Thermodynamic Uncertainty Relations Under Arbitrary Control Protocols

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Thermodynamic uncertainty relations quantifying a trade-off between the current fluctuation and the entropy production have been found in various stochastic systems. Here, we study thermodynamic uncertainty relations for Langevin systems driven by an external control protocol. Using information-theoretic techniques, we derive uncertainty relations for arbitrary observables satisfying a scaling condition in both overdamped and underdamped regimes. We prove that the observable fluctuation is constrained by both the entropy production and a kinetic term. The derived bounds are applicable to both currents and noncurrent observables, and hold for arbitrary time-dependent protocols; thus, providing a wide range of applicability. We illustrate our universal bounds with the help of two systems, a Brownian gyration and a stochastic underdamped heat engine.

Stochastic thermodynamics [1–3] provides a rigorous framework for studying the physical properties of small systems. On theoretical grounds, it is known that thermodynamic cost places fundamental limits on the performance of real-world systems, from living organisms to artificial devices. Investigating such trade-off relations provides insights into design principles of optimum systems.

In recent years, powerful inequalities called thermodynamic uncertainty relations (TURs) have been discovered for nonequilibrium systems [4, 5]. They assert a trade-off between the current fluctuation and dissipation quantified via the entropy production; that is a high precision of currents is unattainable without increasing the associated entropy production. Originally, TURs impose the following bound in steady-state systems:

\[
\frac{\langle \phi \rangle^2}{\langle \phi \rangle} \leq \frac{\langle \sigma \rangle}{2},
\]

where \(\phi\) is an arbitrary time-integrated current, \(\langle \phi \rangle\) and \(\langle \phi \rangle := \langle \phi^2 \rangle - \langle \phi \rangle^2\) are its mean and variance, respectively, and \(\langle \sigma \rangle\) is the average entropy production. This bound was first derived for biomolecular processes [4] and later proven for continuous-time Markov jump processes [5, 6] and overdamped Langevin systems [7, 8]. Subsequently, the violation of the original bound has been found for other dynamics, e.g., for discrete-time Markov chains [9], transport systems [10], and underdamped dynamics [11, 12]. TURs have been refined intensively in other contexts, including both classical and quantum systems [12–23]. A remarkable application of TURs is in the estimation of entropy production [24]. By observing various fluctuating currents, a lower bound on entropy production can be inferred.

In this Letter, we focus on extending the applicability of TURs, which have been derived for currents in steady-state systems. Considering general Langevin systems driven by a possibly time-dependent control protocol, we derive uncertainty relations for both currents and noncurrent observables, which satisfy a scaling condition. We prove for both overdamped and underdamped systems that the observable fluctuation is bounded by the entropy production and a kinetic term. Notably, the derived bounds are not static but dynamic with respect to observables and are tighter than the original one for a broad class of observables. Our results allow investigating arbitrary Langevin systems, from relaxation processes to externally-controlled systems like stochastic heat engines. We apply the results to study two systems, a Brownian gyration and a stochastic underdamped heat engine.

Recent studies have made advances in generalizing TURs. It has been shown that a TUR is a direct consequence of the detailed fluctuation theorem, regardless of the underlying dynamics [25, 26]. This bound is more applicable than the original one, i.e., it holds for arbitrary currents and arbitrary dynamics as long as the fluctuation theorem is provided; however, paying the cost of a weaker predictive power. A generalization to systems with broken symmetry, known as the hysteretic TUR, has been conducted [27, 28]. This bound requires to evaluate currents and entropy production in the backward process, having the form

\[
\frac{\langle (\phi) + (\phi)_b \rangle^2}{\langle (\phi) \rangle + \langle (\phi)_b \rangle} \leq \exp\left(\frac{\langle \sigma \rangle + \langle \sigma \rangle_b}{2}\right) - 1,
\]

where \(\langle \cdot \rangle_b\) denotes averages taken over ensemble in the backward experiment. Another extension which holds for arbitrary dynamics reads [29, 30]

\[
\frac{\langle \phi \rangle^2}{\langle (\phi) \rangle} \leq \frac{e^{\langle \sigma \rangle} - 1}{2},
\]

where \(\langle \sigma \rangle\) is the Kullback–Leibler divergence between distributions of the forward path and its reversed counterpart in the system. However, \(\langle \sigma \rangle\) is not equal to the entropy production \(\langle \sigma \rangle\), except in steady-state systems with time-reversal symmetry. Despite the generalities of...
Eqs. (2) and (3), it is difficult to infer entropy production from these bounds.

