Integral formulas for Painlevé-2 transcendent.

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Abstract

In the work we use integral formulas for calculating the monodromy data for the Painlevé-2 equation. The perturbation theory for the auxiliary linear system is constructed and formulas for the variation of the monodromy data for a non-integrable perturbation are obtained. We also derive a formula for solving the linearized Painlevé-2 equation based on the Fourier-type integral of the squared solutions of the auxiliary linear system of equations.

1 Introduction

The perturbation of the scheme of the isomonodromic deformation method is considered for the perturbed Painleve-2 equation:

\[ u'' = 2u^3 + xu - \varepsilon f(u, u', x), \quad 0 < \varepsilon \ll 1. \]  \hfill (1)

The approach is based on the method of isomonodromic deformations developed in [1], [2], [3] and [4]. One of biggest success of the works sited above is the description of the asymptotics of the equations and the explicit solution of the scattering problem—the formulas of the connection between the asymptotics of the solution of the Painlevé equations at \( x \to -\infty \) and \( x \to \infty \). However, the formulas known in the theory of isomonodromic deformations are inconvenient for the study of another type of problems like problems on non-integrable perturbations of the Painlevé equations.

Solutions to problems of non-integrable perturbations in nonlinear equations are usually based on the modulation of the conservation laws of unperturbed equations. This approach is used in Krylov-Bogolyubov [5] method.

Such method was also developed for finite-zone solutions of equations integrable by the inverse scattering problem [6], [7]. These results are close to the results obtained in the perturbation theory of solitons.
The method of constructing the perturbation theory based on the results for the auxiliary scattering problem was successful for both (1+1) - dimensional partial differential equations [8], [9], when constructing the first corrections of perturbation theory, and when constructing higher corrections to the parameters of the soliton [10], [11], and for corrections from the continuous spectrum [12]. Integral formulas for constructing corrections of perturbed equations are also used for (2+1) - dimensional integrable equations [13], [14].

In spite the importance of the Painlevé equations in perturbation theory and dynamic bifurcation problems [15], [16], [17], [18], and the construction of a perturbation theory based on the asymptotics of solutions of unperturbed equations in [19], the perturbation theory based on the isomonodromic deformation method has not yet been constructed.

One of the reasons is that the technique of the theory of isomonodromic deformations has adopted an approach related to the direct representation of the Painlevé transcendents, based on the asymptotics of solutions to the auxiliary linear problem for the parameter \( \lambda \). Here we use integral formulas for the Stokes coefficients for the auxiliary linear problem.

The approach developed here allows us to study and obtain formulas for the modulation of the Stokes constants, which are the parameters of the Painlevé transcendent. The formulas for the modulated parameter allows to study the global properties of the Painlevé transcendent under perturbation. In particular, and in those areas for which no asymptotic formulas are constructed.

Here is the general structure of the work. In part 2, we present the Stokes theory for solutions of the auxiliary system of equations and obtain integral formulas for Stokes matrices. In the section 3 the integral formulas for the Painlevé-2 transcendent are obtained using the integral representation of the solution of the Riemann problem for the auxiliary system of equations. In the 4 part, formulas for the asymptotics of solutions to an auxiliary linear problem with the parameter \( \lambda \) are given. In the section 5 the formulas for the variation of the Stokes coefficients are derived. The 6 section provides a formula for solving the linearized Painlevé-2 equation.

2 Integral formulas for the Stokes coefficients

In this section the integral formulas for solving the auxiliary system of equations for the parameter \( \lambda \) are given according to the theory from [1], [2], [4]. These integral formulas are used to obtain integral formulas for the Stokes coefficients of the auxiliary system of equations.
Consider an auxiliary system of equations that determines the dependence of the function $\Psi$ on the complex variable $\lambda$:

$$
\frac{d\Psi}{d\lambda} = A\Psi, \quad A = -i(4\lambda^2 + x + 2u^2)\sigma_3 + 4u\lambda\sigma_1 - 2u'\sigma_2.
$$

Here the notation for Pauli matrices is accepted:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

In addition to the system of equations (2) the function $\Psi$ satisfies the system of differential equations for the real variable $x$:

$$
\frac{d\Psi}{dx} = U\Psi, \quad U = -i\lambda\sigma_3 + u\sigma_1.
$$

The Painlevé-2 equation is a condition for the existence of a solution of both systems of equations (2) and (4) [20].

The solution of the system of equations (2) has the singular point at $\lambda = \infty$. The asymptotics of the solution of this system for $\lambda \to \infty$ can be constructed by the WKB [21] method. An explicit form of such an asymptotic [1] has a form:

$$
\Psi_\infty \sim \left( I + \frac{1}{2\lambda} \left( \begin{array}{c} i(u^2x - (u')^2 + u^4) \\ -i(u^2 - (u')^2 + u^4) \end{array} \right) + O(\lambda^{-2}) \right) \times \exp (-i\Omega(\lambda)\sigma_3),
$$

where $\Omega(\lambda) = (4\lambda^3/3 + \lambda x)$.

