Quantum Mechanics of Dynamical Zero Mode in $QCD_{1+1}$ on the Light-Cone

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Abstract

Motivated by the work of Kalloniatis, Pauli and Pinsky, we consider the theory of light-cone quantized $QCD_{1+1}$ on a spatial circle with periodic and anti-periodic boundary conditions on the gluon and quark fields respectively. This approach is based on Discretized Light-Cone Quantization (DLCQ). We investigate the canonical structures of the theory. We show that the traditional light-cone gauge $A_-=0$ is not available and the zero mode (ZM) is a dynamical field, which might contribute to the vacuum structure nontrivially. We construct the full ground state of the system and obtain the Schrödinger equation for ZM in a certain approximation. The results obtained here are compared to those of Kalloniatis et al. in a specific coupling region.

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1 Introduction

Quantum field theory on the light-cone has been recently studied as a new powerful tool for understanding non-perturbative phenomena [1, 2], especially in the theory of strong interaction (QCD) [3, 4, 5, 6, 7]. One of the most remarkable advantages in light-cone formalism is that vacuum is simple or trivial, i.e., Fock vacuum is an eigenstate of the light-cone Hamiltonian [8]. On the other hand, in the usual equal-time formalism the vacuum contains an infinitely large number of particles. However, one simple and naive question arises here: how can we understand phenomena like chiral symmetry breaking and confinement in such simple vacuum?

As has already been indicated by many authors [9, 10, 11], zero modes of the fields might play an essential and important role there. Recently Kalloniatis et al. have investigated about pure glue $QCD_{1+1}$ (an SU(2) non-Abelian gauge theory in 1+1 dimensions with classical sources coupled to the gluons) and have discussed the physical effect of the dynamical zero mode [12]. Note here that there are two kinds of zero modes of the fields. One is called constrained zero mode, which is not independent degrees of freedom. Rather, it is dependent each other through the constraint equation. There have been many works on such a constrained zero mode related to the phenomena of phase transition in scalar field theory [13, 14, 15]. The other is called dynamical zero mode we treat here, which is a true dynamical independent field. Also Kalloniatis et al. have used the specific approach of Discretized Light-Cone Quantization (DLCQ) [2] in their analysis because this approach gives us an infrared regulated theory and the discretization of momenta facilitates putting the many-body problem on the computer. We shall follow them, too.

Our aim in this paper is to study the light-cone quantized $QCD_{1+1}$ with fundamental fermions (quarks) coupled to the gauge fields (gluons) more in detail and give
insight into the nontrivial QCD vacuum structure. The contents of this paper are as follows. In Section 2, we study the canonical structures of $QCD_{1+1}$ on the light-cone (Hamiltonian formalism) based on the Dirac’s treatment of the constraint system. We explicitly obtain canonical light-cone Hamiltonian and Dirac brackets between physical quantities there. We also comment on dynamical zero mode of the gluon fields in this section. In Section 3, we quantize the theory developed in the previous section and construct full ground state of Hamiltonian. Furthermore we derive the Schrödinger equation for zero mode in a specific coupling region. Section 4 is devoted to summary and discussion. The appendix is put in the last part of this paper for the explanations of notations and conventions.

2 Classical Theory - Hamiltonian Formalism -

In this section, we study the canonical structures of light-cone $QCD_{1+1}$, where the space is a circle and the gauge group is $SU(2)$. Let us start with following Lagrangian density

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi.$$  

(2.1)

Here $\Psi(x)$ is a quark field. Especially in two dimensions, the quark field (in a representation in which $\gamma^5$ is diagonal)

$$\Psi(x) = \begin{pmatrix} \Psi_R(x) \\ \Psi_L(x) \end{pmatrix}.$$  

(2.2)

is a two component spinor in the fundamental representation $[2]$. $R$ and $L$ indicates chirality, which specifies only direction for massless fermions. While the field $F_{\mu\nu}^a$ and the covariant derivative are defined as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\epsilon^{abc} A_\mu^b A_\nu^c,$$  

(2.3)

$$iD_\mu = i\partial_\mu - gA_\mu^a T^a,$$  

(2.4)
where $A^a_\mu(x)$ is a “gluon” field and $g$ is the coupling constant and $T^a$ and $\epsilon^{abc}$ are the generators and the structure constant of the $SU(2)$ gauge group defined as

$$[T^a, T^b] = i\epsilon^{abc}T^c,$$

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}.$$  (2.5)

