The Abelian Sandpile Model on Fractal Graphs

Samantha Fairchild, Ilse Haim, Rafael G. Setra, Robert S. Strichartz, and Travis Westura

Abstract

We study the Abelian sandpile model (ASM), a process where grains of sand are placed on a graph’s vertices. When the number of grains on a vertex is at least its degree, one grain is distributed to each neighboring vertex. This model has been shown to form fractal patterns on the integer lattice, and using these fractal patterns as motivation, we consider the model on graph approximations of post critically finite (p.c.f) fractals. We determine asymptotic behavior of the diameter of sites toppled and characterize graphs which exhibit a periodic number of grains with respect to the initial placement.

Keywords: fractal graphs, Abelian sandpile, Sierpinski Gasket, growth model, identity of sandpile group
AMS Classifications: 05C25, 20F65, 91B62, 05C75

1 Introduction

The Abelian Sandpile Model arose as a model of self-organized criticality (SOC) which was popularized by Bak, Tang, and Weisenfield’s paper, [1]. The Abelian Sandpile Model derived it’s name as the algebraic structure upon the “sandpile” configurations is an abelian group. For general reading on this subject, see [2, 3, 4, 5, 6, 7, 17].

In this paper we study the asymptotic and periodic patterns produced in the abelian sandpile model by placing $n$ grains of sand on a single vertex in graph approximations of fractals including the Sierpinski gasket ($SG$, see Fig. 1), the Mitsubishi gasket ($MG$, see Fig. 19), the Pentagasket ($PG$, see Fig. 20), and the Hexagasket ($HG$, see Fig. 21). It has been shown that placing $n$ grains
of sand on the origin of the $d$-dimensional lattice, $\mathbb{Z}^d$, results in an asymptotic growth on the number of sites toppled of the order $n^{d/2}$ \((9)\). Here, distance is measured from the origin with respect to the graph metric. We extend this idea to our fractal approximations by replacing $d$ with the Hausdorff dimension.

Furthermore, if we restrict ourselves to a single vertex on a graph, the number of grains it may possess is between 0 and one less than its degree. We show that under certain conditions the number of grains is periodic with respect to the initial $n$ grains placed.

In section 2 we provide basic definitions of the sandpile model and notation used throughout the paper, in section 3 we provide the boundary growth on the Sierpinski Gasket, in section 4 we prove that graphs exhibit periodicity under certain symmetries, and in section 5 we discuss the identity of the sandpile group for our fractal graphs.

2 Preliminaries

Let $G = (V, E)$ be a connected graph of vertices $V$ and edges $E$. In this model, a vertex is considered a sink if, instead of toppling, it removes the grains of sand placed on it from the graph.

A configuration on $G$ is a function $\eta : V \rightarrow \mathbb{Z}^\oplus$, where $\mathbb{Z}^\oplus$ represent the nonnegative integers. If a vertex has at least as many grains as its degree, it will topple and send one grain to each of its neighbors. If multiple vertices must topple, the order of toppling does not change the final configuration. A configuration is stable if for each $v \in V$, $\eta(v) < \deg(v)$.

We consider the Abelian sandpile model where we only place grains of sand on a central vertex, $v_0$. Also, let $d(v, v_0)$ denote distance from $v_0$ with respect to the graph metric, and $V_r = \{v \in V : d(v, v_0) \leq r\}$.

**Notation 1.** We construct the cell graph of $SG$, denoted $SGC$ (see Fig. 7), by placing a vertex in each cell of the $SG$ and connecting those vertices if the cells are connected in $SG$. Also, $SG_i$
represents an $i^{th}$ level graph approximation; as an example consider $SG_3$ in Figure 1. The level of graph approximation is defined similarly for the other fractals.

## 3 Boundary Growth

Now consider an Abelian sandpile model where we place grains of sand on the vertex at the center of the bottom border of $SG$, denoted $v_0$ (See Figure 1). We act under the assumption that the number of grains we place on $v_0$ is small enough and $i$ large enough for $SG_i$ so that, to reflect the behavior on $SG$, the sand will never reach the boundary.

There are symmetries of $SG$ that we take advantage of when we assume that grains never reach the boundary. From Figure 2 we can see two lines of symmetry: one that divides the left and right portions of the graph, and another that divides the rightmost triangle with respect to grains of sand.

Let the diameter $R$ be the maximum $d(v, v_0)$ over all $v \in V$ such that $v$ contains a grain. We will aim to prove the following theorem:

**Theorem 1.** Upon placing $N$ grains of sand on $v_0$, the diameter $R$ grows asymptotically as $R = \Theta \left( N^{1/D_f} \right)$, where $D_f = \frac{\log 3}{\log 2}$ is the fractal dimension of the $SG$.

We begin by providing a counting argument for a lower bound on the diameter $R$.

