Loop quantum gravity and black hole singularity

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Abstract

In this paper we summarize loop quantum gravity (LQG) and we show how ideas developed in LQG can solve the black hole singularity problem when applied to a minisuperspace model.

Introduction

Quantum gravity is the theory by which we try to reconcile general relativity and quantum mechanics. Because in general relativity the space-time is dynamical, it is not possible to study other interactions on a fixed background. The theory called “loop quantum gravity” (LQG) [1] is the most widespread nowadays and it is one of the non perturbative and background independent approaches to quantum gravity (another non perturbative approach to quantum gravity is called asymptotic safety quantum gravity [2]). LQG is a quantum geometrical fundamental theory that reconciles general relativity and quantum mechanics at the Planck scale. The main problem nowadays is to connect this fundamental theory with standard model of particle physics and in particular with the effective quantum field theory. In the last two years great progresses has been done to connect LQG with the low energy physics by the general boundary approach [3], [4]. Using this formalism it has been possible to calculate the graviton propagator in four [5], [6] and three dimensions [7]. In three dimensions it has been shown that a noncommutative field theory can be obtained from spinfoam models [8]. Similar efforts in four dimension are going in progress [9]. Algebraic quantum gravity, a theory inspired by LQG, contains quantum field theory on curved space-time as low energy limit [10]. About an unified theory of particle physics and gravity authors in a recent paper [11] have showed that spinfoam models [1], including loop quantum gravity, are also unified theories, in which matter degrees of freedom are automatically included and a complete classification of the standard model spectrum is realized.

Early universe and black holes are other interesting places for testing the validity of LQG. In the past years applications of LQG ideas to minisuperspace models lead to some interesting results in those fields. In particular it has been showed in cosmology [12], [13] and recently in black hole physics [14], [15], [16], [17] that it is possible to solve the cosmological singularity problem and the black hole singularity problem by using tools and ideas developed in full LQG. Other recent results concern a semiclassical analysis of the black hole interior [18] and the evaporation process [19].

We can summarize the “loop quantum gravity program” in the following research lines

• the first one dedicated to obtain quantum field theory from the fundamental theory rigorously defined;
• the second one dedicated to apply LQG to cosmology and black holes where extreme energy conditions need to know a quantum gravity theory.

The paper is organized as follow. In the first section we briefly recall loop quantum gravity at kinematical and dynamical level. In the second section we recall a simplified model [14] showing how
quantum gravity solves the black hole singularity problem. In the third section we summarize "loop quantum black hole" (LQBH) [17]. This is a minisuperspace model inspired by LQG where we quantize the Kantowski-Sachs space-time without approximations. This model is useful to understand the black hole physics near the singularity because the space-time inside the event horizon is of Kantowski-Sachs type.

1 Loop quantum gravity in a nutshell

In this section we recall the structure of the theory introducing the Ashtekar’s formulation of general relativity [20], the kinematical Hilbert space, quantum geometry and quantum dynamics.

1.1 Canonical gravity in Ashtekar variables

The classical starting point of LQG [11] is the Hamiltonian formulation of general relativity. In ADM Hamiltonian formulation of the Einstein theory, the fundamental variables are the three-metric \( q_{ab} \) of the spatial section \( \Sigma \) of a foliation of the four-dimensional manifold \( M \equiv \mathbb{R} \times \Sigma \), and the extrinsic curvature \( K_{ab} \). In LQG the fundamental variables are the Ashtekar variables: they consist on an \( SU(2) \) connection \( A^a_i \) and the electric field \( E^a_i \), where \( a, b, c, \ldots = 1, 2, 3 \) are tensorial indices on the spatial section and \( i, j, k, \ldots = 1, 2, 3 \) are indices in the \( su(2) \) algebra. The density weighted triad \( E^a_i \) is related to the triad \( e^i_j \) by the relation \( E^a_i = \frac{1}{2} \epsilon^{abc} e_{ijk} e^b_j e^c_k \). The metric is related to the triad by \( q_{ab} = e^i_a e^j_b \delta_{ij} \). Equivalently,

\[
\sqrt{\text{det}(q)} \, q^{ab} = E^a_i E^b_j \delta_{ij}. \tag{1}
\]

The rest of the relation between the variables \( (A^a_i, E^a_i) \) and the ADM variables \( (q_{ab}, K_{ab}) \) is given by

\[
A^a_i = \Gamma^a_i + \gamma K_{ab} E^b_j \delta_{ij} \tag{2}
\]

where \( \gamma \) is the Immirzi parameter and \( \Gamma^a_i \) is the spin connection of the triad, namely the solution of Cartan’s equation:

\[
\partial_{[a}e^i_{bj]} + e^i_{jk} \Gamma^j_a e^k_b = 0.
\]

The action is

\[
S = \frac{1}{\kappa} \int dt \int d^3x \left[ -2\text{Tr}(E^a_i \dot{A}_a^i) - \mathcal{H} - N^a \mathcal{H}_a - N^i \mathcal{G}_i \right], \tag{3}
\]

where \( N^a \) is the shift vector, \( N \) is the lapse function and \( N^i \) is the Lagrange multiplier for the Gauss constraint \( \mathcal{G}_i \). We have introduced also the notation \( E = E^a_i \partial_a = E^a_i \dot{r}^i \partial_{\dot{r}_a} \) and \( A = A_a dx^a = A^a_i r^i dx^a \). The functions \( \mathcal{H}, \mathcal{H}_a \) and \( \mathcal{G}_i \) are respectively the Hamiltonian, diffeomorphism and Gauss constraints, given by

\[
\mathcal{H}(E^a_i, A^i_a) = -4 \, e^{-1} \text{Tr} \left( F_{ab} E^a_i E^b_i - 2 e^{-1} (1 + \gamma^2) E^a_i E^b_i K^j_{[a} K^j_{b]} \right)
\]

\[
\mathcal{H}_a(E^a_i, A^i_a) = E^a_i F^b_{ab} - (1 + \gamma^2) K^b_a G_i
\]

\[
\mathcal{G}_i(E^a_i, A^i_a) = \partial_a E^a_i + \epsilon_{ij} A^i_a E^k_b,
\]

where the curvature field strength is \( F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \) and \( e = \text{det}(e^a_i) \). The constraints are respectively generators for the foliation reparametrization, for the \( \Sigma \) surface reparametrization and for the gauge transformations. The symplectic structure for the Ashtekar Hamiltonian formulation of general relativity is

\[
\{ E^a_i(x), A^b_j(y) \} = \kappa \gamma \delta^a_b \delta^j_i \delta(x, y), \quad \{ E^a_i(x), E^b_j(y) \} = \{ A^i_a(x), A^b_j(y) \} = 0. \tag{5}
\]

General relativity in metric formulation is defined by the Einstein equations \( G_{\mu\nu} = 8\pi G \mathcal{N} T_{\mu\nu} \). The Ashtekar Hamiltonian formulation of general relativity is instead defined by the constraints \( \mathcal{H} = 0, \mathcal{H}_a = 0, \mathcal{G}_i = 0 \) and by the Hamilton equations of motion: \( \dot{A}^i_a = \{ A^i_a, \mathcal{H} \} \) and \( \dot{E}^a_i = \{ E^a_i, \mathcal{H} \} \).
1.2 The Dirac program for quantum gravity

The general strategy to quantize a system with constraints was introduced by Dirac. The program consist on:

1. find a representation of the phase space variables of the theory as operators in an auxiliary kinematical Hilbert space $H_{kin}$ satisfying the standard commutation relations, i.e., $\{ , \} \to -i/\hbar$, ;

2. promote the constraints to (self-adjoint) operators in $H_{kin}$. For gravity we must quantize a set of seven constraints $\mathcal{G}_i(A,E)$, $\mathcal{H}_a(A,E)$, and $\mathcal{H}(A,E)$ and we must solve the quantum Einstein’s equations (for $\gamma = i$)

$$\hat{\mathcal{H}}|\psi\rangle = \left[ -4 e^{-1} \text{Tr} \left( F_{ij} F^{ij} \right) + \mathcal{H}_M \right]|\psi\rangle = 0,$$

$$\hat{\mathcal{H}}_b|\psi\rangle = \left[ E_i^a E^b_i + \mathcal{H}_{AB} \right]|\psi\rangle = 0,$$

$$\hat{\mathcal{G}}_i|\psi\rangle = \left[ \partial_a E^a_i + \epsilon_{ij}^k A^k_i E_j^i + \mathcal{G}_{M_i} \right]|\psi\rangle = 0 \quad (6)$$

We will consider pure gravity then the matter constraints are identically zero.

