ON THE COMPARISON OF TWO CONSTRUCTIONS OF WITT VECTORS OF NON-COMMUTATIVE RINGS

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Abstract. Let $A$ be any associative ring, possibly non-commutative and let $p$ be a prime number. Let $E(A)$ be the ring of $p$-typical Witt vectors as constructed by Cuntz and Deninger in [1] and $W(A)$ be that constructed by Hesselholt in [3]. The goal of this paper is to answer the following question by Hesselholt: Is $\text{HH}_0(E(A)) \cong W(A)$? We show that in the case $p = 2$, there is no such isomorphism possible if one insists that it be compatible with the Verscheibung operator and the Teichmüller map.

1. Introduction

Let $A$ be an associative unital ring and $p$ be a prime number. When $A$ is commutative, the classical construction of $p$-typical Witt vectors gives us a topological ring $W(A)$, equipped with a Verscheibung operator 

$W(A) \xrightarrow{V} W(A)$

and a Teichmüller map, which we denote by

$A \xrightarrow{\langle \rangle} W(A)$. 

In this paper we consider two generalizations of this construction to the non-commutative case. One of them is a construction of an abelian group $W(A)$ given by Hesselholt in [2] (see also [3]). The other is a construction of a ring $E(A)$ by Cuntz and Deninger, given in [1]. Just as in the commutative case, $E(A)$ and $W(A)$ are topological rings and are equipped with the Verscheibung operator and the Teichmüller map. Moreover, both $W(A)$ and $E(A)$ are isomorphic to the classical construction of Witt vectors when $A$ is commutative. Let $\text{HH}_0(E(A)) := E(A)/[E(A), E(A)]$. The goal of this paper is to answer the following question of Hesselholt.

Question 1.1. Is $W(A)$ isomorphic to $\text{HH}_0(E(A))$?

We note that $\text{HH}_0(E(A))$ inherits the Verschiebung operator $V$ and the Teichmüller map $\langle \rangle$, from $E(A)$. In this paper we only consider maps from $W(A)$ to $\text{HH}_0(E(A))$ which are compatible with $V$ and $\langle \rangle$.

The following is one of the main results of this paper.

Theorem 1.2. Let $A = \mathbb{Z}\{X,Y\}$ and $p = 2$. Then

(i) $W(A)$ is topologically generated by $\{V^n(\langle a \rangle) \mid n \in \mathbb{N}_0, a \in A\}$.

(ii) $\text{HH}_0(E(A))$ is not topologically generated by $\{V^n(\langle a \rangle) \mid n \in \mathbb{N}_0, a \in A\}$.

(iii) there is no continuous surjective map from $W(A) \to \text{HH}_0(E(A))$ which commutes with $V$ and is compatible with $\langle \rangle$.

One can thus slightly modify Question 1.1 and ask for existence of a map (not necessarily surjective) from $W(A) \to \text{HH}_0(E(A))$ compatible with the Verscheibung operator and the Teichmüller map. The following theorem, shows that even this is not possible, at least in the case $p = 2$.

Theorem 1.3. Let $A := \mathbb{Z}\{X,Y\}$ and $p = 2$. Then there is no continuous group homomorphism from $W(A) \to \text{HH}_0(E(A))$ which is compatible with $V$ and $\langle \rangle$. 

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We believe that Theorems 1.2 and 1.3 should hold for all primes \( p \).

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2. Preliminaries

Let \( A \) be a unital associative ring, not necessarily commutative. Fix a prime number \( p \). Let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). We briefly recall a few facts about the constructions of Witt ring by Hesselholt and by Cuntz and Deninger.

2.1. Hesselholt’s construction of \( W(A) \): Let \( A \) be a unital associative ring. Consider the map (called as ghost map)

\[
\omega : A^{\mathbb{N}_0} \rightarrow \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}
\]

\[
\omega(a_0, a_1, a_2, ...) := (\omega_0(a_0), \omega_1(a_0, a_1), \omega_2(a_0, a_1, a_2), ...)
\]

where \( \omega_i \)’s are ghost polynomials defined by

\[
\omega_i(a_0, ..., a_i) := a_0^{p^i} + p a_1^{p^{i-1}} + p^2 a_2^{p^{i-2}} + \cdots + p^i a_i.
\]

\( \omega \) is merely a map of sets and not a homomorphism of groups. For every integer \( n \in \mathbb{N}_0 \), we also have truncated versions of the above map (denoted again by \( \omega \))

\[
\omega : A^n \rightarrow \left( \frac{A}{[A,A]} \right)^n
\]

\[
\omega(a_0, ..., a_{n-1}) := (\omega_0(a_0), \omega_1(a_0, a_1), ..., \omega_{n-1}(a_0, a_1, a_2, ... a_{n-1}))
\]

Hesselholt then inductively defines groups \( W_n(A) \) (see [3]) such that the map \( \omega \) factor through

\[
A^n \xrightarrow{q_n} W_n(A) \xrightarrow{\iota} \left( \frac{A}{[A,A]} \right)^n
\]

and the following are satisfied

1. \( W_1(A) = \frac{A}{[A,A]} \).
2. \( q_n \) is surjective map of sets.
3. \( \iota \) is an additive homomorphism and is injective if \( \frac{A}{[A,A]} \) is \( p \)-torsion free.

