The role of singularities in chaotic spectroscopy*

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Abstract

We review the status of the semiclassical trace formula with emphasis on the particular types of singularities that occur in the Gutzwiller-Voros zeta function for bound chaotic systems. To understand the problem better we extend the discussion to include various classical zeta functions and we contrast properties of axiom-A scattering systems with those of typical bound systems. Singularities in classical zeta functions contain topological and dynamical information, concerning e.g. anomalous diffusion, phase transitions among generalized Lyapunov exponents, power law decay of correlations. Singularities in semiclassical zeta functions are artifacts and enters because one neglects some quantum effects when deriving them, typically by making saddle point approximation when the saddle points are not enough separated. The discussion is exemplified by the Sinai billiard where intermittent orbits associated with neutral orbits induce a branch point in the zeta functions. This singularity is responsible for a diverging diffusion constant in Lorentz gases with unbounded horizon. In the semiclassical case there is interference between neutral orbits and intermittent orbits. The Gutzwiller-Voros zeta function exhibit a branch point because it does not take this effect into account. Another consequence is that individual states, high up in the spectrum, cannot be resolved by Berry-Keating technique.

I. Introduction

Research in quantum chaos has to a large extent been focused on conceptual questions such as how is classical chaos revealed in the $\hbar \to 0$ limit of quantum mechanics and what are the properties of quantum spectra of classically chaotic systems. During the last decade another hot topic has been periodic orbit quantization. Model calculation of quantum spectra using periodic orbit techniques are greatly simplified if one restrict oneself to systems with two degrees of freedom. Applications to three dimensional systems are then restricted to cases

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where some internal symmetry reduces the system by one degree of freedom. One example is the
diamagnetic Kepler problem, see e.g. [1] and references therein. The simplest conceivable two
dimensional systems are billiards. Billiards are of great use in theoretical work on semiclassical
mechanics. But billiard like systems, of real interest, and of any imaginable shape can now be
constructed in the laboratory, there are several examples in the present volume. So the stage
is set for a fruitful interplay between quantum chaos and mesoscopic physics.

To make a successful semiclassical computation one needs to control the semiclassical limit
of the Green function. One can then compute anything from conductance in mesoscopic devices [4]
to wave functions. Another information contained in the Green function is the quantum
spectrum, which can be uncovered by taking its trace. In bound systems the Green function
can be expressed in terms of the eigenfunctions and eigenvalues of the Hamiltonian

\[ G(x, x', E) = \sum_i \frac{\Psi_i^*(x)\Psi_i(x')}{E - E_i}, \quad (1) \]

and the trace reads

\[ \text{tr} G(x, x', E) = \int G(x, x, E)dx = \sum_i \frac{1}{E - E_i}, \quad (2) \]

whose semiclassical limit for chaotic system was derived by Gutzwiller [3]

\[ \text{tr} G(E) = g_0(E) + \frac{1}{i\hbar} \sum_p T_p \sum_{n=0}^{\infty} \frac{1}{|M_p^n - I|^{1/2}} e^{in[S_p/\hbar - \mu_p \frac{2}{\pi}]}. \quad (3) \]

The index \( p \) labels the primitive periodic orbits, \( S_p \) is the action integral along the orbit, \( T_p \)
(= \( \partial S_p / \partial E \)) its period, \( M_p \) is the linearized Poincaré map around the orbit and \( \mu_p \) is a phase
index. Finally, \( g_0(E) \) provides the mean level distribution, which may be interpreted as the
contribution from classical orbits of zero length.

The trace formula can be recasted to the Gutzwiller-Voros zeta function [4]

\[ Z_{GV}(E) = \prod_p \prod_{m=0}^{\infty} \left( 1 - e^{i[S_p/\hbar - \mu_p \frac{2}{\pi}]}/|\Lambda_p|^{1/2} \Lambda_p^m \right). \quad (4) \]

The oscillating part of the trace formula is given by the logarithmic derivative of the zeta
function. The only continuous time systems we are going to consider are billiards. The action
\( S_p \) is then related to the geometric length \( L_p \) of the orbit as \( S_p = L_p k \) where \( k \) is the momentum.
Usually we are going to consider zeta functions as functions of the complex variable \( s = ik \).
This paper is devoted to the study of classical and semiclassical zeta functions. In particular,
we will be interested in the presence of various kinds of singularities. This might seem a highly
technical problem but by putting this theme in focus we will shine some new light on many
of the problems that are associated with semiclassical quantization. The paper is divided into
two parts. First a nonechnical review of the subject. Then follows a series of more technical
discussions to illustrate various aspects of the preceding story.
II. A Story - The rise and fall of periodic orbit theories in quantum mechanics

Our story will begin with Berry’s ‘tour de force’ paper\[5\] *Quantizing a Classically Ergodic System, Sinai’s Billiard and the KKR method*. Centered around a numerical method Berry discusses virtually all issues now known under the phrase of *Quantum Chaos*, like e.g. level statistics and periodic orbit expansions. Taking the semiclassical limit of level density he obtains the Gutzwiller trace formula for the unstable orbits and the Berry-Tabor like expression for the neutral (or marginally stable) orbits.

At the time it was premature to start from the trace formula and compute (semiclassical approximations of) the eigenvalues. Berry’s numerical scheme was successful because of a major trick; a transformation of the badly converging sum over lattice points to a highly convergent sum over points in the dual lattice. One of the major problem with periodic orbit expressions is to make the corresponding trick for the Gutzwiller sum. We will discuss this problem from several points of view in the following.

Later, Gaspard and Rice \[6\] were inspired by Berry’s method to construct a numerical scheme for determining scattering resonances in the 3-disk scatterer, and similar systems. From one point of view this is an easier problem (from another, more conceptual it is not), the disk lattice is finite and there is no need to go over to the dual lattice.

The 3-disk is easy to deal with also from a semiclassical point of view, as was realized by Cvitanović and Eckhardt \[7\]. The Gutzwiller-Voros zeta function\[4\] closely resembles zeta functions studied in classical problems. The 3-disk problem is ideal for application of classical zeta functions. It has a simple topology, provided the disks are enough separated, the dynamics may be mapped to a dynamics among a finite set of symbols. This is known as *symbolic dynamics*. The open 3-disk is also hyperbolic; the stabilities of periodic orbits are exponentially bounded with their length. A system with these two properties is said to be of Axiom-A type.

This property gives nice convergence to the Dirichlet series obtained by expanding the Euler product representation of the zeta function (4), this is usually called a *cycle expansion* \[8, 7\]. The classical resonances as well as the quantum resonances can thus be efficiently computed by a moderate number of periodic orbits.

