Geometry of faithful entanglement

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A typical concept to characterize entanglement is based on the idea that states in the vicinity of some pure entangled state share the same properties; implying that states with a high fidelity must be entangled. States whose entanglement can be detected in this way are also called faithful. We prove a structural result on the corresponding fidelity-based entanglement witnesses, resulting in a simple condition for faithfulness of a two-party state. For the simplest case of two qubits faithfulness can directly be decided and for higher dimensions accurate analytical criteria can be given. Finally, our results show that faithful entanglement is, in a certain sense, useful entanglement; moreover, they simplify several results in entanglement theory.

I. INTRODUCTION

Entanglement is a key ingredient in applications and protocols of quantum information processing, such as quantum key distribution [1] or quantum metrology [2]. Consequently, many works have been devoted to its characterization and quantification [3]. In the experimental context, special methods for the characterization are needed, as often no complete information on the quantum state is available [4,5].

A typical method to detect entanglement is based on the idea that states close to some known entangled pure state $|\psi\rangle$ must be entangled, too. Usually, the distance of a general state $\rho$ to the state $|\psi\rangle$ is measured in terms of the fidelity $F_{\psi} = \langle \psi | \rho | \psi \rangle$. Then, one can formulate the resulting criteria also in the form of entanglement witnesses. That is, one considers the observable

$$W = \alpha \mathbb{I} - |\psi\rangle \langle \psi|.$$  \hspace{1cm} (1)

In general, an entanglement witness is an observable which has a positive mean value on all separable states, hence a negative mean value signals the presence of entanglement. Concretely, if this observable $W$ is measured, one obtains $\text{Tr}(\rho W) = \alpha - F_{\psi}$, so if $F_{\psi}$ is above the threshold value $\alpha$, then the witness detects some entanglement. The fidelity-based witness in Eq. (1) is easy to construct and can be generalized to the multi-particle case. But clearly, it is also limited, and many other approaches to construct witnesses beyond the fidelity approach exist [6-11].

Still, it is an interesting question to ask which states can be detected using the fidelity and what are their physical properties. In Ref. [12], the authors coined the term faithful for states whose entanglement can be characterized using the fidelity, and provided first steps in distinguishing faithful and unfaithful states. For instance, an approach using convex optimization and semidefinite programming was provided which can prove the unfaithfulness of a state, with this one can show that in higher dimensional systems most states are unfaithful.

In this note we go further and deliver analytical results on faithful and unfaithful states. First, we show that one can restrict the attention to a special subclass of fidelity-based criteria. Then, we can derive some general results on faithful states: For two qubits, the faithfulness can directly be decided, and for higher-dimensional systems, an analytical sufficient criterion for unfaithfulness can be given. We also show that faithful entanglement is useful entanglement, in the sense that it leads to a violation of Bell inequalities on multiple copies of a state. Moreover, our results simplify estimates of entanglement measures and we demonstrate that entanglement detection using the fidelity of a general projector $\Pi$ instead of a pure state $|\psi\rangle \langle \psi|$ is fundamentally different. Finally, we discuss possible extensions to the Schmidt number classification of quantum states.

II. ENTANGLEMENT, SEPARABILITY AND WITNESSES

We consider two parties, Alice and Bob, each of which owns a $d$-dimensional quantum system. In general, a mixed state is called separable, if it can be written as a convex combination of product states,

$$\rho_{AB} = \sum_k p_k |a_k, b_k\rangle \langle a_k, b_k|,$$  \hspace{1cm} (2)

where the $p_k$ form a probability distribution, that is, they are positive and sum up to one. States that can not be expressed in this way are called entangled.

There are many criteria for proving entanglement or separability of a given quantum state, although none of them solves the problem completely [13,14]. An important criterion is entanglement witnesses, which are observables that have a positive expectation value on all separable states, and a negative expectation value on some entangled states. It can be straightforwardly

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shown that, in principle, any entangled state can be detected by some suitable witness, but the problem is to construct all witnesses. This is, in fact, not easy and of the same complexity as the separability problem.

