COHERENT SHEAVES ON THE STACK OF LANGLANDS PARAMETERS

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Abstract. We formulate a few conjectures on some hypothetical coherent sheaves on the stacks of arithmetic local Langlands parameters, including their roles played in the local-global compatibility in the Langlands program. We survey some known results as evidences of these conjectures.

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1. Introduction

In recent years, people realize that there should exist certain (complexes of) coherent sheaves $\mathfrak{A}$ on the stacks of local and global arithmetic Langlands parameters, which should largely control the Langlands correspondence, and allow one to formulate local-global compatibilities in the arithmetic Langlands program. In fact, that such objects should exist is already suggested by work of Emerton-Helm [EH14] and Helm [He16] under the idea of local Langlands correspondence in families. This idea is further explored recently by Hellmann [He]. On the other hand, after the work of V. Lafforgue and Genestier-Lafforgue [La18, GL], such ideas become more clear and some powerful tools in the geometric Langlands program are available to realize (part of) them. In fact, even the whole arithmetic local Langlands correspondence over non-archimedean local fields should admit a categorical incarnation (e.g. see [Ga, 4.2] for some indications), and existence of such coherent

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1There are similar $\mathfrak{A}$ appearing in the work of Emerton et. al. in the p-adic local Langlands program but the author is incapable of saying anything in this direction.
sheaves fits nicely in the categorical framework. In another direction, the work of Fargues-Scholze \[FS\] on the geometrization of the local Langlands correspondence is also closely related these ideas, and also leads to a categorical form of arithmetic local Langlands correspondence. In global aspects, the existence of $\mathfrak{A}$ is the guiding principle of the author’s work with Xiao \[XZ\] on the geometric realization of Jacquet-Langlands correspondence via the cohomology of Shimura varieties. In another direction, a very crude form of the coherent sheaf is used in the author’s work with V. Lafforgue \[LZ\] to describe the elliptic part of the cohomology of Shtukas in the framework of Arthur-Kottwitz conjectures.

In this article, we formulate a few precise conjectures related to the hypothetical sheaves $\mathfrak{A}$ and survey some known results, including explicit conjectural descriptions of $\mathfrak{A}$ in some special (but most important) cases and their roles in the local-global compatibility, and some possible categorical forms of the local arithmetic Langlands correspondence, which would give a conceptual explanation why such $\mathfrak{A}$ are expected to exist. In order to formulate these conjectures, we discuss the construction and some properties of the moduli stack of local Langlands parameters ($\ell \neq p$ case) and global Langlands parameters (function field case).

We shall mention that some ideas in this article are shared by experts for years although probably they may not yet exist in literature\(^2\) It is the author’s desire to make some of them more precise and write them down.

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**2. Representation space**

Let $M$ be an affine group scheme over a commutative ring $k$ and $\Gamma$ an abstract group. It is well-known that there is an affine scheme $\mathcal{R}_{\Gamma,M}$ over $k$ such that for every $k$-algebra $A$, $\mathcal{R}_{\Gamma,M}(A)$ classifies the set of group homomorphisms from $\Gamma$ to $M(A)$ . Namely, one first considers the functor over $k$ classifying all maps from $\Gamma$ to $M(A)$ as sets. This is obviously represented by an affine scheme, namely the self product $M^{\Gamma}$ of $M$ over $\Gamma$. Then the condition of set maps being group homomorphisms defines $\mathcal{R}_{\Gamma,M}$ as a closed subscheme of $M^{\Gamma}$.

One would like to apply this idea to construct the moduli space of Langlands parameters. But there are two issues. The first issue is well-known. Namely, the Galois group is a profinite group and one shall only consider continuous representations of $\Gamma$ (satisfying certain additional properties). We will address this issue in Section 2.4. Roughly speaking, by imposing the continuity condition, one obtains an ind-scheme whose completions at closed points recover the usual framed deformation spaces of representations of profinite groups. In general, this space might still not have good global geometry (see Example 2.4.5). But for certain group, it “glues” all the deformation spaces together in a reasonable way. This is indeed the case when $\Gamma$ is the Galois group of a local or global function field, and will be discussed in details in Section 3.

Another issue is that equations defining $\mathcal{R}_{\Gamma,M} \subset M^{\Gamma}$ usually do not form a “regular sequence”, so there might be non-trivial derived structure on $\mathcal{R}_{\Gamma,M}$. At some point in the sequel, we need to

\(^2\)Indeed, when the author was preparing the article, several other works become known through math community: Hellmann’s preprint on his conjectures appeared recently \[Hel\], which is closely related to Conjecture 4.3.1 and 4.4.1. Scholze announced a categorical form of the local Langlands conjecture as part of his joint work with Fargues, which is closely related to Conjecture 4.5.3. On the other hand, the definition of the stack of local Langlands parameters in Section 3.1 will also be the subject of a forthcoming work by Dat-Helm-Kurinczuk-Moss \[DHKM\].
2.1. The derived category of monoids. Our goal is to define a derived object \( \mathcal{R}_{\Gamma,M} \) parameterizing homomorphisms from \( \Gamma \) to \( M \). It is convenient to start with a slightly more general setting by considering homomorphisms of monoids. The basic idea then is to move from the category \( \text{Mon} \) of monoids to its derived category. As \( \text{Mon} \) is non-abelian, one needs the notion of non-abelian derived categories in the sense of Quillen, as developed by Lurie using the language of \( \infty \)-categories [Lu09 5.5.8]. We first recall some general theory and specialize to the examples we need.

In the sequel, we call \((\infty,1)\)-categories just by \( \infty \)-categories, and regard ordinary categories as \( \infty \)-categories in the usual way. Let \( \text{Spc} \) denote the \( \infty \)-category of spaces, containing the category \( \text{Sets} \) of sets as a full subcategory (regarded as discrete spaces). The inclusion \( \text{Sets} \to \text{Spc} \) admits a left adjoint \( \pi_0 : \text{Spc} \to \text{Sets} \) which preserves finite products. If \( x,y \) are two objects in an \( \infty \)-category \( C \), we write \( \text{Map}_C(x,y) \in \text{Spc} \) for the space of maps from \( x \) to \( y \). All functors are understood in the \( \infty \)-categorical setting (and therefore are derived). We let \( \text{Fun}(C,D) \) denote the \( \infty \)-category of functors between two \( \infty \)-categories \( C \) and \( D \). We refer to [Lu09] for foundations of \( \infty \)-categories.

We find it is instructive to adapt Clausen-Scholze’s point of view to start with. For an ordinary category \( C \) admitting colimits, let \( C^{\text{cp}} \) denote its full subcategory of compact projective objects in \( C \), i.e. those \( x \in C \) such that \( \text{Hom}_C(x,-) \) commutes with filtered colimits and reflexive equalizers. This is a category admitting finite coproducts, so one can define its non-abelian derived category \( \mathcal{P}_\Sigma(C^{\text{cp}}) \) ([Lu09 5.5.8.8]), which is the full subcategory of \( \text{Fun}((C^{\text{cp}})^\text{op}, \text{Spc}) \) consisting those functors that preserve finite products. If \( C \) is generated by \( C^{\text{cp}} \) under colimits, \( \mathcal{P}_\Sigma(C^{\text{cp}}) \) is called the \( \infty \)-category of anima of \( C \) by Clausen-Scholze, and is denoted by \( \text{Ani}(C) \). We sometimes also just call it the derived category of \( C \). Now if \( C \) has a symmetric monoidal structure such that the tensor product preserves colimits separately in each variable, and that the symmetric monoidal structure restricts to a symmetric monoidal structure on \( C^{\text{cp}} \), then \( \text{Ani}(C) \) is naturally a symmetric monoidal \( \infty \)-category and the tensor product preserves colimits separately in each variable ([Lu2 4.8.1.10]).

There is a fully faithful embedding \( C \subset \text{Ani}(C) \), by regarding \( C \) as the subcategory of \( \infty \)-categories. Let \( \mathcal{C}^{\text{op}} \to \text{Spc} \) factoring as \( \mathcal{C}^{\text{op}} \to \text{Sets} \subset \text{Spc} \). It admits a left adjoint \( \pi_0 : \text{Ani}(C) \to C \) induced by \( \pi_0 : \text{Spc} \to \text{Sets} \). More generally, for each \( n \geq 0 \), there is the \( n \)-truncation functor \( \tau_{\leq n} : \text{Ani}(C) \to \leq_n \text{Ani}(C) \), where for an \( \infty \)-category \( C \), \( \leq_n C \) denotes the full subcategory of \( n \)-truncated objects of \( C \) ([Lu09 5.5.6.1]), which is a left adjoint of the natural inclusion functor \( \leq_n \text{Ani}(C) \subset \text{Ani}(C) \) ([Lu09 5.5.6.18]). The following are two basic examples.

Example 2.1.1. (1) If \( C = \text{Sets} \), equipped with the Cartesian symmetric monoidal structure (i.e. tensor product is given by product), then \( C^{\text{cp}} \) is the category of finite sets, and \( \text{Ani}(\text{Sets}) \cong \text{Spc} ([Lu09 5.5.8.24]) \), equipped with the Cartesian symmetric monoidal structure.

(2) If \( C = \text{Mod}_k^{\text{c}} \) is the abelian category of \( k \)-modules, equipped with the usual tensor product structure, then \( C^{\text{cp}} \) is the usual category of finite projective \( k \)-modules and \( \text{Ani}(\text{Mod}_k^{\text{c}}) \) is equivalent to the derived category \( \text{Mod}_k^{\text{c}} := D^{\geq 0}(\text{Mod}_k^{\text{c}}) \) of connective complexes of \( k \)-modules (i.e. those complexes whose cohomology vanish in positive degrees), equipped with the usual symmetric monoidal structure ([Lu09 5.5.8.21] and [CS 5.1.6]).

\[ \text{We implicitly assume that } C^{\text{cp}} \text{ is small, which is the case for all examples we encounter.} \]
The example we need is the category of monoids $C = \text{Mon}$. This category admits all small colimits, and is generated under colimits by its compact projective objects, which are finitely freely generated monoids. For a finite set $I$, let $FM(I)$ denote the free monoid generated by $I$. Let $\text{FFM}$ be the full subcategory spanned by these $FM(I)$s. For a monoid $\Gamma$, let $\text{FFM}/\Gamma$ denote the corresponding slice category: i.e. objects are pairs of the form $(FM(I), u : FM(I) \to \Gamma)$ and morphisms from $(FM(I), u)$ to $(FM(J), v)$ are monoid homomorphisms $f : FM(I) \to FM(J)$ such that $u = vf$. We note that the category $\text{FFM}/\Gamma$ is not filtered, but is sifted (see [Lu09, 5.5.8.1] for this notion), as coproducts exist in $\text{FFM}/\Gamma$. There is a canonical isomorphism in $\text{Mon}$

\[
\lim_{\text{FFM}/\Gamma} FM(I) \cong \Gamma.
\]

This isomorphism can also be understood in $\text{Ani(Mon)}$, via the fully embedding $\text{Mon} \subset \text{Ani(Mon)}$, as $\text{Ani(Mon)} = \mathcal{P}_{C}(\text{FFM})$.

On the other hand, for an $\infty$-category $C$ admitting finite products, there is the $\infty$-category $\text{Mon}(C)$ of monoid objects in $C$, which by definition is the full subcategory of the category $C^{\Delta^{op}} := \text{Fun}(\Delta^{op}, C)$ of simplicial objects in $C$, consisting of those $X_*$ such that for every $[n] \in \Delta$, the map

$X([n]) \to X(\{0,1\}) \times X(\{1,2\}) \times \cdots \times X(\{n-1,n\}) = X([1])^n$

induced by $[1] \cong \{i-1, i\} \subseteq \{0,1,\ldots,n\} = [n]$, is an isomorphism in $\text{Spc}$ ([Lu2, 4.1.2.5]). For example, if $C = \text{Sets}$, then $\text{Mon} \cong \text{Mon(Sets)}$ via the usual Milnor construction: for $\Gamma \in \text{Mon}$, the corresponding object in $\text{Mon(Sets)}$ is the nerve of the category with a unique object whose endomorphism monoid is $\Gamma$ ([Lu2, 4.1.2.4]). Then the fully faithful embedding $\text{Sets} \subset \text{Spc}$ induces a fully faithful embedding $\text{Mon} \subset \text{Mon(Spc)}$ (as both of which are full subcategories of $\text{Spc}^{\Delta^{op}}$).

**Lemma 2.1.2.** There is a canonical equivalence $\text{Ani(Mon)} \cong \text{Mon(Ani)}$.

**Proof.** We consider a more general situation. Let $C$ be a(n ordinary) cocomplete symmetric monoidal category as before (i.e. $C$ is generated by $C^{op}$ under colimits and the tensor product preserves colimits separately in each variable). Then it makes sense to talk about the $(\infty)$-category $\text{Alg}(-)$ of its associative (a.k.a $E_1$-)algebra objects in $C$ and $\text{Ani}(C)$ ([Lu2, 2.1.3]). Using [Lu2, 7.2.4.27] and Lemma 2.1.3 below, we obtain a canonical equivalence

$\text{Ani(Alg}(C)) \cong \text{Alg(Ani}(C))$.

The lemma follows by letting $C = \text{Sets}$ and identifying associative algebra objects with monoid objects when the ambient symmetric monoidal structure is Cartesian ([Lu2, 2.4.2, 4.1.2.10]). \qed

To state the following lemma, recall from [Lu2, 3.1.3] that for $(-) = C$ or $\text{Ani}(C)$, the forgetful functor from $\text{Alg}(-) \to (-)$ admits a left adjoint $\text{Fr}_(-)$, given by the free algebra construction.

**Lemma 2.1.3.** For every $X \in C^{op}$, the image of $\text{Fr}_C(X)$ under the functor $\text{Alg}(C) \to \text{Alg(Ani(C))}$ is canonically isomorphic to $\text{Fr}_{\text{Ani}(C)}(X)$.

We note that this lemma is specific to $E_1$-algebras, as the analogous statement for $E_\infty$-algebras is well-known to be false in general.\footnote{We thank Scholze for pointing out this.}

**Proof.** There is a canonical morphism $\text{Fr}_{\text{Ani}(C)}(X) \to \text{Fr}_C(X)$ given by adjunction, and we need to show that it is an isomorphism. As the forgetful functor $\text{Alg(Ani}(C)) \to \text{Ani}(C)$ is conservative ([Lu2, 3.2.2.6]), it is enough to show that it is an isomorphism in $\text{Ani}(C)$. But in this case, both objects are given by $\sqcup_{n \geq 0} X^\otimes n$, by combining [Lu2, 3.1.3.13] with the fact that the embedding $C^{op} \to \text{Ani}(C)$ is monoidal and preserves finite coproducts. \qed
Here is the corollary we need. It can be regarded as a canonical “projective resolution” of an object in \( \text{Mon}(\text{Spc}) \). See [GKRV] 2.1.5 for a closely related statement (with a different proof).

**Corollary 2.1.4.** The isomorphism \([2.1]\) holds in \( \text{Mon}(\text{Spc}) \).

Of course, \([2.1]\) holds for every \( \Gamma \in \text{Mon}(\text{Spc}) \) except that in this case \( \text{FFM}/\Gamma \) might no longer be an ordinary category.

**Remark 2.1.5.** There are variants of the above discussions, by replacing monoid objects by group or semigroup objects in a category \( \mathcal{C} \). Following [Lu2] 5.2.6.2.12, we regard group objects as grouplike monoid objects and semigroup objects as non-unital monoid objects, and denote the corresponding categories by \( \text{Mon}^\text{gp}(\mathcal{C}) \) and \( \text{Mon}^\text{nu}(\mathcal{C}) \) respectively (and omit \( \mathcal{C} \) from the notation if \( \mathcal{C} = \text{Sets} \)). For \( ? = \text{gp} \) or \( \text{nu} \), compact projective objects of \( \text{Mon}^? \) are still finitely generated ones. Following [Wo], we denote the corresponding subcategories by \( \text{FFG} \) and \( \text{FFS} \) respectively. We still have \( \text{Ani}(\text{Mon}^?) \cong \text{Mon}^?(\text{Ani}) \) and therefore analogous Corollary \([2.1.4]\). Indeed, the semigroup case can be proved similarly, and the group case follows from Lemma \([2.1.2]\) and [Lu2] 5.2.6.4 (and in fact is already contained in [Lu2] 5.2.6.10, 5.2.6.21).

There are natural forgetful functors \( \text{Mon}^\text{gp}(\text{Ani}) \to \text{Mon}(\text{Ani}) \to \text{Mon}^\text{nu}(\text{Ani}) \). The first and the composition functors are fully faithful. In our application, we will mainly consider spaces of maps between groups so we can calculate them in any of these three categories.

### 2.2. The derived representation space.

We fix a commutative ring \( k \), and let \( \text{CAlg}_k^\bigotimes \) denote the (ordinary) category of commutative \( k \)-algebras. Let \( \text{CAlg}_k^\bigotimes = \text{Ani}(\text{CAlg}_k^\bigotimes) \) be the derived category of \( \text{CAlg}_k^\bigotimes \). We follow Clausen-Scholze to call objects in \( \text{CAlg}_k^\bigotimes \) animated \( k \)-algebras (instead of the more traditional term of simplicial \( k \)-algebras), and also call objects in \( \text{CAlg}_k^\bigotimes \) classical \( k \)-algebras. Let \( \text{Aff}_k \) (resp. \( \text{DAff}_k \)) denote the opposite of \( \text{CAlg}_k^\bigotimes \) (resp. \( \text{CAlg}_k^\bigotimes \)). Objects in \( \text{Aff}_k \) will be called as classical affine \( k \)-schemes, or simply affine \( k \)-schemes, and objects in \( \text{DAff}_k \) will be called derived affine \( k \)-schemes, or animated \( k \)-affine schemes. Given \( A \in \text{CAlg}_k \), the corresponding object in \( \text{DAff}_k \) will be denoted by \( \text{Spec}A \) as usual, and given \( X \in \text{DAff}_k \), we denote the corresponding object in \( \text{CAlg}_k \) by \( k[X] \), called the ring of regular functions on \( X \). For \( X = \text{Spec}A \), we write \( dX \) for the underlying classical affine scheme \( \text{Spec}A \).

Let \( M \) be an affine monoid scheme flat over \( k \). It is an object in \( \text{Mon}(\text{Aff}_k) \). Then the functor \( \text{CAlg}_k^\bigotimes \to \text{Mon} \) defined by \( M \) extends to a (sifted colimit preserving) functor

\[
\text{CAlg}_k = \text{Ani}(\text{CAlg}_k^\bigotimes) \to \text{Ani}(\text{Mon}) \cong \text{Mon}(\text{Spc}),
\]

still denoted by \( M \). Unveiling the definition, for \( A \in \text{CAlg}_k \), \( M(A) \in \text{Mon}^\text{sp}(\text{Spc}) \) is the simplicial space given by \( [n] \in \Delta \mapsto \text{Map}_{\text{CAlg}_k}(k[M^n], A) \cong \text{Map}_{\text{CAlg}_k}(k[M], A)^n \).

**Definition 2.2.1.** For \( \Gamma \in \text{Mon}(\text{Spc}) \), We define

\[
\mathcal{R}_{\Gamma,M} : \text{CAlg}_k \to \text{Spc}, \quad A \mapsto \text{Map}_{\text{Mon}(\text{Spc})}(\Gamma, M(A)).
\]

**Remark 2.2.2.** Our definition is same as the one given in [To12] §3.2. On the other hand, if \( M \) is a group scheme, by [Lu2] 5.2.6.10, 5.2.6.13, taking the geometric realizations (of simplicial spaces) induces an equivalence

\[
\text{Map}_{\text{Mon}(\text{Spc})}(\Gamma, M(A)) \to \text{Map}_{\text{Spc}}(\Gamma, |M(A)|).
\]

where \( \text{Spc}_\ast \) denote the \( \infty \)-category of pointed spaces ([Lu2] 1.4.2.5). Therefore, this definition agrees with the definition of (framed) derived moduli space of representations as in [GV18] §5. (The geometric realization \(| \cdot |\) is denoted by \( B(\cdot) \) in loc. cit.)

Let us give a presentation of \( \mathcal{R}_{\Gamma,M} \) using the “resolution” of \( \Gamma \) from Corollary \([2.1.4]\) which in particular implies the representability of \( \mathcal{R}_{\Gamma,M} \) as a derived affine scheme.
Example 2.2.3. Lemma 2.1.3 implies that \( R_{\text{FM}(I),M} \cong \text{cl} R_{\text{FM}(I),M} \cong M^I \).

This is consistent with the intuition: since no relation is imposed if \( \Gamma \) is free, there shouldn’t exist non-trivial derived structure of \( \text{cl} R_{\Gamma,M} \) in this case.

**Proposition 2.2.4.** There is a natural isomorphism

\[
R_{\Gamma,M} \cong \lim_{(\text{FFM}/\Gamma)^{\text{op}}} M^I,
\]

where the limit is taken in \( \text{DAff}_k \). As a result, there is the isomorphism in \( \text{CAlg}_k \)

\[(2.4) \quad k[R_{\Gamma,M}] \cong \lim_{\text{FFM}/\Gamma} k[M^I].\]

As mentioned before, \( \text{FFM}/\Gamma \) is not a filtered category, even if \( \Gamma \) is discrete. Therefore, even each \( k[M^I] \) only sits in homological degree zero, this may not be the case for \( k[R_{\Gamma,M}] \).

**Proof.** This follows from Example 2.2.3 and Corollary 2.1.4. \( \square \)

**Remark 2.2.5.** The proposition suggests the following generalization, which will be useful for the discussion of pseudo representations. Let \( A_* : \text{FFM} \to \text{CAlg}_k \) be a functor. We call it an \( \text{FFM} \)-algebra (following [Wg]). We write \( \text{Spec} A_* : \text{FFM}^{\text{op}} \to \text{DAff}_k \) for its opposite, and call it an affine \( \text{FFM} \)-scheme.

For example, every an affine monoid scheme \( M \) over \( k \) defines an \( \text{FFM} \)-algebra by assigning to \( \text{FM}(I) \) the algebra \( k[M^I] = k[R_{\text{FM}(I),M}] \). An \( \text{FFM} \)-algebra \( A_* \) arises in this way if and only if for every \( I \), the map \( A_I \to \otimes_{i \in I} A_{\{i\}} \cong A_{\{i\}}^{\otimes I} \) induced by the inclusion \( \{i\} \subset I \) is an isomorphism.

In any case, for an \( \text{FFM} \)-algebra \( A_* \) and \( \Gamma \in \text{Mon}(\text{Spc}) \), we may define

\[(2.5) \quad R_{\Gamma,\text{Spec} A_*} := \lim_{(\text{FFM}/\Gamma)^{\text{op}}} \text{Spec} A_I, \quad \text{so} \quad k[R_{\Gamma,\text{Spec} A_*}] = \lim_{\text{FFM}/\Gamma} A_I.\]

Now let \( B \in \text{CAlg}_k \). We can attach to it an \( \text{FFM} \)-algebra \( C(\Gamma^I, B) \) sending \( \text{FM}(I) \) to

\[
C(\Gamma^I, B) := \lim_{\text{Map}(\text{FM}(I), I)} B,
\]

the self-product of \( B \) over \( \Gamma^I \) (which is just the \( k \)-algebra of set maps from \( \Gamma^I \) to \( B \) if both are discrete objects). Then the right Kan extension along \( \text{FFM}/\Gamma \to \text{FFM} \) gives a canonical isomorphism

\[(2.6) \quad \text{Map}_{\text{CAlg}_k}(k[R_{\Gamma,\text{Spec} A_*}], B) = \text{Map}_{\text{CAlg}_k}^{\text{FFM}}(A_*, C(\Gamma^I, B)),\]

where the right hand side is calculated in \( \text{CAlg}_k^{\text{FFM}} := \text{Fun}(\text{FFM}, \text{CAlg}_k) \), i.e. is the space of \( \text{FFM} \)-algebra homomorphisms in the sense of [Wg].

**Remark 2.2.6.** There are analogous story by replacing \( \text{FFM} \) by \( \text{FFS} \) or \( \text{FFG} \). We shall not repeat such a remark again.

Let us come back to \( R_{\Gamma,M} \) and discuss certain vector bundles on it. For simplicity, from now on we assume that \( \Gamma \) is discrete, i.e. an object in \( \text{Mon} \). This is enough for our purpose and simplifies the discussions below. As in the preceding discussion, we identify it with a simplicial set via the Milnor construction.

We refer to [Ln3] §25.2.1 for the theory of modules over animated rings (see [CS, 5.1] for some further elaborations). For an animated \( k \)-algebra \( A \), let \( \text{Mod} A \) denote the \( \infty \)-category of \( A \)-modules, and \( \text{Mod}_{\Delta_0} A \) the full subcategory of connective objects. If \( A \) is classical, this is also equivalent to \( \text{Ani}(\text{Mod}_{\Delta_0} A) \), as introduced before. We also call \( A \)-modules as quasi-coherent sheaves on \( \text{Spec} A \).
Now, for a representation \( W \) of \( M \) on a finite projective \( k \)-module, let \( \Gamma W \) denote the (trivial) vector bundle \( k[\mathcal{R}_{\Gamma, M}] \otimes_k W \) on \( \mathcal{R}_{\Gamma, M} \). We sometimes denote \( _{\Gamma W} \) by \( \Gamma W \) for simplicity. Let \( \text{End}(\Gamma W) \in \text{Mon}(\text{Spc}) \) denote the (derived) endomorphism ring of \( \Gamma W \) as a quasi-coherent sheaf. (One may think it is the complete Segal space associated to the full subcategory of \( \text{Mod}_{\leq k[\mathcal{R}_{\Gamma, M}]} \) spanned by \( \Gamma W \).) We will construct a canonical morphism in \( \text{Mon}(\text{Spc}) \)

\[
\Gamma \to \text{End}(\Gamma W).
\]

Note that there is a canonical isomorphism \( \lim_{\text{FFM}/\Gamma} \text{End}(\Gamma W) \to \text{End}(\Gamma W) \) in \( \text{Mon}(\text{Spc}) \). Then by Corollary 2.1.4 it is enough to construct, for every \( u : \text{FM}(I) \to \Gamma \), a morphism \( \text{FM}(I) \to \text{End}(\Gamma W) \), compatible with morphisms in \( \text{FFM}/\Gamma \). We note that this last compatibility can be checked at the ordinary category level.

Next via the inclusion \( \{i\} \subset I \), it is enough to assume that \( I = \{1\} \) and to construct an endomorphism of \( \{1\} \Gamma W \) on \( M \), i.e. a \( k[M] \)-linear endomorphism of \( k[M] \otimes W \). This is nothing by the coaction map

\[
\text{coact} : W \to k[M] \otimes_k W.
\]

This finishes the construction of \( (2.7) \).

**Remark 2.2.7.** (1) Here is a more concrete description of the action \( (2.7) \) of \( \Gamma \) on fibers of \( \Gamma W \). Let \( \text{Spec}\kappa \to \mathcal{R}_{\Gamma, M} \) be a field valued point of \( \mathcal{R}_{\Gamma, M} \), corresponding to a homomorphism \( \rho : \Gamma \to M(\kappa) \). The fiber of \( \Gamma W \) over \( \rho \), usually denoted by \( W_\rho \), is just \( W \otimes_k \kappa \), on which \( \Gamma \) acts via \( \Gamma \overset{\rho}{\to} M(\kappa) \to \text{End}_\kappa(W \otimes_k \kappa) \).

