Metric Redefinitions in Einstein–Æther Theory

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Abstract

‘Einstein–Æther’ theory, in which gravity couples to a dynamical, time-like, unit-norm vector field, provides a means for studying Lorentz violation in a generally covariant setting. Demonstrated here is the effect of a redefinition of the metric and ‘æther’ fields in terms of the original fields and two free parameters. The net effect is a change of the coupling constants appearing in the action. Using such a redefinition, one of the coupling constants can be set to zero, simplifying studies of solutions of the theory.
I. INTRODUCTION

Interest has lately grown in the possibility that Lorentz symmetry is not an exact symmetry of nature. In particular, it may be broken by as-yet-unknown quantum gravity effects. In an effective field theory description, this symmetry-breaking can be realized by a vector field that defines the “preferred” frame. In the flat space-time of the Standard Model, this field can be treated as non-dynamical, background structure. In the context of general relativity, diffeomorphism invariance (a symmetry distinct from that of local Lorentz invariance) can be preserved by elevating this field to a dynamical quantity.

These considerations motivate the investigation of “vector-tensor” theories of gravity. One such model couples gravity to a vector field that is constrained to be everywhere timelike and of unit norm. Theories providing a preferred timelike direction are of most interest as rotational symmetry appears to be soundly preserved. The unit-norm condition embodies the notion that the theory assigns no physical importance to the norm of the vector. For a review of recent investigations of this model, see [1]. Following these authors, I shall refer to this theory as ‘Einstein–Æther’ theory, or ‘Æ-theory’.

The purpose of this work is to demonstrate the effect of a field redefinition on the conventional, second-order Æ-theory action. The redefinition considered is of the form $g_{ab} \to g'_{ab} = A(g_{ab} - (1 - B)u_a u_b)$, $u^a \to u'^a = (1/\sqrt{AB})u^a$, where $g_{ab}$ is a Lorentzian metric and $u^a$ is the ‘æther’ field. The action has the most general form that is generally covariant, second order in derivatives, and consistent with the unit-norm constraint. The redefinition preserves this most-general form, since it preserves covariance of the action, does not introduce higher derivatives, and preserves the unit-norm constraint. The net effect is then a transformation of the coupling constants in the action. The study of Æ-theory systems can be simplified in certain cases by invoking this transformation to give the couplings more convenient values; e.g. by setting one of the constants to zero.

This work generalizes a result of Barbero and Villaseñor [2] that shows equivalence between vacuum general relativity and an Æ-theory system whose coupling constants satisfy certain relations. The four constants must be specific functions of one free parameter for their result to apply. I consider here the general case in which the parameters have arbitrary values. This work also uses a simpler parametrization of the redefinition than that of [2] and works with a now-more-common form of Æ-theory action. The translation between this
work and will be given below.

II. TRANSFORMATION OF THE ACTION

The conventional, second-order $\mathcal{A}$-theory action $S$ has the form

$$S = \frac{-1}{16\pi G} \int \sqrt{|g|} \mathcal{L},$$

(1)

with Lagrangian $\mathcal{L}$

$$\mathcal{L} = R + c_1(\nabla_a u_b)(\nabla^a u^b) + c_2(\nabla_a u^a)(\nabla_b u^b) + c_3(\nabla_a u^b)(\nabla_b u^a) + c_4(u^a\nabla_a u^c)(u^b\nabla_b u_c),$$

(2)

where $R$ is the scalar curvature of the metric $g_{ab}$ (with signature (+−−−)\[8\]), and the $c_i$ are dimensionless constants. This action has the most general form that is covariant, second-order in derivatives, and consistent with the ‘unit-constraint’ $u^a u_a = 1$.

We will assume that the fields are on-shell with respect to this constraint, rather than incorporate it via a Lagrange multiplier. This approach is justified if we view two actions as equivalent if they lead to the same equations of motion. We obtain the same equations of motion either by subjecting the off-shell action with a multiplier term to general variations, then solving for the multiplier in terms of the other fields, or by subjecting the on-shell action to variations that preserve the constraint. It follows that two actions are equivalent if they agree on-shell. The redefinition given below preserves the constraint; hence, it preserves this sense of equivalence.