In the light-cone frame approach, we set the coordinates

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1),$$

and then rewrites all the quantities involved in the Lagrangian density (2.1) in terms of $x^\pm$ instead of the original coordinates $x^0$ (time) and $x^1$ (space). As the usual Discretized Light-Cone Quantization, we define $x^+$ the light-cone “time”, while $x^-$ the light-cone “space”, which is restricted to a finite interval from $-L$ to $L$. Within the interval, we impose periodic and antiperiodic boundary conditions on the gluon field $A^a_\mu(x)$ and the quark field $\Psi(x)$ respectively i.e.,

$$A^a_\mu(x^+, x^- + 2L) = A^a_\mu(x^+, x^-),$$
$$\Psi(x^+, x^- + 2L) = -\Psi(x^+, x^-).$$

In this way the Lagrangian density (2.1) is rewritten as

$$\mathcal{L} = \frac{1}{2}(F^a_{+\mp})^2$$
$$+\sqrt{2}(\Psi_R^\dagger \partial_+ \Psi_R + \Psi_L^\dagger \partial_- \Psi_L) - m(\Psi_L^\dagger \Psi_R + \Psi_R^\dagger \Psi_L)$$
$$-\sqrt{2}g(\Psi_R^\dagger T^a \Psi_R A^a_+ + \Psi_L^\dagger T^a \Psi_L A^a_-),$$

where $\partial_\pm = \frac{\partial}{\partial x^\pm}$. In order to carry out the Hamiltonian formulation, we must compute the canonical momenta

$$\Pi^{+a}(x) = \frac{\partial \mathcal{L}}{\partial (\partial_+ A^a_+(x))} = 0,$$
\[ \Pi^{-a}(x) = \frac{\partial L}{\partial \partial_+ A_+^a(x)} = F_{+-}^a(x), \]  
(2.12)  
\[ P_L(x) = \frac{\partial L}{\partial \partial_+ \Psi_L(x)} = 0, \]  
(2.13)  
\[ P^r_L(x) = \frac{\partial L}{\partial \partial_+ \Psi^r_L(x)} = 0, \]  
(2.14)  
\[ P_R(x) = \frac{\partial L}{\partial \partial_+ \Psi_R(x)} = \sqrt{2i} \Psi_R^\dagger(x), \]  
(2.15)  
\[ P^r_R(x) = \frac{\partial L}{\partial \partial_+ \Psi^r_R(x)} = 0. \]  
(2.16)  

On the other hand, we find that in the DLCQ approach, from the boundary conditions (2.8), the gluon field \( A_\mu^a \) can be decomposed into zero mode (ZM) and particle mode (PM) as follows.

\[ A_\mu^a(x) = A_\mu^a + \tilde{A}_\mu^a(x), \]  
(2.17)  

where

\[ \tilde{A}_\mu^a(x) \equiv A_\mu^a(x) - A_\mu^a, \]  
(2.18)  
\[ \tilde{A}_\mu^a(x) \equiv A_\mu^a(x) - A_\mu^a. \]  
(2.19)  

Here \( \tilde{A}_\mu^a \) and \( \tilde{A}_\mu^a \) denote the zero modes and the particle modes of the gluon field, respectively. Similarly the canonical momenta (2.11) and (2.12) are decomposed into ZM and PM, which leads to

\[ \Pi^{+a} = 0, \]  
(2.20)  
\[ \tilde{\Pi}^{+a}(x) = 0, \]  
(2.21)  
\[ \Pi^{-a} = \tilde{F}^a_{+-}, \]  
(2.22)  
\[ \tilde{\Pi}^{-a}(x) = \tilde{F}^a_{+-}(x), \]  
(2.23)  

where

\[ \tilde{F}^a_{+-} \equiv \frac{1}{2L} \int_{-L}^{L} dx^- F^a_{+-}(x), \]
\[ \partial_+ \tilde{A}_a^+ - g \epsilon^{abc} \tilde{A}_b^+ \tilde{A}_c^+ - \frac{g}{2L} \int_{-L}^L dx^- \epsilon^{abc} \tilde{A}_b^+(x) \tilde{A}_c^+(x), \]  

(2.24)

\[ \tilde{F}_a^+(x) = F_a^+(x) - \tilde{F}_a^+, \]

\[ = \partial_+ \tilde{A}_a^+ - \partial_+ \tilde{A}_a^- - g \epsilon^{abc}(\tilde{A}_b^+ \tilde{A}_c^- + \tilde{A}_+^a \tilde{A}_-^a + \tilde{A}_+^b \tilde{A}_-^c). \]

(2.25)

Following Dirac then [17], we can see that there are six primary constraints such as

\[ \tilde{\Pi}^{+a} \approx 0, \]

(2.26)

\[ \tilde{\Pi}^{+a}(x) \approx 0, \]

(2.27)

\[ P_L(x) \approx 0, \]

(2.28)

\[ P_L^\dagger(x) \approx 0, \]

