Transmission time and resonant tunneling through barriers using localized quantum density soliton waves

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Abstract
In this paper, the interaction and transmission time of quantum density solitons waves passing through finite barrier potentials is investigated. Using the conservation of energy and of quantum density, it is first demonstrated that the soliton waves possess the important particle-like properties, including localization by a finite de Broglie wavelength and constant uniform motion in free space. The passage of quantum density solitons through barriers of finite energies is then shown to lead to the phenomena of resonant tunneling and, in Josephson-like configurations, to the quantization of magnetic flux. A precise general measure for barrier tunneling time is derived which is found to give a new interpretation of the quantum indeterminacy principles.

Keywords: Quantum tunneling; transit time; charge density waves; flux quantization; uncertainty relations; quantum solitons; Josephson junction; NbSe3; coherent microstructures; CDW.

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1 Introduction

Particles in quantum theory are represented by Gaussian wave packets solutions of the linear Schrödinger equation. However it is well known that Gaussian wave packets, in the absence of an external force, exhibit dispersion over time. This lack of localizability is due to the fact that a Gaussian wave packet is formed by a linear superposition of stationary states that decay exponentially with time, hence collectively cause the wave packet dispersion [1]. The only other known free wave packet solution of the linear Schrödinger equation is the Airy-Berry wave packet [2], which although possesses the non-dispersion feature of quantum particles, however exhibits acceleration in the absence of any external force.

It is shown here that non-spreading and other features of particles can be described by a nonlinear solitary wave phenomenon using quantum potential
formulation of basic quantum theory. We show here that these soliton waves not only remain localized in free space, but also possess a finite wavelength equal to the de Broglie wavelength, and move with a constant uniform speed in free space, hence preserve typical particle properties. We demonstrate the physical validity of these soliton waves by applying them to a number of cases that have been lab tested. In particular we derive a precise new measure of particle tunneling time through a finite barrier, where despite many alternative definitions a general measure of tunneling time is still lacking (see for instance Refs. [3], and for further references on this topic).

The paper is organized as follows. First the coupled energy and continuity equations, involving the quantum potential term, are solved for the localized soliton waves in section 2. The solution gives the quantum density function as the localized soliton wave, with wavelength equal to the de Broglie wavelength, which moves at a constant uniform speed. We then apply, in section 3 and 4, the resulting soliton solution to study the case of a free particle interacting with a finite potential well, where now the particle is represented by the quantum density soliton. The correct tunneling condition, as well as the flux quantization condition, are shown to follow simply from the soliton wave behavior of particles. In section 5, a new measure for the tunneling time is derived which can be regarded as a test of the proposed soliton behavior. Finally, it is shown that the indeterminacy principles is a direct consequence of the soliton phenomenon, with the natural assumption that the transversal time must remain non-negative in the tunneling process. In the following we shall keep to the one space dimension only, whose extension to two or three dimensions is rather obvious.

2 Quantum Density Solitons

Quantum description of a system is given by the continuity equation for the quantum density function $\rho(x, t)$:

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial [\rho(x, t)u(x, t)]}{\partial x} = 0, \quad (1)$$

and the energy equation: $E = \frac{1}{2}mu^2 + Q(x, t) + V(x)$, involving the quantum potential $Q(x, t) = -(\hbar^2/2m)(\partial^2 \sqrt{\rho(x, t)}/\partial x^2)/\sqrt{\rho(x, t)}$, and the classical potential function $V(x)$. The energy condition corresponds to the dynamical equation:

$$m \left[ \frac{\partial u(x, t)}{\partial t} + u(x, t) \frac{\partial u(x, t)}{\partial x} \right] = -\frac{\partial Q(x, t)}{\partial x} - \frac{\partial V(x)}{\partial x}, \quad (2)$$

where $u(x, t)$ is the particle/wave speed [4-7]. Here the term $\partial V(x)/\partial x$ is the classical force, which vanishes for the free particle case, thus $V$ is a constant.

Introducing the traveling wave variable $\xi = x - ct$, such that $\rho(x, t) = \rho(\xi)$, and $u(x, t) = u(\xi)$ and re-writing $Q(x, t)$, as the quantum potential divided by the mass $m$, we obtain:

$$\rho(\xi)u'(\xi) + \rho'(\xi)u(\xi) = c\rho'(\xi), \quad (3)$$
where the prime denotes differentiation with respect to the travelling wave variable \( \xi \).

We now solve equations (3) and (4) for the localized soliton waves. Although a large class of soliton waves have been discovered, in a variety of nonlinear problems, localized (or compact) soliton waves correspond to nonspreading waves of finite wavelength, free from spatial extensions, such as exponential tails or wings \[8\]. This is due to the restriction on the domain of periodic functions by a compact support.