(2) If \( W \) is a representation of \( M^J \) for a finite set \( J \), then \( \Gamma W \) admits an action by \( \Gamma^J \), by first applying the above construct to \( \mathcal{R}_{\Gamma^J, M^J} \) and then pulling the \( \Gamma^J \)-action on \( \Gamma W \) back along the morphism \( \mathcal{R}_{\Gamma, M} \to \mathcal{R}_{\Gamma^J, M^J} \).

We can interpret \( (2.7) \) as a functor of \( \infty \)-categories from \( \Gamma \) (regarded as a category with a unique object *) to the category of quasi-coherent sheaves on \( \mathcal{R}_{\Gamma, M} \) by sending * to \( \Gamma W \).

**Definition 2.2.8.** The “universal” homology of \( \Gamma \) with coefficient in \( W \) is the complex of quasi-coherent sheaves on \( \mathcal{R}_{\Gamma, M} \) defined by

\[
C_* (\Gamma, \Gamma W) := \lim_{\Gamma} \Gamma W.
\]

Since tensor product preserves colimits, the (derived) pullback of \( C_* (\Gamma, \Gamma W) \) along \( \text{Spec}\kappa \to \mathcal{R}_{\Gamma, M} \) given by \( \rho : \Gamma \to M(\kappa) \) as in Remark 2.2.7 is just the complex in \( \text{Mod}_\kappa \) computing \( \lim_{\Gamma} W_\rho \), which is nothing but the usual homology of \( \Gamma \) with coefficient \( W_\rho \).

There is a canonical isomorphism

\[
(2.8) \quad C_* (\Gamma, \Gamma W) \cong \lim_{\text{FFM}/\Gamma} k[\mathcal{R}_{\Gamma, M}] \otimes_{k[M]} C_* (\text{FM}(I), I W)
\]

constructed using Corollary 2.1.4

\[
\lim_{\Gamma} \Gamma W \cong \lim_{\text{FFM}/\Gamma} \lim_{\text{FM}(I)} k[\mathcal{R}_{\Gamma, M}] \otimes_{k[M]} I W
\]

\[
\cong \lim_{\text{FFM}/\Gamma} k[\mathcal{R}_{\Gamma, M}] \otimes_{k[M]} \lim_{\text{FM}(I)} I W.
\]

It is convenient to consider a reduced version of \( C_* \). By definition, there is a natural map \( \Gamma W \to C_* (\Gamma, \Gamma W) \). We denote its fiber in the category of quasi-coherent sheaves on \( \mathcal{R}_{\Gamma, M} \) by \( \overline{C}_*(\Gamma, \Gamma W)[-1] \), so we have the distinguished triangle

\[
(2.9) \quad \overline{C}_*(\Gamma, \Gamma W)[-1] \to \Gamma W \to C_* (\Gamma, \Gamma W) \to .
\]
Then \(2.8\) holds with \(C_s\) replaced by \(\overline{C}_s\). The advantage to consider the reduced version is that we have the following canonical isomorphism

\[(2.10) \quad iW^\oplus I \cong \overline{C}_s(FM(I), iW)[-1],\]

obtained from the calculation of homology of free monoids by the following two-term complex (in degree \([-1, 0]\))

\[\bigoplus_{i \in I} iW \xrightarrow{\oplus i(\gamma_i^{-1})} iW,\]

where \(\gamma_i\) denotes the generator of \(FM(I)\) corresponding to \(i \in I\). In particular, \(\overline{C}_s(FM(I), iW)[-1]\) sits in the abelian category of quasi-coherent sheaves on \(\mathcal{R}_{FM(I), M} \cong M^I\), and is a vector bundle on it.

Now let \(f : FM(I) \to FM(J)\) be a monoid morphism. It induces a morphism between homology \(k[M^J] \otimes_k[M^I] \overline{C}_s(FM(I), iW)[-1] \to \overline{C}_s(FM(J), jW)[-1]\). Under the isomorphism \((2.10)\), it is given by a \(k[M^I]\)-linear map

\[(2.11) \quad iW^\oplus I \to jW^\oplus J,\]

which we now describe more explicitly. Note that every such \(f : FM(I) \to FM(J)\) is compositions of maps of the following two types:

- \(f\) sends generators of \(FM(I)\) to generators or the unit of \(FM(J)\), i.e. \(f\) is induced by a map of pointed sets \(I \cup \{\ast\} \to J \cup \{\ast\}\);
- \(f : FM(\{1, \ldots, n\}) \to FM(\{1, \ldots, n+1\})\) sending \(\gamma_i \to \gamma_i\) for \(i \leq n-1\) and \(f(\gamma_n) = \gamma_n \gamma_{n+1}\).

Therefore, it is enough to understand \((2.11)\) in these two cases separately. Unveiling the construction of \((2.10)\), we see that in the first case, it is given by

\[(2.12) \quad (w_i)_{i \in I} \in iW^\oplus I \mapsto (v_j)_{j \in J} \in jW^\oplus J, \quad v_j = \sum_{i \in f^{-1}(j)} 1 \otimes w_i,\]

and in the second case, it is given by

\[(2.13) \quad (w_i) \in \{1, \ldots, n\}W^\oplus_n \mapsto (v_j) \in \{1, \ldots, n+1\}W^\oplus(n+1), \quad v_i = 1 \otimes w_i, i \leq n, \quad v_{n+1} = \gamma_n(1 \otimes w_n).\]

Using the above description, we can compute the cotangent complex on \(\mathcal{R}_{\Gamma, M}\) when \(M\) is an affine smooth group scheme over \(k\). Let \(\text{Ad}^*\) denote the dual adjoint representation of \(M\) on the dual \(m^*\) of the Lie algebra \(m\) of \(M\).

We recall that for an animated \(k\)-algebra \(A\), the (algebraic) cotangent complex \(\mathbb{L}_A\) is a connective \(A\)-module such that for every \(A \to B\) and a connective \(B\)-module \(V\)

\[\text{Map}_{\text{Mod}_{A}^{\infty}}(\mathbb{L}_A, V) \cong \text{Map}_{\text{CAlg}_{k/B}}(A, B \oplus V),\]

where \(B \oplus V \to B\) denotes the trivial square zero extension of \(B\) by \(V\) in \(\text{CAlg}_{k}\), and \(\text{CAlg}_{k/B}\) denotes the category of animated \(k\)-algebras with a map to \(B\). See \([\text{Ln3} \ 25.3.1, 25.3.2]\) for a detailed account. If \(A\) is a classical smooth \(k\)-algebra, then \(\mathbb{L}_A \cong \pi_0(L_A) = \Omega_A\) is just the Kähler differential of \(A\). If \(A \to B\) is a morphism in \(\text{CAlg}_{k}\), there is a natural morphism \(B \otimes_A \mathbb{L}_A \to \mathbb{L}_B\) in \(\text{Mod}_B\) and the relative cotangent complex \(\mathbb{L}_{B/A}\) is defined as its fiber.

**Proposition 2.2.9.** Assume that \(M\) is an affine smooth group scheme over \(k\). For every \(\Gamma\), the cotangent complex of \(\mathcal{R}_{\Gamma, M}\) is canonically isomorphic to \(\overline{C}_s(\Gamma, \text{Ad}^*)[-1]\).

**Proof.** Note that if \(A = \lim_i A_i\) is a colimit in \(\text{CAlg}_{k}\), then

\[(2.14) \quad \mathbb{L}_A \cong \lim_i (A \otimes_{A_i} \mathbb{L}_{A_i}).\]
We apply this to \( k[\mathcal{R}_{\Gamma,M}] = \lim_{\to}^{\mathcal{F}M/F} k[M^I] \). By comparing (2.8) with (2.14), it is enough to establish, for every \( f : FM(I) \to FM(J) \), the following commutative diagram (in the abelian category of \( k[M^I] \)-modules)

\[
\begin{array}{ccc}
  k[M^J] \otimes_{k[M^I]} (\mathfrak{m}^*)^{\oplus I} & \longrightarrow & (\mathfrak{m}^*)^{\oplus J} \\
  \cong \downarrow & & \downarrow \cong \\
  k[M^J] \otimes_{k[M^I]} \Omega_{M^I/k} & \longrightarrow & \Omega_{M^J}.
\end{array}
\]

Now if we identify \( \Omega_M \) with \( k[M] \otimes \mathfrak{m}^* \) by regarding \( \mathfrak{m}^* \) as the space of right invariant differentials, then the vertical isomorphisms become clear and the commutativity of the diagram follows from (2.12) and (2.13). \( \square \)

**Remark 2.2.10.** Sometimes it is convenient to pass to the linear dual of the cotangent complex of \( \mathcal{R}_{\Gamma,M} \), called the tangent complex of \( \mathcal{R}_{\Gamma,M} \), which is isomorphic to \( \mathcal{C}'(\Gamma, \Gamma\text{Ad})[1] \). Here

\[
C^\ast(\Gamma, \Gamma\text{Ad}) := \lim_{\to}^{\Gamma} \Gamma\text{Ad}
\]

is the cohomology of \( \Gamma \) with coefficient in the adjoint representation \( \text{Ad} \) of \( M \), and \( \mathcal{C}'(\Gamma, \Gamma\text{Ad})[1] \) is its reduced version, i.e. the cofiber of \( C^\ast(\Gamma, \Gamma\text{Ad}) \to \Gamma\text{Ad} \).

Note that if \( \Gamma \) is finitely generated and \( k \) is noetherian, then the non-derived space \( \mathcal{O} \mathcal{R}_{\Gamma,M} \) is of finite type over \( k \). Indeed, by choosing a surjective map \( FM(I) \to \Gamma \), \( \mathcal{O} \mathcal{R}_{\Gamma,M} \) is realized as a closed subscheme of \( \mathcal{O} \mathcal{R}_{FM(I),M} \cong M^I \). Now we discuss similar statements for \( \mathcal{R}_{\Gamma,M} \).

Recall that for a compactly generated \( \infty \)-category \( \mathcal{C} \), an object \( c \) is called almost compact if for every \( n \geq 0 \), \( \tau_{\leq n}c \) is compact in \( \leq_n \mathcal{C} \) ([Lu2 7.2.4.8]). Almost compact objects in \( \mathcal{C} \text{Alg}_k \) are also called almost of finite presentation and for an animated \( k \)-algebra \( A \), almost compact objects in \( \text{Mod}_A^{\infty} \) are also called almost perfect \( A \)-modules. If \( k \) is noetherian, \( A \) is almost of finite presentation over \( k \) if and only if \( \pi_0(A) \) is a finitely generated \( k \)-algebra and each \( \pi_i(A) \) is a finitely generated \( \pi_0(A) \)-module ([Lu4 3.1.5]). In particular, if \( A \) is noetherian, a classical \( k \)-algebra of finite type is almost of finite presentation, when regarded as an animated \( k \)-algebra.

On the other hand, recall that a group (even a monoid) \( \Gamma \) is called of type \( FP_\infty(k) \) if the trivial \( k\Gamma \)-module admits a resolution \( P^\ast \to k \) with each term finite projective \( k\Gamma \)-module, where \( k\Gamma \) denotes the group (or monoid) algebra of \( \Gamma \). For example, finite groups are always of type \( FP_\infty(k) \). More generally, if the classifying space of \( \Gamma \) can be realized as a CW complex with finitely many cells in each degree \( n \geq 0 \) (such a group is called of type \( F_\infty \)), then \( \Gamma \) is of type \( FP_\infty(k) \).

**Proposition 2.2.11.** Assume that \( k \) is noetherian, and \( M \) is a smooth affine group scheme over \( k \). Assume that \( \Gamma \) is finitely generated and is of type \( FP_\infty(k) \). Then \( \mathcal{R}_{\Gamma,M} \) is almost of finite presentation over \( k \).

**Proof.** As \( \Gamma \) is finitely generated, \( \mathcal{O} \mathcal{R}_{\Gamma,M} \) is of finite type. Using [Lu4 3.2.18] and Proposition 2.2.9, it is enough to show that \( \mathcal{C}'(\Gamma, \Gamma\text{Ad}^*)[-1] \) is almost perfect. As \( \Gamma \) is of type \( FP_\infty(k) \), the pullback of this complex to every classical \( k \)-algebra \( A \) is a connective complex with each term finite projective \( A \)-module, and therefore is almost perfect. This implies that \( \mathcal{C}'(\Gamma, \Gamma\text{Ad}^*)[-1] \) is almost perfect by [Lu3 2.7.3.2]. \( \square \)

**Remark 2.2.12.** There are also refined notions such as animated \( k \)-algebras of finite generation of order \( n \) and groups of type \( FP_n(k) \). One can use these notions to formulate a refined version of the above proposition.

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Proposition 2.2.13. Assumptions are as above. Let $d$ denote the relative dimension of $M$ over $k$. In addition, assume that for every field valued point $\operatorname{Spec} \kappa \to \mathcal{R}_{\Gamma,M}$, given by a representation $\rho : \Gamma \to M(\kappa)$, $H_i(\Gamma, \text{Ad}_\rho^*) = 0$ for $i > 2$, and

$$\dim_\kappa \partial \mathcal{R}_{\Gamma,M} \leq d - \dim(-1)^i H_i(\Gamma, \text{Ad}_\rho^*),$$

where $\dim_\kappa \partial \mathcal{R}_{\Gamma,M}$ is the relative dimension of $\partial \mathcal{R}_{\Gamma,M}$ over $k$ at $\kappa$. Then $\mathcal{R}_{\Gamma,M} = \partial \mathcal{R}_{\Gamma,M}$ is a local complete intersection. In this case, it is smooth at a geometric point $\rho \in \mathcal{R}_{\Gamma,M}$ if and only if $\mathcal{R}_{\Gamma,M}$ is flat at $\rho$ over $k$ and $H_2(\Gamma, \text{Ad}_\rho^*) = 0$.

Proof. By our assumption, $\mathcal{R}_{\Gamma,M}$ is almost finitely presented over $k$ and its cotangent complex has Tor-amplitude $\leq 1$. So it is quasi-smooth in the sense of [Li1 3.4.15] (see also [AG16 2.1.3] when $k$ is a characteristic zero field). We choose a surjective map $\mathbb{F}(M) \to \Gamma$, inducing a morphism $\mathcal{R}_{\Gamma,M} \to \mathbb{F}(\mathcal{R}_{\Gamma,M})$. It follows from our assumption that $\dim_\kappa \mathcal{R}_{\Gamma,M}$ is a local complete intersection.

By our assumption, $\mathcal{R}_{\Gamma,M}$ is almost finitely presented over $k$ and its cotangent complex has Tor-amplitude $\leq 1$. So it is quasi-smooth in the sense of [Li1 3.4.15] (see also [AG16 2.1.3] when $k$ is a characteristic zero field). We choose a surjective map $\mathbb{F}(M) \to \Gamma$, inducing a morphism $\mathcal{R}_{\Gamma,M} \to \mathbb{F}(\mathcal{R}_{\Gamma,M})$. It follows from arguments as in loc. cit. that Zariski locally on $M$, meaning after replacing $M$ by an open subscheme $\text{Spec} A \subseteq M$ and $\mathcal{R}_{\Gamma,M}$ by $\text{Spec} B := \text{Spec} A \times_M \mathcal{R}_{\Gamma,M}$, there is a morphism $\text{Spec} A \to \mathbb{A}^m := \text{Spec} \mathbb{K}[x_1, \ldots, x_m]$ such that $\text{Spec} B \cong \text{Spec} A \times_{\mathbb{A}^m} \{0\}$. In particular, $\dim_\kappa \partial \mathcal{R}_{\Gamma,M} \geq \dim_\kappa M^I - m$ at every field valued point of $\text{Spec} B$. On the other hand, the distinguished triangles $B \otimes_A \mathbb{L}_A \to \mathbb{L}_B \to \mathbb{L}_{B/A}$ implies that for every point $\kappa$ of $\text{Spec} B$,

$$\dim_\kappa M^I - m = d - \sum (-1)^i \dim H_i(\Gamma, \text{Ad}_\rho^*).$$

It follows from our assumption that $\dim_\kappa \partial \mathcal{R}_{\Gamma,M} = \dim_\kappa M^I - m$. This implies that $\mathcal{R}_{\Gamma,M} = \partial \mathcal{R}_{\Gamma,M}$ is a local complete intersection.

Finally, $\mathcal{R}_{\Gamma,M}$ is smooth at $\rho$ if and only if it is flat and $\dim(\Omega_{\mathcal{R}_{\Gamma,M}} \otimes \kappa) = 0$. But the last condition is equivalent to $H_2(\Gamma, \text{Ad}_\rho^*) = 0$ by the above equality.

Remark 2.2.15. For an algebraically closed field $\kappa$, its $\kappa$-points of $X_{\Gamma,H}$ classify homomorphisms $\Gamma \to H(\kappa)$ up to $H(\kappa)$-conjugacy. In general, $X_{\Gamma,H} : \mathcal{C}\mathcal{A}l\mathcal{G}_k \to \mathcal{S}pc$ is the étale sheafification of the functor sending $A$ to $\text{Map}_{\mathcal{S}pc}(|\Gamma|, |H(A)|)$ (compare with (2.3)).

Now suppose that $W$ is a representation of $M \times H$ (on a finite projective $k$-module), i.e. the coaction morphism (2.2) is an $H$-module morphism. In this case the vector bundle $\Gamma W$ equipped with the action of $\Gamma$ descends to $\mathcal{R}_{\Gamma,M/H}$, denoted by the same notation. In addition, $C_*(\Gamma, \Gamma W)$ also descends to a complex of quasi-coherent sheaves on $\mathcal{R}_{\Gamma,M/H}$. Indeed, this is clear if $\Gamma = \mathbb{F}(M)$, and the general case reduces to the free case by Corollary 2.1.4. Again, in the example $M = H$ with the conjugation action, the coaction map (2.2) is automatically $H$-equivariant for every $H$-module $W$. In particular, the dual adjoint representation of $H$ gives a vector bundle $\Gamma \text{Ad}^*$ on $X_{\Gamma,H}$ equipped with a $\Gamma$-action. We have the isomorphism

$$\mathbb{L}X_{\Gamma,H} \cong C_*(\Gamma, \Gamma \text{Ad}^*)[-1].$$
This follows from Proposition \[2.2.9\] by comparing \([2.9]\) with the usual distinguished triangle of cotangent complexes related to the morphism \(\tau : \mathcal{R}_{\Gamma,H} \to \mathcal{X}_{\Gamma,H}\).

Our last topic of this subsection is the coarse moduli and moduli of pseudo presentations. Let \(\Gamma, M, H\) be as above. We will assume that \(k\) is noetherian and \(H\) is a connected split reductive group over \(k\). Recall that if \(M = H\) acting on itself by conjugation, the GIT quotient of \(\mathcal{R}_{\Gamma,H}\) by \(H\) is usually called the \(H\)-character variety of \(\Gamma\) (at least if \(\Gamma\) is finitely generated and \(k\) is a field).

Similarly, in our more general context, we can make the following definition.

**Definition 2.2.16.** The character variety of \(\mathcal{R}_{\Gamma,M/H}\), denoted by \(\mathcal{C}_{\Gamma,M/H}\), is the geometric realization of \([2.15]\) in \(\text{DAff}_k\). So \(k[\mathcal{C}_{\Gamma,M/H}] = k[\mathcal{R}_{\Gamma,M}]^H\) is the \(H\)-invariants of \(k[\mathcal{R}_{\Gamma,M}]\) in \(\text{CAlg}_k\) (i.e. totalization of the cosimplicial objects in \(\text{CAlg}_k\) obtained from \([2.15]\) by passing to the opposite).

If \(\mathcal{R}_{\Gamma,M}\) is classical, then \(\mathcal{C}_{\Gamma,M/H}\) is classical and is isomorphic to the usual GIT quotient \(\mathcal{R}_{\Gamma,M} / / H\) of \(\mathcal{R}_{\Gamma,M}\) by \(H\) in \(\text{Aff}_k\), so \(k[\mathcal{C}_{\Gamma,M/H}]\) isomorphic to the non-derived \(H\)-invariants of \(k[\mathcal{R}_{\Gamma,M}]\). In general if \(\mathcal{R}_{\Gamma,M}\) is not classical, the underlying \(E_\infty\)-algebra of \(k[\mathcal{C}_{\Gamma,M/H}]\) can be identified with \(\tau_{\geq 0}(\mathcal{R}_{\Gamma,M}/ / H, O)\), where \((\mathcal{R}_{\Gamma,M}/ / H, O)\) is the ring of global functions of \(\mathcal{R}_{\Gamma,M}/ / H\), which is an \(E_\infty\)-algebra isomorphic to the \(H\)-invariants of \(k[\mathcal{R}_{\Gamma,M}]\) in the category of \(E_\infty\)-algebras.

Now, let \(k[M/^\circ/H]\) be the \(\text{FFM}\)-algebra sending \(\text{FM}(\Gamma)\) to \(k[\mathcal{C}_{\text{FM}(\Gamma),M/H}] \cong k[M^\circ/H]\). Its opposite is the \(\text{FFM}\)-scheme \(\text{FM}(\Gamma) \mapsto M^\circ/H\).

**Definition 2.2.17.** The moduli of pseudo representations of \(\mathcal{R}_{\Gamma,M/H}\) is the derived affine scheme over \(k\) as defined by \(\mathcal{R}_{\Gamma,M/^\circ/H} := \lim_k \mathcal{FFM}/\Gamma(M^\circ/H)\) as in \([2.5]\). We call \(k[\mathcal{R}_{\Gamma,M/^\circ/H}] = \lim_k k[M^\circ/H]\) the excursion algebra associated to \(\mathcal{R}_{\Gamma,M/H}\).

**Remark 2.2.18.** If \(M = H\) with the adjoint action, by \([2.6]\) giving a homomorphism \(k[\mathcal{R}_{\Gamma,M/^\circ/H}] \to A\) (say \(A\) classical) is the same as giving an \(H(A)\)-valued pseudo-representation of \(\Gamma\), in the sense of Lafforgue [La18] 11.3, 11.7. This justifies the choice of our terminology.

Tautologically, there are natural morphisms
\[\text{(2.16)}\]
\[\text{Tr} : \mathcal{R}_{\Gamma,M/H} \to \mathcal{C}_{\Gamma,M/H} \to \mathcal{R}_{\Gamma,M/^\circ/H}.\]
If \(M = H\) with the adjoint action, this is just the map sending a representation to its associated pseudo-representation. The induced map of ring of regular functions is explicitly given by
\[\text{(2.17)}\]
\[k[\mathcal{R}_{\Gamma,M/^\circ/H}] = \lim_f \mathcal{FFM}/\Gamma(M^\circ) \to (\lim_f k[M^\circ])^H = k[\mathcal{C}_{\Gamma,M/H}].\]

**Remark 2.2.19.** If \(k\) is a field of characteristic zero, \([2.17]\) is an isomorphism as taking \(H\)-invariants commutes with arbitrary colimits, so \(\mathcal{C}_{\Gamma,M/H} \to \mathcal{R}_{\Gamma,M/^\circ/H}\) is an isomorphism. We have no reason to believe this is the case if char \(k = p > 0\). However, If \(k\) is a perfect field, and \(\mathcal{R}_{\Gamma,M}\) is \(m\)-truncated from some \(m\) (e.g. quasi-smooth), then the induced map \(\mathcal{C}_{\Gamma,M/H}(k) \to \mathcal{R}_{\Gamma,M/^\circ/H}(k)\) is still a bijection.

2.3. Some examples. For later applications, in this subsection we apply the general discussions in the previous subsection to some special cases. Some similar discussions also appear in [DIKM]. We assume that \(k\) is a Dedekind domain (or a field), and the neutral connected component \(M^\circ\) of \(M\) reductive over \(k\).

The following two statements easily follow from Proposition 2.2.13

**Proposition 2.3.1.** Assume that \(\Gamma\) is a finitely generated group and of type \(FP_\infty(k)\) and \(M\) is (finite) étale over \(k\). Then \(\mathcal{R}_{\Gamma,M} = \mathcal{dR}_{\Gamma,M}\) is (finite) étale over \(k\).

**Proposition 2.3.2.** Assume that \(\Gamma\) is finite whose order is invertible in \(k\). Then \(\mathcal{R}_{\Gamma,M} = \mathcal{dR}_{\Gamma,M}\) is smooth of finite type over \(k\). Let \(\rho : \Gamma \to M(\mathcal{O})\) be a homomorphism with \(\mathcal{O}\) an étale \(k\)-algebra,
and let \(Z_M(\rho)\) be its centralizer in \(M_\mathcal{O}\). Then the morphism \(M_\mathcal{O}/Z_M(\rho) \to R_{\Gamma,M} \otimes_k \mathcal{O}\) induced by the conjugation of \(\rho\) by \(M\) is an open and closed embedding.

**Remark 2.3.3.** We keep the assumption of the proposition. In addition, assume that \(M/M^0\) is finite étale over \(k\). Let \(E\) be the fractional field of \(k\). We expect that every conjugacy class of homomorphisms from \(\Gamma \to M(\overline{E})\) admits a representative defined over a finite étale extension of \(k\). If so, there will exist a finite étale extension \(\mathcal{O}\) of \(k\), such that

\[ R_{\Gamma,M} \otimes \mathcal{O} \simeq \bigcup \chi M_\mathcal{O}/Z_M(\rho), \]

where \(\rho\) ranges a set of representatives over \(\mathcal{O}\) of homomorphisms from \(\Gamma\) to \(M(\overline{E})\) up to conjugacy.

We cannot prove this in general. But this is the case if \(M = GL_m\) or if \(\Gamma\) is solvable. The \(GL_m\) case follows from the fact that \(k\Gamma\) is a finite dimensional semisimple algebra over \(k\). For the case \(\Gamma\) solvable, let \(T\) be a maximal torus of \(M\) over \(k\). Then up to conjugation we may assume that \(\rho: \Gamma \to M(\overline{E})\) factors as \(\rho: \Gamma \to N_M(T)(\overline{E})\), where \(N_M(T)\) is the normalizer of \(T\) in \(M\). This follows from [BS53, thm. 2] if \(\text{char } E = 0\) and the general case follows by a lifting argument. Now, let \(m\) be the order of \(\Gamma\). Let \(N_M(T)[m]\) denote the closed subscheme of elements of \(N_M(T)\) of order dividing \(m\). As this is a finite étale scheme over \(k\), our claim follows.

**Example 2.3.4.** If the order of \(\Gamma\) is not invertible in \(k\), then the situation is more complicated. Let \(k = \mathbb{F}_p\). \(R_{\mathbb{Z}/p,\mathbb{G}_m} \neq \mathfrak{d}R_{\mathbb{Z}/p,\mathbb{G}_m} \cong \mathbb{G}_m[p]\) (which is not smooth).

We have the following result about the moduli of pseudo-representations of finite groups over \(k\).

**Proposition 2.3.5.** Assume that \(\Gamma\) is finite, and \(M/M^0\) is finite étale over \(k\). Let \(H = M^0\) act on \(M\) by conjugation. Then \(\mathfrak{d}R_{\Gamma,M^0,H}\) is finite over \(k\). If the order of \(\Gamma\) is invertible in \(k\), then \(\mathfrak{d}R_{\Gamma,M^0,H}\) is finite étale over \(k\).

**Proof.** The second assertion follows from the first by combining Proposition 2.3.2 with the fact that \(R_{\Gamma,M} \to \mathfrak{d}R_{\Gamma,M^0,H}\) is surjective. So we only need to prove that \(\mathfrak{d}R_{\Gamma,M^0,H}\) is finite over \(k\).