Thermodynamic uncertainty relations.—For the sake of simplicity, we will describe our results with one-dimensional systems. The generalization to multidimensional systems is straightforward [31]. Unlike in most previous studies, where the system is assumed to be in a steady state or a transient regime, here, the system starts from an arbitrary distribution at time $t = 0$ and afterward is driven by an external control protocol $\lambda$ up to time $t = \tau$. When $\lambda$ is time-independent, it becomes a relaxation process. Let $\Gamma$ denote the trajectory of the system states during this time interval, and $\phi(\Gamma)$ be a trajectory-dependent observable which can be time-symmetric. We aim to derive a bound on the relative fluctuation of $\phi(\Gamma)$.

We consider observables satisfying the scaling condition: $\phi(\theta \Gamma) = \theta^k \phi(\Gamma)$ for some constant $k > 0$ and for all $\theta \in \mathbb{R}$. Given a trajectory $\Gamma = [x(t)]_{t=0}^{\tau}$, this can be satisfied with a current $\phi(\Gamma) = \int_0^\tau dt x^{\xi-1} \phi$ or a noncurrent observable $\phi(\Gamma) = \int_0^\tau dt x^{\xi}$. Here, $\phi$ denotes the Stratonovich product, and the dot indicates the time derivative. Moreover, $\phi$ can be a discrete-time observable, e.g., $\phi(\Gamma) = \sum_i c_i x(t_i)^k$, where $0 \leq t_i \leq \tau$ is the predetermined time and $c_i$ is an arbitrary coefficient. From a practical perspective, measurements are performed discretely in most cases; thus, the acquisition of continuous-time observables may be difficult. Consequently, a bound on such discrete-time observables provides a useful tool for thermodynamic inference problems. It is noteworthy that these noncurrent observables cannot be applied with TURs reported previously. Hereafter, we consider these three types of observables.

First, let us consider a general overdamped Langevin system, whose dynamics are governed by the following equation:

$$\dot{x} = F(x, \lambda) + \xi,$$  
(4)

where $F(x, \lambda)$ is the total force, and $\xi$ is a zero-mean Gaussian white noise with variance $\langle \xi(t) \xi(t') \rangle = 2D \delta(t - t')$. Here $D > 0$ is the noise intensity. Throughout this work, Boltzmann’s constant is set to $k_B = 1$. Let $\rho(x, t)$ denote the probability distribution function of the system being in state $x$ at time $t$, then its time evolution can be described by the Fokker-Planck equation as $\partial_t \rho(x, t) = -\partial_x j(x, t)$, where $j(x, t) = F(x, \lambda) \rho(x, t) - D \partial_x \rho(x, t)$ is the probability current. The dynamical solution of this differential equation is uniquely determined if the initial distribution $\rho(x, 0) = \rho_0(x)$ is given.

As our first main result, we prove that the observable fluctuation is bounded as

$$\frac{\langle \phi \rangle^2}{\langle \phi \rangle} \leq \frac{1}{\kappa^2} (2\sigma + \chi_\alpha + \psi_\alpha),$$  
(5)

where $\chi_\alpha := \langle \int_0^\tau dt \Lambda_\alpha(x(t)) \rangle / 2D$ is a kinetic term and $\psi_\alpha := \langle (x \partial_x \rho(x)/\rho_0(x))^2 \rangle_{\rho_0} - 1$ is a nonnegative boundary value which can be neglected for long observation time. Here, $\Lambda_\alpha(x, t) = (\partial_x [x F(x, \lambda)])^2 - 4F(x, \lambda) \partial_x [x F(x, \lambda)] - 4D \partial^2_x [x F(x, \lambda)]$ is a function in terms of the force and protocol.

