1. Introduction

We work throughout over the complex numbers \( \mathbb{C} \), i.e. all schemes are over \( \mathbb{C} \) and all maps of schemes are maps of \( \mathbb{C} \)-schemes. A curve, unless otherwise stated, is a smooth complete curve. Points mean geometric points. We will, as is usual in such situations, toggle between the algebraic and analytic categories without warning. For a quasi-projective algebraic variety \( Y \), the (mixed) Hodge structure associated with its \( i \)-th cohomology will be denoted \( H^i(Y) \).

For a curve \( X \), \( SU_X(n, L) \) will denote the moduli space of semi-stable vector bundles of rank \( n \) and determinant \( L \). The smooth open subvariety defining the stable locus will be denoted \( SU_s(X) \). We assume familiarity with the basic facts about such a moduli space as laid out, for example in [21], pp. 51–52, VI.A (see also Theorems 10, 17 and 18 of loc.cit.). Our principal result is the following theorem:

**Theorem 1.1.** Let \( X \) be a curve of genus \( g \geq 3 \), \( n \geq 2 \) an integer, and \( L \) a line bundle of degree \( d \) on \( X \) with \( d \) odd if \( g = 3 \) and \( n = 2 \). Let \( S^* = SU_X(n, L) \). Then \( H^3(S^*) \) is a pure Hodge structure of type \( \{ (1, 2), (2, 1) \} \), and it carries a natural polarization making the intermediate Jacobian

\[
J^2(S^*) = \frac{H^3(S^*, \mathbb{C})}{F_2 + H^3(S^*, \mathbb{Z})}
\]

into a principally polarized abelian variety. There is an isomorphism of principally polarized abelian varieties \( J(X) \simeq J^2(S^*) \).

The word “natural” above has the following meaning: an isomorphism between any two \( S^* \)'s as above will induce an isomorphism on third cohomology which will respect the indicated polarizations. As an immediate corollary, we obtain the following Torelli theorem:

**Corollary 1.1.** Let \( X \) and \( X' \) be curves of genus \( g \geq 3 \), \( L \) and \( L' \) line bundles of degree \( d \) on \( X \) and \( X' \) respectively, and \( n \geq 2 \) an integer. If

\[
SU_X(n, L) \simeq SU_{X'}(n, L') \quad (1.1)
\]

or if

\[
SU_X(n, L) \simeq SU_{X'}(n, L') \quad (1.2)
\]

then

\[
X \simeq X',
\]

except when \( g = 3, n = 2, (n, d) \neq 1 \).
Proof. Since \(SU^*_X(n, L)\) (resp. \(SU^*_X(n, L')\)) is the smooth locus of \(SU_X(n, L)\) (resp. \(SU_X(n, L')\)), therefore it is enough to assume (1.1) holds. By assumption \(J^2(SU^*_X(n, L)) \simeq J^2(SU^*_X(n, L'))\) as polarized abelian varieties. Therefore \(J(X) \simeq J(X')\), and the corollary follows from the usual Torelli theorem.  

The theorem is new for \((n, d) \neq 1\) (the so-called “non-coprime case”). When \((n, d) = 1\) (the “coprime case”), the theorem (and its corollary) has been proven by Narasimhan and Ramanan [17], Tyurin [23] and (for \(n = 2\)) by Mumford and Newstead [15]. In the non-coprime case, Kouvidakis and Pantev [12] have proved the above corollary under the assumption (1.2), and in fact the full result can be deduced from this case. However, the present line of reasoning is extremely natural, and is of a rather different character from that of Kouvidakis and Pantev.

In particular, Theorem 1.1 will not follow from their techniques. In the special case where \(n = 2\) and \(L = O_X\), Balaji [4] has shown a similar Torelli type theorem for Seshadri’s canonical desingularization \(N \rightarrow SU_X(2, O_X)\) in the range \(g > 3\).

In the coprime case, the proofs in [15] and [17] rely on the fact that \(SU^*_X(n, L) = SU_X(n, L)\), and hence \(SU^*_X(n, L)\) is smooth projective, and most importantly the product \(X \times SU_X(n, L)\) possesses a Poincaré bundle. In the non-coprime case \(SU^*_X(n, L)\) is not complete and a result of Ramanan (see [18]) says that there is no Poincaré bundle on \(X \times U\) for any Zariski open subset \(U\) of \(SU_X(n, L)\).

We concentrate primarily on the non-coprime case—the only remaining case of interest. Our strategy is to use a Hecke correspondence to relate the Hodge structure on \(H^3(SU^*_X(n, L))\) to that on \(H^1(X)\). To this extent our proof resembles Balaji’s in [4]. We are able to deduce more than Balaji does by imposing a polarization (which varies well with \(SU^*_X(n, L)\)) on the Hodge structure of \(H^3(SU^*_X(n, L))\). This construction of the polarization needs a version of Lefschetz’s Hyperplane Theorem (for quasi projective varieties. See Theorem 2.3). There is however another approach to the problem of polarization, which uses M. Saito’s theory of polarizations on Hodge modules (see Remark 2.3).

2. The Main Ideas

For the rest of the paper, we fix a curve \(X\) of genus \(g\), \(n \in \mathbb{N}\), \(d \in \mathbb{Z}\) and a line bundle \(L\) of degree \(d\) on \(X\). Assume, as in the main theorem, that if \(n = 2\), then \(g \geq 4\), and that \(g \geq 3\) otherwise. We shall assume, with one brief exception in step 3 below, that \((n, d) \neq 1\).

We will also assume, for the rest of the paper, that \(0 < d \leq n\). This involves no loss of generality, for \(SU_X(n, L)\) is canonically isomorphic to \(SU_X(n, L \otimes \xi^n)\) for every line bundle \(\xi\) on \(X\). Let \(S = SU_X(n, L)\) and \(S' = SU_X(n, L)\) and let \(U \subseteq S\) be a smooth open set containing \(S\).

