GRAPHICAL POTENTIAL GAMES

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Abstract. We study the class of potential games that are also graphical games with respect to a given graph $G$ of connections between the players. We show that, up to strategic equivalence, this class of games can be identified with the set of Markov random fields on $G$.

From this characterization, and from the Hammersley-Clifford theorem, it follows that the potentials of such games can be decomposed to local potentials. We use this decomposition to strongly bound the lengths of strict better-response paths. This result extends to generalized graphical potential games, which are played on infinite graphs.

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1. Introduction

Potential games form an important class of strategic interactions. They include fundamental interactions such as Cournot oligopolies (see, e.g., [12]), congestion games (see, e.g., [12, 16]), routing games (see, e.g., [16]) and many others. The above mentioned interactions are frequently local in nature. Namely, there exists an underlying graph such that the payoff of each player depends on her own action and on the actions of her neighbors, but does not depend on the action of the opponents who are not neighbors. For instance, the locality of the interaction could be geographical: In routing games, the outcome of a driver depends only on her own route and the routes that were chosen by drivers who are geographically close to her. In a Cournot oligopoly where transportation costs are high, a firm is competing only with firms which are geographically close (for instance, this is the case with natural gas market, see [20]). The idea of the locality of an interaction is captured by the notion of a graphical game, introduced by Kearns et al. [11]. These games and similar ones are also sometimes called network games; see Jackson and Zenou [9] for an extensive survey.

The goal of this paper is to understand the class of graphical potential games. First, we address the following questions: What characterizes the potential function of a graphical potential game? What characterizes the payoffs of the players in a potential graphical game? In Theorems 3.1, 3.4 and 3.5, we provide a complete answer to these questions. We show that

(1) The potential function of a graphical potential game can be expressed as an additive function of local potentials, where each local potential corresponds to a maximal clique in the underlying graph, and the value of the local potential is determined by the actions of the players in the maximal clique only. This condition is necessary for a potential game to be graphical, and every such potential is the potential of some graphical game.

(2) Up to strategically equivalent transformations (see definition 3.3), the payoff of a player is the sum of the local potentials of the cliques to which she belongs.

The proof of these results is achieved by showing that, for a fixed graph $G$, the set of potentials of graphical games on $G$ can be identified with the set of Markov random fields on $G$ (see Section 2.4). The latter are a well studied class of probability distributions with certain graphical Markov properties. Having established this correspondence, the Hammersley-Clifford theorem [8], a classical result on Markov random fields, yields the above characterization of graphical potential games.
Next, we use this characterization to study dynamics. A central class of dynamics that has been studied in the context of potential games is (strict) better-response dynamics, where at each period of time a single player updates her action to a (strictly) better one, with respect to the current action of the opponents; the basic observation regarding all potential games is that better-response dynamics always converge. Such a sequence of unilateral improvements is called a better response path. We address the question: Are there properties of better response paths that are unique for graphical potential games? We focus on the number of updates of a single player along a better response path, and we prove in Theorem 4.2 and Corollary 4.5 that under mild uniformity assumptions on the game, and for graphs with slow enough growth (see Definition 4.4), the number of updates of each player is bounded by a constant. This result is general and holds for all the above mentioned examples. Example 4.6 demonstrates that the slow-enough-growth condition is tight, in some sense.

Finally, in Section 5, we define generalized graphical potential games on infinite graphs. These are graphical games which are not necessarily potential games, but still have a local potential structure. We prove here the same bounds on strict better-response paths. We note that this result, as well as the special result on finite graphs, translate to novel results on Markov random fields, which may be of interest outside of game theory.

1.1. Related Literature. Potential games and graphical games are both fundamental classes of games (see e.g., [14]). Recently there is a growing interest in the intersection of these two classes, because many interesting potential games have a graphical structure (see e.g., [3], [4]), and many interesting graphical games have a potential function (see e.g., [1]). However, to the best of our knowledge, this paper is the first to fully characterize the intersection of these two classes.

Best-response and better-response paths in potential games were studied in [6], [18] and [2], where the focus is on the length of the paths. Our result (Theorem 4.2 and Corollary 4.5) focuses on the number of changes of a single player. This aspect of better- and best-response dynamics also plays an interesting role in graphical games of strategic complements [9].

The connection between graphical potential games and Markov random fields was previously observed for specific cases; see [1], who establish a connection between the logit dynamic in a particular class of graphical potential coordination games and Glouber dynamics in the
Markov random field of the Ising model. We show that this connection is much more general and extends to all graphical potential games.

Another intriguing connection between graphical games and Markov random fields was established by Kakade et al. [10], who show that every correlated equilibrium of every graphical game is a Markov random field. Daskalakis and Papadimitriou [5] use Markov random fields to compute pure Nash equilibria in graphical games.

There is a vast literature on majority dynamics, which can be interpreted as best-response dynamics of the majority game on a graph, which is a graphical potential game. Tamuz and Tessler [19] upper bound the number of changes in majority dynamics on slowly growing graphs. We adapt their technique in the proof of Theorem 4.2.

2. Definitions

2.1. Games. Let $G$ be a game played by a finite set of players $V$, and denote $n = |V|$. Let $S_i$ be the finite set of actions available to $i \in V$, and denote the set of action profiles by $S = \prod S_i$. Let $u_i : S \rightarrow \mathbb{R}$ be player $i$’s utility function, which maps action profiles to payoffs. Note that we do not allow payoffs in $\{-\infty, \infty\}$.

