TANGENTIAL STAR PRODUCTS

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Abstract. We establish a necessary and sufficient condition under which there exists a tangential and well graded star product, differential or not, on the dual $g^*$ of a nilpotent Lie algebra $g$. We also give enlightening examples with explicit computations.

Introduction

The theory of deformations and especially the notion of star products have been developed by Flato, Lichnerowicz and their collaborators in [3] with the aim of quantizing a classical system represented by a symplectic or a Poisson manifold $M$. A star product on $M$ is an associative, non commutative product on $C^\infty(M)$ depending formally on a parameter $\nu$ (in physical applications $\nu$ is $\frac{i\hbar}{2}$ where $\hbar$ denotes Planck’s constant). The product should have the form

$$f \ast g = \sum_{n \geq 0} C_n(f, g) \nu^n$$

where $f, g$ are in $C^\infty(M)$, $C_0(f, g) = fg$, $C_1(f, g) = \{f, g\}$ and $C_n(f, g)$ are bilinear operators on $C^\infty(M)$ with values in $C^\infty(M)$.

The main development of this theory went through the proof of the existence of differential star products, that is star products whose cochains $C_n$ are differential operators. In the case of symplectic manifolds, the question has been completely solved, using different approaches [7, 8]. In a recent work, Kontsevich has given a remarkable proof of the existence of differential star products on an arbitrary smooth Poisson manifold [10].

Since every Poisson manifold is “foliated” by symplectic submanifolds, it is quite natural to study star products with nice restrictions to the symplectic leaves. Such star products are called tangential. For regular Poisson manifolds, a tangential version of Vey’s work is introduced in [11] and a proof of the existence of tangential and differential star products can be found in [12]. Unfortunately, for general Poisson manifolds, such tangential and differential star products do not always exist.
Indeed, let \( \mathfrak{g} \) be a Lie algebra and let us consider the dual space \( \mathfrak{g}^* \), endowed with its linear Poisson structure. It is well known that in this case, the symplectic leaves are nothing else but the coadjoint orbits in \( \mathfrak{g}^* \). It turns out that a tangential and differential deformation on \( \mathfrak{g}^* \) is possible only if \( \mathfrak{g} \) satisfies a very strong algebraic condition [5]. No semi-simple Lie algebras satisfy this condition. Moreover, it has been shown that the standard deformation on \( \mathfrak{g}^* \), i.e. the Gutt star product, is very rarely tangential [2].

However, for non-differential star products, the situation is far better. Cahen and Gutt have constructed in [6] an algebraic tangential star product on the set of regular orbits of any semi-simple Lie algebra. Furthermore, there is in [1] a construction of a deformation on \( \mathfrak{g}^* \) in the case where the symmetric algebra \( S(\mathfrak{g}) \) is a free \( I(\mathfrak{g}) \)-module (\( I(\mathfrak{g}) \) denoting the algebra of invariant polynomials on \( \mathfrak{g}^* \)). This deformation generalizes the one which is given in the semi-simple case in [6], but is less explicit.

The purpose of the present paper is to give a simple condition, Theorem 10, for the existence of tangential star products on \( \mathfrak{g}^* \) in the nilpotent case. First, we introduce the needed notions and compute the cohomology related to deformations of the associative and graded algebra \( S(\mathfrak{g}) \). Then, we prove that the construction of a good operator \( C_2 \) is enough to ensure the existence of a tangential, “well” graded, differential or not, star product on \( \mathfrak{g}^* \). We devote the last part to explicit illustrations. In particular, we apply our result to \( \mathfrak{g}_{54} \), the simplest example of a nilpotent Lie algebra \( \mathfrak{g} \) for which there is no tangential and differential deformation of \( S(\mathfrak{g}) \) [1, 5].

**Notation:** Throughout this letter, \( \mathfrak{g} \) denotes a nilpotent Lie algebra and \( \mathfrak{g}^* \) the dual space of \( \mathfrak{g} \). The symmetric algebra \( S(\mathfrak{g}) \) over \( \mathfrak{g} \) is naturally identified with the algebra of real-valued polynomials on the dual \( \mathfrak{g}^* \). Obviously, \( S(\mathfrak{g}) = \oplus S^k(\mathfrak{g}) \) where \( S^k(\mathfrak{g}) \) is the space of homogeneous polynomials of degree \( k \). We denote by \( I(\mathfrak{g}) \) (or \( I \)) the algebra of invariant polynomials on \( \mathfrak{g}^* \).

1. Differential, algebraic and tangential operators

We first recall some essential facts about nilpotent Lie algebras. See [1, 4, 15] for more details.

Suppose that \( \mathfrak{g} \) is an \( m \)-dimensional nilpotent Lie algebra. Denote by \( (X_i) \) a Jordan-Hölder basis of \( \mathfrak{g} \) (that is \( [X_i, X_j] \equiv 0 \mod (X_1, \ldots, X_{j-1}) \) if \( i \geq j \)). Let \( (x_i) \) be the system of coordinates of \( \mathfrak{g}^* \) associated to this basis. Let \( G \) be the simply connected group with Lie algebra \( \mathfrak{g} \). Let also \( 2d \) be the maximal dimension of coadjoint orbits in \( \mathfrak{g}^* \). There exist:

(i) a Zariski open subset \( V \) of \( \mathfrak{g}^* \), invariant by the action of the adjoint group of \( G \), dense in \( \mathfrak{g}^* \), containing only orbits of maximal dimension;

(ii) \( 2d \) rational functions \( (p_1, \ldots, p_d, q_1, \ldots q_d) \) in the variables \( (x_i) \) which are regular on \( V \);

(iii) \( m - 2d \) polynomial functions \( \lambda_1, \ldots, \lambda_{m-2d} \) in the variables \( (x_i) \);
(iv) a Zariski open subset $U$ of $\mathbb{R}^{m-2d}$.

