe $C$, tow be the vector space generated by the elements $[s]$, where $s$ is an element of a set of global sections $S(g_{ij})$. The elements of $L$, are finite sum of global sections.

The Prehilbertian structure of $S(g_{ij})$.
First we remark that $S(g_{ij})$ is a vector space. Let $s$, and $t$ be elements of $S(g_{ij})$, we will denote by $s_i$ and $t_i$ the sections of $e_i$ which define respectively the global sections $s$ and $t$. We have $s'_i = g_{ij}(s'_j)$ and $t'_i = g_{ij}(t'_j)$ this implies that $as'_i + bt'_i = g_{ij}(as'_j + bt'_j)$, where $a$ and $b$ are real numbers.

- The scalar structure of $S(g_{ij})$.
Let $(V_k, f_k)_{k \in K}$ be a partition of unity subordinate to $(U_i)_{i \in I}$, this implies that for each $k$ there exists an $i(k)$ such that $V_k$ is a subset of $U_{i(k)}$. Since the support of $f_k$ is a compact subset of $V_k$, we can calculate $\int_{V_k} f_k < s_{i(k)}, t_{i(k)} >$ where $s_{i(k)}$ and $t_{i(k)}$ are the respective restrictions of $s_{i(k)}$ and $t_{i(k)}$ to $V_k$. Remark that since we have supposed that $s'_i = g_{ij}(s'_j)$ and $g_{ij}$ is a riemannian isomorphism between $e'_i$ and $e'_j$, if $V_k$ is also included in $U_{j(k)}$, then we have the equality $< s_{i(k)}, t_{i(k)} > = < s_{j(k)}, t_{j(k)} >$ on $V_k$. We can define $< s, t > = \sum_k \int < f_k s_{i(k)}, f_k t_{i(k)} >$.
We will denote by $L^2(S(g_{ij}))$ the Hilbert completion of the Pre Hilbert structure of $(S(g_{ij}))[<,>]$.

The scalar structure on the set of formal global sections $L$.
Let $s$ and $t$ be two formal global sections, we have $s = [s_{n_1}] + \ldots + [s_{n_p}]$ and $t = [t_{m_1}] + \ldots + [t_{m_q}]$, where $s_{n_i}$ and $t_{m_j}$ are global sections.
We will define a scalar product on $L$ as follows: if $s$ and $t$ are elements of the same set of global sections $S(g_{ij})$, $< [s], [t] > = < s, t > _{S(g_{ij})}$, where $< s, t > _{S(g_{ij})}$ is the scalar product defined at the paragraph above. If $s$ and $t$ are not elements of the same set of global sections, then $< [s], [t] > = 0$. 
Proposition 7.
An element of $L^2(S(g_{ij}))$ is a family of $L^2$ sections $s_i$ of $e_i$ such that $s_i^* = g_{ij}(s_j^*)$.

Proof.
Let $(s^i)_{t \in \mathbb{N}}$ be a Cauchy sequence of $(S(g_{ij}), < , >)$. We can suppose that the open sets $V_k$ used to construct the riemannian metric are such that the restriction $e_i^k$, of $e_i$, to $V_k$ is a trivial vector bundle. The sequence $(f_k s_i^{(k)})$ is a Cauchy sequence defined on the support $T_k$ of $f_k$. Since this support is compact, we obtain that $(f_k s_i^{(k)})_{t \in \mathbb{N}}$ goes to an $L^2$ section $s_i(k)$ of $e_i^k$. We can define $s_i = \sum_{k, V_k \cap U_i \neq \emptyset} f_k s_i^{(k)}$. The family $(s_i)_{i \in I}$ defines the requested limit.

Suppose that morphisms between objects commute with the laplacian $\Delta$, we can then endow $S(g_{ij})$ with the prehilbertian structure defined by $< u, v > = \int < \Delta^s(u), v >$, where $s$ a positive real number, and $\Delta^s(u)$ is the global section defined by $\Delta^s(u)_i = \Delta^s(u_i)$. We will denote by $H_s(S(g_{ij}))$ the Hilbert completion of this prehilbertian space.

