On the augmented density of a spherical anisotropic dynamic system

J. An*

National Astronomical Observatories, Chinese Academy of Sciences, A20 Datun Road, Chaoyang District, Beijing 100012, PR China

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ABSTRACT

This paper presents a set of new conditions on the augmented density of a spherical anisotropic system that is necessary for the underlying two-integral phase-space distribution function to be non-negative. In particular, it is shown that the partial derivatives of the Abel transformations of the augmented density must be non-negative. Applied for the separable augmented densities, this recovers the result of van Hese et al. (2011).

Key words: galaxies; kinematics and dynamics – methods: analytical

1 INTRODUCTION

Recently, Ciotti & Morganti (2010) raised a question whether the density slope–anisotropy inequality \( \gamma > 2 \beta \) is the necessary condition for the consistency of the underlying two-intphase-space distribution function. Here, \( \gamma \) is the (negative) logarithmic density slope whereas \( \beta \) is the so-called Binney anisotropy parameter. Extending the earlier finding by An & Evans (2006) that the inequality is necessary at the centre (given a finite potential well), Ciotti & Morganti (2010) have shown that, for wide varieties of anisotropic spherical systems built by flexible families of analytic two-integral distributions of certain ansatz (e.g., Cuddeford 1991; Cuddeford & Louis 1995; Baes & van Hese 2007), the forementioned inequality holds everywhere in radial positions given that the distribution function is also non-negative everywhere in the accessible phase space and that the central anisotropy parameter \( \beta_0 \) is restricted to be \( \beta_0 < \frac{1}{4} \). They also presented the equivalent condition to the inequality for the separable augmented density, which has subsequently proven by Van Hese, Baes, & Dejonghe (2011) to be satisfied by such a system if \( \beta_0 < \frac{1}{4} \). The latter authors have also shown that the condition of Ciotti & Morganti (2010) is not valid if \( \beta_0 > \frac{1}{4} \), which they have demonstrated with three counterexamples.

Here, I present new conditions on the augmented density that are necessary for the consistency of the distribution function. It is found that when they are applied for a separable augmented density, one of the conditions recovers the result of van Hese et al. (2011), providing an alternative proof.

2 AUGMENTED DENSITY FOR SPHERICAL SYSTEMS

According to the Jeans theorem, a general spherical dynamical system in equilibrium is described by the phase-space distribution function (DF) of the form of \( f(E, L^2) \) where \( E \) and \( L^2 \) are the two classical isotropic isolating integrals admitted by the spherical potential, namely, \( E = \Psi - \frac{1}{2}v^2 \) is the specific binding energy and

\[
L = \sqrt{|L^2|} = rv_t \text{ is the magnitude of the specific angular momentum. Here the notation of Binney & Tremaine (2008) is followed so that } \Psi \text{ is the relative potential above the boundary (} \Psi = -\Phi \text{ where } \Phi \text{ is the gravitational potential if the system is infinitely extended but the potential is escapable) and the bound particles are given by } E > 0. \text{ In addition, } v^2 = v_t^2 + v_r^2 \text{ where } v_t \text{ and } v_r \text{ are the tangential and radial velocities. This paper concerns an escapable system or a system with a finite boundary, and therefore the local velocity distribution always cuts off (at the escape speed) and } E < 0 \text{ is inaccessible.}
\]

Given the DF, its velocity moments are given by

\[
p_{\rho, m}(\Psi, r^2) = \iiint_{v < 2r} d^3v \int_{r_0}^{r} d^2v_{\perp} f(E = \Psi - \frac{1}{2}L^2, L^2 = r^2 v_t^2) = 4\pi \int_{r_0}^{r} d^2v_{\perp} f v_{\perp}^2 v_t^2 v_{\perp}^{2m+1} dv_{\perp} dv_t. \tag{1}
\]

Here, the zeroth moment \( p_{\rho, 0}(\Psi, r^2) \) as a bivariate function of \( \Psi \) and \( r^2 \) is referred to as the augmented density. Once the potential \( \Psi(r) \) is specified, the local density is found by \( \rho(r) = p_{\rho, 0}(\Psi(r), r^2) \) whilst all the even-integral velocity moments are given by \( \langle v_{\perp}^m v_t^{2m} \rangle = p_{\rho,m}/p_{\rho,0} \) with the specified \( \Psi(r) \). In a self-consistent system on the other hand, the Poisson equation with the augmented density as the source term reduces to an ordinary differential equation on \( \Psi \), and so \( \rho \) becomes entirely specified by \( f \).

