Quantized shells as a tool for studying semiclassical effects in general relativity

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Thin shells in general relativity can be used both as models of collapsing objects and as probes in the space-time outside compact sources. Therefore they provide a useful tool for the analysis of the final fate of collapsing matter and of the effects induced in the matter by strong gravitational fields. We describe the radiating shell as a (second quantized) many-body system with one collective degree of freedom, the (average) radius, by means of an effective action which also entails a thermodynamic description. Then we study some of the quantum effects that occur in the matter when the shell evolves from an (essentially classical) large initial radius towards the singularity and compute the corresponding backreaction on its trajectory.

1. Introduction

A compact object which collapses gravitationally can be used as a probe to test physics at different length scales. In fact, as long as the collapse proceeds, the characteristic size of the object will subsequently cross all the relevant scales, which include the Compton wavelengths $\ell_\phi$ of the particle excitations the object is made of and, eventually, the Planck length $\ell_p$. Starting with a large size, the dynamics can be initially described by classical general relativity, therefore avoiding the issue of the initial conditions which plagues the case of the expansion (e.g., of the universe in quantum cosmology). Then, the quantum nature of the matter in the object will become relevant presumably around $\ell_\phi$ and a quantum theory of gravity is required near $\ell_p$.

In this picture one must also accommodate for the role played by the causal horizon generated by the collapsing object. The horizon is a sphere with radius $R_G = 2M$ determined by the total energy $M/G$ of the system (where $G$ is the Newton constant). If the object does not lose energy, $M$ is constant in time and one may simply take the point of view of a (proper) observer comoving with some particle of the infalling matter, for whom the horizon is totally harmless. The situation is rather different for a radiating object, since one can then consider also an external observer who witnesses the collapse by detecting the emitted radiation. For such an observer $R_G$ is the radius of a (apparent) horizon which shrinks in time with an uncertainty in location related to the probabilistic (quantum mechanical) nature of the emission process (see [2] for an analogy with the Unruh effect). Further, there is an uncertainty (of order $\ell_\phi$) in the position of the particles of the collapsing body and this all renders the issue of (viewing) an emitting source crossing its own horizon of particular relevance, both for the theoretical understanding of gravitation and for astrophysical applications.

It seems appropriate to tackle this hard topic by considering simple cases which capture the basic features of the problem and reduce the technical difficulties. Such an example is given by the spherically symmetric distributions of matter with infinitesimal thickness known in general relativity as thin shells. They can either be pulled just by their own weight or fall in the external gravitational field of a (spherical) source placed at their centre, the latter case being of in-

\footnote{We conceive an observer as some material device and not as an abstract concept.}
terest for studying the accretion of matter onto a black hole \[^{1}\] and the Hawking effect \[^{4}\].

2. Effective action

The equations of motion for thin shells are usually obtained in general relativity from the junction conditions between the embedding four-metric and the intrinsic three-metric on the surface of the shell of radius \( r = R \[^{2}\] \). Their dynamics can also be described by an effective action \[^{5}\] which is obtained by inserting in the general expression for the Einstein-Hilbert action the Schwarzschild solution \[^{1}\] with mass parameter \( M_{0} \) (\( \geq 0 \)) as the space-time inside the shell \( (r < R) \) and the Vaidya solution \[^{6}\] with mass function \( m \) (> \( M_{0} \)) outside the shell \( (r > R) \). When \( M \), the time derivative of \( m \) evaluated on the outer surface of the shell, is negative, the shell emits null dust.

The effective action is given by \[^{5}\]

\[
S_{s} = \int \frac{dt}{G} \left[ R \beta - R \dot{R} \tanh^{-1} \left( \frac{\dot{R}}{\beta} \right) \right]_{\text{in}}^{\text{out}} - \int dt N E + \int \frac{dt}{2G} \left[ \frac{M R^{2}}{R - 2M} - \frac{M R \dot{R}}{R - 2 M} \right],
\]

(1)

and is a functional of \( R, M \) and \( N \) (lapse function of the three-metric on the shell); \( E = E(R, M) \) is the shell energy, \( \beta^{2} = (1 - 2m/R)^{2} + \dot{R}^{2} \) and \( [F]_{\text{in}}^{\text{out}} = F_{\text{in}} - F_{\text{out}} \) denotes the jump of the function \( F \) across the shell. The first two integrals in \[^{5}\] were known from an analogous derivation for non-radiating shells \[^{2}\] and the third integral properly accounts for the fact that a radiating shell defines an open (thermo)dynamical system\[^{2}\].

