Surface measures and integration by parts formula on levels sets induced by functionals of the Brownian motion in $\mathbb{R}^n$

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**Abstract.** On the infinite dimensional space $E$ of continuous paths from $[0, 1]$ to $\mathbb{R}^n$, $n \geq 1$, endowed with the Wiener measure $\mu$, we construct a surface measure defined on level sets of the $L^2$-norm of $n$-dimensional processes that are solutions to a general class of stochastic differential equations, and provide an integration by parts formula involving this surface measure. We follow the approach to surface measures in Gaussian spaces proposed via techniques of Malliavin calculus in Airault and Malliavin (Bull Sci Math 112:3–52, 1988).

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1. Introduction

Let $E = C([0, 1]; \mathbb{R}^n)$ denote the Banach space of continuous functions from $[0, 1]$ to $\mathbb{R}^n$, endowed with the sup-norm $\|f\|_\infty = \sup_{[0,1]}|f(x)|$. We denote by $E^*$ the dual of $E$. Also, we introduce the Hilbert space $H = L^2(0, 1; \mathbb{R}^n)$ of square integrable measurable functions. Let us fix the notation we shall use in the sequel. The norm in $\mathbb{R}^n$ is denoted by $|x|$ and the scalar product as $\langle x, x \rangle_{\mathbb{R}^n}$. In the infinite dimensional spaces $E$ and $H$ we denote the norm respectively by $\|x\|_H$, $\|x\|_E$. Finally, the scalar product in $H$ is $\langle x, x \rangle_H$. By $E^*$ we denote the dual of $E$.

It is known that given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a process $B = \{B(t), t \in [0, 1]\}$ is a standard $n$-dimensional Brownian motion if it a centered Gaussian process with covariance function $\mathbb{E}[\langle B(t), B(s) \rangle] = (s \wedge t)I$, $I$ being the identity matrix in $\mathbb{R}^n$. This process induces a Gaussian measure $\mu$ on the space of trajectories $(E, \mathcal{E})$. This measure is known as the Wiener measure;
the process $B(t)(x) = x(t)$ on the probability space $(E, \mathcal{E}, \mu)$ will denote the standard $n$-dimensional Brownian motion.

On the space $E$ we introduce the Malliavin derivative $D$ with domain $\mathbb{D}^1_p$, which is the closure in $L^p(E, \mu)$ of the class of smooth random variables (see for instance [6,13,14,25]). In Sect. 2 below we explain its construction in more details. The adjoint operator of the Malliavin derivative operator $D$ having domain $\mathbb{D}^1_p$ is the divergence operator, denoted as usual by $\delta$, having domain $D_q(\delta)$, where $q$ is the adjoint exponent of $p$. $\delta$ coincides with the Skorohod integral with respect to the Brownian motion $B$.

In addition to the Sobolev spaces $\mathbb{D}^1_p$, we shall consider the spaces $UC_b(E)$ of uniformly continuous and bounded functions¹ from $E$ to $\mathbb{R}$ and $UC^1_b(E)$, of uniformly continuous and bounded functions which are Fréchet differentiable, with an uniformly continuous and bounded derivative. In the sequel, we simplify the notation to $UC_b$, $UC^1_b$, since no confusion may arise.

Let $u \in L^p(E, \mu; H)$ be a stochastic process, indexed by $t \in [0,1]$, taking values in $\mathbb{R}^n$. The simplest example of such processes is, obviously, the Brownian motion $B$:

$$B(t)(x) = x(t), \quad x \in E, \ t \in [0,1].$$

In this paper we introduce the functional $g : L^p(E, \mu; H) \to \mathbb{R}$ which associates to any such process the random variable

$$g(u)(x) = \frac{1}{2} \|u(x)\|^2_H = \frac{1}{2} \int_0^1 |u(x)(t)|^2 \, dt. \quad (1.1)$$

In case $u = B$, we shall simply write $g(x) = g(B)(x) = \frac{1}{2} \|x\|^2_H$.

The aim of this paper is to construct the surface measure induced by $\mu$ on the level sets $\{g < r\}$ and provide an integration by parts formula involving this surface measure.

The notion of surface measure in infinite dimension has been introduced in the second half of last century by he seminal papers of Skorohod and Uglanov [26,27] and later developed by Fukushima, Hino, Zambotti and several other authors [10,19,20,23,28] in connection with the theory of stochastic differential equations; its importance in the theory of sets of finite perimeter in Gaussian spaces was later emphasised by the papers of Feyel and De la Pradelle, Ambrosio, Figalli, Miranda and others [4,5,18].

We shall mention here that, since the domain $\{g < r\}$ is a convex open set in $E$, our construction is related to that of the recent papers [1,2,12]. In particular, the integration by parts formula that we obtain in Proposition 4.10 is related to formula (1) in [1]. Notice however that our construction is quite different. For instance, they choose the measure $\sigma$ on the level sets of $g$ by appealing to the construction of [18] in order to fix a reference surface measure to use in the integration by parts formula. On the other hand, we construct the measure $\sigma$ by following the approach initiated by Airault and Malliavin [3].

¹ We use indifferently the terms function, functional or random variable to denote a measurable mapping $F : E \to \mathbb{R}$. 
Let $X \in L^1(E, \mu)$ be a random variable (more stringent assumptions on $X$ will be necessary, compare Sect. 3). Then we define the function

$$F_X(r) = \int_{\{g(u)<r\}} X(x) \mu(dx), \quad r \in \mathbb{R}; \quad (1.2)$$

if $F_X$ is differentiable at $r$, its derivative $F_X'(r)$ is candidate to be a surface integral

$$F_X'(r) = \int_{\{g(u)=r\}} X(x) \sigma_r(dx), \quad (1.3)$$

provided that there exists a measure $\sigma_r$, independent of $X$, such that (1.3) holds. Obviously, one further needs to prove that $\sigma_r$ is concentrated on $\{g=r\}$.

This approach was followed, among others, by [6,8,9,14,15]. The main result in this paper is given in the following theorem, whose proof is given in Sect. 4.

**Theorem 1.1.** Let $B$ the standard $n$-dimensional Brownian motion defined on the Wiener probability space $(E, \mathcal{E}, \mu)$. Let $g$ be the random variable defined as

$$g(x) = g(B)(x) = \frac{1}{2} \|x\|^2_H, \quad x \in E, \quad (1.4)$$

and consider the function $F_X$ defined in (1.2). Then, for any $r > 0$ there exists a unique Borel measure $\sigma_r$ on $E$ such that (1.3) holds for any $X \in UC_b \cup D^{1,p}$ and the support of $\sigma_r$ is concentrated on $\{g=r\}$. Moreover, for fixed $r > 0$, for any $X \in D^{1,p}$ and $h \in H$ such that $X\langle Dg, h \rangle$ belongs to $UC_b$ or $D^{1,p}$, the following integration by parts formula holds

$$\int_{\{g=r\}} X \langle Dg, h \rangle_H \sigma_r(dx) = \int_{\{g<r\}} [\langle DX, h \rangle_H - XW(h)] \mu(dx),$$

with $W(h)$ the Gaussian random variable defined in (2.1).

