Emergent Kalb-Ramond fields from a dimer model

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Abstract

The emergence of a Kalb-Ramond field and string charge in the lattice is discussed. The local bosonic model with rotor variables placed on the faces of a cubic lattice is considered. The coupling model consisting of the Maxwell fields and the Kalb-Ramond field is given. This construction naturally incorporates the emerging coupling between both gauge and string fields. In the process, an object that resembles to a D-brane on the lattice is introduced.
1 Introduction

Recently a great deal of work has been done in the subject of the topological phases of matter and topological order (see for instance [1, 2]). Moreover, it is now believed that the diverse types of matter and its interactions would be originated due the existence of a system formed by quantum bits of information (see [1, 3, 4] and references therein). Thus, the matter can be regarded as an emergent object coming from diverse ways of organization of some few degrees of freedom in local lattice bosonic models. In this context, there is a proposal in which electrons and photons and their interaction can emerge from qubits in a string-net liquid. Thus electrons and photons can be viewed as collective models of a string-net model [3].

Beyond the model that gives rise to electrodynamics, in turn, diverse string-net liquids can lead to different types of gauge bosons and fermions with more general properties. Thus, it was possible to find gauge bosons and fermions that behave as gluons and quarks from an appropriate string-net local qubit model. That means that QCD can be obtained as an emergent theory [5, 6]. Later, interesting generalizations to theories of gauge fields and massless fermions in any dimension and for any gauge group were also constructed [6]. In particular, the standard model of particles can be stated in this context. However, the prediction of this description implies the existence of discrete groups, which can be interpreted as cosmic strings in a very early Universe.

On the other hand, superstring theory is a theory from which it is possible as well to incorporate gauge bosons and fermions in a very different way (see, for instance [7, 8, 9]). This procedure requires the introduction of the idea of compactification. This roughly speaking implies to make compact and small the extra dimensions in order to get the theory in four dimensions. These considerations have been, by themselves, problematic and the constructions of sensible field theories requires to avoid the swampland, (for a review see for instance, [10, 11]). Thus it seems natural to search for other alternatives.

In the 90’s with the advent of string dualities and the arising of M-theory, there were some proposals involving the possible origin of the fundamental strings and their properties, as derived objects from more fundamental degrees of freedom. These degrees of freedom were the non-perturbative objects known as D0-branes. A gas of N of these D0-branes is, under certain considerations, described by Matrix N × N quantum mechanics known as Matrix Theory [12]. In AdS/CFT correspondence [13], the gravitational fields are also emergent. Gravity also can be considered an emergent interaction in matrix models of gravity [14]. For a review of different aspects of emergent gravity, see [15]. Thus in superstring theories not only fermions and gauge field emerges but in closed strings gravity arises at low energies.

The claim of deriving theories for a few degrees of freedom localized in some region of spacetime is more general in the sense that not only the diverse types of matter can be derived from the local models of qubits. The idea is that the gravitational degrees of freedom, and moreover spacetime by itself, might be obtained in this way.

With the arrival of many new techniques from Condensed Matter Physics, many efforts have been done to obtain gravitons and soft gravitons as emergent particles from lattice models [16, 17, 18], starting from a symmetric rank-2 tensors immersed on the vertices (diagonal terms) and on the faces of the lattices (off-diagonal terms).
Higher-rank symmetric tensors generalization of these works were also proposed in Refs. [19, 20, 21].

Thus, in the present article we propose a local bosonic model consisting in regarding the fundamental string and some of its properties, as the Kalb-Ramond charge, as emergent objects from a local lattice model. We will work with an anti-symmetric rank-2 tensor (see [9, 22] for a review of Kalb-Ramond fields) and, in particular, we obtain emergent Kalb-Ramond fields from a lattice model. With this purpose, we first introduce the model for electromagnetism, and then we combine both models to obtain the coupling of the Kalb-Ramond field potential to the string charge and to the electric field. Earlier lattice models incorporating Kalb-Ramond fields were proposed in [23, 24]. In [23] a Higgs mechanism for the Kalb-Ramond fields is proposed by coupling them to a string that eventually condensates. Moreover in Ref. [24] a non-abelian tensor gauge theory is implemented in the cubic lattice through the consideration of Chan-Paton colors in each boundary link.

This article is organized as follows, in Section 2 we give some preliminary material concerning some facts about the Kalb-Ramond field, the Maxwell field and their coupling. We also revisited the photon model and the partition function. Section 3 is the main part of our paper and it is devoted to propose our model of emerging Kalb-Ramond field and the string charge. Moreover, in this section we give also the lattice model of the emerging coupling of the mentioned fields, which requires the introduction of the idea of a D-brane in the lattice. Finally, in Section 4 we give our final remarks.

## 2 Preliminaries

In the present section we give some preliminaries for Section 3. Here we will introduce the notation and conventions we will follow in this article. We start by reviewing the field theory of the Kalb-Ramond field including its sources [21, 22, 23].

