A VERY GENERAL QUARTIC DOUBLE FOURFOLD IS NOT STABLY RATIONAL

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Abstract. We prove that a very general double cover of the projective four-space, ramified in a quartic threefold, is not stably rational.

1. Introduction

In this note we consider quartic double fourfolds, i.e., hypersurfaces $X_f$ in the weighted projective space $\mathbb{P}(2,1,1,1,1,1)$, with homogeneous coordinates $(s,x,y,z,t,u)$, given by a degree four equation of the form

$$s^2 + f(x,y,z,t,u) = 0.$$  

The failure of stable rationality for cyclic covers of projective spaces has been considered by Voisin [Voi15], Beauville [Bea16], Colliot-Thélène–Pirutka [CTP16a], and Okada [Oka16]. We work over an uncountable ground field $k$ of characteristic zero. Our main result is

Theorem 1. Let $f \in k[x,y,z,t,u]$ be a very general degree four form. Then $X_f$ is not stably rational.

This note is inspired by [Bea15], which used the new technique of the decomposition of the diagonal [Voi15, CTP16b, Tot16]. The main difficulty is to construct a special $X$ in the family (1.1) with following properties:

(O) Obstruction: the second unramified cohomology group $H^2_{nr}(X)$ (or another birational invariant) does not vanish,

(R) Resolution: there exists a resolution of singularities $\beta : \tilde{X} \to X$, such that the morphism $\beta$ is universally $\text{CH}_0$-trivial,

(see, e.g., Sections 2 and 4 of [HPT16] for definitions). The verification of both properties for potential examples of $X$ is notoriously difficult. The example considered in [Bea15] satisfies the second property, but not the first.

Date: May 12, 2016.
Our main goal here is to produce an $X$ satisfying both. We have a candidate example:

$$(1.2) \quad V : s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0.$$  

The singular locus of $V$ is a connected curve, consisting of 4 components: two nodal cubics, a conic, and a line. How do we find this example? We may transform equation (1.2) to

$$(1.3) \quad yzs^2 + xzt_1^2 + xyu_1^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))v_1^2 = 0.$$  

Precisely, we homogenize via an additional variable $v$, multiply through by $yz$, and absorb the squares into the variables $t_1, u_1,$ and $v_1$. The resulting equation gives a bidegree $(2,2)$ hypersurface $V' \subset \mathbb{P}^2 \times \mathbb{P}^3$, birational to $V$ via the coordinate changes. In [HPT16] we proved that this $V'$ satisfies both properties (O) and (R). In particular, $V$ also satisfies (O), since unramified cohomology is a birational invariant.

A direct verification of property (R) for this $V$ is possible (so that we could take $X = V$), but we found it more transparent to take an alternative approach, applying the specialization argument twice: First we can specialize a very general $X_f$ to a quartic double fourfold $X$ which is singular along a line $\ell$ (contained in the ramification locus); we choose $X$ to be very general subject to this condition. Then we show that the blowup morphism

$$\beta : \tilde{X} := \text{Bl}_\ell(X) \to X$$

is universally $\text{CH}_0$-trivial and that $\tilde{X}$ is smooth, i.e., $X$ satisfies (R). Furthermore, there exists a quadric bundle structure $\pi : \tilde{X} \to \mathbb{P}^2$, with degeneracy divisor a smooth octic curve. In Section 2 we analyze this geometry. We consider a degeneration of these quadric bundles to a fourfold $X'$ which is birational to $V'$, and thus satisfies (O). The singularities of $X'$ are similar to those considered in [HPT16]; the verification of the required property (R) for $X'$ is easier in this presentation. This is the content of Section 3. In Section 4 we give the argument for failure of stable rationality of very general (1.1).

Acknowledgments: The first author was partially supported through NSF grant 1551514.
2. GEOMETRY OF QUARTIC DOUBLE FOURFOLDS

Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. From the equation (1.1) we see that the quartic double fourfold $X$ is singular precisely along the singular locus of the quartic threefold $Y \subset \mathbb{P}^4$ given by $f = 0$.

