Statistical Mechanics of Elastica on Plane as a Model of Supercoiled DNA
——Origin of the MKdV hierarchy——-

Shigeki MATSUTANI
2-4-11 Sairenji, Niihama, Ehime, 792 Japan

Abstract.
In this article, I have investigated statistical mechanics of a non-stretched elastica in two dimensional space using path integral method. In the calculation, the MKdV hierarchy naturally appeared as the equations including the temperature fluctuation. I have classified the moduli of the closed elastica in heat bath and summed the Boltzmann weight with the thermal fluctuation over the moduli. Due to the bilinearity of the energy functional, I have obtained its exact partition function. By investigation of the system, I conjectured that an expectation value at a critical point of this system obeys the Painlevé equation of the first kind and its related equations extended by the KdV hierarchy. Furthermore, I also commented on the relation between the MKdV hierarchy and BRS transformation in this system.

§1. Introduction

Elastica has sometimes appeared in the history of mathematical physics according to refs.[1-4]. The problem of elastica, an ideal thin elastic rod, was proposed by James Bernoulli in 1691. By investigation on the behavior of an elastica, Bernoulli’s family and their related people, Euler, d’Alembert and so on, discovered many non-trivial mathematical and physical facts, e.g., classical field theory, minimal principle, elliptic function, mode analysis, non-linear differential equation and others [1-4]. In fact, James Bernoulli derived the elliptic integral related to the lemniscate function in 1694, before Fagnano found his lemniscate function [1-3], and found that the force of elastica is proportional to inverse of its curvature radius [1]. His nephew, Daniel Bernoulli followed James’s discoveries and discovered the energy functional of elastica and its minimal principle around 1738. Succeeding Daniel’s and James’s discoveries, Euler derived elliptic integral of general modulus as a shape of elastica using Daniel’s minimal principle and numerically integrated it. Then he completely classified shapes of static elasticas by numerical sketch [1], which are, nowadays, known as special cases of loop soliton [5]. In the computations, Euler used the static sine-Gordon equation. These computations essentially imply discovery of the integrable nonlinear differential equation, the investigation of its moduli and first application of algebro-geometrical functions to physics. It should be noted that they came from the studies on the elastica. Furthermore it is well-known that the elastica problem is the simplest prototype $\sigma$-model $S^1 \rightarrow SO(2)$ or $SO(3)$ [6,7], which was found in 18th century and investigated by Kirchhoff in last century [4].

Thus the elastica problem sounds to be legacy before last century but I believe that its properties are not completely understood and its role in mathematical physics is still important. The difficulty to solve the elastica problem is that one must consider the constraint condition such as the isometry (non-stretch) condition and boundary condition. In fact, even though it is not an elastica problem, Goldstein and Petrich naturally rediscovered the MKdV hierarchy through (virtual) dynamics of a space curve with isometry condition [8,9]. (Readers should noted that the time-development of the physical elastic rod is not governed by the MKdV hierarchy in general [7,9-14], even though in some papers it seems to be misunderstood.) Due to their work, the MKdV hierarchy can be geometrically realized. Furthermore, using their construction, I proved that the Hirota bilinear equation and $\tau$-function of MKdV hierarchy [15] can also be translated into the...
geometrical language of a space curve [11]. The classical analogue of the vertex operator naturally appears as a complex tangent vector of the elastica [11].

However these approaches are just mathematical and geometrical ones but are not directly connected with physical problem because in most of works there is little physical reason why the MKdV hierarchy must appear in physical system [8,9].

On the other hand, the study of elastic chain model of deoxyribonucleic acid (DNA) is current [16-22]. DNA usually occurs as a double helix with two complementary nucleotide chains winding around a common axis. In cells, the common axis of the looped DNA often folds into intricate structure, or a supercoiled form [16,17]. Due to the enormous size of DNA, conformation of double helical polymer needs topological and geometrical studies [23]. Thus the double helical polymer is often modeled as a thin elastic isotropic rod or an elastica and studied from the kinematic and topological point of view. Topological invariance such as the linking number classifies the shape of the DNA in three-dimensional euclidean space $\mathbb{R}^3$ [16]. Based on the Kirchhoff’s model of an elastica in $\mathbb{R}^3$ and the nonlinear Schrödinger equation, the possible kinematic conformations of DNA were considered [16-21]. Partial thermal effect on their shapes were argued in ref.[18]. Furthermore the molecular dynamics study combining with the elastic energy model was reported [22] and then an topology change related to the knot was found.

Statistical mechanics of elastic chain as a model of a large polymer like DNA was studied by Saitō et al. using the path integral [24]. They calculated some exact partition function of the energy functional of elastic chain. Such chain model is sometimes called wormlike chain model [25]. Their approach is successive in the polymer physics and influences recent works [26 and references therein]. However they did not pay any attention upon isometry condition in a calculation of the path integration even though they imposed isometry condition after computation (a kind of quantization); they summed allover configuration space without isometry condition. It should be note that the constraint does not commute with quantization and statistical treatment [27]. (An example of such inconsistency is shown in ref.[28].)

It is a natural assumption that flexible polymers which cannot stretch as a classical object (at zero temperature) cannot stretch even in heat bath. Hence it is very important to calculate the partition function of the elastica with isometry condition and sum the Boltzmann weight over only allowed conformations.

The purpose of this article is to investigate the behavior of a closed large polymer like DNA in heat bath of two-dimensional space, whose shape is determined by its elasticity and stretch is negligible, and to clarify the physical meaning of the MKdV hierarchy.

Furthermore it is known that DNA sometimes exhibits topology changes and has topological isomer [22,29]. In this article, as I will deal with an elastica in two-dimensional space, there is no knot invariance but exists writhing number as a topological invariant of the elastica in two-dimensional space if crossings are allowed [28]. Thus in this article, even though the elastica in two-dimensional space will be investigated, I will allow the crossings if it can be realized when I embed it in three-dimensional space. Then I will argue the topological change on this problem.

The organization of this article is as follows. §2 reviews classical shape of an elastica in two-dimensional flat space adding infinity point, i.e., $\mathbb{C}P^1$. In §3, I will investigate the statistical mechanics of an elastica. First, I will consider the thermal fluctuation of the extremum point of a partition function of the elastica in terms of the path integral method. Then I will obtain the MKdV hierarchy by physical requirements. Second, I will investigate the moduli space of the quasi-classical elastica, which is the extremum point of the partition function. Finally I sum the Boltzmann weight over the moduli space and obtain an exact formulation of the partition function. In §4, I will discuss the obtained partition function and I will comment upon the relation between the Goldstein-Petrich method and the BRS quantization of the gauge field [31] and a critical point of this model.

§2: Classical Shape of Elastica
Here I will quickly review a shape of a closed elastica in two-dimensional space [10-12,32]. I will denote by $C$ a shape of the elastica embedded in the projective complex line (or the Riemann sphere) $\mathbb{C}P^1$ and by $X(s)$ its affine vector:

$$S^1 \ni s \mapsto X(s) \in C \subset \mathbb{C}P^1, \quad X(s + L) = X(s), \tag{2-1}$$

where $L$ is the length of the elastica. The Frenet-Serret relation will be expressed as

$$\exp(i\phi) = \partial_s X, \quad \partial_s \exp(i\phi) = i k \exp(i\phi), \tag{2-2}$$

where $\phi$ is a real valued function of $s$ and $k$ is the curvature of the curve $C$, $k := \partial_s \phi$: $\phi(s + L) = \phi(s)$ and $k(s + L) = k(s)$. Here I have chosen the metric of the curve as the induced metric from the natural metric of $C \subset \mathbb{C}P^1$; by the choice $\phi$ is real valued.

As Daniel Bernoulli suggested to Euler [1], the energy integral of the elastica is given as

$$E = \int_0^L ds k^2, \tag{2-3}$$

and shape of the elastica is realized as its stationary point. Here I assume that the elastica does not stretch and preserves its infinitesimal length; since deformation of the elastica is regarded as one parameter transformation, the assumption implies that the transformation is isometric [10-13,32]. Thus I will refer this condition as isometry condition.

Following the minimal principle, I will derive the differential equation. I will consider the variation of the curve $C \to C_\varepsilon$ under isometry condition [8-11],

$$X \to X_\varepsilon = X + \varepsilon(U_1 + iU_2) \exp(i\phi), \quad U_1(L) = U_1(0), \quad U_2(L) = U_2(0) \tag{2-4}$$

where $\varepsilon U_1$ and $\varepsilon U_2$ are infinitesimal real valued functions.

