REALIZABILITY OF TROPICAL CANONICAL DIVISORS

MARTIN MÖLLER, MARTIN ULIRSCH, AND ANNETTE WERNER

Abstract. We use recent results by Bainbridge-Chen-Gendron-Grushevsky-Möller on compactifications of strata of abelian differentials to give a comprehensive solution to the realizability problem for effective tropical canonical divisors in equicharacteristic zero. Given a pair \((\Gamma, D)\) consisting of a stable tropical curve \(\Gamma\) and a divisor \(D\) in the canonical linear system on \(\Gamma\), we give a purely combinatorial condition to decide whether there is a smooth curve \(X\) over a non-Archimedean field whose stable reduction has \(\Gamma\) as its dual tropical curve together with an effective canonical divisor \(K_X\) that specializes to \(D\). Along the way, we develop a moduli-theoretic framework to understand Baker’s specialization of divisors from algebraic to tropical curves as a natural toroidal tropicalization map in the sense of Abramovich-Caporaso-Payne.

Contents

Introduction 1
1. Compactifying the moduli space of algebraic divisors 5
2. Tropical divisors and their moduli 9
3. Specialization versus tropicalization 12
4. Tropicalizing the Hodge bundle 16
5. Twisted differentials and the global residue condition 18
6. The realizability locus 21
References 34

Introduction

The realizability problem in tropical geometry is a metaproblem that underlies many of the successful applications of tropical geometry to other areas of mathematics. It asks whether for a given synthetically defined tropical object, there exists an analogous algebraic geometric object whose tropicalization is precisely the given tropical object.

The realizability problem for divisors is, in general, notoriously difficult, see e.g. [BJ15, Section 10]. In this article we solve it for effective canonical divisors using recent results on the compactification of strata of abelian differentials in [BCGGM].

Realizability of tropical canonical divisors. Let \((\Gamma, D)\) be a tuple consisting of an (abstract) stable tropical curve \(\Gamma\) and a divisor \(D\) in the canonical linear system \(|K_\Gamma|\) on \(\Gamma\). Does there exist a smooth curve \(X\) together with a stable degeneration \(X\) as well as an effective canonical divisor \(K_X\) on \(X\) such that the following two conditions hold:

1. \(\text{condition A}\)
2. \(\text{condition B}\)
• the tropical curve given by the metrized weighted dual graph of the irreducible components in the special fiber of $X$ is $\Gamma$; and
• the specialization of $K_X$, i.e. the multidegree of the special fiber of the closure of $K_X$ in a suitably chosen semistable model of $X$, is equal to $D$.

If that is the case, we say the pair $(\Gamma, D)$ is realizable.

Our main theorem gives an exhaustive answer to this question over an algebraically closed field of characteristic 0. To state it, recall that an element in the tropical canonical series differs from the distinguished element $K_\Gamma$ by the divisor of a piecewise affine function $f$ on $\Gamma$ with integral slopes. We declare the support of $\text{div}(f)$ to be vertices of $\Gamma$ and add to the graph $\Gamma$ legs at the support of $\text{div}(f)$, according to the local multiplicity. Now we simply use the value of such a function $f$ to define an order among the vertices of $\Gamma$ (making it into a level graph). Finally, we provide each half-edge of $\Gamma$ with an enhancement consisting of the (outgoing) slope of $f$. The resulting object is called the enhanced level graph $\Gamma^+(f)$ associated with $f$.

Section 5 explains the algebro-geometric origin of this notion. The correspondence between rational functions and decorations on $\Gamma$ is explained in Section 6. In particular, we introduce the notion of an inconvenient vertex $v$. A vertex $v \in \Gamma^+(f)$ of genus 0 is inconvenient if it has, roughly speaking, an edge with a 'large' positive decoration. For example, trivalent vertices where two edges have decoration less than $-1$ are always inconvenient.

**Theorem 1.** Given a tropical curve $\Gamma$ and an element $D = K_\Gamma + \text{div}(f)$ in the tropical canonical linear series on $\Gamma$, the pair $(\Gamma, D)$ is realizable if and only if the following two conditions hold:

i) For every inconvenient vertex $v$ of $\Gamma^+(f)$ there is a simple cycle $\gamma \subset \Gamma$ based at $v$ that does not pass through any node at a level smaller than $f(v)$.

ii) For every horizontal edge $e$ there is a simple cycle $\gamma \subset \Gamma$ passing through $e$ which does not pass through any node at a level smaller than $f(e)$.

This theorem implies in particular that the canonical divisor $K_\Gamma$ on $\Gamma$ is in general not realizable (see Example 6.4 below). Note that $K_\Gamma$ is always the tropicalization of some (non-effective) canonical divisor by [Bak08, Remark 4.21].

**The realizability locus in the tropical Hodge bundle.** In [LU17] Lin and the second author of this article synthetically constructed a tropical analogue $\mathcal{P}\Omega M^\text{trop}_g$ of the projective Hodge bundle. Set-theoretically it parametrizes isomorphism classes of pairs $(\Gamma, D)$ where $\Gamma$ is a stable tropical curve of genus $g$ and $D$ is an element of the canonical linear system on $\Gamma$. By [LU17, Theorem 1.2] it canonically carries the structure of a generalized (rational polyhedral) cone complex. We denote by $\mathcal{P}\Omega M^\text{an}_g$ the Berkovich analytic space associated to $\mathcal{P}\Omega M_g$ in the sense of [Uli14]. There is a natural tropicalization map

$$\text{trop}_\Omega : \mathcal{P}\Omega M^\text{an}_g \to \mathcal{P}\Omega M^\text{trop}_g$$

that sends an element in $\mathcal{P}\Omega M^\text{an}_g$, represented by a pair $(X, K_X)$ consisting of a smooth algebraic curve $X$ over a non-Archimedean extension of the base field and a canonical divisor $K_X$ on $X$, to the point consisting of the dual tropical curve $\Gamma$ of a stable reduction $\mathcal{X}$ of $X$ together with specialization of $K_X$ to $\Gamma$ (see Section 4 below for details).
Theorem 1 thus gives a complete characterization of the elements in the so-called realizability locus, the image of $\text{trop}_g$ in $P\Omega M_g^{\text{trop}}$.

In general, by the classical Bieri-Groves Theorem (see [BG84, Theorem A] and [EKL06, Theorem 2.2.3]) the tropicalization of a subvariety of a split algebraic torus is a rational polyhedral complex of the same dimension. The Hodge bundle does not embed into an algebraic torus but rather in a suitably defined toroidal compactification (in the sense of [KKMSD]). Consequently, we know in this case a priori only that the dimension of the realizability locus is bounded above by $4g - 4$ by [Uli15a, Theorem 1.1]. Our methods allow us to prove the following stronger result.

**Theorem 2.** The realizability locus in $P\Omega M_g^{\text{trop}}$ naturally has the structure of a generalized rational polyhedral cone complex of pure dimension $4g - 4$.

The main ingredient in the proofs of both Theorem 1 and 2 is the description of compactifications of strata of abelian differentials in [BCGGM], which is achieved using the method of plumbing and gluing. So our proof only works in equicharacteristic zero. It would be highly interesting to find a purely algebraic-geometric proof of these results (and the ones in [BCGGM]) that generalizes to all characteristics.

**Realizability locus for strata.** The Hodge bundle has a natural stratification by locally closed subsets

$$P\Omega M_g = \bigcup_{m_1, \ldots, m_n} P\Omega M_g(m_1, \ldots, m_n)$$

where the strata parametrize canonical divisors whose support has multiplicity profile $(m_1, \ldots, m_n)$ for non-negative integers $m_i$ with $m_1 + \cdots + m_n = 2g - 2$. In our proof we construct a realization of a tropical canonical divisor by an element in the open stratum $P\Omega M_g(1, \ldots, 1)^{an}$ of $P\Omega M_g^{an}$. However, our criterion works exactly the same way for the realizability by an element in a fixed stratum $P\Omega M_g(m_1, \ldots, m_n)^{an}$. In Section 6.3 below we discuss this criterion in detail and show, in an example, how it can be applied to study the realizability problem for Weierstrass points in genus 2.

**Tropicalizing the moduli space of divisors.** We denote by $\text{Div}_{g,d}$ the moduli space parametrizing pairs $(X, \bar{D})$ consisting of a smooth algebraic curve $X$ of genus $g$ and an effective divisor $\bar{D}$ on $X$ of degree $d$, i.e. the symmetric product of the universal curve over $\mathcal{M}_g$. In Section 2 we define a natural tropical analogue $\text{Div}_{g,d}^{\text{trop}}$ which is a set-theoretic moduli space of pairs $(\Gamma, D)$ consisting of a tropical curve $\Gamma$ together with an effective divisor $D$ of degree $d$ on $\Gamma$. Let $\text{Div}_{g,d}^{an}$ be the Berkovich analytic space associated to $\text{Div}_{g,d}$. There is a natural tropicalization map

$$\text{trop}_g,d: \text{Div}_{g,d}^{an} \rightarrow \text{Div}_{g,d}^{\text{trop}}$$

that takes a point in $\text{Div}_{g,d}^{an}$ represented by a tuple $(X, \bar{D})$, consisting of a curve $X$ over a non-Archimedean extension $K$ of $k$ and a divisor $\bar{D}$ of degree $d$, to the dual tropical curve $\Gamma$ of a stable reduction $X$ of $X$ together with the specialization of $\bar{D}$ to $\Gamma$ (see Section 3 for details).

In Section 4 below we give a reinterpretation of this procedure by constructing a modular toroidal compactification $\overline{\text{Div}_{g,d}}$ of $\text{Div}_{g,d}$, using a variant of Hassett’s
moduli spaces of weighted stable curves and inspired by Caporaso’s treatment of the universal Picard variety in [Cap94]. By [Thu07; ACP15], associated with the toroidal coordinates at the boundary, there is a natural strong deformation retraction $p_{g,d}$ onto a non-Archimedean skeleton $\mathcal{S}_{g,d}$ of $\text{Div}^a_{g,d}$ that is functorial with respect to toroidal morphism and therefore, in particular, commutes with the forgetful map to $\mathcal{M}_g$. Expanding on the main result of [ACP15], we prove in Section 3 below the following theorem.

**Theorem 3.** The tropicalization map $\text{trop}_{g,d}: \text{Div}^a_{g,d} \to \text{Div}^{trop}_{g,d}$ has a natural continuous section $J_{g,d}$ that induces an isomorphism between $\text{Div}^{trop}_{g,d}$ and the non-Archimedean skeleton $\mathcal{S}_{g,d}$ of $\text{Div}_{g,d}$ that makes the diagram

\[
\begin{array}{ccc}
\text{Div}^a_{g,d} & \xrightarrow{\text{trop}_{g,d}} & \text{Div}^{trop}_{g,d} \\
p_{g,d} & \downarrow & \downarrow J_{g,d} \\
\mathcal{S}_{g,d} & \cong & \text{Div}^{trop}_{g,d}
\end{array}
\]

commute.

Theorem 3, in particular, implies that $\text{trop}_{g,d}$, as defined above, is well-defined and continuous. Moreover, it gives a moduli-theoretic perspective on Baker’s specialization map from [Bak08, Section 2C] in the spirit of [ACP15]. It has often been suggested that the semicontinuity of the Baker-Norine rank for divisors on curves [Bak08, Corollary 2.11] should be connected to the continuity (or flatness) of the process of tropicalization. Theorem 3 provides one place where we can make the notion of continuity precise. It would be highly interesting to further investigate this connection in more detail.

A moduli-theoretic approach, very similar to the one above, has implicitly appeared in the proof of [JR17, Theorem 4.6], where the authors work with the stacky symmetric product of the universal curve over $\mathcal{M}_g$ (see Remark 1.3 for details).

**Instances of the realizability problem.** The realizability problem for divisors (or divisor classes) of a certain fixed rank on tropical curves is central to the many applications of the tropical approach to limit linear series and has recently received a significant amount of attention (see [BJ15, Section 10] and the references therein). It is a crucial element in the tropical approach to the maximal rank conjecture for quadrics [JP16], which is based on a realizability result coming from [CJP15], as well as in the recent works on the Brill-Noether varieties for curves of fixed gonality [Pfl17; JR17]. We also highlight [Car17], in which the author shows that this realizability problem fulfills a version of Murphy’s Law in the sense of Mnev universality, and [He17], which connects the classical smoothing problem for limit linear series with the divisor theory on metrized curve complexes of Amini and Baker in [AB15].

**Acknowledgements.** M.M. and A.W. acknowledge support from the LOEWE-Schwerpunkt “Uniformisierte Strukturen in Arithmetik und Geometrie”. M.U. would like to thank the Max Planck Institute for Mathematics in the Sciences in Leipzig and, in particular, the Non-Linear Algebra Group (with its director Bernd Sturmfels) for hosting him, while the majority of work on this project was undertaken. The authors thank Matt Bainbridge, Matt Baker, Dawei Chen, Renzo Cavalieri, Quentin Gendron, Sam Grushevsky, Bo Lin, Diane Maclagan, and Dhruv
Ranganathan for useful comments and suggestions. We particularly thank Dave Jensen for pointing out missing cases in an earlier version of Example 6.5.

1. Compactifying the moduli space of algebraic divisors

Fix $k$ to be an algebraically closed field. Let $X$ be a scheme over a $k$. Recall that a Cartier divisor $D$ on $X$ is effective if it admits a representation $(U_i, f_i)$ such that $f_i \in \Gamma(U_i, \mathcal{O}_X)$ and $f_i$ is a non-zero divisor. We may think of $D$ as the closed subscheme of $X$ whose ideal sheaf $I(D) = \mathcal{O}(-D)$ is invertible and generated by $f_i$ on $U_i$.

Given a morphism $X \to S$ of schemes over $k$, we say that an effective divisor $D$ on $X$ is a relative effective Cartier divisor if it is flat over $S$ when regarded as a subscheme of $X$. If $S$ is connected, the function $s \mapsto \deg D_s$ is constant and will be referred to as the degree of $D$.

Inspired by both [Cap94] and [Has03], we define the following moduli space.

**Definition 1.1.** Let $g \geq 2$ and $d \geq 0$. Define $\overline{\text{Div}}_{g,d}$ to be the fibered category over $\overline{M}_g$ whose fiber over a family $\pi : X \to S$ of stable curves is the set of pairs $(X', D)$ consisting of a semistable model $X'$ of $X$, i.e. a semistable curve over $S$ with stabilization $X$, and a relative effective Cartier divisor $D$ on $X'$ such that

(i) the support of $D$ does not meet the nodes in each fiber $X'_i$ of $\pi : X' \to S$; and
(ii) the twisted canonical divisor $K_{X'} + D$ is relatively ample.

Denote by $\text{Div}_{g,d}$ the preimage of the locus $\mathcal{M}_g$ of smooth curves. The goal of this section is to prove the following Theorem 1.2.

**Theorem 1.2.** The fibered category $\overline{\text{Div}}_{g,d}$ is a Deligne-Mumford stack of dimension $N = 3g - 3 + d$ that is smooth and proper over $k$. Its coarse moduli space $\overline{\text{Div}}_{g,d}$ is projective. The complement of $\text{Div}_{g,d}$ in $\overline{\text{Div}}_{g,d}$ is a divisor with (stack-theoretically) normal crossings and the forgetful morphism $\overline{\text{Div}}_{g,d} \to \overline{M}_g$ is toroidal.

Given a smooth curve $X$ over the field $k$, it is well-known that the restriction of $\text{Div}_{g,d}$ to the point $[X]$ in $\mathcal{M}_g$ is representable by the $d$-th symmetric product $\text{Sym}^d X$ of $X$ (see e.g. [Mil86, Theorem 3.1.3]). In this section we generalize this result to all of $\text{Div}_{g,d}$ using a variant of Hassett’s moduli spaces of weighted stable curves [Has03] that automatically provides us with a compactification of $\text{Div}_{g,d}$ with favorable properties. Over $\mathcal{M}_g$ this specializes to an isomorphism between $\text{Div}_{g,d}$ and the relative symmetric product

$$\text{Sym}^d \mathcal{X}_g = \mathcal{X}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{X}_g / S_d$$

of the universal curve $\mathcal{X}_g$ over $\mathcal{M}_g$.

