Local Unitary Equivalence of Arbitrary Dimensional Bipartite Quantum States

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The nonlocal properties of arbitrary dimensional bipartite quantum systems are investigated. A complete set of invariants under local unitary transformations is presented. These invariants give rise to both sufficient and necessary conditions for the equivalence of quantum states under local unitary transformations: two density matrices are locally equivalent if and only if all these invariants have equal values.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

Introduction. — As a fundamental phenomenon in quantum mechanics, the quantum nonlocality has been recently extensively investigated. Nonlocally quantum correlated states, like quantum entangled states\textsuperscript{[1]} or states with nonzero quantum discord\textsuperscript{[2]}, play very important roles in many quantum information processing such as quantum computation\textsuperscript{[3]}, teleportation\textsuperscript{[4]}, dense coding\textsuperscript{[5]}, cryptography\textsuperscript{[6]} and assisted optimal state discrimination\textsuperscript{[7]}. Due to the fact that the nonlocal properties, e.g. the quantum entanglement of two parts of a quantum system, remain invariant under local unitary transformations, they can be characterized in principle by the complete set of invariants under local unitary transformations. For instance, the trace norms of realigned or partial transposed density matrices in entanglement measure\textsuperscript{[8]} and the separability criteria\textsuperscript{[9]} are some of these invariants. A complete set of invariants gives rise to full classification of the quantum states under local unitary transformations.

Many approaches to construct invariants of local unitary transformations have been presented in recent years. The method developed in\textsuperscript{[10, 11]}, in principle, allows one to compute all the invariants of local unitary transformations, though it is not easy to perform it operationally. In\textsuperscript{[12]}, a complete set of 18 polynomial invariants is presented for the locally unitary equivalence of two qubit mixed states. Partial results have been obtained for three qubits states\textsuperscript{[12]}; some generic mixed states\textsuperscript{[13-16]}, tripartite pure and mixed states\textsuperscript{[17]}. The local unitary equivalence problem for multipartite pure qubit states has been solved in\textsuperscript{[18]}. Then the problem for arbitrary dimensional multipartite pure states are also solved recently\textsuperscript{[19]}. However, for mixed states, generally we still have no operational criteria to judge the equivalence of two arbitrary dimensional bipartite states under local unitary transformations.

In this letter, we study the nonlocal properties of arbitrary dimensional bipartite quantum systems and solve the local equivalence problem by presenting a complete set of invariants such that two density matrices are locally equivalent if and only if all these invariants have the equal values in these density matrices. These invariants can be explicitly calculated and give rise to an operational way to judge the local unitary equivalence for nondegenerate density matrices. For degenerate case, due to the eigenvector decompositions of a given state are not unique in the degenerate eigenvectors’ subspace, the approach is no longer operational since the expressions of the set of invariants are not unique. Nevertheless the set of invariants is still complete in the sense that if, in a suitable eigenvector decompositions, two states have the same values of the invariants, they must be equivalent under local unitary transformations. In particular, we also present a set of invariants that are independent on the detailed eigenvector decompositions of a quantum state. These invariants present a necessary criterion on local unitary equivalence.

The linear space formed by the invariants. — We first establish a linear space spanned by matrices whose traces are the invariants under local unitary transformations.

Let $H$ be an $N$-dimensional complex Hilbert space, with $|i\rangle$, $i=1,...,N$, an orthonormal basis. Let $\rho$ be a density matrix defined on $H \otimes H$ with $\text{rank}(\rho) = n \leq N^2$. $\rho$ can be generally written as

$$\rho = \sum_{i=1}^{n} \lambda_i |v_i\rangle \langle v_i|,$$

where $|v_i\rangle$ is the eigenvector with respect to the nonzero eigenvalue $\lambda_i$. $|v_i\rangle$ is a normalized bipartite pure state of the form:

$$|v_i\rangle = \sum_{k,l=1}^{N} a_{kl}^i |kl\rangle, \quad a_{kl}^i \in \mathbb{C}, \quad \sum_{k,l=1}^{N} a_{kl}^i a_{kl}^{\ast i} = 1,$$

where $*$ stands for complex conjugation.

