Quantum Inverse Square Interaction

Kumar. S. Gupta

Theory Division,
Saha Institute of Nuclear Physics,
1/AF Bidhannagar, Calcutta - 700064, India.

Abstract

Hamiltonians with inverse square interaction potential occur in the study of a variety of physical systems and exhibit a rich mathematical structure. In this talk we briefly mention some of the applications of such Hamiltonians and then analyze the case of the N-body rational Calogero model as an example. This model has recently been shown to admit novel solutions, whose properties are discussed.

\[1\] Talk presented at the conference “Space-time and Fundamental Interactions: Quantum Aspects” in honour of Prof. A.P. Balachandran’s 65th birthday, Vietri sul Mare, Italy, 26 - 31 May, 2003.
1 Introduction

Hamiltonians with inverse square type interaction potential appear in the study of a variety of systems and exhibit rich physical and mathematical structures. One of the first applications of such Hamiltonians occurred in the study of electron capture by polar molecules with permanent dipole moments [1]. It has been found that only molecules with a critical value of dipole moment can capture electrons to form stable anions. Various proposals have been put forward to explain such a critical value [2, 3] and extensive theoretical and experimental studies of such molecular systems are found in the literature [4]. It is however a puzzling fact that while most theories of such molecules predict the existence of an infinite number of bound states, experiments so far have typically detected only a single bound state in such systems [4].

Physics of black holes is another interesting area where Hamiltonians with inverse square type interactions occur [5]. These operators appear naturally in the analysis of the near-horizon properties of black holes [6] and their study provides an alternate derivation [7] of the near-horizon conformal structure of black holes [8]. The Bekenstein-Hawking entropy of black holes can also be understood within this framework [7] and recently there has been a novel proposal to understand black hole decay using this formalism [9].

On a more formal side, Hamiltonians with inverse square interactions have been studied within the context of conformal quantum mechanics [10]. The Hamiltonian appears as one of the generators of $SU(1, 1)$ which plays the role of the spectrum generating algebra. Such systems provide a quantum mechanical setting to analyze the phenomena of scaling anomalies [3, 11] and renormalization [12].

Finally, following the seminal works of Calogero [13, 14], exactly solvable many body systems with inverse square interaction have been studies extensively in the literature [15]. The Calogero model and its variants are relevant to the study of many branches of contemporary physics, including generalized exclusion statistics [16], quantum hall effect [17], Tomonaga-Luttinger liquid [18], quantum chaos [19], quantum electric transport in mesoscopic system [20], spin-chain models [21] and Seiberg-Witten theory [22]. The spectrum of the $N$-particle rational Calogero model was first obtained almost three decades ago [14], which has since been analyzed using a variety of different techniques [23]. In the rest of this talk, we shall describe some new solutions to the $N$-body rational Calogero model that have been found recently [24]. These solutions are qualitatively very different compared to what was obtained by Calogero. We shall also see that these new solutions captures many of the interesting physical and mathematical aspects of Hamiltonians with inverse square interactions.
2 N-Body Rational Calogero Model

The rational Calogero model is described by $N$ identical particles interacting with each other through a long-range inverse-square and harmonic interaction on the line. The Hamiltonian of this model is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \left[ \frac{a^2 - \frac{1}{4}}{(x_i - x_j)^2} + \frac{\Omega^2}{16} (x_i - x_j)^2 \right]$$

(1)

where $a$, $\Omega$ are constants, $x_i$ is the coordinate of the $i$th particle and units have been chosen such that $2m\hbar^{-2} = 1$. We are interested in finding normalizable solutions of the eigenvalue problem

$$H \psi = E \psi.$$  

(2)

Following Calogero [14], we consider the above eigenvalue equation in a sector of configuration space corresponding to a definite ordering of particles given by $x_1 \geq x_2 \geq \cdots \geq x_N$. The translation-invariant eigenfunctions of the Hamiltonian $H$ can be written as

$$\psi = \prod_{i<j} (x_i - x_j)^{a + \frac{1}{2}} \phi(r) P_k(x),$$

(3)

where $x \equiv (x_1, x_2, \ldots, x_N)$, $r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2$ and $P_k(x)$ is a translation-invariant as well as homogeneous polynomial of degree $k (\geq 0)$ which satisfies the equation

$$\left[ \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{2(a + \frac{1}{2})}{(x_i - x_j)} \frac{\partial}{\partial x_i} \right] P_k(x) = 0.$$  

(4)

