Distinct Distances in Graph Drawings

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Abstract

The distance-number of a graph $G$ is the minimum number of distinct edge-lengths over all straight-line drawings of $G$ in the plane. This definition generalises many well-known concepts in combinatorial geometry. We consider the distance-number of trees, graphs with no $K_4$-minor, complete bipartite graphs, complete graphs, and cartesian products. Our main results concern the distance-number of graphs with bounded degree. We prove that $n$-vertex graphs with bounded maximum degree and bounded treewidth have distance-number in $O(\log n)$. To conclude such a logarithmic upper bound, both the degree and the treewidth need to be bounded. In particular, we construct graphs with treewidth 2 and polynomial distance-number. Similarly, we prove that there exist graphs with maximum degree 5 and arbitrarily large distance-number. Moreover, as $\Delta$ increases the existential lower bound on the distance-number of $\Delta$-regular graphs tends to $\Omega(n^{0.864138})$.

1 Introduction

This paper initiates the study of the minimum number of distinct edge-lengths in a drawing of a given graph$^1$. A degenerate drawing of a graph $G$ is a function that maps the

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$^1$We consider graphs that are simple, finite, and undirected. The vertex set of a graph $G$ is denoted by $V(G)$, and its edge set by $E(G)$. A graph with $n$ vertices, $m$ edges and maximum degree at most $\Delta$ is an $n$-vertex, $m$-edge, degree-$\Delta$ graph. A graph in which every vertex has degree $\Delta$ is $\Delta$-regular. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of $G$ induced by $S$, and let $G - S := G[V(G) \setminus S]$. For each vertex $v \in V(G)$, let $G - v := G - \{v\}$. Standard notation is used for graphs: complete graphs $K_n$, complete bipartite graphs $K_{m,n}$, paths $P_n$, and cycles $C_n$. A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. Throughout the paper, $c$ is a positive constant. Of course, different occurrences of $c$ might denote different constants.
vertices of $G$ to distinct points in the plane, and maps each edge $vw$ of $G$ to the open straight-line segment joining the two points representing $v$ and $w$. A drawing of $G$ is a degenerate drawing of $G$ in which the image of every edge of $G$ is disjoint from the image of every vertex of $G$. That is, no vertex intersects the interior of an edge. In what follows, we often make no distinction between a vertex or edge in a graph and its image in a drawing.

The distance-number of a graph $G$, denoted by $\text{dn}(G)$, is the minimum number of distinct edge-lengths in a drawing of $G$. The degenerate distance-number of $G$, denoted by $\text{ddn}(G)$, is the minimum number of distinct edge-lengths in a degenerate drawing of $G$. Clearly, $\text{ddn}(G) \leq \text{dn}(G)$ for every graph $G$. Furthermore, if $H$ is a subgraph of $G$ then $\text{ddn}(H) \leq \text{ddn}(G)$ and $\text{dn}(H) \leq \text{dn}(G)$.

1.1 Background and Motivation

The degenerate distance-number and distance-number of a graph generalise various concepts in combinatorial geometry, which motivates their study.

A famous problem raised by Erdős [15] asks for the minimum number of distinct distances determined by $n$ points in the plane\(^2\). This problem is equivalent to determining the degenerate distance-number of the complete graph $K_n$. We have the following bounds on $\text{ddn}(K_n)$, where the lower bound is due to Katz and Tardos [25] (building on recent advances by Solymosi and Tóth [47], Solymosi et al. [46], and Tardos [50]), and the upper bound is due to Erdős [15].

**Lemma 1 ([15, 25]).** The degenerate distance-number of $K_n$ satisfies

$$\Omega(n^{0.864137}) \leq \text{ddn}(K_n) \leq \frac{cn}{\sqrt{\log n}}.$$  

Observe that no three points are collinear in a (non-degenerate) drawing of $K_n$. Thus $\text{dn}(K_n)$ equals the minimum number of distinct distances determined by $n$ points in the plane with no three points collinear. This problem was considered by Szemerédi (see Theorem 13.7 in [37]), who proved that every such point set contains a point from which there are at least $\left\lceil \frac{2n-1}{3} \right\rceil$ distinct distances to the other points. Thus we have the next result, where the upper bound follows from the drawing of $K_n$ whose vertices are the points of a regular $n$-gon, as illustrated in Figure 1(a).

**Lemma 2 (Szemerédi).** The distance-number of $K_n$ satisfies

$$\left\lceil \frac{n-1}{3} \right\rceil \leq \text{dn}(K_n) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$  

Note that Lemmas 1 and 2 show that for every sufficiently large complete graph, the degenerate distance-number is strictly less than the distance-number. Indeed, $\text{ddn}(K_n) \in o(\text{dn}(K_n))$.

\(^2\)For a detailed exposition on distinct distances in point sets refer to Chapters 10–13 of the monograph by Pach and Agarwal [37].
Degenerate distance-number generalises another concept in combinatorial geometry. The unit-distance graph of a set $S$ of points in the plane has vertex set $S$, where two vertices are adjacent if and only if they are at unit-distance; see [23, 35, 36, 39, 42, 45] for example. The famous Hadwiger-Nelson problem asks for the maximum chromatic number of a unit-distance graph. Every unit-distance graph $G$ has $\text{ddn}(G) = 1$. But the converse is not true, since a degenerate drawing allows non-adjacent vertices to be at unit-distance. Figure 2 gives an example of a graph $G$ with $\text{dn}(G) = \text{ddn}(G) = 1$ that is not a unit-distance graph. In general, $\text{ddn}(G) = 1$ if and only if $G$ is isomorphic to a subgraph of a unit-distance graph.

The maximum number of edges in a unit-distance graph is an old open problem. The best construction, due to Erdős [15], gives an $n$-vertex unit-distance graph with $n^{1+c/\log\log n}$ edges. The best upper bound on the number of edges is $cn^{4/3}$, due to Spencer et al. [48]. (Szekely [49] found a simple proof for this upper bound based on the crossing lemma.)

More generally, many recent results in the combinatorial geometry literature provide upper bounds on the number of times the $d$ most frequent inter-point distances can occur.
between a set of \( n \) points. Such results are equivalent to upper bounds on the number of edges in an \( n \)-vertex graph with degenerate distance number \( d \). This suggests the following extremal function. Let \( \text{ex}(n, d) \) be the maximum number of edges in an \( n \)-vertex graph \( G \) with \( \text{ddn}(G) \leq d \).

Since every graph \( G \) is the union of \( \text{ddn}(G) \) subgraphs of unit-distance graphs, the above result by Spencer et al. [48] implies:

**Lemma 3 (Spencer et al. [48]).**

\[
\text{ex}(n, d) \leq cdn^{4/3}.
\]

Equivalently, the distance-numbers of every \( n \)-vertex \( m \)-edge graph \( G \) satisfy

\[
\text{dn}(G) \geq \text{ddn}(G) \geq cnm^{-4/3}.
\]

Results by Katz and Tardos [25] (building on recent advances by Solymosi and Tóth [47], Solymosi et al. [46], and Tardos [50]) imply:

**Lemma 4 (Katz and Tardos [25]).**

\[
\text{ex}(n, d) \in \mathcal{O}(n^{1.457341}d^{0.627977}).
\]

Equivalently, the distance-numbers of every \( n \)-vertex \( m \)-edge graph \( G \) satisfy

\[
\text{dn}(G) \geq \text{ddn}(G) \in \Omega(m^{1.592412}n^{-2.320687}).
\]

Note that Lemma 4 improves upon Lemma 3 whenever \( \text{ddn}(G) > n^{1/3} \). Also note that Lemma 4 implies the lower bound in Lemma 2.

