A note on verification procedures for quantum noiseless subsystems

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We establish conditions under which the experimental verification of quantum error-correcting behavior against a linear set of error operators \( \mathcal{E} \) suffices for the verification of noiseless subsystems of an error algebra \( \mathcal{A} \) contained in \( \mathcal{E} \). From a practical standpoint, our results imply that the verification of a noiseless subsystem need not require the explicit verification of noiseless behavior for all possible initial states of the syndrome subsystem.

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I. INTRODUCTION

Noiseless subsystems (NSs) provide a comprehensive conceptual framework for understanding stabilization strategies for quantum information [1, 2, 3, 4]. NSs include as a special case decoherence-free subspaces (DFSs) [5, 6, 7]. Because, physically, the occurrence of NSs requires the presence of symmetry in the underlying noise process, NSs may not exist for arbitrary error models. However, if the appropriate symmetry requirements are met, the protection that NSs can afford is especially powerful, as encoded information remains immune to errors indefinitely in time. In the language of general quantum error correction (QEC) [1], the latter property characterizes NSs as infinite-distance quantum error-correcting codes.

An important issue in both analyzing error-correcting performance and implementing error-control benchmarks [1, 2, 3, 4] is to establish operational criteria under which NS-behavior may be reliably diagnosed from available data. While it might seem that the implementation of a desired NS simply amounts to verifying that information is preserved once appropriately encoded, two considerations make the procedure less straightforward in practice. On one hand, due to unavoidable operational imperfections, the implemented decoding transformation may differ from the intended one in unknown ways, making the actual subsystem identification potentially inequivalent to the abstract noiseless one. On the other hand, a proper NS is paired with a non-trivial syndrome subsystem in such a way that a distinct infinite-distance quantum error-correcting code can be associated with every initial state of the syndrome. Accordingly, the verification of an NS appears at first to require that noiselessness of the relevant information is checked for all possible syndrome initializations.

It is the purpose of this work to address these issues from an experimentally motivated perspective and to point out a criterion that is applicable whenever stability of encoded information under an error algebra is experimentally verified. The content of the paper is organized as follows. In Sect. II we recall the subsystem view of QEC, by emphasizing in particular the difference between finite- and infinite-distance error-correcting behavior and the association of the latter with noiseless degrees of freedom. Sect. III is devoted to the formulation of the verification problem and to the construction of verification procedures within the assumed setting. This is done by first discussing how error-correcting behavior against a generic linear set of errors may be diagnosed in a typical QEC experiment, and then by showing how stronger conclusions may be reached if an algebra is contained in the error set. In Sect. IV, the general analysis and results are illustrated by revisiting the prototypical example of a three-qubit NS for collective noise as introduced in [1] and implemented in [10, 11]. The example also demonstrates how the algebraic structure may be exploited for deducing the action of errors when the syndrome subsystem is initialized to states other than the one explicitly implemented or, equivalently, for inferring the stability of the relevant information under a larger error set than explicitly checked – both features being advantageous from the practical point of view. The paper concludes with a brief summary in Sect. V.

II. ERROR-CORRECTING SUBSYSTEMS

A general description of QEC, applicable to both finite- and infinite-distance error control, is offered by the subsystem approach [1, 10, 12, 13, 14]. Let \( S \) be a finite-dimensional system, with state space \( \mathcal{S} \), \( \dim(\mathcal{S}) = d \), and let \( \mathcal{A}_S = \text{End}(\mathcal{S}) \) denote the corresponding operator algebra. We may assume \( \mathcal{S} \cong \mathbb{C}^d \), and \( \mathcal{A}_S \cong \text{Mat}_d(\mathbb{C}) \). For example, \( d = 2^n \) for an \( n \)-qubit system. Note that while we will make explicit reference to the usual qubit setting in the present discussion, more general situations (e.g., involving higher-dimensional subsystems) could easily be handled. Suppose that \( S \) is used to protect \( k \) qubits, \( k < \log_2(d) \), against a known family of error operators \( \mathcal{E} = \{ E_{\alpha} \} \). We require that the “no-error” event is correctable, thus \( \mathcal{E} \) contains the identity. Because quantum correctability by a given error control strategy is pre-
served under linear transformations \[12\], we also assume that the error space \( \mathcal{E} \) is a linearly closed subset of operators in \( \mathcal{A}_S \). The subsystems view of QEC relies on separating the degrees of freedom representing the logical state from the ones encoding the effect of the errors on the intended code \( C \subset S \). Thus, \( \mathcal{E} \) is correctable by \( C \) provided that an isomorphism exists,

