Abstract. This paper is the continuation of the work in [14]. In that paper we generalized the definition of $W$-graph ideal in the weighted Coxeter groups, and showed how to construct a $W$-graph from a given $W$-graph ideal in the case of unequal parameters.

In this paper we study the full $W$-graphs for a given $W$-graph ideal. We show that there exist a pair of dual modules associated with a given $W$-graph ideal, they are connected by a duality map. For each of the dual modules, the associated full $W$-graphs can be constructed. Our construction closely parallels that of Kazhdan and Lusztig [6, 10, 11], which can be regarded as the special case $J = \emptyset$. It also generalizes the work of Couillens [2], Deodhar [3, 4], and Douglass [5], corresponding to the parabolic cases.

Introduction

Let $(W, S)$ be a Coxeter system and $\mathcal{H}(W)$ its Hecke algebra over $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in the indeterminate $q$. This is now called the one parameter case (or the equal parameter case). In [9] Howlett and Nguyen introduced the concept of a $W$-graph ideal in $(W, \leq_L)$ with respect to a subset $J$ of $S$, where $\leq_L$ is the left weak Bruhat order on $W$. They showed that a $W$-graph can be constructed from a given $W$-ideal, and a Kazhdan-Lusztig like algorithm was obtained.

In particular, $W$ itself is a $W$-graph ideal with respect to $\emptyset$, and the $W$-graph obtained is the Kazhdan-Lusztig $W$-graph for the regular representation of $\mathcal{H}(W)$ (as defined in [10]). More generally, it was shown that if $J$ is an arbitrary subset of $S$ then $D_J$, the set of distinguished left coset representatives of $W_J$ in $W$, is a $W$-graph ideal with respect to $J$ and also with respect to $\emptyset$, and Deodhars parabolic analogues of the Kazhdan-Lusztig construction are recovered.

In [14] we generalized the definition of $W$-graph ideal in the Coxeter groups with a weight function $L$, we showed that the $W$-graph can also be constructed from a given $W$-graph ideal.

In this paper we continue the work in [14], it grows out of our attempt to understand the "full $W$-graphs" that include $W$-graphs and their dual ones. The duality has appeared in some literatures, for instance, in the original paper [6] Kazhdan and Lusztig implicitly provided a pair of dual bases $C$ and $C'$ for the Hecke algebras, Deodhar introduced a pair of dual modules $M^J$ and $\tilde{M}^J$ in parabolic cases (see [3, 4]).

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The paper is organised as follows. In Section 1 we present some basic concepts and facts concerning the weighted Coxeter groups, Hecke algebras and \( W \)-graphs. In Section 2, we recall the concept of \( W \)-graph ideal. In Section 3, we show that there exist a pair of dual modules \( M(E_f,L) \) and \( \tilde{M}(E_f,L) \) that are associated with a given \( W \)-graph ideal \( E_f \), they are connected by a duality map, this in turn can be used for the construction of the dual bases of the \( W \)-graphs. This construction closely parallels the work of Deodhar [3, 4], Douglass [5], where they focused primarily upon the parabolic cases.

In Section 4 we prove in general the construction of another pair of dual \( W \)-graph bases. This part is motivated by Lusztig's work [11] Ch. 10], the construction is obtained by using the bases of \( \mathcal{H} \)-modules \( \text{Hom}_A(M,A) \) and \( \text{Hom}_A(\tilde{M},A) \).

In Section 5, in the case \( W \) is finite we prove an inversion formula that relates the two versions of the relative Kazhdan-Lusztig polynomials, . In the last section we give some examples and remarks.

1. Preliminaries

Let \( W \) be a Coxeter group, with generating set \( S \). In this section, we briefly recall some basic concepts concerning the general multi-parameter framework of Lusztig [10, 11], which introduces a weight function into Coxeter groups and their associated Hecke algebras on which all the subsequent constructions depend.

We denote by \( \ell : W \to \mathbb{N} = \{0, 1, 2, \ldots \} \) the length function on \( W \) with respect to \( S \). Let \( \leq \) denote the Bruhat order on \( W \).

Let \( \Gamma \) be the totally ordered abelian group which will be denoted additively, the order on \( \Gamma \) will be denoted by \( \leq \). Let \( \{L(s) \mid s \in S\} \subseteq \Gamma \) be a collection of elements such that \( L(s) = L(t) \) whenever \( s, t \in S \) are conjugate in \( W \). This gives rise to a weight function

\[
L : W \longrightarrow \Gamma
\]

in the sense of Lusztig [10, 11]: we have \( L(w) = L(s_1) + L(s_2) + \cdots + L(s_k) \) where \( w = s_1s_2\cdots s_k(s \in S) \) is a reduced expression for \( w \in W \). We assume throughout that

\[
L(s) \geq 0
\]

for all \( s \in S \). (If \( \Gamma = \mathbb{Z} \) and \( L(s) = 1 \) for all \( s \in S \), then this is the original "equal parameter" setting of [4].)

Let \( R \subseteq \mathbb{C} \) be a subring and \( A = R[\Gamma] \) be a free \( R \)-module with basis \( \{q^\gamma \mid \gamma \in \Gamma\} \) where \( q \) is an indeterminant. (The basic constructions in this section are independent of the choice of \( R \) and so we could just take \( R = \mathbb{Z} \).) The flexibility of \( R \) will be useful once we consider representations of \( W \). There is a well-defined ring structure on \( A \) such that \( q^\gamma q^{\gamma'} = q^{\gamma + \gamma'} \) for all \( \gamma, \gamma' \in \Gamma \). We denote \( 1 = q^0 \in A \). If \( a \in A \) we denote by \( a_\gamma \) the coefficient of \( a \) on \( q^\gamma \) so that \( a = \sum_{\gamma \in \Gamma} a_\gamma q^\gamma \). If \( a \neq 0 \) we define the degree of \( a \) as the element of \( \Gamma \) equal to

\[
\deg(a) = \max\{\gamma \mid a_\gamma \neq 0\}
\]

by convention (see [11]), we set \( \deg 0 = -\infty \). So \( \deg : A \to \Gamma \cup \{-\infty\} \) satisfies \( \deg(ab) = \deg(a) + \deg(b) \).

Let \( \mathcal{H} = \mathcal{H}(W,S,L) \) be the generic Hecke algebra corresponding to \( (W,S) \) with parameters \( \{q^{L(s)} \mid s \in S\} \). Thus \( \mathcal{H} \) has an \( A \)-basis \( \{T_w \mid w \in W\} \) and the
multiplication is given by the rules

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w) \\
T_{sw} + (q^{L(s)} - q^{-L(s)})T_w & \text{if } \ell(sw) < \ell(w),
\end{cases}
\]

Let \( \Gamma_{\geq \gamma_0} = \{ \gamma \in \Gamma \mid \gamma \geq \gamma_0 \} \) and denote by \( A_{\geq \gamma_0} \) (or \( R[\Gamma_{\geq \gamma_0}] \)) the set of all \( R \)-linear combinations of terms \( q^\gamma \) where \( \gamma \geq \gamma_0 \). The notations \( A_{\gamma > \gamma_0} \), \( A_{\gamma \leq \gamma_0} \), \( A_{\gamma < \gamma_0} \) have a similar meaning.