The broad strategy of our proof is as follows: Fix a set \(\chi = \{x^1, \ldots, x^{d-1}\} \subset X\) of \(d - 1\) distinct points.

Step 1. First show that there are isomorphisms (modulo torsion), depending only on \((X, L, \chi)\), of Hodge structures

\[
\psi_{X, L, \chi}: H^1(X)(-1) \xrightarrow{\sim} H^3(S^\chi)
\]

1In fact, the exceptional case in the corollary can be eliminated using the results in [25].

2Balaji states the result for \(g \geq 3\), but his proof seems to work only for \(g > 3\). (See Remark 2.3).
Theorem 1.1 will follow by showing that there exists an integer $m$ induced by $\Theta(S_3)$. Step 2. Find a (possibly nonprincipal) polarization $\Theta(S^t)$ on $J^2(S^t)$ which depends only on $S^t$, and varies well with $S^t$. Let $\mu = \mu_{X,L,\chi}$ be the polarization on $J(X)$ induced by $\Theta(S^t)$ and $\varphi_{X,L,\chi}$.

Step 3. In this step we relax the above assumptions, and no longer insist that $(n,d) \neq 1$. Suppose Steps 1 and 2 have been taken (see [17] for the coprime case). Theorem 1.2 will follow by showing that there exists an integer $m$ such that $\frac{1}{m}\Theta$ is principal, and that $J^2(S^t)$ equipped with this polarization is isomorphic to $J(X)$ with its canonical polarization. The essence of the argument will be to show that any natural polarization on $J(X)$ must be a multiple of the standard one. The argument is lifted from [18], §5 where the idea is attributed to S. Ramanan. Pick a curve $X_0$ of genus $g$ such that the Neron-Severi group of its Jacobian, $NS(J(X_0))$ is $\mathbb{Z}$. By [18] such an $X_0$ exists. Pick a line bundle $L_0$ of degree $d$ on $X_0$, and a set of $d-1$ distinct points $\chi_0 = \{x_0^1,\ldots, x_0^{d-1}\}$ in $X_0$. One finds a family of curves $\tilde{X} \to T$, a line bundle $\mathcal{L}$ on $\tilde{X}$, and a set of $d-1$ mutually disjoint $T$-valued points $\tilde{\chi} = \{\tilde{x}_1,\ldots, \tilde{x}^{d-1}\}$, so that $(\tilde{X}, \mathcal{L}, \tilde{\chi})$ interpolates between $(X_0, L_0, \chi_0)$ and $(X,L,\chi)$. To get such a triple, first observe that since the moduli space $\mathcal{M}_{g,d-1}$ of pointed curves is irreducible and quasi-projective, we can find $(\tilde{X} \to T, \tilde{\chi})$ interpolating between $(X_0, \chi_0)$ and $(X,\chi)$. Let $\tilde{\text{Pic}}_d \to T$ be the resulting family of degree $d$ components of the Picard groups. Since $L_0$ and $L$ are points on $\text{Pic}$, one can connect them by a (possibly singular, incomplete) curve $T'$. Base change everything to $T'$. Renaming $T'$ as $T$ and the resulting family of pointed curves as $(\tilde{X}, \tilde{\chi})$, we get a $T$-valued point of the resulting bundle of degree $d$ components of the Picard groups. The line bundle $\mathcal{L}$ on $\tilde{X}$ corresponding to this section completes the triple. We denote the specialization of $(\tilde{X}, \mathcal{L}, \tilde{\chi})$ at $t \in T$ by $(X_t, L_t, \chi_t)$. Let $t_0, t_1 \in T$ be points where $(X_0, L_0, \chi_0)$ and $(X,L,\chi)$ are realized.

The $SU_X(n, L_i)$ string themselves into a family $\tilde{S} \to T$ (one uses Geometric Invariant Theory over the base $T$ to get $\tilde{S}$. The specializations behave well since we are working over $\mathbb{C}$). Similarly we have a family $\tilde{S}^t \to T$ specializing at $t \in T$ to $SU_X(n, L_i)$. The intermediate Jacobians $J^2(SU_X(n, L_i))$ also string together into a family of abelian varieties $A \to T$. Let $\tilde{J} \to T$ be the family $\{J(X_i)\}$ of Jacobians. Step 1 then gives an isomorphism of group schemes $\tilde{\varphi} : \tilde{J} \to A$
which specializes at \( t \in T \) to \( \varphi_{X,L,X} \). By Step 2 we get a family of polarizations 
\[
\{ \mu_t = \mu_{X,L,X} \}_{t \in T} \text{ on } J.
\]
Since \( \NS(J(X_0)) = \mathbb{Z} \), therefore there exists an integer \( m \neq 0 \), such that
\[
m \omega_X = \mu_0
\]
where, for any curve \( C \), \( \omega_C \) denotes the principal polarization on \( J(C) \). Since \( \{ \omega_{X_t} \} \) is a family of polarizations on \( J \) and since the Neron-Severi group is discrete, therefore
\[
m \omega_{X_t} = \mu_t \quad (t \in T).
\]

Theorem \[1\] is now immediate.

2.1. **The isomorphism \( \psi_{X,L,X} \).** One produces \( \psi_{X,L,X} \) as follows: Let
\[
\mathcal{S}_1 = \mathcal{SU}_X(n, L \otimes \mathcal{O}_X(-D))
\]
where \( D \) is the divisor \( \{ x^1 \} + \ldots + \{ x^{d-1} \} \). Since the degree of \( L \otimes \mathcal{O}_X(-D) \) is 1, therefore \( \mathcal{S}_1 \) is smooth and there exists a Poincaré bundle \( \mathcal{W} \) on \( X \times \mathcal{S}_1 \).

