String Dilaton Fluid Cosmology

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Abstract

We investigate $(n + 1)$-dimensional string-dilaton cosmology with effective dilaton potential in presence of perfect-fluid matter. We get exact solutions parametrized by the constant $\gamma$ of the state equation $p = (\gamma - 1)\rho$, the spatial dimension number $n$, the bulk of matter, and the spatial curvature constant $k$. Several interesting cosmological behaviours are selected. Finally we discuss the recovering of ordinary Einstein gravity starting from string dominated regime and a sort of asymptotic freedom due to string effective coupling.

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1 Introduction

The goal to connect the large scale structure of the universe with fundamental particle physics is one of the major issues of modern cosmology. One should be able to relate the great amount of recent observations (COBE, HST, surveys of galaxies, etc. see for example [1]–[10]) with some fundamental theory following the scheme firstly drawn by Sakharov [11]: the primordial quantum fluctuations should have been enlarged to astronomical sizes by some expansion mechanism (e.g. inflation) and then give rise to galaxies, clusters and superclusters of galaxies, when the perturbations grow away from the cosmological background [12]. However, there are some shortcomings in this approach since a very fundamental theory of gravity, connected to the other interactions, does not exist yet (i.e. theories exist but none of them has been, till now, experimentally tested).

All the unification schemes are defective for several reasons when they have to take into account the particle families actually discovered. The standard model of particles (actually detectable) is in agreement with cosmological observations below energies of 1 TeV, while the unification theories require energies from $10^{14}$ to $10^{19}$ GeV of full quantum gravity. Such huge energies are not detectable by the today facilities.

Furthermore, all the present inflationary models have to be fine tuned in order to get perturbation spectra able to reproduce the observed microwave background isotropy and to explain the large scale structures [13].

This situation is unsatisfactory, but some progress have been done in recent years and, very likely, the only available way to understand the large scale structures in the universe and to attempt a superunification with observable consequences is to investigate theories which, at the same time, explain the main riddles of astrophysics (large–scale structures and dark matter) and of high energy physics (hierarchy problem, parity violation, number of families etc.) under the same astroparticle standard.

For example, let us consider the inflationary paradigm. We can change our point of view and taking into consideration the geometric side (without considering matter scalar fields) [14]. In such a way, we can construct classes of cosmological models which avoid the initial singularity, evolve towards the decelerating state of standard cosmology, and can be connected with the perturbation spectra without fine tuning. Alternatively, the so called extended and hyperextended inflations [15] solve the shortcomings of old and chaotic inflations by changing the gravitational sector through a nonminimal coupling. These models allow to complete the graceful exit from the false toward the true vacuum, avoid the overwhelming presence of topological defects after the phase transition, yield realistic power spectra for cosmological perturbations. Besides, all unification theories forecast, in the low energy limit, nonminimal couplings between matter fields and geometry (in particular when a quantum field theory is built on curved spacetimes [16]); then it seems reasonable to invoke a variation of Newton constant $G_N$ toward the early epochs from both astrophysical and particle sides. Furthermore, the cosmological constant $\Lambda$ has to vary from the today observed upper limit ($\Lambda \simeq 10^{-56} cm^{-2}$) [12], in order to allow one
or more than one phase transitions. In general, a lot of theories follow this paradigm: the universe has a fundamental quantum–mechanical nature in which gravity strictly depends on some scalar fields which have to lead the system toward the observed situation \((G_{\text{eff}} \rightarrow G_N, \Lambda_{\text{eff}} \rightarrow \Lambda)\) at present epoch \([17]\).

String theory can be included in this set of theories. It is one of the most serious attempts, in the last thirty years, to get the great unification, since it avoids the shortcomings of quantum field theories essentially due to the pointlike nature of particles (renormalization) and includes gravity in the same conceptual scheme of other fundamental interactions (the graviton is just a string mode as the other gauge bosons). In general, all the elementary particles can be interpreted as different modes of a single string whose oscillation frequencies determine the particle masses.

