ON THE VIOLATION OF THURSTON-BENNEQUIN INEQUALITY FOR A CERTAIN NON-CONVEX HYPERSURFACE

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Abstract. We show that any open subset of a contact manifold of dimension greater than three contains a certain hypersurface $\Sigma$ which violates the Thurston-Bennequin inequality. We also show that no convex hypersurface smoothly approximates $\Sigma$. These results contrasts with the 3-dimensional case, where any surface in a small ball satisfies the inequality (Bennequin[1]) and is smoothly approximated by a convex one (Giroux[5]).

1. Introduction and preliminaries

Tightness is a fundamental notion in 3-dimensional contact topology. It is characterized by the Thurston-Bennequin inequality (see §1.1). Roughly, for a Seifert (hyper)surface $\Sigma$ in a contact manifold, this inequality compares the contact structure along $\Sigma$ with the tangent bundle of $\Sigma$ by means of relative euler number. An overtwisted disk is a 2-disk equipped with a certain germ of 3-dimensional contact structure, for which the inequality fails. A contact manifold is said to be tight if it contains no embedded overtwisted disks. Then the inequality automatically holds for any $\Sigma$ (Eliashberg[3]). The 1-jet space $J^1(\mathbb{R}, \mathbb{R})$ for a function $f : \mathbb{R} \to \mathbb{R}$ is tight (Bennequin[1]). Thus all contact 3-manifolds, which are modelled on $J^1(\mathbb{R}, \mathbb{R})$, are locally tight.

Giroux[5] smoothly approximated a given compact surface in a contact 3-manifold by a surface with certain transverse monotonicity, i.e., a convex surface (see §1.2 for the precise definition). Thus, in 3-dimensional case, we may easily consider that the above inequality is only for convex surfaces. Contrastingly, in higher dimension, we show the following theorem.

Theorem. In the case where $n > 1$, any open subset of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$, which is the model space for contact $(2n+1)$-manifolds, contains a hypersurface $\Sigma$ such that

1. $\Sigma$ violates the Thurston-Bennequin inequality, and
2. no convex hypersurface smoothly approximates $\Sigma$.

This leads us to seriously restrict the inequality to convex hypersurfaces (see [7] for a sequel).

1.1. Thurston-Bennequin inequality. Let $\Sigma$ be a compact oriented hypersurface embedded in a positive contact manifold $(M^{2n+1}, \alpha)$ ($\alpha \land (d\alpha)^n > 0$) which tangents to the contact structure $\ker \alpha$ at finite number of interior points. Let $S_+(\Sigma)$ (resp. $S_-(\Sigma)$) denote the set of positive (resp. negative) tangent points. With respect to the symplectic structure $d\alpha|_{\ker \alpha}$, the symplectic orthogonal of $T\Sigma \cap \ker \alpha$ defines a singular line field $L \subset T\Sigma$. The integral foliation $F_{\Sigma}$ of $L$ on $\Sigma$ is called the characteristic foliation. The singularity of $L$ coincides with $S_+(\Sigma) \cup S_-(\Sigma)$. The restriction $\gamma = \alpha|_{T\Sigma}$ defines a holonomy invariant transverse contact structure of $F_{\Sigma}$ and determines the orientation of $L$ (i.e., $X \in L_{\gamma} \iff \iota_X(\partial/d\alpha)^n = \gamma \land (d\gamma)^n \iff \gamma \land L_X \gamma = 0$). We define the index $\text{Ind} p = \text{Ind}_L p$ of each tangent point $p \in S_\pm(\Sigma)$ by regarding it as a singular point of $L$. Assume that the boundary of each connected component of $\Sigma$ is non-empty and $L$ is outwards transverse to $\partial \Sigma$. Then the boundary $\partial \Sigma$ is said to be contact-type.

The unit 2-disk $D^2$ equipped with the germ of contact structure $\ker \{(2r^2 - 1)dz + r^2(v^2 - 1)d\theta\}$ is called an (the) overtwisted disk, where $(r, \theta, z)$ is the cylindrical coordinates of $D^2 \times \mathbb{R}$. Slightly extending $D^2$, we obtain a disk with contact-type boundary such that the singularity of the characteristic foliation is a single negative sink point. A contact 3-manifold is said to be overtwisted.