As already mentioned, one possible witness construction makes use of the fact that in the vicinity of an entangled pure state there are only entangled states. So, one can define a witness via \( W = \alpha I - |\psi\rangle\langle\psi| \), where \( \alpha \) is the maximal squared overlap between \( |\psi\rangle \) and the separable states. This can be computed as [13]

\[
\alpha = \sup_{\rho \text{ separable}} \langle \psi | \rho | \psi \rangle = \sup_{|a,b\rangle} |\langle a, b | \psi \rangle|^2 = s^2,
\]

where the \( s_i \) for \( i = 1, \ldots, r \) are the decreasingly ordered nonzero Schmidt coefficients of the state \( |\psi\rangle \) in its Schmidt decomposition \( |\psi\rangle = \sum_{i=1}^r s_i |ii\rangle \). Here, the number of terms \( r \) is called the Schmidt rank of the state. The smallest possible \( \alpha \) occurs if the state \( |\psi\rangle \) is a maximally entangled state, for instance \( |\psi\rangle = |\phi^+\rangle = \sum_{i=1}^d |ii\rangle / \sqrt{d} \). Then we have

\[
W = \frac{1}{d} - |\phi^+\rangle \langle \phi^+|.
\]

Clearly, one may also take other maximally entangled states which are locally equivalent to \( |\phi^+\rangle \). For reasons that become apparent later, we call this set of witnesses of the type in Eq. (4) the relevant fidelity witnesses (RFW).

III. MAIN RESULT

In this section we can prove our main result. It states that in the set of fidelity-based witnesses, only the relevant fidelity based witnesses, where \( |\psi\rangle \) is a maximally entangled state, are important. We can directly state and prove the main result:

**Observation 1.** Let \( \varrho_{AB} \) be a faithful entangled state, i.e. its entanglement can be detected by some fidelity-based entanglement witness. Then \( \varrho_{AB} \) can be detected by a relevant entanglement witness. In other words, a state \( \varrho_{AB} \) is faithful if and only if there are local unitary transformations \( U_A \) and \( U_B \) such that

\[
\langle \phi^+ | U_A \otimes U_B \varrho_{AB} U_A^\dagger \otimes U_B^\dagger | \phi^+ \rangle > \frac{1}{d}.
\]

In order to prove this we consider without losing generality the case \( d = 4 \), the general case follows along the same lines with some additional notational effort. Let us assume that the fidelity-based witness detecting \( \varrho_{AB} \) is given by \( W = s_1^2 I - |\psi\rangle\langle\psi| \), with \( |\psi\rangle = \sum_{i=1}^4 s_i |ii\rangle \). We consider eight RFWs, coming from the eight maximally entangled states

\[
|\phi_k\rangle = \frac{1}{2} (|11\rangle + a_2^{(k)} |22\rangle + a_3^{(k)} |33\rangle + a_4^{(k)} |44\rangle),
\]

where \( a_i^k = \pm 1 \) and all eight different possibilities for choosing them are included, so \( k = 1, \ldots, 8 \). This leads to eight RFWs \( W_k = 1/4 - |\phi_k\rangle\langle\phi_k| \).

We aim to show that one can find probabilities \( p_k \) such that the operator

\[
Z = W - 4s_1^2 \sum_{k=1}^8 p_k W_k \geq 0
\]

is positive semidefinite. This would prove already the main claim, since then \( \text{Tr}(\varrho_{AB} W) < 0 \) implies that for at least one \( k \) we have \( \text{Tr}(\varrho_{AB} W_k) < 0 \).

In order to prove Eq. (7) we aim to find \( p_k \) such that \( Z \) is diagonal. This is sufficient for positivity, as the diagonal entries of \( Z \) are independent of the \( p_k \). They are either zero or of the type \( s_1^2 - s_i^2 \geq 0 \), which ensures \( Z \geq 0 \).

