A fast first-order optimization approach to elastoplastic analysis of skeletal structures

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Abstract It is classical that, when the small deformation is assumed, the incremental analysis problem of an elastoplastic structure with a piecewise-linear yield condition and a linear strain hardening model can be formulated as a convex quadratic programming problem. Alternatively, this paper presents a different formulation, an unconstrained nonsmooth convex optimization problem, and proposes to solve it with an accelerated gradient-like method. Specifically, we adopt an accelerated proximal gradient method, that has been developed for a regularized least squares problem. Numerical experiments show that the presented algorithm is effective for large-scale elastoplastic analysis. Also, a simple warm-start strategy can speed up the algorithm when the path-dependent incremental analysis is carried out.

Keywords Elastoplastic problem · incremental analysis · accelerated gradient scheme · proximal gradient method · FISTA

1 Introduction

It has been diversely recognized that the elastoplastic incremental analysis of solids and structures is very linked to theory and algorithms of optimization; see, e.g., [Maier and Munro (1982) for survey. If the small deformation is assumed, the incremental problem of an elastoplastic truss can be formulated as a linear complementarity problem (LCP) (De Donato and Maier 1976; Smith 1978; Kaneko 1979, 1980; Wakefield and Tin-Loi 1990; Tin-Loi and Xia 2001; Tangaranvong and Tin-Loi 2007, 2008). It is well known that, if the hardening modulus is nonnegative (i.e., if the strain softening is not considered), then this LCP can be recast as a (convex) quadratic programming (QP) problem [Maier 1968, 1970, Capurso and Maier 1970, Grierson et

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al. 1979; Spiliopoulos and Patsios 2010). We can solve a QP problem efficiently with a primal-dual interior-point method (Anjos and Lasserre 2012).

This paper attempts to shed new light on this classical problem in computational plasticity from perspective of a recently developed branch of numerical optimization. Namely, in this paper we examine a simpler gradient-based algorithm with acceleration. Instead of QP and LCP, we formulate the incremental problem as an unconstrained nonsmooth convex optimization problem.

Recently, accelerated, or “optimal” (Nesterov 2004), first-order methods have received considerable attention, particularly for solving large-scale optimization problems arising in image processing, regression, etc.; see, e.g., Beck and Teboulle (2009), Goldstein et al. (2014), and O’Donoghue and Candés (2015). Such a method converges in the objective value with rate $O(1/k^2)$, where $k$ is the iteration counter. Also, since it is basically a gradient-like method, the computation at each iteration is very cheap.

In this paper we show that the incremental problem of an elastoplastic truss can be solved efficiently with an accelerated first-order method. Specifically, we adopt an accelerated proximal gradient method (Beck and Teboulle 2009, Calafiore and El Ghaoui 2014, Parikh and Boyd 2014, O’Donoghue and Candés 2015). Computational effort at each iteration of the presented algorithm is dominated by cheap matrix-vector multiplications. When we adopt a variant of the Newton–Raphson method for elastoplastic analysis, a major difficulty is to find a consistent tangent stiffness matrix, because one cannot know a priori each structural element will undergo plastic loading or elastic unloading. As a common attribute among optimization approaches to elastoplastic analysis, the presented algorithm does not use the tangent stiffness matrix, and automatically detects whether each member will undergo plastic loading or elastic unloading. Moreover, unlike other popular efficient optimization algorithms, the presented algorithm does not use a linear-equations solver.

In the course of path-dependent quasi-static analysis, we solve the incremental problem repeatedly with varying the load parameter and updating the state variables. This means that we solve a series of closely related problems. Since the presented approach is based upon the unconstrained optimization formulation, it might possibly employ a simple warm-start strategy that uses the solution at the previous loading step as the initial solution for the present loading step. The effect of this warm-start strategy will be investigated through numerical experiments. In contrast, interior-point methods usually require some specific techniques for warm start; see, e.g., Mitchell (2001), Benson and Shanno (2007), John and Yıldırım (2008), and Yonekura and Kanno (2012).

The paper is organized as follows. Section 2 summarizes fundamentals of the incremental analysis of an elastoplastic truss. As the major contribution, section 3 presents an accelerated proximal gradient method for solving the incremental problem. Section 4 extends the method to a mixed model of isotropic hardening and kinematic hardening. An extension to a piecewise-linear hardening model is presented in section 5. Section 6 reports the results of numerical experiments. We conclude in section 7.

A few words regarding notation. We use $\top$ to denote the transpose of a vector or a matrix. For simplicity, we often write the $(n + m)$-dimensional column vector...
\((x^\top, y^\top)^\top\) consisting of \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\) as \((x, y)\). The Euclidean norm and the \(\ell_\infty\)-norm of \(x = (x_i) \in \mathbb{R}^n\) are denoted by \(\|x\| = \sqrt{x^\top x}\) and \(\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}\), respectively. For a closed convex function \(f : \mathbb{R}^n \to \mathbb{R}\), we define the proximal mapping of \(f\) by

\[
\text{prox}_f(x) = \arg\min_z \left\{ f(z) + \frac{1}{2} \|z - x\|^2 \right\}.
\]

We use \(\partial f(x) \subseteq \mathbb{R}^n\) to denote the subdifferential of \(f\) at a point \(x \in \mathbb{R}^n\). The signum function is denoted by \(\text{sgn}\), i.e.,

\[
\text{sgn}(s) = \begin{cases} 
1 & \text{if } s > 0, \\
0 & \text{if } s = 0, \\
-1 & \text{if } s < 0
\end{cases}
\]

for \(s \in \mathbb{R}\). For vectors \(x = (x_i) \in \mathbb{R}^n\) and \(z = (z_i) \in \mathbb{R}^n\), we use \(|x| \in \mathbb{R}^n\), \(\text{sgn}(x) \in \mathbb{R}^n\), and \(\max\{x, z\} \in \mathbb{R}^n\) to denote

\[
|x| = (|x_1|, \ldots, |x_n|)^\top,
\]

\[
\text{sgn}(x) = (\text{sgn}(x_1), \ldots, \text{sgn}(x_n))^\top,
\]

\[
\max\{x, z\} = (\max\{x_1, z_1\}, \ldots, \max\{x_n, z_n\})^\top.
\]

We use \(\text{diag}(x)\) to denote a diagonal matrix, the vector of diagonal components of which is \(x\).

2 Fundamentals of elastoplastic analysis

In this section we recall the quasi-static analysis of an elastoplastic truss and formulate the incremental problem; see, e.g., Simo and Hughes (1998), de Souza Neto et al. (2008), and Han and Reddy (2013) for fundamentals of computational plasticity.

Consider an elastoplastic truss in the two- or three-dimensional space. Throughout the paper we assume small deformation. We use \(m\) and \(d\) to denote the number of members and the number of degrees of freedom of the displacements, respectively. In this section we consider an isotropic hardening model; see section 4 for kinematic hardening.

Suppose that change in the external forces applied to the truss occurs quite slowly. Then the inertial term of the equations of motion becomes negligibly small. Therefore, we omit the inertial term when we construct the governing equations. The structural behavior modeled in this manner is neither static nor dynamic, and is referred to as quasi-static. We use the term “time” to stand for a parameter with respect to which the evolution process of the quasi-static behavior is described. This parameter, sometimes called pseudo-time, needs not correspond to the actual time, because the quasi-static behavior differs from the dynamic one. Suppose that we shall investigate quasi-static response of the truss within the time interval \([0, T]\). This time interval is subdivided into finitely many intervals. For a specific subinterval, denoted \([t, t + \Delta t]\),
the response at time \( t + \Delta t \) is found by applying the standard backward (or fully implicit) Euler scheme.

Let \( \mathbf{u} \in \mathbb{R}^d \) and \( \mathbf{f} \in \mathbb{R}^d \) denote the vector of displacements and the vector of external forces, respectively. It should be clear that these are values at time \( t + \Delta t \).

We attempt to compute \( \mathbf{u} \) when \( \mathbf{f} \) is specified. With the superscript \( (t) \) we denote the values of variables at time \( t \) (e.g., \( \mathbf{u}^{(t)} \) for the displacement), and with the prefix \( \Delta \) we denote the increments between time \( t \) and \( t + \Delta t \) (e.g., \( \Delta \mathbf{u} \) for the incremental displacement). The values at time \( t \) are supposed to be known, and hence \( \mathbf{u} \) is obtained by finding \( \Delta \mathbf{u} \).

Let \( c_i \) denote the elongation of member \( i \) \( (i = 1, \ldots, m) \). The compatibility relation between the incremental member elongation and the incremental displacements can be written in the form

\[
\Delta c_i = \mathbf{b}_i \top \Delta \mathbf{u},
\]

(1)

where \( \mathbf{b}_i \in \mathbb{R}^d \) is a constant vector. We decompose \( \Delta c_i \) additively as

\[
\Delta c_i = \Delta c_{ei} + \Delta c_{pi},
\]

(2)

where \( \Delta c_{ei} \) and \( \Delta c_{pi} \) are the elastic and plastic parts, respectively.