We will define the formal $s$-distributional global sections $H_s(L)$ as the vector space generated by finite sums $[s_1] + . . . [s_k]$ where $s_i$ is an element of an Hilbert space $H_s(S(g_{ij}))$.

Connection on riemannian gerbes and characteristic classes.

The notion of connection is not well-defined for general vectorial gerbes, nevertheless the existence of a riemannian structure on a vectorial gerbe $C$, defined on the manifold $N$, gives rise to a riemannian connection on each object $e_U$ of $C(U)$, this family of riemannian connections will be the riemannian connection of the gerbe $C$.

Let $(U_p)_{p \in P}$ be an open covering of $N$, and $so(V)$ the Lie-algebra of the orthogonal group $SO(V)$ of $V$. Suppose that the objects of $C(U_p)$ are trivial bundles. The riemannian connection of the object $e_p$ of $C(U_p)$ is defined by a $so(V)$ 1-form $w_p$ on $TU_p$, and the covariant derivative of this connection evaluated at a section $s_p$ of $e_p$ is $ds_p + w_p s_p$. The curvature of this connection is the 2-form $\Omega_p = dw_p + w_p \wedge w_p$.

The 2$k$-form Chern form of $e_p$, $c^{2k}$, is defined by the trace $\text{Trace}[(\frac{1}{2\pi} \Omega_p)^k]$. Let $c'_p$ be another object of $C(U_p)$. There exist isomorphisms $\phi_p : e_p \rightarrow U_p \times V$, $\phi'_p : c'_p \rightarrow U_p \times V$, and $g_p : e_p \rightarrow c'_p$. The map $\phi'_p \circ g_p \circ \phi_p^{-1}$ is an automorphism of $U_p \times V$ defined by the map: $h_p : (x, y) \rightarrow (x, u_p(x, y))$. The riemannian $\phi_p^{-1} < , > = \phi'_p^{-1} < , >$ is preserved by $h_p$. This implies that $c^{2k}$ is equal to the Chern $2k$-form of $c'_p$.

There exists an isomorphism between the respective restrictions $e^j_p$ of $e_j$, and $e^p_j$ of $e_p$ to $U_{pj}$. As above we can show that this implies that the Chern $2k$-forms of $e^j_p$ and $e^p_j$ coincide on $U_p \cap U_j$. We can define a global form $c_{2k}(N)$ on $N$ whose restriction to $U_p$ is $c^{2k}_p$, which will be called the 2$k$-Chern form of the riemannian gerbe.
We can define the \( c(C) = c_1(N) + \ldots + c_n(N) \) the total Chern form of the gerbe, and the form defined locally by \( ch(C)|_{U_\sigma} = Trace(exp(i\frac{H}{2\pi})) \) the total Chern character.

2. Operators on riemannian gerbe.

We begin by recalling the definition of pseudo-differential operators for open subsets of \( \mathbb{R}^n \), and for manifolds. Let \( U \), be an open subset of \( \mathbb{R}^n \), we denote by \( S^m(U) \) the set of smooth functions \( p(x, u) \) defined on \( U \times \mathbb{R}^n \), such that for every compact set \( K \subset U \), and every multi-indices \( \alpha \) and \( \beta \), we have \( \| D^\alpha D^\beta p(x, u) \| \leq C_{\alpha, \beta, K}(1 + \| u \|)^{|\alpha|} \), where \( D^\alpha \) is the partial derivative in respect to \( \alpha \).

Let \( K(U) \) and \( L(U) \) denote respectively the space of smooth functions with compact support defined on \( U \), and the space of smooth functions on \( U \). We can define the map \( P : K(U) \rightarrow L(U) \) by:

\[
P(f) = \int_U p(x, u) \hat{f}(u) e^{i<x,u>} du
\]

where \( \hat{f} \) is the Fourier transform of \( f \).

