Modelling small strain behaviors of overconsolidated clays

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ABSTRACT

Firstly, a new elastic framework, called the elastic hysteretic stress-strain relationship, is established to evaluate the stiffness degradation at small strains and the hysteresis loop generated under cyclic loading condition. Secondly, in order to reasonably describe the phenomenon of isotropic compression curve of OC clay when approaching the normal consolidation line (NCL) under reloading, the unified hardening parameter used in the UH model is revised. Thirdly, a new overconsolidation parameter, which is crucial to make the UH model working with the elastic hysteretic stress-strain relationship effectively, is proposed. Then, a new small strain constitutive model, called the small strain UH (SSUH) model is presented in this paper. The proposed model is found to be able to describe highly nonlinear stress-strain relationship and stiffness degradation at small strains as well as the shear dilatancy and strain hardening/softening behaviors of OC clays at large strains. Besides, it needs only two additional parameter to consider the small strain behaviors compared with the modified Cam-clay (MCC) model.

Keywords: small strain, overconsolidated clays, constitutive model, stiffness degradation, nonlinearity

1 INTRODUCTION

Strains between 0.0001% and 0.1% are called small strain. In urban environments, many foundation pits and tunnels need to be excavated. The deformation of soils that around the excavations are mostly less than 0.1%, and the reasonable estimation of the ground deformation is the crucial element of the successful design and construction of geotechnical projects. Therefore, researchers conducted numerous tests on soils to study the small strain behaviors, and found that the small strain behaviors include highly nonlinear stress-strain relationship and stiffness degradation (e.g., Burland 1989, Viggiani & Atkinson 1995). There are several elasoplastic constitutive models have been proposed to describe some small strain behaviors (e.g., Stallebress & Taylor 1997, Whittle 1994, Benz 2009). However, these models all complex and have plethoric parameters. Hence it is necessary to build a new small strain constitutive model.

2 MODELLING OF ELASTIC HYSTERETIC STRESS-STRAIN BEHAVIOR

2.2 Elastic hysteresis formulation in general stress state

A perfect hysteresis elastic response of clay under cyclic loading is hypothesized, see Fig. 1. The elastic response of reloading curve (from point B to point A) is antisymmetric with the unloading curve (from point A to point B). The elastic hysteresis loop under cyclic loading has following characteristics:

1. At points A and B which are termed stress reversal points, the value of stiffness undergoes a sudden change.
2. Stiffness decreases with unloading (or reloading) and has the maximum value at stress reversal point. The elastic variable \( \kappa_r \) has the minimum value \( \kappa_0 \) at stress reversal point, and increases with unloading (or reloading), see Fig. 1. \( \kappa_0 \) is the slope of the unloading curve in \( e - \ln \rho \) space, \( \kappa_0 \) is the initial slope of the unloading curve.
3. For a cyclic loading, the elastic response is perfect hysteresis, and there is no irrecoverable...
deformation.

The relative volumetric strain \( \varepsilon^* \) and the elastic variable \( \kappa \) are introduced to describe the change of the elastic variable \( \kappa \). They are defined as

\[
\varepsilon^* = \sqrt{\frac{2}{3}(\varepsilon^v_0 - \varepsilon^v_0^*)},
\]

\[
\varepsilon^v_0 = \sqrt{(\varepsilon^v_0^*)^2 + (\varepsilon^s_0^*)^2}
\]

where \( \varepsilon^v_0 \) is the current volumetric strain; \( \varepsilon^v_0^* \) is the volumetric strain at stress reversal point; \( \varepsilon^s_0 \) is the deviator strain tensor; \( \varepsilon^v_0^* \) is the deviator strain tensor at stress reversal point.

Then, the relative strain \( \varepsilon^* \) describing the coupling of volumetric and shear strains can be defined as

\[
\varepsilon^* = \sqrt{\varepsilon^2 + (\varepsilon^s_0^*)^2}
\]

Therefore, the elastic variable \( \kappa \) can be defined as follows

\[
\kappa = \kappa_0 (1 + \delta)
\]

\[
\delta = \frac{\kappa}{\kappa_0} - 1 \tanh \left( w \varepsilon^* \right)
\]

where \( W \) is a material constant reflecting the degradation rate of \( \kappa \) from \( \kappa_0 \) to \( \kappa \) with increasing \( \varepsilon^* \); \( \tanh \) is the hyperbolic tangent function.

2.2 Definition of stress reversal point

Whittle & Michael (1994) suggested a scalar strain amplitude parameter \( \beta \) whose expression changes with the relative shear strain is defined in this paper as follows

\[
\beta = \begin{cases} 
\varepsilon^*_v: \varepsilon^*_v \neq 0 \\
\varepsilon^*_v: \varepsilon^*_v = 0
\end{cases}
\]

The stress reversal point is defined from the direction of the relative strain rate and appears when \( \Delta \beta < 0 \). Only when a stress reversal point has occurred, the relative strain reaches the zero value.