Since the infinitesimal length of the curve is given as

$$ds^2 = d\bar{X}dX = \partial_s \bar{X}\partial_s X ds^2, \tag{2-5}$$

corresponding length of the $C_\varepsilon$,

$$d\bar{X}_\varepsilon dX_\varepsilon = (1 + \varepsilon ((\partial_s - ik)U_1 - i(\partial_s - ik)U_2)) (1 + \varepsilon ((\partial_s + ik)U_1 + i(\partial_s + ik)U_2)) ds^2$$

$$= (1 + 2\varepsilon (\partial_s U_1 - kU_2)) ds^2 + \mathcal{O}(\varepsilon^2), \tag{2-6}$$

must be $ds^2$ modulo $\varepsilon^2$ due to the isometry condition. Hence I obtain the relation,

$$\partial_s U_1 = kU_2. \tag{2-7}$$

The tangential angle of $C_\varepsilon$ is given as

$$\phi_\varepsilon = \frac{1}{i} \log \partial_s X_\varepsilon$$

$$= \phi + \varepsilon(k + \partial_s k^{-1}\partial_s)U_1. \tag{2-8}$$

Its curvature is expressed as

$$k_\varepsilon = \exp(-i\phi_\varepsilon)\partial_s^2 X_\varepsilon$$

$$= \exp(-i\phi)\partial_s^2 X + i k \exp(-i\phi)\partial_s^2 X.$$  \tag{2-9}$$
Finally I obtain the variation of the energy functional

$$\int_0^L \! ds k_s^2 = \int_0^L \! ds \left( k^2 + \varepsilon \left( \frac{1}{2} \partial_s (k^2) + \partial_k \frac{\partial_k^2 k}{k} \right) U_1 \right) + \mathcal{O}(\varepsilon^2).$$  \tag{2-10}$$

From the variational equation,

$$\frac{\delta E[k_s]}{\delta (\varepsilon U_1)} = 0, \tag{2-11}$$

the non-linear differential equation is given as the equation of the shape of the static elastica,

$$\partial_s \left( \frac{1}{2} k^2 + \partial_s^2 k \right) = 0, \tag{2-12-a}$$

and thus

$$a_1 k + \frac{1}{2} k^3 + \partial_s^2 k = 0, \tag{2-12-b}$$

where $a_1$ is the integral constant. This equation is known as the static MKdV equation in the soliton theory if derivate it by $s$ again.

First I will note that (2-12)'s are also equations of the energy functional \[10,11\],

$$E = \int_0^L \! ds \left( k^2 + A_1 \cos \phi + A_2 \right), \tag{2-13}$$

where the second term means the constraint for the relative position of $X(0) - X(L)$ and the third one is for the total length $L$ \[7,10,11,13\]. The sufficiency of the third term is trivial. From (2-9), the second term becomes

$$A_1 \int \! ds \cos(\phi_\varepsilon) = A_1 \int \! ds \left( \cos(\phi) - \varepsilon \left( k U_1 + \partial_s \frac{\partial_s U_1}{k} \right) \sin(\phi) \right) + \mathcal{O}(\varepsilon^2) \tag{2-14}$$

and by the partial integration, the second term in rhs vanishes for any $U_1$. This vanishing occurs owing to the compatibility between the MKdV equation and the static sine-Gordon equation,

$$\partial_s^2 \phi + A_1 \sin \phi = 0. \tag{2-15}$$

(2-15) comes from natural variation of $\phi$ of (2-13) \[1,4,7,10,11,13\]. Hence (2-12)'s can be also regarded as the stationary point of (2-13). In fact it is known that solutions of (2-12) are in agreement with those of (2-15) as I will show later. It should also be noted that (2-13) is the $\sigma$-model with the topological term and was discovered in 18th century. In other words, the system of the elastica can be regarded as $\text{SO}(2)$-principal bundle over $S^1$ and the cosine term in (2-13) is a local version of the fundamental group $\pi_1(S^1) = \mathbb{Z}$ \[6\].

Solutions of (2-12) are completely expressed by the elliptic functions. Multiplying $\partial_s k$ and integrating $s$ \[4,11,13,14,32,33\], I obtain the relation,

$$\left( \partial_s k \right)^2 = -\frac{1}{4} (k^4 - a_1 k^2 + a_2). \tag{2-16}$$

Introducing the quantities, $\beta_2 - \beta_3 := a_1$, $l = \sqrt{\beta_1/(\beta_1 + \beta_2)}$, and $\beta_3 := \sqrt{\beta_1 + \beta_2}/2$, $k$ is expressed by the Jacobi elliptic function \[34\],

$$k = \frac{\sin(\beta_3 \rho)}{\sin(\beta_3 \rho)} \tag{2-17}$$
Transformation from the solutions of (2-12) to those of (2-15) is given by the identities of the integrand in the elliptic integral between trigonometric function and polynomial expressions [34]. After integrating the differential equations, I obtain [1,4,10,11,13,33],

\[ X(s) = \frac{2}{\beta_3} E(\text{am}(\beta_3 s), l) - s - i\frac{2l}{\beta_3} (\text{cn}(\beta_3 s) - 1), \]  

(2-18)

where \( E(\cdot, l) \) is the incomplete elliptic integral of the second kind and \( \text{am} \) is the Jacobi amplitude function [34]. Due to the closed condition,

\[ X(0) = X(L), \]  

(2-19)

there is an eight-figure shape [1,4,11,13,16,33]; the modulus of the elliptic function is \( l = 0.908909 \cdots \) and the ratio of the fundamental parameters is \( K'/K = 0.70946 \cdots \).

Thus in the set of solutions of (2.12), there are only two closed elasticas in \( \mathbb{C} \) up to translation of their centroid, global rotation and scaling; circle \( k = 2\pi / L \) and eight-figure shape. Here though I have chosen solutions such as \( k = 2\pi / L \), there also exist other solutions like \{ \( k = 2\pi n / L \mid n \geq 1 \} \).

On the other hand, by taking limit \( L \to \infty \) and by considering ones in \( \mathbb{C} P^1 \), more various closed elasticas in \( \mathbb{C} P^1 \) are allowed. These solutions were classified by Euler in 18th century [1,4].

In this article the set of elasticas obtained as solutions of (2-12) will be denoted as \( \mathcal{S}_{\text{cls}} \)

\[ \mathcal{S}_{\text{cls}} = \{ C \mid C \text{ is a solution of (2-12)’s} \}, \]  

(2-20)

and the energy functional is expressed by,

\[ E_{\text{cls}}[C] = \int_0^L ds \ k^2, \ C \in \mathcal{S}_{\text{cls}}. \]  

(2-21)

§3. Statistical Mechanics of Elastica

In this section, I will consider statistical mechanics of a closed elastica and investigate its behavior at heat bath. I continue to allow the crossing of the elastica even in two-dimensional space. I set up that there are many independent laboratory dishes in which large polymers like DNA individually exist one by one. A looped elastica is in the liquid whose temperature is determined and viscosity is very large. The liquid is a kind of heat bath. Then the kinetic energy of the elastica is suppressed in equilibrium state due to dissipation but owing to the fluctuation by temperature noise, the elastica sometimes jumps from a quasi-stable state to other quasi-stable states.

The partition function of the elastica is given as [24],

\[ Z = \int DX \exp \left(-\beta \int_0^L ds \ [k^2] \right). \]  

(3-1)

Here I go on to prohibit that the local length of the elastica does change; the isometry condition will be maintained. In [24], it was written that if one also dealt with the kinematic term, it would be decoupled with the potential term (2-3). However as I employ the isometry condition, this statement cannot be guaranteed at all because the kinetic term is also restricted by the isometry condition (2.3) and is also coupled with the shape of the elastica.
Quasi-Classical Motion.