Still, the situation is not ideal. The Gutzwiller-Voros zeta function does exhibit poles \[9\] which limits convergence of cycle expansions far down in the complex \(k\)-plane. A theorem of Rugh\[10\] says that a certain type of zeta functions, Fredholm determinants of evolution operators with multiplicative weights, are entire for Axiom-A systems. A Fredholm determinant may look something like

\[
F(z) = \prod_p \prod_{m=0}^\infty (1 - \frac{w_p e^{-s L_p}}{\Lambda^m})^{m+1}
\]  

(5)

The Gutzwiller-Voros zeta function lacks the right structure among the higher \(m\) factors to make it entire. There is a lot of interesting work \[11, 12\] on improving the Gutzwiller-Voros zeta function by manipulating these higher factors. The by far most interesting is the *quasi*
classical theory of Cvitanović and Vattay [12]. Starting from the hydrodynamical formulation of Quantum Mechanics of Madelung they construct a multiplicative evolution operator whose Fredholm determinant should contain the semiclassical eigenvalues in its spectrum. The derivation avoids the multitude of stationary phase approximations in the standard derivation of the trace formula. Still, the quasiclassical Fredholm determinant contains extra zeros which have nothing to do with the quantum spectrum, and they seriously slow down convergence. So the surprising thing is that, if interpreted as an asymptotic series, the Gutzwiller-Voros zeta function is in many respects still superior to its competitors [13].

Due to the simplicity of the 3-disk scatterer it is an ideal testing ground for various corrections to the Gutzwiller-Voros zeta function, like diffraction-corrections [14, 15] and $\hbar$ corrections [16]. Note that the circumstances that make the 3-disk classically and semiclassically nice also make diffraction corrections small; semiclassical formulas fail if the point particle scatter in an extreme forward angle, or if it passes extremely close to a disk and this is never the case in the (enough separated) 3-disk. The main diffraction effect for this system is then due to creeping and this is indeed a small effect.

When including diffraction effects in axiom-A systems one can say that the classical phase space is enlarged to include diffractive orbits associated with some discontinuity such as a point or wedge or circular disk. This can be done without risking that stationary points come to close. We will see that this is very different from the problem intermittency is going to present us with in bound systems.

We have dwelt on Axiom-A scattering systems at some length, mostly to appreciate how different from bound systems they are. When we now are turning our interest towards bound systems we will encounter singularities that have nothing to do with the higher $m$ factors in Fredholm determinants, so we will frequently omit them. Their subtle role in bound systems is far beyond the scope of this paper.

If the open 3-disk system was an archetype for open chaotic scattering systems the Sinai billiard is a suitable archetype for bound chaotic systems. It is in many ways generic (see below) but is simple enough to allow some analysis. This system is either represented as a circular scatterer inside a square or as an infinite lattice of disks, cf. fig 1. The latter representation is an example of a Lorentz gas. Lorentz gases are among the simplest conceivable systems exhibiting statistical behaviour such as diffusion [23].

Bounded system are generally not of Axiom-A type. We will now discuss various deviations and how they affect the analyticity of the zeta functions.

The topology is generally extremely complicated. Given a bound system, there is a priori no reason to believe that there should exist a simple symbolic dynamics. Often one can find a symbolic coding but the grammar rules are complex, in the sense that the number of forbidden substrings increases exponentially with length. Mathematically, such a symbolic dynamics is known as an infinite subshift. The basic mechanisms for pruning of the symbolic dynamics can be very different. In dispersive billiards pruning enters because a trajectory from one disk to another can be obstructed by an intermediate disk.
Very little is known about how pruning affects the analyticity of zeta functions. In ref. [18] we propose that the lack of finite symbolic dynamics may induce a natural boundary in the complex plane through which analytic continuation is not (even in principle) possible. We discuss this in little more detail in section IIIa.

Intermittency is another abundant property of bound chaotic systems. We will consider a strong form of intermittency associated with the presence of neutral orbits in the system. Orbits accumulating towards this neutral orbit will have Lyapunov exponents tending to zero. Many ergodic billiards, such as the stadium [19] and the Sinai billiard have this property. The effect is relevant for Hamiltonian systems with mixed phase space because the boundaries of stable islands are neutrally stable and will be surrounded by intermittent orbits. So the neutral orbit in a chaotic billiard may be looked upon as a stable island of zero area [19]. Strong intermittency appears to imply a branch point singularity [37, 21] at the origin (s=0) as discussed in section IIIb. It is possible to construct bounded billiards without neutral orbits. One example of is (the unit cell) of a Lorentz gas on a triangular lattice, provided the disks are large enough, see [25]. Such systems often exhibit some weaker form of intermittency.

In classical zeta functions, these singularities play a dual role. They hamper convergence of cycle expansions seriously. But they are also important carriers of dynamical and topological information such as power law decay of correlations, anomalous diffusion and phase transitions among the generalized Lyapunov exponents [23], and should not be considered as undesired mathematical obstacles. In section IIIc we, as an example, discuss the relation between the singularity of the zeta function and anomalous diffusion in the Sinai billiard.

Singularities will be present in virtually any type of zeta function, and in particular the Gutzwiller-Voros zeta function (see section IIIb). This zeta function is supposed to approximate the spectral determinant \( \Delta(k) \) (apart from the smooth Weyl factor). This function obeys a functional equation \( \Delta(k) = \Delta(-k) \) and the singularities we have discussed violate this functional equation. In the Gutzwiller-Voros zeta function they are artifacts and enter because one neglects some quantum effects when deriving it, like making saddle point approximations when the saddle points are not enough separated. This problem is intimately associated with intermittency. In section IIIe we demonstrate that the problem with interfering stationary points actually occur in the Sinai billiard and this is what lets a branch point enter the zeta function.

If the system under study is hyperbolic the situation is somewhat better, but one has still to bother about pruning. The main pruning mechanism in dispersive billiards is, as we said, due to trajectories between two disks being obstructed by a third disk. Orbits that are nearly pruned are therefore greatly affected by diffraction effects so pruning and diffraction goes hand in hand and must be dealt with in a coherent way, let’s call such a hypothetical scheme *Quantum pruning*. Pruning in time reversible system can often be described by a *pruning front* [22]. Intuitively we can say that the quantum counterpart is not sharp as the classical one. Recall that we expect that a generic pruning front may induce a natural boundary in classical zeta functions, so the fuzziness in the quantum counterpart should prevent such a singularity.
to occur.