We denote by \( A \mapsto \overline{A} \) the automorphism of \( A \) induced by the automorphism of \( \Gamma \) sending \( \gamma \) to \(-\gamma\) for any \( \gamma \in \Gamma \). This extends to a ring involution \( \mathscr{H} \mapsto \overline{\mathscr{H}} \), \( h \mapsto \overline{h} \), where

\[
\sum_{w \in W} a_w T_w = \sum_{w \in W} a_{\overline{w}} T_{\overline{w}}^{-1}, a_w \in A \text{ for all } w \in W,
\]

and

\[
T_s = T_{s}^{-1} = T_s + (q^{-L(s)} - q^{L(s)}) \text{ for all } s \in S.
\]

**Definition of \( W \)-graph.**

**Definition 1.1.** (for equal parameter case see [6]; for general \( L \) see [7].) A \( W \)-graph for \( \mathscr{H} \) consists of the following data:

(a) a base set \( \Lambda \) together with a map \( I \) which assigns to each \( x \in \Lambda \) a subset \( I(x) \subseteq S \);

(b) for each \( s \in S \) with \( L(s) > 0 \), a collection of elements

\[
\{ \mu_{x,y}^s \mid x, y \in \Lambda \text{ such that } s \in I(x), s \notin I(y) \};
\]

(c) for each \( s \in S \) with \( L(s) = 0 \) a bijection \( \Lambda \rightarrow \Lambda, x \mapsto s\cdot x \). These data are subject to the following requirements. First we require that, for any \( x, y \in \Lambda \) and \( s \in S \) where \( \mu_{x,y}^s \) is defined, we have

\[
q^{L(s)} \mu_{x,y}^s \in R[\Gamma_{\geq 0}] \text{ and } \overline{\mu_{x,y}^s} = \mu_{\overline{x},\overline{y}}^s.
\]

Furthermore, let \([\Lambda]_A\) be a free \( A \)-module with basis \( \{ b_y \mid y \in \Lambda \} \). For \( s \in S \), define an \( A \)-linear map

\[
\rho_s(b_y) = \begin{cases} 
b_{s,y} & \text{if } L(s) = 0; 
-q^{-L(s)}b_y & \text{if } L(s) > 0, s \in I(y); 
q^{L(s)}b_y + \sum_{x \in A; s \in I(x)} \mu_{x,y}^s b_x & \text{if } L(s) > 0, s \notin I(y).
\end{cases}
\]

Then we require that the assignment \( T_s \mapsto \rho_s \) defines a representation of \( \mathscr{H} \).

**2. W-graph ideals**

For each \( J \subseteq S \), let \( \tilde{J} = S \setminus J \) (the complement of \( J \)) and define \( \mathcal{W}_J = \langle J \rangle \), the corresponding parabolic subgroup of \( W \). Let \( \mathcal{H}_J \) be the Hecke algebra associated with \( \mathcal{W}_J \). As is well known, \( \mathcal{H}_J \) can be identified with a subalgebra of \( \mathscr{H} \).

Let \( D_J = \{ w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J \} \), the set of minimal coset representatives of \( W/\mathcal{W}_J \). The following lemma is well known.
Lemma 2.1. [3] Lemma 2.1(iii)(modified) Let \( J \subseteq S \) and \( s \in S \), and define
\[
D^-_{J,s} = \{ w \in D_J \mid \ell(sw) < \ell(w) \},
D^+_{J,s} = \{ w \in D_J \mid \ell(sw) > \ell(w) \text{ and } sw \in D_J \},
D^0_{J,s} = \{ w \in D_J \mid \ell(sw) > \ell(w) \text{ and } sw \notin D_J \},
\]
so that \( D_J \) is the disjoint union \( D^-_{J,s} \cup D^+_{J,s} \cup D^0_{J,s} \). Then \( sD^+_{J,s} = D^-_{J,s} \), and if \( w \in D^0_{J,s} \) then \( sw = wt \) for some \( t \in J \).

In this section we shall recall [3] Section 5, with some modification.

Let \( \leq_L \) denote the left weak (Bruhat)order on \( W \). We say \( y \leq_L y \) if and only if \( y = zz \) for some \( z \in W \) such that \( \ell(y) = \ell(z) + \ell(x) \). We also say that \( x \) is a suffix of \( y \). The following property of the Bruhat order is useful (see [11] Corollary 2.5), for example).

Lemma 2.2. Let \( y, z \in W \) and let \( s \in S \).

(i) Assume that \( sz < z \), then \( y \leq_L z \iff sy \leq_L z \).
(ii) Assume that \( y < sy \), then \( y \leq_L z \iff y \leq_L sz \).

Definition 2.3. If \( X \subseteq W \), let \( Pos(X) = \{ s \in S \mid \ell(xs) > \ell(x) \text{ for all } x \in X \} \).

Thus \( Pos(X) \) is the largest subset \( J \) of \( S \) such that \( X \subseteq D_J \). Let \( E \) be an ideal in the poset \( (W, \leq_L) \); that is, \( E \) is a subset of \( W \) such that every \( u \in W \) that is a suffix of an element of \( E \) is itself in \( E \). This condition implies that \( Pos(E) = S \setminus E = \{ s \in S \mid s \notin E \} \). Let \( J \) be a subset of \( Pos(E) \), so that \( E \subseteq D_J \).

In contexts we shall denote by \( E_J \) for the set \( E \), with reference to \( J \). For each \( s \in S \) we classify the elements in \( E_J \) as follows:
\[
E^-_{J,s} = \{ w \in E_J \mid \ell(sw) < \ell(w) \text{ and } sw \in E_J \},
E^+_{J,s} = \{ w \in E_J \mid \ell(sw) > \ell(w) \text{ and } sw \in E_J \},
E^0_{J,s} = \{ w \in E_J \mid \ell(sw) > \ell(w) \text{ and } sw \notin D_J \},
E^0_{J,s} = \{ w \in E_J \mid \ell(sw) > \ell(w) \text{ and } sw \in D_J \setminus E_J \}.
\]

Since \( E_J \subseteq D_J \) it is clear that, for each \( w \in E_J \), each \( s \in S \) appears in exactly one of the following four sets \( SA(w) = \{ s \in S \mid w \in E^+_{J,s} \}, SD(w) = \{ s \in S \mid w \in E^0_{J,s} \}, WA_J = \{ s \in S \mid w \in E^0_{J,s} \} \) and \( WD_J = \{ s \in S \mid w \in E^0_{J,s} \} \). We call the elements of these sets the strong ascents, strong descents, weak ascents and weak descents of \( w \) relative to \( E_J \) and \( J \). In contexts where the ideal \( E_J \) and the set \( J \) is fixed we frequently omit reference to \( J \), writing \( WA(w) \) and \( WD(w) \) rather than \( WA_J(w) \) and \( WD_J(w) \). We also define the sets of descents and ascents of \( w \) by \( D(w) = SD(w) \cup WD(w) \) and \( A(w) = SA(w) \cup WA(w) \).

Remark. It follows from [2,1] that
\[
WA_J(w) = \{ s \in S \mid sw \notin E_J \text{ and } w^{-1}sw \notin J \},
WD_J(w) = \{ s \in S \mid sw \notin E_J \text{ and } w^{-1}sw \in J \}.
\]

since \( sw \notin E_J \) implies that \( sw > w \) (given that \( E_J \) is an ideal in \( (W, \leq_L) \)). Note also that \( J = WD_J(1) \).

Definition 2.4. [3] Definition 5.1(modified) Let \( (W, S) \) be a Coxeter group with weight function \( L \) such that \( L(s) \geq 0 \) for all \( s \in S \), \( \mathcal{H} \) be the corresponding Hecke
algebra. The set \( E_J \) is said to be a \( W \)-graph ideal with respect to \( J(\subseteq S) \) and \( L \) if the following hypotheses are satisfied.

(i) There exists an \( A \)-free \( \mathcal{H} \)-module \( M(E_J, L) \) possessing an \( A \)-basis

\[
B = \{ \Gamma_w | w \in E_J \},
\]

for any \( s \in S \) and any \( w \in E_J \) we have

\[
T_s \Gamma_w = \begin{cases} 
\Gamma_{sw} + (q^{L(s)} - q^{-L(s)}) \Gamma_w & \text{if } w \in E_{J,s}^-; \\
\Gamma_{sw} & \text{if } w \in E_{J,s}^+; \\
-q^{-L(s)} \Gamma_w & \text{if } w \in E_{J,s}^0; \\
q^{L(s)} \Gamma_w - \sum_{z < w} r_{z,w} \Gamma_z & \text{if } w \in E_{J,s}^0.
\end{cases}
\]

for some polynomials \( r_{z,w} \in q^{L(s)} A_{>0} \).

(ii) The module \( M(E_J, L) \) admits an \( A \)-semilinear involution \( \alpha \mapsto \overline{\alpha} \) satisfying \( \overline{1} = 1 \) and \( \overline{h \alpha} = \overline{h} \overline{\alpha} \) for all \( h \in \mathcal{H} \) and \( \alpha \in M(E_J, L) \).

An obvious induction on \( \ell(w) \) shows that \( \Gamma_w = T_w \Gamma_1 \) for all \( w \in E_J \).