Two density matrices $\rho$ and $\rho'$ are said to be equivalent under local unitary transformations if there exist unitary operators $U_1$ (resp. $U_2$) on the first (resp. second) space.
of $H \otimes H$ such that

$$\rho' = (U_1 \otimes U_2) \rho (U_1 \otimes U_2)\dagger,$$  \hfill (2)

where $\dagger$ denotes transpose and complex conjugation.

To solve the local equivalence problem of the density matrices $\rho$ and $\rho'$, it is sufficient to find the complete set of invariants under local unitary transformations $U_1 \otimes U_2$.

**Lemma 1.** The following quantities are invariants under local unitary transformations:

$$J^s(\rho) = Tr(T r_1(T r_2 \rho^s),$$

$$Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)],$$  \hfill (3)

where $s = 1, \ldots, N^2$, $i, j, k, l, \ldots, h, p = 1, \ldots, n$, $Tr_1$ and $Tr_2$ stand for the traces over the first and second Hilbert spaces respectively, $A_i$ denotes the matrix with entries given by $[A_i]_{kl} = a_{ikl}$, $i = 1, 2, \ldots, n$.

**Proof.** Let $U_1$ and $U_2$ be unitary transformations satisfying $U_1 U_1^\dagger = U_1^\dagger U_1 = U_2 U_2^\dagger = U_2^\dagger U_2 = 1$. Under the local unitary transformation $U_1 \otimes U_2$, we have $\rho \to \rho' = U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger$. Correspondingly, we have $|\nu_i\rangle \to |\nu'_i\rangle = U_1 \otimes U_2 |\nu_i\rangle$, or equivalently, $A_i$ is mapped to $A'_i = U_1 A_i U_2^\dagger$, where $U_2^\dagger$ is the transpose of $U_2$. Therefore

$$A_i A_j^\dagger = U_1 A_i A_j U_1^\dagger, \quad A_i A_j^\dagger = U_2 A_i^\dagger A_j U_2^\dagger,$$  \hfill (4)

for any $i, j = 1, \ldots, n$. By using relations [1] and taking into account that $Tr_2 |\nu_i\rangle \langle \nu_i| = A_i A_i^\dagger$, it is straightforward to verify that $J^s(\rho') = J^s(\rho)$, $Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)] = Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)]$, hence the quantities in (3) are invariants of local unitary transformations. \qed

If two density matrices are equivalent under local unitary transformations, then their corresponding invariants in (3) must have the same values. Before proving that if two density matrices have the same values of all the invariants in (3), they must be equivalent under local unitary transformations, we first claim that the set of independent invariants in (3) is finite. In fact, each factor $A_i A_j^\dagger$ belongs to the finite dimensional algebra $Mat(N)$, which has a linear basis $E_{ij}$, $i, j = 1, \ldots, N$. Therefore, denoting $\tau$ the number of factors like $(A_i A_j^\dagger)$ in $Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)]$, we have that $\tau$ is at most $N^2$. Subjecting to the variations of the subindices $i, j, k, l, \ldots, h, p$, there could be many invariants of this form. However, $\tau$ may be much less than $N^2$ for given $i, j, k, l, \ldots, h, p$. For instance for the case of $i = j = k = \ldots = h = p$, $\tau$ takes values from 2 to $N$ (the case $\tau = 1$ is trivial since $Tr[(A_i A_j^\dagger)] = 1$ due to the normalization of the state $|\nu_i\rangle$). The same is true also for the invariants $Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)]$.

