Relaxation dynamics of the Kuramoto model with uniformly distributed natural frequencies

Anandamohan Ghosh\textsuperscript{1} and Shamik Gupta\textsuperscript{2}

\textsuperscript{1}Indian Institute of Science Education and Research-Kolkata, Mohanpur 741252, India

\textsuperscript{2}Laboratoire de Physique Théorique et Modèles Statistiques, UMR 8626, Université Paris-Sud 11 and CNRS, Bâtiment 100, Orsay F-91405, France

Abstract

The Kuramoto model describes a system of globally coupled phase-only oscillators with distributed natural frequencies. The model in the steady state exhibits a phase transition as a function of the coupling strength, between a low-coupling incoherent phase in which the oscillators oscillate independently and a high-coupling synchronized phase. Here, we consider a uniform distribution for the natural frequencies, for which the phase transition is known to be of first order. We study how the system close to the phase transition in the supercritical regime relaxes in time to the steady state while starting from an initial incoherent state. In this case, numerical simulations of finite systems have demonstrated that the relaxation occurs as a step-like jump in the order parameter from the initial to the final steady state value, hinting at the existence of metastable states. We provide numerical evidence to suggest that the observed metastability is a finite-size effect, becoming an increasingly rare event with increasing system size.

Key words: Synchronization, Kuramoto model, relaxation dynamics

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Coupled oscillators that have their natural frequencies distributed according to a given distribution, for example, a Gaussian, a Lorentzian, or a uniform distribution, often exhibit collective synchronization in which a finite fraction of the oscillators oscillates with a common frequency. Examples include groups of fireflies flashing in unison \[1,2\], networks of pacemaker cells in the heart \[3,4\], superconducting Josephson junctions \[5,6\], and many others. Understanding the nature and emergence of synchronization from the underlying dynamics

\textsuperscript{1}E-mail: anandamohan@iiserkol.ac.in

\textsuperscript{2}E-mail: shamikg1@gmail.com
of such systems is an issue of great interest. A paradigmatic model in this area is the so-called Kuramoto model involving globally-coupled oscillators [7]. Although studied extensively in the past, the model continues to raise new questions, and has been a subject of active research; for reviews, see [8,9].

One issue that has been explored in recent years, and is also the focus of this paper, concerns the Kuramoto model with uniformly distributed natural frequencies. In this case, it is known that in the limit of infinite system size, where size refers to the number of oscillators, the system in the steady state undergoes a first-order phase transition across a critical coupling threshold $K_c$, from a low-coupling incoherent phase to a high-coupling synchronized phase. For values of the coupling constant slightly higher than $K_c$, non-trivial relaxation dynamics has been reported, based on numerical simulations of large systems [10,11]. Namely, it has been shown that initial incoherent states while evolving in time get stuck in metastable states before attaining synchronized steady states. This phenomenon has been demonstrated by the temporal behavior of the order parameter characterizing the phase transition, which shows a relaxation from the initial value of the order parameter to its final steady state value in step-like jumps. An aspect of the Kuramoto model which is of interest and has been explored in some detail concerns finite-size effects [12,13], which may have important consequences, for example, for $K < K_c$, in stabilizing the incoherent state which in the limit of infinite size is known to be linearly neutrally stable [14]. In this context, it is important to investigate whether the metastable states mentioned above may be attributed to finite-size effects. In this paper, we systematically study this phenomenon of step-like relaxation. We provide numerical evidence to suggest that the observed metastability is indeed a finite-size effect, becoming an increasingly rare event with increasing system size.

The Kuramoto model consists of $N$ phase-only oscillators labeled by the index $i = 1, 2, \ldots, N$. Each oscillator has its own natural frequency $\omega_i$ distributed according to a given probability density $g(\omega)$, and is coupled to all the other oscillators. The phase of the oscillators evolves in time according to [7]

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),$$

where $\theta_i$, the phase of the $i$th oscillator, is a periodic variable of period $2\pi$, and $K \geq 0$ is the coupling constant.