Let \( q_i \) denote the axial force of member \( i \) at time \( t + \Delta t \), which is written as

\[
q_i = q_i^{(t)} + \Delta q_i.
\]

(3)

The constitutive law is written in terms of the increments as

\[
\Delta q_i = k_i \Delta c_{ei},
\]

(4)

where the elongation stiffness, \( k_i > 0 \), is assumed to be constant. Specifically, we have \( k_i = E a_i/l_i \), where \( E \) is the Young modulus, \( a_i \) is the member cross-sectional area, and \( l_i \) is the undeformed member length. The force-balance equation between the external forces and member axial forces at time \( t + \Delta t \) can be written as

\[
\sum_{i=1}^{m} q_i \mathbf{b}_i = \mathbf{f}.
\]

(5)

The yield condition is given by

\[
|q_i| - R_i \leq 0,
\]

(6)

where \( R_i \) corresponds to the magnitude of yield axial force. Define \( \Delta \gamma_i \) by

\[
\Delta \gamma_i = |\Delta c_{pi}|,
\]

(7)

which is the integration of the plastic multiplier between time \( t \) and \( t + \Delta t \). Under the hypothesis of linear isotropic hardening, the evolution of \( R_i \) is written in the form

\[
R_i = R_i^{(t)} + h_i \Delta \gamma_i,
\]

(8)
where $h_i > 0$ is a constant called the isotropic hardening modulus. As usual, we postulate the normality flow rule, that is written as

\begin{align*}
q_i &= R_i \quad \Rightarrow \quad \Delta c_{pi} \geq 0, \\
q_i &= -R_i \quad \Rightarrow \quad \Delta c_{pi} \leq 0, \\
|q_i| &< R_i \quad \Rightarrow \quad \Delta c_{pi} = 0.
\end{align*}

(9a) \hspace{1cm} (9b) \hspace{1cm} (9c)

In other words, $q_i$ should satisfy

\[ q_i \in \arg\max_{\hat{q}_i} \left\{ \hat{q}_i \Delta c_{pi} \mid |\hat{q}_i| \leq R_i \right\}, \]

which is called the principle of maximum plastic work. Here, the objective function is the plastic work due to the incremental plastic elongation, and the constraint is the yield condition. Namely, this principle states that $q_i$ corresponding to $\Delta c_{pi}$ is the one maximizing the plastic work among the axial forces satisfying (6). This is the most fundamental and widely accepted hypothesis in the plasticity theory.

Figure 1 shows the relationship between $q_i$ and $\Delta c_i$ defined by (2), (3), (4), (6), (7), (8), and (9). At time 0, we usually suppose that $c_i(0) = q_i(0) = 0$ as depicted with a filled circle in Figure 1(a). Member $i$ undergoes plastic deformation between time 0 and $\Delta t$ if $q_i$ becomes greater than $R_i(0)$. If this is the case, then $R_i(t) > R_i(0)$ as shown in Figure 1(b).

The following is a key to our formulation.

**Proposition 1** Assume $R_i > 0$. Then $R_i$, $q_i$, $\Delta \gamma_i$, and $\Delta c_{pi}$ satisfy (6), (7), and (9) if and only if they satisfy

\[ R_i \geq |q_i|, \quad \Delta \gamma_i \geq -|\Delta c_{pi}|, \quad \begin{bmatrix} R_i \\ q_i \end{bmatrix}^\top \begin{bmatrix} -\Delta \gamma_i \\ -\Delta c_{pi} \end{bmatrix} = 0. \]

(10)

We omit the proof; see Yonekura and Kanno (2012, Proposition 3). The two inequalities in (10) can be viewed as the second-order cone constraints in the two-dimensional space. The equation in (10) is then understood as a complementarity condition over the second-order cones; see, e.g., Ben-Tal and Nemirovski (2001), Anjos and Lasserre (2012), and Kanno (2011) for fundamentals of second-order cone constraints and complementarity conditions.

We always have $R_i(0) > 0$ and we assume $h_i > 0$. This and (8) imply that the assumption made in Proposition 1 is satisfied at any time $t$. Accordingly, from (1), (2), (3), (4), (5), (8), and Proposition 1, the incremental problem to be solved can be
Formulated as

\[
\begin{align*}
\Delta c_e + \Delta c_p &= \mathbf{b}_i^T \Delta \mathbf{u}, & i &= 1, \ldots, m, \\
q_i &= q_i^{(l)} + k_i \Delta c_e, & i &= 1, \ldots, m, \\
\sum_{i=1}^m q_i \mathbf{b}_i &= f, \\
R_i^{(l)} + h_i \Delta \gamma_i &\geq |q_i|, & \Delta \gamma_i &\geq |\Delta c_p|, \\
\left[ R_i^{(l)} + h_i \Delta \gamma_i \right]^T \left[ \begin{array}{c} \Delta \gamma_i \\ -\Delta c_p \end{array} \right] &= 0, & i &= 1, \ldots, m.
\end{align*}
\]  

(11a) (11b) (11c) (11d)

It should be clear in (11) that \( \Delta \mathbf{u}, \Delta c_e, \Delta c_p, \Delta \gamma, q_i \) \((i = 1, \ldots, m)\) are variables to be found. Problem (11) is a second-order cone linear complementarity problem (SOCLCP). It is known that SOCLCP and the second-order cone programming (SOCP) have diverse applications in applied mechanics, including frictional contact. 

**Fig. 1** The constitutive law at (a) time \( t = 0 \); and (b) time \( t = t \), after plastic deformation has taken place.
2011; Kanno et al. 2006), cable networks (Kanno et al. 2002), and elastoplastic continua (Bisbos et al. 2005; Makrodimitopoulos 2006; Krabbenhøft et al. 2007a; Krabbenhøft and Lyamin 2012; Yonekura and Kanno 2012).

Remark 1 It is known that the incremental problem of an elastoplastic truss can be formulated as a linear complementarity problem (LCP); see, e.g., De Donato and Maier (1976), Kaneko (1979), Smith (1978), Tin-Loi and Xia (2001), and Wakefield and Tin-Loi (1990). Indeed, it is possible to recast (11) as an LCP by splitting variables as

\[ \Delta c^{+}_{pi} = \Delta c^{+}_{pi} + \Delta c^{-}_{pi} \]

and

\[ \Delta \gamma_{i} = \Delta c^{+}_{pi} - \Delta c^{-}_{pi} \]

with \( \Delta c^{+}_{pi} \geq 0 \) and \( \Delta c^{-}_{pi} \geq 0 \) and replacing the complementarity conditions by

\[ \Delta c^{+}_{pi}(q_{i} - R_{i}) = 0 \text{ and } \Delta c^{-}_{pi}(q_{i} + R_{i}) = 0. \]

The resulting LCP has 2m complementarity conditions, while SOCLCP (11) has m complementarity conditions (over the second-order cones). It has been well recognized that this LCP can be recast as (convex) quadratic programming (QP); see, e.g., Maier (1968, 1970), Capurso and Maier (1970), and Grierson et al. (1979). In contrast, the formulations presented below are based upon SOCLCP (11).

A moment’s consideration will show that (11) corresponds to the optimality condition of the following convex optimization problem:

Minimize

\[ \sum_{i=1}^{m} \left( q_{i}^{(t)} \Delta c_{ei} + \frac{1}{2} k_{i} \Delta c_{ei}^{2} \right) + \sum_{i=1}^{m} \left( R_{i}^{(t)} \Delta \gamma_{i} + \frac{1}{2} h_{i} \Delta \gamma_{i}^{2} \right) - f^{T} \Delta u \]  

subject to

\[ \Delta c_{ei} + \Delta c_{pi} = b_{i}^{T} \Delta u, \quad i = 1, \ldots, m, \]  

\[ \Delta \gamma_{i} \geq |\Delta c_{pi}|, \quad i = 1, \ldots, m. \]

It is worth noting that this problem is a variant of the total potential energy minimization formulation.

Problem (12) can be recast as (convex) QP and SOCP; see appendix A for reduction to SOCP. Therefore, it can be solved efficiently with a primal-dual interior-point method (Anjos and Lasserre 2012; Ben-Tal and Nemirovski 2001). As an alternative approach, in this paper we examine a simpler gradient-based algorithm with acceleration.

Remark 2 The formulations presented above can readily be extended to the case in which the tension and compression yield conditions are not symmetric. Suppose that the yield condition is given by

\[ q_{i}^{l} \leq q_{i} \leq q_{i}^{u}, \]

where \( q_{i}^{l} \) and \( q_{i}^{u} \) are constants. This condition is equivalent to

\[ |q_{i} - \bar{\beta}_{i}| \leq R_{i} \]

with \( \bar{\beta}_{i} := (q_{i}^{l} + q_{i}^{u})/2 \) and \( R_{i} := (q_{i}^{u} - q_{i}^{l})/2 \). Here, \( \bar{\beta}_{i} \) and \( R_{i} \) correspond to the center and radius of the yield surface, respectively. This modification from (6) can be realized by adding

\[ \sum_{i=1}^{m} \bar{\beta}_{i} \Delta c_{pi} \]

to the objective function of problem (12). A similar problem setting appears in section A.
3 Accelerated proximal gradient method for elastoplastic analysis

In section 3.1, we reformulate the incremental problem as a form which is tractable within the framework of (accelerated) proximal gradient methods. A proximal gradient method and its accelerated version for this problem are presented in section 3.2 and section 3.3, respectively.

3.1 Unconstrained formulation of minimum potential energy problem

In this section we recast problem (12) as an unconstrained form.

Since \( R_i(t) > 0 \) and \( h_i > 0 \) (\( i = 1, \ldots, m \)), all the inequality constraints of problem (12) become active at the optimal solution. Therefore, by using \( \Delta \gamma_i = |\Delta c_{pi}| \) we can eliminate \( \Delta \gamma_i \) as follows:

\[
\begin{align*}
\text{Minimize} & \quad m \sum_{i=1}^{m} \left( q_i(t) \Delta c_{ei} + \frac{1}{2} k_i \Delta c_{ei}^2 \right) + \frac{1}{2} h_i \Delta c_{pi}^2 \quad \text{subject to} \quad \Delta c_{ei} + \Delta c_{pi} = b_i^\top \Delta u, \quad i = 1, \ldots, m. \\
& \quad \text{(13a)}
\end{align*}
\]

Furthermore, by substituting the equality constraints to the objective function, we can eliminate \( \Delta c_{ei} \) from (13) as

\[
\begin{align*}
\text{Minimize} & \quad m \sum_{i=1}^{m} \left[ q_i(t) (b_i^\top \Delta u - \Delta c_{pi}) + \frac{1}{2} k_i (b_i^\top \Delta u - \Delta c_{pi})^2 \right] \\
& \quad + \frac{1}{2} h_i \Delta c_{pi}^2 - f^\top \Delta u. \\
& \quad \text{(14)}
\end{align*}
\]

This is an unconstrained convex optimization problem.

