PROFINITE MV-ALGEBRAS

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Abstract. We characterize all profinite MV-algebras, these are MV-algebras that are limits of finite MV-algebras. It is shown that these are exactly direct product of finite /suppress Lukasiewicz's chains. We also prove that the category \( M \) of multisets is dually equivalent to the category \( P \) of profinite MV-algebras and homomorphisms that reflect principal maximal ideals. Thus generalizing the corresponding result for finite MV-algebras, and finite multisets.

Key words: MV-algebra, profinite, multiset, dually equivalent, maximal ideal, finitely approximable.

1. Introduction

MV-algebras were introduced by Chang in order to provide an algebraic proof of the completeness theorem of Lukasiewicz many-valued logic [3]. An MV-algebra is an Abelian monoid \((A, \oplus, 0)\) with an involution \(\neg : A \to A\) (i. e.; \(\neg\neg x = x\) for all \(x \in A\)) satisfying the following axioms for all \(x, y \in A\):

\[-0 \oplus x = -0; \neg(-x \oplus y) \oplus y = \neg(-y \oplus x) \oplus x.\]

For any \(x, y \in A\), we write \(x \leq y\) if \(\neg x \oplus y = -0 := 1\). Then, \(\leq\) induces a partial order on \(A\), which is in fact a lattice order where \(x \vee y = \neg(\neg x \oplus y) \oplus y\) and \(x \wedge y = \neg(\neg x \vee \neg y)\). An ideal of an MV-algebra is a nonempty subset \(I\) of \(A\) such that (i) for all \(x, y \in I\), \(x \oplus y \in I\) and (ii) for all \(x \in A\) and \(y \in I\) with \(x \leq y\), then \(x \in I\). A prime ideal of \(A\) is proper ideal \(P\) such that \(x \wedge y \in P\), then \(x \in P\) or \(y \in P\). Maximal ideal has the usual meaning.

The prototype of MV-algebra is the unit real interval \([0, 1]\) equipped with the operation of truncated addition \(x \oplus y = \max\{x + y, 1\}\), negation \(\neg x = 1 - x\), and the identity element 0. For each integer \(n \geq 2\), \(L_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}\) is a sub-MV-algebra of \([01]\) (the Lukasiewicz’s chain with \(n\) elements), and up to isomorphism every finite MV-chain is of this form.

The concept of profiniteness has been investigated on several classes of algebras of logic. It is well known (see, e.g., [10] Sec.VI.2 and VI.3) that a Boolean algebra is profinite if and only if it is complete and atomic, that a distributive lattice is profinite if and only if it is complete and completely join-prime generated [11] Thm.
and that a Heyting algebra is profinite if and only if it is finitely approximable, complete and completely join-prime generated \[ \text{Thm. 3.6}. \] Some other notable works on profinite algebras include: profinite topological orthomodular lattices \[4\], profinite completions of Heyting algebras \[2\], and profinite MV-spaces \[8, 9\].

There is no known simple description of the dual space of MV-algebras comparable to Esakia space for Heyting algebras, or Priestley spaces for distributive lattices. For this reason, we carry a completely algebraic analysis of profinite MV-algebras. We obtain that an MV-algebra is profinite if and only if it is isomorphic to a direct product of finite MV-chains. It follows that an MV-chain is profinite if and only if it is finite.

It is well known that the category \( \mathbf{FMV} \) of finite MV-algebras is dually equivalent to the category of finite multisets. Among the categories that contain \( \mathbf{FMV} \) as a full subcategory, one has the category \( \mathbf{LFMV} \) of locally finite MV-algebras, and the category \( \mathbb{P} \) of profinite MV-algebras. The duality was extended to locally finite MV-algebras in \[6\], yielding an equivalence between generalized multisets and \( \mathbf{LMV}^{\text{op}} \). Very recently, the later duality was extended further to locally weakly finite MV-algebras \[7\]. In the last section of the paper, we extend the duality to profinite MV-algebras, and obtain that the category \( \mathbb{M} \) of multisets is dually equivalent to the category \( \mathbb{P} \) of profinite MV-algebras and homomorphisms that reflect principal maximal ideals.

### 2. Profinite MV-algebras

Recall that an inverse system in a category \( \mathcal{C} \) is a family \( \{ A_i, \varphi_{ij} \}_{i \in I} \) of objects, indexed by a directed poset \( I \) (for every \( i, j \in I \), there exists \( k \in I \) such that \( i \leq k \) and \( j \leq k \)), together with a family of morphisms \( \varphi_{ij} : A_j \to A_i \), for each \( i \leq j \), satisfying the following conditions.