In the low energy limit (that is far from the Planck energy scale), a string dominated universe is described by the tree–level effective action \([18],[19],[20],[21]\) in which only massless modes appear (zero modes). From it, the lowest order string \(\beta\)-function equations can be derived. These equations, for the closed string, describe the dynamics of free long range fields which are: the scalar mode (the dilaton \(\varphi\)), the Kalb–Ramond field strength \(H_{\mu\nu\lambda}\) (usually called the "torsion field"), and the graviton. The last two ones are tensor modes. These three fields are nonminimally coupled in the string effective Lagrangian. In addition, there is a constant related to the central charge of the string theory which vanishes for the critical number of dimensions 10 or 26. In general, one can consider strings propagating in a universe of spatial dimension \(n\) \([19],[21]\). If \(n\) is not the critical dimension (that is \(n < 26\) for bosonic strings or \(n < 10\) for superstrings), we could have one of the two possibilities: either we have some \(n_c\) of compact internal dimensions which make up the rest of the spatial dimensions which we take to be static and to correspond to conformal theories on the world–sheet \([21]\), or we are dealing with a noncritical theory. The difference between the two approaches results in the low energy limit action where we have to consider the constant

\[
\Lambda = \frac{2}{3}(d_c - n - n_c),
\]

where \(d_c\) is the critical spatial dimension. If the theory is critical, we have \(\Lambda = 0\), if it is noncritical \(\Lambda \neq 0\). Usually, the \(n\)–dimensional space is taken to be a box of lengths \(a_i\). In cosmology, that is in the low energy limit, such lengths are the scale factor(s) of an (an)isotropic universe \([22]\).

In any case, string theory is one of the unification scheme able to describe all the fundamental interactions under the same standard \([21]\), so the string–dilaton cosmology could be the way to solve the above astrophysical riddles. However, the fact that only the massless excited states are used suggests that the effective action is not a valid description for probing the highest energies associated with string theory but we may hope that, through the \(\beta\)-function equations, we are investigating physics associated with events from the string scale down to the GUT scales and below.

On the other hand, cosmology is the only available way to test the consequences of the theory due to the high energies requested by every quantitative prediction for
fundamental strings. The today detectable remnants of primordial processes could be a test to which to compare the theory and a lot of open questions of astrophysics, like dark matter and large scale structure, could be solved by strings and their dynamics [22].

In a series of papers, it has been shown that it is possible to obtain exact solutions for string–dilaton cosmology in homogeneous spacetimes (see for example [22]–[28]). Most of them possess the string major feature, the duality, so that when \(a(t)\), the scale factor of the universe, is a solution of the equations of motion, \(1/a(t)\) is a solution too (in Planck’s units). In other words, the whole spacetime behaves like a string. This fact implies that the universe could have a minimal size comparable with the string scale and a maximal one comparable with the inverse of such a scale. In other words, physical size and initial singularity of the universe could be connected to the features of string theory and if the dynamics of early universe is governed by a string gas, it does not begin from a null size but from a finite size that is the Planck scale. In order to determine such a dynamics, string cosmology needs the existence of two kinds of fields, which are dual between them. At small scales, one of them is nonmassive and dominates the dynamics; at large scales, it is the other one which becomes dominant and nonmassive. The first contribution comes from the dilaton which leads the expansion of observed spatial dimensions; the other one is due to the graviton, that is to the geometry.

A fundamental consequence of such a situation is that the gravitational interaction results nonminimally coupled with dilaton; then the effective string theory is a sort of Jordan–Brans–Dicke theory, i.e. an induced gravity theory [22], [23], [27]. We have to note that general relativity is not invariant under such a duality transformation.

Furthermore, in string–dilaton cosmology, there exist the possibility that inflation can occur without relying on a potential energy density. However, the dilaton potential has to be suitable for gaugino condensation. In this case, we need a mechanism (not yet found) able to explain the vanishing of the cosmological constant [29].

From this point of view, the presence of a potential for the dilaton and the fact that the string effective action has to contain matter sources are crucial ingredients. The sources may be represented in a perfect fluid form, but always with an equation of state which consistently follows from the solution of the string equations of motion in the given cosmological background. This description, probably, does not work in the very high curvature regime but it is necessary in order to recover (even through inflation) the standard cosmological picture which we today observe.

In this paper, we want to address this problem: we deal with a \((n+1)\)–cosmological string effective action in which perfect fluid matter is present and we want to investigate how it affects string background dynamics. The method is the same which is used in [30] and in [31]. The dilaton potential is also considered and it is possible to show that in a \((n+1)\) Friedman–Robertson–Walker (FRW) model, the number of spatial dimension

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1The string duality property can be stated as follows: a (closed) string moving on a circle of radius \(r\) is equivalent to one moving on a circle of radius \(\lambda_s^2/r\) where \(\lambda_s^2 = \alpha' \hbar\) is connected to the fundamental length of the string and \(\alpha'\) is a constant called the *Regge slope*. From this fact, strings do not see scales smaller than their proper natural scale.
n, the amount of ordinary matter, the sound speed in the state equation (given by a constant \( \gamma \)), and such a potential are strictly related. The results are easily extensible to any Bianchi model.

