CUBIC HAMILTONIANS

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Abstract. We determine a precise necessary and sufficient condition for completeness of the Hamiltonian vector field associated to a homogeneous cubic polynomial on a symplectic plane.

0. Introduction

The flow of the Hamiltonian vector field generated by a smooth function on a symplectic manifold is a familiar object of study. Let the symplectic manifold be simply a symplectic vector space: the Hamiltonian flow generated by a homogeneous linear function is a one-parameter group of translations; the Hamiltonian flow generated by a homogeneous quadratic function is a one-parameter group of linear symplectic transformations. In each of these two cases, the Hamiltonian flow is complete: each maximal integral curve of the Hamiltonian vector field is defined for all time. The case of cubic Hamiltonian functions is different: for some cubics the flow is complete whereas for others it is incomplete.

Our primary objective in this paper is to establish a simple necessary and sufficient condition for the cubic \( \psi \) on a symplectic plane \( (Z, \Omega) \) to generate a complete Hamiltonian flow. In Section 1 we associate with \( \psi \) a suitably symmetric linear map from \( Z \) to the symplectic Lie algebra \( \text{sp}(Z, \Omega) \); following this map with the determinant yields a quadratic map \( \Delta : Z \to \mathbb{R} \). In Section 2 we analyze an arbitrary integral curve \( z : I \to Z \) of the Hamiltonian vector field \( \xi^\psi \) defined by \( \psi \); we find that the second time-derivative \( \ddot{z} \) equals \( 2Fz \), where the scalar function \( F := \Delta \circ z : I \to \mathbb{R} \) satisfies the equation \( \ddot{F} = 6F^2 \) familiar from the theory of elliptic functions. In Section 3 we achieve our primary objective, proving that the Hamiltonian vector field \( \xi^\psi \) is complete if and only if the determinant \( \Delta \) is identically zero; beyond this, we comment on the nonconstant integral curves of \( \xi^\psi \) in the complete case and the incomplete case. Finally, we assemble several remarks on issues arising from the main body of the paper: in particular, we remark that \( \Delta \) is identically zero if and only if \( \psi \) is a monomial; these remarks we plan to develop more fully in subsequent papers.

In a subsequent paper we also plan to present a similar treatment of quartic Hamiltonian functions; for now, we merely note one difference between the cubic case and the quartic case. In the cubic case, the scalar function \( F \) satisfies the differential equation \( \ddot{F} = 6F^2 \) whose elliptic solutions are always Weierstrass P\(\e\) functions associated to triangular lattices, with \( g_2 \) zero; in the quartic case, the corresponding scalar functions include Weierstrass functions associated to rectangular lattices, with \( g_2 \) nonzero.

1. Symplectic Algebra

Let \( (Z, \Omega) \) be a real symplectic vector space: thus, \( Z \) is a vector space and \( \Omega : Z \times Z \to \mathbb{R} \) a nonsingular alternating bilinear form. Though it is not necessary for some of what we shall say, we suppose throughout that \( Z \) is two-dimensional, so that \( (Z, \Omega) \) is a symplectic plane. The
symplectic algebra $\text{sp}(Z, \Omega)$ is the (commutator bracket) Lie algebra comprising all linear maps $C : Z \to Z$ such that for all $x, y \in Z$

$$\Omega(Cx, y) + \Omega(x, Cy) = 0.$$  
As a vector space, $\text{sp}(Z, \Omega)$ is canonically isomorphic to the space of all symmetric bilinear forms on $Z$: to $C \in \text{sp}(Z, \Omega)$ there corresponds the symmetric bilinear form $Z \times Z \to \mathbb{R} : (x, y) \mapsto \Omega(x, Cy)$.

Now, let $\psi : Z \to \mathbb{R}$ be a homogeneous cubic polynomial. To $\psi$ we associate the (fully) symmetric trilinear function $\Psi : Z \times Z \times Z \to \mathbb{R}$ with value at $(x, y, z) \in Z \times Z \times Z$ given by

$$\Psi(x, y, z) = \psi(x + y + z) - \{\psi(y + z) + \psi(z + x) + \psi(x + y)\} + \psi(x) + \psi(y) + \psi(z).$$

