Liouville theorem on a half-space for biharmonic problem with Dirichlet boundary condition.

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Abstract

We investigate here the nonlinear elliptic Hénon type equation:

\[ \Delta^2 u = |x|^a |u|^{p-1} u \text{ in } \mathbb{R}^n_+ \quad u = \frac{\partial u}{\partial x_n} = 0 \text{ in } \partial \mathbb{R}^n_+, \]

with \( p > 1 \) and \( n \geq 2 \). In particular, we prove some Liouville type theorems for stable at infinity solutions. The main methods used are the integral estimates, the Pohozaev-type identity and the monotonicity formula.

Keywords: Hénon type equations, Morse index, Liouville-type theorems, Pohozaev identity, monotonicity formula.

1. Introduction

In this paper, we consider the following elliptic Hénon type equation

\[ \Delta^2 u = |x|^a |u|^{p-1} u \text{ in } \mathbb{R}^n_+, \quad u = \frac{\partial u}{\partial x_n} = 0 \text{ on } \partial \mathbb{R}^n_+, \quad (1.1) \]

where \( p > 1 \), \( n \geq 2 \), \( \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n, \ x_n > 0 \} \) and \( \partial \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n, \ x_n = 0 \} \).

Our main objective is to classify the non-existence result for \( C^4 \)-solutions for problems \( (1.1) \) belonging to one of the following classes: stable solutions and solutions which are stable outside a compact set.

We now list some known results. We start with the Liouville type theorems for the corresponding nonlinear problem

\[ (-\Delta)^mu = |x|^a |u|^{p-1} u \text{ in } \mathbb{R}^N, \quad (1.2) \]

have been largely studied in the literature (see, e.g.,[2, 3, 4, 5, 6, 8, 3, 4]). In particular, when \( m = 1 \) and \( a = 0 \), the first Liouville theorem was proved by Gidas and Spruck in \( \mathbb{R}^2 \), in which they proved that, for \( 1 < p \leq \frac{n+2}{n-2} \). Soon afterward, similar results were established in \( \mathbb{R}^2 \) for positive solutions of the subcritical problem in the upper half-space

\[ -\Delta u = |u|^{p-1} u \text{ in } \mathbb{R}^n_+, \quad u = 0 \text{ on } \partial \mathbb{R}^n_+, \]

After that, Chen and Li[11] obtained similar nonexistence results for the above two equations by using the moving plane method. In [2], Farina obtained the optimal Liouville type result for solutions stable
at infinity. Indeed, he proved that a smooth nontrivial solution to (1.2) exists, if and only if $p \geq p_{JL}(n)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$. Here $p_{JL}(n)$ denotes the so-called Joseph-Lundgren exponent (see [10, 2]). In addition, similar results were established in [2] for finite Morse index solutions in the upper half-space: $\mathbb{R}^n_+$, with homogeneous Dirichlet boundary conditions on $\partial \mathbb{R}^n_+$. Furthermore, strips provide an interesting example of unbounded domains where, as we shall see, rather sharp.

Furthermore, in a recent paper [3], Dancer, Du and Guo extended some results in [2] have considered (1.2) with $m = 1$ and $a > -2$. It was proved that there is no nontrivial stable solution in $\mathbb{R}^n$ if $1 < p < p_{JL}(n,a)$ and that for $p \geq p_{JL}(n,a)$, admits a positive radial stable solution in $\mathbb{R}^n$, where $p_{JL}(n,a)$ is Joseph-Lundgren exponent for the Hénon type equation. In addition, Wang and Ye [4] obtained a Liouville-type result for finite Morse index solutions in $\mathbb{R}^n$, which is a partial extension of results in [3].

In a very interesting paper, Dávila et al. [1] investigated the bi-harmonic equation i.e. $m = 2$ and $a = 0$, they derived a relevant monotonicity formula and employed blow down analysis to prove a sharp classification of stable at infinity solutions. However, for the fourth-order Hénon type equation i.e. $m = 2$ and $a > 0$, studied by Hu [8]. He proved Liouville-type theorems for solutions belonging to one of the following classes: stable solutions and finite Morse index solutions (whether positive or sign-changing). His proof is based on a combination of the Pohozaev-type identity, monotonicity formula of solutions and a blowing down sequence.

Relying on Hu’s approach [8] and using the technics developed in [2, 1], we give a Liouville-type theorems in the class of stable solution and finite Morse index solutions in the half space $\mathbb{R}^N_+$. Before stating our main results, we first recall the definition of such solutions.

**Definition 1.1.** We say that a solution $u$ of (1.1) belonging to $C^4(\mathbb{R}^N_+)$,

- is stable, if

$$Q_u(\psi) := \int_{\mathbb{R}^N_+} (\Delta \psi)^2 dx - p \int_{\mathbb{R}^N_+} |x|^a |u|^{p-1} \psi^2 dx \geq 0, \quad \forall \psi \in C_c^2(\mathbb{R}^N_+).$$

- is stable outside a compact set $K \subset \mathbb{R}^N_+$, if $Q_u(\psi) \geq 0$ for any $\psi \in C^2_c(\mathbb{R}^N_+ \setminus K)$.

- More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces $E$ of $C^2_c(\mathbb{R}^N_+)$ such that $Q_u(\varphi) < 0$ in $E \setminus \{0\}$. Clearly, a solution stable and only if its Morse index is equal to zero.

**Remark 1.1.** (i). Clearly a solution is stable if and only if its Morse index is equal to zero.

(ii). Any finite Morse index solution $u$ is stable outside a compact set $K \subset \mathbb{R}^N_+$. Indeed there exist $K \geq 1$ and $X_K := \text{span}\{\varphi_1, \ldots, \varphi_K\} \subset C^2_c(\mathbb{R}^N_+)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X_K \setminus \{0\}$. Then, $Q_u(\varphi) \geq 0$ for every $\varphi \in C^2_c(\mathbb{R}^N_+ \setminus K)$, where $K := \bigcup_{j=1}^K \text{supp}(\varphi_j)$.

Now we can state our main results.

**Theorem 1.1.** Let $u \in C^4(\mathbb{R}^N_+)$ be a stable solution of (1.1). If $1 < p < p_{JL}(n,a)$, then $u \equiv 0$

**Theorem 1.2.** Let $u \in C^4(\mathbb{R}^N_+)$ be a solution of (1.1) that is stable outside a compact set.

- If $1 < p < p_{JL}(n,a)$, then $u \equiv 0$.

- If $p = \frac{n+4+2a}{n-1}$, then $u$ has finite energy, i.e.,

$$\int_{\mathbb{R}^N_+} (\Delta u)^2 = \int_{\mathbb{R}^N_+} |x|^a |u|^{p+1} < +\infty.$$

Here the representation of $p_{JL}(n,a)$ in Theorem 1.1 is the fourth-order Joseph–Lundgren exponent which is computed by [6].

The organization of the rest of the paper is as follows. In section 1, we construct a monotonicity formula which is a crucial tool to handle the supercritical case. In section 2, we establish some finer integral estimates for the solutions of (1.1). Also we obtain a nonexistence result for the homogeneous stable solution of (1.1) in $\mathbb{R}^N_+ \setminus \{0\}$, where $p$ belongs to $(\frac{n+4+2a}{n-1}, p_{JL}(n,a))$. Then we prove Liouville-type
theorem for stable solutions of (1.1), this is Theorem 1.1 in section 3. To prove the result, we obtain some estimates of solutions, and show that the limit of blowing down sequence \( u^\infty(x) = \lim_{\lambda \to \infty} \lambda^{\frac{4+a}{p-1}} u(\lambda x) \) satisfies \( E(u, r) \equiv \text{const} \). Here, we use the monotonicity formula of (Proposition 1.1 see below). In section 4, we study Liouville-type theorem of finite Morse index solutions by the use of the Pohozaev-type identity, monotonicity formula and blowing down sequence. In the following, \( C \) denotes always a generic positive constant, which could be changed from one line to another.

