Topological spinor vortex matter on spherical surface induced by non-Abelian spin-orbital-angular-momentum coupling

Jia-Ming Cheng, Ming Gong, Guang-Can Guo, Zheng-Wei Zhou, and Xiang-Fa Zhou

1. CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei, 230026, People’s Republic of China
2. CAS Center for Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, 230026, People’s Republic of China

E-mail: xfzou@ustc.edu.cn

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Abstract
We propose a scheme to implement non-Abelian spin-orbital-angular-momentum (SOAM) coupling in spinor Bose–Einstein condensates using magnetic gradient coupling. For a spherical surface trap addressable using high-order Hermite–Gaussian beams, we show that this system supports various degenerate ground states carrying different total angular momenta \( J \), and the degeneracy can be tuned by changing the strength of SOAM coupling. For spinor condensates with hyperfine spin \( F = 1 \), the system supports various meta-ferromagnetic phases and meta-polar states always described by quantized total mean angular momentum \( | \langle J \rangle | \) in case of weak interactions. Polar states with \( Z_2 \) symmetry and Thomson lattices formed by defects of spin vortices are also discussed. The system can be used to prepare various stable spin vortex states with nontrivial topology, and serve as a platform to investigate strong-correlated physics of neutral atoms with tunable ground-state degeneracy.

1. Introduction
Spin–orbital coupling, originally introduced due to the relativistic effect of the electron’s spin with its orbital angular momentum, is of ubiquitousness now in varying areas of physics. Especially in condensed matter physics, the coupling of spin and linear momentum (SLM) plays a vital role for many novel phenomena such as spin Hall effect \([1]\), topological insulator and topological superconductivity \([2–4]\). For neutral atoms, recent theoretical and experimental progresses show that the atomic spin can be coupled to its momentum degree of freedom with the help of optical Raman coupling \([5–12]\). Since the first experimental realization of SLM coupling in the condensate of \(^{87}\text{Rb}\) atoms \([7]\), various efforts have been made along this direction \([11–35]\).

Although most current investigations focus on SLM coupled quantum gases, as another type of spin–orbit coupled physics, coupling between atomic spin with its orbital angular momentum (OAM) degree of freedom has also received considerable attention recently \([36–45]\). Experimentally, this has been achieved by coupling different atomic spin states with laser beams carrying different OAMs \([42–44]\). However, in current studies, only the Abelian type spin–orbital–angular-momentum (SOAM) coupled interaction \( L_z F_2 \) is considered, where \( L_z \) and \( F_2 \) represent the atomic OAM and internal spin operators in the \( z \)-direction respectively. For general non-Abelian SOAM coupling \( L \cdot F = L_x F_x + L_y F_y + L_z F_z \), experimental implementation using Raman laser beams seems extremely difficult due to the non-commutable properties of spin and OAM operators. In atomic physics, such non-Abelian SOAM coupling can result in various fine structures of atomic levels with different degeneracy \([46]\). Meanwhile, this interaction is also closely linked with the generalization of quantum Hall physics in 3D system \([47, 48]\), and can be viewed as the parent system to generate almost all relevant SLM coupled interactions discussed in current studies \([48, 49]\). However, the difficulty of realizing such non-Abelian SOAM coupling greatly constrains our abilities to explore this novel physics in cold atoms.

In this work, we propose a scheme to implement non-Abelian SOAM coupling for spinful atoms. The method contains a crucial step of constructing a time-dependent Zeeman coupling of spinful atoms subjected to hedgehog-type gradient magnetic fields \( \mathbf{B} \propto \mathbf{r} \) (see figure 1). Although such effective gradient magnetic field is
not divergent-free $\nabla \cdot \mathbf{B} \neq 0$, we show that it can be realized for spinful atoms with internal spin $\mathbf{F}$ through Floquet engineering \cite{35}. In this case, the effect of non-Abelian SOAM coupling appears when the effective dynamics is expanded up to the second-order terms of the driving period. We further show that such interaction physics can be improved by introducing an effective curved spherical surface. Physically, this shell-shaped trap can be implemented using high-order Hermite–Gaussian laser beams, where almost perfect spherical surface potential can be constructed. Using this setting, the strength of SOAM coupling can be tuned in a wide range of parameters.

We notice that currently, there is also a growing interest in the effect of the underlying geometry on various quantum orders in cold atomic system \cite{51–63}. For instance, exotic vortex structures of spinor condensates on a cylindrical surface have been considered recently by Ho and Huang \cite{51}. For condensates subjected to a hollow spherical surface trap, the static and dynamic properties have also been considered in \cite{62, 63}. However, in traditional schemes, the creation of hollow spherical surface magnetic trap is challenging on Earth due to the presence of gravitational sag. To overcome this, micro-gravity conditions are usually expected which can only be satisfied in the space-based Cold Atomic Laboratory \cite{64, 65}. This greatly increases the costs and difficulties for experimentally exploring many novel physics in such curved geometry. Our construction scheme of spherical surface trap involves only high-order Hermite–Gaussian laser beams, which should be relatively easier to be implemented in current experimental setup. In this respect, the method we propose here is very timely and simple. Moreover, the construction of such optical trap also liberates the atom’s internal spin degree of freedom. This also enables the investigation of various novel states carrying nontrivial spin structures in spinor quantum gases.

We stress that in the presence of a spherical surface trap, the single particle ground-states of the SOAM coupled system exhibit tunable degeneracy due to its high symmetry. For weakly interacting spinor condensates with spin $F = 1$ \cite{66–68}, the system supports various meta-ferromagnetic (mFM) and meta-polar phases with quantized magnitudes of the total mean angular momentum $|J| = |(L + F)|$ and non-vanishing spin fluctuations. This is completely different from the usual vortex phases characterized by the quantized angular momentum $L_z$ only. In the case of vanishing spin–exchange interaction, the defects of spin vortices can form stable lattice configurations on sphere characterized by the standard Thomson problem \cite{50, 69}. Meanwhile, in the polar regime with strong spin–exchange interaction, the system supports stable nontrivial polar states \cite{67, 70, 71} characterized by $Z_2$-type topological invariant. The system can be viewed as a vortex zoo of constructing stable spin vortices with novel intrinsic topology, and serve as a desired platform to explore various non-Abelian SOAM coupled physics for both bosons and fermions.

\section*{2. Scheme of implementing SOAM coupling and the model Hamiltonian}

To illustrate the novel physics induced by non-Abelian SOAM coupling, we consider the implementation of the coupling $\mathbf{L} \cdot \mathbf{F}$ in spinor condensates with the total spin $\mathbf{F}$. Such 3D SOAM coupling occurs in atomic physics due to the relativistic effect, where the spin of electrons only take values $S = 1/2$. While for neutral atoms, the spin quantum number $f$ can take integer and half-integer values for boson and fermions, which greatly enriches the underlying physics. However, the implementation of such non-Abelian coupling is nontrivial for cold atoms as the relativistic effect is almost undetectable.

To implement the isotropic SOAM coupling $\mathbf{L} \cdot \mathbf{F}$ for atoms, we consider the following two-step scheme. In the first step, we construct an effective hedgehog-type magnetic gradient fields $\mathbf{B} \propto r$ for spinor atoms which induces a Zeeman coupling term $r \cdot \mathbf{F}$. We note that the strength of such magnetic gradient fields increases linearly along with the radius $|r|$, and cannot be generated directly using traditional electromagnetic coils as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hedgehog.png}
\caption{Schematic plot of a hedgehog-type magnetic field $\mathbf{B} \propto r$.}
\end{figure}
\( \nabla \cdot \mathbf{B} = 0 \). However, for spinor atoms, this monopole-like effective magnetic fields can be generated by employing a Floquet engineered magnetic field, as has recently been proposed in [50]. Here, following the same routine, we consider the following time-dependent oscillating magnetic field as

\[
\mathbf{B} = B_0 \mathbf{e}_z + B_1(t) [1 - 2 \cos(\omega_0 t)] \left[ \cos^2(\omega_0 t)(\mathbf{e}_x \cdot \mathbf{e}_y) + y \mathbf{e}_x - 2 \mathbf{e}_y \right]\sin(2\omega_0 t)(-\mathbf{e}_x + \mathbf{e}_y) \right] \tag{1}
\]

with a strong bias field \( B_0 \). Since \( \nabla \cdot \mathbf{B} = 0 \), this field can be generated using electromagnetic coils in current experiments. Here \( B_1(t) \) represents the slow varying envelope compared with the oscillating terms with frequency \( \omega_0 \) and satisfies \( \partial B_1(t)/\partial t \ll \omega_0 \). The single-particle Hamiltonian can be written as

\[
H' = H_0 - \frac{\mu B_0}{\hbar} \mathbf{B} \cdot \mathbf{F} \quad \text{with} \quad H_0 = -\frac{\hbar^2 \nabla^2}{2\mu} + V(r).
\]

