Power of quantum computing with restricted postselections

Tomoyuki Morimae\textsuperscript{1}\textsuperscript{*} and Harumichi Nishimura\textsuperscript{2}\textsuperscript{†}

\textsuperscript{1}ASRLD Unit, Gunma University, 1-5-1 Tenjin-cho Kiryu-shi Gunma-ken, 376-0052, Japan
\textsuperscript{2}Graduate School of Information Science, Nagoya University, Chikusa-ku, Nagoya, Aichi, 464-8601 Japan

We consider restricted versions of postBQP where postselection probabilities can be efficiently calculated by classical or quantum computers. We show that such restricted postBQP classes are in AWPP (\(\subseteq\) PP = postBQP). This result suggests that postselecting an event with an exponentially small probability does not necessarily boost BQP to PP. The best upperbound of BQP is AWPP, and therefore restricted postBQP classes are other examples of complexity classes "slightly above BQP". In [C. M. Lee and J. Barrett, arXiv:1412.8671], it was shown that the computational capacity of a general probabilistic theory which satisfies the tomographic locality is in AWPP. Our result therefore implies that quantum physics with restricted postselections is an example of a super quantum theory which seems to be outside of their general probabilistic theory but its computational capacity is also in AWPP. Although it is physically natural to expect that such restricted postBQP classes are not equivalent to BQP, we show that UP \(\cap\) coUP, which is unlikely to be in BQP, is contained in the restricted postBQP classes. Finally, we also consider another restricted class of postBQP, where the postselected probability depends only on the size of inputs. We show that this restricted version is contained in APP, where AWPP \(\subseteq\) APP \(\subseteq\) PP = postBQP.

I. INTRODUCTION

Studying computational capacities of physical theories beyond quantum physics is a top-down approach for deepening our understanding of quantum physics and quantum computing. It was shown in Ref. [1] that a nonlinear quantum computer can efficiently solve NP-complete and \#P problems. It was also shown in Ref. [2] that the quantum physics with the probabilistic rule of \(p\)-norm other than \(p = 2\) can efficiently solve PP-complete problems. A quantum computer which can access closed time-like curve (CTC) qubits was characterized to be PSPACE \(\subseteq \text{postBQP}\). All these problems, NP-complete, \#P, PP-complete, and PSPACE, are believed to be intractable for quantum computing. (For definitions of these complexity classes, see Ref. [3].)

Among several generalized physical theories, quantum physics with postselections would be the most well-studied one. Here, a postselection is a fictitious ability that we can choose a specific result of a measurement on a quantum state with probability 1. In Ref. [4], it was shown that the computational complexity of a quantum computer with the ability of a postselection is PP. In other words, if we denote the class of problems efficiently solved by a quantum computer with a postselection by postBQP, this means that postBQP = PP \(\supseteq\) PP = \text{postBQP}. (The definition of postBQP is given later. For the definition of PP and another proof of postBQP = PP, see Appendix.) Since PP is a much stronger class than that efficiently solved with a quantum computer (namely, BQP), this result on the one hand suggests that a fictitious ability can drastically boost the power of quantum computing, and on the other hand explains why the quantum theory is as it is from the computational complexity viewpoint. The relation postBQP = PP was demonstrated to be useful for finding simpler proofs of some properties of PP [6]. Furthermore, the relation has also been used to show the hardness of classically efficiently sampling the output probability distributions of intermediate quantum computing models, such as the constant-depth model [7], the IQP model [8, 9], Boson Sampling [10], and the DQC1 model [11, 12]. These intermediate models are believed to be non-universal, but classical efficient sampling of output probability distributions of these models has been shown to cause the collapse of the polynomial hierarchy. The polynomial hierarchy is a kind of a generalization of P and NP [8, 3], and a collapse of the polynomial hierarchy is believed to be highly unlikely in computer science. Therefore, this belief suggests the impossibility of classical simulations of these intermediate models. The basic idea of results in Refs. [7, 8, 10, 12] is that these intermediate models can simulate BQP circuits with an exponentially small success probability, and therefore if we postselect the successful event, these models can simulate postBQP circuits. Then the possibility of classical sampling means postBQP \(\subseteq\) postBPP, which leads to the collapse of the polynomial hierarchy, since postBQP = PP and postBPP is known to be in the polynomial hierarchy. (Here, postBPP is the class of problems efficiently solved by a probabilistic classical computer with a postselection.)

All the above examples of “fancy quantum physics” have much stronger computational capacity (such as PP) than that of the normal quantum computing, i.e., BQP. Isn’t there any much weaker generalized quantum theory whose computational capacity is only slightly above BQP? Studying such slightly stronger quantum computing models will provide us high-resolution pictures of relations between quantum and classical or quantum and super quantum. Motivated by this question, several researches have been done recently. In Ref. [13], a new

\textsuperscript{*}Electronic address: morimae@gunma-u.ac.jp
\textsuperscript{†}Electronic address: hnishimura@math.cm.is.nagoya-u.ac.jp
complexity class, PDQP, was introduced. It is a class of problems solvable by a quantum computer with the ability of non-destructive measurements. They showed that PDQP contains BQP and SZK, and thus strictly contains BQP relative to an oracle, while it does not contain NP relative to an oracle. They also showed that PDQP ⊆ BPPPP. (It is an open problem whether this bound can be improved to PDQP ⊆ PP.) In Ref. [14], a general probabilistic theory that satisfies linearity and tomographic locality was considered. The theory contains quantum physics and classical physics as special examples. It was shown there that if we denote the class of problems efficiently solved (with a bounded error probability) by a computer based on such a general probabilistic theory by BGP, then BGP ⊆ AWPP. The class AWPP is the best upper bound of BQP known so far. (The definition of AWPP is given later.)