It is necessary to emphasize that the main effort in the proof is required by proving that Hypothesis 1.2 below holds true. It states that the random variable $g$ satisfies, in a suitable sense, the Malliavin condition, see [9]. Such condition was introduced by Nualart [25, Definition 2.1.2] (in a slightly different formulation) in a related context, i.e., the analysis of the density for the law of a random variable. In our construction the law of the random variable $g$ plays a crucial role (we hope to return to this relation in a future work).

**Hypothesis 1.2.** [Malliavin condition on $g$] There exists a process $u \in L^p(E, \mu; H)$ and a real valued random variable $\gamma \in D^{1,p}$, $p > 1$, such that the following identity holds

$$\langle Dg, u \rangle_H = \gamma \quad (1.5)$$

and $\frac{\gamma}{q}$ belongs to $D_q(\delta)$ for every $1 < q < \infty$.

In Sect. 3 we prove that the random variable given in (1.4) satisfies the above Hypothesis. The proof of the main theorem, which is given in Sect. 4, follows quite naturally by the ideas provided in [9,15].
Our construction, in particular, leads to the following identity concerning the surface measure $\sigma_r$:

$$\mathbb{E}[X | g = r] f_1(r) = \int_{\{g=r\}} X(\xi) \sigma_r(d\xi), \quad (1.6)$$

where $f_1$ is the probability density function of the random variable $g = g(B)$. Corollary 4.3 below assures that $f_1$ is a bounded and continuous function and the identity above holds for every $r > 0$.

In the last part of the paper we extend previous results to the analysis of the random variables $g(u)$, where we assume that $u$ is the solution of a stochastic differential equation of gradient form

$$u(t) = -\int_0^t \nabla V(u(s)) \, ds + B(t). \quad (1.7)$$

This is the first step in considering processes whose image law is non Gaussian. In particular, in Sect. 5 we prove the following result.

**Theorem 1.3.** Let $u$ be the solution of equation (1.7), where $V \in C^3_b(\mathbb{R}^n; \mathbb{R}^n)$, and let $g(u)$ be the random variable defined in (1.1).

Then $g(u)$ defined on $(E, \mathcal{E}, \mu)$, has a continuous and bounded density $\varphi_1$ with respect to the Lebesgue measure on $\mathbb{R}_+$. Moreover, there exists a surface measure $\theta_r$ concentrated on $\{g(u) = r\}$ that is the restriction of $\mu$ to the level set $\{g(u) = r\}$.

We notice that the probability density function $\varphi_1(r)$ of the random variable $g(u)$ with respect to the Lebesgue measure can be computed in terms of $f_1$ as follows:

$$\varphi_1(r) = \mathbb{E}[\rho_1(B)^{-1} | g(B) = r] f_1(r),$$

where $\rho_1(B)^{-1}$ is a bounded function which is defined in terms of the coefficient $V$ in (1.7).

Moreover, it follows that $\varphi_1(r) = \theta_r(\{g(u) = r\})$. The proof is based on a Girsanov transformation of the reference Gaussian measure and it exploits the results obtained in the case $u = B$.

2. An introduction to Malliavin calculus

In literature different ways of introducing the Malliavin derivative are present. We work here in the general framework given in [25]. This approach requires to fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an isonormal Gaussian process which provides the Gaussian framework. Here, as reference probability space, we consider the Wiener space $(E, \mathcal{E}, \mu)$. The isonormal Gaussian process is given by the family of Wiener integrals.

We denote, as before, by $B(t)(x) = x(t)$, $t \in [0, 1]$, $x \in E$, the standard $n$-dimensional Brownian motion on the probability space $(E, \mathcal{E}, \mu)$. Given this process, we may introduce the Wiener integral.
\[ W(h) = \int_0^1 (h(s), dB(s))_{\mathbb{R}^n}, \quad h \in H = L^2(0,1;\mathbb{R}^n). \] (2.1)

For any \( h \in H \), \( W(h) \) is a centered Gaussian random variable with variance \( \|h\|_H^2 \). We shall denote by \( \mathcal{H}_1 \) the following subspace of \( L^2(E,\mu) \), called the first Wiener chaos, defined by
\[
\mathcal{H}_1 = \{ F \in L^2(E,\mu) : \exists h \in H, \ F = W(h) \}.
\]
The map \( W \) defines a linear isometry between \( H \) and \( \mathcal{H}_1 \).

Remark 2.1. In the sequel, we shall use the probabilistic notation of expectation for the integral over \( E \)
\[
\mathbb{E}[F] = \int_E F(x) \mu(dx),
\]
for a measurable function (random variable) \( F : (E,\mathcal{E}) \to \mathbb{R} \). In particular,
\[
\mathbb{E}[W(h)] = \int_E W(h)(x) \mu(dx) = 0,
\]
\[
\mathbb{E}[|W(h)|^2] = \int_E |W(h)(x)|^2 \mu(dx) = \|h\|_H^2.
\]

Starting from the space \( \mathcal{H}_1 \) we construct the class of smooth random variables
\[
S = \{ F : (E,\mathcal{E}) \to \mathbb{R} : \exists f \in C_0^\infty(\mathbb{R}^d), \ h_1, \ldots, h_d \in H, \ F = f(W(h_1), \ldots, W(h_d)) \},
\]
where \( C_0^\infty(\mathbb{R}^d) \) is the space of smooth functions on \( \mathbb{R}^d \) with polynomial growth at infinity.

We see that \( S \subset L^p(E,\mu) \) for any \( p \geq 1 \). On the class \( S \) of smooth random variables we consider a functional (actually, a family of functionals indexed by the order of integration \( p \))
\[
D : S \subset L^p(E,\mu) \to L^p(E,\mu;H)
\]
by setting
\[
DF = \sum_{k=1}^d \frac{\partial f}{\partial x_k}(W(h_1), \ldots, W(h_d)) h_k.
\]

Lemma 2.2. Let \( F \in S \) and \( h \in H \). Then it holds
\[
\mathbb{E}[\langle DF, h \rangle_H] = \mathbb{E}[FW(h)].
\]

For the proof we refer to [25]. As a consequence, it is possible to prove that the operator \( D \) is closable from \( L^p(E,\mu) \) to \( L^p(E,\mu;H) \).

Definition 2.3. We define the norm
\[
\|F\|_{1,p}^p = \mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p], \quad F \in S.
\]

Then the domain of the Malliavin derivative \( D \), denoted by \( D^{1,p} \), is the closure of the class \( S \) in \( L^p(E,\mu) \) with respect to the norm \( \| \cdot \|_{1,p} \). We shall denote again by \( D \) this closure.
Let us now introduce the divergence operator. Fix \( 1 < q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). By \( D_q(\delta) \) we denote the domain of the diverge operator \( \delta \). It consists of all \( v \in L^q(E, \mu, H) \) for which there exists a \( G_v \in L^p(E, \mu) \) such that

\[
E[\langle v, DF \rangle_H] = E[G_v F], \quad F \in \mathbb{D}^{1,p}.
\]

The function \( G_v \), if it exists, is uniquely determined. We set

\[
\delta(v) := G_v, \quad v \in D_q(\delta).
\]

The divergence operator is easily seen to be closed and densely defined.

