ABSTRACT: Zamolodchikov’s famous analysis of the RG trajectory connecting successive minimal CFT models $M_p$ and $M_{p-1}$ for $p \gg 1$, is improved by including second order in coupling constant corrections. This allows to compute IR quantities with next to leading order accuracy of the $1/p$ expansion. We compute in particular, the beta function and the anomalous dimensions for certain classes of fields. As a result we are able to identify with a greater accuracy the IR limit of these fields with certain linear combination of the IR theory $M_{p-1}$. We discuss the relation of these results with Gaiotto’s recent RG domain wall proposal.
Introduction

In his famous paper [1] A. Zamolodchikov has investigated Renormalization Group (RG) flow from the minimal model $M_p$ to the $M_{p-1}$ for large $p \gg 1$ caused by the relevant operator $\phi_{1,3}$. Two main circumstances made it possible to investigate this RG flow using a single coupling constant perturbation theory. First, the conformal dimension of the field $\phi_{1,3}$, $\Delta_{1,3} = 1 - \frac{2}{p+1} \equiv 1 - \epsilon$ (see Appendix A) is nearly marginal when $p \gg 1$ and second, the Operator Product Expansion (OPE) of this field with itself produces no relevant field besides the initial one and the unit operator. The method of A. Zamolodchikov not only allowed to identify the IR theory with $M_{p-1}$, but
also provided detailed description how several classes of local fields behave along the
RG trajectory. The analogous RG flow for the $N = 1$ super minimal models has been
investigated in [2].

The main purpose of this paper is a sharpening of Zamolodchikov’s analysis, by the
inclusion of second order perturbative corrections. It is interesting to note that in all
cases we have investigated, the rotation matrix (in the space of fields), that diagonalizes
the matrix of anomalous dimensions, does not receive $1/p$ or $1/p^2$ corrections. So an
interesting question arises, if any higher order corrections appear at all.

As intermediate results, in this paper we have found several four-point correlation
functions in large $p$ limit (see formulae (C.1)).

The initial motivation to carry out these computations came from the recent ap-
proach to this RG flow by D. Gaiotto [3]. Using Goddard-Kent-Olive construction,
Gaiotto has constructed a non-trivial conformal interface between two successive mini-
mal models and made a striking conjecture, that this interface is the exact RG domain
wall which encodes the map between the UV and IR fields. Gaiotto’s conjecture sur-
vives a strong test: it is fully compatible with the first order perturbative calculations
of the mixing amplitudes performed by Zamolodchikov.

In this paper we show that this mixing coefficients computed with the help of the
perturbation theory up to the second order, unlike those obtained from the Gaiotto’s
conjecture, do not receive any corrections up to the order $1/p^2$. Nevertheless, this
discrepancy might be attributed to the renormalization scheme which is adopted here
following Zamolodchikov. Presently the author of this paper does not have any clue
how to take into account possible dependencies on the renormalization schemes in order
to be able to make any conclusive statement about Gaiotto’s conjecture beyond the
leading order.

The paper is organized as follows.

In Section 1, we develope some technical tools, necessary to carry out second order
in coupling constant calculations.

In Section 2 the $\beta$-function and Zamolodchikov’s $c$-function [4] are computed with
next to leading order accuracy. The critical value of the renormalized coupling con-
stant, the slope of the $\beta$-function as well as the $c$-function at the critical point are
calculated. The results of these computations confirm that also the second order con-
tributions perfectly match with the Zamolodchikoved’s conclusion that the IR fixed
point corresponds to the CFT $M_{p-1}$ and that the UV field $\phi_{1,3}$ flows to the field $\phi_{3,1}$ of
the IR theory.

Section 3 is devoted to the renormalization of several series of local fields and to
the calculation of their anomalous dimensions. Thus:
in Section 3.1 we investigate the renormalization of the fields $\phi_{n,n}$.
In Section 3.2 the renormalization of the fields $\phi_{n,n+1}$ and $\phi_{n,n-1}$ is discussed and the matrix of anomalous dimensions is found. At the fixed point the matrix of anomalous dimensions is diagonalized and its eigenvalues are calculated.

In Section 3.3 the same steps are performed for the fields $\phi_{n,n+2}$, $\partial \bar{\partial} \phi_{n,n}$ and $\phi_{n,n+2}$.

In all cases the predictions of Zamolodchikov successfully withstand the next to leading order test.

In Appendix A some basic facts about the minimal models of 2d CFT are reviewed. The Appendices B and C are devoted to computation of the integrals used in the main text. The Appendix D comments how to calculate the large $p$ limit of those four point correlation functions used in the main text.

1 Perturbation theory in second order

Suppose the (Euclidean) action density is given by

$$H(x) = H_0(x) + \lambda \phi(x)$$

with $H_0$ being the UV CFT action density, $\phi$ a relevant local spinless field and $\lambda$ the coupling constant. Then for a two-point function up to second order we’ll have

$$\langle \phi_1(y_1)\phi_2(y_2) \rangle_\lambda = \langle \phi_1(y_1)\phi_2(y_2) \rangle_0 - \lambda \int \langle \phi_1(y_1)\phi_2(y_2)\phi(x) \rangle_0 d^2x$$

$$+ \frac{\lambda^2}{2} \int \langle \phi_1(y_1)\phi_2(y_2)\phi(x_1)\phi(x_2) \rangle_0 d^2x_1 d^2x_2 + O(\lambda^3)$$

In this paper we consider a theory, whose UV limit is given by the minimal CFT model $M_p$ with $p \gg 1$ and the perturbing field is $\phi \equiv \phi_{1,3}$. Leading order corrections in this theory has been investigated by A. Zamolodchikov [1]. Second order computations are more complicated. Indeed, not only the knowledge of four point correlation functions which in general are quite non-trivial in a CFT [5], but also their integrals over two insertion points is required. Fortunately, as we demonstrate below, the conformal invariance allows to perform integration over one of the insertion points explicitly. First let us notice that translational and scale invariance can be exploited to locate the points $y_1$, $y_2$ at $y_1 = 1$ and $y_2 = 0$ without loss of generality:

$$\int \langle \phi_1(y_1)\phi_2(y_2)\phi(x_1)\phi(x_2) \rangle_0 d^2x_1 d^2x_2$$

$$= (y_{12}\bar{y}_{12})^{2-2\Delta-\Delta_1-\Delta_2} \int \langle \phi_1(1)\phi_2(0)\phi(x_1)\phi(x_2) \rangle_0 d^2x_1 d^2x_2$$

$$+ O(\lambda^3)$$

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(here and below I frequently use the shorthand notation \( x_{12} = x_1 - x_2 \), \( y_{12} = y_1 - y_2 \) et al.). Any four-point function of primary fields in a CFT essentially depends only on the cross ratio \( x = \frac{x_{12} x_{34}}{x_{14} x_{23}} \) of the insertion points and its conjugate \[5\]

\[
\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = (x_{14} x_{23} x_{34})^{-2 \Delta_1} (x_{24} x_{23}) \Delta_1 + \Delta_3 - \Delta_2 - \Delta_4 \\
\times (x_{34} x_{34}) \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 (x_{23} \bar{x}_{23}) \Delta_4 - \Delta_1 - \Delta_2 - \Delta_3 G(x, \bar{x}),
\]

where it is assumed that the fields are spin-less (i.e. \( \Delta_i = \bar{\Delta}_i \)). Specifying the insertion points as \( x_1 = x, x_2 = 0, x_3 = 1 \) and \( x_4 = \infty \) we get

\[
G(x, \bar{x}) = \lim_{x_4 \to \infty} (x_4 \bar{x}_4)^{2 \Delta_4} \langle \phi_1(x) \phi_2(0) \phi_3(1) \phi_4(x_4) \rangle \equiv \langle \phi_1(x) \phi_2(0) \phi_3(1) \phi_4(\infty) \rangle
\]

(1.5)

Alternatively specifying \( x_1 = 1/x, x_2 = \infty, x_3 = 1 \) and \( x_4 = 0 \) and comparing with (1.5) we get the identity

\[
\langle \phi_1(x) \phi_2(0) \phi_3(1) \phi_4(\infty) \rangle = (x \bar{x})^{-2 \Delta_1} \langle \phi_1(1/x) \phi_2(0) \phi_3(1) \phi_2(\infty) \rangle
\]

(1.6)

which is useful when investigating the correlation functions at large \( x \). After application of (1.4), (1.5) to the four-point function \( \langle \phi(x_1) \phi_2(0) \phi_1(1) \phi(x_2) \rangle_0 \) and introduction of the new integration variables

\[
\frac{x_1(1 - x_2)}{x_{12}} \to x_1; \quad 1 - x_2 \to x_2,
\]

two integrations on the r.h.s. of eq. (1.3) become partly disentangled

\[
\int \langle \phi_1(1) \phi_2(0) \phi(x_1) \phi(x_2) \rangle_0 d^2 x_1 d^2 x_2 = \int I(x_1) \langle \phi_1(x_1) \phi_2(0) \phi_1(1) \phi(\infty) \rangle_0 d^2 x_1
\]

(1.7)

where

\[
I(x) = \int (y \bar{y})^{a-1} ((1 - y)(1 - \bar{y}))^{b-1} ((x - y)(\bar{x} - \bar{y}))^{c} d^2 y,
\]

(1.8)

with parameters

\[
a = \epsilon_1 + 2 \epsilon; \quad b = \epsilon_2 + 2 \epsilon; \quad c = -2 \epsilon
\]

(1.9)
where \( \epsilon = 1 - \Delta \), \( \epsilon_{1,2} = 1 - \Delta_{1,2} \) are the complementary dimensions. Fortunately the integral (1.8) can be expressed in terms of hyper-geometric functions (see appendix B):

\[
I(x) = \frac{\pi \gamma(b) \gamma(a + c)}{\gamma(a + b + c)} |F(1 - a - b - c, -c, 1 - a - c, x)|^2 \\
+ \frac{\pi \gamma(1 + c) \gamma(a)}{\gamma(1 + a + c)} |x^{a+c} F(a, 1 - b, 1 + a + c, x)|^2 \\
= \frac{\pi \gamma(a) \gamma(b + c)}{\gamma(a + b + c)} |F(1 - a - b - c, -c, 1 - b - c, 1 - x)|^2 \\
+ \frac{\pi \gamma(1 + c) \gamma(b)}{\gamma(1 + b + c)} |(1 - x)^{b+c} F(b, 1 - a, 1 + b + c, 1 - x)|^2 \\
= \frac{\pi \gamma(a) \gamma(b)}{\gamma(a + b)} |x^{c} F(a, -c, a + b, 1/x)|^2 \\
+ \frac{\pi \gamma(1 + c) \gamma(a + b - 1)}{\gamma(a + b + c)} |x^{a+b+c-1} F(1 - a - b - c, 1 - b, 2 - a - b, 1/x)|^2
\]  

(1.10)

where \( \gamma(x) = \Gamma(x)/\Gamma(1 - x) \) and \( F(a, b, c, x) \) is the Gaussian hypergeometric function. Above three expressions for \( I(x) \) are convenient when exploring the regions \( x \sim 0 \), \( x \sim 1 \) and \( x \sim \infty \) respectively. Note also that these expressions make explicit the single-valuedness of \( I(x) \). Specifying the choice of parameters to (1.9) and applying the identity

\[ F(a, b, c, x) = (1 - z)^{-a-b} F(c - a, c - b, c, x) \]

to the second term of the second equality, the eqs. (1.10) can be rewritten as

\[
I(x) = \frac{\pi \gamma (2\epsilon + \epsilon_{21}) \gamma (\epsilon_{12})}{\gamma (2\epsilon)} |F(1 - 2\epsilon, 2\epsilon, 1 + \epsilon_{21}, x)|^2 \\
+ \frac{\pi \gamma (2\epsilon + \epsilon_{12}) \gamma (\epsilon_{21})}{\gamma (2\epsilon)} |(x/(1 - x))^{\epsilon_{12}} F(2\epsilon, 1 - 2\epsilon, 1 + \epsilon_{12}, x)|^2 \\
= \frac{\pi \gamma (2\epsilon + \epsilon_{12}) \gamma (\epsilon_{21})}{\gamma (2\epsilon)} |F(1 - 2\epsilon, 2\epsilon, 1 + \epsilon_{12}, 1 - x)|^2 \\
+ \frac{\pi \gamma (2\epsilon + \epsilon_{21}) \gamma (\epsilon_{12})}{\gamma (2\epsilon)} |(x/(1 - x))^{\epsilon_{12}} F(2\epsilon, 1 - 2\epsilon, 1 + \epsilon_{21}, 1 - x)|^2 \\
= \frac{\pi \gamma (2\epsilon + \epsilon_{12}) \gamma (2\epsilon + \epsilon_{21})}{\gamma (4\epsilon)} |x^{-2\epsilon} F(2\epsilon + \epsilon_{12}, 2\epsilon, 4\epsilon, 1/x)|^2 \\
+ \frac{\pi \gamma (4\epsilon - 1)}{\gamma^2 (2\epsilon)} |x^{2\epsilon - 1} F(1 - 2\epsilon, 1 - 2\epsilon + \epsilon_{12}, 2 - 4\epsilon, 1/x)|^2
\]  

(1.11)

