Mermin-Ho vortex in ferromagnetic spinor Bose-Einstein condensates

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The Mermin-Ho and Anderson-Toulouse coreless non-singular vortices are demonstrated to be thermodynamically stable in ferromagnetic spinor Bose-Einstein condensates with the hyperfine state $F = 1$. The phase diagram is established in a plane of the rotation drive vs the total magnetization by comparing the energies for other competing non-axis-symmetric or singular vortices. Their stability is also checked by evaluating collective modes.

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Topological structure plays an important and decisive role in various research fields, ranging from condensed matter physics to high energy physics. They provide a common framework to connect diverse fields, enhancing mutual understanding. Recent advance of experimental techniques on Bose-Einstein condensation (BEC) prompts us to closely and seriously look into theoretical possibilities which were mere imagination for theoreticians in this field. This is particularly true for spinor BEC where all hyperfine states of an atom Bose-condensed simultaneously, keeping these “spin” states degenerate and active. Recently, Barrett at al [3] have succeeded in cooling $^{87}$Rb with the hyperfine state $F = 1$ by all optical methods without resorting to a usual magnetic trap in which the internal degrees of freedom is frozen. Since the spin interaction of the $^{87}$Rb atomic system is ferromagnetic, based on the refined calculation of the atomic interaction parameters by Klausen at al [4], we now obtain concrete examples of the three component spinor BEC ($F = 1, m_F = 1, 0, -1$) for both antiferromagnetic ($^{23}$Na) and ferromagnetic interaction cases. In the present spinor BEC the degenerate internal degrees of freedom play an essential role to determine the fundamental physical properties. There is a rich variety of topological defect structures, which are already predicted in the earlier studies [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] who examine these topological structures more closely, such as skyrmion, monopole, meron or axis-symmetric or non-axis-symmetric vortices both for antiferromagnetic and ferromagnetic cases.

Superfluid $^3$He is analogous to the spinor BEC where the neutral Cooper pair possesses the orbital and spin degrees of freedom, thus the order parameter is a multi-component [17]. Among various topological structures, the Mermin-Ho (MH) [18] and Anderson-Toulouse (AT) [19] vortices of a coreless and non-singular $l$-vector texture are proposed in $^3$He-A phase. These are an extremely interesting object to study if they exist. The MH vortex is expected to spontaneously appear in a cylindrical vessel without any external rotation as an equilib-
and $i, j, k, l = (0, \pm 1)$ corresponding to the above three species. The chemical potentials for the three components; $\mu_i$ ($i = 0, \pm 1$) satisfy $\mu_1 + \mu_{-1} = 2\mu_0$. The scalar field $V(r) = C r^2$ is the external confinement potential such as an optical potential. The scattering lengths $a_0$ and $a_2$ characterize collisions between atoms with the total spin 0 and 2 respectively. As mentioned, the recent refined estimate for $^{87}$Rb concludes it ferromagnetic ($a_0 > a_2$) and $g_s/g_n = -0.01 \sim -0.005$. The external rotation frequency $\Omega$ is normalized by the harmonic confining frequency. It readily leads to the Gross-Pitaevskii (GP) equation extended to the three components; $\sum_i \delta_{ij} + g_n \sum_k \{ \delta_{i,k} \}$ is fixed. The extended GP equation is solved numerically with two different methods for a two-dimensional disk: One is not to assume axis-symmetry. This calculation is backed up by the computation where the axis-symmetry is assumed. The actual calculations are done by discretizing two-dimensional space into 51x51 mesh. The vortex configuration is characterized by the combination of the winding number of each component. We denote it as $(w_1, w_0, w_{-1})$ where integers $w_1, w_0, w_{-1}$ are the winding numbers of $\phi_1, \phi_0, \phi_{-1}$ respectively where $w$ means the phase change by $2\pi w$ when the wave function goes around the phase singularity.

