RATIONAL CURVES IN THE LOGARITHMIC MULTIPLICATIVE GROUP

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Abstract. The logarithmic multiplicative group is a proper group object in logarithmic schemes, which morally compactifies the usual multiplicative group. We study the structure of the stacks of logarithmic maps from rational curves to this logarithmic torus, and show that in most cases, it is a product of the logarithmic torus with the space of rational curves. This gives a conceptual explanation for earlier results on the moduli spaces of logarithmic stable maps to toric varieties.

1. Introduction

It is a basic fact that a rational function on a \( \mathbb{P}^1 \) is determined up to multiplicative constant by the locations and orders of its zeroes and poles. A combinatorial exercise shows likewise that a balanced piecewise linear function on a tropical tree is determined up to addition of a constant by its slopes along unbounded edges and exists provided the sum of these slopes is zero.

In this note, we point out that both these facts are aspects of a single phenomenon in logarithmic geometry: the space of pre-stable genus 0 maps to the logarithmic torus is a logarithmic torus bundle over the space of genus 0 curves. See Corollary 5 and Corollary 9 for the precise statements of the main results.

The logarithmic torus, defined precisely below, is a non-representable functor on logarithmic schemes that compactifies the algebraic torus. Despite its failure to be representable, one can make sense of its tropicalization as the undivided real line \( \mathbb{R} \), and the fiber of its tropicalization map as the multiplicative group \( \mathbb{G}_m \). Complete toric varieties arise by pullback from subdivisions of the tropicalizations of products of copies of \( \mathbb{G}_{\log} \).

The space of logarithmic stable maps to toric varieties has been studied before, for instance in \([2,3,8]\). In each case, careful polyhedral arguments play a role in determining the geometry of the space of maps to a toric variety. Such arguments are of course necessary in order to obtain the precise geometric descriptions obtained in op. cit., however the result presented here suggests a simple underlying principle behind those results (whose price is to work with a non-representable functor on logarithmic schemes). For instance, the space of logarithmic stable maps to \( \mathbb{P}^1 \) with fixed contact at \( n \geq 3 \) marked points is a toroidal modification of \( \overline{\mathcal{M}}_{0,n} \times \mathbb{P}^1 \), which is a key result in \([2]\). We show here that the analogous space of maps to \( \mathbb{G}_{\log} \) is \( \overline{\mathcal{M}}_{0,n} \times \mathbb{G}_{\log} \), which provides a clear conceptual reason for this result. Analogous statements can be extracted regarding the results of \([2,3]\). For the latter, see Section 4 which studies maps between logarithmic tori.

We note that an incarnation of \( \mathbb{G}_{\log} \) exists with B. Parker’s theory of exploded manifolds, as the exploded manifold attached to the ‘fan’ in \( \mathbb{R} \) with a single non-strictly convex cone equal to \( \mathbb{R} \), see \([7, \text{Section 3}]\).

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2. THE LOGARITHMIC TORUS

The logarithmic multiplicative group seems to have been introduced by Kato [5]. For any logarithmic scheme \( S \) with logarithmic structure \( M_S \), we define \( G_{\log}(S) \) by (1):

\[
G_{\log}(S) = \Gamma(S, M_S^{gp})
\]

This is a contravariant functor on logarithmic schemes. It is not representable by a scheme or even an algebraic stack with a logarithmic structure, but it does have a logarithmically étale cover by \( \mathbb{P}^1 \) (with its toric logarithmic structure). In similar fashion, \( G_n^{\log} \) admits an étale cover by any toric variety of dimension \( n \).

2.1. If the test scheme \( S \) has trivial logarithmic structure, then \( G_{\log} \) coincides with the familiar multiplicative group \( G_m \). Indeed, the sheaf \( M_{gp}^S \) is often thought of as the sheaf of “nearly invertible” or “logarithmically invertible” functions. The functor \( G_{\log} \) is thus a group object that compactifies the multiplicative group. Of course, such a compactification is impossible in the category of schemes.

2.2. Given a test logarithmic scheme \((S, M_S)\) one may also consider \( G_{trop}(S) = \Gamma(S, M_{gp}^S) \), where \( M_S \) is the characteristic sheaf. This is the tropicalization of \( G_{\log} \) and can be identified with the real line, together with its group structure.

3. CURVES IN THE LOGARITHMIC TORUS

3.1. Preliminaries. In the sequel, we use the tropical interpretation of the structures underlying logarithmic schemes. Each logarithmic scheme \( Y \) comes equipped with a polyhedral complex \( Y_{trop} \) known as its tropicalization [4, Appendix B]. Sections of the characteristic abelian sheaf \( M_{gp}^Y \) are identified with piecewise linear functions on \( Y_{trop} \), see [1, Remark 7.3].