\[ \omega : S \rightarrow \mathcal{L} \otimes \mathcal{Z} \oplus \mathcal{D}, \]

such that for every \( E \in \mathcal{E} \) and every \( |\psi\rangle_C \in C \),

\[ E|\psi\rangle_C = \omega^{-1}(|\psi\rangle_L \otimes |\varphi_E\rangle_Z), \]

for a vector \( |\varphi_E\rangle_Z \in \mathcal{Z} \) only dependent upon \( E \). Because \( \mathbb{I} \) is correctable, the code associated to \( \mathcal{L} \) is the subspace

\[ C = \omega^{-1}(\mathcal{L} \otimes |\varphi_0\rangle_Z), \]

with \( |\varphi_0\rangle_Z \) corresponding to no error. Note that \( C \) can detect errors which cause leakage into \( \mathcal{D} \), but not correct them. Thus, under the condition that all errors in \( \mathcal{E} \) are correctable, the mapping \[11\] effectively singles out a subspace \( \mathcal{H} = \omega^{-1}(\mathcal{L} \otimes \mathcal{Z}) \subseteq S \). The correspondence between states in \( \mathcal{H} \) and states in \( \mathcal{L} \otimes \mathcal{Z} \) under the restriction \( \omega[H] \) defines the subsystem identification of the QEC procedure, \( \mathcal{L} \) and \( \mathcal{Z} \) being the information-carrying and syndrome subsystem, respectively. Given any correctable error \( E \in \mathcal{E} \), it is then possible to describe the action of \( E \) directly in the subsystem representation by introducing an operator \( \tilde{E} = \omega \circ E \circ \omega^{-1} \). Equivalently, one may write

\[ E|\Psi\rangle_H = \omega^{-1}\left(\tilde{E}(|\psi\rangle_L \otimes |\varphi\rangle_Z)\right), \]

with \( \omega|\Psi\rangle_H = |\psi\rangle_L \otimes |\varphi\rangle_Z \). We shall use a similar notation for operator sets e.g., \( \mathcal{E} = \{E | E \in \mathcal{E}\} \), etc.

Combining Eqs. \[2\]–\[6\], the correctness of \( \mathcal{E} \) by \( C \) is equivalent to the condition that errors in \( \mathcal{E} \) affect only the syndrome subsystem when the latter is appropriately initialized. Note, however, that the assumed linear structure of \( \mathcal{E} \) does not suffice in general to guarantee that cumulative errors from \( \mathcal{E} \) remain correctable, unless information is properly returned to \( C \) by resetting \( \mathcal{Z} \) to its reference state \( |\varphi_0\rangle_Z \). While active recovery is a necessary feature of finite-distance QEC, codes with stronger error-correcting properties may be designed if \( \mathcal{E} \) is known to have additional structure. Suppose that \( \mathcal{E} \supseteq \mathcal{A} \), where \( \mathcal{A} \) is a \( \mathcal{L} \)-closed sub-algebra of \( \mathcal{A}_S \) containing \( \mathbb{I} \). Because \( \mathcal{A} \) is closed under operator multiplication, arbitrary cumulative errors remain in \( \mathcal{A} \). Under these conditions, \( C \) becomes an infinite-distance quantum error-correcting code for \( \mathcal{A} \).\[11\]

In fact, infinite-distance behavior is associated with the existence of protected degrees of freedom supported by NSs of \( S \) \[1\]. Within the error-algebraic framework, a state space decomposition of the form \[10\] emerges through the reduction of \( \mathcal{A} \) into irreducible components \[1\]–\[3\]. Accordingly, \( \mathcal{H} \) can be identified with a fixed invariant subspace, \( \mathcal{D} \) with its orthogonal complement, and the noiseless factor \( \mathcal{L} \) carries an irreducible representation of the commutant \( \mathcal{A}' \) of \( \mathcal{A} \). Whenever information is protected using a NS, resetting of the syndrome \( \mathcal{Z} \) becomes unnecessary, hence no active intervention is required for maintaining information \[13\]. In the simplest instance, which is realized by a DFS, this happens because the syndrome state effectively does not evolve under the errors, thus the relevant syndrome subsystem is one-dimensional, immediately identifying \( \mathcal{L} \simeq C \) as an infinite-distance quantum error-correcting code. In a generic NS case, where both the logical and the syndrome factors \( \mathcal{L} \) and \( \mathcal{Z} \) are non-trivial subsystems, information encoded in \( \mathcal{L} \) is protected irrespective of the evolution experienced by \( \mathcal{Z} \). This implies that a proper NS is associated with a distinct infinite-distance quantum code for every reference state of the latter.

