Mean field limit for bias voter model on regular trees

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Abstract:
In this paper we are concerned with bias voter models on trees and lattices, where the vertex in state 0 reconsiders its opinion at a larger rate than that of the vertex in state 1. For the process on tree with product measure as initial distribution, we obtain a mean field limit at each moment of the probability that a given vertex is in state 1 as the degree of the tree grows to infinity. Furthermore, for our model on trees and lattices, we show that the process converges weakly to the configuration where all the vertices are in state 1 when the rate at which a vertex in state 0 reconsiders its opinion is sufficiently large. The approach of graphical representation and the complete convergence theorem of contact process are main tools for the proofs of our results.

Keywords: bias voter model, mean field limit, asymptotically independent, contact process, complete convergence theorem.

1 Introduction

In this paper, we are concerned with the bias voter model on regular trees. First we introduce the definition of this process on general graphs. The bias voter model \( \{ \eta_t \}_{t \geq 0} \) on a graph \( S \) is with state space \( \{0, 1\}^S \). That is to say, at each vertex \( x \in S \), there is a spin taking value from \( \{0, 1\} \). For any configuration \( \eta \in \{0, 1\}^S \) and \( x \in S \), we denote by \( \eta(x) \) the value of the spin at \( x \).

At \( t = 0 \), each spin takes a value from \( \{0, 1\} \) according to some probability

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distribution. \( \lambda > \theta > 0 \) are two constants. For each vertex \( x \) in state 0 (resp. 1), it waits for an exponential time with rate \( \lambda \) (resp. \( \theta \)) to choose a neighbor \( y \) uniformly. Then, the value of the spin at \( x \) flips to that of the spin at \( y \). Therefore, \( \{\eta_t\}_{t \geq 0} \) is a spin system (see Chapter 3 of [11]) with flip rates function given by

\[
e(x, \eta) = \begin{cases} 
\frac{\lambda}{\deg(x)} \sum_{y: y \sim x} \eta(y) & \text{if } \eta(x) = 0, \\
\frac{\theta}{\deg(x)} \sum_{y: y \sim x} [1 - \eta(y)] & \text{if } \eta(x) = 1
\end{cases}
\]  
(1.1)

for any \((x, \eta) \in S \times \{0, 1\}^S\), where we denote by \( x \sim y \) that \( x \) and \( y \) are neighbors and denote by \( \deg(x) \) the degree of \( x \).

Intuitively, 0 and 1 are two opposite opinions of a topic. Vertices in state 0 (resp. 1) are individuals holding the opinion 0 (resp. 1). Each individual waits for an exponential time to choose a neighbor randomly and take the neighbor’s opinion as its. The assumption that \( \lambda > \theta \) can be considered as that 0 is a more controversial idea such that individuals holding it prefer to reconsider their opinions.

In this paper, we assume that the initial distribution of the process is the product measure with density \( p \) for \( p \in (0, 1) \), which is denoted by \( \mu_p \). In other words,

\[ \mu_p(\eta : \eta(x) = 1, \forall x \in A) = p^{|A|} \]

for any \( A \subseteq S \).

We denote by \( P^p_S \) the probability measure of the process \( \{\eta_t\}_{t \geq 0} \) on \( S \) with initial distribution \( \mu_p \). It is natural to consider the estimation of \( P^p_S(\eta_t(x) = 1) \), which is the probability that \( x \) takes 1 at the moment \( t \). When \( S \) is the complete graph with \( N \) vertices, which we denote by \( C_N \), it is easy to prove that \( P^p_{C_N}(\eta_t(x) = 1) \) satisfies the following limit theorem such that

\[
\lim_{N \to +\infty} P^p_{C_N}(\eta_t(x) = 1) = \frac{pe^{(\lambda-\theta)t}}{1 - p + pe^{(\lambda-\theta)t}}
\]

(1.2)

for any \( t > 0 \).

Equation (1.2) follows from a classic theory about density dependent population processes constructed by Ethier and Kurtz (see Chapter 11 of [6]). Let \( N_t \) be the number of vertices in state 1 at the moment \( t \), then

\[
N_t \to \begin{cases} 
N_t + 1 \text{ at rate } \frac{\lambda}{N}(N - N_t)N_t, \\
N_t - 1 \text{ at rate } \frac{\theta}{N}(N - N_t)N_t.
\end{cases}
\]

(1.3)
When the initial distribution of $\{\eta_t\}_{t \geq 0}$ is $\mu_p$, then according to Theorem 11.2.1 of [6] and (1.3), $N_t/N$ converges weakly to the solution $f(t, p)$ to the following ODE
\[
\begin{align*}
\frac{d}{dt}f(t, p) &= (\lambda - \theta)f(t, p)[1 - f(t, p)], \\
 f(0, p) &= p
\end{align*}
\]
as $N$ grows to infinity. The mathematical expression of $f(t, p)$ is exactly the right side of (1.2).

It is natural to ask whether (1.2) holds for other homogeneous graph $S$. We manage to prove that the answer is positive when $S$ is a regular tree, which we denote by $T^N$. For mathematical details, see Section 2.