(2.29)

\[ P_R(x) - \sqrt{2} i \Psi_R^\dagger(x) \approx 0, \]

(2.30)

\[ P_R^\dagger(x) \approx 0, \]

(2.31)

where \( \approx \) means the weak equality as usual. Substituting the equation (2.17) into the Lagrangian density (2.10) and using the equations (2.11)-(2.16), total light-cone Hamiltonian can be obtained:

\[ H_{T}^{lc} = \int_{-L}^L dx^- [\Pi^{-a} \partial_+ A_a^- + P_R \partial_+ \Psi_R - \mathcal{L}], \]

\[ = \int_{-L}^L dx^- \left[ \frac{1}{2} (\tilde{\Pi}^{-a})^2 + \frac{1}{2} (\tilde{\Pi}^{-a})^2 + \tilde{\Pi}^{-a} \partial_+ \tilde{A}_a^+ + \tilde{\Pi}^{-a} \tilde{A}_a^- \tilde{A}_a^- + \tilde{\Pi}^{-a} \tilde{A}_a^+ \tilde{A}_a^+ + \tilde{\Pi}^{-a} \tilde{A}_a^+ \tilde{A}_a^- \right] \]

\[ + g \epsilon^{abc} (\Pi^{-a} \tilde{A}_a^+ \tilde{A}_a^+ + \Pi^{-a} \tilde{A}_a^- \tilde{A}_a^- + \Pi^{-a} \tilde{A}_a^+ \tilde{A}_a^- + \Pi^{-a} \tilde{A}_a^- \tilde{A}_a^+ + \Pi^{-a} \tilde{A}_a^+ \tilde{A}_a^- + \Pi^{-a} \tilde{A}_a^- \tilde{A}_a^+) \]

\[ - \sqrt{2} \Psi_L^T i \partial_- \Psi_L + m (\Psi_L^T \Psi_R + \Psi_R^T \Psi_L) \]

\[ \sqrt{2} g \left( \Psi_R^T T^a \Psi_R (A_a^+ + \tilde{A}_a^+) + \Psi_L^T T^a \Psi_L (A_a^+ + \tilde{A}_a^-) \right) \]

\[ + u_0^a \tilde{\Pi}^{+a} + u_1^a \tilde{\Pi}^{+a} + u_2 P_L + u_3 P_L^\dagger \]

\[ + u_4 (P_R - \sqrt{2} i \Psi_R^\dagger) + u_5 P_R^\dagger \],

(2.32)
where $u_0^a$ and $u_1^a$ ($u_2, u_3, u_4$ and $u_5$) are (Grassmann) Lagrange multipliers and we have used the following facts

$$\int_{-L}^{L} dx^- \bar{A}_\pm^a(x) = \int_{-L}^{L} dx^- \bar{\Pi}_\pm^a(x) = 0. \quad (2.33)$$

Once we obtain the expression for the Hamiltonian, we must investigate whether the primary constraints induce the secondary constraints by imposing the consistency conditions. As the result, we find there are four secondary constraints:

$$g \int_{-L}^{L} dx^- (\Pi^- \epsilon^{abc} A^b_\pm + \sqrt{2} \Psi_R^T \Psi_R) \approx 0, \quad (2.34)$$

$$\partial^- \bar{\Pi}^a_\pm - g (\Pi^- \epsilon^{abc} A^b_\pm) \sim - \sqrt{2} g (\Psi_R^T \Psi_R) \sim 0, \quad (2.35)$$

$$\sqrt{2} i \partial^- \Psi_L - m \Psi_R - \sqrt{2} g \Psi_L T^a A^-_\pm \approx 0, \quad (2.36)$$

$$(\sqrt{2} i \partial^- \Psi_L - m \Psi_R - \sqrt{2} g \Psi_L T^a A^-_\pm)^\dagger \approx 0, \quad (2.37)$$

where

$$(A_1 A_2 \cdots A_n)_\sim = A_1(x) A_2(x) \cdots A_n(x) - \frac{1}{2L} \int_{-L}^{L} dx^- A_1(x) A_2(x) \cdots A_n(x). \quad (2.38)$$

