SIGN CHANGING SOLUTIONS OF THE HARDY-SOBOLEV-MAZ'YA EQUATION

DEBDIP GANGULY

Abstract. In this article we will study the existence, multiplicity and Morse index of sign changing solutions for the Hardy-Sobolev-Maz’ya (HSM) equation in bounded domain involving critical growth. We obtain infinitely many sign changing solutions for HSM equation and also for the elliptic problems with critical Sobolev and Hardy terms. We also establish an estimate on the Morse index for the sign changing solutions.

Keywords: Hardy-Sobolev-Maz’ya equation; sign changing solutions; Morse index.

1. Introduction

In this article we will study the equation

$$-\Delta u - \frac{\lambda u}{|y|^2} = \frac{|u|^{2^*(t)-2} u}{|y|^t} + \mu u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (1.1)

where $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k < N$, $\mu > 0$, $0 \leq \lambda < \frac{(k-2)^2}{4}$ when $k > 2$, $\lambda = 0$ when $k = 2$, $0 \leq t < 2$ and $2^*(t) = \frac{2(N-t)}{N-2}$. A point $x \in \mathbb{R}^N$ is denoted as $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $\Omega$ be a open bounded domain in $\mathbb{R}^N$ which contains some points $x^0 = (0, z^0)$.

By a weak solution of the above problem we mean $u \in H^1_0(\Omega)$ satisfying

$$\int_\Omega \left( \nabla u \nabla v - \lambda \frac{uv}{|y|^2} \right) dx = \int_\Omega \frac{|u|^{2^*(t)-2} uv}{|y|^t} dx + \mu \int_\Omega uv dx \quad \forall v \in H^1_0(\Omega).$$  \hspace{1cm} (1.2)

Eq. (1.1) has been used to model several astrophysical phenomenon like stellar dynamics (see [1], [2]). Also, from the mathematical point of view, Eq. (1.1) with $\Omega = \mathbb{R}^N$ has generated lot of interest due to its connection with the Brezis-Nirenberg problem in the Hyperbolic space (see [15], [6], [14], [4]).

In recent years, much attention have been given to the existence of nontrivial solutions for the problem (1.1). In bounded domain, the problem (1.1) does not have a solution in general due to the critical nature of the equation. For the case $\mu = 0$ and $2 \leq k < N$, Bhakta and Sandeep in [3], proved nonexistence of nontrivial solution for the Eq. (1.1), when $\Omega$ is star shaped with respect to the point $(0, z_0)$ using Pohozaev identity. They also discussed the existence result in some special bounded domain. Jannelli in [13], has considered the problem (1.1) with $t = 0$ and $k = N$ and proved the existence of positive solution under some condition on $\lambda$ and $\mu$. In [10], Cao and Han has established that the Eq. (1.1) with $t = 0$ and $k = N$ admits a nontrivial solution for all $\mu > 0$ if $\lambda \in \left(0, \frac{(N-2)^2}{4} - \frac{(N+2)^2}{N} \right)$.

When $\Omega = \mathbb{R}^N$, the existence of positive solution for the Eq. (1.1) has been studied in [17] and [19]. Moreover, the qualitative properties like Cylindrical symmetry, regularity, decay properties and uniqueness of the positive solutions of the Eq. (1.1) are thoroughly
discussed in (15) and (12). Also when \( \Omega = \mathbb{R}^N \), the hyperbolic symmetry of the equation (see 14 15 6) plays a crucial role in the study of nondegeneracy of positive solutions.

The Eq. (1.1) with \( \lambda = 0, t = 0 \) is the well known Brezis-Nirenberg problem and is well studied in (5). One of the most important result obtained, is the existence of infinitely many solutions when \( N \geq 7 \) (see 3). In the same spirit, for the Eq. (1.1) with \( t = 0 \) and \( k = N \) Cao and Yan in 8 has obtained infinitely many solutions if \( \lambda \in \left[ 0, \left( \frac{N-2}{2} \right)^2 - 4 \right] \) and \( \mu > 0 \) and later Wang and Wang in 20 has obtained the same result for the Eq. (1.1) if \( \lambda \in \left[ 0, \left( \frac{N-2}{2} \right)^2 - 4 \right] \) and \( \mu > 0 \). In all of these results one of the main tools used is the compactness of the solutions of the Brezis-Nirenberg problem established by Solimini and Devillanova 9 for \( N \geq 7 \). But 8 or 20 do not have any information about the existence and multiplicity of sign changing solutions. It is also worth mentioning that, one cannot obtain the existence and multiplicity of sign changing solution of the problem (1.1), by adopting the methods introduced in 20.

So the question of existence of infinitely many sign changing solutions for the Eq. (1.1) remains unanswered. In this direction, one of the important results obtained is the existence of infinitely many sign changing solutions for the Brezis-Nirenberg problem (see 18) in higher dimensions and later in 11 proved the same for the Brezis-Nirenberg problem in the Hyperbolic space.

In the literature, the only paper which deals with the existence of sign changing solutions for the Eq. (1.1) with \( t = 0 \) and \( k = N \) is 7, where Cao and Peng obtained a pair of sign changing solutions for \( N \geq 7, 0 \leq \lambda < \left( \frac{N-2}{2} \right)^2 - 4 \) and \( 0 < \mu < \mu_1(\lambda) \).