### 1.1. Monotonicity formula

In this section, we construct a monotonicity formula which plays an important role in dealing to understand supercritical elliptic equations or systems. This approach has been used successfully for the Lane–Emden equation in [1, 8]. Equation (1.1) has two important features. It is variational, with the energy functional given by

\[
E(u, \lambda) = \int_{B_1^+} \left( \frac{1}{2} |\Delta u|^2 - \frac{1}{p+1} |x|^a |u|^{p+1} \right) d\lambda.
\]

For \( \lambda > 0 \), set \( B_1^+ = B_1 \cap \mathbb{R}^n_+ \). Under the scaling transformation

\[
u^\lambda(x) = \lambda^{\frac{4+a}{p-1}} u(\lambda x),
\]

this suggests that the variations of the rescaled energy

\[
\int_{B_1^+} \left( \frac{1}{2} |\Delta u|^2 - \frac{1}{p+1} |x|^a |u^\lambda|^{p+1} \right) d\lambda.
\]

For any given \( x \in \mathbb{R}^n_+ \), choose \( u \in W^{4,2}_loc(\mathbb{R}^n_+) \cap L^{p+1}_loc(\mathbb{R}^n_+) \) and define

\[
E(u, \lambda) = \lambda^{\frac{4(a+1)+2a}{p-1}} \left( \int_{B_1^+} \frac{1}{2} (\Delta u)^2 - \frac{1}{p+1} |x|^a |u|^{p+1} \right) \]

\[+ \frac{4+a}{2(p-1)} \left( n - 2 - \frac{4+a}{p-1} \right) \lambda^{\frac{a+2}{p-1}+1-n} \int_{\partial B_1^+} u^2 \]

\[+ \frac{4+a}{2(p-1)} \left( n - 2 - \frac{4+a}{p-1} \right) \frac{d}{d\lambda} \lambda^{\frac{a+2}{p-1}+2-n} \int_{\partial B_1^+} u^2 \]

\[= \lambda^3 \int_{\partial B_1^+} \left( \frac{4+a}{p-1} \right) \lambda^{-1} u + \frac{\partial u}{\partial r} \right)^2 \]

\[+ \frac{1}{2} \frac{d}{d\lambda} \lambda^{\frac{a+2}{p-1}+3-n} \int_{\partial B_1^+} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \]

\[= \frac{1}{2} \lambda^{\frac{a+2}{p-1}+3-n} \int_{\partial B_1^+} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right),
\]

where derivatives are taken in the sense of distributions. Then, we have the following monotonicity formula.

**Proposition 1.1.** Assume that \( n \geq 5 \), \( a \geq 0 \) and \( p > \frac{n+4+2a}{n-4} \), \( u \in W^{4,2}_loc(\mathbb{R}^n_+) \) and \( |x|^a |u|^{p+1} \in L^1_loc(\mathbb{R}^n_+) \) be a weak solution of (1.1). Then, \( E(u, \lambda) \) is non-decreasing in \( \lambda > 0 \). Furthermore there is a constant \( C(n, p, a) > 0 \) such that

**Proof.** The proof follows the main lines of the demonstration of Theorem 2.1 in [8], with small modifications. Since the boundary integrals in \( E(u, \lambda) \) only involve second order derivatives of \( u \), the boundary integrals in \( \frac{d}{d\lambda} E(u, \lambda) \) only involve third order derivatives of \( u \). Thus, the following calculations can be rigorously verified. Assume that \( x = 0 \) and that the balls \( B_\lambda \) are all centered at 0. Take

\[
\tilde{E}(\lambda) = \lambda^{\frac{4(a+1)+2a}{p-1}} \int_{B_1^+} \left( \frac{1}{2} (\Delta u)^2 - \frac{1}{p+1} |x|^a |u|^{p+1} \right).
\]
Define
\[ v = \Delta u, \quad u^\lambda(x) = \lambda^{\frac{4+a}{p-1}} u(\lambda x), \quad \text{and} \quad v^\lambda(x) = \lambda^{\frac{4+a}{p-1} + 2} v(\lambda x). \]

We still have \( v^\lambda = \Delta u^\lambda \), \( \Delta v^\lambda = |x|^a |u^\lambda|^{p-1} u^\lambda \), and by differentiating in \( \lambda \),
\[ \Delta \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}. \]

Note that differentiation in \( \lambda \) commutes with differentiation and integration in \( x \). A rescaling shows
\[ \tilde{E}(\lambda) = \int_{B^*_1} \frac{1}{2} (v^\lambda)^2 - \frac{1}{p+1} |x|^a |u^\lambda|^{p+1}. \]

hence
\[ \frac{d}{d\lambda} \tilde{E}(\lambda) = \int_{B^*_1} v^\lambda \frac{du^\lambda}{d\lambda} - |x|^a |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \]
\[ = \int_{B^*_1} v^\lambda \Delta \frac{du^\lambda}{d\lambda} - \Delta v^\lambda \frac{du^\lambda}{d\lambda} = \int_{\partial B^*_1} v^\lambda \frac{\partial u^\lambda}{\partial r} \frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r} \frac{du^\lambda}{d\lambda}. \]

Since \( u^\lambda = 0 \) in \( \partial \mathbb{R}^n_+ \) for any \( \lambda > 0 \), then \( \frac{du^\lambda}{d\lambda} = 0 \) in \( \partial \mathbb{R}^n_+ \). Hence, all boundary terms appearing in the integrations by parts vanish under the Dirichlet boundary conditions. So, we get
\[ \frac{d}{d\lambda} \tilde{E}(\lambda) = \int_{\partial B^*_1} \left( v^\lambda \frac{\partial u^\lambda}{\partial r} \frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r} \frac{du^\lambda}{d\lambda} \right). \] (1.5)

In what follows, we express all derivatives of \( u^\lambda \) in the \( r = |x| \) variable in terms of derivatives in the \( \lambda \) variable. In the definition of \( u^\lambda \) and \( v^\lambda \), directly differentiating in \( \lambda \) gives
\[ \frac{du^\lambda}{d\lambda}(x) = \frac{1}{\lambda} \left( \frac{4+a}{p-1} u^\lambda(x) + r \frac{\partial u^\lambda}{\partial r}(x) \right), \] (1.6)

and
\[ \frac{dv^\lambda}{d\lambda}(x) = \frac{1}{\lambda} \left( \frac{2(p+1)+a}{p-1} v^\lambda(x) + \frac{\partial v^\lambda}{\partial r}(x) \right). \] (1.7)

In (1.6), taking derivatives in \( \lambda \) once again, we get
\[ \lambda \frac{d^2 u^\lambda}{d\lambda^2}(x) + \frac{du^\lambda}{d\lambda}(x) = \frac{4+a}{p-1} \frac{du^\lambda}{d\lambda}(x) + r \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda}(x). \] (1.8)

Substituting (1.7) and (1.8) into (1.5), we obtain
\[ \frac{d\tilde{E}}{d\lambda} = \int_{\partial B^*_1} \left( \lambda \frac{d^2 u^\lambda}{d\lambda^2} + p-5 - a \frac{du^\lambda}{d\lambda} \right) - \frac{du^\lambda}{d\lambda} \left( \lambda \frac{dv^\lambda}{d\lambda} - 2 \frac{p+1+a}{p-1} v^\lambda \right) \]
\[ = \int_{\partial B^*_1} \left( \lambda v^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 3 \frac{dv^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} - \lambda \frac{du^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda} \right). \] (1.9)

Observe that \( v^\lambda \) is expressed as a combination of \( x \) derivatives of \( u^\lambda \). So we also transform \( v^\lambda \) into \( \lambda \).
derivatives of $u^\lambda$. By taking derivatives in $r$ in (1.6) and noting (1.8), we get on $\partial B_1^+$,
\[
\frac{\partial^2 u^\lambda}{\partial r^2} = \lambda \frac{\partial}{\partial r} \frac{\partial u^\lambda}{\partial \lambda} - \frac{p + 3 + a}{p - 1} \frac{\partial u^\lambda}{\partial r} 
= \lambda^2 \frac{\partial^2 u^\lambda}{\partial \lambda^2} + \frac{p - 5 - a}{p - 1} \lambda \frac{d u^\lambda}{d \lambda} - \frac{p + 3 + a}{p - 1} \left( \lambda \frac{d u^\lambda}{d \lambda} - \frac{4 + a}{p - 1} u^\lambda \right) 
= \lambda^2 \frac{\partial^2 u^\lambda}{\partial \lambda^2} - \frac{8 + 2 a}{p - 1} \lambda \frac{d u^\lambda}{d \lambda} + \frac{(4 + a)(p + 3 + a)}{(p - 1)^2} u^\lambda.
\]

Then on $\partial B_1^+$,
\[
v^\lambda = \frac{\partial^2 u^\lambda}{\partial r^2} + \frac{n - 1}{r} \frac{\partial u^\lambda}{\partial r} + \frac{1}{r^2} \Delta_\theta u^\lambda 
= \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} - \frac{8 + 2 a}{p - 1} \lambda \frac{d u^\lambda}{d \lambda} + \frac{(4 + a)(p + 3 + a)}{(p - 1)^2} u^\lambda + (n - 1) \left( \lambda \frac{d u^\lambda}{d \lambda} - \frac{4 + a}{p - 1} u^\lambda \right) + \Delta_\theta u^\lambda 
= \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} + \left( n - 1 - \frac{8 + 2 a}{p - 1} \right) \lambda \frac{d u^\lambda}{d \lambda} + \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right) u^\lambda + \Delta_\theta u^\lambda.
\]