**Lemma 2.** Let $N$ be the number of grains of sand placed on $v_0$ and $R$ be the diameter after stabilization. Then \( \left( \frac{N}{60} \right)^{1/D_f} < R \).
Proof. If we place the maximum stable number of grains on all vertices of $V_{R-1}$ and then add one grain of sand to $v_0$, we guarantee a toppling into a vertex of distance $R$. Thus we have an upper bound on the number of grains of sand, $N$, needed for the grains of sand to reach diameter $R$:

$$N \leq 1 + \sum_{v \in V_{R-1}} (\deg(v) - 1).$$  \hfill (1)

But $\deg(v) = 4$ for all vertices, so this is the same as

$$N \leq 1 + 3|V_{R-1}|.$$  \hfill (2)

Note that there exists an integer $k > 0$ such that $2^{k-1} < R \leq 2^k$. So for $k > 0$,

$$|V_{2^k}| = 3^{k+1} + 2 < 5(3^k) = 5(2^k)^{D_f}.$$  \hfill (3)

Combining equations 1, 2, and 3 we observe

$$N < 4|V_{2^k}| < 4(5(2^k)^{D_f}) = 60(2^{k-1})^{D_f} < 60(R^{D_f}).$$  \hfill (4)

This yields our result. \hfill $\square$

Denote the junction point at distance $2^j$ as $v_{2^j}$ (see Figure 4). Obtaining an upper bound makes use of the following lemmas:

**Lemma 3.** If any $v$ with $d(v, v_0) = 2^j$ has toppled, then all vertices with $d(v, v_0) = 2^j - 1$ have the same number of grains on them. If a vertex $v$ with $d(v, v_0) = 2^j$ has toppled, then all vertices in $V_{2^j-1}$ have toppled. Furthermore, if no vertex of distance $2^j$ has toppled then the final configuration will have 1 grain on each $v_{2^j}$, and 2 grains on each vertex at depth $2^j$ in between the two $v_{2^j}$’s.

**Proof.** We proceed by induction and first consider the case of the first level approximation with maximum distance 2. By symmetry the vertices $v_1$ in Figure 3a always have the same value. Once the $v_1$ have 4 grains, each will topple and the final configuration will be as shown in figure 3b. Each $v_1$ still has the same value, and all vertices of distance less than or equal to 1 from $v_0$ have toppled. If we topple each $v_1$ once more, they cause a vertex at distance 2 to topple, so we need not consider this further.

For the induction step, assume that before vertices of depth $2^{j-1}$ topple, all of the vertices at depth $2^{j-1} - 1$ have the same number of grains. If a vertex of depth $2^{j-1}$ topples, each vertex in $V_{2^{j-1}-1}$ has toppled. Now consider what happens at a depth of $2^j$. By the p.c.f. nature of the SG, the only way vertices at distances larger than $2^{j-1}$ receive grains is for both $v_{2^{j-1}}$ to topple. Attached to each $v_{2^{j-1}}$ is a gasket of depth $2^{j-1}$ only connected at depth $2^j$. Applying our assumption to each attached $SG$ of depth $2^{j-1}$ we have that all vertices of depth $2^{j-1} + 2^{j-1} - 1 = 2^j - 1$ will have the same number of grains. In order for a vertex at distance $2^j$ to topple, one vertex and thus all at
Lemma 4. Consider the two boundary points of a $2^j$ graph that connect to the distance $2^j + 1$, i.e. the $v_{2^j}$’s. Given there are grains on vertices $v$ with $d(v, v_0) > 2^j$, then every vertex in $V_{2^j}$ must have toppled.

Proof. First consider a base case where there are grains at a distance greater than or equal to 3. We want to show that every vertex within distance 2 of $v_0$ has toppled. The vertices $v_2$ must have toppled and thus by Lemma 3 all vertices with lesser distance have also toppled. The remaining vertex in between the $v_2$ has toppled because it receives at least as many grains from $v_1$ as $v_2$ does.

For the induction step, we suppose a vertex $v$ with $d(v, v_0) > 2^j$ has grains and by induction every vertex in $V_{2^{j-1}}$ has toppled. We now aim to prove that every vertex of distance $2^j$ has toppled. We use Lemma 3 to observe that every vertex within distance $2^j - 1$ has already toppled, and all vertices at distance $2^j - 1$ have the same value. Adding enough grains so that they all have 4, by
toppling, each vertex at depth $2^j$ that has 2 grains on it will receive another two, thus having 4 and each of those vertices will topple. Then the $v_{2^j}$ vertices must also topple to reach the next level. Thus for the grains to reach a distance greater than $2^j$, vertices in $V_{2^j}$ must topple.

The last idea needed is a theorem from Rossin [9]:

**Theorem 5.** (Rossin) Given a graph with vertices and edges $G = (V, E)$, and $X \subseteq V$ a set of connected vertices, define $C_G(X) = \{(i, j) \in E : i \in X, j \in V \setminus X\}$. The minimum number of grains that must be placed on $X$ so that all vertices of $X$ topple at least once, $M_X$, is $|\text{in}(X)| + |C_G(X)|$ where $\text{in}(X)$ is the number of edges inside $X$.

Thus we have $M_X > |\text{in}(X)|$. In order to relate an $X$ to the diameter $R$, we use the preceding lemmas. Using Lemma 4, we can conclude that $N \geq M_X$ where $X$ is the set of vertices and edges in $SG_k$. Let $E_k$ be the number of edges in $SG_k$. It is also easy to see that $|E_{2^k}| = 6 \times 3^k$.

**Lemma 6.** $R < \left(\frac{N}{2}\right)^{\frac{1}{D_f}}$.

**Proof.** Using Rossin’s theorem we have:

$$N \geq M_X > |\text{in}(X)| \geq |E_{2^{k-1}}| = 2(3^k) = 2(2^k)^{D_f} \geq 2R^{D_f}$$

Using Lemma 2 and 6 we have Theorem 1. For reference, these bounds are shown in Figure 5.

