Hamiltonian walks on Sierpinski and $n$-simplex fractals

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Abstract. We study Hamiltonian walks (HWs) on Sierpinski and $n$–simplex fractals. Via numerical analysis of exact recursion relations for the number of HWs we calculate the connectivity constant $\omega$ and find the asymptotic behaviour of the number of HWs. Depending on whether or not the polymer collapse transition is possible on a studied lattice, different scaling relations for the number of HWs are obtained. These relations are in general different from the well-known form characteristic of homogeneous lattices which has thus far been assumed to hold for fractal lattices too.

Submitted to: J. Phys. A: Math. Gen.

PACS numbers: 05.50.+q, 02.10.Ox, 05.45.Df
1. Introduction

Enumeration of Hamiltonian walks (HWs), i.e. self-avoiding walks (SAWs) that visit every site of a given lattice, is a classic problem in graph theory, but it also has an important role in the study of the configurational statistics of polymers. HWs are used to model collapsed polymers [1], polymer melting [2, 3], as well as protein folding [4, 5]. The number of all possible HWs on a lattice is related to the configurational entropy of a collapsed polymer system, and also to the optimal solutions to the traveling salesman problem [6, 7]. Enumeration of HWs, closed or open, is a difficult combinatorial problem, which has been exactly solved only for few lattices, namely, the two-dimensional Manhattan oriented square lattice [8, 9], the two-dimensional ice lattice [10], the two-dimensional hexagonal lattice [11, 12], the Sierpinski gasket fractal [6], and the 4-simplex fractal [13]. The number of HWs has also been calculated numerically for various lattices by means of direct enumeration [14, 15], transfer matrix methods [16, 17, 18], and Monte Carlo estimates [19, 20]. The field theory representation for this problem was introduced in [21], and further developed in [22] and [23].

The purpose of this paper is to understand the topological properties of Hamiltonian walks on several families of fractal lattices. In order to do this we study the asymptotic behavior of the number of closed HWs \( C_N \) for a large number of vertices \( N \). This analysis yields the values of the so-called connectivity constant \( \omega \) which has the physical meaning of the average number of steps available to the walker having already completed a large number of steps. It also provides insight into the spatial distribution of walks present at large \( N \); this can be related to detailed studies of knot delocalisation in Refs. [24].

The number \( C_N \) is for homogeneous lattices with \( N \gg 1 \) expected to take the form

\[
C_N \sim \omega^N \mu_S N^\sigma N^a.
\] (1.1)

Here \( \omega \) is the connectivity constant and the term with \( \mu_S \) represents a surface correction \( (\mu_S < 1) \), with \( \sigma = (d - 1)/d \) (\( d \) being the dimensionality of the lattice). This differs from the ordinary SAW case, where no surface term \( \mu_S N^\sigma \) is expected, i.e. the number of SAWs of length \( N \) behaves as \( \mu^N N^a \) for large \( N \). Furthermore, the exponent \( a \) is universal in the SAW case, i.e. it depends only on the dimensionality of the lattice, whereas for HWs it may depend on other, not yet identified characteristics of the lattice. To the lowest approximation in Eq. (1.1) \( C_N \sim \omega^N \), and the connectivity constant can be defined as

\[
\ln \omega = \lim_{N \to \infty} \frac{\ln C_N}{N}.
\] (1.2)

For a better understanding of these problems it is helpful to study HWs on fractal lattices. As was first recognized by Bradley [13], the self-similarity of fractal lattices is a useful tool for the exact and computationally fast iterative generation and enumeration of all HWs on an unlimitedly large corresponding fractal structure. In this paper we extend Bradley’s algorithm to two– and three–dimensional Sierpinski fractal families, as well as \( n \)–simplex fractals with \( n > 4 \). We consider several families of fractals in order to compare the obtained results and be able to draw more general conclusions.
about the character of the Hamiltonian walks on different classes of lattices. In the case of the two–dimensional Sierpinski fractal family an exact closed form result for the connectivity constant is obtained due to the simple form of the recursion relations for the numbers of HWs. For the three-dimensional Sierpinski fractals which can model physically more frequently encountered systems a numerical approach is necessary. The study of asymptotics of the number of HWs shows that the surface term in Eq. (1.1) only appears for fractals on which the collapse transition from the polymer coil to the globule phase is possible. This is the same class of lattices for which delocalised HWs dominate over localised ones for large $N$.

The paper is organized as follows. In section 2 we describe the two–dimensional Sierpinski fractal family and obtain exact recursion relations for the number of HWs, as well as the closed exact formula for $\omega$. Recursion relations for HWs on three–dimensional Sierpinski fractals are given and analyzed in section 3. A similar method for analyzing HWs on 5- and 6-simplex lattices is presented in section 4. Finally, in section 5 we discuss all our findings and related results obtained by other authors.

2. Hamiltonian walks on two–dimensional Sierpinski fractals

We begin by defining the two-dimensional (2d) Sierpinski fractal family. Each member of the 2d SF family (labeled by $b$) can be constructed recursively, starting with an equilateral triangle that contains $b^2$ smaller equilateral triangles (generator $G_1^2(b)$). The subsequent fractal stages are constructed self–similarly, by replacing each of the $b(b+1)/2$ upward–oriented small triangles of the initial generator by a new generator. To obtain the $l$th–stage fractal lattice $G_l^2(b)$, which we shall call the $l$th order generator, this process of construction has to be repeated $l - 1$ times, and the complete fractal is obtained in the limit $l \to \infty$. It is easy to see that, for any 2d SF, each closed HW on the $(l + 1)$th generator is comprised of HWs which enter and exit $l$th order generators. Let $C_l$ be the number of closed HWs on the $l$th order generator, whereas $h_l$ and $g_l$ are the numbers of HWs which enter the $l$th order generator at one vertex, and leave it at the other, with or without visiting its third vertex, respectively. Then, it can be shown that a simple relation

$$C_{l+1} = Bh_l^\alpha g_l^\beta$$

is valid for $l \geq 1$ (see Appendix A). Here $B$ is a constant that depends only on SF parameter $b$, whereas exponents $\alpha$ and $\beta$ are equal to

$$\alpha = b + 1, \quad \beta = \frac{(b + 1)(b - 2)}{2}.$$  \hspace{1cm} (2.2)

For instance, the explicit form of the relation (2.1) for $b = 2$ is $C_{l+1} = h_l^3$ and for $b = 3$: $C_{l+1} = 3h_l^4g_l^2$, which is illustrated in Figure 1. One can also show (Appendix A) that numbers $h_l$ and $g_l$ of open HWs obey the following closed set of recursion relations

$$h_{l+1} = Ah_l^x g_l^y, \quad g_{l+1} = Ah_l^{x-1} g_l^{y+1},$$

where $x$ and $y$ are numerical constants.
Figure 1. The \((l+1)\)th order generators \(G_{l+1}(2)\) (a) and \(G_{l+1}(3)\) (b) for \(b = 2\) and \(b = 3\) two-dimensional Sierpinski fractals, with the possible closed Hamiltonian walks configurations depicted. The small up-oriented triangles are the respective \(l\)th order generators, and the lines that traverse them represent the open Hamiltonian walks.