The novelty of this article is to obtain infinitely many sign changing solutions for the Eq. (1.1). We establish an estimate on Morse index of sign changing solutions (see Theorem 3.1) and prove the following existence Theorems:

**Theorem 1.1.** If \( N > 6 + t, \mu > 0 \), and \( \lambda \in \left[ 0, \left( \frac{k-2}{4} \right)^2 - 4 \right] \), then (1.1) has infinitely many sign changing solutions.

We also consider the case when \( k = N \), and prove the following existence Theorem:

**Theorem 1.2.** If \( N \geq 7, k = N, t = 0, \mu > 0 \) and \( \lambda \in \left[ 0, \left( \frac{N-2}{4} \right)^2 - 4 \right], \) then (1.1) has infinitely many sign changing solutions.

As mentioned before, due to the critical nature of the Eq. (1.1), the problem exhibit nonexistence phenomenon. First a definition:

**Definition 1.1.** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with smooth boundary. We say that \( \partial \Omega \) is orthogonal to the singular set if for every \( (0, z_0) \in \partial \Omega \) the normal at \( (0, z_0) \) is in \( \{0\} \times \mathbb{R}^{N-k} \).

Now we remark the following nonexistence result:

**Remark 1.** When \( \mu \leq 0 \), the Eq. (1.1) does not admits a Non-trivial solution if \( \Omega \) is star shaped with respect with respect to some point \( (0, z_0) \) and \( \partial \Omega \) is orthogonal to singular set. See the remark at the end of Section 4 for proof.

**Remark 2.** When \( \lambda = t = 0 \), the Eq. (1.1) is well studied in 18. Hence we assume either \( \lambda > 0 \) or \( t > 0 \).

We divide the article in to four sections. Section 2 of this article dicusses the notations and preliminaries, Section 3 is devoted to the existence and estimate the Morse index of sign changing solutions. The results of Section 3 are used to prove the Theorems 1.1 and 1.2 in Section 4.
2. Notations and preliminaries

We will always denote points in $\mathbb{R}^k \times \mathbb{R}^{N-k}$ as pairs $x = (y, z)$, assuming $2 \leq k < N$ and $0 \leq t < 2$.

Throughout this paper, we denote the norm of $H_0^1(\Omega)$ by $||u|| = \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2}$, the norm of $L^q_t(\Omega)$ ($1 \leq q < \infty$, $0 \leq t < 2$) by $|u|_{q,t,\Omega} = \left( \int_\Omega |u|^{q \cdot t} \, dx \right)^{1/q}$, where $dx$ denote the Lebesgue measure in $\mathbb{R}^N$.

We list here a few integral inequalities, for details we refer to [16]. The first inequality we state is the Hardy Inequality.

**Hardy Inequality:** For $k > 2$ we have

$$\left( \frac{k-2}{2} \right)^2 \int_{\mathbb{R}^N} |y|^{-2} |u|^2 \, dy \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad \forall u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N ; |y|^{-2} \, dy). \quad (2.1)$$

The constant $\left( \frac{k-2}{2} \right)^2$ that appears in (2.1) is the best constant and is not attained.

As a consequence of Hardy Inequality (2.1), for $\lambda < \frac{(k-2)^2}{4}$, $L[\cdot] \equiv -\Delta - \frac{\lambda}{|x|^2}$ is positive definite and has discrete Spectrum in $H_0^1(\Omega)$.

Let $\mu_1(\lambda)$ be the first eigen value of the operator $L[\cdot]$ in $H_0^1(\Omega)$, then it is characterized by the following Variational principle:

$$\mu_1(\lambda) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega \frac{u^2}{|x|^2}}{\int_\Omega u^2}. \quad (2.2)$$

Now it is easy to note that, if $\mu \geq \mu_1(\lambda)$, any nontrivial solution of (1.1) is sign changing. This can be seen by multiplying the first eigen function of the operator $-\Delta - \frac{\lambda}{|x|^2}$ in $H_0^1(\Omega)$ with zero-boundary value problem and integrating both sides. Thus, by the result of [20], the Eq. (1.1) has and only has infinitely many sign changing solutions for this case. Hence, from now onwards we shall only consider $0 < \mu < \mu_1(\lambda)$.

The starting point for studying (1.1) is the Hardy-Sobolev-Maz’ya inequality, that is for the case $k < N$ and was proved by Maz’ya in [16]. Now recall the HSM inequality.

**Hardy-Sobolev-Maz’ya(HSM) Inequality:** Let $p > 2$ and $p \leq \frac{2N}{N-2}$ if $N \geq 3$. Let $t = N - \frac{N-2}{2}p$. Then there is $C = C(N, p)$ such that

$$\left( \int_{\mathbb{R}^k \times \mathbb{R}^{N-k}} \frac{|u|^p}{|y|} \, dy \, dz \right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^k \times \mathbb{R}^{N-k}} \left[ |\nabla u|^2 - \frac{(k-2)^2}{4} \frac{u^2}{|y|^2} \right] \, dy \, dz \quad (2.3)$$

for all $u \in C_\infty^\infty(\mathbb{R}^k \times \mathbb{R}^{N-k})$.

