Computing tensor eigenvalues via homotopy methods

Liping Chen*  Lixing Han†  Liangmin Zhou‡

March 17, 2015

Abstract

We introduce the concept of mode-k generalized eigenvalues and eigenvectors of a tensor and prove some properties of such eigenpairs. In particular, we derive an upper bound for the number of equivalence classes of generalized tensor eigenpairs using mixed volume. Based on this bound and the structures of tensor eigenvalue problems, we propose two homotopy continuation type algorithms to solve tensor eigenproblems. With proper implementation, these methods can find all equivalence classes of isolated generalized eigenpairs and some generalized eigenpairs contained in the positive dimensional components (if there are any). We also introduce an algorithm that combines a heuristic approach and a Newton homotopy method to extract real generalized eigenpairs from the found complex generalized eigenpairs. A MATLAB software package TenEig has been developed to implement these methods. Numerical results are presented to illustrate the effectiveness and efficiency of TenEig for computing complex or real generalized eigenpairs.

Key words. tensors, mode-k eigenvalues, polynomial systems, homotopy continuation, TenEig.

AMS subject classification (2010). 15A18, 15A69, 65H10, 65H17, 65H20.

1 Introduction

Eigenvalues of tensors were first introduced by Qi [34] and Lim [27] in 2005. Since then, tensor eigenvalues have found applications in automatic control, statistical data analysis, diffusion tensor imaging, image authenticity verification, spectral hypergraph theory, and quantum entanglement, etc., see for example, [7, 10, 20, 34, 35, 37, 38, 39] and the references therein. The tensor eigenvalue problem has become an important subject of numerical multilinear algebra.
Various definitions of eigenvalues for tensors have been proposed in the literature, including E-eigenvalues and eigenvalues in the complex field, and Z-eigenvalues, H-eigenvalues, and D-eigenvalues in the real field \([27, 34, 37]\). In \([6]\), Chang, Pearson, and Zhang introduced a notion of generalized eigenvalues for tensors that unifies several types of eigenvalues. Recently this definition has been further generalized by Cui, Dai, and Nie \([11]\).

Unlike the matrix eigenvalue problem, computing eigenvalues of the third or higher order tensors is a difficult problem \([18]\). Nonetheless, several algorithms which aim at computing one or some eigenvalues of a tensor have been developed recently. These algorithms are designed for tensors of certain type, such as entry-wise nonnegative or symmetric tensors.

For nonnegative tensors, Ng, Qi, and Zhou \([31]\) proposed a power-type method for computing the largest H-eigenvalue of a nonnegative tensor. Modified versions of the Ni-Qi-Zhou method have been proposed in \([29, 47, 48]\).

For real symmetric tensors, Hu, Huang, and Qi \([19]\) proposed a sequential semidefinite programming method for computing extreme Z-eigenvalues. Hao, Cui, and Dai \([15]\) proposed a sequential subspace projection method for a similar purpose. Kolda and Mayo \([22]\) proposed a shifted power method (SSHOPM) for computing a Z-eigenvalue. They have improved SSHOPM in \([23]\) by updating the shift parameter adaptively. The resulting method can be used to compute a real generalized eigenvalue. Han \([14]\) proposed an unconstrained optimization method for computing a real generalized eigenvalue for even order real symmetric tensors. The methods in \([14, 22, 23]\) can find more eigenvalues of a symmetric tensor if they are run multiple times using different starting points. Recently, Cui, Dai, and Nie \([11]\) proposed a method for computing all real generalized eigenvalues.

In this paper, we are concerned with computing all eigenvalues of a general real or complex tensor. As can be seen from the next section, finding eigenpairs of a tensor amounts to solving a system of polynomials. Naturally one would consider to use algebraic geometry methods such as the Gröbner basis method and the resultant method \([29]\) for this purpose. These methods can obtain symbolic solutions of a polynomial system, which are accurate. However, they are expensive in terms of computational cost and space. Moreover, they are difficult to parallelize. A class of numerical methods, the homotopy continuation methods, can overcome these shortcomings of the Gröbner basis and the resultant methods. During the past few decades, significant advances have been made on homotopy continuation methods for polynomial systems, see for example, \([3, 24, 25, 30, 40]\). Recently, the homotopy techniques have been used to study tensor decomposition and perfect identification problems \([16]\).

In this paper we investigate computing complex eigenpairs of general tensors using homotopy continuation methods. One attractive feature of the homotopy continuation methods is that they can find all isolated solutions of polynomial systems and some solutions in the positive dimensional solution components. We propose two homotopy type algorithms for computing complex eigenpairs of a tensor. These algorithms allow us to find all equivalence classes of isolated eigenpairs of a general tensor and some eigenpairs in positive dimensional eigenspaces (if there are any). We also present a homotopy method and a heuristic approach to compute real eigenpairs based on the found complex eigenpairs. Numerical examples show that our methods are effective and efficient.
This paper is organized as follows. In Section 2, we define mode-$k$ generalized eigenvalues and eigenvectors which extend the matrix right eigenpairs and left eigenpairs to higher order tensors. Some properties of such eigenpairs are proved. An upper bound for the number of equivalence classes of generalized tensor eigenpairs using mixed volume is derived. In Section 3, we consider computing mode-$k$ generalized complex eigenpairs and present two algorithms. In Section 4, we propose a homotopy method and a heuristic approach to compute real mode-$k$ generalized eigenpairs. Finally in Section 5, some numerical results are provided.

2 Tensor eigenvalues and eigenvectors

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ be the complex field or the real field. Let $m \geq 2$, $m' \geq 2$, and $n$ be positive integers. Denote the set of all $m$th-order, $n$-dimensional tensors on the field $\mathbb{F}$ by $\mathbb{F}^{[m,n]}$. A tensor in $\mathbb{F}^{[m,n]}$ is indexed as $A = (A_{i_1 i_2 \cdots i_m})$, where $A_{i_1 i_2 \cdots i_m} \in \mathbb{F}$, for $1 \leq i_1, i_2, \ldots, i_m \leq n$.

For $x \in \mathbb{C}^n$, the tensor $A$ defines a multilinear form

$$Ax^m = \sum_{i_1, \ldots, i_m=1}^n A_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}. \quad (2.1)$$

For $1 \leq k \leq m$, $A^{(k)}x^{m-1}$ is an $n$-vector whose $j$th entry is defined as

$$(A^{(k)}x^{m-1})_j = \sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m=1}^n A_{i_1 \cdots i_{k-1} j i_{k+1} \cdots i_m} x_{i_1} x_{i_{k+1}} \cdots x_{i_{k-1}} x_{i_{k+1}} \cdots x_{i_m}. \quad (2.2)$$

When $k = 1$, the vector $A^{(1)}x^{m-1}$ is denoted by $Ax^{m-1}$.

A real tensor $A \in \mathbb{R}^{[m,n]}$ is positive definite if the multilinear form $Ax^m$ is positive for all $x \in \mathbb{R}^n \setminus \{0\}$. A tensor $A \in \mathbb{F}^{[m,n]}$ is symmetric if its entries $A_{i_1 i_2 \cdots i_m}$ are invariant under any permutations of their indices $i_1, i_2, \ldots, i_m$.

We now introduce the following mode-$k$ generalized eigenvalue definition for a general tensor $A$.

DEFINITION 2.1 Let $A \in \mathbb{F}^{[m,n]}$ and $B \in \mathbb{F}^{[m',n]}$. Assume that $Bx^{m'}$ is not identical to zero. For $1 \leq k \leq m$, if there exist a scalar $\lambda \in \mathbb{C}$ and a vector $x \in \mathbb{C}^n \setminus \{0\}$ such that

- when $m = m'$,
  $$A^{(k)}x^{m-1} = \lambda Bx^{m-1}, \quad (2.3)$$
- when $m \neq m'$,
  $$A^{(k)}x^{m-1} = \lambda Bx^{m'-1}, \quad Bx^{m'} = 1, \quad (2.4)$$
then $\lambda$ is called a mode-$k$ $\mathcal{B}$-eigenvalue of $\mathcal{A}$ and $x$ a mode-$k$ $\mathcal{B}$-eigenvector associated with $\lambda$. $(\lambda, x)$ is called a mode-$k$ $\mathcal{B}$-eigenpair of $\mathcal{A}$.

If $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$, then $\lambda$ is called a mode-$k$ $\mathcal{B}_R$-eigenvalue of $\mathcal{A}$ and $x$ a mode-$k$ $\mathcal{B}_R$-eigenvector associated with $\lambda$, and $(\lambda, x)$ a mode-$k$ $\mathcal{B}_R$-eigenpair of $\mathcal{A}$.

Denote the set of all mode-$k$ $\mathcal{B}$ eigenvalues of $\mathcal{A}$ by $\sigma_B(\mathcal{A}^{(k)})$.

REMARK 2.1 Let $(\lambda, x)$ be a mode-$k$ $\mathcal{B}$-eigenpair of $\mathcal{A}$. By (2.3) or (2.4), $(\lambda, x)$ is a solution to $\mathcal{A}^{(k)} x^{m-1} = \lambda \mathcal{B} x^{m'-1}$. So is $(\lambda', x')$ with $\lambda' = t^{m-m'} \lambda$ and $x' = tx$ for $t \in \mathbb{C} \setminus \{0\}$. From this point of view, the solution space of $\mathcal{A}^{(k)} x^{m-1} = \lambda \mathcal{B} x^{m'-1}$ consists of different equivalence classes. We denote such an equivalence class by

$$[(\lambda, x)] := \{ (\lambda', x') \mid \lambda' = t^{m-m'} \lambda, x' = tx, t \in \mathbb{C} \setminus \{0\} \}.$$ 

When $m \neq m'$, taking arbitrary $(\lambda', x') \in [(\lambda, x)]$ and substituting $x' = tx$ into $\mathcal{B} x^{m'} = 1$ in (2.3) yields $t^{m'} = 1$, which gives $m'$ different values for $t$. This implies that the normalization $\mathcal{B} x^{m'} = 1$ in (2.3) restricts us to choose $m'$ representative solutions from each equivalence class.

In our later discussions, we often choose only one representative from each equivalence class. And it is only meaningful to count the number of equivalence classes of mode-$k$ $\mathcal{B}$-eigenpairs.

REMARK 2.2 If only one representative is desirable from each equivalence class of eigenpairs, we can solve $\mathcal{A}^{(k)} x^{m-1} = \lambda \mathcal{B} x^{m'-1}$ augmented with an additional linear equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b = 0,$$

where $a_1, \ldots, a_n, b$ are random complex numbers. Then normalize the resulting solution to satisfy $\mathcal{B} x^{m'} = 1$ in the case $m \neq m'$.

In the matrix case when $m = m' = 2$ and $\mathcal{B} = I_n$ (the $n \times n$ identity matrix), the mode-1 eigenvectors are right eigenvectors and the mode-2 eigenvectors are left eigenvectors of $\mathcal{A}$, and the mode-1 and mode-2 eigenvalues are the eigenvalues of matrix $\mathcal{A}$, i.e., $\sigma_B(\mathcal{A}^{(1)}) = \sigma_B(\mathcal{A}^{(2)})$. However, when $m \geq 3$, $\sigma_B(\mathcal{A}^{(k)}) = \sigma_B(\mathcal{A}^{(l)})$ is generally not true when $k \neq l$, unless $\mathcal{A}$ has a certain type of symmetry. The following example illustrates this situation.

EXAMPLE 2.1 Consider the tensor $\mathcal{A} \in \mathbb{R}^{[3,2]}$ whose entries are

$$A_{111} = 1, A_{121} = 2, A_{211} = 3, A_{221} = 4,$$
$$A_{112} = 5, A_{122} = 6, A_{212} = 7, A_{222} = 0.$$ 

Choose $m' = 2$ and $\mathcal{B} = I_2$ (the $2 \times 2$ identity matrix). Note that in this case, if $(\lambda, x)$ is an $\mathcal{B}$-eigenpair of $\mathcal{A}$, so is $(-\lambda, -x)$. We follow [4], regarding $(\lambda, x)$ and $(-\lambda, -x)$ as the same eigenpair. Then

$$\sigma_B(\mathcal{A}^{(1)}) = \{0.4105, 4.3820, 9.8995\},$$
$$\sigma_B(\mathcal{A}^{(2)}) = \{0.2851, 4.3536, 9.5652\},$$
$$\sigma_B(\mathcal{A}^{(3)}) = \{0.2936, 4.3007, 9.4025\}.$$

Clearly, $\sigma_B(\mathcal{A}^{(k)}) \neq \sigma_B(\mathcal{A}^{(l)})$ when $k \neq l$. 

4
PROPOSITION 2.1 Suppose that \((\lambda, x)\) is a mode-\(k\) \(B\)-eigenpair, \((\mu, x)\) is a mode-\(l\) \(B\)-eigenpair of \(A\), and when \(m = m'\), \(Bx^m \neq 0\). Then \(\lambda = \mu\).

Proof: Note that
\[ \lambda Bx^m = Ax^m = \mu Bx^{m'}. \]
This immediately implies that \(\lambda = \mu\), since \(Bx^{m'} = 1\) when \(m \neq m'\) and \(Bx^m \neq 0\) when \(m = m'\).

Let \(A \in \mathbb{F}^{[m,n]}\). For \(1 \leq k < l \leq m\), tensor \(G \in \mathbb{F}^{[m,n]}\) is said to be the \(\langle k, l \rangle\) transpose of \(A\) if
\[ G_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_{l-1} i_{l+1} \cdots i_m} = A_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_{l-1} i_{l+1} \cdots i_m}, \]
for all \(1 \leq i_1, \cdots, i_m \leq m\). Denote the \(\langle k, l \rangle\) transpose of \(A\) by \(A^{(k,l)}\). We say that tensor \(A\) is \(\langle k, l \rangle\) partially symmetric if
\[ A^{(k,l)} = A. \]

PROPOSITION 2.2 Let \(A \in \mathbb{F}^{[m,n]}\) and \(B \in \mathbb{F}^{[m',n]}\). Assume that \(Bx^{m'}\) is not identical to zero. Let \(k, l\) be integers such that \(1 \leq k < l \leq m\). Then

- \((\lambda, x)\) is a mode-\(k\) \(B\)-eigenpair of \(A\) if and only if it is a mode-\(l\) \(B\)-eigenpair of \(A^{(k,l)}\).
- The sets of mode-\(k\) \(B\)-eigenpairs and mode-\(l\) \(B\)-eigenpairs are the same if \(A\) is \(\langle k, l \rangle\) partially symmetric.

The eigenvalues/eigenvectors defined in \([6, 11, 34, 37]\) are mode-1 eigenvalues/eigenvectors. The tensors considered in these papers are primarily real symmetric tensors. For symmetric tensors, the sets of mode-\(k\) \(B\)-eigenpairs and mode-1 \(B\)-eigenpairs are the same for any \(k\). Therefore, mode-1 eigenvalues serve the purpose of those papers. On the other hand, nonsymmetric tensors arise from applications and theoretical studies, see, for example, \([4, 5, 12, 31, 44, 45]\). In \([27]\), Lim defined mode-\(k\) eigenvalues/eigenvectors for nonsymmetric real tensors \(A\) when \(B\) is the \(m'\)th order unit tensor for some \(m' \geq 2\). Definition \([21]\) considers more general \(A\) and \(B\).

