Finite matrix model of quantum Hall fluids on $S^2$

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Abstract

Based on Haldane’s spherical geometrical formalism of two-dimensional quantum Hall fluids, the relation between the noncommutative geometry of $S^2$ and the two-dimensional quantum Hall fluids is exhibited. If the number of particles $N$ is infinitely large, two-dimensional quantum Hall physics can be precisely described in terms of the noncommutative $U(1)$ Chern-Simons theory proposed by Susskind, like in the case of plane. However, for the finite number of particles on two-sphere, the matrix-regularized version of noncommutative $U(1)$ Chern-Simons theory involves in spinor oscillators. We establish explicitly such a finite matrix model on two-sphere as an effective description of fractional quantum Hall fluids of finite extent. The complete sets of physical quantum states of this matrix model are determined, and the properties of quantum Hall fluids related to them are discussed. We also describe how the low-lying excitations are constructed in terms of quasiparticle and quasihole excitations in the matrix model. It is shown that there consistently exists a Haldane’s hierarchical structure of two-dimensional quantum Hall fluid states in the matrix model. These hierarchical fluid states are generated by the parent fluid state for particles by condensing the quasiparticle and quasihole excitations level by level, without any requirement of modifications of the matrix model.

Keywords: matrix model, non-commutative geometry, quantum Hall fluid.

1 Introduction

The planar coordinates of quantum particles in the lowest Landau level of a constant magnetic field provide a well-known and natural realization of noncommutative space [1]. The physics of electrons in the lowest Landau level exhibits many fascinating properties. In particular, when the electron density lies in certain rational fractions of the density corresponding to a fully filled lowest Landau level, the electrons are condensed into special incompressible fluid states whose excitations exhibit unusual phenomena such as fractional charge and fractional statistics. For the filling fractions $\nu = \frac{1}{m}$, the physics of these states is accurately described by certain wave functions proposed by Laughlin [2], and more general wave functions may be used to describe the various types of excitations about the Laughlin states.

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There has recently appeared an interesting connection between quantum Hall effect and noncommutative field theory. In particular, Susskind [3] proposed that noncommutative Chern-Simons theory on the plane may provide a description of the (fractional) quantum Hall fluid and, specifically, of the Laughlin states. Susskind’s noncommutative Chern-Simons theory on the plane describes a spatially infinite quantum Hall system. It gives the Laughlin states at filling fractions \( \nu \) for a system of an infinite number of electrons confined in the lowest Landau level. The fields of this theory are infinite matrices which act on an infinite Hilbert space, appropriate to account for an infinite number of electrons. Subsequently, Polychronakos [4] proposed a matrix regularized version of Susskind’s noncommutative Chern-Simons theory in an effort to describe finite systems with a finite number of electrons in limited spatial extent. This matrix model was shown to reproduce the basic properties of the quantum Hall droplets and two special types of excitations of them. Furthermore, it was shown that there exists a complete minimal basis of exact wave functions for the matrix regularized version of noncommutative Chern-Simons theory at arbitrary level \( \nu^{-1} \) and rank \( N \), and that these are in one to one correspondence with Laughlin wave functions describing excitations of a quantum Hall droplet composed of \( N \) electrons at filling fraction \( \nu \) [5]. It is believed that the matrix regularized version of noncommutative Chern-Simons theory is precisely equivalent to the theory of composite fermions in the lowest Landau level, and should provide an accurate description of fractional quantum Hall states. It does appear an interesting conclusion that they are in agreement with the long distance behavior, but the short distance behavior is different [6]. However, it should be pointed that the Polychronakos’ finite matrix model is still defined on the two-dimensional plane.

It is well known that it is convenient to formulate the quantum Hall system on the two-dimensional sphere for the description of quantum Hall fluids. Such a formulation appears in the work on fractional quantum Hall effect based on the spherical geometry [7]. The Haldane’s model is set up by a two-dimensional electron gas of \( N \) particles on a spherical surface in radial monopole magnetic field. A Dirac’s monopole is at the center of the two-dimensional sphere. The Haldane’s model describes not only a variant of Laughlin’s scheme with fully translationally invariant wave functions, but also a hierarchy of the quantum Hall fluid states. It is a well known fact that the two-dimensional compact spherical space can be mapped to the flat Euclidean space by the standard stereographical mapping. In fixed limit, the connection between this model and the noncommutative Chern-Simons theory can be exhibited clearly. Precisely, the noncommutative property of particle’s coordinates in the Haldane’s model should be described in terms of fuzzy two-sphere [8] ( see below in details ). The different noncommutative manifolds should correspond to finite matrix models with different geometrical properties. Recently, there has been much interest in formulating Chern-Simons theories on noncommutative manifolds [9, 10, 11]. The present problem is what is the finite matrix model for the quantum Hall fluid on two-sphere.

The goal of this paper is to establish the finite regularized-matrix model describing the quantum Hall fluids on \( S^2 \). Based on the Hellerman and Raamsdonk [5]’s discussion for the equivalence of two-dimensional quantum Hall physics and noncommutative field theory, one knows that the second-quantized field theoretical description of quantum Hall fluids for various filling fractions should involve in certain noncommutative field theories. On the 2-dimensional plane, such a noncommutative field theory is the regularized matrix version of the \( U(1) \) noncommutative Chern-Simons theory. For the quantum Hall system on \( S^2 \), which is described by the Haldane’s model, what we want here is to construct a finite matrix model for second-quantized field theory of quantum Hall fluids. On the other hand, we hope to explore the possible hierarchical structure of the quantum Hall fluids in the finite matrix model.

This paper is organized as follows. Section two introduces the two-dimensional quantum Hall
model on the spherical geometry proposed by Haldane [7], and analyzes the noncommutativity of the coordinates on \( S^2 \) by focusing on the lowest Landau level states of the system. It will be shown that this noncommutative geometry is the geometry of fuzzy \( S^2 \). In order to establish the description of effective field theory related with it, we introduce the Hopf mapping of this fuzzy geometry to perform the Hopf fibration. Like the usual Hopf mapping of \( S^2 \) in the presence of Dirac monopole field, the effective field theory is singular-free on the field configurations obtained by the Hopf fibration with fuzzy \( S^2 \) as the base. These configurations are described by the spinor with two complex components. By taking the number of particles infinitely large, it is shown that this effective theory is equivalent to the \( U(1) \) noncommutative Chern-Simons theory proposed by Susskind. However, our matrix regularized version of the effective theory with the finite number of particles is different with the Polychronakos’ finite matrix model, and is related to the matrix fields of the spinor with two complex components since the effective theory is invariant under the \( U(1) \) gauge transformation of such spinors. In the section three, we provide the description of the finite matrix model of the quantum Hall fluids on \( S^2 \). Furthermore, the Fock space structure of this matrix model is analysed, and its complete sets of physical quantum states are determined. The properties of quantum Hall fluids related to them are also discussed. Section four investigates the condensate mechanism of the low-lying excitations in the finite matrix model of quantum Hall fluids on \( S^2 \). It is shown that there exists indeed the Haldane’s hierarchy in the 2-dimensional quantum Hall fluids in our matrix model, and such hierarchy is dynamically generated by condensing of excitations of the quantum Hall fluids level by level. Section five includes a summary of the main results in this paper and remarks on further research in this direction.

2 Haldane’s quantum Hall system and fuzzy \( S^2 \) structure

In the quantum Hall effect problem, it is advantageous to consider compact spherical space which can be mapped to the flat Euclidean space by the standard stereographical mapping[7]. Haldane considered a system where a two-dimensional electron gas of \( N \) particles is placed on a two-sphere \( S^2 \) in a radial Dirac monopole magnetic field \( B \). A point \( x^a \) on \( S^2 \) with radius \( R \) can be described by dimensionless vector coordinates \( n^a = x^a/R \), with \( a = 1, 2, 3 \) which satisfy \( n^a n^a = 1 \). The single particle hamiltonian in this system reads

\[
\mathcal{H} = \frac{1}{2MR^2} \sum_a \Lambda^a \Lambda^a
\]

(1)

where \( M \) is the effective mass, and \( \vec{\Lambda} = \vec{r} \times [\vec{p} - e\vec{A}] = \vec{r} \times [\vec{p} - e\vec{A}] \) is the dynamical angular momentum of the particle. The relation between the vector potential \( \vec{A} \) and the magnetic field is given by \( \vec{\nabla} \times \vec{A} = B\vec{n} \). Due to the presence of the Dirac monopole field, the dynamical angular momentum \( \Lambda^a \) does not obey the algebraic relation of the usual angular momentum. One can easily check that they satisfy the commutation relations

\[
[\Lambda^a, \Lambda^b] = i\hbar \varepsilon^{abc} (\Lambda^c + eBR^2 n^c).
\]

(2)

However, \( L^a \equiv \Lambda^a - eBR^2 n^a \) provide the generators of rotations in the presence of the Dirac monopole field. Indeed, by direct calculation, one can show that

\[
[L^a, L^b] = i\hbar \varepsilon^{abc} L^c, \quad [L^a, \Lambda^b] = i\hbar \varepsilon^{abc} \Lambda^c, \quad [L^a, n^b] = i\hbar \varepsilon^{abc} n^c.
\]

(3)

The vector \( \vec{\Lambda} \) has no component normal to the surface, so we have \( L^a n^a = -eBR^2 = n^a L^a \). As pointed out by Haldane[7], the spectrum of \( \Lambda^a \Lambda^a \) determined by the angular momentum operators
$L^a$ is $\Lambda^a \Lambda^a = (L + eBR^2)^a (L + eBR^2)^a = \hbar^2 [l(l+1) - S^2]$. Because $\Lambda$ is a hermitian operator and the Hamiltonian $H \sim \Lambda^a \Lambda^a$ must be larger than or equal to zero, one determines the spectrum of the algebra $L^a$ as $l = S + n, n = 0, 1, 2, \cdots$. Hence, for a given $S$, the energy eigenvalues of the Hamiltonian Eq.(1) are

$$E_n = \frac{\hbar^2}{2MR^2} [n(n+1) + (2n+1)S].$$

4

The above energy spectrum when $n = 0$ corresponds to the lowest Landau level. Since $S$ is the spin of the particle, the degeneracy of the lowest Landau level is $2S + 1$.