We note this is not limited to the $SG$. By a similar argument to Lemmas 3 and 4 we can show the following proposition.
**Proposition 7.** Let $v_0$ be set in a central point so that the junction points with the next level graph approximation topple symmetrically, let $R$ denote the distance from $v_0$, and $N$ represent the number of grains of sand required to first reach distance $R$. We then have the following results:

- **HG:** 
  \[ R = \Theta \left( N^{\frac{1}{D_f}} \right), \]
  where $D_f = \frac{\log 6}{\log 3}$ is the fractal dimension of HG.

- **MG:** 
  \[ R = \Theta \left( N^{\frac{1}{D_f}} \right), \]
  where $D_f = \frac{\log 6}{\log 2}$ is the fractal dimension of MG.

- **PG:** 
  \[ R = \Theta \left( N^{\frac{1}{D_f}} \right), \]
  where $D_f = \frac{\log 5}{\log \left( \frac{1 + \sqrt{3}}{2} \right)} \approx 1.67$ is the fractal dimension of PG,
  and $D_r = \frac{\log 5}{\log \left( 1 + \sqrt{3} \right)} \approx 1.60$.

Note that PG is different from the other graphs. This is because the fractal dimension is calculated with respect to the euclidean metric, but we make use of the graph metric in the model.

## 4 Periodicity

We have observed that the number of grains of sand on a vertex of the SG graph approximation repeats itself in a periodic pattern with respect to the number of grains of sand added to $v_0$. This extends itself to configurations on subgraphs of the approximations. This periodicity is also apparent in HG, PG, and MG. First we will show the existence of periodicity, and then further characterize its properties on subgraphs.

**Definition 1.** For graphs $A$, and $B \subset A$, and for configurations $\eta_1$ and $\eta_2$

- $A\big|_B$ is the restriction of $A$ to $B$,
- $\eta_A$ is a configuration on $A$ and $\eta_B = \eta_A\big|_B$ is the restriction of $\eta_A$ to $B$,
- $\eta_1 + \eta_2$ is the point-wise sum of the configurations and $k\eta_1$ is the point-wise sums of $k$ copies of $\eta_1$,
- $\eta_1^\circ$ means stabilization of $\eta_1$, 


• $\oplus$ denotes point-wise addition and stabilization. In other words, $\eta_1 \oplus \eta_2 = (\eta_1 + \eta_2)^\circ$.

• $o : V \to \mathbb{Z}^\oplus$ is the odometer function, the number of times a vertex has toppled through stabilizing a given configuration.

• A configuration $\eta_1$ is recurrent if it is stable and if given any configuration $\eta_2$, there exists a configuration $\eta_3$ such that $\eta_2 \oplus \eta_3 = \eta_1$.

The following are known properties of the sandpile model [2]:

**Lemma 8.** For a graph $A$, $((\eta_A + \eta_1 \oplus + \eta_2)^\circ = (\eta_A + \eta_1 + \eta_2)^\circ$

**Lemma 9.** Let $A$ be a finite sinked graph and $\eta$ a nonzero configuration on $A$, then there exists an integer $N(A, \eta)$ such that $(k\eta)^\circ$ is recurrent for $k \geq N(A, \eta)$.

**Definition 2.** For a graph $G = (V, E)$ with $B \subset V$ as sinks, define the configuration $Id_f$ such that for $v \in V$, $Id_f(v)$ is the number of edges that $v$ has connected to sinks.

**Lemma 10.** Consider a graph $G = (V, E)$ with $B \subset V$ as sinks. If $\eta_G$ is a recurrent configuration then $(Id_f + \eta_G)^\circ = \eta_G$, i.e. $Id_f$ acts as an identity configuration on recurrent configurations.

### 4.1 Existence of Periodicity

Now consider a finite subset $B \subset V$, such that removing $B$ and edges connected to elements in $B$ disconnects $G = (V, E)$ into two graphs, a finite graph $F$ and a second graph. Finally, define $S$ to be the graph of $F$ combined with vertices in $B$ as sinks; include edges in $E$ that combine elements of $F$ and $B$. An example of this decomposition is in Figure 6.

![Figure 6](image.png)

Figure 6: Example of a graph $G$ where the white vertices are the subset $B$. The graph $F$ is contained in the central hexagonal outline. The graph $S$ is the graph $F$ with the white vertices from $B$ as sinks.

We now fix a configuration $\eta_F$ on $F$, and use these ideas to prove the following theorem:
Theorem 11. If stabilizing \((k \eta_F + \eta_G)\), for \(k \geq N(F, \eta)\) is possible and causes the vertices of \(B\) to topple equally, then the graphs \(G|_F\) and \(S|_F\) have the same vertex-wise values.

Proof. Let the vertices in \(B\) all topple \(k\) times.

Consider \(S|_F\) first. Because the vertices in \(B\) are sinks as well as the boundary of the graph, we stabilize the graph of \(S\) by only toppling vertices in \(F\). This resulting \(\eta_S\) is recurrent by Lemma \([\ref{lem:recurrence}].\)

Now consider \(G|_F\), recalling that the vertices in \(B\) have all toppled \(k\) times. We then continue to topple only vertices in \(F\) until the entire configuration in \(F\) is stable. That is for all \(v \in F\), \(\eta(v) < \deg(v)\). Now the configuration on \(F\) is the same as \(\eta_S\) as all the same topplings occurred as in the previous case. We have \(\eta_S\) is a recurrent configuration by Lemma \([\ref{lem:recurrence}].\) Thus by Lemma \([\ref{lem:stability}].\), each time we topple all of the vertices in \(B\) and then topple each vertex in \(F\) until each vertex in \(F\) is stable, we will still have \(\eta_S\). That is letting \(Id_F\) be the identity configuration for \(B\), by lemma \([\ref{lem:stability}].\), \((Id_F + \eta_S)^\circ = \eta_S\).