3. introduce an inner product defining the physical Hilbert space $H_{phys}$.

1.3 Kinematical Hilbert space

The fundamental ingredient of LQG is the holonomy of the Ashtekar connection along a path $e$, $h_e[A] = P \exp - \int_e A \in SU(2)$. Given two oriented paths $e_1$ and $e_2$ such that the end point of $e_1$ coincides with the starting point of $e_2$ so that we can define $e = e_1 e_2$ we have the composition rule $h_e[A] = h_{e_1}[A]h_{e_2}[A]$. By a gauge transformation the holonomy transforms as

$$h'_e[A] = g(x(0)) h_e[A] g^{-1}(x(1)), \quad (7)$$

and by a Diffeomorphism of the three dimensional variety $\phi \in \text{Diff}(\Sigma)$ we have

$$h_e[\phi^* A] = h_{\phi^{-1} e}[A], \quad (8)$$

where $\phi^* A$ denotes the action of $\phi$ on the connection. In other words, transforming the connection with a diffeomorphism is equivalent to simply moving the path with $\phi^{-1}$.

We introduce now the space of cylindrical functions (Cyl$_\gamma$) where $\gamma$ denotes a general graph. A graph $\gamma$ is defined as a collection of paths $e \subset \Sigma$ ($e$ stands for edge) meeting at most at their endpoints. If $N_e$ is the number of paths or edges of the graph and $e_i$, for $i = 1, \ldots, N_e$, are the edges of the corresponding graph $\gamma$ a cylindrical function is an application $f : SU(2)^{N_e} \to \mathbb{C}$, defined by

$$\psi_{\gamma,f}[A] := f(h_{e_1}[A], h_{e_2}[A], \ldots h_{e_{N_e}}[A]). \quad (9)$$

Two particular examples of cylindrical functions are the holonomy around a loop, $W_e[A] := \text{Tr}[h_e[A]]$ and the three edges function $\Theta_{e_1 e_2 e_3}^{1/2,1/2,1/2} = \frac{1}{4} \mathcal{D}(h_{e_1}[A])^{ij} \frac{1}{4} \mathcal{D}(h_{e_2}[A])_{AB}^{ij} \mathcal{D}(h_{e_3}[A])_{CD} f_{ijkl}^{ABCD}$, where $\mathcal{D}(h_e)$ is the $SU(2)$ representation for the holonomy along the path $e_i$ and $f_{ijkl}^{ABCD}$ are complex coefficients. The algebra of generalized connections is given by Cyl = $\cup\gamma$ Cyl$_\gamma$.

We introduce now the space of spin networks states. We label the set of edges $e \subset \gamma$ with spins $\{j_e\}$. To each node $n \subset \gamma$ one assigns an invariant tensor, called intertwiner, $\iota_n$, in the tensor product of representations labelling the edges converging at the corresponding node. The spin network function is defined

$$s_{\gamma,\{j_e\},\{\iota_n\}}[A] = \bigotimes_{n \subset \gamma} \iota_n \bigotimes_{e \subset \gamma} j_e \mathcal{D}(h_e[A]), \quad (10)$$

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where the indices of representation matrices and invariant tensors are implicit to simplify the notation. An example of spin network state is
\[
\Theta^{11/2,1/2}_{e1,e2,e3}[A] = 1 \mathcal{D}(h_{e1}[A])^{1/2} \mathcal{D}(h_{e2}[A])_{AB}^{1/2} \mathcal{D}(h_{e3}[A]) \sigma_{AC}^{e1} \sigma_{BD}^{e2},
\]
where for the particular representations converging to the two three-valent nodes of the graph the intertwiner tensor is the Pauli matrix. The spin network states are gauge invariant because for any node of the graph we have invariant tensors (intertwiners), then on the spin network states the Gauss constraint is solved as asked from the Dirac program of the previous subsection.

To complete the Hilbert space definition we must introduce an inner product. The scalar product is defined by the Ashtekar-Lewandowski measure
\[
<\psi_{\gamma,f},\psi_{\gamma',g}> = \mu_{\text{AL}}(\psi_{\gamma,f}\psi_{\gamma',g}) = \int e^{\gamma} \prod_{c \in \Gamma_{\gamma'}} dh_e (h_{e1},...h_{eN_e}) g(h_{e1},...h_{eN_e}),
\]
where we use Dirac notation and \(f(h_{e1},...,h_{eN_e}), g(h_{e1},...,h_{eN_e})\) are cylindrical functions; \(\Gamma_{\gamma'}\) is any graph such that both \(\gamma \subset \Gamma_{\gamma'}\) and \(\gamma' \subset \Gamma_{\gamma}\); \(dh_e\) is the Haar measure of \(SU(2)\). The scalar product \(\text{(12)}\) is non zero only if the two cylindrical functions have support on the same graph. The kinematical Hilbert space \(H_{\text{kin}}\) is the Cauchy completion of the space of cylindrical functions \(\text{Cyl}\) in the Ashtekar-Lewandowski measure. In other words, in addition to cylindrical functions we add to \(H_{\text{kin}}\) the limits of all the Cauchy convergent sequences in the norm defined by the inner product:
\[
\psi = \sum_{n=1}^{\infty} a_n \psi_n, ||\psi||^2 = \sum_{n=1}^{\infty} |a_n|^2 ||\psi_n||^2.
\]
We complete the construction of the theory at kinematical level solving the diffeomorphism constraint. Given \(\psi_{\gamma,f} \in \text{Cyl}\) the finite action of a \(\text{Diff}\) transformation is implemented by an unitary operator \(\mathcal{U}_D\) such that
\[
\mathcal{U}_D[\phi][\psi_{\gamma,f}[A]] = \psi_{\gamma(-1)f}[A].
\]
The states invariant under \(\text{Diff}\) transformations satisfy \(\mathcal{U}_D[\phi]\psi = \psi\) and are distributional states in the dual space of \(H_{\text{kin}}\), \(\psi \in \text{Cyl}^*\)
\[
([\psi_{\gamma,f}] = \sum_{\phi \in \text{Diff}(\Sigma)} <\psi_{\gamma,f} || \mathcal{U}_D[\phi] = \sum_{\phi \in \text{Diff}(\Sigma)} <\psi_{\gamma,f} ||, \quad (14)
\]
where the sum is over all diffeomorphisms which modified the spin network. The brackets in \([\psi_{\gamma,f}]\) denote that the distributional state depends only on the equivalence class \([\psi_{\gamma,f}]\) under diffeomorphisms. Clearly we have \([\psi_{\gamma,f}] || \mathcal{U}_D[\alpha] = ((\psi_{\gamma,f}) || \forall \alpha \in \text{Diff}(\Sigma))\).

We conclude that the Dirac’s program at kinematical level is realized by the Gelfand triple
\[
\text{Cyl} \subset H_{\text{kin}} \subset \text{Cyl}^* \quad \text{SU(2)} \rightarrow \quad \text{Cyl}_0 \subset H_{\text{kin}} \subset \text{Cyl}^*_0 \quad \text{Diff} \rightarrow \quad H_{\text{Diff}} \subset \text{Cyl}^*, \quad (15)
\]
where \(\text{Cyl}_0\) is the subspace of cylindrical functions invariant under \(\text{SU}(2)\).