Next, we consider a general underdamped Langevin system, where inertial effects cannot be neglected. The system consists of a particle being in contact with an equilibrium heat bath. Its dynamics are described by the following equations:

$$\dot{x} = v, \quad m \ddot{v} = -\gamma v + F(x, \lambda) + \xi,$$  
(6)

where $m, \gamma$ are the mass and friction coefficient of the particle, respectively. Let $\rho(x, v, t)$ be the phase-space probability distribution function of the system at time $t$. Suppose that the system evolves from an initial distribution $\rho(x, v, 0) = \rho_1(x, v)$; then, $\rho(x, v, t)$ follows the Fokker-Planck equation, $\partial_t \rho(x, v, t) = -\partial_x j_x(x, v, t) - \partial_v j_v(x, v, t) + v \partial_v \rho(x, v, t)$ and $j_x(x, v, t) = 1/m [-\gamma v + F(x, \lambda) - D / m \partial_v \rho(x, v, t)]$ are probability currents. Since the position $x$ and velocity $v$ are freedom degrees of the system, the trajectory can be written as $\Gamma = [x(t), v(t)]_{t=0}^\tau$.

For observables satisfying the scaling condition, we prove that

$$\frac{\langle \phi \rangle^2}{\langle \phi \rangle} \leq \frac{1}{\kappa^2} (2\sigma + \chi_u + \psi_u),$$  
(7)

where $\chi_u := \langle \int_0^\tau dt \Lambda_u(x, v(t)) \rangle / 2D$ is a kinetic term, and $\psi_u := \langle \left( \int x \partial_x \rho(x, v(x), v) - 4\gamma^2 v^2 + 8\gamma D / m \right) \rangle_{\rho(x, v)} - 4$ is a boundary term which can be neglected for large $\tau$. Here, $\Lambda_u(x, v, t) = [F(x, \lambda) - x \partial_x F(x, \lambda)]^2 - 4\gamma^2 v^2 + 8\gamma D / m$. Inequality (7) is our second main result. The detailed derivations of the bounds are presented at the end of this Letter.

We make several remarks about our main results, Eqs. (5) and (7). These inequalities hold for arbitrary protocol $\lambda$, for arbitrary initial distribution $\rho_0$, and for finite observation time $\tau$; thus, they are also valid for steady-state systems. Interestingly, the derived bounds involve the scaling power $\kappa$; as $\kappa$ is large enough, the bounds become tighter than the original one [Eq. (1)]. Moreover, unlike the reported bounds, which deal with only currents, our bounds are applicable for currents, noncurrent, and discrete-time observables, and for linear combinations of these observables.

In addition to entropy production, the bounds contain kinetic terms $\chi_{(u, v)}$. They are averages of observables, which can be calculated based on the observed trajectories. As will be shown later, these terms play an important role in the bounds; that the observable fluctuation cannot be bounded solely by the entropy production, even with the exponential bound $(e^{\sigma} - 1)/2$. Besides, the fluctuations of a noncurrent observable, $\langle \phi \rangle^2 / \langle \phi \rangle$, may not vanish in equilibrium, for example, for $\phi(\Gamma) = \int_0^\tau dt x^2$, while the entropy production always does, i.e.,
FIG. 1. (a) Schematic diagrams of Brownian gyrator. (b) Schematic diagrams of a stochastic underdamped heat engine. A cyclic period consists of four steps: isothermal expansion for a time $\tau_h \ (1 \rightarrow 2)$, instantaneously cooling the heat bath to temperature $T_h \ (2 \rightarrow 3)$, isothermal compression for a time $\tau_c \ (3 \rightarrow 4)$, and instantaneously heating the heat bath to temperature $T_h \ (4 \rightarrow 1)$. The solid and dashed lines represent the probability distribution $\rho(x, t)$ and the potential $U(x, \lambda)$, respectively.

$\langle \sigma \rangle = 0$. In this scenario, $\chi_{(\sigma, u)}$ are key quantities that constrain fluctuations of such noncurrent observables.

