THE HARDY–MOSER–TRUDINGER INEQUALITY VIA THE TRANSPLANTATION OF GREEN FUNCTIONS

Van Hoang Nguyen
Department of Mathematics, FPT University, Ha Noi, Vietnam

(Communicated by the Xuefeng Wang)

Abstract. We provide a new proof of the Hardy–Moser–Trudinger inequality and the existence of its extremals which are established by Wang and Ye ("G. Wang, and D. Ye, A Hardy–Moser–Trudinger inequality, Adv. Math, 230 (2012) 294–230.") without using the blow-up analysis method. Our proof is based on the transformation of functions via the transplantation of Green functions. This method enables us to compute explicitly the concentrating level of the Hardy–Moser–Trudinger functional over the normalizing concentrating sequences which is crucial to prove the existence of extremals for the Hardy–Moser–Trudinger inequality. Some comments on the applications of this approach to some other Moser–Trudinger type inequalities are given.

1. Introduction. The Sobolev embedding is a basis tool in many aspects of mathematical analysis. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \). For \( p > 1 \), we denote by \( W_0^{1,p}(\Omega) \) the usual first order Sobolev space which is the completion of the space \( C_\infty^0(\Omega) \) under the Dirichlet norm \( \|\nabla u\|_{L^p(\Omega)} := \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \), \( u \in C_\infty^0(\Omega) \). The Sobolev embedding says that the space \( W_0^{1,p}(\Omega) \) with \( 1 < p < n \) can be embedded into the Lebesgue space \( L^q(\Omega) \) for any \( 1 \leq q \leq p^* = np/(n-p) \). However, in the limiting case \( p = n \) (thus \( p^* = \infty \)) the embedding \( W_0^{1,n}(\Omega) \to L^\infty(\Omega) \) fails. In this case, it was proved independently by Yudovič [46], Pohožaev [35], and Trudinger [39] that \( W_0^{1,n}(\Omega) \) can be embedded into an Orlicz space \( L_{\varphi_n}(\Omega) \) generated by the Young function \( \varphi_n(t) = e^{ct(\frac{t}{\omega_n})^{\frac{n-1}{n}}} - 1 \) for some \( c > 0 \). Later, Moser [29] sharpened this result by finding out the sharp exponent \( c \). More precisely, we have the following Moser–Trudinger inequality

\[
\sup_{u \in W_0^{1,n}(\Omega)} \int_{\Omega} e^{\alpha |u|^{\frac{n}{n-1}}} \, dx < \infty,
\]

for any \( \alpha \leq \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}} \) where \( \omega_{n-1} \) denotes the surface area of the unit sphere in \( \mathbb{R}^n \). Furthermore, the inequality (1.1) is sharp in the sense that the supremum above becomes infinity if \( \alpha > \alpha_n \).

The Moser–Trudinger inequality plays the role of the Sobolev inequality in the limiting case. It becomes an interesting subject and has many applications in many branches of mathematics such as analysis, geometry, partial differential equations, 2010 Mathematics Subject Classification. Primary: 26D10; Secondary: 35A23, 46E35. Key words and phrases. Moser–Trudinger inequality, Hardy–Moser–Trudinger inequality, maximizers, Green function.
calculus of variations, etc. There have been many generalizations of the Moser–Trudinger inequality in many directions such as its extension to higher order (or fractional order) Sobolev spaces \cite{[2, 27]}, to unbounded domain in \(\mathbb{R}^n\) \cite{[1, 15, 18, 36]}, to singular weighted case \cite{[4, 6, 33]}, to Riemannian manifolds \cite{[5, 24, 25, 31, 43, 45]}. Beside, there have been some improvements of the Moser–Trudinger inequality given by Adimurthi and Druet \cite{[3]} and by Tintarev \cite{[38]} in dimension two (see \cite{[32]} for the same improvement in any dimension and \cite{[33]} for the improvements of the singular Moser–Trudinger inequality).

An interesting question concerning to the Moser–Trudinger inequality is whether or not the extremal functions exist. This question was first affirmatively answered by Carleson and Chang when \(\Omega\) is the ball in \(\mathbb{R}^n\) (another proofs were given in \cite{[12]} and in \cite{[26]} in dimension two via the energy estimate method). The existence result of Carleson and Chang later was extended to any domain in \(\mathbb{R}^2\) by Flucher \cite{[13]} and to any domain in \(\mathbb{R}^n, n \geq 3\) by Lin \cite{[20]}. In \cite{[16, 17]}, Li developed a blow-up analysis method to establish the existence of extremals for the Moser–Trudinger inequality on Riemannian manifolds. In whole space \(\mathbb{R}^n\), the existence of extremals for the Moser–Trudinger inequality was proved by Ruf \cite{[36]} for \(n = 2\) and by Li and Ruf \cite{[18]} for any \(n\). In \cite{[14]}, Ishiwata established the existence results as well as the non-existence results of extremals for the Moser–Trudinger inequality. For the existence of extremals for the improved Moser–Trudinger inequalities as well as the singular Moser–Trudinger inequality and its improved version, the readers may consult the papers \cite{[9–11, 32, 33]} and the references therein.

Recently, Wang and Ye proved the Hardy–Moser–Trudinger inequality in the unit disk \(B \subset \mathbb{R}^2\) which combines both the Hardy inequality and the classical Moser–Trudinger inequality (1.1). Let us recall the Hardy inequality

\[
H(u) := \int_B |\nabla u|^2 \, dx - \int_B \frac{|u|^2}{(1 - |x|^2)^2} \, dx, \quad u \in C_0^\infty(B) \geq 0.
\]

It is known that \(H(u) \geq C \int_B |u|^2 \, dx\) for some constant \(C > 0\) (see \cite[Remark 1]{[40]}). Thus, the functional \(u \to \sqrt{H(u)}\) defines a norm on \(C_0^\infty(B)\) functions. Let \(\mathcal{H}\) denotes the completion of \(C_0^\infty(B)\) under this norm. Then \(\mathcal{H}\) is a Hilbert space and \(W_{1,2}^{0}(B)\) is a proper subspace of \(\mathcal{H}\). In \cite{[40]}, Wang and Ye proved the following inequality

**Theorem 1.1 (Wang-Ye).** There holds

\[
\sup_{u \in \mathcal{H}, H(u) \leq 1} \int_B e^{4\pi u^2} \, dx < \infty. \tag{1.2}
\]

Moreover, there exists a function \(u^* \in \mathcal{H}\) so that \(H(u^*) = 1\) and

\[
\sup_{u \in \mathcal{H}, H(u) \leq 1} \int_B e^{4\pi u^2} \, dx = \int_B e^{4\pi (u^*)^2} \, dx.
\]

Evidently, we see that the Hardy–Moser–Trudinger inequality is stronger than the classical Moser–Trudinger inequality (1.1) in \(B\). The proof of Theorem 1.1 given in \cite{[40]} is based on the blow-up analysis method which is now a standard method to study the problems of this type. We refer the readers to \cite{[3, 16–18, 32, 33, 36, 40, 42, 44]} and references therein for more details on this method. The Hardy–Moser–Trudinger inequality (1.2) is a special case of the more general improved Moser–Trudinger inequalities established by Tintarev \cite{[38]} in which \(H\) is replaced by the functional \(H_V(u) = \int_B |\nabla u|^2 - \int_B V u^2 \, dx\) for some potential \(V\) so that \(H_V\)
satisfies a weak coercive condition. There have been a lot of generalizations of (1.2) (see [19, 23, 41, 42, 44]). It is very remarkable that the inequality (1.2) can be seen as the analogue of the Hardy–Sobolev–Maz’ya inequality in dimension two. Recall that the Hardy–Sobolev–Maz’ya inequality (see [28, Section 2.1.6, Corollary 3]) says that there exists a constant \( C > 0 \) such that for any \( u \in W^{1,n}_0(B^n) \) with \( n > 2 \) and \( B^n \) being the unit ball in \( \mathbb{R}^n \), it holds

\[
\int_{B^n} |\nabla u|^2 \, dx - \int_{B^n} \frac{|u|^2}{(1 - |x|^2)^2} \, dx \geq C \left( \int_{B^n} |u|^\frac{2n}{n-2} \, dx \right)^{\frac{n-2}{n}}.
\]

Moreover, let \( C_n \) denote the best constant so that the above inequality holds. It is well known that \( C_n < S_n \) and is attained if \( n > 4 \) (see [37]), and \( C_3 = S_3 \) and is not attained (see [7]) where \( S_n \) is the best constant in the Sobolev inequality. The \( L_p \) version of the above inequality in the hyperbolic space was considered in [30] by the author.

