On the number of integral binary $n$-ic forms having bounded Julia invariant

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Abstract
In 1848, Hermite introduced a reduction theory for binary forms of degree $n$ which was developed more fully in the seminal 1917 treatise of Julia. This canonical method of reduction made use of a new, fundamental, but irrational $SL_2$-invariant of binary $n$-ic forms defined over $\mathbb{R}$, which is now known as the Julia invariant. In this paper, for each $n$ and $k$ with $n+k \geq 3$, we determine the asymptotic behavior of the number of $SL_2(\mathbb{Z})$-equivalence classes of binary $n$-ic forms, with $k$ pairs of complex roots, having bounded Julia invariant. Specializing to $(n,k)=(2,1)$ and $(3,0)$, respectively, recovers the asymptotic results of Gauss and Davenport on positive definite binary quadratic forms and positive discriminant binary cubic forms, respectively.

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1 | INTRODUCTION

Let $V_n(\mathbb{R})$ denote the $(n+1)$-dimensional real vector space of binary $n$-ic forms

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$ (1)

having coefficients $a_0, \ldots, a_n \in \mathbb{R}$. The group $SL_2(\mathbb{R})$ acts naturally on $V_n(\mathbb{R})$ via linear substitution of variable; namely, an element $\gamma \in SL_2(\mathbb{R})$ acts on $f(x, y)$ by

$$\gamma \cdot f(x, y) = f((x, y) \cdot \gamma).$$ (2)

This action of $SL_2(\mathbb{R})$ on $V_n(\mathbb{R})$ is a left action, that is, $\langle \gamma_1 \gamma_2 \rangle \cdot f = \gamma_1 \cdot (\gamma_2 \cdot f)$. © 2022 The Authors. The publishing rights in this article are licensed to the London Mathematical Society under an exclusive licence.
In 1917, Julia [18] introduced a natural invariant \( \vartheta(f) \) for this action of \( \text{SL}_2(\mathbb{R}) \) on binary \( n \)-ic forms. The invariant was constructed in terms of the discriminant of a certain canonical but irrational positive-definite \( \text{SL}_2(\mathbb{R}) \)-covariant binary quadratic form \( Q \) of \( f \). More precisely, consider a binary \( n \)-ic form \( f \) with coefficients as in (1). If \( a_0 \neq 0 \), we may write

\[
f(x, y) = a_0(x - \alpha_1 y)(x - \alpha_2 y) \cdots (x - \alpha_n y)
\]

with \( \alpha_i \in \mathbb{C} \), and then, for any vector \( t = (t_1, \ldots, t_n) \) of positive real numbers, we may consider the positive-definite quadratic form

\[
Q_t(x, y) = \sum_{j=1}^{n} t_j^2(x - \alpha_j y)(x - \bar{\alpha}_j y).
\]  

(3)

Julia chose the \( t_j \) so as to minimize the expression

\[
\vartheta = \vartheta(f) = \frac{a_0^2 |\text{Disc} \, Q_t|^{n/2}}{t_1^2 \cdots t_n^2}
\]  

(4)

and proved that with this restriction on the \( t_j \), the form \( Q(x, y) := Q_t(x, y) \) is a covariant of the original form \( f(x, y) \). Julia also proved that the resulting expression (4) for \( \vartheta(f) \) is then an \( \text{SL}_2(\mathbb{R}) \)-invariant of the binary form \( f \). We call the quantity \( \vartheta = \vartheta(f) \) the *Julia invariant* of the binary form \( f(x, y) \).

Julia showed that the quadratic covariant \( Q(x, y) \) enables one to give a natural \( \text{SL}_2(\mathbb{Z}) \)-reduction theory for binary \( n \)-ic forms over \( \mathbb{R} \) (or over \( \mathbb{Z} \)); namely, one says that \( f \) is reduced if \( Q \) is reduced, in the usual sense of Gauss, as a positive-definite binary quadratic form. Furthermore, Julia proved that \( \vartheta \) bounds many quantities of interest for a reduced form \( f(x, y) \); for example, the leading coefficient \( a_0 \) is bounded by a constant times \( \sqrt{\vartheta} \), while the roots \( \alpha_i \) of a reduced form are bounded by a constant times \( \sqrt{\vartheta}/|a_0| \). Julia’s reduction theory has been implemented to great effect in Cremona’s work [8] for cubic and quartic forms (for the purpose of efficient descent on elliptic curves), and in the work of Stoll and Cremona [24] for forms of general degree.

Given the naturality and utility of the Julia invariant of binary forms, the question arises: how many \( \text{SL}_2(\mathbb{Z}) \)-equivalence classes of integral binary \( n \)-ic forms are there having Julia invariant at most \( X \), as \( X \) gets large? More precisely, let \( N_{n,k}(X) \) denote the number of \( \text{SL}_2(\mathbb{Z}) \)-equivalence classes of integral irreducible binary \( n \)-ic forms, having \( k \) pairs of complex roots and \( n - 2k \) real roots, such that \( \vartheta(f) \leq X \). In [27], using the estimates of Julia as well as some additional input from the paper [24], it was shown that \( N_{n,k}(X) = O(X^{n+1/2+\epsilon}) \), for any \( \epsilon > 0 \). The primary objective of this article is to refine the latter estimate to an exact asymptotic, along with a power-saving error term. Specifically, we prove the following theorem:

**Theorem 1.** Let \( n \) and \( k \) be non-negative integers with \( k \in \{0, 1, \ldots, [n/2]\} \) such that \( n + k \geq 3 \). Then there exists a constant \( c_{n,k} > 0 \) such that

\[
N_{n,k}(X) = c_{n,k} X^{n+1/2} + O(X^{n+1/2 - \frac{1}{n}}).
\]
Since the Julia invariant coincides with the discriminant and squareroot of the discriminant in the cases of binary quadratic forms having two complex roots and binary cubic forms having three real roots, respectively, the above theorem includes and extends the Gauss class number summation formula for binary quadratic forms of negative discriminant [15, Article 302], and Davenport’s theorem on the density of discriminants of binary cubic forms of positive discriminant [10]; the above-stated error terms in these two cases were first proven by Shintani, as second-order terms, in [22] and [23], respectively. Theorem 1 thus gives a natural way to count and enumerate SL₂(ℤ)-equivalence classes of integral binary \(n\)-ic forms for any degree \(n\) and any signature, in a uniform manner, extending the results and methods already known for binary quadratic and cubic forms. One recent application of Theorem 1 and the methods behind its proof is seen in the beautiful work of Ho, Shankar, and Varma [17], where it is shown that there are \(S_n\)-number fields of every odd degree \(n\) having odd class number.

As shown in Section 6, the constant \(c_{n,k}\) is the value of a certain integral over a fundamental region. We do not carry out the computation, but for \(n + k \leq 3\) this value is known; Gauss [15] showed that \(c_{2,1} = \pi / 36\), while Davenport [10] showed that \(c_{3,0} = \pi^2 / 36\).