In Sec. 2, we write down and discuss the string–dilaton–fluid matter effective action showing that it can be derived from the general action of a nonminimally coupled theory of gravity. Sec. 3 is devoted to the derivation of equations of motion. In Sec. 4, we discuss some realistic forms of dilaton potential. Sec. 5 is devoted to the exact solutions and to their cosmological classification. In Sec. 6, we discuss the results and draw conclusions.

2 The string–dilaton–fluid matter effective action

The most general action of a scalar field \( \psi \) nonminimally coupled with the Ricci scalar \( R \) in \((n + 1)\) dimensions is

\[
\mathcal{A} = \int d^{n+1}x \sqrt{-g} \left[ F(\psi)R + \frac{1}{2} g^{\mu\nu} \psi_{;\mu} \psi_{;\nu} - W(\psi) + \tilde{\mathcal{L}}_m \right],
\]

where \( F(\psi) \) is the coupling, \( W(\psi) \) is the potential for the field \( \psi \), \( \tilde{\mathcal{L}}_m \) is the ordinary matter contribution, and \( \mu, \nu = 0, ..., n \). The string–dilaton effective action (i.e. the string action in the low energy limit) containing the contributions due to the dilaton \( \varphi \) and to the graviton \( g_{\mu\nu} \) (disregarding the antisymmetric torsion field) is easy recovered by the transformations

\[
\psi = \exp[-\varphi], \quad F(\psi) = \frac{1}{8} \exp[-2\varphi], \quad W(\psi) = U(\varphi) \exp[-2\varphi],
\]

which specify the coupling. The action (2) becomes

\[
\mathcal{A} = \int d^{n+1}x \sqrt{-g} \left\{ \exp[-2\varphi] \left[ R + 4g^{\mu\nu} \varphi_{;\mu} \varphi_{;\nu} - U(\varphi) \right] + \tilde{\mathcal{L}}_m \right\}.
\]

We have supposed the fluid–matter contribution \( \tilde{\mathcal{L}}_m \) minimally coupled with string–dilaton degrees of freedom, and, in principle, we have to specify it with respect to the other fields. Then we take into consideration a \((n + 1)\) dimensional FRW metric since we want to deal with the cosmological problem. In this situation, the Lagrangian in (2) becomes pointlike and we get

\[
\mathcal{L} = n(n-1)a^{n-2}\dot{a}^2 F(\psi) + 2n\dot{a}a^{-1}F'(\psi) - n(n-1)ka^{n-2}F(\psi) + \frac{1}{2}na^2 - a^nW(\psi) + a^n \tilde{\mathcal{L}}_m,
\]

while the Lagrangian in (3) becomes

\[
\mathcal{L} = \frac{1}{8} a^n e^{-2\varphi} \left[ n(n-1) \left( \frac{\dot{a}}{a} \right)^2 - 4n\dot{\varphi} \left( \frac{\dot{a}}{a} \right) - n(n-1) \frac{k}{a^2} + 4\dot{\varphi}^2 - 8U(\varphi) \right] + a^n \tilde{\mathcal{L}}_m.
\]

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Note: This is a continuation of the previous discussion on the string–dilaton–fluid matter effective action.
The constant $k$ is the spatial curvature constant which can be $k = \pm 1, 0$ for spatially closed, open and flat cosmological models, respectively.

The pointlike Lagrangians (5) and (6) can be easily extended to any anisotropic and homogeneous spacetime [32],[33]. Here, for the sake of simplicity, we shall treat only FRW models.

Now we have to determine the fluid matter contribution. In this case, the stress–energy tensor has the form

$$T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} - g_{\mu\nu}p,$$

where $p$ and $\rho$ are, respectively, the pressure and energy–matter density. It has to satisfy the contracted Bianchi identities

$$T_{\mu;\nu}^{\nu} = 0.$$

In FRW $(n + 1)$–dimensional spacetimes, (8) becomes

$$\dot{\rho} + nH(p + \rho) = 0,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Imposing the usual equation of state

$$p = (\gamma - 1)\rho,$$

where $\gamma$ is a constant related with the sound speed, we get

$$\rho = Da^{-n\gamma}.$$

$D$ is a positive integration constant related to the matter density at $t = t_0$ [30],[31].