We have performed extensive search to find a stable vortex, starting with possible vortex configurations, covering a wide range of the ferromagnetic interaction strength $g_s/g_n = 0 \sim -0.02$ and examining various axis-symmetric and non-axis-symmetric vortices (see the classification of possible vortices in the axis-symmetric case). We use the following parameters: the mass of a $^{87}$Rb atom $m=1.44 \times 10^{-25}$kg, the trapping frequency $\nu_r=200$Hz, and the area density $n_s=2.0 \times 10^3/\mu$m. The results described here are $g_s/g_n = -0.02$.

The Mermin-Ho (MH) vortex is described by $|\phi_1, \phi_0, \phi_{-1} \rangle = \sqrt{n(r)}(|0 \rangle \cos \frac{\beta}{2} - i \sqrt{2} \sin \frac{\beta}{2} |1 \rangle \cos \phi \sin \frac{2\nu}{2} |2 \rangle \sin \phi \frac{2\nu}{2}$ where the bending angle $\beta(r)$ is $0 \leq \beta(r) \leq \pi$, $\phi$ is the polar angle in polar coordinates. The spin direction is denoted by the so-called l-vector given by $l(r) = \hat{\mu} \cos \beta + \sin \beta (\cos \phi \hat{x} + \sin \phi \hat{y})$ where $\beta$ varies from $\beta(0) = 0$ to $\beta(R) = \pi/2$ for MH and to $\beta(R) = \pi$ for AT ($R$ is the outer boundary of the cloud). Thus the spin moment is flared out to the radial direction and at the circumference it points outward for MH and downwards for AT (see for schematic l-vector structure Fig.18 in Ref. [17]). These vortices have the winding number combination $(0, 1, 2)$ in our notation.

We show the calculated MH vortex in Fig.1 where the axis-symmetric density profiles for each component and the density map of the vortex $\pm 0.5$ are displayed. It is seen that the central region of the harmonic trap is occupied by $\phi_1$ with zero winding number $w_1 = 0$. The $\phi_0$ component which has a singularity with $w_0 = 1$ at $r = 0$ is pushed outward while the $\phi_{-1}$ component occupies the circumference region because of $w_{-1} = 2$. Note that in the small $r$ region the singular vortex with $w$ behaves like $\propto r^w$. The resulting total density is coreless and non-singular. It has a smooth density variation described by a Gaussian form except for the outermost region.

In Fig.2 we show the spatial dependence of the $l_z$-component along the radial direction, namely, the spatial dependence of the bending angle $\beta(r)$ for the MH vortex. As the magnetization $M$ decreases, the local magnetization changes from positive to negative values through zero. It means that the $l$-vector in the vortex flares out radially to orient almost horizontally $\beta(r) = \pi$ for $M/N \sim 0.5$ and to point downward for $\beta(r) = \pi$ for $M/N \sim 0$. The former (latter) corresponds literally to the Mermin-Ho (Anderson-Toulouse) vortex. This is simply because as $M$ decreases, the spin-down component $\phi_{-1}$ with $w = 2$ increases in the outer region. Thus we can control these MH and AT vortices by merely changing the total magnetization. We notice that in superfluid $^3$He-A phase the stability of the MH and AT vortices is guaranteed by the boundary condition at the wall of a vessel where the $l$-vector is constraint to be parallel to the wall surface. Here the situation is completely different; we impose no constraint on the $l$-vector direction. These vortex configurations are created naturally under the condition of a given total number and magnetization, both of which are well controlled in a harmonic trap experiment.

This difference in the two systems is rather remark-
able and interesting. In our case MH or AT vortex comes about purely due to the spin interaction, which is written as $\propto |g_s| |\vec{r}|/ \phi_{1(r)} \phi_{-1(r)} - \phi_0(r)|^2$. This term favors the mutual phase segregation [15, 16]. If $\phi_1(r) \neq 0$ and $\phi_{-1}(r) \neq 0, \phi_0(r)$ comes in, explaining the concentric layered structure. In fact, in the antiferromagnetic case the similar calculation shows that the MH and AT vortices are never stabilized under a similar condition [16, 21].