It is worth noting that in the case when \( \epsilon_{12} \equiv \epsilon_1 - \epsilon_2 = 0 \) only the third expression is manifestly nonsingular, the first two expressions require a subtle limiting procedure.
Thus for this case it is better to employ the third expression:

\[
I(x) = \frac{\pi \gamma^2(2\epsilon)}{\gamma(4\epsilon)} |x^{-2\epsilon}F(2\epsilon, 2\epsilon, 4\epsilon, 1/x)|^2 \\
+ \frac{\pi \gamma(4\epsilon - 1)}{\gamma^2(2\epsilon)} |x^{2\epsilon - 1}F(1 - 2\epsilon, 1 - 2\epsilon, 2 - 4\epsilon, 1/x)|^2
\]  

Let us investigate the behaviour of (1.12) at \(x \sim 1\). Using standard formulae for the analytic continuation of the hypergeometric function with parameters satisfying the condition \(a + b - c \in \mathbb{Z}\) (see e.g. [7]) one can get convinced that

\[
I(x) \approx \pi(x + \bar{x} - 4) + \pi(1 + 2\epsilon(2\epsilon - 1)(x + \bar{x} - 2)) \\
\times (2 - \log |x - 1|^2 - 2\pi \cot(2\pi\epsilon) - 4\psi(2\epsilon) - 4\gamma)
\]  

where \(\gamma = 0.577216 \cdots\) is the Euler constant and the omitted terms are at most of order \(|x - 1|^2 \log |x - 1|\) in \(x \to 1\) limit. There is no need to investigate the limit \(x \to 0\) separately since the obvious symmetry of \(I(x)\) with respect to \(x \leftrightarrow 1 - x\) at \(x \sim 0\) immediately ensures

\[
I(x) \approx -\pi(2 + x + \bar{x}) + \pi(1 + 2\epsilon(1 - 2\epsilon)(x + \bar{x})) \\
\times (2 - \log |x|^2 - 2\pi \cot(2\pi\epsilon) - 4\psi(2\epsilon) - 4\gamma)
\]

2 \(\beta\)-function

In this section we calculate the \(\beta\)-function up to \(1/p^4 \sim \epsilon^4\) corrections for the small values of the (renormalized) coupling constant (of order \(\epsilon\) or smaller). As it will become quite clear later for this purpose one should evaluate the integral (1.7) in the special case \(\phi_1 = \phi_2 = \phi\) and \(I(y)\) given by (1.12) with the accuracy \(\sim 1/\epsilon\). Our strategy will be as follows: separate in the integration region the discs \(D_{l,0} = \{x \in \mathbb{C} | |x| < l\}\), \(D_{l,1} = \{x \in \mathbb{C} | |x - 1| < l\}\) and \(D_{l,\infty} = \{x \in \mathbb{C} | |x| > 1/l\}\) where \(l\) is an intermediate length scale such that \(0 < l_0 \ll \exp(-1/\epsilon) \ll l \ll 1\) and \(l_0\) is the ultraviolet scale. For the integral outside these discs we will safely use the small \(\epsilon\) limits of the correlation functions given in the appendix while inside the discs we’ll explore (exact in \(\epsilon\)) OPE.

We will see that the trace of the intermediate scale \(l\) will be washed out entirely from the final result. In present case the \(\epsilon = 0\) limit of the four-point function is given by (see appendix C)

\[
\langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle = \left| \frac{1 - 2x + 3x^2 - 2x^3 + \frac{x^4}{1}}{x^2(1-x)^2} \right|^2 + \frac{16}{3} \left| \frac{1 - \frac{3x}{2} + x^2 - \frac{x^3}{1}}{x(1-x)^2} \right|^2 + \frac{5}{9} \left| \frac{x}{1-x} \right|^4
\]

(2.1)
With required accuracy $I(x) \approx \pi/\epsilon$. It is convenient to carry out the integration in radial coordinates $x = r \exp(i\varphi)$, $\bar{x} = r \exp(-i\varphi)$, $d^2x = rdrd\varphi$. The result of integration over the angular variable $\varphi$ will depend on the region where the radial coordinates takes its value

$$\int R(x, \bar{x})d\varphi = \begin{cases} 
Res_{x=0} R(x, r^2/x)/x + Res_{x=r^2} R(x, r^2/x)/x, & \text{if } r < 1 \\
Res_{x=0} R(x, r^2/x)/x + Res_{x=1} R(x, r^2/x)/x, & \text{if } r > 1 
\end{cases}$$

for arbitrary rational function $R(x, \bar{x})$ with poles located at $x = 0$ or $x = 1$. In particular when $R(x, \bar{x})$ is the r.h.s. of the eq. (2.1) we get

$$\frac{3r^{10} - 9r^8 + 25r^6 - 23r^4 + 7r^2 + 3}{3r^4(1 - r^2)^3}, \quad \text{if } r < 1$$

$$\frac{3r^{10} + 7r^8 - 23r^6 + 25r^4 - 9r^2 + 3}{3r^4(r^2 - 1)^3}, \quad \text{if } r > 1$$

After performing the remaining elementary integration over $r$ we finally get

$$\int_{\Omega_{l,l_0}} \langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle d^2x = \frac{2\pi}{l^2} + \frac{\pi}{2l_0^2} - \frac{33\pi}{8} - \frac{32\pi}{3} \log(2l_0l^2) + \cdots$$

where (see Fig. 1)

$$\Omega_{l,l_0} = (D_{1-l_0,0} \setminus D_{l,0}) \cup (D_{1/l,0} \setminus D_{1+l_0,0})$$

and the dots stand for negligible terms of order $l$ or $l_0/l$. There is a subtlety to be treated carefully here. The fact that the part of the white narrow ring (of width $2l_0$) outside of the red circle is missing from the integration region $\Omega_{l,l_0}$ is insignificant since its inclusion would produce only negligible terms of order $l_0$. Instead we have to subtract the contribution of two lens-like regions of $\Omega_{l,l_0}$ included in the red circle (as already said, the contribution coming from the regions around the singular points will be computed separately exploring OPE).

a) Integration over lens-like regions

Expanding (2.1) around $x \sim 1$ we get

$$\frac{1}{|x - 1|^4} + \frac{2}{(x - 1)^2} + \frac{2}{(\bar{x} - 1)^2} + \frac{16}{3|x - 1|^2} + \cdots$$

where only the singular terms, whose integrals over the region around $x = 1$ diverge, are presented. The integrals of such terms have been evaluated in Appendix D. As a result the contribution of the lens-like regions, to be subtracted from the r.h.s. of eq. (2.4), is equal to

$$\frac{\pi}{\epsilon} \left( -\frac{\pi}{l^2} - \frac{\pi}{8} \right) + \frac{\pi}{2l_0^2} + 2 \times 2\pi + \frac{16}{3} \left( 2\pi \log \left( \frac{l}{2l_0} \right) \right)$$

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b) Contributions of the regions around singularities

It remains to calculate the contributions of the regions $D_{l,0}$, $D_{l,1}$, $D_{l,\infty}$ to the integral (1.7). Evidently the first two regions give identical contributions, so let’s concentrate on the region $D_{l,0}$ for definiteness. To calculate the four-point function in this region we apply the OPE (all the structure constants we use in this paper can be extracted from the general formula (A.5))

$$\phi(x)\phi(0) = (x\bar{x})^{-2\Delta} (1 + \cdots) + C^{(1,3)}_{(1,3)(1,3)} (x\bar{x})^{-\Delta} (\phi(0) + \cdots)$$

(2.7)

Taking into account (1.14) and that

$$C^{(1,3)}_{(1,3)(1,3)} C^{(1,3)}_{(1,3)(1,3)} = \frac{16}{3} (1 - 3\epsilon + O(\epsilon^2))$$

(2.8)

one easily gets

$$\int_{D_{l,0}\setminus D_{l,0,0}} I(x) \langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle d^2x$$

$$\approx \frac{\pi^2}{d^2 l_0^{2-4\epsilon}} - \frac{\pi^2}{d^2} + \frac{32\pi^2 (\log(l) - 3)}{3\epsilon} + \frac{32\pi^2}{3\epsilon^2}$$

(2.9)

where the first two terms come from the identity and the last two terms from the $\phi$ field channel of OPE (2.7) respectively.

c) Contribution of $x \sim \infty$
At large $x$ we first make use of eq. (1.6) to pass to the inverse variable $1/x \sim 0$ and then we apply OPE. The calculation is similar to the previous case, the main difference being the fact that at this limit $I(x)$ becomes simply $\frac{\pi}{\epsilon}(x\bar{x})^{-2\epsilon}$. The result is

$$\int_{D_{l,\infty}\setminus D_{l_0,\infty}} I(x)\langle \phi(x)\phi(0)\phi(1)\phi(\infty)\rangle d^2x$$

$$\approx \frac{\pi^2}{\epsilon l_0^{2-4\epsilon}} - \frac{\pi^2}{\epsilon l^2} + \frac{16\pi^2}{3} \left( \frac{1}{\epsilon^2} - \frac{3 - 2 \log \epsilon}{\epsilon} \right)$$

(2.10)

Let’s pick up all the ingredients: (2.4) times $I(x) \approx \frac{2\pi}{\epsilon}$, minus (2.6), plus twice (2.9), and plus (2.10). We get

$$\frac{3\pi^2}{l_0^{2-4\epsilon}} - \frac{88\pi^2}{3\epsilon^2} + \frac{80\pi^2}{3\epsilon^2} + O(\epsilon^0)$$

(2.11)

As expected, the $l$ dependence disappeared. The presence of the divergent term $\frac{3\pi^2}{l_0^{2-4\epsilon}}$ also is not surprising. In our naive regularization scheme we could cancel this infinity by adding an appropriate, proportional to the area counter-term in action. In fact, if we would have been able to treat the integral (1.7) analytically as continuation from a region of parameters where integral converges, then this kind of non-analytic divergent term couldn’t emerge at all. In what follows we’ll simply drop out such terms without further ado.

Thus for the two point function $G(x, \lambda) = \langle \phi(x)\phi(0)\rangle_\lambda$ we get (c.f. (1.2))

$$G(x, \lambda) = (x\bar{x})^{-2+2\epsilon} \left( 1 - \lambda \frac{4\pi}{\sqrt{3}} \left( \frac{2}{\epsilon} - 3 + O(\epsilon) \right) (x\bar{x})^\epsilon \right. + \left. \frac{\lambda^2}{2} \left( \frac{80\pi^2}{3\epsilon^2} - \frac{88\pi^2}{\epsilon} + O(\epsilon^0) \right) (x\bar{x})^{2\epsilon} + \cdots \right)$$

(2.12)

Following A. Zamolodchikov let us introduce a new coordinate $g$ ("renormalized" coupling constant) in the space of one-parameter family of theories (1.1) instead of the initial coupling $\lambda$ and introduce the local field $\phi^{(g)} = \partial_g \mathcal{H}$ (according to (1.1) the initial "bare" perturbing field $\phi = \partial_\lambda \mathcal{H}$). The new coupling $g$ is fixed by the requirement that the two point function $G(x, g) = \langle \phi^{(g)}(x)\phi^{(g)}(0)\rangle_\lambda$ satisfies the normalization condition

$$G(1, g) = 1$$

(2.13)

Then the $\beta$-function can be computed from the identity (see [1])

$$\Theta(x) = \epsilon \lambda \phi(x) = \beta(g)\phi^{(g)}(x)$$

(2.14)
where $\Theta$ is the trace of the energy-momentum tensor. Combining (2.13) and (2.14) one easily finds

$$\partial_\lambda g = \sqrt{G(1, \lambda)}$$  \hspace{1cm} (2.15)

and

$$\beta(g) = \epsilon \lambda \sqrt{G(1, \lambda)}$$  \hspace{1cm} (2.16)

The equation (2.15) allows one to express $g$ in terms of $\lambda$ (the integration constant can be set to zero so that the unperturbed CFT will corresponds to $g = 0$)

$$g = \lambda - \frac{\pi \lambda^2}{\sqrt{3}} \left( \frac{2}{\epsilon} - 3 + O(\epsilon) \right) + \frac{2\pi^2 \lambda^3}{3} \left( \frac{2}{\epsilon^2} - \frac{7}{\epsilon} + O(1) \right) + O(\lambda^4)$$  \hspace{1cm} (2.17)

or, inversely

$$\lambda = g + \frac{\pi g^2}{\sqrt{3}} \left( \frac{2}{\epsilon} - 3 + O(\epsilon) \right) + \frac{2\pi^2 g^3}{3} \left( \frac{2}{\epsilon^2} - \frac{5}{\epsilon} + O(1) \right) + O(g^4)$$  \hspace{1cm} (2.18)

Inverting and replacing in (2.16) $\lambda$ in favour of $g$ we get

$$\beta(g) = \epsilon g - \frac{\pi g^2}{\sqrt{3}} \left( 2 - 3\epsilon + O(\epsilon^2) \right) - \frac{4\pi^2 g^3}{3} (1 + O(\epsilon)) + \cdots$$  \hspace{1cm} (2.19)

The equation

$$\beta(g^*) = 0$$  \hspace{1cm} (2.20)

admits a nonzero solution

$$2\pi g^* = \sqrt{3} \epsilon + \frac{\sqrt{3}}{2} \epsilon^2 + O(\epsilon^3)$$  \hspace{1cm} (2.21)

so we have a non-trivial infrared fixed point. In [1] this fixed point has been identified with the minimal model $M_{p-1}$ and the local field $\phi^{(p-1)}$ with the field $\phi_{3,1}^{(p-1)}$. Now we are in a position to check this identification more accurately. The anomalous dimension of $\phi^{(p-1)}$ is related to the slope of $\beta$-function

$$\Delta^* = 1 - \partial_g \beta(g) \big|_{g=g^*} = 1 + \epsilon + \epsilon^2 + O(\epsilon^3)$$  \hspace{1cm} (2.22)

which matches to the conformal dimension of $\phi_{3,1}^{(p-1)}$ computed from the Kac formula (A.2). Also the shift of the central charge [4]

$$c^* - c_p = -12\pi^2 \int_0^{g^*} \beta(g) dg = -\frac{3\epsilon^3}{2} - \frac{9\epsilon^4}{4} + O(\epsilon^5)$$  \hspace{1cm} (2.23)

neatly matches to the exact expression

$$c_{p-1} - c_p = -\frac{12}{p(p^2 - 1)} = -\frac{3\epsilon^3}{(2 - \epsilon)(1 - \epsilon)}.$$
Field renormalization and the UV - IR map

In this section we calculate the matrices of anomalous dimensions for several classes of fields. Diagonalization of these matrices at the IR fixed point provides a detailed map between the UV local fields and their image under RG flow in the IR theory.