Let \( \mathcal{W}_1, \ldots, \mathcal{W}_{d-1} \) be the \( d-1 \) vector bundles on \( \mathcal{S}_1 \) obtained by restricting \( \mathcal{W} \) to \( \{ x^1 \} \times \mathcal{S}_1 = \mathcal{S}_1, \ldots, \{ x^{d-1} \} \times \mathcal{S}_1 = \mathcal{S}_1 \) respectively. Let \( \mathbb{P}_k = \mathbb{P}(W_k), k = 1, \ldots, d-1 \), and \( \mathbb{P} = \mathbb{P}_{X,L,X} \) be the product \( \mathbb{P}_1 \times \mathcal{S}_1 \times \ldots \times \mathcal{S}_1 \mathbb{P}_{d-1} \). We will show (in \[3\]) that there is a correspondence
\[
\mathcal{S}_1 \xleftarrow{\pi} \mathbb{P} \xrightarrow{f} \mathcal{S}
\]
where \( \pi = \pi_{X,L,X} \) is the natural projection and \( f = f_{X,L,X} \) is defined (via a generalized Hecke correspondence) in \[3\] (see \[3\]). We have isomorphisms of (integral, pure) Hodge structures
\[
H^1(X, \mathbb{Z})(-1) \xrightarrow{\sim} H^3(\mathcal{S}_1, \mathbb{Z}) \xrightarrow{\sim} H^3(\mathbb{P}, \mathbb{Z}).
\]
where the first isomorphism is that in \[17\], p. 392, Theorem 3, and the second is given by Leray-Hirsch. Let \( \mathbb{P}^s = f^{-1}(\mathcal{S}^s) \).

In \[3\] (see Remark \[3\], and \[3\]) we will show

**Proposition 2.1.** (a) If \( n \geq 3 \) and \( g \geq 3 \), the codimension of \( \mathbb{P} \backslash \mathbb{P}^s \) in \( \mathbb{P} \) is at least 3.

(b) The map \( \mathbb{P}^s \to \mathcal{S}^s \) is a \( \mathbb{P}^{n-1} \times \ldots \times \mathbb{P}^{n-1} \) bundle, where the product is \((d-1)\)-fold.

Note that if \( n = 2 \), the codimension of \( \mathbb{P} \backslash \mathbb{P}^s \) in \( \mathbb{P} \) is \( g-1 \) (see \[3\], p. 11, Prop. 7), so that if \( g \geq 4 \) the codimension is at least 3. This fact, along with and Proposition \[2\] implies that the codimension of \( \mathbb{P} \backslash \mathbb{P}^s \) is greater than equal to 3 for \( n, g \) in the range of Theorem \[1\]. It then follows, from Lemma \[2\] below, that the restriction maps
\[
H^3(\mathbb{P}, \mathbb{Z}) \xrightarrow{f^*} H^3(\mathbb{P}^s, \mathbb{Z})
\]
\[
H^1(\mathbb{P}, \mathbb{Z}) \xrightarrow{f^*} H^1(\mathbb{P}^s, \mathbb{Z})
\]
are isomorphisms of Hodge structures. Note that this means:

- The Hodge structure of \( H^3(\mathbb{P}^s) \) is pure of weight 3;
- The cohomology group \( H^1(\mathbb{P}^s, \mathbb{Z}) \) is 0. Indeed, \( \mathbb{P} \) is unirational (for \( \mathcal{S}_1 \) is — see \[21\], pp. 52–53, VI.B), whence \( H^1(\mathbb{P}, \mathbb{Z}) = 0 \).

We can now relate the Hodge structures on \( H^1(\mathcal{S}^s) \) and \( H^3(\mathcal{S}^s) \) with those on \( H^1(\mathbb{P}^s) \) and \( H^3(\mathbb{P}^s) \) using the map \( f \) and part (b) of Proposition \[2\]. For the rest of this section let \( f \) also denote the map \( \mathbb{P}^s \to \mathcal{S}^s \). We claim that
\[
f^* : H^3(\mathcal{S}^s) \to H^3(\mathbb{P}^s)
\]
(2.3)
is an isomorphism of Hodge structures, modulo torsion. This implies that the
Hodge structure on $H^3(S^s, \mathbb{Z})$ is pure of weight 3, a fact that also follows from
Corollary 4.1.

To prove that (2.3) is an isomorphism of Hodge structures, modulo torsion, we
need:

**Lemma 2.1.** $S^s$ is simply connected.

*Proof.* $\mathbb{P}$ is unirational, therefore it is simply connected [19]. Since codim($\mathbb{P} \setminus \mathbb{P}^s$) > 1, it follows that $\mathbb{P}^s$ is also simply connected (purity of the branch locus). The
lemma now follows from the homotopy exact sequence for $f$. □

**Corollary 2.1.** $H^1(S^s, \mathbb{Z}) = 0$.

**Corollary 2.2.** $f_*\mathbb{Z} = \mathbb{Z}$, $R^i f_*\mathbb{Z} = R^i f_*\mathbb{Z} = 0$ and $R^2 f_*\mathbb{Z} = \mathbb{Z}^{d-1}$.

*Proof.* As $S^s$ is simply connected, $R^i f_*\mathbb{Z}$ is just the constant sheaf associated to the $i$-th cohomology of $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$. □

One can now verify (2.3) by using the Leray spectral sequence combined with the
above isomorphisms. It follows that $H^3(\mathbb{P}^s, \mathbb{Z})$ is isomorphic to the cokernel of the
differential

$$H^0(R^2 f_*\mathbb{Z}) \to H^3(f_*\mathbb{Z})$$

but this vanishes mod torsion by [3]. The isomorphisms (2.2) and the map (2.3),
give the desired mod-torsion isomorphism

$$\psi_{X,L,\chi} : H^1(X)(-1) \sim H^3(S^s).$$

**Remark 2.1.** This isomorphism varies well with $(X, L, \chi)$ as the construction of
the correspondence (2.1) will show (see Remark 3.4).

Here then is the promised Lemma:

**Lemma 2.2.** If $Y$ is a smooth projective variety, $Z$ a codimension $k$ closed sub-
scheme, and $U = Y \setminus Z$, then

$$H^j(Y, Z) \sim H^j(U, Z)$$

for $j < 2k - 1$.