We provide an intuitive explanation regarding why the kinetic terms appear in the bounds. The entropy production, which is quantified via irreversible currents of the probability density, characterizes the strength of currents in the system. A zero entropy production implies that there is no current in the system. Therefore, its genuine contribution in the bounds is the constraint on fluctuations of currents. To constrain fluctuations of noncurrent components (e.g., time-symmetric changes), another complement to entropy production, which is identified here as $\chi$, is necessary.

In what follows, we illustrate our results with the aid of two systems.

**Example 1.**—First, we study a Brownian gyration [32], which is a minimal microscopic heat engine and has recently been realized experimentally in an electronic and in a colloidal system [33, 34]. The system consists of a particle with two degrees of freedom $\mathbf{x} = (x_1, x_2)^T$ trapped in an elliptical harmonic potential $U(\mathbf{x}) = [u_1(x_{1}\cos \alpha + x_{2}\sin \alpha)^2 + u_2(- x_{1}\sin \alpha + x_{2}\cos \alpha)^2]/2$, where $u_1, u_2 > 0$ are stiffnesses along its principal axes, and $\alpha$ is the rotation angle. The particle is simultaneously in contact with two heat baths at different temperatures acting along perpendicular directions [Fig. 1(a)]. The particle position follows overdamped Langevin equations

$$\gamma_i \dot{x}_i = -\partial_x U(\mathbf{x}) + \xi_i, \quad (i = 1, 2),$$

where $\gamma_i$ is the friction coefficient, and $\xi_i$ is the zero-mean Gaussian white noise with covariance $\langle \xi_i(t)\xi_j(t') \rangle = 2\delta_{ij}\gamma_i T_i \delta(t - t')$. Here, $T_1 \neq T_2$ are the temperatures of the heat baths. In generic cases, i.e., $u_1 \neq u_2$, the potential is asymmetric, and a systematic gyrating motion of the particle around the potential minimum is induced due to the heat flow. The observable of interest is the accumulated torque exerted by the particle on the potential

$$\Gamma(\phi) = \int_0^\tau dt \{x_1 \partial_{x_2} U(\mathbf{x}) - x_2 \partial_{x_1} U(\mathbf{x})\}.$$  

This observable is time-symmetric; thus, all TURs reported previously cannot be applied. Since $\phi(\theta \Gamma) = \theta^2 \phi(\Gamma)$, the following bound on the torque fluctuation should be satisfied:

$$\frac{\langle \phi_\theta \rangle^2}{\langle \phi_1 \rangle^2} \leq \frac{\langle \sigma \rangle}{2} + \frac{\chi_0 + \psi_0}{4}.$$  

We illustrate Eq. (10) in Fig. 2(a). The fluctuation $\langle \phi_\theta \rangle^2/\langle \phi_1 \rangle$ is numerically evaluated, while $\langle \sigma \rangle$, $\chi_0$, and $\psi_0$ are calculated analytically. As seen, the bound is always satisfied when the observation time $\tau$ is varied. Positive entropy production is needed to generate a nonzero torque; however, the fluctuation cannot be bounded solely by the entropy production, even with the exponential bound $\langle e^{\sigma(t) - 1/2} \rangle$.

**Example 2.**—Next, we consider a stochastic underdamped heat engine consisting of a particle trapped in a harmonic potential $U(x, \lambda) = \lambda x^2/2$ [35] [see Fig. 1(b)]. The particle is embedded in a heat bath, whose temperature $T$ is cyclically varied to operate the system as a heat engine. Its dynamics are described by the Langevin equation,