**Remark 1.3.** In the proof of [JR17, Theorem 4.6] Jensen and Ranganathan work with the symmetric product $\overline{\text{Sym}}^d \mathcal{X}_g$ of the compactified universal curve $\overline{\mathcal{X}}_g$ over $\overline{M}_g$ as a compactification of $\text{Div}_{g,d}$. This compactification is not smooth and not even toroidal, in general; the authors get around this restriction by working with the stacky symmetric product

$$\overline{\text{Sym}}^d \overline{\mathcal{X}}_g = \overline{\mathcal{X}}^d / S_d$$

whose boundary admits (stack-theoretically) toroidal singularities. Our compactification is a relative coarse moduli space of a toroidal resolution of the singularities of $\overline{\text{Sym}}^d \overline{\mathcal{X}}_g$. 
1.1. Moduli of stable curves with unordered points of small weight. Let \( A = (a_1, \ldots, a_d) \in (0,1] \cap \mathbb{Q}^d \) and \( g \geq 0 \) such that
\[
2g - 2 + a_1 + \ldots + a_d > 0.
\]
In [Has03] Hassett studies the moduli stack \( \overline{M}_{g,A} \) of weighted stable \( d \)-marked curves of genus \( g \). The fiber of the stack \( \overline{M}_{g,A} \) over a scheme \( S \) is the groupoid of proper flat morphisms \( \pi : X \to S \) such that each geometric fiber is a nodal curve of genus \( g \) together with \( n \) (marked) sections \( s_1, \ldots, s_d \) that fulfill the following two axioms:

- the sections \( s_1, \ldots, s_d \) lie in the smooth locus of \( \pi \) and for all \( s_{i_1}, \ldots, s_{i_k} \) with non-empty intersections we have \( a_{i_1} + \ldots + a_{i_k} \leq 1 \); and
- the \( \mathbb{Q} \)-divisor \( K_X + a_1 s_1 + \ldots + a_n s_n \) is relatively ample.

If \( A = (1, \ldots, 1) \) this is the classical Deligne-Knudsen-Mumford moduli stack \( \overline{M}_{g,n} \) of \( n \)-marked stable curves of genus \( g \) (see [DM69; Knu83]). By [Has03, Theorem 2.1] the stack \( \overline{M}_{g,A} \) is an irreducible Deligne-Mumford stack that is smooth and proper over \( K \) and its coarse moduli space \( \overline{M}_{g,A} \) is projective. Moreover, it is well-known that the complement of the locus of smooth curves \( M_{g,A} \) in \( \overline{M}_{g,A} \) has (stack-theoretically) normal crossings (see e.g. [Uli15h, Theorem 1.1]).

Let \( g \geq 2, d \geq 0 \), and \( A = (\epsilon, \ldots, \epsilon) = \epsilon^d \) for \( \epsilon > 0 \) such that \( d \cdot \epsilon \leq 1 \), e.g. \( \epsilon = \frac{1}{g} \).

In other words, the sections \( s_1, \ldots, s_d \) of an object in \( \overline{M}_{g,\epsilon^d} \) need not be disjoint, and we require each rational component to have at least two nodes and to contain a marked point if it is has precisely two nodes (so that the underlying nodal curve is semistable).

There is a natural functor \( \overline{M}_{g,\epsilon^d} \to \overline{M}_g \) that forgets about all sections and contracts all rational components that only have two nodes. Moreover, there is a natural operation of \( S_d \) on \( \overline{M}_{g,\epsilon^d} \) that permutes the sections and the forgetful map \( \overline{M}_{g,\epsilon^d} \to \overline{M}_g \) factors through the stack quotient \( \overline{M}_{g,\epsilon^d}/S_d \to \overline{M}_g \).

1.2. Relative coarse moduli spaces. Let \( f : X \to Y \) be a morphism of algebraic stacks that is locally of finite presentation and whose relative inertia stack \( \mathcal{I}(X/Y) = X \times_Y X \times X \) is finite. By [AOV11, Theorem 3.1] there is an algebraic stack \( X \) together with a factorization

\[
f = \overline{f} \circ \pi : X \xrightarrow{\pi} X \xrightarrow{\overline{f}} Y
\]

with \( \overline{f} \) representable that is initial among such factorizations. The algebraic stack \( X \) is called the relative coarse moduli space of \( X \) over \( Y \). We refer the reader to [AOV11, Section 3] for the construction and basic properties of relative coarse moduli spaces.

**Theorem 1.4.** The fibered category \( \overline{\text{Div}}_{g,d} \) is equivalent to the relative coarse moduli space of the stack quotient \( \overline{M}_{g,\epsilon^d}/S_d \) over \( \overline{M}_g \).

In other words, the natural map

\[
\overline{M}_{g,\epsilon^d}/S_d \to \overline{\text{Div}}_{g,d}
\]

makes \( \overline{\text{Div}}_{g,d} \) into the initial object among all factorizations of \( \overline{M}_{g,\epsilon^d}/S_d \to X \to \overline{M}_g \) such that \( X \to \overline{M}_g \) is representable. Theorem [1.4] is a global version of the fact
that the $d$-fold symmetric product of a smooth curve $X$ represents the space of effective divisors of degree $d$ on $X$. Our proof is an adaption of [Mil86, Theorem 3.13] to this situation.

Proof of Theorem 1.4. An object in $\overline{M}_{g,e}$ is a family $\pi: X' \to S$ of semistable curves of genus $g$ with $d$ sections $s_1, \ldots, s_d$ (not necessarily disjoint) such that the twisted canonical divisor $K_{X'} + \varepsilon s_1 + \ldots + \varepsilon s_d$ is relatively ample. Then the sum $D = \sum_{i=1}^{d} s_i$ defines a relative effective divisor of degree $d$ on $X$ whose support does not meet the nodes in each fiber $X'_s$. We call such a relative effective Cartier divisor split. The pair $(X', D)$ is stable in the sense that the twisted canonical divisor $K_{X'} + D$ is ample. This defines a functor $\overline{M}_{g,e} \to Div_{g,d}$ that is invariant under the natural operation of $S_d$ on $\overline{M}_{g,e}$ and so we have a natural induced morphism $[\overline{M}_{g,e}/S_d] \to Div_{g,d}$ of categories fibered in groupoids over $\overline{M}_g$.

Conversely, let $D$ be a relative effective Cartier divisor on a family $\pi: X' \to S$ of semistable curves. Assume first that $D$ is split. Then we may write $D = \sum_{i=1}^{d} s_i$ for sections $s_1, \ldots, s_d$ (not necessarily disjoint) that do not meet the nodes in each fiber and for which the twisted canonical divisor $K_{X'} + D = K_{X'} + s_1 + \ldots + s_d$ is relatively ample. This implies that the divisor $K_{X'} + \varepsilon s_1 + \ldots + \varepsilon s_d$ is also relatively ample and therefore the tuple $(\pi, s_1, \ldots, s_d)$ defines an object in $\overline{M}_{g,e}$.

In general, when $D$ is not split, we may perform a finite and flat base change $T \to S$ so that the pullback $D_T$ of $D$ to $\pi': X'_T \to T$ is split. In this case, both pullbacks of $D'$ to the family $X'_{T'} \to T' = T \times_S T$ are split and naturally isomorphic. This induces a (representable) morphism $T \to [\overline{M}_{g,e}/S_d]$, both of whose restrictions to $T'$ are naturally equivalent.

By finite flat descent [Mil86, Section I, Theorem 2.17], every faithfully flat morphism of finite type is a strict epimorphism. This means that for every algebraic stack $Z$ with a representable morphism to $\overline{M}_g$ the sequence

$$\text{Hom}(\overline{M}_g, Z) \to \text{Hom}([\overline{M}_{g,e}/S_d], T, Z) \to \text{Hom}(\overline{M}_g, (T', Z))$$

is exact. If we take $Z$ to be the relative coarse moduli space of $[\overline{M}_{g,e}/S_d]$ we find an object in the relative coarse moduli space of $[\overline{M}_{g,e}/S_d]$ that maps to $(X', D)$. This shows that $Div_{g,d}$ is equivalent to the relative coarse moduli space of $[\overline{M}_{g,e}/S_d]$.

1.3. Proof of Theorem 1.2. Theorem 1.4 together with [AOV11, Theorem 3.1] implies that $Div_{g,d}$ is an irreducible algebraic stack that is proper over $k$. The universal properties immediately show that the coarse moduli space $\overline{M}_{g,e}$ is isomorphic to the geometric invariant theory quotient $\overline{M}_{g,e}/S_d$ (i.e. the coarse moduli space of the stack quotient $[\overline{M}_{g,e}/S_d]$) and therefore it is projective, since $\overline{M}_{g,e}$ is.

We choose an "exhausting" family for $\overline{M}_{g,e}$ as in [Has03, Section 3.4]: Set

$$L = \omega_n^e(\nu \varepsilon s_1 + \ldots + \nu \varepsilon s_d)$$

where $\nu > 0$ is the smallest integer such that $\nu \varepsilon$ is integral, i.e. $\nu = d$ if we choose $\varepsilon = \frac{1}{d}$. Set $q = \deg(L^\otimes 3) = 3(2g - 2 + d)$ as well as $r = 3g - 3$. Denote by $H_0$ the Hilbert scheme of tuples $(s_1, \ldots, s_d)$ in $\mathbb{P}^r$ and by $H_1$ the Hilbert scheme parametrizing curves $X$ of genus $g$ and degree $q$ in $\mathbb{P}^r$. Consider the locally closed locus $U \subseteq H_1 \times H_0$ of points $([X], s_1, \ldots, s_d)$ such that
• the curve $X$ is reduced and nodal;
• the sections $s_1, \ldots, s_d$ are contained in the smooth locus of $C$; and
• $\mathcal{O}_X(1) = L^{\otimes 3}$.

Notice that there is a natural surjective morphism $U \to \overline{\mathcal{M}}_{g,d}$ that is given by forgetting the embedding into $\mathbb{P}^r$.

The symmetric group $S_d$ acts on $H_0$ by permuting the entries. The induced action on $H_1 \times H_0$ stabilizes $U$ and we denote the quotient $U/S_d$ by $\tilde{U}$. The fiber $\tilde{U}[X]$ over a point $[X]$ in $H_1$ is precisely the open subscheme of the Hilbert scheme of points on $X$ parametrizing closed subschemes $D$ of $X$ of length $d$ whose support does not meet the nodes of $X$, or equivalently the open subset of the symmetric product $\text{Sym}^d X$ whose points do not meet the nodes of $X$.

Consider the product $H_1 \times \tilde{H}_0$ where $\tilde{H}_0$ is the Hilbert scheme of zero-dimensional subscheme of $\mathbb{P}^r$ of length $d$. The quotient $\tilde{U}$ is precisely the locus of tuples $(X, D)$ in $H_1 \times \tilde{H}_0$ such that

• the curve $X$ is reduced and nodal;
• the support of $D$ is contained in the smooth locus of $X$; and
• $\mathcal{O}_X(1) = L^{\otimes 3}$, where $L = \omega_X(D)$.

The natural map $\tilde{U} \to \overline{\text{Div}}_{g,d}$ is representable and surjective by [Has03, Proposition 3.3]. It is smooth, since every automorphism of $(X, D)$ is induced from a projective automorphisms of $\mathbb{P}^r$. In fact, we may realize $\overline{\text{Div}}_{g,d}$ naturally a quotient of $H_1 \times \tilde{H}_0$ by the natural operation of $\text{PGL}_r$. Since at all geometric points this operation has reduced and finite stabilizers, the diagonal morphism of $\overline{\text{Div}}_{g,d}$ is unramified and therefore it is a Deligne-Mumford stack.

Since the symmetric product of a nodal curve is smooth away from the nodes (see e.g. the argument in [Mil86, Proposition 3.2]) each $\tilde{U}[X]$ is smooth. This implies that $\tilde{U}$ is smooth and so is $\overline{\text{Div}}_{g,d}$.

Let $\mathfrak{o}_k$ be the field $k$ if char $k = 0$ and otherwise the unique complete regular local with residue field $k$ and maximal ideal generated by $p$ where char $k = p > 0$ (using Cohen’s structure theorem). A standard deformation-theoretic argument (in the framework of [Has03, Section 3.3]) shows that the complete local ring of $\overline{\text{Div}}_{g,d}$ at a closed point $[X', D]$ (with nodes $x_1, \ldots, x_k$) is given by

$$\tilde{\mathcal{O}}_{\overline{\text{Div}}_{g,d},[X', D]} = \mathfrak{o}_k[t_1, \ldots, t_N]$$

where $N = 3g - 3 + n$ and the locus where $x_i$ stays nodal is given by $t_i = 0$ for $1 \leq i \leq k$. This shows that the complement of the smooth locus $\text{Div}_{g,d}$ in $\overline{\text{Div}}_{g,d}$ has (stack-theoretically) normal crossings.

Denote by $[X]$ in $\overline{\mathcal{M}}_g$ the stabilization of the curve $[X']$. The stability condition in place can be interpreted as saying that $X'$ has no rational tails and that the nodes of $X'$ have to contract to nodes of $X$. By [DM69] the complete local ring of $\overline{\mathcal{M}}_g$ at the point $[X]$ is given by

$$\tilde{\mathcal{O}}_{\overline{\mathcal{M}}_g,[X]} = \mathfrak{o}_k[\tilde{t}_1, \ldots, \tilde{t}_{3g-3}]$$

where the locus where the singularity $\tilde{x}_i$ of stays nodal is given by $\tilde{t}_i = 0$ for $1 \leq i \leq r$. Suppose now that the nodes $x_i$ on $X'$ are being contracted to $\tilde{x}_i$ in $X$. Then the forgetful map $\overline{\text{Div}}_{g,d} \to \overline{\mathcal{M}}_g$ is given by $\tilde{t}_i = t_{i_1} \cdots t_{i_{r_i}}$ on the complete local rings and so it is a toroidal morphism.
2. Tropical divisors and their moduli

Let us first introduce tropical curves (see e.g. [Mik00]). A metric graph is an equivalence class of tuples \((G, |·|)\) consisting of a connected finite graph \(G = (V, E)\) together with an edge length function \(|·| : E(G) \to \mathbb{R}_{>0}\). Two such tuples \((G, |·|)\) and \((G', |·'|)\) are equivalent, if there is a common length preserving refinement. We implicitly identify a metric graph, represented by \((G, |·|)\), with its realization as a metric space, by gluing an interval of length \(|e|\) for every edge \(e\) according to the incidences in \(G\).

A tropical curve \(\Gamma\) is a metric graph together with a function \(h : \Gamma \to \mathbb{Z}_{\geq 0}\) with finite support. We refer to a tuple \((G, |·|)\) as a model of \(\Gamma\) if it represents \(\Gamma\) as a metric graph, and if \(h\) is supported on the vertices of \(G\). The genus of a tropical curve is defined to be

\[
g(\Gamma) = b_1(\Gamma) + \sum_{p \in \Gamma} h(p)
\]  

(1)

A model \(G\) of a tropical curve is said to be semistable, if for every vertex \(v\) of \(G\) we have \(2h(v) - 2 + |v| \geq 0\), where \(|v|\) denotes the valency of the vertex \(v\). It is called stable, if the above inequality is strict, i.e. if we have \(2h(v) - 2 + |v| > 0\) for all vertices \(v\) of \(G\). Notice that, when a tropical curve \(\Gamma\) admits a semistable model, its minimal model is necessarily stable. In this case, we call \(\Gamma\) stable.

