A note on the Waring Ranks of Reducible Cubic Forms

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Abstract

Let $W_3(n)$ be the set of Waring ranks of reducible cubic forms in $n + 1$ variables. We prove that $W_3(n) \subseteq \{1, \ldots, 2n + 1\}$, giving a classification of admissible ranks according to the projective equivalence classes of reducible cubic forms in $\mathbb{P}^n$.

Introduction

Let $K$ be an algebraically closed field of characteristic zero, let $V$ be a $(n + 1)$-dimensional $K$-vector space and $F \in S^d V$, namely a homogeneous polynomial of degree $d$ in $n + 1$ indeterminates. The Waring problem for polynomials asks for the least value $s$ such that there exist linear forms $L_1, \ldots, L_s$, for which $F$ can be written as a sum

$$F = L_1^d + \ldots + L_s^d.$$ (1)

This value $s$ is called the Waring rank, or simply the rank, of the form $F$, and here it will be denoted by $rk(F)$. The Waring problem for a general form $F$ of degree $d$ was solved by Alexander and Hirschowitz, in their celebrated paper [1]. They came out with this result solving the interpolation problem of order 2.

Alexander-Hirschowitz Theorem. A general form $F$ of degree $d$ in $n + 1$ variables is the sum of $\left\lceil \frac{1}{n+1}(\binom{n+d}{d}) \right\rceil$ powers of linear forms, unless

- $d = 2, s = n + 1$ instead of $\left\lceil \frac{n+2}{2} \right\rceil$;
- $d = 3, n = 4$ and $s = 8$ instead of 7;
- $d = 4, n = 2, 3, 4$ and $s = 6, 10, 15$ instead of 5, 9, 14 respectively.
Remark 1. The assumption on the characteristic is not necessary, see [3] for more details.

The Waring problem in the case of a given homogeneous polynomial is far from being solved. A major development in this direction is made in [2] where the rank of any monomial and the rank of any sum of pairwise coprime monomials are computed. The present paper concerns with the Waring rank of reducible cubic forms. The main result of this work is the following theorem.

Theorem 1. Let $W_3(n)$ be the set of ranks of reducible cubic forms in $n + 1$ variables, then

$$W_3(n) \subseteq \{1, \ldots, 2n + 1\}.$$  

The Apolarity

In this section, we recall basic definitions and facts. We recommend to see [3] and [5] for a detailed and deep exposition.

Let $K$ be an algebraically closed field of characteristic zero, $S = K[x_0, \ldots, x_n]$ and $T = K[\partial_0, \ldots, \partial_n]$ be the dual ring of $S$ (i.e. the ring of differential operators over $K$). $T$ is an $S$-module acting on $S$ by differentiation

$$\partial^\alpha(x^\beta) = \begin{cases} \alpha!(\binom{\beta}{\alpha})x^{\beta - \alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha$ and $\beta$ are multi-indices. The action of $T$ on $S$ is classically called **apolarity**. Note that $S$ can also act on $T$ with a (dual) differentiation, defined by

$$x^{\beta}(\partial^\alpha) = \beta!(\binom{\alpha}{\beta})\partial^{\alpha - \beta},$$

if $\alpha \geq \beta$ and 0 otherwise.

In this way, we have a non-degenerate pairing between the forms of degree $d$ and the homogeneous differential operators of order $d$. Let us recall two basic definitions.

**Definition 1.** Let $F \in S$ be a form and $D \in T$ be a homogeneous differential operator. Then $D$ is **apolar** to $F$ if $D(F) = 0$.

**Definition 2.** For any $F \in S^dV$, we define the ideal $F^\perp = \{D \in T | D(F) = 0\} \subseteq T$, called the **principal system** of $F$. If $F \in S^dV$, for every homogeneous operator $D \in T$ of degree $\geq d + 1$, we have $D(F) = 0$, or equivalently $D \in F^\perp$. The principal system of $F$ is a **Gorenstein ideal**.
We recall the major result of this section, which is the main tool in our approach.

**Lemma 1 (Apolarity Lemma).** A form $F \in S^dV$ can be written as

$$F = \sum_{i=1}^{s} L_i^d,$$

where $L_i$ are linear forms pairwise linearly independent, if and only if there exists an ideal $I \subset F^\perp$ such that $I$ is the ideal of a set of $s$ distinct points in $\mathbb{P}^n$, where these $s$ points are the corresponding points of the linear forms $L_i$ in the dual space $\mathbb{P}^{n*}$.