We can show directly that constraints (2.34)-(2.37) do not generate new constraints further. Adding these constraints to the primary constraints, there exist ten constraints, which govern the dynamics of our system. What we have to do next is to classify these primary and secondary constraints to the first class or the second class constraints. A direct calculation shows that the constraints (2.26) and (2.27) belong to the first class and others the second one. But this is not true. As indicated by some authors [18, 19], the minimal set of the second class constraints is found by combining constraints except for (2.26) and (2.27) appropriately and it is easy to show that this set is indeed given by

$$\Omega_0^a = \Pi^+ a, \quad (2.39)$$
\(
\Omega_1^a = \tilde{\Pi}^+ a(x), \tag{2.40}
\)

\[
\Omega_2^b = g \int_{-L}^{L} dx^-[\Pi^{-\epsilon} e^{ab} A^b(x) \nonumber \\
- i(P_L T^a \Psi_L + \Psi_L^+ T^a P_L + P_R T^a \Psi_R + \Psi_R^+ T^a P_R)(x)], \tag{2.41}
\]

\[
\Omega_3^a = \left( \partial_\perp \Pi^{-a}(x) - g \Pi^{-\epsilon} e^{abc} A^b(x) \right. \nonumber \\
\left. + ig(P_L T^a \Psi_L + \Psi_L^+ T^a P_L + P_R T^a \Psi_R + \Psi_R^+ T^a P_R)(x) \right) \sim, \tag{2.42}
\]

\[
\chi_1 = P_L(x), \tag{2.43}
\]

\[
\chi_2 = P_L^+(x), \tag{2.44}
\]

\[
\chi_3 = P_R(x) - \sqrt{2} i \Psi_R^+(x), \tag{2.45}
\]

\[
\chi_4 = P_R^+(x), \tag{2.46}
\]

\[
\chi_5 = \sqrt{2} i \partial_\perp \Psi_L(x) - m \Psi_R(x) - \sqrt{2} g \Psi_L T^a A^+_a(x), \tag{2.47}
\]

\[
\chi_6 = \left( \sqrt{2} i \partial_\perp \Psi_L(x) - m \Psi_R(x) - \sqrt{2} g \Psi_L T^a A^+_a(x) \right)^+, \tag{2.48}
\]

where \(\Omega_\alpha^a(\alpha = 0, 1, 2, 3)\) and \(\chi_\beta(\beta = 1 \sim 6)\) denote the first and second class constraints, respectively. The first class constraints satisfy the algebra

\[
\{ \Omega_\alpha^a(x), \Omega_\beta^b(y) \} = 0, \tag{2.49}
\]

which reflects the gauge invariance of the system. In order to eliminate all the constraints and quantize the system, we need to fix the gauge degrees of freedom and define the Dirac bracket along the usual prescriptions [8]. Here we shall give the gauge fixing conditions as follows:

\[
\omega_0^a \equiv A_+^a \approx 0, \tag{2.50}
\]

\[
\omega_1^a \equiv \tilde{\Pi}^+ a(x) + \partial_\perp \tilde{A}^+_a(x) + g e^{abc} \left( A^a_+(x) A^c_-(x) \right) \sim \approx 0, \tag{2.51}
\]

\[
\omega_2^a \equiv A^+_i \approx 0, \quad for \quad i = 1, 2, \tag{2.52}
\]

\[
\omega_3^a \equiv \tilde{A}^+_a(x) \approx 0. \tag{2.53}
\]
Note here the following remarkable fact. As we see from the gauge-fixing conditions (2.52) and (2.53), we can not impose the traditional light-cone gauge $A^a_\perp = 0$ because we can not put the third component of $A^a_\perp$ to be zero. $SU(2)$ global color rotation symmetry always enables us to choose such a gauge fixing conditions [20]. That is why one of the zero modes of the gluon field $A^3_\perp$ becomes a dynamical variable, which might give insight to the nontrivial structures of the QCD light-cone vacuum [20].

Now we are coming in the stage of evaluating the Dirac bracket. After the straightforward but some tedious calculations, non-zero Dirac brackets are

$$\{ \bar{A}^3_\perp, \Pi^{-3} \}_{DB} = \frac{1}{2L},$$

$$\{ \Psi_R(x), \Psi_R^\dagger(y) \}_{DB} = \frac{i}{\sqrt{2}} \delta(x^--y^-).$$

As far as Dirac brackets have been used, we may put all the constraints and the gauge fixing conditions to strongly zeroes. The result is that total Hamiltonian (2.32) reduces to the form

$$H_{l.c.}^T = \int_{-L}^L dx^- \left[ \frac{1}{2} p^2 + m \Psi_R^\dagger \Psi_L + \frac{g}{\sqrt{2}} (\Psi_R^\dagger T^a \Psi_R^\dagger) \tilde{A}^a_+ \right],$$

where $p \equiv \Pi^{-3}$ and $\Psi_L(x)$ and $\tilde{A}^a_+(x)$ are given as the functions satisfying following equations, i.e.,

$$\sqrt{2} i \partial_- \Psi_L(x) - m \Psi_R(x) - \sqrt{2} g q(x^+) T^3 \Psi_L(x) = 0,$$