By changing integration variables, the augmented density \( p_{\rho,0}(\Psi, r^2) \) is formally expressed to be a bivariate Abel-like integral transformation of the DF, that is,

\[
p_{\rho,0}(\Psi, r^2) = \frac{2\pi}{r^2} \int_{E \leq 0, L^2 > 0, K \geq 0} dE dL^2 f(E, L^2) K(\ell, L^2; \Psi, r^2) \tag{2}
\]

where

\[
K(E, L^2; \Psi, r^2) \equiv 2(\Psi - E) - \frac{L^2}{r^2}. \tag{3}
\]

1 This excludes some systems such as an isothermal sphere. Some of the following results are still valid for such systems if they are not dependent upon the choice of the lower boundary \( E = 0 \) for \( E \)-integral whereas the corresponding proper result may be addressed by choosing \( E = -\infty \) instead.
The integral is over the triangular domain in \((\mathcal{E}, L^2)\) bounded by lines \( \mathcal{E} = 0, L^2 = 0 \) and \( \mathcal{K} = 0 \). The last line is the same as the diagonal line given by \( \mathcal{E} + L^2/(2r^2) = \Psi \). It is known that upon certain restrictions on its regularity, equation (1) can be uniquely inverted at least formally to yield the DF, \( f(\mathcal{E}, L^2) \), provided that \( p_{0,0}(\Psi, r^2) \) is sufficiently well-behving. Two particular such formulae after Hunter & Qian (1993) are provided in Appendix B.

3 ABEL TRANSFORM OF AUGMENTED DENSITY

Let us consider two separate changes of integration variables in equation (2), to \((Q, L^2)\) and \((\mathcal{E}, L^2)\), respectively, where \( Q \equiv \mathcal{E} + L^2/(2r^2) \) and \( R^2 \equiv L^2/[2(\Psi - \mathcal{E})] \).

3.1 on the potential

With \( Q = \mathcal{E} + L^2/(2r^2) \), it is easy to find that the Jacobian determinant for the coordinate change \((\mathcal{E}, L^2) \rightarrow (Q, L^2)\) is simply the unity. Since we then have \( \mathcal{K} = 2(\Psi - Q) \), the augmented density is written to be

\[
p_{0,0}(\Psi, r^2) = \frac{2\pi}{r} \int_{\mathcal{E}(Q)}^{\mathcal{E}(Q)} dQ dL^2 \frac{f(\mathcal{E}_{\text{max}}(Q), L^2)}{\sqrt{2(\Psi - Q)}}
\]

\[
= \frac{\sqrt{2\pi}}{r} \int_{r}^{\Psi} \frac{dQ}{\sqrt{\Psi - Q}} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2)
\]

where \( \mathcal{E}_{\text{max}}(Q) \equiv Q - L^2/(2r^2) \). Here, the inner integral in the last line is independent of \( \Psi \), and furthermore, the outer integral is in the form of the Abel transform. Consequently, the inner integral can be inverted from \( p_{0,0} \) by means of the inverse Abel transform.

