The fundamental invariant of the Hecke algebra $H_n(q)$ characterizes the representations of $H_n(q)$, $S_n$, $SU_q(N)$ and $SU(N)$

J. Katriel*, B. Abdesselam and A. Chakrabarti

Centre de Physique Théorique

Ecole Polytechnique

91128 Palaiseau Cedex, France

*Permanent address: Department of Chemistry, Technion, 32000 Haifa, Israel.
Abstract

The irreducible representations (irreps) of the Hecke algebra $H_n(q)$ are shown to be completely characterized by the fundamental invariant of this algebra, $C_n$. This fundamental invariant is related to the quadratic Casimir operator, $C_2$, of $SU_q(N)$, and reduces to the transposition class-sum, $[(2)]_n$, of $S_n$ when $q \to 1$. The projection operators constructed in terms of $C_n$ for the various irreps of $H_n(q)$ are well-behaved in the limit $q \to 1$, even when approaching degenerate eigenvalues of $[(2)]_n$. In the latter case, for which the irreps of $S_n$ are not fully characterized by the corresponding eigenvalue of the transposition class-sum, the limiting form of the projection operator constructed in terms of $C_n$ gives rise to factors that depend on higher class-sums of $S_n$, which effect the desired characterization. Expanding this limiting form of the projection operator into a linear combination of class-sums of $S_n$, the coefficients constitute the corresponding row in the character table of $S_n$. The properties of the fundamental invariant are used to formulate a simple and efficient recursive procedure for the evaluation of the traces of the Hecke algebra. The closely related quadratic Casimir operator of $SU_q(N)$ plays a similar role, providing a complete characterization of the irreps of $SU_q(N)$ and - by constructing appropriate projection operators and then taking the $q \to 1$ limit - those of $SU(N)$ as well, even when the quadratic Casimir operator of the latter does not suffice to specify its irreps.
1 Introduction

The ordinary representation theory of the symmetric group, whose centennial is soon to be celebrated, is not the most likely arena for the emergence of hitherto unnoticed simplifications. The interest in the Hecke algebra $H_n(q)$, that reduces to the group algebra of the symmetric group in the limit $q \to 1$, is much more recent. It has been pursued as a vehicle for the construction of representations of the braid group and also in the context of the recent study of the quantum unitary groups, with respect to which $H_n(q)$ has been shown to play the role of the $q$-analogue of the Weyl group. For recent expositions of the mathematical structure and of the physical relevance of the Hecke algebra, jointly providing access to many earlier references, see refs. [1-11].

The central theme of the present article is that the Hecke algebra point of view provides a remarkable conceptual simplification in the formulation of the representation theory of the symmetric group. More specifically, the fundamental invariant of $H_n(q)$, that in the limit $q \to 1$ reduces to the transposition class-sum of $S_n$, is shown to be sufficient to effect all the applications that conventionally require the use of the full character table of $S_n$. In fact, the character table of $H_n(q)$, and, a fortiori, that of $S_n$, are shown to be extractable by appropriate manipulations of that fundamental invariant.

A completely analogous situation holds true for the fundamental invariant of $SU_q(N)$, its quadratic Casimir operator. It turns out that the irreducible representations (irreps) of $SU_q(N)$, and, a fortiori, of $SU(N)$, are fully characterized by the quadratic Casimir of the former. In fact, the eigenvalues of the quadratic Casimir operator of $SU_q(N)$ can be expressed in terms of those of the fundamental invariant of $H_n(q)$ in a manner that is the $q$-analogue of the relation noted by Gross [12] between the eigenvalues of the quadratic Casimir operator of $SU(N)$ and the transposition class-sum of $S_n$.

The structure of the present article is as follows: In section 2 we review some pertinent features of the representation theory of the symmetric group. In section 3 we show that the fundamental invariant of the Hecke algebra fully characterizes its irreps. In section 4 we construct projection operators for the irreps of $H_n(q)$ in terms of its fundamental invariant,
and study their limiting forms for \( q \to 1 \), that yield the familiar group-theoretical projection operators for the irreps of \( S_n \). In section 5 we discuss the extraction of the traces of the Hecke algebra from its fundamental invariant, and present an efficient procedure for the evaluation of these traces. The relation between the quadratic Casimir operator of \( SU_q(N) \) and the fundamental invariant of \( H_n(q) \) is presented in section 6, where, for generic \( q \), the former is shown to characterize all the corresponding irreps. Some concluding remarks are made in section 7.

2 Class-sums and classification of irreps of \( S_n \)

Both the irreps and the conjugacy classes of the symmetric group \( S_n \) can be labeled by means of partitions of \( n \). We shall denote irreps by bracketed sets of bold face integers specifying the row lengths of the corresponding Young diagram, and conjugacy classes by sets of individually parenthesized integers that specify the cycle-structure characterising the class. A class-sum will be denoted by bracketing the class-symbol, appending a suffix that specifies \( n \), and usually suppressing cycles of unit length.

As is well-known, the irreps are fully characterized by means of the set of eigenvalues of the class-sums. In fact, it has been shown by Kramer [13] that the single-cycle class-sums, \( \{ [(p)]_n \mid p = 2, 3, \ldots, n \} \), generate the center of the corresponding group-algebra and are therefore sufficient to characterize the irreps. Moreover, it has been noted that the subset of \( k \) class-sums \( \{ [(2)]_n, [(3)]_n, \ldots, [(k + 1)]_n \} \) is sufficient to characterize the irreps of all symmetric groups \( S_n \) with \( n \) not larger than some \( n_{\text{max}}(k) \), that is considerably larger than \( k \). Thus, \( n_{\text{max}}(1) = 5 \), the first instance in which the transposition class-sum has the same eigenvalue for two different irreps being the irreps \([4, 1, 1]\) and \([3, 3]\) of \( S_6 \). Similarly, \( n_{\text{max}}(2) = 14 \), \( n_{\text{max}}(3) = 23 \) and \( n_{\text{max}}(4) = 41 \). The rather conservative bound \( n_{\text{max}}(k) < 2^{2k+1} \) has been found to hold [14] for arbitrary \( k \).

The eigenvalues of the class-sums of the symmetric group, \( i.e., \) its central characters,
have been shown in ref. [15] to be expressible as polynomials in the symmetric power-sums

$$\sigma_k^\Gamma = \sum_{(i,j) \in \Gamma} (j - i)^k.$$

Here, \(\Gamma\) stands for a Young diagram and \((i, j)\) are the row and column indices of the boxes in this diagram, counted matrixlike. The difference \(j - i\) has been referred to as the content of the box \((i, j)\). Algorithms for the construction of expressions for the central characters in terms of these symmetric power-sums were presented in refs. [16] and [17] for single-cycle and arbitrary class-sums, respectively.