*Proof.* We have to show that $H^j_Z(Y, Z)$ vanishes for $j < 2k$. By Alexander duality
(see for e.g. [11], p. 381, Theorem 4.7) we have

$$H^j_Z(Y, Z) \sim H_{2m-j}(Z, Z),$$

where $m = \dim Y$ and $H_*$ is Borel-Moore homology. Now use [11], p. 406, 3.1 to
conclude that the right side vanishes if $j < 2k$ (note that “dim” in loc.cit is
dimension as an analytic space, and in op.cit. it is dimension as a topological (real)
manifold). □

**Remark 2.2.** In view of the above Lemma, it seems that Balaji’s proof of Torelli
(for Seshadri’s desingularization of $SU_X(2, O_X)$) does not work for $g = 3$, for in
this case, the codimension of $\mathbb{P} \setminus \mathbb{P}^s = 2$. (See [4], top of p. 624 and [3],Remark 9.)
2.2. The Polarization on $H^3(S^*)$. It remains to impose a polarization on the Hodge structure of $H^3(S^*)$ which varies well with $S^*$. Note that the map $\psi_{X,L,\chi}$ tells us that the Hodge structure on $H^3(S^*)$ is pure. 

One knows from the results of Drezet and Narasimhan \[7\], that Pic($S^*$) = $\mathbb{Z}$ (see p. 89, 7.12 (especially the proof) of loc. cit.). Moreover, Pic($S$) $\rightarrow$ Pic($S^*$) is an isomorphism. Let $\xi'$ be the ample generator of Pic($S^*$). It is easy to see that there exists a positive integer $r$, independent of $(X, L)$ (with genus $X = g$), such that $\xi = \xi'^r$ is very ample on $S$ (we are not distinguishing between line bundles on $S^*$ and their (unique) extensions to $S$). Embed $S$ in a suitable projective space via $\xi$. Let $e = \text{codim}(S \setminus S^*)$. Let $M$ be the intersection of $k = \dim S - e + 1$ hyperplanes (in general position) with $S^*$. Then $M$ is smooth, projective and contained in $S^*$. Let $p = \dim S$ and $H^p_S$ — cohomology with compact support. We then have a map

\[ l: H^3(S^*) \rightarrow H^{2p-3}_c(S^*) \]

defined by

\[ x \mapsto x \cup c_1(\xi)^{p-k-3} \cup [M]. \]

If $M'$ is another $k$-fold intersection of general hyperplanes, then $[M'] = [M]$. Hence $l$ depends only on $\xi$. According to Proposition 1.1 (see also Remark 1.1), the pairing on $H^3(S^*, \mathbb{C})$ given by

\[ < x, y > = \int_{S^*} l(x) \cup y \]

gives a polarization on the Hodge structure of $H^3(S^*)$. Since $\xi$ “spreads” (for $\xi'$ clearly does), therefore this polarization varies well with $S^*$. Then by arguments already indicated in the beginning of this section, this polarization is a multiple of principal polarization (and the integer factor is necessarily unique). Thus one gets a natural principal polarization on $H^3(S^*)$.

Remark 2.3. There is another approach to this polarization, using Intersection Co-homology (middle perversity) and M. Saito’s theory of Hodge modules \[19\]. The very ample bundle $\xi$ gives rise to Lefschetz operators $L^i: IH^q(S) \rightarrow IH^{q+2i}(S)$ (see \[1\]). Our codimension estimates (see Remark 3.3) are such that $IH^3(S) \xrightarrow{\sim} H^3(S^*)$ and $IH^1(S) = H^1(S^*) = 0$. The group $IH^3(S)$ has a pairing on it given by

\[ < \alpha, \beta > = \int_S Lp-3\alpha \cup \beta \]

where $\int_S(\cdot) \cup \beta: IH^{2p-3}(S) \rightarrow \mathbb{C}$ is the map given by the Poincaré duality pairing between $IH^{2p-3}(S)$ and $IH^3(S)$. According to M. Saito \[19\], 5.3.2, this gives a polarization on the Hodge structure of $IH^3(S)$ (since all classes in $IH^3(S)$ are primitive). This polarization translates to one on $H^3(S^*)$. A little thought shows (say by desingularizing $S$) that the pairing on $H^3(S^*)$ is

\[ < x, y > = \int_{S^*} c_1(\xi)^{p-3} \wedge x \wedge y. \]

Here, on the right side, we are using De Rham theory, and replacing the various elements in cohomology by forms which represent them. The integral above is the usual integral of forms. Note that we could not have defined the pairing by the above formula, for we have no a priori guarantee that the right side (which is an integral over an open manifold) is finite.
3. The correspondence variety $\mathbb{P}$

In this section we define the map $f: \mathbb{P} \rightarrow S$ and prove Proposition 2.1.

3.1. The map $f: \mathbb{P} \rightarrow S$. We need some notations:

- For $1 \leq k \leq d - 1$, $\pi_k: \mathbb{P} \rightarrow \mathbb{P}_k$ is the natural projection;
- $i: Z \hookrightarrow X$ is the reduced subscheme defined by $\chi = \{x^1, \ldots, x^{d-1}\}$;
- $i_k: Z_k \hookrightarrow X$, the reduced scheme defined by $\{x_k\}$, $k = 1, \ldots, d - 1$.

- For any scheme $S$,
  - $(i)\; p_S: X \times S \rightarrow S$ and $q_S: X \times S \rightarrow X$ are the natural projections;
  - $(ii)\; Z^S_S = q_S^{-1}(Z)$;
  - $(iii)\; Z^S_k = q_S^{-1}(Z_k)$, $k = 1, \ldots, d - 1$. Note that $Z^S_k$ can be identified canonically with $S$.

