Some approximate analytical methods in the study of the self-avoiding loop model with variable bending rigidity and the critical behaviour of the strong coupling lattice Schwinger model with Wilson fermions

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Abstract
Some time ago Salmhofer demonstrated the equivalence of the strong coupling lattice Schwinger model with Wilson fermions to a certain 8-vertex model which can be understood as a self-avoiding loop model on the square lattice with bending rigidity $\eta = 1/2$ and monomer weight $z = (2\kappa)^{-2}$. The present paper applies two approximate analytical methods to the investigation of critical properties of the self-avoiding loop model with variable bending rigidity, discusses their validity and makes comparison with known MC results. One method is based on the independent loop approximation used in the literature for studying phase transitions in polymers, liquid helium and cosmic strings. The second method relies on the known exact solution of the self-avoiding loop model with bending rigidity $\eta = 1/\sqrt{2}$. The present investigation confirms recent findings that the strong coupling lattice Schwinger model becomes critical for $\kappa_{cr} \simeq 0.38 - 0.39$. The phase transition is of second order and lies in the Ising model universality class. Finally, the central charge of the strong coupling Schwinger model at criticality is discussed and predicted to be $c = 1/2$.

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1 Introduction

Recently, the strong coupling lattice Schwinger model with Wilson fermions ($N_f = 1$) has received some attention [1]–[5] following work by Salmhofer [6] who has shown that it is equivalent to a certain 8-vertex model (a 7-vertex model, more precisely) which also can be understood as a self-avoiding loop model on the square lattice with bending rigidity $\eta = 1/2$ and monomer weight $z = (2\kappa)^{-2}$. Beyond its toy character interest in the lattice Schwinger model (QED$_2$) mainly derives from the similarity of some of its major features with those of QCD in 4D. However, because the result of Salmhofer [6] is related to the polymer (hopping parameter) expansion of the fermion determinant [7], [8] the strong coupling Schwinger model is also interesting from the point of view of the dynamical fermion problem within lattice gauge theory. While some investigations have been devoted to the polymer expansion of the fermion determinant in the case of staggered fermions [9]–[13] to date almost no attention has been paid to the corresponding case of Wilson fermions [14] due to the additional difficulties involved in general. However, while in the strong coupling limit the system with staggered fermions (QCD, QED) reduces to a pure monomer-dimer system [15] the same is not true for Wilson fermions as the investigation of Salmhofer [6] demonstrates. The equivalence of the strong coupling lattice Schwinger model with Wilson fermions to a self-avoiding loop model enables certain methods used in other branches of physics, e.g. in condensed matter physics (polymers, defect mediated phase transitions) and in cosmic string physics, to be exploited in its investigation [16]–[21]. At the same time, its equivalence (in another language) to some 8-vertex model [22] makes further results available.

Self-avoiding loop models [23], [24] have a long history due to their prominent role in polymer physics as well as their attraction as a simple problem of non-Markovian nature. In addition, systems of closed non-crossing lines or systems which can be approximated by them appear in a variety of contexts ranging from condensed matter physics through cosmology to quantum field theory which generates common interest for appropriate model building [23], [10]. Recently, quantum field theoretic methods have been exploited to study the critical behaviour of self-avoiding loop models in two dimensions [26]–[29]. Somewhat lesser attention has been paid so far to the self-avoiding loop model with variable bending rigidity (while for open chains with bending rigidity a number of investigations exists, e.g., [30] and references therein). Beyond the work of Müser and Rys [21] certain insight in this direction has been obtained in connection with the study of 2D vesicles [32]–[34].

From the point of view of the 8-vertex model, a general solution to the self-avoiding loop model with variable bending rigidity on the square lattice is not known. However, for the special case $\eta = 1/\sqrt{2}$ the free-fermion condition [35], [10] is ful-
filled and it can be solved exactly [37]–[39]. This way one point on the critical line of the self-avoiding loop model with variable bending rigidity is known exactly and consequently one may use methods of perturbative nature in its neighbourhood to approximately find the critical line there by analytical methods.

The plan of the paper is as follows. In section 2 we shortly review the relevant facts concerning the lattice Schwinger model with Wilson fermions and discuss the relation of recent exact studies of its partition function on finite lattices [1], [5] to the earlier MC results of Müser and Rys [31]. Section 3 is devoted to the approximate analytical study of the self-avoiding loop model (SALM) with variable bending rigidity by means of the independent loop approximation. Section 4 then explores the application of the exact solution of the SALM with bending rigidity $\eta = 1/\sqrt{2}$ to the study of the critical behaviour of the SALM in a neighbourhood of it in the relevant parameter space. Section 5 finally discusses the picture emerging from the present investigation paying special attention to the central charge of the SALM with variable bending rigidity on the critical line.

