PROPERTIES OF ANALOGUES OF FROBENIUS POWERS OF IDEALS

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Abstract. Let \( R = \mathbb{K}[X_1, \ldots, X_n] \) be a polynomial ring over a field \( \mathbb{K} \). We introduce an endomorphism \( \mathcal{F}^{[m]} : R \rightarrow R \) and denote the image of an ideal \( I \) of \( R \) via this endomorphism as \( I^{[m]} \) and call it to be the \( m \)-th square power of \( I \). In this article we study some homological invariants of \( I^{[m]} \) such as regularity, projective dimension, associated primes and depth for some families of ideals e.g. monomial ideals. We also relate the asymptotic constants e.g. Waldschmidt constant and resurgence number, of a monomial ideal \( I \) with that of \( I^{[m]} \).

1. Introduction

Let \( R \) be a commutative Noetherian ring with unity. When \( \text{char}(R) \) is a prime number \( p \), then for any positive integer \( e \), one can consider Frobenius endomorphism \( F^e : R \rightarrow R \) which is defined as \( F^e(r) = r^{p^e} \) for \( r \in R \). In this way, \( R \) can be viewed as an \( R \)-module via \( F^e \), and this module is denoted by \( R^e \). For an ideal \( I \subset R \) and \( e \geq 1 \), \( I^{[p^e]} \) denotes the ideal generated by the \( p^e \)-th powers of elements of \( I \) and it is called \( p^e \)-th Frobenius power of \( I \). From now, we fix \( R = \mathbb{K}[X_1, \ldots, X_n] \) to be a polynomial ring over a field \( \mathbb{K} \). Whenever we say \( I \subset R \) is a homogeneous ideal, then we mean that \( R \) is a standard graded polynomial ring and \( I \) is an ideal generated by homogeneous elements in \( R \). When \( \text{char}(\mathbb{K}) = p \), Katzman [Kat98] conjectured that if \( I \subset R \) and \( J \subset R \) are homogeneous ideals, then \( \text{reg}(R/(J + I^{[p^e]})) \) grow linearly in \( p^e \). Katzman [Kat98, Corollary 23] proved this conjecture for certain classes of ideals (e.g., \( I \) a homogeneous ideal and \( J \) a monomial ideal). The behavior of \( \text{reg}(R/I^{[p^e]}) \) has been of great interest throughout the last few decades, see [Cha07, Bre05, DSNnBP18]. Moreover, the growth of \( \text{reg}(R/I^{[p^e]}) \) is of independent interest as this is connected to the localization of tight closure [Kat98]. In this article, we give a precise formula for \( \text{reg}(R/I^{[p^e]}) \) in terms of \( \text{reg}(R/I) \) for any homogeneous ideal \( I \subset R \). We study this problem in a more general setup as follows.

Let \( R = \mathbb{K}[X_1, \ldots, X_n] \) be a polynomial ring over a field \( \mathbb{K} \) (need not be of prime characteristic). For any positive integer \( m \), we define an endomorphism \( \mathcal{F}^{[m]} : R \rightarrow R \) as \( \mathcal{F}^{[m]}(f) = f^{[m]} \) for \( f \in R \), where \( f^{[m]} = f(X_1^m, \ldots, X_n^m) \). In this way, \( R \) can be viewed as an \( R \)-module via \( \mathcal{F}^{[m]} \). Let \( I \subset R \) be an ideal (need not be homogeneous). We define \( I^{[m]} := \langle f^{[m]} : f \in I \rangle \), and call it to be the \( m \)-th square power of \( I \). If \( I \) is a monomial ideal or \( \mathbb{K} \) is the finite field of order \( p \), then the \( p^e \)-th Frobenius power of \( I \) is same as the \( p^e \)-th

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square power of $I$. Whenever we write $I^{[p^e]}$, we mean that $\text{char}(\mathbb{K}) = p$ is prime and $I^{[p^e]}$ is the $p^e$-th Frobenius power of $I$, otherwise the base field is of arbitrary characteristic and $I^{[m]}$ is the $m$-th square power of $I$. We prove that $\text{reg}(R/I^{[m]})$ is a linear function in $m$ by obtaining a precise formula for $\text{reg}(R/I^{[m]})$ in terms of $\text{reg}(R/I)$ (Corollary 2.12). The proof relies on the relation between the graded Betti numbers of $R/I^{[m]}$ and $R/I$ (Corollary 2.7), and the relation between their extremal Betti numbers (Proposition 2.10). We also prove analogous statements for Frobenius powers of homogeneous ideals (Theorem 2.9, Proposition 2.11).

We also study the behavior of $\text{pd}(R/I^{[m]})$ and $\text{Ass}(I^{[m]})$ for $m \geq 2$. The following well-known lemma relates projective dimension and the sets of associated primes of $R/I$ and $R/I^{[p^e]}$.

**Lemma 1.1** (Peskine-Szpiro). [HH02, Lemma 2.2] Let $I \subset R$ be an ideal. Then,

(a) $\text{pd}(R/I^{[p^e]}) = \text{pd}(R/I)$ for all $e \geq 1$.
(b) $\text{Ass}(R/I^{[p^e]}) = \text{Ass}(R/I)$ for all $e \geq 1$.

We prove the analogue of statement (a) (Corollary 2.8) for projective dimension of square powers. The analogous statement for associated primes does not hold in general, see Example 3.3. For monomial ideals, we prove $\text{Ass}(R/I^{[m]}) = \text{Ass}(R/I)$ for all $m \geq 1$ (Theorem 3.2).

The study of regularity of powers of homogeneous ideals has been a central problem in commutative algebra and algebraic geometry. Cutkosky, Herzog and Trung [CHT99], and independently Kodiyalam [Kod00] proved that if $I$ is a homogeneous ideal of $R$, then $\text{reg}(R/I^s)$ is a linear function in $s$, i.e., there exist non-negative integers $a$ and $b$ depending on $I$ such that $\text{reg}(R/I^s) = as + b$ for all $s \gg 0$. Using square powers of homogeneous ideal $I$, we prove that there exist $1 \leq i \leq n$ such that $\beta_{i,i+\text{reg}(R/I^s)}(R/I^s)$ is an extremal Betti number for $s \gg 0$ (Theorem 2.15).