As in \([6, 11]\), Definition \([21]\) adapts a unified approach to define tensor eigenvalues. It covers various types of tensor eigenvalues introduced in the literature, including

- If \(A \in \mathbb{R}^{[m,n]}, m' = 2\), and \(B\) is the identity matrix \(I_n \in \mathbb{R}^{n \times n}\), the mode-1 \(B\)-eigenpairs are the E-eigenpairs and the mode-1 \(B_R\)-eigenpairs are the Z-eigenpairs defined in \([34]\), which satisfy
\[ Ax^{m-1} = \lambda x, \quad x^T x = 1. \quad (2.6) \]

- If \(A \in \mathbb{R}^{[m,n]}, m' = 2\) and \(B = D\), where \(D \in \mathbb{R}^{n \times n}\) the symmetric positive definite matrix, the \(B_R\)-eigenpairs are the D-eigenpairs defined in \([37]\), which satisfy
\[ Ax^{m-1} = \lambda Dx, \quad x^T Dx = 1. \quad (2.7) \]
If $A \in \mathbb{R}^{[m,n]}$, $m = m'$ and $B = I$ is the unit tensor, mode-1 $B$-eigenpairs are the eigenpairs defined in [34], which satisfy

$$Ax^{m-1} = \lambda x^{m-1}, \quad (2.8)$$

where $x^{m-1} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$. For the sake of clarity, we will call such an eigenpair $Q_i$-eigenpair from now on.

If $A \in \mathbb{R}^{[m,n]}$, $m = m'$ and $B = I$ is the unit tensor, mode-1 $B_R$-eigenvalues are the H-eigenvalues defined in [34].

**REMARK 2.3** Theoretical properties of mode-1 eigenvalues of tensors such as the Perron-Frobenius theory ([5][12][44][45]) for nonnegative tensors can be parallely developed for mode-$k$ eigenvalues. However, as Horn and Johnson indicated in [17]: “One should not dismiss left eigenvectors as merely a parallel theoretical alternative to right eigenvectors. Each type of eigenvector can convey different information about a matrix,” we believe that mode-1 through mode-$m$ eigenpairs can convey different information about a general tensor of order $m \geq 3$.

In the rest of this section, we will obtain an upper bound for the number of equivalence classes of mode-$k$ eigenpairs. As shown in Definition 2.1, Remark 2.1 and Remark 2.2, the number of equivalence classes of mode-$k$ generalized eigenpairs for general tensors $A \in \mathbb{C}^{[m,n]}$ and $B \in \mathbb{C}^{[m,n]}$ is equivalent to the number of solutions to the following system of polynomials

$$T(\lambda, x) = \begin{pmatrix}
    (A^{(k)}x^{m-1})_1 - \lambda(Bx^{m-1})_1 \\
    \vdots \\
    (A^{(k)}x^{m-1})_n - \lambda(Bx^{m-1})_n \\
    a_1x_1 + a_2x_2 + \cdots + a_nx_n + b
  \end{pmatrix} = 0, \quad (2.9)
$$

where $\lambda$ and $x := (x_1, \ldots, x_n)^T$ are the unknowns, $a_1, \ldots, a_n, b$ are random complex numbers. This motivates us to use Bernstein’s theorem and its extensions in the field of solving polynomial systems (see [2][28]) to study the number of equivalence classes of eigenpairs.

To initiate our discussion, we first introduce some commonly used notations and definitions. Let $P(x) := (p_1(x), \ldots, p_n(x))^T$ be a polynomial system with $x := (x_1, \ldots, x_n)^T$. For $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, write $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then $P(x)$ can be denoted by

$$P(x) := \begin{pmatrix}
    p_1(x) := \sum_{\alpha \in S_1} c_{1,\alpha}x^\alpha \\
    \vdots \\
    p_n(x) := \sum_{\alpha \in S_n} c_{n,\alpha}x^\alpha
  \end{pmatrix}, \quad (2.10)
$$

where $S_1, \ldots, S_n$ are given finite subsets of $(\mathbb{Z}_{\geq 0})^n$ and $c_{i,\alpha} \in \mathbb{C}^* := \mathbb{C}\setminus \{0\}$ are given coefficients of the corresponding monomials. Here for each $i = 1, \ldots, n$, $S_i$ is called the
support of \( p_i(x) \) and its convex hull \( R_i := \text{conv}(S_i) \) in \( \mathbb{R}^n \) is called the Newton polytope of \( p_i(x) \). \((S_1, \ldots, S_n)\) is called the support of \( P(x) \). For nonnegative variables \( \lambda_1, \ldots, \lambda_n \), let \( \lambda_1 R_1 + \cdots + \lambda_n R_n \) be the Minkowski sum of \( \lambda_1 R_1, \ldots, \lambda_n R_n \), i.e.,

\[
\lambda_1 R_1 + \cdots + \lambda_n R_n := \{ \lambda_1 r_1 + \cdots + \lambda_n r_n | r_i \in R_i, i = 1, \ldots, n \}.
\]

It was shown in [8] that the \( n \)-dimensional volume of \( \lambda_1 R_1 + \cdots + \lambda_n R_n \), denoted by \( \text{Vol}_n(\lambda_1 R_1 + \cdots + \lambda_n R_n) \), is a homogeneous polynomial function of degree \( n \) in \( \lambda_1, \ldots, \lambda_n \). Then the coefficient of the monomial \( \lambda_1 \lambda_2 \cdots \lambda_n \) in \( \text{Vol}_n(\lambda_1 R_1 + \cdots + \lambda_n R_n) \) is called the mixed volume of \( R_1, \ldots, R_n \), denoted by \( \text{MV}_n(R_1, \ldots, R_n) \), or the mixed volume of the supports \( S_1, \ldots, S_n \), denoted by \( \text{MV}_n(S_1, \ldots, S_n) \). Sometimes it is also called the mixed volume of \( P(x) \) if no ambiguity exists. The following theorem relates the number of solutions of a polynomial system to its mixed volume.

**THEOREM 2.1 (Bernstein’s Theorem)** [2] The number of isolated zeros in \( (\mathbb{C}^*)^n \), counting multiplicities, of a polynomial system \( P(x) = (p_1(x), \ldots, p_n(x))^T \) with supports \( S_1, \ldots, S_n \) is bounded by the mixed volume \( \text{MV}_n(S_1, \ldots, S_n) \). Moreover, for generic choices of the coefficients in \( p_i \), the number of isolated zeros is exactly \( \text{MV}_n(S_1, \ldots, S_n) \).

An unexpected limitation of Theorem 2.1 is that it only counts the isolated zeros of a polynomial system in \( (\mathbb{C}^*)^n \) rather than \( \mathbb{C}^n \). To deal with this issue, Li and Wang gave the following theorem.

**THEOREM 2.2** [28] The number of isolated zeros in \( \mathbb{C}^n \), counting multiplicities, of a polynomial system \( P(x) = (p_1(x), \ldots, p_n(x))^T \) with supports \( S_1, \ldots, S_n \) is bounded by the mixed volume \( \text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \).

The following lemma was given as an exercise in [8].

**LEMMA 2.1** Consider a polynomial system \( P(x) = (p_1(x), \ldots, p_n(x))^T \) with supports \( S_1 = S_2 = \cdots = S_n = S \). Then

\[
\text{MV}_n(S, \ldots, S) = n! \text{Vol}_n(\text{conv}(S)).
\]

Recall that a \( n \)-simplex is defined to be the convex hull of \( n + 1 \) points \( z_1, \ldots, z_{n+1} \) such that \( z_2 - z_1, \ldots, z_{n+1} - z_1 \) are linearly independent in \( (\mathbb{R}^n)^T \). Then it can be verified from multivariate calculus that

\[
\text{Vol}_n(\text{conv}(z_2 - z_1, \ldots, z_{n+1} - z_1)) = \frac{1}{n!} \left| \det \begin{pmatrix} z_2 - z_1 \\ \vdots \\ z_{n+1} - z_1 \end{pmatrix} \right|.
\]

An upper bound for the number of equivalence classes of mode-k eigenpairs which generalizes results in [33], [33], [34] is given in the following theorem.

**THEOREM 2.3** Let \( A \in \mathbb{C}^{[m,n]} \) and \( B \in \mathbb{C}^{[m',n]} \). Assume that \( B x^{m'} \) is not identical to zero. Let \( k \) be an integer such that \( 1 \leq k \leq m \). Assume that \( A \) has finitely many equivalence classes of mode-k \( B \)-eigenpairs over \( \mathbb{C} \).
(a) If $m = m'$, then the number of equivalence classes of mode-$k$ $B$-eigenpairs, counting multiplicity, is bounded by

$$n(m - 1)^{n-1}.$$ 

If $A$ and $B$ are generic tensors, then $A$ has exactly $n(m - 1)^{n-1}$ equivalence classes of mode-$k$ $B$-eigenpairs, counting multiplicity.

(b) If $m \neq m'$, then the number of equivalence classes of mode-$k$ $B$-eigenpairs, counting multiplicity, is bounded by

$$\frac{(m - 1)^n - (m' - 1)^n}{m - m'}.$$ 

If $A$ and $B$ are generic tensors, then $A$ has exactly $((m - 1)^n - (m' - 1)^n)/(m - m')$ equivalence classes of mode-$k$ $B$-eigenpairs, counting multiplicity.

**Proof:** Recall that the number of equivalence classes of mode-$k$ $B$-eigenpairs of $A$ is equal to the number of solutions of (2.9). For the random hyperplane $a_1x_1 + \cdots + a_nx_n + b = 0$ in (2.9), without loss of generality, suppose that $a_n \neq 0$. Then $x_n$ can be solved as

$$x_n = c_1x_1 + \cdots + c_{n-1}x_{n-1} + d,$$ \hspace{1cm} (2.11)

where $c_i = a_i/a_n$ for $i = 1, \ldots, n - 1$ and $d = b/a_n$. Notice that the number of solutions of (2.9) in $\mathbb{C}^{n+1}$ is the same as the number of solutions in $\mathbb{C}^n$ of the resulting system $T^*(\lambda, x_1, \ldots, x_{n-1})$ by substituting (2.11) into the first $n$ equations of (2.9). Denote $z := (\lambda, x_1, \ldots, x_{n-1})$ and denote the corresponding supports of $T^*$ by $S_1, \ldots, S_n$. We claim that

$$\text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) = \begin{cases} n(m - 1)^{n-1}, & m = m' \\ \frac{(m - 1)^n - (m' - 1)^n}{m - m'}, & m \neq m' \end{cases}$$ \hspace{1cm} (2.12)

Let $N$ denote the number of equivalence classes of mode-$k$ $B$-eigenpairs of $A$ over $\mathbb{C}$. Then (2.12) implies that

$$N \leq n(m - 1)^{n-1}$$

for $m = m'$ and

$$N \leq \frac{(m - 1)^n - (m' - 1)^n}{m - m'}$$

for $m \neq m'$. When $A$ and $B$ are generic, the above equality holds by using Theorem 2.1 and Theorem 2.2.

To prove (2.12), let $\tilde{A} \in \mathbb{C}^{[m,n]}$ and $\tilde{B} \in \mathbb{C}^{[m',n]}$ be generic tensors. Similar to (2.9) the corresponding polynomial system to solve is

$$\tilde{T}(\lambda, x) = \begin{pmatrix} (\tilde{A}^{(k)}x^{m-1})_1 - \lambda(\tilde{B}x^{m'-1})_1 \\ \vdots \\ (\tilde{A}^{(k)}x^{m-1})_n - \lambda(\tilde{B}x^{m'-1})_n \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n + b \end{pmatrix} = 0.$$ \hspace{1cm} (2.13)
Substituting (2.11) into the first $n$ equations of (2.13) yields a new system $\tilde{T}^*(z) := T^*(\lambda, x_1, \ldots, x_{n-1})$. Let $\tilde{S}_1, \ldots, \tilde{S}_n$ be the corresponding supports of $\tilde{T}^*$. Since $\tilde{A}$ and $\tilde{B}$ are generic, without loss of generality one can assume that all monomials $\lambda x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}}$ will appear in each of the first $n$ equations in (2.13). Therefore, after substituting (2.11) into the first $n$ equations of (2.13), all monomials

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} \mid \alpha_i \in \mathbb{Z}_{\geq 0}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \leq m - 1\}$$

and

$$\{\lambda x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} \mid \alpha_i \in \mathbb{Z}_{\geq 0}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \leq m' - 1\}$$

will be contained in each equation of $\tilde{T}^*$. This implies that $\tilde{S}_1, \ldots, \tilde{S}_n$ are all equal to

$$\tilde{S} := \{(0, \alpha) \mid \alpha \in (\mathbb{Z}_{\geq 0}^{n-1})^T, |\alpha| \leq m - 1 \} \cup \{(1, \alpha) \mid \alpha \in (\mathbb{Z}_{\geq 0}^{n-1})^T, |\alpha| \leq m' - 1\}.$$

Let $\tilde{Q}$ be the convex hull of $\tilde{S}$. Then the vertices of $\tilde{Q}$ are given by the following points in $(\mathbb{Z}^n)^T$:

\[
\begin{align*}
    z_0 &= (0, 0, \ldots, 0), \\
    z_1 &= (0, 0, \ldots, 0, m - 1), \\
    z_2 &= (0, 0, \ldots, 0, m - 1, 0), \\
    \vdots \\
    z_{n-1} &= (0, m - 1, 0, \ldots, 0), \\
    z_n &= (1, 0, \ldots, 0), \\
    z_{n+1} &= (1, 0, \ldots, 0, m' - 1), \\
    z_{n+2} &= (1, 0, \ldots, 0, m' - 1, 0), \\
    \vdots \\
    z_{2n-1} &= (1, m' - 1, 0, \ldots, 0).
\end{align*}
\]

Denote the $i$-th unit vector in $(\mathbb{R}^n)^T$ by $e_i$ for $i = 1, \ldots, n$. Then

$$z_i = \begin{cases} 
    0, & i = 0 \\
    (m - 1)e_{n+1-i}, & 1 \leq i \leq n - 1 \\
    e_1, & i = n \\
    e_1 + (m' - 1)e_{2n+1-i}, & n + 1 \leq i \leq 2n - 1
\end{cases}$$
To compute the volume of $\bar{Q}$, we can divide it to the following $n$ simplicies with

$$
\bar{Q}_1 := \text{conv}(z_0, z_1, \ldots, z_n),
\bar{Q}_2 := \text{conv}(z_1, z_2, \ldots, z_{n+1}),
\vdots
\bar{Q}_i := \text{conv}(z_{i-1}, z_i, \ldots, z_{n+i-1}),
\vdots
\bar{Q}_n := \text{conv}(z_{n-1}, z_n, \ldots, z_{2n-1}).
$$

Thus

$$
\text{Vol}_n(\bar{Q}_1) = \frac{1}{n!} \det \begin{pmatrix}
(z_1 - z_0)
\vdots
(z_n - z_0)
\end{pmatrix} = \frac{1}{n!} \det \begin{pmatrix}
(m - 1)e_n
(m - 1)e_{n-1}
\vdots
(m - 1)e_2
\end{pmatrix} = \frac{(m - 1)^{n-1}}{n!} \det \begin{pmatrix}
e_n
e_{n-1}
\vdots
e_2
e_1
\end{pmatrix}
= \frac{(m - 1)^{n-1}}{n!}.
$$

