Rate of convergence in the maximum likelihood estimation for partial discrete parameter, with applications to the cluster analysis and philology.

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Abstract.

The problem of estimation of the distribution parameters on the sample when the part of these parameters are discrete (e.g. integer) is considered. We prove that the rate of convergence of MLE estimates under the natural conditions on the distribution density is exponentially fast.

We describe also the possible of the applications of the estimates offered in the cluster analysis and consequently in the technical diagnosis, demography and especially in philology.

\textit{Key words and phrases:} Maximum Likelihood Estimation (MLE), metric entropy by Kolmogorov, relative entropy of Kullback and Leibler, Hellinger’s entropy and integral, random process (field), stable distributions, tail of distribution, heavy tail distribution, exponential estimation for random fields and for sums of r.v., rate of convergence, random variables and vectors (r.v.), large deviations, cluster and cluster analysis, objective function, quasi-Gaussian distribution, contrast function, nuisance parameter, action function, mixture, density of distribution, Cartesian and polar coordinates.

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\section{Introduction. Statement of the problem. Notations. Definitions.}

Let $(\Omega, \mathcal{B}, P)$ be a probability space with the expectation $E$, and $(X, \mathcal{A}, \mu)$ be a measurable space with sigma-finite non-trivial measure $\mu$, and $\Theta$ be arbitrary locally compact topological space equipped by the ordinary Borelian sigma-field $G$, whereas $F = \{f\}$, $f = f(x, \theta)$, $\theta \in \Theta$, $x \in X$ be a family of a strictly positive (mod $\mu$) probabilistic densities:

$$\int_X f(x, \theta) \mu(dx) = 1, \ \theta \in \Theta$$

to be assumed continuous relative to the argument $\theta$ for almost all values $x; x \in X$.

We premise also the following natural condition of the identifiability condition:

$$\forall \theta_1, \theta_2 \in \Theta, \ \theta_1 \neq \theta_2 \Rightarrow \mu\{x : f(x, \theta_1) \neq f(x, \theta_2)\} > 0.$$  

Let further $\theta_0$ be some fixed value of the parameter $\theta$. We assume that the r.v. $\xi = \xi(\omega)$ is a random variable (r.v) (or more generally random vector) taking the values in the space $X$ with the density of distribution $f(x, \theta_0)$ relative to the measure $\mu$:

$$P(\xi \in A) = \int_A f(x, \theta_0) \mu(dx).$$

Statistically this means that r.v. $\xi$ is the (statistical) observation (or observations) with the density $f(x, \theta_0)$ relative to the measure $\mu$, where the value $\theta_0$ is the true, but in general case the values of the parameter are unknown.

The $\hat{\theta}$ denotes the Maximum Likelihood Estimation (MLE) of the parameter $\theta$ based on the observation $\xi$:

$$\hat{\theta} = \arg\max_{\theta \in \Theta} f(\xi, \theta), \quad (1.0)$$

or equally

$$\hat{\theta} = \arg\max_{\theta \in \Theta} L(\xi, \theta), \quad (1.1)$$

where the function

$$L = L(\xi, \theta) \overset{df}{=} \log[f(\xi, \theta)/f(\xi, \theta_0)]$$

is termed the contrast function, in contradiction to the function $\theta \rightarrow f(\xi, \theta)$ or $\theta \rightarrow \log f(\xi, \theta)$, which is called the ordinary Likelihood function.

In the case that $\hat{\theta}$ is not unique, we accept any arbitrary but measurable version of $\hat{\theta}$ as a capacity $\hat{\theta}$ provided it satisfies the condition (1.1.)

We will consider in the sequel only the case sample: $\xi = \vec{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}$, where $\xi_i, i = 1, 2, \ldots, n$ are independent identically distributed with density $f(x, \theta)$ with the true value of the parameter $\theta = \theta_0$. 

It is well known that if the set $\Theta$ is convex non-empty smooth submanifold of the whole space $\mathbb{R}^d$, $d = 1, 2, \ldots$ and the density $f(x, \theta)$ is in addition smooth (of a class $C^2$) function in relation to the parameter $\theta$, then for all sufficiently large values $n$ (volume of the sample) the MLE estimate $\hat{\theta} = \hat{\theta}_n$ based on the whole sample does exists. This estimate is asymptotically unbiased, asymptotically normal and is asymptotically effective with the speed of convergence $1/\sqrt{n}$; see for example [24].

Evidently, the MLE estimation $\hat{\theta}_n$ is the solution of system of equations

$$\sum_{i=1}^{n} \frac{\partial \log \left[ \frac{f(\xi_i, \theta)}{f(\xi_i, \theta_0)} \right]}{\partial \theta_k} = 0, \ k = 1, 2, \ldots, d. \quad (1.2)$$

or equally

$$\sum_{i=1}^{n} \frac{\partial \log \left[ \frac{f(\xi_i, \theta)}{f(\xi_i, \theta_0)} \right]}{\partial \theta_k} / (\theta = \hat{\theta}_n) = 0, \ k = 1, 2, \ldots, d. \quad (1.2a)$$

The non-asymptotical estimates for the probability of $\sqrt{n}$ deviation of a form

$$\sup_n P \left( \sqrt{n}||\hat{\theta}_n - \theta_0|| > u \right) \leq C_1 \exp \left( -C_2 u^\gamma \right), \ u \geq 2, \ \gamma = \text{const} > 0 \quad (1.3)$$

in the considered case was obtained in [2].

**We consider in this article the case when some part of the estimated parameters are discrete, (for the sake of definiteness, integer), and investigate the speed of convergence MLE estimation $\hat{\theta}_n$ to the true value $\theta_0$.**

To make the notations clearer, we accept several changes of notations.

$$\Theta = [0, 1, \ldots, N] \times \mathcal{B}, \quad (1.4)$$

where $N \in \{1, 2, \ldots, \infty\}$, $\mathcal{B} = \{\beta\}$ is arbitrary separable compact topological space equipped by the ordinary Borelian sigma-field.

Let the point $\theta_0 = (0, \beta_0)$ be the true value of the parameter $\theta$, so that the sample $\xi = \tilde{\xi} = \{\xi_i\}, \ i = 1, 2, \ldots, n$ consists on the i., i.d. r.v. $\xi_i$ with the density $f = f(x, \theta_0) = f(x, 0, \beta_0)$.