Since
\[
\lim_{t \to +\infty} f(t, p) = 1
\]
when $\lambda > \theta$, it is natural to guess that $\eta_t$ converges weakly to $\delta_1$, the configuration where all vertices are in state 1, as $t$ grows to infinity. For $S$ is a regular tree or a lattice, we can prove that this guess is correct when $\lambda/\theta$ is sufficiently large. For mathematical details, see Section 2.

When $\lambda = \theta$, our model degenerates to the classic voter model introduced by Clifford and Sudbury in [3]. In [10], Holley and Liggett give an important dual relationship between the classic voter model and the coalescent random walks, which shows that any invariant measure of the classic voter model is a convex combination of $\delta_1$ and $\delta_0$ when and only when two independent simple random walks on $S$ will meet with probability one. More details can be found in Section 3.4 and Chapter 5 of [11]. The classic voter model is also a linear system (see Chapter 9 of [11]), which makes the process has some good properties, such as $\sum_{x \in S} \eta_t(x)$ is a martingale. When $\lambda > \theta$, the bias voter model can not be described via a linear system and has no good duality properties, which makes the classic approach to deal with voter models not be valid.

2 Main results

In this section we give the main results of this paper. We denote by $T^N$ the regular tree with degree $N + 1$. We obtain that (1.2) holds for the bias voter model on $T^N$. 
Theorem 2.1. For any \( t > 0 \) and \( p \in (0, 1) \),

\[
\lim_{N \to +\infty} P^p_{T^N} (\eta_t(x) = 1) = \frac{pe^{(\lambda-\theta)t}}{1 - p + pe^{(\lambda-\theta)t}}. \tag{2.1}
\]

Equation (2.1) gives the limit of the probability that a given vertex is in state 1 at the moment \( t \) as the degree of the tree grows to infinity for any \( t > 0 \). The limit function

\[
f(t, p) = \frac{pe^{(\lambda-\theta)t}}{1 - p + pe^{(\lambda-\theta)t}}
\]

is usually called the mean-field limit. Please note that \( P^p_{T^N} (\eta_t(x) = 1) \) does not depend on the choice of \( x \) since \( \mu_p \) is a translation invariant measure on \( \{0, 1\}^{T^N} \) and the flip rate function of \( \{\eta_t\}_{t \geq 0} \) given by (1.1) is also translation invariant.

The proof of Theorem 2.1 is in Section 3. The core step of the proof is to show that \( \eta_t(x) \) and \( \eta_t(y) \) are asymptotically independent for a pair of neighbors \( x \) and \( y \) as the degree of the tree grows to infinity.

It is natural to ask whether the counterpart of Theorem 2.1 for the bias voter model on lattices \( \mathbb{Z}^d, d = 1, 2, \ldots \) holds. We guess the answer is positive but we have not manage to prove that.

We denote by \( \eta_t \Rightarrow \mu \) when the process \( \{\eta_t\}_{t \geq 0} \) converges weakly to a probability measure \( \mu \). That is to say, \( \eta_t \Rightarrow \mu \) when and only when

\[
\lim_{t \to +\infty} Ef(\eta_t) = \int_{\{0,1\}^S} f(\eta) \mu(d\eta)
\]

for any \( f \in C(\{0,1\}^S) \). The mean-field limit \( f(t, p) \) given by (2.1) satisfies that \( f(t, p) \to 1 \) as \( t \to +\infty \). So it is natural to guess that \( P^p_{T^N} (\eta_t(x) = 1) \to 1 \) and therefore \( \eta_t \Rightarrow \delta_1 \), the configuration where all the vertices are in state 1. The following two theorems show that this guess holds for the bias voter model on trees and lattices when \( \lambda/\theta \) is sufficiently large.

Theorem 2.2. For each \( N \geq 2 \), there is a constant \( A(N) > 0 \) such that when \( \lambda/\theta > A(N) \), then

\[
\eta_t \Rightarrow \delta_1
\]

for the bias voter model \( \{\eta_t\}_{t \geq 0} \) on \( \mathbb{T}^N \) with initial distribution \( \mu_p \) with \( p \in (0, 1) \). The sequence \( \{A(N)\}_{N \geq 2} \) satisfies that

\[
\limsup_{N \to +\infty} \frac{A(N)}{\sqrt{N}} \leq 1. \tag{2.2}
\]
The main approach to prove Theorem 2.2 is to compare the bias voter model with a contact process on tree. The fact that the strong survived contact process on tree satisfies the complete convergence theorem is crucial for our proof. The limit theorem (2.2) of \( A(N) \) follows from an important estimation of the second critical value of the contact process on tree. For mathematical details, see Section 4. For more about the contact process on tree, see [13] and Part 1 of [12].

We denote by \( \mathbb{Z}^d \) the lattice with degree \( 2d \). The following theorem is a counterpart of Theorem 2.2 for the bias voter model on \( \mathbb{Z}^d \).

