GLOBAL OPTIMAL REGULARITY FOR VARIATIONAL PROBLEMS WITH NONSMOOTH NON-STRICLY CONVEX GRADIENT CONSTRAINTS

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Abstract. We prove the optimal $W^{2, \infty}$ regularity for variational problems with convex gradient constraints. We do not assume any regularity of the constraints; so the constraints can be non-smooth, and they need not be strictly convex. When the domain is smooth enough, we show that the optimal regularity holds up to the boundary. In this process, we also characterize the set of singular points of the viscosity solutions to some Hamilton-Jacobi equations. Furthermore, we obtain an explicit formula for the second derivative of these viscosity solutions; and we show that the second derivatives satisfy a monotonicity property.

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1. Introduction

Variational problems and differential equations with gradient constraints, has been an active area of study, and has seen many progresses. An important example among them is the famous elastic-plastic torsion problem, which is the problem of minimizing the functional

$$\int_U \frac{1}{2} |Dv|^2 - v \, dx$$
over the set

\[ W_{B_1} := \{ v \in H^1_0(U) : |Dv| \leq 1 \text{ a.e.} \}. \]

Here \( U \) is a bounded open set in \( \mathbb{R}^n \). (In the physical problem \( n = 2 \).) This problem is equivalent
to finding \( u \in W_{B_1} \) that satisfies the variational inequality

\[ \int_U Du \cdot (v - u) - (v - u) \, dx \geq 0 \quad \text{for every } v \in W_{B_1}. \]

Brezis and Stampacchia [1] proved the \( W^{2,p} \) regularity for the elastic-plastic torsion problem. Caffarelli and Rivière [3] obtained its optimal \( W^{2,\infty} \) regularity. Gerhardt [19] proved \( W^{2,p} \) regularity for the solution of a quasilinear variational inequality subject to the same constraint as in the elastic-plastic torsion problem. Jensen [30] proved \( W^{2,p} \) regularity for the solution of a linear variational inequality subject to \( C^2 \) strictly convex gradient constraint. He also obtained \( W^{2,\infty} \) regularity under some additional restrictions. Those restrictions were removed by Wiegner [40], and some extended results were obtained by Ishii and Koike [29]. Choe and Shim [8, 9] proved \( C^{1,\alpha} \) regularity for the solution to a quasilinear variational inequality subject to \( C^2 \) strictly convex gradient constraint, and allowed the operator to be degenerate of the \( p \)-Laplacian type. In [35], following the approach of [5], we obtained \( W^{2,\infty} \) regularity for the minimizers of the functional subject to gradient constraints satisfying some mild regularity.

Recently, there has been new interest in these types of problems. Hynd and Mawi [26] studied fully nonlinear equations with strictly convex gradient constraints, which appear in stochastic singular control. They obtained \( W^{2,p} \) regularity in general, and \( W^{2,\infty} \) regularity with some extra assumptions. Hynd [24, 25] also considered eigenvalue problems for equations with gradient constraints. De Silva and Savin [12] investigated the minimizers of some functionals subject to gradient constraints, arising in the study of random surfaces. In their work, the functionals are allowed to have certain kinds of singularities. Also, the constraints are given by convex polygons; so they are not strictly convex. They showed that in two dimensions, the minimizer is \( C^1 \) away from the obstacles. (Under mild conditions, a variational problem with gradient constraint is equivalent to a double obstacle problem. For the details see Section 3.)

In this paper, we obtain the optimal \( W^{2,\infty} \) regularity for the minimizers of a large class of functionals subject to arbitrary convex gradient constraints. We do not assume any regularity of the constraints; so the constraints can be nonsmooth, and they need not be strictly convex. We also show that the optimal regularity holds up to the boundary, when the domain is smooth enough. Although our functionals are smooth, we hope that our study sheds some new light on the above-mentioned problem about random surfaces.

In addition to the works on the regularity of the elastic-plastic torsion problem, Caffarelli and Rivière [3, 4], Caffarelli and Friedman [2], Friedman and Pozzi [18], and Caffarelli et al. [6], have worked on the regularity and the shape of its free boundary, i.e. the boundary of the set \( \{|Du| < 1\} \). These works can also be found in [17]. In [33, 34], we extended some of these results to the more general case where the functional is unchanged, but the constraint is given by the \( p \)-norm

\[ (|D_1v|^p + |D_2v|^p)^{\frac{1}{p}} \leq 1. \]
Similarly to [33], our results in this paper can be applied to imply the regularity of the free boundary in two dimensions, when the functional (i.e. $F, g$ in (1.4)) is analytic. But the more general case requires extra analysis, so we leave the question of the free boundary’s regularity to future works.

Let us introduce the problem in more detail. Let $K$ be a compact convex subset of $\mathbb{R}^n$ whose interior contains the origin. We recall from convex analysis (see [36]) that the gauge function of $K$ is the convex function

$$\gamma_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}.$$  

The gauge function $\gamma_K$ is subadditive and positively 1-homogenous, so it looks like a norm on $\mathbb{R}^n$, except that $\gamma_K(-x)$ is not necessarily the same as $\gamma_K(x)$. Note that as $K$ is closed, $K = \{\gamma_K \leq 1\}$; and as $K$ has nonempty interior, $\partial K = \{\gamma_K = 1\}$.

Another notion is that of the polar of $K$

$$K^\circ := \{x : \langle x, y \rangle \leq 1 \text{ for all } y \in K\},$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. $K^\circ$, too, is a compact convex set containing the origin as an interior point.

Let $U \subset \mathbb{R}^n$ be a bounded open set. Let $u$ be the minimizer of

$$J[v] = J[v; U] := \int_U F(Dv) + g(v) \, dx,$$

over

$$W_{K^\circ, \varphi} = W_{K^\circ, \varphi}(U) := \{v \in H^1(U) : Dv \in K^\circ \text{ a.e., } v = \varphi \text{ on } \partial U\}.$$  

Here $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and the equality of $v, \varphi$ on $\partial U$ is in the sense of trace. In order to ensure that $W_{K^\circ, \varphi}$ is nonempty we assume that

$$-\gamma_K(y - x) \leq \varphi(x) - \varphi(y) \leq \gamma_K(x - y),$$

for all $x, y \in \mathbb{R}^n$. Then by Lemma 2.1 of [35] this property implies that $\varphi$ is Lipschitz and $D\varphi \in K^\circ$ a.e.; so $\varphi \in W_{K^\circ, \varphi}$. Also note that $Dv \in K^\circ$ is equivalent to $\gamma_{K^\circ}(Dv) \leq 1$. Thus $\gamma_{K^\circ}$ defines the gradient constraint.

We will show that $u$ is also the unique minimizer of $J$ over

$$W_{\rho, \rho} = W_{\rho, \rho}(U) := \{v \in H^1(U) : -\rho \leq v \leq \rho \text{ a.e., } v = \varphi \text{ on } \partial U\},$$

where

$$\rho(x) = \rho_{K, \varphi}(x; U) := \min_{y \in \partial U} [\gamma_K(x - y) + \varphi(y)],$$

$$\bar{\rho}(x) = \bar{\rho}_{K, \varphi}(x; U) := \min_{y \in \partial U} [\gamma_K(y - x) - \varphi(y)].$$

It is well known (see [32, Section 5.3]) that $\rho$ is the unique viscosity solution of the Hamilton-Jacobi equation

$$\gamma_{K^\circ}(Dv) = 1 \quad \text{in } U,$$

$$v = \varphi \quad \text{on } \partial U.$$  

Note that $\bar{\rho}_{K, \varphi} = \rho_{-K, -\varphi}$, since $\gamma_{-K}(\cdot) = \gamma_K(\cdot)$. Thus we have a similar characterization for $\bar{\rho}$ too.
The paper is organized as follows. In Section 2 we introduce our notation, and we state some preliminary results. We also review some well-known facts about the regularity of $K$, and its relation to the regularity of $K^\circ, \gamma_K, \gamma_K^\circ$. In Section 3 we prove that the minimizer $u$ is in $W^{2,\infty}_{\text{loc}}$, when $\partial U$ is Lipschitz, and we have an upper bound on the weak second derivative of $\gamma_K$. To this end, we first show that $u$ is also the minimizer of $J$ over $W^{2,p}_{\text{loc}}$, i.e., it is the solution of a double obstacle problem. Then we mollify the obstacles, and we solve the double obstacle problems with these smooth mollified obstacles. This gives us functions $u_\epsilon$ which approximate $u$. By using penalization technique, we show that $u_\epsilon$ is in $W^{2,p}_{\text{loc}}$. Then we show that $u_\epsilon$’s have a uniform bound in $W^{2,p}_{\text{loc}}$. Hence we can conclude that $u$ is also in $W^{2,\infty}_{\text{loc}}$. The methods employed here are classical, but to the best of author’s knowledge the results have not appeared elsewhere. Nevertheless, we include the proofs here for completeness. Finally in Theorem 4 using the uniform bound on $D^2u_\epsilon$, we can show that $u$ belongs to $W^{2,\infty}_{\text{loc}}$. This last step was made possible by the work of Figalli and Shahgholian [16], and its generalization by Indrei and Minne [27].

In Section 4 following the approach of [33, 35], we investigate the relation between $u$ and the obstacles $\rho, -\bar{\rho}$. Here we introduce the notion of $\rho$-ridge, as the set of all points where $\rho$ is not $C^{1,1}$ in any neighborhood of them. In other words, the $\rho$-ridge is the set of singularities of $\rho$. We will see that $u$ does not touch the obstacles at many of their points of singularity. But our tools in Section 4 are not strong enough to imply that $u$ does not touch the obstacles at all of their points of singularity. To accomplish this, we have to study the function $\rho$ more carefully. This has been done in Section 5. Remember that $\rho$ is the unique viscosity solution of the Hamilton-Jacobi equation (1.9). We will compute the derivatives of $\rho$, and by using them we will characterize the $\rho$-ridge in Theorem 4. Results of this nature have been appeared in the literature before. For example, Crasta and Malusa [10] studied this problem in the case of $\varphi = 0$. (When $\varphi = 0$, $\rho$ is the distance to $\partial U$ with respect to the Minkowski distance defined by $\gamma_K$.) But the novelty of our work is that we were able to find an explicit formula for $D^2\rho$. This has been done in Theorems 2 and 3.

The formula (5.14) for $D^2\rho$ is very crucial in our analysis, and it has been used several times in this paper. To the best of author’s knowledge, formulas of this kind have not appeared in the literature before, except for the simple case where $\rho$ is the Euclidean distance to the boundary. (Although, some special two dimensional cases also appeared in our earlier work [33].) One of the main applications of this formula is in Lemma 14, which implies that $D^2\rho$ attains its maximum on $\partial U$. This interesting property is actually a consequence of a more general property of the solutions to Hamilton-Jacobi equations. Let $v$ be the solution to a Hamilton-Jacobi equation with convex Hamiltonian. Suppose $v$ is smooth enough on a neighborhood of one of its characteristic curves. Then for every vector $\xi$, $D^2_{\xi\xi}v$ decreases along that characteristic curve, as we move away from the boundary. We have derived this in the remark after Lemma 14. Surprisingly, this monotonicity property, although very simple to deduce, has not appeared in the literature before, again to the best of author’s knowledge.

In the rest of Section 5 we obtain several other interesting facts about $\rho$ and the $\rho$-ridge. See Theorem 5, Proposition 4 and the remarks after them. Finally in Section 6 we state and prove the main results of this paper. In Theorem 6 we show that $u$ does not touch the obstacles at all of their points of singularity. Here, we employ our detailed knowledge about the $\rho$-ridge, which we obtained in Theorem 4. Using this result, in Theorem 7 we can show that $u$ belongs to $W^{2,\infty}(U)$,
without assuming any regularity of the gradient constraint defined by $\gamma_{K^\circ}$. The idea of the proof is to approximate $K^\circ$ with smoother convex sets. Then, as it is common in the study of the regularity of PDEs, we have to find uniform bounds for the various norms of the approximations to $u$. Here, among other estimations, we will use the fact that the second derivative of the approximations to $\rho$ attain their maximums on $\partial U$. Let us also mention that in order to get the regularity up to the boundary, we need to use the result of Indrei and Minne [28], which is another generalization of the work of Figalli and Shahgholian [16]. However, their result is only about flat boundaries. So we have to modify it with standard techniques, to be able to handle arbitrary smooth boundaries. At the end of the paper, we will use similar ideas in Theorem 8, to improve the local regularity result of Theorem 1. We will show that when $U$ is convex, $u$ belongs to $W^{2,\infty}_{\text{loc}}(U)$, without assuming any regularity of the gradient constraint, nor of the $\partial U$. The idea of the proof is to approximate both $K^\circ, U$ with smoother convex sets.

2. Notation and Preliminaries

**Notation.** Here we collect the main notations that are used throughout this paper. Other than these, there will be some notations that are used in some sections, and are introduced at the beginning of that section. Also, there will be some notations that are introduced inside a proof or a paragraph, and are only used there.

1. $\varphi, u$: functions; defined in Assumptions 1 and 2 respectively.
2. $U, K, K^\circ$: subsets of $\mathbb{R}^n$; defined in Assumption 1 and equation (1.5) respectively.
3. $\gamma := \gamma_K, \gamma^\circ := \gamma_{K^\circ}, \bar{\gamma} := \gamma_{-K}$: gauge functions; defined in (1.2).
4. $\rho = \rho_{K, \varphi}(:, U), \bar{\rho} = \bar{\rho}_{K, \varphi}(:, U)$: functions; defined in (1.8).
5. $J = J[:, U], F, g$: functional, functions; defined in Assumption 2.
6. $W_{K^\circ, \varphi}(U), W_{\bar{\rho}, \rho}(U)$: function spaces; defined in (1.5) and (1.7) respectively.
7. $R_\rho, R_{\rho, 0}$: the $\rho$-ridges; defined in Definition 2.
8. $E, P^\pm, P, \Gamma$: the elastic and plastic regions, the free boundary; defined in Definition 3.
9. $\nu$: the inward unit normal to $\partial U$.
10. $\lambda, \mu$: functions on $\partial U$; defined in (5.1) and (5.2) respectively.
11. $W, Q, X$: matrix-valued functions, defined in (5.13) and (5.3) respectively.
12. $H_{x, x}, N(K, x)$: supporting hyperplane and normal cone, defined in (6.4) and (6.5) respectively.
13. $\langle, \rangle, I$: the standard inner product on $\mathbb{R}^n$, and the identity matrix.
14. We denote the transpose of a matrix $A$ by $A^T$, and its action on a vector $x$ by $Ax$. We also denote the trace of $A$ by $\text{tr}(A)$. Depending on the circumstances, we consider vectors to be column vectors or row vectors, without further explanation. But it should be clear from the context which one is intended. For example we write $yAx$, instead of $y^TAx$, to denote $\langle y, Ax \rangle$.
15. $x \otimes y$: the rank 1 matrix whose action on a vector $z$ is $\langle z, y \rangle x$. Note that $(x \otimes y)^T = y \otimes x$.
16. $d(x) := \min_{y \in \partial U} \|x - y\|$ : the Euclidean distance to $\partial U$.
17. $[x, y], [x, y[, [x, y[, [x, y]:$ the closed, open, and half-open line segments with endpoints $x, y$. 

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(18) $B_r(x)$ : the open ball of radius $r$ centered at $x$.

(19) $\text{int}(S), \overline{S}, \partial S$ : the interior, the closure, and the boundary of a set $S$, respectively.

(20) $C^{k,\alpha}, W^{k,p}, H^k = W^{k,2}$ : Holder spaces and Sobolev spaces.

(21) We denote by $C^\omega$ the space of analytic functions (or submanifolds); so in the following when we talk about $C^{k,\alpha}$ regularity with $k$ greater than some fixed integer, we are also including $C^\infty$ and $C^\omega$.

(22) $\alpha, \bar{\alpha}$ : arbitrary but fixed Holder exponents; defined in Assumptions 5 and 2 respectively.

(23) We will use the convention of summing over repeated indices.

**Assumption 1.** Throughout this paper we assume that

(a) $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin.

(b) $U \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, i.e. its boundary is locally the graph of a Lipschitz function.

(c) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function that satisfies (1.7).

**Remark.** We may or may not state that we are using this assumption in the hypotheses of our lemmas and theorems, but nevertheless we assume that this assumption always holds.

Recall that the gauge function $\gamma$ satisfies

$$\gamma(rx) = r\gamma(x),$$

$$\gamma(x + y) \leq \gamma(x) + \gamma(y),$$

for all $x, y \in \mathbb{R}^n$ and $r \geq 0$. Also, note that as $B_c(0) \subseteq K \subseteq B_C(0)$ for some $C \geq c > 0$, we have

$$\frac{1}{C}|x| \leq \gamma(x) \leq \frac{1}{c}|x|,$$

for all $x \in \mathbb{R}^n$.

It is well known that for all $x, y \in \mathbb{R}^n$, we have

$$\langle x, y \rangle \leq \gamma(x)\gamma^0(y).$$

In fact, more is true and we have

$$\gamma^0(y) = \max_{x \neq 0} \frac{\langle x, y \rangle}{\gamma(x)}.$$

For a proof of this, see page 54 of [36].

It is easy to see that the the strict convexity of $K$ (which means that $\partial K$ does not contain any line segment) is equivalent to the strict convexity of $\gamma$. By homogeneity of $\gamma$, the latter is equivalent to

$$\gamma(x + y) < \gamma(x) + \gamma(y)$$

when $x \neq cy$ and $y \neq cx$ for any $c \geq 0$.

Now, note that $-K$ is also a compact convex set whose interior contains the origin. Recall that we denote the gauge function of $-K$ by $\bar{\gamma}$. It is easy to see that

$$\bar{\gamma}(x) = \gamma(-x).$$

Hence as we have noted before $\bar{\rho} = d_{-K,-\varphi}$. 

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Moreover, from the definition of $\rho$ we easily obtain (see the proof of Proposition 1)

\begin{equation}
-\gamma(x-y) \leq \rho(y) - \rho(x) \leq \gamma(y-x),
\end{equation}

for all $x, y \in \mathbb{R}^n$. The above inequality also holds if we replace $\rho, \gamma$ with $\bar{\rho}, \bar{\gamma}$. Thus in particular, $\rho, \bar{\rho}$ are Lipschitz continuous.

**Definition 1.** When $\rho(x) = \gamma(x-y) + \varphi(y)$ for some $y \in \partial U$, we call $y$ a $\rho$-closest point to $x$ on $\partial U$. Similarly, when $\bar{\rho}(x) = \gamma(y-x) - \varphi(y)$ for some $y \in \partial U$, we call $y$ a $\bar{\rho}$-closest point to $x$ on $\partial U$.

As shown in the proof of Proposition 1 for $y \in \partial U$ we have $\rho(y) = \varphi(y)$ and $\bar{\rho}(y) = -\varphi(y)$. Therefore $y$ is a $\rho$-closest point and a $\bar{\rho}$-closest point on $\partial U$ to itself.

**Remark.** We will mostly state our results about $\gamma, \rho$, but it is obvious that they also hold for $\bar{\gamma}, \bar{\rho}$.

**Lemma 1.** Suppose $x_i \in \overline{U}$ converges to $x \in \overline{U}$, and $y_i \in \partial U$ is a (not necessarily unique) $\rho$-closest point to $x_i$.

(a) If $y_i$ converges to $\tilde{y} \in \partial U$, then $\tilde{y}$ is one of the $\rho$-closest points on $\partial U$ to $x$.

(b) If $y \in \partial U$ is the unique $\rho$-closest point to $x$, then $y_i$ converges to $y$.

**Proof.** This lemma is a simple consequence of the continuity of $\gamma, \rho$, and compactness of $\partial U$. For (a) we have

\[ \gamma(x - \tilde{y}) = \lim \gamma(x_i - y_i) = \lim \rho(x_i) = \rho(x). \]

Hence $\tilde{y}$ is a $\rho$-closest point to $x$.

Now to prove (b) suppose to the contrary that $y_i \not\rightarrow y$. Then as $\partial U$ is compact, there is a subsequence $y_{i_k}$ that converges to $z \in \partial U$ where $z \neq y$. Then by (a) $z$ must be a $\rho$-closest point to $x$, which is in contradiction with our assumption. \qed

### 2.1. Regularity of the gauge function

Suppose that $\partial K$ is $C^{k,\alpha}$ ($k \geq 1, 0 \leq \alpha \leq 1$). Let us show that as a result, $\gamma$ is $C^{k,\alpha}$ on $\mathbb{R}^n - \{0\}$. Let $r = \sigma(\theta)$ for $\theta \in S^{n-1}$, be the equation of $\partial K$ in polar coordinates. Then $\sigma$ is positive and $C^{k,\alpha}$. To see this note that locally, $\partial K$ is given by a $C^{k,\alpha}$ equation $f(x) = 0$. On the other hand we have $x = rX(\theta)$, for some smooth function $X$. Hence we have $f(rX(\theta)) = 0$; and the derivative of this expression with respect to $r$ is

\[ \langle X(\theta), Df(rX(\theta)) \rangle = \frac{1}{r} \langle x, Df(x) \rangle. \]

But this is nonzero since $Df$ is orthogonal to $\partial K$, and $x$ cannot be tangent to $\partial K$ (otherwise $0$ cannot be in the interior of $K$, as $K$ lies on one side of its supporting hyperplane at $x$). Thus we get the desired by the Implicit Function Theorem. Now, it is straightforward to check that for a nonzero point in $\mathbb{R}^n$ with polar coordinates $(s, \phi)$ we have

\[ \gamma((s, \phi)) = \frac{s}{\sigma(\phi)}. \]

This formula easily gives the smoothness of $\gamma$. On the other hand, note that as $\partial K = \{\gamma = 1\}$ and $D\gamma \neq 0$ by (2.4), $\partial K$ is as smooth as $\gamma$.

Now, suppose in addition that $K$ is strictly convex. Then $\gamma$ is strictly convex too. By Remark 1.7.14 and Theorem 2.2.4 of [33], $K^0$ is also strictly convex and its boundary is $C^1$. Therefore $\gamma^0$
is strictly convex, and it is $C^1$ on $\mathbb{R}^n - \{0\}$. Hence by Corollary 1.7.3 of [36], for $x \neq 0$ we have (Notice that the strict convexity of $K$ is only needed for the existence of $D\gamma^\circ$.)

$$D\gamma(x) \in \partial K^\circ, \quad D\gamma^\circ(x) \in \partial K,$$

or equivalently

$$\gamma^\circ(D\gamma) = 1, \quad \gamma(D\gamma^\circ) = 1.$$

In particular $D\gamma, D\gamma^\circ$ are nonzero on $\mathbb{R}^n - \{0\}$.

Let us also suppose that $k \geq 2$, and the principal curvatures of $\partial K$ are positive everywhere. Then $K$ is strictly convex. We can also show that $\gamma^\circ$ is $C^{k,\alpha}$ on $\mathbb{R}^n - \{0\}$. To see this, let $n_K : \partial K \to \mathbb{S}^{n-1}$ be the Gauss map, i.e. $n_K(y)$ is the outward unit normal to $\partial K$ at $y$. Then $n_K$ is $C^{k-1,\alpha}$ and its derivative is an isomorphism at the points with positive principal curvatures, i.e. everywhere. Hence $n_K$ is locally invertible with a $C^1$ inverse $n_K^{-1}$, around any point of $\mathbb{S}^{n-1}$. Now note that as it is well known, $\gamma^\circ$ equals the support function of $K$, i.e.

$$\gamma^\circ(x) = \sup\{\langle x, y \rangle : y \in K\}.$$  

Thus as shown on page 115 of [36], for $x \neq 0$ we have

$$D\gamma^\circ(x) = n_K^{-1}(\frac{x}{|x|}).$$

Which gives the desired result. As a consequence, since $\partial K^\circ = \{\gamma^\circ = 1\}$ and $D\gamma^\circ \neq 0$ by (2.4), $\partial K^\circ$ is $C^{k,\alpha}$ too. Furthermore, as shown on page 120 of [36], the principal curvatures of $\partial K^\circ$ are also positive everywhere.

Let us recall a few more properties of $\gamma, \gamma^\circ$. Since they are positively 1-homogenous, $D\gamma, D\gamma^\circ$ are positively 0-homogenous, and $D^2\gamma, D^2\gamma^\circ$ are positively $(-1)$-homogenous, i.e.

$$\gamma(tx) = t\gamma(x), \quad D\gamma(tx) = D\gamma(x), \quad D^2\gamma(tx) = \frac{1}{t}D^2\gamma(x),$$

$$\gamma^\circ(tx) = t\gamma^\circ(x), \quad D\gamma^\circ(tx) = D\gamma^\circ(x), \quad D^2\gamma^\circ(tx) = \frac{1}{t}D^2\gamma^\circ(x),$$

for $x \neq 0$ and $t > 0$. As a result, using Euler’s theorem on homogenous functions we get

$$\langle D\gamma(x), x \rangle = \gamma(x), \quad D^2\gamma(x) x = 0,$$

$$\langle D\gamma^\circ(x), x \rangle = \gamma^\circ(x), \quad D^2\gamma^\circ(x) x = 0,$$

for $x \neq 0$. Here $D^2\gamma(x) x$ is the action of the matrix $D^2\gamma(x)$ on the vector $x$. We also recall the following fact from [10], that for $x \neq 0$

$$D\gamma^\circ(D\gamma(x)) = \frac{x}{\gamma(x)}, \quad D\gamma(D\gamma^\circ(x)) = \frac{x}{\gamma^\circ(x)}.$$  

Finally let us mention that by Corollary 2.5.2 of [36], when $x \neq 0$ the eigenvalues of $D^2\gamma(x)$ are 0 with the corresponding eigenvector $x$, and $\frac{1}{\gamma^\circ(x)}$ times the principal radii curvature of $\partial K^\circ$ at the unique point that has $x$ as an outward normal vector. Remember that the principal radii curvature are the reciprocals of the principal curvatures. Thus by our assumption, the eigenvalues of $D^2\gamma(x)$ are all positive except for one 0. We have a similar characterization of the eigenvalues of $D^2\gamma^\circ(x)$.
3. Local Optimal Regularity

In this section we prove the optimal regularity for a special class of variational problems with gradient constraints. Later we will use these results to obtain the optimal regularity in general. Most of the methods employed in this section are classical and well known, but to the best of author’s knowledge the results have not appeared elsewhere. Especially since the results are about the double obstacle problem, and there are far fewer works on this problem compared to the obstacle problem. Nevertheless, we include the proofs here for completeness.