Another natural question is whether analogous results for binary \(n\)-ic forms are known for other invariants, particularly rational invariants. As mentioned above, in the \((n, k) = (2, 1)\) and \((3, 0)\) cases, the Julia variant is essentially the discriminant, and the asymptotics in these cases were known to Gauss and Davenport, respectively. For general binary \(n\)-ic forms, Birch and Merriman [6] proved that the number of binary \(n\)-ic forms having a fixed discriminant is finite. Their result was ineffective, while the first effective bound was proven by Evertse and Győry [13]. It is expected that the number of binary \(n\)-ic forms having absolute discriminant less than \(X\) should be asymptotic to \(d_n X^{(n+1)/(2n-2)}\) for some constant \(d_n > 0\); however, the best-known bounds are currently exponential in \(X\). In the case of binary quartic forms, the ring of polynomial invariants is generated by two invariants commonly denoted \(I\) and \(J\) (in particular, the discriminant is a polynomial in \(I, J\)). In [4], the first-named author and Shankar proved asymptotics of the form

\[
\sum_{\max\{|I|,J|^2/4| < X} h(I, J) = \frac{4}{135} \zeta(2) X^{5/6} + O(X^{3/4+\epsilon}),
\]

where \(h(I, J)\) denotes the number of classes of irreducible binary quartic forms having invariants \(I\) and \(J\) and four real roots; similar asymptotics with different constants in the main term were also obtained for the other two possible real signatures.

The organization of this article is as follows. In Section 2, we review some of the basic facts about the \(SL_2(\mathbb{R})\)-covariants \(Q\) and \(\vartheta\). In Sections 3 and 4, we establish some convenient fundamental domains for the actions of \(SL_2(\mathbb{Z})\) on \(V_n(\mathbb{R})\). As in the classical works of Gauss and Davenport, the primary difficulty in counting points with bounded Julia invariant in these fundamental domains is that they are not compact, but instead have a cuspidal region going off to infinity. To deal with and effectively handle this cusp, in Section 5 we investigate the distribution of reducible and irreducible points inside these fundamental domains. Specifically, we prove that the cusp contains only reducible points, while the remainder of the domain outside the cuspidal region contains primarily irreducible points with Galois group \(S_n\) and, when \(n \geq 5\), having trivial stabilizer. In Section 6, we then develop a refinement of an averaging method introduced in [2] to count irreducible points of bounded Julia invariant in these fundamental domains in terms of the volumes of these domains, via arguments that work uniformly in the degree \(n\). This then allows us to prove the asymptotic formula contained in Theorem 1. Finally, in Section 7, we prove a stronger version
of Theorem 1 where we restrict to counting those binary \( n \)-ic forms whose coefficients satisfy finitely many congruence conditions.

## 2 Preliminaries on the Julia Invariant

In this section, we collect some preliminary facts about the Julia invariant \( \theta \) and the associated quadratic covariant \( Q(x, y) \). The systematic study of these two expressions was begun by Julia in his thesis [18], and recently expanded upon by Stoll and Cremona in [24].

It may not be immediately clear from the definition of either \( \theta \) or \( Q(x, y) \) that \( \theta \) is an invariant of \( f(x, y) \) under the action of \( \text{SL}_2(\mathbb{R}) \), but Julia proved this in his thesis [18]. In fact, this was essentially known to Hermite in the 19th century (see [18, p. 5]). Even though the invariant \( \theta \), unlike the rational invariants of classical invariant theory, is not a polynomial in the coefficients \( a_i \) of \( f(x, y) \), one can still say that \( \theta \) is ‘homogeneous of degree 2’ in the following sense: for any scalar \( \lambda \in \mathbb{R} \) and any binary form \( f \in V_n(\mathbb{R}) \), we have \( \theta(\lambda f) = \lambda^2 \theta(f) \). To see this, note that, if \( f \) is replaced by \( \lambda f \) in (4), then \( a_0^2 \) is multiplied by a factor of \( \lambda^2 \), while the remaining factor in this expression remains unchanged; thus \( \theta \) gets multiplied by \( \lambda^2 \).

As noted earlier, Julia used the definition of \( Q(x, y) \) to develop a theory of reduction for binary \( n \)-ic forms, which generalizes the theory defined by Gauss for positive-definite quadratic forms. Many beautiful aspects of this theory are discussed by Stoll and Cremona in [24]. In particular, this reduction theory coincides with the classical reduction theory for binary cubic forms of positive discriminant, which uses the Hessian as a quadratic covariant. The utility of \( \theta \) arises from the fact that Julia showed that, for reduced binary \( n \)-ics, one can bound the leading coefficient \( a_0 \) in terms of \( \theta \); more precisely, he showed that

\[
a_0^2 \leq \frac{1}{\frac{3^{n/2}}{n^n}} \cdot \theta. \tag{6}
\]

Furthermore, Julia showed that one also can bound the magnitude \( |\alpha_i| \) of the roots of \( f(x, y) \) in terms of \( \theta/a_0^2 \); more precisely, we have

\[
|\alpha_i|^2 \leq \frac{1}{(n - 1)n^{-1}3^{n/2}} \cdot \frac{\theta}{a_0^2}. \tag{7}
\]

Julia provides explicit choices for the parameters \( t_j \) in the case of cubic and quartic forms; for the general case, he does not give as many details, but Stoll and Cremona provide a method for determining the \( t_j \) (and therefore both \( Q(x, y) \) and \( \theta(f) \)) in the general case of a binary form of degree \( n \).

Because \( Q(x, y) \) is a positive-definite quadratic form, there exists a unique point \( z(f) \) in the upper half plane \( \mathbb{H} \) that is a root of \( Q(x, 1) \). We say that \( Q(x, y) \) is reduced if \( z(f) \) lies in the usual fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \), and we say that \( f(x, y) \) is (Julia–)reduced if and only if \( Q(x, y) \) is reduced.

We assume that \( f(x, y) \) is what Stoll and Cremona call a stable form: that is, a form that has no repeated roots of multiplicity \( \geq n/2 \). Since we will only be counting irreducible integral forms, which have no repeated roots, this restriction will not impact our results. In [24], Cremona and Stoll prove that, if we write \( z(f) = t + iu \) where \( t, u \in \mathbb{R} \), then the representative point \( z(f) \) of \( f \)
in the upper half plane $\mathbb{H}$ is the point $(t, u)$ that minimizes the function

$$
\frac{F(t, u)}{u^n} = \frac{|a_0|^2 \prod_{j=1}^n (|t - \alpha_j|^2 + u^2)}{u^n}.
$$

(8)

The Julia invariant $\theta$ is then the minimal value of this function, and it is invariant under the action of $\text{SL}_2(\mathbb{R})$. For proofs of these assertions, as well as an elegant geometric description and alternate formulation of this condition using resultants, see [24, Section 5].

3 | A BOUNDED SEMIALGEBRAIC $\text{SL}_2(\mathbb{R})$-REDUCED REGION $L_n$ FOR REAL BINARY $n$-ics HAVING FIXED JULIA INVARIANT

The objective of this section is to exhibit a fundamental domain $L_n$ for the action of $\text{SL}_2(\mathbb{R})$ on the set of all real stable binary $n$-ics having a fixed Julia invariant (say 1), that is semialgebraic and lies in a bounded set. (Recall that a set in $V_n(\mathbb{R})$, which we identify naturally with $\mathbb{R}^{n+1}$, is called semialgebraic if it defined by finitely many polynomial inequalities.) The construction of $L_n$ will be useful to us in defining convenient fundamental domains for the action of $\text{SL}_2(\mathbb{Z})$ on real stable binary $n$-ic forms.