**Definition 1.**

An operator on \( U \) is pseudo-differential, if it is locally of the above type.

**Definition 2.**

Let \( P \) be a pseudo-differential operator, \( (U_i)_{i \in I} \) an open covering family of \( U \) such that the restriction of \( P \) to \( U_i \) is defined by \( P(f) = \int_{U_i} p_i(x, u) \hat{f}(u) e^{i<x,u>} du \).

The operator is of degree \( m \) if \( \sigma(p)_{|U_i} = \lim_{t \to \infty} \frac{p_i(x, tu)}{t^m} \) exists. In this case \( \sigma(p) \) whose restriction to \( U_i \) is \( \sigma(p)_{|U_i} \) is called the symbol of \( P \).

Let \( E \) be a vector bundle over the riemannian manifold \( N \), endowed with a scalar metric. We denote by \( K(E) \) and \( L(E) \) the respective space of smooth sections of \( E \) with compact support, and the space of smooth sections of \( E \). An operator on the vector bundle \( E \), is a map \( P : K(E) \rightarrow L(E) \) such that there exists an open covering family \( (U_i)_{i \in I} \) of \( N \) which satisfies:

- The restriction of \( E \) to \( U_i \) is trivial, in fact we suppose that \( U_i \) is the domain of a chart.

- The restriction of \( P \), to \( U_i \) is a map \( P_i : K(U_i \times V) \rightarrow L(U_i' \times V) \) where \( V \) is the typical fiber of \( E \), and \( U_i' \) an open chart of \( N \).

  - If we consider charts \( \phi_i \) and \( \psi_i \) whose domains are respectively \( U_i \) and \( U_i' \), and such that \( \phi_i(U_i \times V) = \psi_i(U_i' \times V) = U \times \mathbb{R}^n \), then the map \( P_i \) is defined by a matrix \( (p_{kl}) \), where \( p_{kl} \) defines an operator of degree \( m \). More precisely, if \( s' \) is a section of \( E \) over \( U_i \), and \( s = (s_1, \ldots, s_n) = \phi_i(s') \), we can define \( t_k = \sum_{i=1}^n \int p_{kl}(x, u) \delta_i(u) e^{i<x,u>} du \), and \( P_i(s') = \psi_i^{-1}(t_1, \ldots, t_n) \).

Consider \( SN \) the sphere bundle of the cotangent space \( \pi : T^*N \rightarrow N \), and \( \pi^*E \) the pull-back of \( E \) to \( T^*N \), the symbols defined by the operators \( p_{ij} \) define a map \( \sigma : \pi^*E \rightarrow \pi^*E \). Consider now the projection \( \pi_S : SN \rightarrow N \), then \( \sigma \) induces a map \( \sigma_S : \pi_S^*E \rightarrow \pi_S^*E \).
Let $s$ be a positive integer we denote by $H^\text{loc}_s(N,E)$, the space of distributional sections $u$ of $E$ such that $D(u)$ is a $L^2$-section, where $D$ is any differential operator of order less than $s$, and by $H^\text{comp}_s(N,E)$ the subset of elements of $H^\text{loc}_s(N,E)$ with compact support. Remark that if $N$ is compact, then $H^\text{loc}_s(N,E) = H^\text{comp}_s(N,E)$ in this case, we denote this space by $H_s(N,E)$. We define by $H_{-s}^\text{loc}(N,E)$ to be the dual space of $H^\text{comp}(N,E)$, and by $H_{-s}^\text{comp}(N,E)$ the dual space of $H^\text{loc}(N,E)$.

Suppose that $N$ is compact, then the Sobolev space $H(N,E)$ is a Hilbert space endowed with the norm defined by $(||\int_N <\Delta u, u>||^2)^{1/2}$. Every operator $P$ of order less than $m$ can be extended to a continuous morphism $H_s(N,E) \to H_{s-m}(N,E)$.

We will adapt now the definition of an operator on a manifold to the definition of operators on vectorial gerbes.