\section{III. Verification Setting}

In practice, taking advantage of the ability of a given code to protect information against errors in \( \mathcal{E} \) requires implementing a decoding procedure capable of restoring the information contained in the code after errors in \( \mathcal{E} \) occur. Let \( Q \) be the state space of the physical subsystem \( Q \) of \( S \) that carries the quantum information after decoding. To be specific, if \( S \) consists of \( n \) qubits, \( \dim Q = \dim \mathcal{L} = 2^k \), and we can treat the remaining \( n - k \) qubits as ancillae, with an associated state space \( H_a = (\mathbb{C}^2)^{\otimes(n-k)} \). The QEC procedure is then implemented by first appropriately initializing the ancillae state, next by transferring the resulting \( k \)-dimensional input space to the intended code \( C \) through an encoding operation, and then, after an error event happens, using a decoding operation to extract the state of \( Q \) \[14\].

Let \( U_d, U_s \) denote the experimentally implemented decoding and encoding operations, respectively. While the decoding is designed so as to provide a realization of the abstract mapping \( \omega \) given in Eq. \[11\], in practice, operational errors and inaccuracies will prevent one from exactly knowing the actual \( U_d \). Yet, by construction, \( U_d \) provides a subsystem identification of the form

\[ U_d : S \rightarrow Q \otimes Y \oplus R, \]

where \( Y \) is the state space of a physical syndrome degree of freedom carrying the effect of the errors, and \( R \) collects the states of \( S \) for which the extraction of the relevant information by \( U_d \) effectively fails. Eventually, one would like to claim that \[5\] realizes an error-correcting subsystem equivalent to \( C \), at least in the case where a unique subsystem with the correct behavior is known to exist for \( \mathcal{E} \) except for irrelevant unitary rotations in the underlying factors. But how can we actually verify that the subsystem identified by the implemented decoding is noiseless under the error model of interest, and under what conditions can we conclude that a desired NS has been realized?
Let us consider a verification setting defined by the following assumptions:

1) The ancillae are prepared in a known pure state, say $|0\rangle_a \in \mathcal{H}_a$.

2) The error model $\mathcal{E}$ is known.

3) The implemented decoding transformation, $U_d$, is unitary.

4) The initial state in the code $C$ is recovered perfectly for all $E \in \mathcal{E}$.

While the first requirement is always necessary for the implemented QEC procedure to be meaningful, the remaining conditions may or may not enter the definition of the setting in principle. None of the requirements can be rigorously fulfilled in actual implementations. Apart from the assumed perfect fidelity in both the initialization and the recovery steps, one could naturally encounter situations where either 2) or 3) (or both) would need to be relaxed to some extent. We focus here on the simplest verification scenario, having in mind device technologies capable to meet all the requirements 1) through 4) with sufficiently high accuracy. In particular, the present analysis is directly motivated by the recent experimental implementations of DFSs $^{14, 17, 18}$ and NSs $^{10, 11}$ using single-photon optics, trapped ions, and liquid-state NMR.

A. Verification for finite-distance codes

Let $\mathcal{E}$ be a generic linear error set and suppose that one has experimentally verified that information protected by the implemented code $C$ is recovered perfectly for an error basis $\{E_i\} = \{E_0 = \mathbb{I}, E_1, \ldots \}$ of $\mathcal{E}$ i.e., we have observed that

$$U_d(E_i|\psi\rangle_C) = |\psi\rangle_Q \otimes |\varphi_{E_i}\rangle_Y, \forall \ell,$$  