**Definition 2.5.** ([9, Definition 5.2]) If \( w \in W \) and \( E_J = \{ u \in W \mid u \leq_L w \} \) is a \( W \)-graph ideal with respect to some \( J \subseteq S \) then we call \( w \) a \( W \)-graph determining element.

**Remark.** It has been verified in [9, Section 5] that if \( W \) is finite then \( w_J \), the maximal length element of \( W \), is a \( W \)-graph determining element with respect to \( \emptyset \), and \( d_J \), the minimal length element of the left coset \( w_J^{-1} J \), is a \( W \)-graph determining element with respect to \( J \) and also with respect to \( \emptyset \).

### 3. Duality theorem for \( W \)-graph ideals

Let \( (W, S) \) be a Coxeter group with weight function \( L \) such that \( L(s) \geq 0 \) for all \( s \in S \), \( \mathcal{H} \) be the corresponding Hecke algebra. There exists an algebra map \( \Phi : \mathcal{H} \rightarrow \mathcal{H} \) given by \( \Phi(q^{L(s)}) = q^{L(s)} \) for all \( s \in S \), and \( \Phi(T_w) = \epsilon_w \overline{T_w} \), where the bar is the standard involution in \( \mathcal{H} \). Further, \( \Phi^2 = Id \) and \( \Phi \) commutes with the bar involution.

**Duality theorem.** We now give an equivalent definition of a \( W \)-graph ideal, and the associated module is denoted by \( \overline{M}(E_J, L) \). The following theorem essentially provides the duality between the two set ups.

**Theorem-definition 3.1.** (i) With the above notation, let the set \( E_J \) be a \( W \)-graph ideal with respect to \( J(\subseteq S) \) and \( L \), then the following hypotheses are satisfied.

(i) There exists an \( A \)-free \( \mathcal{H} \)-module \( \overline{M}(E_J, L) \) possessing an \( A \)-basis

\[
\overline{B} = \{ \overline{\Gamma}_w | w \in E_J \},
\]

for any \( s \in S \) and any \( w \in E_J \) we have

\[
T_s \overline{\Gamma}_w = \begin{cases} 
\overline{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)}) \overline{\Gamma}_w & \text{if } w \in E_{J,s}^-; \\
\overline{\Gamma}_{sw} & \text{if } w \in E_{J,s}^+; \\
-q^{-L(s)} \overline{\Gamma}_w & \text{if } w \in E_{J,s}^0; \\
q^{L(s)} \overline{\Gamma}_w - \sum_{z < w} r_{z,w} \overline{\Gamma}_z & \text{if } w \in E_{J,s}^0.
\end{cases}
\]
where \( \overline{r}_{z,w}^s = \epsilon_z \epsilon_w \overline{r}_{z,w}^s \in q^{-L(s)}A_{<0} \). (ii) The module \( \widetilde{M}(E_j, L) \) admits an \( A \)-semilinear involution \( \widetilde{\alpha} \mapsto \overline{\alpha} \) satisfying \( \Gamma_1 = \overline{\Gamma}_1 \) and \( h\alpha = h\overline{\alpha} \) for all \( h \in \mathcal{H} \) and \( \overline{\alpha} \in \widetilde{M}(E_j, L) \).

(II) There exists a unique map \( \eta : M(E_j, L) \to \widetilde{M}(E_j, L) \) such that

\( (i) \eta(\Gamma_1) = \overline{\Gamma}_1; \)

\( (ii) \eta(h\Gamma) = \Phi(h)\eta(\Gamma), \) for all \( h \in \mathcal{H} \) and \( \Gamma \in M(E_j, L) \).

( i.e., \( \eta \) is \( \Phi \)-linear). Further, it has the following properties:

(a) \( \eta \) commutes with the involution on \( M(E_j, L) \) and \( \widetilde{M}(E_j, L) \).

(b) \( \eta \) is one-to-one onto and the inverse \( \theta \) of \( \eta \), satisfies properties (i) and (ii) of \( \eta \).

**Proof.** For \( w \in E_j \), define \( \eta(\Gamma_w) = \epsilon_w \overline{\Gamma}_w \). Extend \( \eta \) to the whole of \( M(E_j, L) \) by \( \Phi \)-linearity. Let \( s \in S \). Then we have,

\[
\eta(T_s \Gamma_w) = \begin{cases} 
\eta[\Gamma_{sw} + (q^{L(s)} - q^{-L(s)})\Gamma_w] & \text{if } w \in E^j_{d,s}, \\
\eta(\Gamma_{sw}) & \text{if } w \in E^+_{j,s}, \\
\eta(-q^{-L(s)}\Gamma_w) & \text{if } w \in E^0_{j,s}, \\
\eta(q^{L(s)}\Gamma_w - \sum_{z \in E_j} r_{z,w}^s \Gamma_z) & \text{if } w \in E^0_{j,s}, 
\end{cases}
\]

which equals to

\[
\begin{cases} 
\epsilon_{sw} \overline{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\epsilon_w \overline{\Gamma}_w & \text{if } w \in E^j_{d,s}, \\
\epsilon_{sw} \overline{\Gamma}_{sw} & \text{if } w \in E^+_{j,s}, \\
-q^{-L(s)}\epsilon_w \overline{\Gamma}_w & \text{if } w \in E^0_{j,s}, \\
q^{L(s)}\epsilon_w \overline{\Gamma}_w - \sum_{z \in E_j} r_{z,w}^s \epsilon_z \overline{\Gamma}_z & \text{if } w \in E^0_{j,s}, 
\end{cases}
\]

for some polynomials \( r_{z,w}^s \in q^{L(s)}A_{>0} \). On the other hand

\[
\Phi(T_s)\eta(\Gamma_w) = -T_s \epsilon_w \overline{\Gamma}_w
\]

\[
= (-1)^{\ell(w)+1} T_s \overline{\Gamma}_w
\]

\[
= (-1)^{\ell(w)+1} \begin{cases} 
\overline{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\overline{\Gamma}_w & \text{if } w \in E^j_{d,s}, \\
\overline{\Gamma}_{sw} & \text{if } w \in E^+_{j,s}, \\
-q^{-L(s)}\overline{\Gamma}_w & \text{if } w \in E^0_{j,s}, \\
q^{L(s)}\overline{\Gamma}_w - \sum_{z \in E_j} r_{z,w}^s \overline{\Gamma}_z & \text{if } w \in E^0_{j,s}, 
\end{cases}
\]

It is easy to check that these two expressions give the same result, and this shows that \( \eta(T_s \Gamma_w) = \Phi(T_s)\eta(\Gamma_w) \). It is also easy to see that \( \eta(h\Gamma_w) = \Phi(h)\eta(\Gamma_w) \) for all \( h \in \mathcal{H} \) and \( \Gamma \in M(E_j, L) \).

If \( \eta' \) is another map satisfying properties (i) and (ii), then

\[
\eta'(\Gamma_w) = \eta'(T_s \Gamma_1) = \Phi(T_w)\overline{\Gamma}_1 = \epsilon_w T_w \overline{\Gamma}_1 = \epsilon_w T_w \overline{\Gamma}_1 = \epsilon_w \overline{\Gamma}_w
\]

It is now clear that \( \eta' = \eta \).
To prove statement (a), observe that for any $\Gamma \in M(E_J, L)$, there exists $h \in \mathcal{H}$ such that $\Gamma = h^\Gamma_1$. Thus

$$\eta(\Gamma) = \eta(h^\Gamma_1) = \Phi(h)^\Gamma_1 = \Phi(h)\bar{\Gamma}_1 = \Phi(h)\tilde{\Gamma}_1 = \eta(h)\tilde{\Gamma}_1 = \eta(\Gamma).$$

This proves (a).

We interchange the role of these two modules to obtain a map

$$\theta : \tilde{M}(E_J, L) \to M(E_J, L)$$

such that $\theta(\tilde{\Gamma}_w) = \epsilon_w\Gamma_w$. It is easy to check that $\theta$ and $\eta$ are inverses of each other. This proves (b).