The main term of this asymptotic oscillates on the lines $\Im(4\lambda^3/3 + \lambda x) = 0$. In the neighborhood of an infinity, such lines have asymptotes – straight lines $\arg(\lambda) = \pi(k - 1)/3$, $k = 1, \ldots, 6$. For each of these six lines in the neighborhood of infinity, one can define a function $\Psi_k$ by the given asymptotic direction $\arg(\lambda) = \pi(k - 1)/3$:

$$
\Psi_k \sim \Psi_\infty, \quad k = 1, 2, 3, 4, 5, 6.
$$

Since each of the $\Psi_k$ matrices is a fundamental solution system for (2), so they can be expressed in terms of each The main term of this asymptotic oscillates on the lines $\Im(4\lambda^3/3 + \lambda x) = 0$. In the neighborhood of an infinitely distant point, such lines have asymptotes – straight lines $\arg(\lambda) = \pi(k - 1)/3$, $k = 1, \ldots, 6$. For each of these six lines in the neighborhood of an infinitely distant point, you can define a function $\Psi_k$ by the given asymptotic direction $\arg(\lambda) = \pi(k - 1)/3$:

$$
\Psi_k \sim \Psi_\infty, \quad k = 1, 2, 3, 4, 5, 6.
$$
Since each of the $\Psi_k$ matrices is a fundamental solution system for (2), so they can be expressed in terms of each other:

$$\Psi_{k+1} = \Psi_k S_k.$$ (6)

Here $S_k$ is a matrix consisting of parameters that depend on the solution of the Painlevé-2 equation, but do not depend on the parameter $\lambda$. These $S_k$ matrices are called Stokes matrices. The symbols correspond to those used in the book [4].

![Stokes rays and curves of integrating](image)

Figure 1: The Stokes rays at the directions $\pi(k - 1)/6$, $k = 1, 2, 3, 4, 5, 6$ and the curves of integrating, which tend to $\infty$ in the following directions: $\infty_k, \infty_{k+1}$.

To derive integral formulas for the Stokes matrix, it is convenient to use the substitution:

$$\Psi_k = \exp (-i\Omega(\lambda)\sigma_3) \Phi_k.$$

Using the system of equations (2), one can derive a similar system of equations for the matrix $\Phi_k$:

$$\frac{d}{d\lambda} \Phi_k = (\exp (i\Omega_3) A \exp (-i\Omega_3) + i\Omega'\sigma_3) \Phi_k.$$ (7)

For the matrix $\Phi_k$ the following condition is true:

$$\Phi_k \to I, \quad \lambda = Re^{i(k-1)\pi/6}, \quad R \to \infty.$$ (8)

It is not difficult to verify that the solution of the scattering problem (7), (8) satisfies to a system of integral equations:

$$\Phi_k(\lambda) = I + \int_{\infty_k}^{\lambda} (\exp (i\Omega_3) A \exp (-i\Omega_3) + i\Omega'\sigma_3) \Phi_k d\mu.$$ (9)
Here the integral is considered as non-proper, where the upper limit is \( \infty \).

According to the formula of the connection of fundamental solutions (6) one can obtain:

\[
I + \int_{\infty}^{\infty} (\exp(i\Omega\sigma_3) A \exp(-i\Omega\sigma_3) + i\Omega'\sigma_3) \Phi_{k+1} d\mu = S_k.
\]

Now \( S_k \) can be expressed using \( \Psi_k \):

\[
S_k = I + \int_{\infty}^{\infty} \exp(i\Omega\sigma_3) (A + i\Omega'\sigma_3) \Psi_{k+1} d\mu.
\]

The integral formula for the matrix \( S_k \) is convenient to consider by components. It is important to take into account the asymptotic properties of the matrix \( \Psi \) from the formula (8).

Integrands in diagonal elements are integer functions of the parameter \( \lambda \), decreasing as \( \lambda^{-2} \) at \( \lambda \to \infty \). Therefore, to calculate the integrals of such functions, one can use Cauchy’s theorem and go to the integrals along the arc of a circle of radius \( R \) at \( R \to \infty \):

\[
(S_k)_{11} = 1 + \lim_{R \to \infty} \int_{R}^{R\exp(i\pi k/3)} \frac{i\mu^2 x - i\mu^2 + i\mu^4}{2\mu^2} + \mathcal{O}(R^{-3}) d\mu = 1.
\]

\[
(S_k)_{22} = 1 + \lim_{R \to \infty} \int_{R}^{R\exp(i\pi k/3)} \frac{u^2 x - iu^2 + iu^4}{2\mu^2} + \mathcal{O}(R^{-3}) d\mu = 1.
\]