Looking at the relevant off-diagonal entries, one finds that for making \( Z \) diagonal, we have to find \( p_k \) such that

\[
\frac{1}{d} = \sum_{k=1}^8 p_k W_k = \sum_{k=1}^8 \left( \times \ a_2^{(k)} a_2^{(k)} a_2^{(k)} a_2^{(k)} a_4^{(k)} a_4^{(k)} a_4^{(k)} a_4^{(k)} \right) \]

where we defined

\[
a_i = \frac{s_i}{s_1} \in [0,1].
\]

With a physical argument one can see now that a decomposition as in Eq. (8) can be found: One can view the three \( a_i \) as expectation values of some observables (say, \( \sigma_z \)) on a three-qubit product state \( \varrho = \varrho_A \otimes \varrho_B \otimes \varrho_C \). The terms \( a_i a_j \) correspond then to a two-body correlation \( \langle \sigma_z \otimes \sigma_z \rangle \) on the same state. On the other hand,
the right-hand side of Eq. (8) can be seen as a local hidden variable model, where the index $k$ is the hidden variable occurring with probability $p_k$, and the $a^{(k)}_i$ are the deterministic assignments for the measurement results of $\sigma_i$ on the different particles. For fully separable states, however, it is well known that all measurements can be explained by a local hidden variable model [14], so there must be $p_k$ such that Eq. (8) is fulfilled.

IV. IMPLICATIONS

The result of the previous section is connected to several other results in entanglement theory. In the following, we will discuss some of these connections and implications.

A. General remarks

First, from Observation 1 it is immediately clear that already for two qubits not all entangled states are faithful. The reason is the following: The RFW in Eq. (3) can be seen as a witnesses for the computable cross-norm or realignment (CCNR) criterion [12,15]. This follows from the fact that one may write it as

$$ W = \frac{1}{d}(\mathbb{1} - \sum_{k=1}^d G_k \otimes G_k^T), $$

where the $\{G_k\}$ form an orthogonal basis of the operator spaces for Alice and Bob and $(\cdot)^T$ denotes the transposition [4]. This means that only states that violate the CCNR criterion can be faithful. But it is known that already for two qubits there are entangled states that do not violate this criterion [7].

Second, using the Jamiołkowski isomorphism [18,19], one can directly see that the RFW in Eq. (3) corresponds to the reduction map. From this one has directly the already known result [10] that only the entanglement in states which violate the reduction criterion can be detected by the fidelity. This further implies that only states that violate the criterion of the positivity of the partial transpose (PPT) can be faithful.

The left hand side of the inequality [3] is related to a well-known quantity called the singlet fraction. This concept was introduced by Horodecki et al. in Ref. [20] in a slightly more general form, where one tries to maximize the fidelity with the maximally entangled state, using trace-preserving local quantum operations and classical communication. Then, it was proved that having a singlet fraction larger than $1/d$ is a necessary and sufficient condition for a state to offer a quantum advantage in teleportation. It was also shown that for every state $\varphi$ with singlet fraction larger than $1/d$, there exists a number of copies $k$ of $\varphi$ such that $\varphi^\otimes k$ is non-local [21]. Together with these results, Observation 1 implies that faithful states are useful for quantum teleportation and that multiple copies of them violate a Bell inequality. This shows that fidelity-based entanglement witnesses detect entanglement that is useful.

B. Deciding faithfulness

Starting from Observation 1 one may ask for direct criteria for faithfulness. In the following, we describe an analytical formula how one can decide for a two-qubit state whether it is faithful or not. For higher dimensions, we provide a similar procedure that gives a strong analytical sufficient criterion to be unfaithful and also a way to prove faithfulness. For randomly generated states, the approaches allow do decide faithfulness in practically all cases.

