The Inductive Kernels of Graphs.

Abstract. It is well known that kernels in graphs are powerful and useful structures, for instance in the theory of games. However, a kernel does not always exist and Chvátal proved in 1973 that it is an NP-Complete problem to decide its existence. We present here an alternative definition of kernels that uses an inductive machinery: the inductive kernels. We prove that inductive kernels always exist and a particular one can be constructed in quadratic time. However, it is an NP-Complete problem to decide the existence of an inductive kernel including (resp. excluding) some fixed vertex.

Introduction.

First, let us recall the notion of kernel. A kernel $K$ of a directed graph $G = (V, E)$ is a stable set of vertices (i.e., no pair of vertices in $K$ is linked by an arc) and such that every vertex $x \in V$ belongs to $K$ or is at distance one of $K$. Here, we will adopt the signification that there is an arc going from $K$ to $x$. Such a kernel does not always exist. For example, a directed cycle of odd length cannot have a kernel. In [1], Chvátal proved that deciding the existence of a kernel is an NP Complete problem. Nevertheless, when a directed graph $G$ has a kernel, one can easily define some strategy for the games related to $G$. Another application is the axiomatization of theories: kernels represent independent propositions that enable to prove all the ones. We are interested in this context to extend the notion of kernels in a way that guarantees their existence and still enable to define an axiomatization. A natural method consists in the introduction of a recursion scheme. The initial lacks of kernels are compensated by an inductive machinery.

Definition. (Inductive Kernel). Let $G = (V, E)$ be a directed graph where $V$ is an ordered set $(x_0, x_1, \ldots, x_{n-1})$. An inductive kernel of $G$ is a set $K \subseteq V$ such that

a. $K$ is stable, i.e., $x \neq y$ and $x \in K$ and $y \in K \implies (x, y) \notin E$

b. the following algorithm will color every vertex $x$ of $V$:

0. Color every vertex $x \in K$

1. For $i$ from 0 to $n - 1$ do if $x_i$ is colored then

   color each $y$ such that $(x_i, y) \in E$

Let us give an example. An inductive kernel for the following graph is $K = \{x_1, x_5\}$. The coloring process is represented by marking vertices $x_i$ for $i = 0, 1, 2, 3, 4, 5$ and coloring $y$ when $x_i$ is colored and $(x_i, y)$ is an arc:
Observe that if a graph $G$ admits a kernel $K$, then $K$ is also an inductive kernel of $G$. The converse is not true. For example:

This example also gives a justification for the terminology "inductive kernel" which can remind the recurrence scheme: in order to prove a property $P$ for every $n \in \mathbb{N}$, just prove $P(0)$ and implications $P(i) \implies P(i+1)$. In a more general way, an inductive kernel corresponds to an independent set of axioms in a theory where propositions are represented by vertices and logical implications are represented by arcs. Now, the generalized recursive scheme is: prove $P(x)$ for every $x$ in the inductive kernel and for $i = 0, 1, \ldots$, if $P(i)$ is proved then deduce all possible $P(j)$ where $P(i) \implies P(j)$, otherwise skip $P(i)$ that will be proved latter.

Now we prove the existence of these inductive kernels.

**Theorem 1. (Existence).** For every $n \geq 0$, every directed graph $G = (V, E)$ with $V = (x_0, x_1, \ldots, x_{n-1})$ has an inductive kernel $K$.

**Proof.** First, one can assume that $G$ is irreflexive since arcs $(x, x)$ do not appear in the conditions. We use an induction on $e = |E|$.

0. For $e = 0$ then $K = V$ is an obvious solution.

1. Assume $e > 0$ and there is an arc $(x_a, x_b)$ with $a < b$. Removing this arc, one obtains a new graph $G'$ and by induction hypothesis, $G'$ has an inductive kernel $K'$. 

1.1. If \( x_a \notin K' \) or \( x_b \notin K' \), then \( K = K' \) is also an inductive kernel for \( G \).

1.2. If \( x_a \in K' \) and \( x_b \in K' \), then take \( K = K' \setminus \{x_b\} \). During the process for \( G \), at step \( i = a \), the vertex \( x_b \) will be colored via the arc \((x_a, x_b)\). Since \( a < b \), the process for \( G \) will color every vertices.

2. Assume \( e > 0 \) and there is no arc \((x_a, x_b)\) with \( a < b \). Hence, every arc has the form \((x_b, x_a)\) with \( b > a \). Remove an arc \((x_b, x_a)\) where \( a \) is taken minimal. Hence, there is no arc \((x_a, y)\). One obtains by induction hypothesis a graph \( G' \) with an inductive kernel \( K' \).