2.3 Elastic stress-strain relationship in \( p-q \) space

Based on Hooker’s law, the elastic strain-stress relationships in \( p-q \) space are expressed as follows

\[
d\varepsilon^*_v = \frac{\kappa}{(1 + \nu)\rho} dp
\]

\[
d\varepsilon^*_s = \frac{2(1 + \nu)\kappa_0}{9(1-2\nu)(1+\nu)\rho} dq
\]

where, \( \nu \) is initial void ratio; \( \nu \) is Poisson’s ratio; \( d\varepsilon^*_v \) is the elastic volumetric strain increment; \( d\varepsilon^*_s \) is the elastic deviator strain increment; \( dq \) is the deviator stress increment.

3 PLASTIC STRESS-STRAIN RELATIONSHIP

3.1 Yield surface and potential failure stress ratio \( M_f \)

![Fig. 3 Yield surface of the SSUH model](image)

The same elliptical yield surface of the UH model (Yao et al. 2009 and 2001) is adopted in this paper, see Fig. 3. The initial yield surface passes through the initial point \( A_0 (p_0, q_0) \) and the current yield surface passes through the current point \( A(p, q) \). The yield surface also can be written as follows

\[
f = \ln \frac{p}{p_0} + \ln \left( 1 + \frac{q^2}{M^2 p^2} \right) - \frac{1}{c_p} H = 0
\]

where, \( c_p = (\lambda - \kappa_0)/(1 + \kappa_0) \); \( \lambda \) is the slope of NCL; \( p_0 \) is corresponds to the length of the major axes of the initial yield ellipses when \( H = 0 \); \( \lambda \) is the deviator stress; \( M \) is the stress ratio at critical state; and \( \eta \) is the stress ratio \( q/p \). \( H \) is the hardening parameter.

As the elastic variable \( \kappa \) is a state variable in this paper, the formula of overconsolidation parameter \( R \) based on the constant elastic material \( \kappa \) is inappropriate here. Therefore, the new definition of \( R \)
needs to be proposed. In e-ln p space, see Fig. 4(a), \( P \) is the mean stress of the current point \( C \); and \( \overline{P} \) is the mean stress of the reference point \( \overline{C} \). Here, the void ratio in the reference point \( \overline{C} \) is the same as that in the current point \( C \). The OC parameter \( R \) is defined as the ratio of the current stress to its corresponding reference stress,

\[
R = \frac{P}{\overline{P}} \tag{9}
\]

From the formula of NCL, \( \overline{P} \) can be solved as

\[
\overline{P} = \exp\left(\frac{N - e}{\overline{\lambda}}\right) \tag{10}
\]

where, \( N \) is the void ratio on NCL when \( p = 1 \); \( e \) is the void ratio of the current point.

![Fig. 4 Current point and reference point in e-ln p space: (a) isotropic compression; (b) general stress compression](image)

Substituting Eq. (10) into Eq. (9), the OC parameter \( R \) of the current point \( C \) when \( \eta = 0 \) can be obtained as follows

\[
R = p \exp\left(\frac{e - N}{\overline{\lambda}}\right) \tag{11}
\]

However, when \( \eta \neq 0 \), like the current point \( B \) shown in Fig. 4(b), the reference stress \( \overline{P} \) of the reference point \( \overline{B} \) is

\[
\overline{P} = \exp\left(\frac{N - e - (\lambda - \kappa)\ln(1 + \eta^2/M^2)}{\overline{\lambda}}\right) \tag{12}
\]

Therefore, the OC parameter \( R \) reflecting the effect of shear stress can be written as follows

\[
R = p \exp\left(\frac{e - N + (\lambda - \kappa)\ln(1 + \eta^2/M^2)}{\overline{\lambda}}\right) \tag{13}
\]

The potential failure stress ratio \( M_f \) is given by Yao et al. [6]. \( M_f \) can be expressed as a function of the OC parameter \( R \) as follows

\[
M_f = \frac{[\frac{1}{R} + \frac{1}{R}] - \frac{1}{R}}{1 - \frac{1}{R}} \tag{14a}
\]

\[
M_f = \frac{M^2}{12(3 - M)} \tag{14b}
\]

When \( R = 1 \), the clay is in normally consolidated state, and \( M_f = M \); when \( 0 < R < 1 \), it depicts the overconsolidation state, and \( M_f > M \); \( M_f \) gradually approaches to \( M \) when \( R \) increases.

### 3.2 The revised unified hardening parameter \( H \)

The isotropic compression curve predicted by the UH model always shows a slow approach to the NCL. Hence, in order to enhance the capability of the UH model describing the phenomenon that the isotropic compression curve of OC clay fast approaches the NCL under reloading, the unified hardening parameter \( H \) is revised here. The formula of the revised unified hardening parameter is

\[
H = \int \frac{dH}{M^2} = \int \frac{1 - M_f^2 - \eta^2}{M^2 - \eta^2} d\epsilon^p \tag{15a}
\]

where

\[
\zeta = \frac{1}{R^{(\lambda - \kappa)}} \tag{15b}
\]

### 4 SSUH MODEL

#### 4.1 Constitutive relationship

The SSUH model is built by combining the elastic hysteretic stress-strain relationship with the plastic stress-strain relationship.