2 The strong coupling Schwinger model with Wilson fermions

The partition function $Z_\Lambda$ of the Schwinger model with Wilson fermions (with Wilson parameter $r = 1$) on a certain lattice $\Lambda$ is given by the standard expression

$$Z_\Lambda = \int DUD\psi D\bar{\psi} e^{-S},$$

where $D$ denotes the multiple integration on the lattice. The action $S$ is defined by

$$S = S_F + \beta S_G,$$

$$S_F = \sum_{x \in \Lambda} \left( \frac{1}{2} \sum_\mu \left( \bar{\psi}(x + \hat{\mu})(1 + \gamma_\mu)U_\mu^\dagger(x)\psi(x) 
\quad + \bar{\psi}(x)(1 - \gamma_\mu)U_\mu^\dagger(x)\psi(x + \hat{\mu}) \right) - M \bar{\psi}(x)\psi(x) \right),$$

and $U_\mu = \exp[-iA_\mu]$, $M = 2 + m$, $\beta = 1/g^2$. $S_G$ is the standard Wilson action and the hopping parameter $\kappa$ is given by $\kappa = 1/2M$. Salmhofer has shown [7] that in the strong (infinite) coupling limit $\beta = 0$ the partition function $Z_\Lambda$ equals that of an
8-vertex model (a 7-vertex model due to eq. (5), more precisely) \[22\] with weights (cf. fig. 1)

\[ w_1 = z = \frac{1}{4\kappa^2} = M^2 \quad , \]

\[ w_2 = 0 \quad , \]

\[ w_3 = w_4 = 1 \quad , \]

\[ w_5 = w_6 = w_7 = w_8 = \eta = \frac{1}{2} \quad . \]

Consequently, one can write

\[ Z_\Lambda = Z_\Lambda \left[ z, \frac{1}{2} \right] \quad , \]

\[ Z_\Lambda[z,\eta] = \sum_L z^{|\Lambda|-|L|} \eta^{C(L)} \quad , \]

where \( L \) denotes any self-avoiding loop configuration, \(|L|\) and \( C(L) \) are the number of links and corners respectively a polymer configuration \( L \) is built of, and \(|\Lambda|\) is the number of lattice points of the lattice \( \Lambda \). \( Z_\Lambda[z,\eta] \) is the partition function of a self-avoiding loop model (SALM) with monomer weight \( z \) and bending rigidity \( \eta \). The same expression can of course also be obtained for non-compact QED\(_2\).

From the point of view of lattice field theory it is interesting to know the phase structure of the lattice Schwinger model. For free fermions \((\beta = \infty)\) the critical value of the hopping parameter reads \( \kappa_{cr}(\beta = \infty) = 1/4 \). In order to pin down the critical line for \( \beta < \infty \) it is of particular interest to know where it ends \((\beta = 0)\). There is a critical point for \( \kappa_{cr}(\beta = 0) = \infty \) because then the strong coupling Schwinger model reduces to a 6-vertex model whose behaviour is known from its exact solution \[4, 22\]. This point however is believed to be isolated and not to be the end point of the critical line starting at \( \kappa_{cr}(\infty) = 1/4 \) \[4\]. Recently, exact studies of the partition function of the strong coupling Schwinger model have been made on finite lattices \[4, 5\]. It has been found \( \kappa_{cr}(0) \approx 0.38 - 0.39 \) and that the phase transition is likely a continuous one (second order or higher) \[5\].

It is worthwhile to compare the result obtained in \[4, 5\] with the MC investigation of the SALM with variable bending rigidity undertaken by Müser and Rys
Müser and Rys employ a different parameter set than \( \{ z, \eta \} \) which we are going to describe first. Their language is thermodynamic in spirit and their parameters temperature and line stiffness \( \{ T, s \} \) are introduced the following way

\[
\begin{align*}
z &= e \left(1 - \frac{s}{\eta}\right)/T, \\
\eta &= e^{-s/T}
\end{align*}
\]

which in turn entails

\[
\begin{align*}
T &= \frac{1}{\ln \frac{z}{\eta}}, \\
\frac{1}{T^3} &= e \left(1 - \frac{s}{T}\right)
\end{align*}
\]