For the first four single-cycle class-sums the eigenvalues are given by

$$\begin{align*}
\lambda^\Gamma_{(2)_n} &= \sigma_1^\Gamma \\
\lambda^\Gamma_{(3)_n} &= \sigma_2^\Gamma - \frac{n(n-1)}{2} \\
\lambda^\Gamma_{(4)_n} &= \sigma_3^\Gamma - (2n-3)\sigma_1^\Gamma \\
\lambda^\Gamma_{(5)_n} &= \sigma_4^\Gamma - (3n-10)\sigma_2^\Gamma - 2(\sigma_1^\Gamma)^2 + \frac{n(n-1)(5n-19)}{6}
\end{align*}$$

The expressions for \(\lambda^\Gamma_{(2)_n}\) and \(\lambda^\Gamma_{(3)_n}\) were originally given by Jucys [18] and by Suzuki [19].

3 The Hecke algebra \(H_n(q)\)

The Hecke algebra \(H_n(q)\) is defined in terms of the generators \(g_1, g_2, \cdots, g_{n-1}\) and the relations

$$\begin{align*}
g_i^2 &= (q-1)g_i + q & i = 1, 2, \cdots, n-1 \\
g_ig_{i+1}g_i &= g_{i+1}g_ig_{i+1} & i = 1, 2, \cdots, n-2 \\
g_ig_j &= g_jg_i & \text{if } |i-j| \geq 2
\end{align*}$$

For \(q = 1\) the relations specified above reduce to the generating relations of the symmetric group, \(S_n\). In particular, \(g_i\) reduces to the transposition \((i, i+1)\).

When \(q\) is neither zero nor a \(k\)th root of unity, \(k = 2, 3, \cdots, n\), the irreps of \(H_n(q)\) are labelled by Young diagrams with \(n\) boxes [2, 3].
The fundamental invariant is

\[ C_n = g_1 + g_2 + \cdots + g_{n-1} + \frac{1}{q} (g_1 g_2 g_1 + g_2 g_3 g_2 + \cdots + g_{n-2} g_{n-1} g_{n-2}) \]

\[ + \frac{1}{q^2} (g_1 g_2 g_3 g_2 g_1 + g_2 g_3 g_4 g_3 g_2 + \cdots + g_{n-3} g_{n-2} g_{n-1} g_{n-2} g_{n-3}) \]

\[ + \cdots \]

\[ + \frac{1}{q^{n-2}} g_1 g_2 \cdots g_{n-2} g_{n-1} g_{n-2} \cdots g_2 g_1 \]  \hspace{1cm} (3)

\( C_n \) is, up to an overall factor of \( q \), the sum of the Murphy operators introduced by Dipper and James [1]. Using their corollary 2.3 it follows that \( C_n \) belongs to the center of \( H_n(q) \), i.e., commutes with all the generators \( g_i \), \( i = 1, 2, \cdots, n-1 \). As an illustration consider \( C_3 = g_1 + g_2 + \frac{1}{q} g_1 g_2 g_1 \). In this case

\[ [g_1, C_3] = g_1 g_2 - g_2 g_1 + \frac{1}{q} ((q-1)g_1 + q) g_2 g_1 - \frac{1}{q} g_1 g_2 ((q-1)g_1 + q) = 0. \]

Using the relation between \( C_n \) and the Murphy operators it follows from corollary 3.13 in ref. [1] that the eigenvalues of the fundamental invariant are given by the expression

\[ \Lambda^n_G(q) = q \sum_{(i,j) \in \Gamma} [j-i]_q \]  \hspace{1cm} (4)

where \([k]_q = \frac{q^k - 1}{q - 1}\). We shall refer to \( q[j-i]_q \) as the q-content of the box \((i,j)\).

For \( q \to 1 \) the fundamental invariant reduces to the symmetric-group transposition class-sum, \( [(2)]_n \), and eq. [4] reduces to the expression for the corresponding eigenvalues,

\[ \lambda^n_{[(2)]} = \sigma^n_1 \] (cf. eq. [4]).

To illustrate the expression for the eigenvalues consider \( C_2 = g_1 \). The corresponding eigenvalues are the roots of the Cayley equation \( \Lambda^2 = (q-1)\Lambda + q \) that is obtained from \( g_1^2 = (q-1)g_1 + q \) by replacing \( g_1 \) by its eigenvalue \( \Lambda \). The roots of this quadratic equation, \( \Lambda = q, -1 \), coincide with \( \Lambda^n_{C_2}^{-2}(q) \) and \( \Lambda^n_{C_2}^{-1,1} \), respectively, evaluated using eq. [4]. A less trivial illustration is provided by \( H_3(q) \). Here,

\[ C_3 = g_1 + g_2 + \frac{1}{q} g_1 g_2 g_1 , \]

so that
\[ C_3^2 = 3q + 2(q-1)C_3 + (q + 1 + \frac{1}{q})D_3 \]

where \( D_3 = g_1g_2 + g_2g_1 + \frac{q-1}{q} g_1g_2g_1 \),

and

\[ C_3^3 = (q^3 - 1) + q(q^2 + q + 5 + \frac{1}{q} + \frac{1}{q^2})C_3 + 2(q-1)C_3^2 + (q^3 + 2q^2 + q - 1 - \frac{2}{q} - \frac{1}{q^2})D_3 \]

Eliminating \( D_3 \) and replacing \( C_3 \) by its eigenvalue we obtain the Cayley equation

\[ \Lambda^3 - (q-1)(q+4 + \frac{1}{q})\Lambda^2 + (q^3 - q^2 - 9q - 1 + \frac{1}{q})\Lambda + (q-1)(2q^2 + 5q + 2) = 0 \]

whose roots, \( \Lambda = q^2 + 2q, q - 1, -2 - \frac{1}{q} \), agree with the values obtained for \( \Lambda^{[3]}_{C_3}, \Lambda^{[2,1]}_{C_3}, \) and \( \Lambda^{[1,1,1]}_{C_3} \), respectively, using eq. 4.

The most important property of the eigenvalues of \( C_n \) from the point of view of the applications to be presented below is that, being polynomials in \( q \) and \( \frac{1}{q} \), for generic \( q \) they obtain distinct values for the various irreps of \( H_n(q) \). As an illustration consider the two irreps \([4, 1, 1]\) and \([3, 3]\) of \( H_6(q) \), whose \( S_6 \) counterparts both correspond to the eigenvalue \( 3 \) of \([2]\)_6. Using eq. 4 we obtain \( \Lambda^{[4,1,1]}_{C_6} = q^3 + 2q^2 + 3q - 2 - \frac{1}{q} \) and \( \Lambda^{[3,3]}_{C_6} = q^2 + 3q - 1 \). These two eigenvalues are equal to one another only when \( q \) is either a square- or a cubic root of unity. Hence, while they are equal when \( q = 1 \), there is some neighbourhood of \( q = 1 \) within which they are distinct.