Denote the subalgebra of $Mat(N)$ spanned by products of $A_i A_j^\dagger$ by

$$\mathcal{R}(\rho) = span\{(A_i A_j^\dagger)(A_k A_l^\dagger) \cdots (A_h A_p^\dagger)\},$$

$i, j, k, l, \ldots, h, p = 1, \ldots, n$. Obviously $\mathcal{R}(\rho)$ is a finite dimensional associative algebra. Set $m = \dim \mathcal{R}(\rho)$. Let $\rho_1, \rho_2, \cdots, \rho_m$ be a basis of the linear space $\mathcal{R}(\rho)$.

**Lemma 2.** The metric tensor matrix $\Omega$, with entries given by $\Omega(\rho)_{ij} = Tr(\rho_i \rho_j)$, $i, j, 1, 2, \cdots, m$, is non-singular.

**Proof.** We prove by contradiction. If $\det(\Omega(\rho)) = 0$, then the $m$ row vectors of the matrix $\Omega(\rho)$ are linear dependent. There exist $c_1, c_2, \cdots, c_m \in \mathbb{C}$ which are not all zero, such that

$$\sum_{i=1}^{m} c_i Tr(\rho_i \rho_j) = 0, \quad j = 1, \ldots, m.$$

It follows that

$$Tr((c_1 \rho_1 + c_2 \rho_2 + \cdots + c_m \rho_m) \rho_j) = 0, \quad j = 1, \ldots, m.$$  \hfill (5)

Since $(A_i A_j^\dagger) \cdots (A_h A_p^\dagger) \in \mathcal{R}(\rho)$ and $\rho_i$ are linear combinations of the terms like $A_i A_j^\dagger \cdots A_h A_p^\dagger$, we have $(c_1 \rho_1 + c_2 \rho_2 + \cdots + c_m \rho_m) \in \mathcal{R}(\rho)$ and

$$(c_1 \rho_1 + c_2 \rho_2 + \cdots + c_m \rho_m)(A_i A_j^\dagger) \cdots (A_h A_p^\dagger) = h_1 \rho_1 + h_2 \rho_2 + \cdots + h_m \rho_m,$$

for some $h_i \in \mathbb{C}, i = 1, 2, \cdots, m$. It follows from (4) that

$$Tr[(c_1 \rho_1 + c_2 \rho_2 + \cdots + c_m \rho_m)(\rho_1 \rho_2 + \cdots + \rho_m)\dagger] = 0,$$

which implies $c_1 \rho_1 + c_2 \rho_2 + \cdots + c_m \rho_m = 0$. Hence one concludes that $\{\rho_i, i = 1, 2, \cdots, m\}$ are linear dependent, which contradicts that $\{\rho_1, \rho_2, \cdots, \rho_m\}$ is a basis of the linear space $\mathcal{R}(\rho)$. \qed

Actually, one can also prove that if $\{\rho_i, i = 1, 2, \cdots, m\}$ is a sequence of matrices such that $\det(\Omega(\rho)) \neq 0$, then $\{\rho_i, i = 1, 2, \cdots, m\}$ is linear independent.

The invariants $Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)]$ can be equivalently written as $Tr[(A_i A_j^\dagger)(A_k A_l^\dagger) \cdots (A_h A_p^\dagger)]$. Correspondingly one has the linear space defined by

$$\mathcal{N}(\rho) = span\{(A_i A_j^\dagger)(A_k A_l^\dagger) \cdots (A_h A_p^\dagger)\},$$

$i, j, k, l, \ldots, h, p = 1, \ldots, n$, with finite dimension $d = \dim \mathcal{N}(\rho)$. If $\{\theta_1, \theta_2, \cdots, \theta_d\}$ is a basis of $\mathcal{N}(\rho)$ like $\mathcal{R}(\rho)$, we similarly have that the matrix $\Theta$ with entries given by $\Theta(\rho)_{ij} = Tr(\theta_i \theta_j)$, $i, j, 1, 2, \cdots, d$, satisfies $\det(\Theta(\rho)) \neq 0$.