3.2. The space of maps. Let \( \mathfrak{M} \) denote the stack of prestable genus 0 logarithmic curves. If \( X \) is a category fibered in groupoids over logarithmic schemes, we denote by \( \mathfrak{M}(X) \) the stack of logarithmic pre-stable maps from genus 0 logarithmic curves to \( X \).

\[
\mathfrak{M}(X) = \left\{ (S, C, \xi) \mid C \text{ is a proper logarithmic curve over } S \text{ and } \xi \in X(C) \right\}
\]

This applies in particular to \( X = G_{\log} \). Since \( G_{\log} \) has a group structure, \( \mathfrak{M}(X) \) is a sheaf of abelian groups over \( \mathfrak{M} \) in the étale topology.

3.3. Contact orders. Let \( C \) be a logarithmic curve. A map \( C \to G_{\log} \) is a section of \( M_C^{gp} \), which in turn induces a section \( \overline{\pi} \) of \( \overline{M}_C^{gp} \). We regard \( \overline{\pi} \) as a linear function on the dual graph of \( C \). The slopes of \( \overline{\pi} \) on the \( n \) infinite legs of the dual graph of \( C \) are locally constant in \( \mathfrak{M}(G_{\log}) \). This gives a homomorphism \( \mathfrak{M}(G_{\log}) \to \mathbb{Z}^n \).

3.4. Maps up to translation. The kernel consists of maps \( C \to G_{\log} \) whose associated linear function has zero slope on the infinite legs. But such a function is effectively a bounded linear function on the complement of the infinite legs in the dual graph. Any such linear function \( \overline{\pi} \) is constant. In that case, \( O_C(\overline{\pi}) \) is the pullback of \( O_S(\overline{\pi}) \) from the base and the map \( C \to G_{\log} \) corresponds to a trivialization of this bundle. Indeed, the fiber of \( M_C^{gp} \) over \( \overline{\pi} \in \overline{M}_C^{gp} \) is \( O_X(-\overline{\pi}) \), by definition, so a section of \( M_C^{gp} \) in the fiber over \( \overline{\pi} \) corresponds to a nowhere vanishing section of \( O_X(\overline{\pi}) \). Since \( X \) is proper over \( S \) with reduced and connected fibers, all sections of \( O_X(\overline{\pi}) \) over \( X \) are pulled back from sections of \( O_S(\overline{\pi}) \). Thus a section
over $S$ of the kernel of $\mathcal{M}(G_{\log}) \to \mathbb{Z}^n$ consists of pairs $(\overline{\alpha}, \alpha)$ where $\overline{\alpha}$ is a section of $\mathcal{M}^\text{gp}\to S$ over $S$ and $\alpha$ is a section of $\mathcal{O}_S(\overline{\alpha})$. This shows that the kernel is isomorphic to $G_{\log}$.

**Theorem 1.** Let $\mathbb{Z}^n_0$ be the subgroup of $\mathbb{Z}^n$ consisting of those $n$-tuples of integers whose sum is zero. There is an exact sequence of sheaves (in the big étale site) of abelian groups over $\mathcal{M}$:

$$0 \to G_{\log} \to \mathcal{M}(G_{\log}) \to \mathbb{Z}^n_0 \to 0$$

**Proof.** Note that $\mathcal{M}(G_{\log}) \to \mathbb{Z}^n$ takes values in $\mathbb{Z}^n_0$ because every section of $M_X^{\text{gp}}$ over a rational curve $X$ induces a linear section of $\mathcal{M}^\text{gp}_X$, which is to say that the sum of the outgoing slopes at any vertex of the dual graph is zero. This implies that the sum of the outgoing slopes along the infinite legs is also zero.

We have already proved the left exactness in the statement of the theorem. To conclude we must prove that $\mathcal{M}(G_{\log}) \to \mathbb{Z}^n_0$ is a smooth surjection.

