Generic rotationally invariant random matrix models satisfy a simple relation: the probability distribution of off-diagonal elements and the one of half the difference between any two diagonal elements coincide. In the spirit of the Eigenstate Thermalization Hypothesis (ETH), we test the hypothesis that the same relation holds in quantum systems that are non-localized, when one considers small energy differences. The relation provides a stringent test of ETH beyond the Gaussian ensemble. We apply it to a disordered spin chain, the SYK model and a Floquet system.

**Introduction.** ETH has become one of the most accepted elements for our understanding of the dynamical and thermalization properties of quantum systems [1, 2]. It relates to the properties of local observables in the basis of the energy eigenstates. The initial assumption, based on work of Berry [3], concerns the leading order of diagonal matrix elements which is assumed to be a smooth function of the energy. Subsequent works by Deutsch and Srednicki [1, 4] extended the assumptions to matrix entries beyond this average result, characterizing the fluctuations from element to element, and under different realizations of the Hamiltonian [2]. In this form, ETH is formulated assuming that the fluctuating part of a given matrix element in the energy eigenbasis $A_{a \beta}$ is represented by exponentially small random numbers. In particular, for matrix elements close or near the diagonal, with energy-differences within the Thouless scale (the timescale of diffusion of a particle across the sample [2, 5]), the matrix is assumed to be essentially random Gaussian.

Recently [6], we argued that the matrix entries $A_{a \beta}$ cannot be in general independent and that the products of certain off-diagonal matrix elements should have small but relevant expectations which enter in the computation of higher order correlation functions. This, together with the fact that the probability distributions of single elements are measurably non-Gaussian [7, 8], calls for a generalization. The extension is however constrained by the argument [4] according to which eigenvectors with near energies of a non-localized Hamiltonian mix with random phases when a small perturbation is applied. This naturally leads one to assume that matrix elements close to the diagonal are well represented by rotationally invariant matrix ensembles.

A class of well-studied models that enjoys this property has matrix elements of the form [9]:

$$
P_{\beta}(A) \propto \exp \left( -\frac{\beta}{2} N \text{Tr} V(A) \right)$$

where $V(A)$ is some function and $\beta = 1, 2, 4$ is the Dyson index which characterizes if the ensemble is orthogonal, unitary or symplectic. The Gaussian ensemble $V(A) = A^2/2$ is the only one which has both rotational symmetry and independence of matrix entries [10]. In [11] it is shown that for a rotationally invariant model the type of diagrams discussed in [6] are the free cumulants of the matrix ensemble [1] which generalize for a random matrix the classical cumulants and encode the departure from the Gaussian statistics [12].

Rotationally invariance and in particular the form [1] imply testable relations between the joint distributions of elements [13] that do not involve determining $V$. These relations are easy to prove for a random ensemble, but for a quantum system remain an assumption, a check of the randomness of the diagonalizing basis below the Thouless energy scale. Here we shall use the simplest of these relations.

**Quenched vs. annealed.** It is important to distinguish these two ensembles, in view of the fact that they give different results and we are not entirely free to choose in the quantum case. Consider a matrix model with distribution [1]. For brevity, here we concentrate on the orthogonal ensemble, the reasoning for other ensembles is the same.

- Quenched ensemble: we consider the set of $2 \times 2$ submatrices $A$ obtained by generating all pairs of mutually orthogonal $s_i$ ($i = 1, 2$) $N$-dimensional vectors with random direction, and for simplicity we...
restrict to the orthogonal ensemble:

\[ P_A(\hat{A}) \propto \int D\sigma_1 D\sigma_2 \prod_{i<j} \delta(\sigma_i^1 A_{ij} - \hat{A}_{ij}) \delta(\sigma_j^1 \sigma_j - \delta_{ij}) \]

\[ \ln P_q(\hat{A}) \propto \int DA e^{-\frac{1}{2} TrV(A)} \ln P_A(\hat{A}) \]

(2)

• Annealed ensemble: here we have fixed vectors \( s_i \) \( (i = 1, 2) \), and we take statistics by changing the realization of \( A \):

\[ P_a(\hat{A}) \propto \int DA e^{-\frac{1}{2} TrV(A)} \prod_{i<j} \delta(\bar{s}_i^1 \bar{A}_{ij} - \hat{A}_{ij}) \]

\[ \delta(\bar{s}_i^1 \sigma_j - \delta_{ij}) \propto \int DA e^{-\frac{1}{2} TrV(A)} P_A(\hat{A}) \]

(3)

where the equality holds because the integral does not change if \( s_i \) are rotated together.