We have done extensive search for determining the region for the $(0, 1, 2)$ vortex of the MH and AT in the plane of $\Omega$ vs $M/N$. We check our computation by starting with a certain initial vortex configuration, allowing non-axis-symmetric MH and AT vortices where these are stable. The stability for these vortices is also examined from the two aspects: one is the global stability and the other is local stability in the energy landscape. The global stability means comparing the relative energy of various vortices and vortex free state to select out the lowest energy state. The local stability is discussed shortly.

The resulting phase diagram is displayed in Fig.3 where a large area is occupied by the $(0, 1, 2)$ vortex, including MH and AT. Near $M \sim 0$ the non-axis-symmetric $(1, 1, 1)$ vortex and near $M/N \sim 1$, $(1, 0, -1)$ vortex are stabilized. We find a large empty region in the intermediate $M/N$ region where no vortex and vortex-free state are stabilized at all because the phase separation in the ferromagnetic case prevents forming a uniform mixture of the three components even when the circulation is absent for the vortex-free state. This is in contrast with the antiferromagnetic case in which everywhere is occupied by a stable phase.

It is noted that the focused frequency region $\Omega = 0.38$ corresponds to the single vortex region in the scalar BEC case [22] beyond which multiple vortex state must be seriously considered, thus the present single vortex consideration is justified in the lower frequencies.

We examine various competing vortices, including a singular vs non-singular and axis-symmetric vs non-axis-symmetric vortices. In particular, since for axis-symmetric vortices the winding number combination of the three components $\langle w_1, w_0, w_{-1} \rangle$ is restricted to $2w_0 = w_1 + w_{-1}$, we exhaust all the possible vortices with $w_1, w_0, w_{-1}$ smaller than unity [16]. Note that MH and AT belong to this axis-symmetry category.

The $(1, 1, 1)$ vortex shown in Fig. 4 is non-singular and non-axis-symmetric which is stable in low $M/N$. This $(1, 1, 1)$ vortex is advantageous because (A) it does not contain the higher winding number. It generally leads to collapse fewer winding number vortex, such as $2 \rightarrow 1 + 1$ vortices. (B) The overall condensation energy is gained by placing their singularities off the trap center where the potential energy is minimum. (C) As $\Omega$ increases, the two separate singularities of $\phi_1$ and $\phi_{-1}$ adjust their mutual distance from the center so as to maximize the angular momentum $\vec{L}$ in order to save the energy $-\Omega \cdot \vec{L}$. In this sense this configuration is flexible against varying $\Omega$. These reasons explain why this vortex survives along the $\Omega$-axis near $M/N \sim 0$ in Fig.3.

In contrast, the $(0, 1, 2)$ vortex contains the higher winding number 2 for $\phi_{-1}$, which makes it less advantageous against the $(1, 1, 1)$ vortex. However, upon varying $M$, this vortex is quite flexible by adjusting the particle numbers for the three components; As $M$ increases, the number of $\phi_1$ grows smoothly relative to the rests. The non-winding $\phi_1$ component works as a “pinning” center for the remaining $\phi_0$ and $\phi_{-1}$. In particular, $\phi_{-1}$ with $w_{-1} = 2$ is stabilized by the presence of $\phi_1$. Therefore as $M$ increases or $\phi_1$ grows, the MH and AT vortices become more and more stable. This explains that the critical frequency, or the lower phase boundary of MH in Fig.3 becomes less as $M$ increases. As for the $(1, 1, 1)$ vortex, in comparison, is not flexible enough for varying $M$ because the symmetric arrangement of the two singularities in $\phi_1$ and $\phi_{-1}$ around the center (see Fig.4) becomes unbalanced as $\phi_1$ component grows while $\phi_{-1}$ shrinks. Thus it is confined in a narrow thin region near $M/N \sim 0$ (see Fig.3).