For $i = 2, \ldots, n$, it is obvious that $z_n$ is contained in each $\bar{Q}_i$, so

$$
\text{Vol}_n(\bar{Q}_i) = \frac{1}{n!} \det \begin{pmatrix}
(z_{i-1} - z_n)
(z_i - z_n)
\vdots
(z_{n-1} - z_n)
z_{n+1} - z_n
\vdots
(z_{n+i-1} - z_n)
\end{pmatrix} = \frac{1}{n!} \det \begin{pmatrix}
(m - 1)e_{n+1-i} - e_1
(m - 1)e_{n+1-i} - e_1
\vdots
(m - 1)e_2 - e_1
(m' - 1)e_n
(m' - 1)e_{n-1}
\vdots
(m' - 1)e_{n+2-i}
\end{pmatrix}
= \frac{(m - 1)^{n-i}(m' - 1)^{i-1}}{n!} \det \begin{pmatrix}
e_1
\vdots
(m - 1)e_{n+1-i}
e_1
\vdots
e_n
\end{pmatrix}
= \frac{(m - 1)^{n-i}(m' - 1)^{i-1}}{n!}.
$$
Comparing (2.14) and (2.15), (2.15) actually also holds for $i = 1$. Thus

$$Vol_n(\bar{Q}) = \sum_{i=1}^{n} Vol_n(\bar{Q}_i) = \sum_{i=1}^{n} \frac{(m-1)^{n-i}(m'-1)^i}{n!}.$$  

Therefore, by Lemma 2.1

$$MV_n(\bar{S}_1, \ldots, \bar{S}_n) = n!Vol_n(\bar{Q}) = \sum_{i=1}^{n} (m-1)^{n-i}(m'-1)^i.$$  

Noting that for $i = 1, \ldots, n$, $S_i \cup \{0\}$ is a subset of $\bar{S}_i$. Hence

$$MV_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \leq MV_n(\bar{S}_1, \ldots, \bar{S}_n) = \sum_{i=1}^{n} (m-1)^{n-i}(m'-1)^i.$$  

This implies that

$$MV_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \leq \sum_{i=1}^{n} (m-1)^{n-1} = n(m-1)^{n-1} \quad (2.16)$$

for $m = m'$ and

$$MV_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \leq \frac{(m-1)^n}{m'-1} \sum_{i=1}^{n} \left( \frac{m'-1}{m-1} \right)^{m'} \leq \frac{(m-1)^n}{m'-1} \frac{1 - (\frac{m'-1}{m-1})^{m'}}{1 - \frac{m'-1}{m-1}} = \frac{(m-1)^n}{m - m'} - \frac{(m'-1)^n}{m - m'} \quad (2.17)$$

for $m \neq m'$.

On the other hand, for $m = m'$ consider the the diagonal tensors $A \in \mathbb{C}^{[m,n]}$ and $B \in \mathbb{C}^{[m,n]}$ such that $A_{ii \ldots i} = i$, $B_{ii \ldots i} = 1$ and all other entries zero. Then the number of equivalence classes of mode-$k$ B-eigenpairs of $A$ is equal to the number of solutions with to the following system of polynomials

$$\begin{pmatrix} x_1^{m-1} - \lambda x_1^{m-1} \\ 2x_2^{m-1} - \lambda x_2^{m-1} \\ \vdots \\ nx_n^{m-1} - \lambda x_n^{m-1} \\ x_1 + x_2 + \cdots + x_n - 1 \end{pmatrix} = 0.$$
The zeros of this system are \((\lambda, x_1, \ldots, x_n) = (1, 1, 0, \ldots, 0), (2, 0, 1, \ldots, 0), \ldots, (n, 0, 0, \ldots, 0)\) with each zero having multiplicity \(m - 1\). By Theorem 2.2
\[
\text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \geq n(m - 1)^{n-1}.
\]
Combining the above inequality with (2.16), we have
\[
\text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) = n(m - 1)^{n-1}.
\]
For \(m \neq m'\), consider the diagonal tensors \(A \in \mathbb{C}^{[m,n]}\) and \(B \in \mathbb{C}^{[m',n]}\) such that \(A_{ii\ldots i} = 1\), \(B_{ii\ldots i} = 1\) and all other entries zero. Assume that \(m > m'\). Then the number of equivalence classes of mode-\(k\) \(B\)-eigenpairs of \(A\) is equal to the number of solutions with to the following system of polynomials
\[
\begin{pmatrix}
x_1^{m-1} - \lambda x_1^{m'-1} \\
\vdots \\
x_n^{m-1} - \lambda x_n^{m'-1} \\
(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b)
\end{pmatrix} = \begin{pmatrix}
x_1^{m'-1}(x_1^{m-m'} - \lambda) \\
\vdots \\
x_n^{m'-1}(x_n^{m-m'} - \lambda) \\
(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b)
\end{pmatrix} = 0,
\]
where \(a_1, \ldots, a_n, b\) are random complex numbers. One can observe that \(x = 0\) cannot be a solution due to the augmented random hyperplane. As discussed in Remark 2.1, the hyperplane is added to ensure that only one representative from each equivalence class will be selected. So the number of equivalence classes of eigenpairs \((\lambda, x)\) needs to be found from the first \(n\) equations of (2.18). For each fixed \(\lambda\), there are \(m - 1\) choices for each \(x_i\) and \(x_i\) is chosen to be 0 for \(m' - 1\) of these choices. Excluding those choices making \(x = 0\), there are totally \((m - 1)^n - (m' - 1)^n\) choices for \(x_1, \ldots, x_n\). Furthermore, for each eigenvalue \(\lambda\), let \(x\) be the corresponding solution of the first \(n\) equations of (2.18). Then \(tx\) with \(t^{m-m'} = 1\) is also a solution associated with \(\lambda\). This implies that for each eigenvalue \(\lambda\), solving the first \(n\) equations of (2.18) results in \(m - m'\) corresponding solutions of \(x\). According to Remark 2.1 these \(m - m'\) solutions are actually equivalent. Therefore, there should be \(((m - 1)^n - (m' - 1)^n)/(m - m')\) equivalence classes of eigenpairs, i.e., (2.18) must have totally \(((m - 1)^n - (m' - 1)^n)/(m - m')\) isolated zeros in \(\mathbb{C}^{n+1}\). By Theorem 2.2
\[
\text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) \geq ((m - 1)^n - (m' - 1)^n)/(m - m').
\]
Combining the above inequality with (2.17), we have
\[
\text{MV}_n(S_1 \cup \{0\}, \ldots, S_n \cup \{0\}) = ((m - 1)^n - (m' - 1)^n)/(m - m').
\]
\(\square\)

**Remark 2.4** A few remarks about Theorem 2.3

(a) If Qi-eigenpairs are desired, then \(m' = m\). Using (a) in Theorem 2.3 the upper bound of the number of equivalence classes of eigenpairs is given by \(n(m - 1)^{n-1}\) equivalence classes of Qi-eigenpairs. This result is consistent with Theorem 1 in [34], in which it is shown that the number of eigenvalues of a symmetric tensor is \(n(m - 1)^{n-1}\) when \(m\) is even.

12
(b) If E-eigenpairs are desired, then $m' = 2$. Using (b) in Theorem 2.3 the upper bound of the number of equivalence classes of E-eigenpairs is given by $((m-1)^n - 1)/(m-2)$, which is consistent with Theorem 1.2 in [4].

(c) The upper bounds given in Theorem 2.3 can be highly useful in designing effective homotopy methods for computing mode-$k$ generalized eigenpairs. In fact, the homotopy method described in Algorithm 3.1 for the case $m = m'$ relies on the bound $n(m-1)^{n-1}$.

3 Computing complex tensor eigenpairs via homotopy methods

Consider $\mathcal{A} \in \mathbb{C}^{[m,n]}$ and $\mathcal{B} \in \mathbb{C}^{[m',n]}$. As discussed in Section 2, the problem of computing mode-$k \mathcal{B}$-eigenpairs of $\mathcal{A}$ in (2.3) or (2.4) is equivalent to the problem of solving (2.9), and if $m \neq m'$, normalize $(\lambda, x)$ to satisfy that $\mathcal{B}x^{m'} = 1$. Since (2.9) is a polynomial system, we consider to use a homotopy continuation method to numerically solve it.

The basic idea of using homotopy continuation method to solve a general polynomial system $P(x) = (p_1(x), \ldots, p_n(x))^T = 0$ as defined in (2.10) is to first deform $P(x)$ to another polynomial system $Q(x) = 0$ with known solutions. Then under certain conditions, a smooth curve that emanates from a solution of $Q(x) = 0$ will lead to a solution of $P(x) = 0$. One can consider construct the following classic linear homotopy [1]:

$$H(x, t) = (1-t)\gamma Q(x) + tP(x) = 0, \quad t \in [0, 1], \quad (3.1)$$

where $\gamma$ is a generic nonzero complex number. If $Q(x)$ is chosen properly, the following properties hold:

- **Property 0 (Triviality)** The solutions of $Q(x) = 0$ are known or easy to solve.
- **Property 1 (Smoothness)** The solution set of $H(x, t) = 0$ for $0 < t < 1$ consists of a finite number of smooth paths, each parameterized by $t$ in $[0, 1]$.
- **Property 2 (Accessibility)** Every isolated solution of $H(x, 1) = P(x) = 0$ can be reached by some path originating at a solution of $H(x, 0) = Q(x) = 0$.

Let $d_1, \ldots, d_n$ be the degrees of polynomials $p_1(x), \ldots, p_n(x)$ respectively. Then $d_1 \times d_2 \times \cdots \times d_n$ is called the total degree or the Bézout number of the system $P(x)$. A typical choice of a starting system $Q(x)$ in (3.1) satisfying Properties 0-2 is

$$Q(x) := \begin{pmatrix} a_1x_1^{d_1} - b_1 \\
\vdots \\

a_nx_n^{d_n} - b_n \end{pmatrix},$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are random complex numbers. The corresponding linear homotopy is called the total degree homotopy, see [25, 30, 42]. Here all the $d_1 \times d_2 \times \cdots \times d_n$ solutions of $Q(x) = 0$ can be easily solved. By tracking $d_1 \times d_2 \times \cdots \times d_n$ number of
solution paths of (3.1) we can find all the isolated solutions of \( P(x) = 0 \). However, most of the polynomial systems in applications usually have far fewer than \( d_1 \times d_2 \times \cdots \times d_n \) solutions. In this case, many of the \( d_1 \times d_2 \times \cdots \times d_n \) paths will diverge to infinity as \( t \to 1 \) resulting in huge wasteful computations.

The polyhedral homotopy continuation method [21] based on Bernstein’s Theorem [2] makes huge progress in this sense. In this method, the number of paths that need to be traced is the mixed volume of a polynomial system, which generally provides a much tighter bound than Bézout’s number for the number of isolated zeros of a polynomial system. Hence the new method reduces a significant amount of extraneous paths than the total degree homotopy in most occasions and thereby is much more efficient. However, the polyhedral homotopy continuation method includes two major stages: mixed volume computation and tracking paths. Sometimes mixed volume computation can be very expensive especially for large polynomial systems. Thus if the mixed volume is far less than the Bézout’s number and an appropriate linear homotopy can be constructed so that only mixed volume number of paths need to be traced, the system should be solved by using a linear homotopy instead of the polyhedral homotopy.

To solve (2.9), one can certainly use the polyhedral homotopy continuation method implemented in HOM4PS [24], PHCpack [41], PHoM [13], PSOLVE [46] (which is a MATLAB implementation of HOM4PS), or the total degree homotopy continuation method implemented in Bertini [3]. However, using these methods to solve (2.9) directly does not take advantage of the special structures of a tensor eigenproblem. We will introduce two homotopy type algorithms here that utilize such structures.

Theorem 2.3 gives us that the mixed volume of (2.9) is

\[
(n - 1)^{n-1} \text{ for } m = m'
\]

and

\[
((m - 1)^n - (m' - 1)^n)/(m - m') \text{ for } m \neq m',
\]

which is far less than the Bézout’s number, which is \( m^n \) for \( m = m' \), max\{\((m - 1)^n, (m')^n \}\) for \( m \neq m' \). From this point of view, we should consider constructing a linear homotopy using which only mixed volume number of paths need to traced.

To construct an appropriate linear homotopy, the concept of multihomogeneous Bézout’s number will be used. For a polynomial system \( P(x) = (p_1(x), \ldots, p_n(x))^T \) as defined in (2.10), where \( x = (x_1, \ldots, x_n) \). Partition the variables \( x_1, \ldots, x_n \) into \( k \) groups \( y_1 = (x_1^{(1)}, \ldots, x_l^{(1)}), y_2 = (x_1^{(2)}, \ldots, x_l^{(2)}), \ldots, y_k = (x_1^{(k)}, \ldots, x_l^{(k)}) \) with \( l_1 + \cdots + l_k = n \). Let \( d_{ij} \) be the degree of \( p_i \) with respect to \( y_j \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, k \). Then the multihomogeneous Bézout’s number of \( P(x) \) with respect to \( (y_1, \ldots, y_k) \) is the coefficient of \( \alpha_1^{l_1} \alpha_2^{l_2} \cdots \alpha_k^{l_k} \) in the product

\[
\prod_{i=1}^n (d_{i1} \alpha_1 + \cdots + d_{ik} \alpha_k).
\]

The following theorem will play a very important role in constructing a proper linear homotopy.

**THEOREM 3.1** [40] Let \( Q(x) \) be a system of polynomials chosen to have the same multihomogeneous form as \( P(x) \) with respect to certain partition of the variables \( (x_1, \ldots, x_n) \).
Assume $Q(x) = 0$ has exactly the multihomogeneous Bézout’s number of nonsingular solutions with respect to this partition, and let

$$H(x, t) = (1 - t)\gamma Q(x) + tP(x) = 0,$$

where $t \in [0, 1]$ and $\gamma \in \mathbb{C}^*$. If $\gamma = re^{i\theta}$ for some positive $r \in \mathbb{R}$, then for all but finitely many $\theta$, Properties 1 and 2 hold.

For (2.9), when $m = m'$ the following polynomial system

$$G(\lambda, x) = \begin{pmatrix}
(A^{(k)}x^{m-1})_1 - \lambda(Bx^{m-1})_1 \\
\vdots \\
(A^{(k)}x^{m-1})_n - \lambda(Bx^{m-1})_n \\
a_1x_1 + a_2x_2 + \cdots + a_nx_n + b
\end{pmatrix} = 0 \quad (3.2)$$

needs to be solved, where $\lambda$ and $x := (x_1, \ldots, x_n)^T$ are the unknowns, $a_1, \ldots, a_n, b$ are random complex numbers. Consider the starting system

$$Q(\lambda, x) = \begin{pmatrix}
(\lambda - \mu_1)(x_1^{m-1} - \beta_1) \\
(\lambda - \mu_2)(x_2^{m-1} - \beta_2) \\
\vdots \\
(\lambda - \mu_n)(x_n^{m-1} - \beta_n) \\
c_1x_1 + \cdots + c_nx_n + d
\end{pmatrix} = 0, \quad (3.3)$$

where $\mu_i, \beta_i, c_i$ for $i = 1, \ldots, n$ and $d$ are random nonzero complex numbers.

