On the spectrum of 1D quantum Ising quasicrystal

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Abstract

We consider one dimensional quantum Ising spin-1/2 chains with two-valued nearest neighbor couplings arranged in a quasi-periodic sequence, with uniform, transverse magnetic field. By employing the Jordan-Wigner transformation of the spin operators to spinless fermions, the energy spectrum can be computed exactly on a finite lattice. By employing the transfer matrix technique and investigating the dynamics of the corresponding trace map, we show that in the thermodynamic limit the energy spectrum is a Cantor set of zero Lebesgue measure. Moreover, we show that local Hausdorff dimension is continuous and nonconstant over the spectrum. This forms a rigorous counterpart of numerous numerical studies.

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1 Introduction

Since the discovery of quasicrystals [40,41,54,55], quasi-periodic models in mathematical physics have formed an active area of research. The method of trace maps, originally introduced in [36,39,45] (see also [3,30,35,38,52,53,56,60] and references therein), has provided a means for rigorous investigation into the physical properties of one-dimensional quasi-periodic structures, leading, for example, to fundamental results in spectral theory of discrete Schrödinger operators and Ising models on one-dimensional quasi-periodic lattices (for Schrödinger operators: [5,9,11–14,16,18,39,50,57], for Ising models: [6,7,10,19,26,59,61,66], and references therein).

Quasi-periodicity and the associated trace map formalism is still an area of active investigation, mostly in connection with their applications in physics. In this paper we use these techniques to investigate the energy spectrum of one-dimensional quantum Ising spin chains with two-valued nearest neighbor couplings arranged in a quasi-periodic sequence, with uniform, transverse magnetic field. We shall concentrate on the quasi-periodic sequence generated by Fibonacci substitution on two symbols.

One-dimensional quasi-periodic quantum Ising spin chains have been investigated (analytically and numerically) over the past two decades [3,6,7,10,19,26,31–34,62,66]. Numerical and some analytic results suggest Cantor structure of the energy spectrum, with nonuniform local scaling (i.e. a multifractal). The multifractal structure and fractal dimensions of the energy spectrum have not been shown rigorously.

Our aim here is to prove rigorously multifractal structure of the energy spectrum and investigate its fractal dimensions. We achieve this by investigating dynamics of a certain polynomial map on $\mathbb{R}^3$ and employing results from the theory of hyperbolic and partially hyperbolic dynamical systems.

2 The 1D quasi-periodic quantum Ising chain

For a general overview of aperiodic (including Fibonacci) Ising models, see, for example, [22].
2.1 The model

Let $J_0, J_1 > 0$. Construct a $\{J_0, J_1\}$-valued sequence $\{\tilde{J}_n\}_{n \in \mathbb{N}}$ by applying repeatedly the Fibonacci substitution rule on two letters:

$$J_0 \mapsto J_0 J_1 \quad \text{and} \quad J_1 \mapsto J_0,$$

starting with $J_0$:

$$J_0 \mapsto J_0 J_1 \mapsto J_0 J_1 J_0 \mapsto J_0 J_1 J_0 J_0 J_1 \mapsto \cdots,$$

at each step substituting for $J_0$ and $J_1$ according to the substitution rule (1). By this procedure an infinite sequence is constructed, which we call $\{\tilde{J}_n\}_{n \in \mathbb{N}}$ (see [49] for more details on substitution sequences).

Let $\hat{J}_k$ be the finite word after $k$ applications of the substitution rule. It is easy to see that the following recurrence relation holds:

$$\hat{J}_{k+1} = \hat{J}_k \hat{J}_{k-1}. \quad (2)$$

The word $\hat{J}_k$ has length $F_k$, where $F_k$ is the $k$th Fibonacci number. The quasiperiodic (Fibonacci in our case) one-dimensional quantum Ising model on the finite one-dimensional lattice of $N$ nodes with transversal external field is given by the Ising Hamiltonian

$$\mathcal{H} = -\sum_{n=1}^{N} \tilde{J}_n \sigma_n^{(x)} \sigma_{n+1}^{(x)} - h \sum_{n=1}^{N} \sigma_n^{(z)},$$

where $h > 0$ is the external magnetic field in the direction transversal to the lattice. The matrices $\sigma_n^{(x),(z)}$ are spin-1/2 operators in the $x$ and $z$ direction, respectively, given by

$$\sigma_n^{(x),(z)} = I \oplus \cdots \oplus I \oplus \sigma_{(x),(z)} \oplus I \oplus \cdots \oplus I,$$

where $I$ is the $2 \times 2$ identity matrix. Here $\sigma_{(x),(z)}$ are the Pauli matrices given by

$$\sigma^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

The magnetic field $h$ can be absorbed into interaction strength couplings $J_0, J_1$, so $\mathcal{H}$ can be rewritten as

$$\mathcal{H} = -\sum_{n=1}^{N} \left( \tilde{J}_n \sigma_n^{(x)} \sigma_{n+1}^{(x)} + \sigma_n^{(z)} \right). \quad (4)$$

The Hamiltonian $\mathcal{H}$ in (4) acts on $\mathbb{C}^{2N}$, where we assume periodic boundary conditions: $\sigma_{N+1}^{(x),(z)} = \sigma_0^{(x),(z)}$. Here $\sigma_n^{(\alpha)}$, $\alpha \in \{x, z\}$, acts on the finite sequence $(c_1, \ldots, c_N) \in \mathbb{C}^{2N}$ by acting by $\sigma^{(\alpha)}$ from (3) on the $j$-th entry, while leaving the other entries unchanged.
2.2 Fermionic representation

For convenience, let us denote by \( \mathcal{H}_k \) the Hamiltonian in (4) on a lattice of size \( F_k \). We can then extend \( \mathcal{H}_k \) periodically to a Hamiltonian \( \hat{\mathcal{H}}_k \) over the periodic infinite lattice with the unit cell of length \( F_k \). The operator \( \mathcal{H}_k \), and hence also \( \hat{\mathcal{H}}_k \), can be cast into fermionic representation by means of the Jordan-Wigner transformation [19,42]:

\[
\mathcal{H}_k = \sum_{i,j} \left[ c_i^\dagger A_{ij} c_j + \frac{1}{2} \left( c_i^\dagger B_{ij} c_j^\dagger + \left\{ c_i^\dagger B_{ij} c_j^\dagger \right\}^* \right) \right],
\]

(5)

where \( c_i \), \( 1 \leq i \leq F_k \), are anticommuting fermionic operators and \( \{ \cdot \}^* \) denotes Hermitian conjugation. The matrices \( A, B \) are given by (see [19,42])

\[
A_{ii} = -2, \quad A_{i,i+1} = A_{i+1,i} = -\tilde{J}_i, \quad A_{1,F_k} = -\tilde{J}_{F_k};
\]

\[
B_{i,i+1} = -\tilde{J}_i, \quad B_{i+1,i} = \tilde{J}_i, \quad B_{1,F_k} = \tilde{J}_{F_k};
\]

all other entries being zero. The energy spectrum of the Hamiltonian in (5) consists of real \( \lambda \) that satisfy

\[
(A - B)\phi = \lambda \psi, \quad (A + B)\psi = \lambda \phi
\]

(see [42]). This equation can be written in the form [6,10,66]

\[
\Phi_{i+1} = M_i(\lambda)\Phi_i,
\]

where \( \Phi_i = (\psi_i, \phi_i)^T \), and

\[
M_i(\lambda) = \begin{pmatrix}
-1/\tilde{J}_i & \lambda/2\tilde{J}_i \\
-\lambda/2\tilde{J}_i & (\lambda^2 - 4\tilde{J}_i^2)/4\tilde{J}_i
\end{pmatrix};
\]

Thus the wave function \( \Phi \) at site \( N \) is given by

\[
\Phi_{N+1} = M_N(\lambda) \times M_{N-1}(\lambda) \times \cdots \times M_0(\lambda)\Phi_0.
\]

Letting \( \hat{M}_k(\lambda) \) denote the transfer matrix over \( F_k \) sites, using the recurrence relation in (2), we obtain

\[
\hat{M}_{k+1}(\lambda) = \hat{M}_k(\lambda) \times \hat{M}_{k-1}(\lambda),
\]

(7)

for \( k \geq 2 \).

Returning to the Hamiltonian \( \hat{\mathcal{H}}_k \), we see that the wave function at site \( nF_k \) is given by

\[
\Phi_{nF_k} = \hat{M}_k(\lambda)^n\Phi_0.
\]
The wave function over the infinite lattice should not diverge exponentially. Hence we allow only those values of $\lambda$ for which the eigenvalues of $\hat{M}_k(\lambda)$ lie in $[-1,1]$. Since $\hat{M}_k(\lambda)$ is unimodular, this is equivalent to the requirement
\[
\frac{1}{2} \left| \text{Tr} \hat{M}_k(\lambda) \right| \leq 1.
\] (8)

Let
\[ x_k(\lambda) = \frac{1}{2} \text{Tr} \hat{M}_k(\lambda). \]

Using the recursion relation (7), one may derive the recursion relation on the traces given by (see [36,39,45] and, for a more general discussion, [52])
\[ x_{k+1} = 2x_k x_{k-1} - x_{k-2}. \]

Thus, in accordance with (8), we require that $|x_k| \leq 1$. Define the so-called Fibonacci trace map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by
\[ f(x,y,z) = (2xy - z,x,y). \] (9)

Set
\[
M_{-1}(\lambda) = \begin{pmatrix} J_0/J_1 & \lambda(J_1^2 - J_0^2)/2J_0J_1 \\ 0 & J_1/J_0 \end{pmatrix},
M_0(\lambda) = \begin{pmatrix} -1/J_0 & \lambda/2J_0 \\ -\lambda/2J_0 & (\lambda^2 - 4J_0^2)/4J_0 \end{pmatrix},
M_1(\lambda) = \begin{pmatrix} -1/J_1 & \lambda/2J_1 \\ -\lambda/2J_1 & (\lambda^2 - 4J_1^2)/4J_1 \end{pmatrix} = M_0(\lambda) \times M_{-1}(\lambda).
\]

Then
\[
x_{-1}(\lambda) = \left( \frac{J_0 + J_1}{J_1} \right)/2, \quad x_0(\lambda) = \frac{\lambda^2 - (4 + 4J_0^2)}{8J_0}, \quad x_1(\lambda) = \frac{\lambda^2 - (4 + 4J_1^2)}{8J_1}.
\]

It is convenient to absorb the factor $1/4$ into $\lambda^2$ and rewrite
\[
x_0(\lambda) = \frac{\lambda^2 - (1 + J_0^2)}{2J_0}, \quad x_1(\lambda) = \frac{\lambda^2 - (1 + J_1^2)}{2J_1}.
\]

Define the line of initial conditions $\gamma(J_0,J_1) : (-\infty,\infty) \to \mathbb{R}^3$ by
\[ \gamma(J_0,J_1)(\lambda) = \left( \frac{\lambda^2 - (1 + J_1^2)}{2J_1}, \frac{\lambda^2 - (1 + J_0^2)}{2J_0}, \frac{J_0 + J_1}{J_1} \right)/2. \] (10)

Let $\pi$ denote projection onto the third coordinate, and define
\[ \sigma_k(J_0,J_1) = \{ \lambda : |\pi \circ f^k \circ \gamma(J_0,J_1)(\lambda)| \leq 1 \}, \] (11)
where $f^k$ denotes $k$-fold composition
\[ f^k = f \circ f \circ \cdots \circ f, \quad k \geq 0. \]

For the sake of simplifying notation, we shall write simply $\sigma_k$, keeping in mind its implicit dependence on the choice of $J_0$ and $J_1$. 
2.3 The problem and main results

We wish to investigate the energy spectrum of $\tilde{H}_k$ in the thermodynamic limit (that is, $k \to \infty$). Since $\pi \circ f^k \circ \gamma_{(J_0, J_1)}(\lambda)$ is a polynomial in $\lambda$, $\sigma_k$ is a union of finitely many compact intervals. Supported by numerical evidence, it is believed that as $k \to \infty$, the sequence $\{\sigma_k\}_{k \in \mathbb{N}}$ shrinks to a Cantor set [7, 10, 19, 66] (i.e. a nonempty, compact, totally disconnected set with no isolated points). In Theorem 2.1 below we make precise the notion of the energy spectrum in the thermodynamic limit, and we rigorously examine its multifractal nature.

We denote Hausdorff metric on $\mathcal{P}(\mathbb{R})$ by $\dist_H$:

$$\dist_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \{|a - b|\}, \sup_{b \in B} \inf_{a \in A} \{|a - b|\} \right\}.$$ 

We denote Hausdorff dimension of a set $A$ by $\dim_H(A)$, and local Hausdorff dimension of $A$ at a point $a \in A$ by $\dim^{loc}_H(A, a)$:

$$\dim^{loc}_H(A, a) = \lim_{\epsilon \to 0^+} \dim_H((a - \epsilon, a + \epsilon) \cap A).$$

**Theorem 2.1.** Fix $J_1 > 0$. There exists $r_0 = r_0(J_1) \in (0, 1)$ such that for all $J_0$ satisfying $J_1/J_0 \in (1 - r_0, 1 + r_0)$, $J_0 \neq J_1$, the following holds.

i. There exists a compact nonempty $B_\infty(J_0, J_1) \in \mathbb{R}$ such that $\sigma_k \xrightarrow{k \to \infty} B_\infty$ in Hausdorff metric;

ii. $B_\infty(J_0, J_1)$ is a Cantor set;

iii. $\dim^{loc}_H(B_\infty(J_0, J_1), b)$ depends continuously on $b \in B_\infty(J_0, J_1)$, is nonconstant and is strictly between zero and one; consequently $\dim_H(B_\infty(J_0, J_1)) \in (0, 1)$, and therefore the Lebesgue measure of $B_\infty(J_0, J_1)$ is zero;

iv. $\dim_H(B_\infty(J_0, J_1))$ is continuous in the parameters $(J_0, J_1)$.

Convergence of the sequence $\{\sigma_k\}$ to a (nonempty, compact) limit and multifractal nature of this limit (statements (i) and (ii) of the theorem) was observed numerically in [19, 65, 66].

We should add that we believe the restrictions on $J_0, J_1$ in the statement of Theorem 2.1 are not necessary; however, our present techniques do not extend to the general case (i.e. to cover all values of $J_0, J_1$). We record this here formally as a conjecture:

**Conjecture 2.2.** The conclusion of Theorem 2.1 holds for all $J_0, J_1 > 0$.

We should remark here that the Ising Hamiltonian above is equivalent, via a unitary transformation, to the tight binding model:

$$(T\theta)_n = \tilde{J}_{n-1}\theta_{n-1} + (1 + \tilde{J}_n^2)\theta_n + \tilde{J}_{n+1}\theta_{n+1}. \quad (12)$$

In fact, it can be seen easily that solving (6) is equivalent to solving the equation

$$(A + B)(A - B)\phi = \lambda^2 \phi.$$
This, on the other hand, is equivalent to solving
\[ T \phi = \lambda^2 \phi. \]
Now, \( T \) is a member of a family of tridiagonal Hamiltonians investigated in our forthcoming paper [64]. From our forthcoming results in [64], combined with Proposition 5.16 from Section 5.1 below, it follows that with \( T \) from (12), we have \( \sigma(T) \subset \mathbb{R}^+ \) and
\[
B_\infty = \{ \pm \sqrt{\lambda} : \lambda \in \sigma(T) \},
\]
where \( \sigma(T) \) is the spectrum of \( T \).