On the other hand, we can discuss the classically canonical dynamics from the Hamiltonians $H$ and $H + V(x^a)$, where $V(x^a)$ is the potential energy with rotational symmetry. By means of the correspondence between classical and quantum physics, one can straightforwardly read off the fundamental Poisson brackets of the classical degrees of freedom from their corresponding commutation relations. In the canonical Hamiltonian formulation, the evolution of dynamical variables with time is described by the canonical Hamilton equation, i.e., $\dot{\Lambda}^a = \{\Lambda^a, H\} = \frac{eB}{M} \epsilon^{abc} \Lambda^b \Lambda^c \neq 0$. This implies that the dynamical angular momentum is not a conservative quantity of the system. In fact, in the presence of the Dirac monopole field, the generator of rotations is modified to $L^a$, which is a conservative quantity since $\dot{L}^a = \{L^a, H\} = \frac{eBR}{M} \epsilon^{abc} \Lambda^b \Lambda^c = 0$. If we consider the system including a term of potential energy $V(x^a)$ with the symmetry of rotations, $L^a$ is still conservative. That is

$$\dot{L}^a = \dot{\Lambda}^a - eBR^2 \dot{n}^a = 0.$$  

So the variation of $n^a$ with time can be given by the canonical Hamiltonian equation of $\Lambda^a$

$$\dot{n}^a = \frac{1}{eBR^2} \frac{eB}{M} \epsilon^{abc} n^b \Lambda^c + \frac{\partial V}{\partial n^b} \epsilon^{abc} n^c.$$  

Since we are interested in the equation of motion in the lowest Landau level, we can take the infinite limit of mass $M \to \infty$. In this limit, we obtain the following equation of motion

$$\dot{n}^a = \frac{1}{eBR^2} \frac{\partial V}{\partial n^b} \epsilon^{abc} n^c.$$  

This implies that the momentum variables can be fully eliminated in the lowest Landau level. The elimination of momentum variables leads to the coordinates on the two-sphere which are noncommutative. Restricted to the lowest Landau level states, the equation of motion can be equivalently derived from the fundamental Poisson bracket

$$\{n^a, n^b\} = \frac{1}{eBR^2} \epsilon^{abc} n^c, \quad n^a n^a = 1.$$  

This Poisson algebra can be realized by the matrix commutator

$$[n^a, n^b] = \frac{1}{eBR^2} \epsilon^{abc} n^c, \quad n^a n^a = 1.$$  

Conclusively, if we focus on the lowest Landau level of the system, the Haldane’s spherical geometry becomes the noncommutative geometry of the fuzzy $S^2[8]$. In order to exhibit the fuzzy property of algebra (9), we finish the isomorphic mapping from algebra (9) to the $SU(2)$ algebra. Set $n^a \mapsto \frac{X^a}{eBR}$, so the equation (9) becomes

$$[X^a, X^b] = i \epsilon^{abc} X^c.$$  

4
which is the standard SU(2) algebra. The quadratic Casimir of SU(2) in the \(N\)-dimensional irreducible representation is given by
\[
X^a X^a = \frac{1}{4} (N^2 - 1).
\]
(11)
The constraint \(n^a n^a = 1\) leads to
\[
e BR^2 = \sqrt{\frac{1}{4} (N^2 - 1)}.
\]
(12)
This relation has shown that the two parameters \(B\) and \(R\) should be quantized, which exhibits the fuzzy property of two sphere. In order to compare them with the usual expression, we rewrite the relation \(e BR^2\) as \(e BR \cdot R \equiv \frac{R}{\theta'} = \sqrt{\frac{1}{4} (N^2 - 1)}\), where \(\theta' = e BR\). Then, we have
\[
[n^a, n^b] = i \frac{\theta'}{R} \epsilon^{abc} n^c \equiv i 2 \theta \epsilon^{abc} n^c, \quad n^a n^a = 1.
\]
(13)
This algebraic relation is the starting point of the following discussion about the Hopf mapping of the fuzzy \(S^2\).

It is well known that for a monopole field, no single vector potential exists which is singularity-free over the entire manifold \(S^2\). The use of two vector potentials living respectively on the north and on the south semi-spheres, which was advocated by Wu and Yang[12], provides a way round the singularity problem, since one can use each in a region where it is singularity-free, and then connects the two in a convenient overlap region by a gauge transformation. However, the Wu-Yang procedure is not well adapted for our later purpose to establish the effective description of the system and its quantization. For the case of \(U(1)\) Dirac’s monopole, one can obtain the related effective Lagrangian, which is singularity-free, by using the Balachandran formalism[13]. The key step is to finish the first Hopf fibration of \(S^2\) to get \(S^3\). The first Hopf map is a mapping from \(S^3\) to \(S^2\) and is related to Dirac’s monopole. In the presence of a Dirac’s monopole, the \(U(1)\) bundle over \(S^2\) is topologically non-trivial. However, since \(S^3\) is parallelizable, one can use first Hopf map to define a non-singular vector potential due to Dirac’s monopole everywhere on \(S^3\), called as the first Hopf fibration.

Let us introduce the notation \(z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\) for the two-component spinor. The spinor \(z\), in principle, has three degrees of freedom since the normalization condition \(z^\dagger z = 1 = |z_1|^2 + |z_2|^2\) is the only constraint on the two complex numbers \(z_1, z_2\). So they are actually defined on the surface of \(S^3\). However, the Hopf projection map which takes us from \(S^3\) to \(S^2\) is given by
\[
\vec{n} = z^\dagger \sigma z,
\]
(14)
where \(\sigma^a\) are three Pauli matrices. It should be noticed that the \(U(1)\) transformation \(z \rightarrow e^{i\alpha} z\) leaves \(n^a\) invariant and so the inverse image of any point on \(S^2\) is a circle on \(S^3\). Now we ask what Poisson relation for \(z\)’s can be used to produce the Poisson algebra (8) of the fuzzy \(S^2\). It can be easily checked that the answer to this problem is
\[
\{z, z^\dagger\} = \theta,
\]
(15)
that is,
\[
\{z_1, \bar{z}_1\} = \theta = \{z_2, \bar{z}_2\}, \{z_1, \bar{z}_2\} = 0 = \{z_2, \bar{z}_1\}.
\]
(16)
Subsequently, we focus on the description of the effective action of particles in the presence of the Dirac monopole field. In the presence of a Dirac monopole, the \(U(1)\) bundle over \(S^2\) is topologically non-trivial. Such a non-trivial topological character leads to the appearance of an additional term,
called the Wess-Zimino term, in the effective action of the system. The effective action had been obtained by Stone[14] in the discussion of the coupling of the $SO(3)$ rotor and spinor and the calculation of the Berry phase. The effective action reads

$$I = \frac{1}{2f} \int dt i n^a i_\alpha + \int A^a d n^a,$$

(17)

where, $f$ is dependent of the parameters of the system, and $A$ is the potential of the Dirac monopole which cannot be globally expressed on $S^2$ due to the singularity of the Dirac string. However, by means of the Hopf fibration of $S^2$ and its $U(1)$ gauge symmetry, the potential of the Dirac monopole can be globally written on $S^3$ as

$$A = i \frac{\tilde{\lambda}}{2} [z^\dagger dz - dz^\dagger z],$$

(18)

where $\tilde{\lambda}$ is related to the magnetic charge of the Dirac monopole. It should be pointed out that the potential $A$ is equivalent to $A^a d n^a$ up to an $U(1)$ gauge transformation, and is non-singular everywhere on $S^3$. Furthermore, the first term in the effective action (17) can be also described by the spinors defined on $S^3$[13, 15]. In fact, if one quantizes the system described by the effective action after finishing the Hopf fibration, he gets the energy spectrum determined by the hamiltonian (1)[15]. Hence, the Haldane’s quantum Hall system on the spherical geometry can be equivalently described by the effective action (17). Restricted on the lowest Landau level state, as mentioned by us above, the contribution of kinetic energy in the effective action should be ignored, which is equivalent to taking the infinite limit of $f$. So the physics in the lowest Landau level is described by the following action

$$I_e = \int A.$$

(19)