We will show — in 3.3 — that there is an exact sequence

$$0 \rightarrow (1 \times \pi)^* W \rightarrow V \rightarrow T \rightarrow 0$$

(3.1)

on $X \times \mathbb{P}$, with $V$ a vector bundle on $X \times \mathbb{P}$ and $T$ a line bundle on the subscheme $Z^\mathbb{P}$, which is universal in the following sense: If $\psi: S \rightarrow S_1$ is a $S_1$-scheme and we have an exact sequence

$$0 \rightarrow (1 \times \psi)^* W \rightarrow E \rightarrow T \rightarrow 0$$

(3.2)

on $X \times S$, with $E$ a vector bundle on $X \times S$ and $T$ a line bundle on the subscheme $Z^S$, then there is a unique map of $S_1$-schemes

$$g: S \rightarrow \mathbb{P}$$

such that,

$$(1 \times g)^* [3.1] \equiv [3.2].$$

The $\equiv$ sign above means that the two exact sequences are isomorphic, and the left most isomorphism $(1 \times g)^* (1 \times \pi)^* \rightarrow (1 \times \psi)^*$ is the canonical one.

There is a way of interpreting this universal property in terms of quasi-parabolic bundles (see [13], p. 211–212, Definition 1.5, for the definitions of quasi-parabolic and parabolic bundles). Taking $\chi$ as our collection of parabolic vertices, we can introduce a quasi-parabolic datum on $X$ by attaching the flag type $(1, n - 1)$ to each point of $\chi$. From now onwards quasi-parabolic structures will be with respect to this datum and on vector bundles of rank $n$ and determinant $L$. One observes that for a vector bundle $V$ (of rank $n$ and determinant $L$), a surjective map $V \rightarrow \mathcal{O}_Z$ determines a unique quasi-parabolic structure, and two such surjections give the same quasi-parabolic structure if and only if they differ by a scalar multiple. The above mentioned universal property says that $\mathbb{P}$ is a (fine) moduli space for quasi-parabolic bundles. More precisely, the family of quasi-parabolic structures

$$V \rightarrow T_0$$

parameterized by $\mathbb{P}$ is universal for families of quasi-parabolic bundles

$$\mathcal{E} \rightarrow \mathcal{T}$$

parameterized by $S$, whose kernel is a family of semi-stable bundles. The points of $\mathbb{P}$ parameterize quasi-parabolic structures $V \rightarrow \mathcal{O}_Z$ whose kernel is semi-stable.

Let $\alpha = (\alpha_1, \alpha_2)$, where $0 < \alpha_1 < \alpha_2 < 1$, and let $\Delta = \Delta_\alpha$ be the parabolic datum which attaches to each parabolic vertex (of our quasi-parabolic datum) weights $\alpha_1, \alpha_2$. We can choose $\alpha_1$ and $\alpha_2$ so small that...
• a parabolic semi-stable bundle is parabolic stable;
• if $V$ is stable, then every parabolic structure on $V$ is parabolic stable;
• the underlying vector bundle of a parabolic stable bundle is semi-stable in the usual sense;
• if $V \to O_Z$ is parabolic stable, then the kernel $W$ is semi-stable.

Showing the above involves some very elementary calculations. Denote the resulting moduli space of parabolic stable bundles $SU_X(n, L, \Delta)$.

Let $P^{ss} \subset P$ be the locus on which $\mathcal{V}$ consists of parabolic semi-stable (=parabolic stable) bundles. One checks that $P^{ss}$ is an open subscheme of $P$ (this involves two things: (i) knowing that the scheme $\bar{R}$ of [13], p. 226 has a local universal property for parabolic bundles and (ii) knowing that the scheme $\bar{R}^{ss}$ of loc.cit. is open).

Clearly $P^{ss}$ is non-empty — in fact if $V$ is stable of rank $n$ and determinant $L$, then any parabolic structure on $V$ is parabolic stable (see above). We claim that $P^{ss} \simeq SU_X(n, L, \Delta)$. To that end, let $S$ be a scheme, and

$$E \to T \to 0$$

a family of parabolic stable bundles parameterized by $S$. The kernel $W'$ of (3.3) is a family of stable bundles of rank $n$ and determinant $L \otimes O_X(-D)$. Since $S_1$ is a fine moduli space, we have a unique map $\xi: S \to S_1$ and a line bundle $\xi$ on $S$ such that $(1 \times g)^*W = W' \otimes p_2^*\xi$. By doctoring (3.3) we may assume that $\xi = O_S$. The universal property of the exact sequence (3.1) on $P$ then gives us a unique map

$$g: S \to P$$

such that $(1 \times g)^*(3.1)$ is equivalent to

$$0 \to W' \to E \to T \to 0.$$

Clearly $g$ factors through $P^{ss}$. This proves that $P^{ss}$ is $SU_X(n, L, \Delta)$. However, $SU_X(n, L, \Delta)$ is a projective variety (see [13], pp. 225–226, Theorem 4.1), whence we have

$$P = SU_X(n, L, \Delta).$$

It follows that $\mathcal{V}$ consists of parabolic stable bundles, and hence of (usual) semi-stable bundles (by our choice of $\alpha$). Since $S$ is a coarse moduli space, we get the map

$$f: P \to S.$$  \hfill (3.4)

Remark 3.1. Note that the parabolic structure $\Delta$ is something of a red herring. In fact $SU_X(n, L, \Delta)$ parameterizes quasi-parabolic structures $V \to O_Z$, whose kernel is semi-stable (cf. [13], p. 238, Remark (5.4), where this point is made for $n = 2, d = 2$). The space $P$ should be thought of as the correspondence variety for a certain Hecke correspondence (cf. [16]).