**Theorem 3.2** Let $G(\lambda, x)$ and $Q(\lambda, x)$ be defined as (3.2) and (3.3) respectively. Then all the isolated zeros $(\lambda, x)$ in $\mathbb{C}^{n+1}$ of $G(\lambda, x)$ can be found by using the homotopy

$$H(\lambda, x, t) = (1 - t)\gamma Q(\lambda, x) + tG(\lambda, x) = 0, \quad t \in [0, 1] \quad (3.4)$$

for almost all $\gamma \in \mathbb{C}^*$.

**Proof:** It is sufficient to verify that $Q(\lambda, x)$ satisfies all the assumptions of Theorem 3.1. Partition the variables $(\lambda, x_1, \ldots, x_n)$ into two groups: $(\lambda)$ and $(x_1, \ldots, x_n)$, we can easily see that each of the first polynomial equations in (3.2) and (3.3) has degree 1 in $(\lambda)$ and degree $m - 1$ in $(x_1, \ldots, x_n)$, and the last equation in both (3.2) and (3.3) has degree 0 in $(\lambda)$ and degree 1 in $(x_1, \ldots, x_n)$. Hence (3.2) and (3.3) have the same multihomogeneous Bézout’s number as the coefficient of $\alpha_1\alpha_2^2$ in the product

$$[1 \cdot \alpha_1 + (m - 1)\alpha_2]^n(0 \cdot \alpha_1 + 1 \cdot \alpha_2).$$

It can be easily computed that this coefficient is equal to

$${\binom{n}{1}}(m - 1)^{n-1} = n(m - 1)^{n-1}.$$
Hence (3.2) and (3.3) have the same multihomogeneous Bézout’s number \( n(m - 1)^{n-1} \) with respect to the partition \((\lambda)\) and \((x_1, \ldots, x_n)\).

We now show that \( Q(\lambda, x) \) in (3.3) has exactly \( n(m - 1)^{n-1} \) zeros. Notice that if \( \lambda \) is equal to none of \( \mu_1, \ldots, \mu_n \), then we end up with a system of \( n + 1 \) equations and \( n \) unknowns, which has no solutions. Thus \( \lambda \) must be equal to one of \( \mu_1, \ldots, \mu_n \). Without loss of generality, assume that \( \lambda = \mu_1 \). Then \( x_1, \ldots, x_n \) can be determined by

\[
\begin{align*}
    x_2^{m-1} - \beta_2 &= 0 \\
    \vdots \\
    x_n^{m-1} - \beta_n &= 0 \\
    c_1x_1 + \ldots + c_nx_n + d &= 0
\end{align*}
\]

Obviously, each \( x_i \) for \( i = 2, \ldots, n \) can be chosen as one of the \((m - 1)\)-th root of \( \beta_i \) and \( x_1 \) will be solved by substituting the chosen \( x_2, \ldots, x_n \) into the last hyperplane equation. So there are \((m - 1)^{n-1}\) solutions corresponding to \( \lambda = \mu_1 \). This argument holds for \( \lambda \) being any \( \mu_i \). Therefore, there are totally \( n(m - 1)^{n-1} \) solutions.

It remains to prove that each solution of \( Q(\lambda, x) = 0 \) in (3.3) is nonsingular. As discussed above, any solution \((\lambda^*, x^*)\) of (3.3) satisfies

\[
\begin{align*}
    \lambda^* &= \mu_i \\
    (x_1^*)^{m-1} - \beta_1 &= 0 \\
    \vdots \\
    (x_{i-1}^*)^{m-1} - \beta_{i-1} &= 0 \\
    (x_i^*)^{m-1} - \beta_i &= 0 \\
    \vdots \\
    (x_n^*)^{m-1} - \beta_n &= 0 \\
    c_1x_1^* + \ldots + c_nx_n^* + d &= 0
\end{align*}
\]

Let \( DQ(\lambda, x) \) be the Jacobian of \( Q(\lambda, x) \) with respect to \((\lambda, x)\). It is sufficient to show that \( DQ(\lambda^*, x^*) \) is nonsingular. Denote

\[
A_j(\lambda, x) := x_j^{m-1} - \beta_j, \quad B_j(\lambda, x) := (\lambda - \mu_j)(m - 1)x_j^{m-2}
\]

for \( j = 1, \ldots, n \). Then

\[
DQ(\lambda, x) = \begin{pmatrix}
A_1 & B_1 \\
\vdots & \ddots \\
A_{i-1} & B_{i-1} \\
A_i & B_i \\
A_{i+1} & B_{i+1} \\
\vdots & \ddots \\
A_n & B_n \\
0 & c_1 & \ldots & c_{i-1} & c_i & c_{i+1} & \ldots & c_n
\end{pmatrix}
\]
Note that
\[ A_j(\lambda^*, x^*) = (x_j^*)^{m-1} - \beta_j = 0, \quad j \neq i \]
and
\[ B_i(\lambda^*, x^*) = (\lambda^* - \mu_i)(m - 1)(x_i^*)^{m-2} = (\mu_i - \mu_i)(m - 1)(x_i^*)^{m-2} = 0 \]
by (3.5). For simplicity, write \( \bullet \) by (3.5). For simplicity, write \( \# \) by (3.5). Then
\[
DQ(\lambda^*, x^*) = \begin{pmatrix}
0 & B_1^* \\
\vdots & \ddots \\
0 & A_i^* & 0 \\
0 & \ddots & \ddots \\
0 & c_1 & c_{i-1} & c_i & c_{i+1} & \cdots & c_n
\end{pmatrix}.
\]
Then
\[
\det(DQ(\lambda^*, x^*)) = (-1)^{i+1} A_i^* (-1)^{n+i} c_i \prod_{j \neq i} B_j^* \neq 0
\]
by (3.5).

Theorem 3.2 suggests us that (3.1) can be used to solve (2.9) in the case of \( m = m' \). For simplicity, write \( u := (\lambda, x) \). In order to improve numerical stability, we apply the transformation \( s = \ln t \) to (3.1) (a strategy first suggested in [24]) and obtain the new homotopy as
\[
\tilde{H}(u, s) = (1 - e^s)\gamma Q(u) + e^s G(u) = 0, \quad s \in [-\infty, 0] \tag{3.6}
\]
where \( Q(u) = Q(\lambda, x) \) and \( G(u) = G(\lambda, x) \) are defined in (3.3) and (3.2) respectively.

We now introduce our algorithm for computing mode-k generalized eigenpairs when \( m = m' \).

**Algorithm 3.1** (Compute complex mode-k \( B \)-eigenpairs of \( A \), where \( A, B \in \mathbb{C}^{[m, n]} \)).

**Step 1.** Compute all solutions of \( Q(u) \) as defined in (3.3).

**Step 2.** Path following: Follow the paths from \( s = -\infty \) to \( s = 0 \). In reality, we certainly cannot start from \( s = -\infty \). In this case, one can choose a very negative \( s_0 \) and obtain a starting point by using Newton’s iterations:
\[
w^{(k+1)} = w^{(k)} - [\tilde{H}_u(w^{(k)}, s_0)]^{-1} \tilde{H}(w^{(k)}, s_0), \quad k = 0, 1, \ldots
\]
until \( \|\tilde{H}(w^{(N)}, s_0)\| \) is very small for some \( N \). Here \( w^{(0)} \) be a solution of \( Q(u) = 0 \). Let \( u_0 := u(s_0) \) and take \( u_0 = w^{(N)} \). Then path following can be triggered.

Path following is done using the prediction-correction method. Let \( (u_k, s_k) := (u(s_k), s_k) \), to find the next point on the path \( \tilde{H}(u, s) = 0 \), we employ the following strategy:
• Prediction Step: Compute the tangent vector \( \frac{du}{ds} \) to \( \bar{H}(u,s) = 0 \) at \( s_k \) by solving the linear system

\[
\bar{H}_u(u_k,s_k) \frac{du}{ds} = -\bar{H}_s(u_k,s_k)
\]

for \( \frac{du}{ds} \). Then compute the approximation \( \tilde{u} \) to \( u_{k+1} \) by

\[
\tilde{u} = u_k + \Delta_s \frac{du}{ds}, \quad s_{k+1} = s_k + \Delta_s,
\]

where \( \Delta s \) is a stepsize.

• Correction Step: Use Newton’s iterations. Initialize \( v_0 = \tilde{u} \). For \( i = 0, 1, 2, \ldots, \)

compute

\[
v_{i+1} = v_i - [\bar{H}_u(v_i, s_{k+1})]^{-1} \bar{H}(v_i, s_{k+1})
\]

until \( \|H(v_J, s_J)\| \) is very small. Then let \( u_{k+1} = v_J \).

Step 3. End game. During Step 3 when \( s_N \) is very close to 0, the corresponding \( u_N \)
should be very close to a zero \( u^* \) of \( G(u) = G(\lambda, x) \). So Newton’s iterations

\[
u^{(k+1)} = u^{(k)} - [DG(u^{(k)})]^{-1} G(u^{(k)}), \quad k = 0, 1, \ldots
\]

will be employed to refine our final approximation \( \tilde{u} \) to \( u^* \). If \( DG(u^*) \) is nonsingular, then \( \tilde{u} \) will be a very good approximation of \( u^* \) with multiplicity 1. If \( DG(u^*) \) is singular, \( \tilde{u} \) is either an isolated singular zero of \( G(u) \) with some multiplicity \( l > 1 \) or in a positive dimensional solution component of \( G(u) = 0 \). We use a strategy provided in Chapter VIII of [25] (see also [40]) to verify whether \( \tilde{u} \) is an isolated zero with multiplicity \( l > 1 \) or in a positive dimensional solution component of \( G(u) = 0 \).

Step 4. For each solution \( u = (\lambda, x) \) obtained in Step 3, normalize \( x \) in the following fashion to get a new eigenvector

\[
y = \frac{x}{x_{i_0}} \quad (3.7)
\]

can be obtained, where \( i_0 := \arg \max_{1 \leq i \leq n} |x_i| \). Hence \( (\lambda, y) \) is an eigenpair.

REMARK 3.1 A few remarks about Step 4 of Algorithm 3.1

(a) As defined in [23] and Remark 2.1 if \( (\lambda, x) \) is an eigenpair, \( (\lambda, tx) \) for \( t \neq 0 \) is also an eigenpair. Therefore, Step 4 is well defined in this sense.

(b) Notice that if \( x \) is a real eigenvector associated with a real eigenvalue \( \lambda \), \( tx \) for any \( t \in \mathbb{C} \setminus \{0\} \) will be a complex eigenvector associated with the same eigenvalue \( \lambda \). If in any case a complex eigenvector like \( tx \) is obtained in Step 3 of Algorithm 3.1 applying (3.7) to \( tx \) will give us a new real eigenvector. In this sense, Step 4 is very helpful for us to detect real eigenpairs.
To compute mode-$k$ generalized tensor eigenpairs when $m \neq m'$, we use the equivalence class structure of the eigenproblem as described in Remark 2.1. We first solve (2.9) to find a representative $(\lambda, x)$ from each equivalence class and then find all $m'$ eigenpairs from each equivalence class by simply using $\lambda' = t^{m-m'} \lambda, x' = tx$, where $t$ is a root of $t^{m'} = 1$. We will use PSOLVE [46] to solve the system (2.9).

One may think of solving (2.4) directly to get $m'$ eigenpairs from each equivalence class. However, this alternative method would have to follow $m'$ times more paths than the approach we described in the previous paragraph and therefore it would need more computation.

Now we present our algorithm for computing mode-$k$ generalized eigenpairs when $m \neq m'$.

**Algorithm 3.2** (Compute complex mode-$k$ B-eigenpairs of $A$, where $A \in \mathbb{C}[m,n], B \in \mathbb{C}[m',n]$ with $m \neq m'$.)

**Step 1.** Using PSOLVE to get all solutions $(\lambda, x)$ of (2.9).

**Step 2.** For each $(\lambda, x)$ obtained in Step 1, if $Bx^{m'} \neq 0$, normalize it to get an eigenpair $(\lambda^*, x^*)$ by

$$
\lambda^* = \frac{\lambda}{(Bx^{m'})^{(m-m')/m'}}, \quad x^* = \frac{x}{(Bx^{m'})^{1/m'}}
$$

to satisfy (2.7).

**Step 3.** Compute $m'$ equivalent eigenpairs $(\lambda', x')$ of $(\lambda^*, x^*)$ by $\lambda' = t^{m-m'} \lambda^*$ and $x' = tx^*$ with $t$ being a root of $t^{m'} = 1$.

### 4 Computing real tensor eigenpairs via homotopy methods

If some applications, tensor $A$ is real and only real eigenpairs (or real eigenvalues) of $A$ are of interest ([11, 34]). In this situation, only real zeros of the polynomial system (2.4) or (3.2) are needed. It is worth noting that there is currently no effective method to find all real zeros for a polynomial system directly. One may think to use a homotopy continuation method to trace only real zeros from the start system to the target system. However, this approach generally does not guarantee a real zero at the end.

For a real tensor $A$, a real eigenvalue may have complex eigenvectors. Sometimes identifying which eigenvalues are real is the only concern. In this case, we can first compute complex zeros $(\lambda, x)$ of (3.2) by Algorithm 3.1 or (2.4) by Algorithm 3.2, then identify the real eigenvalues by checking the size of the imaginary parts of $\lambda$'s. Specifically, let $(\lambda^*, x^*)$ be a computed eigenpair. If

$$
|\text{Im}(\lambda^*)| < \delta_0,
$$

where $\delta_0$ is a threshold for the imaginary part (the default value $\delta_0 = 10^{-8}$), then we take $\text{Re}(\lambda^*)$ as a real eigenvalue.

Note that if $\frac{m'}{m-m'}$ is a nonzero integer multiple of 4 (for example, $m = 5, m' = 4$ or $m = 10, m' = 8$,) and $A$ has an eigenpair $(\lambda^*, x^*)$ with a purely imaginary eigenvalue $\lambda^* = ...
bi, where b ∈ R, then one can easily show that \((b, (−i)^{1/(m−m')}x^*)\) and \((-b, i^{1/(m−m')}x^*)\) are eigenpairs with real eigenvalues. Therefore, when \(\frac{m'}{m−m'}\) is a nonzero integer multiple of 4, if \((\lambda^*, x^*)\) is an eigenpair found by Algorithm 3.2 such that

\[ |\text{Re}(\lambda^*)| < \delta_0, \]

then we take \(\text{Im}(\lambda^*)\) and \(\text{Im}(\lambda^*)\) as real eigenvalues, with corresponding eigenvectors \((-i)^{1/(m−m')}x^*\) and \(i^{1/(m−m')}x^*\).

When looking for real tensor eigenpairs (i.e., both eigenvalues and eigenvectors being real), the situation becomes more complicated. We use a two-step procedure. The first step is to get complex zeros \((\lambda, x)\) of (3.2) by Algorithm 3.1 or (2.4) by Algorithm 3.2. The second step is to extract all real eigenpairs \((\lambda, x)\) from the complex zeros obtained in the first step.