(i) \( \varphi_{kj} = \varphi_{ki} \circ \varphi_{ij} \) for all \( k \leq i \leq j \);

(ii) \( \varphi_{ii} = 1_{A_i} \) for all \( i \in I \).

Given an inverse system \( \{ A_i, \varphi_{ij} \}_{i \in I} \), an inverse limit of this system is an object \( A \) together with a family of morphisms \( \varphi_i : A \to A_i \) satisfying the condition \( \varphi_i \circ \varphi_{ij} = \varphi_i \) when \( i \leq j \) and having the following universal property: for every object \( B \) of \( \mathcal{C} \) together with a family \( \psi_i : B \to A_i \) if \( \varphi_{ij} \circ \psi_j = \psi_i \) for \( i \leq j \), then there exists a unique morphism \( \psi : B \to A \) such that \( \varphi_i \circ \psi = \psi_i \) for all \( i \in I \).

The inverse limit of an inverse system \( \{ A_i, \varphi_{ij} \}_{i \in I} \), when it exists, is unique up to isomorphism and often denoted by \( \varinjlim \{ A_i, \varphi_{ij} \}_{i \in I} \), or simply by \( \varinjlim \{ A_i \}_{i \in I} \) if the transition maps \( \varphi_{ij} \) are understood.

Using the terminology of \[10\], we call an algebra profinite if is isomorphic to the inverse limit of finite algebras of the same type. Let \( \{ \{ A_i, \varphi_{ij} \} \}_{i \in I} \) be an inverse system of MV-algebras. As in many varieties of
algebras, it is easy to see that
\[
\lim_{\leftarrow} \{ A_i \}_{i \in I} \cong \left\{ (a_i) \in \prod_I A_i : \varphi_{ij}(a_j) = a_i \text{ whenever } i \leq j \right\}
\]

Let $A$ be an algebra and $I$ is the set of all congruences $\theta$ of $A$ such that $A/\theta$ is finite. If the class of an element $a \in A$ modulo $\theta$ is denoted by $[a]_\theta$, then for $\phi \subseteq \theta$, there is a canonical projection $\varphi_{\phi\theta} : A/\phi \rightarrow A/\theta$ given by $\varphi_{\phi\theta}([a]_\phi) = [a]_\theta$. It follows easily that $(\{A/\theta, \varphi_{\phi\theta}\})_I$ is an inverse system. Let $\hat{A}$ be the inverse limit of this system. Then, it is well-known that $\hat{A}$ is called the profinite completion of the algebra $A$. Note that there is a canonical homomorphism $e : A \rightarrow \hat{A}$ given by $e(a) = ([a]_\theta)_{\theta \in I}$.

To start our algebraic analysis of profinite MV-algebras, we find necessary conditions to profiniteness. One clear such condition is finite approximability. Recall that an algebra is called finitely approximable if it is (isomorphic) to a sub-algebra of a direct product of finite algebras of the same type. It is known (see, e.g., [1, Prop. 3.2]) that an algebra is finitely approximable if and only if the morphism $e : A \rightarrow \hat{A}$ above is injective. For MV-algebras, given that every finite MV-algebras is the direct product of finite MV-chains, finite approximability means isomorphic to a sub-MV-algebra of a direct product of finite MV-chains.

**Proposition 2.1.** Let $A$ be a profinite MV-algebra. Then,

1. $A$ is complete and finitely approximable.
2. $A$ is simple if and only if $A$ a finite MV-chain.

**Proof.** Suppose $A = \lim_{\leftarrow} A_i \cong \{ (a_i) \in \prod_I A_i : \varphi_{ij}(a_j) = a_i \text{ whenever } i \leq j \}$, where each $A_i$ is a finite MV-algebra.

(1) Let $S$ be a nonempty subset of $A$, then $S$ is a subset of $\prod_I A_i$, which is complete as each of the $A_i$ is. For each $i \in I$, let $S_i$ denotes the projection of $S$ onto $A_i$. Then each $S_i$ is finite and in $\prod_I A_i$, $\forall S = (\forall S_i)_{i \in I}$. Since the transition morphisms preserve finite suprema, then it is readily verified that $\forall S \in A$. On the other hand, it is clear from the definitions that any profinite MV-algebra is finitely approximable.

