Quantitative estimates for the Bakry–Ledoux isoperimetric inequality II

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Abstract
Concerning quantitative isoperimetry for a weighted Riemannian manifold satisfying $\text{Ric}_{\infty} \geq 1$, we give an $L^1$-estimate exhibiting that the push-forward of the reference measure by the guiding function (arising from the needle decomposition) is close to the Gaussian measure. We also show $L^p$- and $W_2$-estimates in the 1-dimensional case.

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1 INTRODUCTION

This short article is devoted to several further applications of the detailed estimates in [15] to quantitative isoperimetry. In [15], on a weighted Riemannian manifold $(M, g, m)$ (with $m = e^{-\Psi} \text{vol}_g$) satisfying $m(M) = 1$ and $\text{Ric}_{\infty} \geq 1$, we investigated the stability of the Bakry–Ledoux isoperimetric inequality [1]:

$$P(A) \geq I_{(\mathbb{R}, \gamma)}(m(A))$$

(1.1)

for any Borel set $A \subset M$, where $P(A)$ is the perimeter of $A$, $\gamma(dx) = (2\pi)^{-1/2} e^{-x^2/2} dx$ is the Gaussian measure on $\mathbb{R}$, and $I_{(\mathbb{R}, \gamma)}$ is its isoperimetric profile written as

$$I_{(\mathbb{R}, \gamma)}(\theta) = \frac{e^{-a_\theta^2/2}}{\sqrt{2\pi}}, \quad \theta = \gamma((-\infty, a_\theta]).$$

(1.2)

It is known by [16, Theorem 18.7] (see also [14, §3]) that equality holds in (1.1) for some $A$ with $\theta = m(A) \in (0, 1)$ if and only if $(M, g, m)$ is isometric to the product of $(\mathbb{R}, | \cdot |, \gamma)$ and a weighted
Riemannian manifold \((\Sigma, g_\Sigma, \mathfrak{m}_\Sigma)\) of \(\text{Ric}_\infty \geq 1\). Moreover, \(A\) is necessarily of the form \((-\infty, a_\theta] \times \Sigma\) or \([-a_\theta, \infty) \times \Sigma\) (so-called a half-space). Then, the stability result [15, Theorem 7.5] asserts that, if equality in (1.1) nearly holds, then \(A\) is close to a kind of half-space in the sense that the symmetric difference between them has a small volume.

The proof as well as the formulation of [15, Theorem 7.5] are based on the needle decomposition paradigm (also called the localization), which was established by Klartag [13] for Riemannian manifolds and has provided a significant contribution specifically in the study of isoperimetric inequalities (we refer to [6] for a generalization to metric measure spaces satisfying the curvature-dimension condition, and to [5] for a stability result). The half-space we mentioned above is in fact a sub-level or super-level set of the guiding function arising in the needle decomposition (see Section 3 and [15] for more details). The needle decomposition enables us to decompose a global inequality on \(M\) into the corresponding 1-dimensional inequalities on minimal geodesics in \(M\) (called needles or transport rays). Therefore, a more detailed 1-dimensional analysis on needles will furnish a better estimate on \(M\).

The 1-dimensional analysis in [15] is concentrated in Proposition 3.2 in it (restated in Proposition 2.1), which gives a very detailed estimate on the difference from the Gaussian measure \(\gamma\). In this article, as an application of the analysis developed in [15], we show an \(L^1\)-bound between \(\gamma\) and the push-forward measure \(u_* \mathfrak{m}\) of \(\mathfrak{m}\) by the guiding function \(u\):

\[
\|\rho \cdot e^{\Psi} - 1\|_{L^1(\gamma)} \leq C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)},
\]

where \(u_* \mathfrak{m} = \rho dx\) and \(\gamma = e^{-\Psi} dx\) (see Theorem 3.1 for the precise statement). In the 1-dimensional case (on intervals), we also prove an \(L^p\)-bound with the improved (and sharp) order \(\delta^{1/p}\) (Proposition 2.2; see Example 2.3 for the sharpness) and an estimate of the \(L^2\)-Wasserstein distance \(W_2\) (Proposition 2.4). The use of \(L^p\) and \(W_2\) (instead of the volume of the symmetric difference) is inspired by stability results for the Poincaré and log-Sobolev inequalities (for example, [2, 4, 8, 11, 12]). We refer to Remark 3.2 for some further related works and open problems.