### 1.2 Our Results

The above results give properties of various graphs defined with respect to the inter-point distances of a set of points in the plane. This paper, which is more about graph drawing than combinatorial geometry, reverses this approach, and asks for a drawing of a given graph with few inter-point distances.

Our first results provide some general families of graphs, namely trees and graphs with no \( K_4 \)-minor, that are unit-distance graphs (Section 2). Here \( K_4^- \) is the graph obtained from \( K_4 \) by deleting one edge. Then we give bounds on the distance-numbers of complete bipartite graphs (Section 3).

Our main results concern graphs of bounded degree (Section 4). We prove that for all \( \Delta \geq 5 \) there are degree-\( \Delta \) graphs with unbounded distance-number. Moreover, for \( \Delta \geq 7 \) we prove a polynomial lower bound on the distance-number (of some degree-\( \Delta \) graph) that tends to \( \Omega(n^{0.864138}) \) for large \( \Delta \). On the other hand, we prove that graphs with bounded degree and bounded treewidth have distance-number in \( \mathcal{O}(\log n) \). Note that bounded treewidth alone does not imply a logarithmic bound on distance-number since \( K_{2,n} \) has treewidth 2 and degenerate distance-number \( \Theta(\sqrt{n}) \) (see Section 3).

Then we establish an upper bound on the distance-number in terms of the bandwidth (Section 5). Then we consider the distance-number of the cartesian product of graphs (Section 6). We conclude in Section 7 with a discussion of open problems related to distance-number.
1.3 Higher-Dimensional Relatives

Graph invariants related to distances in higher dimensions have also been studied. Erdős, Harary, and Tutte [16] defined the dimension of a graph $G$, denoted by $\dim(G)$, to be the minimum integer $d$ such that $G$ has a degenerate drawing in $\mathbb{R}^d$ with straight-line edges of unit-length. They proved that $\dim(K_n) = n - 1$, the dimension of the $n$-cube is 2 for $n \geq 2$, the dimension of the Peterson graph is 2, and $\dim(G) \leq 2 \cdot \chi(G)$ for every graph $G$. (Here $\chi(G)$ is the chromatic number of $G$.) The dimension of complete 3-partite graphs and wheels were determined by Buckley and Harary [10].

The unit-distance graph of a set $S \subseteq \mathbb{R}^d$ has vertex set $S$, where two vertices are adjacent if and only if they are at unit-distance. Thus $\dim(G) \leq d$ if and only if $G$ is isomorphic to a subgraph of a unit-distance graph in $\mathbb{R}^d$. Maehara [32] proved for all $d$ there is a finite bipartite graph (which thus has dimension at most 4) that is not a unit-distance graph in $\mathbb{R}^d$. This highlights the distinction between dimension and unit-distance graphs. Maehara [32] also proved that every finite graph with maximum degree $\Delta$ is a unit-distance graph in $\mathbb{R}^{\Delta(\Delta^2-1)/2}$, which was improved to $\mathbb{R}^{2\Delta}$ by Maehara and Rödl [33]. These results are in contrast to our result that graphs of bounded degree have arbitrarily large distance-number.

A graph is $d$-realizable if, for every mapping of its vertices to (not-necessarily distinct) points in $\mathbb{R}^p$ with $p \geq d$, there exists such a mapping in $\mathbb{R}^d$ that preserves edge-lengths. For example, $K_3$ is 2-realizable but not 1-realizable. Belk and Connelly [6] and Belk [5] proved that a graph is 2-realizable if and only if it has treewidth at most 2. They also characterized the 3-realizable graphs as those with no $K_5$-minor and no $K_{2,2,2}$-minor.

2 Some Unit-Distance Graphs

This section shows that certain families of graphs are unit-distance graphs. The proofs are based on the fact that two distinct circles intersect in at most two points. We start with a general lemma. A graph $G$ is obtained by pasting subgraphs $G_1$ and $G_2$ on a cut-vertex $v$ of $G$ if $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$.

Lemma 5. Let $G$ be the graph obtained by pasting subgraphs $G_1$ and $G_2$ on a vertex $v$. Then:

(a) if $\ddn(G_1) = \ddn(G_2) = 1$ then $\ddn(G) = 1$, and
(b) if $\dn(G_1) = \dn(G_2) = 1$ then $\dn(G) = 1$.

Proof. We prove part (b). Part (a) is easier. Let $D_i$ be a drawing of $G_i$ with unit-length edges. Translate $D_2$ so that $v$ appears in the same position in $D_1$ and $D_2$. A rotation of $D_2$ about $v$ is bad if its union with $D_1$ is not a drawing of $G$. That is, some vertex in $D_2$ coincides with the closure of some edge of $D_1$, or vice versa. Since $G$ is finite, there are only finitely many bad rotations. Since there are infinitely many rotations, there exists a rotation that is not bad. That is, there exists a drawing of $G$ with unit-length edges. \qed

We have a similar result for unit-distance graphs.
Lemma 6. Let $G_1$ and $G_2$ be unit-distance graphs. Let $G$ be the (abstract) graph obtained by pasting $G_1$ and $G_2$ on a vertex $v$. Then $G$ is isomorphic to a unit-distance graph.

Proof. The proof is similar to the proof of Lemma 5, except that we must ensure that the distance between vertices in $G_1 - v$ and vertices in $G_2 - v$ (which are not adjacent) is not 1. Again this will happen for only finitely many rotations. Thus there exists a rotation that works. \hfill \square

Since every tree can be obtained by pasting a smaller tree with $K_2$, Lemma 6 implies that every tree is a unit-distance graph. The following is a stronger result.

Lemma 7. Every tree $T$ has a crossing-free\footnote{A drawing is crossing-free if no pair of edges intersect.} drawing in the plane such that two vertices are adjacent if and only if they are unit-distance apart.