We begin by considering ‘unprimed’ variables—a Lorentzian metric $g_{ab}$ and a timelike vector field $u^a$, satisfying $g_{ab}u^a u^b = 1$. We then define ‘primed’ fields:

$$g'_{ab} = A(g_{ab} - (1 - B)u_a u_b)$$

$$u'^a = \frac{1}{\sqrt{AB}} u^a$$

(3)

where $A$ and $B$ are positive constants. The sign of $A$ merely changes the signature convention of the metric so is irrelevant. A negative value of $B$ results in a primed metric of Euclidean signature. We restrict to positive $B$ to ensure comparison of Lorentzian theories. The primed inverse-metric $g'^{ab}$ and the primed æther one-form $u'^a \equiv g'_{ab} u'^b$ are then uniquely determined.
in terms of unprimed fields:

\[ g_{ab} = \frac{1}{A} \left( g_{ab} - \left(1 - \frac{1}{B}\right) u^a u^b \right) \]

\[ u'_a = \sqrt{AB} u_a. \]  \hspace{1cm} (4)

It follows that \( u'^a u'_a = 1. \)

To observe the effect of this redefinition on the action \( \Pi, \) we shall start with the primed action and express it in terms of unprimed variables. We shall find that the form of the action is left invariant, with new parameters \( G, c_i, \) given as functions of \( A, B, \) and the original \( G', c'_i. \) The calculation is straightforward but lengthy—the demonstration will be explicit to ease the checking of the final results.

Let us begin by considering the role of the parameter \( A, \) whose net effect is a re-scaling of the action. This occurs because \( A \) re-scales the field variables in such a way that each term in the Lagrangian \( 2 \) acquires the same factor. Writing the Lagrangian in terms of primed variables, then invoking the substitutions \( 3 \) and \( 4, \) one finds that each term in the un-primed Lagrangian carries an over-all factor of \( 1/A. \) The ratio of primed-to-unprimed metric determinants will equal \( A^4, \) times a \( B \)-dependent factor given below. Thus, the un-primed action \( \Pi \) will carry a net factor of \( A \) and will have no other \( A \)-dependence. This factor can be absorbed into a redefinition of \( G. \) Having thus accounted for the effect of \( A, \) we will set \( A = 1 \) in the calculations that follow.

We can deduce the full relation between metric determinants \( g, g' \) by evaluating them in a basis, orthonormal with respect to \( g_{ab}, \) of which \( u^a \) is a member. In this basis, \( g = -1. \) From the expression \( g_{ab} = u_a u_b + h_{ab}, \) with \( h_{ab} u^a = 0, \) we have \( g'_{ab} = u'_a u'_b + h_{ab}. \) It follows that \( g' = -(u^a u'_a)^2 = -B \) in this basis. Generalizing to an arbitrary basis, we conclude that

\[ g' = Bg. \]  \hspace{1cm} (5)

The action then re-scales: \( S' = \sqrt{BS}. \) The above rescalings effect a redefinition of Newton’s constant:

\[ G = \frac{G'}{A\sqrt{B}}, \]  \hspace{1cm} (restoring \( A \) temporarily).

4
A. Curvature Term

We turn now to the curvature term in the Lagrangian \( \text{[2]} \). We start by examining properties of the redefined connection coefficients \( \Gamma^a_{bc} \),

\[
(\Gamma^a_{bc})' = \Gamma^a_{bc} + g^{ad} D_{dbc},
\]

where

\[
D_{abc} = \frac{B - 1}{2} \left( \delta^d_a - (1 - 1/B) u^d u_a \right) \left[ \nabla_b (u_a u_c) + \nabla_c (u_a u_b) - \nabla_a (u_b u_c) \right].
\]

Let us define the following quantities:

\[
S_{ab} = \nabla_a u_b + \nabla_b u_a \\
F_{ab} = \nabla_a u_b - \nabla_b u_a \\
\dot{u}^a = u^b \nabla_b u^a.
\]

We can organize \( D_{abc} \) as follows:

\[
D_{abc} = \frac{B - 1}{2} \left( u_a X_{bc} + u_b F_{ca} + u_c F_{ba} \right),
\]

where

\[
X_{bc} = \frac{1}{B} \left( S_{bc} + (B - 1)(\dot{u}_b u_c + u_b \dot{u}_c) \right),
\]

and the unit-constraint has been enforced.