**Definition 2.4.** We define the norm

\[
\|v\|_{D_q(\delta)}^p = E[\|v\|_{H}^q] + E[|\delta(v)|^q], \quad v \in D_q(\delta).
\]

It is known that in the case \( H = L^2(0, 1; \mathbb{R}^n) \), for a given \( F \in \mathbb{D}^{1,p} \), its Malliavin derivative \( DF \in L^p(E, \mu; H) \) can be interpreted as a stochastic process indexed by \( t \in [0, 1] \). In this case we can interpret the divergence operator as a stochastic integral, the Skorohod integral and the following notation becomes meaningful:

\[
\delta(u) = \int_0^1 \langle u(s), \delta B(s) \rangle_{\mathbb{R}^n}.
\]

If the process \( u \) is adapted and Itô integrable then the Skorohod integral coincides with the Itô integral. In the special case \( u = h \in H \), we have \( \delta(u) = W(u) \), with \( W(u) \) given by (2.1).

We conclude this Section with the following proposition, that we will exploit to prove our main result Theorem 1.1. It is an extension of [25, Proposition 1.3.3] and a proof can be found in [24, Proposition 6.9].

**Proposition 2.5.** Let \( 1 < r, q < \infty \) be such that \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \). Assume that \( F \in \mathbb{D}^{1,p} \) and \( u \in D_r(\delta) \). Then \( Fu \in D_q(\delta) \) and

\[
\delta(Fu) = F\delta(u) - \langle u, DF \rangle_H.
\]

3. Verification of the Malliavin condition

In this section we prove that the random variable

\[
g(x) = \frac{1}{2}\|x\|_H^2 = \frac{1}{2} \int_0^1 |x(t)|^2 \, dt, \quad x \in E, \quad (3.1)
\]
satisfies the Malliavin condition stated in Hypothesis 1.2. From a probabilistic point of view, this random variable is strictly related to the Bessel process of order $\nu = \frac{n}{2} - 1$: $X(t) = |B(t)|^2$. Some results on $g$ are given, for instance, in [11].

In the following proposition we state the construction that we aim to prove in order to achieve the verification of the Malliavin condition.

**Proposition 3.1.** Let $\{B(t), t \in [0, 1]\}$ be the standard $n$-dimensional Brownian motion. The random variable $g$, defined in (3.1), satisfies the Malliavin condition in Hypothesis 1.2 with

$$u(s) = D_s g, \quad \gamma = \int_0^1 |D_s g|^2 \, ds.$$  

(3.2)

The proof of Proposition 3.1 is based on a chain of four lemmas that we will prove in the next Subsection.

### 3.1. Preliminary lemmas

The first result we need is the classical Karhunen-Loève expansion for the standard Brownian motion. We refer to the monograph [6] for a discussion of the related literature.

**Lemma 3.2.** Let $\{B(t), t \in [0, 1]\}$ be the standard $n$-dimensional Brownian motion. The Karhunen-Loève expansion of every component $B^i(t)$, $i = 1, \ldots, n$, is

$$B^i(t)(x) = \sum_{k=1}^{\infty} \beta^i_k(x)e_k(t),$$  

(3.3)

where $\{\beta^i_k\}_k$ is a sequence of independent standard Gaussian random variables defined on the same probability space as $B^i$, and $\{e_k\}_k$ is the orthonormal basis in $L^2(0, 1)$ formed by eigenfunctions of the integral operator $Q$ given by

$$Qx(t) = \int_0^1 (s \wedge t)x(s) \, ds.$$  

(3.4)

The eigenfunctions and the eigenvalues of the operator $Q$ are, respectively,

$$e_k(t) = \sqrt{2} \sin \left( \frac{t}{\sqrt{\lambda_k}} \right), \quad k \in \mathbb{N}_+,$$  

(3.5)

and

$$\lambda_k = \frac{4}{(\pi(2k-1))^2}, \quad k \in \mathbb{N}_+.$$  

(3.6)

**Proof.** The result is a consequence of [6, Theorem 3.5.1] (see also the comments at pages 118-119). $\square$

**Remark 3.3.** The operator $Q$ defined in (3.4) is the covariance operator of a standard Brownian motion on the space of trajectories $L^2(0, 1)$. Although it is possible to relate it to the covariance operator of the Gaussian measure $\mu$ on $H$, we do not need this fact in the sequel and we chose to keep the presentation at a simpler level.
Lemma 3.4. Let \( \{B(t), t \in [0,1]\} \) be the standard \( n \)-dimensional Brownian motion. The function \( g = g(B) \) as defined in (3.1) belongs to the space \( D^{1,p} \) for all \( p \geq 1 \) and its derivative (in the direction \( i \)) is given by

\[
D^i_s g = \int_s^1 B^i(t) \, dt. \tag{3.7}
\]

Moreover,

\[
\|Dg\|_H^2 = \sum_{i=1}^n \int_0^1 \int_0^1 (t \wedge s) B^i(t) B^i(s) \, dt \, ds,
\]

Proof. For every \( s \in [0,1] \) we can compute the Malliavin derivative, in the direction \( e_i \), of the function \( g \) as follows

\[
D^i_s g := \langle D_s g, e_i \rangle_{\mathbb{R}^n} = \int_0^1 B^i(t) D_s (B^i_s(t)) \, dt = \int_s^1 B^i(t) \, dt, \tag{3.8}
\]

and it is easy to verify that \( g \in D^{1,p} \), for every \( p > 1 \).

Using repeatedly the Fubini Theorem we obtain

\[
\|Dg\|_H^2 = \int_0^1 |D_s g|^2 \, ds = \sum_{i=1}^n \int_0^1 \left| \int_s^1 B^i(t) \, dt \right|^2 \, ds
\]
\[
= \sum_{i=1}^n \int_0^1 \left( \int_s^1 B^i(t) \, dt \right) \left( \int_s^1 B^i(\sigma) \, d\sigma \right) \, ds
\]
\[
= \sum_{i=1}^n \left[ \int_0^1 \int_0^\sigma B^i(t) B^i(\sigma) \, dt \, d\sigma + \int_0^1 \int_0^\sigma B^i(t) B^i(\sigma) \, dt \, d\sigma \right]
\]
\[
= \sum_{i=1}^n \left[ \int_0^1 \int_0^\sigma t B^i(t) B^i(\sigma) \, dt \, d\sigma + \int_0^1 \int_0^\sigma \sigma B^i(t) B^i(\sigma) \, dt \, d\sigma \right]
\]
\[
= \sum_{i=1}^n \int_0^1 \int_0^1 (t \wedge s) B^i(t) B^i(s) \, dt \, ds.
\]

Lemma 3.5. Let \( \{B(t), t \in [0,1]\} \) be the standard \( n \)-dimensional Brownian motion and let \( \gamma \) be defined as in Proposition 3.1. Then \( \mathbb{E} \left| \frac{1}{\gamma} \right|^p < \infty \) for every \( 1 < p < \infty \).