That is to say,

\[
\frac{\partial}{\partial Q} \int_{0}^{\Psi} p_{0,0}(\Psi, r^2) d\Psi = \frac{\sqrt{2\pi}}{r} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2)
\]

Next, we can show that

\[
\frac{\partial}{\partial r} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2)
\]

\[
= 2Qf(0, 2r^2) + \int_{0}^{2\pi} r^2 dL^2 f^{(1,0)}(\mathcal{E}_{\text{max}}(Q), L^2) L^2 \frac{L^2}{2r^2}
\]

where \( f^{(1,0)}(\mathcal{E}, L^2) \equiv \partial f / \partial \mathcal{E} \). However, we also find that

\[
\frac{\partial}{\partial Q} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2)
\]

\[
= 4r^4 f(0, 2r^2) + \int_{0}^{2\pi} r^2 dL^2 f^{(1,0)}(\mathcal{E}_{\text{max}}(Q), L^2) L^2
\]

\[
= 2r^4 \frac{\partial}{\partial r} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2).
\]

Hence,

\[
\frac{\partial}{\partial r} \left[ \int_{0}^{\Psi} p_{0,0}(\Psi, r^2) d\Psi \right] = \frac{\pi^2}{2r^2} \int_{0}^{2\pi} r^2 dL^2 f(\mathcal{E}_{\text{max}}(Q), L^2)
\]

Strictly, the argument so far only indicates that the difference between the left- and right-hand sides is independent of \( Q \), but both vanish if \( Q = 0 \) and thus the equality holds.

3.2 on the radial distance

Given \( R^2 = L^2/[2(\Psi - \mathcal{E})] \), we first find the Jacobian determinant

\[
\frac{\partial(\mathcal{E}, L^2)}{\partial(\mathcal{E}, R^2)} = 2(\Psi - \mathcal{E})
\]

for the coordinate change \((\mathcal{E}, L^2) \rightarrow (\mathcal{E}, R^2)\) whilst we have that

\[
\mathcal{K} = 2(\Psi - \mathcal{E}) - \frac{2R^2(\Psi - \mathcal{E})}{r^2} = 2(\Psi - \mathcal{E}) \frac{r^2 - R^2}{r^2}.
\]

Then, equation (2) reduces to

\[
p_{0,0}(\Psi, r^2) = 2^{3/2} r \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} \int_{0}^{2\pi} \frac{d\theta}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

Here \( L^2_{\text{max}}(R^2) \equiv 2R^2(\Psi - \mathcal{E}), \) which does not depend on \( r^2 \). The integral is now over the rectangular region and so the double integral can be performed in any order given the absolutely integrable integrand. In addition, the \( R^2 \)-integral is again an Abel transform and its inverse transform results in

\[
\frac{\partial}{\partial R^2} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)] = 2^{3/2} \pi \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

Next we note that

\[
\frac{r^2}{\sqrt{R^2 - r^2}} = \frac{r^2}{\sqrt{R^2 - r^2}} - \frac{r^2}{\sqrt{R^2 - r^2}}
\]

and therefore (provided that \( p_{0,0}d\mathcal{E} \) is integrable over \( r = 0 \))

\[
\frac{\partial}{\partial R^2} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)] = \frac{\partial}{\partial R^2} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

On the other hand,

\[
\frac{\partial}{\partial \mathcal{E}} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)] = \frac{\partial}{\partial \mathcal{E}} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

\[
+ 2R^2 \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f^{(0,1)}[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

where \( f^{(0,1)}(\mathcal{E}, L^2) \equiv \partial f / \partial L^2 \) assuming \( \lim_{L \to \Psi} L f(\mathcal{E}, L^2) = 0 \). However, since

\[
\frac{\partial}{\partial R^2} \left[ \frac{f[\mathcal{E}, L^2_{\text{max}}(R^2)]}{\sqrt{2(\Psi - \mathcal{E})}} \right] = 2\sqrt{2(\Psi - \mathcal{E})} f^{(0,1)}[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

equation (15) further reduces to

\[
\frac{\partial}{\partial \mathcal{E}} \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)] = \left( \frac{\pi^2}{r^2} \right) \int_{0}^{\Psi} \frac{d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]

which is in fact valid provided that \( f dL^2 / \Psi \) is integrable over \( L^2 = 0 \) (c.f., Appendix C). Consequently, we finally find that

\[
\frac{\partial}{\partial \mathcal{E}} \int_{0}^{\Psi} \frac{d\mathcal{E}}{r \sqrt{R^2 - r^2}} f[\mathcal{E}, L^2_{\text{max}}(R^2)] = 2^{3/2} \pi \int_{0}^{\Psi} \frac{d\mathcal{E}}{r \sqrt{R^2 - r^2}} f[\mathcal{E}, L^2_{\text{max}}(R^2)]
\]
4 CONSISTENCY OF DISTRIBUTION FUNCTION