As mentioned above, the $U(1)$ gauge transformation $z \rightarrow e^{i\alpha} z$ leaves $n^a$ invariant, so the effective action after finishing the Hopf fibration is also invariant under such an $U(1)$ gauge transformation. Furthermore, projected in the lowest Landau level state, the effective action $I_e = i \frac{\tilde{\lambda}}{2} \int dt [z^\dagger \partial_t z - \partial_t z^\dagger z]$ should possess this $U(1)$ gauge symmetry. By the standard way of introducing the coupling of gauge field, we can write the effective action in the explicitly gauge invariant form $I_e = i \frac{\tilde{\lambda}}{2} \int dt [z^\dagger (\partial_t + i A_0) z - (\partial_t - i A_0) z^\dagger z]$, where $A_0$ is a $U(1)$ gauge field. Indeed, this action is invariant under the $U(1)$ gauge transformations $z \rightarrow e^{i\alpha} z$ and $A_0 \rightarrow A_0 - \partial_t \alpha$. However, now the spinor $z$ becomes noncommutative since it is from the Hopf mapping of the fuzzy $S^2$. The matrix realization of $z$ is required by the non-trivial algebraic relations (15) [i.e. (16)]. The gauge field $A_0$ should adjointly act on the matrix $z$ in order to make the covariant derivative $\partial_t + i A_0$ satisfy the derivative property. Finally, the effective action projected in the lowest Landau level state is given by

$$I_e = i \frac{\tilde{\lambda}}{2} \int dt Tr \{z^\dagger (\partial_t z + [A_0, z]) - (\partial_t z^\dagger - [A_0, z^\dagger] z)\}.$$

(20)

This is a matrix theory similar to that describing $D0$-branes in the string theory[16, 17]. We can use this theory to investigate the fluctuations of the spherical brane, which describes the excitations of the Hall fluids living on $S^2$, by using the method of expanding the matrix field in terms of the fluctuations around the classical configurations[17].

First of all, let us introduce $\xi^r, r = 1, 2$, as the parameterizing coordinates of the $S^2$. The transformations of area preserving diffeomorphisms on this two-dimensional space are given by

$$\xi^r \rightarrow \xi^r + \beta^r(\xi), \quad \partial_r w(\xi) \beta^r(\xi)) = 0,$$

(21)
where $\beta^r$ can be locally written as
\begin{equation}
\beta^r(\xi) = \frac{\epsilon^{rs}}{w(\xi)} \partial_s \beta(\xi), \tag{22}
\end{equation}
and $w(\xi)$ is a 2-dimensional measure for the normalization. The transformation rules of the fields are determined by introducing the Poisson brackets defined with respect to the measure $w(\xi)$ as
\begin{equation}
\{A, B\} = \frac{\epsilon^{rs}}{w(\xi)} \partial_r A \partial_s B. \tag{23}
\end{equation}
The transformations of the fields are $\delta X^a = \{\beta, X^a\}$ and $\delta A = \partial_r \beta + \{\beta, A\}$. The coordinates $\xi^r, r = 1, 2$, parameterize not only the fields $n^a$ on $S^2$ but also the spinor field $z$ through the Hopf mapping $\overline{n} = z^\dagger \overline{\sigma} z$. However, in order to describe the consistent dynamics of the system, the definition of Poisson brackets (23) should coincide with that of the fundamental Poisson brackets (15). Comparing (23) with (15), we get $W(\xi) = \frac{2}{\theta} \text{det} \left| \frac{\partial(x^{i+}, x^{i-})}{\partial(\xi^r, \xi^s)} \right| = \frac{2}{\theta} \text{det} \left| \frac{\partial(x^{i+}, x^{i-})}{\partial(\xi^r, \xi^s)} \right| \equiv \theta^{-1} W$, where $z_1 = x^1 + ix^2$ and $z_2 = x^3 + ix^4$. According to the transformation rule of the fields, we have
\begin{equation}
\delta x^i = \frac{\theta}{W} \epsilon^{rs} \partial_r \beta \partial_s x^i = \frac{\theta}{W} \epsilon^{rs} \partial_j \beta \partial_r x^j \partial_s x^i = \theta \epsilon^{ij} A_j, \tag{24}
\end{equation}
and
\begin{equation}
\delta x^\tilde{i} = \frac{\theta}{W} \epsilon^{rs} \partial_r \beta \partial_s x^\tilde{i} = \frac{\theta}{W} \epsilon^{rs} \partial_j \beta \partial_r x^\tilde{j} \partial_s x^\tilde{i} = \theta \epsilon^{\tilde{i}\tilde{j}} A^\tilde{j}, \tag{25}
\end{equation}
where $i, j = 1, 2$ and $\tilde{i}, \tilde{j} = 3, 4$. The above transformation relations should be understood as the matrix variables expanded in terms of the fluctuations $A$ around the classical solutions $x^{(0)}$ and $\tilde{x}^{(0)}$, which determine the classical spinor solution $z^{(0)}$. These classical solutions $z^{(0)}$ and $\tilde{z}^{(0)}$ obey the fundamental Poisson relations (15). Substituting the matrix variable expansions with fluctuations into the effective action $I_e$, we get
\begin{equation}
I_e = \bar{\lambda} \int dt \text{Tr} \{2 \theta A_0 + \theta^2 \epsilon^{\mu\nu\lambda}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda) + \theta^2 \epsilon^{\mu\nu\lambda}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda)\}. \tag{26}
\end{equation}

The $x^{(0)}$ and $\tilde{x}^{(0)}$ are the matrices of the classical solution to be related with the noncommutative coordinates of the fuzzy $S^2$ by means of the Hopf mapping. Since any matrix can be expressed in terms of finite sum of products $\prod_i \exp\{ip_i \xi^{(0)}\} \exp\{ip_i \tilde{\xi}^{(0)}\}$, the $N \times N$ matrices $A_\mu$ and $\tilde{A}_\tilde{\mu}$ can be thought of as functions of $x^{(0)}$ and $x^{(0)}$. Based on this fact, we can pass the effective Lagrangian to the continuum limit by taking $N$ large. The changes of the coordinates $\xi^r, r = 1, 2$ parameterizing the spherical geometry induce the variations of the matrix fields $x^{(0)}$ and $x^{(0)}$. In the continuum limit, the $N \times N$ matrices $A_\mu$ and $\tilde{A}_\tilde{\mu}$ will map to smooth functions of the noncommutative coordinates $x^{(0)}$ and $x^{(0)}$. For the fields as the functions of non-commutative coordinates, we can introduce the Weyl ordering to define a suitable ordering for their products in the effective Lagrangian. This implies that the ordinary product should be replaced by the noncommutative $\star$-product. Here, the transition relation from the operator formalism for fields on noncommutative space to the representation in terms of ordinary function with the $\star$-product reads as $[f, g] \rightarrow i \frac{\theta}{W} \epsilon^{rs} \partial_r f \partial_s g$, and $\text{Tr}(f_1 \cdots f_n) \rightarrow \frac{W}{2\pi \theta} \int \{f_1 \star \cdots \star f_n\}$. Finishing all of these, we find the effective action describing the fluctuations
\begin{equation}
I_e = \int d^3 \xi A_0 J^0 + \frac{\lambda \theta}{4\pi} \int d^3 \xi \epsilon^{rst}(A_r \star \partial_s A_t + \frac{2}{3} A_r \star A_s \star A_t), \tag{27}
\end{equation}
where $\lambda = 4\tilde{\lambda}W$ and $J^0 = \lambda/2$.

The first term in the above equation is the chemical potential. The second term is the standard action of the $U(1)$ noncommutative Chern-Simons theroy. Susskind [3] proposed this theory as the description of the quantum Hall fluids on the plane. This Chern-Simons theory on the plane necessarily describes an infinite quantum Hall system since the space noncommuattivity condition requires an infinite diemnsional Hilbert space. In other words, the fields in this theory are infinite matrices corresponding to infinite number of electrons on the infinite plane. However, it is well known that Hal-dane’s description of quantum Hall effect on the spherical geometry is equivalent to that of Laughlin’s on the plane in the thermodynamic limit, taking the number of electrons large. Our conclusion is that in large $N$, the quantum Hall system on two-sphere is also described by the $U(1)$ noncommutative Chern-Simons theory. Physically, such a conclusion is reasonable.