Remark 3.2. Let $V$ be a stable bundle of rank $n$, with $\det V = L$, so that (the isomorphism class of) $V$ lies in $S^*$. Since any parabolic structure on $V$ is parabolic (by our choice of $\alpha$), therefore we see that $f^{-1}(V)$ is canonically isomorphic to $P(V_{x^*}^1) \times \ldots \times P(V_{x^*d-1})$. This gives us part (b) of Proposition 2.1, for it is not

\footnote{One can be more rigorous. Identifying $Z_k^P$ with $P$ for each $k = 1, \ldots, d - 1$, we see that restricting the universal exact sequence to $Z_k^P$ gives us $d - 1$ quotients $O_P \otimes C V_{x^*k} \to O_P Z_k^P$. Let $S$ be a scheme which has $d - 1$ quotients $O_S \otimes C V_{x^*k} \to \mathcal{L}_k k = 1, \ldots, d - 1$, on it, where the $\mathcal{L}_k$ are line bundles. These quotients extend to a family of parabolic structures $\eta_h V \to \mathcal{T}$ (on $V$).}
hard to see that $P^s \to S^s$ is smooth (examine the effect on the tangent space of each point on $P^s$).

3.2. Codimension estimates. We wish to estimate $\text{codim}(P \setminus P^s)$. For any vector bundle $E$ on $X$, let $\mu(E) = \text{rank} E / \text{deg} E$. Let $\mu = d/n$. Let $V \to O_Z$ be a parabolic bundle in $P \setminus P^s$. Then we have a filtration (see [21], p. 18, Théorème 10)

$$0 = V_{p+1} \subset V_p \subset \ldots \subset V_0 = V$$

such that for $0 \leq i \leq p$, $G_i = V_i/V_{i+1}$ is stable and $\mu(G_i) = \mu$. Moreover (the isomorphism class of) the vector bundle $\bigoplus G_i$ depends only upon $V$ and not on the given filtration. We wish to count the number of moduli at $[V \to O_Z] \in P \setminus P^s$.

There are three sources:

a) The choice of $\bigoplus_{i=0}^p G_i$;

b) Extension data;

c) The choice of parabolic structure $V \to O_Z$, for fixed semi-stable $V$.

The source c) is the easiest to calculate — there is a codimension one subspace at each parabolic vertex, contributing

$$(n - 1)(d - 1)$$

moduli.

Let $n_i = \text{rank} G_i$.

The number of moduli arising from a) is evidently

$$\sum_{i=0}^{p} (n_i^2 - 1)(g - 1) + pg.$$ 

Indeed, the bundles $G_i$ have degree $n_i \mu$ and the product of their determinants must be $L$. They are otherwise unconstrained.

It remains to estimate the number of moduli arising from extension data. Each extension

$$0 \to V_{i+1} \to V_i \to G_i \to 0 \quad i = 0, \ldots, p$$

determines a class in $H^1(X, G_i^* \otimes V_{i+1})$. Note that

$$h^0(G_i^* \otimes V_{i+1}) = \dim \text{Hom}_{O_X}(G_i, V_{i+1}) \leq \sum_{j>i} \text{Hom}_{O_X}(G_i, G_j) \leq p - i$$

by the sub-additivity of $\dim \text{Hom}(G_i, \cdots)$ and the stability of $G_i$. By the Riemann-Roch theorem

$$h^1(G_i^* \otimes V_{i+1}) = h^0(G_i^* \otimes V_{i+1}) - n_i(n_{i+1} + \ldots + n_p)(1 - g) \leq (p - i) - n_i(n_{i+1} + \ldots + n_p)(1 - g).$$

parameterized by $S$ in a unique way. The universal property of the exact sequence (3.1) gives us a map $S \to \mathbb{P}$, and this map factors through $f^{-1}(V)$. 
The isomorphism class of $V_i$ depends only on a scalar multiple of the extension class. Therefore the number of moduli contributed by extensions is

$$\sum_{i=0}^{p} [h^1(G^*_i \otimes V_{i+1} - 1)] \leq \sum_{i=0}^{p} [p - i - n_i(n_{i+1} + \ldots + n_p)(1 - g)] - (p + 1)$$

$$= \frac{p(p+1)}{2} - \sum_{i=0}^{p-1} n_i(n_{i+1} + \ldots + n_p)(1 - g) - (p + 1)$$

$$= \frac{(p+1)(p-2)}{2} - \sum_{i<j} n_i n_j (1 - g).$$

Adding the contributions from a), b) and c) and subtracting from

$$\dim \mathbb{P} = (n^2 - 1)(g - 1) + (n - 1)(d - 1)$$

we get

$$\text{codim}(\mathbb{P} \setminus \mathbb{P}^s) \geq (n^2 - 1)(g - 1) - \sum_{i=0}^{p} (n_i^2 - 1)(g - 1) - pg$$

$$- \sum_{i<j} n_i n_j (g - 1) - \frac{(p+1)(p-2)}{2}$$

$$= \sum_{i<j} n_i n_j (g - 1) - \frac{(p-1)(p+2)}{2}$$

$$= B \quad \text{(say)}.$$

Now, $\sum_{i<j} n_i n_j \geq p(p+1)/2$, therefore

$$B \geq \frac{p(p+1)}{2} (g - 1) - \frac{(p+2)(p-1)}{2}.$$ 

It follows that $B \geq 3$ whenever $p \geq 2$ and $g \geq 3$. If $p = 1$ and $n \geq 3$, then $B/(g - 1) = \sum_{i<j} n_i n_j \geq 2$

and one checks that $B \geq 3$ whenever $g \geq 3$. Proposition 2.1(a) may now be considered as proved.

