Recurrent formulas for explicitly finding some minimal polynomials

I G Galyautdinov and E E Lavrentyeva
Kazan Federal University, 18 Kremlyovskaya Street, 420008, Kazan, Russia

E-mail: ialee-4@mail.ru

Abstract. When solving many applied problems, it becomes necessary to solve nonlinear equations, including the need to find the roots of polynomials. Therefore, the study of the properties of polynomials is an important problem. In the present paper, we obtain recurrence formulas for minimal polynomials over circular fields.

1. Introduction
Nonlinear equations arise in solving important problems in many fields, for example, in the theory of underground filtration [1–8], the shells non-linear theory [9–16], the theory of multilayer shells [17–22], the theory of soft shells [23–27], and in modeling low-temperature plasmas [28–35], when solving spectral problems with nonlinear occurrence of a parameter [36–38], etc. At the same time, the functions describing the relationship between the parameters of processes are power functions, including polynomial functions.

In addition, to solve nonlinear, two-layer iterative processes [39–41] are used, the preconditioners of which are duality operators [42–44]. The gauge functions defining these operators are also often polynomial. When solving the above classes of problems, it becomes necessary to repeatedly find the roots of polynomials. Therefore, the problem of studying the properties of polynomial functions is very urgent.

In the present paper, which is a continuation of [45–47], recurrent formulas for minimal polynomials over circular fields are obtained.

2. Basic definitions and notations
We give a number of information we need in the future. The field $K_h = \mathbb{Q}(u_h)$, which is obtained by attaching rational numbers of a primitive root $u_h$ of unit of degree $h$, where $h$ is a natural number, to a field $\mathbb{Q}$, is called a circular field. Since $K_{2h} = K_h$ for odd $h$, in what follows, we will assume that $h$ takes only even values. Moreover, non-isomorphic fields $K_h$ correspond to different $h$. If $h_1$ and $h_2$ are two even positive integers and $h_1$ is divisible by $h_2$, i.e., $h_1 = h_2l$, then $K_{h_2} \subset K_{h_1}$. In the case $h = 2$, we have that and is a real field. When $h > 2$, the fields $K_h$ are complex.

Let $n > 2$ be an arbitrary positive integer and $h = 2n$. Then the element $u_h + u_h^{-1}$ generates a real field $L_n = \mathbb{Q}(u_h + u_h^{-1})$. As is known [48], the field $L_n$ is a subfield of the circular field $K_h$ and $[K_h : L_n] = 2$. Since it follows from the relations $\mathbb{Q} \subset L \subset K_h$ that the degree $[L : \mathbb{Q}]$ of the field
\( L \) is a divisor of the degree \([K_h:Q]\) of the field \( K_h \), equality \([K_h:L_n]=2\) means that \( L_n=Q(2\cos(\pi/n)) \) is the maximum and real subfield of the circular field \( K_h=K_{2n} \).

It is known [49] that the Galois group \( G(K_{2n}/Q) \) over the field \( K_h \) over \( Q \) is isomorphic to the multiplicative group \( UZ_{2^n} \) of residue classes that are mutually simple with module \( 2n \). In addition, the subfield \( Q(2\cos(\pi/n)), 2\cos(\pi/n)=u_h+u_h^{-1}, \) is invariant with respect to the subgroup \( H=\{1,-1\} \subset UZ_{2^n}. \) Therefore, \( G(L_n/Q)\cong UZ_{2^n}/H \), i.e., the Galois group of the field \( L_n \) over \( Q \) is isomorphic to the factor group \( UZ_{2^n}/H \).

Recall that a number \( \alpha \) is called algebraic if it is the root of some polynomial with rational coefficients. Among such polynomials there exists a unique normed polynomial \( f(x)\in Q[x] \) irreducible over a field \( Q \) whose root is a number \( \alpha \). This polynomial is called the minimal polynomial of a number \( \alpha \), and \( \deg f(x) \) is called its degree.

The Euler function \( \varphi(n) \) is the multiplicative arithmetic function, equal to the number of natural numbers, smaller \( n \) and mutually simple with it. For a prime number \( p \), the Euler function is given by the formula \( \varphi(p)=p-1 \). To calculate the Euler function of the degree of a prime number \( p \) the following formula is used: \( \varphi(p^m)=p^m-p^{m-1} \). For an arbitrary natural number \( n \), the value of \( \varphi(n) \) is represented as \( \varphi(n)=n\prod_{p|n}(1-1/p) \).

The extension \( \Omega \) that is obtained from a field \( P \) by attaching to it a single algebraic number \( \alpha \) over a field \( P \) is called a simple algebraic extension of the field \( P \). The number \( \alpha \) for the field \( P \) is called a primitive (generating) element.