## 2 QUANTITATIVE ESTIMATES ON INTERVALS

We first consider the 1-dimensional case (on intervals) and establish quantitative stability estimates in terms of the \(L^p\)-norm and the \(W_2\)-distance. The \(L^1\)-bound will be instrumental to study the Riemannian case in the next section.

### 2.1 An \(L^p\)-estimate

Throughout this section, let \(I \subset \mathbb{R}\) be an open interval equipped with a probability measure \(\mathfrak{m} = e^{-\psi} dx\) such that \(\psi\) is 1-convex in the sense that

\[
\psi((1-t)x + ty) \leq (1-t)\psi(x) + t\psi(y) - \frac{1}{2}(1-t)t|x-y|^2
\]

for all \(x, y \in I\) and \(t \in (0, 1)\). This means that \((I, |\cdot|, \mathfrak{m})\) satisfies \(\text{Ric}_\infty \geq 1\) (or the curvature-dimension condition \(\text{CD}(1, \infty)\)), and (1.1) holds. The 1-dimensional isoperimetric inequality is well investigated in convex analysis. An important fact due to Bobkov [3, Proposition 2.1] is
that an isoperimetric minimizer can be always taken as a half-space of the form \((-\infty, a] \cap I\) or \([b, \infty) \cap I\). Now we restate [15, Proposition 3.2], which is the source of all the estimates. Recall that \(\gamma = e^{-\psi} \, dx\) is the Gaussian measure.

**Proposition 2.1** [15]. Fix \(\theta \in (0, 1)\) and suppose that

\[
m((-\infty, a_{\theta}] \cap I) = \theta
\]

and
\[
e^{-\psi(a_{\theta})} \leq e^{-\psi_+(a_{\theta})} + \delta
\]

hold for sufficiently small \(\delta > 0\) (relative to \(\theta\)). Then we have

\[
\psi(x) - \psi_+(x) \geq (\psi'_+(a_{\theta}) - a_{\theta})(x - a_{\theta}) - C(\theta)\delta
\]

for every \(x \in I\), and

\[
\psi(x) - \psi_+(x) \leq (\psi'_+(a_{\theta}) - a_{\theta})(x - a_{\theta}) + C(\theta)\sqrt{\delta}
\]

for every \(x \in [S, T] \subset I\) such that \(\lim_{\delta \to 0} S = -\infty\) and \(\lim_{\delta \to 0} T = \infty\), where \(\psi'_+\) denotes the right derivative of \(\psi\) and \(C(\theta)\) is a positive constant depending only on \(\theta\).

The first condition (2.1) means that \(I\) is ‘centered’ in comparison with \(\gamma\) which satisfies \(\gamma((-\infty, a_{\theta}]) = \theta\) (as in (1.2)). Note also that \(e^{-\psi(a_{\theta})} \geq e^{-\psi_+(a_{\theta})}\) holds by the isoperimetric inequality (1.1) (since \(P((-\infty, a_{\theta}] \cap I) = e^{-\psi(a_{\theta})}\)), and then (2.2) tells that the deficit of \((-\infty, a_{\theta}] \cap I\) in the isoperimetric inequality is less than or equal to \(\delta\).

Besides the above proposition, we also need the following estimate in its proof (see [15, (3.9)]):  

\[
\limsup_{\delta \to 0} \frac{|\psi'_+(a_{\theta}) - a_{\theta}|}{\delta} \leq C(\theta).
\]

The lower bound (2.3) enables us to obtain the following \(L^p\)-estimate between \(\gamma = e^{-\psi} \, dx\) and \(m = e^{\psi_+ - \psi} \gamma|_I\). (We remark that the upper bound (2.4) will not be used.)

**Proposition 2.2** (An \(L^p\)-estimate on \(I\)). Assume (2.1) and (2.2). Then we have

\[
\|e^{\psi_+ - \psi} - 1\|_{L^p(\gamma)} \leq C(p, \theta)\delta^{1/p}
\]

for all \(p \in [1, \infty)\) and sufficiently small \(\delta > 0\) (relative to \(\theta\) and \(p\)), where we set \(e^{\psi_+ - \psi} := 0\) on \(\mathbb{R} \setminus I\).