### 2.1 String charge density

We first review the string charge by introducing the antisymmetric Kalb-Ramond field potential $A_{\mu\nu} = -A_{\nu\mu}$ on a $(3+1)$-dimensional Minkowski spacetime and its associated field strength $F_{\mu\nu\rho}$ given by

$$F_{\mu\nu\rho} = \partial_{\mu}A_{\nu\rho} + \partial_{\nu}A_{\rho\mu} + \partial_{\rho}A_{\mu\nu}.$$  \hfill (1)

These fields have a great similarity with the Maxwell gauge field potential $A_{\mu}$ and the electromagnetic field strength $F_{\mu\nu}$ (note the ranks). In the electromagnetic theory, the electric current $j^k$ (one index $k = 1, 2, 3$) and the electric charge density $q(= j^0)$ appear in

$$\partial_{\nu}F_{\mu\nu} = j^{\mu}.$$  \hfill (2)

For the Kalb-Ramond field strength we have

$$\frac{1}{\kappa^2} \partial_{\rho}F^{\mu\nu\rho} = j^{\mu\nu},$$  \hfill (3)
where $\kappa$ is a constant needed to keep the units, and $j^{\mu\nu}$ (two indices) is an antisymmetric tensor ($j^{\mu\nu} = -j^{\nu\mu}$). The components $j^{0k} = \vec{j}^0$ are called the Kalb-Ramond charge density, or for simplicity, the string charge density. It satisfies $\nabla \cdot \vec{j}^0 = 0$, and in the case of static strings we have $j^{ik} = 0$, which is the case we adopt in our lattice model, and only $j^{0k}$ will be non-vanishing.

For static strings, we have to consider that $\partial_\rho F_{ik\rho} = 0$, and also that

$$\partial_\ell F_{0k\ell} = \kappa^2 j^{0k},$$

(4)

There is a canonical conjugate variable $\Pi^{k\ell}$ to the string field potential $A^{k\ell}$ which can be obtained as $\Pi^{k\ell} = F^{0k\ell}$ (see the 2.2 and [22]). We can also introduce a vector $\vec{B}_F$ field related to the Kalb-Ramond field strength by

$$F^{0k\ell} = \varepsilon^{k\ell m} B_F^m.$$

(5)

It is called the field strength dual to $F$, and joining these last equations we obtain

$$\varepsilon^{k\ell m} \partial_\ell B_F^m = \kappa^2 j^{0k},$$

(6)

which is like the Ampere’s law but for the string charge densities.

2.2 The Hamiltonian

Now we turn to the Hamiltonian formulation of the free static Kalb-Ramond field strength (see [9, 22, 23]), then first we have to check the Lagrangian density

$$\mathcal{L} = -\frac{1}{6\kappa^2} F^{\mu\nu\rho} F_{\mu\nu\rho},$$

(7)

which is invariant under the gauge transformations $A_{ij} \to A_{ij} + \partial_i f_j - \partial_j f_i$ since we are working in the static case. The canonical momentum conjugate to $A_{ij}$ is found to be (check [22] for details),

$$\Pi^{ij} = \dot{A}^{ij} + \partial_0 A^{0j} + \partial_j A^0i.$$

(8)

This is a two-rank antisymmetric tensor that satisfies $\dot{\Pi}^{0i} = \partial_0 \Pi^{ki}$, and the weak constraints

$$\Pi^{0i} \approx 0, \quad \partial_i \Pi^{ik} \approx 0.$$  

(9)

Thus to make the Hamiltonian density

$$\mathcal{H} = \frac{1}{4} \Pi^{ij} \Pi_{ij} + \frac{1}{2} F_{123} F_{123} + A_{j0} \partial_i \Pi^{ij} - A_{0j} \Pi^{0j}.  

(10)

The first term is analogous to the term $E^2$ for electrodynamics, and the second term for $B^2$. They can be put together in the Hamiltonian as one term like $\propto F_{ijk}^2$. The third and fourth term have been added to the Hamiltonian, so we can modify them to fit the lattice model keeping them as constraints. As can be observed from the last section, the term $\partial_i \Pi^{ij}$ is the string charge $-\kappa^2 j^{0j}$, so if there is a term with $A_{j0}$ as a coefficient, it has to be interpreted as the string charge. Depending on the gauge chosen, this term can be taken as 0, but we leave that for now.
Figure 1: Continuous and lattice D-branes. a) shows how a 0-brane is put on the lattice, b) shows a 1-brane on the lattice, and c) a 2-brane.

2.3 Branes

We would like to introduce a little about D-branes, which are going to be helpful in the section in order to have a better understanding of how the couplings occur. In the continuous case (see [25]), we have that the open strings are free on the space (in our case a 3-dimensional space), but couple to D-branes on their boundaries or endpoints. These objects are D-dimensional bodies that live also free on the space, as a matter of fact, a 0-brane is a point, and a 1-brane is a string. In this way, we can observe that a 2-brane is membrane, which is a 2-dimensional object, and so on for following dimensions.

Objects like these, can be represented on a lattice by restricting the space on which one is allowed to work inside the brane. Different examples are shown on Figure 1. Also, it is important to have in mind how a string couples to a D-brane, as can be seen in Figure 2, where one of the ends of a string is attached to a 2-brane. Closed strings do not attach to D-branes, but open strings attach to D-branes from both sides, it can be to the same D-brane or to a different one. For the coupling model, we will consider this and restrict to the case in which the string carries Kalb-Ramond charge, and only electric fields will be treated on the D-branes.
2.4 Couplings

Now, we want to present the different couplings we will encounter in the lattice model, beginning with the couples between a charge and its corresponding field, with the final objective of coupling the electric field with the string field (or Kalb-Ramond field).