These elements are such that there exists a diffeomorphism $\varphi$ between $V$ and $U \times \mathbb{R}^{2d}$ defined by $\varphi(\xi) = (\lambda(\xi), p(\xi), q(\xi))$ if we note $\lambda = (\lambda_1, \ldots, \lambda_{m-2d})$, $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$, such that each orbit contained in $V$ admits a global Darboux chart defined by the variables $p_i, q_j$ and that each invariant rational function on $\mathfrak{g}^*$ may be written in a unique way as a rational function in the variables $(\lambda_k)$. The orbits contained in $V$ are usually called generic orbits and each polynomial $\lambda_k$ is said to be a generic invariant. Moreover, every $X$ in $\mathfrak{g}$, as a function on $\mathfrak{g}^*$, restricted to $V$, can be written as

$$X = \sum_{1 \leq j \leq d} a_j(q, \lambda)p_j + a_0(q, \lambda)$$

where the coefficients $a_j$ are polynomial in $q$ and rational in $\lambda$.

Let us denote by $\mathbb{R}(\lambda)[p, q]$ the algebra of polynomial functions in $p, q$ with coefficients in the space $\mathbb{R}(\lambda)$ of rational functions in $\lambda$. Thus, every $X$ in $\mathfrak{g}$ can be identified with an element of $\mathbb{R}(\lambda)[p, q]$. Let us now consider $S(\mathfrak{g})_I$, the localized algebra of rational functions with non zero invariant denominators. We see that the quotient field of $I$ is exactly $\mathbb{R}(\lambda)$, thus $S(\mathfrak{g})_I$ coincides with the space $\mathbb{R}(\lambda)[x_1, \ldots, x_m]$ of polynomials on $\mathfrak{g}^*$ with rational coefficients in $\lambda$ and also with $\mathbb{R}(\lambda)[p, q]$. Furthermore, for each element $X$ of $\mathfrak{g}$ considered as a function on $\mathfrak{g}^*$, the derivative $\partial X$ with respect to $X$ can be written in the form

$$\partial X = \sum_{1 \leq i \leq d} a_i \partial_{p_i} + \sum_{1 \leq j \leq d} b_j \partial_{q_j} + \sum_{1 \leq k \leq m-2d} c_k \partial_{\lambda_k}$$

with $a_i, b_j$ and $c_k$ in $S(\mathfrak{g})_I \simeq \mathbb{R}(\lambda)[x_1, \ldots, x_m] \simeq \mathbb{R}(\lambda)[p, q]$.

Let us then fix the system of linear coordinates $(x_i)$ on $\mathfrak{g}^*$ and the local system of coordinates $(p, q, \lambda)$ as above. From now on, the localized algebra $S(\mathfrak{g})_I$ will be identified with $\mathbb{R}(\lambda)[p, q]$.

**Definition 1.**

A multilinear map $F : S(\mathfrak{g}) \times \ldots \times S(\mathfrak{g}) \to S(\mathfrak{g})$ is said to be differential if it is given by differential operators (i.e. of finite order) on each argument. Otherwise, it is called algebraic or non differential.

Now, let $D$ be a differential operator on $S(\mathfrak{g})$. Then, we can write

$$D(u_1, \ldots, u_s) = \sum D_{\alpha_1\ldots\alpha_s} \partial_{\alpha_1}(u_1) \ldots \partial_{\alpha_s}(u_s)$$

where the multi-indexes $\alpha_i$ are relative to the variables $(x_i)$ and where the coefficients $D_{\alpha_1\ldots\alpha_s}$ belong to $S(\mathfrak{g})$. The same operator can be written as a differential operator, say $\bar{D}$, in the variables $(p, q, \lambda)$ just by performing, for the coefficients and for the operators $\partial_{\alpha_i}$, the change of variables from $(x_i)$ to $(p, q, \lambda)$. Such a differential operator $D$ (or $\bar{D}$) can naturally be extended to the localized algebra $S(\mathfrak{g})_I$. Now, let $A$ be
an algebraic operator on $S(\mathfrak{g})$. $A$ can be decomposed into an infinite sum $\sum N A_N$ of differential operators of the form

$$A_N(u_1, ..., u_s) = \sum_{|\alpha_1| + ... + |\alpha_s| = N} A_{\alpha_1...\alpha_s} \partial_{\alpha_1}(u_1) \ldots \partial_{\alpha_s}(u_s)$$

where $\alpha_i = (\alpha_{i1}, ..., \alpha_{im})$ are multi-indexes relative to the variables $(x_i)$ and $|\alpha_i| = \alpha_{i1} + ... + \alpha_{im}^m$. Let $\tilde{A}(l)$ be the operator defined by $\tilde{A}(l) = \sum t^N \tilde{A}_N$. Clearly, $\tilde{A}(1)$ coincides with $A$ on $S(\mathfrak{g})$ and $\tilde{A}(l)$ sends $S(\mathfrak{g})_I$ into the (formal) algebra $\mathbb{R}(\lambda)[p, q][[t]]$ of formal series in $t$ with coefficients in $\mathbb{R}(\lambda)[p, q]$. In the following, we shall use the algebraic notion of tangential operators given in [5].