The results so far are summarized as follows. If we define the Abel transformations of the augmented density as

\[ \hat{\rho}(\Psi, r^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho_{00}(Q, r^2) \, dQ}{\sqrt{\Psi - Q}}; \]

\[ \hat{\rho}(\Psi, r^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho_{01}(\Psi, r^2) \, dR^2}{\sqrt{\Psi - R^2}}; \]

\[ \hat{\rho}(\Psi, r^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho_{02}(\Psi, r^2) \, dR^2}{\sqrt{\Psi - R^2}}. \]

we find that applying the transformation on equation (21) results in

\[ \hat{\rho}(\Psi, r^2) = \frac{\sqrt{\pi}}{r} \int_{E_0(L^2, \rho, 0)} \, dE \, dL^2 \, f(E, L^2); \]

\[ \hat{\rho}(\Psi, r^2) = \frac{2\pi}{r} \int_{E_0(L^2, \rho, 0)} \, dE \, dL^2 \, f(E, L^2); \]

\[ \hat{\rho}(\Psi, r^2) = \frac{\sqrt{\pi}}{r} \int_{E_0(L^2, \rho, 0)} \, dE \, dL^2 \, f(E, L^2). \]

The calculations are performed by interchanging orders of integrations (so that \( Q\) - and \( R^2\)-integrations of the integrals become the inner-most ones) and carrying out the \( Q\) - and \( R^2\) - integrations explicitly. The details are given in Appendix A (see also Qian (1993)). Note that the middle equation (20) is also identifiable to eq. (C10) of Hunter & Qian (1993).

The partial derivatives of equations (20) are related to simple linear integrals of the DF along the diagonal line given by \( E + L^2/(2r^2) = \Psi \). In particular,

\[ \frac{\partial \hat{\rho}(\Psi, r^2)}{\partial \Psi} = \frac{\sqrt{\pi}}{r} \int_0^{\Psi \circ} \, dL^2 \, f(\Psi - L^2/2r^2, L^2); \]

\[ \frac{\partial [\rho(\Psi, r^2)]}{\partial r} = \frac{\pi}{\sqrt{2}r^2} \int_0^{\Psi \circ} \, dL^2 \, f(\Psi - L^2/2r^2, L^2); \]

\[ \frac{\partial \hat{\rho}(\Psi, r^2)}{\partial \Psi} = \frac{2\pi}{r} \int_0^{\Psi \circ} \, dL^2 \, f(\Psi - L^2/2r^2, L^2); \]

\[ \frac{\partial \hat{\rho}(\Psi, r^2)}{\partial r} = \frac{\pi}{r} \int_0^{\Psi \circ} \, dL^2 \, f(\Psi - L^2/2r^2, L^2). \]

Here each integral may alternatively expressed through

\[ \int_0^{\Psi \circ} \, dL^2 \, L^{m-1} f(\Psi - L^2/2r^2, L^2); \]

\[ = (2r^2)^{m-1} \int_0^{\Psi \circ} \, dE \, (\Psi - E)^m f(E, 2r^2(\Psi - E)). \]

This is verified by changing the dummy integration variables. The last line of equations (21) may be expressed in terms of \( \hat{\rho} \) because

\[ \frac{\partial \hat{\rho}(\Psi, r^2)}{\partial r} = (\hat{\rho}(\Psi, r^2))^2 + \frac{\partial \rho(\Psi, r^2)}{\partial r}; \]

which may be shown with equation (14).