Since the only vertices in \(G\) connected to \(F\) are the vertices in \(B\), we can apply the same process as above each time we topple the vertices in \(B\), and thus conclude that \(G|_F\) and \(S|_F\) have the same vertex wise values as desired. \(\square\)

Corollary 12. For a configuration \(\eta\) that satisfies Theorem \([\ref{thm:periodicity}].\) vertices in \(G|_F\) are eventually periodic with respect to the addition and stabilization of \(\eta\).

Proof. After adding \(\eta\) at least \(N(F, \eta)\) times to \(G\), we have that \(G|_F\) is the same as \(S|_F\). However, \(S\) is a finite graph, and thus will enter a cycle of configurations with respect to the addition and stabilization of \(\eta\). \(S|_F\) will then be periodic with respect to \(\eta\), and so \(G|_F\) will as well. \(\square\)

We applied this to a connected graph \(G\), but we may also consider a disconnected graph by applying this idea to each disconnected component separately. This will also work on finite graphs as long as a stabilized state is possible (finite graphs with sinks for example).

So now we can more clearly see the class of graphs in which this is possible. There are two conditions. First, these are the graphs in which stabilizations with respect to a configuration is always possible. Second, these are the graphs in which we can separate into two components, one finite, such that adding a particular configuration causes a removed section to topple equally. This applies to our fractal graph approximations: consider adding grains only to a center point equidistant from all junction points. If we choose to remove the junction points of the graphs, which topple equally due to a configuration on the center due to symmetry, we satisfy Theorem \([\ref{thm:periodicity}].\)
4.2 Periodicity of Nested Graphs

For an understanding of the relationship between subsequent graph approximations, we now consider a sequence of finite subsets $B_n \subseteq V$, such that removing $B_n$ disconnects $G$ where at least one of the components is a finite connected graph, $F_n$, and $F_n$ satisfies the property that $F_1 \subseteq F_2 \subseteq \cdots$. Finally, define $S_n$ to be the graph of $F_n$ combined with vertices in $B_n$ as sinks; include edges in $E_n$ that combine elements of $F_n$ and $B_n$. We now fix a configuration $\eta$ on $F_1$.

We consider the set of configurations formed over $k \in \mathbb{N}$ by taking $(k\eta)\circ$. We take $G$ and $B_n$ as above such that given any $k \in \mathbb{N}$, there exists a constant, $c_k$ such that $o(B_n) = c_k$ in the stabilization from $k\eta$ to $(k\eta)\circ$.

Note that $Id^n_f$ is the same as toppling each $B_n$ once. On $G$, we let $Id_f$ represent the untoppling of the open boundary of the full graph. There are a number of results about $Id_f$ which hold for $Id^n_f$. The most important of which is stated below.

**Proposition 13.** (From [10, 4]) Let $\nu$ be a configuration for a graph $G = (V, E)$. $\nu$ is a recurrent configuration if and only if $\nu \oplus Id_f = \nu$. Furthermore, In the stabilization process from $\nu + Id_f$ to $\nu$, each vertex topples exactly once.

**Definition 3.** Let $C_n = \{\text{Recurrence configurations of } S_n \text{ formed by taking } (k\eta)\circ\}$.

By repeatedly adding $\eta$, we will obtain a cyclic group of recurrent configurations for any $S_n$. Thus $|C_n|$ is the maximum periodicity of each vertex in $S_n$.

**Proposition 14.** For all finite $m, n$ such that $m \geq n \geq 1$,

$$C_m \bigg|_{S_n} = C_n.$$

**Proof.** To show $C_m \bigg|_{S_n} \subseteq C_n$, let $\eta_m$ be a recurrent configuration formed by taking $(k\eta)\circ$ for some $k \in \mathbb{N}$. Then setting $\eta_n = \eta_m \bigg|_{S_n}$, $\eta_n$ must also be a configuration formed by $(k\eta)\circ$. We also have that $\eta_n \oplus Id^n_f = \eta_m$, and in this stabilization process, each vertex topples only once. Then toppling up to and including each $v \in B_n$, we now have the configuration $\eta_n + Id^n_f$. Since $\eta_m$ is recurrent, after toppling each vertex once, $\eta_n$ will also be the same. Thus $\eta_n \oplus Id^n_f = \eta_n$. Thus we have $\eta_n$ is a recurrent configuration formed by adding $\eta$ to itself. Therefore, $\eta_n \in C_n$ as desired.

For the reverse inclusion, let $N = (N_1, N_2, \ldots)$ represent a strictly increasing sequence of the number of times we can add $\eta$ to itself such that $\eta_n \in C_n$ occurs. We note that since $\eta_n$ is recurrent and part of a cyclic group, $N$ will be infinitely long, and thus $N_k$ will go to infinity as $k \to \infty$. We now consider $N_k\eta$ in $S_n$ for any given $N_k \in N$. We topple until stable every vertex up to but not including $v \in B_n$. Since $\eta_n$ is recurrent, it is invariant of topplings of $v \in B_n$. Since $F_m$ is finite, we can thus choose $N_k \in N$ large enough so that the resulting configuration of stabilization, $\eta_m$, is in $C_m$. However since $N_k$ is from $N$, $\eta_m \bigg|_{S_n} = \eta_n$. \hfill \Box
4.2 Periodicity of Nested Graphs

From Proposition 14, since \( |C_m| \leq |C_n| \) for all \( n \leq m < \infty \), \( |C_n| \leq |C_m| \). Thus given any \( v \in G \), let \( m = \min\{i : v \in S_i\} \), then the maximum periodicity of \( v \) is given by \( |C_m| \).