In the following, for notational simplicity, we write problem (14) as

\[
\begin{align*}
\text{Minimize} & \quad m \sum_{i=1}^{m} \left[ q_i(t) (v - p_i) + \frac{1}{2} k_i (v - p_i)^2 \right] \\
& \quad + \frac{1}{2} h_i p_i^2 - f^\top v. \\
& \quad \text{(15)}
\end{align*}
\]

with \( v := \Delta u \) and \( p_i := \Delta c_{pi} \) (\( i = 1, \ldots, m \)). We propose to solve problem (15) by applying an accelerated proximal gradient method.

Remark 3 Problem (15) has a form very similar to the \( \ell_1 \)-regularized least-squares problem, known as the LASSO (Tibshirani 1996, 2011). The LASSO solves

\[
\begin{align*}
\text{minimize} & \quad \|Ax - b\|^2 + \kappa \sum_{j=1}^{n} |x_j|, \\
& \quad \text{(16)}
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is a variable to be optimized and \( \kappa > 0 \) is a constant regularization parameter. A class of proximal gradient methods for solving problem (16) is known as ISTA (iterative shrinkage-thresholding algorithm); see, e.g., Chambolle et al. (1998).
Figueiredo and Nowak (2003), Daubechies et al. (2004), and Combettes and Wajs (2005). An accelerated version of ISTA due to Beck and Teboulle (2009) is called FISTA (fast iterative shrinkage-thresholding algorithm). This paper is motivated by similarity between problems (15) and (16); actually the algorithm presented in section 3.3 is considered essentially an application of FISTA to problem (15). To the best of the author’s knowledge, problem (15) has not been used in literature on computational mechanics.

3.2 Proximal gradient method

In this section we present a proximal gradient method for solving problem (15), which prepares the accelerated version appearing in section 3.3.

Define $g_1 : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ and $g_2 : \mathbb{R}^m \to \mathbb{R}$ by

$$ g_1(v, p) = \sum_{i=1}^{m} \left[ q_i^t (b_i^\top v - p_i) + \frac{1}{2} k_i (b_i^\top v - p_i)^2 \right] + \sum_{i=1}^{m} \frac{1}{2} h_i p_i^2 - f^\top v, \quad (17) $$

$$ g_2(p) = \sum_{i=1}^{m} R_i^t |p_i|, \quad (18) $$

which are closed proper convex functions. Particularly, $g_1$ is differentiable, and $\nabla g_1$ is Lipschitz continuous. We use $L$ to denote the Lipschitz constant of $\nabla g_1$. By making use of $g_1$ and $g_2$, problem (15) can be written as

$$ \minimize g_1(v, p) + g_2(p). \quad (19) $$

A point $(v^*, p^*) \in \mathbb{R}^d \times \mathbb{R}^m$ is optimal for problem (19) if and only if

$$ 0 = \nabla_v g_1(v^*, p^*), \quad (20) $$

$$ 0 \in \nabla_p g_1(v^*, p^*) + \partial g_2(p^*), \quad (21) $$

where

$$ \nabla_v g_1 = \frac{\partial g_1}{\partial v}, \quad \nabla_p g_1 = \frac{\partial g_1}{\partial p}. $$

For any $\alpha > 0$, (20) and (21) hold if and only if the following equalities hold:

$$ v^* = v^* - \alpha \nabla_v g_1(v^*, p^*), \quad (22) $$

$$ p^* = \text{prox}_{\alpha g_2}(p^* - \alpha \nabla_p g_1(v^*, p^*)). \quad (23) $$

Equivalence of (21) and (23) follows from fundamental properties of the proximal mapping (Parikh and Boyd 2014); see appendix B for more accounts. The proximal gradient method can be designed from (22) and (23) as follows; see, e.g., Calafiore and El Ghaoui (2014) and Parikh and Boyd (2014).

Algorithm 1

**Step 0:** Choose $v_0 \in \mathbb{R}^d$, $p_0 \in \mathbb{R}^m$, $\alpha \in [0, 1/L]$, and the termination tolerance $\varepsilon > 0$. Set $l := 0$. 

9
Step 1: Let
\[ \begin{align*}
    v_{l+1} &:= v_l - \alpha \nabla g_1(v_l, p_l), \\
    p_{l+1} &:= \text{prox}_{\alpha g_2}(p_l - \alpha \nabla g_1(v_l, p_l)).
\end{align*} \]

Step 2: If \( \| (v_l, p_l) - (v_{l+1}, p_{l+1}) \| \leq \varepsilon \), then terminate. Otherwise, let \( l \leftarrow l + 1 \), and go to step 1.

3.3 Accelerated proximal gradient method with restart

The convergence analysis of Algorithm 1 can be found in literature, e.g., Calafiore and El Ghaoui (2014) and Parikh and Boyd (2014). It is known that \( g_1(v_l, p_l) + g_2(p_l) \) converges to the optimal value at rate \( \frac{1}{l} \). In this section we introduce the so-called FISTA-type improvement, due to Beck and Teboulle (2009), to achieve an accelerated convergence rate of order \( \frac{1}{l^2} \). Also we incorporate the adaptive restart scheme proposed by O’Donoghue and Candès (2015) to ensure monotonic decrease of the objective function value.

The following is an accelerated proximal gradient method with adaptive restart for the incremental elastoplastic analysis.

**Algorithm 2**

**Step 0:** Choose \( v_0 \in \mathbb{R}^d \), \( p_0 \in \mathbb{R}^m \), \( \alpha \in [0, 1/L] \), and the termination tolerance \( \varepsilon > 0 \). Set \( l := 1, \mu_1 := v_0, \rho_1 := p_0 \), and \( \tau_1 := 1 \).

**Step 1:** Let
\[ \begin{align*}
    v_l &:= \mu_l - \alpha \nabla g_1(\mu_l, \rho_l), \\
    p_l &:= \text{prox}_{\alpha g_2}(\rho_l - \alpha \nabla g_1(\mu_l, \rho_l)).
\end{align*} \]

**Step 2:** Let
\[ \tau_{l+1} := \frac{1}{2} \left( 1 + \sqrt{1 + 4\tau_l^2} \right). \]

**Step 3:** If \( g_1(v_l, p_l) + g_2(p_l) < g_1(v_{l-1}, p_{l-1}) + g_2(p_{l-1}) \), then let
\[ \begin{align*}
    \mu_{l+1} &:= v_l + \frac{\tau_l - 1}{\tau_{l+1}} (v_l - v_{l-1}), \\
    \rho_{l+1} &:= p_l + \frac{\tau_l - 1}{\tau_{l+1}} (p_l - p_{l-1}).
\end{align*} \]
Otherwise, let
\[ \begin{align*}
    \tau_{l+1} &:= 1, \\
    \mu_{l+1} &:= v_l, \\
    \rho_{l+1} &:= p_l.
\end{align*} \]

**Step 4:** If \( \| (v_l, p_l) - (v_{l-1}, p_{l-1}) \| \leq \varepsilon \), then terminate. Otherwise, let \( l \leftarrow l + 1 \), and go to step 1.
Remark 4 If we replace step 2 with $\tau_l + 1 = 1$, then Algorithm 2 reverts to Algorithm 1.

Computation of step 1 can be carried out in an efficient manner as follows. We begin with computation of the proximal mapping of $\alpha g_2$ with $\alpha > 0$, which is defined as

$$\text{prox}_{\alpha g_2}(w) = \arg\min_z \left\{ \alpha \sum_{i=1}^{m} R^{(l)}_i |z_i| + \frac{1}{2} \|z - w\|^2 \right\}. \quad (24)$$

Since we have that

$$\min_z \left\{ \alpha \sum_{i=1}^{m} R^{(l)}_i |z_i| + \frac{1}{2} \|z - w\|^2 \right\} = \min_{z_i} \left\{ \alpha R^{(l)}_i |z_i| + \frac{1}{2} (z_i - w_i)^2 \right\}, \quad (25)$$

the optimal $z$ in the right-hand side of (24) can be found by solving the set of the one-dimensional optimization problems in the right-hand side of (25). Then, it is known that the optimal solution for each $i$ can be obtained via the soft-threshold function (also known as the shrinkage operator) as

$$\arg\min_{z_i} \left\{ \alpha R^{(l)}_i |z_i| + \frac{1}{2} (z_i - w_i)^2 \right\} = \begin{cases} 0 & \text{if } |w_i| \leq \alpha R^{(l)}_i, \\ w_i - \alpha R^{(l)}_i \text{sgn}(w_i) & \text{otherwise}; \end{cases} \quad (26)$$

see, e.g., Combettes and Wajs (2005), Beck and Teboulle (2009), Calafiore and El Ghaoui (2014), and Parikh and Boyd (2014). Consequently, we obtain

$$\text{prox}_{\alpha g_2}(w) = \text{diag}(\text{sgn}(w)) \max\{|w| - \alpha R^{(l)}, 0\}. \quad (27)$$

We next consider computation of $\nabla g_1$. Define $B \in \mathbb{R}^{m \times d}$ by

$$B = \begin{bmatrix} b_1^\top \\ \vdots \\ b_m^\top \end{bmatrix}, \quad (28)$$

which is the compatibility matrix. By using this notation, definition (17) of $g_1$ yields

$$\nabla_{v} g_1(v, p) = B^\top q^{(l)} + B^\top \text{diag}(k)Bv - B^\top \text{diag}(k)p - f, \quad (29)$$

$$\nabla_{p} g_1(v, p) = -q^{(l)} + \text{diag}(k)(p - Bv) + \text{diag}(h)p. \quad (30)$$

For computing (29) and (30), it is convenient to define $e \in \mathbb{R}^m$ by

$$e = Bv - p. \quad (31)$$

It is worth noting that $e$ corresponds to the incremental elastic elongation, $\Delta e$, in problem (12). By using $e$, (29) and (30) can be calculated as

$$\nabla_{v} g_1(v, p) = B^\top (\text{diag}(k)e + q^{(l)}) - f, \quad (32)$$

$$\nabla_{p} g_1(v, p) = \text{diag}(h)p - \text{diag}(k)e - q^{(l)}. \quad (33)$$
Consequently, by using (27), (31), (32), and (33), we see that computation at step 1 of Algorithm 2 can be performed as follows:

\[ \varepsilon_l := B\mu_l - p_l, \]  
\[ v_l := \mu_l - \alpha[B^\top (\text{diag}(k)\varepsilon_l + q^{(i)}) - f], \]  
\[ w_l := p_l - \alpha(\text{diag}(h)p_l - \text{diag}(k)\varepsilon_l - q^{(i)}), \]  
\[ p_l := \text{diag}(\text{sgn}(w_l)) \max\{|w_l| - \alpha R_l^i, 0\}. \]

Here, \( \varepsilon_l \) and \( w_l \) are auxiliary variables.