At quantum level the phase space variables operators are represented on the spin network space by the holonomy operator \(h_e[A]\) that acts multiplicatively on the states and by the smearing of the triad \(E_i^a\) on a two dimensional surface \(S \subset \Sigma\)
\[
\hat{E}[S,a] = \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \alpha^i E_i^a \epsilon_{abc} = -i\hbar \kappa \gamma \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \alpha^i \frac{\delta}{\delta A_i^a} \epsilon_{abc}, \quad (16)
\]
where \(\alpha^i\) is the smearing function with values on the Lie algebra of \(\text{SU}(2)\). The action of \(\hat{E}[S,a]\) on the spin network states can be calculated using
\[
\hat{E}[S,a][h_e(A)] = -i\hbar^2 \gamma \alpha^i h_{e1}(A) \tau_i h_{e2}(A), \quad e = e_1 \cup e_2,
\]
\[
\hat{E}_i^a(x)[h_e(A)] = -i\hbar^2 \frac{\delta}{\delta A_i^a} h_e(A) = \int ds e^a(s) \delta^3(e(s),x) h_{e1}(A) \tau_i h_{e2}(A), \quad (17)
\]
1.4 Quantum geometry and dynamics

We are going to give a physical interpretation of the Hilbert space previously introduced. We consider the spatial section $\Sigma$ of the space-time and we study the spectrum of the area $S$ and volume $R$ in the section $\Sigma$ \cite{21}. We define the area of a surface $S$ as the limit of a Riemann sum

$$ A_S = \lim_{N \to \infty} A_S^N, \quad A_S^N = \sum_{i=1}^{N} \sqrt{E_i(S)} E_i(S) $$

where $N$ is the number of cells. The quantum area operator is $\hat{A}_S = \lim_{N \to \infty} \hat{A}_S^N$. Using (17) we calculate the action

$$ \hat{E}_i(S) \hat{E}_j(S) \hat{D}(h_e[A])_{mn} = (l_P^2 \gamma)^2 (j(j + 1))^j \hat{D}(h_e[A])_{mn}. $$

The area spectrum for spin network without edges and nodes on the surface is

$$ \hat{A}_S |\gamma, e, \iota, n\rangle = \frac{l_P}{P} \sum_{p \in S} \sqrt{j_p(j_p + 1)} |\gamma, e, \iota, n\rangle, $$

where $j_p$ are the representations on the edges that cross the surface $S$. Now we consider a region $R$ with a number $n$ of nodes inside. The spectrum of the volume operator for the region $R$ is

$$ \hat{V}_R |\gamma, e, \iota, n\rangle = \frac{l_P^3}{3^2} \sum_{n \in R} \sqrt{w(n)} |\gamma, e, \iota, n\rangle. $$

We have all the ingredients to give a physical interpretation of the Hilbert space. The states have support on graphs that are a collection of nodes and edges converging in the nodes. The dual of a spin network corresponds to a discretization of the three dimensional surface $\Sigma$. The dual of a set of edges is the 2-dimensional surface crossed by the links and the dual of a set of nodes is the volume \textit{chunk} contained nodes (see Fig.1).

We must now implement the Hamiltonian constraint on the Hilbert space. The Euclidean part of the constraint $S^E(N) = \int_{\Sigma} d^3 x N(x) \mathcal{H}(E_i^a, A_i^b),$

$$ S^E(N) = \int_{\Sigma} d^3 x \, N(x) \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \epsilon^{ij} F_{ab}. $$

Figure 1: In this picture we show the spin network physical meaning. The graph on the left represents a particular spin network. In the center we represent the same spin network and the dual decomposition of the space section in \textit{chunk of space}. In the last picture on the right we consider another spin network and a particular dual volume region. The yellow region is a chunk of space with volume eigenvalues related to the red intertwiners and area eigenvalues given by the $SU(2)$ representations associated to the green edges.

where $\dot{e}^a(s)$ is tangent to the curve $e(s)$ in the graph $\gamma$. The pair $(\dot{h}_e[A], \dot{E}[S, \alpha])$ realizes the first point of the Dirac’s program.
Using the Thiemann’s trick [1] we can express the inverse of $\sqrt{\det(E)}$ by

$$E^i_j E^j_i \epsilon^{ijk} \epsilon_{abc} = \frac{4}{\kappa \gamma} \{A^k_a, V\},$$

and the Hamiltonian constraint is

$$S^E(N) = \frac{4}{\kappa \gamma} \int d^3 x \ N(x) \ e^{\epsilon a b c} \delta_{ij} \ F_{ab}^i \ \{A^j_c, V\}.$$

Now we define the Hamiltonian constraint in terms of holonomies. Given an infinitesimal loop $\alpha_{ab}$ on the $ab$-plane in the surface $\Sigma$ with coordinate area $c^2$, we can define $F^i_{ab}$ in terms of holonomies by $h_{\alpha_{ab}}[A] = h_{\alpha_{ab}}^{-1}[A] = \epsilon^2 F_{ab}^i \gamma_i + \mathcal{O}(\epsilon^4)$ and $h_{\gamma_i}[A]| h_{\alpha_{ab}}[A], V\rangle = \epsilon \ {A^k_a, V} + \mathcal{O}(\epsilon^2) \ (e_a$ is a path along the $a$-coordinate of coordinate length $\epsilon)$. With these ingredients the quantum constraint can formally be written

$$S^E(N) = \frac{4}{\kappa \gamma} \lim_{\epsilon \to 0} \sum \ N_I \ e^{\epsilon a b c} \text{Tr} \left[ \left( \tilde{h}_{\alpha_{ab}}[A] - \hat{h}_{\alpha_{ab}}^{-1}[A] \right) \hat{h}_{\gamma_i}[A], \hat{V} \right],$$

where we have replaced the integral by a Riemann sum over cells of coordinate volume $\epsilon^3$. It is easy to see that the regularized quantum scalar constraint acts only on spin network nodes, because in [21] $F_{ab}$ and $F_{ab}^i$ are respectively antisymmetric and symmetric in indexes on spin network states. In fact $F_{ab}^i \sim e^{\epsilon a b \psi_{\gamma_i}, f}$ (this is a consequence of [17]). The action of (23) on spin network states is $S^E(N) \psi_{\gamma_i, f} = \sum_{\gamma} N_n S^E_n \psi_{\gamma_i, f}$, where $S^E_n$ acts only on the node $n \subset \gamma$ and $N_n$ is the value of the lapse $N(x)$ at the node. The scalar constraint modifies spin networks by creating new exceptional links around nodes. The Euclidean constraint action on 4-valent nodes is [1]

$$\langle s| \hat{S}_e| \psi \rangle = \sum_{op} S_{ijklm,opq} \langle \psi | \hat{S}_e \rangle + \sum_{op} S_{ijklm,opq} \langle \psi | \hat{S}_e \rangle + \sum_{op} S_{ijklm,opq} \langle \psi | \hat{S}_e \rangle.$$

(25)

The only dependence on the regularization parameter $\epsilon$ is in the position of the extra edges in the resulting spin network states, then the limit $\epsilon \to 0$ can be defined on diffeomorphism invariant states in $H_{Diff^C}$. The key property is that in the diffeomorphism invariant context the position of the new link is irrelevant. Therefore, given a diffeomorphism invariant state $\langle | \psi \rangle \in H_{Diff^C} \subset \text{Cyl}^*$, the quantity $\langle \psi | \hat{S}_e(N)| \psi \rangle$ is well defined and independent of $\epsilon$.