**Theorem 15.** For all \( n \leq m < \infty \), \( |C_n| \) divides \( |C_m| \).

**Proof.** Given any \( \eta_1^m, \eta_2^m \in C_m \), we define an equivalence relation, \( \sim \), by saying \( \eta_1^m \sim \eta_2^m \) iff

\[
\eta_1^m \bigg|_{S_n} = \eta_2^m \bigg|_{S_n}.
\]

It is clear that this actually does define an equivalence relation. So we take the equivalence class of \( Id^m \), which is the identity element of \( C_m \), and name it \( N_n \). Since the equivalence class of the identity element is always a normal subgroup of the original group, \( C_m \) mod \( N_n \) is well defined. Thus we have

\[
|C_m| = (|N_n|) \bigg|_{S_n} = (|N_n|) |C_n|
\]

Thus \( |C_n| \) divides \( |C_m| \).

We note that this is not an empty statement as \( SG \), \( PG \), \( HG \), and \( MG \) are examples of graphs that all satisfy the above properties when adding grains to a central point of the fractal graph. We have also observed periodicities which are summarized in the following conjecture.

**Conjecture 16.** Let \( K \) denote any of the following fractal graphs. Let \( m = \min\{i : v \in K_i\} \). The maximum periodicity of any vertex \( v \) is given by:

- \( SG \): \( 4(3^m) \),
- \( HG \): \( 2(3^m) \),
- \( PG \): \( 6(5^m) \),
- \( MG \): \( 6(7^m) \),
- \( SGC \): \( 6(7^m) \).

**Conjecture 17.** Let \( v \in SG \), and let \( m = \min\{i : v \in SG_i\} \). Then the exact periodicity of \( v \) is given by \( |C_m| \).

We believe this conjecture for \( SG \) since there are no vertices that always have the same value once a recurrent configuration is reached. However, for \( SGC \) and the \( HG \), there is evidence of vertices which never change their value in all recurrent configurations.
5 Structure of the Sandpile Group

We can gain insight on the structure of the Sandpile group using the Smith Normal Form of the graph Laplacian \([11]\). In the following tables we identify the structure of the Sandpile group for several levels of the Sierpinski Gasket and for the cell graph.

Vertex Graph: Boundary is a sink:

| Level | Group |
|-------|-------|
| 1     | \(\mathbb{Z}_5 \times \mathbb{Z}_{10} \cong \mathbb{Z}_2 \times (\mathbb{Z}_5)^2\) |
| 2     | \(\mathbb{Z}_2 \times \mathbb{Z}_{30} \times \mathbb{Z}_{150} \times \mathbb{Z}_{150} \cong (\mathbb{Z}_2)^4 \times (\mathbb{Z}_3)^3 \times \mathbb{Z}_5 \times (\mathbb{Z}_{25})^2\) |
| 3     | \(\mathbb{Z}_2 \times (\mathbb{Z}_6)^6 \times (\mathbb{Z}_{30})^3 \times \mathbb{Z}_{450} \times (\mathbb{Z}_{2250})^2 \cong (\mathbb{Z}_2)^{13} \times (\mathbb{Z}_3)^9 \times (\mathbb{Z}_5)^3 \times (\mathbb{Z}_9)^3 \times \mathbb{Z}_{25} \times (\mathbb{Z}_{125})^2\) |

Vertex Graph: Normal Boundary Condition:

| Level | Group |
|-------|-------|
| 0     | \(\mathbb{Z}_5 \times \mathbb{Z}_{10} \cong \mathbb{Z}_2 \times (\mathbb{Z}_5)^2\) |
| 1     | \(\mathbb{Z}_{38} \times \mathbb{Z}_{38} \cong (\mathbb{Z}_2)^2 \times (\mathbb{Z}_{19})^2\) |
| 2     | \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{462} \times \mathbb{Z}_{2310} \cong (\mathbb{Z}_2)^5 \times (\mathbb{Z}_3)^3 \times \mathbb{Z}_5 \times (\mathbb{Z}_7)^2 \times (\mathbb{Z}_{11})^2\) |

Cell Graph: Boundary is a sink:

| Level | Group |
|-------|-------|
| 2     | \(\mathbb{Z}_5 \times \mathbb{Z}_{40} \cong \mathbb{Z}_5 \times (\mathbb{Z}_8)^2\) |
| 3     | \(\mathbb{Z}_5 \times \mathbb{Z}_{15} \times \mathbb{Z}_{735} \times \mathbb{Z}_{3675} \cong (\mathbb{Z}_3)^3 \times (\mathbb{Z}_5)^3 \times \mathbb{Z}_{25} \times (\mathbb{Z}_{49})^2\) |
| 4     | \(\mathbb{Z}_5 \times (\mathbb{Z}_{15})^8 \times \mathbb{Z}_{75} \times \mathbb{Z}_{225} \times \mathbb{Z}_{41200} \times \mathbb{Z}_{306000}
\cong (\mathbb{Z}_3)^9 \times (\mathbb{Z}_5)^9 \times (\mathbb{Z}_9)^3 \times (\mathbb{Z}_{16})^2 \times (\mathbb{Z}_{17})^2 \times (\mathbb{Z}_{25})^3 \times \mathbb{Z}_{125}\) |