In current experiments, the frequency \( \omega_0 \) can be set up to \( 10^8 \) Hz. If we choose \( B_0 \) such that \( \omega_0 = \mu B_0 / \hbar \), and assume that this frequency is much larger than all the other energy scales, then in the rotating frame defined by \( U = e^{\omega_0 \mathbf{B}/\hbar} \), the effective Hamiltonian reads

\[
H = U^+ H U - i\hbar U^+ \partial_t U = H_0 + \frac{\mu B_0}{\hbar} B_1(t) \mathbf{r} \cdot \mathbf{F} + H_F(t) \tag{2}
\]

with the high frequency part

\[
H_F(t) = -\frac{\mu B_0}{\hbar} B_1(t) [(x_F y + y F_1 + 3z E) \cos(\omega_0 t) - (x F y + y F_1 + z E) \cos(2\omega_0 t) + z E \cos(3\omega_0 t)].
\]

In the rotating-wave approximation, this rapid oscillating term \( H_F(t) \) can be safely neglected. Generally, the influence of \( H_F(t) \) on the dynamics of the system can also be estimated using the standard expansion. According to the discussions in [72], for a periodically driven Hamiltonian

\[
H(t) = H_0 + \sum_{j=1}^{\infty} (V(j) e^{i\omega_0 j t} + V^{(-j)} e^{-i\omega_0 j t}),
\]

the dynamics can be described as

\[
U(t_i \rightarrow t_f) = e^{-i\mathbf{K}(t_f)} e^{-i\hbar(t_f - t_i)} e^{-i\mathbf{K}(t_i)} \tag{4}
\]

with the effective time-independent Hamiltonian

\[
H_t = H_0 + \frac{1}{\hbar \omega_0} \sum_{j=1}^{\infty} \frac{1}{j} [V(j), V^{(-j)}] + \frac{1}{2\hbar^2 \omega_0^2} \sum_{j=1}^{\infty} \frac{1}{j^2} ([V(j), H_0], V^{(-j)}] + \text{h.c.})
\]

\[
+ O\left(\frac{1}{\omega_0^3}\right) \tag{5}
\]

and the kick operator \( \mathbf{K}(t) \) depending on the initial and final time \( t_i \) and \( t_f \) of the evolution. If we choose \( t_i = 0 \) and \( t_f - t_i = 2\pi / \omega_0 \) then \( \mathbf{K}(t_f) - \mathbf{K}(t_i) \sim O(1/\omega_0^2) \) which leads to \( e^{-i\mathbf{K}(t)} = e^{-i\mathbf{K}(t_f)} \sim I \). Therefore, in the high frequency limit, we obtain an effective monopole-like magnetic fields up to the second order \( O(1/\omega_0^2) \).

The implementation of 3D non-Abelian SOAM coupling is achieved if we further assume \( B_1(t) = B_1 \cos(\Omega_0 t) \). In this case, we have the following time-dependent Hamiltonian with a hedgehog-type magnetic Zeeman coupling in the rotating frame [50, 72]

\[
H_t(t) = H_0 + \nu_0 \cos(\Omega_0 t) \mathbf{r} \cdot \mathbf{F} + O\left(\frac{1}{\omega_0^3}\right)
\]

with \( \nu_0 = \mu B_0 B_1 / \hbar \). Here, the driven frequency \( \Omega_0 \) (about \( 10^4 \sim 10^5 \) Hz) should be chosen such that \( \Omega_0 \) is much less than \( \omega_0 \) but still much larger the characteristic frequency of the condensates, namely \( \omega_0 \gg \Omega_0 \gg \omega \) (Explicitly, \( \omega \) is the characteristic frequency of spherical trap shown in equation (15)). According to the expansion shown in equation (5) and using the commutation relation

\[
[\mathbf{r} \cdot \mathbf{F}, \mathbf{p} \cdot \mathbf{F}] = i\hbar (\mathbf{F}^2 + \mathbf{L} \cdot \mathbf{F}),
\]

we finally arrive at the following effective interaction

\[
H_{\text{eff}} = H_0 + \lambda (\mathbf{F}^2 + \mathbf{L} \cdot \mathbf{F}) + O\left(\frac{\Omega_0}{\omega_0}\right) + O\left(\frac{1}{\Omega_0^3}\right)
\]

where the SOAM coupling strength reads \( \lambda = \nu_0^2/(4\mu \Omega_0) \sim O(1/\Omega_0^2) \). Since \( \Omega_0 / \omega_0 \ll 1 \), we have succeeded in realizing the desired \( \lambda (\mathbf{L} \cdot \mathbf{F})\)-type coupling up to a constant term \( \mathbf{F}^2 \). Although \( \lambda \) is always positive in the above...
In the case of the electric dipole approximation, the optical potentials induced by these beams can be estimated as

\[ V_{\text{electric}}(x, y, z) = \frac{\mu}{\hbar c} \mathbf{E}(t) \cdot \mathbf{p}(t) \]

where \( \mathbf{E}(t) \) is the electric field amplitude and \( \mathbf{p}(t) \) is the momentum of the atom. For usual Hermite-Gaussian modes, the optical potentials can be expressed as

\[ V_{m,n}(x, y, z) = \frac{\sqrt{2^{m+n} \pi}}{w_{1}^{m+n/2} w_{2}^{m+n/2}} H_{m}(x/w_{1}) H_{n}(y/w_{2}) e^{i(k_{z} z - \omega t)} \]

which is propagating along the \( z \) direction with the waist radius \( w_{1} \). Here \( H_{n}(x) \) is the Hermite polynomial. The first few polynomials are listed as follows for later convenience

\[ H_{0}(x) = 1, \quad H_{1}(x) = 2x, \quad H_{2}(x) = 4x^{2} - 2, \cdots \]

To construct a perfect curved surface trap for spinor atoms, we propose to employ high-order Hermite-Gaussian beams, with which the spin degree of freedom of the atoms can be liberated. For usual Hermite-Gaussian mode denoted as TEM\(_{m,n}\), the electric field amplitude reads

\[ E_{m,n}(x, y, z) = H_{m}(x/w_{1}) H_{n}(y/w_{2}) e^{i(k_{z} z - \omega t)} \]

which is propagating along the \( z \) direction. Here the color-map shows the relative magnitudes in each panel instead of their real values in our scheme.

### Figure 2

(a) Schematic plot about the implementation of the spherical surface trap using Hermite–Gaussian modes along \( x \), \( y \), and \( z \)-axis respectively. (b) Sketch maps of the three optical potentials \( V_{00}^{(z)} \), \( V_{11}^{(z)} \) and \( V_{20}^{(z)} \) along \( x \) direction. Here the color-map shows the relative magnitudes in each panel instead of their real values in our scheme.

3. Creation of spherical surface trap using high-order Hermite–Gaussian beams

To construct a perfect curved surface trap for spinor atoms, we propose to employ high-order Hermite-Gaussian laser beams, with which the spin degree of freedom of the atoms can be liberated. For usual Hermite-Gaussian modes denoted as TEM\(_{m,n}\), the electric field amplitude reads

\[ E_{m,n}(x, y, z) = H_{m}(x/w_{1}) H_{n}(y/w_{2}) e^{i(k_{z} z - \omega t)} \]

where \( \{ w_{1}, w_{2} \} \gg \{ x, y, z \} \). Here \( V_{00}^{(z)} \) and \( V_{11}^{(z)} \) can be realized using TEM\(_{00}\) and TEM\(_{11}\) modes respectively. \( V_{20}^{(z)} \) can also be implemented using combined beams composed of TEM\(_{00}\) and TEM\(_{20}\) modes. The interference effect between different laser beams can be removed by introducing random phases, or using beams with different polarizations and frequencies. The above discussions are also valid for laser beams traveling along the \( x \) and \( y \) direction respectively. The relevant optical potentials can be denoted by \( V_{m,n}^{(x)}(x, y) \), \( V_{m,n}^{(y)}(x, y) \), which take the similar form as \( V_{m,n}^{(z)}(x, y) \) up to a cyclic permutation of the symbols \( \{ x, y, z \} \). The total potential after summing over all those terms along the \( x \), \( y \) and \( z \) directions reads

\[ V_{m,n}(x, y, z) = \frac{\sqrt{2^{m+n} \pi}}{w_{1}^{m+n/2} w_{2}^{m+n/2}} H_{m}(x/w_{1}) H_{n}(y/w_{2}) e^{i(k_{z} z - \omega t)} \]