In a quantum case we shall not be free to rotate the eigenvectors as we please without destroying the structure of the matrix and averages will be essentially annealed: the different realizations may correspond to any change in the Hamiltonian that is small in the thermodynamic limit but still large compared to level-spacing [4].

Symmetries. Let \( T_{ab} \) be a \( 2 \times 2 \) orthogonal matrix. We define \( T\hat{A}T^\dagger = A' \). Using the fact that \( TT^\dagger = I \) and changing variables \( \sigma_a \rightarrow \sum_i T_{ab} \sigma_i \) in the deltas, we obtain:

\[ P(\hat{A}) = P(T\hat{A}T^\dagger) \]

(4)

which implies that \( P(\hat{A}) = F \left( \{ \hat{A}_{\alpha} \} \right) \) a function of the eigenvalues \( \hat{A}_\alpha \) of \( \hat{A} \) (this is valid both for quenched and for annealed averages). In terms of the difference \( A_{\alpha} = (\hat{A}_{11} - \hat{A}_{22})/2 \), the sum \( \hat{A}_+ = (\hat{A}_{11} + \hat{A}_{22})/2 \) and the off diagonal element \( \hat{A}_{12} \), we have:

\[ P(\hat{A}) = F \left( \hat{A}_+ + \sqrt{\hat{A}_+^2 + \hat{A}_{12}^2}, \hat{A}_+ - \sqrt{\hat{A}_+^2 + \hat{A}_{12}^2} \right) \]

(5)

This function is symmetric with respect to exchange of \( A_- \) and \( A_{12} \). This means that their probability marginals are the same function \( F \) evaluated in \( A_- \) and \( A_{12} \), respectively, hence we conclude that they are equally distributed: \( P(\hat{A}_{12}) = F(\hat{A}_{12}) \) and \( P(\hat{A}_-) = F(\hat{A}_-) \). The same may be said of the real and imaginary parts of the off-diagonal elements in the complex case.

Large \( N \) results. (a) Diagonal matrix elements. In order to derive the distribution of diagonal entries we consider the following generating function:

\[ F^\beta_d(s) = \langle e^{\frac{\beta}{N^2} \hat{A}_{ii}} \rangle \]

(6)

with \( \beta = 1.2 \) if \( A \) is symmetric or hermitian. The average is over the probability measure \( P_\beta(A)DA \) which can be decomposed into its eigenvalue and an angular part \( P_\beta(A)DA \propto \exp \left( -\frac{N^2}{2} \sum_i V(\lambda_i) \right) \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_i d\lambda_i \) and similarly the matrix element \( A_{ii} \) can be written \( A_{ii} = \sum k U_{ik} \lambda_k \). This problem has been considered rigorously in [14] but can be done in a more heuristic way considering the normalization of the vector \( \{U_{ik}\} \) and neglecting the orthonormal relation with and of the other vectors [12]. The problem can also be viewed as the study of the partition function of a spherical spin with pairwise disordered interactions, at inverse temperature \( s \) [15]. In [14] only the average over the unitary matrix has been taken, as in Eq [2]. Here we are interested also in the annealed average over the eigenvalues as in [3].

At high temperature the result at leading order in \( N \) is [14]:

\[ F^\beta_d(s) \approx e^{\frac{N\beta}{2} \int_0^\infty R(x) dx} \]

(7)

where \( R \) is the \( R \)-transform of the density \( \rho(\lambda) \) associated to the distribution \( P(A) \). This is defined starting from the Stieltjes transform \( S(z) = \frac{1}{N} \sum_k \frac{1}{z - \lambda_k} \) and taking its inverse \( R(s) = S^{-1}(s) - 1/s \). Note also that the \( R \) transform admits a Taylor expansion \( R(s) = \sum_{k=1}^\infty C_k s^{k-1} \) where \( C_k \) are the free cumulants of the matrix \( A \) which can be defined in terms of the moments \( M_k = \frac{1}{N} \langle \text{Tr} A^k \rangle \) (for a Gaussian ensemble the series stops at \( k = 2 \)) [16]. If one performs the average only over the orthogonal matrices and not over eigenvalues the result [7] is valid up to a critical value of \( s = s_c \) where \( S(\lambda_{\text{max}}) = s_c \). In order to derive the probability distribution of diagonal matrix elements instead we need the annealed average, performing the minimization over eigenvalues. The solution is such that the density of eigenvalues is unperturbed at high temperatures \( s \leq s_c \). At lower temperatures one finds that the bulk of the eigenvalues is not affected except that there is a single eigenvalue \( \lambda_N \) that separates from the bulk. This solution however is consistent with the analytic continuation of the result [7] at small temperatures (large \( s \)) [13].

