TANGENT CONES OF SCHUBERT VARIETIES FOR $A_n$ OF LOWER RANK

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In the paper, the tangent cones of Schubert varieties for series $A_n$ of rank less than or equal to four are calculated, and hypotheses on the structure of tangent cones in the general case are stated. Bibliography: 5 titles.

1. MAIN CONJECTURES

The calculation of tangent cones for Schubert varieties at the origin point is an interesting and extremely difficult problem. One of the reasons is that known methods of calculating the tangent cones are based on the determination of a Gröbner basis of the defining ideal of the Schubert variety (more precisely, of its affine part). Even in the smooth case (where the tangent cone coincides with the tangent space), an interesting and nontrivial theory (see [1]) arises.

In the present paper, we calculate the tangent cones for the series $A_n$, where $1 \leq n \leq 4$, and we state conjectures on the structure of tangent cones in the general case. The description of tangent cones is important for classification of coadjoint orbits of maximal unipotent subgroups (see [2]), since each tangent cone is a subset stable with respect to the coadjoint representation.

Let $G$ be a semisimple $K$-split algebraic group over a field $K$ of characteristic zero. The Lie algebra $\frak{g}$ of the group $G$ admits a decomposition $\frak{g} = \frak{n}_+ \oplus \frak{h} \oplus \frak{n}$, where $\frak{h}$ is the Cartan subalgebra, $\frak{n}$ (respectively, $\frak{n}_+$) is a maximal nilpotent subalgebra, spanned by the root vectors with positive (negative, respectively) roots. Denote $\frak{b} = \frak{h} \oplus \frak{n}$ and $\frak{b}_- = \frak{h} \oplus \frak{n}_-$. As usual, $H, N, N_-, B, B_-$ are the corresponding subgroups of $G$. We denote by $w$ an arbitrarily chosen representative of the element $w$ of the Weyl group $W = \text{Norm}(H)/H$. Using the Killing form, we identify $\frak{n}_-$ with the conjugate space $\frak{n}^*$ of $\frak{n}$.

The group $G$ decomposes into Bruhat classes $G = \bigcup_{w \in W} BwB$. This implies that the flag variety $X = G/B$ decomposes into Schubert cells $X = \bigcup_{w \in W} X^0_w$, where $X^0_w = BwB \bmod B, \ w \in W$. The closure $X_w$ of the Schubert cell $X^0_w$ is called a Schubert variety. Any Schubert variety contains an origin point $p = B \bmod B$.

Denote by $O$ an affine open subset $N_-B \bmod B$ in the flag variety $X$. The set $O$ admits a natural parametrization $\exp(x)B \bmod B$, where $x \in \frak{n}_-$ (for $\frak{g} = A_n$ in the sequel $(1 + x)B \bmod B, x \in \frak{n}_-$).

The subset $O_w = O \cap X_w$ is open in $X_w$ and closed in $O$. The origin point $p$ belongs to $O_w$ and has zero coordinates in the chosen parametrization. We denote by $C_w$ the tangent cone of $X_w$ at the point $p$ (more precisely, it is a tangent cone of $O_w$ at the zero point).

By definition, for a given closed subset $M \subset K^n$ that contains the point $(0, \ldots, 0)$, the tangent cone at the zero is an annihilator of the ideal of lowest terms $f_0$, where $f$ runs through the defining ideal $I = I(M)$ (see [3, Chap. II, §1] or [5, Chap. 9, §7]).

The tangent cone $C_w$ is contained in the tangent space $T_p(X)$ of the flag variety $X$ at the point $p$. Identify $T_p(X) = \frak{g}/\frak{b}$ with $\frak{n}^*$. Since the subgroup $B$ is a stabilizer of $p$ in the group $G$, the subgroup $B$ naturally acts in $T_p(X) = \frak{n}^*$. This action coincides with the coadjoint action of $B$ on $\frak{n}^*$. Any tangent cone $C_w$ is an Ad$^*$-invariant subset in $\frak{n}^*$, closed with respect to Zariski topology. First note that the tangent cones may coincide for different elements $w \in W$. For instance, the tangent cones of Coxeter elements coincide with $[n, n]^\perp$ (see [2]).

It is well known that $\dim X_w = 1(w)$. Since the dimension of an algebraic variety coincides with the dimension of its tangent cone (see Theorem 8 in [5, Chap. 9, §7]), we have $\dim C_w = 1(w)$.

The authors calculated the tangent cones $C_w$ for simple Lie algebras of series $A_n$. The first author developed a computer program that calculates the tangent cones. The algorithm of calculating is presented in the second section. This paper contains the results of calculations for $n \leq 4$, which are obtained by hand and by using a computer. The calculation of tangent cones for different examples provides some general conjectures for arbitrary semisimple Lie algebras.