**Definition 2.**

A multilinear map $F : S(\mathfrak{g}) \times ... \times S(\mathfrak{g}) \to S(\mathfrak{g})$ is said to be tangential if $F$ vanishes on constants and if for each $\Delta$ in $I = I(\mathfrak{g})$, for every $u_1, ..., u_s$ in $S(\mathfrak{g})$ and for all $1 \leq l \leq s$,

$$\Delta F(u_1, ..., u_s) = F(u_1, ..., \Delta u_l, ..., u_s).$$

Such a tangential operator can be uniquely extended to the localized algebra $S(\mathfrak{g})_I$ of rational functions with non zero invariant denominators by

$$\hat{F}\left(\frac{u_1}{Q_1}, ..., \frac{u_s}{Q_s}\right) = \frac{1}{Q_1...Q_s} F(u_1, ..., u_s)$$

where the $u_i$ are in $S(\mathfrak{g})$ and the $Q_i$ are elements of $I$.

Now, it is possible to characterize tangential operators thanks to the variables $(p, q, \lambda)$. Indeed, if $F$ is a tangential map on $S(\mathfrak{g})$, then its extension $\hat{F}$ to $S(\mathfrak{g})_I$ satisfies

$$\hat{F}(v_1, ..., \lambda_k v_i, ..., v_s) = \lambda_k \hat{F}(v_1, ..., v_s)$$

for all generic invariant $\lambda_k$ and for all $v_i$ in $S(\mathfrak{g})$. It follows that $\hat{F}$ is of the form

$$\hat{F}(v_1, ..., v_s) = \sum F_{\tilde{\alpha}_1...\tilde{\alpha}_s}(p, q, \lambda) \partial_{\tilde{\alpha}_1}(v_1) \ldots \partial_{\tilde{\alpha}_s}(v_s)$$

here the $\tilde{\alpha}_i$ are multi-indexes relative to the variables $(p, q, \lambda)$, the coefficients $F_{\tilde{\alpha}_1...\tilde{\alpha}_s}$ belong to $S(\mathfrak{g})_I \simeq \mathbb{R}(\lambda)[p, q]$ and the $\partial_{\tilde{\alpha}_i}$ do not include derivatives with respect to the variables $(\lambda_k)$.

Conversely, suppose that $C = \sum C_N$ is an algebraic operator on $S(\mathfrak{g})$ such that $\tilde{C}(1)$ can be expressed without derivatives with respect to $(\lambda_k)$, then $C$ is tangential.

**Remark 3.**

Frequently, a tangential operator $F$ on $S(\mathfrak{g})$, in the sense of Definition 2, is not only tangential on the set $V$ of generic orbits but also on the set $\Omega$ of all orbits of maximal dimension (see the example of $\mathfrak{g}_{54}$ in Section 4 for instance). More precisely, each
Definition 4.
A s-linear map $C : S(\mathfrak{g}) \times \ldots \times S(\mathfrak{g}) \to S(\mathfrak{g})$ is said to be homogeneous of degree $-n$ if for all $u_1, \ldots, u_s$ in $S^{d_1}(\mathfrak{g}), \ldots, S^{d_s}(\mathfrak{g})$ respectively, $C(u_1, \ldots, u_s)$ is in $S^{d_1+\ldots+d_s-n}(\mathfrak{g})$.

Now, let us recall that each differential operator on $S(\mathfrak{g})$ can naturally be extended to $S(\mathfrak{g})$. If $u$ belongs to $S(\mathfrak{g})$, we will denote by $||u||$ the degree of $u$ as a polynomial in $p$. Then, the following definition makes sense.

Definition 5.
A s-differential operator $D : S(\mathfrak{g}) \times \ldots \times S(\mathfrak{g}) \to S(\mathfrak{g})$ is said to be correct of degree $-n$ if for all $u_i$ in $S(\mathfrak{g})$, such that $||u_i|| = d_i$, $(1 \leq i \leq s)$,

$$||D(u_1, \ldots, u_s)|| \leq (d_1 + \ldots + d_s) - n.$$ 

An algebraic operator $C = \sum C_N$ on $S(\mathfrak{g})$ is said to be correct of degree $-n$ if all the differential operators $C_N$ are correct of degree $-n$.

Let $\mathcal{A}$ be a commutative and associative algebra and $M$ be a $\mathcal{A}$-bimodule. We can introduce the graded $\mathcal{A}$-module of Hochschild cochains $C^*(\mathcal{A}, M)$ that is the $\mathcal{A}$-module of multilinear maps with values in $M$.

Definition 6.
The coboundary of a s-cochain $C : \mathcal{A} \times \ldots \times \mathcal{A} \to M$ is the $(s+1)$-cochain $\delta C$ defined by

$$\delta C(u_1, \ldots, u_{s+1}) = u_1 C(u_2, \ldots, u_{s+1})$$
$$+ \sum_{1 \leq k \leq s} (-1)^k C(u_1, \ldots, u_k u_{k+1}, \ldots, u_{s+1})$$
$$+ (-1)^{s+1} C(u_1, \ldots, u_s) u_{s+1}.$$ 

The $s$th Hochschild cohomology space will be denoted by $H^s(\mathcal{A}, M)$ or $H^s_{diff}(\mathcal{A}, M)$ if we restrict ourselves to differential cochains.

Now, let $C_n^* = C^*_{n, grad, nc}(S(\mathfrak{g}))$ be the space of homogeneous of degree $-n$, correct of degree $-n$, vanishing on constants, s-linear operators on $S(\mathfrak{g})$, differential or not. We denote by $C^*_{n,diff} = C^*_{n,grad,nc,diff}(S(\mathfrak{g}))$ the subspace of operators of $C_n^*$ which are differential. $(C_n^*, \delta)$ and $(C^*_{n,diff}, \delta)$ are subcomplexes of $(C^*(S(\mathfrak{g}), S(\mathfrak{g})), \delta)$. These subcomplexes give rise to well graded cohomology spaces denoted by $H^*_{n,grad,nc}(S(\mathfrak{g}))$.
and $H_{n,\text{grad,nc,}\text{diff}}^*(S(\mathfrak{g}))$. It is useful to know when these spaces vanish. In particular, we have the following result.