When $z \in Z$ is fixed, $\Psi(x, y, z)$ is symmetric bilinear in $(x, y) \in Z \times Z$; it follows that there exists a unique $\Gamma_z \in \text{sp}(Z, \Omega)$ such that for all $x, y \in Z$

$$\Psi(x, y, z) = 2\Omega(x, \Gamma_y z).$$

Full symmetry of $\Psi$ guarantees that the resulting linear map

$$\Gamma^\psi = \Gamma : Z \to \text{sp}(Z, \Omega)$$
is symmetric in the sense that for all $x, y \in Z$

$$\Gamma_x y = \Gamma_y x.$$  

Note that if $z \in Z$ then

$$2\Omega(z, \Gamma_z z) = \Psi(z, z, z) = \{27 - (3 \times 8) + 3\}\psi(z) = 6\psi(z)$$
or

$$\psi(z) = \frac{1}{3}\Omega(z, \Gamma_z z).$$

Differentiation of this formula for $\psi$ yields the result that if $v, z \in Z$ then

$$\psi'_z(v) = \frac{1}{3}\{\Omega(v, \Gamma_z z) + \Omega(z, \Gamma_v z) + \Omega(z, \Gamma_z v)\}$$
whence by symmetry of $\Gamma : Z \to \text{sp}(Z, \Omega)$ it follows that

$$\psi'_z(v) = \Omega(v, \Gamma_z z).$$

Of course, as $\psi$ is a cubic, the first derivative $\psi'_z$ is quadratic in $z \in Z$. As a bilinear form, the second derivative $\psi''_z$ at $z \in Z$ furnishes another means of introducing $\Psi$ and $\Gamma$: indeed, if also $x, y \in Z$ then

$$\psi''_z(y, x) = \Psi(x, y, z) = 2\Omega(x, \Gamma_z y).$$

This equation represents $\psi''_z$ by $2\Gamma_z$ relative to the symplectic form $\Omega$; consequently, the classical Hessian of $\psi$ is $\text{Det}(2\Gamma_z)$.

According to the Cayley-Hamilton theorem, if $z \in Z$ then

$$\Gamma_z \Gamma_z - (\text{Tr} \Gamma_z) \Gamma_z + (\text{Det} \Gamma_z) I = 0$$
whence the fact that $\Gamma_z \in \text{sp}(Z, \Omega)$ is traceless implies that

$$\Gamma_z \Gamma_z = -(\text{Det} \Gamma_z) I.$$  

We define the scalar function $\Delta^\psi = \Delta : Z \to \mathbb{R}$ by requiring that for each $z \in Z$

$$\Delta(z) = -(\text{Det} \Gamma_z)$$
so that

$$\Gamma_z \Gamma_z = \Delta(z) I.$$
Theorem 1. If \( z \in Z \) then \( \Delta(\Gamma_z z) = \Delta(z)^2 \).

Proof. If \( z = 0 \) then both sides of the alleged equation plainly vanish. If \( z \neq 0 \) then apply the special case \( \Gamma_{\Gamma_z z} = \Gamma_z \Gamma_z z \) of symmetry repeatedly: a first application gives
\[
\Delta(\Gamma_z z)z = \Gamma_z \Gamma_z \Gamma_z z = \Gamma_z \Gamma_z \Gamma_z z = \Gamma_z \Delta(z)z
\]
and a second application gives
\[
\Delta(z)\Gamma_z z = \Delta(z)\Gamma_z z = \Delta(z)\Delta(z)z = \Delta(z)^2 z
\]
whence the alleged equation follows by cancellation. \( \square \)

2. Cubic Hamiltonians

We shall now view \((Z, \Omega)\) as a symplectic manifold in the natural way. Thus, the vector space \( Z \) is naturally a smooth manifold; if \( z \in Z \) then there is a natural isomorphism from the vector space \( Z \) to the tangent space \( T_z Z \) sending \( v \in Z \) to the directional derivative operator \( v|_z \in T_z Z \) given by the rule that whenever \( f : Z \to \mathbb{R} \) is a smooth map,
\[
v|_z(f) = f'_z(v) = \frac{d}{dt} f(z + tv)|_{t=0}.
\]
Also, \( \Omega \) serves double duty as a nonsingular alternating bilinear form on the vector space \( Z \) and as a nonsingular closed two-form on the smooth manifold \( Z \); explicitly, if \( x, y, z \in Z \) then the value \( \Omega_z \) of the two-form at \( z \) is given by
\[
\Omega_z(x|_z, y|_z) = \Omega(x, y).
\]