As an obvious consequence of (2.3) is that if $\lambda < \frac{(k-2)^2}{4}$ then

$$||u||_\lambda := \left( \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{\lambda u^2}{|x|^2} \right) \, dx \right)^{1/2}, \quad u \in C_\infty(\Omega) \quad (2.4)$$

is a norm, equivalent to the $H_0^1(\Omega)$ norm.
Let us derive the following weighted $L^p$ embedding.

**Lemma 2.1.** If $\Omega$ is a bounded subset of $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, $t < 2$, then

$$L^p_t(\Omega) \subset L^q_t(\Omega)$$

with the inclusion being continuous, whenever $1 \leq q \leq p < \infty$.

**Proof.** Let $1 \leq q < p < \infty$ and $f \in L^p_t(\Omega)$. Then by Hölder’s Inequality we have

$$\int_{\Omega} \frac{|f|^q}{|y|^t} \, dx \leq \left( \int_{\Omega} \frac{|f|^p}{|y|^t} \, dx \right)^\frac{q}{p} \left( \int_{\Omega} \frac{1}{|y|^t} \, dx \right)^{\frac{p-q}{p}} \leq C \left( \int_{\Omega} \frac{|f|^p}{|y|^t} \, dx \right)^\frac{q}{p}.$$  

Since second term in the RHS is finite as $t < 2$ and $k \geq 2$, hence we have

$$|f|_{q,t,\Omega} \leq C |f|_{p,t,\Omega}.$$  

This completes the proof. \qed

**Remark 3.** If $f \in L^p_t(\Omega)$ for $1 \leq p < \infty$, then clearly $f \in L^p(\Omega)$ with

$$||f||_p \leq C |f|_{p,t,\Omega}.$$  

Let us prove the following Compactness Result.

**Lemma 2.2.** Let $1 \leq q < 2^*(t)$, $0 \leq t < 2$, then the embedding $H^1_0(\Omega) \hookrightarrow L^q_t(\Omega)$ is compact.

**Proof.** Let $\{u_n\}_n$ be a bounded sequence in $H^1_0(\Omega)$. Then upon a subsequence we may assume $u_n \rightharpoonup u$ in $H^1_0(\Omega)$ and pointwise. To complete the proof we need to show $u_n \to u$ in $L^q_t(\Omega)$.

$$\int_{\Omega} \frac{|u_n - u|^q}{|y|^t} \, dx = \int_{|y|<\delta} \frac{|u_n - u|^q}{|y|^t} \, dy \, dz + \int_{|y|\geq\delta} \frac{|u_n - u|^q}{|y|^t} \, dy \, dz$$

$$\leq \int_{|y|<\delta} \frac{|u_n - u|^q}{|y|^t} \, dy \, dz + \frac{1}{\delta^q} \int_{|y|\geq\delta} |u_n - u|^q \, dy \, dz.$$  

The convergence of the 2nd integral follow from Rellich Compactness Theorem, since $2^*(t) < 2^*$. On the other hand, by Hölder’s inequality and Hardy-Sobolev-Maz’ya inequality (2.3), we get

$$\int_{|y|<\delta} \frac{|u_n - u|^q}{|y|^t} \, dy \, dz \leq \left( \int_{|y|<\delta} \frac{dy \, dz}{|y|^t} \right)^{\frac{2^*(t)-q}{2^*(t)}} \left( \int_{\Omega} \frac{|u_n - u|^{2^*(t)}}{|y|^t} \, dx \right)^{\frac{q}{2^*(t)}} \leq C ||u_n - u||^q \leq C$$

for some positive constant $C$. Therefore, for a given $\epsilon > 0$, we can choose a $\delta$ such that

$$\int_{|y|<\delta} \frac{|u_n - u|^q}{|y|^t} \, dx < \frac{\epsilon}{2}.$$  

Therefore,

$$\int_{\Omega} \frac{|u_n - u|^q}{|y|^t} \, dx < \epsilon$$  

for large $n$. Hence this proves the lemma. \qed
We recall that the solutions of (2.1) are the critical points of the energy functional given by

\[ J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \frac{1}{2} \int_\Omega |u|^2 dx - \frac{1}{2 \lambda}(t) \int_\Omega |u|^2 dx - \mu \int_\Omega u^2 dx. \]  

(2.5)

Then \( J_\lambda \) is a well defined \( C^1 \) functional on \( H^1_0(\Omega) \), thanks to Hardy-Sobolev-Maz’ya inequality (2.3).

We use a variational methods in order to prove the theorem (1.1). The main tool used to obtain the multiplicity result is an Abstract theorem proved by Schechter and Zou (see [18], Theorem 2). However the theorem can only be applied when the Palais- Smale condition holds true and this is not the case for the variational problem corresponding to (1.1). Hence, arguing as in [18], for each \( \epsilon_n > 0 \) we obtain a sequence \( \{u_n\}_{n \in \mathbb{N}} \) of sign-changing solutions of

\[ -\Delta u - \frac{\lambda u}{|y|^2} = \frac{|u|^{2^*(t)-2-\epsilon_n} u}{|y|^t} + \mu u \quad \text{in} \; \Omega, \]

\[ u = 0 \quad \text{on} \; \partial \Omega, \]

with a lower bound on the Morse index. Then we prove that for fixed \( l \in \mathbb{N} \), \( \sup_n ||u_n||_{H^1_0(\Omega)} < \infty \). These are discussed in Section 3.