Proof. For a point $v = (x(v), y(v))$ in the plane, let $v_\downarrow$ be the ray from $v$ to $(x(v), -\infty)$. We proceed by induction on $n$ with the following hypothesis: Every tree $T$ with $n$ vertices has the desired drawing, such that the vertices have distinct $x$-coordinates, and for each vertex $u$, the ray $u_\downarrow$ does not intersect $T$. The statement is trivially true for $n \leq 2$. For $n > 2$, let $v$ be a leaf of $T$ with parent $p$. By induction, $T - v$ has the desired drawing. Let $w$ be a vertex of $T - v$, such that no vertex has its $x$-coordinate between $x(p)$ and $x(w)$. Thus the drawing of $T - v$ does not intersect the open region $R$ of the plane bounded by the two rays $p_\downarrow$ and $w_\downarrow$, and the segment $pw$. Let $A$ be the intersection of $R$ with the unit-circle centred at $p$. Thus $A$ is a circular arc. Place $v$ on $A$, so that the distance from $v$ to every vertex except $p$ is not 1. This is possible since $A$ is infinite, and there are only finitely many excluded positions on $A$ (since $A$ intersects a unit-circle centred at a vertex except $p$ in at most two points). Since there are no elements of $T - v$ in $R$, there are no crossings in the resulting drawing and the induction invariants are maintained for all vertices of $T$. \hfill \square

Recall that $K_4^-$ is the graph obtained from $K_4$ by deleting one edge.

Lemma 8. Every 2-connected graph $G$ with no $K_4^-$-minor is a cycle.
Proof. Suppose on the contrary that $G$ has a vertex $v$ of degree at least 3. Let $x,y,z$ be the neighbours of $v$. There is an $xy$-path $P$ avoiding $v$ (since $G$ is 2-connected) and avoiding $z$ (since $G$ is $K_4^-$-minor free). Similarly, there is an $xz$-path $Q$ avoiding $v$. If $x$ is the only vertex in both $P$ and $Q$, then the cycle $(x,P,y,v,z,Q)$ plus the edge $xv$ is a subdivision of $K_4^-$. Now assume that $P$ and $Q$ intersect at some other vertex. Let $t$ be the first vertex on $P$ starting at $x$ that is also in $Q$. Then the cycle $(x,Q,z,v)$ plus the sub-path of $P$ between $x$ and $t$ is a subdivision of $K_4^-$. This contradiction proves that $G$ has no vertex of degree at least 3. Since $G$ is 2-connected, $G$ is a cycle, as desired. 

Theorem 1. Every $K_4^-$-minor-free graph $G$ has a drawing such that vertices are adjacent if and only if they are unit-distance apart. In particular, $G$ is isomorphic to a unit-distance graph and $\text{ddn}(G) = \text{dn}(G) = 1$.

Proof. By Lemma 6, we can assume that $G$ is 2-connected. Thus $G$ is a cycle by Lemma 8. The result follows since $C_n$ is a unit-distance graph (draw a regular $n$-gon).

3 Complete Bipartite Graphs

This section considers the distance-numbers of complete bipartite graphs $K_{m,n}$. Since $K_{1,n}$ is a tree, $\text{ddn}(K_{1,n}) = \text{dn}(K_{1,n}) = 1$ by Lemma 7. The next case, $K_{2,n}$, is also easily handled.

Lemma 9. The distance-numbers of $K_{2,n}$ satisfy

$$\text{ddn}(K_{2,n}) = \text{dn}(K_{2,n}) = \left\lceil \sqrt{\frac{n}{2}} \right\rceil.$$ 

Proof. Let $G = K_{2,n}$ with colour classes $A = \{v,w\}$ and $B$, where $|B| = n$. We first prove the lower bound, $\text{ddn}(K_{2,n}) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil$. Consider a degenerate drawing of $G$ with $\text{ddn}(G)$ edge-lengths. The vertices in $B$ lie on the intersection of $\text{ddn}(G)$ concentric circles centered at $v$ and $\text{ddn}(G)$ concentric circles centered at $w$. Since two distinct circles intersect in at most two points, $n \leq 2 \text{ddn}(G)^2$. Thus $\text{ddn}(K_{2,n}) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil$.

For the upper bound, position $v$ at $(1,0)$ and $w$ at $(0,1)$. As illustrated in Figure 4, draw $\left\lceil \sqrt{\frac{n}{2}} \right\rceil$ circles centered at each of $v$ and $w$ with radii ranging strictly between 1 and 2, such that the intersections of the circles together with $v$ and $w$ define a set of points with no three points collinear. (This can be achieved by choosing the radii iteratively, since for each circle $C$, there are finitely many forbidden values for the radius of $C$.) Each pair of non-concentric circles intersect in at most two points, $n \leq 2 \text{ddn}(G)^2$. Thus $\text{ddn}(K_{2,n}) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil$.

Now we determine $\text{ddn}(K_{3,n})$ to within a constant factor.

Lemma 10. The degenerate distance-number of $K_{3,n}$ satisfies

$$\left\lceil \sqrt{\frac{n}{2}} \right\rceil \leq \text{ddn}(K_{3,n}) \leq 3 \left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1.$$
Figure 4: Illustration for the proof of Lemma 9.

Proof. The lower bound follows from Lemma 9 since \( K_{2,n} \) is a subgraph of \( K_{3,n} \).

Now we prove the upper bound. Let \( A \) and \( B \) be the colour classes of \( K_{3,n} \), where \( |A| = 3 \) and \( |B| = n \). Place the vertices in \( A \) at \((-1,0), (0,0), \) and \((1,0)\). Let \( d := \lceil \sqrt{\frac{n}{2}} \rceil \).

For \( i \in [d] \), let

\[
r_i := \sqrt{1 + \frac{i}{d+1}}.
\]

Note that \( 1 < r_i < 2 \). Let \( R_i \) be the circle centred at \((-1,0)\) with radius \( r_i \). For \( j \in [d] \), let \( S_j \) be the circle centred at \((1,0)\) with radius \( r_j \). Observe that each pair of circles \( R_i \) and \( S_j \) intersect in exactly two points. Place the vertices in \( B \) at the intersection points of these circles. This is possible since \( 2d^2 \geq n \).

Let \((x, y)\) and \((x, -y)\) be the two points where \( R_i \) and \( S_j \) intersect. Thus \((x+1)^2+y^2 = r_i^2\) and \((x-1)^2+y^2 = r_j^2\). It follows that

\[
x^2 + y^2 = \frac{i}{d+1} + 2x = \frac{j}{d+1} - 2x.
\]

Thus \( 2(x^2 + y^2) = \frac{i+j}{d+1} \). That is, the distance from \((x, y)\) to \((0,0)\) equals

\[
\sqrt{\frac{i+j}{2d+2}},
\]

which is the same distance from \((x, -y)\) to \((0,0)\). Thus the distance from each vertex in \( B \) to \((0,0)\) is one of \( 2d-1 \) values (determined by \( i+j \)). The distance from each vertex in \( B \) to \((-1,0)\) and to \((1,0)\) is one of \( d \) values. Hence the degenerate distance-number of \( K_{3,n} \) is at most \( 3d-1 = 3 \lceil \sqrt{\frac{n}{2}} \rceil - 1 \). \( \square \)

Now consider the distance-number of a general complete bipartite graph.
Lemma 11. For all \( n \geq m \), the distance-numbers of \( K_{m,n} \) satisfy
\[
\Omega\left(\frac{mn}{(m+n)^{1.457341}}\right)^{(1/0.627977)} \leq ddn(K_{m,n}) \leq dn(K_{m,n}) \leq \left\lceil \frac{n}{2} \right\rceil .
\]
In particular,
\[
\Omega(n^{0.864137}) \leq ddn(K_{n,n}) \leq dn(K_{n,n}) \leq \left\lceil \frac{n}{2} \right\rceil .
\]

Proof. The lower bounds follow from Lemma 4. For the upper bound on \( dn(K_{n,n}) \), position the vertices on a regular \( 2n \)-gon \((v_1, v_2, \ldots, v_{2n})\) alternating between the colour classes, as illustrated in Figure 1(b). In the resulting drawing of \( K_{n,n} \), the number of edge-lengths is \(|\{(i + j) \mod n : v_i v_j \in E(K_{n,n})\}|\). Since \( v_i v_j \) is an edge if and only if \( i + j \) is odd, the number of edge-lengths is \( \left\lceil \frac{2n}{2} \right\rceil \). The upper bound on \( dn(K_{n,m}) \) follows since \( K_{n,m} \) is a subgraph of \( K_{n,n} \).

