Active-sterile neutrino mixing in the absence of bare active neutrino mass

Yi Liao
Department of Physics, Nankai University, Tianjin 300071, China

Abstract

We investigate a minimal extension of the standard model in which the only new ingredient is the sterile neutrinos. We do not introduce extra Higgs multiplets or high dimensional effective operators to induce mass terms for the active neutrinos, and the model is renormalizable in itself. We show for arbitrary numbers of generations and sterile neutrinos that the independent physical parameters in the leptonic sector are much less than previously anticipated. For instance, with three active and two sterile neutrinos, there are four mixing angles and three CP phases in addition to four non-vanishing neutrino masses. We study phenomenological implications for tritium beta decay, neutrinoless double beta decay and neutrino oscillations. For the most natural see-saw parameters, we find that it is difficult to accommodate in the model the best-fit values of masses and mixing parameters from oscillation data no matter whether we include or not the null short-baseline experiments together with the LSND result. This implies that if the LSND result is confirmed by MiniBooNE, the see-saw parameter region of the model with two sterile neutrinos could be largely excluded.

PACS: 14.60.Pq, 14.60.St, 23.40.-s
Keywords: sterile neutrino, lepton mixing, neutrino oscillation, beta decay
1 Introduction

The experiments on neutrinos have offered the first piece of evidence that points to physics beyond the standard model (SM) of electroweak interactions [1]. The deficits in the solar electron neutrino flux [2] and in the ratio of the atmospheric muon to electron neutrino fluxes [3] can be best and most naturally interpreted in terms of neutrino oscillations [4]. These observations have been confirmed by experiments at accelerators and reactors [5, 6]. They imply that the three neutrinos are non-degenerate and interact non-diagonally with the charged leptons. Nevertheless, the controversial circumstance in the short baseline (SBL) experiments seems to indicate that the complete picture of the neutrino sector may be richer than with the three ordinary neutrinos: while the LSND experiment claimed to observe a statistically significant signal [7] that cannot be accommodated in the three neutrino scheme of solar and atmospheric experiments due to three very different mass gaps, the signal was not confirmed by other SBL experiments [8, 9]. This situation will hopefully be clarified in the near future by the MiniBooNE experiment [10].

The controversy at SBL experiments has stimulated a lot of theoretical attempts. Some of them appeal to certain drastically new physics like CPT violation [11], CPT violating quantum decoherence [12], or extra dimensions [13], to mention a few. A more conventional approach is to introduce additional neutrinos to accommodate the observed at least three mass gaps. Since the number of SM-like neutrinos is severely constrained by electroweak measurements to be three, these new neutrinos must be neutral with respect to the SM gauge groups, i.e., they must be sterile neutrinos. The concept of sterile neutrinos was introduced earlier in the other context [14]. The most economical scenario with one sterile neutrino has been extensively studied in the literature [15]. It is generally difficult, if not impossible, to explain all data in this scenario either because of the rejection of significant involvement of a sterile neutrino by the solar and atmospheric data or because of the tension existing between the positive SBL result at LSND and the negative one at all others. The next simplest would be to add two sterile neutrinos, the so-called (3+2) scenario. Sorel et al. have assessed carefully the compatibility of all SBL experiments, and found that the (3+2) scenario fits the SBL data significantly better than the one with a single sterile neutrino [16]. From the model building point of view, although it is simplest to add one sterile neutrino, there are viable models that contain two or three sterile neutrinos [17]. It has to be left for experiments to decide which is actually realized in Nature.

In this paper, we consider a minimal extension of SM that could potentially accommodate the neutrino data. It is minimal in the sense that only the neutrino sector is extended by adding some sterile neutrinos. We do not introduce additional Higgs particles or higher dimensional effective operators from some high energy scale. In particular, there are no bare masses for the ordinary active neutrinos. The model so extended preserves renormalizability in itself. In an earlier work [18] we found that such a model has a strictly constrained leptonic sector which contains much less physical parameters than previously
anticipated [19, 20]. With one sterile neutrino, for instance, there are only two mixing angles in addition to two non-vanishing neutrino masses, and the leptonic sector preserves CP symmetry automatically. Furthermore, the ratio of the two masses also appears in the mixing matrices, which makes the two otherwise independent mixing angles less effective in generating an experimentally favored mixing matrix than they would have been. The resulting model is even incapable of interpreting the solar and atmospheric data [18, 21].

In this work, we investigate the parametrization of the leptonic sector for any numbers of generations and sterile neutrinos. Since the number of physical parameters is much less than previously expected, it becomes numerically manipulatable to study its implications on neutrino experiments even with two or three sterile neutrinos. For instance, with two sterile neutrinos, there are four mixing angles and three CP violating phases in addition to four non-vanishing neutrino masses, and the model could thus become viable to accommodate the neutrino data including the SBL experiments.

The paper is organized as follows. In the next section, we study the leptonic sector of the minimally extended model, and count in particular the number of physical parameters contained in it for any numbers of generations and sterile neutrinos. To get prepared for phenomenological analyses, we work out analytically in section 3 the neutrino spectrum and leptonic mixing matrices for the most natural see-saw region in parameter space [22]. Then, in section 4 we discuss the phenomenological implications in the case of two sterile neutrinos. We do not attempt here a sophisticated statistical assessment; instead, with the fitting result by Sorel et al. for the general (3+2) scenario in mind, we consider whether it is possible to accommodate their result in the above mentioned parameter region of our minimal model. The answer turns out to be negative and even worse: even if we ignore the tension between the positive and negative SBL experiments, it is difficult to include the LSND result in the see-saw region of the model. We summarize our results in the last section and mention briefly further work worthy to do with the model.

## 2 Mixing matrices for any numbers of active and sterile neutrinos

In this section we first describe the leptonic sector of the $n$-generation SM extended by $n_0$ sterile neutrinos. This is the minimal framework that can accommodate neutrino mass and mixing while preserving renormalizability of SM. By standardizing the neutrino mass matrices, we count independent physical parameters contained in the leptonic sector. Diagonalization of the leptonic mass matrices then yields the mixing matrices in the charged and neutral current interactions. For $n > n_0$, we show that the number of independent physical parameters can be further reduced due to the appearance of massless neutrinos.
2.1 Setup of the model

The only new fields compared to SM are the \( n_0 \) sterile neutrinos that we choose to be right-handed without loss of generality, \( s_{Rx} \), \( x = 1, \ldots, n_0 \). The model contains as usual the \( n \) generations of the lepton doublets, \( F_{La} = (n_{La}, f_{La})^T \), and of the charged lepton singlets, \( f_{Ra} \), \( a = 1, \ldots, n \). Here \( L, R \) refer to the left- and right-handed projections of the fields.

Since the sterile neutrinos are neutral under \( SU(2)_L \times U(1)_Y \) by definition, they are allowed to have bare mass terms of Majorana type,

\[
-\mathcal{L}_{sR} = \frac{1}{2} M_{xy} s_{Rx}^C s_{Ry} + \frac{1}{2} M_{xy}^* s_{Ry}^C s_{Rx}
\]

(1)

where \( \psi^C = C \gamma^0 \psi^* \) stands for the charge-conjugate field of \( \psi \) with \( C = i \gamma^0 \gamma^2 \) satisfying \( C = -C^\dagger = -C^T = -C^{-1} \) and \( C \gamma^\mu T \gamma^C = \gamma^\mu \). We denote \( s_R^C = (s_R)^C \) for brevity. The \( n_0 \times n_0 \) complex matrix \( M \) is symmetric due to anticommutativity of fermion fields, but is otherwise general. The presence of sterile neutrinos also introduces the mixing mass terms between active and sterile neutrinos through the Yukawa interactions,

\[
-\mathcal{L}_Y = y_{ab}^f F_{La} \varphi f_{Rb} + y_{ab}^n \bar{F}_{La} \bar{\varphi} s_{Rx} + \text{h.c.}
\]

(2)

where \( \varphi \) is the Higgs doublet field that develops a vacuum expectation value, \( \langle \varphi \rangle = (0,1)^T v/\sqrt{2} \), and \( \varphi = i \sigma^2 \varphi^* \). Note that no bare mass terms are allowed for the active neutrinos in this minimal extension of SM. The lepton mass terms are summarized by

\[
-\mathcal{L}_m = \left[ \bar{F}_L m_f f_R + \bar{F}_L D r_R + \text{h.c.} \right] + \frac{1}{2} \left[ s_R^C M s_R + \text{h.c.} \right]
\]

(3)

where \( m_f = y^f v/\sqrt{2} \) and \( D = y^n v/\sqrt{2} \) are \( n \times n \) and \( n \times n_0 \) complex matrices respectively.

