The exponentiated Hencky-logarithmic strain energy.  
Part III: Coupling with idealized multiplicative isotropic finite strain plasticity

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\textit{Dedicated to David J. Steigmann, a great scientist and good friend}

Abstract

We investigate an immediate application in finite strain multiplicative plasticity of the family of isotropic volumetric-isochoric decoupled strain energies

\[ F \mapsto W_{\text{sh}}(F) := \tilde{W}_{\text{sh}}(U) := \begin{cases} \frac{\mu}{k} e^{\frac{1}{k} \text{dev}_a \log U}^2 + \frac{\kappa}{2k} e^{\frac{1}{k} \text{ja}(\log U)}^2 & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases} \]

based on the Hencky-logarithmic (true, natural) strain tensor $\log U$. Here, $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu + \lambda}{3} > 0$ is the infinitesimal bulk modulus with $\lambda$ the first Lamé constant, $k, \tilde{k}$ are additional dimensionless material parameters, $F = \nabla \varphi$ is the gradient of deformation, $U = \sqrt{F^T F}$ is the right stretch tensor and $\text{dev}_a \log U = \log U - \frac{1}{n} \text{tr}(\log U) \cdot \mathbb{1}$ is the deviatoric part of the strain tensor $\log U$.

Based on the multiplicative decomposition $F = F_\epsilon F_p$, we couple these energies with some isotropic elasto-plastic flow rules $F_p \frac{\partial}{\partial t} [F_p^{-1}] \in -\bar{\nabla} \chi (\text{dev}_3 \Sigma)$ defined in the plastic distortion $F_p$, where $\bar{\nabla} \chi$ is the subdifferential of the indicator function $\chi$ of the convex elastic domain $E_\epsilon (\Sigma, \frac{1}{k} \Sigma_0^2)$ in the mixed-variant $\Sigma$-stress space, $\Sigma_0 = F_p^T D_{F_p} W_{\text{tot}}(F)$ and $W_{\text{tot}}(F)$ represents the isochoric part of the energy. While $W_{\text{sh}}$ may lose ellipticity, we show that loss of ellipticity is effectively prevented by the coupling with plasticity, since the ellipticity domain of $W_{\text{sh}}$ on the one hand, and the elastic domain in $\Sigma_0$-stress space on the other hand, are closely related. Thus the new formulation remains elliptic in elastic unloading at any given plastic predeformation. In addition, in this domain, the true-stress-true-strain relation remains monotone, as observed in experiments.

\textbf{Key words}: Hencky strain, logarithmic strain, natural strain, true strain, Hencky energy, volumetric-isochoric split, multiplicative decomposition, elasto-plasticity, bounded elastic distortions, ellipticity domain, return mapping algorithm, finite strain plasticity, isotropic formulation, 9-dimensional flow rule, associated plasticity, subdifferential formulation, convex elastic domain, plastic spin

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1 Introduction

1.1 Preliminaries

It is impossible to give an account of all works treating finite strain plasticity based in some way or another on the logarithmic strain space description. The logarithmic description\(^1\) is arguably the simplest approach to finite plasticity, suitable for the phenomenological description of isotropic polycrystalline metals if the structure of geometrically linear theories is used with respect to the Lagrangian logarithmic strain. In this paper we do not consider hypoelastic-plastic models [73, 22, 43, 97] in which, contrary to hyperelastic models, the potential character of the elastic energy is ignored [42, 53]. Otherwise, they are simply the hyperelastic models rewritten in a suitable incremental form. In case of the logarithmic rate, however, the hypoelastic model integrates exactly to the well-known hyperelastic quadratic Hencky model.

In isotropic finite strain computational hyperelasto-plasticity [4, 3, 19, 85, 86, 79, 13, 80, 98] the mostly used elastic energy is the quadratic Hencky logarithmic energy [91, 58, 16, 47, 96, 99, 7, 26] (see also [24, 81, 83, 77, 75, 17, 52])

\[
W_h(F_e) := \frac{\mu}{4} \| \text{dev}_n \log C_e \|^2 + \frac{\kappa}{8} [\text{tr}(\log C_e)]^2 = \mu \| \text{dev}_n \log U_e \|^2 + \frac{\kappa}{2} [\text{tr}(\log U_e)]^2, \tag{1.1}
\]

where \(\mu > 0\) is the infinitesimal shear modulus, \(\kappa > 0\) is the infinitesimal bulk modulus, \(C_e := F_e^T F_e\) is the elastic right Cauchy-Green tensor, \(U_e\) the right stretch tensor, i.e. the unique element of PSym\((n)\) for which \(U_e^2 = C_e\) and