How serious the problems of the Gutzwiller-Voros zeta functions are depend on the method by which one computes the zeros. There are mainly two approaches. The first is the cycle expansion approach, previously discussed, used mainly in open Axiom A systems. This relies heavily on strong convergence properties of the zeta functions. An alternative for bound systems is to use the method of Berry-Keating\[26\]. The main idea is the following. The existence of a functional equation for the spectral determinant imply that one only needs to know the length spectrum (the fourier transform of the level density) up to half the Heisenberg length\[4\] in order to compute the spectrum. The most accurate computation of spectra of the Gutzwiller-Voros zeta functions for bound systems has been performed using this approach\[27, 28\]. The advantage of the method is that one only needs a finite number of periodic orbits to compute a finite number of levels, and one only needs to scan the real energy axis. The approach of Berry-Keating assumes that the spectral determinant may be expressed as a Dirichlet series. There are now growing evidence \[29, 19\] that even the Berry-Keating method will eventually break down for billiards. In ref. \[29\] the authors argue that the majority of orbits in the Sinai billiard with length $L > O(k^{2/3})$ is affected by diffraction and that these corrections are of the same order as the original contribution. Recalling that the Heisenberg length in billiards scales as $L_H = O(k)$ this implies that the Berry-Keating scheme will break down as $k \to \infty$, provided that diffraction effects cannot be accounted for. The occurrence of a branch point singularity is actually just another side of this coin. It indicates that the tail of the Dirichlet series suffers so badly from intermittency and the nearby stationary points so as to make the series divergent (a branch point at $k = 0$ makes the real $k$-axis where we want to find our eigenvalues the border of convergence). The entanglement between neutral and intermittent orbits implies that the representation of the spectral determinant in terms of a Dirichlet series is lost. There is thus no obvious implementation of the Berry-Keating method.

The problem with interfering saddle points has nothing to do with the singular nature of billiards. In smooth chaotic potentials it is even harder to realize a complete symbolic dynamics. The abundance of pruning is deeply associated with the existence of small stability islands. These islands will immediately cause problems with intermittency and close saddle points.

It should be stressed that such a failure of predicting individual eigenstates does not render the trace formula useless. One may take two approaches. One is to take the trace formula more or less as it stands and ask for the limitations of it. What information can be gained from this simple and beautiful formula? To use it as a theoretical tool one also needs a better understanding of the asymptotic properties of periodic orbits \[37, 21\] far beyond the Ozorio-Hannay sum rule \[30\]. The other approach is to try to improve the trace formula in various ways. The two approaches are well motivated and are of course not independent of each other. It is an intellectual challenge to improve periodic orbit theories to predict eigenstates higher

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\[1\] the length corresponding to one mean level spacing
and higher but it is perhaps not always what applications ask for.

Among all systems exhibiting hard chaos, systems with neutral orbits are among the worst, without neutral orbits the situation would probably be better. The conceptual problems of systems with neutral orbits are similar to those of generic systems with mixed phase space, and are thus of fundamental importance. The relative simplicity of the Sinai billiard make it an ideal model system for addressing these issues.

III. Five Illustrations

We will now illustrate the story in the previous section with a series of more technical discussions. We are trying to understand the sense of singularities in zeta functions in the classical case and the nonsense of them in the semiclassical case. We will start our exposé by considering classical zeta functions for discrete time systems (i.e. maps), move on to continuous time systems (in particular the Sinai billiard) and finally discuss semiclassical zeta functions.

a. Symbolic dynamics in bound chaotic systems

We begin by considering one dimensional maps \( x \mapsto f(x) \). Much dynamic information, like different kind of entropies and generalized Lyapunov exponents, is encoded in the following one-parameter family of zeta functions \[ \frac{1}{\zeta(\beta, z)} = \prod_p \left( 1 - \frac{z^{n_p}}{|\Lambda_p|^{1-\beta}} \right). \]

The product in (6) runs over all primitive periodic orbits \( p \), having period \( n_p \) and stability \( \Lambda_p = \frac{d^p f_p}{dx_p}|_{x_p} \) with \( x_p \) being any point along \( p \). If \( \beta = 1 \) no metric information enters the zeta which is now called a topological zeta function

\[ \frac{1}{\zeta_{\text{top}}(z)} = \prod_p (1 - z^{n_p}). \]

The leading zero \( z_0 \) of \( 1/\zeta_{\text{top}}(z) \) (the one with smallest modulus) and the topological entropy \( h \) are related by \( h = -\log z_0 \).

We will restrict our attention to the topological zeta function for unimodal (one-humped) maps with one external control parameter \( f_{\lambda}(x) = \lambda g(x) \). Symbolic dynamics is introduced by mapping a time series \( \ldots x_{i-1}x_i x_{i+1} \ldots \) onto a sequence of symbols \( \ldots s_{i-1} s_is_{i+1} \ldots \) where

\[ s_i = L \quad x_i < x_c \]  
\[ s_i = C \quad x_i = x_c \]  
\[ s_i = R \quad x_i > x_c \]

and \( x_c \) is the critical point of the map (i.e. maximum of \( g \)). The kneading sequence is the itinerary of the critical point. The allowed symbol sequences can be determined from it. All
unimodal maps (obeying some further constraints) with the same kneading sequence have the same set of periodic orbits [32].

If \( \beta \neq 1 \) in (3) the individual details of \( f(x) \) is reflected in the zeta function, but for tent map

\[
x \mapsto f(x) = \begin{cases} 
\lambda \cdot x & x \in [0, 1/2] \\
\lambda \cdot (1 - x) & x \in (1/2, 1]
\end{cases}
\]

(where the parameter \( \lambda \in [0, 2] \)) the general zeta function (3) is obtained from \( 1/\zeta_{\text{top}} \) by simple rescaling,

\[
1/\zeta(\beta, z) = 1/\zeta_{\text{top}}(z/\lambda^{1-\beta})
\]

The topological entropy is \( h = -\log \lambda \).

The set of periodic points of the tent map is countable. A consequence of this fact is that the set of parameter values for which the kneading sequence is periodic or eventually periodic (preperiodic) are countable and thus of measure zero and consequently the kneading sequence is aperiodic for almost all \( \lambda \). For general unimodal maps the corresponding statement is that the kneading sequence is aperiodic for almost all topological entropies.

For a given periodic kneading sequence of period \( n \), \( L_\lambda = PC = s_1s_2\ldots s_{n-1}C \) there is a simple expansion for the topological zeta function. Now let \( a_i = 1 \) if \( s_i = L \), and \( a_i = -1 \) if \( s_i = R \). Then the expanded zeta function is a polynomial of degree \( n \)

\[
1/\zeta_{\text{top}}(z) = \prod_p (1 - z^n_p) = (1 - z) \cdot \sum_{i=0}^{n-1} b_i z^i,
\]

where

\[
b_n = \prod_{i=1}^{n} a_i.
\]

Aperiodic and preperiodic kneading sequences is accounted for by simply replacing \( n \) by \( \infty \). An important consequence of (13) and (14) is that the sequence \( \{b_i\} \) has a periodic tail if and only if the kneading sequence has one (their period may differ by a factor of two though).