To facilitate the discussion, the following notation is introduced. For a vector \(a = (a_1, \ldots, a_n)^T \in \mathbb{C}^n\), let

\[ \text{Im}(a) = (\text{Im}(a_1), \ldots, \text{Im}(a_n))^T, \quad \text{Re}(a) = (\text{Re}(a_1), \ldots, \text{Re}(a_n))^T. \]

Suppose that \((\lambda^*, x^*)\) is an eigenpair found in the first step. Consider two cases:

(i) \((\lambda^*, x^*)\) is an isolated eigenpair;

(ii) \((\lambda^*, x^*)\) is an eigenpair contained in a positive dimensional solution component of system (3.2) or (2.4).

When \((\lambda^*, x^*)\) is an isolated eigenpair, take \((\text{Re}(\lambda^*), \text{Re}(x^*))\) as a real eigenpair if

\[ \|\text{Im}(\lambda^*, x^*)\|_2 < \delta_0. \]

If \((\lambda^*, x^*)\) is an eigenpair in a positive dimensional solution component of system (3.2) or (2.4), in general real eigenvectors are not guaranteed to be found by Algorithm 3.1 or Algorithm 3.2 even if the corresponding eigenvalue \(\lambda^*\) is real. In this case, we will construct the following Newton homotopy [1]

\[ H(\lambda, x, t) := P(\lambda, x) - (1-t)P(\lambda^*, \text{Re}(x^*)), \quad t \in [0, 1] \quad (4.1) \]

to follow curves in \((\lambda, x) \in \mathbb{R}^{n+1}\) in order to get a real eigenpair. Notice that when following curves in the complex space it is proved in [25] that the solution curves of (4.1) can be parameterized by \(t\), but the solution curves of (4.1) is not necessarily to be a function of \(t\) when restricted in the real space. So a different method to follow curves is needed. In this case parameterizing the solution curves by the arc length \(s\) is suggested in [25]. For simplicity, write \(y(s) := (\lambda(s), x(s), t(s))\), then (4.1) becomes \(H(y(s)) = 0\).

We now summarize our algorithm for computing a real eigenpair from a complex eigenpair \((\lambda^*, x^*)\) with real \(\lambda^*\), which is in a positive dimensional solution component of (3.2) or (2.4).

**Algorithm 4.1** (Trace solution paths of (4.1) in the real space to get a real eigenpair.)

**Step 1.** Let \(y_k := y(s_k)\) and let \(y_0 = (\lambda^*, \text{Re}(x^*))\), to find the next point on the solution path \(H(y) = 0\) the following strategy will be used:
• Prediction Step: Compute the tangent vector $\frac{dy}{ds}$ to $H(y) = 0$ at $y_k$ by solving the system

$$DH(y_k) \frac{dy}{ds} = 0$$

$$\left\| \frac{dy}{ds} \right\|_2 = 1$$

for $\frac{dy}{ds}$. Here $DH(y_k)$ is the Jacobian of $H$ with respect to $y$ evaluated at $y_k$. Then compute the approximation $\tilde{y}$ to $y_{k+1}$ by

$$\tilde{y} = y_k + \Delta s \frac{dy}{ds}, \quad s_{k+1} = s_k + \Delta s,$$

where $\Delta s$ is a stepsize.

• Correction Step: Use Newton’s iterations to solve $\xi$ from

$$F(\xi) := \begin{pmatrix} H(\xi) \\ (\xi - \tilde{y}) \cdot \frac{dy}{ds} \end{pmatrix} = 0$$

with the initialization $\xi_0 = \tilde{y}$, i.e., for $i = 0, 1, 2, \ldots$, compute

$$\xi_{i+1} = \xi_i - [DF(\xi_i)]^{-1}F(\xi_i)$$

until $\|F(\xi_J)\|$ is less than a threshold for some $J$. Then let $y_{k+1} = \xi_J$.

**Step 2.** End game. During Step 1 if after sufficiently many Prediction-Correction Steps $t$ does not approach 1, then a real eigenpair cannot be obtained; stop. Otherwise when $t(s_N)$ is very close to 1 for some $N$, the corresponding $(\lambda(s_N), x(s_N))$ is very close to a real zero $u_R := (\lambda_R, x_R)$ of $P(u) := P(\lambda, x)$. So Newton’s iterations

$$u^{(k+1)} = u^{(k)} - [DP(u^{(k)})]^{-1}P(u^{(k)}), \quad k = 0, 1, \ldots$$

with $u^{(0)} = (\lambda(s_N), x(s_N))$ will be employed to refine our final approximation $\tilde{u}$ to $u_R$. Take $\tilde{u}$ as a real eigenpair.

An interesting phenomenon is that in some special cases, the real eigenpairs can be obtained more straightforwardly from the complex eigenpairs found from Algorithm 3.1 or Algorithm 3.2 as illustrated in the following example.

**Example 4.1** Consider the tensor $A \in \mathbb{R}^{[3,5]}$ (Example 4.11 in [11], see also [32]) such that

$$A_{i,j,k} = \frac{(-1)^i}{i} + \frac{(-1)^j}{j} + \frac{(-1)^k}{k}, \quad i, j, k = 1, \ldots, 5.$$
Denote the corresponding polynomial system defined in (2.4) by $P(\lambda, x) = 0$. Using Algorithm 3.2, 62 solutions of this system are found. Among these 62 solutions, 4 are isolated solutions with $\|(\text{Im}(\lambda), \text{Im}(x))\|$ as small as the machine epsilon. Therefore, these isolated zeros can be classified as the $Z$-eigenpairs. Note that in this example if $(\lambda, x)$ is an eigenpair, then so is $(-\lambda, -x)$. Table 1 lists two of these four $Z$-eigenpairs with positive $\lambda$.

| $\lambda$ | $4.2876$ | $9.9779$ |
|-----------|---------|---------|
| $x_1$     | $-0.1859$ | $-0.7313$ |
| $x_2$     | $0.7158$ | $-0.1375$ |
| $x_3$     | $0.2149$ | $-0.4674$ |
| $x_4$     | $0.5655$ | $-0.2365$ |
| $x_5$     | $0.2950$ | $-0.4146$ |

Table 1: Isolated $Z$-eigenpairs of the tensor in Example 4.1

For the remaining 58 solutions of $P(\lambda, x) = 0$, each of them has $\lambda = 0$ and is contained in a positive dimensional solution component of $P(\lambda, x) = 0$. For each of these zeros, it can be verified that $(\lambda, \text{Re}(x)/\|\text{Re}(x)\|_2)$ and $(\lambda, \text{Im}(x)/\|\text{Im}(x)\|_2)$ are both solutions of $P(\lambda, x) = 0$. For example, one of these 58 zeros is

$$
\begin{pmatrix}
\lambda \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-0.7136 - 0.4086i \\
-0.1425 - 0.0266i \\
0.9941 - 0.2291i \\
-0.0180 + 0.1180i \\
-0.1200 + 0.5464i \\
\end{pmatrix}.
$$

Taking the real and imaginary parts of the above $x$ and let

$$
\xi = (-0.7136, -0.1425, 0.9941, -0.0180, -0.1200)^T,
$$

$$
\eta = (-0.4086, -0.0266, -0.2291, 0.1180, 0.5464)^T.
$$

Normalizing $\xi$ and $\eta$ respectively gives

$$
v = (-0.5764, -0.1151, 0.8030, -0.0146, -0.0969)^T,
$$

$$
w = (-0.5599, -0.0365, -0.3139, 0.1617, 0.7487)^T.
$$

It is easy to verify that $\|P(0,v)\|_2 \approx 5.8687 \times 10^{-14}$ and $\|P(0,w)\|_2 \approx 5.4921 \times 10^{-14}$. Therefore, $(0,v)$ and $(0,w)$ are both $Z$-eigenpairs. This implies that $(\lambda, \text{Re}(x)/\|\text{Re}(x)\|_2)$ and $(\lambda, \text{Im}(x)/\|\text{Im}(x)\|_2)$ actually give us two $Z$-eigenpairs.

The above example suggests that if $(\lambda^*, x^*)$ is in a positive dimensional solution component of (2.4) and $\lambda^* \in \mathbb{R}$, then $(\lambda^*, \text{Re}(x^*)/\|\text{Re}(x^*)\|_2)$ and $(\lambda^*, \text{Im}(x^*)/\|\text{Im}(x^*)\|_2)$ may be mode-$k$ $B_R$ eigenpairs of $A$. This gives us a heuristic approach to find real eigenpairs.
for eigenpairs belong to positive dimensional components. We remark that this approach works well for all the examples (e.g., Example 4.8, 4.11, 4.13, 4.14) in [11] when a real Z-eigenvalue has infinitely many real Z-eigenvectors.

This raises a curious question: what is the structure of the eigenspace \( \{ x \in \mathbb{C}^n \mid Ax^{m-1} = \lambda^* Bx^{m'-1}, Bx^{m'} = 1 \} \) associated with a real eigenvalue \( \lambda^* \)? Looking into the eigenspace of \( \lambda^* = 0 \) in Example 4.1, i.e., \( \{ x \in \mathbb{C}^5 \mid P(0, x) = 0 \} \), all the 58 eigenvectors obtained from Algorithm 3.2 lay on the hyperplane \( x_1 + x_2 + x_3 + x_4 + x_5 = 0 \). This observation leads to the following theorem which gives us a way of finding real eigenpairs in special cases.

**Proposition 4.1** Let \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) and \( \mathcal{B} \in \mathbb{R}^{[m',n]} \). Let \( k \) be an integer such that \( 1 \leq k \leq m \). Let \( \lambda \in \mathbb{R} \) be a real mode-\( k \) \( \mathcal{B} \) eigenvalue of \( \mathcal{A} \). If \( U := \{ x \in \mathbb{C}^n \mid \mathcal{A}(k)x^{m-1} = \lambda \mathcal{B}x^{m'-1} \} \) contains a complex linear subspace \( V \) of \( \mathbb{C}^n \), then for any \( x = \xi + i\eta \in V \) such that \( \xi, \eta \in \mathbb{R}^n \) and \( \xi \neq 0, \eta \neq 0 \),

- When \( m = m' \), \( \xi \) and \( \eta \) are both real mode-\( k \) \( \mathcal{B} \) eigenvectors of \( \mathcal{A} \) associated with \( \lambda \).
- When \( m \neq m' \), \( B\xi^{m'} \neq 0 \) and \( B\eta^{m'} \neq 0 \), the normalized vectors

  \[
  v := \frac{\xi}{(B\xi^{m'})^{1/m'}}, \quad w := \frac{\eta}{(B\eta^{m'})^{1/m'}}
  \]

  are real mode-\( k \) \( \mathcal{B} \) eigenvectors of \( \mathcal{A} \) associated with \( \lambda \).

**Proof:** Let \( x \in V \subseteq U \). Then

\[
\mathcal{A}(k)x^{m-1} = \lambda \mathcal{B}x^{m'-1}.
\]

Taking the conjugate of the above equation yields

\[
\overline{\mathcal{A}(k)x^{m-1}} = \overline{\lambda \mathcal{B}x^{m'-1}}.
\]

Since \( \lambda, \mathcal{A} \) and \( \mathcal{B} \) are all real,

\[
\mathcal{A}(k)x^{m-1} = \lambda \mathcal{B}x^{m'-1}.
\]

This implies that \( \bar{x} \in V \). Since \( V \) is a linear subspace, \( \xi = (x + \bar{x})/2 \) and \( \eta = (x - \bar{x})/(2i) \) are also in \( V \). Thus, when \( m = m' \), \( \xi \) and \( \eta \) are both real mode-\( k \) \( \mathcal{B} \) eigenvectors of \( \mathcal{A} \) associated with \( \lambda \). If \( m \neq m' \), we have

\[
B\xi^{m'} = \sum_{i_1, \ldots, i_{m'}=1}^n B_{i_1i_2\cdots i_{m'}} v_{i_1} v_{i_2} \cdots v_{i_{m'}}
\]

\[
= \sum_{i_1, \ldots, i_{m'}=1}^n B_{i_1i_2\cdots i_{m'}} \frac{\xi_{i_1}}{(B\xi^{m'})^{1/m'}} \frac{\xi_{i_2}}{(B\xi^{m'})^{1/m'}} \cdots \frac{\xi_{i_{m'}}}{(B\xi^{m'})^{1/m'}}
\]

\[
= \frac{\sum_{i_1, \ldots, i_{m'}=1}^n B_{i_1i_2\cdots i_{m'}} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m'}}}{B\xi^{m'}}
\]

\[
= \frac{B\xi^{m'}}{B\xi^{m'}} = 1.
\]
This implies that \( v \) is a real mode-\( k \) \( \mathcal{B} \) eigenvector of \( \mathcal{A} \) associated with \( \lambda \). Similarly, \( Bw^m = 1 \) can also be verified. Therefore, \( w \) is also a real mode-\( k \) \( \mathcal{B} \) eigenvector of \( \mathcal{A} \) associated with \( \lambda \).

Consequently, when \( Z \)-eigenpairs of \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) are of interest, let \( \lambda \in \mathbb{R} \) be a real \( \mathcal{E} \)-eigenvalue of \( \mathcal{A} \). If \( U = \{ x \in \mathbb{C}^n \mid Ax^{m-1} = \lambda x \} \) contains a complex linear subspace \( V \), then for any \( x = \xi + i\eta \in V \) with nonzero \( \xi, \eta \in \mathbb{R}^n \), \( \xi/\|\xi\|_2 \) and \( \eta/\|\eta\|_2 \) are both \( Z \)-eigenvalues of \( \mathcal{A} \).

REMARK 4.1 A natural question is when \( U \) defined in Proposition 4.1 contains a linear subspace \( V \). Consider the case when \( \mathcal{A} \) is a symmetric tensor with a low rank decomposition. For simplicity, consider \( m = 3 \). For vectors \( a, b, c \in \mathbb{C}^n \), define the outer product tensor \( a \circ b \circ c = (a_i b_j c_k) \). Suppose that a symmetric tensor \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) can be decomposed as

\[
\mathcal{A} = y_1 \circ y_1 \circ y_1 + y_2 \circ y_2 \circ y_2 + \cdots + y_r \circ y_r \circ y_r,
\]

where \( r < n \), \( y_k \in \mathbb{C}^n, k = 1, 2, \ldots, r \).

Let \( W = \text{span}(y_1, y_2, \ldots, y_r) \) and let \( V \) be the orthogonal complement of \( W \). Then \( V \) is a linear subspace of \( \mathbb{C}^n \). Moreover, for any \( x \in V \setminus \{0\} \),

\[
Ax^{m-1} = 0.
\]

Thus \( x \) is an eigenvector of \( \mathcal{A} \) corresponding to the eigenvalue 0. In this case, \( \text{Re}(x)/\|\text{Re}(x)\| \) and \( \text{Im}(x)/\|\text{Im}(x)\| \) are real \( Z \)-eigenvalues of \( \mathcal{A} \) corresponding to the real \( Z \)-eigenvalue 0. Thus, the set \( V \) is a linear subspace of \( \mathbb{C}^n \) contained in \( U \).

Finally we present an algorithm for computing real eigenpairs based on the heuristic approach and Algorithm 4.1.