2.2. Graphical games. We identify $V$ with the set of nodes of a simple, undirected graph $G = (V, E)$. The game $G$ is a graphical game on $G$ if each player’s utility is a function of the strategies of her neighbors in $G$. This can be formally expressed by the following condition on the utility functions. Choose from each $S_i$ an arbitrary distinguished strategy $o_i$. Given $a \in S$, denote $a^i = (a_1, \ldots, a_{i-1}, o_i, a_{i+1}, \ldots, a_n)$.

**Definition 2.1.** $G$ is a graphical game on $G$ if

\[(i, j) \notin E \Rightarrow \forall a \in S, u_i(a) = u_i(a^j). \tag{2.1}\]

It is easy to verify that this definition does not depend on the choice of $\{o_i\}_{i \in V}$.

2.3. Potential games.

**Definition 2.2.** $G$ is a potential game if there exists a (potential) function $\Phi : S \rightarrow \mathbb{R}$ such that for all $i \in V$ and $a \in S$

\[u_i(a) - u_i(a^i) = \Phi(a) - \Phi(a^i). \tag{2.2}\]

Note that if $\Phi$ is a potential for $G$ then $\Phi + C$ is also a potential for $G$, for any $C \in \mathbb{R}$. We make a canonical choice and assume always that

\[\sum_{a \in S} e^{\Phi(a)} = 1. \tag{2.3}\]
Here we use the fact that utilities (and hence potentials) cannot be infinite. It is easy to see that given a potential game, all its potentials differ by a constant.

2.4. Markov random fields. Let $G = (V, E)$ be a finite, simple, undirected graph. Associate with each $i \in V$ a random variable $X_i$, and denote $X = (X_1, \ldots, X_n)$. $X$ is called a random field on $G$.

A subset $A \subseteq V$ is a $(U, W)$-cut in the graph if every path from $U \subseteq V$ to $W \subseteq V$ must pass through $A$. For any subset $Z \subseteq V$ define the random variable $X_Z = \{X_i : i \in Z\}$ to be the restriction of $X$ to $Z$.

The random field $X$ is said to be positive if its distribution is absolutely continuous with respect to the product of its marginal distributions. For discrete distributions, this is equivalent to requiring the event $(X_1, \ldots, X_n) = (a_1, \ldots, a_n)$ to have positive probability whenever all of the events $X_i = a_i$ have positive probability.

**Definition 2.3.** $X$ is a Markov random field (MRF) if, for every $(U, W)$-cut $A$ it holds that conditioned on $X_A$, $X_U$ is independent of $X_W$.

For positive random fields, Markov random fields can be characterized by a weaker condition, namely that for every pair $i \neq j \in V$ such that $(i, j) \notin E$ it holds that, conditioned on $X_{V \setminus \{i,j\}}$, $X_i$ is independent of $X_j$. While this seems to be a well known fact, we were not able to find a reference for its proof, and therefore provide it (for finite distributions) in Appendix A.

3. Characterization of graphical potential games

3.1. Decomposing graphical potential games. Our first result shows that the potential of a graphical potential game can be decomposed into local potentials.

For $W \subseteq V$ and $a \in \mathcal{S}$, let $a^W \in \prod_{i \in W} \mathcal{S}_i$ be the restriction of $a$ to $W$, given by $(a^W)_i = a_i$, for $i \in W$.

Denote by $\mathcal{C}(G)$ the set of maximal cliques in $G$; these are cliques which are not subsets of strictly larger cliques.

**Theorem 3.1.** Let a game $\mathcal{G}$ be both a graphical game on $G$ and a potential game with potential $\Phi$. Then $\Phi$ can be written as

$$\Phi(a) = \sum_{C \in \mathcal{C}(G)} \Phi^C(a^C),$$

for some functions $\Phi^C: \prod_{i \in C} \mathcal{S}_i \to \mathbb{R}$. 

The functions $\Phi^C$ are called local potentials.

Before proving this proposition we introduce the Hessian $\Phi_{ij}$ and prove a lemma. For $a \in S$ and $i,j \in V$, denote

$$a^{ij} = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n),$$

and note that $(a^i)^j = (a^j)^i = a^{ij}$.

Let $\Phi_i : S \to \mathbb{R}$ be given by

$$\Phi_i(a) = \Phi(a) - \Phi(a^i).$$

Alternatively, by (2.2), $\Phi_i(a) = u_i(a) - u_i(a^i)$.

The Hessian $\Phi_{ij} : S \to \mathbb{R}$ is given by

$$\Phi_{ij}(a) = \Phi_i(a) - \Phi_i(a^j) = \Phi(a) - \Phi(a^i) + \Phi(a^j) + \Phi(a^{ij}).$$

**Lemma 3.2.** Let a game $G$ be both a graphical game on $G$ and a potential game with potential $\Phi$. Then $\Phi_{ij} = 0$ for all $(i,j) \notin E$.

**Proof.** Choose $(i,j) \notin E$. Then by (2.1) we have that $u_i(a) = u_i(a^i)$ and that $u_i(a^i) = u_i(a^{ij})$. Hence

$$0 = \left[u_i(a) - u_i(a^i)\right] - \left[u_i(a^i) - u_i(a^{ij})\right]$$

$$\quad = \left[u_i(a) - u_i(a^i)\right] - \left[u_i(a^i) - u_i(a^{ij})\right]$$

$$\quad = \Phi_i(a) - \Phi_i(a^j)$$

$$\quad = \Phi_{ij}(a).$$

**Proof of Theorem 3.1.** Let $\mathbb{P}$ be a probability distribution over the set of strategy profiles $S$ given by

$$(3.2) \quad \mathbb{P}[a] = e^{\Phi(a)}.$$ 

By (2.3) this is indeed a probability distribution. Let $X = (X_1, \ldots, X_n)$ be a random field on $G$ with law $\mathbb{P}$. Then $X_i$ is a (random according to $\mathbb{P}$) strategy played by player $i$. Note that the $X$ is a positive random field.