In this paper, we provide other examples of complexity classes “slightly above BQP”. We consider restricted versions of postBQP where postselection probabilities are efficiently calculated by classical or quantum computers. We first define the complexity class postBQPFP, which is equivalent to postBQP with the additional condition that the postselected probability can be efficiently calculated by a classical computer. We show that postBQPFP is contained in AWPP.

In Appendix, we define several variants of postBQPFP, namely, postBQPFP and postBQPEXP, and show that postBQPFP = postBQPFP = postBQPEXP. Here, postBQPFP is the restricted postBQP such that the postselection probability can be efficiently calculated by a quantum computer, and postBQPEXP assumes that the postselection probability is equal to 2−poly(n) where n is the input size. The class AWPP is believed to be strictly contained in PP, since

\[ \text{BQP} \subseteq \text{AWPP} \subseteq \text{SBQP} = A_0 \text{PP} \subseteq \text{PP} = \text{postBQP}, \]

and it was shown that A_0 \text{PP} = \text{PP} implies PH \subseteq \text{PP} [13], which is not believed to happen. Therefore, our restricted postBQP classes are expected to be strictly weaker than postBQP = PP, which suggests that restricted postselections can be much weaker than that with the full postselection.

Since an event with a constant or polynomially-small probability can be realized without any postselection (we have only to repeat the measurement in a polynomially many times), the essence of the power of a postselection seems to be the fact that it can “amplify” an exponentially small success probability. In fact, if we define postBQP_{\leq \text{exp}} which is equivalent to postBQP with the additional condition that the postselected probability is upperbounded by 2−poly(n), where n is an input size, we can show postBQP = postBQP_{\leq \text{exp}}. (For a proof, see Appendix.) Furthermore, it is easy to see that a single postselection of an event with the probability 2−poly(n) can promote constant-depth quantum circuits [7] (with a single generalized Toffoli gate) and the DQC1 model [12] to universal quantum computers. Moreover, a postselection with an exponentially small probability can violate the no-signaling principle: if Alice and Bob share the two qutrit state, \[ \frac{1}{\sqrt{2}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}} (\sqrt{1 - \frac{2}{\exp(n)}} |2\rangle \otimes |2\rangle \], Alice can transmit her bit 0 or 1 to Bob by postselecting her qutrit to |0\rangle or |1\rangle. In short, a postselection of an event with an exponentially small probability seems to have strong power even if it can be efficiently calculated by a classical computer. However, our result suggests that as long as the postselection probability can be calculated by a classical (or quantum) computer, no postselection (even one for an exponentially-small success probability) can have enough power to boost BQP to PP.

If the quantum theory with postselections can be described by the general probabilistic theory of Ref. [14], we obtain BGP ⊇ postBQP = PP, which contradicts their result BGP ⊆ AWPP ⊆ PP = postBQP (unless AWPP = PP). Therefore, the quantum theory with postselections is likely to be outside of their general probabilistic theory. It seems that a quantum theory with restricted postselections is also outside of their general probabilistic theory, since the structure of a general theory should be independent of a specific event probability. Therefore, our results imply that quantum physics with restricted postselections is an example of a super quantum theory which seems to be outside of the general probabilistic theory of Ref. [14] but its computational capacity is also in AWPP.

Although it is physically reasonable to expect BQP ≠ postBQPFP, we do not know how to show it. (The separation is very difficult, since the separation leads to that of P and PSPACE.) As a witness supporting the separation between BQP and postBQPFP, we can show UP ∩ coUP ⊆ postBQPFP, which suggests that BQP = postBQPFP is unlikely. (For a proof, see Appendix.) In fact, the proof is valid also with any oracle: UP^A ∩ coUP^A ⊆ postBQPFP for any oracle A. By combining this result with the known result that UP^B ∩ coUP^B ⊆ BQP^B for some oracle B [17], we obtain BQP_C ≠ postBQPF_P for some oracle C.

Finally, we also consider another restricted class of postBQP, namely, postBQP_{\text{size}}, where the postselected probability depends only on the size of inputs, and show that postBQPF_P ⊆ postBQP_{\text{size}} ⊆ AP. (The definition of APP is given later.) Since APP ⊆ PP = postBQP, restricting postselections in such a way can also weaken the power of the full postselection. Since APP is low for PP [10] (i.e., PP^{APP} = PP), postBQP_{\text{size}} is low for PP. This suggests that postBQP_{\text{size}} is useless for PP as an oracle (or a blackbox subroutine), and a strict subset of PP = postBQP.
II. PRELIMINARIES

In this section, we give several definitions which will be used in this paper. We assume that the reader is familiar with basics of computational complexity, such as P, BPP, NP, and non-deterministic Turing machine, etc [5]. Let $C_n$ be a quantum circuit of the input size $n$. We say that a family $\{C_n\}_n$ of quantum circuits is uniform if there is a classical algorithmic polynomial-time algorithm that outputs a description of $C_n$ on input $0^n$. For simplicity, we choose Hadamard and Toffoli gates for a universal gate set.