**Corollary 3.2.** If $R_{x,y}$ and $\tilde{R}_{x,y}$ are the polynomials given by the formula

$$\Gamma_y = \sum_{x \in E_J} R_{x,y}\Gamma_x, \quad \overline{\Gamma}_y = \sum_{x \in E_J} \tilde{R}_{x,y}\overline{\Gamma}_x$$

then

$$\tilde{R}_{x,y} = \epsilon_x\epsilon_y R_{x,y}.$$

**Proof.** Apply the function $\eta$ to both the sides of the formula for $\Gamma_y$ and use the fact that $\eta$ commutes with the involution and then use the formula for $\overline{\Gamma}_y$. We omit the details. \qed

The above result can also be proved by the following recursive formulas.

**Lemma 3.3.** [14 Prop. 4.1] Let $x, y \in E_J$. If $s \in S$ is such that $y \in E_{J,s}$ then

$$R_{x,y} = \begin{cases} 
R_{sx,sy} & \text{if } x \in E_{J,s}^- \\
R_{sx,ys} + (q^{-L(s)} - q^{L(s)})R_{x,ys} & \text{if } x \in E_{J,s}^+ \\
-q^{L(s)}R_{x,ys} & \text{if } x \in E_{J,s}^{0_-} \\
q^{-L(s)}R_{x,ys} & \text{if } x \in E_{J,s}^{0_+}.
\end{cases}$$

Similarly we have

**Lemma 3.4.** Let $x, y \in E_J$. If $s \in S$ is such that $y \in E_{J,s}^-$ then

$$\tilde{R}_{x,y} = \begin{cases} 
\tilde{R}_{sx,ys} & \text{if } x \in E_{J,s}^- \\
\tilde{R}_{sx,ys} + (q^{-L(s)} - q^{L(s)})\tilde{R}_{x,ys} & \text{if } x \in E_{J,s}^+ \\
q^{-L(s)}\tilde{R}_{x,ys} & \text{if } x \in E_{J,s}^{0_-} \\
q^{L(s)}\tilde{R}_{x,ys} & \text{if } x \in E_{J,s}^{0_+}.
\end{cases}$$

We have the further properties of $R_{x,y}$.

**Lemma 3.5.** If $y \in E_{J,s}^{0_-}$ then we have

$$R_{x,y} = \begin{cases} 
-q^{-L(s)}R_{sx,y} & \text{if } x \in E_{J,s}^- \\
-q^{L(s)}R_{sx,y} & \text{if } x \in E_{J,s}^+.
\end{cases}$$

If $y \in E_{J,s}^{0_+}$ then we have

$$R_{x,y} = \begin{cases} 
q^{L(s)}R_{sx,y} & \text{if } x \in E_{J,s}^- \\
q^{-L(s)}R_{sx,y} & \text{if } x \in E_{J,s}^+.
\end{cases}$$
Proof. If \( y \in \mathbf{E}_{J,s}^0 \) then
\[
T_s \Gamma_y = -q^{-L(s)} \Gamma_y
\]
Applying involution bar on both sides. On the left hand side we have
\[
\overline{T_s \Gamma_y} = \overline{T_s \Gamma_y} = [T_s + (q^{-L(s)} - q^{L(s)}) \sum_{x \in \mathbf{E}_J} R_{x,y} \Gamma_x].
\]
while the right hand side is \( -q^{-L(s)} \overline{\Gamma_y} = -q^{L(s)} \sum_{x \in \mathbf{E}_J} R_{x,y} \Gamma_x. \)
Comparing the coefficients of \( \Gamma_x \) in the two expressions, we get the result. The proof for the case \( y \in \mathbf{E}_{J,s}^+ \) is similar with the above. \( \square \)

**Dual bases \( \mathbf{C} \) and \( \mathbf{C}' \).** Recall [14, Th.4.4] that the invariants in \( M(\mathbf{E}_J, L) \) (respectively \( \widetilde{M}(\mathbf{E}_J, L) \)) form a free \( A \)-module with a basis \( \{ \mathbf{C}_w \mid w \in \mathbf{E}_J \} \) (respectively \( \widetilde{\mathbf{C}}_w \mid w \in \mathbf{E}_J \} \), where \( \mathbf{C}_w = \sum_{y,w} P_{y,w} \Gamma_y \) and \( \widetilde{\mathbf{C}}_w = \sum_{y,w} \widetilde{P}_{y,w} \Gamma_y \).

Using the map \( \theta \), we obtain a dual basis \( \{ \mathbf{C}'_w \mid w \in \mathbf{E}_J \} \) for the invariants in \( M(\mathbf{E}_J, L) \). Analogously, using the map \( \eta \) we obtain the dual basis \( \{ \widetilde{\mathbf{C}}'_w \mid w \in \mathbf{E}_J \} \) for the invariants in \( \widetilde{M}(\mathbf{E}_J, L) \).

More precisely, we have:

**Proposition 3.6.** Let \( \mathbf{C}'_w = \theta(\mathbf{C}_w), \widetilde{\mathbf{C}}'_w = \eta(\mathbf{C}_w) \). Then
(a) The \( \mathcal{H} \)-module \( M(\mathbf{E}_J, L) \) has a unique basis \( \{ \mathbf{C}'_w \mid w \in \mathbf{E}_J \} \) such that \( \overline{\mathbf{C}}'_w = \mathbf{C}'_w \) for all \( w \in \mathbf{E}_J \), and \( \mathbf{C}'_w = \sum_{y \in \mathbf{E}_J} e_y \widetilde{P}_{y,w} \Gamma_y \), for some elements \( \widetilde{P}_{y,w} \in A_{\geq 0} \) with the following properties:
   (a1) \( \widetilde{P}_{y,w} = 0 \) if \( y \neq w \);
   (a2) \( \widetilde{P}_{w,w} = 1 \);
   (a3) \( \widetilde{P}_{y,w} \) has zero constant term if \( y \neq w \) and
   \[
   \overline{\widetilde{P}_{y,w}} - \widetilde{P}_{y,w} = \sum_{y < x \leq w} \overline{R}_{y,x} \widetilde{P}_{x,w} \text{ for any } y < w.
   \]
(b) Analogously, the module \( \widetilde{M}(\mathbf{E}_J, L) \) has another basis \( \{ \widetilde{\mathbf{C}}'_w \mid w \in \mathbf{E}_J \} \), where \( \widetilde{\mathbf{C}}'_w = \sum_{y \in \mathbf{E}_J} e_y \overline{\Gamma}_y \).

Proof.
\[
\mathbf{C}'_w = \theta(\sum_{y \in \mathbf{E}_J} \widetilde{P}_{y,w} \overline{\Gamma}_y) = \sum_{y \in \mathbf{E}_J} e_y \widetilde{P}_{y,w} \overline{\Gamma}_y
\]
Hence, \( \overline{\mathbf{C}}'_w = \theta(\mathbf{C}_w) = \theta(\widetilde{\mathbf{C}}_w) = \mathbf{C}'_w \) and the result follows. \( \square \)

**Inversion.** For \( y, w \in \mathbf{E}_J \), we write the matrix \( P = (P_{y,w}) \), where \( P_{y,w} \) are \( \mathbf{E}_J \)-relative Kazhdan-Lusztig polynomials. The formula for \( \mathbf{C}_w \) in [14, Th.4.4] may be written as
\[
\mathbf{C}_w = \Gamma_w + \sum_{y \in \mathbf{E}_J} P_{y,w} \Gamma_y
\]
and inverting this gives
\[
\Gamma_w = \mathbf{C}_w + \sum_{y \in \mathbf{E}_J} \overline{Q}_{y,w} \mathbf{C}_y
\]
where the elements $Q_{y,w}$ (defined whenever $y < w$) are given recursively by

$$Q_{y,w} = -P_{y,w} - \sum_{z \in \mathcal{E}_J|y < z < w} Q_{y,z} P_{z,w}$$

A $\mathcal{E}_J$-chain is a sequence $\zeta: z_0 < z_1 < \cdots < z_n (n \geq 1)$ of elements in $\mathcal{E}_J$, we set $\ell(\zeta) = n$ and $P_\zeta = P_{z_0,z_1} P_{z_1,z_2} \cdots P_{z_{n-1},z_n}$. $z_0$ is called the initial element of $\zeta$ and $z_n$ is called the final element of $\zeta$. For $y < w$, let $\tau(y,w)$ denote the set of all $\mathcal{E}_J$-chains with $y$ as the initial element and $w$ as the final element.