For a proof of Apolarity Lemma 1 and its applications see for instance [3]. We will refer to the $s$ points of this Lemma as decomposition points.

**Classification of Ranks of Reducible Cubic Forms in $\mathbb{P}^n$**

In this section we give the classification of the ranks of reducible cubic forms. Since the rank is invariant under projective transformations, we only need to check the projective equivalence classes of cubic forms. Let $W_3(n)$ be the set of values of ranks of reducible cubic forms in $n + 1$ variables, namely forms of type $F = LQ$, where $L, Q \in S$ are linear and quadratic forms respectively. In order to give a classification, note that $W_3(n - 1) \subset W_3(n)$. Indeed, every form in $n$ indeterminates is also a form in the ring of polynomials in $n + 1$ indeterminates and the ranks as polynomial in $n$ variables and as polynomial in $n + 1$ variables are equal. The subset $W_3(n - 1) \subset W_3(n)$ is the set of the ranks of reducible cones in $n + 1$ variables. The forms $F = LQ$ which are not cones (up to projective equivalence) are the following.

- (Type A) $Q$ is not a cone and $L$ is not tangent to $Q$;
- (Type B) $Q$ is a cone and $L$ does not pass through any vertex of $Q$;
- (Type C) $Q$ is not a cone and $L$ is tangent to $Q$.

The aim of this section is to prove this classification result.

**Theorem 2.** The ranks of reducible cubic forms $A$, $B$ and $C$ in $n + 1$ variables are the following.

| Type | Rank       |
|------|------------|
| $A$  | $= 2n$     |
| $B$  | $= 2n$     |
| $C$  | $\geq 2n, \leq 2n + 1$ |

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In [4], the ranks of cubic forms of type $A$ and $B$ are also computed (see Proposition 7.2), using Theorem 1.3, which gives a classical lower bound on the Waring rank of a given form. We provide an alternative proof of this fact. First, we recall an useful scheme-theoretic lemma and then, in the next three subsections (Type $A$, Type $B$ and Type $C$), we compute the ranks of cubic forms of type $A$, $B$ and $C$ resp., in order to give a proof of Theorem [2]. This lemma will be used throughout the first two subsections as a tool for counting decomposition points outside suitable hyperplanes.

**Lemma 2.** Let $\mathcal{X}$ be a reduced zero-dimensional scheme in the space $\text{Proj}(T)$. Then the scheme of zeros of $(I_\mathcal{X} : \partial)$ is the reduced zero-dimensional scheme of the points not lying on $\{\partial = 0\}$.

**Proof.** Let $I = I_\mathcal{X}$ be the ideal of the reduced zero-dimensional scheme $\mathcal{X}$ and let $J$ be the ideal of points not lying on $\{\partial = 0\}$. We want to show the equality $(I : \partial) = J$. Take $G \in (I : \partial)$, then by definition $G(p)\partial(p) = 0$ for every $p \in \mathcal{X}$. If $p \notin \{\partial = 0\}$ then $G(p) = 0$. This implies that $(I : \partial) \subseteq J$. Vice versa, if $\partial \in I$ then $(I : \partial) = (1) = T$. Thus $J \subseteq (I : \partial) = T$. Suppose $\partial \notin I$, then $\partial^2 \notin I$, or equivalently $\partial \notin (I : \partial)$, since $I$ is a radical ideal. Take $H \in J$, then $H\partial$ vanishes on $\mathcal{X}$. Hence $H \in (I : \partial)$ and $J \subseteq (I : \partial)$. 

**Notation.** We denote by $\int G dx_i$ a suitable choice of a primitive of $G$ (that will be specified any time it is needed), namely a form $H$ such that $\partial_i H = G$, where $\partial_i$ denotes the usual partial derivative with respect to the variable $x_i$.

**Type $A$**

Cubic forms of type $A$ are projectively equivalent to the cubic form

$$F = x_0(x_0^2 + x_1^2 + \ldots + x_n^2).$$

We have this upper bound on the rank of cubic forms of type $A$. Furthermore, using the inductive argument one could obtain an explicit decomposition of $F$.