3.1 Primary fields $\phi_{n,n}$

This is the simplest case to analyze since the fields $\phi_{n,n}$ never get mixed with other fields [1]. This follows from the structure of the OPE involving the perturbing field $\phi_{1,3}$. The subspace of fields which is generated by the field $\phi_{n,n}$ and is closed w.r.t. OPE with $\phi_{1,3}$, doesn’t contain any other field with a dimension close to $\Delta_{n,n} = O(\epsilon^2)$. We are going to calculate corrections to the anomalous dimension up to the order $\epsilon^4$. That is why for the present purpose the knowledge of the four point function $\langle \phi(x)\phi_{n,n}(0)\phi_{n,n}(1)\phi(\infty) \rangle$ up to $\epsilon^2$ correction is required. As in previous case, to find this correlation function we first used AGT relation to find the relevant conformal blocks up to sufficiently large level (actually the computations were performed up to the order $x^6$ terms). Expanding a conformal block up to $\epsilon^2$ and examining first few coefficients of the resulting power series in $x$ it is possible to guess the entire power series and identify it with some elementary function. Having in our disposal the expression for the correlation function we then checked that it satisfies all the nontrivial physical requirements: the single-valuedness and the compatibility with OPE around the points $x \sim 1$ and $x \sim \infty$. Here is the final expression (see Appendix C)

$$\langle \phi(x)\phi_{n,n}(0)\phi_{n,n}(1)\phi(\infty) \rangle = 1 + \frac{\epsilon^2(n^2 - 1)}{12} \left( \frac{1}{2x(x - 1)} + \frac{1}{2\bar{x}(\bar{x} - 1)} + 4\log^2 \left| \frac{1 - x}{x} \right| \right) + O(\epsilon^3)$$

From eq. 1.12, up to order $\epsilon$, $I(x)$ is equal to

$$I(x) = \frac{\pi}{\epsilon} - 2\pi \log |(1 - x)x| + 8\pi \epsilon \log |x| \log |1 - x| + O(\epsilon^2)$$

Now we are ready to perform integration over the region $\Omega_{l_0,l}$ (see Fig. 1). Since the singularities at $x \sim 0$ and $x \sim 1$ are integrable, we can put $l_0 = 0$. As in Section 2 the integration over the angular variable should be performed separately for the cases $0 < |x| < 1$ and $|x| > 1$. Integration of rational expressions we have already discussed earlier. As about the logarithmic terms, they can be easily handled first expanding into power series in $x$ if $|x| < 1$ or in $1/x$ if $|x| > 1$. Then we proceed with the radial
integration. Both steps are elementary and we present only the final result:

\[
\int_{\Omega_{l,0}} I(x) \langle \phi(x) \phi_{n,n}(0) \phi_{n,n}(1) \phi(\infty) \rangle d^2x \approx \frac{\epsilon \pi^2 (n^2 - 1)}{3} \left( \log \frac{1}{l} + 1 \right) (3.3)
\]

\[
+ \frac{\pi^2}{l^2} \left( 2 + 4\epsilon + (4 + 8\epsilon) \log l + 8\epsilon \log^2 l \right) - \pi^2 (1 + 4\epsilon) - \frac{\epsilon \pi^2 (n^2 - 1)}{12} + \frac{\pi^2}{\epsilon l^2}
\]

Due to the already mentioned mildness of singularities at 0 and 1 the only remaining contribution to be taken into account comes from the neighbourhood of \( \infty \), i.e. from \( D_{l,\infty} \setminus D_{l,0} \).

At large \( x \) it is convenient to employ eq. (1.6)

\[
\langle \phi(x) \phi_{n,n}(0) \phi_{n,n}(1) \phi(\infty) \rangle = (\bar{x}x)^{-2\Delta} \langle \phi(1/x) \phi(0) \phi_{n,n}(1) \phi_{n,n}(\infty) \rangle (3.4)
\]

and apply the OPE (2.7) with \( x \) replaced by \( 1/x \). The correlation function decomposes into a sum of two partial amplitudes one corresponding to the identity and the other to the field \( \phi \).

a) \textit{Contribution of identity}

The prefactor \( (\bar{x}x)^{-2\Delta} \) in (3.4) compensates the factor \( (\bar{x}x)^{2\Delta} \) accompanying the identity operator in OPE and with sufficient accuracy we can replace this partial amplitude by 1. It is straightforward to expand \( I(x) \) given by (1.12) at large \( x \) keeping only those terms which after integration may produce non-vanishing terms in small \( l \) limit

\[
I(x) \approx \frac{\pi \gamma^2 (2\epsilon)}{\gamma (4\epsilon)} (\bar{x}x)^{-2\epsilon} \left( 1 + \frac{\epsilon}{x} \right) \left( 1 + \frac{\epsilon}{\bar{x}} \right) + \frac{\pi \gamma (4\epsilon - 1)}{\gamma (2\epsilon)^2} (\bar{x}x)^{2\epsilon - 1} (3.5)
\]

Integrating this expression over the region \( D_{l,\infty} \setminus D_{l,0} \), dropping out, as earlier, all singular in \( l_0 \) terms and expanding the result up to the linear in \( \epsilon \) terms we get

\[
- \frac{\pi^2}{\epsilon l^2} + \pi^2 - \frac{2\pi^2 (2\log(l) + 1)}{l^2} + \frac{4\pi^2 \epsilon (l^2 - 2\log^2(l) - 2\log(l) - 1)}{l^2} (3.6)
\]

b) \textit{Contribution of the field \( \phi_{1,3} \)}

This contribution is

\[
\int_{D_{l,\infty} \setminus D_{l,0,\infty}} \frac{\pi}{\epsilon} C^2_{(1,3)(n,n)(n,n)} (\bar{x}x)^{-2\epsilon - 2 + 2\epsilon + 1 - \epsilon} d^2x (3.7)
\]

where \( \frac{\pi}{\epsilon} (\bar{x}x)^{-2\epsilon} \) is just the function \( I(x) \) with required accuracy, \( (\bar{x}x)^{-2 + 2\epsilon} \) is the prefactor of (3.4), \( (\bar{x}x)^{1 - \epsilon} \) comes from OPE and the squared structure constant is equal to

\[
C^2_{(1,3)(n,n)(n,n)} = \frac{\epsilon^2 (1 + \epsilon) (n^2 - 1)}{6} + O(\epsilon^4) (3.8)
\]
The integral is converging at the limit \( l_0 \to 0 \), so we may perform integration over the entire region \( D_{l,\infty} \). The result reads

\[
\frac{\pi^2(n^2-1)(1-\epsilon + 2\epsilon \log(l))}{6} \tag{3.9}
\]

The sum of all contributions (3.3), (3.6), and (3.9) is

\[
\frac{\pi^2(n^2-1)(2+5\epsilon)}{12} + O(\epsilon^2) \tag{3.10}
\]

Combining this with the first order in coupling constant contribution

\[
\int \langle \phi_{n,n}(1)\phi_{n,n}(0)\phi(x) \rangle d^2x = \frac{\pi(n^2-1)(2+5\epsilon)}{8\sqrt{3}} + O(\epsilon^4) \tag{3.11}
\]

where the value

\[
C_{(1,3),(n,n),(n,n)} = \frac{(n^2-1)(2+5\epsilon)\epsilon^2}{16\sqrt{3}} + O(\epsilon^4) \tag{3.12}
\]

for the structure constant is inserted, we get

\[
G_n(x, \lambda) \equiv \langle \phi_{n,n}(x)\phi_{n,n}(0) \rangle_{\lambda} = (x\bar{x})^{-2\Delta_{n,n}} \left( 1 - \lambda \frac{\pi(n^2-1)(2+5\epsilon + O(\epsilon^2))\epsilon}{8\sqrt{3}} (x\bar{x})^\epsilon 
+ \frac{\lambda^2 \pi^2(n^2-1)(2+5\epsilon + O(\epsilon^2))}{12} (x\bar{x})^{2\epsilon} + \ldots \right) \tag{3.13}
\]

Let’s introduce the renormalized field \( \phi^{(g)}_{n,n} = B(\lambda)\phi_{n,n} \) by requiring that the two point function \( G_n(x, g) = \langle \phi^{(g)}_{n,n}(x)\phi^{(g)}_{n,n}(0) \rangle_{\lambda} \) satisfies the normalization condition

\[
G_n(1, g) = 1 \tag{3.14}
\]

so that

\[
B(\lambda) = \frac{1}{\sqrt{G_n(1, \lambda)}} \tag{3.15}
\]

Then for the anomalous dimension we get (cf. eq. (3.48), derived for a more general situation)

\[
\Delta^{(g)}_{n,n} = \Delta_{n,n} + \epsilon \lambda \partial_{\lambda} \log B = \Delta_{n,n} - \frac{\epsilon \lambda}{2} \partial_{\lambda} G_n(1, \lambda) \tag{3.16}
\]
In view of (2.18) we find
\[ \Delta_{n,n}^{(g)} = \Delta_{n,n} + \frac{\pi g (n^2 - 1) \epsilon^2 (2 + 5\epsilon + O(\epsilon^2))}{16\sqrt{3}} - \frac{\pi^2 g^2 (n^2 - 1) \epsilon^2 (1 + O(\epsilon))}{8} + O(g^3) \] (3.17)
So that at the fixed point
\[ \Delta_{n,n}^{(g^*)} = (n^2 - 1) \left( 4\epsilon^2 + 6\epsilon^3 + 7\epsilon^4 + O(\epsilon^5) \right) \] (3.18)
which completely agrees with the dimension \( \Delta_{n,n}^{(p-1)} \) of the field \( \phi_{n,n}^{(p-1)} \) in the minimal model \( M_{p-1} \). Thus the conclusion of A. Zamolodchikov that under RG the UV field \( \phi_{n,n}^{(p)} \) flows to IR \( \phi_{n,n}^{(p-1)} \) is robust also against our second order test.

### 3.2 Renormalization of the fields \( \phi_{n,n+1} \) and \( \phi_{n,n-1} \)

Already in this case one encounters with the phenomenon of mixing. The OPE \( \phi_1 \phi_{n,n+1} \) produces besides \( \phi_{n,n+1} \) also the primary field \( \phi_{n,n-1} \), both having dimensions close to \( 1/4 \) in large \( p \) limit. Thus we have to consider the correlation functions \( \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty) \rangle \) with all four possible choices of signs. The strategy is exactly the same as in previous sections and for each choice we will follow the steps performed in Section 3.1.

#### 3.2.1 Correlation function \( \langle \phi_{n,n+1}(1)\phi_{n,n+1}(0) \rangle_x \)

a) **Contribution of the region \( \Omega_{l,l_0} \)**

This is given by the integral
\[ \int_{\Omega_{l,l_0}} I(x) \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty) \rangle d^2x \] (3.19)

The large \( p \) limit of the four-point function found from AGT relation is (See Appendix C):
\[ \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty) \rangle = \frac{2(n+2)}{3n} \left| \frac{x - \frac{1}{2}}{x(x-1)} \right|^2 + \left| \frac{x^2 - x + \frac{1}{2}}{x(x-1)} \right|^2 + O(\epsilon) \] (3.20)

With required accuracy \( I(x) \) can be replaced by \( \frac{\pi}{\epsilon} \). The integral (3.19) can be performed using the technique already explored in computing (2.4) or (3.3). The result is
\[ \int_{\Omega_{l,l_0}} I(x) \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty) \rangle d^2x \approx \frac{\pi}{l^2} - \pi - \frac{\pi(13n+20) \log(l)}{6n} - \frac{\pi(5n+4) \log(2l_0)}{6n} \] (3.21)
b) Contribution of lens-like regions
Near $x \sim 1$ the r.h.c. of eq. (3.20) becomes \( \frac{5n+4}{12n^2|x-1|^2} \), hence the contribution of the lens-like regions near $x \sim 1$ to be subtracted from the r.h.s. of eq. (3.21) is equal to (see (D.9))

\[
\frac{\pi^2(5n+4)}{6n\epsilon} \log \left( \frac{l}{2l_0} \right)
\]  
(3.22)

c) Contributions of the regions $D_{l,0}, D_{l,1}$ and $D_{l,\infty}$
Contributions of $D_{l,0}$ and $D_{l,1}$ obviously are identical so we will concentrate on $D_{l,0}$ only. The relevant OPE is

\[
\phi(x)\phi_{n,n+1}(0) = (x\bar{x})^{-\Delta} C^{(n,n+1)}_{(1,3)(n,n+1)} (\phi_{n,n+1} + \cdots)
\]
\[+ C^{(n,n-1)}_{(1,3)(n,n+1)} (x\bar{x})^{-\Delta-n-\Delta+\Delta} (\phi_{n-1,0} + \cdots)
\]  
(3.23)

Taking into account that in this region $I(x) \approx \frac{\pi}{\epsilon} - \pi \log (|x|^2)$ (see (1.14)) and that

\[
C^2_{(1,3)(n,n+1)(n,n+1)} = \frac{(n+2)^2(1-(2n-1)\epsilon)}{12n^2} + O(\epsilon^2)
\]
\[
C^2_{(1,3)(n,n+1)(n,n-1)} = \frac{(n^2-1)(1+\epsilon)}{3n^2} + O(\epsilon^2)
\]  
(3.24)

we get

\[
\int_{D_{l,0}\setminus D_{0,0}} I(x)\langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle d^2x \approx \frac{\pi^2(n+2)^2}{6n^2} \left( \frac{1}{\epsilon^2} + \frac{1-2n+2\log l}{\epsilon} \right)
\]
\[+ \frac{2\pi^2(n^2-1)}{3n^2(n+2)^2} \left( \frac{n+4}{\epsilon^2} + \frac{n+4+(n+2)^2\log l}{\epsilon} \right)
\]  
(3.25)

Above two terms come from two primaries $\phi_{n\pm1}$ appearing on the r.h.s. of the OPE (3.23).