One can therefore apply this results to the computation of \( P(A_{ii}) \), by taking the Fourier transform
of $F_d(s)$ which at the saddle point level amounts to computing a Legendre transform \[17\]:

$$P_d^{\beta}(A_{ii}) \approx e^{-\frac{N\beta}{2} \text{extr}_x \left( sA_{ii} - \int_0^x R(x) dx \right)} \quad (8)$$

These results are compatible with the trivial marginals of a Gaussian ensemble. For a Wishart matrix ensemble of parameter $\alpha R(s) = \frac{1}{N}$ \[10\] and this implies: $P_W^{\beta}(A_{ii}) \propto e^{\frac{N\beta}{2} (-A_{ii} + \alpha \log A_{ii})}$ which (for $\beta = 2$) agrees with the leading order in $N$ of the distribution obtained in \[18\].

(b) Off-diagonal matrix elements. In order to compute the distribution of off-diagonal matrix elements we consider the generating function for $\beta = 1$:

$$F_{\alpha}^{\beta=1}(s) = \left \langle e^{NsA_{ij}} \right \rangle = \left \langle e^{\frac{N}{2} \text{Tr} SUAU^\dagger} \right \rangle \quad (9)$$

with $S_{ij} = s(\delta_{i,j}\delta_{l,j} + \delta_{i,j}\delta_{l,i})$ and $A = UAU^\dagger$. (From here one can also show Eq. \[4\]). Diagonalizing the matrix $S$ therefore leads us to the problem of two diagonal matrix entries with parameter $s$ and $-s$. This leads to the high temperature result:

$$F_{\alpha}^{\beta=1}(s) \approx F_{d}^{\beta=1}(s) F_{d}^{\beta=1}(-s) \quad (10)$$

however again one has to be careful with the continuation of this result at small temperatures. In this limit two eigenvalues detach from the to the left and to the right of the bulk, respectively, when $s = S(\lambda_{\text{max}})$ and $s = -S(\lambda_{\text{min}})$. As in the previous case one can show that the expression at high temperatures \[10\] is analytically continued in the low temperature regime \[13\].

From Eq. \[10\] one can compute the cumulant $c_{2k}^d$ of $A_{ij}$ in terms of the free cumulant $C_k$ of matrix ensemble $P(A)$ and from \[7\] compare with the cumulant $c_k^d$ of $A_{ii}$:

$$c_{2k}^d = (2k - 1)! \left( \frac{1}{N} \right)^{2k-1} C_{2k} = \left( \frac{1}{2} \right)^{2k-1} c_{2k}^d . \quad (11)$$

From the result \[10\], by taking the Fourier transform, one can read the distribution of the off-diagonal matrix elements in terms of the diagonal one, in particular:

$$P_{\alpha}^{\beta=1}(A_{ij}) \simeq 2 \int dAP_{\alpha}^{\beta=1}(A) P_d^{\beta=1}(A - 2A_{ij}) \quad (12)$$

which is the large $N$ limit of our result on the equivalence between the distribution of $A_{ij}$ and $(A_{ii} - A_{jj})/2$ when the correlations between $A_{ii}$ and $A_{jj}$ are lost. For a hermitian matrix one should consider the real part:

$$F_r^{\beta=2}(s) = \left \langle e^{NsR_{A,j}} \right \rangle \simeq e^{NW(s)} \quad (13)$$

with $W(s) = \int_0^{s/2} R(w) + \int_0^{s/2} R(w)$ and a similar result holds for the observable $A = i\psi_\alpha \psi_\beta$ over all pair $\alpha \beta$. We choose $i$ in the middle of the spectrum and $j = i + 1$. Comparison with a Gaussian is shown with black lines. In the inset we show the collapse of $\log P_d(A_{ii})/N + C$ with $N = 2^{L/2}$ and $C$ is an arbitrary constant for $L = 10$ and $L = 14$.

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Applications to quantum systems.

We now check if in ergodic quantum systems the result we have obtained relating the full distribution of diagonal and off-diagonal matrix elements applies. For a true matrix model it is valid for any $N$ while for a quantum system, one may expect that a random matrix ETH regime sets in for $N$ sufficiently large. The particular case of variances of elements in a Gaussian case was already discussed in Refs. \[19\] \[20\].