**Proposition 3.** The cubic form $F = x_0(x_0^2 + x_1^2 + \ldots + x_n^2)$ has rank $\leq 2n$.

**Proof.** We prove the statement by induction on $n$. For $n = 1$, the form is $F = x_0(x_0^2 + x_1^2)$, whose rank is 2. Suppose the thesis holds for $i \leq n - 1$. We have to prove it for $i = n$. Take $F = x_0(x_0^2 + x_1^2 + \ldots + x_n^2)$. Now $\partial_i F = 2x_0x_1 = \frac{1}{2}[(x_0 + x_1)^2 - (x_0 - x_1)^2]$. Set $K_1 = \int \partial_i F dx_1 = \frac{1}{8}[(x_0 + x_1)^3 + (x_0 - x_1)^3] = \frac{1}{3}x_0^3 + x_0x_1^2$. Then $F = K_1 + K_2$, where $K_2 = x_0(-\frac{1}{3}x_0^2 + x_2^2 + \ldots + x_n^2)$. 


The form $K_2$ is projectively equivalent to the cubic form of type $A$, and, by inductive assumption, it has rank at most $2(n - 1)$. Then $rk(F) \leq rk(K_1) + rk(K_2) \leq 2 + 2(n - 1)$. Repeating the argument above, one obtains a decomposition of $F$.

\[\Box\]

**Proposition 4.** The cubic form $F = x_0(x_0^2 + x_1^2 + \ldots + x_n^2)$ has rank $\geq 2n$.

**Proof.** Let $I_X \subset F$, where $X$ is a scheme of decomposition points. Consider the ideal $(F : \partial_i)$. Let $I_{X_i}$ be the ideal of the set $X_i$, namely the set of points in $X$ not lying on $\{\partial_i = 0\}$. By Lemma 2, $I_{X_i} = (I_X : \partial_i) \subseteq (F : \partial_i)$. Note that $(F : \partial_i) = (\partial_i F)^\perp = (x_0 x_i)^\perp$. This means that $X_i$ is a scheme of decomposition points for the monomial $x_0 x_i$, and so $X_i$ contains at least 2 points for any $1 \leq i \leq n$. Consider the union $X_{ij} = X_i \cup X_j$, where $1 \leq i \leq n$ and $i \neq j$.

Then $(F : \partial_i) \cap (F : \partial_j) = (x_0 x_i)^\perp \cap (x_0 x_j)^\perp = (\partial_i^2, \partial_i, \partial_j, \partial_i \partial_j, \partial_i, \partial_j) = (x_0 x_i x_j)^\perp$ and $I_{X_{ij}} \subset (F : \partial_i) \cap (F : \partial_j) = (x_0 x_i x_j)^\perp$. Hence $X_{ij}$ is a scheme of decomposition points for the monomial $x_0 x_i x_j$, whose rank is 4 (see [2]). Since the rank is 4, $X_{ij}$ do not contain a number of points $m < 4$; indeed, if so, then $rk(x_0 x_i x_j) \leq m < 4$, that is a contradiction. This is equivalent to saying that $X_i$ and $X_j$ do not have points in common, then there are at least $2n$ points. By Apolarity Lemma 1, the thesis is proved.

\[\Box\]

**Type B**

Cubic forms of type $B$ are projectively equivalent to the cubic form

$$F = x_0(x_1^2 + x_2^2 + \ldots + x_n^2).$$

**Proposition 5.** The cubic form $F = x_0(x_1^2 + x_2^2 + \ldots + x_n^2)$ has rank $\leq 2n$.

**Proof.** The first step is to compute the rank of $F = x_0(x_1^2 + x_2^2)$, which is 4 (see [2]). Then, suppose the thesis true for $3 \leq i \leq n - 1$. Take the form $F = x_0(x_1^2 + x_2^2 + \ldots + x_n^2)$ and set $K_1 = \int \partial_1 F dx_1 = \frac{1}{3}x_0^3 + x_0 x_1^2$. $F = K_1 - \frac{1}{3}x_0^3 + x_0 x_2^2 + \ldots + x_0 x_n^2$ and set $K_2 = \int \partial_2 F dx_2 = -\frac{1}{3}x_0^3 + x_0 x_2^2$. $F = K_1 + K_2 + K_3$, where $K_3 = x_0(x_2^2 + \ldots + x_n^2)$, which is projectively equivalent to the cubic form of type $B$ and so, by induction, $K_3$ has rank at most $2(n - 2)$. The rank of $K_1 + K_2$ is 4, hence $rk(F) \leq 4 + 2(n - 2) = 2n$. Repeating the argument above, one obtains a decomposition of $F$.

\[\Box\]

The converse inequality holds as stated in the following proposition, where we use the same arguments as in the proof of Proposition 4.
**Proposition 6.** The cubic form $F = x_0(x_1^2 + x_2^2 + \ldots + x_n^2)$ has rank $\geq 2n$.