Proof. Thanks to Lemma 3.4 we have

\[
\gamma = \|Dg\|_H^2 = \sum_{i=1}^n \int_0^1 \int_0^1 (t \wedge s) B^i(t) B^i(s) \, dt \, ds.
\]
Thanks to Lemma 3.2 [see (3.3) and (3.4)], we can rewrite every component \(i\) of the above expression as (here we drop the index \(i\) for simplicity)

\[
\int_0^1 \int_0^1 (t \wedge s) \left( \sum_{k=1}^{\infty} \beta_k e_k(s) \right) \left( \sum_{j=1}^{\infty} \beta_j e_j(t) \right) \, dt \, ds
\]

\[
= \sum_{k,j=1}^{\infty} \beta_k \beta_j \int_0^1 \int_0^1 (t \wedge s) e_k(s) \, ds \, e_j(t) \, dt
\]

\[
= \sum_{k=1}^{\infty} \lambda_k \beta_k^2,
\]

where in the last equality we used the fact that \(\{e_k\}_k\) is the orthonormal basis in \(L^2(0,1)\) formed by eigenfunctions of the integral operator \(Q\) with corresponding eigenvalues \(\lambda_k\). The above series converges [see (3.6)] and, for a fixed \(N \geq 1\), it holds

\[
\sum_{k=1}^{\infty} \lambda_k \beta_k^2 \geq \lambda_N \sum_{k=1}^{N} \beta_k^2.
\]

Therefore,

\[
\gamma \geq \lambda_N \sum_{i=1}^{n} \sum_{k=1}^{N} \underbrace{(\beta_k^i)}_{Z}^2.
\]

Since \(\{\beta_k^i\}_{i,k}\) are independent standard Gaussian random variables, the random variable \(Z\) on the right hand side is a \(\chi^2\)-distribution with \(nN\) degrees of freedom, hence, for a fixed \(1 < p < \infty\),

\[
\mathbb{E} \left| \frac{1}{\gamma} \right|^p \leq \lambda_N^{-p} \mathbb{E} [Z^{-p}] = \lambda_N^{-p} \int_0^{\infty} x^{-p} x^{nN-1} e^{-\frac{x}{2}} \, dx,
\]

which converges provided we choose a \(N > 2p + 1\). This concludes the proof.

Notice that the bound in the right hand side of (3.9) explodes, as \(p \to \infty\), as fast as \(p^p\).

**Lemma 3.6.** Let \(\{B(t), t \in [0,1]\}\) be the standard \(n\)-dimensional Brownian motion and let \(\gamma\) be defined as in Proposition 3.1. Then \(\gamma \in D^{1,p}\) for every \(p > 1\) and its Malliavin derivative (in the direction \(i\)) is

\[
D^i \gamma = 2 \int_0^1 \int_s^1 B^i(r) \, dr \int_s^1 1_{(0,t)}(\theta) \, dt \, ds.
\]

Moreover \(\frac{1}{\gamma} \in D^{1,p}\) for any \(p > 1\) and its Malliavin derivative is given by

\[
D^i \left( \frac{1}{\gamma} \right) = -\frac{D^i \gamma}{\gamma^2}.
\]
Proof. By the chain rule one easily obtains

\[
D_t^i \gamma = D_t^i \left( \int_0^1 \sum_{i=1}^n \left| \int_s^1 B^i(t) \, dt \right|^2 \, ds \right) = 2 \int_0^1 \int_s^1 B^i(r) \, dr \int_s^1 1_{(0,t)}(\theta) \, dt \, ds.
\]

In order to show that \( \gamma \in \mathbb{D}^{1,p} \), we need to prove that

\[
\mathbb{E} \left[ |\gamma|^p \right] + \mathbb{E} \left[ \| D\gamma \|_{H}^p \right] < \infty.
\]

The boundedness of the first term in the above expression immediately follows from Lemma 3.4, since \( \gamma = \| Dg \|_{H}^2 \). We can estimate the second term by means of the Hölder’s inequality. We have

\[
|D_t^i \gamma|^2 = \left( \int_0^1 \sum_{i=1}^n \left| \int_s^1 B^i(r) \, dr \int_s^1 1_{(0,t)}(\theta) \, dt \right|^2 \, ds \right) \leq 4 \int_0^1 \sum_{i=1}^n \left( \int_s^1 B^i(r) \, dr \right)^2 \, ds = 4 \| Dg \|_{H}^2.
\]

Therefore,

\[
\mathbb{E} \| D\gamma \|_{H}^p \leq \mathbb{E} \left[ \left( \int_0^1 |D_t^i \gamma|^2 \, d\theta \right)^p \right] \leq 2^p \mathbb{E} \left[ \left( \int_0^1 \| Dg \|_{H}^2 \, d\theta \right)^p \right]
\]

which is finite thanks to Lemma 3.4.

Equality (3.11) is straightforward to prove. It remains to prove that \( \frac{1}{\gamma} \in \mathbb{D}^{1,p} \).

In view of Lemma 3.5 it is sufficient to show that

\[
\mathbb{E} \left[ D \left( \frac{1}{\gamma} \right) \right]_{H}^p < \infty.
\]

From (3.11), thanks to Hölder’s inequality, we get

\[
\mathbb{E} \left[ D \left( \frac{1}{\gamma} \right) \right]_{H}^p = \mathbb{E} \left[ \left( \frac{D \gamma}{\gamma^2} \right) \right]_{H}^p = \mathbb{E} \left[ \| D\gamma \|_{H} \left| \gamma \right|^2 \right]^p \leq \left[ \mathbb{E} \| D\gamma \|_{H}^{p+\gamma} \right]^p \mathbb{E} \left| \frac{1}{\gamma} \right|^{2p},
\]

which is finite thanks to Lemma 3.5 and estimate (3.12). This concludes the proof.

We are now ready to prove Proposition 3.1.

3.2. Proof of Proposition 3.1

Thanks to Lemma 3.4 we know that \( g \in \mathbb{D}^{1,p} \), for every \( p > 1 \). By definition, compare (3.2), we verify that

\[
\langle Dg, u \rangle_H = \int_0^1 \langle Ds g, u(s) \rangle_{\mathbb{R}^n} \, ds = \gamma.
\]

It remains to prove that

\[
\frac{u}{\gamma} \in D_q(\delta) \text{ for all } 1 < q < \infty.
\]

In order to factor out a scalar random variable from a Skorohod integral we can appeal to Proposition 2.5. From Lemma 3.6 we know that \( \frac{1}{\gamma} \in \mathbb{D}^{1,p} \) for every \( 1 < p < \infty \); we claim that \( u \in D_r(\delta) \) for every \( r > 1 \). Therefore, by previous
proposition, we have \( u \gamma \in D_q(\delta) \) for all \( 1 < q < \infty \) such that \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \). It follows from [25, Proposition 1.3.1] (see also pag. 42) that, in order to prove that \( u \in D_r(\delta) \), it is sufficient to show

\[
E \left[ \int_0^1 |u(s)|^2 \, ds \right]^{\frac{2}{q}} + E \left[ \int_0^1 \int_0^1 |D_s u(t)|^2 \, ds \, dt \right]^{\frac{2}{r}} < \infty.
\]