We consider the smoothness first. Since $\mathbb{Z}^n_0$ is étale over $\mathcal{M}$, it is equivalent to demonstrate that $\mathcal{M}(G_{\log})$ is smooth over $\mathcal{M}$. Consider a first order deformation of a logarithmic curve $C \subset C'$ and a section $\alpha$ of $M'_C$. Let $\overline{\alpha}$ be the image of $\alpha$ in $\mathcal{M}^{\text{gp}}_C$. Then $\overline{\alpha}$ extends uniquely to $C'$ since $\mathcal{M}^{\text{gp}}_C = \mathcal{M}^{\text{gp}}_{C'}$ when their étale sites are identified. We can view $\alpha$ as a trivialization of $\mathcal{O}_C(\overline{\alpha})$ and we wish to extend this to a trivialization of $\mathcal{O}_{C'}(\overline{\alpha})$. The obstructions to doing so lie in $H^1(C, \mathcal{O}_C(\overline{\alpha}))$. But $\overline{\alpha}$ is a linear function on the tropicalization of $C$, so $\mathcal{O}_C(\overline{\alpha})$ has multidegree $0$. As $C$ is a tree of rational curves, this implies $H^1(C, \mathcal{O}_C(\overline{\alpha})) = 0$.

To prove the surjectivity, we fix a genus 0 logarithmic curve $C$ with tropicalization $\Gamma$ and a vector $\sigma \in \mathbb{Z}^n_0$. We can construct a unique linear function $\overline{\sigma}$ on $\Gamma$ whose slopes on the legs of $\Gamma$ are given by $\sigma$. To lift this section to an element of $\mathcal{M}(G_{\log})$ in the fiber over $\sigma$, we must give a nowhere vanishing section of $\mathcal{O}_C(\overline{\sigma})$. But $\mathcal{O}_C(\overline{\sigma})$ has multidegree $0$ and the components of $C$ are rational, so $\mathcal{O}_C(\overline{\sigma}) \simeq \mathcal{O}_C$ has a nowhere vanishing section. \qed

**Corollary 2.** Let $\mathcal{M}_{\Gamma+1}(G_{\log})$ denote the moduli space of $1$-marked genus 0 pre-stable maps to $G_{\log}$ with combinatorial data $\Gamma$ and one additional marked point of contact order 0. Then evaluation at the final marked point furnishes an isomorphism $\mathcal{M}_{\Gamma+1}(G_{\log}) \simeq \mathcal{M}_{\Gamma+1} \times G_{\log}$.

**Corollary 3.** If $n \geq 3$ then the exact sequence (3) splits and $\mathcal{M}_{0,n}(G_{\log}) \simeq \mathcal{M}_{0,n} \times G_{\log} \times \mathbb{Z}^n_0$.

**Proof.** For $n \geq 3$, let $X$ be a logarithmic curve over $S$, let $\alpha \in \Gamma(X, M_X^{\text{gp}})$ give an $S$-point of $\mathcal{M}_{0,3}(G_{\log})$, and let $x$ be a marked point of $X$, viewed as a non-logarithmic section over $S$. Write $\overline{\alpha}$ for the image of $\alpha$ in $\mathcal{M}^{\text{gp}}_S$. Then $x^{-1}\mathcal{M}_X = N \times \mathcal{M}_S$, canonically, and $x^*\overline{\alpha} = (\overline{\alpha}(x), c(x))$ where $c$ denotes the contact order of $\alpha$ at $x$. We can view $x^*\alpha$ as a nowhere vanishing section of $\mathcal{O}_S((\overline{\alpha}(x), c(x))) = \mathcal{O}_S(\overline{\alpha}(x)) \otimes N_{x/X}^{\otimes c(x)}$. But, as $n \geq 3$, the universal tangent line $N_{x/X}$ is canonically isomorphic to the line bundle associated to a boundary divisor, so the $M_X^{\text{gp}}$-torsor associated to $N_{x/X}^{\otimes c(x)}$ is canonically trivial. Likewise the $M_S^{\text{gp}}$-torsor associated to $\mathcal{O}_S(\overline{\alpha})$ is canonically trivial, so $x^*\alpha$ is thus identified with a section of $G_{\log}$ and we obtain a morphism $\phi : \mathcal{M}(G_{\log}) \to G_{\log}$. It follows from the canonical isomorphism (4)

$$\mathcal{O}_S(\overline{\alpha}(x) + \overline{\alpha}'(x)) \otimes N_{x/X}^{\otimes c(x)+c'(x)} = \mathcal{O}_S(\overline{\alpha}(x)) \otimes N_{x/X}^{\otimes c(x)} \otimes \mathcal{O}_S(\overline{\alpha}'(x)) \otimes N_{x/X}^{\otimes c'(x)}$$

that $\phi$ is a homomorphism with respect to the group structure of $\mathcal{M}(G_{\log})$. It is immediate that this homomorphism splits the inclusion of $G_{\log}$ in $\mathcal{M}(G_{\log})$, and therefore that $\mathcal{M}_{0,n}(G_{\log}) \simeq \mathcal{M} \times \mathbb{Z}^n_0 \times G_{\log}$. \qed
Warning 4. The proof of Corollary 3 gives a canonical splitting of the extension (3), for each of the \( n \) marked points of the curve. These splittings genuinely depend on the markings and distinct markings give distinct splittings.