Figure 2. Schematic representation of the recursion relation for the number of open Hamiltonian walks (2.3) of \(h\)-type on \(b = 3\) two-dimensional Sierpinski fractal structures.

for all \(l \geq 1\), where \(A\) is again a constant, different for every \(b\), and

\[
x = b, \quad y = \frac{b(b-1)}{2}.
\]

(2.4)

For \(b = 2\) these relations have the form: \(h_{l+1} = 2h_l^2 g_l\), \(g_{l+1} = 2h_l g_l^2\) and for \(b = 3\):

\(h_{l+1} = 8h_l^3 g_l^3\), \(g_{l+1} = 8h_l^2 g_l^4\) (see Figure 2). From the relations (2.3) it follows straightforwardly that

\[
\frac{g_l}{h_l} = \frac{g_1}{h_1} = K,
\]

(2.5)

for any \(l \geq 1\), so that from (2.1) and (2.2) one gets \(C_{l+1} = BK^l h_l^{b+1}/2\). Since the number \(N_l\) of sites on the \(l\)th order generator satisfies the recursion relation

\[
N_{l+1} = \frac{b(b+1)}{2}N_l - (b^2 - 1),
\]

(2.6)

according to (2.2) it then follows that

\[
\ln \omega = \lim_{l \to \infty} \frac{\ln C_{l+1}}{N_{l+1}} = \lim_{l \to \infty} \frac{\ln h_l}{N_l}.
\]
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Table 1. Values of the number $A$ of Hamiltonian configurations relevant for the recursion relations (2.3), numbers $h_1$ and $g_1$ of open HWs on the generators of two-dimensional Sierpinski fractals, connectivity constant $\omega$ for HWs, and connectivity constant $\mu$ for SAWs (obtained via RG method - $\mu^{\text{RG}}$ and numerically estimated $\mu^{\text{num}}$ in [28]), for $2 \leq b \leq 8$.

| $b$ | $A$ | $h_1$ | $g_1$ | $\omega$ | $\mu^{\text{RG}}(\mu^{\text{num}})$ |
|-----|-----|-------|-------|----------|-----------------------------|
| 2   | 2   | 2     | 3     | 1.31798  | 2.288(2.282 ± 0.007)        |
| 3   | 8   | 10    | 11    | 1.39157  | 2.491(2.49 ± 0.02)          |
| 4   | 40  | 92    | 112   | 1.46186  | 2.656(2.686 ± 0.004)        |
| 5   | 360 | 1 852 | 2 286 | 1.52155  | 2.791(2.82 ± 0.01)          |
| 6   | 3 872 | 78 032 | 94 696 | 1.56895  | 2.904(2.92 ± 0.02)          |
| 7   | 62 848 | 6 846 876 | 8 320 626 | 1.61011 | 3.005(2.99 ± 0.05)          |
| 8   | 1 287 840 | 1 255 156 712 | 1 527 633 172 | 1.64528 | ——-(3.13 ± 0.07)          |

On the other hand, from (2.3), (2.4), and (2.5) one has $h_{l+1} = A K^y h_l^{b(b+1)/2}$, i.e. $\ln h_l$ satisfies the difference equation

$$\ln h_{l+1} = \frac{1 - [b(b+1)/2]^l}{1 - [b(b+1)/2]} \ln(AK^y) + \left(\frac{b(b+1)}{2}\right)^l \ln h_1.$$  \hspace{1cm} (2.7)

From this equation, together with the explicit expression for the number of sites

$$N_{l+1} = \frac{4}{b+2} \left(\frac{b(b+1)}{2}\right)^{l+1} + 2 \frac{b+1}{b+2},$$  \hspace{1cm} (2.8)

which follows directly from the relation (2.6), one can derive the general form

$$\omega = A \frac{4}{b(b+4)(b^2-1)} \frac{4}{h_1} \frac{2}{g_1} \frac{1}{(b+1)(b+4)}$$  \hspace{1cm} (2.9)

of the connectivity constant (1.2) for HWs on two-dimensional SFs. Consequently, in order to calculate $\omega$ for any particular 2d SF, one should find the numbers $h_1$ and $g_1$ of open HWs on the generator, and the number $A$ of all Hamiltonian configurations which are relevant for recursion relations (2.3). In Table 1 we present these numbers, together with the values of $\omega$, for $2 \leq b \leq 8$. As one can see, for $b = 2$, and $b = 3$ the numbers $g_1$, $h_1$, and $A$ are small and can be directly enumerated, whereas for larger values of $b$ they quickly increase, so that enumeration should be computerized (calculation of the numbers $A$, $g_1$, and $h_1$ required 13 minutes for $b = 7$ case, and about 100 hours for $b = 8$, both on a computer with a processor MIPS R10000, Rev 2.6 on 180 MHz). One should mention that the connectivity constant for the Sierpinski gasket ($b = 2$) has already been calculated in a different way by Bradley [6].

Combining equations (2.7), (2.8) and (2.9) it is not difficult to see that $h_l = G \omega^{N_l}$, where $G$ depends only on $b$, and correspondingly $C_l \sim \omega^{N_l}$. Comparing with (1.1) one can conclude that neither surface nor power correction terms are present in the scaling form for the number of closed HWs on two-dimensional Sierpinski fractals.
Comparison with the self-avoiding walk case

It is interesting to compare the value of the connectivity constant obtained for the case of Hamiltonian walks to that corresponding to all possible self-avoiding configurations on 2d SFs. Our algorithm for enumerating HWs is easily adjusted for that purpose. By means of exact renormalization group (RG) approach [25, 26], these configurations can be used for calculating the connectivity constant \( \mu \) for ordinary SAWs, which was done earlier only for the \( b = 2 \) case [26]. Here we extend an exact RG calculation of the connectivity constant \( \mu \) for any \( b \).

The connectivity constant \( \mu \) for the SAW model is equal to \( \mu = \lim_{N \to \infty} \left( c_{N+1}/c_N \right) = \lim_{N \to \infty} \left( p_{N+1}/p_N \right) \), where \( c_N \) (\( p_N \)) is the average number of distinct open (closed) \( n \)-step SAWs. In order to calculate \( \mu \) within the exact RG approach, one should introduce two generating functions \( B^{(l)} \) and \( B^{(1)}_1 \):

\[
B^{(l)} = \sum_N B^{(l)}_N x^N, \quad B^{(1)}_1 = \sum_N B^{(1)}_{1,N} x^N,
\]

where \( x \) is the statistical weight of each step of the SAW (fugacity), whereas \( B^{(l)}_N \) (\( B^{(1)}_{1,N} \)) is the number of SAWs which enter the \( l \)th order generator \( G^2_l(b) \) at one vertex, and leave it at the second, without (with) visiting the third one. For every 2d SF lattice functions \( B \) and \( B_1 \) obey recursion relations of the following form

\[
B^{(l+1)} = \sum_{i,j} f_{i,j}(b) \left( B^{(l)} \right)^i \left( B^{(1)}_1 \right)^j, \quad B^{(1)}^{(l+1)} = \sum_{i,j} g_{i,j}(b) \left( B^{(l)} \right)^i \left( B^{(1)}_1 \right)^j,
\]

(2.10)

where \( f_{i,j}(b) \) and \( g_{i,j}(b) \) are coefficients that do not depend on \( l \), but do depend on the fractal parameter \( b \). The initial conditions are \( B^{(0)} = x, \ B^{(1)}_1 = x^2 \) and the connectivity constant \( \mu \) is equal to \( 1/x^* \), where \( x^* \) is the value of the fugacity for which one approaches the fixed point \((B^*, B_1^*)\) of (2.10), after a large (infinite) number of iterations.

The explicit RG recursion relations for 2d SFs with \( b = 2 \) are

\[
B' = B^2 + B^3 + 2BB_1 + 2B^2B_1 + B_1^2, \quad B'_1 = B^2B_1 + 2BB_1^2
\]

and for \( b = 3 \):

\[
B' = B^3 + 3B^4 + B^5 + 2B^6 + 3B^2B_1 + 12B^3B_1 + 4B^4B_1 + 8B^5B_1 + 3BB_1^2 + 16B^2B_1^2 + 5B^3B_1^2 + 8B^4B_1^2 + +B_1^3 + 8BB_1^2 + 2B^2B_1^3 + B_1^4
\]

\[
B'_1 = B^4B_1 + 2B^5B_1 + 4B^3B_1^2 + 8B^4B_1^2 + 5B^2B_1^3 + 8B^3B_1^3 + 2BB_1^4
\]

whereas relations for \( b = 4 \) and \( 5 \) are given in Appendix B. For \( b = 6 \) and \( 7 \) RG relations are too cumbersome to be quoted here, but they are available upon request. Underlined terms in quoted RG relations correspond to the Hamiltonian configurations, as one can check by direct comparison with relations (2.3); replacing \( g_l \) and \( h_l \) in (2.3) by \( B \) and \( B_1 \), respectively, one obtains underlined terms in the RG relations (values of coefficient \( A \) are given in table 1). It is obvious that for larger \( b \) values the number of SAW configurations is much larger than the number of Hamiltonian configurations, and for that reason we were not able to enumerate all SAW configurations on 2d SF beyond \( b = 7 \).
For all \( b \) considered here recursive relations (2.10) have only one nontrivial fixed point \( (B^*_b, 0) \), where \( B^*_b \) lies in the interval \( 0 < B^*_b < 1 \). One can check that by setting \( B_1 = 0 \) in (2.10) RG equations used in [27] for calculating critical exponent \( \nu \) (connected with the mean end-to-end distance) for SAWs on 2d SFs are recovered. Of course, SAW model treated in [27] is slightly different (each unit triangle within the fractal can be traversed only along one side) from the usual one, treated here, but both of them belong to the same universality class, i.e. the critical exponent \( \nu \) is equal for both considered SAW models. This is not the case with the connectivity constant, which is a nonuniversal quantity, so an extra RG parameter \( B_1 \) had to be introduced.