Remark 3.3. We could use similar techniques to estimate $\text{codim}(S \setminus S^s)$, but our task is made easier by the exact answers in [21], p. 48, A. For just this remark, assume $d > n(2g - 1)$, and let $a = (n, d)$. Then $a \geq 2$. Let $n_0 = n/a$. Then according to loc.cit.,

$$\text{codim}(S \setminus S^s) = \begin{cases} (n^2 - 1)(g - 1) - \frac{n^2}{2}(g - 1) - 2 + g & \text{if } a \text{ is even} \\ (n^2 - 1)(g - 1) - \frac{n^2 + n_0^2}{2}(g - 1) - 2 + g & \text{if } a \text{ is odd.} \end{cases}$$

It now follows that

$$\text{codim}(S \setminus S^s) > 5$$

if $n, g$ are in the range of Theorem 1.3.
3.3. The universal exact sequence on $X \times \mathbb{P}$. We begin by reminding the reader of some elementary facts from commutative algebra. If $A$ is a ring (commutative, with 1), $t \in A$ a non-zero divisor, and $M$ an $A$-module, then each element $m_0 \in M$ gives rise to an equivalence class of extensions

$$0 \longrightarrow M \longrightarrow E_{m_0} \longrightarrow A/tA \longrightarrow 0 \quad (3.5)$$

where $E_{m_0} = (A \bigoplus M) / A(t, m_0)$, and the arrows are the obvious ones. Moreover, if $m_0 - m_1 \in tM$, say

$$m_0 - m_1 = tm'$$

then the extension given by $m_0$ is equivalent to that given by $m_1$. In fact, one checks that

$$E_{m_0} \sim \xrightarrow{E_{m_1}} (a, m) \mapsto (a, m - am') \quad (3.6)$$

gives the desired equivalence of extensions. This is another way of expressing the well known fact that each element of $M/tM = \text{Ext}^1(A/t, M)$ gives rise to an equivalence class of extensions.

One globalizes to get the following: Let $S$ be a scheme, $T \hookrightarrow S$ a closed immersion, $F$ a quasi-coherent $\mathcal{O}_S$-module, $U$ an open neighbourhood of $T$ in $S$, and $t \in \Gamma(U, \mathcal{O}_S)$ an element which defines $T \hookrightarrow U$, and which is a non-zero divisor for $\Gamma(V, \mathcal{O}_S)$ for any open $V \subset U$. Then every global section $s$ of $t^*F = F \otimes \mathcal{O}_T$ gives rise to an equivalence class of extensions

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_T \longrightarrow 0. \quad (3.7)$$

Indeed, we are reduced immediately to the case $S = U$. We build up exact sequences $(3.5)$ on each affine open subset $W \subset S$, by picking a lift $s_W \in \Gamma(W, F)$ of $s|W$. One patches together these exact sequences via $(3.6)$.

Now consider $\mathbb{P} = \mathbb{P}_1 \times_S \ldots \times_S \mathbb{P}_{d-1}$. For each $k = 1, \ldots, d-1$, let $p_k : \mathbb{P}_k \to S_1$ be the natural projection. We have a universal exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow p_k^*M_k \longrightarrow B \longrightarrow 0$$

whence a global section $s_k \in \Gamma(\mathbb{P}_k, p_k^*M_k(1))$. However, note that

$$p_k^*M_k = (1 \times p_k)^*M \mid Z_k^P$$

where we are identifying $Z_k^P$ with $\mathbb{P}_k$. By $(3.7)$ we get exact sequences

$$0 \longrightarrow (1 \times \pi)^*\mathcal{W} \longrightarrow \mathcal{V}_k \longrightarrow \mathcal{O}_{Z_k^P} \otimes L_k \longrightarrow 0$$

where $L_k$ is the line bundle obtained by pulling up $\mathcal{O}_{\mathbb{P}_k}(-1)$. It is not hard to see that $\mathcal{V}_k$ is a family of vector bundles parameterized by $\mathbb{P}$. Glueing these sequences together — the $k$-th and the $l$-th agree outside $Z_k^P$ and $Z_l^P$ — we obtain $(3.1)$.

Now suppose we have a $S_1$-scheme $\psi : S \to S_1$ and the exact sequence $(3.2)$

$$0 \longrightarrow (1 \times \psi)^*\mathcal{W} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0$$

on $X \times S$. Restricting $(3.2)$ to $Z_k^S$ ($1 \leq k \leq d-1$) one checks that the kernel of $(1 \times \psi)^*\mathcal{W} \cap Z_k^S \to \mathcal{E}|Z_k^S$ is a line bundle $\mathcal{L}_k$. Identifying $Z_k^S$ with $S$, we see that $(1 \times \psi)^*\mathcal{W}|Z_k^S = \psi^*\mathcal{W}_k$. Thus $\mathcal{L}_k$ is a line sub-bundle of $\psi^*\mathcal{W}_k$. By the universal property of $\mathbb{P}_k$, we see that we have a unique map of $S_1$-schemes

$$g_k : S \longrightarrow \mathbb{P}_k$$
such that $O(-1)$ gets pulled back to $L_k$. The various $g_k$ give us a map
\[ g: S \to \mathbb{P} \]
One checks that $g$ has the required universal property. The uniqueness of $g$ follows from the uniqueness of each $g_k$.

**Remark 3.4.** It is clear from the construction that the map
\[ f = f_{X,L,\chi}: \mathbb{P}_{X,L,\chi} \to SU_X(n, L) \]
varies well with $(X, L, \chi)$. This implies that the correspondence (2.1) also varies well with $(X, L, \chi)$ and hence so do $\psi_{X,L,\chi}$ and $\varphi_{X,L,\chi}$.

### 4. Polarizations

Let $Y$ be an $m$-dimensional projective variety. Suppose that $U$ is a smooth Zariski open subset. One then has the following version of the Lefschetz theorem.

**Theorem 4.1.** If $H$ is a hyperplane section of $Y$ such that $U \cap H$ is non-empty, then
\[ H^i(U, \mathbb{Q}) \to H^i(U \cap H, \mathbb{Q}) \]
is an isomorphism for $i < m - 1$ and injective when $i = m - 1$.