(2) If $A$ is simple, every homomorphism with domain $A$ is injective. For each $i$, there is a projection $p_i : A \rightarrow A_i$ that must be one-to-one, and forcing $A$ to be finite since $A_i$ is. But, finite simple MV-algebras are MV-chains. It is also known that finite MV-chains are simple. \[\square\]

The following result can be derived from the theory of vector lattices, but for the convenience of the reader, we give a direct self-contained proof here.
Lemma 2.2. Let $A$ be a direct product of copies of $[0, 1]$, that is $A = [0, 1]^X$ for some nonempty set $X$, and $M$ be a maximal ideal of $A$. Then,

$$A/M \cong [0, 1]$$

Proof. For any MV-algebra $A$, let $\mathcal{H}(A)$ denotes the set of MV-algebra homomorphisms from $A$ into $[0, 1]$ and $Max(A)$ denotes the set of maximal ideals of $A$. It is well known [6] that $\chi \mapsto \ker \chi$ defines a one-to-one correspondence between $\mathcal{H}(A)$ and $Max(A)$, where $\ker \chi = \{a \in A : \chi(a) = 0\}$. As a consequence, if $A$ is simple, there is a unique (injective) homomorphism from $A \to [0, 1]$; in particular $\mathcal{H}([0, 1]) = \{Id\}$. Now, suppose $A = [0, 1]^X$ and $M \in Max(A)$, then $M = \ker \chi$ for some $\chi \in \mathcal{H}(A)$. We need to justify that $A/\ker \chi$ is isomorphic to $[0, 1]$. Consider the map $\tau : [0, 1] \to A$ defined by $\tau(t)(x) = t$ for all $t \in [0, 1]$ and $x \in X$, then $\tau$ is clearly a homomorphism. Thus, $\chi \circ \tau$ is a homomorphism $[0, 1] \to [0, 1]$ and it follows from $\mathcal{H}([0, 1]) = \{Id\}$ that $\chi \circ \tau = Id$. Now, consider the map $\theta : [0, 1] \to A/\ker \chi$ defined by $\theta(t) = \tau(t)/\ker \chi$, in other words $\theta$ is the composition of $\tau$ followed by the natural projection $A \to A/\ker \chi$. Then $\theta$ is a homomorphism, and we claim that $\theta$ is an isomorphism. Since $[0, 1]$ is simple, then $\theta$ is injective. For the surjectivity, let $f \in A$ and $t = \chi(f)$. Then, since $\chi \circ \tau = Id$, $(\chi \circ \tau)(t) = t$. So, $-((\chi \circ \tau)(t)) \otimes \chi(f) = 0$ and $-(\chi(f)) \otimes (\chi \circ \tau)(t) = 0$. Therefore, $-f \otimes \tau(t), -(\tau(t)) \otimes f \in \ker \chi$ and $f/\ker \chi = \tau(t)/\ker \chi = \theta(t)$. Hence, $\theta$ is an isomorphism as claimed. □

Proposition 2.3. For every non-empty set $X$, the MV-algebra $[0, 1]^X$ is not finitely approximable.

Proof. Let $A = [0, 1]^X$ and suppose by contradiction that $A$ is finitely approximable, then there is a homomorphism from $A$ into a finite MV-algebra (any of the projections). Since every finite MV-algebra is a product of finite MV-chains [5, Prop. 3.6.5], then there exists an integer $n \geq 2$ and a homomorphism $p : A \to L_n$. Thus, $A/\ker p$ is isomorphic to a subalgebra of $L_n$ and therefore by [5, Thm. 3.5.1], $A/\ker p$ is simple, from which it follows that $\ker p$ is a maximal ideal of $A$. But, $A/\ker p$ is infinite by Lemma 2.2, which contradicts the fact that $A/\ker p$ is isomorphic to a subalgebra of $L_n$. □

Proposition 2.4. The MV-algebra $[0, 1] \times 2$ (where 2 is the 2-element Boolean algebra) is not finitely approximable.

Proof. By contradiction suppose that $[0, 1] \times 2$ is finitely approximable. Then there exists finite MV-chains $L_{n_i}; i \in I$ and a one-to-one homomorphism $\tau : [0, 1] \times 2 \to \prod_{i \in I} L_{n_i}$. For each $i \in I$, let $p_i$ denotes the natural projection $\prod_{i \in I} L_{n_i} \to L_{n_i}$, and consider $\phi_i = p_i \circ \tau$. Then each $\phi_i$ is a homomorphism from $[0, 1] \times 2 \to [0, 1]$ and it follows that $\ker \phi_i$ is a maximal ideal of $[0, 1] \times 2$. But, $[0, 1] \times 2$ has exactly two maximal ideals: $[0, 1] \times \{0\}$ and $\{0\} \times 2$. Note that it is not possible to have $\ker \phi_i = \{0\} \times 2$, for this would imply by the homomorphism theorem that $Im \phi_i \cong$
And this would contradict the fact that $Im\phi_i$ is finite as it is a sub-MV-algebra of $L_{n_i}$. Therefore, $ker\phi_i = [0,1] \times \{0\}$ for all $i \in I$. Thus, for every $t \in [0,1]$, and every $i \in I$, $\phi_i(t,0) = 0$, that is $p_i(\tau(t,0)) = 0$. Hence, $\tau(t,0) = 0$ and $t = 0$, which is contradiction. Hence, $[0,1] \times 2$ is not finitely approximable as claimed.