After \( B_\infty \) has been tied to the spectrum of \( T \) as above, results of [2] yield a proof of statement (i) of Theorem 2.1. Moreover, in this case the restrictions given in the statement of Theorem 2.1 are not necessary; these restrictions are, however, still a necessity to our techniques for proving the remaining statements (ii)–(iv).

Since previous numerical methods have relied entirely on the dynamics of the Fibonacci trace map, motivated by providing a rigorous counterpart, we provide an alternative proof of Theorem 2.1-i in Section 5.1, based entirely on dynamical properties of the trace map.

3 Strategy for the proof of main results

Given that the proof of Theorem 2.1 is quite technical, in the present section we give a somewhat detailed outline of the main ideas involved in our arguments below. Thus the intention of this section is to serve as a roadmap for the reader throughout Sections 4 and 5.

3.1 Definition of \( B_\infty \) and its topological properties

The forward-stable set of the map \( f \) from (9) is defined to be the set of all points \( x \in \mathbb{R}^3 \), whose forward semi-orbit under iterations of \( f \) is bounded; that is,
\[
\{ f^n(x) \}_{n \in \mathbb{N}} \text{ is bounded.}
\]
It turns out that the set \( B_\infty \) from Theorem 2.1 is precisely the intersection of \( \gamma(J_0, J_1) \) from (10) with the forward-stable set of \( f \). Therefore, in order to investigate properties of \( B_\infty \), it is natural to first understand the geometry of the forward-stable set of \( f \), and in order to achieve the latter, we investigate the dynamics of \( f \) on \( \mathbb{R}^3 \); in particular, in Section 4 the following facts are established:

- It is recalled that \( f \) preserves certain two-dimensional submanifolds of \( \mathbb{R}^3 \), and a region of \( \mathbb{R}^3 \) (which is of interest to us in the sense that it contains a line segment along \( \gamma(J_0, J_1) \) along which intersection with forward-stable set occurs) is smoothly foliated by these surfaces. In order to study the dynamics of \( f \) on \( \mathbb{R}^3 \), it is enough to study the dynamics of \( f \) on the invariant surfaces. The geometry and topology of these invariant surfaces are discussed;
• Proposition 4.1 establishes necessary and sufficient conditions for a point to have bounded forward semi-orbit;

• Lemma 4.4 uses Proposition 4.1 to establish that the forward-stable set is a closed subset of \( \mathbb{R}^3 \);

• Proposition 4.5 establishes that there are only two types of behavior for points under forward iterations of \( f \): either the semi-orbit is bounded, or it escapes to infinity in every coordinate. This is important, since sometimes it is beneficial (and even necessary) to pass from the dynamical system given by \( f \), to one given by \( f^k \), for some positive integer \( k \). It is then important to know that no forward-stable points are lost, nor does \( f^k \) have forward-stable points that are not forward-stable for \( f \);

• In Theorem 4.6 the fundamental result pertaining to hyperbolicity of \( f \) when restricted to invariant surfaces is recalled: the nonwandering set for \( f \), when restricted to an invariant surface, is a compact hyperbolic set. In Corollary 4.8 we record an immediate consequence of Theorem 4.6: when restricted to an invariant surface, the forward-stable set is the disjoint union of stable manifolds to points in the nonwandering set for \( f \) on the given surface (for definitions, see Appendix B);

• Based on hyperbolicity of the nonwandering set for \( f \), when \( f \) is restricted to a two-dimensional invariant surface, in Proposition 4.9 we establish partial hyperbolicity for \( f \) when \( f \) is viewed as a three-dimensional map acting on the region of \( \mathbb{R}^3 \) that is foliated by the aforementioned two-dimensional surfaces. As a consequence, we establish that the forward-stable set of \( f \) is formed by a family of injectively immersed pair-wise disjoint smooth connected two-dimensional submanifolds of \( \mathbb{R}^3 \), and properties of this family are investigated in Proposition 4.9. Members of this family are called center-stable manifolds;

• Based on Proposition 4.9, in Proposition 4.11 it is established that if \( \gamma \) is a compact regular curve in \( \mathbb{R}^3 \) that intersects the center-stable manifolds transversally, then this intersection forms a Cantor set. This is important, since it is one of the main ingredients in our proof of Theorem 2.1. Indeed, in Section 5 we prove that for \( J_0, J_1 \) satisfying the hypothesis of Theorem 2.1, \( \gamma_{J_0,J_1} \) intersects the center-stable manifolds transversally, hence \( B_\infty \) is a Cantor set.

3.2 Measure-theoretic and fractal-dimensional properties of \( B_\infty \)

According to the last point of the previous section, part of the problem (an essential part) is showing that \( \gamma_{(J_0,J_1)} \) intersects the center-stable manifolds transversally. Indeed, then \( B_\infty \) is a Cantor set. It turns out that transversality is also sufficient (and to an extend necessary) to establish all the other properties of \( B_\infty \). Hence most of Section 5 is dedicated to proving transversality.
3.2.1 Proof of transversality: main ideas

The formal statement about transversal intersection of $\gamma_{(J_0,J_1)}$ with the center-stable manifolds is given as Proposition 5.1. To prove this proposition, we concentrate on a subfamily of four manifolds in the family of center-stable manifolds, which form a dense sublamination of the lamination by center-stable manifolds, and prove uniform transversality of $\gamma_{(J_0,J_1)}$ with these four manifolds. We denote these manifolds by $W_s^i$, $i = 1, \ldots, 4$. These four manifolds are constructed in Section 4.1.2. Then Proposition 4.1 (necessary and sufficient conditions for a point to be forward-stable) is used to show that forward-stable points on $\gamma_{(J_0,J_1)}$ form a compact set; so by compactness, uniform transversality of $\gamma_{(J_0,J_1)}$ with $\{W_s^i\}$, and density of $\{W_s^i\}$ in the lamination of center-stable manifolds, we obtain transversality of $\gamma_{(J_0,J_1)}$ with all center-stable manifolds.

The formal statement about density of the sublamination formed by $\{W_s^i\}$ is given in Lemma 5.15. The precise statement about transversality of $\gamma_{(J_0,J_1)}$ with the aforementioned four manifolds is recorded in Lemma 5.2. Below is a summary of main ideas involved in the proof of Lemma 5.2.

- As was mentioned in the previous section, $f$ preserves two dimensional surfaces that foliate the region of $\mathbb{R}^3$ that is of interest to us. Moreover, the intersection of the four manifolds with these surfaces forms a family of eight one-dimensional manifolds on the respective surface, which are stable manifolds to some periodic points in the nonwandering set for $f$ on the same surface;

- Using this, on each of the invariant surfaces, we construct unstable cone field in the tangent bundle of the surface, which is transversal to the aforementioned eight one-dimensional manifolds, and invariant under the action of the derivative cocycle of $f$, $(f, Df)$. Moreover, we investigate scaling properties (under the action of $Df$) of vectors in these cones. We refer to this scaling as scaling in the horizontal direction. This is done in (23), Lemma 5.3, Lemma 5.4, Lemma 5.5, Lemma 5.6 and finally Corollary 5.7;

- Next we construct a three-dimensional cone field on the three-dimensional region of $\mathbb{R}^3$ that is foliated by the invariant surfaces, by adding a positive component to the previously constructed unstable cone field in the direction normal to the invariant surfaces. This construction is given in equation (24). In this construction the opening angle in the normal direction of the three-dimensional cones depends on the given invariant surface;

- By controlling the vertical angle of the three-dimensional cones, we show that on each one of the four two-dimensional manifolds $\{W_s^i\}_{1 \leq i \leq 4}$, there exists an open neighborhood, called the fundamental domain, along which the three-dimensional cones are transversal to $W_s^i$. Moreover, for any $i = 1, \ldots, 4$ and $p \in W_s^i$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $f^k(p) \in W_s^i$ belongs to the fundamental domain. Hence in order to show that the constructed three-dimensional cones are transversal to $\{W_s^i\}_{1 \leq i \leq 4}$ everywhere, it is enough to prove invariance of the three-dimensional cones under the derivative cocycle $(f, Df)$;
The precise statement about invariance of the three-dimensional cones under 
\((f,Df)\) is given in Lemma 5.8. The main ingredients in the proof of this
lemma is scaling properties in the horizontal direction investigated earlier,
and the result of Lemma 5.9, which shows that scaling in the normal direction
is much weaker than scaling in the horizontal direction;

Finally, it is shown that if \(J_0, J_1\) satisfy the hypothesis of Theorem 2.1, then
\(\gamma(J_0, J_1)\) is tangent to the three-dimensional cones, and hence is transversal
(uniformly) to \(\{W_i\}_{1 \leq i \leq 4}\).

3.2.2 Properties of \(B_\infty\) based on transversality

After transversality of \(\gamma(J_0, J_1)\) is shown, we use it to prove the properties of \(B_\infty\) given in the statement of Theorem 2.1.

In addition to \(\gamma(J_0, J_1)\) being transversal to the center-stable manifolds, a simple
computation in (27) shows that \(\gamma(J_0, J_1)\) is also transversal to the two-dimensional
invariant surfaces. Take \(x_0 \in B_\infty\) and let \(U\) be a small neighborhood of \(x_0\) in \(\mathbb{R}^3\).
With \(U\) sufficiently small, there exists a plane \(\Pi\) containing the line \(\gamma(J_0, J_1)\) in \(\mathbb{R}^3\) such that in \(U\), \(\Pi\) is transversal to the invariant surfaces as well as the center-stable
manifolds. Figure 11 shows intersection of \(\Pi\) with the invariant surfaces and center-
stable manifolds inside \(U\). More precisely, in Figure 11, \(\{\tau^V\}_{V \in [V_0 - \epsilon, V_0 + \epsilon]}\) is the
family of smooth one-dimensional curves obtained by intersecting \(\Pi\) with invariant
surfaces (the family of invariant surfaces is parameterized by a real parameter
\(V > 0\)). In fact, \(\{\tau^V\}\) foliate \(\Pi\) smoothly. The "vertical curves" in Figure 11 are
formed by the intersection of \(\Pi\) with center-stable manifolds, and form a smooth
one-dimensional lamination. This family of curves is denoted by \(\{\vartheta^x\}_{x \in C}\) such
that \(\vartheta^x\) is the curve passing through \(x\), where \(C\) is the set formed by intersecting
the surface that contains \(x_0\) with the center-stable manifolds and the plane \(\Pi\).
Finally, \(\Gamma\) is a compact line segment along \(\gamma(J_0, J_1)\) inside \(U\) containing \(x_0\) in its
interior. That the structure shown in Figure 11 is in fact present follows from
Proposition 4.9 and transversality of \(\gamma(J_0, J_1)\) with the center-stable manifolds,
which is established earlier.

Now, in order to investigate \(B_\infty\) in a neighborhood of \(x_0\), it is of course enough
to investigate the set

\[ \Gamma \cap \left( \bigcup_{x \in C} \vartheta^x \right). \]

The strategy here is to relate \(B_\infty\) to \(C\) via a sufficiently regular map. Indeed,
we define the holonomy map from \(B_\infty \cap U\) to \(C\) by "sliding" points along the
curves \(\{\vartheta^x\}_{x \in C}\). It turns out that this map is a homeomorphism and, in fact, is
Hölder continuous. Moreover, the Hölder constant can be controlled (i.e. taken
arbitrarily close to one) by localizing sufficiently closely around \(x_0\). This is the
subject of Propositions 5.19 and 5.20, and Lemma 5.21.

This is then used to relate local fractal-dimensional and measure-theoretic
properties of \(B_\infty\) to those of \(C\). On the other hand, \(C\) is a well-known object
(it is a so-called dynamically defined Cantor set). The details are discussed in
Section 5.3, and further properties of $C$ are investigated in Propositions 5.22 and 5.23.

3.2.3 Convergence of $\{\sigma_k\}$ to $B_\infty$ in Hausdorff metric

To prove convergence of the sequence $\{\sigma_k\}$ to $B_\infty$ in Hausdorff metric, we prove that for given $\epsilon > 0$, if $B_\infty$ is covered by finitely many open intervals (intervals in $\gamma(J_0,J_1)$) of radius not exceeding $\epsilon$, then for all sufficiently large $k$, $\sigma_k$ is covered by these intervals and intersects each of the intervals in a nonempty set. The details are given in Section 5.1, and are not too difficult to follow.

3.2.4 Continuity of the Hausdorff dimension of $B_\infty$ as a function of parameters $(J_0, J_1)$

Finally, Theorem 2.1-iv follows quite easily from earlier geometric considerations; the details are presented in Section 5.4.

4 Dynamics of the Fibonacci trace map

In the proof of Theorem 2.1 we shall employ dynamical properties of the Fibonacci trace map $f$, which we discuss in this section. For a brief overview of basic notions and notation from the theory of hyperbolic and partially hyperbolic dynamical systems, see Appendix B.

For convenience henceforth we shall refer to the sequence $\{f^k(x)\}_{k \in \mathbb{N}}$ as the positive semi-orbit of $x$, and denote it by $O_+^f(x)$. The negative semiorbit, for $k \in \mathbb{Z}_{<0}$, and the full orbit, for $k \in \mathbb{Z}$, are defined similarly and denoted, respectively, by $O_-^f$ and $O_f$.

Define the so-called Fr"{i}cke-Vogt character [20, 21, 63]

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1. \quad (13)$$

Consider the family of cubic surfaces $\{S_V\}_{V \geq 0}$ given by

$$S_V = \{x \in \mathbb{R}^3 : I(x) = V\}. \quad (14)$$

For all $V > 0$, $S_V$ is a smooth, connected 2-dimensional submanifold of $\mathbb{R}^3$ without boundary. For $V = 0$, $S_V$ has four conic singularities:

$$P_1 = (1, 1, 1), \quad P_2 = (-1, -1, 1), \quad P_3 = (1, -1, -1), \quad P_4 = (-1, 1, -1) \quad (15)$$

(see Figure 1).

One can easily check that $f$ preserves the Fr"{i}cke-Vogt character, and hence also the surfaces $\{S_V\}$. For convenience we will write $f_V$ for $f|_{S_V}$.

Proposition 4.1. Let $x_k = \pi \circ f^k(x_1, x_0, x_{-1})$. We have the following.

1. Assume $|x_{-1}| \leq C$ for some $C \geq 1$. The sequence $\{x_k\}_{k \geq 1}$ is unbounded if and only if there exists $k_0 \geq 0$ such that

$$|x_{k_0-1}| \leq C \quad \text{and} \quad |x_{k_0}|, |x_{k_0+1}| > C.$$
2. A sufficient condition for \( \{ x_k \}_{k \geq -1} \) to be unbounded is that there exists \( k_0 \geq 0 \) such that
\[
|x_{k_0}|, |x_{k_0+1}| > 1 \quad \text{and} \quad |x_{k_0}| |x_{k_0+1}| > |x_{k_0-1}|.
\]

Remark 4.2. As defined earlier, \( \pi \) denotes projection onto the third coordinate.

Proof of Proposition 4.1. For the proof of (1), see, for example, [12, Proposition 5.2] (replace 1 with \( C \)). For (2), see [37, 39] ((2) also follows from the aforementioned proof of (1)).