**Definition:** A language $L$ is in BQP iff there exists a uniform family $\{V_n\}$ of polynomial-size quantum circuits such that

1. If $w \in L$ then $P_{V_n}(o = 1) \geq \frac{2}{3}$.
2. If $w \notin L$ then $P_{V_n}(o = 1) \leq \frac{1}{3}$.

Here, $P_{V_n}(o = 1)$ is the probability of obtaining $o = 1$ if we measure the single output qubit of the circuit $V_{|w|}$ on input $w$. The pair of the thresholds $(\frac{2}{3}, \frac{1}{3})$ is rather arbitrary. For example, we can take $(2^{-r(|w|)}, 1 - 2^{-r(|w|)})$ for any polynomial $r$ [3].

**Definition:** A function $f : \{0, 1\}^* \to \mathbb{Z}$ is a GapP function [16] if there exists a polynomial-time non-deterministic Turing machine $M$ such that $f(x)$ is the number of accepting paths of $M(x)$ minus the number of rejecting paths of $M(x)$, where $M(x)$ denotes the non-deterministic computation of $M$ on input $x$.

AWPP was defined in Ref. [21]. The inclusion BQP ⊆ AWPP was shown in Ref. [17]. A simplified definition of AWPP is given in Ref. [18]. See also Ref. [19].

**Definition:** A language $L$ is in AWPP iff for any polynomial $r$, there exist $f \in \text{FP}$ and $g \in \text{GapP}$ such that $f(w) > 0$ for all $w$ and

1. If $w \in L$ then $1 - 2^{-r(|w|)} \leq \frac{g(w)}{f(w)} \leq 1$.
2. If $w \notin L$ then $0 \leq \frac{g(w)}{f(w)} \leq 2^{-r(|w|)}$.

Here, FP is the class of functions from bit strings to integers that are computable in polynomial time by a Turing machine.

**Definition:** A language $L$ is in APP iff for any polynomial $r$, there exist $f, g \in \text{GapP}$ such that $f(1^{|w|}) > 0$ for all $w$ and

1. If $w \in L$ then $1 - 2^{-r(|w|)} \leq \frac{g(w)}{f(1^{|w|})} \leq 1$.
2. If $w \notin L$ then $0 \leq \frac{g(w)}{f(1^{|w|})} \leq 2^{-r(|w|)}$.

It was shown in Ref. [18] that AWPP ⊆ APP ⊆ PP.

The class postBQP was defined in Ref. [3]. (See also Ref. [20].)

**Definition:** A language $L$ is in postBQP iff there exist a uniform family $\{V_n\}$ of polynomial-size quantum circuits with the ability of a postselection and a polynomial $u$ such that

1. $P_{V_n}(p = 1) \geq 2^{-u(|w|)}$.
2. If $w \in L$ then $P_{V_n}(o = 1 | p = 1) \geq \frac{2}{3}$.
3. If $w \notin L$ then $P_{V_n}(o = 1 | p = 1) \leq \frac{1}{3}$.

Here, $p \in \{0, 1\}$ is the measurement result of the postselected qubit of the circuit $V_{|w|}$. Like BQP, the pair of the thresholds $(\frac{1}{3}, \frac{2}{3})$ is arbitrary. In particular, it can be $(2^{-r(|w|)}, 1 - 2^{-r(|w|)})$ for any polynomial $r$. Furthermore, without loss of generality, we can assume that only a single qubit is postselected, since postselections on more than two qubits can be transformed to that on a single qubit by using the generalized Toffoli gate, which can be implemented in a polynomial-size quantum circuit.

Now we define our restricted versions of postBQP.

**Definition:** A language $L$ is in postBQP$_{\text{FP}}$ iff it is in postBQP and there exist a polynomial $h \geq 0$ and $f \in \text{FP}$ such that $P_{V_n}(p = 1) = \frac{f(w)}{g^h(|w|)}$.

Like postBQP, the pair of the thresholds $(\frac{1}{3}, \frac{2}{3})$ is arbitrary, and we can assume that only a single qubit is postselected without loss of generality.

**Definition:** A language $L$ is in postBQP$_{\text{size}}$ iff it is in postBQP and $P_{V_n}(p = 1)$ depends only on $|w|$.

In Appendix, we show postBQP$_{\text{FP}} \subseteq$ postBQP$_{\text{size}}$.

III. MAIN RESULTS

In this section, we show our main results.

**Theorem:** postBQP$_{\text{FP}} \subseteq$ AWPP.

**Proof:** Let us assume that a language $L$ is in postBQP$_{\text{FP}}$. Then, there exist a uniform family $\{V_n\}$ of polynomial-size quantum circuits such that $1 - 2^{-r(|w|)} \leq P_{V_n}(o = 1 | p = 1) \leq 1$ if $w \in L$ and $0 \leq P_{V_n}(o = 1 | p = 1) \leq 2^{-r(|w|)}$ if $w \notin L$ for any polynomial $r$, and a polynomial $h \geq 0$ and $f \in \text{FP}$ such that $P_{V_n}(p = 1) = f(w)2^{-h(|w|)}$. From the postBQP$_{\text{FP}}$ circuit $V_n$, we construct a polynomial-size quantum circuit $W_n$ of Fig. 1.