2.1. Equations of motion

The Euler-Lagrange equations of motion,

\[
\frac{\delta S_{s}}{\delta N} = -[H_{G} + E] = 0,
\]

(2)

\[
\frac{\delta S_{s}}{\delta R} = 0,
\]

(3)

\[^{2}\]This term becomes dynamically irrelevant when the radiation ceases \[^{3}\].

The meaning of the identity \[^{3}\] is that the total energy of the system is conserved and it can also be cast in the form of the first principle of thermodynamics \[^{3}\],

\[
dE = P dA + dQ,
\]

(7)

where \( \dot{Q} \sim \dot{M} \) is the luminosity. A second principle can also be introduced, at least in the quasi-static limit \( \dot{R}^{2} \ll 1 - 2M/R \), by defining an entropy \( S \) and a temperature \( T \) such that

\[
dS = \frac{dQ}{T},
\]

(8)

is an exact differential, which yields \[^{3}\]

\[
T = \frac{a}{8 \pi k_{B} \left( \hbar M \right)^{b}} \frac{1}{\sqrt{1 - 2M/R}} \equiv \frac{T_{a,b}}{\sqrt{1 - 2M/R}},
\]

(9)

where \( k_{B} \) is the Boltzmann constant, \( a \) and \( b \) are constants and \( T/T_{a,b} \) is the Tolman factor. We note in passing that \( T_{1,1} \) is the Hawking temperature of a black hole of mass \( M \[^{3}\]. \)
2.3. Microstructure

A microscopic description of the shell can be obtained by considering \( n \) close microshells of Compton wavelength \( \ell_\phi = \hbar / m_\phi \). One then finds that such a (many-body) system is gravitationally confined within a thickness \( \Delta \) around the mean radius \( R \) and can estimate \( \Delta \) from an Hartree-Fock approximation for the wavefunction of each microshell. This yields

\[
\left( \frac{\Delta}{R} \right)^{3/2} \sim Gr_\hbar m_\phi \frac{\ell_\phi}{R^2} \sim \frac{\ell_\phi R_G}{R^2} .
\] (10)

which, for \( R \geq R_G \), is negligibly small provided

\[
R \gg \ell_\phi ,
\] (11)

in agreement with the naive argument that the location of an object cannot be quantum mechanically defined with an accuracy higher than its Compton wavelength.

In the limit (11) one can second quantize the shell by introducing a (scalar) field \( \phi \) with support within a width \( \Delta \) around \( R \) and Compton wavelength \( \ell_\phi \) and obtains (neglecting terms of order \( \Delta \) and higher)

\[
E \simeq \frac{1}{2 \ell_\phi} \left[ \pi_\phi^2 + R^2 \dot{\phi}^2 \right] + H_{int}(\phi, M, \dot{M}, R) ,
\] (12)

where \( \pi_\phi \) is the momentum conjugated to \( \phi \) and \( H_{int} \) describes the local interaction between the matter in the shell and the emitted radiation. When \( H_{int} \neq 0 \), one expects that \( M = M(t) \) becomes a dynamical variable which cannot be freely fixed and the luminosity should then be determined by the corresponding Euler-Lagrange equation (1) from purely initial conditions for \( R \) and \( M \) (in any gauge \( N = N(t) \))

3. Semiclassical description

Lifting the time-reparametrization invariance of the shell to a quantum symmetry yields the Wheeler-DeWitt equation (10) corresponding to the classical Hamiltonian constraint (2). For the non-radiating case (\( \dot{M} = H_{int} = 0 \)) and in the proper time gauge (\( N = 1 \)) it is given by

\[
\left[ \hat{H}_G(P_R,R) + \hat{E}(P_\phi,\phi;R) \right] \Psi = 0 .
\] (13)