Let us first consider qubits and recall that we have to maximize the overlap with a maximally entangled state in some basis. This problem has been solved before [22], but in order to generalize it later, we formulate it in a different language. Starting from a general two-qubit state $\varrho_{AB}$, we can decompose it into Pauli matrices,

$$ \varrho_{AB} = \frac{1}{4} \sum_{i,j=0}^3 \lambda_{ij} \sigma_i \otimes \sigma_j, $$

where $\lambda_{ij} = \text{Tr}(\varrho \sigma_i \otimes \sigma_j)$ and $\varrho_0 = \mathbb{1}$ denotes the identity matrix. Then we consider the operator

$$ X_{d}(\varrho_{AB}) = \varrho_{AB} - \frac{1}{d} (\varrho_{A} \otimes \mathbb{1} + \mathbb{1} \otimes \varrho_{B}) + \frac{2}{d^2} \mathbb{1} \otimes \mathbb{1}, $$

where, for the case of two qubits, we take $d = 2$. In terms of the representation in Eq. (11) this operator has a block-diagonal $\lambda_{ij}$, the terms $\lambda_{ij}$ and $\lambda_{00}$ for $i,j = 1,2,3$ corresponding to the marginals have been removed. Note that we have for any maximally entangled state in any basis $(\varphi^+) X_2(\varrho) (\varphi^+) = (\varphi^+) (\varrho_{AB}) (\varphi^+)$ as for maximally entangled states the marginal terms are maximally mixed.

By local unitary transformations we can diagonalize the remaining $3 \times 3$ matrix $\lambda_{ij}$ for $i,j = 1,2,3$ for the operator $X_2$. In this basis, $X_2$ is Bell diagonal, and we can directly read of the maximal overlap with Bell states by computing the maximal eigenvalue. So we can summarize:

**Observation 2.** A two-qubit state $\varrho_{AB}$ is faithful if and only if the maximal eigenvalue of $X_2(\varrho_{AB})$ in Eq. (12) is larger than $1/2$.

We add that the results of Ref. [23] imply that any two-qubit state is faithful after suitable local filtering operations. For a general two-qudit state $\varrho_{AB}$, one can proceed as follows: First, we can consider $X_{d}(\varrho_{AB})$ from above, and if its largest eigenvalue is smaller than $1/d$, the state cannot be faithful. The proof of this statement follows along the same lines as for qubits, just needs to
replace the Pauli matrices by some other basis of the operator space, where one basis element is proportional to the identity and the other elements are traceless. Then, the block of the matrix $\lambda$ can, in general, not be diagonalized anymore. Still, the maximal eigenvalue of $X_d$ is an upper bound on the overlap with maximally entangled states.

Alternatively, instead of maximizing the overlap over all maximally entangled states, one can maximize the overlap over all states with maximally mixed reduced states. That is, one considers the relaxed optimization

$$\max: \text{Tr}(\rho_{AB}\chi),$$

subject to: $\text{Tr}(\chi) = 1, \quad \chi \geq 0,$

$$\text{Tr}_A(\chi) = \text{Tr}_B(\chi) = \frac{I}{d}. \quad (15)$$

This is a simple SDP, and if a result smaller than $1/d$ is found, the state cannot be faithful. In fact, one can show that the dual of this SDP is equivalent to the SDP2 presented in Ref. [12].

The formulation in Eq. (13) has, however, two advantages. First it can also be used to prove that a state is faithful. If the SDP returns a value larger than $1/d$ one can check whether the optimal $\chi$ is a pure state. If this is the case, this state is maximally entangled and $\rho_{AB}$ must be faithful. Second, one can systematically improve the SDP in Eq. (13) by adding rank constraints on $\chi$. These can be implemented by a hierarchy of SDPs and detect indeed more states; see also below. So we can summarize:

**Observation 3.** Consider a general two-qubit state $\rho_{AB}$. (a) If the largest eigenvalue of the operator $X_d(\rho_{AB})$ in Eq. (12) is not larger than $1/d$, then $\rho_{AB}$ is unfaithful. (b) If the semidefinite program from Eq. (13) has an optimal value not larger than $1/d$, then $\rho_{AB}$ is unfaithful. (c) If the optimization in Eq. (15) returns a value larger than $1/d$ and the optimal $\chi$ is a pure state, then $\rho_{AB}$ is faithful.