**Conjecture 1.1.** If $C_{w_1} = C_{w_2}$, then the elements $w_1$ and $w_2$ are conjugate in the Weyl group. The converse statement is false.

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Conjecture 1.2. \( C_w = C_{w-} \).

Let \( g_0 \) be a semisimple regular subalgebra in \( g \) (a regular subalgebra is a subalgebra stable with respect to the adjoint action of the Cartan subgroup). The subalgebra \( g_0 \) admits a decomposition \( g = g_0- \oplus h_0 \oplus n_0 \), where \( h_0, n_0- \), and \( n_0 \) are subalgebras in \( g, n- \), and \( n \), respectively. Identify the conjugate space \( n'_0 \) with the subspace of \( n^* \) that consists of all linear forms that annihilate all root vectors in \( n \setminus n_0 \). The Weyl group \( W_0 \) of \( g_0 \) is a subgroup of \( W \). For any \( w \in W_0 \), one can define both the tangent cone \( C_{w,0} \) in \( n'_0 \) and the tangent cone \( C_w \) in \( n^* \); note that \( C_{w,0} \subset C_w \).

**Conjecture 1.3.** For any \( w \in W_0 \) the closure of \( \text{Ad}^* C_{w,0} \) is an irreducible component in \( C_w \).

**Theorem 1.4.** The above conjectures are true for all Lie algebras \( A_n \) for \( n \leq 4 \).

The proof follows from Tables 1–4.

2. Algorithm of Calculating the Tangent Cones and The Results

For any root \( \gamma \), we denote by \( e_\gamma \) a root vector, \( x_\gamma(t) = \exp(te_\gamma) \), \( N_\gamma = \{ x_\gamma(t) : t \in K \} \), \( N'_\gamma = \{ x_\gamma(t) : t \in K^* \} \).

**Remark.** If \( w = r_{\alpha_1} \cdots r_{\alpha_l} \) is a reduced decomposition, where \( \alpha \) is a simple root, then

\[
BwB = N_\alpha r_\alpha Bw'B \supset N'_{\alpha_1} r_\alpha Bw'B = N'_{\alpha_1} Bw'B
\]

(see [4, §3, formulas R1–R8 and Lemma 25]). The subset \( N'_{\alpha_1} Bw'B \) is dense in \( BwB \).

For any \( w \in W \) consider a reduced decomposition

\[
w = r_{\alpha_1} \cdots r_{\alpha_l},
\]

where \( l = l(w) \) and \( \alpha_1, \ldots, \alpha_l \) are simple roots.

According to the above remark, the subset

\[
N'_{\alpha_1} \cdots N'_{\alpha_l} B
\]

is dense in the Bruhat class \( BwB \).

Therefore the subset \( O_w \) is the closure of the image of the mapping \( F : K^l \rightarrow O \), where

\[
F(t_1, \ldots, t_l) = x_{-\alpha_1}(t_1) \cdots x_{-\alpha_l}(t_l) \bmod B.
\]

The elimination theory (see [5, §3]) provides a method of constructing a Gröbner basis of the defining ideal \( I_w \) of the subset \( O_w \). Further, using the standard procedure (see Proposition 4(i) of Chap. 9, §7 and Theorem 4 of Chap. 8, §4 in book [5]) one can construct a Gröbner basis of the ideal \( I_{w,0} \) the annihilator of which coincides with \( C_{w,0} \). Here is an example of calculation of a tangent cone.

**Example.** \( g = A_3, \ w = (13)(24) \). Identify \( O \) with

\[
N_+ = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ x_{31} & x_{21} & 1 & 0 \\ x_{41} & x_{42} & x_{43} & 1 \end{pmatrix} \right\}.
\]

To the reduced decomposition \( w = (23)(12)(34)(23) \) there corresponds a mapping \( F : K^4 \rightarrow O \), defined by the formulas

\[
x_{21} = t_2, \quad x_{31} = t_1 t_2, \quad x_{41} = 0, \quad x_{32} = t_1 + t_4, \quad x_{42} = t_3 t_4, \quad x_{43} = t_3.
\]

Eliminating \( t_1, t_2, t_3, \) and \( t_4 \), we find the generators \( x_{41}, x_{43} x_{21} + x_{42} x_{21} - x_{43} x_{32} x_{21} \) of \( I(O_w) \). We conclude that the tangent cone is determined by the system of equations \( x_{41} = 0, x_{43} x_{31} + x_{42} x_{21} = 0 \).

Below we present the results of calculations of the tangent cones for \( g = A_n, 1 \leq n \leq 4 \).

**Tangent cones for \( A_2 \).** The Weyl group coincides with \( S_3 \). The equations that define the tangent cones in

\[
n^* = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & 0 & 0 \\ x_{31} & x_{32} & 0 \end{pmatrix}
\]

are given in Table 1.