Define \( W(A) := \varprojlim_n W_n(A) \). Clearly one also has a factorization of \( A^{\mathbb{N}_0} \xrightarrow{\omega} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0} \) as

\[
A^{\mathbb{N}_0} \xrightarrow{q} W(A) \xrightarrow{\iota} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}
\]

where \( q \) is always surjective and where \( \iota \) is injective if \( \frac{A}{[A,A]} \) has no \( p \)-torsion. Thus every element of \( W(A) \) is of the form \( q(a_0, a_1, ...) \) for some (not necessarily unique) \( (a_0, a_1, ...) \in A^{\mathbb{N}_0} \). We will often abuse notation to denote an element of \( W(A) \) simply by a tuple \( (a_0, a_1, ...) \) instead of the cumbersome \( q(a_0, a_1, ...) \) while taking care to handle well definedness issues should they arise.

We have the Verschiebung operator

\[ V : W(A) \rightarrow W(A) \]

and the Teichmüller map

\[ \langle \rangle : A \rightarrow W(A) \]

which satisfy

\[ V(a_0, a_1, ...) = (0, a_0, a_1, ...) \]
and

\[ \langle a \rangle = (a, 0, 0, \ldots) \]

One can show that \( V \) and \( \langle \rangle \) are well defined and that \( V \) is an additive homomorphism. Similarly for \( n \in \mathbb{N}_0 \), we have truncated versions (denoted by the same notation) \( W_n(A) \rightarrow W_n(A) \) and \( A \rightarrow W_n(A) \), satisfying

\[ V(a_0, \ldots, a_{n-1}) = (0, a_0, a_1, \ldots, a_{n-2}) \]

and

\[ \langle a \rangle = (a, 0, 0, \ldots, 0) \].

### 2.2. Cuntz and Deninger’s construction of \( E(A) \):

For any associative ring \( R \)

(i) Let \( V : R_{\mathbb{N}_0} \rightarrow R_{\mathbb{N}_0} \) be the map defined by \( V(a_0, a_1, \ldots) := p(0, a_0, a_1, \ldots) \).

(ii) For an element \( a \in R \), define \( \langle a \rangle \in R_{\mathbb{N}_0} \) by \( \langle a \rangle := (a, a^p, a^{p^2}, \ldots) \).

(iii) Let \( X(R) \subset R_{\mathbb{N}_0} \) be the closed subgroup generated by

\[ \left\{ V^m(\langle a_1 \rangle \cdots \langle a_r \rangle) \mid m \in \mathbb{N}_0, r \in \mathbb{N}, a_i \in R \forall i \right\} \]

Similarly, if \( I \subset R \) is an ideal, we let \( X(I) \) denote the closed subgroup generated by

\[ \left\{ V^m(\langle a_1 \rangle \cdots \langle a_r \rangle) \mid m \in \mathbb{N}_0, r \in \mathbb{N}, a_i \in I \forall i \right\} \]

For \( n \in \mathbb{N}_0 \), we also have the following truncated versions of the definitions.

(i) Let \( V : \prod_{i=0}^n R \rightarrow \prod_{i=0}^n R \) be the map defined by \( V(a_0, a_1, \ldots, a_n) := p(0, a_0, a_1, \ldots, a_{n-1}) \).

(ii) For an element \( a \in R \), define \( \langle a \rangle \in \prod_{i=0}^n R \) by \( \langle a \rangle := (a, a^p, a^{p^2}, \ldots, a^{p^n}) \).

(iii) Let \( X_n(R) \subset \prod_{i=0}^n R \) be the subgroup generated by

\[ \left\{ V^m(\langle a_1 \rangle \cdots \langle a_r \rangle) \mid m \in \mathbb{N}_0, r \in \mathbb{N}, a_i \in R \forall i \right\} \]

Note that the above definitions depend on the chosen prime number \( p \). Now let \( A \) be any associative ring. Let \( ZA \) be the monoid algebra of the multiplicative monoid underlying \( A \). Thus the elements of \( ZA \) are formal sums of the form \( \sum_{r \in \mathbb{Z}} n_r [r] \) with almost all \( n_r = 0 \). We have a natural epimorphism of rings from \( ZA \rightarrow A \) and we let \( I \) denote its kernel. One now defines

\[ E(A) := \frac{X(ZA)}{X(I)} \quad \text{and} \quad E_n(A) := \frac{X_n(ZA)}{X_n(I)} \]

We have a set map, called the Teichmüller map

\[ (\ ) : A \rightarrow E(A) \]

\[ \langle a \rangle := ([a], [a]^p, [a]^{p^2}, \ldots) \ \text{mod} \ X(I). \]

It is elementary to check that

(i) \( X(ZA) \) is a subring of \( (ZA)_{\mathbb{N}_0} \) and \( X(I) \subset X(ZA) \) is a two sided ideal. Thus \( E(A) \) has the structure of an associative ring. Similarly \( E_n(A) \) has a ring structure \( \forall n \in \mathbb{N}_0 \).

(ii) \( V : (ZA)_{\mathbb{N}_0} \rightarrow (ZA)_{\mathbb{N}_0} \) induces maps (denoted by the same notation for simplicity) called Verschiebung operators

\[ E(A) \xrightarrow{V} E(A) \quad \text{and} \quad E_n(A) \xrightarrow{V} E_n(A). \]