4 Bounded degree graphs

Lemma 9 implies that if a graph has two vertices with many common neighbours then its distance-number is necessarily large. Thus it is natural to ask whether graphs of bounded degree have bounded distance-number. This section provides a negative answer to this question.

4.1 Bounded degree graphs with \( \Delta \geq 7 \)

This section proves that for all \( \Delta \geq 7 \) there are \( \Delta \)-regular graphs with unbounded distance-number. Moreover, the lower bound on the distance-number is polynomial in the number of vertices. The basic idea of the proof is to show that there are more \( \Delta \)-regular graphs
than graphs with bounded distance-number; see [4, 13, 14, 38] for other examples of this paradigm.

It will be convenient to count labelled graphs. Let $\mathcal{G}(n, \Delta)$ denote the family of labelled $\Delta$-regular $n$-vertex graphs. Let $\mathcal{G}(n, m, d)$ denote the family of labelled $n$-vertex $m$-edge graphs with degenerate distance-number at most $d$. Our results follow by comparing a lower bound on $|\mathcal{G}(n, \Delta)|$ with an upper bound on $|\mathcal{G}(n, m, d)|$ with $m = \frac{\Delta n}{2}$, which is the number of edges in a $\Delta$-regular $n$-vertex graph.

The lower bound in question is known. In particular, the first asymptotic bounds on the number of labelled $\Delta$-regular $n$-vertex graphs were independently determined by Bender and Canfield [7] and Wormald [52]. McKay [34] further refined these results. We will use the following simple lower bound derived by Barát et al. [4] from the result of McKay [34].

**Lemma 12 ([4, 7, 34, 52]).** For all integers $\Delta \geq 1$ and $n \geq c\Delta$, the number of labelled $\Delta$-regular $n$-vertex graphs satisfies

$$|\mathcal{G}(n, \Delta)| \geq \left(\frac{n}{3\Delta}\right)^{\Delta n/2}. $$

The proof of our upper bound on $|\mathcal{G}(n, m, d)|$ uses the following special case of the Milnor-Thom theorem by Rónyai et al. [43]. Let $\mathcal{P} = (P_1, P_2, \ldots, P_t)$ be a sequence of polynomials on $p$ variables over $\mathbb{R}$. The zero-pattern of $\mathcal{P}$ at $u \in \mathbb{R}^p$ is the set $\{i : 1 \leq i \leq t, P_i(u) = 0\}$.

**Lemma 13 ([43]).** Let $\mathcal{P} = (P_1, P_2, \ldots, P_t)$ be a sequence of polynomials of degree at most $\delta \geq 1$ on $p \leq t$ variables over $\mathbb{R}$. Then the number of zero-patterns of $\mathcal{P}$ is at most $(\delta^t)^{(\binom{p}{\delta})}$.

Recall that $\text{ex}(n, d)$ is the maximum number of edges in an $n$-vertex graph $G$ with $\text{ddn}(G) \leq d$. Bounds on this function are given in Lemmas 3 and 4. Our upper bound on $|\mathcal{G}(n, m, d)|$ is expressed in terms of $\text{ex}(n, d)$.

**Lemma 14.** The number of labelled $n$-vertex $m$-edge graphs with $\text{ddn}(G) \leq d$ satisfies

$$|\mathcal{G}(n, m, d)| \leq \left(\frac{\text{end}}{2}\right)^{2n+d} \binom{\text{ex}(n, d)}{m},$$

where $e$ is the base of the natural logarithm.

**Proof.** Let $V(G) = \{1, 2, \ldots, n\}$ for every $G \in \mathcal{G}(n, m, d)$. For every $G \in \mathcal{G}(n, m, d)$, there is a point set

$$S(G) = \{(x_i(G), y_i(G)) : 1 \leq i \leq n\}$$

and a set of edge-lengths

$$L(G) = \{\ell_k(G) : 1 \leq k \leq d\},$$

such that $G$ has a degenerate drawing in which each vertex $i$ is represented by the point $(x_i(G), y_i(G))$ and the length of each edge in $E(G)$ is in $L(G)$. Fix one such degenerate drawing of $G$.

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For all $i, j, k$ with $1 \leq i < j \leq n$ and $1 \leq k \leq d$, and for every graph $G \in \mathcal{G}(n, m, d)$, define

$$P_{i,j,k}(G) := (x_j(G) - x_i(G))^2 + (y_j(G) - y_k(G))^2 - \ell_k(G)^2.$$ 

Consider $\mathcal{P} := \{P_{i,j,k}: 1 \leq i < j \leq n, 1 \leq k \leq d\}$ to be a set of $\binom{n}{2}d$ degree-2 polynomials on the set of $2n + d$ variables $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, \ell_1, \ell_2, \ldots, \ell_d\}$. Observe that $P_{i,j,k}(G) = 0$ if and only if the distance between vertices $i$ and $j$ in the degenerate drawing of $G$ is $\ell_k(G)$.

Recall the well-known fact that $\binom{n}{2} \leq \left(\frac{en}{2}\right)^d$. By Lemma 13 with $t = \binom{n}{2}d$, $\delta = 2$ and $p = 2n + d$, the number of zero-patterns determined by $\mathcal{P}$ is at most

$$\left(\frac{2\binom{n}{2}d}{2n + d}\right)^{2n+d} \leq \left(\frac{2e\binom{n}{2}d}{2n + d}\right)^{2n+d} < \left(\frac{en^2d}{2n + d}\right)^{2n+d} < \left(\frac{en^2d}{2n}\right)^{2n+d} = \left(\frac{en}{2}\right)^{2n+d}.$$ 

Fix a zero-pattern $\sigma$ of $\mathcal{P}$. Let $\mathcal{G}_\sigma$ be the set of graphs $G$ in $\mathcal{G}(n, m, d)$ such that $\sigma$ is the zero-pattern of $\mathcal{P}$ evaluated at $G$. To bound $|\mathcal{G}(n, m, d)|$ we now bound $|\mathcal{G}_\sigma|$. Let $H_\sigma$ be the graph with vertex set $V(H_\sigma) = \{1, \ldots, n\}$ and edge set $E(H_\sigma)$ where $ij \in E(H_\sigma)$ if and only if $ij \in E(G)$ for some $G \in \mathcal{G}_\sigma$. Consider a degenerate drawing of an arbitrary graph $G \in \mathcal{G}_\sigma$ on the point set $S(G)$. By $(\ast)$, $S(G)$ and $L(G)$ define a degenerate drawing of $H$ with $d$ edge-lengths. Thus $d \cdot d \cdot n \cdot (H_\sigma) \leq d$ and by assumption, $|E(H_\sigma)| \leq ex(n, d)$. Since every graph in $\mathcal{G}_\sigma$ is a subgraph of $H_\sigma$, $|\mathcal{G}_\sigma| \leq \binom{|E(H_\sigma)|}{m}$.