The final RG results \( \mu_{RG} \) for the SAW connectivity constant are given in the last column of Table 1, together with the corresponding values \( \mu_{num} \) obtained in [28] using a graph counting technique. As one can expect, the connectivity constant \( \mu \) for SAW is larger than \( \omega \) for HW model, for every considered SF, since the physical meaning of the connectivity constant is the average number of steps available to the walker after \( N \) steps completed, for large \( N \). One can also see that the values of \( \mu \) obtained by two different methods for \( b = 4 \) and 5 are not in good agreement. The RG method applied here is exact, implying that numerical estimations within the graph counting technique used in [28] were not accurate enough.

### 3. Hamiltonian walks on three–dimensional Sierpinski fractals

We proceed by analyzing the properties of HWs on the three–dimensional (3d) SF family. A 3d SF can be constructed recursively as in the 2d case, the only difference being that the generator \( G^3_1(b) \) of the fractal with the parameter \( b \) is no longer a triangle, but a tetrahedron that contains \( b(b+1)(b+2)/6 \) upward oriented smaller tetrahedrons. Consequently, one should observe four types of open HWs that traverse the \( l \)–th order generator in order to obtain the overall number of closed HWs on 3d SF. The first three types are HWs which enter the generator at one vertex and leave it at another, meanwhile

- visiting the third, but not the fourth vertex (\( g \)–type),
- visiting both the third and the fourth vertex (\( h \)–type),
- visiting neither the third nor the fourth vertex (\( i \)–type).

HWs of the fourth possible type (\( j \)–type) consist of two self-avoiding branches, and correspond to the walks that enter the generator and leave it without visiting the remaining two vertices, then, after visiting other parts of the lattice, enter the same generator again at the third corner vertex and finally leave it at the fourth corner vertex. Examples of these types of walks are sketched in figure 3. In principle, it is possible to establish a closed set of recursion relations for the numbers \( g_l, h_l, i_l, \) and \( j_l \) of the corresponding walks for any 3d SF in the following form

\[
g_{l+1} = \sum_{n,m,k} G(n, m, k) g_l^n h_l^m i_l^k j_l^{b(b+1)(b+2)/6} + (n+m+k),
\]
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Figure 3. Examples of four possible types of open HWs on $G_3^3(2)$. Vertices not visited by the Hamiltonian walker are encircled.

\[ h_{l+1} = \sum_{n,m,k} H(n,m,k)g^n_h l^m_i j^k l_{ij} \frac{b(b+1)(b+2)}{6} - (n+m+k), \]
\[ i_{l+1} = \sum_{n,m,k} I(n,m,k)g^n_i l^m_h j^k l_{ij} \frac{b(b+1)(b+2)}{6} - (n+m+k), \]
\[ j_{l+1} = \sum_{n,m,k} J(n,m,k)g^n_j l^m_i j^k l_{ij} \frac{b(b+1)(b+2)}{6} - (n+m+k), \]

where $G(n,m,k)$, $H(n,m,k)$, $I(n,m,k)$, and $J(n,m,k)$ are the numbers of open Hamiltonian configurations of the corresponding types, with $n$ branches of the $g$–type, $m$ branches of the $h$–type, $k$ branches of the $i$–type, and \([b(b+1)(b+2)/6 - (n+m+k)]\) branches of the $j$–type. The number $C_{l+1}$ of all closed HWs within the $(l+1)$th order generator for any $b$ is equal to

\[ C_{l+1} = \sum_{n,m,k} B(n,m,k)g^n_l h^m_i j^k l_{ij} \frac{b(b+1)(b+2)}{6} - (n+m+k), \]

where $B(n,m,k)$ is the number of all closed Hamiltonian configurations with $n$ branches of the $g$–type, $m$ branches of the $h$–type, $k$ branches of the $i$–type, and \([b(b+1)(b+2)/6 - (n+m+k)]\) branches of the $j$–type. As one can see, the recursion relations (3.1) for the number of open HWs on 3d Sierpinski fractals are much more complicated than the corresponding equations (2.3) for 2d SFs. Consequently, it is not possible to find explicit expression, similar to (2.9), for the connectivity constant $\omega$. Instead, one should perform a numerical analysis of the recursion relations (3.1) in order to find the value of $\omega$. We shall demonstrate the method on the particular case of the $b = 2$ fractal.

By computer enumeration of the possible HW configurations within the $(l+1)$th-order generator $G_{l+1}^3(2)$ of the $b = 2$ 3d Sierpinski fractal, we found the following recursion relations:

\[ g' = 6g^2j^2 + 4g^3j + 2g^4 + 12i^2h + 24ijh + 24ig^2h + 8i^2h^2, \]
\[ h' = 24j^2hg + 16h^2ij + 16h^3 + 32h^2gi + 24g^2hj, \]
\[ i' = 12igj^2 + 12ig^2j + 8ig^3 + 8i^2jh + 16i^2gh, \]
\[ j' = 8i^2h^2 + 48igjh + 22j^4 + 2g^4 + 8g^3j + 24ig^2h, \]

where we have used the prime symbol as a superscript for the numbers of HWs on the $(l+1)$th-order generator $G_{l+1}^3(2)$ and no indices for the $l$th-order numbers. From the
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definition (1.2) of the connectivity constant and the formula for the number $C_{l+1}$ of closed HWs within $G^3_{l+1}(2)$:

$$C_{l+1} = 16g_l^2h_l^2,$$

(3.6)
it then follows that

$$\ln \omega = \lim_{l \to \infty} \frac{\ln C_{l+1}}{N_{l+1}} = \frac{1}{2} \lim_{l \to \infty} \frac{\ln g_l}{N_l} + \frac{1}{2} \lim_{l \to \infty} \frac{\ln h_l}{N_l},$$

(3.7)
where $N_l = 2(4^l + 1)$ is the number of sites in $G^3_l(2)$. On the other hand, from the recursion relation (3.2) for the numbers $g_l$ one obtains

$$\lim_{l \to \infty} \frac{\ln g_{l+1}}{N_{l+1}} = \frac{1}{2} \lim_{l \to \infty} \frac{\ln g_l}{N_l} + \frac{1}{2} \lim_{l \to \infty} \frac{\ln j_l}{N_l} + \frac{1}{2} \lim_{l \to \infty} \frac{1}{4^{l+1}} \ln \left( 6 + 4x_l + 2x_l^2 + 24y_l z_l + \frac{12y_l z_l}{x_l^2} + \frac{24y_l z_l}{x_l} + \frac{8y_l^2 z_l^2}{x_l^2} \right),$$

(3.8)
where $x_l = g_l/j_l$, $y_l = h_l/j_l$ and $z_l = i_l/j_l$. In a similar way, from (3.3) it follows that:

$$\lim_{l \to \infty} \frac{\ln h_{l+1}}{N_{l+1}} = \frac{1}{4} \lim_{l \to \infty} \frac{\ln h_l}{N_l} + \frac{1}{4} \lim_{l \to \infty} \frac{\ln g_l}{N_l} + \frac{1}{2} \lim_{l \to \infty} \frac{\ln j_l}{N_l} + \frac{1}{2} \lim_{l \to \infty} \frac{1}{4^{l+1}} \ln 8 \left( 3 + 3x_l + 2x_l^2 + 4y_l z_l + \frac{2y_l z_l}{x_l} \right).$$