We first consider the case \(M = GL_m\). Let \(\chi_i \in k[GL_m]^GL_m\) be the character of the \(i\)th wedge representation of \(GL_m\). For each \(\gamma \in \Gamma\), let \(\chi_i,\gamma \in k[\mathfrak{d}R_{\Gamma,M^0,H}]\) be the image of \(\chi_i\) under the map \(k[GL_m]^GL_m \to k[\mathfrak{d}R_{\Gamma,M^0,H}]\) corresponding to the map \(FM\{1\} \to \Gamma\) induced by \(\gamma\). As the \(FM\)-algebra \(k[GL_m]^GL_m\) is generated by \(\chi_i\) (this is proved in [Dj02] when \(k\) is a field but the arguments work for \(k\) being a Dedekind domain), \(k[\mathfrak{d}R_{\Gamma,M^0,H}]\) is generated by these \(\chi_i,\gamma\) as a \(k\)-algebra. Given a positive integer \(r\), \(\chi_{i,\gamma}^r\) can be expressed as a \(k\)-linear combinations of \(\{\chi_{j,s,\gamma}, j \leq m, s \leq r\}\). As \(\Gamma\) is finite, this implies that each \(\chi_{i,\gamma}\) is integral over \(k\). Therefore, \(k[\mathfrak{d}R_{\Gamma,M^0,H}]\) is finite over \(k\).

Now assume that \(M\) is general. We choose a faithful representation \(\phi: M \to GL_m\) over \(k\). Then the proposition follows if we show that the induced map \(\phi_n: M^n/H \to GL_m^n/GL_m\) is finite for any \(n\), as this will imply that \(k[\mathfrak{d}R_{\Gamma,M^0,H}]\) is finite over \(k[\mathfrak{d}R_{\Gamma,GL_m,H}]\).

Passing to a finite étale extension of \(k\) we may assume that \(M/M^0\) is finite constant. Choose \(g = (a_1,\ldots,a_n) \in (M/M^0)^n\) and let \(M^n_g\) be the corresponding connected component in \(M^n\), on which \(H\) still acts. It is easy to see that \(\phi_{n,g}: M^n_g/H \to GL_m^n/GL_m\) is a quasi-finite morphism between finite type (integral) normal schemes over \(k\), and therefore admits the factorization \(M^n_g/H \overset{\gamma}{\to} Z \overset{\pi}{\to} X \overset{j}{\to} GL_m^n/GL_m\) with \(j\) open, \(\pi\) finite surjective, and \(i\) closed embedding, and \(Z\) affine normal.

If \(k\) has a characteristic zero point, then over the generic point of \(k\), \(j\) is an isomorphism by [V96]. Now let \(s\) be a closed point of \(k\), and let \(\eta\) be the generic point of \(X_s\). Its preimage in \(M^n_g/H\) is the generic point \(\tilde{\eta}\) of \((M^n_g/H)_s\). Then the irreducibility and normality of \((M^n_g/H)_s\) implies that the complement of \(j\) has codimension \(\geq 2\) and therefore is empty. Therefore, \(\phi_{n,g}\) is finite.

If the fractional field \(E\) of \(k\) is of characteristic \(p > 0\), then we can lift \(M \to GL_m\) to \(W(\overline{E})\). The above argument implies that \(\phi_{n,g}\) if finite over \(W(\overline{E})\) and therefore over \(E\), and repeating the argument implies that it is finite over \(k\).
Remark 2.3.6. If the order of $\Gamma$ is not invertible in $k$, then unlike the expectation in Remark 2.3.3 $\cl{R}_{\Gamma,M}$ is complicated. Let us assume that $k = \mathbb{F}_\ell$. Then it follows from [BHKT, 4.5] that $\mathbb{F}_\ell$-points of $\cl{R}_{\Gamma,M,0}/H$ classifies $M$-completely reducible representation of $\Gamma$ (in the sense of [BHKT, 3.5]) up to $H$-conjugacy. Then the above proposition implies that there is a decomposition

$$\cl{R}_{\Gamma,M} = \sqcup_{\rho_0} \cl{R}^{\rho_0}_{\Gamma,M}$$

into open and closed subschemes, where $\rho_0$ ranges over $H$-conjugacy classes of $M$-completely reducible representation of $\Gamma$, such that for every geometric point $x \in \cl{R}^{\rho_0}_{\Gamma,M}$, the semisimplification of $\rho_x : \Gamma \to M$ is $\rho_0$. Note that, however, $\rho_x$ itself may not be $M$-completely reducible. For example, if $\rho_0$ is the trivial representation, then $\cl{R}^{\rho_0}_{\Gamma,M}$ classifies those $\rho_x$ such that the image $\rho_x(\Gamma)$ is contained in a unipotent subgroup of $M$.

We remark the above decomposition is a toy model of decomposition of the stack of Langlands parameters as we shall see later.

Let $q = p^r$ for some $r \in \mathbb{Z}_{>0}$. We consider the following group (sometimes called the $q$-tame group)

$$(2.18) \quad \Gamma_q := \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^q \rangle.$$ 

It contains a normal subgroup $\tau^{\mathbb{Z}[1/p]} \langle m \rangle$ and the quotient of $\Gamma_q$ by this subgroup is $\langle \sigma \rangle \cong \mathbb{Z}$.

Proposition 2.3.7. Let $k$ be a Dedekind domain over $\mathbb{Z}[1/p]$. Then $\cl{R}_{\Gamma,q,M} = \cl{dR}_{\Gamma,q,M}$. It is equidimensional of dimension $\dim M^q$, flat over $k$, and is a local complete intersection.

Proof. Except $\cl{R}_{\Gamma,M} = \cl{dR}_{\Gamma,M}$, this is proved in [LT+] Prop. E.4.2] in this generality. We briefly review some ingredients needed later, and explain how to apply Proposition 2.2.13 in this situation.

Let $\chi : M \to M//M = \text{Speck}[M]^M$ denote the adjoint quotient map. For every $m \in \mathbb{Z}_{>0}$, the $m$-power morphism $M \to M$, $h \mapsto h^m$ is equivariant with respect to conjugation action and therefore induces a morphism

$$[m] : M//M \to M//M.$$ 

Let $(M//M)^{[m]}$ denote the fixed point subscheme of $[m]$, and let $M^{[m]} := \chi^{-1}((M//M)^{[m]})$, which is a closed subscheme of $M$ stable under conjugation. Note that the morphism $\cl{R}_{\Gamma,q,M} \to M$ induced by the inclusion $\langle \tau \rangle \subset \Gamma_q$ factors through $\cl{R}_{\Gamma,q,M} \to M^{[0]} \subset M$.

As explained in [LT+] Prop. E.4.2], over an algebraically closed field $K$ over $k$, there are only finitely many conjugacy classes in $M^{[0]}(K)$, and from this one deduces that over $K, \dim \cl{dR}_{\Gamma,M} \otimes K = \dim M_K$. It follows that $\dim \cl{dR}_{\Gamma,M} = \dim M$.

On the other hand, we have the following resolution of $k$ as right $k\Gamma_q$-modules

$$(2.19) \quad 0 \to k\Gamma_q \xrightarrow{(1-(\sum_i <q \tau_i)\sigma, \sigma^{-1})} k\Gamma_q \oplus k\Gamma_q \xrightarrow{(1-\tau, 1-\sigma)} k\Gamma_q \to k \to 0.$$ 

Therefore, $H_i(\Gamma_q, \text{Ad}_\rho^*) = 0$ for every $i > 2$ and $\dim(-1)^i H_i(\Gamma_q, \text{Ad}_\rho^*) = 0$. We now apply Proposition 2.2.13 to conclude that $\cl{R}_{\Gamma,M} = \cl{dR}_{\Gamma,M}$ is a local complete intersection. As fibers of $\cl{dR}_{\Gamma,M}$ over $k$ are equidimensional of the same dimension, $\cl{dR}_{\Gamma,M}$ is flat over $k$.

Remark 2.3.8. The argument in Proposition 2.3.7 implies that for a not necessarily reductive affine algebraic group $M$ over a field $k$, if $\dim \cl{dR}_{\Gamma,q,M} > \dim M$, then $\cl{R}_{\Gamma,q,M} \neq \cl{dR}_{\Gamma,q,M}$. For example, let $M = B_n$ be the group of determinant one $n \times n$-upper triangular matrices. Then the derived structure on $\cl{dR}_{\Gamma,q,B_n}$ is non-trivial when $n$ is large, even for $r > 0$ and $k = \mathbb{C}$. Indeed, the underlying classical scheme $\cl{dR}_{\Gamma,q,B_n}$ has dimension $> \dim B_n$. This is essentially due to the fact that the number of $B_n$-orbits in the set of strictly upper triangular matrices is not finite when

\[\text{The prototype of the argument is probably due to D. Helm.}\]
is equidimensional of dimension \( \text{dim} \). Proposition 2.3.9. scheme of \( R \) compact Riemann surface. Then \( \mathcal{R}_{\Gamma_g,M} = \text{cl} \mathcal{R}_{\Gamma_g,M} \) if \( g \geq 2 \) and \( M \) is semisimple. Otherwise, \( \mathcal{R}_{\Gamma_g,M} \) has non-trivial derived structure. In particular, the scheme \( \mathcal{R}_{\Gamma,1,M} \), usually called the commuting scheme of \( M \), is always derived.

Finally, we put Proposition 2.3.2 and 2.3.7 together.

**Proposition 2.3.9.** Let \( \Gamma = \mathbb{Q} \rtimes \Gamma_q \) where \( \mathbb{Q} \) is a finite \( p \)-group. Let \( k = \mathbb{Z}[1/p] \) and assume that \( M/M^0 \) is finite étale over \( k \). Then \( \mathcal{R}_{\Gamma,M} \) is classical, of finite type, and flat over \( k \). In addition, it is equidimensional of dimension \( \text{dim} M \), and is a local complete intersection.

**Proof.** The inclusion \( Q \subset \Gamma \) induces a morphism \( \mathcal{R}_{\Gamma,M} \to \mathcal{R}_{Q,M} \). Using Proposition 2.2.13 and the fact that \( H_i(\Gamma, \text{Ad}^+_\rho) \cong H_i(\Gamma_q, (\text{Ad}^+_\rho)^{p(Q)}) \), it is enough to show that for every \( \rho_0 : Q \to M(\mathcal{O}) \) defined over some étale \( \mathbb{Z}[1/p] \)-algebra \( \mathcal{O} \),

\[
\text{cl} \mathcal{R}_{\Gamma,M}^{\rho_0} := \text{cl} \mathcal{R}_{\Gamma,M} \times \text{cl} \mathcal{R}_{Q,M} \{ \rho_0 \}
\]

is of finite type and flat over \( \mathcal{O} \), is equidimensional of dimension \( = \text{dim} Z_M(\rho_0) \), and is a local complete intersection.

Let \( N_M(\rho_0) \) be the normalizer of \( \rho_0 \) in \( M_\mathcal{O} \). It is a smooth affine group scheme over \( \mathcal{O} \) and \( N_M(\rho_0)^0 = Z_M(\rho_0)^0 \) is connected reductive ([PY02] thm. 2.1]). The quotient \( \pi_0(N_M(\rho_0)) = N_M(\rho_0)/N_M(\rho_0)^0 \) is étale over \( \mathcal{O} \), which acts on the constant group \( \rho_0(Q) \) over \( \mathcal{O} \). Consider the subfunctor \( U \subset \mathcal{R}_{\Gamma_q,\pi_0(N_M(\rho_0))} \) consisting of those \( \rho : \Gamma_q \to \pi_0(N_M(\rho_0)) \) such that the composition \( \Gamma_q \to \pi_0(N_M(\rho_0)) \to \text{Aut}(\rho_0(Q)) \) is compatible with the action of \( \Gamma_q \) on \( Q \). This is open in \( \mathcal{R}_{\Gamma_q,\pi_0(N_M(\rho_0))} \). Then \( \text{cl} \mathcal{R}_{\Gamma,M}^{\rho_0} \cong \text{cl} \mathcal{R}_{\Gamma_q,N_M(\rho_0)} \times \mathcal{R}_{\Gamma_q,\pi_0(N_M(\rho_0))} U \) is open. Therefore, the desired statement follows from Proposition 2.3.7.

\( \square \)

### 2.4. Continuous representations and deformation spaces.

In Langlands program, we need to study continuous representations of profinite groups, rather than arbitrary representations of abstract groups. We address this issue in this subsection. Let \( k = \mathcal{O}_E \) be a finite extension of \( \mathbb{Z}_\ell \), and \( \varpi \) a uniformizer of \( \mathcal{O}_E \). Let \( \kappa_E \) denote the residue field. Let \( M \) be a smooth affine group scheme over \( \mathcal{O}_E \) and let \( H \) be smooth affine group scheme over \( \mathcal{O}_E \) that acts on \( M \) by group automorphisms. Let \( M_n = M \otimes \mathcal{O}_E/\varpi^n \), \( H_n = H \otimes \mathcal{O}_E/\varpi^n \).

We consider locally profinite groups \( \Gamma \) such that open normal subgroups form a neighborhood base at the identity and the quotient of \( \Gamma \) by any open normal subgroup is finitely generated. Examples include Galois groups, as well as Weil groups of non-archimedean local fields and global function fields. Then we may regard \( \Gamma \) as a pro-object in the category of finitely generated monoids by writing \( \Gamma = \lim \Gamma_i \) with each \( \Gamma_i \) discrete and finitely generated.

The embedding \( \text{Mon} \to \text{Ani}(\text{Mon}) \) extends to an embedding of the categories of pro-objects. Then for each \((i,n)\), we have the framed representation space \( \mathcal{R}_{\Gamma_i,M_n} \) over \( \mathcal{O}_E/\varpi^n \), as defined in Definition 2.2.1. For \( i \leq i', n \leq n' \), the map \( \mathcal{R}_{\Gamma_i,M_n} \to \mathcal{R}_{\Gamma_{i'},M_{n'}} \) is a closed embedding of derived schemes almost of finite presentation over \( \mathcal{O}_E/\varpi^n \). Roughly speaking, the representation space \( \mathcal{R}_{\Gamma,M}^c \) classifying continuous homomorphisms from \( \Gamma \) to \( M \) is the inductive limit of these \( \mathcal{R}_{\Gamma_i,M_i} \)'s. To be precise, we use the following (non-standard) definition.

**Definition 2.4.1.** Let \( I \) be a filtered category and \( I \to \text{DAff}_k \) be a filtered diagram of derived affine schemes over \( k \) with transition maps closed embedding. Then we define

\[
\lim_i X_i : \text{CAlg}_k \to \text{Spc}, \quad (\lim_i X_i)(A) = \lim_m \lim_i X_i(\tau_{\leq m} A),
\]
where we recall \( \tau_{\leq m} : \mathbf{CAlg}_k \to \leq_n \mathbf{CAlg}_k \) is the \( m \)th truncation functor. By definition \( \lim_i X \) is nilcomplete ([Lu4, 3.4.1]) (or sometimes called convergent). We call a functor \( \mathbf{CAlg}_k \to \mathbf{Spc} \) of the above form a derived affine ind-scheme over \( k \).

For example, let \( \text{Spf} \mathcal{O}_E := \varinjlim_n \text{Spec} \mathcal{O}_{E^n} / \varpi^n \). Then \( \text{Spf} \mathcal{O}_E(A) \) consists of those \( \mathcal{O}_E \)-algebras \( A \) in which \( \ell \) is nilpotent. (So our notation is consistent with the one in [Lu3, 8.1.5.2].)

**Remark 2.4.2.** Note that a priori \( (\lim_i X_i)(A) \) may not be isomorphic to \( \lim_i X_i(A) \) if \( A \) is not \( m \)-truncated for some \( m \). But there always exists another presentation \( \lim_i X_i = \lim_j Y_j \) with \( Y_j \) derived affine schemes and transitioning maps closed embeddings, such that \( (\lim_i X_i)(A) = \lim_{i,j} Y_j(A) \) for all \( A \).

Now we define the representation spaces over \( \mathcal{O}_E / \varpi^n \) and over \( \text{Spf} \mathcal{O}_E \) as

\[
\mathcal{R}^c_{\Gamma,M_n} := \varinjlim_i \mathcal{R}_{\Gamma_i,M_n}, \quad \mathcal{R}^c_{\Gamma,M} := \varinjlim_n \mathcal{R}^c_{\Gamma,M_n} = \varinjlim_{i,n} \mathcal{R}_{\Gamma_i,M_n}
\]

and the representation stacks over \( \mathcal{O}_E / \varpi^n \) and over \( \text{Spf} \mathcal{O}_E \) as

\[
\mathcal{R}^c_{\Gamma,M_n} / H_n, \quad \mathcal{R}^c_{\Gamma,M} / H^\wedge_n := \varinjlim_n \mathcal{R}^c_{\Gamma,M_n} / H_n.
\]

Here \( H^\wedge \) stands for the \( \varpi \)-adic completion of \( H \).

If \( A \) is a classical \( \mathcal{O}_E \)-algebra in which \( \varpi \) is nilpotent, then \( A \)-points of \( \mathcal{R}^c_{\Gamma,M} \) form the set of continuous homomorphisms from \( \Gamma \) to \( M(A) \) (equipped with the discrete topology). More generally, for a classical \( \mathcal{O}_E \)-algebra, let \( \text{Spf} A = \varinjlim_n \text{Spec} \mathcal{O}(A / \varpi^n) \) be over \( \text{Spf} \mathcal{O}_E \). Then

\[
\text{Map}(\text{Spf} A, \mathcal{R}^c_{\Gamma,M}) = \varinjlim_n \mathcal{R}^c_{\Gamma,M}(A / \varpi^n).
\]

**Remark 2.4.3.** We may take the rigid generic fiber of \( \mathcal{R}^c_{\Gamma,M} \), or the adic space over \( \text{Spa}(E, \mathcal{O}_E) \) (as in [SW13, 2.2]), denoted by \( \mathcal{R}^c_{\Gamma,M}^{\text{ad}} \). It is the sheafification (with respect to the Zariski topology on the category of affinoid \( (E, \mathcal{O}_E) \)-algebras) of the presheaf on the category of affinoid \( (E, \mathcal{O}_E) \)-algebras:

\[
(A, A^+) \mapsto \varinjlim_{A_0 \subset A^+} \mathcal{R}^c_{\Gamma,M}(A_0) = \varinjlim_{A_0 \subset A^+} \varinjlim_j \mathcal{R}^c_{\Gamma,M}(A_0 / \varpi^j),
\]

where \( A_0 \) range over open and bounded subrings of \( A^+ \). For example, if \( \Gamma \) is a profinite group, then \( E \)-points of \( \mathcal{R}^c_{\Gamma,M}^{\text{ad}} \) are the set of continuous homomorphisms from \( \Gamma \) to \( M(E) \), where the latter is equipped with the usual \( \ell \)-adic topology. So \( \mathcal{R}^c_{\Gamma,M}^{\text{ad}} \) probably coincides with the space considered in [An, §2].

There is a definition of cotangent complex for a very general class of functors \( F : \mathbf{CAlg}_k \to \mathbf{Spc} \) (e.g. see [Lu4]). But as \( \mathcal{R}^c_{\Gamma,M} \) is in general just an ind-scheme, the cotangent complex is in general just a pro-object in the category of quasi-coherent sheaves on \( \mathcal{R}^c_{\Gamma,M} \). Therefore, it is more convenient to pass to its dual to consider the tangent complex (Remark 2.2.10). For every \( \rho : \text{Spec} A \to \mathcal{R}^c_{\Gamma,M_n} \), the tangent space of \( \mathcal{R}^c_{\Gamma,M_n} \) at \( \rho \) is an \( A \)-module \( \mathbb{T}_\rho \mathcal{R}^c_{\Gamma,M_n} \) characterized by the existence of a canonical equivalence

\[
\Omega^\infty(\mathbb{T}_\rho \mathcal{R}^c_{\Gamma,M_n} \otimes_A V) \simeq \text{Map}(\text{Spec}(A \oplus V), \mathcal{R}^c_{\Gamma,M_n}) \times_{\text{Map}(\text{Spec} A, \mathcal{R}^c_{\Gamma,M_n})} \{ \rho \},
\]

whenever \( V \) is a perfect, connective \( A \)-module. Then by Proposition 2.2.9 and Remark 2.2.10 if \( M \) is a smooth affine group scheme over \( \mathcal{O}_E \) and \( A \) is \( m \)-truncated from some \( m \), then

\[
(2.20) \quad \mathbb{T}_\rho \mathcal{R}^c_{\Gamma,M_n} \cong \varinjlim_i \mathcal{O}^*_c(\Gamma_i, \text{Ad}_\rho)[1] = \mathcal{O}^*_c(\Gamma, \text{Ad}_\rho)[1].
\]
Therefore, (\(A\))

Next we relate \(R^c_{\Gamma,M}\) with the usual deformation space (and its derived version as in \([GV18]\)). In the rest of this subsection, we assume that \(M\) is an affine smooth group scheme over \(O_E\).

We fix a closed point \(x\) of \(\mathcal{R}^c_{\Gamma,M}\), corresponding to \(\bar{\rho} : \Gamma \to M(\kappa)\), where \(\kappa\) is the residue field of \(x\), which is algebraic over \(\kappa_E\). Let \(\text{Art}_{O_{E,\kappa}}\) denote the category of local Artinian \(O_{E,\kappa}\)-algebras with residue field algebraic over \(\kappa\), and \(\text{CAlg}_{O_{E,\kappa}} \subset \text{CAlg}_{O_E}\) the \(\infty\)-category of animated \(O_{E,\kappa}\)-algebras \(A\), such that \(\pi_0(A) \in \text{Art}_{O_{E,\kappa}}\), and such that \(\bigoplus_n \pi_n(A)\) is a finitely generated \(\pi_0(A)\)-module.

Following \([Lu3\,8.1.6.1]\), we denote the formal completion \((\mathcal{R}^c_{\Gamma,M})_x\) of \(\mathcal{R}^c_{\Gamma,M}\) at \(x\) as the functor sending an animated ring \(A\) over \(O_E\) to the subspace of \((\mathcal{R}^c_{\Gamma,M})(A)\) consisting of those \(\text{Spec}A \to \mathcal{R}^c_{\Gamma,M}\) such that every point of \(\text{Spec}(\pi_0(A))\) maps to \(x\). Its restriction to \(\text{CAlg}_{O_{E,\kappa}} \subset \text{CAlg}_{O_E}\), also denoted by \(\text{Def}^c_{\bar{\rho}}\), is the functor

\[
\text{CAlg}_{O_{E,\kappa}} \to \text{Spc}, \quad A \mapsto \mathcal{R}^c_{\Gamma,M}(A) \times \mathcal{R}^c_{\Gamma,M}(\kappa_A) \{\bar{\rho}\}, \quad \mathcal{R}^c_{\Gamma,M}(A) = \lim_{i,n} \mathcal{R}_{\Gamma_i,M_n}(A).
\]

This recovers the deformation functor defined in \([GV18\,\S 5]\). Its further restriction to \(\text{Art}_{O_{E,\kappa}}\), denoted by \(\text{Def}^c_{\bar{\rho}}\), is identified with the functor

\[
\text{Art}_{O_{E,\kappa}} \to \text{Sets}, \quad A \mapsto \left\{ \text{Continuous homomorphism } \rho : \Gamma \to M(A) \mid \rho \otimes_A \kappa_A = \bar{\rho} \otimes_\kappa \kappa_A \right\}.
\]

This is the classical framed deformation space of \(\bar{\rho}\).

Similarly, we have \((\mathcal{R}_{\Gamma_i,M_n})^\wedge_x\). By \([Lu3\,8.1.2.2]\), each \((\mathcal{R}_{\Gamma_i,M_n})^\wedge_x \simeq \lim_{i,n} \text{Spec}A_j\) is represented by a derived affine ind-scheme with \(A_j \in \text{CAlg}_{O_{E,\kappa}}\), and

\[
(\mathcal{R}^c_{\Gamma,M})^\wedge_x \simeq \lim_{i,n}(\mathcal{R}_{\Gamma_i,M_n})^\wedge_x.
\]

Therefore, \((\mathcal{R}^c_{\Gamma,M})^\wedge_x\) is also represented by a derived affine ind-scheme over \(\text{Spf}O_E\). Combining the above discussions with \([2.20]\), we recover the following statement from \([GV18]\).

**Proposition 2.4.4.** The functor \(\text{Def}^c_{\bar{\rho}}\) is prorepresentable, whose tangent complex is \(\mathcal{C}_{cts}^{\wedge}(\Gamma, \Gamma \text{Ad})[1]\).

Let \(H^\wedge\) denote the formal completion of \(H\) at the unit of \(H_\kappa\). We also define

\[
\text{Def}^c_{\bar{\rho},H} = \text{Def}^c_{\bar{\rho}} / H^\wedge : \text{CAlg}_{O_{E,\kappa}} \to \text{Spc}.
\]

If \(A\) is a classical artinian ring with residue field \(\kappa_A\) algebraically closed, then \(\text{Def}^c_{\bar{\rho},H}(A)\) is the groupoid with objects being continuous homomorphisms \(\rho : \Gamma \to M(A)\) together with an element \(g \in H(\kappa_A)\) such that \(g(\rho \otimes A \kappa_A) = \bar{\rho} \otimes_\kappa \kappa_A\) and morphisms between \((\rho_1, g_1)\) and \((\rho_2, g_2)\) being elements \(g \in H(A)\) such that \(g\rho_1 = \rho_2\) and \(g_2 \bar{g} = g_1\) where \(\bar{g}\) denotes the image of \(g\) under the reduction map \(H(A) \to H(\kappa_A)\). This is the classical deformation space of \(\bar{\rho}\).

While the formal completion of \(\mathcal{R}^c_{\Gamma,M}\) at a closed point gives the usual (framed) deformation spaces, the global geometry of \(\mathcal{R}^c_{\Gamma,M}\) is usually poorly behaved, as its closed points usually do not “connect” into a good family.

**Example 2.4.5.** Let us the simplest case when \(\Gamma = \widehat{\mathbb{Z}}\). If \(M = \mathbb{G}_m\), then \(\mathcal{R}^c_{\Gamma,M}\) is just the union of all torsion points of \(\mathbb{G}_m\), and therefore is isomorphic to \(\sqcup_x (\mathbb{G}_m)^\wedge_x\), where \(x\) ranges over all closed points of \(\mathbb{G}_m/\mathbb{O}_E\). So this space is quite disconnected! For a slightly more complicated example, we let \(M\) be a split connected reductive group over \(\mathcal{O}_E\), and denote \(M/\mathbb{M}\) its adjoint quotient. Then \(\mathcal{R}^c_{\Gamma,M} \cong M \times_{M/\mathbb{M}} (\sqcup_x (M/\mathbb{M})^\wedge_x)\), where \(x\) ranges over all closed points of \(M/\mathbb{M}\).

---

*The proof is written for \(E\)-rings, but it works for animated rings, with \(A[t_n]\) in *loc. cit.* replaced by the usual polynomial ring \(A[t_n]\). In addition, in this case each \(A_n\) in *loc. cit.* is perfect as an \(A\)-module.*
However, we have the following observation. Recall that a finitely generated group $\Gamma$ is called good (e.g. see Zd 1.2.6) if the map $\Gamma \to \hat{\Gamma}$ from $\Gamma$ to its profinite completion $\hat{\Gamma}$ induces an isomorphism of group cohomology $H^i(\hat{\Gamma}, V) \cong H^i(\Gamma, V)$ for every finite $\Gamma$-module $V$ (which automatically extends to a continuous $\hat{\Gamma}$-module) and every $i \geq 0$. Examples of good groups include finite groups, finitely generated free groups, and extensions of such. (In particular, extensions of $\Gamma_q$ by finite groups are good.)