We add that the condition $3(b)$ detects strictly more states as unfaithful than condition $3(a)$. The reason is that in Eq. (13) one can directly replace $\rho_{AB}$ with $X_d$ from Eq. (12), without changing the result of the SDP. Then, the optimization can further be relaxed by considering only the largest eigenvalue of $X_d$.

Armed with these insights, one can now generate random states in the Hilbert-Schmidt distribution (HSD) and the Bures metric (BM) and estimate which fraction is separable, entangled but unfaithful, or faithful. For two qubits, we generated $10^6$ states randomly for both distributions. In HSD, we find 24.35% of all states separable via the condition of the positivity of the partial transpose (PPT) and 21.14% are entangled, but unfaithful, the remaining 54.51% are faithful. In the BM, 7.32% are separable, and 15.44% of all states are entangled, but not faithful. These values coincide up to statistical fluctuations with the values reported for the sufficient criterion SDP2 for unfaithfulness in Ref. [12] (or Eq. (13)). Indeed, using the fact that for $d = 2$ the extreme points of the unital maps are unitary maps [26], one can see that in this case the extreme points of the matrices $\chi$ considered in Eq. (13) are the projectors onto maximally entangled states, hence for $d = 2$ the SDP2 in Ref. [12] or Observation 3(b) are necessary and sufficient for faithfulness.

For higher dimensions, we also generated $10^6$ states randomly for both distributions. The results are given in Tables I and II. In the underlying samples of states it turned out that any state not obeying the criterion in Observation 3(b) is faithful. This, however, is not generally true. Consider a highly entangled state $\rho_{AB}$, where the relaxed optimization in Eq. (13) gives a strictly larger value than the optimization of the overlap with all maximally entangled state, see Eq. (13), but both values are larger than $1/d$. Such states exist and can easily be found by random search. Then, mixing the state with white noise leads to a linear decrease of the results in both optimizations. For a proper amount of noise, the value of Eq. (13) will be smaller than $1/d$ and the value of Eq. (13) strictly larger than $1/d$. So, the state will be unfaithful, but the criterion of Observation 3(b) does not detect it.

In an additional numerical analysis, we also identified one state for $d = 4$ which escapes the detection as unfaithful via the criterion 3(b), but it could not be shown to be faithful with criterion 3(c) or the optimization in Eq. (13). This state could then, however, detected as unfaithful by the SDP in Eq. (13) with additional rank constraints on $\chi$ [24].

### Table I: Fraction of states in the Hilbert-Schmidt distribution that can be detected with the various criteria. First, we consider the PPT states, which are always unfaithful. Then, we consider the NPT states, which are unfaithful (UFF) due to Observation 3(a). Then, the NPT states, which are unfaithful due to Observation 3(b). The fourth column is the fraction of states that can be detected as faithful via Observation 3(c).

|   | PPT | UFF [3(a)] | UFF [3(b)] | FF [3(c)] | FF [Eq. (3)] |
|---|-----|------------|------------|-----------|-------------|
| 3 | 0.01%| 83.05%     | 94.55%     | 5.44%     |             |
| 4 | 0%  | 99.93%     | 99.999%    | 0.001%    |             |
| 5 | 0%  | 100%       | 100%       | 0%        |             |
| 6 | 0%  | 100%       | 100%       | 0%        |             |

### Table II: Fraction of states in the Bures metric that can be detected with the various criteria. For a detailed explanation see the caption of Table I. For $d = 4$ it happened that few states were left where faithfulness could not be decided with the previous criteria. All of these states could be shown to be faithful via a direct optimization in Eq. (5) (fifth column).