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or tight depending on whether it contains an embedded overtwisted disk (with the same germ as above) or not. Let Σ be any surface with contact-type boundary embedded in the 1-jet space $J^1(\mathbb{R}, \mathbb{R})(\approx \mathbb{R}^3)$ equipped with the canonical contact form. Then Bennequin [1] proved the following inequality which immediately implies the tightness of $J^1(\mathbb{R}, \mathbb{R})$:

\textbf{Thurston-Bennequin inequality.} \[ \sum_{p \in S_-(\Sigma)} \text{Ind} \ p \leq 0. \]

Eliashberg proved the same inequality for symplectically fillable contact 3-manifolds ([2]), and finally for all tight contact 3-manifolds ([3]). The inequality can be written in terms of relative euler number. Let $X$ be the above vector field on a hypersurface $\Sigma \subset (M^{2n+1}, \alpha)$ with contact-type boundary. Then, since $X \in T\Sigma \cap \ker \alpha$, we can regard $X$ as a section of $\ker \alpha|\Sigma$ which is canonical near the boundary $\partial \Sigma$. Thus we can define the relative euler number of $\ker \alpha|\Sigma$ by

\[ \langle e(\ker \alpha), [\Sigma, \partial \Sigma] \rangle = \sum_{p \in S_+(\Sigma)} \text{Ind} \ p - \sum_{p \in S_-(\Sigma)} \text{Ind} \ p. \]

Then the Thurston-Bennequin inequality may be expressed as

\[ -\langle e(\ker \alpha), [\Sigma, \partial \Sigma] \rangle \leq -\chi(\Sigma). \]

There is also an absolute version of the Thurston-Bennequin inequality which is expressed as $|\langle e(\ker \alpha), [\Sigma] \rangle| \leq -\chi(\Sigma)$, or equivalently

\[ \sum_{p \in S_-(\Sigma)} \text{Ind} \ p \leq 0 \quad \text{and} \quad \sum_{p \in S_+(\Sigma)} \text{Ind} \ p \leq 0 \]

for any closed hypersurface $\Sigma$ with $\chi(\Sigma) \leq 0$. This holds if the euler class $e(\ker \alpha)$ is a torsion, especially if $H^{2n}(M; \mathbb{R}) = 0$. Note that the inequality and its absolute version can be defined for any oriented plane field on $M^3$ (see Eliashberg-Thurston [4]). They are originally proved for codimension 1 foliations on $M^3$ without Reeb components by Thurston (see [9]).

\begin{enumerate}
\item \textbf{1.2. Convex hypersurfaces.} A vector field $X$ on $(M^{2n+1}, \alpha)$ is said to be contact if the Lie derivative $\mathcal{L}_X \alpha$ vanishes on $\ker \alpha$. Let $\mathcal{V}$ denote the space of contact vector fields on $(M^{2n+1}, \alpha)$. We can see that the linear map $\alpha(\cdot) : \mathcal{V} \to C^\infty(M^{2n+1})$ is an isomorphism. The function $\alpha(X)$ is called the contact Hamiltonian function of $X$. A closed oriented hypersurface $\Sigma \subset (M^{2n+1}, \alpha)$ is said to be convex if there exists a contact vector field $Y$ on a neighbourhood $\Sigma \times (-\varepsilon, \varepsilon)$ of $\Sigma$ for $\varepsilon > 0$ with $Y = \partial / \partial z$ ($z \in (-\varepsilon, \varepsilon)$), i.e., $Y$ is positively transverse to $\Sigma$ (Giroux [2]). By perturbing the contact Hamiltonian function if necessary, we may assume that $\Gamma = \{ \alpha(Y) = 0 \}$ is a hypersurface transverse to $\Sigma$. Then $\Gamma$ separates $\Sigma$ into the \textit{positive region} $\Sigma_+ = \{ \alpha(Y) \geq 0 \}$ and the \textit{negative region} $\Sigma_- = \{ \alpha(Y) \leq 0 \}$ so that $\Sigma = \Sigma_+ \cup (-\Sigma_-)$. Each interior int $\Sigma_\pm$ has the positive exact symplectic form $\omega = \pm \alpha(Y) \alpha$. We can modify the function $\frac{1}{\alpha(Y)}$ near $\Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}_{>0} \cup \{ \infty \}$ near $\Gamma$ to obtain a function $f : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}_{>0}$ such that $d(f \alpha)|\text{int} \Sigma_\pm$ are symplectic and $f \alpha$ is $\mathbb{R}$-invariant. (This is the “transverse monotonicity” of $\Sigma$.) Note that the dividing set $\Gamma \cap \Sigma$ is then the convex ends of the exact symplectic manifolds int $\Sigma_\pm$.

A \textit{convex hypersurface with contact-type boundary} is a connected hypersurface $\Sigma$ which admits a transverse contact vector field $X$ such that, for the associated decomposition $\Sigma = \Sigma_+ \cup (-\Sigma_-)$, int $\Sigma_\pm$ are also convex exact symplectic manifolds, and the contact-type boundary $\partial \Sigma_\pm = \partial \Sigma_+ \cup \partial \Sigma_-$ is non-empty. (Changing $X$ if necessary, we can assume moreover that the dividing set $\Gamma \cap \Sigma$ contains $\partial \Sigma_\pm$.) Then the Thurston-Bennequin inequality can be expressed as follows.