2.2 Standardization and diagonalization of mass matrices

To facilitate counting the independent physical parameters contained in the leptonic mixing matrices after diagonalizing the lepton mass matrices, we first convert the neutrino mass matrices into a standard form. Since \( M \) is symmetric, it can be diagonalized by a unitary transformation \( s_R \rightarrow Y_0 s_R \) to the real positive \( Y_0^T M Y_0 = M_{\text{diag}} = \text{diag}(r_1, \ldots, r_{n_0}) \). The only other change in the total Lagrangian occurs in \( \mathcal{L}_m: \ D \rightarrow D Y_0 \). Denote \( D Y_0 = (a_1, \ldots, a_{n_0}) \) where \( a_x \) \( (x = 1, \ldots, n_0) \) are general \( n \)-component complex vectors. Making a unitary transformation \( n_L \rightarrow X^\dagger n_L \), we convert \( D Y_0 \) into \( X D Y_0 = D_\Delta \) which has a zero triangle at its upper-left corner, and when \( n > n_0 \) an additional \( (n - n_0) \times n_0 \) zero rectangle over the triangle. To keep the partnership of \( n_L \) and \( f_L \), we also transform \( f \rightarrow X^\dagger f \) so that the only other change is again in \( \mathcal{L}_m \) which now becomes

\[
-\mathcal{L}_m = \left[ \bar{F}_L X m_f X^\dagger f_R + \bar{F}_L D_\Delta r_R + \text{h.c.} \right] + \frac{1}{2} \left[ s_R^C M_{\text{diag}} s_R + \text{h.c.} \right]
\]

(4)
The matrix $D_{\Delta}$ is found successively and in a way applicable to both cases $n > n_0$ and $n \leq n_0$. First, we choose the unitary matrix $X_1$ so that $X_1 a_1 = (0, \ldots, 0, |a_1|)^T$ with $|a_1| = \sqrt{a_1^1 a_1^1}$. Invariance of the inner product of vectors implies that $(X_1 a_x)_n = \frac{a_1^x a_x}{|a_1|}$ which determines all entries in the $n$-th row of $X_1(a_1, \ldots, a_{n_0})$. The first $(n-1)$ components of the column vector $X_1 a_x$ are normalized to $\sqrt{a_1^1 P_1 a_x}$ with the projector $P_1 = 1 - \frac{a_1^1 a_x}{a_x a_1}$. Next, we choose the unitary matrix $X_2$ that leaves all entries in the $n$-th row of $X_1(a_1, \ldots, a_{n_0})$ untouched and that rotates the first $(n-1)$ components of $X_1 a_2$ to its $(n-1)$-th component. Invariance of the inner product then determines the $(n-1)$-th row of $X_2 X_1(a_1, \ldots, a_{n_0})$, and so on. The procedure continues by induction until the $n_{\text{min}}$-th column vector where $n_{\text{min}} = \min(n, n_0)$. Defining $a_0 = 0$ and the projectors

$$P_0 = 1, \quad P_x = 1 - \begin{bmatrix} x-1 \prod_{y=0}^x y \hspace{1cm} a_x \hspace{1cm} x-1 \prod_{y=0}^x y \end{bmatrix} a_x \begin{bmatrix} x-1 \prod_{y=0}^x y \hspace{1cm} a_x \hspace{1cm} x-1 \prod_{y=0}^x y \end{bmatrix},$$

which have the properties

$$P_x^+ = P_x, \quad P_x P_x = P_x, \quad \prod_{y=1}^x y \cdot a_x = 0, \quad [P_x, P_y] = 0,$$

the end result, $D_{\Delta} = X(a_1, a_2, \ldots, a_{n_0})$ with $X = X_{n_{\text{min}}} \cdots X_1$, has the following non-vanishing entries,

$$(D_{\Delta})_{ax} = \frac{a_{n+1-a}^\dagger \prod_{y=0}^{n-a} P_y a_x}{\sqrt{a_{n+1-a}^\dagger \prod_{y=0}^{n-a} P_y a_{n+1-a}}}$$

where the indices are restricted to $a \in [1, n]$ and $x \in [n + 1 - a, n_0]$ when $n \leq n_0$, and to $x \in [1, n_0]$ and $a \in [n + 1 - x, n]$ when $n > n_0$.

Now we diagonalize the lepton mass matrices. The charged part is done as usual by the bi-unitary transformations $f_{L,R} = X_{L,R} \ell_{L,R}$ with $X_{L}^\dagger (X_{L} f_{L}) X_{R} = m_\ell$ being real positive. The matrix $X_{L}^\dagger$ then appears in the charged current interactions of leptons. To diagonalize the neutral part, we first rewrite the Lagrangian in terms of the fields $(n_L^C, s_R^C)$ and their charge conjugates $(n_L^C, s_R^C)$. Then the neutrino mass terms become

$$-\mathcal{L}_m^\nu = \frac{1}{2} (n_L^C, s_R^C) m_n \begin{pmatrix} n_L^C \\ s_R^C \end{pmatrix} + \frac{1}{2} (n_L^C, s_R^C) m_n^\dagger \begin{pmatrix} n_L^C \\ s_R^C \end{pmatrix}$$

where the $(n + n_0)$-dimensional, symmetric mass matrix in the new basis is

$$m_n = \begin{pmatrix} 0_n & D_{\Delta} \\ D_{\Delta}^T & M_{\text{diag}} \end{pmatrix}$$
with $0_n$ being the zero matrix of $n$ dimensions. Finally, the matrix $m_n$ is diagonalized by the unitary transformation

$$
\begin{pmatrix}
  n_L^C \\
  s_R
\end{pmatrix} = Y \nu_R
$$

(10)

such that

$$
Y^T m_n Y = m_\nu
$$

(11)

is real and non-negative. The $n \times (n + n_0)$ submatrix, $y$, composed of the first $n$ rows of $Y$ appears in the charged and neutral current interactions of leptons. We shall denote the mass eigenstate fields of the neutral leptons, $\nu$, by the Latin indices, $j, k = 1, 2, \cdots, n + n_0$, and those of the charged leptons, $\ell$, by the Greek indices, $\alpha, \beta = 1, \cdots, n$. Introducing the Majorana neutrino fields $\nu = \nu_R + \nu_R^C$, the changes in the total Lagrangian are summarized as follows,

$$
\mathcal{L}_m = -m_{\ell j} \ell j - \frac{1}{2} m_{\nu j} \nu_j
$$

$$
\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \left[ V_{\beta j}^C W_{\mu j} \ell \gamma^\mu \nu_j + V_{\beta j}^C W_{\mu j} \nu_j \ell \gamma^\mu \right]
$$

(12)

$$
\mathcal{L}_{NC} = \frac{g}{4c_W} Z_{\mu j} \nu_j [i \text{ Im } V_{kj}^N - \gamma^5 \text{ Re } V_{kj}^N] \nu_j
$$

where upon redefining $X_L^\dagger = V$ for brevity

$$
V^C = V y^*, \quad V^C V^C_\dagger = 1_n, \quad V^N = V^C_\dagger V^C = y^T y^*
$$

(13)

2.3 Counting of physical parameters

We note first of all that $D_\Delta$ has different zero textures according to $n > n_0$ or $n \leq n_0$. This affects the number of physical parameters in the model. We start with the easier case of $n \leq n_0$. The matrix $D_\Delta$ has $n$ real entries lying at the rows and columns fulfilling $a + x = n + 1$ and $\frac{1}{2}n(2n_0 - n - 1)$ complex ones to the right of the real entries. There are thus $(n + n_0)$ real parameters and $\frac{1}{2}n(2n_0 - n - 1)$ complex ones in $m_n$, which are traded for $(n + n_0)$ neutrino masses, $\frac{1}{2}n(2n_0 - n - 1)$ mixing angles and $\frac{1}{2}n(2n_0 - n - 1)$ CP violation phases in $Y$. Since for general original matrices $M$, $D$ there are no further unitary transformations on $s_R$, $n_L$ that leave the structure of $m_n$ invariant, all those angles and phases must appear in the submatrix $y$ of $Y$, and thus in $V^N$ and $V^C$. The additional parameters introduced to $V^C$ on diagonalizing the charged lepton masses are counted as usual: $\frac{1}{2}n(n - 1)$ angles and $\frac{1}{2}n(n - 1)$ phases after absorbing the $n$ phases by the charged lepton fields. Again, for general original matrices, they are not expected to combine with or cancel those in $Y$. In summary, there are $n$ massive charged leptons, $(n + n_0)$ massive neutrinos, $(n_0 - 1)$ mixing angles and $(n_0 - 1)$ CP phases in the charged current matrix $V^C$. Out of those, only $\frac{1}{2}n(2n_0 - n - 1)$ angles and $\frac{1}{2}n(2n_0 - n - 1)$ phases appear in the neutral current matrix $V^N$ of the neutrinos.

When $n > n_0$, in addition to the indicated zero triangle, the first $(n - n_0)$ rows of $D_\Delta$ also vanish, which correspond to $(n - n_0)$ massless neutrinos after diagonalization.
Arbitrary unitary transformations, \( y_0 \), amongst the massless modes are allowed without changing the matrix \( m_n \). Thus, \( y \) has the structure

\[
y = \left( \begin{array}{ccc} y_0 & 0_{n-n_0} \times 2n_0 \\ 0_{n_0 \times (n-n_0)} & \bar{y} \end{array} \right), \quad y_0^{-1} = y_0^* \tag{14}\]

While both \( \bar{y} \) and \( y_0 \) appear in \( V^C \), only \( \bar{y} \) appears in \( V^N \). In this case, there are \( n_0 \) real and \( \frac{1}{2} n_0(n_0 - 1) \) complex parameters in \( D_\Delta \); together with \( M_{\text{diag}} \), they are traded for \( 2n_0 \) neutrino masses, \( \frac{1}{2} n_0(n_0 - 1) \) mixing angles and \( \frac{1}{2} n_0(n_0 - 1) \) CP phases in \( \bar{y} \) and thus \( V^N \).