In this paper, recurrent formulas are obtained for calculating the minimum polynomial \( q_n(x) \) of a primitive element \( \alpha=2\cos(\pi/n) \) of the circular field \( L_n=Q(2\cos(\pi/n)) \). Thus we are using the results previously obtained for the Chebyshev polynomials. Examples of using recurrent formulas for constructing the minimal polynomials are given.

3. Minimal polynomials of numbers \( 2\cos(\pi/n) \)

Let \( T_n(x), \; n=0,1,2, \ldots, \) be the Chebyshev polynomials [50]. As is known, they have the property \( T_n(\cos(\phi))=\cos(n\phi) \) and are calculated by the recurrence formulas \( T_{n+1}(x)=2xT_n(x)−T_{n−1}(x), T_0(x)=1, T_1(x)=x, \; n=0,1,2, \ldots \).

The following statements are true [45].

**Theorem 1.** Let \( n=2k+1 \) be a positive integer number and \( \varphi \) be an Euler function. Then there are polynomials \( p_n(x)\in Q[x] \) of degree \( \varphi(n)/2 \) with roots \( \cos(s\pi/n), \) where odd \( s \) is less than \( n \) and takes all values that are mutually simple with \( n \). These polynomials are found recursively from the formulas:

\[
p_1(T_n(x))=p_1(x)(p_1(x))^2, \quad n=2k+1 \text{ is a prime,} \quad p_1(x)=x+1, \quad (1)
\]

\[
p_m(T_q(x))=p_n(x), \quad n=mq, \quad m\div q, \quad q \text{ is a prime,} \quad (2)
\]

\[
p_m(T_q(x))=p_n(x)p_n(x), \quad n=mq, \quad \text{LCD}(m,q)=1, \quad q \text{ is a prime.} \quad (3)
\]

The numbers \( \cos(t\pi/n), \) where even \( t<n \) takes all values that are mutually simple with \( n \), are the roots of the polynomial \((-1)^d p_n(x) \) where \( d=\deg(p_n(x)) \).

**Theorem 2.** Let \( n=2k \) be a positive integer number and \( \varphi \) be an Euler function. Then there are polynomials \( p_n(x)\in Q[x] \) of degree \( \varphi(n) \) with roots \( \cos(s\pi/n), \) where odd \( s \) is less than \( n \) and takes all values that are mutually simple with \( n \). These polynomials are found recursively from the formulas:

\[
\]
$p_n(x) = p_k(T_2(x)), \quad n = 2k, \quad p_2(x) = x. \quad (4)$

**Theorem 3.** Polynomials $p_n(x)$ constructed by formulas (1)–(4) are irreducible over a field $Q$.

By virtue of Theorem 3, the polynomials $p_n(x)$ constructed in Theorems 1 and 2 are minimal polynomials of numbers $\cos(\pi / n)$ up to a constant factor. This means that formula (5)

$q_n(x) = p_n(x/2), \quad n = 1, 2, \ldots, \quad (5)$

enables us to calculate the minimum polynomials of numbers $2\cos(\pi / n)$ but one can write recurrent formulas for explicitly calculating polynomials $q_n(x)$ with the roots $2\cos(\pi / n)$. This will be discussed in the next section.

4. Recurrent formulas minimal polynomials

In order to obtain recurrence formulas for the explicit computation of polynomials $q_n(x)$ with roots $2\cos(\pi / n)$, we turn from the Chebyshev polynomials $T_n(x)$ to the polynomials $S_n(x)$ given by the formula

$S_n(x) = 2T_n(x/2), \quad n = 0, 1, 2, \ldots \quad (6)$

Polynomials satisfy the recurrence relation

$S_{n+1}(x) = 2S_n(x) - S_{n-1}, \quad n = 0, 1, 2, \ldots \quad (7)$

From formulas (6) and (7) it follows that the polynomials $S_n(x)$ have properties [49]:

1. $\deg S_n(x) = n$;
2. the leading coefficient is 1;
3. all coefficients of $S_n(x)$ are integers;
4. $S_n(2\cos \alpha) = 2\cos(n\alpha)$.

Using formula (6) and the fact that $T_0(x) = 1$, $T_1(x) = x$, let’s construct the several polynomials $S_n(x)$. We have $S_0(x) = 2$, $S_1(x) = x$, $S_2(x) = x^2 - 2$, $S_3(x) = x^3 - 3x$, $S_4(x) = x^4 - 4x^2 + 2$, $S_5(x) = x^5 - 5x^3 + 5x$, $S_6(x) = x^6 - 6x^4 + 9x^2 - 2$.