The Kuramoto model has been mostly studied for a unimodal $g(\omega)$, i.e., one which is symmetric about the mean frequency $\omega = \Omega$, and which decreases monotonically and continuously to zero with increasing $|\omega - \Omega|$ [8,9]. Then, it is known that in the limit $N \to \infty$, the system of oscillators in the steady state undergoes a continuous transition at the critical threshold $K_c = 2/\pi g(0)$. For
For $K < K_c$, each oscillator tends to oscillate independently with its own natural frequency. On the other hand, for $K > K_c$, the coupling synchronizes the phases of the oscillators, and in the limit $K \to \infty$, they all oscillate with the mean frequency $\Omega$. The degree of synchronization in the system at time $t$ is measured by the complex order parameter

$$r(t) = r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)},$$

with magnitude $r(t)$ and phase $\psi(t)$, in terms of which the time evolution (1) may be written as

$$\frac{d\theta_i}{dt} = \omega_i + Kr(t) \sin(\psi(t) - \theta_i).$$

Here $r(t)$ with $0 \leq r(t) \leq 1$ measures the phase coherence of the oscillators, while $\psi(t)$ gives the average phase. When $K$ is smaller than $K_c$, the quantity $r(t)$ while starting from any initial value relaxes in the long-time limit to zero, corresponding to an incoherent phase in the steady state. For $K > K_c$, on the other hand, $r(t)$ grows in time to asymptotically saturate to a non-zero steady state value $r_{st} = r_{st}(K) \leq 1$ that increases continuously with $K$. The relaxation of $r(t)$ to steady state is exponentially fast for $K > K_c$. For $K < K_c$, however, the nature of relaxation depends on $g(\omega)$. When $g(\omega)$ has a compact support, $r(t)$ while starting from any initial value decays to zero more slowly than any exponential as $t \to \infty$ [15]. When $g(\omega)$ is supported on the whole real line, $r(t)$ as a function of time is known only in particular cases. For example, for a Lorentzian $g(\omega)$, and a specific initial condition, $r(t)$ decays exponentially to zero [15]. For other choices of $g(\omega)$ in this class and for other initial conditions, the dependence of $r(t)$ on time is not known analytically, and it has been speculated that $r(t)$ is a sum of decaying exponentials [15].

In the limit $N \to \infty$, the state of the oscillator system at time $t$ is described by the probability distribution $f(\theta, t, \omega)$ that gives for each natural frequency $\omega$ the fraction of oscillators with phase $\theta$ at time $t$. The time evolution of $f(\theta, t, \omega)$ satisfies the continuity equation for the conservation of the number of oscillators with natural frequency $\omega$, and is given by a non-linear partial integro-differential equation [8]. Recent analytical studies for a unimodal $g(\omega)$ (specifically, a Lorentzian) and for two different bimodal $g(\omega)$’s (given by a suitably defined sum and difference of two Lorentzians) demonstrated by considering a restricted class of $f(\theta, t, \omega)$, and by employing an ansatz due to Ott and Antonsen that the time evolution in terms of the integro-differential equation may be exactly reduced to that of a small number of ordinary differential equations (ODEs) [16,17,18]. Interestingly, the ODEs for the reduced system contain the whole spectrum of dynamical behavior of the full system.
The Ott-Antonsen ansatz has also been applied to various globally and non-linearly coupled oscillators with uniformly distributed frequencies [19].

A uniform $g(\omega)$ with a compact support does not qualify as a unimodal distribution. In this case, it is known that in the limit $N \to \infty$, the Kuramoto model in the steady state exhibits a first-order phase transition between an incoherent and a synchronized phase at the critical coupling $K_c = 2/\pi g(0)$ [20]. For large $N$, numerical studies of the relaxation of an initial state with uniformly distributed phases have demonstrated that for values of $K$ around $K_c$ in the supercritical regime, the process occurs as a step-like jump in $r(t)$ from its initial to the steady state value. One may interpret this behavior as suggesting the existence of metastable states in the system [10,11]. Our motivation is to investigate the implication of the existence of the step-like relaxation, and whether such relaxation can be seen only in finite-sized systems.

![Fig. 1](image_url)

Fig. 1. For a given realization of the $\omega_i$’s, independently and uniformly distributed in $[-2, 2]$, the figure shows the time evolution of the $r(t)$ for different realizations of the $\theta_i$’s, independently and uniformly distributed in $[-\pi, \pi]$. Here, the system size is $N = 1000$, while the coupling constant is $K = 2.62$, the critical value being $K_c \approx 2.55$.