\[
F = F_e \cdot F_p \tag{1.2}
\]

is the multiplicative decomposition of the deformation gradient [33, 34, 35, 38, 61, 68, 9, 50, 49]. Here we adopt the usual abbreviations of Lie-group theory and we let Sym\((n)\) and Sym\((n)\) denote the symmetric and positive definite symmetric tensors respectively. We have used the Frobenius tensor norm \(\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}\), where \(\langle X, Y \rangle_{\mathbb{R}^{n \times n}}\) is the standard Euclidean scalar product on \(\mathbb{R}^{n \times n}\). The identity tensor on \(\mathbb{R}^{n \times n}\) will be denoted by \(\mathbb{1}\), so that \(\text{tr}(X) = \langle X, \mathbb{1} \rangle\), while \(\text{dev}_n X = X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{1}\) is the n-dimensional deviatoric part of a second order tensor \(X \in \mathbb{R}^{n \times n}\).

Among the works which use the Hencky strain in elasto-plasticity we may also mention [87, 47, 75, 8, 77, 76, 14, 18]. The expression \(W_h\) is the energy considered by J.C. Simo (see Eq. (3.4), page 147, from [88] and also [1]) because

\[
W_h(F_e) := \mu \| \text{dev}_n \log F_e^T F_e \|^2 + \frac{\kappa}{8} [\text{tr}(\log F_e^T F_e)]^2 = \frac{\mu}{4} \| \text{dev}_n \log F_e^T F_e \|^2 + \frac{\kappa}{2} [\text{tr}(\log F^T F)]^2. \tag{1.3}
\]

As J.C. Simo already pointed out [89, page 392], the Hencky energy \(W_h\) "has the correct behaviour for extreme strains in the sense that" \(W(F_e) \to \infty\) as \(\text{det} F_e \to 0\) and, likewise \(W(F_e) \to \infty\) as \(\text{det} F_e \to \infty\), but \(W_h\)

\(^1\)According to Hanin and Reiner [23, page 384]: "... there are problems in large plastic deformation. Here the only adequate measure in the Hencky measure, because this is the only measure in which the extensions form a group as can be seen from the relation

\[
\log \frac{\ell_3}{\ell_1} = \log \left( \frac{\ell_3}{\ell_2} \frac{\ell_2}{\ell_1} \right) = \log \frac{\ell_3}{\ell_2} + \log \frac{\ell_2}{\ell_1},
\]

This property of forming a group is required in plasticity because in (ideal) plasticity the amount of finite deformation reached at any time is of no physical significance. As a matter of fact no definite meaning can be attached to such deformation because while in elasticity there exists an 'unstrained state' to which the length \(\ell_0\) is referred, no "undeformed state" can be defined. In plasticity, as in viscosity, the increase in length \(d\ell\), which takes place during the time increment \(dt\), can only be referred to the instantaneous length \(\ell\) so that the extension

\[
\varepsilon = \int_{t_0}^{t_{n+1}} \frac{d\ell}{\ell} = \log \frac{\ell_n}{\ell_{n+1}}
\]

which is Hencky's measure. At the same time, only in the Hencky measure can the cubical dilatation be measured by the first invariant as can be seen from

\[
\varepsilon_v = \log \frac{V}{V_0} = \log \left( \frac{\ell_1}{\ell_{n0}} \frac{\ell_2}{\ell_{n0}} \frac{\ell_3}{\ell_{n0}} \right) = \log \frac{\ell_1}{\ell_{n0}} + \log \frac{\ell_2}{\ell_{n0}} + \log \frac{\ell_3}{\ell_{n0}}
\]

so that the deviator is of physical significance; and plasticity relations must be expressed in terms of deviators."
"is not a convex function of" $\det F_c$ "and hence $W_H$ "cannot be a polyconvex function of the deformation gradient [...]. Therefore, the stored energy function" $W_H$ "cannot be accepted as a correct model of elasticity for extreme strains. Despite this shortcoming, the model provides an excellent approximation for moderately large elastic strains, vastly superior to the usual Saint-Venant-Kirchhoff model of finite elasticity". Furthermore, this limitation has negligible practical implications in realistic models of classical plasticity, which are typically restricted to small elastic strains, and is more than offset by the simplicity of the return mapping algorithm in stress space, which takes a format identical to that of the infinitesimal theory\( ^{\text{2}} \). The last statement is the core argument why the Hencky energy is favoured in computational metal elasticity\( ^{\text{3}} \).

