Gravitational waves in geometric scalar gravity

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Abstract

We investigate the gravitational waves phenomena in the geometric scalar theory of gravity (GSG) that belongs to a class of theories such that gravity is described by a single scalar field. The associated physical metric describing the spacetime is constructed from a disformal transformation of Minkowski geometry. In this theory, a weak field approximation gives rise to a description similar to that one obtained in general relativity, although the gravitational waves in GSG have a characteristic longitudinal polarization mode, besides others modes that are observer dependent. We also analyze the energy carried by the gravitational waves as well as how their emission affects the orbital period of a binary system.
I. INTRODUCTION

Although general relativity (GR) has been a very successful gravitational theory during the last century, many proposals for modification of Einstein original formulation appeared in the literature over the past decades. Most of these ideas coming up within the cosmological scenario, where GR only works if unrecognized components, such as dark matter or dark energy, are introduced. To cite a few examples, we can quote the Horndeski theories, which add a scalar field to GR, and the so called $f(R)$, a modification of Einstein-Hilbert action for more general Lagrangians that includes fourth-order theories \[1, 2\].

Unlike those variations of GR, it was recently proposed a theory of gravity in the realm of purely scalar theories, introducing some crucial modifications from the previous attempts that took place before the emergence of GR \[3\]. It represents the gravitational field with a single scalar function $\Phi$, that obeys a non-linear dynamics\[1\]. Interaction with matter fields is given only though the minimal coupling to the physical metric $q_{\mu\nu}$, constructed from a disformal transformation of the Minkowski metric $\eta_{\mu\nu}$,

$$q_{\mu\nu} = A(\Phi, w) \eta_{\mu\nu} + B(\Phi, w) \partial_\mu \Phi \partial_\nu \Phi,$$

with,

$$w = \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi,$$

where we are using the short notation $\partial_\mu = \partial/\partial x^\mu$. A complete theory can only be set if one defines the functions $A$ and $B$, and also the Lagrangian of the scalar field. Then, a field equation, characterizing the theory, can be derived. We refer to this class of gravitational theories as geometric scalar gravity (GSG).

In the early communications on GSG, it was explored a specific set of those functions defining the theory, which shows that it is possible to go further in representing the gravitational field as a single scalar, giving realistic descriptions of the solar system and cosmology \[3, 4\]. In a more recent paper, it was done an analyses of GSG within the so called parametrized post-Newtonian description and, although the theory is not covered by the formalism, a limited situation indicate a good agreement with the observations \[5\].

\[1\] The non linearity of the field must be specifically in the kinetic term of the Lagrangian, namely $w = \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$. Thus, the Lagrangian density of the scalar field can be described as $L = F_1(\Phi, w)w + F_2(\Phi)$, with the condition that $F_1$ can not be a constant.
In the present work we develop the description and characterization of gravitational waves (GW) according to GSG.

The direct detection of GW by LIGO and Virgo collaborations initiated a new era of testing gravitational theories [6, 7]. It enables to construct constraints over a series of theoretical mechanisms associated with GW’s physics [8], but the crucial point relies on the observed waveform and how a theory can reproduce it. Notwithstanding, this is not the scope of this work. We are mainly focused here in analyzing the GW fundamentals on the perspective of GSG; studying their propagation, polarization modes, the energy carried by them and the orbital variation formula of a binary system that is predicted by the theory.

The paper is organized as follows. In section II is presented a brief overview of GSG in order to introduce to the reader the main features of this theory. The following section describes the theory’s weak field approximation and the study of the propagation of gravitational waves are made in section IV. The discussion on the polarization modes admitted by the theory is in section V and, the definition of the energy-momentum tensor of the waves are presented in section VI. The generation of waves, including the deduction of the orbital variation of binary system due to the emission of GW, are discussed in section VII. Also, two appendices was introduced in order to better describe the calculations appearing in the section VII.

II. OVERVIEW OF GEOMETRIC SCALAR GRAVITY

GSG is a class of gravitational theories which identifies the gravitational field to a single real scalar function $\Phi$, satisfying a non-linear dynamic described by the action,

$$S_\Phi = \frac{1}{\kappa c} \int \sqrt{-\eta} L(\Phi, w) d^4 x ,$$

where $c$ is the velocity of light in vacuum, $\kappa = 8\pi G/c^4$, $\eta$ is the determinant of the Minkowski metric and $w$ is defined in Eq. (2). The metric signature convention is $(+, -, -, -)$. The physical metric is constructed from the gravitational field according to the expression (1) and its contravariant form is obtained from the definition of the inverse, $q^{\mu\nu}q_{\alpha\nu} = \delta^\mu_\nu$, namely,

$$q^{\mu\nu} = a(\Phi, w) \eta^{\mu\nu} + b(\Phi, w) \eta^{\mu\alpha} \eta^{\nu\beta} \partial_\alpha \Phi \partial_\beta \Phi ,$$

where,

$$A = \frac{1}{a} \quad \text{and} \quad B = -\frac{b}{a(a + bw)} .$$

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In order to describe the interaction of the scalar gravitational field with matter, GSG makes the fundamental hypothesis, according to Einstein’s proposal, that gravity is a geometric phenomenon. Thus, it is assumed that the interaction with Φ is given only through a minimal coupling with the gravitational metric \( q_{\mu\nu} \). The matter action in GSG is then described as

\[
S_m = \frac{1}{c} \int \sqrt{-q} L_m d^4x. \tag{6}
\]

A complete theory should specify the metric’s functions \( A \) and \( B \) together with the Lagrangian of the scalar field \( L \), in order to be possible to derive its field equation. Up to now in the literature, it has been explored the case in which the following choice is made,

\[
a = e^{-2\Phi}, \tag{7}
\]

\[
b = \frac{(a-1)(a-9)}{4w}, \tag{8}
\]

\[
L = \frac{(a-3)^2}{4a^3} w = V(\Phi) w. \tag{9}
\]