Later we will also use the notion of a tropical curve \(\Gamma\) with legs for a tropical curve \(\Gamma\) decorated with a collection \(L\) of infinite half-edges, i.e. of legs, emanating from the vertices of \(G\). In this case, we modify the definition of stability by also counting the legs, when determining the valency of a vertex. Whenever it is clear from the context, we refer to a tropical curve with legs simply as a tropical curve.

Let \(g \geq 2\). The moduli space \(M^\text{trop}_g\) of stable tropical curves is defined to be the set of isomorphism classes of stable tropical curves (without legs) of genus \(g\). By [ACP15] it has the structure of a generalized (rational polyhedral) cone complex, i.e. it arises as a colimit of a diagram of (not necessarily proper) face morphisms of rational polyhedral cones.

The goal of this section is to construct a moduli space \(\text{Div}^\text{trop}_{g,d}\) of tropical divisors of degree \(d\) over \(M^\text{trop}_g\).

2.1. Divisors on tropical curves. A divisor on a tropical curve \(\Gamma\) is a finite formal sum \(D = \sum a_i p_i\) of points \(p_i \in \Gamma\) with integral coefficients \(a_i\). We let \(\deg(D) = \sum a_i\) be the degree of the divisor and write \(D(p) = \sum_{p_i \to p} a_i\) for a point \(p \in \Gamma\). A divisor \(D\) is said to be effective, denoted by \(D \geq 0\), if \(D(p) \geq 0\) for all points \(p\) of \(\Gamma\). Given a tropical curve \(\Gamma\), we denote by \(\text{Div}(\Gamma)\) the group of divisors on \(\Gamma\).

A rational function on \(\Gamma\) is a continuous function \(f : \Gamma \to \mathbb{R}\) whose restriction to every edge \(\Gamma\) (thought of as an interval \([0, |e|]\)) is a piece-wise linear function whose slopes are integral. Given a rational function \(f\) on \(\Gamma\) and \(P \in \Gamma\), we define the order \(\text{ord}_P(f)\) of \(f\) at \(P\) to be the sum of the outgoing slopes of \(f\) over all edges emanating from \(P\). This defines a map

\[
div : \text{Rat}(\Gamma) \to \text{Div}(\Gamma) \\
f \mapsto \sum_{p \in \Gamma} \text{ord}_p(f) \cdot p
\]

that assigns to any rational function its divisor. The image of the map \(\text{div}\) is called the subgroup \(P\text{Div}(\Gamma) \subset \text{Div}(\Gamma)\) of principal divisors. The divisors \(D\) and \(D'\) are called equivalent (denoted by \(D \sim D'\)) if \(D - D' \in P\text{Div}(\Gamma)\).
We can now define the linear system of a divisor $D$ to be

$$|D| = \{ D' \in \operatorname{Div}(\Gamma) : D \geq 0 \text{ and } D \sim D' \}.$$  

It is convenient to also introduce the tropical analogue

$$R(D) = \{ f \in \operatorname{Rat}(\Gamma) : D + \operatorname{div}(f) \geq 0 \}$$  

of the global sections of $\mathcal{O}(D)$. Note that we can shift any element in $R(D)$ by adding a real number and that $|D| = R(D)/\mathbb{R}$.

For any divisor $D$ the space $R(D)$ has the structure of a polyhedral complex (see e.g. [GK08, Lemma 1.9], [MZ08], and [LU17, Proposition 3.2]). However, this polyhedral complex is not equidimensional in general, as we will see in the case of the canonical linear system in Section 6.

2.2. Moduli of tropical divisors.

**Definition 2.1.** Let $g \geq 2$. The moduli space $\operatorname{Div}_{g,d}^{\text{trop}}$ is the set of isomorphism classes of tuples $(\Gamma, D)$ consisting of a stable tropical curve of genus $g$ and an effective divisor $D$ on $\Gamma$ of degree $d$.

**Proposition 2.2.** The moduli space $\operatorname{Div}_{g,d}^{\text{trop}}$ naturally has the structure of a generalized cone complex of dimension $3g - 3 + d$.

Consider a pair $(G', D)$ consisting of a finite semistable vertex-weighted graph $G'$ of genus $g$ and an effective divisor $D$ on $G'$ of degree $d$ supported on the vertices of $G'$. We say that the pair $(G', D)$ is stable if for every vertex $v$ of $G'$ we have $2h(v) - 2 + |v| + D(v) > 0$.

**Proof of Proposition 2.2.** Denote by $J_{g,d}$ the category of stable pairs $(G', D)$ where $G'$ is of genus $g$ and $D$ has degree $d$. The morphisms in $J_{g,d}$ are generated by

- automorphisms $\phi$ of the weighted graph $G'$ such that $\phi^* D = D$; and
- weighted edge contractions $\pi : (G'_1, D_1) \rightarrow (G'_2, D_2)$ (i.e. edge contractions for which $g(\pi^{-1}(v)) = h(v)$ for all vertices $v$ of $G'_2$) that fulfill $\pi_* D_1 = D_2$.

There is a natural functor from $J_{g,d}$ to the category $\mathbf{RPC}_{g,d}^{\text{face}}$ of (rational polyhedral) cones with (not-necessarily proper) face morphisms, given by $(G', D) \mapsto \sigma_{G'} = \mathbb{R}_{\geq 0}^{E(G')}$. Recall that a face morphism is a morphism of rational polyhedral cones $\sigma \rightarrow \sigma'$ that induces an isomorphism between $\sigma$ and a (not necessarily proper) face of $\sigma'$; the class of face morphisms, in particular, includes all automorphisms.

For a fixed $(G', D)$, the open cone $\sigma_{G'} = \mathbb{R}_{>0}^{E(G')}$ parametrizes the space of triples consisting of

- a tropical curve $\Gamma$ in $M_{g}^{\text{trop}}$,
- an effective divisor $D$ on $\Gamma$ of degree $d$, and
- an isomorphism between $G'$ and the unique minimal semistable model of $\Gamma$ whose vertices contain the support of $D$.

The automorphism group $\operatorname{Aut}(G', D)$ acts on $\sigma_{G'} = \mathbb{R}_{>0}^{E(G')}$ by permuting the entries of the vectors accordingly and the natural map $\sigma_{G'} \rightarrow \operatorname{Div}_{g,d}^{\text{trop}}$ factors through the injection $\sigma_{G'}/\operatorname{Aut}(G', D) \rightarrow \operatorname{Div}_{g,d}^{\text{trop}}$. Thus the set $\operatorname{Div}_{g,d}^{\text{trop}}$ arises as a colimit of the diagram $J_{g,d} \rightarrow \mathbf{RPC}_{g,d}^{\text{face}}$ and therefore carries the structure of a generalized cone complex.
Finally, a maximally degenerate object \((G', D)\) in \(J_{g,d}\) with all vertex weights equal to zero, has precisely \(3g - 3 + d\) finite edges. Therefore the dimension of every maximal cone in \(\text{Div}^{\text{trop}}_{g,d}\) is \(3g - 3 + d\).

**Remark 2.3.** Let \(\Gamma\) be a stable tropical curve. The set \(\text{Div}^\ast_d(\Gamma)\) of effective divisors of degree \(d\) on \(\Gamma\) admits a natural rational polyhedral subdivision that arises by subdividing \(\text{Sym}^3 \Gamma\) along the folds.

Let \(G\) be the unique minimal stable model of \(\Gamma\) and denote by \(J_{g,d}/G\) the category of triples consisting of a stable pair \((G', D)\) in \(J_{g,d}\) and an isomorphism between the stabilization of \(G'\) with \(G\). This datum induces a functor \(J_{g,d}/G \to \text{RPC}^\text{face}_G\) given by \(G' \mapsto \sigma_{G'}\) together with a natural transformation to the constant functor \(G \mapsto \sigma_G\). The set \(\text{Div}^\ast_d(\Gamma)\) is the preimage of the point \([\Gamma] \in \sigma_G\) in the colimit of all \(\sigma_{G'}\).

**Remark 2.4.** Using the language of tropical moduli stacks developed in [CCUW], we may consider the natural moduli functor \(\text{Div}^{\text{trop}}_{g,d}\) that associates to a rational polyhedral cone \(\sigma\) the groupoid of semistable pairs \((G'', D)\) together with non-zero edge length on \(G''\) in the dual monoid \(S_{\sigma}\). The proof of Proposition 2.2 actually shows that \(\text{Div}^{\text{trop}}_{g,d}\) is representable by a cone stack in the sense of [CCUW] (see [CCUW], Section 3.4) for an analogous argument for the moduli stack \(\mathcal{M}^{\text{trop}}_{g,n}\) of \(n\)-marked stable tropical curves of genus \(g\).

A stable tropical curve \(\Gamma\) of genus \(g\) (with real edge lengths) then corresponds to a morphism \(\mathbb{R}_{\geq 0} \to \mathcal{M}^{\text{trop}}_g\). Expanding on [CCUW], Section 4, one can show that pullback \(\mathbb{R}_{\geq 0} \times_{\mathcal{M}^{\text{trop}}_g} \text{Div}^{\text{trop}}_{g,d}\) is representable by a cone space and its fiber over \(1 \in \mathbb{R}_{\geq 0}\) is exactly the polyhedral decomposition of \(\text{Div}^\ast_d(\Gamma)\) we have considered in Remark 2.3.

**Remark 2.5.** Given a tropical curve \(\Gamma\) with \(n\) legs, a marking is an ordering \(l_1 < \ldots < l_n\) of the legs of \(\Gamma\). Let \(g \geq 0\) and \(\mathcal{A} = (a_1, \ldots , a_n) \in (0, 1] \cap \mathbb{Q}^n\) such that \(2g - 2 + a_1 + \ldots + a_n > 0\). Following [CHMR; Uli15b], we say that a marked tropical curve \(\Gamma\) is stable of weight \(\mathcal{A}\) if we have

\[
2h(v) - 2 + |v|_E + |v|_A > 0
\]

for every vertex \(v \in V(G)\). Here \(|v|_E\) denotes the inner valency of the vertex \(v\), i.e. the number finite edges emanating from \(v\) (counting loops twice), and \(|v|_A\) denotes the sum \(a_{i_1} + \ldots + a_{i_k}\), where \(l_{i_1}, \ldots , l_{i_k}\) are the legs emanating from \(v\).

Denote by \(\mathcal{M}^{\text{trop}}_g,\mathcal{A}\) the space of tropical curves of genus \(g\) with \(n\) marked legs that are stable of type \(\mathcal{A}\). It has the structure of a generalized cone complex by [Uli15b], Section 3]. If \(\mathcal{A} = (\epsilon, \ldots , \epsilon)\) with \(\epsilon \leq \frac{1}{n}\), there is a natural morphism

\[
\mathcal{M}^{\text{trop}}_{g, \epsilon^d} \to \text{Div}^{\text{trop}}_{g,d}
\]

of generalized cone complexes, where \(\sum_{i=1}^n l_i\) denotes the sum over the vertices the legs are emanating from. This map induces a homeomorphism

\[
\mathcal{M}^{\text{trop}}_{g, \epsilon^d}/\mathcal{S}_d \sim \text{Div}^{\text{trop}}_{g,d}
\]

of the underlying topological spaces.
3. Specialization versus tropicalization

Let $k$ be an algebraically closed field $k$ endowed with the trivial absolute value. Denote by $\text{Div}^{an}_{g,d}$ the non-Archimedean analytification of the moduli space of smooth curves together with an effective divisor of degree $d$. In this section we define a natural tropicalization map

$$\text{trop}_{g,d} : \text{Div}^{an}_{g,d} \rightarrow \text{Div}^{trop}_{g,d}$$

from the Berkovich analytic space $\text{Div}^{an}_{g,d}$ to the tropical moduli space $\text{Div}^{trop}_{g,d}$ and show that this map can be identified with a natural strong deformation retraction onto the non-Archimedean skeleton of $\text{Div}^{an}_{g,d}$ in the sense of [Thu07].

Note that for every algebraic stack $\mathcal{X}$ which is locally of finite type over $k$, there is an associated analytic stack $\mathcal{X}^{an}$ defined by pullback with respect to the usual analytification functor on $k$-schemes, see [Uli14, Definition 2.18]. We usually abuse notation and denote by $\mathcal{X}^{an}$ the associated topological space as defined in [Uli14, Definition 3.3]. Hence, if $\mathcal{X}$ is a separated algebraic Deligne-Mumford stack, by [Uli14, Proposition 3.8], the space $\mathcal{X}^{an}$ can be identified with the Berkovich analytification of the coarse moduli space associated to $\mathcal{X}$.

3.1. Tropicalization of $\text{Div}^{an}_{g,d}$. We begin by defining the tropicalization map

$$\text{trop}_{g,d} : \text{Div}^{an}_{g,d} \rightarrow \text{Div}^{trop}_{g,d}.$$  

A point in $\text{Div}^{an}_{g,d}$ is represented by a proper, smooth algebraic curve $X$ of genus $g$ over a field $K$ that is a non-Archimedean extension of $k$ together with an effective divisor $D$ on $X$ of degree $d$. Possibly after replacing $K$ by a finite extension, there is a semistable model $\mathcal{X}/S$ of $X$ over the spectrum $S$ of the valuation ring $R$ of $K$ together with a relative effective divisor $\mathcal{D}$ on $\mathcal{X}$ that does not meet the singularities in the special fiber $\mathcal{X}_s$ and makes the divisor $K_X + \mathcal{D}$ relatively ample. Here we use that the moduli stack $\text{Div}^{an}_{g,d}$ is proper. Its special fiber $(\mathcal{X}_s, \mathcal{D}_s)$ (as a Cartier divisor) is an element in $\text{Div}^{an}_{g,d}(\bar{K})$, where $\bar{K}$ denotes the residue field of $R$. On the level of points, the tropicalization map

$$\text{trop}_{g,d} : \text{Div}^{an}_{g,d} \rightarrow \text{Div}^{trop}_{g,d}$$

associates to the pair $(X, D)$ the dual tropical curve $\Gamma$ of $\mathcal{X}_s$ together with the specialization of $D$ to $\Gamma$, an effective divisor of degree $d$ on $\Gamma$. More precisely, the weighted dual graph $G'$ is the incidence graph of $\mathcal{X}_s$ together with the vertex weights $h(v)$ given by the genus of (the normalization of) the corresponding irreducible component of $\mathcal{X}_s$. The dual tropical curve of $\mathcal{X}$ is the tropical curve with semistable model $G'$ for which the length $|e|$ of the edge $e \in E(\Gamma)$ is defined to be $\text{val}_R(f)$, where $xy = f$ is the local equation of the node corresponding to $e$, and where $\text{val}_R$ denotes the valuation. For $v \in V(\Gamma)$ we denote the normalization of the component $C_v$ of the special fiber $\mathcal{X}_s$ by $\bar{C}_v$. The specialization of $D$ to $\Gamma$ is then defined as the multidegree

$$\text{mdeg}(D_s) = \sum_{v \in V(\Gamma)} \text{deg}(D_s|_{\bar{C}_v}) \cdot [v]$$

of the special fiber $D_s$ of $\mathcal{D}$, thought of as a divisor on $\Gamma$ (with support contained in the vertices of $G'$).

The independence of the choices made in this construction is checked in [Viv13] for the moduli space of stable curves. It also follows a posteriori from Theorem 3.2 below.
3.3. The retraction to the skeleton. Let $X_0 \to X$ be a toroidal embedding, i.e., an open immersion of normal schemes locally of finite type over $k$ that étale locally on $X$ admits an étale morphism $\gamma : X \to Z$ into a $T$-toric variety $Z$ such that $\gamma^{-1}(T) = X_0$. Moreover, suppose for notational simplicity that $X$ is proper over $k$.