Here $\Delta_\theta$ is the Beltrami–Laplace operator on $\partial B_1$ and below $\nabla_\theta$ represents the tangential derivative on $\partial B_1$. For notational convenience, we also define the constants
\[
\alpha = n - 1 - \frac{8 + 2 a}{p - 1}, \quad \beta = \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right).
\]

Now (1.9) reads
\[
\frac{d}{d \lambda} \tilde{E}(\lambda) := I_1 + I_2,
\]
where
\[
I_1 := \int_{\partial B_1^+} \lambda \left( \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} + \alpha \lambda \frac{d u^\lambda}{d \lambda} + \beta u^\lambda \right) \frac{d^2 u^\lambda}{d \lambda^2} 
+ 3 \left( \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} + \alpha \lambda \frac{d u^\lambda}{d \lambda} + \beta u^\lambda \right) \frac{d u^\lambda}{d \lambda} - \lambda \frac{d u^\lambda}{d \lambda} \frac{d}{d \lambda} \left( \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} + \alpha \lambda \frac{d u^\lambda}{d \lambda} + \beta u^\lambda \right),
\]
and
\[
I_2 := \int_{\partial B_1^+} \lambda \Delta_\theta u^\lambda \frac{d^2 u^\lambda}{d \lambda^2} + 3 \Delta_\theta u^\lambda \frac{d u^\lambda}{d \lambda} - \lambda \frac{d u^\lambda}{d \lambda} \Delta_\theta \frac{d u^\lambda}{d \lambda}.
\]

Let $\lambda > 0$. Since $\frac{d u^\lambda}{d \lambda} = 0$ in $\partial \mathbb{R}^n_+$ then, all boundary terms appearing in the integrations by parts vanish under the Dirichlet boundary conditions, hence the calculations are even easier. The integral $I_2$ can be estimated as
\[
I_2 = \int_{\partial B_1^+} -\lambda \nabla_\theta u^\lambda \nabla_\theta \frac{d^2 u^\lambda}{d \lambda^2} - 3 \nabla_\theta u^\lambda \nabla_\theta \frac{d u^\lambda}{d \lambda} + \lambda \left| \nabla_\theta \frac{d u^\lambda}{d \lambda} \right|^2 
= -\frac{\lambda}{2} \frac{d^2}{d \lambda^2} \left( \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right) - \frac{3}{2} \frac{d}{d \lambda} \left( \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right) + 2 \lambda \int_{\partial B_1^+} \left| \nabla_\theta \frac{d u^\lambda}{d \lambda} \right|^2 
= -\frac{1}{2} \frac{d^2}{d \lambda^2} \left( \lambda \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right) - \frac{1}{2} \frac{d}{d \lambda} \left( \lambda \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right) + 2 \lambda \int_{\partial B_1^+} \left| \nabla_\theta \frac{d u^\lambda}{d \lambda} \right|^2 
\geq -\frac{1}{2} \frac{d^2}{d \lambda^2} \left( \lambda \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right) - \frac{1}{2} \frac{d}{d \lambda} \left( \lambda \int_{\partial B_1^+} \left| \nabla_\theta u^\lambda \right|^2 \right).
Furthermore, a direct calculation implies that

\[
I_1 = \int_{\partial B^+} \lambda^3 \left( \frac{d^3 u^\lambda}{d \lambda^2} \right)^2 + \lambda \frac{d^2 u^\lambda}{d \lambda^2} \frac{d u^\lambda}{d \lambda} + \beta \lambda u^\lambda \frac{d^2 u^\lambda}{d \lambda^2} + 3 \beta u^\lambda \frac{d u^\lambda}{d \lambda} + (2 \alpha - \beta) \lambda \left( \frac{d u^\lambda}{d \lambda} \right)^2 - \lambda^3 \frac{d^4 u^\lambda}{d \lambda^4} - \frac{1}{2} \frac{d}{d \lambda} \left[ \lambda^3 \frac{d}{d \lambda} \left( \frac{d u^\lambda}{d \lambda} \right)^2 \right].
\]

Here we have used the relations (writing \( f' = \frac{d}{d \lambda} f \) etc.)

\[
\lambda f'' = \left( \frac{\lambda}{2} f'^2 \right)'' - 2 f' - \lambda (f')^2, \quad \text{and} \quad -\lambda^3 f' f'' = -\left[ \frac{\lambda^3}{2} (f')^2 \right]' + 3 \lambda^2 f' f'' + \lambda^3 (f'')^2.
\]

Since \( p > \frac{n+1+2a}{n-1} \), direct calculations show that

\[
\alpha - \beta = \left( n - 1 + \frac{8 + 2a}{p - 1} \right) - \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right) > 1.
\]

Consequently,

\[
2 \lambda^3 \left( \frac{d^2 u^\lambda}{d \lambda^2} \right)^2 + 4 \lambda^2 \frac{d^2 u^\lambda}{d \lambda^2} \frac{d u^\lambda}{d \lambda} + (2 \alpha - 2 \beta) \lambda \left( \frac{d u^\lambda}{d \lambda} \right)^2
\]

\[
= 2 \lambda \left( \lambda \frac{d^2 u^\lambda}{d \lambda^2} + \frac{d u^\lambda}{d \lambda} \right)^2 + (2 \alpha - 2 \beta - 2) \lambda \left( \frac{d u^\lambda}{d \lambda} \right)^2 \geq 0.
\]

We we conclude then

\[
I_1 \geq \int_{\partial B^+} \frac{\beta}{2} \frac{d^2}{d \lambda^2} [\lambda (u^\lambda)^2] - \frac{d}{d \lambda} \left[ \lambda^3 \frac{d}{d \lambda} \left( \frac{d u^\lambda}{d \lambda} \right)^2 \right] + \frac{\beta}{2} \frac{d}{d \lambda} (u^\lambda)^2.
\]

Now, rescaling back, we can write those \( \lambda \) derivatives in \( I_1 \) and \( I_2 \) as follows.

\[
\int_{\partial B^+} \frac{d}{d \lambda} (u^\lambda)^2 = \frac{d}{d \lambda} \left( \frac{\lambda^{\frac{n+2a}{p-1}+1-n}}{p-1} \int_{\partial B^+} u^2 \right),
\]

\[
\int_{\partial B^+} \frac{d^2}{d \lambda^2} [\lambda (u^\lambda)^2] = \frac{d^2}{d \lambda^2} \left( \lambda^{\frac{n+2a}{p-1}+2-n} \int_{\partial B^+} u^2 \right),
\]

\[
\int_{\partial B^+} \lambda^3 \frac{d}{d \lambda} \left( \frac{d u^\lambda}{d \lambda} \right)^2 = \frac{d}{d \lambda} \left[ \lambda^3 \frac{d}{d \lambda} \left( \lambda^{\frac{n+2a}{p-1}+1-n} \int_{\partial B^+} \left( \frac{4 + a}{p - 1} \lambda^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right) \right],
\]

\[
\frac{d^2}{d \lambda^2} \left( \lambda \int_{\partial B^+} |\nabla u\lambda|^2 \right) = \frac{d^2}{d \lambda^2} \left[ \lambda^{1 + \frac{n+2a}{p-1}+2+1-n} \int_{\partial B^+} \left( |\nabla u|^2 - \frac{\partial u}{\partial r} \right)^2 \right],
\]

and

\[
\frac{d}{d \lambda} \left( \int_{\partial B^+} |\nabla u\lambda|^2 \right) = \frac{d}{d \lambda} \left[ \lambda^{\frac{n+2a}{p-1}+2+1-n} \int_{\partial B^+} \left( |\nabla u|^2 - \frac{\partial u}{\partial r} \right)^2 \right].
\]

Substituting these into \( \frac{d}{d \lambda} E(u, \lambda) \) we finish the proof.
2. Main technical tool.