$$\partial_-^2 \tilde{A}^a_+(x) + 2 \epsilon^{abc} g q(x^+) \partial_- \tilde{A}^b_+(x) - g^2 q^2(x^+) (\tilde{A}^a_+ - \delta^{a,3} \tilde{A}^3_+)(x) + \sqrt{2} g \Psi_R^\dagger T^a \Psi_R(x) = 0,$$

where $q(x^+) \equiv \tilde{A}^3_\perp$. While $\Psi_R(x)$ has been chosen to satisfy charge neutrality condition

$$Q^3 \equiv g \int_{-L}^L dx^- \Psi_R^\dagger T^3 \Psi_R(x) = 0.$$

This corresponds to the third component of ZM of Gauss law, which is necessary whenever the system is in a finite interval [21]. The fields $\Psi_L$ and $\tilde{A}^a_+$ are expressed
in terms of $\Psi_R$ and $q(x^+)$ by solving the equations (2.57) and (2.58). As the result, we find that the physical degrees of freedom in this system are only the diagonal part of zero modes of the gluon field $q(x^+)$ and right-handed quark field $\Psi_R$.

3 Quantum Theory - Dynamical ZM Equation -

In this section, we discuss the quantum aspects of the theory studied in the previous section at the classical level in detail. First we start by discussing eigenstates of the matter part in the Hamiltonian (2.56) in the fixed background gauge field and then we construct the full ground state including ZM of the gauge field $q(x^+)$. Before doing that, we must solve equations (2.57) and (2.58) for $\Psi_L$ and $\tilde{A}_a^+$. Equation (2.57) for $\Psi_L$ is easy to be solved as follows:

$$\Psi_L^c(x) = \frac{m}{2\sqrt{2}L} \sum_{k=0}^{\infty} \int_{-L}^{L} dy e^{-\frac{i\pi}{2L}(k+\frac{1}{2})(x^- - y^-)} \tilde{\Psi}_R^c(y; k),$$  \hspace{1cm} (3.1)

where

$$\tilde{\Psi}_R^c(x; k) = \begin{pmatrix} \Psi_R^1(x) \\ \Psi_R^2(x) \\ \Psi_R^3(x) \end{pmatrix}^c,$$  \hspace{1cm} (3.2)

with $c$ being the color indices.

On the other hand, equations for $\tilde{A}_a^+$ consist of the following three components:

$$\partial^2 \tilde{A}_1^+(x) + 2gq(x^+)\partial_- \tilde{A}_2^+ - g^2 q^2(x^+) \tilde{A}_1^+ + \rho^1(x) = 0,$$  \hspace{1cm} (3.3)

$$\partial^2 \tilde{A}_2^+(x) - 2gq(x^+)\partial_- \tilde{A}_1^+ - g^2 q^2(x^+) \tilde{A}_2^+ + \rho^2(x) = 0,$$  \hspace{1cm} (3.4)

$$\partial^2 \tilde{A}_3^+(x) + \rho^3(x) = 0,$$  \hspace{1cm} (3.5)

where $\rho^a(x) \equiv \sqrt{2}\frac{g}{\Psi_R^i T^a \Psi_R}$. Clearly a solution for equation (3.5) is formally written of the form $\tilde{A}_3^+(x) = -\frac{\partial}{\partial x} \rho^3(x)$. Thus, we will concentrate to the remained equations (3.3) and (3.4). For brevity, we rewrite equations (3.3) and (3.4) as

$$\frac{d^2 f(x)}{dx^2} + 2a \frac{dg(x)}{dx} - a^2 f(x) + \rho^1(x) = 0,$$
\[
\frac{d^2 g(x)}{dx^2} - 2a \frac{df(x)}{dx} - a^2 g(x) + \rho^2(x) = 0, \tag{3.6}
\]

where we are putting now as follows:

\[
x \equiv x^-, \\
a \equiv gq(x^+), \\
\bar{A}_1(x) \equiv f(x), \\
\bar{A}_2(x) \equiv g(x). \tag{3.7}
\]

It is easy to find that we can express equation (3.6) in matrix representation

\[
M^{ij}(x) \varphi^i(x) = -\rho^i(x). \tag{3.8}
\]

Here 2 \times 2 matrix \(M^{ij}(x)\) and vectors \(\phi^i(x)\) and \(\rho^i(x)\) are defined by

\[
M^{ij}(x) = \left( \begin{array}{cc} \frac{\sigma}{dx} - a^2 & 2a \frac{d}{dx} \\ -2a \frac{d^2}{dx^2} - a^2 \end{array} \right)^{ij}, \\
\varphi^i(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}^i, \\
\rho^i(x) = \begin{pmatrix} \rho^1(x) \\ \rho^2(x) \end{pmatrix}^i. \tag{3.9}
\]

