The parametric resonance features for theory of energy transfer in dusty plasma

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Abstract. One of the mechanisms of energy transfer between degrees of freedom of dusty plasma system can be described by equations similar to Mathieu equation with account of stochastic forces. Such equation is studied by analytical approach. The solutions for higher order of accuracy are obtained. The method for numerical solution and resonance zone detection is proposed. The solution for the extended Mathieu equation is obtained for wide range of parameter values. The results of numerical solution are compared with analytical solutions of different order and known analytical results for Mathieu equation.

1. Introduction
The mechanisms of energy transfer between degrees of freedom of a dusty plasma system are of great interest [1,2]. One of such mechanisms [2] is based on parametric resonance. The charged dust particles oscillate in the sheath of the gas discharge. The dust particle charge is determined by electron and ion fluxes on the surface of dust grain. The concentration of electrons and ions depend on the distance from the electrode, so periodical vertical motion of dust particles in the sheath leads to synchronous oscillations of the dust particles charge [3]. The mechanism of energy transfer based on such phenomena and parametric resonance is studied in [2, 4]. The equation of motion of dust particles in the gas discharge sheath with account of the friction, charge fluctuations and assumptions due to the conditions of a standard laboratory experiment has the form close to the extended Mathieu equation [2, 4]:

\[ \ddot{x} + 2\lambda \dot{x} + \omega_0^2 (1 + h \cos \omega_p t) = \eta(t), \] (1)

where \( \lambda \) is a friction coefficient, \( h \) is an amplitude of modulation, \( \omega_0 \) is an eigen frequency of system and \( \omega_p \) is a frequency of parameter, \( \eta(t) \) is a stochastic force with zero mean value. The parameters are close to the experimental ones [5–7]. Mathieu equation [8] is used for dust particles motion description in various articles [9,10]. In these articles the equation is derived by assuming, that the fluctuating dust particle charge varies in time as \( Q(t) = Q_0 (1 + h_q \cos \nu t)^{1/2} \), where \( h_q \) is a small parameter and \( \nu \) is a frequency of charge fluctuations. In this article, extended Mathieu equation is used to describe energy transfer between degrees of freedom of dusty plasma system. The equation is derived from fundamentally different assumptions. The amplitude and
frequency of modulation in this case have values significantly different from the ones considered in [9, 10].

Classical Mathieu equation is studied for $h \ll 1$, $\lambda = 0$ and $|\omega_p - 2\omega_0/n| \ll \omega_0$ on the level of the first order of accuracy [11]. This range of parameters is not enough for dusty plasma due to significant friction force, $h$ may be close and even above 1. In this article, extended Mathieu equation is studied for wider range of parameters and bigger orders of accuracy by analytical and numerical approaches.

In the second section, the mechanism of energy transfer between vertical and horizontal oscillations is described and the extended Mathieu equation is derived. In the third section, the analytical solutions with different orders of accuracy are described. In the fourth section, the method for numerical solution is proposed. The numerical solution results are shown in the fifth part. Conclusions are performed in the last section.

2. Energy transfer between vertical and horizontal oscillations

The motion of dust particle system is described by model presented in [4]. Since $x$ and $y$ axes are symmetrical in such system, so only motion along $x$ and $z$ axis is considered:

$$\begin{cases} m\ddot{x} = F_{\text{inter}} + F_{\text{trap}} + F_{\text{fr}}, \\ m\ddot{z} = F_{\text{inter}} + F_{\text{fr}} + F_{\text{grav}} + F_{\text{el}}, \end{cases}$$

where $F_{\text{inter}}$ is the force of interaction between dust particles, $F_{\text{fr}}$ is the friction force, $F_{\text{el}}$ is electrostatic force, $F_{\text{grav}}$ is the gravity force and $F_{\text{trap}}$ is the force of potential trap [2, 12].