The following results are motivated by Lusztig [11, Ch. 10]. For the sake of completeness we attach the proofs.

**Proposition 3.7.** For any $y, w \in \mathcal{E}_J$ we have

$$Q_{y,w} = \sum_{\zeta \in \tau(y,w)} (-1)^{\ell(\zeta)} P_\zeta$$

We have $Q_{y,w} \in A_{\geq 0}$ with the following properties:

(a1) $Q_{y,w} = 0$ if $y \not\leq w$;

(a2) $Q_{w,w} = 1$;

**Proof.** If $\ell(w) - \ell(y) = 1$, by Eq.(7) we have $Q_{y,w} = -P_{y,w}$. The statement is true. Applying induction on $\ell(w) - \ell(y) \geq 1$. For any $z \in \mathcal{E}_J, y < z < w$, in the sum of Eq.(7) we use the induction hypothesis.

$$Q_{y,z} = \sum_{\zeta' \in \tau(y,z)} (-1)^{\ell(\zeta')} P_{\zeta'}$$

We have

$$Q_{y,w} = -P_{y,w} - \sum_{\zeta' \in \tau(y,z)} (-1)^{\ell(\zeta')} P_{\zeta'} P_{z,w}$$

$$= \sum_{\zeta \in \tau(y,w)} (-1)^{\ell(\zeta)} P_\zeta$$

where the sequence $\zeta = (y,w) \in \tau(y,w)$ is with $\ell(\zeta) = 1$ and $(\zeta', w) \in \tau(y,w))$ with the length $\ell(\zeta') + 1$. The listed properties of $Q'$s are by Eq.(7). The result is proved. \(\square\)

We define

$$Q'_{y,w} = sgn(y) sgn(w) Q_{y,w}$$

**Proposition 3.8.** For any $y, w \in \mathcal{E}_J$ we have $Q_{y,w} = \sum_{z: y \leq L z \leq L w} Q_{y,z}^\perp R_{z,w}$

**Proof.** The triangular matrices $Q = (Q_{y,w}), P = (P_{y,w}), R = (R_{y,w})$ are related by

$$PQ = QP = 1, \mathcal{T} = \mathcal{R} P, \overline{\mathcal{R}} = \mathcal{R} \mathcal{R} = 1$$

where the bar involution over a matrix is the matrix obtained by applying $\bar{\cdot}$ to each entry. We deduce that

$$PQ = 1 = Q P = \overline{Q} \mathcal{T} = \overline{Q} \mathcal{R} P$$

Multiplying on the right by $Q$ and using the fact $PQ = 1$ we deduce $Q = \overline{Q} \mathcal{R}$. Multiplying on the right by $R$ gives

$$\overline{Q} = QR$$
Let $S$ be the matrix whose $(y,w)$-entry is $\text{sgn}(y)\delta_{y,w}$. We have $S^2 = 1$. Note that $Q' = SQS$. By Corollary 3.2 we have $\widehat{R} = S\widehat{R}S$. Hence

$$Q' = SQS = S(QR)S = SQS \cdot SRS = Q\widehat{R}$$

The result follows. $\square$

4. W-graphs for the modules $\hat{M}$ and $\tilde{M}$

Denote by $M := M(E_J, L)$ and $\tilde{M} := \tilde{M}(E_J, L)$. Let $\hat{M} := Hom_A(M, A)$ and $\hat{\tilde{M}} := Hom_A(\tilde{M}, A)$.

Define an left $\mathcal{H}$-module structure on $\hat{M}$ by

$$hf(m) = f(hm)$$

with $f \in \hat{M}$, $m \in M$, $h \in \mathcal{H}$.

We define a bar operator $\hat{\tilde{M}} \rightarrow \hat{\tilde{M}}$ by $\hat{\tilde{f}}(m) = \overline{f(m)}$ (with $f \in \hat{M}$, $m \in M$); in $\overline{f(m)}$ the lower bar is that of $M$ and the upper bar is that of $A$.

$$\overline{h \cdot f(m)} = \overline{hf(m)} = \overline{f(hm)} = \overline{f(hm)} = \overline{f(m)}$$

Hence we have $\overline{h \cdot f} = \overline{h} \cdot \overline{f}$ for $f \in \hat{M}, h \in \mathcal{H}$.

In the following contexts we focus on the module $\hat{M}$, and usually omit the analogous details for $\hat{\tilde{M}}$.

If $P$ is a property we set $\delta_P = 1$ if $P$ is true and $\delta_P = 0$ if $P$ is false. We write $\delta_{x,y}$ instead of $\delta_{x=y}$.

The basis of $\hat{M}$. We firstly introduce two bases for the module $\hat{M}$. For any $z \in E_J$ we define $\hat{\Gamma}_z \in \hat{M}$ by $\hat{\Gamma}_z(w) = \delta_{z,w}$ for any $w \in E_J$. Then $\tilde{B} := \{\hat{\Gamma}_z; z \in E_J\}$ is an $A$-basis of $\hat{M}$.

Further, for any $z \in E_J$ we define $\hat{D}_z \in \hat{M}$ by $\hat{D}_z(C_w) = \delta_{z,w}$ for any $w \in E_J$. Then $D := \{\hat{D}_z; z \in E_J\}$ is an $A$-basis of $\hat{M}$.

Obviously we have

$$D_z = \sum_{y \in E_J, z < y} Q_{z,y} \hat{\Gamma}_y.$$

An equivalent definition of the basis element $D_w \in \hat{M}$ is

$$D_z(\Gamma_y) = Q_{z,y}$$

for all $y \in E_J$. In fact, we have

$$D_z(C_w) = D_z \sum_{y \in E_J} P_{y,w}(\Gamma_y) = \sum_{y \in E_J} Q_{z,y} P_{y,w} = \delta_{z,w}$$

Lemma 4.1. For any $y \in E_J$ we have

$$\overline{\Gamma}_y = \sum_{w \in E_J, y \leq w} R_{y,w} \hat{\Gamma}_w.$$
Proof. For any \( x \in E_J \) we have

\[
\hat{\Gamma}_y(\Gamma_x) = \hat{\Gamma}_y(\Gamma_x) \\
= \hat{\Gamma}_y \left( \sum_{x' \in E_J, x' \leq x} R_{x',x} \Gamma_{x'} \right) = \delta_{y \leq x} R_{y,x} = \delta_{y \leq x} R_{y,x} \\
= \sum_{w \in E_J, y \leq w} R_{y,w} \hat{\Gamma}_w(\Gamma_x)
\]

\( \square \)

**Theorem 4.2.** [14, Th. 4.7] The basis elements \( \{ C_v \mid v \in E_J \} \) give the module \( M(E_J, L) \) the structure of a \( W \)-graph module such that

\[
T_s C_v = \begin{cases} 
q^{L(v)} C_v + C_{sv} + \sum_{z \in E_J, s \leq z < v} m_{z,v} C_z & \text{if } s \in SA(v), \\
-q^{-L(v)} C_v & \text{if } s \in D(v), \\
q^{L(v)} C_v + \sum_{z \in E_J, z < v} m_{z,v} C_z & \text{if } s \in WA(v).
\end{cases}
\]

**Theorem 4.3.** The \( \mathcal{H} \)-module \( \hat{M}(E_J, L) \) has a unique basis \( \{ D_z \mid z \in E_J \} \) such that \( D_z = D_z \) for all \( z \in E_J \), and \( D_z = \sum_{y \in E_J} Q_{z,y} \hat{\Gamma}_y \) for some elements \( Q_{z,y} \in A \geq 0 \) with the following properties:

1. \( Q_{z,y} = 0 \) if \( z \not\leq y \);
2. \( P_{z,z} = 1 \);
3. \( Q_{z,y} \) has zero constant term if \( z \not= y \) and

\[
Q_{z,y} - Q_{z,y} = \sum_{z \leq x < y, x \in E_J} Q_{z,x} R_{x,y}
\]

for any \( z < y \).