The analytic structure of the function represented by the series \( \sum b_i z_i \) depends on whether the tail of \( \{b_i\} \) is periodic or not. If the period of the tail is \( N \) we can write

\[
1/\zeta_{\text{top}}(z) = p(z) + q(z)(1 + z^N + z^{2N} + \ldots) = p(z) + \frac{q(z)}{1 - z^N},
\]

for some polynomials \( p(z) \) and \( q(z) \). The result is a set of poles spread out along the unit circle. This applies to the preperiodic case. An aperiodic sequence of coefficients would formally correspond to an infinite \( N \) and it is natural to assume that the singularities will fill the unit circle. There is indeed a theorem ensuring that this in a sense is the case (provided the \( b_i \)'s can only take an finite number of values). The unit circle becomes a natural boundary. A function with a natural boundary lacks an analytic continuation outside it.
What happens with $1/\zeta(\beta, z)$ for $\beta \neq 1$ if we make our unimodal map nonlinear but still hyperbolic (that is $|f'| \geq C > 1$)? All that is known is that the zeta function is analytic inside a certain radius and an obvious guess is that this radius is limited by a natural boundary in the generic case. It is natural to expect that the natural boundary moves outward from the unit circle as $\beta$ is decreased, and probably it gets deformed.

Even less is known if $f$ is intermittent, some results have recently been made \cite{34}. Except for the branch cut (see section IIIb) the zeta functions ($\beta = 0$) is holomorphic in a certain region enclosing the unit circle.

If the system is time reversible one expects pruning to be described by a pruning front which is the two-dimensional generalization of the kneading sequences\cite{22}. Moreover, if the system is time continuous the expanded zeta function is a Dirichlet series rather than a power series. Knowing that natural boundaries are abundant also in Dirichlet series we have at least to focus the possibility of the existence of natural boundaries in zeta functions of systems with no simple symbolic description.

\section*{b. Intermittency}

Intermittency means that the system switches between chaotic behaviour and quasi-integrable behaviour. We will consider a strong form of intermittency associated with the presence of (one or more) neutrally stable orbits. To illustrate how this affects the analyticity of the zeta function it is instructive to consider a one dimensional map $x \mapsto f(x)$ on the unit interval, with

$$f(x) = \begin{cases} x + 2sx^{1+s} & 0 \leq x < 1/2 \\ 2x - 1 & 1/2 \leq x \leq 1 \end{cases} ,$$

(16)

where $s > 0$, see fig 2. For $s = 0$ the map is just the binary shift map, which is uniformly hyperbolic, but for $s > 0$ it is intermittent; the fix point $x = 0$ is neutrally stable: $f'(0) = 1$.

The map admits a binary coding, as before we associate the letter $L$ with the left leg, and $R$ with the right leg. The neutral fix point now corresponds to the periodic orbit $\overline{L}$. The symbolic dynamics is by construction complete.

We will consider the zeta function (6) with the neutral fix point pruned

$$1/\zeta_f(z) = \prod_{\nu \neq \overline{L}} (1 - z^{\nu p} / |\Lambda_p|^{1-\beta}) .$$

(17)

We now suppress the parameter $\beta$. The new index indicates what map the zeta function refer to.

A power series representation of $1/\z$ is obtained by expanding this product: $1/\zeta_f(z) = \sum_n a_n z^n$. The nature of the leading singularity will be reflected in the asymptotics of the sequence $\{a_n\}$. To get an idea what this asymptotic behaviour may be we consider the zeta function of closely related map $\hat{f}$, a piecewise linear approximation of $f$. We define $\hat{f}$ as a continuous function, coinciding with $f$ on a sequence of points $\hat{f}(c_n) = f(c_n)$ where the $c_n$’s are
the inverse images of the critical point (cf fig 2)

\[ c_0 = 1/2 \]
\[ c_{n+1} = f_L^{-1}(c_n) \]

and linear in the intervals \([c_{n+1}, c_n]\).

It is relatively easy (see e.g. [35]) to show that the sequence \(\{c_n\}\) has the asymptotic behaviour

\[ c_n \sim n^{-1/s} \quad n \to \infty \quad . \]  

The construction of \(\hat{f}\) gives it a very simple cycle expansion

\[ 1/\zeta_f(z) = 1 - \sum_{n=0}^{\infty} \frac{z^{n+1}}{\Lambda_{L \cdot R}} \equiv \sum_n \hat{a}_n z^n \quad . \]

The stabilities \(\Lambda_{L \cdot R}\) are simply related to the \(c_n\)'s

\[ \Lambda_{L \cdot R} = 2 \frac{c_n - c_{n+1}}{c_0 - c_1} \quad , \]

with the asymptotic behaviour

\[ \Lambda_{L \cdot R} \sim n^{(s+1)/s} \quad . \]

The asymptotic behaviour of the coefficients are thus \(\hat{a}_i = O(i^{-(s+1)/s})\) and it seems likely that the same holds for the sequence \(\{a_i\}\). This suggest that \(1/\zeta_f(z)\) contains a singularity of the type

\[ (1 - z)^{\alpha - 1} \quad \alpha \notin N \]
\[ (1 - z)^{\alpha - 1} \log(1 - z) \quad \alpha \in N^+ \quad , \]

with

\[ \alpha(\beta, s) = \frac{(1 - \beta)(s + 1)}{s} \quad , \]

as can be realized through the Tauberian theorems for power series.

We will now move on to the Sinai billiard. In billiards with continuous time one cannot study zeta functions in complex \(z\) since the lengths \(L_p\) are not integer multiples of some unit length. Instead we replace \(z\) by \(\exp(-s)\) and formulate the the zeta functions

\[ Z(s) = \prod_p \left( 1 - \frac{e^{-sL_p}}{|\Lambda_p|^{1-\beta}} \right) \]

where \(p\) runs over all primitive unstable cycles. A semiclassical zeta function is formally obtained as \(\beta = 1/2\), we only need to reinsert the Maslov indices (which can be done properly by a multiplicative weight).