We will consider quartic threefolds $Y$ double along $\ell$. These form a linear series of dimension

$$\binom{8}{4} - 5 - 12 = 53$$

and taking into account changes of coordinates—automorphisms of $\mathbb{P}^4$ stabilizing $\ell$—we have 34 free parameters.

Let $\beta : \tilde{X} \to X$ be the blowup of $X$ along $\ell$. We will analyze its properties by embedding it into natural bundles over $\mathbb{P}^2$.

We start by blowing up $\ell$ in $\mathbb{P}^4$. Projection from $\ell$ gives a projective bundle structure

$$\pi : \text{Bl}_\ell(\mathbb{P}^4) \to \mathbb{P}^2$$

where we may identify

$$\text{Bl}_\ell(\mathbb{P}^4) \cong \mathbb{P}(\mathcal{E}), \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

Write $h$ for the hyperplane class on $\mathbb{P}^2$ and its pullbacks and $\xi$ for the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Taking global sections

$$\mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \to \mathcal{E}^\vee$$

induces morphisms

$$\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}) \cong \mathbb{P}^4 \times \mathbb{P}^2,$$

projecting onto the first factor gives the blow up. Its exceptional divisor

$$E \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \cong \mathbb{P}^3 \times \mathbb{P}^2$$

has class $\xi - h$.

Let $\tilde{Y} \subset \mathbb{P}(\mathcal{E})$ denote the proper transform of $Y$, which has class

$$4\xi - 2E = 2\xi + 2h.$$

Conversely, divisors in this linear series map to quartic hypersurfaces in $\mathbb{P}^4$ singular along $\ell$. Since $2\xi + 2h$ is very ample in $\mathbb{P}(\mathcal{E})$ the generic such divisor is smooth. The morphism $\pi$ realizes $\tilde{Y}$ as a conic bundle over $\mathbb{P}^2$; its defining equation $q$ may also be interpreted as a section of the vector bundle $\text{Sym}^2(\mathcal{E}^\vee)(2h)$. Let $\gamma : \tilde{Y} \to Y$ denote the resulting resolution; its exceptional divisor $F = \tilde{Y} \cap E$ is a divisor of bidegree...
(0, 2) in $E \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Hence $F \rightarrow \ell$ is a trivial conic bundle and $\gamma$ is universally CH$_0$-trivial.

Let $\tilde{X} \rightarrow \mathbb{P}(\mathcal{E})$ denote the double cover branched over $\tilde{Y}$, i.e., $s^2 = q$. This naturally sits in the projectization of an extension
\[ 0 \rightarrow \mathcal{L} \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0, \]
where $\mathcal{L}$ is a line bundle. Note the natural maps
\[ \text{Sym}^2(\mathcal{E}^\vee) \hookrightarrow \text{Sym}^2(\mathcal{F}^\vee) \rightarrow \mathcal{L}^{-2}, \]
and their twists
\[ \text{Sym}^2(\mathcal{E}^\vee)(2h) \hookrightarrow \text{Sym}^2(\mathcal{F}^\vee)(2h) \rightarrow \mathcal{L}^{-2}(2h); \]
the last sheaf corresponds to the coordinate $s$. Since we are over $\mathbb{P}^2$ the extension above must split; furthermore, the coordinate $s$ induces a trivialization
\[ \mathcal{L}^{-2}(2h) \simeq \mathcal{O}_{\mathbb{P}^2}. \]
Thus we conclude
\[ \mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1). \]

The divisor $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ is generically smooth; let $\beta : \tilde{X} \rightarrow X$ denote the induced resolution of $X$. Its exceptional divisor is a double cover of $E$ branched over $F$, isomorphic to a product of a smooth quadric surface with $\mathbb{P}^1$. (A double cover of $\mathbb{P}^2$ branched along a conic curve is a smooth quadric surface.) It follows that $\beta$ is universally CH$_0$-trivial.

We summarize the key elements we will need:

**Proposition 2.** Let $X \rightarrow \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. Assume that $Y$ is singular along a line $\ell$ and generic subject to this condition. Let $\beta : \tilde{X} \rightarrow X$ be the blowup of $X$ along $\ell$. Then $\tilde{X}$ is smooth and $\beta$ universally CH$_0$-trivial.