for arbitrary encoded states $|\psi\rangle_C$. Then stability under all error operators in $\mathcal{E}$ can be immediately inferred using linearity. Note that the existence of a non-trivial summand $\mathcal{R}$ in the correspondence $^{1}$ is signaled by the fact that the span $\{|\varphi_{E_i}\rangle_Y\}$ does not cover the full ancilla state space $\mathcal{H}_a$. In the assumed qubit setting, this means that the syndrome qubits may be in general a proper subset of the ancillary ones. From Eq. $^{3}$, one knows that, in particular, the $\mathbb{I}$ is correctable, namely

$$U_d(|\psi\rangle_C) = |\psi\rangle_Q \otimes |\varphi_r\rangle_Y, \quad (7)$$

for some reference state $|\varphi_r\rangle_Y \in \mathcal{Y}$. This makes two remarks possible: (i) since the prepared state $|0\rangle_a$ of the ancillae is pure, and $U_d$ is unitary, one can check to what extent the implemented $U_e$ is unitary by verifying the purity of the decoded state in Eq. $^{3}$. Suppose that based on this observation we can take $U_e$ to be unitary with high accuracy henceforth. (ii) In the identification provided by $U_d$, the encoding operation $U_e$ implies the initialization of the syndrome subsystem in the state $|\varphi_r\rangle_Y$, and the code $C$ may be represented as

$$C = U_d^{-1}(Q \otimes |\varphi_r\rangle_Y). \quad (8)$$

If desired, the accuracy to which $U_e$ avoids transferring unintended information to $\mathcal{R}$ may be checked by measuring the amplitude in states orthogonal to the span $\{U_d^{-1}(|\psi\rangle_Q \otimes |\varphi_{E_i}\rangle_Y)\}$. Finally, under the identification given by $U_d$, Eqs. $^{4}$ and $^{5}$ together imply that errors in $\mathcal{E}$ have an identity action on the logical subsystem when $\mathcal{Y}$ is initialized to $|\varphi_r\rangle_Y$ i.e., one can conclude that for all $E \in \mathcal{E}$

$$\hat{E}(|\psi\rangle_Q \otimes |\varphi_r\rangle_Y) = |\psi\rangle_Q \otimes |\hat{E}(\varphi_r)\rangle_Y, \forall |\psi\rangle_Q \in \mathcal{Q}, \quad (9)$$

or, equivalently,

$$U_d(E|\mathcal{C}) = \hat{E}Q \otimes |\varphi_r\rangle_Y = \mathbb{I}_Q \otimes \hat{E}, \quad (10)$$

with $\dagger$ denoting restriction, $\hat{E} = U_d E U_d^{-1}$, and $\hat{E}(\varphi_r) = |\hat{E}(\varphi_r)\rangle_Y$. This means that experimentally establishing Eq. $^{4}$ or $^{5}$ suffices for claiming that $C$ is a quantum $\mathcal{E}$-correcting code. Whenever a unique $k$-dimensional error-correcting code is known to exist for given $n$ and $\mathcal{E}$, the implemented code is effectively the one abstractly described by Eqs. $^{11}$-$^{2}$.

B. Verification for infinite-distance codes

In addition to having verified the validity of Eqs. $^{4}$-$^{10}$, assume now the stronger condition that the errors include a known, non-trivial error algebra $\mathcal{A}$, with $\mathcal{E} \supseteq \mathcal{A}$ and $\mathbb{I} \in \mathcal{A}$. Then the algebraic structure of $\mathcal{A}$ further enables one to infer a trivial action of errors on the logical subsystem when the syndrome subsystem is initialized to certain states other than $|\varphi_r\rangle_Y$. Let

$$\mathcal{V} = \{|\hat{E}(\varphi_r)\rangle_Y | E \in \mathcal{A}\} \subseteq \mathcal{Y} \quad (11)$$

denote the states of $\mathcal{Y}$ reachable from $|\varphi_r\rangle_Y$ under the effect of error operators in $\mathcal{A}$. Thus, $\mathcal{V}$ depends upon $|\varphi_r\rangle_Y$ and $\mathcal{A}$. The fact that $\mathcal{V}$ is a linear space follows from the property that, for $E_1, E_2 \in \mathcal{A}$ and for arbitrary complex $\alpha, \beta$, one may write

$$\alpha|\hat{E}_1(\varphi_r)\rangle_Y + \beta|\hat{E}_2(\varphi_r)\rangle_Y = |\hat{E}(\varphi_r)\rangle_Y, \quad (12)$$

with $E = \alpha E_1 + \beta E_2 \in \mathcal{A}$. We can then prove the following