The elements of the matrix \( S_k \) lying on the inverse diagonal have exponents in the integrand for large values of \( \lambda \): (11)

\[
(S_k)_{12} = \lim_{R \to \infty} \int_{R}^{R\exp(i\pi k/3)} (4i\mu + (2u^3 x - 2uw^2 - 2w + 2u^5) + \mathcal{O}(1/R)) \exp(2i(4\mu^3/3 + x\mu)) d\mu
\]

\[
(S_k)_{21} = \lim_{R \to \infty} \int_{R}^{R\exp(i\pi k/3)} (-4i\mu + (2u^3 x - 2uw^2 - 2w + 2u^5) + \mathcal{O}(1/R)) \exp(-2i(4\mu^3/3 + x\mu)) d\mu
\]

The values of the integrals in the formulas (11) and (12) depend on the sign \( \Re(i\mu^3) \) on the integration path. Therefore, it is convenient to make calculations for different values of \( k \).

If \( k = 1, 3, 5 \), then on the arc \( \pi(k - 1)/3 < \arg(\mu) < \pi k/3 \) we get \( \Re(i\mu^3) = -R \sin(3\arg(\mu)) < 0 \). In this case, it can be shown that:

\[
(S_k)_{12} = 0.
\]
Similarly, for $k = 2, 4, 6$ on the arc $\pi(k - 1)/3 < \arg(\mu) < \pi k/3$ we get $\Re(-i\mu^3) = R \sin(3 \arg(\mu)) < 0$, that is:

$$(S_k)_{21} = 0.$$ 

As a result, we get:

$$S_k = \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix}, \quad k = 1, 3, 5;$$

$$S_k = \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}, \quad k = 2, 4, 6.$$ 

In the terms of [1] we rewrite: $s_1 = a$, $s_2 = b$, $s_3 = c$, $s_4 = d$, $s_5 = e$, $s_6 = f$ and $s_1 = s_4$, $s_2 = s_5$, $s_3 = s_6$.

The main result of the section 2 – explicit formulas for monodromy data:

$$s_k = 2 \int_{\infty k+1}^{\infty k} \left( (2u\mu - iu')(\Psi_k)_{11} + iu^2(\Psi_k)_{21} \right) e^{-i(4\mu^3/3 + x\mu)} d\mu, \quad k = 1, 3, 5;$$

$$s_k = 2 \int_{\infty k+1}^{\infty k} \left( (2u\mu + iu')(\Psi_k)_{22} - iu^2(\Psi_k)_{12} \right) e^{i(4\mu^3/3 + x\mu)} d\mu, \quad k = 2, 4, 6.$$ 

### 3 Integral formula for the Painlevé transcendent

The analytical properties of the functions $\Psi_k$ allow us to formulate the problem of conjugation of functions for the analytical continuation of the function $\Psi_k$ into neighboring sectors of the complex plane of the parameter $\lambda$. To obtain integral equations in this case, it is convenient to use the formulas of Sokhotsky [22]. Such constructions were made, in particular, in the work [1]. As a result, we obtained a system of equations for the first and second columns of analytical equations in the complex plane $\lambda$:

$$\Psi^{(1)} e^{i\Omega} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \text{Res}_{\mu=0} \frac{\Psi^{(1)} e^{i\Omega}}{\mu - \lambda} + \frac{s_1}{2\pi i} \int_{C_{42}} \frac{\Psi^{(2)} e^{i\Omega}}{\mu - \lambda} d\mu +$$

$$\frac{s_2}{2\pi i} \int_{C_{46}} \frac{\Psi^{(2)} e^{i\Omega}}{\mu - \lambda} d\mu + \frac{s_2 s_3}{2\pi i} \int_{C_{64}} \frac{\Psi^{(1)} e^{i\Omega}}{\mu - \lambda} d\mu,$$

$$\Psi^{(2)} e^{-i\Omega} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \text{Res}_{\mu=0} \frac{\Psi^{(2)} e^{-i\Omega}}{\mu - \lambda} + \frac{s_2}{2\pi i} \int_{C_{53}} \frac{\Psi^{(1)} e^{-i\Omega}}{\mu - \lambda} d\mu +$$

$$\frac{s_4}{2\pi i} \int_{C_{51}} \frac{\Psi^{(1)} e^{-i\Omega}}{\mu - \lambda} d\mu + \frac{s_1 s_2}{2\pi i} \int_{C_{53}} \frac{\Psi^{(2)} e^{-i\Omega}}{\mu - \lambda} d\mu.$$
The solution of the Painlevé-2 equation is usually represented using the asymptotics at \( \lambda \to \infty \) for the components of the matrix \( \Psi \) lying on the inverse diagonal \([1]\):

\[
\begin{align*}
    u(x) &= \lim_{\lambda \to \infty} \lambda i \Psi_{12} e^{-i\Omega} \\
    u(x) &= -\lim_{\lambda \to \infty} \lambda i \Psi_{21} e^{i\Omega}.
\end{align*}
\]