Using the usual Green function’s method, the form of \(\varphi^i(x)\) would be given by

\[
\varphi^i(x) = \int_{-L}^{L} dy G^{ij}(x, y) \rho^j(y), \tag{3.10}
\]

where \(G^{ij}(x, y)\) is the Green function defined by

\[
M^{ij}(x)G^{jk}(x, y) = -\delta^{ik} \delta(x - y). \tag{3.11}
\]

By solving equation (3.11) with \(M^{ij}(x)\) given by (3.9), we obtain the explicit forms of \(G^{ij}(x, y)\), \(\bar{A}_1(x)\) and \(\bar{A}_2(x)\) such that

\[
G^{ij}(x, y) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{e^{\frac{\pi n}{L}(x-y)}}{(\frac{\pi n}{L} + a)^2(\frac{\pi n}{L} - a)^2} \left( \frac{(\frac{\pi n}{L})^2 + a^2}{-2\pi na/L} \right)^{ij}, \tag{3.12}
\]
\begin{align}
\tilde{A}_1^+(x) &= \frac{1}{L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} dy \left[ \frac{(\pi n L)^2 + a^2}{(\pi n L + a)^2(\pi n L - a)^2} \rho_1(y) + \frac{2\pi ina L}{(\pi n L + a)^2(\pi n L - a)^2} e^{\frac{\pi n a}{L}(x-y)} \right], \quad (3.13) \\
\tilde{A}_2^+(x) &= \frac{1}{L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} dy \left[ \frac{(\pi n L)^2 + a^2}{(\pi n L + a)^2(\pi n L - a)^2} \rho_2(y) - \frac{2\pi ina L}{(\pi n L + a)^2(\pi n L - a)^2} e^{\frac{\pi n a}{L}(x-y)} \right]. \quad (3.14)
\end{align}

As the result, by substituting these equations and \( \tilde{A}_3^+(x) = -\frac{1}{\tilde{\rho}_3} \rho_3(x) \) into the electrostatic Coulomb energy part in the Hamiltonian, then we find that

\begin{align}
H_{\text{Coulomb}}^{l.c.} &= \int_{-L}^{L} dx \frac{g}{\sqrt{2}} (\bar{\Psi}_R T \rho \Psi_R) \tilde{A}_3^+(x), \\
&= \frac{1}{2} \int_{-L}^{L} dx \left[ \rho_1(x) \tilde{A}_1^+ + \rho_2(x) \tilde{A}_2^+ + \rho_3(x) \tilde{A}_3^+ \right], \\
&- \frac{1}{2} \int_{-L}^{L} dx \rho_3(x) \frac{1}{\tilde{\rho}_3(x)} \tilde{A}_3^+(x), \\
&= \int_{-L}^{L} dx \int_{-L}^{L} dy \sum_{i=1,2} \rho_i^+(x) \rho_i^-(y) e^{-igq(x^+)} T \delta(x^--y^-) \\
& \times \left( \frac{L}{2 \sin^2(gqL)} + i(x^--y^-) \cot(gqL) - |x^--y^-| \right) \\
& - \frac{1}{2} \int_{-L}^{L} dx \rho_3^+(x) \frac{1}{\tilde{\rho}_3^3(x)}. \quad (3.15)
\end{align}

This is consistent with the result from [22].

In order to quantize the Hamiltonian (2.56), we replace the Dirac brackets to the commutators. Quantization conditions for the fields \( \tilde{\Psi}_R(x) \) and \( q(x^+) \) are defined as

\begin{align}
[\tilde{q}(x^+), \tilde{p}(x^+)] &= i, \quad (3.16) \\
\{ \tilde{\Psi}_R^c(x), \tilde{\Psi}_R^{c\dagger}(y) \} &= \frac{1}{\sqrt{2}} \delta_{c'c} \delta(x^- - y^-), \quad (3.17) \\
\{ \tilde{\Psi}_R^c(x), \tilde{\Psi}_R^c(y) \} &= \{ \tilde{\Psi}_R^{\dagger}(x), \tilde{\Psi}_R^{\dagger}(y) \} = 0. \quad (3.18)
\end{align}

where \( \hat{A} \) means an operator and we have rescaled \( 2Lq \rightarrow q \). Thus, quantum light-cone Hamiltonian is composed of following three parts:

\begin{align}
\hat{H}^{l.c.} = \hat{H}^{l.c.}_{ZM} + \hat{H}^{l.c.}_F + \hat{H}^{l.c.}_{\text{Coulomb}}, \quad (3.19)
\end{align}
where

\[
\hat{H}_{\text{ZM}}^{\text{l.c.}} = \int_{-L}^{L} dx - \frac{1}{2} \hat{\rho}^2 ,
\]

\[
\hat{H}_{F}^{\text{l.c.}} = \int_{-L}^{L} dx m \hat{\Psi}_{R}^{\dagger} \hat{\Psi}_{L} ,
\]