Lemma 2.4.6. Let $\Gamma$ be a finitely generated good group and let $M$ be as above. Let $\hat{\Gamma}$ be the profinite completion of $\Gamma$. Regarding $\Gamma$ as an abstract group, we have the derived affine scheme $\mathcal{R}_{\Gamma,M}$ over Spec$\mathcal{O}_E$. Then the natural morphism $\mathcal{R}_{\Gamma,M}^c \to \mathcal{R}_{\Gamma,M}$ over Spf$\mathcal{O}_E$ induces isomorphisms after completing at closed points.

This suggests that sometimes $\mathcal{R}_{\Gamma,M}$ is an algebraization of the space $\mathcal{R}_{\Gamma,M}^c$. Note that if one only compares the underlying classical formal schemes after completions, no assumption on $\Gamma$ is needed.

Proof. We fix a closed point $x$ of $\mathcal{R}_{\Gamma,M}$, corresponding to $\hat{\rho}$. We need to show that for $A \in \text{CA} \text{Alg}_{\mathcal{O}_E,\kappa}$, the natural map

$$\lim_j (\mathcal{R}_{\Gamma_j,M}(A) \times_{\mathcal{R}_{\Gamma_j,M}(\kappa A)} \{\hat{\rho}\}) \to \mathcal{R}_{\Gamma,M}(A) \times_{\mathcal{R}_{\Gamma,M}(\kappa A)} \{\hat{\rho}\}$$

is an isomorphism in $\text{Spc}$, where $\Gamma_j$ are finite quotients of $\Gamma$. We may factor $A \to \kappa A$ as a sequence of maps $A = A_0 \to A_{i-1} \to \cdots \to A_0 = \kappa A$ with each $A_i \to A_{i-1}$ a square zero extension with kernel being $\kappa A[n_i]$ for some $n_i$. Then we may prove (2.21) by induction on $A_i$.

Suppose that (2.21) is an isomorphism for $i-1$. As argued in [CS, 5.1.13], there is the following fiber sequence $M(A_i) \to M(A_{i-1}) \to m \otimes \kappa[n_i + 1]$ in $\text{Mon(\text{Spc})}$, which induces a fiber sequence

$$\text{Map}_{\text{Mon(\text{Spc})}}(\Gamma, M(A_i)) \to \text{Map}_{\text{Mon(\text{Spc})}}(\Gamma, M(A_{i-1})) \to \text{Map}_{\text{Mon(\text{Spc})}}(\Gamma, m \otimes \kappa[n_i + 1]),$$

and similarly fiber sequences for $\Gamma_j$. As elements of $m \otimes \kappa$ are of finite order, $\lim_j \text{Map}_{\text{Mon(\text{Spc})}}(\Gamma_j, m \otimes \kappa[n_i + 1]) \to \text{Map}_{\text{Mon(\text{Spc})}}(\Gamma, m \otimes \kappa[n_i + 1])$ is an equivalence by our goodness assumption on $\Gamma$. Then one can deduce (2.21) for $A_i$ from the case for $A_{i-1}$.

3. The stack of arithmetic Langlands parameters

In this section, we apply the constructions from the previous section to understanding the moduli of Langlands parameters. The picture is relatively well understood in the local field case, which will be discussed in [Zhu] §3.1 and §3.2. Much less can be said in the global field case, but we are still able to construct the moduli space in the global function field case in [Zhu] §1.1. Here we allow $F$ to be any field and $G$ is a connected reductive group over $F$. Let $\Gamma_F$ denote the Galois group of $F$, and $\hat{G}$ the dual group of $G$, regarded as a group scheme over $\mathbb{Z}$. It is equipped with a pinning $(\hat{B}, \hat{T}, \hat{e})$, and an action of $\Gamma_F$ via the homomorphism $\xi : \Gamma_F \to \text{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{e})$. Let $\hat{G}_{ad}$ be the adjoint group of $\hat{G}$, and $\rho_{ad} : \mathbb{G}_m \to \hat{G}_{ad}$ the cocharacter given by the half sum of positive coroots of $\hat{G}$. Let $\text{pr} : \Gamma_F \to \Gamma_{\hat{F}/F}$ be the finite quotient of $\Gamma_F$ by $\ker \xi$. Let

$$^c G := \hat{G} \times (\mathbb{G}_m \times \Gamma_{\hat{F}/F}),$$

be the $C$-group of $G$, regarded as a group scheme over $\mathbb{Z}$, where $\mathbb{G}_m$ acts on $\hat{G}$ via the homomorphism $\mathbb{G}_m \xrightarrow{\rho_{ad}} \hat{G}_{ad} \subset \text{Aut}(\hat{G})$, and $\Gamma_{\hat{F}/F}$ acts via $\xi$. Let $d : ^c G \to \mathbb{G}_m \times \Gamma_{\hat{F}/F}$ denote the natural projection.
Remark 3.0.1. If $F$ is a local field with residue field $\mathbb{F}_q$ or a global function field with $\mathbb{F}_q$ its field of constant, upon a choice of $q^{1/2}$, $c^G$ and $L^G \times \mathbb{G}_m$ are isomorphic over $\mathbb{Z}[q^{\pm 1/2}]$, where $L^G = \mathcal{G} \times \Gamma_{\bar{F}/F}$ is the usual Langlands dual group of $G$. So one can replace $c^G$ by $L^G$ in most discussions below (with small modifications). However, we prefer to use $C$-group rather than $L$-group in our formulation. On the one hand, it is more canonical. On the other hand using $\hat{C}$ does not seem to simplify the formulation if $\hat{F} \neq F$.

On the other hand, if the cocharacter $\rho_{ad}$ can be lifted to a $\Gamma_{\bar{F}/F}$-invariant cocharacter $\hat{\rho} : \mathbb{G}_m \to \mathcal{G}$, then one can also use $L^G$ instead of $c^G$ in the discussions below. For example, this is the case if $G = \text{GL}_n$ or odd unitary group. See [Zhu, Example 2].

3.1. The stack of local Langlands parameters. In the next two subsections, we discuss the stack of local Langlands parameters over a base in which $p$ is invertible, for a connected reductive $G$ over a local field $F$ of residue characteristic $p$. Most discussions in these two subsections are also contained in the work of Dat-Helm-Kurinczuk-Moss [DHKM], and are also independently carried out by Scholze, sometimes by different methods.

We fix a connected reductive group $G$ over a local field $F$ of residue field $\mathbb{F}_q$ with $q = p^r$. Let $\Gamma_F$ be the Galois group of $F$, $P_F \subset \Gamma_F$ be the wild inertia and the inertia, corresponding to Galois extensions $F^i \supset F^u \supset F$. Recall that the tame inertia

$$I^t_F := I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1) =: \hat{\mathbb{Z}}^p(1)$$

is prime-to-$p$, while $P_F$ is a pro-$p$-group. Then $\Gamma^t_F := \Gamma_F/I_F \cong \Gamma_F/P_F$ fits into the following short exact sequence

$$1 \to I^t_F \to \Gamma^t_F \to \hat{\mathbb{Z}} \to 1.$$

Let $W_F \subset \Gamma_F$ be the Weil group of $F$. We normalize the map

$$|| \cdot || : W_F \to \mathbb{Z}$$

so it is trivial on $I_F$ and $||\Phi|| = 1$ for a lifting of the arithmetic Frobenius. Similarly, we have the tame Weil group $W^t_F := W_F/P_F$, which is an extension of $\mathbb{Z}$ by $I^t_F$. We let

$$\chi = (q^{-|| \cdot ||}, \text{pr}) : W_F \to \mathbb{Z}[1/p]^\times \times \Gamma_{\bar{F}/F}.$$

Note that $q^{-|| \cdot ||}$ is the restriction of the inverse cyclotomic character of $\Gamma_F$ to $W_F$.

There are several versions of the moduli of local Langlands parameters.

First, there is the moduli $\mathcal{R}^c_{W_F, c^G}$ of continuous representations of $W_F$ over $\text{Spf} \mathbb{Z}_\ell$, defined via the general recipe as in §2.4. The homomorphism $d : c^G \to \mathbb{G}_m \times \Gamma_{\bar{F}/F}$ induces a morphism $\mathcal{R}^c_{W_F, c^G} \to \mathcal{R}^c_{W_F, c^G \times \Gamma_{\bar{F}/F}}$. We may regard $\chi$ as a $\text{Spf} \mathbb{Z}_\ell$-point of $\mathcal{R}^c_{W_F, c^G \times \Gamma_{\bar{F}/F}}$ and define

$$\text{Loc}_{c^G, F}^\wedge := \mathcal{R}^c_{W_F, c^G} \times \mathcal{R}^c_{W_F, c^G \times \Gamma_{\bar{F}/F}} \{ \chi \}, \quad \text{Loc}_{c^G, F}^\wedge = \text{Loc}_{c^G, F}^\wedge / \hat{G}^\wedge,$$

where $\hat{G}^\wedge$ is the $\ell$-adic completion of $\mathcal{G}$. As $\Gamma_F$ is the profinite completion of $W_F$, a slight variant of Lemma 2.4.6 implies that the completion of $\text{Loc}_{c^G, F}^\wedge$ at a closed point corresponding to $\hat{\rho} : \Gamma_F \to c^G(\mathbb{F})$ is the space $\text{Def}^\wedge_{\hat{\rho}}$ of framed deformations $\rho$ of $\hat{\rho}$ such that $d \circ \rho = \chi$.

Remark 3.1.1. As mentioned above, $\text{Loc}_{c^G, F}^\wedge \cong \mathcal{R}^c_{W_F, c^G} \times \mathcal{R}^c_{W_F, c^G \times \Gamma_{\bar{F}/F}} \{ \text{pr} \}$ over $\text{Spf} \mathbb{Z}_\ell[q^{\pm 1/2}]$.

Remark 3.1.2. The analogue of $\text{Loc}_{c^G, F}^\wedge$ over $\text{Spf} \mathbb{Z}_p$ probably should the Emerton-Gee stack [EG] (whose definition is much more involved).
Second, there is the stack
\[ \text{Loc}^{\text{WD}}_{G,F} := \text{Loc}^{\text{WD}}_{c,G,F} / \hat{G} \]
of Weil-Deligne representations of \( F \) as an algebraic stack over \( \mathbb{Q} \) (see e.g. [BG11, 2.1]). Here \( \text{Loc}^{\text{WD}}_{c,G,F} \) is the presheaf over \( \text{CAlg}_{\mathbb{Q}} \) defined as follows. Let \( \hat{N}_\mathbb{Q} \subset \text{Lie}\hat{G}_\mathbb{Q} \) denote the nilpotent cone of \( \hat{G}_\mathbb{Q} \). For a \( \mathbb{Q} \)-algebra \( A \), we equip \( {}^c G(A) \) with the discrete topology, and let
\[ \text{Loc}^{\text{WD}}_{c,G,F}(A) = \left\{ (r,X) \mid r : W_F \to \right. \begin{array}{l} \left. \text{continuous} \end{array} \right. \left. , \begin{array}{l} X \in \hat{N}_\mathbb{Q}(A) \mid d \circ \rho = \chi, \right. \begin{array}{l} \text{Ad}_{r(\gamma)} X = q^{\| \gamma \|} X \right\} . \]
We note that there is a natural \( \mathbb{G}_m \) action on \( \text{Loc}^{\text{WD}}_{c,G,F} \), by scaling the nilpotent element \( X \).

One sees that
\[ \text{Loc}^{\text{WD}}_{c,G,F} = \lim_{L} \text{Loc}^{\text{WD}}_{c,G,L/F}, \]
where \( L \) ranges over all finite extensions of \( F^{\text{unr}} \hat{F} \) that are Galois over \( F \), and \( \text{Loc}^{\text{WD}}_{c,G,L/F} \) is the (open and closed) subfunctor of \( \text{Loc}^{\text{WD}}_{c,G,F} \) consisting of those \((r,X)\) such that \( r \) factors through \( W_{L/F} \to {}^c G(A) \), where \( W_{L/F} \) denotes the Weil group of \( L/F \).

Note that as \( W_{L/F} \) is a finitely generated group, namely an extension of \( \hat{W} \), for a \( \mathbb{Q} \)-algebra \( A \), we equip \( \hat{N}_A \) with the discrete topology, and let \( \text{Loc}^{\text{WD}}_{c,G,L/F} := \text{Loc}^{\text{WD}}_{c,G,F}(\hat{N}_A) \) with the discrete topology, while \( G,F \) is a finite \( l \)-group.

Remark 3.1.3. Here we only define \( \text{Loc}^{\text{WD}}_{c,G,F} \) as a classical scheme as this is what we need in the sequel. Of course, one can define it as a derived scheme in a natural way, but it turns out the derived structure will be trivial. In fact, we have such kind of discussions in the sequel when we discuss integral versions of \( \text{Loc}^{\text{WD}}_{c,G,F} \).

Finally, we can glue the above two moduli spaces into algebraic stacks over \( \mathbb{Z}[1/p] \), once we make a choice. Recall the following basic facts ([Iw55]).

- There exists a topological splitting \( \Gamma^t_F \to \Gamma_F \) so that \( \Gamma_F \cong P_F \times \Gamma^t_F \).
- Let \( \Gamma_q = \langle \tau, \sigma \rangle \) be as in (2.18). Then there exists an embedding
\[ \iota : \Gamma_q \to \Gamma^t_F \]
such that \( \iota(\tau) \) is a generator of the tame inertia, and that \( \iota(\sigma) \) is a lifting of the Frobenius. Then \( \iota \) induces an isomorphism of the profinite completion of the projection \( \Gamma_q \to \hat{Z} \) with \( \Gamma^t_F \to \hat{Z} \).

For a choice of \( \iota \), we write \( \Gamma_{F,\iota} \) be the pullback of \( \Gamma_F \) via \( \iota \) (we will not consider the topology on these groups). Then we have inclusions \( \Gamma_{F,\iota} \to W_F \to \Gamma_F \). By abuse of notations, we still use \( \iota \) to denote both inclusions \( \Gamma_{F,\iota} \subset W_F \) and \( \Gamma_{F,\iota} \subset \Gamma_F \). We have the short exact sequence
\[ 1 \to P_F \to \Gamma_{F,\iota} \to \Gamma_q \to 1. \]
The homomorphism \( \| \cdot \| \) from (3.1) restricts to \( \Gamma_{F,\iota} \). Similarly, if \( L \) is finite over \( F^t \) and is Galois over \( F \), let \( \Gamma_{L/F,\iota} \) be the pullback of \( \Gamma_{L/F} \) (the Galois group for \( L/F \)) along \( \iota \). We have the short exact sequence
\[ 1 \to Q_L := \Gamma_{L/F^t} \to \Gamma_{L/F,\iota} \to \Gamma_q \to 1, \]
where \( Q_L \) is a finite \( p \)-group.

Remark 3.1.4. (1) Note that for two choices \( \iota_1, \iota_2 \), there is in general no isomorphism between \( \Gamma_{F,\iota_1} \) and \( \Gamma_{F,\iota_2} \) that restricts to the identity map of \( P_F \).

(2) All possible choices of \( \iota \) as in (3.3) form a torsor under \( \text{Aut}^0 \), the group of continuous automorphisms of \( \Gamma^t_F \) that restricts to an automorphism of \( \Gamma^t_F \) and induces the identity map on \( \Gamma^t_F / \Gamma^t_F \). The group \( \text{Aut}^0 \) itself is an extension of \( \hat{Z} \perp : = \prod_{\ell \neq p} \hat{Z}_\ell \) by \( \hat{Z} \perp(1) \).
Now we define the stack of local Langlands parameters over $\mathbb{Z}[1/p]$. We first choose an $\ell$ as in (3.3). If $L/F^\ell F$ is finite such that $L/F$ is Galois, then $\chi: \Gamma_{F,\ell} \to \mathbb{Z}[1/p]^{\times} \times \hat{\Gamma}_{F/F}$ factors through $\Gamma_{L/F,\ell} \to \mathbb{Z}[1/p]^{\times} \times \hat{\Gamma}_{F/F}$, denoted by the same notation, which can be regarded as a $\mathbb{Z}[1/p]$-point of $\mathcal{R}_{\Gamma_{L/F,\ell}} \mathbb{G}_m \times \hat{\Gamma}_{F/F}$. We define the scheme

$\text{Loc}_{G,L/F,\ell}^{\square} := \mathcal{R}_{\Gamma_{L/F,\ell}} \mathbb{G}_m \times \mathcal{R}_{\Gamma_{L/F,\ell}} \mathbb{G}_m \times \hat{\Gamma}_{F/F} \{ \chi \}$,

Explicitly, for a classical $\mathbb{Z}[1/p]$-algebra $A$,

$\text{Loc}_{G,L/F,\ell}^{\square}(A) := \left\{ \rho : \Gamma_{L/F,\ell} \to \mathfrak{c}G(A) \mid d \circ \rho = \chi: \Gamma_{L/F,\ell} \to \mathbb{G}_m \times \hat{\Gamma}_{F/F} \right\}$.

Now, we define the scheme of framed $\ell$-local Langlands parameters as

$\text{Loc}_{G,F,\ell}^{\square} := \lim_{\to} \text{Loc}_{G,L/F,\ell}^{\square}$.

By Lemma 2.4.6 its formal completion at $\hat{\rho}$ is the framed deformation space $\text{Def}_{\hat{\rho}}^{\square}$.

**Proposition 3.1.5.** The derived ind-scheme $\text{Loc}_{G,F,\ell}^{\square}$ is a disjoint union of classical affine schemes of finite type and flat over $\mathbb{Z}[1/p]$. It is equidimensional of dimension = dim $\hat{G}$, and is a local complete intersection.

**Proof.** We apply Proposition 2.3.9 to $\Gamma = \Gamma_{L/F,\ell} \simeq \mathbb{Q}_L \times \Gamma_q$, and $M = \mathfrak{c}G$ and $M = \mathbb{G}_m \times \hat{\Gamma}_{F/F}$. We have the projection $\mathcal{R}_{\Gamma_{L/F,\ell}} \mathbb{G}_m \to \mathcal{R}_{\Gamma_{L/F,\ell}} \mathbb{G}_m \times \hat{\Gamma}_{F/F}$. Taking the fiber over $\chi$ shows that $\text{Loc}_{G,F,\ell}^{\square}$ is a classical affine scheme of finite type and flat over $\mathbb{Z}[1/p]$, is equidimensional of dimension = dim $\hat{G}$, and is a local complete intersection. In addition, clearly if $L'/L$ is finite such that $L'/F$ is Galois, then $\text{Loc}_{G,L/F,\ell}^{\square} \subset \text{Loc}_{G,L'/F,\ell}^{\square}$ is an open and closed embedding. The proposition follows.

Now we can define the stack of $\ell$-local Langlands parameters as

$\text{Loc}_{G,F,\ell} = \text{Loc}_{G,F,\ell}^{\square}/\hat{G}$.

It is the union of open and closed substacks $\text{Loc}_{G,L/F,\ell} = \text{Loc}_{G,L/F,\ell}^{\square}/\hat{G}$, each of which is of finite presentation over $\mathbb{Z}[1/p]$.

**Remark 3.1.6.** There are two ways to view $\text{Loc}_{G,F,\ell}$ (and $\text{Loc}_{G,F}^{\text{WD}}$) as an algebraic stack. The first is by viewing it as a stack locally of finite type, and the second is by viewing it as an ind-finite type stack. We will adapt the second point of view. So its ring of regular functions (see (3.4) below) is regarded as pro-algebra. In addition, later on we will consider the category $\text{Coh}(\text{Loc}_{G,F,\ell})$ of coherent sheaves on $\text{Loc}_{G,F,\ell}$. According our definition, these are complexes of quasi-coherent sheaves that only support on finitely connected components of $\text{Loc}_{G,F,\ell}$, and are coherent complexes on these component. In particular, the structure sheaf of $\text{Loc}_{G,F,\ell}$ itself is not regarded as a coherent sheaf. It lies in the ind-completion $\text{IndCoh}(\text{Loc}_{G,F,\ell})$ of $\text{Coh}(\text{Loc}_{G,F,\ell})$.

We have discussed three versions of moduli of local Langlands parameters: one over $\text{Spf} \mathbb{Z}_\ell$, one over $\mathbb{Q}$ and one over $\text{Spec} \mathbb{Z}[1/p]$. Our next task is to relate them and to analyze how $\text{Loc}_{G,L/F,\ell}$ depends on the choice of $\ell$.

**Lemma 3.1.7.** The map $\ell: \Gamma_{F,\ell} \to W_F$ induces a natural isomorphism

$\phi_{\ell,\ell} : \text{Loc}_{G,F}^{\square,\ell} \xrightarrow{\cong} (\text{Loc}_{G,F,\ell}^{\square})^{\ell}$,

where $(\text{Loc}_{G,F,\ell}^{\square})^{\ell}$ is the $\ell$-adic completion of $\text{Loc}_{G,F,\ell}^{\square}$.
Proof. The map \( \phi_{i,t} \) is just sending a continuous representation \( W_F \to {}^cG(A) \) to its restriction to \( \Gamma_{F,t} \), where \( A \) is some \( \mathbb{Z}/\ell^n \)-algebra. We need to show that \( \phi_{i,t} \) is an isomorphism. Let us prove this at the level of classical schemes. Using that \( \Gamma_{L/F,t} \) is good, one can extend this at the derived level as well, in the way similar to Lemma 2.4.6.

As above, we write \( \Gamma_{F,t} \cong P_F \times \Gamma_q \) by choosing a topological splitting \( \Gamma_{t} \to \Gamma_{F} \). Then it is enough to show that for every \( \mathbb{Z}/\ell^n \)-algebra \( A \), and every morphism \( \rho : \Gamma_{L/F,t} \to {}^cG(A) \), there is some \( N \) such that \( \rho(\tau^N) = 1 \). (The integer \( N \) might depend on the choice of the topological splitting.) Namely, if this holds, then every homomorphism \( \rho : \Gamma_{F,t} \to {}^cG(A) \) automatically (and uniquely) extends to a continuous homomorphism from \( W_F \to {}^cG(A) \).

Recall that the restriction \( \langle \tau \rangle \subset \Gamma_q \) induces \( \text{Loc}_{G,F,t} \to {}^cG[q] \) (see the proof of Proposition 2.3.7). So it is enough to show that for every \( \mathbb{Z}/\ell^n \)-algebra \( A \), there is some \( N \) such that the \( N \)th power map \( {}^cG \to {}^cG, \ g \mapsto g^N \) sends \( {}^cG[q](A) \) to 1. By choosing a faithful representation \( {}^cG \to GL_m \), it is enough to show a similar statement for \( GL_m \). By this holds as for every element in \( X \in GL_m[q](A) \), some power of it is unipotent. I.e. there are \( r, s \) such that \( (X^r - 1)^s = 0 \). Raising this equation to \( \ell^t \)th power for some \( t \gg 0 \), we see \( X^N = 1 \) for some \( N \). \( \square \)



On the other hand, we have the following.

Lemma 3.1.8. The map \( \Gamma_{F,t} \to W_F \) induces a natural isomorphism

\[
\phi_{i,Q} : \text{Loc}_{G,F,t}^{WD} \cong \text{Loc}_{G,F,t}^{\square} \otimes \mathbb{Q},
\]

Proof. The morphism \( \phi_{i,Q} \) is given by send \( (r, X) \in \text{Loc}_{G,F}^{WD}(A) \) to

\[
\rho : \Gamma_{F,t} \to {}^cG(A), \quad \rho(\gamma) = r(\gamma) \exp(|\gamma|_r X),
\]

where \( |\gamma|_r \in \mathbb{Z}[1/p] \) such that the image of \( \gamma \in \Gamma_{F,t} \) in \( \Gamma_q \) can be written as \( \sigma^{|\gamma|_{(t)} \tau^{|\gamma|_{(t)}}} \), and

\[
\exp : \mathcal{U}_Q \cong \hat{\mathcal{U}}_Q
\]

is the usual exponential map inducing isomorphisms between the nilponent variety and the unipo-otent variety of \( \hat{G} \) (over \( \mathbb{Q} \)). Let \( \log : \hat{\mathcal{U}}_Q \cong \mathcal{U}_Q \) be its inverse.

Next we define the morphism in another direction. Let \( \rho : \Gamma_{F,t} \to {}^cG(A) \) be an \( A \)-point of \( \text{Loc}_{G,F,t}^{\square} \). We assume that it factors through some \( \Gamma_{L/F,t} \). Note that there is some \( m \) such that the image of \( \tau^m \in \Gamma_q \) in \( \Gamma_{L,F,t} \) is independent of the choice of the splitting \( \Gamma_q \to \Gamma_{L/F,t} \). In addition, by replacing \( m \) by a multiple, we may assume that \( \rho(\tau)^m \in \mathcal{U}_Q(A) \). Then we take \( X = \frac{1}{m} \log(\rho(\tau)^m) \).

Clearly \( X \) is independent of the choice of \( m \). Then we obtain a well-defined homomorphism

\[
r : \Gamma_{F,t} \to {}^cG(A), \quad r(\gamma) = \rho(\gamma) \exp(-|\gamma|_r X).
\]

As \( r(\tau^m) = 1 \), we may regard \( r \) as a continuous map \( W_{L/F} \to {}^cG(A) \), where \( A \) is equipped with the discrete topology. Then \( \rho \mapsto (r, X) \) gives the inverse of \( \phi_{i,Q} \). \( \square \)

Before continuing, we observe that as a byproduct we obtain the following.

Corollary 3.1.9. The scheme \( \text{Loc}_{G,F,t}^{\square} \) is reduced.

Note that the fiber of \( \text{Loc}_{G,F,t}^{\square} \) over some prime \( \ell \) could be non-reduced. More detailed study of reducedness of the mod \( \ell \) fiber of \( \text{Loc}_{G,F,t}^{\square} \) is contained in [DHKM].

Proof. As \( \text{Loc}_{G,F,t}^{\square} \) is a local complete intersection flat over \( \mathbb{Z}_\ell \) (Proposition 3.1.5), the statement follows from the generic smoothness of \( \text{Loc}_{G,F,t}^{\square} \otimes \mathbb{Q} \cong \text{Loc}_{G,F,t}^{WD} \) as proved in [BG11]. \( \square \)
Now we can compare $\text{Loc}^{\square}_{G,F,\ell}$ for different choices of $\ell$. Let $\iota_1, \iota_2 : \Gamma_q \to \Gamma'_F$ be two embeddings. Recall from Remark 3.1.4 that there is $\psi \in \text{Aut}^0$ such that $\iota_2 = \psi \iota_1 : \Gamma_q \to \Gamma'_F$, and there is a projection $\text{Aut}^0 \to \mathbb{Z}_\ell^\times$. Let $\tilde{\psi} \in \mathbb{Z}_\ell^\times$ denote the image of $\psi$. As $\mathbb{G}_m$ acts on $\text{Loc}^{\square}_{G,F}$ by scaling the nilpotent element, $\tilde{\psi}$, regarded as an element in $\mathbb{G}_m(\mathbb{Q}_\ell)$, acts on $\text{Loc}^{\square}_{G,F,\ell}$. 