Several models of such a type have been considered in [72, 32]. The decisive advantage of using the energy $W_H$ compared to other elastic energies stems from the fact that computational implementations of elasto-plasticity [17] based on the additive decomposition $\varepsilon = \varepsilon_c + \varepsilon_p$ in infinitesimal models [69, 15, 71, 62, 27], can be used with nearly no changes also in isotropic finite strain problems [89, page 392].

The computation of the elastic equilibrium at given plastic distortion $F_p$ suffers, however, under the well-known non-ellipticity\( ^{\text{4}} \) of $W_H$ [5, 2. 55, 28].

Recently, it has been discovered that the elastic Hencky energy does have a fundamental differential geometric meaning, not shared by any other elastic energy (see [63, 64, 39]). For this investigation new mathematical tools had to be discovered [70, 36] also having consequences for the classical polar decomposition [30, 29]. We denote by $C = F^T F$ the right Cauchy-Green tensor, $U$ the right stretch tensor, $B = F F^T$ the left Cauchy-Green (or Finger) tensor, and by $V$ the left stretch tensor.

In the remaining part of this paper, after a paragraph giving some information on the results obtained for the exponentiated Hencky energy, we will consider the coupling to finite plasticity based on a 9-dimensional flow rule [92, 21].

1.2 The exponentiated Hencky energy

With a view to overcome the shortcomings of the quadratic Hencky energy, in a previous work [66] we have modified the Hencky energy and we considered

$$ W_{\text{ex}}(F) = W_{\text{ex}}^\text{iso}(\frac{F}{\det F^2}) + W_{\text{ex}}^\text{vol}(\det F^2 \cdot \mathbb{I}) = \begin{cases} \frac{H}{k} k \| \varepsilon_c \| \log U \|^2 + \frac{K}{2k} \| \varepsilon_p \| (\log \det U)^2 & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0. \end{cases} \tag{1.4} $$

We have called this the exponentiated Hencky energy. For the two-dimensional situation $n = 2$ and for $\mu > 0, \kappa > 0$, we have established that the functions $W_{\text{ex}}$ : $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}^+$ from the family of exponentiated Hencky type energies are rank-one convex [66] for $k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$, while they are polyconvex [67] for $k \geq \frac{1}{3}$ and $\hat{k} \geq \frac{1}{8}$.

Regarding the three-dimensional case we have proved [66] that, for all $k > 0$, the function $F \mapsto e^{k \| \varepsilon_c \| \log U \|^2}$, $F \in \text{GL}^+(3)$ is not rank-one convex (and therefore not polyconvex). However, from this shortcoming, in the next section we will discuss an interesting relation between non-ellipticity of $W_{\text{ex}}$ in three-dimensions and finite plasticity models.

Moreover, in the first part [66] it is shown that the proposed energies have some other very useful properties: analytical solutions are in agreement with Bell’s experimental data; planar pure Cauchy shear stress produces

\[ \text{The isotropic Saint-Venant-Kirchhoff elastic energy } W_{\text{SVK}}(F_c) \text{ reads: } W_{\text{SVK}}(F_c) = \frac{H}{k} \| F^T F_c - \mathbb{I} \|^2 + \frac{K}{2k} \| \varepsilon_p \| \| F^T F_c - \mathbb{I} \|^2 \]

and does not satisfy the Baker-Ericksen-inequalities, and is not separably convex. Therefore $W_{\text{SVK}}(F_c)$ is not rank-one convex [55, 78, 84]. Moreover, in the neighborhood of the identity it has the wrong nonlinear second order correction compared to all known experimental facts. For this reason, $W_{\text{SVK}}(F_c)$ is not a useful strain energy expression and should therefore be avoided in simulations.

\[ \text{We need to be a little more specific. For the additive model in the format } \| \log C - \log C_0 \|^2 \text{ the complete systems of equations of the plastic flow rule are identical to the infinitesimal additive model, while for the truly multiplicative model the return mapping algorithm is similar to the infinitesimal case.} \]

\[ \text{We know that } W_H \text{ is L2-elliptic in a large neighbourhood of the identity if } \lambda, \mu > 0, \lambda_i \in [0.2162, 1.39561], \text{ [see 5, 6], therefore } F \mapsto W_H(F) \text{ is L2-elliptic for small elastic strains. Since the elasto-plastic model should secure small elastic strains anyway it seems that the non-ellipticity occurring for large elastic strains is not essential. However, in numerical FEM-implementation it is necessary to compute the so-called elastic trial stress. The corresponding elastic trial deformation states } F_c \text{ may well be far outside the L2-ellipticity range. The model using the Hencky energy } W_H \text{ cannot guarantee that the computation of the elastic trial state is well-posed!} \]
bimodal pure shear strain and the value 0.5 of Poisson’s ratio corresponds to exact incompressibility, the analytical expression of the pressure is in concordance with the classical Bridgman’s compression data for natural rubber etc. An immediate application to rubber-like materials is proposed in [51].