It is, however, of interest to also describe finite systems of limited spatial extent with a finite number of electrons. If we want to describe the quantum Hall fluids on two-sphere, we must regularize the noncommutative Chern-Simons theory for an infinite number of electrons. By means of the Hopf fibration, the spinor $z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)$ can be used to describe the dynamics of electrons on $S^2$. So, unlike the Polychronakos’ [4] finite matrix model on the plane, the regularized matrix model for particles on $S^2$ should correspond to the $U(1)$ noncommutative Chern-Simons theory, and be described by the spinor matrix fields. It should be pointed out that in such a model, the spinor $z$ must be regarded as a field with single particle rather than that with particles $z_1$ and $z_2$.

3 Regularized version of the $U(1)$ noncommutative Chern-Simons theory on $S^2$

Now we describe the regularized version of the $U(1)$ noncommuattive Chern-Simons theory on $S^2$. This regularized matrix model should recover the $U(1)$ noncommutative Chern-Simons model in the large $N$ limit. Explicitly, in the large $N$ limit, the equation of motion for $A_0$ as the constraint in the $U(1)$ noncommutative Chern-Simons model will provide the noncommutaivity of the coordinates, which equivalently produces the classical matrix commutator (9). Such a regularized matrix model associated with the spinor matrix field $z$ can be obtained by following the Polychronakos’ construction of the finite matrix model on the plane[4]. Notice that the Hopf mapping makes the normal vector on $S^2$ be related to the coordinates of $S^3$ described by the spinor of two components $z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)$, i.e., $n^a = z^\dagger \sigma^a z$. This mapping relation is invariant under the $U(1)$ gauge transformation $z \to z' = e^{i\alpha} \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)$. So it is natural for us to propose the following action

$$S_p = \frac{\lambda}{4} \int dt Tr \{ iZ^\dagger D_t Z + 2\theta A_0 - \omega Z^\dagger Z \} + \frac{1}{2} \Psi^\dagger (i\dot{\Psi} - A_0 \Psi) + h.c. \quad (28)$$

to describe the finite number of electrons living on the two-dimensional sphere, where the covariant derivative is defined as $D_t = \partial_t + i[A_0, \quad ]$. In the above equation, $\Psi$ and $Z$ are spinors with two components, defined by $\Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right)$ and $Z = \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right)$, respectively. $Z_\alpha, \alpha = 1, 2$ are $N \times N$ complex matrices, $A_0$ is a $N \times N$ hermitian matrix, and $\Psi_\alpha, \alpha = 1, 2$ are complex $N$-vectors. They, in the
fundamental representation of the gauge group $U(N)$, are transformed as

$$Z_\alpha \to UZ_\alpha U^{-1}, \quad \Psi_\alpha \to U\Psi_\alpha, \quad A_0 \to UA_0U^{-1} + i\dot{U}U^{-1}. \quad (29)$$

It is obvious that the action $S_\alpha$ is invariant under the $U(N)$ gauge transformation. Due to the gauge invariance of the action, we can choose gauge $A_0$ and impose the equation of motion of $A_0$ as the constraint

$$\frac{\lambda}{2}[Z, Z^\dagger] + \Psi\Psi^\dagger = \lambda \theta \quad (30)$$

which is from the variation of the action with respect to $A_0$. If we rescale $Z$ into $\sqrt{\frac{\lambda}{2}}Z'$ and denote $Z'$ as $Z$, the above equation can be rewritten as

$$[Z, Z^\dagger] + \Psi\Psi^\dagger = \lambda \theta. \quad (31)$$

One can see from the action (28) that the conjugate momenta of $Z$ and $\Psi$ are $Z^\dagger$ and $\Psi^\dagger$, respectively. So they obey the classical matrix commutators $[(Z_\alpha)_{mn}, (Z^\dagger_\beta)_{kl}] = -i\delta_{mk}\delta_{nl}\delta_{\alpha\beta}$ and $[(\Psi_\alpha)_m, (\Psi^\dagger_\beta)_n] = -i\delta_{mn}\delta_{\alpha\beta}$. Since the spinor describes a particle moving on the two-dimensional sphere, we should regard such spinor as a single oscillator. So there exist $N^2 + N$ uncoupled oscillators in the present system. Their hamiltonian is

$$H = \omega Tr Z^\dagger Z = \omega \sum_{m,n,\alpha} (Z^\dagger_\alpha)_{mn}(Z_\alpha)_{nm}. \quad (32)$$

The constraint equation can be used to reduce the space of quantum physical states. Since the constrained matrix $G \equiv [Z, Z^\dagger] + \Psi\Psi^\dagger$ is the generator of unitary transformations of both $Z$ and $\Psi$, it must obey the commutation relations of the $U(N)$ algebra. In terms of the basis $\{I, T^a\}$ of the $U(N)$ algebra, where $T^a$ are the $N^2 - 1$ normalized $SU(N)$ generators, the matrix fields $Z$ and $Z^\dagger$ can be expanded as

$$Z = z_0 + \sum_{a=1}^{N^2-1} z_a T^a, \quad Z^\dagger = z_0^\dagger + \sum_{a=1}^{N^2-1} z^\dagger_a T^a. \quad (33)$$

By using these expansions, we can express the constrained matrix as

$$G^a = Tr(GT^a) = -if^{abc}z^\dagger_b z_c + \Psi^\dagger T^a \Psi, \quad (34)$$

where $f^{abc}$ are the structure constants of $SU(N)$ algebra, i.e., $[T^a, T^b] = if^{abc}T^c$.

After quantization, the elements of the matrix fields $Z, Z^\dagger$ and the vector fields $\Psi, \Psi^\dagger$ become operators, and satisfy the fundamental commutation relations of operators

$$[(Z_\alpha)_{mn}, (Z^\dagger_\beta)_{kl}] = \delta_{mk}\delta_{nl}\delta_{\alpha\beta}, \quad [(\Psi_\alpha)_m, (\Psi^\dagger_\beta)_n] = \delta_{mn}\delta_{\alpha\beta}. \quad (35)$$

Furthermore, after quantization, the expanded modes in the constrained matrix $G$ also become operators. The constrained operators $\hat{G}^a$ satisfy the $SU(N)$ algebra, and can be regarded as the generators of the $SU(N)$ algebra. Because the generators $T^a$ of the $SU(N)$ algebra are traceless, the constrained matrix $G$ gives the traceless part of the constraint equation (31). After quantization, the operators $\hat{G}^a$ become the projected operators of the quantum physical states in the matrix model

$$\hat{G}^a|\text{Phys} > = (\hat{G}^a_Z + \hat{G}^a_\Psi)|\text{Phys} > = 0. \quad (36)$$
On the other hand, the trace part of the constraint equation (31) produces the following constrained condition of the quantum physical states

\[
(\Psi_n^\dagger \Psi_n - 2N\lambda \theta)|\text{Phys} > = 0.
\] (37)

Since we are considering the matrix model of finite number of particles moving on the two-dimensional sphere, we must also add the geometrical constraint to the quantum physical states to map the manifold parameterized by the coordinates \( Z \) to the two-sphere \( S^2 \). As mentioned above, \( z^\dagger z = 1 \) together with the \( U(1) \) gauge transformations of \( z \), i.e., \( z \to e^{i\alpha}z \), implies that the geometrical condition \( n^a n^a = 1 \) of \( S^2 \) is satisfied. However, in our matrix model, this condition becomes

\[
[\text{Tr}(Z^\dagger Z) - g]|\text{Phys} > = 0,
\] (38)

where \( g \) is a parameter dependent of the model. Here \(|\text{Phys} >_s\) stands for the geometrically stable configuration among the quantum physical states. In fact, the quantum physical states including the excitations do not belong to such stable configurations, but the Laughlin-type states of quantum Hall fluids do.

From the constraint condition (36), we know that the physical states must be the singlet representations of the \( SU(N) \) group, of which \( \hat{G}^a \) are the generators. However, \( \hat{G}^a_Z \) are only realized by the representations arising from products of the adjoint representations of \( SU(N) \). Furthermore, \( Z_1 \) and \( Z_2 \) form a spinor, and describe the spin degree of freedom of particles. So they should appear in pairs in the singlet representations. Therefore, the representation of \( \hat{G}^a_Z \) contains only the irreducible representations whose total number of boxes in their Young tableau is an integer multiple of \( 2N \). Since the physical states are invariant under the sum of \( G^a_Z \) and \( G^a_{\Psi} \), the representations of \( G_Z \) and \( G_{\Psi} \) must be conjugate to each other so that their product contains the singlet of the \( SU(N) \) group. Hence, the irreducible representations of \( G_{\Psi} \) must also have a number of boxes which is a multiple of \( 2N \). Following the Polychronakos’ arguments[4], from the other constraint condition (37), one knows that the number of boxes equals to the total number of operators of the spinor oscillators \( \Psi^\dagger \Psi \). Thus, we conclude

\[
\lambda \theta = k,
\] (39)

where \( k \) is an integer. This conclusion is the same as that of Polychronakos for the finite matrix model on the plane. In fact, in the large \( N \) limit, Haldane’s quantum Hall effect model on the spherical geometry is equivalent to Laughlin’s quantum Hall system on the plane. So, the level of \( U(1) \) noncommutative Chern-Simons action is not changed by the geometry on which the particles move.