The contribution from the region $D_{l,\infty}$ is completely analogous to the case of the correlation function $\langle \phi\phi \rangle_{\lambda}$ discussed in Section 2. The only difference is that now the contribution of the field $\phi$ which appears in u-channel OPE is proportional to

\[
C^2_{(1,3)(1,3)} C^1_{(1,3)(n+1)(n+1)} = \frac{2(n+2)(1-(n+1)\epsilon)}{3n} + O(\epsilon^2)
\]  
(3.26)

instead of $C^2_{(1,3)(1,3)} \approx \frac{16(1-3\epsilon)}{3}$. The result is (c.f. eq. (2.10))

\[
\int_{D_{l,\infty}\setminus D_{0,\infty}} I(x)\langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty) \rangle d^2x
\]
\[\approx -\frac{\pi^2}{\epsilon l^2} + \frac{2\pi^2(n+2)}{3n} \left( \frac{1}{\epsilon^2} - \frac{n+1-2\log l}{\epsilon} \right)
\]  
(3.27)
Picking up all the contributions: (3.21) minus (3.22) plus twice (3.25) and plus (3.27), we get
\[
\frac{\pi^2 (3n^3 + 24n^2 + 64n + 44)}{3n(n + 2)^2\epsilon^2} - \frac{4\pi^2 (n + 1) (n^3 + 7n^2 + 14n + 5)}{3n(n + 2)^2\epsilon} + O(\epsilon^0)
\] (3.28)

3.2.2 Correlation function \( \langle \phi_{n,n-1}(1)\phi_{n,n+1}(0) \rangle_\lambda \)

a) Contribution of the region \( \Omega_{l,0} \)

is given by the integral
\[
\int_{\Omega_{l,0}} I(x) \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n-1}(1)\phi(\infty) \rangle d^2x
\] (3.29)

The large \( p \) limit of the four-point function now is (see Appendix C):
\[
\langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n-1}(1)\phi(\infty) \rangle = \frac{\sqrt{n^2 - 1}}{3n} \left| \frac{2x - 1}{x(x - 1)} \right|^2 + O(\epsilon)
\] (3.30)

Using the last equality in (1.11) we see that \( I(x) \approx \frac{16\pi}{(16 - n^2)\epsilon} \) and for the result of the integral (3.29) we get
\[
\int_{\Omega_{l,0}} I(x) \langle \phi(x)\phi_{n,n+1}(0)\phi_{n,n-1}(1)\phi(\infty) \rangle d^2x
\approx \frac{32\pi^2 \sqrt{n^2 - 1} (5 \log(l) + \log(2l_0))}{3n(n^2 - 16)\epsilon}
\] (3.31)

b) Contribution of lens-like regions

Near \( x \sim 1 \) the r.h.s. of eq. (3.30) behaves as \( \frac{\sqrt{n^2 - 1}}{3n} |x - 1|^2 \) and the contribution of the lens-like regions near \( x \sim 1 \), which should be subtracted from the r.h.s. of eq. (3.31) is (see (D.9))
\[
\frac{32\pi^2 \sqrt{n^2 - 1}}{3n(16 - n^2)\epsilon} \log \left( \frac{l}{2l_0} \right)
\] (3.32)

c) Contributions of the regions \( D_{l,0}, D_{l,1} \) and \( D_{l,\infty} \)

We will treat contributions of \( D_{l,0} \) and \( D_{l,1} \) separately.

i) \( D_{l,0} \) contribution.

The relevant OPE is
\[
\phi(x)\phi_{n,n+1}(0) = C^{(n,n+1)}_{(1,3)(n,n+1)}(x\bar{x})^{-\Delta} (\phi_{n,n+1} + \cdots)
+ C^{(n,n-1)}_{(1,3)(n,n+1)}(x\bar{x})^{\Delta_{n,n-1} - \Delta_{n,n+1}} (\phi_{n,n-1}(0) + \cdots)
\]
It follows from (1.11) that in this region with sufficient accuracy

\[ I(x) \approx \frac{8\pi}{n\epsilon} \left( -\frac{1}{n+4} - \frac{(x\bar{x})^{\Delta_{n,n+1}-\Delta_{n,n-1}}}{n-4} \right) \]

From (A.5)

\[ C_{(1,3)(n,n+1)}^{(n,n+1)} C_{(1,3)(n,n+1)(n,n-1)}^{(1,3)} = \frac{\sqrt{n^2 - 1} (n + 2)(1 - (n - 1)\epsilon)}{6n^2} + O(\epsilon^2) \]

\[ C_{(1,3)(n,n+1)}^{(n,n-1)} C_{(1,3)(n,n+1)(n,n-1)}^{(1,3)(n,n+1)} = \frac{\sqrt{n^2 - 1} (n - 2)(1 + (n + 1)\epsilon)}{6n^2} + O(\epsilon^2) \tag{3.33} \]

and for the \( D_{l,0} \) contribution we get

\[
\int_{D_{l,0}\setminus D_{l,0}} I(x) \langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle d^2x
\approx - \frac{4\pi^2(n + 2)\sqrt{n^2 - 1}((n - 8)(1 - (n - 1)\epsilon) + 4(n - 2)\epsilon \log l)}{3(n^2 - 16)(n^2 - 16)\epsilon^2}
- \frac{4\pi^2(n - 2)\sqrt{n^2 - 1}((n + 8)(1 + (n + 1)\epsilon) + 4(n + 2)\epsilon \log l)}{3(n^2 - 16)n^2(n^2 + 2)\epsilon^2} \tag{3.34} \]

where two terms correspond to the two intermediate primaries \( \phi_{n\pm 1} \).

ii) \( D_{l,1} \) contribution.

The relevant OPE is

\[ \phi(x)\phi_{n,n-1}(1) = C_{(1,3)(n,n-1)}^{(n,n+1)} ((x - 1)(\bar{x} - 1))^{\Delta_{n,n+1}-\Delta_{n,n-1}}(\phi_{n,n+1} + \cdots)
+ C_{(1,3)(n,n-1)}^{(n,n-1)} ((x - 1)(\bar{x} - 1))^{-\Delta} (\phi_{n,n-1}(1) + \cdots) \]

Considering \( x \rightarrow 1 \) limit of (1.11) we get

\[ I(x) \approx \frac{8\pi}{n\epsilon} \left( -\frac{1}{n+4} - \frac{(x - 1)(\bar{x} - 1)^{\Delta_{n,n+1}-\Delta_{n,n-1}}}{n+4} \right) \]

The combinations of structure constants relevant for this case are those already presented in (3.33). For the \( D_{l,1} \) contribution we get

\[
\int_{D_{l,0}\setminus D_{l,0}} I(x) \langle \phi(x)\phi(0)\phi(1)\phi(\infty) \rangle d^2x
\approx - \frac{4\pi^2(n + 2)\sqrt{n^2 - 1}((n - 8)(1 - (n - 1)\epsilon) + 4(n - 2)\epsilon \log l)}{3(n^2 - 16)(n - 2)n^2\epsilon^2}
- \frac{4\pi^2(n - 2)\sqrt{n^2 - 1}((n + 8)(1 + (n + 1)\epsilon) + 4(n + 2)\epsilon \log(l))}{3(n^2 - 16)n^2(n + 2)\epsilon^2} \tag{3.35} \]

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Again the two terms correspond to two intermediate primaries $\phi_{n\pm 1}$. Notice that due to some subtle interplay among quantities involved, for the contribution of $D_{l,1}$ we got exactly the same result as for the contribution of $D_{l,0}$.

iii) $D_{l,\infty}$ contribution.

Since the structure constant $C_{(1,1)(n,n-1)(n,n+1)} = 0$ only the field $\phi$ which appears in the u-channel OPE gives a nonzero contribution. This contribution is proportional to

$$C_{(1,3)(1,3)(n,n-1)(n,n+1)}^{(1,3)} = \frac{4\sqrt{n^2 - 1} (1 - \epsilon)}{3n} + O(\epsilon^2)$$

Approximating $I(x)$ by $I(x) \approx \frac{\pi}{\epsilon} |x|^{-4\epsilon}$ we get

$$\int_{D_{l,\infty} \setminus D_{l,0}} I(x) \langle \phi(x) \phi_{n,n+1}(0) \phi_{n,n+1}(1) \phi(\infty) \rangle d^2 x$$

$$\approx -\frac{64\pi^2 \sqrt{n^2 - 1} (1 - \epsilon (1 - 2 \log l))}{3n (n^2 - 16) \epsilon^2}$$

It remains to collect all the contributions together to get

$$-\frac{16\pi^2 \sqrt{n^2 - 1} \left(5n^2 - 44 + (n^2 + 20) \epsilon\right)}{3n (n^2 - 16) (n^2 - 4) \epsilon^2} + O(\epsilon^0)$$

### 3.2.3 The matrix of anomalous dimensions

There is no need to calculate the remaining two point functions $\langle \phi_{n,n+1}(1) \phi_{n,n-1}(0) \rangle$ and $\langle \phi_{n,n-1}(1) \phi_{n,n-1}(0) \rangle$ since the former is identical with $\langle \phi_{n,n-1}(1) \phi_{n,n+1}(0) \rangle$ and the latter can be obtained from $\langle \phi_{n,n+1}(1) \phi_{n,n+1}(0) \rangle$ by simply replacing $n \rightarrow -n$. For simplicity of notation let us denote $\phi_{n,n+1} \equiv \phi_1$ and $\phi_{n,n-1} \equiv \phi_2$, then the two-point functions can be represented as

$$G_{\alpha,\beta}(x, \lambda) \equiv \langle \phi_\alpha(x) \phi_\beta(0) \rangle \lambda$$

$$= (x\bar{x})^{-\Delta_\alpha - \Delta_\beta} \left( \lambda C_{(1),(1)}^{(1)}(x\bar{x})^{\epsilon} + \frac{\lambda^2}{2} C_{(1),(1)}^{(2)}(x\bar{x})^{2\epsilon} + \cdots \right)$$

The first order coefficients $C_{(1),\alpha,\beta}$ are given by

$$C_{(1),\alpha,\beta}^{(1)} = \int \langle \phi_\alpha(1) \phi_\beta(0) \phi(x) \rangle d^2 x$$

$$= C_{(1,3)(\alpha)(\beta)}^{(1)} \pi \gamma (\epsilon + \Delta_\alpha - \Delta_\beta) \gamma (\epsilon + \Delta_\beta - \Delta_\alpha) \gamma (2\epsilon)$$

From eq. (A.2) for the dimensions we have

$$\Delta_1 \equiv \Delta_{n,n+1} = \frac{1}{4} - \left( \frac{n}{4} + \frac{1}{8} \right) \epsilon + \frac{1}{16} (n^2 - 1) \epsilon^2 + O(\epsilon^3)$$

$$\Delta_2 \equiv \Delta_{n,n-1} = \frac{1}{4} + \left( \frac{n}{4} - \frac{1}{8} \right) \epsilon + \frac{1}{16} (n^2 - 1) \epsilon^2 + O(\epsilon^3)$$