So first, we remind the coupling between the gauge field potential $A_\mu$ and the electric charge density vector $j^\mu$, which is given by

$$A_\mu j^\mu$$  \hspace{1cm} (11)

in the Hamiltonian density. The conservation of the electric charge density $\partial_\mu j^\mu = 0$ is implied by the gauge invariance under the transformation

$$A_\mu = A_\mu + \partial_\mu f,$$  \hspace{1cm} (12)

as can be seen from the variation of the coupling action (see [9]). Here $f$ is an arbitrary scalar field.

For the Kalb-Ramond charge and field we have the general form of the coupling as

$$A_{\mu\nu} j^{\mu\nu},$$  \hspace{1cm} (13)

and, as in the electromagnetic case, invariance under gauge transformation

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu,$$  \hspace{1cm} (14)

where the $\lambda_\mu$'s are arbitrary vector fields, implies the conservation of string charge $\partial_\mu j^{\mu\nu} = 0$.

When the string is open, there is another type of coupling on the boundaries given between the Kalb-Ramond field and the electric field. For this matter, first we have to split the coordinates in: coordinates normal or perpendicular to the electric field ($\mu_\bot$), and coordinates along the electric field or parallel to it ($\mu_\parallel$), as $\mu = (\mu_\bot, \mu_\parallel)$. Following this, we also have to couple the gauge transformations as

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu,$$  \hspace{1cm} (15)
Figure 3: Continuous and lattice string charge attached to 2-branes. The string charge is shown in blue, while the electric field is shown in green.

\[ A_{\mu \parallel} \rightarrow A_{\mu \parallel} - \lambda_{\mu \parallel}, \quad (16) \]

where the term added for the Maxwell gauge transformation is equivalent to the equation (12).

In order to keep the string action and the Maxwell action invariant, we have to include the invariant quantity

\[ -\frac{1}{4} (F + A)_{\mu \parallel \nu} (F + A)_{\mu \parallel \nu}, \quad (17) \]

in the Hamiltonian, which, by expansion, gives rise to the term

\[ -F^{0k}A_{0k}. \quad (18) \]

This is the coupling between the electric field \( F^{0k} = E^k \), which takes the place of the string charge away from the string, and the string field potential \( A^{0k} \) (see [9, 23]).

It has to be clear that the string charge couples to the string field potential along the string (which is the place where the string charge exists). On the other hand, the electric field couples to the string field potential only through the D-brane to which the string is attached, as can be seen in Figure 3. It can also be appreciated how the string charge goes along the string, and how the electric field goes on the 2-brane.

\subsection{2.5 The photon model}

Now we will present a dimer model that gives rise to photons as emergent particles (there have been many other models, see for example [16, 17, 26, 27, 19], and many generalizations to higher order symmetric tensors, see [16, 17, 20, 21, 25]).

Let’s first introduce the quantum dimer model (QDM see [28]) with the simplest kinetic and potential energy terms written as

\[ H_{QD} = \sum_{\square} \left\{ -T_1 (|=\rangle\langle|| + h.c.) + T_2 (||\rangle \langle|| + |-=\rangle\langle=-|) \right\}, \quad (19) \]
where the summation runs over all the plaquettes (□) of the lattice, in which the plaquettes are the same as the faces.

In this model, the kinetic term $T_1$ flips pairs of nearest-neighbor parallel dimers, which are links on the lattice (see Figure 4), and the potential term $T_2$ creates a repulsion between them. This model has been used widely with different purposes (see [17, 26, 20, 29, 30, 31]), but we will mostly follow the meaning for the electrical part used in [17] with a slight variation in the notation as in [32].

We are going to define on each link $(i, \alpha)$ $(i = (i_x, i_y, i_z))$ denotes the site in which the dimer begins, and $\alpha = x, y, z$ the direction to which it grows) an number operator analog to the electric field $\hat{E}_{i\alpha}$ and its conjugate angular phase operator analog to the potential field $\hat{A}_{i\alpha}$ (as defined in [17, 26, 30, 32]), so these variables satisfy $[\hat{A}_{i\alpha}, \hat{E}_{j\beta}] = i\delta_{ij}\delta_{\alpha\beta}$. It is important to notice that the notation for the operators defined on the links implies that $\hat{E}_{i\alpha} = \hat{E}_{i+\hat{\alpha}, -\alpha}$.

There is a constraint on the system that we have to impose on the Hilbert space because it represents the discrete form of Gauss’s law for electric fields (see [16, 18, 17])

$$\nabla_\alpha \hat{E}_{i\alpha} = 0,$$

(20)

where the symbol $\nabla_\alpha$ means lattice differentiation or difference, and is defined as $\nabla_\alpha \hat{E}_{i\beta} = \hat{E}_{i+\hat{\alpha}, \beta} - \hat{E}_{i\beta}$. Because of this constraint, the low-energy Hamiltonian has to be invariant under the gauge transformation

$$\hat{A}_{i\alpha} \rightarrow \hat{A}_{i\alpha} + \nabla_\alpha f_i,$$

(21)

where $f_i$ is an arbitrary scalar field defined on the sites. This last equation is the discrete version of the equation (12).