**Lemma 3.** If two density matrices $\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle \langle \nu_i|$ and $\rho' = \sum_{i=1}^{n} \lambda'_i |\nu'_i\rangle \langle \nu'_i|$ have the same values for the following invariants:

$$Tr[(A_i A_j^\dagger)](A_k A_l^\dagger) \cdots (A_h A_p^\dagger)],$$  \hfill (6)
i, j, k, l, ..., h, p = 1, ..., n, then the corresponding linear spaces $R(\rho)$ and $R(\rho')$ have the same dimension. Moreover, the bases of $R(\rho)$ and $R(\rho')$ have a one to one correspondence: if $(A_i A_j')(A_h A_p')$ is a basis element of $R(\rho)$, then $(A_i j'(A_h A_p')$ is a basis element of $R(\rho')$. Similar results hold between the linear spaces $N(\rho)$ and $N(\rho')$.

[Proof]. Assume that $\{\rho_\alpha, \alpha = 1, 2, \cdots, m\}$ is a basis of $R(\rho)$. Each $\rho_\alpha$ has a form of $(A_i A_j')(A_h A_p')$. Denote $\rho'_\alpha = (A_i j'(A_h A_p')$, $\alpha = 1, 2, \cdots, m$. Since $\det[\Omega(\rho)] = \det[\Omega(\rho')] \neq 0$ from Lemma 1 and the condition (5), we have $\{\rho_\alpha, \alpha = 1, 2, \cdots, m\}$ is linearly independent. Hence $\dim R(\rho') \geq \dim R(\rho)$. In a similar way one can prove that $\dim R(\rho) \geq \dim R(\rho')$. Therefore $\dim R(\rho') = \dim R(\rho)$ and $\{\rho_\alpha, \alpha = 1, 2, \cdots, m\}$ is a basis of $R(\rho')$.

Similar results between the linear spaces $N(\rho)$ and $N(\rho')$ can be obtained from the expression of invariants $Tr[(A_i A_j')(A_h A_p')]$.

Local equivalence of bipartite states. — We now give the necessary and sufficient condition for the equivalence of bipartite states under local unitary transformations. Theorem Two arbitrary dimensional bipartite density matrices are equivalent under local unitary transformations if and only if there exist eigenstate decompositions (1) such that the following invariants have the same values for both density matrices:

$$J^s(\rho) = Tr_2(Tr_1 \rho^s), \quad s = 1, \cdots, N^2;$$
$$Tr[(A_i A'_j)(A_h A_p')],$$

where $i, j, k, l, ..., h, p = 1, ..., n$.

[Proof]. We have proved that the quantities in (7) are invariants under local unitary transformations. We now prove that these invariants are complete. Suppose that the states $\rho = \sum_{i=1}^n \lambda_i |\psi_i\rangle \langle \psi_i|$ and $\rho' = \sum_{i=1}^n \lambda'_i |\psi'_i\rangle \langle \psi'_i|$ have the same values to the invariants in (7). From $J^s(\rho) = J^s(\rho')$, $s = 1, \cdots, N^2$, we have that the two density matrices $\rho$ and $\rho'$ have the same set of eigenvalues, i.e. $\lambda_i = \lambda'_i$ for $i = 1, 2, \cdots, n$.

Next we introduce the dual basis $\rho^*_i$ in $R(\rho)$ such that $Tr(\rho_i \rho^*_j) = \delta_{ij}$. In fact, let $\Omega(\rho)^{-1} = [\Omega]^j(\rho)$, then

$$\rho^*_i = \sum_{j=1}^m \Omega^j(\rho) \rho_j.$$  

By Cramer’s rule $\Omega(\rho)^{-1} = [\Omega(\rho)]^{-1} \text{Adj}(\Omega(\rho))$, so $\Omega^j(\rho)$ are given by polynomials in $\Omega_{ij}(\rho)$, consequently $\rho^*_i$ are expressed as polynomials in the basis element $\rho_i$.