If we use the integral equations, for the matrix \( \Psi \), then we can get another expression for the second transcendent of Painlevé, through the components of the functions \( \Psi_k \):

\[
\begin{align*}
    u(x) &= -\frac{s_1}{2\pi} \int_0^\infty (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_1}{2\pi} \int_0^\infty (\Psi_2)_{21} e^{i\Omega} d\mu - \frac{s_2}{2\pi} \int_0^\infty (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_2}{2\pi} \int_0^\infty (\Psi_6)_{21} e^{i\Omega} d\mu - \frac{s_2s_3}{2\pi} \int_0^\infty (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_2s_3}{2\pi} \int_0^\infty (\Psi_6)_{21} e^{i\Omega} d\mu. \\
    &+ \frac{s_1s_2}{2\pi} \int_0^\infty (\Psi_5)_{12} e^{-i\Omega} d\mu - \frac{s_1s_2}{2\pi} \int_0^\infty (\Psi_3)_{12} e^{-i\Omega} d\mu + \frac{s_3}{2\pi} \int_0^\infty (\Psi_5)_{12} e^{-i\Omega} d\mu + \frac{s_3}{2\pi} \int_0^\infty (\Psi_1)_{12} e^{-i\Omega} d\mu + \frac{s_1s_2}{2\pi} \int_0^\infty (\Psi_5)_{12} e^{-i\Omega} d\mu + \frac{s_1s_2}{2\pi} \int_0^\infty (\Psi_3)_{12} e^{-i\Omega} d\mu. \\
\end{align*}
\]

The formulas (13) and (14) are the main result of the section \([3]\).
4 Asymptotics near Stokes rays

Here we construct the asymptotics for $|\lambda| \to \infty$ of the function $\Psi$ near the Stokes rays $\text{arg}(\lambda) = k\pi/3$, $k = 1, 2, 3, 4, 5, 6$.

In the system of equations (2), let us replace the derivative of the complex variable $\lambda$ with the derivatives of $r = \sqrt{\lambda\lambda}$ and $\alpha = 1/(2i) \log(\lambda/\overline{\lambda})$:

$$\frac{d\Psi}{d\lambda} \equiv e^{-i\alpha} \frac{d\Psi}{dr} + e^{-i\alpha} \frac{2i\alpha}{d\alpha}.$$  

The function $\Psi$ is analytic by $\lambda$, so its derivative by $\lambda$ is zero:

$$\frac{d\Psi}{d\lambda} \equiv \frac{e^{i\alpha}}{2} \frac{d\Psi}{dr} - \frac{e^{i\alpha}}{2i\alpha} \frac{d\Psi}{d\alpha} = 0, \quad \frac{d\Psi}{dr} = \frac{1}{ir} \frac{d\Psi}{d\alpha}.$$  

This results in a system of ordinary equations for the real variable $r$:

$$\frac{d\Psi}{dr} = (-i(4r^2e^{3i\alpha} + (x + 2u^2)e^{i\alpha})\sigma_3 + 4ure^{2i\alpha}\sigma_1 - 2ue^{i\alpha}\sigma_2)\Psi \quad (15)$$

In the system (15) $\alpha$ is a parameter.

Denote the matrix on the right side (15) $\bar{A}$. Diagonalization of this matrix

$$T^{-1}\bar{A}T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

yields the eigenvalues of the matrix from the right side (15):

$$\lambda_{1,2} = \mp e^{i\alpha} \sqrt{-16r^4e^{4i\alpha} - 8r^2xe^{2i\alpha} - x^2 - 4u^2x + 4(u')^2 - 4u^4}$$

Asymptotics of the matrix $T$:

$$T \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2r} \begin{pmatrix} 0 & -iue^{-i\alpha} \\ iue^{-i\alpha} & 0 \end{pmatrix} + \frac{1}{4r^2} \begin{pmatrix} 0 & u'e^{-2i\alpha} \\ u'e^{-2i\alpha} & 0 \end{pmatrix}.$$  

Asymptotics of eigenvalues is:

$$\lambda_{1,2} \sim \mp 4i \left( e^{3i\alpha} r^2 - ix e^{i\alpha} - \frac{iux^2 - i(u')^2 + iu^4}{2e^{i\alpha} r^2} \right)$$