Now we count the physical parameters contained in \( V^C \). An \( n \) dimensional unitary matrix \( V \) may be parameterized as a product in any arbitrarily specified order of the \( n \) diagonal phase matrices, \( e_\alpha(u_\alpha) (\alpha = 1, \cdots, n) \), and the \( \frac{1}{2} n(n - 1) \) complex rotation matrices in the \((\alpha, \beta)\) plane, \( \omega_{\alpha\beta}(\theta_{\alpha\beta}, \varphi_{\alpha\beta}) (n \geq \beta > \alpha \geq 1) \) \cite{19}. Here \( e_\alpha(z) \) is obtained by replacing the \( \alpha \)-th entry in \( 1_n \) by the phase \( z \), and

\[
\omega_{\alpha\beta}(\theta_{\alpha\beta}, \varphi_{\alpha\beta}) = e_\alpha(e^{i\varphi_{\alpha\beta}})R_{\alpha\beta}(\theta_{\alpha\beta})e_\alpha(e^{-i\varphi_{\alpha\beta}}) \tag{15}\]

where \( R_{\alpha\beta}(\theta_{\alpha\beta}) \) is the usual real rotation matrix through angle \( \theta_{\alpha\beta} \) in the \((\alpha, \beta)\) plane. We choose the order of products in such a way that it fits our purpose here:

\[
V = V_2 V_1 V_0 \tag{16}\]

where \( V_0 \) is the general unitary matrix in the subspace spanned by the first \((n - n_0)\) axes, \( V_2 \) the unitary matrix in the subspace spanned by the last \( n_0 \) axes, and \( V_1 \) is the one mixing the two subspaces. In \( V^C \), \( V_0 \) can be cancelled by \( y_0^* \) in \( y^* \), which remains free until now. The \( n_0 \) diagonal phases in \( V_2 \) can be arranged to its very left to get absorbed by the charged lepton fields. We are thus left in \( V^C \) with the \( \frac{1}{2} n_0(n_0 - 1) \) complex rotations in \( V_2 \) and \((n - n_0)n_0\) ones in \( V_1 \), in addition to the \( \frac{1}{2} n_0(n_0 - 1) \) complex parameters contained in \( \bar{y} \).

However, not all parameters in \( V_1 \) are physical. To see this, we write

\[
V_1 = \prod_{a,z} \omega_{az}(\theta_{az}, \varphi_{az}) \tag{17}\]

with \( a \in [1, n - n_0], z \in [n - n_0 + 1, n] \). The order of products is specified as follows: factors with the same \( z \) are grouped together with \( z \) increasing from left to right; within each group, \( a \) also increases from left to right. Denoting for brevity \( \omega_{az}(\theta_{az}, \varphi_{az}) = \omega_{az}, R_{az}(\theta_{az}) = R_{az}, e_\alpha(e^{i\varphi_{az}}) = e_{az} \) and \( e_\alpha(e^{-i\varphi_{az}}) = e_{az}^* \), it is clear that

\[
\omega_{1,z} \cdots \omega_{n, n_0, z} = \prod_{a=1}^{n-n_0} e_{az} \left[ \prod_{b=1}^{n-n_0} R_{bz} \right] \left[ \prod_{c=1}^{n-n_0} e_{cz}^* \right] \tag{18}\]
Then,

\[
V_1 = \left[ \prod_{a=1}^{n-n_0} e_{a,n-n_0+1} \right] \left[ \prod_{b=1}^{n-n_0} R_{b,n-n_0+1} \right] \left[ \prod_{c=1}^{n-n_0} e_{c,n-n_0+1} e_{c,n-n_0+2} \right] \\
\times \left[ \prod_{b=1}^{n-n_0} R_{b,n-n_0+2} \right] \left[ \prod_{c=1}^{n-n_0} e_{c,n-n_0+2} \right] \cdots \left[ \prod_{a=1}^{n-n_0} e_{a,n-1} \right] \left[ \prod_{b=1}^{n-n_0} R_{b,n-1} \right]
\]

(19)

The left phase matrix in the first line can be pushed through \(V_2\) to be absorbed by the charged lepton fields, while the right phase matrix in the last line passes through \(y\) in which \(y_0 = 1_{n-n_0}\) to get absorbed by the massless neutrino fields. This leaves with us the \((n-n_0)(n_0-1)\) phase differences and all of the real rotations in \(V_1\) that cannot be removed. To summarize, there are \(n_0(n-1)\) physical mixing angles and \(n(n_0-1)\) physical CP phases in \(V^C\).

There is another way to count the physical parameters in \(V^C\). A general \(n \times (n+n_0)\) complex matrix \(V^C\) has \(2n(n+n_0)\) real parameters. First, \(V^C V^C = 1_n\) offers \(n^2\) real constraints. Second, \(V^C\) may be multiplied from the left by an arbitrary diagonal phase matrix by redefining the charged lepton fields without affecting physics. This amounts to another \(n\) real parameters. Finally, since there is no bare mass for the active neutrinos, we have \(y^* m_\nu y = 0_n\) which implies \(V^C m_\nu V^{CT} = 0_n\) [23]. The constraint is symmetric, and thus imposes \(2n\) real conditions from the diagonal and \(n(n-1)\) from the off-diagonal.

Thus, the number of real physical parameters in \(V^C\) is \(2n(n_0-1)\). However, this counting is not as advantageous as the above one. It does not distinguish between mixing angles and CP phases, or tell how many of them enter in \(V^N\). More importantly, it does not take into account the further reduction of physical parameters due to the appearance of massless neutrinos when \(n > n_0\).

The counting of physical parameters is summarized in Table 1 where the result of Ref. [19] is also shown for comparison. The difference arises from the fact that the zero bare mass for active neutrinos has been completely exploited here to remove all unphysical parameters while it was only partially applied in Ref. [19] to delete unitary transformations within the massless neutrinos for the case \(n > n_0\).

### 3 Approximate results for two sterile neutrinos

The number of active neutrinos has been constrained by experiments to be \(n = 3\). To accommodate neutrino mass and mixing, one would introduce sterile neutrinos as few as possible. For \(n_0 = 1\), we have two massless neutrinos and two massive ones; and there are two mixing angles in \(V^C\). This has been shown in Refs. [18, 21] to be even impossible to explain the solar and atmospheric experiments that call for two mass squared differences \((\Delta m^2_{ji} = m^2_j - m^2_i)\) and two mixing angles, let alone the LSND results hinting at a third \(\Delta m^2_{ji}\). The next simplest is the case with two sterile neutrinos which is studied in this and
Table 1: The numbers of independent physical parameters are shown for the mixing matrices $V^C$ and $V^N$. Note that all parameters in $V^N$ are already included in $V^C$. Also shown are the results of Ref. [19] for $V^C$.

|     | $n_0 \geq n \geq 1$ | $n \geq n_0 \geq 1$ |
|-----|---------------------|---------------------|
| $V^C$: angles | $n(n_0 - 1)$         | $n_0(n - 1)$         |
|     | $n(n_0 - 1)$         | $n(n_0 - 1)$         |
|     | $2n(n_0 - 1)$        | $2n_0n - (n + n_0)$  |
| $V^N$: angles | $n_0n - \frac{1}{2}n(n + 1)$ | $\frac{1}{2}n_0(n_0 - 1)$ |
|     | $n_0n - \frac{1}{2}n(n + 1)$ | $\frac{1}{2}n_0(n_0 - 1)$ |
|     | $2n_0n - n(n + 1)$   | $n_0(n_0 - 1)$       |
| $V^C$: angles | $n_0n + \frac{1}{2}n(n - 1)$ | $2n_0n - \frac{1}{2}n_0(n_0 + 1)$ |
|     | $n_0n + \frac{1}{2}n(n - 1)$ | $2n_0n - n - \frac{1}{2}n_0(n_0 - 1)$ |
|     | $2n_0n + n(n - 1)$   | $4n_0n - n - n_0^2$  |

the next section. Now we have one massless neutrino ($\nu_1$) and four massive ones ($\nu_2, \ldots, \nu_5$). Separating out the massless mode from the matrix $D_\Delta$, we have

$$D_\Delta = \begin{pmatrix} 0 & d_2 \\ d_1 & z \end{pmatrix}, \quad M_{\text{diag}} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

(20)

where generally $d_{1,2} > 0$, $r_{1,2} > 0$ and $z$ is complex. These parameters are traded for the four neutrino masses $m_2, \ldots, 5$ and one mixing angle plus one CP phase in $V^N$. Both the angle and the phase enter in $V^C$ which includes additional three angles and two phases coming from the matrices $V_1$, $V_2$. We thus should have enough degrees of freedom to accommodate oscillation experiments that require at least three $\Delta m^2_{ji}$ and four mixing angles.