**Proposition 7.**

\[ H_{n,\text{grad,nc}}^3(S(\mathfrak{g})) = H_{n,\text{grad,nc,}\text{diff}}^3(S(\mathfrak{g})) = \{0\} \quad \forall n \geq 4. \]

**Proof:** Let $E$ be an element of $C^3_{n,\text{grad,nc}}(S(\mathfrak{g}))$ such that $\delta E = 0$. Clearly, $E(u, v, w)$ can be decomposed into a sum of two cocycles $E_1 + E_2$ with $E_1$ symmetric in $u, v, w$ and $E_2$ skew-symmetric in $u, v, w$. In [14] p.148, G. Pinczon shows that if $N$ denotes the algebra of smooth functions over $\mathfrak{g}^*$, then

\[ H^3(S(\mathfrak{g}), N) = H^3_{\text{diff}}(S(\mathfrak{g}), N) = H^3_{\text{diff}}(N, N). \]

It is well known [16] that $H^*_{\text{diff}}(N, N)$ is isomorphic to the space of skew multivectors fields over $\mathfrak{g}^*$. Thus, there exist two Hochschild cochains $C_1$ and $C_2$ in $C^2(N, N)$ such that

- $E_1 = \delta C_1$ with $C_1(u, v)$ skew-symmetric in $u, v$
- $E_2 = \delta C_2 + A$ with $C_2(u, v)$ symmetric in $u, v$ and where $A(u, v, w)$ is

\[ A(u, v, w) = \sum a_{ijk} \partial_i(u)\partial_j(v)\partial_k(w) \]

with completely skew-symmetric coefficients.

Since

\[ \oint_{(u,v,w)} E_2(u, v, w) := E_2(u, v, w) + E_2(v, w, u) + E_2(w, u, v) = 3A(u, v, w), \]

$A$ is necessarily homogeneous of degree $-n$ and the coefficients $a_{ijk}$ are polynomials of degree $3 - n$. Thus, for $n \geq 4$, $n - 3 < 0$ and $A \equiv 0$. Moreover, since $E$ vanishes on constants, we can suppose it is the same for $C_1$ and $C_2$ just by replacing $C_i$ by $C_i - \delta T_i$ where $T_i$ is defined by $T_i(u) = C_i(u, 1)$ ($i = 1, 2$).

2) Let us now prove that $C_1$ can be chosen in $C_{n,\text{grad,nc}}^2(S(\mathfrak{g}))$: $E_1$ and $C_1$ can be decomposed into an infinite sum $\sum_{N \geq 0} E_{1,N}$ ($\sum C_{1,N}$ respectively) of differential operators in the variables $(x_i)$ of the form

\[ E_{1,N}(u, v, w) = \sum_{a+b+c=N} E_{k_1,...k_a,l_1,...l_b,m_1,...,m_c} \partial_{k_1,...,k_a}(u)\partial_{l_1,...,l_b}(v)\partial_{m_1,...,m_c}(w). \]

Respectively,

\[ C_{1,N}(u, v) = \sum_{a+b=N,a \geq b} C_{k_1,...,k_a,l_1,...l_b}(\partial_{k_1,...,k_a}(u)\partial_{l_1,...,l_b}(v) - \partial_{k_1,...,k_a}(v)\partial_{l_1,...,l_b}(u)) \quad \text{(*)} \]

where the coefficients $C_{k_1,...,k_a,l_1,...l_b}$ are supposed to be symmetric in the indexes $k_i$ and in the indexes $l_j$, and such that $C_{k_1,...,k_a,l_1,...l_b} = -C_{l_1,...,l_b,k_1,...k_a}$ (if $a = b$).
Since $E_1$ vanishes on constants, $E_{1,N} = 0$ if $N < 3$. Thus,

$$E_1 = \sum_{N \geq 3} E_{1,N} := \sum_{N \geq 3} (E_1)_N = \sum_{N \geq 3} (\delta C_1)_N = \sum_{N \geq 3} \delta(C_{1,N}).$$

The last equality directly comes from the definition of the Hochschild coboundary. In the following, we shall assume that $C_1 = \sum_{N \geq 3} C_{1,N}$ because $C_{1,1}$ and $C_{1,2}$ are not involved in the expression of $E_1 = \delta C_1$.

Then, we want to prove that every $C_{1,N}$ ($N \geq 3$) sends $S(g) \times S(g)$ into $S(g)$ and is homogeneous and correct of degree $-n$, or equivalently, to show that every $C_{k_1...k_a,l_1...l_b}$ is an element of $S(g)$ homogeneous of degree $a + b - n = N - n$ and that

$$||C_{k_1...k_a,l_1...l_b}|| \leq ||X_{k_1}|| + \ldots + ||X_{k_a}|| + ||X_{l_1}|| + \ldots + ||X_{l_b}|| - n.$$

To this end, we use a technique which can be found in [11] p.238-242 or in Gutt’s thesis [9].

By (*), $C_{1,N}$ is a finite sum of terms of type $(a,b)$ ($a \geq b$). Let $(r,s)$ be the highest of the types $(a,b)$ ($(a,b) > (a',b')$ if $a > a'$ or if $a = a'$ and $b > b'$). Let also $C_{k_1...k_r,l_1...l_s}$ be the topmost coefficient with respect to lexicographical order in the indexes. We shall call principal part of $C_{1,N}$ the unique term $P$ of type $(r,s)$ defined by

$$P(u,v) = C_{k_1...k_r,l_1...l_s}(\partial_{k_1...k_r}(u)\partial_{l_1...l_s}(v) - \partial_{k_1...k_r}(v)\partial_{l_1...l_s}(u)).$$

The principal part $P$ of $C_{1,N}$ becomes in $\delta(C_{1,N})$

$$-C_{k_1...k_r,l_1...l_s}(\partial_{k_1...k_r}(uv)\partial_{l_1...l_s}(v) - \partial_{k_1...k_r}(w)\partial_{l_1...l_s}(uv))$$

$$+ C_{k_1...k_r,l_1...l_s}(\partial_{k_1...k_r}(u)\partial_{l_1...l_s}(vw) - \partial_{k_1...k_r}(vw)\partial_{l_1...l_s}(u))$$

up to terms without any derivatives of $u$ or of $w$.