The presence of a simple symbolic dynamics in the previous example made it easy for us since (at least in the piecewise linear approximation) we only needed to consider one sequence
of periodic orbits accumulating at the neutral fixpoint. In the Sinai billiard there are neutral orbits in a (finite) number of directions. A sequence of periodic orbits accumulating towards a neutral orbit is indicated in fig 1, let us denote the sequence $q_n$. The stability of the cycles obey a power law $\Lambda_{q_n} \sim n^2$ asymptotically and since $T_{q_n} \sim n$ the local Lyapunov exponent $\log \Lambda_{q_n}/T_{q_n}$ goes to zero and we can again expect a branch cut. But the order of this singularity cannot be estimated as easily as in the previous one dimensional map. The reason is the lack of simple symbolic dynamics. There is a simple symbolic coding [37] but it is heavily pruned for cycles having segments close to the neutral orbits. We must rely on more elaborate methods to investigate the singularity.

In a series of paper [36, 37, 21, 38, 22, 24] we have investigated this problem within the framework of the BER (acronym for Baladi-Eckmann-Ruelle) approximation [39]. The basic idea is that for intermittent systems one can usually find a surface of section whose associated Poincaré map is not intermittent, ideally it is exponentially mixing. An approximate zeta function for the flow through this map can be formulated as a average over the surface of section

$$Z_{BER}(s) = 1 - \langle e^{-s \Delta_s(x_s)}|\Lambda(x_s)|^{\beta}\rangle_{x_s \in \text{sos}}, \quad (27)$$

where $\Delta_s(x_s)$ is the traveling length to the next intersection with the surface of section and $\Lambda(x_s)$ its associated instability, both being functions of the surface of section coordinate $x_s$.

In Hamiltonian chaotic system this average is easily performed since the invariant density is uniform. It is convenient to define

$$p_\beta(\Delta) = \langle \delta(\Delta - \Delta_s(x_s))|\Lambda(x_s)|^{\beta}\rangle_{x_s \in \text{sos}}. \quad (28)$$

For $\beta = 0$ this is just the (probability) distribution of recurrence times to the surface of section. The BER zeta function is obtained by a simple Laplace transform

$$Z_{BER}(s) = 1 - \int_0^\infty e^{-s\Delta} p_\beta(\Delta) d\Delta \quad (29)$$

In the Sinai billiard the obvious choice of surface of section is the disk. The function $p_\beta(\Delta)$ of the disk-to-disk map can be controlled well: for $\beta = 0$ we get in the limit $R \to 0$ [38]

$$p_0(\Delta) \sim 
\begin{cases} 
\frac{12R}{\pi^2} \frac{\delta}{\delta \xi} (2\xi + \xi(4 - 3\xi) \log(\xi) + 4(\xi - 1)^2 \log(\xi - 1) - (2 - \xi)^2 \log |2 - \xi|) & \xi < 1 \\
\frac{1}{\Delta^{3 - 3\beta/2}} & \xi > 1
\end{cases} \quad (30)$$

where $\xi = \Delta 2R$.

For finite $R$ we only have simple expression for the tail

$$p_\beta(\Delta) \sim \frac{1}{\Delta^{3 - 3\beta/2}} \quad (31)$$
The prefactor can be computed exactly for any disk radius $R$ but this is not our concern here. It is also well known the the mean is given by

$$\int_0^\infty \Delta p_{\beta=0}(\Delta) d\Delta = \frac{1}{2R} + O(R),$$

an expression we will need in section IIIc.

A power law tail of $p \sim \Delta^{-\alpha}$ thus gives a leading singularity

$$s^{\alpha-1} \quad \alpha \not\in \mathbb{N}$$

$$s^{\alpha-1} \log s \quad \alpha \in \mathbb{N}^+$$

to the BER zeta function.

How accurate the BER approximation is depend on the underlying map. For the Sinai billiard the disk to disk map is hyperbolic. It is not known if it is exponentially mixing but the correlation function can at least be bounded by a stretched exponential. We expect that the first few terms in the generalized series expansion is exactly reproduced by the BER approximation, at least we expect that

$$Z(s) = Z_{BER}(s) + O(s^2),$$

but higher terms are also described well by the BER approximation. For $\beta = 0$ this would imply that the leading singularity $s^2 \log s$ is exactly reproduced by the BER approximation. For the semiclassical case $\beta = 1/2$ we get a leading singularity of the form $s^{5/4}$ and thus an algebraic branch point at $s = 0$. The introduction of Maslov indices will affect the size but not the order of the singularity.

c. The role of singularities in classical zeta functions

What are classical zeta functions good for?

The most well known application is the relation between the zeros of $1/\zeta_{\beta=0}$ and the decay of correlations. This problem is rather complicated mathematically and we will instead consider the somewhat simpler problem of computing averages of chaotic systems [41]. In particular we will be interested in computing the diffusion constants but we’ll start by considering more general averages.

We assign a weight $w(x_0, t)$ to the trajectory starting at phase space point $x_0$ and evolving during time $t$ to point $x(x_0, t)$. There will be an important technical restriction on $w(x_0, t)$: it must be multiplicative along the flow, which means that $w(x_0, t_1 + t_2) = w(x_0, t_1)w(x(x_0, t_1), t_2)$. The phase space average of $w(x_0, t)$ may be expanded in terms of periodic orbits as

$$\lim_{t \to \infty} \langle w(x_0, t) \rangle = \lim_{t \to \infty} \sum_p T_p \sum_{r=1}^\infty w_p \delta(t - rT_p) |A_p|^r,$$

(35)
where \( r \) is the number of repetitions of primitive orbit \( p \) and \( w_p \) is the weight integrated along cycle \( p \). This trace may be formulated in terms of zeta functions in the following way

\[
\lim_{t \to \infty} \langle w(x_0,t) \rangle = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} e^{st} \frac{Z_w'(s)}{Z_w(s)} ds ,
\]

with the zeta function

\[
Z_w(s) = \prod_p \left( 1 - w_p e^{-sT_p} |\Lambda_p| \right) .
\]

In the previous section we studied the thermodynamic weight \( w_p = |\Lambda_p|^\beta \) but we will now turn to the problem of diffusion of interest for the present volume. To this end we follow \[42\] and introduce the weight

\[
w_{\text{diff}}(x_0,t) = e^{\bar{\beta} \cdot (\bar{x}(x_0,t) - \bar{x}_0)} ,
\]

where \( \bar{x} \) is the configuration space part of the phase space vector \( x \). The diffusion constant may now be expressed in terms of the associated zeta function \[42, 23\]

\[
D = \lim_{t \to \infty} \frac{1}{2t} \langle (\bar{x}(x_0,t) - \bar{x}_0)^2 \rangle = \lim_{t \to \infty} \frac{1}{2t} \left( \frac{d^2}{d\bar{\beta}_1^2} + \frac{d^2}{d\bar{\beta}_2^2} \right) (e^{\bar{\beta} \cdot (\bar{x}(x_0,t) - \bar{x}_0)}) |_{\beta=0}
\]

\[
= \lim_{t \to \infty} \frac{1}{2t} \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} e^{st} \left( \frac{d^2}{d\bar{\beta}_1^2} + \frac{d^2}{d\bar{\beta}_2^2} \right) \frac{d}{ds} \log Z_{\text{diff}}(s) |_{\beta=0} ds .
\]

The subscript \( \text{diff} \) will be suppressed form now on.