For the QHD equations, we assume that the quantum density function \( \rho \) represents the particle-like localization, therefore, we take the ansatz for the soliton solution:

\[
\rho(\xi) = \rho_0 \cos(\beta \mu \xi), \quad |\mu \xi| < \pi/2, \\
= 0, \quad \text{elsewhere.} \tag{5}
\]

Here the inequality \(|\mu \xi| < \pi/2\) represents the compact support for the cosine function, which localizes the quantum density. Putting \( \rho(\xi) \) from equation (5) in equations (3) and (4) yields \( u(\xi) = c \) and \( \beta = 0, 2 \) and \( \mu = \sqrt{A} \) where the constant of integration \( A \) is to be identified, using equation (4), as the initial form of the quantum potential \( Q(0, 0) = Q_0 \).

This implies that the quantum hydrodynamic equations allow a constant quantum density (for \( \beta = 0 \)), as in the usual linear quantum theory. However there is another possible solution corresponding to \( \beta = 2 \), which yields a localized soliton density wave, traveling with a constant uniform speed \( c \). Thus in the cosine representation of the solitary wave, the localized soliton solution is given by:

\[
\rho(\xi) = \rho_0 \cos^2 \left( \frac{\sqrt{2mQ_0}}{\hbar} \xi \right), \quad |\mu \xi| < \pi/2, \\
= 0, \quad \text{elsewhere.} \tag{6}
\]

It is easily verified that the quantum density function (6), with the wave speed \( u(\xi) = c \), is a solution to the QHD equations (3) and (4), and equivalently equations (1) and (2).

Indeed equation (2) implies that \( Q_0 = E - V \), and thus the wavelength of the soliton must be equal to the de Broglie wavelength \( h/\sqrt{2m(E - V)} \). The localized soliton has a constant amplitude, equal to \( \rho_0 \), which is independent of the soliton wavelength. Also the soliton wave moves in free space, along the \( x \) direction, with a constant uniform speed \( c \). Notice that the quantum potential for the quantum density soliton comes out a constant, therefore the quantum force corresponding to the quantum density soliton is identically zero.

### 3 Application to Resonant Tunneling

We now apply the above obtained results to the case of resonant tunneling through a barrier of finite height. It is demonstrated that without making
any extra assumptions, only the requirement that the quantum density remains continuous throughout the process, especially at the barrier walls, it is possible to deduce the correct resonant tunneling conditions. In what follows we use the terms particle and quantum density soliton interchangeably.

Consider a freely moving quantum density soliton interacting with a barrier potential of width \( a \) and height \( V_0 \):

\[
V(x) = \begin{cases} 
V_0, & 0 < x < a, \\
0, & \text{otherwise}.
\end{cases}
\] (7)

Refer as region II to the region \( 0 < x < a \), where the potential has the constant value \( V_0 \), and region I and III as free regions to the left and right side of the barrier, respectively. Such a potential occurs, for instance, in the Josephson junction with the superconductor-insulator-superconductor configuration [9].

The quantum density soliton inside the insulating region has a momentum \( \hbar k_2 = \sqrt{2m(E - V_0)} \). Now two different cases may arise: either \( E > V_0 \), or \( E < V_0 \).

For \( E > V_0 \), the quantum potential in region II is \( Q = \hbar^2 k_2^2 / 2m > 0 \), therefore the quantum density in region II is a soliton wave given by:

\[
\rho_{II}(\xi) = \rho_2 \cos^2(k_2 \xi).
\] (8)

In the left and right hand regions the density is given respectively by the free quantum solitons:

\[
\rho_I(\xi) = \rho_1 \cos^2(k_1 \xi),
\] (9)

\[
\rho_{III}(\xi) = \rho_3 \cos^2(k_3 \xi),
\] (10)

with the particle/soliton momenta \( \hbar k_1 \), and \( \hbar k_3 \) respectively. Notice that, if however \( E < V_0 \), the density function in region II is \( \rho_{II}(\xi) = \rho_2 \cosh^2(k_2 \xi) \), where \( \hbar k_2 = \sqrt{2m(V_0 - E)} \).