**Definition 3.**

Let $C$ be a riemannian gerbe defined on the manifold $N$, an operator $D$ of degree $m$ on $C$, is a family of operators $D_e$ of degree $m$ defined on each object $e$, of the category $C(U)$, for each open subset $U$ of $N$. We suppose that for each morphism $g : e \to f$, $D_f g^* = g^* D_e$, where $g^*(s) = g(s)$.

**Remark.**

The last condition in the previous definition implies that $D_e$ is invariant by the automorphisms of $e$. The operators considered in the sequel will be assumed to be continue, and we will assume that they preserve $C^\infty$-sections. Thus an operator defines a map $D^H_e : H^\text{comp}(U,e) \to H^\text{loc}_{s-m}(U,e)$.

**Proposition 4.** Let $D$ be an operator of degree $n$ defined on the riemannian gerbe $C$, then $D$ induces a map $D_{S(g_{ij})} : H_s(S(g_{ij})) \to H_{s-n}(S(g_{ij}))$ and a map $D_L : H_s(L) \to H_{s-n}(L)$.

**Proof.**

Consider a global distributional section $s$ which is an element of $H_s(S(g_{ij}))$, we have $g_{ij}(D_{ij}(s^j)) = D_{ij}(s^j)$. This implies the result.

The symbol of an operator defined on a gerbe.

Let $C$ be a vectorial gerbe defined on a compact manifold $N$ endowed with the operator $D$ of degree $m$, for each object $e$ of $C(U)$, we can pull back the bundle $e$ by the projection map $\pi_{SU} : SU \to U$ to a bundle $\pi^*_{SU,e}$ over $SU$, where $SU$ is the restriction of the cosphere bundle defined by a fixed riemannian metric of $T^*N$. The set $C_S(U)$ which elements are $\pi^*_{SU,e}$ where $e$ is an object of $C(U)$ is a category. The maps between objects of this category are induced by maps between elements of $C(U)$. The map $U \to C_S(U)$ is a gerbe which has the same band than $C$. Now on the object $e$, we can define a symbol $\sigma_{D_e} : \pi^*_{SU,e} \to \pi^*_S U e$.

Remark that for every automorphism $g$ of $e$, the fact that $g^* \circ D_e = D_e \circ g^*$ implies that $\sigma_{gD_e g^{-1}} = \sigma_{D_e}$. 

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Proposition 5. Rellich Lemma for the family $S(g_{ij})$.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of $H_s(S(g_{ij}))$, we suppose that there exists a constant $L$ such that $\|f_n\|_s < L$, then for every $s > t$, there exists a subsequence $f_{n_k}$ which converges in $H_t$.

Proof.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of sections which satisfy the condition of the proposition, and $(V_\alpha, f_\alpha)$ a partition of unity subordinate to $(U_i)_{i \in I}$. We suppose that the support of $f_\alpha$ is a compact space $K_\alpha$. We denote by $s^i_\alpha$ the section of $e_i$ which defines $s_n$, and by $s^i_{n\alpha}$ the restriction of $s^i_\alpha$ to the restriction of $e_i$ to $V_\alpha$. The family $(f_\alpha s^i_{n\alpha})$ $H_t$-converges towards the element $s_{i\alpha}$ in $V_\alpha$ by the classical Rellich lemma. We can write then $s^i = \sum_{V_\alpha \cap U_i} f_\alpha s_{i\alpha}$. This is an $H_t$ map since the family of $V_\alpha$ can be supposed to be finite, since $N$ is compact.

The family $(s^i)$ defines a global $H_t$-section which is the requested limit.

Remark.

A compact operator between Hilbert spaces, is an operator which transforms bounded spaces to compact spaces. The previous lemma implies that if $s > t$, then the inclusion $H_s(S(g_{ij})) \rightarrow H_t(S(g_{ij}))$ is compact, indeed since $H_s(S(g_{ij}))$ is a separate space, a compact subspace of $H_s(S(g_{ij}))$ is a set such that we can extract a convergent sequence from every sequence.