The amplitude of oscillations is small comparing to inter-particle distance, so these forces can be expanded in Taylor series:

$$\begin{cases} \ddot{x} = -a_1 x + a_2 x^2 + a_3 x^3 + a_4 x z^2 + \ldots - \gamma \dot{x}, \\ \ddot{z} = -b_1 z + b_2 z^2 + b_3 z^3 + b_4 z x^2 + \ldots - \gamma \dot{z} + g \delta q(t). \end{cases}$$

Consideration of only the most significant terms in equation (3) [4] leads to the equation:

$$\ddot{x}_i \approx -(a_1 x - a_4 x z^2) - \gamma \dot{x}_i.$$  

Substitution of one harmonic of vertical motion

$$z \approx A_z \cos (\omega_z t)$$

in equation (4) leads to extended Mathieu equation (1).

So one of the mechanisms of energy transfer between vertical and horizontal motion can be described by extended Mathieu equation.

3. Analytical solution

The analytical approach for solution of extended Mathieu equation solution is based on several iterations [11] which can give solutions of different orders of accuracy. The extended Mathieu equation

$$\ddot{x} + \omega_0^2 (1 + h \cos \omega_p t) = \eta(t)$$

is considered for the assumptions $h \ll 1$ and $\varepsilon \ll \omega_0$, where $\omega_p = 2\omega_0 + \varepsilon$ for the study of first resonance zone. In this section the method for the first order of accuracy can give the presentation of the method for obtaining of solutions of bigger orders of accuracy. For the solution of the first order of accuracy the assumption of solution looks like

$$x(t) = a(t) \cos \left(\omega_0 + \frac{\varepsilon}{2}\right) t + b(t) \sin \left(\omega_0 + \frac{\varepsilon}{2}\right) t.$$
The solution terms with frequencies

\[ 3 \left( \omega_0 + \frac{\varepsilon}{2} \right), 5 \left( \omega_0 + \frac{\varepsilon}{2} \right) \]  

are not considered for the first order of accuracy. These terms and the ones with bigger frequencies are taken into account for the solutions of bigger orders of accuracy.

The substitution of equation (7) in equation (6) gives the equations

\[
\left( 2\omega_0 \dot{b} - a \omega_0 \varepsilon + \frac{1}{2} \omega_0^2 ha \right) \cos \left( \omega_0 + \frac{\varepsilon}{2} \right) t - 
\left( 2\omega_0 \dot{a} + b \omega_0 \varepsilon + \frac{1}{2} \omega_0^2 hb \right) \sin \left( \omega_0 + \frac{\varepsilon}{2} \right) t = \eta(t).
\]

The assumption \( x(t) \sim e^{st} \) and thus \( a(t) \sim b(t) \sim e^{st} \) is valid in the resonance area. This assumption and averaging over an ensemble of distributions of \( \eta(t) \) turn equation (9) into

\[
\left( 2\omega_0 \dot{b} - a \omega_0 \varepsilon + \frac{1}{2} \omega_0^2 ha \right) \cos \left( \omega_0 + \frac{\varepsilon}{2} \right) t - 
\left( 2\omega_0 \dot{a} + b \omega_0 \varepsilon + \frac{1}{2} \omega_0^2 hb \right) \sin \left( \omega_0 + \frac{\varepsilon}{2} \right) t = 0.
\]

The substitution of equation (7) in equation (6). Next three sets of equations specify the second resonance boundaries approximation of second, third and fourth orders of accuracy respectively:

\[
- \omega_0 \varepsilon + \frac{\omega_0^2 h}{2} - \frac{\varepsilon^2}{4} + \frac{\omega_0^2 h^2}{32} = 0,
\]

\[
- \omega_0 \varepsilon + \frac{\omega_0^2 h}{2} - \frac{\varepsilon^2}{4} + \frac{\omega_0^2 h^2}{32} - \frac{9 \omega_0 \varepsilon h^2}{28} = 0,
\]

\[
- \omega_0 \varepsilon + \frac{\omega_0^2 h}{2} - \frac{\varepsilon^2}{4} + \frac{\omega_0^2 h^2}{32} + \frac{h^4}{3 \cdot 215} + \frac{9 \varepsilon^2 h^2}{213} = 0.
\]