(iii) The morphism \( E(A) \rightarrow E_n(A) \) is a continuous epimorphism of rings, where target has discrete topology and \( E(A) \) has topology inherited from the product topology on \( (ZA)_{\mathbb{N}_0} \). Moreover this map commutes with \( V \).
Recall that for any associative ring $R$, $HH_0(R) := R/[R,R]$. However, as $E(A)$ has a topology, we use the notation $HH_0(E(A))$ to denote $E(A)/[E(A),E(A)]$ where $[E(A),E(A)]$ is the closure of $[E(A),E(A)]$. The Verschiebung operator and the Teichmüller map induce (see (3.2))

$$V : HH_0(E(A)) \rightarrow HH_0(E(A))$$

and

$$\langle \rangle : A \rightarrow HH_0(E(A)).$$

2.3. Comparison of $E(A)$ and $X(A)$: In the case when $A$ is commutative and has no $p$-torsion, $E(A)$ is isomorphic to $X(A)$. In the non-commutative case we do not know a sufficient condition for $E(A)$ to be isomorphic to $X(A)$. Nevertheless, in this paper we will often find it easier to work in the ring $X(A)$ instead of $E(A)$. We note that exact sequence

$$0 \rightarrow I \rightarrow ZA \rightarrow A \rightarrow 0$$

induces a map $X(ZA) \rightarrow X(A)$, with $X(I)$ in its kernel. Thus we have a continuous ring homomorphism from

$$E(A) := X(ZA)/X(I) = \pi \rightarrow X(A).$$

Like $E(A)$, $X(A)$ also has a Verschiebung operator $V : X(A) \rightarrow X(A)$ given by

$$V(a_0, a_1, ...) = p(0, a_0, a_1, ...)$$

and the Teichmüller map $\langle \rangle : A \rightarrow X(A)$ given by

$$\langle a \rangle := (a, a^p, a^{p^2}, ...).$$

Clearly, the map $E(A) \xrightarrow{\pi} X(A)$ is compatible with the Verschiebung operator and Teichmüller map on both sides.

3. Proof of Theorem 1.2

Lemma 3.1. Let $A$ be any associative ring. Then $[X(A),X(A)]$ is the closed subgroup of $X(A)$ generated by

$$\left\{ p^nV^n([\langle a_1 \rangle \cdots \langle a_s \rangle, \langle b_1^{n^m} \cdots b_t^{n^m} \rangle]) \mid m, n \in \mathbb{N}_0, m \leq n, s, t \in \mathbb{N}, a_i, b_j \in A \right\}.$$

Proof. Since $X(A)$ is the closed subgroup of $A^{\mathbb{N}_0}$ generated by

$$\left\{ V^n((a_1) \cdots (a_s)) \mid s \in \mathbb{N}, n \in \mathbb{N}_0, a_i \in A \right\},$$

$[X(A),X(A)]$ is the closed subgroup generated by

$$\left\{ [V^n((a_1) \cdots (a_s)), V^m((b_1) \cdots (b_t))] \mid s, t \in \mathbb{N}, m, n \in \mathbb{N}_0, a_i, b_j \in A \right\}.$$

Since

$$[V^n((a_1) \cdots (a_s)), V^m((b_1) \cdots (b_t))] = -[V^m((b_1) \cdots (b_t)), V^n((a_1) \cdots (a_s))],$$

$[X(A),X(A)]$ is also the closed subgroup generated by

$$\left\{ [V^n((a_1) \cdots (a_s)), V^m((b_1) \cdots (b_t))] \mid s, t \in \mathbb{N}, a_i, b_j \in A, m, n \in \mathbb{N}_0 \text{ with } m \leq n \right\}.$$

When $m \leq n$, a straightforward calculation shows that

$$[V^n((a_1) \cdots (a_s)), V^m((b_1) \cdots (b_t))] = p^nV^n([\langle a_1 \rangle \cdots \langle a_s \rangle, \langle b_1^{n^m} \cdots b_t^{n^m} \rangle]).$$

This proves the lemma. \hfill \square

Corollary 3.2. $V([X(A),X(A)]) \subset [X(A),X(A)]$. Thus $V$ induces a map (to be also denoted by $V$)

$$HH_0(X(A)) \xrightarrow{V} HH_0(X(A)).$$

Thus like the Witt rings and like $X(A)$, $HH_0(X(A))$ also comes equipped with Verschiebung operator and a Teichmüller map.
Lemma 3.3. Let $A$ be any associative unital ring such that $\frac{A}{[A,A]}$ has no $p$-torsion. Then

(i) for any element $(a_0, a_1, \ldots) \in W(A)$

$$(a_0, a_1, \ldots) = \sum_{i=0}^{\infty} V^i(a_i).$$

(ii) $W(A)$ is topologically generated by the set $\left\{ V^n(a) \mid a \in A, n \in \mathbb{N}_0 \right\}$, i.e. $W(A)$ is the closure of the subgroup generated by this set.

Proof. (ii) is a direct consequence of (i). Thus it is enough to prove (i). The ghost map $W(A) \xrightarrow{\omega} (\frac{A}{[A,A]})^{\text{h0}}$ is a continuous group homomorphism, where addition in $(\frac{A}{[A,A]})^{\text{h0}}$ is componentwise. Since $\frac{A}{[A,A]}$ has no $p$-torsion, $\omega$ is injective. Thus, to prove (i), it is enough to show

$$\omega(a_0, a_1, \ldots) = \sum_{i=0}^{\infty} \omega(V^i(a_i)).$$

This is checked by explicit calculation.

$$\sum_{i=0}^{\infty} \omega(V^i(a_i))) = \sum_{i=0}^{\infty} \omega(0, 0, \ldots, 0, a_i, 0, \ldots)$$

(where $a_i$ is in the $i$-th position)

$$= \sum_{i=0}^{\infty} (0, 0, \ldots, 0, p^i a_1, p a_0^i, p a_0^i, \ldots)$$

$$= \sum_{i=0}^{\infty} (0, 0, \ldots, 0, p^i a_1, p^i a_1^p, p^i a_1^p, \ldots)$$

$$= (a_0, a_0^p + pa_1, a_0^p + pa_1^p, a_0^p + pa_1^p, \ldots)$$

$$= \omega(a_0, a_1, a_2, \ldots)$$

□

Remark 3.4. It is possible that the above lemma holds without the assumption that $\frac{A}{[A,A]}$ has no $p$-torsion.