Notation. First we collect some additional notations that are used in this section.

1. \(c_1, \ldots, c_9\) : positive constants; defined in Assumption 2.
2. \(\phi_\epsilon, \psi_\epsilon\) and \(\delta_\epsilon\) : smooth functions and positive number; defined in (3.3).
3. \(u_\epsilon\) : function; defined in (3.10).
4. \(D^2_{h,\xi}\) : operator on functions; defined in (3.6).
5. \(U_\epsilon, E_\epsilon\) : open sets; defined in (3.4), (3.17) respectively.
6. \(C_0, C_1, \ldots, C_4\) : positive constants; defined in (3.2), (3.6), (3.8), (3.18) respectively.
7. \(J_\epsilon\) and \(W_{\phi_\epsilon, \psi_\epsilon}\) : functional and function space; defined in (3.10).
8. \(\eta_\epsilon\) : the standard mollifier, i.e. \(\eta_\epsilon \in C_\infty(B_\epsilon(0)), \eta_\epsilon \geq 0\) and \(\int_{B_\epsilon(0)} \eta_\epsilon \, dx = 1\).
9. \(f^*\eta_\epsilon\) : the mollification of the function \(f\), i.e. the convolution of \(f\) and \(\eta_\epsilon\).

Assumption 2. We assume that

(a) The functional \(J\) is given by

\[
J[v] = J[v; U] := \int_U F(Dv) + g(v) \, dx,
\]

where \(F : \mathbb{R}^n \to \mathbb{R}\) and \(g : \mathbb{R} \to \mathbb{R}\) are \(C^{2,\bar{\alpha}}\) convex functions satisfying

\[
- c_1 |z|^q \leq g(z) \leq c_2 |z|^2, \quad c_3 |Z|^2 \leq F(Z) \leq c_4 |Z|^2, \quad |g'(z)| \leq c_5 (|z| + 1), \quad |DF(Z)| \leq c_6 |Z|, \quad 0 \leq g'' \leq c_7, \quad c_8 |\xi|^2 \leq D^2_{ij} F(Z) \xi_i \xi_j \leq c_9 |\xi|^2,
\]

for all \(z \in \mathbb{R}\) and \(Z, \xi \in \mathbb{R}^n\). Here, \(c_i > 0, 0 < \bar{\alpha} \leq 1, \text{ and } 1 \leq q < 2\).

(b) The function \(u\) is the minimizer of \(J\) over \(W_{K^\circ, \varphi} = W_{K^\circ, \varphi}(U) := \{v \in H^1(U) : Dv \in K^\circ \text{ a.e., } v = \varphi \text{ on } \partial U\}\).

Remark. Note that by our assumption, \(F\) is strictly convex, and \(F(0) = 0\) is its unique global minimum.

Remark. Since \(W_{K^\circ, \varphi}\) is a nonempty closed convex set, we can apply the direct method of the calculus of variations to conclude the existence of a unique minimizer \(u\). See, for example, the proof of Theorem 3.30 in [11].

Remark. Note that we do not require \(K\) to be strictly convex; thus \(\gamma^0\), which defines the gradient constraint, need not be \(C^1\).
Proposition 1. Suppose the Assumptions hold. Then, is also the minimizer of over
\[ W_{\rho,\rho} = W_{\rho,\rho}(U) := \{ v \in H^1(U) : -\bar{\rho} \leq v \leq \rho \text{ a.e., } v = \varphi \text{ on } \partial U \} . \]

Remark. In fact we can weaken the hypothesis of this theorem, and only assume that are convex and at least one of them is strictly convex. See [35] for details.

Proof. As shown in [38], is also the minimizer of over
\[ \{ v \in H^1(U) : u^- \leq v \leq u^+ \text{ a.e., } v = \varphi \text{ on } \partial U \} , \]
where satisfy for all . We claim that
\[ u^+(x) = \rho(x) = \min_{y \in \partial U} [\gamma(x-y) + \varphi(y)] , \]
\[ u^-(x) = -\bar{\rho}(x) = -\min_{y \in \partial U} [\gamma(y-x) - \varphi(y)] . \]

First, let us show that , . By (1.6), for a given and every we have
\[ \gamma(x-x) + \varphi(x) = \varphi(x) \leq \gamma(x-y) + \varphi(y) , \]
\[ \gamma(x-x) - \varphi(x) = -\varphi(x) \leq \gamma(y-x) - \varphi(y) . \]

Hence , . Next let , . Then due to the compactness of , there is such that . Therefore we have
\[ \rho(x) - \rho(z) \leq \gamma(x-y) + \varphi(y) - \gamma(z-y) - \varphi(y) \leq \gamma(x-z) . \]

Then by Lemma 2.1 of [38], is Lipschitz and . The case of is similar.

Finally, let us show that for an arbitrary we have . Note that is Lipschitz, since is bounded and are Lipschitz. Then similarly to the proof of Lemma 2.2 of [38], we can mollify and use the mean value theorem together with the inequality (2.1) to obtain
\[ v(x) - v(y) \leq \gamma(x-y) , \]
when . Using continuity we can allow to be on too. Hence whenever . Now let in , and consider the line segment . This line segment might not be entirely in , but if we let to be the closest point to on , then we must have . Then
\[ v(x) \leq \gamma(x-y) + \varphi(y) \leq \gamma(x-y) + \gamma(y-z) + \varphi(z) = \gamma(x-z) + \varphi(z) . \]

Hence . Similarly we have and . Therefore \[ -v \leq d_{-K,-\varphi} = \bar{\rho} , \text{ or } v \geq -\bar{\rho} . \]

Remark. The above proof also shows that if and , and , then . In addition it shows that for all (Note that and are continuous). In fact the next lemma implies that
\[ -\bar{\rho}(x) < \rho(x) \]
for .
Remark. Note that as \( u \) has bounded gradient, it is Lipschitz continuous, i.e. belongs to \( C^{0,1}(U) \). Thus, for every \( x \in U \) (not just for a.e. \( x \)) we have
\[
-\rho(x) \leq u(x) \leq \rho(x).
\]
As a result for every \( x \in \partial U \) we have \( u(x) = \varphi(x) \).

Remember that for some \( C_1 \geq C_0 > 0 \), we have
\[
C_0|x| \leq \gamma(x) \leq C_1|x|,
\]
for all \( x \in \mathbb{R}^n \). Obviously, this inequality also holds if we replace \( \gamma \) with \( \bar{\gamma} \).

**Lemma 2.** We have
\[
2C_0d(x) \leq \rho(x) + \bar{\rho}(x) \leq 2C_1d(x),
\]
where \( d \) is the Euclidean distance to \( \partial U \).

Remark. Note that the above inequality implies that the two obstacles do not touch inside \( U \).

**Proof.** Let \( y \in \partial U \). Then we have
\[
\rho(x) + \bar{\rho}(x) \geq \gamma(x - y) + \varphi(y) + \gamma(y - x) - \varphi(y) \geq 2C_0|x - y| \geq 2C_0d(x).
\]
On the other hand \( \rho + \bar{\rho} \) vanishes on \( \partial U \). Hence by (2.3) for every \( y \in \partial U \) we have
\[
\rho(x) + \bar{\rho}(x) = \rho(x) + \bar{\rho}(x) - \rho(y) - \bar{\rho}(y) \leq \gamma(x - y) + \bar{\gamma}(x - y) \leq 2C_1|x - y|.
\]
Now as \( y \) is arbitrary we get \( \rho(x) + \bar{\rho}(x) \leq 2C_1d(x) \).

Let \( \eta_\varepsilon \) be the standard mollifier. Then we define
\[
\psi_\varepsilon(x) := (\eta_\varepsilon * \rho)(x) := \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)\rho(x - y)\,dy,
\]
(3.3)
\[
\phi_\varepsilon(x) := -(\eta_\varepsilon * \bar{\rho})(x) + \delta_\varepsilon,
\]
where \( 4C_1\varepsilon < \delta_\varepsilon < 6C_1\varepsilon \) is chosen such that \( \partial\{\phi_\varepsilon < \psi_\varepsilon\} \) is \( C^\infty \), which is possible by Sard’s Theorem. Note that since \( \rho, \bar{\rho} \) are defined on all of \( \mathbb{R}^n \), \( \psi_\varepsilon, \phi_\varepsilon \) are smooth functions on \( \mathbb{R}^n \). Also
\[
|\psi_\varepsilon(x) - \rho(x)| \leq \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)|\rho(x - y) - \rho(x)|\,dy
\]
\[
\leq \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)\max\{\gamma(-y), \gamma(y)\}\,dy \leq \int_{|y| \leq \varepsilon} C_1|y|\eta_\varepsilon(y)\,dy \leq C_1\varepsilon.
\]
Notice that we used (2.3) in the second inequality. Similarly we have
\[
3C_1\varepsilon < \phi_\varepsilon - (-\bar{\rho}) < 7C_1\varepsilon.
\]

Now, let
(3.4)
\[
U_\varepsilon := \{x \in U : \phi_\varepsilon(x) < \psi_\varepsilon(x)\}.
\]
Then we have
(3.5)
\[
\{x \in U : d(x) > \frac{4C_1}{C_0}\varepsilon\} \subset \{x \in \overline{U} : \phi_\varepsilon(x) \leq \psi_\varepsilon(x)\} \subset \{x \in U : d(x) > \varepsilon\}.
\]
Thus \( \delta \) have chosen (3.6)

\[
D \text{ Assumption 3.}
\]

We assume that

\[
\text{Lemma 3. Assumption 3 holds when}
\]

\[\text{Note that} \]

\[\text{Remark. The above inclusions show that} \ U_\epsilon \subset U, \text{and} \ U = \bigcup_{\epsilon > 0} U_\epsilon. \text{In addition, remember that we have chosen} \ \delta \ \text{so that} \ \partial U_\epsilon \text{is} C^\infty.\]

**Assumption 3.** We assume that

\[
(3.6) \quad D^2_{h,\xi} \gamma(x) := \frac{\gamma(x + h\xi) + \gamma(x - h\xi) - 2\gamma(x)}{h^2} \leq \frac{C_2}{\gamma(x) - h},
\]

for some \( C_2 > 0 \), and every nonzero \( x, \xi \in \mathbb{R}^n \) with \( \gamma(\xi), \gamma(-\xi) \leq 1 \), and every \( 0 < h < \gamma(x) \).

**Remark.** Note that \( \gamma \) satisfies Assumption 3 if and only if \( \bar{\gamma} \) does.

**Lemma 3.** Assumption 3 holds when \( \gamma \text{ is} C^2 \text{ on} \ \mathbb{R}^n - \{0\}, \text{or equivalently when} \ \partial K \text{ is} C^2. \)

**Proof.** First note that \( \gamma \) is nonzero on the segment \( \{x + \tau\xi : -h \leq \tau \leq h\} \). Because \( \gamma(x) > h \) and \( \gamma(\xi), \gamma(-\xi) \leq 1 \), so by the triangle inequality we get

\[
(3.7) \quad \gamma(x + \tau\xi) \geq \gamma(x) - \gamma(-\tau\xi) = \gamma(x) - |\tau|\gamma(\pm\xi) \geq \gamma(x) - h > 0.
\]

Thus \( \gamma \) is twice differentiable on this segment. Therefore, we can apply the mean value theorem to the restriction of \( \gamma \) and \( D\xi \gamma \) to the segment. Hence we get

\[
D^2_{h,\xi} \gamma(x) = \frac{\gamma(x + h\xi) - \gamma(x) + \gamma(x - h\xi) - \gamma(x)}{h^2} = \frac{hD\xi \gamma(x + s\xi) - hD\xi \gamma(x - t\xi)}{h^2} = \frac{(s + t)}{h}D^2_{\xi\xi} \gamma(x + r\xi) \leq 2D^2_{\xi\xi} \gamma(x + r\xi).
\]

Here, \( 0 < s, t < h \) and \( -t < r < s \); and we used the fact that \( D^2_{\xi\xi} \gamma \geq 0 \), due to the convexity of \( \gamma \). Now, let \( C_2 > 0 \) be the maximum of the continuous function

\[
(w, v) \mapsto 2D^2_{wv} \gamma(v) = 2 \langle D^2 \gamma(v) w, w \rangle
\]

over the compact set \( (K \cap (-K)) \times \partial K \). Then by \((-1)\)-homogeneity of \( D^2 \gamma \) we get

\[
2D^2_{\xi\xi} \gamma(x + r\xi) = \frac{2}{\gamma(x + r\xi)}D^2_{\xi\xi} \gamma\left(\frac{x + r\xi}{\gamma(x + r\xi)}\right) \leq \frac{C_2}{\gamma(x + r\xi)} \leq \frac{C_2}{\gamma(x) - h}.
\]

Which is the desired result. Note that in the last inequality above, we used (3.7).

**Lemma 4.** Suppose the Assumptions 3 hold. Then we have

\[
D\phi, D\psi \in K^\circ.
\]
Furthermore, for any unit vector $\xi$, and every $x \in U$ with $d(x) > \epsilon$ we have

\[
D_{\xi}^2 \psi_{\epsilon}(x) \leq \frac{C_3}{d(x) - \epsilon},
\]

(3.8)

\[
D_{\xi}^2 \phi_{\epsilon}(x) \geq \frac{-C_3}{d(x) - \epsilon},
\]

where $C_3 := C_0^{-1}C_1^2C_2$, and $d$ is the Euclidean distance to $\partial U$.

**Proof.** To show the first part, note that $\rho, \bar{\rho}$ are Lipschitz functions and $D\rho, -D\bar{\rho} \in K^0$ a.e., as shown in [38] using the property (2.3). Then because of Jensen's inequality, and convexity and homogeneity of $\gamma^0$, we have

\[
\gamma^0(D\psi_{\epsilon}(x)) \leq \int_{|y| \leq \epsilon} \gamma^0(\eta_{\epsilon}(y)D\rho(x - y)) \, dy
\]

\[
= \int_{|y| \leq \epsilon} \eta_{\epsilon}(y)\gamma^0(D\rho(x - y)) \, dy \leq \int_{|y| \leq \epsilon} \eta_{\epsilon}(y) \, dy = 1.
\]

The case of $\phi_{\epsilon}$ is similar.

Next, we assume initially that $\gamma(\xi), \gamma(-\xi) \leq 1$. Let $x \in U$, then

\[
\rho(x) = \gamma(x - y) + \varphi(y)
\]

for some $y \in \partial U$. We also have $\rho(\cdot) \leq \gamma(\cdot - y) + \varphi(y)$. Thus by (3.6) we get

\[
\mathcal{D}_{\eta_{\epsilon},\xi}^2 \rho(x) := \frac{\rho(x + h\xi) + \rho(x - h\xi) - 2\rho(x)}{h^2}
\]

(3.9)

\[
\leq \frac{\gamma(x + h\xi - y) + \varphi(y) + \gamma(x - h\xi - y) + \varphi(y) - 2(\gamma(x - y) + \varphi(y))}{h^2}
\]

\[
= \mathcal{D}_{\eta_{\epsilon},\xi}^2 \gamma(x - y) \leq \frac{C_2}{\gamma(x - y) - h} \leq \frac{C_2}{C_0|x - y| - h} \leq \frac{C_2}{C_0d(x) - h},
\]

for $0 < h < C_0d(x)$.

Now suppose $d(x) > C_0^{-1}h + \epsilon$. Then due to the Lipschitz continuity of $d$, for $|y| \leq \epsilon$ we have

\[
C_0d(x - y) \geq C_0d(x) - C_0|y| \geq C_0d(x) - C_0\epsilon > h.
\]

Hence by (3.3) we get

\[
\mathcal{D}_{\eta_{\epsilon},\xi}^2 \psi_{\epsilon}(x) = \int_{|y| \leq \epsilon} \eta_{\epsilon}(y)\mathcal{D}_{\eta_{\epsilon},\xi}^2 \rho(x - y) \, dy
\]

\[
\leq \int_{|y| \leq \epsilon} \eta_{\epsilon}(y)\frac{C_2}{C_0d(x) - C_0\epsilon - h} \, dy = \frac{C_2}{C_0d(x) - C_0\epsilon - h}.
\]

Let $h \to 0^+$. Then for $x \in U$ with $d(x) > \epsilon$ we get

\[
D_{\xi}^2 \psi_{\epsilon}(x) \leq \frac{C_0^{-1}C_2}{d(x) - \epsilon}.
\]
Now assume that $|\xi| = 1$. Then for $\hat{\xi} := \frac{1}{C_1} \xi$ we have $\gamma(\hat{\xi}), \gamma(-\hat{\xi}) \leq 1$. We can apply the above inequality to $\hat{\xi}$ to get

$$D^2_{\xi \xi} \psi_\epsilon(x) = C_1^2 D^2_{\hat{\xi} \hat{\xi}} \psi_\epsilon(x) \leq \frac{C_0^{-1} C_1^2 C_2}{d(x) - \epsilon}.$$  

The inequality for $\phi_\epsilon$ follows similarly. \qed

Next we define:

Let $u_\epsilon$ be the minimizer of

$$J_\epsilon[v] := J[v; U_\epsilon] = \int_{U_\epsilon} F(Dv) + g(v) \, dx$$

over $W_{\phi_\epsilon, \psi_\epsilon} := \{v \in H^1(U_\epsilon) : \phi_\epsilon \leq v \leq \psi_\epsilon \text{ a.e.}\}$. Take an arbitrary $v$ in the above space. Then $u_\epsilon + t(v - u_\epsilon)$ is in this space for $0 \leq t \leq 1$. Thus

$$\frac{d}{dt} \bigg|_{t=0} J_\epsilon[u_\epsilon + t(v - u_\epsilon)] \geq 0.$$  

By using the bounds (3.1), we arrive at the variational inequality

$$\int_{U_\epsilon} D_iF(Du_\epsilon) D_i(v - u_\epsilon) + g'(u_\epsilon)(v - u_\epsilon) \, dx \geq 0.$$  

For the details see, for example, the proof of Theorem 3.37 in [11].

**Lemma 5.** Suppose the Assumptions 1,2,3 hold. Then we have

$$u_\epsilon \in \bigcap_{p<\infty} W^{2,p}(U_\epsilon) \subset \bigcap_{i<1} C^{1,\tilde{\alpha}}(U_\epsilon).$$

**Proof.** For $\delta > 0$, let $\tilde{\beta}_\delta$ be a smooth increasing convex function on $\mathbb{R}$, that vanishes on $(-\infty, 0]$, and equals $\frac{1}{2\delta} t^2$ for $t \geq \delta$. Set $\beta_\delta := \tilde{\beta}_\delta'$. Then $\beta_\delta$ is a smooth increasing function that vanishes on $(-\infty, 0]$, and equals $\frac{1}{\delta} t$ for $t \geq \delta$. We further assume that $\beta_\delta$ is convex too. Let $u_{\epsilon, \delta}$ be the minimizer of

$$J_{\epsilon, \delta}[v] := \int_{U_\epsilon} F(Dv) + g(v) + \tilde{\beta}_\delta(\phi_\epsilon - v) + \tilde{\beta}_\delta(v - \psi_\epsilon) \, dx,$$

over $\phi_\epsilon + H^1_0(U_\epsilon)$. By Theorems 3.30, 3.37 in [11], $u_{\epsilon, \delta}$ exists and is the unique weak solution to the Euler-Lagrange equation

$$-D_i(D_i F(Du_{\epsilon, \delta})) + g'(u_{\epsilon, \delta})(\phi_\epsilon - u_{\epsilon, \delta}) + \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon) = 0,$$

$$u_{\epsilon, \delta} = \phi_\epsilon \text{ on } \partial U_\epsilon.$$  

As proved in [21], $u_{\epsilon, \delta} \in C^{1,\alpha_0}(\overline{U_\epsilon})$ for some $\alpha_0 > 0$. On the other hand, as shown in Chapter 2 of [20], by using the difference quotient technique we get $u_{\epsilon, \delta} \in H^2_{\text{loc}}(U_\epsilon)$. Hence we have

$$-a_{ij, \delta}(x) D^2_{ij} u_{\epsilon, \delta}(x) = b_\delta(x),$$

where $a_{ij, \delta}$ is the matrix defined in (3.12), $b_\delta$ is the function defined in (3.13), and $\delta > 0$. We also assume that $\delta$ is small enough so that $a_{ij, \delta}$ is positive definite. Then we can apply the inequality (3.1) to $u_{\epsilon, \delta}$ to get

$$D^2_{\xi \xi} \psi_\epsilon(x) = C_1^2 D^2_{\hat{\xi} \hat{\xi}} \psi_\epsilon(x) \leq \frac{C_0^{-1} C_1^2 C_2}{d(x) - \epsilon}.$$  

The inequality for $\phi_\epsilon$ follows similarly.
for a.e. \( x \in U_\epsilon \). Where \( a_{ij,\delta}(x) := D_{ij}^2 F(Du_{e,\delta}(x)) \), and
\[
b_{\delta} := -g'(u_{e,\delta}) + \beta_\delta(\phi_\epsilon - u_{e,\delta}) - \beta_\delta(u_{e,\delta} - \psi_\epsilon).
\]
Note that \( a_{ij,\delta} \in C^{0,\alpha_1}(\overline{U}_\epsilon) \), \( b_{\delta} \in C^{1,\alpha_1}(\overline{U}_\epsilon) \), where \( \alpha_1 = \min\{\bar{\alpha}, \alpha_0\} \). Thus by using Schauder estimates (see Theorem 6.14 of \([22]\)) we deduce that \( u_{e,\delta} \in C^{2,\alpha_1}(\overline{U}_\epsilon) \).

We can easily show that \( u_{e,\delta} \) is uniformly bounded, independently of \( \delta \). Suppose \( \delta \leq \min\{1, \frac{1}{4\epsilon_0}\} \), and \( C^+ \geq 1 + 2 \max_{x \in \overline{U}_\epsilon} |\psi_\epsilon(x)| \). Then by the comparison principle (Theorem 10.1 of \([22]\)) to show that \( u_{e,\delta} \leq C^+ \), it is enough to show that
\[
-a_{ij} D_{ij}^2 C^+ + g'(C^+) \geq -c_5(C^+ + 1) + \frac{1}{\delta}(C^+ - \psi_\epsilon) \geq \frac{1}{2\delta} - c_5 \geq c_5 C^+ - c_5 \geq 0.
\]
Similarly we can obtain a uniform lower bound for \( u_{e,\delta} \).

Now, add \( D_i(D_i F(Du_{e,\delta})) \) to the both sides of (3.13), and multiply the result by \((\beta_\delta(u_{e,\delta} - \psi_\epsilon))^{p-1}\) for some \( p > 2 \), and integrate over \( U_\epsilon \) to obtain
\[
\int_{U_\epsilon} [\frac{D_i(D_i F(Du_{e,\delta})) + D_i(D_i F(D\psi_\epsilon))}{\beta_\delta(u_{e,\delta} - \psi_\epsilon)}]^{p-1} dx + \int_{U_\epsilon} (\beta_\delta(u_{e,\delta} - \psi_\epsilon))^{p} dx = \int_{U_\epsilon} [D_i(D_i F(D\psi_\epsilon)) - g'(u_{e,\delta})]^{p-1} dx.
\]
Note that \( \beta_\delta(\phi_\epsilon - u_{e,\delta})\beta_\delta(u_{e,\delta} - \psi_\epsilon) = 0 \). After integration by parts, the first term becomes
\[
(p-1) \int_{U_\epsilon} [D_i F(Du_{e,\delta}) - D_i F(D\psi_\epsilon)][D_i u_{e,\delta} - D_i \psi_\epsilon] \beta_\delta(u_{e,\delta} - \psi_\epsilon) \geq 0.
\]
Note that we used the facts that \( F \) is convex, and \( u_{e,\delta} - \psi_\epsilon \) vanishes on \( \partial U_\epsilon \). By employing this inequality in (3.14) we get
\[
\int_{U_\epsilon} (\beta_\delta(u_{e,\delta} - \psi_\epsilon))^{p} dx \leq C_\epsilon \int_{U_\epsilon} (\beta_\delta(u_{e,\delta} - \psi_\epsilon))^{p-1} dx \leq C_\epsilon |U| \left( \int_{U_\epsilon} (\beta_\delta(u_{e,\delta} - \psi_\epsilon))^{p} dx \right)^{\frac{p-1}{p}}.
\]
Here \( C_\epsilon \) is a constant independent of \( \delta \), and \( |U| \) is the Lebesgue measure of \( U \). Also in the second line we used the uniform boundedness of \( u_{e,\delta} \), and in the last line we used Holder’s inequality. Thus we have
\[
\|\beta_\delta(u_{e,\delta} - \psi_\epsilon)\|_{\mathcal{L}^p(U_\epsilon)} \leq C_\epsilon |U|^{\frac{1}{p}}.
\]
By sending \( p \to \infty \) we get
\[
\|\beta_\delta(u_{e,\delta} - \psi_\epsilon)\|_{\mathcal{L}^\infty(U_\epsilon)} \leq C_\epsilon.
\]
Similarly we obtain \( \| \beta_\delta(\phi_\epsilon - u_{\epsilon,\delta}) \|_{L^\infty(U_\epsilon)} \leq C_\epsilon \). Consequently we have
\[
(3.15) \quad u_{\epsilon,\delta} - \psi_\epsilon \leq \delta(C_\epsilon + 1), \quad \phi_\epsilon - u_{\epsilon,\delta} \leq \delta(C_\epsilon + 1).
\]
Utilizing these bounds, and the fact that \( u_{\epsilon,\delta} \) is uniformly bounded, in equation (3.13), gives us
\[
\| D_i(D_iF(Du_{\epsilon,\delta})) \|_{L^\infty(U_\epsilon)} \leq C,
\]
for some \( C \) independent of \( \delta \). Equivalently we have the quasilinear elliptic equation
\[
-D_i(D_iF(Du_{\epsilon,\delta})) = b_\delta(x),
\]
and \( \| b_\delta \|_{L^\infty(U_\epsilon)} \leq C \). Then Theorem 15.9 of [22] implies that \( \| Du_{\epsilon,\delta} \|_{C^0(\overline{U}_\epsilon)} \leq C \), for some \( C \) independent of \( \delta \). Thus by Theorem 13.2 of [22] we have \( \| u_{\epsilon,\delta} \|_{C^{1,\alpha_2}(\overline{U}_\epsilon)} \leq C \), for some \( C, \alpha_2 > 0 \) independent of \( \delta \).