**Proof.** In this proof, we denote by \(C\) a positive constant depending on \(\theta\), and put \(a := a_{\theta}\) for brevity. Since \(e^{\psi_+ - \psi} - 1 \geq -1\) and \(m(I) = \gamma(\mathbb{R}) = 1\), we find
\[ \| e^{\Phi_e - \psi} - 1 \|_{L^p(\mathcal{F})}^p = \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy + \int_{-\infty}^{\infty} \left( 1 - e^{\Phi_e - \psi} \right)^+_p dy \]
\[ \leq \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy + \int_{-\infty}^{\infty} \left( 1 - e^{\Phi_e - \psi} \right)^+_p dy \]
\[ = \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy + \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy, \]

where \( [r]_+ := \max\{r, 0\} \). Thus, we need to estimate only \( [e^{\Phi_e - \psi} - 1]_+ \). Observe that

\[ \left[ e^{(\Phi_e - \psi)(x)} - 1 \right]^+_p \leq (e^{C\delta|x-a|+C\delta} - 1)^p \leq e^{p(C\delta|x-a|+C\delta)} - 1 \]

from (2.3) and (2.5), and hence

\[ \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy \leq \int_{-\infty}^{\infty} \left( e^{p(C\delta|x-a|+C\delta)} - 1 \right) \mathcal{Y}(dx) \]
\[ = \frac{e^{pC\delta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{2} + pC\delta |x-a| \right) dx - 1. \]

Dividing the integral into \((-\infty, a]\) and \([a, \infty)\), we continue the calculation as

\[ \int_{-\infty}^{a} \exp \left( -\frac{x^2}{2} - pC\delta(x-a) \right) dx + \int_{a}^{\infty} \exp \left( -\frac{x^2}{2} + pC\delta(x-a) \right) dx \]
\[ = \int_{-\infty}^{a} \exp \left( -\frac{(x+pC\delta)^2}{2} + \frac{(pC\delta)^2}{2} + pCa\delta \right) dx \]
\[ + \int_{a}^{\infty} \exp \left( -\frac{(x-pC\delta)^2}{2} + \frac{(pC\delta)^2}{2} - pCa\delta \right) dx \]
\[ \leq \exp \left( \frac{(pC\delta)^2}{2} + pCa\delta \right) \left\{ \int_{-\infty}^{a} e^{-x^2/2} dx + pC\delta \right\} \]
\[ + \exp \left( \frac{(pC\delta)^2}{2} - pCa\delta \right) \left\{ \int_{a}^{\infty} e^{-x^2/2} dx + pC\delta \right\} \]
\[ \leq \exp \left( \frac{(pC\delta)^2}{2} + pC|a|\delta \right) \left( \sqrt{2\pi} + 2pC\delta \right). \]

Therefore, we obtain

\[ \int_I \left[ e^{\Phi_e - \psi} - 1 \right]^+_p dy \leq \exp \left( pC\delta + pC|a|\delta + \frac{(pC\delta)^2}{2} \right) \left( 1 + \frac{2pC\delta}{\sqrt{2\pi}} \right) - 1 \]
\[ \leq C(p, \delta)\delta. \]

This completes the proof. \( \square \)
We remark that since
\[
\left\{ \exp \left( pC\delta + \frac{(pC\delta)^2}{2} \right) - 1 \right\}^{1/p} \geq \exp \left( C\delta + \frac{p(C\delta)^2}{2} \right) - 1,
\]
the constant \( C(p, \theta) \) given by the above proof necessarily depends on \( p \). The order \( \delta^{1/p} \) in Proposition 2.2 may be compared with \( L^p \)-estimates in [11] for the log-Sobolev inequality on Gaussian spaces. One can see that the order \( \delta^{1/p} \) is optimal from the following example.