2.1. Integral estimates

For $\beta > 0$, set $B^+_\beta = B_\beta \cap \mathbb{R}^n_+$ and $A^+_\beta = \{ x \in \mathbb{R}^n, a_1 \beta < |x| < a_2 \beta \}$, for some $0 < a_1 < a_2$. Let $u$ be a solution of (1.1), which is stable outside a compact set $\mathcal{K} \subset B^+_R$. For all $R > 4R_0$, we define a family of test functions $\psi = \psi(R, R_0) \in C^2_c(\mathbb{R}^N)$ satisfying

$$
\begin{cases}
0 \leq \psi \leq 1 & \text{if } |x| < R_0 \text{ or } |x| > 2R, \\
\psi \equiv 1 & \text{if } 2R_0 < |x| < R, \\
|\nabla^q \psi| \leq CR_0^{-q} & \text{if } R_0 < |x| < 2R_0, \\
|\nabla^q \psi| \leq CR^{-q} & \text{if } R < |x| < 2R, \text{ and } 1 \leq q \leq 4.
\end{cases}
$$

(2.1)

Similarly, if $u$ is a stable solution of (1.1), then $\psi = \psi(R)$, with $R > 0$ verifying (2.1) with $R_0 = 0$ that is $\psi = 1$ if $|x| < R$.

First of all, we need the following lemma which plays an important role in dealing with Theorem 1.1 and Theorem 1.2.

**Lemma 2.1.** Let $u \in C^4(\mathbb{R}^n_+)$ be a solution of (1.1), which is stable outside a compact set $\mathcal{K}$. Let $R_0 > 0$ such that $\mathcal{K} \subset B^+_R$ and set $v = \Delta u$, there hold

$$
\int_{B^+_R} v^2 + \int_{B^+_R} |x|^a|u|^{p+1} \leq C_0 + CR^{-4} \int_{A^+_R} u^2 + CR^{-2} \int_{A^+_R} |uv|, \quad \forall R > 4R_0,
$$

(2.2)

and

$$
\int_{B^+_R} v^2 + \int_{B^+_R} |x|^a|u|^{p+1} \leq C(1 + R^{n - \frac{4(p+1)+2a}{p+2}}), \quad \forall R > 4R_0.
$$

(2.3)

**Proof.**

**Proof of (2.2).** First, for $\epsilon \in (0, 1)$ and $\eta \in C^2(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^n_+} [\Delta (u\eta)]^2 = \int_{\mathbb{R}^n_+} (u\Delta \eta + 2\nabla u \nabla \eta + \eta \Delta u)^2
$$

$$
\leq (1 + C\epsilon) \int_{\mathbb{R}^n_+} v^2 \eta^2 + \frac{C}{\epsilon} \int_{\mathbb{R}^n_+} u^2 (\Delta \eta)^2 + \frac{C}{\epsilon} \int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla \eta|^2.
$$

Using $\Delta (u^2) = 2|\nabla u|^2 + 2u \Delta u$,

$$
2 \int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla \eta|^2 = \int_{\mathbb{R}^n_+} u^2 \Delta (|\nabla \eta|^2) - 2 \int_{\mathbb{R}^n_+} uv |\nabla \eta|^2.
$$

(2.4)

So, we get

$$
\int_{\mathbb{R}^n_+} [\Delta (u\eta)]^2 \leq (1 + C\epsilon) \int_{\mathbb{R}^n_+} v^2 \eta^2 + C\epsilon \int_{\mathbb{R}^n_+} u^2 (\Delta \eta)^2 + C\epsilon \int_{\mathbb{R}^n_+} |uv| |\nabla \eta|^2.
$$

(2.5)

Take $\eta = \eta^m$ with $m \geq 2$. Apply Cauchy-Schwarz’s inequality, we get

$$
\int_{\mathbb{R}^n_+} |uv| |\nabla \eta|^m \leq C\epsilon^2 \int_{\mathbb{R}^n_+} v^2 \eta^2 + C\epsilon^m \int_{\mathbb{R}^n_+} u^2 |\nabla \eta|^4 \eta^{2m-4}.
$$

(2.6)

Substitute $\eta$ by $\psi^m$ in (2.5), then from (2.6) and (2.1), we obtain
\[
\int_{B_{2R}^+} |\Delta(u\psi^m)|^2 \leq C_0 + (1 + C\epsilon) \int_{B_{2R}^+} v^2 \psi^{2m} + C\epsilon R^{-4} \int_{A_R^+} u^2,
\]

where
\[
C_0 = CR^{-4} \int_{A_R^+} u^2, \quad \text{and} \quad A_0^+ = \{x \in \mathbb{R}^n_+, \ R_0 < |x| < 2R_0\}.
\]

Let \( u \) be a solution of (1.1), which is stable outside a compact set \( K \subset B_{R_0}^+ \). Clearly \( u\psi^m \in H^2_0(B_{2R}^+ \setminus B_{R_0}^+) \) so after a standard approximation argument, the main inequality of stability (1.3) implies
\[
p \int_{B_{2R}^+} |x|^a |u|^{p+1} \psi^{2m} - \int_{B_{2R}^+} (\Delta(u\psi^m))^2 \leq 0, \quad \forall \ R > 4R_0.
\]

Therefore, we conclude then
\[
p \int_{B_{2R}^+} |x|^a |u|^{p+1} \psi^{2m} - (1 + C\epsilon) \int_{B_{2R}^+} v^2 \psi^{2m} \leq C_0 + C\epsilon R^{-4} \int_{A_R^+} u^2. \tag{2.7}
\]

On the other hand, recall that \( u = \frac{\partial u}{\partial \eta} = 0 \) in \( \partial \mathbb{R}^n_+ \). Multiply the equation (1.1) by \( u\eta \), \( \eta \in C^2(\mathbb{R}^N) \) and integrate by parts, using again (2.3)
\[
\int_{\mathbb{R}^n_+} \left[ v^2 \eta^2 - |x|^a |u|^{p+1} \eta^2 \right] = -4 \int_{\mathbb{R}^n_+} \eta v \nabla u \cdot \nabla \eta - 2 \int_{\mathbb{R}^n_+} \eta uv \Delta \eta - 2 \int_{\mathbb{R}^n_+} uv |\nabla \eta|^2 \leq C\epsilon \int_{\mathbb{R}^n_+} v^2 \eta^2 + C\epsilon \int_{\mathbb{R}^n_+} u^2 (\Delta \eta)^2 + C\epsilon \int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla \eta|^2 - 2 \int_{\mathbb{R}^n_+} uv |\nabla \eta|^2 \leq C\epsilon \int_{\mathbb{R}^n_+} v^2 \eta^2 + C\epsilon \int_{\mathbb{R}^n_+} u^2 \left[ (\Delta \eta)^2 + |\nabla (|\nabla \eta|^2)| \right] + C\epsilon \int_{\mathbb{R}^n_+} |uv||\nabla \eta|^2. \tag{2.8}
\]

Using the above inequality (where one substitutes \( \eta \) by \( \psi^m \)), it follows from (2.6) and (2.1) that
\[
(1 - C\epsilon) \int_{B_{2R}^+} v^2 \psi^{2m} - \int_{B_{2R}^+} |x|^a |u|^{p+1} \psi^{2m} \leq C_0 + C\epsilon R^{-4} \int_{A_R^+} u^2. \tag{2.9}
\]

Taking \( \epsilon > 0 \) but small enough, multiplying (2.9) by \( \frac{1 + 2C_0}{1 - C\epsilon} \), adding it with (2.7) we get then
\[
C\epsilon \int_{B_{2R}^+} v^2 \psi^{2m} + \left( p - \frac{1 + 2C\epsilon}{1 - C\epsilon} \right) \int_{B_{2R}^+} |x|^a |u|^{p+1} \psi^{2m} \leq C_0 + C\epsilon R^{-4} \int_{A_R^+} u^2.
\]

As \( p > 1 \) and \( A_R^+ \subset B_{2R}^+ \), using \( \epsilon > 0 \) small enough, there holds
\[
\int_{B_R^+} u^2 + \int_{B_R^+} |x|^a |u|^{p+1} \leq C_0 + CR^{-4} \int_{A_R^+} u^2.
\]

Applying Young’s inequality, we deduce then for any \( \epsilon' > 0 \)
\[
\int_{B_R^+} u^2 + (1 - \epsilon') \int_{B_R^+} |x|^a |u|^{p+1} \leq C_0 + CR^{n- \frac{2(p+1)+2n}{p}} , \quad \forall \ R > 4R_0.
\]

Take \( \epsilon' > 0 \) small enough, the estimate (2.3) is proved.

Let us now prove (2.2). Invoking now (2.5) where we substitute \( \eta \) by \( \psi^m \), we obtain
\[
\int_{B_{2R}^+} |\Delta(u\psi^m)|^2 \leq C_0 + (1 + C\epsilon) \int_{B_{2R}^+} v^2 \psi^{2m} + C\epsilon R^{-4} \int_{A_R^+} u^2 + C\epsilon R^{-2} \int_{A_R^+} |uv|.
\]
Adopting the similar argument as above where we use the equality \(2.8\) and inequality of stability \((1.3)\), we obtain readily the estimate \(2.2\). Thus, Lemma \(2.1\) is well proved. \(\square\)

2.2. Homogeneous solutions

In this section, we obtain a nonexistence result for a homogeneous stable solution of \((1.1)\).