One can study (13) in the Born-Oppeheimer approach (11) by writing

\[
\Psi(R,\phi) = \psi(R) \chi(\phi;R) \simeq \psi_{WKB}(R) \chi(\phi;R) ,
\] (14)

where \( \psi_{WKB} \) is the semiclassical (WKB) wavefunction for the radius of the shell. This allows one to retrieve the semiclassical limit in which \( R \) is a collective (semi)classical variable driven by the expectation value of the scalar field Hamiltonian operator \( \hat{E} \) over the quantum state \( \chi \),

\[
H_G + \langle \hat{E} \rangle = 0 ,
\] (15)

while \( \chi \) evolves in time according to the Schrödinger equation

\[
i \hbar \frac{\partial \chi}{\partial t} = \hat{E} \chi .
\] (16)

In general, in order to obtain (13) and (14) from (13), one needs to assume that certain fluctuations (corresponding to quantum transitions between different trajectories of the collective variable \( R \)) be negligible (11). For the present case one can check \textit{a posteriori} that this is true if the condition (11) holds (that is, \( n \) is sufficiently big, see (10))

3.1. Particle production and backreaction

The equation (16) can be solved beyond the adiabatic approximation by making use of invariant operators (12). In particular, one can choose \( \chi \) as the state with initial (proper) energy \( E_0 = m_0 / G \) and radius \( R_0 \) and expand in the parameter of non-adiabaticity \( \delta^2 = \ell_\phi^2 M/R_G^3 \) to obtain (to first order in \( \delta^2 \))

\[
m = G \langle \hat{E} \rangle \simeq m_0 \left[ 1 + \frac{R_0^3}{R_G R^2} \delta^2 \right] ,
\] (17)

which is an increasing function for decreasing \( R \). This signals a (non-adiabatic) production of matter particles in the shell along the collapse. Since the total energy \( M/G \) is conserved, such a production can be viewed as a transfer of energy from
the collective degree of freedom $R$ to the microscopic degree of freedom $\phi$ and one therefore expects a slower approach towards the horizon.

In fact, the equation of motion (15) for $R$,

$$\dot{R}^2 = \frac{R_G}{2R} + 2 \left(1 - \frac{2m}{R_G}\right) - \frac{\ell_p^2m^2}{R_G^2 R^2} + \frac{\ell_p^2m^2}{R^2 R} \phi,$$

(18)

can be integrated numerically along with (17) to compute $m = m(t)$ and the corresponding back-reaction on the trajectory $R = R(t)$ confirms the above qualitative argument [8].

### 3.2. Gravitational fluctuations

One can also study the effects due to higher WKB order terms in the gravitational wavefunction by defining

$$\psi(R) = f(R) \psi_{WKB}(R),$$

(19)

where $f$ must then satisfy

$$i\hbar \frac{\partial f}{\partial t} \simeq \hbar \frac{\ell_p^2}{2R} \frac{\partial^2 f}{\partial R^2} \bigg|_{R_c},$$

(20)

in which $R_c = R_c(t)$ is the (semi)classical trajectory $\psi_{WKB}$ is peaked on and we have neglected terms of order $\dot{R}/R$ and higher.

Acceptable solutions [13] to (20) are given by plane waves with wave numbers $\lambda \geq \ell_p$ and negative “energy”

$$E_\lambda = -\frac{\hbar}{2R_c} \frac{\ell_p^2}{\lambda^2},$$

(21)

which agrees with the fact that the gravitational contribution to the total (super)Hamiltonian has the opposite sign with respect to matter.

Another basic feature of (21) is that $E_\lambda$ is proportional to $R_c^{-1}$, which makes it negligible with respect to $\langle \dot{E} \rangle$ for large radius, but one then expects appreciable corrections as the collapse proceeds. One can indeed take $E_\lambda$ into account in an improved semiclassical equation [13] for the trajectory of the shell

$$H_G + \langle \dot{E} \rangle + E_\lambda = 0,$$

(22)

which predicts a breakdown of the semiclassical approximation for values of $R$ larger than the limit (11) and possibly larger than $R_G$. This would imply that the whole shell becomes a quantum object far before reaching the (quantum mechanically unacceptable) singularity $R = 0$.

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