**Remark 5** The most expensive part of Algorithm 2 is computation at step 1. As seen in (34), (35), (36), and (37), this essentially amounts to two matrix-vector products and four component-wise vector products. Here, \( B \) is a sparse matrix, and hence the two matrix-vector products may exploit this sparsity effectively.

**Remark 6** Algorithm 2 does not contain any process of solving a system of linear equations. Therefore, Algorithm 2 does not require any linear-equations solver. If a conventional method for elastoplastic analysis is applied to large-scale problems, then a linear-equations solver dominates the computational cost. Hence, usually iterative methods are used for solving the equilibrium equation with the tangent stiffness matrix. Also, parallel computing, such as domain decomposition methods (Nineb et al. 2007; Cermák et al. 2014), is often required. When we adopt an approach based upon mathematical programming, an interior-point method solves a system of linear equations to find the search direction at each iteration. Hence, to solve a large-scale problem an iterative solver is usually employed for computation of the search direction; see, e.g., [Johnson et al. 2000], [Portugal et al. 2000], [Kim et al. 2007], and [Bergamaschi et al. 2007]. In contrast, Algorithm 2 does not use a linear-equations solver at all. In other words, Algorithm 2 is explicit (and also simple), although it solves a problem discretized with a fully implicit Euler scheme.

**Remark 7** Like other approaches based upon mathematical programming, Algorithm 2 does not resort to a consistent tangent stiffness matrix. Moreover, it does not require any procedure to determine whether each member undergoes plastic loading or elastic unloading.

At step 0 of Algorithm 2, we can determine the step size, \( \alpha \), as follows. From (29) and (30), the Hessian matrix of \( g_1 \) can be obtained as:

\[ \nabla^2 g_1(v, p) = \begin{bmatrix} B^\top \text{diag}(k)B & -B^\top \text{diag}(k) \\ -\text{diag}(k)B & \text{diag}(k) + h \end{bmatrix} \]

\[ = \begin{bmatrix} B^\top O & \text{diag}(k) \\ -I & O \end{bmatrix} \begin{bmatrix} O & \text{diag}(h) \\ O & I \end{bmatrix} = \begin{bmatrix} B & -I \end{bmatrix}. \]

Recall that \( k_i > 0 \) and \( h_i > 0 \) (\( i = 1, \ldots, m \)). Moreover, for a stable (more precisely, kinematically determinate) truss, \( B \) is of row full rank. Therefore, from (39) we see that \( \nabla^2 g_1(v, p) \) is positive definite, which implies that \( g_1 \) is strongly convex. Furthermore, the maximum eigenvalue of \( \nabla^2 g_1(v, p) \) is equal to \( L \), i.e., the Lipschitz
constant of $\nabla g_1$. One obvious choice for determining $\alpha$ is, therefore, to find the maximum eigenvalue of the matrix in (38) and set $\alpha := 1/L$. Another choice is to find an upper bound for $L$ that can be performed much faster than its exact value. For notational convenience, let $H = (H_{ij}) = \nabla^2 g_1(\mathbf{v}, p)$. It follows from the Gershgorin disc theorem that $L'$ defined by

$$L' = \max \left\{ H_{ii} + \sum_{j \neq i} |H_{ij}| \right\}$$

(40)

satisfies $L' \geq L$; see, e.g., Horn and Johnson (2013, Theorem 6.1.1). Then we may set $\alpha := 1/L'$.

### 4 Mixed isotropic/kinematic hardening

In this section we consider a plasticity model that combines linear isotropic hardening and linear kinematic hardening.

We begin by formulating the incremental problem. To incorporate the kinematical hardening, the yield condition, (6), is replaced with

$$|q_i - \beta_i| \leq R_i.$$ 

Here, $\beta_i \in \mathbb{R}$ is an internal force corresponding to the back stress. Let $\theta \in [0, 1]$ be a constant. The evolutions of $R_i$ and $\beta_i$ are given by

$$R_i = R_i^{(t)} + \theta h_i \Delta \gamma_i,$$

(41)

$$\beta_i = \beta_i^{(t)} + (1 - \theta) h_i \Delta c_{pi}.$$ 

(42)

Here, $\theta$ is the ratio of the effect of isotropic hardening to the total strain hardening. Particularly, $\theta = 1$ corresponds to the pure isotropic hardening, and $\theta = 0$ corresponds to the pure kinematic hardening. Thus, (8) in section 2 is replaced with (41), and (42) is newly added. Consequently, the incremental problem can be formulated as

$$\Delta c_{ei} + \Delta c_{pi} = \mathbf{b}^T \Delta \mathbf{u}, \quad i = 1, \ldots, m,$$

(43a)

$$q_i = q_i^{(t)} + k_i \Delta c_{ei}, \quad i = 1, \ldots, m,$$

(43b)

$$\beta_i = \beta_i^{(t)} + (1 - \theta) h_i \Delta c_{pi}, \quad i = 1, \ldots, m,$$

(43c)

$$\sum_{i=1}^m q_i \mathbf{b}_i = \mathbf{f},$$

(43d)

$$R_i^{(t)} + \theta h_i \Delta \gamma_i \geq |q_i - \beta_i|, \quad \Delta \gamma_i \geq |\Delta c_{pi}|,$$

(43e)

Like (11) in section 2, this is an SOCLCP.
It is easy to verify that (43) is the optimality condition of the following convex optimization problem:

Minimize \( \sum_{i=1}^{m} \left( q_i^{(i)} \Delta c_{ei} + \frac{1}{2} k_i \Delta c_{ei}^2 \right) \)
\[ + \sum_{i=1}^{m} \left[ R_i^{(i)} \Delta \gamma_i + \frac{1}{2} \theta h_i \Delta \gamma_i^2 + \beta_i^{(i)} \Delta c_{pi} + \frac{1}{2} (1 - \theta) h_i \Delta c_{pi}^2 \right] \]
\[ - f^\top \Delta u \] (44a)

subject to \( \Delta c_{ei} + \Delta c_{pi} = b_i^\top \Delta u, \quad i = 1, \ldots, m, \) (44b)
\( \Delta \gamma_i \geq |\Delta c_{pi}|, \quad i = 1, \ldots, m. \) (44c)

Since \( R_i^{(i)} > 0 \) and \( \theta h_i \geq 0 \) \( (i = 1, \ldots, m) \), the constraints in (44c) become active at the optimal solution. Therefore, without changing the optimal solution we can substitute \( \Delta c_{pi}^2 = \Delta \gamma_i^2 \) into (44a). This results in

\[ \sum_{i=1}^{m} \left( q_i^{(i)} \Delta c_{ei} + \frac{1}{2} k_i \Delta c_{ei}^2 \right) + \sum_{i=1}^{m} \left( R_i^{(i)} \Delta \gamma_i + \frac{1}{2} h_i \Delta \gamma_i^2 + \beta_i^{(i)} \Delta c_{pi} + \frac{1}{2} (1 - \theta) h_i \Delta c_{pi}^2 \right) - f^\top \Delta u, \] (45)

which is similar to (12a). More precisely, the difference is only the presence of \( \beta_i^{(i)} \Delta c_{pi} \) \( (i = 1, \ldots, m) \). Therefore, Algorithm 2 can be applied in a very similar manner. Namely, we just replace (36) used at step 1 with

\[ w_i := \rho_i - \alpha (\text{diag}(h) p_i + \beta^{(i)} - \text{diag}(k) e_i - q^{(i)}) \]

to find the solution. Subsequently, \( R_i^{(i)} \) and \( \beta_i^{(i)} \) should be updated according to (41) and (42) for computation of the next time increment.

5 Piecewise-linear hardening

In this section we consider an accelerated proximal gradient method for a piecewise-linear hardening model. It is worth noting that QP formulations for piecewise-linear model is known in literature, e.g., [Maier 1968].

Suppose that the evolution of \( R_i \) with respect to the plastic multiplier, is given as illustrated in Figure 2. That is, when \( R_i \) attains at \( R_i^f \), which is a given positive constant, the hardening modulus decreases from \( h_{1i} \) to \( h_{2i} \), where \( h_{1i} > 0 \) and \( h_{2i} \in [0, h_{1i}] \) are given constants. Under this hypothesis we formulate a quasi-static incremental problem.