In the Haldane’s description of quantum Hall effect in terms of the spherical geometry, the spinors are the fundamental elements in the description of electrons on the two-dimensional sphere of which a Dirac monopole lies at the center, and are the dynamical degrees of freedom of the electrons. On the other hand, it can be easily seen from the constraint conditions (36) and (37) that only if the quantum physical states are the spin singlets the constraint of the \( SU(N) \) invariance (36) is consistent with the vanishing condition of the total \( U(1) \) charge (37).

To summarize, the quantum physical states of our matrix model must possess the following properties. (a) They are the singlet representations of the \( SU(N) \) group. (b) They must be the spin singlets. This implies that the same number of spin-up and spin-down components will be present in the quantum physical states. That is, they are the \( SU(2) \) invariant states associated with the spin. (c) There exist \( kN \) number of \( \Psi_1^\dagger \) and \( kN \) number of \( \Psi_2^\dagger \) in the quantum physical states, where \( \Psi_1^\dagger \)
and $\Psi_2^\dagger$ form the spinor $\Psi^\dagger$. (d) The geometrically stable states among the quantum physical states should also satisfy the geometrical constraint condition (38).

Subsequently, we shall determine the quantum physical states of the matrix model, which build up the physical Fock space of the matrix model. Recall that the Hamiltonian (32) of the system can be expressed as $\omega \hat{N}_Z$ in terms of the number operator $\hat{N}_Z \equiv \sum_{m,n} Z_{nm}^\dagger Z_{nm}$ of the spinor oscillators $Z$. From this expression, we know that energy eigenstates will be linear combination of terms with a fixed number of $Z^\dagger$ creation operators acting on the Fock space vacuum $|0\rangle$, which is defined by

$$Z_{mn}|0\rangle = \Psi_n|0\rangle = 0.$$  \hspace{1cm} (40)

The constraint conditions of quantum physical states require that all physical states must have a fixed number $Nk$ of $\Psi_1^\dagger$ creation operators and the same number of $\Psi_2^\dagger$ creation operators acting on the Fock vacuum. Furthermore, the number of $Z^\dagger$ creation operators appearing in the physical state should be the same as that of $Z^\dagger$. Thus, any physical state describing an energy eigenstate will be a sum of terms of the form

$$\prod_{m=1}^M (Z_1^\dagger)_{j_m}^{\nu_m} (Z_2^\dagger)_{j_m}^{\nu_m} \prod_{n=1}^{Nk} (\Psi_1^\dagger)_{i_n}^{\nu_n} (\Psi_2^\dagger)_{i_n}^{\nu_n} |0\rangle,$$

where, the fundamental indices of $SU(N)$ are written as upper indices, and the anti-fundamental indices as lower indices.

Now, the problem is how we can construct a singlet of both $SU(N)$ and spin from the equation (41) through contracting all indices by the covariant tensors of $SU(N)$. Since the product of an upper index epsilon tensor and a power index epsilon tensor may be rewritten as a sum of products of the delta functions, we may only use one type of epsilon tensor to finish the contraction. To continue the construction of the quantum physical states, first of all, we shall establish a few of lemmas, which are the generalizations of the facts given by Hellerman and Raamsolok in the appendix of their paper [5].

**Lemma 1:** Setting $\chi(u,v) \equiv \prod_{i<j} (u_i v_j - u_j v_i)$ where $u$ and $v$ are two components of spinor $z$, i.e.,

$$z = \begin{pmatrix} u \\ v \end{pmatrix},$$

we have

$$\chi(u,v) = \epsilon^{i_1 \cdots i_N} \prod_{n=1}^N (u^{N-n} v^{n-1})_{i_n},$$

where we have abbreviated $\epsilon^{(i_1 \cdots i_N} \prod_{n=1}^N (u^{N-n})_{i_n} (v^{n-1})_{i_n}$ as $\chi(u,v)$.

**Proof:** From the definition of $\chi(u,v)$ and its expression (42), one can see that they all are completely antisymmetric, and have the same order of $u$ and $v$ power $N(N-1)$. hence, the definition of $\chi(u,v)$ must be equals its expression up to a numerical factor. Taking a fixed $N$, e.g., $N = 3$, we can check that the numerical factro is equal to 1.

**Lemma 2:** Any polynomial $D(u,v) = \epsilon^{i_1 \cdots i_N} \prod_{m=1}^N (u^{n_m} v^{\tilde{n}_m})_{i_m}$, where $\sum_{i=1}^N n_i = \sum_{i=1}^N \tilde{n}_i$, may be written as a sum of terms of the form

$$F(u,v) = \prod_{n=1}^N S_{n}^{c_n} \hat{S}_{n}^{c_n} \chi(u,v),$$

where $S_l = \sum_{i=1}^N u_i^l$ and $\hat{S}_l = \sum_{i=1}^N v_i^l$. The equality $\sum_{i=1}^N n_i = \sum_{i=1}^N \tilde{n}_i$ is the conclusion of spin singlet, which implies that $\sum_{i=1}^N i c_i = \sum_{n=1}^N \tilde{i} c_i$.
Lemma 3: Let $\Psi_1^\dagger$, $\Psi_2^\dagger$, and $Z_1^\dagger$, $Z_2^\dagger$ be the $N$-dimensional vectors and the $N \times N$ matrices of commuting variables, respectively. Thus, any expression of the form

$$D(\Psi_1^\dagger, \Psi_2^\dagger; Z_1^\dagger, Z_2^\dagger) = \epsilon^{(i\bar{i})} \prod_{l=1}^{N} (\Psi_1^\dagger Z_1^{\dagger_{n_l}} Z_2^{\dagger_{\bar{n}_l}} \Psi_2^\dagger)_{(i\bar{i})}$$

may be uniquely written as a sum of terms of form

$$F(\Psi_1^\dagger, \Psi_2^\dagger, Z_1^\dagger, Z_2^\dagger) = \prod_{i=1}^{N} (Tr Z_1^{\dagger_i}) c_i (Tr Z_2^{\dagger_i}) \epsilon^{i\bar{i}} \prod_{n=1}^{N} (\Psi_1^\dagger Z_1^{\dagger_{N-n}} Z_2^{\dagger_{N-n-1}} \Psi_2^\dagger)_{i_n}, \quad (44)$$

conversely, the conclusion of the lemma also is true.

The proofs of the lemma 2 and the lemma 3 are completely parallel to those of the fact 2 and the fact 3 provided by Hellerman and Raamsolokin. Here we shall omit them. The above lemmas are the base of constructing the quantum physical states of the matrix model presented here.

Now, we return to the contraction of the Fock states (41). The problem is to determine all ways of combining the symmetrized anti-fundamentals $\Psi_1^\dagger$, $\Psi_2^\dagger$ with any fixed number of adjoints to form a singlet of $SU(N)$. Unlike the Polychronakos’ finite matrix model, this singlet of $SU(N)$ is also a spin singlet. Let us notice that $Z_1^\dagger$, $\Psi_1^\dagger$ should be regarded as the spin-up components and $Z_2^\dagger$, $\Psi_2^\dagger$ as spin-down components. In order to avoid the spin confusion, we shall make $Nk$ creation operators $\Psi_1^\dagger$ contract with a fixed number of creation operators $Z_1^\dagger$ and $Nk$ operators $\Psi_2^\dagger$ with certain number of $Z_2^\dagger$ to form a singlet of $SU(N)$. Precisely, let us consider first the indices of the $Nk$ number of $\Psi_1^\dagger$. The lowr index on each $\Psi_1^\dagger$ must contract either with the upper index on a $Z_1^\dagger$ or with an epsilon tensor. If the $\Psi_1^\dagger$ contracts with a $Z_1^\dagger$, the resulting object will again have a single lower index, i.e., $(\Psi_1^\dagger)_{i_1} (Z_1^{\dagger_{i_1}})_{j_1} \rightarrow (\Psi_1^\dagger Z_1^{\dagger_{i_1}})_{j_1}$. The lower index on the object may again contract either with the upper index on a $Z_1^\dagger$ or with an epsilon tensor. Repeating this logic, we conclude that each $\Psi_1^\dagger$ will contract with a number of $Z_1^\dagger$ and that the resulting object will have its single lower index contracted with an upper index epsilon tensor. So the result is $(\Psi_1^\dagger Z_1^{\dagger_{i_1}})_{i_1}$. Similarly, we have $(\Psi_2^\dagger Z_2^{\dagger_{i_1}})_{i_1}$. However, the indices of $(\Psi_1^\dagger Z_1^{\dagger_{i_1}})_{i_1}$ and $(\Psi_2^\dagger Z_2^{\dagger_{i_1}})_{i_1}$ belong to the same particle label since $Z_1^\dagger$ and $Z_2^\dagger$ are associated with the components of spin up and spin down, respectively, of the particle. This implies that for the fixed particle, $\Psi_1^\dagger Z_1^{\dagger_{i_1}}$ and $\Psi_2^\dagger Z_2^{\dagger_{i_1}}$ should appear in one contracted element, i.e., $(\Psi_1^\dagger Z_1^{\dagger_{i_1}} Z_2^{\dagger_{\bar{i}_{i_1}}} \Psi_2^\dagger)_{(i\bar{i})}$. Such $N$ elements are contracted with the upper indices of an epsilon tensor to produce the fundamental contraction block

$$\epsilon^{i_1\bar{i}_1} \prod_{l=1}^{N} (\Psi_1^\dagger Z_1^{\dagger_{n_l}} Z_2^{\dagger_{\bar{n}_l}} \Psi_2^\dagger)_{i_l}, \quad (46)$$

where we have admitted the abbreviated symbol appeared in the lemmas.