**Proposition 3.1.10.** There is a unique isomorphism $\psi = \psi_{\iota_1,\iota_2} : \text{Loc}^{\square}_{G,F,\iota_1} \otimes \mathbb{Z}_\ell \cong \text{Loc}^{\square}_{G,F,\iota_2} \otimes \mathbb{Z}_\ell$ of schemes over $\mathbb{Z}_\ell$ making the following diagram commutative

$$
\begin{array}{ccc}
\text{Loc}_{G,F,\ell}^{\square} & \xrightarrow{\phi_{1,\ell}} & \text{Loc}_{G,F,\iota_1}^{\square} \otimes \mathbb{Z}_\ell \\
\downarrow & & \downarrow \psi \\
\text{Loc}_{G,F,\ell}^{\square} & \xrightarrow{\phi_{1,\ell}} & \text{Loc}_{G,F,\iota_2}^{\square} \otimes \mathbb{Z}_\ell
\end{array}
$$

It follows that $\text{Loc}^{\square}_{G,F,\ell} \otimes \mathbb{Z}_\ell$ is independent of the choice of $\ell : \Gamma_q \to \Gamma'_F$ up to canonical isomorphism. Therefore, we denote it by $\text{Loc}^{\square}_{G,F,\ell}$.

**Proof.** First, if we replace $\text{Spec}\mathbb{Z}_\ell$ by $\text{Spf} \mathbb{Z}_\ell$ in the middle column of the above diagram, $\phi_{1,\ell}$ becomes isomorphism and therefore there is a unique isomorphism $\tilde{\psi} : \text{Loc}^{\square}_{G,F,\iota_1} \otimes \text{Spf} \mathbb{Z}_\ell \cong \text{Loc}^{\square}_{G,F,\iota_2} \otimes \text{Spf} \mathbb{Z}_\ell$ of formal schemes compatible with $\phi_{1,\ell}$. Then passing to the rigid generic fiber, and by tracing the construction, we see that $\tilde{\psi} \circ \phi_{1,\ell} = \phi_{1,\ell} \circ \tilde{\psi}$. It follows that $\tilde{\psi}$ algebraizes to a unique desired isomorphism $\psi$. \hfill \square

As we learned from Scholze, $\text{Loc}^{\square}_{G,F,\ell}$ admits the following moduli interpretation which is obviously independent of the choice of $\ell$.

**Proposition 3.1.11.** The scheme $\text{Loc}^{\square}_{G,F,\ell}$ assigns every $\mathbb{Z}_\ell$-algebra $A$ the set of continuous homomorphism $\rho : W_F \to {}^c G(A)$ such that $\alpha \rho = \chi$, where now $A$ is equipped with the topology in which a subgroup $U \subset A$ is open if and only if its intersection with every finitely generated $\mathbb{Z}_\ell$-submodule of $A$ is open in the $\ell$-adic topology.

**Proof.** To see this, we need to show that every $\rho : \Gamma_{L/F,\ell} \to {}^c G(A)$ extends uniquely to a continuous homomorphism $\Gamma_{L/F} \to {}^c G(A)$. We can proceed as in the proof of Lemma 3.1.7 and 3.1.8 to choose a splitting $\Gamma_q \to \Gamma_{L/F,\ell}$ and some $N$, so that $\rho(\tau^N) \in \hat{U}(A)$. Then the claim follows as every $u \in \hat{U}(A)$ extends to a continuous map $\mathbb{Z}_\ell \to \hat{U}(A)$, $a \mapsto u^a$. \hfill \square

Now let

$$Z_{cG,F} := H^0 \Gamma(\text{Loc}^{\square}_{G,F,\ell}, \mathcal{O}),$$

be the ring of regular functions on $\text{Loc}^{\square}_{G,F,\ell}$. Here according to our convention, $\Gamma(\text{Loc}^{\square}_{G,F,\ell}, -)$ standards for the derived functor, while $H^0 \Gamma$ denotes its zeroth cohomology. Note that we leave out the subscript $\ell$ as it is independent of the choice $\ell$ up to canonical isomorphism. Indeed, the $\mathbb{G}_m$-action on $\text{Loc}^{\square}_{G,F,\ell}$ (by scaling the nilpotent element) induces the trivial action on its ring of regular functions. Therefore $\tilde{\psi}$ in Proposition 3.1.10 induces the identity map after taking $G$-invariants.

This algebra is usually called the stable center of $G^*$ (the quasi-split inner form of $G$), at least when base changed to $\mathbb{C}$ (see [Ha14]). It admits idempotent decompositions indexed by connected components of $\text{Loc}^{\square}_{G,F,\ell}$. For a finite union of connected components $D$, let $Z_{cG,F,D}$ denote the corresponding ring of regular functions, which is a finitely generated $k$-algebra. In particular, if $D = \text{Loc}^{\square}_{G,L/F,\ell}$, we denote $Z_{cG,F,D}$ by $Z_{cG,L/F}$. 


As taking $\hat{G}$-invariants on $\hat{G}$-representations over $k$ is not exact if $k$ is not a field of characteristic zero, a priori the higher cohomology $H^i\Gamma\left(\text{Loc}_{G,F,\iota}, \mathcal{O}\right)$ may not vanish for $i > 0$. But Conjecture 4.4.1 suggests this is not the case. In fact, we make the following conjecture.

**Conjecture 3.1.12.** For every $i \geq 1$, $H^i\Gamma\left(\text{Loc}_{G,F,\iota}, \mathcal{O}\right) = 0$.

**Remark 3.1.13.** Let $\kappa$ be an algebraically closed field over $\mathbb{Z}[1/p]$. By [La18, 11.7] and [BHKT, 4.5], and Remark 2.2.19, there is a bijection between $\kappa$-points of $Z_{G,F}$ and $\hat{G}(\kappa)$-conjugacy classes of homomorphisms $\rho : \Gamma_{F,\iota} \to ^cG(\kappa)$ satisfying

- $d \circ \rho = \chi$;
- $\rho$ factors through $\Gamma_{L/F,\iota} \to ^cG(\kappa)$ for some finite extension $L/F$;
- $\rho$ is completely reducible (see [BHKT, 3.5] for the terminology).

Giving Conjecture 3.1.12 one may further conjecture that a slight variant of (2.17) in the current setting is an isomorphism (after taking $\pi_0$).

At the end of this subsection, we discuss the behavior of these stacks under tensor induction.

Let $F'/F$ be a finite separable extension. Let $G'$ be a connected reductive group over $F'$ and $G = \text{Res}_{F'/F} G'$. As explained in [Bo79, 5.1.4], the dual group $\hat{G}$ of $G$ equipped with an action of $\Gamma_F$ is canonically isomorphic to the tensor induction $\text{Ind}_{G'}^{G,F} \hat{G}'$, which by definition is the space of all $\Gamma_{F'}$-equivariant maps from $\Gamma_{F'}$ to $\hat{G}'$. There is the $\Gamma_{F'}$-equivariant maps ([Bo79, 4.1])

$$
\hat{G}' \overset{i}{\to} \hat{G} \overset{\text{ev_e}}{\to} \hat{G}'
$$

whose composition is the identity, where the first map sends $g$ to the unique map $f : \Gamma_F \to \hat{G}'$ that is supported on $\Gamma_{F'}$ and such that $f(1) = g$, and the second map sends $f : \Gamma_F \to \hat{G}'$ to $f(e)$. Then there is a canonical homomorphism $^c(G') \to ^cG$ compatible with $i$ and with $\mathbb{G}_m \times \Gamma_{F'/F} \to \mathbb{G}_m \times \Gamma_{F/F}$ as in ([Bo79, 5.1 (5)]). A choice of $\iota : \Gamma_{q} \to \Gamma_{F}$ gives $\iota' : \Gamma_{q} \to \Gamma_{F'}$. Note that

$$
\text{Ind}_{\Gamma_{F,\iota}}^{\Gamma_{F',\iota'}} \hat{G}' = \text{Ind}_{\Gamma_{F',\iota'}}^{\Gamma_{F,\iota}} \hat{G}'.
$$

**Lemma 3.1.14.** There is the canonical isomorphism

$$
\text{Loc}_{G,F,\iota} \cong \text{Loc}_{G',F',\iota'}, \quad \rho \mapsto \text{ev_e} \circ (\rho|_{\Gamma_{F',\iota'}}).
$$

**Proof.** This is a geometric version of the Shapiro’s lemma. We generalize the argument from [XZ19, 4.1.2] to explicitly construct the inverse map. For simplicity, we write $\Gamma(\cdot)$ for $\Gamma(\cdot)$. Let $s : \Gamma_{F}/\Gamma \to \Gamma$ be a section (sending the unit coset to $1 \in \Gamma$) of the projection $\Gamma \to \Gamma_{F}/\Gamma, \gamma \mapsto \gamma\iota$. Then we have the map

$$
\Xi_s : \Gamma \to \Gamma', \quad \Xi(s)(\gamma) := s^{-1}.\gamma
$$

Note that $\Xi_s(\gamma') = \gamma'\Xi_s(\gamma)$ for $\gamma' \in \Gamma'$. In addition, let

$$
\Delta_s : \hat{G}' \to \hat{G}, \quad \Delta_s(g) : \Gamma \to \hat{G}', \quad \Delta_s(g)(\delta) = \chi(\Xi_s(\delta))(g).
$$

Now we construct a morphism $I_s : \text{Loc}_{G',F',\iota'} \to \text{Loc}_{G,F,\iota}$ as follows. Let $\rho' = (\varphi, \chi) : \Gamma' \to ^c(G')(\mathcal{O}) = \hat{G'}(\mathcal{O}) \times (A^x \times \Gamma_{F'/F'}).$ We define $I_s(\rho') = (\varphi, \chi) : \Gamma \to ^cG(\mathcal{O}) = \hat{G}(\mathcal{O}) \times (A^x \times \Gamma_{F/F})$, where

$$
\varphi(\gamma) : \Gamma \to \hat{G}'(\mathcal{O}), \quad \varphi(\gamma)(\delta) = \varphi'(\Xi_s(\delta)(\delta)).
$$

One verifies that

- $\varphi(\gamma') = \chi(\gamma') \varphi(\gamma)$ for $\gamma' \in \Gamma'$ so $\varphi(\gamma) \in \hat{G}(\mathcal{O})$;
- $I_s(\rho')$ is a homomorphism $\Gamma \to ^cG(\mathcal{O})$, and that $\text{ev_e} \circ (I_s(\rho')|_{\Gamma'}) = \rho'$;
- $I_s(g^{-1}\rho'g) = \Delta_s(g^{-1})I_s(\rho') \Delta_s(g)$ for any $g \in \hat{G}'(\mathcal{O})$.

Therefore we construct a morphism $\text{Loc}_{G,F,\iota} \to \text{Loc}_{G,F,\iota}$ inverse to the map in the lemma. \hfill $\Box$
3.2. **Spectral parabolic induction.** Let \( \hat{P} \) be a parabolic subgroup of \( \hat{G} \) containing \( \hat{B} \) and stable under the action of \( \Gamma_{\hat{F}/\hat{F}} \) on \( \hat{G} \), and let \( \hat{M} \) be its standard Levi (the one containing \( \hat{T} \)). Then the action of \( \mathbb{G}_m \times \Gamma_{\hat{F}/\hat{F}} \) on \( \hat{G} \) preserves \( \hat{P} \) and \( \hat{M} \) and we can form \( ^c P \) and \( ^c M \) respectively, and define \( \text{Loc}^c_{P,F,t} \) and \( \text{Loc}^c_{M,F,t} \) similarly. Note that unlike \( \text{Loc}_{G,F,t} \) and \( \text{Loc}^c_{M,F,t} \), \( \text{Loc}^c_{P,F,t} \) may not be not classical (see Remark 2.3.8), although it is still quasi-smooth. We emphasize that we need to remember the derived structure of \( \text{Loc}^c_{P,F,t} \) in the following discussions. There is the following commutative diagram over \( \mathbb{Z}[1/p] \)

\[
\begin{array}{ccc}
\text{Loc}^c_{M,F,t} & \xrightarrow{\pi} & \text{Loc}^c_{G,F,t} \\
\downarrow i & & \downarrow \pi \\
\text{Spec} \mathcal{Z}_{M,F} & \to & \text{Spec} \mathcal{Z}_{G,F}.
\end{array}
\]

where \( \pi, r, i \) are induced by the corresponding morphisms between \( \hat{G}, \hat{P}, \hat{M} \), and where the bottom map is induced by \( \pi \circ i : \text{Loc}^c_{M,F,t} \to \text{Loc}^c_{G,F,t} \). To see this diagram is commutative, it is enough to show that \( r \) induces an isomorphism

\[
H^0 \Gamma(\text{Loc}^c_{M,F,t}, \mathcal{O}) \to H^0 \Gamma(\text{Loc}^c_{P,F,t}, \mathcal{O}).
\]

Let \( 2\rho_{G,\hat{M}} = 2\rho - 2\rho_{\hat{M}} \), where \( 2\rho \) (resp. \( 2\rho_{\hat{M}} \)) is the sum of positive coroots of \( \hat{G} \) (resp. \( \hat{M} \)). Then the conjugation action of \( 2\rho_{G,\hat{M}}(\mathbb{G}_m) \) on \( ^c P \) contracts it into \( ^c M \). Equivalently, the weight zero part of \( k[\hat{F}] \) with respect to \( 2\rho_{G,\hat{M}}(\mathbb{G}_m) \) is just \( k[\hat{M}] \). It follows that (3.6) is an isomorphism.

If we let \( W_{G,^c M} \) be the quotient of the normalizer of \( ^c M \subset ^c G \) in \( \hat{G} \) by \( \hat{M} \), then it follows that the map \( \mathcal{Z}_{^c G,F} \to \mathcal{Z}_{^c M,F} \) factors through

\[
\mathcal{Z}_{^c G,F} \to (\mathcal{Z}_{^c M,F})^{W_{G,^c M}}.
\]

We have the following lemma (compare with [AG16, 13.2.2]).

**Lemma 3.2.1.** The morphism \( r \) is quasi-smooth and \( \pi \) is proper and schematic.

**Proof.** That \( \pi \) is proper and schematic is clear. For quasi-smoothness of \( r \), it is enough to note that the relative cotangent complex at \( \rho \in \text{Loc}^c_{P,F,t} \) is \( \mathcal{C}_r(\Gamma_{F,t}, \text{Ad}^{\ast}_{\rho})[-1] \) which concentrates in degree \([-1, 1]\) if \( \rho \) is a classical point. Here \( \text{Ad}^{\ast} \) is the coadjoint representation of \( ^c P \) on the dual of the Lie algebra of its unipotent radical.

Recall that Arinkin-Gaitsgory (in [AG16]) attach, to a quasi-smooth derived algebraic stack \( X \) over a field of characteristic zero a classical stack \( \text{Sing}(X) \) of singularities of \( X \), and to a coherent sheaf \( F \) on \( X \) a conic subset \( \text{Sing}(F) \subset \text{Sing}(X) \) as its singular support. One checks that such constructions carry through for quasi-smooth stacks over \( \text{CAlg}_k \) without change. In particular, by definition

\[
\text{Sing}(\text{Loc}_{^c G,F,t}) = \left\{ (\rho, \xi) \mid \rho \in \text{Loc}_{^c G,F,t}, \xi \in H_2(\Gamma_{F,t}, \text{Ad}^{\ast}_{\rho}) \right\},
\]

where \( \text{Ad}^{\ast} \) denote the coadjoint representation of \( ^c G \) on the dual of the Lie algebra of \( \hat{G} \).

As explained in [AG16], a particular conic subset \( \mathcal{N}_{^c G,F,t} \) of \( \text{Sing} (\text{Loc}_{^c G,F,t}) \) plays an important role in the Langlands correspondence. Using (2.19) (or a version of local Tate duality), we have

\[
H_2(\Gamma_{F,t}, \text{Ad}^{\ast}_{\rho}) \cong \langle \mathcal{G}^{\ast}\rangle_{\rho(1_{\hat{F},t})}^{\rho(\sigma)=q^{-1} \subset \text{Ad}^{\ast}_{\rho}}.
\]
Let \( \hat{N}^* \subset \hat{g}^* \) be the nilpotent cone of \( \hat{g}^* \). We define
\[
(3.8) \quad \hat{N}_{G,F,t} = \left\{ (\rho, \xi) \in \text{Sing}(\text{Loc}^c_{G,F,t}), \xi \in \hat{N}_\rho^* \right\}.
\]

The following proposition can be proved exactly the same as [AG16, 13.2.6]. Recall our convention of coherent sheaves on \( \text{Loc}^c_{G,F,t} \) (see Remark 3.1.6).

**Proposition 3.2.2.** There is a well-defined functor (called the spectral parabolic induction)
\[
\pi_\ast r^!: \text{Coh}(\text{Loc}^c_{M,F,t}) \to \text{Coh}(\text{Loc}^c_{G,F,t}),
\]
which restricts to a functor \( \pi_\ast r^!: \text{Coh}_{N\subset M,F,t}(\text{Loc}^c_{M,F,t}) \to \text{Coh}_{N\subset G,F,t}(\text{Loc}^c_{G,F,t}) \).

We have the following observation.

**Lemma 3.2.3.** Over \( \Q \), \( \text{Sing}(\text{Loc}^c_{G,F,t} \otimes \Q) = \hat{N}_{G,F,t} \otimes \Q \).

However, over \( \F_\ell \) when \( \ell \mid q - 1 \), \( \text{Sing}(\text{Loc}^c_{G,F,t}) \) is strictly larger than \( \hat{N}_{G,F,t} \).

**Proof.** Using the identification between \( \text{Loc}^c_{G,F,t} \otimes \Q \) and \( \text{Loc}^\text{WD}_{G,F} \) as in Lemma 3.1.8, we identify \( H_2(\Gamma_{F,t}, \text{Ad}^*_\rho) \) with
\[
\left\{ \xi \in (\hat{g}^*)^r(I_F) \mid \text{ad}^*_X(\xi) = 0, r(\sigma)(\xi) = q^{-1}\xi \right\},
\]
where \((r, X)\) corresponds to \( \rho \) as in Lemma 3.1.8. We need to show such \( \xi \) is automatically nilpotent. Let \( \h := \hat{g}^r(I_F) \), which is a reductive Lie algebra. We can identify \((\hat{g}^*)^r(I_F)\) with its dual \( (r(\sigma), \h) \)-module. Then \( \text{ad}_X^j(\xi) \) is an eigenvector of \( r(\sigma) \) with eigenvalue \( q^{-j-1} \). This will force \( \text{ad}_X^j(\xi) = 0 \) for some \( j \) large enough. That is, \( \xi \) is nilpotent. \( \square \)

In the remaining part of this subsection, we assume that \( \bar{F}/F \) is tamely ramified, i.e. the image of \( P_F \subset G_F \to \Gamma_{\bar{F}/F} \) is trivial. Then we have the stack \( \text{Loc}^\text{tame}_{G,F,t} \), called the stack of tame Langlands parameters, also denoted as \( \text{Loc}^\text{tame}_{G,F,t} \). This is an open and closed substack of \( \text{Loc}^c_{G,F,t} \).

Let \( \text{Loc}^\text{tame}_{G,F,t} \) denote the framed version. Explicitly, if we denote the image of \( \tau \) (resp. \( \sigma \)) under the map \( \Gamma_{q^t} \to \Gamma_{F,t} \to \Gamma_{\bar{F}/F} \) by \( \bar{\tau} \) (resp. \( \bar{\sigma} \)), then
\[
\text{Loc}^\text{tame}_{G,F,t} \cong (\{ (\tau, \sigma) \in \bar{\gi} \times \bar{\gi} q^{-1} \bar{\sigma} \mid \sigma \tau \sigma^{-1} = \tau^q \} \subset \bar{\gi} \times \bar{\gi} \}.
\]

**Remark 3.2.4.** One can compare \( \text{Loc}^\text{tame}_{G,F,t} \) with the commuting scheme of \( \hat{G} \), classifying pairs of elements in \( \hat{G} \) that commute with each other. They behave quite differently over \( \Q \), but share some similar properties over \( \F_\ell \) when \( \ell \mid q - 1 \).

We can similarly define \( \text{Loc}^\text{tame}_{B,F,t} \) and \( \text{Loc}^\text{tame}_{T,F,t} \). There is a diagram similar to (3.5), with the superscript \((-)^\text{tame}\) added everywhere. As in Lemma 3.2.1, \( \pi^\text{tame} \) is quasi-smooth, and \( \pi^\text{tame} \) is proper, schematic.

The inclusion \( \langle \tau \rangle \subset \Gamma_{q^t} \) induces a morphism
\[
\text{Loc}^\text{tame}_{G,F,t} \to \bar{\gi}^\text{tame} / \hat{G} \to \bar{\gi}^\text{tame} / \hat{G} \cong \hat{A} / W_0,
\]
where the second map is taking the GIT quotient, and the last isomorphism is the Chevalley restriction isomorphism, where \( \hat{A} = \hat{T} / (1 - \tau) \hat{T} \), and \( W_0 \) is the \( \bar{\tau} \)-invariants of the Weyl group of \( \hat{G} \) (e.g. see [XZ19, 4.2.3]). This morphism factors through \( \text{Loc}^\text{tame}_{G,F,t} \to (\hat{A} / W_0)^{[q]} \), where \( (\hat{A} / W_0)^{[q]} \)

\( ^7 \)This is also observed by Scholze.
is the preimage of \((^cG/\overline{^cG})[q]\) (as defined in the proof of Proposition 2.3.7). Note that \((\hat{A}/W_0)[q]\) is finite over \(\mathbb{Z}[1/p]\), and is étale over \(\mathbb{Q}\). We denote by

\[
\text{Loc}_{c,G,F,\ell}^{\text{unip}} := \text{Loc}_{c,G,F,\ell}^{\text{tame}} \times (\hat{A}/W_0)[1]\]

called the stack of unipotent parameters.

**Remark 3.2.5.**

1. When base changed to \(\mathbb{Q}\), \(\text{Loc}_{c,G,F,\ell}^{\text{unip}} \otimes \mathbb{Q}\) is open and closed in \(\text{Loc}_{c,G,F,\ell}^{\text{tame}} \otimes \mathbb{Q}\). In particular, it is still a local complete intersection. We have not checked whether this is the case over \(\mathbb{Z}[1/p]\).

2. Our terminology could be potentially misleading as for \(\rho \in \text{Loc}_{c,G,F,\ell}^{\text{unip}}\), \(\rho(\tau) \in \hat{G}\tau\) may not be a unipotent element (as \(\tau\) may not be trivial). On the other hand, if \(\tau = 1\), i.e. \(\hat{F}/F\) is unramified, then

\[
\text{Loc}_{c,G,F,\ell}^{\text{unip}} \cong \text{Loc}_{c,G,F,\ell}^{\text{unip}}/\hat{G},
\]

where as before \(\hat{U}\) is the unipotent variety of \(\hat{G}\). So the image of \(\tau\) in \(\hat{G}\) is indeed unipotent.

We let \(\text{Loc}_{c,G,F,\ell}^{\text{unip}} \rightarrow \text{Loc}_{c,T,F,\ell}^{\text{unip}}\) be the base change of \(r^{\text{tame}}\) along \(\text{Loc}_{c,T,F,\ell}^{\text{unip}} \rightarrow \text{Loc}_{c,G,F,\ell}^{\text{unip}}\). Then there is a diagram similar to (3.5), with the subscript \((-)^{\text{unip}}\) added everywhere. Finally, if \(\hat{F}/F\) is unramified, then inside \(\text{Loc}_{c,G,F,\ell}^{\text{unip}}\) there is the stack of unramified parameters.

\[
\text{Loc}_{c,G,F,\ell}^{\text{ur}} \cong \hat{G}^{-1}q^{-1}q \subset ^cG, \quad \text{Loc}_{c,G,F,\ell}^{\text{ur}} = \text{Loc}_{c,G,F,\ell}^{\text{ur}}/\hat{G}.
\]

We note that this stack is independent of the choice of \(\ell\).

### 3.3. Stack of global Langlands parameters

Now we turn to global Langlands parameters. Currently, we do not have the analogue of \(\text{Loc}_{c,G,F,\ell}\) in the global case. In fact, we are not aware of any possible way to define a stack of global Langlands parameters over \(\mathbb{Z}\) (or over \(\mathbb{Z}[1/p]\) for a function field of characteristic \(p\)). However, the general recipe as in Section 2.4 provides of a reasonable definition of the stack over \(\text{Spf } \mathbb{Z}_\ell\), at least in the global function field case, as we shall see below. \(^8\)

We fix a few notations. Let \(F\) be a global field. We regard the Galois group \(\Gamma_F\) as a profinite group, and in the global function field case the Weil group \(W_F\) as a locally profinite group. Let \(k = \mathbb{Z}_\ell\), where \(\ell \neq \text{char } F\) if \(F\) is a function field. For a place \(v\), let \(F_v\) denote the corresponding local field, and \(q_v\) the cardinality of the residue field. Let \(\Gamma_v\) (resp. \(W_v\)) denote the Galois (resp. Weil) group of \(F_v\). Let \(G\) be a connected reductive group over \(F\). We write \(G_v\) for either \(\text{G}(F_v)\) or \(G(F_v)\). The \(C\)-group of \(G\) is denoted by \(^cG\) and the \(\overline{C}\)-group of \(G_v\) is denoted by \(^cG_v\). For a place \(v\) not lying above \(\ell\), let \(\text{Loc}_{c,G,F,v,\ell}\) denote \(\text{Loc}_{c,G,F,\ell}\) for simplicity, where \(? \in \{\text{tame, unip, ur}\}\), etc. We will fix a non-empty finite set of places \(S\) (containing all the infinite places, the places above \(\ell\), and the places ramified in \(\hat{F}/F\)) and consider the quotient \(\Gamma_{F,S}\) corresponding to the maximal Galois extension of \(F\) that is unramified outside \(S\). Similarly, we have \(W_{F,S}\) in the global function field setting. Let \(U\) be the affine Dedekind scheme with fractional field \(F\) and étale fundamental group \(\Gamma_{F,S}\).

Now let \(F\) be a function field. Let \(\mathbb{F}_q\) be the algebraic closure of \(\mathbb{F}_p\) in \(F\). Then \(U\) is a smooth curve over \(\mathbb{F}_q\) and let \(\overline{U}\) be the base change of \(U\) to \(\mathbb{F}_q\). Let \(\pi_1(\overline{U})\) denote the geometric fundamental group. (We ignore the choice of base point on \(\overline{U}\) as it plays little role in the sequel.) We have

\[
1 \rightarrow \pi_1(\overline{U}) \rightarrow W_{F,S} \rightarrow \mathbb{Z} \rightarrow 1.
\]

\(^8\)Note that [Ga, GKRV] also suggest that there is a way to define the corresponding category of quasi-coherent sheaves in the global function field case.
We replace the local Weil group $W_F$ in (3.2) by $W_{F,S}$ and define
\[ \text{Loc}_{G,F,S}^\square := \mathcal{R}_{W_{F,S}, cG} \times_{\mathcal{R}_{W_{F,S}, \mathcal{O}_m \times \Gamma_{F'}}} \{ \chi \}, \quad \text{Loc}_{G,F,S}^\square = \text{Loc}_{G,F,S}^\square / \hat{G}^\square, \]
Recall that it is the inductive limit of its restriction $\text{Loc}_{G,F,S,n}$ from $\text{Spf} \mathbb{Z}_\ell$ to $\text{Spec} \mathbb{Z}/\ell^n$, and that $\text{Loc}_{G,F,S,n} = \text{Loc}_{G,F,S,n} \Gamma_{G,F,S,n}$, where $\text{Loc}_{G,F,S,n}$ classifies, for every $\mathbb{Z}/\ell^n$-algebra $A$, the space of continuous homomorphisms $\rho$ from $W_{F,S}$ to $cG(A)$ such that $d \circ \rho = \chi$, and $\hat{G}_n = \hat{G} \otimes \mathbb{Z}/\ell^n$. A priori $\text{Loc}_{G,F,S,n}$ itself is still an ind-stack. However, we have the following result.