The operator (21) defines the dynamics and an unitary implementation of the constraint $S^E(N)$ gives the evolution amplitude from a spin network $s$ to a new spin network $s'$. Introducing the projector operator $P = \int D[N] \exp (i \int N(x) S^E(x))$ we can characterize the solutions of quantum Einstein equations by $P|s\rangle$, $\forall |s\rangle \in H_{kin}$. The matrix elements of $P$ define the physical scalar product, $W(s, s') := \langle s'| P|s\rangle$. The amplitude $W(s, s')$ solves the Hamiltonian constraint in the following sense. If $|\Psi_{phys}\rangle := P|s\rangle$ we have that $S^E(N)|\Psi_{phys}\rangle = 0$, but $\langle s' | \Psi_{phys}\rangle := \Psi_{phys}(s) = W(s, s')$, then we obtain

$$\langle s| S^E(N)| \Psi_{phys}\rangle = \sum_{s''} \langle s| S^E(N)| s''\rangle \langle s''| \Psi_{phys}\rangle =$$

$$= \sum_{s''} S^E(N)| s'' \rangle \Psi_{phys}(s'') = \sum_{s''} S^E(N)| s'' \rangle W(s'', s').$$

(26)

Relation (26) shows the amplitude $W(s, s')$ is in the Hamiltonian constraint kernel. $W(s, s')$ realizes the Dirac’s program because corresponds to the finite implementation of the scalar constraint on the kinematical states and defines a class of models called spin foam models [1].
2 Avoidance black hole singularity in quantum gravity

In this section we study the black hole system inside the event horizon in ADM variables considering a simplified minisuperspace model \cite{14} and in particular using the fundamental ideas suggested by full LQG and introduced in the first section. The simplification consist on a semiclassical condition which reduce the phase space from four to two dimensions. This is an approximate model but it is useful to understand the ideas before to quantize the Kantowski-Sachs system in Ashtekar variables.

2.1 Classical theory

Consider the Schwarzschild solution inside the event horizon; the metric is homogeneous and it reads

\[
\text{d}s^2 = -\left(\frac{2MG_N}{T} - 1\right)\text{d}t^2 + T^2(\sin^2\theta\text{d}\phi^2 + \text{d}\theta^2).
\]

This metric is a particular representative of the Kantowski-Sachs class \cite{25}

\[
\text{d}s^2 = -\text{d}t^2 + a(t)^2\text{d}r^2 + b(t)^2(\sin^2\theta\text{d}\phi^2 + \text{d}\theta^2).
\]

Introducing (28) in the Einstein-Hilber action we obtain

\[
S = -\frac{R}{2G_N} \int dt \left[ a\dot{b}^2 + 2\dot{a}b\dot{b} - a \right] \quad \text{(where } R = \int dr \text{ is a cut-off on the radial cell) \cite{14}.}
\]

We introduce in the action the classical relation

\[
a^2 = 2\frac{MG_N}{b(t)} - 1,
\]

obtaining

\[
S = \frac{R}{2G_N} \int dt \left[ \frac{\sqrt{b}}{\sqrt{2MG_N}} \left(1 - \frac{b}{2MG_N}\right)^{-1/2} \dot{b}^2 + \frac{\sqrt{2MG_N}}{\sqrt{b}} \left(1 - \frac{b}{2MG_N}\right)^{1/2} \right].
\]

The momentum conjugates to the variable \(b(t)\) is

\[
p = \frac{R\sqrt{b}}{G_N\sqrt{2MG_N}} \left(1 - \frac{b}{2MG_N}\right)^{-1/2} \dot{b},
\]

and the Hamiltonian constraint, by Legendre transform, is

\[
H = p\dot{b} - L = \left(\frac{G_N p^2}{2R} - \frac{R}{2G_N}\right) \left[ \frac{\sqrt{2MG_N}}{\sqrt{b}} \left(1 - \frac{b}{2MG_N}\right)^{1/2} \right].
\]

We introduce a further approximation. In quantum theory, we will be interest in the region of the scale the Planck length \(l_P\) around the singularity. We assume that the Schwarzschild radius \(r_s = 2MG_N\) is much larger than this scale.In this approximation we can write \(1 - b/2MG_N \sim 1\) and \(H\) becomes

\[
H = \left(\frac{G_N p^2}{2R} - \frac{R}{2G_N}\right) \sqrt{2MG_N} \sqrt{b}.
\]

In the same approximation the volume of the space section is \(V = 4\pi R \sqrt{2MG_N} b^{3/2} := l_o b^{3/2}\).

The canonical pair is given by \(b \equiv x\) and \(p\), with Poisson brackets \(\{x, p\} = 1\). We now assume that \(x \in \mathbb{R}\). This choice it not correct classically, because for \(x = 0\) we have the singularity, but it allows us to open the possibility that the situation be different in the quantum theory. We introduce an algebra of classical observables and we write the quantities of physical interest in terms of those variables. We are motivated by loop quantum gravity to use the fundamental variables \(x\) and

\[
U_{\delta}(p) \equiv \exp \left(\frac{8\pi G_N \delta \cdot i}{L} p\right)
\]

where \(\delta\) is a real parameter (see next paragraph for a rigorous mathematical definition of \(\delta\)) and \(L\) fixes the unit of length. The operator (31) can be seen as the analog of the holonomy variable of loop quantum gravity.
A straightforward calculation gives
\[
\{x, U_\delta(p)\} = 8\pi G_N \frac{i\delta}{L} U_\delta(p),
\]
\[
U_\delta^{-1}\{V^n, U_\delta\} = l_0^n U_\delta^{-1}\{|x|^{2n}, U_\delta\} = i 8\pi G_N l_0^n \frac{\delta}{L} \frac{3n}{2} \text{sgn}(x)|x|^{3n-1}.
\]
These formulas allow us to express inverse powers of \(x\) in terms of a Poisson bracket between \(U_\delta\) and the volume \(V\), following Thiemann’s trick \[20\]. For \(n = 1/3\) \[32\] gives
\[
\frac{\text{sgn}(x)}{\sqrt{|x|}} = -\frac{2Li}{(8\pi G_N)^{l_0^n\delta}} U_\delta^{-1}\{V^{1/3}, U_\delta\}.
\]
We will use this relation in quantum mechanics to define the physical operators. We are interested to the quantity \(|x|\) because classically this quantity diverge for \(|x| \to 0\) and produce the singularity. We are also interested to the Hamiltonian constraint and the dynamics and we will use \[33\] for writing the Hamiltonian.

2.2 Polymer quantization

In this section we recall the polymer representation \[22\] of the Weyl-Heisenberg algebra and we compare this representation with full LQG. The polymer representation of the Weyl-Heisenberg algebra is unitarily inequivalent to the Schröedi-nger representation. Now we construct the Hilbert space \(H_{\text{Poly}}\).

First of all we define a graph \(\gamma\) as a finite number of points \(\{x_i\}\) on the real line \(\mathbb{R}\). We denote by \(\text{Cyl}_\gamma\) the vector space of function \(f(k)\) (\(f : \mathbb{R} \to \mathbb{C}\)) of the type
\[
f(k) = \sum_j f_j e^{-ikx_jk}
\]
where \(k \in \mathbb{R}, x_j \in \mathbb{R}\) and \(f_j \in \mathbb{C}\) and \(j\) runs over a finite number of integer (labelling the points of the graph). We will call cylindrical with respect to the graph \(\gamma\) the function \(f(k)\) in \(\text{Cyl}_\gamma\). The real number \(k\) is the analog of the connections in loop quantum gravity and the plane wave \(e^{-ikx_j}\) can be thought as the holonomy of the connection \(k\) along the graph \(\{x_j\}\).

Now we consider all the possible graphs (the points and their number can vary from a graph to another) and we denote \(\text{Cyl}\) the infinite dimensional vector space of functions cylindrical with respect to some graph: \(\text{Cyl} = \bigcup_\gamma \text{Cyl}_\gamma\). A basis in \(\text{Cyl}\) is given by \(e^{-ikx_j}\) with \(|e^{-ikx_j}| e^{-ikx_j}\rangle = \delta_{x_i, x_j}\). \(H_{\text{Poly}}\) is the Cauchy completion of \(\text{Cyl}\) or more succinctly \(H_{\text{Poly}} = L_2(\mathbb{R}_{\text{Bohr}}, d\mu_0)\), where \(\mathbb{R}_{\text{Bohr}}\) is the Bohr-compactification of \(\mathbb{R}\) and \(d\mu_0\) is the Haar measure on \(\mathbb{R}_{\text{Bohr}}\).