Remark 4.3. For detailed analysis of orbits of trace maps, see [51].

A consequence of Proposition 4.1 that will be used later is

**Lemma 4.4.** For \( V_0 \geq 0 \) and \( \infty \geq V_1 \geq V_0 \), the set of all points of \( \bigcup_{V \in [V_0, V_1]} S_V \) whose forward orbit is bounded is a closed set.

Proof of Lemma 4.4. If \( O_f^+(x_1, x_0, x_{-1}) \) is unbounded, then for some \( k_0 \geq 0 \), the point \( f^{k_0}(x_1, x_0, x_{-1}) \) satisfies (1) of Proposition 4.1, and hence also satisfies (2), which is an open condition.
Another consequence of Proposition 4.1 that will be used later is Proposition 4.5 below; for a proof see [12, Proposition 5.2], or (in a more general context) [51].

**Proposition 4.5.** The positive semi-orbit is unbounded if and only if \(\{f^k(x)\}_{k \in \mathbb{N}}\) escapes to infinity in every coordinate.

### 4.1 Dynamics of \(f_V\) for \(V \geq 0\)

In this and the following sections, we shall use the notation and terminology from Appendix B.

#### 4.1.1 Hyperbolicity of \(f_V\) for \(V > 0\)

The following result on hyperbolicity of the Fibonacci trace map will serve as the main tool for us.

**Theorem 4.6** (M. Casdagli in [9], D. Damanik and A. Gorodetski in [14], and S. Cantat in [8]). For \(V > 0\) let

\[
\Omega_V = \{ p \in S_V : \mathcal{O}_f(p) \text{ is bounded} \}.
\]

Then \(\Omega_V\) is a Cantor set, it is \(f_V\)-invariant compact locally maximal transitive hyperbolic set in \(S_V\) (with \((1,1)\) splitting). Moreover, \(\Omega_V\) is precisely the set of nonwandering points of \(f_V\) (a point \(p\) is nonwandering if for any neighborhood \(U\) of \(p\) and \(N \in \mathbb{N}\), there exists \(n \geq N\) such that \(f^n(U) \cap U \neq \emptyset\)).

**Remark 4.7.** It follows that for \(V > 0\), for any \(x \in \Omega_V\), \(W^s_{\text{loc}}(x) \cap \Omega_V\) and \(W^u_{\text{loc}}(x) \cap \Omega_V\) are Cantor sets (\(W^s_{\text{loc}}(x)\) denotes the local stable/unstable manifold at \(x\) – see Appendix B).

An immediate consequence of Theorem 4.6 is the following (which follows from general principles in hyperbolic dynamics).

**Corollary 4.8.** For \(V > 0\), \(x \in S_V\), \(\mathcal{O}_f^+(x)\) is bounded if and only if \(x \in W^s(\Omega_V)\).

#### 4.1.2 Dynamics of \(f_V\) for \(V = 0\)

As mentioned above, the surface \(S_0\) is smooth everywhere except for the four singularities \(P_1, \ldots, P_4\) (see (15) and Figure 1). Let us set, and henceforth fix, the following notation

\[
S = \{(x,y,z) \in S_0 : |x|, |y|, |z| \leq 1\}. \tag{16}
\]

It is easily seen that \(S\) is invariant under \(f\). Moreover, \(f|_S\) is a factor of the hyperbolic diffeomorphism on \(\mathbb{T}^2\)

\[
\mathcal{A}(\theta, \phi) = ((\theta + \phi), \theta)(\text{mod } 1) \tag{17}
\]

[1] The special case of \(V \geq 64\) was done by M. Casdagli in [9]. D. Damanik and A. Gorodetski extended the result to all \(V > 0\) sufficiently small in [14]. Finally, S. Cantat proved the result for all \(V > 0\) in [8] (D. Damanik and A. Gorodetski, and S. Cantat obtained their results independently, and used different techniques).
Figure 2: Per$_2$ in a neighborhood of $P_1$.

via the map

$$F : (\theta, \phi) \mapsto (\cos 2\pi(\theta + \phi), \cos 2\phi, \cos 2\pi\phi).$$

(18)

By a factor we mean

$$f|_{S} \circ F = F \circ A.$$

The map $F$ is not, however, a conjugacy in the sense of (46), since $F$ is a two-to-one map. The dynamics of $f$ on $\{P_1, \ldots, P_4\}$ is as follows.

$$f : P_1 \mapsto P_1; \quad f : P_2 \mapsto P_3 \mapsto P_4 \mapsto P_2.$$

We now concentrate on a neighborhood of $P_1$; similar results hold for the other singularities due to the symmetries of $f$ (see Appendix A for details).

Let $\text{Per}_2(f)$ denote the set of period-two periodic points for $f$. A direct computation shows that

$$\text{Per}_2(f) = \left\{ (x, y, z) : x \in (-\infty, \frac{1}{2}) \cup \left( \frac{1}{2}, \infty \right), \ y = \frac{x}{2x-1}, \ z = x \right\}.$$

Let

$$\vartheta(x) = \left( x, \frac{x}{2x-1}, x \right), \quad \vartheta : (-\infty, \frac{1}{2}) \cup \left( \frac{1}{2}, \infty \right) \rightarrow \mathbb{R}^3$$

(19)

be the curve of these periodic points. Then in a neighborhood of $P_1$, $\vartheta$ is a smooth curve (see Figure 2). Also,

$$I(\vartheta(x)) \geq 0,$$

(20)

with $I(\vartheta(x)) = 0$ if and only if $x = 1$, where $\vartheta(1) = P_1$. On the other hand,

$$Df_{P_1} = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is similar to

$$\begin{pmatrix} \frac{3-\sqrt{5}}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{3+\sqrt{5}}{2} \end{pmatrix}.$$
Hence $T_P \mathbb{R}^3$ splits as

$$T_P \mathbb{R}^3 = E^s_{P_i} \oplus E^c_{P_i} \oplus E^u_{P_i}$$  \hspace{1cm} (21)

(see Section B.2). Now using invariance of $\text{Per}_2(f)$ under $f$, and combining (20) with Theorem 4.6 for $x \neq 1$ and (21) for $x = 1$, we get that the curve $\vartheta$ is normally hyperbolic in a neighborhood of $P_1$. Hence we can apply Theorem B.2. We get that $W^{cs, cu}_{\text{loc}}(\text{Per}_2(f))$ is a smooth two-dimensional submanifold of $\mathbb{R}^3$. Moreover, by invariance of the surfaces $S_V$ under $f$, it follows that $W^{cs, cu}_{\text{loc}}(\text{Per}_2(f)) \cap S_V$ is precisely the one-dimensional manifold $W^{s, u}_{\text{loc}}(\text{Per}_2(f) \cap S_V)$ on $S_V$ (see Section B.1), for $V > 0$. When $V = 0$, $W^{cs, cu}_{\text{loc}}(\text{Per}_2(f)) \cap S_0 \setminus \{P_1\}$ forms the local strong-stable/unstable one-dimensional submanifold (as defined in Section B.3) of $S_0$ consisting of two smooth branches that connect smoothly (when viewed as submanifolds of $\mathbb{R}^3$) at $P_1$. Points of the strong-stable manifold converge to $P_1$ under iterations of $f$; the same happens under iterations of $f^{-1}$ for points on the strong-unstable manifold. The strong-stable and strong-unstable manifolds intersect at $P_1$.

We have

$$\frac{d(I \circ \vartheta)}{dx} \neq 0$$

for all $x \neq 1$. Hence $\vartheta$, and therefore also $W^{cs, cu}_{\text{loc}}(\text{Per}_2(f))$, intersect $S_V$ transversally for $V > 0$. On the other hand, $W^{cs, cu}_{\text{loc}}(\text{Per}_2(f))$ has quadratic tangency with $S_0$ along the strong-stable/unstable submanifold (see also [14, Section 4]).

The manifolds $W^{cs}_{\text{loc}}(\text{Per}_2)$ and $W^{cu}_{\text{loc}}(\text{Per}_2)$ can be extended globally by

$$W^{cs}(\text{Per}_2) = \bigcup_{n \in \mathbb{N}} f^{-n}(W^{cs}_{\text{loc}}(\text{Per}_2(f)))$$

and

$$W^{cu}(\text{Per}_2) = \bigcup_{n \in \mathbb{N}} f^n(W^{cu}_{\text{loc}}(\text{Per}_2(f))).$$

In this case $W^{s,u}_{\text{loc}}(\text{Per}_2(f) \cap S_V) = W^{cs,cu}_{\text{loc}}(\text{Per}_2(f)) \cap S_V$ for $V > 0$. For $V = 0$, these form branches of global strong-stable and strong-unstable submanifolds of $S_0$ that connect at $P_1$ - these branches are injectively immersed one-dimensional submanifolds of $S_0$.

Similar results hold for $P_2, P_3, P_4$. Indeed, as $V$ takes on positive values, the points $P_2, P_3$ and $P_4$ bifurcate from three cycles to six cycles. These six cycles form three smooth curves, one through each $P_2, P_3$ and $P_4$. Considering $f^6$ in place of $f$, it is easy to show, as in the case of $P_1$ above, that each curve is normally hyperbolic.

Let us fix the following notation. Denote by $\mathcal{W}^{s, u}_i$ (as in Appendix B) the stable/unstable 2-dimensional invariant manifold to the normally hyperbolic curve through $P_i$ for the map $f^6$. We denote by $\mathcal{W}^{s, u}_{i, \text{loc}}$ a small neighborhood of the normally hyperbolic curve in $\mathcal{W}^{s, u}_i$.

In particular, the orbit $\mathcal{O}^+(f^6(x))$ (respectively, $\mathcal{O}^-(f^6(x))$), for $x \in S_0$, is bounded if and only if either $x \in S$ or $x \in \mathcal{W}^{s, u}_i \cap S_0$ (respectively, $x \in \mathcal{W}^{s, u}_i \cap S_0$) for $i = 1, \ldots, 4$.  

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4.2 Partial hyperbolicity: center-stable and center-unstable manifolds

In the previous section we proved normal hyperbolicity of \( f \) on a certain submanifold and derived the existence of two-dimensional analogs of stable and unstable manifolds. The result of the previous section is a special case of a more general fact:

**Proposition 4.9.** Let

\[
\mathcal{M} = \bigcup_{V \in (0, \infty)} S_V.
\]

For \( V_1 \geq V_0 > 0 \) let

\[
\Omega = \bigcup_{V \in [V_0, V_1]} \Omega_V.
\]

Then \( \mathcal{M} \) is a smooth, connected, \( f \)-invariant 3-dimensional submanifold of \( \mathbb{R}^3 \), \( \Omega \subset \mathcal{M} \) is compact, \( f \)-invariant partially hyperbolic set with \((1,1,1)\) splitting, and the following holds.

There exist two families, denoted by \( \mathcal{W}^s \) and \( \mathcal{W}^u \), of smooth 2-dimensional connected manifolds injectively immersed in \( \mathcal{M} \), whose members we denote by, respectively, \( \mathcal{W}^{cs} \) and \( \mathcal{W}^{cu} \), and call center-stable and center-unstable manifolds, with the following properties.

1. The family \( \mathcal{W}^{cs,u} \) is \( f \)-invariant;
2. For every \( x \in \Omega \) there exist unique \( W^{cs}(x) \in \mathcal{W}^s \) and \( W^{cu}(x) \in \mathcal{W}^u \) such that \( x \in W^{cs}(x) \cap W^{cu}(x) \);
3. Conversely, for every \( W^{cs} \in \mathcal{W}^s \) and \( W^{cu} \in \mathcal{W}^u \), there exist \( x, y \in \Omega \) such that \( x \in W^{cs} \) and \( y \in W^{cu} \). In fact, if \( W^{cs}_1, W^{cs}_2 \in \mathcal{W}^s \) and \( W^{cu}_1, W^{cu}_2 \in \mathcal{W}^u \) with \( W^{cs}_1 \cap W^{cs}_2 \neq \emptyset \) and \( W^{cu}_1 \cap W^{cu}_2 \neq \emptyset \), then \( W^{cs}_1 = W^{cs}_2 = W^{cs}(x) \) for some \( x \in \Omega \), and \( W^{cu}_1 = W^{cu}_2 = W^{cu}(y) \) for some \( y \in \Omega \).
4. For any \( V > 0 \) and any \( W^{cs,cu} \in \mathcal{W}^{s,u} \), \( W^{cs,cu} \cap S_V = W^{s,u}(x) \) for every \( x \in \Omega_V \cap W^{cs,cu} \); moreover, this intersection is transversal.

**Proof of Proposition 4.9.** All statements about \( \mathcal{M} \), as well as compactness of \( \Omega \), are trivially true. The surfaces \( \{S_V\} \), \( V > 0 \), are all diffeomorphic. Fix \( V > 0 \), and let \( \pi_V : S_{\tilde{V}} \to S_V \) be a diffeomorphism, depending smoothly on \( V \), with \( \pi_{\tilde{V}} = \text{Id}|_{S_{\tilde{V}}} \). Now \( f : \mathcal{M} \to \mathcal{M} \) may be considered, up to smooth conjugacy, as a skew product of identity on an interval \( I \) with a map on \( S_{\tilde{V}} \):

\[
\mathcal{G} : I \times S_{\tilde{V}} \to I \times S_{\tilde{V}}, \quad \mathcal{G}(V, x) = (V, \pi_V^{-1} \circ f_V \circ \pi_V(x));
\]

that is, the following diagram commutes

\[
\begin{array}{ccc}
I \times S_{\tilde{V}} & \xrightarrow{\mathcal{G}} & I \times S_{\tilde{V}} \\
\downarrow & & \downarrow \\
\bigcup V S_V & \xrightarrow{f} & \bigcup V S_V
\end{array}
\]
where \( \tilde{\pi}(V, x) = \pi_V(x) \). Now all statements about \( \Omega \) follow.

We now construct the family \( \mathcal{W}^s \). The family \( \mathcal{W}^u \) can be constructed similarly by considering \( f^{-1} \) in place of \( f \).

Fix \( \tilde{V} > 0 \) and \( x \in \Omega_{\tilde{V}} \). Take \( \delta > 0 \) small (in particular so that \( \tilde{V} - \delta > 0 \)) and for \( V \in [\tilde{V} - \delta, \tilde{V} + \delta] \) let \( H_V : \Omega_{\tilde{V}} \to \Omega_V \) be the topological conjugacy (see Section B.1.1). Then \( \bigcup_{V \in (\tilde{V} - \delta, \tilde{V} + \delta)} \mathcal{W}_{loc}^{cs}(H_V(x)) \) gives a smooth two-dimensional manifold (see [29, Section 6] for proof of smoothness of \( \mathcal{W}_{loc}^{cs} \)) that can be extended to all \( V \), and the sought \( \mathcal{W}^{cs}(x) \) is then given by

\[
\mathcal{W}^{cs}(x) = \bigcup_{n \in \mathbb{N}} f^{-n}(\mathcal{W}_{loc}^{cs}(x)).
\]

The collection \( \{\mathcal{W}^{cs}(x)\}_{x \in \Omega_{\tilde{V}}} \) gives the family \( \mathcal{W}^s \).