The first term in the above expression is bounded. This is an immediate consequence of the definition of \( u \) and the fact that \( g \in D_1^{1,r} \), for every \( r > 1 \). For what concerns the second term we have

\[
E \left[ \int_0^1 \int_0^1 |D_s u(t)|^2 \, ds \, dt \right]^{\frac{2}{r}} = E \left[ \int_0^1 \int_0^1 |D_s \int_0^1 B(\theta) \, d\theta|^2 \, ds \, dt \right]^{\frac{2}{r}} \leq E \left[ \int_0^1 \int_0^1 \int_0^1 1_{\{\theta > s\}} \, d\theta \, ds \, dt \right]^{\frac{2}{r}} \leq 1.
\]

4. Proof of Theorem 1.1: existence of the surface measure for sets defined by the Brownian motion

In this section, by mimicking the construction provided in [9] we construct the surface measure induced by \( \mu \) on the level sets \( \{ g = r \} \). Recall that \( g \) is the random variable defined by

\[
g(x) = \frac{1}{2} \|x\|_H^2, \quad x \in E;
\]

by construction, \( g \geq 0 \). Moreover, thanks to Proposition 3.1, \( g \) satisfies the Malliavin condition of Hypothesis 1.2.

As stated in the Introduction, we study the family of functions \( F_X(s) \) indexed by (suitably regular) random variables \( X \), where

\[
F_X(s) = E[1_{\{g \leq s\}} X], \quad s > 0.
\]

In this section, we first assume that \( X \) is a random variable in \( D^{1,p} \), for some \( 1 < p < \infty \). Let us introduce the family of functions \( \phi_s \) indexed by \( s > 0 \) by setting

\[
\phi_s(u) = \int_0^u 1_{\{y \leq s\}} \, dy.
\]

\( \phi_s \) is a Lipschitz continuous function, hence it is possible to compute the Malliavin derivative

\[
D\phi_s(g) = \phi'_s(g) Dg; \quad (4.1)
\]

scalar multiplying both sides of (4.1) by \( \frac{Xu}{\gamma} \) implies, after a little algebra

\[
X \left\langle D\phi_s(g), \frac{u}{\gamma} \right\rangle_H = \phi'_s(g) X = X 1_{\{g \leq s\}};
\]
thus, the duality relationship between Malliavin derivative and Skorohod integral leads to

$$F_X(s) = \mathbb{E}[X \mathbb{1}_{\{g \leq s\}}] = \mathbb{E}\left[X \left< D\phi_s(g), \frac{u}{\gamma} \right>_H \right] = \mathbb{E}\left[\delta\left(X \frac{u}{\gamma}\right) \phi_s(g) \right]. \quad (4.2)$$

Notice that, in order for the last term in (4.2) to be well defined, we need to have $X \frac{u}{\gamma} \in D_\theta(\delta)$ for some $\theta$. We postpone the verification of this fact to Sect. 4.2 and we start by considering the special case $X \equiv 1$. This case allows us to study the probability density function of $g$.

4.1. Existence of the probability density function for $g$

By taking $X \equiv 1$, the above reasoning leads to the existence of a density for the cumulative distribution function of the random variable $g$, as already proved by Nualart [25, Proposition 2.1.1]:

$$\mathbb{P}(g \leq s) = F_1(s) = \mathbb{E}\left[\delta\left(\frac{u}{\gamma}\right) \int_0^s \mathbb{1}_{\{y < g\}} \, dy \right]$$

and, by an application of Fubini’s theorem, we get the following expression, which easily led to the existence of a density:

$$F_1(s) = \int_0^s \mathbb{E}\left[\delta\left(\frac{u}{\gamma}\right) \mathbb{1}_{\{g > y\}} \right] \, dy. \quad (4.3)$$

**Proposition 4.1.** Assume that there exist $u$ and $\gamma$ such that Hypothesis 1.2 holds. Then the mapping $s \mapsto F_1(s)$ is continuous.

**Proof.** The integrand function $G : y \mapsto \mathbb{E}\left[\delta\left(\frac{u}{\gamma}\right) \mathbb{1}_{\{g > y\}} \right]$, defined in (4.3), is measurable and bounded, by assumption, hence the statement is obvious. □

As a consequence of previous proposition, the mapping $G$ is also continuous, since

$$|G(y + \varepsilon) - G(y)| \leq \mathbb{E}\left[\left|\delta\left(\frac{u}{\gamma}\right) \mathbb{1}_{\{y < g < y + \varepsilon\}} \right| \right] \leq \mathbb{E}\left[\left|\delta\left(\frac{u}{\gamma}\right) \right|^q \left(1 + q^{-1}\right)^{1/q} \right],$$

where $q'$ is the conjugate exponent of $q$; therefore, we can apply the integral mean value theorem to get the following.

**Proposition 4.2.** For $s > 0$ there exists the derivative $f_1(s) = F_1'(s)$ and it is equal to

$$F_1'(s) = \mathbb{E}\left[\delta\left(\frac{u}{\gamma}\right) \mathbb{1}_{\{g > s\}} \right] = G(y).$$

**Proof.** Since

$$\frac{1}{\varepsilon} \left( F_1(s + \varepsilon) - F_1(s) \right) = \frac{1}{\varepsilon} \int_s^{s + \varepsilon} G(y) \, dy$$

and the integrand function $G$ is continuous, the thesis follows by letting $\varepsilon \to 0$. □
Corollary 4.3. The random variable $g$ has a probability density function $f_1(s)$ that is continuous and bounded.

Actually, the existence of this density is already known in the literature, as well as the explicit form of this function, see [11, Part II.4, formula (1.9.4), page 377].

4.2. Differentiability of $F_X$

Lemma 4.4. If $X \in D^{1,p}$, for some $1 < p < \infty$ and $\frac{p}{p'} \in D_q(\delta)$, with $1 < q < \infty$, then $X \frac{u}{\gamma} \in D_q(\delta)$ for some $1 < \theta < \infty$.