The MLE estimate $\hat{\theta}_n$ of the parameter $\theta$ will denoted by $(\hat{\tau}_n, \hat{\beta}_n)$:

$$(\hat{\tau}_n, \hat{\beta}_n) := \arg\max_{m, \beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \log [f(\xi_i, m, \beta)] \right\} = \arg\max_{m, \beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \log \left[ \frac{f(\xi_i, m, \beta)}{f(\xi_i, 0, \beta_0)} \right] \right\}. \quad (1.5)$$

Note that the discrete parameter estimates, without nuisance parameters, was considered in many works; see for example, [16], [22], [35], [36] etc.

The example of these statement of problem is described in the articles [1], [3], where some problems of cluster analysis are considered, as well as their applications in technical diagnosis, demography and especially in philology.
In these cases the number of clusters acts as a discrete parameters.

Namely, there are some grounds to accept the function \( f(x; \bar{\theta}) \) as a density in technical diagnosis, demography, philology etc. \([1], [3]\), whereas the density has a form of the so-called mixed quasy-Gaussian distribution (1.3)

\[
f(x; \bar{\theta}) = \sum_{k=1}^{N} W_k G \left( x_1 - a^{(k)}_1, x_2 - a^{(k)}_2, \ldots, x_d - a^{(k)}_d; \vec{\alpha}^{(k)}, \{\sigma_j^{(k)}\}, \vec{C}_1^{(k)} \right).
\]

Here

\[
\bar{\theta} = \theta \overset{\text{def}}{=} \{N; \{\vec{a}_d\}, \{\vec{\sigma}_j\}, \{\vec{C}_1^{(k)}\}\}, \ d = \dim X, \ j, k = 0, 1, 2, \ldots, N,
\]

\(W_k, \ k = 1, 2, \ldots, N\) be positive numbers (weights) such that \(\sum_{k=1}^{N} W_k = 1\).

The quasi-Gaussian distribution was defined as follows.

We denote as trivial for any measurable set \(A, A \subset \mathbb{R}\) its indicator function by \(I_A(x) = I_A(x)\):

\[
I_A(x) = 1, \ x \in A; \quad I_A(x) = 0, \ x \notin A.
\]

Let us introduce a family of functions

\[
\omega_\alpha(x) = \omega_\alpha(x; C_1, C_2) := C_1 |x|^{\alpha(1)} I_{(-\infty,0)}(x) + C_2 x^{\alpha(2)} I_{(0,\infty)}(x),
\]

\(x \in \mathbb{R}, \ C_{1,2} = \text{const} \geq 0, \ \alpha = \vec{\alpha} = (\alpha(1), \alpha(2)), \ \alpha(1), \alpha(2) = \text{const} > -1,\)

so that \(\omega_\alpha(0) = 0\), and a family of a correspondent probability densities of a form

\[
g_{\alpha,\sigma}(x) = g_{\alpha,\sigma}(x; C_1, C_2) \overset{\text{def}}{=} \omega_\alpha(x; C_1, C_2) f_\sigma(x),
\]

\[
f_\sigma(x) = (2\pi)^{-1/2} \sigma^{-1} \exp \left(-x^2/(2\sigma^2)\right).
\]

Since

\[
I_{\alpha(k)}(\sigma) := \int_{0}^{\infty} x^{\alpha(k)} \exp \left(-x^2/(2\sigma^2)\right) \ dx = 2^{(\alpha(k)-1)/2} \sigma^{(\alpha(k)+1)} \Gamma((\alpha(k) + 1)/2),
\]

where \(\Gamma(\cdot)\) is ordinary Gamma function, there is the interrelation between the constants \(C_1, C_2:\)

\[
C_1 I_{\alpha(1)}(\sigma) + C_2 I_{\alpha(2)}(\sigma) = \sigma \ (2\pi)^{1/2},
\]

has only one degree of freedom. In particular, the constant \(C_1\) may be equal to zero; in this case the r.v. \(\xi\) possess only non-negative values.

We will denote in the sequel by \(C_i, K_j\) some finite non-negative constants that are not necessary to be the same in different places.
The one-dimensional distribution of a r.v. ξ with density function of a form

\[ x \rightarrow g_{\alpha,\sigma}(x - a; C_1, C_2), \ a = \text{const} \in \mathbb{R} \]

is said to be quasi-Gaussian or equally quasi-normal. Notation:

\[ \text{Law}(\xi) = QN(a, \alpha, \sigma, C_1, C_2). \]

Let us explain the "physical" sense of introduced parameters of these distributions. The value "a" may be named quasi-center by analogy with normal distribution; the value "α" expresses the degree of concentration of this distribution about the center and the value of "σ" which may be called quasi-standard of the r.v. ξ expressed alike in the classical Gaussian r.v. the degree of scattering.

Many properties of these distributions are previously studied in [1]: moments, bilateral tail behavior etc. In particular, it is proved that if the r.v. (ξ, η) are independent and both have the quasi-Gaussian distribution with parameters \( a = 0, b = 0 \) ("quasi-centered" case):

\[ \text{Law}(\xi) = QN(0, \alpha, \sigma, C_1, C_2), \quad \text{Law}(\eta) = QN(0, \beta, \sigma, C_3, C_4) \]

may occur with different parameters \( \alpha \neq \beta, C_1 \neq C_3, C_2 \neq C_4 \) but with the same value of the standard σ, \( \sigma > 0 \), then their polar coordinates \( (\rho, \zeta) \) are also independent.

The opposite conclusion was also proved in [1]: the characterization of quasi-Gaussian distribution in the demography and philology: if the polar and Decart (cartesian) coordinates are independent, then under some natural conditions the random variables ξ, η have quasi-Gaussian distribution, and is explained why this property denotes this distribution of the words parameters in many languages.