In [Thu07] Thuillier has constructed a strong deformation retraction

$$p_{X_0 \to X} : X_0^{an} \to X^{an}$$

onto a closed subset $\mathcal{G}(X_0 \to X)$ of $X_0^{an}$ with the structure of a generalized cone complex, the non-Archimedean skeleton of $X_0$ (defined with respect to the toroidal compactification $X_0 \to X$). We refer the reader to [ACP15, Section 6] for a generalization of this construction to separated toroidal Deligne-Mumford stacks.

In fact, Thuillier’s construction in [Thu07] extends to the analytification of the toroidal compactification $X^{an}$ and we obtain a strong deformation retraction to a compactified skeleton $\overline{\mathcal{G}}_{X_0 \to X}$ of $X^{an}$. The resulting strong deformation retraction is proper and closed as a map $X^{an} \to \overline{\mathcal{G}}_{X_0 \to X}$, since $\overline{\mathcal{G}}_{X_0 \to X}$ is compact. Therefore $p_{X_0 \to X}$ is proper and closed as well.

By Theorem 1.2 the open immersion $\text{Div}_{g,d} \to \overline{\text{Div}}_{g,d}$ is toroidal and hence there is a natural strong deformation retraction

$$p_{g,d} : \overline{\text{Div}}^{an}_{g,d} \to \text{Div}^{an}_{g,d}$$
We begin by showing that the skeleton characteristic monoid of the natural divisorial logarithmic structure on the generic point of \(D\) is contained in the closure of \(D\). This implies Theorem 3.2 from the introduction.

**Theorem 3.2.** There is a natural isomorphism \(\Phi_{g,d} : \text{Div}^{\text{trop}}_{g,d} \rightarrow \mathfrak{S}_{g,d}\) that makes the diagram

\[
\begin{array}{ccc}
\text{Div}^{\text{an}}_{g,d} & \xrightarrow{\text{trop}_{g,d}} & \text{Div}^{\text{trop}}_{g,d} \\
\downarrow \Phi_{g,d} & & \downarrow \text{id} \\
\mathfrak{S}_{g,d} & \xrightarrow{z} & \text{Div}^{\text{trop}}_{g,d}
\end{array}
\]

commute.

Theorem 3.2 in particular shows that the tropicalization map \(\text{trop}_{g,d}\) defined above is well-defined, continuous, proper, and closed.

**Proof of Theorem 3.2** This is yet another version of the main result of [ACP15]. We begin by showing that the skeleton \(\mathfrak{S}_{g,d}\) is naturally isomorphic to \(\text{Div}^{\text{trop}}_{g,d}\).

Since \(\text{Div}_{g,d} \rightarrow \text{Div}^{\text{trop}}_{g,d}\) is a toroidal embedding by Theorem 1.2, the stack \(\text{Div}_{g,d}\) admits a natural stratification by locally closed substacks (as in [Tha07, Section 3.1]). The locally closed strata are precisely the locally closed substacks parametrizing pairs \((X', D)\) for which the pair consisting of its weighted dual graph and the multidegree of \(D\) is constant. Consequently, there is a stratum \(\mathcal{M}(G', D)\) for every object \((G', D)\) of \(J_{g,d}\) (as in the proof of Proposition 2.2).

For a pair \((G', D)\) in \(J_{g,d}\), set

\[
\text{Div}_{(G', D)} = \bigcap_{v \in V(G')} \text{Div}_{h(v), |v|, d_v}
\]

where \(|v|\) denotes the valency of \(v\) in \(G'\) and \(d_v = D(v)\) denotes the degree of \(D\) at the vertex \(v\). Here we make use of the stack \(\text{Div}_{g,n,d}\) of effective divisors over \(\mathcal{M}_{g,n}\), which is constructed as the \(d\)-fold symmetric product of the universal curve over \(\mathcal{M}_{g,n}\), i.e. as the relative coarse moduli space of the morphism \([\mathcal{M}_{g,1^n,\epsilon}/S_d] \rightarrow \mathcal{M}_g\).

We denote by \(\Lambda^+_{(G', D)}\) the monoid of effective Cartier divisors on \(\text{Div}_{g,d}\) whose support is contained in the closure of \(\text{Div}_{(G', D)}\). In other words, \(\Lambda^+_{(G', D)}\) is the characteristic monoid of the natural divisorial logarithmic structure on \(\text{Div}_{(G', D)}\). Notice that \(\Lambda^+_{(G', D)} \cong \mathbb{N}^{E(G')}\), since \(E(G')\) is precisely the set of nodes of an element in \(\text{Div}_{(G', D)}\).

We observe:

- A stratum \(\mathcal{M}_{(G'_1, D_1)}\) is in the closure of another stratum \(\mathcal{M}_{(G'_2, D_2)}\) if and only if there is a weighted edge contraction \(\pi : G'_1 \rightarrow G'_2\) such that \(\pi_* D_1 = D_2\). In this case, the \(\mathcal{E}\)-specialization \(\Lambda^+_{(G'_1, D_1)} \rightarrow \Lambda^+_{(G'_2, D_2)}\) is given by the projection \(\mathbb{N}^{E(G'_1)} \rightarrow \mathbb{N}^{E(G'_2)}\) and the induced map

\[
\mathbb{R}^{E(G'_1)} = \text{Hom}(\Lambda^+_{(G'_1, D_1)}, \mathbb{R}_{\geq 0}) \rightarrow \text{Hom}(\Lambda^+_{(G'_2, D_2)}, \mathbb{R}_{\geq 0}) = \mathbb{R}^{E(G'_2)}
\]

is precisely the face morphism induced from \(\pi : G'_1 \rightarrow G'_2\).
- As in [ACP15, Proposition 3.4.1], we have an equivalence

\[
\text{Div}_{(G', D)} \cong \left[\text{Div}_{(G', D)}/\text{Aut}(G', D)\right]
\]
and therefore the action of the étale fundamental group of $\text{Div}_{(G',D)}$ on $\mathbb{R}^{E(G')}_{\geq 0} = \text{Hom}(\Lambda^+(G',D), \mathbb{R}_{\geq 0})$ surjectively factors through the action of $\text{Aut}(G', D)$ that permutes the entries.

The above observations together with [ACP15, Proposition 6.2.6] show that the non-Archimedean skeleton $\mathcal{S}_{g,d}$ arises as the colimit

$$\lim_{(G', D) \in J_{g,d}} \text{Hom}(\Lambda^+(G',D), \mathbb{R}_{\geq 0}) = \lim_{(G', D) \in J_{g,d}} \mathbb{R}^{E(G')}_{\geq 0} = \lim_{(G', D) \in J_{g,d}} \sigma(G', D)$$

and therefore $\mathcal{S}_{g,d}$ is naturally isomorphic to $\text{Div}^{trop}_{g,d}$ (as a generalized cone complex).

Let us now show that above diagram commutes, i.e. that $p_{g,d} = \Phi_{g,d} \circ \text{trop}_{g,d}$. A point $x$ in $\text{Div}^{an}_{g,d}$ gives rise to a morphism $S = \text{Spec } R \to \text{Div}^{an}_{g,d}$ and the image of the closed point lies exactly in a stratum $\text{Div}(G',D)$. Thus the stable pair $(G', D)$ is the underlying combinatorial type of both $p_{g,d}(x)$ and $\text{trop}_{g,d}(x)$ and so they are in the interior of the same cone $\sigma(G', D) = \mathbb{R}^{E(G')}_{\geq 0}$ in $\mathcal{S}_{g,d} \simeq \text{Div}^{trop}_{g,d}$.

The edge lengths (of the edges in $G'$) of both of the resulting tropical curves agree, since $p_{g,d}$ is given by taking valuations of the elements in $\Lambda^+(G',D)$ and these precisely correspond to the deformation parameters in the family $\mathcal{X}$ over $S$, as $\mathcal{X}$ is the pullback of the universal curve over $\text{Div}_{g,d}$ to $S$.

**Remark 3.3.** Let $0 < \epsilon \leq \frac{1}{2}$. On the level of underlying topological spaces, the two analytic stacks $[\mathcal{M}^{an}_{g,e,d}/\mathcal{S}_d]$ and $\text{Div}^{an}_{g,d}$ are homeomorphic, since they have the same coarse moduli space (using [Uli14, Proposition 3.9]), and in Remark 2.5 we have seen that $\text{Div}^{trop}_{g,d}$ is naturally homeomorphic to the quotient $M_{g,e,d}^{trop}/\mathcal{S}_d$. By [Uli15b, Theorem 1.2] we have a natural identification of the non-Archimedean skeleton of $\mathcal{M}^{an}_{g,d}$ with the tropical moduli space $M_{g,d}^{trop}$ of weighted stable tropical curves. Therefore, since this identification is invariant under the $\mathcal{S}_d$-operations on both sides, we may also reduce Theorem 3.2 to this earlier result.

**Proposition 3.4.** The tropicalization naturally commutes with the forgetful map, i.e. we have a commutative diagram

$$\begin{array}{ccc}
\text{Div}^{an}_{g,d} & \xrightarrow{\text{trop}_{g,d}} & \text{Div}^{trop}_{g,d} \\
\downarrow & & \downarrow \\
\mathcal{M}^{an}_g & \xrightarrow{\text{trop}} & M^{trop}_g
\end{array}$$

**Proof.** This is in essence the argument in the proof of [ACP15, Theorem 1.2.2]. Let $\mathcal{X}$ be a stable degeneration of a smooth curve $X$ over a discretely valued non-Archimedean field $K$ and let $X'$ be a semistable degeneration of $X$ whose stabilization is equal to $\mathcal{X}$. Let $xy = f$ be the local equation of a node corresponding to an edge $e$ of the dual tropical curve $\Gamma$ of $\mathcal{X}$ and let $x_i y_i = f_i$ (for $i = 1, \ldots, k$) be equations of the nodes of $\mathcal{X}$ that lie above the node $xy = f$. Then we have $f = f_1 \cdots f_k$ and thus $\text{val}(f) = \text{val}(f_1) + \cdots + \text{val}(f_k)$ and so the edges $e_i$ in the dual tropical curve $\Gamma'$ corresponding to $x_i y_i = f_i$ form a subdivision of $e$.

This proves the commutativity of the above diagram for points that can be represented by a semistable family over a discrete valuation ring. Since these points
are dense in $\Div^{an}_{g,d}$ and $\cM_{g}^{an}$ respectively, and both $\trop_{g}$ and $\trop_{g,d}$ are continuous maps, the commutativity of the above diagram follows. \qed

**Remark 3.5.** Assume that $X$ is a smooth curve over a non-Archimedean and algebraically closed extension $K$ of $k$. Expanding on Section 3.2 we may define a specialization map $\tau_{*} : \Div^{\dagger}_{g}(X^{an}) \to \Div^{\dagger}_{g}(\Gamma)$ and, using an argument analogous to the one in the proof of Theorem 3.2 one can show that $\tau_{*}$ has a natural section $\Phi_{X,d} : \Div^{\dagger}_{d}(\Gamma) \to \Div^{\dagger}_{d}(X^{an})$ making the composition $\text{P}_{X,d} = \Phi_{X,d} \circ \tau_{*}$ into a strong deformation retraction onto a closed subset $\mathfrak{S}_{g}(X)$ of $\Div^{\dagger}_{g}(X^{an})$. If $X$ is a regular stable model of $X$ over $R$, corresponding to a morphism $S \to \cM_{g}$ with $S = \text{Spec} R$, the closed subset $\mathfrak{S}_{g}(X)$ is the non-Archimedean skeleton of $\Div^{\dagger}_{g}(X^{an})$ in the sense of Berkovich [Ber95, Section 5] associated to the regular semistable model $S \times \cM_{g} \Div_{g,d}$ of $\Div_{d}(X)$.

4. Tropicalizing the Hodge bundle

From now on we specialize from general divisors to canonical divisors and the Hodge bundle. Contrary to the case of algebraic curves, the canonical linear system on a tropical curve $\Gamma$ without legs comes with a distinguished element

$$K_{\Gamma} = \sum_{v \in V(\Gamma)} (2h(v) + |v| - 2) \cdot v$$

with support at the vertices of $\Gamma$. We denote by $|K_{\Gamma}|$ the canonical linear series.

In [LU17] Lin and the second author introduce a tropical analogue of the Hodge bundle $\Omega M_{g}$ and of its projectivization $\mathbb{P} \Omega M_{g}$ on the moduli space $M_{g}$. As a set, the **tropical Hodge bundle** $\Omega M_{g}^{trop}$ is defined to be the set of isomorphism classes of pairs $(\Gamma, f)$ consisting of a stable tropical curve $\Gamma$ of genus $g$ and a rational function $f \in \text{Rat}(\Gamma)$ with $K_{\Gamma} + \text{div}(f) \geq 0$. Its projectivization $\mathbb{P} \Omega M_{g}^{trop}$ parametrizes pairs $(\Gamma, D)$ consisting of a stable tropical curve $\Gamma$ of genus $g$ and an effective divisor $D = K_{\Gamma} + \text{div}(f)$ in $|K_{\Gamma}|$. Both spaces come with a natural forgetful map to $M_{g}^{trop}$.

**Proposition 4.1** ([LU17], Theorem 1). The tropical Hodge bundle $\mathbb{P} \Omega M_{g}^{trop}$ is a closed subset of $\Div^{trop}_{g,2g-2}$ that canonically carries the structure of a generalized cone complex of (maximal) dimension $5g - 5$.

The Hodge bundle is not equidimensional, see Example 6.7 below. Proposition 4.1 shows in particular that $\mathbb{P} \Omega M_{g}^{trop}$ is a closed subset of $\Div^{trop}_{g,2g-2}$ that is a subcomplex of a subdivision of $\Div^{trop}_{g,2g-2}$.

**Proof of Proposition 4.1.** Proposition 4.1 has already been proved as part of [LU17, Theorem 1] (and building upon the polyhedral description of tropical linear systems from [GK08, MZ08]). We rephrase the main insights of this proof using the language developed in Section 2.

Let $G'$ be a semistable finite vertex-weighted graph of genus $g$ and an effective divisor $D \in \Div_{g,2g-2}(G')$ making $(G', D)$ into a stable pair. Write $G$ for the stabilization of $G'$. We will show that the pullback of the tropical Hodge bundle to $\sigma_{G'} : \cE_{g,2g-2}(G')$ is given by a finite union of linear subspaces of $\sigma_{G'}$.

\footnote{This space was called $\cH_{g}^{\trop}$ in [LU17], and $\Omega M_{g}^{trop}$ was called $\cA_{g}^{\trop}$ there. Here we mainly follow the notation conventions of [BCGGM].}
For simplicity choose an orientation on every edge of the graph $G$; the resulting structure will not depend on this choice. Consider a tropical curve $\Gamma$ whose underlying graph is $G'$. In order to specify a rational function $f$ on $\Gamma$ such that $D = K_\Gamma + \text{div}(f)$ (up to a global additive $\mathbb{R}$-operation) we need to specify a collection of integers $(m_e) \in \mathbb{Z}^{E(G')}$ (one for each edge of the stabilization $G$ of $G'$), the initial slopes of $f$ at the origin of the edge $e$, subject to the condition

$$2h(v) - 2 + |v| = \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -\deg(D|_e + m_e).$$

Notice that by [GK08, Lemma 1.8] there are, in fact, only finitely many choices for the initial slopes $m_e$.

In each of the finitely many cases that such an $f \in \text{Rat}(\Gamma)$ exists, the continuity on $f$ imposes a collection of linear conditions on the coordinates of $\sigma_{G'} = \mathbb{R}^{E(G')}$ (i.e. the edge lengths of $G'$). The intersection of $\sigma_{G'}$ with such a linear subspace is a cone in the generalized cone complex structure on $\mathbb{P}\Omega M_{g,\text{trop}}^\text{an}$.