Then, $P_{W_n}(o' = 1) = P_{V_n}(o = 1, p = 1) = P_{V_n}(o = 1 | p = 1)P_{V_n}(p = 1)$.

![FIG. 1: The circuit $W_n$.](image)

For any polynomial-size quantum circuit $C$, there exist $g \in \text{GapP}$ and a polynomial $q$ such that

$$\Pr(C(x) \text{ accepts}) = \frac{g(x)}{2^q(|x|)}$$

Therefore, for any polynomial $r$,

1. If $w \in L$ then

$$1 - 2^{-r(|w|)} \leq P_{W_n}(o' = 1 | p = 1) \leq 1$$
which means
\[ 1 - 2^{-r(|w|)} \leq \frac{g(w)2^h(|w|)}{f(w)2^q(|w|)} \leq 1, \]

2. If \( w \notin L \) then
\[ 0 \leq \frac{P_{w^c}(o' = 1)}{P_{w}(p = 1)} \leq 2^{-r(|w|)}, \]

which means
\[ 0 \leq \frac{g(w)2^h(|w|)}{f(w)2^q(|w|)} \leq 2^{-r(|w|)}. \]

Since \( g(w)2^h(|w|) \in \text{GapP} \) (due to the closure properties 1 and 4 of Ref. [14]), \( f(w)2^q(|w|) \in \text{FP} \) and \( f(w)2^q(|w|) > 0 \) for any \( w, L \in \text{AWPP} \). Therefore we have shown \( \text{postBQP}_{\text{FP}} \subseteq \text{AWPP} \).

Theorem: \( \text{postBQP}_{\text{size}} \subseteq \text{APP} \).

Proof: Let us assume that a language \( L \) is in \( \text{postBQP}_{\text{size}} \). From the \( \text{postBQP}_{\text{size}} \) circuit \( V_n \), we construct a polynomial-size quantum circuit \( W_n \) of Fig. 4.

Then, for any polynomial \( r \),

1. If \( w \in L \) then
\[ 1 - 2^{-r(|w|)} \leq \frac{g(w)2^h(|w|)}{2^q(|w|)f(1^{|w|})} \leq 1, \]

2. If \( w \notin L \) then
\[ 0 \leq \frac{g(w)2^h(|w|)}{2^q(|w|)f(1^{|w|})} \leq 2^{-r(|w|)}, \]

for some polynomials \( g \) and \( h \), and \( f, g \in \text{GapP} \). Since \( g(w)2^h(|w|) f(1^{|w|}) 2^q(|w|) \in \text{GapP} \) and \( 2^q(|w|) f(1^{|w|}) > 0 \) for any \( w, L \in \text{APP} \). Therefore we have shown \( \text{postBQP}_{\text{size}} \subseteq \text{APP} \).

IV. DISCUSSION

In this paper, we have studied the computational capacity of quantum theory with restricted postselections. We have defined the complexity class \( \text{postBQP}_{\text{FP}} \), which characterizes the quantum computing with an FP probability postselection, and shown that \( \text{postBQP}_{\text{FP}} \subseteq \text{AWPP} \). We have also shown that \( \text{postBQP}_{\text{size}} \subseteq \text{APP} \), where \( \text{postBQP}_{\text{size}} \) characterizes the quantum computing with postselections whose probabilities depend only on the size of inputs. Since \( \text{AWPP} \subseteq \text{APP} \subseteq \text{postBQP} = \text{PP} \), our results suggest that quantum computing with restricted postselections can be much weaker than that with the full postselection, although restricted postselections can have exponentially small probabilities, or can violate the no-signaling principle.

Let us give one example to understand why the restricted postselections considered in this paper cannot promote BQP to PP. In the algorithm of Ref. [4] to simulate PP with postBQP, the probability \( p \) of a postselection is given by \( p = (s^2 + \frac{1}{2}(2^n - 2s)^22^{2^n})((2^n - s)^2 + s^2)^{-1} \), where \( i \) is an integer we can freely choose, and \( s \in [0, 2^n] \) is the number of \( x \in \{0, 1\}^n \) that satisfy \( f(x) = 1 \) for a function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \). Since to know \( p \) means to know \( s \), which is in \#P, \( p \) is unlikely in FP. Furthermore, \( p \) depends on not only the input length \( n \) but also input \( f \) itself.

In Ref. [14], it was shown that the output probability distribution of a BGP computer can be written as a GapP function divided by \( 2^{poly(n)} \), where \( n \) is the size of inputs. Therefore, \( \text{postBGP}_{\text{FP}} \subseteq \text{AWPP} \), where \( \text{postBGP}_{\text{FP}} \) is the class of languages recognized by postselected BGP computers with FP postselection probabilities.

It is known that there exists an oracle \( A \) such that \( \text{P}^A = \text{AWPP}^A \) and the polynomial hierarchy is infinite [21]. Furthermore as in a similar way of Ref. [14], we can show \( \text{postBQP}^A_{\text{FP}} \subseteq \text{AWPP}^A \) for any oracle \( A \). Therefore, there exists an oracle \( A \) relative to which \( \text{postBQP}^A_{\text{FP}} \subseteq \text{P}^A \) and the polynomial hierarchy is infinite, which means that there exists an oracle relative to which \( \text{NP} \) is not contained in \( \text{postBQP}_{\text{FP}} \). It remains open if we can improve this fact from \( \text{NP} \) to \( \text{UP} \) or we can prove that \( \text{UP} \) is in \( \text{postBQP}_{\text{FP}} \).