It is possible to generalize our distributions on the multidimensional case. Actually, let us consider the random vector \( \xi = \vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_d) \) with the density

\[ f_\xi(x_1, x_2, \ldots, x_d) = G(x_1, x_2, \ldots, x_d; \vec{\alpha}, \vec{\sigma}, \vec{C}_1, \vec{C}_2) \overset{\text{def}}{=} \prod_{j=1}^{d} g_{\alpha_j,\sigma_j}(x_j; C_{1j}, C_{2j}), \]

where \( \alpha_j > -1, \sigma_j = \text{const} > 0, C_{ij} = \text{const} \geq 0 \),

\[ C_{1j} I_{\alpha(1)}(\sigma_j) + C_{2j} I_{\alpha(2)}(\sigma_j) = \sigma_j (2\pi)^{1/2}, \]

An important note: during the investigation of these discrete estimates the so-called Large Deviations Principle (LDP) was used [17], [20], [25], [52]; [12], [15], [18], [19], [43], [44] etc.

We introduce some notations. Let \( f = f(x) \) and \( g = g(x) \) be two densities in relation to the measure \( \mu \), i.e. measurable non-negative functions such as

\[ \int_{\mathcal{X}} f(x) \, d\mu = \int_{\mathcal{X}} g(x) \, d\mu = 1. \]
A relative entropy by Kullback and Leibler [32] $H_r(f;g)$ is defined as ordinary by the equality

$$H_r(f;g) = \int_X f(x) \ln \left( \frac{f(x)}{g(x)} \right) \mu(dx). \quad (1.6)$$

It is well known that $H_r(f;g) \geq 0$ and $H_r(f;g) = 0$ iff $f(x) = g(x)$ almost everywhere.

Let $f = f(x)$, $g = g(x)$ and $h = h(x)$ be three densities relative the measure $\mu$. We define a three term relative entropy $H_R(f;g,h)$ as follows:

$$H_R(f;g,h) = \int_X f(x) \ln \left[ \frac{g(x)}{h(x)} \right] \mu(dx). \quad (1.7)$$

Evidently, $H_R(f;f,h) = H_R(f;h)$, $H_R(f;g,h) = -H_R(f;h,g)$.

Entropy by Hellinger or Hellinger’s integral defined for any real number $\lambda \in \mathbb{R}$ and two densities $f(x)$, $g(x)$ is by definition the following integral (if there exists)

$$H_{ell}(\lambda; f,g) := \int_X f^{\lambda}(x) \ g^{1-\lambda}(x) \mu(dx).$$

Hellinger [7] originally introduced this concept for the value $\lambda = 1 - \lambda = 0.5$. The general notion was for the first time introduced most likely in [2].

This notion is closely related with the so-called R’enyi and Tsallis divergences, see [47], [51]. The consistent statistical estimation of $H_{ell}(\lambda; f,g)$ is obtained, e.g. in [31]; see also reference therein.

We offer here a slight modification of this notion, namely, a three term Hellinger’s integral:

$$H_{ell}^{(3)}(\lambda; f,g,h) := \int_X f^{\lambda}(x) \ g^{-\lambda}(x) \ h(x) \mu(dx), \quad (1.8)$$

where $\lambda \in \mathbb{R}$, $f, g, h$ be three densities. Of course, $H_{ell}^{(3)}(\lambda; f,g,g) = H_{ell}(\lambda; f,g)$.

At last, let $(T,d)$ be a metric space equipped with distance $d = d(t,s)$, $t, s \in T$. The entropy by Kolmogorov [29] $H(T,d,\epsilon)$ is named the natural logarithm of the minimal numbers of closed balls $B(y,\epsilon)$, $y \in T, \epsilon \in (0,\infty)$ in the distance $d$ which cover all the set $T$.

Obviously, $\forall \epsilon > 0 \Rightarrow H(T,d,\epsilon) < \infty$ iff the set $T$ is precompact set relative the distance $d$.

2 Main result: exponential convergence for discrete parameter.

We need to introduce some notations and conditions. $\xi := \xi_1$,

$$Q_n = Q_n(\theta) := \mathbf{P}(\hat{\tau}_n \neq 0) = \mathbf{P}(\hat{\tau}_n \geq 1). \quad (2.0)$$

This probability for confidence interval (confidence probability) play a very important role in our considerations.
Also, let us admit

\[
a(\theta) = a(m, \beta) := \mathbb{E} \ln \left[ \frac{f(\xi_i, m, \beta)}{f(\xi_i, 0, \beta_0)} \right] = \\
\int_X \ln \left[ \frac{f(x, m, \beta)}{f(x, 0, \beta_0)} \right] \cdot f(x, 0, \beta_0) \mu(dx).
\]

The function \(-a = -a(\theta) = -a(m, \beta)\) is relative entropy of the density \(f(x, m, \beta)\) in relation to other density \(f(x, 0, \beta_0)\):

\[
-a(m, \beta) = H_r(f(x, 0, \beta_0); f(x, m, \beta)) = H_r(f(x, \theta_0); f(x, \theta)).
\]

Therefore \(\forall m \geq 1 \ a(m, \beta) < 0\). Obviously, \(a(0, \beta_0) = 0\).

We suppose in addition the function \(\theta \to a(\theta)\) there exists and is continuous:

\[
a(\cdot) \in C(\Theta_1); \text{ and we denote by } \Theta_1 \text{ the (closed) subspace of the space } \Theta \text{ of the form}
\]

\[
\Theta_1 = \{1, 2, \ldots, N\} \otimes \mathcal{B}.
\]

Moreover, we assume the following random processes (field) (r.f.)

\[
\eta(m, \beta) := \eta_1(m, \beta) = \eta(\theta), \ \theta \in \Theta_1 = \{m, \beta\}, \ m = 1, 2, \ldots, N
\]

belong also to the space of all continuous functions \(C(\Theta_1)\) equipped by ordinary norm:

\[
\forall g(\cdot) \in C(\Theta_1) \quad \|g\| = \max_{\theta \in \Theta_1} |g(\theta)|.
\]

Let \(Z = Z(A)\) be an element of conjugate space \(C^*(\Theta_1)\), i.e. countable additive signed measure defined on the Borelian sigma-field \(G_1\) (charge) with finite variation, which we denote by

\[
\|Z\| = ||Z||C^* = ||Z||C^*(\Theta_1) = \text{Variation}(Z).
\]

We postulate the finiteness of the logarithm of generating function \(\Psi(\cdot)\) for all the charges \(Z = Z(\cdot)\)

\[
\Psi(Z) := \ln \mathbb{E} \exp \left( \int_{\Theta_1} \eta(\theta) \ Z(d\theta) \right) < \infty.
\]