\textbf{Thurston-Bennequin inequality for convex hypersurfaces.} $\chi(\Sigma_-) \leq 0$.

Slightly extending the overtwisted disk $D^2$, we obtain a convex disk $\Sigma$ which is the union $\Sigma_+ \cup (-\Sigma_-)$ of a disk $\Sigma_-$ and an annulus $\Sigma_+$ surrounding $\Sigma_-$. Then $\Sigma$ violates the Thurston-Bennequin inequality ($\chi(\Sigma_-) = 1 > 0$), and is called a convex overtwisted disk. A possible higher dimensional overtwisted convex hypersurface would also satisfy $\chi(\Sigma_-) > 0$ and $\partial \Sigma_+ \setminus \partial \Sigma_- \neq \emptyset$. Particularly $\partial \Sigma_+$ would have to be disconnected (see [7]).
2. Proof of Theorem

We show the following Proposition.

Proposition. Let \((M^3, \alpha)\) be an overtwisted contact 3-manifold and \(B^2_{\epsilon}\) the \(\epsilon\)-ball in \(\mathbb{R}^{2n-2}\) \((0 < \epsilon \ll 1)\). Then there exist a closed hypersurface \(\Sigma\) and a hypersurface \(\tilde{\Sigma}\) with contact-type boundary in the product contact \((2n+1)\)-manifold

\[
\left( M^3 \times B^2_{\epsilon} \right) \ni (p, (x_1, y_1, \ldots , x_{n-1}, y_{n-1})) \Rightarrow \beta = \pi^* \alpha + \sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i)
\]

such that \(\tilde{\Sigma}\) and \(\Sigma\) are not convex, \(\Sigma \subset \tilde{\Sigma}\), and \(\Sigma\) violates the Thurston-Bennequin inequality, where \(\pi\) denotes the natural projection to \(M^3\).

Proof. Let \((r, \theta, z)\) be the cylindrical coordinates of \(\mathbb{R}^3\), and consider the functions

\[
\lambda(r) = 2r^2 - 1 \quad \text{and} \quad \mu(r) = r^2(r^2 - 1).
\]

Then we see that the contact structure on \(\mathbb{R}^3\) defined by the contact form

\[
\alpha' = \lambda(r) dz + \mu(r) d\theta
\]

is overtwisted. An overtwisted disk in \((M^3, \alpha)\) has a neighbourhood which is contactomorphic to \(U = \{\epsilon^{-2}z^2 + r^2 < 1 + 2\epsilon\} \subset (\mathbb{R}^3, \alpha')\). Thus, by using the formula

\[
f^2 \sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i) = \sum_{i=1}^{n-1} (f x_i d(f y_i) - f y_i d(f x_i)) \quad (\forall f \in C^\infty (M^3 \times \mathbb{R}^{2n-1}))
\]

we can replace \((M^3, \alpha)\) in Proposition with \((U, \alpha'|U)\). Then we take the hypersurface

\[
\tilde{\Sigma} = \left\{ (z, r, \theta, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \mid r^2 + \epsilon^{-2} \left( z^2 + \sum_{i=1}^{n-1} (x_i^2 + y_i^2) \right) = 1 + \epsilon \right\}
\]

and its subset

\[
\Sigma = \left\{ (z, r, \theta, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in \tilde{\Sigma} \mid r - z \leq 1 \right\}.
\]

We orient \(\tilde{\Sigma}\) so that the characteristic foliation \(\mathcal{F}_{\tilde{\Sigma}}\) is presented by the vector field

\[
X = \epsilon^{-2} r (r^2 - 1) z \partial_r + (1 + 2\epsilon - \epsilon^{-2}z^2) \partial_\theta + \left\{ (r^2 - 1)^2 + (2r^2 - 1)(\epsilon^{-2}z^2 - \epsilon) \right\} \partial_z
\]

\[
+ \epsilon^{-2} (2r^2 - 1) z \sum_{i=1}^{n-1} (x_i \partial_{x_i} + y_i \partial_{y_i}) + \epsilon^{-2} (2r^4 - 2r^2 + 1) \sum_{i=1}^{n-1} (-y_i \partial_{x_i} + x_i \partial_{y_i}).
\]

Indeed the following calculations shows that the vector field \(X\) satisfies \(X \in T\tilde{\Sigma}\), \(\beta(T\tilde{\Sigma})(X) = 0\), and \(\mathcal{L}_X (\beta|T\tilde{\Sigma}) = 2\epsilon^{-2}(2r^2 - 1)z\beta|T\tilde{\Sigma}\).