We then move on to study square powers of monomial ideals. First, we compute the primary decomposition of square powers of monomial ideals (Theorem 3.2). Then, we study normally torsion-freeness of square powers of monomial ideals. We prove that for a monomial ideal $I$, $I^{[m]}$ is normally torsion-free if and only if $I$ is normally torsion-free (Theorem 3.5). Gitler, Valencia and Villarreal [GVV07] classified normally torsion-free quadratic square-free monomial ideals. Given a normally torsion-free monomial ideal, one can construct a family of normally torsion-free monomial ideals using Theorem 3.5. Brodmann [Bro79] proved that the sequence $\{\text{Ass}(R/I^s)\}_{s \geq 1}$ of associated prime ideals is stationary for $s \gg 0$. The least positive integer $s_0$ such that $\text{Ass}(R/I^s) = \text{Ass}(R/I^{s_0})$ for all $s \geq s_0$ is called the index of stability of $I$. We prove that the index of stability of monomial ideal $I$ and $I^{[m]}$ are the same (Theorem 3.6).

Next, we study the depth of symbolic powers of monomial ideals. Nguyen and Trung proved that $\text{depth}(R/I^{(s)})$ is periodic for $s \gg 0$, [NT19, Proposition 3.1]. Moreover, they
proved that for an extensive class of monomial ideals (including all square-free monomial ideals) \( \text{depth}(R/I^{(s)}) \) is always a convergent function. They proved:

**Theorem 1.2.** [NT19 Theorem 3.1] Let \( I \) be a monomial ideal in \( R \) such that \( I^{(s)} \) is integrally closed for \( s \gg 0 \). Then, \( \text{depth}(R/I^{(s)}) \) is a convergent function with

\[
\lim_{s \to \infty} \text{depth}(R/I^{(s)}) = \dim(R) - \dim(F_s(I))
\]

which is also the minimum of \( \text{depth}(R/I^{(s)}) \) among all integrally closed symbolic powers \( I^{(s)} \).

In Lemma 3.7 we prove that \( (I^{[m]})^{(s)} \) is not integrally closed for \( m \geq 2, s \geq 2 \). If \( I \) is a monomial ideal such that \( I^{(s)} \) is integrally closed for \( s \gg 0 \), then for all \( m \geq 2 \), the sequence \( \{\text{depth}(R/(I^{[m]})^{(s)})\}_{s \geq 1} \) is a convergent sequence (Theorem 3.9) and

\[
\lim_{s \to \infty} \text{depth}(R/(I^{[m]})^{(s)}) = \dim(R) - \dim(F_s(I^{[m]})).
\]

During the last decade, there has been a lot of interest in the “containment problem,” i.e., given a homogeneous ideal of \( I \) of \( R \), determine all pairs \((s, t)\) of positive integer such that \( I^{(s)} \subseteq I^t \). This problem was motivated by the fundamental results of the [ELS01] and [HH02], which shows that the ideal containment problem holds whenever \( s \geq t(n - 1) \). To capture more precise information about the containment, Bocci and Harbourne [BH10a] introduced the resurgence number of \( I \), denoted by \( \rho(I) \). Bounds on \( \rho(I) \) are much sought after and one such bound is the Waldschmidt constant, denoted by \( \tilde{\alpha}(I) \), which appears in many areas of mathematics e.g., number theory ([Wal76], [Wal79]), complex analysis ([Sko77]), algebraic geometry ([BH10a], [BH10b]) and commutative algebra ([HH13]). We relate the Waldschmidt constant and resurgence number of the monomial ideal \( I \) with that of \( I^{[m]} \) (Theorem 3.11).

### 2. Square Powers of an Ideal

Throughout the article, we fix \( \mathbb{K} \) to be any field, and \( R = \mathbb{K}[X_1, \ldots, X_n] \) to be a polynomial ring. We assume that \( 0 \in \mathbb{N} \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), set \( X^\alpha = \prod_{i=1}^n X_i^{\alpha_i} \). Set \( \tilde{R} = \mathbb{K}[X_1^m, \ldots, X_n^m] \) for any positive integer \( m \).

**Definition 2.1.** Let \( f \in R \) be an element. We set \( f^{[m]} = f(X_1^m, \ldots, X_n^m) \). Let \( I \) be an ideal in \( R \). Then, for any positive integer \( m \), we define \( I^{[m]} := \langle f^{[m]} : f \in I \rangle \). We call \( I^{[m]} \) to be the \( m \)-th square power of \( I \).

The following remark forms the basis for the proofs of this article.

**Remark 2.2.** The square power operation preserves products and sums, i.e., \((f + g)^{[m]} = (f)^{[m]} + (g)^{[m]} \) and \((fg)^{[m]} = (f)^{[m]}(g)^{[m]} \) and hence defines a ring isomorphism from \( R \) to \( \tilde{R} \). Thus, \( I^{[m]} \) is the extension of the ideal \( (F^{[m]}(I)) \).

**Remark 2.3.** If \( \mathbb{K} \) is the finite field of order \( p \), and \( I \) is a homogeneous ideal in \( R \), then \( p^e \)-th square power of \( I \) coincides with the \( p^e \)-th Frobenius power of \( I \).
We first prove some results that are needed for our next main theorem.

**Lemma 2.4.** Let $I$ be an ideal of $R$ and $f \in R$. Then, $f \in I$ if and only if $f^{[m]} \in I^{[m]}$.

**Proof.** Let $I$ be generated by polynomials $f_1, \ldots, f_k$. We claim that $I^{[m]} = \langle f_1^{[m]}, \ldots, f_k^{[m]} \rangle$. Clearly, $\langle f_1^{[m]}, \ldots, f_k^{[m]} \rangle \subset I^{[m]}$. Now, let $g \in I^{[m]}$ be any element. Then, $g = \sum_{i=1}^{r} h_i g_i^{[m]}$ for some $h_i \in R$ and $g_i \in I$. Now, for each $i$, $g_i = \sum_{j=1}^{k} h_{ij} f_j$ for some $h_{ij} \in R$. Therefore, $g_i^{[m]} = \sum_{j=1}^{k} h_{ij}^{[m]} f_j^{[m]}$. Thus, $g = \sum_{i=1}^{r} \sum_{j=1}^{k} h_i h_{ij}^{[m]} f_j^{[m]} \in \langle f_1^{[m]}, \ldots, f_k^{[m]} \rangle$. Hence, $I^{[m]} = \langle f_1^{[m]}, \ldots, f_k^{[m]} \rangle$.