Cell Graph: Normal Boundary Condition:

| Level | Group |
|-------|-------|
| 1     | \(\mathbb{Z}_3 \times \mathbb{Z}_4 \cong (\mathbb{Z}_4)^2\) |
| 2     | \(\mathbb{Z}_{17} \times \mathbb{Z}_{485} \cong \mathbb{Z}_5 \times (\mathbb{Z}_{17})^2\) |
| 3     | \(\mathbb{Z}_5 \times \mathbb{Z}_{15} \times \mathbb{Z}_{1140} \times \mathbb{Z}_{5700} \cong (\mathbb{Z}_3)^3 \times (\mathbb{Z}_4)^3 \times (\mathbb{Z}_5)^3 \times (\mathbb{Z}_{19})^2 \times \mathbb{Z}_{25}\) |
| 4     | \(\mathbb{Z}_5 \times (\mathbb{Z}_{15})^8 \times \mathbb{Z}_{75} \times \mathbb{Z}_{225} \times \mathbb{Z}_{79425} \times \mathbb{Z}_{397125}
\cong (\mathbb{Z}_3)^9 \times (\mathbb{Z}_5)^9 \times (\mathbb{Z}_9)^3 \times (\mathbb{Z}_{25})^3 \times \mathbb{Z}_{125} \times (\mathbb{Z}_{353})^2\) |

The order of the Sandpile group when associated to a directed graph is given by the number of spanning trees rooted at a fixed vertex \([12, 13]\). In the \(SG\), we calculate the number of spanning trees of the sequence of graph approximations to the \(SG\). We conjecture that my collapsing the three boundary vertices of the graph into a single vertex and treating this vertex as the sink, then
we can calculate the order of the resulting graph’s Sandpile group. We conjecture the order of this
group is given by $2^{f_n} 3^{g_n} 5^{h_n}$, where $f_n$, $g_n$, and $h_n$ are

$$f_n = \frac{1}{2}(3^n - 1)$$
$$g_n = \frac{1}{4}(3^{n+1} - 6n - 3)$$
$$h_n = \frac{1}{4}(3^n + 6n - 1)$$

This conjecture is based on results from [14], which uses a modified version of Kirchhoff’s matrix
tree theorem to calculate the number of spanning trees as a product of the nonzero eigenvalues of
the graph’s probabilistic Laplacian.

6 Identities

There are multiple papers on attempting to characterize the identity element of the Sandpile Group.
(See [10, 4, 15, 16]). There is no current proof that any of the following are in fact the identity ele-
ment. However, by the way they are generated, we know computationally that the approximations
shown are the identity element of that particular graph.

From [6], we know there exists minimum value $k \in \mathbb{Z}^+$ such that $(kId_f)^{\circ} = Id_r$, where $Id_r$ is the
identity element of the group of recurrent configurations formed by the Abelian sandpile model.
We present the level 3 approximation of each fractal’s identity element, and the calculated $k$ values
for each level of the approximation as well as a conjectured formula for $SG$ and $SGC$.

| Fractal | Figure | $k$ Values for levels 1,2,3,... | Conjectured value of $k$ for the $m^{th}$ level |
|---------|--------|-------------------------------|------------------------------------------|
| $SGC$   | ![7](image) | 1,8,49,272, ... | $\frac{5^m - 3^m}{2}$ |
| $SG$    | ![8](image) | 0,2,12,62,... | $\frac{5^m - 1}{2}$ |
These two identities have clear patterns. The identity on the $SGC$ is a configuration where all vertices have 2 grains of sand. The identity on the $SG$ has a pattern when we group vertices with respect to the horizontal lines connecting them. The initial vertex has zero grains, and the next two groups have 3. This periodic pattern then continues.

Consider a Sierpinski Gasket Cell graph, where each cell is a vertex with a single edge to each of its neighbors. The three boundary cells each have a single edge to the sink. We show that the identity of this graph’s Sandpile Group is the maximal configuration with 2 chips on each cell.
Theorem 18. For all levels $n > 0$, the identity of SGC is the configuration with 2 chips on each vertex.

We can check the first few levels by hand, and the proof will proceed by a strong induction.

Letting $e$ denote the configuration with 2 chips on each vertex, we show that $e \oplus e = e$. This process entails stacking two 2-chip configurations on top of each other, resulting in 4 chips on each vertex, and then showing that this configuration stabilizes to the 2-chip configuration. Since we can topple chips in any order, we just need to find an order of topplings that is simple enough for us to explain.

Before beginning the proof, let’s examine the structure of the cell graph so that we can understand how the subgraphs interact with each other. A cell graph contains an inner ring of level 1 triangles.
This ring has an important property.

**Lemma 19** (Ring of Triangles). Consider a chain of \( t \) level 1 triangles in which each outer vertex has one edge to the sink, as in fig. 11. Place 2 chips on every inner vertex of each triangle, and 3 chips on the outer vertices of each triangle. The stable configuration consists of 2 chips on each vertex. Further, every corner vertex topples exactly once, resulting in \( t \) chips toppling into the sink.

Moreover, we can start with any number of chips \( m \geq 3 \) on each of the corner vertices. The stable configuration has 2 chips on each vertex. Every corner vertex topples exactly \( m - 2 \) times, and \( t(m - 2) \) chips in total topple into the sink.

\[ \text{Figure 11: 3 chips on the corner vertices, 2 chips on the inside vertices.} \]

**Proof.** First topple all of the corner vertices, losing one chip each to the sink and leaving us with 0 chips on these corners and 3 chips on all of the other vertices. Now topple these inner vertices, producing the desired 2 chip configuration.