The Weyl-Heisenberg algebra is represented on \(H_{\text{Poly}}\) by the two unitary operators
\[
\hat{V}(\lambda)f(k) = f(k - \lambda), \quad \hat{U}(\delta)f(k) = e^{i\delta k} f(k),
\]
where \(\lambda, \delta \in \mathbb{R}\). In terms of eigenkets of \(\hat{V}(\lambda)\) (we associate a ket \(|x_j\rangle\) with the basis elements \(e^{-ikx_j}\)) we obtain
\[
\hat{V}(\lambda)|x_j\rangle = e^{i\lambda x_j} |x_j\rangle, \quad \hat{U}(\delta)|x_j\rangle = |x_j - \delta\rangle.
\]
It is easy to verify that \(\hat{V}(\lambda)\) is weakly continuous in \(\lambda\), whence exists a self-adjoint operator \(\hat{x}\) such that \(\hat{x}|x_j\rangle = x_j |x_j\rangle\) \[22\], \[24\].

The operator analogy between loop quantum gravity and polymer representation is the following: the basic operator of loop quantum gravity, holonomies and electric field fluxes, are respectively analogous to the operators \(\hat{U}(\delta)\) and \(\hat{x}\) with commutator \([\hat{x}, \hat{U}(\delta)] = -\delta \hat{U}(\delta)\). The commutator is parallel to the commutator between electric fields and holonomies. As, in the polymer representation,
the unitary operator $\hat{U}(\delta)$ is well defined but the operator $\hat{p}$ doesn’t exist, in loop quantum gravity the holonomies operators are unitary represented self-adjoint operators but the connection operator doesn’t exist. As $\hat{x}$, the electric flux operators are unbounded self-adjoint operators with discrete eigenvalues.

After this short review on polymer representation of the Weyl-Heisenberg algebra we return to our system.

2.3 Polymer black hole quantization

Following the previous section we quantize the Hamiltonian constraint and the inverse volume operator in the Polymer representation of the Weyl-Heisenberg algebra. The operators are $\hat{x}$, acting on the basis states according to

$$\hat{x}|\mu\rangle = L|\mu\rangle, \quad \langle \mu | \nu \rangle = \delta_{\mu\nu}$$

(37)

(we have redefined the continuum eigenvalues of the position operator of the previous section $x_i \rightarrow \mu$)

and the operator corresponding to the classical momentum function $U_\delta = e^{i\frac{8\pi G N \delta p}{L}}$. We define the action of $\hat{U}_\delta$ on the basis states using the definition (37) and using a quantum analog of the Poisson bracket between $x$ and $\hat{U}_\delta$

$$\hat{U}_\delta|\mu\rangle = |\mu - \delta\rangle, \quad [\hat{x}, \hat{U}_\delta] = -\delta L \hat{U}_\delta.$$  

(38)

Using the standard quantization procedure $[ , ] \rightarrow i\hbar\{ , \}$, the Poisson bracket (32) and (38) we obtain the value of the length scale $L = \sqrt{8\pi G_N \hbar} = \ell_p$.

2.3.1 Avoidance black hole singularity and regular dynamics

We recall that the dynamics is all in the function $b(t)$, which is equal to the radial Schwarzschild coordinate inside the horizon. The important point is that $b(t=0) = 0$ and this is the Schwarzschild singularity. We now show that the spectrum of the operator $\frac{1}{|x|}$ does not diverge in quantum mechanics and therefore there is no singularity in the quantum theory.

Using the relation (33), and promoting the Poisson brackets to commutators, we obtain (for $\delta = 1$) the operator

$$\frac{1}{|x|} = \frac{1}{2\pi G_N \hbar} \left( \hat{U}^{-1} \left[ \hat{V}^\frac{\gamma}{2}, \hat{U} \right] \right)^2.$$  

(39)

The action of this operator on the basis states is (the volume operator is diagonal on the basis states, $\hat{V}|\mu\rangle = l_0|\mu\rangle|\frac{\mu}{2}|\mu\rangle = l_0|\mu\rangle|\frac{\mu}{2}|\mu\rangle$)

$$\frac{1}{|x|}|\mu\rangle = \sqrt{\frac{2}{\pi G_N \hbar}} \left( |\mu\rangle - |\mu - 1|\frac{\mu}{2}|\mu\rangle \right)^2 |\mu\rangle.$$  

(40)

We can now see that the spectrum is bounded from below and so we have not singularity in the quantum theory. In fact the curvature invariant $\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} = 48M^2G_N^2/x(t)^6$ is finite in quantum mechanics in $\mu = 0$. The eigenvalue of the operator $1/|x|$ for the state $|0\rangle$ corresponds to the classical singularity and in the quantum case it is $4/l_p^2$, which is the largest possible eigenvalue. For this particular value the curvature invariant it is not infinity

$$\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma}|0\rangle = \frac{48M^2G_N^2}{|x|^6}|0\rangle = \frac{384M^2G_N^2}{\pi^3 l_p^2}|0\rangle.$$  

(41)
If we consider the \( \hbar \rightarrow 0 \) limit we obtain the classical singularity so the result is a genuinely quantum gravity effect). On the other hand, for \( |\mu| \rightarrow \infty \) the eigenvalues go to zero, which is the expected behavior of \( 1/|x| \) for large \( |x| \).

Now we study the quantization of the Hamiltonian constraint near the singularity, in the approximation \( \delta \sim 1 \). There is no operator \( p \) in polymer quantum representation that we have chosen, hence we choose the following alternative representation for \( p^2 \). Consider the classical expression

\[
p^2 = \frac{L^2}{(8\pi G_N)^2} \lim_{\delta \rightarrow 0} \left( 2 - \frac{U_\delta - U_\delta^{-1}}{\delta^2} \right).
\]

(42)

We have can give a physical interpretation to \( \delta \) as \( \delta = l_p/L_{phys} \), where \( L_{phys} \) is the characteristic size of the system. Using (42) we write the Hamiltonian constraint as

\[
\hat{H} = \frac{C}{l_0^{1/3} L^{1/2}} \left[ \hat{U}_\delta + \hat{U}_\delta^{-1} - (2 - C') \hat{I} \right] \text{sgn}(x) \left( \hat{U}^{-1} \left[ \hat{V}^4, \hat{U} \right] \right)
\]

(43)

where \( C = \frac{L^{5/2} G_N}{(8\pi G_N)^{7/2} R_0^{3/2} \hbar} \) and \( C' = \frac{8\pi R^2 \delta^2}{l_p} \). The action of \( \hat{H} \) on the basis states is

\[
\hat{H} |\mu\rangle = C V(\mu) [|\mu - \delta\rangle + |\mu + \delta\rangle - (2 - C')|\mu\rangle],
\]

\[
V(\mu) = \begin{cases} 
-|\mu - \delta|^{1/2} - |\mu|^{1/2} & \text{for } \mu \neq 0 \\
|\delta|^{1/2} & \text{for } \mu = 0
\end{cases}
\]

(44)

We now calculate the solutions of the the Hamiltonian constraint. The solutions are in the dual space of \( H_{Poly} \). A generic element of this space is \( \langle \psi | = \sum_\mu \psi(\mu) |\mu\rangle \). The constraint equation \( \hat{H}|\psi\rangle = 0 \) is now interpreted as an equation in the dual space \( \langle \psi | \hat{H}^\dagger \rangle \); from this equation we can derive a relation for the coefficients \( \psi(\mu) \)

\[
V(\mu + \delta) \psi(\mu + \delta) + \psi(\mu - \delta) \psi(\mu - \delta) - (2 - C') V(\mu) \psi(\mu) = 0.
\]

(45)

This relation determines the coefficients for the physical dual state. We can interpret this states as describing the quantum spacetime near the singularity. From the difference equation (45) we obtain physical states as combinations of a countable number of components of the form \( \psi(\mu + n\delta)|\mu + n\delta\rangle \) \( (\delta \sim l_p/L_{phys} \sim 1) \); any component corresponds to a particular value of volume, so we can interpret \( \psi(\mu + \delta) \) as the wave function describing the black hole near the singularity at the time \( \mu + \delta \). A solution of the Hamiltonian constraint corresponds to a linear combination of black hole states for particular values of the volume or equivalently particular values of the time.