3. Existence and Morse Index of Sign-Changing Critical Points

In this section we will prove the existence of sign changing solutions for the perturbed compact problem (2.6) with an estimate on the Morse index. This is done by using the abstract theorem of Schechter and Zou (see [18], Theorem 2). However, we cannot directly apply it due to the presence of the singular Hardy term and Hardy-Sobolev-Maz’ya term in Eq. (2.6). We need some precise estimates in order to satisfy the assumptions of the Abstract Theorem.

In the sequel we assume that \( H^1_0(\Omega) \) is endowed with \(||.||_\lambda\) norm as defined in (2.4) unless and otherwise mentioned.

Let \( 0 < \mu_1(\lambda) < \mu_2(\lambda) \leq \mu_3(\lambda) \ldots \leq \mu_l(\lambda) \leq \ldots \) be the eigen values of \((-\Delta - \frac{\lambda}{|y|^2})\) on \( H^1_0(\Omega) \) and \( \phi_i(x) \) be the eigen function corresponding to \( \mu_i(\lambda) \). Denote \( E_l := \text{span}\{\phi_1, \phi_2, \ldots, \phi_l\} \). Then \( H^1_0(\Omega) = \bigcup_{l=1}^{\infty} E_l, \; \dim E_l = l \) and \( E_l \subset E_{l+1} \).

We fix a \( \epsilon_0 > 0 \) small enough and choose a sequence \( \epsilon_n \) in \((0, \epsilon_0)\) such that \( \epsilon_n \to 0 \). Now consider the problem

\[ -\Delta u - \frac{\lambda u}{|y|^2} = \frac{|u|^{2^*(t)-2-\epsilon_n} u}{|y|^t} + \mu u, \quad u \in H^1_0(\Omega) \]

(3.1)

then we have:

**Theorem 3.1.** Fix \( \lambda \in [0, \frac{(k-2)^2}{4} - 4), \mu > 0 \), then for every \( n \) the Eq. (3.1) has infinitely many sign changing solutions \( \{u_n\}_{n=1}^{\infty} \) such that for each \( l \), the sequence \( \{u_n\}_{n=1}^{\infty} \) is bounded in \( H^1_0(\Omega) \) and the augmented Morse index of \( u_n^\lambda \) on the space \( H^1_0(\Omega) \) is greater than equal to \( l \).

Let us denote the energy functional corresponding to (3.1) by

\[ J_{\lambda, \epsilon_n}(u) = \frac{1}{2} \int_\Omega \left( |\nabla u|^2 - \frac{\lambda}{|y|^2} u^2 - \mu u^2 \right) dx - \frac{1}{2 \lambda}(t) \int_\Omega |u|^2 dx - \epsilon_n \int_\Omega \frac{|u|^{2^*(t)-\epsilon_n}}{|y|^t} dx, \]

(3.2)
then the singular term $\int_\Omega \frac{|u|^{2^*(t) - 4}}{|y|^t} dx$ is finite by Lemma 2.1 and Hardy-Sobolev-Maz’ya inequality (2.3). Hence $J_{\lambda, \epsilon_n}$ is a $C^1$, even functional on $H^1_0(\Omega)$. In view of Lemma 2.2, $J_{\lambda, \epsilon_n}$ also satisfies the Palais- Smale condition. In order to prove the Theorem 3.1, it is enough to obtain sign-changing critical points for the functional $J_{\lambda, \epsilon_n}$.

Recall the Augmented Morse index of $u^n_0$ denoted $m^*(u^n_0)$ in the space $H^1_0(\Omega)$ is defined as

$$m^*(u^n_0) = \max \{ \dim H : H \subset H^1_0(\Omega) \text{ is a subspace such that } J'_{\lambda, \epsilon_n}(h, h) \leq 0 \text{ for all } h \in H \}. $$

For each $\epsilon_n \in (0, \epsilon_0)$ fixed, we define,

$$||u||_\epsilon = \left( \int_\Omega \frac{|u|^{2^*(t) - \epsilon_n}}{|y|^t} dx \right)^{\frac{1}{2^*(t) - \epsilon_n}},$$

then from Lemma 2.1 and Hardy-Sobolev-Maz’ya inequality (2.3), we get $||u||_\epsilon \leq C ||u||_\lambda$ for all $u \in H^1_0(\Omega)$ for some constant $C > 0$. Moreover we have $||v_n - v||_\epsilon \to 0$ whenever $v_n \rightharpoonup v$ weakly in $H^1_0(\Omega)$, thanks to Lemma 2.2.

We write $P := \{ u \in H^1_0(\Omega) : u \geq 0 \}$ for the convex cone of positive functions in $H^1_0(\Omega)$.

Define for $\mu > 0,$

$$D(\mu) := \{ u \in H^1_0 : \text{dist}(u, P) < \mu \}.$$ 

Denote the set of all critical points by

$$K^\lambda_\mu := \{ u \in H^1_0(\Omega) : J'_{\lambda, \epsilon_n}(u) = 0 \}.$$ 

The important properties of $J_{\lambda, \epsilon_n}$ needed in the proof of Theorem 3.1 are collected below.