By the quasi-classical method in path integration [35,36], I will evaluate the partition function (3-1). I will expand the affine vector around an extremum point of the integral,

\[ X = X_{qcl} + \epsilon(u_1(s) + iu_2(s)) \exp(i\phi_{qcl}) + O(\epsilon^2), \]

where \( \epsilon \) is an infinitesimal parameter, \( \epsilon \propto 1/\sqrt{\beta} \) and \( \phi_{qcl} \) is the tangent angle of the quasi-classical curve of elastica. \( X_{qcl} \) is an affine vector of the extremum point of the functional integral and will be determined later. In the path integral, terms with higher orders of \( \epsilon \) also play important role and thus I must pay attentions upon the higher perturbations of \( \epsilon \) here. Hence I will assume that \( X \) is parametrized by a parameter \( t \) and the difference between \( X \) and \( X_{qcl} \) can be expressed by,

\[ X(s,t) := e^{\epsilon \partial_t} X_{qcl}(s,t), \quad \epsilon \partial_t X_{qcl} = X - X_{qcl} + O(\epsilon^2). \]  

(3-3)

Since for an analytic function \( f(x) \), \( e^{a\partial_x} f(x) = f(x + a) \), \( X(s,t) \) can be expressed as \( X(s,t) = X_{qcl}(s,t + \epsilon) \), where direction \( t \) differs from that of \( s \); the domain of functional integration (3-1) deviates from the domain \( S^1 \) of the classical map (2-1). Then (3-2) becomes

\[ \partial_t X_{qcl} = (u_1 + iu_2) \exp(i\phi_{qcl}), \quad u_1(L) = u_1(0), \quad u_2(L) = u_2(0). \]  

(3-4)

This is virtual dynamics of the curve [8-10]. As well as the argument in §1, due to the isometry condition, I require \( [\partial_t, \partial_s] = 0 \) for \( X \). Then the isometry condition exactly preserves, \( ds = ds_{qcl} \) by the definition (3-3). This isometry relation should be compared with (2-6) which is isometry modulo \( \epsilon^2 \). It also should be noted that although \( \epsilon \) is constant, dependence of the variation upon the position \( s \) is given though (3-4) and \( u_a(s), \ a = 1, 2 \). Thus (3-3) is not trivial deformation in general.

I have the relation,

\[ -\partial_t \exp(i\phi_{qcl}) = ((u_{1s} - u_{2k_{qcl}}) + i(u_{2s} + u_{1k})) \exp(i\phi_{qcl}). \]  

(3-5)

Noting that \( \phi \) and \( u \)'s are real valued, I obtain (2-7) again from the first term and solve the differential equation between \( u_1 \) and \( u_2 \) [8,10],

\[ \partial_s u_1 = k_{qcl} u_2, \quad u_1 = \int_0^s ds u_2 k_{qcl} =: \partial_s^{-1} u_2 k_{qcl}. \]  

(3-6)

Here \( \partial_s^{-1} \) has a parameter \( c \in \mathbb{R} \) as an integral constant and coincides with the pseudo-differential operator.

Then (3-5) is reduced to the equations,

\[ \partial_t k_{qcl} = -\Omega u_2, \quad \Omega := \partial_s^2 + \partial_s k_{qcl} \partial_s^{-1} k_{qcl}. \]  

(3-7)

From (3-3) and \( [\partial_t, \partial_s] = 0 \) for \( X \), \( \phi \) is calculated as,

\[ \phi(s,t) = \phi_{qcl}(s,t + \epsilon) = e^{\epsilon \partial_t} \phi_{qcl}(s,t) = \phi_{qcl} + \epsilon \partial_t \phi_{qcl} + \frac{1}{2!} \epsilon^2 \partial_t^2 \phi_{qcl} + \cdots. \]  

(3-8)

Then noting \( k^2(s,t) = k_{qcl}^2(s,t + \epsilon) \), the energy functional is expressed as

\[ \int k^2 ds = \int (k_{qcl}^2 + 2\epsilon k_{qcl} \partial_t k_{qcl} + \epsilon^2 ((\partial_t k_{qcl})^2 + k_{qcl} \partial_t^2 k_{qcl}) + \cdots) ds \]

\[ = \int (k_{qcl}^2 + 2\epsilon k_{qcl} \Omega u_2 + \epsilon^2 ((\partial_t k_{qcl})^2 + k_{qcl} \partial_t^2 k_{qcl}) + \cdots) ds \]

\[ E = \epsilon^2 \int \Omega u_2 + \cdots. \]  

(3-9)
Thus for the variations along the directions, the energy of the system is invariant modulo 1
initial state as
\[ \epsilon \]
Using (3-6), if I will perform the functional derivative of \( E \) in \( u_1 \), I obtain the classical equations (2-12)'s again. Since the quasi-classical configuration is realized as the extremum point of the functional space, I must impose the relation,
\[ \delta^{(1)} E = 0. \quad (3-10) \]
Noting the relation (3-6), if \( \Omega u_2 \) could be regarded as another function \( u'_2 \) of the variation of the normal direction in (3-2), I might find the relation
\[ \int ds k_{\text{qcl}} \Omega u_2 \sim \int ds k_{\text{qcl}} u'_2 = \int ds \delta_s u'_2 = 0. \quad (3-11) \]
Accordingly I will introduce the sequence for mathematical times \( t := (t_1, t_3, t_5, \ldots, t_{2n+1}, \ldots) \) so that (3-11) is satisfied. I will redefine the fluctuation (3-2) and introduce infinite parameters family, which can sometimes become finite set as I will show later,
\[ X = e^{(1/\beta) \sum_n \delta t_{2n+1} \partial_{t_{2n+1}} X_{\text{qcl}}} = X_{\text{qcl}} + (1/\beta) \sum_n \delta t_{2n+1} \partial_{t_{2n+1}} X_{\text{qcl}} + \mathcal{O}(1/\beta). \quad (3-12) \]
Here \( \epsilon \) was replaced with \( (1/\beta) \delta t_{2n+1} \). Hence (3-11) should also include such translation symmetry. Of course, they contain other equations as quasi-stable shapes as I show following.

Then the variation of the energy functional is calculated as,
\[
\int k^2 ds = \int (k_{\text{qcl}}^2 + (2/\beta) \sum_n \delta t_{2n+1} k_{\text{qcl}} \partial_{t_{2n+1}} k_{\text{qcl}}) ds + \mathcal{O}(1/\beta)
= \int (k_{\text{qcl}}^2 + (2/\beta) \sum_n \delta t_{2n+1} k_{\text{qcl}} \Omega u_2^{(n)}) ds + \mathcal{O}(1/\beta)
= \int ds k_{\text{qcl}}^2 + (2/\beta) \sum_n \delta t_{2n+1} \int ds \partial_s u_1^{(n+1)} + \mathcal{O}(1/\beta)
= \int ds k_{\text{qcl}}^2 + \mathcal{O}(1/\beta). \quad (3-14)\]
Thus for the variations along the directions, the energy of the system is invariant modulo \( 1/\beta \). Without work, we can move it for these directions \( \delta t_{2n+1} X \) \[10,37,38\].

However for this sequence, the infinite differential equations appear \[8,9\],
\[
\partial_{t_{2n+1}} k_{\text{qcl}} = -\Omega^n u_2^{(0)}. \quad (3-15)\]
The recursion equations (3-13) are determined by the initial condition \( u_2^{(0)} \). By following the thought of the quasi-classical method of the path integral, this sequence must contain the classical equations (2-12)'s. On the other hand, for a close elastica, there is a trivial continuous symmetry which is the translation of the coordinate system \( s \) along the curve \( C \). Hence (3-15) should also include such translation symmetry. Of course, they contain other equations as quasi-stable shapes as I show following.