Recall that, in section 2 we have decomposed the incremental elongation by (2). Instead, in this section we consider the following decomposition:

\[ \Delta c_i = \Delta c_{ei} + \Delta c_{pi} + \Delta c_{si}, \] (46)
Here, $\Delta c_{\text{pl}}$ is a variable used to assess the plastic elongation after the axial force attains $R_s^i$; see (48) and (53) for more precise interpretation. Define $\Delta \gamma_1$ and $\Delta \gamma_2$ by

\[ \Delta \gamma_1 = | \Delta c_{\text{pl}} |, \]
\[ \Delta \gamma_2 = | \Delta c_{\text{pl}} |. \]

The evolution of $R_i$ defined as Figure 2 can be written in terms of $\Delta \gamma_1$ and $\Delta \gamma_2$ as

\[ R_i \leq R_s^i \Rightarrow R_i = R_s^i + h_1 \Delta \gamma_1, \quad \Delta \gamma_2 = 0, \]
\[ R_i > R_s^i \Rightarrow R_i = R_s^i + h_2 (\Delta \gamma_1 + \Delta \gamma_2). \]

For simplicity, define $\eta_i$ by

\[ \eta_i = \frac{h_1 h_2}{h_{11} - h_{12}}, \]

which is a positive constant. A moment’s consideration will show that (49) and (50) are equivalent to

\[ R_i = R_s^i + h_1 \Delta \gamma_1 + \eta_i \Delta \gamma_2 \]

and

\[ R_i \leq R_s^i \Rightarrow \Delta \gamma_2 = 0, \]
\[ R_i > R_s^i \Rightarrow R_i = R_s^i + \eta_i \Delta \gamma_2. \]

It can be readily verified that (48), (52), and (53) are equivalent to the following second-order cone complementarity condition.

\[ \| R_i - R_s^i \| + \eta_i \Delta \gamma_2 \leq 0, \quad \Delta \gamma_2 \geq 0. \]
Proposition 2 Assume $R_i > 0$. Then $R_i, \Delta y_{i2},$ and $\Delta c_{si}$ satisfy

\begin{align*}
R_i \leq R_i^0 & \Rightarrow \Delta y_{i2} = \Delta c_{si} = 0, \\
R_i > R_i^0 & \Rightarrow \Delta y_{i2} = |\Delta c_{si}|, \ R_i = R_i^0 + \eta_i \Delta y_{i2}
\end{align*}

if and only if they satisfy

\[ R_i^0 + \eta_i \Delta y_{i2} \geq |R_i|, \quad \Delta y_{i2} \geq |\Delta c_{si}|, \quad \left[ R_i^0 + \eta_i \Delta y_{i2} \right] \left[ \begin{array}{l} \Delta y_{i2} \\ \Delta c_{si} \end{array} \right] = 0. \]

Proposition 1 and Proposition 2 combine to give the following formulation of the incremental problem:

\begin{align*}
\Delta c_{si} + \Delta c_{pi} + \Delta c_{ci} &= b_i^T \Delta u, \quad i = 1, \ldots, m, \quad (54a) \\
q_i &= q_i^{(i)} + k_i \Delta c_{ci}, \quad i = 1, \ldots, m, \quad (54b) \\
\sum_{i=1}^m q_i b_i &= f, \quad (54c) \\
R_i^{(i)} + h_{1i} \Delta y_{1i} &\geq |q_i|, \quad \Delta y_{1i} \geq |\Delta c_{pi}|, \quad (54d) \\
R_i^0 + \eta_i \Delta y_{i2} &\geq |q_i|, \quad \Delta y_{i2} \geq |\Delta c_{si}|, \quad (54e)
\end{align*}

We can show that (54) corresponds to the optimality condition of the following convex optimization problem:

\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^m \left( q_i^{(i)} \Delta c_{si} + \frac{1}{2} k_i \Delta c_{ci}^2 \right) \\
& \quad + \sum_{i=1}^m \left( R_i^{(i)} \Delta y_{1i} + \frac{1}{2} h_{1i} \Delta y_{1i}^2 + R_i^0 \Delta y_{i2} + \frac{1}{2} \eta_i \Delta y_{i2}^2 \right) \\
& \quad - f^T \Delta u \quad (55a) \\
\text{subject to} & \quad \Delta c_{si} + \Delta c_{pi} + \Delta c_{ci} = b_i^T \Delta u, \quad i = 1, \ldots, m, \quad (55b) \\
& \quad \Delta y_{1i} \geq |\Delta c_{pi}|, \quad i = 1, \ldots, m, \quad (55c) \\
& \quad \Delta y_{i2} \geq |\Delta c_{si}|, \quad i = 1, \ldots, m. \quad (55d)
\end{align*}

In a manner similar to section 3.1, we can recast problem (55) as an unconstrained nonsmooth convex optimization problem. Firstly, since $R_i^{(i)} > 0, R_i^0 > 0, h_{1i} > 0,$ and $\eta_i > 0 (i = 1, \ldots, m)$, all the inequality constraints of problem (55) become active at an optimal solution. Therefore, $\Delta y_{1i}$ and $\Delta y_{i2}$ can be eliminated. Next, by making use
of the equality constraints, we can eliminate $\Delta c_{gi}$. As a result, we see that problem (55) is equivalent to

$$\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} \left( q_i^{(i)}(b_i^T \Delta u - \Delta c_{pi} - \Delta c_{ui}) + \frac{1}{2} \eta_i (b_i^T \Delta u - \Delta c_{pi} - \Delta c_{ui})^2 \right) \\
& + \sum_{i=1}^{m} \left( R_i^{(i)} |\Delta c_{pi}| + \frac{1}{2} h_i |\Delta c_{ui}| + \frac{1}{2} \eta_i |\Delta c_{ui}| + \frac{1}{2} \eta_i |\Delta c_{ui}| - f^T \Delta u. \right)
\end{align*}$$

(56)

For notational simplicity, we write problem (56) as

$$\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} \left( q_i^{(i)}(b_i^T v - p_i - s_i) + \frac{1}{2} \eta_i (b_i^T v - p_i - s_i)^2 \right) \\
& + \sum_{i=1}^{m} \left( R_i^{(i)} |p_i| + \frac{1}{2} h_i |s_i| + \frac{1}{2} \eta_i |s_i|^2 - f^T v \right)
\end{align*}$$

(57)

with $v := \Delta u$, $p_i := \Delta c_{pi}$, and $s_i := \Delta c_{ui}$ ($i = 1, \ldots, m$).

Define convex functions $g_1 : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and $g_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$\begin{align*}
g_1(v, p, s) &= \sum_{i=1}^{m} \left( q_i^{(i)}(b_i^T v - p_i - s_i) + \frac{1}{2} \eta_i (b_i^T v - p_i - s_i)^2 \right) \\
& + \sum_{i=1}^{m} \frac{1}{2} h_i |s_i|^2 + \sum_{i=1}^{m} \frac{1}{2} \eta_i |s_i|^2 - f^T v, \quad (58)
\end{align*}$$

$$g_2(p, s) = \sum_{i=1}^{m} R_i^{(i)} |p_i| + \sum_{i=1}^{m} R_i^{(i)} |s_i|. \quad (59)$$

Here, $g_1$ is differentiable, and $\nabla g_1$ is Lipschitz continuous. Problem (57) can be written as

$$\begin{align*}
\text{minimize} & \quad g_1(v, p, s) + g_2(p, s).
\end{align*}$$

(60)

Then we can design an accelerated proximal gradient method in a manner similar to section 3.3; details appear in appendix C.

6 Numerical experiments

The presented algorithms were implemented with MATLAB ver. 8.4.0. Comparison is performed with QUADPROG (The MathWorks, Inc. 2014), IPOPT ver. 3.11.3 (Wachter and Biegler 2006) via the Matlab interface (Carbonetto 2014), and PATH ver. 4.7.03 (Dirkse and Ferris 1995; Ferris and Munson 2000) via the Matlab Interface (Ferris and Munson 1999). QUADPROG is a MATLAB built-in function for convex quadratic programming (QP). We use an implementation of an interior-point method by setting the parameter Algorithm to interior-point-convex. IPOPT is a primal-dual interior-point method with a filter line-search method for nonlinear programming. We set the parameters options.ipopt.hessian.approximation and options.ipopt.tol to limited-memory and $10^{-1}$, respectively. PATH
is a nonsmooth Newton method to solve mixed complementarity problems. We apply PATH to solve the KKT condition for the QP problem in (12). The termination criterion of Algorithm 2 and Algorithm 3 is $\|{(v_{l}, p_{l})} - {(v_{l-1}, p_{l-1})}\|_{\infty} \leq \varepsilon$ with $\varepsilon = 10^{-8}$ m. Computation was carried out on a 2.6GHz Intel Core i5 processor with 8GB RAM.

In the following numerical experiments, we consider a truss shown in Figure 3, where $N_X$ and $N_Y$ are varied to generate problem instances with diverse sizes. This barrel vault truss is a two-way space grid with square pyramids. In the direction of the $X$-axis, the nodes are aligned with regular intervals, as shown in Figure 3(c). In the $YZ$-plane, as seen in Figure 3(d), the top layer nodes are aligned on a circle equiangularly. Also, the bottom layer nodes are equiangularly aligned on a circle with the same radius as the one of the top layer nodes. All the lowest nodes of the top layer are pin-supported. The number of members, $m$, and the number of degrees of freedom of displacements, $d$, are listed in Table 1. The cross-sectional area of each member is $a_{i} = 500 \text{mm}^2$ and Young’s modulus is $E = 200\text{GPa}$.

Remark 8 It is well known that vectorizing MATLAB code often increases computational efficiency drastically (The MathWorks, Inc. 2014). All the computations at step 1 of Algorithm 2, i.e., (34), (35), (36), and (37), can be implemented in vectorized forms. Namely, a component-wise vector product can be carried out with the MATLAB function times. Also, for calculation of (37), we can apply abs, max,
and sign functions to vectors. Similarly, step 1 of Algorithm 3 can also be implemented in vectorized forms.