Since the minimal consistency of the DF indicates that \( f \geq 0 \) in the accessible region of the phase space and also \( L^2 \geq 0 \) and \( E \leq \Psi \), equations (21) are all non-negative, provided that they are indeed well-defined. In fact, \( \hat{\rho} \) and \( \rho \) are well-defined given that the DF is integrable – which further implies that \( \rho_{00}(\Psi, r^2) \, d\Psi \) and \( \rho_{01}(\Psi, r^2) \, d\Psi \) are integrable over \( \Psi = 0 \) and \( r^2 = 0 \), respectively. However, the definition of \( \hat{\rho} \) and subsequently the third line of equations (21) require \( f(E, L^2) \, dL^2/L \) to be integrable over the region containing \( L^2 = 0 \), which is a strictly stronger condition than the mere integrability of the DF. For instance, if \( f(E, L^2) = L^{-2} g(E) \) and \( \rho_{00}(\Psi, r^2) = r^{-2} A(\Psi) B(r^2) \) with \( \frac{1}{2} < \beta < 1 \), it is clear that \( \hat{\rho} \) cannot be defined as in equation (19) whilst the right-hand side of the third line of equations (21) diverges.

4.1 separable augmented density

Next, we consider the particular cases for which the \( \Psi \) - and \( r^2 \) - dependences of the augmented density are multiplicatively separable, that is, \( \rho_{00}(\Psi, r^2) = A(\Psi) B(r^2) \). It is a fairly straightforward exercise to establish (c.f., Deimelhe 1986)

\[ \frac{\partial p_{1,0}}{\partial \Psi} = p_{0,0} \Rightarrow \rho_{1,0}(\Psi, r^2) = \int_0^{\Psi} p_{0,0}(Q, r^2) \, dQ, \]

and

\[ \frac{\partial r^2 p_{1,0}}{\partial r} = \frac{1}{r} p_{0,1} \Rightarrow 1 - \frac{p_{0,1}}{2p_{1,0}} = \frac{\partial \log p_{1,0}}{\partial \log r^2}; \]

from equation (1). The last result is also directly related to the Binney anisotropy parameter (Binney & Mamon 1982), i.e.,

\[ \beta(r) = \int_{r^2}^{1} \frac{\partial \rho(\Psi, r^2)}{\partial r} \, dr = \frac{1}{2} \left( \frac{\rho_{0,1}(\Psi, r^2)}{2p_{1,0}(\Psi, r^2)} \right) = \frac{\partial \log p_{1,0}}{\partial \log r^2} \left|_{\Psi = 1/2} \right. \]

Therefore, with a separable augmented density, we further have

\[ p_{1,0}(\Psi, r^2) = B(r^2) \left( \int_{0,0}^{\Psi} \, A(Q) \, dQ \right), \]

which is also separable. In addition,

\[ \beta = -\frac{\partial \log B(r^2)}{\partial \log r^2}; \quad \log B = -\int \frac{2\rho(r)}{r} \, dr, \]

which is thus completely specified (up to a constant) given the anisotropy parameter.

The first and last of equations (21) being non-negative for a separable augmented density then indicates that

\[ \frac{dA(\Psi)}{d\Psi} \geq 0; \quad \frac{dB(r^2)}{dr^2} \geq 0, \]

where

\[ A(\Psi) \equiv \int_0^{\Psi} A(Q) \, dQ; \quad B(r^2) \equiv \int_0^{r^2} R B(R^2) \, dR^2. \]

Here we have also used \( A(\Psi) \geq 0 \) and \( B(r^2) \geq 0 \), which are direct consequences of the DF being non-negative. It is also obvious that \( A(\Psi) \geq 0 \), and so we find from the second line of equation (21) that

\[ \frac{d[r^2 B(r^2)]}{dr^2} = (1 - \beta) B > 0. \]