The approximations of boundaries of the second resonance area can be obtained considering \( \omega_p = \omega_0 + \varepsilon \) in equation (6). Next three sets of equations specify the second resonance boundaries approximation of second, fourth and sixth orders of accuracy respectively:

\[
\begin{aligned}
-2 \omega_0 \varepsilon + \frac{\omega_0^2 h^2}{12} &= 0, \\
-2 \omega_0 \varepsilon - \frac{5 \omega_0^2 h^2}{12} &= 0;
\end{aligned}
\]

\[
\begin{aligned}
-2 \omega_0 \varepsilon + \frac{\omega_0^2 h^2}{12} - \frac{\varepsilon^2}{4} - \frac{2 \omega_0 \varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2!} &= 0, \\
-2 \omega_0 \varepsilon - \frac{5 \omega_0^2 h^2}{12} - \frac{\varepsilon^2}{4} - \frac{2 \omega_0 \varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2!} &= 0;
\end{aligned}
\]

\[
\begin{aligned}
-2 \omega_0 \varepsilon + \frac{\omega_0^2 h^2}{12} - \frac{\varepsilon^2}{4} - \frac{2 \omega_0 \varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2!} - \frac{\varepsilon^2 h^2}{9} - \frac{\omega_0 \varepsilon h^4}{9 \cdot 2!} + \frac{h^6}{2 \cdot 3! \cdot 5!} &= 0, \\
-2 \omega_0 \varepsilon - \frac{5 \omega_0^2 h^2}{12} - \frac{\varepsilon^2}{4} - \frac{2 \omega_0 \varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2!} - \frac{\varepsilon^2 h^2}{9} - \frac{\omega_0 \varepsilon h^4}{9 \cdot 2!} + \frac{h^6}{2 \cdot 3! \cdot 5!} &= 0.
\end{aligned}
\]
It is usually considered that friction in system causes amplitude to decay as $e^{-\lambda t}$. Such approach leads to serious differences with the numerical solution of the equation in the presence of friction ($\lambda \neq 0$). This is because this approach takes into account only terms of zero-order of accuracy with $\lambda$. So this approximation works only with assumption of with $\lambda \ll 1$ and for solution of the first order of accuracy.

The other way to describe resonance area in the presence of friction is solving equation (1) by an analogy with equation (6) without any additional assumptions. Such approach can be applied for any value of $\lambda$ and gives other results, which are closer to the data obtained numerically. Thus for first resonance area in approximation of first order of accuracy equation for $s$ is

$$4 \left( \omega_0^2 - \lambda^2 \right) s^2 + \left( 8 \omega_0 \lambda \left( \omega_0 + \frac{\varepsilon}{2} \right) - 2 \omega_0^2 \lambda h \right) s = \frac{\omega_0^4 h^2}{4} - \omega_0^2 \varepsilon^2 - 4 \lambda^2 \left( \omega_0 + \frac{\varepsilon}{2} \right)^2$$

instead of

$$s^2 = \frac{1}{4} \left( \left( \frac{h \omega_0}{2} \right)^2 - \varepsilon^2 \right) - \lambda^2. \tag{19}$$

This explains such phenomenon as the shift of the $\omega_0/\omega_p$ value wherein the resonance occurs with a minimum value of $h$. In equation (19) $h$ is minimum for $s = \varepsilon = 0$, while in equation (18) $h$ is minimum for $s = 0$ and $\varepsilon = -\frac{2 \omega_0 \lambda}{\omega_0 + \lambda^2}$.

### 4. Method of numerical solution

In case of numerical approach for the study of the resonance area equation (1) is solved by numerical integration for each fixed parameters set. The velocity Verlet algorithm is used for numerical integration. Initial values is $x(0) = 0$ and $\dot{x}(0) = 1$, time step is $\Delta t = 10^{-5}$.

Two methods for resonance detection are proposed. The first method is based on the amplitude multiplication by at least $N$ times during fixed period of time. The second method is based on the assumption that $K$ amplitude values in a row are growing almost at the same rate before fixed period of time $T$ has passed.

It is found that second approach is better. This method give the same results if $h < 0.6$ but first method takes a lot of extra time to give result if $h > 0.6$. Even after this additional time, results obtained by first method are less stable then ones obtained by second method. So the second algorithm is chosen for resonance detection.