Let \( \mathcal{M}_\Gamma(G_{\log}) \) denote genus 0 stable maps to \( G_{\log} \) with fixed contact orders \( \Gamma \), where stability means that if \( \alpha \in \Gamma(X, M^p_X) \) is constant on a component of \( X \) then that component has at least 3 special points.

**Corollary 5.** Fix a vector of \( n \) contact orders and genus 0 in the combinatorial datum \( \Gamma \). We have isomorphisms:

\[
\mathcal{M}_\Gamma(G_{\log}) \simeq \begin{cases} 
\emptyset & n \leq 1 \\
\mathcal{M}_{0,n} \times G_{\log} & n \geq 3
\end{cases}
\]

**Proof.** The statement for \( n \leq 1 \) is immediate because there are no nonconstant linear functions on a genus 0 tropical curve with only one infinite leg and for \( n \geq 3 \) is an immediate consequence of Corollary 3.

We have not included a statement for \( n = 2 \) because the space ‘stable’ maps from 2-marked rational curves with contact orders \((r, -r)\) to \( G_{\log} \) is the nonseparated stack \((G_{\log}/\mathbb{G}_m) \times \text{Bj}_r\).

The difficulty is that the unique semistable component of \( P^1 \) is ‘contracted’ by any morphism \( P^1 \to G_{\log} \) and it is therefore necessary to contract it in the source to obtain a reasonable parameter space. We explain how this works in the next section.

4. Maps between logarithmic tori

The following lemma is well known, but we include a proof for completeness.

**Lemma 6.** Let \( S \) be a scheme. Then \( \mathcal{O}_S[t, t^{-1}]^* = \mathcal{O}_S^* \times t^{\mathbb{Z}}_S \).

**Proof.** The assertion is straightforward to check when \( S \) is integral. Let \( \alpha \) be a section of \( \mathcal{O}_S[t, t^{-1}]^* \). For each point \( p \) of \( S \), we have \( \alpha(p) = u(p)t^{n(p)} \) for some \( u \in k(p)^* \) and \( n \in \mathbb{Z} \). Let \( S_n \) be the set of points \( p \) of \( S \) where \( n(p) = n \). It is a quick exercise to see that \( S_n \) contains a constructible neighborhood of each of its points. As valuation rings are integral, it follows that \( S_n \) is also stable under generization, so each \( S_n \) is open.

This implies that \( n(p) \) is a constructible function on \( S \). Therefore \( t^{-n} \) is a section of \( t^{\mathbb{Z}}_S \) and \( \alpha(p)t^{-n(p)} \in k(p)^* \) for every \( p \in X \). This implies \( \alpha(p) \in \mathcal{O}_S^* \), as required.

**Proposition 7.** Let \( t \) denote the identity function on \( G_{\log} \). Any map \( S \times G_{\log} \to G_{\log} \) can be represented uniquely as \( \alpha t^n \) where \( \alpha \) is a section of \( M_{S^\text{gp}}^\text{gp} \) and \( n : S \to \mathbb{Z} \) is a locally constant function.

**Proof.** Suppose that \( \beta : S \times G_{\log} \to G_{\log} \) is a map. There is a map \( P^1 \to G_{\log} \) (where \( P^1 \) has its toric logarithmic structure) given in local coordinates by \((x, y) \mapsto x^{-1}y \). Restricting \( \beta \) along this map, we obtain a section \( \beta' \in \Gamma(S \times P^1, M_{S \times P^1}^\text{gp}) \). We note that \( M_{S \times P^1}^\text{gp} \) is the maximal extension of \( M_{S \times \mathbb{G}_m}^\text{gp} \) (where \( \mathbb{G}_m \) has the trivial logarithmic structure) to \( P^1 \).

Therefore \( M_{S \times P^1}^\text{gp} = j_*M_{S \times \mathbb{G}_m}^\text{gp} \).