For instance, three modes such as TEM\(_{00}\), TEM\(_{11}\), and TEM\(_{20}\) along the \( z \)-axis can induce the following optical potentials (see figure 2)

\[ V_{00}^{(z)}(x, y) = U_{0} e^{2x^{2}+y^{2}/w_{1}^{2}} = U_{0} \left[ 1 - \frac{2(x^{2} + y^{2})}{w_{1}^{2}} \right] + O \left( \frac{U_{0}}{w_{1}^{2}} \right) \]

\[ V_{11}^{(z)}(x, y) = U_{1} x^{2} e^{2x^{2}+y^{2}/w_{2}^{2}} = U_{1} x^{2} \left[ 1 - \frac{2(x^{2} + y^{2})}{w_{2}^{2}} \right] + O \left( \frac{U_{1}}{w_{2}^{2}} \right) \]

\[ V_{20}^{(z)}(x, y) = U_{2} x^{4} e^{2x^{2}+y^{2}/w_{2}^{2}} = U_{2} x^{4} \left[ 1 - \frac{2(x^{2} + y^{2})}{w_{2}^{2}} \right] + O \left( \frac{U_{2}}{w_{2}^{2}} \right) \]

where we have used the paraxial approximation such that \( \{ w_{1}, w_{2} \} \gg \{ x, y, z \} \). Here \( V_{00}^{(z)} \) and \( V_{11}^{(z)} \) can be realized using TEM\(_{00}\) and TEM\(_{11}\) modes respectively. \( V_{20}^{(z)} \) can also be implemented using combined beams composed of TEM\(_{00}\) and TEM\(_{20}\) modes. The interference effect between different laser beams can be removed by introducing random phases, or using beams with different polarizations and frequencies. The above discussions are also valid for laser beams traveling along the \( x \) and \( y \) direction respectively. The relevant optical potentials can be denoted by \( V_{m,n}^{(x)}(x, y) \), \( V_{m,n}^{(y)}(x, y) \), which take the similar form as \( V_{m,n}^{(z)}(x, y) \) up to a cyclic permutation of the symbols \( \{ x, y, z \} \). The total potential after summing over all those terms along the \( x \), \( y \) and \( z \) directions reads
The total potential can be simplified by approximating \( r^2 = x^2 + y^2 + z^2 \), and other terms proportional to \( r^6 \) or higher are safely neglected due to the paraxial condition. Then if we set \( U_1 = 2U_2 = 2U \) and \( R^2 = 2U_0/(Uw_1^2) \), the total potential field can be written as

\[
V = U(r^2 - R^2)^2 - UR^4 + 3U_0 - \frac{2U}{w_1^2}[x^2(2y^2 + x^2)(x^2 + y^2) + y^2(2z^2 + y^2)(z^2 + y^2) + z^2(2x^2 + z^2)(x^2 + z^2)] + O\left(\frac{U_0}{w_1^2}\right) + O\left(\frac{U}{w_2^2}\right).
\]

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V = U(r^2 - R^2)^2 - UR^4 + 3U_0 - \frac{2U}{w_1^2}[x^2(2y^2 + x^2)(x^2 + y^2) + y^2(2z^2 + y^2)(z^2 + y^2) + z^2(2x^2 + z^2)(x^2 + z^2)] + O\left(\frac{U_0}{w_1^2}\right) + O\left(\frac{U}{w_2^2}\right).
\]

Moreover, if we set \( R \sim 1 \) and \( \{w_1, w_2\} \gg R \), then \( U_0 \sim Uw_1^2/2 \gg U \). In this case, the total potential can be approximated as

\[
V \approx U(r^2 - R^2)^2 - UR^4 + 3U_0 - \frac{4U_0}{w_1^2}x^2(2y^2 + x^2)(x^2 + y^2)
+ y^2(2z^2 + y^2)(z^2 + y^2) + z^2(2x^2 + z^2)(x^2 + z^2)] + O\left(\frac{U_0}{w_1^2}\right) + O\left(\frac{U}{w_2^2}\right).
\]

which indicates that the minimal value of the total potential is obtained at \( r = R \). Around this minimal point, the total potential can be simplified as (up to a constant term)

\[
V(r) \approx 4UR^2(r - R)^2 = \frac{1}{2}\mu\omega^2(r - R)^2
\]

with \( \mu \) the mass of the atom, and \( \omega = \sqrt{8UR^2/\mu} \) the characteristic frequency. Therefore, we have succeeded to implement a desired spherical surface trap using high-order Hermite–Gaussian beams.

We stress that to ensure the paraxial conditions, appropriate \( U_0, U, w_1 \) and \( w_2 \) should be chosen so that \( R = [2U_0/(Uw_1^2)]^{1/2} \ll \{w_1, w_2\} \) is satisfied. Meanwhile, to ensure \( U_0 \gg U > 0 \), laser modes with different blue-detuning and Rabi coupling are also required. Figure 3(a) shows the the local minimal points \( r_{\text{min}} \) of \( V(r) \) along different radial direction defined by \( (\theta, \phi) \), which indicates that \( r_{\text{min}} \) is almost fixed satisfying \( r_{\text{min}} \approx R = 1 \). The contour plot of the spherical potential \( V(r) \) around the surface \( r = R \) is also depicted in figure 3(b), which is also fixed for different \( (\theta, \phi) \). As an example, the radial dependence of \( V(r) \) along the \( x \)-axis is also shown in figure 3(c), which exhibits a desired local minimal around \( r \approx R \). Therefore, using these setting, we can obtain an almost perfect spherical surface trap. Meanwhile, the radius \( R \) and the trapping frequency \( \omega \) can be well adjusted by using laser beams with different strength and detuning. This provides an ideal platform of investigating various novel physics for cold atoms subjected to such boundaryless curved geometry. Especially, for SOAM coupled condensates, the system supports exotic spinor vortex phases with intrinsic topological properties.
The above construction of non-Abelian SOAM coupling in cold atoms provides an avenue to explore various novel physics with high flexibility. First, the usual SLM coupled subsystem in 2D can be easily obtained by cutting the system appropriately, as shown in [48, 49]. For instance, by fixing $z = z_0$, we can have a 2D subsystem with Rashba-type SLM coupling. Meanwhile, quantum Hall physics in 3D space can also be induced by non-Abelian SOAM coupling [47, 48]. Second, as will be shown later, the implementation of a spherical surface trap with tunable radius allows us to modify the strength of SOAM coupling in a wide range of parameters. Therefore, the degeneracy of the ground states subspace can be well controlled in a much flexible manner. This also opens up the possibility of exploring strong-correlated physics with only a few particles. Finally, for spinor condensates, this non-Abelian SOAM coupling results in various spin vortices with intrinsic topological properties, as will be discussed in the following.

4. Single particle spectra in the presence of surface trap

In the case of deep traps and low temperature, the radial motion of atoms is frozen and atoms are mainly confined around the spherical surface with radius $R$. The field operator can be assumed to be $\sqrt{N}\varphi(r)\hat{\psi}(\theta, \phi)$. The radial wavefunction $\varphi(r)$ reads $\varphi(r) = (\pi l_f^2 r^4)^{-1/4} \exp\left[-(r - R)^2/(2l_f^2)\right]$, where $l_f = \sqrt{\hbar/\mu_0} \ll R$ is the characteristic length of spherical trap in radial direction. After integrating out the radial degree-of-freedom, we obtain a reduced dimensionless single-particle Hamiltonian in a spherical surface trap as (see appendix B)

$$\mathcal{H}_0 = \mathbf{L}^2 + \lambda \mathbf{L} \cdot \mathbf{F},$$

where $\lambda = 2\mu R^2 \tilde{\lambda} = 2\mu F_0^2 B_0^2 R^2/(\hbar^2 \Omega_0^2)$, which thus can be tuned in a wide range by changing the radius $R$, or the ratio $B_0/\Omega_0$ respectively.