Though there might be other choices, I will select minimal subspace of the functional space in order to satisfy above requirement. As I performed the variational computation in §1, I choose initial state as
\[
\delta_{t_{2n+1}} k_{\text{qcl}} = -\Omega^n u_2^{(0)} \quad (3-16)\]
and thereby, \( \delta^{(1)} \epsilon = \Omega^n \delta^{(1)} u_2^{(0)} \). Hence (3-15) is satisfied. Of course, we can move it for these directions \( \delta t_{2n+1} X \) \[10,37,38\].
Then the minimal set of the virtual equations of motion, which satisfies the physical requirements, is
\[ \partial_{t_{2n+1}} k_{\text{qcl}} = -\Omega^n \partial_s k_{\text{qcl}}, \quad \partial_{t_{2n+3}} k_{\text{qcl}} = \Omega \partial_{t_{2n+1}} k_{\text{qcl}}. \] (3-17)

First several equations and \( u \)'s are given as follows,
\[ u_1^{(0)} = 1, \quad u_2^{(0)} = 0, \] (3-18-a)
\[ u_1^{(1)} = \frac{1}{2} k_{\text{qcl}}^2, \quad u_2^{(1)} = \partial_s k_{\text{qcl}}, \] (3-18-b)
\[ u_1^{(2)} = \frac{3}{8} k_{\text{qcl}}^4 - \frac{1}{2} (\partial_s k_{\text{qcl}})^2 + k \partial_s^2 k_{\text{qcl}}, \quad u_2^{(2)} = \frac{3}{2} k_{\text{qcl}}^2 \partial_s k_{\text{qcl}} + \partial_s^3 k_{\text{qcl}}, \] (3-18-c)

\[ n = 0 : \quad \partial_{t_1} k_{\text{qcl}} + \partial_s k_{\text{qcl}} = 0, \] (3-19-a)
\[ n = 1 : \quad \partial_{t_3} k_{\text{qcl}} + \partial_s^3 k_{\text{qcl}} + \frac{3}{2} k_{\text{qcl}}^2 \partial_s k_{\text{qcl}} = 0, \] (3-19-b)
\[ n = 2 : \quad \partial_{t_5} k_{\text{qcl}} + \partial_s^5 k_{\text{qcl}} + \frac{15}{8} k_{\text{qcl}}^4 \partial_s k_{\text{qcl}} + \frac{5}{2} (\partial_s k_{\text{qcl}})^3 \]
\[ + \frac{5}{2} k_{\text{qcl}}^2 \partial_s^2 k_{\text{qcl}} + 10 k_{\text{qcl}} \partial_s k_{\text{qcl}} \partial_s^2 k_{\text{qcl}} = 0. \] (3-19-c)

Since \( \Omega \) is identified with the Gel’fand-Dikii operator for the MKdV equation, (3-17) is regarded as the MKdV hierarchy and \( u_1^{(m)} \)'s are hamiltonian of the MKdV hierarchy [8-10,39].

Next I will consider the solutions of these equations. (3-19-a) means the freedom of choice of the origin of \( s \) and has only trivial solution \( k(s - t_1) \); \( t_1 \) is the origin of \( s \). Hence I must pay my attention only upon properties of (3-17) for \( n \geq 1 \). Derivative of (2-12-b) can be described as
\[ c \partial_s k = \Omega \partial_s k. \] (3-20)

By \( t_3 = s/c \), this is identified with \( n = 1 \) equation (3-19-b). Here \( \partial_s k \) of the solutions of (2-12-b) is interpreted as the eigenvectors of \( \Omega \) and \( c \) is an eigenvalue of the operator \( \Omega \). Thus (3-17) becomes
\[ \partial_{t_{2n+1}} k = -\Omega^n \partial_s k = -c^n \partial_s k. \] (3-21)

Hence the solutions of (2-12) can be solutions of all equations in (3-17). In fact for a stable solutions \( k = \text{constant} \), all \( u_2^{(n)} \equiv 0 \) and thus it satisfies all equations in (3-17).

It should be noted that (3-21) comes from the fact that \( u_1^{(m)} \)'s agree with the hamiltonians of the MKdV hierarchy and are regarded as conservation quantities for \( n \)-th equations of \( n < m \) [39]. Hence using soliton theory, any solutions of (3-19-b) are solutions of higher equations in (3-17) \( n \geq 2 \). Due to the requirement of the quasi-classical solutions, any solutions must satisfy all of \( n \)-th equations \( n \geq 1 \) (3-17). Hence I should deal with only the solutions of the MKdV equation (3-19-b) as the quasi-classical solutions of this system.

Then the sequence (3-17) fulfills the physical requirements. In other words, the solution space of the MKdV equation (3-19-b) is the minimal space containing the classical solutions and translation symmetry and filling the quasi-stable condition (3-10).

Since for the variations along the directions \( s, t_{2n+1} \) (3-17), the energy of the system (3-14) is invariant, the fluctuation of the quasi-classical shape \( k_{\text{qcl}}(s, t_{2n+1}) \) should be regarded as (generalized) jacobi-fields or the Goldstone mode [37,38] even though they do not obey a linear differential equation in general.

Here I will remark that the MKdV hierarchy naturally appears by physical requirement. It is very surprising because the MKdV hierarchy has infinite hamiltonians and time axes; in classical theory, these quantities cannot be physically interpreted. Furthermore it should be contrasted with the works related to space curves in [8,9,33], in which the authors chose (3-16) by hand without any physical requirement.
Due to the fluctuation of the heat noise, the equation of the elastica in heat bath obeys the MKdV equation (3-19-b) rather than the static MKdV equation (2-12). Let the set of solutions referred as 

$$S_{qcl} = \{ C | C \text{ is a solution of (13-19-b)} \}.$$  

(3-22)

Of course $S_{qcl}$ contains $S_{cls}$. In $S_{qcl}$, various shapes appear as the quasi-classical solutions in heat bath. For example, there should exist a deformed circle as a solutions of (3-19-b). As another example, there are other topological solutions of the different sector of the fundamental group,

$$\frac{1}{2\pi} \int d\phi_{qcl} \in \mathbb{Z}.$$  

(3-23)

**Moduli of Closed Quasi-Classical Elastica.**

Here I will go back to compute the partition function (3-1), which was formally defined. First the problem aries how (3-1) should be interpreted. According to the philosophy of the canonical ensemble, the partition function should be calculated by the sum allover distinguishable and possible curves satisfying the isometry condition with Boltzmann weight. In the calculus, different topological class of (3-23) will be summed over.

However the partition function (3-1) naturally diverges because the energy function is invariant for the affine transformation i.e. for the translation and the global rotation. Fixing $C$, if I denote $X$ as

$$X(s) = X_g + X_r(s), \quad \int ds X_r(s) = 0,$$  

(3-24)

where $X_g$ is the centroid of the curve $C$, the measure of the partition function can be rewritten as

$$\int DX e^{-\beta E} = \sum C \int DX_g \int DX_r e^{-\beta E(C)}.$$  

(3-25)

The $\int DX_g$ is volume of the base space $\mathbb{C}P^1$ and diverges. Including the rotational symmetry and the inner translation symmetry $s \rightarrow s - t_1$, I will redefine the partition function which is divided by the volume of the affine transformation of the base space and length of the elastica,

$$Z_{reg} := \frac{\int DX e^{-\beta E}}{\text{vol(Aff($\mathbb{C}P^1$)))L}}.$$  

(3-26)

Then I concentrate the shape of the elastica. I must classify the shape of the elastica and sum over the possible shapes. In other words, I must investigate the moduli space of the quasi-classical elastica,

$$\mathcal{M}_{qcl} := \frac{S_{qcl}}{\text{Aff($\mathbb{C}P^1$)) \times S^1}.$$  

(3-27)

First I will consider the moduli space of the MKdV equation. The moduli of the MKdV equation was investigated as the KP-hierarchy using Sato-theory [40-42]. (By the Miura map, the MKdV hierarchy is transformed to the KdV hierarchy and the KdV hierarchy is a subset of the KP hierarchy [41].) The moduli of the MKdV equation is classified with the genus $g \in \mathbb{N}$ of the hyperelliptic Riemannian surface (hyperelliptic curve) $R_g$, which is the finite gap energy manifold (Bloch band spectrum) of the wave functions in the inverse scattering method of the MKdV equation [40-46]. Hence the moduli of the closed elastica is also classified by the genus,

$$\mathcal{M}_{qcl} = \bigcap \mathcal{M}_{qcl}^{(g)}.$$  

(3-28)
I will call the genus of the MKdV equation with the boundary condition genus of the elastica \( g \). In fact, the classical solutions of (2-12)’s and (3-20) correspond to the elasticas of genus zero and one because these energy manifolds appearing in its inverse scattering method exhibit a Riemannian sphere and an elliptic curve respectively [43]. (It should be noted that even in the quasi-classical equation, the solutions of the circle and the eight-figure are unique up to homothety for \( g = 0 \) and \( g = 1 \) respectively.) Using the knowledge of the properties of the universal Grassmannian manifold (UGM), I will consider the moduli of the closed elastica.

First I will consider the simplest case (\( g = 0 \)), a circle \( k = 2\pi n/L, \ n \geq 1 \). In (3-9), the quasi-classical action of these circles,

\[
E_{\text{qcl}}[C_n] = \frac{2\pi n^2}{L}, \quad k(C_n) = \frac{2\pi n}{L}
\]

(3-29)

Hence for large \( n \), the Boltzmann weight \( \exp(-\beta E_{\text{qcl}}[C_n]) \) rapidly decreases. This situation preserves for the elasticas of higher genus.