\[
\left\{ 2rdr + \epsilon^{-2} \left( 2zdz + 2 \sum_{i=1}^{n-1} (x_i dx_i + y_i dy_i) \right) \right\} (X)
\]

\[
= 2\epsilon^{-2}(2r^2 - 1)z \left\{ r^2 + \epsilon^{-2} \left( z^2 + \sum_{i=1}^{n-1} (x_i^2 + y_i^2) \right) - 1 - \epsilon \right\},
\]

\[
\beta = (2r^2 - 1)dz + r^2(r^2 - 1)d\theta + \sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i),
\]

\[
\beta(X) = (2r^2 - 1) \left\{ (r^2 - 1)^2 + (2r^2 - 1)(\epsilon^{-2}z^2 - \epsilon) \right\}
\]

\[
+ r^2(r^2 - 1)(1 - 2(\epsilon^{-2}z^2 - \epsilon)) + \epsilon^{-2} (2r^4 - 2r^2 + 1) \sum_{i=1}^{n-1} (x_i^2 + y_i^2)
\]

\[
= (2r^4 - 2r^2 + 1) \left\{ r^2 + \epsilon^{-2} \left( z^2 + \sum_{i=1}^{n-1} (x_i^2 + y_i^2) \right) - 1 - \epsilon \right\},
\]
The assumption implies that the singularity consists of the following five points; two (quarter-) elliptic points of the hyperbolic singular point of \( \Sigma \) with attention to the vector field which is the self-intersection of the leaf corresponding to \( C \). They are respectively a source point and a sink point (see Figure 1). Since the indices of these points are equal to 1, the hypersurface \( \Sigma \) violates the Thurston-Bennequin inequality.

Precisely, Figure 1 depicts (the fourfold covering of) the well-defined push-forward \( \mathcal{F}_\Sigma \) of \( X \) under the natural projection \( p \) from \( \Sigma \) to the quarter-sphere

\[
\Sigma' = \left\{ (z, r, |(x, y)|) \mid r^2 + \varepsilon^{-2}(z^2 + |(x, y)|^2) = 1 + \varepsilon \right\} \quad (r \geq 0, \ |(x, y)| \geq 0).
\]

The vector field \( X' \) defines the singular foliation \( \mathcal{F}' = \{ \varepsilon^{-2}z^2 = (Cr^2 - 1)(r^2 - 1) + \varepsilon \}_{-\infty \leq C \leq +\infty} \). The singularity consists of the following five points; two (quarter-)elliptic points \( \pm \varepsilon \sqrt{1 + \varepsilon}, 0, 0 \) whose preimages under \( p \) are the above singular points; other two (half-)elliptic points \( \pm \varepsilon \sqrt{1 + \varepsilon}, 1, 0 \) whose preimages are the periodic orbits \( P_\pm = \{ \pm \varepsilon \sqrt{1 + \varepsilon} \times S^1(1) \times \{ 0 \} \subset \Sigma \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2n-2} \} \) of \( X \) \((P_+ \subset \Sigma, P_- \subset \Sigma - \Sigma)\); and a hyperbolic point \( \left( 0, r_0 = \sqrt{1 + \varepsilon - \sqrt{\varepsilon(1 + \varepsilon)}}, n_0 = \sqrt{\varepsilon^2 \sqrt{\varepsilon(1 + \varepsilon)}} \right) \), which is the self-intersection of the leaf corresponding to \( C = 1 + 2\varepsilon + 2\sqrt{\varepsilon(1 + \varepsilon)} \). Slightly changing \( \varepsilon \) if necessary, we may assume that the preimage \( H = p^{-1}\left( \{ (0, r_0, n_0) \} \right) = \{ 0 \} \times S^1(r_0) \times S^{2n-1}(n_0) \) of the hyperbolic singular point of \( X' \) is a union of periodic orbits of \( X \).

Now we assume that \( \Sigma \) is (approximately) convex in order to prove Proposition by contradiction. The assumption implies

\[
S_i(\Sigma), \ P_i \subset \tilde{\Sigma}_i \quad (i = +, -) \quad \text{and} \quad \Gamma \cap H = \emptyset
\]
where $\tilde{\Sigma}_\pm$ are the $\pm$ regions of the convex surface $\tilde{\Sigma}$ divided by the transverse intersection with the level set $\Gamma$ of the contact Hamiltonian function described in §1.2. Then we can see that the intersection $\Gamma \cap \tilde{\Sigma}$ contain a spherical component. However the Eliashberg-Floer-McDuff theorem implies that $S^{2n-1} \bigsqcup$ (other components) can not be realized as the boundary of a connected convex symplectic manifold (see McDuff[6]). This contradiction proves Proposition. Here we omit a similar proof of the non-convexity of the Seifert hypersurface $\Sigma$.

Theorem in §1 is deduced from Proposition and the following easy lemma (see [8] for a proof).

**Lemma.** There exists an embedded overtwisted contact $S^3$ topologically unknotted in $J^1(\mathbb{R}^2, \mathbb{R})$.

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