When \( f : Z \to \mathbb{R} \) is a smooth (Hamiltonian) function, the corresponding Hamiltonian vector field \( \xi_f \in \text{Vec}Z \) on \( Z \) is defined by the requirement
\[
\xi_f \lrcorner \Omega = -df
\]
where \( \lrcorner \) signifies contraction as usual. An integral curve of the vector field \( \xi_f \) is a smooth map \( z : I \to Z \) (on some open interval \( I \ni t \)) satisfying the Hamilton equations: for each \( t \in I \) the tangent vector to \( z \) at \( t \) equals the value of \( \xi_f \) at \( z_t \), thus
\[
\frac{dz_t}{dt} = \xi_f(z_t).
\]

We shall focus on the case of a homogeneous cubic \( \psi : Z \to \mathbb{R} \) as Hamiltonian function. The value of \( \xi_{\psi} \) at \( z \in Z \) is a vector made tangent at \( z \); say
\[
\xi_{\psi} = x_{\psi}(z)|_z
\]
with \( x_{\psi} : Z \to Z \) a smooth vector-valued function. Now, let \( v, z \in Z \): on the one hand,
\[
(\xi_f \lrcorner \Omega)_z(v|_z) = \Omega_z(\xi_{\psi}^z, v|_z) = \Omega_z(x_{\psi}(z)|_z, v|_z) = \Omega(x_{\psi}(z), v);
\]
on the other hand,
\[
-\,d\psi_z(v|_z) = -\psi'_z(v) = -\Omega(v, \Gamma_z z) = \Omega(\Gamma_z z, v).
\]
As the symplectic form \( \Omega \) is nonsingular, it follows that
\[
x_{\psi}(z) = \Gamma_z z.
\]
Accordingly, the Hamilton equation for \( z : I \to Z \) reads
\[
\frac{\dot{z}}{\dot{z}} = \Gamma_z z.
\]

Let \( z : I \to Z \) be a solution of this Hamilton equation. Take a further derivative: as \( \Gamma \) is symmetric,
\[
\frac{\ddot{z}}{\dot{z}} = \Gamma_z z + \Gamma_z \frac{\dot{z}}{\dot{z}} = 2\Gamma_z \frac{\dot{z}}{\dot{z}} = 2\Gamma_z \Gamma_z z
\]
by a further application of the Hamilton equation. Recall that if \( w \in Z \) then \( \Gamma_w \Gamma_w = \Delta(w)I \) and write
\[
F := \Delta \circ z : I \to \mathbb{R}.
\]
It then follows that \( z : I \to Z \) satisfies the second-order equation
\[
\overset{\circ}{\overset{\circ}{z}} = 2Fz.
\]
Note here that \( \Delta \) is defined on the whole space \( Z \) while \( F \) is defined only along the integral curve \( z \).

**Theorem 2.** The scalar function \( F \) satisfies the second-order equation
\[
\overset{\circ}{\overset{\circ}{F}} = 6F^2.
\]

**Proof.** From the definition
\[
FI = \Gamma_z \Gamma_z
\]
we deduce by repeated differentiation that
\[
\overset{\circ}{\overset{\circ}{F}} I = \Gamma_z \Gamma_z + \Gamma_z \Gamma_z
\]
and
\[
\overset{\circ}{\overset{\circ}{F}} I = \Gamma_z \Gamma_z + 2\Gamma_z \Gamma_z + \Gamma_z \Gamma_z.
\]
Here, the first and last terms on the right both equal \( 2FTP \Gamma_z = 2F^2I \) on account of \( \overset{\circ}{\overset{\circ}{z}} = 2Fz \) while \( \Gamma_z \Gamma_z \) equals \( F^2I \) on account of \( \overset{\circ}{z} = \Gamma_z z \) and Theorem 1. \( \square \)

We may at once deduce a first-order integral of this second-order equation: multiply through by \( 2F \) to obtain
\[
2F \overset{\circ}{\overset{\circ}{F}} I = 12F^2I \overset{\circ}{\overset{\circ}{F}}
\]
from which there follows
\[
(F)^2 = 4F^3 - g_3
\]
for some real constant \( g_3 \). This notation is deliberately chosen to accord with the theory of elliptic functions. In fact, the solutions to this first-order differential equation are as follows:

- if \( g_3 \) is nonzero then \( F(t) = \varphi(t - a) \) for some real \( a \) where \( \varphi \) is the Weierstrass Pe function associated to a triangular lattice (the so-called equianharmonic case);
- if \( g_3 \) is zero then either \( F(t) = (t - a)^{-2} \) for some real \( a \) or \( F \) is identically zero.