We will prove the claim by showing that for every $(i,j) \notin E$, the random variables $X_i$ and $X_j$ are independent, conditioned on $\{X_k\}_{k \notin \{i,j\}}$. This will prove that $X$ is a Markov random field over the graph $G$, and so the claim will follow immediately from the Hammersley-Clifford Theorem (see [8], and also [7,15,17]) for positive MRFs, which states that $\mathbb{P}$ be can decomposed as

$$\log \mathbb{P}[X = a] = \sum_{C \in \mathcal{C}(G)} \Phi^C(a^C),$$

for some functions $\Phi^C : \prod_{i \in C} S_i \to \mathbb{R}$.
Choose \((i,j) \not\in E\). Then by Lemma 3.2 we have that, for every \(a \in S\),
\[
\Phi(a) + \Phi(a^{ij}) = \Phi(a^i) + \Phi(a^j).
\]

By (3.2), this can be written as
\[
(3.3) \quad \mathbb{P}[X = a] \cdot \mathbb{P}[X = a^{ij}] = \mathbb{P}[X = a^i] \cdot \mathbb{P}[X = a^j].
\]

Denote by \(X^{ij}(a) = \{X_k = a_k\}_{k \not\in \{i,j\}}\) the event that \(X_k = a_k\) for all \(k \not\in \{i,j\}\). Then we can write (3.3) as
\[
\mathbb{P}[X = a_i, X = a_j, X^{ij}(a)] = \mathbb{P}[X = o_i, X = o_j, X^{ij}(a)].
\]

Hence
\[
\mathbb{P}[X = a_i, X = a_j, X^{ij}(a)] \cdot \mathbb{P}[X = o_i, X = o_j] = \mathbb{P}[X = a_i, X = o_j, X^{ij}(a)].
\]

Summing over all possible values of \(a_i\) and \(a_j\) we arrive at
\[
\mathbb{P}[X = o_i, X = o_j] = \mathbb{P}[X = a_i, X = o_j].
\]

Finally, we recall that the choice of \(\{o_i\}_{i \in V}\) was arbitrary, and so \(X_i\) and \(X_j\) are independent, conditioned on \(\{X_k\}_{k \not\in \{i,j\}}\).

3.2. Potentials of graphical potential games. Theorem 3.1 states that if a potential game is graphical, then the potential has the form (3.1). We cannot hope to have the opposite direction (i.e., if the potential function has the form (3.1) then the game is graphical) because of the following observation: If we add a constant payoff of \(c\) to player \(i\) whenever player \(j \neq i\) plays a certain action \(a_j\), then the potential function does not change; on the other hand, if in the original game player’s \(j\) action does not influence player’s \(i\) payoff (i.e., there is no edge \((i,j)\) in \(G\)), then in the new game (with the additional payoff of \(c\)) this is no longer the case. Such a change preserves the potential function but not the graphical structure. Therefore, we cannot deduce a result on the graphical structure of the game from the potential function.

However, the above mentioned change is a special type of change of the game in the following sense: it does not cause any strategic change in the game, because whether or not player \(i\) receives an additional payoff of \(c \neq 0\) does not depend on her own behavior. This motivates the notion of strategically equivalent games introduced by [12]. Denote by \(S^{-i}\) the set of strategy profiles of players in \(V \setminus \{i\}\).
Definition 3.3. Two games $G$ and $G'$ with payoff functions $u$ and $w$ over the same action profile set $S$ are called strategically equivalent if there exist functions $f_1, \ldots, f_n$ where $f_i : S^{-i} \to \mathbb{R}$ such that $u_i(a) = w_i(a) + f_i(a_{-i})$. We say that $G$ is a strategically equivalent transformation of $G'$.

Note again, that since the additional payoff of $f_i(a_{-i})$ does not depend on player’s $i$ behavior, the transformation cause no strategic change. In particular, any (mixed) Nash equilibrium of $G$ is a (mixed) Nash equilibrium of $G'$.

We next show that up to a strategically equivalent transformation of the game, the decomposition (3.1) is a necessary and sufficient condition for a potential game to be graphical.

Theorem 3.4. A potential game $G$ with potential $\Phi$ is strategically equivalent to a graphical game on $G$ if and only if the potential $\Phi$ has the form

$$\Phi(a) = \sum_{C \in \mathcal{C}(G)} \Phi^C(a^C).$$

Proof. If $G$ is strategically equivalent to a graphical game $G'$ on $G$, then $G'$ is also a potential game with the same potential function $\Phi$, because a strategic equivalent transformation preserves the potential function (see [12]). By Theorem 3.1 $\Phi$ has the form (3.4).

For the opposite direction, let $\Phi$ be of the form (3.4), and consider the game $G'$ where player’s $i$ payoff is defined by

$$u_i(a) = \sum_{C \in \mathcal{C}_{\max}(G) : i \in C} \Phi^C(a^C).$$

We show that $G'$ is a potential game on the graph $G$ with the potential $\Phi$. This will complete the proof, because $G$ and $G'$ have the same potential, and therefore are strategically equivalent (see [12] again).