Therefore,

$$|\mathcal{G}(n, m, d)| \leq \left(\frac{en}{2}\right)^{2n+d} \binom{|E(H_\sigma)|}{m} \leq \left(\frac{en}{2}\right)^{2n+d} \binom{ex(n, d)}{m},$$

as required. 

By comparing the lower bound in Lemma 12 and the upper bound in Lemma 14 we obtain the following result.

**Lemma 15.** Suppose that for some real numbers $\alpha$ and $\beta$ with $\beta > 0$ and $1 < \alpha < 2 < \alpha + \beta$,

$$ex(n, d) \in \mathcal{O}(n^\alpha d^\beta).$$

Then for every integer $\Delta > \frac{4}{2-\alpha}$, for all $\epsilon > 0$, and for all sufficiently large $n > n(\alpha, \beta, \Delta, \epsilon)$, there exists a $\Delta$-regular $n$-vertex graph $G$ with degenerate distance-number

$$ddn(G) > \frac{n^{2-\alpha} - (2-n+\beta)(4+2\epsilon)}{\beta^2 \Delta^{1+\beta}}.$$ 

**Proof.** In this proof, $\alpha$, $\beta$, $\Delta$ and $\epsilon$ are fixed numbers satisfying the assumptions of the lemma. Let $d$ be the maximum degenerate distance number of a graph in $\mathcal{G}(n, \Delta)$. The result will follow by showing that for all sufficiently large $n > n(\alpha, \beta, \Delta, \epsilon)$,

$$d > \frac{n^{2-\alpha} - (2-n+\beta)(4+2\epsilon)}{\beta^2 \Delta^{1+\beta}}.$$
By the definition of $d$, and since every $\Delta$-regular $n$-vertex graph has $\frac{n^2}{2}$ edges, every graph in $G(n, \Delta)$ is also in $G(n, \frac{\Delta n}{2}, d)$. By Lemma 12 with $n \geq c\Delta$, and by Lemma 14,
\[
\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq |G(n, \Delta)| \leq |G(n, \frac{\Delta n}{2}, d)| \leq \left(\frac{\text{end}}{2}\right)^{2n+d} \left(\frac{\text{ex}(n, d)}{\Delta n/2}\right)^{\Delta n/2}.
\]
Since $\text{ex}(n, d) \in \mathcal{O}(n^\alpha d^\beta)$, and since $d$ is a function of $n$, there is a constant $c$ such that $\text{ex}(n, d) \leq cn^\alpha d^\beta$ for sufficiently large $n$. Thus (and since $\left(\frac{a}{b}\right) \leq \left(\frac{an}{bn}\right)^b$),
\[
\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{\text{end}}{2}\right)^{2n+d} \left(c n^\alpha d^\beta\right)^{\Delta n/2} \leq \left(\frac{\text{end}}{2}\right)^{2n+d} \left(\frac{2ecn^\alpha d^\beta}{\Delta n}\right)^{\Delta n/2}.
\]
Hence
\[
n^\Delta \leq 3\Delta \left(\frac{\text{end}}{2}\right)^{4n+2d} \left(2ecn^\alpha d^\beta\right)^{\Delta n}.
\]
By Lemma 2, $d \leq \text{ddn}(K_n) \leq \frac{cn}{\log n}$, implying $2d \leq \varepsilon n$ for all large $n > n(\varepsilon)$. Thus
\[
n^\Delta \leq 3\Delta \left(\frac{\text{end}}{2}\right)^{4+\varepsilon} \left(2ecn^\alpha d^\beta\right)^{\Delta n}.
\]
Hence
\[
n^{(2-\alpha)\Delta-4-\varepsilon} \leq 3\Delta \left(\frac{e}{2}\right)^{4+\varepsilon} \left(2ec\right)^{\Delta d^\beta+4+\varepsilon}.
\]
Observe that $3\Delta \left(\frac{e}{2}\right)^{4+\varepsilon} \left(2ec\right)^{\Delta d^\beta} \leq n^\varepsilon$ for all large $n > n(\Delta, \varepsilon)$. Thus
\[
n^{(2-\alpha)\Delta-4-2\varepsilon} \leq d^{\beta\Delta+4+\varepsilon}.
\]
Hence
\[
d \geq n^{\frac{(2-\alpha)\Delta-4-2\varepsilon}{\beta\Delta+4+\varepsilon}} = n^{\frac{2-\alpha}{\beta} \cdot \frac{(2-\alpha)\Delta(4+\varepsilon)+\beta\varepsilon}{\beta\Delta+4+\varepsilon}} > n^{\frac{2-\alpha}{\beta} \cdot \frac{(2-\alpha)\Delta(4+2\varepsilon)}{\beta^2\Delta+4+\varepsilon}},
\]
as required. \hfill \Box

We can now state the main results of this section. By Lemma 3, the conditions of Lemma 15 are satisfied with $\alpha = \frac{4}{3}$ and $\beta = 1$; thus together they imply:

**Theorem 2.** For every integer $\Delta \geq 7$, for all $\varepsilon > 0$, and for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a $\Delta$-regular $n$-vertex graph $G$ with degenerate distance-number
\[
\text{ddn}(G) > n^\frac{2-0.105}{5\Delta+7+5\varepsilon}.
\]

By Lemma 4, the conditions of Lemma 15 are satisfied with $\alpha = 1.457341$ and $\beta = 0.627977$; thus together they imply:

**Theorem 3.** For every integer $\Delta \geq 8$, for all $\varepsilon > 0$, and for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a $\Delta$-regular $n$-vertex graph $G$ with degenerate distance-number
\[
\text{ddn}(G) > n^{0.864138 \cdot \frac{4.683544+2.341272}{0.394355\Delta+2.511998}}.
\]

Note that the bound given in Theorem 3 is better than the bound in Theorem 2 for $\Delta \geq 17$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 15 (2008), #R107

12
4.2 Bounded degree graphs with $\Delta \geq 5$

Theorem 2 shows that for $\Delta \geq 7$ and for sufficiently large $n$, there is an $n$-vertex degree-$\Delta$ graph whose degenerate distance-number is at least polynomial in $n$. We now prove that the degenerate distance-number of degree-5 graphs can also be arbitrarily large. However, the lower bound we obtain in this case is polylogarithmic in $n$. The proof is inspired by an analogous proof about the slope-number of degree-5 graphs, due to Pach and Pálvölgyi [38].

Theorem 4. For all $d \in \mathbb{N}$, there is a degree-5 graph $G$ with degenerate distance-number $\text{ddn}(G) > d$.