The elastic strain increment \( \Delta \epsilon^p \) is calculated by the Hooke’s law:

\[
\Delta \epsilon^p = \frac{1 + \nu}{3K(1 - 2\nu)} \Delta \sigma - \frac{\nu}{3K(1 - 2\nu)} \Delta \sigma_{\text{m}} \epsilon^p \tag{16}
\]

where, \( K \) is the elastic bulk modulus.

The plastic strain increment \( \Delta \epsilon^p \) can be expressed as

\[
\Delta \epsilon^p = \Lambda \frac{\partial f}{\partial \sigma^s} \tag{17}
\]

where the plastic factor \( \Lambda \) can be derived from the consistency condition

\[
\Lambda = \frac{1}{\zeta} \frac{M_f^2 - \eta^2}{M^2 - \eta^2} c\left(\frac{\partial f}{\partial \sigma^s} + \frac{2\eta}{M^2 - \eta^2} \Delta \sigma_{\text{m}} \right) \tag{18}
\]

Comparing Eq. (19) with the plastic factor of the UH model, one can visualize that an additional coefficient \( \zeta \) is included in Eq. (19). When \( R = 1 \), the same plastic factor as that in the UH model is obtained.

#### 4.2 Model parameters

Compared with the MCC model, the SSUH model needs only two additional parameters. They are initial slope of the unloading curve \( K_0 \) and material constant \( w \). The \( K_0 \) defines the initial unloading slope at very small strains in e-ln p space and determines the elastic bulk modulus at small strains. The initial shear stiffness \( G_0 \) can be derived as

\[
G_0 = \frac{3(1 - 2\nu)}{2(1 + \nu)} \frac{(1 + \epsilon_0)}{\kappa_0 (1 + \delta)} \tag{19}
\]

Therefore, an estimate of the value of \( K_0 \) can be obtained by Eq.(19) with the initial shear modulus measured immediately after reversal of loading in oedometer or triaxial test.

The material constant \( w \) is used to control the speed that stiffness decreases with increasing strain. An estimate of the value of \( w \) can be obtained by fitting the stiffness degradation curve from small strain measurements in triaxial tests. The stiffness at small
strains is almost a constant when $w = 0$, see Fig. 5, which means stiffness degradation at small strains is not considered in the model. At small strains, the larger the value of $w$ is, the faster the stiffness decays with increasing strain.

![Fig. 5 Curves of normalized stiffness decays with shear strain](image)

### 5 VERIFICATION

#### Table 1 Parameters of Speswhite kaolin clay

| $M$ | $A$  | $K'$ | $K''$ | $e_p$ | $V$ | $W$ |
|-----|------|------|-------|-------|-----|-----|
| 0.89| 0.18 | 0.035| 0.0042| 0.81  | 0.3 | 800 |

#### Table 2 Parameters of Shanghai soft clay

| $M$ | $A$  | $K'$ | $K''$ | $e_p$ | $V$ | $W$ |
|-----|------|------|-------|-------|-----|-----|
| 1.55| 0.173| 0.034| 0.0005| 1.2   | 0.3 | 150 |

![Fig. 6 Comparison between predicted and measured stress-strain response (data after Stallebrass & Taylor 1997)](image)

![Fig. 7 Stress-strain relationship at small strains of Shanghai soft clay (data after Jiang 2009)](image)

The SSUH model can describe the behaviors of OC clay at large strains just like the UH model can describe. This section only shows the capabilities of the SSUH model simulating the small strain behaviors. Fig. 6 shows the predicted and test results of the triaxial cyclic test on Speswhite kaolin clay with OCR=2.4. The parameters are shown in Table 1. One can see that the hysteresis loop is well simulated by the SSUH model.

Fig. 7 illustrates the predicted and test results of the triaxial compression/extension ($P = \text{const}$) test data on Shanghai soft clay. The parameters are shown in Table 2. The Highly stress-strain relationship and stiffness degradation of Shanghai soft clay at small strains are generally simulated by the SSUH model.

### 6 CONCLUSIONS

This paper proposed a new small strain framework by establishing the elastic hysteretic stress-strain relationship. Furthermore, the hardening parameter used in the UH model is revised to predict the phenomenon of isotropic compression curve of OC clay when approaching the normal consolidation line (NCL) under reloading. With the new overconsolidation parameter, the elastic hysteretic stress-strain relationship can efficiently work with the UH model and then the SSUH model is built. The SSUH model can describe highly nonlinear stress-strain relationship and stiffness degradation at small strains of OC clay. Compared with MCC, the proposed model needs only two additional parameter and the two parameters easily obtained from laboratory experiments.

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