2.1. If \( x_a \notin K' \) or \( x_b \notin K' \), then \( K = K' \) is also an inductive kernel for \( G \).

2.2 If \( x_a \in K' \) and \( x_b \in K' \), then take \( K = K' \setminus \{x_a\} \). During the process for \( G \), at step \( i = b \), the vertex \( x_a \) will be colored. During the process for \( G' \), the colored vertex \( x_a \in K' \) contributed to no other coloring. Hence, the process for \( G \) will also color every vertices.

\[ \text{\begin{center}
\begin{tikzpicture}
  \node[shape=circle,draw=red] (A) at (2,0) {\text{YES}};
  \node[shape=circle] (B) at (0,0) {\text{NO}};
  \node[shape=circle] (C) at (4,0) {\text{YES}};
  \node[shape=circle] (D) at (0,2) {\text{YES}};
  \node[shape=circle] (E) at (2,2) {\text{YES}};
  \node[shape=circle] (F) at (4,2) {\text{YES}};

  \draw[blue, thick, ->] (A) to (B);
  \draw[blue, thick, ->] (B) to (C);
  \draw[blue, thick, ->] (C) to (D);
  \draw[blue, thick, ->] (D) to (E);
  \draw[blue, thick, ->] (E) to (F);
\end{tikzpicture}
\end{center} \]

Observe that an inductive kernel \( K \) of \( G \) may be not one for \( G' \) obtained by a permutation of vertices : if one reverses the order of vertices in the previous example, \( K = \{x_0\} \) is not an inductive kernel anymore. However when \( K \) is a kernel, it is an inductive kernel for every reordering of vertices.

\textbf{Computational Complexities.}

First, we are going to show how to construct a particular inductive kernel in polynomial time.

\textbf{Theorem 2.} (generic). There exists a polynomial time algorithm that constructs an inductive kernel \( K \) of a given directed graph \( G = (V, E) \).

\textbf{Proof.} The method is directly based on the proof of existence.

0. Begin with \( K = V \)

1. For \( j = n, n - 1, \ldots, 1 \) do for \( i = n, n - 1, \ldots, j + 1 \) do

if \((x_i, x_j) \in E \) and \( x_i \in K \) and \( x_j \in K \) then remove \( x_j \) from \( K \)
2. For $i = 1, 2, \ldots, n$ do for $j = i + 1, i + 2, \ldots, n$ do
   if $(x_i, x_j) \in E$ and $x_i \in K$ and $x_j \in K$ then remove $x_j$ from $K$
Return $K$

Hence, it is easy to compute an inductive kernel of a given graph $G$. However, the problem becomes NP-Complete if one expects some vertex to be or not to be in the inductive kernel. In order to prove this result, we begin to introduce some tool.

**Definition. (gadget).** Given three vertices $x, x', x''$, the *gadget* $g(x, x', x'')$ is the directed graph:

```
X  X'  X''
```

Let us make several remarks that will be useful for the next proof:

- every inductive kernel $K$ of a directed graph $G$ that contains such a gadget $g(x, x', x'')$ as a subgraph can contain at most one of the vertices $(x, x', x'')$ because these vertices are pairwise linked.
- if $x \in K$, then $x$ will color $x''$ that will color $x'$.
- if $x' \in K$, then $x'$ will color $x$ and $x''$.
- if $x'' \in K$, then $x''$ will color $x'$ but not $x$. Hence $x$ must receive another arc.

**Theorem 3. (include).** The problem to decide if a directed graph $G$ has an inductive kernel $K$ including $x_0$ is NP-complete.

**Proof.**

Obviously, this problem is in NP: chose a subset $K \subseteq V$ with $x_0 \in K$ and check the conditions (in polynomial time).

To prove that it is an NP-Hard problem, we reduce the SAT problem to it. Let $\Phi$ be a conjunction of $m$ clauses $C_1, \ldots, C_m$ on $n$ variables $z_1, \ldots, z_n$. Construct a directed graph $G = (V, E)$ with $V = (x_0, C_1, \ldots, C_m, z_1, z'_1, z''_1, \ldots, z_n, z'_n, z''_n)$ and the arcs are for every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$:

\[
\begin{align*}
&\{(C_j, x_0)\} \\
&\{(z_i, z''_i), (z'_i, z''_i), (z''_i, z_i), (z'_i, z_i)\} \quad \text{arcs of the gadget } g(z_i, z'_i, z''_i) \\
&\{(z_i, C_j)\} \quad \text{if } z_i \in C_j \\
&\{(z'_i, C_j)\} \quad \text{if } z'_i \in C_j
\end{align*}
\]

We claim that $\Phi$ is satisfiable if and only if $G$ has an inductive kernel $K$ with $x_0 \in K$. 