We now consider the Lorentz gas obtained by infinitely unfolding the Sinai billiard. The relevant zeta function has a BER approximation \[27\]

\[
Z_{\text{BER}}(s) = 1 - \langle e^{-s\Delta} e^{x_1 \beta_1 + x_2 \beta_2} \rangle = 1 - \langle e^{-s\Delta} \frac{1}{2\pi} \int_0^{2\pi} e^{\Delta \cos(\theta)} d\theta \rangle_{\text{sos}} ,
\]

where we assumed isotropy (which is not really necessary cf. refs. \[23, 17\]), and \( \Delta = \sqrt{x_1^2 + x_2^2} \). We only need to expand to second order in \( \beta \)

\[
Z_{\text{BER}}(s) = 1 - \langle e^{-s\Delta} \rangle - \frac{\beta^2}{4} \langle \Delta^2 e^{-s\Delta} \rangle \ldots
\]

We can now evaluate the averages by means of the distribution \( p_{\beta=0} \) discussed in section IIb.

\[
Z_{\text{BER}}(s) = 1 - \left( \frac{s}{2R} + O(s^2 \log s) \right) - \left( \frac{\beta^2}{4\pi^2 R^2} (\log s + O(1)) \right) \ldots .
\]

keeping only the leading terms in \( R \). Inserting tis expression into \(\[23\] we get as a result a diverging diffusion constant

\[
D = \lim_{t \to \infty} \frac{1}{2t} \frac{2}{\pi^2 R^2} \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} e^{st} \log s + O(1) \frac{ds}{s^2} ds = \frac{1}{\pi^2 R} (\log(t) + O(1)) ,
\]
in agreement with the suggested exact result \[17\] and with numerical simulations \[43\]. This is one of the reasons why we expect that 
\[ Z(s) = Z_{BER}(s) + O(s^2) . \]

This anomalous diffusion behaviour of this Lorentz gas is a consequence of the strongly intermittent properties of the Sinai billiard. Viewed as a Lorentz gas the diverging diffusion constant is due to the unbounded horizon, the point particle can make arbitrary long jumps between disks.

d. A short cut to the trace formula

In investigations of the accuracy of the trace formula for billiards the natural starting point is the boundary integral method \[14, 29\]. Below we will, from this method, briefly derive the trace formula for the Sinai billiard. The reason for this exercise is that it will help us understand the interplay between neutral and unstable orbits in the next section.

According to the boundary integral method the eigenvalues of the problem are those for which the following integral equation has a solution

\[ u(r(s)) = 2 \int_S \frac{\partial G}{\partial \hat{n}_s}(r(s), r(s')) u(r(s')) ds' . \]  \hspace{1cm} (44)

The function \( u(r(s)) \) is related to the wave function according to

\[ u(r(s)) = \frac{\partial \Psi(r(s))}{\partial \hat{n}_s} . \]  \hspace{1cm} (45)

The Green function is arbitrary but we now adopt the idea of ref. \[29\] and use the one-disk Green function, the integral \((44)\) should thus be performed only along the square boundary. The one disk Green function reads

\[ G(r_1, r_2, \Delta \theta) = \frac{i}{8} \sum_{\ell=-\infty}^{\infty} \left( H_{\ell}^{-}(kr_1) + S_{\ell}(kR) H_{\ell}^{+}(kr_1) \right) H_{\ell}^{+}(kr_2) e^{i\ell(\Delta \theta)} , \]  \hspace{1cm} (46)

where \( H_{\ell}^{\pm}(z) \) are Hankel functions and \( r_1, r_2 \) and \( \Delta \theta \) are explained in fig 3a. The phase shift function \( S_{\ell}(kR) \) is defined by

\[ S_{\ell}(kR) = \frac{H_{\ell}^{-}(kR)}{H_{\ell}^{+}(kR)} . \]  \hspace{1cm} (47)

Using Poisson resummation we get

\[ G(r_1, r_2, \Delta \theta) = \sum_{m=-\infty}^{\infty} G^{(m)}(r_1, r_2, \Delta \theta) , \]  \hspace{1cm} (48)

where

\[ G^{(m)}(r_1, r_2, \Delta \theta) = \frac{i}{8} \int_{-\infty}^{\infty} \left( H_{\ell}^{-}(kr_1) + S_{\ell}(kR) H_{\ell}^{+}(kr_1) \right) H_{\ell}^{+}(kr_2) e^{i\ell(\Delta \theta + 2\pi m)} d\ell . \]  \hspace{1cm} (49)
The standard semiclassical result are obtained by replacing all Hankel functions by their Debye approximation and taking the integral approximation $G^{(m=0)}$ of the sum (the higher terms in the Poisson resummation corresponds to creeping trajectories). The integral is then performed by stationary phase.

We will now introduce an unnecessary approximation. The reason is just to make the following reasoning more transparent. We will assume that $r_1, r_2 \gg R$. This works well for small radii $R$ because the end points in $G(r_1, r_2, \Delta \theta)$ is kept on the square boundary. In the semiclassical limit $k \to \infty$ the Green function is

$$G(r_1, r_2, \Delta \theta) \approx$$

$$\begin{cases} 
G_d(D) + G_d(R_1) \sqrt{4\pi k R} \cos \phi e^{-i\pi/4} G_d(R_2) & \Delta \theta < \arccos(R/r) + \arccos(R/r') \\
0 & \Delta \theta > \arccos(R/r) + \arccos(R/r')
\end{cases}$$

expressing that in the lit region the Green function is a sum between the direct trajectory and one reflected on the disk. The lengths $D$, $R_1$ and $R_2$ are explained in fig 3a. $G_d$ denote the asymptotic limit of the free (outgoing) Green function

$$G_d(L) = -\frac{i}{4} \sqrt{\frac{2}{\pi}} e^{i(kL-\pi/4)} \sqrt{kL}.$$  

In (50) we have neglected the transitional behaviour around $\Delta \theta \approx \arccos(R/r_1) + \arccos(R/r_2)$ which is referred to as the penumbra in [29], this is the subject of section IIIe.