Note that when \( F \) is a (shifted) Weierstrass Pe function, \( \overset{\circ}{\overset{\circ}{z}} = 2Fz \) is a (vectorial) Lamé equation and may be solved accordingly; for example, see page 285 of [Forsyth].

### 3. Completeness Characterized

We continue to let \( \Gamma : Z \to \text{sp}(Z, \Omega) \) be the symmetric linear map corresponding to the homogeneous cubic \( \psi : Z \to \mathbb{R} \) on the symplectic plane \( (Z, \Omega) \); we also continue to let \( z : I \to Z \) be an integral curve of the associated Hamiltonian vector field \( \xi^\psi \). We shall suppose that the curve \( z \) has initial point \( z_0 \) and hence initial velocity \( \overset{\circ}{z}_0 = \Gamma_{z_0} z_0 \). Our aim in this section is to decide precisely when such an integral curve may be defined for all time; that is, precisely when the maximal domain of definition \( I \) is \( \mathbb{R} \) itself.

The critical case is decided immediately. Let \( \xi^\psi \) (equivalently, \( d\psi \)) vanish at \( z_0 \); thus, \( z \) has initial velocity \( \overset{\circ}{z}_0 = \Gamma_{z_0} z_0 = 0 \). In this critical case, the solution \( z : I \to Z \) is plainly given by \( z_t = z_0 \) for all \( t \in I \) and the maximal \( I \) is indeed \( \mathbb{R} \). In this connexion, note further that if an integral curve \( z : I \to Z \) vanishes at any point then so does its velocity vector and hence \( z \) itself is identically zero.

Now let the integral curve \( z : I \to Z \) be other than critical; thus, \( \Gamma_{z_0} z_0 = \overset{\circ}{z}_0 \neq 0 \) and of course \( z_0 \neq 0 \). We distinguish two cases.
For the first case, suppose there exists some \( s \in I \) such that \( 0 \neq F(s) = \Delta(z_s) \) and therefore \( \bar{F}'(s) = \bar{F}(s)^2 > 0 \). The comments after Theorem 2 show that \( F \) has a double pole at some real \( a \); thus \( \Gamma_z \Gamma_z = F(t)I \) is unbounded as \( t \to a \) and so \( z_t \) itself is unbounded as \( t \to a \). In this case, the maximal domain of \( z \) omits \( a \) and thereby falls short of \( \mathbb{R} \).

For the second case, suppose that \( F(t) = 0 \) whenever \( t \in I \). Note that the linear map \( \Gamma_{z_0} \) kills \( \Gamma_{z_0}z_0 \) (because \( \Gamma_{z_0}\Gamma_{z_0} = F(0)I = 0 \)) but does not kill \( z_0 \) (because \( \Gamma_{z_0}z_0 = z_0 \neq 0 \)); thus \( z_0 \) and \( z_0 \) constitute a basis for the plane \( Z \) and so

\[
\{s(z_0 + t \bar{z}_0) : s, t \in \mathbb{R}\} = (Z \setminus \mathbb{R} \bar{z}_0) \cup \{0\}.
\]

The supposition \( F \equiv 0 \) implies that \( \bar{z} = 2Fz \equiv 0 \) so that \( z_t = z_0 + t \bar{z}_0 \) for all \( t \in I \); essentially as in the critical case, the maximal \( I \) is therefore \( \mathbb{R} \). Now \( \Delta \) vanishes on \( z_0 + t \bar{z}_0 \) whenever \( t \in \mathbb{R} \) (as \( F \) is identically zero) and hence vanishes on \( s(z_0 + t \bar{z}_0) \) whenever \( s, t \in \mathbb{R} \) (as \( \Delta \) is homogeneous); the continuous function \( \Delta \) now vanishes on the dense set \( (Z \setminus \mathbb{R} \bar{z}_0) \cup \{0\} \) and therefore vanishes on the whole of \( Z \). This proves that if \( \Delta \) vanishes on the image of some non-critical integral curve then \( \Delta \) vanishes identically.

We may now marshal these facts towards the following result.

**Theorem 3.** Let \( \psi : Z \to \mathbb{R} \) be a homogeneous cubic and \( \Delta^{\psi} \) the associated determinant.