With all this information, we have that the Hamiltonian for our system is given by

$$H_e = \frac{K_1}{2} \sum_{i\alpha} \hat{E}^2_{i\alpha} - K_2 \sum_{i\gamma} \cos(\varepsilon_{\gamma\alpha\beta} \nabla_\alpha \hat{A}_{i\beta}).$$

(22)

The notation employed is that of [32]. This Hamiltonian is the free three dimensional compact QED model (see [16]) with a deconfined photon phase (see [17, 33]). The term with coefficient $K_1$ keeps a uniform density of dimers, and the term with coefficient $K_2$ flips dimers around a plaquette. The term $\varepsilon_{\gamma\alpha\beta} \nabla_\alpha \hat{A}_{i\beta}$ is shown in Figure 5 (see [32]).
Figure 5: The argument of the cosine function in equation (22) is visible, and how it gives rise to a dual magnetic field $-B_{az}$. The positive field potentials are shown in blue, while the negative ones are shown in red. The magnetic field generated is on the opposite direction and negative, shown in green.

In order to include currents and charges in our model, we have to include a new phase angle operator $\hat{\phi}_i$ on the sites, and its conjugate number operator $\hat{n}_i$. Also, there is an important point about the Gauss’ constraint, and it is that if it is violated, some defects will be generated on the sites $i$ where $\nabla_\alpha \hat{E}_{i\alpha} \neq 0$, and these defects carry charges of the gauge field potential $\hat{A}_{i\alpha}$ (see Figure 6 and Ref. [17]). The way in which the defects couple to the gauge field potential is as a term in the Hamiltonian as follows:

$$H_{A,q} = -c_1 \sum_\alpha \cos(\partial_\alpha \phi^{(q)} - A_\alpha).$$  \hspace{1cm} (23)

Considering the constraint (20) and the gauge transformation (21) we obtain a new Hamiltonian as:

$$H_\varepsilon = \frac{K_1}{2} \sum_{i\alpha} \hat{E}_{i\alpha}^2 - K_2 \sum_{i\gamma} \cos(\varepsilon_{\gamma\alpha\beta} \nabla_\alpha \hat{A}_{i\beta}) - K_3 \sum_i \hat{n}_{i\tau} \hat{A}_{i\tau} - K_4 \sum_{i\alpha} \cos(\nabla_\alpha \hat{\phi}_i - \hat{A}_{i\alpha}).$$ \hspace{1cm} (24)

The term with coefficient $K_3$ is the coupling between the electric charge and the electric field potential. It has to be clear that this term is imposed by the Gauss’ law in equation (20), in which the term $\hat{A}_{i\tau}$ can be seen as a Lagrange multiplier defined on the vertices. The term with coefficient $K_4$ is the gauge transformation of equation (23) defined on each link. It is important to notice that for the gauge transformation inside the cosine of the $K_4$ term to be valid in equation (25), there has to be no current or $K_3 \rightarrow \infty$ as mentioned above, and we work with the low-energy physics.

We now follow the line of Ref. [32], so we write the path integral representation of the partition function by inserting the eigenstates of $\hat{E}_{i\alpha}$ at small imaginary time intervals $\Delta \tau$. The cosine term of $K_2$ can be replaced by the Villain form approximation as

$$\exp \left\{ K_2 \Delta \tau \cos(\varepsilon_{\gamma\alpha\beta} \nabla_\alpha \hat{A}_{i\beta}) \right\} \rightarrow \sum_{\{B_{a\gamma}\}} \exp \left\{ -\frac{B_{a\gamma}^2}{2K_2\Delta \tau} + iB_{a\gamma} \varepsilon_{\gamma\alpha\beta} \nabla_\alpha \hat{A}_{i\beta} \right\}. \hspace{1cm} (25)$$
Figure 6: A defect at site \( i \) when the Gauss’ constraint is violated in the \( \hat{x} \) direction. Again, blue is positive, red negative, and green the charge. One can see that the 2 in the current on the equation (28) comes from the difference of the links.

The corresponding to \( K_4 \) is given by

\[
\exp \left\{ K_4 \Delta \tau \cos(\nabla \alpha \hat{\phi}_i - \hat{A}_{i\alpha}) \right\} \rightarrow \sum_{\{j_{i\mu}\}} \exp \left\{ -\frac{j_{i\mu}^2}{2K_4 \Delta \tau} + ij_{i\alpha}(\nabla \alpha \hat{\phi}_i - \hat{A}_{i\alpha}) \right\}, \quad (26)
\]

where we have to keep in mind that the Gauss’ law along with the gauge transformation keep this term as 0.

Here, \( B_{a\gamma} \) is an integer dual magnetic field defined on the links \((a, \gamma)\) of the dual lattice (see Figure 5 and Ref. [34]). The relationship between the sites of the direct lattice (with indexes \( i, j, \ldots \)) and the sites of the dual lattice (with indexes \( a, b, \ldots \)) will not be taken into account explicitly, but it is well understood from the example of the dual magnetic field that a link in the dual lattice represents a face in the direct lattice, and sites on the dual lattice are located at the center of the cubes on the direct lattice and vice versa.

Furthermore, we define an integer dual electromagnetic tensor \( \tilde{F}_{a\mu\nu} \) on the dual lattice (as a 3-dimensional analog to that one in [32]), with its row components as

\[
\begin{align*}
\tilde{F}_{ax\nu} &= (B_{ax}, 0, E_{iz}, 0), \\
\tilde{F}_{ay\nu} &= (0, -E_{iz}, E_{ix}, 0), \\
\tilde{F}_{az\nu} &= (B_{az}, E_{ix}, E_{iy}, 0),
\end{align*}
\]

where \( B_{a,\cdot z} \) is the field generated as in Figure 5. We also define the electric current on the direct lattice as \( j_{i\mu} = (-n_i, k_{iz}, k_{iy}, k_{iz}) \), where the \( k_{i\alpha} \) can be written in terms of the variation of the defects \( \hat{\phi} \) between the vertex \( i \) and the vertex \( i + \alpha \). Note that for both the dual electromagnetic tensor, and the current, we use \( \lambda, \mu, \nu, \rho = \tau, x, y, z; \) and \( \alpha, \beta, \gamma, \delta = x, y, z \).