The asymptotics of the solution for $r \to \infty$ is follow:

$$\Psi \sim T e^{\int \lambda_1 dr}\sigma_3.$$
Consider the behavior of the solution in the neighborhood of the ray $\alpha = \text{org}(\lambda) = \pi/3 + \beta$. As a result of this replacement, we get:

$$\Psi \sim \begin{pmatrix} \frac{1}{\lambda} \exp(-\frac{iue^{-i\beta}}{(1+i\sqrt{3})r}) & -iue^{-i\beta} \exp(\Omega) \\ iue^{-i\beta} \exp(-\frac{iue^{-i\beta}}{(1+i\sqrt{3})r}) & 0 \end{pmatrix},$$

where

$$\Omega = \frac{4}{3} r^3 (\sin(3\beta) + i \cos(3\beta)) + \frac{rx}{2} ((\sin(\beta) + \sqrt{3}\cos(\beta)) - i(\cos(\beta) - \sqrt{3}\sin(\beta)) + O\left(\frac{1}{r}\right)).$$

5 Variation of the Stokes coefficients

Consider the effect of perturbations on the Stokes coefficients associated with the scattering problem (2). For an infinitesimal perturbation of the coefficients of the system (2) $u = u + \delta u$, we obtain a system of equations for the variation of $\delta \Psi$:

$$\frac{d\delta \Psi}{d\lambda} = A\delta \Psi + \delta A \Psi, \quad \delta A = -i(4\lambda^2 + x + 4u\delta u)\sigma_3 + 4\delta u\lambda \sigma_1 - 2\delta u' \sigma_2. \quad (16)$$

The general solution of the system of equations (16) can be represented as:

$$\delta \Psi = \Psi C + \Psi \int \Psi^{-1} \delta A \Psi d\mu, \quad (17)$$

where $C$ is a matrix composed of arbitrary constants, which are parameters of the solution of the system (16). Consider the matrix in the integrand of (17):

$$(\Psi^{-1} \delta A \Psi)_{11} = (\Psi_{2,1} \Psi_{2,2} - \Psi_{1,1} \Psi_{1,2}) 4\lambda \delta u + (\Psi_{2,1} \Psi_{2,2} + \Psi_{1,1} \Psi_{1,2}) 2i \frac{d}{dx} \delta u - (\Psi_{1,1} \Psi_{2,2} + \Psi_{1,2} \Psi_{2,1}) 4i u \delta u,$$

$$(\Psi^{-1} \delta A \Psi)_{21} = (\Psi_{1,1}^2 - \Psi_{2,1}^2) 4\lambda \delta u - (\Psi_{2,1}^2 + \Psi_{1,1}^2) 2i \left( \frac{d}{dx} \delta u \right) + 8i \Psi_{1,1} \Psi_{2,1} uv.$$
\[(\Psi^{-1} \delta A \Psi)_{12} = (\Psi_{2,2}^2 - \Psi_{1,2}^2) 4\lambda \delta u + (\Psi_{2,2}^2 + \Psi_{1,2}^2) 2i \left(\frac{d}{dx} \delta u\right)\]
\[= 8i \Psi_{1,2} \Psi_{2,2} u \delta u,\]
\[(\Psi^{-1} \delta A \Psi)_{22} = (\Psi_{1,1} \Psi_{1,2} - \Psi_{2,1} \Psi_{2,2}) 4\lambda \delta u\]
\[= (\Psi_{2,1} \Psi_{2,2} + \Psi_{1,1} \Psi_{1,2}) 2i \left(\frac{d}{dx} \delta u\right)\]
\[+ (\Psi_{1,1} \Psi_{2,2} + \Psi_{1,2} \Psi_{2,1}) 4iu \delta u.\]

It is important to note here that the integrals of the diagonal elements along the integration paths marked in Fig. gives zeros. It is convenient to define:

\[\psi_1^+ = \Psi_{11}^2 + \Psi_{21}^2, \quad \psi_1^- = \Psi_{11}^2 - \Psi_{21}^2, \quad \psi_1 = \Psi_{11} \Psi_{21};\]
\[\psi_2^+ = \Psi_{12}^2 + \Psi_{22}^2, \quad \psi_2^- = \Psi_{22}^2 - \Psi_{12}^2, \quad \psi_2 = \Psi_{12} \Psi_{22};\]

For reasons that completely repeat the calculations of the Stokes coefficients from the section we get:

\[\delta s_k = \int_{\infty}^{\infty} 4\mu \delta u \psi_1^- - 2i \left(\frac{d}{dx} \delta u\right) \psi_1^+ + 8i \delta u \psi_1 d\mu, \quad k = 1, 3, 5; \quad (18)\]
\[\delta s_k = \int_{\infty}^{\infty} 4\mu \delta u \psi_2^- + 2i \left(\frac{d}{dx} \delta u\right) \psi_2^+ - 8i \delta u \psi_2 d\mu, \quad k = 2, 4, 6. \quad (19)\]