Proposition 6.

The space $\text{Op}(C)$ of continuous linear maps of $H_s(S(g_{ij}))$ is a Banach space.

Proof.

Let $(D_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of elements of $\text{Op}(C)$, for each global section $s$, the sequence $D_n(s)$ is a Cauchy sequence in respect to the norm of $H_s(S(g_{ij}))$, we conclude that it converges towards an element $D(s)$. The map $D : s \rightarrow D(s)$ is the requested limit. It is bounded since $(D_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

The previous proposition allows us to define $O^n$, the completion of the pseudo-differential operators in $\text{OP}(C)$ of order $n$, and to extend the symbol $\sigma$ to $O^n$. Now we will show that the kernel of the extension of the symbol to $O^n$ contains only compact operators.

Proposition 7.

The kernel of the symbol map contains only compact operators.

Proof.

The symbol $\sigma(P)$ of the operator $P$ is zero if and only if the order $m$ of the operator is less than $-1$. This implies that the operator $P$ is compact. To see this, we can suppose our operator to be an $L^2(S(g_{ij}))$ operator, by composing it by the inclusion map $H_{2-m}(S(g_{ij})) \rightarrow L^2(S(g_{ij}))$, we conclude by using the previous Rellich lemma.

We end this section by defining elliptic operators for gerbes.

Definition 8.
We say that an operator is elliptic if the family of symbols $\sigma_{D_n}$ are invertible maps.

3. $K$-theory and the index.

In this part we will give the definitions of the $K$-theory groups $K_0$, and $K_1$, and show how we can use them to associate to a symbol of an operator on a

We will denote by $M_n$ the vector space of $n \times n$ complex matrices. For $n \leq m$, we consider the natural injection $M_n \to M_m$. We will call $M_\infty$, the inductive limit of the vector spaces $M_n, n \in \mathbb{N}$.

Let $R$ be a ring, $p$, and $q$ be two idempotents of $R_\infty = R \otimes M_\infty$, we will say that $p \sim q$ if and only if there exists elements $u$ and $v$ of $R_\infty$ such that $p = uv$, and $q = vu$. We denote by $[p]$ the class of $p$, and by $Idem(R_\infty)$ the set of equivalence classes. If $[p]$ and $[q]$ are represented respectively by elements of $R \otimes M_n$ and $R \otimes M_m$, we can define an idempotent of $R \otimes M_{n+m}$ represented by the matrix $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ that we denote $[p + q]$.

**Definition 1.**

We will denote by $K_0(R)$, the semi-group $Idem(R_\infty)$, endowed with the law $[p] + [q] = [p + q]$.

Let $N$ be a compact manifold, and $C(N)$ the set of complex valued functions on $N$. It is a well-known fact that for a complex vector bundle $V$ on $N$, there exists a bundle $W$, such that $V \oplus W$ is a trivial bundle isomorphic to $N \times \mathbb{C}^d$. We can thus identify a vector bundle over $N$, to an idempotent of $C(X) \otimes M_l$ which is also an idempotent of $C(X)_\infty$. This enables to identify $K_0(N)$ to $K_0(C(X))$. In fact the semi-group $K_0(C(X))$ is a group.

Let $Gl_n(R)$ be the group of invertible elements of $M_n(R)$, if $l \leq n$ we have the canonical inclusion map $Gl_l(R) \to Gl_n(R)$. We will denote by $Gl_\infty(R)$, the inductive limit of the groups $Gl_n(R)$.

**Definition 2.**

Let $Gl_\infty(R)_0$ be the connected component of $Gl_\infty(R)$. We will denote by $K_1(R)$ the quotient of $Gl_\infty(R)$ by $Gl_\infty(R)_0$.

For a compact manifold $N$, we denote $K_1(C(N))$ by $K_1(N)$.