Regarding $\tilde{X} \subset \mathbb{P}(\mathcal{F})$, there is an induced quadric surface fibration
\[ \pi : \tilde{X} \rightarrow \mathbb{P}^2. \]
Let $D$ denote the degeneracy curve, naturally a divisor in
\[ \det(\mathcal{F}^\vee(2h)) \simeq \mathcal{O}_{\mathbb{P}^2}(8). \]

The analysis above gives an explicit determinantal description of the defining equation of $D$. Choose homogeneous forms
\[ c \in \Gamma(\mathcal{O}_{\mathbb{P}^2}), \ F_1, F_2, F_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(2)), \ G_1, G_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(3)), H \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(4)) \]
so that the symmetric matrix associated with $\tilde{X}$ takes the form:

$$
\begin{pmatrix}
  c & 0 & 0 & 0 \\
  0 & F_1 & F_2 & G_1 \\
  0 & F_2 & F_3 & G_2 \\
  0 & G_1 & G_2 & H
\end{pmatrix}
$$

We fix coordinates to obtain a concrete equation for $\tilde{X}$. Let $(x, y, z)$ denote coordinates of $\mathbb{P}^2$, or equivalently, linear forms on $\mathbb{P}^4$ vanishing along $\ell$. Let $s$ denote a local coordinate trivializing $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$, $t$ and $u$ coordinates corresponding to $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{F}$. Then we have

$$(2.1) \tilde{X} = \{cs^2 + F_1 t^2 + 2F_2 tu + G_1 tv + 2G_2 uv + Hv^2 = 0\},$$

where $F_1, F_2, F_3, G_1, G_2$, and $H$ are homogeneous in $x, y, z$.

Finally, we interpret the degeneration curve in geometric terms. Ignoring the constant, we may write

$$D = (F_1 F_3 - F_2^2)H - F_3 G_1^2 + 2F_2 G_1 G_2 - F_1 G_2^2 = 0.$$ 

Modulo $F_1 F_3 - F_2^2$ we have

$$-F_3 G_1^2 + 2F_2 G_1 G_2 - F_1 G_2^2 = 0$$

which is equal to

$$\frac{-1}{F_1}(F_2 G_1 - F_1 G_2)^2 = \frac{-1}{F_3}(F_3 G_1 - F_2 G_2)^2.$$ 

Thus we conclude that $D$ is tangent to a quartic plane curve

$$C = \{F_1 F_3 - F_2^2\} = 0$$

at 16 points. Every smooth quartic plane curve admits multiple such representations: Surfaces

$$\{a^2 F_1 + 2ab F_2 + b^2 F_3 = 0\} \subset \mathbb{P}^1_{a,b} \times \mathbb{P}^2$$

are precisely degree two del Pezzo surfaces equipped with a conic bundle structure, the conic structures indexed by non-trivial two-torsion points of the branch curve $C$. One last parameter check: The moduli space of pairs $(C, D)$ consisting of a plane quartic and a plane octic tangent at 16 points depends on

$$14 + 44 - 16 - 8 = 34$$

parameters. This is compatible with our first parameter count.
Remark 3. Smooth divisors $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ as above necessarily have trivial Brauer group. This follows from Pirutka’s analysis [Pir16]: if the degeneracy curve is smooth and irreducible then there cannot be unramified second cohomology. It also follows from a singular version of the Lefschetz hyperplane theorem. Let $\zeta = c_1(O_{\mathbb{P}(\mathcal{F})}(1))$ so that $[\tilde{X}] = 2\zeta + 2h$. This is almost ample: the line bundle $\zeta + h$ contracts the distinguished section $s: \mathbb{P}^2 \to \mathbb{P}(\mathcal{F})$ associated with the summand $O_{\mathbb{P}^2}(1) \subset \mathcal{F}$ to a point but otherwise induces an isomorphism onto its image. In particular, $\zeta + h$ induces a small contraction in the sense of intersection homology. The homology version of the Lefschetz Theorem of Goresky-MacPherson [GM88, p. 150] implies that $0 \cong H^3(\mathbb{P}(\mathcal{F}), \mathbb{Z}) \to H^3(\tilde{X}, \mathbb{Z})$.