Let $A = \mathbb{Z}\{X, Y\}$. Then $A = \oplus_{i \geq 0} A_i$ is naturally a graded ring where $A_i$ is the free abelian group generated by words in $X, Y$ of length $i$. The center of $A$ is $A_0 = \mathbb{Z}$. Consider the filtration $F^n A$ on $A$ defined by

$$F^n A = \oplus_{i \geq n} A_i.$$  

Proof of Theorem 1.2. (iii) is a direct consequence of (i) and (ii). Lemma 3.3 proves (i). Thus to prove the theorem, it remains to show that $HH_0(E(A))$ is not topologically generated by $\left\{ V^n(a) \mid a \in A, n \in \mathbb{N}_0 \right\}$. The surjectivity of $E(A)$ to $X(A)$ implies the surjectivity of the map $HH_0(E(A)) \to HH_0(X(A))$. As this map commutes with the $V$ and $\langle \rangle$, it suffices to show that $HH_0(X(A))$ is not topologically generated by $\left\{ V^n(a) \mid a \in A, n \in \mathbb{N}_0 \right\}$. More specifically, we will show that the element $\langle X \rangle \langle Y \rangle \in HH_0(X(A))$ is not in the closure of the subgroup generated by $\left\{ V^n(a) \mid a \in A, n \in \mathbb{N}_0 \right\}$. Suppose it is. Then, there exists elements $c_i \in A$ and $\ell_i \in \mathbb{N}_0$, such that in $X(A)$ we have a congruence of the form

$$\langle X \rangle \langle Y \rangle = \sum_i V^{\ell_i} \langle c_i \rangle \mod [X(A), X(A)].$$
Clearly the integers $\ell_i$ are such that the summation above converges. In particular, only finitely many of the $\ell_i$’s can be zero. We may rewrite the above as

$$
\langle X \rangle \langle Y \rangle - \sum_{i} V^{\ell_i} \langle c_i \rangle \in \left[ X(A), X(A) \right].
$$

By Lemma 3.1, we have an equality of the form (for suitable choices of integers and elements of $A$)

$$
\langle X \rangle \langle Y \rangle - \sum_{i=0}^{\infty} V^{\ell_i} \langle c_i \rangle = \sum_{j=0}^{\infty} 2^{m_j} V^{n_j} \left( \langle a_{j,1} \rangle \cdots \langle a_{j,s_j} \rangle, \langle b_{j,1}^{2^{m_j}-m_j} \rangle \cdots \langle b_{j,t_j}^{2^{m_j}-m_j} \rangle \right). \quad (\ast)
$$

Without loss of generality, we assume that

$$
0 = \ell_0 = \cdots = \ell_r < \ell_{r+1} \leq \ell_{r+2} \leq \cdots
$$

$$
0 = n_0 = \cdots = n_\kappa < n_{\kappa+1} \leq n_{\kappa+2} \leq \cdots
$$

Modulo 2, most of the terms in $(\ast)$ become zero and we have the following congruence in $A^{N_0}$

$$
\langle X \rangle \langle Y \rangle - \sum_{i=0}^{r} \langle c_i \rangle = \sum_{j=0}^{\kappa} \left( \langle a_{j,1} \rangle \cdots \langle a_{j,s_j} \rangle, \langle b_{j,1} \rangle \cdots \langle b_{j,t_j} \rangle \right) \mod 2(A^{N_0}).
$$

Looking at the second component (i.e. component indexed by 1) of the tuples we get the following congruence in the ring $A = \mathbb{Z}[X, Y]$,

$$
X^2Y^2 - \sum_{i=0}^{r} c_i^2 = \sum_{j=0}^{\kappa} \left[ a_{j,1}^2 \cdots a_{j,s_j}^2, b_{j,1}^2 \cdots b_{j,t_j}^2 \right] \mod 2A
$$

i.e.

$$
X^2Y^2 = \sum_{i=0}^{r} c_i^2 \mod (2A + [A, A]).
$$

Observe that for any two elements $a, b \in A$ we have

$$(a + b)^2 = a^2 + b^2 + ab + ba = a^2 + b^2 \mod (2A + [A, A]).$$

Thus if we let $c := \sum_{i=0}^{r} c_i$, then

$$
c^2 = \sum_{i=0}^{r} c_i^2 \mod (2A + [A, A]).
$$

This implies

$$
X^2Y^2 = c^2 \mod (2A + [A, A]).
$$

By Lemma 3.5 below, this is impossible. This proves the proposition.

\[ \square \]

**Lemma 3.5.** Let $A = \mathbb{Z}\{X, Y\}$. Then, $\forall c \in A$

$$
X^2Y^2 \neq c^2 \mod (2A + [A, A] + F^5 A).
$$

**Proof.** Let $\overline{A} := \mathbb{Z}/2\{X, Y\}$. As in the case of $A$, $\overline{A} = \bigoplus_{i \geq 0} \overline{A}_i$, where $\overline{A}_i$ is $\mathbb{Z}/2$ span of words in $X, Y$ of length precisely $i$. We let $F^n \overline{A} := \bigoplus_{i \geq n} \overline{A}_i$. In order to prove the lemma it is enough to arrive at a contradiction by assuming the existence of an element $c \in \overline{A}$, such that

$$
X^2Y^2 = c^2 \mod (\overline{A}, \overline{A} + F^5 \overline{A}). \quad (\ast\ast)
$$

Without loss of generality we may assume that $c = a_0 + a_1 + a_2 + a_3 + a_4$ where $a_i \in \overline{A}_i$ are its homogeneous components. But then

$$
c^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 \mod (\overline{A}, \overline{A} + F^5 \overline{A})
$$

$$
= a_0^2 + a_1^2 + a_2^2 \mod (\overline{A}, \overline{A} + F^5 \overline{A}).
$$
Thus, (**) continues to hold with \( c \) replaced by \( a_0 + a_1 + a_2 \). Hence without loss of generality we may assume \( c = a_0 + a_1 + a_2 \).