Consider now an exact sequence $0 \to R_1 \to R_2 \to R_3 \to 0$ of $C^*$-algebras, we have the following exact sequence in $K$-theory:

$$K_1(R_1) \to K_1(R_2) \to K_1(R_3) \to K_0(R_1) \to K_0(R_2) \to K_0(R_3).$$

Let $\mathcal{H}$ be an Hilbert space, we denote by $B(\mathcal{H})$ the space of continuous operators defined on $\mathcal{H}$, and $\mathcal{K}$ the subspace of compact continuous operators of $B(\mathcal{H})$. We have the following exact sequence:

$$0 \to \mathcal{K} \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K} = C_0 \to 0.$$
The algebra $Ca$ is called the Calkin algebra of $\mathcal{H}$. It is a well-known fact that $K_0(K) = \mathbb{Z}$.

Let $N$ be a riemannian manifold, and $C$ a riemannian gerbe defined on $N$. Consider an elliptic operator $D$ defined on $C$, of degree $l$. The operator $D$ induces a morphism: $D : L^2(S(g_{ij})) \to H_{2-l}(S(g_{ij}))$. Consider the operator $(1 - \Delta)^{-m}$ of degree $-l$. The operator $D' = (1 - \Delta)^{-m}D$ is a morphism of $L^2(S(g_{ij}))$. The symbol of $(1 - \Delta)^{-m}D$ is also $\sigma(D)$. This implies that the image of the operator $D$ in the Calkin algebra of $L^2(S(g_{ij}))$ is invertible. It thus defines a class $[\sigma(D')]$ of $K_1(Ca)$. The image of $[\sigma(D')]$ in $K_0(K)$ is the index of $D$. We remark that the index of the operator depends only of the symbol.

For every object $e$ of $C(U)$, the symbol $\sigma(D_e)$ is an automorphism of $\pi_{SU}^*e$, it defines an element $[\sigma(D_e)]$ of $K_1(C(S(U)))$ (recall that $S(U)$ is the cosphere bundle over $U$ defined by the riemannian metric).

**Remark.**

Let $U$ be an open subset such that the objects of $C(U)$ are trivial bundles. Consider an object $e$ of $C(U)$, and a trivialization map $\phi_e : e \to U \times V$. For every object $f$ of $C(U)$, we have $\phi_e^{-1*}(\sigma_{D_e}) = \phi_f^{-1*}(\sigma_{D_f})$.

**Proposition 3.**

Let $C$ be a vectorial gerbe defined over the compact manifold $N$, and $U$ an open subset of $N$, there exists a trivial complex bundle $f_n = N \times \mathbb{C}^n$, such that each object of $C(U)$ is isomorphic to a sub-bundle of the restriction of $f_n$ to $U$.

**Proof.**

Let $(U_i)_{i \in I}$ be an open finite covering family of $N$ such that for each $i$ the objects of the category $C(U_i)$ are trivial bundles. Let $e_U$ be an object of $C(U)$, the restriction $e_i$ of $e$ to $U_i$ is a trivial vectorial bundle. We consider a fixed object $e^0_i$ of $C(U_i)$. Consider a finite partition of unity $(f_p)_{p \in 1, \ldots, l}$ subordinate to the covering family $(U_i)_{i \in I}$, and $g_i : e^0_i \to V$ the composition of the trivialization and the second projection. Let $h_i : e_i \to e^0_i$ be an isomorphism, we can define the map $k : e_U \to L^{ldim V}$ such that $k(x) = (f_1(\pi_{e_U}(x))g_1h_1(x), \ldots, f_l(\pi_{e_U}(x))g_lh_l(x))$. $k$ induces a map $K : N \to G_p(\mathbb{C}^n)$, where $p = dim V$ and $n = ldim(V)$, and $G_p(\mathbb{C}^n)$ is the Grassmanian of $p$-complex plane of $\mathbb{C}^n$; $e_U$ is the pull-back of the canonical $p$-vector bundle over $G_p(\mathbb{C}^n)$. We remark that it is a subbundle of the pull-back of the trivial bundle $G_p(L^m) \times L^m$.