As we said, $\gamma$ determines the thermodynamical state of matter fluid. In standard Friedman–Einstein cosmology, we have, for example, $\gamma = 1$ for dust, $\gamma = 4/3$ for radiation, $\gamma = 2$ for stiff matter and $\gamma = 0$ for scalar field matter. The last case is particular since yields a repulsive gravity. It is found in inflationary epoch. This scheme takes into consideration only stationary states but it does not consider the phase transitions between one regime to another. Actually, we should take into consideration a sort of step–function which, at equilibrium, assumes certain values. In $(n + 1)$–dimensional string–dilaton cosmology, as we shall see below, $\gamma$ can depend on $n$ and, for any dimension, it can assume different values which means a different thermodynamical interpretation.

The scale–factor duality symmetry is present in the lowest order string effective action and, as we said, it means that the transformation of scale factor of a homogeneous and isotropic target space metric, $a(t) \rightarrow a^{-1}(t)$, leaves the model invariant, provided that the dilaton field is transformed as

$$\Phi = \varphi - \frac{n}{2}\ln a.$$
Finally, taking into account the matter contribution (11) and the duality transformation (12), Lagrangian (6) becomes

\[ L = \frac{1}{2} e^{-2\Phi} \left[ 4 \dot{\Phi}^2 - n \left( \frac{\dot{a}}{a} \right)^2 - n(n-1)\frac{k}{a^2} - 8V(\Phi, \alpha) \right] + Da^{n(1-\gamma)}. \]  

(13)

where the potential \( U(\varphi) \rightarrow V(\Phi(\varphi, \alpha)) \). However, the duality invariance strictly depends on the form of the potential \( V \) and on \( k \). Lagrangian (13) can be furtherly simplified introducing the variable

\[ Z = \ln a, \]  

(14)

so that

\[ L = \frac{1}{2} e^{-2\Phi} \left[ 4 \dot{\Phi}^2 - n \dot{Z}^2 - n(n-1)ke^{-2Z} - 8V(\Phi, Z) \right] + De^{n(1-\gamma)Z}. \]  

(15)

We have to note that in such a new variables the duality invariance becomes a sort of parity invariance since \( Z \) and \( -Z \) are solutions [26, 27].

3 The equations of motion

From (15), the Euler–Lagrange equations are

\[ n\ddot{Z} - 2n\dot{\Phi}\dot{Z} + n(n-1)ke^{-2Z} - \frac{\partial V}{\partial Z} + n(1-\gamma)De^{n(1-\gamma)Z}e^{2\Phi} = 0, \]  

(16)

and

\[ \ddot{\Phi} - \dot{\Phi}^2 - \frac{n}{4} \dot{Z}^2 - \frac{n(n-1)}{4}ke^{-2Z} + \frac{\partial V}{\partial \Phi} = 0. \]  

(17)

The first one is the Friedman–Einstein equation, the second one is the Klein–Gordon equation. The energy function connected with (13) is

\[ E_L = \frac{\partial L}{\partial \dot{Z}} \dot{Z} + \frac{\partial L}{\partial \dot{\Phi}} \dot{\Phi} - L, \]  

(18)

and then

\[ 4\dot{\Phi}^2 - n\dot{Z}^2 + n(n-1)ke^{-2Z} + 8V - 2De^{n(1-\gamma)Z}e^{2\Phi} = 0, \]  

(19)

which corresponds to the (0, 0)–Einstein equation. The dynamical system (16), (17), and (19) is that we are going to study. Actually, such a system is a parametric system depending on the form of the potential \( V \), the bulk of fluid matter \( D \), the thermodynamical state of fluid \( \gamma \), the spatial curvature \( k \) and the spatial dimension \( n \). We shall show that all these parameters are connected giving specific cosmological features.
4 The effective potential

As we said, the presence of the dilaton potential is a crucial ingredient in string effective action for several reasons. From a cosmological point of view, it is needed in order to achieve some kind of inflation (e.g. chaotic inflation) and to solve the cosmological constant problem. From a fundamental physics point of view, it is needed in order to break supersymmetry taking into account the gaugino condensation, to compensate the contributions due to bosons and fermions in the vacuum energy, to lead the dynamics of particles like the axions \[12\]. In general, it appears as a nontrivial combination of exponentials. This feature expresses the fact that string–loop interactions have an expansion in term of their coupling constants \[34\]. It is related also to the string central charge as we mentioned above. By taking into account the transformations \([12]\) and \([14]\), the dilaton potential becomes a combination of geometrical and matter degrees of freedom, that is of \(Z\) and \(\Phi\).