The existence of complete solutions of (4) has been discussed by Calogero [14]. Substituting Eqn. (3) in Eqn. (2) and using Eqns (4) we get

$$\tilde{H} \phi = E \phi,$$

(5)

where

$$\tilde{H} = \left[ -\frac{d^2}{dr^2} - (1 + 2\nu) \frac{1}{r} \frac{d}{dr} + w^2 r^2 \right]$$

(6)

with $w^2 = \frac{1}{8} \Omega^2 N$ and

$$\nu = k + \frac{1}{2} (N - 3) + \frac{1}{2} N(N - 1)(a + \frac{1}{2}).$$

(7)

$\tilde{H}$ is the effective Hamiltonian in the “radial” direction. It can be shown that $\phi(r) \in L^2[\mathbb{R}^+, d\mu]$ where the measure is given by $d\mu = r^{1+2\nu} dr$.

The Hamiltonian $\tilde{H}$ is a symmetric (Hermitian) operator on the domain $D(\tilde{H}) \equiv \{ \phi(0) = \phi'(0) = 0, \phi, \phi' \text{ absolutely continuous} \}$. To determine whether $\tilde{H}$ is self-adjoint [25] in $D(\tilde{H})$, we have to first look for square integrable solutions of the equations

$$\tilde{H} \phi_{\pm} = \pm i \phi_{\pm},$$

(8)
where $\tilde{H}^*$ is the adjoint of $\tilde{H}$ (note that $\tilde{H}^*$ is given by the same differential operator as $\tilde{H}$ although their domains might be different). Let $n_+ (n_-)$ be the total number of square-integrable, independent solutions of (8) with the upper (lower) sign in the right hand side. Now $\tilde{H}$ falls in one of the following categories [25] : 1) $\tilde{H}$ is (essentially) self-adjoint iff $(n_+, n_-) = (0, 0)$; 2) $\tilde{H}$ has self-adjoint extensions iff $n_+ = n_- \neq 0$; 3) If $n_+ \neq n_-$, then $\tilde{H}$ has no self-adjoint extensions.

For the Calogero model, the solutions of Eqn. (8) are given by

$$\phi_{\pm}(r) = e^{-\frac{\nu r^2}{2}}U\left(d_{\pm}, c, w r^2\right),$$

where $d_{\pm} = \frac{1+\nu}{2} \pm \frac{i}{2w}$, $c = 1 + \nu$ and $U$ denotes the confluent hypergeometric function of the second kind [26]. The asymptotic behaviour of $U$ [26] together with the exponential factor in Eqn. (9) ensures that $\phi_{\pm}(r)$ vanish at infinity. The solution in Eqn. (9) have different short distance behaviour for $\nu \neq 0$ and $\nu = 0$. From now onwards, we shall restrict our discussion to the case for $\nu \neq 0$, the analysis for $\nu = 0$ being similar. When $\nu \neq 0$, $U(d_{\pm}, c, w r^2)$ can be written as

$$U\left(d_{\pm}, c, w r^2\right) = C\left[\frac{M(d_{\pm}, c, w r^2)}{\Gamma(b_{\pm})\Gamma(c)} - \left(w r^2\right)^{1-c}\frac{M(b_{\pm}, 2-c, w r^2)}{\Gamma(d_{\pm})\Gamma(2-c)}\right],$$

where $b_{\pm} = \frac{1-\nu}{2} \pm \frac{i}{2w}$, $C = \frac{\pi}{\sin(\pi+\nu \pi)}$ and $M$ denotes the confluent hypergeometric function of the first kind [26]. In the limit $r \to 0$, $M(d_{\pm}, c, w r^2) \to 1$. This together with Eqns. (9) and (10) implies that as $r \to 0$,

$$|\phi_{\pm}(r)|^2d\mu \to \left[A_1 r^{(1+\nu)} + A_2 r + A_3 r^{(1-\nu)}\right] dr,$$

where $A_1, A_2$ and $A_3$ are constants independent of $r$. From Eqn. (11) it is now clear that in the limit $r \to 0$, the functions $\phi_{\pm}(r)$ are not square-integrable if $|\nu| \geq 1$. In that case, $n_+ = n_- = 0$ and $\tilde{H}$ is essentially self-adjoint in the domain $D(\tilde{H})$. However, if either $0 < \nu < 1$ or $-1 < \nu < 0$, the functions $\phi_{\pm}(r)$ are indeed square-integrable. Thus if $\nu$ lies in these ranges, we have $n_+ = n_- = 1$ and Hamiltonian $\tilde{H}$ is not self-adjoint in $D(\tilde{H})$ but admits self-adjoint extensions. The domain $D_2(\tilde{H})$ in which $\tilde{H}$ is self-adjoint contains all the elements of $D(\tilde{H})$ together with elements of the form $\phi_+ + e^{iz}\phi_-$, where $z \in R \ (\text{mod} \ 2\pi)$ [25]. We can similarly show that $n_+ = n_- = 1$ for $\nu = 0$ as well. Thus the self-adjoint extensions of this model exist when $-1 < \nu < 1$.