We will now note some useful relations involving \( D_{abc} \). To begin, we find that

\[
u^a S_{ab} = u^a X_{ab} = u^a F_{ab} = \dot{u}_b.
\]

We then find that contraction once with \( u^a \) gives

\[
u^a D_{abc} = \frac{(B - 1)}{2B} \left( S_{bc} - (\dot{u}_b u_c + u_b \dot{u}_c) \right),
\]

\[
u^c D_{abc} = \frac{B - 1}{2} \left( F_{ba} + (\dot{u}_a u_b + u_a \dot{u}_b) \right),
\]

and that contraction twice gives

\[
u^b u^c D_{abc} = (B - 1) \ddot{u}_a,
\]

\[
u^a u^b D_{abc} = 0.
\]
In addition,

\[ X^{ab} u^c D_{abc} = (B - 1) \dot{u}_a \dot{u}^a, \]
\[ F^{ab} u^c D_{abc} = \frac{(1 - B)}{2} F_{ab} F^{ab}. \]  

(15)

As for the trace of \( D_{abc} \), we find that

\[ D^b_{bc} \equiv g^{ab} D_{abc} = 0. \]  

(16)

Let us now examine the transformation of the curvature tensor. A short calculation reveals that

\[ (R_{abc})' = R_{abc} + 2 \nabla [b D_{a]c} + 2 D_{e[b} D_{a]d}, \]

so that

\[ (R_{ab})' = R_{ab} + W_{ab}, \]  

(17)

where

\[ W_{ab} = \nabla_a D_{ab} - D_{ea} D_{eb}. \]  

(19)

The scalar curvature \( R' = R_{ab} g^{ab} \) takes the form

\[ R' = R_{ab} g^{ab} + \frac{1 - B}{B} R_{ab} u^a u^b + W_{ab} \left( g^{ab} + \frac{1 - B}{B} u^a u^b \right). \]  

(20)

The second term on the right-hand-side can be re-expressed via the definition of the curvature tensor:

\[ R_{ab} u^a u^b = u^a \nabla_b \nabla_a u^b - u^a \nabla_a \nabla_b u^b \]
\[ = (\nabla_a u^a)(\nabla_b u^b) - (\nabla^a u^b)(\nabla_b u_a) + \nu, \]  

where \( \nu \) represents a total divergence. We can discard this, with the same justification given above for taking the fields as on-shell. The symbol \( \nu \) will continue to represent other total divergences that appear in the calculations below, but the specific form of the divergence will differ by equation. The third term on the right-hand-side of (20) has the form

\[ W_{ab} g^{ab} = -D^{cba} D_{abc} + \nu \]
\[ = \frac{(1 - B)}{2} \left( u^c X^{ab} + u^b F^{ac} \right) D_{abc} + \nu \]
\[ = -\frac{(1 - B)^2}{2} \left( \dot{u}_a u_a - \frac{1}{2} F_{ab} F^{ab} \right) + \nu. \]  

(21)
As for the last term in (20), we have

\[ W_{ab}u^a u^b = u^a u^b (\nabla^c D_{cab} - D_{cda} D^{dc}_b) \]

\[ = -D_{abc} u^c (2\nabla^a u^b + D^{bad} u_d) + \nu \]

\[ = \frac{1 - B}{2} (\dot{u}_a u_b + u_a \dot{u}_b) + F_{ba} \]

\[ \times (S^{ab} + \frac{B - 1}{2} (\dot{u}_a u_b + u_a \dot{u}_b) + (\frac{B + 1}{2}) F^{ab}) + \nu \]

\[ = \frac{(B^2 - 1)}{2} (\dot{u}_a \dot{u}_a - \frac{1}{2} F_{ab} F^{ab}) + \nu. \tag{23} \]

Combining the above and suppressing a total divergence, we can express the transformation of the scalar curvature as

\[ R' = R + \frac{1 - B}{B} ((\nabla^a u^b)(\nabla_b u^b) - (\nabla^a u^b)(\nabla_b u_a)) + \frac{(1 - B)^2}{2B} (\dot{u}_a \dot{u}_a - \frac{1}{2} F_{ab} F^{ab}) \]

\[ = R - \frac{1 - B}{2B} \left\{ (1 - B) (\nabla^a u_b)(\nabla_a u^b) - 2(\nabla^a u_a)(\nabla_b u^b) \right\} \tag{24} \]

\[ + (1 + B) (\nabla^a u^b)(\nabla_b u_a) - (1 - B) (\dot{u}_a \dot{u}_a) \right\} \]