Now the density wave amplitude \( \rho_2 \), in region II, can be calculated from the incident wave amplitude, and the barrier parameters, as follows. First using the continuity condition, at \( x = 0 \), we have \( \rho_I = \rho_{II} \), from which it follows that at the initial wall,

\[
\rho_2 = \rho_1,
\] (11)

Similarly at the other end of the barrier at wall \( x = a \), we obtain:

\[
\rho_3 = \frac{\cos^2(k_2 a)}{\cos^2(k_1 a)} \rho_2,
\] (12)

Putting the matching conditions (11) and (12), in equations (8) to (10), it follows that the quantum density soliton wave of given energy, after interacting with the potential barrier, identically recovers itself (that is \( \rho_I = \rho_{III} \), and \( k_1 = k_3 \)), only when \( k_1 = k_2 = n\pi \) for \( n = 0, 1, 2, \ldots \). These are the conditions for resonant tunneling through the barrier. Similarly when \( E < V_0 \), we obtain the same tunneling conditions.
4 Flux Quantization

The Josephson junction provides another important example where the above considerations of quantum density solitons are of rather direct significance. In the Josephson junction [9], the superconductor region is modeled by the macroscopic wave function $\psi(x,t) = \sqrt{n_e} e^{i\theta(x,t)}$, where $n_e$ is the number density of the superconducting electrons. In the quantum potential formalism, this wave function corresponds to the quantum density $\rho(x,t) = n_e(x,t)$.

Thus with an initially normalized number density $n_e$, in the superconducting region III, it follows from equations (8) to (12) that the transmitted current density for $E > V_0$ is:

$$n_e(x,t) = \cos^2\left(\sqrt{2m(E-V_0)a/h}\right) \cos^2\left(\frac{\sqrt{2mEa}}{h}(x - ct)\right), \quad (13)$$

and similarly if $E < V_0$,

$$n_e(x,t) = \coth^2\left(\sqrt{2m(V_0-E)a/h}\right) \cos^2\left(\frac{\sqrt{2mEa}}{h}(x - ct)\right), \quad (14)$$

Since energy is the time derivative of the phase function, we have $E = -\hbar \partial \theta_1 / \partial t$ and $(E - V_0) = -\hbar \partial \theta_2 / \partial t$, whereas $V_0$ is the junction potential.

The above formulas imply that there will be a conduction current across the junction without loss for $E > V_0$ if,

$$\sqrt{2mEa/h} = \sqrt{2m(E-V_0)a/h} \pm n\pi, \quad (15)$$

and for $E < V_0$ if,

$$\cos^2\left(\frac{\sqrt{2mEa}}{h}\right) = \coth^2\left(\frac{\sqrt{2m(V_0-E)a}}{h}\right). \quad (16)$$

We now show that flux quantization follows from equation (15), or equivalently from equation (16). Writing equation (15) in terms of momenta $p_1 = \sqrt{2mE}$ and $p_2 = \sqrt{2m(E-V_0)}$ implies that $(p_1 - p_2)a = \pm n\pi\hbar$, where for a full loop $a = 2\pi$. Using now $p_i = \hbar \nabla \theta_i$, where the index $i$ takes on values 1 and 2, we have on integrating along the path from initial point with momentum $p_1$ and final point with momentum $p_2$:}

$$\int_1^2 \nabla \theta_1 \cdot ds - \int_1^2 \nabla \theta_2 \cdot ds = \pm n \int_1^2 ds. \quad (17)$$

Then for a complete loop, starting from say point 1, going to the point 2 and then ending at the starting point 1, we have

$$\oint \nabla \theta \cdot ds = \pm n\pi. \quad (18)$$
The left hand side of equation (18) defines the flux $\Phi$ through the closed loop multiplied by $q/\hbar$. Equation (18) is thus identical to the flux quantization condition:

$$\Phi = \pm \frac{n\pi\hbar}{q},$$

(19)

where the flux $\Phi$ defined in terms of the vector potential $A$ is given by $\Phi = \oint A \cdot ds$. Notice that in deducing condition (19), the usual factor of 2 does not appear, hence we need not re-define charge in terms of Cooper pairs.

5 Tunneling Time and the Indeterminacy Principle

Quantum tunneling time through barriers has been investigated in different contexts [10-14]. We now apply the soliton wave representation to estimate the time of tunneling through a potential barrier.

Referring to the potential barrier considered in section 3 above, let $x_1$, $x_2$, and $x_3$ be the particle/soliton positions at three arbitrary points in region I, II, and III, respectively. Let the respective time instants of soliton arrival at these positions be $t_1$, $t_2$, and $t_3$. After interaction the soliton is free, therefore has the same quantum density $\rho_I = \rho_{III}$, however the wave number, as well as the position and the time coordinated are different. Thus we have in this case $\rho_I(\xi_1) = \rho_{III}(\xi_3)$, or explicitly, $\rho_I \cos^2 k_1 \xi_1 = \rho_3 \cos^2 k_3 \xi_3$. Then employing the matching conditions (11) and (12), it follows, that the time for particle/soliton to arrive at point $x_3$ is given by:

$$t_3 = \frac{x_3}{c} - \frac{1}{k_3c} \cos^{-1} \left[ \frac{\cos(k_1 a)}{\cos(k_2 a)} \cos k_1 (x_1 - ct_1) \right],$$