**Proposition 3.3.1.** Assume that $\ell > 2$. Then the ind-stack $\text{Loc}_{G,F,S,n}$ is a quasi-smooth algebraic stack over $\mathbb{Z}/\ell^n$. Over $W(\mathbb{F}_\ell)/\ell^n$, it decomposes as a disjoint union of its open and closed substacks
\[
\text{Loc}_{G,F,S,n} \otimes W(\mathbb{F}_\ell)/\ell^n = \bigsqcup_{\rho_0} \text{Loc}_{G,F,S,n}^{(\rho_0)},
\]
where $\rho_0$ ranges over $\hat{G}$-conjugacy classes of $cG$-completely reducible representation of $\pi_1(\mathcal{U}) \to cG(\mathbb{F}_\ell)$ satisfying $d \circ \rho_0 = \chi$. Each $\text{Loc}_{G,F,S,n}^{(\rho_0)}$ is quasi-compact, and for every geometric point $x : \text{Spec} \mathbb{F}_\ell \to \text{Loc}_{G,F,S,n}^{(\rho_0)}$ the semisimplification of $\rho_x : \pi_1(\mathcal{U}) \to cG(\mathbb{F}_\ell)$ is $\hat{G}$-conjugate to $\rho_0$.

The restriction of $\ell$ is due to the fact that we make use of de Jong’s conjecture, which is only proved under the current assumption. Certainly such restriction is expected to be removed.

**Proof.** Unlike the local situation we do not have an explicit presentation of $\text{Loc}_{G,F,S,n}$ as an algebraic stack “by hand” as we know very little about the structure of the global Weil group. Instead, we use the Artin-Lurie representability theorem [Li4] 7.5.1. The only conditions to check are Condition (6) (7) of loc. cit.. As in (2.20), the tangent space of $\text{Loc}_{G,F,S,n}$ at a point $\rho : W_{F,S} \to cG(A)$ is $C_{cts}(W_{F,S}, \text{Ad}_\rho)[1]$, where $A$ is a classical $\mathbb{Z}/\ell^n$-algebra. As the continuous group cohomology of $C_{cts}(\pi_1(\mathcal{Y}), \text{Ad}_\rho)[1]$ coincides with the étale cohomology of $\mathcal{Y}$ (which is affine as $S$ is non-empty), we see that $C_{cts}(W_{F,S}, \text{Ad}_\rho)[1]$ concentrates in degree $[-1, 1]$, and is a finite $A$-module in each degree. This verifies Condition (7) of loc. cit.. In addition, it shows that if $\text{Loc}_{G,F,S,n}$ is representable, then it is quasi-smooth.

It turns out Condition (6) is the deepest part in this situation. We need to check that for every (classical) noetherian complete local $\mathbb{Z}/\ell^n$-algebra $(A, m, \kappa)$, where $\kappa$ is finite over $\mathbb{F}_\ell$, and $\bar{\rho} : \Gamma \to cG(\kappa)$, $\text{Loc}_{G,F,S,n}(A) \to \varprojlim_j \text{Loc}_{G,F,S}(A/m^j)$ is an isomorphism. This follows from the next proposition by choosing a faithful representation $cG \to \text{GL}_m$.

**Proposition 3.3.2.** Assume that $\ell > 2$. Let $(A, m, \kappa)$ be as above, and let us equip $A$ with the $m$-adic topology. Let $\rho : W_{F,S} \to \text{GL}_m(A)$ be a continuous homomorphism. Then $\rho(\pi_1(\mathcal{U}))$ is finite.

**Proof.** If $A \simeq \kappa[[t]]$, this is de Jong’s conjecture [JJ01], proved in [Ga07] when $\ell > 2$ (see also [BK06]).

Next, we assume that $A$ is reduced (in particular $A$ is an $\mathbb{F}_\ell$-algebra). We write $W = A^m$ for simplicity. We claim that for every $j = 1, \ldots, m$, the continuous function
\[ F_j : \pi_1(\mathcal{U}) \to A_{\text{red}}, \quad \gamma \mapsto \text{Tr}(\rho(\gamma) | \wedge^j W) \]
takes value in $\kappa$. If not, there would be a map $A \to \kappa[[t]]$ such that $\text{Tr}(\rho(\gamma) | \wedge^j W) \in \kappa[[t]]$ is not algebraic over $\mathbb{F}_\ell$. This contradicts with de Jong’s conjecture.

Now the functions $F_j$ are locally constant. Therefore, there is an open subgroup $H \subset \pi_1(\mathcal{U})$ such that $F_j|_H = 0$ for all $j = 1, 2, \ldots, m$. Therefore, $\rho(H) \subset \mathcal{U}(A)$, where $\mathcal{U}$ is the variety of unipotent $m$ by $m$ matrices. We may replace $A$ by its quotient ring, and then proceed as in [JJ01 2.8-2.10] to finish the proof.
Finally, we consider general $A$. Let $A_{\text{red}}$ be its quotient by the nilradical. Then as $A$ is noetherian, the kernel $\text{GL}_n(A) \to \text{GL}_n(A_{\text{red}})$ is a nilpotent group of exponent some power of $\ell$. Then we can again proceed as in [dJ01] 2.9-2.10 to conclude.

We have proved the representability of $\text{Loc}^{c}_{G,F,S,n}$. Let $Y = \text{Spec} A \to \text{Loc}^{c}_{G,F,S,n}$ be a morphism, where $Y$ is classical connected affine scheme. Then there is some quotient $\Gamma$ of $W_{F,S}$, which is an extension of $\mathbb{Z}$ by a finite quotient $\overline{\Gamma}$ of $\pi_1(\overline{U})$ such that the representation factors as $\rho : \Gamma \to \hat{c} G(A)$. Let $cR_{\Gamma,\hat{G}_n}$ be the moduli of pseudo-presentations of $\Gamma$ over $\mathbb{Z}/\ell^n$ (see Definition 2.2.17). Then we have the natural morphism

$$Y \to cR_{\Gamma,\hat{G}_n}, \quad \rho \mapsto \text{Tr}(\rho|_{\Gamma}).$$

as in (2.16). By Proposition 2.3.5 $cR_{\Gamma,\hat{G}_n}$ is finite over $\mathbb{Z}/\ell^n$. Therefore, $Y$ maps to one point of $cR_{\Gamma,\hat{G}_n}$. It follows that $\text{Loc}^{c}_{G,F,S,n}$ admits the desired decomposition (3.9).

Remark 3.3.3. One may think the set of $\rho_0$ in the proposition as the global analogue of mod $\ell$ inertia types in the local case. The Frobenius of $\mathbb{F}_q$ acts on this set as it acts on $\pi_1(\overline{U})$ by outer automorphisms. Clearly, $\text{Loc}^{c}_{G,F,S,n}$ is non-empty if and only if $\rho_0$ is fixed under this action.

Lemma 3.3.4. If $\rho_0$ is absolutely irreducible, then $\text{Loc}^{c}_{G,F,S,n}$ is classical and is a local complete intersection of dimension 0.

Proof. Using Proposition 2.2.13 and a calculation of the Euler characteristic of the cohomology of $W_{F,S}$, it is enough to show that $\dim \text{Loc}^{c}_{G,F,S,1} = \dim \hat{G}_1$. Assume that $\rho_0$ factors through $\Gamma \to cG(\mathbb{F}_q)$ for some finite group $\Gamma$. As $\rho_0$ is absolutely irreducible, the fiber of $R_{\Gamma,cG_1} \to R_{\Gamma,\hat{G}_1}$ consists of a single $\hat{G}$-orbit of dimension $= \dim \hat{G}_1$. The desired dimension estimate follows.

One may expect that the whole stack $\text{Loc}^{c}_{G,F,S,n}$ is classical, as in the local situation. As mentioned in Remark 2.3.8, $\text{Loc}^{c}_{G,F,S,n}$ is classical if and only if $\dim \text{Loc}^{c}_{G,F,S,n} = 0$. Unfortunately, this is not always the case.

Example 3.3.5. Consider the case $G = \text{PGL}_2$ (so $cG = \text{GL}_2$), and let $\rho_0$ be the trivial representation of $\pi_1(\overline{U})$. Then $\text{Loc}^{c}_{G,F,S,1}$ consists of those $\rho : W_{F,S} \to \text{GL}_2$ such that $\rho|_{\pi_1(\overline{U})}$ is a self extension of the trivial character. Note that there is an $H^1(U,\mathbb{F}_q)$-family of self extensions of the trivial character of $\pi_1(\overline{U})$. It follows that if the multiplicity of one Frobenius eigenvalue on $H^1(U,\mathbb{F}_q)$ is greater than one, then $\dim \text{Loc}^{c}_{G,F,S,1} > \dim \hat{G}_q$, and $\text{Loc}^{c}_{G,F,S,n}$ is non-classical.

The embedding $W_{F,v} \to W_F$ up to conjugacy induces a well-defined morphism

$$\text{res} : \text{Loc}^{c}_{G,F,S} \to \prod_{v \in S} \text{Loc}_v \times \prod_{w \notin S} \text{Loc}_w^{\text{unr}}. \tag{3.10}$$

If we enlarge $S$ by adding one place $S \cup \{w_0\}$, then we have the following Cartesian diagram

$$\begin{array}{ccc}
\text{Loc}^{c}_{G,F,S} & \overset{\text{res}}{\longrightarrow} & \prod_{v \in S} \text{Loc}_v \times \text{Loc}_w^{\text{unr}} \\
\downarrow & & \downarrow \\
\text{Loc}^{c}_{G,F,S,\cup \{w_0\}} & \overset{\text{res}}{\longrightarrow} & \prod_{v \in S} \text{Loc}_v \times \text{Loc}_w^{\text{unr}}.
\end{array} \tag{3.11}$$

Indeed, it is obvious Cartesian if one forgets the derived structure. To see that it is a Cartesian even in the derived sense, one compares the tangent complexes. We leave the details to readers. (See [GV18, §8] for an argument in a closely related context.)
For every place \( v \in S \), we choose a finite extension \( L_v/F_v \) that is Galois over \( F_v \). Let
\[
\text{Loc}^\wedge_{G,F,\{L_v\}} := \text{Loc}^\wedge_{G,F,S} \times \prod_{v \in S} \text{Loc}_v \prod_{v \in S} \text{Loc}_{G_v,L_v/F,v,\ell}
\]
As \( \text{Loc}_{G_v,L_v/F,v,\ell} \) is open and closed in \( \text{Loc}_v \), the stack \( \text{Loc}^\wedge_{G,F,\{L_v\}} \) is also open closed in \( \text{Loc}^\wedge_{G,F,S} \). It is expected that its restriction to \( \mathbb{Z}/\ell^n \) is quasi-compact.

**Remark 3.3.6.** Let \( \text{Loc}^\wedge_{G,F} := \varinjlim_S \text{Loc}^\wedge_{G,F,S} \). It is very interesting to understand its structure.

4. Coherent sheaves on the stack of Langlands parameters

In this section, we formulate some conjectures about the coherent sheaves on the stack of local Langlands parameters. We will also formulate a (possibly imprecise) categorical form of the arithmetic local Langlands correspondence, which would give a theoretical explanations of the existence of the coherent sheaves with the desired properties. In the last subsection, we formulate some local-global compatibility conjectures using these coherent sheaves. We also survey some known results, which provide some evidences of these conjectures. In this section, \( k \) will also denote a noetherian commutative ring.

4.1. The category of representations of \( G(F) \). Let \( F \) be a non-archimedean local field, with \( \mathcal{O}_F \) its ring of integers, \( \kappa_F \) its residue field and let \( q = \sharp \kappa_F = p^r \). Let \( G \) be a reductive group over \( F \). Let \( \text{Rep}(G(F), k) \) denote the abelian category of smooth representations of \( G(F) \) on \( k \)-modules. It is a Grothendieck abelian category (a set of generators being given below). For a closed subgroup \( K \subset G(F) \), we similar have \( \text{Rep}(K, k) \). We always denote by \( 1 \) the trivial representation. Let
\[
c\text{-ind}^{G(F)}_K : \text{Rep}(K, k) \otimes \to \text{Rep}(G(F), k) \otimes
\]
denote the usual compact induction functor, and write
\[
\delta_K := c\text{-ind}^{G(F)}_K 1 \cong C_c^\infty(G(F)/K, k),
\]
which is the space of \( k \)-valued locally constant functions on \( G(F)/K \) with compact support, on which \( G(F) \) acts by left translation.

If \( K \) is open, then \( c\text{-ind}^{G(F)}_K \) is the left adjoint of the forgetful functor. By definition of smooth representations, the collection \( \{ \delta_K \}_K \), with \( K \) open, form a set of generators of \( \text{Rep}(G(F), k) \). We say an open compact subgroup \( K \) of \( G(F) \) is \( k \)-admissible (or just admissible if \( k \) is clear from the context) if the index of any open subgroup of \( K \) is invertible in \( k \). Note that if \( p \) is invertible in \( k \), \( k \)-admissible open compact subgroups always exist. E.g. the pro-\( p \) Sylow subgroup \( I(1) \) of an Iwahori subgroup (sometimes also called prop-\( p \)-Iwahori subgroup) of \( G(F) \) is \( k \)-admissible. On the other hand, every open compact subgroup is \( \mathbb{Q} \)-admissible. If \( K \) is \( k \)-admissible, then \( \delta_K \) is a projective object in \( \text{Rep}(G(F), k) \).

Next, let \( \text{Rep}(G(F), k) \) denote its (unbounded) \( \infty \)-derived category of \( \text{Rep}(G(F), k) \) (\cite[1.3.5]{Lu2}). This category behaves quite differently depending on whether \( p \) is invertible in \( k \) or not. For our purpose, we assume that \( p \) is invertible in \( k \) throughout this section. In this case \( c\text{-ind}^{G(F)}_K \) is an exact functor. If \( K \) is a \( k \)-admissible open compact subgroup, \( \delta_K \) is a compact object in \( \text{Rep}(G(F), k) \). It follows that \( \text{Rep}(G(F), k) \) is compactly generated, with a set of generators given by \( \{ \delta_K \}_K \), where \( K \) are admissible.

**Remark 4.1.1.** If \( F \) is of characteristic zero and \( k \) is a field of characteristic \( p \) (which is not the case we consider), then \( \delta_{I(1)} \) itself is a compact generator of \( \text{Rep}(G(F), k) \) (see \cite{Sc15}).

In general if an open compact subgroup \( K \) is not \( k \)-admissible, then \( \delta_K \) may not be compact in \( \text{Rep}(G(F), k) \).
Example 4.1.2. If $G = \mathbb{G}_m$, $K = \mathcal{O}_F^\times$, and $k = \mathbb{F}_\ell$ where $\ell$ is a prime dividing $q - 1$, then $\delta_K \simeq C_c(\mathbb{Z}, \mathbb{F}_\ell)$ is not compact in $\text{Rep}(\mathfrak{F}_\ell)$. Tautologically, for any open compact subgroup $K$ in $\text{Rep}(\mathcal{O}_F^\times)$, $\delta_K$ is invertible in $k$, retracts, and let $\delta$ be the full subcategory generated by these $\delta_K$ (for all open compact $K$) under finite colimits and retracts, and let

$$\text{Rep}_{f,g}(G(F), k) \subset \text{Rep}(G(F), k)$$

be the full subcategory generated by these $\delta_K$ (for all open compact $K$) under finite colimits and retracts, and let

$$\text{Rep}^{\text{ren}}(G(F), k) = \text{Ind} \text{Rep}_{f,g}(G(F), k)$$

be its ind-completion. Tautologically, for any open compact subgroup $K \subset G(F)$, $\delta_K$ is compact in $\text{Rep}^{\text{ren}}(G(F), k)$, and there is a colimit preserving functor

$$\text{Rep}^{\text{ren}}(G, k) \to \text{Rep}(G, k).$$

If $k$ is a field of characteristic zero, this is an equivalence, as $\text{Rep}(G, k)^\mathcal{O}$ has finite global cohomological dimension by a result of Bernstein. In general, this functor induces an equivalence $\text{Rep}^{\text{ren}}(G, k)^+ \cong \text{Rep}(G, k)^+$ when restricted to the bounded from below subcategories (w.r.t. the natural t-structure). More details will appear in [HZ].

For open compact subgroup $K \subset G(F)$, we define the corresponding $k$-coefficient derived Hecke algebra as

$$H_{G,K,k} := (\text{End} \delta_K)^{\mathcal{O}_K},$$

where the (derived) endomorphism is taken in $\text{Rep}(G(F), k)$. (So $H_{G,K,k}$ is an object in $\text{Alg}(\text{Mod}_k)$, i.e. an $E_1$-algebra.) Sometimes we omit $G$ or $k$ from the subscript, if they are clear from the context. Note that its zeroth cohomology

$$H^0 H_K \cong C_c(K \setminus G(F)/K, k),$$

is just the usual Hecke algebra with $k$-coefficient, with algebra structure given by convolution product. In addition, as $k$-modules,

$$H_K \cong \bigoplus_{g \in K \setminus G(K)} C^*(K \cap gKg^{-1}, k),$$

where the right hand side denotes the (pro-finite) group cohomology of $K \cap gKg^{-1}$ with trivial coefficient $k$. In particular, if $K$ is $k$-admissible, then $H_{G,K,k}$ concentrates in degree zero.

Remark 4.1.3. By choosing an invariant Haar measure on $G(F)$ assigning the volume of the pro-unipotent radical of one Iwahori subgroup (and therefore every Iwahori subgroup) to be 1, one can define the usual full Hecke algebra $H_G$ of $G(F)$. Namely, the underlying space is $\delta_{\{1\}} \simeq C_c^\infty(G)$, with the multiplication given by the usual convolution. If $K$ is $k$-admissible, its volume $\text{vol}(K)$ is invertible in $k$ and therefore there is an idempotent $e_K = \frac{1}{\text{vol}(K)} \text{ch}_K$ of $H_G$ as usual, where $\text{ch}_K$ is the characteristic function of $K$. It follows that as usual there is an equivalence of categories between $\text{Rep}(G(F), k)^\mathcal{O}$ and the category of non-degenerate $H_G$-modules. We have $\delta_K \cong H_G e_K$ as left $H_G$-modules, and $H_{G,K} \cong e_K H_G e_K$.

Let $\text{Mod}_{H_K}$ denote the $\mathcal{O}$-category of left $H_K$-modules. It follows from general nonsense that there is the pair of adjoint functors

$$\delta_K \otimes_{H_K} (-) : \text{Mod}_{H_K} \leftarrow \text{Rep}(G(F), k) : \text{Hom}(\delta_K, -).$$

If $K$ is admissible, then $W \mapsto \delta_K \otimes_{H_K} W$ is fully faithful. (It is fully faithful for any $K$ if we replace $\text{Rep}(G(F), k)$ by $\text{Rep}^{\text{ren}}(G(F), k)$.)

For two open compact subgroups $K_1$ and $K_2$ of $G(F)$, there is the $(H_{K_2} \times H_{K_1})$-bi-module

$$K_1 H_{K_2} := \text{Hom}(\delta_{K_1}, \delta_{K_2}),$$
where again the Hom is taken in the derived sense. Its degree zero cohomology is given by
\[ H^0(K_1, H_{K_2}) \cong C_c(G(F)/K_2)^{K_1} = C_c(K_1 \backslash G(F)/K_2), \]
the space of \( K_1 \times K_2 \)-invariant, compactly supported functions on \( G(F) \). If either \( K_1 \) and \( K_2 \) is \( k \)-admissible, then \( K_1 H_{K_2} = H^0(K_1, H_{K_2}) \).

Tautologically, under the above identification, the map \( \iota_{K_1, K_2} : \delta_{K_1} \to \delta_{K_2} \) sending \( \text{ch}_{K_1} \in \delta_{K_1} \) to \( \text{ch}_{K_1, K_2} \in \delta_{K_2} \) corresponds to \( \text{ch}_{K_1, K_2} \in C_c(K_1 \backslash G(F)/K_2) \). On the other hand,
\[ \text{Av}_{K_1, K_2} : \delta_{K_1} \to \delta_{K_2}, \quad (\text{Av}_{K_1, K_2} f)(g) = \int_{K_2} f(gk)dk. \]
corresponds to \( \text{vol}(K_2)\text{ch}_{K_1, K_2} \).

Tautologically, there is a \( G(F) \)-module homomorphism
\[ (4.1) \quad \delta_{K_1} \otimes_{H_{K_1}} K_1 H_{K_2} \to \delta_{K_2}. \]
If \( K_1 \subset K_2 \), and \( K_2 \) is a \( k \)-admissible open compact subgroup (so is \( K_1 \)), then \((4.1)\) is an isomorphism. But this may not be the case in general.

**Example 4.1.4.** Let \( G = SL_2 \), \( K_2 = K = SL_2(O_F) \), and \( K_1 = I \) the standard Iwahori subgroup. Let \( k = \mathbb{F}_\ell \) with \( \ell > 2 \) and \( \ell \mid p + 1 \). Then \( I \) is \( k \)-admissible, but \( K \) is not. In this case, \((4.1)\) is not an isomorphism. In fact, \( \delta_I \otimes_{H_I} I H_K \) does not even concentrate in degree zero.

As is well-known, usually the (local) Langlands correspondence depends on a choice of Whittaker datum. Our last topic of this subsection is to briefly review this notion. Assume that \( G \) is quasi-split over \( F \) and \( k \) is a \( \mathbb{Z}[\mu_{p^n}][1/p] \)-algebra. A Whittaker datum of \( G \) is a choice of the unipotent radical \( U \) of an \( F \)-rational Borel subgroup of \( G \), and a non-degenerate character \( \Psi : U(F) \to (U/[U, U])(F) \to k^\times \).

The set of Whittaker data up to conjugation action by \( G(F) \), denoted by \( \mathcal{W} \), form a torsor under the finite group
\[ \Omega := G_{ad}(F)/(G(F)/Z_G(F)) \cong \ker(H^1(F, Z_G) \to H^1(F, G)). \]
Fix a Whittaker datum \((U, \Psi)\), let
\[ \text{Whit}_{U, \Psi} := c \cdot \text{ind}_{U(F)}^{G(F)} \Psi \in \text{Rep}(G(F), k) \hat{\otimes}. \]
We note that \( \text{Whit}_{U, \Psi} \) is not finitely generated as a \( G(F) \)-module. However, it can be written as a filtered colimit of finitely generated projective objects in \( \text{Rep}(G(F), k) \hat{\otimes} \) [Ro75, Prop. 3].

Note that for every open compact subgroup \( K \subset G(F) \), the \( H_K \)-module
\[ \text{Hom}(\delta_K, \text{Whit}_{U, \Psi}) \cong \text{Whit}_{U, \Psi}^K \]
concentrates in degree zero, and can be identified with space \( C_c(K \backslash G(F)/(U(F), \Psi)) \) of \((U(F), \Psi)\)-invariant functions on \( K \backslash G(F) \) that are compactly supported modulo \( U(F) \).

If \( G \) is unramified, then the set of hyperspecial subgroups up to conjugation by \( G(F) \), denoted by \( \mathcal{H}_s \), is torsor under
\[ \Omega' := \ker(H^1(F, Z_G)/H^1(O_F, Z_G) \to H^1(F, G)). \]
In other words, \( \Omega' = \Omega/\Omega' \). In particular, if \( Z_G \) is connected, \( \Omega = \Omega' \).

There is a map from \( \mathcal{W} \) to \( \mathcal{H}_s \), compatible with the actions of \( \Omega \) and \( \Omega' \). It is characterized by the following property: given \((U, \Psi)\), there is \( K = G(O) \) in the associated conjugacy class of hyperspecial subgroups such that \( K_U := K \cap U(F) \) is the \( O_F \)-points of the unipotent radical of a Borel of \( G \), and the conductor of \( \Psi : (U/[U, U])(F) \to k^\times \) is \( K_U/[K_U, K_U] \). In this case, \( \text{Whit}_{U, \Psi}^K \) is a free \( H^0 H_K \)-module of rank one. This is known as the Casselman-Shalika formula.
4.2. Derived Satake isomorphism. Let $k$ be a $\mathbb{Z}[1/p]$-algebra. We will fix $\iota : \Gamma_q \to \Gamma_F^t$ so we have the stack $\text{Loc}_{c,G,F,t}$ over $\mathbb{Z}[1/p]$. In this subsection, we assume that $G$ is unramified. Then we have the stacks $\text{Loc}_{c,G,F} \subset \text{Loc}_{c,G,F,t}$. Our first conjecture can be regarded as the derived Satake isomorphism.\footnote{The author came up with this conjecture during conference on “Modularity and Moduli Spaces” in Oaxaca, inspired by Emerton’s hope to “see” the action of derived Hecke algebra on the cohomology of modular curves (and general Shimura varieties), and encouraged by Feng’s result on spectral Hecke algebra [Fe]. See Remark 4.6.10 for a discussion.} We remind readers that all functors are derived.

**Conjecture 4.2.1.** Let $K$ be a hyperspecial subgroup of $G$. Then there is a natural isomorphism of $k$-algebras

$$H_K \cong (\text{End}_{\text{Loc}_{c,G,F,t}}(\mathcal{O}_{\text{Loc}_{c,G,F}}))^{\text{op}},$$

which induces the classical Satake isomorphism after taking $H^0$:

$$C_c(K\backslash G(F)/K,k) \cong H^0H_K \cong H^0\text{End}_{\text{Loc}_{c,G,F,t}}(\mathcal{O}_{\text{Loc}_{c,G,F}}) \cong H^0\Gamma(\text{Loc}_{c,G,F}^\text{ur},\mathcal{O}_{\text{Loc}_{c,G,F}}).$$

In addition, this isomorphism is compatible with the isomorphism $\psi$ from Proposition 3.1.10 for different choices of $\iota$.

As $\text{Loc}_{c,G,F,t}$ is an open and closed substack in $\text{Loc}_{c,G,F}$, we may replace $\mathcal{O}_{\text{Loc}_{c,G,F,t}}$ by $\mathcal{O}_{\text{Loc}_{c,G,F}}$ in the above conjecture.

**Remark 4.2.2.**

1. Note that this conjecture is a non-trivial even if $k = \mathbb{C}$. It amounts to saying that $\text{End}_{\text{Loc}_{c,G}}(\mathcal{O}_{\text{Loc}_{c,G}^\text{ur}}) = \text{End}_{\text{Loc}_{c,G}^\text{unip}}(\mathcal{O}_{\text{Loc}_{c,G}^\text{ur}})$ concentrates in degree zero. It can be deduced from Theorem 4.3.3 below. But we invite readers to check this for different choices of $\iota$.

2. It would be interesting to formulate a mod $p$ derived Satake isomorphism (or even an integral derived Satake isomorphism) in this style. The non-derived version with integral coefficients appears in [Zhu], and its formulation involves the Vinberg monoid of $\hat{G}$.

One can check this conjecture by hands when $G = T$ is an unramified torus.

**Proposition 4.2.3.** Conjecture 4.2.1 holds for unramified tori.

**Proof.** We have the action of $\sigma$ on $\hat{T}$. We write $F_n$ (instead of $\hat{F}$) for the degree $n$ unramified extension such that $T$ is split, and let $\kappa_n$ be the residue field of $F_n$, on which $\sigma$ also acts.