Proof. By definition, this requires to show that for any smooth random variable $Y$

$$\left| E\left\langle DY, X\frac{u}{\gamma} \right\rangle_H \right| \leq c E[|Y|^{\theta'}]^{1/\theta'},$$

with $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Recall the integration by parts formula

$$E\left\langle DY, X\frac{u}{\gamma} \right\rangle_H = E\left[ \left\langle D(XY), \frac{u}{\gamma} \right\rangle_H - \left\langle DX, Y\frac{u}{\gamma} \right\rangle_H \right],$$

it follows

$$\left| E\left\langle DY, X\frac{u}{\gamma} \right\rangle_H \right| \leq E \left| X \delta \left( \frac{u}{\gamma} \right) - \left\langle DX, \frac{u}{\gamma} \right\rangle_H \right|$$

and by Hölder’s inequality (in the sequel we exploit the assumption $\theta > p'$, that is equivalent to $\theta' < p$)

$$\leq c E[|Y|^{\theta'}]^{1/\theta'} \left( \left[ E \left| X \delta \left( \frac{u}{\gamma} \right) \right|^{\theta} \right]^{1/\theta} + \left[ E \left| DX, \frac{u}{\gamma} \right|_H^{\theta} \right]^{1/\theta} \right),$$

$$\leq c E[|Y|^{\theta'}]^{1/\theta'}
\left( \left[ E|X|^p \right]^{1/p} \left[ E \left| \delta \left( \frac{u}{\gamma} \right) \right|^{\frac{p\theta}{p-q}} \right]^{\frac{p-q}{p\theta}} + \left[ E\|DX\|_H^{p} \right]^{1/p} \left[ E \left\| \frac{u}{\gamma} \right\|_H^{\frac{p\theta}{p-q}} \right]^{\frac{p-q}{p\theta}} \right),$$

$$\leq c E[|Y|^{\theta'}]^{1/\theta'} \left\| X \right\|_{1,p} \left\| \frac{u}{\gamma} \right\|_{D_{\frac{p\theta}{p-q}}(\delta)}$$

and the thesis follows by taking $\theta = \frac{pq}{p+q}$. \hfill $\Box$

Let us now come back to formula (4.2). Previous lemma guarantees the well posedness of the last term in (4.2). Proceeding now in the same way we did in previous subsection, an application of Fubini’s theorem implies that

$$F_X(s) = \int_0^s E \left[ \delta \left( X\frac{u}{\gamma} \right) 1_{\{g>y\}} \right] dy.$$
Proposition 4.5. Let $X$ belong to $\mathbb{D}^{1,p}$ for some $1 < p < \infty$. Then there exists the derivative $f_X(s) = F'_X(s)$ and it is equal to
\[
F'_X(s) = \mathbb{E}\left[\delta\left(\frac{X}{\gamma}\right) 1_{\{g>s\}}\right].
\] (4.6)
Moreover, $f_X(r)$ is a continuous and bounded function and there exists a constant $c > 0$ such that
\[
|F'_X(s)| \leq c\|X\|_{\mathbb{D}^{1,p}}.
\] (4.7)

Proof. We proceed as in Propositions 4.1 and 4.2, by taking into account that the mapping $G_X : y \mapsto \mathbb{E}\left[\delta\left(\frac{X}{\gamma}\right) 1_{\{g>y\}}\right]$ is bounded and continuous [compare the computations in (4.4)] thanks to the integrability of $X \frac{u}{\gamma}$ provided in Lemma 4.4.

We consider further the estimate (4.7). By the integration by parts formula for Malliavin derivative,
\[
|F'_X(r)| \leq \mathbb{E}\left[ \left|X\delta\left(\frac{u}{\gamma}\right)\right| + |\langle DX, \frac{u}{\gamma}\rangle_H| \right] 1_{\{g>r\}}
\]
and the thesis follows by Hölder’s inequality and Hypothesis 1.2. □

Actually, the existence of a continuous density for the functional $g$ implies that we can write formula (1.2) as follows (we use a probabilistic notation, since it seems more expressive)
\[
F_X(s) = \mathbb{E}[X 1_{\{g<s\}}] = \int_{-\infty}^s \mathbb{E}[X \mid g = t] f_1(t) \, dt.
\] (4.8)

Therefore, by comparing with the results in Proposition 4.5, we obtain that the identity
\[
f_X(s) = F'_X(s) = \mathbb{E}[X \mid g = s] f_1(s)
\] (4.9)
holds for almost every $s$ and, since the left-hand side is continuous, we conclude that there exists a continuous version of the function $s \mapsto \mathbb{E}[X \mid g = s] f_1(s)$.

Notice that expression (4.9) for $F'_X(s)$ is more meaningful than (4.6). In particular it provides “a candidate” to be the surface measure.

4.3. The surface measure
The results in this section mimic the construction in [9,15] and we shall skip some minor detail. Notice however that these papers only address the Hilbert setting, while we work in the Banach space $E$.

Let us notice that on the probability space $(E, \mathcal{E}, \mu)$, identity (4.9) formally reads
\[
F'_X(r) = \int_{\{g=r\}} X(x) f_1(r) \mu(dx).
\]

We are interested in proving that there exists a surface measure $\sigma_r$ on the boundary surface $\{g=r\}$ such that previous expression simplifies to
\[
F'_X(r) = \int_{\{g=r\}} X(x) \sigma_r(dx).
\] (4.10)
In order to achieve this result, we need to extend previous construction to the class of functionals $X \in UC_b$.

Since functions in $UC^1_b$ can be uniformly approximated by elements in $UC^1_b$ (see [17, Sect. 2.2]), for every $X \in UC_b$ there exists a sequence $X_n \in UC^1_b$ such that $X_n \to X$. Moreover, since $UC^1_b \subset D^{1,p}$ for every $p$, results in previous section applies to the elements of the approximating sequence.

**Proposition 4.6.** For every $X \in UC^1_b$, $s \mapsto F_X(s)$ is continuously differentiable and there exists a constant $c > 0$ such that

$$|f_X(s)| \leq c \|X\|_{\infty}, \quad \text{where } \|X\|_{\infty} \text{ is the sup-norm in } E. \quad (4.11)$$

**Proof.** Only (4.11) needs to be proven. By (4.9) we get

$$|f_X(s)| \leq |E[X \mid g = s]| f_1(s) \leq E|X| f_1(s) \leq c \|X\|_{\infty}$$

since we know, by Corollary 4.3, that $f_1$ is a continuous and bounded function. \qed

By an approximation argument we obtain that the same result holds for $X \in UC_b$.

**Proposition 4.7.** For any $X \in UC_b$ the functional $s \mapsto F_X(s)$ is continuously differentiable and there exists a constant $c > 0$ such that

$$|f_X(s)| \leq c \|X\|_{\infty}. \quad (4.12)$$

We are finally in the position to conclude the proof of the main result of Theorem 1.1. For fixed $r$, consider a sequence $\varepsilon_n \to 0$ and define the family of measures

$$\sigma_n := \frac{1}{\varepsilon_n} \mathbb{1}_{\{r < g \leq r + \varepsilon_n\}} \mu.$$

Let us consider $X \in UC_b$. We have

$$\int_E X(x) \sigma_n(dx) = \int_E \frac{1}{\varepsilon_n} \mathbb{1}_{\{r < g \leq r + \varepsilon_n\}} X(x) \mu(dx) = \frac{1}{\varepsilon_n} [F_X(r + \varepsilon_n) - F_X(r)];$$

thanks to Proposition 4.7 we can pass to the limit in the above formula to get

$$\lim_{n \to \infty} \int_E X(x) \sigma_n(dx) = F_X'(r).$$

By an application of the Prokhorov’s theorem (see [7, Corollary 8.6.3]) we finally obtain that the sequence $\sigma_n$ converges to a measure $\sigma_r$ such that

$$F_X'(r) = \int_E X(x) \sigma_r(dx).$$

Finally, by taking suitable approximations of $X = \mathbb{1}_{\{|g-r| > \delta\}}$ we check that $\sigma_r$ is concentrated on $\{g = r\}$ and the proof is complete.

