Sums of fractions modulo $p$

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Abstract

Let $\mathbb{F}_p$ be the field of residue classes modulo a large prime $p$. The present paper is devoted to the problem of representability of elements of $\mathbb{F}_p$ as sums of fractions of the form $x/y$ with $x, y$ from short intervals of $\mathbb{F}_p$.

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1 Introduction

Throughout the paper $\varepsilon$ is a small fixed positive constant, $p$ is a prime number sufficiently large in terms of $\varepsilon$. As usual, $\mathbb{F}_p$ denotes the field of residue classes modulo $p$. The elements of $\mathbb{F}_p$ we will frequently associate with the set $\{0, 1, \ldots, p-1\}$. Given an integer $x$ coprime to $p$ (or an element $x$ from $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$) we use $x^*$ or $x^{-1}$ to denote its multiplicative inverse modulo $p$.

Let $\lambda \in \mathbb{F}_p$ be fixed and let $\mathcal{I}$ and $\mathcal{J}$ be two intervals in $\mathbb{F}_p$. We assume that $\mathcal{I}$ and $\mathcal{J}$ are nonzero, that is $\mathcal{I} \neq \{0\}$, $\mathcal{J} \neq \{0\}$. Motivated by the recent work of Shparlinski [7], we consider the equation

$$\sum_{i=1}^{n} \frac{x_i}{y_i} = \lambda,$$

where $x_i, y_j$ are variables that run through the intervals $\mathcal{I}$ and $\mathcal{J}$ respectively. Using exponential sum estimates Shparlinski obtained an asymptotic formula for the number of solutions of general linear congruences. In the case of (1) his results imply nontrivial estimates under some conditions imposed on the
cardinalities of $\mathcal{I}$ and $\mathcal{J}$ (see Lemma 4 below). In particular, if $n \geq 3$ and $|\mathcal{I}| = |\mathcal{J}| > p^{n/(3n-2)+\varepsilon}$, then the asymptotic formula obtained by Shparlinski becomes nontrivial for any fixed constant $\varepsilon > 0$ (here and below, for a given set $\mathcal{X}$ we use $|\mathcal{X}|$ to denote its cardinality).

In the present paper we consider the problem of solvability of (1). Our results are based on combinatorial and analytical tools. Although we do not get an asymptotic formula for the number of solutions, our results give the solvability of (1) under weaker conditions on the sizes of $|\mathcal{I}|$ and $|\mathcal{J}|$.

**Theorem 1.** Let $\mathcal{I}$ and $\mathcal{J}$ be intervals of $\mathbb{F}_p$ such

$$|\mathcal{I}|^2 \cdot |\mathcal{J}| > p^{1+\varepsilon}, \quad |\mathcal{I}| \cdot |\mathcal{J}|^2 > p^{1+\varepsilon}.$$ 

Then for any $\lambda \in \mathbb{F}_p$ the equation

$$\sum_{i=1}^{8} \frac{x_i}{y_i} = \lambda \quad (2)$$

has a solution with $(x_1, \ldots, x_8) \in \mathcal{I}^8$ and $(y_1, \ldots, y_8) \in \mathcal{J}^8$.

From Theorem 1 it follows, in particular, that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $\mathcal{I}$ and $\mathcal{J}$ are intervals of $\mathbb{F}_p$ with

$$|\mathcal{I}| > p^{1/3+\varepsilon}, \quad |\mathcal{J}| > p^{1/3-\delta},$$

then any element $\lambda \in \mathbb{F}_p$ can be represented in the form (2) for some $(x_1, \ldots, x_8) \in \mathcal{I}^8$ and $(y_1, \ldots, y_8) \in \mathcal{J}^8$.