Our main aim in this paper is to provide a new proof of the Hardy–Moser–Trudinger inequality (1.2). Our proof below is based on the transplantation of Green functions. This method was previously used by Flucher [13] to prove the existence of maximizer for the Moser–Trudinger inequality in dimension two, and then was used by Lin [20] in any dimension. It also was successfully applied to prove the existence of maximizers for the singular Moser–Trudinger inequality [9–11]. Recall that the function \( G_B(x) = -\frac{1}{2^*} \ln |x| \) is Green function of \( -\Delta \) in \( B \) with pole at 0 and the Dirichlet boundary condition, i.e., \( -\Delta G_B = \delta_0 \) in the distributional sense, and \( G_B = 0 \) on \( \partial B \). Denote by \( \mathcal{L}_H \) the operator \( \mathcal{L}_H = -\Delta - \frac{1}{(1 - |x|^2)^2} \). It was shown in [40] that there exists a unique radial function \( G \in \mathcal{H} + W^{1,p}_0(B_{1/2}) \) with \( p \in [1, 2) \) such that \( \mathcal{L}_H G = \delta_0 \) in the distributional sense. We will show that \( G \) is strict decreasing and positive in \( B \). Our approach is as follows. By the symmetrization argument, it is enough to prove (1.2) for radial functions. Let \( u \) be a radial function in \( \mathcal{H} \), we define a new radial function \( v \) on \( B \) such that \( u(x) = v(e^{-2\pi G(x)}) \) (here we write \( v(r) \) for the value of \( v(x) \) with \( |x| = r \)). We will show that \( v \in W^{1,2}_0(B) \) and \( H(u) \geq \int_B |\nabla v|^2 \, dx \). The inequality (1.2) is then followed from the classical Moser–Trudinger inequality in \( B \) and some estimates related to \( G \) and \( G_B \) near the pole 0. It is interesting that this approach gives us an explicit value of the concentrating level of the Hardy–Moser–Trudinger functional \( \mathcal{F}(u) = \int e^{4\pi u^2} \, dx \) on the set of normalizing concentrating sequences ((NCS) for shorty) in \( \mathcal{H} \). More precisely, a sequence \( \{u_k\}_k \subset \mathcal{H} \) is called a (NCS) if \( u_k \) is non-increasing radial function and \( H(u_k) \leq 1 \) for each \( k \), \( u_k \rightharpoonup 0 \) weakly in \( \mathcal{H} \) and \( |\nabla u_k|^2 \, dx \rightarrow \delta_0 \) weakly in the measure sense. We will show that

\[
\sup \{ \limsup_{k \to \infty} \mathcal{F}(u_k) \mid \{u_k\} \text{ is (NCS)} \} = \pi \left( 1 + e^{1+4\pi C_G} \right)
\]

where \( C_G \) is the constant appearing in (2.4) below. The above estimate is crucial in proving the attainability of the Hardy–Moser–Trudinger inequality (1.2).

As a final comment, it was shown in [22] that the inequality (1.2) is equivalent to the following inequality

\[
\sup_{u \in \mathcal{H}, H(u) \leq 1} \int_B e^{4\pi u^2} - \frac{1 - 4\pi u^2}{(1 - |x|^2)^2} \, dx < \infty.
\]

(1.3)

The higher dimension version of (1.3) was conjectured in [25] (see the Conjecture 5.2). In the recent work [34], this conjecture was affirmatively confirmed by the
from the proof of Proposition 2 in [40], the function $\phi$ with
some relevant facts as well as prove several useful results which will be used in
the proof of Theorem 1.1 such as the hyperbolic symmetrization method, the Green
functions of $\mathcal{L}_H$, the change of functions via the transplantation of Green functions
and its properties. We also compute the concentrating level of the Hardy–Moser–
Trudinger functional in this section. Section §3 is devoted to prove Theorem 1.1.
We also give some further remarks in this approach at the end of this section.

2. Preliminaries. In this section, we recall some useful facts and prove some useful
results which will be used in the proof of Theorem 1.1. First, we set

$$
\Sigma = \{ u \in C_0^\infty (B) : u(x) = u(r) \text{ with } r = |x|, u' \leq 0 \},
$$

and let $\mathcal{H}_1$ be the closure of $\Sigma$ in $\mathcal{H}$. Using the symmetrization argument with
respect to the standard hyperbolic measure $d\text{vol}_H = \left( \frac{2}{1-|x|^2} \right)^\frac{1}{2} dx$, Wang and Ye show that

$$
\sup_{u \in \mathcal{H}, H(u) \leq 1} \int_B e^{4\pi u^2} dx = \sup_{u \in \mathcal{H}_1, H(u) \leq 1} \int_B e^{4\pi u^2} dx.
$$

Hence, it is enough to prove Theorem 1.1 for functions $\mathcal{H}_1$.

We next recall the Poincaré–Sobolev inequality in $B$ (see [25, Lemma 2.1]): for
any $p > 2$ there exists the constant $C$ depending only on $p$ such that

$$
H(u) \geq C \left( \int_B |u|^p d\text{vol}_H \right)^\frac{1}{p}, \quad u \in \mathcal{H}. \tag{2.1}
$$

Since $u \in \mathcal{H}_1$ is non-increasing, hence it is easy to see that

$$
u(r) \leq Dr^{-\frac{2}{p}} H(u)^{\frac{1}{2}}, \quad r \in (0, 1/2), \tag{2.2}
$$

for any function $u \in \mathcal{H}_1$ for some constant $D$ depending only on $p$. Moreover, from
Lemma 5.3 in [25], we have that for any $p > 2$ there exists the constant $B$ depending
only on $p$ so that

$$
|u(r)| \leq B(1 - r^2)^{\frac{1}{2}} H(u)^{\frac{1}{2}}, \quad \forall r \in [1/2, 1), \tag{2.3}
$$

for any radial function $u \in \mathcal{H}_1$.

Recall the operator $\mathcal{L}_H$ in the introduction. It was proved by Wang and Ye
(see [40, Proposition 2]) that there exists a unique function $G \in \mathcal{H} + W_0^{1,p}(B_{1/2})$
with $p \in [1, 2)$ such that $\mathcal{L}_H(G) = 0$ in the distributional sense. Moreover, $G$ is
radial function and there is a constant $C_G$ such that

$$
G(r) = -\frac{1}{2\pi} \ln r + C_G + \phi(r), \tag{2.4}
$$

with $\phi \in C_{\text{loc}}^{1,\alpha}(B)$ and $\phi(r) = O(r^{1+\alpha})$ as $r \to 0$ for any $\alpha \in (0, 1)$. Furthermore,
from the proof of Proposition 2 in [40], the function $G$ can be decomposed as $G =
G_1 + G_2$ with $G_1 \in \mathcal{H}_1$ and $G_2 = -\frac{2\pi}{\sqrt{r}} \psi(r) \ln r$ with a cut-off function $\psi \in C_0^\infty (B_{1/2})$.
Using (2.3), we have $G = 0$ on $\partial B$. By the maximum principle we have $G \geq 0$ in $B$.
Notice that $\Delta G = -G/(1 - |x|^2)^2$ in $B \setminus \{0\}$. Integrating this equation on $B_r \setminus B_\epsilon$
we obtain

$$
-2\pi G'(r) r + 2\pi G'(\epsilon) \epsilon = \int_{B_r \setminus B_\epsilon} \frac{G(x)}{(1 - |x|^2)^2} dx.
$$
From (2.4), we have \( \lim_{\epsilon \to 0} 2\pi G'(\epsilon)\epsilon = -1 \). Therefore, we have
\[
-2\pi rG'(r) = 1 + \int_{B_r} \frac{G}{(1 - |x|^2)^2} dx > 1,
\]
for any \( r > 0 \). Hence \( G : (0, 1] \to [0, \infty) \) is strictly decreasing. Let \( G^{-1} \) denote the inverse function of \( G \). Set \( G^{-1}(\infty) = 0 \) and for \( t \in [0, 1] \) denote
\[
a(t) = G^{-1}\left(-\frac{1}{2\pi}\ln t\right).
\]
Then \( a \) is strictly increasing, continuous function on \( [0, 1] \) with \( a(0) = 0 \) and \( a(1) = 1 \).