$$m\dot{v} = -\gamma v - \lambda x + \xi,$$
where the noise variance is \( \langle \xi(t)\xi(t') \rangle = 2\gamma T \delta(t-t') \). We employ a time-linear protocol \[36\]
\[
\lambda(t) = \begin{cases} 
\lambda_h + (\lambda_c - \lambda_h) t / \tau_h, & 0 \leq t < \tau_h, \\
\lambda_c + (\lambda_c - \lambda_h)(t - \tau_h) / \tau_c, & \tau_h \leq t < \tau,
\end{cases}
\] (12)
where \( \tau_h, \tau_c \) are the coupling times to the hot and cold heat baths, respectively, and \( \tau = \tau_h + \tau_c \) is the total observation time. The work \( w(\Gamma) = \int_0^\tau dt \partial_\theta U(x(t), \lambda(t)) \). We consider two observables: the power output \( \phi_p = -w / \tau \) and the accumulated kinetic energy \( \phi_v = \int_0^\tau dt v^2 \). Noting that these observables are not time-antisymmetric; thus, they cannot be applied with existing bounds. Since \( \phi(\theta_\Gamma) = \theta^2 \phi(\Gamma) \), fluctuations of these observables are bounded as
\[
\langle \phi \rangle^2 / \langle \langle \phi \rangle \rangle \leq \sigma^2 / 2 + \chi_u + \psi_u / 4
\] (13)
for \( \phi \in \{ \phi_p, \phi_v \} \). We assume that the initial distribution \( \rho_0(x,v) \) is Gaussian and illustrate Eq. (13) in Fig. 2(b).

As shown, the derived bound is always satisfied, while the fluctuations cannot be constrained by the exponential bound.

We illustrate the implication of our results to the power output of heat engines. The original TUR has been exploited to derive a bound on the fluctuation of power output in steady-state heat engines \[37\]. It indicates that a steady-state heat engine working with Carnot’s efficiency \( \eta_C = 1 - T_c / T_h \) and delivering work with a finite fluctuation is impossible. However, our bound does not imply this consequence as does the original bound. It has been shown that one can construct a cyclic Brownian heat engine operating with efficiency asymptotically close to \( \eta_C \) at nonzero power output with vanishing fluctuations \[38\]. Our bound is applicable to such engine and arbitrary heat engines described by Langevin dynamics.

We also analytically verified the derived bound for a dragged Brownian particle with three observables: the displacement, the final position, and the area under the trajectory of the particle. We confirmed that our bound is always satisfied, while the exponential bound is violated (see [31]).

Derivation.—To obtain Eqs. (5) and (7), we employ the information-theoretic inequality with the perturbation technique \[8\]. We modify the force in the original system with a perturbation parameter \( \theta \) and obtain a new auxiliary dynamics. For a given trajectory \( \Gamma \), let \( P_\theta(\Gamma) \) denote the path probability of observing \( \Gamma \) in the auxiliary dynamics. According to the Cramér–Rao inequality \[19\], the precision of the observable \( \phi \) is bounded by the Fisher information as
\[
\langle \partial_\theta (\phi) \rangle^2 / \langle \langle \phi \rangle \rangle \leq \mathcal{I}(\theta).
\] (14)

Here \( \mathcal{I}(\theta) := \langle (\partial_\theta \ln P_\theta(\Gamma))^2 \rangle_\theta = -\langle \partial^2_\theta \ln P_\theta(\Gamma) \rangle_\theta \) is the Fisher information. Inequality (14) can be proven by applying the Cauchy–Schwartz inequality to \( \langle \partial_\theta (\phi) \rangle^2 \) as follows:
\[
\langle \partial_\theta (\phi) \rangle^2 = \left( \partial_\theta \int D\Gamma P_\theta(\Gamma) \phi(\Gamma) \right)^2
\]
\[
\leq \left( \int D\Gamma P_\theta(\Gamma) \phi(\Gamma) \right)^2 \mathcal{I}(\theta).
\] (15)

For overdamped systems, let us consider the auxiliary dynamics, \( \dot{x} = H_\theta(x,v,t) + \xi \), where
\[
H_\theta(x,v,t) = \theta F(x / \theta, \lambda) + (1 - \theta^2) \frac{\partial v \rho(x,v) / \rho(x / \theta, v)}{\partial (x / \theta)}.
\] (16)

Analogously, for underdamped systems, the dynamics are modified as \( \dot{m} \dot{v} = H_\theta(x,v,t) + \xi \), where
\[
H_\theta(x,v,t) = -\gamma v + \theta F(x / \theta, \lambda) + \frac{D}{m} (1 - \theta^2) \frac{\partial v \rho(x,v) / \rho(x / \theta, v)}{\partial (x / \theta)}.
\] (17)