We begin by exhibiting a semialgebraic fundamental domain $E$ for the action of the usual compact group $K = \text{SO}_2(\mathbb{R})$ on the whole space $V_n(\mathbb{R})$ of real binary $n$-ic forms. Namely, we define $E$ as the set of all real binary $n$-ic forms $f(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \in V_n(\mathbb{R})$ such that the associated sequence $S(f)$ given by $|a_0|, -a_0, |a_1|, -a_1, \ldots, |a_n|, -a_n$ is minimal, with respect to the lexicographic ordering, among all forms $f' \in K \cdot f$ because $K$ is compact. The set $E$ is clearly a fundamental domain for the action of $K$ on $V_n(\mathbb{R})$. Moreover, this set $E \in V_n(\mathbb{R})$ may evidently be defined by polynomial equations and inequalities using the logical connectors $\lor$, $\land$, $\neg$ and the quantifiers $\forall$, $\exists$, and hence is semialgebraic by the theorem of Tarski and Seidenberg on quantifier elimination (see [25] and [21]).

To construct a bounded semialgebraic fundamental domain $L_n$ for the action of $\text{SL}_2(\mathbb{R})$ on real stable binary $n$-ics $f$ having Julia invariant 1, recall that the representative point $z(f)$ of $f$ in the upper half plane $\mathbb{H}$ is the point $t + iu$ that minimizes the function $\bar{F}(t, u)$, where $\bar{F}$ is as defined in (8); furthermore, $\theta(f)$ is the minimal value of this function. Let $L'_n$ denote the set of all real stable binary $n$-ic forms $f(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ satisfying $z(f) = i$ and $\theta(f) = 1$. Then the orthogonal group $K = \text{SO}_2(\mathbb{R})$, the stabilizer in $\text{SL}_2(\mathbb{R})$ of $i \in \mathbb{H}$, acts on $L'_n$. Let $L_n$ denote the fundamental domain $L'_n \cap E$ for the action of $K$ on $L'_n$.

Proposition 2. The set $L_n$ is a fundamental domain for the action of $\text{SL}_2(\mathbb{R})$ on the set of real stable binary $n$-ics having Julia invariant 1 and, moreover, $L_n$ is bounded and semialgebraic.

Proof. $L_n$ is a fundamental domain. Let $f$ be any real stable binary $n$-ic form having Julia invariant 1. Then there exists an element $\gamma \in \text{SL}_2(\mathbb{R})$ that sends the representative point $z(f)$ to $i$, because $\text{SL}_2(\mathbb{R})$ acts transitively on the upper half plane. Furthermore, since $z(f)$ is a covariant of $f$, if we act on $f$ by this same element $\gamma$, the resulting binary $n$-ic form will have $z(f) = i$ as its representative point in the upper half plane. In addition, $\gamma$ is uniquely determined up to left multiplication by elements of $K$, the stabilizer in $\text{SL}_2(\mathbb{R})$ of $i \in \mathbb{H}$. Thus, for any real stable binary $n$-ic
form $f$ with Julia invariant 1, by the definition of $L_n$ there exists a unique associated element $\gamma \cdot f$ ($\gamma \in \text{SL}_2(\mathbb{R})$) such that $\gamma \cdot f \in L_n$; hence $L_n$ is a fundamental domain for the action of $\text{SL}_2(\mathbb{R})$ on real stable binary $n$-ic forms having Julia invariant 1.

$L_n$ is bounded. It suffices to show that $L_n'$ lies in a bounded subset of $\mathbb{R}^{n+1}$. Suppose that $f \in L_n'$, that is, $f$ is a form with $z(f) = i$ and $\theta(f) = 1$. Then

$$\theta(f) = \frac{F(t, u)}{u^n} = |a_0|^2 \prod_{j=1}^n (|\alpha_j|^2 + 1) = 1,$$  

(9)

which is obtained by setting $t = 0$ and $u = 1$ in (8). In particular, this implies

$$\prod_{j=1}^n (|\alpha_j|^2 + 1) = \frac{1}{a_0^2}.$$  

(10)

If we expand the product in the expression on the left-hand side of (10), we see that the square of the absolute value of each (distinct) $k$-fold product of the $n$ roots of $f$ appears, for every $k \in \{0, \ldots, n\}$. Since each of the terms appearing in this expanded product is nonnegative, each is then bounded by $1/a_0^2$. For example, in the case $k = 1$, note that each $|\alpha_i|^2$ appears in this product, and so we have a bound of the form

$$|\alpha_1|^2 + \cdots + |\alpha_n|^2 \leq \frac{1}{a_0^2}.$$  

Since $a_k/a_0$ is, up to sign, the sum of the distinct $k$-fold products of the roots $\alpha_i$ of $f(x, 1)$, by the Cauchy–Schwarz inequality we obtain

$$\left|\frac{a_k}{a_0}\right| = \left|\sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1} \cdots \alpha_{i_k}\right| \leq \left(\binom{n}{k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |\alpha_{i_1} \cdots \alpha_{i_k}|^2\right)^{1/2} \leq \frac{1}{|a_0|} \left(\binom{n}{k}\right)^{1/2},$$

which implies that $|a_k| \leq \left(\binom{n}{k}\right)^{1/2} < 2^{n/2}$. This shows that the forms $f$ in $L_n'$ have the property that all coefficients are less than $2^{n/2}$ in absolute value; thus the set $L_n'$ (and hence $L_n$) is indeed contained in a bounded set.

$L_n$ is semialgebraic. Again, it suffices to show that the set $L_n'$ is semialgebraic. By [24, Equations 4.5], the condition that $z(f) = i$ is equivalent to the condition that the roots $\alpha_1, \ldots, \alpha_n$ of $f(x, 1)$ satisfy the two equations

$$\sum_{j=1}^n \frac{1}{|\alpha_j|^2 + 1} = \frac{n}{2},$$  

(11)

$$\sum_{j=1}^n -\alpha_j |\alpha_j|^2 + 1 = 0.$$  

(12)

In addition, when $z(f) = i$, by equation (9) the condition that $\theta(f) = 1$ is equivalent to

$$|a_0|^2 \prod_{j=1}^n (|\alpha_j|^2 + 1) = 1.$$  

(13)
These three equations taken together define a semialgebraic set in the space whose coordinates are \((a_0, \alpha_1, \ldots, \alpha_n)\). (It is possible that some of the \(\alpha_i\) are complex, in which case we think of each such \(\alpha_i\) as an element of \(\mathbb{R}^2\).) Since there is a polynomial map from the space with coordinates \((a_0, \alpha_1, \ldots, \alpha_n)\) to the space of coefficients \((a_0, \ldots, a_n)\) of \(f(x, y)\) (namely, the polynomial map which expresses each coefficient \(a_i\) as a function of \(a_0\) and the \(\alpha_i\)), and polynomial images of semialgebraic sets are semialgebraic by the theorem of Tarski and Seidenberg, this shows that \(L'_n\) is also semialgebraic. The set \(L_n = L'_n \cap E\), being the intersection of two semialgebraic sets, is then also semialgebraic.