Using the standard definition of the energy momentum tensor in terms of a metric structure, we set

\[
T_{\mu\nu} \equiv \frac{2}{\sqrt{-q}} \frac{\delta(\sqrt{-q} L_m)}{\delta q^{\mu\nu}}. \tag{10}
\]

Then, the dynamics of the scalar field is described by the equation

\[
\sqrt{V} \Box \Phi = \kappa \chi, \tag{11}
\]

where the \( \Box \) indicates the d’Alembertian operator constructed with the physical metric \( q_{\mu\nu} \), and the source term \( \chi \) is provided by

\[
\chi = -\frac{1}{2} \left[ T + \left(2 - \frac{V'}{2V}\right) E + C^\lambda_{\mu} \right], \tag{12}
\]

where \( V' = dV/d\Phi \), and

\[
T = T^\mu_\nu q_{\mu\nu}, \tag{13}
\]

\[
E = \frac{T^\mu_\nu \partial_\mu \Phi \partial_\nu \Phi}{a^3 V w}, \tag{14}
\]

\[
C^\lambda = \frac{b}{a^4 V} \left( T^{\lambda\mu} - E q^{\lambda\mu} \right) \partial_\mu \Phi. \tag{15}
\]
The choices made in (7)-(9) are such that the resulting theory satisfies the Newtonian limit, the classical gravitational tests (bending of light, perihelion of Mercury, etc), and the spherically symmetric vacuum solution is given by the Schwarzschild geometry. Moreover, in the absence of any matter fields, $\Phi$ is a free wave propagating in the metric $q_{\mu\nu}$ [9]. More details concerning the fundamentals of GSG and how this specific model can successfully describe the solar system physics can be found in [3, 5]. In the present work we will consider only this model. To work with different functions $a$, $b$ and $L$, all the process of constructing the field equation of the theory has to be redone, as well as it should be checked the feasibility of the new theory.

III. WEAK FIELD APPROXIMATION

Following the standard way to describe GW, we shall apply the weak field approximation to GSG. This procedure consider an observer who is ideally situated in a region far away from gravitational sources, such that the metrical structure in his surroundings is nothing but a small perturbation of flat spacetime. In GSG, the metric structure is constructed from the scalar field according to expressions (4), (7) and (8), so the weak field scenario is just a configuration where the gravitational scalar field $\Phi$ is a small quantity. In order to distinguish this approximation method to the original notation of GSG, we set

$$\Phi \approx \phi, \quad \text{with} \quad |\phi|^2 \ll |\phi|^3. \quad (16)$$

Keeping only first order terms in $\phi$, we expand the metric coefficients $a$ and $b$,

$$a \approx 1 - 2\phi, \quad b \approx 4\phi. \quad (17)$$

The gravitational metric takes the form

$$q^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (18)$$

where $h^{\mu\nu}$ is of the same order of $\phi$ and it is defined as follows,

$$h^{\mu\nu} = 2\phi \left( \eta^{\mu\nu} - 2 \eta^{\mu\alpha} \eta^{\rho\beta} \frac{\partial_{\alpha}\phi \partial_{\beta}\phi}{w^{(2)}} \right). \quad (19)$$

The subindex $X^{(N)}$ means that the quantity $X$ is of order $\phi^N$, so the $w^{(2)}$ is the second-order expansion of $w$,

$$w^{(2)} = \eta^{\alpha\beta} \partial_{\alpha}\phi \partial_{\beta}\phi. \quad (20)$$
Note that we are taking the ratio appearing in the second term of (19) as a zeroth-order quantity, i.e.
\[
\frac{\partial_\alpha \phi \partial_\beta \phi}{w(2)} \sim \mathcal{O}(1). \tag{21}
\]
The asymptotic limit, where \(\phi \to 0\), contains an indetermination when taken in general, because of the above term. It is necessary to verify that \(q_{\mu \nu}\) becomes asymptotically flat after solving the linear field equation.

The corresponding covariant expression for (18) is obtained from the definition \(q_{\mu \alpha}q^{\alpha \nu} = \delta^\mu_\nu\). It reads
\[
q_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, \tag{22}
\]
where,
\[
h_{\mu \nu} = \eta_{\mu \alpha} h^{\alpha \beta} \eta_{\nu \beta}, \tag{23}
\]
Equations (18) and (22) shows that the weak field limit in GSG follows similar lines as in GR, describing the gravitational metric as a small perturbation of flat spacetime. In this approximation the indices are lowered and raised by the Minkowski background metric.

A. The linear field equation

Once the metric structure in the weak field limit is established, we can use it to obtain the linear approximation of the GSG’s dynamical equation. Expanding the left hand side of Eq. (11), we get,
\[
\sqrt{V} \square \phi \approx (1 + 4\phi)(\eta^{\mu \nu} - h^{\mu \nu}) \left(\partial_\mu \partial_\nu \phi - \Gamma^\lambda_{\mu \nu} \partial_\lambda \phi\right), \tag{24}
\]
where \(\Gamma^\lambda_{\mu \nu}\) is the Levi-Civita connection associated with the gravitational metric. It can be separated in two parts as follows,
\[
\Gamma^\lambda_{\mu \nu} = \Gamma^\lambda_{M \mu \nu} + \Gamma^\lambda_{(1)\mu \nu}; \tag{25}
\]
where \(\Gamma^\lambda_{M \mu \nu}\) refers to the Minkowski connection and,
\[
\Gamma^\lambda_{(1)\mu \nu} = \frac{1}{2} \eta^{\lambda \alpha} \left(\partial_\nu h_{\alpha \mu} + \partial_\mu h_{\alpha \nu} - \partial_\alpha h_{\mu \nu}\right). \tag{26}
\]
Neglecting second orders terms, equation (24) reduces to
\[
\sqrt{V} \square \phi \approx \eta^{\mu \nu} \left(\partial_\mu \partial_\nu \phi - \Gamma^\lambda_{M \mu \nu} \partial_\lambda \phi\right) \equiv \square_M \phi, \tag{27}
\]
where we indicate the Minkowskian d’Alembertian operator as $\Box_M$.