Let $f \in R$ be such that $f^{[m]} \in I^{[m]}$. Then, $f^{[m]} \in I^{[m]} \cap \tilde{R}$, and hence, $f^{[m]} = \sum_{i=1}^{k} h_i^{[m]} f_i^{[m]} = (\sum_{i=1}^{k} h_i f_i)^{[m]}$ for some $h_i \in R$. Therefore, $f = \sum_{i=1}^{k} h_i f_i \in I$. Hence, the assertion follows. □

**Lemma 2.5.** If $f_1, \ldots, f_r \in R$ is an $R$-regular sequence, then, for any $m \geq 2$, $f_1^{[m]}, \ldots, f_r^{[m]}$ is an $R$-regular sequence.

**Proof.** Let $m \geq 2$. Observe that $\{X^a : a \in \mathbb{N}^n, a_i < m\}$ is a free basis of $\tilde{R}$-module $R$. By [BH93, Proposition 1.6.7], the Koszul homology of $f_1^{[m]}, \ldots, f_r^{[m]}$ is $H_{\bullet}(f_1^{[m]}, \ldots, f_r^{[m]}; R) = H_{\bullet}(f_1^{[m]}, \ldots, f_r^{[m]}; R) \otimes_{\tilde{R}} R$. We define $\Phi : R \to \tilde{R}$ as $\Phi(X_i) = X_i^{m}$. Observe that $\Phi$ is a ring isomorphism. Thus, $f_1^{[m]}, \ldots, f_r^{[m]}$ is an $\tilde{R}$-regular sequence. Now, by [BH93, Corollary 1.6.14], $f_1^{[m]}, \ldots, f_r^{[m]}$ is an $R$-regular sequence. □

Let $I \subset R$ be a homogeneous ideal. We aim to study algebraic invariants associated with the minimal free resolution of $R/I^{[m]}$. First, we recall some facts about the minimal free resolution.

Let

$$(\mathcal{F}_{\bullet}, \phi_{\bullet}) : 0 \to \mathcal{F}_p \xrightarrow{\phi_p} \mathcal{F}_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_2} \mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_0 \to R/I \to 0$$

be the minimal graded free resolution of $R/I$, where $\mathcal{F}_i = \bigoplus_k R(-k)^{\beta_{i,k}(R/I)}$ for $i \geq 1$, $\mathcal{F}_0 = R$ and the matrix $\phi_1$ is given by minimal homogeneous set of generators. Here, $p$ is the projective dimension of $R/I$, and it is denoted by $pd(R/I)$. Also, $R(-k)$ is the free $R$-module of rank 1 generated in degree $k$. The number $\beta_{i,k}(R/I)$ is called the $(i,k)$-th graded Betti number of $R/I$, and this is uniquely determined by $I$, i.e., $\beta_{i,k}(R/I) = \dim_k \text{Tor}_i^R(\mathbb{K}, R/I)_k$, for $i \geq 0$, and $k \in \mathbb{Z}$. For more details on the graded minimal free resolution, we refer the readers to [BH93] and [Pee11].

We now compute the graded minimal free resolution of $R/I^{[m]}$ in terms of graded minimal free resolution of $R/I$.

**Theorem 2.6.** Let $I \subset R$ be a homogeneous ideal and $m \geq 2$. If $(\mathcal{F}_{\bullet}, \phi_{\bullet})$ is the graded minimal free resolution of $R/I$, then $(\mathcal{F}_{\bullet}^{[m]}, \phi_{\bullet}^{[m]})$ is the graded minimal free resolution of $R/I^{[m]}$, where for all $i \geq 1$,

$$\mathcal{F}_i^{[m]} = \bigoplus_k R(-mk)^{\beta_{i,k}(R/I)}.$$
and the matrix $\phi_i^{[m]}$ is given by

$$(\phi_i^{[m]})_{k,l} = ((\phi_i)_{k,l})^{[m]}.$$  

**Proof.** Let $(\mathfrak{F}_*, \phi_*)$ be the graded minimal free resolution of $R/I$. By Remark 2.2 for each $i \geq 1$, $\phi_i^{[m]} \circ \phi_i^{[m]} = (\phi_i \circ \phi_i)^{[m]} = 0$. Now, we prove that $(\mathfrak{F}_*, \phi_*)$ is a free resolution of $R/I^{[m]}$. For that we use Buchsbaum-Eisenbud acyclicity criterion [BH93, Theorem 1.4.13]. Set $\phi_i$ and the matrix $\phi_i^{[m]}$

For that we use Buchsbaum-Eisenbud acyclicity criterion [BH93, Theorem 1.4.13]. Set $r_i = \sum_{j=1}^{p} (-1)^{j-i} \text{rank}(\mathfrak{F}_j)$. Since $(\mathfrak{F}_*, \phi_*)$ is a free resolution, grade $I_{p_i}(\phi_i) \geq i$ for $i \geq 1$, where $I_{p_i}(\phi_i)$ is an ideal of $R$ generated by $p_i \times p_i$ minors of $\phi_i$. Note that $r_i = \sum_{j=1}^{p} (-1)^{j-i} \text{rank}(\mathfrak{F}_j^{[m]})$, and $(I_{p_i}(\phi_i))^{[m]} = I_{p_i}(\phi_i^{[m]})$. By Lemma 2.5 for $i \geq 1$, grade $I_{p_i}(\phi_i^{[m]}) \geq i$. Thus, by [BH93, Theorem 1.4.13], $(\mathfrak{F}_*, \phi_*)$ is a free resolution of $R/I^{[m]}$. Note that for each $i \geq 1$, $\phi_i^{[m]}$ is a graded homomorphism of degree 0. Therefore, $(\mathfrak{F}_*, \phi_*)$ is a graded free resolution of $R/I^{[m]}$. Hence, the assertion follows from [BH93, Theorem 1.3.1]. □

As an immediate consequence, we compute graded Betti numbers of $R/I^{[m]}$ in terms of graded Betti numbers of $R/I$.