For each fixed \(n\), we have thus obtained a fundamental domain \(L_n\), for the action of \(\text{SL}_2(\mathbb{R})\) on real stable binary \(n\)-ic forms having Julia invariant 1, that is bounded and is defined by some fixed set of polynomial equalities and inequalities. More generally, by restricting the above construction to just those real binary \(n\)-ic forms having nonzero discriminant and \(n-2k\) real roots (which is also a semialgebraic subset of \(V_n(\mathbb{R}) \cong \mathbb{R}^{n+1}\)), we obtain a fundamental domain \(L_{n,k} \subset L_n\) for the action of \(\text{SL}_2(\mathbb{R})\) on real binary \(n\)-ic forms having nonzero discriminant, \(n-2k\) real roots, and Julia invariant 1, which is again bounded and semialgebraic.

4 | REDUCTION THEORY FOR THE ACTION OF \(\text{SL}_2(\mathbb{Z})\) ON BINARY \(n\)-ics

Let \(k \in \{0, 1, \ldots, \lfloor n/2 \rfloor\}\), and let \(L_{n,k}\) denote a fundamental domain for the action of \(\text{SL}_2(\mathbb{R})\) on the open subset \(V_{n,k} \subset V_n(\mathbb{R})\) of those nondegenerate binary \(n\)-ic forms \(f\) with coefficients in \(\mathbb{R}\) having \(n-2k\) real roots, and satisfying \(\theta(f) = 1\); here, a binary \(n\)-ic form is called nondegenerate if it has nonzero discriminant. By the previous section, we may assume that \(L_{n,k}\) is bounded and semialgebraic. For convenience, we will assume for now (until Remark 9) that \(n \geq 3\).

Let \(\mathcal{F}\) denote Gauss’s usual fundamental domain for \(\text{GL}_2^+(\mathbb{Z})\) acting on \(\text{GL}_2^+(\mathbb{R})\), where \(\text{GL}_2^+(\mathbb{R})\) is the subgroup of \(\text{GL}_2(\mathbb{R})\) of elements having positive determinant, and \(\text{GL}_2^+(\mathbb{Z})\) is simply \(\text{GL}_2^+(\mathbb{R}) \cap \text{GL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})\). Then \(\mathcal{F}\) may be expressed in the form \(\mathcal{F} = \{\nu \alpha \lambda : \nu = \nu(u) \in N'(\alpha), \alpha = \alpha(t) \in A', \lambda \in K\},\) where

\[
N'(\alpha) = \left\{ \nu(u) = \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} : u \in I(\alpha) \right\},
\]

\[
A' = \left\{ \alpha(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} : t^2 \geq \sqrt{3}/2 \right\},
\]

\[
\Lambda = \left\{ \lambda = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} : \lambda > 0 \right\},
\]

and \(K\) is the usual (compact) real orthogonal group \(\text{SO}_2(\mathbb{R})\); here \(I(\alpha)\) is a union of one or two subintervals of \([-\frac{1}{2}, \frac{1}{2}]\) depending only on the value of \(\alpha \in A'\). We use \(N \subset \text{SL}_2(\mathbb{R})\) to denote the subgroup of all matrices of the form \(\nu(u) (u \in \mathbb{R})\) and \(A \subset \text{SL}_2(\mathbb{R})\) to denote the subgroup of all diagonal matrices \(\alpha(t) (t \in \mathbb{R}^\times)\) of determinant 1, so that \(N' \subset N\) and \(A' \subset A\). In this notation, we also have the Iwasawa decomposition \(\text{SL}_2(\mathbb{R}) = NAK\).
Let $m = m(n, k)$ denote the size of $\#\text{Stab}_{\text{SL}_2(\mathbb{R})}(v)/\#\text{Stab}_{\text{SL}_2(\mathbb{Z})}(v)$ for a generic element $v \in V_{n,k}$ (that is, for $v$ outside a set of measure 0 in $V_{n,k}$). Then it is easy to see and well known that $m = 3$ if $(n, k) = (3, 0); m = 4$ if $(n, k) = (4, 0)$ or $(4, 2); m = 2$ if $(n, k) = (4, 1);$ and $m = 1$ otherwise.

Let $L := L_{n,k}$. For $h \in \text{GL}_2(\mathbb{R})$, we regard $FhL$ as a multiset, where the multiplicity of a point $v$ in $FhL$ is the cardinality of the set $\{g \in F : v = ghL\}$. By the argument of [4, §2.1], the $\text{SL}_2(\mathbb{Z})$-equivalence class of $v \in V_{n,k}$ is represented $m_v = \#\text{Stab}_{\text{SL}_2(\mathbb{R})}(v)/\#\text{Stab}_{\text{SL}_2(\mathbb{Z})}(v)$ times in $FhL$. It follows, as in [4, §2.1], that away from a measure zero set (where $m_v \neq m$), the multiset $FhL$ is the union of $m$ fundamental domains for the action of $\text{SL}_2(\mathbb{Z})$ on $V_{n,k}$.

Thus for any $h \in \text{GL}_2(\mathbb{R})$, if we let $R_X(hL)$ denote the multiset $\{v \in FhL : \theta(w) < X\}$, then the product $mN_{n,k}(X)$ is equal to the number of irreducible integer points in $R_X(hL)$, with the slight caveat that the (relatively rare — see Corollary 6) integer points $v \in V_{n,k}$ with $m_v \neq m$ are counted with weight $m/m_v$.

Thus, to determine the asymptotic behavior of $N_{n,k}(X)$, it suffices to count the number of lattice points in $R_X(hL)$. However, one major obstacle to counting integer points of bounded height in $R_X(hL)$ is that it is not bounded, but rather has a cusp going off to infinity. We simplify the counting in this cuspidal region by ‘thickening’ the cusp; more precisely, we compute the number of integer points in $R_X(hL)$ by averaging over a compact continuum of such fundamental domains, where $h$ ranges over some suitable compact subset $G_0 \subset \text{GL}_2(\mathbb{R})$. This adaptation of the method of [2] is described in more detail in §6.

However, in §5 we first examine the problem of estimating the number of reducible points in the main bodies (that is, away from the cusps) of our fundamental domains.

## 5 ESTIMATES ON REDUCIBILITY

We first consider the integral elements in the region $R_X(hL) := \{f \in FhL : \theta(f) < X\}$ that are reducible over $\mathbb{Q}$, where $h$ is any element in a fixed compact subset $G_0$ of $\text{GL}_2(\mathbb{R})$. Let $V_n(\mathbb{Z})$ denote the lattice of integral binary $n$-ic forms in $V_n(\mathbb{R})$. Note that if a binary $n$-ic form $a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \in V_n(\mathbb{Z})$ satisfies $a_0 = 0$, then it is automatically reducible over $\mathbb{Q}$, since $y$ is a factor. The following lemma shows that for integral binary $n$-ic forms in $R_X(hL)$, reducibility with $a_0 \neq 0$ does not occur very often:

**Lemma 3.** Let $h \in G_0$ be any element, where $G_0$ is any fixed compact subset of $\text{GL}_2(\mathbb{R})$. Then the number of integral binary $n$-ic forms $a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \in R_X(hL)$ that are reducible over $\mathbb{Q}$ with $a_0 \neq 0$ is $O(X^{n+1}2^{n-1}+\varepsilon)$, where the implied constant depends only on $n, G_0$, and $\varepsilon$.