ALGORITHM 4.2 (Compute real mode-k \( \mathcal{B} \)-eigenpairs of \( \mathcal{A} \), where \( \mathcal{A} \in \mathbb{R}^{[m,n]} \), \( \mathcal{B} \in \mathbb{R}^{[m',n]} \))

1. **Step 1.** Compute all complex eigenpairs using Algorithm 3.1 or Algorithm 3.2. Let \( K \) denote the set of found eigenpairs \((\lambda, x)\) such that \( |\text{Im}(\lambda)| < \delta_0 \).

2. **Step 2.** For each eigenpair \((\lambda^*, x^*) \in K\): if \((\lambda^*, x^*)\) is in a positive dimensional solution component of (3.2) or (2.4), go to Step 3. Otherwise, \((\lambda^*, x^*)\) is an isolated eigenpair. If \( ||\text{Im}(x^*)||_2 < \delta_0 \), then take \((\text{Re}(\lambda^*), \text{Re}(x^*))\) as a real eigenpair and stop.

3. **Step 3.** Set \( \lambda = \text{Re}(\lambda^*) \). If \( m = m' \), set \( v := \text{Re}(x^*) \) (if \( \text{Re}(x^*) \neq 0 \)) and \( w := \text{Im}(x^*) \) (if \( \text{Im}(x^*) \neq 0 \)); otherwise, set

\[
v := \frac{\text{Re}(x^*)}{(\mathcal{B} \text{Re}(x^*)^{m'})^{1/m'}} \quad \text{(if } \mathcal{B} \text{Re}(x^*)^{m'} \neq 0),
\]

and

\[
w := \frac{\text{Im}(x^*)}{(\mathcal{B} \text{Im}(x^*)^{m'})^{1/m'}} \quad \text{(if } \mathcal{B} \text{Im}(x^*)^{m'} \neq 0).
\]

If \((\lambda, v)\) or \((\lambda, w)\) is a mode-\( k \) \( \mathcal{B} \)-eigenpair of \( \mathcal{A} \), then we have obtained a real eigenpair and stop. Otherwise, goto Step 4.

4. **Step 4.** Starting from \((\lambda^*, x^*)\), use Algorithm 4.1 to find a real eigenpair.
5 Implementation and numerical results

Based on the algorithms introduced in Section 3 and Section 4, a MATLAB package TenEig has been developed. The current version is TenEig 1.1. The numerical results reported in this paper were obtained using this version. The package can be downloaded from

\[http://www.math.msu.edu/~chenlipi/TenEig.html\]

Consider the tensors \( A \in \mathbb{C}^{[m,n]} \) and \( B \in \mathbb{C}^{[m',n]} \). In the TenEig package, function \texttt{teig} can be used to compute the general mode-\( k \) \( B \) eigenvalues and eigenvectors of a tensor \( A \) for \( m = m' \). The input of this function is: tensor \( A \) or the polynomial form (if \( A \) is symmetric) \( Ax^m \), tensor \( B \) (the default is the identity tensor), mode \( k \) (the default value is 1), and the output is: mode-\( k \) \( B \) eigenvalues and eigenvectors of \( A \). By default, \texttt{teig} finds Qi-eigenvalues and Qi-eigenvectors.

The function \texttt{teneig} computes the general mode-\( k \) \( B \) eigenvalues and eigenvectors of a tensor \( A \) for \( m \neq m' \). The input of this function is: tensor \( A \) or the polynomial form (if \( A \) is symmetric) \( Ax^m \), tensor \( B \), mode \( k \) (the default value is 1), and the output is: mode-\( k \) \( B \) eigenvalues and eigenvectors of \( A \). If \( B \) is chosen as the identity matrix, the \texttt{teneig} computes the E-eigenvalues and E-eigenvectors of \( A \) as defined in Qi [34].

Since E-eigenpairs of a tensor are frequently needed, our package includes a separate function \texttt{eeig}, which only computes E-eigenpairs of a tensor.

The package also includes two functions \texttt{heig} and \texttt{zeig} to compute real eigenpairs of a tensor: The first one computes H-eigenpairs and the second one computes Z-eigenpairs.

In the next two subsections, numerical results are reported to illustrate the effectiveness and efficiency of our methods for computing tensor eigenpairs. All the numerical experiments were done on a Thinkpad T400 Laptop with an Intel(R) dual core CPU at 2.80GHz and 2GB of RAM, running on a Windows 7 operating system. The package TenEig was run using MATLAB 2013a. In our examples, we used \texttt{teig} or \texttt{teneig} to compute generalized eigenpairs, \texttt{teig} to compute Qi-eigenpairs, \texttt{eeig} to compute E-eigenvalues, \texttt{heig} to compute (real) H-eigenpairs, and \texttt{zeig} to compute (real) Z-eigenpairs, respectively.

5.1 Examples for computing complex eigenpairs

In this subsection, some numerical examples illustrating the performance of TenEig for computing complex tensor eigenpairs are provided.

A numerical solver \texttt{NSolve}, based on the Gröbner basis, for solving systems of algebraic equations is provided by Mathematica. We will compare the performance of TenEig and \texttt{NSolve} in computing complex tensor eigenpairs. Denote

\[
T(m,n) := n(m-1)^{n-1},
\]

\[
E(m,n) := ((m-1)^n - 1)/(m-2),
\]

\[
G(m,m',n) := ((m-1)^n - (m'-1)^n)/(m-m').
\]
Recall that Theorem 2.3 explains that for tensors \( A \in \mathbb{C}^{[m,n]} \) and \( B \in \mathbb{C}^{[m',n]} \), the number of equivalence classes of isolated \( B \)-eigenpairs of \( A \) is bounded by \( T(m,n) \) for \( m = m' \) and \( G(m,m',n) \) for \( m \neq m' \). In particular, Remark 2.4 states that the number of equivalence classes of isolated \( \Psi \)-eigenpairs is bounded by \( T(m,n) \) and the number of equivalence classes of isolated \( E \)-eigenpairs of \( A \) is bounded by \( E(m,n) \).

**EXAMPLE 5.1** In this example, we compare the performance of our TenEig with NSolve and PSOLVE. teig, eeig, NSolve and PSOLVE are used to compute \( \Psi \)-eigenpairs or \( E \)-eigenpairs of a generic tensor \( A \in \mathbb{C}^{[m,n]} \). We remark the following:

(a) teig is based on Algorithm 3.1
(b) eeig is based on Algorithm 3.2
(c) When computing \( \Psi \)-eigenpairs, NSolve and PSOLVE solve the polynomial system defined by (3.2).
(d) When computing \( E \)-eigenpairs, NSolve and PSOLVE solve the polynomial system defined by (2.4).

The tensors \( A \) were generated using \( \text{randn}(n, \cdots , n) + i \text{randn}(n, \cdots , n) \) in MATLAB. The results are given in Table 2. In this table, \( N \) denotes the number of equivalence classes of \( \Psi \)-eigenpairs or \( E \)-eigenpairs found by \( \text{teig, eeig, PSOLVE or NSolve} \), the reported CPU times are in seconds, “-” denotes that no results were returned after 12 hours.

| \((m, n)\) | \((T(m,n))\) | Alg | \(N\) | time (s) | \((E(m,n))\) | Alg | \(N\) | time (s) |
|---------|-----------|-----|------|--------|-----------|-----|------|--------|
| (4,5)   | 1280      | teig | 405  | 15.8   | -         | -   | -    | -      |
|         |           | PSOLVE | 404  | 14.0   | -         | -   | -    | -      |
|         |           | NSolve | 405  | 3136.4 | -         | -   | -    | -      |
| (5,5)   | 6144      | teig | 1280 | 65.5   | -         | -   | -    | -      |
|         |           | PSOLVE | 1280 | 65.5   | -         | -   | -    | -      |
|         |           | NSolve | -    | -      | -         | -   | -    | -      |
| (5,6)   | 18750     | teig | 6144 | 606.5  | -         | -   | -    | -      |
|         |           | PSOLVE | 6144 | 604.2  | -         | -   | -    | -      |
|         |           | NSolve | -    | -      | -         | -   | -    | -      |
| (6,6)   | 18750     | teig | 18750| 3721.3 | -         | -   | -    | -      |
|         |           | PSOLVE | 18748| 4636.0 | -         | -   | -    | -      |
|         |           | NSolve | -    | -      | -         | -   | -    | -      |

Table 2: Comparison of \( \text{teig and eeig} \) with PSOLVE and NSolve

From table 2 we see that \( \text{teig and eeig} \) successfully find all equivalence classes of \( \Psi \)-eigenpairs or \( E \)-eigenpairs using reasonable amount of time in all the cases. NSolve cannot get any results in 12 hours in some cases (we terminated it after 12 hours). Although PSOLVE successfully finds all equivalence classes in many cases, it does miss a few equivalence classes in some cases. Regarding the CPU time usage, PSOLVE is comparable to \( \text{teig} \) but takes more time than \( \text{eeig} \).

**EXAMPLE 5.2** In this example we show the effectiveness and efficiency of \( \text{teig} \) for finding all equivalence classes of isolated \( \Psi \)-eigenpairs of a generic tensor \( A \in \mathbb{C}^{[m,n]} \). Each tensor was generated using \( \text{randn}(n, \cdots , n) + i \text{randn}(n, \cdots , n) \) in MATLAB. The
results are reported in Table 3, in which \( N \) denotes the number of equivalence classes of isolated Qi-eigenpairs found by \( \text{teig} \) and \( T(m, n) \) denotes the bound of the number of equivalence classes of isolated Qi-eigenpairs (see Remark 2.4(a)).

\[
\begin{array}{cccccc}
(m, n) & T(m, n) & N & \text{time(s)} & (m, n) & T(m, n) & N & \text{time(s)} \\
(3, 5) & 80 & 80 & 2.4 & (3, 6) & 192 & 192 & 6.8 \\
(3, 7) & 448 & 448 & 18.3 & (3, 8) & 1024 & 1024 & 53.0 \\
(3, 9) & 2304 & 2304 & 145.9 & (3, 10) & 5120 & 5120 & 409.2 \\
(4, 3) & 27 & 27 & 0.7 & (4, 4) & 108 & 108 & 2.9 \\
(4, 5) & 405 & 405 & 15.8 & (4, 6) & 1458 & 1458 & 80.0 \\
(4, 7) & 5103 & 5103 & 385.9 & (4, 8) & 17496 & 17496 & 2115.5 \\
(5, 3) & 48 & 48 & 1.2 & (5, 4) & 256 & 256 & 8.8 \\
(5, 5) & 1280 & 1280 & 73.8 & (5, 6) & 6144 & 6144 & 606.5 \\
(5, 7) & 28672 & 28672 & 5394.2 & (5, 8) & 17496 & 17496 & 2115.5 \\
(6, 4) & 500 & 500 & 21.0 & (6, 5) & 3125 & 3125 & 287.7 \\
(6, 6) & 18750 & 18750 & 3721.3 & (6, 7) & 108 & 108 & 3.6 \\
(7, 4) & 864 & 864 & 51.5 & (7, 5) & 6480 & 6480 & 981.3 \\
\end{array}
\]

Table 3: Performance of \( \text{teig} \) on computing Qi-eigenpairs of complex random tensors

**EXAMPLE 5.3** In this example we show the effectiveness and efficiency of \( \text{eeig} \) for finding all equivalence classes of isolated \( \mathcal{E} \)-eigenpairs of a generic tensor \( \mathcal{A} \in \mathbb{C}^{[m,n]} \). Each generic tensor was generated using \( \text{randn}(n, \cdots, n) + i \ast \text{randn}(n, \cdots, n) \) in MATLAB. The results are reported in Table 4, in which \( N \) denotes the number of equivalence classes of \( \mathcal{E} \)-eigenpairs found by \( \text{eeig} \) and \( E(m, n) \) denotes the bound of the number of equivalence classes of isolated \( \mathcal{E} \)-eigenpairs (see Remark 2.4(b)).

According to [34], [7], and [4], for a randomly generated tensor \( \mathcal{A} \in \mathbb{C}^{[m,n]} \), it has \( T(m, n) \) number of equivalence classes of Qi-eigenpairs and \( E(m, n) \) number of equivalence classes of \( \mathcal{E} \)-eigenpairs. Moreover, its eigenpairs and \( \mathcal{E} \)-eigenpairs are isolated. From Tables 3 and 4, we observe that \( \text{teig} \) and \( \text{eeig} \) can find all equivalence classes of Qi-eigenpairs and \( \mathcal{E} \)-eigenpairs of such an tensor in the examples we tested.

**EXAMPLE 5.4** In this example we show the effectiveness and efficiency of \( \text{teig} \) and \( \text{teneig} \) for finding all equivalence classes of isolated \( \mathcal{B} \)-eigenpairs of a tensor \( \mathcal{A} \), where \( \mathcal{A} \in \mathbb{C}^{[m,n]}, \mathcal{B} \in \mathbb{C}^{[m',n]} \) are generic tensors. Each generic tensor was generated using \( \text{randn}(n, \cdots, n) + i \ast \text{randn}(n, \cdots, n) \) in MATLAB. The results are reported in Table 5, in which \( N \) denotes the number of equivalence classes of eigenpairs found by \( \text{teig} \) or \( \text{teneig} \), \( T(m, n) \) denotes the bound of the number of equivalence classes of isolated \( \mathcal{B} \)-eigenpairs for \( m = m' \), and \( G(m, m', n) \) denotes the bound of the number of equivalence classes of isolated \( \mathcal{B} \)-eigenpairs for \( m \neq m' \) (see Theorem 2.3).