Clearly $J_{\lambda, \epsilon_n}$ is a $C^2$ even functional which maps bounded sets to bounded sets in terms of the norm $||.||_\lambda$. The gradient $J'_{\lambda, \epsilon_n}$ is of the form $J'_{\lambda, \epsilon_n}(u) = u - K_{\lambda, \epsilon_n}(u)$, where $K_{\lambda, \epsilon_n} : H^1_0(\Omega) \to H^1_0(\Omega)$ is a continuous operator. Now we are going to study how the operator $K_{\lambda, \epsilon_n}$ behaves on $D(\mu).$ Let us prove the following proposition.

**Proposition 3.2.** For any $\rho_0 > 0$ small enough, we have that $K_{\lambda, \epsilon_n}(D(\rho_0)) \subset D(\rho) \subset D(\rho_0)$ for some $\rho \in (0, \rho_0)$ for each $\lambda, n$ with $\lambda \in [0, \frac{(k-2)^2}{4} - 4).$ Moreover, $D(\rho_0) \cap K^\lambda_\mu \subset P.$

**Proof.** First note that $K_{\lambda, \epsilon_n}(u)$ can be decomposed as $K_{\lambda, \epsilon_n}(u) = L(u) + W(u)$ where $L(u), W(u) \in H^1_0(\Omega)$ are the unique solution of the equations

$$-\Delta(L(u)) = \mu u \quad \text{and} \quad -\Delta(W(u)) = \frac{|u|^{2^*(t) - \epsilon_n - 2} u}{|y|^t}.$$ 

In other words, $L(u)$ and $W(u)$ are uniquely determine by the relations

$$\langle Lu, v \rangle_\lambda = \mu \int_\Omega uv, \quad \langle W(u), v \rangle_\lambda = \int_\Omega \frac{|u|^{2^*(t) - \epsilon_n - 2} uv}{|y|^t}. \quad (3.3)$$

Now by Maximum Principle, $L(u) \in P$ and $W(u) \in P$ if $u \in P.$

Using the above relation we have

$$\langle Lu, Lu \rangle_\lambda = \mu \int_\Omega uvu \leq \mu \left( \int_\Omega |u|^2 \right)^{\frac{1}{2}} \left( \int_\Omega |Lu|^2 \right)^{\frac{1}{2}} \leq \frac{\mu}{\mu_1(\lambda)} \left( \int_\Omega \nabla u^2 - \frac{\lambda}{2} \frac{|u|^2}{|y|^t} \right)^{\frac{1}{2}} \left( \int_\Omega |Lu|^2 - \lambda \frac{|Lu|^2}{|y|^t} \right)^{\frac{1}{2}}$$

w
Now since \( \mu < \mu_1(\lambda) \), we get,
\[
||Lu||_\lambda \leq \alpha ||u||_\lambda,
\]
where \( \alpha < 1 \). Let \( u \in H_0^1(\Omega) \) and \( v \in P \) be such that \( \text{dist}(u, P) = ||u - v||_\lambda \), then
\[
\text{dist}(Lu, P) \leq ||Lu - Lv||_\lambda \leq \alpha ||u - v||_\lambda \leq \alpha \text{dist}(u, P).
\]
Next we shall estimate the distance between \( W(u) \) and \( P \), set \( u^- := \{u, 0\} \), \( p_n(t) = 2^t(\lambda) - \epsilon_n \). Then
\[
\text{dist}(W(u), P)||W(u)^-||_\lambda \leq ||W(u)^-||_\lambda^2 \leq (W(u), W(u)^-)\lambda
\]
\[
= \int_\Omega \frac{|u|^{2^t(\lambda) - \epsilon_n} - 2^t u W(u)^-}{|y|^t} \leq \int_\Omega \frac{|u|^{2^t(\lambda) - \epsilon_n} - 1 W(u)^-}{|y|^t}
\]
\[
= \int_\Omega \frac{|u|^{-p_n(t)}}{|y|^{p_n(t)}} \frac{|W(u)^-|}{|y|^{p_n(t)}}
\]
\[
\leq \left( \int_\Omega \frac{|u|^{-p_n(t)}}{|y|^t} \right) \left( \int_\Omega \frac{|W(u)^-|}{|y|^{p_n(t)}} \right) \frac{1}{p_n(t)},
\]
the second term of the right hand side could be estimated by using Lemma 2.1 and Hardy-Sobolev-Maz’ya inequality \( 2.4 \), hence we get,
\[
\left( \int_\Omega \frac{|W(u)^-|}{|y|^{p_n(t)}} \frac{1}{p_n(t)} \right) \leq C ||W(u)^-||_\lambda.
\]
Using Lemma 2.1 and inequality \( 2.3 \), we obtain,
\[
\int_\Omega \frac{|u|^{-p_n(t)}}{|y|^t} = \min_{v \in P} \int_\Omega \frac{|u - v|^{-p_n(t)}}{|y|^t} \leq C \min_{v \in P} ||u - v||_\lambda^{p_n(t)}.
\]
Thus we get,
\[
\text{dist}(W(u), P) \leq C[\text{dist}(u, P)]^{p_n(t)-1} \forall u \in H_0^1(\Omega).
\]
Choose \( \alpha < \nu < 1 \), then there exists \( \rho_0 \) such that, if \( \rho \leq \rho_0 \)
\[
\text{dist}(W(u), P) \leq (\nu - \alpha)\text{dist}(u, P) \quad \forall u \in D(\rho).
\]
Fix \( \rho \leq \rho_0 \). Inequalities \( 3.4 \) and \( 3.5 \) yield
\[
\text{dist}(K_{\lambda,\epsilon_n}(u), P) \leq \text{dist}(L(u), P) + \text{dist}(W(u), P) \leq \nu \text{dist}(u, P)
\]
for all \( u \in D(\rho) \). This proves the Proposition \( \square \)