### 3 Loop quantum black hole

In this section we quantize the Kantowski-Sachs space-time in Ashtekar variables and without approximations [17].

#### 3.1 Ashtekar variables for Kantowski-Sachs space-time

The Kantowski-Sachs space-time is a simplified version of an homogeneous but anisotropic spacetime, written in coordinates \( (t, r, \theta, \phi) \). An homogeneous but anisotropic space-time of spatial section \( \Sigma \) of topology \( \Sigma \cong \mathbb{R} \times S^2 \) is characterized by an invariant connection 1-form \( A_{[1]} \) of the form [27], [28]

\[
A_{[1]} = A_t(t) \tau_3 \, dt + (A_1(t) \tau_1 + A_2(t) \tau_2) \, d\theta + (A_1(t) \tau_2 - A_2(t) \tau_1) \sin \theta \, d\phi + \tau_3 \cos \theta \, d\phi.
\]

(46)
The $\tau_i$ are the generators of the $SU(2)$ fundamental representation. They are related to the Pauli $\sigma_i$ matrix by $\tau_i = -\frac{i}{2}\sigma_i$. On the other side the dual invariant densitized triad is

$$E_{[i]} = E^r(t) \tau_i \sin \theta \frac{\partial}{\partial r} + (E^2(t) \tau_1 + E^3(t) \tau_2) \sin \theta \frac{\partial}{\partial \theta} + (E^3(t) \tau_2 - E^2(t) \tau_1) \frac{\partial}{\partial \phi}. \quad (47)$$

Since spacetime is homogeneous, the diffeomorphism constraint is automatically satisfied.

In this paper we study the Kantowski-Sachs space-time with space section of topology $\mathbb{R} \times S^2$; the connection $A_{[i]}$ is more simple than in (46) with $A_2 = A_1$, and in the triad (47) we can choose the gauge $E^2 = E^3 = 0$. There is a residual gauge freedom on the pair $(A_1, E^1)$. This is a discrete transformation $P : (A_1, E^1) \rightarrow (-A_1, -E^1)$; we have to fix this symmetry on the Hilbert space.

The Gauss constraint is automatically satisfied and the Euclidean part of the Hamiltonian constraint becomes

$$\mathcal{H}_E = \frac{2 \sqrt{2} \sin \theta \text{sgn}(E^r)}{\sqrt{|E^r||E^1|}} \left[ 2 A_r E^r A_1 E^1 + (2(A_1)^2 - 1)(E^1)^2 \right] \quad (48)$$

we redefine $E^r \equiv E$ and $A_r \equiv A$. The connection between the metric $g_{ab}$ and the density triad is

$$q_{ab} = \text{diag}(2(E^1)^2/[|E|, |E|, |E| \sin \theta]).$$

Another useful quantity is the volume of the spatial section $\Sigma$

$$V = \int dr \, d\phi \, d\theta \sqrt{\det(q)} = 4\pi\sqrt{2}R \sqrt{|E||E^1|}, \quad (49)$$

where $R$ is a cut-off on the space radial coordinate. The spatial homogeneity enable us to fix a linear radial cell $L_r$ and restrict all integrations to this cell $[12]$. To simplify notations we restrict the linear radial cell to the Planck length $l_P$ and we can take $\int dr = R \equiv l_P$ in the action functional $[19]$. Now we are going to use $R \equiv l_P$ in all the paper.

The classical symplectic structure of the phase space can be obtained by inserting the symmetry reduced variables in the action (53). This gives

$$S = \frac{1}{\kappa \gamma} \int dt \int dr \, d\phi \, d\theta \sin \theta \left[ \text{Tr} \left( -2E^r \dot{A}_r \right) + \ldots \right] = \frac{4\pi l_P}{\kappa \gamma} \int dr \left[ E \dot{A} + 4E^1 \dot{A}_1 + \ldots \right]. \quad (50)$$

We can read the symplectic structure of the classical phase space directly from the reduced action (50). The phase space consists of two canonical pairs $A, E$ and $A_1, E^1$ and from (50) we can obtain the symplectic structure. The simplectic structure is given by the poisson brackets, $\{A, E\} = \frac{2\pi \gamma}{4\pi l_P}$ and $\{A_1, E^1\} = \frac{\pi \gamma}{4\pi l_P}$. The coordinates and the momenta have dimensions: $[A] = L^{-1}$, $[A_1] = L^0$, $[E] = L^2$ and $[E^1] = L$.

The elementary configuration variables used in LQG are given by the holonomies along curves in the spatial section $\mathbb{R} \times S^2$ and the fluxes of triads on a two-surface in $\mathbb{R} \times S^2$. We restrict our attention to three sets of curves. More precisely, we consider only spin networks $[1]$ based on graph made just of radial edges, and of edges along circles in the $\theta$-direction or at $\theta = \pi/2$.

Let us introduce the fiducial triad $^{a}e^f_j = \text{diag}(1, 1, \sin^{-1} \theta)$ and co-triad $^o\omega^f_i = \text{diag}(1, 1, \sin \theta)$. The holonomy along a curve in the direction “$I$” is given by $h_I = \exp \int A_I^f \tau_i$, $h_2 = \exp[A_1 \mu_0 \tau_3]$, $h_3 = \exp[A_1 \mu_0 \tau_2]$ and $h_3 = \exp[A_1 \mu_0 \tau_1]$, (51)

where $A_I^f_1 = (0, 0, A), A_I^f_2 = (A_1, A_1, 0)$ and $A_I^f_3 = (-A_1, A_1, 0)$. The connection in (51) is integrated in the direction “$I$”; $\mu_0 l_P$ is the length of the curve along the direction $r$, $\mu_0$ is the length of the curve along the directions $\theta$ and $\phi$. The length are defined using the fiducial triad $^{a}e^f_j$.

Recall that the Hamiltonian constraint can be written in terms of the curvature $F_{ab}$ and the Poisson bracket between $A_a$ and the volume $V$ $[29]$. The Euclidean part of the Hamiltonian constraint becomes

$$H_E = -\frac{4}{\kappa \gamma} \int d^3x \, N e^{abc} \text{Tr} [F_{ab} \{ A_c, V \}]. \quad (52)$$
Because of homogeneity we can assume that the lapse function \( N \) is constant and in the rest of the paper we will set \( N = 1 \).

We can express the curvature \( F_{ab} \) and the Poisson bracket \( \tau_i \{ A_a, V \} \) in terms of holonomies [17] obtaining the following form for the Hamiltonian constraint

\[
H_E = -\frac{16\pi}{\kappa \gamma \mu_0^2} \sum_{IJK} \epsilon^{IJK} \text{Tr} \left[ h_{I} h_{J} h_{K}^{-1} \right],
\]

where \( h_{IJ} = \exp(-\mu_0^2 C_{IJ} \tau_3) \) and \( C_{IJ} = \delta_{2I} \delta_{3J} - \delta_{3I} \delta_{2J} \).