We performed extensive numerical simulations involving integration of Eq. (3) by a 4th-order Runge-Kutta algorithm. We considered a system of $N = 1000$ oscillators, with the $\omega_i$’s independently and uniformly distributed in $[-2, 2]$, so that $g(\omega) = 1/4$ for $\omega \in [-2, 2]$, and is zero otherwise. We chose the initial state to be homogeneous in phases: $\theta_i$’s are independently and uniformly distributed in $[-\pi, \pi]$. We took $K = 2.62$, the critical value being $K_c = 2/\pi g(0) \approx 2.55$. In simulations, we monitored the evolution of $r(t)$ in time. To discuss our results, let us note that in the Kuramoto model, one simulation run of the dynamics differs from another in that it corresponds to (i) a different realization of the set of initial phases $\{\theta_i\}_{1 \leq i \leq N}$, and (ii) a different realization of the set of natural frequencies $\{\omega_i\}_{1 \leq i \leq N}$. In order to distinguish between the effects of the two, we performed two sets of simulations, by fixing the initial $\theta_i$’s and
running simulations for different realizations of the $\omega_i$'s, and vice versa.

(a) For a given realization of the $\omega_i$'s, Fig. 1 shows $r(t)$ as a function of time for different realizations of the initial $\theta_i$'s. We see that in all cases, $r(t)$ shows similar relaxation behavior, jumping in a step-like manner from the initial to the final steady state value corresponding to a synchronized phase.

![Figure 1](image1.png)

Fig. 2. For a given realization of the $\theta_i$'s, independently and uniformly distributed in $[-\pi, \pi]$, the figure shows the time evolution of the $r(t)$ for different realizations of the $\omega_i$'s, independently and uniformly distributed in $[-2, 2]$. Here, the system size is $N = 1000$, while the coupling constant is $K = 2.62$, the critical coupling being $K_c \approx 2.55$.

(b) In the second set of simulations, we focused on different realizations of the frequency distribution, while keeping the set of initial $\theta_i$'s fixed. In Fig. 2, we see that, similar to Fig. 1, the quantity $r(t)$ jumps in a step from the initial to the final steady state value corresponding to a synchronized phase. However, an important difference is that across realizations, there is a wide range of values of the jump time or the relaxation time $\tau$. For realization 2, the system settles into a partially synchronized state. However, we have checked that this state is not the true steady state, the latter being a synchronized state, the relaxation to which takes place at very long times. Thus, the partially synchronized state may be interpreted as only a metastable state. From the above numerical experiment, it is clear that the occurrence of the metastable state is dependent on the realization of the frequency distribution only, as we had kept the initial $\theta_i$’s fixed in the experiment.

Let us mention a different way to demonstrate the reluctance of some frequency realizations to relax to a synchronized state, as observed in Fig. 2, by performing the following numerical experiment. From the realizations depicted in Fig. 2 that relax to the synchronized state within the time duration shown, the typical relaxation time $\tau_{\text{typical}}$ may be estimated. In simulations, we prepared the system in a homogeneous state, as in Fig. 2, and tuned the
coupling parameter $K$ cyclically, from low to high values and back, while simultaneously measuring the order parameter $r$. We tuned $K$ at a rate much smaller than the inverse of $\tau_{\text{typical}}$, thereby ensuring that the system during the course of tuning of $K$ remains close to the instantaneous steady state at all times. The result is the hysteresis loop depicted in Figs. 3 and 4. For one of the realizations depicted in Fig. 2 that is relaxing within the time duration shown, say, the realization 5, Fig. 3 shows the corresponding plot of $r$ vs. $K$, illustrating hysteretic behavior. Such a behavior is expected of a first-order transition, as is the case here in the model with uniformly distributed frequencies. One may observe from the hysteresis plot that the value of $K$ used in Fig. 2, namely, $K = 2.62$, is well outside the hysteresis loop where the only stable state is the synchronized state. This is consistent with Fig. 2. For a realization in Fig. 2 that is not relaxing within the time duration shown, say, the realization 6, the value $K = 2.62$ lies within the corresponding hysteresis loop displayed in Fig. 4. The latter shows that at this value of $K$, the stable state of the system is indeed not a synchronized state, but a non-synchronized state, consistent with Fig. 2. However, note that for this realization, the actual relaxation time evident from Fig. 2 is actually much larger than $\tau_{\text{typical}}$, thus demanding that $K$ be tuned at a slower rate than the one used to obtain Fig. 4.