To see this property in the general setting we note that the principal diagonal of a Young diagram consists of boxes with equal row and column indices, \( i.e., \) of the boxes whose contents are zero. The boxes with a common content \( k \) lie along a diagonal that is parallel to the principal diagonal at a distance \( k \) above or below it, depending on the sign of \( k \). Denoting the number of boxes with content \( k \) in the Young diagram \( \Gamma \) by \( \beta_k^\Gamma \), the eigenvalue of \( C_n \) can be written in the form

\[ \Lambda_{C_n}(q) = \sum_{k>0} (q + q^2 + \cdots + q^k) \beta_k^\Gamma - \sum_{k<0} (1 + \frac{1}{q} + \cdots + \frac{1}{q^{k-1}}) \beta_k^\Gamma \]
\[
\sum_{k>0} q^k \sum_{\ell \geq k} \beta_\ell^\Gamma - \sum_{k<0} \frac{1}{q^{-k-1}} \sum_{\ell \leq k} \beta_\ell^\Gamma \\
= \sum_{k>0} q^k \pi_k^\Gamma - \sum_{k<0} \frac{1}{q^{-k-1}} \nu_k^\Gamma
\]  
(5)

i.e., \( \pi_k^\Gamma \), the coefficient of \( q^k, k > 0 \), in \( \Lambda_G(q) \), is equal to the number of boxes with contents larger than or equal to \( k \), and \( \nu_k^\Gamma \), the coefficient of \( \frac{1}{q^{-k-1}}, k < 0 \), is equal to the number of boxes with contents smaller than or equal to \( k \).

Denoting the largest and smallest contents of a given Young diagram by \( k_M \) and \( k_m \), respectively, we note that the former must correspond to a single box, located at the right end of the top row, and the latter to a single box located at the bottom of the first column. Thus, \( \beta_{kM}^\Gamma = \beta_{km}^\Gamma = 1 \). For \( 0 < k < k_M \) we have from eq. (5)

\[
\beta_k^\Gamma = \pi_k^\Gamma - \sum_{\ell > k} \beta_\ell^\Gamma = \pi_k^\Gamma - \pi_{k+1}^\Gamma
\]

and for \( k_m < k < 0 \)

\[
\beta_k^\Gamma = \nu_k^\Gamma - \sum_{\ell < k} \beta_\ell^\Gamma = \nu_k^\Gamma - \nu_{k-1}^\Gamma
\]

Finally, the number of boxes along the principal diagonal (boxes whose contents are zero) is

\[
n - \left( \sum_{k>0} \beta_k^\Gamma + \sum_{k<0} \beta_k^\Gamma \right) = n - (\pi_1^\Gamma + \nu_{-1}^\Gamma).\]

In conclusion, given an eigenvalue of \( C_n \) as a polynomial in \( q \) and \( \frac{1}{q} \), the structure of the corresponding Young diagram, expressed in terms of the lengths of its diagonals, can readily be recovered. This conclusion presupposes that different integral powers of \( q \) with positive powers up to and including \( k_M \) and negative powers up to an including \( k_m \) are distinct. For a Young diagram with \( n \) boxes \( k_M \leq n-1 \) and \( k_m \geq -(n-1) \). Therefore, to assure that different relevant powers of \( q \) are distinct, it is sufficient to require that \( q \) is not a root of unity of order less than or equal to \( n \).

It will be convenient to define the operator \( \tilde{C}_n \equiv \frac{q-1}{q} C_n \) whose eigenvalues are

\[
\Lambda_{G,n}^\Gamma(q) = \sum_{(i,j) \in \Gamma} (q^j - q^i - 1).
\]

Substituting \( q = \exp(\delta) \) we obtain

\[
\Lambda_{G,n}^\Gamma(\exp(\delta)) = \sum_{k=1}^{\infty} \delta^k \sigma_k^\Gamma
\]  
(6)
Hence, using eq. 1,
\[ \Lambda_{C_n}^{\Gamma}(\exp(\delta)) = \delta \lambda_{(2)^n}^{\Gamma} + \frac{\delta^2}{2!}\left(\lambda_{(3)^n}^{\Gamma} + \frac{n(n-1)}{2}\right) + \frac{\delta^3}{3!}\left(\lambda_{(4)^n}^{\Gamma} + (2n-3)\lambda_{(2)^n}^{\Gamma}\right) + \cdots \]  

(7)

In fact, eq. 3 implies that, for generic \( q \), the eigenvalue \( \Lambda_{C_n}^{\Gamma}(q) \) determines all the symmetric power sums \( \{\sigma_k^{\Gamma} | k = 1, 2, \ldots, n-1\} \) and, consequently, the central characters of all the class-sums of the corresponding symmetric group.

Eq. 7 implies the correspondence
\[ \tilde{C}_n \rightarrow \delta [(2)^n] + \frac{\delta^2}{2!}\left(\sigma_{(3)^n} + \frac{n(n-1)}{2}\right) + \frac{\delta^3}{3!}\left(\sigma_{(4)^n} + (2n-3)\sigma_{(2)^n}\right) + \cdots \]  

(8)

that should be understood to imply that the operator on the right has the same eigenvalue for a Young diagram corresponding to an irrep of \( S_n \) that \( \tilde{C}_n \) has for the same Young diagram taken as an irrep of \( H_n(q) \). This correspondence will be used in the next section.

4 Projection operators

The group-theoretical projection operators for the subspace carrying any desired irrep can easily be written down in terms of the characters corresponding to that representation. This procedure appears to suggest that full knowledge of the character table is required, or, more specifically, the projection operator corresponding to any particular irrep requires the use of the complete row of characters belonging to that irrep.

In fact, when a subset of class-sums is sufficient to characterize the irreps, one only needs the eigenvalues corresponding to these class-sums. As a very simple illustration consider \( S_3 \), that has three irreps, corresponding to the Young diagrams \([3] \), \([2,1]\), and \([1,1,1]\). The eigenvalues of the transposition class-sum corresponding to these irreps are easily evaluated using the first of eqs. 1, obtaining the respective values 3, 0, and -3. The projection operator for any of the three irreps can now be constructed in terms of the transposition class-sum. Thus,
\[ P_{[2,1]} = \frac{((2)_3 - 3)}{(0 - 3)} \cdot \frac{((2)_3 + 3)}{(0 + 3)} \]
where the left factor annihilates the irrep \([3]\) and the right factor annihilates \([1,1,1]\). Expanding the product and using the identity \([3][2][3] = 3 + [3][3]\) we obtain

\[ P_{[2,1]} = \frac{1}{3} (2 - [3][3] \).

By comparison with the group-theoretical form of the projection operator this expression provides the row in the character table corresponding to the irrep \([2,1]\), i.e., \(\chi((1)^3) = 2\); \(\chi((1)(2)) = 0\), and \(\chi((3)) = -1\).