Moreover, we impose the classical in the theory of great deviations condition:

\[
\forall t \in (0, \infty) \Rightarrow \mathbb{E} \exp (t||\eta(\cdot)||) < \infty.
\]

The following function, which usually called action function, plays a very important role in the theory of great deviations is defined by the Young-Fenchel, or Legendre transform of \(\Psi(\cdot)\) in the space \(C^*(\Theta_1)\), which we will denote also by \(\Psi^*:\)

\[
I(g) := \sup_{Z \in C^*(\Theta_1)} \left\{ \int_{\Theta_1} g(\theta) \ Z(d\theta) - \Psi(Z) \right\} =
\]
\[
\sup_{Z \in C^*(\Theta_1)} \{(g(\cdot), Z(\cdot)) - \Psi(Z)\} = \Psi^*(g). \tag{2.5}
\]

We denote also by \( U = C^+(\Theta_1) \) the set of all continuous functions \( \Theta \rightarrow R \) where
\[
g \in U \ (= C^+(\Theta_1)) \iff \max_{\theta \in \Theta_1} g(\theta) \geq 0. \tag{2.6}
\]

Evidently, \( U \) is closed set in the space \( C(\Theta_1) \), and we denote by \( U^o \) its interior:
\[
U^o = \{ g, \max_{\theta \in \Theta_1} g(\theta) > 0 \}. \tag{2.7}
\]

**Theorem 2.1.** Let the listed above conditions are fulfilled. Suppose in addition
\[
\lim_{||Z||C^* \rightarrow \infty} \Psi(Z)/||Z||C^* = \infty. \tag{2.8}
\]
Then
\[
- \inf_{g \in U^o} \Psi^*(g) \leq \lim_{n \rightarrow \infty} n^{-1} \ln Q_n \leq \lim_{n \rightarrow \infty} n^{-1} \ln Q_n \leq - \inf_{g \in U} \Psi^*(g). \tag{2.9}
\]

**Proof** is the same as in the article of Choirat Ch. and Seri R. [16], where the case of the complete discrete parametric space \( \Theta \) is considered. We need only to replace the finite-dimensional LDP (Large Deviation Principle) used in [16] by infinite-dimensional version one, see e.g. [15], [43], [45].

Several details. Let us consider the partial sum denoting it as follows:
\[
S_n(m, \beta) = \sum_{i=1}^{n} \eta_i(m, \beta) : \quad Q_n = P \left( \sup_{m \geq 1} \sup_{\beta} S_n(m, \beta) > 0 \right) = P \left( \sup_{m \geq 1} \sup_{\beta} \frac{S_n(m, \beta)}{n} > 0 \right) = \quad P (S_n(\cdot, \cdot)/n \in U^o). \tag{2.10}
\]

It is easy to verify that all the conditions for LDP in the space \( C(\Theta_1) \) are satisfied.

This completes the proof of Theorem 2.1.

**Corollary 2.1.** Assume that the density \( f(x, \theta) \) and the space \( \Theta_1 \) are such as
\[
\inf_{g \in U^o} \Psi^*(g) = \inf_{g \in U} \Psi^*(g). \tag{2.11}
\]
Then obviously
\[
\lim_{n \rightarrow \infty} n^{-1} \ln Q_n = - \inf_{g \in U} \Psi^*(g). \tag{2.12}
\]

We will prove in the next section in particular that both the inequalities in theorem 2.1 are in general case non-trivial.
3 Non-asymptotical estimates.

A. Lower bound.

We have:

\[ Q_n = \mathbb{P} \left( \sup_{m \geq 1} \beta \sum_{i=1}^{n} \eta_i^{(0)}(m, \beta) + a(m, \beta) > 0 \right) \geq \]

\[ \sup_{m \geq 1} \beta \mathbb{P} \left( \sum_{i=1}^{n} \eta_i^{(0)}(m, \beta) > 0 \right) = \sup_{m \geq 1} \beta \mathbb{P} \left( \sum_{i=1}^{n} \eta_i^{(0)}(m, \beta) > n \mid a(m, \beta) \right) \]

\[ \overset{\text{def}}{=} \sup_{m \geq 1} \beta Q_n^{(m, \beta)}, \quad (3.1) \]

where

\[ Q_n^{(m, \beta)} = \mathbb{P} \left( \sum_{i=1}^{n} \eta_i^{(0)}(m, \beta) > n \mid a(m, \beta) \right). \quad (3.2) \]

For the lower estimates of the variable \( Q_n^{(m, \beta)} \) we can apply the one-dimensional LDP, see for example the book of O.Kallenberg [25], p. 538 - 541.

To implement this plan we will use the (generalized) Hellinger’s integral (entropy) \( H_{\text{ell}}(\lambda; f, g) \), \( \lambda = \text{const.} \). We observe that the deviation function \( \Lambda(\lambda) = \Lambda(\lambda; m, \beta) \) for the sequence \( \eta_i(m, \beta) \) is closely related to Hellinger’s integral:

\[ \exp \Lambda(\lambda; m, \beta) = \mathbb{E} \exp(\lambda \eta(\lambda; m, \beta)) = \mathbb{E} \exp(\lambda \log(f(\xi; m, \beta)/f(\xi; 0, \beta_0))) = \]

\[ \int_{X} f(x; 0, \beta_0) \cdot \exp(\lambda \log(f(\xi; m, \beta)/f(\xi; 0, \beta_0))) \ d\mu = \]

\[ \int_{X} f^{\lambda}(x; m, \beta) f^{1-\lambda}(x; 0, \beta_0) \ d\mu = H_{\text{ell}}(\lambda; f(x; 0, \beta_0), f(x; m, \beta)). \]

Therefore as \( n \rightarrow \infty \)

\[ \ln Q_n^{(m, \beta)} = -n(1 + o(1)) \Lambda^*(0; m, \beta) \]

and following

\[ \ln Q_n \geq -n(1 + o(1)) \inf_{m \geq 1, \beta} \Lambda^*(0; m, \beta) \geq -1.5 \inf_{m \geq 1, \beta} \Lambda^*(0; m, \beta). \quad (3.3) \]

B. Upper bound.

We need to introduce some new notations.