When \( \theta = 1 \), these auxiliary dynamics become the original ones. The distributions of auxiliary dynamics in the overdamped and underdamped cases are \( \rho_\theta(x,t) = \rho(x/\theta,t)/\theta \) and \( \rho_\theta(x,v) = \rho(x,v) / (\theta) \), respectively. In both cases, the observable average is scaled as \( \langle \phi \rangle_\theta = \theta \langle \phi \rangle \); thus, \( \partial_\theta \langle \phi \rangle_\theta |_{\theta=1} = \kappa \langle \phi \rangle \) the path probability using the pre-point discretization can be expressed via the path-integral representation as
\[
P_\theta(\Gamma) = N_u \rho_0(x(0),0) \exp \left( -\int_0^\tau dt \frac{(\dot{x} - H_\theta(x,t))^2}{4D} \right)
\] (18)
for the overdamped case and
\[
P_\theta(\Gamma) = N_u \rho_0(x(0),v(0),0) \exp \left( -\int_0^\tau dt \frac{(m \dot{v} - H_\theta(x,v,t))^2}{4D} \right)
\] (19)
for the underdamped case. Here, \( N_u \) and \( N_a \) are terms independent of \( \theta \). Noting that the entropy production \( \sigma \) is
\[
\int_0^\tau dt \int dx j(x,t)^2 / [D \rho(x,t)] \text{in overdamped systems and is}
\]
\[
\int_0^\tau dt \int dx dv j^2(x,v,t) / [D \rho(x,v,t)] \text{in underdamped systems.}
\]

Here, if \( j^2(x,v,t) = -1/m [\gamma v + D (m \partial_x \rho) \rho(x,v,t)] \) is the irreversible probability current. Consequently, by simple algebraic calculations, one can show that \( \mathcal{I}(1) \) is equal to \( 2\sigma + \chi_\phi + \psi_\phi \) for the overdamped case and to \( 2\sigma + \chi_\phi + \psi_\phi \) for the underdamped case. By letting \( \theta = 1 \) in Eq. (14), we obtain the uncertainty relations given in Eqs. (5) and (7).

Based on information theory, we have derived bounds for both currents and noncurrent observables in overdamped and underdamped regimes. These bounds universally hold for arbitrary protocols and arbitrary initial distributions. Our results serve as a useful tool for estimation tasks in general Langevin systems. Information inequalities have successfully been applied to derive many important thermodynamic bounds, such as the
sensitivity-precision trade-off [19], a quantum TUR [39], and the speed limit [40]. Extending our approach to other classical and quantum systems or finding a hyperaccurate observable [41] would be interesting.

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This supplementary material describes the calculations introduced in the main text. Equation and figure numbers are prefixed with S [e.g., Eq. (S1) or Fig. S1]. Numbers without this prefix [e.g., Eq. (1) or Fig. 1] refer to items in the main text.

S1 Uncertainty relations for multidimensional systems

Considering observables satisfying the scaling condition: \( \phi(\theta \Gamma) = \theta^\kappa \phi(\Gamma) \), where \( \kappa > 0 \) is a real constant. Specifically, we focus on three types of observables: a current \( \phi(\Gamma) = \int_0^\tau dt \Lambda_c(\mathbf{x}) \cdot \mathbf{v} \), a noncurrent observable \( \phi(\Gamma) = \int_0^\tau dt \Lambda_{nc}(\mathbf{x}) \), and a discrete-time observable \( \phi(\Gamma) = \sum_i c_i \Lambda_{nc}(\mathbf{x}(t_i)) \), where \( \Lambda_c(\mathbf{x}) \) and \( \Lambda_{nc}(\mathbf{x}) \) satisfy that \( \Lambda_c(\theta \mathbf{x}) = \theta^{\kappa-1} \Lambda_c(\mathbf{x}) \) and \( \Lambda_{nc}(\theta \mathbf{x}) = \theta^\kappa \Lambda_{nc}(\mathbf{x}) \). The probability currents and distribution in the auxiliary dynamics are scaled as

\[
\rho_0(\mathbf{x}, t) = \rho(\mathbf{x}/\theta, t)/\theta^n, \quad j_0(\mathbf{x}, t) = j(\mathbf{x}/\theta, t)/\theta^{n-1} \quad \text{(overdamped cases)},
\]

\[
\rho_\theta(\mathbf{x}, \mathbf{v}, t) = \rho(\mathbf{x}/\theta, \mathbf{v}/\theta, t)/\theta^{2n}, \quad j_\theta(\mathbf{x}, \mathbf{v}, t) = j(\mathbf{x}/\theta, \mathbf{v}/\theta, t)/\theta^{2n-1} \quad \text{(underdamped cases)}.
\]