**Proof.** We need some results involving Verdier duality. The standard references are [8] and [11]. Let $S$ be an analytic space and $p_S$ the map from $S$ to a point. For $F \in D_{\text{const}}(S, \mathbb{Q})$ (the derived category of bounded complexes of $\mathbb{Q}_S$-sheaves whose cohomology sheaves are $\mathbb{Q}_S$-constructible), set
\[ D_S(F) = R\text{Hom}_S(F, p_S! \mathbb{Q}) \]
We then have by Verdier duality
\[ \mathbb{H}^i(S, F) \sim \mathbb{H}^{-i}(S, D_S(F))^* \] (4.1)
Here $\mathbb{H}^*$ denotes “hypercohomology”.

For an open immersion $h: S' \hookrightarrow S$, one has canonical isomorphisms
\[ Rh_* D_S' G \sim D_S(h_* G) \] (4.2)
and
\[ Rh_! D_S' G \sim D_S(Rh_* G) \] (4.3)
Here $G \in D_{\text{const}}^b(S', \mathbb{Q})$. The first isomorphism is easy (using Verdier duality for the map $h$) and the second follows from the first and from the fact that $D_S$ is an involution. We have used (throughout) the fact that $h_!$ is an exact functor.

If $S$ is smooth, we have
\[ p_S^! \mathbb{Q} = \mathbb{Q}_S[2 \dim S]. \] (4.4)

In order to prove the theorem, let $V = U \setminus H$ and $W = Y \setminus H$. We then have a cartesian square
\[
\begin{array}{ccc}
V & \xrightarrow{\iota'} & U \\
\downarrow{\iota'} & & \downarrow{\iota} \\
W & \xrightarrow{\iota} & Y
\end{array}
\]
where each arrow is the obvious open immersion. We have, by (4.2) and (4.3), the identity
\[ \gamma_! \mathbb{R}i_*'D_VQ_V = D_V(\mathbb{R}j_*'Q_V). \] (4.5)

Consider the exact sequence of sheaves
\[ 0 \longrightarrow i'_!Q_V \longrightarrow Q_U \longrightarrow g_*Q_{H \cap U} \longrightarrow 0 \]
where \( g: H \cap U \rightarrow U \) is the natural closed immersion. It suffices to prove that \( H^i(U, i'_!Q_V) = 0 \) for \( i \leq m - 1 \). Now,
\[ H^i(U, i'_!Q_V) = \mathbb{H}^i(Y, \mathbb{R}j_*'Q_V). \]

Using (4.1), (4.5) and (4.4), the above is dual to
\[ \mathbb{H}^{-i}(Y, \gamma_!\mathbb{R}i_*'D_VQ_V) = \mathbb{H}^{2m-i}(Y, \gamma_!\mathbb{R}i_*'Q_V) \]
But \( \gamma_!\mathbb{R}i_*' = \mathbb{R}i_*'\gamma_! \), and hence the above is
\[ = \mathbb{H}^{2m-i}(Y, \mathbb{R}i_*'(j'_!Q_V)) \]
\[ = \mathbb{H}^{2m-i}(W, j'_!Q_V) \]
\[ = H^{2m-i}(W, j'_!Q_V). \]

Now, \( W \) is an affine variety, and therefore, according to M. Artin, its constructible cohomological dimension is less than or equal to its dimension \( [2] \). Consequently, the above chain of equalities vanish whenever \( i < m \) (see also \( [9] \)).

We immediately have:

**Corollary 4.1.** Let \( e = \text{codim}(Y \setminus U) \). For \( i < e - 1 \), the Hodge structure \( H^i(U) \) is pure of weight \( i \).

**Proof.** This is true if \( U \) is projective. In general proceed using Bertini’s theorem, induction, Theorem 4.1 and the fact that submixed Hodge structures of pure Hodge structures are pure \( [6] \). \( \square \)

Let \( i \in \mathbb{N} \) and \( \mathcal{L} \) a line bundle on \( Y \) be such that
(a) \( H^j(U, \mathbb{Q}) = 0 \) for \( j = i - 2, i - 4, \ldots \);
(b) \( i < e - 1 \);
(c) \( \mathcal{L} \) is very ample.

**Remark 4.1.** Note that if \( Y = S, U = S^s \), then \( i = 3 \) and \( \mathcal{L} = \xi \) (\( \xi = \) the very ample bundle of \( [2] \)) satisfy the above conditions by the results of \( [1] \) and Remark 3.3.

Let \( M \) be the intersection of \( k = m - e + 1 \) hyperplanes in general position. Then \( M \) is a smooth variety contained in \( U \). Let
\[ l: H^i(U) \longrightarrow H^2m-i(U) \]
be the composite of
\[ H^i(U) \longrightarrow H^i(M) \]
\[ \longrightarrow H^{2m-2k-i}(M) \]
\[ \longrightarrow H^{2m-i}(U) \]
where the first map is restriction, the second is “cupping with $c_1(L)^{m-k-i}$” and the third is the Poincaré dual to restriction. The map $l$ is also described as

$$x \mapsto x \cup c_1(L)^{m-k-i} \cup [M].$$

One then has (easily)

**Lemma 4.1.** If $M'$ is another $k$-fold intersection of general hyperplanes, then $[M'] = [M]$. Therefore $l$ depends only on $L$.

**Proposition 4.1.** The pairing

$$<x, y> = \int_U l(x) \cup y$$

on $H^i(U, \mathbb{C})$ gives a polarization on the pure Hodge structure $H^i(U)$.

**Proof.** By Theorem 4.1, we have an isomorphism

$$r: H^i(U) \rightarrow H^i(M).$$

The latter Hodge structure carries a polarization given by

$$<\alpha, \beta> = \int_M c_1(L)^{m-k-i} \cup \alpha \cup \beta.$$

The conditions on $i$ and the Hodge-Riemann bilinear relations on the primitive part of $H^i(M, \mathbb{C})$, assure us that the above is indeed a polarization (see [10] or Chap. V, §6 of [24]). In fact, our conditions on $i$ imply that the primitive part of $H^i(M)$ is everything. This translates to a polarization on $H^i(U)$ given by

$$<x, y> = \int_U l(x) \cup y.$$

This gives the result.

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