**Example 2.3.** Let \( I = (-D, D) \) and \( m = (1 + \delta) \cdot \gamma \), where \( \delta > 0 \) is given by \( \gamma(I) = (1 + \delta)^{-1} \). Then, at \( \theta = 1/2 \), we have \( a_{1/2} = 0 \), \( m((-\infty, 0] \cap I) = 1/2 \),
\[
e^{-\psi(0)} - e^{-\psi(0)} = \frac{\delta}{\sqrt{2\pi}},
\]
and
\[
\|e^{\psi_g - \psi} - 1\|_{L^p(\gamma)} = \left( \frac{\delta^p}{1 + \delta} + \frac{\delta}{1 + \delta} \right)^{1/p} = \left( \frac{1 + \delta^{p-1}}{1 + \delta} \right)^{1/p} \delta^{1/p}.
\]

### 2.2 A \( W_2 \)-estimate

From Proposition 2.1, one can also derive an upper bound of the \( L^2 \)-Wasserstein distance between \( m \) and \( \gamma \). We refer to [21] for the basics of optimal transport theory. What we need is only the following Talagrand inequality with \( \gamma \) as the base measure (see [20], [21, Theorem 22.14]):

\[
W_2^2(m, \gamma) \leq 2 \text{Ent}_\gamma(m) = 2 \int_I (\psi_g - \psi)e^{\psi_g - \psi} \, d\gamma,
\]
(2.6)
where \( \text{Ent}_\gamma(m) \) is the relative entropy of \( m \) with respect to \( \gamma \). We remark that both \( \gamma \) and \( m \) have finite second moment (by the 1-convexity of \( \psi \)).

**Proposition 2.4 (A \( W_2 \)-estimate on \( I \)).** Assume (2.1) and (2.2). Then we have
\[
W_2(m, \gamma) \leq C(\theta) \sqrt{\delta}
\]
for sufficiently small \( \delta > 0 \) (relative to \( \theta \)).

**Proof.** We again denote \( a_0 \) by \( a \), and \( C \) will be a positive constant depending only on \( \theta \). Similarly to the proof of Proposition 2.2, we observe from (2.3) and (2.5) that
\[
\int_I (\psi_g - \psi)e^{\psi_g - \psi} \, d\gamma \leq \int_{-\infty}^{\infty} (C\delta|x - a| + C\delta)e^{C\delta|x - a| + C\delta \gamma}(dx)
\]
\[
= \frac{C\delta}{\sqrt{2\pi}} e^{C\delta} \int_{-\infty}^{\infty} (|x - a| + 1) \exp \left( -\frac{x^2}{2} + C\delta|x - a| \right) dx
\]
\[
\leq C\delta \left\{ \int_{-\infty}^{\infty} |x - a| \exp \left( -\frac{x^2}{2} + C\delta|x - a| \right) dx + C \right\},
\]
where we used
\[ \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{2} + C\delta |x - a| \right) dx \leq C \]

from the proof of Proposition 2.2. Then we have

\[
\int_{-\infty}^{a} (a - x) \exp \left( -\frac{x^2}{2} - C\delta (x - a) \right) dx \\
= \exp \left( Ca\delta + \frac{(C\delta)^2}{2} \right) \int_{-\infty}^{a} (a - x) \exp \left( -\frac{(x + C\delta)^2}{2} \right) dx \\
\leq (1 + C\delta) \left\{ \int_{-\infty}^{a} \exp \left( -\frac{(x + C\delta)^2}{2} \right) dx + \left[ \exp \left( -\frac{(x + C\delta)^2}{2} \right) \right]_{-\infty}^{a} \right\} \\
\leq (1 + C\delta) \left\{ a \int_{-\infty}^{a} e^{-x^2/2} dx + C\delta + \exp \left( -\frac{(a + C\delta)^2}{2} \right) \right\} \\
\leq a \int_{-\infty}^{a} e^{-x^2/2} dx + e^{-a^2/2} + C\delta.
\]

We similarly find

\[
\int_{a}^{\infty} (x - a) \exp \left( -\frac{x^2}{2} + C\delta (x - a) \right) dx \\
= \exp \left( -Ca\delta + \frac{(C\delta)^2}{2} \right) \int_{a}^{\infty} (x - a) \exp \left( -\frac{(x - C\delta)^2}{2} \right) dx \\
\leq (1 + C\delta) \left\{ (a - C\delta) \int_{a}^{\infty} \exp \left( -\frac{(x - C\delta)^2}{2} \right) dx - \left[ \exp \left( -\frac{(x - C\delta)^2}{2} \right) \right]_{a}^{\infty} \right\} \\
\leq (1 + C\delta) \left\{ -a \int_{a}^{\infty} e^{-x^2/2} dx + C\delta + \exp \left( -\frac{(a - C\delta)^2}{2} \right) \right\} \\
\leq -a \int_{a}^{\infty} e^{-x^2/2} dx + e^{-a^2/2} + C\delta.
\]