Because there exist $Nk$ number of $\Psi_1^\dagger$ and $Nk$ number of $\Psi_2^\dagger$ in the quantum physical states, the physical states are composed of $k$ fundamental contraction blocks. So, using the lemmas mentioned by us above, we can write the minimal basis of the physical energy eigenstates being both the $SU(N)$ singlets and the spin singlets as

$$|\{n_i\}, \{\bar{n}_i\}, k\rangle = \epsilon^{i_1\bar{i}_1} \prod_{l=1}^{N} (\Psi_1^\dagger Z_1^{\dagger_{n_l}} Z_2^{\dagger_{\bar{n}_l}} \Psi_2^\dagger)_{i_l} \epsilon^{i_1\bar{i}_1} \prod_{n=1}^{N} (\Psi_1^\dagger Z_1^{\dagger_{N-n}} Z_2^{\dagger_{N-n-1}} \Psi_2^\dagger)_{i_n})^{k-1} |0\rangle, \quad (47)$$
where \( \{n_i\} \) and \( \{\tilde{n}_i\} \) satisfy the relation \( \sum_i n_i = \sum_i \tilde{n}_i \) from the requirement of the spin singlet. On the other hand, there exist some additional \( Z_i^\dagger \), for \( i = 1, 2 \), which are contracted amongst themselves as the terms of products \( \text{Tr}(Z_i^\dagger), \cdots \text{Tr}(Z_i^{N_i}) \) and \( \text{Tr}(Z_1^\dagger), \cdots \text{Tr}(Z_1^{N_1}) \) to form the physical states. By means of the Lemma 3, they can be used to build up another set of minimal basis of the physical energy eigenstates as

\[
|\{c_i\}, \{\tilde{c}_i\}, k\rangle = \prod_{i=1}^N (\text{Tr}Z_1^{i\dagger})^{c_i}(\text{Tr}Z_2^{i\dagger})^{\tilde{c}_i}(e^{i_{1\cdots N}} \prod_{n=1}^N (\Psi_1^{\dagger} Z_1^{N-n} Z_2^{\sum_{n=1}^N \Psi_2^{\dagger} i_n})^k |0\rangle. \tag{48}
\]

The spin singlet condition of the physical states leads to \( \sum_{i=1}^N ic_i = \sum_{i=1}^N i\tilde{c}_i \). By using the expression of hamiltonian (32), we can easily read off the energy eigenvalues of the above bases. The states of the former basis have the structure of energy levels as the following

\[
E(\{n_i\}, \{\tilde{n}_i\}, k) = \omega[(k-1)N(N-1) + \sum_i n_i + \sum_i \tilde{n}_i] = \omega[(k-1)N(N-1) + 2 \sum_i n_i]. \tag{49}
\]

The energy eigenvalues of the latter basis are given by

\[
E(\{c_i\}, \{\tilde{c}_i\}, k) = \omega[kN(N-1) + \sum_{i=1}^N ic_i + \sum_{i=1}^N i\tilde{c}_i] = \omega[kN(N-1) + 2 \sum_{i=1}^N ic_i]. \tag{50}
\]

Furthermore, from the expressions (47) and (48) of the minimal bases of the physical energy eigenstates, we find that the physical ground state of the present finite matrix model is expressed by

\[
|\{0\}, \{0\}, k\rangle = (e^{i_{1\cdots N}} \prod_{n=1}^N (\Psi_1^{\dagger} Z_1^{N-n} Z_2^{\sum_{n=1}^N \Psi_2^{\dagger} i_n})^k |0\rangle \equiv L^{\dagger k}|0\rangle. \tag{51}
\]

Roughly speaking, after finishing the formal substitutions \( \Psi_1^{\dagger} \rightarrow 1 \), \( \Psi_2^{\dagger} \rightarrow 1 \) and \( Z_1^{ij} \rightarrow \delta_{ij} A_{1j}, Z_2^{ij} \rightarrow \delta_{ij} A_{1j} \) in (51), we can get \( |0, k\rangle = (e^{i_{1\cdots N}} \prod_{n=1}^N (A_{1\dagger}^{N-n} A_{2}^{n-1} i_n)^k |0\rangle \). Furthermore, if \( u \) and \( v \) are regarded as the eigenvalue parameters of \( A_1 \) and \( A_2 \) in the coherent state picture respectively, we find that

\[
|u, v, 0, k\rangle = (e^{i_{1\cdots N}} \prod_{n=1}^N (u^{N-n} v^{n-1} i_n)^k |0\rangle = \prod_{i<j} (v_i v_j - u_i v_j)^k. \tag{52}
\]

It is just the same as the ground state wavefunction of two-dimensional quantum Hall fluid on the Haldane’s spherical geometry [7]. However, as pointed by Polychronakos [4], the classical value of the inverse filling fraction is shifted quantum mechanically if one uses the finite matrix Chern-Simons theory to describe the fractional quantum Hall states. This can be equivalently viewed as a renormalization of the Chern-Simons coefficient. In fact, this level shift of the matrix Chern-Simons model can be read off from the well known quantum mechanical level shift of the corresponding Chern-Simon theory. The renormalization of the level of Chern-Simons theory has been finished by using a biparameter family of BRS invariant regularization methods of Chern-Simons theory [18]. This renormalization leads to the level shift \( k \rightarrow k + \text{sign}(k)c_V \), where \( k \) is the bare Chern-Simons level parameter and \( c_V \) the quadratic Casimir in the adjoint representation of the gauge group of Chern-Simons theory. As mentioned previously, our finite Chern-Simons matrix model on \( S^2 \) corresponds to the \( U(1) \) Chern-Simons theory. Hence, physical states in the matrix model presented here at level \( k \) should be identified with the quantum Hall states at the filling fraction 1/(\( k+1 \)) rather than 1/k, like the Polychronakos’ finite matrix model on the plane. That is, the Laughlin type wavefunction on the two-dimensional Haldane’s spherical geometry for filling fraction 1/(\( k+1 \)) can be equivalently described by the physical ground state of the finite Chern-Simons matrix model on \( S^2 \).
The reason that the state (51) is regarded as the physical ground state becomes clear from the following discussion. The system considered by us is a priori $2N(N+1)$ uncoupled oscillators, which are composed of $2N^2$ harmonic oscillators coordinated by $Z_1$, $Z_2$ and $2N$ harmonic oscillators done by $\Psi_1$, $\Psi_2$. However, they should be regarded as $N(N+1)$ uncoupled spinor oscillators since $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ and $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ must be viewed as the spinors describing the particles on $S^2$. Furthermore, what couples the spinor oscillators is $N^2 - 1$ constraint equations in the traceless part of the Gauss constraint (31). Effectively, we can describe the system with $N+1$ independent oscillators. All $SU(N)$ invariant states can be spanned by the operators, $Q_{1n}^\dagger = Tr(Z_1^{\dagger n})$, $Q_{2n}^\dagger = Tr(Z_2^{\dagger n})$ with $n = 1, 2, \cdots, N$, and $L_l^k$ acting on the Fock vacuum. However, the spin singlet condition of physical states must result in the balance of the numbers of operators $\{Q_{1n}^\dagger\}$ and $\{Q_{2n}^\dagger\}$ appearing in the physical states. So they can be regarded as the $N$ independent spinor oscillators together with the operator $L_l^k$ composed of the $N+1$ independent oscillators physically describing the system. A useful conclusion in mathematics is that the operators $Q_{1l}^\dagger$ and $Q_{2l}^\dagger$ for $l > N$ can be expressed as the homogeneous polynomials of total order $l$ in $\{Q_{11}^\dagger, Q_{12}^\dagger, \cdots, Q_{1N}^\dagger\}$ and $\{Q_{21}^\dagger, Q_{22}^\dagger, \cdots, Q_{2N}^\dagger\}$, respectively, with constant coefficients which are common to all operators. Based on this conclusion and the commutation relations between $Z$ and $Z^{\dagger}$, we have $Q_{1l}Q_{2l}L_l^k|0\rangle = 0$, for all $l$. This means that the state $L_l^k|0\rangle \equiv |0, k\rangle$ is the physical vacuum with respect to all operators $Q_{1l}$ and $Q_{2l}$. Equivalently, the Laughlin-type state $|0, k\rangle$ is the physical ground state of our finite matrix model. In the next section, we shall discuss the excitation states produced by the creation operators $Q_{1l_1}^\dagger$ and $Q_{2l_2}^\dagger$ acting on the ground state.