**Theorem:** Let $\mathcal{E} \supseteq \mathcal{A}$, $\mathcal{A}$ being an error algebra on $S$, and let $\mathcal{V}$ be defined as above. Assume that stability under $\mathcal{E}$ has been verified as in Eq. $^{10}$. Then

$$U_d(\mathcal{A}|U_d^{-1}(Q \otimes \mathcal{V})\rangle) = \hat{A}Q \otimes \mathcal{V} \subseteq \mathbb{I}_Q \otimes \text{End}(\mathcal{V}) \quad (13)$$
Proof: We need to show that any error operator in $\mathcal{A}$ has no effect on $Q$ whenever the state of $Y$ is in $\mathcal{V}$. Let $|\psi\rangle_Y \in \mathcal{V}$. Then $|\psi\rangle_Y = \tilde{E}_a|\varphi_r\rangle_Y$ for some $\tilde{E}_a \in \mathcal{A}$. If $\tilde{E}_a$ and $|\psi\rangle_Q$ are any error operator in $\mathcal{A}$ and state in $\mathcal{Q}$, respectively, one has: $\tilde{E}_a|\psi\rangle_Q \otimes |\psi\rangle_Y = \tilde{E}_a|\varphi_r\rangle_Q \otimes \tilde{E}_b|\varphi_r\rangle_Y = \tilde{E}_a|\psi\rangle_Q \otimes \tilde{E}_b|\varphi_r\rangle_Y = |\psi\rangle_Q \otimes \tilde{E}_b|\varphi_r\rangle_Y$, for some $\tilde{E}_b = \tilde{E}_a \tilde{E}_b \in \mathcal{A}$.

According to the above Theorem, $V$ effectively determines the portion of the syndrome’s state space $\mathcal{V}$ relative to which noiselessness of $Q$ against $\mathcal{A}$ may be inferred from a verification procedure based on a fixed reference state $|\varphi_r\rangle_Y$ or, equivalently, a fixed encoding $U_e$. Because the error model is assumed to be known, the dimensionality of $\mathcal{V}$ may be inferred from the observed behavior of the syndrome subsystem upon decoding. Three different possibilities may arise:

- $1 = \dim(\mathcal{V}) < \dim(\mathcal{Y})$. This implies that $\mathcal{V} = \text{span}\{|\varphi\rangle_Y\}$ for a fixed state in $\mathcal{V}$ independent (up to an irrelevant phase factor) of the error operator in $\mathcal{A}$. Because $\mathcal{A}$ contains the $\mathbb{1}$, then $|\varphi\rangle_Y = |\varphi_Y\rangle_Y$, meaning that the state of the syndrome subsystem is invariant under $\mathcal{A}$. Having verified Eq. (10), one knows that $\mathcal{C}$ realizes an infinite-distance error-correcting code for $\mathcal{A}$. With $Q \otimes V \cong Q$ and $\text{End}(\mathcal{V}) \cong \mathbb{C}$, Eq. (13) reads

$$U_d(\mathcal{A}U_d^{-1}(Q)) = \mathcal{A}|Q \subseteq \mathbb{1}Q,$$

which is exactly the characterization of a DFS against $\mathcal{A}$ [11, 12, 20]. Thus, observing that information is robustly encoded against $E \supseteq \mathcal{A}$, and that $\mathcal{A}$ preserves the syndrome’s state, implies the verification of $Q$ as an infinite-distance DFS-code for $\mathcal{A}$. Note that establishing $Q$ as a (proper) NS under $\mathcal{A}$ would require verifying DFS-behavior for a set of linearly-independent reference states spanning $\mathcal{Y}$.

- $1 < \dim(\mathcal{V}) < \dim(\mathcal{Y})$. In this case, the above Theorem implies that verifying a trivial action of errors in $\mathcal{A}$ on $\mathcal{C}$ according to Eqs. (9)–(10) suffices for inferring a trivial action of $\mathcal{A}$ whenever the reference state of $\mathcal{Y}$ is an arbitrary state in $\mathcal{V}$. Thus, one can conclude that any quantum code $U_d^{-1}(Q \otimes |\varphi\rangle_Y)$, $|\varphi\rangle_Y \in \mathcal{V} \subseteq \mathcal{Y}$, provides infinite-distance error protection against $\mathcal{A}$. In a procedure, the procedure establishes $Q$ as a (proper) NS against $\mathcal{A}$ conditionally on initialization of $\mathcal{Y}$ in $\mathcal{V}$.