The system possesses conserved quantities including $\mathbf{L}^2, \mathbf{F}^2, \mathbf{J}^2$, and $J_z$ with the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{F}$. The single-particle eigenstates can be labeled using quantum numbers $(l, f, j_+, j_-)$ as

$$\psi_{j_+ j_-}^l(\theta, \phi) = |j_+, j_-\rangle = \sum_{m=-l}^{l} \sum_{f=-f}^{f} C_{j_+ j_-}^{l(f)}|l, m; f, j_+\rangle$$

with the Clebsch–Gordan coefficients $C_{j_+ j_-}^{l(f)} \equiv \langle l, m; f, j_+ | l, m \rangle = Y_{j, m}(\theta, \phi)$ the usual spherical harmonics, and $|f, j_+\rangle$ the internal state of spinful atoms. The corresponding single-particle energy is degenerate for different $j_+$ and reads

$$E_0 = \lambda (l + 1) + \frac{\lambda}{2} [j(j+1) - l(l+1) - f(f+1)]$$

with $j = |l - f|, |l - f| + 1, \ldots, l + f$. When $\lambda > 0$, $\mathbf{L}$ is anti-parallel to $\mathbf{F}$, and we have $j = |l - f|$ for the ground state. Otherwise, we have $j = l + f$. In both cases, the explicit values of $l$ and $j$ for ground states depend on the coupling $\lambda$. Therefore, the degeneracy of ground states can be tuned in a much flexible manner. The mean values of $\mathbf{F}$ and $\mathbf{L}$ is proportional to $|\langle \mathbf{J} \rangle|$, and can be computed [73] as

$$\langle \mathbf{F} \rangle = \frac{1 - \alpha}{2} |\langle \mathbf{J} \rangle|, \quad \langle \mathbf{L} \rangle = \frac{1 + \alpha}{2} |\langle \mathbf{J} \rangle|, \quad |\langle \mathbf{J} \rangle| = j^2 \mathbf{e}_z,$$

with $\alpha = |l(l + 1) - f(f + 1)|/j(j + 1)$. Therefore, $|\langle \mathbf{F} \rangle|$ and $|\langle \mathbf{L} \rangle|$ can take fractional values for different $j_z$.

5. Phase diagram of SOAM-coupled spinor condensates with spin $F = 1$

For spinor condensates ($F = 1$) with low-energy $s$-wave contact scattering, the interaction Hamiltonian in a spherical trap can be written as

$$H_{int} = \int_{R-\delta}^{R+\delta} r^2 dr \int d\Omega \left[C_0 \hat{n}^2(\mathbf{r}) + C_1 \hat{F}_z^2(\mathbf{r})\right]$$

with $d\Omega = \sin \theta d\theta d\phi$. Here the interaction strengths read $C_0 = \frac{g_0 + 2g_0}{6}$ and $C_1 = \frac{g_0 - 2g_0}{6}$ with $g_0 = \frac{4\pi \hbar^2}{\mu} a_0$ and $g_z = \frac{4\pi \hbar^2}{\mu} a_z$ describing the strengths of the two-body scattering channels for the total spin $f = 0$, and $f = 2$ respectively, and $a_0$ and $a_z$ are the corresponding $s$-wave scattering lengths. $\hat{n} = \sum_{j = 0}^{l} \hat{n}_{j+}^l \hat{n}_{j-}^l$ is the total particle number. The spin operators are defined as

$$\hat{F}_z^2 = (\hat{F}_+ \hat{F}_- + \hat{F}_- \hat{F}_+)/2 + \hat{F}_z^2$$

with $\hat{F}_+ = \hat{F}_+^0 = \sqrt{2}(\psi_{1+}^l \hat{\psi}_{0+} + \psi_{0+}^l \hat{\psi}_{1+}^l)$, and $\hat{F}_- = \psi_{1-}^l \hat{\psi}_{1-} - \psi_{0-}^l \hat{\psi}_{0-}^l$ represents the normal ordering of the relevant operators when all creation operators are moved to the left of all annihilation operators. For low-energy physics considered here, the radial motion of bosons is frozen. Therefore the field operator can be written as
\[ \psi(\mathbf{r}) = \sqrt{N} \psi(\theta, \phi) \varphi(r) \] with \( N \) the total number of particles and
\[ \psi(\theta, \phi) = [\psi_\uparrow(\theta, \phi) \psi_\downarrow(\theta, \phi)]^T \] satisfying \( \int d\Omega \psi^\dagger(\theta, \phi) \psi(\theta, \phi) = 1 \). After integrating out the radial part, the reduced contact interaction Hamiltonian in the spherical surface turns into
\[ H_{\text{int}} = N \int d\Omega \left[ c_0 \hat{n}^2(\theta, \phi) + c_1 \hat{F}^2(\theta, \phi) \right]. \]

Here \( c_0 \) and \( c_1 \) define the reduced dimensionless strengths of density-density and spin-dependent interactions on the surface trap, whose explicit forms can be written as
\[
\begin{align*}
c_0 &= \frac{\xi_0 + 2\xi_2}{6c_0} N \int r^3 |\varphi(r)|^4 \, dr \approx \frac{2\sqrt{2\pi}}{3} \frac{a_0 + 2a_2}{l_f} N, \\
c_1 &= \frac{\xi_2 - \xi_0}{6c_0} N \int r^3 |\varphi(r)|^4 \, dr \approx \frac{2\sqrt{2\pi}}{3} \frac{a_2 - a_0}{l_f} N.
\end{align*}
\]

In the mean-field level, we find the phase diagrams using both the imaginary-time-evolution and variational methods. Since the single-particle eigenstates are degenerate, the ground states \( \psi(\theta, \phi) \) exhibit complex spin patterns even for condensates with weak contact interaction. When \( f = 1 \), the interaction energy of each atom is
\[ E_{\text{int}} = \int d\Omega \left[ c_0 n(\theta, \phi)^2 + c_1 \hat{F}(\theta, \phi)^2 \right]. \]

Here \( n(\theta, \phi) = |\psi(\theta, \phi)|^2 \) is the local density and \( \hat{F}(\theta, \phi) = \psi^\dagger \hat{F} \psi \) represents the local spin-density vector. Due to the symmetry, the ground state \( \psi(\theta, \phi) \) is equivalent up to a global rotation defined as \( R(\alpha, \beta, \gamma) = \exp(-i\hat{L}_z \alpha) \exp(-i\hat{L}_y \beta) \exp(-i\hat{L}_x \gamma) \).

### 5.1. Phase diagram for weak interaction \( c_0 = 1 \)

Figure 4 shows the phase diagram in \( c_1/c_0 - \lambda \) plane for small quantum numbers of \( l \) and \( j \), where the explicit ground-state configurations within different regimes of \( \lambda \) are also provided in tables 1 and 2. In these cases, the ground states can be determined quantitatively as the single-particle eigenstates have much lower degeneracy. The phase diagram shows many novel features, which will be discussed below.

First, due to the presence of interaction, the degeneracy of \( \psi_{ij}^{lf} \) with different \( j \) is broken even in the presence of very weak interaction \( c_0 = 1 \). For given \( \lambda \) and \( j \), the ground state carries different \( |f\rangle \) (or \( |L\rangle \) and \( |F\rangle \)), and supports various exotic spin patterns depending on the ratio \( c_1/c_0 \) (see tables 1 and 2 for details). A special case occurs when \( \lambda \in (1, 4) \) with \( j = 0 \) (\( l = 1 \)). In this case, the spin and OAM are along opposite directions and

|\( \lambda \)| | \( \lambda | F(0) \) | \( \lambda | P(0) \) | \( \lambda | P(+1) \) | \( \lambda | P(-1) \) |
|---|---|---|---|---|
| 0 | FM(0) | FM(1) | mFM(1) | P(0) |
| 1 | FM(0) | FM(1) | mFM(1) | P(0) |
| 2 | FM(0) | FM(1) | mFM(1) | P(0) |
| 3 | FM(0) | FM(1) | mFM(1) | P(0) |
| 4 | FM(0) | FM(1) | mFM(1) | P(0) |

\[ \psi(r) = \sqrt{N} \psi(\theta, \phi) \varphi(r) \] with \( N \) the total number of particles and
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\begin{align*}
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c_1 &= \frac{\xi_2 - \xi_0}{6c_0} N \int r^3 |\varphi(r)|^4 \, dr \approx \frac{2\sqrt{2\pi}}{3} \frac{a_2 - a_0}{l_f} N.
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In the mean-field level, we find the phase diagrams using both the imaginary-time-evolution and variational methods. Since the single-particle eigenstates are degenerate, the ground states \( \psi(\theta, \phi) \) exhibit complex spin patterns even for condensates with weak contact interaction. When \( f = 1 \), the interaction energy of each atom is
\[ E_{\text{int}} = \int d\Omega \left[ c_0 n(\theta, \phi)^2 + c_1 \hat{F}(\theta, \phi)^2 \right]. \]

Here \( n(\theta, \phi) = |\psi(\theta, \phi)|^2 \) is the local density and \( \hat{F}(\theta, \phi) = \psi^\dagger \hat{F} \psi \) represents the local spin-density vector. Due to the symmetry, the ground state \( \psi(\theta, \phi) \) is equivalent up to a global rotation defined as \( R(\alpha, \beta, \gamma) = \exp(-i\hat{L}_z \alpha) \exp(-i\hat{L}_y \beta) \exp(-i\hat{L}_x \gamma) \).