Acknowledgments

TM is supported by the Tenure Track System by MEXT Japan, and KAKENHI 26730003. HN is supported by KAKENHI 24106009, 24240001, and 25330012.

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Appendix A: Another proof of postBQP = PP

In this section, we give another proof of postBQP = PP. Before showing the proof, we will give a theorem, which is used for the proof, and two definitions of PP.

**Theorem** (Fenner, et al. [24]): For any \( g \in \text{GapP} \), there exist a polynomial \( s \) and a uniform family \( \{ V_n \} \) of polynomial-size quantum circuits such that

\[
P_{V_w}(o = 1) = \frac{g(w)^2}{2^s(|w|)}
\]

where \( P_{V_w}(o = 1) \) is the probability that the output of the circuit \( V_w \) is \( o = 1 \) on input \( w \).

For the purpose of the self-consistency, we give a proof of the theorem.

**Proof**: Let \( M \) be the polynomial-time non-deterministic Turing machine corresponding to \( g \). Then, for a given input \( w \), the state

\[
\frac{1}{\sqrt{2^s(|w|)}} \sum_{x \in \{0, 1\}^{s(|w|)}} (-1)^{M(w,x)} |x\rangle \otimes |M(w,x)\rangle
\]

can be generated in \( \text{poly}(|w|) \) steps, where \( s \) is a polynomial, and \( M(w,x) = 0(1) \) if the path \( x \) is the accepting (rejecting) path on the input \( w \). Let \( V_{|w|} \) be the circuit that generates this state. We define that its output is \( o = 1 \) when the above state is projected onto \( |+\rangle^{\otimes (s(|w|)+1)} \), where \( |+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2} \). Then, the probability of obtaining \( o = 1 \) is

\[
P_{V_w}(o = 1) = \frac{g(w)^2}{2^{2s(|w|)+1}},
\]

where \( g(w) \) is the number of accepting paths of \( M \) on input \( w \) minus that of rejecting paths of \( M \) on input \( w \). This completes the proof.

A standard definition of PP is as follows.

**Definition**: A language \( L \) is in PP iff there exists a polynomial-time non-deterministic Turing machine such that

1. If \( w \in L \) then at least 1/2 of computation paths accept.
2. If \( w \notin L \) then less than 1/2 of computation paths accept.

There is another definition of PP that we will use:

**Definition** (Fortnow [19, Theorem 6.4.16]): A language \( L \) is in PP iff for any polynomial \( r \), there exist \( f, g \in \text{GapP} \) such that \( f > 0 \) and

1. If \( w \in L \) then \( 1 - 2^{-r(|w|)} \leq \frac{g(w)}{f(w)} \leq 1 \).
2. If \( w \notin L \) then \( 0 \leq \frac{g(w)}{f(w)} \leq 2^{-r(|w|)} \).
Theorem (Aaronson [6]): PP = postBQP.

Proof: First we show postBQP ⊆ PP. We assume that a language $L$ is in postBQP. Then, for any polynomial $r$, there exists a uniform family $\{V_n\}$ of polynomial-size quantum circuits. As in the proof of postBQP$_{FP} \subseteq$ AWPP, if $w \in L$,

\[
1 - 2^{-r(|w|)} \leq P_{V_w}(o = 1|p = 1) \leq 1
\]

\[
\Leftrightarrow 1 - 2^{-r(|w|)} \leq \frac{P_{V_w}(o = 1,p = 1)}{P_{V_w}(p = 1)} \leq 1
\]

\[
\Leftrightarrow 1 - 2^{-r(|w|)} \leq \frac{g(w)2^{q(|w|)}}{2^{g(|w|)}f(w)} \leq 1
\]

for $f, g \in \text{GapP}$ and polynomials $q$ and $q'$. Here, we have used the fact that

\[
P_{V_w}(o = 1, p = 1) = \frac{g(w)}{2^{g(|w|)}}
\]

\[
P_{V_w}(p = 1) = \frac{f(w)}{2^{g(|w|)}}
\]

for some $g, f \in \text{GapP}$ and polynomials $q$ and $q'$.

If $w \notin L$

\[
0 \leq P_{V_w}(o = 1|p = 1) \leq 2^{-r(|w|)}
\]

\[
\Leftrightarrow 0 \leq \frac{P_{V_w}(o = 1,p = 1)}{P_{V_w}(p = 1)} \leq 2^{-r(|w|)}
\]

\[
\Leftrightarrow 0 \leq \frac{g(w)2^{q'(|w|)}}{2^{g(|w|)}f(w)} \leq 2^{-r(|w|)}.
\]

Since $2^{q(|w|)}g(w), 2^{q(|w|)}f(w) \in \text{GapP}$, $L$ is in PP.