(3.41)
Explicitly, up to $O(\epsilon)$ terms we get
\[
C_{1,1}^{(1)} = \frac{\pi(n + 2)(2 - (2n - 1)\epsilon)}{2\sqrt{3} n \epsilon} + O(\epsilon); \quad C_{2,2}^{(1)} = \frac{\pi(n - 2)(2 + (2n + 1)\epsilon)}{2\sqrt{3} n \epsilon} + O(\epsilon)
\]
\[
C_{1,2}^{(1)} = C_{2,1}^{(1)} = -\frac{4\pi \sqrt{n^2 - 1}(\epsilon + 2)}{\sqrt{3} n (n^2 - 4)\epsilon} + O(\epsilon)
\]
(3.42)
and for the second order coefficients we have (see (3.28), (3.38))
\[
C_{1,1}^{(2)} = \frac{\pi^2 (3n^3 + 24n^2 + 64n + 44)}{3n(n + 2)^2 \epsilon^2} - \frac{4\pi^2(n + 1)(n^3 + 7n^2 + 14n + 5)}{3n(n + 2)^2 \epsilon} + O(\epsilon^0)
\]
\[
C_{2,2}^{(2)} = \frac{\pi^2 (3n^3 - 24n^2 + 64n - 44)}{3n(n - 2)^2 \epsilon^2} + \frac{4\pi^2(n - 1)(n^3 - 7n^2 + 14n - 5)}{3n(n - 2)^2 \epsilon} + O(\epsilon^0)
\]
\[
C_{1,2}^{(2)} = C_{2,1}^{(2)} = -\frac{16\pi^2 \sqrt{n^2 - 1} (5n^2 - 44 + (n^2 + 20)\epsilon)}{3n(n^2 - 16)(n^2 - 4)\epsilon^2} + O(\epsilon^0)
\]
(3.43)
Obviously the correlation function (3.39) satisfies the Callan-Symanzik equation
\[
(x\partial_x + \Delta_\alpha + \Delta_\beta - \epsilon\lambda \partial_\lambda) G_{\alpha,\beta}(x, \lambda) = 0 \quad (3.44)
\]
As in Section 3.1 let us introduce renormalized fields
\[
\phi_{\alpha}^{(g)} = B_{\alpha,\beta}(\lambda) \phi_\beta
\]
and require that the two point functions $G_{\alpha,\beta}^{(g)}(x) = \langle \phi_{\alpha}^{(g)}(x) \phi_{\beta}^{(g)}(0) \rangle_\lambda$ satisfy the normalization condition
\[
G_{\alpha,\beta}^{(g)}(1) = \delta_{\alpha,\beta} \quad (3.45)
\]
In matrix notations we may write
\[
G^{(g)}(x) = B \cdot G(x) \cdot B^T \quad (3.46)
\]
Comparing with (3.44) we see that the renormalized two-point function satisfies the equation
\[
(x\partial_x - \beta(g)\partial_y) G_{\alpha,\beta}^{(g)} + \sum_{\rho=1}^{2} (\Gamma_{\alpha,\rho} G_{\rho,\beta}^{(g)} + \Gamma_{\beta,\rho} G_{\alpha,\rho}^{(g)}) = 0 \quad (3.47)
\]
where the $\beta$ function and the renormalized coupling $g$ have been introduced in Section 2 and the matrix of anomalous dimensions $\Gamma$ is defined as
\[
\Gamma = BA^{-1} - \epsilon \lambda B \partial_\lambda B^{-1} \quad (3.48)
\]
where
\[
\hat{\Delta} = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}
\] (3.49)

Expanding the matrix \( B \) up to second order in \( \lambda \)
\[
B = 1 + \lambda B_1 + \lambda^2 B_2 + O(\lambda^3)
\]
\[
B^{-1} = 1 - \lambda B_1 + \lambda^2 (B_1^2 - B_2) + O(\lambda^3)
\] (3.50)

imposing the normalization condition (3.45) and requiring that the matrix of the anomalous dimensions (3.48) be symmetric, we find
\[
B_1 = \frac{1}{2} C^{(1)} + \frac{1}{2\epsilon} \left[ \hat{\Delta}, C^{(1)} \right]
\]
\[
B_2 = \frac{3}{8} \left( C^{(1)} \right)^2 - \frac{1}{4} C^{(2)} - \frac{1}{8\epsilon} \left[ \hat{\Delta}, C^{(2)} \right] + \frac{1}{8\epsilon} \left[ \left[ \hat{\Delta}, C^{(1)} \right], C^{(1)} \right] + \frac{1}{4\epsilon} \left[ \hat{\Delta}, (C^{(1)})^2 \right] + \frac{1}{8\epsilon^2} \left[ \hat{\Delta}, C^{(1)} \right]^2
\] (3.51)

Now all the ingredients to calculate the matrix of anomalous dimensions (3.48) are at our disposal. Taking also into account the \( \lambda-g \) relation (2.18), we get
\[
\Gamma_{1,1} = \Delta_1 + \frac{\pi g(n+2)(2 - (2n - 1)\epsilon)}{4\sqrt{3} n} + \frac{\pi^2 g^2}{2}
\]
\[
\Gamma_{1,2} = \Gamma_{2,1} = \frac{\pi g\sqrt{n^2 - 1} (2 + \epsilon)}{2\sqrt{3} n}
\]
\[
\Gamma_{2,2} = \Delta_2 + \frac{\pi g(n - 2)(2 + (2n + 1)\epsilon)}{4\sqrt{3} n} + \frac{\pi^2 g^2}{2}
\] (3.52)

Notice that all the matrix elements are regular at \( \epsilon = 0 \), all double and single poles in \( \epsilon \) disappeared. At the fixed point \( g = g^* \) (see (2.21))
\[
\Gamma^{(g^*)}_{1,1} = \frac{1}{4} - \frac{(2n^2 - n - 4)\epsilon}{8n} + \frac{(n^3 - 4n^2 + n + 8)\epsilon^2}{16n}
\]
\[
\Gamma^{(g^*)}_{1,2} = \Gamma^{(g^*)}_{2,1} = \frac{\sqrt{n^2 - 1} \epsilon (1 + \epsilon)}{2n}
\]
\[
\Gamma^{(g^*)}_{2,2} = \frac{1}{4} + \frac{(2n^2 + n - 4)\epsilon}{8n} + \frac{(n^3 + 4n^2 + n - 8)\epsilon^2}{16n}
\] (3.53)

It is easy to get the eigenvalues of this matrix
\[
\Delta^{(g^*)}_1 = \frac{1}{4} + \left( \frac{n}{4} + \frac{1}{8} \right) \epsilon + \frac{1}{16} \left( n^2 + 4n + 1 \right) \epsilon^2
\]
\[
\Delta^{(g^*)}_2 = \frac{1}{4} - \left( \frac{n}{4} - \frac{1}{8} \right) \epsilon + \frac{1}{16} \left( n^2 - 4n + 1 \right) \epsilon^2
\] (3.54)
Up to $O(\epsilon^3)$ terms they coincide with the dimensions $\Delta_{n+1,n}^{(p-1)}$ and $\Delta_{n-1,n}^{(p-1)}$ of the IR CFT $M_{p-1}$. We can easily identify also the corresponding normalized eigenvectors and establish the explicit map

$$
\phi_{n+1,n}^{(p-1)} = \frac{1}{n} \phi_1^{(g^*)} + \frac{\sqrt{n^2 - 1}}{n} \phi_2^{(g^*)}
$$

$$
\phi_{n-1,n}^{(p-1)} = -\frac{\sqrt{n^2 - 1}}{n} \phi_1^{(g^*)} + \frac{1}{n} \phi_2^{(g^*)}
$$

Remarkably the coefficients in (3.55) did not receive neither $\epsilon$ nor $\epsilon^2$ corrections. Thus it is quite conceivable that under the renormalization scheme (3.45), which we have adopted following A. Zamolodchikov, the relation (3.55) is exact. The same phenomenon we will encounter in the next section where a more involved case of mixing of the three fields $\phi_{n,n}^{\pm 2}$ and $\partial \bar{\partial} \phi_{n,n}$ will be considered.

### 3.3 Renormalization of the fields $\phi_{n,n+2}$, $\partial \bar{\partial} \phi_{n,n}$ and $\phi_{n,n-2}$

The OPE $\phi_1, \phi_{n,n+2}$ includes fields from the conformal families $[\phi_{n,n+4}]$, $[\phi_{n,n+2}]$ and $[\phi_{n,n}]$. Similarly the product $\phi_1, \phi_{n,n-2}$ produces fields from the families $[\phi_{n,n-4}]$, $[\phi_{n,n-2}]$ and $[\phi_{n,n}]$. Since the dimensions of the primary fields $\phi_{n,n}^{\pm 2}$ and the descendant field $\partial \bar{\partial} \phi_{n,n}$ are close to 1 in large $p$ limit, we have a situation when these three fields effectively get mixed along the RG flow\(^1\) [1]. To find the matrix of anomalous dimensions one has to calculate all the two point correlators of these fields.

#### 3.3.1 Correlation function $\langle \phi_{n,n+2}(1)\phi_{n,n+2}(0) \rangle_\lambda$

a) **Contribution of the region $\Omega_{1,0}$**

is given by the integral

$$
\int_{\Omega_{1,0}} I(x) \langle \phi(x)\phi_{n,n+2}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle d^2 x
$$

At large $p$ from the AGT relation we have found that (see Appendix C)

$$
\langle \phi(x)\phi_{n,n+2}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle = \left| \frac{3x^4 - 6x^3 + 9x^2 - 6x + 1}{3(x-1)^2x^2} \right|^2
$$

$$
+ \frac{8(3+n)}{3(1+n)} \left| \frac{(2x-1)(2x^2-2x+1)}{4(x-1)^2x^2} \right|^2 + \frac{(3+n)(4+n)}{18n(n+1)} \left| (x-1)^2x^2 \right|^{-2} + O(\epsilon)
$$

\(^1\)The fields $\phi_{n,n}^{\pm 4}$ have larger dimensions $\sim 4$ and do not get mixed with these three fields.
For present purposes $I(x)$ can be simply replaced by $\frac{\pi}{\epsilon}$. Performing the integration (3.56) we get

$$
\int_{\Omega_{l,0}} I(x)\phi(x)\phi_{n,n+2}(0)\phi_{n,n+2}(1)\phi(\infty) \, d^2x
\approx -\frac{16\pi^2 (2n^2 + 5n + 1) \log(l)}{3n(n+1)\epsilon} - \frac{16\pi^2 (n+1) \log(l_0)}{3n\epsilon} - \frac{\pi^2 (66 + 33n + 128(n+1) \log(2))}{24n\epsilon} + \frac{2\pi^2 (2n+1)}{3nl^2\epsilon} + \frac{\pi^2 (n+2)}{6nl_0^2\epsilon}
$$

b) **Contribution of lens-like regions**

Expanding (3.57) near $x \sim 1$ we get

$$
\frac{2 + n}{3n|x-1|^4} + \frac{2(n^2 + n - 2)(x + \bar{x} - 2)}{3n(n+1)|x-1|^4} + \frac{(n^2 + 5n + 6)((x-1)^2 + (\bar{x}-1)^2)}{3n(n+1)|x-1|^4} + \frac{8(n+1)}{3n|x-1|^2}
$$

So the contribution of the lens-like regions near $x \sim 1$ is (see Appendix D)

$$
\frac{\pi}{\epsilon} \left( \frac{2 + n}{3n} \left( \frac{\pi}{2l_0^2} - \frac{\pi}{l^2} - \frac{\pi}{8} \right) + \frac{2(n^2 + n - 2)\pi}{3n(n+1)} + \frac{2\pi(n^2 + 5n + 6)}{3n(n+1)} + \frac{8(n+1)}{3n} \frac{2\pi \log l}{2l_0} \right)
$$

(3.59)

c) **Contributions of the regions $D_{l,0}, D_{l,1}$ and $D_{l,\infty}$**

The contributions of $D_{l,0}$ and $D_{l,1}$ are identical so we compute only the $D_{l,0}$ part. Here are the relevant terms of the OPE

$$
\phi(x)\phi_{n,n+2}(0) = (x\bar{x})^{-\Delta} C^{(n,n+2)}_{(1,3)(n,n+2)} \phi_{n,n+2} + C^{(n,n)}_{(1,3)(n,n+2)} (x\bar{x})^{-\Delta_n - \Delta_{n+2} - \Delta} \times \left( 1 + \frac{\Delta + \Delta_n - \Delta_{n+2}}{2\Delta_n} xL_{-1} \right) \left( 1 + \frac{\Delta + \Delta_n - \Delta_{n+2}}{2\Delta_n} \bar{x}\bar{L}_{-1} \right) \phi_{n,n}(0) + \cdots
$$

(3.61)

where $L_k, \bar{L}_k$ are the left and right Virasoro generators. To proceed let us notice that the effect of the Virasoro generator $L_{-1}$ or $\bar{L}_{-1}$ in the three point function is rather simple. Namely the relation

$$
\langle L_{-1} \phi_{n,n}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle = (\Delta_n + \Delta_{n+2} - \Delta) \langle \phi_{n,n}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle
$$

and a similar relation with $L_{-1}$ replaced by $\bar{L}_{-1}$ hold. We see from (1.14) that in this region $I(x)$ can be approximated by

$$
I(x) \approx \frac{\pi}{\epsilon} + \pi \left( x + \bar{x} - \log |x|^2 \right) - 2\pi\epsilon(x + \bar{x}) \log |x|^2
$$

(3.62)

(3.63)
We need also the combination of structure constants
\[
C_{(1,3)(n,n+2)(n,n+2)}^{2} = \frac{4(n+3)^2(1-(n+2)\epsilon)}{3(n+1)^2} + O(\epsilon^2)
\]
\[
C_{(1,3)(n,n+2)}^{(n,n)(n,n+2)(1,3)} = \frac{n+2}{3n} + O(\epsilon^2)
\]
(3.64)

Taking into account that
\[
\Delta_{n,n} - \Delta_{n,n+2} - \Delta = -2 + \frac{n+3}{2} \epsilon
\]
\[
\frac{(\Delta + \Delta_{n,n} - \Delta_{n,n+2}) (\Delta_{n,n} + \Delta_{n,n+2} - \Delta)}{2\Delta_{n,n}} = -\frac{2(n-1)}{n+1} + \frac{(n-1)(n+3)\epsilon}{2(n+1)}
\]
we get that the piece of integrand corresponding to the contribution of the family \([\phi_{n,n}]\) is equal to
\[
\frac{2+n}{3n} |x|^{-4+(3+n)\epsilon} \left| 1 - \left( \frac{2(n-1)}{n+1} - \frac{(n-1)(n+3)\epsilon}{2(n+1)} \right) x \right|^2
\]
\[
\times \left( \frac{\pi}{\epsilon} + \pi \left( x + \bar{x} - \log |x|^2 \right) - 2\pi \epsilon (x + \bar{x}) \log |x|^2 \right)
\]
(3.65)