We are now ready to obtain the following partition function after working with the Villain form for the cosine terms as:

\[
Z_{\tilde{c}} = \sum_{\{\tilde{F}_{a\mu\nu}, j_{i\mu}\}} \exp \left\{ -\frac{e_1^2}{2} \tilde{F}_{a\mu\nu}^2 + \frac{g_1}{2} j_{i\mu} A_{i\mu} \right\}, \quad (27)
\]
restricted to
\[ \varepsilon_{\lambda\mu\nu\rho} \nabla_\mu \tilde{F}_{\alpha
u\rho} + 2 j_{i\lambda} = 0, \]
and
\[ \nabla_\lambda \tilde{F}_{a\lambda\mu} = 0, \]
where the time interval is chosen to give \( e_1^2 = K_1 \Delta \tau = \frac{1}{K_3 \Delta \tau} \), and \( g_1 = K_3 \Delta \tau = \frac{1}{K_4 \Delta \tau} \).

It is also important to observe that equations (28) and (29) give rise to \( \nabla_\lambda j_{i\lambda} = 0 \), where in every case we are using all the subindices without taking assumptions on the time differentiation, which can be taken as 0 for simplicity.

Notice that the phases obtained in this case can be compared to those obtained in Refs. [17, 26, 32], but the main difference is that we are not working with holes because our Hamiltonian does not include a term taking it into account, which means 0 hole average, as reflected on the partition function and the Gauss’ constraint.

In order to give some solutions, we use the restrictions to obtain that
\[ j_{i\lambda} = \varepsilon_{\lambda\mu\nu\rho} \nabla_\mu a_{i\nu\rho}, \]
and
\[ \tilde{F}_{a\nu\rho} = \nabla_\nu N_{a\rho} - 2 a_{a\nu\rho}. \]
These equations are like those of Ref. [32], but on 3 dimensions (+ 1 if the time is taken into account as evolution). It can be observed that the current \( j_{i\lambda} \) is generated by a field on the faces of the direct lattice \( a_{i\nu\rho} \), which is an integer field on the links of the dual lattice.

The important phase to consider for the photon model is when the term \( g_1 \to \infty \) we can see that either the the gauge \( a \) or the field potential \( A \) have to be 0, which means that there is no coupling. On the other side, the other term is always giving rise to a stable phase with a maximum on the energy when equation (31) is 0, which implies that the creation of current is directly given by the integer \( N \) variable. In Ref. [32] it is called the Higgs scalar and is defined on the vertices, but on our case it is defined on the links, so it can be taken as a scalar number quantity that gives rise to the electric and magnetic fields as this Higgs quantity fluctuates. The term \( g_1 \) serves as an order parameter between these two phases as \( g_1 > 0 \) on the phase with no couplings, to the phases on which there are couplings and currents with \( g_1 < 0 \).

### 3 The Kalb-Ramond model

Turning now to the model for the string charges which is the same lattice of the model above for electromagnetism, but now we define different variables. First, we introduce the integer number operators or boson numbers as \( \hat{n}_{i\alpha\beta} \) on each face \( i \) with the constraint \( \alpha \neq \beta \). Conjugate to these variables, we have the phase angle operators \( \hat{\theta}_{i\alpha\beta} \), defined on the same faces, related to the boson \( (\hat{n}_{i\alpha\beta}) \) creation operators by \( \hat{b}_{i\alpha\beta} \propto e^{-i\hat{\theta}_{i\alpha\beta}} \) and by the commutation relations on the face \( i \) by
\[ [\hat{n}_{i\alpha\beta}, \hat{\theta}_{i\gamma\delta}] = i \delta_{\alpha\gamma} \delta_{\beta\delta}. \]

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We also define an antisymmetric tensor $\hat{\Pi}_{i\alpha\beta} = \varepsilon_{\alpha\beta} (\hat{n}_{i\alpha\beta} - \bar{n})$ and its conjugate antisymmetric tensor $\hat{A}_{i\alpha\beta} = \varepsilon_{\alpha\beta} \hat{b}_{i\alpha\beta}$, where both are defined purely on the faces because of the antisymmetry ($\hat{\Pi}_{\alpha\beta} = -\hat{\Pi}_{\beta\alpha}$). The average density of bosons per site and face is written as $\bar{n}$.

The Hamiltonian of our system is written as follows
\begin{equation}
H_k = H_t + H_u + H_0,
\end{equation}
with
\begin{align}
H_t &= -t \sum_{i\alpha\beta\gamma\delta} \hat{b}_{i\alpha\beta}^\dagger \hat{b}_{i\gamma\delta}, \\
H_u &= u \sum_{\alpha\beta} (\hat{n}_{i\alpha\beta} - \bar{n})^2,
\end{align}
\begin{equation}
H_0 \mid_{(i,x)} = U \left( \hat{n}_{i+\frac{1}{2},\frac{1}{2}} + \hat{n}_{i+\frac{1}{2},-\frac{1}{2}} + \hat{n}_{i+\frac{1}{2},\frac{1}{2},xy} - \hat{n}_{i+\frac{1}{2},-\frac{1}{2},xy} \right)^2.
\end{equation}

The first term $H_t$ is a hopping term between the bosons that are on the nearest neighbor faces, the second term $H_u$ is a repulsive interaction between them, and the last term $H_0 \mid_{(i,x)}$ is the component along the link in the $x$ direction beginning at the site $i$ relating the bosons that are on the faces touching that link. For the term $H_0$, the terms along the $y$ and the $z$ direction are defined similarly (see Figure 7 for a reference).