Since $(a^C)_i = a^C$ whenever $i \notin C$, we have that $\Phi^C(a^C) - \Phi^C((a^i)^C) = 0$ whenever $i \notin C$. Hence

$$u_i(a) - u_i(a^i) = \sum_{C \in \mathcal{C}(G) : i \in C} \Phi^C(a^C) - \Phi^C((a^i)^C)$$

$$= \sum_{C \in \mathcal{C}(G)} \Phi^C(a^C) - \Phi^C((a^i)^C)$$

$$= \Phi(a) - \Phi(a^i),$$

which completes the proof.
where the first equality follows from (3.5), the second equality follows from the observation above the display, and the third from the fact that Φ has the form (3.4).

Finally, to see that $G$ is a graphical game over $G$, note that for $(i, j) \notin E$ we have that $(a^j)^C = a^C$ for all $C \in \mathcal{C}$ such that $i \in C$. Hence

$$u_i(a) - u_i(a^j) = \sum_{C \in \mathcal{C}(G) : i \in C} \Phi_C(a^C) - \Phi_C((a^j)^C)$$

$$= \sum_{C \in \mathcal{C}(G) : i \in C} \Phi_C(a^C) - \Phi_C(a^C)$$

$$= 0.$$  

\[ \square \]

3.3. Payoffs of graphical potential games. Theorem 3.1 characterizes the potential function of a graphical potential game, but it does not characterize the payoff functions of the players in such a game. We next present an exact characterization of the payoff functions in a graphical potential game. For every player $i \in V$ let $N(i)$ be the set of neighbors of player $i$, excluding $i$. Denote by $\mathcal{S}^{N(i)}$ the set of strategy profiles of players in $N(i)$.

**Theorem 3.5.** Let $G$ be a graph, and let $\Phi$ be a potential of the form (3.1). For every choice of functions $f_1, \ldots, f_n$ where $f_i : \mathcal{S}^{N(i)} \to \mathbb{R}$, the game with the payoffs

$$u_i(a) = \sum_{C \in \mathcal{C}(G) : i \in C} \Phi_C(a^C) + f_i(a^{N(i)})$$

(3.6)

is a graphical potential game over $G$ with the potential $\Phi$.

Conversely, for every graphical potential game over $G$ with potential $\Phi$ there exist functions $f_1, \ldots, f_n$ such that the payoffs are given by (3.6).

**Proof.** Let $G$ be a game with payoffs $u_i$, and let $G'$ be the game with payoffs

$$w_i(a) = \sum_{C \in \mathcal{C}(G) : i \in C} \Phi_C(a^C).$$

(3.7)

$G'$ is a graphical potential game over $G$ with the potential $\Phi$; this was proved in Theorem 3.4. The strategically equivalent transformation $u_i = w_i + f_i(a^{N(i)})$ preserves the potential function $\Phi$. Moreover it preserve the graphical structure as well, because $f_i$ depends only on the action of the neighbors of $i$. Therefore $G$ is a graphical potential game over $G$ with the potential $\Phi$. 

For the opposite direction, given a graphical potential game $\mathcal{G}$ over $G$ with the potential $\Phi$, we know that the payoffs can be written as:

$u_i(a) = \sum_{C \in \mathcal{C}(G): i \in C} \Phi_C(a_C) + g_i(a_{-i})$  \hspace{1cm} (3.8)

for some functions $g_i : S^{-i} \rightarrow \mathbb{R}$. This follows from the fact that the the games $\mathcal{G}$ and $\mathcal{G}'$ have the same potential function $\Phi$, and therefore are strategically equivalent.

The game $\mathcal{G}$ is graphical, therefore for every $j \notin N(i) \cup \{i\}$ it holds that $g(a_{-i}) = g((a_{-i})^j)$. Otherwise, $j$ will have influence on $i$’s payoff. Therefore $g$ does not depend on $a_j$, and it can be written as $g(a_{-i}) = f(a^{N(i)})$, which completes the proof. $\Box$

3.4. Equivalence of graphical potential games and Markov random fields. In the proof of Theorem 3.1 we showed how every graphical potential game on $G$ can be mapped to a Markov random field on $G$. We next show that this map is a bijection, so that every Markov random field on $G$ can be mapped back to the potential of a graphical game on $G$.

Let $\text{MRF}(G, S) \subset \Delta(S)$ denote the set of probability measures on $S$ which describe positive Markov random fields with underlying graph $G$. Let $\text{PGG}(G, S) \subset \mathbb{R}^S$ be the set of normalized (as in (2.3)) potentials of graphical potential games over $G$ with strategy profiles in $S$. Define $\psi : \text{PGG}(G, S) \rightarrow \text{MRF}(G, S)$ by

$[\psi(\Phi)](a) = e^{\Phi(a)}$. \hspace{1cm} (3.9)

This is the same mapping that we use in the proof of Theorem 3.1.

**Proposition 3.6.** The map $\psi : \text{PGG}(G, S) \rightarrow \text{MRF}(G, S)$ is a bijection.

**Proof.** It was shown in the proof of Theorem 3.1 that the image of $\psi$ is indeed in $\text{MRF}(G, S)$. Since $\psi$ is clearly one-to-one, it remains to be shown that it is onto.

Let $\mathbb{P} \in \text{MRF}(G, S)$ be a distribution over $S$ that is a positive MRF over $G$. Then, by the Hammersley-Clifford Theorem it can be written as

$\mathbb{P} [a] = \prod_{C \in \mathcal{C}(G)} e^{\Phi_C(a_C)}$, for some functions $\Phi_C : \prod_{i \in C} S_i \rightarrow \mathbb{R}$. 