**Proof.** Let $f(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ be any element in $R_X(hL) \cap V_n(\mathbb{Z})$ with $a_0 \neq 0$. Since coefficients of forms in $hL$ are uniformly bounded, and since $R_X(hL) \subset N' A' K NhL$ (where $0 < \lambda \ll X^{1/2n}$, with the absolute constant only depending on $G_0$), we see that

$$\prod_{0 \leq i \leq n-1} a_0 \prod_{0 \leq i \leq n-1} a_i = O(X^{\frac{n}{2}}),$$

implying that that the number of points in $R_X(hL)$ with $a_0 \neq 0$ and $a_n = 0$ is $O(X^{\frac{n}{2}+\varepsilon})$. Indeed, the actions of $N$ and $K$ only change coefficients by an absolute constant, while a generic element
of $A'$ sends the coefficients $(a_0, ..., a_n)$ to $(a_0 a^{-n}, a_1 a^{-(n-2)}, ..., a_n a^n)$; the bound above follows (recall that we chose $A$ such that $a$ is bounded from below). Hence we may assume that $a_0 \neq 0$ and $a_n \neq 0$.

Now suppose that $f$ factors as $f = rs$, where $r, s$ are binary forms where $r$ has degree $k \geq 1$ and $s$ has degree $n - k$, such that $k \leq n - k$. We write $r(x, y) = b_0 x^k + b_1 x^{k-1} y + \cdots + b_k y^k$ and $s(x, y) = c_0 x^{n-k} + c_1 x^{n-k-1} y + \cdots + c_{n-k} y^{n-k}$. Then the assumption that $a_0, a_n \neq 0$ implies that we also must have $b_0, c_0 \neq 0$.

Since $f \in R_X(hL)$, we may write $f = \nu x \lambda h f_0$, where $\nu \in N'(\alpha), \alpha \in A', \lambda \in \mathbb{R}_{>0}$ with $\lambda = O(X^{1/2n})$, and $f_0 \in L$. If we define the height $H(F)$ of a binary form $F$ as the maximum of the absolute values of its coefficients, since $L$ lies in a compact set, we have $H(f_0) \ll 1$. Furthermore, the factorization of $f$ as $f = rs$ corresponds to a factorization $f_0 = r_0 s_0$, so that just as $f = \nu x \lambda h f_0$, we also have $r = \nu x \lambda h r_0$ and $s = \nu x \lambda h s_0$, where $r_0$ and $s_0$ are real polynomials of degree $k$ and $n - k$, respectively. By Gelfond’s inequality (see [20, Theorem 4.2.2]), since $f_0 = r_0 s_0$,

$$H(r_0)H(s_0) \leq 2^{n-2} \sqrt{n + 1}H(f_0) = O(1). \quad (15)$$

Since $\nu$ acts by a bounded lower triangular transformation, $\alpha$ acts by $(t^{-1}, t)$ for some $t \gg 1$, $K$ is compact, and $\lambda = O(X^{1/2n})$, it follows from (15) that

$$\left[ \prod_{0 \leq i \leq k} |b_i| \prod_{b_i \neq 0} |b_i| \right]^{1\over k+1} \cdot \left[ \prod_{0 \leq j \leq n-k} |c_j| \prod_{c_j \neq 0} |c_j| \right]^{1\over n-k+1} = O \left( X^{1\over 2} \right),$$

or equivalently,

$$\left[ \prod_{0 \leq i \leq k} |b_i| \prod_{b_i \neq 0} |b_i| \right]^{n-k+1\over k+1} \cdot \left[ \prod_{0 \leq j \leq n-k} |c_j| \prod_{c_j \neq 0} |c_j| \right]^{n-k+1\over n-k+1} = O \left( X^{n-k+1\over 2} \right). \quad (16)$$

The number of integer possibilities for the $b_i$ and $c_j$, subject to (16), is evidently at most $O(X^{n-k+1\over 2} + \varepsilon)$. Since by assumption $k \geq 1$ (that is, $f$ factors nontrivially), we obtain the desired estimate. □

In fact, we may prove the stronger statement that most (that is, 100%) of binary $n$-ic forms in the fundamental domain $R_X(hL)$ with $a_0 \neq 0$ are not only irreducible but also have associated Galois group $S_n$. For monic polynomials ordered by the maximum of the absolute values of their coefficients, this is a well-known result of van der Waerden [26]. Specifically, van der Waerden showed that among the $\sim (2H)^n$ monic integral polynomials of degree $n$ whose coefficients are bounded in absolute value by $H$, at most $O(H^{n-6/(n-2) \log \log n})$ have associated Galois group not $S_n$. This was subsequently improved by Gallagher [14] to $O(H^{n-1/2} \log H)$, by Zywina [28] to $O(H^{n-1/2})$, by Dietmann [12] to $O(H^{n-2+\sqrt{2}+\varepsilon})$, by Anderson, Gafni, Lemke Oliver, Lowry-Duda, Shakan, and Zhang [1] to $O(H^{n-\frac{3}{2}+\frac{2}{3\pi\sqrt{3}}+\varepsilon})$, and most recently to the optimal $O(H^{n-1})$ in [3].
These results do not directly apply to the situation at hand, as we are counting polynomials in a noncompact fundamental domain for $\text{SL}_2(\mathbb{Z})$ rather than in a compact box having equal-length sides. Nevertheless, the methods of Dietmann [11] can be adapted to our situation to yield the following:

**Theorem 4.** Let $h \in G_0$ be any element, where $G_0$ is any fixed compact subset of $\text{GL}_2(\mathbb{R})$. Then the number of integral binary $n$-ic forms $a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \in \mathcal{R}_X(hL)$ with $a_0 \neq 0$ whose Galois group over $\mathbb{Q}$ is not isomorphic to $S_n$ is $O(X^{\frac{n+1}{2} - \frac{1}{4} + \varepsilon})$, where the implied constant depends only on $n, G_0,$ and $\varepsilon$.

**Proof.** While the methods of either [7] or [11] can be adapted to prove this result, we use the methods of [11] as they are technically simpler.

First, we note that the ideas of [11] can be applied even to integral polynomials $g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, that are not necessarily monic, that is, for which $a_0 \neq 1$, so long as $a_0$ is nonzero. The reason is that [11, Lemma 2] holds also for such nonmonic polynomials $g(x)$: simply apply the proof there to $h(x) = a_0^{n-1}g(x/a_0)$, which is monic, and then the identical result is then seen to hold true for $g(x)$. The definition of resolvent in [11, Lemma 5] can also be modified similarly, again by replacing the resolvent $r(x)$ as given by $a_0^{\text{deg}(r)-1}r(x/a_0)$, so that the modified resolvent is again integral. All arguments then apply in the identical manner.