Here we shall show that its dynamics is strongly connected with that of standard fluid matter. We shall consider three representative cases:

\[ V(Z, \Phi) = \frac{D}{4} e^{n(1-\gamma)Z} e^{2\Phi}, \]  
\( n \)  \( \Phi \)

\[ i.e. \]  \( \)  \( = \)

\[ V(Z, \Phi) = 0, \]  
\( n \)  \( \Phi \)

\[ i.e. \]  \( \)  \( = \)

\[ V(Z, \Phi) = \Lambda, \]  
\( n \)  \( \Phi \)

\[ i.e. \]  \( \)  \( = \)

\[ V(Z, \Phi) = 0, \]  
\( n \)  \( \Phi \)

\[ i.e. \]  \( \)  \( = \)

\[ V(Z, \Phi) = \Lambda, \]

\[ \text{the constant case, which is the properly called dilaton potential. Actually, the second case is nothing else but a subcase of the third one when the constant } \Lambda \text{ is chosen to be zero or it is extremely small.} \]

5 The solutions and their cosmological interpretation

Let us now discuss the various situations that can be parametrized by \( n, D, k, \gamma \) and the form of the potential.

5.1 The compensating potential

In the first of above cases, we are assuming that the amount of potential energy due to the dilaton is exactly comparable with the amount of fluid matter so that the two contributions annihilate each other. It could be interpreted as a sort of energy conversion. The dynamical system \([16], \) \([17]\), and \([19]\) becomes

\[ n\ddot{Z} - 2n\dot{\Phi}\dot{Z} + n(n-1)ke^{-2Z} = 0, \]  
\( n \)  \( \)  \( = \)

\[ 7 \]
\[
\ddot{\Phi} - \Phi^2 - \frac{n}{4} \dot{Z}^2 - \frac{n(n-1)}{4} ke^{-2Z} = 0, \tag{24}
\]
\[
4\ddot{\Phi}^2 - n\dot{Z}^2 + n(n-1)ke^{-2Z} = 0. \tag{25}
\]

We have to distinguish the cases with \( k = 0 \) and \( k \neq 0 \). If \( n = 1 \), the two cases coincide since a bidimensional theory (i.e. a spatial dimension plus time) is not affected by spatial curvature. This is true also for the considerations below.

In the flat case, the general solution is

\[
\Phi(t) = -\frac{1}{2} \ln |t - t_0| + \Phi_0, \tag{26}
\]
\[
Z(t) = \pm \frac{1}{\sqrt{n}} \ln |t - t_0| + Z_0, \tag{27}
\]

and by using the inverse transformations of (12) and (14), we get the cosmological behaviours

\[
a(t) = a_0(t - t_0)^{\pm 1/\sqrt{n}}, \tag{28}
\]
\[
\varphi(t) = \varphi_0 - \left( \frac{1 \pm \sqrt{n}}{2} \right) \ln |t - t_0|. \tag{29}
\]

Clearly they are invariant for duality and the ordinary fluid matter (i.e. \( D \) and \( \gamma \)) is not present in the evolution. They are the same as in a theory without potential (a critical or a free theory) and without ordinary matter [26]. Being \( \frac{1}{\sqrt{n}} \leq 1 \) in any case, the evolution is Friedmanian for any spatial dimension. In the case of \( n = 1 \), the scale factor becomes linear in \( t \) while its dual solution pole–like.

If the model is not spatially flat (i.e. \( k \neq 0 \)), a solution is

\[
\Phi(t) = \Phi_0 - \frac{n}{2} \ln |t - t_0|, \tag{30}
\]
\[
Z(t) = \frac{1}{2} \ln |k| + \ln |t - t_0|, \tag{31}
\]

which means

\[
a(t) = a_0 \sqrt{|k|(t - t_0)}, \quad \varphi(t) = \varphi_0 = \varphi_0 + \frac{n}{4} \ln |k|. \tag{32}
\]

The universe evolves linearly and the gravitational coupling is a constant. The curvature constant must be \( k < 0 \), that is \( k = -1 \). Considering the results we have got, the potential (20) can be regarded as a sort effective cosmological constant which is, at any time, comparable with the bulk of ordinary fluid matter and annihilate each other.
5.2 The constant potential