**Corollary 2.5.** If $A$ is the direct product of MV-chains, among which $[0,1]$, then $A$ is not finitely approximable.

**Proof.** Every such MV-algebra contains a sub-MV-algebra isomorphic to $[0,1] \times 2$. In fact, suppose that $A = \prod_{i \in I} C_i$, where $C_{i_0} = [0,1]$ for some $i_0 \in I$. Let $S_{i_0} = \{f \in A : f(i) = 0, \text{ for all } i \neq i_0\} \cup \{f \in A : f(i) = 1, \text{ for all } i \neq i_0\}$. Then $S_{i_0}$ is a sub-MV-algebra of $A$, that is clearly isomorphic to $[0,1] \times 2$.

The next result offers a simple algebraic characterizations of profinite MV-algebras.

**Theorem 2.6.** For every non-trivial MV-algebra $A$, the following assertions are equivalent:

1. $A$ is profinite
2. $A$ is complete and finitely approximable
3. $A$ is isomorphic to the direct product of finite MV-chains.

**Proof.** (1)⇒(2): Obvious.
(2)⇒(3): Suppose that $A$ is a sub-MV-algebra of $\prod I A_i$, where $A_i$ is a finite MV-algebra. Since each $A_i$ is a finite MV-algebra, then it is isomorphic to a (finite) product of finite MV-chains. So, $\prod I A_i$ is isomorphic to a direct product of finite MV-chains, and by [3] Thm. 6.8.1, $\prod I A_i$ is complete and completely distributive. Since $A$ is a complete sub-MV-algebra of $\prod I A_i$, then $A$ is completely distributive. Therefore, by [3] Thm. 6.8.1 again, $A$ is a direct product of complete MV-chains.
But, every complete MV-chain is isomorphic to a finite MV-chain, or to $[0,1]$. Moreover, since $A$ is finitely approximable, it follows from Corollary 2.5 that $A$ is a direct product of finite MV-chains.
(3)⇒(1): Is clear.

It follows that profinite MV-chains are finite.

**Corollary 2.7.** A (non-trivial) MV-chain $A$ is profinite if and only if $A$ is isomorphic to $L_n$ for some $n \geq 2$.

### 3. Maximal ideals of profinite MV-algebras

For any MV-algebra $A$, let $\mathcal{H}(A)$ denotes the set of MV-algebra homomorphisms from $A$ into $[0,1]$ and $Max(A)$ denotes the set of maximal ideals of $A$. It is well known [3] that $\chi \mapsto ker\chi$ defines a one-to-one correspondence between $\mathcal{H}(A)$ and $Max(A)$, where $ker\chi = \{a \in A : \chi(a) = 0\}$. We will use the following notations
through out the paper. For $A := \prod_{x \in X} L_{n_x}$ a profinite MV-algebra, and each $x \in X$, $p_x : A \to L_{n_x}$ denotes the natural projection. In addition $M_x$ will denote $\ker p_x$, it follows that each $M_x$ is a maximal ideal of $A$. It is easy to see that $\oplus_{x \in X} L_{n_x} := \{ f \in A : f(x) = 0 \text{ for all, but finitely many } x \in X \}$ is an ideal of $A$. Recall that a principal ideal of an MV-algebra $A$ is any ideal $I$ that is generated by a single element, that is there exists $a \in A$, such that $I = \langle a \rangle$. It is well known that $x \in \langle a \rangle$ if and only if $x \leq na$ for some integer $n \geq 1$.

**Lemma 3.1.** Let $A := \prod_{x \in X} L_{n_x}$ be a profinite MV-algebra. For any maximal ideal $M$ of $A$, the following conditions are equivalent.

(i) $M$ is principal;
(ii) There exists a unique $x_0 \in I$, such that $M = \ker p_{x_0}$;
(iii) $M$ does not contain $\oplus_{x \in X} L_{n_x}$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose that $M$ is principal, then $M = \langle a \rangle$ for some $a \in A$. We claim that there exists $x_0 \in X$ with $a(x_0) = 0$. By contradiction suppose that $a(x) \neq 0$ for all $x \in X$, then for each $x \in X$, $a(x) = \frac{k_{n_x}}{n_x - 1}$ for some $1 \leq k_{n_x} \leq n_x - 1$.