Therefore, together with the Talagrand inequality (2.6), we obtain the desired estimate

\[ W_2^2(m, \gamma) \leq C\delta. \]

We do not know whether the order $\sqrt{\delta}$ in Proposition 2.4 is optimal. Since $W_p (m, \gamma) \leq W_2 (m, \gamma)$ for any $p \in [1, 2)$ by the Hölder inequality, we have, in particular, a bound of the $L^1$-Wasserstein distance:

\[ W_1 (m, \gamma) \leq C(\theta) \sqrt{\delta}. \]

One can alternatively infer this estimate from the Kantorovich–Rubinstein duality (see [21]); in fact,

\[ W_1 (m, \gamma) \leq \int_{-\infty}^{\infty} |x - a| \cdot |e^{(\psi_\delta - \psi)(x)} - 1| \gamma(dx) \leq C(\theta) \sqrt{\delta}. \]
We also remark that, when we take a detour via the reverse Poincaré inequality in [15, Proposition 5.1] and the stability result [8, Theorem 1.2], we arrive at a weaker estimate
\[ W_1(\mathfrak{m}, \gamma) \leq C(\delta, \epsilon)\delta^{(1-\epsilon)/4}. \]

We refer to [7, 9] for stability results for the Poincaré inequality (equivalently, the spectral gap) on CD(\(N - 1, N\))-spaces and RCD(\(N - 1, N\))-spaces with \(N \in (1, \infty)\).

\section{AN L\(^1\)-ESTIMATE ON WEIGHTED RIEMANNIAN MANIFOLDS}

Next, we consider a weighted Riemannian manifold, namely a connected, complete \(C^\infty\)-Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) equipped with a probability measure \(\mathfrak{m} = e^{-\Psi} \text{vol}_g\), where \(\Psi \in C^\infty(M)\) and \(\text{vol}_g\) is the Riemannian volume measure. Assuming \(\text{Ric}_\infty \geq 1\), we have the Bakry–Ledoux isoperimetric inequality (1.1).

We begin with an outline of the proof of (1.1) via the needle decomposition (see [13]). Given a Borel set \(A \subset M\) with \(\theta = \mathfrak{m}(A) \in (0, 1)\), we employ the function \(f := \chi_A - \theta\) (\(\chi_A\) denotes the characteristic function of \(A\)) and an associated 1-Lipschitz function \(u : M \to \mathbb{R}\) attaining the maximum of \(\int_M f \phi \, d\mathfrak{m}\) among all 1-Lipschitz functions \(\phi\). Then, analyzing the behavior of \(u\), one can build a partition \(\{X_q\}_{q \in Q}\) of \(M\) consisting of (the image of) minimal geodesics (called needles), and \(Q\) is endowed with a probability measure \(\nu\). For \(\nu\)-almost every \(q \in Q\), \(u|_{X_q}\) has slope 1 (\(|u(x) - u(y)| = d(x, y)\) for all \(x, y \in X_q\)) and \(X_q\) is equipped with a probability measure \(\mathfrak{m}_q\) such that \(\mathfrak{m}_q(A \cap X_q) = \theta\) and \((X_q, |\cdot|, \mathfrak{m}_q)\) satisfies \(\text{Ric}_\infty \geq 1\). Moreover, we have
\[
\int_M h \, d\mathfrak{m} = \int_Q \left( \int_{X_q} h \, d\mathfrak{m}_q \right) \, \nu(dq),
\]
for all \(h \in L^1(\mathfrak{m})\). Then, (1.1) for \(A\) is obtained by integrating its 1-dimensional counterparts for \(A \cap X_q\) with respect to \(\nu\).