Consequently, it is easy to verify that \( \langle \phi \rangle_\theta = \theta^\kappa \langle \phi \rangle \). For \( n \)-dimensional overdamped systems described as

\[
\dot{x}_i = F_i(\mathbf{x}, \lambda) + \xi_i, \quad (i = 1, \ldots, n),
\]

the uncertainty relation reads

\[
\frac{\langle \phi \rangle^2}{\langle \langle \phi \rangle \rangle} \leq \frac{1}{\kappa^2} (2(\sigma) + \chi_\circ + \psi_o),
\]

where the terms in the right-hand side of Eq. (S4) are defined by

\[
\chi_\circ := \int_0^\tau dt \int d\mathbf{x} \Lambda_c(\mathbf{x}, t) \rho(\mathbf{x}, t),
\]

\[
\psi_o := \left( \left( \sum_{i=1}^n x_i \partial_{x_i} \rho_\circ(x)/\rho_\circ(x) \right)^2 \right)^{1/2} - n^2.
\]

Here,

\[
\Lambda_{nc}(\mathbf{x}, t) = \sum_{i=1}^n \frac{1}{2D_i} \left( G_i(\mathbf{x}, \lambda)^2 - 4F_i(\mathbf{x}, \lambda)G_i(\mathbf{x}, \lambda) - 4D_i \partial_{x_i} G_i(\mathbf{x}, \lambda) \right),
\]

\[
G_i(\mathbf{x}, \lambda) = F_i(\mathbf{x}, \lambda) + \sum_{j=1}^n x_j \partial_{x_j} F_i(\mathbf{x}, \lambda).
\]

Analogously, for \( n \)-dimensional underdamped systems described as

\[
\dot{x}_i = v_i, \quad \dot{v}_i = -\gamma_i v_i + F_i(\mathbf{x}, \lambda) + \xi_i, \quad (i = 1, \ldots, n),
\]

the bound has the following form:

\[
\frac{\langle \phi \rangle^2}{\langle \langle \phi \rangle \rangle} \leq \frac{1}{\kappa^2} (2(\sigma) + \chi_u + \psi_u),
\]
where the terms in the right-hand side of Eq. (S10) are defined by

$$\chi_u := \int_0^\tau dt \int dx dv \Lambda_u(x, v, t) \rho(x, v, t),$$  \hspace{1cm} (S11)

$$\psi_u := \left\{ \left( \sum_{i=1}^n \left[ x_i \partial_x \rho_i(x, v) + v_i \partial_v \rho_i(x, v) \right] / \rho_i(x, v) \right) \right\}^2 \rho_i - 4n^2.$$  \hspace{1cm} (S12)

Here,

$$\Lambda_u(x, v, t) = \frac{1}{2D_i} \left( \left[ F_i(x, \lambda) - \sum_{j=1}^n x_j \partial_x F(x, \lambda) \right]^2 - 4\gamma_i v_i^2 + \frac{8\gamma_i D_i}{m_i} \right).$$  \hspace{1cm} (S13)

## S2 Dragged Brownian particle

We study a dragged Brownian particle confined in a harmonic potential $U(x, \lambda) = c(x - \lambda)^2/2$, where $c > 0$ is a constant. The total force is $F(x, \lambda) = -\partial_x U(x, \lambda)$, and the particle position is governed by the following equation:

$$\dot{x} = c(\lambda - x) + \xi.$$  \hspace{1cm} (S14)