From Table 5 we see that our \( \text{teig} \) and \( \text{teneig} \) find all equivalence classes of isolated \( \mathcal{B} \)-eigenpairs of \( \mathcal{A} \) for the generic tensors \( \mathcal{A} \) and \( \mathcal{B} \) we tested in a reasonable amount of time.
Table 4: Performance of \texttt{eeig} on computing E-eigenpairs of complex random tensors

| (m, n) | E(m, n) | N | time(s) | (m, n) | E(m, n) | N | time(s) |
|--------|---------|---|---------|--------|---------|---|--------|
| (3, 5) | 31      | 31| 1.4     | (3, 6) | 63      | 63| 3.1     |
| (3, 7) | 127     | 127| 7.5     | (3, 8) | 255     | 255| 20.3    |
| (3, 9) | 511     | 511| 48.5    | (3, 10)| 1023    | 1023| 133.9   |
| (4, 3) | 13      | 13| 0.4     | (4, 4) | 40      | 40| 1.7     |
| (4, 5) | 121     | 121| 5.4     | (4, 6) | 364     | 364| 26.9    |
| (4, 7) | 1093    | 1093| 119.5   | (4, 8) | 3280    | 3280| 555.8   |
| (5, 3) | 21      | 21| 0.7     | (5, 4) | 85      | 85| 4.2     |
| (5, 5) | 341     | 341| 22.3    | (5, 6) | 1365    | 1365| 166.5   |
| (5, 7) | 5461    | 5461| 1330.7  | (6, 3) | 31      | 31| 1.2     |
| (6, 4) | 156     | 156| 9.5     | (6, 5) | 781     | 781| 100.4   |
| (6, 6) | 3906    | 3906| 990.2   | (7, 3) | 43      | 43| 1.9     |
| (7, 4) | 259     | 259| 21.3    | (7, 5) | 1555    | 1555| 245.0   |

Table 5: Performance of \texttt{teig} and \texttt{teneig} on computing generalized eigenpairs of complex random tensors

| (m, n) | T(m, n) | N | time(s) | (m, m', n) | G(m, m', n) | N | time(s) |
|--------|---------|---|---------|------------|-------------|---|--------|
| (3, 7) | 448     | 448| 23.7    | (3, 2, 7)  | 127         | 127| 10.3   |
| (3, 8) | 1024    | 1024| 68.3    | (3, 4, 6)  | 665         | 665| 68.1   |
| (3, 9) | 2304    | 2304| 210.3   | (3, 5, 5)  | 496         | 496| 49.5   |
| (4, 5) | 405     | 405| 20.8    | (4, 2, 6)  | 364         | 364| 28.9   |
| (4, 6) | 1458    | 1458| 110.4   | (4, 3, 5)  | 211         | 211| 13.2   |
| (4, 7) | 5103    | 5103| 737.5   | (4, 5, 4)  | 175         | 175| 9.5    |
| (5, 5) | 1280    | 1280| 97.9    | (5, 4, 5)  | 781         | 781| 83.9   |
| (5, 6) | 6144    | 6144| 1178    | (5, 6, 3)  | 61          | 61| 2.6    |
| (6, 4) | 500     | 500| 29.9    | (6, 5, 4)  | 369         | 369| 30.7   |
| (6, 5) | 3125    | 3125| 449.4   | (6, 7, 3)  | 91          | 91| 6.0    |
| (7, 3) | 108     | 108| 4.4     | (7, 6, 4)  | 671         | 671| 77.1   |
| (7, 4) | 864     | 864| 77.6    | (7, 8, 3)  | 127         | 127| 9.4    |
5.2 Examples for Computing Real Eigenpairs

In this subsection, numerical examples are provided to illustrate the effectiveness and efficiency of \texttt{zeig} or \texttt{heig} for computing real Z-eigenpairs or H-eigenpairs of a tensor $A \in \mathbb{R}^{[m,n]}$. By Definition 2.1, $(\lambda, x)$ is a Z-eigenpair if and only if $((-1)^{m-2}\lambda, -x)$ is a Z-eigenpair, and $(\lambda, x)$ is an H-eigenpair if and only if $(\lambda, tx)$ is an H-eigenpair for any nonzero $t \in \mathbb{R}$. Only one representative from each equivalence class of eigenpairs will be listed in our examples. The notation $\lambda^{(l)}$ is used to denote $l$ eigenvectors counting multiplicities are found for the eigenvalue $\lambda$. In the following tables, the multiplicity of an eigenpair means the multiplicity of this eigenpair as a zero of the corresponding defining polynomial system. For the sake of conciseness, the polynomial system resulted from the tensor eigenvalue problem will be omitted.

**EXAMPLE 5.5** Consider the symmetric tensor $A \in \mathbb{R}^{[6,3]}$ whose corresponding polynomial form is the Motzkin polynomial:

$$Ax^6 = x_2^6 + x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2x_3^2.$$  

In Example 5.9 of [4], it states that this tensor has 25 equivalence classes of Z-eigenpairs. Using \texttt{zeig} exactly 25 equivalence classes of Z-eigenpairs are found as shown in Table 6, which confirms the results of [4]. \texttt{zeig} takes about 0.9 seconds to carry out the entire computation.

| $\lambda$ | 0$^{(14)}$ | 0.0156$^{(8)}$ | 0.2500$^{(2)}$ | 1  |
|-----------|-----------|----------------|----------------|----|
| $x_1$     | 0.5774    | 0.8253         | 0.2623         | 0  |
| $x_2$     | $\pm0.5774$ | $\pm0.2623$   | $\pm0.8253$   | $\pm0.7071$ |
| $x_3$     | $\pm0.5774$ | $\pm0.5000$   | $\pm0.5000$   | 0  |
| multiplicity | 1       | 5             | 5             | 1  |

Table 6: Z-eigenpairs of the tensor in Example 5.5

All the H-eigenpairs found by Example 4.10 of [11] are also found by \texttt{heig} as shown in Table 7. \texttt{heig} takes about 1.7 seconds to carry out the entire computation.

| $\lambda$ | 0$^{(14)}$ | 0.0555$^{(8)}$ | 1$^{(15)}$ |
|-----------|-----------|----------------|-----------|
| $x_1$     | $\pm1$    | $\pm0.4568$   | 1         |
| $x_2$     | 1         | 1              | $\pm0.4568$ |
| $x_3$     | $\pm1$    | $\pm0.6856$   | 1         |
| multiplicity | 1       | 5             | 5         |

Table 7: H-eigenpairs of the tensor in Example 5.5

So far the only available method for computing all real eigenvalues of a symmetric tensor is Algorithm 3.6 in [11]. In the next two examples, we compare the performance of our methods with that of Algorithm 3.6 ([11]).
EXAMPLE 5.6 In this example, we use our \texttt{zeig} to compute the Z-eigenvalues of 12 symmetric tensors from [11]. The test problems and numerical results are given in the Appendix. From the numerical results we see that \texttt{zeig} finds all the Z-eigenvalues found by Algorithm 3.6 ([11]) on this set of test problems. We now summarize the CPU times (in seconds) used by \texttt{zeig} and by Algorithm 3.6 ([11]) in Table 8. The CPU times by Algorithm 3.6 are from [11].

| Problem | \texttt{zeig} time(s) | Algorithm 3.6 ([11]) time(s) |
|---------|-----------------------|-----------------------------|
| 1       | 0.3                   | 9                           |
| 2       | 4                     | 400                         |
| 3       | 0.3–0.4               | 5–20                        |
| 4       | 0.1                   | 1                           |
| 5       | 0.6                   | 9                           |
| 6       | 1.8                   | 10870                       |
| 7       | 15.7                  | 280                         |
| 8       | 6.1                   | 320                         |
| 9       | 0.3                   | 1                           |
| 10      | 6.3                   | 370                         |
| 11      | 27.3                  | 170                         |
| 12      | 4.5                   | 420                         |

Table 8: \texttt{zeig} vs Algorithm 3.6 ([11]): CPU time

EXAMPLE 5.7 Consider the symmetric tensor $A \in \mathbb{R}^{[4,n]}$ (Example 4.16 in [11]) with the polynomial form

$$Ax^4 = (x_1 - x_2)^4 + \cdots + (x_1 - x_n)^4 + (x_2 - x_3)^4 + \cdots + (x_2 - x_n)^4 + \cdots + (x_{n-1} - x_n)^4.$$  

For different $n$, all the Z-eigenvalues found by Algorithm 3.6 in [11] are also found by \texttt{zeig}, which are given in Table 9. We want to point out that when $n = 8, 9, 10$, our \texttt{zeig} can find all the Z-eigenvalues in a reasonable amount of time, but [11] reports that Algorithm 3.6 can only find the first three largest Z-eigenvalues. The CPU times used by \texttt{zeig} and Algorithm 3.6 ([11]) are reported in the table. The CPU times by Algorithm 3.6 ([11]) are from [11]. For the sake of conciseness, the corresponding Z-eigenvectors are not displayed.

\footnote{The computer used in [11] is a Thinkpad W520 laptop with an Intel dual core CPU at 2.20GHz and 8 GB RAM. The computer used in this paper is a Thinkpad T400 laptop with an Intel dual core CPU at 2.80GHz and 2GB RAM. The reader should be cautious when comparing the CPU time usages of the methods since different computers were used.}

\footnote{Again, we should be cautious when comparing the CPU times used by the two methods because of different computers were used.}

30
Table 9: Z-eigenvalues of the tensor in Example 5.7 (* denotes that the CPU time used by Algorithm 3.6 ([11]) when it finds the first three largest Z-eigenvalues)

| n  | λ                      | time(s) | 
|----|------------------------|---------|
|    | zeig                   | Algorithm 3.6 ([11]) |
| 4  | 0.0000, 4.0000, 5.0000, 5.3333 | 1.7     |
| 5  | 0.0000, 4.1667, 4.2500, 5.5000, 6.2500 | 5.4     |
| 6  | 0.0000, 4.0000, 4.5000, 6.0000, 7.2000 | 15.5    |
| 7  | 0.0000, 4.0833, 4.1667, 4.7500, 4.8846, 4.9000, 6.5000, 8.1667 | 58.3    |
| 8  | 0.0000, 4.0000, 4.2667, 4.2727, 4.3333, 5.0000, 5.2609, 5.3333, 5.3333, 7.0000, 9.1429 | 244.1   |
| 9  | 0.0000, 4.0500, 4.1250, 4.5000, 5.2500, 5.6250, 5.7857, 7.5000, 10.1250 | 788.0   |
| 10 | 0.0000, 4.0000, 4.1667, 4.1818, 4.2500, 4.6667, 4.7500, 4.7593, 4.7619, 4.7619, 5.5000, 5.9808, 6.2500, 8.0000, 11.1111 | 2665.6  |

Acknowledgment. The authors are very grateful to Professor T. Y. Li for his encouragement and help. His constructive comments and suggestions have helped improve the content and presentation of the paper.

References

[1] E.L. Allgower and K. Georg, *Numerical Continuation Methods, an Introduction*, Springer Series in Comput. Math., Vol 13, Springer-Verlag (Berlin, Heidelberg, New York), 1990.

[2] D. N. Bernstein, The number of the roots of a system of equations, *Funct. Anal. Appl.*, 1975, 9:183-185.

[3] D.L. Bates, J.D. Hauenstein, A.J. Sommese and C.W. Wampler, *Numerically Solving Polynomial Systems with Bertini*, Society for Industrial and Applied Mathematics, Philadelphia, 2013.

[4] D. Cartwright and B. Sturmfels, The number of eigenvalues of a tensor, *Linear Algebra and its Applications*, 2013, 438: 942–952.

[5] K.C. Chang, K. Pearson and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, *Commun. Math. Sci.*, 2008, 6(5): 507–520.

[6] K.C. Chang, K. Pearson and T. Zhang, On eigenvalues of real symmetric tensors, *Journal of Mathematical Analysis and Applications*, 2009, 350: 416–422.
[7] K.C. Chang, L. Qi, and T. Zhang, A survey of the spectral theory of nonnegative tensors, *Numerical Linear Algebra with Applications*, 2013, 20: 891–912.

[8] D. A. Cox, J. Little, and D. O’Shea, *Using Algebraic Geometry*, 2nd ed., Springer-Verlag, New Yor, NY, 2005.

[9] D. A. Cox, J. Little, and D. O’Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 3rd ed., Springer-Verlag, Secucus, NJ, USA, 2007.

[10] J. Cooper and A. Duttle, Spectra of uniform hypergrphs, *Linear Algebra and its Applications*, 2012, 436: 3268–3292.

[11] C. Cui, Y.-H. Dai and J. Nie, All real eigenvalues of symmetric tensors, arXiv:1403.3720, 2014.

[12] S. Friedland, S. Gaubert and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, *Linear Algebra and Applications*, 2013, 438: 738–749.

[13] T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa and T. Mizutani, PHoM - A polyhedral homotopy continuation method, *Computing*, 2004, 73: 53-57.

[14] L. Han, An unconstrained optimization approach for finding real eigenvalues of even order symmetric tensors, *Numerical Algebra, Control and Optimization*, 2013, 3: 583–599.

[15] C. Hao, C. Cui, and Y.-H. Dai, A sequential subspace projection method for extreme Z-eigenvalues of supersymmetric tensors, *Numerical Linear Algebra with Applications*, 2015, 22(2): 287–298.

[16] J, Hauenstein, L. Oeding, G. Ottaviani, and A. Sommese, Homotopy techniques for tensor decomposition and perfect identifiability, arXiv1501.00090, 2015.

[17] R. Horn and C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

[18] C. Hillier and L.-H. Lim, Most tensor problems are NP-hard, *Journal of the ACM*, 2013, 60, no. 6, art. 45.

[19] S. Hu, Z. Huang, and L. Qi, Finding the extreme Z-eigenvalues of tensors via a sequential semidefinite programming method, *Numerical Linear Algebra with Applications*, 2013, 20: 972–984.

[20] S. Hu, Li. Qi, and B. Zhang, The geometric measure of entanglement of pure states with nonnegative amplitudes and the spectral theory of nonnegative tensors, arXiv:1203.3675, 2012.
[21] B. Huber and B. Sturmfels, A polyhedral method for solving sparse polynomial systems, *Math. Comp.*, 1995, 64: 1541-1555.

[22] T.G. Kolda and J.R. Mayo, Shifted power method for computing tensor eigenpairs, *SIAM Journal on Matrix Analysis and Applications*, 2011, 32: 1095–1124.

[23] T.G. Kolda and J.R. Mayo, An adaptive shifted power method for computing generalized tensor eigenpairs, arXiv: 1401.1183, 2014.

[24] T.L. Lee, T.Y. Li, and C.H. Tsai, Hom4PS-2.0, a software package for solving polynomial systems by the polyhedral homotopy continuation method, *Computing*, 2008, 83:109–133.

[25] T.Y. Li, Solving polynomial systems by the homotopy continuation method, *Handbook of Numerical Analysis*, XI, 2003, 209–304.

[26] G. Li, L. Qi, and G. Yu, The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory, *Numerical Linear Algebra with Applications*, 2013, 20:1001-1029.

[27] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP’05)*, 2005, 1: 129–132.

[28] T.Y. Li and X. Wang, The BKK root count in $\mathbb{C}^n$, *Math. Comp.*, 1996, 65: 1477-1484.

[29] Y. Liu, G. Zhou, and N. F. Ibrahim, An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor, *Journal of Computational and Applied Mathematics*, 2010, 235: 286–292.

[30] A.P. Morgan, *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems*, Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[31] M. Ng, L. Qi, and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SIAM Journal on Matrix Analysis and Applications*, 2009, 31: 1090–1099.

[32] J. Nie and L. Wang, Semidefinite relaxations for best rank-1 tensor approximations, *SIAM Journal on Matrix Analysis and Applications*, 2014, 35: 1155–1179.

[33] L. Oeding and G. Ottaviani, Eigenvectors of tensors and algorithms for Waring decomposition, *Journal of Symbolic Computation*, 2013, 54: 9–35.

[34] L. Qi, Eigenvalues of a real supersymmetric tensor, *Journal of Symbolic Computation*, 2005, 40: 1302–1324.

[35] L. Qi, W. Sun, and Y. Wang, Numerical multilinear algebra and its applications, *Frontiers of Mathematics in China*, 2007, 2: 501–526.
[36] L. Qi, F. Wang, and Y. Wang, Z-eigenvalue methods for a global optimization polynomial optimization problem, *Mathematical Programming*, 2009, 118: 301–306.

[37] L. Qi, Y. Wang, and E.X. Wu, D-eigenvalues of diffusion kurtosis tensors, *Journal of Computational and Applied Mathematics*, 2008, 221: 150–157.

[38] L. Qi, G. Yu, and E.X. Wu, Higher order positive semi-definite diffusion tensor imaging, *SIAM Journal on Imaging Sciences*, 2010, 3: 416–433.