If we now take into account $X \in D^{1,p}$, $p > 1$, formula (4.10) makes sense provided $X$ possesses a trace on the surface $\{g = r\}$. When the Malliavin condition is fulfilled this problem has been investigated in [12], see also [14].
Definition 4.8. Let $r > 0$ and $p > 1$. We say that $X \in \mathbb{D}^{1,p}$ possesses a trace $Tr(X)$ on $\{ g = r \}$ if there exists a sequence $\{ X_n \} \subset C^1_b$ such that, if $X_n \to X$ in $\mathbb{D}^{1,p}$, then $X_n|_{\{ g = r \}} \to Tr(X)$ in $L^1(E, \sigma_r)$.

The following result ensures the well-posedness of (4.10) (and the validity of Theorem 1.1) in the case $X \in \mathbb{D}^{1,p}$.

Proposition 4.9. Let $X \in \mathbb{D}^{1,p}$, $p > 1$, and let $\{ X_n \} \subset C^1_b$ be a sequence converging to $X$ in $\mathbb{D}^{1,p}$. Then $\{ X_n \}$ is Cauchy in $L^1(E, \sigma_r)$, so that $X$ possesses a trace $Tr(X)$ on $\{ g = r \}$.

Proof. Exploiting equalities (4.9) and (4.10) (passing, if necessary, to an approximation sequence in $UC^1_b$), we can write

$$\int_E |X_n - X_m| d\sigma_r = \mathbb{E}[|X_n - X_m| |g = r] f_1(r) \leq C(r) \|X_n - X_m\|_{1,p},$$

where the last inequality is a consequence of Corollary 4.3. This shows that $\{X_n\}$ is a Cauchy sequence in $L^1(E, \sigma_r)$. Since $L^1(E, \sigma_r)$ is a complete space, the thesis immediately follows. \qed

4.4. The integration by parts formula

In this section we discuss the integration by parts formula on the level sets of the mapping $g$. Similar results have been obtained by [1,12] with different techniques, see also [9, Sect. 4].

Proposition 4.10. Let $r > 0$ be fixed. For any $X \in \mathbb{D}^{1,p}$ and $h \in H$ such that $X \langle Dg, h \rangle_H$ belongs to $UC^1_b$ or $\mathbb{D}^{1,p}$, it holds

$$\int_{\{ g < r \}} X \langle Dg, h \rangle_H \sigma_r(dx) = \int_{\{ g < r \}} [\langle DX, h \rangle_H - XW(h)] \mu(dx) \quad (4.13)$$

where $W(h)$ is the Gaussian random variable defined in (2.1).

Proof. The starting point is the integration by parts formula [compare (4.5)]

$$\mathbb{E}[X \langle DY, h \rangle_H] = \mathbb{E}[XY \delta(h - Y \langle DX, h \rangle_H)]; \quad (4.14)$$

which holds for random variables $X$ and $Y$ in the domain $\mathbb{D}^{1,p}$ of the Malliavin derivative and $h \in H$.

In a sense, we aim to apply this formula to the random variable $Y = 1_{\{ g < r \}}$, but this cannot be obtained directly due to the lack of regularity of this mapping. We thus approximate $Y$ by the following procedure.

Let

$$\theta_\varepsilon(a) = \frac{1}{\varepsilon} \int_a^{+\infty} 1_{(r-\varepsilon,r)}(s) \, ds, \quad a > 0;$$

$\theta_\varepsilon$ is a Lipschitz continuous function, hence the mapping $Y_\varepsilon = \theta_\varepsilon(g)$ is a smooth approximation of $Y$, in the sense that $Y_\varepsilon \to Y = 1_{\{ g \leq r \}}$ in $L^2(E, \mu)$.

The right-hand side of (4.14), with $Y_\varepsilon$ instead of $Y$, converges as $\varepsilon \downarrow 0$ to

$$\int_{\{ g < r \}} [XW(h) - \langle DX, h \rangle_H] \mu(dx).$$
On the other hand, we have
\[ DY_\varepsilon = \psi_\varepsilon'(g) \, Dg = -\frac{1}{\varepsilon} \mathbf{1}_{(r-\varepsilon,r)}(g) \, Dg \]
hence
\[ \mathbb{E} [ X \langle DY_\varepsilon, h \rangle_H ] = -\frac{1}{\varepsilon} \int_X X \mathbf{1}_{(r-\varepsilon,r)}(g) \langle Dg, h \rangle_H \, \mu(dx). \]
Since we assume that \( X \langle Dg, h \rangle \) belongs to \( UC_b \) or \( D^{1,p} \), reasoning as in Sect. 4.3, we have that \( \frac{1}{\varepsilon} \mathbf{1}_{(r-\varepsilon,r)}(g) \, \mu \) converges to the measure \( \sigma_r \) concentrated on \( \{ g = r \} \). We have thus proved the thesis. \( \square \)

**Remark 4.11.** In the special case \( X = 1 \), formula (4.13) reads
\[ F_W(h)(r) = \int_{\{g < r\}} W(h) \, \mu(dx) = -\int_{\{g = r\}} \langle Dg, h \rangle_H \, \sigma_r(dx), \quad h \in H. \]
This is a sort of *divergence theorem* (in infinite dimensions) for the vector \( h \); we remark that similar results are already present in the literature, compare for instance [21].

Let us notice that \( Dg \) is explicitly known [see formula (3.8)], hence
\[ \langle Dg, h \rangle_H = \sum_{i=1}^{n} \int_{0}^{1} h_i(s) \int_{0}^{1} B_i(t) \, dt \, ds. \]
Let us define
\[ \tilde{h}(t) = \int_{0}^{1} (1 - (t \vee r))h(r) \, dr, \quad t \in (0,1). \]
Then it holds \( \langle Dg, h \rangle_H = W(\tilde{h}) \), which shows that this term belongs to \( D^{1,q} \) for every \( q > 1 \).

**Remark 4.12.** Since \( \langle Dg, h \rangle \in D^{1,q} \) for every \( q > 1 \), the requirement \( X \langle Dg, h \rangle \in D^{1,p} \), \( p > 1 \), is satisfied, for instance, requiring \( X \in UC_b^1 \) or \( X \in D^{1,\theta} \), for some \( \theta > p \).

5. **Existence of the surface measure for sets defined by the solution of gradient systems**

In this section we extend previous results to cover the case of a (multidimensional) gradient system SDE (see [22]). Let \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) be a potential energy function; we assume that
\[ V \in C^3_b(\mathbb{R}^n), \quad (5.1) \]
i.e., it is continuous and bounded together with its first three derivatives.
Then we define \( u \) to be the solution of the following equation:
\[ du(t) = -\nabla V(u(t)) \, dt + dB(t), \quad u_0 = 0 \in \mathbb{R}^n. \quad (5.2) \]
Under our assumptions, the solution \( u \) belongs to \( L^2(E, \mu; E) \) (notice that we can solve the equation in a pathwise sense).
Recall from Corollary 4.3 that $g(B)$ has a density function $f_1(r)$ that is continuous and bounded for $r > 0$. In this section we aim to study the same property for the random variable $g(u)$, where $u$ is the solution of the equation (5.2).