Let $\mathbb{P}\Omega M_{g,\text{an}}$ be the analytification of the projective Hodge bundle over an algebraically closed field $k$ endowed with the trivial absolute value. In this section we recall in detail the construction of the tropicalization map on the Hodge bundle from [LU17, Proposition 6] and elaborate on its properties.

The moduli space $\mathbb{P}\Omega M_{g,\text{an}}$ parametrizes pairs $(X/K, K_X)$ consisting of a point $X/K \in \mathcal{M}_g$ as recalled above, together with a divisor $K_X$ that is equivalent to the canonical bundle $\omega_{X/K}$. We define a natural tropicalization map

$$\text{trop}_\Omega : \mathbb{P}\Omega M_{g,\text{an}} \to \mathbb{P}\Omega M_{g,\text{trop}}$$

by setting

$$\text{trop}_\Omega(X/K, K_X) = \text{trop}_{g,2g-2}(X/K, K_X),$$

where $\text{trop}_{g,2g-2}$ is the tropicalization map introduced in Section 5.1.

**Proposition 4.2.** The tropicalization map $\text{trop}_\Omega$ is well-defined, continuous, proper, and closed.

**Proof.** The fact that $\text{trop}_\Omega$ is well-defined can be shown using a moving lemma as in [Bak08, Lemma 4.20]. We will see an alternative proof of this fact in Section 6.1 in the framework of this article.

Let $X$ be a semistable model of $X$ over a discrete valuation ring $R$ (or a finite extension thereof) extending $k$. We may assume that $X$ is regular; otherwise we blow up accordingly. By a moving lemma, such as [Lin02, Proposition 1.11], we may find a (not necessarily effective) canonical divisor $K_X$ on $X$ that does not meet the singularities in the special fiber. It is well-known that the multidegree of $K_X$ in the special fiber is equal to $K_\Gamma$ (see e.g. [Bak08, Remark 4.18]). Any effective canonical divisor $K_X$ on $X$ is equivalent to the generic fiber of $K_X$ and therefore the specialization of $K_X$ to $\Gamma$ is equivalent to $K_\Gamma$.

The discretely valued points in $\mathbb{P}\Omega M_{g,\text{an}}$ are dense and, since $\text{trop}_{g,2g-2}$ is continuous and $\mathbb{P}\Omega M_{g,\text{trop}}$ is closed by Proposition 4.1 above, we obtain that $\text{trop}_{g,2g-2}(x)$ is in $\mathbb{P}\Omega M_{g,\text{trop}}$ for every (not necessarily discretely valued) point $x \in \mathbb{P}\Omega M_{g,\text{an}}$. Since $\mathbb{P}\Omega M_{g,\text{trop}}$ is naturally a closed subset of $\text{Div}_{g,2g-2}^\text{trop}$, the properness and closedness of $\text{trop}_\Omega$ follow from the corresponding properties of $\text{trop}_{g,2g-2}$.

\qed
Definition 4.3. The realizability locus $\mathbb{P}R_\Omega$ in $\mathbb{P}\Omega M_g^{\text{trop}}$ is the image
\[ \mathbb{P}R_\Omega = \text{trop}_\Omega(\mathbb{P}\Omega M_g^{\text{an}}) \]
of the tropicalization map.

The realizability locus $\mathbb{P}R_\Omega$ is the locus of tuples $(\Gamma, D)$ consisting of a stable tropical curve $\Gamma$ and a canonical divisor $D$ for which there is a stable family $\mathcal{X}$ of curve over a valuation ring $R$ together with an effective canonical divisor $K_X$ on the generic fiber $X$ of $\mathcal{X}$ such that $\Gamma$ is the dual tropical curve of $\mathcal{X}$ and $D$ is the specialization of $K_X$.

5. Twisted differentials and the global residue condition

From now on we work over the field $\mathbb{C}$ of complex numbers. Let $X/\mathbb{C}$ be a smooth and proper algebraic curve, i.e. a compact Riemann surface. We let $\Omega M_g \to M_g$ be the space of pairs $(X, \omega)$ consisting of an algebraic curve together with a non-zero holomorphic one-form $\omega$ on $X$. This is the Hodge bundle over the complex-analytic moduli space of curves, deprived of the zero section. The multiplicities of the zeros of $\omega$ define a partition $\mu$ of $2g - 2$ and the subspaces $\Omega_{g, \mu}$ with fixed partition $\mu$. We will focus most of the time on the principal stratum corresponding to the partition $\mu = (1, \ldots, 1)$ and we usually put $n = |\mu|$.

In this section we recall from [BCGGM] the description of a compactification of the strata of $\Omega M_g$. More concretely, observe that there is a natural map $\varphi : \Omega M_g(\mu) \to M_{g,[\mu]}$ sending $(X, \omega)$ to the curve marked by the zeros of $\omega$. Here $M_{g,[\mu]}$ is the quotient of $M_{g,n}$ by the symmetric group that permutes the entries of $\mu$. The main theorem of [BCGGM] is a characterization of the closure of $\varphi$. The version given here highlights the possible scaling parameters of one-parameter families approaching the boundary. These scaling parameters will reflect the location of the support of the corresponding tropical divisors. We need to set up some notation to recall the theorem for the case of holomorphic abelian differentials.

Definition 5.1. We call any tuple $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $\sum m_i = 2g - 2$ and that $m_1 \geq \ldots \geq m_n > m_{r+1} = 0 = \ldots = m_{r+s} > m_{r+s+1} \geq \ldots m_{r+s+p}$ a type. We denote by $p_1$ the number of $-1$ occurring in this tuple.

We assign a type to any meromorphic differential via the multiplicities of its associated divisor.

The moduli space $\Omega M_g(\mu)$ parametrizes meromorphic one-forms whose divisor is of type $\mu$. We may view these spaces as strata of a twisted Hodge bundle (see [BCGGM], but we do not need this viewpoint here).

5.1. Level graphs. Let $\Gamma = (V, E)$ be an (unmetrized) graph. A full order $\Gamma$ on $\Gamma$ is an order $\succ$ on the vertices $V$ that is reflexive, transitive, and such that for any $v_1, v_2 \in V$ at least one of the statements $v_1 \succ v_2$ or $v_2 \succ v_1$ holds. We call any function $\ell : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ such that $\ell^{-1}(0) \neq \emptyset$ a level function on $\Gamma$. Note that a level function induces a full order on $\Gamma$ by setting $v \preceq w$, if $\ell(v) \leq \ell(w)$. A level graph is a graph together with a choice of a level function. Abusing notation, we use the symbol $\Gamma$ also for level graphs.

For a given level $L$ we call the subgraph of $\Gamma$ that consists of all vertices $v$ with $\ell(v) > L$ along with edges between them the graph above level $L$ of $\Gamma$, and denote it by $\Gamma_{>L}$. We similarly define the graph $\Gamma_{\geq L}$ above or at level $L$, and the graph $\Gamma_{=L}$.
at level $L$. An edge $e \in E(\Gamma)$ of a level graph $\Gamma$ is called horizontal if it connects two vertices of the same level, and it is called vertical otherwise. Given a vertical edge $e$, we denote by $v^+(e)$ (resp. $v^-(e)$) the vertex that is its endpoint of higher (resp. lower) level.

5.2. **Twisted differentials.** Let $C$ be a nodal, in general non-smooth curve over the complex numbers. Let $\mu = (m_1, \ldots, m_n)$ be a type. A twisted differential of type $\mu$ on a stable $n$-pointed curve $(C, s)$ is a collection of (possibly meromorphic) differentials $\eta_v$ on the irreducible components $C_v$ of $C$ such that no $\eta_v$ is identically zero with the following properties.

(0) **(Vanishing as prescribed)** Each differential $\eta_v$ is holomorphic and nonzero outside of the nodes and marked points of $C_v$. Moreover, if a marked point $s_i$ lies on $C_v$, then $\text{ord}_{s_i} \eta_v = m_i$.

(1) **(Matching orders)** For any node of $C$ that identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$, the vanishing orders satisfy $\text{ord}_{q_1} \eta_{v_1} + \text{ord}_{q_2} \eta_{v_2} = -2$.

(2) **(Matching residues at simple poles, MRC)** If at a node of $C$ that identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$ the condition $\text{ord}_{q_1} \eta_{v_1} = \text{ord}_{q_2} \eta_{v_2} = -1$ holds, then $\text{Res}_{q_1} \eta_{v_1} + \text{Res}_{q_2} \eta_{v_2} = 0$.

Let $\Gamma$ be the dual graph of $C$. Recall that the vertices $v$ in $\Gamma$ correspond to the irreducible components $C_v$ of $C$. If $\ell$ is a level function on $\Gamma$, we write $C_{\ell,L}$ for the subcurve of $C$ containing only the components $C_v$ with $v$ of level strictly bigger than $L$. Similarly, we define $C_{\ell,L}$. If two components $C_v$ and $C_w$ with $\ell(v) < \ell(w)$ intersect in the point $q$, we denote by $q^-$ the corresponding point on $C_v$, and we write $v = v^-(e)$ for the edge $e$ in $\Gamma$ connecting $v$ and $w$.

Denote by $\Gamma$ the full order on the dual graph $\Gamma$ given by a level function. We say that a twisted differential $\eta$ of type $\mu$ on $C$ is called compatible with $\Gamma$ if in addition it also satisfies the following two conditions.

(3) **(Partial order)** If a node of $C$ identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$, then $v_1 \succeq v_2$ if and only if $\text{ord}_{q_1} \eta_{v_1} \geq -1$. Moreover, $v_1 \succeq v_2$ if and only if $\text{ord}_{q_1} \eta_{v_1} = -1$.

(4) **(Global residue condition, GRC)** For every level $L$ and every connected component $Y$ of $C_{\ell,L}$ the following condition holds: Let $q_1, \ldots, q_b$ denote the set of all nodes where $Y$ intersects $C_{\ell,L}$. Then

$$\sum_{j=1}^{b} \text{Res}_{q_j}^{-} \eta^{-}(q_j) = 0,$$

where we recall that $q_j^- \in C_{\ell,L}$ and $v^{-}(q_j) \in \Gamma_{\ell,L}$.

5.3. **The characterization of limit points.** Suppose that $S$ is the spectrum of a discrete valuation ring $R$ with residue field $\mathbb{C}$, whose maximal ideal is generated by $t$. Let $X/S$ be a family of semi-stable curves with smooth generic fiber $X$ and special fiber $C$. Let $\omega$ be a section of $\omega_{X/S}$ of type $\mu$ whose divisor is given by the sections $s = (s_1, \ldots, s_n)$ with multiplicity $m_i$. The triple $(X/S, s, \omega)$ is called a pointed family of stable differentials, if moreover $(X/S, s)$ is stable. Then we define the scaling factor $\ell(v)$ for the node $v$ as the non-positive integer such that the restriction of the meromorphic differential $t^{-\ell(v)} \cdot \omega$ to the component $C_v$ of the special fiber corresponding to $v$ is a well-defined and generically non-zero differential $\eta_v$ on $C_v$ (see [BCGGM, Lemma 4.1]). The $\eta_v$ are called the scaling limits of $\omega$. 

---

REALIZABILITY OF TROPICAL CANONICAL DIVISORS 19
Theorem 5.2 (BCGGM). If \((X/S,s,\omega)\) is as above, then the function \(\ell(v)\) defines a full order on the dual graph \(\Gamma\) of the special fiber of \(X\) and the collection \(\eta_v|_{X_s}\) is a twisted differential of type \(\mu\) compatible with the level function \(\ell\).

Conversely, suppose that \(C\) is a stable \(n\)-pointed curve with dual graph \(\Gamma\) and \(\eta = \{\eta_v\}_{v \in V}\) is a twisted differential of type \(\mu\) compatible with a full order \(\Gamma\) on \(\Gamma\). Then for every level function \(\ell : \Gamma \to \mathbb{Z}\) defining the full order \(\Gamma\) and for every assignment of integers \(n_v\) to horizontal edges there is a stable family \(X/S\) over \(S = \text{Spec}(\mathbb{C}[[t]])\) with smooth generic fiber and special fiber \(C\) that satisfies the following properties:

i) There exists a global section \(\omega\) of \(\omega_{X/S}\) whose horizontal divisor \(\text{div}_{\text{hor}}(\omega) = \sum_{i=1}^n m_i \Sigma_i\) is of type \(\mu\) and whose scaling limits are the collection \(\{\eta_v\}_{v \in V}\).

ii) The intersections \(\Sigma_i \cap C = \{s_i\}\) are smooth points of the special fiber and \(\eta\) has a zero of order \(m_i\) in \(s_i\).

iii) There exists a positive integer \(N\) such that a local equation near every node corresponding to a horizontal edge \(e\) is \(xy = t^N q^+(\sigma(e))\), and it is \(xy = t^N(q^-(\sigma(e))\)) for every vertical edge \(e\).

Proof. The first statement is the necessity of the Theorem 1.3 of BCGGM, proven in Section 4.1. Note that the arguments given in loc. cit. for this direction hold over any discrete valuation ring.

For the second statement one has to trace the proof of sufficiency of this theorem, given in Section 4.4 of loc. cit. As stated there (see equation (4.8) and the last two paragraphs of the proof of Addendum 4.8), there are no constraints for the plumbing fixtures to be used for plumbing horizontal nodes, whereas for the plumbing fixtures used for every vertical nodes, given by an edge \(e\), the level function \(\ell_0\) used for plumbing has to satisfy the divisibility constraint

\[ (\text{ord}_{\sigma(e)}(\eta^+ + 1) | (\ell_0(q^+(\sigma(e))) - \ell_0(q^-(\sigma(e)))) ). \]

Multiplying the prescribed function \(\ell\) by a sufficiently divisible \(N\), the resulting level function \(\ell_0 = N \cdot \ell\) satisfies this divisibility property. \(\square\)

5.4. Dimension and period coordinates. In preparation for the dimension statements in Section 6.2 we recall here two results about the geometry of strata of meromorphic differentials. Consider the neighborhood of a point \((X,\omega) \in \Omega M_g(\mu)\). We denote by \(Z\) the \(r+s\) zeros and marked points of \(\omega\) and let \(P\) be the \(p\) poles of \(\omega\). On such a neighborhood, integration of the meromorphic one-form against a basis of the relative cohomology group \(H^1(X - P, Z; \mathbb{Z})\) gives local coordinates, called period coordinates. See BCGGM for a proof of this statement (including the case of \(k\)-differentials) and for references to the history of this result. The fact that these functions are local coordinates also proves the following dimension statement.

Theorem 5.3. The stratum \(\Omega M_g(\mu)\) has dimension \(2g - 1 + n\) if the type \(\mu\) is holomorphic (i.e. if \(p = 0\)), and it has dimension \(2g - 2 + n\) if the type \(\mu\) is strictly meromorphic (i.e. if \(p > 0\)).

The following result is the special case for one-forms of a main result of Che17.

Theorem 5.4 (Che17). The projectivisation of a stratum \(\mathbb{P} \Omega M_g(\mu)\) of strictly meromorphic type (i.e. with \(p > 0\)) does not contain a complete curve.

5.5. The image of the residue map. Since the global residue condition imposes strong constraints on the residues, we need a criterion on which residues can actually be realized. Since a twisted differential is a collection of meromorphic differentials,
rather than just holomorphic differentials, we have to deal more generally with types of meromorphic differentials. Recall from the beginning of Section 5 the conventions used to denote types of meromorphic differentials. In particular, $p_1 \leq p$ denotes the number of simple poles. For every type $\mu$ with $p \neq 0$ we let

$$\text{Res} : \Omega \mathcal{M}_g(\mu) \to H$$

be the residue map, whose range is contained by the residue theorem in

$$H = \left\{ \mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{C}^p : \sum_{i=1}^p x_i = 0 \right\}.$$ 

Moreover we define the 'non-zero set' $N \subset H$ to consist of those $\mathbf{x}$ with $x_i \neq 0$ whenever $m_i = -1$. By definition of a stratum, the image of Res is obviously contained in $H \cap N$.