By construction, there are only three cases to consider:

- if $r \geq s, r \geq 2, s \geq 2$ : There is only one term of type $(r,s-1,1)$ in $\delta(C_{1,N})$ which corresponds to the principal part and which can be written, up to some constant coefficient,

$$C_{k_1...k_r,l_1...l_s}\partial_{k_1...k_r}(u)\partial_{l_1...l_s-1}(v)\partial_{l_s}(w).$$

- if $r \geq 3, s = 1$ : The only term of type $(r-1,1,1)$ in $\delta(C_{1,N})$ can be written, up to an eventual constant coefficient,

$$C_{k_1...k_r,l_1}\partial_{k_1...k_{r-1}}(u)\partial_{k_r}(v)\partial_{l_1}(w).$$

In these two cases, the coefficients $C_{k_1...k_r,l_1...l_s}$ are convenient.

- if $r = 2, s = 1$ : We get the following terms of type $(1,1,1)$ in $\delta(C_{1,N})$, up to some constant coefficient,

$$(C_{ij,k} + C_{jk,i})\partial_{i}(u)\partial_{j}(v)\partial_{k}(w).$$

Thus, $(C_{ij,k} + C_{jk,i})$ is polynomial of degree $3 - n$. For $n \geq 4$, $n - 3 < 0$ thus $C_{ij,k} + C_{jk,i} = 0$. By cyclic summation, we find $C_{ij,k} = 0$. In other words, there are no terms of type $(2,1)$ in $C_{1,N}$. Then, as we see, the principal part $P$ of $C_{1,N}$ is
homogeneous and correct of degree $-n$. We can now repeat the proof for the principal part of $(C_{1,N} - P)$... A step-by-step application of the same arguments finally shows that all the $C_{1,N}$ (thus also $C_1 = \sum C_{1,N}$) belong to $C_n^2$.

3) We apply the same method for $C_2$: we start by decomposing $E_2$

$$E_2 = \sum_{N \geq 3} E_{2,N} = \sum_{N \geq 3} (\delta C_2)_N = \sum_{N \geq 3} \delta(C_{2,N}).$$

We can suppose that $C_2 = \sum_{N \geq 3} C_{2,N}$.

Now, the principal part of $C_{2,N}$ of type $(r, s)$ can be written as follows

$$C_{k_1,\ldots,k_r,l_1,\ldots,l_s}(\partial_{k_1,\ldots,k_r}(u)\partial_{l_1,\ldots,l_s}(v) + \partial_{k_1,\ldots,k_r}(v)\partial_{l_1,\ldots,l_s}(u)).$$

- The cases $r \geq s \geq 2$ and $r \geq 3, s = 1$ are the same as above.
- If $r = 2, s = 1$: The terms of type $(1,1,1)$ in $\delta(C_{2,N})$, up to some constant coefficients, are

$$(C_{i,j,k} - C_{j,k,i})\partial_i(u)\partial_j(v)\partial_k(w).$$

Since $n \geq 4$, $(C_{i,j,k} - C_{j,k,i})$ is polynomial of degree $n - 3 < 0$ and $C_{i,j,k} = C_{j,k,i}$. Therefore, the $C_{i,j,k}$ do not appear in $\delta(C_{2,N})$ and we can remove every terms of type $(2,1)$ from the expression of $C_{2,N}$. As before, we succeed in proving that all the $C_{2,N}$ with $N \geq 3$ (thus also $C_2 = \sum_{N \geq 3} C_{2,N}$) are elements of $C_{n,\text{grad,nc}}^2(S(g))$.

This ends the proof for non-differential cochains. Obviously, the same can be done for differential cochains.

**Remark 8.**

If $E_1$ is a symmetric 3-cocycle in $C_3^3$, 1) and 2) are still valid. There exists $C_1$ in $C_3^2$ such that $E_1 = \delta C_1$. The only difference is that the terms of type $(2,1)$ of $C_1$ have now constant coefficients.

3. Tangential and well graded deformation of $S(g)$

**Definition 9.**

A graded star product of $S(g)$ is a bilinear map from $S(g) \times S(g)$ to $S(g)[[\nu]]$ defined by

$$(u, v) \rightarrow u \ast v = uv + \{u, v\} \nu + \sum_{n \geq 2} C_n(u, v) \nu^n$$

where the cochains $C_n$ are operators on $S(g)$ with values in $S(g)$ satisfying the following properties.
Moreover, if $C$ and well graded star product of $S$, then $C$ up to order $n$.

Clearly, $E$ is said to be a well graded star product of $S$ if the cochains $C_n$ are both homogeneous of degree $-n$ and correct of degree $-n$.

Moreover, $*$ is said to be a tangential star product of $S$ if the $C_n$ are tangential operators on $S$.

**Theorem 10.**

Suppose we know a tangential operator $C_2$ on $S$ homogeneous and correct of degree $-2$, such that $u \ast v = vw + \{u, v\}v + C_2(u, v) v^2$ is associative up to order 3 in $v$. Then $C_2$ is the second order term of a tangential, well graded star product of $S$.

Moreover, if $C_2$ is differential, $C_2$ is the second order term of a differential, tangential and well graded star product of $S$.