Let us denote the set of irreps of \(S_n\) (and eventually of \(H_n(q)\)) by \(\mathcal{IR} \mathcal{R}_n\). For any irrep \(\Gamma_0\) that satisfies

\[ \lambda^{\Gamma_0}_{[(2)]_n} \neq \lambda^{\Gamma}_{[(2)]_n} \]  

for all \(\Gamma \in \mathcal{IR} \mathcal{R}_n \setminus \{\Gamma_0\}\) we write the projection operator in the form

\[ P_{\Gamma_0} = \prod_{\Gamma \in \mathcal{IR} \mathcal{R}_n \setminus \{\Gamma_0\}} \frac{[(2)]_n - \lambda^{\Gamma}_{[(2)]_n}}{\lambda^{\Gamma_0}_{[(2)]_n} - \lambda^{\Gamma}_{[(2)]_n}} \]

The procedure illustrated above needs a slight modification for irreps that do not satisfy eq. (9), i.e., are not fully characterized by the corresponding eigenvalue of the transposition class-sum. Thus, the two irreps \([4,1,1]\) and \([3,3]\) of \(S_6\) both correspond to the eigenvalue 3 for \([2]\). These two irreps can be distinguished by means of the class-sum \([3]\), for which their eigenvalues are 4 and -8, respectively. Furthermore, no other irrep of \(S_6\) corresponds to the same eigenvalue of \([2]\). Therefore, to construct a projection operator for any of the above two irreps one can first annihilate all the irreps but these two by means of an operator, \(P^*\), that depends only on the transposition class-sum. Explicitly,

\[ P^* = \prod_{\Gamma \in \mathcal{IR} \mathcal{R}_6} \frac{[(2)]_6 - \lambda^{\Gamma}_{[(2)]_6}}{\lambda^{\Gamma_0}_{[(2)]_6} - \lambda^{\Gamma}_{[(2)]_6}} \]

where \(\lambda^{\Gamma_0}_{[(2)]_6} = \lambda^{[4,1,1]}_{[(2)]_6} = \lambda^{[3,3]}_{[(2)]_6} = 3\) and \(\mathcal{IR} \mathcal{R}^*_6 = \mathcal{IR} \mathcal{R}_6 \setminus \{[4,1,1], [3,3]\}\). This should be followed by annihilation of the undesired one of the two remaining irreps, using \([3]\). Explicitly,

\[ P_{[4,1,1]} = \frac{([3]+8)}{12} \cdot P^* \]
and
\[ P_{[3,3]} = \frac{\left(\left\lfloor \frac{3}{2} \right\rfloor \right)_6 - 4}{-12} \cdot P^* \]

The construction of the corresponding projection operators within \( H_6(q) \) avoids the complication noted above, since all eigenvalues of the fundamental invariant, \( C_6 \), are distinct. Thus,
\[ P_{[4,1,1]}(q) = \prod_{\Gamma \in \text{IRR}_6 \setminus \{[4,1,1]\}} \frac{C_6 - \Lambda^\Gamma_{C_6}}{\Lambda_{C_6}^{[4,1,1]} - \Lambda^\Gamma_{C_6}} \]
and
\[ P_{[3,3]}(q) = \prod_{\Gamma \in \text{IRR}_6 \setminus \{[3,3]\}} \frac{C_6 - \Lambda^\Gamma_{C_6}}{\Lambda_{C_6}^{[3,3]} - \Lambda^\Gamma_{C_6}} \]

Taking the \( q \to 1 \) limit of \( P_{[4,1,1]}(q) \) we observe that all the factors except \( \frac{C_6 - \Lambda^{[3,3]}_{C_6}}{\Lambda_{C_6}^{[4,1,1]} - \Lambda^{[3,3]}_{C_6}} \) reduce to their \( S_6 \) counterparts. To obtain the limiting form of this last factor we note that
\[ \frac{C_6 - \Lambda^{[3,3]}_{C_6}}{\Lambda_{C_6}^{[4,1,1]} - \Lambda^{[3,3]}_{C_6}} = \frac{\tilde{C}_6 - \Lambda^{[3,3]}_{\tilde{C}_6}}{\Lambda_{\tilde{C}_6}^{[4,1,1]} - \Lambda^{[3,3]}_{\tilde{C}_6}} \]

Moreover,
\[ \Lambda_{\tilde{C}_6}^{[4,1,1]} \sim 3\delta + 19\frac{\delta^2}{2!} + \cdots \]
and
\[ \Lambda_{\tilde{C}_6}^{[3,3]} \sim 3\delta + 7\frac{\delta^2}{2!} + \cdots \]

Therefore, using eq. 8
\[ \tilde{C}_6 - \Lambda_{\tilde{C}_6}^{[3,3]} \sim \delta \left( \left\lfloor \frac{1}{2} \right\rfloor \right)_6 - 3 \right) + \frac{\delta^2}{2!} \left( \left\lfloor \frac{3}{2} \right\rfloor \right)_6 + 8 \right) + \cdots \]
and
\[ \Lambda_{\tilde{C}_6}^{[4,1,1]} - \Lambda_{\tilde{C}_6}^{[3,3]} \sim 12\frac{\delta^2}{2!} + \cdots \]

The factor \( \frac{\tilde{C}_6 - \Lambda^{[3,3]}_{\tilde{C}_6}}{\Lambda_{\tilde{C}_6}^{[4,1,1]} - \Lambda^{[3,3]}_{\tilde{C}_6}} \) is multiplied by the projection operator
\[ \prod_{\Gamma \in \text{IRR}_6^*} \frac{C_6 - \Lambda^\Gamma_{C_6}}{\Lambda_{C_6}^{[4,1,1]} - \Lambda^\Gamma_{C_6}} \]
that annihilates all irreps but \([4, 1, 1]\) and \([3, 3]\), that have a common eigenvalue of \([(2)]_6\). Therefore, within that factor one can replace \([(2)]_6\) by its relevant eigenvalue, 3, to obtain

\[
\tilde{C}_6 - \Lambda^{[3,3]}_{\hat{C}_6} \sim \frac{([(3)]_6 + 8) + O(\delta)}{12 + O(\delta)}
\]

or, finally,

\[
P_{[4,1,1]} \equiv \lim_{q \to 1} P_{[4,1,1]}(q) = \frac{([(3)]_6 + 8)}{12} \prod_{r \in \mathcal{R}_6} \frac{([(2)]_6 - \lambda^{r}_{[(2)]_6})}{\lambda^{r}_{[(2)]_6}}
\]

that coincides with the projection operator constructed directly within \(S_6\), utilizing both \([2]_6\) and \([3]_6\) to characterize the irreps of interest.

Clearly, if two or more irreps of \(S_n\) were to have common eigenvalues for \([(2)]_n, [(3)]_n, \ldots, [(k)]_n\), the relevant factor in the projection operator constructed within \(H_n(q)\) in terms of \(C_n\) would, in the limit \(q \to 1\), depend on \([(k+1)]_n\), the lowest single-cycle class-sum distinguishing between these irreps.