**Theorem 2.3.** For each \( d \geq 1 \) and the bias voter model \( \{ \eta_t \} _{t \geq 0} \) on \( \mathbb{Z}^d \) with initial distribution \( \mu_p \) with \( p \in (0, 1) \), when \( \lambda/\theta > 4 \), then

\[
\eta_t \Rightarrow \delta_1.
\]

The proof of Theorem 2.3 is nearly the same analysis as that of Theorem 2.2. The assumption \( \lambda/\theta > 4 \) relies on the fact that the critical value for the contact process on \( \mathbb{Z}^d \) is at most \( 2/d \). For more details, see Section 4.

3 Mean-field limit

In this section, we will prove Theorem 2.1. For any \( t > 0 \) and \( p \in (0, 1) \), we define

\[
f(t, p) = \frac{pe^{(\lambda-\theta)t}}{1 - p + pe^{(\lambda-\theta)t}}.
\]

Since we are focused on the case where \( S = \mathbb{T}^N \) in this section, we rewrite \( P^p_T \) as \( P_N^p \). First it is easy to show that \( f(t, p) \) is an upper bound of \( P_N^p(\eta_t(x) = 1) \) for any \( t \geq 0 \) and \( N \geq 1 \).

**Lemma 3.1.** For any \( t \geq 0 \) and \( N \geq 1 \),

\[
P_N^p(\eta_t(x) = 1) \leq f(t, p). \tag{3.1}
\]

**Proof.** According to the flip rate function \( c(x, \eta) \) of \( \{ \eta_t \}_{t \geq 0} \) given by (1.1) and Hille-Yosida Theorem,

\[
\frac{d}{dt} P_N^p(\eta_t(x) = 1) = \frac{\lambda}{N+1} \sum_{y \sim x} P_N^p(\eta_t(x) = 0, \eta_t(y) = 1) - \frac{\theta}{N+1} \sum_{y \sim x} P_N^p(\eta_t(x) = 1, \eta_t(y) = 0) \tag{3.2}
\]
for any \( t > 0 \).

Since \( \mu_p \) and \( c(x, \eta) \) are translation invariant,

\[
P^p_N(\eta_t(x) = 0, \eta_t(y) = 1) = P^p_N(\eta_t(x) = 1, \eta_t(y) = 0)
\]

and does not rely on the choose of the neighbor \( y \).

Therefore,

\[
\frac{d}{dt} P^p_N(\eta_t(x) = 1) = (\lambda - \theta) P^p_N(\eta_t(x) = 1, \eta_t(y) = 0),
\]

(3.3)

where \( y \) is a fixed neighbor of \( x \).

It is easy to check that the bias voter model is an attractive spin system (see Section 3.2 of [11]). Therefore, the two events \( \{\eta_t(x) = 1\} \) and \( \{\eta_t(y) = 0\} \) are negative correlated when the initial distribution is \( \mu_p \) according to Theorem 2.2.14 of [11].

As a result,

\[
P^p_N(\eta_t(x) = 1, \eta_t(y) = 0) \leq P^p_N(\eta_t(x) = 1) P^p_N(\eta_t(y) = 0)
\]

\[
= P^p_N(\eta_t(x) = 1)[1 - P^p_N(\eta_t(x) = 1)]
\]

and hence

\[
\frac{d}{dt} \left[ \log \frac{P^p_N(\eta_t(x) = 1)}{1 - P^p_N(\eta_t(x) = 1)} \right] \leq (\lambda - \theta)
\]

(3.4)

by (3.3).

According to (3.4),

\[
\log \frac{P^p_N(\eta_t(x) = 1)}{1 - P^p_N(\eta_t(x) = 1)} - \log \frac{p}{1 - p} \leq (\lambda - \theta)t
\]

(3.5)

for any \( t > 0 \).

Equation (3.1) follows from (3.5) directly.

To give a lower bound of \( P^p_N(\eta_t(x) = 1) \), we give another description of the bias voter model \( \{\eta_t\}_{t \geq 0} \). We are inspired by the approach of graphical representation introduce by Harris in [9] and the construction of stochastic processes of spin systems with exchange dynamics introduced by Durrett and Neuhauser in [11]. For any \( x, y \in \mathbb{N}^N \), \( x \sim y \), we assume that \( \{N(x,y)(t) : t \geq 0\} \) is a Poisson process with rate \( (\lambda + \theta)/(N+1) \). Please note that we care the order of \( x \) and \( y \), so \( N(x,y) \neq N(y,x) \). We assume that all these Poisson processes are independent. At \( t = 0 \), each spin takes a value from \( \{0,1\} \) according to the distribution \( \mu_p \).
Then, the spin at \( x \) may change its value only at event times of \( N(x,y) \), \( y \sim x \).
For any \( t > 0 \), we define

\[
\eta_{t-}(x) = \lim_{s \uparrow t, s < t} \eta_s(x)
\]
as the value of the spin at \( x \) at the moment just before \( t \).
For any event time \( s \) of \( N(x,y) \) for some \( y \sim x \), we flip a coin with head probability \( \frac{\lambda}{\lambda + \theta} \) and tail probability \( \frac{\theta}{\lambda + \theta} \) at the moment \( s \). If \( \eta_{s-}(x) = 0 \) (resp. \( \eta_{s-}(x) = 1 \)) and the result of the coin flipping is head (resp. tail), then \( \eta_s(x) = \eta_{s-}(y) \), otherwise, \( \eta_s(x) = \eta_{s-}(x) \). According to the properties of exponential distribution, it is easy to check that the process \( \{ \eta_t \}_{t \geq 0} \) evolving according to the rules above is a bias voter model with flip rate function \( c(x, \eta) \) given by (1.1).