We can extract from this expression contributions \( a_i \) to the redefined \( c_i \):

\[ a_1 = -\frac{(1 - B)^2}{2B} \]

\[ a_2 = \frac{1 - B}{B} \]

\[ a_3 = -\frac{1 - B^2}{2B} \]

\[ a_4 = \frac{(1 - B)^2}{2B}. \tag{25} \]

The constants \( a_i \) are characterized by the relations

\[ 0 = a_1 + a_4 = a_1 + a_2 + a_3 = a_1 (a_1 - 2) - (a_3)^2, \tag{26} \]

and \( a_1 < 0 \). If the \( c_1 \) satisfy these conditions, then the \( \mathcal{E} \)-system is equivalent to pure gravity via a field redefinition. The translation from this result to that of [2] is made by choosing \( A = -\sqrt{\alpha(\alpha + \beta)} / 2 \) and \( B = -\alpha / (\alpha + 2\beta) \) [compare the first line of (24) with Eqn. (6) of [2]].
B. Æther Term

We now proceed to examine the transformation of the æther portion of the Lagrangian. From the form of the covariant derivative

\[ (\nabla_a u^b)' = \frac{1}{\sqrt{B}}(\nabla_a u^b + D_{ac}^b u^c), \]  

and the relations (16) and (14), we can deduce the transformation of the \( c_2 \) and \( c_4 \) terms:

\[ (\nabla_a u^a)' = (1/\sqrt{B})(\nabla_a u^a), \quad (\dot{u}^a)' = \dot{u}^a \]

and further \((\dot{u}^a)' = \ddot{u}^a\). Thus, we have

\[ ((\nabla_a u^a)(\nabla_b u^b))' = \frac{1}{B}((\nabla_a u^a)(\nabla_b u^b)), \]  

and

\[ (\dot{u}^a \ddot{u}_a)' = (\dot{u}^a \ddot{u}_a). \]

These results indicate contributions of \( c_2' / B \) to \( c_2 \) and \( c_4' / B \) to \( c_4 \).

It will be convenient to reorganize the \( c_1 \) and \( c_3 \) terms:

\[ c_1(\nabla_a u_b)(\nabla^a u^b) + c_3(\nabla_a u_b)(\nabla^b u^a) = \frac{c_+}{4} S_{ab} S_{ab} + \frac{c_-}{4} F_{ab} F_{ab}, \]

where \( c_\pm = c_1 \pm c_3 \). We then need the form of the covariant derivative of \( u'_a \),

\[ (\nabla_a u_b)' = \sqrt{B}(\nabla_a u_b - D_{cab} u^c) \]

\[ = \frac{1}{2\sqrt{B}}(S_{ab} + (B - 1)(\dot{u}_a u_b + u_a \dot{u}_b)) + \frac{\sqrt{B}}{2} F_{ab}. \]

Raising an index on the symmetrized derivative,

\[ (S_{cb} g^{ac})' = \frac{1}{\sqrt{B}}(S_{b}^a + (B - 1)u_b \dot{u}_a), \]

leads to

\[ (S_{ab} S^{ab})' = (S_{b}^a S_{a}^b)' = \frac{1}{B} \left( S_{ab} S^{ab} + 2(B - 1)\dot{u}_a \ddot{u}_a \right), \]

indicating contributions of \( c_+' / B \) to \( c_+ \) and \((B - 2)c_+' / 2B \) to \( c_4 \). Raising an index on the anti-symmetrized derivative,

\[ (F_{cb} g^{ac})' = \sqrt{B} \left( F_{b}^a + \frac{1 - B}{B} u_a \dot{u}_b \right), \]

leads to

\[ (F_{ab} F^{ab})' = -(F_{b}^a F_{a}^b)' = B \left( F_{ab} F^{ab} + 2\frac{1 - B}{B} \dot{u}_a \ddot{u}_a \right), \]

indicating contributions of \( c_2' / B \) to \( c_2 \) and \( c_4' / B \) to \( c_4 \).
indicating contributions of $Bc'_- \to c_-$ and $(1 - B)c'_- / 2 \to c_4$.