We note that the **Kirchhoff stress tensor** \(\tau_{\text{st}}\) corresponding to the exponentiated energies is given [74] by

\[
\tau_{\text{st}} = 2\mu e^k \| \text{dev} \log V \|^2 - \text{dev} \log V + \kappa e^k |\text{tr}(\log V)|^2 \text{tr}(\log V) \cdot \mathbb{1},
\]

while the **Cauchy stress tensor** is given by

\[
\sigma_{\text{st}} = e^{-\text{tr}(\log V)} \cdot \tau_{\text{st}}.
\]

Both tensors \(\sigma_{\text{st}}\) and \(\tau_{\text{st}}\) differ from their classical Hencky-counterparts \(\sigma_{\text{H}}\) and \(\tau_{\text{H}}\) only by some nonlinear scalar factors. Moreover, by orthogonal projection onto the Lie-algebra \(\mathfrak{s}(3)\) and \(\mathbb{R} \cdot \mathbb{1}\), respectively, we find

\[
\text{dev} \sigma_{\text{st}} = e_k \| \text{dev} \log V \|^2 \text{dev} \sigma_{\text{H}}, \quad \text{tr}(\sigma_{\text{st}}) = e_k |\text{tr}(\log V)|^2 \text{tr}(\sigma_{\text{H}}).
\]

Therefore, the deviatoric part of the Cauchy stress \(\text{dev} \sigma_{\text{st}}\) and the trace of the Cauchy stress \(\text{tr}(\sigma_{\text{st}})\) are in a simple relation with the corresponding quantities for the quadratic Hencky energy \(\mathcal{W}_H\). Hence, the change of a given FEM-implementation of \(\mathcal{W}_H\) into \(\mathcal{W}_{\text{st}}\) is nearly free of costs [93, 40, 54, 25].

We also need to introduce the convex elastic domain in the Kirchhoff-stress space

\[
\mathcal{E}_c(\tau_{\text{st}}, \frac{2}{3} \sigma^2) := \left\{ \tau_c \in \text{Sym}(3) \mid \| \text{dev} \tau_c \|^2 \leq \frac{2}{3} \sigma_e^2 \right\} \subset \text{Sym}(3).
\]

Incidentally, the set \(\mathcal{E}_c(\tau_{\text{st}}, \frac{2}{3} \sigma_e^2)\) coincides with the set considered in the study of the monotonicity properties of the map \(\log U \mapsto \mathcal{W}_{\text{st}}(\log U)\) which we have called the true-stress-true-strain (TSTS-M+) monotonicity condition [66, 31]. The monotonicity of the Cauchy stress tensor as a function of \(\log B\) or \(\log V\) means

\[
\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \rangle > 0, \quad \forall B_1, B_2 \in \text{PSym}(3), \quad B_1 \neq B_2,
\]

which implies the **true-stress-true-strain-invertibility** (TSTS-I), i.e. the invertibility of the map \(\log B \mapsto \sigma(\log B)\). This means that for our \(\mathcal{W}_{\text{st}}\)-formulation, the **true-stress-true-strain** relation is **monotone inside the elastic domain** \(\mathcal{E}_c(\tau_{\text{st}}, \frac{2}{3} \sigma_e^2)\). This is a feature of \(\mathcal{W}_{\text{st}}\) not shared with any other known elastic energy. For more constitutive issues regarding the interesting properties of \(\mathcal{W}_{\text{st}}\) we refer the reader to [66].

2 Multiplicative isotropic elasto-plasticity directly in terms of the non-symmetric plastic distortion \(F_p\)

In planar elasto-plasticity\(^6\) our development suggests to replace the energy \(\mathcal{W}_H\) by

\[
\mathcal{W}_{\text{st}}(F_c) := \frac{\mu}{k} e^k \| \text{dev} \log C_e \|^2 + \frac{\kappa}{2k} e^k |\text{tr}(\log C_e)|^2 = \frac{\mu}{k} e^k \| \text{dev} \log F_p^{-T} F_p^{-T} F F^{-1} \|^2 + \frac{\kappa}{2k} e^k |\text{tr}(\log (F F^{-1}))|^2 + \frac{\kappa}{2k} e^k |\log(|\det F_p|)|^2,
\]