**Corollary 2.7.** Let $I \subset R$ be a homogeneous ideal and $m \geq 2$. Then, $\beta_{i,j}(R/I^{[m]})$ is a non-zero graded Betti number only if $j = mk$ for some $k$, and $\beta_{i,k}(R/I^{[m]})$ is a non-zero graded Betti number. Moreover, $\beta_{i,mk}(R/I^{[m]}) = \beta_{i,k}(R/I)$, for every $k$, and $i \geq 0$.

**Proof.** The assertion follows from Theorem 2.6. □

We can conclude the analogue of statement (a) of Lemma 1.1 for projective dimension of square powers as an immediate consequence of Theorem 2.6.

**Corollary 2.8.** Let $I \subset R$ be a homogeneous ideal and $m \geq 2$. Then, $\text{pd}(R/I) = \text{pd}(R/I^{[m]})$, for all $m \geq 1$.

We now compute graded Betti numbers of $p^r$-th Frobenius power of $I$ in terms of graded Betti numbers of $I$.

**Theorem 2.9.** Assume that $\text{char}(\mathbb{K}) = p > 0$. Let $I \subset R$ be a homogeneous ideal. Then, $\beta_{i,j}(R/I^{[p^r]})$ is a non-zero graded Betti number only if $j = pk$ for some $k$, and $\beta_{i,k}(R/I^{[p^r]})$ is a non-zero graded Betti number. Moreover, $\beta_{i,p^rk}(R/I^{[p^r]}) = \beta_{i,k}(R/I)$, for every $k$, and $i \geq 0$.

**Proof.** Let $(\mathfrak{F}_*, \phi_*)$ be the graded minimal free resolution of $R/I$. For $i \geq 0$, set

$$\mathfrak{F}_i^{[p^r]} = \bigoplus_k R(-p^rk)^{\beta_{i,k}(R/I)},$$

and the matrix $\phi_i^{[p^r]}$ given by

$$(\phi_i^{[p^r]})_{k,l} = ((\phi_i)_{k,l})^{p^r}.$$  

Similar to the proof of Theorem 2.6, one can prove that $(\mathfrak{F}_*, \phi_*)$ is the minimal free resolution of $R/I^{[p^r]}$. Hence, the assertion follows. □
One important algebraic invariant which is associated with the minimal free resolution of finitely generated graded module is the Castelnuovo-Mumford regularity, henceforth called regularity. The regularity of \( R/I \), denoted by \( \text{reg}(R/I) \), is defined as

\[
\text{reg}(R/I) := \max\{j - i : \beta_{i,j}(R/I) \neq 0\}.
\]

A nonzero graded Betti number \( \beta_{i,j}(R/I) \) is called an extremal Betti number, if \( \beta_{r,s}(R/I) = 0 \) for all pairs \((r, s) \neq (i, j)\) with \( r \geq i \) and \( s \geq j \). Observe that \( R/I \) admits a unique extremal Betti number if and only if \( \beta_{p,p+\ell}(R/I) \neq 0 \), where \( p = \text{pd}(R/I) \) and \( \ell = \text{reg}(R/I) \).

Now, we study extremal Betti numbers of \( R/I^{[m]} \) in terms of extremal Betti numbers of \( R/I \).

**Proposition 2.10.** Let \( I \) be a homogeneous ideal of \( R \), and \( m \geq 2 \). If \( \beta_{i,j}(R/I^{[m]}) \) is an extremal Betti number, then \( j = ml \) for some \( l \), and \( \beta_{i,l}(R/I) \) is an extremal Betti number. Conversely, if \( \beta_{i,j}(R/I) \) is an extremal Betti number, then \( \beta_{i,mj}(R/I^{[m]}) \) is an extremal Betti number.

**Proof.** Since \( \beta_{i,j}(R/I^{[m]}) \) is an extremal Betti number, \( \beta_{i,j}(R/I^{[m]}) \neq 0 \) and \( \beta_{k,l}(R/I^{[m]}) = 0 \) for \( (k, l) \neq (i, j) \) and \( k \geq i, l \geq j \). By Corollary 2.7, \( \beta_{i,j}(R/I^{[m]}) = \beta_{i,j/m}(R/I) \neq 0 \). Let \((k, l) \neq (i, j/m)\) such that \( k \geq i, l \geq j/m \). Then, by Corollary 2.7, \( \beta_{k,l}(R/I) = \beta_{k,lm}(R/I^{[m]}) = 0 \) as \( k \geq i \) and \( lm \geq j \). Hence, \( \beta_{i,j/m}(R/I) \) is an extremal Betti number.

Now, assume that \( \beta_{i,j}(R/I) \) is an extremal Betti number. By Corollary 2.7, \( \beta_{i,mj}(R/I^{[m]}) = \beta_{i,j}(R/I) \neq 0 \). Let \((k, l) \neq (i, mj)\) such that \( k \geq i \) and \( l \geq mj \). If \( l \) is not multiple of \( m \), then, by Corollary 2.7, \( \beta_{k,l}(R/I^{[m]}) = 0 \). Thus, we may assume that \( l = ml' \), for some \( l' \). Then, \( \beta_{k,l}(R/I^{[m]}) = \beta_{k,l'}(R/I) = 0 \) as \( k \geq i \) and \( l' \geq j \). Hence, \( \beta_{i,mj}(R/I^{[m]}) \) is an extremal Betti number.

The above result also holds for Frobenius powers of homogeneous ideals.

**Proposition 2.11.** Assume that \( \text{char}(K) = p > 0 \). Let \( I \) be a homogeneous ideal of \( R \). If \( \beta_{i,j}(R/I^{[p^e]}) \) is an extremal Betti number, then \( j = p^el \) for some \( l \), and \( \beta_{i,l}(R/I) \) is an extremal Betti number. Conversely, if \( \beta_{i,j}(R/I) \) is an extremal Betti number, then \( \beta_{i,p^el}(R/I^{[p^e]}) \) is an extremal Betti number.