6 Formula for solution of the linearized Painlevé-2 equation

Consider the equations for quadratic expressions \(\psi_1^+, \psi_1^-, \psi_1\). To do this, it is convenient to use the system of equations:

\[\frac{d\Psi_{11}}{d\lambda} = -i(4\lambda^2 + x + 2u^2)\Psi_{11} + (4u\lambda + 2iu')\Psi_{21}, \quad (20)\]
\[\frac{d\Psi_{21}}{d\lambda} = (4u\lambda - 2iu')\Psi_{11} + i(4\lambda^2 + x + 2u^2)\Psi_{21} \quad (21)\]

The differential equation for \(\psi_1^+\) by \(\lambda\) is obtained if the equation (20) is multiplied by \(2\Psi_{11}\), the equation (21) is multiplied by \(2\Psi_{21}\) and the resulting equations are added. As a result, we get:

\[\frac{d\psi_1^+}{d\lambda} = -2i(4\lambda^2 + x + 2u^2)\psi_1^- + 16\lambda u \psi_1. \quad (22)\]
The differential equation for $\psi_1^-$ by $\lambda$ is obtained if the first equation of the system equation (20) is multiplied by $2\Psi_{11}$, the second equation of the system (21) is multiplied by $2\Psi_{21}$ and the resulting equations are subtracted. As a result, we get:

$$\frac{d\psi_1^-}{d\lambda} = -2i(4\lambda^2 + x + 2u^2)\psi_1^+ + 8iu'\psi_1. \quad (23)$$

The differential equation for $\psi_1$ by $\lambda$ is obtained if the first equation of the system equation (20) is multiplied by $\Psi_{21}$, the second equation of the system (21) is multiplied by $\Psi_{11}$ and the resulting equations are added. As a result, we get:

$$\frac{d\psi_1}{d\lambda} = 4\lambda u\psi_1^- - 2iu'\psi_1^- \quad (24)$$

Similar expressions are obtained for the derivatives of the same quadratic expressions with respect to $x$. To obtain differential equations for $x$, it is convenient to use the second auxiliary system of equations (4 for $\Psi$ or component-by-component):

$$\frac{d\Psi_{11}}{dx} = -i\lambda \Psi_{11} + u\Psi_{21}, \quad (25)$$

$$\frac{d\Psi_{21}}{dx} = u\Psi_{11} + i\lambda \Psi_{21}. \quad (26)$$

Similar transformations give a system of differential equations for the variable $x$:

$$\frac{d\psi_1^+}{dx} = -2i\lambda\psi_1^- + 4u\psi_1, \quad (27)$$

$$\frac{d\psi_1^-}{dx} = -2i\lambda\psi_1^+, \quad (28)$$

$$\frac{d\psi_1}{dx} = u\psi_1^+, \quad (29)$$

We assume that the variation $\delta u$ is the solution of the linearized equation:

$$\delta u'' = (6u^2 + x)\delta u + f, \quad (30)$$

Differentiating (18) by $x$ by virtue of the equations (25), (26), and the linearized equation (30) gives (here and below, $s_1$ is considered for certainty):

$$\frac{d\delta s_1}{dx} = \int_{\infty}^{\infty} ((-2i(4\mu^2 + x + 2u^2)\psi_1^+ + 8iu'\psi_1)\delta u - 2if\psi_1^+) \, d\mu$$

$$= \int_{\infty}^{\infty} \frac{d\psi_1^-}{d\mu} \, d\mu - 2if \int_{\infty}^{\infty} \psi_1^+ \, d\mu. \quad (31)$$
To calculate the integral of the derivative of $\lambda$, let us consider the representation of $\psi_1^-$ through the squares of the first column of the $\Psi$-function. For $\lambda \to \infty$ and $r = |\lambda|$, $\alpha = \text{Arg}(\lambda)$ we obtain:

$$\Psi_{11}^2 \sim \exp \left(-2i \left(\frac{4}{3}r^3 e^{3i\alpha} + xre^{i\alpha}\right)\right),$$
$$\Psi_{21}^2 \sim \frac{e^{-2i\alpha}}{r^2} \exp \left(-2i \left(\frac{4}{3}r^3 e^{3i\alpha} + xre^{i\alpha}\right)\right), \quad \lambda \to \infty.$$

As $\lambda \to \infty_1$ and $r = |\lambda|$, $\beta = \text{Arg}(\lambda) - \pi/3$ we get:

$$\Psi_{11}^2 \sim \exp \left(2i \left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right) \left(1 + O(r^{-1})\right) + O(1/r),$$
$$+ s_1^2 \exp \left(-2i \left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right) \left(\frac{e^{-2i\pi/3 - 2i\beta}}{r^2} + O(r^{-3})\right),$$

$$\Psi_{21}^2 \sim s_1^2 \exp \left(-2i \left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right) \left(1 + O(r^{-1})\right) + O(1/r),$$
$$+ \exp \left(2i \left(\frac{4}{3}r^3 e^{3i\beta} + xre^{i\pi/3}e^{i\beta}\right)\right) \left(\frac{e^{2i\pi/3 + 2i\beta}}{r^2} + O(r^{-3})\right).$$