3. Singularities of the special fiber

We specialize (2.1) to:

\[(3.1) \quad s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0.\]

Proposition 4. The fourfold $X' \subset \mathbb{P}(\mathcal{F})$ defined by (3.1) admits a resolution of singularities $\beta': \tilde{X}' \to X'$ such that $\beta'$ is universally $\text{CH}_0$-trivial.

The remainder of this section is a proof of this result.

3.1. The singular locus. A direct computation in Magma (or an analysis as in [HPT16, Section 5]) yields that the singular locus of (3.1) is a connected curve consisting of the following components:

- Singular cubics:
  
  $E_z := \{v^2y(y-x)^2 + u^2x = z = s = t = 0\}$
  
  $E_y := \{v^2z(z-x)^2 + t^2x = y = s = u = 0\}$

- Conics:
  
  $R_x := \{u^2 - 4v^2 + t^2 = x = z - y = s = 0\}$
  
  $C_x := \{zu^2 + yt^2 = s = v = x = 0\}$

The nodes of $E_z$ and $E_y$ are

$\text{n}_z := \{z = s = t = y - x = u = 0\}$

$\text{n}_y := \{y = s = u = z - x = t = 0\}$,

respectively. Here $R_x$ and $C_x$ intersect transversally at two points,

$\text{r}_\pm := \{u \pm it = v = s = z - y = x = 0\}$;
$R_x$ is disjoint from $E_z$ and $E_y$, and the other curves intersect transversally in a single point (in coordinates $(x, y, z) \times (s, t, u, v)$):

$$
E_z \cap E_y = q_x := (1, 0, 0) \times (0, 0, 1),
E_z \cap C_x = q_y := (0, 1, 0) \times (0, 0, 1, 0),
E_y \cap C_x = q_z := (0, 0, 1) \times (0, 1, 0).
$$

This configuration of curves is similar to the one considered in \[HPT16\], but the singularities are different.

### 3.2. Local étale description of the singularities and resolutions.

The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

The main normal form is

\begin{equation}
(3.2) \quad a^2 + b^2 + c^2 = p^2 q^2
\end{equation}

which is singular along the locus

$$
\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}.
$$

This is resolved by successively blowing up along these components in either order. Indeed, after blowing up the first component, using \{A, B, C, P\} for homogeneous coordinates associated with the corresponding generators of the ideal, we obtain

$$
A^2 + B^2 + C^2 = P^2 q^2.
$$

The exceptional fibers are isomorphic to a non-singular quadric hypersurface (when $q \neq 0$) or a quadric cone (over $q = 0$). Dehomogenizing by setting $P = 1$, we obtain

$$
A^2 + B^2 + C^2 = q^2
$$

which is resolved by blowing up \{A = B = C = q = 0\}. This has ordinary threefold double points at each point, so the exceptional fibers are all isomorphic to non-singular quadric hypersurfaces.

There are cases where

$$
\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}
$$

are two branches of the same curve. For example, this could arise from

\begin{equation}
(3.3) \quad a^2 + b^2 + c^2 = (m^2 - n^2 - n^3)^2
\end{equation}

by setting $p = m - n \sqrt{1 + n}$ and $q = m + n \sqrt{1 + n}$. Of course, we cannot pick one branch to blow up first. We therefore blow up the origin first,
using homogeneous coordinates $A, B, C, D, P, Q$ corresponding to the generators to obtain

$$A^2 + B^2 + C^2 = P^2q^2 = Q^2p^2.$$ 

The resulting fourfold is singular along the stratum

$$A = B = C = q = p = 0$$

as well as the proper transforms of the original branches. Indeed, on dehomogenizing $P = 1$ we obtain local affine equation

$$A^2 + B^2 + C^2 = Q^2p^2;$$

this is singular along $\{A = B = C = p = 0\}$, the locus where the exceptional divisor is singular, and $\{A = B = C = Q = 0\}$, and proper transform of $\{a = b = c = q = 0\}$. The local affine equation is the same as (3.2); we resolve by blowing up the singular locus of the exceptional divisor followed by blowing up the proper transforms of the branches. This descends to a resolution of (3.3).