For positive temperatures \( T \) negative values of the line stiffness \( s \) correspond to values of the bending rigidity \( \eta > 1 \) (i.e., bending preferred) and positive values of \( s \) to \( \eta < 1 \) (i.e., bending is costly). The Jacobian \( F \) of this coordinate transformation from \( (z, \eta) \in ([0, \infty), [0, \infty)) \) to \( (T, s) \in ([0, \infty), (-\infty, \infty)) \) reads

\[
F = z\eta \left[\ln \frac{z}{\eta}\right]^3
\]

and has consequently singular lines beyond the given range of the map. Figure 2 displays the result of Müser and Rys (adapted from their fig. 2) for the critical line of the loop model with line stiffness and fig. 3 displays the same information in \( \{ z, \eta \} \) coordinates (for further comments see the figure captions). In regions II and III the system at criticality is found to exhibit Ising-like behaviour while in region I some non-universal behaviour is seen. For better orientation the ordinary loop gas \( (\eta = 1) \) result is specially shown in figs. 2, 3. One immediately recognizes that the results found for the strong coupling Schwinger model fits well onto the critical line given by Müser and Rys. Moreover, the MC result of Müser and Rys also well agrees with the exactly known critical point for the free fermion model \( (\eta = 1/\sqrt{2}) \) to be discussed in section 4. For the ordinary loop gas it has also been
found numerically \[43\] that critical exponents (and amplitudes, in part) agree well with those of the Ising model. The free fermion model \((\eta = 1/\sqrt{2}, \text{see section 4})\) of course also lies in the Ising universality class. This immediately suggest (for a further discussion see section 5) that in general the SALM with variable bending rigidity at criticality lies in the Ising universality class (in the parameter regions II, III). From this we immediately infer that also the strong coupling lattice Schwinger model \((\eta = 1/2)\) belongs to this class. In [3] however a critical exponent \(\nu \approx 0.63\) has been reported for the strong coupling Schwinger model which is quite off the Ising result \(\nu = 1\). The discrepancy very likely stems from finite size effects of the small lattices on non-quadratic domains investigated. These non-quadratic domain lattices however can be exploited in other ways as we will see in section 5.

3 The independent loop approximation

Inasmuch as exact expressions for the partition function \([\mathbf{9}]\) for general \(\eta\) are not available analytical attempts to understand the phase structure of the self-avoiding loop model with variable bending rigidity have to rely on certain approximations. A method also applied in related situations in condensed matter physics and cosmic string physics is the so-called ’independent loop approximation’ \([16 - 21], [44, 45]\). The approximation is approached by writing the partition function \(Z[\Lambda, z, \eta]\) as a sum over partition functions with a fixed number \(l\) of (polymer) loops.

\[
Z[\Lambda, z, \eta] = z^{\mid \Lambda \mid} \sum_{l=0}^{\infty} Z[l, z, \eta] \tag{16}
\]

The approximation now made is to express the \(l\)-loop partition function \(Z[l, z, \eta]\) exclusively by means of the single loop partition function \(Z[1, z, \eta]\)

\[
Z[l, z, \eta] = \frac{1}{l!} [ Z[1, z, \eta] ]^l , \tag{17}
\]

leading to

\[
Z[\Lambda, z, \eta] = z^{\mid \Lambda \mid} e^{Z[1, z, \eta]} . \tag{18}
\]

This approximation can be expected to give reasonable results for those parameter regions where the loop system is sufficiently dilute (filling in the average only a certain fraction of the lattice \(\Lambda\)). According to eq. \([18]\) the investigation now may concentrate on the single loop partition function \(Z[1, z, \eta]\). One can easily convince
oneself that in the independent loop approximation the average number of loops in
the system is given by the value of the single loop partition function [17], eq. (56).
The free energy density \( f \) reads in the independent loop approximation \((\beta_T = 1/T)\)

\[
\beta_T f(z, \eta) = - \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \ln Z_\Lambda[z, \eta] = - \ln z - \lim_{|\Lambda| \to \infty} \frac{Z_\Lambda[1, z, \eta]}{|\Lambda|} . \tag{19}
\]

To proceed further, the single loop partition function can now be written as sum
over the loop length

\[
Z_\Lambda[1, z, \eta] = \sum_{k=1}^\infty z^{-2k} Z_\Lambda[2k, \eta] \tag{20}
\]

where \( Z_\Lambda[2k, \eta] \) is the conformational partition function of a single loop of length
2\( k \) (Here we already have taken into account that on a square lattice the length of
a loop is always even.) [20]. The conformational partition function is represented
then as sum over all single loop configurations \( L \) with length 2\( k \)

\[
Z_\Lambda[2k, \eta] = \sum_{L, |L|=2k} \eta^{C(L)} = \sum_{C=0}^{2k} N(2k, C) \eta^C . \tag{21}
\]

Let us start with the consideration of the ordinary loop model \((\eta = 1)\). In this
case \( Z_\Lambda[2k, 1] \) denotes the total number of possibilities to place a self-avoiding loop
with length 2\( k \) on the lattice \( \Lambda \). It can be expressed by means of the number \( p_{2k} \)
of 2\( k \)-step self-avoiding loops per lattice site which is a standard quantity that has
been investigated in the literature.

\[
p_{2k} = \lim_{|\Lambda| \to \infty} \frac{Z_\Lambda[2k, 1]}{|\Lambda|} \tag{22}
\]