The $(1, 0, -1)$ vortex, (not shown here, see Fig.3 in...
Ref. [15] is stabilized in larger $M/N$ and higher $\Omega$ region because in this vortex the dominant component $\phi_1$ has the winding number 1, which can effectively lower the rotation energy.

Having finished the “global” stability of the MH and AT vortices, we now turn to the “local” stability, that is, the stability against a small perturbation. This is done by solving the extended Bogoliubov equations to the three components under the axis-symmetric situation \[ \sum_j (A_{ij} u_j(r,j) - B_{ij} v_j(r,j)) = \varepsilon_g u_{ij}(r,i) \]
and \[ \sum_j (B_{ij} u_j(r,j) - A_{ij} v_j(r,j)) = \varepsilon_g v_{ij}(r,i) \]
where
\[
A_{ij} = h(r) \delta_{ij} - \mu_i \delta_{ij} + g_{\alpha} \left\{ \sum_k [\phi_k]_l \delta_{ij} + \phi_i \phi_j^* + g_\alpha \sum_k \sum_l [F_\alpha]_{ik} \phi_k \phi_l \right\}, \]
\[
B_{ij} = g_{\alpha} \phi_i \phi_j + g_\alpha \sum_l \sum_k [F_\alpha]_{ik} \phi_k \phi_l \right\}, u_{ij}(r,i) \]
and $v_{ij}(r,i)$ are the $q$-th eigenfunctions with the spin $i$ and $\varepsilon_g$ corresponds to the $q$-th eigenvalue. Since this gives the excitation spectrum of the system, the negative energy relative to the condensation energy at zero implies the local intrinsic instability of the putative vortex in the energy landscape. We have performed the extensive computation to check this local stability for MH vortex and other axis-symmetric vortices against a small perturbation. As expected, the lower phase boundary of the MH in Fig.3 coincides almost completely with the local stability region, below which the lowest excitation mode with the angular momentum $q_\theta = -1$ for the positive external rotation ($\Omega > 0$) becomes negative (see the inset in Fig.3). The upper boundary of the $\langle 0,1,2 \rangle$ vortex in Fig.3 indicates that the lowest excitation mode with mainly $q_\theta = +1$ becomes negative (see the inset in Fig.3). Thus the global stability mentioned roughly corresponds to the present local stability. Therefore, it is concluded that the MH and AT vortices are stable and robust in the ferromagnetic spinor BEC.

It is easy to calculate the total angular momentum $L$ in MH vortex which is given by $L/\hbar = N_0 + 2N_{-1}$ ($N_i$ is the particle number of the $i$-component). Since $M = N_1 - N_{-1}$ and $N = N_1 + N_0 + N_{-1}$, the total angular momentum per particle is found to be $\frac{L}{\hbar} = 1 - \frac{M}{N}$. This simple formula means that at $M = 0$, $\phi_0$ with $w_0 = 1$ carries all the angular momentum as in the usual scalar vortex, and at $M = N$, $L = 0$ because $\phi_1$ has no winding. At $\frac{M}{N} = \frac{1}{2}$, $L/\hbar$ is half, implying that in the MH vortex the angular momentum per particle is exactly $\hbar/2$.

In summary, we have shown that the Mermin-Ho and Anderson-Toulouse vortices are thermodynamically stable under a certain rotation drive and the total magnetization of the system, both of which are experimentally well-controlled parameters. We have examined their stability by two methods, local and global ones in the energy landscape, coinciding with the earlier conclusion that these are unstable under no rotation drive \[ \Omega \]. These intriguing objects might be detected by various ways, such as optically by utilizing the Faraday rotation which images the spatial magnetic pattern of these vortices. A favorable experiment is to use an oblate ellipsoidal shaped system to mimic our disk calculation. An elongated cigar system may not be appropriate, which induces the phase separation along the long axis.

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