For any \( x \sim y \) and \( T > 0 \), we define

\[
A_{x,y}(T) = \{ N(x,y)(T) = N(y,x)(T) = 0 \}
\]
as the random event that the first event time of \( N(x,y) \) and \( N(y,x) \) does not come before \( T \).
Then,

\[
P_N^p(A_{x,y}(T)) = e^{-\frac{2(\lambda + \theta)}{\lambda + \theta} T}
\]
and hence

\[
\lim_{N \to +\infty} P_N^p(A_{x,y}(T)) = 1 \tag{3.6}
\]
for any \( T > 0 \) and \( p \in (0, 1) \).

After all the prepared work, we can give the proof of Theorem 2.1 now.

**Proof of Theorem 2.1** According to (3.3), \( P_N^p(\eta_t(x) = 1) \) is increasing with \( t \), therefore by Lemma 3.1

\[
p \leq P_N^p(\eta_t(x) = 1) \leq f(T, p) < 1 \tag{3.7}
\]
for any \( t \in [0, T] \) and \( N \geq 1 \).

For any \( t \in [0, T] \), \( x \sim y \in \mathbb{T}^N \).

\[
P_N^p(\eta_t(x) = 1, \eta_t(y) = 0) \geq P_N^p \left( \eta_t(x) = 1, \eta_t(y) = 0, A_{x,y}(T) \right)
= P_N^p \left( \eta_t(x) = 1, \eta_t(y) = 0 \big| A_{x,y}(T) \right) P_N^p \left( A_{x,y}(T) \right) \tag{3.8}
\]
For \( x \sim y \in \mathbb{T}^N \), let

\[
C_y(x) = \{ z \in \mathbb{T}^N : \text{there is a path avoiding } y \text{ from } x \text{ to } z \}.
\]
Conditioned on $A_{x,y}(T)$, $\{\eta_t(z) : z \in C_y(x)\}_{t \leq T}$ and $\{\eta_t(w) : w \in C_x(y)\}_{t \leq T}$ are independent when the initial distribution is $\mu_p$, since vertices in $C_x(y)$ can not exchange opinions with vertices in $C_y(x)$ before the moment $T$.

As a result,

$$
P^p_N\left(\eta_t(x) = 1, \eta_t(y) = 0 \mid A_{x,y}(T)\right) = P^p_N\left(\eta_t(x) = 1 \mid A_{x,y}(T)\right) P^p_N\left(\eta_t(y) = 0 \mid A_{x,y}(T)\right) \geq \left[ P^p_N\left(\eta_t(x) = 1\right) - P^p_N\left(A_{x,y}(T)^c\right)\right] \left[ P^p_N\left(\eta_t(y) = 0\right) - P^p_N\left(A_{x,y}(T)^c\right)\right]
$$

for any $t \in [0, T]$, where $A_{x,y}(T)^c$ is the complementary set of $A_{x,y}(T)$.

By (3.3), (3.8) and (3.9), for $t \in [0, T]$,

$$
\frac{d}{dt} P^p_N(\eta_t(x) = 1) \geq (\lambda - \theta) P^p_N(\eta_t(x) = 1) \left[ 1 - P^p_N(\eta_t(x) = 1) \right] G^p_t(x, y, N),
$$

where

$$
G^p_t(x, y, N) = \frac{1 - P^p_N\left(\eta_t(x) = 1\mid A_{x,y}(T)^c\right)}{P^p_N\left(A_{x,y}(T)\right)} \left[ 1 - \frac{P^p_N\left(\eta_t(x) = 0\mid A_{x,y}(T)^c\right)}{P^p_N\left(\eta_t(x) = 0\right)} \right]
$$

(3.10)

$$
\geq \frac{1 - P^p_N\left(\eta_t(x) = 1\mid A_{x,y}(T)^c\right)}{P^p_N\left(A_{x,y}(T)\right)} \left[ 1 - \frac{P^p_N\left(\eta_t(x) = 0\mid A_{x,y}(T)^c\right)}{1 - f(T, p)} \right]
$$

(3.11)

according to (3.7).

By (3.10) and (3.11), for any $\epsilon > 0$ and $T > 0$, there exists $N(\epsilon, T) > 0$ such that

$$
G^p_t(x, y, N) \geq 1 - \epsilon
$$

(3.12)

for any $N \geq N(\epsilon, T)$ and $t \in [0, T]$.

By (3.10) and (3.12),

$$
\frac{d}{dt} P^p_N(\eta_t(x) = 1) \geq (\lambda - \theta)(1 - \epsilon) P^p_N(\eta_t(x) = 1) \left[ 1 - P^p_N(\eta_t(x) = 1) \right] \left[ 1 - \frac{P^p_N(\eta_t(x) = 1)}{1 - f(T, p)} \right]
$$

(3.13)

for $N \geq N(\epsilon, T)$ and $t \in [0, T]$.