To illustrate the problem of determining the image of Res, consider differentials $\eta$ of type $\mu = (a, -b, b - 2 - a)$ with $a \geq 0$, $b \geq 2$ and $b - a - 2 \leq -2$ on a projective line with coordinate $z$. We may assume that the zero of $\omega$ is at $z = 1$, while the poles are at $z = 0$ and $z = \infty$. Consequently, $\eta = C(z - 1)^{a} dz/z^b$ with $C \neq 0$. This implies that the residue is non-zero, in fact $\text{Im}(\text{Res}) = H \setminus \{(0, 0)\} \subset H$. The image of the map Res in the general case was determined by Gendron and Tahar in [GT17]. We restate a simplified version of their main result that is sufficient for our purposes.

**Proposition 5.5.** (i) If $g \geq 1$ then Res is surjective onto $N \cap H$, which is a non-empty set unless $p = p_1 = 1$.

(ii) If $g = 0$ and $p = p_1 > 0$ or $p_1 = 0$ and there does not exist an index $1 \leq i \leq n$ with

$$m_i > \left( \sum_{j=r+1}^{r+s+1} -m_j \right) - p - 1 \quad (4)$$

then Res is surjective onto $N \cap H$.

(iii) If $g = 0$ and $p_1 = 0$ and there exists an index $1 \leq i \leq n$ with $4$, then Res is surjective onto $N \cap H \setminus \{(0, \ldots, 0)\}$.

(iv) If $g = 0$ and $p = p_1 = 2$ then Res is surjective onto $N \cap H$.

(v) If $g = 0$ and $p = p_1 > 2$, then the image of Res contains all tuples in $N \cap H$ consisting of $p \mathbb{R}$-linearly independent vectors.

**Proof.** All the conditions except for the case $g = 0$ and $p = p_1$ (only simply poles) are restatements of the case of abelian differentials in [GT17, Theorem 1.1-1.5]. The case $p = p_1 = 2$ follows directly from the preceding discussion. The case $p = p_1 > 2$ is stated for $r = 1$ in [GT17, Proposition 1.6]. For $r > 1$ it can be easily deduced from this by the procedure of splitting a zero, or it can be derived from the more involved formulation in [GT17, Proposition 1.7].

6. **The realizability locus**

In this section we prove our main Theorem 5.3 describing the realizability locus of tropical canonical divisors over an algebraically closed field of characteristic zero. For tropical curves with rational sides lengths and canonical divisors with rational coordinates we use the complex-analytic techniques of the previous section to characterize the image of the tropicalization map. For general points the results follow from the general properties of trop$\Omega$. 


6.1. From rational functions to enhanced level graphs. To formalize the correspondence between rational functions and level graphs, we introduce the following enhancement of the notion of a level graph. We consider vertex weighted graphs of the form \( \Gamma = (V, E, L, h) \) without edge lengths, where \((V, E)\) is a classical graph, \(L\) is a finite set of legs, i.e. infinite half-edges starting at a vertex, and \(h\) is a marking of the vertices with integer numbers. After choosing a level function on the underlying graph \((V, E)\), we call \(\mathbf{T}\) a level graph. We divide every edge \(e\) in two half-edges and write \(\Lambda\) for the set containing all half-edges and all legs in \(\Gamma\). Note that every \(\lambda \in \Lambda\) is adjacent to a unique vertex. An enhanced level graph \(\Gamma^+\) is a level graph \(\mathbf{T}\) as above together with an assignment \(k : \Lambda \to \mathbb{Z}\) such that the following compatibility conditions hold.

i) If \(e^+\) and \(e^-\) are two half edges forming an edge \(e\), then \(k(e^+) + k(e^-) = -2\). An edge is horizontal if and only if \(k(e^+) = -1\) for both its half-edges. Moreover, if \(e\) is a vertical edge consisting of the half-edges \(e^+\) leading downwards and \(e^-\) leading upwards, then \(k(e^+) > k(e^-)\).

ii) For each vertex \(v \in V\)

\[
\sum k(\lambda) = 2h(v) - 2,
\]  

where the sum is over all \(\lambda \in \Lambda\) which are adjacent to \(v\).

Let \(v\) be a vertex in an enhanced level graph \(\Gamma^+\). Then we define the type \(\mu(v)\) as the ordered tuple consisting of all \(k(\lambda)\), where \(\lambda\) is a half-edge adjacent to \(v\). Note that \(\mu(v)\) is a type in the sense of Definition 6.1 if we replace \(g\) by \(h(v)\).

This definition is motivated from the notion of twisted differentials. In fact, given a stable curve \(X\), let \(\Gamma\) be the associated graph consisting of the dual graph of \(X\) with legs attached for each marked point and \(h\) given by the genera of the irreducible components \(X_v\) for \(v \in V\). Note that up to the choice of a metric this is the same construction as in Section 5.1. Every \(\lambda\) in \(\Lambda\) gives rise to a point \(z_\lambda\) in \(X_v\), where \(v\) is the vertex adjacent to \(\lambda\). If \(\lambda\) is a half-edge, we let \(z_\lambda\) be the node corresponding to the edge containing \(\lambda\), and if \(\lambda\) is a leg, we let \(z_\lambda\) be the associated smooth point. For any twisted differential \(\eta\) on \(X\), we decorate all \(\lambda \in \Lambda\) with the order of \(\eta\) at \(z_\lambda\). This defines an enhanced level graph structure on \(\Gamma\).

**Lemma 6.1.** Let \(\Gamma\) be a tropical curve. To every element \(D = K_\Gamma + \text{div}(f) \in |K_\Gamma|\) we can associate a natural structure of an enhanced level graph \(\Gamma^+ = \Gamma^+(f)\) on some realization of \(\Gamma\).

**Proof.** Let \(\Gamma_0\) be the minimal realization of \(\Gamma\) subdivided with vertices at the places where \(f\) is not differentiable. We use the function \(f\) itself to give \(\Gamma_0\) a full order i.e. for nodes \(v, w\) of \(G_0\) we declare \(v \succeq w\) if and only if \(f(v) \geq f(w)\).

We provide each vertex \(v\) of \(\Gamma_0\) with \(2h(v) - 2 + \sum (1 + s(e))\) legs, each given the decoration \(k = 1\). Here the sum runs over all non-leg half-edges adjacent to \(v\) and \(s(e)\) denotes the slope of \(f\) along \(e\), oriented to be pointing away from \(v\). The fact that \(D \in |K_\Gamma|\) is equivalent to this number of legs being indeed non-negative for all vertices. We provide each half-edge \(e\) which is not a leg, with \(k = -s(e) - 1\), using the same orientation convention. The conditions for an enhanced level graph now follow immediately.

We need two more notions to state our main theorem.
**Definition 6.2.** A vertex $v$ of an enhanced level graph is called inconvenient if $h(v) = 0$ and if its type $\mu(v) = (m_1, \ldots, m_n)$ has the property that $p_1 = 0$ and there exists an index $i$ such that the inequality $1$ holds.

A cycle is called simple, if it does not visit any vertex more than once. Recall the tropicalization map $\text{trop}_\Omega : \mathbb{P}\Omega M_{g}^{an} \to \mathbb{P}\Lambda_{g}^{\text{trop}}$ from Proposition 4.2.

**Theorem 6.3.** Suppose that $k$ is an algebraically closed field of characteristic zero. A pair $(\Gamma, D)$ with $D = K_{\Gamma} + \text{div}(f)$ in the tropical canonical linear series lies in the image of $\text{trop}_\Omega$ if and only if the following two conditions hold:

i) For every inconvenient vertex $v$ of $\Gamma^+(f)$ there is a simple cycle $\gamma \subset \Gamma$ based at $v$ that does not pass through any node at a level smaller than $f(v)$.

ii) For every horizontal edge $e$ there is a simple cycle $\gamma \subset \Gamma$ passing through $e$ which does not pass through any node at a level smaller than $f(e)$.

![Figure 1. Illustration of the edge condition](image)

Figure 1 illustrates the conditions of the theorem. The value of $f$ is given by the height of the point in $\Gamma^+(f)$ over its image point in $\Gamma$. In this example there is a simple cycle through the horizontal edge in the foreground. However all the possible simple cycles through this edge pass through the vertex with two markings, which is at a lower level. Consequently, this graph is not realizable. Note also that realizability depends on the edge lengths here: If the edge containing the vertex with two markings were shorter (and all the other lengths remained the same), the corresponding vertex could be at a level above the horizontal edge and the corresponding divisor would be realizable.

**Proof of Theorem 6.3.** Let us first assume that $k = \mathbb{C}$. The conditions i) and ii) define a closed subset of $\mathbb{P}\Omega M_{g}^{\text{trop}}$. Tropical curves $\Gamma$ with rational edge lengths and divisors $D = K_{\Gamma} + \text{div}(f)$ associated to a function $f$ are dense in this subset. Since $\text{trop}_\Omega$ is continuous and closed by Proposition 4.2, the image of $\text{trop}_\Omega$ is the closure of this locus. We may therefore assume that $\Gamma$ has rational edge lengths. Moreover, if $(\Gamma, K_{\Gamma} + \text{div}(f))$ is realizable, we may rescale the edge lengths by a global constant and still obtain a realizable object in $\mathbb{P}\Omega M_{g}^{\text{trop}}$. Therefore we may assume that $\Gamma$ has integral edge lengths.

Suppose that the enhanced level graph $\Gamma^+(f)$ associated with $D$ satisfies i) and ii) and the integrality hypothesis made above on $\Gamma$. We want to show that there is a twisted differential of type $\mu = (1, 1, \ldots, 1)$ on a stable pointed curve $C$ with dual graph $\Gamma$, compatible with the enhanced level structure $\Gamma^+(f)$ and then apply the ‘converse’ implication in Theorem 5.2. For every vertex $v$ this amounts to finding
a differential of type $\mu(v)$ on some smooth curve $C_v$. This is indeed the type of a meromorphic differential on $C_v$ by property ii) of an enhanced level graph. The matching order condition and the partial order condition of a twisted differential are built into the condition i) of an enhanced level graph.

Hence the main point is to choose the curves $C_v$ and the differential $\eta_v$ such that the matching residue condition (MRC) and the GRC can be satisfied. For this purpose, we want to apply Proposition 5.3. By the following procedure we specify residues which on the one hand lie in $H \cap N$ at every node and satisfy both MRC and GRC, and which on the other hand are non-zero at inconvenient nodes, and match the conditions of the last item in Proposition 5.3 at nodes with only simple poles.

Take the cycles $\gamma_i, i \in I_1$ corresponding to inconvenient vertices $v_i$ by condition i) and the cycles $\delta_j, j \in J_h$ corresponding to horizontal edges $e_j$ by condition ii) and provide them with some orientation. Let $\{\alpha_i, i \in I_1\} \cup \{\beta_j, j \in J_h\}$ be a collection of complex numbers such that no sum of a subset of the $\pm \alpha_i$ and $\pm \beta_j$ is real. (i.e. the $\alpha_i$ and $\beta_j$ lie in a complement of a finite union of real codimension one hyperplanes in $\mathbb{C}^{[h]_i + |J_h|}$.) Starting from residue zero at each edge we increase, for every $i \in I_1$, the prescribed residue at all the half-edges $e$ with $f(e) = f(v_i)$ by $\alpha_i$ if the outward pointing orientation of $e$ agrees with the orientation of the cycle and by $-\alpha_i$ otherwise. For every $j \in J_h$ we increase the prescribed residue at all the half-edges $e$ with $f(e) = f(e_j)$ by $\beta_j$. The collection of residues prescribed in this way is non-zero for every horizontal edge (by the choice of the cycles $\delta_j$ and since the choice of the $\alpha_i$ and $\beta_j$ avoids unintended cancellations), it is non-zero at every inconvenient node (by the choice of the cycles $\gamma_i$) and satisfies the residue theorem (since a cycle enters and exits any vertex the same number of times), i.e. the prescribed residues lie in the image of the residue map at each inconvenient node by Proposition 5.3. At each of the vertices with only simple poles, i.e. with $p = p_1$, the residues are non-zero and $\mathbb{R}$-linearly independent (if there is more than one pair of such poles, i.e. if $p_1 > 2$ at such a vertex). Consequently, by Proposition 5.3 the residues lie in the image of the residue map at each vertex. Finally, we check that the GRC continues to hold at each step of adding the contributions along a cycle $\gamma_i$ or $\delta_j$. We give the details for the first case, the second being the same, replacing $f(v_i)$ by $f(e_j)$ everywhere. In fact, the addition procedure prescribes a zero total sum of residues to each of the component of $C_{2f(v_i)}$ the cycle passes through, so the GRC holds for $C_{2f(v_i)}$. Since the cycle does not pass through levels below $f(v_i)$, the GRC for those levels remains valid. If $w$ is a vertex with level $f(w) > f(v_i)$ then all the edges to level $f(v_i)$ are unseen in the GRC for $C_{2f(w)}$ and hence the GRC for these levels continues to hold, too.

Consequently, we can now use Theorem 5.2 with the level function $\ell = f$ and with $n_e = |e|$, the length in $\Gamma$ for any horizontal edge. The conclusion of the theorem is precisely that the divisor $D + \text{div}(f)$ is the specialization of an effective canonical divisor on a graph equivalent to $\Gamma$, with all the lengths rescaled by the integer $N$ of Theorem 5.2 iii). Hence $(\Gamma, D)$ lies in the image of $\text{trop}_\Omega$ by definition of this map in Equation 8.

Let us now show the converse implication. Points of the form $(X, D)$, where $X$ is the smooth generic fiber of a stable curve $\mathcal{X}$ over the valuation ring $R$ of a finite extension of $\mathbb{C}(t)$ are dense in $\mathbb{P}\Omega M^a_{\mathbb{C}}$. Since $\text{trop}_\Omega$ is continuous by Proposition 4.2 it suffices to show that $\text{trop}_\Omega(X, D)$ satisfies conditions i) and ii) in our claim.
Denote by \( S \) the spectrum of \( R \). Moreover, let \( \omega \) be a stable differential on \( X \) such that the divisor of its generic fiber is \( D \). We may assume, by the density of the principal stratum, that \( \text{div}(\omega)_{\text{hor}} = \sum_{i=1}^{2g-2} s_i(S) \) consists of \( 2g-2 \) images of sections.

Let \( \ell \) be the level function on the dual graph \( \Gamma \) of the special fiber given by the scaling parameters of this family (cf. Section 5.3). Let \( \eta \) be the twisted differential on the special fiber \( C \) of \( \X \), obtained as the scaling limit of \( \omega \), and let \( \Gamma^+(\ell) \) be the enhanced level graph given by \( \ell \) and the enhancement given by \( \eta \), as described before Lemma 6.1. We want to show that \( \text{trop}_\Omega(X, D) = (\Gamma, K_\Gamma + \text{div}(\ell)) \) that the enhanced level graph \( \Gamma^+(\ell) \) satisfies i) and ii).

Concerning the first claim, we note that by definition of \( \eta \) exactly \( 2h(v) - 2 - \sum k(e) \) sections of \( \omega \) lie in the irreducible component \( C_v \) of the special fiber associated to the vertex \( v \), counted with multiplicity. Here the sum runs over all half-edges adjacent to \( v \) which are not legs at \( v \). By definition of \( \text{trop}_\Omega \) in 6.1 the first claim thus amounts to showing that the slope defined by \( \ell \) of non-leg half edges \( e \) is equal to \( -k(e) - 1 \). This in turn is a consequence of the way degenerating families are built by plumbing (cf. BCGGM Theorem 4.5). The core observation is that at a node \( xy = \alpha \) the differential on the two ends of the node is \( (x^k + \alpha^{k+1} y^2) dx \) and \( -\alpha^{k+1}, (y^{k-2} + y) dy \), where \( \alpha^{k+1} \) is the period of \( \omega \) along the vanishing cycle. Consequently, the difference of scaling parameters (or, equivalently, values of the level function) is \( a(k+1) \), proving the slope claim.