This Hamiltonian is the lattice version of the equation (10) with the modifications mentioned below it. The $H_0$ term is a local constraint like the one of electromagnetism (similar to the one exposed in [16, 17, 26] for soft gravitons and linear gravity) which is given by
\begin{equation}
\nabla_{\alpha} \Pi_{\alpha\beta} = 0
\end{equation}
(with summation over the repeated index). This can take us back to the canonical momentum ($\Pi_{ij}$) conjugate to the string field potential ($A_{ij}$) in the continuous case.
which had a gauge transformation \((14)\). In our lattice system, if we impose \(U \to \infty\),
the constraint has to be kept equal to 0, and in the low-energy Hamiltonian, this
constraint imposes the Hamiltonian to be invariant under a gauge transformation
\[
\hat{A}_{i\alpha\beta} \to \hat{A}_{i\alpha\beta} + \nabla_{\alpha} \lambda_{i\beta} - \nabla_{\beta} \lambda_{i\alpha},
\]
(38)
with the \(\lambda_{i\alpha}\)'s arbitrary vector fields defined on the links.

In the low-energy physics, the \(H_t\) term of the Hamiltonian is compactified (see \([16, 17, 26]\),
and writing the \(H_u\) with the antisymmetric tensor \(\Pi_{i\alpha\beta}\), we have the
following effective Hamiltonian
\[
H_{k_{eff}} = -\tilde{t} \sum_{i\alpha,\beta \neq i\gamma,\beta} \cos(K\nabla_{[i\alpha} \hat{A}_{i\beta\gamma]}) + u \sum_{i\alpha,\beta \neq i} \hat{\Pi}_{i\alpha\beta}^2,
\]
(39)
where the term inside the cosine is related to the term \(\sim F_{123} F_{123}\) by a constant \(K\)
(a modification is shown below), and the summation is intended along the face. \(\tilde{t}\) is
a constant related to a high order perturbation of \(t/U\) derived in the compactification
process. Comparing our model to that of electromagnetism of the last section, we have
plaquettes instead of dimers for the terms inside the cosine, visibly as in Figure 5 (see
\([20]\)).

We can observe that if the equation \((37)\) is violated, we obtain the string charge
(Kalb-Ramond charge) as can be seen in the continuous case in equations \((4)\) and \((8)\).
This string charge is a defect and is taken by the string field potential \(\hat{A}_{i\alpha\beta}\). As can
be observed from the constraints \((35)\) and \((37)\), and from the behavior of the string
charge \((6)\), we can see that these defects have to travel along the links. To couple
these defects to the string field in a gauge invariant way, we need to consider the gauge
transformation \((38)\) and observe that the defects can be added by arbitrary vector
fields working there, so we have the following gauge invariant coupling term for the
Hamiltonian (see \([17]\))
\[
H_{A,j} = -c_2 \sum_{\alpha,\beta} \cos(\partial_{\alpha} \phi_{\beta}^{(j)} - \partial_{\beta} \phi_{\alpha}^{(j)} - A_{\alpha\beta}),
\]
(40)
where the $\phi^{(j)}_a$'s are interpreted as the creation operators of the string charge, and the sum is along all the links surrounding the face $\alpha\beta$.

In order to obtain a dimer model like the electromagnetic one, we first have to define the variables required for equation (40). In this way, we have the boson number operator $\hat{n}_i\dot{\alpha}$ on the link $(i\dot{\alpha})$, as well as the conjugate operator the phase angle operator $\hat{\phi}_{i\dot{\alpha}}$ defined likewise. Then, we can work with the following description:

$$H_k = \frac{K_5}{2} \sum_{i\alpha\beta} \hat{n}_{i\alpha\beta}^2 - K_6 \sum_i \cos(\nabla_x \hat{A}_{iyz})$$

$$- K_7 \sum_{i\alpha} \hat{n}_{i\alpha} \hat{A}_{i\tau\alpha} - K_8 \sum_{i\alpha\beta} \cos(\nabla_\alpha \hat{\phi}_{i\beta} - \nabla_\beta \hat{\phi}_{i\alpha} - \hat{A}_{i\alpha\beta}).$$

(41)

The term with coefficient $K_5$ of equation (41) is kept as it was, but the one with coefficient $K_6$ was modified as mentioned. The term $K_7$ uses $\hat{A}_{i\tau\alpha}$ as a Lagrange multiplier defined on the links, and we need $K_7 \rightarrow \infty$ in order to keep the Gauss-like constraint. The last term with coefficient $K_8$ is the modification for the lattice of equation (40), implied from the $K_7$ restrictions.