(3.9)
The new variables $x_l$, $y_l$ and $z_l$ fulfill recursion relations which are easy to deduce from their definitions and Eqs. (3.2, 3.3), and are not difficult to iterate (starting with initial values $x_1 = \frac{1}{11}$, $y_1 = 1$ and $z_1 = \frac{4}{7}$), following from the numbers $g_1 = 88$, $h_1 = 112$, $i_1 = 64$ and $j_1 = 112$, found by direct computer enumeration of the corresponding HWs within the generator $G^3_1(2)$). One quickly finds that $x_l$, $y_l$, and $z_l$ tend to zero, and $z_l \ll x_l \ll y_l$, $x_l^2 \sim y_l z_l$ for large $l$, meaning that the last terms on the right–hand sides of equations (3.8) and (3.9) tend to zero. It is then straightforward to see that from (3.8) and (3.9) it follows that

$$\lim_{l \to \infty} \frac{\ln g_l}{N_l} = \lim_{l \to \infty} \frac{\ln h_l}{N_l} = \lim_{l \to \infty} \frac{\ln j_l}{N_l},$$

and, according to (3.7), one finds

$$\ln \omega = \lim_{l \to \infty} \frac{\ln j_l}{N_l},$$

(3.10)
Instead of the number $j_l$, which rapidly grows with $l$, it is convenient to introduce yet another variable

$$u_l = \frac{\ln j_l}{4^l} - \frac{\ln 22}{3} \left( \frac{1}{4} - \frac{1}{4^l} \right),$$

(3.11)
which has the initial value $u_1 = \frac{\ln 112}{4}$. Numerically iterating its recursion relation, together with those for $x_l$, $y_l$ and $z_l$, one can show that $u_l$ tends to 1.2507788499... when $l \to \infty$. Finally, since $\ln \omega = \frac{1}{2} \lim_{l \to \infty} u_l + \frac{1}{24} \ln 22$, the connectivity constant for the $b = 2$ 3d Sierpinski fractal is equal to $\omega = 2.12587...$. 
In order to find the first correction to the leading-order behavior of $C_l$ one needs to know the asymptotic behavior of the numbers $x_l$, $y_l$ and $j_l$, according to the formula

$$\frac{\ln C_{l+1}}{N_{l+1}} = \frac{\ln 16}{N_{l+1}} + 2 \frac{\ln x_l}{N_{l+1}} + 2 \frac{\ln y_l}{N_{l+1}} + 4 \frac{\ln j_l}{N_{l+1}}, \quad (3.12)$$

following from (3.6) and the definition of $x_l$ and $y_l$. Keeping only the leading-order terms in the recursion relation for $x_l$ one gets

$$x_{l+1} \approx \text{const} x_l^2,$$

which means that $x_l$ behaves as

$$x_l \sim \lambda^{2l}, \quad (3.13)$$

for large $l$. Numerically iterating the recursion relations for $x_l$, $y_l$ and $z_l$ one finds

$$\lambda = \lim_{l \to \infty} \frac{\ln x_l^2}{2l} = 0.9055...$$

It then follows that $y_{l+1} \approx \frac{12}{11} x_l y_l$, $z_{l+1} \approx \frac{6}{11} x_l z_l$, implying that $y_l z_l \sim 2^l$, which, together with the numerically established relation $x_l^2 \sim y_l z_l$, gives

$$z_l \sim 2^{-1/2} x_l \sim 2^{-1/2} \lambda^{2l}, \quad y_l \sim 2^{1/2} \lambda^l. \quad (3.14)$$

On the other hand, from the definition of $u_l$ (3.11) and the corresponding recursion relation it follows

$$\ln j_l = \ln \omega - \frac{1}{4l} \ln 22 - \frac{1}{4l} \ln (\frac{1}{4} - \frac{1}{4l}) + \sum_{k=1}^{l-1} \frac{1}{4k+1} \ln \left( 1 + \frac{4}{11} x_k^3 + \frac{1}{11} x_k^4 + \frac{24}{11} x_k y_k z_k + \frac{12}{11} x_k^2 y_k z_k + \frac{4}{11} y_k^2 z_k^2 \right),$$

and consequently, since

$$\sum_{k=1}^{\infty} \frac{1}{4k+1} \ln \left( 1 + \frac{4}{11} x_k^3 + \frac{1}{11} x_k^4 + \frac{24}{11} x_k y_k z_k + \frac{12}{11} x_k^2 y_k z_k + \frac{4}{11} y_k^2 z_k^2 \right) \leq \ln \frac{104}{11} \sum_{k=1}^{\infty} \frac{1}{4k+1} = \frac{1}{3} \ln 104 \frac{1}{11},$$

which is not difficult to show, one obtains

$$\ln j_l = 2 \ast 4l \ln \omega + O(1). \quad (3.15)$$

Finally, from (3.12)–(3.15) it follows that

$$\ln C_l = N_l \ln \omega + N_l^{1/2} \ln \lambda \sqrt{2} + \frac{1}{2} \ln N_l + O(1),$$

which means that the behavior (1.1) of the number of HWs, expected for homogeneous lattices, is also satisfied for this fractal lattice, with the following values of the exponents:

$$\sigma = \frac{1}{2}, \quad a = \frac{1}{2}.$$

The number of all possible HW configurations within a generator of the 3d Sieprinski fractal grows rapidly with $b$. In Appendix C we give the corresponding
recursion relations for the numbers $g_l$, $h_l$, $i_l$ and $j_l$ found for the $b = 3$ 3d SF, together with their initial values, and the formula for the number $C_{l+1}$ of closed HWs. The CPU time required for the enumeration and classification of HW configurations was so long for the $b = 3$ case that we could not go beyond it. However, the method used for $b = 2$ 3d SF in principle could be applied for any $b$ with no qualitative difference. Analyzing recursion relations given in Appendix C in that manner, for the $b = 3$ 3d SF we found that the number $C_l$ of closed HWs obeys a scaling form similar to that of the $b = 2$ 3d case with the following values for the connectivity constant $\omega$ and exponents $\sigma$ and $a$:

$$\omega = 2.2722364..., \quad \sigma = \frac{\ln 3}{\ln 10}, \quad a = 0.386...$$

4. Hamiltonian walks on n–simplex fractals

To complete our analysis of the nature of Hamiltonian walks on 2d and 3d Sierpinski fractal lattices, we now turn to n-simplexes, which are in some ways a generalization of SFs for $b = 2$ in $n - 1$ dimensions. To obtain an $n$-simplex lattice one starts with a complete graph of $n$ points and replaces each of these points by a new complete graph of $n$ points. The subsequent stages are constructed self-similarly, by repeating this procedure. After $l$ such iterations one obtains an $n$-simplex of order $l$, whereas the complete $n$-simplex lattice is obtained in the limit $l \to \infty$. It is trivial to see that the connectivity constant $\omega$ for HWs on 3-simplex lattice is equal to 1, whereas Bradley found that $\omega = 1.399710...$ for the 4-simplex. In principle, it is possible to establish an exact set of recursion relations for the numbers of suitably chosen prerequisite HWs on any $n$-simplex, as Bradley did for $n = 4$. Here we'll demonstrate the method on $n = 5$ and $n = 6$ cases.
4.1. 5-simplex

Any closed HW of order \((l+1)\) on this fractal can be decomposed into 5 open HWs through 5-simplices of order \(l\). There are two possible types of these open HWs, as depicted in Figure 4(a). The first corresponds to walks which enter the simplex at one corner, visit all vertices inside it, and leave it - we shall denote the number of these walks by \(C_{1,l}\). A walk of the second type enters the simplex at one of its five corners, wanders around it visiting some of the vertices inside it, leaves it through the second corner, and afterwards enters it again at the third corner, visits all the remaining vertices, and finally leaves it - let the total number of these walks on the \(l\)-th order 5-simplex be \(C_{2,l}\).