Fix \( x \in \Omega \) and consider the curve \( V \mapsto H_V(x) \) in a neighborhood of \( x \). This curve is the intersection of \( \mathcal{W}_{loc}^{cs}(x) \) with \( \mathcal{W}_{loc}^{cu}(x) \), hence is smooth. Since \( f_V \) depends smoothly, and hence Lipschitz-continuously (when restricted to a compact subset), on \( V \), by [27, Theorem 7.3], \( H_V \) is the fixed point of a contracting map on a certain Banach space that depends Lipschitz-continuously on \( V \). So by [27, Theorem 1.1], if \( V_0 > 0 \) is fixed, there exists \( C > 0 \) such that for all \( V \) close to \( V_0 \),

\[
\frac{|H_V(x) - H_{V_0}(x)|}{|V - V_0|} \leq C,
\]

proving transversality of intersection of \( \{H_V(x)\}_{V} \) with \( S_{V_0} \). Hence \( \mathcal{W}^{cs, cu}(x) \) intersects \( S_{V}, V > 0 \), transversally, as claimed.

**Remark 4.10.** From the proof of Proposition 4.9 and Section B.1.2 it is evident that the local center-stable and center-unstable manifolds at \( x \), \( \mathcal{W}_{loc}^{cs}(x) \) and \( \mathcal{W}_{loc}^{cu}(x) \), depend continuously on the point \( x \). Consequently by compactness we obtain uniform transversality of \( \mathcal{W}_{loc}^{cs, cu} \) with the surface \( S_V, V > 0 \).

As a consequence, we obtain

**Proposition 4.11.** Let \( \mathcal{M} \) be as in Proposition 4.9. Suppose \( \gamma : [0, 1] \to \mathcal{M} \) is smooth and regular. Suppose further that \( \gamma \) intersects the members of \( \mathcal{W}^s \) transversally with its endpoints not lying on center-stable manifolds. Then the set of all points \( x \in [0, 1] \) for which \( \mathcal{O}^+_f(f \circ \gamma(x)) \) is bounded is a Cantor set.

**Proof of Proposition 4.11.** By construction of the center-stable manifolds, for \( s \in [0, 1] \), \( \gamma(s) \) has a bounded forward orbit if and only if \( \gamma(s) \) belongs to a center-stable manifold.

Compactness follows from Lemma 4.4. Now absence of isolated points and total disconnectedness follows from Remark 4.7 and Proposition 4.9. \( \square \)

We are ready to prove Theorem 2.1.
5 Proof of main results

In this section we prove Theorem 2.1. Below we use terminology, notation and results of Appendix B.

The appearance of $\lambda^2$ in $\gamma_{(J_0, J_1)}$ in (10) makes $\gamma$ symmetric in $\lambda$ with respect to the origin. By abuse of notation, let us write $\lambda$ in place of $\lambda^2$, where $\lambda$ is allowed to take values in $[0, \infty)$.

Take $r = J_0/J_1$ and, for convenience, let us also write $\gamma_r$ in place of $\gamma_{(J_0, J_1)}$.

Hence

$$\gamma_r(\lambda) = \left(\frac{\lambda - (1 + J_1^2)}{2J_1}, \frac{\lambda - (1 + r^2 J_1^2)}{2r J_1}, \frac{1 + r^2}{2r}\right).$$

(22)

Proposition 5.1. For every $J_1 > 0$, there exists $r_0 = r_0(J_1) \in (0, 1)$, such that for all $r \in (1-r_0, 1+r_0)$ and $r \neq 1$, the curve $\gamma_r$ in (22) intersects the center-stable manifolds transversally.

Proof of Proposition 5.1. We begin by showing that transversality holds for the stable manifolds $W^s_i$, $i = 1, \ldots, 4$:

Lemma 5.2. For all $r$ sufficiently close to one and not equal to one, $\gamma_r$ intersects $W^s_i$, for $i = 1, \ldots, 4$, uniformly transversally.

Proof of Lemma 5.2. Returning to the map $A$ in (17), we see that $A$ is hyperbolic and is given by the matrix $A = \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right)$ with eigenvalues

$$\mu = \frac{1 + \sqrt{5}}{2}, \quad -\mu^{-1} = \frac{1 - \sqrt{5}}{2}.$$

Let us denote by $v^s, v^u \in \mathbb{R}^2$ the stable and unstable eigenvectors of $A$:

$$Av^s = -\mu^{-1}v^s, \quad Av^u = \mu v^u, \quad \|v^s\| = \|v^u\| = 1.$$

Fix some small $\zeta > 0$ and define the stable and unstable cone fields on $\mathbb{R}^2$ in the following way:

$$K^s(p) = \{v \in T_p\mathbb{R}^2 : v = v^s v^u + v^s v^s, |v^s| \geq \zeta^{-1} |v^u| \},$$

$$K^u(p) = \{v \in T_p\mathbb{R}^2 : v = v^u v^u + v^u v^s, |v^u| \geq \zeta^{-1} |v^s| \}.$$

(23)

These cone fields are invariant:

$$A(K^s(p)) \subset \text{Int}(K^u(Ap)) \quad \text{and} \quad A^{-1}(K^s(p)) \subset \text{Int}(K^s(A^{-1}p)).$$

Also, the iterates of the map $A$ expand vectors from the unstable cones, and the iterates of the map $A^{-1}$ expand vectors from the stable cones. That is, there exists a constant $C > 0$ such that

$$\forall v \in K^u(p) \quad \forall n \in \mathbb{N} \quad \|A^n v\| > C \mu^n \|v\|,$$

$$\forall v \in K^s(p) \quad \forall n \in \mathbb{N} \quad \|A^{-n} v\| > C \mu^n \|v\|.$$
The families of cones \( \{K^s(p)\}_{p \in \mathbb{R}^2} \) and \( \{K^u(p)\}_{p \in \mathbb{R}^2} \) can also be considered on \( \mathbb{T}^2 \).

The differential of the semiconjugacy \( F \) in (18) sends these cone families to \( Df \)-invariant stable and unstable cone families on \( S \setminus \{P_1, \ldots, P_4\} \). Let us denote these images by \( \{\mathcal{K}^s\} \) and \( \{\mathcal{K}^u\} \), respectively.

**Lemma 5.3** ([14, Lemma 3.1]). The differential of the semiconjugacy \( DF \) induces a map of the unit bundle of \( \mathbb{T}^2 \) to the unit bundle of \( S \setminus \{P_1, \ldots, P_4\} \). The derivatives of the restrictions of this map to a fiber are uniformly bounded. In particular, the sizes of cones in families \( \{\mathcal{K}^s\} \) and \( \{\mathcal{K}^u\} \) are uniformly bounded away from zero.

Fix \( j_0 \in \{1, \ldots, 4\} \). Let \( \alpha \) be a smooth curve in \( \mathbb{W}_{j_0} \) that is \( C^1 \) close to, and disjoint from, the curve of periodic points. Assume also that \( \alpha \cap S_0 \subset S \). Let \( \mathbb{P}_{j_0} \) be the fundamental domain of \( \mathbb{W}_{j_0} \) bounded by \( \alpha \) and \( f^{-1}(\alpha) \). Let \( U \) be a neighborhood of \( \{P_1, \ldots, P_4\} \) in \( \mathbb{R}^3 \) so small, that \( U \cap \mathbb{P}_{j_0} = \emptyset \) (see Figure 3).

Given \( V_0 > 0 \) sufficiently small, for all \( V \in [0, V_0) \), the surface \( S_V \setminus U \) consists of five smooth connected components (with boundary), one of which is compact. Let \( S_{V,U} \) denote the compact component. The family \( \{S_{V,U}\}_{V \in [0, V_0)} \) depends smoothly on \( V \), that is, there exists a family of smooth projections, depending smoothly on \( V \):

\[
\{\pi_V : S_{0,U} \to S_{V,U}\}_{V \in [0, V_0)}; \quad \pi_0 = \text{Id}_{S_{0,U}}.
\]

Assuming \( V_0 \) is sufficiently small, \( D\pi_V \) carries the cones \( \{\mathcal{K}^{s,u}\} \) to nonzero cones on \( S_{V,U} \); denote these cones by \( \{\mathcal{K}^{s,u}_V\} \).

**Lemma 5.4.** For all \( N \in \mathbb{N} \) there exists \( V_0 > 0 \) sufficiently small such that for all \( k \in \mathbb{Z}_{\geq 0} \) with \( k \leq N \) and all \( V \in [0, V_0) \), if \( x \in S_{V,U} \) and \( f^k(x) \in S_{V,U} \), then

\[
Df_x^k(\mathcal{K}^{s}_V(x)) \subset \text{Int}(\mathcal{K}^{s}_V(f^k(x))).
\]
Proof of Lemma 5.4. Since $\pi_V$ depends smoothly on $V$, for $x \in S_{0,U}$, the cones $K^u(x)$ and $K^u_V(\pi_V(x))$ are close provided that $V$ is close to zero. Since $f_V$ depends smoothly on $V$, $Df^k_V$ and $Df^k_{0}$ are close. In particular, for a given $x \in S_{0,U}$ with $f^k(x) \in S_{0,U}$, if $\pi_V(x) \in S_{V,U}$ and $f^k(\pi_V(x)) \in S_{V,U}$, then $Df^k_V(K^u_V(\pi_V(x)))$ and $Df^k(\pi_V(x))$ are close. Thus by compactness of the surfaces $S_{V,U}$, we can choose $V_0$ suitably small, so that the conclusion of the Lemma holds. \hfill $\square$

**Lemma 5.5** ([14, Lemma 5.4]). Assuming $U$ is sufficiently small, there exists sufficiently large $N \in \mathbb{N}$ and sufficiently small $V_0 > 0$, such that for all $V \in [0, V)$ the following holds. If $x \in S_{V,U}$ and $k$ is the smallest number such that $f^k(x) \in S_{V,U}$ and $k \geq N$, then

$$Df^k_x(K^u_V(x)) \subset \text{Int}(K^u_V(f^k(x))).$$

**Lemma 5.6** ([14, Lemma 5.2]). There exists $V_0 > 0$ sufficiently small, $C > 0$ and $\mu > 1$, such that for all $V \in [0, V_0)$ the following holds. If $x \in S_{V,U}$ and for $k \in \mathbb{N}$, $f^k(x) \in S_{V,U}$, and if $v \in K^u_V(x)$, then

$$\|Df^k_x(v)\| \geq C\mu^k \|v\|.$$

Combination of Lemmas 5.4, 5.5, 5.6 gives

**Corollary 5.7.** Assuming $U$ is small, there exists $V_0 > 0$ sufficiently small, $C > 0$ and $\mu > 1$ such that for all $V \in [0, V_0)$, the following holds. If $x \in S_{V,U}$, $v \in K^u_V(x)$, $k \in \mathbb{N}$ and $f^k(x) \in S_{V,U}$, then

$$Df^k_x(K^u_V(x)) \subset \text{Int}(K^u_V(f^k(x))) \quad \text{and} \quad \|Df^k_x(v)\| \geq C\mu^k \|v\|.$$ 

With $U$ and $V_0$ satisfying the hypothesis of Corollary 5.7, let us construct the following cone field on $S_{V,U}$, for $V \in [0, V_0)$ and $\eta > 0$:

$$K^\eta_V(x) = \left\{ (u,v) \in T_xS_{V,U} \oplus (T_xS_{V,U})^\perp : u \in K^u_V(x), \|v\| \leq \eta \sqrt{V} \|u\| \right\}. \quad (24)$$

**Lemma 5.8.** For every $\tilde{\eta} > 0$ there exists $\eta = \eta(\tilde{\eta}) > 0$, $\eta < \tilde{\eta}$, and $V_0 > 0$ sufficiently small, such that for any $V \in [0, V_0)$, any $x \in S_{V,U}$, $k \in \mathbb{Z}_{\geq 0}$, if $f^k(x) \in S_{V,U}$, then

$$Df^k_x(K^\eta_V(x)) \subset K^{\tilde{\eta}}_V(f^k(x)).$$

Proof of Lemma 5.8. Smooth dependence of the surfaces $\{S_V\}_{V>0}$ on $V$ and invariance under $f$ implies the following.

**Lemma 5.9.** For any $V > 0$, $x \in S_V$ and $k \in \mathbb{Z}$, if $v \in (T_xS_V)^\perp$, then

$$\|\text{Proj}_{(T_{f^k(x)}S_V)^\perp} \left( Df^k_x(v) \right) \| = \frac{\|\nabla I(x)\|}{\|\nabla I(f^k(x))\|} \|v\|, \quad (25)$$

where $\nabla I$ is the gradient of the Fricke-Vogt character (see (13)). In particular, there exists $D > 0$ such that for all $V \in (0, V_0)$ and any $x \in S_{V,U}$, if $f^k(x) \in S_{V,U}$, then for every $v \in (T_xS_V)^\perp$, we have

$$\|\text{Proj}_{(T_{f^k(x)}S_V)^\perp} \left( Df^k_x(v) \right) \| \leq D \|v\|.$$
In fact, we can take

\[ D = \sup \left\{ \frac{\| \nabla I(x) \|}{\| \nabla I(y) \|} : x, y \in S_{V,U}, V \in [0, V_0] \right\} < \infty. \]

Proof of Lemma 5.9. Let

\[ \mathcal{M} = \bigcup_{V > 0} S_V. \]

Integrate the gradient vector field \( x \mapsto \nabla I(x) \) on \( \mathcal{M} \), and let \( \alpha_x \) denote a compact arc along the integral curve through \( x \), say parameterized on \([-1, 1]\) with \( \alpha_x(0) = x \). Let \( \beta = f^k(\alpha_x) \). Then

\[ \| \nabla I(x) \|^2 = (I \circ \alpha_x)'(0) = (I \circ \beta)'(0) = C \nabla I(f^k(x)) \cdot \nabla I(f^k(x)), \]

where \( C \nabla I(f^k(x)) \) is the projection of \( \beta'(0) \) onto \( (T_{f^k(x)}S_V)^\perp \), \( C > 0 \) a constant. Hence

\[ \| C \nabla I(f^k(x)) \| = \frac{\| \nabla I(x) \|}{\| \nabla I(f^k(x)) \|} \cdot \| \nabla I(x) \|. \]

Let \( D \) be as in Lemma 5.9 and \( \mu \) and \( C \) as in Corollary 5.7. Let \( k_0 \in \mathbb{N} \) be the smallest number such that \( C \mu^{k_0} > D \). Fix \( N \in \mathbb{N} \) with \( N > k_0 \). Let \( U^* \) be a neighborhood of \( \{P_1, \ldots, P_4\} \) in \( \mathbb{R}^3 \) such that \( U^* \subset U \), so small that if \( x \in U^* \) and \( l \in \mathbb{N} \) is the smallest number such that \( f^l(x) \notin U \), then \( l > N \).

Case (i). Suppose \( x \in S_{V,U}, f^k(x) \in S_{V,U} \) and \( N > k \geq k_0 \). Then \( C \mu^k > D \), hence the expansion in the cone \( K^k_V(x) \) dominates the expansion along the normal.
On the other hand, the normal at $x$, under the action of $Df^k_x$, may tilt to the side away from $K_V^n(f^k(x))$. However, since $k < N$, $\{x, f(x), \ldots, f^k(x)\} \subset S_{V,U}$. By compactness of $S_{V,U}$, the angle between the image under $Df^k_x$ of the normal at $x$ and $S_{V,U}$ remains uniformly bounded away from zero. This, together with the fact that $K_V^n(x)$ is mapped into the interior of $K_V^n(f^k(x))$, allows us to choose $V_0$ sufficiently small to compensate for the tilt in the normal. Hence for any $\eta > 0$, there exists $V_0 > 0$ small, such that $Df^k_x(K_V^n(x)) \subset K_V^n(f^k(x))$.