For the right hand side of the dynamical equation, the corresponding terms yields the approximated values:

\[
\frac{V'}{2V} \approx 4 + 3\phi, \quad (28)
\]

\[
T \approx T^{\mu\nu} \eta_{\mu\nu} + T^{\mu\nu} h_{\mu\nu} \equiv T_L, \quad (29)
\]

\[
E \approx (1 - 2\phi) \frac{T^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{w_2} \equiv E_L, \quad (30)
\]

\[
C^\lambda \approx \frac{4\phi}{w_2} \left( T^\lambda{}^\mu - E_L \eta^\lambda{}^\mu \right) \partial_\mu \phi \equiv C^\lambda_L. \quad (31)
\]

Thus, using expressions (27)–(31) in Eq. (11), the linear dynamical equation of GSG takes the form

\[
\Box_M \phi = -\frac{\kappa}{2} \left[ T_L - (2 + 3\phi)E_L + \partial_\lambda C^\lambda_L \right]. \quad (32)
\]

This is the equation that we will examine, firstly for the vacuum case and then taking matter sources into account.

IV. PROPAGATION OF GRAVITATIONAL WAVES

Consider the linearized dynamics of GSG, Eq. (32), without the presence of sources,

\[
\Box_M \phi = 0. \quad (33)
\]

The scalar field has oscillatory solutions. Once the metric is constructed with the field and its first derivatives, such solutions yields oscillations as GW in the geometric structure of the spacetime.

The general solution of (33) can be written as a combination of plane null waves,

\[
\phi(t, \vec{x}) = \sum_n C_n e^{ik^{(n)}_\mu x^\mu}, \quad (34)
\]

where $C_n$ are constants and each one of the $k^{(n)}$ is a (constant) null vector, i.e. $k^{(n)}_\mu k^{(n)}_\nu \eta^{\mu\nu} = 0$.

It is worth to note that, from equation (19), the ideal case of monochromatic wave is out of
this description. This implies that at least two of the $C_n$’s must be different from zero. The simplest exact solution of (33) thus is given by,

$$\phi(t, \vec{x}) = C_1 e^{ik_{(1)}^\mu x^\mu} + C_2 e^{ik_{(2)}^\mu x^\mu},$$

(35)

with

$$\eta^{\mu\nu} k_{\mu}^{(1)} k_{\nu}^{(1)} = \eta^{\mu\nu} k_{\mu}^{(2)} k_{\nu}^{(2)} = 0 \quad \text{and} \quad \eta^{\mu\nu} k_{\mu}^{(1)} k_{\nu}^{(2)} \neq 0.$$

(36)

Consider now the Cartesian coordinate system $x^\mu = (t, x, y, z)$, where the Minkowski background metric assumes the form $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and set $\phi$ as a function of $t$ and $z$,

$$\phi = \phi(t, z).$$

(37)

The Eq. (33) then describes the gravitational field as a scalar wave propagating in the $z$ axis. For this $z$-oriented wave, the non vanishing components of the perturbed metric are [cf. Eq.(19)],

$$h_{00} = h_{zz} = -\frac{2\phi}{w(z)} \left[ (\partial_t \phi)^2 + (\partial_z \phi)^2 \right],$$

(38)

$$h_{0z} = -\frac{4\phi}{w(z)} \partial_t \phi \partial_z \phi,$$

(39)

$$h_{xx} = h_{yy} = -2\phi.$$

(40)

Using the solution (35), conditions (36) imply

$$k_{\mu}^{(1)} = k^{(1)}(1, 0, 0, 1), \quad k_{\mu}^{(2)} = k^{(2)}(1, 0, 0, -1),$$

(41)

where $k^{(1)}$ and $k^{(2)}$ are constants. To further simplify the solution we can consider $k^{(1)} = k^{(2)} = k$ and $C_1 = C_2 = C$. Thus, when the scalar field depends only on the coordinates $t$ and $z$, the simplest solution of the linearized dynamical equation in vacuum is a superposition of two plane null waves with opposite direction of propagation,

$$\phi(t, z) = C \left( e^{ik(t+z)} + e^{ik(t-z)} \right).$$

(42)
The metric components can be obtained following Eqs. (38)-(40),

\[ h_{00} = h_{zz} = -\phi - C (e^{ik(t-3z)} + e^{ik(t+3z)}) , \]
\[ h_{0z} = -C (e^{ik(t-3z)} + e^{ik(t-z)} - e^{ik(t+z)} - e^{ik(t+3z)}) , \]
\[ h_{xx} = h_{yy} = -2\phi . \]

A. Spherical waves

Although plane waves are simple to be analyzed, we have to look for spherical waves in order to get more realistic description of GW carrying energy from a gravitational bounded source to infinity. Consider the spherical coordinates \( x^\mu = (t, r, \theta, \varphi) \), where the Minkowski background metric assumes the form,

\[ ds^2_M = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) . \] (43)

Let us restrict the scalar field as function only of \( t \) and \( r \). Now, we can consider a monochromatic spherical wave as a solution of (33),

\[ \phi(t, r) = \frac{C}{r} e^{ik(t-r)} , \] (44)

where \(|C| \ll 1\) is the amplitude of the wave and \( k \) is the constant wavenumber. This solution gives \( w_2 \neq 0 \), and no divergence appears in the metric. In the wave zone we can neglect terms of order higher than \( 1/r \), and the unique non-null components of the perturbed metric are given by

\[ h_{tt} = h_{rr} = -h_{tr} = 2iCKe^{ik(t-r)} + \frac{C}{r} e^{ik(t-r)} , \] (45)
\[ h_{\theta\theta} = 2\phi \eta_{\theta\theta} , \quad h_{\varphi\varphi} = 2\phi \eta_{\varphi\varphi} . \] (46)

Note that the components \( h_{tt}, h_{rr} \) and \( h_{tr} \) are not asymptotically flat, but this is only a matter of the coordinate system. Consider an infinitesimal coordinate transformation,

\[ \tilde{x}^\mu = x^\mu + \xi^\mu . \] (47)