The part corresponding to the intermediate field \(\phi_{n,n+2}\) is simpler. For this one we can restrict us with a less accurate expression for \(I(x)\)
\[
I(x) \approx \frac{\pi}{\epsilon} - \pi \log |x|^2
\]
(3.66)

and the integrand is simply
\[
\frac{4(n+3)^2(1-(n+2)\epsilon)}{3(n+1)^2} \left( \frac{\pi}{\epsilon} - \pi \log |x|^2 \right) |x|^{-2+2\epsilon}
\]

Performing integrations we get
\[
\int_{D_{1,0}\backslash D_{0,0}} I(x) (\phi(x)\phi_{n,n+2}(0)\phi_{n,n+2}(1)\phi(\infty)) d^2 x
\]
\[
= \frac{8\pi^2(n+2)(n+5)(n-1)^2}{3n(n+1)^2(n+3)^2\epsilon^2} - \frac{\pi^2(n+2)}{3l^2n\epsilon}
\]
\[
+ \frac{4\pi^2(n+2)(n-1) ((2n^3 + 10n^2 + 6n - 18) \log(l) - n^3 - 9n^2 - 23n + 1)}{3n(n+1)^2(n+3)^2\epsilon}
\]
\[
+ \frac{8\pi^2(n+3)^2}{3(n+1)^2\epsilon} - \frac{8\pi^2(n+3)^2(n+2 - \log l)}{3(n+1)^2\epsilon}
\]
(3.67)

where the second and the third lines come from the family \([\phi_{n,n}]\) and the last line, from the \(\phi_{n,n+2}\) field of the OPE (3.61).
Also in this case the contribution coming from the region $D_{l,\infty}$ is quite similar to the case discussed in Section 2. We should simply take into account that the contribution of the field $\phi$ appearing in the u-channel OPE is proportional to

$$C^{(1,3)} C^{(1,3)} = \frac{4(n + 3)(2 - (n + 5)\epsilon)}{3(n + 1)} + O(\epsilon^2)$$

(3.68)

The result (c.f. eq. (2.10)) is

$$\int_{D_{l,\infty}\setminus D_{l_0,\infty}} I(x) (\phi(x)\phi_{n,n+1}(0)\phi_{n,n+1}(1)\phi(\infty)) d^2x \approx -\frac{\pi^2}{\epsilon l^2} + \frac{4\pi^2(n + 3)}{3(n + 1)} \left( \frac{2}{\epsilon^2} - \frac{n + 5 - 4 \log l}{\epsilon} \right)$$

(3.69)

Finally, (3.58) minus (3.60) plus twice (3.67) and plus (3.69) gives

$$\frac{8\pi^2}{3n(n + 1)(n + 3)^2\epsilon^2} \frac{3n^4 + 33n^3 + 121n^2 + 143n + 20}{3n(n + 1)(n + 3)^2\epsilon^2} - \frac{4\pi^2(n + 5)}{3n(n + 1)(n + 3)^2\epsilon^2} \left( \frac{5n^4 + 45n^3 + 143n^2 + 151n + 8}{3n(n + 1)(n + 3)^2\epsilon} \right) + O(\epsilon^0)$$

(3.70)

### 3.3.2 Correlation function $\langle \phi_{n,n}(1)\phi_{n,n+2}(0)\rangle$  

a) Contribution of the region $\Omega_{l,l_0}$

is given by the integral

$$\int_{\Omega_{l,l_0}} I(x) (\phi(x)\phi_{n,n+2}(0)\phi_{n,n}(1)\phi(\infty)) d^2x$$

(3.71)

The large $p$ limit of the four-point function is very simple (see Appendix C)

$$\langle \phi(x)\phi_{n,n+2}(0)\phi_{n,n}(1)\phi(\infty) \rangle = \frac{4}{3} \sqrt{\frac{n + 2}{n}} |x|^{-2} + O(\epsilon)$$

(3.72)

$I(x)$ can be replaced by

$$I(x) \approx -\frac{2\pi(n + 1)\epsilon}{n + 5} \left[ 1 + \frac{4x}{(n + 1)(1 - x)} \right]^2 - \frac{2\pi(n - 3)\epsilon}{n + 1} \frac{x}{1 - x}$$

(3.73)

and the result of integration is

$$\int_{\Omega_{l,l_0}} I(x) (\phi(x)\phi_{n,n+2}(0)\phi_{n,n}(1)\phi(\infty)) d^2x \approx \frac{16\pi^2\epsilon}{3(n + 5)\sqrt{\frac{n + 2}{n}}} \left( (3n - 5) \log(l) + (n + 1) \log(2l_0) \right)$$

(3.74)
b) Contribution of lens-like regions

We see from (3.72), (3.73) that near \( x \sim 1 \) the integrand in eq. (3.71) behaves as

\[
- \frac{8\pi \epsilon (n+1) \sqrt{\frac{n+2}{n}}}{3(n+5)|x-1|^2}
\]

So according to the Appendix D the contribution of the lens-like regions near \( x \sim 1 \) which should be subtracted from the r.h.s. of eq. (3.74) is

\[
- \frac{16\pi^2(n+1)\epsilon \sqrt{\frac{n+2}{n}}}{3(n+5)} \log \left( \frac{l}{2l_0} \right)
\]

(3.75)

c) Contributions of the regions \( D_{l,0}, D_{l,1} \) and \( D_{l,\infty} \)

Let us compute the contributions of \( D_{l,0} \) and \( D_{l,1} \) separately.

i) \( D_{l,0} \) contribution.

The relevant OPE has already appeared in (3.61). Instead of (3.62) we now need the analogous relation

\[
\langle L_{-1}\phi_{n,n}(0)\phi_{n,n}(1)\phi(\infty) \rangle = (2\Delta_{n,n} - \Delta) \langle \phi_{n,n}(0)\phi_{n,n}(1)\phi(\infty) \rangle
\]

(3.76)

The function \( I(x) \) will be determined using the first equality in (1.11). For the calculation of the contribution of the field \( \phi_{n,n+2} \) it is safe to replace the hypergeometric functions simply by 1. Instead for the contribution of the family \([\phi_{n,n}]\) also the first order in \( x \) and in \( \bar{x} \) terms should be taken into account. Below we present expressions for the relevant combinations of structure constants with required accuracy

\[
C^{(n,n)}_{(1,3)(n,n+2)}C^{(1,3)(n,n)}_{(n,n)} = \sqrt{\frac{n+2}{n}} \frac{(n^2 - 1)(2 + 5\epsilon)\epsilon^2}{48} + O(\epsilon^4)
\]

\[
C^{(n,n+2)}_{(1,3)(n,n+2)}C^{(1,3)(n+2,n)}_{(n,n+2)} = \sqrt{\frac{n+2}{n}} \frac{(n + 3)(2 - (n+2)\epsilon)}{3(n+1)} + O(\epsilon^2)
\]

(3.77)

For the final result of the \( D_{l,0} \) contribution we get

\[
\int_{D_{l,0}|D_{l,0}} I(x) \langle \phi(x)\phi_{n,n+2}(0)\phi_{n,n}(1)\phi(\infty) \rangle d^2x \\
\approx - \frac{8\pi^2(n-1)\sqrt{\frac{n+2}{n}}}{3(n+3)(n+5)} \left( 1 + \epsilon \left( n + 5 \right) \left( \frac{5}{2} + (n+3) \log l \right) \right)
\]

\[
- \frac{4\pi^2(n+3)\sqrt{\frac{n+2}{n}}}{3(n+5)} \left( 1 + \epsilon \left( \frac{n}{2} + 4 + 2 \log l \right) \right)
\]

(3.78)
where the two lines correspond to the \([\phi_{n,n}]\) and \(\phi_{n,n+2}\) contributions respectively.

ii) \(D_{l,1}\) contribution.

The relevant OPE is

\[
\phi(x)\phi_{n,n}(1) = |x - 1|^2(\Delta_{n,n+2} - \Delta_{n,n} - \Delta)C_{(1,3)(n,n)}^{(n,n+2)}\phi_{n,n+2}(1) + \cdots 
\]

\[
+ C_{(1,3)(n,n)}^{(n,n)}|x - 1|^{-2\Delta} \left( 1 + \frac{\Delta(x - 1)}{2\Delta_{n,n}} L_{-1} \right) \left( 1 + \frac{\Delta(\bar{x} - 1)}{2\Delta_{n,n}} \bar{L}_{-1} \right) \phi_{n,n}(1) + \cdots 
\]

The relation between the three point functions relevant for this case is

\[
\langle L_{-1}\phi_{n,n}(1)\phi_{n,n+2}(0)\phi(\infty) \rangle = (\Delta - \Delta_{n,n} - \Delta_{n,n+2}) \langle \phi_{n,n}(1)\phi_{n,n+2}(1)\phi(\infty) \rangle 
\] (3.80)

Notice the flip of sign compared to (3.62) due to the rearrangement of the points 0 and 1. For the function \(I(x)\) the second equality in (1.11) should be used. The hypergeometric functions should be expanded around \(x = 1\). When calculating the contribution of \(\phi_{n,n+2}\) it would suffice to keep the constant term only while for the contribution of \([\phi_{n,n}]\) also the terms linear in \(x - 1\) (or \(\bar{x} - 1\)) should be taken into account. During the calculation one encounters the same combinations of the structure constants as in (3.77). Finally we get for the \(D_{l,1}\) contribution a result identical to that of \(D_{l,0}\) given by (3.34). Remember that a similar phenomenon we have encountered earlier in Section 3.2.2.

iii) \(D_{l,\infty}\) contribution

Only the field \(\phi\) appearing in u-channel OPE gives a nonzero contribution. Since

\[
C_{(1,3)(1,3)}^{(1,3)}C_{(1,3)(n,n)(n,n+2)} = \sqrt{\frac{n + 2}{n}} \frac{2(2 - 3\epsilon)}{3} + O(\epsilon^2), \quad (3.81)
\]

and, from the third equality in (1.11),

\[
I(x) \approx -\frac{4\pi\epsilon(n - 3)}{n + 5} \left( 1 + (n + 5)\epsilon \right) |x|^{-4\epsilon} \quad (3.82)
\]

we get

\[
\int_{D_{l,\infty}\setminus D_{l,\infty}} I(x)\langle \phi(x)\phi_{n,n+2}(0)\phi_{n,n}(1)\phi(\infty) \rangle d^2x 
\]

\[
\approx -\sqrt{\frac{n + 2}{n}} \frac{8\pi^2(n - 3) (2 + (2n + 7 + 4 \log l) \epsilon)}{3(n + 5)} \quad (3.83)
\]

It remains to combine all the contributions. The result is

\[
-\sqrt{\frac{n + 2}{n}} \frac{4\pi^2(n - 1)(6n + 22 + (5n^2 + 37n + 64)\epsilon)}{3(n + 3)(n + 5)} + O(\epsilon^2) \quad (3.84)
\]
3.3.3 Correlation function $\langle \phi_{n,n+2}(1)\phi_{n,n-2}(0) \rangle_\lambda$

a) Contribution of the region $\Omega_{l,0}$

\[
\int_{\Omega_{l,0}} I(x) \langle \phi(x)\phi_{n,n-2}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle d^2x
\]  

(3.85)

The large $p$ limit of the four-point function is (see Appendix C)

\[
\langle \phi(x)\phi_{n,n-2}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle = \frac{\sqrt{n^2 - 4}}{3n|x(1 - x)|^4} + O(\epsilon)
\]  

(3.86)

$I(x)$ can be replaced by

\[
I(x) \approx -\frac{4\pi}{(n^2 - 4)\epsilon}
\]  

(3.87)

and the result of the integration is

\[
\int_{\Omega_{l,0}} I(x) \langle \phi(x)\phi_{n,n-2}(0)\phi_{n,n+2}(1)\phi(\infty) \rangle d^2x
\]

\[
\approx \frac{\pi^2}{6n\epsilon\sqrt{n^2 - 4}} \left(64\log(2l_0) + 33 - \frac{8}{l^2} - \frac{4}{l_0^2}\right)
\]  

(3.88)

b) Contribution of lens-like regions

It follows from (3.86) and (3.87) that near $x \sim 1$ the integrand in eq. (3.85) up to less singular terms behaves as

\[
-\frac{4\pi (1 - 2(x + \bar{x} - 2) + (x - 1)^2 + (\bar{x} - 1)^2 + 4|x - 1|^2)}{3n\epsilon\sqrt{n^2 - 4}|x - 1|^4}
\]

Consequently, from the Appendix D, we see that the contribution of the lens-like regions near $x \sim 1$ is

\[
-\frac{4\pi}{3n\epsilon\sqrt{n^2 - 4}} \left(\frac{\pi}{2l_0^2} - \frac{\pi}{l^2} - \frac{\pi}{8} - 2(\pi) + (2\pi) + 8\pi \log \left(\frac{l}{2l_0}\right)\right)
\]  

(3.89)

c) Contributions of the regions $D_{l,0}$, $D_{l,1}$ and $D_{l,\infty}$

i) $D_{l,0}$ contribution

The relevant OPE:

\[
\phi(x)\phi_{n,n-2}(0) = (x\bar{x})^{\Delta_{n,n} - \Delta_{n,n-2} - \Delta C^{(n,n)}_{(1,3)(n,n-2)}} \times
\]

\[
\left(1 + \frac{\Delta + \Delta_{n,n} - \Delta_{n,n-2}}{2\Delta_{n,n}} xL_{-1}\right) \left(1 + \frac{\Delta + \Delta_{n,n} - \Delta_{n,n-2}}{2\Delta_{n,n}} \bar{x}L_{-1}\right) \phi_{n,n}(0) + \cdots
\]

(3.90)
The impact of $L_{-1}$ on the three-point function:

\[ \langle L_{-1} \phi_{n,n}(0) \phi_{n,n+2}(1) \phi(\infty) \rangle = (\Delta_{n,n} + \Delta_{n,n+2} - \Delta) \langle \phi_{n,n}(0) \phi_{n,n+2}(1) \phi(\infty) \rangle \]  \hspace{1cm} (3.91)

In the first expression of (1.11) for $I(x)$, the hypergeometric functions should be expanded up to the linear order in $x$ (or $\bar{x}$) terms.