The surjective map $I_F = I_{F_n} \to \kappa_n^\times$ from the local class field theory induces an isomorphism

$$(\overline{c_{\mathcal{R}_{\kappa_n^{\times},T}}})^{\sigma} \times (\hat{T})/\hat{T} \cong \text{Loc}_{c,T,F,t}^{\text{tame}},$$

compatible with the isomorphism $\psi$ from Proposition 3.1.10 for different choices of $\iota$. Here $(\overline{c_{\mathcal{R}_{\kappa_n^{\times},T}}})^{\sigma}$ is the (classical) moduli of $\sigma$-equivariant homomorphisms from $\kappa_n^{\times}$ to $\hat{T}$. It follows that

$$\text{End}_{\text{Loc}_{c,T,F,t}}(\mathcal{O}_{\text{Loc}_{c,T,F,t}^{\text{unip}}}) \cong \text{End}_{\overline{c_{\mathcal{R}_{\kappa_n^{\times},T}}}}^{\sigma}(\mathcal{O}_{\{1\}} \otimes \Gamma(\hat{T}/(\sigma - 1),\mathcal{O})), $$

where $\mathcal{O}_{\{1\}}$ denotes the skyscraper sheaf at the point of $(\overline{c_{\mathcal{R}_{\kappa_n^{\times},T}}})^{\sigma}$ corresponding to the trivial representation. On the other hand, there is the canonical isomorphism $H_K \cong C^*(T(\kappa_F),k) \otimes H^0H_K$.

Then the desired isomorphism follows from the classical Satake isomorphism

$$\Gamma(\hat{T}/(\sigma - 1),\mathcal{O}) \cong H^0H_K$$

and the canonical isomorphism (constructed below)

$$k[(\overline{c_{\mathcal{R}_{\kappa_n^{\times},T}}})^{\sigma}] \simeq kT(\kappa_F),$$

$$(4.3)$$
where we recall the l.h.s is the ring of regular functions of \((clR_{\kappa,F})^\sigma\), and the r.h.s is the group ring of \(T(\kappa_F)\).

To construct (4.3), we first assume that \(T\) is split, so \(\sigma\) acts trivially on \(\hat{T}\) and \(n = 1\). Then
\[
k[(clR_{\kappa,F})^\sigma] = k[X_*(T) \otimes \kappa^\times],
\]
and \(X_*(T) \otimes \kappa^\times \cong T(\kappa_F)\), where \(X_*(T)\) denote the cocharacter lattice of \(T\) (defined over \(F\)). Using the norm map \(\text{Res}_{\kappa,F} T_{\kappa} \to T(\kappa_F)\), the construction (4.3) for general unramified tori reduces to the split case. \(\square\)

**4.3. Coherent Springer sheaf.** In this subsection, we describe a (complex of) coherent sheaf on the stack of tame Langlands parameters, whose definition is reminiscent of the definition of the Springer sheaf. Therefore, it is called the coherent Springer sheaf\(^{10}\). We describe some of its (conjectural) properties.

We will assume that \(\tilde{F}/F\) is tamely ramified. Recall the morphism \(\pi^{\text{tame}} : \text{Loc}_{c,t}^{\text{tame}} G,F,\iota \to \text{Loc}_{c,t}^{\text{tame}} G,F,\iota\) and \(\pi^{\text{unip}} : \text{Loc}_{c,t}^{\text{unip}} G,F,\iota \to \text{Loc}_{c,t}^{\text{tame}} G,F,\iota\). For \(? = \text{tame}\) and \(\text{unip}\), let
\[
\text{CohSpr}^?_{c,t} := \pi^?_* \mathcal{O}_{\text{Loc}_{c,t}^{?} G,F,\iota} \in \text{Coh}(\text{Loc}_{c,t}^{\text{tame}} G,F,\iota).
\]
Again, we recall all the functors are derived. We have the following conjecture\(^{11}\).

**Conjecture 4.3.1.** Assume that \(G\) is quasi-split over \(F\). Let \(I\) (resp. \(I(1)\)) be the Iwahori (resp. pro-\(p\) Iwahori) subgroup of \(G(F)\). Then for a choice of a Whittaker datum \((U, \Psi)\), there are natural isomorphisms of \(k\)-algebras
\[
H_I \cong (\text{End}_{\text{Loc}_{c,t}^{\text{tame}} G,F,\iota} \text{CohSpr}_{c,t}^{\text{unip}} G,F,\iota)^{\text{op}}, \quad H_I(1) \cong (\text{End}_{\text{Loc}_{c,t}^{\text{tame}} G,F,\iota} \text{CohSpr}_{c,t}^{\text{tame}} G,F,\iota)^{\text{op}},
\]
which are compatible with the isomorphism \(\psi\) from Proposition 3.1.10 for different choices of \(\iota\). In particular, there is a fully faithful embedding
\[
\text{Mod}_{H_I(1)} \to \text{IndCoh}(\text{Loc}_{c,t}^{\text{tame}} G,F,\iota), \quad M \mapsto \text{CohSpr}_{c,t}^{\text{tame}} G,F,\iota \otimes H_I(1) M.
\]

Note that when computing the endomorphisms, \(\text{CohSpr}_{c,t}^{\text{unip}} G,F,\iota\) is still considered as a coherent sheaf on \(\text{Loc}_{c,t}^{\text{tame}} G,F,\iota\), similar to the unramified case as in Conjecture 4.2.1.

**Remark 4.3.2.** The conjecture in particular implies that there should exist a natural morphism
\[
Z_{c,t}^{\text{tame}} := H^0 \Gamma(\text{Loc}_{c,t}^{\text{tame}} G,F,\iota, \mathcal{O}) \to Z(H_I(1)),
\]
where \(Z(H_I(1))\) is the center of \(H_I(1)\), which should fit into the following commutative diagram
\[
\begin{array}{ccc}
Z_{c,t}^{\text{tame}} & \longrightarrow & Z(H_I(1)) \\
\downarrow & & \downarrow \cong \\
(Z_{c,t}^{\text{tame}})_{W_0} & \cong & (H_{T,I(1)})^{W_0}.
\end{array}
\]

\(^{10}\)We learned this name from D. Ben-Zvi.

\(^{11}\)Let us comment on the history of this conjecture, according to our knowledge. Some form of the conjecture was first studied by Ben-Zvi, Helm and Nadler a few years ago, as a natural continuation/combination of their previous works. Hellmann came up with a similar conjecture independently when studying \(p\)-adic automorphic forms and \(p\)-adic Galois representations (see his article [He] for an account). We came up with these ideas when trying to find the generalization of the work [YZ] to the Iwahori level structure (see [16] for a discussion). The emphasis of general coefficients in our formulation is our hope to understand the arithmetic level rising/lowering in this framework. It is quite remarkable that people from different considerations are led to study the same object.
Here $T$ denotes the abstract Cartan of $G$, and $W_0$ is the relative Weyl group of $G$. The left vertical map is from (3.7). (Note that $W_0^\dagger \cong W_{G,T}$.) The right vertical isomorphism comes from [Vï15] 5.1, and the bottom isomorphism is Conjecture 4.3.1 for tamely ramified tori (in this case $\text{CohSpr}^\text{tame}_{T,F,\iota} \cong \mathcal{O}_{\text{LocSpr}^\text{tame}_{T,F,\iota}}$). In fact, the proof of Proposition 4.2.3 already verifies this when $T$ is an unramified torus.

For evidences of the conjecture, we just mentioned that it holds for unramified tori. In addition, in a forthcoming work with Hemo ([HZ]), we will prove the following result. Let $T$ be an unramified torus. $CohSpr$ [Vi15, 5.1], and the bottom isomorphism is Conjecture 4.3.1 for tamely ramified tori (in this case $\text{CohSpr}^\text{tame}_{T,F,\iota} \cong \mathcal{O}_{\text{LocSpr}^\text{tame}_{T,F,\iota}}$). In fact, the proof of Proposition 4.2.3 already verifies this when $T$ is an unramified torus.

For evidences of the conjecture, we just mentioned that it holds for unramified tori. In addition, in a forthcoming work with Hemo ([HZ]), we will prove the following result. Let $k = \overline{\mathbb{Q}}_\ell$. We write $\text{Loc}^\text{unip}_G$ for $\text{Loc}^\text{unip}_{G,F,\iota}$ (see Proposition 3.1.11).

**Theorem 4.3.3.** Assume that $G$ is unramified and $k = \overline{\mathbb{Q}}_\ell$. Let $I$ be an Iwahori subgroup of $G(F)$. Then for a choice of Whittaker datum $(U, \Psi)$, there is a natural isomorphism

$$H_I \cong \text{End}_{\text{Loc}^\text{unip}_G} \text{CohSpr}^\text{unip}_G.$$  

inducing a fully faithful embedding

$$\text{Mod}_{H_I} \rightarrow \text{IndCoh}(\text{Loc}^\text{unip}_G), \quad M \mapsto \text{CohSpr}^\text{unip}_G \otimes_{H_I} M.$$  

This functor sends

- $\text{Whit}^I_{U,\Psi}$ to $\mathcal{O}_{\text{Loc}^\text{unip}_G}$,
- $\mathbb{I}H_K$ to $\mathcal{O}_{\text{Loc}^\text{unip}_{G,F}}$, if $K$ is a hyperspecial subgroup of $G$ associated to $(U, \Psi)$.

The first statement in fact follows from Theorem 4.5.8 stated below. We remark that Hellmann has obtained partial results in this direction (see [Hel]). In addition, Ben-Zvi-Chen-Helm-Nadler also obtained the very similar results ([BCHN]).

### 4.4. Conjectural coherent sheaves.

With the conjectures in the previous two subsections in mind, it is natural to go one step further to conjecture that for every open compact subgroup $K \subset G(F)$, there is a coherent sheaf $\mathfrak{A}_{G,K}$ on $\text{Loc}^\text{unip}_{G,F,\iota}$, whose (opposite) endomorphism algebra $\text{End}_{\text{Loc}^\text{unip}_G} \mathfrak{A}_{G,K}$ in $\text{Coh}(\text{Loc}^\text{unip}_{G,F,\iota})$ in $H_K$. The best way to formulate this is as follows. For simplicity, we will assume that the center of $G$ is connected, which is equivalent to asking the derived group of $G$ to be simply-connected. Recall our convention of the category of coherent sheaves on $\text{Loc}^\text{unip}_{G,F,\iota}$ in Remark 3.1.6.

**Conjecture 4.4.1.** We fix a Whittaker datum $(U^\ast, \Psi)$ of the quasi-split inner form $G^\ast$ of $G$. There is an exact fully faithful functor

$$\mathfrak{A}_G : \text{Rep}_{\text{unip}}^\text{ren}(G(F), k) \rightarrow \text{Coh}(\text{Loc}^\text{unip}_{G,F,\iota}),$$

compatible with the isomorphism $\psi$ in Proposition 3.1.10 for different choices of $\iota$. The induced colimit preserving functor $\text{Rep}_{\text{ren}}(G(F), k) \rightarrow \text{IndCoh}(\text{Loc}^\text{unip}_{G,F,\iota})$ is still denoted by $\mathfrak{A}_G$. For every open compact subgroup $K$ of $G(F)$, let $\mathfrak{A}_{G,K} := \mathfrak{A}_G(\delta_K)$. Then the following should hold.

- Then sheaf $\mathfrak{A}_{G,\delta(1)} \simeq \mathfrak{A}_G(\text{lim}_{\rightarrow K} \delta_K) = \text{lim}_{\rightarrow K} \mathfrak{A}_{G,K}$ is coherent when restricted to each connected component of $\text{Loc}^\text{unip}_{G,F,\iota}$, and has full support.
- If $G = G^\ast$ is unramified and $K$ is a hyperspecial subgroup associated to $(U, \Psi)$, then $\mathfrak{A}_{G,K} \cong \mathcal{O}_{\text{Loc}^\text{unip}_{G,F}}$.
- If $G = G^\ast$ is quasi-split and is tamely ramified, and $K = I(1)$ (resp. $K = I$), then $\mathfrak{A}_{G,I(1)} \cong \text{CohSpr}^\text{tame}_{G,F,\iota}$ (resp. $\cong \text{CohSpr}^\text{unip}_{G,F,\iota}$).
- If $G = G^\ast$ is quasi-split, $\mathfrak{A}_G(\text{Whit}_{U,\Psi}) \cong \mathcal{O}_{\text{Loc}^\text{unip}_{G,F,\iota}}$.  

\(^{12}\)A closely related conjecture also appeared in [Hel].

\(^{13}\)If $G$ is quasi-split, such restriction is not necessary.
To give an idea of the content of this conjecture, we record the following immediate consequences.

Recall the stable center $Z_{G,F}$ as in (3.4), and the Hecke algebra $H_G$ of $G$ as in Remark 4.1.3. Let $Z_{G,F} := Z(H_G)$ denote the center of $H_G$ (the Bernstein center of $G(F)$).

**Corollary 4.4.2.** Assuming this conjecture, there exists a natural map

$$Z_{G,F} \to Z_{G,F}. \tag{4.6}$$

For a connected component $D$ of $\text{Loc}_{G,F}$, let $Z_{G,F,D}$ and $Z_{G,F,D}$ be the corresponding idempotent components. Then $Z_{G,F,D}$ is finite over $Z_{G,F,D}$. If $G = G^*$, then (4.6) is split injective.

**Remark 4.4.3.** In the case of $GL_n$ over a $p$-adic field and $k = \overline{Q}$, the map in the corollary is constructed earlier by Scholze [Sch13]. Using the local Langlands for $GL_n$, such map is constructed by Helm and Helm-Moss [He16, HM, HM18] for $k = \mathbb{Z}_p$. Note that for $GL_n$, (4.6) is an isomorphism. For general $G$, a map from the excursion algebra (see Remark 3.1.13) to $Z_{G,F}$ is constructed by Genestier-Lafforgue [GL] (in equal characteristic and after $\ell$-adic completion). The map (4.6) in general (for $k = \mathbb{Z}_p$) is expected to appear in the work of Fargues-Scholze, without the construction of $\mathfrak{A}$. But as far as we know, for general $G$, it is not known yet that $Z_{G,F} \to Z_{G,F}$ is finite (when restricted to each component $D$ of $\text{Loc}_{G,F}$) and is injective when $G$ is quasi-split.

**Example 4.4.4.** If $G = T$ is a torus, it should be able to construct (4.6) by hand, which should be an isomorphism, and which in turn would induce the functor $\text{Rep}(T(F), k) \cong \text{Mod}_{Z_{T,F}} \subset \text{Qcoh}(\text{Loc}_{T,F})$, sending $\text{Rep}_{\mathfrak{A}_T}(T(F), k)$ to $\text{Coh}(\text{Loc}_{T,F})$. This should be the desired functor $\mathfrak{A}_T$. We illustrate this in the simplest case when $G = G_m$. (The case of an unramified torus is not more difficult as in Proposition 4.2.3, but some works seem needed to deal with general ramified tori.)

Let $U^{(n)} \subset \mathcal{O}_F$ be the $n$th unit group. By the local class field theory, $\lim_n \mathcal{R}_{F^{\times}/U^{(n)}, \mathbb{G}_m}$ is isomorphic to $\text{Loc}_{G,F}$ compatible with isomorphism $\psi$ from Proposition 3.1.10 for different choices of $\iota$. Note that there are natural isomorphisms $H_{G,U^{(n)}} \cong k[\mathcal{R}_{F^{\times}/U^{(n)}, \mathbb{G}_m}]$ compatible for $n$, which induce a natural equivalence

$$\text{Rep}(F^{\times}, k) \cong \lim_n \text{Qcoh}(\mathcal{R}_{F^{\times}/U^{(n)}, \mathbb{G}_m}) \cong \text{Ind}(\text{Perf}(\text{Loc}_{G,F})).$$

sending $\text{Rep}_{\mathfrak{A}_T}(F^{\times}, k) \cong \text{Coh}(\text{Loc}_{G,F}) \subset \text{Coh}(\text{Loc}_{G,F})$, where the last inclusion follows as $G_n$ acts trivially on $\text{Loc}_{G,F}$. Passing to ind-completion gives $\text{Rep}^\text{en}(F^{\times}, k) \cong \text{IndCoh}(\text{Loc}_{G,F}) \subset \text{IndCoh}(\text{Loc}_{G,F})$.

**Remark 4.4.5.** Conjecture 4.4.1 should be compatible with parabolic induction in the representation side and spectral parabolic induction from Proposition 3.2.2. So in particular, (4.6) should be compatible with parabolic induction. This would in particular imply (4.5). Note that the conjectural description for $\mathfrak{A}_{G,I(1)}$ and $\mathfrak{A}_{G,I}$ in Conjecture 4.4.1 is indeed compatible with parabolic induction. This amounts to saying that

$$\delta_{I(1)} \cong \text{Ind}_{B(F)}^{G(F)} \delta_{T,I(1)}, \quad \delta_{I} \cong \text{Ind}_{B(F)}^{G(F)} \delta_{T,I}$$

as $G(F)$-representations, where $\delta_{T,I(1)}$ and $\delta_{T,I}$ are the representations of $T(F)$ compactly induced from its pro-$p$-Iwahori and Iwahori subgroup. These isomorphisms are probably well-known if $k = \mathbb{C}$, and they are implicitly contained in [Da09, 3.6, 6.2, 6.3] for general $k$ in which $p$ is invertible.\footnote{We thank Vigneras for pointing out this.}

**Remark 4.4.6.** Unfortunately, we do not have an explicit conjectural description of $\{\mathfrak{A}_{G,K}\}_K$ for general $K$ at the moment. Here are some remarks.
(1) We expect that if $K$ is the pro-unipotent radical of a parahoric subgroup, then $\mathfrak{A}_{G,K}$ is supported on $\text{Loc}^{\text{tame}}_{G,F,\ell}$. In particular, there should exist a map

$$Z^{\text{tame}}_{G,F} \to Z(H_{G,K}).$$

generalizing (4.4).

(2) By the conjecture, if $G$ is quasi-split, $\Gamma(\text{Loc}_{G,F,\ell}, \mathfrak{A}_{G,K}) \cong \text{Hom}(\text{Whit}_{U,\varphi}, \delta_K)$.

(3) Using Drinfeld’s formalism (see Theorem [4.6.1] in the global setting), it should be possible to extract candidates of $\{\mathfrak{A}_{G,K}\}_K$ (as quasi-coherent sheaves) from the cohomology of moduli of local Shtukas, although certain technical difficulties must be overcome. In our opinion, it is still important to have an explicit (conjectural) construction of $\{\mathfrak{A}_{G,K}\}_K$ purely in the spectral side.

(4) Using the fact that some connected component of $\text{Loc}_{G,F,\ell}$ “looks like” the tame stack of local Langlands parameters for another group, it might be possible to relate the restriction of $\mathfrak{A}_G$ to this component with the coherent Springer sheaf of the other group. For $G = \text{GL}_n$, this might give a construction of $\mathfrak{A}_G$ “by hand”.

(5) Even if we understand $\{\mathfrak{A}_{G,K}\}_K$ for various $K$ (so knowing that the functor $\mathfrak{A}_G$ is well-defined), it is still quite important (and challenging) to understand the (ind)-coherent sheaves on $\text{Loc}_{G,F,\ell}$ corresponding to specific $G(F)$-representations. To give an example, let $X$ be a $G$-variety over $F$. Then $C_c(X(F))$ is a natural $G(F)$-representation, and therefore should correspond to an ind-coherent sheaf $\mathfrak{A}_X := \mathfrak{A}_G(C_c(X(F)))$ on $\text{Loc}_{G,F,\ell}$. The recent conjectures of Ben-Zvi-Sakellaridis-Venkatesh in relative Langlands program should have analogue in the current setting, giving conjectural construction of $\mathfrak{A}_X$ (for some $X$) purely from the spectral side (at least for $k$ being a field of characteristic zero).

4.5. Categorical local Langlands correspondence. In this subsection, written in a slightly informal style, we briefly explain how the conjectural sheaf $\mathfrak{A}_G$ fits into a hypothetical categorical form of the local Langlands conjecture. In this subsection, let $k$ to be finite over $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, where $\ell \neq \text{char } k_F$. Then the stack $\text{Loc}_{G,F,\ell} \otimes k$ is independent of the choice of $\ell : \Gamma_q \to \Gamma'_F$ up to canonical isomorphism (Remark [3.1.11]). So we write $\text{Loc}_{G,F,\ell} \otimes k$ in this subsection.

Recall that for a connected reductive group $G$ over a local field $F$, Kottwitz introduced a set $B(G)$ ([Ko85]) as follows. Let $L$ be the completion of the maximal unramified extension of $F$ and $\sigma$ is the Frobenius element in $\text{Gal}(L/F)$. Then $\sigma$ acts on $G(L)$ (through the action on $L$). The set $B(G)$ is defined as the quotient of $G(L)$ by the $\sigma$-conjugation action:

$$\text{Ad}_\sigma : G(L) \times G(L) \to G(L), \ (h, g) \mapsto h g \sigma(h)^{-1}.$$ 

It turns out $B(G)$ should be thought as the set of $\overline{\pi}_F$-points of some algebro-analytic geometric structure. As the first evidence, the quotient by $G(L)$ of the $\sigma$-conjugacy class containing $1 \in G(L)$ is not merely a point, but the classifying stack $[*/G(F)]$ of the pro-finite group $G(F)$. More generally, for every basic element $b \in B(G)$ (see [Ko85 5.1] for the definition), the quotient by $G(L)$ of the $\sigma$-conjugacy class containing $b \in G(L)$ should be the classifying stack $[*/J_b(F)]$ of $J_b(F)$, where $J_b$ is an inner form of $G$ associated to $b$ ([Ko85 5.2]).

Continuing with this philosophy, it is realized that a fundamental object to study in the arithmetic local Langlands program is the category $\text{Shv}(B(G), k)$ of sheaves with $k$-coefficient on $B(G)$ in appropriate sense. For example, the subcategory of sheaves supported on the classifying stack of $G(F)$ should give the category $\text{Rep}(G, k)$. More generally, for each basic element $b \in B(G)$, there should exist a pair of adjoint functors

$$i_b! : \text{Rep}(J_b(F), k) \rightleftharpoons \text{Shv}(B(G), k) : i_b^!$$

where $i_b : [*/J_b(F)] \to B(G)$ is the above mentioned embedding.
As far as we know, there are two ways to define the category \( \text{Shv}(B(G), k) \). One approach is due to Fargues-Scholze. In this approach, \( B(G) \) is regarded as the set of points of the \( v \)-stack \( \text{Bun}_G \) of \( G \)-bundles on the Fargues-Fontaine curve and \( \text{Shv}(B(G), k) \) is defined as category \( D(\text{Bun}_G, k) \) of appropriately defined étale sheaves on \( \text{Bun}_G \) [FS]. The definition of \( \text{Shv}(B(G), k) \) in this way is quite sophisticated, relying on Scholze’s work on \( \ell \)-adic formalism of diamond and condensed mathematics.

In another approach [Zhu18], which might be less sophisticated and stays in the realm of traditional \( \ell \)-adic formalism of schemes [Ga], \( B(G) \) is regarded as the set of points of the quotient prestack of \( LG/\text{Ad}_\sigma LG \), where \( LG \) denotes the loop group of \( G \), which is a (perfect) group ind-scheme over \( \kappa_F \), and \( \text{Ad}_\sigma \) denotes the Frobenius twisted conjugation given by (4.8) (e.g. see [Zhu18] 2.1 for a review). Then \( \text{Shv}(B(G), k) \) is defined as the category of \( k \)-sheaves on the prestack \( LG/\text{Ad}_\sigma LG \) in appropriate sense.

More precisely, this category can be also realized (via “\( h \)-descent”) as the category of sheaves on the moduli \( \text{Sht}^{\text{loc}} \) of local Shtukas (with the leg at the closed point \( 0 \in \text{Spec}\mathcal{O}_F \)) with morphisms given by cohomological correspondences. A discussion is sketched at the end of [Zhu18] (see also [Ga, 4.1]), and a detailed study of this category will appear in [IZ]. Here we repeat the outline given in [Zhu18]. Let \( L^+\mathcal{G} \) be the positive loop group of a parahoric model \( \mathcal{G} \) of \( G \) over \( \mathcal{O}_F \). We let

\[
\text{Sht}^{\text{loc}} := \frac{LG}{\text{Ad}_\sigma L^+\mathcal{G}}
\]

be the moduli of local Shtukas (with the leg at \( 0 \in \text{Spec}\mathcal{O}_F \), see [Zhu18] (4.1.1)), and let

\[
\text{Hk}(\text{Sht}^{\text{loc}}) := \frac{LG}{\text{Ad}_\sigma L^+\mathcal{G}} \times_{\text{Ad}_\sigma L^+\mathcal{G}} \frac{LG}{\text{Ad}_\sigma L^+\mathcal{G}},
\]

be the Hecke stack of local Shtukas (see [Zhu18] (4.1.2) with \( s = t = 1 \)). Then we have the groupoid \( \text{Hk}(\text{Sht}^{\text{loc}}) \rightrightarrows \text{Sht}^{\text{loc}} \), with both morphism ind-(perfectly) proper, and one can form a simplicial diagram (with degeneracy maps omitted)

\[
\cdots \rightrightarrows \text{Hk}(\text{Sht}^{\text{loc}}) \times_{\text{Sht}^{\text{loc}}} \text{Hk}(\text{Sht}^{\text{loc}}) \rightrightarrows \text{Hk}(\text{Sht}^{\text{loc}}) \rightrightarrows \text{Sht}^{\text{loc}}
\]

with morphisms ind-(perfectly) proper. Although \( \text{Sht}^{\text{loc}} \) and \( \text{Hk}(\text{Sht}^{\text{loc}}) \) (and each term in the above diagram) are not algebraic, they can be nevertheless approximated by nice (perfect) algebraic stacks (perfectly) of finite type over \( \kappa_F \) (see [Zhu18] for a detailed discussion and [Zhu18] 4.1 for a summary), and one can associate the \( \infty \)-category of \( k \)-sheaves \( \text{Shv}(\cdot, k) \) to them. Then we can define \( \text{Shv}(B(G), k) \) as the geometric realization of a simplicial \( \infty \)-category

\[
\cdots \rightrightarrows \text{Shv}(\text{Hk}(\text{Sht}^{\text{loc}}) \times_{\text{Sht}^{\text{loc}}} \text{Hk}(\text{Sht}^{\text{loc}}), k) \rightrightarrows \text{Shv}(\text{Hk}(\text{Sht}^{\text{loc}}), k) \rightrightarrows \text{Shv}(\text{Sht}^{\text{loc}}, k),
\]

where connecting morphisms are proper push-forward ([Zhu18] Remark 6.2). As explained in [Zhu18], its homotopy category can be expressed as the category of sheaves on \( \text{Sht}^{\text{loc}} \) with morphisms given by cohomological correspondences supported on \( \text{Hk}(\text{Sht}^{\text{loc}}) \). The latter was constructed in details in [XZ]. In particular, for every open compact subgroup \( K \), there is an object \( \delta_K \) in this category, whose endomorphism algebraic is the derived Hecke algebra \( H_K \) (see [XZ] Remark 5.4.5)). This gives a functor

\[
\text{Mod}_{H_K} \to \text{Shv}(B(G), k).
\]

which is a full embedding if \( K \) is \( k \)-admissible. By varying \( K \) for \( k \)-admissible open compact subgroups, we obtain an embedding

\[
\text{Rep}(G(F), k) \to \text{Shv}(B(G), k).
\]

\[\text{This approach has been the folklore among the geometric Langlands community for a while.}\]

\[\text{But this approach probably is insufficient for some purposes such as the } p \text{-adic local Langlands program.}\]

\[\text{This notion is chosen purposely to be same as the notion for } c\text{-}\text{ind}_K^G(F) 1.\]
More generally, for every $b \in B(G)$, there is a pair of adjoint functors (4.9) as promised, with $i_{b,\ast}$ fully faithful embedding.