**Lemma 2.1.** The function \( t \to \frac{a(t)}{t} \) is strictly decreasing and
\[
\lim_{t \to 0} \frac{a(t)}{t} = e^{2\pi C_G}.
\]
Consequently, we have \( C_G > 0 \).

**Proof.** Differentiating the function \( a(t)/t \), we have
\[
\left( \frac{a(t)}{t} \right)' = \frac{a'(t)t - a(t)}{t^2}.
\]
Since \( G(a(t)) = (-\ln t)/2\pi \), then it holds \( a'(t) = -1/(2\pi tG'(a(t))) \). Inserting this equality into the previous equality, we get
\[
\left( \frac{a(t)}{t} \right)' = \frac{-1 - 2\pi G'(a(t))a(t)}{2\pi t^2 G'(a(t))} < 0,
\]
for any \( t > 0 \), here we use (2.5) and \( G' < 0 \). Thus, the function \( t \to a(t)/t \) is strictly decreasing. Moreover, we have from (2.4) that
\[
-\frac{1}{2\pi} \ln t = G(a(t)) = -\frac{1}{2\pi} \ln a(t) + C_G + \phi(a(t)),
\]
which implies
\[
a(t) = e^{2\pi C_G + \phi(a(t))}.
\]
Notice that \( \lim_{t \to 0} a(t) = 0 \), hence by letting \( t \to 0 \) in the previous equality, we get the limit (2.6). The conclusion \( C_G > 0 \) comes from the monotonicity of \( t \to a(t)/t \), \( a(1) = 1 \) and the limit (2.6). \( \square \)

The next ingredient in the proof of Theorem 1.1 is a result due to Carleson and Chang [8] which is crucial to prove the existence of maximizer for the Moser–Trudinger inequality (see [8–11,13,20]). For our purpose, we only state their result in dimension 2.

**Lemma 2.2.** Let \( v_k \) be a sequence of radially non-increasing functions in \( W^{1,2}_0(B) \) such that \( \|
abla v_k\|_{L^2(B)} \leq 1 \) for each \( k \), \( v_k \to 0 \) weakly in \( W^{1,2}_0(B) \) and \( \|
abla v_k\|^2 dx \to \delta_0 \) weakly in the measure sense. Then
\[
\limsup_{k \to \infty} \int_B e^{4\pi v_k^2} dx \leq \pi + \pi e.
\]

(2.7)
From (2.7), we easily derive the following result
\[
\lim_{\delta \downarrow 0} \lim_{k \to \infty} \int_{B_{\delta}} e^{4\pi v_k^2} dx = \pi e. \tag{2.8}
\]
Indeed, since \(\delta \to \limsup_{k \to \infty} \int_{B_{\delta}} e^{4\pi v_k^2} dx\) is non-decreasing, then the limit exists.
For each \(\delta > 0\), suppose
\[
\limsup_{k \to \infty} \int_{B_{\delta}} e^{4\pi v_k^2} dx = \lim_{l \to \infty} \int_{B_{\delta}} e^{4\pi v_{k_l}^2} dx,
\]
for a subsequence \(\{v_{k_l}\}_l\) of \(\{v_k\}_k\). By extracting a subsequence, we can assume that \(v_{k_l} \to 0\) a.e. in \(B\). Since \(v_k\) is radial, then
\[
|v_k(r)| = \left| \int_r^1 v_k'(s) ds \right| \leq \left( 2\pi \int_r^1 (v_k'(s))^2 s ds \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \int_r^1 s^{-1} ds \right)^{\frac{1}{2}} \leq \left( \frac{-\ln r} {2\pi} \right)^{\frac{1}{2}},
\]
for any \(r \in (0,1)\). Hence, the Lebesgue dominated convergence theorem yields
\[
\lim_{l \to \infty} \int_{B_{\delta}} e^{4\pi v_{k_l}^2} dx = |B_{\delta}|.
\]
Notice that
\[
\int_{B_{\delta}} e^{4\pi v_{k_l}^2} dx = \int_B e^{4\pi v_{k_l}^2} dx - \int_{B_{\delta}} e^{4\pi v_{k_l}^2} dx.
\]
Letting \(l \to \infty\) and using (2.7), we obtain
\[
\limsup_{k \to \infty} \int_{B_{\delta}} e^{4\pi v_k^2} dx = \lim_{l \to \infty} \int_{B_{\delta}} e^{4\pi v_{k_l}^2} dx \leq \pi + \pi e - |B_{\delta}|.
\]
Letting \(\delta \downarrow 0\), we get (2.8).

We next discuss about the transformation of functions between \(\mathcal{H}_1\) and \(W_0^{1,2}(B)\) via the transplantation of Green functions. Recall that \(G_B = -\frac{1}{2\pi} \ln r\) is the Green function of \(-\Delta\) in \(B\) with pole at 0. Let \(u \in \Sigma\), we define the new radial function \(v\) in \(B\) by \(v(r) = u(G_B^{-1}(G_B(r))) = u(a(r))\) or \(u(r) = v(e^{-2\pi G(r)}).\) Since \(e^{-2\pi G(r)} = r e^{G_B + \phi_G(r)}\), then \(v \in C^0_0(B)\). By a direct computation, we have
\[
u'(r) = -2\pi v'(e^{-2\pi G(r)}) e^{-2\pi G(r)} G'(r).
\]
Therefore, for any \(0 \leq \rho < 1\) we have
\[
\int_{B_{\rho} \setminus B_\rho} |
abla u|^2 dx = 2\pi \int_\rho^1 |u'(r)|^2 r dr = 2\pi \int_\rho^1 (v'(e^{-2\pi G(r)}))^2 e^{-4\pi G(r)} 4\pi^2 (G'(r))^2 r dr,
\]
with convention that \(B_0 = \emptyset\). Making the change of variable \(t = e^{-2\pi G(r)}\) or equivalently \(r = a(t)\), we have
\[
\int_{B_{\rho} \setminus B_\rho} |
abla u|^2 dx = 2\pi \int_{e^{-2\pi G(\rho)}}^{e^{-2\pi G(\rho)}} (v'(t))^2 t^2 4\pi^2 (G'(a(t)))^2 a(t) a'(t) dt.
\]
Notice that \(G(a(t)) = -\frac{1}{2\pi} \ln t\). Hence it holds \(-2\pi G'(a(t)) a'(t) = 1/t\) and
\[
\int_{B_{\rho} \setminus B_\rho} |
abla u|^2 dx = 2\pi \int_{e^{-2\pi G(\rho)}}^{e^{-2\pi G(\rho)}} (v'(t))^2 t (-2\pi G'(a(t)) a(t)) dt.
\]
Define
\[ \Phi(t) = -2\pi G'(a(t))a(t) - 1. \]
Using (2.5) we have \( \Phi(t) \geq 0 \), and
\[ \int_{B \setminus B_r} |\nabla u|^2 dx = \int_{B \setminus B_{e^{-2\pi G(r)}}} |\nabla v|^2 dx + 2\pi \int_{e^{-2\pi G(r)}} (v'(t))^2 t \Phi(t) dt. \tag{2.9} \]
We next compute \( \int_{B \setminus B_r} \frac{u^2}{(1-|x|^2)^2} dx \). Using polar coordinate, we have
\[ \int_{B \setminus B_r} \frac{u^2}{(1-|x|^2)^2} dx = 2\pi \int_0^1 \frac{u(r)^2}{(1-r^2)^2} r dr = 2\pi \int_0^1 \frac{v(e^{-2\pi G(r)})^2}{(1-r^2)^2} r dr \]
\[ = 2\pi \int_{e^{-2\pi G(r)}} v(t)^2 \frac{a(t)'}{a(t)^2} a(t) dt, \]
here the last equality comes from the change of variable \( r = a(t) \). Furthermore, since \( L_H(G) = 0 \) on \( B \setminus \{0\} \) and \( G(a(t)) = -\frac{1}{2\pi} \ln t \), then we have
\[ \Phi'(t) = -2\pi \left( G''(a(t)) + \frac{G'(a(t))}{a(t)} \right) a(t) dt = 2\pi \frac{G(a(t))}{(1-a(t)^2)^2} a(t) = -\frac{\ln t}{(1-a(t)^2)^2} a(t) dt. \]
Consequently, we get
\[ \int_{B \setminus B_r} \frac{u^2}{(1-|x|^2)^2} dx = 2\pi \int_{e^{-2\pi G(r)}} v(t)^2 \frac{\Phi'(t)}{-\ln t} dt. \tag{2.10} \]
We next prove a Hardy type inequality as follows