**Proof.** The proof is similar to the proof of Proposition 2.10.

We now compute the regularity of square power of homogeneous ideals and further show that it matches with the regularity of Frobenius power of the same when the characteristic of the underlying field is prime.

**Corollary 2.12.** Let \( I \) be a homogeneous ideal of \( R \). Let \( r = \text{reg}(R/I) \) and \( \beta_{i,i+r}(R/I) \) be an extremal Betti number. Then, for any \( m \geq 2 \), \( \text{reg}(R/I^{[m]}) = mr + (m-1)i \). Furthermore, if \( \text{char}(K) = p > 0 \), then \( \text{reg}(R/I^{[p^e]}) = p^er + (p^e - 1)i \), where \( I^{[p^e]} \) denotes the Frobenius power of the ideal \( I \).
Proof. It follows from Proposition 2.10 that $\beta_{i,m_i}(R/I^{[m_i]})$ is an extremal Betti number, and hence, $\beta_{i,m_i}(R/I^{[m_i]}) \neq 0$. Therefore, $\text{reg}(R/I^{[m_i]}) \geq mr + (m - 1)i$. Set $r' = \text{reg}(R/I^{[m_i]})$. Then, for some $k \leq i$, $\beta_{k,k+r'}(R/I^{[m_i]})$ is an extremal Betti number. Thus, by Proposition 2.10

$$
\beta_{k,k+r'}(R/I^{[m_i]}) = \beta_{k,(k+r')/m}(R/I) \neq 0.
$$

Therefore, $r \geq (k+r')/m - k$, and hence, $r' \leq rm + (m - 1)k \leq rm + (m - 1)i$ as $k \leq i$. Hence, $\text{reg}(R/I^{[m_i]}) = mr + (m - 1)i$. The second assertion follows similarly by using Proposition 2.11.

We illustrate Corollaries 2.7 and 2.12 with the following example.

Example 2.13. Let $R = \mathbb{K}[X_1, X_2, X_3, X_4]$ and $I = (X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4)$. The following is the Betti diagram of $R/I$:

|      | 0 | 1 | 2 | 3 |
|------|---|---|---|---|
| total| 1 | 6 | 8 | 3 |
| 0    | . | . | . | . |
| 1    | . | 6 | 8 | 3 |

The Betti diagram of $R/I^{[2]}$ is

|      | 0 | 1 | 2 | 3 |
|------|---|---|---|---|
| total| 1 | 6 | 8 | 3 |
| 0    | . | . | . | . |
| 1    | . | . | 6 | . |
| 2    | . | . | . | . |
| 3    | . | 6 | . | . |
| 4    | . | . | 8 | . |
| 5    | . | . | . | 3 |

One can note that $R/I$ has linear resolution, but the resolution of $R/I^{[2]}$ is not linear.

Now, we prove an auxiliary result.

Lemma 2.14. Let $I \subset R$ be a homogeneous ideal. Then, for all $m \geq 2$ and $s \geq 2$,

$$(I^{[m]})^s = (I^s)^{[m]}.$$

Proof. Let $\{f_1, \ldots, f_r\}$ be a minimal generating set of $I$. Then, $I^{[m]} = \langle f_1^{[m]}, \ldots, f_r^{[m]} \rangle$. Also, $\{f_1^{a_1} \ldots f_r^{a_r} : a_i \in \mathbb{N}, \sum_{i=1}^r a_i = s \}$ generates $I^s$. Note that $(f_1^{a_1} \ldots f_r^{a_r})^{[m]} = (f_1^{[m]})^{a_1} \ldots (f_r^{[m]})^{a_r}$ for all $a_1, \ldots, a_r \in \mathbb{N}$. Therefore, $(I^s)^{[m]} = \langle (f_1^{a_1} \ldots f_r^{a_r})^{[m]} : a_i \in \mathbb{N}, \sum_{i=1}^r a_i = s \rangle = (I^{[m]})^s$. Hence, the assertion follows.

We use square powers of homogeneous ideals to prove the following result:
Theorem 2.15. Let $I$ be a homogeneous ideal. Then, there exists $i$ such that $\beta_{i,i+\text{reg}(R/I^s)}(R/I^s)$ is an extremal Betti number for $s \gg 0$.

Proof. It follows from [CHT99, Kod00] that there exist non-negative integers $a, b$ and $s_0$ such that $\text{reg}(R/I^s) = as + b$ for all $s \geq s_0$. For each $s \geq s_0$, there exist $i_s$ such that $\beta_{i_s,i_s+as+b}(R/I^s)$ is an extremal Betti number. Fix $m \geq 2$. Then, by Corollary 2.14,

$$\text{reg}(R/(I^s)^{[m]}) = m(as + b) + (m - 1)i_s$$

for $s \geq s_0$.

Now, $I^{[m]}$ is a homogeneous ideal. Therefore, it follows from [CHT99, Kod00] that there exist non-negative integers $a_m, b_m$ and $s_m$ such that $\text{reg}(R/(I^{[m]})^s) = a_m s + b_m$ for all $s \geq s_m$. By Lemma 2.14, $\text{reg}(R/(I^{[m]})^s) = \text{reg}(R/(I^s)^{[m]})$. Thus, $a_m s + b_m = m(as + b) + (m - 1)i_s$ for $s \geq \max\{s_0, s_m\}$. By comparing, we get $a_m = am$ and $b_m = mb + (m - 1)i_s$. Since $b, b_m$ are constant for $s \geq \max\{s_0, s_m\}$, $i_s$ is a constant for $s \geq \max\{s_0, s_m\}$. Hence, the assertion follows.

3. Square Powers of Monomial Ideals

In this section, we consider monomial ideals. First note that if $\text{char}(K) = p > 0$, then for a monomial ideal $I$, the $p^s$-th square power and the $p^s$-th Frobenius power are same. We now prove an auxiliary result about square powers of monomial ideals.

Lemma 3.1. Let $I$ and $J$ be monomial ideals of $R$. Then, $(I \cap J)^{[m]} = I^{[m]} \cap J^{[m]}$, for $m \geq 2$.