Now we have
\[
\| a_{ij,\delta} D^2_{ij} u_{\epsilon,\delta} \|_{L^\infty(U_\epsilon)} = \| D_i(D_iF(Du_{\epsilon,\delta})) \|_{L^\infty(U_\epsilon)} \leq C,
\]
where \( a_{ij,\delta} = D^2_{ij} F(Du_{\epsilon,\delta}) \). Then by Theorem 9.13 of [22] we have
\[
\| u_{\epsilon,\delta} \|_{W^{2,p}(U_\epsilon)} \leq C_p,
\]
for all \( p < \infty \), and some \( C_p \) independent of \( \delta \). Here we used the fact that \( a_{ij,\delta} \)'s have a uniform modulus of continuity independently of \( \delta \), due to the uniform boundedness of the \( C^{\alpha_2} \) norm of \( Du_{\epsilon,\delta} \).

Therefore there is a sequence \( \delta_i \to 0 \) such that \( u_{\epsilon,\delta_i} \) weakly converges in \( W^{2,p}(U_\epsilon) \) to a function \( \tilde{u}_\epsilon \). In addition, we can assume that \( u_{\epsilon,\delta_i}, Du_{\epsilon,\delta_i} \) uniformly converge to \( \tilde{u}_\epsilon, D\tilde{u}_\epsilon \), since \( \| u_{\epsilon,\delta} \|_{C^{1,\alpha_2}(\overline{U}_\epsilon)} \) is bounded independently of \( \delta \).

Finally, we want to show that \( \tilde{u}_\epsilon = u_\epsilon \). Note that by (3.15) we have \( \phi_\epsilon \leq \tilde{u}_\epsilon \leq \psi_\epsilon \). Hence, it suffices to show that \( \tilde{u}_\epsilon \) is the minimizer of \( J_\epsilon \) over \( W_{\phi_\epsilon,\psi_\epsilon} \). Take \( v \in W_{\phi_\epsilon,\psi_\epsilon} \subset \phi_\epsilon + H^1_0(U_\epsilon) \). Then we have
\[
J_\epsilon[u_{\epsilon,\delta_i}] \leq J_\epsilon[u_{\epsilon,\delta_i}] \leq J_\epsilon[v] = J_\epsilon[v].
\]
Note that the extra terms in \( J_{\epsilon,\delta} \) (defined in (3.12)) vanish for this \( v \), since \( \phi_\epsilon \leq v \leq \psi_\epsilon \). Now sending \( i \to \infty \) gives the desired due to the uniform convergence of \( u_{\epsilon,\delta_i}, Du_{\epsilon,\delta_i} \) to \( \tilde{u}_\epsilon, D\tilde{u}_\epsilon \).

Since \( u_\epsilon \in H^2(U_\epsilon) \), we can integrate by parts in (3.11), and use appropriate test functions in place of \( v \), to obtain
\[
(3.16) \quad \begin{cases}
-D_i(D_iF(Du_\epsilon)) + g'(u_\epsilon) = 0 & \text{if } \phi_\epsilon < u_\epsilon < \psi_\epsilon, \\
-D_i(D_iF(Du_\epsilon)) + g'(u_\epsilon) \leq 0 & \text{a.e. if } \phi_\epsilon < u_\epsilon \leq \psi_\epsilon, \\
-D_i(D_iF(Du_\epsilon)) + g'(u_\epsilon) \geq 0 & \text{a.e. if } \phi_\epsilon \leq u_\epsilon < \psi_\epsilon.
\end{cases}
\]
Note that \( u_\epsilon \) is \( C^{2,\alpha} \) on the open set
\[
E_\epsilon := \{ x \in U_\epsilon : \phi_\epsilon(x) < u_\epsilon(x) < \psi_\epsilon(x) \},
\]
due to the Schauder estimates (see Theorem 6.13 of [22]).

**Lemma 6.** Suppose the Assumptions hold. Then we have
\[
Du_\epsilon \in K^0 \quad \text{in } U_\epsilon.
\]
Proof. First note that $Du_\epsilon$ is continuous on $\overline{U}_\epsilon$. Now since $u_\epsilon = \phi_\epsilon = \psi_\epsilon$ on $\partial U_\epsilon$, we have $D_\xi u_\epsilon = D_\xi \phi_\epsilon = D_\xi \psi_\epsilon$ for any direction $\xi$ tangent to $\partial U_\epsilon$. Also as $\phi_\epsilon \leq u_\epsilon \leq \psi_\epsilon$ in $U_\epsilon$, we have $D_\nu \phi_\epsilon \leq D_\nu u_\epsilon \leq D_\nu \psi_\epsilon$ on $\partial U_\epsilon$, where $\nu$ is the inward normal to $\partial U_\epsilon$. Hence by (2.2), and the fact that $D\phi_\epsilon, D\psi_\epsilon \in K^\gamma$, we get

$$\gamma^\circ(Du_\epsilon) \leq 1 \quad \text{on } \partial U_\epsilon.$$  

This bound also holds on the sets $\{u_\epsilon = \psi_\epsilon\}$ and $\{u_\epsilon = \phi_\epsilon\}$, as either $\psi_\epsilon - u_\epsilon$ or $u_\epsilon - \phi_\epsilon$ attains its minimum there, so $Du_\epsilon$ equals $D\psi_\epsilon$ or $D\phi_\epsilon$ over them.

To obtain the bound on the open set $E_\epsilon$, note that for any vector $\xi$ with $\gamma(\xi) = 1$, $D_\xi u_\epsilon$ is a weak solution to the elliptic equation

$$-D_i(a_{ij}D_j D_\xi u_\epsilon) + bD_\xi u_\epsilon = 0 \quad \text{in } E_\epsilon,$$

where $a_{ij} := D^2_{ij}F(Du_\epsilon)$, and $b := g''(u_\epsilon)$. Now suppose that $D_\xi u_\epsilon$ attains its maximum at $x_0 \in E_\epsilon$ with $D_\xi u_\epsilon(x_0) > 1$. Then the strong maximum principle (Theorem 8.19 of [22]) implies that $D_\xi u_\epsilon$ is constant over $E_\epsilon$. This contradicts the fact that $D_\xi u_\epsilon \leq 1$ on $\partial E_\epsilon$. Thus we must have $D_\xi u_\epsilon \leq 1$ on $E_\epsilon$; and as $\xi$ is arbitrary, we get the desired bound using (2.2). \qed

**Lemma 7.** Suppose the Assumptions [123] hold. Then for $\epsilon < 1$ we have

$$|D_i(D_jF(Du_\epsilon))| \leq C_4 + \frac{nc_3 C_3}{d - \epsilon} \quad \text{a.e. on } U_\epsilon,$$

(3.18)

$$|D^2 \psi_\epsilon| \leq \frac{1}{c_8} \left( C_4 + \frac{nc_3 C_3}{d - \epsilon} \right) \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\},$$

$$|D^2 \phi_\epsilon| \leq \frac{1}{c_8} \left( C_4 + \frac{nc_3 C_3}{d - \epsilon} \right) \quad \text{a.e. on } \{u_\epsilon = \phi_\epsilon\},$$

where $d$ is the Euclidean distance to $\partial U$, and $C_4 := c_5(\max|\rho|, |\bar{\rho}| + 7C_1 + 1)$.

**Remark.** Note that for a function $f$

$$|D^2 f| = \max_{|\xi| = 1} |D^2_{\xi\xi} f| = \max\{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } D^2 f\}.$$

**Proof.** On the open set $E_\epsilon \subset U_\epsilon$, we have

$$|D_i(D_jF(Du_\epsilon))| = |g'(u_\epsilon)| \leq c_5(|u_\epsilon| + 1) \leq c_5(\max\{|\phi_\epsilon|, |\psi_\epsilon|\} + 1) \leq c_5(\max\{|\rho|, |\bar{\rho}|\} + 7C_1 \epsilon + 1) \leq C_4.$$

Next consider the closed subset of $U_\epsilon$ over which $u_\epsilon = \psi_\epsilon$. By (3.15), we have

$$D_i(D_jF(Du_\epsilon)) \geq g'(u_\epsilon) \geq -C_4 \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\}.$$

Since both $u_\epsilon, \psi_\epsilon$ are twice weakly differentiable, we have (see Theorem 4.4 of [13])

$$D_i(D_jF(Du_\epsilon)) = D_i(D_jF(D\psi_\epsilon)) \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\}.$$

But we have

$$D_i(D_jF(D\psi_\epsilon)) = D^2_{ij} F(D\psi_\epsilon) D^2 \psi_\epsilon = \text{tr}[D^2 F(D\psi_\epsilon) D^2 \psi_\epsilon]$$

$$= \sum_{i \leq n} D^2_{\xi_i \xi_i} F(D\psi_\epsilon) D^2_{\xi_i \xi_i} \psi_\epsilon,$$
where $\xi_1, \cdots, \xi_n$ is an orthonormal basis of eigenvectors of $D^2\psi_c$. Thus, by using (3.1), (3.8) we get

$$D_i(D_i F(D u_\epsilon)) \leq \sum_{i \leq n} D^2_{i,\xi_i} F(D\psi_c) \frac{C_3}{d(x) - \epsilon} \leq n c_9 C_3 \frac{C_3}{d(x) - \epsilon} \quad \text{a.e. on } \{u_\epsilon = \psi_c\}.$$  

We have similar bounds on the set $\{u_\epsilon = \phi_\epsilon\}$. These bounds easily give the first inequality of (3.18).

On the other hand for a.e. $x \in \{u_\epsilon = \psi_c\}$ we have

$$\sum_{i \leq n} D^2_{i,\xi_i} F(D\psi_c) D^2_{i,\xi_i} \psi_\epsilon = D_i(D_i F(D\psi_c)) = D_i(D_i F(D u_\epsilon)) \geq g'(u_\epsilon) \geq -C_4.$$  

Thus again by using (3.1), (3.8) we get

$$D^2_{i,\xi_i} F(D\psi_c) D^2_{i,\xi_i} \psi_\epsilon \geq -C_4 - \sum_{i \neq j} D^2_{i,\xi_i} F(D\psi_c) D^2_{i,\xi_i} \psi_\epsilon \geq -C_4 - \frac{(n-1)c_9 C_3}{d(x) - \epsilon}.$$  

Hence

$$D^2_{i,\xi_i} \psi_\epsilon \geq -\frac{C_4}{c_8} - \frac{(n-1)c_9 C_3}{d(x) - \epsilon}.$$  

The reverse inequality is given by (3.8) (Keep in mind that $\frac{a_8}{c_8} \geq 1$). Note that the numbers $D^2_{i,\xi_i} \psi_\epsilon$ are the eigenvalues of $D^2\psi_c$, so we get the desired bound. The case of $D^2\phi_\epsilon$ is similar. \qed

**Theorem 1.** Suppose the Assumptions (3.4) hold. Then we have

$$u \in W^{2,\infty}(U) = C^{1,1}_{\text{loc}}(U).$$

**Proof.** We choose a decreasing sequence $\epsilon_k \to 0$ such that $\overline{U}_{\epsilon_k} \subset U_{\epsilon_{k+1}}$ (this is possible by (3.5)). For convenience we use $U_k, u_k, \phi_{k}, \psi_k$ instead of $U_{\epsilon_k}, u_{\epsilon_k}, \phi_{\epsilon_k}, \psi_{\epsilon_k}$. Consider the sequence $u_k|_{U_3}$ for $k > 3$. By (3.18), (3.5) we have

$$\|D_i(D_i F(D u_k))\|_{L^\infty(U_3)} \leq C,$$  

for some $C$ independent of $k$. Let $g_k := D_i(D_i F(D u_k))$. Then $D u_k$ is a weak solution to the elliptic equation

$$-D_i(a_{ij,k} D_j D u_k) + D g_k = 0,$$  

where $a_{ij,k} := D^2_{ij} F(D u_k)$. Thus by Theorem 8.24 of [22] we have

$$\|D u_k\|_{C^{\alpha_0}(\overline{U}_3)} \leq C,$$  

for some $C, \alpha_0 > 0$ independent of $k$. Here we used the fact that $D u_k, g_k, a_{ij,k}$ are uniformly bounded independently of $k$ (Remember that $D u_k \in K^\circ$).

Now we have

$$\|a_{ij,k} D^2_{ij} u_k\|_{L^\infty(U_2)} = \|D_i(D_i F(D u_k))\|_{L^\infty(U_2)} \leq C.$$  

Then by Theorem 9.11 of [22] we have

$$\|u_k\|_{W^{2,p}(U_1)} \leq C_p,$$  

for all $p < \infty$, and some $C_p$ independent of $k$. Here we used the fact that $a_{ij,k}$’s have a uniform modulus of continuity independently of $k$, due to the uniform boundedness of the $C^{\alpha_0}$ norm of $D u_k$.

Consequently, as $\partial U_1$ is smooth, for every $\alpha < 1$, $\|u_k\|_{C^{1,\alpha}(\overline{U}_1)}$ is bounded independently of $k$.  

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Therefore there is a subsequence of $u_k$’s, which we denote by $u_{k_l}$, that weakly converges in $W^{2,p}(U_1)$ to a function $\tilde{u}_1$. In addition, we can assume that $u_{k_l}, Du_{k_l}$ uniformly converge to $\tilde{u}_1, D\tilde{u}_1$. Now we can repeat this process with $u_{k_l}|_{U_2}$ and get a function $\tilde{u}_2$ in $W^{2,p}(U_2)$, which agrees with $\tilde{u}_1$ on $U_1$. Continuing this way with subsequences $u_{k_l}$ for each positive integer $l$, we can finally construct a $C^1$ function $\tilde{u}$ in $W^{2,p}(U)$. It is obvious that $D\tilde{u} \in K^\circ$ and $-\bar{\rho} \leq \tilde{u} \leq \rho$, since $Du_k \in K^\circ$ and $\phi_k \leq u_k \leq \psi_k$ for every $k$. In particular we have $\tilde{u} \in W^{2,p}(\bar{U}_1, \bar{U}_2)$.

Now we want to show that $u = \tilde{u}$. Due to the uniqueness of the minimizer, it is enough to show that $\tilde{u}$ is the minimizer of $J$ over $W^{2,p}(U)$. As it is well known, it suffices to show that (see, for example, the proof of Theorem 3.37 in [11])

\[
(3.20) \quad \int_U D_i F(D\tilde{u})D_i(v - \tilde{u}) + g'(\tilde{u})(v - \tilde{u}) \, dx \geq 0,
\]

for every $v \in W^{2,p}(U)$. Note that $v$ is Lipschitz continuous. First suppose that $v > -\bar{\rho}$ on $U$, and $v(\rho(x) + \rho(x) \leq \delta)$ for some $\delta > 0$. Let $v_k := \eta_{\epsilon_k} \ast v$ be the mollification of $v$. Then for large enough $k$ we have $\phi_k \leq v_k \leq \psi_k$ on $U_k$. Because for large enough $k$ we have $\phi_k \leq v_k = \psi_k$ on $\bar{\Omega}_k \cap \{\rho + \bar{\rho} \leq \delta/2\}$. (Note that for $x \in U \cap \{\rho + \bar{\rho} \leq \delta/2\}$ we have $B_{\epsilon_k}(x) \subset U \cap \{\rho + \bar{\rho} \leq \delta\}$ for large enough $k$, due to the Lipschitz continuity of $\rho, \bar{\rho}$.) On the other hand, $v - (-\bar{\rho})$ has a positive minimum on the compact set $\{\rho + \bar{\rho} \geq 4\} \cap U$, so by (3.3) we have $v_k - \phi_k = (v - (-\bar{\rho})) \ast \eta_{\epsilon_k} - \delta_{\epsilon_k} \geq 0$ on $\{\rho + \bar{\rho} \geq \delta/2\} \cap U$, for large enough $k$. Hence by (3.11) we must have

\[
\int_{U_k} D_i F(Du_k)D_i(v_k - u_k) + g'(u_k)(v_k - u_k) \, dx \geq 0.
\]

By taking the limit through the diagonal sequence $u_{l_l}$, and using the Dominated Convergence Theorem, we get (3.20) for this special $v$.

It is easy to see that an arbitrary test function $v$ in $W^{2,p}(\Omega)$ can be approximated by such special test functions. Just consider the functions $v_{\delta} := \min\{v + \delta, \rho\}$. Then we have $v_{\delta} > -\bar{\rho}$ on $U$, since $v \geq -\bar{\rho}$, and $\rho > -\bar{\rho}$ on $U$. Also on $\{\rho + \bar{\rho} \leq \delta\} \cap U$ we have $\rho \leq -\bar{\rho} + \delta \leq v + \delta$, so $v_{\delta} = \rho$ there. It is also easy to see that $v_{\delta} \in W^{2,p}(U)$, and $v_{\delta} \rightarrow v$ in $H^1(U)$. Therefore we get (3.20) for all $v \in W^{2,p}(\Omega)$, as desired.

It remains to show that $u$ belongs to $W^{2,\infty}_{\text{loc}}(U)$. First note that $D^2 u_k = D^2 \phi_k$ a.e. on $\{u_k = \phi_k\}$, hence by (3.13) $D^2 u_k$ is bounded there independently of $k$. Similarly, $D^2 u_k$ is bounded on $\{u_k = \psi_k\}$ independently of $k$. Now take $x_0 \in U$ and suppose that $B_r(x_0) \subset U$. Let $l$ be large enough so that $B_l(x_0) \subset U_l$. Set $u_k(y) := u_k(x_0 + r y)$ for $y \in B_l(0)$, and $k \geq l$. Then by (3.13) and the above argument we have

\[
\left\{ \begin{array}{l}
D^2_{ij} F\left(\frac{1}{r} D u_k \right) D^2_{ij} v_k = r^2 g'(v_k) \quad \text{a.e. in } B_l(0) \cap \Omega_k, \\
|D^2 v_k| \leq C \quad \text{a.e. in } B_l(0) - \Omega_k,
\end{array} \right.
\]

for some $C$ independent of $k$. Here $\Omega_k := \{y \in B_l(0) : u_k(x_0 + r y) \in E_{\epsilon_k}\}$.

Now recall that $\|u_k\|_{W^{2,\infty}(B_l(x_0))}$, $\|g'(u_k)\|_{L^\infty(B_l(x_0))}$ are bounded independently of $k$, due to (3.19), and the fact that $\phi_k \leq u_k \leq \psi_k$ and $\phi_k, \psi_k$ are bounded independently of $k$. Therefore $\|u_k\|_{W^{2,\alpha}(B_l(0))}$ and $\|g'(v_k)\|_{L^\infty(B_l(0))}$ are bounded independently of $k$. Also note that the Holder norms of $D^2_{ij} F\left(\frac{1}{r} D u_k \right)$, $r^2 g'(v_k)$ are bounded independently of $k$, since for every $\bar{\alpha} < 1$, $\|u_k\|_{C^{1,\alpha}(\bar{U})}$ is bounded
independently of \( k \). Thus we can apply the result of [27] to deduce that
\[
|D^2v_k| \leq \tilde{C} \quad \text{a.e. in } B_{1/2}(0),
\]
for some \( \tilde{C} \) independent of \( k \). Therefore
\[
|D^2u_k| \leq C \quad \text{a.e. in } B_{3/2}(x_0),
\]
for some \( C \) independent of \( k \). Hence, \( u_k \) is a bounded sequence in \( W^{2,\infty}(B_{3/2}(x_0)) \). Consider the diagonal subsequence \( u_{lj} \). Then a subsequence of it converges weakly star in \( W^{2,\infty}(B_{3/2}(x_0)) \). But the limit must be \( u \); so we get \( u \in W^{2,\infty}(B_{3/2}(x_0)) \), as desired. \( \square \)

4. The ridge, and the elastic and plastic regions

In the sequel we need a stronger version of (1.6):

**Assumption 4.** We assume that for all \( x \neq y \in \mathbb{R}^n \) we have
\[
(4.1) \quad -\gamma(y-x) < \varphi(x) - \varphi(y) < \gamma(x-y).
\]
Remember that we say \( y \in \partial U \) is a \( \rho \)-closest point to \( x \) if \( \rho(x) = \gamma(x-y) + \varphi(y) \). Similarly, we say \( y \in \partial U \) is a \( \rho \)-closest point to \( x \) if \( \rho(x) = \gamma(y-x) - \varphi(y) \). An obvious consequence of the above assumption is that every \( y \in \partial U \) is the unique \( \rho \)-closest point and \( \rho \)-closest point on \( \partial U \) to itself.

**Lemma 8.** Suppose \( y \) is one of the \( \rho \)-closest points on \( \partial U \) to \( x \in U \). Then
(a) \( y \) is a \( \rho \)-closest point on \( \partial U \) to every point of \( |x,y| \). Therefore \( \rho \) varies linearly along the line segment \( [x,y] \).
(b) If in addition Assumption 4 holds, then we have \( |x,y| \subseteq U \).
(c) If in addition \( \gamma \) is strictly convex, and Assumption 4 holds, then \( y \) is the unique \( \rho \)-closest point on \( \partial U \) to the points of \( |x,y| \).

**Proof.** (a) Let \( z \in |x,y| \). Suppose to the contrary that there is \( w \in \partial U - \{y\} \) such that
\[
\gamma(z-w) + \varphi(w) < \gamma(z-y) + \varphi(y).
\]
Then we have
\[
\gamma(x-w) + \varphi(w) \leq \gamma(x-z) + \gamma(z-w) + \varphi(w) \\
< \gamma(x-z) + \gamma(z-y) + \varphi(y) = \gamma(x-y) + \varphi(y),
\]
which is a contradiction. Hence \( y \) is a \( \rho \)-closest point to \( z \).

Therefore the points in the segment \( [x,y] \) have \( y \) as a \( \rho \)-closest point on \( \partial U \). Hence for \( 0 \leq t \leq \gamma(x-y) \) we have
\[
\rho(x - \frac{t}{\gamma(x-y)}(x-y)) = \gamma(x - \frac{t}{\gamma(x-y)}(x-y) - y) + \varphi(y) \\
= (1 - \frac{t}{\gamma(x-y)})\gamma(x-y) + \varphi(y) = \gamma(x-y) - t + \varphi(y).
\]
Thus \( \rho \) varies linearly along the segment.
(b) Suppose to the contrary that there is \( v \in \{x, y\} \cap \partial U \). But then we have
\[
\gamma(x - v) + \varphi(v) < \gamma(x - v) + \gamma(v - y) + \varphi(y) = \gamma(x - y) + \varphi(y),
\]
which is a contradiction.

(c) Suppose \( z \in \{x, y\} \) and \( w \in \partial U - \{y\} \) is another \( \rho \)-closest point to \( z \). Hence we have
\[
\gamma(z - w) + \varphi(w) = \gamma(z - y) + \varphi(y).
\]
If \( w \) belongs to the line containing \( x, z, y \), then there are two cases. If \( w \) is on the same side of \( x \) as \( y \), then \( w \) cannot lie between \( x, y \), since \( \{x, y\} \subseteq U \). Thus \( y \) is between \( w, x \), and therefore it is also between \( w, z \). But this is a contradiction since \( \{w, z\} \subseteq U \). On the other hand if \( w, y \) are on different sides of \( x \), we have
\[
\gamma(x - w) + \varphi(w) < \gamma(z - w) + \varphi(w) = \gamma(z - y) + \varphi(y) < \gamma(x - y) + \varphi(y),
\]
which is also a contradiction. Finally suppose that \( x, z, w \) are not collinear. Then by strict convexity of \( \gamma \) we get
\[
\gamma(x - w) + \varphi(w) < \gamma(x - z) + \gamma(z - w) + \varphi(w) = \gamma(x - z) + \gamma(z - y) + \varphi(y) = \gamma(x - y) + \varphi(y),
\]
which is a contradiction too. Thus \( y \) is the unique \( \rho \)-closest point to \( z \).

Lemma 9. Suppose the Assumptions \( 1, 4 \) hold, and \( \gamma \) is strictly convex. If for some point \( x \in U \) there are two different points \( y, z \in \partial U \) so that
\[
\rho(x) = \gamma(x - y) + \varphi(y) = \gamma(x - z) + \varphi(z),
\]
them \( \rho \) is not differentiable at \( x \).

Proof. We know that the points in the segment \( \{x, y\} \) have \( y \) as a \( \rho \)-closest point on \( \partial U \). Hence as we have seen in the proof of the previous lemma, for \( 0 \leq t \leq \gamma(x - y) \) we have
\[
\rho(x - \frac{t}{\gamma(x - y)}(x - y)) = \gamma(x - y) - t + \varphi(y).
\]
Now suppose to the contrary that \( \rho \) is differentiable at \( x \). Then by differentiating the above equality (and the similar formula for \( z \)) with respect to \( t \), we get
\[
\langle D\rho(x), \frac{x - y}{\gamma(x - y)} \rangle = 1 = \langle D\rho(x), \frac{x - z}{\gamma(x - z)} \rangle.
\]
On the other hand, it is easy to show that \( \gamma(\langle D\rho(x) \rangle) \leq 1 \). To do this just note that
\[
\rho(x + tv) - \rho(x) \leq \gamma(x + tv) - x = t\gamma(v).
\]
By taking the limit as \( t \to 0^+ \), we get \( \langle D\rho(x), v \rangle \leq \gamma(v) \). Then we get the desired result by \( |2, 2| \).