The proof is very similar to that of [11, Th. 5.2] or [10, Section 2]. It uses induction on \( \ell(w) - \ell(y) \), the equation \( Q = QR \) in Proposition 3.8, and the fact: If \( f = \sum_{y \in E_J} Q_{z,y} R_{z,y} \) then \( f = -f \). We omit further details of the proof.

The (left) ascent set of \( z \in E_J \) is

\[
A(z) = \{ s \in S \mid z \in E_J^+ \cup E_J^{0,+} \}
\]

**Theorem 4.4.** Let \( s \in S \) and assume that \( L(s) > 0 \). The basis elements \( \{ D_z \mid z \in E_J \} \)

give \( \hat{M} \) the structure of a \( W \)-graph module such that

\[
T_s D_z = \begin{cases} 
-q^{-L(z)} D_z + D_{sz} + \sum_{z < u, s \in A(u)} m_{z,u} D_u & \text{if } s \in SD(z), \\
q^{L(s)} D_z & \text{if } s \in A(z), \\
-q^{-L(z)} D_z + \sum_{z < u, s \in A(u)} m_{z,u} D_u & \text{if } s \in WD(z),
\end{cases}
\]
Proof. In the case $s \in SD(z)$, $T_s D_z(C_w) = D_z(T_s C_w)$ gives

$$T_s D_z(C_w) = \begin{cases} D_z(q^{L(s)}C_w + C_{sw} + \sum_{x \in E_j, s \leq x < w} m^s_{x,w} C_x) & \text{if } s \in SA(w), \\ D_z(-q^{-L(s)}C_w) & \text{if } s \in D(w), \\ D_z(q^{L(s)}C_w + \sum_{x \in E_j, s \leq x < w} m^s_{x,w} C_x) & \text{if } s \in WA(w), \end{cases}$$

$$= \begin{cases} \delta_{z,sw} + \sum_{x \in E_j, s \leq x < w} m^s_{x,w} \delta_{z,x} & \text{if } s \in SA(w), \\ -q^{-L(s)} \delta_{z,w} & \text{if } s \in SD(w), \\ 0 & \text{if } s \in WD(w), \\ \sum_{x \in E_j, s \leq x < w} m^s_{x,w} \delta_{z,x} & \text{if } s \in WA(w), \end{cases}$$

$$= \begin{cases} (D_{sz} + \sum_{z < u, w \in E_j} m^s_{z,u} D_u)(C_w) & \text{if } s \in SA(w), \\ -q^{-L(s)} D_z(C_w) & \text{if } s \in SD(w), \\ 0 & \text{if } s \in WD(w), \\ \sum_{z < u, w \in E_j} m^s_{z,u} D_u(C_w) & \text{if } s \in WA(w), \end{cases}$$

Hence, we obtain

$$T_s D_z(C_w) = (-q^{-L(s)} D_z + D_{sz} + \sum_{z < u, s \in A(u)} m^s_{z,u} D_u)(C_w)$$

for all $w \in E_J$. The desired formula follows in this case.

In other cases the computation is similar with the above, we omit the details. \(\square\)

The following is by \([13, \text{Prop.4.8}]\).

**Corollary 4.5.** For $s \in S$ with $L(s) = 0$, $z \in E_J$, we have

$$T_s D_z = \begin{cases} D_{sz} & \text{if } s \in SD(z) \text{ or } s \in SA(z), \\ -D_z & \text{if } s \in WD(z), \\ D_z & \text{if } s \in WA(z), \end{cases}$$

**The $D'$-basis for $M$.**

**Theorem 4.6.** The $\mathcal{H}$-module $\hat{M}(E_J, L)$ has a unique basis $\{ D'_z \mid z \in E_J \}$ such that $\overline{D'_z} = D'_z$ for all $z \in E_J$, and $D'_z = \sum_{y \in E_J} \epsilon_y \overline{Q_{z,y} \Gamma}_{y,v}$, where $\overline{Q_{z,y}} \in A_{\geq 0}$, are the
analogous elements in the case of \( \tilde{M} \).

\[
T_s D'_z = \begin{cases} 
q^{L(s)} D'_z + D'_z + \sum_{z < u, s \in \Delta(u)} m_{z,u} D'_{u} & \text{if } s \in SD(z), \\
-q^{-L(s)} D'_z & \text{if } s \in A(z), \\
q^{L(s)} D'_z + \sum_{z < u, s \in \Delta(u)} m_{z,u} D'_{u} & \text{if } s \in WD(z),
\end{cases}
\]

(10)

For the \( \mathcal{H} \)-module \( M(E_J, L) \), two pairs of dual bases \( C, C' \) and \( D, D' \) give the structures of the "full W-graphs".

The module \( \tilde{M}(D_J, L) \). Set \( E_J := D_J \). If \( D_J \) is regarded as a \( W \)-graph ideal with respect to \( \emptyset \) (see Deodhar's construction in Section 6), we have

Lemma 4.7. The modules \( \tilde{M}(D_J, L) \) and \( M(D_J, L) \) are identical.

Proof. For any basis element \( \hat{\Gamma}_w \) of \( \tilde{M}(D_J, L) \) and element \( \Gamma_y \) of \( M(D_J, L) \), we have

\[
T_s \hat{\Gamma}_w(\Gamma_y) = \hat{\Gamma}_w(T_s \Gamma_y)
\]

\[
= \delta_{y \in D_{J,s}} \delta_{w,y} + (q^{L(s)} - q^{-L(s)}) \delta_{y \in D_{J,s}} \delta_{w,y} + \delta_{y \in D_{J,s}^+} \delta_{w,y}
\]

\[
+ q^{L(s)} \delta_{y \in D_{J,s}^+} \delta_{w,y}
\]

\[
= \delta_{w \in D_{J,s}^+} \delta_{w,y} + (q^{L(s)} - q^{-L(s)}) \delta_{w \in D_{J,s}^+} \delta_{w,y} + \delta_{w \in D_{J,s}^+} \delta_{w,y}
\]

\[
+ q^{L(s)} \delta_{w \in D_{J,s}^+} \delta_{w,y}
\]

\[
= (\delta_{w \in D_{J,s}^+} \hat{\Gamma}_{w,y} + (q^{L(s)} - q^{-L(s)}) \delta_{w \in D_{J,s}^+} \hat{\Gamma}_{w,y} + \delta_{w \in D_{J,s}^+} \hat{\Gamma}_{w,y})
\]

\[
+ q^{L(s)} \delta_{w \in D_{J,s}^+} \hat{\Gamma}_{w,y}(\Gamma_y)
\]

hence we have

\[
T_s \hat{\Gamma}_w = \begin{cases} 
\hat{\Gamma}_{w,y} & \text{if } w \in D_{J,s}^+ \\
\hat{\Gamma}_{w,y} + (q^{L(s)} - q^{-L(s)}) \hat{\Gamma}_{w,y} & \text{if } w \in D_{J,s}^0 \\
q^{L(s)} \hat{\Gamma}_{w,y} & \text{if } w \in D_{J,s}^0
\end{cases}
\]

The result follows. \( \square \)

Corollary 4.8. The \( \mathcal{H} \)-module \( M(D_J, L) \) has basis \( \{ D_z \mid z \in D_J \} \), where \( D_z = \sum_{z \in D_J} Q_{z,y} \Gamma_y \). This basis gives the structure of \( W \)-graph module such that

\[
T_s D_z = \begin{cases} 
-q^{-L(s)} D_z + D_{sz} + \sum_{z < u, u \in D_{J,s}^+ \cup D_{J,s}^0} m_{z,u} D_u & \text{if } z \in D_{J,s}^+, \\
q^{L(s)} D_z & \text{if } z \in D_{J,s}^0
\end{cases}
\]

5. IN THE CASE \( W \) IS FINITE

Let \( (W, S) \) be a finite Coxeter system and \( w_0 \) be the longest element in \( W \). Define the function \( \pi : W \rightarrow W \) by \( \pi(w) = w_0 w w_0 \), it satisfies \( \pi(S) = S \) and it extends to a \( C \)-algebra isomorphism \( \pi : C[W] \rightarrow C[W] \). We denote by \( s_0 = \pi(s) \). For \( s \in S \) we have \( \ell(w_0) + \ell(w_0 s) + \ell(s) = \ell(s) + \ell(\pi(s) w_0) \), hence

\[
L(w_0) = L(w_0 s) + L(s) = L(s) + \ell(\pi(s)) + L(\pi(s) w_0) = L(s) + L(w_0 s)
\]
so that \( L(\pi(s)) = L(s) \). It follows that \( L(\pi(w)) = L(w) \) for all \( w \in W \) and that we have an \( A \)-algebra automorphism \( \pi : \mathcal{H} \to \mathcal{H} \) where \( \pi(T_w) = T_{\pi(w)} \) for any \( w \in W \).