**Step 1:** Let \( \overline{A} := \mathbb{Z}/2[X, Y] \) be the commutative polynomial ring in variables \( X, Y \). This is quotient of the non-commutative polynomial ring \( \overline{A} = \mathbb{Z}/2\{X, Y\} \), i.e. we have a natural surjective map

\[
\overline{A} \xrightarrow{\phi} \overline{A}
\]

with \( [\overline{A}, \overline{A}] \subseteq \text{Ker}(\phi) \). The homomorphism \( \phi \) preserves grading. We let \( F^n \overline{A} := \bigoplus_{i \geq n} \overline{A}_i \). Applying \( \phi \) to the congruence (**) above, we get the following congruence in \( \overline{A} \)

\[
X^2Y^2 = \phi(c)^2 \mod F^5\overline{A}
= \phi(a_0)^2 + \phi(a_1)^2 + \phi(a_2^2) \mod F^5\overline{A}
\]

Note that \( \phi(a_i) \in \overline{A}_i \). Thus both sides of the congruence belong to the subgroup \( \bigoplus_{i=0}^4 \overline{A}_i \), which has trivial intersection with \( F^5\overline{A} \). Therefore the congruence gives us an equality in \( \overline{A} \)

\[
X^2Y^2 = \phi(a_0)^2 + \phi(a_1)^2 + \phi(a_2^2) \mod \overline{A}
\]

Further, since \( X^2Y^2 \in \overline{A}_4 \), the only possibility is \( \phi(a_0) = \phi(a_1) = 0 \) and \( \phi(a_2) = XY \). Note that \( \phi|_{\overline{A}_i} : \overline{A}_i \to \overline{A}_i \) is an isomorphism for \( i = 0, 1 \). Thus

\[
a_0 = a_1 = 0.
\]

Moreover,

\[
\text{Ker}(\overline{A}_2 \xrightarrow{\phi|_{\overline{A}_2}} \overline{A}_2) = \mathbb{Z}/2 \text{ span of } XY - YX.
\]

Thus \( \phi(c) = \phi(a_2) = XY \) implies that

\[
c = XY + \epsilon[X, Y] \quad \text{for some } \epsilon \in \mathbb{Z}/2.
\]

This implies

\[
c^2 = (XY)^2 + \epsilon^2(XY - YX)^2 \mod [\overline{A}, \overline{A}].
\]

**Step 2:**

\[
(XY - YX)^2 = (XY)^2 + (YX)^2 \mod [\overline{A}, \overline{A}]
= [X, YXY] \mod [\overline{A}, \overline{A}]
= 0 \mod [\overline{A}, \overline{A}]
\]

Thus

\[
c^2 = (XY)^2 \mod [\overline{A}, \overline{A}].
\]

Therefore, without loss of generality, we may assume \( c = XY \).

**Step 3:** Now (***) gives us

\[
X^2Y^2 - XYXY = u + v
\]

for some \( u \in [\overline{A}, \overline{A}] \) and \( v \in F^5\overline{A} \). Write \( u = \sum_i u_i \) with \( u_i \in \overline{A}_i \) \( \forall i \). Comparing the degree 4 homogeneous components of the above equation, we get

\[
X^2Y^2 - XYXY = u_4.
\]

Note that \( [\overline{A}, \overline{A}] \) is graded subgroup of \( \overline{A} \). Thus \( u_4 \in [\overline{A}, \overline{A}] \). Let \( S \subseteq \overline{A}_4 \) be a subset defined by

\[
S := \{ \text{ all words in } X, Y \text{ of length } 4, \text{ except } XYXY \text{ and } YXYX \} \bigcup \{ XYXY - YXYX \}.
\]

Let \( K \subseteq \overline{A}_4 \) be the \( \mathbb{Z}/2 \)-subspace spanned by \( S \). Clearly

\[
X^2Y^2 - XYXY \not\in K.
\]

Thus to finish the proof, it suffices to show \( u_4 \in K \). This will be done in the next step.
Step 4: $u_4 \in |A| \cap |A_4|$. It is elementary to check that $|A| \cap |A_4|$ is generated by the set 
\[ \{a_ib_j - b_ja_i \mid (i, j) = (1, 3) \text{ or } (2, 2), \text{ with } a_i, b_j \text{ are words of length } i, j \text{ respectively} \}. \]

It therefore suffices to show that all elements of the form $a_ib_j - b_ja_i$ of the above set belong to $K$.