Now suppose that we start with \( m \) chips on each corner vertex instead of three. We can topple in any order, so we can iteratively apply the 3 chip case. Each vertex topples exactly \( m - 2 \) times, so \( t(m - 2) \) chips in total topple into the sink.

Next we want to identify the behavior of a larger triangle, as each level \( n + 1 \) graph contains an inner ring of 6 level \( n - 1 \) graphs (see fig. 12).
We show inductively that the same result holds for triangles of level $n \geq 1$. That is, if we start with 3 chips on one corner and 2 chips on all other vertices, we will end up with 2 chips everywhere except for the other corners, each of which have 1.

We now follow the same construction for a cell graph of arbitrary size.

**Proposition 20 (A Larger Triangle).** Lemma 14 holds for all cell graphs of level $n \geq 1$. That is, consider a chain of $t$ level $n \geq 1$ cell graphs in which each corner vertex has a single edge to the sink. Place $m \geq 3$ chips on each of these corner vertices and 2 chips on each of the other vertices. The stable configuration then consists of 2 chips on each vertex, and the odometer of each corner vertex is $m - 2$.

**Proof.** Our result will follow once we note that for a cell graph of any level, if we place 2 chips on every vertex, except for a single corner where we instead place 3 chips, then the stable configuration consists of 2 chips on every vertex except for the other corners, which each have 1 chip (see fig. 13).
of three level $n$ cell graphs. Applying the level $n$ case to the subgraph containing the corner with three chips, we see that toppling just this subgraph produces one chip on the corner vertices of this subgraph and three chips on the neighboring vertices of each of the two other subgraphs of level $n$, as in the middle of fig. 13. These subgraphs each topple one chip back to the previous subgraph, giving it 2 chips everywhere, and the level $n$ case applied to these subgraphs gives the desired result.

The original claim now follows using the same argument as in lemma 19.

Now we move on to the main proof of Theorem 18, that the identity of a level $n$ cell graph is the configuration with 2 chips on every vertex. We need to construct a sequence of topplings that stabilizes the 4-chip configuration.

Proof. The first few levels can be checked by hand. For the induction case we show that the configuration with 4 chips on every vertex topples to the configuration with 2 chips on every vertex. We also need to keep track of the odometer functions of the boundary vertices.

Assume that for all $k \leq n$, the identity element of the level $k$ cell graph consists of 2 chips on each vertex. Further assume that when the configuration stabilizes from 4 chips everywhere, the odometer of each boundary vertex is $2 \cdot 3^{k-1}$. Note that the vertex odometers can be obtained by counting the number of vertices, which is $3^k$, multiplying by 2 for the 2 chips per vertex that must topple, and dividing by 3 for symmetry.

Consider a level $n+1$ subgraph with 4 chips on each vertex. We partially carry out a stabilization in which we treat each level $n$ subgraph independently. That is, if a chip topples off a subgraph and onto another subgraph, we simply ignore it for the time being. Using our assumption that the identity of a level $n$ cell graph has two chips on every vertex, we obtain a configuration with 2 chips everywhere except for the vertices that join the level $n$ subgraphs together. Each such neighboring vertex has $2 + 2 \cdot 3^{n-1}$ chips: 2 from the identity configuration and $2 \cdot 3^{n-1}$ from the neighboring subgraph’s toppling (see fig. 14 and fig. 15).

![Figure 14: Topple the 3 subtriangles individually.](image-url)
We would like to show that this configuration stabilizes to having 2 chips on each vertex, but we cannot produce a simple enough sequence of topples that shows how this configuration stabilizes. Therefore we further break down the level $n + 1$ graph into nine level $n - 1$ subgraphs, the identity of each we have assumed to be 2 chips on every vertex.

As before, we topple each of these level $n - 1$ subgraphs individually, again leaving some chips left over on their corner vertices. Since each of these subgraphs has 4 chips on every vertex, each stabilizes to 2 chips everywhere and loses $2 \cdot 3^{(n-1)-1} = 2 \cdot 3^{n-2}$ chips from each of its corner vertices.

Our graph now has an inner ring of six level $n - 1$ subgraphs, each of which has two chips on every vertex except for its corners, which each have $2 + 2 \cdot 3^{n-2}$ chips. We want to show that this inner
ring stabilizes to 2 chips everywhere. First we apply Proposition 20 to clear the chips from the boundary of this ring, removing $2 \cdot 3^{n-2}$ chips from each such vertex.

![Figure 17: Moving chips off of the inside.](image)

We continue clearing chips off of the six inner triangles, but we note that by the symmetry of these graphs it is sufficient to show that a ring of three such triangles stabilizes to the 2-chip configuration. This is exactly the situation that occurs in fig. 15, but with a level $n$ graph instead of a level $n + 1$ graph. This configuration can be reached by starting with 4 chips on every vertex of a level $n$ graph and toppling the three level $n - 1$ subgraphs individually. Now the previous problem that we have with a level $n + 1$ graph in fig. 15 is solved by our induction hypothesis. We assumed that the identity of the level $n$ graph is the 2-chip configuration, and since we can topple chips in any order we know that this configuration must stabilize to the desired 2-chip configuration.