We consider three cases: (a) a time-linear protocol $\lambda(t) = at$, (b) a time-periodic protocol $\lambda(t) = \alpha \sin(\beta t)$, and (c) a time-symmetric protocol

$$\lambda(t) = \begin{cases} 
\alpha t, & 0 \leq t \leq \tau/2, \\
\alpha(\tau - t), & \tau/2 < t \leq \tau,
\end{cases}$$  \hspace{1cm} (S15)

where $\alpha$ and $\beta$ are positive constants. We suppose that the system is initially in equilibrium with the distribution $\rho_0(x) = \exp[-cx^2/(2D)]$. We consider three observables: a current representing the particle’s displacement $\phi_c(\Gamma) = x(\tau) - x(0)$, the final position $\phi_{pos}(\Gamma) = x(\tau)$, and a noncurrent observable $\phi_{nc}(\Gamma) = \int_0^\tau dt x$, which represents the area under the trajectory. These observables satisfy the scaling condition with $k = 1$, i.e., $\phi(\theta \Gamma) = \theta \phi(\Gamma)$. According to the derived bound [Eq. (5) in the main text], inequality

$$\frac{\langle \phi \rangle^2}{\langle \dot{\phi} \rangle^2} \leq 2(\sigma) + \chi_\alpha + \psi_\sigma$$  \hspace{1cm} (S16)

should be satisfied for all $\phi \in \{\phi_c, \phi_{pos}, \phi_{nc}\}$. All the terms in this bound can be calculated analytically as in the following.

Let $\rho(x, t)$ be the probability density distribution of $x$ at time $t$. Since the force is linear, this distribution is Gaussian, i.e., $\rho(x, t) = N(x; \mu(t), \vartheta(t))$, where $\mu(t)$ and $\vartheta(t)$ are the mean and variance, respectively. We have the initial conditions $\mu(0) = 0, \vartheta(0) = D/c$. From the Fokker–Planck equation, we obtain

$$\dot{\mu}(t) = c [\lambda(t) - \mu(t)], \quad \dot{\vartheta}(t) = \frac{D}{c}.$$  \hspace{1cm} (S17)

Solving the differential equation with respect to $\mu(t)$, we obtain

$$\mu(t) = c \int_0^t ds e^{-c(t-s)} \lambda(s).$$  \hspace{1cm} (S18)

Using the Laplace transform, the analytical solution of Eq. (S14) can be expressed as

$$x(t) = \mu(t) + x_0 e^{-ct} + \int_0^t ds e^{-c(t-s)} \xi(s).$$  \hspace{1cm} (S19)

From Eq. (S19), we can obtain

$$\{ [x(t) - \mu(t)] [x(t') - \mu(t')] \} = \frac{D}{c} e^{-c|t-t'|}.$$  \hspace{1cm} (S20)
Figure S1: Bounds on fluctuations of observables under (a) time-linear, (b) time-periodic, and (c) time-symmetric protocols. The observation time $\tau$ is varied, while the remaining parameters are fixed as $\alpha = 1, \beta = 1, \text{ and } c = 1$. The derived bound is always satisfied, while the exponential bound is violated for all three cases.

The observable averages can be analytically calculated, i.e., $\langle \phi_c \rangle = \langle \phi_{pos} \rangle = \mu(\tau)$ and $\langle \phi_{nc} \rangle = \int_0^\tau dt \mu(t)$. Analogously, the variances of observables are obtained as follows:

$$\langle \langle \phi_c \rangle \rangle = \frac{2D}{c} (1 - e^{-c\tau}),$$
(S21)

$$\langle \langle \phi_{pos} \rangle \rangle = \frac{D}{c},$$
(S22)

$$\langle \langle \phi_{nc} \rangle \rangle = \frac{2D}{c^3} (e^{-c\tau} + c\tau - 1).$$
(S23)

The terms in the bound can be calculated analytically as

$$\langle \sigma \rangle = \frac{c^2}{D} \int_0^\tau dt \left( \lambda(t) - \mu(t) \right)^2,$$
(S24)

$$\chi_o = 2c\tau - \frac{c^2}{2D} \int_0^\tau dt \left[ 4\mu(t)^2 + 3\lambda(t)^2 - 8\lambda(t)\mu(t) \right],$$
(S25)

$$\psi_o = 2.$$  
(S26)

We illustrate the bound in Eq. (S16) in Fig. S1. As seen, the derived bound is always satisfied, while the exponential bound is violated.