By (3.13),

$$
\frac{d}{dt} \left[ \log \frac{P^p_N(\eta_t(x) = 1)}{1 - P^p_N(\eta_t(x) = 1)} \right] \geq (\lambda - \theta)(1 - \epsilon)
$$
and hence
\[ P_N^p(\eta_t(x) = 1) \geq \frac{pe^{(\lambda - \theta)(1 - \epsilon)t}}{1 - p + pe^{(\lambda - \theta)(1 - \epsilon)t}} \] (3.14)
for \( N \geq N(\epsilon, T) \) and \( t \in [0, T] \).

By (3.14),
\[ \liminf_{N \to +\infty} P_N^p(\eta_t(x) = 1) \geq \frac{pe^{(\lambda - \theta)(1 - \epsilon)t}}{1 - p + pe^{(\lambda - \theta)(1 - \epsilon)t}} \] (3.15)
for any \( t > 0 \) and \( \epsilon > 0 \).

Theorem 2.1 follows from (3.1) and (3.15) directly.

In the proof above, we show that \( \eta_t(x) \) and \( \eta_t(y) \) are asymptotically independent as \( N \) grows to infinity, since \( \eta_t(x) \) and \( \eta_t(y) \) are independent conditioned on \( A_{x,y}(t) \) and the probability of \( A_{x,y}(t) \) converges to 1 as \( N \) grows to infinity. If we could show that \( \eta_t(x) \) and \( \eta_t(y) \) are asymptotically independent for \( x \sim y, x, y \in \mathbb{Z}^d \) as \( d \) grows to infinity, then we could extend Theorem 2.1 to the case where the bias voter model is on the lattice. We will work on this problem as a further study.

## 4 Weak convergence

In this section we will give the proofs of Theorem 2.2 and Theorem 2.3. After a scaling of the time, it is easy to see that the limit behavior of \( \{\eta_t\}_{t \geq 0} \) only depends on \( \lambda/\theta \), so in this section we assume that \( \theta = 1 \).

The proofs of Theorem 2.2 and 2.3 are very similar, so we only give details of the proof of Theorem 2.2. For Theorem 2.3, we only give a sketch of the proof.

First we introduce the definition of the contact process \( \{\zeta_t\}_{t \geq 0} \) on \( \mathbb{T}^N \). \( \{\zeta_t\}_{t \geq 0} \) is a spin system with state space \( \{0, 1\}^\mathbb{T}^N \) and flip rate function given by
\[ c_1(x, \zeta) = \begin{cases} 1 & \text{if } \zeta(x) = 1, \\ \frac{1}{N+1} \sum_{y \sim x} \zeta(y) & \text{if } \zeta(x) = 0 \end{cases} \] (4.1)
for any \( (x, \zeta) \in \{0, 1\}^\mathbb{T}^N \).

The contact process is first introduced by Harris in [8]. Chapter 6 of [11] and Part 1 of [12] give a detailed summary of main properties of the contact process. Intuitively, the contact process describes the spread of an infection disease. Vertices in state 1 are infected while vertices in state 0 are healthy.
An infected vertex waits for an exponential time with rate 1 to become healthy and a healthy vertex is infected at a rate proportional to the number of infected neighbors.

According to the basic coupling of spin systems (see Section 3.1 of [11]), we can also use $P^p_N$ to denote the probability measure of the contact process $\{\zeta_t\}_{t \geq 0}$ on $\mathbb{T}^N$ with initial distribution $\mu_p$. We write $P^p_N$ as $P^p_{N,\lambda}$ when we need to distinguish $\lambda$.

The following lemma shows that we can control the evolution of the bias voter model $\{\eta_t\}_{t \geq 0}$ from below by the contact process $\{\zeta_t\}_{t \geq 0}$, which is crucial for us to prove Theorem 2.2.

**Lemma 4.1.** Assume that $\{\eta_t\}_{t \geq 0}$ is the bias voter model with flip rate function $c(x, \eta)$ given by (1.1) with $\theta = 1$ and $\{\zeta_t\}_{t \geq 0}$ is the contact process with flip rate function $c_1(x, \zeta)$ given by (4.1), then

$$P^p_{N,\lambda}(\eta_t(x) = 0, \forall x \in A) \leq P^p_{N,\lambda}(\zeta_t(x) = 0, \forall x \in A) \quad (4.2)$$

for any $A \subseteq \mathbb{T}^N$ and any $t \geq 0$.

**Proof.** For any $\eta, \zeta \in \{0, 1\}^{\mathbb{T}^N}$, we write $\eta \geq \zeta$ when and only when $\eta(x) \geq \zeta(x)$ for any $x \in \mathbb{T}^N$.

By direct calculation, it is easy to check that

$$c(x, \eta) \geq c_1(x, \zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 0,$$

$$c(x, \eta) \leq c_1(x, \zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 1 \quad (4.3)$$

for any $\eta \geq \zeta$.