Proof. Consider the following degree-5 graph $G$. For $n \equiv 0 \pmod{6}$, let $F$ be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_j : |i - j| \leq 2\}$. Let $S := \{v_i : i \equiv 1 \pmod{3}\}$. No pair of vertices in $S$ are adjacent in $F$, and $|S| = \frac{n}{3}$ is even.

Let $\mathcal{M}$ denote the set of all perfect matchings on $S$. For each perfect matching $M_k \in \mathcal{M}$, let $G_k := F \cup M_k$. As illustrated in Figure 6, let $G$ be the disjoint union of all the $G_k$.

Thus the number of connected components of $G$ is $|\mathcal{M}|$, which is at least $\left(\frac{n}{3}\right)^{n/6}$ by Lemma 12 with $\Delta = 1$. Here we consider perfect matchings to be 1-regular graphs. (It is remarkable that even with $\Delta = 1$, Lemma 12 gives such an accurate bound, since the actual number of matchings in $S$ is $\sqrt{2}(\frac{n}{3e})^{n/6}$ ignoring lower order additive terms).

Figure 6: The graph $G$ with $n = 18$.  

Suppose, for the sake of contradiction, that for some constant $d$, for all $n \in \mathbb{N}$ such that $n \equiv 0 \pmod{6}$, $G$ has a degenerate drawing $D$ with at most $d$ edge-lengths.

Label the edges of $G$ that are in the copies of $F$ by their length in $D$. Let $\ell_k(i, j)$ be the label of the edge $v_i v_j$ in the copy of $F$ in the component $G_k$ of $G$. This defines a

---

\text{For even $n$, let $f(n)$ be the number of perfect matchings of $[n]$. Here we determine the asymptotics of $f$. In every such matching, $n$ is matched with some number in $[n-1]$, and the remaining matching is isomorphic to a perfect matching of $[n-2]$. Every matching obtained in this way is distinct. Thus $f(n) = (n-1) \cdot f(n-2)$, where $f(2) = 1$. Hence $f(n) = (n-1)!! = (n-1)(n-3)(n-5) \ldots 1$, where $!!$ is the double factorial function. Now $(2n-1)!! = \frac{(2n)!}{2^{n-1}n!}$. Thus $f(n) = \frac{n!}{2^{n-1}n/2} \approx \sqrt{2}(\frac{n}{e})^{n/2}$ by Stirling’s Approximation.}
labelling of the components of $G$. Since $F$ has $2n - 3$ edges and each edge in $F$ receives one of $d$ labels, there are at most $d^{2n-3}$ distinct labellings of the components of $G$.

Let $D_k$ be the degenerate drawing of $G_k$ obtained from $D$ by a translation and rotation so that $v_1$ is at $(0,0)$ and $v_2$ is at $(\ell_k(1,2),0)$. We say that two components $G_q$ and $G_r$ of $G$ determine the same set of points if for all $i \in [n]$, the vertex $v_i$ in $D_q$ is at the same position as the vertex $v_i$ in $D_r$.

Partition the components of $G$ into the minimum number of parts such that all the components in each part have the same labelling and determine the same set of points.

Observe that two components of $G$ with the same labelling do not necessarily determine the same set of points. However, the number of point sets determined by the components with a given labelling can be bounded as follows. For each component labelling of the components of $G$.

Let $D_k$ be the degenerate drawing of $G_k$ obtained from $D$ by a translation and rotation so that $v_1$ is at $(0,0)$ and $v_2$ is at $(\ell_k(1,2),0)$. We say that two components $G_q$ and $G_r$ of $G$ determine the same set of points if for all $i \in [n]$, the vertex $v_i$ in $D_q$ is at the same position as the vertex $v_i$ in $D_r$.

Hence the components with the same labelling determine at most $2^{n-2}$ distinct point sets. Therefore the number of parts in the partition is at most $d^{2n-3} \cdot 2^{n-2} < (2d^2)^n$.

Finally, we bound the number of components in each part, $R$, of the partition. Let $H_R$ be the graph with vertex set $V(H_R) = \{v_1, \ldots, v_n\}$ where $v_iv_j \in E(H_R)$ if and only if $v_iv_j \in E(G_k)$ for some component $G_k \in R$. Since the graphs in $R$ determine the same set of points, the union of the degenerate drawings $D_k$, over all $G_k \in R$, determines a degenerate drawing of $H_R$ with $d$ edge-lengths. Thus $|E(H_R)| \leq dcn^4/d$ and by Lemma 3, $|E(H_R)| \leq c\frac{dn^{4/3}}{6}$ for some constant $c > 0$. Every component in $R$ is a subgraph of $H_R$, and any two components in $R$ differ only by the choice of a matching on $S$. Each such matching has $\frac{n}{6}$ edges. Thus the number of components of $G$ in $R$ is at most

\[ \left( \frac{|E(H_R)|}{n/6} \right) \leq \left( \frac{c\frac{dn^{4/3}}{6}}{n/6} \right)^{n/6} \leq (6cde)^{n/6}n^{n/18}. \]

Hence $|\mathcal{M}| < (2d^2)^n \cdot (6cde)^{n/6}n^{n/18}$, and by the lower bound on $|\mathcal{M}|$ from the start of the proof,

\[ \left( \frac{n}{6} \right)^{n/6} \leq |\mathcal{M}| < (2d^2)^n \cdot (6cde)^{n/6}n^{n/18}. \]

The desired contradiction follows for all $n \geq (3456cde^{13})^{3/2}$. \[ \square \]

### 4.3 Graphs with bounded degree and bounded treewidth

This section proves a logarithmic upper bound on the distance-number of graphs with bounded degree and bounded treewidth. Treewidth is an important parameter in Robertson and Seymour’s theory of graph minors and in algorithmic complexity (see the surveys [8, 41]). It can be defined as follows. A graph $G$ is a $k$-tree if either $G = K_{k+1}$, or $G$ has a vertex $v$ whose neighbourhood is a clique of order $k$ and $G - v$ is a $k$-tree. For
example, every 1-tree is a tree and every tree is a 1-tree. Then the \textit{treewidth} of a graph $G$ is the minimum integer $k$ for which $G$ is a subgraph of a $k$-tree. The \textit{pathwidth} of $G$ is the minimum $k$ for which $G$ is a subgraph of an interval\footnote{A graph $G$ is an \textit{interval graph} if each vertex $v \in V(G)$ can be assigned an interval $I_v \subset \mathbb{R}$ such that $I_w \cap I_v \neq \emptyset$ if and only if $vw \in E(V)$.} graph with no clique of order $k + 2$. Note that an interval graph with no $(k + 2)$-clique is a special case of a $k$-tree, and thus the treewidth of a graph is at most its pathwidth.

Lemma 7 shows that (1-)trees have bounded distance-number. However, this is not true for 2-trees since $K_{2,n}$ has treewidth (and pathwidth) at most 2. By Theorem 3, there are $n$-vertex graphs of bounded degree with distance-number approaching $\Omega(n^{0.864138})$.