Next I will consider elastica of genus one and why the number of the genus one solutions of closed elastica is only one, i.e., eight-figure shape, up to scaling. The moduli of compact Riemannian surface of genus one (or elliptic curve) is conventionally expressed as \((1, \tau); \ \tau \in \mathbb{M}_{R_1}; \)

\[
\mathbb{M}_{R_1} = H_+/\text{PSL}(2, \mathbb{Z}),
\]

\[
H_+ := \{ m \in \mathbb{C}| \text{Im}(m) \geq 0 \},
\]

\[
\text{PSL}(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\}/(\pm 1).
\]

(3-30)

However there is a dilatation freedom \((\tilde{K}, \tilde{K}':= \tilde{K})\) and thus I will denote \( \mathbb{M}_{R_1} := \mathbb{R}_{>0} \times \mathbb{M}_{R_1} \) to include the freedom: \( \tilde{K} \in \mathbb{R}_{>0} := \{ x \in \mathbb{R}| x > 0 \} \). The Jacobi variety of an elliptic curve is given as \( J_{1, \mathfrak{m}} := \mathbb{C}/(K \mathbb{Z} \oplus \bar{K}' \mathbb{Z}) \) for \( \mathfrak{m} := (\tilde{K}, \tilde{K}') \). Since \( \phi_{\text{qcl}} \) is a real analytic function of \( s \in S^1 = \mathbb{R}/L \mathbb{Z} \), its domain embedded in \( J_{1, \mathfrak{m}} \) must be real. Thus only one-dimensional parameterization of \( \phi_{\text{qcl}}, S^1 \subset J_{1, \mathfrak{m}} \), is allowed, which is direct line in \( J_{1, \mathfrak{m}} \) and passes its origin, because \( J_{1, \mathfrak{m}} \) is complex one-dimensional manifold. Since the moduli was divided by \( \text{PSL}(2, \mathbb{Z}) \), there are \( \text{PSL}(2, \mathbb{Z}) \) choices how such \( S^1 \) is embedded \( J_{1, \mathfrak{m}} \). I choose a function with period \( L/n, \ n \in \mathbb{N} := \{ n \in \mathbb{Z}| n \geq 1 \} \) \( \phi_{\text{qcl}}(s + L/n) = \phi_{\text{qcl}}(s) \). By the periodicity of \( \phi_{\text{qcl}}(s) \), I will fix the \( \tilde{K} \) for each embedding of \( S^1 \) into \( J_{1, \mathfrak{m}} \), then the moduli of the MKdV equation with period \( L \) is \( \mathbb{M}_{R_1} \times \mathbb{N} \times \text{PSL}(2, \mathbb{Z})/\mathbb{R}_{>0} \), which is equivalent with \( \mathbb{N} \times H_+ \).

On the other hand, the closed condition (2-19) in \( \mathbb{C}P^1 \) restricts the moduli of the elastica. I will introduce a real analytic map,

\[
f_1: \frac{\mathbb{M}_{R_1} \times \mathbb{N} \times \text{PSL}(2, \mathbb{Z})}{\mathbb{R}_{>0}} \approx \mathbb{N} \times H_+ \to \mathbb{C}P^1,
\]

\[
f_1(m) = X(L) - X(0).
\]

(3-31)

Both \( H_+ \) and \( \mathbb{C}P^1 \) are complex one-dimensional spaces. The moduli of closed elastica with genus one is given as the inverse image of zero point of \( f_1 \),

\[
\mathbb{M}^{(1)}_{\text{qcl}} = f_1^{-1}(0).
\]

(3-32)

Due to the analyticity of the map \( f_1 \), \( \mathbb{M}^{(1)}_{\text{qcl}} \) is zero-dimensional manifold. Thus the kind of the shapes are countable and due to the uniformity, there is only one-solution for each \( n \in \mathbb{N} \).

In ordinary computations [4,10-14,16,32,33], by reparameterizing above \( S^1 \) as \( \mathbb{R}/(L/n) \mathbb{Z} \), one starts with \( \mathbb{C}/((L/n) \mathbb{Z} \oplus \bar{K}' \mathbb{Z}) \), \( n \in \mathbb{N} \) and \( \bar{K}' \in H_+ \) without dividing \( \text{PSL}(2, \mathbb{Z}) \) and searches for the solutions satisfying the closed condition (2-19).
Similarly properties of the moduli of the closed elastica with genus \( g > 1 \) will be investigated. It is well-known that by the Sato theory, the characteristic of the KdV hierarchy in the KP hierarchy is to characterize its energy manifold in the inverse scattering method as hyperelliptic curve in general (compact) Riemannian surfaces [40-42]. The Miura map from the MKdV hierarchy to the KdV hierarchy are bijective. Thus I will deal only with the hyperelliptic curves in this article. First I will denote the moduli of the hyperelliptic curve of \( g(> 1) \) as \( \mathcal{M}_{R_g} \). Its element is conventionally expressed as \((I_g, T_g)\), where \( I_g \) and \( T_g \) are \( g \times g \) matrices; \( T_g = (\tau_1, \ldots, \tau_g) = (\tau_{ij}) \) and \( I_g = (e_1, \ldots, e_g) \) is the unit matrix. As I did in \( g = 1 \) case, I will deal with \( \tilde{K}(I_g, T_g) \), \( \tilde{K} \in \mathbb{R}_{>0} \) rather than \((I_g, T_g)\) itself. It is known that the dimension of the moduli of the hyperelliptic curves, \( \mathcal{M}_{R_g} \), is \( 2g - 1 \). Then I will also introduce a real 2\( g \)-dimensional lattice for a point of the moduli \( m \in \mathcal{M}_{R_g} := \mathbb{R}_{>0} \times \mathcal{M}_{R_g} [44-46] \),

\[
\Gamma_m = \left\{ \sum_{j=1}^{g} m_j \tilde{K} e_j + \sum_{j=1}^{g} n_j \tilde{K} \tau_j \mid (m_i, m_j \in \mathbb{Z}) \right\},
\]

and the Jacobi variety \( J_{g,m} := \mathbb{C}^g/\Gamma_m \). If I determine a point \( m \) of \( \mathcal{M}_{R_g} \), I can uniquely construct the Jacobi variety; \( J_{g,m} := \mathbb{C}^g/\Gamma_m \). From the soliton theory, if the coordinates of \( J_{g,m} \) as real manifold are expressed by \( t_{KP} = (t_1, t_2, t_3, \ldots, t_{2g}) \) [40-42], its subset with odd indices \( t_g := (t_1, t_3, \ldots, t_{2g-1}) \) can be identified with the part of \( t \) in (3-12). This identification can be guaranteed by the Krichever construction of the solution of the KP hierarchy [44] and Sato theory [40-42]. By the Krichever construction, it is known that each parameterization \( t_n \in t_{KP} \) is direct line passing the origin in the Jacobi variety. (3-17) and (3-19) are reduced to the linear differential equations in the Jacobi variety [44-46]. (Thus (3-17) can be recognized as the Jacobi equation of the Jacobi-field of the system (3-14).)