Given that \( B(r^2) \geq 0 \), this is equivalent to \( \beta \leq 1 \), which is physically obvious. None the less, it is a nontrivial necessary condition on \( B(r^2) \) for the consistency of the corresponding DF. Furthermore, we also find that

\[ \frac{d}{dr^2} \left( \int_0^{r^2} \frac{R B(R^2) \, dR^2}{\sqrt{R^2 - R^2}} \right) = \left( \int_0^{r^2} \frac{R B(R^2) \, dR^2}{R \sqrt{R^2 - R^2}} \right), \]

and thus equation (29) is in fact a sufficient condition for the second part of equation (21). Finally, let us suppose that

\[ B(r^2) \equiv \int_0^{r^2} \frac{R^2 B(R^2) \, dR^2}{R \sqrt{R^2 - R^2}} = \int_0^{r^2} B(r^2 \sin^2 \theta) \, d\phi \]
converges (and so is well-defined). Then since it is also non-negative, the non-negativity of the DF indicates that
\[
\frac{dA(\Psi)}{d\Psi} > 0; \quad \frac{d[rB(r^2)]}{dr^2} = \frac{1}{r} \frac{d[rB(r^2)]}{dr^2} > 0. \tag{32}
\]

Here we note that
\[
\frac{dA(\Psi)}{d\Psi} = A(0) + \int_0^\Psi \frac{dQ}{\sqrt{\Psi - Q}} \frac{dA(Q)}{dQ}, \tag{33}
\]
and so equation (32) is also a sufficient condition for equation (27). However, equation (32) is the necessary condition for the consistency of the DF provided that \(B(r^2)\) is well-defined in equation (31) whereas equation (27) is the necessary condition for the non-negativity of any integrable DF that produces a separable augmented density.

The partial result of van Hese et al. (2011) is also deduced from these. If \(\beta_0 = \lim_{r \to 0} \beta(r)\), this implies that \(B(r^2) \sim r^{-2\beta_0}\) as \(r \to 0\). Hence, if \(2\beta_0 < 1\), then \(B\) converges and is well-defined. Consequently, equation (32), in particular, \(dA/d\Psi > 0\) is the necessary condition for the consistency of the DF. Following Ciotti & Morganti (2010), this is also equivalent to the density slope–anisotropy inequality \((\gamma > 2\beta)\) for the system with a separable augmented density. The result here is however not directly applicable for the case that \(\beta_0 = \frac{1}{2}\), for which equation (31) diverges.

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**APPENDIX A: ABEL TRANSFORMATION OF \(p_{nm}(\Psi, r^2)\)**

It is easy to establish that the moment function \(p_{nm}(\Psi, r^2)\) defined in equation (1) is all expressed as an integral transformation of the DF in \((E, L^2)\)-space, i.e.,
\[
p_{nm}(\Psi, r^2) = \frac{2\pi}{r^2} \int_{rT} dE dL^2 K^{\frac{n}{2}} L^{2n} f(E, L^2) \tag{A1}
\]
where the integral is over the domain in the \((E, L^2)\) defined to be \(T = \{(E, L^2) | E \geq 0, L \geq 0, K \geq 0\}\).

Here, \(L_{2n} = (L^2)^n\) is itself a function of only \(L^2\) and so we may set \(m = 0\) without any loss of generality in the following discussion – formally, \(F(E, L^2) = L_{2n} f(E, L^2)\), etc.

Next, we define the Abel transformations of equation (A1),
\[
\tilde{p}_n(\Psi, r^2) = \frac{1}{\pi} \int_0^\Psi \frac{p_{nm}(Q, r^2) dQ}{(\Psi - Q)^{1/2}}, \tag{A2}
\]
\[
\tilde{p}_n(\Psi, r^2) = \frac{1}{\pi} \int_0^\Psi \frac{p_{nm}(\Psi, R^2) dR^2}{R(\Psi - R^2)^{1/2}}; \tag{A3}
\]
\[
\tilde{p}_n(\Psi, r^2) = \frac{1}{\pi} \int_0^\Psi \frac{R_{nm-1}(\Psi, R^2)^{1/2} dR^2}{(\Psi - R^2)^{1/2}}. \tag{A4}
\]