## 5 Traces of the Hecke algebra

The problem of evaluating the traces of the Hecke algebra has received some attention in recent years. We refer to King and Wybourne [7] for the definition and the discussion of the properties of these traces, as well as for a presentation of the pertinent earlier references. By way of illustration of some properties that we shall need let us demonstrate the observation made by King and Wybourne [7], according to which the traces of elements of the same connectivity class are equal. In addition to the defining relations of the Hecke algebra, eq. 2 we only use the property \(tr(AB) = tr(BA)\).

Thus, consider \(tr(g_i g_{i+1} g_i)\), that we evaluate in two different ways

\[
tr(g_i g_{i+1} g_i) = tr(g_i^2 g_{i+1}) = (q - 1)tr(g_i g_{i+1}) + q tr(g_i)
\]

\[
= tr(g_{i+1} g_i g_i) = tr(g_i^2 g_{i+1}) = (q - 1)tr(g_i g_{i+1}) + q tr(g_i)
\]

Hence, \(tr(g_i) = tr(g_{i+1})\).
Similarly,
\[
tr(g_{i+2}g_{i+1}g_{i+2}) = (q - 1)tr(g_{i+1}g_{i+2}) + q tr(g_{i+1})
\]
\[
= tr(g_{i+2}g_{i+1}g_{i+2}) = tr(g_{i+1}g_{i+2}) =
\]
\[
= tr(g_{i+1}g_{i+2}) = (q - 1)tr(g_{i+1}) + q tr(g_{i+1})
\]
from which it follows that \(tr(g_{i+1}) = tr(g_{i+1}g_{i+2})\). The generalization to Hecke algebra elements of any simply-connected class is now obvious.

As an example of a non-simply connected element consider
\[
tr(g_{i+k}g_{i+k+1}) = tr(g_{i+k}g_{i+k+1}) = (q - 1)tr(g_{i+k}) + q tr(g_{i+k+1})
\]
where \(k > i + 1\). This identity implies that \(tr(g_{i+k}) = tr(g_{i+k+1})\), and suggests how similar connections can be established for arbitrary traces of terms belonging to common connectivity classes.

It is of some interest to note that the fundamental invariant determines these traces as well. To illustrate this point consider the simplest non-trivial case, corresponding to \(H_3(q)\). In this case \(C_3 = g_1 + g_2 + \frac{1}{q}g_1g_2g_1\). Obviously,
\[
tr(C_3) = 3tr(g_1) + \frac{q - 1}{q}tr(g_1g_2) . \tag{10}
\]
Furthermore, \(C_3^2 = 3qI + 2(q - 1)C_3 + \left(\frac{1}{q} + 1 + q\right)(g_1g_2 + g_2g_1 + \frac{(q - 1)}{q}g_1g_2g_1)\) so that
\[
tr(C_3^2) = 3qtr(I) + (q^2 + 6q - 6 - \frac{1}{q})tr(g_1) + (q^2 + 3q - 2 + \frac{3}{q} + \frac{1}{q^2})tr(g_1g_2) , \tag{11}
\]
where \(I\) is the unit operator. Since both the eigenvalues of \(C_3\) and the dimensionalities of the corresponding irreps are known, equations [10] and [11] can be solved for \(tr(g_1)\) and \(tr(g_1g_2)\).

Thus,
\[
tr(g_1) = \frac{(q^2 + 3q - 2 + \frac{3}{q} + \frac{1}{q^2})tr(C_3) - 2(q - 1)(tr(C_3^2) - 3qtr(I))}{2(q^2 + 2q + 3 + \frac{2}{q} + \frac{1}{q^2})}
\]
and
\[ tr(g_1g_2) = \frac{-(q^2 + 6q - 6 + \frac{1}{q})tr(C_3) + 3(tr(C_3^2) - 3q tr(I))}{2(q^2 + 2q + 3 + \frac{2}{q} + \frac{1}{q^2})} \]

For the one dimensional irrep \([3]\) \(\Lambda_{C_3} = q^2 + 2q\) so \(tr(I) = 1\), \(tr(C_3) = q^2 + 2q\), and \(tr(C_3^2) = (q^2 + 2q)^2\). Therefore, \(tr(g_1) = q\) and \(tr(g_1g_2) = q^2\).

Similarly, for the two dimensional irrep \([2, 1]\) \(tr(I) = 2\), \(tr(C_3) = 2(q - 1)\), and \(tr(C_3^2) = 2(q - 1)^2\), so that \(tr(g_1) = q - 1\) and \(tr(g_1g_2) = -q\), and for the irrep \([1, 1, 1]\) \(tr(I) = 1\), \(tr(C_3) = -(2 + \frac{1}{q})\), and \(tr(C_3^2) = (2 + \frac{1}{q})^2\), hence \(tr(g_1) = -1\) and \(tr(g_1g_2) = 1\), all in agreement with King and Wybourne [7].

While this procedure can be followed to obtain the traces of higher Hecke algebras, a simpler procedure can be formulated in the following manner:

For a representation \(\Gamma_n\) of \(H_n(q)\) consider the basis states specified by means of all the sequences of Young diagrams of the form \(\Gamma_2 \Gamma_3 \cdots \Gamma_n\), where each \(\Gamma_{i+1}\) is obtained from the preceding \(\Gamma_i\) by the addition of one box. Since each such state is a common eigenstate of the sequence of mutually commuting fundamental invariants of \(H_2(q) \subset H_3(q) \subset \cdots \subset H_n(q)\), i.e., \(C_2, C_3, \cdots, C_n\), it is also an eigenstate of the Murphy operators \([1]\) \(L_2 = C_2,\)

\(L_3 = C_3 - C_2, \cdots, L_n = C_n - C_{n-1}\). Obviously, the eigenvalue of \(L_i\), that we shall denote \(\lambda_{L_i}\), is equal to the q-content of the box added to \(\Gamma_{i-1}\) to form \(\Gamma_i\). It is now a simple matter to construct the (diagonal) representation matrices of the Murphy operators, from which their traces are readily obtained.

A straightforward computation yields

\[ tr(C_n) = \sum_{i=2}^{n} \binom{n}{i} \left( \frac{q - 1}{q} \right)^{i-2} \tau_i \] \hfill (12)

where \(\tau_i = tr(g_1g_2 \cdots g_{i-1})\).