Although it is possible to diagonalize algebraically the mass matrix $m_n$, it is not instructive for our phenomenological analysis. Experimentally, we need three well-separated $\Delta m^2_{ji}$ to explain the positive oscillation results, $\Delta m^2_{\odot} \ll \Delta m^2_{\text{ATM}} \ll \Delta m^2_{\text{LSND}}$. Since the massless neutrino $\nu_1$ is purely active and the solar and atmospheric experiments are more in favor of active neutrino mixing than the involvement of sterile neutrinos, the other two mainly active neutrinos $\nu_{2,3}$ have to lie close to $\nu_1$ while the two mainly sterile neutrinos $\nu_{4,5}$ must be well above it; i.e., $m_{4,5} \gg m_{2,3}$. Furthermore, since the sterile mass terms are not constrained by the low energy symmetries of SM, it is natural that they may be linked to some new physics at a higher scale. Thus, we shall assume the hierarchy $r_{1,2} \gg d_{1,2}, |z|$ in our phenomenological analysis. As a special case, we shall also consider the possibility of $z = 0$ in which analytic results can be easily worked out for masses and mixing matrices without assuming the hierarchy $r_{1,2} \gg d_{1,2}$.
3.1 See-saw case: $r_{1,2} \gg d_{1,2}, |z|$

Excluding the massless neutrino, the mass matrix to be diagonalized via $Y^T m_n Y = \text{diag}(m_2, \ldots, m_5)$ is

$$m_n = \begin{pmatrix} d_2 & d_1 & z \\ d_1 & r_1 & z \\ d_2 & z & r_2 \end{pmatrix}$$

(21)

The eigenvalues exact to the second order in the expansion of $d_{1,2}, z$ over $r_{1,2}$ are found to be

$$m_{2,3} = \frac{1}{1 + |\rho_\pm|^2} \left| d_1^2 + \frac{(z + d_2 \rho_+^*)^2}{r_2} \right|$$

$$m_4 = r_1 + \frac{r_1}{d_1^2} + \frac{|z|^2}{r_2}$$

$$m_5 = r_2 + \frac{r_1}{d_2} + \frac{|z|^2}{r_2}$$

(22)

where $m_2 < m_3$, $m_2 m_3 = \frac{d_1^2 d_2^2}{r_1 r_2}$ and

$$\rho_\pm = \frac{1}{2z} \left[ -\zeta \pm \sqrt{\zeta^2 + 4|\xi|^2} \right]$$

$$\xi = d_2 \left| d_1 z + \frac{r_1}{r_2} (d_2^2 + |z|^2) \right|$$

$$\zeta = d_2^2 (z^2 + z^{*2}) + \frac{r_1}{r_2} (|z|^4 - d_2^4) + \frac{r_1}{r_2} d_1^4$$

(23)

When $z$ is real, there is an additional useful relation,

$$m_2 + m_3 = \frac{d_1^2}{r_1} + \frac{d_2^2 + |z|^2}{r_2}$$

(24)

The first two rows of $Y$ give the submatrix $\bar{y}$ in $y$. Including the massless degree of freedom corresponding to $y_0 = 1$ after removing unphysical parameters, we have

$$y^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{-\rho} & v_+ \rho & 0 \\ 0 & v_- & v_+ & \frac{d_1}{r_1} \\ 0 & \frac{z}{r_2} & \frac{z}{r_2} & 0 \end{pmatrix}$$

(25)

where the second and third columns are exact to the first order and the last two to the second order, and

$$v_\mp = \frac{ie^{i\alpha_\mp}}{\sqrt{1 + |\rho_\pm|^2}}$$

(26)

where $2\alpha_\pm$ are the phases of the complex numbers, $\left( \frac{d_1^2}{r_1} + \frac{(z + d_2 \rho_+^*)^2}{r_2} \right)$.

Parameterizing the $3 \times 3$ unitary matrix $V = (V_{\alpha\beta})$ with $\alpha, \beta = e, \mu, \tau$, we have

$$V^C = \begin{pmatrix} V_{ee} & v_- (V_{e\mu} \rho_+ + V_{e\tau}) & v_+ (V_{e\mu} \rho_+ + V_{e\tau}) \\ V_{\mu e} & v_+ (V_{\mu \mu} \rho_+ + V_{\mu \tau}) & v_+ (V_{\mu \mu} \rho_+ + V_{\mu \tau}) \\ V_{\tau e} & v_- (V_{\tau \mu} \rho_+ + V_{\tau \tau}) & v_+ (V_{\tau \mu} \rho_+ + V_{\tau \tau}) \end{pmatrix}$$

(27)

Since $y^* y = 1_3$ is exact at the first order in the expansion, so is $V^C V^C = 1_3$. 

10
3.2 Case: $z = 0$

The neutrino mass matrix can be easily diagonalized in this case. The masses are

$$m_{2,5} = \frac{1}{2} \left[ \sqrt{r^2 + 4d^2} \mp r_2 \right], \quad m_{3,4} = \frac{1}{2} \left[ \sqrt{r^1 + 4d^1} \mp r_1 \right]$$  \hspace{1cm} (28)

There is no free angle or phase in the diagonalizing matrix $Y$ which is completely fixed by the masses, so that

$$y^* = \begin{pmatrix} 1 & is_2 & 0 & 0 & c_2 \\ 0 & is_1 & c_1 & 0 \end{pmatrix}$$  \hspace{1cm} (29)

where

$$c_1 = \sqrt{\frac{m_3}{m_3 + m_4}} < s_1 = \sqrt{\frac{m_3}{m_3 + m_4}}$$

$$c_2 = \sqrt{\frac{m_2}{m_2 + m_5}} < s_2 = \sqrt{\frac{m_2}{m_2 + m_5}}$$  \hspace{1cm} (30)

Note that the $i$ factors are introduced to make $m_{2,3}$ real positive and do not by themselves signal CP violation [24]. Then

$$V_C = \begin{pmatrix} V_{ee} & iV_{e\mu} & iV_{e\tau} & V_{e\mu} & V_{e\tau} \\ iV_{\mu e} & iV_{\mu\mu} & iV_{\mu\tau} & V_{\mu\mu} & V_{\mu\tau} \\ iV_{\tau e} & iV_{\tau\mu} & iV_{\tau\tau} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}$$  \hspace{1cm} (31)

We have $V_C V_C^\dagger = 1_3$ exactly.

4 Implications on neutrino oscillations

We consider now the phenomenological implications of the previous section. Before we move to neutrino oscillations, we discuss briefly the results on the tritium decay [25] and the neutrinoless double $\beta$ decay [26]. Both decays are currently available experiments sensitive to the absolute neutrino mass. The first one measures the following effective mass through the distorted decay spectrum,

$$m_{\nu e}^2 = \sum_{j=1}^{5} m_j^2 |V_{ej}|^2 = (V y^* m^2 y^T V^\dagger)_ee$$  \hspace{1cm} (32)

Using $Y^T m_n Y = m_\nu$, $y^* m^2 y^T$ is just the upper-left $3 \times 3$ submatrix of $m_n m_n^\dagger$, i.e., $D_\Delta D_\Delta^\dagger$. Thus the effective mass is only sensitive to the Dirac mass terms,

$$m_{\nu e}^2 = d_2^2 |V_{e\mu}|^2 + (d_1^2 + |z|^2)|V_{e\tau}|^2 + d_2 (z V_{e\mu}^* V_{e\mu} + z^* V_{e\tau} V_{e\tau})$$  \hspace{1cm} (33)

which is of order the product of a light neutrino mass and a heavy one if $V_{ee}$ is not very close to unity.

The neutrinoless double $\beta$ decay violates the lepton number conservation and can occur only when the neutrino is of Majorana character. At the leading order in the expansion
of neutrino mass over the characteristic momentum transfer in the nuclear transition, the decay is proportional to the effective mass

$$m_{ee} = \left| \sum_{j=1}^{5} m_j (V_{e j}^C)^2 \right|$$

(34)

Note that there is interference amongst different terms. In the model considered here where the only extension is to add sterile neutrinos to SM, the above mass vanishes. Actually we can show that the effective mass for neutrinoless double like-sign charged leptons ($\ell_\alpha^+ \ell_\beta^-$) decays

$$m_{\alpha\beta} = \left| \sum_j m_j V_{\alpha j}^C V_{\beta j}^C \right|$$

(35)

vanishes generally for any number of generations and any number of sterile neutrinos. Using $V_C = V y^*$, we have

$$\sum_{j=1}^{n+n_0} m_j V_{\alpha j}^C V_{\beta j}^C = \sum_{\gamma,\delta} V_{\alpha\gamma} V_{\beta\delta} \sum_j m_j y_{\gamma j}^* y_{\delta j}^* = \sum_{\gamma,\delta} V_{\alpha\gamma} V_{\beta\delta} (y^* m_\nu y^\dag)_{\gamma\delta}$$

(36)

where $(y^* m_\nu y^\dag)_{\gamma\delta}$ constitutes the upper-left $n \times n$ submatrix of $Y^* m_\nu Y^\dag = m_n$, which vanishes however due to the absence of Majorana-type mass terms for active neutrinos in the current model. While this does not necessarily mean that such decays are forbidden, because they can be induced at the next orders of the expansion and weak interactions, it does imply they are strongly suppressed. It could also be the case that one or more neutrinos are much heavier than the momentum transfer so that the expansion does not apply to them and the sum over $j$ is not complete. Even in this case, the decays are still suppressed since the vanishing of the complete sum and very large masses of those neutrinos imply very small couplings between them and the charged leptons.