To obtain Theorem 4, we now proceed as follows. Suppose we are given the coefficients $a_0, \ldots, a_{n-2} \in \mathbb{Z}$ of $f(x, y; a_{n-1}, a_n) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$, where $a_0 \neq 0$. Then [11, Lemma 2], as modified above, implies that there are only at most $n^2 + n$ integral values of $a_{n-1}$ such that $f$ does not have associated Galois group $S_n$ over $\mathbb{Q}(a_n)$. Since

$$\prod_{\substack{i \neq n-1 \ni \in 0 \ni \neq 0 \ni \neq 0}} |a_i| = O((X^{\frac{n+1}{2} - \frac{1}{2} + \varepsilon}))$$

for integral binary forms $f(x, y)$ in $\mathcal{R}_X(hL)$, the total number of such binary $n$-ic forms with $a_0 \neq 0$ is at most $O((X^{\frac{n+1}{2} - \frac{1}{2} + \varepsilon})^n)$.

Next, suppose again that $a_0, \ldots, a_{n-2}$ are given, and furthermore suppose that $a_{n-1} \in \mathbb{Z}$ is not among the above $n^2 + n$ distinguished values, so that the associated Galois group of the binary form $f$ over $\mathbb{Q}(a_n)$ is in fact $S_n$. Since

$$a_n = O((X^{-\frac{1}{2}}/A)$$

where

$$A = \prod_{1 \leq i \leq n-1 \text{ } a_i = 0} |a_i| \prod_{0 \leq i \leq n-1 \text{ } a_i \neq 0} |a_i|,$$

the argument of the proof of [11, Theorem 1] shows that at most $O((X^{\frac{n+1}{2}}/A)^{1/2})$ of these values of $a_n$ can yield binary forms $f(x, y)$ having associated Galois group smaller than $S_n$. Since the number of values of $a_0, \ldots, a_{n-1}$ yielding a given value of $A$ is $O(A^\varepsilon)$, the number of possible
values of $a_0, \ldots, a_n$ is thus at most

$$\sum_{A=O(X^{n/2})} O\left( \mathcal{A}^2 \left( X^{n+1/2} \right) \right) = O\left( X^{n+1/2} - \frac{1}{4+\varepsilon} \right),$$

yielding the desired result.

The $O$-estimate in Theorem 4 can be further improved to $O(X^{n+1/2} - \frac{1}{2})$ using the methods of [3], although we shall not require this improvement here when $n > 3$. For $n = 3$, the further improved estimate $O(X)$ can be deduced from [5] (see also [10, Lemma 2] for a proof of the estimate $O(X^{3/2+\varepsilon})$), while for $n = 2$, we observe that every definite integral binary quadratic form has Galois group $S_2$.

One interesting and useful consequence of a binary $n$-ic form $f$ having associated Galois group $S_n$ ($n \geq 5$) is that in that case $f$ cannot have any nontrivial projective linear automorphisms over $\mathbb{Q}$, that is, there cannot exist elements in $SL_2(\mathbb{Q})$ that stabilize $f$ and induce a nontrivial permutation of the roots of $f$:

**Theorem 5.** Suppose $n \geq 5$. If a binary $n$-ic form $f(x, y) \in V_n(\mathbb{Z})$ is irreducible with Galois group $S_n$, then $f$ has no projective linear automorphisms over $\mathbb{Q}$.

**Proof.** Suppose $n \geq 5$. Let $f$ be an integral binary $n$-ic form having associated Galois group $S_n \cong G \subset \text{Gal}(\mathbb{Q}/\mathbb{Q})$. Let $H \subset G$ denote the subgroup of those symmetries of the roots of $f$ in $P^1(\mathbb{Q})$ that come from symmetries of $P^1(\mathbb{Q})$ in $PGL_2(\mathbb{Q})$. Then $H$ is normal in $G$; indeed, if $h \in H$ and $g \in G$, then $ghg^{-1}$ is again in $H$, for if we write $h(x) = \frac{ax+b}{cx+d}$, then

$$ghg^{-1}(x) = g\left( \frac{ag^{-1}(x) + b}{cg^{-1}(x) + d} \right) = \frac{g(a)x + g(b)}{g(c)x + g(d)}.$$

It follows from a result of Olver [19, Corollary 8.68] that for $n \geq 5$, we have $|H| \leq 4n - 8$. However, for $n \geq 5$, the only subgroup of $S_n$ that is normal and of cardinality at most $4n - 8$ is the trivial subgroup, and Theorem 5 follows.

**Corollary 6.** Let $h \in G_0$ be any element, where $G_0$ is any fixed compact subset of $SL_2(\mathbb{R})$. Then all but $O(X^{n+1/2} - \frac{1}{4+\varepsilon})$ of the integral binary $n$-ic forms $f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n \in R_X(hL)$ with $a_0 \neq 0$ are irreducible over $\mathbb{Q}$, have associated Galois group $S_n$, and satisfy $m_f = m$. (Here again the implied constant depends only on $n, G_0, \text{and } \varepsilon$.)

**Proof.** In the case of $n = 3$, this follows directly from Theorem 4 and [10, Lemma 2], while in the case $n = 4$, the argument is identical to the proofs of [4, Lemmas 2.2 and 2.4]. For $n = 5$, the assertion follows from Theorems 4 and 5; indeed, the stabilizer in $GL_2(C)$ of a binary $n$-ic form $f$ is an extension of the projective automorphism group of $f$ over $C$ by the group of $n$th roots of unity in $C^\times$, and the only $n$th roots of unity in $R_X^\times$ (or $Z^\times$) are $1$ or $\pm 1$ depending on whether $n$ is odd or even. This completes the proof.
6 AVERAGING AND CUTTING OFF THE CUSP

Let $G_0$ be a compact, semialgebraic, left $K$-invariant set in $GL_2(\mathbb{R})$ that is the closure of a nonempty open set and in which every element has determinant greater than or equal to 1. Then we may write

$$N_{n,k}(X) = \frac{\int_{h \in G_0} \# \{ x \in FH \cap V_n(\mathbb{Z})^{irr} : \theta(x) < X \} dh}{m \cdot \int_{h \in G_0} dh}, \quad (17)$$

where $L := L_{n,k}$, $V_n(\mathbb{Z})^{irr}$ denotes the set of irreducible elements in $V_n(\mathbb{Z})$, and $dh$ is Haar measure on $GL_2(\mathbb{R})$. The denominator of the latter expression is an absolute constant $C_{G_0}^{n,k}$ greater than 0.