3.3. Computation in local charts. We exploit the symmetry under the involution exchanging $y \leftrightarrow z$ and $t \leftrightarrow u$. It suffices then to analyze $E_z, C_x$, and $R_x$ and the distinguished points $n_z, q_x, q_y$, and $r_+$. 

**Analysis along the curve $C_x$.** Recall the equation of $X'$:

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0$$

and the equation of $C_x$: $zu^2 + yt^2 = s = v = x = 0$. We order coordinates $(x, y, z), (s, t, u, v)$ and write intersections

- $C_x \cap R_x = (0, 1, 1) \times (0, 1, \pm i, 0)$;
- $C_x \cap E_z = (0, 1, 0) \times (0, 0, 1, 0)$;
- $C_x \cap E_y = (0, 0, 1) \times (0, 1, 0, 0)$.

We use the symmetry between $t$ and $u$ to reduce the number of cases.

**Chart $u = 1, z = 1$.** We extract equations for the exceptional divisor $E$ obtained by blowing up $C_x$. In this chart, $C_x$ takes the form

$$1 + yt^2 = s = v = x = 0$$

and $X'$ is

$$s^2 + x(yt^2 + 1) + v^2(y - 1)^2 + v^2xG = 0,$$

where $v^2xG$ are the 'higher order terms'.

Now we analyse the local charts of the blow up:
(1) \( E : yt^2 + 1 = 0, \ s = s_1(yt^2 + 1), \ x = x_1(yt^2 + 1), \ v = v_1(yt^2 + 1), \) the equation for \( X' \), up to removing higher order terms, in new coordinates is:

\[
s_1^2 + x_1 + v_1^2(y - 1)^2 = 0,
\]

this is smooth and rational. The exceptional divisor

\[
s_1^2 + x_1 + v_1^2(y - 1)^2 = 0, \ yt^2 + 1 = 0
\]

is rational.

(2) \( E : x = 0, \ s = s_1x, v = v_1x, \ yt^2 + 1 = wx, \) equation of \( X' \):

\[
s_1^2 + w + v_1(y - 1)^2 = 0, \ yt^2 + 1 = wx,
\]

smooth;

(3) \( E : s = 0, \ x = x_1s, v = v_1s, \ yt^2 + 1 = ws: \)

\[
1 + wx_1 + v_1(y - 1)^2 = 0, \ yt^2 + 1 = sw,
\]

smooth.

(4) \( E : v = 0, \ s = s_1v, x = x_1v, \ yt^2 + 1 = wv, \) equation of \( X' \) is

\[
s_1^2 + wx_1 + (y - 1)^2 = 0, \ yt^2 + 1 = wv,
\]

which has at most ordinary double singularity (corresponding to \( C_x \cap R_x = \tau_\pm \)) of type

\[
a^2 + b^2 + cd = 0, \ a = b = c = d = 0.
\]

This is resolved by one blowup.

Chart \( u = 1, \ y = 1 \). In this chart \( C_x \) is \( z + t^2 = s = v = x = 0 \) and \( X' \) is

\[
s^2 + x(t^2 + z) + v^2(z - 1)^2 + v^2xG = 0,
\]

where \( v^2xG \) are the ‘higher order terms’. We analyze local charts of the blow up:

(1) \( E : t^2 + z = 0, \ s = s_1(t^2 + z), \ x = x_1(t^2 + z), \ v = v_1(t^2 + z), \) the equation for \( X' \), up to removing higher order terms, in new coordinates is:

\[
s_1^2 + x_1 + v_1^2(z - 1)^2 = 0,
\]

this is smooth and rational. The exceptional divisor

\[
s_1^2 + x_1 + v_1^2(z - 1)^2 = 0, \ t^2 + z = 0
\]

is rational.
(2) $E : x = 0, s = s_1 x, v = v_1 x, t^2 + z = wx$, equation of $X'$:

$$s_1^2 + w + v_1(z - 1)^2 = 0, t^2 + z = wx,$$

smooth;

(3) $E : s = 0, x = x_1 s, v = v_1 s, t^2 + z = ws$:

$$1 + wx_1 + v_1(z - 1)^2 = 0, t^2 + z = sw,$$

smooth.