In order to determine the spectrum we note that the solution to Eqn. (5) which is bounded at infinity is given by

$$\phi(r) = Be^{-\frac{\nu r^2}{2}}U(d, c, w r^2),$$

where $d = \frac{1+\nu}{2} - \frac{E}{4w}$ and $B$ is a constant. In the limit $r \to 0$,

$$\phi(r) \to BC\left[\frac{1}{\Gamma(b)\Gamma(c)} - \frac{w^{-\nu} r^{-2\nu}}{\Gamma(d)\Gamma(2-c)}\right],$$

where $C$ is a constant.
where \( b = \frac{1-\nu}{2} - \frac{E}{4w} \). On the other hand, as \( r \to 0 \),

\[
\phi_+ + e^{iz}\phi_- \to C \left[ \frac{1}{\Gamma(c)} \left( \frac{1}{\Gamma(b_+)} + \frac{e^{iz}}{\Gamma(b_-)} \right) - \frac{w^{-\nu}r^{-2\nu}}{\Gamma(2-c)} \left( \frac{1}{\Gamma(d_+)} + \frac{e^{iz}}{\Gamma(d_-)} \right) \right].
\]

(14)

The requirement of self-adjointness demands that \( \phi(r) \in \mathcal{D}_\varepsilon(\tilde{H}) \), which implies that the coefficients of different powers of \( r \) in Eqns. (13) and (14) must match. Comparing the coefficients of the constant term and \( r^{-2\nu} \) in Eqns. (13) and (14) we get

\[
f(E) = \frac{\Gamma \left( \frac{1-\nu}{2} - \frac{E}{4w} \right)}{\Gamma \left( \frac{1+\nu}{2} - \frac{E}{4w} \right)} = \frac{\xi_2 \cos \left( \frac{z}{2} - \eta_1 \right)}{\xi_1 \cos \left( \frac{z}{2} - \eta_2 \right)},
\]

(15)

where \( \Gamma \left( \frac{1+\nu}{2} + \frac{i}{4w} \right) \equiv \xi_1 e^{i\eta_1} \) and \( \Gamma \left( \frac{1-\nu}{2} + \frac{i}{4w} \right) \equiv \xi_2 e^{i\eta_2} \). For given values of the parameters \( \nu \) and \( w \), the bound state energy \( E \) is obtained from Eqn. (15) as a function of \( z \). The corresponding eigenfunctions are obtained by substituting \( \phi(r) \) from Eqn. (12) into Eqn. (3). Different choices of \( z \) thus leads to inequivalent quantizations of the many-body Calogero model. Moreover from Eqn. (15) we see that for fixed value of \( z \), the Calogero model with parameters \((w, \nu)\) and \((w, -\nu)\) produces identical energy spectrum although the corresponding wavefunctions are different.
The following features about the spectrum may be noted:

1) We have obtained the spectrum analytically when the r.h.s. of Eqn. (15) is either 0 or $\infty$. When the r.h.s. of Eqn. (15) is 0, we must have the situation where $\Gamma \left( \frac{1+\nu}{2} - \frac{E}{4w} \right)$ blows up, i.e. $E_n = 2w(2n + \nu + 1)$ where $n$ is a positive integer. This happens for the special choice of $z = z_1 = \pi + 2\eta_1$. These eigenvalues and the corresponding eigenfunctions are analogous to those found by Calogero although for a different parameter range. Similarly, when the r.h.s. of Eqn. (15) is $\infty$, an analysis similar to the one above shows that $E_n = 2w(2n - \nu + 1)$. This happens for the special value of $z$ given by $z = z_2 = \pi + 2\eta_2$.