- If \( \Delta^{\psi} = 0 \) then \( \xi^{\psi} \) is complete; each non-constant integral curve is an affine line.
- If \( \Delta^{\psi} \neq 0 \) then \( \xi^{\psi} \) is incomplete; only the constant integral curves are defined for all time.

**Proof.** If \( \Delta \equiv 0 \) then each maximal integral curve \( z \) has \( F \equiv 0 \) so that \( \bar{z} = 2Fz \equiv 0 \) and \( z \) on \( \mathbb{R} \) is affine, as we have seen. If \( \Delta \neq 0 \) and the integral curve \( z \) is not critical, then \( F \neq 0 \) so that \( z \) experiences finite-time blow-up, as we have seen.

Looking ahead to the next section, we remark that \( \Delta^{\psi} \) is identically zero if and only if \( \psi \) is monomial in the sense that there exists \( w \in Z \) such that for all \( z \in Z \)

\[
\psi(z) = \frac{1}{3} \Omega(w, z)^3.
\]

4. Remarks

In this closing section, we record a number of miscellaneous remarks that stem from the body of this paper.

**COORDINATE EXPRESSIONS**

Though our whole approach has been intentionally coordinate-free, it is also of interest to see the development in terms of linear symplectic coordinates, not least because this may offer glimpses of a fresh perspective on classical invariant theory.

To this end, let \( u, v \in Z \) satisfy \( \Omega(u, v) = 1 \) and so constitute a symplectic basis for \((Z, \Omega)\). Decompose \( z \in Z \) as

\[
z = pu + qv
\]

with

\[
p = p(z) = \Omega(z, v), \quad q = q(z) = \Omega(u, z).
\]

Write

\[
a = \Omega(u, \Gamma_u u), \quad b = \Omega(u, \Gamma_v u),
\]

\[
c = \Omega(v, \Gamma_u v), \quad d = \Omega(v, \Gamma_v v).
\]

With these conventions, the cubic

\[
\psi(z) = \frac{1}{3} \Omega(z, \Gamma z z)
\]
has coordinate form
\[ \psi(z) = \frac{1}{3}(ap^3 + 3bp^2q + 3cpq^2 + dq^3) \]
and the (vector) Hamilton equation
\[ \circ z = \Gamma_z z \]
becomes the familiar scalar pair
\[ \circ p = -\frac{\partial \psi}{\partial q}, \quad \circ q = \frac{\partial \psi}{\partial p}. \]
The associated determinant
\[ \Delta(z) = -(\text{Det } \Gamma_z) \]
assumes the form
\[ \Delta(z) = (b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2 \]
and is the Hessian of \( \psi \) (up to scale). We are not the first to observe that the discriminant
\[ (bc - ad)^2 - 4(b^2 - ac)(c^2 - bd) \]
of this quadratic is precisely the discriminant
\[ a^2d^2 - 3b^2c^2 + 4ac^3 + 4b^3d - 6abcd \]
of the cubic
\[ ap^3 + 3bp^2q + 3cpq^2 + dq^3; \]
for example, see page 60 of [Salmon].

Of course, a purely coordinate-based approach is possible. Let us indicate partial derivatives more succinctly by means of subscripts. With the cubic
\[ \psi(z) = \frac{1}{3}(ap^3 + 3bp^2q + 3cpq^2 + dq^3) \]
as above, direct computation reveals that \( \psi_{pq}\psi_q - \psi_p\psi_{qq} \) is divisible by \( p \) and \( \psi_{pp}\psi_p - \psi_q\psi_{qp} \) is divisible by \( q \); in each case, the quotient is precisely \( 2\{(b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2\} \) and we recover (twice) the determinant \( \Delta \) in coordinate form. In fact, when the Hamilton equations
\[ \circ \circ p = \psi_{pq}\psi_q - \psi_p\psi_{qq}, \quad \circ \circ q = \psi_{pp}\psi_p - \psi_q\psi_{qp} \]
are differentiated by time once more, they yield precisely
\[ \circ \circ z = 2Fz. \]