Since the moduli of the hyperelliptic curves has been also divided by a discrete group \( \text{Sp}(g, \mathbb{Z}) \) [44] like \( \text{PSL}(2, \mathbb{Z}) \) of \( g = 1 \) case, for fixing \( m \in \mathcal{M}_{R_g} \), there are \( N \times \text{Sp}(g, \mathbb{Z}) \) ways to embed \( S^1 \), as a period of \( \phi \), into \( J_{g,m} \). The number of the ways are equivalent to cardinal of \( N \times \text{Sp}(g, \mathbb{Z}) \) as a set. As I impose its periodicity of \( L/n \ (n \in \mathbb{N}) \) on \( S^1 \), the dilation parameter \( \tilde{K} \) is determined. Let such moduli space denoted as

\[
\mathcal{M}_{R_g,S^1} := \mathcal{M}_{R_g} \times N \times \text{Sp}(g, \mathbb{Z}) / \mathbb{R}_{>0}.
\]

By choosing a point of the moduli space \( \mathcal{M}_{R_g,S^1} \), a \( g \times g \) lattice \( \Gamma_{m,S^1} \) is uniquely determined and Jacobi variety is given as

\[
\mathcal{M}_{R_g,S^1} \to \{ \Gamma_{m,S^1} \}, \quad J_{g,m,S^1} := \mathbb{C}^g/\Gamma_{m,S^1}.
\]

I will fix a point of the modulus \( m \in \mathcal{M}_{R_g,S^1} \) for a while. From the properties of the MKdV hierarchy, \( \phi_{qcl} \) is a real analytic function of \( t_g \). As I can expand it around a point \( t_g \) using the properties of its real analyticity,

\[
\phi_{qcl}(t_g') = \sum_{n_0, \ldots, n_g} a_{n_0, \ldots, n_g} (t_1 - t_1)^{n_0}(t_3 - t_3)^{n_1} \cdots (t_{2g-1} - t_{2g-1})^{n_g},
\]

\( t_g \) must be a system of real parameters in \( J_{g,m,S^1} \). By analytic continuation, \( t_g \) can be locally complexified. On the other hand, the Jacobi variety has a canonical complex structure,

\[
\mathcal{J} : J_{g,m,S^1} \to J_{g,m,S^1}, \quad \mathcal{J}^2 = -1,
\]

which consists with its affine (vector) structure. By the structure \( \mathcal{J} \), there are set of the real \( g \)-dimensional submanifolds \( \{ \Sigma_{g,m,S^1} \} \) which includes the orbit of \( s (\in S^1) \) as its one-dimensional submanifold. Due to the analyticity of \( \phi_{qcl} \) over \( J_{g,m,S^1} \), these complexification of (3-36) cannot contradict with this complex structure \( \mathcal{J} \). Then \( t_g \) can be regarded as an element of \( \Sigma_{g,m,S^1} \). I will refer such embedding,
and the set of $\sigma_0$ is expressed as $\text{Em}_0(\Sigma_{g,m,S^1}, J_{g,m,S^1})$. Since $(t_3, \cdots, t_{2g-1})$ need not be periodic, this embedding is measurable and $\text{Em}_0(\Sigma_{g,m,S^1}, J_{g,m,S^1})$ can be regarded as a subset of the Grassmann manifold, $\text{Em}_0(\Sigma_{g,m,S^1}, J_{g,m,S^1}) \subset \text{GL}(\mathbb{R}, 2g - 1)/\text{GL}(\mathbb{R}, g)\text{GL}(\mathbb{R}, g - 1)$.

Then I can construct the fiber structure for $\text{Em}_0(\Sigma_{g,m,S^1}, J_{g,m})$, because for a way to such embedding, there is the trajectory space $\Sigma_{g,m,S^1}/S^1$ as fiber space, where $(t_3, \cdots, t_{2g-1}) \in \Sigma_{g,m,S^1}/S^1$. I refer the fiber bundle as $\mathcal{F}_m$,

$$\pi_{\text{traj}} : \mathcal{F}_m \to \text{Em}_0(\Sigma_{g,m,S^1}, J_{g,m,S^1}). \quad (3-39)$$

This fiber space also depends upon the point of the moduli space of the Riemannian surfaces $\mathcal{M}_{R_0,S^1}$. Hence the moduli of the periodic solutions of $\phi_{qcl}(s, t_g)$, which is written as $\mathcal{M}_{\text{period}}(g)$, has also fiber structure,

$$\pi_{\text{period}} : \mathcal{M}_{\text{period}}(g) \to \mathcal{M}_{R_0,S^1} \quad (3-40)$$

For each point $m \in \mathcal{M}_{R_0,S^1}$, the fiber bundle $\mathcal{F}_m$ (3-39) stands up as a fiber of $\mathcal{M}_{\text{period}}(g)$.

By the closed condition, I must restrict the moduli space. Here I will consider a real analytic map like (3-31),

$$f_g : \mathcal{M}_{\text{period}}(g) \to \mathbb{C}P^1,$$

$$f_g(\mu) = X(L) - X(0). \quad (3-41)$$

Consequently I obtain the moduli of the closed elastica, which is expressed as

$$\mathcal{M}_{qcl}^{(g)} = f_g^{-1}(0) \subset \mathcal{M}_{\text{period}}^{(g)} \quad (3-42)$$

Since image of $f_g$ is real two-dimensional manifold, for $g > 2$, $\dim(\mathcal{M}_{qcl}^{(g)}) \geq 1$ and $\mathcal{M}_{qcl}^{(g)}$ is measurable.

For simplicity, I will introduce the notations:

$$\mathcal{M}_{t_m=0}^{(g)} := \pi_{\text{traj}}(\mathcal{M}_{qcl}^{(g)}),$$

$$\mathcal{X}^{(g)} \ni n : \mathcal{M}_{t_m=0}^{(g)} \to \mathcal{M}_{t_m=0,n}^{(g)}, \quad \text{or} \quad \mathcal{M}_{t_m=0}^{(g)} = \bigoplus_{n \in \mathcal{X}^{(g)}} \mathcal{M}_{t_m=0,n}^{(g)}$$

$$\quad \text{for} \ (n, m) \in \mathcal{M}_{t_m=0}^{(g)}, \quad \tilde{\Sigma}_{m,n} := \pi_{\text{traj}}^{-1}(n, m) \quad (3-43)$$

where $\mathcal{X}^{(g)}$ is countable part of $\mathcal{M}_{t_m=0}^{(g)}$ and $\mathcal{M}_{t_m=0,n}^{(g)}$, the restriction of $\mathcal{M}_{t_m=0}^{(g)}$ for a point $n \in \mathcal{X}^{(g)}$, is measurable part. Here $\tilde{\Sigma}_{m,n}$ has coordinate $(t_3, \cdots, t_{2g-1}) =: t_m$. Hence there is a map from the moduli to the shape of the elastica.

$$h : \mathcal{M}_{qcl}^{(g)} \ni (n, m, t_m) \mapsto C_m^{(n)}(t_m^{(n)}) \subset \mathbb{C}P^1 \quad (3-44)$$

For a perturbative deformation like distortion to an ellipse from a circle, the energy manifold in the inverse scattering method has infinite gaps (or genera) in general. Thus such deformation is expressed in the moduli of $g \to \infty$ and due to the integrability of the MKdV equation, the deformation can be predicted like harmonic oscillator around a stable point. This picture is supported by the linearized method of the nonlinear equation and is also built in the above property of the limit.
Partition Function.

As I finish to classify the solution space formally, I will consider the fluctuation of the elastica again. It should be noted that there is above limit of the sequence $u_2^{(m)}$ corresponding to the genus of the elastica. If I encounter for $n = N$

$$\Omega u_2^{(N)} \equiv \lambda u_2^{(N)}, \quad \lambda \in \mathbb{R},$$

like (3-20), then $\delta t_{2(N+m)+1} \propto \delta t_{2N+1}$ for $m > 0$ because of (3-12) and

$$\partial t_{2(N+m)+1} u_1^{(N+m)} = k_{qcl} u_2^{(N+m)} = k_{qcl} \Omega u_2^{(N)} = \lambda \partial t_{2N+1} u_1^{(N)}.$$  \hfill (3-47)

Accordingly there needs no other fluctuation parameter $n > N$ because these fluctuation vectors are linearly dependent. The sequence of (3-9) should be truncated according to the philosophy of the canonical ensemble. Thus I will denote such minimal integer, which is a function of the solution, as $\text{ind}_0 : C \rightarrow N(C) \in \mathbb{Z}$. \hfill (3-48)

However from the soliton theory \[40-42\] and above argument, for $C \in \mathcal{M}_{qcl}^{(g)}$, I conclude that $\text{ind}_0(C) = g$. Avoiding meaningless divergence, I will replace the infinite series in (3-12) with finite sum from 1 to $g$ depending upon the shape of elastica.

Since the direction of $\delta t_1$ is along the tangential direction of the elastica $C_{qcl}$, its effect has been treated as the integral of $\delta t_1 \propto s$ in (3-26). On the other hand, $\delta t_{2n+1}$ ($n > 1$) includes the normal direction fluctuation and I must integrate the Boltzmann weight over $\delta t_{2n+1}$ space depending upon the genus of the elastica. Linear independence of these bases are guaranteed by above truncation.