|   | PPT | UFF [3(a)] | UFF [3(b)] | FF [3(c)] | FF [Eq. (5)] |
|---|-----|------------|------------|-----------|-------------|
| 3 | 0%  | 25.79%     | 54.68%     | 45.32%    |             |
| 4 | 0%  | 71.40%     | 96.959%    | 3.04%     | 0.001%      |
| 5 | 0%  | 99.33%     | 99.998%    | 0.002%    |             |
| 6 | 0%  | 100%       | 100%       | 0%        |             |
C. Further implications

Among the numerous works on entanglement estimation, many approaches use the comparison with the fidelity of some pure entangled states. For example, in Ref. [27] several lower bounds of entanglement measures given, which were all of the type \( E(\rho) \geq f[S(\rho_{AB}) - 1] \), where \( E(\cdot) \) is some entanglement measure, \( f[\cdot] \) is some function, and

\[
S(\rho_{AB}) = \max_{|\psi\rangle} \langle \psi | \rho_{AB} | \psi \rangle / s_{\max}^{2}, 1, \tag{16}
\]

which demands the maximization over all pure states. Clearly, \( S > 1 \), if and only if \( \rho \) is faithful. From Eq. (7) it follows that for computing \( S \) one needs only to optimize over maximally entangled states. Furthermore, we can conclude that \( S > 1 \) can only happen if a state violates the CCNR criterion, and Observation 3 can also be applied to estimate \( S \).

Finally, note that pure states are rank-one projectors and the fidelity of a bipartite state \( \rho_{AB} \) and a pure state \( |\psi\rangle \) is nothing but the expectation value \( \langle \psi | \rho_{AB} | \psi \rangle \). One can thus generalize the definition of the witness in Eq. (11) to

\[
W_{V} = \epsilon I - \Pi_{V}, \tag{17}
\]

where \( \Pi_{V} \) is the projector to some subspace \( V \). The minimal value of \( \epsilon \) while \( W \) having positive expectation value on all separable states is given by \( \epsilon_{\min} = \sup_{|\psi\rangle \in V_{\text{sep}}} \langle \psi | \rho_{AB} | \psi \rangle \), which is the maximal value of the largest Schmidt coefficient of the pure states in \( V \) [28]. With this construction, also entanglement measures can be estimated [29]. It should be noted, however, that this generalization allows the detection of some bound entangled states with positive partial transpose, e.g., if \( V \) is the subspace complementary to an extendible product basis [6, 28]. In contrast, as we have seen, the witness in Eq. (11) can only detect entangled states with negative partial transpose. So, the witnesses in Eq. (17) are structurally quite different from fidelity-based entanglement witnesses.

V. SCHMIDT NUMBER WITNESSES

Finally, let us discuss the notion of the Schmidt number and Schmidt number witnesses. It turns out, however, that Observation 1 cannot so easily be generalized to this case.

A. The concept of the Schmidt number

The concept of entanglement and separability can be generalized to the notion of the Schmidt number. For that, one defines states to be of Schmidt number \( r \), if they can be written as a convex combination of pure states with Schmidt rank \( r \), \( \rho = \sum k p_{k} |\phi_{k}\rangle \langle \phi_{k}| \), where all the states \( |\phi_{k}\rangle \) have a Schmidt rank \( r \). Clearly, the separable states are exactly the states with Schmidt number one.

Then, Schmidt witnesses can be defined in analogy to entanglement witnesses: A Schmidt witness for Schmidt number \( \ell \) is an observable \( S_{\ell} \) with positive expectation value on all states with Schmidt number \( \ell \). So, observing a negative expectation value proves that the state has at least Schmidt number \( \ell + 1 \). Again, such witnesses can be based on the projector

\[
S_{\ell} = \beta(\ell) I - |\psi\rangle \langle \psi|, \tag{18}
\]

where the maximal squared overlap is now given by the sum of the \( \ell \) biggest squared Schmidt coefficients,

\[
\beta(\ell) = \sum_{k=1}^{\ell} s_{k}^{2}. \tag{19}
\]

Again, the theory of Schmidt witnesses is well developed [28, 30, 31]. The direct generalization of the RFW to the Schmidt witnesses is the set of Schmidt witnesses with \( |\psi\rangle = \sum_{i} a_{i} |i\rangle / \sqrt{d} \), where \( a_{i} = \pm 1 \). All these witnesses have \( \beta(\ell) = r / d \).