Proof. Let $I$ be minimally generated by the set $\{X^{a_1}, \ldots, X^{a_k}\}$, where $a_1, \ldots, a_k \in \mathbb{N}^n$. Then, for $m \geq 1$, $I^{[m]} = \langle X^{ma_1}, \ldots, X^{ma_k} \rangle$. Let $J$ be minimally generated by the set $\{X^{b_1}, \ldots, X^{b_l}\}$, where $b_1, \ldots, b_l \in \mathbb{N}^n$. Therefore, by [HH11, Proposition 1.2.1], $I \cap J$ is minimally generated by the set $\{\text{lcm}(X^{a_i}, X^{b_j}) : 1 \leq i \leq k, 1 \leq j \leq l\}$. Thus,

$$(I \cap J)^{[m]} = \langle \text{lcm}(X^{a_i}, X^{b_j})^m : 1 \leq i \leq k, 1 \leq j \leq l \rangle$$

$$= \langle \text{lcm}(X^{ma_i}, X^{mb_j}) : 1 \leq i \leq k, 1 \leq j \leq l \rangle$$

$$= I^{[m]} \cap J^{[m]}$$

where the last equality follows from [HH11, Proposition 1.2.1]. Hence, the assertion follows.

Now, we compute primary decomposition of $I^{[m]}$ for any monomial ideal $I$.

Theorem 3.2. Let $I$ be a monomial ideal with primary decomposition $I = \bigcap_{i=1}^{r} Q_i$. Then, $I^{[m]} = \bigcap_{i=1}^{r} Q_i^{[m]}$ is primary decomposition of $I^{[m]}$ for all $m \geq 2$. Moreover, $\text{Ass}(I) = \text{Ass}(I^{[m]})$ and $\text{Min}(I) = \text{Min}(I^{[m]})$, for all $m \geq 2$. 
Proof. We prove this assertion by induction on $r$. If $r = 1$, then by [HH11, Theorem 1.3.1], $I$ is generated by pure powers of the variables, which implies that $I^m$ is also generated by pure powers of the variables. By [HH11, Theorem 1.3.1], the assertion follows. Assume that $r \geq 2$ and the result is true for $r - 1$. Set $J = \bigcap_{i=1}^{r-1} Q_i$. Note that $I = J \cap Q_r$. By Lemma 3.1, $I^m = J^m \cap Q_r^m$ for any $m \geq 2$. By induction, $J^m = \bigcap_{i=1}^{r-1} Q_i^m$ is primary decomposition of $J^m$ for all $m \geq 2$. Therefore, $I^m = \bigcap_{i=1}^{r} Q_i^m$. Since $Q_i^m$ is generated by pure powers of variables, $\bigcap_{i=1}^{r} Q_i^m$ is primary decomposition of $I^m$, by [HH11, Theorem 1.3.1]. □

The following example shows that Ass($I$) and Ass($I^m$) need not be the same in general.

Example 3.3. Let $I = \langle X - Y, X - Z \rangle \subset \mathbb{K}[X,Y,Z]$. Observe that $I$ is a prime ideal. If char$(\mathbb{K}) \neq 2$, then the associated primes of $I^{[2]} = \langle X^2 - Y^2, X^2 - Z^2 \rangle$ are $\langle -X + Z, -X + Y \rangle, \langle -X + Z, X + Y \rangle, \langle X + Z, -X + Y \rangle, \langle X + Z, X + Y \rangle$.

The following result is an immediate consequence of Corollary 2.8 and Theorem 3.2.

Corollary 3.4. Let $I \subset R$ be a monomial ideal and $m \geq 2$. If $R/I$ is Cohen-Macaulay, then $R/I^m$ is Cohen-Macaulay.

Proof. By Auslander-Buchsbaum formula and Corollary 2.8 we get that depth$(R/I) = \text{depth}(R/I^m)$, for all $m \geq 2$. Now, it follows from Theorem 3.2 that dim$(R/I) = \text{dim}(R/I^m)$. Hence, the assertion follows. □

An ideal $I$ in $R$ is called normally torsion-free if Ass($I^s$) $\subseteq$ Ass($I$) for all $s \geq 1$. Given a normally torsion-free monomial ideal, one can construct a family of normally torsion-free monomial ideals in the following way:

Theorem 3.5. Let $I \subset R$ be a monomial ideal. If $I$ is normally torsion-free, then $I^m$ is normally torsion-free, for all $m \geq 2$.

Proof. Since $I$ is a normally torsion-free ideal, Ass($I^s$) $\subseteq$ Ass($I$) for all $s$. Let $s \geq 1$ and $m \geq 2$. By Lemma 2.14, $(I^m)^s = (I^s)^m$. Now, by Theorem 3.2, Ass($((I^m)^s)$) = Ass($((I^s)^m)$) = Ass($I^s$) $\subseteq$ Ass($I$) = Ass($I^m$). Hence, $I^m$ is normally torsion-free for all $m$. □

In the following result, we compute the index of stability of $I^m$ in terms of index of stability of $I$.

Theorem 3.6. The index of stability of $I$ and $I^m$ are same for all $m \geq 2$.

Proof. Let $k \geq 2$ and $m \geq 2$. Then, by Lemma 2.14, $(I^m)^k = (I^k)^m$. Therefore, by Theorem 3.2, Ass($((I^m)^k)$) = Ass($((I^k)^m)$) = Ass($I^k$). Hence, the assertion follows. □
Given an ideal $I \subset R$, its $s$-th symbolic power of $I$ is defined as

$$I^{(s)} = \bigcap_{p \in \text{Min}(I)} (I^s R_p \cap R),$$

where $R_p$ is the localisation of $R$ at $p$.