To show i), we work with the complex topology. Note that there exists a disc \( \Delta \) in \( \C \) such that \( \X(\C) \) can be extended to a complex analytic space \( X_\Delta(\C) \) over \( \Delta \). At every inconvenient vertex \( v \) the restriction of \( \eta \) to \( v \) has non-zero residue at \( q^{-1}(e) \) for some edge \( e \) with \( k(e) < 0 \) by Proposition 6.3. To illustrate the idea for constructing the necessary cycle, let \( e_1, \ldots, e_m \) be the edges adjacent to \( v \) with \( k(e) < 0 \). Choose a continuously varying family \( \beta_j(s) \) for \( s \in \Delta \) of simple closed curves in the fibers \( \X_s \) belonging to the homotopy class which is pinched to the node \( e_j \). If all the curves \( \beta_j(s) \) are separating for one (hence every) \( s \in \Delta \), the period of \( \omega_s \) around \( a_0 \) is zero by Stokes’ theorem. This gives a contradiction to the non-zero residue in the limit. Consequently, there is some non-trivial cycle \( \gamma \) passing through \( v \).

To deal with the general case and to derive the claimed property of \( \gamma \) we revisit the proof of the GRC, compare BCGGM Section 4.1. Let \( A = \alpha_1(s) \cup \ldots \cup \alpha_m(s) \) be the union of simple closed curves which are pinched (when \( s \to 0 \)) to the nodes joining a level \( \geq \ell(v) \) to a level \( < \ell(v) \). If \( \beta_j(s) \) is separating the connected component of \( X_s \setminus A \) it lies in, we obtain the same contradiction from Stokes’ theorem as before. In fact, let \( I \subset \{0, \ldots, s\} \) be the index set of curves bounding a component of \( X_s \setminus (A \cup \beta_j) \). Then

\[
\int_{\beta_j} t^{\ell(v)} \omega(t) + \sum_{i \in I} \int_{\alpha_i} t^{\ell(v)} \omega(t) = 0
\]

by Stokes. The first term of this sum tends to the residue we are interested in. The other terms tend to the residue of the limiting twisted differential at level \( \geq \ell \) at a node corresponding to an edge to level \( \ell \), which is zero since the limiting differential is holomorphic there (see the condition ‘partial order’ in the definition of twisted differential in Section 5.2). Consequently, if some \( \beta_j(s) \) is not separating its connected component in \( X_s \setminus A \), there exists a cycle as claimed in i). The argument for horizontal edges is the same and gives ii).
Hence we have proved the theorem in the case $k = \mathbb{C}$. If $K \subset L$ is a field extension of two trivially valued algebraically closed fields of characteristic zero, we have a natural surjective projection map $(\mathbb{P} \Omega M_{g,L})^{\text{an}} \to (\mathbb{P} \Omega M_{g,K})^{\text{an}}$ which is compatible with the tropicalization map, i.e. the diagram

\[
\begin{array}{ccc}
(\mathbb{P} \Omega M_{g,L})^{\text{an}} & \xrightarrow{\text{trop}_\Omega} & \mathbb{P} \Omega M_{g}^{\text{trop}} \\
\downarrow & & \\
(\mathbb{P} \Omega M_{g,K})^{\text{an}} & \xrightarrow{\text{trop}_\Omega} & \mathbb{P} \Omega M_{g}^{\text{trop}}
\end{array}
\]

is commutative. Hence the realizability locus does not depend on the choice of the algebraically closed ground field of characteristic zero, which implies our claim. □

**Example 6.4.** Figure 2 shows points in the realizability locus over the dumbbell graph in genus two, i.e. all vertex genera zero. Those points in the realizability locus consist of two symmetrically placed marked points on either dumbbell end (left picture) or double point anywhere on the central edge (including the ends) of the dumbbell (picture on the right). A canonical divisor whose support consists of two different points on the central edge of the dumbbell (see Figure 3) is not in the realizability locus, since the edge between those two points is horizontal in the enhanced level graph and separating, thus violating condition ii) in Theorem 6.3. These two figures are reinterpretation of the corresponding figures in [GK08] in our viewpoint of level graphs.

**Figure 2.** Realizability locus over the dumbbell graph.

**Figure 3.** A non-realizable configuration over the dumbbell graph.
Proof of Proposition \ref{prop:alternative_tropical_canonical_divisor}, alternative proof that the image of $\text{trop}_{Ω}$ belongs to $|K_Γ|$.

We need to show that for any given graph $Γ$ there exists a rational function $f ∈ \text{Rat}(Γ)$ with $K_Γ + \text{div}(f) ≥ 0$ such that $Γ^+(f)$ satisfies conditions i) and ii) of the preceding theorem. We prove this by induction on the genus. Since adding marked points and increasing the vertex genus can only improve the situation concerning inconvenient nodes, it suffices to treat the case that all vertex genera $h(v) = 0$.

For $g = 2$, there are two cases. For the graph with three nodes joining the two vertices (and in general: for any graph $Γ$ without separating edges), the canonical divisor $K_Γ$ is in the image of $\text{trop}_{Ω}$. For the dumbbell graph, we take a function $f$ that is constant on the edges and has a global minimum on the separating edge, see the picture on the right hand side of Figure 2.

In the induction step, we consider a graph $Γ$ of genus $g$ and remove a non-separating edge $e$. Let $Δ = Γ - e$ be the resulting graph. There are two cases to consider. First, suppose that the two ends of $e$ are different nodes in $Δ$. Then $Δ$ is semistable and we start with the $f_0$ given by induction on the stable graph equivalent to $Δ$. We complete this to a function $f$ on $Γ$ having slope $≥ -1$ on each half-edge of $e$. (This is possible for all values of $f_0$ at the ends of $e$.) Together with the induction hypothesis this condition implies $K_Γ + \text{div}(f) ≥ 0$. Neither a horizontal separating edge nor a trivalent vertex with negative decorations has been added, hence conditions i) and ii) continue to hold.

Second, suppose that $e$ is a cycle, adjacent to some vertex $v$.

If $Δ = Γ - e$ is semi-stable, we simply declare $f$ to be constant on $e$. Otherwise, there is a separating edge $e_s$ ending at $v$ and $Δ \setminus e_s$ is semi-stable. In this case we take $f_0$ from $Δ \setminus e_s$ by induction and complete it to $f$ constant on $e$ and with slopes $-1$ on the two half-edges of $e_s$ (i.e. $\text{div}(f)$ contains twice the midpoint of $e_s$). The conditions i), and ii) and $K_Γ + \text{div}(f) ≥ 0$ follow from the construction. □

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure.pdf}
\caption{Realizable configurations of maximal dimension on $K_4$: 3-cycles on top level, edges with two points}
\end{figure}

\textbf{Example 6.5.} To give a more involved example we discuss the realizability locus over the complete graph $K_4$. We first claim that there are five types of maximal dimensional cones, as given in Figure 4, Figure 5 and Figure 6.

To prove this claim, we establish some notation. Suppose that $Γ^+$ is the enhanced level graph corresponding to a canonical divisor in the realizability locus. Let $v_1, \ldots, v_4$ be the four vertices of the original $K_4$ and let $w_1, \ldots, w_n$ with $n ≤ 4$ be the remaining vertices of $Γ$, each of them having at least one leg. Consider the
vertices on the top level of $\Gamma^+$. A vertex $w_i$ cannot be on top level, since its two non-leg half edges have $k = -1$, since it has at least one leg and since the sum of decorations is equal to $-2$. Suppose one of the $v_i$ on top level is decorated with a leg. This requires $v_i$ to have (at least) three horizontal edges, otherwise the sum of decorations cannot be $-2$. Together with the previous argument this implies that all $v_i$ lie on top level, and that each of them is decorated with a single leg. By Lemma 6.8 below the cone with this configuration has dimension 6, strictly less than the maximal dimension 8.

Consequently the top level consists of vertices $v_i$ without legs, hence each of them is adjacent to precisely two horizontal edges. Thus the subgraph on top level is a simple cycle. In the $K_4$ the length of the cycle might be three or four. In the case of a 4-cycle on top, the case of a single $w_i$ with just one leg on one of the edges is ruled out by the sum of decorations being equal to $-2$. The configuration in Figure 6 remains and attains the maximal dimension.

Suppose the top level is a 3-cycle consisting of $v_1, v_2, v_3$ and that $v_4$ is on lower level. Consider the edges $e_i$ for $i = 4, 5, 6$ joining $v_{i-3}$ to $v_4$. For each partition $(4, 0, 0), (3, 1, 0), (2, 2, 0)$ and $(2, 1, 1)$ of the four vertices $w_i$ on these three edges there is a unique solution to the enhancement conditions, leading to the graphs in Figure 4 and Figure 5, all of them of maximal dimension. Graphs with less than four $w_i$ are degenerations hereof, and hence of strictly smaller dimension.

Note that for a given tropical curve $\Gamma$ with underlying graph $K_4$ the preimage $\pi^{-1}([\Gamma])$ does not meet all of these maximal cones. For example the graph in
Figure 6 is possible for any edge lengths, with the canonical divisor supported on an arbitrary pair of disjoint edges. However, the graph pictured in Figure 4 on the left, with the canonical divisor supported on a pair of adjacent edges is possible if and only if $|e_6| < |e_4|, |e_6| < |e_5|$. The realizability locus is connected in codimension one over the closure of the $K_4$-cone. To see this, note that contracting one of the horizontal edges on the top level 4-cycle in Figure 6 and reopening it as a vertical edge, connects this cone to the cone in Figure 3 (left). This cone is connected to the cone in Figure 5 (left) by pushing one of the $w_i$ adjacent to $v_4$ into $v_4$. This cone is connected to the cone in Figure 5 (right) by pushing the isolated $w_i$ on $e_6$ through $v_4$ onto the edge $e_5$. Finally, the cone in Figure 5 (left) is also connected to the cone Figure 4 (right) by pushing the vertex $w_i$ on $e_5$ adjacent to $v_4$ through $v_4$.

It is an interesting combinatorial question whether in general the maximal cones of the realizability locus are connected in codimension one.

6.2. Dimensions. The fundamental Theorem of Bieri-Groves [BG84, Theorem A] (also see [EKL06, Theorem 2.2.3]) shows that, given a closed subvariety of a split algebraic torus, its tropicalization admits the structure of a polyhedral complex of the same dimension. In our situation, the tropical Hodge bundle does not admit a natural embedding into a toric variety, but rather a toroidal embedding in the sense of [KKMS], the compactification $\overline{\text{Div}}_{g,2g−2}$ of $\text{Div}_{g,2g−2}$ over $\overline{M}_g$. In this situation a weaker version of the Bieri-Groves Theorem holds (see [Uli15a, Theorem 1.1]) and we only know that the realizability locus (i.e. the tropicalization of $P\Omega M_{g,trop}$) is a generalized cone complex of dimension $≤ 4g−4$. This result technically only applies in the case when the boundary has no self-intersection, but the arguments immediately generalize to our situation. Our methods allow us to prove the following much stronger statement.

**Theorem 6.6.** The realizability locus $P\Omega$ is a generalized cone complex, all whose maximal cones have dimension $4g−4$.

The fiber in $P\Omega$ over a maximal-dimensional cone $\sigma_G$ in $M_{g,trop}^+ (i.e. for a trivalent graph $G$ with all vertex-weights $h(v) = 0$) is a generalized cone complex, all whose maximal cones have relative dimension $g−1$.

Recall that in Figure 4 we have seen that the realizability locus is not a subcomplex of $\overline{\text{Div}}_{g,2g−2}$. **Example 6.7.** We revisit Example 6.4. The dumbbell graph is one of the two trivalent genus two graphs. For any edge lengths assigned to the dumbbell, the fiber of $P\Omega M_{g,trop}$ over the corresponding tropical curve is the folded square with two ends pictured in Figure 7. The realizability locus corresponds to the thickened line segments, drawn horizontally. Notice here that the canonical divisor $K_T$ (which corresponds to the third corner in the triangle) is not in the realizability locus.

The dimension estimates are based on the following lemma. The contraction procedure in the lemma stems from the fact that the length information encoded in those genus zero nodes is not recorded when passing to the associated tropical curves with divisor. Note that for all enhanced level graphs that appear in Theorem 6.3 i.e. those resulting from Lemma 6.1 we have $Γ^* = Γ_0^*$ in the following statement.
Lemma 6.8. For every level graph $\Gamma^+$ let $\Gamma^+_0$ be the graph obtained by successively contracting edges in $\Gamma^+$ that have a $(n+1)$-valent genus zero node with $n \geq 1$ marked points at one of its ends. The dimension of a cone $\sigma(\Gamma^+)$ in the realizability locus with associated level graph $\Gamma^+$ is equal to one less than the number of levels $\text{Lev}(\Gamma^+_0)$ plus the number of horizontal edges $E_H(\Gamma^+_0)$, i.e.

$$\dim(\sigma(\Gamma^+)) = |\text{Lev}(\Gamma^+_0)| - 1 + |E_H(\Gamma^+_0)|.$$ 

Proof. Assign a real number $d_i \leq 0$ (‘depth’) to each level $i \in \text{Lev}(G, \ell)$, such that $d_0 = 0$ and $d_i < d_j$ if $i < j$. Then endow any edge $e$ joining the vertices $v_1$ and $v_2$ with $\ell(v_1) > \ell(v_2)$ with length $(d_{\ell(v_1)} - d_{\ell(v_2)})/(k(e^+) + 1)$ and endow horizontal edges with arbitrary lengths. By the construction this tropical curve admits a unique continuous function $f$ (up to addition of a global constant) that is linear of slope zero on horizontal edges and linear of slope $-k(e^+) - 1$ on each edge (as viewed from the top end). This implies that $\dim(\Gamma^+)$ $\geq |\text{Lev}(\Gamma^+_0)| - 1 + |E_H(\Gamma^+_0)|$.

On the other hand, every rational function $f$ on a tropical curve with enhanced level graph $\Gamma^+_0$ determines uniquely a collection of real numbers $d_i$ with $d_0 = 0$ and $d_{\ell(v_1)} - d_{\ell(v_2)} = |e|s(e)$ whenever $\ell(v_1) > \ell(v_2)$. This implies the converse estimate. 

Proof of Theorem 6.6. To prove the upper bound $4g - 4$, we compare with the complex dimension of the moduli space of twisted differentials (denoted by $\mathcal{M}_{\text{ab}}(\overline{\mathcal{G}})$ in [BCGGM2]) compatible with a level graph $\overline{\mathcal{G}}$. (The dimension does not depend on the enhancement.) Each level contributes at least one to the dimension of $\mathcal{M}_{\text{ab}}(\overline{\mathcal{G}})$, namely by rescaling the differentials on that level by a scalar and defining the dimension of $\mathcal{M}_{\text{ab}}(\overline{\mathcal{G}})$ is the sum of the dimensions of the spaces of twisted differentials at each level. Consequently, the maximal dimension is bounded above by the number of horizontal edges plus $\dim_\mathbb{C} \mathcal{M}_{\text{ab}}(\overline{\mathcal{G}}) - 1$. This sum is computed in [BCGGM2, Theorem 6.1] to be equal to $\dim_\mathbb{C} \Sigma M_g(\mu) - 1$, where $\mu$ is the type of the twisted differential. This quantity is maximized for the principal stratum $\mu = (1, \ldots, 1)$ and gives $\dim_\mathbb{C} \Sigma M_g(\mu) - 1 = 2g - 2 + |\mu| = 4g - 4$ by Theorem [AW], and thus the claimed upper bound.