**Proof.** Let $I_X \subset F^\perp$, where $X$ is a scheme of decomposition points. Consider the ideal $(F^\perp : \partial_i)$. Let $I_{X_i}$ be the ideal of the set $X_i$, namely the set of points in $X$ not lying on $\{\partial_i = 0\}$. By Lemma 2, $I_{X_i} = (I_X : \partial_i) \subseteq (F^\perp : \partial_i)$. As before in Proposition 4, note that $(F^\perp \partial_i) = (\partial_i F^\perp = (x_0 x_i)^\perp$. This means that $X_i$ is a scheme of decomposition points for the monomial $x_0 x_i$, and so $X_i$ contains at least 2 points for any $1 \leq i \leq n$. Consider the union $X_{ij} = X_i \cup X_j$, where $1 \leq i \leq n$ and $i \neq j$. We have $(F^\perp : \partial_i) \cap (F^\perp : \partial_j) = (x_0 x_i)^\perp \cap (x_0 x_j)^\perp = (\partial_0^2, \partial_1, \partial_2, \ldots, \partial_i^2, \partial_j^2, \ldots, \partial_n) = (x_0 x_i x_j)^\perp$, and $I_{X_{ij}} \subset (F^\perp : \partial_i) \cap (F^\perp : \partial_j)$. The scheme $X_{ij}$ is a scheme of decomposition points for the monomial $x_0 x_i x_j$, whose rank is 4 (see [2]). Since the rank is 4, $X_{ij}$ do not contain a number of points $m < 4$; indeed, if so, then $rk(x_0 x_i x_j) \leq m < 4$, that is a contradiction. This is equivalent to saying that $X_i$ and $X_j$ do not have points in common, then there are at least $2n$ points. By Apolarity Lemma 1, the thesis is proved.

**Remark 2.** Propositions 4 and 6 are also a consequence of Theorem 1.3, in [4], which implies that every reducible cubic form in $S^3 \mathbb{C}^{n+1}$ has rank at least $2n$.

**Type C**

Cubic forms of type C are projectively equivalent to the cubic form

$$F = x_0(x_1 x_3 + x_2 x_4 + \ldots + x_n^2).$$

(5)

First, note that if $n = 2$, we have this proposition.

**Proposition 7.** The cubic form $F = x_0(x_1 x_3 + x_2^2)$ has rank $\leq 5$.

**Proof.** Consider the coordinate system given by the following linear transformation.

$$
\begin{align*}
\begin{cases}
x_0 = y_1 \\
x_1 = \frac{1}{3} y_1 + y_3 \\
x_2 = y_2
\end{cases}
\end{align*}
$$

(6)

By this, we have $F = \frac{1}{3} y_1^3 + y_2^2 y_3 + y_1 y_2^2$. Let $K_1 = \int \partial_2 F d y_2$ be the primitive of $\partial_2 F$ given by $K_1 = \int [y_1^3 + (y_1 + y_2)^3 + (y_1 - y_2)^3] = \frac{1}{3} y_1^3 + y_1 y_2^2$. Thus $F = K_1 + y_1 y_3$, where $K_2 = y_1^2 y_3 = \frac{1}{6} [(y_1 + y_3)^3 - (y_1 - y_3)^3 - 2y_3^3]$. Then $rk(F) \leq 5$, which proves the statement.

It is straightforward to generalize this fact as follows.
Proposition 8. The cubic form $F = x_0(x_0x_1 + x_2x_3 + x_4^2 + \ldots + x_n^2)$ has rank $\leq 2n + 1$.