The orthogonal bundle of $e_U$ in the previous result can be chosen canonically by considering the orthogonal bundle of the canonical $L^p$-bundle over $G_p(\mathbb{C}^n)$ in $G_p(L^m) \times L^m$. We will suppose that this bundle is chosen canonically in the sequel.

**Proposition 4.**

Let $C$ be a riemannian gerbe defined over the compact manifold $N$. Then we can associate naturally to the symbol of the elliptic operator $D$, a class $[\sigma_D]$ in $K_1(T^*N)$. 

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We can define the boundary $d$ between the set of differentiable functions of $U$ and $N$ operators an element of the given operators $D$. The symbols of $K$ and $e$ endowed with a gerbe implies that this symbol is homotopic to a function which depends only of the unit ball. We consider the bundle $B^*N$ be the compactification of $T^*N$ whose fibers are isomorphic to the unit ball. We consider the bundle $B^*N/T^*N$ that we can identify to the sphere bundle $S^*N$. We have the following exact sequence

$$0 \to C_0(T^*N) \to C(B^*N) \to C(S^*N) \to 0.$$ 

This sequence gives rise to the following exact sequence in $K$-theory:

$$K_1(C(S^*N) \otimes M_n) \to K_0(C(T^*N) \otimes M_n) \to K_0(C(B^*N) \otimes M_n) \to K_0(C(S^*N) \otimes M_n) \to 0.$$ 

We can define the boundary $\delta([\sigma_D])$ which is an element of $K_0(T^*N \otimes M_n) \simeq K_0(T^*N)$.

**Proposition 5.**

The index of $D$ depends only of the class of $\delta([\sigma_D])$ in $K_0(T^*N)$.

**Proof.**

We remark that the image of a symbol by the map $K_1(S^*N \otimes M_n) \to K_0(T^*N \otimes M_n)$ is zero, if it is the restriction of a map defined on $B^*N$. This implies that this symbol is homotopic to a function which depends only of $N$. Thus the operator it defines is homotopic to a multiplication by a function whose index is zero.

We end this part by answering the following question: let $N$ be a manifold endowed with a gerbe $C$, which is the union of two open sets $U_1$ and $U_2$ such that there exists objects $e_1$ and $e_2$ of the respective categories $C(U_1)$ and $C(U_2)$. Given operators $D_{e_1}$ and $D_{e_2}$ on $e_1$ and $e_2$, is it possible to associate to these operators an element of the $K$-theory? We do not request any compatibility between $D_{e_1}$ and $D_{e_2}$.

The vectors bundles $e_1$ and $e_2$ are sub-bundles of the respective trivial bundles $U_1 \times \mathcal{L}^m$ and $U_2 \times \mathcal{L}^p$. Let $SU_1$ and $SU_2$ be the restriction of the sphere bundle of the cotangent space of $N$ to $U_1$ and $U_2$, we can canonically extends the symbols of $D_{e_1}$ and $D_{e_2}$, to respective automorphisms $\sigma_{D_1}$ and $\sigma_{D_2}$ of $F_1 = SU_1 \times \mathcal{L}^k$, and $F_2 = SU_2 \times \mathcal{L}^p$, thus elements of $K_1(C(U_1) \otimes M_n) = K_1(C(U_1))$, and $K_1(C(U_2) \otimes M_n) = K_1(C(U_2))$, where $C(U_1)$ and $C(U_2)$ are respectively the set of differentiable functions of $U_1$ and $U_2$. 

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The Mayer-Vietoris sequence for algebraic $K$-theory gives rise to the sequence

$$K_2(C(S(U_1 \cap U_2))) \to K_1(C(SN)) \to K_1(C(SU_1)) \oplus K_1(C(SU_2)) \to K_1((S(U_1 \cap U_2)))$$

if the image of $[\sigma_{D_1}] + [\sigma_{D_2}]$ in the previous sequence is zero, then there exists a class $[\sigma_D]$ of $K_1(SN)$, whose image by the map of the previous exact sequence is the element $[\sigma_{D_1}] + [\sigma_{D_2}]$ of $K_1(C(SU_1) \oplus K_1(C(SU_2)))$. The class $[\sigma_D]$ is not necessarily unique.