**Tangent cones for \( A_3 \).** The Weyl group coincides with \( S_4 \). Introduce the notation

\[
D = \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix}, \quad P = x_{43} x_{31} + x_{42} x_{21}.
\]

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### Table 1

| $w$       | $C(w)$          |
|-----------|-----------------|
| $(13)$    | $n^*$           |
| $(123),(132)$ | $x_{31} = 0$   |
| $(12)$    | $x_{31} = x_{32} = 0,$ |
| $(23)$    | $x_{31} = x_{21} = 0,$ |
| $e$       | $x_{31} = x_{32} = x_{21} = 0$ |

### Table 2

| $w$       | $C(w)$          |
|-----------|-----------------|
| $(14),(23)$ | $n^*$           |
| $(14)$    | $D = 0$         |
| $(1324),(1423)$ | $x_{41} = 0,$ |
| $(13),(24)$ | $x_{41} = 0,$ $P = 0$ |
| $(134),(143)$ | $x_{41} = x_{42} = 0$ |
| $(13)$    | $x_{41} = x_{42} = x_{43} = 0$ |
| $(124),(142)$ | $x_{41} = x_{31} = 0$ |
| $(24)$    | $x_{41} = x_{31} = x_{21} = 0$ |
| $(1234),(123),(1342),(1432)$ | $x_{41} = x_{31} = x_{42} = 0$ |
| $(234),(243)$ | $x_{41} = x_{31} = x_{21} = x_{42} = 0$ |
| $(12),(34)$ | $x_{41} = x_{31} = x_{42} = x_{32} = 0$ |
| $(123),(132)$ | $x_{41} = x_{31} = x_{42} = x_{43} = 0$ |
| $(12)$    | $x_{41} = x_{31} = x_{21} = x_{42} = x_{43} = 0$ |
| $(23)$    | $x_{41} = x_{31} = x_{21} = x_{42} = x_{43} = 0$ |
| $(34)$    | $x_{41} = x_{31} = x_{21} = x_{42} = x_{43} = 0$ |
| $e$       | $x_{41} = x_{31} = x_{21} = x_{42} = x_{43} = 0$ |

The equations that define the tangent cones in

$$n^* = \begin{pmatrix}
0 & 0 & 0 & 0 \\
x_{21} & 0 & 0 & 0 \\
x_{31} & x_{32} & 0 & 0 \\
x_{41} & x_{42} & x_{43} & 0
\end{pmatrix}$$

are given in Table 2.

**Tangent cones for $A_4$.** The Weyl group coincides with $S_5$. The equations that define the tangent cones in

$$n^* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
x_{21} & 0 & 0 & 0 & 0 \\
x_{31} & x_{32} & 0 & 0 & 0 \\
x_{41} & x_{42} & x_{43} & 0 & 0 \\
x_{51} & x_{52} & x_{53} & x_{54} & 0
\end{pmatrix}$$

are given in Tables 3 and 4.

### Table 3

| $w$       | $C(w)$          |
|-----------|-----------------|
| $(13425),(15243)$ | $x_{51} = x_{41} = 0$ |
| $(14235),(15324)$ | $x_{52} = x_{51} = 0$ |
| $(14325),(15234)$ | $x_{51} = 0,x_{41}x_{52} = 0$ |
| $(12345),(15432),(12453)$, $(12354),(13542),(15432)$, $(12543),(13452),(14532),(13542)$ | $x_{31} = x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = 0$ |
| (12345), (15342), (12534), (14352) | \( x_{31} = x_{41} = x_{51} = 0 = x_{52} = 0 \) |
| (13245), (15423), (14523), (13254) | \( x_{41} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (13524), (14253) | \( x_{41} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (1425), (1524) | \( x_{54} = 0 \) |
| (125), (152) | \( x_{51} = x_{41} = 0 = x_{53} x_{42} - x_{52} x_{43} = 0 \) |
| (145), (154) | \( x_{51} = x_{52} = 0 = x_{42} x_{31} - x_{41} x_{32} = 0 \) |
| (124), (142) | \( x_{51} = x_{41} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (134), (143) | \( x_{41} x_{42} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (235), (253) | \( x_{21} = x_{31} = x_{41} = x_{42} = 0 = x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (234), (234), (134), (142), (132) | \( x_{31} = x_{41} = x_{42} = 0 = x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (135), (153) | \( x_{41} = x_{42} = x_{51} = x_{52} = 0 \) |
| (125), (152) | \( x_{51} = x_{41} = 0 = x_{53} x_{42} - x_{52} x_{43} = 0 \) |
| (145), (154) | \( x_{51} = x_{52} = 0 = x_{42} x_{31} - x_{41} x_{32} = 0 \) |
| (124), (142) | \( x_{51} = x_{41} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (134), (143) | \( x_{41} x_{42} = x_{51} = x_{52} = 0 = x_{53} = 0 \) |
| (235), (253) | \( x_{21} = x_{31} = x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (123), (132) | \( x_{31} = x_{41} = x_{42} = 0 = x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (345), (354) | \( x_{21} = x_{31} = x_{32} = x_{41} = x_{42} = 0 = x_{51} = x_{52} = x_{53} = 0 \) |
| \( w \) | \( C(w) \) |
| (15), (24) | \( n^* \) |
| (15), (15), (23), (12) | \( x_{52} x_{41} - x_{51} x_{42} = 0 \) |