**Lemma 5.1.** The \( \mathcal{H} \)-modules \( M \) and \( \widetilde{M} \) have basis \( \Gamma^\pi = \{ T_{w_0} \Gamma_w \mid w \in E_J \} \) and \( \widetilde{\Gamma}^\pi = \{ T_{w_0} \widetilde{\Gamma}_w \mid w \in E_J \} \) respectively. Moreover we have \( \eta(T_{w_0} \Gamma_w) = \epsilon_{w_0 w} T_{w_0} \Gamma_w \).

**Proof.** Since the involution is square 1 and \( T_{w_0} \) is invertible in \( \mathcal{H} \), the statement follows. Furthermore

\[
\eta(T_{w_0} \Gamma_w) = \Phi(T_{w_0}) \eta(\Gamma_w) = \epsilon_{w_0 w} T_{w_0} \epsilon_w \Gamma_w = \epsilon_{w_0 w} T_{w_0} \Gamma_w.
\]

□

In the following, for the sake of convenience we primarily focus on the module \( M \) and omit the analogous details for \( \widetilde{M} \), unless it is needed. For any \( w \in E_J \) we denote by \( w' := w_0 w \) and \( \Gamma^\pi_w := T_{w_0} \Gamma_w \in M(E_J, L) \).

**Remark** Generally \( w_0 E_J \neq E_J \). We emphasize that, in the following contexts, the set \( w_0 E_J \) will be just used as the index set for the objects involved.

Direct computation gives the following multiplication rules for the basis \( \Gamma^\pi \).

\[
T_{s_0} \Gamma_w^\pi = \begin{cases} \Gamma^\pi_{s_0 w'} + (q^L(s) - q^{-L(s)}) \Gamma^\pi_{s_0 w'} & \text{if } w \in E_J^+, \\
\Gamma^\pi_{s_0 w'} & \text{if } w \in E_J^-, \\
-q^{-L(s)} \Gamma^\pi_{w'} & \text{if } w \in E_J^{0, -}, \\
q^L(s) \Gamma^\pi_w - \sum_{z < w} r_{w, z}^s \Gamma^\pi_z & \text{if } w \in E_J^{0, +}, \end{cases}
\]

where \( r_{w, z}^s = \frac{r}{r_{w, w}} \in q^{-L(s)} A_{<0} \).

**Lemma 5.2.** For any \( y' \in w_0 E_J \) there exist coefficients \( R^\pi_{x', y'} \in A \), defined for \( x' \in w_0 E_J \) and \( x' < y' \), such that \( \Gamma^\pi_{y'} = \sum_{x' \in w_0 E_J} R^\pi_{x', y'} \Gamma^\pi_{x'} \). If \( R^\pi_{x', y'} \neq 0 \) then \( x' \leq y' \); particularly \( R^\pi_{y', y'} = 1 \).

The proof is trivial.

We have further properties of \( R^\pi_{x', y'} \).

**Lemma 5.3.** If \( y' \in w_0 E_J^{0, -} \) then we have

\[
R^\pi_{x', y'} = \begin{cases} -q^L(s_0) R^\pi_{s_0 x', y'} & \text{if } x' \in w_0 E_J^{0, -}, \\
-q^{-L(s_0)} R^\pi_{s_0 x', y'} & \text{if } x' \in w_0 E_J^+ \end{cases}
\]

If \( y' \in w_0 E_J^{0, +} \) then we have

\[
R^\pi_{x', y'} = \begin{cases} q^{-L(s_0)} R^\pi_{s_0 x', y'} & \text{if } x' \in w_0 E_J^{0, +}, \\
q^L(s_0) R^\pi_{s_0 x', y'} & \text{if } x' \in w_0 E_J^- \end{cases}
\]

**Proof.** The proof is similar with that of Lemma 3.5. □

5.1. The bases \( C^\pi \) for \( M \). The elements \( R^\pi_{w', y'} \), where \( w', y' \in w_0 E_J \), lead to the construction of another set of elements \( P^\pi_{w', y'} \) and the following basis of \( M(E_J, L) \).

**Theorem 5.4.** The \( \mathcal{H} \)-module \( M(E_J, L) \) has a unique basis \( \{ C^\pi_{y'} \mid y' \in w_0 E_J \} \) such that \( C^\pi_{y'} = C^\pi_{y'} \) for all \( y \in w_0 E_J \), and \( C^\pi_{y'} = \sum_{w' \in w_0 E_J} P^\pi_{w', y'} \Gamma^\pi_{w'} \) for some elements \( P^\pi_{w', y'} \in A_{\geq 0} \) with the following properties:
\(P_{w',y'} = 0\) if \(w' \neq y'\);

(a2) \(P_{y',y'} = 1\);

(a3) \(P_{w',y'}\) has zero constant term if \(y' \neq w'\) and

\[
P_{w',y'} - P_{w',y'} = \sum_{w' < x' \leq y'} R_{x',y'} P_{x',y'} \text{ for any } w' < y'.
\]

The proof is very similar to that of [11, Th. 5.2] or [10, Section 2]. It uses induction on \(\ell(w') - \ell(y')\), and the fact:

\[
\text{If } f = \sum_{w' < x' \leq y'} R_{x',y'} P_{x',y'} \text{ then } \overline{f} = -f.
\]

We omit further details of the proof.

**Lemma 5.5.** For \(y, w \in E_J\), we have (i) \(y \leq L w \iff w' \leq L y'\);

(ii) \(R_{w',y'} = R_{y,w} ; \overline{R}_{w',y'} = \overline{R}_{y,w} \);

(iii) for any \(w', y' \in w_0 E_J\) and \(w' < y'\) we have

\[
\overline{P}_{w',y'} = \sum_{w' < x' \leq y'} R_{x',y'} P_{x',y'}
\]

**Proof.** (a) is obvious. We prove (b) by induction on \(\ell(w)\). If \(\ell(w) = 0\) then \(w = 1\). We have \(R_{y,1} = \delta_{y,1}\). Now \(R_{w_0,y} = 0\) unless \(w_0 \leq L w_0 y\). On the other hand we have \(w_0 y \leq L w_0\). Hence \(R_{w_0,y} = 0\) unless \(w_0 y = w_0\), that is \(y = 1\) in which case it is 1. The desired equality holds when \(\ell(w) = 0\). Assume that \(\ell(w) \geq 1\). We can find \(s \in S\) such that \(sw < w\). The proof of the following cases (a) and (b) is similar with Lusztig,...

In the case (a) \(y \in E_J^{-}\). By the induction hypothesis we have

\[
R_{y,w} = R_{s y,sw} = R_{w_0,w_0 y w_0} = R_{w_0,w_0 w_0 y} = R_{w_0,w_0 y}
\]

In the case (b) \(y \in E_J^{+}\). Using Lemma 3.3, by the induction hypothesis we have

\[
R_{y,w} = R_{s y,sw} + (q^{-L(s)} - q^{L(s)}) R_{y,sw}
\]

\[
= R_{w_0,w_0 y} + (q^{-L(s)} - q^{L(s)}) P_{w_0,w_0 y}
\]

\[
= R_{s_{w_0},w_0 y} + (q^{-L(s)} - q^{L(s)}) R_{s_{w_0},w_0 y'}
\]

\[
= R_{s_{w_0},w_0 y'} + (q^{-L(s)} - q^{L(s)}) R_{s_{w_0},w_0 y'}
\]