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First, due to the presence of interaction, the degeneracy of \( \psi_{ij}^{lf} \) with different \( j \) is broken even in the presence of very weak interaction \( c_0 = 1 \). For given \( \lambda \) and \( j \), the ground state carries different \( |f\rangle \) (or \( |L\rangle \) and \( |F\rangle \)), and supports various exotic spin patterns depending on the ratio \( c_1/c_0 \) (see tables 1 and 2 for details). A special case occurs when \( \lambda \in (1, 4) \) with \( j = 0 \) (\( l = 1 \)). In this case, the spin and OAM are along opposite directions and
forms a singlet state. The single-particle ground-state $\psi_{1,0}^{1,1}$ (see equation (B.11)) is non-degenerate, which is also the ground-state of the condensate with weak interaction strength. Meanwhile, the system possesses homogeneous density distribution with its spin mean-value $\langle F \rangle = 0$ and spin-density $\mathcal{F}_{\phi, \theta}(\theta, \phi) = 0$ ($\lambda = x, y, z$). So it inherently belongs to a polar state ($P_{-}(1)$) no matter whether spin-exchange interaction is antiferromagnetic or not. Coreless vortices with vorticity $n_{\phi} = -1$ and $n_{\theta} = 1$ appear in component $f_{z} = 1$ and $f_{z} = -1$ respectively, which thus also induces pure spin currents in such system. For $\lambda < 1$ and $\lambda > 4$, the interaction ground-states are listed in table 1 and table 2 respectively. Interestingly, the phase diagram exhibits similar structures for the same $j_{z}$ regardless of whether $j = l + f$ or $j = l - f$.

Table 1. Explicit information of different phases in the phasediagram for $\lambda < 1$ within different regimes $c_{i}/c_{0}$. Here ‘WF’ is short for ‘wavefunction’. Others symbols are defined as follows: $\eta = (41 - \sqrt{5233})/192 \approx -0.16$, $\alpha = (\sqrt{11c_{0} - 28c_{0}})/(47c_{0} - 80c_{0})$ and $\beta = \sqrt{1 - 2\alpha^{2}}$.

| Phase | $c_{i}/c_{0}$ | WF | $\langle |F| \rangle$ | $\Delta \mathcal{F}$ |
|-------|----------------|-----|---------------------|----------------------|
| FM(0) | $(-\infty, 0)$ | $\psi_{0,0}^{1,1}$ | (1,1) | $0_{\lambda < 1}$ |
| P(0)  | $(0, \infty)$  | $\psi_{0,0}^{1,1}$ | (1,0) | $0_{\lambda < 1}$ |
| FM_{1}(0) | $(-\infty, -1/3)$ | $\psi_{0,1}^{1,1}$ | (2.1) | $0_{\lambda < 1}$ |
| mP_{(1)} | $(-1/3, 1/3)$ | $\left[ \sqrt{\mathcal{T}_{1,1,1}} \psi_{1,1}^{1,1} + \psi_{1,1}^{1,1} \right]/\sqrt{\mathcal{T}}$ | (0.0) | $\text{diag}[1, 1, 1]/208$ |
| P_{+}(1) | $(1/3, \infty)$ | $\psi_{1,1}^{1,1}$ | (0.0) | $0_{\lambda < 1}$ |
| FM_{2}(2) | $(-\infty, -2/3)$ | $\psi_{0,1}^{1,1}$ | (3.1) | $0_{\lambda < 1}$ |
| mFM_{(2)} | $(-2/3, 2/3)$ | $\psi_{1,1}^{1,1}$ | (2.2/3) | $\text{diag}[3, 3, 3]/128$ |
| mP_{(2)} | $(\eta, 1/4)$ | $\alpha[\psi_{1,1}^{1,1} + \psi_{1,1}^{1,1}] + \beta \psi_{1,1}^{1,1}$ | (0.0) | $\text{diag}[3, 3, 3]/208$ |
| P_{+}(2) | $(1/4, \infty)$ | $\left[ \psi_{1,1}^{1,1} + \psi_{1,1}^{1,1} \right]/\sqrt{\mathcal{T}}$ | (0.0) | $0_{\lambda < 1}$ |

Table 2. Explicit information of different phases in the phasediagram for $\lambda > 4$ within different regimes $c_{i}/c_{0}$. Here ‘WF’ is short for ‘wavefunction’. Others symbols are defined as follows: $\eta_{1} = -1453/2379 \approx -0.61$, $\eta_{2} = (880213 - 33\sqrt{1848514055})/2816736 \approx -0.19$, and $n_{\phi} = 674/1443 \approx 0.47$.

| Phase | $c_{i}/c_{0}$ | WF | $\langle |F| \rangle$ | $\Delta \mathcal{F}$ |
|-------|----------------|-----|---------------------|----------------------|
| mFM_{(2)} | $(-\infty, 1/2)$ | $\psi_{1,1}^{1,1}$ | (1,1) | $\text{diag}[3, 3, 3]/128$ |
| P_{+}(2) | $(1/2, \infty)$ | $\psi_{0,0}^{1,1}$ | (0.0) | $0_{\lambda < 1}$ |
| mFM_{(3)} | $(-\infty, -1/2)$ | $\psi_{1,1}^{1,1}$ | (2,1) | $\text{diag}[125, 25, 124]/640$ |
| mP_{(3)} | $(-1/2, \pi)$ | $\sqrt{\mathcal{T}} \psi_{1,1}^{1,1} + \sqrt{\mathcal{T}} \psi_{1,1}^{1,1}$ | (0.0) | $\text{diag}[4, 4, 4]/640$ |
| P_{+}(3) | $(\pi, \infty)$ | $\psi_{1,1}^{1,1}$ | (0.0) | $0_{\lambda < 1}$ |
| mFM_{(4)} | $(-\infty, \eta_{1})$ | $\psi_{1,1}^{1,1}$ | (3,1) | $\text{diag}[273, 323, 850]/208$ |
| mFM_{(4)} | $(\eta_{1}, \eta_{2})$ | $\psi_{1,1}^{1,1}$ | (2,1) | $\text{diag}[315, 415, 925]/640$ |
| mP_{(4)} | $(\eta_{2}, \pi)$ | $\alpha[\psi_{1,1}^{1,1} + \psi_{1,1}^{1,1}] + \beta \psi_{1,1}^{1,1}$ | (0.0) | $\text{diag}[331, 331, 330]/208$ |
| P_{+}(4) | $(\pi, \eta_{2})$ | $\sqrt{\mathcal{T}} \psi_{1,1}^{1,1} + \psi_{1,1}^{1,1}$ | (0.0) | $0_{\lambda < 1}$ |

Second, in the absence of SOAM coupling (see figure 4 for regimes with $l = 0$ and $\lambda \in (-2, 1)$), it is well-known that the condensates can be in the FM and polar phases depending on the sign of $c_{i}$, where the magnitude of normalized local vector $|\mathcal{F}(\theta, \phi)|/n(\theta, \phi)$ takes the value 1 and 0 respectively. However, the presence of SOAM coupling can result in new ground states with $l = 0$, which supports novel vortex patterns and spin textures. In addition, the abovementioned FM and polar phases appear only for strong spin-exchanging interaction $|c_{i}| \sim c_{0}$ and $\lambda < 0$. These phases also exhibit many new features, which are listed as follows:

(i) $c_{i}/c_{0} \sim -1$ with $\lambda < 0$ and $j = l + f$. In this case, the ground-state wavefunction reads $\psi_{1,0}^{1,1}(\theta, \phi)$. Since $j_{z} = j$, this corresponds to the usual FM phases with maximized local vector $\mathcal{F}_{\phi, \theta}(\theta, \phi)$ satisfying $|\mathcal{F}_{\phi, \theta}(\theta, \phi)|/n(\theta, \phi) = 1$. In addition, the spin fluctuation defined by $\Delta \mathcal{F}_{\phi} = \langle \mathcal{F}_{\phi, \theta} \rangle - \langle \mathcal{F}_{\phi} \rangle$ with $(i, j) \in (x, y, z)$ also vanishes. These states are also denoted by FM_{(1)}(l).
(ii) $c/c_0 \sim -1$ with $\lambda > 0$ and $j = |l - f|$. Here the ground states also reads $\psi^{lf}_{j, j}(\theta, \phi)$. Since $j < l$, the normalized vector $|\mathcal{F}(\theta, \phi)/n(\theta, \phi)| < 1$ and changes its direction over the surface. Therefore, the state possesses nonzero spin fluctuation $\Delta \mathcal{F} \neq 0$, and is then called as the meta-FM phase.