In the subsections to follow, we will consider the potential of accommodating neutrino oscillation experiments in our simple model. If the LSND signal does exist at all, it must point to a much larger mass squared splitting than those in the solar and atmospheric experiments. Since the lightest active neutrino is massless in our model, it implies that the masses of the neutrinos responsible for the LSND signal are much larger than those of mainly active neutrinos involved in the solar and atmospheric experiments. It is thus adequate to work in the see-saw limit ($r_1, r_2 \gg d_1, d_2, |z|$) to simplify our numerical analysis. We will also consider a special case with $z = 0$ in which analytic results have been readily worked out in the last section.

### 4.1 See-saw case: $d_1, d_2, |z| \ll r_1, r_2$

#### 4.1.1 Including LSND and negative SBL results

A difficulty to accommodate all oscillation results by adding sterile neutrinos is the tension existing between the negative and positive SBL experiments themselves. The statistical compatibility of the two sides has been analyzed in Ref. [16] amongst others [27]. It
was found in Ref. [16] that the SBL results can be significantly better reconciled with two sterile neutrinos than with a single one. Assuming the SBL compatibility and CP conservation, they found the following best-fit point with two sterile neutrinos, in our notation,

\[ (1) \quad m_1^2 = 0.92 \text{ eV}^2, \quad m_2^2 = 22 \text{ eV}^2; \]
\[ |V_{e1}^C| = 0.121, \quad |V_{\mu 1}^C| = 0.204, \quad |V_{e5}^C| = 0.036, \quad |V_{\mu 5}^C| = 0.224 \]  \tag{37}

Restricting the neutrino masses to sub-eV, the best-fit point moves to

\[ (2) \quad m_1^2 = 0.46 \text{ eV}^2, \quad m_2^2 = 0.89 \text{ eV}^2; \]
\[ |V_{e4}^C| = 0.090, \quad |V_{\mu 4}^C| = 0.226, \quad |V_{e5}^C| = 0.125, \quad |V_{\mu 5}^C| = 0.160 \]  \tag{38}

The above results were obtained for a general model of three active and two sterile neutrinos, in particular without parametric constraints on the mixing matrix \( V^C \). In our minimal model, the independent physical parameters in \( V^C \) are much less than the general case as detailed in the last section; and in addition, \( V^C \) involves the neutrino mass ratios. It is thus interesting to ask whether the above best-fit points can be accommodated with the minimal number of physical parameters in the model.

With two sterile neutrinos, there are four mixing angles in \( V^C = V y^* \), three of them from \( V \) and one from \( y^* \). Assuming CP conservation, \( z \) and \( V \) become real so that no phase can appear. Although the second and third columns of \( y^* \) are pure imaginary, their \( i \) factors have no effect on oscillations. Since the above best-fit points fall in the see-saw limit, it is good enough to identify \( m_4 \approx r_1, m_5 \approx r_2 \) without loss of generality. For the solar and atmospheric mass gaps that enter our analysis through the matrix \( y^* \), we use

\[ \Delta m^2_\odot = 8 \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{\text{ATM}} = 2.4 \times 10^{-3} \text{ eV}^2 \]  \tag{39}

A slight deviation from those numbers will not change our qualitative conclusion. Since \( m_3 > m_2 > m_1 = 0 \), we must have \( m_3^2 \approx \Delta m^2_{\text{ATM}} \) while there are two possible ways to arrange for the solar gap: either (a) \( m_2^2 \approx \Delta m^2_\odot \) or (b) \( \Delta m^2_{23} \approx \Delta m^2_\odot \). Combining with the two best-fit points shown in eqns. (37, 38), we have four schemes to consider, named below as (1a), (1b), etc.

We find that schemes (1a,1b,2b) can be excluded without using any constraints from the solar and atmospheric mixing angles. Take scheme (1a) for instance. The best-fit values for the matrix entries in eqn. (37) translate into

\[ |V_{ee}| \frac{d_{11}}{r_1} = 0.121, \quad |V_{\mu r}| \frac{d_{12}}{r_1} = 0.204, \quad |V_{e2} \frac{d_{21}}{r_2} + V_{\mu 2} \frac{d_{22}}{r_2}| = 0.036, \quad |V_{\mu \mu} \frac{d_{13}}{r_1} + V_{\mu \tau} \frac{d_{14}}{r_1}| = 0.224 \]  \tag{40}

The first two give \( \frac{d_{11}}{r_1} \geq \sqrt{0.121^2 + 0.204^2} = 0.237. \) Using eqn. (24) we have the bound, \( z^2 \leq (\sqrt{m_2} - \sqrt{m_3})^2 \), which implies then \( \frac{|z|}{r_2} \leq \frac{1}{\sqrt{m_2 - m_3}} = 5.85 \times 10^{-2}. \) Now we multiply the last one in eqn. (40) by \( \frac{d_{11}}{r_1} \), use the second and \( \frac{d_{12}d_{21}}{r_1r_2} = \frac{m_{\mu\mu}}{m_{\mu\mu}m_{\tau\tau}} = 0.987 \times 10^{-2} \) to arrive at the following result

\[ 0.987 \times 10^{-2} V_{\mu \mu} \pm 0.204 \frac{|z|}{r_2} \frac{d_{11}}{r_1} = 0.224 \]  \tag{41}
where the sign in the second term refers to $V_{\mu \tau \bar{z}}$. Even if the second term takes its largest magnitude of $1.20 \times 10^{-2}$ and adds to the first one, it still requires $|V_{\mu \mu}| \gg 1$ to reach the lowest value of the right-hand side of $5.31 \times 10^{-2}$. This scheme is thus excluded, and similarly with the schemes (1b,2b).

For the scheme (2a), we have to appeal to a constraint from the solar mixing angle. The best-fit matrix entries correspond to

$$
|V_{ee}| \frac{d_1}{r_1} = 0.090, \quad |V_{\mu r}| \frac{d_1}{r_1} = 0.226,
$$

$$
|V_{\mu \mu} \frac{d_2}{r_2} + V_{\mu r} \frac{z}{r_2}| = 0.125, \quad |V_{\mu \mu} \frac{d_2}{r_2} + V_{\mu r} \frac{z}{r_2}| = 0.160
$$

The third one gives

$$
|0.0262 V_{ee} \pm 0.090 \frac{|z|}{r_2}| = 0.125 \frac{d_1}{r_1} \quad \text{(42)}
$$

where $\frac{d_1}{r_1} \geq 0.243$, $\frac{|z|}{r_2} \leq 0.131$ and we have used $\frac{d_1 d_{1r}}{r_1 r_2} = 2.62 \times 10^{-2}$. Since the best-fit solar angle $\theta_\odot$ is about $32^\circ$, using $V_{ee}^C = V_{ee} \approx \sqrt{1 - \sin^2 \theta_\odot}$ we have $|V_{ee}| \leq s_\odot = \sin \theta_\odot \approx 0.53$. Then the left-hand side in the above equation cannot exceed $(1.39 + 1.18) \times 10^{-2}$, significantly deviating from the lowest value of the right-hand side, $3.04 \times 10^{-2}$. The difficulty can be better viewed the other way around. Assuming $V_{ee}^2 + V_{ee}^2 = s_\odot^2$ and using

$$
\frac{d_2^2 + z^2}{r_2^2} = \frac{m_2 + m_3}{m_5} - \left(\frac{d_1}{r_1}\right)^2 \frac{m_4}{m_5} \leq 1.896 \times 10^{-2} \quad \text{(44)}
$$

the LHS of the third equality in eqn. (42) cannot exceed the value, $\sqrt{V_{ee}^2 + V_{ee}^2} \frac{d_2^2 + z^2}{r_2^2} \leq 0.138 s_\odot$. Then, the equality requires that $s_\odot \geq \frac{0.125}{0.138} = 0.908$, or $\theta_\odot \geq 65^\circ$, which deviates far away from the best-fit solar angle.

### 4.1.2 Excluding negative SBL results

Assuming the positive and negative SBL experimental results are compatible, we have seen in the above that the best-fit points obtained in Ref. [16] cannot be accommodated in the see-saw parameter region of the minimally extended model which has much less physical parameters than a general nonrenormalizable model. The deviations are so large that we suspect it may be difficult to find a parameter region to encompass all SBL results that is consistent with other oscillation experiments. Although this issue should better be assessed by an extensive statistical analysis, it goes beyond the scope of this work. Rather, we would ask the question: since it is generally difficult to relax the tension in SBL experiments, what will happen if we take seriously the LSND signal but ignore the negative SBL results?