\[
\hat{H}_{\text{Coulomb}}^{\text{l.c.}} = \int_{-L}^{L} dx - \int_{-L}^{L} dy \sum_{i=1,2} \hat{\rho}^i(x) \hat{\rho}^i(y) e^{-\frac{q}{\hbar} \hat{q}(x^-)(x^- - y^-)}
\]

\[
\times \left( \frac{L}{2 \sin^2 \left( \frac{q L}{2} \right)} + i (x^- - y^-) \cot \left( \frac{q L}{2} \right) - |x^- - y^-| \right)
\]

\[- \frac{1}{2} \int_{-L}^{L} dx \hat{\rho}^3(x) \frac{1}{\partial^2} \hat{\rho}^3(x) ,
\]

and

\[
\hat{\Psi}_{c}^{\dagger}(x) = \frac{m}{2 \sqrt{2L}} \sum_{k=0}^{\infty} \int_{-L}^{L} dy e^{-\frac{i q}{\hbar} (k+\frac{1}{2})(x^- - y^-)} \hat{\Psi}_{R}^{\dagger}(y) ,
\]

\[
\hat{\rho}^a(x) = \sqrt{2g} : \hat{\Psi}_{R}^{\dagger} T^a \hat{\Psi}_{R}(x) : .
\]

Here \( : : \) means a normal-ordered product.

In this stage, our treatment is still exact. Since we could obtain the complete forms of quantum light-cone Hamiltonian \( \hat{H}_{\text{l.c.}}^{\text{l.c.}} \), we will next discuss eigenstates of \( \hat{H}_{F}^{\text{l.c.}} + \hat{H}_{\text{Coulomb}}^{\text{l.c.}} \). To do that, we first construct the ground state of our system in the presence of fixed background values of ZM of the gauge field \( q(x^+) \) \[22\]. This is corresponding to a kind of the adiabatic approximation. Then we can find easily a fermionic part of the light cone vacuum eigenstate as \( |0\rangle_f \), defined by

\[
\hat{a}_{k}^{\dagger}|0\rangle_f = \hat{d}_{k}^{\dagger}|0\rangle_f = 0 ,
\]

for all \( c \) and \( k \). Here the creation (annihilation) operator \( \hat{a}_{k}^{\dagger}, \hat{d}_{k}^{\dagger}(\hat{a}_{k}, \hat{d}_{k}) \) are defined through the following mode expansion of \( \hat{\Psi}_{R}(x) \), which comes from the antisymmetric boundary condition (2.9) and the anticommutation relation (3.17) and (3.18):

\[
\hat{\Psi}_{R}^{\dagger}(x) = \frac{1}{2 \sqrt{2L}} \sum_{k=0}^{\infty} \{ \hat{a}_{k}^{\dagger} e^{-\frac{iq}{\hbar} (k+\frac{1}{2}) x^-} + \hat{d}_{k}^{\dagger} e^{\frac{iq}{\hbar} (k+\frac{1}{2}) x^-} \} ,
\]

(3.25)
where

\[ \{\hat{a}_k^c, \hat{a}_{k'}^{c'}\} = \delta^{c,c'} \delta_{k,k'} = \{\hat{d}_k^{c}, \hat{d}_{k'}^{c'}\}, \]
\[ \{\hat{a}_k^c, \hat{a}_{k'}^{c'}\} = \{\hat{a}_{k'}^{c'}, \hat{a}_k^c\} = 0, \]
\[ \{\hat{d}_k^{c}, \hat{d}_{k'}^{c'}\} = \{\hat{d}_{k'}^{c'}, \hat{d}_k^{c}\} = 0. \]  \hspace{1cm} (3.26)

Of course, we can easily show that the states \(|0\rangle_f\) satisfies the constraints (2.59), that is,

\[ \hat{Q}^3|0\rangle_f \equiv g \int_{-L}^{L} dx e^{-\hat{\Psi}^\dagger_R(x)} T^3 \hat{\Psi}_R(x) |0\rangle_f = 0. \]  \hspace{1cm} (3.27)

This is nothing but the physical state condition. In physical meaning, this is saying that physical states be charge neutral as a whole.

As the result, the full ground state of the system can be written by

\[ |\text{vacuum}\rangle \cong |0\rangle_f \otimes \Phi_0(q). \]  \hspace{1cm} (3.28)

\(\Phi_0(q)\) is the zero mode wave function in the \(q\) representation, satisfying the following Schrödinger equation for a free particle with a unit mass \((m = 1)\)

\[ \frac{1}{2} \left(-i \frac{d}{dq}\right)^2 \Phi_0(q) = \mathcal{E} \Phi_0(q), \]  \hspace{1cm} (3.29)

where \(\mathcal{E} \equiv E/(2L)\) is an energy density.