![Hysteresis Loop Diagram](image)

Fig. 3. Starting from an initial homogeneous state, the figure shows $r$ as a function of adiabatically and cyclically tuned $K$ for the realization 5 in Fig. 2.

An average of $r(t)$ over different realizations of the $\omega_i$’s shows the following behavior. Figure 5(a) shows the temporal evolution of $r(t)$ for 8 realizations with different $\omega_i$’s, illustrating that $r(t)$ jumps in a step at times $\sim 100$ from its initial to its steady state value of $\approx 1$ (with fluctuations that decrease with increasing system size $N$). For the realization shown in Fig. 5(b), however, the jump to the steady state value takes place at time $\approx 2000$, while for the realization in Fig. 5(c), $r(t)$ stays close to the initial value, and the jump does not take place in the time duration shown. The average of $r(t)$ over
Fig. 4. Starting from an initial homogeneous state, the figure shows $r$ as a function of adiabatically and cyclically tuned $K$ for the realization 6 in Fig. 2.

Fig. 5. (a) Evolution of $r(t)$ in time for 8 different realizations of the initial homogeneous state. Here, $N = 1000, K = 2.62$, while the critical coupling is $K_c \approx 2.55$. The plot shows that for these realizations, $r(t)$ jumps in a step at times $\sim 100$ from its initial to its steady state value. (b) and (c) also show the time evolution of $r(t)$ for two other realizations of the initial state: for the realization in (b), $r(t)$ shows a jump to the steady state value at time $\approx 2000$, while for the one in (c), $r(t)$ remains close to its initial value and does not relax to the steady state value in the time shown. (d) and (e) show respectively the time evolution of $r(t)$ when averaged over the realizations in (a)-(c), and over typical 1000 realizations.

The realizations in (a)-(c) is shown in Fig. 5(d), which exhibits a step-like relaxation, and may be interpreted as suggesting the existence of metastable states. Similar behavior of the average $r(t)$, where the averaging is over a few
realizations, was reported in \cite{10}. Figure 5(e) shows the time evolution of \( r(t) \) when averaged over a large (\( \sim 1000 \)) number of realizations, which however does not show any step-like relaxation. Now, as \( N \) tends to \( \infty \), it is reasonable to expect that the system becomes self-averaging. This observation, combined with our result that \( r(t) \) when averaged over a large number of realizations of \( \omega_i \)'s for a finite system does not show any step-like relaxation, suggests that in the thermodynamic limit, the relaxation of \( r(t) \) occurs as a smooth process. Indeed, the probability distribution of the steady state value \( r_{st} \) of the order parameter, displayed in Figure 6, shows that with increasing system size, the distribution is more and more peaked at a value close to one, while the probability to obtain any other value gets negligibly smaller. The form of \( P(r_{st}) \) suggests that with increasing \( N \), metastable states during the relaxation occur with increasing rarity, so that in the thermodynamic limit, relaxation of \( r(t) \) is a smooth process; had the occurrence of metastable states not been rarer with increasing \( N \), one would not have had \( P(r_{st}) \) peaked at a single value of \( r_{st} \).

![Distribution of steady state value](image)

**Fig. 6.** Starting from an initial homogeneous state, the figure shows the distribution \( P(r_{st}) \) of the steady state value \( r_{st} \) of the order parameter for \( K = 2.62 \).

It is worthwhile to study the distribution \( P(\tau) \) of the jump time \( \tau \geq 1 \). Our numerical data, displayed in Fig. 7, show that \( P(\tau) \) has a power-law tail with an exponent greater than 1; the scaling collapse of the data, shown in Fig. 8, suggests the following scaling form:

\[
P(\tau) \sim \sqrt{N} f \left( \frac{\tau}{\sqrt{N}} \right),
\]

where \( f(x) \) is the scaling function.