The possibilities for $a_i$ and $b_j$ appearing in the above set are listed in the table below:

| $a_i$ | $X, Y$ |
|-------|--------|
| $b_j$ | $X^3, Y^3, X^2Y, XY^2, YX^2, Y^2X, XXY, YXY$ |
| $a_2$ or $b_2$ | $X^2, Y^2, XY, YX$ |

For all possible combinations of $a_1$ and $b_3$, one explicitly checks that $a_1b_3 - b_3a_1$ always belongs to $K$. Similarly for all possible combinations of $a_2$ and $b_2$ from the above table, one checks that $a_2b_2 - b_2a_2$ always belongs to $K$. This finishes the proof.

\[
\square
\]

4. Proof of Theorem 1.3

The goal of this section is to prove Theorem 1.3. Define a set map $\Omega: A^{N_0} \rightarrow A^{N_0}$ by 
\[ \Omega(\mathbf{a}) = (\omega_0(\mathbf{a}), \omega_1(\mathbf{a}), ...) \]
where $\omega_n(\mathbf{a}) = a_0^{a_n} + pa_1^{a_n-1} + \cdots + p^na_n$ are the Witt polynomials. Recall that $X(A) \subset A^{N_0}$ is the closed subgroup generated by elements of the form $V^i(\langle a_1 \cdots a_r \rangle)$ where $V$ and $\langle \rangle$ are as in the construction by Cuntz and Deninger summarized in the previous section.

**Lemma 4.1.** Image of $A^{N_0} \xrightarrow{\Omega} A^{N_0}$ is contained in $X(A)$. In fact, we have the following equality in $X(A)$:

\[ \Omega(a_0, a_1, ...) = \sum_{i=0}^{\infty} V^i(\langle a_i \rangle) \quad \forall (a_0, a_1, ...) \in A^{N_0}. \]

**Proof.**

\[ \Omega(a_0, a_1, ...) = (\omega_0(\mathbf{a}), \omega_1(\mathbf{a}), ...) \]
\[ = (a_0, a_0^p + pa_1, a_0^{p^2} + p^2a_2, ...) \]
\[ = (a_0, a_0^p, a_0^{p^2}, ...) + p(a_1, a_1^p, a_1^{p^2}, ...) + p^2(0, 0, a_2, a_2^p, a_2^{p^2}, ...) + \cdots \]
\[ = \sum_{i=0}^{\infty} V^i(\langle a_i \rangle) \]

Thus, by the above lemma, $\Omega$ induces a map from $A^{N_0} \rightarrow X(A)$ which will also be denoted by $\Omega$.

**Lemma 4.2.** Let $A$ be such that $\frac{A}{[A, A]}$ is p-torsion free. Suppose there exists a continuous map $\Phi: W(A) \rightarrow HH_0(X(A))$ which commutes with $\langle \rangle$ and $V$. Then the following diagram commutes

\[ \begin{array}{ccc}
X(A) & \xrightarrow{\Phi} & HH_0(X(A)) \\
\downarrow \Omega & & \downarrow \\
A^{N_0} & \xrightarrow{q} & W(A)
\end{array} \]

In other words, $\Phi$ is necessarily given by

\[ \Phi(q(\mathbf{a})) = \Omega(\mathbf{a}) \mod [X(A), X(A)]. \]
Proof. For \( a = (a_0, a_1, \ldots) \in A^\mathbb{N}_0 \) we have

\[
\Phi(q(a)) = \Phi\left( \sum_{i=0}^{\infty} V^i(a_i) \right) \quad \text{(by lemma 3.3(i))}
\]

\[
= \sum_{i=0}^{\infty} \Phi(V^i(a_i)) \quad \text{(by continuity of \( \Theta \))}
\]

\[
= \Omega(a) \mod [X(A), X(A)] \quad \text{(by Lemma 4.1 and since \( \Phi \) is continuous and preserves \( V, \langle \rangle \))}
\]

This proves the lemma. \( \square \)

To state the next result we fix the following notation. Let \( X(A) \overset{\gamma}{\to} (A[A, A])^\mathbb{N}_0 \) denote the composition

\[
X(A) \hookrightarrow A^\mathbb{N}_0 \to (A[A, A])^\mathbb{N}_0.
\]

Let \( E(A) \overset{\eta}{\to} (A[A, A])^\mathbb{N}_0 \) denote the composition

\[
E(A) \overset{\pi}{\to} X(A) \overset{\gamma}{\to} (A[A, A])^\mathbb{N}_0
\]

where \( \pi \) is the surjection induced by \( X(\mathbb{Z}A) \to X(A) \) (see subsection 2.3). Since \( \eta \) and \( \gamma \) are ring homomorphisms, and the target is a commutative ring, we have induced maps

\[
HH_0(X(A)) \overset{\pi}{\to} (A[A, A])^\mathbb{N}_0,
\]

\[
HH_0(E(A)) \overset{\pi}{\to} (A[A, A])^\mathbb{N}_0.
\]

The maps \( \eta \) and \( \Phi \) are analogous to the ghost map

\[
W(A) \overset{\pi}{\to} (A[A, A])^\mathbb{N}_0.
\]

In the case when \( A[A, A] \) is p-torsion free, \( \Phi \) is always injective (see [3, page 56]). However as the following theorem shows, this is not the case for \( \Phi \).

**Theorem 4.3.** Let \( A = \mathbb{Z}\{X, Y\} \), and \( p = 2 \). The map

\[
HH_0(E(A)) \overset{\pi}{\to} (A[A, A])^\mathbb{N}_0
\]

is not injective.

To prove this theorem we need the following lemma and a construction from [2] (see 4.5).