2) For choices of $z$ other than $z_1$ or $z_2$, the nature of the spectrum can be understood from Figure 1, which is a plot of Eqn. (15) for specific values of $\nu, z$ and $w$. In that plot, the horizontal straight line corresponds to the r.h.s of Eqn. (15). The energy eigenvalues are obtained from the intersection of $f(E)$ with the horizontal straight line. Note that the spectrum generically consists of infinite number of positive energy solutions and at most one negative energy solution. The existence of the negative energy states can be understood in the following way. For large negative values of $E$, the asymptotic value of $f(E)$ is given by $\left( \frac{E}{4w} \right)^{-\nu}$, which monotonically tends to 0 or $+\infty$ for $\nu > 0$ or $\nu < 0$ respectively. When $\nu > 0$, the negative energy state will exist provided r.h.s. of Eqn. (15) lies between 0 and $\Gamma \left( \frac{1+\nu}{2} \right)$ and $\infty$. Similarly, when $\nu < 0$, the negative energy state will exist when the r.h.s. of Eqn. (15) lies between $\Gamma \left( \frac{1-\nu}{2} \right)$ and $+\infty$. For any given values of $\nu$ and $w$, the position of the horizontal straight line in Fig. 1 can always be adjusted to lie anywhere between $-\infty$ and $+\infty$ by suitable choices of $z$. Thus the spectrum would always contain a negative energy state for some choice of the parameter $z$.

3) Contrary to the spectrum of the rational Calogero model, the energy spectrum obtained from Eqn. (15) is not equispaced for finite values of $E$ and for generic values of $z$. This may seem surprising with the presence of $SU(1,1)$ as the spectrum generating algebra in this system \[10\], which demands that the eigenvalues be evenly spaced. In order to address this issue, we consider the action of the dilatation generator $D = \frac{1}{2} \left( r \frac{d}{dr} + \frac{d}{dr} r \right)$ on an element $\phi(r) = \phi_+(r) + e^{iz} \phi_-(r)$. In the limit $r \to 0$, we have

$$D\phi = \frac{C}{2} \left[ \frac{1}{\Gamma(c)} \left( \frac{1}{\Gamma(b_+)} + \frac{e^{iz}}{\Gamma(b_-)} \right) - r^{-2\nu}(1 - 4\nu) \left( \frac{1}{\Gamma(d_+)} + \frac{e^{iz}}{\Gamma(d_-)} \right) \right].$$

We therefore see that $D\phi(r) \in D_z(\hat{H})$ only for $z = z_1$ or $z = z_2$. Thus the generator of dilatations does not in general leave the domain of the Hamiltonian invariant \[27\ \[24\]. Consequently, $SU(1,1)$ cannot be implemented as the spectrum generating algebra except for $z = z_1, z_2$.

We have thus presented a new quantization scheme for the rational Calogero model. The non-equispaced nature of the energy levels and the existence of a negative energy
bound state are some of the salient features that emerge from our analysis. These features of the solutions presented here are qualitatively different from those obtained by Calogero.

3 Conclusion

In this talk we have mentioned some of the interesting areas in which Hamiltonian operators with inverse square interaction occur. In some of the applications that we have analyzed, including the Calogero model discussed above, the self-adjoint extension of the relevant Hamiltonian operator plays an important role. Such Hamiltonians are also generically related to the Virasoro algebra, which hints at the existence of an underlying conformal symmetry. While the presence of the conformal symmetry has been established in some of the examples, it is not clear if such a symmetry plays any role in a generic case. Much more work needs to be done to understand the role of such a symmetry, which is a task for the future.

Acknowledgments

The importance of self-adjoint extension in physics has often been emphasized by Professor A. P. Balachandran which has served as an inspiration to me in pursuit of this line of research. I am very grateful to Bal for this. The material presented here is based partly on work done in collaboration with B. Basu-Mallick, D. Birmingham, P. K. Ghosh and S. Sen. I would like to thank the organizers of the conference in general, and F. Lizzi and G. Marmo in particular, for their hospitality in Vietri. I also thank the Associateship Scheme of the Abdus Salam ICTP, Trieste, Italy for their financial support which made my participation in the conference possible.
References

[1] E. Fermi and E. Teller, Phys. Rev. 72, 406 (1947);

[2] J.-M. Levy-Leblond, Phys. Rev. 153, 1 (1967).

[3] H. E. Camblong, L. N. Epele, H. Fanchiotti and C. A. Garcia Canal, Phys. Rev. Lett. 87, 220402 (2001).

[4] O. H. Crawford and W. R. Garret, J. Chem. Phys. 66, 4969 (1977); W. R. Garret, J. Chem. Phys. 69, 2622 (1978); Amy S. Mullin, Ph.D. Thesis, University of Colorado, Boulder, (1991); Amy S. Mullin, K. K. Murray, C. P. Schulz and W. C. Lineberger, J. Phys. Chem. 97, 10281 (1993); C. Desfrancois, H. Abdoul-Carime, N. Khelifa and J.P. Schermann, Phys. Rev. Lett. 73, 2436 (1994); C. Desfrancois, H. Abdoul-Carime, N. Khelifa and J.P. Schermann, Int. Jour. Mod. Phys. B 10, 1339 (1996); F. Guthe, M. Tulej, M. V. Pachkov and J. P. Maier, The Astrophysical Jour. 555, 466 (2001).