\[ \Delta H = H_R = H_R(m_1, \beta_1; m_2, \beta_2) = H_R(\theta_1; \theta_2) = \]

\[ \Delta H(m_1, \beta_1; m_2, \beta_2) := H_r(f(x; m_2, \beta_2); f(x; 0, \beta_0)) - H_r(f(x; m_1, \beta_1); f(x; 0, \beta_0)); \]
then $\Delta H =$

$$
\int_X f(x; 0, \beta_0) \cdot \ln \left[ \frac{f(x; 0, \beta_0)}{f(x, m_1, \beta_1)} \right] \, d\mu - \int_X f(x; 0, \beta_0) \cdot \ln \left[ \frac{f(x; 0, \beta_0)}{f(x, m_2, \beta_2)} \right] \, d\mu =
$$

$$
\int_X f(x; 0, \beta_0) \cdot \ln \left[ \frac{f(x; m_2, \beta_2)}{f(x, m_1, \beta_1)} \right] \, d\mu = H_R(f(\cdot; 0, \beta_0); f(\cdot; m_2, \beta_2), f(\cdot; m_1, \beta_1)). \quad (3.4)
$$

Further, let us introduce the following functions:

$$
\phi = \phi(\lambda; \theta_1, \theta_2), \quad \bar{\phi} = \bar{\phi}(\lambda; \theta_1, \theta_2),
$$

$$
\nu = \nu(\lambda), \quad \gamma = \gamma(\lambda; \theta_1, \theta_2), \quad \gamma = \gamma(\lambda; \theta_1, \theta_2),
$$

$$
\lambda \in \mathbb{R}, \quad \theta_{1,2} \in \Theta_1,
$$
as follows:

$$
\phi(\lambda, \theta) := \lambda H_r + \ln \left[ H_{all}(\lambda; f(x, m, \beta), f(x, 0, \beta_0)) \right]; \quad (3.5)
$$

$$
\bar{\phi}(\lambda, \theta) := \sup_n \left[ n \phi(\lambda/\sqrt{n}, \theta) \right]; \quad \nu(\lambda) := \sup_{\theta \in \Theta_1} \bar{\phi}(\lambda, \theta); \quad (3.6)
$$

$$
\gamma(\lambda; \theta_1, \theta_2) := \lambda H_R(\theta_1, \theta_2) + \ln H_{all}^{(3)}(\lambda; f(x, 0, \beta_0), f(x, m_1, \beta_1), f(x, m_2, \beta_2)); \quad (3.7)
$$

$$
\gamma(\lambda; \theta_1, \theta_2) := \sup_n \left[ n \gamma(\lambda/\sqrt{n}; \theta_1, \theta_2) \right].
$$

The function $\bar{\phi}(\lambda, \theta)$ and analogously $\gamma(\lambda, \theta)$ means as will be described. If the centered random variable $Y$ is such that

$$
\mathbb{E} e^{\lambda Y} \leq e^{\phi(\lambda)},
$$

and $\{Y_j\}$ are independent copies of $Y$, then

$$
\sup_n \mathbb{E} e^{\lambda n^{-1/2} \sum_{j=1}^n Y_j} \leq e^{\bar{\phi}(\lambda)}. \quad (3.8)
$$

It will be assumed later that the function $\nu(\cdot)$, and as well as the function $\gamma(\cdot; \cdot, \cdot)$ are finite at least in some non-trivial neighborhood of origin:

$$
\exists \lambda_0 > 0 \Rightarrow \nu(\lambda_0) < \infty. \quad (3.8)
$$

We will accept

$$
\lambda_0 = \sup \{ z : \nu(z) < \infty \},
$$
as a capacity of the value $\lambda_0$ its maximal value; may be $\lambda_0 = \infty$. 

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Let us define a distance \( d = d(\theta_1, \theta_2) \) on the set \( \Theta_1 \) as follows:

\[
d(\theta_1, \theta_2) := \sup_{\lambda: 0 < \lambda < \lambda_0} \left[ \frac{\nu^{-1}(\pi(\lambda; \theta_1, \theta_2))}{\lambda} \right],
\]

so that

\[
\pi(\lambda; \theta_1, \theta_2) \leq \nu(\lambda \cdot d(\theta_1, \theta_2)).
\]

Further, define

\[
G(\delta) := \sum_{m=1}^{\infty} \delta^{m-1} H(\Theta_1, d, \delta^m).
\]

\[
H_r := H_r(f(x; 0, \beta_0); f(x, m, \beta));
\]

\[
M(u) := \inf_{\delta \in (0,1)} [G(\delta) - \gamma^*(H_r \cdot (1 - \delta))],
\]

where \( \gamma^*(\cdot) \) denotes the classical Young-Fenchel, or Legendre transform for the function \( \gamma(\cdot) : \)

\[
\gamma^*(u) = \sup_{z > 0} (uz - \gamma(z)).
\]

**Theorem 3.1.** Assume that \( H_r > 0 \) and that \( G(\delta) < \infty, \delta \in (0,1) \). Then

\[
Q_n \leq \exp\{-M(H_r \cdot \sqrt{n})\}.
\]

**Corollary 3.1.** Suppose in addition \( M(u) \geq K \cdot u^2 \) for all sufficiently large values \( u; u \geq u_0 = \text{const} > 0 \). Then it follows from the assertion of theorem 3.1 the exponential non-asymptotical estimation for \( Q_n : \)

\[
Q_n \leq \exp\{-KH_r^2 \cdot n\}, \ n \geq n_0.
\]

**Remark 3.1.** The condition \( H_r > 0 \) is automatically satisfied if for example the set \( \Theta_1 \) is compact set relative to the distance \( d \).

The second condition \( G(\delta) < \infty \) in turn is satisfied if the space \( \Theta_1 \) has finite dimension relative the distance \( d \).