Case (ii). If $x \in S_{V,U}$, $f^k(x) \in F_{j_0} \subset S_{V,U}$ and $k < N$, then for sufficiently small $\eta$ (depending only on $N$ and independent of $x$), $K_V^n(x) \subset K_V^n(f^k(x))$; that is, given that the number of iterations does not exceed a given constant, the distortion can be controlled.

Case (iii). We now handle the case when, under iterates of $f$, passes through $U^*$. By symmetries of the map $f$, it is enough to consider only a neighborhood of $P_1$.

Say $U^*$ is a neighborhood of $P_1$. If $U^*$ is sufficiently small, there exists a smooth change of coordinates $\Phi : U^* \rightarrow \mathbb{R}^3$ such that $\Phi(P_1) = (0,0,0)$ and the following holds.

Denote by $\mathcal{W}_1^{s/u}(P_1)$ a small neighborhood of the point $P_1$ on the manifold $\mathcal{W}_1^{s/u}$ (the stable/unstable manifold of $\text{Per}_2(f)$). We have

$\Phi(\text{Per}_2(f))$ is part of the line $\{x = 0, z = 0\}$;

$\Phi(\mathcal{W}_1^u(P_1))$ is part of the plane $\{z = 0\}$;

$\Phi(\mathcal{W}_1^s(P_1))$ is part of the plane $\{x = 0\}$;

$\Phi(\mathcal{W}_1^s \cap S_0)$ is part of the plane $\{y = 0, z = 0\}$;

$\Phi(\mathcal{W}_1^u \cap S_0)$ is part of the plane $\{x = 0, y = 0\}$.

Denote $\mathcal{S}_V = \Phi(S_V)$. Then $\{\mathcal{S}_V\}_{V > 0}$ is a family of smooth surfaces depending smoothly on $V$, $\mathcal{S}_0$ is diffeomorphic to a cone, contains lines $\{y = 0, z = 0\}$ and $\{x = 0, y = 0\}$, and at each nonzero point on those lines it has a quadratic tangency with the $xy$- and $zy$-plane (see Figure 4).

For a point $p$, denote its coordinates by $(x_p, y_p, z_p)$.

**Lemma 5.10** ([16, Propositions 3.12 and 3.13]). Given $C_1, C_2 > 0$, $\rho > 1$, there exists $\delta_0 > 0$, $C > 0$, $\mu > 1$ and $N_0 \in \mathbb{N}$, such that for any $\delta \in (0, \delta_0)$, the following holds.

Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a $C^2$ diffeomorphism such that

(i) $\|g\|_{C^2} \leq C_1$;

(ii) The plane $\{z = 0\}$ is invariant under the iterates of $g$;

(iii) $\|Dg(p) - A\| < \delta$ for every $p \in \mathbb{R}^3$, where

$$ A = \begin{pmatrix} \rho^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} $$

is a constant matrix.
Introduce the following cone field on $\mathbb{R}^3$:

$$
K_p = \left\{ v \in T_p \mathbb{R}^3, v = v_{xy} + v_z : |v_z| \geq C_2 \sqrt{|z_p| \|v_{xy}\|} \right\}.
$$

(26)

Then for any $p \in \mathbb{R}^3$ satisfying $|z_p| \leq 1$,

1. $Dg(K_p) \subset K_{g(p)}$;

2. if $|z_{g(p)}| > 1$ with $N \geq N_0$, then for any $v \in K_p$, if $Dg^N(v) = u_{xy} + u_z$, then

$$
\|u_{xy}\| < 2 \delta^{1/2} |u_z| \quad \text{and} \quad \|Dg^N(v)\| \geq C \mu^N \|v\|.
$$

For a given $\delta > 0$, if the neighborhood $U^*$ of singularities is small enough, then at every point $p \in U^*$ the differential $D(\Phi \circ f \circ \Phi^{-1})(p)$ satisfies condition (iii) of Lemma 5.10. Since the tangency of $S$ at every point $x$ is quadratic, there exists $C_2 > 0$ such that every vector tangent to $S$ from the cone $D\Phi(K^u)$ also belongs to the cone in (26). The same holds for vectors tangent to $S_V$ from the cones $D\Phi(K^u)$ for $V$ small enough. Therefore, Lemma 5.10 can be applied to all those vectors. In particular, we have

**Lemma 5.11.** Assume $U$ is so small that Lemma 5.10 can be applied. There exists $C > 0$ such that if $x \in S_{V,U}$, $f(x) \in U$ and $l \in \mathbb{N}$ is the smallest number such that $f^{l-1}(x) \in U$ and $f^l(x) \notin U$, then if $l > N_0$, then for any $v \in T_x \mathbb{R}^3$ with $D\Phi(Df_x(v)) \in K_{\Phi(f(x))}$, we have

1. $\Proj_{f^{l-1}(U)} K_{f^l(U)}(f^l(x))$;

2. $\|Df^l_x(v)\| \geq C \mu^l \|v\|$, where $\mu$ is as in Lemma 5.10.

Now, if $U$ is sufficiently small and $V_0 > 0$ is sufficiently small, and $x \in S_{V,U}$ with $f(x) \in U$, then for any $\eta > 0$ we have

$$
D\Phi(Df_x(K^u_V(x))) \subset K_{\Phi(f(x))}.
$$

Hence Lemma 5.11 can be applied to vectors in $Df(K^u_V(x))$. In particular, choosing $U^*$ so small that $N > N_0$, we see that if $x \in S_{V,U}$, $f(x) \in U$ and $k \in \mathbb{N}$ is the smallest number such that $f^k(x) \notin U$, and $\{x, f(x), \ldots, f^{k-1}(x)\} \cap U^* \neq \emptyset$, then $Df_x^k(K^u_V(x)) \subset K^u_V(f^k(x))$, for any $\eta > 0$.

The proof of Lemma 5.8 follows by combining cases (i) and (ii), with

$$
\eta = \min \left\{ \frac{C}{D}, 1 \right\},
$$

where $C$ is as in Corollary 5.7 and $D$ is as in Lemma 5.9.

We immediately obtain, as a consequence of Lemma 5.8, the following
Corollary 5.12. Since the fundamental domain $F_{j_0}$ has quadratic tangency with $S_0$, there exists $V_0 > 0$ and $\tilde{\eta} > 0$ such that for all $V \in [0, V_0)$ and $x \in S_{V,U} \cap F_{j_0}$, the cone $K_V^0(x)$ is transversal to $F_{j_0}$. Let $\eta < \tilde{\eta}$ be as in Lemma 5.8. Then for all $x \in S_{V,U} \cap W_{j_0}$ and $v \in K_V^0(x)$, $v$ is transversal to $W_{j_0}$.

Proof of Corollary 5.12. Since $x \in S_{V,U} \cap W_{j_0}$, there exists $k \in \mathbb{N}$, such that $f^k(x) \in F_{j_0}$. The result follows by Lemma 5.8.

Now recall the definition of $\gamma_r(\lambda)$ from (22). With $I$ denoting the Fricke-Vogt character (see (13)), we have

$$I(\gamma_r(\lambda)) = \frac{\lambda}{4} \left( \frac{1}{r} - r \right)^2;$$

$$\frac{\partial I(\gamma_r(\lambda))}{\partial \lambda} = \frac{1}{4} \left( \frac{1}{r} - r \right)^2.$$

Consequently $\gamma_1(\lambda)$ is the line that contains the singularities $P_1$ and $P_2$ (see (15)) and lies entirely on $S_0$. For $r \neq 1$, $\gamma_r$ intersects the surfaces $\{S_V\}_{V \geq 0}$ transversally, intersecting each surface in a unique point, and intersects $S_0$ when $\lambda = 0$.

Case (i) (Assuming $J_1 \neq 1$). Assume $J_1 \neq 1$. If $\lambda = 0$ then by (2) of Proposition 4.1, $O_f^+(\gamma_1(0))$ escapes. By continuity, there exists $r_0 \in (0, 1)$ such that for all $r \in (1 - r_0, 1 + r_0)$, the set

$$\left\{ x \in \gamma_r : O_f^+(x) \text{ is bounded} \right\}$$

lies on a line segment $\Lambda_r$ whose endpoints belong to $U$(the neighborhood of singularities) (see Figures 5 and 6). Let $V_0$ be so small that Corollary 5.12 can be applied, and $r_0 \in (0, 1)$ such that for all $r \in (1 - r_0, 1 + r_0)$, the line segment $\Lambda_r$
lies entirely in $U \cup \left( \bigcup_{V \in [0, V_0]} S_{V,U} \right)$. Let $\eta$ and $\hat{\eta}$ be as in Corollary 5.12. If $\lambda > 0$ is such that $\gamma_r(\lambda) \in S_V$, from (27) we have

$$\frac{\partial I(\gamma_r(\lambda))}{\partial \lambda} = \frac{1}{\lambda} I(\gamma_r(\lambda)).$$

Since the set $\{\lambda : \gamma_r(\lambda) \in \Lambda_r, r \in (1 - r_0, 1 + r_0)\}$ is bounded away from zero uniformly in $r$, there exists $C_0 > 0$ such that for all such $\lambda$, if $\gamma_r(\lambda) \in S_{V,U}$, then

$$\mathcal{L}(\gamma_r(\lambda), S_V) \leq C_0 V.$$  \hfill (29)

On the other hand, $F^{-1}(\gamma_1 \cap S) = \{\theta = -\phi\}$ (see (18)), and $(1, -1)$ is not an eigenvector of $A$ (see proof of Lemma 5.2), hence by taking $K^u(p)$ in (23) wider as necessary, we have that for all $\lambda$ such that $\gamma_r(\lambda) \in S\{P_1, \ldots, P_4\}$, $\gamma_r(\lambda) \in \text{Int}(K^u)$. By continuity we have, for all $r$ sufficiently close to 1 and $V_0$ close to 0, if $\lambda$ is such that $\gamma_r(\lambda) \in S_{V,U}$, then

$$\text{Proj}_{T_{\gamma_r(\lambda)} S_V} (\gamma_r) \in K^u_V.$$  \hfill (30)

Combined with (29), this gives $\gamma_r(\lambda) \in K^u_V(\gamma_r(\lambda))$.

Now suppose $\lambda$ is such that $\gamma_r(\lambda) \in \Lambda_r \cap U$. Let $k \in \mathbb{N}$ be the smallest number such that $f^k(\gamma_r(\lambda)) \in S_{V,U}$. If $k < N_0$, where $N_0$ is as in Lemma 5.10, then $\gamma_r(\lambda) \notin U^*$. The cones $K^u_V$ defined on $S_{V,U}$, $V \in [0, V_0]$, can be defined on $S_{V,U^*}$ in the same way, by taking $V_0$ smaller as necessary. Hence by taking $r_0$ closer to 1 as necessary, for all $r \in (1 - r_0, 1 + r_0)$, if $\lambda$ is such that $\gamma_r(\lambda) \in S_{V,U^*}$, then (30) again holds. Assuming $V_0$ was initially taken sufficiently small, we must have

$$Df^k_{\gamma_r(\lambda)}(\gamma'_r(\lambda)) \in K^u_V(f^k(\gamma_r(\lambda))).$$

Now suppose $k > N_0$.

**Lemma 5.13.** For all sufficiently small $C^1$-perturbations of $\gamma_1$, $\Phi(\gamma_1)$ is tangent to the cones in (26) on $\Phi(U)$, with $U$ a sufficiently small neighborhood of $P_1$.

**Proof of Lemma 5.13.** Observe that $\gamma_1$ lies in the plane $\{z = 1\}$. Notice, from (19), that the curve of periodic points, $\text{Per}_2(f)$, is transversal to $\{z = 1\}$ at the point $P_1$. A simple calculation shows that the eigenvector corresponding to the smallest eigenvalue of $Df_{P_1}$ is also transversal to $\{z = 1\}$. Hence $\Phi(\gamma_1)$ is transversal to $\mathcal{W}^u_{\gamma_1}(P_1)$ (a neighborhood of $P_1$ in $\mathcal{W}^u_{\gamma_1}$), and so also to the plane $\{z = 0\}$ in the rectified coordinates. Hence all sufficiently small $C^1$-perturbations of $\Phi(\gamma_1)$ are (uniformly) transversal to $\{z = 0\}$, and therefore tangent to the cones in (26) in $\Phi(U)$, for sufficiently small $U$. \hfill $\square$

Now Lemma 5.11 can be applied. We get

$$\text{Proj}_{T_{\gamma_r(\lambda)} S_V} \left( Df^k_{\gamma_r(\lambda)}(\gamma'_r(\lambda)) \right) \in K^u_V(f^k(\gamma_r(\lambda))).$$  \hfill (31)

On the other hand, by (29) we have

$$\left\| \text{Proj}_{T_{\gamma_r(\lambda)} S_V} (\gamma'_r(\lambda)) \right\| \leq \frac{C_0 V}{\|\nabla I(\gamma_r(\lambda))\|}.$$
and after applying (25) (Lemma 5.9) we obtain
\[
\| \text{Proj}_{(T^k_r(\gamma_r(\lambda))) S_V^1)} (DF_r^k(\gamma_r(\lambda))) \| = \frac{\| \nabla I(\gamma_r(\lambda)) \|}{\| \nabla I(f^k(\gamma_r(\lambda))) \|} \| \text{Proj}_{(T^k_r(\gamma_r(\lambda))) S_V^1)} (\gamma_r(\lambda)) \| \leq \frac{C_0}{\| \nabla I(f^k(\gamma_r(\lambda))) \|} V.
\]
(32)

By compactness, for all \( V \in [0, V_0] \), \( x \in S_{V^1} \), \( \| \nabla I(x) \| \) is uniformly bounded away from zero. Thus, assuming \( V_0 \) is sufficiently small, combining equations (31) and (32), we get
\[
DF_r^k(\gamma_r(\lambda))) \in K^s_{\gamma_r}(f^k(\gamma_r(\lambda))).
\]

We can now use Corollary 5.12 to conclude that \( A_r \) intersects \( \mathbb{W}^s_{\gamma_r} \) transversally.

Case (ii) (Assuming \( J_1 = 1 \)). When \( J_1 = 1 \), \( \gamma_r(\lambda) \) has the form
\[
\gamma_r(\lambda) = \left( \frac{\lambda - 2}{2}, \frac{\lambda - (1 + r^2)}{2r}, \frac{1 + r^2}{2r} \right).
\]
(33)

Observe that if \( r \neq 1 \), then \( O^+_r(\gamma_r(0)) \) escapes, since \( f^2(\gamma_r(0)) \) satisfies (2) of Proposition 4.1. Hence for all \( r \neq 1 \) (and sufficiently close to 1) there exists \( \lambda_0(r) > 0 \) such that for all \( \lambda < \lambda_0(r) \), \( O^+_r(\gamma_r(\lambda)) \) escapes, by Lemma 4.4. We have

**Lemma 5.14.** There exists \( C_0 > 0 \) such that for all \( r \neq 1 \) sufficiently close to 1, and \( \lambda_0(r) \) as above, if \( \lambda \geq \lambda_0(r) \) and \( \gamma_r(\lambda) \in S_V \), then
\[
\frac{V}{\lambda} \leq C_0 \sqrt{V}.
\]

**Proof of Lemma 5.14.** Let \( \pi_1 : \mathbb{R}^3 \to \mathbb{R} \) denote projection onto the first coordinate. Since \( r \approx 1 \), \( \lambda_0(r) \neq 0 \) is close to zero, hence from (33),
\[
|\pi_1(\gamma_r(\lambda_0))| < 1.
\]

So by Proposition 4.1, \( O^+_r(\gamma_r(\lambda_0)) \) will diverge if \( |\pi_1 \circ f(\gamma_r(\lambda_0))| > 1 \), \( i = 1, 2 \).

