The Parameter Rigid Flows on Orientable 3-Manifolds

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Abstract. A flow defined by a nonsingular smooth vector field $X$ on a closed manifold $M$ is said to be parameter rigid if given any real valued smooth function $f$ on $M$, there are a smooth function $g$ and a constant $c$ such that $f = X(g) + c$ holds. We show that the parameter rigid flows on closed orientable 3-manifolds are smoothly conjugate to Kronecker flows on the 3-torus with badly approximable slope.

1. Introduction

Throughout this paper we work in the $C^\infty$-category: any manifold, function, diffeomorphism, form, vector field e.t.c. are to be of class $C^\infty$. Let $X$ be a nonsingular vector field on a closed manifold $M$ that defines a flow $\phi^t$.

Definition 1.1. The flow $\phi^t$ is called parameter rigid if for any function $f$ on $M$, there are a function $g$ and a constant $c$ such that $f = X(g) + c$ holds.

It is well known and easy to show that the parameter rigidity is equivalent to the following property: if $\psi^t$ be another nonsingular flow defined by a vector field $fX$, where $f$ is a nowhere vanishing function, then there are an orbit preserving diffeomorphism $F$ of $M$ and a nonzero constant $c$ such that

$$\psi^t(F(x)) = F(\phi^{ct}(x)).$$

The only known examples of parameter rigid flows are Kronecker flows on tori with badly approximable (sometimes called Diophantine or non-Liouville) slope, and A. Katok has conjectured that in fact they are the all ([K]). In this paper we show a partial result supporting this conjecture.

Theorem 1.2. A parameter rigid flow on a closed orientable 3-manifold is smoothly conjugate to a linear flow on the 3-torus with badly approximable slope.

At Paulfest, A. Kocsard has announced the same result ([Ko]).

The method of this paper cannot be applied to the nonorientable 3-manifolds. The difficulty lies in showing that the lift of a parameter rigid flow to the orientable double cover is again parameter rigid.

Thanks are due to the unanimous referee, whose valuable comments are helpful for the shorter and clearer arguments.

1991 Mathematics Subject Classification. Primary 37C10, secondary 37A20, 37E99.

Key words and phrases. Nonsingular flows, Rigidity, Smooth conjugacy, Kronecker flows.

The author was partially supported by Grant-in-Aid for Scientific Research (A) No. 17204007.
2. General properties of parameter rigid flows

Here we collect some basic facts needed in the proof of Theorem 1.2. Let $\varphi^t$ be a parameter rigid flow on a closed $(n+1)$-dimensional manifold, defined by a nonsingular vector field $X$.

(1) The flow $\varphi^t$ is uniquely ergodic, leaves a volume form $\Omega$ invariant, and hence is minimal.

Indeed the Birkhoff average of any smooth function tends to a constant, which is enough for the unique ergodicity, since the smooth functions are dense in the space of continuous functions. For the second statement, let $\Omega_0$ be an arbitrary volume form and define a function $f$ by $\mathcal{L}_X \Omega_0 = f \Omega_0$. Then $\Omega = e^{-g} \Omega_0$ is the desired form, where $g$ is the function obtained by Definition 1.1.

(2) The function $g$ in Definition 1.1 is unique up to a constant sum, and the constant $c$ is given by $c = \int_M f \Omega$.

In fact if $X(h)$ is constant, then it should be 0, and the minimality of the flow implies that $h$ is constant.

(3) The vector space $\Lambda^n(X)$ consisting of n-forms $\omega$ such that $i_X \omega = i_X d\omega = 0$ is one dimensional, spanned by $i_X \Omega$.

Indeed $i_X \Omega$ belongs to $\Lambda^n(X)$ and any n-form in $\Lambda^n(X)$ is a function multiple of $i_X \Omega$. Taking the Lie derivative, one can show the function is constant.

A 1-form $\alpha$ is called normal if $\alpha(X)$ is constant. The normalization $\alpha$ of any 1-form $\alpha'$ is defined to be $\alpha = \alpha' - dg$, where $g$ is a function (unique up to constant sum) such that $\alpha'(X) = X(g) + c$.

(4) A closed normal 1-form $\alpha$ is invariant by the flow i.e. $\mathcal{L}_X \alpha = 0$. By the minimality of the flow, it is either identically zero or nonsingular.

Let $\Lambda^1(X)$ be the space of closed normal 1-forms and let $\epsilon : \Lambda^1(X) \rightarrow H^1(M; \mathbb{R})$ be the map assigning the cohomology class to each form.

(5) The homomorphism $\epsilon$ is an isomorphism.

Indeed the normalization of each closed form belongs to $\Lambda^1(X)$, showing the surjectivity of $\epsilon$. On the other hand an exact normal form $dg$ is identically zero, since if $X(g)$ is a constant, then $g$ is a constant.

3. Proof of the main theorem

Let $\varphi^t$ be a parameter rigid flow defined by a vector field $X$ on a closed orientable 3-manifold $M$. We shall prove Theorem 1.2 dividing into cases.