The \( n \to 0 \) limit of the lattice \( O(n) \) spin model provides us now just with the
information necessary to study the critical behaviour [23] (see also, e.g., [26], sect.
2). For large \( k \) \( p_{2k} \) reads [27]

\[
p_{2k} \xrightarrow{k \to \infty} B \mu^{2k} [2k]^{-2\nu-1} + ... \tag{23}
\]

Here \( \mu \) denotes the connective constant (effective coordination number) for the self-
avoiding walk problem on the given lattice \( \Lambda \) [46] and \( B \) is some lattice dependent
constant. The (universal) critical exponent $\nu$ is believed to be given in two dimensions by $\nu = \frac{3}{4}$ (obtained on a honeycomb lattice) \cite{47}, \cite{48}. Inserting (23) into eq. (20) one finds

$$Z_{\Lambda} = B \sum_{k=1}^{\infty} [2k]^{-5/2} \left( \frac{\mu}{z} \right)^{2k}.$$  

(24)

This is a justified approximation because we are mainly interested in the critical domain which is related to the $k \to \infty$ behaviour. From eq. (24) one easily recognizes that the critical point is given by $z_{cr} = \mu$. Most recent (precise) estimates for $\mu$ on the square lattice can be found in \cite{49}–\cite{51}. We keep here only a few digits and write $z_{cr} = \mu = 2.638$ ($T_{cr} = 1.031$). Eq. (24) inserted into eq. (19) gives immediately the free energy and one recognizes that the phase transition at $z_{cr} = \mu = 2.638$ found within the independent loop approximation is of second order. Using \cite{52}, \cite{53}

$$F(x,k) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \Gamma(1-k) (-\ln x)^{k-1} + \sum_{n=0}^{\infty} \zeta(k-n) \frac{(\ln x)^n}{n!}$$

(25)

one reobtains for the critical exponent of the specific heat $\alpha$ the hyperscaling relation $\alpha = 2 - 2\nu$ entailing in the independent loop approximation $\alpha = 1/2$ which is to be confronted with the expected Ising result $\alpha = 0$ \cite{43}.

We are now prepared to study the general case with variable bending rigidity $\eta$. First we have to find an appropriate expression for the number $N(2k,C)$ of self-avoiding loops with length $2k$ and $C$ corners. Let us count first the number of random non-backtracking walks of length $2k$ with $C$ corners \cite{20}. It should be stressed that the following argument does not depend on the dimension of the lattice. There are $2k-1$ vertices available the $C$ corners can be placed at, i.e., there are $\binom{2k-1}{C}$ possibilities to do so. To each prospective corner exist $h = (2d-1) - 1$ ways of bending where $d$ is the dimension of a (hyper)cubic lattice (in our case of a square lattice $d = 2$). $(2d - 1)$ is here the non-backtracking dimension of the lattice and it has to be diminished by 1 corresponding to the straight line choice. Consequently, we find

$$N_{NB}(2k,C) = |\Lambda| \binom{2k-1}{C} h^C.$$  

(26)

Using eq. (21) the corresponding (non-backtracking) conformational partition function reads then
\[ Z_{\Lambda}[2k, \eta]_{NB} = |\Lambda| \left[ 1 + h\eta \right]^{2k-1}. \] (27)

We now obtain an approximation to the 2k-step self-avoiding walk (SAW) conformational partition function by simply replacing \( h = (2d - 1)[1 - 1/(2d - 1)] \) by \( h = \bar{\mu}[1 - 1/(2d - 1)] \) (this is based on the assumption that the self-avoidance constraints effectively encoded in \( \bar{\mu} \) are independent of whether propagation is straight or bent) where \( \bar{\mu} \) is a certain effective coordination number to be determined in a moment. This yields in our case

\[ Z_{\Lambda}[2k, \eta]_{SAW} = |\Lambda| U(2k) \left[ 1 + \frac{2\bar{\mu}\eta}{3} \right]^{2k-1}. \] (28)

The additional factor \( U(2k) \) also to be determined below takes care of some additional length dependence which might show up in the transition from non-backtracking to SAW. Specializing eq. (28) to \( \eta = 1 \) we find the total number of SAW of length 2k on the square lattice which has to be confronted with the standard expectation [27] for large \( k \)

\[ c_{2k} = \lim_{|\Lambda| \to \infty} \frac{Z_{\Lambda}[2k, 1]_{SAW}}{|\Lambda|} \xrightarrow{k \to \infty} A \mu^{2k} [2k]^{\gamma-1} + \ldots \] (29)

where \( A \) is some lattice dependent constant and \( \gamma = 43/32 \) [17], [18]. From eq. (29) we immediately find

\[ \bar{\mu} = \frac{3}{2} (\mu - 1), \] (30)

\[ U(2k) = A \mu [2k]^{\gamma-1} \] (31)