Now note that if two vectors \( v, w \) satisfy \( \gamma(v) = 1 = \gamma(w) \) and
\[
\langle D\rho(x), v \rangle = 1 = \langle D\rho(x), w \rangle,
\]
then if }\]
then one of them is a positive multiple of the other, and consequently the two vectors are equal. Since otherwise by strict convexity of $\gamma$ and inequality (2.1), we get

$$\langle D\rho(x), \frac{v + w}{2} \rangle \leq \gamma(\rho(x)) \gamma(\frac{v + w}{2}) < \gamma(\rho(x)) \frac{\gamma(v) + \gamma(w)}{2} = 1.$$ 

However we must have $\langle D\rho(x), \frac{v + w}{2} \rangle = \frac{1 + 1}{2} = 1$, which is a contradiction.

Therefore we must have $x - y = \frac{x - z}{\gamma(x - y)} \gamma(x - z)$.

This implies that $x, y, z$ are collinear, and $y, z$ are on the same side of $x$. Then either $y$ is between $x, z$, or $z$ is between $x, y$. But both of these cases are impossible since $\|x, y\|$ and $\|x, z\|$ are subsets of $U$. Thus $\rho$ cannot be differentiable at $x$. $\square$

Now, we generalize the notion of ridge introduced by Ting [37], and Caffarelli and Friedman [2].

**Definition 2.** The $\rho$-ridge of $U$ is the set of all points $x \in U$ where $\rho(x)$ is not $C^{1,1}$ in any neighborhood of $x$. We denote it by $R_{\rho}$.

When $\gamma$ is strictly convex and Assumptions [14] hold, the subset of the $\rho$-ridge consisting of the points with more than one $\rho$-closest point on $\partial U$, is denoted by $R_{\rho,0}$. Similarly we define $R_{\bar{\rho}}, R_{\rho,0}$.

The following definition is motivated by the physical properties of the elastic-plastic torsion problem.

**Definition 3.** Let

$$P^+ := \{x \in U : u(x) = \rho(x)\}, \quad P^- := \{x \in U : u(x) = -\bar{\rho}(x)\}.$$ 

Then $P := P^+ \cup P^-$ is called the plastic region; and

$$E := \{x \in U : -\bar{\rho}(x) < u(x) < \rho(x)\}$$

is called the elastic region. We also define the free boundary to be $\Gamma := \partial E \cap U$.

**Remark.** Note that $E$ is open, and $P$ is closed in $U$. It is also obvious that $\Gamma \subset P$.

Remember that by Theorem [1], $u$ is locally $C^{1,1}$. Thus similarly to (3.10), we obtain

$$\begin{cases}
-D_i(D_i F(Du)) + g'(u) = 0 & \text{in } E, \\
-D_i(D_i F(Du)) + g'(u) \leq 0 & \text{a.e. on } P^+, \\
-D_i(D_i F(Du)) + g'(u) \geq 0 & \text{a.e. on } P^-.
\end{cases}
$$

(4.2)

Note that $u$ is $C^{2,\delta}$ on $E$, due to the Schauder estimates (see Theorem 6.13 of [22]).

**Lemma 10.** Suppose the Assumptions [1,2,3,4] hold. If $x \in P^+$, and $y$ is a $\rho$-closest point on $\partial U$ to $x$, then $[x, y] \subset P^+$. Similarly, if $x \in P^-$, and $y$ is a $\bar{\rho}$-closest point on $\partial U$ to $x$, then $[x, y] \subset P^-$. 

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Proof. Note that $u$ is $C^1$, and $[x,y] \subset U$. Suppose $x \in P^-$; the other case is similar. We have
\[ u(x) = -\bar{\rho}(x) = -\gamma(y - x) + \varphi(y). \]
Let $v := u - (-\bar{\rho}) \geq 0$, and $\xi := \frac{y-x}{\gamma(y-x)} = -\frac{x-y}{\bar{\gamma}(x-y)}$. Then $\bar{\rho}$ varies linearly along the segment $[x,y]$, since $y$ is a $\bar{\rho}$-closest point to the points of the segment. So we have $D_\xi(-\bar{\rho}) = D_{-\xi}\bar{\rho} = 1$ along the segment, as shown in the proof of Lemma 9. Note that we do not assume the differentiability of $\bar{\rho}$; and $D_{-\xi}\bar{\rho}$ is just the derivative of the restriction of $\bar{\rho}$ to the segment $[x,y]$. Now since
\[ D_\xi u = (Du, \xi) \leq \gamma^\circ(Du)\gamma(\xi) \leq 1, \]
we have $D_\xi v \leq 0$ along $[x,y]$. Thus as $v(x) = v(y) = 0$, and $v$ is continuous on the closed segment $[x,y]$, we must have $v \equiv 0$ on $[x,y]$. Therefore $u = -\bar{\rho}$ along the segment as desired. \hfill \Box

Now recall that we have $Du \in K^\circ$, which is equivalent to $\gamma^\circ(Du) \leq 1$. The next lemma tells us when we hit the gradient constraint, i.e. when $Du \in \partial K^\circ$, or equivalently when $\gamma^\circ(Du) = 1$. The answer is that we hit the gradient constraint exactly when we hit one of the obstacles $-\bar{\rho}, \rho$.

**Lemma 11.** Suppose the Assumptions 1234 hold, and $\gamma$ is strictly convex. Then we have
\[ P = \{ x \in U : \gamma^\circ(Du(x)) = 1 \}, \]
\[ E = \{ x \in U : \gamma^\circ(Du(x)) < 1 \}. \]

Proof. First suppose $x \in P^-$; the case of $P^+$ is similar. Then we have
\[ u(x) = -\bar{\rho}(x) = -\gamma(y - x) + \varphi(y), \]
for some $y \in \partial U$. Thus by Lemma 10 $u = -\bar{\rho}$ along the segment $[x,y]$. We also know that $\bar{\rho}$ varies linearly along the segment $[x,y]$, since $y$ is a $\bar{\rho}$-closest point to the points of the segment. Hence we have $D_\xi u(x) = 1$ for $\xi := \frac{y-x}{\gamma(y-x)}$, as shown in the proof of Lemma 9. Therefore $\gamma^\circ(Du(x))$ can not be less than 1 due to the equation (2.2).

Next, assume that $\gamma^\circ(Du(x)) = 1$. Then by (2.2), there is $\tilde{\xi}$ with $\gamma(\tilde{\xi}) = 1$ such that $D_\tilde{\xi} u(x) = 1$. Suppose to the contrary that $x \in E$, i.e. $-\bar{\rho}(x) \leq u(x) < \rho(x)$. By (4.2) we know $D_\tilde{\xi} u$ is a weak solution to the elliptic equation
\[ -D_i(a_{ij}D_jD_\xi u) + bD_\xi u = 0 \quad \text{in } E, \]
where $a_{ij} := D^2_{ij}F(Du)$, and $b := g''(u)$. On the other hand
\[ D_\xi u = (Du, \xi) \leq \gamma^\circ(Du)\gamma(\xi) \leq 1 \]
on $U$. Let $E_1$ be the connected component of $E$ that contains $x$. Then the strong maximum principle (Theorem 8.19 of [22]) implies that $D_\xi u \equiv 1$ over $E_1$. Note that we can work in open subsets of $E_1$ which are compactly contained in $E_1$; so we do not need the global integrability of $D^2u$ to apply the maximum principle.

Now consider the line passing through $x$ in the $\tilde{\xi}$ direction, and suppose it intersects $\partial E_1$ for the first time in $y := x - \tau \tilde{\xi}$ for some $\tau > 0$. If $y \in \partial U$, then for $t > 0$ we have
\[ \frac{d}{dt}[u(y + t\xi)] = D_\xi u(y + t\xi) = 1 = \frac{d}{dt}[\gamma(\xi)] = \frac{d}{dt}[\gamma(y + t\tilde{\xi} - y)]. \]
Thus as \( u(y) = \varphi(y) \), we get \( u(x) = u(y + \tau \tilde{\xi}) = \gamma(x - y) + \varphi(y) \geq \rho(x) \); which is a contradiction. Now if \( y \in U \), then as it also belongs to \( \partial E \) we have \( y \in P \). If \( u(y) = \rho(y) = \gamma(y - \tilde{y}) + \varphi(\tilde{y}) \) for some \( \tilde{y} \in \partial U \), similarly to the above we obtain
\[
    u(x) = \gamma(x - y) + u(y) = \gamma(x - y) + \gamma(y - \tilde{y}) + \varphi(\tilde{y}) \geq \gamma(x - \tilde{y}) + \varphi(\tilde{y}) \geq \rho(x),
\]
which is again a contradiction.

On the other hand, if \( u(y) = -\bar{\rho}(y) = -\gamma(\tilde{y} - y) + \varphi(\tilde{y}) \) for some \( \tilde{y} \in \partial U \), then by Lemma 10 we have \( u = -\bar{\rho} \) on the segment \([y, \tilde{y}]\); and consequently \( D \xi u(y) = 1 \), where \( \hat{\xi} := \frac{\tilde{y} - y}{\gamma(\tilde{y} - y)} \). Since \( u \) is differentiable we must have \( \hat{\xi} = \xi \), as shown in the proof of Lemma 9. Therefore \( x, y, \tilde{y} \) are collinear, and \( x, \tilde{y} \) are on the same side of \( y \). But \( \tilde{y} \) cannot belong to \([y, x] \subset E \). Hence we must have \( x \in [y, \tilde{y}] \subset P^- \), which means \( u(x) = -\bar{\rho}(x) \); and this is a contradiction.

**Remark.** In the above proof, we only used the strict convexity of \( \gamma \) in the last paragraph. So without this assumption, i.e. just under the Assumptions [1][2][3] we have
\[
    P \subset \{ x \in U : \tau^0(Du(x)) = 1 \}, \quad E \supset \{ x \in U : \gamma^0(Du(x)) < 1 \}.
\]
Furthermore, if we can drop one of the obstacles, then we do not need the argument given in the last paragraph, and we can conclude that the above lemma holds without assuming the strict convexity of \( \gamma \). (Note that if we only have the obstacle \(-\bar{\rho}\), then in the above proof we have to look for a point of the form \( x + \tau \hat{\xi} \in \partial E \) for some \( \tau > 0 \).) For example, when \( g \) is decreasing, we can show that \( u \geq 0 \) (since \( J[u^+] \leq J[u] \)). Thus if in addition \( \varphi = 0 \), then \( u \) does not touch the lower obstacle, since in this case we have \(-\bar{\rho} < 0\).

**Proposition 2.** Suppose the Assumptions [1][2][3] hold, and \( \gamma \) is strictly convex. Then we have
\[
    R_{\rho,0} \cap P^+ = \emptyset, \quad R_{\bar{\rho},0} \cap P^- = \emptyset.
\]

**Proof.** Let us show that \( R_{\rho,0} \cap P^- = \emptyset \); the other case is similar. Suppose to the contrary that \( x \in R_{\rho,0} \cap P^- \). Then there are at least two distinct points \( y, z \in \partial U \) such that
\[
    \bar{\rho}(x) = \gamma(y - x) - \varphi(y) = \gamma(z - x) - \varphi(z).
\]
Now by Lemma 10 we have \([x, y] \subset [x, z] \subset P^- \). In other words, \( u = -\bar{\rho} \) on both of these segments. Therefore, we can argue as in the proof of Lemma 9 to obtain
\[
    \langle Du(x), \frac{y - x}{\gamma(y - x)} \rangle = 1 = \langle Du(x), \frac{z - x}{\gamma(z - x)} \rangle;
\]
and to get a contradiction with the fact that \( \gamma^0(Du(x)) \leq 1 \). □

5. **Regularity of the Obstacles**

In this section, we are going to study the singularities of the functions \( \rho, \bar{\rho} \). Since \( \rho, \bar{\rho} \) are viscosity solutions of Hamilton-Jacobi equations of type [1][9], we can regard this section as the study of the regularity of solutions to some Hamilton-Jacobi equations. We know that \( \rho, \bar{\rho} \) are Lipschitz functions (see [2][3]). We want to characterize the set over which they are more regular. In order to do that, we need to impose some additional restrictions on \( K, U \) and \( \varphi \).
**Assumption 5.** Suppose that $k \geq 2$ is an integer, and $0 \leq \alpha \leq 1$. We assume that

(a) $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin. In addition, $\partial K$ is $C^{k,\alpha}$, and its principal curvatures are positive at every point.
(b) $U \subset \mathbb{R}^n$ is a bounded open set, and $\partial U$ is $C^{k,\alpha}$.
(c) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{k,\alpha}$ function, such that $\gamma^\circ(D\varphi) < 1$.

Finally, from now on, we restrict the domain of $\rho, \bar{\rho}$ to $U$. So that we can extend them to have different values beyond $\partial U$.

**Remark.** Note that we have incorporated Assumption 1 into the above assumption. It is also obvious that under the above assumption, $\varphi$ will satisfy the inequality (1.6) and its stronger version given in Assumption 4, as shown for example in Lemma 2.2 of [33]. Furthermore, the above assumption implies Assumption 3 as we have shown in Lemma 3.

**Remark.** As we discussed in Subsection 2.1, the above assumption implies that $K, \gamma$ are strictly convex. In addition, $K^\circ, \gamma^\circ$ are strictly convex, and $\partial K^\circ, \gamma^\circ$ are also $C^{k,\alpha}$. Furthermore, the principal curvatures of $\partial K^\circ$ are also positive at every point. Similar conclusions obviously hold for $-K, -\varphi$ and $(-K)^\circ = -K^\circ$ too. Hence in the sequel, whenever we prove a property for $\rho$, it also holds for $\bar{\rho}$ too.

Suppose $\nu$ is the inward unit normal to $\partial U$. By the above assumption, $\nu$ is a $C^{k-1,\alpha}$ function of $y \in \partial U$. It is well known (see [22, Section 14.6]) that we have $\nu = Dd$, where $d$ is the Euclidean distance to $\partial U$. So $Dd$ is a $C^{k-1,\alpha}$ extension of $\nu$ to a neighborhood of $\partial U$. We always assume that we are working with this extension of $\nu$.

**Lemma 12.** Suppose the Assumption 5 holds. Then for every $y \in \partial U$ there is a unique scalar $\lambda(y) > 0$ such that

$$\gamma^\circ(D \varphi(y) + \lambda(y) \nu(y)) = 1. \tag{5.1}$$

Furthermore, $\lambda : \partial U \to (0, \infty)$ is a $C^{k-1,\alpha}$ function.

**Proof.** Since $\gamma^\circ(D \varphi) < 1$, the vector $D \varphi(y)$ is in the interior of $K^\circ$, and therefore the ray emanating from it in the direction of $\nu(y)$ must intersect $\partial K^\circ$. Also, this ray cannot intersect $\partial K^\circ$ in more than one point due to the convexity of $K^\circ$. So there is a unique $\lambda(y) > 0$ such that $D \varphi(y) + \lambda(y) \nu(y) \in \partial K^\circ$, as desired.

By the implicit function theorem we can show that $\lambda(y)$ is a $C^{k-1,\alpha}$ function of $y$. Consider the $C^{k-1,\alpha}$ function

$$(y, \lambda) \mapsto \gamma^\circ(D \varphi(y) + \lambda \nu(y)).$$

We need the derivative with respect to $\lambda$ to be nonzero, i.e. we need $\langle D \gamma^\circ(D \varphi + \lambda \nu), \nu \rangle \neq 0$ when $\gamma^\circ(D \varphi + \lambda \nu) = 1$. But $D \gamma^\circ(D \varphi + \lambda \nu)$ is normal to the surface of $\partial K^\circ$ at the point $D \varphi + \lambda \nu$. On the other hand, the line $t \mapsto D \varphi + t \nu$ that passes through $D \varphi + \lambda \nu$ cannot be tangent to $\partial K^\circ$, since it intersects the interior of the convex set $K^\circ$ at $D \varphi$. Hence $D \gamma^\circ(D \varphi + \lambda \nu)$ cannot be orthogonal to the direction of the line, i.e to $\nu$. Thus we get the desired.

**Definition 4.** For $y \in \partial U$ we define

$$\mu(y) := D \varphi(y) + \lambda(y) \nu(y). \tag{5.2}$$
By the above lemma, we know that \( \mu \) is a \( C^{k-1, \alpha} \) function, and \( \gamma^\circ(\mu) = 1 \). In addition, it is easy to compute \( D\mu \). First we need to compute \( D\lambda \) using implicit function theorem. We have

\[
0 = D(\gamma^\circ(D\varphi + \lambda \nu)) = D\gamma^\circ(\mu)(D^2 \varphi + \lambda D\nu + \nu \otimes D\lambda)
\]

which gives us \( D\lambda \). Thus we get

\[
D\mu = D^2 \varphi + \lambda D\nu + \nu \otimes D\lambda
\]

\[
= D^2 \varphi + \lambda D\nu - \frac{1}{D\gamma^\circ(\mu), \nu} \nu \otimes (D\gamma^\circ(\mu)D^2 \varphi + \lambda D\gamma^\circ(\mu)D\nu).
\]

Note that \( \langle D\gamma^\circ(\mu), \nu \rangle \neq 0 \) as shown in the above proof. To simplify the above expression, we set

\[
(5.3) \quad X := \frac{1}{\langle D\gamma^\circ(\mu), \nu \rangle} D\gamma^\circ(\mu) \otimes \nu.
\]

Note that if \( w \) is orthogonal to \( \nu \), i.e., if it is tangent to \( \partial U \), then \( (I - X)w = w - 0 = w \). In addition we have \( (I - X)D\gamma^\circ(\mu) = D\gamma^\circ(\mu) - D\gamma^\circ(\mu) = 0 \). So \( I - X \) is the projection on the tangent space to \( \partial U \) parallel to \( D\gamma^\circ(\mu) \). (Note that \( D\gamma^\circ(\mu) \) is not tangent to \( \partial U \) due to \( \langle D\gamma^\circ(\mu), \nu \rangle \neq 0 \).) Now it is easy to check that

\[
(5.4) \quad D\mu = (I - X^T)(D^2 \varphi + \lambda D^2 d).
\]

Here we also used the fact that \( \nu = Dd \). Let us recall that the eigenvalues of \( D^2 d(y) = D\nu(y) \) are minus the principal curvatures of \( \partial U \) at \( y \), and 0. For the details see [22, Section 14.6].

**Lemma 13.** Suppose the Assumption [3] holds. Then for every \( y \in \partial U \) we have

\[
(5.5) \quad \langle D\gamma^\circ(\mu(y)), \nu(y) \rangle > 0.
\]

Furthermore, let \( x \in U \), and suppose \( y \) is one of the \( \rho \)-closest points to \( x \) on \( \partial U \). Then we have

\[
(5.6) \quad \frac{x - y}{\gamma(x - y)} = D\gamma^\circ(\mu(y)).
\]

Or equivalently

\[
(5.7) \quad x = y + (\rho(x) - \varphi(y)) D\gamma^\circ(\mu(y)).
\]

**Proof.** First note that \( \mu \in \partial K^\circ \), since \( \gamma^\circ(\mu) = 1 \). Hence \( D\gamma^\circ(\mu) \) is the outward normal to \( \partial K^\circ \) at \( \mu \). On the other hand, \( D\varphi = \mu - \lambda \nu \) belongs to the ray passing through \( \mu \) in the direction \( -\nu \), because \( \lambda > 0 \). But we know that \( D\varphi \) is in the interior of \( K^\circ \), since \( \gamma^\circ(D\varphi) < 1 \). Thus the ray \( t \mapsto \mu - t \nu \) for \( t > 0 \), passes through the interior of \( K^\circ \). Therefore this ray and \( K^\circ \) must lie on the same side of the tangent space to \( \partial K^\circ \) at \( \mu \), because \( K^\circ \) is strictly convex. Hence we must have \( \langle D\gamma^\circ(\mu), \nu \rangle = -\langle D\gamma^\circ(\mu), -\nu \rangle > 0 \), as desired.

Now let \( z \mapsto Y(z) \) be a smooth parametrization of \( \partial U \) around \( Y(0) = y \). Then due to \( \rho \)'s definition, the function \( \varphi(Y(z)) + \gamma(x - Y(z)) \) has a minimum at \( z = 0 \). Hence

\[
0 = D_{z^j} [\varphi(Y(z)) + \gamma(x - Y(z))] = \sum_{i \leq n} D_{z^j} Y^i [D_i \varphi(y) - D_i \gamma(x - y)].
\]
Thus $D\varphi(y) - D\gamma(x - y)$ is orthogonal to every $D_{s_j}Y$ for every $j$, and therefore it is orthogonal to $\partial U$. Hence we have $D\varphi(y) - D\gamma(x - y) = cv(y)$ for some scalar $c$. First, let us show that $c < 0$. Suppose to the contrary that $c \geq 0$. Note that $\langle v(y), x - y \rangle \geq 0$, because by Lemma 8 we know that $[x, y] \subset U$. Therefore by (2.10) and (2.11) we have

$$\gamma(x - y) = \langle D\gamma(x - y), x - y \rangle = \langle D\varphi(y) - cv(y), x - y \rangle$$

$$= \langle D\varphi(y), x - y \rangle - c \langle v(y), x - y \rangle$$

$$\leq \langle D\varphi(y), x - y \rangle \leq \gamma_0(D\varphi)\gamma(x - y) < \gamma(x - y),$$

which is a contradiction. So $c < 0$.

On the other hand we know that $\gamma_0(D\gamma) = 1$. Thus we must have

$$\gamma_0(D\varphi(y) - cv(y)) = 1.$$

Therefore by the previous lemma $-c = \lambda(y)$, since $-c > 0$. Hence we obtain

$$D\gamma(x - y) = D\varphi(y) - cv(y) = D\varphi(y) + \lambda(y)v(y) = \mu(y).$$

Now if we apply $D\gamma_0$ to both sides of the above equation, then by (2.7) we obtain

$$\frac{x - y}{\gamma(x - y)} = D\gamma_0(D\gamma(x - y)) = D\gamma_0(\mu(y)),$$

as desired. The other equation follows immediately, since by (1.8) we know that $\gamma(x - y) = \rho(x) - \varphi(y)$.

Proposition 3. Suppose the Assumption 3 holds. Let $x \in U$. Then $\rho$ is differentiable at $x$ if and only if $x \in U - R_{p,0}$. And in that case we have

$$D\rho(x) = D\gamma(x - y) = \mu(y),$$

where $y$ is the unique $\rho$-closest point to $x$ on $\partial U$. In particular we have $D\rho(x) \neq 0$.

Remark. This theorem is actually true if we merely assume that $\varphi$ and $\partial K$ are $C^1$. In addition, we only need $\partial U$ to be compact, and the boundedness of $U$ is not actually needed in the following proof. So for example, the same result holds on the domain $\mathbb{R}^n - \overline{U}$, with its corresponding $\rho, \mu$.

Proof. This is a trivial consequence of Theorem 3.4.4 of [7], since $\rho$ is the minimum of a family of smooth functions. (Such functions are called marginal functions.) Just note that we have to restrict ourselves to a neighborhood of $x$, so that $D\gamma(\cdot - \tilde{y})$ exists and is continuous for every $\tilde{y} \in \partial U$. Also note that if $x$ has more than one $\rho$-closest point on $\partial U$, then $\rho$ is not differentiable at $x$ by Lemma 9. Finally note that $D\gamma(x - y) = \mu(y)$, as we have shown in the proof of the previous lemma.

Theorem 2. Suppose the Assumption 2 holds. Let $y \in \partial U$. Then there is an open ball $B_r(y)$ such that $\rho$ is $C^{k,\alpha}$ on $\overline{U} \cap B_r(y)$. Furthermore, $y$ is the $\rho$-closest point to some points in $U$, and we have

$$D\rho(y) = \mu(y).$$

In addition we have

$$D^2\rho(y) = (I - X^T)(D^2\varphi(y) + \lambda(y)D^2d(y))(I - X),$$

where $X$ is a $\rho$-closest point to some points in $U$, and we have
where $I$ is the identity matrix, $d$ is the Euclidean distance to $\partial U$, and $X$ is given by (5.13). Furthermore we have

\begin{equation}
D^2 \rho(y) D\gamma^o(\mu(y)) = 0.
\end{equation}

**Remark.** As a consequence of this theorem we get that $R_\rho$, and therefore $R_{\rho,0}$, have a positive distance from $\partial U$. Thus $\overline{R}_\rho = R_\rho \subset U$, since $R_\rho$ is closed in $U$ by definition.

**Proof.** We will show that $\rho$ has a $C^{k,\alpha}$ extension to an open neighborhood of $y$. Note that if we consider $\rho$ as a function on all of $\mathbb{R}^n$, then it is not differentiable on $\partial U$. However, we will show that the following extension of $\rho$, which can be considered a signed version of $\rho$ on $\mathbb{R}^n$, is $C^{k,\alpha}$ on a neighborhood of $\partial U$:

$$
\rho_s(x) := \begin{cases} 
\rho(x) & \text{if } x \in \overline{U}, \\
-\rho(x) & \text{if } x \in \mathbb{R}^n - \overline{U}.
\end{cases}
$$

Note that for $x \in \partial U$ we have $-\bar{\rho}(x) = -(\varphi(x)) = \varphi(x) = \rho(x)$. So in particular, $\rho_s$ is a continuous function. In addition, note that $\partial(\mathbb{R}^n - \overline{U}) = \partial U$, but the inward unit normal to $\partial(\mathbb{R}^n - \overline{U})$ is $-\nu$.

Let $\bar{\mu}$ be the function constructed from $-\varphi, -\nu$, as $\mu$ is constructed from $\varphi, \nu$. Then we have $\bar{\mu} = -D\varphi + \lambda(-\nu) = -D\varphi + \lambda \nu$, for some uniquely determined $\lambda > 0$. But the gauge function of $(-K)^o = -K^o$ is $\gamma^o(\cdot)$, so we must have $1 = \gamma^o(-\bar{\mu}) = \gamma^o(D\varphi + \lambda \nu)$. Hence $\lambda = \lambda$, and therefore $\bar{\mu} = -\mu$. Thus if we incorporate this in (5.7), we obtain

\begin{equation}
x = y + (\bar{\rho}(x) - \varphi(y)) \left( - D\gamma^o(-\bar{\mu}) \right) = y + ( - \bar{\rho}(x) - \varphi(y) ) D\gamma^o(\mu(y)),
\end{equation}

where $x \in \mathbb{R}^n - \overline{U}$ has $y$ as its $\bar{\rho}$-closest point on $\partial U$. Note that the derivative of the gauge function of $(-K)^o$ is $-D\gamma^o(\cdot)$.