**CANONICAL FORMS**

The simplest type of homogeneous cubic is a monomial: for \( w \in \mathbb{Z} \) define \( \psi^w : \mathbb{Z} \rightarrow \mathbb{R} \) by requiring that for all \( z \in \mathbb{Z} \)
\[ \psi^w(z) = \frac{1}{3}\Omega(w, z)^3. \]
For this cubic, the corresponding symmetric linear map \( \Gamma^w : \mathbb{Z} \rightarrow \text{sp}(\mathbb{Z}, \Omega) \) is given by
\[ \Gamma^w v = \Omega(z, w)\Omega(w, v)w \]
whenever \( z, v \in \mathbb{Z} \), and the associated determinant \( \Delta^w \) is identically zero.
Conversely, let the cubic $\psi$ with corresponding symmetric linear map $\Gamma$ be such that the associated determinant $\Delta$ is identically zero. We claim that $\psi = \psi^w$ for a unique $w \in Z$; to justify this claim, we may of course assume that $\Gamma$ is not itself identically zero. Note that if $z \in Z$ then $\Gamma_z \Gamma_z = 0$ so that $\text{Ran} \Gamma_z \subseteq \text{Ker} \Gamma_z$ with equality precisely when $\Gamma_z \neq 0$. Note also that if $x, y \in Z$ then

$$\Gamma_y \Gamma_y + \Gamma_y \Gamma_x = \{\Delta(x+y) - \Delta(x) - \Delta(y)\}I = 0.$$  

When $x, y, z \in Z$ let us write

$$\gamma(x, y, z) = \Gamma_x \Gamma_y \Gamma_z.$$  

Observe that this expression is now antisymmetric in its first pair of variables and was already symmetric in its last pair; thus

$$\gamma(x, y, z) = \gamma(x, z, y) = -\gamma(z, x, y) = -\gamma(z, y, x) = \gamma(y, z, x) = \gamma(y, x, z) = -\gamma(x, y, z)$$

and so $\gamma$ vanishes identically. This proves that if $x, y \in Z$ then

$$\text{Ran} \Gamma_y \subseteq \text{Ker} \Gamma_x$$

and choosing any $z \in Z$ with $\Gamma_z \neq 0$ then gives

$$\text{Ran} \Gamma_z \subseteq \cup_{y \in Z} \text{Ran} \Gamma_y \subseteq \cap_{x \in Z} \text{Ker} \Gamma_x \subseteq \text{Ker} \Gamma_z$$

with equality of the end terms and hence equality throughout, whence

$$\cup_{y \in Z} \text{Ran} \Gamma_y = \cap_{x \in Z} \text{Ker} \Gamma_x$$

is a distinguished line in the plane $Z$. Let $w \in Z$ be a basis vector for this line. If $z \in Z$ then $\Gamma_z = \lambda_z(\cdot)w$ for some linearly $z$-dependent $\lambda_z$ in the dual $Z^*$: as $\Gamma_z$ kills $w$ so does $\lambda_z$ and therefore $\lambda_z = \mu_z \Omega(w, \cdot)$ for some $\mu_z \in \mathbb{R}$ also linear in $z$; this shows that

$$\Gamma_z = \mu_z \Omega(w, \cdot)w$$

for some $\mu \in Z^*$. Symmetry of $\Gamma$ forces $\mu$ to kill $w$ so that $\mu = \nu \Omega(\cdot, w)$ for some $\nu \in \mathbb{R}$. In the resulting formula

$$\Gamma_z = \nu \Omega(z, w) \Omega(w, \cdot)w$$

the cube root of the scalar $\nu$ may be absorbed into $w$; this renders $w$ unique and we conclude that $\Gamma = \Gamma^w$ as claimed.

Thus, the assignment $w \mapsto \Gamma^w$ is a (cubic!) bijection from $Z$ to the set of all symmetric linear maps $Z \to \text{sp}(Z, \Omega)$ for which the associated determinant $\Delta$ is identically zero.

The same conclusion may be reached efficiently (though prosaically) using coordinates. From the identical vanishing of $\Delta$ in the form

$$(b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2 \equiv 0$$

we deduce (by setting $q = 0, p = 0$, and $pq \neq 0$ in turn) that $b^2 = ac, c^2 = bd$, and $ac = bd$. Let $\lambda$ be the cube root of $a$ and $\mu$ the cube root of $d$; then

$$(\lambda^2 \mu)^3 = a^2 d = a \cdot ad = a \cdot bc = b \cdot ac = b \cdot b^2 = b^3$$

so that $\lambda^2 \mu = b$ and $\lambda \mu^2 = c$ likewise; it follows that the cubic is a monomial, namely

$$ap^3 + 3bp^2q + 3cpq^2 + dq^3 = (\lambda p + \mu q)^3.$$  

When the determinant $\Delta$ is not identically zero, there are three possibilities:

- $\Delta(z) = 0$ for $z$ on a line-pair through 0 and $\Delta$ takes values of each sign elsewhere;
- $\Delta(z) = 0$ for $z$ on a line through 0 and $\Delta$ is positive elsewhere;
- $\Delta(0) = 0$ and $\Delta$ is positive elsewhere;

and canonical forms may be developed for each of these. In connexion with these possibilities, we remark (from Theorem 1) that if $\Delta$ takes negative values then it also takes positive values.
EVALUATION OF $g_3$

Let $\psi : Z \to \mathbb{R}$ be a homogeneous cubic and let the Hamiltonian vector field $\xi^\psi$ have $z : I \to Z$ as an integral curve. As we have seen, $\overset{\circ}{\psi} = 2Fz$ where the scalar function $F : I \to \mathbb{R}$ satisfies $(\overset{\circ}{F})^2 = 4F^3 - g_3$ for some constant $g_3$ that depends on the integral curve $z$.

Let the initial point $z_0$ be such that $\psi(z_0) = 0$; as the Hamiltonian $\psi$ is constant along the integral curve, it follows that $\psi(z_t) = 0$ for all $t \in I$. If $z_0$ itself is zero, then of course $F \equiv 0$ and $g_3 = 0$. Now assume that $z_0$ is nonzero, so that $z_t$ is nonzero for all $t \in I$. For each $t \in I$ we have $0 = 3\psi(z_t) = \Omega(z_t, \overset{\circ}{z}_t)$ whence (as $Z$ is a plane) $\overset{\circ}{z}_t$ is parallel to $z_t$; say $\overset{\circ}{z} = \lambda z$ for some scalar function $\lambda : I \to \mathbb{R}$. On the one hand,

$$Fz = \Gamma_z z = \lambda z = \lambda z = \lambda \overset{\circ}{z} = \lambda \overset{\circ}{z} + \lambda Fz.$$  

on the other hand,

$$2Fz = \overset{\circ}{z} = \lambda z + \lambda \overset{\circ}{z} = \overset{\circ}{z} + \lambda Fz.$$  

Thus

$$\overset{\circ}{\lambda} = F = \lambda^2$$  

and so

$$\overset{\circ}{F} = (\lambda^2)\overset{\circ}{\lambda} = 2\lambda \overset{\circ}{\lambda} = 2\lambda F = 2\lambda^3.$$  

It follows that in this case,

$$g_3 = 4F^3 - (\overset{\circ}{F})^2 = 4(\lambda^2)^3 - (2\lambda^3)^2 = 0.$$  

In short, an initial point $z_0$ with $\psi(z_0) = 0$ spawns an integral curve for which $g_3 = 0$.

Let us offer some sample computations in coordinates. If $\psi = \frac{1}{3}(p^3 - q^3)$ then $\overset{\circ}{p} = -\psi_q = q^2$ and $\overset{\circ}{q} = \psi_p = p^2$ so that $\overset{\circ}{p} = 2(pq)p$ and $\overset{\circ}{q} = 2(pq)q$; thus $F = pq$ so $\overset{\circ}{F} = F\psi_p - F_p\psi_q = p^3 + q^3$ and $g_3 = 4F^3 - (\overset{\circ}{F})^2 = 4p^3q^3 - (p^3 + q^3)^2 = -(p^3 - q^3)^2$ or $g_3 = 9\psi^2 \leq 0$. Similarly, if $\psi = p^2q + pq^2$ then $F = p^2 + pq + q^2$ and $\overset{\circ}{F} = (q - p)(2p + q)(p + 2q)$; after considerable simplification, $g_3 = 4F^3 - (\overset{\circ}{F})^2$ yields $g_3 = 27\psi^2 \geq 0$.

Finally, we remark (without proof - but see page 100 of [Salmon]) that classical invariant theory reappears in general: if

$$\delta = a^2d^2 - 3b^2c^2 + 4ac^3 + 4b^3d - 6abcd$$

denotes the discriminant of the cubic $3\psi$ then

$$g_3 = -9\delta \psi^2$$

so

$$(\overset{\circ}{F})^2 = 4F^3 + 9\delta \psi^2.$$  

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