We consider two cases:

(a) $\left\{ \frac{n_x - 1}{k_{n_x}} \right\}_{x \in X}$ is bounded, then there exists an integer $m \geq 1$ such that $\frac{n_x - 1}{k_{n_x}} \leq m$ for all $x \in X$. It follows that $ma = 1$, and so $M = A$, which is a contradiction.

(b) $\left\{ \frac{n_x - 1}{k_{n_x}} \right\}_{x \in X}$ is unbounded. Then $\{n_x\}_{x \in X}$ is unbounded. We can write $X$ as the disjoint union of of two sets $X'$ and $X''$ such that $\{n_x\}_{x \in X'}$, and $\{n_x\}_{x \in X''}$ are unbounded. Define $f, g \in A$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in X' \\ a(x) & \text{if } x \in X'' \end{cases} \quad \text{and} \quad g(x) = \begin{cases} a(x) & \text{if } x \in X' \\ 1 & \text{if } x \in X'' \end{cases}$$

Then $f \wedge g = a$, in particular $f \wedge g \in M$. Since $M$ is prime, as every maximal ideal is, then $f \in M$ or $g \in M$. Assume $f \in M = \langle a \rangle$, then there exists an integer $r \geq 1$ such that $f \leq ra$. Therefore, $1 \leq \frac{rk_{n_x}}{n_x - 1}$ for all $x \in X''$, and so $\frac{n_x - 1}{k_{n_x}} \leq r$ for all $x \in X''$. This contradicts the fact that $\{n_x\}_{x \in X''}$ is unbounded. In a similar argument, $g \in M$ would contradict the fact that $\{n_x\}_{x \in X'}$ is unbounded.

Thus $a(x_0) = 0$ for some $x_0 \in X$. For every $f \in M = \langle a \rangle$, there exists $k \geq 1$ such that $f \leq ka$, and it follows that $f(x_0) = 0$ for all $f \in M$. Hence, $M \subseteq M_{x_0}$. Since $M$ and $M_{x_0}$ are maximal, then $M = M_{x_0} = \ker p_{x_0}$. The uniqueness is clear.

(ii) $\Rightarrow$ (i) This is clear as each $M_{x_0}$ is principal as it is generated by $f(x) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x \neq x_0 \end{cases}$.
(ii) ⇒ (iii): Suppose that there exists a unique $x_0 \in I$, such that $M = \ker p_{x_0}$. Consider $f \in A$ defined by $f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$ Then $f \in \bigoplus_{x \in X} L_{n_x}$ and $f \notin M$.

(iii) ⇒ (ii): Suppose that for all $x \in X$, $M \neq M_x$. For each $x \in X$, let $b_x \in A$ defined by $b_x(t) = \begin{cases} 0 & \text{if } t = x \\ 1 & \text{if } t \neq x \end{cases}$ Then for every $x \in X$, since $M_x = \langle b_x \rangle$, then $b_x \notin M$ and since $M$ is maximal, by [5 Prop.1.2.2] there exists an integer $k_x \geq 1$ such that $-k_x b_x = -b_x \in M$. It follows that $M$ contains $\bigoplus_{x \in X} L_{n_x}$.

\begin{proof}
Corollary 3.2. Let $A := \prod_{x \in X} L_{n_x}$ be a profinite MV-algebra. A maximal ideal $M$ of $A$ is not principal if and only if $\bigoplus_{x \in X} L_{n_x} \subseteq M$.

4. A Stone type duality

We say that a homomorphism $\varphi : A \to B$ of MV-algebras reflect principal ideals if for every principal ideal $J$ of $B$, $\varphi^{-1}(J)$ is a principal ideal of $A$. It is clear that the identity reflects principal maximal ideals, and that the composition of two homomorphisms that reflect principal maximal ideals also reflects principal maximal ideals. Let $\mathbb{P}$ denotes the category of profinite MV-algebras and homomorphisms that reflect principal maximal ideals. We recall the definition of the category $\mathcal{M}$ of multisets. A multiset is a pair $\langle X, \sigma : X \to \mathbb{N} \rangle$, where $X$ is a set and $\sigma$ is a map. Given two multisets $\langle X, \sigma \rangle$ and $\langle Y, \mu \rangle$, a morphism from $\langle X, \sigma \rangle$ to $\langle Y, \mu \rangle$ is a map $\varphi : X \to Y$ such that $\mu(\varphi(x))$ divides $\sigma(x)$ for all $x \in X$.