As the mass gap required by LSND is much higher than those by other experiments, a simple but plausible way to answer the question is to take the best-fit values for the latter and consider whether there is any room for the LSND signal. Since only LSND hints at a third mass gap besides the known solar and atmospheric ones, it is natural to
simplify our analysis further by arranging either $r_1 \ll r_2$ or $r_1 \approx r_2$. In the first case, the neutrino $\nu_5$ essentially decouples whose only effect is to provide a small mass to $\nu_2$. The decoupling is verified by the small entries in the 5th column of $V^C$. In the second case, $\nu_{4,5}$ effectively form a Dirac neutrino when $m_4 = m_5$ holds exactly. Even if they have a gap that happens to be a solar or an atmospheric one, their contributions to other experiments can still be ignored because of the smallness of the relevant mixing entries.

We assume CP conservation so that $V$ and $z$ are real. For definiteness, we assume $z > 0$ so that $\rho_+ > 0, \rho_- = -\rho_+^1 < 0$. The conclusion does not change for $z < 0$. For oscillations, we can also ignore $i$ factors in the 2nd and 3rd columns of $V^C$. Then all matrix entries are effectively real. The oscillation amplitude responsible for the LSND signal appropriate for both cases of $r_1 \ll r_2$ and $r_1 \approx r_2$ is

$$A_{\text{LSND}} = 4|V^{C*}_{\mu 4}V_{e4} + V^{C*}_{\mu 5}V_{e5}|^2$$

$$= 4 \left[ \left( \frac{d_1^2}{r_1^2} + \frac{z^2}{r_2^2} \right) V_{e\nu}V_{\mu \nu} + \frac{d_2^2}{r_2^2} V_{e\mu}V_{\mu \nu} + \frac{d_2 z}{r_2^2} (V_{e\nu}V_{\mu \nu} + V_{e\mu}V_{\nu \mu}) \right]^2$$ \hspace{1cm} (45)

which involves the upper-right $2 \times 2$ submatrix in $V$. To determine it, we apply the best-fit point ($\theta_\odot = 32^\circ, \theta_{\text{ATM}} = 45^\circ, \theta_{\text{Chooz}} = 0$) to the relevant $2 \times 2$ submatrix in $V^C$:

$$V^C_{\nu 2} = \frac{V_{e\nu}V_{\mu \nu} - V_{e\mu}V_{\nu \mu}}{\sqrt{1 + \rho_+^2}} = s_\odot, \quad V^C_{\nu 3} = \frac{V_{e\nu}V_{\mu \nu} + V_{e\mu}V_{\nu \mu}}{\sqrt{1 + \rho_+^2}} = 0,$$

$$V^C_{\mu 2} = \frac{V_{\mu \nu}V_{\mu \nu} - V_{e\mu}V_{\nu \mu}}{\sqrt{1 + \rho_+^2}} = c_\odot, \quad V^C_{\mu 3} = \frac{V_{\mu \nu}V_{\mu \nu} + V_{e\mu}V_{\nu \mu}}{\sqrt{1 + \rho_+^2}} = 1$$ \hspace{1cm} (46)

which solves the $V$ entries in terms of $\theta_\odot$ and $\rho_+$. The amplitude becomes

$$A_{\text{LSND}} = 2s_\odot^2 \left[ \frac{d_1^2}{r_1^2} + \frac{z^2}{r_2^2} \frac{\rho_+ (1 + c_\odot \rho_+)}{1 + \rho_+^2} + \frac{d_2^2}{r_2^2} c_\odot - \rho_+ \frac{d_2 z}{r_2^2} + \frac{\rho_+^2 - 2 c_\odot \rho_+ - 1}{1 + \rho_+^2} \right]$$ \hspace{1cm} (47)

When $r_1 \ll r_2$, we have approximately

$$m_2 \approx \frac{d_1^2}{r_2}, \quad m_3 \approx \frac{d_1^2}{r_1}, \quad m_4 \approx r_1, \quad \rho_+ \approx \frac{d_2 z}{d_1} \frac{r_1}{r_2} \ll 1$$ \hspace{1cm} (48)

The amplitude is dominated by the first term

$$A_{\text{LSND}} \approx 2s_\odot^2 \frac{d_1^4}{r_1^2} \rho_+^2 \approx 2s_\odot^2 \frac{d_2^2 z^2}{r_1^2 r_2^2} \approx 2s_\odot^2 \frac{m_2^2 z^2}{r_2^2} \frac{z^2}{r_1^2}$$ \hspace{1cm} (49)

which is very small because $z \ll r_1 (\sim 1 \text{ eV } ) \ll r_2$ and $m_2 = \sqrt{\Delta m_{\odot}^2} < 0.01 \text{ eV }$. For an order of magnitude estimate, we note that $z$ should be about the same order of $d_{1,2}$. Then, the order of $A_{\text{LSND}}$ should be roughly from $2s_\odot^2 \frac{\Delta m_{\odot}^2}{\Delta m_{\text{LSND}} r_2^2} \approx 4 \times 10^{-5}$ to $2s_\odot^2 \sqrt{\frac{\Delta m_{\odot}^2 \Delta m_{\odot}^3}{\Delta m_{\text{LSND}}^2}} \ll 2 \times 10^{-4}$, which is well below the required LSND level of $\sim 3 \times 10^{-3}$.

For $m_4 \approx r_1 \approx r_2 \approx \sqrt{\Delta m_{\text{LSND}}^2}$, we have the relations

$$\rho_+^{-1} - \rho_+ \approx \frac{d_1^2 - d_2^2 + z^2}{d_2 z}, \quad m_2 m_4 \approx \frac{(d_1^2 + z^2) \rho_+^2 + d_2^2 - 2d_2 z \rho_+}{1 + \rho_+^2}$$ \hspace{1cm} (50)
which, after some algebra, simplify considerably the amplitude to

\[ A_{\text{LSND}} \approx 2c_\odot^2 s_\odot^2 \frac{m_2^2}{m_4^2} \approx 2c_\odot^2 s_\odot^2 \frac{\Delta m_\odot^2}{\Delta m_{\text{LSND}}} \sim 3 \times 10^{-5} \]  

(51)

which is again far below the desired level.

In summary, even if we ignore the negative SBL results to avoid the potential incompatibility in SBL experiments, we cannot naturally accommodate simultaneously the results of solar, atmospheric, reactor and accelerator experiments including LSND in this model with see-saw parameters \( d_1, d_2, |z| \ll r_1, r_2 \).

4.2 Case: \( z = 0 \)

As we argued in the above, the see-saw hierarchy \( r_{1,2} \gg d_{1,2}, |z| \) is natural both theoretically and phenomenologically. The conclusion reached in the last subsection is thus quite general. Nevertheless, as a good illustration we consider below a special case where all relevant quantities can be calculated without working perturbatively. Since

\[ \prod_{j=2}^{5} m_j^2 = \text{det}(m_n^1 m_n) = (d_1 d_2)^4, \]

neither of \( d_1, d_2 \) can vanish. We therefore study the case of \( z = 0 \) in a way parallel to the last subsection.

4.2.1 Including LSND and negative SBL results

Although \( m_3 < m_4, m_2 < m_5 \) but otherwise arbitrary, theoretical and phenomenological considerations require that \( r_{1,2} \gg d_{1,2} \). Thus we generally have \( m_{4,5} \gg m_{2,3} \). As the problem is symmetric under the simultaneous interchanges \( \nu_2 \leftrightarrow \nu_3 \) and \( \nu_4 \leftrightarrow \nu_5 \), we can choose without loss of generality \( m_5 \geq m_4 \). Since the relative ordering of \( m_{2,3} \) is still free, we have four schemes to arrange for the solar and atmospheric mass gaps:

\[
\begin{align*}
\text{(S1)} & \quad m_2^2 = \Delta m_{\odot}^2, \quad m_3^2 = \Delta m_{\text{ATM}}^2; \\
\text{(S2)} & \quad m_{22}^2 = \Delta m_{\odot}^2, \quad m_3^2 = \Delta m_{\text{ATM}}^2; \\
\text{(S3)} & \quad m_2^2 = \Delta m_{\odot}^2, \quad m_{23}^2 = \Delta m_{\text{ATM}}^2; \\
\text{(S4)} & \quad m_{23}^2 = \Delta m_{\odot}^2, \quad m_2^2 = \Delta m_{\text{ATM}}^2.
\end{align*}
\]  

(52)

The parameters \( c_j, s_j \) are collected in Table 2 that are computed for the best-fit values shown in eqn. (39) and the best-fit values \( m_2^2, m_3^2 \) of Ref. [16] shown in eqs. (37, 38). A slight shift of the best-fit values does not change much those parameters because of the double square root dependence, and in particular our qualitative conclusion below is rather stable.