Now we study the algebra structure of $R(\rho)$. First we define the bilinear form $\langle \cdot, \cdot \rangle$ on $R(\rho)$ by

$$\langle \sigma, \tau \rangle = Tr(\sigma \tau^\dagger), \quad \sigma, \tau \in R(\rho).$$

Then $R(\rho)$ becomes a Hilbert space and each operator $\rho_i$ is bounded under the norm $||\rho|| = (\rho, \rho)^{1/2}$. Next we have

$$\rho_i \rho_j = \sum_{k=1}^m \Omega^{ij}(\rho) \rho_k.$$  

The invariance of the traces (1) implies that in the algebra $R(\rho)$ the element $\rho^*_i$ is given by the same formula as $\rho^*_i$:

$$\rho^*_i = \sum_{j=1}^m \Omega^{ij}(\rho') \rho_j' = \sum_{j=1}^m \Omega^{ij}(\rho) \rho_j.$$  

Moreover $\rho_i \rho_j'$ are also given by the same structure constants:

$$\rho_i \rho_j' = \sum_{k=1}^m Tr(\rho_i \rho_j \rho_k') \rho_k = \sum_{k=1}^m Tr(\rho_i \rho_j \rho_k') \rho_k.$$  

Eqs (10) and (12) imply that the map $\rho_i \rightarrow \rho_i'$ gives an isomorphism from the algebra $R(\rho)$ onto the algebra $R(\rho')$. Therefore $R(\rho)$ and $R(\rho')$ are two equivalent representations of the same underlying associative algebra. Thus there exists a non-singular matrix $T$ such that $\rho_i = T \rho_i T^{-1}$ for all $i = 1, ..., m$. In particular, we have $A_i A'_j = T A'_j A_i T^{-1}$, $i = 1, ..., m$. As $A_i A'_j$ are hermitian, due to the algebraic property (11) and using Theorem 12.36 in (20) we have

$$A_i A'_j = u A'_j A_i u^\dagger,$$  

where $u$ is the unitary part of $T$ in the polar decomposition.

Similarly from isomorphism of the algebras $N(\rho) \simeq N(\rho')$, one has

$$A_i A'_j = w A'_j A_i w^\dagger,$$  

for some unitary matrix $w$ and all $i$.

By using relations (13) and (14) we can show

$$A_i = u A'_i (w^*)^t, \quad i = 1, ..., n.$$  

In fact, let $u_i$ and $u'_i$ be the unitary matrices that diagonalize the hermitian matrices $A_i A'_j$ and $A'_i A_j$, respectively, $u_i A_i A'_i u_i^\dagger = \text{diag}(\eta_{i1}, ..., \eta_{in})$, $u'_i A'_i A_i u'_i^\dagger = \text{diag}(\eta'_{i1}, ..., \eta'_{in})$. From the procedure of the singular value decomposition of matrices (21), we have $u_i A_i A' u_i^\dagger = \text{diag}(\eta_{i1}, ..., \eta_{in})$ with $\eta_i \geq 0$. From (13) and (14), we have $u_i A_i A'_i u_i^\dagger = u_i u_i A_i A'_i u_i^\dagger = \text{diag}(\eta'_{i1}, ..., \eta'_{in})$ and $u'_i A'_i A_i u'_i^\dagger = u'_i u'_i A'_i A_i u'_i^\dagger = \text{diag}(\eta^*_{i1}, ..., \eta^*_{in})$. Therefore from singular value decomposition we have $u_i u_i A_i A' u_i^\dagger = \text{diag}(\eta_i, ..., \eta_i)$. Hence we obtain $u_i A_i A'_i u_i^\dagger = u_i u_i A_i A' u_i^\dagger$, i.e. $A_i = u_i A_i A' u_i^\dagger = u A_i (w^*)^t$. Since $A_i = u A_i (w^*)^t$, we have $|\lambda_i'\rangle = u^\dagger \otimes (w^*)^t |\lambda_i\rangle$, $i = 1, ..., n$, and $\rho = u^\dagger \otimes (w^*)^t \rho u w^*$. Therefore $\rho$ and $\rho'$ are equivalent under local unitary transformations. □
From the proof one sees that the invariants (7) are complete, finite and can be easily calculated. For non-degenerate states $\rho$ and $\rho'$, the eigenvector $|\nu_i\rangle$ of $\rho$ corresponds uniquely to the eigenvector $|\nu'_i\rangle$ of $\rho'$. Hence the correspondence between $A_i$ and $A'_i$ is also unique. It is straightforward to judge the local unitary equivalence of $\rho$ and $\rho'$ by simply comparing the values of invariants from $A_i$ and $A'_i$ one by one.