Let $z \mapsto Y(z)$ be a $C^{k,\alpha}$ parametrization of $\partial U$ around $Y(0) = y$, where $z$ varies in an open set $V \subset \mathbb{R}^{n-1}$. Consider the map $G : V \times \mathbb{R} \to \mathbb{R}^n$ defined by

$$
G(z,t) := Y(z) + (t - \varphi(Y(z))) D\gamma^o(\mu(Y(z))).
$$

Note that $G$ is a $C^{k-1,\alpha}$ function. Since $\mu$ is a $C^{k-1,\alpha}$ function on the $C^{k,\alpha}$ manifold $\partial U$, we can extend it to a $C^{k-1,\alpha}$ function on a neighborhood of $\partial U$, by Lemma 6.38 of [22]. Also note that we have $G(0,\varphi(y)) = y$. Now we have

$$
\begin{align*}
Dz_j G &= Dz_j Y - Dz_j Y^i D_i \varphi D\gamma^o(\mu) + (t - \varphi) Dz_j Y^i D^2 \gamma^o(\mu) D_i \mu, \\
D_1 G &= D \gamma^o(\mu).
\end{align*}
$$

Note that $DY$ is evaluated at $z$, and $\mu, \varphi, D\mu, D\varphi$ are evaluated at $Y(z)$. Also note that in the last term of $Dz_j G$, we evaluate the action of the matrix $D^2 \gamma^o(\mu)$ on the vector $D_i \mu$. In addition, remember that we are using the convention of summing over repeated indices. Hence we have

$$
\begin{align*}
Dz_j G(0,\varphi(y)) &= Dz_j Y - Dz_j Y^i D_i \varphi D\gamma^o(\mu), \\
D_1 G(0,\varphi(y)) &= D \gamma^o(\mu),
\end{align*}
$$

where $\mu, D\varphi$ are evaluated at $y$. Let $w_j := Dz_j Y$. Note that $w_1, \ldots, w_{n-1}$ is a basis for the tangent space to $\partial U$ at $y$. Let $w$ be the orthogonal projection of $D \gamma^o(\mu)$ on this tangent space. Then we
have (we represent a matrix by its columns)

$$\det DG(0, \varphi(y))$$

$$= \det \begin{bmatrix} w_1 - \langle w_1, D\varphi \rangle D\gamma^0(\mu) & \cdots & w_{n-1} - \langle w_{n-1}, D\varphi \rangle D\gamma^0(\mu) & D\gamma^0(\mu) \end{bmatrix}$$

$$= \det \begin{bmatrix} w_1 & \cdots & w_{n-1} & D\gamma^0(\mu) \end{bmatrix} = \det \begin{bmatrix} w_1 & \cdots & w_{n-1} & w + \langle D\gamma^0(\mu), \nu \rangle \nu \end{bmatrix}$$

$$= \langle D\gamma^0(\mu), \nu \rangle \det \begin{bmatrix} w_1 & \cdots & w_{n-1} & \nu \end{bmatrix} \neq 0.$$  

Note that in the last line we have used (5.5), and the fact that \(w_1, \ldots, w_{n-1}, \nu\) are linearly independent.

Therefore by the inverse function theorem, \(G\) is invertible on an open set of the form \(W \times (\varphi(y) - h, \varphi(y) + h)\), and it has a \(C^{k-1,\alpha}\) inverse on a neighborhood of \(y\). Let \(B_r(y)\) be contained in that neighborhood, and suppose \(r\) is small enough so that for every \(x \in U \cap B_r(y)\), the \(\rho\)-closest points on \(\partial U\) to \(x\) belong to \(\partial U \cap Y(W)\). This is possible due to Lemma II and the fact that \(y\) is the unique \(\rho\)-closest point to \(y\) because of (4.1). Similarly, suppose that \(r\) is small enough so that for every \(x \in (\mathbb{R}^n - \overline{U}) \cap B_r(y)\), the \(\hat{\rho}\)-closest points on \(\partial U\) to \(x\) belong to \(\partial U \cap Y(W)\). Also suppose that \(r\) is small enough so that for every \(x \in B_r(y)\) we have \(\rho_s(x) \in (\varphi(y) - h, \varphi(y) + h)\), which is possible due to the continuity if \(\rho_s\). Now we know that \(G : (z, t) \mapsto x\) has an inverse, denoted by \(z(x), t(x)\), where \(z(\cdot), t(\cdot)\) are \(C^{k-1,\alpha}\) functions of \(x\). Let \(\tilde{y} := Y(z(x))\). Then we have

$$x = G(z(x), t(x)) := \tilde{y} + (t(x) - \varphi(\tilde{y})) D\gamma^0(\mu(\tilde{y})).$$

On the other hand, (5.7) and (5.12) imply that

$$x = \hat{y} + (\rho_s(x) - \varphi(\hat{y})) D\gamma^0(\mu(\hat{y})), $$

where \(\hat{y}\) is one of the \(\rho\)-closest or \(\hat{\rho}\)-closest points on \(\partial U\) to \(x\), depending on whether \(x \in \overline{U}\) or \(x \in \mathbb{R}^n - \overline{U}\). (Note that when \(x = \hat{y} \in \partial U\), the equation holds trivially.) But by our assumption about \(B_r(y)\), there is \(\hat{z} \in W\) such that \(\hat{y} = Y(\hat{z})\). Hence \((\hat{z}, \rho_s(x)) \in W \times (\varphi(y) - h, \varphi(y) + h)\), and we have \(G(\hat{z}, \rho_s(x)) = x\). Therefore due to the invertibility of \(G\) we must have

$$\hat{y} = Y(\hat{z}) = Y(z(x)), \quad \rho_s(x) = t(x).$$

Thus in particular, \(\rho_s\) is a \(C^{k-1,\alpha}\) function of \(x\). Hence \(\rho\) is a \(C^{k-1,\alpha}\) function on \(\overline{U} \cap B_r(y)\).

Next, consider the line segment \(t \mapsto y + (t - \varphi(y)) D\gamma^0(\mu(y))\), where \(t \in (\varphi(y), \varphi(y) + \tilde{h})\). If \(\tilde{h} > 0\) is small enough, then this segment lies inside \(U \cap B_r(y)\), since we know that \(\langle D\gamma^0(\mu(y)), \nu(y) \rangle > 0\). Now similarly to the last paragraph, we can show that if \(x\) belongs to this segment, then \(y\) is the \(\rho\)-closest point on \(\partial U\) to \(x\). Note that \(x \notin R_{\rho,0}\), since \(\rho\) is differentiable at \(x\). Thus we have \(D\rho(x) = \mu(y)\). Hence if we let \(x\) approaches \(y\) along this segment, we get

$$D\rho(y) = \lim_{x \to y} D\rho(x) = \lim_{x \to y} \mu(y) = \mu(y),$$

because \(D\rho\) is continuous.

Finally note that for every \(x \in \overline{U} \cap B_r(y)\) we have

$$D\rho_s(x) = D\rho(x) = \mu(Y(z(x))).$$
Similarly we have \( D\rho_s(x) = -D\rho(x) = -\mu(Y(z(x))) = \mu(Y(z(x))), \) when \( x \in (\mathbb{R}^n - \overline{U}) \cap B_r(y). \) Furthermore we know that \( z(\cdot), Y, \mu \) are \( C^{k-1,0} \) functions. Therefore \( \rho_s \) is a \( C^{k,\alpha} \) function. Consequently, \( \rho \) is a \( C^{k,\alpha} \) function on \( \overline{U} \cap B_r(y), \) as desired.

Now let us compute \( D^2\rho(y) \). We know that \( D\rho = \mu \) on \( \partial U \). So for every vector \( w \) which is tangent to \( \partial U \) we have \( D_w D\rho = D_w \mu \). Remember that \( I - X \) is the projection on the tangent space to \( \partial U \) parallel to \( D\gamma(\mu) \). Hence by \eqref{eq:5.4}, for every vector \( \tilde{w} \) we have
\[
\tilde{w}(D^2\rho)w = \tilde{w}(D\mu)w = \tilde{w}(I - X^T)(D^2\varphi + \lambda D^2d)w
= \tilde{w}(I - X^T)(D^2\varphi + \lambda D^2d)(I - X)w.
\]
Next let us show that \( D^2\rho(y)D\gamma(\mu) = 0 \). The reason is that as we have seen before, when \( s \geq 0 \) is small, the point \( y + sD\gamma(\mu) \in U \) has \( y \) as its unique \( \rho \)-closest point on \( \partial U \). Thus by \eqref{eq:5.8} we have \( D\rho(y + sD\gamma(\mu)) = \mu \). Hence if we differentiate with respect to \( s \) we get
\[
D^2\rho(y + sD\gamma(\mu))D\gamma(\mu) = 0.
\]
So if we let \( s \to 0 \) we get the desired. Therefore we have
\[
\tilde{w}(D^2\rho)D\gamma(\mu) = 0 = \tilde{w}(I - X^T)(D^2\varphi + \lambda D^2d)0
= \tilde{w}(I - X^T)(D^2\varphi + \lambda D^2d)(I - X)D\gamma(\mu).
\]
Thus we get the desired formula for \( D^2\rho(y). \)

**Theorem 3.** Suppose the Assumption \( \Box \) holds. Suppose \( x \in U - R_{\rho,0} \), and let \( y \) be the unique \( \rho \)-closest point to \( x \) on \( \partial U \). Let
\[
W = W(y) := -D^2\gamma(\mu(y))D^2\rho(y), \quad Q = Q(x) := I - (\rho(x) - \varphi(y))W,
\]
where \( I \) is the identity matrix. If \( \det Q \neq 0 \) then \( \rho \) is \( C^{k,\alpha} \) on a neighborhood of \( x \). In addition we have
\[
D^2\rho(x) = D^2\rho(y)Q(x)^{-1}.
\]

**Remark.** As a result, when \( \det Q \neq 0 \) we have \( x \not\in R_{\rho} \). Thus by definition of \( R_{\rho} \), a neighborhood of \( x \) does not intersect \( R_{\rho} \), and hence it does not intersect \( R_{\rho,0} \) either.

**Proof.** Let \( z \mapsto Y(z) \) be a \( C^{k,\alpha} \) parametrization of \( \partial U \) around \( Y(0) = y \), where \( z \) varies in an open set \( V \subset \mathbb{R}^{n-1} \). Consider the map \( G : V \times \mathbb{R} \to \mathbb{R}^n \) defined by
\[
G(z,t) := Y(z) + (t - \varphi(Y(z)))D\gamma(\mu)(D\rho(Y(z))).
\]
Note that \( G \) is a \( C^{k-1,\alpha} \) function. Also note that by \eqref{eq:5.7} and \eqref{eq:5.9} we have
\[
G(0,\rho(x)) = y + (\rho(x) - \varphi(y))D\gamma(\mu)(D\rho(y)) = y + (\rho(x) - \varphi(y))D\gamma(\mu(y)) = x.
\]
We wish to compute \( DG \) around the point \( (0,\rho(x)). \) We have
\[
\begin{cases}
D_{z_j} G = D_{z_j}Y - (D\varphi, D_{z_j}Y)D\gamma(\mu) + (\rho(x) - \varphi)D^2\gamma(\mu)D^2\rho(y)D_{z_j}Y,
D_\rho G = D\gamma(\mu).
\end{cases}
\]
Note that \( DY \) is evaluated at \( z = 0 \); \( \mu, \varphi, D\varphi \) are evaluated at \( y = Y(0) \); and we used the fact that \( D\rho(y) = \mu(y) \).
Next note that we have $QD\gamma^\circ(\mu) = D\gamma^\circ(\mu)$, since by (5.31) we have $D^2\rho(y)D\gamma^\circ(\mu) = 0$. Now let $w_j := D_{\mu}Y(0)$. Note that $w_1, \ldots, w_{n-1}$ is a basis for the tangent space to $\partial U$ at $y$. Let $w$ be the orthogonal projection of $D\gamma^\circ(\mu)$ on this tangent space. Then we have (we represent a matrix by its columns)

$$\det DG(0, \rho(x)) = \det [Qw_1 - (D\varphi, w_1)D_1\gamma^\circ(\mu) \cdots Qw_{n-1} - (D\varphi, w_{n-1})D_{n-1}\gamma^\circ(\mu) \ D\gamma^\circ(\mu)]$$

$$= \det Q \det [w_1 \cdots w_{n-1} \ D\gamma^\circ(\mu)] = \det Q \det [w_1 \cdots w_{n-1} w + (D\gamma^\circ(\mu), \nu)\nu]$$

$$= (D\gamma^\circ(\mu), \nu) \det Q \det [w_1 \cdots w_{n-1} \nu] \neq 0.$$

Note that in the last line we have used (5.31), and the fact that $w_1, \ldots, w_{n-1}, \nu$ are linearly independent. Therefore by the inverse function theorem, $G$ is invertible on a neighborhood of $(0, \rho(x))$, and it has a $C^{k,\alpha}$ inverse on a neighborhood of $x$. Then as in the proof of the previous theorem, we can show that the inverse of $G$ is of the form $G^{-1}(\cdot) = (z(\cdot), \rho(\cdot))$, where $z(\cdot)$ is a $C^{k,\alpha}$ function of $x$. Note that here we need the fact that $x \notin R_{\rho,0}$. In addition, we can similarly conclude that $\rho$ is $C^{k,\alpha}$ on a neighborhood of $x$.

Now let us compute $DG^{-1}$ at $x$. To simplify the notation set $s_j := \langle D\varphi, w_j \rangle$. Then we have

$$DG(0, \rho(x)) = [Qw_1 - s_1D_1\gamma^\circ(\mu) \cdots Qw_{n-1} - s_{n-1}D_{n-1}\gamma^\circ(\mu) \ D\gamma^\circ(\mu)]$$

$$= [Qw_1 - s_1QD_1\gamma^\circ(\mu) \cdots Qw_{n-1} - s_{n-1}QD_{n-1}\gamma^\circ(\mu) \ QD\gamma^\circ(\mu)]$$

$$= Q [w_1 - s_1D_1\gamma^\circ(\mu) \cdots w_{n-1} - s_{n-1}D_{n-1}\gamma^\circ(\mu) \ D\gamma^\circ(\mu)]$$

$$= QA [e_1 - s_1e_n \cdots e_{n-1} - s_{n-1}e_n \ e_n],$$

where $e_j$ is the $j$-th column vector in the standard basis of $\mathbb{R}^n$, and

$$A := [w_1 \cdots w_{n-1} \ D\gamma^\circ(\mu)].$$

Note that the $j$-th column of every matrix is equal to the action of the matrix on $e_j$. Also note that $w_1, \ldots, w_{n-1}, D\gamma^\circ(\mu)$ are linearly independent due to (5.3). Thus $A$ is invertible. Therefore we have

$$DG^{-1}(x) = (DG(0, \rho(x)))^{-1}$$

$$= (QA [e_1 - s_1e_n \cdots e_{n-1} - s_{n-1}e_n \ e_n])^{-1}$$

$$= [e_1 - s_1e_n \cdots e_{n-1} - s_{n-1}e_n \ e_n]^{-1} A^{-1}Q^{-1}$$

$$= [e_1 + s_1e_n \cdots e_{n-1} + s_{n-1}e_n \ e_n] A^{-1}Q^{-1}.$$
Now note that $D\rho(x) = \mu(y) = D\rho(y) = D\rho(Y(0)) = D\rho(Y(z(x)))$. Thus we have

$$D^2\rho(x) = D^2\rho(y)DY(0)Dz(x) = D^2\rho(y)DY(0)I^{-1}A^{-1}Q^{-1}.$$  

On the other hand we know that $DY(0) = [w_1 \ \cdots \ w_{n-1}]$, i.e. the $j$-th column of $DY(0)$ is the $j$-th column of $A$, for $j < n$. Then it is easy to check that $DY(0)\tilde{I} = AI$, where $\tilde{I}$ is the $n \times n$ matrix whose first $n-1$ columns are the same as $I$, and its $n$-th column is $0$. Next note that the $n$-th row of $A^{-1}$ is $1_{\langle \gamma^\circ(\mu), \nu \rangle} \nu$. Because it must be orthogonal to the first $n-1$ columns of $A$, i.e. $w_j$'s; so it is a multiple of $\nu$. In addition, its inner product with the $n$-th column of $A$, i.e. $D\gamma^\circ(\mu)$, must be 1; thus we get the desired. Hence we have

$$\tilde{A}A^{-1} = A(I - [0 \ \cdots \ 0 \ e_n])A^{-1} = I - [0 \ \cdots \ 0 \ Ae_n]A^{-1} = I - [0 \ \cdots \ 0 \ D\gamma^\circ(\mu)]A^{-1} = I - \frac{1}{\langle D\gamma^\circ(\mu), \nu \rangle}D\gamma^\circ(\mu) \otimes \nu = I - X.$$  

Therefore we get

$$D^2\rho(x) = D^2\rho(y)\tilde{A}A^{-1}Q^{-1} = D^2\rho(y)(I - X)Q^{-1}.$$  

However, by (5.10) we know that $D^2\rho(y)$ has a factor of $I - X$. Also, as explained after equation (5.3) we know that $I - X$ is a projection. Thus $(I - X)^2 = I - X$, and we get the desired formula for $D^2\rho(x)$. \hfill \Box

At this point we have the tools to completely characterize the $\rho$-ridge $R_\rho$. Note that under the Assumption 5 we have $R_{\rho,0} \subset R_\rho$, due to Lemma 9. The next theorem specifies those points which are in $R_\rho - R_{\rho,0}$.

**Theorem 4.** Suppose the Assumption 5 holds. Suppose $x \in U - R_{\rho,0}$, and let $y$ be the unique $\rho$-closest point to $x$ on $\partial U$. Then

$$x \in R_\rho \ if \ and \ only \ if \ det Q(x) = 0,$$

where $Q$ is defined by (5.4).  

**Proof.** In the previous theorem, we have shown that if $det Q \neq 0$ then $x \notin R_\rho$. So we only need to show that if $det Q = 0$ then $x \in R_\rho$. Suppose $det Q(x) = 0$. Then the definition of $Q$ implies that

$$\tilde{\kappa} := \frac{1}{\rho(x) - \rho(y)} = \frac{1}{\gamma(x - y)} > 0$$

is an eigenvalue of $W(y)$. Suppose $\zeta$ is the corresponding eigenvector of $W$. Let $z \in [x, y]$. Then by Lemma 8 we have $z \in U$, and $y$ is the unique $\rho$-closest point on $\partial U$ to $z$. In addition we have $z - y = t(x - y)$ for some $t \in (0, 1)$. Thus $\gamma(z - y) = t\gamma(x - y)$. Therefore if $z$ is close enough to $x$, then we must have $det Q(z) \neq 0$. Because otherwise $\frac{1}{\gamma(z - y)} = \frac{\kappa}{t}$ must be an eigenvalue of $W$ for infinitely many $t$'s, which is a contradiction. Hence by Theorem 8 $\rho$ is $C^{k,\alpha}$ on a neighborhood of $z$.

Now suppose to the contrary that $x \notin R_\rho$. Then $\rho$ is $C^{1,\alpha}$ on a neighborhood of $x$. Thus $\rho$ belongs to $W^{2,\infty}$ on a neighborhood of $x$. Consequently, $D^2\rho$ belongs to $L^\infty$ on a neighborhood of $x$. Therefore $D^2\rho$ is bounded on a neighborhood of an open line segment $[x, z_0]$ for some $z_0 \in [x, y]$.  


Theorem 5. Suppose the Assumption 3 holds. Suppose \( x \in U \), and \( y \) is one of the \( \rho \)-closest points to \( x \) on \( \partial U \). Suppose \( \bar{k} \) is an eigenvalue of \( W(y) \), where \( W \) is defined by (5.14). Then we have
\[
\bar{k}(\rho(z) - \varphi(y)) = \bar{k}\gamma(z - y) < 1,
\]
for every \( z \in [x, y] \). As a consequence we have
\[
\det Q(z) = \det (I - \gamma(z - y)W(y)) > 0.
\]

Remark. As a result, due to the continuity we have
\[
\bar{k}(\rho(x) - \varphi(y)) = \bar{k}\gamma(x - y) \leq 1,
\]
and \( \det (I - \gamma(x - y)W(y)) \geq 0 \). Note that \( x \) can have \( \rho \)-closest points on \( \partial U \) other than \( y \).

Proof. Note that by Lemma 8, \( y \) is the unique \( \rho \)-closest point to \( z \) on \( \partial U \). Also note that by (1.8) we have \( \gamma(z - y) = \rho(z) - \varphi(y) \). So \( Q(z) = I - \gamma(z - y)W \). Now let \( z \mapsto Y(z) \) be a smooth parametrization of \( \partial U \) around \( Y(0) = y \), where \( z \) varies in an open set \( V \subset \mathbb{R}^{n-1} \). Then due to \( \rho \)'s definition, the function \( \varphi(Y(z)) + \gamma(x - Y(z)) \) has a minimum at \( z = 0 \). Hence its second derivative must be positive semidefinite at \( z = 0 \). We have
\[
D_{z_j} [\varphi(Y(z)) + \gamma(x - Y(z))] = \langle D_{z_j} Y, D\varphi(Y(z)) - D\gamma(x - Y(z)) \rangle.
\]
Therefore at \( z = 0 \) we have
\[
D^2_{z_j, z_k} [\varphi(Y(z)) + \gamma(x - Y(z))] = \langle D^2_{z_j, z_k} Y, D\varphi(y) - D\gamma(x - y) \rangle
\]
\[
+ D_{z_j} Y (D^2 \varphi(y) + D^2 \gamma(x - y)) D_{z_k} Y \geq 0.
\]
However by (5.6) we know that \( x - y = \gamma(x - y)D\gamma(\mu(y)) \). Hence by (2.5),(2.7) we have
\[
D\gamma(x - y) = D\gamma(D\gamma(\mu)) = \mu,
\]
since \( \gamma(\mu) = 1 \). Thus by (5.2) we have
\[
D\varphi(y) - D\gamma(x - y) = D\varphi(y) - \mu(y) = -\lambda(y)\nu(y).
\]
On the other hand we have \( \langle D_{x_j} Y, \nu(Y(z)) \rangle = 0 \). So if we differentiate this equality we get
\[
\langle D^2_{x_j} Y, \nu \rangle + D_{x_j} Y(D\nu)D_{x_j} Y = 0.
\]
However we know that \( D\nu = D^2 d \), where \( d \) is the Euclidean distance to \( \partial U \). Therefore we obtain
\[
\langle D^2_{x_j} Y, D\varphi(y) - D\gamma(x-y) \rangle = \langle D^2_{x_j} Y, -\lambda \nu \rangle = D_{x_j} Y(\lambda D^2 d)D_{x_j} Y.
\]
Thus by inserting this equality in (5.16) we get
\[
D_{x_j} Y(\lambda D^2 d(y) + D^2 \varphi(y) + D^2 \gamma(x-y))D_{x_j} Y \geq 0.
\]
To simplify the notation set \( w_j := D_{x_j} Y \). Remember that we have \((I-X)w_j = w_j\), since \(I-X\) is the projection on the tangent space to \( \partial U \) parallel to \( D\gamma^\circ(\mu) \), as explained after equation (3.3).
Hence we have
\[
w_j(I-X^T)(\lambda D^2 d(y) + D^2 \varphi(y))(I-X)w_j + w_j D^2 \gamma(x-y)w_j \geq 0.
\]
Then by (5.10) we get
\[
w_j(D^2 \rho(y) + D^2 \gamma(x-y))w_j \geq 0.
\]
In addition, by (2.5) we have
\[
D^2 \gamma(x-y) = D^2 \gamma(\gamma(x-y)D\gamma^\circ(\mu)) = \frac{1}{\gamma(x-y)}D^2 \gamma(D\gamma^\circ(\mu)).
\]
Thus the symmetric matrix \( D^2 \rho(y) + \frac{1}{\gamma(x-y)}D^2 \gamma(D\gamma^\circ(\mu)) \) is positive semidefinite, since its action on \( D\gamma^\circ(\mu) \) is zero due to (5.11), (2.6), and \( w_1, \ldots, w_{n-1}, D\gamma^\circ(\mu) \) form a basis for \( \mathbb{R}^n \) due to (5.5).
Now note that the eigenvalues of \( D^2 \gamma(D\gamma^\circ(\mu)) \) are all positive except for one 0 which corresponds to the eigenvector \( D\gamma^\circ(\mu) \), as explained in Subsection 2.1. We also know that \( \langle D\gamma^\circ(\mu), \mu \rangle = \gamma^\circ(\mu) = 1 \neq 0 \). Let \( \{\mu\}^\perp \) be the subspace orthogonal to \( \mu \). Then for every nonzero \( w \in \{\mu\}^\perp \) we must have \( wD^2 \gamma(D\gamma^\circ(\mu))w > 0 \). Hence for every \( t > \frac{1}{\gamma(x-y)} \) we have
\[
w(D^2 \rho(y) + tD^2 \gamma(D\gamma^\circ(\mu)))w = w(D^2 \rho(y) + \frac{1}{\gamma(x-y)}D^2 \gamma(D\gamma^\circ(\mu)))w + (t - \frac{1}{\gamma(x-y)})wD^2 \gamma(D\gamma^\circ(\mu))w > 0.
\]
On the other hand, if we differentiate (2.7) we get
\[
D^2 \gamma(D\gamma^\circ(\cdot))D^2 \gamma^\circ(\cdot) = \frac{1}{\gamma^\circ(\cdot)}I - \frac{1}{\gamma^\circ(\cdot)^2(\cdot)} \otimes D\gamma^\circ(\cdot).
\]
Hence we have \( D^2 \gamma(D\gamma^\circ(\mu))D^2 \gamma^\circ(\mu) = I - \mu \otimes D\gamma^\circ(\mu) \). By taking the transpose of this equation we get
\[
D^2 \gamma^\circ(\mu)D^2 \gamma(D\gamma^\circ(\mu)) = I - D\gamma^\circ(\mu) \otimes \mu.
\]
Next suppose that \( \kappa \) is an eigenvalue of \( W \) corresponding to the eigenvector \( w \). If \( \kappa = 0 \) then (5.15) holds trivially. So suppose \( \kappa \neq 0 \). Then we have
\[
\langle w, \mu \rangle = \frac{1}{\kappa} \langle Ww, \mu \rangle = -\frac{1}{\kappa} \langle D^2 \gamma^\circ(\mu)D^2 \rho(y)w, \mu \rangle
\]
\[
= -\frac{1}{\kappa} \langle D^2 \rho(y)w, D^2 \gamma^\circ(\mu)\mu \rangle = -\frac{1}{\kappa} \langle D^2 \rho(y)w, 0 \rangle = 0.
\]
Hence \( w \in \{ \mu \}^\perp \). On the other hand, note that the eigenvalues of \( D^2 \gamma^\circ(\mu) \) are all positive except for one 0 which corresponds to the eigenvector \( \mu \). In addition we know that the other eigenvectors of \( D^2 \gamma^\circ(\mu) \) are orthogonal to \( \mu \), since it is a symmetric matrix. Thus the image of \( D^2 \gamma^\circ(\mu) \) is \( \{ \mu \}^\perp \), and its restriction to \( \{ \mu \}^\perp \) is positive definite and invertible. Let \( A \) be the symmetric matrix whose action on \( \{ \mu \}^\perp \) is the inverse of the action of \( D^2 \gamma^\circ(\mu) \) restricted to \( \{ \mu \}^\perp \); and \( A \mu = 0 \). Then the restriction of \( A \) to \( \{ \mu \}^\perp \) is also positive definite. It is easy to check that \( AD^2 \gamma^\circ(\mu) = I - \frac{1}{|\mu|^2} \mu \otimes \mu \).