The total number \(C_{l+1}\) of closed HWs is equal to

\[
C_{l+1} = 12 C_{1,l}^5 + 30 C_{1,l} C_{2,l} + 60 C_{1,l}^3 C_{2,l}^2 + 132 C_{2,l}^5
\]  (4.1)

which we found by computer enumeration, together with the recursion relations

\[
C_{1,l+1} = 6 C_{1,l}^5 + 30 C_{1,l}^4 C_{2,l} + 78 C_{1,l}^3 C_{2,l}^2 + 96 C_{1,l}^2 C_{2,l}^3 + 132 C_{1,l} C_{2,l}^4 + 132 C_{2,l}^5,
\]
\[
C_{2,l+1} = 2 C_{1,l}^5 + 13 C_{1,l}^4 C_{2,l} + 32 C_{1,l}^3 C_{2,l}^2 + 88 C_{1,l}^2 C_{2,l}^3 + 220 C_{1,l} C_{2,l}^4 + 186 C_{2,l}^5.
\]  (4.2)

Initial values for these numbers are \(C_{1,1} = 6\) and \(C_{2,1} = 2\). From the recursion relation (4.1) it follows that

\[
\ln C_{l+1} \bigg/ \frac{-1}{5^{l+1}} = \ln C_{2,l} \bigg/ \frac{-1}{5^{l}} + \frac{1}{5^{l+1}} \left(132 + 60 x_i^3 + 30 x_i^4 + 12 x_i^5\right),
\]  (4.3)

where the new variable \(x_i = C_{1,l} / C_{2,l}\) satisfies the recursion relation

\[
x_{i+1} = \frac{22 + 22 x_i + 16 x_i^2 + 13 x_i^3 + 5 x_i^4 + x_i^5}{186 + 220 x_i + 88 x_i^2 + 32 x_i^3 + 13 x_i^4 + 2 x_i^5},
\]  (4.4)

obtained from (4.2). Numerically iterating this relation, starting with \(x_1 = 3\), we find that \(x_l \to 0.802318837...\) when \(l \to \infty\), and consequently, since \(N_l = 5^l\), it follows that

\[
\ln \omega = \lim_{l \to \infty} \ln C_{l+1} \bigg/ \frac{-1}{5^{l+1}} = \lim_{l \to \infty} \ln C_{2,l} \bigg/ \frac{-1}{5^{l}} + \frac{1}{5^{l+1}} \ln \left(1 + \frac{5}{11} x_i^3 + \frac{5}{22} x_i^4 + \frac{1}{11} x_i^5\right) = \lim_{l \to \infty} \ln C_{2,l} \bigg/ \frac{-1}{5^{l}}.
\]

The last limiting value can be quickly calculated if we introduce the variable

\[
y_l = \ln C_{2,l} \bigg/ \frac{-1}{5^{l}} - \ln \frac{186}{4} \left(\frac{1}{5} - \frac{1}{5}\right) - \ln \frac{2}{5},
\]

which, according to (4.2), obeys the recursion relation

\[
y_{l+1} = y_l + \frac{1}{5^{l+1}} \ln \left(1 + \frac{110}{93} x_l + \frac{44}{93} x_l^2 + \frac{16}{93} x_l^3 + \frac{13}{186} x_l^4 + \frac{1}{93} x_l^5\right),
\]  (4.5)

and has the initial value \(y_1 = 0\). Then, iterating (4.5) simultaneously with (4.4) we find \(y_l \to 0.141065489481...\) for large \(l\), and finally, since \(\ln \omega = \lim_{l \to \infty} y_l + \ln 186/20 + \ln 2/5\), we obtain \(\omega = 1.717769...\)

To examine the leading order correction to the number of Hamiltonian walks on the 5-simplex fractal, we note that for any \(k\) the quantity \(y_k\) can be written as

\[
y_l = y_k + \sum_{m=k}^{l-1} (y_{m+1} - y_m).
\]

Taking the \(l \to \infty\) limit and keeping \(k\) fixed in that equation, one obtains:

\[
\ln \omega = \ln C_{2,k} \bigg/ \frac{-1}{5^{k}} + \ln \frac{186}{4} \sum_{m=k}^{\infty} \frac{1}{5^{m+1}} \ln \left(1 + \frac{110}{93} x_m + \frac{44}{93} x_m^2 + \frac{16}{93} x_m^3 + \frac{13}{186} x_m^4 + \frac{1}{93} x_m^5\right).
\]
Since \( x_1 = 3 \) and the array \( x_m \) is monotonically decreasing, the sum on the right hand side of the above equation is bounded from above by

\[
\ln(21.72) \sum_{m=k}^{\infty} \frac{1}{5^{m+1}} = \ln(21.72) \frac{1}{5^{k+1}} (1 + \frac{1}{5} + \ldots) = \frac{1}{4} \ln(21.72) \frac{1}{5^k}.
\]

Therefore we can conclude that

\[
\ln C_{2,k} = 5^k \ln \omega + O(1) \quad \text{and} \quad \ln C_{1,k} = \ln C_{2,k} + \ln x_k = 5^k \ln \omega + O(1).
\] (4.6)

Finally, for the number \( C_l \) of closed HWs, from (4.3), we obtain the same behavior:

\[
\ln C_l = 5^l \ln \omega + O(1).
\] (4.7)

Comparing with equation (1.1) we see that in the case of the 5-simplex fractal both the surface and the power correction to the number of HWs are absent.

4.2. 6-simplex

In addition to the \( C_{1,l} \) and \( C_{2,l} \)-type walks, already defined for the 5-simplex, one should introduce one other type of walks for complete enumeration of HWs on the 6-simplex lattice. These walks enter the 6-simplex three times, as depicted in Figure 4(b). Let’s denote their number on the 6-simplex of order \( l \) by \( C_{3,l} \). The total number \( C_{l+1} \) of closed HWs within the 6-simplex of order \( (l + 1) \) is equal to

\[
C_{l+1} = 60 C_{1,l}^6 + 360 C_{1,l}^5 C_{2,l} + 1170 C_{1,l}^4 C_{2,l}^2 + 1920 C_{1,l}^3 C_{3,l} + 3960 C_{1,l}^2 C_{2,l}^4 + 7920 C_{1,l} C_{2,l}^5 + 5580 C_{2,l}^6,
\] (4.8)

as we found by computer enumeration. Numbers \( C_{1,l} \), \( C_{2,l} \) and \( C_{3,l} \) satisfy closed set of recursion relations, which can be numerically analyzed in a way similar, although more complicated than that used in the case of the 5-simplex lattice (see Appendix D). In contrast to the 5-simplex case, where numbers \( C_{1,l} \), \( C_{2,l} \) and \( C_l \) have the same asymptotic behaviour (4.6) and (4.7), here we obtained

\[
\ln C_{1,l} = 6^l \ln \omega + 2^{l+1} \ln \lambda + O(1),
\] (4.9)

\[
\ln C_{2,l} = 6^l \ln \omega + 2^l \ln \lambda + O(1),
\] (4.10)

\[
\ln C_{3,l} = 6^l \ln \omega + O(1),
\] (4.11)

with \( \omega = 2.0550..., \lambda = 0.9864 \) and

\[
\ln C_l = N_l \ln \omega + N_l^0 \ln \mu_S + O(1),
\] (4.12)

with \( N_l = 6^l, \sigma = \ln 2/\ln 6 \) and \( \mu_S = \lambda^3 \). The surface correction to the number of HWs is present here, but still there is no term proportional to \( \ln N_l \), which would imply the presence of the power correction. The same conclusion was obtained by Bradley [13] for the 4-simplex lattice, with \( \sigma = 1/2 \).
5. Discussion and conclusions

The values of the connectivity constant $\omega$ found by us, as well as some values previously found by other authors [13] are depicted in Figure 5(a), as functions of the coordination number $z$ of the lattice, together with the Flory [28] and Orland [21] approximations, $\omega_F = (z - 1)/e$ and $\omega_O = z/e$, respectively. One can clearly see that $\omega$ increases with $z$, which is in accord with Flory $\omega_F = (z - 1)/e$ and Orland et al $\omega_O = z/e$ formulas, but it is obvious that $\omega$ depends on other lattice properties too. Furthermore, all values of $\omega$ lie between the values predicted by these two formulas, and it seems that these kind of approximations give satisfying results for fractal lattices studied here. Based on the information in this figure one may conclude that $\omega_O$ and $\omega_F$ are good upper and lower bounds for fractal HW connectivity constants.

In Figure 5(b) the connectivity constant $\omega$ for 2d SFs is presented as a function of the reciprocal of the scaling parameter $b$, where line connecting triangles serves merely as the guide to the eye. Open up-oriented triangle on vertical axes depicts the value of $\omega$ for triangular lattice.