**Lemma 2.3.** Let \( v \in C^1_0(B) \). Then it holds
\[ \int_{e^{-2\pi G(r)}} (v'(t))^2 t \Phi(t) dt \geq \int_{e^{-2\pi G(r)}} v(t)^2 \frac{\Phi'(t)}{-\ln t} dt. \]

**Proof.** Define \( w(t) = v(t)/(-\ln t) \) if \( t > 0 \) and \( w(0) = 0 \). Thus \( w \in C^1((0, 1)) \). For \( t > 0 \), we have
\[ v'(t) = w'(t)(-\ln t) - \frac{w(t)}{t}. \]
For any \( \epsilon \in (0, 1) \), we have
\[ \int_{e^{-2\pi G(r)}} (v'(t))^2 t \Phi(t) dt \]
\[ = \int_{e^{-2\pi G(r)}} \left( w'(t)(-\ln t) - \frac{w(t)}{t} \right)^2 t \Phi(t) dt \]
\[ = \int_{e^{-2\pi G(r)}} (w'(t))^2 (\ln t)^2 t \Phi(t) dt + \int_{e^{-2\pi G(r)}} \frac{w(t)^2}{t} t \Phi(t) dt + \int_{e^{-2\pi G(r)}} (w(t))^2' \ln t \Phi(t) dt \]
\[ = \int_{e^{-2\pi G(r)}} (w'(t))^2 (\ln t)^2 t \Phi(t) dt + \int_{e^{-2\pi G(r)}} v(t)^2 \frac{\Phi'(t)}{-\ln t} dt + v(e^{-2\pi G(r)})^2 \frac{\Phi(e^{-2\pi G(r)})}{2\pi G(\rho)} \frac{1}{\ln t} dt \]
\[ \geq \int_{e^{-2\pi G(r)}} v(t)^2 \frac{\Phi'(t)}{-\ln t} dt, \]
here the last equality comes from the integration by parts and the change of function \( v(t) = w(t)(-\ln t) \). This finishes the proof of Lemma 2.3. \( \square \)
Combining (2.9), (2.10) and Lemma 2.3 together, we arrive
\[
\int_{B\setminus B_\rho} |\nabla u|^2 dx - \int_{B\setminus B_\rho} \frac{u^2}{(1-|x|^2)^2} dx \geq \int_{B\setminus B_{\rho-2\pi G(\rho)}} |\nabla v|^2 dx,
\]
for any \( u \in \Sigma \). Especially, for any \( \rho \in [0, 1) \), we have
\[
\int_{B_\rho} |\nabla u|^2 dx - \int_{B_\rho} \frac{u^2}{(1-|x|^2)^2} dx \leq H(u),
\]
for any \( u \in \Sigma \). Using density argument, we easily obtain the following results.

**Lemma 2.4.** Let \( u \in \mathcal{H}_1 \) and define the function \( v(x) = u(a(|x|)) \). Then \( v \in W^{1,2}_0(B) \) and
\[
H(u) \geq \int_B |\nabla u|^2 dx.
\]
Moreover, for any \( \rho \in (0, 1) \) we have
\[
\int_{B_\rho} |\nabla u|^2 dx - \int_{B_\rho} \frac{u^2}{(1-|x|^2)^2} dx \leq H(u).
\]

Let us define the Hardy–Moser–Trudinger functional on \( \mathcal{H} \) by
\[
\mathcal{F}(u) = \int_B e^{4\pi u^2} dx, \quad u \in \mathcal{H}.
\]
The next lemma gives us the concentrating level of the functional \( \mathcal{F} \) over the (NCS) sequences in \( \mathcal{H}_1 \).

**Lemma 2.5.** It holds,
\[
\sup_{k \to \infty} \{	ext{lim sup } \mathcal{F}(u_k) : \{u_k\} \text{ is (NCS)}\} = \pi \left( 1 + e^{1+4\pi C_{2\mathcal{H}}} \right). \tag{2.11}
\]

**Proof.** Let \( \{u_k\}_k \) be a (NCS) sequence in \( \mathcal{H}_1 \) and \( A = \limsup_{k \to \infty} \mathcal{F}(u_k) \). By extracting a subsequence, we can assume that \( \lim_{k \to \infty} \mathcal{F}(u_k) = A \) and \( u_k \to 0 \) a.e. in \( B \). Let \( v_k(r) = u_k(a(r)) \). By Lemma 2.4, we have \( v_k \in W^{1,2}_0(B) \) and \( \|\nabla v_k\|_{L^2(B)} \leq 1 \). Moreover, \( v_k \to 0 \) a.e. in \( B \). Hence \( v_k \rightharpoonup 0 \) weakly in \( W^{1,2}_0(B) \). If \( \|\nabla v_k\|_{L^2(B)} \neq 0 \) in the measure sense. Then there exists \( \delta < 1 \) and \( t_0 \in (0, e^{-2\pi C_{1,2}}) \) such that
\[
\int_{B_{t_0}} |\nabla v_k|^2 dx \leq \delta < 1,
\]
for any \( k \). For any \( t_1 \in (0, t_0) \), define \( w_k(t) = v_k(t) - v_k(t_1) \) if \( t \leq t_1 \) and \( w_k(t) = 0 \) if \( t > t_1 \). Then \( w_k \in W^{1,2}_0(B) \) and \( \|\nabla w_k\|_{L^2(B)} \leq \delta \) for any \( k \). By the classical Moser–Trudinger inequality, we have \( e^{4\pi w_k^2} \) is bounded in \( L^q(B) \) for some \( q > 1 \). In the other hand, we have
\[
v_k(t)^2 = (w_k(t))^2 + v_k(t_1)^2 \leq (1 + \epsilon)w_k(t)^2 + 1 + \epsilon^{-1}v_k(t_1)^2,
\]
for any \( \epsilon > 0 \). Choosing \( \epsilon > 0 \) small enough so that \( 1 + \epsilon < q \), we then have that \( e^{4\pi v_k^2} \) is bounded in \( L^p(B_{t_1}) \) for some \( p > 1 \), here we use the fact that \( \{v_k(t_1)\}_k \) is bounded by (2.2). Since \( v_k \to 0 \) a.e. in \( B \), hence it holds
\[
\lim_{k \to \infty} \frac{1}{t_1} \int_{B_{t_1}} e^{4\pi v_k^2} dx = \pi t_1^2. \tag{2.12}
\]
Making the change of variable $r = a(t)$ and using the fact $a'(t) = -1/(2\pi t G'(a(t)))$, we have
\[
\int_{B_{a(t_1)}} e^{4\pi u_k^2} dx = 2\pi \int_0^{a(t_1)} e^{4\pi v_k(t)^2} r dr
= 2\pi \int_0^{t_1} e^{4\pi v_k(t)^2} t \left(\frac{a(t)}{t}\right)^2 \frac{1}{-2\pi G'(a(t))a(t)} dt,
\] (2.13)