Interchanging the orders of integrations indicates that
\[
\tilde{p}_n(\Psi, r^2) = 2 \int_{rT} dE dL^2 f \int_0^\Psi \frac{[K(Q)]^{1/2} dQ}{(\Psi - Q)^{1/2}}, \tag{A5}
\]
where \[K(Q) \equiv K(E, L^2; Q, r^2)\] and \(K(R^2) \equiv K(E, L^2; \Psi, R^2)\) with \(K\) being defined in equation (A5). The inner-most integrals can be evaluated analytically. In particular,
\[
\int_{r^2/2}^\Psi \frac{[K(Q)]^{1/2} dQ}{(\Psi - Q)^{1/2}} = \frac{\gamma_0}{\sqrt{2}} B(n + \frac{1}{2}, \frac{1}{2}); \\
\int_{r^2/2}^\Psi \frac{[K(R)]^{1/2} dR^2}{R(\Psi - R^2)^{1/2}} = \frac{\gamma_0}{\sqrt{2}} B(n + \frac{1}{2}, \frac{1}{2}); \tag{A6}
\]
where \(\gamma \equiv \gamma(E, L^2; \Psi, r^2)\), and
\[
B(n + \frac{1}{2}, \frac{1}{2}) \equiv \int_0^1 x^{n+\frac{1}{2}} (1 - x)^{\frac{1}{2}} dx = \frac{\Gamma(n + \frac{1}{2}) \Gamma(1/2)}{\Gamma(n + 1)} = \frac{(1/2)n!}{n!}; \tag{A7}
\]

The last equality is valid for a non-negative integer \(n\), and \(\Gamma(n + i) = \prod_{j=0}^{n+i-1} (n + j)\) is the Pochhammer symbol. This may be proven by the changes of variables given by
\[
\tau(Q) = \frac{K(Q)}{\gamma}; \quad \tau(R^2) = \frac{K(R^2)}{\gamma}; \quad \tau(R^2) = \frac{K(R^2)}{\gamma}; \tag{A5}
\]
respectively. Finally, gathering the results so far results in
\[
\tilde{p}_n(\Psi, r^2) = \frac{(1/2)n! \sqrt{\pi}}{n!} \int_{rT} dE dL^2 K^{\frac{n}{2}} L^{2n} f(E, L^2); \tag{A6}
\]
\[
\tilde{p}_n(\Psi, r^2) = \frac{(1/2)n! \sqrt{\pi}}{n!} \int_{rT} dE dL^2 K^{\frac{n}{2}} L^{2n} f(E, L^2); \tag{A7}
\]
\[
\tilde{p}_n(\Psi, r^2) = \frac{(1/2)n! \sqrt{\pi}}{n!} \int_{rT} dE dL^2 K^{\frac{n}{2}} L^{2n} f(E, L^2). \tag{A8}
\]
APPENDIX B: HUNTER-QIAN INVERSION FOR $f(E, L^2)$

The Abel transformations of the augmented density examined in this paper indicate that the particular problem of inverting the augmented density for the DF, $f(E, L^2)$, is also closely related to finding the (even part of) two-integral distribution function $f'(E, L)$ from the axisymmetric density $\rho[\Psi(R^2, z^2), R^2]$, as the integral transform that relates them can be formally identified with equation (20). Hunter & Qian (1993) had presented the complete solution for this latter problem by means of the complex contour integral, and also discussed its relation to the spherical anisotropic case. Although they did not provide explicitly the corresponding general inversion formulae for anisotropic spherical systems, they are in fact no more complicated than the axisymmetric case (Hunter & Qian 1993, eq. 3.1). That is to say (see also Qian 1993),

$$f(E, L^2) = \frac{1}{2\pi^2} \frac{\partial}{\partial E} \int_{E_{\text{min}}}^{E_{\text{max}}} \frac{dZ}{Z - E} \hat{\rho}[\Psi(L^2), \frac{L^2}{2(Z - E)}]$$