The coordination number $z$ (the average number of nearest neighbours per site) for the $n$-simplex lattice is equal to $n$, whereas it can be shown that $z = 6(b + 2)/(b + 4)$ for 2d SF with the scaling parameter $b$, $z = 6$ for $b = 2$ 3d SF and $z = 6.75$ for $b = 3$ 3d SF.
of the reciprocal of the fractal scaling parameter $b$. It seems that for $b \to \infty$ the connectivity constant might approach its triangular lattice value. This is not surprising, since for $b = \infty$ the first step of the construction of the corresponding 2d SF is already the wedge of the triangular lattice. In the same limit, 3d SFs approach the corresponding three-dimensional Euclidean lattice, so it would be useful to obtain recursion relations for HWs on these fractals for larger $b$, since there are fewer results for more realistic three-dimensional lattices. This task requires faster computers, as well as establishing a better algorithm for enumerating the HW configurations within a generator of a fractal, which is something we are planning to do in the nearest future.

As for the correction to the leading-order asymptotic behavior of the number of HWs, we have shown that the surface correction $\mu_s N^\sigma$ appears for neither the two-dimensional Sierpinski fractals nor for the 5-simplex lattice. In contrast, for both three-dimensional Sierpinski fractals considered here, as well as for the 4-simplex \[13\] and the 6-simplex lattice, the surface correction is present, with the value of the exponent $\sigma = 1/d_f$, where $d_f$ is the fractal dimension of the corresponding lattice\[§\]. This result is certainly not a simple generalization of the formula proposed for the homogeneous lattices: $\sigma = (d - 1)/d$. This is actually not surprising: The correction term $\mu_s N^\sigma$ in \[1.1\] was originally introduced in order to take into account possible surface tension effects, since at low temperatures a SAW forms a compact globule (see, for instance, \[36\]). The value $\sigma = (d - 1)/d$ for homogeneous lattices then follows from the fact that the surface of such a globule is proportional to $N^{(d-1)/d}$. In the case of fractal lattices it is, however, questionable whether such surface effects exist at all. For instance, all sites of the $b = 2$ 3d SF lie on the surface, which is not the case for the $b = 3$ 3d SF, but for both lattices the number of HWs has the correction term $\mu_s N^\sigma$. Or, in the case of an $n$-simplex lattice all sites have the same number $n$ of neighbors, and one should not expect any surface correction, but still, for some of them the correction $\mu_s N^\sigma$ was found. So, it seems that the term $\mu_s N^{1/d_f}$, obtained for some of the studied fractals, does not originate from the surface effects, which is also supported by the fact that $N^{1/d_f}$ is proportional to the mean radius of the globule formed by a HW, and not to its surface.

It is interesting to note here that the existence of the term $\mu_s N^{1/d_f}$ in the scaling form for the number of HWs, coincides with the existence of the polymer coil to globule transition on the corresponding lattice. As was shown earlier, a polymer chain in a solvent, modeled by SAWs on the 2d Sierpinski fractals \[27\] and 5-simplex \[32\] can exist only in the swollen phase; on the 3d Sierpinski fractals \[33\] \[34\] \[35\], 4-simplex \[25\] and 6-simplex \[32\] lattices, when the temperature is lowered the polymer undergoes a collapse transition from an expanded state to a globule state (compact or semi-compact \[34\]). Analyzing the asymptotic behaviour of different types of open HWs on fractals one can observe that the collapse transition exists on lattices whose topology allows for the statistical domination of HWs which are not localised. In particular, HWs on the 2d SG fractals cannot enter a generator of any order more than once, i.e. all the walks

\[§\] Fractal dimension for 3d SF is $d_f = \ln(b(b + 1)(b + 2)/6)/\ln b$, and for the $n$-simplex lattice $d_f = \ln n/\ln 2$. 
are localised. On the 5-simplex fractal this is possible (see Figure 4(a)), but the number $C_{1,l}$ of localised HWs and the number $C_{2,l}$ of delocalised HWs (walks that enter every $l$th order 5-simplex twice) are of the same order (see (4.6), i.e. the delocalised HWs do not dominate. On the other hand, the delocalised HWs on the 3d SG fractals ($j$-type, see Figure 3) and on the 6-simplex ($C_{2,l}$– and $C_{3,l}$–type, see Figure 4(b)), as well as on the 4–simplex $[13]$, are possible, and furthermore, the number of these walks is much larger than the number of the localised HWs (see section 3 and subsection 4.2). This observation strongly resembles the conclusion obtained in a series of recent papers $[24]$ about the delocalisation of knots in the low-temperature globular phase. Although the term 'delocalisation' was not used in quite the same sense in these two contexts, it seems that the same effect is in question, and this problem deserves further investigation.

Finally, the power dependence of the overall number $C_N$ of closed HWs on the number of sites $N$ of the lattice was found only for the 3d SFs. A more detailed inspection of the calculation of the exponent $a$ for these lattices reveals that

$$a = \text{const} \lim_{l \to \infty} \frac{\ln h_l}{l}.$$

The numbers $h_l$ and $i_l$ correspond to the localised open Hamiltonian configurations that visit the maximal (4) and the minimal (2) number of vertices, respectively, within the generator of order $l$ (see Figure 3). In the case of 2d SFs, numbers of open configurations visiting the maximal or the minimal number of vertices were $h_l$ and $g_l$, respectively, and it was shown that the ratio $h_l/g_l=\text{const}$ for every $l$. Consequently, $\lim_{l \to \infty} \ln(h_l/i_l)/l = 0$, which may be the formal explanation for the absence of the power correction to the number of HWs on 2d SF. On the other hand, on the $n$-simplex lattices only one type of localised HW configurations is possible, so it appears that the power term in the scaling form for the number of HWs is obtained on lattices where a larger number of different types of localised configurations is possible.

In conclusion we can say that the method of exact recursion relations turned out to be very powerful for the generation and the enumeration of extremely long Hamiltonian walks on the two- and three-dimensional Sierpinski and $n$-simplex fractals. Furthermore, it allows for a detailed numerical analysis of HWs of different topologies. This enabled us to find various scaling forms for the number of closed HWs on these lattices. In the case of two-dimensional Sierpinski fractals, a closed-form expression is obtained for the connectivity constant. Very interesting results were obtained for the three-dimensional Sierpinski fractals. This should be utilized for attaining deeper insight into the realistic physical problems which can be modeled by Hamiltonian walks.

Acknowledgments

We thank Mr Antun Balaz and the Institute of Physics, Belgrade, for providing the computing facilities at the initial stage of this work. SEH acknowledges the financial support from the Serbian Ministry of Science and Environmental Protection (Project
Appendix A. Recursion relations for HWs on 2d Sierpinski fractals

In this appendix we present the derivation of relations (2.1) and (2.3) for open and closed HWs on 2d SFs, for general $b$.

It is obvious that the number of closed HWs on the $(l+1)$th order generator is of the form $C_{l+1} = \sum_i B_i h_i^{\alpha_i} g_i^{\beta_i}$, where $B_i$ is the number of all closed Hamiltonian configurations consisting of $\alpha_i$ steps of $h$-type and $\beta_i$ steps of $g$-type ("step" is here a part of HW that traverses $l$th order generator within the considered $(l+1)$th order generator). By definition, closed HW visits all sites of the $(l+1)$th generator, consequently it traverses all $b(b+1)/2$ $l$th order generators within it, so that

$$\alpha_i + \beta_i = \frac{b(b+1)}{2}. \quad (A.1)$$

On the other hand, every $h$-type step occupies all three vertices of the $l$th generator it traverses, whereas a $g$-type step occupies only two vertices (the entering and the exiting one). This means that the numbers $\alpha_i$ and $\beta_i$ also have to satisfy the following equation

$$2\alpha_i + \beta_i = \frac{(b+1)(b+2)}{2}, \quad (A.2)$$

since the number of $l$th order vertices inside the $(l+1)$th generator is $(b+1)(b+2)/2$. The system of equations (A.1) and (A.2) has a unique solution $\alpha_i = b+1$, $\beta_i = (b+1)(b-2)/2$ which completes the derivation of the relation (2.1).