We now apply Lemma 2.1 and the estimate (2.5) to get
\[
\int_{B_{a(t_1)}} e^{4\pi u_k^2} dx \leq e^{4\pi C_G} \int_{B_{t_1}} e^{4\pi v_k^2} dx.
\]
Letting $k \to \infty$ and using (2.12), we get
\[
\limsup_{k \to \infty} \int_{B_{a(t_1)}} e^{4\pi u_k^2} dx \leq \pi e^{4\pi C_G} t_1^2.
\] (2.14)

Since $u_k$ is non-increasing and (2.2), we have $u_k$ is uniformly bounded in $B_{a(t_1)}$. By Lebesgue dominated convergence theorem, we get
\[
\lim_{k \to \infty} \int_{B_{a(t_1)}} e^{4\pi u_k^2} dx = \pi (1 - a(t_1)^2).
\] (2.15)

Combining (2.15) together with (2.14) implies
\[
A = \lim_{k \to \infty} \int_B e^{4\pi u_k^2} dx \leq \pi (1 + e^{4\pi C_G} t_1^2 - a(t_1)^2)
\]
for any $t_1 \in (0, t_0)$. Letting $t_1 \to 0$, we obtain $A \leq \pi (1 + e^{4\pi C_G})$.

We next consider the case $|\nabla v_k|^2 dx \to \delta_0$ in the measure sense. Similar with the previous case, for any $t_1 < e^{-2\pi G(1/2)}$, we have
\[
\int_{B_{a(t_1)}} e^{4\pi u_k^2} dx = 2\pi \int_0^{t_1} e^{4\pi v_k(t)^2} t \left(\frac{a(t)}{t}\right)^2 \frac{1}{-2\pi G'(a(t))a(t)} dt \leq e^{4\pi C_G} \int_{B_{t_1}} e^{4\pi v_k^2} dx.
\]
Letting $k \to \infty$ and then $t_1 \to 0$ and using (2.8), we get
\[
\lim_{t_1 \downarrow 0} \sup_{k \to \infty} \int_{B_{a(t_1)}} e^{4\pi u_k^2} dx \leq \pi e^{1+4\pi C_G}.
\] (2.16)

Combining (2.16) together with (2.15), we arrive
\[
A \leq \pi \left(1 + e^{1+4\pi C_G}\right).
\]

Thus, we have proved
\[
\sup\{\limsup_{k \to \infty} F(u_k) : \{u_k\} \text{ is (NCS)}\} \leq \pi \left(1 + e^{1+4\pi C_G}\right).
\]

For the reverse inequality, let us consider the sequence of test functions $f_\epsilon(r)$ used in the proof of Lemma 10 in [40],
\[
f_\epsilon(r) = \begin{cases} 
\beta_\epsilon \frac{\xi(\epsilon^{-1}r) + \gamma_\epsilon}{\beta_\epsilon} & \text{if } |x| \leq \epsilon R_\epsilon \\
G(r) & \text{if } \epsilon R_\epsilon \leq r < 1,
\end{cases}
\]
with $R_\epsilon = -\ln \epsilon$, 

\[\xi(r) = -\frac{1}{4\pi} \ln(1 + \pi r^2)\] and \(\beta_\epsilon\) and \(\gamma_\epsilon\) chosen such that \(f_\epsilon \in \mathcal{H}\) and \(H(f_\epsilon) = 1\). We first need
\[
\frac{G(\epsilon R_\epsilon)}{\beta_\epsilon} = \beta_\epsilon + \frac{\xi(R_\epsilon)}{\beta_\epsilon} + \frac{\gamma_\epsilon}{\beta_\epsilon},
\]
such that \(f_\epsilon\) is continuous. Notice that \(f_\epsilon\) is non-increasing and on \(B_{\epsilon R_\epsilon}\) we have
\[
f_\epsilon(r) \leq f_\epsilon(0) = \beta_\epsilon + \frac{\gamma_\epsilon}{\beta_\epsilon} = -\frac{\xi(R_\epsilon)}{\beta_\epsilon} + \frac{G(\epsilon R_\epsilon)}{\beta_\epsilon} = O(G(\epsilon R_\epsilon)),
\]
here we used (2.4) and \(R_\epsilon = -\ln \epsilon\). Therefore, we have
\[
\int_{B_{\epsilon R_\epsilon}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx = \frac{1}{\beta_\epsilon^2} O(G(\epsilon R_\epsilon)^2(\epsilon R_\epsilon)^2).
\]
Fix a \(\rho \in (0, 1)\), using the second inequality in Lemma 2.4, we have
\[
1 = H(f_\epsilon) \geq \int_{B_{\rho}} |\nabla f_\epsilon|^2 dx - \int_{B_{\rho}} f_\epsilon^2 dx
\]
\[
= \int_{B_{\rho} \setminus B_{\epsilon R_\epsilon}} |\nabla f_\epsilon|^2 dx - \int_{B_{\rho}} f_\epsilon^2 dx + \int_{B_{\epsilon R_\epsilon}} |\nabla f_\epsilon|^2 dx - \int_{B_{\rho}} f_\epsilon^2 dx
\]
\[
= \frac{1}{4\pi \beta_\epsilon^2} \left(-2 \ln \epsilon + 4\pi C_G - 1 + 8\pi^2 \rho G(\rho)G'(\rho) + O(R_\epsilon^2)\right),
\]
here the last equality used some estimates in the proof of Lemma 10 in [40]. Therefore
\[
\beta_\epsilon \geq \frac{1}{4\pi} \left(-2 \ln \epsilon + 4\pi C_G - 1 + 8\pi^2 \rho G(\rho)G'(\rho) + O(R_\epsilon^2)\right).
\]
Combining this together with the estimate for \(\beta_\epsilon\) in the proof of Lemma 10 in [40], we get
\[
\lim_{\epsilon \to 0} \frac{\beta_\epsilon^2}{-\ln \epsilon} = \frac{1}{2\pi}.
\]
Hence \(f_\epsilon \to 0\) a.e. in \(B\), so \(f_\epsilon \to 0\) weakly in \(\mathcal{H}\). We next check that \(|\nabla f_\epsilon|^2 dx \to \delta_0\) in the measure sense. Indeed, for any \(\rho \in (0, 1)\), we have
\[
\int_{B_{\rho}} |\nabla f_\epsilon|^2 dx = \int_{B_{\rho}} |\nabla f_\epsilon|^2 dx - \int_{B_{\rho}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx + \int_{B_{\rho}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx
\]
\[
= \frac{1}{4\pi \beta_\epsilon^2} \left(-2 \ln \epsilon + 4\pi C_G - 1 + 8\pi^2 \rho G(\rho)G'(\rho) + O(R_\epsilon^2)\right)
\]
\[
+ \int_{B_{\rho}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx.
\]
Moreover
\[
\int_{B_{\rho}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx = \frac{1}{\beta_\epsilon^2} \int_{B_{\rho} \setminus B_{\epsilon R_\epsilon}} \frac{G^2}{(1 - |x|^2)^2} dx + \int_{B_{\epsilon R_\epsilon}} \frac{f_\epsilon^2}{(1 - |x|^2)^2} dx
\]
\[
\leq \frac{1}{\beta_\epsilon^2} \int_{B_{\rho}} \frac{G^2}{(1 - |x|^2)^2} dx + \frac{1}{\beta_\epsilon^2} O(G(\epsilon R_\epsilon)^2(\epsilon R_\epsilon)^2) \to 0,
\]
as \(\epsilon \to 0\). Letting \(\epsilon \to 0\) and using (2.17) we get
\[
\lim_{\epsilon \to 0} \int_{B_{\rho}} |\nabla f_\epsilon|^2 dx = 1
for any $\rho > 0$. In other hand, for any compact subset $K \subset B \setminus \{0\}$, there exist $0 < \rho_1 < \rho_2 < 1$ such that $K \subset B_{\rho_2} \setminus B_{\rho_1}$. For $\epsilon > 0$ small enough, we have $\epsilon R_\varepsilon < \rho_1$ and hence
\[
\int_K |\nabla f_\varepsilon|^2 dx \leq \frac{1}{\beta_\varepsilon} \int_{B_{\rho_2} \setminus B_{\rho_1}} |\nabla G|^2 dx \to 0
\]
as $\epsilon \to 0$. Consequently, $|\nabla f_\varepsilon|^2 dx \to \delta_0$ weakly in the measure sense. Thus, $f_\varepsilon$ is a (NCS) sequence in $H^1$. It was proved in the proof of Lemma 10 in [40] that
\[
\mathcal{F}(f_\varepsilon) \geq \pi + \pi e^{4\pi C_G} + \frac{4\pi}{\beta_\varepsilon} \left( \int_B G^2 dx + o_\varepsilon(1) \right).
\]
By letting $\epsilon \to 0$, we get
\[
\sup\{\limsup_{k \to \infty} \mathcal{F}(u_k) : \{u_k\} \text{ is (NCS)}\} \geq \limsup_{\epsilon \to 0} \mathcal{F}(f_\varepsilon) \geq \pi + \pi e^{4\pi C_G} + 1.
\]
The proof of this lemma is completed.