Then for a curve $C \in \mathcal{M}_{qcl}^{(g)}$, the heat fluctuation of higher order is expressed as

$$\delta^{(n)} E[C, \delta t_{2m+1}] = \sum_{0 < m_i \leq g} \frac{1}{\sqrt{\beta}} \prod_i (\delta t_{2m_i+1}) \int ds \prod_i (\partial t_{2m_i+1}) k^2.$$ \hfill (3-49)

Here $m_i = 0$ part vanishes due to the periodicity

$$\int ds \partial_s \left( \prod_i (\partial t_{2m_i+1}) k^2 \right) = 0.$$ \hfill (3-50)

On the other hand, if the set $\{m_i\}$ does not contain $m_i = 0$ component, the integral commutes with these derivatives,

$$\int ds \prod_i (\partial t_{2m_i+1}) k^2 = \prod_i (\partial t_{2m_i+1}) \int ds k^2.$$ \hfill (3-51)

Since $\int k^2 ds$ is invariant for the time $t_{2n+1}$ ($n > 0$) development from the soliton theory, (3-51) vanishes. Hence I obtain that all higher order fluctuations vanish,

$$\delta^{(n)} E[C, \delta t_{2m+1}] \equiv 0, \quad \text{for } n > 0.$$ \hfill (3-52)

In other words, the effect of heat fluctuation is given only through the energy functional of the quasi-classical motions. Since for a quasi-motion of genus $g$, the curvature is precisely given as

$$k(s, t_1, t_2, \cdots, t_{2g-1}) = k_{qcl}(s, t_1 + \frac{1}{\sqrt{\beta}} \delta t_1, t_2 + \frac{1}{\sqrt{\beta}} \delta t_2, \cdots, t_{2g-1} + \frac{1}{\sqrt{\beta}} \delta t_{2g-1}),$$ \hfill (3-53)
I must integrate the Boltzmann weight \( \exp(-\beta \int ds k^2) \) over all \( \delta t \)'s except \( \delta t_1 \). Using the translation symmetry and freedom of the integration variable, I regard that \( t_{2n+1} = \delta t_{2n+1}/\sqrt{\beta} \).

Consequently I obtain an explicit form of the regularized partition function (3-26), which is expressed by

\[
Z_{\text{reg}}[\beta] = \sum_{g} \sum_{C \in \mathcal{W}_{qcl}^{(g)}} \left( \exp(-\beta E_{qcl}[C]) \right)
\]

\[
= \sum_{g=0}^{1} \sum_{n \in \mathcal{X}_{g}} \left( \exp(-\beta E_{qcl}[C^n]) \right)
\]

\[
+ \sum_{g=2}^{\infty} \sum_{n \in \mathcal{X}(g)} \int_{\mathcal{W}_{t_0}^{(g)}} \mathcal{M} \left( \prod_{n=2}^{g} \int_{\Sigma_{m,n}} dt_{2n-1} \right) \exp \left( -\beta E_{qcl}[C^{(n)}(t^{(n)})_{m}] \right).
\]

This is the exact form of the partition function (3-1) of the non-stretched elastica without divergence. In the second term, there appear the integration of the type of \( \int dx e^{-\beta f(x)} \). Thus it is expected that the prefactor of the second term begins with the negative power of \( \beta \). For large \( \beta \), the second term is less than the first term. Hence for the zero temperature limit \( \beta \to \infty \), the second term disappears and only the contribution of the genus 0 and 1 survives. Noting that the moduli of the quasi-classical elastica with \( g \leq 1 \) is equivalent with the that of the classical, I obtain

\[
\lim_{\beta \to \infty} Z_{\text{reg}}[\beta] = \max_{C \in \mathcal{S}_{\text{cls}}} \exp(-\beta E_{\text{cls}}[C])
\]

\[
= \exp(-\beta \min_{C \in \mathcal{S}_{\text{cls}}} E_{\text{cls}}[C]).
\]

Depending upon the boundary condition, the classical solutions appear as minimal points of the partition function \( Z_{\text{reg}}[\beta] \). Hence this partition function (3-54) does not contradict with the discovery of Daniel Bernoulli [1].

§4. Discussion

It is worth while noting that due to the isometric condition, I have derived the MKdV hierarchy. In the elastic body, the lagrangian coordinate system should be employed rather than the Eulerian coordinate system when I will use the terms of the fluid mechanics. In the elastic body theory by marking some points on an elastic body and by estimating variation of distance among the marking points which is measured using the induced metric, the force will be locally evaluated as linear response for its certain deformation. The marking points corresponds to the Lagrangian test particle in the language of fluid mechanics. On the other hand, as I have used the metric induced from the base space and calculated the deformation, my calculation corresponds to the Eulerian one. Here it should be noted that if one uses the induced metric or Eulerian picture, any stretching (physical) curve can be regarded as a non-stretching (mathematical) curve; it is a trivial trick between the lagrangian picture and Eulerian picture and such recognition has few physical meanings. If stretching has physical meaning like an elastic body, Eulerian picture does not exhibit dynamical situation and unless stretch plays important role like boundary curve of binary fluid, dealing with stretch has less physical meaning. Accordingly the isometric condition I employed plays the central role in this scheme. In other words, in above computation, the reason why I could physically use the Eulerian picture even in the elastic body problem is owing to this isometric condition.
It should be also noted that even though there appears non-linear differential equation in this scheme, I have used the energy functional which is locally given in the framework of linear response of the force for the deformation [4,11]; if one uses the non-linear energy functional, he must evaluate it from basic elastic body theory because it is beyond the ordinary elastic body theory. It is remarked that due to the bilinearity of the energy functional, which is established in the framework of the ordinary elastic body theory, I could find the exact partition function (3-54) in this model.

Furthermore the origin of the MKdV hierarchy in ref.[8,9,33] was artificial and was not physically supported. If one physically sets up problem of time development of the elastica for real physical time, he concludes that its motion is not governed by the MKdV equation nor the MKdV hierarchy in general [10-14]. However in this article, I obtained the MKdV hierarchy from the physical requirement and a (mathematical) parameter time \( \delta t_{2n+1} \) appears variational direction as I pointed out in ref.[10]. In other words, by virtue of the novel investigation of the properties of isometric curve of Goldstein and Petrich [8,9], I conclude that the virtual dynamics is realized as thermal fluctuation of an elastica in heat bath. Due to the isometry condition, these equations become non-linear differential equations. In the linear differential equation such as the harmonic oscillator, the mode, which is determined by the global feature of the system, is represented by a vector of momentum space. As well as mode analysis of the linear system, these parameters exhibit the global deformation of the elastica due to the thermal fluctuation and is expressed as a vector in the Jacobi variety.

It is remarked that the obtained partition function (3-54) differs from that in ref.[24], which is obtained by summing the weight function over the configuration including non-isometry deformation. Due to the isometry condition, non-linear terms appears in the quasi-classical curve equation while the partition function proposed by Saitô et al. [24] is essentially linear. However for perturbative deformation, e.g. from the circle, the non-linear term might be negligible. Thus as long as the deformation is in perturbative, their partition function can be applicable even for a polymer which cannot be stretched even in thermal fluctuation.

On the other hand, my partition function is justifiable even for large deformation. The partition function is summed over different topology \( g \), which is related to the writhing number of the configuration. Hence there is a possibility of the topology change due to the thermal fluctuation. It is of interesting to calculate the possibility (or kernel function) from \( g = 1 \) conformation to \( g = 2 \) conformation. Even though the partition function (3-54) has not been concretely calculated, such computation, in principle, can be performed.