It is clear from the table that it is far from possible to get close to the large best-fit values for \( |V_{\mu\tau}^C|, |V_{\mu\mu}^C| \) in eqn. (37). For the sub-eV fit, schemes (S2,S4) seem to have a chance, so let us take a closer look. We find \( |V_{ee}| \approx 0.35, \quad |V_{e\mu}| \approx 0.56, \quad |V_{\mu\tau}| \approx 0.87, \quad |V_{\mu\mu}| \approx 0.73 \). The unitarity of \( V \) is violated by \( |V_{\mu\tau}|^2 + |V_{\mu\mu}|^2 \approx 1.3 > 1 \). More seriously, large \( |V_{\mu\tau}| \) and \( |V_{\mu\mu}| \) imply a very small \( |V_{\mu\tau}| = |V_{\mu\mu}^C| \) which cannot be tolerated.
by the data of $\Delta m^2_{\text{ATM}}$ oscillations; the circumstance with $|V_{e\tau}|$ and $|V_{\mu\tau}|$ is conflicting; while their intermediate size implies a too large $|V_{e\ell}|$ that is excluded by Chooz, they are not large enough to explain $\Delta m^2_{\odot}$ oscillation data. We can thus safely conclude that the best-fit points of Ref. [16] cannot be realized in the current model.

### 4.2.2 Excluding negative SBL results

Now we ignore the negative SBL experiments and consider the consistency to accommodate all other experiments in the current model. It is sufficient to restrict ourselves to those that can be reasonably well described by vacuum oscillations, i.e., KamLAND, K2K, Chooz, and LSND that cover all three mass gaps. We take as our input the mass gaps shown in eqn. (39) and $\Delta m^2_{\text{LSND}} \approx 1.2 \text{eV}^2$, and study the possibility to obtain desired oscillation amplitudes for the mentioned experiments. Again we consider the natural case $m_{4,5} \gg m_{2,3}$ so that $\nu_{4,5}$ are mostly sterile and $\nu_{2,3}$ mostly active, and simplify the matter further by working with the ‘only three gaps’, i.e., either $m_4^2 = m_5^2 = \Delta m^2_{\text{LSND}}$ or $m_5 \gg m_4 = \Delta m^2_{\text{LSND}}$. The masses $m_{2,3}$ are classified into the four schemes shown in eqn. (52).

For $m_4 = m_5$, schemes (S1) and (S3) are equivalent because they are related by a renumbering of indices; similarly with (S2) and (S4). Thus we only need to consider the schemes (S1,S2) below. The masses yield

$$\begin{align*}
\text{(S1)}: \quad & c_1 \approx 0.21, \quad s_1 \approx 0.98, \quad c_2 \approx 0.09, \quad s_2 \approx 1.0 \\
\text{(S2)}: \quad & c = c_1 \approx c_2 \approx 0.21, \quad s = s_1 \approx s_2 \approx 0.98
\end{align*}$$

The oscillation amplitudes that are multiplied to the oscillating factors are, in scheme (S1):

$$\begin{align*}
\text{LSND} : & \quad 4|V_{\mu}^C e^C + V_{\mu}^C e^C|^2 \approx 4|V_{e\mu}^C|^2 (1 - |V_{e\mu}|^2) s_1^2 \\
\text{Chooz} : & \quad 4|V_{e\mu}^C|^2 (|V_{e\mu}^C|^2 + |V_{e\mu}^C|^2) \approx 4|V_{e\mu}^C|^2 (1 - |V_{e\mu}|^2) s_1^2 \\
\text{K2K} : & \quad 4|V_{e\mu}^C|^2 (|V_{e\mu}^C|^2 + |V_{e\mu}^C|^2) \approx 4|V_{e\mu}^C|^2 (1 - |V_{e\mu}|^2) s_1^2 \\
\text{KamLAND} : & \quad 4|V_{e\mu}^C|^2 |V_{e\mu}^C|^2 \approx 4|V_{e\mu}^C|^2 |V_{e\mu}^C|^2
\end{align*}$$

---

Table 2: The parameters $c_j, s_j$ are computed for the best-fit values $\Delta m^2_{\odot} = 8 \times 10^{-5} \text{eV}^2$, $\Delta m^2_{\text{ATM}} = 2.4 \times 10^{-3} \text{eV}^2$ and the best-fit values $m^2_4, m^2_5$ shown in eqs. (37, 38), corresponding to the left and right part of the table.

| Schemes | $c_1$ | $s_1$ | $c_2$ | $s_2$ | $c_1$ | $s_1$ | $c_2$ | $s_2$ |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|
| (S1)    | 0.22  | 0.98  | 0.04  | 1.00  | 0.26  | 0.97  | 0.10  | 0.99  |
| (S2, S4)| 0.22  | 0.98  | 0.10  | 0.99  | 0.26  | 0.97  | 0.22  | 0.98  |
| (S3)    | 0.10  | 0.99  | 0.10  | 0.99  | 0.10  | 0.99  | 0.22  | 0.98  |
and in scheme (S2):

\[
\begin{align*}
\text{LSND} & : 4|V_{\mu 4}^C V_{e 4}^C + V_{\mu 5}^C V_{e 5}^C|^2 \approx 4|V_{ee} V_{\mu e}|^2 c^4 \\
\text{Chooz} & : 4(|V_{e 2}^C|^2 + |V_{e 3}^C|^2)|V_{C e}^C|^2 \approx 4|V_{ee}|^2 (1 - |V_{ee}|^2) s^2 \\
\text{K2K} & : 4(|V_{\mu 2}^C|^2 + |V_{\mu 3}^C|^2)|V_{C \mu}^C|^2 \approx 4|V_{\mu e}|^2 (1 - |V_{\mu e}|^2) s^2 \\
\text{KamLAND} & : 4|V_{e 2}^C|^2 |V_{e 3}^C|^2 \approx 4|V_{e \mu} V_{e \tau}|^2 s^4
\end{align*}
\]

Consider scheme (S1) first. Positive K2K and negative Chooz results imply \( |V_{\mu \tau}|^2 \sim |V_{\tau \tau}|^2 \sim \frac{1}{2} \) and tiny \( |V_{e \tau}|^2 \). The order \( 10^{-3} \) exclusion level at Chooz restricts the \( c_2^2 \) term for LSND to be less than \( \sim 10^{-6} \), thus subdominant to the \( c_1^2 \) term, which however cannot exceed \( c_1^2 \sim 6.6 \times 10^{-5} \), well below the desired LSND signal. Scheme (S2) is not better. K2K and Chooz imply \( |V_{\mu e}|^2 \sim \frac{1}{2} \) and tiny \( |V_{ee}|^2 \). But in that case, the LSND result should be more negative than the Chooz because the ratio of their amplitudes is, \( \frac{\text{LSND}}{\text{Chooz}} \approx \frac{c_4^2}{s^2} |V_{\mu e}|^2 \sim 10^{-3} \), in sharp conflict with the observations.

When \( m_5 \gg m_4 \gg m_{2,3} \), we have \( c_2 \approx 0 \), \( s_2 \approx 1 \) and \( v_5 \) decouples from the oscillation. Since \( m_2 \approx \frac{d_2^2}{r_2^2} \), it is more appropriate to assume \( m_2^2 = \Delta m_\odot^2 \) than any other arrangements. Thus we only have to consider the scheme (S1). This is just a special case of the above discussions with the difference that the LSND signal becomes more difficult to accommodate since it arises from the first term that was ignored in the above.

5 Conclusion

If the LSND experiment is confirmed by MiniBooNE, the scenario of three ordinary active neutrinos will be insufficient to explain all neutrino oscillation data. A natural approach to the problem is to add sterile neutrinos to allow for more independent mass squared differences that could potentially accommodate three well-separated mass gaps. We have investigated systematically the simplest type of such models where the only extension to SM is the addition of sterile neutrinos. In particular, we do not introduce extra Higgs multiplets or higher dimensional effective operators to induce masses for the active neutrinos, and the extended model keeps renormalizable as SM. We found that for any numbers of generations and sterile neutrinos its leptonic sector contains much less physical parameters than previously expected, upon exploiting completely the texture zero in the neutrino mass matrix. The model thus becomes quite viable for phenomenological analysis even if it contains two or three sterile neutrinos. We demonstrated that the mixing matrix in the leptonic charged current interactions has a factorized form and that the factor containing the neutrino mass ratios makes the neutral current interactions of neutrinos non-diagonal as well.

We have studied the phenomenological feasibility to accommodate all oscillation results in the extended model with two sterile neutrinos. We have restricted ourselves in this work to the see-saw region of the parameter space which is most natural considering the experimentally found well-separated mass gaps and for which all relevant results can be worked out analytically. We have used as our reference points the best-fit values of
Ref. [16] based on the analysis of the complete SBL data. Unfortunately, the answer is negative, and even more: even if we take the LSND result seriously but ignore other null SBL results, it is difficult to accommodate the best-fit values for the solar (plus KamLAND), atmospheric (plus K2K), Chooz and LSND results in the see-saw parameter region. A slight shift of the best-fit values does not alter our qualitative conclusion. We attribute this to a new feature of the minimally extended model exposed here: the leptonic charged current mixing matrix is strongly modified by mass ratios of neutrinos. Adding more free mixing angles does not necessarily improve the simultaneous fitting of masses and mixing angles. If the LSND signal is confirmed by MiniBooNE, the see-saw region of the model with two sterile neutrinos could largely be excluded.