Using the classical identity

\[
\frac{\text{sgn}(E)}{(\det(q_a))^{3/2}} = \frac{64(4\pi p)^{3/2}}{3\kappa^2 \gamma^3 (\sin \theta)^{3/2}} \epsilon_{ijk} \epsilon_{ABC} \epsilon_{DEF} \epsilon_{GHI} \epsilon_{JMN} \epsilon_{OPQ}
\]

we can define the inverse of the volume in terms of holonomies. The result is [17]

\[
\text{det}(q_a) = \frac{256(4\pi)^{3/2} \sqrt{p}}{3\kappa^2 \gamma^3 (\sin \theta)^{3/2}} \epsilon_{ijk} \epsilon_{ABC} \epsilon_{DEF} \epsilon_{GHI} \epsilon_{JMN} \epsilon_{OPQ} \frac{1}{(\det(q_a))^{3/2}}.
\]

The definition (55) will be useful to calculate the inverse volume spectrum.

3.2 Quantum Theory

We construct the kinematical Hilbert space \( H_{\text{kin}} \) for the Kantowski-Sachs minisuperspace model in analogy with the full theory. As in the second section we define a graph \( \Gamma \) as a finite number of couple of points \( (\mu_{E_i}, \mu_{E_j}) \), where \( \mu_{E_i}, \mu_{E_j} \in \mathbb{R} \). Denote by \( \text{Cyl}_r \) the vector space of function \( f(A, A_1) \) (\( f : \mathbb{R}^2 \to \mathbb{C} \)) of the type

\[
f(A, A_1) = \sum_{ij} f_{ij} e^{i \mu_{E_{ij}} A_i A_j}.
\]

where \( A, A_1 \in \mathbb{R}, \mu_{E_i}, \mu_{E_j} \in \mathbb{R}, f_{ij} \in \mathbb{C} \) and \( i, j \) run over a finite number of integers (labeling the points of the graph). We call the function \( f(A, A_1) \) in \( \text{Cyl}_r \) cylindrical with respect to the graph \( \Gamma \). We consider all possible graphs (the points and their number can vary from a graph to another) and denote by \( \text{Cyl} \) the infinite dimensional vector space of functions cylindrical with respect to some graph: \( \text{Cyl} = \bigcup_\Gamma \text{Cyl}_r \). Thus, any element \( f(A, A_1) \) of \( \text{Cyl} \) can be expanded as in (56), where the uncountable basis \( e^{i \mu_{E_{ij}} A_i A_j} \) is now labeled by arbitrary real numbers \( (\mu_{E_i}, \mu_{E_j}) \). A basis in \( \text{Cyl} \) is given by \( |\mu_E, \mu_{E_1} \rangle \equiv |\mu_E \rangle \otimes |\mu_{E_1} \rangle \). Introducing the standard bra-ket notation we can define a basis [13] in the Hilbert space via

\[
\langle A | \mu_E \rangle \otimes \langle A_1 | \mu_{E_1} \rangle = e^{i \mu_{E_{ij}} A_i A_j} \otimes e^{i \mu_{E_{ij}} A_i A_j}.
\]

The basis states [57] are normalizable in contrast to the standard quantum mechanical representation and they satisfy

\[
\langle \mu_E, \mu_{E_1} | \mu_E, \mu_{E_1} \rangle = \delta_{\mu_E, \mu_{E_1}} \delta_{\mu_{E_1}, \mu_{E_1}}.
\]

The Hilbert space \( H_{\text{kin}} \) is the Cauchy completion of \( \text{Cyl} \) or more succinctly \( H_{\text{kin}} = L_2(\mathbb{R}^2_{\text{Bohr}}, d\mu_0) \), where \( \mathbb{R}^2_{\text{Bohr}} \) is the Bohr-compactification of \( \mathbb{R} \) and \( d\mu_0 \) is the Haar measure on \( \mathbb{R}^2_{\text{Bohr}} \). In LQG (or in polymer representation) the fundamental operators are \( \hat{E}, \hat{E}^+, h_I \) and...
the momentum operators can be represented on the Hilbert space by
\[
\hat{E} \to -i \frac{\gamma l_P}{4\pi} \frac{d}{dA}, \quad \hat{E}^1 \to -i \frac{\gamma l_P}{16\pi} \frac{d}{dA_1},
\]
and the he spectrum of these two momentum operators on the Hilbert space basis is
\[
\hat{E}|\mu_E,\mu_{E^1}\rangle = \frac{\mu_E \gamma l_P}{8\pi}|\mu_E,\mu_{E^1}\rangle, \quad \hat{E}^1|\mu_E,\mu_{E^1}\rangle = \frac{\mu_{E^1} \gamma l_P}{16\pi \sqrt{2}}|\mu_E,\mu_{E^1}\rangle;
\]

the holonomy operators \(\hat{h}_I\) in the directions \(r, \theta, \phi\) of the space section \(\mathbb{R} \times S^2\) are : \(\hat{h}_1(\mu_E), \hat{h}_2(\mu_E)\) and \(\hat{h}_3(\mu_{E^1})\), where \(\mu_E l_P\) is the length along the radial direction \(r\) and \(\mu_{E^1}\) is the length along the directions \(\theta\) and \(\phi\) (all the length are define using the fiducial triad \(e_i^a\)). The holonomies operators act on the Hilbert space \(\mathcal{H}_{kin}\) by multiplication.

We have to fix the residual gauge freedom on the Hilbert space. We consider the operator \(\hat{P} : |\mu_E,\mu_{E^1}\rangle \rightarrow |\mu_E,-\mu_{E^1}\rangle\) and we impose that only the invariant states (under \(\hat{P}\)) are in the kinematical Hilbert space. The states in the Hilbert space are : \(\frac{1}{\sqrt{2}}[|\mu_E,\mu_{E^1}\rangle + |\mu_E,-\mu_{E^1}\rangle]\) for \(\mu_{E^1} \neq 0\) and the states \(|\mu_E,0\rangle\) for \(\mu_{E^1} = 0\).

### 3.2.1 Inverse volume spectrum

In this section we study the black hole singularity problem in loop quantum gravity calculating the spectrum of the inverse volume operator. The operator version of the quantity \(\text{sgn}(E)/(\det(q))^{\frac{1}{4}}\) defined in the formula (55) is

\[
\frac{\text{sgn}(E)}{(\det(q))^{\frac{1}{4}}} = \frac{256 i (4\pi l_P)^{\frac{3}{2}}}{3 l_P^3 \gamma^3 \mu_0^3 (\sin \theta)^{\frac{3}{2}}} \sum_{IJK} \epsilon_{ijk} \text{Tr} \left[ \tau^i \hat{h}^{-1}_I \hat{h}_J \hat{V} \right] \text{Tr} \left[ \tau^j \hat{h}^{-1}_J \hat{h}_K \hat{V} \right] \text{Tr} \left[ \tau^k \hat{h}^{-1}_K \hat{h}_L \hat{V} \right].
\]

To calculate the action of (61) on the Hilbert space basis we Introduce the normalized vectors \(n_1^i = (0, 0, 1), n_2^i = \frac{1}{\sqrt{2}}(1, 1, 0), n_3^i = \frac{1}{\sqrt{2}}(-1, 1, 0)\). Using the vectors \(n_i^j\) we can rewrite the holonomies \(h_I\) of (51) as \(h_I \rightarrow \exp(\hat{A}_I n_i^j \tau_i) = \cos(\hat{A}_I / 2) + 2n_i^j \sin(\hat{A}_I / 2)\), where \(\hat{A}_{I=1} = \hat{A}_I l_P \mu_0\) and \(\hat{A}_{I=2} = \hat{A}_{I=3} = \hat{A}_I l_P \mu_0 \sqrt{2}\). The action of the multiplicative operator \(\hat{h}_I\) on the bases states is

\[
\hat{h}_I|\mu_I\rangle = \frac{I - in_i^j \tau_i}{2}|\mu_I + \mu_0\rangle + \frac{I + in_i^j \tau_i}{2}|\mu_I - \mu_0\rangle.
\]

The spectrum of the inverse volume operator is

\[
\frac{1}{(\det(q))^{\frac{1}{4}}} |\mu_E,\mu_{E^1}\rangle = \frac{2\pi}{\gamma} \frac{8}{l_P \mu_0^3 (\sin \theta)^{\frac{3}{2}}} |\mu_E|^{\frac{1}{4}} |\mu_{E^1}|^{\frac{1}{4}} |\mu_E + \mu_0|^{\frac{1}{4}} - |\mu_E - \mu_0|^{\frac{1}{4}} \left( |\mu_{E^1} + \mu_0|^{\frac{1}{4}} - |\mu_{E^1} - \mu_0|^{\frac{1}{4}} \right)^2 |\mu_E,\mu_{E^1}\rangle.
\]

The spectrum of the inverse volume operator is bounded above, near the classical singularity which is in \(E = 0\) or \(\mu_E = 0\), and reproduces the correct classical spectrum of \(1/(\det(q))^{\frac{1}{4}}\) for large volume eigenvalues (Fig[2]).
An element of this space is \[ \langle \subset H \rangle \] non-trivially constrained systems, we expect that the physical states are not normalizable in the Hilbert space of) the Hilbert space.