We now will take the Villain form approximation in the path integral representation for the $K_8$ term of equation (41). This will result in the following process

$$\exp \left\{ \frac{K_8}{2} \Delta \tau \cos(\nabla_\alpha \hat{\phi}_{i\beta} - \nabla_\beta \hat{\phi}_{i\alpha} - \hat{A}_{i\alpha\beta}) \right\}$$

$$\rightarrow \sum \exp \left\{ -\frac{\hat{J}_{i\mu\beta}}{2K_8\Delta \tau} + i \hat{J}_{i\mu\beta}(\nabla_\alpha \hat{\phi}_{i\beta} - \nabla_\beta \hat{\phi}_{i\alpha} - \hat{A}_{i\alpha\beta}) \right\},$$

(42)

where we have to take into account that both terms will sum up to 0 as well as we keep the Gauss-like constraint (37) and the gauge transformation.

Now, we can proceed by defining the string field by components as $F_{i\tau\alpha\beta} = \Pi_{i\alpha\beta}$ and $F_{i\alpha\beta\gamma} = \nabla_\alpha A_{i\beta\gamma} + \nabla_\beta A_{i\gamma\alpha} + \nabla_\gamma A_{i\alpha\beta}$, where we are working with the eigenvalues of the operators. As in the electromagnetic case, we can work with the Villain form approximation for the last term of equation (41) to obtain the partition function (see [32])

$$Z_k = \sum_{\{F_{a\mu\alpha\beta,j_{i\mu\alpha}}\}} \exp \left\{ -\frac{e^2}{2} \frac{F_{a\mu\alpha\beta}^2}{2} + \frac{g_2}{2} \hat{J}_{i\tau\alpha} A_{i\tau\alpha} \right\},$$

(43)

restricted to

$$\nabla_\alpha F_{i\mu\alpha\beta} + m \hat{J}_{i\mu\beta} = 0,$$

(44)

with conserved string charge. The index $\mu$ runs over $\tau, x, y, z$, and the other indices only on the spatial dimensions. The term $m = 2, 4$ depending on how many variables create the charge (see Figure [7]). Only the $j_{i\tau\alpha}$ components survive for the string charge, the other terms appear when we have a D-brane with $D > 1$ and with time evolution. The time interval is chosen to give $e_2^2 = K_7\Delta \tau = \frac{1}{K_8\Delta \tau}$, and $g_2 = K_7\Delta \tau = \frac{1}{K_8\Delta \tau}$.

We can make an analysis similar to the one on Ref. [17], but our interest is the stable phase on the partition function of equation (43). It is observed how the string
charge $j_{i\tau\alpha}$ appears in the last step of the process as in the electromagnetic case. The spatial components of this charge are canceled by the constraint because we are working in a quasi-static case (see Ref. [9]). Also, it has to be clear that this phase is made only of closed strings formed along the links of the direct lattice, at the links where the string charge is defined.

Here we could also do the same analysis as we did for the photon model, but it can be observed that the pure Kalb-Ramond field can be seen as a photon model but worked on a dual lattice since all the variables defined on the links for QED are now defined on the faces. So pure electrodynamics and pure anti-symmetric fields are analogous or ”dual”. A different analysis will be obtained when working on the coupling model.

### 3.1 The coupling model

Now, we can work on the coupling of the Kalb-Ramond charge with the electric field, but there will be no electric charges or currents on the system, so we will have to modify the term $K_4$ of the Hamiltonian $H_k$. For this, we go back to equations (15) and (16), where we can see that the coupling is through the potentials, and that the defects created will be along the links. Following this, we have that the way in which these defects couple to the electromagnetic field potential is:

$$H_{A,j} = -c_3 \sum_{\alpha} \cos(\phi_{(j)}^{\parallel} + A_{\alpha}),$$

(45)

where the sum is along the coordinates parallel to direction of the electric field. With its modification to fit in the lattice, we obtain the $H_e'$ for our total Hamiltonian which is

$$H_{e'} = \frac{K_{e'}}{2} \sum_{i\beta,\alpha} \tilde{E}_{i\beta,\alpha}^2 - K_{3'v} \sum_{i\beta,\alpha} \tilde{n}_{i\beta,\alpha} \tilde{E}_{i\beta,\alpha},$$

(46)

where the $K_{3'v}$ is the term that couples the electric field with the string charge. The coupling term with the electric field is only along the terms parallel to the electric field, on the D-branes as

$$H_{int} = -K_{e'} \sum_{i\beta,\alpha} \cos(\phi_{i\beta,\alpha}^{\parallel} + \tilde{A}_{i\beta,\alpha}^{\parallel}),$$

(47)

where the vertices $i\beta$ are the vertices that belong to the D-branes where the electric fields are defined.

The Hamiltonian for the Kalb-Ramond charge and fields is kept exactly as in equation (41). Then, the total Hamiltonian will be

$$H = H_k + H_{e'} + H_{int},$$

(48)

and, given that we have the modification for the electric part, we will obtain a Villain form approximation in the path integral representation for the $K_{e'}$ term as

$$\exp \left\{ K_{e'} \Delta \tau \cos(\phi_{i\beta,\alpha}^{\parallel} + \tilde{A}_{i\beta,\alpha}^{\parallel}) \right\} \rightarrow \sum \exp \left\{ iA_{i\beta,\tau\alpha}^{\parallel} \tilde{\phi}_{i\beta,\alpha}^{\parallel} + iA_{i\beta,\tau\alpha}^{\parallel} \tilde{A}_{i\beta,\alpha}^{\parallel} \right\}. $$

(49)
Here, the defect $\hat{\phi}_{ib\alpha}$ can be interpreted as the electric field, given the fact that away from the open string, the string potential couples to the electric field through a $\tau\alpha$ component. In this way, we obtain the partition function as

$$Z = \sum_{\{E_{ib\alpha\parallel}, F_{a\mu\alpha\beta}, j_{i\tau\alpha}\}} \exp \left\{ -\frac{e_1^2}{2} E_{ib\alpha\parallel}^2 - \frac{e_2^2}{2} F_{a\mu\alpha\beta}^2 + \frac{g_1}{2} E_{ib\alpha\parallel} A_{ib\tau\alpha\parallel} + \frac{g_2}{2} j_{i\tau\alpha} A_{i\tau\alpha} \right\}, \quad (50)$$

restricted to

$$\nabla_\alpha F_{i\mu\alpha\beta} + m j_{i\mu\beta} = 0, \quad (51)$$

and the Maxwell equations without charges (see equations (28) and (29)).