Next, we study the depth function $\text{depth}(R/I^{(s)})$. Nguyen and Trung proved [NT19, Corollary 3.2] that if $I$ is a monomial ideal, then $\text{depth}(R/I^{(s)})$ is asymptotically periodic i.e. it is a periodic function for $s \gg 0$. They also proved that [NT19, Theorem 3.3] if $I$ is a monomial ideal such that $I^{(s)}$ is integrally closed for $s \gg 0$, then $\text{depth}(R/I^{(s)})$ is a convergent function, and

$$\lim_{s \to \infty} \text{depth}(R/I^{(s)}) = n - \dim(F_s(I)),$$

where

$$F_s(I) := \bigoplus_{s \geq 0} I^{(s)}/mI^{(s)}$$

and $m$ is the homogeneous maximal ideal of $R$. Also, they presented a large class of monomial ideals $I$ with the property that $I^{(s)}$ is integrally closed for $s \gg 0$, [NT19, Lemma 3.5]. Here, we give a class of monomial ideals $I$ with the property that $I^{(s)}$ is not integrally closed for all $s \geq 2$ and $\text{depth}(R/I^{(s)})$ is a convergent function, and

$$\lim_{s \to \infty} \text{depth}(R/I^{(s)}) = n - \dim(F_s(I)).$$

First, we prove the following lemma showing that the integral closure of the ordinary power and that of the square power is same for a monomial ideal.

**Lemma 3.7.** Let $I \subset R$ be a monomial ideal. Then, for $m \geq 1$

$$I^{[m]} = I^m.$$

**Proof.** Since $I^{[m]} \subset I^m$, $I^{[m]} \subset \overline{I^m}$. Let $u \in I^m$ be a monomial. By Lemma 2.14, $(I^m)^{[m]} = (I^{[m]})^m$ which implies that $u^m \in (I^{[m]})^m$, and hence, by [HH11, Theorem 1.4.2], $u \in \overline{I^{[m]}}$. Therefore, $I^m \subset \overline{I^{[m]}}$. Hence, the assertion follows.

It is clear from Lemma 3.7 that if $I$ is a monomial ideal, then $I^{[m]}$ is never integrally closed for $m \geq 2$.

**Lemma 3.8.** Let $I \subset R$ be a monomial ideal. Then, for all $m \geq 2$ and $s \geq 2$,

$$(I^{[m]})^{(s)} = (I^{(s)})^{[m]}.$$

**Proof.** The assertion is immediate from Theorem 3.2 and the definition of the symbolic power of ideals.
Theorem 3.9. Let $I \subset R$ be a monomial ideal such that $I^{(s)}$ is integrally closed for $s \gg 0$. Then, the sequence $\{\text{depth}(R/(I^{[m]})^{(s)})\}_{s \geq 1}$ is a convergent sequence for any $m \geq 2$, and
\[
\lim_{s \to \infty} \text{depth}(R/(I^{[m]})^{(s)}) = n - \dim(F_s(I^{[m]})).
\]

Proof. It follows from Lemma 3.8 that for any $s \geq 1$, $(I^{[m]})^{(s)} = (I^{(s)})^{[m]}$. Therefore, depth$(R/(I^{[m]})^{(s)}) = \text{depth}(R/(I^{(s)})^{[m]})$. Now, by Corollary 2.8
\[
\text{depth}(R/(I^{[m]})^{(s)}) = \text{depth}(R/I^{(s)}).
\]
Therefore,
\[
\lim_{s \to \infty} \text{depth}(R/(I^{[m]})^{(s)}) = \lim_{s \to \infty} \text{depth}(R/I^{(s)}) = n - \dim(F_s(I)),
\]
where the last equality follows from [NTT19, Theorem 3.3]. Now, it is enough to prove that \(\dim(F_s(I^{[m]})) = \dim(F_s(I))\). Note that for each $s \geq 0$,
\[
\ell((I^{(s)})/mI^{(s)}) = \mu(I^{(s)}) = \mu((I^{(s)})^{[m]}) = \mu((I^{[m]})^{(s)}) = \ell((I^{[m]})^{(s)}/m(I^{[m]})^{(s)}).
\]
Hence, the assertion follows. \qed

We now study the growth of multiplicity of $R/I^{[m]}$ as $m$ increases.

Theorem 3.10. Let $I \subset R$ be a monomial ideal. Then, for all $m \geq 1$, $e(R/I^{[m]}) = m^n - e(R/I)$, where $d = \dim(R/I)$.

Proof. By Theorem 3.2, $d = \dim(R/I) = \dim(R/I^{[m]})$. It follows from [BH93, Corollary 4.1.8] that there exists $Q_{R/I}(t) \in \mathbb{Z}[t]$ such that $Q_{R/I}(1) \neq 0$ and
\[
\text{Hilb}_{R/I}(t) = \frac{Q_{R/I}(t)}{(1-t)^d}.
\]
Similarly, there exists $Q_{R/I^{[m]}}(t) \in \mathbb{Z}[t]$ such that $Q_{R/I^{[m]}}(1) \neq 0$ and
\[
\text{Hilb}_{R/I^{[m]}}(t) = \frac{Q_{R/I^{[m]}}(t)}{(1-t)^d}.
\]
By [BH93, Theorem 4.1.13],
\[
\text{Hilb}_{R/I}(t) = \frac{\sum_{i,k}(-1)^i\beta_{i,k}(R/I)t^k}{(1-t)^n} = \frac{Q_{R/I}(t)}{(1-t)^d} \quad \text{and}
\]
\[
\text{Hilb}_{R/I^{[m]}}(t) = \frac{\sum_{i,j}(-1)^i\beta_{i,j}(R/I^{[m]})t^j}{(1-t)^n} = \frac{Q_{R/I^{[m]}}(t)}{(1-t)^d}.
\]
It follows from the proof of Corollary 2.7 that
\[
\text{Hilb}_{R/I^{[m]}}(t) = \frac{\sum_{i,k}(-1)^i\beta_{i,k}(R/I)t^{km}}{(1-t)^n}.
\]
Therefore,
\[
(1-t)^{n-d} \cdot Q_{R/I^{[m]}}(t) = \sum_{i,k}(-1)^i\beta_{i,k}(R/I)t^{km} = (1-t^m)^{n-d} \cdot Q_{R/I}(t^m),
\]
which implies that

\[ Q_{R/I}^{[n]}(t) = \left( \sum_{i=0}^{m-1} t^i \right)^{n-d} \cdot Q_{R/I}(t^m). \]

Thus, \( Q_{R/I}^{[n]}(1) = m^{n-d} \cdot Q_{R/I}(1) \), and hence, the assertion follows from [BH93 Proposition 4.1.9]. \( \square \)