4
First, given a truth assignment \( T \) of the variables for the satisfaction of \( \Phi \), the set
\[
K = \{ x_0 \} \cup \{ z_i : T(z_i) = \text{true} \} \cup \{ z'_i : T(z_i) = \text{false} \}
\]
is an inductive kernel of \( G \) : by the properties of gadgets, each \( z_i \in K \) will color \( z''_i \) and after \( z''_i \) will color \( z'_i \). Moreover, \( z_i \) will color the clauses \( C_j \) that contain \( z_i \). Conversely, each \( z'_i \in K \) will color \( z_i \) and \( z''_i \) and the clauses \( C_j \) that contain \( \overline{z_i} \). By definition of \( T \), all the vertices will be colored.

For the other direction, assume there is an inductive kernel \( K \) with \( x_0 \in K \). By stability, no \( C_j \) can belong to \( K \). Hence, one must take in \( K \) some vertex \( z_i \) or \( z'_i \) that enables to color \( C_j \). Moreover, one must have exactly one of the vertices \( z_i \) or \( z'_i \) or \( z''_i \) in \( K \). Since \( z_i \) receives no other arc than \( (z'_i, z_i) \), the third case \( z''_i \in K \) is not possible. That defines a truth assignment for the satisfaction of \( \Phi \) : if \( z_i \in K \) then \( T(z_i) = \text{true} \), if \( z'_i \in K \) then \( T(z_i) = \text{false} \).

One could think that the previous problem is difficult because \( x_0 \) is the first vertex in \( G \) in the ordering of vertices. However, the dual problem is still difficult.

**Theorem 4. (exclude).** The problem to decide if a directed graph \( G \) has an inductive kernel \( K \) excluding \( x_0 \) is NP-complete.

**Proof.**

Obviously, this problem is in NP : chose a subset \( K \subseteq V \) with \( x_0 \notin K \) and check the conditions (in polynomial time).

To prove that it is an NP-Hard problem, we reduce the previous problem to it.

Given a graph \( G \) with vertices \( (X_0, \ldots, X_{p-1}) \), we construct a graph \( G' \) with vertices \( (x_0, X_0, \ldots, X_{p-1}) \) such that \( G \) has an inductive kernel \( K \) with \( X_0 \in K \) if and only if \( G' \) has an inductive kernel \( K' \) with \( x_0 \notin K' \). The construction just consists in adding to \( G \) a new vertex \( x_0 \) and a new arc \( (X_0, x_0) \).

If \( G \) has an inductive kernel \( K \) that contains \( X_0 \), then \( K \) is also an inductive kernel for \( G' \) that does not contain \( x_0 \) (by stability).

If \( G' \) has an inductive kernel \( K' \) that does not contain \( x_0 \), then \( K' \) must contain \( X_0 \) since the only possibility to color \( x_0 \) is via the arc \( (X_0, x_0) \). Hence, \( K' \) is also an inductive kernel for \( G \) that contains \( X_0 \).

**Conclusion.**

There are other variants of kernels that always exist. For instance, a semi-kernel is a stable set of vertices such that every vertex \( x \in V \) is at distance at most two of \( K \). In 1974, V. Chvátal and L. Lovász proved that such a semi-kernel always exist [2]. Perhaps they defined this variant of kernels with the same motivations than in this paper. However, the notion of semi-kernel is
different of inductive kernels. For example, here is an inductive kernel which is not a semi-kernel:

[Diagram of an inductive kernel]

and here is a semi-kernel which is not an inductive kernel:

[Diagram of a semi-kernel]

However, like for a kernel (when it exists), one can find an ordering of vertices such that a semi-kernel $K$ becomes an inductive kernel. Just take an ordering $\prec$ such that $x \prec y$ when the distance of $x$ from $K$ is strictly smaller than the distance of $y$ from $K$.

At last, we must point out that an inductive kernel is not intrinsically a graph but depends on the chosen order of vertices. We conjecture that a subset of vertices is an inductive kernel for every possible orders if and only if it is also a kernel.

References.

[1] V. Chvátal, *On the computational complexity of finding a kernel*, Report No. CRM-300 (1973), Centre de Recherches Mathématiques, Université de Montréal.

[2] V. Chvátal and L. Lovász, *Every directed graph has a semi-kernel*, edts C. Berge and D.K. Ray-Chaudhuri, Lecture Notes in Math, Springer-Verlag (1974).

Serge Burckel.
INRIA-LORIA,
Campus Scientifique
BP 239
54506 Vandoeuvre-lès-Nancy Cedex
serge.burckel@loria.fr