Using eq. (12) we obtain

\[ tr(L_i) = tr(C_i) - tr(C_{i-1}) = \sum_{j=2}^{i} \binom{i - 1}{j - 1} \left( \frac{q - 1}{q} \right)^{j-2} \tau_j \]

or, inverting,

\[ \tau_k = \left( \frac{q}{q - 1} \right)^{k-2} \sum_{i=0}^{k-2} (-1)^i \binom{k - 1}{i} tr(L_{k-i}) \] \hfill (13)
As an illustration we note that in $H_3(q)$ we have:

1. For the one-dimensional irrep $[3]$ possessing the unique sequence $[1] [2] [3]$: 
   \[ \lambda_{L_2} = q, \lambda_{L_3} = q + q^2. \] Hence, using eq. 13, $tr(g_1) = q$ and $tr(g_1g_2) = q^2$.

2. For the two-dimensional irrep $[2,1]$, in the basis specified by
   \[
   \begin{bmatrix}
   [1] & [2] & [2,1] : & \lambda_{L_2} = q, & \lambda_{L_3} = -1 \\
   [1] & [1,1] & [2,1] : & \lambda_{L_2} = -1, & \lambda_{L_3} = q
   \end{bmatrix}
   \]

   $L_2$ and $L_3$ are represented by the matrices
   \[
   \begin{pmatrix}
   q & 0 \\
   0 & -1
   \end{pmatrix}
   \quad \text{and} \quad
   \begin{pmatrix}
   -1 & 0 \\
   0 & q
   \end{pmatrix},
   \]
   respectively.

   Consequently, $tr(L_2) = tr(L_3) = q - 1$ and, finally, $tr(g_1) = q - 1$, $tr(g_1g_2) = -q$.

3. For $[1] [1,1] [1,1,1]$: $\lambda_{L_2} = -1, \lambda_{L_3} = -1 - \frac{1}{q}$. Hence, $tr(g_1) = -1$ and $tr(g_1g_2) = 1$.

Similarly, in $H_4(q)$:

1. For $[1] [2] [3] [4]$: $\lambda_{L_2} = q, \lambda_{L_3} = q + q^2$ and $\lambda_{L_4} = q + q^2 + q^3$. Hence, $tr(g_1) = q$, $tr(g_1g_2) = q^2$, and $tr(g_1g_2g_3) = q^3$.

2. For the three-dimensional irrep $[3,1]$
   \[
   \begin{bmatrix}
   [1] & [2] & [3] & [3,1] : & \lambda_{L_2} = q, & \lambda_{L_3} = q + q^2, & \lambda_{L_4} = -1 \\
   [1] & [2] & [2,1] & [3,1] : & \lambda_{L_2} = q, & \lambda_{L_3} = -1, & \lambda_{L_4} = q + q^2 \\
   [1] & [1,1] & [2,1] & [3,1] : & \lambda_{L_2} = -1, & \lambda_{L_3} = q, & \lambda_{L_4} = q + q^2
   \end{bmatrix}
   \]

   Hence, $tr(L_2) = 2q - 1$, $tr(L_3) = q^2 + 2q - 1$, and $tr(L_4) = 2q^2 + 2q - 1$, i.e., $tr(g_1) = 2q - 1$, $tr(g_1g_2) = q(q - 1)$, and $tr(g_1g_2g_3) = -q^2$.

3.-5. etc.

To obtain the traces of the non-simply-connected element $g_1g_3$ we note that 
\[ L_2L_4 = g_1(g_3 + \frac{1}{q}g_2g_3g_2 + \frac{1}{q^2}g_1g_2g_3g_2g_1), \]
so that
\[ tr(L_2L_4) = tr(g_1g_3) + 2\frac{q-1}{q}(g + \frac{1}{q})tr(g_1g_2g_3) + 2(q - 1 + \frac{1}{q})tr(g_1g_2) + (q - 1)tr(g_1). \]
Thus, for the irrep $[4]$ of $H_4(q)$, using the values of $\lambda_{L_2}$, $\lambda_{L_4}$, $tr(g_1)$, $tr(g_1g_2)$ and $tr(g_1g_2g_3)$ obtained above, $tr(g_1g_3) = q^2$.

Similarly, for the irrep $[3,1]$

$$L_2L_4 \rightarrow \begin{pmatrix} -q & 0 & 0 \\ 0 & q^2 + q^3 & 0 \\ 0 & 0 & -q - q^2 \end{pmatrix}$$

so that $tr(L_2L_4) = q^3 - 2q$ and $tr(g_1g_3) = q^2 - 2q$.

etc.

Evaluating the trace of the product $L_2L_5$ we find that it can be expressed in terms of $tr(g_1g_3)$, $tr(g_1g_3g_4)$, and traces of simply connected terms. Thus,

$$tr(L_2L_5) = 2 tr(g_1g_3) + \left( \frac{q - 1}{q} \right) tr(g_1g_3g_4) + \left( \frac{q - 1}{q} \right)^2 \left( q + \frac{1}{q} \right) tr(g_1g_2g_3g_4) + \left( \frac{q - 1}{q} \right) \left( 3q - 2 + \frac{3}{q} \right) tr(g_1g_2g_3) + \left( 3q - 4 + \frac{3}{q} \right) tr(g_1g_2) + (q - 1) tr(g_1)$$

Since $tr(g_1g_3)$, as well as all the simply-connected traces, have already been evaluated, $tr(L_2L_5)$ provides the next non-simply connected term, $tr(g_1g_3g_4)$.

Continuing, we observe that products of pairs of Murphy operators provide the traces of all doubly-connected terms: $tr(L_2L_6)$ provides $tr(g_1g_3g_4g_5)$, $tr(L_3L_6)$ provides $tr(g_1g_2g_4g_5)$, thus exhausting the doubly-connected classes involving four Hecke generators, etc. Triply-connected terms are generated by triple products of non-consecutive Murphy operators. Thus, $tr(L_2L_4L_6)$ provides $tr(g_1g_3g_5)$, etc. A more systematic study of the relations between traces of products of Murphy operators and traces of reduced non-simply connected terms will be presented elsewhere.

The following recursive procedure can be implemented to obtain the traces of the simply-connected elements systematically:

For an irrep $\Gamma_n$ of $H_n(q)$ consider the set of irreps $\Gamma_{n-1} \subset \Gamma_n$ of $H_{n-1}(q)$, obtained by
eliminating one box from $\Gamma_n$ in all possible ways. Clearly,

$$\text{tr}(L_i)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} \text{tr}(L_i)_{\Gamma_{n-1}} \quad i = 2, 3, \ldots, n - 1.$$ 

and

$$|\Gamma_n| = \sum_{\Gamma_{n-1} \subset \Gamma_n} |\Gamma_{n-1}|$$

where $|\Gamma_n|$ is the dimensionality of the irrep $\Gamma_n$.