We shall define two functors $\mathcal{H} : \mathbb{P}^{\text{op}} \to \mathcal{M}$ and $\mathcal{F} : \mathcal{M} \to \mathbb{P}^{\text{op}}$

(1) $\mathcal{H} : \mathbb{P}^{\text{op}} \to \mathcal{M}$. For any profinite MV-algebra $A$, set

$\mathcal{H}_F(A) := \{ \chi : A \to [0,1] : \chi \text{ is a homomorphism and } \ker \chi \text{ is principal (maximal) ideal} \}$

and $\sigma_A : \mathcal{H}_F(A) \to \mathbb{N}$ defined by $\sigma_A(\chi) = \#(\chi(A)) - 1$.

Note that $\sigma_A$ is well-defined because as $\ker \chi$ is principal, by Lemma 3.1 and the homomorphism theorem, $\chi(A)$ is finite.

- On objects: Given a profinite MV-algebra $A$, define $\mathcal{H}(A) = \langle \mathcal{H}_F(A), \sigma_A \rangle$.
- On morphisms: let $\varphi$ be a homomorphism in $\mathbb{P}^{\text{op}}$ from $A \to B$, that is $\varphi : B \to A$ is an MV-algebra homomorphism that reflects principal ideals. Define $\mathcal{H}(\varphi) : \mathcal{H}_F(A) \to \mathcal{H}_F(B)$ by $\mathcal{H}(\varphi)(\chi) = \chi \circ \varphi$. Note that since $\ker \chi(A)$ is a principal maximal ideal of $A$, $\ker(\chi \circ \varphi) = \varphi^{-1}(\ker \chi)$, and $\varphi$ reflects principal maximal ideals, then $\ker(\chi \circ \varphi)$ is principal maximal. So, $\chi \circ \varphi \in \mathcal{H}_F(B)$ and $\mathcal{H}(\varphi)$ is well-defined.

On the other hand, note by [5 Cor. 3.5.4, Cor. 7.2.6] that for each $\chi \in \mathcal{H}_F(A)$, $\chi(A) = L_{\#(\chi(A))}$. Thus, $L_{\#(\chi(\varphi))(A)} \subseteq L_{\#(\chi(A))}$; and it follows that $\#(\chi \circ \varphi)(A) - 1$ divides $\#(\chi(A)) - 1$. Thus, $\sigma_B(\mathcal{H}(\varphi)(\chi))$ divides $\sigma_A(\chi)$ for all $\chi \in \mathcal{H}_F(A)$. Therefore, $\mathcal{H}(\varphi)$ is a morphism in $\mathcal{M}$ from
\[ \mathcal{H}_F(A) \to \mathcal{H}_F(B). \]

(2) \( \mathcal{F} : \mathbb{M} \to \mathbb{P}^{\text{op}}. \) For any multiset \( \langle X, \sigma \rangle, \prod_{x \in X} L_{\sigma(x)+1} \) is clearly a profinite MV-algebra, that shall be denoted by \( A_{X,\sigma} \).

- On objects: Given a multiset \( \langle X, \sigma \rangle \), define \( \mathcal{F}(\langle X, \sigma \rangle) := A_{X,\sigma} \).
- On morphisms: Let \( \varphi : \langle X, \sigma \rangle \to \langle Y, \mu \rangle \) be a morphism in \( \mathbb{M} \). Define \( \mathcal{F}(\varphi) : A_{Y,\mu} \to A_{X,\sigma} \) by \( \mathcal{F}(\varphi)(f)(x) = f(\varphi(x)) \) for all \( f \in A_{Y,\mu} \) and all \( x \in X \). To see that \( \mathcal{F}(\varphi) \) is well-defined, first note that for all \( f \in A_{Y,\mu} \) and all \( x \in X \), \( f(\varphi(x)) \in L_{\mu(\varphi(x))+1} \). On the other hand, \( \mu(\varphi(x)) \) divides \( \sigma(x) \), hence \( L_{\mu(\varphi(x))+1} \subseteq L_{\sigma x+1} \). Thus, \( f(\varphi(x)) \in L_{\sigma x+1} \). In addition, let \( M \) be a principal maximal ideal of \( A_{X,\sigma} \), then by Lemma 3.1 there exists \( x_0 \in X \) such that \( M = M_{x_0} \). It is easy to see that \( \mathcal{H}(\varphi)^{-1}(M_{x_0}) = M_{\varphi(x_0)} \), which is a principal maximal ideal of \( A_{Y,\mu} \). Finally, it is easy to see that \( \mathcal{H}(\varphi) \) is a MV-homomorphism from \( A_{Y,\mu} \to A_{X,\sigma} \).