Now, we need to know the conformational partition function for the self-avoiding loop problem. In order to be able to make further progress let us assume that \( N(2k, C) \) for self-avoiding loops is just a certain fraction \( M(2k) \) of \( N_{SAW}(2k, C) \) at least for large \( k \) independent of the number of corners \( C \). According to eq. (21) then we can write

\[ Z_{\Lambda}[2k, \eta] = M(2k) Z_{\Lambda}[2k, \eta]_{SAW} \] (32)

which reads after having taken into account eqs. (28), (30), (31)
\[ Z_{\Lambda}[2k, \eta] = |\Lambda| M(2k) A \mu \left[ 1 + (\mu - 1)\eta \right]^{2k-1} [2k]^{\gamma-1} . \tag{33} \]

\( M(2k) \) is the fraction to be determined. We here simply ignore the fact that for any loop the number of corners is even, necessarily. This is justified for the study of the \( k \to \infty \) behaviour we are primarily interested in. For \( \eta = 1 \) we already have displayed an expression (eq. (23)) which now serves as reference expression to determine \( M(2k) \). We obtain

\[ M(2k) = \frac{B}{A} [2k]^{-2\nu-\gamma} \tag{34} \]

leading to

\[ Z_{\Lambda}[1, z, \eta] = |\Lambda| \frac{B}{1 + (\mu - 1)\eta} \sum_{k=1}^{\infty} [2k]^{-5/2} \left( \frac{1 + (\mu - 1)\eta}{z} \right)^{2k} . \tag{35} \]

Consequently, the critical line is found to be

\[ \eta_{cr}(z_{cr}) = \frac{(z_{cr} - 1)}{(\mu - 1)} . \tag{36} \]

This translates into the \( \{T, s\} \) coordinate system as

\[ s_{cr}(T_{cr}) = T_{cr} \ln \left[ \frac{e^{1/T_{cr}}}{\mu} - \mu + 1 \right] . \tag{37} \]

It should be emphasized that the result of our approximate consideration (eq. (37)) entails \( s_{cr} \to 1 \) for \( T_{cr} \to 0 \). This is well in line with expectations spelled out in \[31\]. From the above equations we obtain for the strong coupling Schwinger model \( z_{cr} = 1.819 \) (\( T_{cr} = 0.774 \), \( s_{cr} = 0.537 \)). We find for the critical hopping parameter

\[ \kappa_{cr}(0) = \frac{1}{\sqrt{2(\mu + 1)}} = 0.371 . \tag{38} \]

The critical line (eqs. (36), (37)) obtained within the independent loop approximation is plotted in figs. 4, 5. One recognizes that the critical line found analytically agrees qualitatively quite well with the result of the MC calculation of Müser and
However, it is clear that the validity of the independent loop approximation is confined to the low (polymer) loop density domain. The high density result of Müser and Rys [31] displayed in region I of figs. 2, 3 cannot be obtained within the present scheme. It is also well known [16], [44] that while within the independent loop approximation the critical line can be determined in a qualitatively correct way, results for critical exponents are correct to a lesser degree. This also applies to our case as we have seen above. While we would have expected, e.g., for the critical exponent $\alpha$ the Ising result ($\alpha = 0$) within the independent loop approximation we see $\alpha = 1/2$. Finally, it should also be mentioned that the relative simplicity of the independent loop approximation has its price because so far no way of its systematic improvement is known and one consequently has no quantitative control over the approximation made. Perhaps, this drawback is offset by the applicability of the approximation to systems in any dimension.

4 The exact solution of the self-avoiding loop model with bending rigidity $\eta = 1/\sqrt{2}$ and its use

While in general the self-avoiding loop model (SALM) with variable bending rigidity (which is equivalent to a 7-vertex model due to $w_2 = 0$) cannot be studied exactly so far, there exists an exact solution to it for $\eta = 1/\sqrt{2}$ first investigated by Priezzhev [37] (see also [38]). This solution has later been rediscovered by Blum and Shapir [39] who apparently were unaware of the earlier work of Priezzhev. The solution relies on the general study of the 8-vertex model by Fan and Wu [35], [36]. They found that the 8-vertex model is exactly solvable if the free fermion condition

$$ w_1 w_2 + w_3 w_4 = w_5 w_6 + w_7 w_8 $$

(39)

is fulfilled (cf. fig. 1 for the labelling of the vertices). Inserting eqs. (4)-(7) into (39) ($\eta$ taken arbitrary here) one immediately finds that the free fermion condition is fulfilled for $\eta = 1/\sqrt{2}$. The partition function for the SALM with bending rigidity $\eta = 1/\sqrt{2}$ has been found in [37], [39] by standard methods. The free energy density $f$ reads

$$ \beta_T f(z, 1/\sqrt{2}) = -\frac{1}{8\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln \left[ 2 + z^2 + 2z \cos \theta + 2z \cos \phi + 2 \cos \theta \cos \phi \right] . $$