where we have imposed the condition of plastic incompressibility \(\det F_p = 1\). Let us remark that for small deformations

\[
\mathcal{W}_{\text{st}}(F_c) = \mu e^k \| \text{dev} \log U_c \|^2 + \frac{\kappa}{2k} e^k |\text{tr}(\log U_c)|^2 = \mu \| \text{dev} \log U_c \|^2 + \frac{\kappa}{2k} |\text{tr}(\log U_c)|^2 + \text{h.o.t.}
\]

\[
\mathcal{W}_{\text{st}}(F_c) = \mu \| \log U_c \|^2 + \frac{\kappa}{2k} |\text{tr}(\log U_c)|^2 + \text{h.o.t.} = \mu \| \varepsilon_c \|^2 + \frac{\kappa}{2k} |\text{tr}(\varepsilon_c)|^2 + \text{h.o.t.,}
\]

where \(\varepsilon_c\) is the symmetric elastic strain. A direct identification of the constitutive coefficients gives us that

\[
\mu = \mu_{\text{sd}}, \quad \kappa - \mu = \lambda_{\text{sd}}.
\]

\(^6\)In order to model plane strain with this model, the coefficient \(\kappa\) has to be modified in order to be consistent with plane strain linear elasticity in the infinitesimal limit, see (2.2).
Lemma 2.1. (rank-one convexity and multiplicative decomposition) If the elastic energy \( F \mapsto W(F) \) is rank-one convex of class \( C^2 \), it follows that the elasto-plastic formulation
\[
F \mapsto W(F, F_p) := W(F F_p^{-1}) = W(F_c)
\]
remains rank-one convex w.r.t \( F \) \cite{[58, 56, 37]} at given plastic distortion \( F_p \).

Proof. This is clear, because
\[
D_{F_p}^2[W(F F_p^{-1})].(\xi \otimes \eta, \xi \otimes \eta) = D_{F_c}^2[W(F_c)].((\xi \otimes \eta)F_p^{-1}, (\xi \otimes \eta)F_p^{-1}) = D_{F_c}^2[W(F_c)].(\xi \otimes \tilde{\eta}, \xi \otimes \tilde{\eta}),
\]
where \( \tilde{\eta} = F_{p}^{-T}\eta \) and \( \xi \otimes \tilde{\eta} \) is the (dyadic) tensor product of \( \xi, \eta \in \mathbb{R}^3 \).

Remark 2.2. The same constitutive invariance property is true for convexity, polyconvexity and quasiconvexity \cite{[58, 32]}.

Therefore, the multiplicative approach is ideally suited as far as preservation of ellipticity properties for elastic unloading is concerned. Note that this feature is not true for some additive approaches, see \cite{[65]}.

Definition 2.3. (reduced dissipation inequality-thermodynamic consistency) We say that the reduced dissipation inequality along the plastic evolution is satisfied if and only if
\[
\frac{d}{dt}[W(F F_p^{-1}(t)) \leq 0, \quad (2.5)]
\]
for all constant in time \( F \).

Let us further remark that for fixed \( F \) and for an energy for which the decomposition into isochoric and volumetric parts
\[
W = W_{\text{iso}}(F_c) + W_{\text{vol}}(F_c) = W_{\text{iso}}(FF_p^{-1}) + W_{\text{vol}}(F) \quad (2.6)
\]
holds true, in view of Sansour’s result \cite{[82]} (see also \cite{[46, 48, 45] and [90, page 305]}), we have for the reduced dissipation inequality
\[
\frac{d}{dt}[W_{\text{iso}}(FF_p^{-1})] = \langle DF_cW_{\text{iso}}(F_c), F \frac{d}{dt}[F_p^{-1}] \rangle = \langle DF_cW_{\text{iso}}(F_c), FF_p^{-1}F_p \frac{d}{dt}[F_p^{-1}] \rangle
\]
\[
= \langle FF_p^{-1}F_c \frac{d}{dt}[F_p^{-1}] \rangle = \langle \Sigma_c, F_p \frac{d}{dt}[F_p^{-1}] \rangle = -\langle \Sigma_c, \frac{d}{dt}[F_p]F_p^{-1} \rangle \leq 0, \quad (2.7)
\]
where
\[
\Sigma_c = F_c^TDF_cW_{\text{iso}}(F_c) = 2C_cD_{C_c}[\widehat{\Theta}_{\text{iso}}(C_c)] = F_c^T \tau_c F_c^{-T}
\]
is the mixed variant (transformed) Kirchhoff tensor and
\[
\tau_c := 2D_{B_c}[\widehat{\Theta}_{\text{iso}}(B_c)]B_c = 2D_{\log B_c}[\widehat{\Theta}_{\text{iso}}(\log B_c)] = D_{\log V_c}[\widehat{\Theta}_{\text{iso}}(\log V_c)]
\]
is the elastic Kirchhoff stress-tensor. Note that \( \Sigma_c \) is symmetric in case of elastic isotropy, while \( \tau_c \) is always symmetric. The tensor \( \Sigma = C \cdot S_2(C) \) where \( S_2 = 2D_C[W(C)] \) is the second Piola-Kirchhoff stress tensor, is sometimes called the Mandel stress tensor and \( \text{dev} \Sigma_c = \text{dev} \Sigma_E \), where \( \Sigma_E \) is the elastic Eshelby tensor
\[
\Sigma_E : = F_c^TDF_c[W(F_c)] - W(F_c) \cdot \mathbb{1} = D_{\log C_c}[\widehat{\Theta}(\log C_c)] - \widehat{\Theta}(\log C_c) \cdot \mathbb{1}, \quad (2.8)
\]
driving the plastic evolution (see e.g. \cite{[20, 61, 41, 12, 10, 11]}).