On the other hand, no polynomial lower bound holds for graphs of bounded degree and bounded treewidth, as shown in the following theorem.

\textbf{Theorem 5.} Let $G$ be a graph with $n$ vertices, maximum degree $\Delta$, and treewidth $k$. Then the distance-number of $G$ satisfies

$$dn(G) \in O(\Delta^4 k^3 \log n).$$

To prove Theorem 5 we use the following lemma, the proof of which is readily obtained by inspecting the proof of Lemma 8 in \cite{14}. An \textit{$H$-partition} of a graph $G$ is a partition of $V(G)$ into vertex sets $V_1, \ldots, V_t$ such that $H$ is the graph with vertex set $V(H) := \{1, \ldots, t\}$ where $ij \in E(H)$ if and only if there exists $v \in V_i$ and $w \in V_j$ such that $v_i, v_j \in E(G)$. The \textit{width} of an $H$-partition is $\max\{|V_i| : 1 \leq i \leq t\}$.

\textbf{Lemma 16 (\cite{14}).} Let $H$ be a graph admitting a drawing $D$ with $s$ distinct edge-slopes and $\ell$ distinct edge-lengths. Let $G$ be a graph admitting an $H$-partition of width $w$. Then the distance-number of $G$ satisfies

$$dn(G) \leq s\ell w(w - 1) + \left\lfloor \frac{w}{2} \right\rfloor + \ell.$$

\textit{Sketch of Proof.} The general approach is to start with $D$ and then replace each vertex of $H$ by a sufficiently scaled down and appropriate rotated copy of the drawing of $K_w$ on a regular $w$-gon. The only difficulty is choosing the rotation and the amount by which to scale the $w$-gons so that we obtain a (non-degenerate) drawing of $G$. Refer to \cite{14} for the full proof. \hfill $\square$

\textit{Proof of Theorem 5.} Let $w$ be the minimum width of a $T$-partition of $G$ in which $T$ is a tree. The best known upper bound is $w \leq \frac{k}{2}(k + 1)(\frac{k}{2} \Delta(G) - 1)$, which was obtained by Wood \cite{51} using a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski \cite{12}. For each vertex $x \in V(T)$, there are at most $w\Delta$ edges of $G$ incident to vertices mapped to $x$. Hence we can assume that $T$ is a forest with maximum degree $w\Delta$, as otherwise there is an edge of $T$ with no edge of $G$ mapped to it, in which case the edge of $T$ can be deleted. Similarly, $T$ has at most $n$ vertices. Scheffler \cite{44} proved that $T$ has pathwidth at most $\log(2n + 1)$; see \cite{8}. Dujmović et al.
[14] proved that every tree $T$ with pathwidth $p \geq 1$ has a drawing with $\max\{\Delta(T) - 1, 1\}$ slopes and $2p - 1$ edge-lengths. Thus $T$ has a drawing with at most $\Delta w - 1$ slopes and at most $2\log(2n + 1) - 1$ edge-lengths. By Lemma 16,

$$dn(G) \leq (\Delta w - 1)(2\log(2n + 1) - 1)w(w - 1) + \left\lfloor \frac{w}{2} \right\rfloor + 2\log(2n + 1) - 1,$$

which is in $O(\Delta w^3 \log n) \subseteq O(\Delta^4 k^3 \log n)$.  

**Corollary 1.** Any $n$-vertex graph with bounded degree and bounded treewidth has distance-number $O(\log n)$.

Since a path has a drawing with one slope and one edge-length, Lemma 16 with $s = \ell = 1$ implies that every graph $G$ with a $P$-partition of width $k$ for some path $P$ has distance-number $dn(G) \leq k(k - \frac{1}{2}) + 1$.

## 5 Bandwidth

This section finds an upper bound on the distance-number in terms of the bandwidth. Let $G$ be a graph. A *vertex ordering* of $G$ is a bijection $\sigma : V(G) \to \{1, 2, \ldots, |V(G)|\}$. The *width* of $\sigma$ is defined to be $\max\{|\sigma(v) - \sigma(w)| : vw \in E(G)\}$. The *bandwidth* of $G$, denoted by $bw(G)$, is the minimum width of a vertex ordering of $G$. The *cyclic width* of $\sigma$ is defined to be $\max\{|\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\} : vw \in E(G)\}$. The *cyclic bandwidth* of $G$, denoted by $cbw(G)$, is the minimum cyclic width of a vertex ordering of $G$; see [11, 20, 28, 30, 53]. Clearly $cbw(G) \leq bw(G)$ for every graph $G$.

**Lemma 17.** For every graph $G$,

$$dn(G) \leq cbw(G) \leq bw(G).$$

**Proof.** Given a vertex ordering $\sigma$ of an $n$-vertex $G$, position the vertices of $G$ on a regular $n$-gon in the order $\sigma$. We obtain a drawing of $G$ in which the length of each edge $vw$ is determined by

$$\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\}.$$

Thus the number of edge-lengths equals

$$|\{\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\} : vw \in E(G)\}|,$$

which is at most the cyclic width of $\sigma$. The result follows.

**Corollary 2.** The distance-number of every $n$-vertex degree-$\Delta$ planar graph $G$ satisfies

$$dn(G) \leq \frac{15n}{\log_\Delta n}.$$

**Proof.** Böttcher et al. [9] proved that $bw(G) \leq \frac{15n}{\log_\Delta n}$. The result follows from Lemma 17.
6 Cartesian Products

This section discusses the distance-number of cartesian products of graphs. For graphs $G$ and $H$, the cartesian product $G \Box H$ is the graph with vertex set $V(G \Box H) := V(G) \times V(H)$, where $(v, w)$ is adjacent to $(p, q)$ if and only if (1) $v = p$ and $wq$ is an edge of $H$, or (2) $w = q$ and $vp$ is an edge of $G$.

Thus $G \Box H$ is the grid-like graph with a copy of $G$ in each row and a copy of $H$ in each column. Type (1) edges form copies of $H$, and type (2) edges form copies of $G$. For example, $P_n \Box P_n$ is the planar grid, and $C_n \Box C_n$ is the toroidal grid.

The cartesian product is associative and thus multi-dimensional products are well defined. For example, the $d$-dimensional product $K_2 \Box K_2 \Box \ldots \Box K_2$ is the $d$-dimensional hypercube $Q_d$. It is well known that $Q_d$ is a unit-distance graph. Horvat and Pisanski [24] proved that the cartesian product operation preserves unit-distance graphs. That is, if $G$ and $H$ are unit-distance graphs, then so is $G \Box H$, as illustrated in Figure 7. The following theorem generalises this result.

![Figure 7: A unit-distance drawing of $K_3 \Box K_3 \Box K_2$](image)

**Theorem 6.** For all graphs $G$ and $H$, the distance-numbers of $G \Box H$ satisfy

$$\max\{ \dn(G), \dn(H) \} \leq \dn(G \Box H) \leq \dn(G) + \dn(H) - 1, \text{ and}$$

$$\max\{ \ddn(G), \ddn(H) \} \leq \ddn(G \Box H) \leq \ddn(G) + \ddn(H) - 1.$$
Proof. The lower bounds follow since $G$ and $H$ are subgraphs of $G \Box H$. We prove the upper bound for $\dn(G \Box H)$. The proof for $\ddn(G \Box H)$ is simpler.