(20)

where $a$ is the barrier width. Thus if initially the particle/soliton was at $x_1 = 0$, at time $t = 0$, then the time it arrives, at the barrier wall $x_3 = a$, is:

$$t_3 = \frac{a}{c} - \frac{1}{k_3c} \cos^{-1} \left( \frac{\cos(k_1 a)}{\cos(k_2 a)} \right).$$

(21)

This is the total time a quantum density soliton takes to transverse a barrier of length $a$. In this formula the first term corresponds to the classical transit time, whereas the second term involves the effects of the barrier. Since tunneling time is the time during which the soliton-barrier interaction takes place, it must be the residual (in absolute measures) $| t_3 - \frac{a}{c} |$, that is, the difference between the total transversal time $t_3$, and the time $a/c$ it takes to travel the same distance with the same given speed, when free. Therefore according to equation (21) the quantum tunneling time, denoted by $\tau$, is given by

$$\tau = \frac{1}{k_3c} \cos^{-1} \left[ \frac{\cos(k_1 a)}{\cos(k_2 a)} \right].$$

(22)
In formula (22), two points are noticed:

(1) $\tau$ is an oscillatory function, thus it can delay as well as shorten the tunneling time. Also, due to this oscillatory behavior, it can be zero under appropriate conditions that are the same as for resonant tunneling, that is $k_1 = k_2 \pm n\pi$ where $n = 0, 1, 2, \ldots$.

(2) The denominator in this formula corresponds to the frequency of the outgoing free particle, that is $k_3 c = \nu_3$. This has implications regarding whether tunneling can be an instantaneous process. Excluding the resonant tunneling case, this requires that $\tau = 0$. For the free particle speed $c$ is finite, this requirement implies $k_3$, thus the frequency for free outgoing particle, is infinite, hence it must carry an infinite energy. Thus particles/quantum density solitons cannot transverse a barrier of finite length instantaneously.

The measure of tunneling time (21) consistently recovers the classical transversal time.

5.1 The Indeterminacy Principle

A direct consequence of formula (20) is the indeterminacy principle, which is now deduced from the condition that the total time $t_3$ in formula (20) (or equivalently formula (21)) cannot be negative or zero for a barrier of finite, non-zero width $a$. Thus we take $t_3 \geq 0$ in equation (20). This gives the condition

$$x_3 \geq \frac{1}{k_3} \cos^{-1} \left[ \frac{\cos(k_1 a)}{\cos(k_2 a)} \cos k_1 (x_1 - ct_1) \right]$$  \hspace{1cm} (23)

for each value of $k_1, k_2, k_3$ and $a$, and for each choice of $x_1$, and $t_1$. This implies that maximally $x_3 \geq \pi/k_3$. Replacing coordinate $x_3$ by $\Delta x$, that is the distance measured from the reference point $x_1$, of the point $x_3$, we obtain $\Delta x \geq \pi/\Delta k$, and if the quantum density soliton after interaction has momentum $\hbar \Delta k = \Delta p$, it follows that:

$$\Delta x \Delta p \geq \hbar / 2,$$  \hspace{1cm} (24)

where $\hbar = 2\pi\hbar$ is the Planck’s constant. In circular measures (23) has the standard form: $\Delta x \Delta p \geq \hbar / 2$. Similar considerations give the energy-time indeterminacy relation also.

6 Conclusions

The above analysis shows how particle-like phenomena at the quantum scale can be described in terms of localized soliton waves of de Broglie wavelength. These soliton waves do not form as a linear superposition of the plane wave solutions (stationary states) of the Schrödinger equation, hence cannot be represented as a Fourier sum, but must be considered as a (non-identical) consequence of the quantum potential based formalism (1) and (2). The validity of the soliton wave solutions is established by the correct tunneling, and the flux quantization conditions. Furthermore, we have derived a new measure of the time for particle
to tunnel through a barrier, which leads to the indeterminacy principle as a consequence.

Finally, inequality (23) can now be interpreted as follows. The quantum density soliton has wavelength $\hbar/\Delta p$, whereas the distance covered by the soliton is $\Delta x$. The factor $1/2$, on the right hand side of expression (23) is due to the symmetry (in cosine function) of soliton wavelength $\lambda$, about the mean (reference) position, whereas $\Delta x$ measures length only on one side of the mean position. Inequality (23) thus states that: the length to be transversed must be greater than or equal to the wavelength of the quantum density wave itself.

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