We can write eq (44) symbolically as the matrix equation

$$(I - A)U = 0,$$  

having a solution only when $\det(I - A) = 0$. We can now express the integrated density of states as

$$N(k) = \sum_{i=1}^{\infty} \Theta(k - k_i) = -\frac{1}{\pi} \text{Im} \log \det(I - A)$$  

$$= -\frac{1}{\pi} \text{Im} \text{tr} \log(I - A) = -\frac{1}{\pi} \text{Im} \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}(A^n),$$

where

$$\text{tr}(A^n) = 2^n \int ds_1 \ldots ds_n \frac{\partial G}{\partial n_{s_1}}(r(s_1), r(s_2)) \ldots \frac{\partial G}{\partial n_{s_{n-1}}}(r(s_{n-1}), r(s_n)) \frac{\partial G}{\partial n_{s_n}}(r(s_n), r(s_1)).$$

Performing the normal derivative on (50) we get (for large $k$)

$$\frac{\partial G(r_1, r_2, \Delta \theta)}{\partial \theta} \sim (-ik) \cos(\theta_D)G_d(D) + (-ik) \cos(\theta_R)G_d(R_1) \sqrt{4\pi k R} \cos \phi e^{-i\pi/4} G_d(R_2),$$

provided of course that we are in the lit region $\Delta \theta < \arccos(R/r_1) + \arccos(R/r_2)$. The angles $\theta_D$ and $\theta_R$ are explained in fig 3b.
The integrals in (54) will select classical orbits obeying the classical reflection law. After performing the last integral in (54) only the periodic orbits will remain. To compute the contribution from classical orbits we will need to perform the following integrals by stationary phase analysis around a stationary point \( s = s_0 \)

\[
\int ds \ G_d(|r' - r(s)|)2(-ik)\cos(\theta)G_d(|r(s) - r''|) \sim -G_d(|r' - r(s_0)| + |r(s_0) - r''|). \tag{56}
\]

Periodic orbits come in two different types. Orbits never bouncing on a disk are neutrally stable (neutral orbits for short) and enter in one-parameter families. Orbits bouncing at least once on a disk are unstable. In the small \( R \) limit the stability eigenvalues are approximately

\[
\Lambda_p = \prod_{i=1}^{n_p} \frac{2l_i}{R \cos \phi_i} \tag{57}
\]

where \( l_i \) is the traveling length between the hits on the disk and \( \phi_i \) are the scattering angles for these hits.

The contribution to \( \text{tr}(A^n)/n \) from a primitive unstable periodic \( p \) with length \( L_p \) hitting the square boundary \( n_p \) times and traversed \( r = n/n_p \) times is

\[
\frac{1}{n} n_p \left( \prod_{i=1}^{n_p} \sqrt{4kR \cos(\phi_i)} \right) \sqrt{\frac{1}{8\pi k l_i}} r \ e^{ikrL_p} = \frac{1}{r |\Lambda_p|^{r/2}} e^{ikrL_p + \chi_p r\pi}. \tag{58}
\]

The factor \( n_p \) in front accounts for the number of periodic points on the square boundary and \( \chi_p \) counts the number of bounces along the orbit. The result above was obtained in the small \( R \) limit. For finite \( R \) the factor \(|\Lambda_p|^{r/2}\) should be replaced by \(|\Lambda_p|^{r/2}(1 - 1/\Lambda_p^r)\). The unstable orbits give rise to Gutzwiller’s trace formula which can be recasted to the Gutzwiller-Voros zeta function. The neglect of the factor \((1 - 1/\Lambda_p^r)\) is equivalent to the neglect of the higher \( m \) factors in (5) which is not our primary concern.

A primitive neutral orbit (or rather a one-parameter family of neutral orbits) of length \( L_q \) and geometric width \( D_q \) and repetition number \( r = n/n_q \) give the following contribution to \( \text{tr}(A^n)/n \)

\[
\frac{1}{n} n_q D_q \left( -\frac{i}{4} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{kL_q r}} \right) e^{i(kL_q r - \pi/4) + \chi_q r\pi}. \tag{59}
\]

Now only \( n - 1 \) integrals are computed by stationary phase, the last integral gives rise to the factor \( D_q \).

The important lesson from this section is that the dichotomy between unstable and neutral orbits in the trace formula is a direct consequence of eq. (50). In the next section we will see how the neglect of the quantum fuzziness in the twilight zone between the reflection and direct region in the Green function (50) allows singularities to enter the Gutzwiller-Voros zeta function.

e. How singularities sneaks into the Gutzwiller-Voros zeta function
We will consider scatterings in extreme forward angles so only the second term in the integral \((m=0)\) contributes. This means that we only need to evaluate the integral
\[
G^{(0)}(r_1, r_2, \Delta \theta) = \frac{i}{8} \int_{-\infty}^{\infty} S_\ell(kR) H^+_\ell (kr_1) H^+_\ell (kr_2) e^{i\ell \Delta \theta} d\ell ,
\]
where \(0 \ll \Delta \theta < \pi\). We can still safely use the Debye approximation for the Hankel functions \(H^+_\ell (kr)\) and \(H^+_\ell (kr')\)
\[
H^+_\ell (z) \sim \sqrt{\frac{2}{\pi \sqrt{z^2 - \ell^2}}} e^{i\sqrt{z^2 - \ell^2 - \ell \arccos(\ell/z) - \pi/4}} ,
\]
because the end points stay on the square boundary. However, the phase shift function needs a more careful analysis. The phase shift function \(S_\ell(kR)\) is of unit modulus and we call the phase \(\gamma(kR, \ell)\):
\[
S_\ell(kR) = -\frac{H^-_\ell (kR)}{H^+_\ell (kR)} \equiv e^{i\gamma(kR, \ell)} .
\]
The Green function now reads
\[
G^{(0)}(r_1, r_2, \Delta \theta) = \int_{-\infty}^{\infty} A(r_1, r_2) e^{i(\ell \Delta \theta - \Psi(\ell))} d\ell ,
\]
where \(A(r_1, r_2)\), coming from the Hankel functions \(H^+_\ell (kr_1)\) and \(H^+_\ell (kr_2)\), is a slowly varying amplitude so that the integral is in principle determined by the phase function
\[
\Psi(\ell) = -\gamma(kR, \ell) - [\sqrt{(kr)^2 - \ell^2 - \ell \arccos(\ell/(kr))] - [\sqrt{(kr')^2 - \ell^2 - \ell \arccos(\ell/(kr'))]} + \pi/2 ,
\]
and the stationary phase condition will simply read \(\Psi(\ell) = \Delta \theta\) (subscripts denote differentiation).