**Proposition 2.1.** Let \(u \in W^{2,2}_\text{loc}(\mathbb{R}^n_+ \setminus \{0\})\) be a homogeneous, stable solution of \((1.1)\) in \(\mathbb{R}^n_+ \setminus \{0\}\), \(p \in \left(\frac{n+4+2a}{n-2}, p_{JL2}(n,a)\right)\). Assume that \(|x|^a|u|^{p+1} \in L^1_\text{loc}(\mathbb{R}^n_+ \setminus \{0\})\). Then \(u \equiv 0\).

**Proof.** Let \(u\) be a homogeneous solution of \((1.1)\), that is there exists a \(w \in W^{2,2}(\mathbb{S}^{n-1}_+)\) such that in polar coordinates

\[
u(r, \theta) = r^{-\frac{n-a}{2}} w(\theta).
\]

Denote \(A^+_R = B^+_2 \setminus B^+_R\). Since \(u \in W^{2,2}(A^+_1)\) and \(|x|^a|u|^{p+1} \in L^1(A^+_1)\), it implies that

\(w \in W^{2,2}(\mathbb{S}^{n-1}_+) \cap L^{p+1}(\mathbb{S}^{n-1}_+)\).

A direct calculation gives

\[
\Delta^2 \theta w(\theta) - J_1 \Delta \theta w(\theta) + J_2 w(\theta) = |w|^{p-1} w \quad \text{in} \quad \mathbb{S}^{n-1}_+, \quad w = \frac{\partial w}{\partial \theta} = 0 \quad \text{on} \quad \partial \mathbb{S}^{n-1}_+, \quad (2.10)
\]

where

\[
J_1 = \left(\frac{4 + a}{p - 1} + 2\right) \left(n - 4 - \frac{4 + a}{p - 1}\right) + \frac{4 + a}{p - 1} \left(n - 2 - \frac{4 + a}{p - 1}\right),
\]

and

\[
J_2 = \frac{4 + a}{p - 1} \left(\frac{4 + a}{p - 1} + 2\right) \left(n - 4 - \frac{4 + a}{p - 1}\right) \left(n - 2 - \frac{4 + a}{p - 1}\right).
\]

Because \(w \in W^{2,2}(\mathbb{S}^{n-1}_+)\), we can test \((2.10)\) with \(w\), and we obtain

\[
\int_{\mathbb{S}^{n-1}_+} (\Delta \theta w)^2 + J_1 |\nabla \theta w|^2 + J_2 w^2 \ d\theta = \int_{\mathbb{S}^{n-1}_+} |w|^{p+1} \ d\theta. \quad (2.11)
\]

As in \[1\], for any \(\epsilon > 0\), choose an \(\eta_\epsilon \in C_0^\infty((\frac{\epsilon}{2}, \frac{\epsilon}{4}))\), such that \(\eta_\epsilon \equiv 1\) in \((\epsilon, \frac{1}{\epsilon})\), and

\[
|\frac{\partial \eta_\epsilon}{\partial r}(r)| + r^2 |\frac{\partial^2 \eta_\epsilon}{\partial r^2}(r)| \leq 64, \quad \text{for all} \ r > 0.
\]

We assume that \(\Omega_k = B_{2k/\epsilon} \setminus B_{\epsilon/2k}\). Since \(w \in W^{2,2}(\mathbb{S}^{n-1}_+) \cap L^{p+1}(\mathbb{S}^{n-1}_+)\), \(r^{-\frac{n-4}{2}} w(\theta) \eta_\epsilon(r)\) can be approximated by \(C_0^\infty(\Omega_2 \cap \mathbb{R}^n_+)\) functions in \(W^{2,2}(\Omega_1 \cap \mathbb{R}^n_+) \cap L^{p+1}(\Omega_1 \cap \mathbb{R}^n_+)\). Hence, in the stability condition for \(u\), we are allowed to choose a test function of the form

\[
r^{-\frac{n-4}{2}} w(\theta) \eta_\epsilon(r).
\]

Direct calculations show that

\[
\Delta \left(r^{-\frac{n-4}{2}} w(\theta) \eta_\epsilon(r)\right) = -\frac{n(n - 4)}{4} r^{-\frac{2}{2}} w(\theta) \eta_\epsilon(r) + 3 r^{-\frac{2}{2} + 1} w(\theta) \eta'_\epsilon(r) + r^{-\frac{2}{2} + 2} w(\theta) \eta''_\epsilon(r) + r^{-\frac{2}{2}} \Delta \theta w(\theta) \eta_\epsilon(r).
\]
Substituting this into the stability condition for $u$, we deduce that

$$p \left( \int_{S^+_{n-1}} |w|^{p+1} d\theta \right) \left( \int_0^{+\infty} r^{-1} \eta_r(r)^2 dr \right) \leq \left( \int_{S^+_{n-1}} \left( (\Delta \theta w)^2 + \frac{n(n-4)}{2} |\nabla \theta w|^2 + \frac{n^2(n-4)^2}{16} w^2 \right) d\theta \right) \left( \int_0^{+\infty} r^{-1} \eta_r(r)^2 dr \right)$$

$$+ O \left[ \int_0^{+\infty} \left( r \eta_r(r)^2 + r^3 \eta''_r(r)^2 + \eta_r(r) |\eta'_r(r)| + r \eta_r(r) |\eta''_r(r)| \right) dr \times \int_{S^+_{n-1}} |\nabla \theta w(\theta)|^2 + w(\theta)^2 d\theta \right].$$

Note that

$$\int_0^{+\infty} r^{-1} \eta_r(r)^2 dr \geq |\log \epsilon|,$$

$$\int_0^{+\infty} \left( r \eta'_r(r)^2 + r^3 \eta''_r(r)^2 + \eta_r(r) |\eta'_r(r)| + r \eta_r(r) |\eta''_r(r)| \right) dr \leq C,$$

for some constant $C$ independent of $\epsilon$. By letting $\epsilon \to 0$, we obtain

$$p \int_{S^+_{n-1}} |w|^{p+1} d\theta \leq \int_{S^+_{n-1}} (\Delta \theta w)^2 + \frac{n(n-4)}{2} |\nabla \theta w|^2 + \frac{n^2(n-4)^2}{16} w^2 d\theta.$$

Substituting (2.11) into this we derive

$$\int_{S^+_{n-1}} (p-1) (\Delta \theta w)^2 + \left( pJ_1 - \frac{n(n-4)}{2} \right) |\nabla \theta w|^2 + \left( pJ_2 - \frac{n^2(n-4)^2}{16} \right) w^2 d\theta \leq 0.$$

If $\frac{n+4+2a}{n-4} < p < p_{JL}(n,a)$, implies that $pJ_1 - \frac{n(n-4)}{2} > 0$ and $pJ_2 - \frac{n^2(n-4)^2}{16} > 0$.

The proof for the last inequality is very similar to that of [8, Theorem 3.1], we leave the details for interested readers. So, it follows that $u \equiv 0$. \hfill \Box

3. Classification of stable solutions.

For the case, $1 < p \leq \frac{n+4+2a}{n-4}$, we apply the integral estimates. For the case, $\frac{n+4+2a}{n-4} < p < p_{JL}(n,a)$, with the energy estimates and the desired monotonicity formula under the condition $\frac{n+4+2a}{n-4} < p < p_{JL}(n,a)$, we can show that the stable solutions must be homogeneous solutions, hence by applying the classification of the homogeneous solutions (see Proposition 2.1), the solutions must be zero.

3.1. Proof of Theorem 1.1

Since we assume that $u$ is a stable solution, then the integral estimate (2.3) holds with $C_0 = 0$. We divide the proof in three parts.

Step 1. Subcritical case: $1 < p < \frac{n+4+2a}{n-4}$.