The constants $K$, $T$ and the criteria of almost the same grow of amplitudes are to be determined before solving equation. Amplitude value is detected many times during numerical solving of equation (1). The growth of amplitude assumes to be exponential:

$$s_{1,2} = \frac{\ln (x(t_2)) - \ln (x(t_1))}{t_2 - t_1}. \tag{20}$$

The evaluation of each two amplitudes give the growth of amplitude:

$$s_{1,3} = \frac{\ln (x(t_3)) - \ln (x(t_1))}{t_3 - t_1}. \tag{21}$$

If the condition $|s_{1,3} - s_{1,2}|/s_{1,2} < l$ is valid than these three amplitude values is considered to grow almost at the same rate, where $l$ is some constant to be determine.

The constant $l$ evaluated as $0.02$. The reduction of $l$ to $0.01$ causes loss of many resonance cases while increase of it up to $0.1$ doesn’t lead to any differences.

$K$ is determined to be $10$. The reduction of $K$ to $6$ causes many cases when there are no resonance to be considered as resonance cases, while increase of it up to $100$ does not lead to any differences.

$T$ is determined to be $2000$ seconds cause the time of onset is often between $1500$ and $2000$ seconds and it can exceed $2000$ seconds only with insignificant number of cases.
5. Results of numerical solution

For system without friction boundaries of the resonance area does not depend on $\omega_0$ value they depend only on $\omega_0/\omega_p$ value. First resonance area is obtained by numerical solution and is compared with four approximations of it’s boundaries (figure 1).

Figure 1 shows that differences between approximations of third and fourth orders of accuracy are insignificant. However, they are much closer to numerically obtained data than ones of first and second orders of accuracy. So third approximation can be used if $h < 1$.

Second resonance area is obtained by numerical solution and is compared with three approximations of it is boundaries (figure 2).

![Figure 1](image1.png)

**Figure 1.** The first resonance region evaluated by numerical solution and analytical solution for four orders of accuracy.

![Figure 2](image2.png)

**Figure 2.** The second resonance region evaluated by numerical solution and analytical solution for three orders of accuracy.
The first resonance region is evaluated by numerical solution and analytical solution of first orders of accuracy with and without assumption that friction in system causes amplitude to decay as $e^{-\lambda t}$, $\omega_0 = 8$, $\lambda = 0.85$.

Second, fourth and sixth orders of accuracy are used here since $\varepsilon \sim h^2$ on the boundaries of second resonance region (approximation of second order of accuracy shows this). Differences between approximations of fourth and sixth orders of accuracy are insignificant, while they are closer to numerically obtained data than approximation of second order of accuracy. Unlike for the first resonance region, for second resonance region significant differences between numerically obtained data and analytical solutions of any order of accuracy starts from $h = 0.8$.

As it was previously mentioned approach used in [11] doesn’t show such phenomena as the shift of the $\omega_0/\omega_p$ value wherein the resonance occurs with a minimum value of $h$. Value of this shift depends on $\lambda$ and $\omega_0$ values. First and second resonance areas are obtained by numerical solution for various $\lambda$ and $\omega_0$ values.

Figure 3 shows that the assumption that friction in system causes amplitude to decay as $e^{-\lambda t}$ leads to significant differences with numerically obtained data on any values of $h$ even for $\lambda < 1$.

6. Conclusions
The disruption of mechanisms of energy transfer between vertical and horizontal oscillations of dust particles by extended Mathieu equation is shown. The analytical solution of extended Mathieu equation is proposed for different orders of accuracy. The program for numerical solution of considered equation is created. Two methods for resonance detection are formulated. The boundaries for the first and second resonance regions of system without friction are shown on the graph and are compared with analytical solution for several orders of accuracy. The boundaries for resonance regions are extended to wide range of parameter of modulation and difference of frequencies. Phenomenon of the shift of the $\omega_0/\omega_p$ value wherein the resonance occurs with a minimum value of $h$ is shown and the way to describe it analytically is proposed.

The results of extended Mathieu equation research allow to describe energy transfer in dusty plasma more accurate.

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