(4) $E : v = 0, s = s_1 v, x = x_1 v, t^2 + z = vw$, equation of $X'$ is

$$s_1^2 + wx_1 + (z - 1)^2 = 0, t^2 + z = vw,$$

or, up to removing the higher order terms

$$s_1^2 + wx_1 + (t^2 + 1)^2 = 0, z = -t^2 + vw,$$

this has at most ordinary double singularities

$$s_1 = w = x_1 = 0, t = \pm i$$

(where we meet the proper transform of $R_x$) of type

$$a^2 + b^2 + cd = 0, a = b = c = d$$

resolved as above by one blowup.

*Analysis near $n_z$. Center the coordinates by setting $\xi = y - 1$*

$$s^2 + (\xi + 1) t^2 + zw^2 + (\xi + 1) z(\xi^2 + z^2 - 2z(\xi + 2)) = 0.$$

Note that $E_z$ is given by

$$(\xi + 1)\xi^2 + u^2 = z = s = t = 0.$$

We regroup terms

$$s^2 + (\xi + 1) t^2 + z(u^2 + (\xi + 1)\xi^2) + (\xi + 1)z^2(z - 2\xi - 4) = 0.$$ 

Provided $\xi \neq -1, -2$ this is étale-locally equal to

$$s_1^2 + t_1^2 + z_1(u^2 + (\xi + 1)\xi^2) + z_1^2 = 0$$

which is equivalent to normal form [3.3]. When $\xi = -1$ we are at the point $q_x$, which we analyze below. A local computation at $\xi = -2$ shows that the singularity is resolved there by blowing up $E_z$ and the exceptional fiber there is isomorphic to $\mathbb{P}_0$. In other words, we have ordinary threefold double points there as well.
Blowing up the singular point $n_z$ of $E_z$. The point $n_z$ lies in the chart $x = 1, v = 1$, where we now make computations. The equation of the point (and the locus we blow up) is

$$s = t = u = z = y - 1 = 0.$$ 

The equation of $X$ can be written as:

$$s^2 + yt^2 - 2z^2y(y + 1) + zu^2 + z^3y + yz(y - 1)^2 = 0.$$ 

The curve $E_z$ has equations

$$y(y - 1)^2 + u^2 = z = s = t = 0.$$ 

Now we compute the charts for the blow up:

1. $E : s = 0$. The change of variables is $u = su_1, t = st_1, z = sz_1, y = 1 + y_1 s$. Then the equation of $X'$ (resp. the exceptional divisor $E$), up to removing the higher order terms, is:

$$1 + t_1^2(1 + y_1 s) - 2z_1^2(1 + sy_1)(2 + sy_1) = 0$$

(resp. $1 + t_1^2 - 2z_1^2 = 0$), so that the blow up and the exceptional divisor are smooth, and $E$ is rational.

2. $E : t = 0$. The change of variables is $s = s_1 t, u = u_1 t, z = z_1 t, y = 1 + y_1 t :$ the equations are

$$s_1^2 + (1 + y_1 t) - 2z_1^2(1 + y_1 t)(2 + y_1 t) = 0,$$

and $E$ is given by

$$s_1^2 + 1 - 4z_1^2 = 0,$$

so that the blowup is smooth at any point of the exceptional divisor.

3. $E : z = 0$, the change of variables is $s = s_1 z, u = u_1 z, y = 1 + y_1 z; \text{ we obtain}$

$$s_1^2 + (1 + y_1 z)t_1^2 - 2(1 + y_1 z)(2 + y_1 z) = 0$$

and the equation of $E$ is $s_1^2 + t_1^2 - 4 = 0$, so that the blow up is smooth at any point of the exceptional divisor.