The 1-Lipschitz function \(u\) is called the guiding function. We can assume \(\int_M u \, d\mathfrak{m} = 0\) without loss of generality, and \(X_q\) will be identified with an interval via \(u\) (in other words, \(X_q\) is parametrized by \(u\)). Denote \(\mathfrak{m}_q = e^{-\sigma_q} \, dx\) and \(\mu := u_* \mathfrak{m} = \rho \, dx\). Note that \(\text{supp} \mu\) is an interval and may not be the whole \(\mathbb{R}\). Through the parametrization of \(X_q\) by \(u\), we deduce from (3.1) that
\[
\rho(x) = \int_Q e^{-\sigma_q(x)} \, \nu(dq),
\]
where we set \(e^{-\sigma_q(x)} := 0\) if \(x \notin X_q\).

**Theorem 3.1** (An \(L^1\)-estimate on \(M\)). Assume \(\text{Ric}_\infty \geq 1\) and fix \(\epsilon \in (0, 1)\). If \(P(A) \leq I_{(\mathbb{R}, \gamma)}(\theta) + \delta\) holds for some Borel set \(A \subset M\) with \(\theta = \mathfrak{m}(A) \in (0, 1)\) and sufficiently small \(\delta\) (relative to \(\theta\) and \(\epsilon\)), then \(u_* \mathfrak{m} = \rho \, dx\) satisfies
\[
\|\rho \cdot \phi - 1\|_{L^1(\gamma)} \leq C(\theta, \epsilon)\delta^{(1-\epsilon)/(\theta - 3\epsilon)},
\]
where \(\rho\) is the guiding function associated with \(A\) such that \(\int_M u \, d\mathfrak{m} = 0\).
Proof. First of all, by (3.2) and Fubini’s theorem, we have

$$\|\rho \cdot e^{\Psi_\theta} - 1\|_{L^1(\gamma)} = \int_{-\infty}^{\infty} \left| \int_{Q} (e^{\Psi_\theta - \sigma q} - 1) \nu(dq) \right| dy \leq \int_{Q} \|e^{\Psi_\theta - \sigma q} - 1\|_{L^1(\gamma')} \nu(dq).$$

We shall estimate $\|e^{\Psi_\theta - \sigma q} - 1\|_{L^1(\gamma)}$ by dividing into ‘good’ needles and ‘bad’ needles. Note that $\nu(Q_{r'}) \geq 1 - \sqrt{\delta}$ holds for

$$Q_{r'} := \{ q \in Q \mid m_q(A \cap X_q) = \theta, \; P(A \cap X_q) < I_{(R, \gamma)}(\theta) + \sqrt{\delta} \}$$

by [15, Lemma 7.1], where $P(A \cap X_q)$ denotes the perimeter of $A \cap X_q$ in $(X_q, |\cdot|, m_q)$. Moreover, it follows from [15, Proposition 7.3] that there exists a measurable set $Q_c \subset Q$ such that $\nu(Q_c) \geq 1 - \delta^{(1-\varepsilon)/(9-3\varepsilon)}$ and

$$\max \{ |a_\theta - r_q^-|, |a_{1-\theta} - r_q^+| \} \leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)}$$

for all $q \in Q_c \cap Q_{r'}$, where $m_q((-\infty, r_q^-] \cap X_q) = m_q([r_q^+, \infty) \cap X_q) = \theta$ (recall that $\gamma((-\infty, a_\theta]) = \gamma([a_{1-\theta}, \infty)) = \theta$).

On the one hand, for $q \in Q_c \cap Q_{r'}$, note that either $P(A \cap X_q) \geq e^{-\sigma q(r_q^-)}$ or $P(A \cap X_q) \geq e^{-\sigma q(r_q^+)}$ holds by [3, Proposition 2.1] (recall Subsection 2.1). When $P(A \cap X_q) \geq e^{-\sigma q(r_q^-)}$, we put

$$\gamma_q(dx) = e^{-\Psi_{\theta q}(x)} dx := e^{-\Psi_{\theta q}(x+a_\theta-r_q^-)} dx,$$

which is a translation of $\gamma$ satisfying $\gamma_q((-\infty, r_q^-]) = \theta$. Then, it follows from Proposition 2.2 (with $e^{-\sigma q(r_q^-)} \leq P(A \cap X_q) \leq e^{-\Psi_{\theta q}(r_q^-)} + \sqrt{\delta}$) and Cavalieri’s principle that