We have no magnetic field here because we are on a stationary case with no electric currents and charges. The string field potential couples to the electric field (on the D-brane where the electric field can be defined) and to the string charge (only along the string). In this stable phase, we have open strings attached to D-branes where electric fields are defined up to short distances (because it costs energy to keep electric fields), and we also have closed strings only with string charge. The string charge may only have a value different from 0 for the coordinates $\tau\alpha$.

Now we see that the phases on the Hamiltonian for this coupling model are diverse, but focusing on the different locations where these couplings happen we can obtain the stable phases. First, we can see a solution obtained from the restrictions for the electric fields as

$$E_{ib\alpha\parallel} = \varepsilon_{\alpha\parallel}^{\beta\mu\nu} \nabla_\beta e_{ib\mu\nu}, \quad (52)$$

where the electric field on the D-branes can be seen to be generated from fluctuations of another integer variable $b$. The current has to be conserved, and given that it travels only along the strings we have

$$j_{i\tau\beta} = \varepsilon_{\beta\mu\nu} \nabla_\mu b_{i\nu\beta}, \quad (53)$$

where the term $b$ is an integer gauge field that gives rise to the string charge. The difference between the string charge and the electric field is where they are located, as is said on the subindices. The string charge is only defined on the strings, and the electric field is only defined on the D-branes. Finally, for the restriction on the equation (51) we obtain that

$$F_{i\mu\alpha\beta} = \nabla_\mu N_{i\alpha\beta} - m \delta_{\tau\mu} b_{i\alpha\beta}, \quad (54)$$

where the term $N$ can be seen as an integer defined on the faces of the direct lattice.

We can observe that along the strings $e_1 \to \infty, g_1 = 0$, and we will be working only with the string variables. Here, the important phase is when $g_2 \to \infty$, which is when there is no coupling, so the Kalb-Ramond field potential and its conjugate field are given by the fluctuations of the integer quantity $N$, which can be considered as the Higgs variable on the faces. At the maximum for the Hamiltonian this term has to be 0, so there can exist a balance between this fluctuations and the gauge fields $b$ that generate the string charge, but this can only happen when the term $g_2$ is finite or close to 0. When this happens, the current is created by the fluctuations of the variable $b$ which also affects the creation of the string fields.
The term $g_2$ can work as an order parameter that takes the system from a stable phase with pure string fields $g_2 > 0$, to a stable phase with couplings between the string charges and the gauge potential fields $g_2 < 0$.

When we take into consideration the D-branes, we have that $g_2 = 0$, where the electric fields are generated, as has been said, by the fluctuations of an integer gauge field variable $b$. Here, the string fields are only generated by the fluctuations of the Higgs term $N$ when $g_1$ is finite. As the term $g_1 \to \infty$ there is no coupling, so the $b$ does not generate the electric field and only the string terms are generated.

## 4 Final Remarks

In the present article we have proposed a model to obtain electromagnetism (3 + 1)-dimensions from a dimer model. We have found also its corresponding partition function, finding in the way the dual electromagnetic tensor in 3+1 dimensions. The model is assumed to be quasi-static in 3 spatial dimensions, but the extensions to evolve on time have been marked along the construction of the model. It is also important to remark that we used as the source for the electric and magnetic fields the electric charges and currents instead of the field potentials (see Refs. [18, 17, 35]). This has to be taken into account when calculating the correlation functions.

In addition, we have obtained Kalb-Ramond fields and charges from a dimer model in 3+1 dimensions. We again assumed that the system is quasi-static, but it has been marked through the construction how to evolve on time and the places where the field potentials grow. Its source has been taken to be the string charge, and because the model is quasi-static, only the components that run along the strings $j_{\tau \alpha}$ can be different from 0.

In the last section, we have been able to couple both models into one 3+1-dimensional dimer model, this time taking as the source the string charges. We have also pointed out that the coupling of the string charges with the string field potentials has to be along the strings (open or closed), while the coupling of the string field potential with the electric field has to be on the D-branes attached at the endpoints of the open strings.

In Ref. [35], the correlation functions were obtained using Monte Carlo simulations. It would be interesting to apply these methods within this context and compute some quantum observables. We would like for the near future to search for a relation to the results involving Kalb-Ramond fields [23, 24]. Moreover, the model of the antisymmetric field studied in the present article could be coupled to the symmetric model (as in Refs. [16, 18, 17]) as a solution to the problem of finding linearized gravity as an emergent theory from lattice models. The resulting model would be interesting in the study of gravitational models with torsion [36]. It is also of our interest to study the possible relationship of our results in the context of fermion-fermion duality [37]. Some of these issues will be reported elsewhere.
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