This situation shows a lot of interesting subcases in which the amount $D$ and the thermodynamical state $\gamma$ of ordinary matter strongly influence dynamics. The equations of motion become

\[ n\dddot{Z} - 2n\dot{\Phi}\dot{Z} + n(n-1)ke^{-2Z} - n(1-\gamma)De^{n(1-\gamma)Ze^{2\Phi}} = 0, \]  
\[ \ddot{\Phi} - \dot{\Phi}^2 - \frac{n}{4}\dot{Z}^2 - \frac{n(n-1)}{4}ke^{-2Z} - 2\Lambda = 0, \]  
\[ 4\dot{\Phi}^2 - n\dot{Z}^2 + n(n-1)ke^{-2Z} + 8\Lambda - 2De^{n(1-\gamma)Ze^{2\Phi}} = 0. \]

The critical case (21) is just a subcase. The summary of all the situations is given in Tab.1.

Due to the structure of the equations of motion, the case with $\gamma = 1$ deserves more attention. For $n = 3$, it corresponds to a dust dominated universe. In Tab.2, all combinations of the parameters in this particular case are summarized.

Before discussing the various situations, we have to stress again that the cases $\Lambda > 0$, which is a positive defined potential, $\Lambda < 0$, which is a negative defined potential, and $\Lambda = 0$ have all physical interest as pointed out, for example, in [35]. As mentioned above, $\Lambda$ is the string central charge depending on the number of spatial dimensions (see Eq.(1)) and on the string–loop interactions [34]. At a very fundamental level, the combinations of these two facts can determine the sign of the effective dilaton potential changing, from a cosmological point of view, the overall evolution of the universe.

Let us now show the various situations, that, as in Tab.1, can be classified by $\Lambda$, $k$, and $D$. As further parameters, we use $n$ and $\gamma$ which are always related. It is interesting to compare the cases with $D = 0$ and $D \neq 0$ in order to see how the bulk of ordinary matter influences the evolutions.

5.2.1 The cases with $D = 0$

If $k = D = 0$, by solving the system (33)–(35) and inverting Eqs.(12), (14) we get the solutions [19], [22], [26]

\[ a(t) = a_0 \left[ \tan \sqrt{2\Lambda(t-t_0)} \right]^{\pm 1/\sqrt{n}}, \]  
\[ \varphi(t) = \pm \frac{\sqrt{n}}{2} \ln \left| \tan \sqrt{2\Lambda(t-t_0)} \right| - \frac{1}{2} \ln \left| \sin \sqrt{2\Lambda(t-t_0)} \right| + \varphi_0, \]  
for $\Lambda > 0$;

\[ a(t) = a_0 \left[ \tanh \sqrt{2|\Lambda|(t-t_0)} \right]^{\pm 1/\sqrt{n}}, \]  
\[ \varphi(t) = \pm \frac{\sqrt{n}}{2} \ln \left| \tanh \sqrt{2|\Lambda|(t-t_0)} \right| - \frac{1}{2} \ln \left| \sinh \sqrt{2|\Lambda|(t-t_0)} \right| + \varphi_0, \]  
for $\Lambda < 0$ and

\[ a(t) = a_0(t-t_0)^{\pm 1/\sqrt{n}}, \]  
for $\Lambda = 0$. 


\[ \varphi(t) = \varphi_0 - \left(\frac{1 \pm \sqrt{n}}{2}\right) \ln |t - t_0| . \] (41)

for \( \Lambda = 0 \). Duality is evident. Eq. (39) shows an inflationary behaviour while (38) goes towards a stationary universe for \( t \to +\infty \). For \( \Lambda = 0 \), the situation is completely equivalent to that above.

If \( \Lambda = D = 0 \) and \( k \neq 0 \), we have, also here as above,

\[ a(t) = a_0 \sqrt{|k|}(t - t_0), \quad \varphi(t) = \varphi_0 = \varphi_0 + \frac{n}{4} \ln |k| . \] (42)

5.2.2 The cases with \( D \neq 0 \)

The presence of fluid matter can strongly influence the cosmological evolution. If \( \Lambda \neq 0, k \neq 0, \) and \( D \neq 0 \), an exact solution is

\[ a(t) = a_0 \cos \sqrt{|k|}(t - t_0), \quad \varphi(t) = \varphi_0 = \varphi_0 + \frac{1}{2} \ln \left[ \frac{n|k|}{D} \right] . \] (43)

The constant \( \Lambda \) has to be

\[ \Lambda = \frac{1}{8} n(n + 1)|k| > 0 , \] (44)

with \( \gamma = 0 \) and \( k = -1 \). This is an oscillating universe where the gravitational coupling is constant and depend on the bulk of fluid matter.