**Proof of theorem 3.1.** Note that as before

\[
Q_n = P \left( \max_{\theta \in \Theta_1} \sum_{i=1}^{n} \eta_i(\theta) > 0 \right) = P \left( \max_{\theta \in \Theta_1} \left( \sum_{i=1}^{n} \eta_i^0(\theta) - H_r(\theta) \right) > 0 \right) \leq
\]

\[
P \left( \max_{\theta \in \Theta_1} \sum_{i=1}^{n} \eta_i^0(\theta) > nH_r \right) = P \left( \max_{\theta \in \Theta_1} n^{-1/2} \sum_{i=1}^{n} \eta_i^0(\theta) > \sqrt{n}H_r \right).
\]
Let us introduce the centered random field (more exactly, the sequence of centered random fields)
\[ \zeta_n(\theta) = n^{-1/2} \sum_{i=1}^{n} \eta_i^0(\theta), \; \theta \in \Theta_1; \]
then
\[ Q_n \leq P \left( \max_{\theta \in \Theta_1} \zeta_n(\theta) > \sqrt{nH_r} \right). \]

The exact exponential bounds for tail of distribution of maximum for random fields may be found, e.g. in [41]; see also [40], chapter 2. We have:
\[ E e^{\lambda \eta_0} = E e^{\lambda(\eta + H_r)} = e^{\lambda H_r} E \exp \left( \lambda \ln \left( \frac{f(\xi, m, \beta)}{f(\xi, 0, \beta_0)} \right) \right) = \]
\[ e^{\lambda H_r} \int_X f(x, 0, \beta_0) \cdot f^\lambda(x, m, \beta) f^{-\lambda}(x, 0, \beta_0) \mu(dx) = \]
\[ e^{\lambda H_r} H_{\text{ell}}(\lambda; f(x, m, \beta), f(x, 0, \beta_0)) = e^{\phi(\lambda, \theta)}. \]

Therefore,
\[ E e^{\lambda \zeta_n(\theta)} \leq e^{n \phi(\lambda, \theta)} \leq e^{\sup_n \left[ n \phi(\lambda, \theta) \right]} = e^{\phi(\lambda, \theta)} \leq e^{\nu(\lambda)}. \quad (3.13) \]

Let us estimate the exponential moment for the difference
\[ \Delta \eta = \Delta \eta(\theta_1, \theta_2) = \eta(\theta_1) - \eta(\theta_2) = H_R(f(\xi, \theta_0), f(x, \theta_2), f(x, \theta_1)) + \ln \left[ \frac{f(\xi; \theta_1)}{f(\xi; \theta_2)} \right]. \]
Thus
\[ E e^{\lambda \Delta \eta} = e^{\lambda H_R} H_{\text{ell}}^{(3)}(\lambda; f(x, \theta_0), f(x, \theta_1), f(x, \theta_2)) = e^{\gamma(\lambda, \theta_1, \theta_2)}. \]

Following,
\[ E e^{\lambda \zeta_n(\theta_1 - \zeta_n(\theta_2))} \leq e^{\gamma(\lambda, \theta_1, \theta_2)}. \]

It follows immediately from the direct definition of the $d$ - distance that
\[ E e^{\lambda \zeta_n(\theta_1 - \zeta_n(\theta_2))} \leq e^{\nu(\lambda, d(\theta_1, \theta_2))}. \quad (3.14) \]

The inequalities (3.13) and (3.14) may be rewritten on the language $B(\phi)$ spaces, see [30], [40], chapter 1, as follows
\[ \sup_n \sup_{\theta \in \Theta_1} \| \zeta_n(\theta) \|_{B(\nu)} \leq 1, \quad (3.15a) \]
\[ \sup_n \| \zeta_n(\theta_1) - \zeta_n(\theta_2) \|_{B(\nu)} \leq d(\theta_1, \theta_2). \quad (3.15b) \]
It remains to apply the main result of [40], chapter 3, section 3.4.
4 Examples.

A. Regular case. If \( \xi_i \) are Gaussian distributed with parameters \( m = \mathbb{E}\xi_i = 0, 1, \ldots \) and \( \beta = \text{Var} \xi_i \geq 1 \) we conclude:

\[
Q_n = 1 - \Phi(\sqrt{n}), \quad \Phi(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-y^2/2) \, dy.
\]

As a consequence: there holds for suitable greatest values \( n \), for instance, \( n \geq 4 \)

\[
(2\pi)^{-1/2} n^{-1/2} e^{-n/2} \left( 1 - \frac{c_-}{n} \right) \leq Q_n \leq (2\pi)^{-1/2} n^{-1/2} e^{-n/2} \left( 1 + \frac{c_+}{n} \right).
\]

holds true for suitable greatest values \( n \), for instance, \( n \geq 4 \)

We will prove further that if the conditions of theorem 2.1 are not satisfied, the speed of convergence \( Q_n \to 0 \) may differ from the exponential.

Obviously, if for all the values \( m = 1, 2, \ldots N \) \( f_0(x) \neq f_m(x) \) on the set of positive measure, then \( \lim_{n \to \infty} Q_n = 0 \).

B. Stretched exponential random variables.

The distribution of a r.v. \( \xi \) for which

\[
c_1(t) \exp(-b(t)t^r) \leq P(\xi > t) \leq c_2(t) \exp(-b(t)t^r),
\]

\[
c_3(t) \exp(-b(t)t^r) \leq P(\xi < -t) \leq c_4(t) \exp(-b(t)t^r), \tag{4.1}
\]

\[ t > t_0 = \text{const} > 0, \quad r = \text{const} \in (0, 1), \]

where \( c_k(t), \, b(t) \) are positive continuous slowly varying functions, is named in the article [21] stretched exponential distribution.

Let for definiteness \( c_k(t) = \text{const} > 0, \, b(t) = 1 \). Denote \( m = \mathbb{E} \xi, \, \xi_i \) be independent copies \( \xi \). It is proved in particular in [21] that

\[
\lim_{n \to \infty} \frac{1}{n} \ln P \left( \sum_{i=1}^{n} \xi_i > x \right) = -(x - m)^r, \quad x > m. \tag{4.2}
\]

See also earlier publication of Nagaev S.V. [38].

Assume in addition that the r.v. \( \xi \) has a positive even density \( f_0 = f_0(x) = f_0(|x|) \), so that \( \xi_i \) are symmetrically distributed and hence \( m = 0 \).

Introduce a second density \( f_1 = f_1(x) = f_1(|x|) \) as follows:

\[
f_1(x) = C \, e^{-|x|} f_0(x), \quad \int_{\mathbb{R}} f_1(x) \, dx = 1,
\]

and consider the following estimation problem \( \Theta = \{0, 1\}, \, \Theta_1 = \{1\} \), in other words, testing of statistical hypotheses. It follows from the cited main result of [21] that

\[
\ln Q_n = -C_0 n^r (1 - o(1)), \quad n \to \infty, \quad C_0 = \text{const} \in (0, \infty). \tag{4.3}
\]
Note that in the article [21] is considered the case of weighted sums of independent random variables; see also [26].