The relevant combination of the structure constants:

\[ C_{(1,3)(n,n-2)}^{(n,n)} C_{(1,3)(n,n)(n,n+2)}^{(1,3)} = \frac{\sqrt{n^2 - 4}}{3n} + O(\epsilon^2) \]  \hspace{1cm} (3.92)

So, the final result for the $D_{l,0}$ contribution is

\[
\int_{D_{l,0}\setminus D_{0,0}} I(x) \langle \phi(x) \phi_{n,n-2}(0) \phi_{n,n+2}(1) \phi(\infty) \rangle d^2 x \\
\approx \frac{16\pi^2}{3n(n^2 - 9)\sqrt{n^2 - 4}} \left( \frac{10}{\epsilon^2} + \frac{n^2 - 9}{4l^2 \epsilon} - \frac{n^2 + 1 + 2(n^2 - 9) \log l}{\epsilon} \right) \]  \hspace{1cm} (3.93)

ii) $D_{l,1}$ contribution

The relevant OPE:

\[
\phi(x) \phi_{n,n+2}(1) = |x - 1|^2 (\Delta_{n,n-\Delta_{n,n+2}-\Delta}) C_{(1,3)(n,n+2)}^{(n,n)} C_{(1,3)(n,n)}^{(n,n)} \]

\[ \times \left( 1 + \frac{\Delta + \Delta_{n,n} - \Delta_{n,n+2}}{2\Delta_{n,n}} (x - 1)L_{-1} \right) \]

\[ \times \left( 1 + \frac{\Delta + \Delta_{n,n} - \Delta_{n,n+2}}{2\Delta_{n,n}} (\bar{x} - 1)\bar{L}_{-1} \right) \phi_{n,n}(1) + \cdots \]  \hspace{1cm} (3.94)

The impact of $L_{-1}$ on the three-point function:

\[ \langle L_{-1} \phi_{n,n}(1) \phi_{n,n-2}(0) \phi(\infty) \rangle = (\Delta - \Delta_{n,n} - \Delta_{n,n-2}) \langle \phi_{n,n}(1) \phi_{n,n-2}(1) \phi(\infty) \rangle \]  \hspace{1cm} (3.95)

The combination of structure constants required for this computation coincides with that given by eq. (3.92). The explicit calculation shows that in this case too, the $D_{l,1}$ contribution is identical to that of $D_{l,0}$ given by (3.93).

Note also that the contribution of $D_{l,\infty}$ is negligible. Combining all the contributions for the case at hand we get

\[
320\pi^2 (1 - \epsilon) \\
3\epsilon^2 n(n^2 - 9)\sqrt{n^2 - 4} + O(\epsilon^2) \]  \hspace{1cm} (3.96)

\[ -28 - \]
3.3.4 The matrix of anomalous dimensions

The remaining two point functions \( \langle \phi_{n,n-2}(1)\phi_{n,n-2}(0) \rangle_\lambda \) and \( \langle \phi_{n,n}(1)\phi_{n,n-2}(0) \rangle_\lambda \) can be obtained from \( \langle \phi_{n,n+2}(1)\phi_{n,n+2}(0) \rangle_\lambda \) and \( \langle \phi_{n,n}(1)\phi_{n,n+2}(0) \rangle_\lambda \) replacing \( n \) by \( -n \). So we have all necessary material to repeat the steps of Section 3.2.3 and calculate the matrix of anomalous dimensions for the fields

\[
\phi_1 \equiv \phi_{n,n+2}; \quad \phi_2 \equiv (2\Delta_{n,n}(2\Delta_{n,n} + 1))^{-1} \partial \bar{\partial} \phi_{n,n}; \quad \phi_3 \equiv \phi_{n,n-2}
\]

The two-point functions of these fields can be represented as in (3.39), but the indices now take the values \( \alpha, \beta = 1, 2, 3 \). The replacement of the field \( \phi_{n,n} \) by \( \phi_2 \) in a two point function, at a given order \( k \) of the perturbation theory, results in an extra multiplier which is easy to calculate. Here is the rule: the coefficients \( C^{(k)}_{\alpha,2} = C^{(k)}_{2,\alpha} \), for \( \alpha \neq 2 \), and \( C^{(k)}_{2,2} \) should be endowed with the extra multipliers

\[
\frac{(k\epsilon - \Delta_\alpha - \Delta_2)^2}{2\Delta_{n,n}(2\Delta_{n,n} + 1)}
\]

and

\[
\left( \frac{(k\epsilon - 2\Delta_2)(k\epsilon - 2\Delta_2 - 1)}{2\Delta_{n,n}(2\Delta_{n,n} + 1)} \right)^2
\]

respectively. The numerators come from the derivatives and the denominators from the normalization factor, present in the definition of the field \( \phi_2 \). The dimensions at the zero coupling \( \lambda = 0 \) are

\[
\begin{align*}
\Delta_1 &= \Delta_{n,n+2} = 1 - \frac{n+1}{2} \epsilon + \frac{n^2 - 1}{16} \epsilon^2 + O(\epsilon^3) \\
\Delta_2 &= 1 + \Delta_{n,n} = 1 + \frac{n^2 - 1}{16} \epsilon^2 + O(\epsilon^3) \\
\Delta_3 &= \Delta_{n,n-2} = 1 + \frac{n-1}{2} \epsilon + \frac{n^2 - 1}{16} \epsilon^2 + O(\epsilon^3) 
\end{align*}
\]

(3.97)

Computation of the first order coefficients as in previous cases is quite easy and
with desired accuracy we get

\[ C_{1,1}^{(1)} = \frac{2\pi(n + 3)(2 - (n + 2)\epsilon)}{\sqrt{3}(n + 1)\epsilon} + O(\epsilon) \]

\[ C_{1,2}^{(1)} = C_{2,1}^{(1)} = -\frac{8\pi\sqrt{n + 2}}{3n} (2 - \epsilon) + O(\epsilon) \]

\[ C_{1,3}^{(1)} = C_{3,1}^{(1)} = 0; \quad C_{2,2}^{(1)} = \frac{4\pi (4 - (n^2 + 1)\epsilon)}{\sqrt{3}(n^2 - 1)\epsilon} + O(\epsilon) \]

\[ C_{2,3}^{(1)} = C_{3,2}^{(1)} = -\frac{8\pi\sqrt{n - 2}}{3n} (2 - \epsilon) + O(\epsilon) \]

\[ C_{3,3}^{(1)} = \frac{2\pi(n - 3)(2 + (n - 2)\epsilon)}{\sqrt{3}(n - 1)\epsilon} + O(\epsilon) \]

From (3.70), (3.84), (3.10), (3.96) and the above presented considerations, for the second order coefficients \( C_{\alpha,\beta}^{(2)} = C_{\beta,\alpha}^{(2)} \) we find

\[ C_{1,1}^{(2)} = \frac{8\pi^2 (3n^4 + 33n^3 + 121n^2 + 143n + 20)}{3n(n + 1)(n + 3)^2\epsilon^2} - \frac{4\pi^2(n + 5) (5n^4 + 45n^3 + 143n^2 + 151n + 8)}{3n(n + 1)(n + 3)^2\epsilon} + O(\epsilon^0) \]

\[ C_{1,2}^{(2)} = -\frac{64\pi^2 \sqrt{\frac{n + 2}{n} (3n + 11)}}{3(n + 1)(n + 3)(n + 5) \epsilon^2} + \frac{32\pi^2 \sqrt{\frac{n + 2}{n} (n^2 + 18n + 57)}}{3(n + 1)(n + 3)(n + 5)\epsilon} + O(\epsilon^0) \]

\[ C_{1,3}^{(2)} = \frac{320\pi^2}{3n(n^2 - 9)\sqrt{n^2 - 4} \epsilon^2} - \frac{320\pi^2}{3n(n^2 - 9)\sqrt{n^2 - 4} \epsilon} + O(\epsilon^0) \]

\[ C_{2,2}^{(2)} = \frac{128\pi^2}{3(n^2 - 1)\epsilon^2} - \frac{16\pi^2 (n^2 + 19)}{3(n^2 - 1) \epsilon} + O(\epsilon^0) \]

\[ C_{2,3}^{(2)} = -\frac{64\pi^2 \sqrt{\frac{n + 2}{n} (3n - 11)}}{3(n - 1)(n - 3)(n - 5)\epsilon^2} - \frac{32\pi^2 \sqrt{\frac{n + 2}{n} (n^2 - 18n + 57)}}{3(n - 1)(n - 3)(n - 5)\epsilon} + O(\epsilon^0) \]

\[ C_{3,3}^{(2)} = \frac{8\pi^2 (3n^4 - 33n^3 + 121n^2 - 143n + 20)}{3n(n - 1)(n - 3)^2\epsilon^2} + \frac{4\pi^2(n - 5) (5n^4 - 45n^3 + 143n^2 - 151n + 8)}{3n(n - 1)(n - 3)^2\epsilon} + O(\epsilon^0) \] (3.99)

With this input we can repeat the procedure of the Section 3.2.3 and compute the
matrix of anomalous dimensions. Here is the final result:

\[
\Gamma_{1,1} \approx \Delta_1 + \frac{\pi g(n + 3)(2 - (n + 2)\epsilon)}{\sqrt{3}(n + 1)} + \frac{8\pi^2 g^2(n + 2)}{3(n + 1)}
\]

\[
\Gamma_{1,2} = \Gamma_{2,1} \approx \frac{\pi g(n - 1)\sqrt{\frac{n+2}{3n}}(2 - \epsilon)}{n + 1} + \frac{4\pi^2 g^2(n - 1)\sqrt{\frac{n+2}{n}}}{3(n + 1)}
\]

\[
\Gamma_{1,3} = \Gamma_{3,1} \approx 0
\]

\[
\Gamma_{2,2} \approx \Delta_2 + \frac{2\pi g(4 - (n^2 + 1)\epsilon)}{\sqrt{3}(n^2 - 1)} + \frac{4\pi^2 g^2(n^2 + 3)}{3(n^2 - 1)}
\]

\[
\Gamma_{2,3} = \Gamma_{3,2} \approx \frac{\pi g(n + 1)\sqrt{\frac{n-2}{3n}}(2 - \epsilon)}{n - 1} + \frac{4\pi^2 g^2(n + 1)\sqrt{\frac{n-2}{n}}}{3(n - 1)}
\]

\[
\Gamma_{3,3} \approx \Delta_3 + \frac{\pi g(n - 3)(2 + (n - 2)\epsilon)}{\sqrt{3}(n - 1)} + \frac{8\pi^2 g^2(n - 2)}{3(n - 1)}
\]

(3.100)

Again we see that all matrix elements are regular at \(\epsilon = 0\). All double and single poles in \(\epsilon\) disappeared. At the fixed point \(g = g^*\) (see (2.21))

\[
\Gamma_{1,1}^{(g^*)} = 1 - \frac{(n^2 - 5)\epsilon}{2(n + 1)} + \frac{(n^3 - 7n^2 - n + 39)\epsilon^2}{16(n + 1)} + O(\epsilon^3)
\]

\[
\Gamma_{1,2}^{(g^*)} = \Gamma_{2,1}^{(g^*)} = \frac{(n - 1)\sqrt{\frac{n+2}{n}}\epsilon(n + 1)}{n + 1} + O(\epsilon^3)
\]

\[
\Gamma_{1,3}^{(g^*)} = \Gamma_{3,1}^{(g^*)} = O(\epsilon^3)
\]

\[
\Gamma_{2,3}^{(g^*)} = \Gamma_{3,2}^{(g^*)} = \frac{(n + 1)\sqrt{\frac{n-2}{n}}\epsilon(n + 1)}{n - 1} + O(\epsilon^3)
\]

\[
\Gamma_{3,3}^{(g^*)} = 1 + \frac{(n^2 - 5)\epsilon}{2(n - 1)} + \frac{(n^3 + 7n^2 - n - 39)\epsilon^2}{16(n - 1)} + O(\epsilon^3)
\]

(3.101)

Here are the eigenvalues of this matrix

\[
\Delta_1^{(g^*)} = 1 + \frac{(n + 1)\epsilon}{2} + \frac{(n + 1)(n + 7)\epsilon^2}{16} + O(\epsilon^3)
\]

\[
\Delta_2^{(g^*)} = 1 + \frac{(n^2 - 1)\epsilon^2}{16} + O(\epsilon^3)
\]

\[
\Delta_3^{(g^*)} = 1 - \frac{(n - 1)\epsilon}{2} + \frac{(n - 1)(n - 7)\epsilon^2}{16} + O(\epsilon^3)
\]

(3.102)

which, up to \(O(\epsilon^3)\) terms coincide with the dimensions \(\Delta_{n+2, n}^{(p-1)}\), \(1 + \Delta_{n, n}^{(p-1)}\) and \(\Delta_{n-2, n}^{(p-1)}\) of the IR CFT \(M_{p-1}\). It is easy to find the orthogonal matrix which diagonalizes the
matrix of anomalous dimensions (3.101) and to establish the explicit map

\[
\phi^{(p-1)}_{n+2,n} = \frac{2}{n^2 + n} \phi^{(g^*)}_1 + \frac{2\sqrt{\frac{n+2}{n} - 1}}{n} \phi^{(g^*)}_2 + \frac{\sqrt{n^2 - 4}}{n} \phi^{(g^*)}_3
\]

\[
\phi^{(p-1)}_2 = -\frac{2\sqrt{\frac{n+2}{n}}}{n+1} \phi^{(g^*)}_1 + \frac{5-n^2}{n^2-1} \phi^{(g^*)}_2 + \frac{2\sqrt{\frac{n-2}{n}}}{n-1} \phi^{(g^*)}_3
\]

\[
\phi^{(p-1)}_{n-2,n} = \frac{\sqrt{n^2 - 4}}{n} \phi^{(g^*)}_1 - \frac{2\sqrt{\frac{n-2}{n}}}{n-1} \phi^{(g^*)}_2 + \frac{2}{n(n-1)} \phi^{(g^*)}_3
\]

(3.103)

where

\[
\phi^{(p-1)}_2 \equiv (2\Delta^{(p-1)}_{n,n}(2\Delta^{(p-1)}_{n,n} + 1))^{-1} \partial \bar{\partial} \phi^{(p-1)}_{n,n}
\]

(3.104)

In this case too we see that the coefficients in (3.103) do not receive \(\epsilon\) or \(\epsilon^2\) corrections.