More generally, for any $SL_2(\mathbb{Z})$-invariant subset $S \subset V_{n,k}(\mathbb{Z}) := V_n(\mathbb{Z}) \cap V_{n,k}$, let $N(S; X)$ denote the number of irreducible $SL_2(\mathbb{Z})$-orbits in $S$ having Julia invariant less than $X$. Let $S^{irr}$ denote the subset of irreducible points of $S$. Then $N(S; X)$ can be similarly expressed as

$$N(S; X) = \frac{\int_{h \in G_0} \# \{ x \in FH \cap S^{irr} : \theta(x) < X \} dh}{C_{G_0}^{n,k}}. \quad (18)$$

Now, given $x \in V_{n,k}$, let $x_L$ denote the unique point in $L$ that is equivalent by an element of $GL_2^+(\mathbb{R})$ to $x$. Then

$$N(S; X) = \frac{1}{C_{G_0}^{n,k}} \sum_{x \in S^{irr}} \int_{h \in G_0} \# \{ g \in \mathcal{F} : x = ghx_L \} dh. \quad (19)$$

For a given $x \in S^{irr}$, since $n \geq 3$, there exist a finite number of elements $g_1, \ldots, g_r \in GL_2^+(\mathbb{R})$ satisfying $g_jx_L = x$. We then have

$$\int_{h \in G_0} \# \{ g \in \mathcal{F} : x = ghx_L \} dh = \sum_j \int_{h \in G_0} \# \{ g \in \mathcal{F} : gh = g_j \} dh = \sum_j \int_{h \in G_0 \cap \mathcal{F}^{-1}g_j} dh.$$

As $dh$ is an invariant measure on $GL_2(\mathbb{R})$, we have

$$\sum_j \int_{h \in G_0 \cap \mathcal{F}^{-1}g_j} dh = \sum_j \int_{g \in G_0 g_j^{-1} \cap \mathcal{F}^{-1}} dg = \sum_j \int_{g \in \mathcal{F}} \# \{ h \in G_0 : gh = g_j \} dg$$

$$= \int_{g \in \mathcal{F}} \# \{ h \in G_0 : x = ghx_L \} dg.$$

Therefore,

$$N(S; X) = \frac{1}{C_{G_0}^{n,k}} \sum_{x \in S^{irr}} \int_{g \in \mathcal{F}} \# \{ h \in G_0 : x = ghx_L \} dg.$$
\[
\begin{aligned}
&= \frac{1}{C_{n,k}^{G_0}} \int_{g \in F} \# \{ x \in S^\text{irr} \cap gG_0L : \vartheta(x) < X \} dg \\
&= \frac{1}{C_{n,k}^{G_0}} \int_{g \in N'(t)A'\Lambda K} \# \{ x \in S^\text{irr} \cap n(t^{-1}L)\lambda \lambda G_0L : \vartheta(x) < X \} t^{-2}dn dt d\lambda dx.
\end{aligned}
\]

Let us write \( B(u, t, \lambda, X) = \nu(u)\left( t^{-1}L \right)\lambda \lambda G_0L \cap \{ x \in V_{n,k} : \vartheta(x) < X \} \). Then since \( G_0 \) is left \( K \)-invariant, and we may normalize Haar measure so that \( \int_{\times K} dx = 1 \), we obtain

\[
N(S; X) = \frac{1}{C_{n,k}^{G_0}} \int_{g \in N'(t)A'\Lambda} \# \{ x \in S^\text{irr} \cap B(u, t, \lambda, X) \} t^{-2}du dt d\lambda.
\] (20)

To estimate the number of lattice points in \( B(u, t, \lambda, X) \), we have the following proposition due to Davenport [9].

**Proposition 7.** Let \( R \) be a bounded, semialgebraic multiset in \( \mathbb{R}^n \) having maximum multiplicity \( m \), and that is defined by at most \( k \) polynomial inequalities each having degree at most \( \ell \). Let \( R' \) denote the image of \( R \) under any (upper or lower) triangular, unipotent transformation of \( \mathbb{R}^n \). Then the number of integer lattice points (counted with multiplicity) contained in the region \( R' \) is

\[
\text{Vol}(R) + O(\max\{\text{Vol}(\bar{R}), 1\}),
\]

where \( \text{Vol}(R) \) denotes the greatest \( d \)-dimensional volume of any projection of \( R \) onto a coordinate subspace obtained by equating \( n - d \) coordinates to zero, where \( d \) takes all values from 1 to \( n - 1 \). The implied constant in the second summand depends only on \( n, m, k, \) and \( \ell \).

Although Davenport states the above lemma only for compact semialgebraic sets \( R \subset \mathbb{R}^n \), his proof adapts without significant change to the more general case of a bounded semialgebraic multiset \( R \subset \mathbb{R}^n \), with the same estimate applying also to any image \( R' \) of \( R \) under a unipotent triangular transformation.

By our construction of \( L \), the coefficients of the binary \( n \)-ic forms in \( G_0L \) are all uniformly bounded. Let \( C^n \) be a constant that bounds the absolute value of the leading coefficient \( a_0 \) of all the forms \( a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \) in \( G_0L \). (We choose \( C^n \) instead of \( C \) to simplify the exponents of \( C \) in the calculations which follow.)

We then have the following lemma on the number of irreducible lattice points in \( B(u, t, \lambda, X) \):

**Proposition 8.** The number of lattice points \( (a_0, \ldots, a_n) \) in \( B(u, t, \lambda, X) \) with \( a_0 \neq 0 \) is

\[
\left\{ \begin{array}{ll}
0 & \text{if } \frac{C^1}{t} < 1; \\
\text{Vol}(B(u, t, \lambda, X)) + O(\max\{t^n\lambda^{n^2}, 1\}) & \text{otherwise},
\end{array} \right.
\]

where the implied constant in the big-O expression depends only on \( n \) and \( G_0 \).
Proof. If $C\lambda / t < 1$, then $a_0 = 0$ is the only possibility for an integral binary $n$-ic form $a_0 x^n + \cdots + a_n y^n$ in $B(u, t, \lambda, X)$, and any such form is reducible. Indeed, note that any element in $B(u, t, \lambda, X)$ has first coordinate bounded by $C^n \lambda^n / t^n$, which is $< 1$ if $C\lambda / t < 1$. If $C\lambda / t \geq 1$, then $\lambda$ and $t$ are positive numbers bounded from below by $(\sqrt{3}/2)^{1/2} / C$ and $(\sqrt{3}/2)^{1/2}$, respectively. In this case, one sees that the projection of $B(u, t, \lambda, X)$ onto $a_0 = 0$ has volume $O(t^n \lambda^{n^2})$: the coefficients of forms in $G_0 L$ are uniformly bounded, and acting by the scalar $\lambda$ scales each coefficient by a factor of $\lambda^n$. Acting by the scalar matrix $(t^{-1})^t$ then multiplies the $k$th coefficient by $t^{-n+2k}$. Thus, after acting by these two elements, $a_k$ is multiplied by a factor of $\lambda^n t^{-n+2k}$. The product of these numbers, for $0 \leq k \leq n$, is $\lambda^{n(n+1)} t^0$, and represents a big-O upper bound for the volume of $B(0, t, \lambda, X)$ and therefore also $B(u, t, \lambda, X)$. Therefore, if we project this region onto $a_0 = 0$, an upper bound for the volume of this projection is given by $\lambda^n n^2 t^n$, as claimed.