It is well known that the metric \( h_{\mu\nu} \) changes according to the relation

\[ \tilde{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu\gamma} \xi^\gamma - \eta_{\mu\gamma} \xi^\gamma \eta_{\nu\mu} - \eta_{\nu\gamma} \xi^\gamma \eta_{\mu\nu} . \] (48)
Then, consider the specific transformation where
\[ \xi^\mu = \phi r (1, 1, 0, 0). \] (49)

The scalar field remains the same, \( \tilde{\phi} = \phi \), since we are neglecting second order terms. The resulting metric is then given by
\[ \tilde{h}_{00} = \tilde{h}_{rr} = -\tilde{h}_{0r} = \phi, \] (50)
\[ \tilde{h}_{\theta\theta} = 2\phi \eta_{\theta\theta}, \quad \tilde{h}_{\varphi\varphi} = 2\phi \eta_{\varphi\varphi}, \] (51)
and the above expressions show a consistent asymptotic behavior: when \( r \to \infty \) (or \( \phi \to 0 \)) it results that \( g_{\mu\nu} \to \eta_{\mu\nu} \).

V. POLARIZATION STATES

The most general (weak) gravitational wave that any metric theory of gravity is able to predict can contain six modes of polarization. Considering plane null waves propagating in a given direction, these modes are related to tetrad components of the irreducible parts of the Riemann tensor, or the Newmann-Penrose quantities (NPQ): \( \Psi_2, \Psi_3, \Psi_4 \) and \( \Phi_{22} \) (\( \Psi_3 \) and \( \Psi_4 \) are complex quantities and each one represents two modes of polarization) [11]. The others NPQ are negligible by the weak field approximation, or are described in terms of these four ones.

The linearized dynamic equations of a gravitational theory can automatically vanish some of these NPQ, specifying then the number of polarization states predicted by it. For instance, in GR only \( \Psi_4 \) is not identically zero, which characterizes two transversal polarization modes, called “+” and “×” states.

In general, the six polarization modes can not be specified in a observer-independent way because of their behavior under Lorentz transformations. Nevertheless, if we restrict our attention to a set of specific observers, which agree with the GW on the frequency and on the direction of propagation, then is possible to make some observer-invariant statements about the NPQ. Such assertions are on the basis of the so called \( E(2) \)-classification of gravitational theories, introduced in ref. [12]:

- Class II_6: If \( \Psi_2 \neq 0 \), all the standard observers agree with the same nonzero \( \Psi_2 \) mode, but the presence or absence of the other modes is observer-dependent.
• **Class III** : If $\Psi_2 = 0$ and $\Psi_3 \neq 0$, all the standard observers measure the absence of $\Psi_2$ and the presence of $\Psi_3$, but the presence or absence of all other modes is observer dependent.

• **Class N** : If $\Psi_2 = \Psi_3 = 0$, $\Psi_4 \neq 0$ and $\Phi_{22} \neq 0$, this configuration is independent of observer.

• **Class N** : If $\Psi_2 = \Psi_3 = \Psi_2 = 0$ and $\Psi_4 \neq 0$, this configuration is independent of observer.

• **Class O** : If $\Psi_2 = \Psi_3 = \Psi_4 = 0$ and $\Phi_{22} \neq 0$, this configuration is independent of observer.

• **Class O** : If $\Psi_2 = \Psi_3 = \Psi_4 = \Phi_{22} = 0$, this configuration is independent of observer.

The $E(2)$-classification of GSG is easily obtained by noticing that the linearized field equation does not impose any restriction to the metric components $h_{\mu\nu}$. Consequently, the Ricci scalar is not null in general, which implies $\Psi_2 \neq 0$ (cf. equation (A4) of [12]) and GSG is from the class $II_6$. This $\Psi_2$ represents a pure longitudinal polarization state (see figure 1) that is always present in the GW, although others modes can be detected depending on the observer.

GSG belongs to the most general class of theories with respect to the $E(2)$-classification, where is always possible to find an observer that measures all six gravitational wave modes. The authors in [12] have already said that the number of polarization states predicted by a gravitational theory does not necessarily match with the numbers of degrees of freedom inside the theory. They also give an example of this with the so called stratified theories. Another example of such a discrepancy is the $f(R)$ theories, that is also classified as $II_6$, just like GSG [13, 15].

Thus, the description of GW by GSG carries a substantial distinction from GR, as it predicts the presence of a longitudinal mode of polarization. Up to now, the recent detections of GW can not exclude the existence of any one of the six modes of polarization [16]. But in the future, with the appropriate network of detectors, with different orientations, this information can be used to restrict gravitational theories.
VI. ENERGY OF THE GRAVITATIONAL WAVE

In order to associate an energy-momentum tensor to the gravitational waves in GSG we follow a standard procedure, like in GR, identifying the relation between the second and the first order gravitational field \( \Phi \approx \phi + \psi^{(2)} \), where \( \psi^{(2)} \) represents the second order terms of the gravitational field, and compute the second order vacuum field equation, namely

\[
\Box M \psi^{(2)} = \eta^{\mu\nu} \Gamma^\lambda_{(1)\mu\nu} \partial_\lambda \phi - h^{\mu\nu} \partial_\mu \partial_\nu \phi,
\]

Developing the above expression, using equations (19) and (26), it follows

\[
\Box M \psi^{(2)} = -4w^{(2)}.
\]  

The right hand side of this equation contains only the first order field \( \phi \), thus it can be interpreted as the source for the second order field generated by the linear waves.