A simple thermodynamically admissible perfect plasticity model \cite{[44, page 67]} (see also \cite{[90, 57, 59, 60, 61]}) is obtained by defining the plastic evolution
\[
F_p \frac{d}{dt}[F_p^{-1}] = -\frac{d}{dt}[F_p]F_p^{-1} \in -\partial \chi(\text{dev} \Sigma_c), \quad (2.8)
\]
where $\partial X$ is the subdifferential of the indicator function $X$ of the convex elastic domain
\[
E_e(\Sigma_e, \frac{1}{3} \sigma_e^2) := \{ \Sigma_e \in \mathbb{R}^{3 \times 3} \mid \| \text{dev}_3 \Sigma_e \|^2 \leq \frac{1}{3} \sigma_e^2 \} \tag{2.9}
\]
in the mixed-variant $\Sigma_e$-stress space. Let us remark that in the isotropic case
\[
E_e(\Sigma_e, \frac{1}{3} \sigma_e^2) := \{ \Sigma_e \in \text{Sym}(3) \mid \| \text{dev}_3 \Sigma_e \|^2 \leq \frac{1}{3} \sigma_e^2 \}. \tag{2.10}
\]
The choice $(2.8)$ ensures $\frac{d}{dt} [W_{\text{iso}} (F F_p^{-1})] \leq 0$ at fixed $F$, therefore the reduced dissipation inequality $(2.7)$ is satisfied and the deviatoric formulation together with the use of $F_p \frac{d}{dt} [F F_p^{-1}] = -\frac{d}{dt} [F_p] F_p^{-1}$ as conjugate variable guarantees $\det F_p = 1$.

Next, a (for us at first surprising) algebraic estimate is introduced.

**Lemma 2.4.** Let $F_e \in \text{GL}^+(3)$ be given. Then it holds $\| F_e^T S F_e^{-T} \|^2 \geq \frac{1}{2} \| S \|^2$ for all $S \in \text{Sym}(3)$, the constant being independent of $F_e$.

**Proof.** Let us define the left Cauchy-Green tensor $B_e = F_e F_e^T \in \text{PSym}(3)$. We have
\[
\| F_e^T S F_e^{-T} \|^2 = \langle F_e^T S F_e^{-T}, F_e^T S F_e^{-T} \rangle = \langle F_e F_e^T S, S F_e^{-T} F_e^{-1} \rangle = \langle B_e S, S B_e^{-1} \rangle. \tag{2.11}
\]
Since $B_e = F_e F_e^T \in \text{PSym}(3)$, there is $Q \in \text{SO}(3)$ such that $D_e = Q^T B_e Q = \text{diag}(d_1, d_2, d_3)$. Hence, we obtain
\[
\| F_e^T S F_e^{-T} \|^2 = \langle Q^T D_e Q S, S Q^T D_e^{-1} Q \rangle = (D_e Q S Q^T, Q S Q^T D_e^{-1}). \tag{2.12}
\]
Moreover, considering $Q S Q^T := \tilde{S} = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} & \tilde{S}_{13} \\ \tilde{S}_{21} & \tilde{S}_{22} & \tilde{S}_{23} \\ \tilde{S}_{31} & \tilde{S}_{32} & \tilde{S}_{33} \end{pmatrix}$ and using that $1 \leq \frac{d_i}{d_j} + \frac{d_j}{d_i}$ for $d_i, d_j > 0$, we deduce
\[
\| F_e^T S F_e^{-T} \|^2 = \tilde{S}_{11}^2 + \tilde{S}_{22}^2 + \tilde{S}_{33}^2 + \tilde{S}_{12} \left( \frac{d_1}{d_2} + \frac{d_2}{d_1} \right) + \tilde{S}_{23} \left( \frac{d_2}{d_3} + \frac{d_3}{d_2} \right) + \tilde{S}_{31} \left( \frac{d_3}{d_1} + \frac{d_1}{d_3} \right) \geq \frac{1}{2} \| \tilde{S} \|^2 = \frac{1}{2} \| S \|^2,
\]
due to the symmetry of $\tilde{S}$ and the proof is complete. \hfill \Box