Proof: Let us assume that we have found tangential, homogeneous and correct $C_2, \ldots, C_n$ ($n \geq 3$) such that $u \ast v = vw + \{u, v\}v + \sum_{2 \leq k \leq n-1} C_k(u, v) v^k$ is associative up to order $n$ in $v$. Consider then the Hochschild cocycle $E_n$ defined by

$$E_n(u, v, w) = \sum_{r \geq 1, s \geq 1, r + s = n} C_r(C_s(u, v), w) - C_r(u, C_s(v, w)).$$

Clearly, $E_n$ is homogeneous and correct of degree $-n$. Thanks to Proposition 7 and Remark 8, we can already say that there exists an operator $C_n$ on $S$ such that $E_n = \delta C_n$, $C_n(u, v) = (-1)^n C_n(v, u)$, $C_n(1, v) = 0$ for all $u, v$ in $S$ and so that $C_n$ is both homogeneous of degree $-n$ and correct of degree $-n$. It remains to show that $C_n$ is tangential.

First, by transposing the equality $E_n = \delta C_n$ in coordinates $(p, q, \lambda)$, one obtains: $\tilde{E}_n = \delta(\tilde{C}_n(1))$ where $\tilde{\ }$ and $\tilde{\ }$ have the same meaning as in Section 1. To make the writing simpler, we forget the $n$ and we note $\tilde{E} = E_n$ and $\tilde{C} = C_n$. As in Proposition 7, we decompose $\tilde{E}$ and $\tilde{C}_n(1)$ in an infinite sum $\sum_{K \geq 0} \tilde{E}_K (\sum_{K \geq 0} \tilde{C}_K$ respectively) of operators in the variables $(p, q, \lambda)$ of the form

$$\tilde{E}_K(u, v, w) = \sum_{a+b+c=K} E_{k_1 \ldots k_a l_1 \ldots l_b m_1 \ldots m_c}(u) \partial_{k_1 \ldots k_a}(v) \partial_{l_1 \ldots l_b}(w).$$

Respectively,

$$\tilde{C}_K(u, v) = \sum_{a+b=K, a \geq b} C_{k_1 \ldots k_a l_1 \ldots l_b}(\partial_{k_1 \ldots k_a}(u) \partial_{l_1 \ldots l_b}(v) + (-1)^n \partial_{k_1 \ldots k_a} v \partial_{l_1 \ldots l_b} u).$$
\( E \) vanishes on constants, \( \hat{E}_K = 0 \) for \( K < 3 \). Thus,
\[
\sum_{K \geq 3} \hat{E}_K = \sum_{K \geq 3} \delta(\tilde{C}_K).
\]

**First case:** \( n \) is odd \((n \geq 3)\)

\( \tilde{C}_{(1)} = \sum_{K \geq 3} \tilde{C}_K + \tilde{C}_2 \). But, since \( C \) is correct of degree \(-n\) and since \( n \geq 3 \), \( \tilde{C}_2 = 0 \). Let us now prove that all the \( \tilde{C}_K \) \( (K \geq 3) \) do not involve derivatives with respect to \( (\lambda_k) \). To this end, we proceed as in Proposition 7. We consider first the principal part \( \hat{C} \) of \( \tilde{C}_K \) of type \((r, s)\) in the variables \((p, q, \lambda)\). Three cases have to be considered. The cases \( r \geq s \geq 2 \) and \( r \geq 3, s = 1 \) are directly solved. Now, if \( r = 2 \) and \( s = 1 \), the terms of type \((1, 1, 1)\) in \( \delta \tilde{C}_K \) up to some constant coefficients are
\[
(C_{ij,k} - C_{jk,i}) \partial_i(u) \partial_j(v) \partial_k(w).
\]
Suppose that some derivative with respect to \((\lambda_k)\) appears, then \( (C_{ij,k} - C_{jk,i}) \) should be zero. And, by cyclic summation, we find \( C_{ij,k} = 0 \). Therefore, we conclude that there are no derivatives with respect to \((\lambda_k)\) in the principal part. Then, we repeat the proof step by step and finally get that all the \( \tilde{C}_K \) (thus also \( \tilde{C}_{(1)} = \sum_{K \geq 3} \tilde{C}_K \)) do not involve derivatives with respect to the variables \((\lambda_k)\). Thus, \( C = C_n \) is tangential if \( n \) is odd.

**Second case:** \( n \) is even \((n \geq 4)\)

\( \tilde{C}_{(1)} = \sum_{K \geq 4} \tilde{C}_K + \tilde{C}_3 + \tilde{C}_2 \). But, since \( C \) is correct of degree \(-n\) and \( n \geq 4 \), \( \tilde{C}_2 = \tilde{C}_3 = 0 \).

As before, we use principal parts to show that all the \( \tilde{C}_K \) for \( K \geq 4 \) do not include derivatives with respect to \((\lambda_k)\). In other words, \( C = C_n \) is also tangential if \( n \) is even. This ends the proof.