3. The index formula for operator on gerbes.

Now, we will deduce an index type theorem for riemannian gerbe. We know that the Chern character of the cotangent bundle induces an isomorphism:

$$K_0(T^*N) \otimes \mathbb{R} \to H^*_c(N, \mathbb{R})$$

$$x \otimes t \to tch(x)$$

Consider $Vect(Ind)$ the subspace of $K_0(T^*N)$ generated by $\sigma_P$, where $P$ is an operator on the riemannian gerbe. It can be considered as a subspace of $H^*_c(N, \mathbb{R})$. The map $Vect(\sigma_P) \to \mathbb{R}$ determined by $ch([\sigma_P]) \to \text{ind}(P)$ can be extended to a linear map $H^*_c(N, \mathbb{R}) \to \mathbb{R}$.

The Poincare duality implies the existence of a class $t(N)$ such that

$$Ind(P) = \int_{T^*N} ch([\sigma_P]) \wedge t(N)$$

4. Applications.

We will apply now this theory to the problem which has motivated its construction.

Let $N$ be a riemannian manifold, consider the Clifford gerbe on $N$, Let $(U_i)_{i \in I}$ be an open covering of $N$, the riemannian connection $w_c$ is defined by a family of $so(n)$ 1-forms $w_i$ on $U_i$ which satisfy $w_j = \text{ad}(g_{ij}^{-1})w_i + g_{ij}^{-1}dg_{ij}$. The covariant derivative of the Levi-Civita connection is $d + w_i$. We will fixes an orthogonal basis $(e_1, \ldots, e_n)$ of the tangent space $TU_i$ of $U_i$, and write $w_i = \sum_{k=1}^n w_{ikk}e_k$. we can define the spinorial covariant derivative by setting $\phi_{ij} = -\frac{i}{4}w_{ij}$. In the orthogonal basis $(e_1, \ldots, e_n)$ we have $\phi_{ij} = \sum_{k,l} \phi_{klk}e_ke_l$, with $\phi_{kl} = -\phi_{lk}$.

The Dirac operator.

Let $e_U$ be an object of $Cl(U)$, $U_i$ a trivialization of $Cl(U)$, we will define $D_{e_U} = \sum_{k=1}^n e_i \nabla_{spin e_i}$, On each object $e_U$ of $C(U)$, we have the Lichnerowicz-Weitzenbock formula: $D^2 = \nabla^* \nabla + \frac{1}{4}s$, where $s$ is the scalar curvature. In
this formula $\nabla^* \nabla$ is the connection laplacian. We will say the global spinor is harmonic if $D_{e_i}(s_i) = 0$, for each $s_i$.

**Proposition 1.**

*Suppose that the scalar curvature $s$ of $N$ is strictly positive, and $N$ is compact then every harmonic global spinor is $0$.***

**Proof.**

Let $\psi$ be an harmonic global spinor, we can represent $\psi$ by a family of spinors $s_i$ defined on an open cover $(U_i)_{i \in I}$ of $N$, we have on each $U_i$, $D_{e_i}(s_i) = 0$, this implies that $D_{e_i}^2(s_i) = 0$, we can write

$$\int_{U_i} < \nabla \nabla^* s_i, s_i > + \frac{1}{4} s s_i, s_i > = 0$$

this implies that $\int_N s = 0$, which contradicts the fact that the scalar curvature is strictly positive.

**Corollary 2.** *Suppose that the sectional curvature of a compact riemannian manifold is strictly positive then the class $\tau(N)$ associated to the index formula for operators on the gerbe $\text{Cl}(N)$ is zero.*

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