Define a game $G$ on $V$ with strategies in $S$ by

$$u_i(a) = \sum_{C \in \mathcal{C}(G), i \in C} \Phi^C(a^C).$$

In the proof of Theorem 3.4 we showed that $G$ is a graphical potential game with the potential

$$(3.10) \quad \Phi(a) = \sum_{C \in \mathcal{C}(G)} \Phi^C(a^C),$$

and therefore $\psi(G) = \mathbb{P}$. 

\[ \square \]

4. Better-response paths in graphical potential games

In this section we make the simplifying assumption that utilities take integer values, and in fact demand that each local potential is integer. The extension to real values is straightforward if instead of considering better-response paths (see definition below) one considers $\varepsilon$-better-response paths, which are paths where each updating player improves her payoff by at least $\varepsilon$.

A game $G$ with potential $\Phi$ over a graph $G$ has $M$-bounded local potentials if in the decomposition of the potential (3.1) all the local potentials satisfy $|\Phi^C(a^C)| \leq M$ for all $a^C$. Here we do not require $\Phi$ to be normalized as in (2.3). $G$ is called an integral game if all the local potentials are always integer.

A better-response path of length $L$ is defined to be a sequence of action profiles $(a(t))_{t=0}^L$ such that $a(t + 1)$ defers from $a(t)$ in exactly one coordinate $i = i(t)$ and $a_i(t + 1)$ is a strict better-response to $a_{-i}(t)$. Player $i = i(t)$ is called the updating player at time $t$. Note that strict best-response dynamics are a special case, and therefore our results will apply to them, too.

An immediate observation regarding better-response dynamics is that in an integral game, the length of a better-response path is bounded by the range of the potential. This follows from the fact that in each update the potential increases by at least one.

We analyze general better-response paths, without committing to a particular order in which the player sequence $\{i(t)\}$ is chosen. This analysis, in particular, applies to continuous time better-response dynamics (or best-response dynamics) in which the path is drawn by Poisson arrivals.
We define the *clique-degree* of a graph as the largest number of maximal cliques that any vertex in the graph participates in.

**Claim 4.1.** For every $n$-player potential game on the graph $G$ with $M$-bounded local potentials, where the clique-degree of $G$ is $D$, the potential of the game is bounded by $|\Phi(a)| < nDM$ for all $a \in S$.

The claim follows from the fact that each vertex may participate in at most $D$ cliques.

An immediate consequence is that the length of any better-response path is at most $nDM$. Hence, the average number of updates performed by a player is at most $DM$, regardless of the size of the graph. The next claim shows that for graphs of slow enough growth (in a particular sense) we can bound by a constant the total number of updates performed by any player, without having to average over all players.

Given a graph $G$, denote by $\Delta(i, j)$ the graph distance between two nodes $i$ and $j$ in $G$. This is the length of a shortest path between $i$ and $j$. Denote by $B_r(G, i)$ the number of vertices at exactly distance $r$ from $i$:

$$B_r(G, i) = \# \{j : \Delta(i, j) = r\}.$$

**Theorem 4.2.** Let $G$ be an integral potential game on the graph $G$ with $M$-bounded local potentials, where $G$ has clique-degree $D$. Then the number of times that a player $k$ updates her action in any better-response path is at most

$$2DM \sum_{r=0}^{\infty} \left(1 - \frac{1}{2DM}\right)^r B_r(G, k).$$

The proof appears in Section 4.1. Before that, we give an example of an application, introduce a consequence of this theorem on graphs that satisfy a “slow enough growth” condition, and an example that demonstrates the tightness of this condition.

**Example 4.3.** Let $\mathbb{Z}_n^2$ be the $n \times n$ two dimensional grid, and consider the following game $G_n$, in which each player has two strategies. Label some subset of the edges blue and the rest red. The utility of each player is equal to the number of neighbors along blue edges that match her strategy, minus the number of neighbors along red edges that match her strategy. It is easy to see that $G$ is a graphical potential game on

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1Note that in a graph of constant degree $d$, the clique-degree is trivially bounded by the constant $2^d$, because the number of cliques that a vertex $i$ participates in is bounded by the number of subsets of neighbors of $i$. More interestingly, the number of maximal cliques is sharply bounded by $4 \cdot 3^{d/3}$ [13].
that local potentials are 1-bounded, that the clique-degree of \( \mathbb{Z}_n^2 \) is four (since all maximal cliques are of size two and correspond to the edges), and that \( B_r(\mathbb{Z}_n^2, i) \leq 4r \).

Hence by Theorem 4.2, in any better-response path, no player changes her strategy more than 1792 times, independently of \( n \).

We generalize this example to a larger class of games and graphs.

**Definition 4.4.** Given a function \( f : \mathbb{N} \to \mathbb{R} \) we say that the growth of a graph \( G \) is bounded by \( f \) if for every vertex \( i \) and every \( r \in \mathbb{N} \) it holds that \( B_r(G, i) \leq f(r) \).

In order to gain some intuition about the notion of the growth of the graph let us mention several graphs and their growths:

- For a line or a cycle with \( n \) vertices, the growth is bounded by \( f(r) = 2 \).
- The growth of the \( m \)-dimensional grid is bounded by \( f(r) = (2mr)^{m-1} \), which is polynomial in \( r \) for a constant \( m \).
- The growth of a binary tree is at least \( 2^r \), and therefore is exponential in \( r \).

A straightforward corollary from Theorem 4.2 is the following:

**Corollary 4.5.** Fix \( M \), and let \( G \) be a graph with click-degree \( D \) and with growth that is bounded from above by the following exponentially increasing function

\[
f(r) = c \left( 1 + \frac{1}{4DM} \right)^r
\]

for some constant \( c > 0 \). Then for every potential game on the graph \( G \) with \( M \)-bounded local potentials, the number of updates of every player along every better-response path is bounded by the constant \( 8cD^2M^2 \). In particular, this bound does not depend on the size of the graph.