Now for \( z \in ]x, y[ \) let \( t := \frac{1}{\gamma(z-y)} \). Then \( t > \frac{1}{\gamma(x-y)} \). Therefore by (5.17) and (5.18) we have

\[
0 < w(D^2 \rho(y) + tD^2 \gamma(D\gamma^\circ(\mu)))w = w(I - \frac{1}{|\mu|^2} \mu \otimes \mu)(D^2 \rho(y) + tD^2 \gamma(D\gamma^\circ(\mu)))w
\]

\[
= wAD^2 \gamma^\circ(\mu)(D^2 \rho(y) + tD^2 \gamma(D\gamma^\circ(\mu)))w = wA(-\tilde{\kappa} + t)w = \frac{1}{\gamma(z-y)}(-\tilde{\kappa}\gamma(z - y) + 1)wAw.
\]

Note that since \( w \) is orthogonal to \( \mu \), we have \( w(\mu \otimes \mu) = (D\gamma^\circ(\mu) \otimes \mu)w = 0 \). Hence we get the desired relation (5.15), because \( wAw > 0 \) due to the positive definiteness of \( A \) restricted to \( \{ \mu \}^\perp \).

Finally note that every eigenvalue of \( W \) must be real. Because \( W \) is the product of two symmetric matrices \( D^2 \gamma^\circ(\mu) \), \( -D^2 \rho(y) \); and \( D^2 \gamma^\circ(\mu) \) is positive semidefinite. Thus in particular, \( W \) is similar to a real triangular matrix. Hence the determinant of \( Q(z) = I - \gamma(z-y)W \) is the product of \( 1 - \tilde{\kappa}\gamma(z - y) \), where \( \tilde{\kappa} \) varies among the eigenvalues of \( W \). Therefore \( \det Q > 0 \) as desired.

**Remark.** Suppose the Assumption 5 holds. Suppose \( x \in U \), and \( y \) is one of the \( \rho \)-closest points to \( x \) on \( \partial U \). It can be shown that the segment \( ]x, y[ \) is the characteristic curve associated to the first order PDE (1.9). We do not use this fact in this article, but let us summarize what we have proved so far about the segment \( ]x, y[ \).

1. In Lemma 8 we have shown that \( ]x, y[ \subset U \), and \( y \) is the unique \( \rho \)-closest point to every point of \( ]x, y[ \). In particular we have \( ]x, y[ \cap \rho_{p,0} = \emptyset \).
2. We have also seen that \( \rho \) varies linearly along \( ]x, y[ \), and by (5.8) we have \( D\rho(z) = \mu(y) \) for every \( z \in ]x, y[ \).
3. By (5.6) we know that \( ]x, y[ \) is parallel to \( D\gamma^\circ(\mu(y)) \). We also have

\[
D^2 \rho(z)D\gamma^\circ(\mu(y)) = 0,
\]

for every \( z \in ]x, y[ \). Because we have \( W(y)D\gamma^\circ(\mu) = 0 \) by (5.13),(5.11). So we have

\[
Q(z)D\gamma^\circ(\mu) = D\gamma^\circ(\mu) \text{ due to (5.13).}
\]

Hence we get the desired by (5.14) and (5.11). Furthermore due to (5.14), \( D^2 \gamma^\circ(\mu)D^2 \rho(z) \) can be triangulated, and its triangular form is

\[
\begin{pmatrix}
\frac{-\tilde{\kappa}_1}{1-\gamma(z-y)\kappa_1} & * & * \\
0 & \ddots & * \\
0 & 0 & \frac{-\tilde{\kappa}_{n-1}}{1-\gamma(z-y)\kappa_{n-1}}
\end{pmatrix},
\]

where \( \kappa_1, \ldots, \kappa_{n-1}, 0 \) are the eigenvalues of \( W(y) \). Remember that the eigenvalues of \( W \) are all real, and hence it can be triangulated, as we explained in the previous proof. Also remember that \( 1 - \gamma(z - y)\tilde{\kappa}_j > 0 \) by the previous theorem. Notice the similarity between
the above form and the classical formula for $D^2 d$ derived in [22, Section 14.6]. Although, we cannot necessarily find a similar form for $D^2 \rho(z)$ itself.

(4) By the above theorem we know that $\det Q(z) \neq 0$ for every $z \in [x, y]$. Thus in particular we have $\langle x, y \rangle \cap R_{\rho} = \emptyset$. Hence by Theorem [3] we have $\det G > 0$. The third factor is also positive, since it can be shown that

$$\det G = \langle D\gamma^0(\mu), \nu \rangle \det Q \det [w_1, \ldots, w_{n-1}, \nu],$$

and we know that the first two factors are positive. The third factor is also positive, since it can be shown that

$$\det [w_1, \ldots, w_{n-1}, \nu] = \sqrt{\det(g_{ij})},$$

where $g_{ij} := \langle w_i, w_j \rangle$ are the components of the Riemannian metric on $\partial U$. For the proof see for example Lemma 4.10 of [10].

**Remark.** When $\varphi = 0$, the function $\rho$ is the distance to $\partial U$ with respect to the Minkowski distance defined by $\gamma$. So this case has a geometric interpretation. An interesting simplification that happens here is that $\mu$ becomes a multiple of $\nu$. Another interesting fact is that in this case the eigenvalues of $W$ coincide with the notion of curvature of $\partial U$ with respect to some Finsler structure. For the details see [10]. Let us also mention that in this case we have

$$D\rho(x) = \mu(y) + \frac{1}{\gamma^0(\nu)}\nu(y) = \frac{1}{\gamma^0(Dd)}Dd(y)$$

$$= \frac{1}{\gamma^0(Dd)}Dd(x - \rho(x)D\gamma^0(\mu(y))) = \frac{1}{\gamma^0(Dd)}Dd(x - \rho(x)D\gamma^0(\rho(x))).$$

Note that we have used [5], [5.8] to eliminate $y$ from the above equation; and this is not possible when $\varphi$ is nonzero. Now we can differentiate the above equation to find $D^2 \rho(x)$. This approach is how we first found $D^2 \rho(x)$.

**Proposition 4.** Suppose the Assumption [3] holds. Then we have $R_{\rho} = \overline{R}_{\rho, 0}$.

**Proof.** Note that $R_{\rho} \subset U$ is closed in $U$ by definition, and it has a positive distance from $\partial U$ due to Theorem [2]. Hence $R_{\rho}$ is closed. We also know that $R_{\rho, 0} \subset R_{\rho}$. So we have $\overline{R}_{\rho, 0} \subset R_{\rho}$. Thus we only need to show that $R_{\rho} \subset \overline{R}_{\rho, 0}$. Let $x \in U - \overline{R}_{\rho, 0}$. We will show that $x \notin R_{\rho}$. There is $r > 0$ such that $B_r(x) \subset U - \overline{R}_{\rho, 0}$. For every $z \in B_r(x)$ let $p(z) \in \partial U$ be the unique $\rho$-closest point to $z$ on $\partial U$. Then Lemma [1] implies that $p$ is continuous on $B_r(x)$.

Let $z \mapsto Y(z)$ be a smooth parametrization of $\partial U$ around $Y(0) = p(x)$, where $z$ varies in an open set $V \subset \mathbb{R}^{n-1}$. Then $Y^{-1} : Y(V) \to V$ is a continuous bijection. There is $0 < s < r$ such that $p(B_s(x)) \subset Y(V)$, since $p$ is continuous. Now $Y^{-1} \circ p : \partial B_s(x) \to V \subset \mathbb{R}^{n-1}$ is a continuous map. Hence by Borsuk–Ulam theorem (see Corollary 2B.7 of [23]) there are $x^- := x - sw$ and $x^+ := x + sw$, for some $w \in \partial B_1(0)$, such that

$$Y^{-1} \circ p(x^-) = Y^{-1} \circ p(x^+).$$

But $Y^{-1}$ is one to one, so $p(x^-) = p(x^+)$. Let $y := p(x^+)$. Then by (5.7) we have

$$x^+ = y + (\rho(x^+) - \varphi(y)) D\gamma^0(\mu(y)).$$
Thus $x^-, x^+$ are on the ray emanating from $y$ in the direction of $D \gamma^\circ(\mu)$. Suppose for example $x^- \in [y, x^+]$. On the other hand, it is obvious that $x \in [x^-, x^+]$. So we get $x \in [y, x^+]$. Hence by Theorem 5 we have $\det Q(x) \neq 0$, and therefore $x \notin R_\rho$ due to Theorem 5 as desired. Note that by Lemma 8 it also follows that $p(x) = y$. This fact is not needed in this proof, but we will use it in the next remark.

**Remark.** Suppose the Assumption 5 holds. Let $y \in \partial U$. Then by Theorem 2 we know that $y$ is the $\rho$-closest point to some points in $U$. By Lemma 8, these points must lie on the ray emanating from $y$ in the direction of $D \gamma^\circ(\mu(y))$. By Lemma 8, we know that if $x$ on this ray has $y$ as a $\rho$-closest point on $\partial U$, then every point between $x, y$ belongs to $U$, and has $y$ as its unique $\rho$-closest point on $\partial U$. On the other hand, this ray will intersect $\partial U$ at a point other than $y$, because $U$ is bounded. Let $\tilde{y}$ be the closest of such intersection points to $y$. Then by Lemma 1, the points on the ray near $\tilde{y}$ (which are inside $U$) cannot have $y$ as their $\rho$-closest point on $\partial U$, since their $\rho$-closest points must converge to $\tilde{y}$ as they approach $\tilde{y}$.

Let $x$ be the supremum of the points on the ray that are inside $U$, and have $y$ as a $\rho$-closest point on $\partial U$. Note that by Lemma 1, $y$ is one of the $\rho$-closest points to $x$ on $\partial U$, since we can approach $x$ with points having $y$ as their $\rho$-closest point on $\partial U$. On the other hand, $x$ must belong to $R_\rho = \overline{R}_{\rho,0}$. Because as shown in the above proof, if $x \notin R_{\rho,0}$ then there is another point $x^+$ on the ray that has $y$ as a $\rho$-closest point on $\partial U$, and $x \in [y, x^+]$; which contradicts our choice of $x$. Hence to summarize, along the ray emanating from $y$ in the direction of $D \gamma^\circ(\mu(y))$, the only points that have $y$ as their $\rho$-closest point on $\partial U$ are those points which lie between $y$ and the closest point to $y$ on the intersection of $R_\rho$ and the ray.

The next lemma is needed in the next section, when we deal with the regularity of the minimizers of the functional $J$. It states that the pure second order partial derivatives of $\rho$ satisfy a monotonicity property. In the remark after the lemma, we will see that this property is indeed true for the solutions of quite general Hamilton-Jacobi equations, provided that those solutions are smooth enough.

**Lemma 14.** Suppose the Assumption 5 holds. Let $x \in U - R_\rho$, and let $y$ be the unique $\rho$-closest point to $x$ on $\partial U$. Let $A$ be a symmetric positive semidefinite matrix. Then we have

$$\text{tr}[AD^2 \rho(x)] \leq \text{tr}[AD^2 \rho(y)],$$

In particular we have $D_{\xi \xi}^2 \rho(x) \leq D_{\xi \xi}^2 \rho(y)$, for every $\xi \in \mathbb{R}^n$.

**Proof.** By (5.7), we know that $t \mapsto y + tD \gamma^\circ(\mu)$ parametrizes the segment $[y, x]$, when $t$ goes from 0 to $\gamma(x - y)$. We also know that $\rho$ is $C^2$ on a neighborhood of $[y, x]$, due to the Theorems 5 and 5. Thus to prove the lemma it suffices to show that

$$\frac{d}{dt}\text{tr}[AD^2 \rho(y + tD \gamma^\circ(\mu))] \leq 0.$$
To simplify the notation set \( \tilde{\mu} := D\gamma^\circ(\mu) \). Also let \( \sqrt{A} \) be the unique symmetric positive semidefinite matrix whose square is \( A \). Then by (5.13), (5.14) we have

\[
\frac{d}{dt} \text{tr}[AD^2 \rho(y + t\tilde{\mu})] = \text{tr}[A \frac{d}{dt} D^2 \rho(y + t\tilde{\mu})] = \text{tr}[A \frac{d}{dt} (D^2 \rho(y)(I - tW)^{-1})] = \text{tr}[AD^2 \rho(y)(I - tW)^{-1}W(I - tW)^{-1}] = -\text{tr}[AD^2 \rho(y)(I - tW)^{-1}D^2\gamma^\circ(\mu)D^2 \rho(y)(I - tW)^{-1}] = -\text{tr}[\sqrt{A}\sqrt{AD^2 \rho(y + t\tilde{\mu})D^2 \rho(y + t\tilde{\mu})}] = -\text{tr}[\sqrt{AD^2 \rho(y + t\tilde{\mu})D^2 \rho(y + t\tilde{\mu})}\sqrt{A}] \leq 0.
\]

Note that the matrix in the last line of the above formula is positive semidefinite, since \( D^2\gamma^\circ(\mu) \) is positive semidefinite, and the other factors are symmetric. Finally, to get the last statement of lemma we just need to set \( A = \xi \otimes \xi \), which is trivially a symmetric positive semidefinite matrix. Then it is easy to check that \( \text{tr}[AD^2 \rho] = D^2\xi \xi^T \rho \).

**Remark.** The above monotonicity property is true because \( \rho \) satisfies the Hamilton-Jacobi equation (1.9); and as we mentioned before, the segment \([x, y]\) is the characteristic curve associated to it. In general, suppose \( v \) satisfies the equation \( H(x, v, Dv) = 0 \), where \( H \) is a convex function. Let \( p \) be the variable in \( H \) for which we substitute \( Dv \). Let \( x(s) \) be a characteristic curve of the equation. Then we have \( \dot{x} = DpH \). (See for example [14, Section 3.2].) Let us assume that \( v \) is \( C^3 \) on a neighborhood of the image of \( x(s) \). Let

\[
q(s) := D^2\xi \xi^T v(x(s)) = \xi_i \xi_j D^3_{ijk} v D_{ijk} H.
\]

for some vector \( \xi \). Then we have

\[
\dot{q} = \xi_i \xi_j D^3_{ijk} v D_{ijk} \dot{H}.
\]

On the other hand, if we differentiate the equation we get \( D_{x, i} H + D_{v, i} HD_i v + D_{p, k} HD^2_{ik} v = 0 \). And if we differentiate one more time we get

\[
D^2_{x, x, j} H + D^2_{x, j} HD_j v + D_{x, p, x} HD^2_{j, k} v + D_{v, x, j} HD_j v + D_{v, v, j} HD_j v + D_{v, p, v} HD^2_{j, k} v + D_{p, k, x} HD^2_{j, v} v + D_{p, p, v} HD^2_{j, v} v + D_{p, k} HD^3_{j, k, v} = 0.
\]

Now if we multiply the above expression by \( \xi_i \xi_j \), and sum over \( i, j \), we obtain

\[
(5.19) \quad \dot{q} = -[\xi^T \langle \xi, Dv \rangle \xi^T D^2 v] \begin{bmatrix} D^2_{xx} H & D^2_{xv} H & D^2_{xp} H \\ D^2_{xv} H & D^2_{vv} H & D^2_{vp} H \\ D^2_{xp} H & D^2_{vp} H & D^2_{pp} H \end{bmatrix} \begin{bmatrix} \xi \\ \langle \xi, Dv \rangle \\ D^2 v \xi \end{bmatrix} - D_v Hq
\]

\[
\dot{q} = -\eta^T D^2 H \eta - D_v Hq,
\]

where \( \eta := [\xi^T \langle \xi, Dv \rangle \xi^T D^2 v]^T \). Hence we have \( \dot{q} \leq -D_v Hq \), since \( H \) is convex. Thus by Gronwall’s inequality (see for example [14, Appendix B]) we obtain

\[
q(s) \leq q(0)e^{-\int_0^s D_v Hdr}.
\]
In particular when $D_v H \geq 0$, i.e. when $H$ is increasing in $v$, we have

$$D_{\xi_0}^2 v(x(s)) = q(s) \leq q(0) = D_{\xi_0}^2 v(x(0)).$$

The general case of $\text{tr}[AD^2 v(x(s))] \leq \text{tr}[AD^2 v(x(0))]$ for a positive semidefinite matrix $A$ follows easily; because we have $\text{tr}[AD^2 v] = \sum a_j D_{\xi_j}^2 \xi_j v$, where $\xi_1, \cdots, \xi_n$ is an orthonormal basis of eigenvectors of $A$ corresponding to the nonnegative eigenvalues $a_1, \ldots, a_n$. Finally note that in the previous lemma we did not need the $C^3$ regularity of $\rho$. We suspect that in the general case, it is possible to weaken the regularity assumption on $v$ too.

Also notice that the ODE system $[5.19]$ can be used to compute an explicit formula for $D^2 v$. For example when $H$ does not depend on $x, v$, we have

$$\frac{d}{ds} D^2 v(x(s)) = -D^2 v D_{pp} H D^2 v.$$  

Then it is easy to show that $D^2 v$ must be given by a formula similar to $[5.14]$. However, as we said above, this approach requires imposing extra regularity on $v$. Also, solving the system $[5.19]$ in general seems to be a daunting task.

**Remark.** At the end of this section, we would like to comment on the structure of the set $R_\rho$, particularly its Hausdorff dimension. Suppose additionally that $\gamma^0, \varphi$ are $C^\infty$. Moreover suppose that $U$ is connected, and $\partial U$ is $C^{2,1}$. Let $H(x, p) := \gamma^0(p + D\varphi(x))$. Then $v(x) := \rho(x) - \varphi(x)$ satisfies the following Hamilton-Jacobi equation

$$\begin{cases}
    H(x, Dv) = \gamma^0(Dp) = 1 & \text{in } U, \\
    v = 0 & \text{on } \partial U.
\end{cases}$$

In addition, the sets $\{ p \in \mathbb{R}^n : H(x, p) < 1 \} = -D\varphi(x) + \text{int}(K^\circ)$ are smooth strictly convex sets containing $0$; because we have $\gamma^0(D\varphi) < 1$. Furthermore, by $[44]$, for every $x \in U$ we have

$$v(x) = \rho(x) - \varphi(x) = \gamma(x - y) + \varphi(y - \varphi(x) > 0,$$

where $y$ is the $\rho$-closest point on $\partial U$ to $x$. Also note that the set of singularities of $v$ equals $R_\rho$, since $\varphi$ is smooth. Therefore we can apply the result of Li and Nirenberg $[31]$, and conclude that

$$H^{n-1}(R_\rho) < \infty,$$

where $H^{n-1}$ is the $n-1$-dimensional Hausdorff measure. It also follows that $R_\rho$ can be covered by countably many $n-1$-dimensional $C^1$ submanifolds, except for a set whose $H^{n-1}$ measure is zero. In other words, $R_\rho$ is $C^1$-rectifiable. It is also shown in $[31]$ that $R_\rho$ is path connected. Finally let us mention that if we merely assume that $\gamma^0, \varphi, \partial U$ are $C^{2,\alpha}$, for some $\alpha > 0$, then we can repeat the arguments in $[10]$, and conclude that the Hausdorff dimension of $R_\rho$ is at most $n-\alpha$. However, we will not use these properties of $R_\rho$, so we will not provide the details here.

### 6. Global Optimal Regularity

In this final section we present our main results. We will prove that $u$, i.e. the minimizer of the functional $J$ over $W_{K^\circ, \varphi}(U)$, belongs to $C^{1,1}(\overline{U})$, without assuming any regularity about $K$. To this end, first we show that when $\partial K$ is smooth enough, $u$ does not touch the obstacles $\rho, -\bar{\rho}$ at their singularities.
Theorem 6. Suppose the Assumptions hold. Then we have

\[ R_\rho \cap P^+ = \emptyset, \quad R_\rho \cap P^- = \emptyset. \]

Proof. We have already shown in Proposition that \( R_{\rho,0}, R_{\bar{\rho},0} \) do not intersect \( P^+, P^- \) respectively. Note that Assumption implies the assumptions of that proposition. So we only need to show that \( R_\rho - R_{\rho,0}, R_\bar{\rho} - R_{\bar{\rho},0} \) do not intersect \( P^+, P^- \) respectively. Suppose to the contrary that there is a point \( x \in U \) which belongs to \( (R_\rho - R_{\rho,0}) \cap P^+ \); the other case is similar. Let \( y \) be the unique \( \rho \)-closest point to \( x \) on \( \partial U \). Then by Theorem we must have \( \det Q(x) = 0 \). Now the definition of \( Q \) implies that \( \tilde{\kappa} := \frac{1}{\rho(x) - \rho(y)} = \frac{1}{\gamma(x-y)} > 0 \) is an eigenvalue of \( W(y) \). Suppose \( \zeta \) is the corresponding eigenvector of \( W \), and \( |\zeta| = 1 \). Note that \( \zeta \) is not parallel to the segment \( |x, y| \), i.e. to the vector \( D\gamma^0(\mu) \); because we have \( WD\gamma^0(\mu) = 0 \).

Let \( z \in \rho, \mu \). Then by Lemma we have \( z \in U \), and \( y \) is the unique \( \rho \)-closest point on \( \partial U \) to \( z \). In addition, by Theorem we have \( \det Q(z) \neq 0 \). Hence by Theorem \( \rho \) is \( C^{k,\alpha} \) on a neighborhood of the line segment \( |x, y| \). We call this neighborhood \( V \). Furthermore, for \( z \in \rho, \mu \) we have

\[ Q(z) = I - \gamma(z-y)W = \frac{1}{s\tilde{\kappa}}(s\tilde{\kappa}I - W), \]

where \( s > 1 \) is such that \( s(z-y) = x-y \). Hence we have \( Q(z)\zeta = \frac{1}{s\tilde{\kappa}}(s-1)\tilde{\kappa}\zeta = \frac{s-1}{s}\zeta \). Then by we have

\[ D^2\zeta\rho(z) = \zeta D^2\rho(z)\zeta = \zeta D^2\rho(y)Q(z)^{-1}\zeta = \frac{s}{s-1} \zeta D^2\rho(y)\zeta. \]

We claim that \( \zeta D^2\rho(y)\zeta < 0 \). The reason is that for \( \xi := D^2\rho(y)\zeta \) we have \( D^2\gamma^0(\mu)\xi = -W\zeta = -\tilde{\kappa}\zeta \). On the other hand, we know that the eigenvalues of \( D^2\gamma^0(\mu) \) are all positive except for one which corresponds to the eigenvector \( \mu \). We also know that the other eigenvectors of \( D^2\gamma^0(\mu) \) are orthogonal to \( \mu \), since it is a symmetric matrix. In addition, as shown in the proof of Theorem \( \zeta \) is orthogonal to \( \mu \), since it is an eigenvector of \( W \) corresponding to a nonzero eigenvalue. Let \( (\zeta_1, \ldots, \zeta_{n-1}, 0) \) and \( (\xi_1, \ldots, \xi_n) \) be the coordinates of \( \zeta, \xi \) in the orthonormal basis consisting of the eigenvectors of \( D^2\gamma^0(\mu) \). Let \( \tau_1, \ldots, \tau_{n-1}, 0 \) be the corresponding eigenvalues of \( D^2\gamma^0(\mu) \). Then we have

\[ (\tau_1\zeta_1, \ldots, \tau_{n-1}\zeta_{n-1}, 0) = D^2\gamma^0(\mu)\xi = -\tilde{\kappa}\zeta = (-\tilde{\kappa}\zeta_1, \ldots, -\tilde{\kappa}\zeta_{n-1}, 0). \]

Hence we have \( \xi_j = \frac{-\tilde{\kappa}}{\tau_j}\zeta_j \) for \( j < n \). Therefore

\[ \zeta D^2\rho(y)\zeta = \langle \zeta, \xi \rangle = -\tilde{\kappa} \sum_{j<n} \frac{1}{\tau_j} |\zeta_j|^2 < 0, \]

as desired. As a consequence we have

\[ D^2\zeta\rho(z) = \frac{s}{s-1} D^2\rho(y)\zeta \rightarrow -\infty \quad \text{as} \quad z \rightarrow x, \]

since \( s \rightarrow 1^+ \).