In the limit of large \( N \), and in the power-law regime of \( P(\tau) \), let us write

\[
P(\tau) = A\tau^{-\alpha},
\]
Fig. 7. Distribution $P(\tau)$ of the relaxation time $\tau$. Here, $K = 2.62$. The inset shows power-law fit to the tail of $P(\tau)$ according to Eq. (5), giving values 2.2, 2.7, and 2.8 for system sizes $N = 1000$, 2000 and 4000, respectively.

Fig. 8. Scaling collapse of the data for the distribution $P(\tau)$ shown in Fig. 7. where $\alpha > 1$ is the decay exponent, and $A$ is a constant. In the long-time regime, $r(t)$ will have the steady state value $r_{\text{st}} \approx 1$ (the exact value of $r_{\text{st}}$ may be estimated by using results from [20]), provided that the jump from the initial close-to-zero value has already taken place by this time; otherwise, $r(t) \approx 0$. It then follows that the average order parameter, $\langle r \rangle(t)$, for large $t$ will be

$$\langle r \rangle(t) = r_{\text{st}} \int_{t}^{1} d\tau P(\tau)$$
$$= r_{\text{st}} \left[ \int_{1}^{\infty} d\tau P(\tau) - \int_{t}^{\infty} d\tau P(\tau) \right].$$

(6)

Note that in writing down Eq. (6), we made use of the fact that the fluctuations
in the steady state value $r_{\text{st}}$ in the limit of large $N$ are negligibly small. Now, $P(\tau)$ being normalized, the first integral on the right hand side of Eq. (6) equals one. Moreover, since $t$ is large, the second integral may be evaluated by using the form (5) for $P(\tau)$. We get

$$\langle r \rangle(t) = r_{\text{st}} \left[ 1 - \frac{A}{\alpha - 1} t^{1-\alpha} \right],$$  \hspace{1cm} (7)$$

where we have used the fact that $\alpha > 1$. It then follows that

$$\ln(r_{\text{st}} - \langle r \rangle(t)) = \ln \left( \frac{r_{\text{st}} A}{\alpha - 1} \right) - \beta \ln t,$$

$$\hspace{1cm} (8)$$

with

$$\beta = \alpha - 1.$$ \hspace{1cm} (9)$$

The prediction of Eq. (8) is easily checked from our simulation results. Figure 9(a) shows the power-law fit to the tail of $P(\tau)$ for $N = 1000$, giving $\alpha \approx 2.2$. On the other hand, Fig. 9(b) shows the power-law fit to the quantity $r_{\text{st}} - \langle r \rangle(t)$ at long times, giving $\beta \approx 0.9$ in Eq. (8), while the value predicted by Eq. (9) is 1.2. One may hope to obtain a closer agreement with better statistics and larger $N$.

To summarize and conclude, we considered the Kuramoto model with uniformly distributed frequencies, and studied the relaxation of a homogeneous non-steady state in time to the steady state for values of $K$ close to the phase transition ($K \approx K_c$). Our numerical simulations for finite systems showed that for fixed initial phases, but for different realizations of the natural frequencies, the order parameter relaxes as a step-like jump from its initial ($\approx 0$) to its steady state value ($\approx 1$), and that there is a wide range of values of the relaxation time $\tau$ across realizations. We demonstrated that the distribution $P(\tau)$
has a power-law tail. In a finite system, averaging $r(t)$ over a few realizations naturally shows a step-like relaxation but which disappears when averaged over a sufficiently large number of realizations. In the thermodynamic limit, when the system becomes self-averaging, our observation that $r(t)$ when averaged over a large number of realizations for a finite system does not show any step-like relaxation suggests that in this limit, the relaxation of $r(t)$ occurs as a smooth process. The observed metastability is more a finite-size effect, its occurrence being an increasingly rare event with increasing $N$. Our results on the distribution $P(r_{st})$ of the steady state order parameter clearly show that with the increase of system size $N$, the distribution gets sharply peaked around a single value close to 1. This is in conformity with our observation that metastable states become rarer with increasing $N$. In order to answer quantitatively the issues raised in this work, a rigorous mathematical analysis of the relaxation dynamics of the Kuramoto model with uniform frequency distribution is very much desirable.

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