Choosing $\epsilon > 0$ small enough in (2.18), we obtain
\[
\sup_{u \in H, H(u) \leq 1} \int_B e^{4\pi u^2} dx > \pi e^{1 + 4\pi C_G}. \tag{2.19}
\]

Finally, we need the following Concentration–Compactness principle of Lions type [21] to prove the existence of maximizers for the Hardy–Moser–Trudinger inequality,

**Lemma 2.6.** Let $\{u_k\}_k \subset H_1$ such that $H(u_k) = 1$ for any $k$ and $u_k \rightharpoonup u \neq 0$ weakly in $H$, then
\[
\limsup_{k \to \infty} \int_B e^{4\pi pu_k^2} dx < \infty
\]
for any $p < (1 - H(u))^{-1}$.

In order to prove Lemma 2.6, we need the following weaker (or subcritical) Hardy–Moser–Trudinger inequality (see [40, Theorem 3]): for any $\alpha < 4\pi$
\[
\sup_{u \in H, H(u) \leq 1} \int_B e^{\alpha u^2} dx < \infty. \tag{2.20}
\]
The proof of (2.20) in [40] is quite complicated by using a non-trivial claim (see claim (17) in that paper). It was mentioned in [40, Remark 5] that finding an elementary proof of (2.20) is an interesting question. For the completeness of our approach, we give a new proof of (2.20) here.

**Proof of (2.20).** Let $u \in H_1$ with $H(u) \leq 1$. For any $r \in (0, 1/2)$, the second inequality in Lemma 2.4, the Hölder inequality and the Poincaré–Sobolev inequality (2.1) imply
\[
\int_{B_r} |\nabla u|^2 \leq 1 + \int_{B_r} \frac{u^2}{(1 - |x|^2)^2} dx
\]
\[
\leq 1 + \left( \int_{B_r} \frac{u^4}{(1 - |x|^2)^2} dx \right)^{1/2} \left( \int_{B_r} \frac{1}{(1 - |x|^2)^2} dx \right)^{1/2}
\]
\[
\leq 1 + Cr,
\]
for some constant $C$ independent of $u$ and $r \in (0, 1/2)$. Define the function $v(t) = u(t) - u(r)$ for $t \leq r$ and $v(t) = 0$ for $t > r$. We have $v \in W^{1,2}_0(B)$ and

$$||\nabla v||_{L^2(B)} = \int_{B_r} |\nabla u|^2 dx \leq 1 + Cr.$$ 

Choosing $r > 0$ small enough such that $\alpha(1 + Cr) < 4\pi$. From (2.2) and (2.3), we get $u(s) \leq C_1$ for any $s \geq r$ and for some constant $C_1$ depending only on $r$. Consequently, we get

$$\int_{B \setminus B_r} e^{\alpha u^2} dx \leq \pi e^{4\pi C_1^2}. \quad (2.21)$$

Since $u(t) = v(t) + u(r)$ for $t \leq r$, then $u(t)^2 \leq (1 + \epsilon)v(t)^2 + \frac{1 + \epsilon}{\epsilon}u(r)^2$. Taking $\epsilon > 0$ small enough such that $(1 + \epsilon)(1 + Cr)\alpha < 4\pi$ and using the classical Moser–Trudinger inequality, we then have

$$\int_{B_r} e^{\alpha u^2} dx \leq \int_B e^{\alpha v^2} dx \leq e^{2\alpha C_1^2} \sup_{w \in W^{1,2}_0(B)} \int_B e^{4\pi w^2} dx. \quad (2.22)$$

Combining (2.21) together with (2.22) proves (2.20).

We are now ready to prove Lemma 2.6.

**Proof of Lemma 2.6.** We first claim that $\int_B e^{w^2} dx < \infty$ for any $v \in H_1$. Indeed, fix a $r_0 \in (1/2, 1)$ and define $w(r) = v(r) - v(r_0)$ if $r \leq r_0$ and $w(r) = 0$ if $r \geq r_0$. Hence $w \in W^{1,2}_0(B)$ and $\int_B e^{w^2} dx < \infty$ for any $v$. We have

$$\int_{B_{r_0}} e^{w^2} dx \leq \int_{B_{r_0}} e^{2w^2 + 2v(r_0)^2} dx < \infty.$$

In other hand, we have $v(r) \leq C$ for any $r \in (r_0, 1)$ by (2.3). The claim is then proved.

Since $u_k \rightharpoonup u \not\equiv 0$, then it holds

$$H(u_k - u) = 1 - H(u) + o_k(1).$$

For any $p < (1 - H(u))^{-1}$, choosing $\epsilon > 0$ small enough such that $(1 + \epsilon)p < (1 - H(u))^{-1}$. We have

$$u_k^2 \leq (1 + \epsilon)(u_k - u)^2 + \frac{1 + \epsilon}{\epsilon} u^2.$$

Taking $r > 1$ such that $(1 + \epsilon)pr < (1 - H(u))^{-1}$, and using Hölder inequality, we have

$$\int_B e^{4\pi u_k^2} dx \leq \left( \int_B e^{4\pi p(1+\epsilon)(u_k - u)^2 + \frac{1 + \epsilon}{\epsilon} u^2} dx \right)^\frac{1}{1 + \epsilon} \left( \int_B e^{4\pi p(1 + \epsilon)rH(u_k - u)\frac{(u_k - u)^2}{H(u_k - u)}} dx \right)^\frac{1}{r} \leq C \left( \int_B e^{4\pi p(1 + \epsilon)(1 - H(u) + o_k(1))(\frac{u_k - u)^2}{H(u_k - u)^2})} dx \right)^\frac{1}{r},$$

by the claim. For $k$ large enough, we have $p(1 + \epsilon)r(1 - H(u) + o_k(1)) \leq 1$. Hence, the lemma follows from the weaker Hardy–Moser–Trudinger inequality (2.20). \qed
A new proof of Theorem 1.1. This section is devoted to prove Theorem 1.1. Our proof is based on the transformations of functions via the transplantation of Green functions introduced in Section §2. We first prove the Hardy–Moser–Trudinger inequality (1.2).