Now since \( x \in P^+ \) we have \( u(x) = \rho(x) \). Hence by lemma we have \( |x, y| \subset P^+ \). Thus \( u(z) = \rho(z) \) for every \( z \in |x, y| \). Also remember that \( u \leq \rho \) everywhere, since \( u \in W_{\rho, \rho} \). Hence \( \rho - u \) is a \( C^1 \) function on \( V \), which attains its maximum, 0, on \( |x, y| \). Thus \( Du = D\rho \) on the segment \( |x, y| \).
Now we claim that for any \( z \in ]x, y[ \) there are points \( z_i := z + \epsilon_i \zeta \) in \( V \) converging to \( z \), at which we have
\[
D_\zeta u(z_i) \leq D_\zeta \rho(z_i).
\]
Since otherwise we would have \( D_\zeta u > D_\zeta \rho \) on a segment of the form \( ]z, z + r\zeta[ \), for some small \( r > 0 \). But as \( u(z) = \rho(z) \) and \( Du(z) = D\rho(z) \), this implies that \( u > \rho \) on \( ]z, z + r\zeta[ \); which is a contradiction. Thus we get the desired. As a consequence we have
\[
D_\zeta u(z_i) - D_\zeta u(z) \leq D_\zeta \rho(z_i) - D_\zeta \rho(z).
\]
By applying the mean value theorem to the restriction of \( \rho \) for distinct \( z, z_i \) we get
\[
\text{(6.2)} \quad D_\zeta u(z_i) - D_\zeta u(z) \leq |z_i - z| D^2_\zeta \rho(w_i),
\]
for some \( w_i \in ]z, z_i[ \).

On the other hand, \( u \) is a \( C^{1,1} \) function on a neighborhood of \( x \), due to Theorem 1. Consequently there is \( M > 0 \) such that
\[
\text{(6.3)} \quad -M \leq \frac{D_\zeta u(z_i) - D_\zeta u(z)}{|z_i - z|},
\]
for distinct \( z, z_i \) sufficiently close to \( x \). Now let \( z \in ]x, y[ \) be close enough to \( x \) so that \( D^2_\zeta \rho(z) < -3M \), which is possible due to (6.1). Then let \( z_i = z + \epsilon_i \zeta \) be close enough to \( z \) so that we have \( D^2_\zeta \rho(w_i) < -2M \), which is possible due to the continuity of \( D^2 \rho \) on \( V \). But this is in contradiction with (6.2) and (6.3).

Let us review some well-known facts from convex analysis, which are needed in the next theorem. Consider a compact convex set \( K \). Let \( x \in \partial K \), and \( v \in \mathbb{R}^n - \{0\} \). We say the hyperplane
\[
\text{(6.4)} \quad H_{x,v} := \{x + y : \langle y, v \rangle = 0\}
\]
is a \textit{supporting hyperplane} of \( K \) at \( x \) if \( K \subset \{x + y : \langle y, v \rangle \leq 0\} \). (Note that we always have \( x \in H_{x,v} \).) In this case we say \( v \) is an \textit{outer normal vector} of \( K \) at \( x \). The \textit{normal cone} of \( K \) at \( x \) is the closed convex cone
\[
\text{(6.5)} \quad N(K, x) := \{0\} \cup \{v \in \mathbb{R}^n - \{0\} : v \text{ is an outer normal vector of } K \text{ at } x\}.
\]
It is easy to see that when \( \partial K \) is \( C^1 \) we have
\[
N(K, x) = \{tD\gamma(x) : t \geq 0\}.
\]
For more details see [36, Sections 1.3 and 2.2].

\textbf{Theorem 7.} Suppose the Assumption 2 holds. Also suppose that
\begin{enumerate}[(a)]
  \item \( K \subset \mathbb{R}^n \) is a compact convex set whose interior contains the origin.
  \item \( U \subset \mathbb{R}^n \) is a bounded open set, and \( \partial U \) is \( C^{2,\alpha} \), for some \( \alpha > 0 \).
  \item \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is a \( C^{2,\alpha} \) function, such that \( \gamma^0(D\varphi) \leq 1 \). And if for some \( y \in \partial U \) we have \( \gamma^0(D\varphi(y)) = 1 \) then we must have
\end{enumerate}
\[
\text{(6.6)} \quad \langle v, \nu(y) \rangle \neq 0,
\]
for every nonzero \( v \in N(K^0, D\varphi(y)) \).
Let $u$ be the minimizer of $J$ over $W_{K^0,\varphi}(U)$. Then we have

$$u \in W^{2,\infty}(U) = C^{1,1}(\overline{U}).$$

Remark. Note that we are not assuming any regularity about $\partial K$ or $\partial K^0$. In particular, $K^0$, which defines the gradient constraint, need not be strictly convex.

Remark. Note that the assumptions of the theorem also hold when we replace $K, \varphi, K^0$ by $-K, -\varphi$ and $(-K)^0 = -K^0$. In particular note that if $D\varphi \in \partial K^0$, i.e. if $\gamma^0(D\varphi(y)) = 1$, then we have $-D\varphi \in -\partial K^0 = \partial(-K^0)$; and vice versa. In addition, it is easy to see that

$$v \in N(K^0, D\varphi(y)) \iff -v \in N(-K^0, -D\varphi(y)).$$

So as a result, $\rho, \bar{\rho}$ will have the same properties.

Remark. When $\gamma^0(D\varphi(y)) = 1$, and $v \in N(K^0, D\varphi(y))$ is nonzero, then we do not necessarily have $\langle v, \nu(y) \rangle > 0$. Because $D\varphi(y)$ can be different from $\mu(y)$, if we define $\mu$ in a continuous way.

Remark. As we will see in the following proof, in order to show that $u \in W^{2,p}(U) \cap W^{2,\infty}_{\text{loc}}(U)$ for every $p < \infty$, we only need $\partial U, \varphi$ to be $C^2$. But we need their $C^{2,\alpha}$ regularity to be able to apply the result of [28], and conclude the optimal regularity of $u$ up to the boundary.

Proof. As shown in [32], a compact convex set with nonempty interior can be approximated, in the Hausdorff metric, by a sequence of compact convex sets with nonempty interior which have $C^2$ boundaries with positive curvature. We can scale each element of such approximating sequence, to make the sequence a shrinking one. We apply this result to $K^0$. Thus there is a sequence $K^0_k$ of compact convex sets, that have $C^2$ boundaries with positive curvature, and

$$K^0_{k+1} \subset \text{int}(K^0_k), \quad K^0 = \bigcap K^0_k.$$ 

Notice that we can take the approximations of $K^0$ to be the polar of another convex set, because the double polar of a compact convex set with 0 in its interior is itself. Also note that $K^0_k$'s are strictly convex compact sets with 0 in their interior, which have $C^2$ boundaries with positive curvature. Furthermore we have $K = (K^0)^0 \supset K_{k+1} \supset K_k$. For the proof of these facts see [32, Sections 1.6, 1.7 and 2.5].

To simplify the notation we use $\gamma_k, \gamma^0_k, \rho_k, \bar{\rho}_k$ instead of $\gamma_{K_k}, \gamma^0_{K_k}, \rho_{K_k,\varphi}, \bar{\rho}_{K_k,\varphi}$, respectively. Note that $K_k, U, \varphi$ satisfy the Assumption 3. In particular we have $\gamma^0_k(D\varphi) < 1$, since $D\varphi \in K^0 \subset \text{int}(K^0_k)$. Let $u_k$ be the minimizer of $J$ over $W_{K^0_k,\varphi}(U)$. Then by Theorem 1 we have $u_k \in W^{2,\infty}_{\text{loc}}(U)$. We also have

$$-\bar{\rho}_k \leq -\bar{\rho}_k \leq u_k \leq \rho_k \leq \rho_1, \quad Du_k \in K^0_k \subset K^0 \quad \text{a.e.}$$

Note that $\rho_k \leq \rho_1$ and $\bar{\rho}_k \leq \bar{\rho}_1$, since $\gamma_k \leq \gamma_1$ due to $K_k \supset K_1$. Therefore $u_k$ is a bounded sequence in $W^{1,\infty}(U) = C^{0,1}(\overline{U})$. Hence by the Arzela-Ascoli Theorem a subsequence of $u_k$, which we still denote by $u_k$, uniformly converges to a continuous function $\tilde{u} \in C^0(\overline{U})$. Note that $\tilde{u}|_{\partial U} = \varphi$, because $u_k|_{\partial U} = \varphi$ for every $k$.

We break the rest of this proof into two parts. In Part I we derive the uniform bound [6,7]. And in Part II we obtain the regularity of $u$ by using this bound.

PART I:
Let $R_k$ be the $\rho_k$-ridge, and let $E_k, P_k^+$ be the elastic and plastic regions of $u_k$. Let us show that
\begin{equation}
\|D_i(D_i F(Du_k))\|_{L^\infty(U)} \leq C,
\end{equation}
for some $C$ independent of $k$. To see this, note that by (4.2) on $E_k$ we have
\[ D_i(D_i F(Du_k)) = g'(u_k). \]
But $u_k$ is uniformly bounded independently of $k$. So by (3.1) we get the desired bound on $E_k$. Next consider $P_k^+$.

Again by (4.2) we have
\[ D_i(D_i F(Du_k)) \geq g'(u_k) \quad \text{a.e. on } P_k^+. \]
Thus similarly we have a lower bound for $D_i(D_i F(Du_k))$ on $P_k$, which is independent of $k$.

On the other hand, since $P_k^+$ does not intersect $R_k$ due to Theorem (6) $\rho_k$ is at least $C^2$ on $P_k^+$. Let $\lambda_k, \mu_k$ be defined by (5.1), (5.2), respectively, when we use $\gamma_k^\circ$ instead of $\gamma^\circ$. Now as $u_k = \rho_k$ on $P_k^+$, Theorem 4.4 of [13] implies that for a.e. $x \in P_k^+$ we have $D_i(D_i F(Du_k(x))) = D_i(D_i F(D\rho_k(x)))$. Hence by Lemma (14) we have
\begin{align*}
D_i(D_i F(Du_k(x))) &= D_i(D_i F(D\rho_k(x))) = D_k^2 F(D\rho_k(x)) D_{ij}\rho_k(x) \\
&= \text{tr}[D^2 F(\mu_k) D^2 \rho_k(x)] \leq \text{tr}[D^2 F(\mu_k) D^2 \rho_k(y)],
\end{align*}
where $y$ is the $\rho_k$-closest point on $\partial U$ to $x$. But by (3.1) we know that $D^2 F$ is bounded. So we only need to show that $D^2 \rho_k$ is bounded on $\partial U$, independently of $k$. However by (5.10), for every $y \in \partial U$ we have
\[ D^2 \rho_k(y) = (I - X_k^T)(D^2 \varphi(y) + \lambda_k(y) D^2 d(y))(I - X_k), \]
where $X_k := \frac{1}{(D\gamma_k^\circ(\mu_k), \nu)} D\gamma_k^\circ(\mu_k) \otimes \nu$. It is obvious that $D^2 \varphi, D^2 d$ are bounded on $\partial U$, independently of $k$. In addition, note that $\gamma_k^\circ \geq \gamma_k^\circ$, since $K_k^\circ \subset K_i^\circ$. Thus by (5.1) we have
\[ \gamma_k^\circ(D\varphi + \lambda_k \nu) \leq \gamma_k^\circ(D\varphi + \lambda_k \nu) = 1. \]
Hence by (3.2) applied to $\gamma_1^\circ$, we have $|D\varphi + \lambda_k \nu| \leq C$, for some $C > 0$. Therefore we get $|\lambda_k| = |\lambda_k \nu| \leq C + |D\varphi|$. Thus $\lambda_k$ is bounded on $\partial U$, independently of $k$.

Hence we only need to show that the entries of $I - X_k$ are bounded on $\partial U$ independently of $k$.

Note that we have $\gamma_k(D\gamma_k^\circ(\mu_k)) = 1$ due to (2.3). Thus $\gamma(D\gamma_k^\circ(\mu_k)) \leq 1$ for every $k$, since $\gamma \leq \gamma_k$ due to $K \supset K_k$. So $D\gamma_k^\circ(\mu_k)$ is bounded independently of $k$. Therefore it only remains to show that $\langle D\gamma_k^\circ(\mu_k), \nu \rangle$ has a positive lower bound on $\partial U$ independently of $k$. Note that for every $k$, $\langle D\gamma_k^\circ(\mu_k), \nu \rangle$ is a continuous positive function on the compact set $\partial U$, due to (5.3). Hence there is $c_k > 0$ such that $\langle D\gamma_k^\circ(\mu_k), \nu \rangle \geq c_k$. Suppose to the contrary that $c_k$ has a subsequence $c_k \to 0$.

Let us denote $k_j$ by $j$ for simplicity. Then there is a sequence $y_j \in \partial U$ such that
\begin{equation}
\langle D\gamma_j^\circ(\mu_j(y_j)), \nu(y_j) \rangle \to 0.
\end{equation}
By passing to another subsequence, we can assume that $y_j \to y \in \partial U$, since $\partial U$ is compact. Now remember that
\[ \mu_j(y_j) = D\varphi(y_j) + \lambda_j(y_j) \nu(y_j), \]
and
where \( \lambda_j > 0 \). As we have shown in the last paragraph, \( \lambda_j \) is bounded on \( \partial U \) independently of \( j \). Hence by passing to another subsequence, we can assume that \( \lambda_j \to \lambda^* \geq 0 \). Therefore we have

\[
\mu_j(y_j) \to \mu^* := D\varphi(y) + \lambda^* \nu(y).
\]

On the other hand we have \( \gamma_j^\circ(\mu_j(y_j)) = 1 \). Hence \( \gamma^\circ(\mu_j(y_j)) \geq 1 \), since \( \gamma_j^\circ \leq \gamma^\circ \) due to \( K^\circ \subset K_j^\circ \). Thus we get \( \gamma^\circ(\mu^*) \geq 1 \). However we cannot have \( \gamma^\circ(\mu^*) > 1 \). Because then \( \mu^* \) will have a positive distance from \( K^\circ \), and therefore it will have a positive distance from \( K_j^\circ \) for large enough \( j \). But this contradicts the facts that \( \mu_j(y_j) \to \mu^* \) and \( \mu_j(y_j) \in K_j^\circ \). Thus we must have \( \gamma^\circ(\mu^*) = 1 \), i.e. \( \mu^* \in \partial K^\circ \).

Now note that \( v_j := D\gamma_j^\circ(\mu_j(y_j)) \in N(K_j^\circ, \mu_j(y_j)) \). In addition we have \( \gamma_j(v_j) = 1 \) due to (6.9). Hence we have \( v_j \in K_j \subset K \). Thus by passing to yet another subsequence, we can assume that \( v_j \to v \in K \). We also have \( \gamma_1(v_j) \geq 1 \), since \( \gamma_j \leq \gamma_1 \) due to \( K_j \subset K_1 \). So we get \( \gamma_1(v) \geq 1 \). In particular \( v \neq 0 \). We claim that \( v \in N(K^\circ, \mu^*) \). To see this, note that we have

\[
K^\circ \subset K_j^\circ \subset \{ \mu_j(y_j) + z : \langle z, v_j \rangle \leq 0 \}.
\]

Hence for every \( x \in K^\circ \) there is \( z_j \) such that \( x = \mu_j(y_j) + z_j \), and \( \langle z_j, v_j \rangle \leq 0 \). Thus as \( j \to \infty \) we get \( z_j \to z := x - \mu^* \). So we have \( \langle z, v \rangle \leq 0 \). Therefore

\[
K^\circ \subset \{ \mu^* + z : \langle z, v \rangle \leq 0 \},
\]

as desired. Now by (6.8) we obtain

\[
(6.9) \quad \langle v, \nu(y) \rangle = \lim \langle v_j, \nu(y_j) \rangle = 0.
\]

But if \( \gamma^\circ(D\varphi(y)) < 1 \) then we must have \( \lambda^* > 0 \). So \( D\varphi = \mu^* - \lambda^* \nu \) belongs to the ray passing through \( \mu^* \in \partial K^\circ \) in the direction \( -\nu \). However, we know that \( D\varphi \) is in the interior of \( K^\circ \), since \( \gamma^\circ(D\varphi) < 1 \). Thus the ray \( t \mapsto \mu^* - t \nu \) for \( t > 0 \), passes through the interior of \( K^\circ \). Therefore this ray and \( K^\circ \) must lie on the same side of the supporting hyperplane \( H_{\mu^*, v} \). In addition, the ray cannot lie on the supporting hyperplane, since it intersects the interior of \( K^\circ \). Hence we must have \( \langle v, \nu \rangle = -\langle v, -\nu \rangle > 0 \), which contradicts (6.9).

Thus we must have \( \gamma^\circ(D\varphi(y)) = 1 \), i.e. \( D\varphi \in \partial K^\circ \). If \( \lambda^* = 0 \) then \( \mu^* = D\varphi \). Hence \( v \in N(K^\circ, D\varphi) \). Then (6.9) is in contradiction with our assumption (6.6), since we showed that \( v \neq 0 \). So suppose \( \lambda^* > 0 \). Then the ray \( t \mapsto \mu^* - t \nu \) for \( t > 0 \), passes through the two points \( D\varphi, \mu^* \in \partial K^\circ \). Furthermore, (6.9) implies that the ray lies on the supporting hyperplane \( H_{\mu^*, v} \). Therefore \( D\varphi(y) \in H_{\mu^*, v} \). Hence \( H_{\mu^*, v} \) is also a supporting hyperplane of \( K^\circ \) at \( D\varphi(y) \). So \( v \in N(K^\circ, D\varphi) \), and again we arrive at a contradiction with (6.9).

Thus \( \langle D\gamma_k^\circ(\mu_k), \nu \rangle \) must have a positive lower bound on \( \partial U \) independently of \( k \), as desired. Therefore \( D^2 p_k \) is bounded on \( \partial U \) independently of \( k \); and consequently we have an upper bound for \( D_i(D_i F(Du_k)) \) on \( P_k^+ \), which is independent of \( k \). Similarly, we can show that \( D_i(D_i F(Du_k)) \) is bounded on \( P_k^- \), independently of \( k \). Hence we obtain the desired bound (6.7).

PART II:

Now let \( g_k := D_i(D_i F(Du_k)) \). Then \( u_k \) is a weak solution to the quasilinear elliptic equation

\[
D_i(D_i F(Du_k)) = g_k, \quad u_k|_{\partial U} = \varphi.
\]
Thus as shown in [21] we have \( u_k \in C^{1,\alpha_0}(U) \) for some \( \alpha_0 > 0 \). Therefore we have
\[
a_{i,j,k} D^2_{ij} u_k = g_k, \quad u_k|_{\partial U} = \varphi,
\]
where \( a_{i,j,k} := D^2_{ij} F(Du_k) \) is continuous on \( U \). Hence by Theorem 9.15 and Lemma 9.17 of [22] we have \( u_k \in W^{2,p}(U) \) for every \( p < \infty \), and there exists \( C_p > 0 \) independent of \( k \) such that
\[
\|u_k\|_{W^{2,p}(U)} \leq C_p(\|g_k\|_{L^p(U)} + \|\varphi\|_{C^2(U)}).
\]
Therefore \( u_k \) is a bounded sequence in \( W^{2,p}(U) \) due to (6.17). Consequently for every \( \tilde{\alpha} < 1 \), \( \|u_k\|_{C^{1,\tilde{\alpha}}(\overline{U})} \) is bounded independently of \( k \), because \( \partial U \) is \( C^2 \).

Hence there is a subsequence of \( u_k \), which we still denote by \( u_k \), that is weakly convergent in \( W^{2,p}(U) \), and strongly convergent in \( C^1(\overline{U}) \). Also remember that \( u_k \) uniformly converges to a continuous function \( \tilde{u} \in C^0(\overline{U}) \) that satisfies \( \tilde{u}|_{\partial U} = \varphi \). Thus all the above limits must be equal to \( \tilde{u} \). As a result, \( \tilde{u} \) belongs to \( W^{2,p}(U) \) for every \( p < \infty \). Furthermore we have \( D\tilde{u} \in K^\infty \); because \( Du_k \in K^0 \), and thus \( Du \in K^0 \) for every \( k \). So we have \( \tilde{u} \in W^{2,p}(U) \), since \( \tilde{u}|_{\partial U} = \varphi \). Now we will show that \( \tilde{u} \) is the minimizer of \( J \) over \( W^{2,p}(U) \). Let \( v \in W^{2,p}(U) \). Then \( v \in W^{2,p}(U) \) for every \( k \). Thus we get \( J[u_k] \leq J[v] \). But by the Dominated Convergence Theorem we have \( J[u_k] \to J[\tilde{u}] \). Hence \( \tilde{u} \) is the minimizer, and we must have \( \tilde{u} = u \). Therefore \( u \in W^{2,p}(U) \) for every \( p < \infty \).

Finally let us show that \( u \) belongs to \( W^{2,\infty}(U) \). First we show that \( D^2 u_k \) is bounded on \( P^+ \) independently of \( k \). To see this, consider \( P^+ \); the other case is similar. We have shown in the Part I of the proof that \( D^2 \rho_k \) is bounded on \( \partial U \), independently of \( k \). Hence by Lemma [13] when \( y \) is the \( \rho_k \)-closest point on \( \partial U \) to \( x \in P^+ \), and \( \xi \in \mathbb{R}^n \), we have
\[
D^2_{\xi\xi} \rho_k(x) \leq D^2_{\xi\xi} \rho_k(y) \leq \tilde{C},
\]
for some \( \tilde{C} \) independent of \( k \). In addition, for a.e. \( x \in P^+ \) we have
\[
D_i(D_i F(Du_k(x))) = D_i(D_i F(D\rho_k(x))) = D^2_{ij} F(D\rho_k(x)) D^2_{ij} \rho_k(x) = \text{tr}[D^2 F(\mu_k)] D^2 \rho_k(x) = \sum_{i \leq n} D^2_{\xi_i \xi_i} F(\mu_k) D^2_{\xi_i \xi_i} \rho_k(x),
\]
where \( \xi_1, \ldots, \xi_n \) is an orthonormal basis of eigenvectors of \( D^2 \rho_k(x) \). Hence by (4.2) we get
\[
\sum_{i \leq n} D^2_{\xi_i \xi_i} F(\mu_k) D^2_{\xi_i \xi_i} \rho_k(x) = D_i(D_i F(Du_k(x))) \geq g'(u_k(x)) \geq -C,
\]
for some \( C \) independent of \( k \). Note that here we are using (3.4), and the fact that \( u_k \) is bounded independently of \( k \). Therefore by (3.4), (6.15) we obtain
\[
D^2_{\xi_j \xi_j} F(\mu_k) D^2_{\xi_j \xi_j} \rho_k(x) \geq -C - \sum_{i \neq j} D^2_{\xi_i \xi_i} F(\mu_k) D^2_{\xi_i \xi_i} \rho_k(x) \geq -C + C(1) \tilde{C}.
\]
Hence again by (3.4) we obtain
\[
D^2_{\xi_j \xi_j} \rho_k(x) \geq -\frac{\bar{C}}{c_8} - (n - 1) c_9 \tilde{C}.
\]
Thus by (6.11) and the above inequality, $|D^2_{k_i k_j} \rho_k(x)|$ is bounded independently of $k$, for a.e. $x \in P_1^+$. Now note that the numbers $D^2_{k_i k_j} \rho_k(x)$ are the eigenvalues of $D^2 \rho_k(x)$. So $D^2 \rho_k$ is bounded on $P_1^+$ independently of $k$. But on $P_k^+$ we have $D^2 u_k = D^2 \rho_k$ a.e.. Therefore $D^2 u_k$ is bounded on $P_k^+$ independently of $k$. The case of $P_k^-$ can be treated similarly.

Now let $x_0 \in U$, and suppose that $B_r(x_0) \subset U$. Set $v_k(y) := u_k(x_0 + ry)$ for $y \in B_1(0)$. Then by (4.2), and the arguments of the above paragraph, we have

$$
\begin{cases}
D_{ij}^2 F(\frac{1}{r} Dv_k) D_{ij}^2 v_k = r^2 g'(v_k) & \text{a.e. in } B_1(0) \cap \Omega_k, \\
|D^2 v_k| \leq C & \text{a.e. in } B_1(0) - \Omega_k,
\end{cases}
$$

for some $C$ independent of $k$. Here $\Omega_k := \{ y \in B_1(0) : u_k(x_0 + ry) \in E_k \}$. (Note that on $B_1(0) - \Omega_k$ we have $D^2 v_k = r^2 D^2 u_k$, and $u_k \in P_k$; so $D^2 v_k$ is bounded there.) Next recall that $\|u_k\|\_2(B_r(x_0)), \|g'(u_k)\|_L^\infty(B_r(x_0))$ are bounded independently of $k$, due to (6.11), (6.7), and the fact that $u_k$ is bounded independently of $k$. Therefore $\|v_k\|\_2(B_1(0))$ and $\|g'(v_k)\|_L^\infty(B_1(0))$ are bounded independently of $k$ too. Also note that the Holder norms of $D_{ij}^2 F(\frac{1}{r} Dv_k), r^2 g'(v_k)$ are bounded independently of $k$, since for every $\alpha < 1, \|u_k\|_{C^{1, \alpha}(\bar{\Omega})}$ is bounded independently of $k$. Thus we can apply the result of (27) to deduce that

$$
|D^2 v_k| \leq \tilde{C} \quad \text{a.e. in } B_{1/2}^+(0),
$$

for some $\tilde{C}$ independent of $k$. Therefore

$$
|D^2 u_k| \leq \tilde{C} \quad \text{a.e. in } B_{1/2}^+(x_0),
$$

for some $\tilde{C}$ independent of $k$. Hence $u_k$ is a bounded sequence in $W^{2, \infty}(B_{1/2}^+(x_0))$. Therefore a subsequence of them converges weakly star in $W^{2, \infty}(B_{1/2}^+(x_0))$. But the limit must be $u$; so we get $u \in W^{2, \infty}(B_{1/2}^+(x_0))$, as desired.