From the general structure of the field equation of GSG, the influence of any energy-momentum tensor enters in the equation of motion uniquely through the quantity \( \chi \) [cf. equation (12)]. Thus, the energy-momentum tensor of the GWs must be consistent with Eq. (53), i.e.

\[
\chi^{(2)}(\Theta_{\mu\nu}) = -\frac{4}{\kappa} w^{(2)},
\]

where \( \chi^{(2)}(\Theta_{\mu\nu}) \) means the second order approximation of the source term calculated with the energy-momentum tensor of the gravitational field \( \Theta_{\mu\nu} \), instead of \( T_{\mu\nu} \). Therefore, we
write
\[ \Box_M \psi_{(2)} = \kappa \chi_{(2)}(\Theta_{\mu\nu}) . \] (55)

To describe the energy and momentum carried by the linear waves, the second-order approximation of \( \Theta_{\mu\nu} \) must be quadratic in the first derivatives of \( \phi \). This lead us to a specific form for it,
\[ \Theta_{(2)\mu\nu} = \frac{1}{\kappa} (\sigma \psi_{(2)} \eta_{\mu\nu} + \gamma \partial_\mu \phi \partial_\nu \phi) , \] (56)

with \( \sigma \) and \( \gamma \) being arbitrary constants. The condition (54) returns the relation
\[ 2\sigma - \gamma = 8 . \] (57)

Any tensor, described like in Eq. (56) and satisfying the above relation, can be used as the energy-momentum tensor of the linear GWs in GSG. This ambiguity already appeared in reference [10], where the authors show how to construct the energy-momentum tensor of the gravitational field in GSG, without using approximate methods. In that occasion, they fixed the functions defining the energy tensor by requiring that \( \Theta_{\mu\nu} \) can be derived from the Lagrangian.

However, as in the case of the metric’s functions \( a \) and \( b \), where the choice made was guided by the classical gravitational tests [3], we will look again to observational data as the second constraint necessary to entirely determine \( \Theta_{(2)\mu\nu} \). Actually, we will see in the next section that the examination of the orbital variation of a binary system can provide us a value for \( \gamma \), due to the data obtained from the observation of pulsars.

VII. ORBITAL VARIATION OF A BINARY SYSTEM

In order to obtain the energy rate emitted by a binary system we consider the influence of the source into the dynamics in the linear approximation. From the method of Green functions we immediately write down the general solution of Eq. (32),
\[ \phi(t, \vec{x}) = -\frac{\kappa}{8\pi} \iint \frac{H(t_r, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x' , \] (58)

where \( t_r = t - r/c \), with \( r = |\vec{x} - \vec{x}'| \), is the retarded time and we have defined the function
\[ H = T^{\mu\nu} \eta_{\mu\nu} + T^{\mu\nu} h_{\mu\nu} - (2 + 3\phi) E_L + \partial_\lambda C^\lambda_L . \] (59)
By considering that the source is far away from the point where we calculate the scalar field \((R \gg r')\), where \(R = |\vec{x}|\), and \(r' = |\vec{x}'|\) is the typical distances between the source’s components, it is possible to expand the denominator to obtain,

\[
\phi(t, \vec{x}) \approx -\frac{\kappa}{8\pi R} \int H(t, \vec{x}') \, d^3 x',
\]

(60)

where we have neglected terms of order \(1/R^2\). Further assuming that the typical velocities of the source components are non relativistic, it is also possible to expand the time dependent terms of the integrand in a Taylor series. For our purpose here it is sufficient to take only the first term of this expansion. The only modification in Eq. (60) will be in the argument of the integrand, changing to \((t_R, \vec{x}')\), with \(t_R = t - R/c\).

Before trying to solve the these integrals, let us note that \(\partial_\lambda C^\lambda_L\) can be split in a time derivative plus a total divergent, where the latter does not contribute to the solution (by the Gauss law). Moreover, from the conservation law, \(\partial_\mu T^{\mu\nu} = 0\), we derive the following relations:

\[
\frac{d}{dt} \int T^{00} \, d^3 x = 0,
\]

(61)

\[
\int T^{ij} \, d^3 x = \frac{1}{2c^2} \frac{d^2}{dt^2} \int T^{00} \, x^i x^j \, d^3 x.
\]

(62)

Quantities that are constant in time does not contribute to the radiation, as we will see. Thus we write,

\[
\phi(t, \vec{x}) = -\frac{G}{c^4 R} \left\{ -\frac{\ddot{I}}{2c^2} + \int \left[ T^{\mu\nu} h_{\mu\nu} - (2 + 3\phi) E_L + \partial_\nu C^\nu_L \right] \right\} \, d^3 x + C,
\]

(63)

where \(C\) accounts for time independent terms. The \(\dddot{I}\) is the second time derivative of the trace of the source’s second momenta of mass distribution,

\[
\dddot{I}^{ij} = \int T^{00} \, x^i x^j \, d^3 x.
\]

(64)

Since the integral in (63) contains the scalar field explicitly, we must expand these terms using the correspondent post-Newtonian approximation of the field in the near-zone region \([18]\). However, to keep \(\phi\) up to order \(G^2/c^4\), it is only necessary the Newtonian approximation of the near-zone scalar field \(\Phi_N\). By the viral theorem, we know that, for slow motions, \(v^2 \sim G\), where \(v\) is the typical velocity of the source’s components. Thus, \(\Phi_N \sim v^2/c^2\),
∂₀Φ_N ∼ v^3/c^3 and ∂ᵢΦ_N ∼ v^2/c^2 (see appendix A for more details). The energy-momentum tensor is also expanded in terms of the velocity, yielding

\[ T^{00} \approx T^{00}_0 + T^{00}_2, \]

\[ T^{0i} \approx T^{0i}_1, \]

\[ T^{ij} \approx T^{ij}_2, \]

where \( T^{\mu\nu} \sim v^N/c^N \). Keeping terms up to order \( v^2/c^2 \), we get

\[ T^{00}_1 \approx 2Φ_N T^{00}_0, \]

\[ (2 + 3φ)E_L \approx \frac{(0) T^{00}}{w_N} + 2 \frac{(1) T^{00}}{w_N} \frac{∂₀Φ_N}{w_N} + \frac{(2) T^{ij}}{w_N} \frac{∂ᵢΦ_N}{w_N}, \]

\[ C^0_L \approx \frac{4Φ_N}{w_N} \left( \frac{(0) T^{00}}{w_N} \frac{∂₀Φ_N}{w_N} + \frac{(1) T^{0i}}{w_N} \frac{∂ᵢΦ_N}{w_N} \right). \]