6.1 Holonomic (path-independent) analysis

In sections 6.1.1 and 6.1.2, we perform holonomic analysis, i.e., we assume that no elastic unloading takes place and consider a relatively large loading step. With these examples we attempt to evaluate the efficiency of the presented algorithm when it is applied to moderately large-scale problem instances. Two variants of the step size discussed in section 3.3 are examined. In the following, by APGM-E we mean that the step size is set to \( \alpha = 1/L \) with Lipschitz constant \( L \) of \( \nabla g_1 \), and by APGM-G we mean that \( \alpha = 1/L' \) with \( L' \) defined by (40).

We set the current axial forces to \( q^{(0)}_i = 0 \) \((i = 1, \ldots, m)\). The hardening moduli are \( h_i = 0.1k_i \) \((i = 1, \ldots, m)\). The yielding stress is \( \sigma_y = 200\text{MPa} \) and, accordingly, \( R^{(0)}_i = \sigma_y a_i = 100\text{kN} \) \((i = 1, \ldots, m)\).

6.1.1 Example (I): linear hardening model

In this section we assume a linear hardening model and solve problem (15) with Algorithm 2. The initial point chosen at step 0 is \( v = 0 \) and \( p = 0 \). For comparison, we also solve QP (12) with QUADPROG, IPOPT, and PATH. It is worth noting that problem (15) has \( d + m \) variables, while the QP has \( d + 3m \) variables to be optimized, \( m \) linear equality constraints, and \( 2m \) linear inequality constraints. As for the external load, \( f \), a vertical downward force of \( 250/N_X \) in kN is applied at each of the top layer nodes.

The computational results are listed in Table 2 and Table 3. Here, “iter.” means the number of iterations required before termination, “time” is the total computational time required by each algorithm, and “eigs” is the computational time required by the MATLAB function eigs to compute the maximum eigenvalue of \( \nabla^2 g_1(v, p) \).

Table 1 Characteristics of the problem instances.

| \((N_X, N_Y)\)   | \(m\)  | \(d\)  |
|-----------------|-------|-------|
| (10, 10)        | 800   | 597   |
| (20, 20)        | 3,200 | 2,397 |
| (30, 30)        | 7,200 | 5,397 |
| (40, 40)        | 12,800| 9,597 |
| (50, 50)        | 20,000| 14,997|
| (60, 60)        | 28,800| 21,597|
| (70, 70)        | 39,200| 29,397|
| (80, 80)        | 51,200| 38,397|
| (90, 90)        | 64,800| 48,597|
| (100, 100)      | 80,000| 59,997|
| (110, 110)      | 96,800| 72,597|
| (120, 120)      | 115,200| 86,397|
| (130, 130)      | 135,200| 101,397|
The accuracy of the computed solutions are compared in “rel. diff.” This reports the value defined by $\phi = (\hat{f} - f^*) / f^*$, where $f^*$ is the objective value computed by QUADPROG and $\hat{f}$ is the one computed by the respective method. By definition, $\phi < 0$ means that the computed solution has a better objective value than the one obtained by a standard QP solver, QUADPROG. It should be clear that “time” of APGM-E includes the computational time of eigs. For PATH, only the number of major iterations is listed in Table 3, although the number of minor iterations is also reported by the solver. It is observed from Table 2 and Table 3 that the computational time required by PATH is extremely large compared with the other four methods.

Figure 4 shows the computational time of APGM-E, APGM-G, QUADPROG, and IPOPT. The computational time required by IPOPT is very large compared with the other three methods. QUADPROG spent two or three times larger time than APGM-E and APGM-G. APGM-E and APGM-G are comparable from the view

| Table 2 | The computational results of the proposed methods in example (I). |
|---------|------------------|------------------|------------------|
|         | Iter. | Time (s) | eigs (s) | Rel. diff. | Iter. | Time (s) | Rel. diff. |
| APGM-E  |       |         |         |           | APGM-G |         |           |
| (10, 10) | 404   | 0.1     | (0.0)   | -8.6e-12  | 350    | 0.1     | 1.2e-08   |
| (20, 20) | 549   | 0.3     | (0.1)   | 5.3e-08   | 732    | 0.3     | 5.0e-08   |
| (30, 30) | 669   | 0.8     | (0.3)   | 6.1e-07   | 893    | 0.6     | 6.7e-07   |
| (40, 40) | 450   | 2.4     | (0.9)   | 7.9e-09   | 1,927  | 2.1     | 8.9e-09   |
| (50, 50) | 2,112 | 5.2     | (1.7)   | 2.4e-08   | 2,778  | 4.5     | 2.5e-08   |
| (60, 60) | 3,700 | 10.1    | (1.5)   | 1.6e-07   | 4,900  | 11.2    | 1.6e-07   |
| (70, 70) | 7,772 | 27.5    | (4.1)   | -1.7e-09  | 4,853  | 14.6    | 2.3e-05   |
| (80, 80) | 6,237 | 30.0    | (6.3)   | 5.2e-06   | 7,075  | 27.0    | 3.9e-05   |
| (90, 90) | 8,784 | 53.0    | (11.8)  | 1.9e-07   | 9,235  | 43.3    | 9.5e-06   |
| (100, 100) | 11,158 | 84.7 | (17.9) | 1.4e-08   | 14,724 | 88.2    | 1.6e-08   |
| (110, 110) | 10,807 | 101.8 | (20.8) | 1.5e-05   | 14,251 | 107.1   | 1.5e-05   |
| (120, 120) | 12,973 | 159.8 | (41.5) | 1.4e-05   | 17,097 | 156.3   | 1.4e-05   |
| (130, 130) | 15,309 | 253.5 | (45.3) | 8.2e-06   | 20,165 | 269.0   | 8.2e-06   |

| Table 3 | The computational results of QUADPROG, IPOPT, and PATH in example (I) |
|---------|------------------|------------------|------------------|
|         | Iter. | Time (s) | IPOPT | Rel. diff. | PATH | Iter. | Time (s) | Rel. diff. |
| QUADPROG |       |         |       |           |       |         |           |
| (10, 10) | 10    | 0.4     | 36    | 0.7       | 4.0e-06 | 12 | 0.5     | -4.8e-10  |
| (20, 20) | 11    | 1.7     | 49    | 3.8       | 3.6e-05 | 12 | 8.7     | -3.1e-10  |
| (30, 30) | 11    | 6.2     | 74    | 14.8      | 2.9e-05 | 12 | 95.0    | -8.6e-10  |
| (40, 40) | 11    | 12.8    | 192   | 70.3      | 4.5e-06 | 12 | 699.2   | -6.2e-09  |
| (50, 50) | 11    | 26.2    | 423   | 236.4     | 1.8e-06 | 12 | (> 1200.0) | - |
| (60, 60) | 11    | 44.9    | 508   | 425.5     | 1.5e-05 | 12 | (> 1200.0) | - |
| (70, 70) | 11    | 74.0    | 958   | 1,116.7   | 1.3e-04 | 12 | (> 1200.0) | - |
| (80, 80) | 11    | 112.2   | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
| (90, 90) | 11    | 167.0   | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
| (100, 100) | 11 | 241.3 | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
| (110, 110) | 11 | 353.4 | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
| (120, 120) | 11 | 495.8 | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
| (130, 130) | 11 | 633.3 | (> (> 1200.0)) | - | (> (> 1200.0)) | - |
point of computational time. The number of iterations required by these two methods are shown in Figure 5. It is worth noting that the difference of these two methods is only the step size, $\alpha$. Figure 6 shows $1/\alpha$ of these two methods. The step size of APGM-E is about 1.75 times larger than that of APGM-G. It is observed in Figure 5 that, for large-scale problems, APGM-E requires less iterations; the number of iterations required by APGM-G is about 1.3 times larger. Nevertheless, the total computational time is comparable as seen in Figure 4, because in APGM-E the computation of the maximum eigenvalue of the Hessian matrix requires relatively large computational time. In contrast, the computational time required for computing the Gershgorin disc bound is negligible (e.g., less than 0.2s for the instance with $(N_X,N_Y) = (130,130)$). It is observed in Table 3 that the number of iterations of the interior-point method (QUADPROG) is independent of the problem size. For solving a large-scale instance, the interior-point method has to solve a large-scale system.
Fig. 6 The reciprocal of the step size of example (I). “□” The Lipschitz constant of $\nabla g$ (used in APGM-E); and “×” its upper bound (used in APGM-G).

Fig. 7 Convergence history of the objective value for $(N_X, N_Y) = (40, 40)$. —— APGM-E with restart; ····· APGM-E without restart; and - - - PGM (Algorithm 1).

of linear equations to find the search direction at every iteration, and this computation dominates the computational time.

Figure 7 reports the convergence history of the objective value of APGM-E with respect to the iteration count. It also shows the result of APGM-E without restart scheme, and that of Algorithm 1 (i.e., a proximal gradient method without acceleration). It is observed that the acceleration and restart schemes drastically speed up the convergence.