We have an exact sequence (5):

\[
0 \to \mathcal{O}_{S \times \mathbb{G}_m}^* \to M_{S \times \mathbb{G}_m}^\text{gp} \to q^{-1}M_S^\text{gp} \to 0
\]
Pushing forward to $S$ and applying Lemma 6, we obtain (6):

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}_S^* & \rightarrow \ M_{S}^{\gp} & \rightarrow \ M_{S}^{\gp} & \rightarrow 0 \\
0 & \rightarrow & \mathcal{O}_S^* \times t^Z_S & \rightarrow q_\ast M_{S \times \mathbb{P}^1}^{\gp} & \rightarrow q_\ast q^{-1} M_S^{\gp} & \rightarrow 0 \\
\end{array}
\]

(6)

It follows that there is an exact sequence (7),

\[
0 \rightarrow M_{S}^{\gp} \rightarrow q_\ast M_{S \times \mathbb{P}^1}^{\gp} \rightarrow t^Z_S \rightarrow 0
\]

that is split by the pullback of $t$ along the second projection $S \times \mathbb{P}^1 \rightarrow \mathbb{G}_{\log}$. We therefore have $q_\ast M_{S \times \mathbb{P}^1}^{\gp} = M_{S}^{\gp} \times t^Z_S$. In particular, $\beta'$ can be represented uniquely as $\alpha t^n$ where $\alpha \in M_{S}^{\gp}$ and $n : S \rightarrow \mathbb{Z}$ is a locally constant function.

To see that this formula actually describes $\beta$, consider $\beta^{-1} \alpha^{-1} t^{-n}$. This is now a map $S \times \mathbb{G}_{\log} \rightarrow \mathbb{G}_{\log}$ whose restriction to $S \times \mathbb{P}^1$ is trivial. Let $f : T \rightarrow S \times \mathbb{G}_{\log}$ be any map. Then $f^{-1}(S \times \mathbb{P}^1)$ is a logarithmic modification $p : T' \rightarrow T$ and $p^* f^* \beta = 1$ by construction. But $\Gamma(T, M_{T}^{\gp}) \rightarrow \Gamma(T', M_{T'}^{\gp})$ is an injection (in fact an isomorphism) [6, Theorem 4.4.1], so we conclude $f^* \beta = 1$, as required.

**Corollary 8.** We have $\text{End}(\mathbb{G}_{\log}) = \mathbb{Z}$ and every $S$-morphism $\mathbb{G}_{\log} \rightarrow \mathbb{G}_{\log}$ is uniquely representable as the product of a translation and a homomorphism.

**Corollary 9.** Let $\mathcal{M}_{0,2}(\mathbb{G}_{\log})$ denote the stack on logarithmic schemes whose $S$-points consist of a $\mathbb{G}_{\log}$-torsor $X$ on $S$ and a map $X \rightarrow \mathbb{G}_{\log}$. Then $\mathcal{M}_{0,2}(\mathbb{G}_{\log}) \simeq \coprod_{r \in \mathbb{Z}} B \mu_r$ with the understanding that $B \mu_0 = \mathbb{G}_{\log} \times B \mathbb{G}_{\log}$.

**Proof.** Locally in $S$, there is an isomorphism $X \simeq S \times \mathbb{G}_{\log}$. By Proposition 7, a map $f : X \rightarrow \mathbb{G}_{\log}$ is therefore representable locally as $\alpha t^n : S \times \mathbb{G}_{\log} \rightarrow \mathbb{G}_{\log}$ with $\alpha \in M_{S}^{\gp}$ and $r \in \mathbb{Z}$. It follows that $X \rightarrow \mathbb{G}_{\log}$ is equivariant with respect to the map $[r] : \mathbb{G}_{\log} \rightarrow \mathbb{G}_{\log}$. If $r \neq 0$, the fiber of $X \rightarrow \mathbb{G}_{\log}$ over the origin is therefore a torsor under $\ker [r] = \mu_r$. This gives a map $\mathcal{M}_{0,2}(\mathbb{G}_{\log}) \rightarrow \coprod_{r \in \mathbb{Z}} B \mu_r$ sending $f : X \rightarrow \mathbb{G}_{\log}$ to $(r, f^{-1}(1))$. Conversely, given any $\mu_r$-torsor $X_0$ on $S$ we may extend $X_0$ along $\mu_r \rightarrow \mathbb{G}_{\log}$ to obtain a $\mathbb{G}_{\log}$-torsor $X$ along with a map $X \rightarrow \mathbb{G}_{\log}$. These operations are easily seen to be inverse to one another.

If $r = 0$ then the map to $X \rightarrow \mathbb{G}_{\log}$ factors uniquely through $S$, giving a factor $\mathbb{G}_{\log}$. The choice of $X$ is parameterized by $B \mathbb{G}_{\log}$.

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