Applying (2.3) and $1 < p < \frac{n+4+2a}{n-4}$, we deduce that

$$\int_{B_R^n} v^2 + \int_{B_R^n} |x|^n |u|^{p+1} \leq CR^{n-\frac{2(p+1)+2a}{p-1}} \to 0, \text{ as } R \to +\infty.$$
Step 2. Subcritical case: \( p = \frac{n+4+2a}{n-4} \). Applying again (2.3) we have
\[
\int_{\mathbb{R}^n_+} v^2 + |x|^a|u|^{p+1} < +\infty.
\]
So, we get
\[
\lim_{R \to +\infty} R^{-4} \int_{A_R^+} v^2 + |x|^a|u|^{p+1} = 0.
\]
Now, applying Hölder inequality, we derive
\[
R^{-4} \int_{A_R^+} v^2 \leq CR^{-4} \left( \int_{A_R^+} |x|^a|u|^{p+1} \right)^{\frac{2}{p+1}} \left( \int_{A_R^+} |x|^{\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}}.
\]
Therefore, from (2.2) we conclude then
\[
\int_{B_R^+} v^2 + |x|^a|u|^{p+1} \leq CR^{-4} \left( \int_{A_R^+} |x|^a|u|^{p+1} \right)^{\frac{2}{p+1}} + C \int_{A_R^+} v^2.
\]
Under the assumptions \( p = \frac{n+4+2a}{n-4} \), tending \( R \to +\infty \), we obtain \( u \equiv 0 \).

Step 3. Supercritical case: \( \frac{n+4+2a}{n-4} < p < p_{JL,2}(n, a) \). We define blowing down sequences
\[
u^\lambda(x) = \lambda^{\frac{4+a}{n-4}} u(\lambda x), \quad v^\lambda(x) = \lambda^{\frac{4+a}{n-4}+2} v(\lambda x), \quad \forall \; \lambda > 0.
\]
\( u^\lambda \) is also a smooth stable solution of (1.1) on \( \mathbb{R}^n_+ \). By rescaling (2.3) for all \( \lambda > 0 \) and balls \( B_r \subset \mathbb{R}^n \),
\[
\int_{B_r^+} (v^\lambda)^2 + |x|^a|u^\lambda|^{p+1} \leq CR^{-4} \left( \int_{A_R^+} |x|^a|u|^{p+1} \right)^{\frac{2}{p+1}}.
\]
In particular, \( u^\lambda \) are uniformly bounded in \( L^{p+1}\text{loc}(\mathbb{R}^n_+) \). By elliptic estimates, \( u^\lambda \) are also uniformly bounded in \( W^{2,2}\text{loc}(\mathbb{R}^n_+) \). Hence, up to a subsequence of \( \lambda \to +\infty \), we can assume that \( u^\lambda \to u_\infty \) weakly in \( W^{2,2}\text{loc}(\mathbb{R}^n_+) \cap L^{p+1}\text{loc}(\mathbb{R}^n_+) \). By compactness embedding, one has \( u^\lambda \to u_\infty \) strongly in \( W^{2,2}\text{loc}(\mathbb{R}^n_+) \). Then for any ball \( B_R^+(0) \), by interpolation between \( L^q \) spaces and noting (2.3), for any \( q \in (1, p+1) \), as \( \lambda \to +\infty \)
\[
||u^\lambda - u_\infty||_{L^q(B_R^+(0))} \leq ||u^\lambda - u_\infty||_{L^q(B_R^+(0))} ||u^\lambda - u_\infty||_{L^{p+1}(B_R^+(0))}^{1-\frac{1}{q}} \to 0,
\]
where \( \frac{1}{q} = \mu + \frac{1-\nu}{p+1} \). That is, \( u^\lambda \to u_\infty \) in \( L^{q}\text{loc}(\mathbb{R}^n_+) \) for any \( q \in (1, p+1) \).
For any function \( \zeta \in C_0^\infty(\mathbb{R}^n_+) \),
\[
\int_{\mathbb{R}^n_+} \Delta u^\lambda \Delta \zeta - |x|^a|u_\infty|^{p-1} u_\infty \zeta = \lim_{\lambda \to +\infty} \int_{\mathbb{R}^n_+} \Delta u^\lambda \Delta \zeta - |x|^a|u^\lambda|^{p-1} u^\lambda \zeta,
\]
\[
\int_{\mathbb{R}^n_+} (\Delta \zeta)^2 - p|x|^a|u_\infty|^{p-1}(\zeta)^2 = \lim_{\lambda \to +\infty} \int_{\mathbb{R}^n_+} (\Delta \zeta)^2 - p|x|^a|u^\lambda|^{p-1}(\zeta)^2 \geq 0.
\]
Thus \( u_\infty \in W^{2,2}\text{loc}(\mathbb{R}^n_+) \cap L^{p+1}\text{loc}(\mathbb{R}^n_+) \) is a stable solution of (1.1).

Now, we can follow exactly the proof of Lemmas 3.1–3.3 in Hu [8] (see also Lemmas 4.4–4.6 in Dávila et al. [11]), to obtain

Lemma 3.1. 1. \( \lim_{\lambda \to +\infty} E(u, \lambda) < +\infty \).
2. \( u_\infty \) is homogeneous.
3. \( \lim_{r \to +\infty} E(u, r) = 0. \)

Therefore, by the monotonicity formula we know that \( u \) is homogeneous, then \( u \equiv 0 \), by Proposition 1.1. This finishes the proof of Theorem 1.1. □

4. Classification of the finite Morse index solutions

We proceed based on a Pohozaev-type identity, the decay estimates from the doubling lemma 7, the monotonicity formula, the classification of the homogeneous solutions and stable solutions we obtained before.

4.1. Subcritical and critical case

Our approach consists in testing the equation (1.1) against \( \nabla u \cdot x \psi \) where \( \psi \in C^2_c(\mathbb{R}^N) \), \( 0 \leq \psi \leq 1 \) is a cut-off function satisfying

\[
\begin{cases}
\psi \equiv 1 & \text{if } |x| < R, \\
\psi \equiv 1 & \text{if } |x| > 2R,
|\nabla^q \psi| \leq CR^{-q}, & \text{if } x \in A_R = \{ R < |x| < 2R \} & q \leq 2.
\end{cases}
\]

We provide the following variant of the Pohozaev identity. In view of the cut-off functions \( \psi \), we can avoid the spherical integrals raised in [9, 11], which are very difficult to control. Precisely, we have

Lemma 4.1. Let \( u \) be a solution of (1.1) and set \( v = \Delta u \). Then for any \( \psi \in C^2_c(\Omega) \),

\[
\frac{n + a}{p + 1} \int_{\Omega} |x|^a |u|^{p+1} \psi - \frac{n - 4}{2} \int_{\Omega} \psi
= - \frac{1}{p + 1} \int_{\Omega} |x|^a |u|^{p+1} (\nabla \psi \cdot x) + \frac{1}{2} \int_{\Omega} (\nabla \psi \cdot x) v^2
- \int_{\Omega} \left[ 2v(\nabla u \cdot \nabla \psi) + 2v \nabla^2 u(x, \nabla \psi) + v(\nabla u \cdot x) \Delta \psi \right].
\]

Proof. Let \( \psi \in C^2_c(\Omega) \), multiplying equation (1.1) by \( \nabla u \cdot x \psi \) and integrating by parts, we get

\[
\int_{\Omega} |x|^a |u|^{p-1} u(\nabla u \cdot x) \psi
= \int_{\Omega} \Delta u \Delta(\nabla u \cdot x) \psi = \int_{\Omega} v \left[ (\nabla (v) \cdot x) \psi + 2v \psi + 2(\nabla (v) \cdot x) \cdot \nabla \psi + (\nabla u \cdot x) \Delta \psi \right].
\]

Direct calculation yields \( \nabla (v) \cdot x \cdot \nabla \psi = \nabla^2 u(x, \nabla \psi) + (\nabla u \cdot \nabla \psi) \) and

\[
\int_{\Omega} v \left[ (\nabla (v) \cdot x) \psi + 2v \psi \right] = \int_{\Omega} \frac{\nabla (u^2)}{2} \cdot x \psi + 2 \int_{\Omega} v^2 \psi
= \frac{4 - n}{2} \int_{\Omega} v^2 \psi - \frac{1}{2} \int_{\Omega} v^2 (\nabla \psi \cdot x).
\]

Moreover,

\[
\int_{\Omega} |x|^a |u|^{p-1} u(\nabla u \cdot x) \psi = - \frac{n + a}{p + 1} \int_{\Omega} |u|^{p+1} \psi - \frac{1}{p + 1} \int_{\Omega} |x|^a |u|^{p+1} x \cdot \nabla \psi.
\]

Therefore, the claim follows by regrouping the above equalities. □

We claim then
Lemma 4.2. Let $u \in C^4(\mathbb{R}_+^n)$ be a solution of (1.1) which is stable a compact set of $\mathbb{R}_+^n$. If $p \in (1, \frac{n+4+2a}{n-4})$, then $|x|^\frac{n-4}{p+1} u \in L^{p+1}(\mathbb{R}_+^n)$, $v \in L^2(\mathbb{R}_+^n)$.