The trace of $L_n$ can now be evaluated very conveniently using either

$$\text{tr}(L_n)_{\Gamma_n} = |\Gamma_n| A_{\Gamma_n}^n - \sum_{i=2}^{n-1} \text{tr}(L_i)_{\Gamma_n},$$

where $[\Gamma_n \setminus \Gamma_{n-1}]_q$ denotes the $q$-content of the box that has been eliminated from $\Gamma_n$ to obtain $\Gamma_{n-1}$, or the equivalent expression

$$\text{tr}(L_n)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} |\Gamma_{n-1}| [\Gamma_n \setminus \Gamma_{n-1}]_q.$$ 

This recursive procedure is ideally suited to implementation using symbolic programming. The traces of the Murphy operators of $H_2(q)$, $H_3(q)$, $\ldots$, $H_7(q)$, evaluated (manually!) using this recursive procedure, are presented in Tables 1-5.

Having obtained the traces of the Murphy operators, the traces of the simply-connected elements of the Hecke algebra can be evaluated using eq. [13]. For $H_2(q)$, $H_3(q)$, $\ldots$, $H_6(q)$ these traces agree with the values presented by King and Wybourne [7]. The traces of the simply-connected terms in $H_7(q)$ are presented in Tables 6-7.

As illustrated above, to obtain traces of non-simply-connected elements of the Hecke algebra we have to evaluate traces of products of Murphy operators. For $\prod_{i=1}^{\ell} L_{\alpha_i}$,

$$2 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_\ell \leq n,$$

we consider all the sequences of irreps leading to the desired irrep $\Gamma_n$ of $H_n(q)$ from all possible irreps of $H_{\alpha_1-1}(q)$. Let $\Gamma_{\alpha_1-1} \subset \Gamma_{\alpha_1} \subset \Gamma_{\alpha_1+1} \subset \cdots \subset \Gamma_n$ be such a sequence and let $\{\cdots \subset \Gamma_n\}$ denote the complete set of sequences. Obviously,

$$\text{tr} \left( \prod_{i=1}^{\ell} L_{\alpha_i} \right) = \sum_{\{\cdots \subset \Gamma_n\}} |\Gamma_{\alpha_1-1}| \prod_{i=1}^{\ell} [\Gamma_{\alpha_i} \setminus \Gamma_{\alpha_i-1}]_q.$$
6 Correspondence between the eigenvalues of the fundamental invariants of $SU_q(N)$ and $H_n(q)$

To introduce the relation between the fundamental invariant of the Hecke algebra $H_n(q)$ and that of the quantum-unitary group $SU_q(N)$ we first recall that the space of an irrep of the latter is spanned by the Gelfand-Zetlin states [20]

$$|h⟩ = |h_{1,N}, h_{2,N}, \ldots, h_{N-1,N}, h_{N,N}⟩$$

For generic $q$ the parameters $h_{i,j}$ are integers satisfying the triangular inequalities $h_{i,j+1} \geq h_{i,j} \geq h_{i+1,j+1}$ ($i \leq j$). The top row specifies the irrep, $(h_{i,N} - h_{N,N})$ giving the length of the $i$th row of the Young diagram. A convenient convention is $h_{N,N} = 0$. The number of rows in a Young diagram corresponding to an irrep of $SU_q(N)$, for generic $q$, is at most $N - 1$, and the dimension is given by the number of distinct sets of integers $\{h_{i,j}; 1 \leq i \leq j \leq N - 1\}$ satisfying the triangular inequalities.

We will not reproduce the standard Drinfeld-Jimbo q-deformation of the $SU(N)$ algebra. The matrix elements of the Chevalley triplets $(h_k, e_k, f_k; k = 1, 2, \ldots, N - 1)$ are given by

$$q^{\pm h_k} |h⟩ = q^{\pm (2 \sum_{i=1}^{k} h_{i,k} - \sum_{i=1}^{k+1} h_{i,k+1} - \sum_{i=1}^{k-1} h_{i,k-1})} |h⟩$$

$$⟨h_j,k - 1|f_k|h⟩ = ⟨h|e_k|h_j,k - 1⟩ = \left( - \frac{P_1(j, k)P_2(j, k)}{P_3(j, k)} \right)^{1/2}$$

where $|h_{j,k} - 1⟩$ differs from $|h⟩$ only by the translation $h_{j,k} \rightarrow (h_{j,k} - 1)$, and

$$P_1(j, k) = \prod_{i=1}^{k+1} [h_{i,k+1} - h_{j,k} - i + j + 1]_s$$

$$P_2(j, k) = \prod_{i=1}^{k-1} [h_{i,k-1} - h_{j,k} - i + j]_s$$
\[ P_3(j, k) = \prod_{1 \leq i \leq k, i \neq j} [h_{i,k} - h_{j,k} - i + j + 1]_s [h_{i,k} - h_{j,k} - i + j]_s \]

where \([x]_s\) is the symmetric \(q\)-bracket

\[ [x]_s \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \]

Define

\[ h_k = A_k^k - A_{k+1}^{k+1} \]

\[ e_k = A_{k+1}^{k+1}, \quad f_k = A_k^k \]

\[ A_{k-p}^{k+1} = A_{k-p}^{k+1-p} A_{k+1-p}^{k+1} - q^{-1} A_{k+1-p}^{k+1-p} A_{k-p}^{k+1} \]

\[ A_{k+1}^{k-p} = A_{k+1}^{k-p} A_{k+1-p}^{k+1} - q^{-1} A_{k+1-p}^{k+1} A_{k+1}^{k-p} \]

The operator

\[ C_2 = \sum_{j>i} A_j^i A_i^j q^{(A_i^i + A_j^j) - 2j} + \frac{q}{(q - q^{-1})^2} \sum_{i=1}^N q^{2(A_i^i + i)} \]

commutes with all the generators. Subtracting the invariant terms \([21]\)

\[ \frac{1}{(q - q^{-1})^2} \left( q^2 \sum_{i=1}^N A_i^i + N(N+1) + (N-1) \right) \]

one obtains the correct classical \((q = 1)\) limit (this corresponds to a modified version of \(C_2\) of ref. \([21]\)). However, for our purposes it is more interesting to note the result (obtained immediately by using the minimal state, that is annihilated by \(A_i^i\) for \(j > i\)) that on the space of any irrep

\[ (q - q^{-1})^2 q^{-(2N+3)} C_2 = \left( \sum_{k=1}^N q^{2(h_{k,n} - k)} \right) I \]

\[ (15) \]

Defining

\[ \hat{C}_2 \equiv (q - q^{-1})^2 q^{-(2N+3)} C_2 - q^{-2N} I \]

we find that for \(h_{N,N} = 0\) the eigenvalue of \(\hat{C}_2\) on the irrep \(\Gamma\) is

\[ \Lambda_\Gamma^{\hat{C}_2} = \left( \sum_{k=1}^{N-1} q^{2(t_k - k)} \right) I \]
where $\ell_k$ is the length of the $k$th row of the Young diagram. Note also that the linear invariant $C_1 \equiv \sum_{i=1}^{N} A_i^i$ of eq. [4] has, for $h_{N,N} = 0$, the eigenvalue

$$\sum_{k=1}^{N-1} h_{k,N} = n$$

where $n = \sum_i \ell_i$ is the number of boxes in the Young diagram $\Gamma$ that labels the irrep.