Note that Corollary 4.5 holds, in particular, for the case where the graph has any subexponential growth.

\(^2\)Note that this is not tight; a tighter bound for this particular game and graph can be (essentially) found in [19].
Proof of Corollary 4.5. By Theorem 4.2 we can bound the number of updates by

\[ 2DM \sum_{r=0}^{\infty} \left( 1 - \frac{1}{2DM} \right)^r \left( 1 + \frac{1}{4DM} \right)^r \]

\[ \geq 2cDM \sum_{r=0}^{\infty} \left( 1 - \frac{1}{4DM} \right)^r = 8cD^2M^2. \]

\[ \square \]

The bound on the growth of the graph is crucial for the result of Theorem 4.2 (and Corollary 4.5), as demonstrated by the following example.

Example 4.6. We denote by \( BT_k = (V, E) \) the binary tree graph of depth \( k \) and \( n = 2^{k+1} - 1 \) vertices. For \( l = 0, ..., k \) we denote by \( V_l = \{v_{l1}, ..., v_{2^l}\} \) the set of vertices at depth-level \( l \).

We consider the majority game \( G \) on \( BT_k \) in which each player has two strategies, and her utility is equal to the number of neighbors who played the same strategy. As in Example 4.3, it is easy to see that \( G \) is a graphical potential game on \( BT_k \), that local potentials are 1-bounded, and that the clique-degree of \( G \) is three, since again all maximal cliques correspond to edges. However, in this case \( B_r(BT_k, i) \geq 2^r \) for \( r < k \), and so we cannot hope to use Theorem 4.2 to bound the number of changes without dependence on \( n \). Indeed, we show that no such bound exists.

Consider the following better-response path\(^3\) In the initial configuration \( a(0) \) we set the action of all players to be 1 at even depth-levels and (-1) at odd depth-levels (i.e., \( a(0)_v = (-1)^l \) for \( v \in V_l \)). The players update their actions according to the following order:

\(^3\)In fact, the presented path is also a best-response path.
\[
\begin{align*}
\{ & \text{Step 1.1: All the players in } V_0. \\
& \text{Step 2.1: All the players in } V_1. \\
& \text{Step 2.2: All the players in } V_0. \\
& \text{Step 3.1: All the players in } V_2. \\
& \text{Step 3.2: All the players in } V_1. \\
& \text{Step 3.3: All the players in } V_0. \\
& \vdots \\
& \text{Step } l.j \text{ for } 1 \leq j \leq l \leq k: \text{ All the players in } V_{l-j}. \\
& \vdots \\
& \{ \text{Step } k.k: \text{ All the players in } V_0. \\
\end{align*}
\]

where at each step \( l.j \) the players in \( V_{l-j} \) update their actions in an arbitrary order.

Note that at each step \( l.j \), every player \( v \in V_{l-j} \) indeed updates her action, because the order is designed in such a way that both \( v \)'s neighbors at depth-level \( l - j + 1 \) play the opposite action, and player \( v \) has at most three neighbors. Note also that the root player \( v_1^0 \) updates her action \( k = \log_2(\frac{n+1}{2}) \) times. Namely the number of her updates is \textit{not} bounded by a constant. This is possible on a sequence of growing binary trees because their growth rate is too high. As Example 4.3 above shows, no such construction is possible on a sequence of (polynomially) growing grids.

4.1. \textbf{Proof of Theorem 4.2} Let \( G \) be a graph of clique-degree \( D \) with \( n \) nodes, and let \( i \) be a vertex in \( G \). Let \( C \) be a clique in \( G \). We define \( \Delta(i,C) \), the distance between \( i \) and \( C \), as the minimal graph distance between \( i \) and a vertex in \( C \).

Choose \( \lambda \) satisfying

\[
1 - \frac{1}{2DM} < \lambda < 1.
\]

Given a potential \( \Phi \) with \( M \)-bounded local potentials on \( G \), fix a vertex \( k \) in \( G \), and define the potential \( \Theta \) by

\[
\Theta = \sum_{C \in C(G)} \lambda^{\Delta(k,C)} \Phi^C.
\]
By the definition of better-response dynamics, $\Phi$ increases along any better-response path. We claim that the same holds for $\Theta$. This follows from the next claim.

**Lemma 4.7.** For every $a \in \mathcal{S}$ and $b = (b_i, a_{-i}) \in \mathcal{S}$ it holds that if $\Phi(b) > \Phi(a)$ then $\Theta(b) > \Theta(a)$.

In other words, the lemma claims that $\Theta$ is ordinal potential (see [12]) of the game $\mathcal{G}$.

**Proof.** Fix $a$ and $b$ such that $\Phi(b) > \Phi(a)$. Since they differ only in the strategy of $i$ then

\[
\Phi(b) - \Phi(a) = \sum_{C \in \mathcal{C}(G) : i \in C} \left[ \Phi^C(b) - \Phi^C(a) \right].
\]

and likewise

\[
\Theta(b) - \Theta(a) = \sum_{C \in \mathcal{C}(G) : i \in C} \lambda^{\Delta(k,C)} \left[ \Phi^C(b) - \Phi^C(a) \right].
\]