(40)
A second order phase transition occurs for $z_{cr} = 2$, $T_{cr} = 2/(3 \ln 2) = 0.962$, $s_{cr} = 1/3$ which will be of main interest to us. There is of course also a critical point at $z = 0$ related to the exactly solvable 6-vertex model [22], [3]. Because the system can be represented by means of free fermions [54] the SALM with bending rigidity $\eta = 1/\sqrt{2}$ lies in the Ising universality class [39]. In accordance with this it has been shown (for $w_2$ chosen arbitrarily) that the partition function of the free fermion model can be expressed in terms of that of the regular Ising model [55]. It finally deserves mention that the result of the MC calculation of M"user and Rys [31] is in complete agreement with the exact solution of the free fermion model (cf. figs. 2, 3).

The above exact solution lying on the critical line of the SALM with variable bending rigidity is quite useful because this way one may take advantage of universality arguments to draw conclusions about the model at criticality for fairly wide ranges of the bending rigidity $\eta$. This will be discussed further in section 5. Here we will study the approximate calculation of the critical line in a neighbourhood of the model for $\eta = 1/\sqrt{2}$. This discussion is in a certain sense a generalization of that given in [38], [37]. Let us write the partition function (9) in the following way.

\begin{align}
Z_A[z, \eta] &= \sum_{l=0}^{[A]} z^{[A]-2l} \sum_{L, |L|=2l} \eta^{C(L)} \\
&= \sum_{l=0}^{[A]} z^{[A]-2l} \sum_{L, |L|=2l} \sum_{k=0}^{\infty} \frac{1}{k!} \left[C(L) \ln \eta \right]^k \\
&= \sum_{l=0}^{[A]} z^{[A]-2l} \sum_{k=0}^{\infty} \frac{[\ln \eta]^k}{k!} \langle C(L)^k \rangle_{2l} \\
\end{align}

Here,

\begin{align}
\langle C(L)^k \rangle_{2l} &= \sum_{L, |L|=2l} C(L)^k \\
\end{align}

One may now express $\langle C(L) \rangle_{2l}$ the following way.

\begin{align}
\langle C(L) \rangle_{2l} &= C_{2l} N(2l) \\
C_{2l} &= 2l \ n_C(2l)
\end{align}
\( N(2l) = \langle 1 \rangle_{2l} \) denotes the number of (multi) loop configurations of total length \( 2l \) on the lattice \( \Lambda \) and \( n_C(2l), 0 \leq n_C(2l) \leq 1 \), stands for the average relative density of corners in the considered loop ensemble of total length \( 2l \). The following of course holds for the higher moments of \( C \).

\[
0 \leq \langle (C(L)^k)_{2l} \rangle \leq (2l)^k N(2l) \quad (47)
\]

One can now write

\[
Z_\Lambda[z, \eta] = \sum_{l=0}^{\left| \Lambda \right|} N(2l) \left[ \frac{\eta n_C(2l)}{z} \right]^{2l} \left[ 1 + \frac{\langle (C(L) - \bar{C}_{2l})^2 \rangle_{2l}}{2 N(2l)} \ln \eta + \ldots \right] \quad (48)
\]

where \( \ldots \) stands for a series in higher order correlation functions of the corner number \( C \) and \( \ln \eta \). The critical behaviour of the system is related to large \( l \). For \( l \to \infty \) \( n_C \) tends to some value \( \bar{n}_C \) and consequently points \( \{ z_1, \eta_1 \}, \{ z_2, \eta_2 \} \) on the critical line not to far away from each other should obey to leading approximation the equation

\[
\frac{\bar{n}_C}{z_1} = \frac{\bar{n}_C}{z_2} \quad . \quad (49)
\]

The contribution of correlation functions of \( C \) should be expected to be of minor importance in eq. (48), leading to corrections to the leading behaviour only. Inserting into eq. (19) the exactly known critical point \( \{ z, \eta \} = \{ 2, 1/\sqrt{2} \} \) of the free fermion model leads in a neighbourhood of it to the equation for the critical line

\[
\eta_{cr}(z_{cr}) = 2^{-(\bar{n}_C+2)/2\bar{n}_C} z_{cr}^{1/\bar{n}_C} \quad . \quad (50)
\]

The only unknown quantity in this expression is the average relative corner density \( \bar{n}_C \). Its value \( n_C(2l \to \infty) \) is related to the high density polymer limit which is reached for \( z \to 0 \). \( \bar{n}_C \) has been calculated in [37], [38] for the free fermion model and found to have the value 1/2. Values for the correlation functions of \( C \) for \( l \to \infty \) can also be obtained along the same lines by tedious, but standard methods ([50], [51]; the latter is the English original of ref. no. 16 in [37]). So, we end up with the following equation for the critical line of the SALM with variable bending rigidity in the neighbourhood of the free fermion point (cf. also figs. 4, 5).
\[ \eta_{cr}(z_{cr}) = 2^{-5/2} z_{cr}^2. \quad (51) \]