For open $h$-type HWs, in a similar way, we have $h_{l+1} = \sum_i A_i h_i^{x_i} g_i^{y_i}$, where exponents $x_i$ and $y_i$ satisfy the system $x_i + y_i = \frac{b(b+1)}{2}$, $2x_i + y_i = \frac{(b+1)(b+2)}{2} - 1$, whose only solution is $x_i = b$, $y_i = b(b-1)/2$. At the same time, every $g$-type HW on a generator $G_{l+1}^2$ can be obtained from one and only one $h$-type HW (and vice versa), by substituting one $h$-step with a $g$-step, as depicted in Figure A1. This means that all of these walks have exactly $(b-1)$ $h$-steps and $[b(b-1)/2 + 1]$ $g$-steps, i.e. relations (2.3) are correct.
Appendix B. RG equations for SAW model on 2d SF

In this appendix we give exact RG equations \(2.10\) for calculating the connectivity constant for the ordinary SAW model on the \(b = 4\) and the \(b = 5\) two-dimensional Sierpinski fractals. These equations were obtained via computer enumeration and classification of all SAW configurations within the corresponding fractal generator. RG relations \(2.10\) for \(b = 6\) and 7 were also found, but they are too cumbersome to be quoted here and they are available upon request.

\[
b = 4: \quad B' = B^4 + 6B^5 + 6B^6 + 9B^7 + 9B^8 + 9B^9 + 4B^{10} + 4B^3B_1 + 30B^4B_1 + 36B^5B_1 + 56B^6B_1 + 56B^7B_1 + 26B^9B_1 + 6B^2B_1^2 + 57B^3B_1^2 + 84B^4B_1^2 + 134B^5B_1^2 + 143B^6B_1^2 + 128B^7B_1^2 + 56B^8B_1^2 + 4BB_1^3 + 51B^2B_1^3 + 96B^3B_1^3 + 156B^4B_1^3 + 168B^5B_1^3 + 128B^6B_1^3 + 40B^7B_1^3 + 4 + 21BB_1^4 + 55B^2B_1^4 + 93B^3B_1^4 + 9B^4B_1^4 + 48B^5B_1^4 + 3B_1^5 + 14BB_1^5 + 28B^2B_1^5 + 20B^3B_1^5 + B_1^6 + 4BB_1^6 \]

\[
b' = B^6B_1 + 6B^7B_1 + 7B^8B_1 + 4B^9B_1 + 6B^5B_2 + 38B^6B_2 + 44B^7B_2 + 26B^8B_2 + 14B^3B^3 + 92B^5B^3 + 102B^6B^3 + 56B^7B^3 + 16B^3B_1^4 + 106B^4B_1^4 + 104B^5B_1^4 + 40B^6B_1^4 + 9B^2B_1^5 + 58B^3B_1^5 + 40B^4B_1^5 + 2BB_1^6 + 12B^2B_1^6 \]

\[
b = 5: \quad B' = B^5 + 10B^6 + 20B^7 + 30B^8 + 54B^9 + 68B^{10} + 98B^{11} + 94B^{12} + 86B^{13} + 38B^{14} + 16B^{15} + 5B^4B_1 + 60B^5B_1 + 140B^6B_1 + 228B^7B_1 + 443B^8B_1 + 586B^9B_1 + 867B^{10}B_1 + 854B^{11}B_1 + 786B^{12}B_1 + 348B^{13}B_1 + 140B^{14}B_1 + 10B^3B_1^2 + 146B^4B_1^2 + 402B^5B_1^2 + 718B^6B_1^2 + 1521B^7B_1^2 + 2137B^8B_1^2 + 3203B^9B_1^2 + 3240B^{10}B_1^2 + 2918B^{11}B_1^2 + 1268B^{12}B_1^2 + 458B^{13}B_1^2 + 10B^2B_1^3 + 184B^3B_1^3 + 610B^4B_1^3 + 1218B^5B_1^3 + 2846B^6B_1^3 + 4316B^7B_1^3 + 6433B^8B_1^3 + 6648B^9B_1^3 + 5630B^{10}B_1^3 + 2306B^{11}B_1^3 + 664B^{12}B_1^3 + 5BB_1^4 + 126B^2B_1^4 + 523B^3B_1^4 + 1209B^4B_1^4 + 3170B^5B_1^4 + 5307B^6B_1^4 + 7678B^7B_1^4 + 7960B^8B_1^4 + 5960B^9B_1^4 + 2104B^{10}B_1^4 + 360B^{11}B_1^4 + 360B^{12}B_1^4 + 2B_1^5 + 4BB_1^5 + 249B^2B_1^5 + 71OB^3B_1^5 + 2159B^4B_1^5 + 4118B^5B_1^5 + 5604B^6B_1^5 + 5554B^7B_1^5 + 3292B^8B_1^5 + 776B^9B_1^5 + 6B^6 + 59BB_1^6 + 234B^2B_1^6 + 891B^3B_1^6 + 2031B^4B_1^6 + 2479B^5B_1^6 + 2086B^6B_1^6 + 744B^7B_1^6 + 5B_1^7 + 36B^2B_1^7 + 214B^3B_1^7 + 630B^4B_1^7 + 626B^5B_1^7 + 324B^6B_1^7 + 28B^8B_1^7 + 28B^3B_1^8 + 72B^3B_1^8 + 117B^2B_1^9 + 2BB_1^9 + 10BB_1^9 \]

\[
b' = B^8B_1 + 12B^9B_1 + 39B^{10}B_1 + 48B^{11}B_1 + 60B^{12}B_1 + 34B^{13}B_1 + 16B^{14}B_1 + 8B^7B_1^2 + 102B^8B_1^2 + 344B^9B_1^2 + 432B^{10}B_1^2 + 556B^{11}B_1^2 + 314B^{12}B_1^2 + 140B^{13}B_1^2 + 27B^6B_1^3 + 366B^7B_1^3 + 1278B^8B_1^3 + 1616B^9B_1^3 + 2098B^{10}B_1^3 + 1156B^{11}B_1^3 + 458B^{12}B_1^3 + 50B^{13}B_1^4 + 722B^6B_1^4 + 2600B^7B_1^4 + 3254B^8B_1^4 \]
Here we give the recursion relations for the numbers \( g_l \) and \( h_l \), obtained via computer enumeration. With symbols \( g_{l+1}, h_{l+1}, i_{l+1}, j_{l+1}, g_l, h_l, j_l \) and \( i_l \) abbreviated to \( g', h', i', j', g, h, i \) and \( j \), respectively, these relations have the following form:

\[
g' = 6120i^2g^3j^5 + 3312i^2g^4j^4 + 8176i^2g^5j^3 + 5068i^2g^6j^2 + 2964i^2g^7j + 776i^2g^8 + 13296i^3gj^5h + 12688i^3g^2j^4h + 36832i^3g^3j^3h + 36504i^3g^4j^2h + 27768i^3g^5jh + 9200i^3g^6h + 2080i^4g^4j^2h + 20224i^4gj^3h^2 + 40464i^4g^2j^2h^2 + 46064i^4g^3jh^2 + 21856i^4g^4h^2 + 2848i^5j^2h^3 + 12192i^5gjh^3 + 11328i^5g^2h^3 + 512i^6h^4
\]

\[
h' = 2928i^4g^4j^5 + 1288i^4g^5j^4 + 3296i^4g^6j^3 + 1760i^4g^7j^2 + 936i^4g^8j + 232i^4g^9 + 13296i^2g^2j^5h + 10608i^2g^3j^4h + 28160i^2g^4j^3h + 23840i^2g^5j^2h + 16632i^2g^6jh + 5168i^2g^7h + 12768i^3gj^5h^2 + 14336i^3g^2j^4h^2 + 50176i^3g^3j^3h^2 + 63232i^3g^4j^2h^2 + 56624i^3g^5jh^2 + 21856i^3g^6h^2 + 13824i^3g^7h^3 + 34432i^3gj^2h^3 + 48928i^3g^2j^2h^3 + 27968i^3g^3h^3 + 5824i^5j^3h^2 + 8192i^5gj^3h^2 + 13232i^5g^2j^3h^2 + 2224i^5g^3j^3h^2 + 2184i^5g^4j^3h^2 + 512i^6j^4h^3
\]

\[
i' = 12504i^3g^2j^5 + 7880i^3g^3j^4 + 19240i^3g^4j^3 + 13232i^3g^5j^2 + 8224i^3g^6j + 2184i^3g^7 + 528i^4j^5h + 4592i^4gj^4h + 24224i^4g^2j^3h + 36928i^4g^3j^2h + 33728i^4g^4jh + 13296i^4g^5h + 1184i^5j^3h^2 + 8384i^5gj^2h^2 + 18880i^5g^2j^2h^2 + 13024i^5g^3j^2h^2 + 640i^6j^3h^3 + 1600i^6gj^3h^3
\]

\[
j' = 4308i^2j^6 + 5808i^2gj^7 + 17424i^2g^2j^6 + 11936i^2g^3j^5 + 19164i^2g^4j^4 + 14096i^2g^5j^3 + 9208i^2g^6j^2 + 3360i^2g^7j + 544i^4g^2j^6 + 11616i^3j^5h + 21440i^3gj^5h + 51024i^3g^2j^4h + 66096i^3g^3j^3h + 50565i^3g^4j^2h + 28864i^3g^5jh + 6400i^3g^6h + 10312i^4j^4h^2 + 35296i^4gj^3h^2 + 53248i^4g^2j^2h^2 + 45440i^4g^3j^2h^2 + 14080i^4g^4j^2h^2 + 5728i^5j^2h^3 + 12544i^5gj^2h^3 + 7168i^5g^2j^2h^3 + 512i^6j^4h^4
\]