A simple calculation shows the existence of a constant \( D > 0 \), independent of \( r \), such that
\[
|\pi_1 \circ f(\gamma_r(\lambda_0))| \leq 1 \quad \text{or} \quad |\pi_1 \circ f^2(\gamma_r(\lambda_0))| \leq 1 \quad \Rightarrow \quad \lambda_0 \geq \frac{(r - 1)^2}{D}.
\]

Now, say \( \gamma_r(\lambda) \in S_V \), and its forward orbit is bounded. Then \( \lambda \geq \lambda_0(r) \). From (27),
\[
V = \lambda \left( \frac{1}{r} - r \right)^2.
\]

So
\[
\frac{\sqrt{V}}{\lambda} \leq \frac{r^2 - 1}{2r} \leq \frac{r^2 - 1}{2r^2} \leq \frac{\sqrt{D}(r + 1)}{2r}.
\]

The right side above is uniformly bounded for all \( r \) away from zero. \( \square \)
Suppose $\gamma_r(\lambda) \in S_V$, $\gamma_r(\lambda) \in U^* \cap \Lambda_r$ and $k \in \mathbb{N}$ is the smallest number such that $f^k(\gamma_r(\lambda)) \in S_{V,U}$. Then $\lambda \neq 0$ and $k > N_0$. From equation (28) and a calculation similar to (32), it follows that

$$\left\| \text{Proj}_{T^k f^k(\gamma_r(\lambda)) S_V} \left( Df^k f^k(\gamma_r(\lambda)) \right) \right\| = \left\| \text{Proj}_{T^k f^k(\gamma_r(\lambda)) S_V} \left( Df^k f^k(\gamma_r(\lambda)) \left( \text{Proj}_{T^k f^k(\gamma_r(\lambda)) S_V} \gamma_r(\lambda) \right) \right) \right\| \leq \frac{D_0}{\lambda} V \leq D_0 C_0 \sqrt{V},$$

where $C_0$ is as in Lemma 5.14 and

$$D_0 = \sup \left\{ \frac{1}{\| \nabla f(x) \|} : x \in \bigcup_{V \in [0, V_0]} S_{V,U} \right\}.$$

On the other hand, $\| \gamma_r'(\lambda) \| > 1$ independently of $r$. Now by Lemma 5.13, Lemma 5.11 can be applied, so that by Part (2), with $C > 0$ and $\mu > 1$ as in the Lemma, we obtain

$$\left\| Df^k f^k(\gamma_r(\lambda)) \gamma_r'(\lambda) \right\| \geq C \mu^k \| \gamma_r'(\lambda) \| > C \mu^k.$$

Hence we have

1. $\text{Proj}_{T^k f^k(\gamma_r(\lambda)) S_V} \left( Df^k f^k(\gamma_r(\lambda)) \right) \in K_{\eta}^V(f^k(\gamma_r(\lambda)))$,

2. and from (34) and (35),

$$\frac{\left\| \text{Proj}_{T^k f^k(\gamma_r(\lambda)) S_V} \left( Df^k f^k(\gamma_r(\lambda)) \right) \right\|}{\left\| Df^k f^k(\gamma_r(\lambda)) \gamma_r'(\lambda) \right\|} \leq \frac{D_0 C_0}{C \mu^k} \sqrt{V}.$$

Hence

$$\angle(Df^k f^k(\gamma_r(\lambda)) \gamma_r'(\lambda), S_V) \leq \tilde{\eta} \sqrt{V},$$

where $\tilde{\eta}$ can be made arbitrarily small if $k$ is sufficiently large (i.e. if $U^*$ is initially chosen sufficiently small). Hence

$$Df^k f^k(\gamma_r(\lambda)) \gamma_r'(\lambda) \in K_{\eta}^V,$$

and Corollary 5.12 can be applied.

Case (iii) (If a point doesn’t enter $F_{j_0}$). Assume that the point $\gamma_r(\lambda)$ lies on the region of $W_{s_i}^s$ bounded by the curve of periodic points and the curve $\alpha$ (see discussion following Lemma 5.3). This is a finite region, hence contains at most finitely many points of intersection. So there exists $r_0 \in (0, 1)$, such that for all $r \in (1 - r_0, 1 + r_0)$, if $\gamma_r$ intersects this region, then this intersection is transversal.

Combination of Cases (i), (ii) and (iii) gives transversality in Lemma 5.2. To prove uniform transversality, observe that by Lemma 5.8, the cones $\{K_{\eta}^V\}$ are transversal to $\{W_{s_i}^s\}_{i=1,\ldots,4}$ for $V > 0$ sufficiently small. It follows that the cones
\{K_{V/2}^2\} are uniformly transversal to \{W_i^s\}_{i=1,...,4} on surfaces away from \(S_0\). Hence by taking \(r\) closer to one as necessary, but not equal to one, we can ensure that \(\gamma_r\) lies in the cones \(\{K_{V/2}^2\}\), and hence intersects \(\{W_i^s\}_{i=1,...,4}\) uniformly transversally.

For justification of the following result, see [9, Section 2] and [8, Section 5.4].

**Lemma 5.15.** For \(V > 0\), \(\{W_i^s \cap S_V\}_{i=1,...,4}\) is dense in the lamination \(W^s(\Omega_V)\).

The conclusion of Proposition 5.1 now follows by combination of Lemmas 5.2 and 5.15, and Remark 4.10.

### 5.1 Proof of Theorem 2.1-(i)

**Proposition 5.16.** With \(\gamma_r\) from (22), define
\[
\tilde{B}_{\infty,r} = \left\{ \lambda \in [0, \infty) : \mathcal{O}^{+}_f(\gamma_r(\lambda)) \text{ is bounded} \right\}.
\]
Define also
\[
\tilde{\sigma}_{k,r} = \left\{ \lambda \in [0, \infty) : |\pi \circ f^k(\gamma_r(\lambda))| \leq 1 \right\},
\]
where \(\pi\), as above, denotes projection onto the third coordinate. Then \(\tilde{\sigma}_k \xrightarrow{k \to \infty} \tilde{B}_\infty\) in Hausdorff metric.

**Proof of Proposition 5.16.** For convenience we drop \(r\) and simply write \(\tilde{B}_\infty\) and \(\tilde{\sigma}_k\), keeping in mind implicit dependence of these sets on \(r\).

We begin the proof with the following simple observation.

**Lemma 5.17.** For any \(V > 0\), if \(x = (x_1, x_2, x_3) \in \Omega_V\), then one of the coordinates of \(x\) is strictly smaller than one in absolute value.

**Proof of Lemma 5.17.** If all three coordinates are one in absolute value, then \(x\) has \(-1\) as one of its coordinates, and \(1\) for the other two (otherwise, \(x\) is one of the four singularities of \(S_0\), but by hypothesis \(V > 0\)). Then iterating forward twice gives a point that satisfies (2) of Proposition 4.1, hence \(x\) cannot be an element of \(\Omega_V\).

If at least one but not all of its coordinates is equal to one in absolute value, and the rest are strictly greater than one in absolute value, then by applying either \(f\) or \(f^{-1} = (y, z, 2yz - x)\), we’ll obtain a point \((y_1, y_2, y_3)\) with \(|y_1|, |y_2| > 1\) and \(|y_1y_2| > |y_3|\) or \(|y_2|, |y_3| > 1\) and \(|y_2y_3| > |y_1|\). In the first case, by (2) of Proposition 4.1, the point \((y_1, y_2, y_3)\) has unbounded forward semiorbit; in the second case, by a similar result applied to \(f^{-1}\), \((y_1, y_2, y_3)\) has unbounded backward semiorbit. In either case, \(x\) cannot belong to \(\Omega_V\).

By construction of the center-stable manifolds, it follows that \(\tilde{B}_\infty\) is precisely the intersection of \(\gamma_r\) with the center-stable manifolds. As in the proof of Corollary 5.12, this intersection occurs on a compact line segment \(\Lambda_r\) along \(\gamma_r\). Now application of Lemma 4.4 shows that \(\tilde{B}_\infty\) is compact.
By Lemma 5.17 it follows that for all \( \lambda \in \tilde{B}_\infty \), there exists \( N_\lambda \in \mathbb{N} \), such that for all \( k \geq N_\lambda \), \( f^k(\gamma(\lambda)) \) belongs to \( \tilde{\sigma}_k \cup \tilde{\sigma}_{k+1} \cup \tilde{\sigma}_{k+2} \). By compactness of \( \tilde{B}_\infty \), there exists such \( N \in \mathbb{N} \) uniformly for all \( \lambda \in \tilde{B}_\infty \). Define

\[
\Sigma_k = \tilde{\sigma}_{N+k} \cup \tilde{\sigma}_{N+k+1} \cup \tilde{\sigma}_{N+k+2}.
\]

It is a simple observation that follows from Proposition 4.1 (see, for example, [12]) that \( \Sigma_k \supset \Sigma_{k+1} \) for all \( k \geq 0 \), and \( \tilde{B}_\infty = \bigcap_{k \geq 0} \Sigma_k \). It follows that

\[
\lim_{k \to \infty} \text{dist}_H(\tilde{\sigma}_{N+k}, \Sigma_k) = 0. 
\] (36)

Recall that \( f|_S \) is a factor of the toral automorphism \( \mathcal{A} : \mathbb{T}^2 \to \mathbb{T}^2 \) defined in (17), and the factor map \( F \) is given in (18). The Markov partition for \( \mathcal{A} \) on \( \mathbb{T}^2 \) is given in Figure 7 (see [9] and [14] for more details). This Markov partition is carried to a Markov partition on \( S \) for \( f \) by the factor map \( F \).

Let \( \Lambda_1 \) be the line segment along \( \gamma_1 \) that connects the singularities \( P_1 \) and \( P_2 \). Then \( \Lambda_1 \) is precisely the set of those points on \( \gamma_1 \) whose forward orbit under iterations of \( f|_{S_6} \) is bounded. The set \( F^{-1}(\Lambda_1) \) is the line segment connecting \((0,0)\) and \((0,1/2)\) in the Markov partition shown in Figure 7. This set is densely intersected by the stable manifold on \( \mathbb{T}^2 \) of the point \((0,0)\), and these intersections are carried by \( F \) to a dense subset of \( \Lambda_1 \) formed by intersections of \( \Lambda_1 \) with the strong-stable manifold on \( S \) of the point \( P_1 \).

Let \( \vartheta \) be the curve of period-two periodic points for \( f \), passing through \( P_1 \), as defined in (19). Recall that \( W^s_1 \) denotes the stable manifold to \( \vartheta \). Since \( \vartheta \) has two smooth branches connecting at \( P_1 \), \( W^s_1 \) can be realized as two smooth manifolds, call them \( W^s_{1,j}, j = 1,2 \), that connect smoothly along the strong-stable manifold of \( P_1 \) on \( S_6 \).
Lemma 5.18. Let $\Lambda_r$ be a compact line segment along $\gamma_r$ which contains the intersection of $\gamma_r$ with the center-stable manifolds. Assume also that the endpoints of $\Lambda_r$ belong to this intersection. Then for all $r \approx 1$ but not equal to one, there exists a set $\{G_i^r\}_{i \in \mathbb{N}}$ of open, mutually disjoint subintervals of $\Lambda_r$ (we call them gaps), such that $B_\infty \subset \Lambda_r \setminus \bigcup_i G_i^r$, and the collection of endpoints of all $G_i^r$ is a dense subset of $B_\infty$. Moreover, for each $i$, one of the endpoints of $G_i^r$ belongs to $\mathbb{W}_{s1,1}^r$, and the other to $\mathbb{W}_{s1,2}^r$.

Proof of Lemma 5.18. Let $r_0 \in (0, 1)$, such that for all $r \in (1 - r_0, 1 + r_0)$, $r \neq 1$, $\Lambda_r$ intersects the center-stable manifolds transversally. Since $\Lambda_{r-1}$ intersects the strong-stable manifold of $P_1$ transversally (in a dense set of points), as soon as $r$ is slightly perturbed, a gap opens with one endpoint in $\mathbb{W}_{s1,1}^r$, the other in $\mathbb{W}_{s1,2}^r$ (see Figure 8). This gap persists for all $r \in (1 - r_0, 1 + r_0)$ (i.e. as long as $\Lambda_r$ intersects the center-stable manifolds transversally).

In order to show that the endpoints of these gaps form a dense subset of $\tilde{B}_\infty$, it is enough to show that no point inside of a gap belongs to $\tilde{B}_\infty$. This follows from, for example, [8, Theorem 5.22] (in fact, the strong-stable and strong-unstable manifolds of the eight points that are born from the singularities $\{P_1, \ldots, P_4\}$ form boundaries of the Markov partition on $S_V$, $V > 0$—see also [9]).

Fix $\epsilon > 0$ and let $C_1, \ldots, C_m$ be an open cover of $\tilde{B}_\infty$, with $\text{diam}(C_i) \leq \epsilon$, $i = 1, \ldots, m$. It follows that for all $k$ sufficiently large, $\Sigma_k$ is entirely contained in $\bigcup_i C_i$.

Now, for any $j \in \{1, \ldots, m\}$, pick a gap whose endpoints lie inside of $C_j$. Call one endpoint $e_1$, and the other $e_2$. Assume that $e_1$ lies on $S_{V_1}$, and $e_2$ on $S_{V_2}$ (of course, $V_1, V_2 > 0$). Let $\vartheta \cap S_{V_1} = \{p_1, q_1\}$, and $\vartheta \cap S_{V_2} = \{p_2, q_2\}$ (here $f(p_1) = q_1$, $f(q_1) = p_1$). Say $p_1$ and $p_2$ lie on the same one of the two branches of $\vartheta$, and $e_1 \in W^s(p_1)$. Then $e_2 \in W^s(q_2)$.

Observe that if $x = (x, x/(2x - 1), x) \in \vartheta$, then $f(x) = (x/(2x - 1), x, x/(2x - 1))$. Hence if $x \neq P_1$, either $|\pi(x)| < 1$ or $|\pi \circ f(x)| < 1$. It follows that $|\pi(p_1)| < 1$ or $|\pi(q_1)| < 1$. Since

$$|f^k(p_1) - f^k(e_1)| \xrightarrow{k \to \infty} 0 \quad \text{and} \quad |f^k(q_2) - f^k(e_2)| \xrightarrow{k \to \infty} 0,$$

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and \(|V_1 - V_2|\) is small provided that \(\epsilon\) is sufficiently small (hence \(|p_1 - p_2|\) and \(|q_1 - q_2|\) are small), it follows that, for all \(k\) sufficiently large, either \(|\pi \circ f^k(x_1)| < 1\) or \(|\pi \circ f^k(x_2)| < 1\). Therefore, for all \(k\) sufficiently large, \(\tilde{\sigma}_{N+k} \cap C_j \neq \emptyset\), proving (36).