- $1 < \dim(\mathcal{V}) = \dim(\mathcal{Y})$. Because $\mathcal{V} \subseteq \mathcal{Y}$, one has $\mathcal{V} = \mathcal{Y}$, implying that every state in $\mathcal{Y}$ is effectively reachable from $|\varphi_Y\rangle_Y$ through the action of an error in $\mathcal{A}$. Under these circumstances, the verification of stability under $E$ implies via the Theorem that noiselessness can be inferred irrespective of the state of $\mathcal{Y}$. If a unique $k$-dimensional NS with dim $\mathcal{Y} = \dim \mathcal{Z}$ is known to exist, the procedure enables one to conclude that $Q$ is effectively the intended NS against $\mathcal{A}$.

IV. EXAMPLE

Let us briefly illustrate the above ideas on the simplest instance of a non-trivial quantum NS, which arises when a system of three qubits is used to protect a qubit in the presence of arbitrary collective noise [1, 13]. In this case, $S = C^a$, $A_S = \text{Mat}_8(C)$, $k = 1$, and the relevant subsystem decomposition [11] applies to the subspace $H_{1/2}$ of states carrying total spin angular momentum $J^Z = j(j + 1)$, $j = 1/2$. $\mathcal{L}$ and $\mathcal{Z}$ are both two-dimensional, with $\mathcal{L} = \text{span}\{|\ell\rangle_L | \ell = 0, 1\}$ and $\mathcal{Z} = \text{span}\{|j_z\rangle_Z | j_z = \pm 1/2\}$, $\ell$ and $j_z$ being a logical quantum number and the total $z$-angular momentum eigenvalue, respectively. The summand $D = H_{1/2} = H_{3/2}$ is the four-dimensional subspace of states symmetric under qubit exchange, corresponding to $j = 3/2$. Explicit realizations of the correspondence $\omega : S = \mathcal{L} \otimes \mathcal{Z} \otimes H_{3/2}$ are given in [11, 13, 14, 20].

$\mathcal{L}$ is designed as a NS against the collective error algebra $A_c$, which contains all possible permutation-invariant error operators. For three qubits, $A_c$ is a twenty-four-dimensional non-abelian sub-algebra of $A_S$, supporting $\mathcal{L}$ as a unique NS (up to unitary transformations). Practically relevant abelian sub-algebras of $A_c$ include $A_{z}, A_{\varphi}, A_{\psi}$, describing collective error processes about a fixed spatial axis. Each of the latter sub-algebras is linearly spanned by the set of four Hermitian Kraus operators describing a full-strength phase damping channel $E_u$ along the direction $u$, e.g., $A_{\varphi} = \text{span}\{K_u^x | a = 0, \ldots, 3\}$, where $E_u = E_u(\varphi_{\text{in}}) = \sum_{a=0}^3 K_u^x \varphi_{\text{in}} K_u^x$, and so forth. While complete expressions for $K_u^x | a = 0, \ldots, 3, u = x, y, z$ may be found in [11], the representation in the collective error-correcting subsystem decomposition is especially transparent. For collective $z$ errors, for instance, one obtains that $K_0^z = K_1^z = 0$, and

$$K_2^z |H_{1/2} = \mathcal{L} \otimes |+1/2\rangle_Z,$$

$$K_3^z |H_{1/2} = \mathcal{L} \otimes |-1/2\rangle_Z,$$

(15)

corresponding to full-strength phase damping on the syndrome subsystem $\mathcal{Z}$ alone. Similar representations hold for $u = x, y$ [11]. Let us also denote by $E_{\text{env}}$ a composite error process obtained by cascading $E_u$ and $E_v$ in sequence [11]. A set of operation elements for such a composite process can be constructed by multiplication of the sets describing the individual error components.