Second, let us show PP ⊆ postBQP. We assume that a language $L$ is in PP. If $w \in L$, for any polynomial $r$, there exist $g, f \in \text{GapP}$ such that

\[
(1 - 2^{-r(|w|)})^2 \leq \frac{g(w)^2}{f(w)^2}.
\]

Then we have

\[
(1 - 2^{-r(|w|)})^2 \leq \frac{2^{q(|w|)}P_{V_w}(o = 1)}{2^{g(|w|)}P_{V_w}(o = 1)}
\]

for some polynomials $q$ and $q'$, and uniform families $\{V'_n\}$ and $\{W'_n\}$ of polynomial-size quantum circuits. Let us define $V'_{|w|}$ and $W'_{|w|}$ such that

\[
P_{V_w}(o = 1) = P_{V'_w}(o = 1)2^{-q'(|w|)},
\]

\[
P_{W_w}(o = 1) = P_{W'_w}(o = 1)2^{-q(|w|)}.
\]

The circuit $V'_{|w|}$ ($W'_{|w|}$) can be constructed by simulating $V'_{|w|}$ ($W'_{|w|}$) and outputting $o = 1$ with probability $2^{-q'(|w|)}$ ($2^{-q(|w|)}$) if $V'_{|w|}$ ($W'_{|w|}$) outputs $o = 1$. Then, we obtain

\[
(1 - 2^{-r(|w|)})^2 \leq \frac{P_{V_w}(o = 1)}{P_{W_w}(o = 1)}.
\]

Similarly, if $w \notin L$, we have

\[
\frac{g(w)^2}{f(w)^2} \leq 2^{-2r(|w|)}
\]

\[
\Leftrightarrow \frac{P_{V_w}(o = 1)}{P_{W_w}(o = 1)} \leq 2^{-2r(|w|)}.
\]

Let us consider the following quantum circuit $R_n$: It first flips two unbiased coins. If both are heads, $R_n$ simulates $W_n$. 
1. If \( W_n \) outputs \( o = 1 \), then \( R_n \) outputs \( o = 0 \) and \( p = 1 \).
2. If \( W_n \) outputs \( o = 0 \), then \( R_n \) outputs \( o = 0 \) and \( p = 0 \).

Otherwise, \( R_n \) simulates \( V_n \).

1. If \( V_n \) outputs \( o = 1 \), then \( R_n \) outputs \( o = 1 \) and \( p = 1 \).
2. If \( V_n \) outputs \( o = 0 \), then \( R_n \) outputs \( o = 0 \) and \( p = 0 \).

Then,

\[
PR_w(p = 1) = 3^\frac{3}{4}PV_w(o = 1) + 1^\frac{1}{4}PW_w(o = 1)
\]

\[
\geq f(w)\frac{1}{24^{2q(|w|)+q'(|w|)+2}}
\]

and

\[
PR_w(o = 1|p = 1) = \frac{PR_w(o = 1, p = 1)}{PR_w(p = 1)}
\]

\[
= \frac{\frac{3}{4}PV_w(o = 1)}{\frac{3}{4}PV_w(o = 1) + \frac{1}{4}PW_w(o = 1)}.
\]

If \( w \in L \),

\[
PR_w(o = 1|p = 1) = \frac{3PV_w(o = 1)}{3PV_w(o = 1) + PW_w(o = 1)}
\]

\[
\geq \frac{3PV_w(o = 1)}{3PV_w(o = 1) + \frac{PV_w(o = 1)}{(1 - 2^{r(|w|)})^2}}
\]

\[
\geq \frac{1}{2} + \frac{1}{22} - \frac{12}{11} \times 2^{-r(|w|)}.
\]

If \( w \notin L \),

\[
PR_w(o = 1|p = 1) = \frac{3PV_w(o = 1)}{3PV_w(o = 1) + PW_w(o = 1)}
\]

\[
\leq \frac{3PV_w(o = 1) + \frac{PV_w(o = 1)}{2^{r(|w|)}}}{3PV_w(o = 1) + \frac{PV_w(o = 1)}{2^{2r(|w|)}}}
\]

\[
\leq 3 \times 2^{-2r(|w|)}.
\]

Therefore, \( L \in \text{postBQP} \).

**Appendix B:** \( \text{postBQP} = \text{postBQP} \leq \text{exp} \)

**Definition:** A language \( L \) is in \( \text{postBQP} \leq \text{exp} \) if it is in \( \text{postBQP} \) and there exists a polynomial \( q > 0 \) such that for any \( w \)

\[
PV_w(p = 1) \leq 2^{-q(|w|)},
\]

where \( PV_w(p = 1) \) is the postselection probability \( p = 1 \) of the postBQP circuit \( V_{|w|} \) on input \( w \).

**Theorem:** \( \text{postBQP} = \text{postBQP} \leq \text{exp} \)

**Proof:** \( \text{postBQP} \supseteq \text{postBQP} \leq \text{exp} \) is obvious. Let us show \( \text{postBQP} \subseteq \text{postBQP} \leq \text{exp} \). We assume that a language \( L \) is in \( \text{postBQP} \). Then, from the quantum circuit \( V_{|w|} \) of \( \text{postBQP} \), we construct the circuit \( W_{|w|} \) which runs as follows: \( W_{|w|} \) generates a random bit \( b \) which is \( b = 1 \) with the probability \( 2^{-q(|w|)} \), where \( q > 0 \) is any polynomial.
Then, $W_{|w|}$ simulates $V_{|w|}$ and outputs $p = 1$ if $b = 1$ and $V_{|w|}$ outputs $p = 1$. Then,

$$P_{V_w}(p = 1) = P_{V_w}(p = 1)2^{-q(|w|)} \leq 2^{-q(|w|)}$$

and

$$P_{W_w}(o = 1|p = 1) = P_{V_w}(o = 1|p = 1).$$

Therefore, $L$ is in $\text{postBQP} \subseteq \text{exp}$.