Consider now the fundamental invariant of $H_n(q)$, introduced before. The corresponding eigenvalues are (cf. eq. [4])

$$\Lambda^{\Gamma}_{C_n} = q \left( \sum_{(i,j) \in \Gamma} \frac{q^{j-i} - 1}{q-1} \right)$$

or,

$$\left( \frac{q-1}{q} \right)^2 \Lambda^{\Gamma}_{C_n} = \sum_{i \in \Gamma} \left( q^{\ell_i-i} - q^{-i} - \frac{q^{-1\ell_i}}{q} \right)$$

If $\Gamma$ corresponds to the Young diagram that labels the irrep of $SU_q(N)$ considered before then

$$\sum_i \ell_i = n$$

$$\sum_{i \in \Gamma} q^{-i} = \sum_{i=1}^{N-1} q^{-i} = -\frac{q^{-N+1} - 1}{q-1}$$

Hence, if $q$ in $C_n(q)$ is replaced by $q^2$, than for irreps $\Gamma$ with $n$ boxes the precise correspondence between the eigenvalues of the fundamental invariant of the Hecke algebra and those of the quadratic Casimir of $SU_q(N)$ is

$$\left( \frac{q^2-1}{q^2} \right)^2 \Lambda^{\Gamma}_{C_n(q^2)} + \frac{q^2-1}{q^2} n = \Lambda^{\Gamma}_{C_2} + \frac{q^{2(-N+1)} - 1}{q^2 - 1}$$

Thus, through this relation $C_n(q^2)$ also characterizes the irreps of $SU_q(N)$ corresponding to Young diagrams with $n$ boxes.
The relation between the transposition class-sum of the symmetric group and the quadratic Casimir of $SU(N)$, that is the classical ($q = 1$) limit of eq. 1, was used by Gross [12] to study the $\frac{1}{N}$ expansion. For us the crucial terms are not the $n$ or $N$ dependent constants but

$$\sum_k q^{2(\ell_k - k)}$$

The fact that one needs $C_n(q^2)$ in eq. 1 can be viewed in a more instructive way as follows. In the relation 2

$$g_i^2 = (q - 1)g_i + q$$

setting $q = (q')^2$, $g_i = q'g'_i$ and then suppressing the primes for simplicity, one obtains

$$g_i^2 = (q - q^{-1})g_i + 1$$

This is the form used, for example, by Pan and Chen [10]. Using this form $C_n$ can be defined without explicit appearance of $q$ (or $q^{-1}$) and the correspondence with the $SU_q(N)$ Casimir is more direct.

For $SU_q(N)$ all the higher order Casimir operators are available [23, 24, 26]. We will not reproduce them here. However, a few comments on the structure of the eigenvalues can help to better appreciate the special role of $C_2$ (or $\hat{C}_2$). The construction of Faddeev-Reshetikhin-Takhtadzhyan [23] (suitably normalized) can be shown to lead for the $p$th order invariant to the eigenvalue

$$\sum_{i_1 < i_2 < \ldots < i_p} q^{2\sum_{k=1}^p(h_{i_k,N} - i_k)}$$

This is not yet the most convenient form. An operator giving the simplest symmetric polynomial of order $p$ in $q^{h_{k,N}-k}$ namely,

$$\sum_k q^{2p(h_{k,N}-k)}$$

would be desirable for relating to a corresponding invariant of $H_n(q)$. But such operators remain to be constructed, both for $SU_q(N)$ and for $H_n(q)$. For $SU_q(N)$ the simplest, most compact construction of the Casimir operators is that due to Bincer [24], but the corresponding eigenvalues, for all Casimirs but $C_2$, are much more complicated.
Anyhow, since for generic $q \neq 1$, once $n$ is determined by the linear invariant, the fundamental Casimir $\hat{C}_2$ completely specifies the irreps, the necessity of the higher order Casimirs is not evident.

Since in eq. [15] $h_{k,N} \geq h_{k+1,N}$, so $h_{k,n} - k > h_{k+1,N} - (k + 1)$. Hence, given an eigenvalue of $C_2$, simply arranging the powers of $q$ in decreasing order one gets an expression of the type

$$q^{2L_1} + q^{2L_2} + \cdots + q^{2L_{N-1}} \quad (L_i > L_{i+1})$$

Now,

$$\ell_k = L_k + k \quad (k = 1, 2, \cdots, N - 1)$$

completely specifies the Young diagram, giving the lengths of successive rows. Again, since (setting $q^2 = \exp(\delta)$)

$$\sum_k q^{L_k} = 1 + \delta(\sum_k L_k) + \frac{\delta^2}{2!}(\sum_k L_k^2) + \cdots$$

the coefficient of $\delta^p$ gives the eigenvalue of a suitably defined $p$th order classical invariant. Thus all the essential information is contained in $C_2$ (or $\hat{C}_2$) in a simple way. In fact, this is presumably true for the $q$-deformations of all algebras.

### 7 Conclusions

The fundamental invariant of the Hecke algebra $H_n(q)$ has been shown to fully characterize the corresponding irreps. In fact, when $q$ is neither zero nor a root of unity of order less than $n$, the eigenvalues of the fundamental invariant are polynomials in $q$ and $\frac{1}{q}$ whose coefficients specify the structure of the corresponding Young diagram. This is remarkable in view of the fact that the transposition class-sum, to which this fundamental invariant reduces in the limit $q \to 1$, exhibits (for $n \geq 6$) eigenvalue degeneracies. The projection operators constructed in terms of the fundamental invariant onto the various irreps are well behaved in the limit $q \to 1$, even for irreps that are not uniquely specified by the transposition class-sum. These limiting projection operators generate the characters of all the classes of $S_n$. 
The eigenvalues of the quadratic Casimir operator of $SU_q(N)$ corresponding to Young diagrams with $n$ boxes have been shown to be expressible in terms of those of the fundamental invariant of $H_n(q)$. The former suffice to specify the irreps of $SU_q(N)$, i.e., to determine the eigenvalues of all the higher order Casimir operators. We expect that the fundamental Casimir operators of other quantum groups possess similar properties.

While the applications to the symmetric group and to the unitary and quantum-unitary groups may be of mostly conceptual interest, the results are demonstrated to lead to useful procedures for evaluating the traces of the Hecke algebra.

In this article we have considered only generic $q$. For $q$ a root of unity the situation changes radically since the centre becomes suddenly much larger. But then the formalism of “fractional parts” can be used to unify the construction of the $SU_q(N)$ representations [22, 26] (for $q$ generic and root of unity). A similar approach to the $H_q(n)$ representations for $q$ a root of unity might be fruitful. This aspect will be explored elsewhere.

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