[39] L. Qi, G. Yu, and Y. Xu, Nonnegative diffusion orientation distribution function, *Journal of Mathematical Imaging and Vision*, 2013, 45: 103-113.

[40] A.J. Sommese and W.W. Wampler, *The Numerical Solution of Systems of Polynomials Arising in Engineering And Science* World Scientific Pub Co Inc, 2005.

[41] J. Verschelde, Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation, *ACM Trans. Math. Softw.*, 1999, 25:251-276.

[42] A.H. Wright, Finding all solutions to a system of a polynomial equations, *Math. Comp.*, 1985, 44: 125-133.

[43] J. Xie and A. Chang, On the Z-eigenvalues of the signless Laplacian tensor for an even uniform hypergraph, *Numerical Linear Algebra with Applications*, 2013, 20: 1030-1045.

[44] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, *SIAM Journal on Matrix Analysis and Applications*, 2010, 31: 2517–2530.

[45] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, *SIAM Journal on Matrix Analysis and Applications*, 2011, 32: 1236–1250.

[46] Z. Zeng and T. Li, NACLab: A Matlab Toolbox for Numerical Algebraic Computation, *ACM Communications in Computer Algebra*, 2013, 47: 170-173.

[47] L. Zhang, L. Qi, and Y. Xu, Finding the largest eigenvalue of an irreducible tensor and linear convergence for weakly positive tensors, *Journal of Computational Mathematics*, 2012, 30: 24–33.

[48] G. Zhou, L. Qi, and S.-Y. Wu, On the largest eigenvalue of a symmetric nonnegative tensor, *Numerical Linear Algebra with Applications*, 2013, 20: 913–928.
Appendix

PROBLEM 1 Consider the symmetric tensor \( A \in \mathbb{R}^{[4,3]} \) (Example 4.1 in [11], see also [34]) with the polynomial form
\[
A x^4 = x_1^4 + 2x_2^4 + 3x_3^4.
\]
Using \texttt{zeig} all the Z-eigenpairs found in [11] are obtained (see Table 10). \texttt{zeig} takes about 0.3 seconds to carry out the entire computation.

| \( \lambda \) | 0.5455\(^{(1)} \) | 0.6667\(^{(2)} \) | 0.7500\(^{(2)} \) | 1.2\(^{(2)} \) | 2 | 3 |
|---|---|---|---|---|---|---|
| \( x_1 \) | 0.7385 | 0.8165 | 0.8660 | 1 | 0 | 0 |
| \( x_2 \) | \( \pm 0.5222 \) | \( \pm 0.5774 \) | 0 | 0 | 0.7746 | 1 |
| \( x_3 \) | \( \pm 0.4264 \) | 0 | \( \pm 0.5000 \) | 0 | \( \pm 0.6325 \) | 1 |
| multiplicity | 1 | 1 | 1 | 1 | 1 | 1 |

Table 10: Z-eigenpairs of the tensor in Problem 1

PROBLEM 2 For the diagonal tensor \( D \in \mathbb{R}^{[5,4]} \) (Example 4.2 in [11]) such that \( D x^5 = x_1^5 + 2x_2^5 - 3x_3^5 - 4x_4^5 \). Consider the symmetric tensor \( A \in \mathbb{R}^{[5,4]} \) such that \( A x^5 = D (Q x)^5 \) where
\[
Q = (I - 2w_1 w_1^T)(I - 2w_2 w_2^T)(I - 2w_3 w_3^T)
\]
and \( w_1, w_2, w_3 \) are randomly generated unit vectors. All the 30 Z-eigenpairs found in [11] are also found by using \texttt{zeig}. The 15 nonnegative Z-eigenvalues are listed below
\[
0.2518, \ 0.3261, \ 0.3466, \ 0.3887, \ 0.4805, \ 0.5402, \ 0.5550, \ 0.6057, \ 0.8543, \ 0.9611, \ 1.0000, \ 1.2163, \ 2.0000, \ 3.0000, \ 4.0000.
\]
For conciseness, the corresponding Z-eigenvectors are not displayed here. \texttt{zeig} takes about 4.0 seconds to do the entire computation.

PROBLEM 3 Consider the symmetric tensor \( A \in \mathbb{R}^{[4,3]} \) (Example 4.3 in [11], see also [34]) with the polynomial form
\[
A x^4 = 2x_1^4 + 3x_2^4 + 5x_3^4 + 4ax_4^2 x_2 x_3,
\]
where \( a \) is a parameter. All Z-eigenvalues found in [11] are also found by \texttt{zeig} for different values of \( a \), as shown in Table 11. For conciseness, the corresponding Z-eigenvectors are not displayed here. The CPU time used by \texttt{zeig} for each \( a \) is also reported in the table.

PROBLEM 4 Consider the symmetric tensor \( A \in \mathbb{R}^{[4,2]} \) (Example 4.4 in [11], see also [34]) with the polynomial form
\[
A x^4 = 3x_1^4 + x_2^4 + 6ax_4^2 x_2^2,
\]
where \( a \) is a parameter. All Z-eigenvalues found in [11] are also found by \texttt{zeig} for different values of \( a \), which are listed in Table 12. The CPU time used by \texttt{zeig} for each \( a \) is also given in the table. For conciseness, the corresponding Z-eigenvectors are not displayed here.
Table 11: Z-eigenvalues of the tensor in Problem 3

| \(a\) | \(\lambda\) | \(\text{time(s)}\) |
|--------|------------|------------------|
| 0.25   | 0.8464(2), 1.0881(2), 1.2150(2), 1.4412(2), 1.8750(2), 2, 3, 5 | 0.4 |
| 0.5    | 0.7243(2), 1.2069(2), 1.2593(2), 1.4783(2), 1.8750(2), 2, 3, 5 | 0.4 |
| 1      | 0.4787(2), 1.6133(2), 1.8750(2), 2, 3, 5 | 0.3 |
| 3      | −0.5126(2), 1.8750(2), 2, 2.2147(2), 3, 5 | 0.3 |

Table 12: Z-eigenvalues of the tensor in Problem 4

| \(a\) | \(\lambda\) | \(\text{time(s)}\) |
|--------|------------|------------------|
| −1     | −0.6000(2), 1, 3 | 0.1 |
| 0      | 0.7500(2), 1, 3 | 0.1 |
| 0.25   | 0.9750(2), 1, 3 | 0.1 |
| 0.5    | 1, 3 | 0.1 |
| 2      | 1, 3, 4.1250(2) | 0.1 |
| 3      | 1, 3, 5.5714(2) | 0.1 |

PROBLEM 5 Consider the symmetric tensor \(A \in \mathbb{R}^{[4, 3]}\) (Example 4.5 in [11], see also [22] or [32]) such that

\[
\begin{align*}
A_{1111} &= 0.2883, \quad A_{1112} = -0.0031, \quad A_{1113} = 0.1973, \quad A_{1122} = -0.2485, \\
A_{1123} &= -0.2939, \quad A_{1133} = 0.3847, \quad A_{1222} = 0.2972, \quad A_{1223} = 0.1862, \\
A_{1233} &= 0.0919, \quad A_{1333} = -0.3619, \quad A_{2222} = 0.1241, \quad A_{2223} = -0.3420, \\
A_{2233} &= 0.2127, \quad A_{2333} = 0.2727, \quad A_{3333} = -0.3054.
\end{align*}
\]

All the Z-eigenpairs found in [11] are also found by \texttt{zeig}, as given in Table 13. \texttt{zeig} takes about 0.6 seconds to do the entire computation.

Table 13: Z-eigenpairs of the tensor in Problem 5

| \(\lambda\) | \(x_1\) | \(x_2\) | \(x_3\) |
|------------|--------|--------|--------|
| -1.0954    | -0.5629 | -0.0451 | 0.1735 |
| -0.5915    | -0.1762 | 0.7797 | 0.3357 |
| 0.7467     | 0.1796 | 0.6135 | 0.9073 |
| 0.3045     | -0.9678 | 0.1250 | 0.2531 |
| -1.0888    | 0.8870 | 0.7160 | -0.5150 |
| -0.8169    | -0.4722 | 0.7027 | 0.8893 |

PROBLEM 6 Consider the symmetric tensor \(A \in \mathbb{R}^{[3, 6]}\) (Example 4.6 in [11], see also [36]) such that \(A_{iii} = i\) for \(i = 1, \ldots, 6\) and \(A_{i,i,i+1} = 10\) for \(i = 1, \ldots, 5\) and zero otherwise. All the Z-eigenvalues found in [11] are also found by \texttt{zeig}. The 19 nonnegative
Z-eigenvalues are listed below:

|       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|
| 3.9992 | 4.0225 | 4.2464 | 4.3358 | 5.1402 | 5.4817 | 5.5218 |
| 5.5674 | 6.0000 | 7.2165 | 8.1889 | 8.5979 | 8.6596 | 8.7347 |
| 15.4298 | 15.4552 | 16.2345 |       |       |       |       |

For conciseness, the corresponding Z-eigenvectors are not displayed here. \texttt{zeig} takes about 1.8 seconds to carry out the entire computation.

**PROBLEM 7** Consider the symmetric tensor $A \in \mathbb{R}^{[4,6]}$ (Example 4.7 in [11], see also [26]) with the polynomial form

$$-Ax^4 = (x_1 - x_2)^4 + (x_1 - x_3)^4 + (x_1 - x_4)^4 + (x_1 - x_5)^4 + (x_1 - x_6)^4$$
$$+ (x_2 - x_3)^4 + (x_2 - x_4)^4 + (x_2 - x_5)^4 + (x_2 - x_6)^4$$
$$+ (x_3 - x_4)^4 + (x_3 - x_5)^4 + (x_3 - x_6)^4$$
$$+ (x_4 - x_5)^4 + (x_4 - x_6)^4 + (x_5 - x_6)^4.$$

All the 5 Z-eigenvalues found in [11] are also found by \texttt{zeig}, which are given in Table 14.

As pointed out in [11], every permutation of a Z-eigenvector is also a Z-eigenvector. Only

| $\lambda$     | $x^4$                              | multiplicity |
|--------------|------------------------------------|--------------|
| $-7.2000^{(6)}$ | (0.1826, 0.1826, 0.1826, 0.1826, 0.1826, 0.1826, -0.9129) | 1            |
| $-6.0000^{(15)}$ | (0.7071, 0, 0, 0, 0, -0.7071)       | 1            |
| $-4.5000^{(*)}$ | (0.5774, 0.5774, -0.2887, -0.2887, -0.2887, -0.2887) | -            |
| $-4.0000^{(10)}$ | (0.4082, 0.4082, 0.4082, -0.4082, -0.4082, -0.4082) | 1            |
| 0$^{(*)}$    | (0.4082, 0.4082, 0.4082, 0.4082, 0.4082, 0.4082)      | -            |

Table 14: Z-eigenpairs of the tensor in Problem 7

the Z-eigenvector with $x_1 \geq x_2 \geq \cdots \geq x_6$ corresponding to one Z-eigenvalue is listed. We remark that the Z-eigenpairs corresponding to Z-eigenvalues 0 and $-4.5$ are in a positive dimensional solution component of the corresponding polynomial system. Therefore, there are infinitely many Z-eigenvectors associated with 0 and $-4.5$. \texttt{zeig} finds 484 Z-eigenvectors associated with 0 and 180 Z-eigenvectors associated with $-4.5$. Only one of these Z-eigenvectors for each case is listed in Table 14. \texttt{zeig} takes about 15.7 seconds to do the entire computation.

**PROBLEM 8** Consider the symmetric tensor $A \in \mathbb{R}^{[4,5]}$ (Example 4.8 in [11], see also [43]) with the polynomial form

$$Ax^4 = (x_1 + x_2 + x_3 + x_4)^4 + (x_2 + x_3 + x_4 + x_5)^4.$$

All the 3 Z-eigenvalues found in [11] are also found by \texttt{zeig}, which are shown in Table 15.
We remark that the Z-eigenpairs corresponding to Z-eigenvalue 0 are in a positive dimensional solution component of the corresponding polynomial system. Thus, there are infinitely many Z-eigenvectors associated with Z-eigenvalue 0. 

PROBLEM 9 Consider the symmetric tensor $A \in \mathbb{R}^{[3,3]}$ (Example 4.9 in [11], see also [4]) with the polynomial form

$$Ax^3 = 2x_1^3 + 3x_1x_2^2 + 3x_1x_3^2.$$ 

We remark that the Z-eigenpairs corresponding to Z-eigenvalue 2 are in a positive dimensional solution component of the corresponding polynomial system. Thus, there are infinitely many Z-eigenvectors associated with Z-eigenvalue 0. 

\[zeig\text{ finds 7 of them. Only one of them is listed in Table 16. \textit{zeig} uses about 0.3 seconds to do the entire computation.}\]

\[
\begin{array}{|c|c|c|}
\hline
\lambda & x^T & \text{multiplicity} \\
\hline
2^{(*)} & (1,0,0) & - \\
\hline
\end{array}
\]

Table 16: Z-eigenpairs of the tensor in Problem 9

PROBLEM 10 Consider the tensor $A \in \mathbb{R}^{[4,n]}$ (Example 4.12 in [11], see also [32]) such that

$$A_{i_1,...,i_4} = \sin(i_1 + i_2 + i_3 + i_4).$$

When $n = 5$, all the 5 Z-eigenvalues found in [11] are also found by \textit{zeig}, which are given in Table 17.

We remark that the Z-eigenpairs corresponding to Z-eigenvalue 0 are in a positive dimensional solution component of the corresponding polynomial system. Thus, there are infinitely many Z-eigenvectors associated with 0. 

\[\textit{zeig} finds 234 of them. Only one of them is listed in Table 17. \textit{zeig} takes about 6.3 seconds to carry out the entire computation.\]

PROBLEM 11 Consider the tensor $A \in \mathbb{R}^{[4,n]}$ (Example 4.13 in [11]) such that

$$A_{i_1,...,i_4} = \tan(i_1) + \cdots + \tan(i_4).$$
When \( n = 6 \), all the 3 Z-eigenvalues found in [11] are also found by \texttt{zeig}, which are given in Table 18.

We remark that the Z-eigenpairs corresponding to Z-eigenvalue 0 are in a positive dimensional solution component of the corresponding polynomial system. Thus, there are infinitely many Z-eigenvectors associated with 0. \texttt{zeig} finds 724 of them. Only one of them is listed in Table 18. It takes \texttt{zeig} about 27.3 seconds to carry out the entire computation.

**PROBLEM 12** Consider the tensor \( A \in \mathbb{R}^{[5,n]} \) (Example 4.14 in [11]) such that

\[
A_{i_{1},...,i_{5}} = \ln(i_{1}) + \cdots + \ln(i_{5}).
\]

For \( n = 4 \), all the 3 Z-eigenvalues found in [11] are also found by \texttt{zeig}, which are shown in Table 19.

We remark that the Z-eigenpairs corresponding to Z-eigenvalue 0 are in a positive dimensional solution component of the corresponding polynomial system. Thus, there are infinitely many Z-eigenvectors associated with 0. \texttt{zeig} finds 166 of them. Only one of them is listed in Table 18. The entire computation takes \texttt{zeig} about 4.5 seconds.