[5] P. Claus, M, Derix, R. Kallosh, J. Kumar, P. K. Townsend and A. V. Proeyen, Phys. Rev. Lett. 81, 4553 (1998); G. W. Gibbons and P. K. Townsend, Phys. Lett. B454, 187 (1999); K. Srivivasan and T. Padmanabhan, Phys. Rev. D60, 024007 (1999); Basu-Mallick, Pijush K. Ghosh, Kumar S. Gupta, hep-th/0304071.

[6] T. R. Govindarajan, V. Suneeta and S. Vaidya, Nucl. Phys. B583, 291 (2000).

[7] Danny Birmingham, Kumar S. Gupta and Siddhartha Sen, Phys. Lett. B505, 191 (2001); Kumar S. Gupta and Siddhartha Sen, Phys. Lett. B526, 121 (2002); Kumar S. Gupta, hep-th/0204137.

[8] A. Strominger, JHEP 9802, 009 (1998); S. Carlip, Phys. Rev. Lett. 82, 2828 (1999).

[9] Kumar S. Gupta and S. Sen, hep-th/0203094.

[10] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. 34A, 569 (1976); C. Leiva and M. S. Plyushchay, Ann. Phys. 307, 372 (2003).

[11] G. N.J. Ananos, H. E. Camblong and C. R. Ordonez, Phys. Rev. D68, 025006 (2003); H. E. Camblong and C. R. Ordonez, hep-th/0303166.

[12] Kumar S. Gupta and S. G. Rajeev, Phys. Rev. D48, 5940 (1993); S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. van Kolck, Phys. Rev. A64, 042103 (2001); H. E. Camblong and C. R. Ordonez, hep-th/0305035.

[13] F. Calogero, Jour. Math. Phys. 10, 2191 (1969); F. Calogero, Jour. Math. Phys. 10, 2197 (1969).
[14] F. Calogero, Jour. Math. Phys. 12, 419 (1971).

[15] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71, 314 (1981); ibid 94, 6 (1983).

[16] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994); Z. N. C. Ha, Quantum Many-Body Systems in One Dimension, Series on Advances in Statistical Mechanics, Vol. 12, (World Scientific, 1996); A. P. Polychronakos, hep-th/9902157 B. Basu-Mallick and A. Kundu, Phys. Rev. B62, 9927 (2000).

[17] H. Azuma and S. Iso, Phys. Lett. B331, 107(1994).

[18] N. Kawakami and S.-K. Yang, Phys. Rev. Lett. 67, 2493 (1991).

[19] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. 72, 64(1994); S. Jain, Mod. Phys. Lett. A11, 1201(1996).

[20] C. W. J. Beenakker and B. Rejaei, Phys. Rev. B49, 7499 (1994); M. Caselle, Phys. Rev. Lett. 74, 2776 (1995).

[21] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988); B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988); A. P. Polychronakos, Phys. Rev. Lett. 70, 2329 (1993).

[22] E. D’Hoker and D. H. Phong, hep-th/9912271; A. Gorsky and A. Mironov, hep-th/0011197; A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 100, 1107 (1998).

[23] A. P. Polychronakos, Phys. Rev. Lett. 69, 703 (1992); L. Brink, T. H. Hansson and M. A. Vasiliev, Phys. Lett. B286, 109 (1992); N. Gurappa and P. K. Panigrahi, Phys. Rev. B 59, R2490 (1999).

[24] B. Basu-Mallick and Kumar S. Gupta, Phys. Lett. A 292, 36 (2001; B. Basu-Mallick, Pijush K. Ghosh and Kumar S. Gupta, Nucl. Phys. B659, 437 (2003); B. Basu-Mallick, Pijush K. Ghosh and Kumar S. Gupta, Phys. Lett. A311, 87 (2003).

[25] M. Reed and B. Simon, Methods of Modern Mathematical Physics, volume 2, (Academic Press, New York, 1972).

[26] Handbook of Mathematical Functions, M. Abromowitz and I. A. Stegun (Dover Publications, New York, 1974).

[27] E. D’Hoker and L. Vinet, Comm. Math. Phys. 97, 391 (1985); J. G. Esteve, Phys. Rev. D 34, 674 (1986); R. Jackiw in M.A.B. Beg Memorial Volume, A. ALi and P. Hoodbhoy, eds. (World Scientific, Singapore, 1991); C. Aneziris, A. P. Balachandran and Diptiman Sen, Int. Jour. Mod. Phys. 6, 4721 (1991); J. G. Esteve, Phys. Rev. D 66, 125013 (2002).