$$\frac{n-4}{2} \int_{\mathbb{R}_+^n} v^2 = \frac{n+a}{p+1} \int_{\mathbb{R}_+^n} |x|^a |u|^{p+1}. \quad (4.3)$$

and

$$\int_{\mathbb{R}_+^n} v^2 = \int_{\mathbb{R}_+^n} |x|^a |u|^{p+1}. \quad (4.4)$$

Proof. Using (2.3) and tending $R \to \infty$, we obtain

$$|x|^\frac{n-4}{p+1} u \in L^{p+1}(\mathbb{R}_+^n) \quad \text{and} \quad v \in L^2(\mathbb{R}_+^n). \quad (4.5)$$

By Hölder’s inequality, there holds

$$R^{-4} \int_{A_R^+} |u|^2 \leq CR^{(\frac{n-4(p+1)+2a}{p+1})+1} \left( \int_{A_R^+} |x|^a |u|^{p+1} \right)^{\frac{p+1}{p+4}}. \quad (4.6)$$

On the other hand, by standard scaling argument, there exists $C > 0$ such that for any $R > 0$, any $u \in C^4(A_R^+)$ with $A_R^+ = B_{2R} \setminus B_R$,

$$R^{-2} \int_{A_R^+} |\nabla u|^2 \leq C \int_{A_R^+} v^2 + CR^{-4} \int_{A_R^+} u^2. \quad (4.7)$$

Therefore, as $p$ is subcritical, we deduce that

$$CR^{-4} \int_{A_R^+} u^2 + R^{-2} \int_{A_R^+} |\nabla u|^2 \to 0 \quad \text{as} \quad R \to \infty. \quad (4.8)$$

Now we shall estimate the integral

$$\int_{A_R^+} |\nabla^2 u|^2. \quad (4.9)$$

Since $u \zeta = 0$ on $\partial \mathbb{R}_+^n$, by standard elliptic theory, there exists $C > 0$ such that

$$\int_{A_R^+} |\nabla^2 (u \zeta)|^2 \leq C \int_{A_R^+} |\Delta (u \zeta)|^2 \leq C \int_{A_R^+} \left[ |u|^2 |\Delta \zeta|^2 + |\nabla u|^2 |\nabla \zeta|^2 + v^2 \right]. \quad (4.10)$$

So, we get

$$\int_{A_R^+} |\nabla^2 u|^2 \zeta^2 \leq C \int_{A_R^+} |\nabla^2 (u \zeta)|^2 + C \int_{A_R^+} |\nabla u|^2 |\nabla \zeta|^2 + C \int_{A_R^+} u^2 \left( |\Delta \zeta|^2 + |\nabla^2 \zeta|^2 \right) \quad (4.11)$$

$$\leq C \int_{A_R^+} v^2 + CR^{-4} \int_{A_R^+} u^2 + R^{-2} \int_{A_R^+} |\nabla u|^2. \quad (4.12)$$

Using (4.5)–(4.6), there holds

$$\int_{\mathbb{R}_+^n} |\nabla^2 u|^2 < \infty. \quad (4.13)$$

Now, to prove (4.3), we will show that any terms on the right hand side of (1.2) (Denote by $I_R$), tends
to 0 as $R \to +\infty$. Remark that $\nabla \psi \neq 0$ only in $A_R^+ = B_{2R}^+ \setminus B_R^+$ and $\|\nabla^k \psi\|_{\infty} \leq C_k R^{-k}$, there holds

$$|I_R| \leq C \int_{A_R^+} (|x|^a |u|^{p+1} + v^2) + \frac{C}{R} \int_{A_R^+} |v||\nabla u| + C \int_{A_R^+} |v||\nabla^2 u|$$

Thanks to the estimates (4.5)-(4.9) and Hölder’s inequality, clearly $\lim_{R \to \infty} I_R = 0$, hence we get (4.3).

On the other hand, using $u \psi$ as test function in (4.1), we have

$$\int_{B_{2R}^+} v^2 \psi - \int_{B_{2R}^+} |x|^a |u|^{p+1} \psi \leq C \int_{B_{2R}^+} |uv||\Delta \psi| + C \int_{B_{2R}^+} |v||\nabla u||\nabla \psi| dx \leq \frac{C}{R^2} \int_{A_R^+} |uv| + \frac{C}{R} \int_{A_R^+} |v||\nabla u|.$$

Apply Hölder’s inequality, (4.5)-(4.6) and tending $R$ to $\infty$, so we obtain (4.3). The proof is completed.

\[ \square \]

**Proof of Theorem 1.2**

**Step 1. Subcritical case:** $1 < p < \frac{n+4+2a}{n-4}$.

Combining (4.3) and (4.4), there holds

$$\left( \frac{n-4}{2} - \frac{n+a}{p+1} \right) \int_{A_R^+} |u|^{p+1} = 0.$$

We are done, since $n < \frac{4(p+1)+2a}{p-1}$ implies that $\frac{n-4}{2} - \frac{n+a}{p+1} < 0$.

**Step 2. Subcritical case:** $p = \frac{n+4+2a}{n-4}$.

We can proceed as in the proof of equality (4.4), to derive that

$$\int_{\mathbb{R}^n_+} v^2 = \int_{\mathbb{R}^n_+} |x|^a |u|^{p+1} < +\infty.$$

\[ \square \]

### 4.2. Supercritical case

To classify finite Morse index solutions in the supercritical case, applying the doubling lemma in [7], we get the following crucial lemma.

**Lemma 4.3.** Let $n \geq 1$, $1 < p < p_{JL2}(n,0)$ and $\tau \in (0,1]$. Let $c \in C^r(B_1^+)$ satisfy

$$\|c\|_{C^r(B_1^+)} \leq C_1 \quad \text{and} \quad c(x) \geq C_2, \quad x \in B_1^+, \quad (4.10)$$

for some constants $C_1, C_2 > 0$. There exists a constant $C$, depending only on $\alpha, C_1, C_2, p, n$, such that, for any stable solution $u$ of

$$\Delta^2 u = c(x)|u|^{p-1}u \quad \text{in } B_1^+ \quad \text{and} \quad u = \frac{\partial u}{\partial x_n} = 0 \quad \text{on } \partial B_1^+, \quad (4.11)$$

$u$ satisfies

$$|u(x)|^{\frac{p-1}{2}} \leq C(1 + \text{dist}^{-1}(x, \partial B_1^+)).$$

**Proof.** Arguing by contradiction, we suppose that there exist sequences $c_k, u_k$ verifying (4.10)-(4.11) and points $y_k$, such that the functions

$$M_k = |u_k|^{\frac{p-1}{2}}$$

satisfy

$$M_k(y_k) > 2k(1 + \text{dist}^{-1}(y_k, \partial B_1^+)) \geq 2k\text{dist}^{-1}(y_k, \partial B_1^+)).$$
By the doubling lemma in [1], there exists $x_{k}$ such that

$$M_{k}(x_{k}) \geq M_{k}(y_{k}), \quad M_{k}(x_{k}) \geq 2kdist^{-1}(x_{k}, \partial \mathcal{B}^{+}_{1}),$$

and

$$M_{k}(z) \leq 2M_{k}(x_{k}), \quad \text{for all } z \in \mathcal{B}^{+}_{1} \text{ such that } |z - x_{k}| \leq kM_{k}^{-1}(x_{k}). \quad (4.12)$$

We have

$$\lambda_{k} = M_{k}^{-1}(x_{k}) \rightarrow 0, \quad k \rightarrow \infty \quad (4.13)$$
due to $M_{k}(x_{k}) \geq M_{k}(y_{k}) > 2k$.