(iii) $c/c_0 \sim 1$ for all $\lambda$. In this case, the system supports various polar states with $\mathcal{F}(\theta, \phi) = 0$, which exhibits novel intrinsic topological properties. To show this, we write the wavefunction using the nematic order $\hat{d}$ as

$$\psi(\theta, \phi) = \hat{d} \cdot |\mathcal{F}| = d_x|x| + d_y|y| + d_z|z|,$$

(25)

where $\{|x|, |y|, |z|\}$ are the Cartesian basis formed by the eigenvectors of $F_{x,y,z}$ with zero eigenvalues $[67,70]$, namely

$$E_x|x| = E_y|y| = E_z|z| = 0.$$

(26)

The complex vector $\hat{d}$ can be written as $\hat{d} = \hat{u} + i\hat{v}$. Since the state $\psi(\theta, \phi)$ is equivalent up to a gauge factor $e^{\phi}$, we can always choose $\gamma$ such that $\hat{u} \cdot \hat{v} = 0$. The average spin vector new reads $\langle \hat{F} \rangle = 2\hat{u} \times \hat{v}$. In the case of polar state with $\langle \hat{F} \rangle = 0$, $\hat{d}$ can be chosen to be real with $\hat{v} = 0$. Therefore, for a close spherical surface, the unit vector $\hat{d} = \hat{d}/|\hat{d}|$ exhibits nontrivial distribution which can be described by the topological charge $W$ as

$$W = \frac{1}{4\pi} \left| \int d\theta \, d\phi \, \hat{d} \cdot (\partial_\theta \hat{d} \times \partial_\phi \hat{d}) \right|,$$

(27)

where we have introduced the absolute value to avoid the global ambiguity of $\hat{d}$ and $-\hat{d}$. Figure 5(a) shows that only two values 0 and 1 are allowed for the charge $W$. This $Z_2$ feature of $W$ is directly related to the parity of $\hat{d} (\theta, \phi)$, as we have $\hat{d} (\pi - \theta, \pi + \phi) \rightarrow (-1)^{\lambda} \hat{d} (\theta, \phi)$ due to $Y^m_l (\pi - \theta, \pi + \phi) \rightarrow (-1)^m Y^m_l (\theta, \phi)$.

Especially, when $j = 0$, the polar phase $P. (1)$ survives for all $c/c_0$, and the relevant vector $\hat{d}$ exhibits a stable hedgehog-like pattern with nonzero topological charge $W = 1$. Figure 5(b) shows the patterns formed by vectors $\hat{d}$ on spherical surface. We note that for condensates without SOAM coupling, such polar state is unstable towards the formation of Alice ring, as shown in $[67]$.

Finally, across the intermediate regimes of $c_1/c_0 \in (-1, 1)$, the system transits between the FM and polar states, and various new phases arise. These phases support quantized mean values of $\langle |J| \rangle$ or $\langle |L| \rangle$ or $\langle |\mathcal{F}| \rangle$ as shown in figures 5(b)–(c) with different density and local spin distributions (see top panels in figures 6(a) and (c)), which represent another key feature of such SOAM coupled condensates. Since both $\langle |\mathcal{F}| \rangle$ and the fluctuation $\Delta \mathcal{F}$ take nonzero values, they still belong to the meta-FM phases. Beside this, the system also supports another meta-polar states with $\langle |J| \rangle = \langle |\mathcal{F}| \rangle = 0$ and nonzero local spin-density vector $\hat{F}(\theta, \phi) \neq 0$. For instance, when $j = 2, \lambda \in (-4, -2)$, and $c_1/c_0 \in (-1/3, 1/3)$, the meta-polar state reads

$$\psi_a(\theta, \phi) = \frac{1}{\sqrt{3}} \left[ \sqrt{2} \psi^{1,1}_{2,-1}(\theta, \phi) + \psi^{1,1}_{2,1}(\theta, \phi) \right].$$
and supports a homogeneous density distribution over the spherical surface with isotropic spin fluctuation $\Delta \mathcal{F} \sim 1_{0.3}$. While for $c_1/c_0 > 1/3$, polar phase with non-homogeneous distribution is preferred so that the spin-dependent interaction is minimized. We note that all the abovementioned transitions are of first-order.

### 5.2. Phase diagram for strong interaction $c_0 = 10^3$

For larger contact interaction $c_0 = 10^3$, eigenstates $\psi_{jl}^{WF}$ with different $j$ can be mixed to form new ground states so that atoms can distribute around the whole spherical surface uniformly (see bottom panels in figures 6(a) and (c)). This leads to quantitative changes of all the previous results.

First, the boundaries for the FM and polar phases move leftwards in the $c_1/c_0-\lambda$ plane for all phases with $l \gg 0$, as shown in figure 5 with magenta dotted--dashed and red dotted lines respectively.

Second, topological charge $W$ defined in the polar regimes around $c_1/c_0 = 1$ shift and even vanishes when $\lambda \in (-4, -2)$, as shown in figure 5(a). This is a direct evidence that the ground state can no longer be written as the superposition of different $\psi_{jl}^{WF}$ with fixed $j = 2$ and $l = 1$. Additional components with different $j$ and $l$ should be involved.

Finally, the regime of meta-polar phases shrinks for large $c_0$ and distribute mainly around the line with $c_1 = 0$. On the other hand, the meta-FM regimes in the phase diagram becomes larger. These intermediate meta-FM phases show complex patterns, and can not be simply characterized using quantized $\langle J \rangle$ (or $\langle L \rangle$ and $\langle \mathcal{F} \rangle$) any more. For instance, new meta-FM phase with fixed $\langle \mathcal{F} \rangle = 1$ appears for intermediate $c_1/c_0$, as shown in figure 5(b). While for larger $|\lambda| = 5$, the quantized feature of $\langle \mathcal{F} \rangle$ breaks (figure 5(c)), which makes the discrimination of different meta-FM states to be a numerically challenging task. We leave this for further investigations.

### 5.3. Thomson lattices formed by topological defects

In the case of negligible spin-exchanging interactions around $c_1/c_0 \sim 0$, which is fulfilled in most current experiments, the condensates spread almost homogeneously over the surface so that the contact interaction is minimized. The local vector $\mathcal{F}(\theta, \phi)$ changes its magnitude and direction around the closed surface, with its tangential component forming different-types of defects. These defects can be characterized using the topological Poincaré index (or winding number) $Q$ [74], which counts the number of field rotations while traveling in an anti-clockwise direction along a closed curve around the defect. In our case, the index $Q$ only takes the value $+1$ or $-1$, as shown in figure 7. For a closed compact 2-manifold with isolated defects, the Poincaré–Hopf theorem [75] says that the sum of all the indices equals to the Euler characteristic $\chi(S^2) = 2$ of the surface. In our case, this means

$$N_+ - N_- = 2.$$

Therefore, the system also provides a novel method to explore the nontrivial connection between topology and phase transition physics.

Figures 8–12 show the representative spin-textures we obtained numerically for both weak interaction $c_0 = 1$ and strong interaction $c_0 = 100$. The spin-density vector $\mathcal{F}(\theta, \phi) = \mathcal{F}_r \mathbf{e}_r + \mathcal{F}_\theta \mathbf{e}_\theta + \mathcal{F}_\phi \mathbf{e}_\phi$ exhibits nontrivial patterns after projecting on the tangent plane of the surface. These tangential components $\mathcal{F}_r \mathbf{e}_r + \mathcal{F}_\phi \mathbf{e}_\phi$, are also depicted using white arrows in the figures, where defects for $Q = 1$ are also marked with the symbol ‘+’. One can see that, around each defect center, the spin texture forms a coreless vortex [67, 71, 76].