**Remark 4.5.1.** The most optimal guess would be the category $D(\text{Bun}_G, k)$ defined by Fargues-Scholze and the one outlined above are equivalent. A striking feature is in the above two interpretations of $B(G)$, the partial order on $B(G)$ gets reversed. For example, that the functor $i_b^\ast$ for basic $b$ admits right adjoint in Fargues-Scholze’s category is clear but this is not so obvious in the above outlined setting (although it is nevertheless true).

**Remark 4.5.2.** As mentioned in [Zhu18], exactly the same construction allows one to define and study the category of sheaves on the adjoint quotient space $LG/AdLG$.

In any case, there is the following “meta” conjecture (as at the moment we are not sure which version of $\text{Shv}(B(G), \mathbb{Z}_\ell)$ to be put in the conjecture). Let $\mathcal{N}_{G,F}$ denote the conic subset of $\text{Sing}(\text{Loc}_{G,c})$ as in (3.8). Recall our convention of the category of coherent sheaves on $\text{Loc}_{G,F,i}$ in Remark 3.1.6.

**Conjecture 4.5.3.** We assume that $G$ is quasi-split over $F$. Then for every choice of a Whittaker datum $(U, \Psi)$ of $G$, there is a natural equivalence of $\infty$-categories

$$\mathbb{L}_G : \text{Shv}(B(G), \mathbb{Z}_\ell) \to \text{IndCoh}_{\mathcal{N}_{G,F}}(\text{Loc}_{G,c})$$

sending $\text{Whit}(U, \Psi)$ to the structural sheaf $\mathcal{O}_{\text{Loc}_{G,c}}$. In addition, for every basic element $b \in B(G)$, the conjectural functor $\mathfrak{A}_{J_b}$ in Conjecture 4.4.1, when tensored with $\mathbb{Z}_\ell$, fits into the following commutative diagram

$$\text{Rep}_{\text{Ig}, b}(J_b, \mathbb{Z}_\ell) \xrightarrow{\mathfrak{A}_{J_b}} \text{Coh}(\text{Loc}_{G,c}) \xrightarrow{i_{b,\ast}} \text{Shv}(B(G), \mathbb{Z}_\ell) \xrightarrow{\mathbb{L}_G} \text{IndCoh}_{\mathcal{N}_{G,F}}(\text{Loc}_{G,c}).$$

**Remark 4.5.4.** In Fargues-Scholze approach defining $\text{Shv}(B(G), \mathbb{Z}_\ell)$ as $D(\text{Bun}_G, \mathbb{Z}_\ell)$, this conjecture formally looks like the global geometric Langlands conjecture as proposed by Arinkin-Gaitsgory [AG16]. Indeed, Fargues-Scholze independently announced the same conjecture using $D(\text{Bun}_G, \mathbb{Z}_\ell)$ in the formulation.

**Remark 4.5.5.** For $\mathbb{Z}_\ell$-coefficient and $\ell$ the so-called non banal prime, the existence of $\mathfrak{A}_{J_b}$ does not follow directly from the existence of $\mathbb{L}_G$, as $\text{Rep}_{\text{Ig}, b}(J_b, \mathbb{Q}_\ell)$ does not belong to the subcategory of compact objects of $\text{Shv}(B(G), \mathbb{Q}_\ell)$. However, there is a renormalized version $\text{Shv}^{\text{ren}}(B(G), \mathbb{Z}_\ell)$ of $\text{Shv}(B(G), \mathbb{Z}_\ell)$, which will contain $\text{Rep}_{\text{Ig}, b}(J_b, \mathbb{Q}_\ell)$ inside its subcategory of compact objects (the definition is similar to [AG16, 12.2.3] and will be given in [HZ]). One would expect that $\mathbb{L}_G$ extends to an equivalence

$$\mathbb{L}_G^{\text{ren}} : \text{IndCoh}(\text{Loc}_{G,c}) \cong \text{Shv}^{\text{ren}}(B(G), \mathbb{Z}_\ell),$$

which would imply the existence of $\mathfrak{A}_{J_b}$. If we replace $\mathbb{Z}_\ell$ by $\mathbb{Q}_\ell$, then $\text{Shv}^{\text{ren}}(B(G), \mathbb{Q}_\ell) = \text{Shv}(B(G), \mathbb{Q}_\ell)$, and the nilpotent singular support condition is automatic by Lemma 3.2.3. So $\mathbb{L}_G^{\text{ren}}$ would coincide with $\mathbb{L}_G$.

**Example 4.5.6.** Let us analyze the conjecture in the simplest case when $G = \mathbb{G}_m$. As $B(\mathbb{G}_m)$ is just $\mathbb{Z}$-copies of $[\ast/F\ast]$, Example 4.4.4 gives

$$\mathbb{L}_G : \text{IndCoh}_{\mathcal{N}_{G,F}}(\text{Loc}_{G,c}) = \text{Ind(Perf}(\text{Loc}_{G,c})) \cong \text{Shv}(B(\mathbb{G}_m), \mathbb{Z}_\ell),$$

\[^{18}\text{As suggested by Scholze, even the two versions of Shv}(B(G), k)\text{ are equivalent, the natural functors }i_{b,\ast}\text{ in these two settings might differ by a duality.}\]
where the first equivalence is due to the fact $\hat{N}_{G,F}$ is just the zero section of $\text{Sing} (\text{Loc}_{c} G)$ when $G$ is a torus. As mentioned in the previous remark, there is also a version

$$L_{G}^{\text{ren}} : \text{Ind} (\text{Coh} (\text{Loc}_{c} G)) \cong \text{Shv}^{\text{ren}} (B(G_{m}), \mathbb{Z}_{\ell})$$

induced from $\text{Coh} (\text{Loc}_{c} G) \cong \text{Rep}_{f.g.} (F_{\times}, \mathbb{Z}_{\ell})$. (Here $\text{Shv}^{\text{ren}} (B(G_{m}), \mathbb{Z}_{\ell})$ is just $\mathbb{Z}$-graded version of $\text{Rep}^{\text{ren}} (F_{\times}, \mathbb{Z}_{\ell})$).

**Remark 4.5.7.** The conjectural equivalence is supposed to satisfy a set of compatibility conditions as in [AG16, Ga15]. In particular implies that there should exist an action of the category $\text{Qcoh} (\text{Loc}_{c} G)$ of quasi-coherent sheaves on $\text{Loc}_{c} G$ on $\text{Shv} (B(G), \mathbb{Z}_{\ell})$, usually called the spectral action. Fargues-Scholze have announced a construction of such action in their setting. But the existence of such spectral action in the setting outlined above is not known.

On the other hand, an evidence that the category outlined above might also be the correct input in the conjecture is the following statement which will be established in [HZ].

**Theorem 4.5.8.** Assume that $G$ is unramified over an equal characteristic local field $F$, and that $k = \mathbb{Q}_{\ell}$. Then after choosing a Whittaker datum $(U, \Psi)$, there is a full embedding

$$\text{Coh} (\text{Loc}_{c}^{\text{unip}} G) \to \text{Shv} (B(G), \mathbb{Q}_{\ell})$$

into the subcategory of compact objects of $\text{Shv} (B(G), \mathbb{Q}_{\ell})$. It sends $\text{CohSpr}_{c}^{\text{unip}} G$ to $\delta_{1}$. In fact, for every basic element $b \in B(G)$, let $J_{b}$ denote the corresponding inner form of $G$ and $H_{I_{b}}$ the corresponding Iwahori-Hecke algebra. Then there is the following commutative diagram

$$\begin{array}{ccc}
\text{Mod}_{H_{I_{b}}} & \rightarrow & \text{Rep} (J_{b} (F), \mathbb{Q}_{\ell}) \\
\downarrow & & \downarrow i_{b, !} \\
\text{IndCoh} (\text{Loc}_{c}^{\text{unip}} G) & \rightarrow & \text{Shv} (B(G), \mathbb{Q}_{\ell})
\end{array}$$

Further properties of the embedding in the theorem will be studied in [HZ].

**Remark 4.5.9.** (1) The proof is an exercise of calculation of the Frobenius-twisted categorical trace of the two versions of affine Hecke categories [Be16], generalizing the calculation of the Frobenius-twisted categorical trace of the geometric Satake as in [XZ, Zhu18].

(2) As Bezrukavnikov’s equivalence [Be16] is only available for $\mathbb{Q}_{\ell}$-sheaves and for reductive groups over equal characteristic local fields at the moment, we need to put the same assumptions in the theorem. If such equivalence becomes available in modular coefficients and/or in mixed characteristic setting, the above theorem should generalize as well. We refer to [BRR] for the progress of such equivalence for modular coefficients in equal characteristic. On the other hand, currently Bezrukavnikov’s equivalence in mixed characteristic is not available, as the crucial input of Gaitsgory’s central sheaf construction ([Ga01]) is not available yet.

### 4.6. Local-global compatibility

In this last subsection, we use the conjectural coherent sheaves in Conjecture 4.4.1 to formulate the local-global compatibility in the Langlands program and to give some evidences. We will first consider the global function field case as the picture is more complete. Then we will move to the number field case.

Let $F$ be a global field, and $G$ a connected reductive group over $F$. We will let $k$ be finite over $\mathbb{Z}_{\ell}$ or over $\mathbb{Q}_{\ell}$ in this subsection, where $\ell \neq \text{char} F$ if $F$ is a function field. In addition, we assume that the center of $G$ is connected for simplicity. We will use notations from [3.3]. In addition, for an open compact subgroup $K_{v} \subset G_{v}$, let $\mathfrak{A}_{K_{v}}$ denote $\mathfrak{A}_{G_{v}, K_{v}}$ as appearing in Conjecture 4.4.1.
First, let $F$ be a global function field, and $G$ a connected reductive group over $F$. We need to assume some familiarity with [La18]. Let $\mathbb{F}_q \subset F$ be the field of constants in $F$, and $X$ be the connected smooth projective curve $X$ over $\mathbb{F}_q$ with $F$ its fractional field. Let $W_F$ be the Weil group of $F$. We extend $G$ to a Bruhat-Tits integral model $\mathcal{G}$ over $X$. Let $\mathcal{O} = \prod_v \mathcal{O}_{F_v}$. Let $K \subset \mathcal{G}(\mathcal{O})$ be a level structure, and let $H_K = C_c(K \backslash G(\mathcal{O})/K, k)$ be the corresponding global Hecke algebra (with coefficients in $k$). For a finite set $I$, let $\text{Sht}_{\eta,I}$ denote the moduli of $\mathcal{G}$-shtukas with $K$-level structures over the generic point $\eta^I$ of $X^I$. Recall that for every representation $V$ of $(\mathcal{G})^I$ on a finite projective $k$-module, the geometric Satake provides a perverse sheaf $\text{Sat}(V)$ on $\text{Sht}_{\eta,I}$, supported on a finite dimensional closed substack of $\text{Sht}_{\eta,I}$.

For a representation $V$ of $(\mathcal{G})^I$ on a finite free $k$-module, let $\Gamma_c(\text{Sht}_{\eta,I}, \text{Sat}(V))$ denote the total compactly supported cohomology of $\text{Sht}_{\eta,I}$ with coefficient in $\text{Sat}(V)$. By a theorem of Hemo ([Hem], based on [Xu1] [Xu2]), it admits an action of $(H_K \times W_F^I)$. In addition, Hemo proves the following theorem.

**Theorem 4.6.1.** There is a quasi-coherent sheaf $\mathfrak{A}_K$ on $\mathcal{R}_{W_F, cG}/\hat{G}$, equipped with an action of $H_K$, such that for every finite dimensional representation $V$ of $(\mathcal{G})^I$, there is a natural $(H_K \times W_F^I)$-equivariant isomorphism

$$
\Gamma_c(\text{Sht}_{\eta,I}, \text{Sat}(V)) \cong \Gamma(\mathcal{R}_{W_F, cG}/\hat{G}, \mathfrak{A}_K \otimes (w_F V))
$$

where $w_F V$ is equipped with an action by $W_F^I$ as in Remark 2.2.7.

**Remark 4.6.2.**

1. In the above theorem, $W_F$ is regarded as an abstract group in the definition of $\mathcal{R}_{W_F, cG}/\hat{G}$. So this is a huge space, much bigger than the stack of global Langlands parameters $\text{Loc}^\wedge_{G,F}$ as we considered in [3.3]. Of course, $\mathfrak{A}_K$ should be supported in $\text{Loc}^\wedge_{G,F}$.

2. The idea that something like (4.10) should exist due to Drinfeld, as an interpretation of certain construction of [La18]. As explained in [Gal, GKRV], (4.10) should follow by taking categorical trace of the geometric Langlands correspondence. (But for technical reasons, it is probably not easy to literally deduce Theorem 4.6.1 from such considerations.)

3. When $k = E$ is finite over $\mathbb{Q}_{\ell}$, let $H_{I,V}$ denote the degree zero cohomology of $\Gamma_c(\text{Sht}_{\eta,I}, \text{Sat}(V))$. Then a non-derived version of (4.10) is used in [LZ] to give a multiplicity formula of the the elliptic part of $H_{I,V}$, in light of the Arthur-Kottwitz conjecture.

Although the space $\mathcal{R}_{W_F, cG}/\hat{G}$ is huge, there is still a map $\mathcal{R}_{W_F, cG}/\hat{G} \rightarrow \prod_v \text{Loc}_v$, after choosing $\nu_v : G_0 \rightarrow \Gamma^I_{F_v}$, for each local place $v$. In [LZ], in light of the Arthur-Kottwitz conjecture, we conjecture that $\mathfrak{A}_K$ (or more precisely its non-derived version) factorizes as a tensor product of local factors. Now, we further conjecture that these local factors should exactly be the coherent sheaves appearing in Conjecture 4.4.1.

Here is the precise conjecture. We formulate it for $k = \mathbb{Z}/\ell^n$. Recall that $\text{Loc}_{cG,F,S,n}$ is an algebraic stack locally of finite presentation over $k$ (Proposition 3.3.1). We denote by $\boxtimes_{v \in S} \mathfrak{A}_{K_v}$ the external tensor product of those coherent sheaves on $\prod_{v \in S} \text{Loc}_v$, and by $\text{res}^* (\boxtimes_{v \in S} \mathfrak{A}_{K_v})$ its (derived) pullback to $\text{Loc}_{cG,F,S,n}$ via (3.10). For a representation $V$ of $(\mathcal{G})^I$ on a finite free $k$-module, we have the vector bundle $w_{F,S} V$ on $\text{Loc}_{cG,F,S,n}$, equipped with an action of $W_{F,S}$.

To avoid discussing the Whittaker normalization, we assume that the center of $G$ is a split torus.

**Conjecture 4.6.3.** Let $k = \mathbb{Z}/\ell^n$. Let $S$ be a finite set of places of $F$ such that $K_w$ is hyperspecial if $w \notin S$. Then for every representation $V$ of $(\mathcal{G})^I$ on a finite free $k$-module, there is a canonical $(H_K \times W_F^I)$-equivariant isomorphism

$$
\Gamma_c(\text{Sht}_{\eta,I}, \text{Sat}(V)) \cong \Gamma(\text{Loc}_{cG,F,S,n}, \text{res}^* (\boxtimes_{v \in S} \mathfrak{A}_{K_v}) \otimes (w_F V)).
$$

Here as before, all the functors are derived in the above formula.
Note that the conjecture is consistent with enlarging $S$, as when $K_{v}$ is hyperspecial, $\mathfrak{A}_{K_{v}} \cong \mathcal{O}_{\text{Loc}^\text{sur}}$, and we have the Cartesian diagram as in Remark 4.6.4.

This conjecture seems to be widely open. For example, it would imply modularity. In addition, it can be regarded as an integral version of Arthur-Kottwitz multiplicity formula. Therefore let us formulate a conjectural consequence, which is more accessible and might be useful to attack the original conjecture. Let $k$ be finite over $\mathbb{Z}_\ell$ or over $\mathbb{Q}_\ell$. Let $v \in |X|$ be a place. We assume that $\mathcal{G}^1_{|O_v}$ is reductive for an open subset $U \subset X$ containing $v$, and $\mathcal{G}^1|_{O_v}$ is a parahoric group scheme of $G_{F_v}$. We assume that the level $K_w = \mathcal{G}(O_{F_w})$ for all $w \in |U|$ (including $w = v$). We write $K = K_vK_v^e$, where $K_v = \mathcal{G}(O_{F_v})$, and $K_v^e$ is the level away from $v$. The Hecke algebra $H_K$ also decomposes as $H_K = H_{K_v} \otimes H_{K_v^e}$.

Let $\text{Sht}_{U^f}$ be the stack of global Langlands parameters in the number field case. Nevertheless, currently we are missing the description of $\mathfrak{A}_{K_v}$ at places above $\ell$ and $\infty$. (In particular, the sheaf at $\ell$ or $\infty$ should encode the information of the “weights”.) In addition, we do not yet have the stack of global Langlands parameters in the number field case. Nevertheless, currently Liang Xiao and the author are verifying the cohomology of the modular curve indeed admits such a formulation of $\mathfrak{g}$. Let $G$ be finite over $\mathbb{Z}_\ell$, and we have the Cartesian diagram as in Remark 3.11.

Conjecture 4.6.4. Let $W$ and $W'$ be two representations of $\mathfrak{c}G$ on finite projective $k$-modules, and $V$ a representation of $(\mathfrak{c}G)^f$ on finite projective $k$-module. Then there is a natural $(H_{K_v} \times H_{K_v^e})$-bimodule map

$$\text{Hom}_{\text{Coh}(\text{Loc}_v)}(\mathfrak{A}_{K_v} \otimes \tau_q W, \mathfrak{A}_{K_v} \otimes \tau_q W') \to \text{Hom}_{\text{Z}^\text{tame}(\mathfrak{c}G)}(\mathcal{H}_f(W \boxtimes V), \mathcal{H}_f(W' \boxtimes V)).$$

Theorem 4.6.5. Such map exits if either $K_v$ is hyperspecial or Iwahori, and $k = E$.

When $K_v$ is hyperspecial, this follows from our previous work with Xiao [XZ]. When $K_v$ is Iwahori, this will be established in the forthcoming work [HZ].

Now we move to the number field case. In fact, it is a Shimura variety version of Theorem 4.6.4 that was first proved in [XZ] when $K_v$ is hyperspecial, which motivated all the conjectures. Therefore, there must be analogue of Conjecture 4.6.3 for the cohomology of Shimura varieties except currently we are missing the description of $\mathfrak{A}_{K_v}$ at places above $\ell$ and $\infty$. (In particular, the sheaf at $\ell$ or $\infty$ should encode the information of the “weights”.) In addition, we do not yet have the stack of global Langlands parameters in the number field case. Nevertheless, currently Liang Xiao and the author are verifying the cohomology of the modular curve indeed admits such a description, at least when localized at maximal ideals of the Hecke algebra given by “good” residual representations. To give a flavor, we present a conjecture in the simplest case.

Let $X = X_0(N)$ be the modular curve with $N = \prod p_i$ square free, containing at least two prime factors. We assume that $\ell \nmid 2N \prod (p_i^2 - 1)$. At each prime $p_i$, let $K_{p_i} = \text{GL}_2(\mathbb{Z}_{p_i})$ and $I_{p_i}$ its standard Iwahori subgroup.

$$H_{f,p_i} := C_{c}(I_{p_i}, K_{p_i}/I_{p_i}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell[T_i]/(T_i - p_i)(T_i + 1)$$

acts on $H^1(X_{\mathbb{Q}_\ell}, \mathbb{Z}_\ell)$, where $T_i$ is the operator corresponding to the double coset containing $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Under our assumption $\ell \nmid p_i^2 - 1$, the action is semisimple. Let $H^1(X_{\mathbb{Q}_\ell}, \mathbb{Z}_\ell)^{st} \subset H^1(X_{\mathbb{Q}_\ell}, \mathbb{Z}_\ell)$ denote the direct summand on which each $T_i$ acts by $-1$.  

\[ \text{It would be very interesting to see whether the cohomology of locally symmetric spaces admit similar descriptions.} \]
Let $\tilde{\rho} : \Gamma_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p)$ be an irreducible residue Galois representation, giving a maximal ideal $m$ of the global Hecke algebra $T$. We would like to give a description of $H^1(X_{\mathbb{Q}^p}, \mathbb{Z}_\ell)^{\text{st}}_m$ (which is assumed to be non-zero), the localization of $H^1(X_{\mathbb{Q}^p}, \mathbb{Z}_\ell)^{\text{st}}$ at $m$.

Let $R_{\tilde{\rho}}$ be universal deformation ring for those deformations $\rho$ of $\tilde{\rho}$ satisfying:

- $\rho$ is unramified outside $N\ell$;
- $\rho$ is crystalline at $\ell$ of Hodge-Tate weights $\{0, 1\}$;
- $\rho$ sends the generator of the tame inertia at $p_i$ to a unipotent element;
- the determinant of $\rho$ is the inverse cyclotomic character.

There is the tautological rank two free $R_{\tilde{\rho}}$-module $V$, affording the action of $\Gamma_{\mathbb{Q}}$. The analogue of Conjecture 4.6.3 in this case would be the following.

**Conjecture 4.6.6.** As $\mathbb{T} \times \Gamma_{\mathbb{Q}}$-modules, $H^1(X_{\mathbb{Q}^p}, \mathbb{Z}_\ell)^{\text{st}}_m = V$.

Of course, this conjecture is mostly interesting when there are several places $p_i$ such that $\tilde{\rho}$ is unramified at $p_i$ and $\tilde{\rho}($Frob$_{p_i}) \sim \begin{pmatrix} p_i & \ast \\ 1 & 1 \end{pmatrix}$.

**Remark 4.6.7.** If $\tilde{\rho}$ in addition satisfies the Taylor-Wiles condition, such statement should be known to experts. But let us roughly explain why this is the analogue of Conjecture 4.6.3. The relevant group is $G = \text{PGL}_2$ so $\hat{G} = \text{SL}_2$ and $\hat{G} \cong \text{GL}_2$. Under our assumption that $\ell \nmid p_i^2 - 1$, the unipotent coherent Springer sheaf on $\text{Loc}_{p_i}^{\text{unip}}$ as defined in §4.3 is just $\mathcal{O}_{\text{Loc}_{p_i}^{\text{unip}}} \oplus \mathcal{O}_{\text{Loc}_{p_i}'}$. Requiring that $T_i$ acts by $-1$ amounts to picking out the direct summand $\mathcal{O}_{\text{Loc}_{p_i}^{\text{unip}}}$. It follows that in this case the coherent sheaf $\mathfrak{a}_K$ on the hypothetical stack of global Langlands parameters should just be the structural sheaf. Then taking the formal neighborhood at $\tilde{\rho}$ should give the above conjecture.

The more complicated cases, when $\ell \nmid p_i^2 - 1$, will also be studied in the future with Liang Xiao.

Next, we formulate analogue of Conjecture 4.6.4 for the Shimura varieties, which would be a generalization of one of the main results of [XZ], and would imply the geometric realization of the Jacquet-Langlands correspondence between inner forms that are different at finite places (the work [XZ] only gives JL transfers between inner forms that are different at $\infty$). We use the notations as in loc. cit.. Let $(G, X)$ and $(G', X')$ be two Shimura data. We assume that there is an inner twist $\Psi : G \to G'$ which is trivial over $\mathbb{A}_p$, and we choose a trivialization $\theta : G(\mathbb{A}_p) \cong G'(\mathbb{A}_p)$. Then we may identify the dual group of $G$ and $G'$ via $\Psi$ (see [XZ] for the detailed discussions about these data.)

Let $V$ and $V'$ be the irreducible representations of $\hat{G}$ associated to the Shimura cocharacters of $G$ and $G'$ in the usual way. We choose a prime-to-$p$ level $K^p \subset G(\mathbb{A}_p)$, and let $K'^p = \theta(K^p)$. Let $K_p \subset G(\mathbb{Q}_p)$ and $K'_p \subset G'(\mathbb{Q}_p)$ be parahoric subgroups. Let $\text{Sh}_V$ (resp. $\text{Sh}_{V'}$) be the Shimura variety of $(G, X)$ of level $K$ (resp. $(G', X')$ of level $K'$), base changed to $\overline{\mathbb{Q}}_p$. Recall that we let $Z_p^{\text{tame}} = H^0\Gamma(\text{Loc}_{p}^{\text{tame}}, \mathcal{O})$.

**Conjecture 4.6.8.** For every choice of $\text{Spec} \overline{\mathbb{Q}}_p \to \text{Spec} \mathbb{Z}_p^{\text{ur}}$ (a specialization map), there is an $(H_{K_p} \times H_{K'_p})$-bimodule map

$$\text{Hom}_{\text{Coh}(\text{Loc}_{p})}(\mathfrak{a}_{K_p} \otimes_{\mathbb{Q}_p} V, \mathfrak{a}_{K'_p} \otimes_{\mathbb{Q}_p} V') \to \text{Hom}_{Z_p^{\text{tame}} \otimes H_{K_p}}(\Gamma_c(\text{Sh}_V, k[d]), \Gamma_c(\text{Sh}_{V'}, k[d'])),$$

compatible with compositions, where $d = \dim \text{Sh}_V$ (resp. $d' = \dim \text{Sh}_{V'}$). In the particular case when $G = G'$ and $\Psi$, $\theta$ are the identity map, one obtains an action

$$S : \text{End}_{\text{Coh}(\text{Loc}_{p})}(\mathfrak{a}_{K_p} \otimes_{\mathbb{Q}_p} V) \to \text{End}_{Z_p^{\text{tame}} \otimes H_{K_p}}(\Gamma_c(\text{Sh}_V, k)).$$

The composition

$$H_{K_p} \cong \text{End}(\mathfrak{a}_{K_p}) \to \text{End}(\mathfrak{a}_{K_p} \otimes_{\mathbb{Q}_p} V) \xrightarrow{S} \text{End}_{\text{tame}}(\Gamma_c(\text{Sh}_V, k))$$

is...
coincides with the natural Hecke action of $H_{K_p}$ on $\Gamma_c(Sh_V, k)$.

**Remark 4.6.9.** If the Shimura data $(G, X)$ and $(G', X')$ are of abelian type, then there are canonical integral models of $Sh_V$ and $Sh_{V'}$, as constructed in [KP18] (under some mild restrictions on $p$). Then instead of choosing $\text{Spec} \mathbb{Q}_p \to \text{Spec} \mathbb{Z}_p$, one can formulate the conjecture using the compactly supported cohomology of special fibers with coefficients in the sheaves of nearby cycles.

**Remark 4.6.10.** The works of [XZ], [Zhu2] confirm a weak form of this conjecture in the case $G \otimes \mathcal{A} \cong G' \otimes \mathcal{A}$ and $K_p$ is hyperspecial. But we note that even in this case, the conjecture is stronger. Namely, the derived Hecke algebra $H_{K_p}$ acts on $\Gamma_c(Sh_V, k)$, when $\Gamma_c(Sh_V, k)$ is regarded as a $\mathbb{Z}_p^{\text{tame}}$-module. So the conjecture includes a derived $S = T$ statement.

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20 Unlike the cohomology of general locally symmetric space as considered in [Ve19], the derived Hecke action is invisible when $R^1\Gamma_c(Sh_V, k)$ is merely regarded as a $k$-module.