As a first example we consider a paradigmatic model of chaos, the Sachdev-Ye-Kitaev (SYK) model \[21\]. This is defined considering $N$ Majorana
fermions \( \{ \psi_\alpha, \psi_\beta \} = \delta_{\alpha \beta} \) interacting via a disordered multibody term:

\[
H = \sum_{0 \leq \alpha < \beta < \gamma < \delta \leq L} J_{\alpha \beta \gamma \delta} \psi_\alpha \psi_\beta \psi_\gamma \psi_\delta \quad (14)
\]

where \( J_{\alpha \beta \gamma \delta} \) are Gaussian random variables with zero mean and variance \( \langle J^2_{\alpha \beta \gamma \delta} \rangle = 6/L^3 \). We consider the observable \( A = i \psi_\alpha \psi_\beta \) for all pairs of \( \alpha \) and \( \beta \). The Hamiltonian commutes with the parity operator \( P = i^{-L/2} \prod_{\alpha=1}^L \psi_\alpha \) and we restrict ourselves to the sector with eigenvalue \( \lambda_P = -1 \). In Fig. 3 we show the results for \( L = 10 \) where we compare the distribution of off-diagonal matrix elements \( P(\text{Re}A_{ij}) \) with the difference of two diagonal matrix elements \( P((A_{ii} - A_{jj})/2) \) in the middle of the spectrum. We also show the distribution of diagonal matrix elements \( P(A_{ii}) \) that is far from a Gaussian distribution in the wings, where the agreement between \( P(\text{Re}A_{ij}) \) and \( P((A_{ii} - A_{jj})/2) \) is still good. In the inset we show the collapse of the large deviation function \( P(A_{ii}) \) for two sizes \( L = 10 \) and \( L = 14 \).

Next we consider a Floquet system as the one implemented in [22]. We take a one dimensional chain and we consider as evolution operator \( U = W_1 W_2 \) with \( W_1 = U_{1,2} \times U_{3,4} \times \cdots \times U_{L-1,L} \) and \( W_2 = S^T (U_{1,2} \times U_{3,4} \times \cdots \times U_{L-1,L}) S \) where \( U_{i,j} \) are random unitary matrix and \( S \) is the shift operator which translate the spins of one site implementing periodic boundary condition. We consider as observable \( A = \sigma^z_{L/2} \) in the basis of the operator \( U \). In Fig. 3 we show the result of the distribution of diagonal and off-diagonal matrix elements for a system of size \( L = 6 \). The deviation from a Gaussian are clear as well as the agreement between \( P(\text{Re}A_{ij}) \) and \( P((A_{ii} - A_{ij})/2) \).

We finally study a disordered spin chain:

\[
H = \sum_{i=1}^{L} [JS_i \cdot S_{i+1} + h_i S_i^z] \quad (15)
\]

where \( S_i \) is a spin 1/2, the sum is over periodic boundary conditions and \( h_i \) are random uniform variables between \([-h, h] \). In Fig. 3 we show the results for \( L = 12 \) and \( h = 1 \) and we choose \( S_i^z \). In our procedure to construct the histogram we generate a configuration of fields and then we vary it by a small amount \( \delta h_i \in [-0.05, 0.05] \). We obtain a good agreement between \( P(A_{ij}) \) and \( P((A_{ii} - A_{ij})/2) \). The deviations from the Gaussian are not very marked for this particular sample but are inevitable given that \( P(A_{ii}) \) is clearly not Gaussian. Allowing the fields to vary freely over the disorder gives different distributions (as we show in the inset of Fig. 3) because rare values of fields in the statistics dominate in the large deviations.

Let us note that the distributions of off-diagonal matrix elements in many models (also considered here) are observed to be Gaussian [7] [8] [22] [23], if relatively small deviations are considered.

Perhaps the most interesting situation is when the distributions do not coincide. This can be seen for instance increasing the disorder in the model [15]. Already for \( h = 2 \) and \( L = 12 \) the distributions
FIG. 4. Imperfect eigenstate thermalization: disordered chain with $L = 12$ spins at larger disorder $h = 2$. In the main panel we show the distributions for a given sample, in the inset the distributions averaged over the disorder and the spins. As before we take 20 states in the middle of the spectrum.

$P(A_{ij})$ and $P((A_{ii} - A_{jj})/2)$ differ as shown in Fig. 4. This value of $h$ is expected to be below the MBL transition [7, 25], but at these sizes eigenstate thermalization does not hold. For large $h$, in the MBL phase, the relation is expected to break down for every size.

Conclusions. We have introduced a test of eigenstate thermalization inspired by matrix models that can be used in conjunction with level-statistics to detect subtle localization properties.

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