B. Fidelity-based Schmidt witnesses

One may wonder whether a similar result as in Observation 1 also holds for Schmidt witnesses. In the following, we show that a similar result holds for special cases, but not in general. Before going into details, we note a few general results on the order of witnesses for general convex sets.

We say a witness \( W \) is weaker than a set of witnesses \( \{W_{k}\}_{k} \), denoted as \( W < \{W_{k}\}_{k} \), if for any \( \rho \) such that \( \text{Tr}(W_{k} \rho) < 0 \), there exists a \( W_{k} \) such that \( \text{Tr}(W_{k} \rho) < 0 \), or equivalently, if \( \text{Tr}(W_{k} \rho) \geq 0 \) for all \( k \), then \( \text{Tr}(W \rho) \geq 0 \). This relation can be analyzed by considering the following optimization problem:

\[
\text{min: } \text{Tr}(W \rho) \tag{20}
\]

subject to: \( \text{Tr}(W_{k} \rho) \geq 0 \) for all \( k \),

\( \rho \geq 0. \tag{21} \)

More precisely, \( W < \{W_{k}\}_{k} \) if and only if the solution of the optimization in Eq. (20) is zero. The dual problem of Eq. (20) reads

\[
\text{max: } 0 \tag{23}
\]

subject to: \( W - \sum_{k} x_{k} W_{k} \geq 0, \tag{24} \)

\( x_{k} \geq 0 \) for all \( k \). \tag{25} \)

According to the general theory of convex optimization [32], weak duality gives that if there exists \( x_{k} \geq 0 \) such
that Eq. (24) holds, then \( W < \{ W_k \}_k \). When the strong duality holds, e.g., in the case that \( \{ W_k \}_k \) is a finite set, then Eq. (24) gives a necessary and sufficient condition. This can be viewed as a generalization of the result in Eq. (7).

Now, we want to investigate whether any fidelity-based Schmidt witness \( S_\ell \) defined by Eqs. (15) and (19), 
\[
S_\ell = \beta(\ell) \mathbb{1} - |\psi\rangle \langle \psi | \quad \text{with} \quad \beta(\ell) = \sqrt{\sum_{i=1}^{d} s_i^2},
\]
is always weaker than the set witnesses based on maximally entangled states, i.e., whether 
\[
S_\ell < \{ S_{\ell}^\phi \}. \quad \text{Here,} \quad \{ S_{\ell}^\phi \} \quad \text{denotes the set of all Schmidt number} \ \ell \ \text{witnesses based on the maximally entangled state:}
\]
\[
\ell \mathbb{1} - U_A \otimes U_B |\phi^+\rangle \langle \phi^+ | U_A^\dagger \otimes U_B^\dagger, \quad (26)
\]

for all local unitary transformations \( U_A \) and \( U_B \). In order to see that there exists \( S_\ell \) such that \( S_\ell \notin \{ S_{\ell}^\phi \} \), we have the following observation.

**Observation 4.** Let \( |\psi\rangle = \sum_{i=1}^{d} s_i |ii\rangle \) be a pure state with Schmidt number larger than \( \ell \), i.e., \( s_{\ell+1} > 0 \). Then it can be detected by \( \{ S_{\ell}^\phi \} \) if and only if \( \sum_{i=1}^{d} s_i > \sqrt{\ell} \).