4 Quasiparticle excitations and hierarchy of the quantum Hall fluids on $S^2$ in the finite matrix model

The low-lying excitations in our matrix model can be described in terms of quasiparticles and quasiholes by following [4, 19]. A quasiparticle state is obtained by peeling a 'particle' from the surface of the Fermi sea. That is, one quasiparticle obtained by exciting a 'particle' at Fermi level by energy amount $n\omega$ is described by

$$p_{n}^{\dagger}|0, k\rangle = (e^{i\pi \cdots iN} \prod_{m=1}^{N} (\Psi_1^{\dagger}Z_1^{\dagger N-m}Z_2^{\dagger m-1}\Psi_2^{\dagger})_{im})^{k-1} \quad \epsilon^{i\cdots iN} \quad (\Psi_1^{\dagger}Z_1^{\dagger N-1+n}Z_2^{\dagger 0}\Psi_2^{\dagger})_{i_1} \prod_{m=2}^{N} (\Psi_1^{\dagger}Z_1^{\dagger N-m}Z_2^{\dagger m-1}\Psi_2^{\dagger})_{im}|0\rangle. \quad (52)$$

The quasiholes correspond to the minimal excitations of the ground state inside the quantum Hall fluid. One quasihole excitation is obtained by creating a gap inside the QHF with the energy increase $m\omega$

$$h_{m}^{\dagger}|0, k\rangle = (e^{i\cdots iN} \prod_{n=1}^{N} (\Psi_1^{\dagger}Z_1^{\dagger N-n}Z_2^{\dagger n-1}\Psi_2^{\dagger})_{in})^{k-1} \quad \epsilon^{i\cdots iN} \quad m \prod_{n=1}^{N} (\Psi_1^{\dagger}Z_1^{\dagger N-n+1}Z_2^{\dagger n-1}\Psi_2^{\dagger})_{in} \prod_{n=m+1}^{N} (\Psi_1^{\dagger}Z_1^{\dagger N-n}Z_2^{\dagger n-1}\Psi_2^{\dagger})_{in}|0\rangle. \quad (53)$$
Obviously, \( p_{1}^{\dagger} = h_{1}^{\dagger} \). So there is no fundamental distinction between 'particles' and 'holes' in the matrix model. Similarly, one can describe the quasiparticle \( p_{n}^{\dagger} \) and quasi-hole \( h_{n}^{\dagger} \) of excitations corresponding to the oscillator field \( Z_{2} \). They read as

\[
p_{n}^{\dagger} |0, k\rangle = (\epsilon^{i_{1} \cdots i_{N}} \prod_{m=1}^{N} (\Psi_{1}^{\dagger} Z_{1}^{N-m} Z_{2}^{\dagger m-1} \Psi_{2}^{\dagger})_{i_{m}})^{k-1} \epsilon^{i_{1} \cdots i_{N}} \prod_{m=1}^{N-1} (\Psi_{1}^{\dagger} Z_{1}^{N-m} Z_{2}^{\dagger m-1} \Psi_{2}^{\dagger})_{i_{m}} (\Psi_{1}^{\dagger} Z_{1}^{0} Z_{2}^{N-1-n} \Psi_{2}^{\dagger})_{i_{1}} |0\rangle, \tag{54}
\]

and

\[
h_{n}^{\dagger} |0, k\rangle = (\epsilon^{i_{1} \cdots i_{N}} \prod_{n=1}^{N-m} (\Psi_{1}^{\dagger} Z_{1}^{N-n} Z_{2}^{\dagger n-1} \Psi_{2}^{\dagger})_{i_{n}})^{k-1} \epsilon^{i_{1} \cdots i_{N}} \prod_{n=1}^{N-m} (\Psi_{1}^{\dagger} Z_{1}^{N-n} Z_{2}^{\dagger n} \Psi_{2}^{\dagger})_{i_{n}} \prod_{n=N-m+1}^{N} (\Psi_{1}^{\dagger} Z_{1}^{N-n} Z_{2}^{\dagger n} \Psi_{2}^{\dagger})_{i_{n}} |0\rangle. \tag{55}
\]

Although all of these excitations are the fundamental excitations in the finite matrix model here, they can not be regarded directly as the physical low-lying excitations in the matrix model. The physical exciting states must obey the constraint condition of physical states of the matrix model, which is just the spin singlet condition of the physical states. By using the lemmas shown by us in the previous section, one can find that all of the fundamental excitations as mentioned above can be equivalently expressed in terms of the following states

\[
P_{n_{1}, n_{2}}^{\dagger} |0, k\rangle = \prod_{j=1}^{N} (Tr Z_{1}^{(j)c_{j}} (Tr Z_{2}^{(j)\tilde{c}_{j}}) \tilde{c}_{j}) |0, k\rangle = \prod_{j=1}^{N} (Q_{1j}^{\dagger})^{c_{j}} (Q_{2j}^{\dagger})^{\tilde{c}_{j}} |0, k\rangle, \tag{56}
\]

where, \( n_{1} = \sum_{i=1}^{N} i c_{i} \) and \( n_{2} = \sum_{i=1}^{N} i \tilde{c}_{i} \). Indeed, they are exciting states with respect to the Laughlin-type ground state \(|0, k\rangle\) clearly from the discussion of the last paragraph in the above section. The physical excitation states can be constructed by using the expression (56) of fundamental excitations and by adding the spin singlet condition of physical states to restrict them.

Following Haldane [7], we can construct the collective ground state of two-dimensional quantum Hall fluid from the present matrix model by the condensing of the quasiparticle and quasi-hole excitations. The condensate here means that the physical state including the excitations becomes the Laughlin-type fluid state, like the physical ground state \(|0, k\rangle\). This Laughlin-type fluid state is given by

\[
|0, p_{1}, k\rangle = (\epsilon^{i_{1} \cdots i_{N_{1}}} \prod_{n=1}^{N_{1}} (P_{1}^{\dagger N_{1}} - n P_{2}^{\dagger n-1})_{i_{n}})^{p_{1}} |0, k\rangle, \tag{57}
\]

where \( N_{1} = N/p_{1} + 1 \) and \( N \) is divisible by \( p_{1} \). \( P_{1}^{\dagger i_{n}} \) and \( P_{2}^{\dagger i_{n}} \) stand for \( P_{i, 0}^{\dagger} \) and \( P_{0, i}^{\dagger} \), respectively, defined by the expression (56). In fact, the condition \( N_{1} = N/p_{1} + 1 \) is required by the fact that if one uses the operators \( \{Q_{1n}^{\dagger}\} \) and \( \{Q_{2n}^{\dagger}\} \) to be a set of independent creation operators, the condition of operator powers \( n \leq N \) must be satisfied, otherwise, the constructed creation operators will be dependent. By using the lemmas in the section three, one can easily check that the excitation fluid state indeed satisfies the spin singlet condition of the physical states in the matrix model.

We can also construct the excitation states of the two-dimensional excitation fluid in a way similar to the construction of the excitation states of the quantum Hall fluid state \(|0, k\rangle\). It will be convenient
to use the latter basis of the Fock space of the matrix model to do this. Introducing operators
\[ P_s^{1|j} = \sum_{n=1}^{N_1} F_n^{1|j} \] and \[ P_s^{2|j} = \sum_{n=1}^{N_1} F_n^{2|j}, \]
we can write the excitation states of the excitation fluid as
\[
|\{c_{1i}\}, \{\tilde{c}_{1i}\}, p_1, k\rangle = \prod_{j=1}^{N_1} (P_s^{1|j})^{c_{1j}} (P_s^{2|j})^{\tilde{c}_{1j}} |0, p_1, k\rangle. \tag{58}
\]