Using the relations in (62) we can calculate the action of the Hamiltonian constraint on the Hilbert space evolution does not stop at the classical singularity. The "other side" of the singularity corresponds to a new domain where the triad reverses its orientation.

### 3.2.2 Quantum dynamics

In this section we study the dynamics of the model solving the Hamiltonian constraint in (the dual space of) the Hilbert space.

The quantum version of the Hamiltonian constraint defined in (53) can be obtained promoting the classical holonomies to operators and the poisson bracket to the commutator

\[
\hat{H}_E = \frac{16\pi i}{\mu_0^3} \sum_{IJK} \epsilon^{IJK} \text{Tr} \left[ \hat{h}_I \hat{h}_J \hat{h}_K^{-1} \hat{h}_L^{-1} \hat{h}_{[I,J]} \hat{h}_{L]^{-1} \hat{h}_{K} \hat{V} \right].
\]

Using the relations in (62) we can calculate the action of the Hamiltonian constraint on the Hilbert space basis \[ |\mu, E\rangle \] and solve the Hamiltonian constraint to obtain the physical states. As in loop quantum cosmology, we again have the Gelfand triple \( \mathcal{Cyl} \subset \mathcal{H}_{kin} \subset \mathcal{Cyl}^* \) and the physical states will be in \( \mathcal{Cyl}^* \), which is the algebraic dual of \( \mathcal{Cyl} \). An element of this space is \( \langle \psi \rangle = \sum_{\mu, E} \psi_{\mu, E} \langle \mu, E \rangle \). The constraint equation \( \hat{H} |\psi\rangle = 0 \) is now interpreted as an equation in the dual space \( \langle \psi | \hat{H}^\dagger \rangle \); from this equation we can derive a relation for the coefficients \( \psi_{\mu, E} \).

\[
-\alpha_{\mu, E} - 2\mu_0 \psi_{\mu, E} - 2\mu_0 \mu_0 \psi_{\mu, E} + 2\mu_0 \mu_0 \psi_{\mu, E} + 2\mu_0 \mu_0 \psi_{\mu, E} - 2\mu_0 \\
+ \alpha_{\mu, E} + 2\mu_0 \mu_0 \psi_{\mu, E} - 2\mu_0 \mu_0 \psi_{\mu, E} + 2\mu_0 \mu_0 \psi_{\mu, E} + 2\mu_0 \mu_0 \psi_{\mu, E}
\]

\[
+ \frac{\sin(\mu_0^2/2)}{2} \left( \beta_{\mu, E} \psi_{\mu, E} - 4\mu_0 \psi_{\mu, E} - 4\mu_0 - 2\beta_{\mu, E} \psi_{\mu, E} + \beta_{\mu, E} \psi_{\mu, E} + 4\mu_0 \psi_{\mu, E} \psi_{\mu, E} + 4\mu_0 \right)
\]

\[
-2 \sin(\mu_0^2/2) \left( \beta_{\mu, E} \psi_{\mu, E} - 2\mu_0 \psi_{\mu, E} - 2\mu_0 + \beta_{\mu, E} \psi_{\mu, E} + 2\mu_0 \psi_{\mu, E} \psi_{\mu, E} + 2\mu_0 \right) = 0,
\]

where the functions \( \alpha_{\mu, E} \) and \( \beta_{\mu, E} \) are defined by

\[
\alpha_{\mu, E} := |\mu, E \rangle \langle \mu, E | (|\mu, E | + |\mu, E |) \quad \beta_{\mu, E} := |\mu, E | (|\mu, E | + |\mu, E |)
\]

How can be seen from equation (65) the quantization program produce a difference equation and imposing a boundary condition we can obtain the wave function \( \psi_{\mu, E} \) for the black hole. We can interpret \( \psi_{\mu, E} \) as the wave function of the anisotropy \( E^1 \), at the time \( E \). It is evident from (66) that the dynamics is regular in \( \mu = 0 \) where the classical singularity is localized. As in loop quantum cosmology also in this case the state \( \psi_{0, 0} \) decouples from the dynamics and the quantum evolution does not stop at the classical singularity. The "other side" of the singularity corresponds to a new domain where the triad reverses its orientation.
Conclusions

In this paper we have summarized loop quantum gravity theory and we have applied the ideas to study the space-time region inside the Schwarzschild black hole horizon. Because the space-time region inside the horizon is spatially homogeneous of Kantowski-Sachs type \[25\], we have studied this minisuperspace model. This is an homogeneous but anisotropic minisuperspace model with spatial topology \( \mathbb{R} \times S^2 \). We have analyzed the model firstly in ADM variables with some drastic simplification and then in Ashtekar variables. We have quantized the reduced model using a quantization procedure induced by the full “loop quantum gravity”. Our analysis it was useful in order to understand what happens close to the black hole singularity where quantum gravity effects are dominant and the classical Schwarzschild solution is not correct.

The main results are:

1. the curvature invariant and the inverse volume operator have a finite spectrum in all the region inside the horizon and we can conclude that the classical singularity disappears at the kinematical level; on the other side for large eigenvalues of the volume operator we find the classical inverse volume behavior,

2. the solution of the Hamiltonian constraint gives a difference equation for the coefficients of the physical states defined in the dual space of some dense subspace of the kinematical Hilbert space. All the coefficients in the difference equation are regular in the classical singular point then we have a solution of the singularity problem also at the dynamical level.

An important consequence of the quantization is that, unlike the classical evolution, the quantum evolution does not stop at the classical singularity and the “other side” of the singularity corresponds to a new domain where the triad reverses its orientation. From the difference equation we obtain physical states as combinations of a countable number of components of the form \( \psi_{\mu_E + n, \mu_0, \mu_{E1} + m \mu_0} \) (where \( \mu_0 \sim 1 \) at the Plank scale and \( n, m \in \mathbb{Z} \)); any component corresponds to a particular value of the volume of the space section. We can interpret \( \mu_E \) as the time and the anisotropy \( \mu_{E1} \) as the space partial observable \[23\] that defines the quantum fluctuations around the Schwarzschild solution. We recall that \( |E| = b^2 \), therefore in the classical theory and in quantum mechanics, we can regard \( |E| \) as an internal time. So the function \( \psi_{\mu_E + n, \mu_0, \mu_{E1} + m \mu_0} \) is the wave function of the Black Hole inside the horizon at the time \( \mu_E + n \mu_0 \) and we have a natural and regular evolution beyond the classical singularity point which is in \( \mu_E = 0 \) localized. A solution of the Hamiltonian constraint corresponds to a linear combination of black hole states for particular values of the anisotropy \( \mu_{E1} \) at the time \( \mu_E \).

It is interesting to recall that beyond the classical singularity the eigenvalue \( \mu_E \) is negative and so we can suggest a new universe was born from the black hole formation process. In LQBH scenario pure states which fall into black hole emerge in a new universe as pure states and the information loss problem is avoided. Information is not lost in the black hole but it exists again in the space-time region in the future of the avoided singularity.

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