By (4.3) and Theorem 3.1.5 of [11],

$$\eta_t \geq \zeta_t \quad (4.4)$$

for any $t > 0$ in the sense of coupling when $\eta_0$ and $\zeta_0$ have the same distribution $\mu_p$.

Equation (4.2) follows from (4.4) directly. \qed

Now we introduce the second critical value of the contact process on tree, $\lambda$ above which makes the complete convergence theorem hold.
The contact process \( \{ \zeta_t \}_{t \geq 0} \) is an attractive spin system (see Section 3.2 of [11]), therefore

\[
P^1_{N, \lambda_1} ( \exists \ t_n \uparrow +\infty, \eta_{t_n}(x) = 1, \forall \ n \geq 1 ) \geq P^1_{N, \lambda_2} ( \exists \ t_n \uparrow +\infty, \eta_{t_n}(x) = 1, \forall \ n \geq 1 )
\]

for \( \lambda_1 > \lambda_2 \). As a result, it is reasonable to define the following critical value for each \( N \geq 2, \)

\[
A(N) = \sup \{ \lambda : P^1_{N, \lambda} ( \exists \ t_n \uparrow +\infty, \eta_{t_n}(x) = 1, \forall \ n \geq 1 ) = 0 \}.
\]

(4.5)

\( A(N) \) is called the second critical value of the contact process on \( T_N \). When \( \lambda > A(N) \), the contact process is called strong survived. For more details, see Section 1.4 of [11].

According to Theorem 1.4.65 of [11],

\[
\limsup_{N \to +\infty} \frac{\sqrt{N} A(N)}{N + 1} \leq 1,
\]

which is exactly equation (2.2).

The following lemma is a corollary of the complete convergence theorem of strong survived contact process on tree. Please note that we denote by \( \delta_0 \) the configuration in \( \{0,1\}^{T_N} \) where all the vertices are in state 0.

**Lemma 4.2.** When \( \lambda > A(N) \), then there is a probability measure \( \nu_\lambda \) on \( \{0,1\}^{T_N} \) such that

\[
\nu_\lambda ( \zeta : \zeta = \delta_0 ) = 0
\]

(4.6)

and

\[
\zeta_t \Rightarrow \nu_\lambda
\]

(4.7)

when \( \zeta_0 \) has probability distribution \( \mu_p \) with \( p \in (0,1) \).

**Proof.** We denote by \( \zeta_t^1 \) the contact process with \( \zeta_0 = \delta_1 \). According to Theorem 3.2.3 and Theorem 6.1.6 of [11], when \( \lambda > A(N) \), there exists probability measure \( \nu_\lambda \) such that

\[
\zeta_t^1 \Rightarrow \nu_\lambda
\]

and

\[
\nu_\lambda ( \zeta : \zeta = \delta_0 ) = 0.
\]

Let

\[
\tau = \inf \{ t : \zeta_t = \delta_0 \},
\]

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then Theorem 1 of [16] shows that for any probability measure \( \mu \) on \( \{0, 1\}^{\mathbb{T}_N} \) and \( \{\zeta_t\}_{t \geq 0} \) with initial distribution \( \mu \),

\[
\zeta_t \Rightarrow P_\mu(\tau < +\infty)\delta_0 + P_\mu(\tau = +\infty)\nu_\lambda
\]

(4.8)

when \( \lambda > A(N) \).

When \( \mu = \mu_p \) for \( p \in (0, 1) \), there are infinite many vertices in state 1 at \( t = 0 \) with probability one and hence

\[
P_N^p(\tau < +\infty) = 0.
\]

(4.9)

Lemma 4.2 follows from (4.8) and (4.9) directly.

Equation with form as (4.8) is called the complete convergence theorem, which shows that the process with any initial distribution converges weakly to a convex combination of invariant measures. In [1], Bezuidenhout and Grimmett show that the complete convergence theorem holds for the contact process on \( \mathbb{Z}^d \). References [16] authored by Zhang and [14] authored by Salzano and Schonmann give two different proofs of the complete convergence theorem of the strong survived contact process on trees. In [2] and [15], Chen and Yao show that the complete convergence theorem holds for contact process in a random environment on \( \mathbb{Z}^+ \times \mathbb{Z}^d \). In [7], Handjani shows that the complete convergence theorem holds for the threshold-one voter model on \( \mathbb{Z}^d \) such that the process with any initial distribution converges weakly to a convex combination of three invariant measures.

By (3.3), \( P_N^p(\eta_t(x) = 1) \) is increasing with \( t \), so it is reasonable to define

\[
h(N, p) = \lim_{t \to +\infty} P_N^p(\eta_t(x) = 1)
\]

for each \( N \geq 1 \) and \( p \in (0, 1) \). It is easy to see that \( \{\eta_t\}_{t \geq 0} \) with initial distribution \( \mu_p \) converges weakly to \( \delta_1 \) when and only when \( h(N, p) = 1 \). The following lemma shows that there is a subsequence of \( \{\eta_t\}_{t \geq 0} \) converges weakly to a convex combination of \( \delta_1 \) and \( \delta_0 \).