**Theorem 2.** Let $\mathcal{I}$ and $\mathcal{J}$ be nonzero intervals of $\mathbb{F}_p$ such that

$$|\mathcal{J}| > p^{5/119}, \quad |\mathcal{I}| \cdot |\mathcal{J}|^{21/20} > p^{3/4+\varepsilon}.$$ 

Then for any $\lambda \in \mathbb{F}_p$ the equation

$$\sum_{i=1}^{12} \frac{x_i}{y_i} = \lambda \quad (3)$$

has a solution with $(x_1, \ldots, x_{12}) \in \mathcal{I}^{12}$ and $(y_1, \ldots, y_{12}) \in \mathcal{J}^{12}$. 
From Theorem 2 it follows, in particular, that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $\mathcal{I}$ and $\mathcal{J}$ are intervals of $\mathbb{F}_p$ with

$$|\mathcal{I}| > p^{9/40 + \varepsilon}, \quad |\mathcal{J}| > p^{1/2 - \delta},$$

then any element $\lambda \in \mathbb{F}_p$ can be represented in the form (3) for some $(x_1, \ldots, x_{12}) \in \mathcal{I}^{12}$ and $(y_1, \ldots, y_{12}) \in \mathcal{J}^{12}$.

**Theorem 3.** Let $k$ be a fixed positive integer constant, $\mathcal{I}$ and $\mathcal{J}$ be intervals of $\mathbb{F}_p$ such that

$$|\mathcal{I}| \cdot |\mathcal{J}|^{2^{k/(k+1)}} > p^{1+\varepsilon}.$$

Then for any $\lambda \in \mathbb{F}_p$ the equation

$$\sum_{i=1}^{4k} \frac{x_i}{y_i} = \lambda$$

has a solution with $(x_1, \ldots, x_{4k}) \in \mathcal{I}^{4k}$ and $(y_1, \ldots, y_{4k}) \in \mathcal{J}^{4k}$.

In particular, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, k) > 0$ such that if $\mathcal{I}$ and $\mathcal{J}$ be intervals of $\mathbb{F}_p$ with

$$|\mathcal{I}| > p^{4/11 + \varepsilon}, \quad |\mathcal{J}| > p^{1/2 - \delta},$$

then any element $\lambda \in \mathbb{F}_p$ can be representable in the form

$$\sum_{i=1}^{4k} \frac{x_i}{y_i} = \lambda,$$

for some $(x_1, \ldots, x_{4k}) \in \mathcal{I}^{4k}$ and $(y_1, \ldots, y_{4k}) \in \mathcal{J}^{4k}$.

It is to be mentioned that if the interval $\mathcal{J}$ starts from the origin and $|\mathcal{J}| > p^\varepsilon$, then there is a positive integer $n = n(\varepsilon)$ such for any element $\lambda \in \mathbb{F}_p$ the equation

$$\sum_{i=1}^{n} \frac{1}{y_i} = \lambda$$

has a solution with $y_i \in \mathcal{J}$, see Shparlinski [8]. However, the problem is still open for intervals $\mathcal{J}$ of arbitrary positions.
2 Lemmas

Given sets $X \subset \mathbb{F}_p$ and $Y \subset \mathbb{F}_p$, the product set $X \cdot Y$ is defined by

$$X \cdot Y = \{xy; x \in X, y \in Y\}.$$ 

For a positive integer $k$, the $k$-fold sum of $X$, is defined by

$$kX = \{x_1 + \ldots + x_k; x_i \in X\}.$$ 

We also use the notation $X^{-1} = \{x^{-1}; x \in X \setminus \{0\}\}$.

From the results of Glibichuk [5] it is known if $|X| |Y| > 2p$ then $8X \cdot Y = \mathbb{F}_p$. Here we need its version given by Garaev and Garcia [3] (see also Garcia [4] for even a more general statement).

**Lemma 1.** Let $A, B, C, D$ be subsets of $\mathbb{F}_p^*$ such that

$$|A||C| > (2 + \sqrt{2})p, \quad |B||D| > (2 + \sqrt{2})p.$$ 

Then

$$(2A)(2B) + (2C)(2D) = \mathbb{F}_p.$$ 

We remark that the constant $2 + \sqrt{2}$ that appears in the condition of the lemma can be substituted by a smaller one, but we do not need it here.

Next, we need the following result from Cilleruelo and Garaev [2] which is based on the idea of Heath-Brown [6].