Fix a drawing of $G$ with $\dn(G)$ edge-lengths. Let $(x(v), y(v))$ be the coordinates of each vertex $v$ of $G$ in this drawing. Fix a drawing of $H$ with $\dn(H)$ edge-lengths, scaled so that one edge-length in the drawing of $G$ coincides with one edge-length in the drawing of $H$. Let $\alpha$ be a real number in $[0, 2\pi)$. Let $(x_\alpha(w), y_\alpha(w))$ be the coordinates of each vertex $w$ of $G$ in this drawing of $H$ rotated by $\alpha$ degrees about the origin.

Position vertex $(v, w)$ in $G \Box H$ at $(x(v) + x_\alpha(w), y(v) + y_\alpha(w))$. This mapping preserves edge-lengths. In particular, the length of a type-(1) edge $(v, u)(v, w)$ equals the length of the edge $uw$ in $H$, and the length of a type-(2) edge $(u, v)(w, v)$ equals the length of the edge $uw$ in $G$. Thus for each $\alpha$, the mapping of $G \Box H$ has $\dn(G) + \dn(H) - 1$ edge-lengths.

It remains to prove that for some $\alpha$ the mapping of $G \Box H$ is a drawing. That is, no vertex intersects the closure of an incident edge. An angle $\alpha$ is bad for a particular vertex/edge pair of $G \Box H$ if that vertex intersects the closure of that edge in the mapping with rotation $\alpha$.

Observe that the trajectory of a vertex $(v, w)$ of $G \Box H$ (taken over all $\alpha$) is a circle centred at $(x(v), y(v))$ with radius $\dist_H(0, w)$.

Now for distinct points $p$ and $q$ and a line $\ell$, there are only two angles $\alpha$ such that the rotation of $p$ around $q$ by an angle of $\alpha$ contains $\ell$ (since the trajectory of $p$ is a circle that only intersects $\ell$ in two places), and there are only two angles $\alpha$ such that the rotation of $\ell$ around $q$ by an angle of $\alpha$ contains $p$.

It follows that there are finitely many bad values of $\alpha$ for a particular vertex/edge pair of $G \Box H$. Hence there are finitely many bad values of $\alpha$ in total. Hence some value of $\alpha$ is not bad for every vertex/edge pair in $G \Box H$. Hence $D_\alpha$ is a valid drawing of $G \Box H$. \qed

Note that Loh and Teh [31] proved a result analogous to Theorem 6 for dimension.

Let $G^d$ be the $d$-fold cartesian product of a graph $G$. The same construction used in Theorem 6 proves the following:

**Theorem 7.** For every graph $G$ and integer $d \geq 1$, the distance-numbers of $G^d$ satisfy

$$\ddn(G^d) = \ddn(G) \quad \text{and} \quad \dn(G^d) = \dn(G).$$

## 7 Open Problems

We conclude by mentioning some of the many open problems related to distance-number.

- What is $\dn(K_n)$? Pach and Agarwal [37] write that “it can be conjectured” that $\dn(K_n) = \left\lceil \frac{n}{2} \right\rceil$. That is, every set of $n$ points in general position determine at least $\left\lceil \frac{n}{2} \right\rceil$ distinct distances. Note that Altman [1, 2] proved this conjecture for points in convex position.

- What is the relationship between distance-number and degenerate distance-number? In particular, is there a function $f$ such that $\dn(G) \leq f(\ddn(G))$ for every graph $G$?
Theorems 2, 3 and 4 establish a lower bound for the distance-number of bounded degree graphs. But no non-trivial upper bound is known. Do $n$-vertex graphs with bounded degree have distance-number in $o(n)$?

- Outerplanar graphs have distance-number in $O(\Delta^4 \log n)$ by Theorem 5. Do outerplanar graphs (with bounded degree) have bounded (degenerate) distance-number?

- Non-trivial lower and upper bounds on the distance-numbers are not known for many other interesting graph families including: degree-3 graphs, degree-4 graphs, 2-degenerate graphs with bounded degree, graphs with bounded degree and bounded pathwidth.

- As described in Section 1.1, determining the maximum number of times the unit-distance can appear among $n$ points in the plane is a famous open problem. We are unaware if the following apparently simpler tasks have been attempted: Determine the maximum number of times the unit-distance can occur among $n$ points in the plane such that no three are collinear. Similarly, determine the maximum number of edges in an $n$-vertex graph $G$ with $dn(G) = 1$.

- Determining the maximum chromatic number of unit-distance graphs in $\mathbb{R}^d$ is a well-known open problem. The best known upper bound of $(3 + o(1))^d$ is due to Larman and Rogers [29]. Exponential lower bounds are known [17, 40]. Unit-distance graphs in the plane are 7-colourable [19], and thus $\chi(G) \leq 7^{dn(G)}$. Can this bound be improved?

- Degenerate distance-number is not bounded by any function of dimension since $K_{n,n}$ has dimension 4 and unbounded degenerate distance-number. On the other hand, $\dim(G) \leq 2 \cdot \chi(G) \leq 2 \cdot 7^{dn(G)}$. Is $\dim(G)$ bounded by a polynomial function of $ddn(G)$?

- Every planar graph has a crossing-free drawing. A long standing open problem involving edge-lengths, due to Harborth et al. [21, 22, 26], asks whether every planar graph has a crossing-free drawing in which the length of every edge is an integer. Geelen et al. [18] recently answered this question in the affirmative for cubic planar graphs. Archdeacon [3] extended this question to nonplanar graphs and asked what is the minimum $d$ such that a given graph has a crossing-free drawing in $\mathbb{R}^d$ with integer edge-lengths. Note that every $n$-vertex graph has such a drawing in $\mathbb{R}^{n-1}$.

- The slope number of a graph $G$, denoted by $sn(G)$, is the minimum number of edge-slopes over all drawings of $G$. Dujmović et al. [13] established results concerning the slope-number of planar graphs. Keszegh et al. [27] proved that degree-3 graphs have slope-number at most 5. On the other hand, Barát et al. [4] and Pach and Pálvölgyi [38] independently proved that there are 5-regular graphs with arbitrarily large slope number. Moreover, for $\Delta \geq 7$, Dujmović et al. [14] proved that there are $n$-vertex degree-$\Delta$ graphs whose slope number is at least $n^{1-\frac{1}{\Delta+1}}$. The proofs of these
results are similar to the proofs of Theorems 2, 3 and 4. Given that Theorem 5 also depends on slopes, it is tempting to wonder if there is a deeper connection between slope-number and distance-number. For example, is there a function $f$ such that $sn(G) \leq f(\Delta(G), dn(G))$ and/or $dn(G) \leq f(sn(G))$ for every graph $G$. Note that some dependence on $\Delta(G)$ is necessary since $sn(K_{1,n}) \to \infty$ but $dn(K_{1,n}) = 1$.

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