Specifying the source for the case of a binary system, we have

\[ T^{00} = \sum_n m_n c^2 \left( 1 + \frac{v_n^2}{2c^2} + Φ_N \right) \delta^3(\vec{x} - \vec{x}_n) + O(v^4/c^4), \]

\[ T^{0i} = \sum_n m_n c v_i^n \delta^3(\vec{x} - \vec{x}_n) + O(v^3/c^3), \]

\[ T^{ij} = \sum_n m_n v_i^n v_j^n \delta^3(\vec{x} - \vec{x}_n) + O(v^4/c^4), \]

where summation is over the two particles of the system, i.e. \( n = (1, 2) \). Using the above expressions the quadrupole term is directly calculated,

\[ I = \int T^{00}_0 r^2 d^3x = \sum_n m_n c^2 r_n^2 = \frac{c^2 r^2}{M} m_1 m_2. \]

We are adopting the usual center of mass notation such that the position \( \bar{r}_n \) of the particle \( n \) is described as,

\[ \bar{r}_1 = \frac{m_2}{M} \bar{r} \quad \text{and} \quad \bar{r}_2 = -\frac{m_1}{M} \bar{r}, \]

with \( \bar{r} = \bar{r}_1 - \bar{r}_2 \) and \( M = m_1 + m_2 \). Applying the second derivative in time, we obtain

\[ \frac{\ddot{r}}{c^2} = \frac{2m_1 m_2}{M} (\dot{r}^2 + r \ddot{r}). \]
Computing the remaining terms in eq. (63) is tedious but straightforward (see the details in the appendix B). The results are given by

$$\int \Phi_{N T}^{00} d^3 x = - 2G \frac{m_1 m_2}{r},$$

(77)

$$\int (2 + 3\phi) E_L d^3 x = - M \dot{r}^2,$$

(78)

$$\int \partial_0 C_L^0 d^3 x = 4M \left ( \dot{r}^2 + r \ddot{r} \right ),$$

(79)

Substituting the above integrals in eq. (63), it follows

$$\phi(t, \vec{x}) = \frac{G}{c^4 R} \left [ 4G \frac{m_1 m_2}{r} + \left ( \frac{m_1 m_2}{M} - 4M \right ) (\dot{r}^2 + r \ddot{r}) - 2M \dot{r}^2 \right ] + C.$$  

(80)

Once we are dealing with a binary system as the source of the gravitational field, we can use the Keplerian orbital parameters to simplify the above expression [19]. The distance between the two masses are,

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta},$$

(81)

where $a$ is the semimajor axis and $e$ is the eccentricity of the orbit. They are related with the total energy $E$ and the angular momentum $L$ by

$$a = - \frac{G m_1 m_2}{2E}, \quad e^2 = 1 + 2 \frac{EL^2}{G^2 (m_1 m_2)^3},$$

(82)

with $E < 0$. The fact that $L = (m_1 m_2/M)r^2 \dot{\theta}$, allow us to derive the following relation,

$$\dot{\theta} = \sqrt{\frac{GM}{a^3 (1 - e^2)^3}} (1 + e \cos \theta)^2.$$  

(83)

Then, from eqs. (81) and (83), we calculate the time derivatives of $r$ and use them in (80) to write

$$\phi = \frac{G^2 e}{c^4 a (1 - e^2) R} \left [ f_1(m_1, m_2) \cos \theta + f_2(m_1, m_2) e \sin^2 \theta \right ] + C,$$

(84)

where we have defined the functions,

$$f_1(m_1, m_2) = 5m_1 m_2 - 4M^2, \quad \text{and} \quad f_2(m_1, m_2) = - 2M^2.$$  

(85)

Now we calculate the energy-flux that has been carried off by the GW. Using the energy-momentum tensor given in eq. (56), the flux in the radial direction will be $c \Theta^{0r}$. Thus, the energy radiated per unit time, that is passing through a sphere of radius $R$, is given by

$$\frac{dE}{dt} = \gamma \frac{c^3 R^2}{2G} \dot{\phi}^2,$$

(86)
where we have used the fact that
\[ \partial_0 \phi = \frac{1}{c} \partial_{tR} \phi \quad \text{and} \quad \partial_i \phi = -\frac{x^i}{cR} \partial_{tR} \phi + O \left( \frac{1}{R^2} \right). \quad (87) \]

Noting that
\[ \dot{\phi} = \frac{G^2 e}{c^4 a(1 - e^2) R} \left[ -f_1(m_1, m_2) \sin \theta + 2 f_2(m_1, m_2) e \sin \theta \cos \theta \right] \dot{\theta}, \quad (88) \]
we write,
\[ \frac{dE}{dt} = \frac{\gamma G^3 e^2 \sin^2 \theta}{2c^5 a^2(1 - e^2)^2} \left( f_1^2 - 4 f_1 f_2 e \cos \theta + 4 f_2^2 e^2 \cos^2 \theta \right) \dot{\theta}^2. \quad (89) \]

Averaging the energy loss over an orbital period \( T \), where
\[ T = \frac{2\pi a^{3/2}}{\sqrt{GM}}, \quad (90) \]
we have,
\[ \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{T} \int_0^T \frac{dE}{dt} \, dt = \frac{1}{T} \int_0^{2\pi} \frac{dE}{dt} d\theta \]
\[ = \frac{\gamma M G^4 e^2}{4\pi c^3 a^5 (1 - e^2)^{7/2}} \int_0^{2\pi} \left( f_1^2 - 4 f_1 f_2 e \cos \theta + 4 f_2^2 e^2 \cos^2 \theta \right) (1 + e \cos \theta)^2 \sin^2 \theta d\theta. \]

The above integral is directly solved yielding,
\[ \left\langle \frac{dE}{dt} \right\rangle = \frac{\gamma M G^4}{4c^3 a^5} F(m_1, m_2, e), \quad (91) \]
where we have defined,
\[ F(m_1, m_2, e) = e^2(1 - e^2)^{-7/2} \left[ f_1^2 + \left( f_1^2 - 4f_1 f_2 + f_2^2 \right) e^2 + \frac{f_2^2}{2} e^4 \right]. \quad (92) \]