Proof. We prove it by induction on $n$. The proposition holds for $n = 2$ by Proposition 7. Let us suppose the proposition true for all $i \leq n - 1$ and prove the case $i = n$. Introduce the coordinate system given by the following linear transformation.

$$
\begin{align*}
x_0 &= y_1 \\
x_1 &= y_3 \\
x_2 &= y_0 + y_2 \\
x_3 &= y_0 - y_2 \\
x_4 &= y_4 \\
&\vdots \\
x_n &= y_n
\end{align*}
$$

(7)

Then, the cubic becomes $F = y_0^2y_1 - y_1y_2^2 + y_1^2y_3 + y_1y_4^2 + \ldots + y_1y_n^2$. Setting $G = \int \partial_0 F d y_0 = y_0^2y_1 + \frac{1}{3}y_1^2$, we take $F = G - \frac{1}{3}y_1^3 - y_1y_2^2 + y_1^2y_3 + y_1y_4^2 + \ldots + y_1y_n^2$. We have that $rk(G) = 2$. Let $H = -\frac{1}{3}y_1^3 - y_1y_2^2 + y_1^2y_3 + y_1y_4^2 + \ldots + y_1y_n^2$. Since $H$ is a cubic form in $\mathbb{P}^{n-1}$ decomposed into a smooth quadric $Q$ and a tangent space $L$ to a point of $Q$ (and hence it is of type $C$), by inductive assumption $rk(H) \leq 2(n - 1) + 1$. Thus $rk(F) \leq rk(G) + rk(H) \leq 2 + 2(n - 1) + 1 = 2n + 1$. Repeating the argument, one obtains a decomposition for $F$. 

\[\square\]

Remark 3. By Theorem 1.3 in [4], the Waring rank of the cubic forms of type $C$ is $\geq 2n$.

Remark 4. The ranks of the reducible cubic forms are quite different from the generic rank of cubic forms given by the Alexander-Hirschowitz Theorem. For sufficiently large values of $n$, the ranks of reducible cubics are smaller than the rank of the generic cubic.

Proof of Theorem 1

Proof. We prove it by induction on $n$. If $n = 1$, it is well known that cubic forms (actually, forms of any degree) in 2 variables have rank at most their degree; in this case the set of ranks is exactly $W_3(1)$. Suppose that the thesis holds for $i \leq n - 1$ and we want to show it for $i = n$. Consider $W_3(n) \setminus W_3(n - 1)$; applying Theorem 2 there exist forms of ranks $2n$ and of rank at most $2n + 1$. By induction, $W_3(n - 1) \subseteq \{1, \ldots, 2n - 1\}$, and so $W_3(n) \subseteq \{1, \ldots, 2n + 1\}$.

\[\square\]

Conjecture. The Waring rank of the reducible cubic forms of type $C$ in $n + 1$ variables is $2n + 1$. 

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Remark 5. The Conjecture states that \( F = y_0^2 y_1 - y_1 y_2^2 + y_1^2 y_3 + y_1 y_2^2 + \ldots + y_1 y_n^2 \) has rank \( \geq 2n + 1 \). The ideal \( F^1 \) is generated by \( \partial_i \partial_3 \) (for \( i \neq 1 \)), \( \partial_i \partial_3 - \partial_i^2 \) (for \( i \neq 1, 2, 3 \)), \( \partial_i \partial_3 + \partial_2^2 \), \( \partial_i \partial_j \) (for \( i, j \neq 1, 3 \)), \( \partial_1^3 \), \( \partial_3^3 \) (for \( i \neq 1, 3 \)), \( \partial_1^2 \partial_i \) (for \( i \neq 1, 3 \)).

We remark that we can estimate the degree of a zero-dimensional scheme using the Hilbert Function. Let \( H \) of the difference function. We do not know how to show this conclusion for any \( n \) on \( \mathbb{C} \) of these decomposition points. The Conjecture for \( n=2 \). Let us assume that \( X \) has no points on \( \{ \partial_3 = 0 \} \). In this case, \( \partial_3 \) is not a zero-divisor in \( T/I \), which is crucial here. Then

\[
\deg X = \sum_{i \geq 0} \Delta HF(T/I, i) = \sum_{i \geq 0} HF(T/(I + (\partial_3)), i) \geq \sum_{i \geq 0} HF(T/(F^1 + (\partial_3)), i) = 2n + 1,
\]

where the Hilbert Function \( HF \) of \( F^1 + (\partial_3) \) is the sequence \( (1, n, n, 0, -\cdots) \) and \( \Delta HF \) is the first difference function of the \( HF \).