Now we want to examine how the functional \( J_{\lambda,\epsilon_n} \) behaves on each finite dimensional space \( E_l \).

**Proposition 3.3.** For each \( l \), \( \lim_{||u|| \to \infty, u \in E_l} J_{\lambda,\epsilon_n}(u) = -\infty \)

*Proof.* For each \( n \), \( ||.|| \) defines a norm on \( H_0^1(\Omega) \), now since \( E_k \) is finite dimensional, there exists a constant \( C > 0 \) such that \( ||u||_\lambda \leq C ||u||_\star \) for all \( u \in E_k \). Thus
\[
J_{\lambda,\epsilon_n}(u) \leq \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 - \frac{1}{2} \int_\Omega \frac{|u|^{2^t(\lambda) - \epsilon_n}}{|y|^t}
\]
\[
\leq \frac{1}{2} ||u||_\star^2 - C ||u||_\lambda^{2^t(\lambda) - \epsilon_n}.
\]
Since \( 2^t(\lambda) - \epsilon_n \), we have \( \lim_{||u|| \to \infty, u \in E_k} J_{\lambda,\epsilon_n}(u) = -\infty \). \( \square \)
Proposition 3.4. For any \( \alpha_1, \alpha_2 > 0 \), there exist an \( \alpha_3 \) depending on \( \alpha_1 \) and \( \alpha_2 \) such that \( \|u\| \leq \alpha_3 \) for all \( u \in J_{\lambda, \epsilon}^{0, \alpha} \cap \{ u \in H_0^1(\Omega) : \|u\|_s \leq \alpha_2 \} \) where \( J_{\lambda, \epsilon}^{0, \alpha} = \{ u \in H_0^1(\Omega) : J_{\lambda, \epsilon}(u) \leq \alpha_1 \} \).

Proof. Using Hardy’s inequality [2,1] we have,
\[
J_{\lambda, \epsilon}(u) + \frac{1}{2^*(t) - \epsilon_n} \|u\|^{2^*(t) - \epsilon_n} = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\mu}{2} \int_{\Omega} u^2 - \frac{\lambda}{2} \int_{\Omega} \frac{|u|^2}{|y|^2} \\
= \frac{1}{2} \|u\|_\lambda^2 - \frac{\mu}{2} \int_{\Omega} |u|^2 \\
\geq \frac{1}{2} \left[ \|u\|_\lambda^2 - \frac{\mu}{\mu_1(\lambda)} \|u\|_\lambda^2 \right].
\]
Thus we have \( J_{\lambda, \epsilon}(u) + \frac{1}{2^*(t) - \epsilon_n} \|u\|^{2^*(t) - \epsilon_n} \geq \frac{1}{2} \left[ \frac{\mu_1(\lambda) - \mu}{\mu_1(\lambda)} \right] \|u\|_\lambda^2 \).
Hence the proposition follows. \( \square \)

Now we are in a situation to proof the theorem 3.1.

Proof of Theorem 3.1 The propositions 3.2, 3.3, and 3.4 tell us that \( J_{\lambda, \epsilon} \) satisfies all the conditions of Theorem 2 in [18]. Thus \( J_{\lambda, \epsilon} \) has a sign changing critical point \( u^n \in H_0^1(\Omega) \) at a level \( C(n, \lambda, l) \) where \( C(n, \lambda, l) \leq \inf_{E_{t+1}} J_{\lambda, \epsilon} \) and the augmented Morse index \( m^*(u^n) \) of \( u^n \) is greater than equal to \( l \). The only things remains to show is that the sequence \( \{u^n\}_{n=1}^\infty \) is bounded for each \( l \).