4. $E : y_1 := y - 1 = 0$, the change of variables is $z = z_1 y_1, s = s_1 y_1, u = u_1 y_1, t = t_1 y_1; \text{ the equations are}$

$$s_1^2 + t_1^2(1 + y_1) - 2z_1^2(1 + y_1)(2 + y_1) + u_1^2 y_1 z + z_1^3 y_1 (1 + y_1) + z_1 (1 + y_1) = 0,$$

this is smooth, as well as the exceptional divisor ($y_1 = 0$).
(5) $E : u = 0$, the change of variables is $s = s_1u, t = t_1u, z = z_1u, y = 1 + y_1u$, the equations for the proper transform of $X'$ are

$$s_1^2 + (1 + y_1u)t_1^2 - 2z_1^2(1 + y_1u)(2 + y_1u) + uz_1 + uz_1^3(1 + y_1u) + z_1uy_1^2(1 + y_1u) = 0,$$

and the proper transform of $E_z$ is given by

$$(1 + y_1u)y_1^2 + 1 = z_1 = s_1 = t_1 = 0.$$

The exceptional divisor

$$E : s_1^2 + t_1^2 - 4z_1^2 = 0$$

is singular along $s_1 = t_1 = z_1 = u = 0$ (and $y_1$ is free). The resulting curve is denoted $R_z \simeq \mathbb{P}^1$; note that $R_z$ meets the proper transform of $E_z$ at two points $y_1 = \pm i$.

**Blowing up $R_z$.** For the analysis of singularities we can remove higher order terms, so that the equation of the variety (resp. $R_z$) is given by:

$$s_1^2 + t_1^2 - 4z_1^2 + uz_1 + uz_1^3(1 + y_1) = 0,$$

and $s_1 = t_1 = z_1 = u = 0$.

The charts for the new blow up with exceptional divisor $E'$ are:

1. $E' : s_1 = 0$, then after the usual change of variables for a blow up, we obtain the equation:

$$1 + t_2^2 - 4z_2^2 + u_2z_2 + u_2z_2y_1^2 = 0,$$

which is smooth.

2. $E' : t_1 = 0$ is similar to the previous case.

3. $E' : z_1 = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4 + u_2 + u_2y_1^2 = 0,$$

that is smooth;

4. $E' : u = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4z_2^2 + z_2(1 + y_1^2) = 0,$$

which has ordinary double points at $s_2 = t_2 = z_2 = y_1^2 + 1 = 0$.

These are resolved by blowing up the proper transform of $E_z$. 
**Analysis near \( q_x \).** Dehomogenize

\[
s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0
\]

by setting \( v = 1 \) and \( x = 1 \) to obtain

\[
s^2 + yt^2 + zu^2 + yz(1 + y^2 + z^2 - 2(y + z + yz)) = 0.
\]

We first analyze at \( q_x \), the origin in this coordinate system. Note that

\[1 + y^2 + z^2 - 2(y + z + yz) \neq 0\]

here and thus its square root can be absorbed (étale locally) into \( s, t, \) and \( u \) to obtain

\[
s_1^2 + yt_1^2 + zu_1^2 + yz = 0.
\]

Setting \( y_1 = y + u_1^2 \) and \( z_1 = z + t_1^2 \) gives

\[
s_1^2 + y_1z_1 = t_1^2u_1^2,
\]

which is equivalent to normal form (3.2). (The blow up over the generic point of \( E_z \) was analyzed previously.)