In most case for the defects with $Q = +1$, we have coreless FM-centered vortices with the normalized vector...
While for defect with $Q = -1$, a polar-core spin vortex with $qf = 0$ at the center is favored. We stress that an exception occurs when $\lambda = (4, 6)$. In this case, the ground state is meta-ferromagnetic and reads $y_{1,1}^{2,1}$ with maximum mean spin $F = 1/2$ and $|J| = 1$. The tangential local spin-density vector is $\tilde{F}(\theta, \phi) = -[\cos(\theta)e_x + 2\sin(\theta)e_y]/(8\pi)$ as shown in figure 8. So two meta-FM-centered spin vortices with $|\tilde{F}|/n < 1$ appear in the two poles.

To minimize the interaction effect, these vortex defects form regular patterns around the spherical surface, as depicted in figure 13. Interestingly, in most cases, the defects with $Q = -1$ and $\tilde{F}(\theta, \phi) = 0$ form stable configurations characterized by the solution of Thomson problem for $N_e$ electrons [50, 69] on sphere. This also verifies the well-known charge-vortex duality for magnetic vortices in 2D system. In the case of small $c_0 = 1$, these defects exhibit the same patterns when the single-particle eigenstates share the same total angular

$|\tilde{F}(\theta, \phi)|/n(\theta, \phi) \sim 1$. While for defect with $Q = -1$, a polar-core spin vortex with $\tilde{F}(\theta, \phi) = 0$ at the center is favored. We stress that an exception occurs when $\lambda \in (4, 6)$. In this case, the ground state is meta-ferromagnetic and reads $\psi^{1,1}_{\lambda,1}$ with maximum mean spin $F = 1/2$ and $|J| = 1$. The tangential local spin-density vector is $\tilde{F}(\theta, \phi) = -[\cos(\theta)e_x + 2\sin(\theta)e_y]/(8\pi)$ as shown in figure 8. So two meta-FM-centered spin vortices with $|\tilde{F}|/n < 1$ appear in the two poles.

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momentum $j$ for given $\lambda$, as shown in figures 13(a)–(c). For instance, when $\lambda \in (-4, -2)$ and (6,8), the total angular momentum of the single-particle ground states takes the value $j = 2$ in both cases. In this case, we have 8 defects with $Q = 1$ and 6 defects with $Q = -1$, as shown in figure 13(b) using green and blue dots respectively. The distribution of the defects with $Q = -1$ forms a standard octahedron, which corresponds to the solution of Thomson problem for 6 electrons. Similarly, for $j = 3$ with $\lambda \in (-6, -4)$ and (8,10), we have 14 defects with $Q = -1$, which forms bipyramidal hexagonal prism (see figures 13(c) or (h) for $c_0 = 100$). For larger $c_0 = 100$, the ground states can be the superposition of eigenstates with different $j$. This results in new lattice patterns for different $\lambda \in (8.50, 9.78)$ and $\lambda \in (-4.38, -4) \cup (8, 8.5)$ respectively, as explicitly listed and depicted in
Figure 13. Topological defects of spin vortices on spherical surface with charge $Q = \pm 1$ (see figure 7) for different contact interaction $c_0 = 1$ and 100 respectively. The defects are obtained by plotting the spin textures of the mean-field ground state on the curved surface for given parameters, where the indices of defects are determined based on the patterns shown in figure 7. In all cases shown above, defects with $Q = -1$ form the standard Thomson lattices except figures (c) and (b). When $c_0 = 1$, only three kinds of lattice pattern exist in the phase diagram as shown in (a)–(c) using linked dots. When $c_0 = 100$, two new types of Thomson lattices emerge, as shown in (d) and (g).

6. Experimental consideration and conclusion

For $^{87}\text{Rb}$ atoms which have been widely studied in current experiments, the relevant parameters chosen in the paper are summarized as follows: specifically, by choosing suitable $U_0$, $U$, $w_1$ and $w_2$ we can have a spherical surface trap with the spherical radius $R \approx 10 \mu$m, which can be much larger than the characteristic length $l_T \approx 0.48 \mu$m of the trap along the radial direction with the frequency $\omega = 2\pi \times 0.5$ kHz. The oscillating frequency of magnetic gradient $B_1$ can be set to satisfy $\Omega_0 = 2\pi \times 50$ kHz $\gg \omega$. In this case, the strength of SOAM coupling reads $\lambda \in (0.98, 15.7)$ when $B_1 \in (50, 200)$ G cm$^{-1}$. Meanwhile, the constant bias magnetic field $B_0 = 6.0$ G can result in a Zeeman splitting $\omega_0 = 2\pi \times 4.2$ MHz. This ensures that the following constraints $\omega_0 \gg \Omega_0$ can also be well-satisfied. In concrete experiments, the density $\rho$ of $^{87}\text{Rb}$ BECs can be $10^{11}$–$10^{13}$ cm$^{-3}$. The total number of particles $N \sim 4\pi R^2 l_T \rho$ in our spherical trap can be $10^2$ to $10^3$. This leads to a dimensionless interaction strength $c_0$ ranging from $10^1$ to $10^2$, as required by our calculations.

To summarize, we have proposed a promising route to explore non-Abelian SOAM coupling in cold atomic system with the help of synthetic monopole fields. The flexibility of the system allows us to construct an effective spherical surface trap, where its ground-state degeneracy can be tuned in a wide parameter regime. For spinor condensates with $f = 1$, we show that the system supports various exotic meta-FM, meta-polar, and polar phases with nontrivial intrinsic topology. The proposed method works for both bosons and fermions, which thus opens up an avenue to explore various spin vortices on curved surfaces, and may provide a new routine to investigate strong-correlated physics using ultra-cold atoms with tunable ground-state degeneracy.

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Appendix A. SOAM coupling with negative sign

The method proposed in the main text can be used to generate non-Abelian SOAM with positive coupling strength. To obtain SOAM coupling with negative coefficient, an appropriately organized gradient magnetic pulse sequences can be employed. The construction method can be divided into two steps.

In the first step, using the standard magnetic pulses, such as $B \propto x e_x, y e_y, z e_z$, we can implement an intermediate 3D spin-momentum coupling as

$$H_{\text{eff}}' \approx H_0 - \frac{\tau}{2\mu} \mathbf{F} \cdot \mathbf{P}$$

(A.1)

with $H_0 = -\frac{\mu}{2\mu} + V(\tau)$. Here $\tau$ is the duration of the strong gradient pulses, which is much less than the modulation period $T$ as $\tau \ll T$. This is possible if we consider the following sequence of magnetic pulses as (see figure A1)

$$U_0(T) = \exp \left[ -\frac{\tau}{\hbar} H_0 - \frac{\tau}{3\hbar} \mathbf{F} \cdot \mathbf{P} + \frac{\lambda^2 \tau^2}{18\mu} \mathbf{F}^2 \right] + O(T^3),$$

(A.2)

with $\lambda = (x, y, z)$. Then the 3D spin-momentum coupling can be implemented as

$$U(T) = U_0(T/2)U_0(T/2)U_0(T)U_0(T/2)U_0(T/2)$$

$$= \exp \left[ -\frac{\tau}{\hbar} H_0 - \frac{\lambda^2 \tau^2}{18\mu} \mathbf{F}^2 + O(T^3) \right],$$

(A.3)

where $T$ is the period of the driving sequence. In the above derivation, we have assumed that the magnetic pulse is strong enough so that during the time interval $\tau \ll T$, the free evolution of the system can be neglected.

In the second step, we employ a hedgehog like magnetic pulses to realize the desired SOAM coupling. The corresponding pulse sequence reads (see figure A1)

$$U(T) = \exp \left[ -\frac{\tau}{\hbar} H_{\text{eff}} \right] = e^{i\frac{\tau}{\hbar} H_{\text{eff}}}$$

$$= e^{i\frac{\tau}{\hbar} H_{\text{eff}}} \exp \left[ -\frac{\tau}{\hbar} \left( H_0 - \frac{\lambda^2 \tau^2}{18\mu} \mathbf{F}^2 + O(T^3) \right) e^{-i\frac{\tau}{\hbar} H_{\text{eff}}} \right]$$

$$= \exp \left\{ -\frac{\tau}{\hbar} H_0 - \left( \frac{\lambda^2 + 3\lambda}{3\mu} \mathbf{F} \cdot \mathbf{P} + \frac{\tau^2 \lambda^2}{\mu} \left( \frac{\lambda^2 + \lambda}{2} \right) \mathbf{L} \cdot \mathbf{F} + \frac{\tau^2}{2\mu} \left( \frac{\lambda^2}{3} + \frac{2\lambda^3}{9} \mathbf{F}^2 + O(T^3) \right) \right\} + O(T^3).$$

(A.4)
So if we set \( \lambda' = -3\lambda \), then the evolution operator becomes

\[
\mathcal{U}(T) = \exp\left(-\frac{T}{\hbar} H_{\text{eff}}\right) \approx \exp\left[-\frac{T}{\hbar} (H_0 - \frac{\pi^2 \lambda^2}{2\mu} \mathbf{L} \cdot \mathbf{F})\right],
\]

(A.5)

thus, we have that

\[
H_{\text{eff}} \approx H_0 - \frac{\pi^2 \lambda^2}{2\mu} \mathbf{L} \cdot \mathbf{F},
\]

(A.6)

which is the desired SOAM coupled Hamiltonian with negative coupling coefficient.