Claim There exists a constant \( T_1 > 0 \) independent of \( l \) and \( n \) such that
\[
\sup_{E_{t+1}} J_{\lambda, \epsilon}(u) \leq T_1 \mu_{l+1}(\lambda)^{\frac{2}{2^*(t) - \epsilon_n}}.
\]

Proof of Claim The definition of \( E_{t+1} \) implies that \( \|u\|_{L^2}^2 \leq \lambda_{l+1} \|u\|_{L^2}^2 \leq C \lambda_{l+1} \|u\|_{p,t, \Omega}^2 \). Note we have \( \|u\|_{2^*(t) - \epsilon_n, \Omega} \leq D_1 \|u\|_{2^*(t) - \epsilon_n, \Omega} \), where \( D_1 > 0 \) is a constant, independent of \( n \) and \( k \). Thus we have
\[
J_{\lambda, \epsilon}(u) \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda}{2} \int_{\Omega} |u|^2 - \frac{1}{2} \int_{\Omega} \frac{|u|^{2^*(t) - \epsilon_n}}{|y|^t} \\
= \frac{1}{2} \|u\|_\lambda^2 - \frac{\lambda}{2} \int_{\Omega} |u|^2 - \epsilon_n \int_{\Omega} \frac{|u|^{2^*(t) - \epsilon_n}}{|y|^t}. \\
\]
Now using the inequality
\[
|u|^{2^*(t) - \epsilon_n} \leq c_1 |u|^{2^*(t) - \epsilon_n} + c_2,
\]
and the fact \( t < 2 \), we get,
\[
J_{\lambda, \epsilon}(u) \leq \frac{1}{2} \|u\|_\lambda^2 - D_2 \int_{\Omega} \frac{|u|^{2^*(t) - \epsilon_n}}{|y|^t} + D_3,
\]
where \( D_2 > 0, D_3 > 0 \) are constant independent of \( n \) and \( l \). Also there exist a constant \( D_4 \) such that \( |u|_{2^*(t) - \epsilon_n, \Omega} \leq D_4 \|u\|_{2^*(t) - \epsilon_n, t, \Omega} \), therefore we may have \( D_5 \) such that
\[
|u|_{\lambda}^{2^*(t) - \epsilon_0} \leq D_5 \lambda_{l+1}^{2^*(t) - \epsilon_0} \|u|^{2^*(t) - \epsilon_0}_{2^*(t) - \epsilon_0, t, \Omega} \text{ for all } u \in E_{l+1}. \]
Then
\[
J_{\lambda, \epsilon}(u) \leq \frac{1}{2} \|u\|_\lambda^2 - D_6 \mu_{l+1}(\lambda)^{-(2^*(t) - \epsilon_0)/2} \|u|^{2^*(t) - \epsilon_0}_{\lambda} + D_3 \\
\leq \frac{2^*(t) - \epsilon_0}{2^*(t) - \epsilon_0 - 2} D_4 \lambda_{l+1}^{2^*(t) - \epsilon_0} + D_3 \\
\leq T_1 \mu_{l+1}(\lambda)^{\frac{2}{2^*(t) - \epsilon_0}}.
\]
where \( D_i (i = 1, \ldots, 7) \) and \( T_1 \) are positive constants independent of \( l \) and \( n \).
Also we know that energy of any critical point of \( J_{\lambda, \epsilon} \) is nonnegative. Thus \( J_{\lambda, \epsilon}(u^n) \)
of brevity, we have omitted the detailed verification.

Theorem 1.1 with an account of propositions similar to Propositions 3.2-3.4. For the sake we show that energy of $u$ for a proof): First we recall the following compactness results by C. Wang and J. Wang (see [20], Theorem 1.3 for a proof):

Lemma 4.1. Suppose that $\mu > 0$, $N > 6 + t$ and $\lambda \in [0, (k - 2)^2 - 4]$. Then for any sequence $u_n$, which is a solution of (2.6) with $\epsilon = \epsilon_n \to 0$, satisfying $||u_n|| \leq C$ for some constant independent of $n$, $u_n$ has a subsequence, which converges strongly in $H^1_0(\Omega)$ as $n \to +\infty$.

Proof of Theorem 1.1: The proof is divided into two steps. Step-1: By combining Lemma 4.1 and Theorem 3.1, we get a sequence $\{u_l\}_{l=1}^\infty$ of solutions of the problem (1.1) with energy $C(\lambda, l) \in [0, T\mu_{l+1}(\lambda) \frac{2^*(\lambda)-\epsilon_n}{4^*(\lambda)-\epsilon_n}]$. Moreover, we claim that $u_l$ is still sign-changing. Since $\{u_l^\pm\}_{l=1}^\infty$ is a sign-changing solutions to (1.1), let

$$\{u_l^\pm\} := \max\{\pm u_l^\pm, 0\}.$$ 

Then we have

$$\int_\Omega |\nabla(u_l^\pm)|^2 = \lambda \int_\Omega \frac{|u_l^\pm|^2}{|y|^2} + \mu \int_\Omega |(u_l^\pm)^2|^2 + \int_\Omega \frac{|(u_l^\pm)^2|^{2^*(\lambda)-\epsilon_n}}{|y|^\mu}.$$ 

Thus

$$|||u_l^\pm||_\lambda^2 \leq \alpha |||u_l^\pm||_\lambda^2 + \int_\Omega \frac{|(u_l^\pm)^2|^{2^*(\lambda)-\epsilon_n}}{|y|^\mu},$$ 

where $\alpha < 1$. Then using Hardy-Sobolev-Maz’ya inequality [23] it follows that

$$|||u_l^\pm||_\lambda \geq C_0 > 0.$$ 

where $C_0$ is a constant independent of $n$. This implies that the limit $u_l$ of the subsequence $\{u_l^\pm\}$ is still sign-changing.

Proof of Theorem 1.2: The proof of this theorem has the similar lines of proof of Theorem 1.1 with an account of propositions similar to Propositions 3.2, 3.3. For the sake of brevity, we have omitted the detailed verification.