**Lemma 2.** Let $J$ be an interval in $\mathbb{F}_p$ and $\lambda \in \mathbb{F}_p^*$. Then the number $W_\lambda$ of solutions of the congruence

$$xy = \lambda, \quad x \in J, \; y \in J,$$

satisfies

$$W_\lambda < \frac{|J|^{3/2+o(1)}}{p^{1/2}} + |J|^{o(1)}.$$ 

Observe that for $\lambda \in \mathbb{F}_p^*$ the equation $x^{-1} + y^{-1} = \lambda$ implies that

$$(x - \lambda^{-1})(y - \lambda^{-1}) = \lambda^2.$$ 

Hence, we have the following consequence of Lemma 2.
Corollary 1. Let $J$ be an interval in $\mathbb{F}_p$ and $\lambda \in \mathbb{F}_p^*$. Then the number $W_\lambda$ of the solutions of the congruence

$$x^{-1} + y^{-1} = \lambda, \quad x \in J, \ y \in J,$$

satisfies

$$W_\lambda < \frac{|J|^{3/2+\omega(1)}}{p^{1/2}} + |J|^\omega(1). \quad (4)$$

We recall that (4) is equivalent to the claim that for any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that

$$W_\lambda < c\left(\frac{|J|^{3/2+\varepsilon}}{p^{1/2}} + |J|^\varepsilon\right).$$

We also need the following result of Bourgain and Garaev [1].

Lemma 3. Let $J$ be an arbitrary nonzero interval in $\mathbb{F}_p$. For any fixed positive integer constant $k$ the number $T_k$ of solutions of the congruence

$$y_1^{-1} + \ldots + y_k^{-1} = y_{k+1}^{-1} + \ldots + y_{2k}^{-1}, \quad y_1, \ldots, y_{2k} \in J, \quad (5)$$

satisfies

$$T_k < \left(|J|^{2k/(k+1)} + \frac{|J|^{2k}}{p}\right)|J|^\omega(1). \quad (6)$$

Finally, we state the result of Shparlinski [7] which will be used to deal with Theorem 2 for relatively small intervals $J$.

Lemma 4. Let $I$ and $J$ be two nonzero intervals in $\mathbb{F}_p$. Then the number $R = R(\lambda, I, J)$ of solutions of (1) with $x_i \in I$ and $y_i \in J$ satisfies

$$|R - \frac{|I|^n |J|^n}{p}| < |I||J|\left(|I|^{n-2} + (p|J|)^{(n-2)/2}\right)p^{\omega(1)}.$$

3 Proofs

3.1 Proof of Theorem 1

We can assume that $|I| > 10$, $|J| > 10$. Let $I_0 \subset \mathbb{F}_p$ be an interval such that

$$|I_0| > 0.3|I|, \quad 2I_0 = I_0 + I_0 \subset I.$$
Such an interval obviously exists. Let $W_\lambda$ be the number of solutions of the congruence

$$x^{-1} + y^{-1} = \lambda, \quad x \in \mathcal{J}, y \in \mathcal{J}.$$ 

Using Corollary 1, we have

$$|\mathcal{J}|^2 \ll \sum_{\lambda \in \mathcal{J}^{-1} + \mathcal{J}^{-1}} W_\lambda \leq |\mathcal{J}^{-1} + \mathcal{J}^{-1}| \cdot \left( \frac{|\mathcal{J}|^{3/2+o(1)}}{p^{1/2}} + |\mathcal{J}|^{o(1)} \right).$$

It follows that

$$|\mathcal{J}^{-1} + \mathcal{J}^{-1}| > \min \left\{ |\mathcal{J}|^{2+o(1)}, p^{1/2} |\mathcal{J}|^{1/2+o(1)} \right\}.$$

Denote

$$A = D = \mathcal{I}_0 \setminus \{0\}, \quad B = C = \left( \mathcal{J}^{-1} + \mathcal{J}^{-1} \right) \setminus \{0\}.$$ 

We have

$$|A||C| = |B||D| \geq 0.1 |\mathcal{I}_0| \cdot |\mathcal{J}^{-1} + \mathcal{J}^{-1}| \geq \min \left\{ |\mathcal{I}| |\mathcal{J}|^{2+o(1)}, (p |\mathcal{I}|^2 |\mathcal{J}|)^{1/2+o(1)} \right\} \geq p^{1+0.1\varepsilon} > 4p.$$ 

Thus, the condition of Lemma 4 is satisfied. Therefore, we get

$$(2\mathcal{I}_0)(4\mathcal{J}^{-1}) + (2\mathcal{I}_0)(4\mathcal{J}^{-1}) = \mathbb{F}_p.$$ 

Since $2\mathcal{I}_0 \subset \mathcal{I}$, the result follows.