**Lemma 4.3.** For \( \{\eta_t\}_{t \geq 0} \) with initial distribution \( \mu_p \) on \( \mathbb{T}^N \), there is a sequence \( \{t_n\}_{n \geq 1} \) increasing to infinity such that

\[
\eta_{t_n} \Rightarrow h(N, p)\delta_1 + [1 - h(N, p)]\delta_0.
\]
Proof. By (3.3), it is easy to see that

\[
\liminf_{t \to +\infty} P^p_N(\eta_t(x) = 1, \eta_t(y) = 0) = 0
\]

for \(x \sim y\). Otherwise, there would be \(\alpha > 0\) such that \(P^p_N(\eta_t(x) = 1, \eta_t(y) = 0) \geq \alpha\) for any \(t > T_0\), where \(T_0\) is a sufficiently large number. Then, by (3.3),

\[
P^p_N(\eta_t(x) = 1) - P^p_N(\eta_{T_0}(x) = 1) \geq \alpha(t - T_0) \to +\infty
\]
as \(t\) grows to infinity, which is contradictory.

Therefore, there exists sequence \(\{t_n\}_{n \geq 1}\) increasing to infinity such that

\[
\lim_{n \to +\infty} P^p_N(\eta_{t_n}(x) = 1, \eta_{t_n}(y) = 0) = 0 \quad (4.10)
\]
for \(x \sim y\).

Since \(\{0, 1\}^\mathbb{N}\) is a compact space, there is a subsequence of \(\{\eta_{t_n}\}_{n \geq 1}\) that converges weakly to a probability measure on \(\{0, 1\}^\mathbb{N}\) according to the Helly’s selection theorem (see Theorem 3.2.6 of [5]). Without loss of generality, we can assume that \(\{\eta_{t_n}\}_{n \geq 1}\) is a convergent sequence itself.

We denote by \(\varphi\) the limit distribution of \(\eta_{t_n}\) as \(n\) grows to infinity. Then, according to (4.10),

\[
\varphi(\eta(x) = 1, \eta(y) = 0) = 0 \quad (4.11)
\]
for any \(x \sim y\).

By (4.11),

\[
\varphi(\eta : \eta \neq \delta_0, \delta_1) = \varphi(\exists \ x \sim y, \eta(x) \neq \eta(y)) \\
\leq \sum_{x \sim y} \varphi(\eta(x) = 1, \eta(y) = 0) + \sum_{x \sim y} \varphi(\eta(x) = 0, \eta(y) = 1) = 0.
\]

As a result, \(\varphi\) is a convex combination of \(\delta_1\) and \(\delta_0\). Since

\[
\varphi(\eta(x) = 1) = \lim_{n \to +\infty} P^p_N(\eta_{t_n}(x) = 1) = h(N, p)
\]
according to definition of \(h\),

\[
\varphi = h(N, p)\delta_1 + [1 - h(N, p)]\delta_0.
\]

Finally we can give the proof of Theorem 2.2.
Proof of Theorem 2.2. We only need to show that \( h(N, p) = 1 \) when \( \lambda > A(N) \). When \( \lambda > A(N) \), for any \( \epsilon > 0 \), by (4.6) in Lemma 4.2, there exists finite subset \( D \) of \( T^N \) such that

\[
\nu_\lambda(\zeta(x) = 0, \forall x \in D) \leq \epsilon. \tag{4.12}
\]

By Lemma 4.3, there exists sequence \( \{t_n\}_{n \geq 1} \) increasing to infinity such that

\[
\lim_{n \to +\infty} P_p^{N,\lambda}(\eta_{t_n}(x) = 0, \forall x \in D) = 1 - h(N, p). \tag{4.13}
\]

By Lemma 4.2 and (4.12),

\[
\lim_{n \to +\infty} P_p^{N,\lambda}(\zeta_{t_n}(x) = 0, \forall x \in D) = \nu_\lambda(\zeta(x) = 0, \forall x \in D) \leq \epsilon. \tag{4.14}
\]

By (4.2), (4.13) and (4.14),

\[
1 - h(N, p) \leq \epsilon
\]

for any \( \epsilon > 0 \).

As a result,

\[
h(N, p) = 1
\]

when \( \lambda > A(N) \) and the proof is complete.

At the end of this section, we give a sketch of the proof of Theorem 2.3.

Proof of Theorem 2.3. Let \( \{\xi_t\}_{t \geq 0} \) be contact process on \( \mathbb{Z}^d \) with flip rate function given by

\[
c_2(x, \xi) = \begin{cases} 
1 & \text{if } \xi(x) = 1, \\
\frac{\lambda}{2d} \sum_{y: y \sim x} \xi(y) & \text{if } \xi(x) = 0
\end{cases}
\]

for any \( (x, \xi) \in \mathbb{Z}^d \times \{0, 1\}^{\mathbb{Z}^d} \).