This equation reads in \( \{T, s\} \) coordinates

\[ T_{cr}(s_{cr}) = \frac{2 (2 - s_{cr})}{5 \ln 2}. \quad (52) \]

Consequently, we obtain for the strong coupling Schwinger model \( z_{cr} = 2^{3/4} \approx 1.682 \) \( (T_{cr} = 4/(7 \ln 2) \approx 0.824, s_{cr} = 4/7 \approx 0.571) \). This yields for the critical hopping parameter

\[ \kappa_{cr}(0) = 2^{-11/8} \approx 0.386. \quad (53) \]

We see (cf. also figs. 4, 5) that the approximation based on the exactly solvable free fermion model yields a numerical value of the critical hopping parameter fairly close to the results of previous computer studies \[ 31 \], \[ 4 \], \[ 5 \]. As mentioned above, systematic improvements can be obtained by taking into account correlation functions of \( C \). This apparently is necessary as one learns from figs. 4, 5 if one wants to find the critical line beyond the region defined by the critical points of the ordinary loop model and the strong coupling Schwinger model respectively.

## 5 Discussion and conclusions

Let us first have a look at the larger picture emerging for the critical behaviour of the self-avoiding loop model with variable bending rigidity. There is one point on the critical line known exactly from the solution of the free fermion model (\( \eta = 1/\sqrt{2}, z_{cr} = 2 \)) \[ 37 \]–\[ 39 \]. For this model it is established that the phase transition is Ising-like, i.e., the model experiences a second order phase transition with exactly the same critical exponents as the regular Ising model. By the argument of universality we may conclude that neighbouring models which lie on the same critical line exhibit the same behaviour. This in particular concerns the ordinary loop model (\( \eta = 1 \)) and the strong coupling Schwinger model (\( \eta = 1/2 \)). For the ordinary loop model this has been confirmed by MC investigations in the past \[ 13 \]. For the strong coupling Schwinger model this consideration specifies the so far unknown character of the phase transition and confirms the recent suggestion that the transition might be a continuous one \[ 3 \].
In order to extend the understanding of the self-avoiding loop model with variable bending rigidity at criticality let us consider the central charge $c$ of the corresponding conformal field theory (CFT). Helpful information can be obtained most easily for the free fermion model considered in sect. 4. First, it seems worthwhile mentioning that the regular Ising model can be understood as a special free fermion model [54]. A preliminary investigation along the lines given for the Ising model in [58] indicates that the self-avoiding loop model with bending rigidity $\eta = 1/\sqrt{2}$ can be represented at the critical point $z_{cr} = 2$ by just one massless (continuum) Majorana fermion (As in the special case of the Ising model just one half of the fermionic modes needed to express the partition function becomes massless at the critical point.). Consequently, this suggests that the critical self-avoiding loop model at the free fermion point is equivalent to a $c = 1/2$ CFT. The central charge cannot change continuously on the critical line in the neighbourhood of the free fermion model, therefore CFT’s corresponding to the self-avoiding loop model with variable bending rigidity should all be expected to exhibit $c = 1/2$. This of course entails that the strong coupling Schwinger model at criticality should be equivalent to a $c = 1/2$ CFT. Consequently, in accordance with Zamolodchikov’s c-theorem [59], [60], $z \neq 0$ is related to a flow from the 6-vertex model ($z = 0$) having $c = 1$ [61] towards a model with $c = 1/2$ (as discussed in general terms by Salmhofer [6]).

We are going to test the above insight now by calculating the central charge for the strong coupling Schwinger model. This can be done most easily by considering the model on a strip of width $a$ and length $b \to \infty$ [63], [64]. The central charge is related to the partition function (on a torus) by the formula ($f$ is the (bulk) free energy density on the infinite plane)

$$\lim_{b \to \infty} \frac{\ln Z_{\Lambda(a \times b)}[z_{cr}, 1/2]}{b} = af(z_{cr}, 1/2) + c \frac{\pi}{6} \frac{1}{a}. \quad (54)$$