**Remark 2.5.** Note that for $S \notin \text{Sym}(3)$ there is always an estimate similar to that given by Lemma 2.4 but which involves constants depending on $F_e$, i.e. $\| F_e^T S F_e^{-T} \|^2 \geq c(F_e) \| S \|^2$.

**Remark 2.6.** It is easy to see the relations
\[
\Sigma_e = F_e^T \tau_e F_e^{-T}. \tag{2.13}
\]
Note that $(2.13)$, as opposed to appearance, is not at variance with symmetry of $\Sigma_e$ in case of isotropy.

**Remark 2.7.** Since
\[
\text{dev}_3 \Sigma_e = \text{dev}_3 (F_e^T \tau_e F_e^{-T}) = F_e^T \tau_e F_e^{-T} - \frac{1}{3} \text{tr}(F_e^T \tau_e F_e^{-T}) \cdot \mathbb{1} = F_e^T (\tau_e - \frac{1}{3} \text{tr}(\tau_e)) \cdot \mathbb{1}) F_e^{-T}, \tag{2.14}
\]
we can note
\[
\text{dev}_3 \Sigma_e = F_e^T (\text{dev}_3 \tau_e) F_e^{-T}, \quad \text{dev}_3 \tau_e = F_e^{-T} (\text{dev}_3 \Sigma_e) F_e^T, \quad \text{tr}(\Sigma_e) = \text{tr}(\tau_e). \tag{2.15}
\]
Thus, using Lemma 2.4 we obtain the estimate
\[
\| \text{dev}_3 \Sigma_e \| = \| F_e^T (\text{dev}_3 \tau_e) F_e^{-T} \| \geq \frac{1}{\sqrt{2}} \| \text{dev}_3 \tau_e \|, \tag{2.16}
\]
which is valid for general anisotropic materials and it explains our choice of factors in $\mathcal{E}_c(\Sigma_{\text{el}}, \frac{1}{3} \sigma_0^2)$ and $\mathcal{E}_c(\tau_{\text{el}}, \frac{2}{3} \sigma_0^2)$, respectively. Moreover, numerical tests suggest that the LH-ellipticity domain of the distorsional energy function $F \mapsto W_{\text{el}}(F) = \frac{2}{k} e \| \text{dev}_3 \log U \|^2$, $F \in \text{GL}^+(3)$, with $k \geq \frac{1}{10}$ (the necessary condition for separate convexity (SC) of $e^k \| \text{dev}_3 \log U \|^2$ in 3D) is an extremely large cone

$$\mathcal{E}(W_{\text{el}}, \text{LH}, U, 27) = \{ U \in \text{PSSym}(3) \mid \| \text{dev}_3 \log U \|^2 < 27 \};$$

Therefore we have the inclusion of domains

$$\{ U \in \text{PSSym}(3), \Sigma_{\text{el}} \in \mathcal{E}_c(\Sigma_{\text{el}}, \frac{1}{3} \sigma_0^2) \} \subseteq \{ U \in \text{PSSym}(3), \tau_{\text{el}} \in \mathcal{E}_c(\tau_{\text{el}}, \frac{2}{3} \sigma_0^2) \} \subseteq \mathcal{E}(W_{\text{el}}, \text{LH}, U, 27).$$

![Figure 1: Elastic domains expressed in the mixed variant symmetric stress tensor $\Sigma_{\text{el}}$ and the symmetric Kirchhoff stress tensor $\tau_{\text{el}}$ related to the ellipticity domain $\mathcal{E}(W_{\text{el}}, \text{LH}, U, 27)$. The elastic domains are viewed as subsets of PSSym(3).](image)