$$\|e^{\Psi_\theta - \sigma q} - 1\|_{L^1(\gamma)} \leq \|e^{\Psi_{\theta q} - \sigma q} - 1\|_{L^1(\gamma_q)} + \|e^{-\Psi_{\theta q}} - e^{-\Psi_\theta}\|_{L^1(dx)}$$

$$\leq C(\theta) \sqrt{\delta} + 2 \frac{|a_\theta - r_q^-|}{\sqrt{2\pi}}$$

$$\leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)}.$$

We have the same bound also in the case where $P(A \cap X_q) \geq e^{-\sigma q(r_q^+)}$ by reversing $I$ in Proposition 2.2.

On the other hand, for $q \in Q \setminus (Q_c \cap Q_{r'})$, we have the trivial bound

$$\|e^{\Psi_\theta - \sigma q} - 1\|_{L^1(\gamma)} \leq \|e^{\Psi_\theta - \sigma q}\|_{L^1(\gamma)} + \|1\|_{L^1(\gamma)} = 2.$$

Therefore, we obtain

$$\|\rho \cdot e^{\Psi_\theta} - 1\|_{L^1(\gamma)} \leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)} + 2\left(1 - \nu(Q_c \cap Q_{r'})\right) \leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)}. \quad \square$$

Note that $q \in Q_c \cap Q_{r'}$ is well behaved and can be handled by the 1-dimensional analysis, whereas one has a priori no information of $q \in Q \setminus (Q_c \cap Q_{r'})$. This could be a common problem
for stability estimates via the needle decomposition (see, for example, [15, Theorem 6.2] showing a reverse Poincaré inequality on a manifold from a sharper estimate on intervals). In particular, it may be difficult to achieve the same order $\delta$ as in the 1-dimensional case (Proposition 2.2) by the needle decomposition. In the $L^p$-case, it is unclear (to the authors) with what we can replace the trivial bound $\|e^{\psi_g-q} - 1\|_{L^1(\gamma)} \leq 2$. For the Wasserstein distance $W_2$ or $W_1$, we have the same problem on the control of $q \in Q \setminus (Q_c \cap Q_r)$.

Remark 3.2 (Further related works and open problems).

(a) Theorem 3.1 holds true also for reversible Finsler manifolds by the same proof (see [15, Remark 7.6(c)] and [17, 18]).

(b) As we mentioned in the introduction, our $L^p$- and $W_2$-estimates are inspired by the quantitative stability for functional inequalities. We refer to [4, 10–12] for the study of the log-Sobolev inequality on the Gaussian space:

$$\text{Ent}_\gamma(f \gamma) \leq \frac{1}{2} I_\gamma(f \gamma) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, d\gamma,$$

where $I_\gamma(f \gamma)$ is the Fisher information of a probability measure $f \gamma$ with respect to $\gamma$. They investigated the difference between $\gamma$ and $f \gamma$, in terms of the additive deficit $\delta(f) = I_\gamma(f \gamma)/2 - \text{Ent}_\gamma(f \gamma)$. For instance, $W_2$-bounds (under certain convexity and concavity conditions on $f$) were given in [4, 12], and $L^1$- and $L^p$-bounds can be found in [11]. In the setting of weighted Riemannian manifolds satisfying $\text{Ric}_\infty \geq 1$ (as in Theorem 3.1), we have only the rigidity (see [19]) and the stability is an open problem.

(c) We have seen in [15, §6] that the reverse forms of the Poincaré and log-Sobolev inequalities can be derived from the isoperimetric deficit. The reverse Poincaré inequality then implies a $W_1$-estimate for the push-forward by an eigenfunction due to [2, Theorem 1.3] (see also [9]). We also expect a direct $W_1$- or $W_2$-estimate for the push-forward by the guiding function, which remains an open question (see [15, Remark 7.6(g)]).

(d) Another direction of research is a generalization to negative effective dimension, that is, $\text{Ric}_N \geq K > 0$ with $N < -1$. We have established rigidity in the isoperimetric inequality in [14], thereby it is natural to consider quantitative isoperimetry, though it seems to require longer calculations.

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