Let \( \lambda(d) \) be the first critical value of \( \{\xi_t\}_{t \geq 0} \), that is to say,

\[
\lambda(d) = \sup\{\lambda : \lim_{t \to +\infty} P_{Z^d,\lambda}^1(\xi_t(x) = 1) = 0\}.
\]

It is shown in [1] that the complete convergence theorem holds for \( \{\xi_t\}_{t \geq 0} \) when \( \lambda > \lambda(d) \). Then, according to a similar analysis with that in the proof of Theorem 2.2,

\[
\eta_t \Rightarrow \delta_1
\]

for the bias voter model \( \{\eta_t\}_{t \geq 0} \) on \( \mathbb{Z}^d \) with initial distribution \( \mu_p \) with \( p \in (0, 1) \) when \( \lambda > \lambda(d) \).
According to Corollary 6.4.4 of [11],
\[
\frac{\lambda(d)}{2d} \leq \frac{2}{d}
\]
and hence
\[
\lambda(d) \leq 4.
\]
Therefore, when \( \lambda > 4 \), the bias voter model on \( \mathbb{Z}^d \) with initial distribution \( \mu_p \) with \( p \in (0,1) \) converges weakly to \( \delta_1 \).

\(\square\)

5 Two conjectures

In this section we propose two conjectures. The first one is about the mean field limit of the bias voter model on lattices.

**Conjecture 5.1.** For \( p \in (0,1) \),
\[
\lim_{d \to +\infty} P_{\mathbb{Z}^d}^p(\eta_t(x) = 1) = \frac{pe^{(\lambda - \theta)t}}{1 - p + pe^{(\lambda - \theta)p}}
\]
for any \( t > 0 \).

As we introduced in Section 3, the main difficulty to prove Conjecture 5.1 is to show that \( \eta_t(x) \) and \( \eta_t(y) \) are asymptotically independent for \( x \sim y \), \( x, y \in \mathbb{Z}^d \) as \( d \) grows to infinity. Since there are infinite many paths on the lattice from \( x \) to \( y \) avoiding the edge connecting \( x \) and \( y \), our proof of Theorem 2.1 is not applicable for the case where the process is on the lattice.

The second conjecture is about the weak convergence of the process. We guess that Theorem 2.2 and Theorem 2.3 hold under a generalized condition.

**Conjecture 5.2.** For any \( \lambda > \theta \), \( S = \mathbb{T}^d \) or \( \mathbb{Z}^d \) with \( d \geq 1 \),
\[
\eta_t \Rightarrow \delta_1
\]
for \( \{\eta_t\}_{t \geq 0} \) on \( S \) with initial distribution \( \mu_p \) with \( p \in (0,1) \).

According to the proof of Theorem 2.2, the core step to prove Conjecture 5.2 is to verify a claim that the limit distribution of any convergent subsequence of \( \{\eta_t\}_{t \geq 0} \) puts no mass on \( \delta_0 \). However, for \( \lambda \) not large enough for the complete convergence theorem of the contact process to hold, we have not find a way to prove this claim yet.
We will work on this two conjectures as a further study and hope to discuss with readers who are interested in them.

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**References**

[1] Bezuidenhout, C. and Grimmett, G. (1990). The critical contact process dies out. *The Annals of Probability.* 18, 1462-1482.

[2] Chen, XX and Yao, Q. (2009). The complete convergence theorem holds for contact processes on open clusters of $Z^d \times Z^+$. *Journal of Statistical Physics.* 135, 651-680.

[3] Clifford, P. and Sudbury, A. (1973). A model for spatial conflict. *Biometrika.* 60, 581-588.

[4] Durrett, R. and Neuhauser, C. (1994). Particle systems and reaction-diffusion equations. *The Annals of Probability.* 22, 289-333.

[5] Durrett, R. (2010). *Probability: Theory and Examples.* 4th edition, Cambridge U. Press.

[6] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes Characterization and convergence.* John Wiley and Sons.

[7] Handjani, S. (1999). The complete convergence theorem for coexistent threshold voter models. *The Annals of Probability.* 27, 226-245.

[8] Harris, T. E. (1974). Contact interactions on a lattice. *The Annals of Probability.* 2, 969-988.

[9] Harris, T. E. (1978). Additive set-valued Markov processes and graphical methods. *The Annals of Probability.* 6, 355-378.

[10] Holley, R. and Liggett, T. M. (1975). Ergodic theorems for weakly interacting systems and the voter model. *The Annals of Probability.* 3, 643-663.
[11] Liggett, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

[12] Liggett, T. M. (1999). *Stochastic interacting systems: contact, voter and exclusion processes*. Springer, New York.

[13] Pemantle, R. (1992). The contact process on trees. *The Annals of Probability*. **20**, 2089-2116.

[14] Salzano, M. and Schonmann, R. H. (1998). A new proof that for the contact process on homogeneous trees local survival implies complete convergence. *The Annals of Probability*. **26**, 1251-1258.

[15] Yao, Q. and Chen XX. (2012). The complete convergence theorem holds for contact processes in a random environment on $\mathbb{Z}^d \times \mathbb{Z}^+$. *Stochastic processes and their applications*. **122**, 3066-3100.

[16] Zhang, Y. (1996). The complete convergence theorem of the contact process on trees. *The Annals of Probability*. **24**, 1408-1443.