In experimental realizations of the above NS as in [10, 11], the implemented decoding $U_d$ effectively maps the abstract $\mathcal{L}, \mathcal{Z}$ degrees of freedom to a physical information-carrying qubit $Q$ and a physical syndrome qubit $\mathcal{Y}$, respectively. In the resulting identification, the initialization of the syndrome subsystem $\mathcal{Y}$ is typically constrained to a fixed state $|\varphi_Y\rangle_Y$ determined by the implemented encoding. In the setting of [10, 11], the presence of unintentional amplitude in the $H_{3/2}$ subspace is reflected in the state of the non-syndrome ancilla qubit upon decoding.
Various verification procedures for the intended NS may be considered depending on the experimentally available class of error processes. Suppose, for instance, that we have verified Eq. (11) under arbitrary single-axis collective errors, namely under the error set

$$\mathcal{E} = \text{span}\{K_x^a, K_y^b, K_z^c | a, b, c = 0, \ldots, 3\},$$

(16)
in terms of the above-mentioned collective Kraus operators. Suppose, in addition, that by looking at the behavior of the decoded syndrome qubit one is able to determine that \(\dim(\mathcal{V}_x) = \dim(\mathcal{V}_y) = 2\), whereas \(\dim(\mathcal{V}_z) = 1\). This effectively implies initialization of the system in a \(j_z\)-eigenstate, say \(|\varphi_r\rangle_y = |+1/2\rangle_y\). Thus, by using the Theorem, one can conclude that the decoded signal originates from a proper NS under \(\mathcal{A}_x\) and \(\mathcal{A}_y\), and from a DFS under \(\mathcal{A}_z\). However, by the same argument used in the proof of the Theorem, the fact that stability under the two error processes \(\mathcal{E}_x, \mathcal{E}_y\) has been verified irrespective of the initial syndrome state implies the possibility to enlarge the relevant error set to include arbitrary products of \(x, y\) error operators. This effectively enables one to deduce the validity of the condition (10) under a linear set \(\mathcal{E}' \supseteq \mathcal{E}\) larger than the one explicitly tested \(i.e.,\)

$$\mathcal{E}' = \text{span}\{K_x^a K_y^b, K_y^b K_x^a, K_z^a K_y^b, K_y^b K_z^a, K_z^a K_y^b K_z^a K_y^b\},$$

(17)

where errors of the form \(K_z^a K_y^b, K_z^a K_y^b, K_z^a K_y^b, K_z^a K_y^b\) are absent because stability under \(\mathcal{E}_z\) can only be assumed conditionally on the initial invariant state \(|+1/2\rangle_y\). Finally, one can show that \(\mathcal{E}' \supseteq \mathcal{A}_z\), hence by applying again the Theorem it is possible in fact to infer noiselessness of the implemented subsystem \(Q\) against the full \(\mathcal{A}_c\).

A second, more direct, verification procedure consists of checking stability of the encoded information under two composite, conjugate error processes \(\mathcal{E}_{uv}\) and \(\mathcal{E}_{vu}\), \(\mathcal{E}_{uv} = \mathcal{E}_{vu}^\dagger\), and by using the fact that the resulting error set,

$$\mathcal{E}'' = \text{span}\{K_x^a K_y^b, K_z^a K_y^b\},$$

(18)

again satisfies the property that \(\mathcal{E}'' \supseteq \mathcal{A}_c\). Finally, if DFS-behavior with initialization into the orthogonal state \(|\varphi_r\rangle_y = |−1/2\rangle_y\) is observed as well, then verification of robust behavior under \(\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z\) again directly translates into verification of the desired NS-behavior against \(\mathcal{A}_c\) via the Theorem.

VI. CONCLUSION

We have outlined verification procedures for quantum NSs in a simple experimentally motivated setting. As a main practical implication of our analysis, establishing a NS need not require the complete verification of the initial syndrome space provided that sufficient access to the final decoded states is available. This may be practically advantageous to avoid the need of checking different encodings for the same error model. Verification procedures designed under the assumptions of unitary decoding and known error behavior, as well as perfect fidelity as invoked throughout here, may be expected to remain valid if the relevant conditions can be met with sufficiently high accuracy. However, it is not \textit{a priori} obvious that procedures that are equivalent (as in the above NS Example) in such an idealized scenario will remain applicable and equally reliable when some of the assumptions are relaxed \(e.g.,\) implementation is not perfect. In general, identifying and characterizing verification procedures for quantum NSs and error-correcting codes under realistic constraints is an interesting issue which deserves further investigation.

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