**Appendix C: Relations among** \(\text{postBQP}_{\text{FP}}, \text{postBQP}_{\text{FQP}}, \text{postBQP}_{\text{exp}} \text{ and } \text{postBQP}_{\text{size}}\)

**Definition:** A language $L$ is in $\text{postBQP}_{\text{FQP}}$ iff it is in $\text{postBQP}$ and there exist a polynomial $h \geq 0$ and a function $f : \{0, 1\}^* \rightarrow \mathbb{N}$, which can be calculated by a uniform family of polynomial-size quantum circuits, such that for any input $w$

$$P_{V_w}(p = 1) = \frac{f(w)}{2^{h(|w|)}}.$$

**Definition:** A language $L$ is in $\text{postBQP}_{\text{exp}}$ iff it is in $\text{postBQP}$ and there exists a polynomial $h \geq 0$ such that

$$P_{V_w}(p = 1) = \frac{1}{2^{h(|w|)}}.$$

**Theorem:** $\text{postBQP}_{\text{FQP}} = \text{postBQP}_{\text{FP}} = \text{postBQP}_{\text{exp}} \subseteq \text{postBQP}_{\text{size}}$.

**Proof:** The inclusions $\text{postBQP}_{\text{exp}} \subseteq \text{postBQP}_{\text{size}}$ and $\text{postBQP}_{\text{FQP}} \supseteq \text{postBQP}_{\text{FP}} \supseteq \text{postBQP}_{\text{exp}}$ are obvious. Let us show $\text{postBQP}_{\text{FQP}} \subseteq \text{postBQP}_{\text{exp}}$. Its proof uses the idea of an additive adjustment of the acceptance probability from Ref. [25] with a standard multiplicative adjustment.

Let us assume that a language $L$ is in $\text{postBQP}_{\text{FQP}}$. Then, there exist a uniform family $\{V_n\}$ of polynomial-size quantum circuits, a function $f : \{0, 1\}^* \rightarrow \mathbb{N}$ whose $f(w)$ can be calculated by a uniform family of polynomial-size quantum circuits for any input $w$, and a polynomial $h \geq 0$ such that

$$P_{V_w}(p = 1) = \frac{f(w)}{2^{h(|w|)}}$$

and

1. If $w \in L$, then $\frac{0}{10} \leq P_{V_w}(o = 1|p = 1) \leq 1$.
2. If $w \notin L$, then $0 \leq P_{V_w}(o = 1|p = 1) \leq \frac{1}{10}$.

We can take a function $t : \{0, 1\}^* \rightarrow \mathbb{N} \cup \{0\}$ such that

$$2^{t(w)} \leq f(w) < 2^{t(w)+1}$$

for any input $w$. Note that $t(w)$ can be calculated by a uniform family of polynomial-size quantum circuits. From $V_{|w|}$, we construct the following polynomial-size quantum circuit $W_{|w|}$:

1. $W_{|w|}$ flips a coin. If heads, it simulates $V_{|w|}$.
2. If tails, $W_{|w|}$ outputs $o = 1$ with probability $1/2$, and $p = 1$ with probability

$$\frac{2^{t(w)+1} - f(w)}{2^{h(|w|)}}.$$

Since

$$2^{h(|w|)} - 2^{t(w)+1} + f(w) \geq f(w) - 2^{t(w)+1} + f(w) = 2(f(w) - 2^{t(w)}) \geq 0,$$
\[ \frac{2^{t(w)+1} - f(w)}{2^{h(|w|)}} \leq 1. \]

Then,
\[ P_{W_w}(p = 1) = \frac{1}{2} P_{V_w}(p = 1) + \frac{1}{2} \frac{2^{t(w)+1} - f(w)}{2^{h(|w|)}} \]
and
\[ P_{W_w}(o = 1|p = 1) = \frac{P_{W_w}(o = 1, p = 1)}{P_{W_w}(p = 1)} \]
\[ = \frac{1}{2} P_{V_w}(o = 1|p = 1) P_{V_w}(p = 1) + \frac{1}{2} \frac{2^{t(w)+1} - f(w)}{2^{2h(|w|)}} \]
\[ = \frac{f(w)}{2^{t(w)+1}} P_{V_w}(o = 1|p = 1) + \frac{1}{2} \frac{2^{t(w)+1} - f(w)}{2^{t(w)+1}}. \]

If \( w \in L \),
\[ P_{R_w}(o = 1|p = 1) = \frac{f(w)}{2^{t(w)+1}} P_{V_w}(o = 1|p = 1) + \frac{1}{2} \frac{2^{t(w)+1} - f(w)}{2^{t(w)+1}} \geq \frac{1}{2} \frac{9}{10} + \frac{1}{2} \frac{1}{2} \]
\[ = \frac{7}{10}. \]

If \( w \notin L \),
\[ P_{R_w}(o = 1|p = 1) = \frac{f(w)}{2^{t(w)+1}} P_{V_w}(o = 1|p = 1) + \frac{1}{2} \frac{2^{t(w)+1} - f(w)}{2^{t(w)+1}} \leq \frac{1}{2} \frac{1}{10} + \frac{1}{2} \frac{1}{2} \]
\[ = \frac{3}{10}. \]

Here, we have used the fact that
\[ \alpha \frac{9}{10} + (1 - \alpha) \frac{1}{2} \geq \frac{1}{2} \frac{9}{10} + \frac{1}{2} \frac{1}{2} \]
and
\[ \alpha \frac{1}{10} + (1 - \alpha) \frac{1}{2} \leq \frac{1}{2} \frac{1}{10} + \frac{1}{2} \frac{1}{2} \]
for \( \alpha \geq 1/2 \). Note that \( f(w)/2^{t(w)+1} \geq 1/2 \), since \( f(w) \geq 2^{t(w)} \).