This result seems to suggest that in a sense, the ground state structure of the correctly normal-ordered light-cone QCD Hamiltonian is almost trivial in the adiabatic approximation. However, it is difficult to construct the full ground state beyond the adiabatic approximation. Rather, we are interested in how the effects of ZM change the spectrum of the excited states. But we can not answer this question in this paper.

Instead of it, we shall see the relation between our result and that of Kalloniatis et al.. In order to do so, we shall neglect the fermion mass term (3.21) and replace the currents \(\hat{\rho}(x)(i = 1, 2)\) in eq.(3.22) with classical external source terms \(\rho^i\) independent...
of \( x \) (note that \( \rho^3 \) automatically vanishes because of the charge neutrality condition (2.59)). They have assumed in their paper that only zero mode external sources excite ZM of the gauge fields. After some straightforward calculations of \( x^- \) and \( y^- \) integrations, we find that the light-cone Hamiltonian in the \( q \) representation would be of the form as

\[
H_{l.c.} = H_{ZM}^{l.c.} + 4L \left[ (\rho^1 L)^2 + (\rho^2 L)^2 \right] + \frac{1}{gq} \left[ \cot \left( \frac{gq}{2} \right) - 2 \cot \left( \frac{gq}{2} \right) \cos(gq) - 4 \sin(gq) \right] + \frac{4}{g^2 q^2} \cos^2 \left( \frac{gq}{2} \right).
\]

Moreover in a specific coupling region, i.e., weak coupling region \((gq \ll 1)\), the above equation reduces to

\[
H_{l.c.} = H_{ZM}^{l.c.} + 8L \left[ (\rho^1 L)^2 + (\rho^2 L)^2 \right] \frac{1}{g^2 q^2} = 2L \left[ \frac{1}{2} \left( -i \frac{d}{dq} \right)^2 + \frac{(2wL)^2}{2q^2} \right],
\]

where

\[
w^2 \equiv \frac{\rho^+ \rho^-}{g^2}, \quad \rho_{\pm} \equiv \sqrt{2}(\rho^1 \pm i\rho^2).
\]

Therefore Schrödinger equation for dynamical zero mode is now given by

\[
\frac{1}{2} \left[ -\frac{d^2}{dq^2} + \frac{(2wL)^2}{q^2} \right] \Phi_0(q) = \mathcal{E} \Phi_0(q).
\]

This is the same result Kalloniatis et al. have obtained in ref. [12]. They have also used a kind of weak coupling approximation to obtain their result.

4 Summary and Discussion

In this paper, we have studied QCD\(_{1+1}\) with fundamental fermions based on Discretized Light-Cone Quantization (DLCQ) formalism. We have discussed both classical and quantum aspects of the theory in detail and obtained the full ground state
wave function of the theory by the method of “separation of variables” mentioned in ref. [22] and we could see the light-cone QCD ground state has almost trivial structure in the range of the adiabatic approximation we have used here. Also we could find the relation between the original work by Kalloniatis et al. and ours. The physical effects of ZM for the essentially nonperturbative phenomena e.g. chiral symmetry breaking and confinement etc., however, remains unclear. More precise considerations for these would be the future work. Moreover, what we would like to really understand is QCD bound-state problem in 3+1 dimensions [6]. But as there are many difficulties to get there, especially renormalization problem [10], it seems far long way. Future works will be concentrated on these points.

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Appendix: Notations and Conventions

We describe here some notations and conventions in light-cone formalism. They are essentially the same as those by Harada et al [8]. The coordinates are set \( x^\pm = (x^0 \pm x^1)/\sqrt{2} \), where \( x^+ \) is taken as “time”. The light-cone metric is given by

\[
g_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g^{\mu\nu}, \quad \mu, \nu = +, - \tag{4.1}
\]

The derivatives are also defined as

\[
\partial_\pm \equiv \frac{\partial}{\partial x^\pm},
\]

with \( \partial_\pm = \partial^\mp \). \( \gamma \) matrices in a representation in which \( \gamma^5 \) is diagonal are as follows:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^5 &= \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^+ &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \\
\gamma^- &= \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \\
\gamma^+ \gamma^- &= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \\
\gamma^- \gamma^+ &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\tag{4.3}
\]

The SU(2) gauge fields are represented by

\[
A_\mu \equiv A^a_\mu T^a, \quad T^a = \frac{1}{2} \sigma^a, \quad a = 1, 2, 3 \tag{4.4}
\]

where \( \sigma^a \) is ordinary Pauli matrices such that

\[
\begin{align*}
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\tag{4.5}
\]
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