Equation \( (91) \) gives the energy radiated by a binary system due to the emission of GW. It is also possible to derive how this lost of energy changes the orbital period of the system. From \( (90) \), one gets that
\[ \frac{\dot{T}}{T} = \frac{3 \dot{a}}{2a} = \frac{3a}{G m_1 m_2} \left\langle \frac{dE}{dt} \right\rangle = \frac{3\gamma G^3 M}{4c^5 a^4 m_1 m_2} F(m_1, m_2, e). \quad (93) \]

Note that eq. \( (93) \) must be negative, otherwise it would imply that the masses are moving away from each other, in other words, they would be increasing their energy by the emission of GW, an unrealistic situation. The function \( F \) is positive, as it can be verified by comparison between the \( f_1^2 \) and the term inside the parenthesis multiplying \( e^2 \),
\[ f_1^2 + \left( f_1^2 - 2f_1 f_2 + f_2^2 \right) = 8 m_1^4 + \frac{77}{4} m_1^2 m_2^2 + 2 m_1^3 m_2 + 2 m_1 m_2^3 + 8 m_2^4 > 0, \quad (94) \]

Note that eq. \( (93) \) must be negative, otherwise it would imply that the masses are moving away from each other, in other words, they would be increasing their energy by the emission of GW, an unrealistic situation. The function \( F \) is positive, as it can be verified by comparison between the \( f_1^2 \) and the term inside the parenthesis multiplying \( e^2 \),
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where we have used the definitions made in eqs. (85). Since $e^2 < 1$ for elliptical orbits, it follows that $F$ is always positive. Thus to guarantee $\dot{T} < 0$, we must have $\gamma < 0$.

The Keplerian and post-Keplerian orbital parameters of a binary system can be extracted from the timing observation of a pulsar in a theory-independent way, but the determination of the masses of the pulsar and its companion is only obtained by making use of specific equations relating them to this set of parameters. These relations are particular for each gravitational theory [20]. Thus, a confrontation between the orbital variation of a binary system, as predicted by GSG, and the observational data is only possible after obtaining the so called post-Keplerian parametrization of the theory to extract the mass values according to GSG. We will leave this task to be addressed in a future work.

VIII. CONCLUDING REMARKS

We have presented a discussion on gravitational waves (GW) in the context of the geometric scalar gravity (GSG), a class of theories describing the effects of gravity as a consequence of a modification of spacetime metric in terms of a single scalar field. GSG overcomes the serious drawbacks present in all previous attempts to formulate a scalar theory of gravity. Its fundamental idea rests on the proposal that the geometrical structure of the spacetime is described by a disformal transformation of the flat Minkowski metric. The model analyzed here has already showed several advances within the scalar gravity program, featuring a good representation of the gravitational phenomena both in the solar system domains as well as in cosmology.

For what concerns the GW, we have shown that the weak field limit in GSG, similar as in GR, describes the geometry of the spacetime as a small perturbation of the Minkowski metric. An important distinction appears in the polarization states of the waves, which is characterized by the presence of a longitudinal mode in GSG. Within the $E(2)$-classification of gravitational theories, GSG is of the type $II_6^0$, since $\Psi_2 \neq 0$. This is the most general class, where the detection of all the other five polarization modes depends on the observer.

GSG has been constructed in order to be in agreement with observations that must be satisfied by any gravitational theory. The functions $a$ and $b$, defining the metric $q_{\mu\nu}$, do not

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2 A phenomenological parametrization for binary pulsars introduced by T. Damour [21], where the Keplerian and post-Keplerian parameters can be read off.
come from axiomatic principles, they were chosen mainly in order to satisfy the standard tests of gravity. A more fundamental formulation shall be pursued in the future.

Following such a phenomenological line of reasoning, we present an orbital variation formula for a binary system due to the emission of GW. It shows that is possible to use the data coming from the observed pulsars to provide the more appropriate energy-momentum tensor for the gravitational waves. The next task is to perform the post-Keplerian approximation of GSG in order to be able to determine the component masses of the binary system. This will complete the theory’s confrontation with the observations concerning the pulsars. We will come back to this question soon.

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Appendix A: The near zone scalar field

The dynamical equation (32) derived from the weak field limit is a traditional wave equation, which possesses some properties depending whether $R$ (the point where the field is being calculated) is larger or smaller than the typical wavelength $\lambda$ of the solution [22]. In the wave zone, where $R \gg \lambda$, the difference between the retarded time $t_R$ and $t$ is large, so the time derivative of the field is comparable to the spatial derivative. This is the region where the radiation effects are influential in determining the metric. On the other hand, in the region where $R \ll \lambda$, called near zone, the difference between the $t_R$ and $t$ are small and the time derivatives becomes irrelevant in front of the spatial derivatives.

The near zone region is covered by the post-Newtonian approximation of the gravitational field, expanding it in orders of $v/c$, where $v$ is the typical velocities of the source’s components, and considering also slow motion. This is the approximation required for the scalar field when integrating the wave equation. Once the scalar field appears multiplied by $T^{\mu\nu}$ in the integrand, we only need to know its near zone behavior up to order $v^2/c^2$, i.e. its Newtonian approximation. So, noticing that the constant $\kappa = 8\pi G/c^4$ is of
order \( v^2/c^2 \) (by virial theorem), eq. (32) reduces to

\[ \nabla^2 \Phi_N = \frac{4\pi G(0) T_{00}}{c^4}. \]  

(A1)

The solution of this equation is the Newtonian potential and, for the specific case of binary system as a source,

\[ \Phi_N(t, \vec{x}) = -\frac{G}{c^2} \sum_n \frac{m_n}{|\vec{x} - \vec{x}_n|}. \]  

(A2)

When calculating the Newtonian gravitational potential in the position of one of the particles we have to neglect the infinity self potential, thus

\[ \Phi_N(t, \vec{x}_n) = -\frac{G}{c^2} \sum_{p \neq n} \frac{m_n}{|\vec{x}_n - \vec{x}_p|}, \]  

(A3)

where the summation above is taken excluding terms when \( p = n \). This can be interpreted as a mass renormalization \[19\]. To finish, from \[19\] and \(22\), one can easily sees that the metric assumes the form

\[ g_{00} = 1 + 2\Phi_N + \mathcal{O}\left(\frac{v^4}{c^4}\right), \]  

(A4)

since the time derivatives of \( \Phi_N \) are of higher orders.