C. Random variables with heavy/power tails.

We consider in this subsection the case when $\xi$, $\{\xi_k\}$ have symmetrical (even) density and are i.i.d. so as for some $p = \text{const} > 2$ ⇒

$$|\xi|^p \overset{\text{def}}{=} E|\xi|^p < \infty.$$  

For instance,

$$f_0(x) = \frac{C_0(p)}{(1+|x|^{p+1})(\ln(e+|x|)^2), \quad \int_R f_0(x) \, dx = 1.}$$

We denote also $a = E|\xi_k| = \text{const} < \infty$. Let, as before, in any case the alternative density $f_1 = f_1(x) = f_1(|x|)$ looks like

$$f_1(x) = C \, e^{-|x|} \, f_0(x), \quad \int_R f_1(x) \, dx = 1.$$  

Then

$$Q_n = P \left( \sum_{k=1}^n \ln \frac{f_1(\xi_k)}{f_0(\xi_k)} > 0 \right) = P \left( \sum_{k=1}^n (b - |\xi_k|) > 0 \right) =$$

$$P \left( \sum_{k=1}^n (|\xi_k| - a) < -nd \right), \quad d = \text{const} > 0.$$  

We deduce the following inequality using the Rosenthal’s and Tchebychev’s inequalities

$$Q_n \leq C_R^p \frac{|\xi|^p}{n^{p/2} \, d \ln^p p}, \quad (4.4)$$

where $C_R \approx 1.773682$ is the absolute known constant, see [5].

The more exact estimate for $Q_n$ in the case when $p > 4$ may be obtained from the famous theorem of Baum and Katz [14] under at the same condition $|\xi|^p < \infty$:

$$Q_n \leq C_3(p, |\xi|^p),$$

but with an unknown constant $C_3 = C_3(p, |\xi|^p)$.

D. Random variables from Grand Lebesgue spaces.

Let $\psi = \psi(p)$, $p \in (a, b)$, $a = \text{const} \geq 2$, $b = \text{const} \in (a, \infty)$ be continuous positive function so that the function $p \to p \cdot \log \psi(p)$ would be convex.

In the case when $b = \infty$ we impose on the function $\psi(\cdot)$ in addition the restriction

$$\lim_{p \to \infty} \psi(p) = \infty.$$  

For instance, $\psi(p) = p^l$, $l = \text{const} > 0$ or $\psi(p) = p^{l_1} \log^{l_2}(p), l_j = \text{const}$, $l_1 > 0$. 

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The Banach space $G_\psi$ consists by definition on all the r.v. $\{\eta\}$ defined on the fixed probability space with finite norm

$$||\eta||_{G_\psi} := \sup_{p \in (a,b)} \left[ \frac{|\eta|_p}{\psi(p)} \right].$$

(4.5)

This spaces were introduced in [30], more detailed investigation of these spaces may be found in the monograph [40], chapters 1,2.

We define for all such a function $\psi(\cdot)$ a new functions

$$\psi_1(p) = C_R d^{-1} \psi(p) p/ \ln p, \ p \in (a,b);$$

$$\psi_2(p) = p \ln \psi_1(p); \ \psi_3(p) = \psi_2^*(z) = \sup_{p \in (a,b)} (pz - \psi_2(p)).$$

(4.6)

The operator $\psi \to \psi^*$ is called the Young-Fenchel, (it is finite, in general case) or Legendre transform.

We suppose as a continuation of the subsection C that the random variables $\xi, \xi_k$ belong to some $G_\psi$ space. We can, for instance, choose this function $\psi(\cdot)$ by a so-called natural way:

$$\psi(p) := |\xi|_p,$$

if, of course, $|\xi|_p < \infty$ for any value $p, \ p > 2$. As a capacity of the boundaries $a, b$ we put

$$a := 2, \quad b := \sup\{p : \psi_p < \infty\},$$

may be $b = \infty$. Naturally, in this case $||\xi||_{G_\psi} = 1$.

We assert if $\xi \in G(\psi)$ :

$$Q_n \leq \exp \left\{ -\psi_3[\ln(\sqrt{n}/||\xi||_{G_\psi})] \right\},$$

(4.7)

the so-called subexponential estimate.

Indeed, it follows from the direct definition of the $G_\psi$ norm

$$|\xi|_p \leq \psi(p) ||\xi||_{G_\psi}, \ p \in (a,b).$$

We deduce after substituting into (4.4)

$$Q_n \leq \frac{\psi_2^p(p)}{np^{p/2}} ||\xi||_{G_\psi}^p = \exp\{ -[p \ln(\sqrt{n}/||\xi||_{G_\psi}) - p \ln \psi_2(p)] \}.$$

It remains to take the minimum over $p$.

**E. Stable distributed variables.**

Let now $\xi, \xi_k, \ k = 1,2,\ldots,n$ be i.i.d. random variables with the density $f_0 = f_0(x) = f_0(|x|)$ with symmetric stable distribution:

$$\mathbb{E}e^{it\xi_k} = \int_{R} e^{itx} f_0(x) \, dx = e^{-|t|^\alpha}, \ t \in R, \ \alpha = \text{const} \in (1, 2).$$

The condition $\alpha > 1$ guarantees the finiteness of the first moment: $\mathbb{E}|\xi| < \infty$. 

Let us introduce as before a second density \( f_1 = f_1(x) = f_1(|x|) \) as follows:

\[
f_1(x) = C e^{-|x|} f_0(x), \quad \int_R f_1(x) \, dx = 1,
\]

and consider again the following estimation problem \( \Theta = \{0, 1\}, \, \Theta_1 = \{1\} \), to put it differently, testing of statistical hypotheses. It follows from the main result of articles [11], [23], [37], [38], [42] that

\[
Q_n = \frac{C_\alpha}{n^{\alpha-1}} (1 + o(1)), \quad n \to \infty, \quad C_\alpha = \text{const} \in (0, \infty). \tag{4.8}
\]

The case of non-symmetrical distribution that will be treated later was investigated in [27], [37].