Next let $x_0 \in \partial U$. Let $\Phi$ be a $C^{2, \alpha}$ change of coordinates on a neighborhood of $x_0$, that flattens $\partial U$ around $x_0$. More specifically, we assume that $\Phi: x \mapsto y$ maps a neighborhood of $x_0$ onto a neighborhood of $0$ that contains $\bar{B}_1(0)$, and the $\Phi$-image of $U, \partial U$ lie respectively in the half-space $\{ y_n > 0 \}$ and on the plane $\{ y_n = 0 \}$. Let $\Psi$ be the inverse of $\Phi$. Then we have $y = \Phi(x)$ and $x = \Psi(y)$. Let $B_1^+ := B_1(0) \cap \{ y_n > 0 \}$ and $B_1^- := B_1(0) \cap \{ y_n = 0 \}$. Now set

$$
\hat{u}_k(y) := u_k(\Psi(y)) - \varphi(\Psi(y)) = u_k(x) - \varphi(x).
$$

It is obvious that $\hat{u}_k = 0$ on $B_1^-$. We also have $\hat{u}_k \in W^{2, \infty}(B_1^+ \cap C_1(\bar{B}_1^+))$ (see [22, Section 7.3]). In addition we have

$$
\begin{align*}
D_1 \hat{u}_k(y) &= D_1 u_k(x) D_1 \Psi^i(y) - D_1 \varphi(x) D_1 \Psi^i(y), \\
D_{ij}^2 \hat{u}_k(y) &= D_{ij}^2 u_k(x) D_1 \Psi^i(y) D_1 \Psi^j(y) + D_1 u_k(x) D_{ij}^2 \Psi^i(y) - D_1 \varphi(x) D_{ij}^2 \Psi^i(y).
\end{align*}
$$

Therefore we get

$$
\|\hat{u}_k\|_{W^{2, \infty}(B_1^+)} \leq C (\|u_k\|_{W^{2, \infty}(U)} + \|\varphi\|_{C^2(\bar{U})}),
$$

for some $C$ independent of $k$. Hence $\|\hat{u}_k\|_{W^{2, \infty}(B_1^+)}$ is bounded independently of $k$, due to (6.11).
Now note that by \((4.2)\) we have \(D_{ij}^2 F(Du_k)D_{ij}^2 u_k = D_i(D_i F(Du_k)) = g'(u_k)\) in \(E_k\). Let \(a_{ij,k} := D_{ij}^2 F(Du_k)\), and
\[
\hat{a}_{ij,k}(y) := a_{lm,k}(x) D_l \Phi^i(x) D_m \Phi^j(x),
\]
where \(x = \Psi(y)\). Then we have (Note that we sum over all repeated indices except for \(k\).)
\[
\hat{a}_{ij,k} D_{ij}^2 \hat{u}_k(y) = a_{lm,k} D_i \Phi^i D_m \Phi^j (D_{ij}^2 u_k D_i \Psi^j D_j \Psi^j + D_i u_k D_{ij}^2 \Psi^i) - D_{ij} \hat{\varphi} D_i \Phi^i D_j \Psi^j - D_i \hat{\varphi} D_{ij}^2 \Psi^j)
\]
\[
= a_{ij,k} D_{ij}^2 u_k - a_{ij,k} D_{ij}^2 \hat{\varphi} + a_{lm,k} D_i \Phi^i D_m \Phi^j (D_i u_k D_{ij}^2 \Psi^i - D_i \hat{\varphi} D_{ij}^2 \Psi^i)
\]
\[
= g'(u_k) - a_{ij,k} D_{ij}^2 \hat{\varphi} + a_{lm,k} D_i \Phi^i D_m \Phi^j (D_i u_k D_{ij}^2 \Psi^i - D_i \hat{\varphi} D_{ij}^2 \Psi^i) =: \hat{f}_k(y),
\]
for every \(y \in \Omega_k := \{y \in B_1^+ : \Psi(y) \in E_k\}\). Note that in the 2nd equality we used the fact that \(D \Psi D \Phi = I\). Also note that \(\hat{f}_k \in C^{\alpha_0}(\mathbb{B}_1^+)\) for some \(\alpha_0 > 0\), and \(\|f_k\|_{C^{\alpha_0}(\mathbb{B}_1^+)}\) is bounded independently of \(k\); since \(\|u_k\|_{C^{\alpha_0}(\mathbb{B}_1^+)}\) is bounded independently of \(k\), for every \(\alpha < 1\). Similarly, \(\|\hat{a}_{ij,k}\|_{C^{\alpha_0}(\mathbb{B}_1^+)}\) is bounded independently of \(k\).

On the other hand, \(D^2 \hat{u}_k\) is bounded on \(B_1^+ - \Omega_k := \{y \in B_1^+ : \Psi(y) \in P_k\}\) independently of \(k\) due to \((6.12)\); because \(D^2 u_k\) is bounded on \(P_k\) independently of \(k\), and \(Du_k\) is bounded independently of \(k\). Therefore we have
\[
\begin{cases}
\hat{a}_{ij,k} D_{ij}^2 \hat{u}_k = \hat{f}_k & \text{a.e. in } B_1^+ \cap \Omega_k, \\
|D^2 \hat{u}_k| \leq C & \text{a.e. in } B_1^+ - \Omega_k, \\
|u_k| = 0 & \text{on } B_1^+ \\
\end{cases}
\]
for some \(C\) independent of \(k\). Hence by the result of \((28)\) we get
\[
|D^2 \hat{u}_k| \leq \tilde{C} \quad \text{a.e. in } B_1^+(0) \cap \{y_n > 0\},
\]
for some \(\tilde{C}\) independent of \(k\). Thus
\[
|D^2 u_k| \leq \tilde{C} \quad \text{a.e. in } B_r(x_0) \cap U,
\]
for some \(r > 0\) and some \(\tilde{C}\) independent of \(k\). Because we have
\[
u_k(x) = \hat{u}_k(y) + \varphi(x) = \hat{u}_k(\Psi(x)) + \varphi(x);
\]
so we can compute \(D^2 u_k\) in terms of \(D^2 \hat{u}_k\), similarly to \((6.12)\). (Also note that \(D \hat{u}_k\) is bounded independently of \(k\) due to \((6.12)\).) Hence \(u_k\) is a bounded sequence in \(W^{2,\infty}(B_r(x_0) \cap U)\). Therefore a subsequence of them converges weakly star in \(W^{2,\infty}(B_r(x_0) \cap U)\). But the limit must be \(u;\) so we get \(u \in W^{2,\infty}(B_r(x_0) \cap U)\). Finally note that we can cover \(\partial U\) with finitely many open balls of the form \(B_r(x_0)\) for \(x_0 \in \partial U\), over which \(u\) is \(W^{2,\infty}\). Also, there is an open subset of \(U\) whose union with these balls cover \(U\), and over it \(u\) is \(W^{2,\infty}\) too. Thus we can conclude that \(u \in W^{2,\infty}(U)\), as desired.

Our final theorem is a generalization of our local regularity result, i.e. Theorem \((11)\). We will drop the Assumption \((3)\) which is some sort of regularity assumption about \(K\). But we need to impose extra regularity on \(U\) and \(\varphi\). In particular, we assume that \(U\) is also convex.
Theorem 8. Suppose the Assumption 2 holds. Also suppose that
(a) $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin.
(b) $U \subset \mathbb{R}^n$ is a bounded convex open set.
(c) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$ function, such that $\gamma_\varphi(D\varphi) \leq 1$. And either $\varphi$ is linear, or the following
condition holds: If for some $y \in \partial U$ we have $\gamma_\varphi(D\varphi(y)) = 1$ then we must have
\begin{equation}
\langle v, w \rangle \neq 0,
\end{equation}
for every nonzero $v \in N(K^\circ, D\varphi(y))$ and $w \in N(U, y)$.
Let $u$ be the minimizer of $J$ over $W_{K^\circ, \varphi}(U)$. Then we have
\begin{equation}
u \in W^{2, \infty}(U) = C^{1,1}_{\text{loc}}(U).
\end{equation}

Remark. Note that as we mentioned after the last theorem, the assumptions of this theorem also
hold when we replace $K, \varphi, K^\circ$ by $-K, -\varphi$ and $(-K)^\circ = -K^\circ$. So as a result, $\rho, \bar{\rho}$ will have the
same properties.

Proof. The idea of the proof is to approximate both $K^\circ$ and $U$ with smooth convex sets. As we
explained in the proof of last theorem, there is a sequence $K_k^\circ$ of compact convex sets, that have $C^2$
boundaries with positive curvature, and
\begin{equation}
K_k^\circ \subset \text{int}(K_{k+1}^\circ), \quad K^\circ = \bigcap K_k^\circ.
\end{equation}
Then it follows that $K_k$’s are strictly convex compact sets with 0 in their interior, which have $C^2$
boundaries with positive curvature. Furthermore we have $K = (K^\circ)^\circ \supset K_{k+1} \supset K_k$. Similarly,
there is a sequence $U_k$ of bounded convex open sets with $C^2$ boundaries such that
\begin{equation}
U_{k+1} \subset U_k, \quad U = \bigcap U_k.
\end{equation}

To simplify the notation we use $\gamma_k, \gamma_k^\circ, \rho_k, \bar{\rho}_k$ instead of $\gamma_{K_k}, \gamma_{K_k^\circ}, \rho_{K_k, \varphi}(:; U_k), \bar{\rho}_{K_k, \varphi}(:; U_k)$, respectively. Note that $K_k, U_k, \varphi$ satisfy the Assumption 5. In particular we have $\gamma_k^\circ(D\varphi) < 1$, since
$D\varphi \in K^\circ \subset \text{int}(K_k^\circ)$. Let $u_k$ be the minimizer of $J[\cdot; U_k]$ over $W_{K_k^\circ, \varphi}(U_k)$. Then by Theorem 1
we have $u_k \in W^{2, \infty}(U_k)$. We also have
\begin{equation}
-\bar{\rho}_k \leq u_k \leq \rho_k, \quad Du_k \in K_k^\circ \subset K_k^1 \quad \text{a.e.}
\end{equation}
Now note that for every $x$ we have $\rho_k(x) = \gamma_k(x - y_k) + \varphi(y_k)$ for some $y_k \in \partial U_k$. Also note that
$\gamma_k \leq \gamma_1$, since $K_k \supset K_1$. Hence by (1.6), (3.3), for every $y \in \partial U$ we have
\begin{equation}
\rho_k(x) \leq \gamma_1(x - y_k) + \varphi(y_k)
\leq \gamma_1(x - y) + \gamma_1(y - y_k) + \varphi(y) + \varphi(y_k) - \varphi(y)
\leq \gamma_1(x - y) + \varphi(y) + \gamma_1(y - y_k) + \gamma_1(y_k - y)
\leq \gamma_1(x - y) + \varphi(y) + 2C|y - y_k|,
\end{equation}
for some $C > 0$. Therefore we get $\rho_k(x) \leq \rho_{K_1, \varphi}(x; U) + 2C \text{dist}(\partial U, \partial U_k)$. We have a similar bound
for $\bar{\rho}_k$. So we obtain
\begin{equation}
-\bar{\rho}_{K_1, \varphi}(\cdot; U) - 2C \text{dist}(\partial U, \partial U_k) \leq u_k \leq \rho_{K_1, \varphi}(\cdot; U) + 2C \text{dist}(\partial U, \partial U_k).
\end{equation}
Thus $u_k, Du_k$ are bounded independently of $k$. Therefore $u_k$ is a bounded sequence in $W^{1, \infty}(U) = C^{0,1}(U)$. Note that here we are using the fact that $\partial U$ is Lipschitz, since locally it is the graph of a
convex function. Hence by the Arzela-Ascoli Theorem a subsequence of \( u_k \), which we still denote by \( u_k \), uniformly converges to a continuous function \( \tilde{u} \in C^0(\overline{U}) \). In addition we have \( \tilde{u}\vert_{\partial U} = \varphi \), because in the limit, the bounds \( \ref{6.14} \) become \( -\hat{\rho}_{K,\varphi}(\cdot; U) \leq \tilde{u} \leq \rho_{K,\varphi}(\cdot; U) \). Thus we get the desired, since

\[
\begin{align*}
-\hat{\rho}_{K,\varphi}(\cdot; U)|_{\partial U} = \rho_{K,\varphi}(\cdot; U)|_{\partial U} = \varphi,
\end{align*}
\]

as shown in the proof of Proposition \( \ref{1} \).

Now we argue as we did in the proof of Theorem \( \ref{7} \). Let \( R_k \) be the \( \rho_k \)-ridge, and let \( E_k, P_k^{\pm} \) be the elastic and plastic regions of \( u_k \). We want to show that

\[
\|D_i(D_iF(\partial U_k))\|_{L^\infty(U_k)} \leq C,
\]

for some \( C \) independent of \( k \). As we have shown before by using \( \ref{12} \), it is easy to see that \( D_i(D_iF(\partial U_k)) \) is bounded on \( E_k \), and it is bounded above on \( P_k^- \), and bounded below on \( P_k^+ \). The hard part is to obtain upper bound for \( D_i(D_iF(\partial U_k)) \) on \( P_k^+ \), and lower bound for it on \( P_k^- \), which are independent of \( k \). We will obtain the upper bound on \( P_k^+ \); the other case is similar.

Since \( P_k^+ \) does not intersect \( R_k \) due to Theorem \( \ref{1} \), \( \rho_k \) is at least \( C^2 \) on \( P_k^+ \). Let \( \nu_k \) be the inward unit normal to \( \partial U_k \). Also let \( \lambda_k, \mu_k \) be defined by \( \ref{3.4}, \ref{3.5} \) respectively, when we use \( \gamma_k^\circ \) instead of \( \gamma^\circ \), and \( \partial U_k, \nu_k \) instead of \( \partial U, \nu \). Hence as before, by Lemma \( \ref{13} \) for a.e. \( x \in P_k^+ \) we have

\[
D_i(D_iF(\partial U_k(x))) = D_i(D_iF(\partial u_k(x))) = D_i^2F(\partial u_k(x))D_i^2\rho_k(x)
\]

\[
= \text{tr}[D^2F(\mu_k)D^2\rho_k(x)] \leq \text{tr}[D^2F(\mu_k)D^2\rho_k(y)],
\]

where \( y \) is the \( \rho_k \)-closest point on \( \partial U_k \) to \( x \). But by \( \ref{5.10} \), for every \( y \in \partial U_k \) we have

\[
D^2\rho_k(y) = (I - X_k^T)(D^2\varphi(y) + \lambda_k(y)D^2d_k(y))(I - X_k),
\]

where \( X_k := \frac{1}{\partial \gamma_k^\circ(\mu_k) \cdot \nu_k} \cdot \rho_k \). Thus \( D^2d_k \) is the Euclidean distance to \( \partial U_k \). However, we know that the eigenvalues of \( D^2d_k(y) \) are minus the principal curvatures of \( \partial U_k \) at \( y \), and 0; as shown in \( \ref{22} \), Section 14.6. So \( D^2d_k(y) \) is negative semidefinite, since \( U_k \) is convex. Thus \( \lambda_k(I - X_k^T)D^2d_k(I - X_k) \) is also negative semidefinite, since \( \lambda_k > 0 \). On the other hand by \( \ref{3.4} \), we know that \( D^2F \) is positive definite. Let \( \sqrt{D^2F} \) be the unique symmetric positive definite matrix whose square is \( D^2F \). Then we get

\[
D_i(D_iF(\partial U_k(x))) - \text{tr}[D^2F(I - X_k^T)D^2\varphi(I - X_k)] \leq \text{tr}[D^2F\lambda_k(I - X_k^T)D^2d_k(I - X_k)]
\]

\[
= \lambda_k\text{tr}[\sqrt{D^2F}(I - X_k^T)D^2d_k(I - X_k)\sqrt{D^2F}] \leq 0,
\]

because \( \sqrt{D^2F}(I - X_k^T)D^2d_k(I - X_k)\sqrt{D^2F} \) is negative semidefinite too.

Hence we only need to find a bound for \( \text{tr}[D^2F(I - X_k^T)D^2\varphi(I - X_k)] \). When \( \varphi \) is linear we have \( D^2\varphi = 0 \); so we have the desired bound. Thus let us assume that \( \varphi \) is not necessarily linear. By \( \ref{3.4} \) we know that \( D^2F \) is bounded. It is also obvious that \( D^2\varphi \) is bounded on \( \partial U_k \subset \overline{U} \), independently of \( k \). Hence we only need to show that the entries of \( I - X_k \) are bounded on \( \partial U_k \) independently of \( k \). Note that we have \( \gamma_k(\partial \gamma_k^\circ(\mu_k)) = 1 \) due to \( \ref{2.4} \). Thus \( \gamma_k(D^2\gamma_k^\circ(\mu_k)) \leq 1 \) for every \( k \), since \( \gamma \leq \gamma_k \) due to \( K \supset K_k \). So \( D^2\gamma_k^\circ(\mu_k) \) is bounded independently of \( k \). Also, \( \nu_k \) is bounded independently of \( k \), since \( |\nu_k| = 1 \). Therefore it only remains to show that \( \langle D^2\gamma_k^\circ(\mu_k), \nu_k \rangle \) has a positive lower bound on \( \partial U_k \) independently of \( k \).
Note that for every k, \( \langle D\gamma_\omega^0(\mu_k), \nu_k \rangle \) is a continuous positive function on the compact set \( \partial U_k \), due to (5.3). Hence there is \( c_k > 0 \) such that \( \langle D\gamma_\omega^0(\mu_k), \nu_k \rangle \geq c_k \). Suppose to the contrary that \( c_k \) has a subsequence \( c_{k_j} \to 0 \). Let us denote \( k_j \) by \( j \) for simplicity. Then there is a sequence \( y_j \in \partial U_j \subset \overline{U}_1 \) such that

\[
\langle D\gamma_\omega^0(\mu_j(y_j)), \nu_j(y_j) \rangle \to 0.
\]

By passing to another subsequence, we can assume that \( y_j \to y \), since \( \overline{U}_1 \) is compact. We claim that \( y \in \partial U \). First note that \( y \notin U \), since otherwise \( y_j \) must be in \( U \) for large enough \( j \), which contradicts the fact that \( y_j \in \partial U_j \). On the other hand, we have \( y \in \overline{U}_k \) for every \( k \); because \( y_j \in \overline{U}_k \) for large enough \( j \). So we must have \( y \in \bigcap \overline{U}_k = \overline{U} \).

Now remember that

\[
\mu_j(y_j) = D\varphi(y_j) + \lambda_j(y_j)\nu_j(y_j),
\]

where \( \lambda_j > 0 \). In addition, note that \( \gamma_j^0 \geq \gamma_1^0 \), since \( K_j^0 \subset K_1^0 \). Thus by (5.1) we have

\[
\gamma_j^0(D\varphi + \lambda_j\nu_j) \leq \gamma_1^0(D\varphi + \lambda_1\nu_j) = 1.
\]

Hence by (3.2) applied to \( \gamma_1^0 \), we have \( |D\varphi + \lambda_j\nu_j| \leq C \), for some \( C > 0 \). Therefore we get \( |\lambda_j| = |\lambda_j\nu_j| \leq C + |D\varphi| \). Thus \( \lambda_j \) is bounded on \( \partial U_j \) independently of \( j \). Hence by passing to another subsequence, we can assume that \( \lambda_j \to \lambda^* \geq 0 \). Also, note that \( -\nu_j(y_j) \in N(U_j, y_j) \). In addition we have \( |\nu_j(y_j)| = 1 \). Thus by passing to yet another subsequence, we can assume that \( -\nu_j(y_j) \to w \), with \( |w| = 1 \). Therefore similarly to the Part I of the previous proof, we can show that \( w \in N(U, y) \). Thus we have

\[
\mu_j(y_j) \to \mu^* := D\varphi(y) - \lambda^*w.
\]

It also follows similarly that \( \gamma^0(\mu^*) = 1 \), i.e. \( \mu^* \in \partial K^0 \). In addition, we can similarly conclude that the sequence \( v_j := D\gamma_j^0(\mu_j(y_j)) \in N(K_j^0, \mu_j(y_j)) \) converges to a nonzero vector \( v \in N(K^0, \mu^*) \), after we pass to one further subsequence.

Now by (6.16) we obtain

\[
(v, w) = \lim (v_j, -\nu_j(y_j)) = 0.
\]

But if \( \gamma^0(D\varphi(y)) < 1 \) then we must have \( \lambda^* > 0 \). So \( D\varphi = \mu^* + \lambda^*w \) belongs to the ray passing through \( \mu^* \in \partial K^0 \) in the direction \( w \). However, we know that \( D\varphi \) is in the interior of \( K^0 \), since \( \gamma^0(D\varphi) < 1 \). Thus the ray \( t \mapsto \mu^* + tw \) for \( t > 0 \), passes through the interior of \( K^0 \). Therefore this ray and \( K^0 \) must lie on the same side of the supporting hyperplane \( H_{\mu^*, v} \). In addition, the ray cannot lie on the hyperplane, since it intersects the interior of \( K^0 \). Hence we must have \( (v, w) < 0 \), which contradicts (6.17). Thus we must have \( \gamma^0(D\varphi(y)) = 1 \), i.e. \( D\varphi \in \partial K^0 \). If \( \lambda^* = 0 \) then \( \mu^* = D\varphi \). Hence \( v \in N(K^0, D\varphi) \). Then (6.17) is in contradiction with our assumption (6.13), since \( v \neq 0 \). So suppose \( \lambda^* > 0 \). Then the ray \( t \mapsto \mu^* + tw \) for \( t > 0 \), passes through the two points \( D\varphi, \mu^* \in \partial K^0 \). Furthermore, (6.17) implies that the ray lies on the supporting hyperplane \( H_{\mu^*, v} \). Therefore \( D\varphi(y) \in H_{\mu^*, v} \). Hence \( H_{\mu^*, v} \) is also a supporting hyperplane of \( K^0 \) at \( D\varphi(y) \). So \( v \in N(K^0, D\varphi) \), and again we arrive at a contradiction with (6.13).

Thus \( (D\gamma_k^0(\mu_k), \nu_k) \) must have a positive lower bound on \( \partial U_k \) independently of \( k \), as desired. Therefore \( D^2\rho_k \) is bounded on \( \partial U_k \) independently of \( k \); and consequently we have an upper bound.
for $D_i(D_iF(Du_k))$ on $P_k^+$, which is independent of $k$. Similarly, we can show that $D_i(D_iF(Du_k))$ is bounded on $P_k^-$, independently of $k$. Hence we obtain the desired bound $6.15$.

Now let $V_l \subset \overline{V}_l \subset U$ be an expanding sequence of open sets with $C^2$ boundaries, such that $U = \bigcup V_l$. Consider the sequence $u_k|_{V_{l+2}}$. Similarly to the proof of Theorem 11 we can show that for every $p < \infty$ there is $C_{p,l}>0$, which is independent of $k$, such that

$$
\|u_k\|_{W^{2,p}(V_l)} \leq C_{p,l}.
$$

Consequently, as $\partial V_l$ is $C^2$, for every $\tilde{\alpha} < 1$, $\|u_k\|_{C^{1,\alpha}(\overline{V}_l)}$ is bounded independently of $k$. Therefore, we can inductively construct subsequences $u_{ki}$ of $u_k$, such that $u_{ki}$ is a subsequence of $u_{kl-1}$; and $u_{ki}$ is weakly convergent in $W^{2,p}(V_l)$, and strongly convergent in $C^1(\overline{V}_l)$. Also remember that $u_k$ uniformly converges to a continuous function $\tilde{u} \in C^0(U)$ that satisfies $\tilde{u}|_{\partial U} = \varphi$. Thus all the limits of the subsequences $u_{ki}$ must be equal to $\tilde{u}$. As a result, $\tilde{u}$ belongs to $W^{2,p}_loc(U)$ for every $p < \infty$. Furthermore we have $D\tilde{u} \in K^2$; because $Du_k \in K^2$, and thus $Du \in K^2$ for every $k$. So we have $\tilde{u} \in W^{-,\varphi}(U)$, since $\tilde{u}|_{\partial U} = \varphi$.

Now we will show that $\tilde{u}$ is the minimizer of $J[\cdot; U]$ over $W^{-,\varphi}(U)$. Let $v \in W^{-,\varphi}(U)$. Then for every $k$ we have

$$
v_k := \begin{cases} v & \text{in } U \\ \varphi & \text{in } U_k - U \end{cases} \in W^{2,\varphi}(U_k).
$$

(Note that $v_k$ is Lipschitz, so it belongs to $H^1(U_k)$.) Thus we get

$$
J[u_k;U_k] \leq J[v_k;U_k] = \int_{U_k} F(Dv_k) + g(v_k) \, dx
$$

$$
= \int_U F(Dv) + g(v) \, dx + \int_{U_k - U} F(D\varphi) + g(\varphi) \, dx \leq J[v;U] + C\mathcal{L}^n(U_k - U),
$$

where $C > 0$ is an upper bound for $F(D\varphi) + g(\varphi)$ on $\overline{U}_1$, and $\mathcal{L}^n(U_k - U)$ is the Lebesgue measure of $U_k - U$. But since $u_k, Du_k$ are bounded independently of $k$, by the Dominated Convergence Theorem we have $J[u_k;U_k] \to J[\tilde{u};U]$, where the limit is taken through the diagonal subsequence $u_{kl}$, constructed in the previous paragraph. Also, $\mathcal{L}^n(U_k - U) \to 0$ as $k \to \infty$. Hence $\tilde{u}$ is the minimizer of $J[\cdot; U]$ over $W^{-,\varphi}(U)$, and therefore we must have $\tilde{u} = u$. Thus $u \in W^{2,p}_loc(U)$ for every $p < \infty$. Then similarly to the proof of Theorem 7 we can conclude that $u \in W^{2,\infty}_loc(U)$, as desired.

Remark. The relations $(6.12)$ hold under the more general assumptions of Theorems 7 and 8. Also, if in addition Assumption $(\text{H})$ holds, then Lemma 10 holds too. And if we furthermore require $K$ to be strictly convex, then Lemma 11 and Proposition 2 hold too. The reason is that we only used the regularity of $u$ in these results’ proofs; and we did not use Assumption $(\text{K})$ directly in their proofs. On the other hand, in the proof of Theorem 7 we used the formula $(5.14)$ for $D^2p$. So it is not obvious whether this theorem holds under the more general assumptions of Theorems 7 and 8.

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