Next, we study asymptotic invariants associated with the containment problem. In order to capture the precise information about the containment \( I^{(s)} \subseteq I^{t} \), Bocci and Harbourne [BH10a] introduced the resurgence number of homogeneous ideals. The *resurgence number* of \( I \), denoted by \( \rho(I) \), is \( \rho(I) = \sup\{s/t \mid I^{(s)} \not\subseteq I^{t}\} \). If \( s > \rho(I) \cdot t \), then \( I^{(s)} \not\subseteq I^{t} \). There are very few classes of ideals for which the value of resurgence number is known. For example, by Macaulay's unmixedness theorem [Hoc73, LS06], it follows that \( \rho(I) = 1 \) when \( I \) is a complete intersection. In general, the computation of \( \rho(I) \) is quite tricky. Bocci and Harbourne gave a bound for \( \rho(I) \) in terms of another invariant of \( I \), known as Waldschmidt constant. Let \( \alpha(I) \) denote the least generating degree of homogeneous ideal \( I \). The *Waldschmidt constant* of \( I \), denoted by \( \hat{\alpha}(I) \), is \( \hat{\alpha}(I) = \lim_{s \to \infty} \alpha(I^{(s)})/s \). This limit exists and was first defined by Waldschmidt [Wal76] for ideals of finite point sets. The bound for \( \rho(I) \) in terms of \( \hat{\alpha}(I) \) stated in [BH10a] is \( \alpha(I)/\hat{\alpha}(I) \leq \rho(I) \). In [GHVT13] Guardo, Harbourne, and Van Tuyl introduced the *asymptotic resurgence number* of \( I \), \( \rho_a(I) = \sup\{s/t \mid I^{(sr)} \not\subseteq I^{tr} \text{ for all } r \gg 0\} \), and studied it for smooth projective schemes. It is evident that \( \rho_a(I) \leq \rho(I) \). Moreover there are several examples [BDRH+19, Theorem 1.4 and Theorem 1.5] where \( \rho_a(I) \neq \rho(I) \). Bocci and Harbourne [BH10a] showed that \( 1 \leq \alpha(I)/\hat{\alpha}(I) \leq \rho_a(I) \leq \rho(I) \). If \( I \) defines a smooth scheme, then \( \rho_a(I) \leq \omega(I)/\hat{\alpha}(I) \), where \( \omega(I) \) denotes the largest degree of a minimal generator of \( I \) [GHVT13]. Hence, these three asymptotic invariants are quite naturally related with each other and are of great interest.

The following theorem relates the Waldschmidt constant, the resurgence number, and the asymptotic resurgence number of monomial ideal \( I \) with that of \( I^{[m]} \).

**Theorem 3.11.** Let \( I \subset R \) be a monomial ideal. Then, we have the following,

- (a) \( \hat{\alpha}(I^{[m]}) = m \hat{\alpha}(I) \)
- (b) \( \rho(I^{[m]}) = \rho(I) \)
- (c) \( \rho_a(I^{[m]}) = \rho_a(I) \)

**Proof.** (a) As \( \alpha(I) \) denotes the least generating degree of \( I \), then \( \alpha(I^{[m]}) = m \cdot \alpha(I) \). By Lemma 3.8,

\[ \hat{\alpha}(I^{[m]}) = \lim_{s \to \infty} \frac{\alpha((I^{[m]})^{(s)})}{s} = \lim_{s \to \infty} \frac{\alpha(I^{(s)})^{[m]}}{s} = \lim_{s \to \infty} \frac{m \cdot \alpha(I^{(s)})}{s} = m \cdot \hat{\alpha}(I), \]

which proves (a).

(b) It is clear that if \( I^{(s)} \not\subseteq I^{t} \), then \( (I^{(s)})^{[m]} \not\subseteq (I^{t})^{[m]} \), for any \( m \geq 2 \). Also, if \( I^{(s)} \subseteq I^{t} \), then \( (I^{(s)})^{[m]} \subseteq (I^{t})^{[m]} \), for any \( m \geq 2 \). Thus, \( I^{(s)} \not\subseteq I^{t} \) iff \( (I^{(s)})^{[m]} \not\subseteq (I^{t})^{[m]} \), for any \( m \geq 2 \).
Hence, by Lemmas 2.14 and 3.8
\[ \rho(I^{[m]}) = \sup \{ s/t \mid (I^{[m]})^{(s)} \nsubseteq (I^{[m]})^{t} \} \]
\[ = \sup \{ s/t \mid (I^{(s)})^{[m]} \nsubseteq (I^{[m]})^{[m]} \} \]
\[ = \sup \{ s/t \mid I^{(s)} \nsubseteq I^{t} \} \]
\[ = \rho(I) \]

(c) By Lemmas 2.14 and 3.8
\[ \rho_a(I^{[m]}) = \sup \{ s/t \mid (I^{[m]})^{(sr)} \nsubseteq (I^{[m]})^{tr}, \text{ for all } r \gg 0 \} \]
\[ = \sup \{ s/t \mid (I^{(sr)})^{[m]} \nsubseteq (I^{tr})^{[m]}, \text{ for all } r \gg 0 \} \]
\[ = \sup \{ s/t \mid I^{(sr)} \nsubseteq I^{tr}, \text{ for all } r \gg 0 \} \]
\[ = \rho_a(I) \]

\[ \square \]

We conclude this article with the following remark:

**Remark 3.12.** Let \( I \subset R \) be a monomial ideal such that \( I \) defines a smooth scheme. Then \( I^{[m]} \) does not define smooth scheme for all \( m \geq 2 \) as the ring \( R/I^{[m]} \) is not reduced. Note that \( \omega(I^{[m]}) = m \cdot \omega(I) \) for all \( m \geq 2 \) (where \( \omega(I) \) denotes the largest degree of a minimal generator of \( I \), as mentioned earlier). This shows that, if \( I \) defines a smooth scheme, then for \( m \geq 2 \),
\[ \rho_a(I^{[m]}) \leq \omega(I^{[m]})/\hat{\alpha}(I^{[m]}) \quad \text{for all } m \geq 2, \]
by Theorem 3.11.

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