Next I will comment on the physical meaning of \( \delta t_{2n+1} \) and the relation between the BRS transformation [31] and the Sato coordinate [40-42]. Since I have dealt with the SO(2) principal bundle over \( S^1 \), the gauge group is expressed as

\[
\mathfrak{G} \subset \prod_{s \in S^1} \text{SO}(2),
\]

where \( \prod \) means the disjoint union. \( \mathfrak{G} \) is infinite dimensional Lie group. It acts upon the shape of the curve, which corresponds to a section of the principal bundle, and deforms it. For a given shape of elastica, there is a unique group element which acts the elastica to become the shape with constant curvature, i.e., the simplest classical solution with \( g = 0 \). Thus the genus is well-defined, which is induced from the genus of curve (the quasi-classical section). There is a decomposition of \( \mathfrak{G} \) as a family of subgroups \( \mathfrak{G}_g \) respect to the genus, whose action on the elastica preserves its genus. The representation of each group \( \mathfrak{G}_g \) will be realized as \( \mathfrak{G}_g \) module in the set of corresponding Jacobi varieties. However in the soliton theory, instead of dealing with individual sets of Jacobi varieties of genus \( g \), it is natural to consider the UGM if one wishes to formally treat a soliton equation. In fact there are singular elements in \( \mathfrak{G} \), which change the genus of the elastica; the transformation are known as the global gauge transformation. Corresponding to UGM, \( \mathfrak{G} \) should be taken to be the inductive limit of the filtration of \( \mathfrak{G}_g \) as well as the set of corresponding Jacobi varieties.
and then $\mathfrak{g}$ naturally contains the singular elements due to the natural extension of the group action. Thus $\mathfrak{g}$ is represented as a subset of $\text{GL}(\infty)$ in the UGM. The quasi-classical curve (a section of the $\text{SO}(2)$ principal bundle) is embedded in the UGM. The infinitesimal deformation of the curve in the UGM can be expressed by a vector in the UGM. In other words, such deformation exhibits (mathematical) velocity of a trajectory in the UGM and can be represented as subset of the infinite dimensional general linear Lie algebra $\text{gl}(\infty)$, which is known as the affine Lie algebra $A_1^{(1)}$ [41]; $A_1^{(1)}$ is the Lie algebra associated with the Lie group $\mathfrak{g}$. I will introduce the extrinsic differential operator in the UGM

$$\delta := \sum_n dt_{2n+1} \partial_{t_{2n+1}}, \quad \delta^2 = 0. \quad (4.2)$$

Then (3-15) and (3-17) are expressed as for $A := k_{\text{qcl}} ds$,

$$\delta A = \tilde{\Omega} u_1, \quad (4.3)$$

where

$$u_1 = \sum_n u_1^{(n)} dt_{2n+1}, \quad \tilde{\Omega} := \Omega k_{\text{qcl}}^{-1} \partial_s. \quad (4.4)$$

Noting the fact that $u_1^{(n)}$ is hamiltonian density of the MKdV hierarchy,

$$\delta u_1 \approx 0, \quad (4.5)$$

where $\approx$ means equivalence after integration the both sides over $s$ like (3-50) and (3-51). Since $u_1$ obeys the Grassmannian algebra, $u_1$ can be regarded as fermionic field over $S^1$. Consequently (4-3) and (4-5) can be regarded as the BRS transformation of this system. Hence $\delta t_{2n+1}$ in the path integration may be naturally understood in the framework of the Faddeev-Popov integration scheme [31]. In fact the square root of the Frenet-Serret system (2-2) can be regarded as the Dirac operator [11,12], which is realized by confining the free Dirac field into a thin elastic rod. Using the Dirac field confined in the elastica, I constructed the MKdV hierarchy and $\tau$-function as the partition function of the Dirac field [47]. Thus I expect that the partition function (3-54) should also be expressed by the $\tau$-function of the MKdV hierarchy.

Here I will mention a conjecture associated with the critical phenomenon of this elastica model. The critical point must be determined as a topological discontinuity of the moduli space of the quasi-classical elastica. At the point, physical quantity sometimes diverges and becomes meaningless. I expect that the length of the elastica becomes less important around the critical point, e.g. the topological change, as a kernel function at an ordinary second order critical point becomes scale-invariant [48]. I will consider a dilatation of a quasi-classical elastica for the normal direction of elastica,

$$X_c = X + it e^{i\phi_c}, \quad (4.6)$$

where $X_c$ and $\phi_c$ are an affine vector and tangential phase of the quasi-classical elastica at the critical point and $t$ is a real deformation parameter. The infinitesimal length of the elastica $ds_c$ becomes

$$ds_c = \sqrt{dX_c dX_c} \approx (1 - k_c t) ds, \quad k_c := \partial_s \phi_c, \quad (4.7)$$

and then the length of elastica is

$$\int ds_c \approx L - t(\phi_c(L) - \phi_c(0)). \quad (4.8)$$
Noting that $\phi_c(L) - \phi_c(0) = 2\pi n$, $n \in \mathbb{Z}$, this deformation makes the length of the elastica change. However, this deformation must be compatible with the isometric condition since I have been dealing with the non-stretching elastica. Both requires seem to contradict with but the critical point is an irregular point at which the contradicted objects coexist. From (3-2), $it$ in (4-6) must be proportional to $u_c := u_1 + iw_2$ with (3-18-b), and $u_1$ satisfies (3-19-b) in respect of the deformation parameter $t$.

$$ict = u_c = \frac{1}{4}k_c^2 - \frac{1}{2}\partial_s k_c,$$  \hspace{1cm} (4-9)$$

where $c$ is a real proportional constant. Since this relation is the Miura map, from (3-19-b), $u_c$ obeys the KdV equation

$$ic = \partial_t u_c = 6u_c\partial_s u_c + \partial_s^3 u_c.$$  \hspace{1cm} (4-10)$$

By integrating it in $s$, (4-10) becomes

$$ics = 3u_c^2 + \lambda \partial_s^2 u_c.$$  \hspace{1cm} (4-11)$$

For $z := is$ $w := u_c/2$ and $c = -2$, (4-11) can be rewritten as

$$\partial_z^2 w = 6w^2 + z.$$  \hspace{1cm} (4-12)$$

This is the Painlevé equation of the first kind. Thus I will conjecture that at a critical point of the elastica, an expectation value obeys the Painlevé equation of the first kind [49]. In our scheme, I can formally obtain the series of the ordinary differential equations related to the KdV hierarchy (from the Miura map) and the Painlevé transcendents. Thus the moduli space must be closely related to the quantum two-dimesnional gravity, in which (4-12) and the KdV hierarchy naturally appear [50-52]. In fact, the loop soliton partially appears in the immersed surface in three dimensional space $\mathbb{R}^3$ [38,53]. By the fermionic study [47,53-57], the immersed surface system is interpreted as a natural extension of the elastica system and also that of the Liouville surface system whose quatum version is known as the quantum two-dimesnional gravity [55-56]. Thus I plane to investigate the immersed surface system and reveal the relation between the elastica system and the quantum two-dimesnional gravity. (Here it should be noted that elastica problem is not directly related to the string problem in the string theory because the action of the elastica is biharmonic for $X$ while that of string is harmonic [55]. In non-relativistic space, thickness is more important and biharmonic equation is very natural.)

As I dealt with the kinetic properties of the large closed polymer and investigated the moduli of the MKdV equation in $\mathbb{C}P^1$, recently another statistical mechanics model of a large polymer was reported [58]. Partition function of non-contractible self-avoiding two-dimensional polymers in the topological torus,

$$T = \mathbb{C}/(L_1\mathbb{Z} + L_2\mathbb{Z}),$$  \hspace{1cm} (4-13)$$

was studied associated with the MKdV equation. The partition function of such polymers was also solved by the MKdV equation [58] and is recognized as its $\tau$-function [59]. As I assumed the base space as $\mathbb{C}P^1$, I can also replace it with the topological torus. Then the ratio $L : L_1 : L_2$ becomes important as the boundary conditions but I can formally calculate it. Then obtained partition function may be also closely related to the $\tau$-function of the MKdV equation. It is a very interesting fact that the MKdV equation appears and plays central roles in both theories even though they are not directly concerned with as models.

Finally I will mention farther possibilities and development of this theory. I investigated the elastica in two-dimensional space but can extend my theory to that in three-dimensional space if I can classify solutions of quasi-classical curves of elastica in three-dimensional space. Thus I need investigation on the moduli of quasi-classical elastica in three-dimensional space including topological properties like a knot invariance. Furthermore even though I formally classified the moduli of the quasi-classical elastica in two-dimensional space, I cannot explicitly draw the shape.
of the closed elastica of \( g > 1 \) now. Accordingly it is very important for this study to find explicit shapes of closed elastica of \( g > 1 \). If I could explicitly draw any shapes of closed quasi-classical elastica in plane of \( g = 2 \) or 3 using hyperelliptic functions, I think, they have enormous effects on this problem. Since the summation of the partition function might be converges on the genus \( g \), to determine the shapes even for small \( g \) means important steps on this problem. Moreover, it is also of interests to deal with an elastic rod which can be stretched as more physical model [14] and higher dimensional objects, such as an immersed surface [38,53,56,57].

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