The integral of the derivative $\psi_1^-$ by $\lambda$ is written as the sum of the integrals:

$$\int_{\infty_1}^{\infty_6} \frac{d\psi_1^-}{d\mu} d\mu = \int_{\mathcal{L}_{11}} \frac{d\psi_{11}^2}{d\mu} d\mu + \int_{\mathcal{L}_{21}} \frac{d\psi_{21}^2}{d\mu} d\mu.$$

For each of the integrals, we deform the contour so that at its ends the functions $\Psi_{11}^2$ and $\Psi_{21}^2$, respectively, vanish.

For the integral of the derivative of the function $\Psi_{11}^2$, the integration contour $\mathcal{L}_{11}$ is shown in FigurereffigPathOfIntegrating11. The path starts at the point $\mathcal{L}_{11}^-$ inside the sector $-\Delta < \text{Arg}(\lambda) < 0$, where $\Delta > 0$ and ends at the point, $\pi/3 + 2\log(r) < \text{Arg}(\mathcal{L}_{11}^+) < \pi/3$, where $r = |\lambda|$.

The contour for $\mathcal{L}_{21}$ can be obtained by the similar manner. It is shown in the figure 4.

Then the formula is valid:

$$\frac{d\delta s_1}{dx} = 2i \int_{\infty_6}^{\infty_1} \psi_1^+ d\mu. \quad (31)$$

The formulas for the squares of $\Psi$ allow us to represent the solution of the linearized Painleve-2 equation in terms of quadratic expressions from $\Psi$. 

12
\[ \text{Arg}(\lambda) = \pi/3 + 2 \log(|\lambda|) \]

\[ \text{Arg}(\lambda) = \pi/3 \]

\[ \text{Arg}(\lambda) = \pi/3 + 2 \log(|\lambda|) \]

\[ \text{Arg}(\lambda) = \pi/3 \]

\[ -\Delta < \text{Arg}(\lambda) < 0 \]

\[ \mathcal{L}_{11} \]

\[ \text{Arg}(\lambda) = 0 \]

Figure 3: The endpoints of the integration path \( \mathcal{L}_{11} \) lie inside the sectors \( \pi/3 < \text{Arg}(\lambda) < \pi/3 + 2 \log(r) \) and \( -\Delta < \text{Arg}(\lambda) < 0 \), for some \( \Delta > 0 \).

\[ \text{Arg}(\lambda) = \pi/3 \]

\[ \text{Arg}(\lambda) = \pi/3 - 2 \log(|\lambda|) \]

\[ \text{Arg}(\lambda) = \pi/3 \]

\[ \text{Arg}(\lambda) = 0 \]

\[ -\Delta < \text{Arg}(\lambda) < 0 \]

\[ \mathcal{L}_{21} \]

Figure 4: The endpoints of the integration path \( \mathcal{L}_{21} \) lie inside the sectors \( \pi/3 - 2 \log(r) < \text{Arg}(\lambda) < \pi/3 \) and \( -\Delta < \text{Arg}(\lambda) < 0 \), for some \( \Delta > 0 \).
Indeed, we differentiate by \( x \) the equation (27) by virtue of the equations (28) and (29):

\[
d^2\psi^+ + \frac{1}{dx^2} = 4(-\lambda^2 + u^2)\psi^+ + 4u'\psi_1.
\]

In this equation, the last term on the right is replaced by the equation (23):

\[
4u'\psi_1 = (4\lambda^2 + x + 2u^2)\psi^+ - \frac{i}{2} \frac{d}{d\lambda} \psi^-.
\]

As a result, we get:

\[
d^2\psi^+ = (x + 6u^2)\psi^+ - i \frac{1}{2} \frac{d}{d\lambda} \psi^-.
\]

The same calculations for \( \psi^- \) give:

\[
d^2\psi^- = 4(-\lambda^2)\psi^- - 8i\lambda u\psi_1.
\]

Replacement:

\[-8i\lambda \psi_1 = -i \frac{1}{2} \frac{d}{d\lambda} + (4\lambda^2 + x + 2u^2)\psi^-\]

yields:

\[
\frac{d^2\psi^-}{dx^2} = (x + 2u^2)\psi^- - i \frac{1}{2} \frac{d}{d\lambda} \psi^+.
\]

These formulas are useful for deriving an integral representation of the solution of the linearized Painlevé-2 equation.