Proof of the inequality (1.2): As discussed in Section §2, it is enough to prove (1.2) for functions in $\mathcal{H}_1$. Let $u \in \mathcal{H}_1$ with $H(u) \leq 1$. Define $v(x) = u(a(|x|))$. Moreover, by the same computations in (2.13), we have

$$\int_B e^{4\pi u^2} dx = 2\pi \int_0^1 e^{4\pi v(t)^2} t \left(\frac{a(t)}{t}\right)^2 \frac{1}{-2\pi G'(a(t))a(t)} dt.$$ 

Now, using (2.5) and Lemma 2.1, we get

$$\int_B e^{4\pi u^2} dx \leq 2\pi e^{4\pi C_G} \int_0^1 e^{4\pi v(t)^2} t dt = e^{4\pi C_G} \int_B e^{4\pi v^2} dx.$$ 

By the classical Moser–Trudinger inequality (1.1), we obtain

$$\sup_{w \in \mathcal{H}_1, H(u) \leq 1} \int_B e^{4\pi u^2} dx \leq e^{4\pi C_G} \sup_{w \in W^{2,2}_0(B)} \int_B e^{4\pi w^2} dx < \infty.$$ 

This proves the inequality (1.2).

We next move to prove the existence of maximizers for (1.2). Again, from Section §2, it is enough to find the maximizers for (1.2) in $\mathcal{H}_1$.

The existence of maximizers: Let $\{u_k\}_k \subset \mathcal{H}_1$ be a maximizing sequence for the Hardy–Moser–Trudinger inequality, i.e., $H(u_k) = 1$ and

$$\lim_{k \to \infty} \mathcal{F}(u_k) = \sup_{u \in \mathcal{H}_1, H(u) \leq 1} \int_B e^{4\pi u^2} dx.$$ 

By extracting a subsequence, we can assume that $u_k \rightharpoonup u$ weakly in $\mathcal{H}$ and $u_k \to u$ a.e. in $B$ for some function $u \in \mathcal{H}_1$ with $H(u) \leq 1$. By (2.19), the sequence $\{u_k\}_k$ is not (NCS). Suppose that $u \equiv 0$. By the second inequality in Lemma 2.4, we have

$$\int_{B_r} |\nabla u_k|^2 dx \leq 1 + \int_{B_r} \frac{u_k^2}{(1 - |x|^2)^2} dx,$$

and hence

$$\lim_{k \to \infty} \int_{B_r} |\nabla u_k|^2 dx \leq 1.$$ 

for any $r \in (0, 1)$. Moreover, since $\{u_k\}_k$ is not a (NCS) sequence, then there exist $\delta < 1$, $r_0 \in (0, \frac{1}{2})$ and a subsequence (which we still denote by $\{u_k\}$) such that

$$\int_{B_{r_0}} |\nabla u_k|^2 dx \leq \delta < 1,$$

for any $k$. Note that $u_k \to 0$ a.e. in $B$. Repeating the proof of (2.11) (the case $|\nabla u_k|^2 dx \not\rightharpoonup \delta_0$ in the measure sense), we have

$$\sup_{u \in \mathcal{H}_1, H(u) \leq 1} \int_B e^{4\pi u^2} dx = \lim_{k \to \infty} \mathcal{F}(u_k) = \pi,$$

which is impossible since (2.19). Hence, $u \not\equiv 0$. By Lemma 2.6, we have $e^{4\pi u_k^2}$ is bounded in $L^p(B)$ for some $p > 1$. Hence, it holds

$$\sup_{u \in \mathcal{H}_1, H(u) \leq 1} \int_B e^{4\pi u^2} dx = \lim_{k \to \infty} \mathcal{F}(u_k) = \int_B e^{4\pi u^2} dx.$$
Using a simple argument, it is easy to see that $H(u) = 1$, and hence $u$ is an extremal for the Hardy–Moser–Trudinger inequality. The proof of Theorem 1.1 is then completed.

We conclude this paper by some further comments on our approach to prove Theorem 1.1. In [38], Tintarev proved some improved version of the classical Moser–Trudinger inequality in $B$ by replacing the condition $\|\nabla u\|_{L^2(B)}^2 \leq 1$ by the condition $H_V(u) = \|\nabla u\|_{L^2(B)}^2 - \int_B V u^2 dx \leq 1$ with the radial potential $V$ satisfying some suitable conditions: $V > 0$, the function $r \to (1 - r^2)V(r)$ is non-increasing,

$$\lim_{r \to 0} r^2(- \ln r)^{2+\alpha}V(r) = 0,$$

for some $\alpha > 0$ and the quadratic form $H_V$ is weakly coercive, i.e., there exists a positive, continuous function $W$ on $B$ such that

$$H_V(u) \geq \int_B W(\|\nabla u\|^2 + u^2) dx.$$ 

The Hardy–Moser–Trudinger inequality (1.2) is a special cases of the Tintarev’s results by choosing $V(r) = (1 - r^2)^{-2}$. When $V \equiv \lambda$ with $\lambda < \lambda_1(B)$ where $\lambda_1(B)$ is the first eigenvalue of $-\Delta$ on $B$ under the Dirichlet boundary condition, we get a refine of the inequality of Adimurthi and Druet in $B$ (see [3]). It is worthy to note that the approach in this paper could be applied to prove the results of Tintarev for several potentials $V$ such that $V$ positive and continuous in $B$ and $H_V$ weakly coercive. These conditions on $V$ guarantee the existence of the Green function $G_V$ of the operator $L_V = -\Delta - V$, i.e., $L_V(G_V) = \delta_0$ in the distributional sense in $B$ and $G_V = 0$ on $\partial B$. Furthermore, $G_V$ is radially symmetric, strictly decreasing in $B$ and satisfies the decomposition

$$G_V(r) = -\frac{1}{2\pi} \ln r + C_V + \psi(r),$$

where $C_V$ is a constant and $\psi \in C^{1,\alpha}_{\text{loc}}(B)$ for any $\alpha \in (0,1)$ with $\psi(r) = O(r^{1+\alpha})$ as $r \to 0$. Indeed, we can mimic the proof of Wang and Ye in [40] for the existence and the decomposition of $G_V$. By using the method in this paper, we can prove the results of Tintarev for such potentials $V$ by using $G_V$ instead of $G$. Furthermore, this method also can be applied to prove the existence of maximizers for the results of Tintarev which was not considered in [38]. We leave the details of the proof for the interesting readers.

The last remark concerns to the singular Hardy–Moser–Trudinger inequality recently proved by Wang [42],

$$\sup_{u \in \mathcal{H}, |\mathcal{H}(u)| \leq 1} \int_B e^{4\pi(1-\beta/2)u^2 |x|^{-\beta}} dx < \infty,$$

for any $0 \leq \beta < 2$. Moreover, Wang also proved the existence of maximizers for the above inequality. Wang’s proof follows the ideas of Wang and Ye by using the blow-up analysis method. We notice that our approach can be used to prove the singular Hardy–Moser–Trudinger inequality and the existence of its maximizers without using the blow-up analysis. Indeed, using the change of functions introduced in Section 2, we see that for any $u \in \mathcal{H}_1$ and $v = u(a(r))$ we have

$$\int_B e^{4\pi(1-\beta/2)u^2 |x|^{-\beta}} dx = 2\pi \int_0^1 e^{4\pi(1-\beta/2)v(t)^2} t^{1-\beta} \left(\frac{a(t)}{t}\right)^{2-\beta} \frac{1}{-2\pi G'(a(t))a(t)} dt.$$
The singular Hardy–Moser–Trudinger inequality follows from the preceding equality by using the singular Moser–Trudinger inequality in $B$ (see [4]). Furthermore, we can compute explicitly the concentrating level of the singular Hardy–Moser–Trudinger functional
\[
F_\beta(u) = \int_B e^{4\pi (1 - \beta/2) u^2} |x|^{-\beta} \, dx
\]
as \[
\frac{2\pi}{2-\beta} \left(1 + e^{1+2\pi(2-\beta)C_G}\right).
\]
Using test functions, we can verify that
\[
\sup_{ \mathcal{H}_N, H(u) \leq 1} \int_B e^{4\pi (1 - \beta/2) u^2} |x|^{-\beta} \, dx > \frac{2\pi}{2-\beta} \left(1 + e^{1+2\pi(2-\beta)C_G}\right).
\]

Using this inequality, we can prove the existence of maximizers for the singular Hardy–Moser–Trudinger inequality by following the argument in the proof of Theorem 1.1 (the existence part).