Next we let

$$v_{k}(y) = \frac{1}{\lambda_{k}^{k-1}} u_{k}(x_{k} + \lambda_{k}y) \quad \text{and} \quad \bar{c}_{k}(y) = c_{k}(x_{k} + \lambda_{k}y), \quad \text{for } y \in \mathcal{B}_{k} \text{ and } y_{n} > -\frac{y_{k,n}}{\lambda_{k}},$$

where $y_{k} = (y_{k,1}, \ldots, y_{k,n})$. Then, $v_{k}(y)$ is the solution of

$$\begin{cases}
\Delta^{2}v_{k}(y) = \bar{c}_{k}(y)v_{k}(y)|v_{k}(y)|^{p-1}v_{k}(y), \quad |y| < k, \quad y_{n} > -\frac{y_{k,n}}{\lambda_{k}}, \\
v_{k}(y) = \frac{\partial v_{k}(y)}{\partial y_{n}} = 0, \quad |y| < k, \quad y_{n} = -\frac{y_{k,n}}{\lambda_{k}},
\end{cases} \quad (4.14)$$

with

$$|v_{k}(0)| = 1 \quad \text{and} \quad |v_{k}(y)| \leq 2^{\frac{k}{p-1}}, \quad |y| < k, \quad y_{n} > -\frac{y_{k,n}}{\lambda_{k}}.$$

Two cases may occur as $k \rightarrow \infty$, either case (1)

$$\frac{y_{k,n}}{\lambda_{k}} \rightarrow +\infty$$

for a subsequence still denoted as before, or case (2)

$$\frac{y_{k,n}}{\lambda_{k}} \rightarrow c \geq 0.$$

If case (1), after extracting a subsequence, $\bar{c}_{k} \rightarrow c \in C_{loc}(\mathbb{R}^{n})$ with $C > 0$ a constant and we may also assume that $v_{k} \rightarrow v$ in $C^{4}_{loc}(\mathbb{R}^{n})$, and $v$ is a stable solution of

$$\Delta^{2}v = C|v|^{p-1}v \quad \text{in } \mathbb{R}^{n} \quad \text{and } |v(0)| = 1.$$

By the Liouville type Theorems in [1] for stable solutions, we derive that $v \equiv 0$. This is a contradiction.

If case (2) we can prove that $c > 0$, thus we get a stable solution of $\Box$ in $\mathbb{R}^{+}$ and $|v(c)| = 1$, which contradict Theorem [1.1] for $1 < p < p_{0}(n,4)$

**Proposition 4.1.** Let $u$ be a (positive or sign changing) solution to $\Box$ which is stable outside a compact set of $\mathbb{R}_{+}^{n}$. There exist constants $C$ and $R_{0}$ such that

$$|u(x)| \leq C|x|^{-\frac{n+\alpha}{2}}, \quad \text{for all } x \in \mathcal{B}_{R_{0}}^{+}(0)^{c}, \quad (4.15)$$

$$\sum_{k \leq 3} |x|^{\frac{\alpha}{p-1} + k} |\nabla^{k}u(x)| \leq C, \quad \text{for all } x \in \mathcal{B}_{R_{0}}^{+}(0)^{c}. \quad (4.16)$$

**Proof.** Assume that $u$ is stable outside $\mathcal{B}_{R_{0}}^{+}$ and $|x_{0}| > 2R_{0}$. We denote

$$R = \frac{1}{2}|x_{0}|$$
and observe that, for all \( y \in B_1^+ \), \( \frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2} \), so that \( x_0 + Ry \in B_1^+(0)^c \). Let us thus define

\[
U(y) = R^{\frac{4+a}{p-1}} u(x_0 + Ry).
\]

Then \( U \) is a solution of

\[
\Delta^2 U = c(y)|U|^{p-1}U \text{ in } B_1^+ \quad \text{and} \quad U = \frac{\partial U}{\partial y_n} = 0 \text{ on } \partial B_1^+, \quad \text{with } c(y) = \left| y + \frac{x_0}{R} \right|^a.
\]

Notice that \( |y + \frac{x_0}{R}| \in [1, 3] \) for all \( y \in B_1^+ \). Moreover \( \|c\|_{C^1(B_1^+)} \leq C(a) \). Then applying Lemma 4.3 we have \( |U(0)| \leq C \), hence

\[
|u(x_0)| \leq CR^{-\frac{4+a}{p-1}},
\]

which yields the inequality (4.16).

Next we only prove the inequality (4.16). For any \( x_0 \) with \( |x_0| > 3R_0 \), take \( \lambda = \frac{|x_0|}{2} \) and define

\[
\overline{u}(x) = \lambda^{\frac{4+a}{p-1}} u(x_0 + \lambda x).
\]

From (4.15), \( |\overline{u}| \leq C_0 \) in \( B_1^+(0) \). Standard elliptic estimates give

\[
\sum_{k \leq 5} |\nabla^k \overline{u}(0)| \leq C.
\]

Rescaling back we get (4.16). \( \square \)

**Proof of Theorem 1.2** Supercritical case: \( p > \frac{4+a+2a}{n-4} \) and \( p < p_{1L2}(n, 0) \).

**Lemma 4.4.** There exists a constant \( C_2 \), such that for all \( r > 3R_0 \), \( E(u, r) \leq C_2 \).

**Proof.** From the monotonicity formula, combining the derivative estimates (4.16), we have then

\[
E(u, r) \\
\leq Cr^{\frac{4(p+1)+2a}{p-1}} - n \left( \int_{B_1^+} v^2 + |x|^a |u|^{p+1} \right) + Cr^{\frac{8+2a}{p-1} + 1 - n} \int_{\partial B_1^+} u^2 + Cr^{\frac{8+2a}{p-1} + 2 - n} \int_{\partial B_1^+} |u||\nabla u| \\\n+ Cr^{\frac{8+2a}{p-1} + 3 - n} \int_{\partial B_1^+} |\nabla u|^2 + Cr^{\frac{8+2a}{p-1} + 3 - n} \int_{\partial B_1^+} |u||\nabla^2 u| + Cr^{\frac{8+2a}{p-1} + 4 - n} \int_{\partial B_1^+} |u|^2 \leq C.
\]

where \( C \) depends on the constant appeared in (4.10). \( \square \)

We claim then

**Corollary 4.1.**

\[
\int_{(B_{2R_0}^+(0))^c} \left( \frac{4+a}{p-1} |x|^{-1} u(x) + \frac{2a}{p-1} (x) \right)^2 \frac{|x|^{n-2} \cdot \frac{8+2a}{p-1}}{< \infty}.
\]

As before, we define a blowing down sequence

\[
u^\lambda(x) = \lambda^{\frac{4+a}{p-1}} u(\lambda x).
\]

By Proposition 4.1, \( u^\lambda \) are uniformly bounded in \( C^6(B_{1}^+(0) \setminus B_{1/r}^+(0)) \) for any fixed \( r > 1 \).

\( u^\lambda \) is stable outside \( B_{R_0/\lambda}^+(0) \). There exists a function \( u^\infty \in C^6_c(\mathbb{R}^n \setminus \{0\}) \), such that up to a subsequence of \( \lambda \to +\infty \), \( u^\lambda \) converges to \( u^\infty \in C^4_{loc}(\mathbb{R}^n \setminus \{0\}) \). \( u^\infty \) is a stable solution of (1.1) in \( \mathbb{R}^n \setminus \{0\} \).
Using Corollary 4.1, we obtain for any $r > 1$,

$$
\int_{B^+_r \setminus B^+_1} \left( \frac{4+ap}{p-1} |x|^{-1} u^\infty(x) + \frac{\partial u^\infty}{\partial r}(x) \right)^2 \left| x \right|^{n-2-\frac{8+2a}{p-1}} = \lim_{\lambda \to +\infty} \int_{B^+_r \setminus B^+_1} \left( \frac{4+ap}{p-1} |x|^{-1} u^\lambda(x) + \frac{\partial u^\lambda}{\partial r}(x) \right)^2 \left| x \right|^{n-2-\frac{8+2a}{p-1}} = 0.
$$

Hence, $u^\infty$ is homogeneous, and from Proposition 2.1, $u^\infty \equiv 0$. This holds for every limit of $u^\lambda$ as $\lambda \to +\infty$, thus we get

$$
\lim_{|x| \to +\infty} \frac{|x|^{4+ap}}{|u(x)|} = 0.
$$

From (4.16), we derive

$$
\lim_{|x| \to +\infty} \sum_{k \leq 4} |x|^{4+ap-k} |\nabla^k u(x)| = 0.
$$

For $\varepsilon > 0$, take an $R$ such that for $|x| > R$,

$$
\sum_{k \leq 4} |x|^{4+ap-k} |\nabla^k u(x)| \leq \varepsilon.
$$

Then for $r >> R$,

$$
E(u, r) \leq Cr^{4(p+1)+2a-p-1-n} \left( \int_{B^+_r(0)} v^2 + |x|^n |u|^{p+1} \right) + C \int_{\partial B^+_r(0) \setminus \partial B^+_1(0)} |x|^{-\frac{8+2a}{p-1}-4} + C \int_{\partial B^+_1(0)} |x|^{-\frac{8+2a}{p-1}-4} \leq C(R) \left( r^{4(p+1)+2a-p-1-n} + \varepsilon \right).
$$

Since $4(p+1)+2a - n < 0$ and $\varepsilon$ can be arbitrarily small, we derive $\lim_{r \to +\infty} E(u, r) = 0$. Because $\lim_{r \to 0} E(r, u) = 0$ (by the smoothness of $u$), the same argument for stable solutions implies that $u \equiv 0$. 

\[ \blacksquare \]

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