Now, note that for any clique $C$ that includes $i$ it holds that either $\Delta(k,C) = \Delta(k,i)$ or else $\Delta(k,C) = \Delta(k,i) - 1$. Hence, if we denote

\[
C_1 = \{ C \in \mathcal{C}(G) : i \in C, \Delta(k,C) = \Delta(k,i) \}
\]

and

\[
C_2 = \{ C \in \mathcal{C}(G) : i \in C, \Delta(k,C) = \Delta(k,i) - 1 \}
\]

then we can write (4.2) as

\[
\frac{\Theta(b) - \Theta(a)}{\lambda^{\Delta(k,C) - 1}} = \lambda \sum_{C_1} \left[ \Phi^C(b) - \Phi^C(a) \right] + \sum_{C_2} \lambda^{\Delta(k,C)} \left[ \Phi^C(b) - \Phi^C(a) \right]
\]

\[
= \sum_{C \in \mathcal{C}(G) : i \in C} \left[ \Phi^C(b) - \Phi^C(a) \right] - (1 - \lambda) \sum_{C_1} \left[ \Phi^C(b) - \Phi^C(a) \right].
\]

By (4.1) this implies that

\[
\frac{\Theta(b) - \Theta(a)}{\lambda^{\Delta(k,C) - 1}} \geq \Phi(b) - \Phi(a) - (1 - \lambda) \sum_{C_1} \left[ \Phi^C(b) - \Phi^C(a) \right],
\]

and by the definition of $\lambda$

\[
\geq \Phi(b) - \Phi(a) - \frac{1}{2DM} \sum_{C_1} \left[ \Phi^C(b) - \Phi^C(a) \right].
\]
Now, $\Phi(b) > \Phi(a)$ implies $\Phi(b) - \Phi(a) \geq 1$, since the potential is integer. Hence

$$ \geq 1 - \frac{1}{2DM} \sum_{C_1} [\Phi^C(b) - \Phi^C(a)]. $$

But the summand has at most $D$ terms, each one of which is strictly less than $2M$ (because the local potentials are bounded by $M$). Hence

$$ \frac{\Theta(b) - \Theta(a)}{\lambda^{\Delta(k,C)-1}} \geq 0. $$

By the definition of $\Theta$ we have that

$$ |\Theta(a)| \leq \sum_{C \in \mathcal{C}(G)} \lambda^{\Delta(k,C)}|\Phi^C(a)|. $$

Since $|\Phi^C(a)| \leq M$ then

$$ |\Theta(a)| \leq \sum_{C \in \mathcal{C}(G)} \lambda^{\Delta(k,C)} M 
= M \cdot \sum_{r=0}^{\infty} \lambda^r \cdot \#\{C \in \mathcal{C}(G) : \Delta(k,C) = r\}. $$

Now, the number of cliques at distance $r$ from $k$ is at most $D$ times the number of vertices at distance $r$ from $k$. Hence

$$ |\Theta(a)| \leq DM \sum_{r=0}^{\infty} \lambda^r B_r(G,k). $$

Whenever player $k$ updates her strategy then $\Phi$ increases. By Claim 4.7, $\Theta$ then also increases. By the definition of $\Theta$, it increases by at least one, because $\Delta(k,C) = 0$ for all $C \ni k$. Hence the total number of times that player $k$ updates her action is at most

$$ 2DM \sum_{r=0}^{\infty} \lambda^r B_r(G,k). $$

Finally, since we let $\lambda$ have any value $> 1 - 1/(2DM)$ then it follows that the total number of times that player $k$ updates her strategy is at most

$$ 2DM \sum_{r=0}^{\infty} \left(1 - \frac{1}{2DM}\right)^r B_r(G,k). $$

This completes the proof of Theorem 4.2.
5. Generalization and open problem

5.1. Infinite graphs. Let $G = (V, E)$ be a countably infinite graph, where the degree of each node is finite.

**Definition 5.1.** A graphical game $G$ on $G$ is a generalized graphical potential game if its utilities are given by

$$u_i(a) = \sum_{C \in \mathcal{C}(G): i \in C} \Phi^C(a^C)$$

for some local potentials $\{\Phi^C\}_{C \in \mathcal{C}}$.

Note that $G$ is not necessarily a potential game, as the sum of the local potentials may diverge. However, given the decomposition of graphical potential games into local potentials (Theorem 3.1), this is indeed a natural generalization.

The clique-degree of an infinite graph can be defined as for finite graphs, although in this case it may diverge. Likewise, the definition of $M$-boundedness can be used, since it only depends on the local potentials.

The statement of the next theorem is identical to that of Theorem 4.2, with the exception that it refers more widely to generalized graphical potential games. An inspection of the proof of Theorem 4.2 will reveal that it applies here too.

**Theorem 5.2.** Let $G$ be an integral generalized potential game on the graph $G$ with $M$-bounded local potentials, where $G$ has clique-degree $D$. Then the number of times that a player $k$ updates her action in any better-response path is at most

$$2DM \sum_{r=0}^{\infty} \left(1 - \frac{1}{2DM}\right)^r B_r(G, k).$$

We state without proof (which is standard) that this result extends to better-response dynamics governed by Poisson arrivals on the players.

5.2. Open problem. Consider the class of graphical potential games on a graph with bounded degree $d$. For integral $M$-bounded games and for a constant $M$, the length of every strict best-response path is bounded by $O(n)$, since the potential is bounded by $O(n)$. Consider the best-response dynamics in which, at each step, a single player is chosen uniformly at random to update her action, and makes no change if a strictly better action is not available. The expected time of reaching a pure Nash equilibrium is bounded by $O(n^2)$. This follows from the fact that at each step in which the current strategy profile is not a pure Nash
equilibrium, there is a probability of at least \(1/n\) to choose a player who will update her action, and the maximal number of updates is bounded by \(O(n)\).

However, if we drop the integer-valued assumption, then the above argument fails because we cannot bound the length of a best-response path.