6.1.2 Example (II): piecewise-linear hardening model

In this section we solve problem instances with a piece-wise linear hardening model. Specifically, we solve problem (57) with Algorithm 3. The initial point is $v_0 = 0$ and $p_0 = s_0 = 0$. The corresponding QP is problem (55), which is solved with an
Table 4 The computational results of example (II).

| \((N_X, N_Y)\) | APGM-E | APGM-G | QUADPROG |
|---------------|---------|---------|----------|
|               | Iter. | Time (s) | Rel. diff. | Iter. | Time (s) | Rel. diff. | Iter. | Time (s) |
| \((10, 10)\)  | 473   | 0.1 (0.0) | 1.6e-09   | 394   | 0.1     | 1.2e-06   | 12    | 0.5      |
| \((20, 20)\)  | 1061  | 0.7 (0.2) | 4.3e-07   | 1526  | 0.8     | 4.4e-07   | 16    | 3.6      |
| \((30, 30)\)  | 2159  | 2.5 (0.6) | 3.1e-08   | 3102  | 2.7     | 3.4e-08   | 14    | 10.2     |
| \((40, 40)\)  | 6181  | 10.1 (1.3) | 1.0e-07  | 8861  | 12.5    | 1.1e-07   | 17    | 26.3     |
| \((50, 50)\)  | 7763  | 22.3 (3.1) | 3.1e-07  | 8118  | 18.1    | 1.4e-05   | 14    | 47.9     |
| \((60, 60)\)  | 12391 | 45.7 (4.7) | 4.2e-08  | 12981 | 40.1    | 6.2e-06   | 15    | 91.0     |
| \((70, 70)\)  | 13263 | 63.8 (7.6) | 2.9e-06  | 18971 | 76.4    | 2.9e-06   | 16    | 154.5    |
| \((80, 80)\)  | 24767 | 144.6 (14.6) | 9.4e-09  | 25646 | 134.1   | 7.6e-06   | 15    | 244.8    |
| \((90, 90)\)  | 25616 | 168.7 (19.0) | 4.2e-05  | 33734 | 210.9   | 4.2e-05   | 14    | 371.3    |
| \((100, 100)\)| 30782 | 289.3 (34.8) | 3.7e-05  | 43946 | 346.1   | 3.7e-05   | 15    | 566.7    |
| \((110, 110)\)| 39675 | 405.0 (32.1) | 3.3e-06  | 56613 | 535.9   | 3.3e-06   | 16    | 796.9    |

Fig. 8 The computational time of example (II). “□” APGM-E; “×” APGM-G; and “◦” QUADPROG.

Fig. 9 The reciprocal of the step size of example (II). “□” The Lipschitz constant of ∇g (used in APGM-E); and “×” its upper bound (used in APGM-G).
interior-point method. It is worth noting that problem (57) has \( d + 2m \) variables, while the QP has \( d + 5m \) variables, \( m \) linear equality constraints, and \( 4m \) linear inequality constraints. As for the external load, \( \mathbf{f} \), a vertical downward force of \( 200/N_X \) (in kN) and a horizontal force of \( 40/N_Y \) (in kN) in the positive direction of the \( Y \)-axis are applied at each of the top layer nodes. The parameters of the hardening model are \( R_i^{(0)} = 100 \text{kN}, R_i^0 = 1.3R_i^{(0)}, h_{i1} = 0.1k_i, \) and \( h_{i2} = 0.5h_{i1} \) \((i = 1, \ldots, m)\).

Table 4 lists the computational results. The computational time is compared also in Figure 8. QUADPROG spent about twice larger time than APGM-E. APGM-G seems to be comparable with APGM-E. However, for large instances, APGM-G spent about 1.2 times larger computational time than APGM-E, because the number of iterations required by APGM-G is about 1.4 times larger than that of APGM-E. Figure 9 depicts the maximum eigenvalue of \( \nabla^2 g_1 \) used in APGM-E and its upper bound used in APGM-G. For all the instances, the upper bound is about twice larger than its true value. Figure 10 shows the distribution of member stress and strain at the equilibrium solution for \( (N_X, N_Y) = (50, 50) \). It is observed that the piecewise-linear hardening model is simulated correctly.

### 6.2 Path-dependent incremental analysis

In sections 6.2.1 and 6.2.2 we perform the path-dependent incremental analysis. We adopt a combined isotropic and kinematic hardening model studied in section 4, where \( \theta = 0.5 \). Problem (15) is solved with Algorithm 2. We begin by solving the incremental problem for the first loading step from an initial point \( \mathbf{v}_0 = 0 \) and \( \mathbf{p}_0 = 0 \). At each subsequent loading step, we make use of the solution of the previous loading step as the initial point. The efficiency of this simple warm-start strategy is examined in the following examples. In the course of incremental analysis, the Hessian matrix of \( g_1 \) is independent of the loading step count, \( t \). Therefore, we need to compute the maximum eigenvalue of the Hessian matrix only at the beginning of the analysis at
the first loading step. Hence, using the maximum eigenvalue outperforms using its Gershgorin disc bound.

6.2.1 Example (III)

In this section we consider \((N_X, N_Y) = (10, 10)\) for the truss shown in Figure 3. A vertical downward force of 5 kN is applied at each node of the top layer. Simultaneously, a horizontal force shown in Figure 11 is applied in the positive direction of the \(Y\)-axis, where \(t\) is the loading step count.

Figure 12(a) shows the load versus the displacement relation in the \(Y\)-direction of the node that is located on the \(XZ\)-plane and on the boundary of the top layer. It is observed that the truss gradually shows elastic shakedown, because the maximum magnitude of the load is fixed and the isotropic hardening is unlimited. Figure 12(b) shows the stress–strain relation of a typical member. The effect of combination of isotropic and kinematic hardening can be observed. Figure 12(c) shows the number of iterations required to solve the incremental problem at the loading step \(t\). Similarly, the computational time is shown in Figure 12(d). The total computational time was 49.7 s. If the incremental solutions both at the \(t\)th and \((t + 1)\)th loading steps involve no plastic deformation, then these two solutions coincide. In such a case, the number of iteration required at the \((t + 1)\)th loading step is negligibly small (and is often one). The maximum computational time required for solving one incremental problem is 1.87 s.

6.2.2 Example (IV)

We next consider a larger instance, \((N_X, N_Y) = (20, 20)\). At each node of the top layer, a vertical downward force of 4 kN is applied. Simultaneously, a horizontal force shown in Figure 13 is applied in the \(Y\)-direction.

Like Figure 12(a) in section 6.2.1, Figure 14(a) shows the load versus the displacement relation in the \(Y\)-direction of the the middle node on the boundary of the
Fig. 12 The result of example (III). (a) The load versus displacement relation; (b) the typical stress versus strain relation; (c) the number of iterations; and (d) the computational time.

Fig. 13 Loading history of example (IV).
Fig. 14 The result of example (IV). (a) The load versus displacement relation; (b) the typical stress versus strain relation; and (c) the number of iterations; and (d) the computational time.

top layer. A typical member stress–strain relation is shown in Figure 14(b). The hysteresis loop expands, because the magnitude of the horizontal load is gradually increased.

Figure 14(c) shows the number of iterations required to solve the incremental problem at each loading step. There are two cases that required more than 10,000 iterations. Loosely speaking, the computational cost increases as the number of members with incremental plastic deformations increases. The computational time required at each loading step is shown in Figure 14(d). Since the solution at the previous loading step is used as an initial solution for the present loading step, the computational cost becomes negligible if no member undergoes plastic deformation.

The total computational time was 103.2 s. Almost all problems were solved within 0.5 s; there exist 19 problems that require more than 0.5 s. If we use an interior-point method (QUADPROG) from cold start, it is estimated from Table 3 that the total computational time might approximately become $1.7 \times 1200 = 2040$ s. The computational time required by the proposed method is much smaller than this estimate.
7 Concluding remarks

In this paper we have presented a fast first-order optimization approach to the quasi-static incremental analysis of elastoplastic structures. The algorithm is free from numerical solution of linear equations. The most expensive computation of the algorithm consists of some matrix-vector multiplications with sparse matrices, such as the compatibility matrix. Also, the presented algorithm does not involve any conditional branching procedures stemming from the disjunctive nature of plastic loading and elastic unloading processes. The algorithm is a version of the accelerated gradient-based methods, and converges in potential energy function value as $O(1/k^2)$, where $k$ is the iteration counter. More precisely, it is essentially viewed as an application of FISTA, an accelerated proximal gradient method for the $\ell_1$-regularized least-squares problem, to the elastoplastic analysis. Owing to these attributes, the algorithm is easy to implement and applicable to large-scale problems. Indeed, the numerical experiments suggest that the algorithm outperforms interior-point methods for convex quadratic programming and nonlinear programming.

In the course of path-dependent quasi-static analysis, we solve a series of closely related optimization problems. It has been shown that the presented approach can drastically speed up by employing a simple warm-start strategy that uses the solution at the previous loading step as the initial solution for the present loading step.

It is well known that the incremental problem studied in this paper can be recast as a convex quadratic programming problem and a linear complementarity problem. In contrast, a key to the proposed approach is formulating the incremental problem as an unconstrained nonsmooth convex optimization problem. For simplicity of presentation, in this paper we have restrict ourselves to truss structures. The presented methodology can be readily applied to other types of structures when the yield function can be approximated by a piecewise-linear function. An example is a frame structure with a piecewise-linear yield condition incorporating interaction between the member axial force and end moment.

This paper has been intended to be the first attempt to apply an accelerated gradient-like method to applied mechanics. Much remains to be explored. For instance, extensions to yield criteria other than the piecewise-linear model can be studied; optimization-based approaches to such problems can be found in, e.g., Krabbenhøft et al. (2007a), Krabbenhøft et al. (2007b), Tangaramvong et al. (2012), and Yonekura and Kanno (2012). Also, applications of other fast first-order optimization methods can be examined. Parallelization of the presented method—which is probably quite easy to implement because no linear-equations solver is required—has not been considered. Extension to strain-softening models might be challenging, because it requires to deal with a nonconvex objective function, as considered, e.g., in Li and Lin (2015). Recently, it has been discussed that accelerated gradient-like method can be viewed as a finite difference approximation of an ordinary differential equation (Su et al., 2014; Krichene et al., 2015). With reference to these results, the physical interpretation of the method presented in this paper might be analyzed. Furthermore, besides problems in plasticity theory, extensions to complementarity problems arising in diverse fields of nonsmooth mechanics can be considered. Possible examples include
cable networks (Kanno et al. 2002), static and dynamic contact problems (Acary and Brogliato 2008, Wriggers 2006), masonry structures (Kanno 2011), etc.