Finally we discuss briefly how the current work could be extended. Although the see-saw parameters are most natural from theoretical point of view, it is highly desirable to make a complete scanning of the parameter space. Since the mainly sterile neutrinos are generally light or at least not very heavy in the model, fitting data in regions other than the see-saw one may not cause too serious fine-tuning of parameters. Suppose the LSND result will be confirmed by MiniBooNE, a negative result of the scanning would rule out the simplest extension of SM in its leptonic sector and call for something really new to SM.

References

[1] For reviews on neutrino oscillations, see e.g.: S.M. Bilenky, S.T. Petcov, Rev. Mod. Phys. 59(1987)671; T.K. Kuo, J. Pantaleone, Rev. Mod. Phys. 61(1989)937; M.C. Gonzalez-Garcia, Y. Nir, Rev. Mod. Phys. 75 (2003) 345; M. Maltoni, T. Schwetz, M.A. Tortola, J.W.F. Valle, New J. Phys. 6 (2004) 122.

[2] B.T. Cleveland et al., Astrophys. J. 496(1998)505; SAGE Collaboration, J.N. Abdurashitov et al., Phys. Rev. C60(1999)055801; GALLEX Collaboration, W. Hampel et al., Phys. Lett. B447(1999)127; SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 87(2001)071301; Phys. Rev. Lett. 89(2002)011301; S.N. Ahmed et al., Phys. Rev. Lett. 92(2004)181301; Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Lett. B539(2002)179.

[3] Kamiokande Collaboration, S.H. Hirata et al., Phys. Lett. B280(1992)146; Y. Fukuda et al., Phys. Lett. B335(1994)237; Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Rev. Lett. 81(1998)1562; Soudan Collaboration, W.W.M. Allison et al., Phys. Lett. B449(1999)137; MACRO Collaboration, M. Ambrosio et al., Phys. Lett. B517(2001)59.

[4] B. Pontecorvo, Sov. Phys. JETP 6(1958)429; Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28(1962)870; L. Wolfenstein, Phys. Rev. D 17(1978)2369; S.P. Mikheyev, A.Yu. Smirnov, Sov. J. Nucl. Phys. 42(1985)913.
[5] K2K Collaboration, M.H. Ahn et al., Phys. Rev. Lett. 90(2003)041801.

[6] KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. 90 (2003) 021802; T. Araki et al., Phys. Rev. Lett. 94(2005)081801.

[7] LSND Collaboration, C. Athanassopoulos et al., Phys. Rev. Lett. 75(1995)2650; Phys. Rev. Lett. 77(1996)3082; Phys. Rev. Lett. 81(1998)1774; A. Aguilar et al., Phys. Rev. D64(2001)112007.

[8] Bugey Collaboration, Achkar et al., Nucl. Phys. B434(1995)503; CCFR Collaboration, Romosan et al., Phys. Rev. Lett. 78(1997)2912; CDHS Collaboration, F. Dydak et al., Phys. Lett. 134B(1984)281; KARMEN Collaboration, B. Armbruster et al., Phys. Rev. D65(2002)112001; NOMAD Collaboration, P. Astier et al., Phys. Lett. B570(2003)19; V. V. V. Valuev et al., J. High Energy Phys., Conference Proceedings, PRHEP-hep2001/190.

[9] Chooz Collaboration, M. Apollonio et al., Phys. Lett. B420(1998)397; Phys. Lett. B466(1999)415; Eur. Phys. J. C27(2003)331.

[10] BooNE Collaboration: E.D. Zimmerman et al., hep-ex/0211039; M.H. Shaevitz, Nucl. Phys. Proc. Suppl. 145(2005)208.

[11] H. Murayama, T. Yanagida, Phys. Lett. B520(2001)263; G. Barenboim, L. Borissov, J.D. Lykken, A.Y. Smirnov, J. High Energy Phys. 0210(2002)001; A. Strumia, Phys. Lett. B539(2002)91; G. Barenboim, L. Borissov, J. Lykken, hep-ph/0212116; M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz, Phys. Rev. D68(2003)053007; V. Barger, D. Marfatia, K. Whisnant, Phys. Lett. B576(2003)303.

[12] G. Barenboim, N.E. Mavromatos, J. High Energy Phys. 0501(2005)034; G. Barenboim, N.E. Mavromatos, S. Sarkar, A. Waldron-Lauda, hep-ph/0603028.

[13] H. Pas, S. Pakvasa, T.J. Weiler, Phys. Rev. D72(2005)095017.

[14] J.T. Peltoniemi, D. Tommasini, J.W.F. Valle, Phys. Lett. B298(1993)383; D.O. Caldwell, R.N. Mohapatra, Phys. Rev. D48(1993)3259; Z.G. Berezhiani, R.N. Mohapatra, Phys. Rev. D52(1995)6607; R. Foot, R.R. Volkas, Phys. Rev. D52 (1995)6595; D. Suematsu, Phys. Lett. B 392(1997)413.

[15] J.J. Gomez-Cadenas, M.C. Gonzalez-Garcia, Z. Phys. C71(1996)443; S. Goswami, Phys. Rev. D55(1997)2931; S.M. Bilenky, C. Giunti, W. Grimus, Eur. Phys. J. C1(1998)247; V.D. Barger et al., Phys. Lett. B489(2000)345; O.L.G. Peres, A.Y. Smirnov, Nucl. Phys. B599(2001)3; A. Strumia, cited in Ref.[11]; W. Grimus, T. Schwetz, Eur. Phys. J. C20(2001)1; J.M. Gonzalez-Garcia, M. Maltoni, C. Pena-Garay, Phys. Rev. D64(2001)093001; C. Giunti, M. Laveder, J. High Energy Phys. 0102(2001)1; M. Maltoni, T. Schwetz, J.W.F. Valle, Phys. Rev. D 65(2002)093004.
and its update cited in Ref. [1]; M. Maltoni, T. Schwetz, M.A. Tortola, J.W.F. Valle, Nucl. Phys. B643(2002)321; C. Giunti, Mod. Phys. Lett. A18(2003)1179; P.C. de Holanda, A.Y. Smirnov, Phys. Rev. D69(2004)113002; M. Cirelli et al., Nucl. Phys. B708(2005)215; A.Y. Smirnov, R. Zukanovich Funchal, hep-ph/0603009; S. Goswami, W. Rodejohann, hep-ph/0512234.

[16] M. Sorel, J.M. Conrad, M.H. Shaevitz, Phys. Rev. D70 (2004)073004.

[17] G.J. Stephenson, T. Goldman, B.H.J. McKellar, M. Garbutt, hep-ph/0307245; K.S. Babu, G. Seidl, Phys. Lett. B591(2004)127; K.L. McDonald, B.H.J. McKellar, A. Mastrano, Phys. Rev. D70(2004)053012; W. Krolikowski, Acta Phys. Polon. B35(2004)1675; hep-ph/0506099; R.N. Mohapatra, S. Nasri, H.B. Yu, Phys. Rev. D72(2005)033007; J. Sayre, S. Wiesenfeldt, S. Willenbrock, Phys. Rev. D72(2005)015001; A.G. Dias, C.A. de S. Pires, P.S. Rodrigues da Silva, Phys. Lett. B628(2005)85; W. Krolikowski, Acta Phys. Polon. B35(2005)2241; A. de Gouvea, Phys. Rev. D72(2005)033005.

[18] Y. Liao, hep-ph/0504018.

[19] J. Schechter, J.W.F. Valle, Phys. Rev. D 22 (1980) 2227.

[20] J. Schechter, J.W.F. Valle, Phys. Rev. D 21 (1980) 309; see also, J.F. Donoghue, Phys. Rev. D 18 (1978) 1632.

[21] F. del Aguila, J. Gluza, M. Zralek, Acta Phys. Polon. B 30 (1999) 3139.

[22] M. Gell-Mann, P. Ramond, R. Slansky, in Supergravity (Ed., D. Freedman, P. van Nieuwenhuizen) (North-Holland, Amsterdam, 1979), p.315; T. Yanagida, in Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe (Ed. O. Sawada, A. Sugamoto) (KEK, Japan, 1979); R.N. Mohapatra, G. Senjanovic, Phys. Rev. Lett. 44 (1980) 912.

[23] A. Pilaftsis, Z. Phys. C55(1992)275; B.A. Kniehl, A. Pilaftsis, Nucl. Phys. B474(1996)286.

[24] B. Kayser, A.S. Goldhaber, Phys. Rev. D28(1983)2341; B. Kayser, Phys. Rev. D30(1984)1023; S.M. Bilenky, N.P. Nedelcheva, S.T. Petcov, Nucl. Phys. B247(1984)61.

[25] C. Kraus et al., Eur. Phys. J. C40(2005)447.

[26] H.V. Klapdor-Kleingrothaus et al., Eur. Phys. J. A12(2001)147; IGEX Collaboration, C.E. Aalseth et al., Phys. Rev. D65(2002)092007.

[27] K. Eitel, New J. Phys. 2(2000)1; E.D. Church, K. Eitel, G.B. Mills, M. Steidl, Phys. Rev. D66(2002)013001.