This can be seen in the following manner: When maximizing the overlap between two states over local unitaries, its best is to take them both in the same Schmidt basis [35]. So, the maximization over all maximally entangled states \( |\phi\rangle \) gives
\[
\max_{\langle \phi |} |\langle \phi | \psi \rangle |^2 = \left( \sum_{i=1}^{d} \frac{s_i}{\sqrt{d}} \right)^2 = \frac{1}{d} \left( \sum_{i=1}^{d} s_i \right)^2. \quad (27)
\]
The relation \( \sum_{i=1}^{d} s_i > \sqrt{\ell} \) can always be satisfied when \( \ell = 1 \). However, it can be violated when \( \ell \geq 2 \). On the other hand, \( |\psi\rangle \) can always be detected by \( S_{\ell} \) defined by Eq. (18), as long as \( s_{\ell+1} > 0 \). This implies that there exists \( S_\ell \) such that \( S_\ell \notin \{ S_{\ell}^\phi \} \).

In general, we have the following necessary and sufficient condition for \( S_\ell \notin \{ S_{\ell}^\phi \} \):

**Observation 5.** The Schmidt witness \( S_\ell \) defined in Eq. (18) satisfies that \( S_\ell \notin \{ S_{\ell}^\phi \} \) if and only if \( s_1 = s_2 = \cdots = s_\ell \).

The sufficient part follows similarly to Observation 1. To prove the necessity part, we assume that not all \( s_1, s_2, \ldots, s_\ell \) are equal and construct a state that is detected by \( S_\ell \), but not by any \( S_{\ell}^\phi \). First, since not all \( s_1, s_2, \ldots, s_\ell \) are equal, we have \( \sum_{i=1}^{\ell} s_i \langle i | i \rangle / \sqrt{\sum_{i=1}^{\ell} s_i^2} < \sqrt{\ell} \). So, there exists \( 0 < \epsilon \leq s_{\ell+1} \), such that
\[
\frac{1}{\sqrt{\sum_{i=1}^{\ell} s_i^2 + \epsilon^2}} \left( \sum_{i=1}^{\ell} s_i + \epsilon \right) \leq \sqrt{\ell}. \quad (28)
\]

We define the state \( |x\rangle = \sum_{i=1}^{\ell+1} x_i |ii\rangle \), with the coefficients
\[
x_i = \frac{s_i}{\sqrt{s_{\ell+1}^2 + \epsilon^2}} \quad \text{for} \quad i = 1, 2, \ldots, \ell, \
x_{\ell+1} = \frac{\epsilon}{\sqrt{s_{\ell+1}^2 + \epsilon^2}}, \quad (29)
\]

Then \( |x\rangle \) cannot be detected by \( \{ S_{\ell}^\phi \} \) due to Eq. (25) and Observation 4. On the other hand, by taking advantage of the relation that \( 0 < \epsilon \leq s_{\ell+1} \), we have
\[
|\langle \phi | x \rangle |^2 = \left( \sum_{i=1}^{\ell+1} s_i x_i \right)^2 \geq \sum_{i=1}^{\ell} s_i^2 + \epsilon^2 = \beta(\ell) + \epsilon^2. \quad (30)
\]

This means that \( |x\rangle \) can be detected by \( S_\ell \) but not by witnesses in the set \( \{ S_{\ell}^\phi \} \).

**VI. CONCLUSION**

The question whether the entanglement of a state can be characterized by the fidelity with some pure state is practically relevant and is closely related to the geometry of entangled states. We have provided a structural result on such faithful states for bipartite systems. For the case of two qubits, we provided a simple analytical criterion for faithfulness, and for higher dimensional systems strong necessary criteria and sufficient criteria were developed. Our results showed that fidelity-based entanglement witnesses detect a form of useful entanglement, moreover, they shed light on several other results in entanglement theory, such as the estimation of entanglement measures.

For further work, it would be very interesting to generalize the approach discussed here to multiparticle systems. Also, one may consider other quantum resources (such as measurements or quantum channels) and discuss the question, which of their properties can be inferred by comparing it with a desired “pure” resource (e.g., a projective measurement or a unitary channel). This may lead to new insights into the geometry of these quantum resources.

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