**Remark 11.**

It is possible to define cohomology spaces related to tangential and well graded deformations of \( S(\mathfrak{g}) \). For the moment, we denote by \( C^*_{n,\text{tang,grad}} \) the space of \( s \)-linear operators on \( S(\mathfrak{g}) \), which are tangential, homogeneous of degree \(-n\) and correct of degree \(-n\). Endowed with the Hochschild coboundary, \( C^*_{n,\text{tang,grad}} \) becomes a complex. Let \( H^*_{n,\text{tang,grad}}(S(\mathfrak{g})) \) be the corresponding cohomology. Then, we see that Proposition 7 together with Theorem 10 contain the computation of \( H^3_{n,\text{tang,grad}}(S(\mathfrak{g})) \). Similarly, one can prove the vanishing of the second spaces of this cohomology, \( H^2_{n,\text{tang,grad}}(S(\mathfrak{g})) \), for \( n \geq 2 \). More exactly, the following facts hold

(i) if \( C(u, v) \) is a cocycle, skew-symmetric in \( u, v \), tangential, homogeneous of degree \(-(2k - 1)\) and correct of degree \(-(2k - 1)\), \( k \geq 2 \), then \( C \equiv 0 \);

(ii) if \( C(u, v) \) is a cocycle, symmetric in \( u, v \), tangential, homogeneous of degree \(-2k\) and correct of degree \(-2k\), \( k \geq 1 \), then we can suppose that \( C = \delta R \) where \( R \) is tangential, homogeneous of degree \(-2k\) and correct of degree \(-2k\).
Theorem 12.
Two tangential and well graded star products of $S(\mathfrak{g})$ are always tangentially equivalent. One can find an equivalence operator of the form

$$T = Id + \sum_{k \geq 1} T_{2k} \nu^{2k}$$

where all the $T_{2k}$ are tangential operators from $S(\mathfrak{g}) \times S(\mathfrak{g})$ to $S(\mathfrak{g})$, homogeneous of degree $-2k$ and correct of degree $-2k$. (The homogeneity property also implies that for all $k$, each term of $T_{2k}$ is of order $\geq 2k$.)

Proof: The result is a straightforward consequence of the previous remark. Indeed, let $\ast, \ast'$ be two tangential and well graded star products of $S(\mathfrak{g})$. Let $k \geq 1$. Assume that we found $T_0, \ldots, T_{2k-2}$, such that $T_0 = Id$, that every $T_{2j} (j \geq 1)$ is tangential, homogeneous of degree $-2j$, correct of degree $-2j$ and that the star product $\ast''$ defined by

$$u \ast'' v = H^{-1}(H(u) \ast' H(v))$$

where $H = Id + \ldots + T_{2k-2} \nu^{2k-2}$, satisfies $C''_j(u, v) = C_j(u, v)$ for all $j \leq 2k-2$. The associativity condition leads to

$$\delta(C''_{2k-1} - C_{2k-1}) = 0.$$  

Afterwards, either $k = 1$ and $C''_1(u, v) = C_1(u, v) = \{u, v\}$ or $k \geq 2$ and $C''_{2k-1} = C_{2k-1}$ (Remark 11, (i)). Then, we obtain:

$$\delta(C''_{2k} - C_{2k}) = 0.$$  

Thus, there exists $T_{2k}$ as announced (Remark 11, (ii)) so that $C''_{2k} = C_{2k} + \delta T_{2k}$. A simple induction enables us to construct the equivalence operator $T$ and thereby ends the proof.

4. Applications and examples

Let us first recall the construction of the Gutt star product $\ast_G$ defined on the symmetric algebra $S(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map. Denote by $[u]_k$ the $k$th component of an element $u$ of $U(\mathfrak{g})$ relative to the canonical decomposition $U(\mathfrak{g}) = \oplus \sigma(S^k(\mathfrak{g}))$. If $P, Q$ are homogeneous polynomials of degree $r, s$ respectively, then

$$P \ast_G Q = \sum_{n \geq 0} C_{n,G}(P, Q) \nu^n = \sum_{n \geq 0} \sigma^{-1}([\sigma(P), \sigma(Q)]_{(r+s-n)}) (2\nu)^n.$$  