**Lemma 4.4.** Let \( A = \mathbb{Z}\{X, Y\} \) and \( p = 2 \). Let \( A_0^0 \subset A_4 \) be the free subgroup generated by all words of length 4 in \( X, Y \) except the words \( XYXY \) and \( YXXY \). Let \( H := A_4^0 + F^5 A + 2A \). Then

\[
(x_0, x_1, x_2, \ldots) \in [X(A), X(A)] \implies x_1 = 0 \mod H.
\]

**Proof.** By Lemma 3.1, \([X(A), X(A)]\) is the closed subgroup generated by the following elements of \( X(A)\):

\[
\left\{ 2^n V^m ([a_1] \cdots [a_s], \langle b_1^{2^{n-m}} \cdots b_t^{2^{n-m}} \rangle) \mid 0 \leq m \leq n \leq 1, s, t \in \mathbb{N}, a_i, b_j \in A \right\}
\]

where recall that for \( a \in A \), \( \langle a \rangle = (a, a^2, a^2, \ldots) \in X(A) \). As \( A_0 \) is in the center of \( A \) we have

\[
\langle c \rangle (d) = (cd) \forall c \in A_0 \text{ and } d \in A.
\]
Thus, without loss of generality, we may restrict the \( a_i, b_j \) in the generating set above to elements of \( A \setminus A_0 \). In other words \([X(A), X(A)]\) is the closed subgroup generated by the following elements of \( X(A)\):

\[
\left\{ 2^n V^n \left( \langle a_1 \rangle \cdots \langle a_s \rangle, \langle b_1^{2^{n-m}} \cdots b_t^{2^{n-m}} \rangle \right) \mid 0 \leq m \leq n \leq 1, s, t \in \mathbb{N}, a_i, b_j \in A \setminus A_0 \right\}
\]

Thus it is enough to prove the statement of the lemma when \((x_0, x_1, \ldots)\) is an element of the generating set, i.e.

\[(x_0, x_1, \ldots) = 2^n V^n \left( \langle a_1 \rangle \cdots \langle a_s \rangle, \langle b_1^{2^{n-m}} \cdots b_t^{2^{n-m}} \rangle \right).
\]

Clearly if \( m > 0 \) or if \( n > 0 \), the RHS is divisible by 2 and since \( 2A \subset H \), we have \( x_1 \in H \). It thus remains to show that for

\[(x_0, x_1, \ldots) = \left[ \langle a_1 \rangle \cdots \langle a_s \rangle, \langle b_1 \rangle \cdots \langle b_t \rangle \right],
\]

where \( s, t \geq 1, x_1 \in H \). Equivalently, we need to show the following:

\[ [a_1^2 \cdots a_s^2, b_1^2 \cdots b_t^2] = 0 \mod H \quad \forall \ a_i, b_i \in A.
\]

To see this, we first observe that \( H \) is a two sided ideal of \( A \). The above statement is equivalent to showing that images of the elements \((a_1^2 \cdots a_s^2)\) and \((b_1^2 \cdots b_t^2)\) in the quotient ring \( A/H \) commute. For this it is enough to show that for all \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \), the images of the elements \( a_i^2 \) and \( b_j^2 \) in \( A/H \) commute. In other words, we are reduced to showing that for elements \( a, b \in A \),

\[ a^2 b^2 = b^2 a^2 \mod H.
\]

Write

\[ a = a_0 + a_1 + a_2 \]
\[ b = b_0 + b_1 + b_2 \]

where \( a_0, b_0 \in A_0, a_1, b_1 \in A_1 \) and \( a_2, b_2 \in F^2 A \). Then, since \( 2A \subset H \), we have

\[ a^2 = a_0^2 + a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1 \mod H \]
\[ b^2 = b_0^2 + b_1^2 + b_2^2 + b_1 b_2 + b_2 b_1 \mod H \]

We now explicitly calculate \( a^2 b^2 - b^2 a^2 \) modulo \( H \), while keeping in mind that \( a_0, b_0 \) are in the center of \( A \).

\[ a^2 b^2 - b^2 a^2 = (a_0^2 + a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1) (b_0^2 + b_1^2 + b_2^2 + b_1 b_2 + b_2 b_1) \]
\[ - (b_0^2 + b_1^2 + b_2^2 + b_1 b_2 + b_2 b_1) (a_0^2 + a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1) \mod H \]
\[ = a_1^2 b_1^2 - a_1^2 b_1^2 + (\text{deg} \geq 5 \text{ terms}) \mod H \]
\[ = a_1^2 b_1^2 - b_1^2 a_1^2 \mod H \]

Thus to finish the proof, it suffices to show that \( a_1^2 b_1^2 - b_1^2 a_1^2 \in A_1^0 \). Since \( a_1, b_1 \in A_1 \), there exists integers \( m_i, n_i \) for \( i = 1, 2 \) such that

\[ a_1 = m_1 X + n_1 Y \]
\[ b_1 = m_2 X + n_2 Y \]

It is now elementary to check that in the expression of \( a_1^2 b_1^2 - b_1^2 a_1^2 \), which is a linear combination of words of length 4 in \( X \) and \( Y \), the coefficients of the words \( XXYY \) and \( XYXY \) are zero. By definition of \( A_1^0 \), this shows that \( a_1^2 b_1^2 - b_1^2 a_1^2 \in A_1^0 \).

\[ \square \]

4.5. Hesselholt’s construction of the map \( [A, A]^n \to A^n \): In this subsection we quickly recall the construction of a map from \( [A, A]^n \to A^n \) by Hesselholt (see [2, Page 114]). Everything recalled here is taken directly from [2] and nothing is new, except that instead of the non-unital polynomial
ring $\mathbb{Z}[X,Y]$ we deal with the unital polynomial ring $\mathbb{Z}[X,Y]$. Let $A = \mathbb{Z}[X,Y]$. Fix a prime $p$. An element $x \in A$ can be uniquely written as

$$x = \sum w n_w w, \quad n_w \in \mathbb{Z} \text{ with almost all zero}$$

where the summation is over words $w$ in $X,Y$ (including the empty word). Define two words to be equivalent, if one can be obtained by a cyclic permutation of the other. An equivalence class of words will be called a circular word. One can show that $A/[A,A]$ is a free abelian group generated by circular words. Define a additive section of the quotient map $A \to A/[A,A]$

$$\sigma_0 : \frac{A}{[A,A]} \to A$$

which takes a circular word to the unique word in its equivalence class which is least for the lexicographic order. Define an additive group homomorphism $A \to A$ by

$$\phi(\sum w n_w w) := \sum w n_w w^p.$$ 

As observed in [2], $\phi$ satisfies the following.