**Appendix B: More detailed calculation**

In this section we aim to be more clear on the calculation of expressions (77) to (79). The first one is easily obtained from expressions (71) and (A3),

\[ \int \Phi_N^{(0)} T_{00} \, d^3x = -G \sum_n \sum_{p \neq n} \frac{m_n m_p}{r_{np}} = -2G \frac{m_1 m_2}{r}, \]  

(B1)

where we are using the short notation \( \vec{r}_{np} = \vec{x}_n - \vec{x}_p \), and \( r_{np} = |\vec{r}_{np}| \). The sub-indexes \((p, q, n)\) indicates each one of the particles, assuming the values 1 or 2.

To solve the integral in (78) we need the derivatives of the Newtonian scalar field, namely

\[ \partial_t \Phi_N(t, \vec{x}_n) = -\frac{G}{c^2} \sum_{p \neq n} m_p \frac{(\vec{r}_{np} \cdot \vec{v}_p)}{r_{np}^3}, \]  

(B2)

\[ \partial_i \Phi_N(t, \vec{x}_n) = \frac{G}{c^2} \sum_{p \neq n} m_p \frac{(x^i_n - x^i_p) \delta_{ji}}{r_{np}^3}, \]  

(B3)
with \( \vec{v}_p = \vec{r}_p' \). The upper-indexes \((i,j,k)\) refer to the usual components of a three-vector and they run from 1 to 3. Also, the kinect term reads

\[
w_N(t, \vec{x}_n) = -\frac{G^2}{c^4} \sum_{p,q \neq n} m_pm_q \frac{(\vec{r}_np \cdot \vec{r}_nq)}{r_{np}^3 r_{nq}^3}.
\] (B4)

Then, from (69) one has

\[
\int (2 + 3\phi) E^L dt \approx \sum_n m_n \left[ (\partial_t \Phi_N)^2 \right] w_N + \frac{2 \partial_t \Phi_N \partial_i \Phi_N v^i_n}{w_N} + \frac{\partial_i \Phi_N \partial_j \Phi_N v^i_n v^j_n}{w_N} \bigg|_{x = x_n}.
\] (B5)

Let us calculate one of this sums explicitly,

\[
\sum_n m_n \frac{(\partial_t \Phi_N)^2}{w_N} = \sum_n \sum_{p,q \neq n} \frac{m_pm_q}{w_N(t, \vec{x}_n)} \frac{(\vec{r}_np \cdot \vec{v}_p)}{r_{np}^3} \frac{(\vec{r}_nq \cdot \vec{v}_q)}{r_{nq}^3},
\] (B6)

where the symbol \( \Sigma_{p,q \neq n} \) means the product of two summations, one in \( p \) and other in \( q \), with both never assuming the value of \( n \). Using that

\[
w_N(t, \vec{x}_1) = -\frac{m_2^2}{r_1^4}, \quad w_N(t, \vec{x}_2) = -\frac{m_1^2}{r_4^4} \quad \text{and} \quad \vec{r} \cdot \vec{v}_n = r \dot{r}_n, \] (B7)

we have

\[
\sum_n m_n \frac{(\partial_t \Phi_N)^2}{w_N} = -m_1 \dot{r}_1^2 - m_2 \dot{r}_1^2 = -\frac{\dot{r}^2}{M^2} \left( m_1^3 + m_2^3 \right).
\] (B8)

In the last equality we use the relations (75). For the remaining terms in (B5), paying attention to \( \vec{r}_{21} = -\vec{r} \), it follows

\[
\sum_n m_n \frac{\partial_i \Phi_N \partial_j \Phi_N v^i_n v^j_n}{w_N} = \sum_n \sum_{p,q \neq n} \frac{m_pm_q}{w_N(t, \vec{x}_n)} \frac{(\vec{r}_np \cdot \vec{v}_n)}{r_{np}^3} \frac{(\vec{r}_nq \cdot \vec{v}_n)}{r_{nq}^3} = -\frac{\dot{r}^2}{M} m_1 m_2,
\] (B9)

and,

\[
\sum_n m_n \frac{\partial_i \Phi_N \partial_j \Phi_N v^i_n v^j_n}{w_N} = \sum_n \sum_{p,q \neq n} \frac{m_pm_q}{w_N(t, \vec{x}_n)} \frac{(\vec{r}_np \cdot \vec{v}_n)}{r_{np}^3} \frac{(\vec{r}_nq \cdot \vec{v}_n)}{r_{nq}^3} = -\frac{\dot{r}^2}{M} m_1 m_2.
\] (B10)

Putting all these terms together, following (B5), we finally obtain the relation (78).
To compute eq. (79), we first note that the time derivative can be put outside the integral. Then,

\[
\int C^0_L \, d^3x = 4 \sum_n m_n \left[ \frac{\Phi_N}{w_N} \left( \partial_t \Phi_N + \partial_i \Phi_N v^i_n \right) \right]_{x=x_n}
\]

\[= 4 \sum_n \sum_{p,q \neq n} \frac{m_n m_p m_q}{w_N(t, \vec{x}_n)} \left[ \frac{(\vec{r}_{nq} \cdot \vec{v}_{q})}{r_{np} r_{nq}^3} - \frac{(\vec{r}_{nq} \cdot \vec{v}_{n})}{r_{np} r_{nq}^3} \right] \] (B12)

\[= 4Mcr \dot{r} \] (B13)

Deriving in time we get expression (79).

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