Using linearity to extend the above expression to all polynomials, we get the Gutt star product. In the literature, the same star product is sometimes called the star product coming from the enveloping algebra via Poincaré-Birkhoff-Witt. One checks that $\ast_G$ is differential and graded.
Now, we are interested in the example of $\mathfrak{g}_{54}$. This nilpotent Lie algebra is defined by the following brackets
\[
[X_5, X_4] = X_3, [X_5, X_3] = X_2, [X_4, X_3] = X_1.
\]
The quotient field of $I(\mathfrak{g}_{54})$ is generated by two central elements, namely $X_1$ and $X_2$, and by $\Delta = \frac{X_3^2}{2} + X_1X_5 - X_2X_4$.
A simple calculation shows that, up to a normalization, the second order term $C_{2,G}$ satisfies
\[
C_{2,G}(\Delta, .) = \frac{x_2^2}{6} \partial_{44} + \frac{x_1x_2}{3} \partial_{45} + \frac{x_2^2}{6} \partial_{55}.
\]
It is thus clear that $*_{G}$ is not tangential. If it was, $C_{2,G}(\Delta, .)$ would be reduced to zero.
A natural idea to know whether a tangential and well graded star product of $S(\mathfrak{g}_{54})$ exists or not, is to try to correct $C_{2,G}$ by means of an operator $T$ on $S(\mathfrak{g}_{54})$ such that
\[
C_{2,G}(\Delta, v) + \delta T(\Delta, v) = 0 \text{ for all } v \text{ in } S(\mathfrak{g}_{54}).
\]
A possible $T$ is
\[
T = \sum_{n \geq 4} (-1)^n \frac{2n-3}{6(n-2)!} x_3^{n-4} \left( x_2^2 \partial_3^{n-2} \partial_{55} + x_1^2 \partial_3^{n-2} \partial_{44} + 2x_1x_2 \partial_3^{n-2} \partial_{45} \right).
\]
If we note $\sigma_3 = \sum_{n \geq 0} \frac{(-2x_3)^n}{n!} \partial_3^n$, $T$ can be written in the form
\[
T = A_{55} \partial_{55} + A_{45} \partial_{45} + A_{44} \partial_{44} + A_{355} \partial_{355} + A_{345} \partial_{345} + A_{344} \partial_{344},
\]
where
\[
A_{55} = \frac{x_2^2}{12x_3^2} (\sigma_3 - Id)
\]
\[
A_{45} = \frac{x_1x_2}{x_3^2} (\sigma_3 - Id)
\]
\[
A_{44} = \frac{x_1^2}{12x_3^2} (\sigma_3 - Id)
\]
\[
A_{355} = \frac{x_2^2}{6x_3}
\]
\[
A_{345} = \frac{x_1x_2}{3x_3}
\]
\[
A_{344} = \frac{x_1^2}{6x_3}.
\]
One immediately sees that $C_2 = C_{2,G} + \delta T$ is tangential and homogeneous of degree $-2$. Now, we need to prove that $C_2$ is also correct of degree $-2$. Let us first introduce the canonical variables
\[
(p = x_4, q = \frac{x_3}{x_1}, \lambda_1 = x_1, \lambda_2 = x_2, \lambda_3 = (\frac{x_3^2}{2} + x_1x_5 - x_2x_4)).
\]
Then, changing variables \( ((x_i) \to (p, q, \lambda)) \) and using the notation of Section 1, we obtain
\[
\tilde{C}_{2,G}(u, v) = \partial_{pp}(u)\partial_{qq}(v) - 2\partial_{pq}(u)\partial_{pq}(v) + \partial_{qq}(u)\partial_{pp}(v) \\
+ \frac{1}{3}\lambda_1^2(\partial_{\lambda_3}(u)\partial_{p}(v) + \partial_{\lambda_3}(v)\partial_{p}(u)) \\
- \frac{1}{3}\lambda_1^2(\partial_{pp}(u)\partial_{\lambda_3}(v) + \partial_{pp}(v)\partial_{\lambda_3}(u)) \\
\tilde{T}_{(1)}(u) = \sum_{n \geq 4} (-1)^n \frac{2n-3}{6(n-2)!} q^{n-4}(q \lambda_1^2 \partial_{\lambda_3} + \partial_{q})^{n-2}\partial_{pp}(u).
\]
Recall now that \( \tilde{C}_{2(1)} = \tilde{C}_{2, G} + \delta\tilde{T}_{(1)} \) coincides with \( C_2 \) on \( S(g) \). Thus, the above expressions mean that \( C_2 \) is correct of degree \(-2\). By Theorem 10 (see Section 3), this is sufficient to show the existence of an algebraic, tangential and well graded star product on \( \mathfrak{g}_5^* \). That is the best we can do, because there is no deformation on \( \mathfrak{g}_5^* \) which is both tangential and differential [1, 5]. Remark also that, since the only polynomials on \( \mathfrak{g}_5^* \), whose restriction to an orbit of maximal dimension is zero, are invariant (see [13] p.23), our deformation is tangential to all the regular orbits (i.e. orbits of maximal dimension).

Nevertheless, as differential operators are more convenient to handle than algebraic maps, let us mention the possibility of constructing a differential and tangential deformation on the subset \( \Omega \) of all the orbits of maximal dimension. Note that \( \Omega \) is a regular Poisson manifold and that
\[
\Omega = \{ \xi = (\xi_1, ..., \xi_5) \in \mathfrak{g}_5^* \text{ such that } \xi_1^2 + \xi_2^2 + \xi_3^2 \neq 0 \}.
\]
We found an explicit expression of an operator \( C'_2 \) with homogeneous coefficients in \( C^\infty(\Omega) \), which is both tangential and differential. Here it is
\[
C'_2 = C_{2,G} + \delta T',
\]
where
\[
T' = A_{453}\partial_{453} + A_{355}\partial_{355} + A_{455}\partial_{455} + A_{344}\partial_{344} + A_{445}\partial_{445} \\
+ A_{555}\partial_{555} + A_{444}\partial_{444} \\
r = x_1^2 + x_2^2 + x_3^2 \\
A_{453} = \frac{x_1x_2x_3}{3r} \\
A_{355} = \frac{x_3x_2^2}{6r} \\
A_{455} = \frac{-x_3^3 + 2x_1^2x_2}{6r} \\
A_{344} = \frac{x_1^2x_3}{6r}
\]
\[ A_{445} = \frac{x_1^3 - 2x_1x_2^2}{6r} \]
\[ A_{555} = \frac{x_1x_2^2}{6r} \]
\[ A_{444} = \frac{-x_1^2x_2}{6r} . \]

Further examples are given by Pedersen in [13]. Let us say a few words about \( g_{6,12} \) ([13] p.87) and \( g_{6,14} \) ([13] p.99).

- The Lie algebra structure of \( g_{6,12} \) is defined by the non vanishing brackets
  \[ [X_6, X_5] = X_4, [X_6, X_4] = X_3, [X_6, X_3] = X_2, [X_5, X_2] = -X_1, [X_4, X_3] = X_1. \]
  The quotient field of \( I(g_{6,12}) \) is generated by \( X_1 \) and \( \frac{x_2^2}{2} - X_2X_4 + X_1X_6. \)

- The Lie algebra structure of \( g_{6,14} \) is defined by the following brackets
  \[ [X_6, X_5] = X_4, [X_6, X_4] = X_3, [X_6, X_3] = X_2, [X_5, X_4] = X_2, [X_5, X_2] = -X_1, \]
  \[ [X_4, X_3] = X_1. \]
  Moreover, the quotient field of \( I(g_{6,14}) \) is generated by \( X_1 \) and \( \frac{x_3^2}{3} - \frac{x_1x_2^2}{2} + X_1X_2X_4 - X_1^2X_6. \)

For these two examples, we may explicitly define an algebraic, tangential \( C_2 \) on the symmetric algebra, and a differential, tangential \( C'_2 \) on \( C^\infty(\Omega) \), \( \Omega \) denoting the open set of regular orbits, with similar argument as for \( g_{5,4} \).

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