Now define
\[
\tilde{B}_\infty = \{ \lambda \in (-\infty, 0] : |\lambda| \in \tilde{B}_\infty \}
\quad \text{and} \quad
\tilde{\sigma}_k = \{ \lambda \in (-\infty, 0] : |\lambda| \in \tilde{\sigma}_k \}.
\]

Take \(B_\infty = \tilde{B}_\infty \cup \tilde{B}_\infty\) Then \(\sigma_k\) in (11) is precisely \(\tilde{\sigma}_k \cup \tilde{\sigma}_k\), and so
\[
\sigma_k \xrightarrow[k \to \infty]{} B_\infty \text{ in Hausdorff metric.}
\]

This completes the proof of (i) of Theorem 2.1. In what follows, we prove properties of \(B_\infty\) stated in (ii)–(iv) of Theorem 2.1.

5.2 Proof of Theorem 2.1-(ii)

This is a direct consequence of Propositions 4.11 and 5.1.

5.3 Proof of Theorem 2.1-(iii)

Let \(M\) be a smooth two-dimensional Riemannian manifold and \(\Lambda \subset M\) a basic set for \(f \in \text{Diff}^2(M)\). Let \(\{f_\alpha\} \subset \text{Diff}^2(M)\) depending continuously on \(\alpha \in \mathbb{R}^+\), with \(f = f_0\). Then there exists \(\beta > 0\) such that for all \(\alpha \leq \beta\), \(f_\alpha\) has a basic set \(\Lambda_\alpha\) near \(\Lambda_0 = \Lambda\). Let \(\\{\tau_\alpha\}, \alpha \in \mathbb{R}^+\), be a family of smooth compact regular curves depending continuously on \(\alpha\) in the \(C^1\)-topology. Assume also that \(\tau_0\) intersects \(W^s(\Lambda_0)\) transversally. Then there exists \(\beta \in \mathbb{R}^+\), such that for all \(\alpha \in [0, \beta]\), \(\tau_\alpha\) intersects \(W^s(\Lambda_\alpha)\) transversally. Hence we may define the holonomy map
\[
h_\alpha : \tau_\alpha \cap W^s(\Lambda_\alpha) \to \Lambda_\alpha
\]
by sliding points along the stable manifolds to an unstable one (see Figure 9).

Then locally \(h_\alpha\) and its inverse are well-defined, it is a homeomorphism onto its image, and, together with its inverse, is Lipschitz (Lipschitz continuity follows easily from Section B.1.4).

Proposition 5.19. There exists \(\beta > 0\) such that the Lipschitz constant for \(h_\alpha\) and its inverse can be chosen uniformly for all \(\alpha \in [0, \beta]\).

To prove Proposition 5.19, one proceeds by a (rather standard) technique that was first introduced in [1]. The proof is sketched in Appendix C.

Recall that a morphism of metric spaces \(H : (M_1, d_1) \to (M_2, d_2)\) is said to be \((C, \nu)\)-Hölder continuous provided that for all \(x, y \in M_1\), \(d_2(H(x), H(y)) \leq Cd_1(x, y)\nu\). We have the following, due to J. Palis and M. Viana [47, Theorem B].

Proposition 5.20. Let \(f : M \to M\) be a \(C^1\) diffeomorphism on a Riemannian 2-manifold and \(\Lambda \subset M\) a basic set for \(f\) with \((1, 1)\) splitting. Then there exists \(C > 0\) and for any \(\nu \in (0, 1)\) there exists \(U \subset \text{Diff}^1(M)\) an open neighborhood of \(f\) such that for all \(g \in U\) and \(x \in \Lambda\), \(H_g|_{W^s(x) \cap \Lambda_0}\) and its inverse are \((C, \nu)\)-Hölder continuous. Here \(H_g : \Lambda \to \Lambda_g\) is the topological conjugacy (see Section B.1.1).
Under the hypothesis of and with the notation from Proposition 5.19, we get

**Lemma 5.21.** Let $\beta > 0$ satisfying Proposition 5.19. There exists $C > 0$ and for any $\nu \in (0, 1)$ there exists $\beta_0 \in (0, \beta)$, such that for any $\alpha_1, \alpha_2 \in [0, \beta_0)$, the map

$$[h_{\alpha_2}^{\text{loc}}]^{-1} \circ H_{\alpha_1, \alpha_2} \circ h_{\alpha_1}^{\text{loc}} : \tau_{\alpha_1} \cap W^s(\Lambda_{\alpha_1}) \rightarrow \tau_{\alpha_2} \cap W^s(\Lambda_{\alpha_2}),$$

where $H_{\alpha_1, \alpha_2} : \Lambda_{\alpha_1} \rightarrow \Lambda_{\alpha_2}$ is the topological conjugacy, is $(C, \nu)$-Hölder continuous (see Figure 10).

**Proof of Lemma 5.21.** Let $x_1 \in \tau_1 \cap W^s(\Lambda_{\alpha_1})$ and $h_{\alpha_1}^{\text{loc}}(x_1) \in W^u(y_1)$. Since the stable and unstable manifolds depend continuously on the point and on the diffeomorphism, if $\alpha_2$ is sufficiently close to $\alpha_1$, then $\tau_{\alpha_2} \cap W^s(H_{\alpha_1, \alpha_2}(y_1)) \neq \emptyset$. Let $x_2 \in W^s(H_{\alpha_1, \alpha_2}(y_1)) \cap \tau_{\alpha_2}$ and $y_2 = H_{\alpha_1, \alpha_2}(y_1)$. Now $h_{\alpha_2}^{\text{loc}} : U \rightarrow W^u(y_2)$, where $U$ is a neighborhood of $x_2$ in $\tau_{\alpha_2}$, is defined and Propositions 5.19 and 5.20 can be applied together. \qed

Recall from (22):

$$\gamma_r(\lambda) = \left( \frac{\lambda - (1 + J_1^2)}{2J_1}, \frac{\lambda - (1 + r^2 J_1^2)}{2r J_1}, \frac{1 + r^2}{2r} \right).$$

Hence $\gamma_r$ lies in the plane

$$\Pi_r = \left\{ z = \frac{1 + r^2}{2r} \right\}.$$

Let $r_0 \in (0, 1)$ be as in Proposition 5.1. Let $r \in (1 - r_0, 1 + r_0)$, $r \neq 1$. Fix $x_0 \in \gamma_\rho$ whose forward orbit under $f$ is bounded. Pick $\delta$ small, with $0 < \delta < V_0$, and let $\Gamma$ be a compact segment along $\gamma_\rho$ containing $x_0$, with endpoints lying on $S_{V_0 - \delta}$ and $S_{V_0 + \delta}$ (note, $x_0$ may be an endpoint of $\Gamma$). Then $\Gamma$ intersects the center-stable manifolds, as well as the surfaces $S_V$, $V \in [V_0 - \delta, V_0 + \delta]$, transversally.
For $V \in [V_0 - \delta, V_0 + \delta]$, let $\tau_V$ denote the projection of $\Gamma$ onto $S_V$ along the plane $\Pi_r$. Then $\tau_V$ is a smooth, compact regular curve in $S_V$ intersecting $W^s(\Omega_V)$ transversally. Let $\mathcal{C} = \tau_{V_0} \cap W^s(\Omega_V)$. For every $x \in \mathcal{C}$, let

$$\vartheta_x = W^{cs}(x) \cap \Pi_r \cap \bigcup_{V \in [V_0 - \delta, V_0 + \delta]} S_V,$$

where $W^{cs}(x)$ is the center-stable manifold containing $x$. Then (see Figure 11)

C1. $\{\tau_V\}$ form a smooth foliation of $\Pi$;

C2. $\vartheta_x$ is a smooth compact regular curve with endpoints lying in $S_{V_0 - \delta}$ and $S_{V_0 + \delta}$;

C3. $\{\vartheta_x\}_{x \in \mathcal{C}}$ intersects $\{\tau_V\}_{V \in [V_0 - \delta, V_0 + \delta]}$ and $\Gamma$ uniformly transversally, and the curves $\vartheta_x$ depend continuously on $x$ in the $C^1$-topology (see Remark 4.10);

C4. For $\delta > 0$ sufficiently small, $\{\tau_V\}_{V \in [V_0 - \delta, V_0 + \delta]}$ and $\{f_V\}_{V \in [V_0 - \delta, V_0 + \delta]}$ satisfy the hypothesis of Lemma 5.21. In particular, there exists $\tilde{C} > 0$ and for any $\nu \in (0,1)$ there exists $0 < \epsilon < \delta$, such that the map $\mathcal{C} \ni x \mapsto \tau_V \cap W^s(\Omega_V)$ defined by projecting points along the curves $\{\vartheta_x\}_{x \in \mathcal{C}}$ is $(\tilde{C}, \nu)$-H"{o}lder continuous for all $V \in [V_0 - \epsilon, V_0 + \epsilon]$.

Now, from C1–C4 it follows that there exists a sufficiently small neighborhood $U$ of $x_0$ in $\tau_{V_0}$ and $\tilde{C} > 0$, such that the map $U \cap \mathcal{C} \ni x \mapsto \Gamma$ defined by projecting points along the curves $\{\vartheta_x\}_{x \in U \cap \mathcal{C}}$ is $\tilde{C}$-$\nu$-H"{o}lder continuous (see [64, Lemma 3.5] for technical details). Hence

$$\dim_{H}^{\text{loc}}(\Gamma, x_0) = \dim_{H}^{\text{loc}}(\mathcal{C}, x_0).$$

(38)
On the other hand,
\[
\dim_{H}^{\text{loc}}(C, x_0) = \dim_{H}(W_{\text{loc}}^u(h(x_0))) = h^u(\Omega_{V_0}),
\]
where \( h : C \rightarrow \Omega_{V_0} \) is a holonomy map as defined in (37), and \( h^u \) is defined in Section B.1.4 as the Hausdorff dimension of \( \Omega_V \) along leaves of the unstable lamination \( W^u(\Omega_V) \). Now, \( h^u(\Omega_V) \) depends continuously (in fact analytically, as we shall see below) on \( V \). It follows that the local Hausdorff dimension is continuous over \( B_{\infty} \).

**Proposition 5.22** ([8, Theorem 5.23]). Let \( \gamma \) be an analytic curve in \( \bigcup_{V > 0} S_V \) parameterized on \( (0, 1) \). Then \( h^{s,u}(\Omega_{\gamma(t)}) : (0, 1) \rightarrow \mathbb{R} \) is analytic with values strictly between zero and one.

**Proposition 5.23** ([15, Theorem 1]). The Hausdorff dimension of \( \Omega_V \) is right-continuous at \( V = 0 \):
\[
\lim_{V \to 0^+} h^u(\Omega_V) = 1.
\]

Now, for \( r \in (1 - r_0, 1 + r_0) \), let
\[
\Gamma_r = \left\{ \lambda : O^+_f(\gamma_r(\lambda)) \text{ is bounded} \right\}.
\]
As we have already seen, \( \Gamma_r \) is a Cantor set; in particular it contains limit points. On the other hand, \( \gamma_r \) is an analytic curve that intersects \( S_V \) for every \( V \in [0, \infty) \). Combining this with (38), Propositions 5.22 and 5.23, we get that the local Hausdorff dimension is nonconstant over the spectrum. Also, (38) in combination with Proposition 5.22 shows that the local Hausdorff dimension at every point of \( B_{\infty} \) is strictly between zero and one, hence so is the global Hausdorff dimension. It follows that the Lebesgue measure of \( B_{\infty} \) is zero.

### 5.4 Proof of Theorem 2.1-iv

Observe that the line \( \gamma(J_0, J_1) \) is continuous in the parameters \( (J_0, J_1) \). From constructions carried out in the previous section, it is evident that \( \dim_{H}(B_{\infty}(J_0, J_1)) \) is also continuous in the parameters \( (J_0, J_1) \). Also, by Proposition 5.23, continuity extends to the pure case \( J_0 = J_1 \).

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A Symmetries of the Fibonacci trace map

The following discussion is taken from our forthcoming paper [17].

Let us denote the group of symmetries of $f^6$ by $G_{\text{sym}}$, and the group of reversing symmetries of $f^6$ by $G_{\text{rev}}$; that is,

$$G_{\text{sym}} = \{ s \in \text{Diff}(\mathbb{R}^3) : s \circ f^6 \circ s^{-1} = f^6 \},$$

and

$$G_{\text{rev}} = \{ s \in \text{Diff}(\mathbb{R}^3) : s \circ f^6 \circ s^{-1} = f^{-6} \},$$

where $\text{Diff}(\mathbb{R}^3)$ denotes the set of diffeomorphisms on $\mathbb{R}^3$.

Observe that $G_{\text{rev}} \neq \emptyset$. Indeed,

$$s(x, y, z) = (z, y, x)$$

is a reversing symmetry of $f$, and hence also of $f^6$. Hence $f^6$ is smoothly conjugate to $f^{-6}$. It follows (see Appendix B) that forward-time dynamical properties of $f^6$, as well as the geometry of dynamical invariants (such as stable manifolds) are mapped smoothly and rigidly to those of $f^{-6}$. That is, forward-time dynamics of $f^6$ is essentially the same as its backward-time dynamics.

The group $G_{\text{sym}}$ is also nonempty, and more importantly, it contains the following diffeomorphisms:

$$s_2 : (x, y, z) \mapsto (-x, -y, z),$$

$$s_3 : (x, y, z) \mapsto (x, -y, -z),$$

$$s_4 : (x, y, z) \mapsto (-x, y, -z).$$

Notice that $s_i$ are rigid transformations. Also notice that

$$s_i(P_i) = P_i,$$

and since $s_i$ is in fact a smooth conjugacy, we must have

$$s_i(\rho_i) = \rho_i,$$

where $\rho_i$ is the curve of periodic points passing through $P_i$.

For a more general and extensive discussion of symmetries and reversing symmetries of trace maps, see [4].

B Background on uniform, partial and normal hyperbolicity

B.1 Properties of locally maximal hyperbolic sets

A more detailed discussion can be found in [23, 24, 27–29].
A closed invariant set $\Lambda \subset M$ of a diffeomorphism $f : M \to M$ of a smooth manifold $M$ is called \textit{hyperbolic} if for each $x \in \Lambda$, there exists the splitting $T_x\Lambda = E^s_x \oplus E^u_x$ invariant under the differential $Df$, and $Df$ exponentially contracts vectors in $E^s_x$ and exponentially expands vectors in $E^u_x$. The set $\Lambda$ is called \textit{locally maximal} if there exists a neighborhood $U$ of $\Lambda$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

(44)

The set $\Lambda$ is called \textit{transitive} if it contains a dense orbit. It isn’t hard to prove that the splitting $E^s_x \oplus E^u_x$ depends continuously on $x \in \Lambda$, hence $\dim(E^s_x \oplus E^u_x)$ is locally constant. If $\Lambda$ is transitive, then $\dim(E^s_x \oplus E^u_x)$ is constant on $\Lambda$. We call the splitting $E^s_x \oplus E^u_x$ a $(k^s_x, k^u_x)$ splitting if $\dim(E^s_x \oplus E^u_x) = k^s_x$, respectively. In case $\Lambda$ is transitive, we’ll simply write $(k^s, k^u)$.