The properties of polynomials $S_n(x)$ make it possible to prove the following result.

**Lemma 1.** For any natural $k$, the degree $x^k$ is representable as a linear combination with integer rational coefficients of the polynomials $1$, $S_1(x)$, $S_2(x)$, $\ldots$, $S_k(x)$.

**Proof.** The proof is carried out by induction on $k$. For $k = 1, 2, 3$ we have $x = S_1(x)$, $x^2 = S_2(x) + 2$, $x^3 = S_3(x) + 3S_1(x)$. If we assume that the required representations are true for all $i = 1, 2, \ldots, k$, then from properties 2 and 3 of polynomials $S_n(x)$ for $S_{k+1}(x)$ we obtain

$x^{k+1} = S_{k+1}(x) + a_{k-1}x^{k-1} + a_{k-3}x^{k-3} + \ldots$, where $a_{k-i}$ are rational integers. Hence, taking into account the induction hypothesis, we find

$x^{k+1} = S_{k+1}(x) + b_{k-1}S_{k-1}(x) + b_{k-3}S_{k-3}(x) + \ldots$, where $b_{k-i}$ are also whole rational numbers. Lemma is proved.

If in equations (1)–(4), using formulas (5) and (6), we replace the polynomials $p_2(x)$ and $T_2(x)$, respectively, by the polynomials $q_2(x)$ and $S_2(x)$, then Theorems 1 and 2 take the following form.

**Theorem 4.** Let $n = 2k + 1$ be a positive integer number and $\varphi$ be an Euler function. Then there are polynomials $q_n(x) \in \mathbb{Q}[x]$ of degree $\varphi(n) / 2$ with roots $2\cos(s\pi / n)$, where odd $s$ is less than $n$ and takes all values that are mutually simple with $n$. These polynomials are found recursively from the formulas:

$q_1(S_2(x)) = q_1(x)(q_2(x))^2, \quad n = 2k + 1 \text{ is a prime}, \quad q_1(x) = x + 2, \quad (8)$

$q_m(S_q(x)) = q_n(x), \quad n = mq, \quad m \mid q, \quad q \text{ is a prime}, \quad (9)$
\[ q_n(S_q(x)) = q_n(x)q_m(x), \quad n = m \cdot q, \quad \text{LCD}(m, q) = 1, \quad q \text{ is a prime.} \quad (10) \]

The numbers \(2\cos(t \pi / n)\), where even \(t < n\) takes all values that are mutually simple with \(n\), are the roots of the polynomial \((-1)^d q_n(-x)\) where \(d = \deg(q_n(d))\).

**Theorem 5.** Let \(n = 2k\) be a positive integer number and \(\varphi\) be an Euler function. Then there are polynomials \(q_s(x) \in \mathbb{Q}[x]\) of degree \(\varphi(n)\) with roots \(2\cos(s\pi / n)\), where odd \(s\) is less than \(n\) and takes all values that are mutually simple with \(n\). These polynomials are found recursively from the formulas:

\[ q_n(x) = q_k(S_2(x)), \quad n = 2k, \quad q_2(x) = x. \quad (11) \]

From Theorems 3–5 and formula (5), taking into account the properties of polynomials \(S_n(x)\), it follows that the following theorem holds.

**Theorem 6.** If \(n\) is a natural number, then \(2\cos(\pi / n)\) is an algebraic integer, and \(q_n(x)\) is its minimal polynomial.

**Proof.** As previously noted, the polynomial \(S_n(x)\) has integer rational coefficients and its leading coefficient is 1. Therefore, from the recurrent formulas (8)–(11) it follows that the polynomial \(q_n(x)\) also has integer rational coefficients and its leading coefficient is 1. This means that its root is algebraic integer. It follows from Theorem 3 and formula (5) that \(q_n(x)\) is also irreducible over a field. Hence, \(q_n(x)\) is the minimal polynomial of numbers \(2\cos(\pi / n)\). The theorem is proved.

Let’s give examples of the polynomials \(q_n(x)\). We have \(q_1(x) = x + 2, q_2(x) = x, q_3(x) = x - 1, q_4(x) = x^2 - 2, q_5(x) = x^2 - x - 1, q_6(x) = x^2 - 3, q_7(x) = x^3 - x^2 - 2x + 1, q_8(x) = x^4 - 4x^2 + 2, q_9(x) = x^3 - 3x - 1, q_{10}(x) = x^4 - 5x^2 + 5, q_{11}(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x + 1, q_{12}(x) = x^4 - 4x^2 + 1\) etc.

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