Consider the integral:

\[
v(x) = \int_{\infty}^{\infty} \psi^+(\lambda, x) d\lambda.
\]

The second derivative of this integral is:

\[
\frac{d^2}{dx^2} \int_{\infty}^{\infty} \psi^+(\lambda, x) d\lambda = (6u^2 + x) \int_{\infty}^{\infty} \psi^+(\lambda, x) d\lambda - \frac{i}{2} \int_{L_{11}} \frac{d}{d\lambda} \psi^2_{11} d\lambda + \frac{i}{2} \int_{L_{21}} \frac{d}{d\lambda} \psi^2_{21} d\lambda.
\]

Then, for the same reasons as in the derivation of the formula (31), we obtain that the solution of the linearized Painlevé-2

\[
v'' = (6u^2 + x)v
\]

can be represented as:

\[
v(x) = \int_{\infty}^{\infty} \psi^+(\lambda, x) d\lambda. \tag{32}
\]
References

[1] H. Flaschka and A.C. Newell. Monodromy- and spectrum preserving deformations. *Comm. Math. Phys.*, 76:65–116, 1980.

[2] A.R. Its and V.Yu. Novokshenov. *The isomonodromic deformation method in the theory of Painleve equations*, volume 1191 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[3] A.A. Kapaev. Asymptotic behaviour of solutions for painleve equation of first kind. *Differentsial′nye uravneniya*, 24:1684, 1988.

[4] A.A. Kapaev V.Yu. Novokshenov A.S. Fokas, A.R. Its. *Painlevé Transcendents The Riemann-Hilbert Approach*, volume 128 of *mathematical Surveys and Monographs*. Am.Math.Soc., 2006.

[5] N.N. Bogolyubov and Yu.A. Mitropolskii. *Asymptotic methods in the theory of non-linear oscillations*. Gordon and Breach science publishers, New York, 1961.

[6] I.M. Krichever. Method of averaging for two-dimensional integrable equations. *Funct Anal Its Appl*, 22:200–213, 1988.

[7] V.Yu. Novokshenov. Anzatz boutroux for the second equation the painleve equation in complex area. *zv. AN USSR, ser. matem.*, 54:1229–1251, 1990.

[8] D. J. Kaup. A perturbation expansion for the zakharov-shaba t inverse scattering transform. *SIAM J. Appl. Math.*, 31:121–133, 1976.

[9] V.E.Maslov V.I.Karpman. Perturbation thery for solitons. *Sov.Phys.JETPH*, 46(2):281–291, 1977.

[10] E.M. Maslov. Perturbation theory for solitons in the second approxima-tion. *Theor Math Phys*, 42:237–245, 1980.

[11] L.A. Kalyakin. Perturbation of the korteweg-de vries soliton. *Theor Math Phys*, 92:736–747, 1992.

[12] O.M. Kiselev. Kink asymptotics of the perturbed sine-gordon equation. *Theor. Math. Phys.*, 93:1106–1111, 1992.

[13] O.M. Kiselev. Perturbation theory for the dirac equation in the two-dimensional space. *Journal of Math. Phys.*, 39:2333–2345, 1998.
[14] O.M. Kiselev. Basic functions associated with a two-dimensional dirac system. *Funct Anal Its Appl*, 32:56–59, 1998.

[15] R. Haberman. Nonlinear transition layers - second painlevé transcendent. *Studies in Applied Mathematics*, 57:247–270, 1977.

[16] G.J.M. Maree. Slow passage through a pitchfork bifurcation. *SIAM J. Math. Appl.*, 56:889–918, 1996.

[17] O.M. Kiselev and S.G. Glebov. An asymptotic solution slowly crossing the separatrix near a saddle-center bifurcation point. *Nonlinearity*, 16:327–362, 2003.

[18] O.M. Kiselev and S.G. Glebov. The capture into parametric autoresonance. *Nonlinear Dynamics*, 48(1):217–230, 2007.

[19] O.M. Kiselev. Stabilization of the wheeled inverted pendulum on a soft surface. *Russian Journal of Nonlinear Mechanics*, 16(3).

[20] R. Garnier. Sur les equations differentielles du troisieme ordre don’t l’integrale est uniforme et sur une classe d’équations nouvelles d’ordre superieuer dont l’integrale generale a les points criticues fixes. *Annales scientifiques de l’Eqole Normale Superieure*, 29:1–126, 1912.

[21] W. Wasow. *Asymptotic expansion for ordinary differential equations*, volume XIV of *Pure andapplied mathematics series*. Interscience, NY, 1965.

[22] Yu.V. Sokhotskii. *Ob opredelyennykh integralakh i funkcyakh, upotre-blyaemykh pri razlozheniyakh v ryady*. tip. M.Stasyulevicha., 1873.