**Appendix B. The reduced model and eigenstates of free Hamiltonian**

We consider a spin-\( f \) bosonic gas confined in a spherical surface trap \( V(r) \) around \( (R - \delta, R + \delta) \) with \( \delta \ll R \). The condensates suffer from a SOAM coupling defined by \( \tilde{\lambda} \mathbf{L} \cdot \mathbf{F} \). For low-energy physics considered here, the radial motion of bosons is frozen and its field operator can be written as \( \sqrt{\mathcal{N}} \varphi(r) \psi(\theta, \phi) \). The total Hamiltonian can be divided into two parties

\[
H_{\text{tot}} = H_0 + H_{\text{int}},
\]

in which single-particle Hamiltonian \( H_0 \) has following form

\[
H_0 = N \int_{R-\delta}^{R+\delta} r^2 \, dr \int d\Omega \varphi^\dagger(r) \psi^\dagger(\theta, \phi) \tilde{\mathcal{H}}_0 \varphi(r) \psi(\theta, \phi),
\]

(B.2)

where \( d\Omega = \sin \theta d\theta d\phi \), and the Hamiltonian \( \tilde{\mathcal{H}}_0 \) contains the SOAM coupling and reads

\[
\tilde{\mathcal{H}}_0 = -\frac{\hbar^2}{2\mu r^2} \left( \frac{\partial^2}{\partial r^2} + \frac{\mathbf{L}^2}{2\mu r^2} + \tilde{\lambda} \mathbf{L} \cdot \mathbf{F} + V(r) \right).
\]

(B.3)

Here \( \mu \) is atomic mass, \( \tilde{\lambda} \) stands for SOAM coupling strength. After integrating out the radial degree-of-freedom, we obtain an effective single-particle Hamiltonian subject to a spherical surface trap

\[
H_0 = \frac{B \hbar^2 N}{2\mu R^2} \int d\Omega \varphi^\dagger(\theta, \phi) \tilde{\mathcal{H}}_0 \varphi(\theta, \phi) + NE_0.
\]

(B.4)

where \( E_0 = \int_{R-\delta}^{R+\delta} r^2 \varphi^\dagger(r) \left( -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial r^2} + V(r) \right) \varphi(r) \, dr \) is the energy arising from radial motion, \( N \) is the total particle number, and \( B = \int_{R-\delta}^{R+\delta} r^2 \varphi^2(r) \, dr \approx 1 \) due to the normalization of \( \varphi(r) \). Hereafter, we neglect the last term \( NE_0 \) by selecting a new zero energy point and set \( \epsilon_0 = \frac{B \hbar^2}{2\mu R^2} \) as energy unit. The free particle Hamiltonian on the spherical surface with radius \( R \) then reads

\[
\mathcal{H}_0 = \mathbf{L}^2 + \lambda \mathbf{L} \cdot \mathbf{F}, \quad \text{with} \quad \lambda = 2\mu R^2 \tilde{\lambda}.
\]

(B.5)

Since the total angular-momentum \( \mathbf{J} = \mathbf{L} + \mathbf{F} \) of the system is invariant, and we have \( [\mathcal{H}_0, \mathbf{J}] = 0 \), \( [\mathcal{H}_0, \mathbf{L}] = 0 \), and \( [\mathcal{H}_0, \mathbf{F}] = 0 \). Using the equality \( \mathbf{L} \cdot \mathbf{F} = \frac{J_y}{2}(J^2 - L^2 - F^2) \), we can derive the single-particle energy

\[
E_0 = l(l + 1) + \lambda \frac{1}{2} j(j + 1) - l(l + 1) - f(f + 1)
\]

with \( j = |l - f|, |l - f| + 1, \ldots, l + f \). So its ground state configuration is determined by \( \lambda \) with the degeneracy given by \( 2j + 1 \). The energy of the lowest energy band for \( \lambda > 0 \) then reads

\[
E_0 = \begin{cases} 
(l - a)^2 - a^2 & \text{for } l \leq f, \quad \lambda \in \left( \frac{2l}{l + 1}, \frac{2(l + 1)}{f + 1} \right) \\
(l - b)^2 - b^2 - \lambda f & \text{for } l \geq f, \quad \lambda \in \left( \frac{2l}{f}, \frac{2l + 1}{f + 1} \right)
\end{cases}
\]

(B.7)

with \( a = \frac{\lambda f + 1}{2}, b = \frac{\lambda - 1}{2} \). When \( \lambda < 0 \), \( \mathbf{L} \) is parallel to \( \mathbf{F} \), and we have \( j = l + f \). The ground states energy is

\[
E_0 = \left( l + \frac{\lambda + 1}{2} \right) \frac{1}{2} - \frac{(f + 1)l}{4} \text{ when } \lambda \in \left( -\frac{2l + 1}{f}, -\frac{2l}{f} \right).
\]

(B.8)

In this way, we can figure out relations between OAM quantum number \( l_0 \) of ground states and the corresponding spin–orbit coupling strength \( \lambda \) for spin \( f = 1 \) (see figure B1).
Specifically, for spinor condensates with $f = 1$, when $\lambda \in (-2(l + 1), -2l)(l \geq 1)$, the single-particle ground-states are organized such that $L$ is parallel with $F$. Therefore we have $j = l+1$ with $j_z = -l - 1, -l, \ldots, l + 1$. The ground-state is of $2j + 1$-fold degeneracy and can be written as

$$
\psi^{l+1}_{-l} (\theta, \phi) = \begin{pmatrix}
\sqrt{\frac{(l+j)(l+j+1)}{(2l+1)(2l+2)}} Y_{l,-j} (\theta, \phi) \\
\sqrt{\frac{2l-(l+j)(l+j+1)}{(2l+1)(2l+2)}} Y_{l,j} (\theta, \phi) \\
\sqrt{\frac{(l-j)(l-j+1)}{(2l+1)(2l+2)}} Y_{l,j+1} (\theta, \phi)
\end{pmatrix}.
$$

(B.9)

When $\lambda \in (-2, 1)$, we have $j = f = 1$ and $l = 0$. The ground-state is $2j + 1 = 3$-fold degenerate and reads

$$
\psi^{0,1}_{0} (\theta, \phi) = Y_{0,0}(\theta, \phi) |f_z = j_z\rangle.
$$

(B.10)

When $\lambda \in (1, 4)$, the OAM $L$ of the atoms is anti-parallel with $F$ for the ground states, so we have $j = l - 1 = 0$. The ground-state has no degeneracy and is in form of

$$
\psi^{1,1}_{1,0} (\theta, \phi) = \frac{1}{\sqrt{3}} (Y_{l,-1}(\theta, \phi) - Y_{l,0}(\theta, \phi)) Y_{l,1}(\theta, \phi)\Gamma.
$$

(B.11)

We also stress that this state supports a homogeneous density distribution, and the nematic vector $\vec{d}$ exhibits hedgehog like pattern over the spherical surface.

When $\lambda \in (2l, 2(l+1)) \cap (l \geq 1)$, we have $j = l - 1$ for the single-particle ground states with $j_z = -l + 1, -l + 2, \ldots, l - 1$. The ground-state is of $2j + 1$-fold degeneracy and reads

$$
\psi^{l+1}_{l-1} (\theta, \phi) = \begin{pmatrix}
\sqrt{\frac{(l-j)(l-j+1)}{2l(2l+1)}} Y_{l,-j} (\theta, \phi) \\
-\sqrt{\frac{2l-(l-j)(l-j+1)}{2l(2l+1)}} Y_{l,j} (\theta, \phi) \\
\sqrt{\frac{(l+j)(l+j+1)}{2l(2l+1)}} Y_{l,j+1} (\theta, \phi)
\end{pmatrix}.
$$

(B.12)

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Figure B1. Orbital-angular-momentum quantum number $l_0$ of single-particle ground-states verses spin-orbital-angular-momentum coupling $\lambda$ for spin $f = 1$. 
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