Collecting the above results, we find contributions $b_i$ to the redefined $c_i$:

$$b_1 = \frac{1}{2B} (c'_+ + B^2 c'_-)$$
$$= \frac{1}{2B} ( (1 + B^2) c'_1 + (1 - B^2) c'_3 )$$

$$b_2 = \frac{c'_2}{B}$$

$$b_3 = \frac{1}{2B} (c'_+ - B^2 c'_-)$$
$$= \frac{1}{2B} ( (1 - B^2) c'_1 + (1 + B^2) c'_3 )$$

$$b_4 = c'_4 - \frac{1 - B}{2B} (c'_+ - Bc'_-)$$
$$= c'_4 - \frac{1 - B}{2B} ((1 - B)c'_1 + (1 + B)c'_3).$$

The redefined $c_i$ are given by the sum of $a_i$ (25) and $b_i$ (36):

$$c_1 = \frac{1}{2B} (c'_+ + B^2 c'_- - (1 - B)^2)$$
$$= \frac{1}{2B} ( (1 + B^2) c'_1 + (1 - B^2) c'_3 - (1 - B)^2 )$$

$$c_2 = \frac{1}{B} (c'_2 + 1 - B)$$

$$c_3 = \frac{1}{2B} (c'_+ - B^2 c'_- - (1 - B^2))$$
$$= \frac{1}{2B} ( (1 - B^2) c'_1 + (1 + B^2) c'_3 - (1 - B^2) )$$

$$c_4 = c'_4 - \frac{1 - B}{2B} (c'_+ - Bc'_- - (1 - B))$$
$$= c'_4 - \frac{1}{2B} ((1 - B)^2 c'_1 + (1 - B^2) c'_3 - (1 - B)^2).$$

In addition,

$$c_+ = \frac{1}{B} (c'_+ - (1 - B))$$

$$c_- = Bc'_- + (1 - B).$$

III. DISCUSSION

The redefinition (3) can simplify the problem of characterizing solutions for a specific set of $c_i$. This is done by transforming that set into one in which the $c_i$ take on more convenient values. It was noted in [2] that a system with restricted values of the coefficients, equivalent
to \(c_i\) that satisfy \(26\), can be transformed into æther-free general relativity. The current work extends this result by allowing for general values of the \(c_i\). Using this result, different sets of \(c_i\) are seen to be equivalent. For example, it follows from the relations \(37\) that a set of \(c_i\) is equivalent to one in which one of \(c_+, c_-\), or \(c_2\) vanishes if the original values satisfy, respectively, \(c_+ < 1\), \(c_- < 1\), or \(c_2 > -1\).

An extra constant can be eliminated in the case of spherically-symmetric configurations \([4]\). In this case, the hyper-surface orthogonality and unit norm of the æther imply the vanishing of the twist \(\omega_a = \epsilon_{abcd}u^b\nabla^c u^d\), so that
\[
\omega_a\omega^a = \dot{u}^a\dot{u}_a - 1/2 F^{ab}F_{ab} = 0.
\] (38)

The redefinition of a particular configuration preserves any Killing symmetries shared by the metric and æther fields, so it preserves the relation \(38\). One can then eliminate, for instance, \(c_+\) by redefinition and \(c_4\) by absorption into \(c_-\). The Lagrangian is reduced to the form
\[
\mathcal{L} = R + c_-^{-4}F_{ab}F^{ab} + c_2 (\nabla_a u^a)^2.
\] (39)

This is considerably simpler than the general form \(2\), since the connection enters the æther action only through the divergence of \(u^a\).

Spherically-symmetric, static \(\mathcal{E}\)-theory black hole solutions have been shown to exist \([1]\). One can apply the above to simplify their study. The question arises, though, of how to define the location of the horizon. Initial results \([5]\) indicate that a solution with a horizon can be equivalent to one without, where the horizon is defined via the fastest speed of linearized wave modes.

Once non-æther matter is included, a metric redefinition not only changes the \(c_i\) coefficients, but also modifies the matter action. The fact that Lorentz violating effects in non-gravitational physics are already highly constrained \([6]\) means that, to a very good approximation, there is a universal metric to which matter couples. Within the validity of this approximation, one can identify the field \(g_{ab}\) with this universal metric, thus excluding any æther dependence from the matter action. This identification then eliminates the freedom to redefine the metric. Recent studies of observational bounds on the values of the \(c_i\), such as \([7]\), have adopted this convention.
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