We do not know how to show this conclusion for any \( n \), under the assumption that \( X \) has points on \( \{ \partial_3 = 0 \} \). We ask for a technique of counting decomposition points for the cubic form of type \( C \) and, more generally, for arbitrary fixed forms, exploring the geometry and combinatorics of these decomposition points.

As an evidence of the Conjecture, we prove it for \( n = 2 \).

The Conjecture for \( n=2 \). Let us denote \( T = \mathbb{C}[\partial_1, \partial_2, \partial_3] \). In this case, we have \( F = y_1(y_1 y_3 + y_2^2) \). The principal system of \( F \) is the ideal \( F^1 = \langle \partial_1 \partial_3 - \partial_2^2, \partial_2 \partial_3, \partial_3^2, \partial_1^2 \partial_2, \partial_2^3 \rangle \). Let \( X \) be a set of decomposition points of \( F \) and let us set \( I = I(X) \).

If \( X \) has no points on \( \{ \partial_3 = 0 \} \) then \( \partial_3 \) is not a zero-divisor of \( T/I \). Then

\[
\deg X = \sum_{i \geq 0} \Delta HF(T/I, i) = \sum_{i \geq 0} HF(T/(I + (\partial_3)), i) \geq \sum_{i \geq 0} HF(T/F^1 + (\partial_3), i) = 5.
\]

Indeed, \( I + (\partial_3) \subset F^1 + (\partial_3) = (\partial_3, \partial_1 \partial_2, \partial_2^2, \partial_1^2, \partial_2^3) \) and the Hilbert Function of \( T/(F^1 + (\partial_3)) \) is the sequence \((1, 2, 2, 0, -\cdots)\), as in the Remark 5 above.

Let us assume that \( X \) has some point on \( \{ \partial_3 = 0 \} \). If \( \dim I_2 \leq 1 \) then the Hilbert Function of \( T/I \) is the sequence \((1, 3, m \geq 5, \ldots)\) and hence again \( \deg X \geq 5 \). So let us assume \( \dim I_2 \geq 2 \). Note that \( I_2 \subset F_2 = (\partial_1 \partial_3 - \partial_2^2, \partial_2 \partial_3, \partial_3^2) \). There exists a two-dimensional subspace of conics \( L \subset I_2 \subset F_2 \).

Either this space \( L \) is the pencil \( a\partial_2^2 + b\partial_2 \partial_3 \), and the base locus of this pencil is \( \{ \partial_3 = 0 \} \), or \( I_2 \) contains some irreducible conic of equation \( \partial_1 \partial_3 - \partial_2^2 + a\partial_3^2 + b\partial_2 \partial_3 \), whose only common intersection with \( \{ \partial_3 = 0 \} \) is the point \((1 : 0 : 0)\). The first case is not possible, since otherwise \( X \subset \{ \partial_3 = 0 \} \), namely \( \partial_3 F = 0 \), which is false. Hence we have \( X \cap \{ \partial_3 = 0 \} = \{(1 : 0 : 0)\} \). This implies that \( X \cap \{ \partial_3 = 0 \} \subset X \cap \{ \partial_2 = 0 \} \). Then \( \partial_3 \) does not vanish at any point of \( X \cap \{ \partial_2 \neq 0 \} = X' \).
Note that \( \deg X' \leq \deg X - 1 \) because the point \((1:0:0)\) does not belong to \(X'\). Setting \(J = (I : \partial_2)\) the ideal of \(X'\), we have that \(\partial_3\) is not a zero-divisor of \(T/J\), so we can compute
\[
\deg X' = \sum_{i \geq 0} HF(T/J + \langle \partial_3 \rangle, i) \geq \sum_{i \geq 0} HF(T/((F^1 : \partial_2) + \langle \partial_3 \rangle), i) \geq 4,
\]
since \((F^1 : \partial_2) + \langle \partial_3 \rangle = \langle \partial_3, \partial_1^2, \partial_2^2 \rangle\). Finally \(\deg X \geq \deg X' + 1 \geq 5\), which says that the rank of \(F\) is at least 5 using the Apolarity Lemma \(\text{[1]}\). Hence the Conjecture holds for \(n = 2\).

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