**Lemma 4.6.** $\phi([A,A]) \subset [A,A]$ and for all $x \in A$, $x_{p^n} = \phi(x_{p^{n-1}}) \mod \langle p^n A + [A,A] \rangle$.

We now define a map

$$\mathcal{R} : [A,A]_{N_0} \to A_{N_0}$$

$$\mathcal{R}(\epsilon_0, \epsilon_1, ...) := (r_0, r_1, ...)$$

where $r_0 = \epsilon_0$ and for $n \geq 1$, $r_n$'s are recursively defined as:

$$r_n = \epsilon_n - \sigma_0(\sum \omega (\omega_n(0, r_0, r_1, ..., r_n-1, 0) - \phi(\omega_{n-1}(0, ..., r_{n-1})))).$$ 

In the above formula, we note that by $(\omega_n(0, r_0, r_1, ..., r_n-1, 0) - \phi(\omega_{n-1}(0, ..., r_{n-1})))$ we actually mean its image in $A/[A,A]$, which is divisible by $p^n$ by Lemma 4.6. Moreover, since $A/[A,A]$ has no $p$-torsion, $p^{-n}(\omega_n(0, r_0, r_1, ..., r_n-1, 0) - \phi(\omega_{n-1}(0, ..., r_{n-1})))$ is a well defined element of $A/[A,A]$.

Finally we recall the part of [2, Lemma 1.3.7] which remains true.

**Lemma 4.7.** For any $\xi \in [A,A]_{N_0}$, $\mathcal{R}(\xi)$ maps to zero via the ghost map

$$A_{N_0} \to (\frac{A}{[A,A]})_{N_0}.$$ 

**Proof of Theorem 4.3.** As the map $HH_0(E(A)) \to HH_0(X(A))$ is surjective, it is enough to show that the map

$$HH_0(X(A)) \to (\frac{A}{[A,A]})_{N_0}$$

is not injective. Consider the following diagram

$$\begin{array}{ccc}
X(A) & \longrightarrow & HH_0(X(A)) \\
\downarrow \Omega & & \downarrow \tau \\
A_{N_0} & \longrightarrow & (\frac{A}{[A,A]})_{N_0}
\end{array}$$

In order to prove the theorem it suffices to find an element $r = (r_0, r_1, ...) \in A_{N_0}$ such that $\Omega(r) \notin [X(A), X(A)]$ and $\omega(r) = 0$. Let

$$r := (r_0, r_1, r_2, ...) := \mathcal{R}(XY - YX, 0, 0, ...) \in A_{N_0}.$$
By Lemma 4.7, $\omega(r) = 0$. It remains to show that $\Omega(r) \notin [X(A), X(A)]$. By Lemma 4.4, it suffices to show that the 1-th component of $\Omega(r)$ (i.e. $\omega_1(r)$) is not in $A_4^0 + F^5 A + 2A$.

$$\omega_1(r) = r_0^2 + 2r_1$$
$$= (XY - YX)^2 + 2(-XYXY + XXYY)$$
$$= -XYXY + YXYX - YYXY - XYY + 2XXYY$$
$$= -XYXY + YXYX \mod A_4^0 + F^5 A + 2A.$$

As a direct consequence of the definition of $A_4^0$ and $F^5 A$, one thus sees that $\omega_1(r) \notin A_4^0 + F^5 A + 2A$. This finishes the proof.  

\[\Box\]

**Proof of Theorem 1.3.** Suppose there exists a continuous group homomorphism $W(A) \to HH_0(E(A))$. Composing with the natural homomorphism $HH_0(E(A)) \to HH_0(X(A))$ we get a map

$$\Phi : W(A) \to HH_0(X(A)).$$

Let us denote the quotient map $X(A) \to HH_0(X(A))$ by $\pi$. By lemma 4.2, the following diagram must commute:

$$\begin{array}{ccc}
X(A) & \xrightarrow{\pi} & HH_0(X(A)) \\
\downarrow{\Omega} & & \downarrow{\Phi} \\
A^{N_0} & \xrightarrow{q} & W(A)
\end{array}$$

Using surjectivity of $q$ and the fact that both $\Omega$ and $\gamma \circ q$ are given by the same Witt polynomials, one checks that the above commutative square can be extended to the following commutative diagram.

$$\begin{array}{ccc}
X(A) & \xrightarrow{\pi} & HH_0(X(A)) & \xrightarrow{\gamma} & \left(\frac{A}{[A, A]}\right)^{N_0} \\
\downarrow{\Omega} & & \downarrow{\Phi} & & \downarrow{\gamma} \\
A^{N_0} & \xrightarrow{q} & W(A) & \xrightarrow{\gamma} & \left(\frac{A}{[A, A]}\right)^{N_0}
\end{array}$$

However, such a commutative diagram is not possible as there exists an element $r \in A^{N_0}$ (see proof of Theorem 4.3) such that $\gamma \circ \pi \circ \Omega(r) \neq 0$ but $\gamma \circ q(r) = 0$.  

\[\Box\]

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