**Open question:** Let \(G\) be a graph on \(n\) vertices with bounded degree \(d\). Let \(G^\star\) be a real-valued graphical potential game on \(G\). How fast does best-response dynamics reach a pure Nash equilibrium in \(G^\star\)?

An interesting aspect of this question is to understand whether the rate of convergence is polynomial in \(n\), or whether there are examples of a game and an initial strategy profile \(a(0)\) such that the expected time for reaching a pure-Nash equilibrium from \(a(0)\) is exponential in \(n\). It is known (see [6]) that in general potential games an exponential rate of convergence is possible. The question is whether the graphical structure of the game improves significantly the rate of convergence to polynomial.

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Appendix A. Markov properties of positive random fields

Let $G = (V,E)$ be a finite simple graph with $|V| = n$, and let $X = (X_1, \ldots, X_n)$ be random field on $G$. Recall that we say that $X$ is positive if $P[X_1 = a_1, \ldots, X_n = a_n] > 0$ whenever $P[X_i = a_i] > 0$ for $i = 1, \ldots, n$. We say that $X$ has the pairwise Markov property if for all $(i,j) \not\in E$ it holds that $X_i$ is independent of $X_j$, conditioned on $X_{V\setminus\{i,j\}}$. In this section we prove the following proposition.

**Proposition A.1.** If $X$ is positive and has the pairwise Markov property then it is a Markov random field.

To prove this proposition we will need the following lemma. Let $G^{1,2}$ be the graph derived from $G$ by amalgamating the nodes 1 and 2 into a node $(1,2)$; the vertices of $G^{1,2}$ are $V^{1,2} = \{(1,2), 3, \ldots, n\}$, and the edges of $G^{1,2}$ are

$E^{1,2} = \{(i,j) \in E : i,j > 2\} \cap \{(1,2), j : j > 2, (1,j) \in E \text{ or } (2,j) \in E\}$.

Accordingly, let $X^{1,2} = ((X_1, X_2), X_3, \ldots, X_n)$ be the associated random field, where we unite the two random variables $X_1$ and $X_2$ into the random variable $(X_1, X_2)$, and associate it to the node $(1,2)$. Generally, define $G^{i,j}$ and $X^{i,j}$ likewise.
Lemma A.2. When $X$ is positive and has the pairwise Markov property with respect to $G$, then $X^{i,j}$ is positive and has the pairwise Markov property, with respect to $G^{i,j}$.

Proof. Without loss of generality, we assume that $i = 1$ and $j = 2$. It is immediate that $X^{1,2}$ is positive.

Every vertex $k$ that is connected to either 1 or 2 in the graph $G$, is connected to $(1, 2)$ in $G^{1,2}$. Therefore, for proving the pairwise property on the graph $G^{1,2}$ we will consider a vertex $k$ such that neither $(1, k)$ nor $(2, k)$ is an edge in $G$, so that $((1, 2), k)$ is not an edge in $G^{1,2}$. Without loss of generality, we assume that $k = 3$. We will show that, conditioned on $Y = X_{V\setminus\{1,2,3\}}$, $(X_1, X_2)$ is independent of $X_3$. This will prove the claim.

Denote by $S_i$ the (finite) support of $X_i$. We would like to show that for all $a_1 \in S_1, a_2 \in S_2$ and $a_3 \in S_3$ it holds that
\[
\mathbb{P}[(X_1, X_2, X_3) = (a_1, a_2, a_3) | Y] = \mathbb{P}[(X_1, X_2) = (a_1, a_2) | Y] \cdot \mathbb{P}[X_3 = a_3 | Y].
\]
(A.1)

Since $X$ is positive $\mathbb{P}[X_3 = a_3 | Y, (X_1, X_2) = (a_1, a_2)]$ is well defined, and we can apply the pairwise Markov property of $X$ once to $X_1, X_3$ and once to $X_2, X_3$ to arrive at
\[
\mathbb{P}[X_3 = a_3 | Y, (X_1, X_2) = (a_1, a_2)] = \mathbb{P}[X_3 = a_3 | Y, X_1 = a_1]
\]
(A.2) \[
= \mathbb{P}[X_3 = a_3 | Y, X_2 = a_2].
\]

Since $X$ is positive we can apply the same argument with any $b_2 \in S_2$ substituted for $a_2$, and conclude, as in (A.2), that
\[
\mathbb{P}[X_3 = a_3 | Y, X_1 = a_1] = \mathbb{P}[X_3 = a_3 | Y, X_2 = b_2]
\]
for all $b_2 \in S_2$, and that therefore
\[
\mathbb{P}[X_3 = a_3 | Y, X_1 = a_1] = \mathbb{P}[X_3 = a_3 | Y].
\]
Applying (A.1) now yields
\[
\mathbb{P}[X_3 = a_3 | Y, (X_1, X_2) = (a_1, a_2)] = \mathbb{P}[X_3 = a_3 | Y],
\]
which proves the claim.

Proof of Proposition A.1. Let $A$ be a $(U, W)$-cut. We would like to show that $X_U$ is independent of $X_W$, conditioned on $X_A$.

Amalgamate all the vertices in $U$ to a single vertex, and likewise amalgamate $A$ and $W$. The resulting graph is (a subgraph of)
\[
G' = (V', E') = (\{U, A, W\}, \{(U, A), (A, W)\}),
\]
and the associated random field is precisely \((X_U, X_A, X_W)\). By repeated applications of Lemma A.2 this graph has the pairwise Markov property, and hence \(X_U\) is independent of \(X_W\), conditioned on \(X_A\). □

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