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**A Minimal models**

For the readers convenience we present here few facts about unitary series of the minimal models [5, 8] denoted by \(M_p, p = 3, 4, \ldots\). The central charge is given by

\[
c_p = 1 - \frac{6}{p(p+1)}
\]

(A.1)

This theory contains finitely many spinless\(^2\) primary fields denoted by \(\phi_{n,m}\) with conformal dimensions (the famous Kac spectrum [6])

\[
\Delta_{n,m} = \frac{(n-m)^2}{4} + \frac{n^2 - 1}{4p} - \frac{m^2 - 1}{4(p+1)}
\]

(A.2)

where \(n \in \{1, 2, \ldots, p-1\}, m \in \{1, 2, \ldots, p\}\). There is an identification \(\phi_{p-n,p+1-m} \equiv \phi_{n,m}\) so that the number of primary fields is equal to \(p(p-1)/2\). The field \(\phi_{1,1}\) with

\(^2\)We consider here the so called diagonal series only.
dimension 0 is the identity operator. The operator product expansions satisfy the fusion rules
\[ \phi_{n_1,m_1} \phi_{n_2,m_3} \in \bigoplus_{n_3=|n_1-n_2|+1}^{n_1+n_2-1} \bigoplus_{m_3=|m_1-m_2|+1}^{m_1+m_2-1} \left[ \phi_{n_3,m_3} \right] \] (A.3)

The main subject of this paper is the minimal model \( M_p \) perturbed by the relevant field \( \phi_{1,3} \). It’s dimension
\[ \Delta_{1,3} = 1 - \frac{2}{p+1} \equiv 1 - \epsilon < 1 \] (A.4)

For large \( p \) this field becomes nearly marginal which is the main reason why in this region the non-trivial RG behavior can be investigated by means of the perturbation theory. The structure constants of the OPE have been computed in [9]. A slightly more compact expression which we present below is taken from [10]
\[ C_{(n_1,m_1)(n_2,m_2)(n_3,m_3)} = \rho^{4s+2t-2s-1} \] (A.5)
\[ \times \left[ \frac{\Gamma(x)}{\Gamma(1-x)} \right] s = \frac{n_1 + n_2 - n_3 - 1}{2}; \quad t = \frac{m_1 + m_2 - m_3 - 1}{2} \]

**B Computation of \( I(x) \)**

One way to get the result (1.10) for the integral (1.8) is to notice that \( I(x) \) satisfies the hypergeometric differential equation independently with respect to the both variables \( x \) and \( \bar{x} \). The starting point is the identity
\[ [x(1-x)\partial_x^2 + (1-a-c + (a+b+2c-2)x)\partial_x + c(1-a-b-c)](y^{a-1}(1-y)^{b-1}(y-x)^c) = \partial_y (c y^a(1-y)^b(y-x)^c) \] (B.1)
which shows that as a function of the variable $x$, $I(x)$ is a linear combination of the
hypergeometric functions

$$F(1-a-b-c,-c,1-a-c,x)$$

and

$$x^{a+c}F(a,1-b,1+a+c,x)$$

The same conclusion is true also for the conjugate variable $\bar{x}$. The condition that
the function $I(x)$ is single valued around the points $x = 0$ and $x = 1$ fixes a specific
combination of holomorphic and anti-holomorphic parts up to a constant which in its
turn can be easily evaluated considering the special case $x = 0$. The final result is
presented in eq. (1.10).

C  Four-point functions at large $p$ limit

Since the structure constants of OPE for the minimal models are known (see A.5),
to construct the correlation functions it remains to calculate related conformal blocks.
According to AGT relation [11] this conformal blocks in a simple fashion are related to
the instanton part of the Nekrasov partition function of $N = 2$ SYM theory with the
gauge group $SU(2)$ and with four fundamental hypermultiplets. In the large $p$ limit
the minimal models approach to a free theory (the central charge $c \approx 1$), so it is not
surprising that in this limit conformal blocs of degenerated primary fields become very
simple and can be expressed in terms of rational (and also logarithmic in the cases when
the leading corrections in $1/p$ is required to be taken into account) functions of the the
cross ratio of the coordinates. It is straightforward to compute Nekrasov partition [12]
function up to desired order in instanton expansion using combinatorial formula found
in [13] and extended to the case with extra hypermultiplets in [14]. Computing the first
few coefficients of the instanton expansion (for more confidence we made calculations
up to 6th order ), adjusting appropriately the parameters in order to get the required
conformal block and finally taking the large $p$ limit one can easily guess the exact
dependence of the conformal block on the cross ratio of the insertion points (which is
the same as the instanton counting parameter, from the gauge theory point of view).
In this way we got expressions\(^3\) for the correlation functions

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n,n}(1)\phi_{n,n}(\infty) \rangle = |x|^{4\epsilon - 4} + \frac{(n^2 - 1) \epsilon^2}{12|x|^4} \left( \frac{x^2}{2(1-x)} + \frac{x^2}{2(1-\bar{x})} + \log^2 |1-x|^2 \right) + O(\epsilon^3)
\]

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n,n+1}(1)\phi_{n+1,n}(\infty) \rangle = \left[ \frac{1-x + \frac{x^2}{2}}{x^2(1-x)} \right]^2 + \frac{2(n+2)}{3n} \left| \frac{1-\frac{x}{2}}{x(1-x)} \right|^2 + O(\epsilon)
\]

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n,n-1}(1)\phi_{n,n+1}(\infty) \rangle = \frac{4\sqrt{n^2 - 1}}{3n} \left| \frac{1-\frac{x}{2}}{x(1-x)} \right|^2 + O(\epsilon)
\]

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n+2,n}(1)\phi_{n+2,n}(\infty) \rangle = \frac{1-2x + 3x^2 - 2x^3 + \frac{x^4}{3}}{x^2(1-x)^2} + \frac{8(n+3)}{3(n+1)} \left| \frac{1-\frac{3x}{2} + x^2 - \frac{x^3}{4}}{x(1-x)^2} \right|^2
\]

\[
+ \frac{(n+3)(n+4)}{18n(n+1)} \left| \frac{x}{1-x} \right|^4 + O(\epsilon)
\]

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n,n+2}(1)\phi_{n,n+2}(\infty) \rangle = \frac{4}{3} \sqrt{\frac{n+2}{n}} |x|^{-2} + O(\epsilon)
\]

\[
\langle \phi_{1,3}(x)\phi_{1,3}(0)\phi_{n,n-2}(1)\phi_{n,n+2}(\infty) \rangle = \frac{\sqrt{n^2 - 4}}{3n} \left| \frac{x}{1-x} \right|^4 + O(\epsilon)
\]

\[\text{(C.1)}\]

As a nontrivial check, we have tested the crossing invariance of all these correlation functions. Of course the interested reader can get convinced in correctness of our expressions also by examining the third order differential equation satisfied by any conformal block which includes the degenerated field \(\phi_{1,3}\) \[5\].

Performing the conformal map \(x \to 1/x\) with the help of the eq. (1.6) we get the correlation functions (2.1), (3.2), (3.20), (3.2), (3.30), (3.57), (3.72), (3.86) used in the main text.

\[\text{D} \quad \text{Integrations over lens-like regions}\]

Here we compute the contributions of the lens-like regions

\[
D_L = D_{1-l_0,0} \cap D_{l_1,1} \\
D_R = D_{1+l_0,0} \cap D_{l_1,1}
\]

\[\text{(D.1)}\]

\[\footnote{Some particular conformal blocks in large \(p\) limit have been computed earlier in \[15\] using more traditional approach.} \]
Using Green’s theorem the integrals over lens-like regions can be easily transformed to the contour integrals over their boundaries. The integrals over the arcs which belong to the circle $|x - 1| = l$ are trivial. Instead, the integrals along remaining parts of the boundary which lay on $|x| = 1 + l_0$ (for the right lens-like region $D_R$) or on $|x| = 1 - l_0$ (for the left lens-like region $D_L$) seem more complicated, but fortunately these contour integrals too (with an exception to be considered later) admit exact treatment. Below we give the details on the integration along the arc $|x| = 1 + l_0$. The formulae for the other arc $|x| = 1 - l_0$ can be found by a simple replacement $l_0 \leftrightarrow -l_0$. During the calculations we heavily employ the formulae (we use the notation $r \equiv |x - 1|$ and the angles $\phi$, $\alpha$ are depicted in Fig.2)

\[ (1 + l_0) \sin(\alpha) = \sin(\varphi); \quad r = (1 + l_0) \cos(\alpha) - \cos(\varphi) \quad \text{(D.2)} \]
\[ r^{-1} = \frac{(1 + l_0) \cos(\alpha) + \cos(\varphi)}{l_0(l_0 + 2)} \quad \text{(D.3)} \]

First note that for the arbitrary region $D$

\[
\int_D \frac{d^2x}{|x - 1|^4} = \int_{\partial D} \frac{dr d\varphi}{r^3} = -\frac{1}{2} \int_{\partial D} \frac{d\varphi}{r^2} \\
\int_D d^2x \left( \frac{1}{(x - 1)^2} + \frac{1}{(\bar{x} - 1)^2} \right) = -\int_{\partial D} \frac{\sin(2\varphi) dr}{r} \\
\int_D d^2x \left( \frac{1}{x - 1} + \frac{1}{\bar{x} - 1} \right) = -\int_{\partial D} \frac{2d\sin(\varphi)}{r} \quad \text{(D.4)}
\]

When restricted on the circle $|x| = 1 + l_0$ the one forms appearing on the r.h.s. can be
represented as total derivatives:

\[
\frac{d\varphi}{r^2} = \frac{(1 + l_0)^2 \cos^2(\alpha) + \cos^2(\varphi) + 2(1 + l_0) \cos(\alpha) \cos(\varphi)}{l_0^2(1 + l_0)^2} d\varphi
\]

\[
= \frac{(1 + l_0)^2 d(\alpha + \varphi) + (1 + l_0) d \sin(\alpha + \varphi)}{l_0^2(2 + l_0)^2}
\]

\[
\frac{\sin(2\varphi) dr}{r} = \frac{(1 + l_0)^2}{2} d(2\alpha - \sin(2\alpha))
\]

\[
\frac{2d \sin(\varphi)}{r} = d \left( \frac{(1 + l_0)^2 \alpha + \varphi + (1 + l_0) \sin(\alpha + \varphi)}{l_0(l_0 + 2)} \right)
\]

(D.5)

With these formulae at hand it is easy to evaluate the integrals over lens-like regions in the limit \(l_0/l \to 0\) and \(l \to 0\)

\[
\int_{D_L \cup D_R} \frac{d^2 x}{|x - 1|^2} \approx \left( -\frac{\pi}{l^2} - \frac{\pi}{8} \right) + \frac{\pi}{2l_0^2}
\]

(D.6)

\[
\int_{D_L \cup D_R} d^2 x \left( \frac{1}{(x - 1)^2} + \frac{1}{(\bar{x} - 1)^2} \right) \approx 2\pi
\]

\[
\int_{D_L \cup D_R} d^2 x \left( \frac{1}{x - 1} + \frac{1}{\bar{x} - 1} \right) \approx \pi
\]

(D.7)

We need also the integral

\[
\int_{\partial D} \frac{d^2 x}{|x - 1|^2} = \int_{\partial D} \log(r) d\varphi
\]

(D.8)

Unlike the previous cases this integral can not be evaluated exactly in terms of elementary functions. Nevertheless it is not difficult to show that up to terms vanishing in the limit \(l_0/l \to 0\) and \(l \to 0\) it is equal to

\[
\int_{D_L \cup D_R} \frac{d^2 x}{|x - 1|^2} \approx 2\pi \log \left( \frac{l}{2l_0^2} \right)
\]

(D.9)

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