| Table 4 |

| (14)(25) | \( x_{51} = 0 = x_{54} x_{41} + x_{53} x_{31} + x_{52} x_{21} = 0 \) |
| (15)(23) | \( x_{52} x_{41} - x_{51} x_{42} = 0 = x_{41} x_{33} - x_{43} x_{31} = 0 = x_{42} x_{32} - x_{43} x_{31} = 0 \) |
| (15)(34) | \( x_{52} x_{41} - x_{51} x_{42} = 0 = x_{52} x_{31} - x_{51} x_{32} = 0 = x_{42} x_{31} - x_{41} x_{32} = 0 \) |
| (15) | \( x_{42} x_{31} - x_{41} x_{32} = 0 = x_{52} x_{41} - x_{51} x_{42} = 0 = x_{53} x_{42} - x_{52} x_{43} = 0 = x_{53} x_{31} - x_{51} x_{32} = 0 = x_{53} x_{41} - x_{51} x_{43} = 0 \) |
| (34)(125), (34)(152) | \( x_{51} = x_{41} = x_{31} = 0 \) |
| (23)(154), (23)(145) | \( x_{51} = x_{52} = x_{53} = 0 \) |
| (29)(134), (29)(143) | \( x_{51} = x_{41} = 0 = x_{54} x_{41} + x_{53} x_{31} + x_{52} x_{21} = 0 \) |
| (14)(23), (14)(235) | \( x_{51} = x_{32} = 0 = x_{54} x_{41} + x_{53} x_{31} + x_{52} x_{21} = 0 \) |
| (24)(135), (24)(153) | \( x_{41} = x_{51} = x_{52} = 0 \) |
| (13)(25) | \( x_{21} x_{42} + x_{43} x_{31} = 0 = x_{31} x_{33} + x_{21} x_{32} = 0 = x_{53} x_{42} - x_{52} x_{43} = 0 \) |
| (14)(35) | \( x_{32} x_{53} + x_{42} x_{54} = 0 = x_{54} x_{41} + x_{53} x_{31} = 0 = x_{43} x_{31} - x_{41} x_{32} = 0 \) |
| (14)(23) | \( x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (25)(34) | \( x_{41} = x_{31} = x_{41} = x_{51} = 0 \) |
| (13)(245), (13)(254) | \( x_{41} = x_{51} = x_{52} = x_{53} = 0 = x_{31} x_{43} + x_{21} x_{42} = 0 \) |
| (35)(124), (35)(142) | \( x_{31} x_{41} = x_{51} = x_{52} = 0 = x_{42} x_{54} + x_{32} x_{53} = 0 \) |
| (14) | \( x_{51} = x_{52} = x_{53} = x_{54} = 0 = x_{42} x_{31} - x_{41} x_{32} = 0 \) |
| (25) | \( x_{21} = x_{31} = x_{41} = x_{51} = 0 \) |
| (12)(35) | \( x_{31} = x_{32} = x_{41} = x_{42} = x_{51} = x_{52} = 0 \) |
| (13)(45) | \( x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \) |
| (13)(24) | \( x_{41} = x_{51} = x_{52} = x_{53} = x_{54} = 0 = x_{41} x_{43} + x_{21} x_{42} = 0 \) |
| (24)(35) | \( x_{21} = x_{51} = x_{41} = x_{51} = x_{52} = 0 = x_{42} x_{54} + x_{32} x_{53} = 0 \) |
\[
\begin{array}{ll}
(12)(345), (12)(354) & x_{31} = x_{32} = x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = 0 \\
(45)(132), (45)(123) & x_{31} = x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = 0 \\
(13) & x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(35) & x_{21} = x_{31} = x_{32} = x_{41} = x_{42} = x_{51} = x_{52} = 0 \\
(24) & x_{21} = x_{31} = x_{41} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(12)(45) & x_{31} = x_{32} = x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = 0 \\
(12)(34) & x_{21} = x_{31} = x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(23)(45) & x_{21} = x_{31} = x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(12) & x_{31} = x_{32} = x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(23) & x_{21} = x_{31} = x_{41} = x_{42} = x_{43} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(34) & x_{21} = x_{31} = x_{32} = x_{41} = x_{42} = x_{51} = x_{52} = x_{53} = x_{54} = 0 \\
(e) & 0
\end{array}
\]

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