\textbf{Definition B.1.} We call $\Lambda \subset M$ a \textit{basic set} for $f \in \text{Diff}^r(M)$, $r \geq 1$, if $\Lambda$ is a locally maximal invariant transitive hyperbolic set for $f$.

Suppose $\Lambda$ is a basic set for $f$ with $(1,1)$ splitting. Then the following holds.

\textbf{B.1.1 Stability}

Let $U$ be as in (44). Then there exists $\mathcal{U} \subset \text{Diff}^1(M)$ open, containing $f$, such that for all $g \in \mathcal{U}$,

$$\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$$

(45)

is $g$-invariant transitive hyperbolic set; moreover, there exists a (unique) homeomorphism $H_g : \Lambda \to \Lambda_g$ such that

$$H_g \circ f|_{\Lambda} = g|_{\Lambda_g} \circ H_g.$$  

(46)

Also $H_g$ can be taken arbitrarily close to the identity by taking $\mathcal{U}$ sufficiently small. In this case $g$ is said to be \textit{conjugate} to $f$, and $H_g$ is said to be the conjugacy.

\textbf{B.1.2 Stable and unstable invariant manifolds}

Let $\epsilon > 0$ be small. For each $x \in \Lambda$ define the \textit{local stable} and \textit{local unstable} manifolds at $x$:

$$W^s_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \leq \epsilon \text{ for all } n \geq 0\},$$

$$W^u_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \leq \epsilon \text{ for all } n \leq 0\}.$$  

We sometimes do not specify $\epsilon$ and write

$$W^s_{\text{loc}}(x) \quad \text{and} \quad W^u_{\text{loc}}(x)$$

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for $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$, respectively, for (unspecified) small enough $\epsilon > 0$. For all $x \in \Lambda$, $W^s_{\text{loc}}(x)$ is an embedded $C^r$ disc with $T_x W^s_{\text{loc}}(x) = E^s_{x,u}$. The global stable and global unstable manifolds

$$W^s(x) = \bigcup_{n \in \mathbb{N}} f^{-n}(W^s_{\text{loc}}(x)) \quad \text{and} \quad W^u(x) = \bigcup_{n \in \mathbb{N}} f^n(W^u_{\text{loc}}(x))$$

are injectively immersed $C^r$ submanifolds of $M$. Define also the stable and unstable sets of $\Lambda$:

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x) \quad \text{and} \quad W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x).$$

If $\Lambda$ is compact, there exists $\epsilon > 0$ such that for any $x, y \in \Lambda$, $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of at most one point, and there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, $x, y \in \Lambda$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \neq \emptyset$. If in addition $\Lambda$ is locally maximal, then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \in \Lambda$.

The stable and unstable manifolds $W^s_{\text{loc}}(x)$ depend continuously on $x$ in the sense that there exists $\Phi^{s,u} : \Lambda \to \text{Emb}^r(\mathbb{R}, M)$ continuous, with $\Phi^{s,u}(x)$ a neighborhood of $x$ along $W^s_{\text{loc}}(x)$, where $\text{Emb}^r(\mathbb{R}, M)$ is the set of $C^r$ embeddings of $\mathbb{R}$ into $M$ [28, Theorem 3.2].

The manifolds also depend continuously on the diffeomorphism in the following sense. For all $g \in \text{Diff}^r(M)$ $C^r$ close to $f$, define $\hat{\Phi}^{s,u}_g : \Lambda_g \to \text{Emb}^r(\mathbb{R}, M)$ as we defined $\Phi^{s,u}$ above. Then define

$$\hat{\Phi}^{s,u}_g : \Lambda \to \text{Emb}^r(\mathbb{R}, M)$$

by

$$\hat{\Phi}^{s,u}_g = \Phi^{s,u}_g \circ H_g.$$ 

Then $\hat{\Phi}^{s,u}_g$ depends continuously on $g$ [28, Theorem 7.4].

### B.1.3 Fundamental domains

Along every stable and unstable manifold, one can construct the so-called fundamental domains as follows. Let $W^s(x)$ be the stable manifold at $x$. Let $y \in W^s(x)$. We call the arc $\gamma$ along $W^s(x)$ with endpoints $y$ and $f^{-1}(y)$ a fundamental domain. The following holds.

- $f(\gamma) \cap W^s(x) = y$ and $f^{-1}(\gamma) \cap W^s(x) = f^{-1}(y)$, and for any $k \in \mathbb{Z}$, if $k < -1$, then $f^k(\gamma) \cap W^s(x) = \emptyset$; if $k > 1$ then $f^k(\gamma) \cap W^s(x) = \emptyset$ iff $x \neq y$;

- For any $z \in W^s(x)$, if for some $k \in \mathbb{N}$, $f^k(z)$ lies on the arc along $W^s(x)$ that connects $x$ and $y$, then there exists $n \in \mathbb{N}$, $n \leq k$, such that $f^n(z) \in \gamma$.

Similar results hold for the unstable manifolds.
B.1.4 Invariant foliations

A stable foliation for $\Lambda$ is a foliation $F^s$ of a neighborhood of $\Lambda$ such that

1. for each $x \in \Lambda$, $F(x)$, the leaf containing $x$, is tangent to $E^s_x$;
2. for each $x$ sufficiently close to $\Lambda$, $f(F^s(x)) \subset F^s(f(x))$.

An unstable foliation $F^u$ is defined similarly.

For a locally maximal hyperbolic set $\Lambda \subset M$ for $f \in \text{Diff}^1(M)$, dim$(M) = 2$, stable and unstable $C^0$ foliations with $C^1$ leaves can be constructed; in case $f \in \text{Diff}^2(M)$, $C^1$ invariant foliations exist (see [46, Section A.1] and the references therein).

B.1.5 Local Hausdorff and box-counting dimensions

For $x \in \Lambda$ and $\epsilon > 0$, consider the set $W^s,u_\epsilon \cap \Lambda$. Its Hausdorff dimension is independent of $x \in \Lambda$ and $\epsilon > 0$.

Let

$$h^{s,u}(\Lambda) = \dim_H(W^s,u_\epsilon(x) \cap \Lambda).$$

For properly chosen $\epsilon > 0$, the sets $W^s,u_\epsilon(x) \cap \Lambda$ are dynamically defined Cantor sets, so

$$h^{s,u}(\Lambda) < 1$$

(see [46, Chapter 4]). Moreover, $h^{s,u}$ depends continuously on the diffeomorphism in the $C^1$-topology [44]. In fact, when dim$(M) = 2$, these are $C^{r-1}$ functions of $f \in \text{Diff}^r(M)$, for $r \geq 2$ [43].

Denote the box-counting dimension of a set $\Gamma$ by $\dim_{\text{Box}}(\Gamma)$. Then

$$\dim_H(W^s,u_\epsilon(x) \cap \Lambda) = \dim_{\text{Box}}(W^s,u_\epsilon(x) \cap \Lambda)$$

(see [44,58]).

B.2 Partial hyperbolicity

For a more detailed discussion, see [25,48].

An invariant set $\Lambda \subset M$ of a diffeomorphism $f \in \text{Diff}^r(M)$, $r \geq 1$, is called partially hyperbolic (in the narrow sense) if for each $x \in \Lambda$ there exists a splitting $T_xM = E^s_x \oplus E^c_x \oplus E^u_x$ invariant under $Df$, and $Df$ exponentially contracts vectors in $E^s_x$, exponentially expands vectors in $E^u_x$, and $Df$ may contract or expand vectors from $E^c_x$, but not as fast as in $E^s,u_x$. We call the splitting $(k^s,k^c,k^u)$ splitting if dim$(E^s,c,u) = k^s,c,u$, respectively. We’ll write $(k^s,k^c,k^u)$ if the dimension of subspaces does not depend on the point.
B.3 Normal hyperbolicity

For a more detailed discussion and proofs see [29] and also [48].

Let $M$ be a smooth Riemannian manifold, compact, connected and without boundary. Let $f \in \text{Diff}^r(M)$, $r \geq 1$. Let $N$ be a compact smooth submanifold of $M$, invariant under $f$. We call $f$ normally hyperbolic on $N$ if $f$ is partially hyperbolic on $N$. That is, for each $x \in N$,

$$T_xM = E_x^s \oplus E_x^c \oplus E_x^u$$

with $E_x^c = T_xN$. Here $E_x^{s,c,u}$ is as in Section B.2. Hence for each $x \in N$ one can construct local stable and unstable manifolds $W^s(x)$ and $W^u(x)$, respectively, such that

1. $x \in W^s_{\text{loc}}(x) \cap W^u_{\text{loc}}(x)$;
2. $T_xW^s_{\text{loc}}(x) = E^s(x)$, $T_xW^u_{\text{loc}}(x) = E^u(x)$;
3. for $n \geq 0$,

$$d(f^n(x), f^n(y)) \xrightarrow{n \to \infty} 0 \text{ for all } y \in W^s_{\text{loc}}(x),$$

$$d(f^{-n}(x), f^{-n}(y)) \xrightarrow{n \to \infty} 0 \text{ for all } y \in W^u_{\text{loc}}(x).$$

(For the proof see [48, Theorem 4.3]). These can then be extended globally by

$$W^s(x) = \bigcup_{n \in \mathbb{N}} f^{-n}f(W^s_{\text{loc}}(x));$$
$$W^u(x) = \bigcup_{n \in \mathbb{N}} f^n(W^u_{\text{loc}}(x)).$$

The manifold $W^s(x)$ is referred to as the strong-stable manifold, while $W^u(x)$ is called the strong-unstable manifold; sometimes to emphasize the point $x$, we add at $x$.

Set

$$W^{cs}_{\text{loc}}(N) = \bigcup_{x \in N} W^s_{\text{loc}}(x) \quad \text{and} \quad W^{cu}_{\text{loc}}(x) = \bigcup_{x \in N} W^u_{\text{loc}}(x).$$

**Theorem B.2** (Hirsch, Pugh and Shub [29]). The sets $W^{cs}_{\text{loc}}(N)$ and $W^{cu}_{\text{loc}}(N)$, restricted to a neighborhood of $N$, are smooth submanifolds of $M$. Moreover,

1. $W^{cs}_{\text{loc}}(N)$ is $f$-invariant and $W^{cu}_{\text{loc}}$ is $f^{-1}$-invariant;
2. $N = W^{cs}_{\text{loc}}(N) \cap W^{cu}_{\text{loc}}(N)$;
3. For every $x \in N$, $T_xW^{cs,cu}_{\text{loc}}(N) = E^s_{x,u} \oplus T_xN$;
4. $W^{cs}_{\text{loc}}(N)$ (or $W^{cu}_{\text{loc}}(N)$) is the only $f$-invariant ($f^{-1}$-invariant) set in a neighborhood of $N$;
5. \( W^{cs}_{\text{loc}}(N) \) (respectively, \( W^{cu}_{\text{loc}}(N) \)) consists precisely of those points \( y \in M \) such that for all \( n \geq 0 \) (respectively, \( n \leq 0 \)), \( d(f^n(x), f^n(y)) < \epsilon \) for some \( \epsilon > 0 \).

6. \( W^{cs, cu}_{\text{loc}}(N) \) is foliated by \( \{W^{cs}_{\text{loc}}(x)\}_{x \in N} \).

C Background results

We prove here some background results that follow from rather general principles in dynamical systems.

C.1 Proof of Proposition 5.19: sketch of main ideas

To prove Proposition 5.19, one proceeds by a (rather standard) technique that was first introduced in [1]. Let us sketch the proof below.

First suppose \( \tau_{\alpha} = \tau_0 = \tau \) for all \( \alpha \). Let \( x_0 \in \tau \cap W^s(\Lambda) \) and \( \gamma \) an open arc along \( \tau \) containing \( x_0 \) such that \( h^\text{loc} \) and its inverse are defined along \( \gamma \). Let \( U \) be an open neighborhood of \( \Lambda \) such that \( \Lambda \) is maximal in \( U \), and \( U \) can be foliated into stable and unstable foliations. There exists \( k_0 \in \mathbb{N} \) such that for all \( \alpha, f_{k_0}^\alpha(x_0) \in U \). Assuming \( \gamma \) is sufficiently short, we also have \( f_{k_0}^\alpha(\gamma) \subset U \).

To simplify notation, let us write \( f \) for \( f_0 \). Let \( \gamma^s = f^{k_0}(\gamma) \) and \( \gamma^u \) the unstable manifold such that \( f^{k_0} \circ h(x_0) \in \gamma^u \). Let \( \hat{h} : \gamma^s \to \gamma^u \) be the induced holonomy map:

\[
\hat{h}(x) = f^{k_0} \circ h^\text{loc} \circ f^{-k_0}.
\]

By the \( C^1 \) stable foliation, \( \hat{h} \) may be considered as the restriction of a \( C^1 \) map \( F : \gamma^s \to \gamma^u \) to the set \( f^{k_0}(\gamma \cap W^s(\Lambda)) \). Then for all \( x, y \in \gamma^s \) sufficiently close, there exists \( k = k(x, y) \in \mathbb{N} \) such that the arc along \( f^k(\gamma^s) \) (respectively, \( f^k(\gamma^u) \)) connecting the points \( f^k(x), f^k(y) \) (respectively, \( f^k(F(x)), f^k(F(y)) \)) belongs to \( U \), and

\[
\left[ \frac{\text{dist}_{f^k(\gamma^s)}(f^k(x), f^k(y))}{\text{dist}_{f^k(\gamma^u)}(f^k(F(x)), f^k(F(y)))} \right]^{\pm 1} \leq 2 \tag{50}
\]

(the number 2 is not significant; anything larger than 1 will work). Hence it is enough to provide an estimate, independent of \( k \), for

\[
\left[ \frac{\text{dist}_{\gamma^s}(x, y)}{\text{dist}_{f^k(\gamma^s)}(f^k(x), f^k(y))} \right]^{\pm 1} \left[ \frac{\text{dist}_{f^k(\gamma^s)}(f^k(F(x)), f^k(F(y)))}{\text{dist}^\gamma_{\gamma^u}(F(x), F(y))} \right]^{\pm 1} \tag{51}
\]

In order to estimate (51), it is enough to estimate

\[
\frac{\|Df^k|_{\gamma^s(x, y)}\|}{\|Df^k|_{\gamma^u(F(x), F(y))}\|}.
\]
where $\gamma^{s,u}(a,b)$ is the arc along $\gamma^{s,u}$ with endpoints $a, b$. After taking log, one estimates the latter by estimating

$$\sum_{j=0}^{k} \left| \| Df^j \|_{\gamma^s(x,y)} - \| Df^j \|_{\gamma^u(F(x),F(y))} \right|.$$ 

The sum above is majorized by a geometric series, and hence admits an upper bound $L$ for all $k$. One shows that the bound in (50) and the bound $L$, for $L$ sufficiently large, hold for all $f_\alpha, \alpha \in (0, \beta)$, with $\beta$ sufficiently small (this follows from continuous dependence of $f_\alpha$ and $Df_\alpha$ on $\alpha$).

Finally, small $C^1$ perturbations of $\tau$ do not destroy these bounds.

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