Thus we can clear the inner ring of the level $n + 1$ graph of all excess chips, leaving 2 chips on every vertex except for the vertices neighboring this inner ring, which each have $2 + 8 \cdot 3^{n-2}$ chips each (see fig. 18).
Since 8 is a multiple of 2, it is sufficient to show that this configuration with stacks of \(2 + 2 \cdot 3^{n-2}\) chips topples to the identity configuration. If we didn’t have the inner ring of triangles and instead just had a level \(n\) triangle with chips in the inner vertices, then we could simply reapply our induction hypothesis, as this is the exact situation we just encountered.

We conclude by using Proposition 20 to ignore this inner ring. Consider toppling each vertex that has excess chips in fig. 18 exactly once. Then each triangle in the inner ring would have 3 chips on its outer boundary vertex, and stabilizing just this inner ring produces 2 chips on every vertex of the inner ring. Hence the inner ring is unaffected by this stabilization of the outer three triangles, and we reach the desired configuration of a level \(n + 1\) triangle with two chips on each vertex. 

We provide the identities on the \(MG_3\), \(PG_3\), and \(HG_3\) below.
Figure 19: Identity Element for $MG$ level 3.

Figure 20: Identity Element for $PG$ level 3.

Figure 21: Identity Element for $HG$ level 3.
7 Acknowledgements

Ilse, Sam, and Rafael would like to acknowledge support by the National Science Foundation through the Research Experiences for Undergraduates Programs at Cornell, grant DMS-1156350. Robert Strichartz would like to acknowledge support by the National Science Foundation, grant DMS-1162045. We are all grateful to Lionel Levine for his many useful suggestions.

References

[1] O. Bak, C. Tang, and K. Wiesenfeld, “Self-organized criticality,” Physical review A, vol. 38, no. 1, p. 364, 1988.

[2] G. Paoletti, Deterministic Abelian Sandpile Models and Patterns. Springer Science & Business Media, 2013.

[3] G. Paoletti, “Abelian sandpile models and sampling of trees and forests,” Master in Physics at University of Milan, defending thesis (July 11 2007) supervisor: Prof. S. Caracciolo http://pteserver.mi.infn.it/caraccio/index.html, 2007.

[4] D. Dhar, P. Ruelle, S. Sen, and D.-N. Verma, “Algebraic aspects of abelian sandpile models,” 1995.

[5] R. Meester, F. Redig, and D. Znamenski, “The abelian sandpile; a mathematical introduction,” arXiv preprint cond-mat/0301481, 2003.

[6] M. Creutz, “Abelian sandpiles,” Computers in Physics, vol. 5, no. 2, pp. 198–203, 1991.

[7] D. Dhar, “The abelian sandpile and related models,” 1999.

[8] A. Fey, L. Levine, and Y. Peres, “Growth rates and explosions in sandpiles,” Journal of Statistical Physics, vol. 138, no. 1-3, pp. 143–159, 2010.

[9] D. Rossin, Propriétés combinatoires de certaines familles d’automates cellulaires. PhD thesis, L’ecole Polytechnique, 2000.

[10] S. Caracciolo, G. Paoletti, and A. Sportiello, “Explicit characterization of the identity configuration in an abelian sandpile model,” Journal of Physics A: Mathematical and Theoretical, vol. 41, no. 49, p. 495003, 2008.

[11] D. Lorenzini, “Smith normal form and laplacians,” Journal of Combinatorial Theory, Series B, vol. 98, no. 6, pp. 1271–1300, 2008.

[12] L. Levine, “Sandpile groups and spanning trees of directed line graphs,” Journal of Combinatorial Theory, Series A, vol. 118, no. 2, pp. 350–364, 2011.
[13] L. Levine, “The sandpile group of a tree,” *European Journal of Combinatorics*, vol. 30, no. 4, pp. 1026–1035, 2009.

[14] J. A. Anema, “Counting spanning trees on fractal graphs,” *arXiv preprint arXiv:1211.7341*, 2012.

[15] Y. L. Borgne and D. Rossin, “On the identity of the sandpile group,” *Discrete Mathematics*, vol. 256, no. 3, pp. 775–790, 2002.

[16] A. Dartois and C. Magnien, “Results and conjectures on the sandpile identity on a lattice,” pp. 89–102, 2003.

[17] A. A. Járai, “Sandpile models,” *arXiv preprint arXiv:1401.0354*, 2014.

[18] S.-C. Chang, L.-C. Chen, and W.-S. Yang, “Spanning trees on the sierpinski gasket,” *J. Stat. Phys.*, vol. 126, no. 3, pp. 649–667, 2007.

Websites: http://www.math.cornell.edu/ skayf/ and http://www.math.cornell.edu/ twestura/

S. Fairchild  
Mathematics Department, Houghton College, Houghton NY 14744  
Current: Mathematics Department, University of Washington, Seattle WA 98125  
samantha.fairchild15@houghton.edu, skayf@uw.edu

I. Haim  
Department of Computer Science, University of Maryland, College Park MD 20742  
Department of Mathematics, University of Maryland, College Park MD 20742  
Current: Google Inc., 1600 Amphitheatre Parkway, Mountain View CA 94043  
ilsehaim11@gmail.com

R. G. Setra  
Department of Electrical and Computer Engineering, University of Maryland, College Park MD 20742  
Current: Department of Electrical Engineering, Stanford University, Stanford CA 94305  
rafael.g.setra@gmail.com

R. S. Strichartz  
Mathematics Department, Cornell University, Ithaca NY 14852  
str@math.cornell.edu

T. Westura  
Mathematics Department, Cornell University, Ithaca NY 14852  
Current: Epic Systems Corporation, 1979 Milky Way, Verona WI 53593  
tsw52@cornell.edu