If \( \Lambda = 0, k \neq 0, \) and \( D \neq 0 \), we have

\[ a(t) = a_0(t - t_0), \quad \varphi(t) = \varphi_0 + \frac{1}{2} \ln |t - t_0| , \] (45)

with \( \gamma < 2/n, k = -1, \) and \( n \neq 1 \). The evolution is linear and the number of spatial dimensions is connected with the sound speed of fluid matter, that is the thermodynamical state is determined by the dimensionality.

When \( \Lambda = k = 0 \) and \( D \neq 0 \), we have

\[ a(t) = a_0 \left[ \frac{D(t - t_0) - \sqrt{n}}{D(t - t_0) + \sqrt{n}} \right]^{\pm 1/\sqrt{n}} , \] (46)

\[ \varphi(t) = - \left( \pm \frac{\sqrt{n} - 1}{2} \right) \ln |D(t - t_0) - \sqrt{n}| - \left( \pm \frac{\sqrt{n} + 1}{2} \right) \ln |D(t - t_0) + \sqrt{n}| + \varphi_0 , \] (47)

but it has to be \( \gamma = 1 \).

For \( \Lambda \neq 0, k = 0, \) and \( D \neq 0 \), we have

\[ a(t) = a_0 \left[ \cos 2 \sqrt{\Lambda}(t - t_0) \right]^{\pm 1/\sqrt{n}} , \] (48)

\[ \varphi(t) = - \left( \frac{1 \pm \sqrt{n}}{2} \right) \ln \left| \cos 2 \sqrt{\Lambda}(t - t_0) \right| + \varphi_0 , \] (49)
when $\Lambda > 0$ and
\[ a(t) = a_0 \left[ \cosh 2\sqrt{|\Lambda|(t-t_0)} \right]^{\pm 1/\sqrt{n}}, \] (50)
\[ \varphi(t) = - \left( \frac{1 \pm \sqrt{n}}{2} \right) \ln \left[ \cosh 2\sqrt{|\Lambda|(t-t_0)} \right] + \varphi_0, \] (51)
when $\Lambda < 0$. The bulk of fluid matter can be any but
\[ \gamma = \frac{\sqrt{n} \pm 1}{\sqrt{n}}. \] (52)

A particular treatment deserves this case if $\gamma = 1$. In fact we have
\[ a(t) = a_0 \left[ \frac{D \tan[\sqrt{2\Lambda}(t-t_0)] - c_1}{D \tan[\sqrt{2\Lambda}(t-t_0)] - c_2} \right]^{\pm 1/\sqrt{n}}, \] (53)
\[ \varphi(t) = - \frac{1}{2} \ln \left| c_3 - c_4 \sin[2\sqrt{2\Lambda}(t-t_0)] \right| \pm \frac{\sqrt{n}}{2} \ln \left| \frac{D \tan[\sqrt{2\Lambda}(t-t_0)] - c_1}{D \tan[\sqrt{2\Lambda}(t-t_0)] - c_2} \right|^{\pm 1/\sqrt{n}}, \] (54)
where
\[ c_{1,2} = \sqrt{D^2 + 8\Lambda n \dot{Z}_0^2} \pm \dot{Z}_0 \sqrt{8\Lambda n}, \quad c_3 = \frac{D}{8\Lambda}, \quad c_4 = \frac{\sqrt{D^2 + 8\Lambda n \dot{Z}_0^2}}{8\Lambda}, \] (55)
if $\Lambda > 0$, and
\[ a(t) = a_0 \left[ \frac{d_1 e^{2\sqrt{2\Lambda}(t-t_0)} - d_2}{d_1 e^{2\sqrt{2\Lambda}(t-t_0)} - d_3} \right]^{\pm 1/\sqrt{n}}, \] (56)
\[ \varphi(t) = - \frac{1}{2} \ln \left| d_1 \sinh[2\sqrt{2\Lambda}(t-t_0)] - D \right| \pm \frac{\sqrt{n}}{2} \ln \left| \frac{d_1 e^{2\sqrt{2\Lambda}(t-t_0)} - d_2}{d_1 e^{2\sqrt{2\Lambda}(t-t_0)} - d_3} \right|^{\pm 1/\sqrt{n}}, \] (57)
where
\[ d_1 = \sqrt{8|\Lambda|n \dot{Z}_0^2 - D^2}, \quad d_{2,3} = D \pm \dot{Z}_0 \sqrt{8|\Lambda|n}. \] (58)
if $\Lambda < 0$. $\dot{Z}_0$ is an integration constant.