Remark 4. We give a proof of the claim in Remark 2. The proof has the similar lines of proof of (Theorem 4.1, [3]). It is based on the Pohozaev identity. The difficulty in applying this identity is because of the presence singular terms. We can overcome this difficulty by using the partial $H^2$-regularity. With an obvious modification from (3, Theorem 2.4), we can prove if $u$ is a solution to the Eq. (1.1), then $u_{\lambda_i} \in H^1(\Omega)$ for all $1 \leq i \leq N - k$.

To make the test function smooth we introduce cut-off functions and pass to the limit with help of above regularity result. We will assume without loss of generality that $\Omega$ is star shaped with respect to the origin.
For $\epsilon > 0$ and $R > 0$, define $\varphi_{\epsilon,R} = \varphi_{\epsilon}(x)\psi_R(x)$ where $\varphi_{\epsilon}(x) = \phi(|y|/\epsilon)$, $\psi_R = \psi(|x|/R)$, $\varphi$ and $\psi$ are smooth functions in $\mathbb{R}$ with the properties $0 \leq \varphi, \psi \leq 1$, with supports of $\varphi$ and $\psi$ in $(1, \infty)$ and $(-\infty, 2)$ respectively and $\varphi(t) = 1$ for $t \geq 2$, and $\psi(t) = 1$ for $t \leq 1$.

Assume that (1.1) has a nontrivial solution $u$. Then $u$ is smooth away from the singular set and hence $(x, \nabla u)\varphi_{\epsilon,R} \in C^2_0(\Omega)$. Multiplying Eq. (1.1) by this test function and integrating by parts, we have

\[
\int_{\Omega} \nabla u \cdot \nabla ((x, \nabla u)\varphi_{\epsilon,R}) - \lambda \int_{\Omega} \frac{u(x, \nabla u)}{|y|^2} \varphi_{\epsilon,R} - \int_{\Omega} \frac{\partial u}{\partial \nu} (x, \nabla u)\varphi_{\epsilon,R} = 0.
\]

Using integration by parts and the fact that $u$ is smooth in $\mathbb{R}^n$, we get

\[
\int_{\Omega} \frac{|u|^{2^*(t) - 2} u}{|y|^t} (x, \nabla u)\varphi_{\epsilon,R} + \mu \int_{\Omega} u (x, \nabla u)\varphi_{\epsilon,R}.
\]

Now, RHS of (4.1) can be simplified as

\[
\int_{\Omega} \frac{|u|^{2^*(t) - 2} u}{|y|^t} (x, \nabla u)\varphi_{\epsilon,R} + \mu \int_{\Omega} u (x, \nabla u)\varphi_{\epsilon,R}
\]

\[
= \frac{1}{2^*(t)} \int_{\Omega} (\nabla |u|^{2^*(t)} x) \frac{\varphi_{\epsilon,R}}{|y|^t} + \frac{\mu}{2} \int_{\Omega} (\nabla |u|^2 x) \varphi_{\epsilon,R}
\]

\[
= -\frac{(N - 2)}{2} \int_{\Omega} \frac{|u|^{2^*(t)}}{|y|^t} \varphi_{\epsilon,R} - \frac{1}{2^*(t)} \int_{\Omega} \frac{|u|^{2^*(t)}}{|y|^t} [x.(\psi_R \nabla \varphi_{\epsilon} + \varphi_{\epsilon} \nabla \psi_R)]
\]

\[
- \frac{N \mu}{2} \int_{\Omega} |u|^2 \varphi_{\epsilon,R} - \frac{\mu}{2} \int_{\Omega} |u|^2 [x.(\psi_R \nabla \varphi_{\epsilon} + \varphi_{\epsilon} \nabla \psi_R)].
\]

Note that $|x.(\psi_R \nabla \varphi_{\epsilon} + \varphi_{\epsilon} \nabla \psi_R)| \leq C$ and hence using the dominated convergence theorem we get

\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} RHS = -\frac{(N - 2)}{2} \int_{\Omega} \frac{|u|^{2^*(t)}}{|y|^t} - \frac{N \mu}{2} \int_{\Omega} |u|^2. \tag{4.2}
\]

For LHS, using integration by parts and the fact that $u_{z_\nu} \in H^1(\Omega)$, (see [3], Theorem 4.1 for detail), we get

\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} LHS = -\frac{(N - 2)}{2} \left[\int_{\Omega} |\nabla u|^2 - \frac{1}{2} \int_{\partial \Omega} (\frac{\partial u}{\partial \nu})^2 (x, \nu)\right]. \tag{4.3}
\]

Substituting (4.2) and (4.3) in (1.1), and using Eq. (1.1), we get

\[
\int_{\partial \Omega} (\frac{\partial u}{\partial \nu})^2 (x, \nu) + 2|\mu| \int_{\Omega} u^2 = 0
\]

which implies $u = 0$ in $\Omega$ by the principle of unique continuation. This proves the Remark.

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(Debidip Ganguly)

CENTRE FOR APPLICABLE MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, P.O. BOX 6503, GKV POST OFFICE, BANGALORE 560065, INDIA

E-mail address: debdip@math.tifrbng.res.in