From the circuit \( W_{|w|} \), we construct the polynomial-size circuit \( R_{|w|} \) in the following way:
1. \( R_{|w|} \) simulates \( W_{|w|} \).
2. \( R_{|w|} \) outputs \( o = 1 \) if \( W_{|w|} \) outputs \( o = 1 \).
3. \( R_{|w|} \) generates a random bit \( b \) which takes \( b = 1 \) with probability \( 2^{-t(w)} \). (Note that \( t(w) \leq h(|w|) \).)
4. \( R_{|w|} \) outputs \( p = 1 \) if \( b = 1 \) and \( W_{|w|} \) outputs \( p = 1 \).

Then,
\[ P_{R_w}(o = 1|p = 1) = P_{W_w}(o = 1|p = 1) \]
and
\[ P_{R_w}(p = 1) = P_{W_w}(p = 1)2^{-t(w)} = 2^{-h(|w|)}. \]

Therefore, \( L \) is in postBQP\textsubscript{exp}. 

Appendix D: UP \cap \text{coUP} is in postBQP_{FP}

**Theorem:** UP \cap \text{coUP} \subseteq \text{postBQP}_{FP}.

**Proof:** Let us assume that a language \( L \) is in UP \cap \text{coUP}. Since \( L \in \text{UP} \), there exists a polynomial-time non-deterministic Turing machine \( M \) such that

1. If \( w \in L \) then \( M(w) \) has one accepting path.
2. If \( w \notin L \) then \( M(w) \) has no accepting path.

Furthermore, since \( L \in \text{coUP} \), which means \( \overline{L} \in \text{UP} \), there exists a polynomial-time non-deterministic Turing machine \( N \) such that

1. If \( w \in L \) then \( N(w) \) has no accepting path.
2. If \( w \notin L \) then \( N(w) \) has one accepting path.

For a given input \( w \), a polynomial-size quantum circuit can generate

\[
\frac{1}{\sqrt{2^{q(|w|)}}} \sum_{x \in \{0,1\}^{|w|}} |x\rangle \otimes |M(w,x)\rangle_a \otimes |N(w,x)\rangle_b \otimes |0\rangle_p
\]

for a polynomial \( q \), where we can assume without loss of generality that both \( M \) and \( N \) have computation trees on input \( w \) whose paths are represented by \( \{0,1\}^{|w|} \). Here, \( M(w,x) = 1(0) \) if \( x \) is the accepting (rejecting) path of \( M \) on input \( w \), and \( N(w,x) \) is defined in a similar way. Let us apply CNOT between \( a \) (control) and \( p \) (target). We also apply CNOT between \( b \) (control) and \( p \) (target). Then, if \( w \in L \), the state becomes

\[
\frac{1}{\sqrt{2^{q(|w|)}}} |x_w\rangle \otimes |1\rangle_a \otimes |0\rangle_b \otimes |1\rangle_p + \frac{1}{\sqrt{2^{q(|w|)}}} \sum_{x:x \neq x_w} |x\rangle \otimes |0\rangle_a \otimes |0\rangle_b \otimes |0\rangle_p,
\]

for certain \( x_w \) and if \( w \notin L \),

\[
\frac{1}{\sqrt{2^{q(|w|)}}} |x'_w\rangle \otimes |0\rangle_a \otimes |1\rangle_b \otimes |1\rangle_p + \frac{1}{\sqrt{2^{q(|w|)}}} \sum_{x:x \neq x'_w} |x\rangle \otimes |0\rangle_a \otimes |0\rangle_b \otimes |0\rangle_p,
\]

for certain \( x'_w \).

Let us postselect the register \( p \) to \( |1\rangle \). Then, if \( w \in L \), the state becomes

\[
\frac{1}{\sqrt{2^{q(|w|)}}} |x_w\rangle \otimes |1\rangle_a \otimes |0\rangle_b \otimes |1\rangle_p,
\]

and therefore

\[
P(p = 1) = \frac{1}{2^{q(|w|)}}
\]

and

\[
P(a = 1|p = 1) = 1.
\]

If \( w \notin L \), the state becomes

\[
\frac{1}{\sqrt{2^{q(|w|)}}} |x'_w\rangle \otimes |0\rangle_a \otimes |1\rangle_b \otimes |1\rangle_p,
\]

and therefore

\[
P(p = 1) = \frac{1}{2^{q(|w|)}}
\]

and

\[
P(a = 1|p = 1) = 0.
\]

Therefore, \( L \) is in postBQP_{FP}. 

FIG. 2: Relations among several complexity classes.

Appendix E: Summary

In Fig. 2 we provide the summary of relations among the complexity classes studied in this paper.