Now consider any other projection of $B(u, t, \lambda, X)$ onto one of the subspaces $a_k = 0$, say. The volume of this projection is given by $O(\lambda^n t^{n-2k})$. This is evidently $O(\lambda^n t^n)$, since $t$ is uniformly bounded from below. If we want to project to a space defined by an additional condition $a_\ell = 0$, say, this space will have volume bounded by $O(\lambda^n t^n / (\lambda^n t^{n+2\ell})) = O(\lambda^n t^{n-2-2\ell})$. Although the exponent of $t$ might increase, we use the fact that $CA > t$ to exchange $n - 2\ell$ (which is $\leq n$) factors of $t$ for $n - 2\ell \leq n$ factors of $\lambda$, which shows that this expression is still $O(\lambda^n t^n)$. It is clear that we may interchange powers of $t$ for powers of $\lambda$ in any projection of the original region onto any proper subspace spanned by coordinate axes to get an upper bound of $O(\lambda^n t^n)$ on their volumes. The lemma then follows from Proposition 7. □

In (20), since $L$ (and therefore also $G_0 L$) only contains points with Julia invariant at least 1, we observe (by the definition of $B(u, t, \lambda, X)$) that the integrand will be nonzero only if $t < C\lambda$ and $\lambda < X^{1/2n}$. Thus we may write

\[
N(V_n(Z) \cap V_{n,k}; X) = \frac{1}{C_{G_0}^{n,k}} \int_{\lambda=(\sqrt{3}/2)^{1/2}}^{C\lambda} \int_{t=(\sqrt{3}/2)^{1/2}}^{C\lambda} (\text{Vol}(B(u, t, \lambda, X)) + O(max\{t^n\lambda^{n^2}, 1\})) t^{-2} du dX dY
\]

\[+ O(X^{n+\epsilon}),
\]

where the latter error term is due to the estimate on reducible forms in Lemma 3.

Let us first consider the evaluation of the integral of the second summand in (21). First, we observe that $t^n \lambda^{n^2} \gg 1$, so that the integral of the second summand is bounded from above by (a constant factor times)

\[
\frac{1}{C_{G_0}^{n,k}} \int_{\lambda=(\sqrt{3}/2)^{1/2}}^{X^{1/2n}} \int_{t=(\sqrt{3}/2)^{1/2}}^{C\lambda} t^n \lambda^{n^2} t^{-2} d\lambda dt \ll \int_{\lambda=(\sqrt{3}/2)^{1/2}}^{X^{1/2n}} \lambda^n t^{n-2} d\lambda dt \ll \int_{t=(\sqrt{3}/2)^{1/2}}^{C\lambda} dX d\lambda
\]

\[\ll \frac{1}{C_{G_0}^{n,k}} \int_{\lambda=(\sqrt{3}/2)^{1/2}}^{X^{1/2n}} C^{n-2} \lambda^{n^2+n-2} d\lambda = C^{n-2} \lambda^{n^2+n-2} \Big|_{\lambda=(\sqrt{3}/2)^{1/2}}^{X^{1/2n}} = O(X^{-n/2 - 1}),
\]

(23)
Meanwhile, the integral of the first summand is

\[
\int_{n \in G_0} \frac{1}{C_{n,k}} \int_{\lambda = (\sqrt{3}/2)^{1/2} / C}^{X^{1/2n}} \lambda \int_{t = C\lambda}^{N'(t)} Vol(B(u, t, \lambda, X)) t^{-2} d\lambda d\lambda d\lambda d\lambda - O(\int_{\lambda = (\sqrt{3}/2)^{1/2} / C}^{X^{1/2n}} \lambda \lambda^2 + n - 2 |X^{1/2n}|(\sqrt{3}/2)^{1/2} / C) = O(X^{n+1/2 - \frac{1}{n}}).
\]

(24)

However, Vol(\(R_X(hL)\)) is independent of \(h\), so that the first term in (24) is simply Vol(\(R_X(L)\)). Next, using the fact that Vol(B(u, t, \lambda, X)) = O(\(\lambda^{n+1}\)), and carrying out the integration in the second term of (24) exactly as in (22)–(23), we find that this term is also O(\(X^{n+1/2 - \frac{1}{n}}\)):

\[
\int_{\lambda = (\sqrt{3}/2)^{1/2} / C}^{X^{1/2n}} \lambda^{n+1} t^{-2} d\lambda d\lambda d\lambda d\lambda \ll - \int_{\lambda = (\sqrt{3}/2)^{1/2} / C}^{X^{1/2n}} \lambda^{n+1} t^{-2} |t = C\lambda| d\lambda d\lambda d\lambda d\lambda = O(X^{n+1/2 - \frac{1}{n}}).
\]

(25)

We conclude that

\[
N_{n,k}(X) = Vol(R_X(L)) + O(X^{n+1/2 - \frac{1}{n}}).
\]

(27)

This proves Theorem 1.

Remark 9. The proof we have given for \(n \geq 3\) also adapts easily to the case \(n = 2, k = 1\). Indeed, rather than being finite, the stabilizer in SL_2(\(\mathbb{R}\)) of a definite binary quadratic form is compact and conjugate to \(K\). In the usual way, we may then replace occurrences of cardinalities of sets of group elements with integrals over \(K\); for example, \#Stab_{SL_2(\mathbb{R})}(v) is replaced by \(\int_{x \in K} dx\) (which we may normalize to be 1). All other arguments then hold without any essential change, yielding Theorem 1 for \(n = 2\) as well.

7 | CONGRUENCE CONDITIONS

We may prove a version of Theorem 1 for a set in \(V_{n,k}(\mathbb{Z})\) that is defined by a finite number of congruence conditions:

**Theorem 10.** Suppose \(S\) is an SL_2(\(\mathbb{Z}\))-invariant subset of \(V_{n,k}(\mathbb{Z})\) that is defined by congruence conditions modulo finitely many prime powers. Then we have

\[
N(S; X) = c_{n,k} \cdot \prod_p \mu_p(S) \cdot X^{\frac{n+1}{2}} + O_S(X^{\frac{n+1}{2} - \frac{1}{n}}),
\]

(28)

where \(\mu_p(S)\) denotes the density of the \(p\)-adic closure of \(S\) in \(V_n(\mathbb{Z}_p)\).
To obtain Theorem 10, note that the set \( S \subset V_{n,k}(\mathbb{Z}) \) in Theorem 10 may be viewed for some fixed integer \( m \) as the intersection of \( V_{n,k} \) with the union \( U \) of (say) \( \tau \) translates \( L_1, \ldots, L_\tau \) of the lattice \( m \cdot V_n(\mathbb{Z}) \). For each such lattice translate \( L_j \), we may use formula (20) and the discussion following that formula to compute \( N(L_j \cap V_{n,k}; X) \), where each \( d \)-dimensional volume is scaled by a factor of \( 1/m^d \) to reflect the fact that our new lattice has been scaled by a factor of \( m \). Proceeding as in § 6 then gives by the identical arguments:

\[
N(S; X) = \tau m^{-(n+1)} \text{Vol}(R_\mathcal{X}(v)) + O_S(X^{-\frac{n+1}{2}} \frac{1}{n}).
\]  

(29)

Finally, the identity \( \tau m^{-(n+1)} = \prod_p \mu_p(S) \) yields (28).

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