**Acknowledgments.** The author are grateful to thank the referees for their careful reading and useful comments which improve the presentation of the paper.

**REFERENCES**

[1] S. Adachi and K. Tanaka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, *Proc. Amer. Math. Soc.*, **128** (2000), 2051–2057.

[2] D. R. Adams, A sharp inequality of J. Moser for higher order derivatives, *Ann. Math.*, **128** (1988), 385–398.

[3] Adimurthi and O. Druet, Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality, *Commun. Partial Differ. Equ.*, **29** (2004), 295–322.

[4] Adimurthi and K. Sandeep, A singular Moser–Trudinger embedding and its applications, *NoDea Nonlinear Differ. Equ. Appl.*, **13** (2007), 585–603.

[5] Adimurthi and C. Tintarev, On a version of Trudinger-Moser inequality with Möbius shift invariance, *Calc. Var. Partial Differ. Equ.*, **39** (2010), 203–212.

[6] Adimurthi and Y. Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications, *Int. Math. Res. Not. IMRN*, (2010), 2394–2426.

[7] R. D. Benguria, R. L. Frank and M. Loss, The sharp constant in the Hardy–Sobolev–Maz’ya inequality in the three dimensional upper half-space, *Math. Res. Lett.*, **15** (2008), 613–622.

[8] L. Carleson and S. Y. A. Chang, On the existence of an extremal function for an inequality of J. Moser, *Bull. Sci. Math.*, **110** (1986), 113–127.

[9] G. Csató and P. Roy, Extremal functions for the singular Moser-Trudinger inequality in 2 dimensions, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2341–2366.

[10] G. Csató and P. Roy, Singular Moser-Trudinger inequality on simply connected domains, *Commun. Partial Differ. Equ.*, **41** (2016), 838–847.

[11] G. Csató, V. H. Nguyen and P. Roy, Extremals for the singular Moser-Trudinger inequality via $n$-harmonic transplantation, preprint, [arXiv:1801.03932v3](https://arxiv.org/abs/1801.03932v3).

[12] D. G. De Figueiredo, J. M. do O and B. Ruf, On an inequality by N. Trudinger and J. Moser and related elliptic equations, *Commun. Pure Appl. Math.*, **55** (2002), 135–152.

[13] M. Flucher, Extremal functions for the Trudinger-Moser inequality in 2 dimensions, *Comment. Math. Helv.*, **67** (1992), 471–497.

[14] M. Iwashita, Existence and nonexistence of maximizers for variational problems associated with Trudinger–Moser inequalities in $\mathbb{R}^N$, *Math. Ann.*, **351** (2011), 781–804.

[15] N. Lam and G. Lu, A new approach to sharp Moser-Trudinger and Adams type inequalities: A rearrangement–free argument, *J. Differ. Equ.*, **255** (2013), 298–325.

[16] Y. Li, Moser-Trudinger inequality on compact Riemannian manifolds of dimension two, *J. Partial Differ. Equ.*, **14** (2001), 163–192.

[17] Y. Li, Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds, *Sci. China Ser. A.*, **48** (2005), 618–648.

[18] Y. Li and B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$, *Indiana Univ. Math. J.*, **57** (2008), 451–480.
[19] J. Li, G. Lu and Q. Yang, Fourier analysis and optimal Hardy-Adams inequalities on hyperbolic spaces of any even dimension, *Adv. Math.*, 333 (2018), 350–385.
[20] K. Lin, Extremal functions for Moser's inequality, *Trans. Amer. Math. Soc.*, 348 (1996), 2663–2671.
[21] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, *Rev. Mat. Iberoam.*, 1 (1985), 45–121.
[22] G. Lu and Q. Yang, A sharp Trudinger-Moser inequality on any bounded and convex planar domain, *Calc. Var. Partial Differ. Equ.*, 55 (2016), Art. 153, 16 pp.
[23] G. Lu and Q. Yang, Sharp Hardy–Adams inequalities for bi-Laplacian on hyperbolic space of dimension four, *Adv. Math.*, 319 (2017), 567–598.
[24] G. Mancini and K. Sandeep, Moser-Trudinger inequality on conformal discs, *Commun. Contemp. Math.*, 12 (2010), 1055–1068.
[25] G. Mancini, K. Sandeep and C. Tintarev, Trudinger-Moser inequality in the hyperbolic space $\mathbb{H}^n$, *Adv. Nonlinear Anal.*, 2 (2013), 309–324.
[26] G. Mancini and L. Martinazzi, The Moser-Trudinger inequality and its extremals on a disk via energy estimates, *Calc. Var. Partial Differ. Equ.*, 56 (2017), Art. 94, 26 pp.
[27] L. Martinazzi, Fractional Adams-Moser-Trudinger type inequalities, *Nonlinear Anal.*, 127 (2015), 263–278.
[28] V. G. Maz'ya, *Sobolev spaces*, Springer Verlag, Berlin, New York, 1985.
[29] J. Moser, A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.*, 20 (1970/71), 1077–1092.
[30] V. H. Nguyen, The sharp Poincaré–Sobolev type inequalities in the hyperbolic spaces $\mathbb{H}^n$, *J. Math. Anal. Appl.*, 462 (2018), 1570–1584.
[31] V. H. Nguyen, Improved Moser-Adams type inequalities in the hyperbolic space $\mathbb{H}^n$, *Nonlinear Anal.*, 168 (2018), 67–80.
[32] V. H. Nguyen, Improved Moser–Trudinger inequality of Tintarev type in dimension $n$ and the existence of its extremal functions, *Ann. Global Anal. Geom.*, 54 (2018), 237–256.
[33] V. H. Nguyen, Improved singular Moser-Trudinger inequalities and their extremal functions, *Potential Anal.*, in press.
[34] V. H. Nguyen, The sharp Hardy–Moser–Trudinger inequality in dimension $n$, preprint, arXiv:1909.12887.
[35] S. I. Pohožaev, On the eigenfunctions of the equation $\Delta u + Af(u) = 0$ (Russian), *Dokl. Akad. Nauk. SSSR*, 165 (1965), 36–39.
[36] B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$, *J. Funct. Anal.*, 219 (2005), 340–367.
[37] A. Tertikas and C. Tintarev, On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality, *Ann. Mat. Pura Appl. (4)*, 186 (2007), 645–662.
[38] C. Tintarev, Trudinger–Moser inequality with remainder terms, *J. Funct. Anal.*, 266 (2014), 55–66.
[39] N. S. Trudinger, On imbedding into Orlicz spaces and some applications, *J. Math. Mech.*, 17 (1967), 473–483.
[40] G. Wang and D. Ye, A Hardy-Moser-Trudinger inequality, *Adv. Math.*, 230 (2012), 294–320.
[41] X. Wang, Improved Hardy-Adams inequality on hyperbolic space of dimension four, *Nonlinear Anal.*, 182 (2019), 45–56.
[42] X. Wang, Singular Hardy-Moser-Trudinger inequality and the existence of extremals on the unit disc, *Commun. Pure Appl. Anal.*, 18 (2019), 2741–2757.
[43] Y. Yang, A sharp form of the Moser–Trudinger inequality on a compact Riemannian surface, *Trans. Amer. Math. Soc.*, 359 (2007), 5761–5776.
[44] Y. Yang and X. Zhu, An improved Hardy-Trudinger-Moser inequality, *Ann. Global Anal. Geom.*, 49 (2016), 23–41.
[45] Q. Yang, D. Su and Y. Kong, Sharp Moser-Trudinger inequalities on Riemannian manifolds with negative curvature, *Ann. Mat. Pura Appl.*, 195 (2016), 459–471.
[46] V. I. Yudovič, Some estimates connected with integral operators and with solutions of elliptic equations (Russian), *Dokl. Akad. Nauk. SSSR*, 138 (1961), 805–808.

Received April 2019; revised January 2020.

E-mail address: hoangnv47@fe.edu.vn
E-mail address: vanhoang0610@yahoo.com