We will now investigate the phase of \(S_\ell(kR)\) in detail. To this end we will use the uniform approximation for Hankel functions relating the phase of Hankel functions to the phase of Airy functions \(\theta\) according to [45]
\[
S_\ell(kR) = -\frac{\text{Ai}(-x) + i\text{Bi}(-x)}{\text{Ai}(-x) - i\text{Bi}(-x)} \equiv e^{2\theta(x) - \pi} \equiv e^{i\gamma(\ell, kR)}
\]
where
\[
x(kR, \ell) = \begin{cases} 
\frac{3}{2} (\sqrt{(kR)^2 - \ell^2 - \ell \arccos(\ell/(kR))})^{2/3} & \ell < kR \\
-\frac{3}{2} (\ell \cdot \log(\ell + \sqrt{\ell^2 - (kR)^2})) - \sqrt{\ell^2 - (kR)^2})^{2/3} & \ell > kR
\end{cases} .
\]
If \(|kR/\ell - 1| \ll 1\) we can use
\[
x = \left(\frac{2}{\ell}\right)^{1/3} (kR - \ell) .
\]
Using this expression for \(x\) one gets the so called transition region approximation. This approximation fails to yield the Debye approximation as its asymptotic limit but the nice thing
is that there is a considerable overlap because whenever $x \ll \ell^{2/3}$ we can use the transition region approximation and when $x \gg 1$ we can use Debye. (The relevant $\ell$ for our consideration scales as $\ell \sim k$).

$\theta(x)$ is a complicated function but one has the following asymptotic expressions

\[
\theta(x) = \frac{\pi}{3} - \frac{3^{4/3}\Gamma(2/3)^2}{4\pi} x - \frac{3^{5/3}\Gamma(2/3)^3}{8\pi\Gamma(1/3)} x^2 + O(x^4) \quad x \to \pm 0
\]

\[
\theta(x) \sim \frac{\pi}{2} - \frac{1}{2} e^{-\frac{4}{3}(-x)^{3/2}} (1 + O(1/(-x)^{3/2})) \quad x \to -\infty ,
\]

\[
\theta(x) = \frac{\pi}{4} - \frac{2}{3} |x|^{3/2} + O(x^{-3/2}) \quad x \to +\infty
\]

so we obtain the Debye approximation when $x \to \infty$ as expected.

In fig 4 we plot the function $\Psi_\ell(\ell)$ for some arbitrary choose values of $r_1/R = r_2/R = 3$ and $kR = 10$ together with its Debye approximation. We have marked some critical values of $\Psi_\ell(\ell)$ in the plot:

- $\Delta \theta_{cl} = \arccos(R/r_1) + \arccos(R/r_2)$ denotes the limit of the classically illuminated region.
- $\Delta \theta_d$ is the limit of the standard semiclassical result on the direct side. It scales with $kR$ as $\Delta \theta_d = \Delta \theta_{cl} - O((kR)^{-2/3})$.
- $\Delta \theta_r$ is the corresponding limit on the reflection side. It scales as $\Delta \theta_r = \Delta \theta_{cl} - O((kR)^{-1/3})$.
- The maximum of the curve occurs at $\Delta \theta_{max} = \Delta \theta_{cl} - O((kR)^{-2/3} \log(kR)^{2/3})$ and at $\ell = \ell_{max}$. The second derivative of $\Psi_\ell$ at the maximum is $\Psi_{\ell\ell\ell}(\ell_{max}) = O((kR)^{-4/3} \log(kR)^{1/3})$.

If $\Delta \theta = \Delta \theta_{max} + \delta$ and $\delta$ is small, the Green function is still non zero but cannot be evaluated by stationary phase. This means that we cannot in an unambiguous way divide the Green function into one part associated with the direct ray and one with the reflected, as in (50), and something in the semiclassical description is definitely lost.

Close to the maximum the result can be approximated by the integral

\[
\int_{-\infty}^{\infty} e^{i(\delta \ell + \frac{1}{6} |\Psi_{\ell\ell\ell}(\ell-\ell_{max})|^3)} d\ell = 2\pi |\Psi_{\ell\ell\ell}/2|^{-1/3} \text{Ai}(\ell_{max}/2)^{-1/3} \delta
\]

(71)

From this we can get an estimate of the critical value of $\delta$ when stationary phase approximation ceases to be valid. This happens when the argument of the Airy function is $O(1)$. We can define a critical $\delta$ by

\[
\delta_{crit} |\Psi_{\ell\ell\ell}(\ell_{max})| = O(1) ,
\]

(72)

and we find that $\delta_{crit}$ are of the order

\[
\delta_{crit} = O((kR)^{-4/9} \log(kR)^{1/9}) .
\]

(73)

We only expect the Airy function approximation to work well close to the maximum since the curve $\Psi_{\ell}$ is very skew and badly approximated by a parabola.
In ref. [29], the authors choose to approximate the direct side $\ell > k$ by a Fresnel integral (this assuming constant slope of $\Psi_\ell$) but the treatment on the reflections side is incomplete.

Let us now consider the family $q_n$ of periodic orbits indicated in fig 1. We see that if $n$ is big enough the unstable periodic orbit will interfere with the neutral orbit and for large enough $n$ the corresponding terms in the Gutzwiller trace formula are suppressed. More precisely, it is the stationary points closest to the disk (cf. the one marked with a cross in fig 1) that suffers most of interference with the edge of the corresponding family of neutral orbits. In the same way as there is no longer an unambiguous division between a direct and a reflected ray there is no longer a corresponding division between neutral and unstable orbits in the trace formula, cf. section IIId. It is the neglect of this effect that allows the branch point singularity in the Gutzwiller-Voros zeta function.

The critical $n = n_{\text{crit}}$ when this interference problem gets crucial scales as

$$n_{\text{crit}} \sim (kR)^{2/3}.$$  

(74)

which is the same as the critical threshold of ref. [29] above which the majority of periodic orbits suffer from ‘diffraction’ effects.

It is this type of entanglement which makes semiclassical quantization of systems with mixed phase space so difficult. Our findings are very similar to those of ref.[19] concerning the Stadium billiard.

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Figure captions

Figure 1: The Sinai billiard (upper left corner) is the unit cell of the associated Lorentz gas. Two transparent direction are indicated (shaded areas). The family $q_n$ of periodic orbits accumulating towards the horizontal transparent direction is also indicated.

Figure 2: The intermittent map discussed in section IIIb. The sequence \( \{c_n\} \) is defined as the inverse images of the critical point as indicated.

Figure 3: The semiclassical limit of the Green function between two points on the border can be divided into two parts. One associated with the direct trajectory and one that is reflected on the disk.

Figure 4: The function $\Psi_\ell$ discussed in section IIIe together with its Debye approximation.