More than 30 years ago Giulio Maier wrote (Maier 1984): "Why nonlinear boundary value problems, such as incremental elastoplastic analysis, are routinely solved in several areas of engineering practice fully ignoring the fact that they can be cast in the form of nonlinear or quadratic programs? Obviously, the popular, often merely heuristic, solution schemes resting on iterated use of linear solvers are favoured by the fact that they gradually evolved from the enormous amount of experience accumulated in linear elastic analysis. But their intrinsic superiority over mathematical programming approaches is doubtful, and by no means ensured, in several situations."

Until today, however, mathematical programming approaches have not been used very widely by practitioners. The approach presented in this paper has solid background of mathematical programming, while computation can be performed without knowledge of optimization. It might possibly encourage widespread use of various mathematical programming approaches to computational and applied mechanics.

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A SOCP formulation of problem \([12]\)

In this section, we explain how problem \([12]\) is recast as a second-order cone programming (SOCP) problem.

The second-order cone in \(\mathbb{R}^n\) is defined by

\[
\mathcal{L}^n = \left\{ (x_0, x_1, \ldots, x_{n-1})^T \in \mathbb{R}^n \mid x_0 \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2} \right\}.
\]

SOCP is a minimization (or maximization) of a linear objective function under some second-order cone constraints and affine constraints.
The inequality constraints in (12c) can be written as second-order cone constraints as
\[
[\Delta \gamma_i] \in \mathbb{L}^2, \quad i = 1, \ldots, m.
\]
The constraints in (12b) are affine (i.e., linear equality) constraints. To convert the objective function to a linear one, we introduce auxiliary variables, \( \xi, \zeta \in \mathbb{R} \) that serve as upper bounds for the quadratic terms in (12a). Namely, we consider the following constraints:
\[
\xi \geq m \sum_{i=1}^{m} k_i \Delta c_e^2_i, \quad (61)
\]
\[
\zeta \geq m \sum_{i=1}^{m} h_i \Delta p_i^2. \quad (62)
\]
The convex quadratic inequality constraint in (61) can be rewritten equivalently as (Ben-Tal and Nemirovski 2001)
\[
\xi + 1 \geq \begin{bmatrix} \xi - 1 \\ \sqrt{2k_1 \Delta c_e^1} \\ \vdots \\ \sqrt{2k_m \Delta c_e^m} \end{bmatrix} \in \mathbb{L}^{m+1},
\]
This is a second-order cone constraint. Constraint (62) can be rewritten in the same manner.

The upshot is that problem (12) can be converted to the following SOCP problem:
\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{m} q_i(t) \Delta c_e^2_i + \xi + m \sum_{i=1}^{m} R_i(t) \Delta \gamma_i \\
\text{subject to} & \quad \Delta c_e^1 + \Delta c_p^1 = b_e^1 \Delta u, \quad i = 1, \ldots, m, \\
& \quad [\Delta \gamma_i \Delta c_p^i] \in \mathbb{L}^2, \quad i = 1, \ldots, m, \\
& \quad \begin{bmatrix} \xi + 1 \\ \zeta + 1 \\ \sqrt{2h_1 \Delta \gamma_1} \\ \vdots \\ \sqrt{2h_m \Delta \gamma_m} \end{bmatrix} \in \mathbb{L}^{m+1}.
\end{align*}
\]
Here, \( \Delta c_e^1, \ldots, \Delta c_e^m, \xi, \Delta \gamma_1, \ldots, \Delta \gamma_m, \zeta, \) and \( \Delta u \) are variables to be optimized.

B Equivalence of (21) and (23)

As one of fundamental properties of the proximal mapping, we can show, for any \( \alpha > 0 \), that \( p \in \mathbb{R}^m \) satisfies
\[
0 \in \nabla_p g_1(v, p) + \partial g_2(p) \quad (63)
\]
if and only if it satisfies
\[
p = \text{prox}_{\alpha g_2} (p - \alpha \nabla_p g_1(v, p)). \quad (64)
\]
See, e.g., Parikh and Boyd (2014). For the reader’s convenience, essentials of the proof are repeated here.

1 Conversion to SOCP is not unique.
Suppose that \( p \) satisfies (63). This is equivalent to
\[
0 \in \alpha \nabla \varepsilon G_1(v, p) + \alpha \partial G_2(p) \\
= \alpha \nabla p G_1(v, p) - p + \alpha \partial G_2(p).
\]  
(65)

Let \( s := p - \alpha \nabla p G_1(v, p) \) for notational simplicity. Then (65) is rewritten as
\[
0 \in \partial G_2(p) + (p - s),
\]
which is equivalent to
\[
p = \arg \min_{z} \left\{ \alpha G_2(z) + \frac{1}{2} \| z - s \|^2 \right\}.
\]  
(66)

By definition, (66) is equivalent to (64).

### C Algorithm for piecewise-linear hardening

We begin with computation of the gradient of \( g_1 \) defined by (58). In a manner similar to section 3.3, it is convenient to define \( e \in \mathbb{R}^m \) by
\[
e = Bv - p - s,
\]
which corresponds to the vector of incremental elastic elongation, \( \Delta e \), in problem (55). Then the gradient of \( g_1 \) can be calculated as
\[
\nabla v g_1(v, p, s) = B^\top (\text{diag}(k)e + q^{(0)}) - f,
\]
\[
\nabla p g_1(v, p, s) = \text{diag}(h_1)p - \text{diag}(k)e - q^{(0)},
\]
\[
\nabla s g_1(v, p, s) = \text{diag}(\eta)s - \text{diag}(k)e - q^{(0)},
\]

where
\[
\nabla v g_1 = \frac{\partial g_1}{\partial v}, \quad \nabla p g_1 = \frac{\partial g_1}{\partial p}, \quad \nabla s g_1 = \frac{\partial g_1}{\partial s}.
\]

Moreover, the Hessian matrix of \( g_1 \) is written as
\[
V^2 g_1(v, p, s) = \begin{bmatrix}
B^\top & 0 & 0 \\
0 & \text{diag}(k) & O \\
0 & O & \text{diag}(h_1)
\end{bmatrix} - \begin{bmatrix}
B & I & -I \\
-I & O & O \\
O & O & O
\end{bmatrix}.
\]  
(67)

Since \( k_i > 0, h_i > 0, \eta_i > 0 \) \((i = 1, \ldots, m)\) and \( B \) is of row full rank for a kinematically determinate truss, \( V^2 g_1(v, p, s) \) is positive definite. In a manner similar to section 3.3, the proximal mapping of \( \alpha g_1 \) with \( \alpha > 0 \) can be computed as
\[
\text{prox}_{\alpha g_1}(w, z) = \begin{bmatrix}
\text{diag}(\text{sgn}(w)) \max \{ |w| - \alpha R, 0 \} \\
\text{diag}(\text{sgn}(z)) \max \{ |z| - \alpha R, 0 \}
\end{bmatrix}.
\]

We are now in position to describe an accelerated proximal gradient method for solving problem (57).

**Algorithm 3**

**Step 0:** Let \( L \) denote the maximum eigenvalue of \( V^2 g_1(v, p, s) \) in (67). Choose \( v_0 \in \mathbb{R}^d, p_0 \in \mathbb{R}^m, s_0 \in \mathbb{R}^n, \alpha \in [0, 1/L], \) and termination tolerance \( \varepsilon > 0 \). Set \( l := 1, \mu_1 := v_0, \beta_1 := p_0, \sigma_1 := s_0, \) and \( \tau_1 := 1 \).
Step 1: Let
\[
\begin{align*}
\epsilon_l &:= B \mu_l - \rho_l - \sigma_l, \\
v_l &:= \mu_l - \alpha [h^T (\text{diag}(k) e_l + q^k) - f], \\
w_l &:= \rho_l - \alpha (\text{diag}(h) \mu_l - \text{diag}(k) e_l - q^k), \\
p_l &:= \text{diag}(\text{sgn}(w_l)) \max \{|w_l| - \alpha R^k, 0\}, \\
z_l &:= \sigma_l - \alpha (\text{diag}(\eta) \sigma_l - \text{diag}(k) e_l - q^k), \\
s_l &:= \text{diag}(\text{sgn}(z_l)) \max \{|z_l| - \alpha R^k, 0\}.
\end{align*}
\]

Step 2: Let
\[
\tau_{l+1} = \frac{1}{2} \left(1 + \sqrt{1 + 4 \tau_l^2}\right).
\]

Step 3: If \(g_1(v_l, p_l, s_l) + g_2(p_l, s_l) < g_1(v_{l-1}, p_{l-1}, s_{l-1}) + g_2(p_{l-1}, s_{l-1})\), then let
\[
\begin{align*}
\mu_{l+1} &:= v_l + \frac{\tau_l - 1}{\tau_{l+1}} (v_l - v_{l-1}), \\
\rho_{l+1} &:= p_l + \frac{\tau_l - 1}{\tau_{l+1}} (p_l - p_{l-1}), \\
\sigma_{l+1} &:= s_l + \frac{\tau_l - 1}{\tau_{l+1}} (s_l - s_{l-1}).
\end{align*}
\]

Otherwise, let \(\tau_{l+1} := 1, \mu_{l+1} := v_l, \rho_{l+1} := p_l, \) and \(\sigma_{l+1} := s_l\).

Step 4: If \(\|v_l, p_l, s_l - (v_{l-1}, p_{l-1}, s_{l-1})\| \leq \varepsilon\), then terminate. Otherwise, let \(l := l + 1\), and go to step 1.

At step 1 of Algorithm 3, auxiliary variables \(e_l, w_l, \) and \(s_l\) are used for convenience of computation.