In (1.7) the considered convex "elastic domain", in which monotonicity and/or ellipticity for $W_{\text{el}}$ is considered, is defined in terms of $\| \text{dev}_3 \Sigma_{\text{el}} \|$, not in terms of $\| \text{dev}_3 \Sigma_{\text{el}} \|$. However, adapting Lemma 2.4 to $W_{\text{el}}$, we see that, in the three-dimensional case, our previous results indicating the loss of ellipticity only for extreme distorsional strains suggest that the coupling with plasticity is most natural: permanent deformation sets in, based on a criterion of distorsional energy (Huber-Hencky-von Mises-type) $\| \text{dev}_3 \Sigma_{\text{el}} \|^2 \leq \frac{1}{3} \sigma_0^2$, and our former results suggest that $W_{\text{pl}}(F_0)$ never reaches the non-elliptic domain in any elasto-plastic process. This is in sharp contrast to the loss of ellipticity of the quadratic Hencky energy $W_{\text{pl}}$, which is not related to the distorsional energy alone. As it turns out, for the overall non-elliptic energy $W_{\text{pl}}$ (in three dimensions) plasticity provides a natural relaxation mechanism, which prevents loss of ellipticity in the elastic domain. Moreover, in the above defined elastic domain $\mathcal{E}_c(\Sigma_{\text{el}}, \frac{1}{3} \sigma_0^2)$, the constitutive relation $\log B_{\text{el}} \mapsto \sigma(\log B_{\text{el}})$ remains monotone, i.e. the true-stress-true-strain monotonicity condition (TSTS-M) is satisfied in $\mathcal{E}_c(\Sigma_{\text{el}}, \frac{1}{3} \sigma_0^2)$ (see [66, 94, 95, 31]).

**Remark 2.8.** For the isotropic case we have $\tau_{\text{el}} B_{\text{el}} = B_{\text{el}} \tau_{\text{el}}$, which implies

$$\| \text{dev}_3 \Sigma_n \|^2 = \langle F_{\text{el}}^T (\text{dev}_3 \tau_{\text{el}}) F_{\text{el}}^{-T}, F_{\text{el}}^T (\text{dev}_3 \tau_{\text{el}}) F_{\text{el}}^{-T} \rangle = \langle B_{\text{el}} (\text{dev}_3 \tau_{\text{el}}), (\text{dev}_3 \tau_{\text{el}}) B_{\text{el}}^{-1} \rangle = \| \text{dev}_3 \tau_{\text{el}} \|^2.$$

This fact can be also proved using the fact that, in the isotropic case, both tensors $\Sigma_{\text{el}}$ and $\tau_{\text{el}}$ are symmetric and they have the same invariants. Therefore, using (2.13) we obtain that in the isotropic case we have $\sqrt{2} \mathcal{E}_c(\Sigma_{\text{el}}, \frac{1}{3} \sigma_0^2) = \mathcal{E}_c(\tau_{\text{el}}, \frac{2}{3} \sigma_0^2)$.

Summarizing the properties of the 9-dimensional flow rule for the plastic distortion (2.8) we have:

1) it is thermodynamically correct ($\frac{d}{dt} \| W(F F_p^{-1}) \| \leq 0$);
ii) the right hand side is a function of $F$ and $F_p^{-1}$;

iii) plastic incompressibility: the constraint $\det F_p(t) = 1$ for all $t \geq 0$ follows from the flow rule in combination with the initial condition $\det F_p(t)|_{t=0} = 1$;

iv) the above properties imply that the flow rule (2.8) is consistent;

v) it satisfies the principle of maximum dissipation and is an associated plasticity model;

vi) elastic unloading remains rank-one convex under arbitrary plastic predeformation.

3 Conclusion and open problems

We have shown that the multiplicative plasticity models preserve ellipticity in purely elastic processes at frozen plastic variables provided that the initial elastic response is elliptic. Preservation of LH-ellipticity is, in our view, a property which should be satisfied by any hyperelastic-plastic model since the elastically unloaded material specimen should respond reasonable under further purely elastic loading. In contrast to multiplicative models, the much used additive logarithmic model does not preserve LH-ellipticity in general [65].

An interesting question concerns the requirements that one should impose on the elastic response for arbitrary large distortional strains. One may reasonably argue that these requirements are void of any relevance, since the material can never be observed in a state of large elastic distortional strain: prior to that, dissipative processes will occur. In the case of the energy $W_{\text{st}}$, which is not rank-one elliptic for extreme elastic distortional strains, we have explicitly shown that elastic unloading will remain rank-one convex and the true-stress-true-strain relation remains monotone. This is a remarkable feature in geometrically nonlinear material models. In a future contribution we will provide the analytical proof for the rank-one convexity domain for $W_{\text{st}}$ in $n = 3$.

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