The initial values of these numbers are \( g_1 = 497000, \ h_1 = 728480, \ i_1 = 340476, \ j_1 = 811468 \) and the formula for the number \( C_{l+1} \) of closed HWs within \( G^3_{l+1}(3) \) is:

\[
C_{l+1} = 92g^8_{l+1} + 48g^9_{l+1} + 8g^{10}_{l+1} + 1792i_{l+1}g^6_{l+1}h_{l+1} + 1248i_{l+1}g^7_{l+1}h_{l+1} + 384i_{l+1}g^8_{l+1}h_l + 7568i_{l+1}g^4j^2_{l+1} + 7104i_{l+1}g^5j^2_{l+1} + 3008i_{l+1}g^6j^2_{l+1} + 10560i_{l+1}g^2j^2_{l+1} + 13440i^3g^3j^3_{l+1} + 7680i^3g^4j^3_{l+1} + 4480i^4j^3_{l+1} + 7680i^4j^2_{l+1} + 6016i_{l+1}g^2j^2_{l+1} + 512i_{l+1}g^5j^5_{l+1}.
\]
Appendix D. Analysis of recursion relations for HWs on 6-simplex lattice

Recursion relations for the numbers $C_{1,l}$, $C_2$, and $C_{3,l}$ of open HW configurations within the 6-simplex of order $l$ have the following form:

$$
C_1' = 5544C_1^2C_2 + 1728C_1^3C_2^3 + 2592C_1^4C_2^4 + 480C_1^4C_2C_3 + 960C_1^4C_2^2
+ 48C_1^5C_3 + 216C_1^6C_2 + 24C_1^6 + 25008C_1^4C_2^3 + 20544C_1^5C_3 + 6576C_1^6
+ 11328C_1^2C_2^2 + 15264C_1C_2^3 + 8688C_1^2C_3^2 + 4992C_1C_3^2C_2C_3
$$

(D.1)

$$
C_2' = 94336C_2^4 + 76800C_2^5C_3 + 48160C_2^6 + 23520C_5C_3^2 + 6576C_1C_2^5
+ 17120C_2C_3^2 + 2832C_1^2C_2^2 + 5088C_1^2C_3^2 + 832C_2^2C_3 + 3620C_1C_2
+ 1232C_1^2C_2^2 + 144C_1C_2C_3 + 324C_1^2C_2 + 64C_1C_2 + 6C_1 + 16C_1C_3
$$

(D.2)

$$
C_3' = 541568C_3^6 + 94336C_2^5C_3 + 43200C_2^6 + 2944C_2^5C_3 + 2940C_2^6 + 6252C_1C_2C_3
+ 2568C_1C_2 + 1416C_2C_3^2 + 954C_2C_3 + 208C_1^2C_2 + 54C_1C_2^2
+ 6C_1^2C_2 + 12C_1C_2 + C_1^2,
$$

(D.3)

where $C_i' = C_{i+1}$ and $C_i = C_{i,l}$, with the initial values $C_{1,1} = 24$, $C_{2,1} = 6$, $C_{3,1} = 1$.

Introducing new variables

$$
x_l = \frac{C_{1,l}}{C_{2,l}}, \quad y_l = \frac{C_{2,l}}{C_{3,l}}, \quad z_l = \frac{\ln C_{3,l}}{6l} - \frac{\ln 541568}{5} \left( \frac{1}{6} - \frac{1}{6l} \right),
$$

one can obtain closed set of recursion relations which iterates towards $x_l, y_l \to 0, z_l \to 0.280204...$ for $l \to \infty$. On the other hand, from (D.2) it follows that

$$
\ln C_{l+1} = \ln \frac{y_l}{6l} + \ln 5580 \cdot \frac{1}{6l+1} + \frac{1}{6l+1} \ln \left( 1 + \frac{44}{31} x_l + \frac{22}{31} x_l^2 + \frac{32}{93} x_l^3 + \frac{13}{62} x_l^4 + \frac{2}{31} x_l^5 + \frac{1}{93} x_l^6 \right),
$$

whereas from (D.2) one gets

$$
\ln C_{l+1} = \ln \frac{y_l}{6l} + \ln 94336 \cdot \frac{1}{6l+1} + \frac{1}{6l+1} \ln \left( 1 + \frac{600}{737} y_l + \frac{1505}{2948} y_l^2 + \frac{521}{2948} x_l y_l^2 + \frac{177}{5896} x_l^2 y_l^2 + \frac{735}{2948} y_l^3 + \frac{535}{2948} x_l y_l^3 + \frac{159}{2948} x_l^2 y_l^3 + \frac{1}{1474} x_l^3 y_l^3 + \frac{3}{47168} x_l^4 y_l^4 \right),
$$

and, consequently,

$$
\ln \omega = \lim_{l \to \infty} \frac{\ln C_{3,l}}{6l} + \frac{4}{3} \lim_{l \to \infty} \frac{\ln y_l}{6l}.
$$

Since the numerical analysis shows that $\ln y_l/6l \to 0$, we finally obtain $\omega = 2.0550...$

To find the leading order asymptotic behavior of the number of Hamiltonian walks on the 6-simplex fractal ($C_{1,l}, C_{2,l}, C_{3,l}$ and $C_l$), we conduct an analysis similar to the one used in the case of the 5-simplex. However, we can tell right away that $C_{1,l}, C_{2,l}$ and $C_{3,l}$ will have mutually different asymptotics, since their ratios $x_l = \frac{C_{1,l}}{C_{2,l}}$ and $y_l = \frac{C_{2,l}}{C_{3,l}}$ were found to be approaching zero for large $l$. By iterating the recursion relations for $x_l$ and $y_l$, one finds that the ratio of $x_l$ and $y_l$ quickly approaches a constant equal to 1.521868... Keeping only the terms with the lowest sum of powers in $x_l$ and $y_l$ in the recursion relations, we find $x_{l+1} = 12 1042y_l^2$, $y_{l+1} = 2 47168 y_l^4$, for $l \gg 1$ and the ratio $x_{l+1}/y_{l+1}$ is indeed 1.521868... From the above equation for $y_{l+1}$, one can see that

$$
\ln y_l \approx 2^l \ln \lambda, \quad \lambda = \lim_{l \to \infty} \frac{\ln y_l}{2^l} = 0.9864...
$$

(D.4)
The asymptotic relation (4.11) can be obtained starting with \( z_l = z_k + \sum_{m=k}^{l-1} (z_{m+1} - z_m) \) and following a procedure completely analogous to the one used in the 5-simplex case. Relations (4.9) and (4.10) then follow from (4.11), (D.4) and the fact that \( x_l \) and \( y_l \) are proportional in the large \( l \) limit. From (4.9)-(4.11) it is apparent that in the limiting case \( C_{1,l} \ll C_{2,l} \ll C_{3,l} \), since \( \lambda < 1 \). Therefore, it holds that \( C_{l+1} \approx 5580C_{2,l}^6 \), so finally \( \ln C_l = 6^l \ln \omega + 3 * 2^l \ln \lambda \), which is equivalent to (4.12).

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