For degenerate states, it becomes less operational since the eigenvector decompositions (1) is no longer unique. If $|v_i\rangle$, $i = 1, \ldots, r$, are the orthogonal eigenvectors with respect to a same eigenvalue, then $|\tilde{v}_i\rangle = \sum_j U_{ij} |v_j\rangle$ is also a set of linearly independent orthogonal eigenvectors to the same eigenvalue for any unitary matrix $(U)_{ij} = U_{ij}$. Therefore the expressions of the invariants (7) are not unique in general. However, by linear combinations of the invariants in (7) we can get a set of invariants which does not depend on the detailed eigenvector decompositions,

$$\sum_{i',i,j',\ldots,k,k'=1}^r Tr(A_i^A A_{i}^A A_{j'}^A A_{j'}^A \cdots A_{k}^A A_{k}^A),$$

$$\sum_{i',i,j',\ldots,k,k'=1}^r Tr(A_{i'}^A A_{i'}^A A_{j'}^A A_{j'}^A \cdots A_{k}^A A_{k}^A),$$

where the indices $\{i',j',\ldots,k',k\}$ is any given permutation of the indices $\{i,j,\ldots,k\}$. Different given permutation gives different invariants. Since the invariants (15) are independent on the detailed eigenvector decompositions, they gives an operational necessary condition for the local equivalence, with respect to the subspaces spanned by the degenerate eigenvectors. If two density matrices with the same degenerate eigenvalue are equivalent under local unitary transformations, they must have the same values of all the invariants in (15).

As an example, consider $\rho = \text{diag}\{1/2, 1/2, 0, 0\}$, $\rho' = \text{diag}\{1/2, 0, 1/2, 0\}$. We have $A_1 = A'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = (A_2^2)' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

From (15) we have that $\rho$ and $\rho'$ are not local unitary equivalent, since $Tr(A_1 A_1 A_1 A_1 + A_1 A_2 A_2 A_1 + A_2 A_1 A_1 A_2 + A_2 A_2 A_2 A_2) = 2$, while $Tr(A'_1 A'_1 A'_1 A'_1 + A'_1 A'_2 A'_1 A'_2 + A'_2 A'_1 A'_2 A'_2 + A'_2 A'_2 A'_2 A'_2) = 4$.

**Conclusion and remarks.** We have investigated the nonlocal properties of arbitrary dimensional bipartite quantum systems and solved the local equivalence problem by presenting a complete set of invariants such that two density matrices are locally equivalent if and only if all these invariants have the equal values. Although the independent invariants may vary with the detailed bipartite states, the number of the invariants one needs to check is finite. Here we have dealt with the case that the dimensions of both Hilbert spaces are the same for simplicity. Nevertheless the case that the Hilbert spaces have different dimensions can be similarly discussed. Our approach of constructing local invariants and algebraic proof of the sufficiency may also shed light on multipartite case, for which only the multipartite pure state case has been extensively studied [18, 19].

**Acknowledgments** We thank K.L. Lin for useful discussions. C.Q. Zhou thanks the Max Planck Institute for Mathematics in the Sciences for hospitality. This work is supported by the NSFC 10875081 and PHR201007107. N. Jing thanks the support from Simons foundation, NSF and MPI für Mathematik in Bonn and MPI für Mathematik in Wissenschaft, Leipzig for hospitality.

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