The only missing aspects of the proof of the following results are simple computations, which we shall omit.

**Proposition 4.1.** \( \mathcal{H} : \mathbb{P}^{\text{op}} \to \mathbb{M} \) and \( \mathcal{F} : \mathbb{M} \to \mathbb{P}^{\text{op}} \) are functors.

**Proposition 4.2.** Let \( \langle X, \sigma \rangle \) be a multiset, define \( \eta_X : \langle X, \sigma \rangle \to \langle \mathcal{H}_F(A_{X,\sigma}), \sigma_{A_{X,\sigma}} \rangle \) by \( \eta_X(x)(f) = f(x) \), for all \( x \in X \) and all \( f \in A_{X,\sigma} \). Then \( \eta_X \) is an isomorphism in \( \mathbb{M} \).

**Proof.** Note that for each \( x \in X \), \( \eta_X(x) \) is a homomorphism from \( A_{X,\sigma} \to L_{\sigma(x)+1} \), in particular \( \eta_X(x) \in \mathcal{H}_F(A_{X,\sigma}) \) and \( \eta_X \) is well-defined. To see that \( \eta_X \) is a morphism, let \( x \in X \), then \( \eta_X(x)(A_{X,\sigma}) \subseteq L_{\sigma(x)+1} \). Thus, \( L_{\#\eta_X(x)(A_{X,\sigma})} \subseteq L_{\sigma(x)+1} \), hence \( \#\eta_X(x)(A_{X,\sigma}) - 1 \) divides \( \sigma(x) \). Whence, \( \sigma_{A_{X,\sigma}}(\eta_X(x)) \) divides \( \sigma(x) \) for all \( x \in X \). It remains to prove that \( \eta_X \) is bijective. Injectivity: Let \( x_1, x_2 \in X \) such that \( x_1 \neq x_2 \). Define \( f \in A_{X,\sigma} \) by \( f(x_1) = 0 \) and \( f(x) = 1 \) for \( x \neq x_1 \). Then \( \eta_X(x_1)(f) = 0 \), while \( \eta_X(x_2)(f) = 1 \). Therefore \( \eta_X(x_1) \neq \eta_X(x_2) \) and \( \eta_X \) is injective. Surjectivity: Let \( \chi \in \mathcal{H}_F(A_{X,\sigma}) \), then \( ker\chi \) is a principal maximal ideal of \( A_{X,\sigma} \). By Lemma 3.1 there exists \( x \in X \) such that \( ker\chi = M_x = ker p_x \). Hence, \( \chi = p_x \), and it follows that \( \eta_X(x) = \chi \). Thus, \( \eta_X \) is an isomorphism in \( \mathbb{M} \).

**Proposition 4.3.** Let \( A \) be a profinite MV-algebra. Define \( \varepsilon_A : A \to \prod_{\chi \in \mathcal{H}_F(A)} L_{\#\chi(A)} \) by \( \varepsilon_A(f)(\chi) = \chi(f) \) for all \( f \in A \) and all \( \chi \in \mathcal{H}_F(A) \).

Then \( \varepsilon_A \) is an isomorphism in \( \mathbb{P}^{\text{op}} \).
Proof. Since $\chi(A) = L_{#\chi(A)}$ for all $\in H_F(A)$, it follows that $\varepsilon_A$ is well-defined. In addition, let $M$ be a principal maximal ideal of $\prod_{\chi \in H_F(A)} \varepsilon_A$, then by Lemma 3.1 there exists $\chi_0 \in H_F(A)$ such that $M = M_{\chi_0}$. But, it is clear that $\varepsilon_A^{-1}(M_{\chi_0}) = ker\chi_0$, which is principal maximal ideal of $A$. Thus, $\varepsilon_A$ reflects principal maximal ideals. It is straightforward to verify that $\varepsilon_A$ is a homomorphism of MV-algebras. It remains to prove that $\varepsilon_A$ is bijective.

Injectivity: Let $f, g \in A$ such that $\varepsilon_A(f) = \varepsilon_A(g)$, then for all $\chi \in H_F(A)$, $\chi(f) = \chi(g)$. Since $A$ is profinite, by Theorem 2.6 there exists a set $X$ and a sequence of integers $\{n_x\}_{x \in X}$ such that $A = \prod_{x \in X} L_{n_x}$. We have $p_x(f) = p_x(g)$ for all $x \in X$, hence $f(x) = g(x)$ for all $x \in X$ and $f = g$.