To show that this upper bound is always attained we have to split vertices whose contribution to $\mathcal{M}_{\text{ab}}(\overline{\mathcal{G}})$ is greater than one. The claim follows from the more precise statement in the subsequent proposition.

Proposition 6.9. Maximal dimensional cones of the realizability locus correspond precisely to the enhanced level graphs $\Gamma^+$ with the following properties.

(i) All the vertices have vertex genus zero.

(ii) Each vertex is either
(ii.1) \(n\)-valent \((n \geq 3)\) with precisely two edges which are legs or edges to lower level.

(ii.2) \(n\)-valent \((n \geq 3)\) with precisely one edge which is a leg or an edge to lower level.

(iii) Each level \(L\) contains either

(iii.1) precisely one vertex as in (ii.1). All the edges of this vertex to higher level disconnect the subgraph \(\Gamma^*_{>L}\), or

(iii.2) only vertices as in (ii.2). At each of these nodes moreover \(|v|\geq 2\) edges disconnect the subgraph \(\Gamma^*_{>L}\) while the remaining edges of the nodes at level \(\ell\) together with the connected components of the subgraph \(\Gamma^*_{\geq L}\) form a simple cycle.

In (iii.2) the valence \(|v|\) refers to the valence of the subgraph \(\Gamma^*_{>L}\), i.e. an edge to lower level does not contribute. A simple cycle is a graph isomorphic to some \(n\)-gon. In the proof we will see that the conditions in the theorem can be explained using period coordinates (see Section 5.4).

Proof. In order to show that these cones are maximal we need to show that the contribution of each level to the dimension of \(M_{ab}(\Gamma)\) is at most one. Then we conclude using [BCGGM2, Theorem 6.1], since then the number of levels has to be at least \(4g - 4 - |E_H(\Gamma)|\).

We start by discussing the dimension contribution for the cones in the statement of the proposition. Recall from Theorem 5.3 that the space of differentials corresponding to a \(n\)-valent vertex of genus zero has dimension \(n - 2\).

If there are two marked points (or edges to lower level) as in (ii.1) this space is parametrized by the relative period between the two marked points and \(n - 1\) residues. Moreover the condition in (iii.1) and the GRC implies that all the residues are zero and the resulting contribution of that level \(L\) is of dimension one.

If there is only one marked point (or edges to lower level), the space of differentials is parametrized by the \(n - 1\) residues with one constraint given by the residue theorem. The condition in (iii.2) and the GRC imply again that \(n - 3\) residues vanish. Hence each node contributes again individually one to the complex dimension of the space of differentials at that level. Moreover, the cycle constraint in (iii.2) implies that this residue is the same for each vertex at the given level. Consequently the total contribution of that level to the space of twisted differentials is one, as claimed.

To show that the cones listed in the proposition are the only cones of maximal dimension, we show that we can split the vertices in an enhanced level graph until the condition of the proposition are met, while maintaining the conditions Theorem 6.3 (i) and (ii) of the realizability locus along the splitting procedure. E.g. while some vertex genus is positive, we apply the splitting of Figure 8 where \(a_i \geq 0\) and where \(b_i \leq -1\).

We may thus assume from now on that all vertex genera are zero. In order to show that the nodes at a given level \(i\) can be split until their contribution to the dimension of \(M_{ab}(\Gamma)\) is one (in the sense computed in the beginning of the proof) we could argue combinatorially but arguing geometrically as follows seems more enlightening. Consider the space of twisted differentials \(\eta = \{\eta_v\}\) compatible with the level graph currently under consideration. Suppose that the subspace of twisted differentials with all \(\eta_v\) fixed except for those with \(\ell(v) = i\) has dimension greater than one. To put it differently, we assume that the projectivisation of this subspace
has positive dimension. This subspace is cut out inside \( \Omega M_g(\mu) \) by a collection of residue conditions. Since \( \Omega M_g(\mu) \) does not contain a projective curve by Theorem 5.4, a subspace defined by residue conditions does not contain such a curve either. Consequently, there is some way to degenerate the meromorphic differential, thus increasing either the number of levels or the number of horizontal edges. This also increases the dimension of the corresponding cone in the realizability locus and we can repeat the process until each cone has dimension one, with the caveat given in Lemma 6.8 that trees of marked points are contracted. If we replace each such tree as in Figure 9 then this is also a degeneration of the graph where all

![Diagram](https://via.placeholder.com/150)

**Figure 8.** Splitting positive genus

trees of marked points are contracted, the number of levels and horizontal edges is the same and this graph satisfies the conditions of Theorem 6.3 if the graph prior to the replacement did. This concludes the proof of the existence of a splitting procedure. \( \square \)

### 6.3. The realizability locus for strata of abelian differentials

Let \( \mu \) be a partition of \( d \). We say that an effective divisor \( D \) of degree \( d \) on a tropical curve \( \Gamma \) has type \( \mu \) if the multiplicities at its support define the partition \( \mu \). Notice, in particular, that, in complete analogy with the situation for the projective algebraic Hodge bundle \( \mathbb{P}\Omega M_g \), the tropical Hodge bundle \( \mathbb{P}\Omega M_g^{trop} \) admits a stratification by strata \( \mathbb{P}\Omega M_g^{trop}(\mu) \) that are indexed by partitions \( \mu \) of \( 2g-2 \).

Theorem 6.3 contains also a characterization of the realizability locus \( \mathbb{P}\mathcal{R}_\Gamma(\mu) \) of the stratum of type \( \mu \), defined as the image of \( \text{trop}_\Gamma|_{\mathbb{P}\Omega M_g(\mu)^{an}} \) of the restriction of the tropicalization map to the corresponding stratum \( \mathbb{P}\Omega M_g(\mu)^{an} \) of abelian differentials. In fact, the proof of our main theorem applies verbatim to give the following criterion.

**Proposition 6.10.** An element \( D = K_\Gamma + \text{div}(f) \) in the tropical canonical linear series lies in \( \mathbb{P}\mathcal{R}_\Gamma(\mu) \) if and only if \( D \) is a divisor of type \( \mu \) and for the enhanced level graph \( \Gamma^*(f) \) the two conditions i) and ii) of Theorem 6.3 hold.
Example 6.11. We give for example the realizability locus $\mathbb{P} \mathcal{R}_\Omega(2)$ if the underlying graph is the dumbbell graph, i.e. as a subset of Example 6.4. Restricted to this graph, the tropical Hodge bundle is disconnected and consists of (isolated) double zeros at the midpoint of either of the dumbbell cycles and of a one-dimensional component with double zero on the central edge of the dumbbell. The midpoints of the dumbbell cycles satisfy the criteria i) and ii). A point of multiplicity two in the interior of the the central edge (as Figure 2 on the right) does not satisfy criterion i), since a vertex with one zero (of order two) and two poles (of order two) is inconvenient. However, if the double is located on the vertices of the dumbbell, the vertices are no longer inconvenient and the criteria are satisfied.

In conclusion, the realizability locus $\mathbb{P} \mathcal{R}_\Omega(2)$ on the dumbbell graph consists of four 'Weierstrass' points as in Figure 10.

6.4. Algorithmic aspects. Our main theorem can be turned into an algorithm to compute the simplicial structure of the realizability locus.

i) For each genus $g$ construct the finitely many abstract graphs $G = (E, V, L)$ with a genus function $h : V \to \mathbb{Z}_{>0}$ of genus $g$ (in the sense of (1)) and with $|L| = 2g - 2$ that are stable.

ii) There is a finite number of partial orders on $G$ such that any two vertices joined by an edge are comparable (equality permitted). For each of those partial orders there is a finite number of enhancements $k$ with the properties that $k(e^+) = -1$ for both half edges of $e$ joining $v_1$ and $v_2$ with $v_1 \simeq v_2$ and such that whenever there is an edge $e$ joining $v_1$ and $v_2$ with $v_1 \succ v_2$ then $k(e_1) \geq 0$, where $e_1$ is the half-edge of $e$ adjacent to $v_1$.

To see this, we attribute $k(e^+) = -1$ to all edges with $v_1 \simeq v_2$ and argue inductively top-down: for each vertex $v$ such that all the upward pointing half-edges have already been assigned an enhancement, there is a finite number of possibilities to assign a non-negative enhancement to each of the downward pointing half-edges $e^+$ such that the genus formula (5) holds. We complete this enhancement on each of the complementary half edges $e^-$ using the condition $k(e^+) + k(e^-) = -2$ and proceed to another vertex.

iii) For each of the partial orders there is a finite number of full orders that refine the partial order. We assume for notational convenience that the full order is given by the level function $\ell$.

iv) For each of the horizontal edges $e$ (i.e. both half edges are decorated with $k(e^+) = -1$) check if $e$ disconnects the graph $G_{\ell(e)}$ and discard the graph if this is the case. Here $\ell(e) := \ell(v)$ for any of the two vertices adjacent to $e$.

v) Using the enhancement we can determine the set of inconvenient vertices $I \subset V$.

For each vertex $v \in I$ check if $v$ disconnects the graph $G_{\ell(v)}$ and discard the graph if this is the case.

vi) The realizability locus consists of a cone $\sigma = \sigma_{(G, h, k, \ell)}$ for each tuple $(G, h, k, \ell)$ not discarded. The cone $\sigma_{(G, h, k, \ell)}$ parametrizes the following tropical curves.

Assign as in the proof of Lemma 6.8 a real number $d_i \leq 0$ ('depth') to each
level \( i \in \text{Lev}(G, \ell) \), such that \( d_0 = 0 \) and \( d_i < d_j \) if \( i < j \). Then endow any edge \( e \) joining the vertices \( v_1 \) and \( v_2 \) with \( \ell(v_1) > \ell(v_2) \) with length
\[
|e| = \left( d_{\ell(v_1)} - d_{\ell(v_2)} \right) / \left( k(e^+) + 1 \right).
\]
This algorithm is effective but not efficient, since most of the enhanced level graphs that are built in the process will be discarded in the end.

References

[AB15] O. Amini and M. Baker. “Linear series on metrized complexes of algebraic curves”. In: *Math. Ann.* 362.1-2 (2015), pp. 55–106.

[ACP15] D. Abramovich, L. Caporaso, and S. Payne. “The tropicalization of the moduli space of curves”. In: *Ann. Sci. Éc. Norm. Supér.* (4) 48.4 (2015), pp. 765–809.

[AOV11] D. Abramovich, M. Olsson, and A. Vistoli. “Twisted stable maps to tame Artin stacks”. In: *J. Algebraic Geom.* 20.3 (2011), pp. 399–477.

[Bak08] M. Baker. “Specialization of linear systems from curves to graphs”. In: *Algebra Number Theory* 2.6 (2008). With an appendix by Brian Conrad, pp. 613–653.

[BCGGM] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. *Compactification of strata of abelian differentials*. (2016). arXiv: 1604.08834

[BCGGM2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. *Strata of \( k \)-differentials*. (2016). arXiv: 1610.09238

[Ber90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Vol. 33. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990, pp. x+169.

[Ber99] V. G. Berkovich. “Smooth \( p \)-adic analytic spaces are locally contractible”. In: *Invent. Math.* 137.1 (1999), pp. 1–84.

[BG84] R. Bieri and J. R. J. Groves. “The geometry of the set of characters induced by valuations”. In: *J. Reine Angew. Math.* 347 (1984), pp. 168–195.

[BJ15] M. Baker and D. Jensen. *Degeneration of Linear Series From the Tropical Point of View and Applications*. 2015. arXiv: 1504.05544

[Cap94] L. Caporaso. “A compactification of the universal Picard variety over the moduli space of stable curves”. In: *J. Amer. Math. Soc.* 7.3 (1994), pp. 589–660.

[Car15] D. Cartwright. “Lifting matroid divisors on tropical curves”. In: *Res. Math. Sci.* 2 (2015), Art. 23, 24.

[CCUW] R. Cavalieri, M. Chan, M. Ulirsch, and J. Wise. *A moduli stack of tropical curves*. 2017. arXiv: 1704.03806

[Che17] D. Chen. *Affine geometry of strata of differentials*. (2017). arXiv: 1706.01142

[CHMR] R. Cavalieri, S. Hampe, H. Markwig, and D. Ranganathan. “Moduli spaces of rational weighted stable curves and tropical geometry”. In: *Forum Math. Sigma* 4 (2016), e9, 35.

[CJP15] D. Cartwright, D. Jensen, and S. Payne. “Lifting divisors on a generic chain of loops”. In: *Canad. Math. Bull.* 58.2 (2015), pp. 250–262.
[DM69] P. Deligne and D. Mumford. “The irreducibility of the space of curves of given genus”. In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 75–109.

[EKL06] M. Einsiedler, M. Kapranov, and D. Lind. “Non-Archimedean amoebas and tropical varieties”. In: *J. Reine Angew. Math.* 601 (2006), pp. 139–157.

[GK08] A. Gathmann and M. Kerber. “A Riemann-Roch theorem in tropical geometry”. In: *Math. Z.* 259.1 (2008), pp. 217–230.

[GT17] Q. Gendron and G. Tahar. *Différentielles à singularités prescrites.* (2017). arXiv: [1705.03240].

[Has03] B. Hassett. “Moduli spaces of weighted pointed stable curves”. In: *Adv. Math.* 173.2 (2003), pp. 316–352.

[He17] X. He. *Smoothing of limit linear series on curves and metrized complexes of pseudocompact type.* 2017. arXiv: [1707.04824].

[JP16] D. Jensen and S. Payne. “Tropical independence II: The maximal rank conjecture for quadrics”. In: *Algebra Number Theory* 10.8 (2016), pp. 1601–1640.

[JR17] D. Jensen and D. Ranganathan. *Brill-Noether theory for curves of a fixed gonality.* 2017. arXiv: [1701.06579].

[KKMSD] G. Kempf, F.F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973, pp. viii+209.

[Knu83] F. F. Knudsen. “The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$”. In: *Math. Scand.* 52.2 (1983), pp. 161–199.

[Liu02] Q. Liu. *Algebraic geometry and arithmetic curves.* Vol. 6. Oxford Graduate Texts in Mathematics. Translated from the French by Reinie Erné. Oxford Science Publications. Oxford University Press, Oxford, 2002, pp. xvi+576.

[LU17] B. Lin and M. Ulirsch. *Towards a tropical Hodge bundle.* (2017). arXiv: [1701.04385].

[Mik06] Grigory Mikhalkin. “Tropical geometry and its applications”. In: *International Congress of Mathematicians. Vol. II.* Eur. Math. Soc., Zürich, 2006, pp. 827–852.

[Mil80] J. S. Milne. *Étale cohomology.* Vol. 33. Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1980, pp. xiii+323.

[Mil86] J. S. Milne. “Jacobian varieties”. In: *Arithmetic geometry (Storrs, Conn., 1984).* Springer, New York, 1986, pp. 167–212.

[MZ08] G. Mikhalkin and I. Zharkov. “Tropical curves, their Jacobians and theta functions”. In: *Curves and abelian varieties.* Vol. 465. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 203–230.

[Pfl17] N. Pflueger. “Brill-Noether varieties of $k$-gonal curves”. In: *Adv. Math.* 312 (2017), pp. 46–63.

[Thu07] A. Thuillier. “Géométrie toroidale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels”. In: *Manuscripta Math.* 123.4 (2007), pp. 381–451.

[Uli14] M. Ulirsch. *Tropicalization is a non-Archimedean analytic stack quotient.* Math. Res. Lett., to appear. 2014. arXiv: [1410.2216].
M. Ulirsch. “Tropical compactification in log-regular varieties”. In: *Math. Z.* 280.1-2 (2015), pp. 195–210.

M. Ulirsch. “Tropical geometry of moduli spaces of weighted stable curves”. In: *J. Lond. Math. Soc. (2)* 92.2 (2015), pp. 427–450.

F. Viviani. “Tropicalizing vs. compactifying the Torelli morphism”. In: *Tropical and non-Archimedean geometry*. Vol. 605. Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 181–210.