Although these states do not generally satisfy all of constraint conditions of the physical states in the
matrix model, they are the mediate states to construct the physical ground state of the excitation fluid in
the next level obeying the constraint conditions. Here, the further constraint condition of the physical
excitation states is the spin singlet condition. One can get the next level of the Laughlin-type fluid
state with the excitations derived by the excitation fluid by further condensing the 'quasiparticle' and
'quasihole' excitations from the excitation fluid. Furthermore, similar to Haldane's hierarchical scheme
of the quantum Hall fluids, in our matrix model the procedure of constructing the two-dimensional
excitation fluids can be iterated, and leads to the hierarchy of the two-dimensional quantum Hall
fluid states. We give the result of the iterated construction of the quantum Hall states as following
\[
|0, p_m, \cdots, p_1, k\rangle = \prod_{q=1}^{m} (\epsilon^{j_1, \cdots, j_{N_q}} \prod_{n=1}^{N_q} \langle P_{q-1}^{1|n} - \cdots, p_1, k\rangle, \tag{59}
\]
where the iterated relation is given by \( p_q(N_q - 1) + N_{q+1} = N_q \) with \( N_q = 0 \) for \( q > m \)
and \( N_0 = N \). The fundamental excitation operators of the excitation fluids \( P_m^{1|j} \equiv P_{j,0}^{(m)} \) and \( P_m^{2|j} \equiv P_{0,j}^{(m)} \)
are determined by the set of equations
\[
P_{n_1, n_2}^{(m)} |0, p_m, \cdots, p_1, k\rangle = \prod_{j=1}^{N_m} (P_{s,m-1}^{1|j})^{c_{1j}} (P_{s,m-1}^{2|j})^{\tilde{c}_{1j}} |0, p_m, \cdots, p_1, k\rangle, \tag{60}\]
where \( n_1 = \sum_{j=1}^{N_m} j c_j \) and \( n_2 = \sum_{j=1}^{N_m} j \tilde{c}_j \). The symmetric operators in the equation (60) are defined
as \( P_s^{1|j} = \sum_{i=1}^{N_m} (P_{m-1}^{1|j})_i \) and \( P_s^{2|j} = \sum_{i=1}^{N_m} (P_{m-1}^{2|j})_i \). In fact, in the procedure of constructing
the hierarchy of two-dimensional quantum Hall fluids in the matrix model, we have finished the
construction of all quasiparticle and quasihole excitations of the quantum Hall excitation fluids.

Haldane's idea for the hierarchy of fluid states is to consider a slightly different field strength with
corresponding to Haldane’s low-energy states. Physically, the imbalance of quasiparticle and quasihole
excitations leads to that the degeneracy of the lowest Landau level states in the hierarchical fluids becomes
\[
2S + 1 = k(N - 1) + N_1 + 1, \tag{61}
\]
where \( N_1 \) in the expression of degeneracy is from the contribution of the imbalance of quasiparticle
and quasihole excitations with respect to the background of the quantum Hall fluid for the electrons.
Such \( N_1 \) can be obtained by solving the hierarchical iterated equations \( p_q(N_q - 1) + N_{q+1} = N_q \).
The result is
\[
N_1 = \frac{N}{p_1 + \frac{1}{p_2 + \cdots + \frac{1}{p_m}}} + \frac{1}{p_1 + \frac{1}{p_2 + \cdots + \frac{1}{p_m}}} \left[ p_1 - \frac{1}{p_2 + \cdots + \frac{1}{p_m}} \left[ \cdots \left[ p_{m-2} - \frac{1}{p_{m-1} + \frac{1}{p_m}} \right] \right] \right]. \tag{62}
\]
The filling factor means the lowest Landau level occupation factor. For the general hierarchical fluid states, the imbalance of excitations with respect to the background of the electron’s quantum Hall fluid can be affected by the imbalances of the excitations of the excitation fluids. This results in different $N_1$ for the variation of hierarchical fluid states. But the spin of electrons in the hierarchical fluid state is given by the formula $\frac{1}{2} \tilde{k} (N - 1) + \frac{1}{2} N_1$, where we have involved in the level shifting $k \to \tilde{k} = k + 1$ of the Chern-Simons matrix model, while the number of electrons is still $N$. So we obtain the filling factor of the $m$-th hierarchical fluid state in the thermodynamic limit

$$\nu = \lim_{N \to \infty} \frac{N}{k(N - 1) + N_1 + 1} = \frac{1}{\tilde{k} + \frac{1}{p_1 + \frac{1}{p_2 + \cdots + \frac{1}{p_m}}}}. \quad (63)$$

Since the hierarchical quantum Hall fluid states are directly from the condensation of excitations of the electron’s quantum Hall fluid in the matrix model, we find the energies of these hierarchical fluid states by using the hamiltonian expression of the matrix model. Explicitly,

$$H[0, p_m, \ldots, p_1, k] = \omega[kN(N - 1) + \sum_{q=1}^{m} p_q N_q (N_q - 1)] |0, p_m, \ldots, p_1, k\rangle. \quad (64)$$

Substituting the iterated equations $p_q (N_q - 1) + N_{q+1} = N_{q-1}$, we obtain the energy of the $m$-th hierarchical fluid state

$$E(p_m, \ldots, p_1, k) = \omega[kN(N - 1) + NN_1]. \quad (65)$$

The term for $N_1$ in the energy is from the contribution of condensing of the excitations. This implies that these hierarchical fluid states are the substable states of the matrix model since their energies are higher than that of the parent fluid state, i.e., the physical ground state of the matrix model. It should be emphasized that the hierarchical quantum Hall fluids are dynamically formed by condensing the ‘quasiparticle’ and ‘quasihole’ excitations level by level. In other words, there consistently exist such hierarchical fluid states in our matrix model without any requirement of modifications of the matrix model.

5 Summary and outlook

If one considers particle’s motion on two-sphere in a radial monopole magnetic field, the configurations of particle’s coordinates can not be smoothly defined globally over the entire $S^2$ due to the singularity of the Dirac monopole. The effective action of the system can be described in a singularity-free way by using a nontrivial bundle over $S^2$, which can be obtained by the Hopf fibration with base $S^2$. The existence of the Dirac monopole in the 2-dimensional quantum Hall system makes the coordinates of particles moving on the two-sphere become noncommutative. The appearance of such monopole also results in the irreducible representations of $SU(2)$ belonging to the Hilbert space composed of the lowest Landau level states of the system to be truncated. This phenomenon occurs in the description of fuzzy two-sphere. We have explicitly shown that the noncommutative structure of fuzzy $S^2$ appears indeed in the Haldane’s model of quantum Hall effect by restricting to the lowest Landau level states. In order to establish the description of noncommutative field theory for the quantum Hall system on $S^2$, we have provided the Hopf mapping of the fuzzy $S^2$, i.e. (14) and (15). This mapping between the fuzzy manifolds plays the essential role in the description of noncommutative field theory of quantum Hall fluids on $S^2$. It results in that the finite matrix model of quantum Hall fluids on $S^2$
is related to the matrix fields of the spinor with two complex components, which are from the matrix regularized-version of the spinor in the original Hopf map. In the first Hopf map, the Hopf fibration of $S^2$ can be regarded as a principal fibre bundle with the base space $S^2$ and a $U(1)$ structure group. This $U(1)$ gauge group is iterated in the formulation of noncommutative field theory of the quantum Hall fluids on $S^2$. This implies that the finite matrix model given by us is the finite matrix regularized version of the $U(1)$ noncommutative Chern-Simons theory on $S^2$. The finite matrix model (28) on $S^2$ involves in the matrix and vector fields of the spinor, different from the Polychronakos' finite matrix model on the plane. In fact, the Hopf mapping of the fuzzy manifolds related to the second Hopf map is very important for the descriptions of the quantum Hall effect on $S^4$ and of the open two-brane in M theory[20, 21]. It should be interesting to further recognize the mathematical implication of the Hopf mapping between the fuzzy manifolds and their applications in physics.

Hellerman and Raamsdonk [5] had speculated the second-quantized field theoretical description of the quantum Hall fluid for various filling fractions, which is given by the regularized matrix version of the noncommutative $U(1)$ Chern-Simons theory on the plane, by determining the completely minimal basis of exact wavefunctions for the Polychronakos' finite matrix model. For the quantum Hall fluids on $S^2$, we have determined the complete set of the physical quantum states of the finite matrix model on $S^2$, and shown the correspondence between the physical ground state of our model and the Laughlin-type wavefunction on the Haldane's spherical geometry. Although such a determination of physical states is the generalization of Hellerman and Raamsdonk's work for the finite matrix model on the plane, it is nontrivial because the spinor matrix fields and the spinor vector fields are included in the finite matrix model on $S^2$. On the other hand, we have attempted to establish the second-quantized field theoretical description of the quantum Hall fluids for various filling fractions on the Haldane's spherical geometry, and determined some essential physical elements in the quantum Hall fluid. However, further investigations are needed. For example, Karabali and Sakita[6] used the technique of coherent state to realize the Laughlin wave functions on the plane in the Polychronakos' finite matrix model on the plane. The technique of $SU(2)$ coherent states can be used to investigate the explicit relation between the physical states and the Haldane's quantum Hall wavefunctions.

It is an interesting conclusion of our work that the hierarchical Hall fluid states can be dynamically generated in the finite matrix model on $S^2$. The formation of these hierarchical Hall fluid states originates from the condensing of excitations of the quantum Hall fluids level by level. This dynamical mechanism is consistent with the original idea of Haldane's hierarchy. In the procedure of the construction of the hierarchical Hall fluid states, we have found the explicit forms of 'quasiparticle' and 'quasihole' excitations in each level of the hierarchy. We believe that these results are helpful in studying the correlations of excitations in the quantum Hall effect and the interaction behaviour of them.

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