6 Discussion and conclusions

In this paper, we have discussed the string-dilaton cosmology for a FRW spacetime in presence of minimally coupled perfect fluids. Such fluids are needed if we want to connect string cosmology with standard Einstein cosmology. As shown above the presence of fluids significantly modifies the evolution of the scale factor and the dilaton and, in some cases, it is connected with the dimensionality of the string background. However, the main role in the evolution is played by $\Lambda$ which determines the nature of solutions. We
get power law behaviours for $\Lambda = 0$, oscillating behaviours for $\Lambda > 0$, and hyperbolic ones for $\Lambda < 0$. Since $\Lambda$ is connected to the dimensionality (see Eq.(11)), dynamics of compactification should leave a specific imprint in the today observed evolution. The fluid matter plays a role in the rapidity by which the cosmological system evolves since it modulates the slope of $a(t)$.

Another important point has to be stressed. The effective gravitational constant, for a nonminimally coupled theory as that in (2), can be defined, in Planck units, as

$$G_{\text{eff}} = -\frac{1}{2F(\psi)}.$$  \hspace{1cm} (59)

To recover the Newton constant, it has to be $F(\psi) = -1/2$.

By using the choices (3) and specifying the action as in (4), we have that

$$G_{\text{eff}} = -4 \exp[2\varphi],$$  \hspace{1cm} (60)

and

$$\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} = 2\dot{\varphi}.$$  \hspace{1cm} (61)

This fact means that in various cases (see for example the last three ones) it is the bulk of fluid matter (i.e. $D$) which modulates the variation of the gravitational coupling and then the strength of gravity. Furthermore, for several solutions, when $t \to \pm \infty$, the gravitational coupling disappears (see e.g. all the solutions in which $\varphi(t)$ diverges toward $\pm \infty$). Then, by string cosmology, we can recover, in some cases, the standard gravity toward present epoch ($G_{\text{eff}} \to G_N$) and an asymptotically free theory toward the past or toward the future (see for example Eq.(47)) \[37\]. The fact that $G_N$ can be essentially modulated by $D$, $\Lambda$, and $\gamma$ means that the today observed gravitational interaction depends upon very fundamental primordial processes which are connected with the dimensionality, the string–loop interactions ($\Lambda$), the thermodynamical state of matter ($\gamma$), and the effective content of fluids in the universe ($D$)\[36\]. All these parameters can be related to the cosmological observables in order to reproduce the experimental limits on $G_N$ variation \[38\]. In a forthcoming paper, we will face this task.

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Tab.1: The possible combinations of \( \Lambda, k \) and \( D \). We are considering any \( \gamma \) and \( n \). In a lot of cases these two parameters are related. For \( \Lambda \neq 0 \), we mean positive and negative values; for \( k \neq 0 \), we mean \( k = \pm 1 \).

| \( \Lambda \neq 0 \) | \( k \neq 0 \) | \( D \neq 0 \) |
|---------------------|-----------------|----------------|
| \( \Lambda \neq 0 \) | \( k = 0 \) | \( D = 0 \) |
| \( \Lambda = 0 \)  | \( k \neq 0 \) | \( D \neq 0 \) |
| \( \Lambda = 0 \)  | \( k = 0 \) | \( D \neq 0 \) |

Tab.2: For \( \gamma = 1 \), we have particularly interesting cases which we summarize here. \( D \) is different from zero and positive.

| \( \gamma = 1 \) | \( \Lambda \neq 0 \) | \( k = \pm 1,0 \) | \( n = 1 \) |
|-----------------|-----------------|----------------|
| \( \gamma = 1 \) | \( \Lambda = 0 \) | \( k = \pm 1,0 \) | \( n = 1 \) |
| \( \gamma = 1 \) | \( \Lambda \neq 0 \) | \( k = 0 \) | \( n \neq 1 \) |
| \( \gamma = 1 \) | \( \Lambda \neq 0 \) | \( k \neq 0 \) | \( n \neq 1 \) |
| \( \gamma = 1 \) | \( \Lambda = 0 \) | \( k = 0 \) | \( n \neq 1 \) |
| \( \gamma = 1 \) | \( \Lambda = 0 \) | \( k \neq 0 \) | \( n \neq 1 \) |