In the Case (c) \(y \in E_J^{0,-}\). Using Lemma 3.5 and Lemma 5.3, by the induction hypothesis we have

\[
R_{y,w} = -q^{L(s)} R_{y,sw} = -q^{L(s)} R_{w_0,(sw),w_0 y} = -q^{L(s)} R_{s,w_0',y'}
\]

\[
= -q^{L(s)} (q^{-L(s)} R_{w_0',y'}) = R_{w_0',y'}.
\]
Case (d) $y \in E_{J_+}^0$. Using Lemma 3.5 and 5.3, by the induction hypothesis we have

$$R_{y,w} = q^{-L(s)}R_{y,sw} = q^{-L(s_0)}R_{s_0w,y'} = R_{w',y'}.$$  

(iii) follows (ii).

\[\square\]

**Proposition 5.6.** For any $y, w \in E_J$ we have $Q_{y,w} = \epsilon_y \epsilon_w \tilde{P}_{w',y'}^\pi$. (Analogously $\tilde{Q}_{y,w} = \epsilon_y \epsilon_w \tilde{P}_{w',y'}^\pi$).

\[\text{Proof.}\] We argue by induction on $\ell(w) - \ell(y) \geq 0$. If $\ell(w) - \ell(y) = 0$ we have $y = w$ and both sides are 1. Assume that $\ell(w) - \ell(y) > 0$. Subtracting the identity in ...from that in ...and using induction hypothesis, we obtain

$$\epsilon_y \epsilon_w Q_{y,w} - \tilde{P}_{w',y'}^\pi = \epsilon_y \epsilon_w Q_{y,w} - \tilde{P}_{w',y'}^\pi$$

The right hand side is in $A_{>0}$; since it is fixed by the involution bar, it is 0. \[\square\]

More precisely, we have the following inversion formulas

**Corollary 5.7.** In the above situation,

$$\sum_{z \in E_J, x \leq z \leq w} \epsilon_w \epsilon_z P_{x,z} \tilde{P}_{w',z'}^\pi = \delta_{x,w};$$

$$\sum_{z \in E_J, x \leq z \leq w} \epsilon_w \epsilon_z P_{x,z} \tilde{P}_{w',z'}^\pi = \delta_{x,w}$$

for all $x, w \in E_J$.

**Corollary 5.8.** If $W$ is finite, for any $y, w \in E_J$ we have

$$m^s_{y,w} = -\epsilon_{w_0y} \epsilon_{w_0w} m^\pi_{w_0w, w_0y};$$

where $m^s_{y,w}$ are the elements involved in the multiplication formulas for $C$-basis, $m^\pi_{w_0w, w_0y}$ are the analogous in the formulas for $C^\pi$-basis.

**Corollary 5.9.** If $W$ is finite, for the bases $D$ and $C^\pi$ in $M(D_J, L)$, and the $\tilde{D}$-basis and $\tilde{C}^\pi$-basis for $\tilde{M}(D_J, L)$ we have

$$T_{w_0} D_z = \epsilon_{w_0z} \theta(\tilde{C}^\pi_{w_0z})$$

and

$$T_{w_0} \tilde{D}_z = \epsilon_{w_0z} \eta(C^\pi_{w_0z}).$$

6. Some remarks

**An example: the dual Solomon modules.** In this subsection, let $(W,S)$ be a finite Coxeter group system. Assume that $L(s) > 0$ for all $s \in S$. In [14] we introduced the $A$-free $\mathcal{H}$-module $\mathcal{H}C_{w,J} C'_{w,J}$, which is called the **Solomon module** with respect to $J$ and $L$, and where

$$C_{w,J} = \epsilon_{w,J} \sum_{w \in W_J} \epsilon_w q^{L(ww_J)} T_w = \epsilon_{w,J} q^{L(wJ)} \sum_{w \in W_J} \epsilon_w q^{-L(s)} T_w;$$

$$C'_{w,J} = \sum_{w \in W_J} q^{-L(ww_J)} T_w = q^{-L(wJ)} \sum_{w \in W_J} q^{L(w)} T_w.$$ 

that is, $C_{w,J}$ is the $C'$-basis element corresponding to $w_J$, the maximal length element of $W_J$, or $c$-basis element corresponding to $w_J$ (see [14 Corollary 12.2]). $C_{w,J}$ is the $C$-basis element corresponding to $w_J$. 

In [13] we showed that $\mathcal{H}C_{w,J}C_{w,J}'$ has basis $\{T_xC_{w,J}C_{J}' \mid x \in F_J\}$. This basis admits the multiplication rules listed in the Definition 2.4, and $F_J$ is a $W$-graph ideal with respect to $J$ and weight function $L$.

Similarly, the $\mathcal{H}$-module $\mathcal{H}C_{w,J}'C_{w,J}$ has basis $\{T_xC_{w,J}'C_{J} \mid x \in F_J\}$. We can easily prove that this basis admits the multiplication rules listed in the Definition 3.1. We call this the dual module of $\mathcal{H}C_{w,J}C_{w,J}'$.

**The Kazhdan-Lusztig construction.** Assume that $J = \emptyset$. Then $D_J = W$ and the sets $WD_J(w)$ and $WA_J(w)$ are empty for all $w \in W$.

(1). If $L(s) > 0$ (for all $s \in S$), both modules $M(E_J, L)$ and $\tilde{M}(E_J, L)$ are with $A$-basis $(X_w \mid w \in E_J)$ such that,

$$T_sX_w = \begin{cases} X_{sw} & \text{if } \ell(sw) > \ell(w) \\ X_{sw} + (q^{L(s)} - q^{-L(s)})X_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where the elements $X_w$ stand for $\Gamma_w$ or $\tilde{\Gamma}_w$. If we let $X_w = T_w$ for all $w \in W$, then both modules are the regular module $\mathcal{H}$ with weight function $L$. Thus we can recover some of Lusztig’s results (for example, see [11, Ch.5, 6, 10, 11]) for the regular case.

**Deodhar’s construction: the parabolic cases.** Let $J$ be an arbitrary subset of $S$ and $L(s) = 1$ for all $s \in S$, we can now turning to Deodhar’s construction.

Set $E_J := D_J$, then $D_J$ is a $W$-graph ideal with respect to $J$, and also it is a $W$-graph ideal with respect with $\emptyset$.

In the latter case we have $D_\emptyset = W$, if $w \in E_J$ then

$$SA(w) = \{s \in S \mid sw > w \text{ and } sw \in D_J\},$$

$$SD(w) = \{s \in S \mid sw < w\},$$

$$WD_\emptyset(w) = \{s \in S \mid sw \notin D_\emptyset\} = \emptyset,$$

$$WA_\emptyset(w) = \{s \in S \mid sw \in D_\emptyset \text{ and } D_J\} = \{s \in S \mid sw = wt \text{ for some } t \in J\}.$$

Let $\mathcal{H}_J$ be the Hecke algebra associated with the Coxeter system $(W_J, J)$. Let $M_\psi = \mathcal{H} \otimes_{\mathcal{H}_J} A_\psi$, where $A_\psi$ is $A$ made into an $\mathcal{H}_J$-module via the homomorphism $\psi : \mathcal{H}_J \to A$ that satisfies $\psi(T_u) = q^{\ell(u)}$ for all $u \in W_J$, it is a $A$-free with basis $B = \{b_w \mid w \in D_J\}$ defined by $b_w = T_w \otimes 1$. This corresponds to $M^J$ in [4] in the case $u = q$ (we note that this is denoted by $\tilde{M}^J$ in [4]).

Let $M_\phi = \mathcal{H} \otimes_{\mathcal{H}_J} A_\phi$, where $A_\phi$ is $A$ made into an $\mathcal{H}_J$-module via the homomorphism $\phi : \mathcal{H}_J \to A$ that satisfies $\psi(T_u) = (-q)^{-\ell(u)}$ for all $u \in W_J$, again it is a $A$-free with basis $B = \{b_w \mid w \in D_J\}$ defined by $b_w = T_w \otimes 1$. This corresponds to $M^J$ in [4] in the case $u = -1$ (this is denoted by $M^J$ in [4]).

Our module $M(E_J, L)$ is now essentially reduced to be the module $M_\psi$, while $\tilde{M}(E_J, L)$ is reduced to be the module $M_\phi$, the only differences being due to our non-traditional definition of $\mathcal{H}$.

In the case $D_J$ is a $W$-graph ideal with respect to $J$, the discussion is similar with the above. For more details see [4] Sect. 8.

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