**Blowing up \( R_z \).** Similar to the analysis of singularities near \( R_z \), see also [HPT16, Section 5.2 (4)]

### 3.4. Summary of the resolution.

**Blowup steps.** The resolution \( \beta' \) is a sequence of blowups:

1. Blow up the nodes \( n_z \) and \( n_y \); the resulting fourfold is singular along rational curves \( R_z \) and \( R_y \) in the exceptional locus, meeting the proper transforms of \( E_z \) and \( E_y \) transversally in two points sitting over \( n_x \) and \( n_y \), respectively.
2. The exceptional divisors are quadric threefolds singular along \( R_z \) and \( R_y \).
3. At this stage the singular locus consists of six smooth rational curves, the proper transforms of \( E_z, E_y, R_x, C_x \) and the new curves \( R_z \) and \( R_y \), with a total of nine nodes. (This is the configuration appearing in [HPT16, Section 5].)
4. The local analytic structure is precisely as indicated in Section 3.2. Thus we can blow up the six curves in any order to obtain a resolution of singularities. The fibers are either the Hirzebruch surface \( F_0 \) or a union of Hirzebruch surfaces \( F_0 \cup \Sigma; F_2 \) where \( \Sigma \simeq \mathbb{P}^1 \) with self intersections \( \Sigma_{F_0}^2 = 2 \) and \( \Sigma_{F_2}^2 = -2 \).

For concreteness, we blow up in the order

\[ R_z, R_y, E_z, E_y, C_x, R_x \].
Exceptional fibers. The following fibers arise:

- Over the nodes $n_z$ and $n_y$: The exceptional fiber has two three-dimensional components. One is the standard resolution of a quadric threefold singular along a line, that is,

$$F' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\otimes 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)).$$

The other is a quadric surface fibration $F'' \to \mathbb{P}^1$, over $R_z$ and $R_y$ respectively, smooth except for two fibers corresponding to the intersections with $E_z$ and $E_y$; the singular fibers are unions $F_0 \cup F_2$ as indicated above. The intersection $F' \cap F''$ is along the distinguished subbundle

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\otimes 2}) \subset F'$$

which meets the smooth fibers of $F'' \to \mathbb{P}^1$ in hyperplanes and the singular fibers in smooth rational curves in $F_2$ with self-intersection 2.

- Over $E_z$: the exceptional divisor is a quadric surface fibration over $\mathbb{P}^1$, with two degenerate fibers of the form $F_0 \cup F_2$ corresponding to the intersections with $C_x$ and $E_y$.

- Over $E_y$: the exceptional divisor is a quadric surface fibration with one degenerate fiber, corresponding to the intersection with $C_x$.

- Over $C_x$: a quadric surface fibration with two degenerate fibers corresponding to the intersections with $R_x$.

- Over $R_x$: a smooth quadric surface fibration.

In each case, the fibers of $\beta'$ are universally $\text{CH}_0$-trivial.

4. Proof of the Theorem

We recall implications of the “integral decomposition of the diagonal and specialization” method, following [CTP16b, Voi15].

Theorem 5. [Voi15, Theorem 2.1], [CTP16b, Theorem 1.14 and Theorem 2.3] Let

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ so that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:
• $X$ admits a desingularization
\[ \beta : \tilde{X} \to X, \]
where the morphism $\beta$ is universally $\text{CH}_0$-trivial,

• $\tilde{X}$ is not universally $\text{CH}_0$-trivial.

Then a very general fiber of $\phi$ is not universally $\text{CH}_0$-trivial and, in particular, not stably rational.

We apply this twice: Consider a family of double fourfolds $X_f$ ramified along a quartic threefold $f = 0$, as in (1.1). Let $X'$ be the fourfold given by (3.1) and let $V'$ be the bidegree $(2, 2)$ hypersurface defined in (1.3).

(1) As mentioned in the introduction, $V'$ satisfies property (O); this is an application of Pirutka’s computation of unramified second cohomology of quadric surface bundles over $\mathbb{P}^2$ [Pir16]. By construction, $X'$ is birational to $V'$. Proposition 4 and Section 3.4 yield property (R) for $X'$. We conclude that very general hypersurfaces $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ given by Equation 2.1 in Section 2, following Proposition 2, fail to be universally $\text{CH}_0$-trivial.

(2) By Proposition 2, the resolution morphism $\beta : \tilde{X} \to X$ is universally $\text{CH}_0$-trivial; here $X$ is a double fourfold, ramified along a quartic which is singular along a line. A second application of Theorem 5 to the family of double fourfolds ramified along a quartic threefold completes the proof of Theorem 1.

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