Surjectivity: Let $g \in \prod_{\chi \in H_F(A)} \varepsilon_A$. Since $A$ is profinite, by Theorem 2.6 there exists a set $X$ and a sequence of integers $\{n_x\}_{x \in X}$ such that $A = \prod_{x \in X} L_{n_x}$. Then, by Lemma 3.1 $x \mapsto p_x$ is a one-to-one correspondence between $X$ and $H_F(A)$. Now define $f \in A$ by $f(x) = g(p_x)$. Then, it follows clearly that $\varepsilon_A(f) = g$.

Thus, $\varepsilon_A$ is an isomorphism in $\mathbb{P}^{op}$.

Theorem 4.4. The composite $H \circ F$ is naturally equivalent to the identity functor of exists a natural isomorphism $M$. In other words, for all multisets $(X, \sigma)$, $(Y, \mu)$ and $\varphi: (X, \sigma) \to (Y, \mu)$ a morphism in $M$, we have a commutative diagram

$$
\begin{array}{ccc}
\langle X, \sigma \rangle & \xrightarrow{\varphi} & \langle Y, \mu \rangle \\
\eta_X & & \eta_Y \\
H(F((X, \sigma))) & \xrightarrow{H(F(\varphi))} & H(F((Y, \mu)))
\end{array}
$$

in the sense that, for each $x \in X$, $H(F(\varphi)(\eta_X(x)) = \eta_Y(\varphi(x))$.

Proof. Let $x \in X$, then $H(F(\varphi))(\eta_X(x)) = \eta_X(x) \circ F(\varphi)$. For every $g \in A_{Y, \mu}$,

$$(\eta_X(x) \circ F(\varphi))(g) = \eta_X(x)(F(\varphi)(g))$$

$$= F(\varphi)(g)(x)$$

$$= g(\varphi(x))$$

$$= \eta_Y(\varphi(x))(g)$$

Hence $H(F(\varphi)(\eta_X(x)) = \eta_Y(\varphi(x))$ for all $x \in X$ as claimed.

Theorem 4.5. The composite $F \circ H$ is naturally equivalent to the identity functor of exists a natural isomorphism $\mathbb{P}^{op}$. In other words, for all all profinite MV-algebras.
A, B and \( \varphi : A \rightarrow B \) a homomorphism in \( \mathcal{P}^{\text{op}} \), we have a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi} & A \\
\varepsilon_B & & \downarrow \varepsilon_A \\
\mathcal{F}(\mathcal{H}(B)) & \xrightarrow{\mathcal{F}(\mathcal{H}(\varphi))} & \mathcal{F}(\mathcal{H}(A))
\end{array}
\]

in the sense that, for each \( f \in B \), \( \mathcal{F}(\mathcal{H}(\varphi))(\varepsilon_B(f)) = \varepsilon_A(\varphi(f)) \)

Proof. Let \( f \in B \) and \( \chi \in \mathcal{H}_F(A) \), then

\[
\mathcal{F}(\mathcal{H}(\varphi))(\varepsilon_B(f))(\chi) = \varepsilon_B(f)(\mathcal{H}(\varphi)(\chi)) = \varepsilon_B(f)(\chi \circ \varphi) = (\chi \circ \varphi)(f) = \chi(\varphi(f)) = \varepsilon_A(\varphi(f))(\chi)
\]

Hence, \( \mathcal{F}(\mathcal{H}(\varphi))(\varepsilon_B(f)) = \varepsilon_A(\varphi(f)) \) for all \( f \in B \), as desired.

Combining Theorem 4.4 and Theorem 4.5, we obtain the long sought duality.

**Corollary 4.6.** The category \( \mathcal{M} \) of multisets is dually equivalent to the category \( \mathcal{P} \) of profinite MV-algebras and homomorphisms that reflect principal maximal ideals.

**Remark 4.7.** While every MV-homomorphism reflects maximal ideals [5, Prop. 1.2.16], MV-homomorphism may not reflect principal maximal ideals. For instance, consider the simplest infinite profinite MV-algebra, namely \( A = 2^X \) for some fixed infinite set \( X \). Then \( \bigoplus_X 2 \) is an ideal of \( A \), and is contained in a maximal ideal \( M \) of \( A \), which is not principal by Lemma 3.1. But, \( A/M \) is a Boolean algebra that is isomorphic to a Boolean subalgebra of \([0, 1]\). Hence, \( A/M \cong 2 \). Now consider the natural projection \( p : A \rightarrow 2 \), then \( p^{-1}(0) = M \), which is not principal.

**Remark 4.8.** The category \( \mathcal{FMV} \) of finite MV-algebras is a full subcategory of \( \mathcal{P} \) and when restricted \( \mathcal{FMV} \), the equivalence yields the well known duality between finite MV-algebras and finite multisets.

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