### 3.2 Proof of Theorem 2

Let $R$ be the number of solutions of the congruence (3) with $x_i \in \mathcal{I}$, $y_j \in \mathcal{J}$. There are three cases to consider.

**Case 1.** $p^{5/19} < |\mathcal{J}| < p^{15/37}$.

In view of Lemma 4, applied with $n = 12$, the number $R$ satisfies

$$R > \frac{|\mathcal{I}|^{12} |\mathcal{J}|^{12}}{p} - |\mathcal{I}|^{11} |\mathcal{J}| p^{0.1\varepsilon} - |\mathcal{I}| |\mathcal{J}|^{6} p^{5+0.1\varepsilon}.$$ 

From the condition of the theorem it follows that

$$|\mathcal{I}|^{11} |\mathcal{J}| p^{0.1\varepsilon} < \frac{0.1 |\mathcal{I}|^{12} |\mathcal{J}|^{12}}{p}, \quad |\mathcal{I}| |\mathcal{J}|^{6} p^{5+0.1\varepsilon} < \frac{0.1 |\mathcal{I}|^{12} |\mathcal{J}|^{12}}{p}.$$
Therefore, $R > 0$ and the result follows in this case.

**Case 2.** $|\mathcal{J}| > p^{5/8}$.

We fix a nonzero element $x_0 \in \mathcal{I}$ and denote by $R_1$ the number of solutions of the equation

$$\sum_{i=1}^{12} y_i^{-1} = \lambda x_0^{-1}, \quad y_i \in \mathcal{J}.$$ 

It suffices to show that $R_1 > 0$. Let $\mathcal{J}_1 = \mathcal{J} \setminus \{0\}$. Expressing $R_1$ via exponential sums and following the standard procedure, we get

$$|R_1 - \frac{|\mathcal{J}_1|^{12}}{p}| \leq \frac{1}{p} \sum_{a=1}^{p-1} \left| \sum_{y \in \mathcal{J}_1} e_p(ay^*) \right|^{12}.$$ 

Here and below, we use the abbreviation $e_p(z) = e^{2\pi iz/p}$. By the well-known estimate for incomplete Kloosterman sums we have

$$\max_{\gcd(a,p)=1} \left| \sum_{y \in \mathcal{J}_1} e_p(ay^*) \right| \ll 2p^{1/2} \log p.$$ 

We also have

$$\frac{1}{p} \sum_{a=0}^{p-1} \left| \sum_{y \in \mathcal{J}_1} e_p(ay^*) \right|^2 = |\mathcal{J}_1|.$$ 

Therefore,

$$R_1 > \frac{|\mathcal{J}_1|^{12}}{p} - 2^{10} |\mathcal{J}_1| p^5 (\log p)^{10}.$$ 

Since $|\mathcal{J}_1| \geq |\mathcal{J}| - 1 > 0.5p^{5/8}$, we get that $R_1 > 0$ and the result follows in this case.

**Case 3.** $p^{15/37} < |\mathcal{J}| < p^{5/8}$.

Following the notation of Lemma 3, we denote by $T_k$ the number of solutions of the congruence (3). From the well-known application of the Cauchy-Schwarz inequality it follows that

$$T_3 \leq (T_2 T_4)^{1/2}. \quad (7)$$ 

From Corollary 1 we easily obtain that

$$T_2 \leq \left( \frac{|\mathcal{J}|^{3/2+o(1)}}{p^{1/2}} + |\mathcal{J}|^{o(1)} \right) |\mathcal{J}|^2.$$
Since $|\mathcal{J}| > p^{15/37} > p^{1/3}$, we get that

$$T_2 \leq \frac{|\mathcal{J}|^{7/2 + o(1)}}{p^{1/2}}.$$  \hfill (8)