We however will approach the study of the central charge of the strong coupling Schwinger model by means of the exact partition functions calculated on finite lattices in [4], [5]. For our purpose the exact partition functions available on a $8 \times 8$ torus [4], and on $2 \times 32$, $3 \times 16$ tori [5] are suited. In fig. 6 we have plotted the function

$$c(z) = \frac{a}{b} \frac{6}{\pi} \left\{ \ln Z_{\Lambda(a \times b)}[z, 1/2] - ab \ f(z, 1/2) \right\} \quad (55)$$

for the $2 \times 32$, $3 \times 16$ tori in dependence on $\kappa$ ($z = (\kappa)^{-2}$ has been inserted). $f(z, 1/2)$ has been calculated by means of the $8 \times 8$ partition function. For sufficiently large $b$ the function $c(z)$ should be expected to approach the value of the central charge.
at the critical point. However, one has to be aware of the fact that on the very narrow (with respect to $a$) tori considered massless and massive fields can contribute comparable amounts to the Casimir energy. Inasmuch as the central charge is calculated by means of eqs. (54), (55) from the Casimir energy results obtained from very narrow tori may turn out misleading. In part, this is what we observe from fig. 6. The result for the $a = 2$ torus (solid line) rather suggests $c = 1$ (or some value close to it), however the torus is so narrow that massless and massive fields contribute comparably to the Casimir energy. Consequently, in agreement with our expectation $c = 1/2$ for the wider $a = 3$ torus (dashed line) we already observe a much smaller value of $c(z)$ at the critical point. However, it turns out that the sizes of the tori for which the exact partition functions have been calculated so far in the literature are too small to allow any final conclusions for the central charge of the strong coupling Schwinger model at criticality. Only partition functions calculated on considerably larger lattices will allow to numerically test the prediction $c = 1/2$ in a reliable way.

To conclude, the study of the self-avoiding loop model with variable bending rigidity presented in this article enhances the understanding of the critical behaviour of the strong coupling Schwinger model with Wilson fermions. We find that a second order phase transition which lies in the Ising model universality class takes place at some finite value of the hopping parameter $\kappa_{cr}(0)$. Using certain approximate analytic methods the value of the critical hopping parameter is confirmed to lie at $\kappa_{cr} \simeq 0.38 - 0.39$ in accordance with earlier numerical investigations [4], [5]. Certain arguments considered suggest that the strong coupling Schwinger model at criticality is equivalent to a $c = 1/2$ CFT.

From a technical point of view, the present paper studies the application of the independent loop approximation to the qualitative and in part quantitative exploration of the phase structure of the self-avoiding loop model with variable bending rigidity in 2D. Comparison with known numerical results [31] shows that this method delivers a fairly correct picture for sufficiently low (polymer) loop densities. This is encouraging because the method is equally applicable to higher dimensions, while the analytic approach based on the exactly solvable free fermion model presented in section 4 is at least in part specific to 2D. This suggests that the independent loop approximation might successfully be applied also to analogous systems in higher dimensions (e.g., to strong coupling QCD in 4D where the critical hopping parameter has recently been studied by other methods [54]).
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Figure captions

Fig. 1  
Vertices of the 8-vertex model and their weights (cf. eqs. (4)-(7)).

Fig. 2  
Critical line of the self-avoiding loop model with variable bending rigidity as found by MC calculations on a 64×64 lattice by Müser and Rys [31]. This figure has been copied and adapted from fig. 2 of [31]. The dashed line indicates some interpolated curve to the point $s = 1, T = 0$ (specially emphasized by a black half disk). The domains I – VI are mapped to the correspondingly labelled domains in the $\{z, \eta\}$-plane (see fig. 3, boundary lines are plotted in the same style in both figures). ff denotes the exactly known critical point $\{T_{cr} = 2/(3 \ln 2) \approx 0.962, s_{cr} = 1/3\} \ (z_{cr} = 2)$ of the free fermion model ($\eta = 1/\sqrt{2}$) [37]-[39]. lg stands for the ordinary self-avoiding loop model ($\eta = 1, s = 0$) with the critical point $T_{cr} = 1.157 \ (z_{cr} = 2.373)$ [42]-[43]. sm denotes the critical point $\kappa_{cr}(0) = 0.38 \ (T_{cr} = 0.80, s_{cr} = 0.56, z_{cr} = 1.73)$ of the strong coupling Schwinger model ($\eta = 1/2$) as found in [4], [5].

Fig. 3  
This is the equivalent of fig. 2 shown here for the $\{z, \eta\}$ coordinate system. For further explanations refer to fig. 2.

Fig. 4  
The critical line according to the results of the independent loop approximation (37) (dotted line) and of the free fermion model related approach (52) (dashed line) in comparison with the MC result (solid line) of Müser and Rys [31]. For further explanations refer to fig. 2.

Fig. 5  
This is the equivalent of fig. 4 shown here for the $\{z, \eta\}$ coordinate system and relating to eqs. (36) (dotted line) and (51) (dashed line).

Fig. 6  
The function $c$ (see eq. (53)) in dependence on $\kappa(0)$. The solid line is the result for the $2 \times 32$ lattice while the dashed line stands for the result on the $3 \times 16$ lattice. The value of $c$ at the critical point $\kappa_{cr}(0) \simeq 0.38 - 0.39$ has to be compared with the expectation for the central charge (For a discussion see the main text.)