It is interesting to note by our opinion that when the true distribution of the sample has a density \( f_1(x) \), then the error probability \( \tilde{Q}_n \) has as ordinary an exponential form:

\[
|\ln \tilde{Q}_n| \asymp C_1(\alpha) n, \quad n \to \infty.
\]

Indeed, we have

\[
\sum_{k=1}^{n} \ln \left[ \frac{f_1(\xi_k)}{f_0(\xi_k)} \right] = nC_2 - \sum_{k=1}^{n} |\xi_k|.
\]

But it is known from the cited articles that for the random sequences from the domain of stable attraction under considered condition \( \alpha > 1 \)

\[
P(S_n - \mathbf{E}S_n > x) \sim nP(\xi - \mathbf{E}\xi > x), \quad x \asymp n.
\]

In the more general case when a symmetric distributed i., i.d. random variables \( \{\xi_k\} \) having a heavy regular varying tail of distribution:

\[
P(\xi_k > x) \sim x^{-\alpha} L(x), \quad x \to \infty, \quad \alpha = \text{const} \in (1, 2),
\]

where \( L = L(x) \) is positive continuous slowly varying function as \( x \to \infty \), we can conclude

\[
\lim_{n \to \infty} \frac{P(S_n > x + a)}{n^{1-\alpha} L(n)} = x^{-\alpha}, \quad a = \mathbf{E}\xi, \quad x \asymp n.
\]

Therefore we deduce in the considered case

\[
Q_n \sim C_3(\alpha) n^{1-\alpha} L(n). \tag{4.9}
\]

Let us consider now the case \( \alpha \in (0, 1) \). More exactly, let the r.v. \( \xi \) obeys a standard symmetric stable distribution with such a value of the parameter \( \alpha \). We derive analogously using the particular case of the results of Amosova [11]

\[
Q_n \sim \frac{C(\alpha)}{n^{1/\alpha-1}}, \quad n \to \infty. \tag{4.10}
\]
At last, in symmetrical case with \( \alpha = 1 \), i.e. when \( \xi \) has a classical Cauchy distribution
\[
f_0(x) = \frac{\pi^{-1}}{1 + x^2}, \quad x \in (-\infty, +\infty)
\]
and as before \( f_1(x) = Ce^{-|x|}f_0(x) \), then
\[
Q_n \sim \frac{C}{\ln n}, \quad n \to \infty, \quad n \geq 3.
\] (4.11)

F. Martingale generalization.

Let us assume again \( f_1(x) = Ce^{-|x|}f_0(x) \). We continue to accept \( \theta = 0 \) if
\[
\sum_{i=1}^{n} \ln \frac{f_0(\xi_i)}{f_1(\xi_i)} > 0
\]
and \( \theta = 1 \) otherwise.

But we suppose in this subsection that the sequence of the random variables
\( \{\eta_i\} := \{|\xi_i| - E|\xi_i|\}, \quad i = 1, 2, \ldots \) forms the sequence of centered martingale differences under certain filtration \( \{F_i\} \).

As before
\[
Q_n = P(\sum_{i=1}^{n} \eta_i > nd), \quad d = \text{const} > 0.
\]

The exact non-asymptotic estimations for these probabilities for martingales can be found in [4]; see also [33], [34].

For example, if for some \( p \geq 2 \) \( \forall i \Rightarrow |\eta_i|_p < \infty \), then
\[
Q_n \leq d^{-p} (p - 1)^p n^{-p/2} \left\{ n^{-1} \sum_{i=1}^{n} |\eta_i|_p^2 \right\}^{p/2}.
\] (4.12)

Another example. Introduce the tail function \( T = T(x), \ x > 0 \) for the sequence \( \{\eta_i\} \) as follows:
\[
T(x) := \sup_i \max(\mathbf{P}(\eta_i > x), \mathbf{P}(\eta_i < -x))
\]
and define
\[
W[T](x) = \min \left( 1, \inf_{v>0} \left[ e^{-x^2/(8v^2)} - \int_{v}^{\infty} x^2 dT(x) \right] \right),
\]
if of course \( \int_{0}^{\infty} x^2 |dT(x)| < \infty \).

Proposition:
\[
Q_n \leq W[T](d\sqrt{n}).
\] (4.13)

A particular case for some \( q = \text{const} > 0, \ K = \text{const} > 0 \)
\[
T(x) \leq \exp\left( -(x/K)^q \right), \quad x > 0,
\]
\[ Q_n \leq \exp \left( -C(d) n^{q/(q+2)} K^{-2q/(q+2)} \right), \quad (4.14) \]

and the last estimate is unimprovable.

5 Concluding remarks.

Confidence region for "continuous" parameters.

Suppose the set \( B = \{ \beta \} \) is compact smooth \((C^3)\) subset of the Euclidean space \( \mathbb{R}^d \) equipped with ordinary norm \( |\beta| \). We consider the confidence probability

\[ W_n = P(\sqrt{n}|\hat{\beta}_n - \beta_0| > u), \quad u = \text{const} > 0, \quad (5.1) \]

where as before \((\hat{\tau}_n, \hat{\beta}_n)\) is MLE estimation for \((m, \beta)\).

We get:

\[ W_n = P(\hat{\tau}_n = 0, \sqrt{n}|\hat{\beta}_n - \beta_0| > u) + P(\hat{\tau}_n \geq 1, \sqrt{n}|\hat{\beta}_n - \beta_0| > u) \overset{\text{def}}{=} V_1(n) + V_2(n, u). \]

If all the conditions of theorem 2.1 are satisfied, then

\[ V_2(n, u) \leq P(\hat{\tau}_n \geq 1) \leq e^{-C n}. \quad (5.2) \]

As for the probability \( V_2(n, u) \), that

\[ V_2(n, u) \leq P \left( \max_{|\beta-\beta_0| > u/\sqrt{n}} \sum_{i=1}^{n} \left[ \ln \frac{f(\xi_i, 0, \beta)}{f(\xi_i, 0, \beta_0)} \right] > 0 \right) \leq \]

\[ \leq e^{-C_2 u^*}, \quad \kappa = \text{const} \in (0, 2], \quad C_2 = \text{const} > 0, \quad (5.3) \]

see [2].

Eventually,

\[ W_n \leq e^{-C n} + e^{-C_2 u^*}, \quad \kappa = \text{const} \in (0, 2], \quad C, C_2 = \text{const} > 0. \quad (5.4) \]

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