(B1)

where

$$\hat{\rho}(\Psi, r^2) = \frac{\partial \hat{\rho}(\Psi, r^2)}{\partial \Psi} = \frac{1}{\pi} \frac{\partial}{\partial \Psi} \int_0^\Psi \rho_{\text{00}}(\Psi', r^2) d\Psi'$$

and

$$\hat{\varphi}(\Psi, r^2) = \frac{\partial \hat{\varphi}(\Psi, r^2)}{\partial \Psi} = \frac{1}{\pi} \frac{\partial}{\partial \Psi} \int_0^\Psi \rho_{\text{00}}(\Psi, R^2) dR^2$$

and $\hat{\rho}(\Psi, r^2)$ and $\hat{\varphi}(\Psi, r^2)$ are as defined in equation (19). Here, the outer integrals of equation (B1) are contour integrals in the complex $Z$-plane whose path is given such that it starts from $Z = E_{\text{min}}$ to the below of the real axis, winds along the positive (counterclockwise) orientation, crosses the real axis upwards (from the negative to positive imaginary) at the right side of $Z = \Psi_{\text{min}}(E)$, and returns to $Z = E_{\text{min}}$ from the positive imaginary side. (Note that, given that $E < 0$ is inaccessible, $E_{\text{min}} = 0$.) For more details including the definition of $\Psi_{\text{min}}(E)$, please refer Hunter & Qian (1993) or Qian (1993). We note that the formula reduces to the Cuddeford formula if the $r^2$-dependence of the augmented density is given by a pure power law [i.e., $\rho_{\text{00}}(\Psi, r^2) = r^{-2\beta}A(\Psi)$]. Of course, this further reduces to the Eddington formula if $\beta = 0$.

APPENDIX C: ALTERNATIVE DERIVATION OF Eq(17)

In equation (15), we have assumed $\lim_{E \to 0} L f(E, L^2) = 0$ in order to omit the term $\lim_{E \to 0} \sqrt{\Psi - E} f(E, L_{\text{max}}^2(R^2))$ in its right-hand side. However, this assumption is only incidental here and the result in equation (17) and subsequently equation (18) are valid provided that the integrals there are well-defined and differentiable. This may be argued based on the fact that equation (18) is rederived as the third line of equations (21) by differentiating equation (20) directly and using equation (22). Alternatively, using equation (22), we first find that

$$\int_0^\Psi dE \sqrt{\Psi - E} f(E, 2R^2(\Psi - E)) = \frac{1}{(2R^2)^{1/2}} \int_0^{2R^2} dL^2 L f\left(\Psi - \frac{L^2}{2R^2}, L^2\right)$$

and

$$\int_0^\Psi dE \frac{L^2}{\sqrt{\Psi - E}} f(E, 2R^2(\Psi - E)) = \frac{1}{(2R^2)^{1/2}} \int_0^{2R^2} dL^2 L f\left(\Psi - \frac{L^2}{2R^2}, L^2\right)$$

Then, the direct differentiations indicate that

$$\frac{\partial}{\partial \Psi} \int_0^\Psi dE \sqrt{\Psi - E} f(E, 2R^2(\Psi - E)) = \Psi^{1/2} f(0, 2R^2\Psi) + \frac{1}{(2R^2)^{1/2}} \int_0^{2R^2} dL^2 L f^{1/0}\left(\Psi - \frac{L^2}{2R^2}, L^2\right)$$

and

$$\frac{\partial}{\partial R^2} \int_0^\Psi dE \frac{L^2}{\sqrt{\Psi - E}} f(E, 2R^2(\Psi - E)) = \frac{\Psi^{1/2}}{R} f(0, 2R^2\Psi) + \frac{1}{2R^2} \int_0^{2R^2} dL^2 L f^{1/0}\left(\Psi - \frac{L^2}{2R^2}, L^2\right)$$

That is to say,

$$\frac{\partial}{\partial R^2} \int_0^\Psi dE \frac{L^2}{\sqrt{\Psi - E}} f(E, 2R^2(\Psi - E)) = \frac{1}{R} \frac{\partial}{\partial \Psi} \int_0^\Psi dE \sqrt{\Psi - E} f(E, 2R^2(\Psi - E))$$

which is equivalent to equation (17).