Furthermore, by Lemma 3 and the condition $|\mathcal{J}| < p^{5/8}$, we have

$$T_4 < \frac{|\mathcal{J}|^{32/5 + o(1)}}{p} < |\mathcal{J}|^{32/5 + o(1)}.$$

Combining this estimate with (7) and (8), we obtain that

$$T_3 < \frac{|\mathcal{J}|^{99/20 + o(1)}}{p^{1/4}}.$$  

From the relationship between the number of solutions of a symmetric equation and the cardinality of the corresponding set, we have

$$|3\mathcal{J}^{-1}| = |\mathcal{J}^{-1} + \mathcal{J}^{-1} + \mathcal{J}^{-1}| \gg \frac{|\mathcal{J}|^6}{T_3},$$

implying that

$$|3\mathcal{J}^{-1}| \geq |\mathcal{J}|^{21/20}p^{1/4 - 0.1\varepsilon}.$$  

As in the proof of Theorem 1, let $\mathcal{I}_0 \subset \mathbb{F}_p$ be an interval such that

$$|\mathcal{I}_0| > 0.3|\mathcal{I}|,$$

$$2\mathcal{I}_0 \subset \mathcal{I}.$$  

Denote

$$\mathcal{A} = \mathcal{D} = \mathcal{I}_0 \setminus \{0\}, \quad \mathcal{B} = \mathcal{C} = \left(3\mathcal{J}^{-1}\right) \setminus \{0\}.$$  

We have

$$|\mathcal{A}| \cdot |\mathcal{C}| = |\mathcal{B}| \cdot |\mathcal{D}| \geq 0.1|\mathcal{I}_0| \cdot |3\mathcal{J}^{-1}| \geq |\mathcal{I}| \cdot |\mathcal{J}|^{21/20}p^{1/4 - 0.2\varepsilon} \geq p^1 + 0.1\varepsilon > 4p.$$  

Thus, the condition of Lemma 4 is satisfied. Therefore, we get

$$(2\mathcal{I}_0)(6\mathcal{J}^{-1}) + (2\mathcal{I}_0)(6\mathcal{J}^{-1}) = \mathbb{F}_p.$$  

Since $2\mathcal{I}_0 \subset \mathcal{I}$, this concludes the proof of Theorem 2.
3.3 Proof of Theorem 3

There are two cases to consider.

**Case 1.** $|\mathcal{J}| > p^{(k+1)/2k}$.

We fix a nonzero element $x_0 \in \mathcal{I}$ and denote by $R_2$ the number of solutions of the equation

$$\sum_{i=1}^{4k} y_i^{-1} = \lambda x_0^{-1}, \quad y_i \in \mathcal{J}.$$ 

It suffices to show that $R_2 > 0$. Denoting $\mathcal{J}_1 = \mathcal{J} \setminus \{0\}$ and following exactly the same argument as in the Case 2 of Theorem 2, we get

$$R_2 > \frac{|\mathcal{J}_1|^{4k}}{p} - 2^{4k-2}|\mathcal{J}_1|p^{2k-1}(\log p)^{4k-2}.$$

Since $|\mathcal{J}_1| \geq |\mathcal{J}| - 1 > 0.5p^{(k+1)/2k}$, we have $R_2 > 0$ and the claim follows in this case.

**Case 2.** $|\mathcal{J}| < p^{(k+1)/2k}$.

We recall that $T_k$ is the number of solutions of the congruence (5). From Lemma 3 it follows that in our case we have the bound

$$T_k < \left(||\mathcal{J}|^{2k/(k+1)} + \frac{|\mathcal{J}|^{2k}}{p}\right)|\mathcal{J}|^{o(1)} < |\mathcal{J}|^{2k/(k+1)} + o(1).$$

Hence, from the relationship between the number of solutions of a symmetric equation and the cardinality of the corresponding set, we have

$$|k\mathcal{J}^{-1}| > \frac{|\mathcal{J}|^{2k}}{|\mathcal{J}|^{2k/(k+1)} + o(1)} > |\mathcal{J}|^{2k/(k+1)}p^{-0.1\varepsilon}.$$ 

Hence, denoting by $\mathcal{I}_0 \subset \mathbb{F}_p$ an interval such that $|\mathcal{I}_0| > 0.3|\mathcal{I}|$, $2\mathcal{I}_0 \subset \mathcal{I}$ we verify that the condition of Lemma 1 is satisfied with $A = C = \mathcal{I}_0$ and $B = D = k\mathcal{J}^{-1}$. Thus, we get that

$$\mathcal{I}(2k\mathcal{J}^{-1}) + \mathcal{I}(2k\mathcal{J}^{-1}) = \mathbb{F}_p$$

which finishes the proof of Theorem 3.
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