**Theorem 5.1.** The cumulative distribution function of $g(u)$ admits a probability density function

$$
\varphi_1(r) = \int_{\{g(B)=r\}} \rho_1(B)^{-1} \sigma_r(dx)
$$

where for every process $h \in L^2(E,\mu;E)$ we let $\rho_1(h)$ be the Girsanov’s density defined by

$$
\rho_1(h) = \exp \left( \int_0^1 \langle \nabla V(h(s)), dB(s) \rangle - \frac{1}{2} \int_0^1 |\nabla V(h(s))|^2 ds \right), \tag{5.3}
$$

and the support of $\sigma_r$ is concentrated on $\{g(B) = r\}$.

**5.1. Change of measure**

First, we notice that our assumptions on $V$ implies that

$$
\sup_{t \in [0,1]} \mathbb{E} \left( \exp \left( |\nabla V(u(t))|^2 \right) \right) < +\infty,
$$

therefore by [16, Theorem 10.14 & Proposition 10.17] we get that the process

$$
uu(t) = B(t) - \int_0^t \nabla V(u(s)) \, ds
$$

is a Brownian motion in $(E,\mathcal{E},\nu)$, where $\nu$ is a centered Gaussian measure such that

$$d\nu = \rho_1(\cdot) \, d\mu.$$

Let $F : E \to \mathbb{R}$ be a bounded and Borel function; then we have that

$$
\mathbb{E}_\nu[F(u)] = \mathbb{E}_\mu[F(B)].
$$

**Lemma 5.2.** The following representation of $\rho_1(u)$ holds:

$$
\rho_1(u) = \exp \left( V(u(1)) + \frac{1}{2} \int_0^1 |\nabla V(u(t))|^2 \, dt - \frac{1}{2} \int_0^1 \text{Tr}(\nabla^2 V(u(t))) \, dt \right). \tag{5.4}
$$

**Proof.** Let us compute the Itô differential of $V(u)$:

$$dV(u(t)) = \langle \nabla V(u(t)), [-\nabla V(u(t)) \, dt + dB(t)] \rangle + \frac{1}{2} \text{Tr}(\nabla^2 V(u(t))) \, dt$$

Therefore, using the integral form of previous differential and recalling that $u(0) = 0$, we get

$$V(u(1)) = -\int_0^1 |\nabla V(u(t))|^2 \, dt + \int_0^1 \langle \nabla V(u(s)), dB(s) \rangle$$

$$+ \frac{1}{2} \int_0^1 \text{Tr}(\nabla^2 V(u(t))) \, dt$$

We substitute this expression in (5.3) to get the thesis. □
Proposition 5.3. The mapping \( x \mapsto \rho_1(B)^{-1}(x) \) belongs to \( UC_b \).

Proof. By lemma 5.2 we can write
\[
\rho_1(B)^{-1} = \exp \left[ - \left( V(B(1)) + \frac{1}{2} \int_0^1 |\nabla V(B(t))|^2 \, dt - \frac{1}{2} \int_0^1 \text{Tr}(\nabla^2 V(B(t))) \, dt \right) \right].
\]

Then the assumption that \( V \in C^3_b(\mathbb{R}^n) \) implies that \( \rho_1(B)^{-1} \) is bounded.

Now, we exploit that \( B \) is the canonical Brownian motion on the Wiener space \((E, \mathcal{E}, \mu)\), hence
\[ B(t)(x) - B(t)(y) = x(t) - y(t); \]

notice again that the assumption on \( V \) implies that the mappings on \( E \) defined by
\[
\begin{align*}
  x \mapsto V(x(1)),
  x \mapsto \int_0^1 \text{Tr}[\nabla^2 V(x(t))] \, dt,
  x \mapsto \int_0^1 |\nabla V(x(t))|^2 \, dt
\end{align*}
\]
are Lipschitz continuous. Therefore, if \( \|x - y\|_\infty < \delta \), then \( |\rho_1(B)^{-1}(x) - \rho_1(B)^{-1}(y)| \leq e^{3L\delta} \) and the proof is complete. \( \square \)

5.2. The main result

We have now all the ingredients to prove Theorem 5.1. The proof of the existence of the density for \( g(u) \) can be obtained as a corollary to the results of Sect. 4.2. To see this, we propose the following computation.

Using Girsanov’s transform we have
\[
\begin{align*}
  \mu(g(u) \leq r) &= \mathbb{E}_\mu[\mathbb{1}_{\{g(u) \leq r\}}] = \mathbb{E}_\nu[\mathbb{1}_{\{g(u) \leq r\}} \rho_1(u)^{-1}] \\
  &= \mathbb{E}_\mu[\mathbb{1}_{\{g(B) \leq r\}} \rho_1(B)^{-1}].
\end{align*}
\]

More generally, it holds
\[
\Phi_X(r) = \int_{\{g(u) \leq r\}} X(x) \, \mu(dx) = F_{X\rho(B)^{-1}}(r).
\]

Lemma 5.4. The random variable \( g(u) \), defined on the space \((E, \mathcal{E}, \mu)\) with values in \( \mathbb{R} \), admits a probability density function with respect to the Lebesgue measure that is continuous and bounded.

Proof. Using Proposition 5.3 we are able to apply Theorem 1.1 to obtain that the distribution function \( \Phi_1(r) \) of \( g(u) \) admits a derivative
\[
\varphi_1(r) = f_{\rho_1(B)^{-1}}(r) = \int_E \rho_1(B)^{-1} \sigma_r(dx),
\]
where, as stated in Theorem 1.1 the support of the measure \( \sigma_r \) is concentrated on \( \{g(B) = r\} \). Now, the thesis follows from Proposition 4.5. \( \square \)

Next, we prove that there exists a surface measure on \( \{g(u) = r\} \) for \( r > 0 \) that is the restriction of the Gaussian measure \( \mu \) to the given surface. Proceeding as in Sect. 4 we obtain that
\[
\frac{1}{\epsilon_n} [\Phi_X(r + \epsilon_n) - \Phi_X(r)] = \int_E X(x) \frac{1}{\epsilon_n} \mathbb{1}_{\{r < g(u) \leq r + \epsilon_n\}} \, \mu(dx)
\]
and we can pass to the limit in previous formula, since the left hand side converges to \( \Phi'_X(r) = F_{X^{P_1(B)}_1}^\prime(r) \) by Proposition 4.7.

Therefore, by mimicking the procedure in Sect. 4 we get that the sequence of measures

\[
\theta_n := \frac{1}{\epsilon_n} \mathbb{1}_{ \{ r < g(u) \leq r + \epsilon_n \} } \mu(dx)
\]

converges to a measure \( \theta_r \) and this measure is concentrated on \( \{ g(u) = r \} \).

In particular,

\[
\varphi_1(r) = \int_{\{ g(u) = r \}} \theta_r(dx) = \theta_r(\{ g(u) = r \}).
\]

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