Case 1. $H^1(M; \mathbb{R}) \neq 0$.

Let $\alpha \in \Lambda^1(X)$ be a closed normal 1-form representing an integral class. Then the equation $\alpha = 0$ defines a fibration of $M$ over the circle. The constant $\alpha(X)$ cannot be 0, since the flow is minimal. Thus the flow has a global cross section, say $\Sigma$. The first return map of $\Sigma$ must be minimal, and especially it does not admit any periodic point. Then by a theorem of Jiang [J], one can show that $\Sigma$ is diffeomorphic to the 2-torus. Now the first return map is cohomologically rigid in the sense of [LS] and is shown in that paper to be conjugate to a translation by a badly approximable vector. We have done with this case.
Case 2. \( H^1(M; \mathbb{R}) = 0 \).

Let \( \Omega \) be the volume form which is left invariant by \( X \). Then since
\[
\mathcal{L}_X \Omega = d\iota_X \Omega = 0,
\]
there is a 1-form \( u \) such that \( \iota_X \Omega = du \). Taking the normalization of the previous section, one may assume that \( u \) is normal, i.e. \( u(X) = c_1 \) is a constant. Then since \( \iota_X (u \wedge du) = c_1 du = \iota_X (c_1 \Omega) \), we have \( u \wedge du = c_1 \Omega \).

Case 2.1. \( c_1 \neq 0 \).

In this case the vector field \( c_1^{-1} X \) generates the Reeb flow of a contact form \( u \). The solution of the Weinstein conjecture in [T] shows that the flow admits a closed orbit, contrary to the minimality.

Case 2.2. \( c_1 = 0 \).

We have \( \iota_X \Omega = du \) and \( u(X) = 0 \). First of all notice that \( u \) is nonsingular. Indeed we have \( \mathcal{L}_X u = 0 \), that is, \( u \) is invariant by the flow \( \varphi^t \). By the minimality of the flow \( \varphi^t \), vanishing of \( u \) at some point would imply that \( u \) is identically zero, which is not the case since \( du \) is nonsingular. As noted before, we have \( u \wedge du = 0 \), that is, the 1-form \( u \) is integrable, and \( du = \eta' \wedge u \) for some 1-form \( \eta' \). Notice that \( \eta'(X) = 0 \). We get
\[
0 = d(\iota_X du) = \mathcal{L}_X du = \mathcal{L}_X \eta' \wedge u + \eta' \wedge \mathcal{L}_X u.
\]
Since \( \mathcal{L}_X u = 0 \), we have \( \mathcal{L}_X \eta' \wedge u = 0 \). That is, \( \mathcal{L}_X \eta' = f_2 u \) for some function \( f_2 \).

Write \( f_2 = X(g_2) + c_2 \) and let \( \eta = \eta' - g_2 u \). Then we have
\[
du = \eta \wedge u, \quad \eta(X) = 0 \quad \text{and} \quad \mathcal{L}_X \eta = c_2 u.\]

Case 2.2.1. \( c_2 = 0 \).

In this case we have \( d\eta \in \Lambda^2(X) \), and thus by (3) of the previous section, \( d\eta = r \iota_X \Omega \) for some constant \( r \). Since \( \eta \wedge u = du = \iota_X \Omega \) is nonvanishing, \( ru - \eta \) is a nonzero element of \( \Lambda^1(X) \cong H^1(M; \mathbb{R}) \), contrary to the assumption of Case 2.

Case 2.2.2. \( c_2 \neq 0 \).

Changing \( X \) and \( u \) by a scalar multiple at the same time one may assume that \( \mathcal{L}_X \eta = -2u \) and still \( du = \iota_X \Omega \). In summary, there are two 1-forms \( u \) and \( \eta \) such that
\[
du = \eta \wedge u = \iota_X \Omega, \quad u(X) = \eta(X) = 0, \quad \iota_X d\eta = -2u.
\]

Since \( \eta \wedge u \) is nonvanishing, that is, \( \eta \) and \( u \) are linearly independent everywhere, there is a 1-form \( \sigma' \) such that \( \Omega = \eta \wedge u \wedge \sigma' \). Then the triplet \( \langle \eta, u, \sigma' \rangle \) is a basis of the space of 1-forms as a module over the ring of functions, and likewise \( \langle \eta \wedge u, u \wedge \sigma', \sigma' \wedge \eta \rangle \) is a basis of the space of 2-forms.

Note that \( \sigma'(X) = 1 \) since \( \iota_X \Omega = du \). Now we have:
\[
0 = \mathcal{L}_X \Omega = \mathcal{L}_X (du \wedge \sigma') = \mathcal{L}_X du \wedge \sigma' + du \wedge \mathcal{L}_X \sigma'.
\]
But \( \mathcal{L}_X du = 0 \), and thus we have \( du \wedge \mathcal{L}_X \sigma' = 0 \). Thus one can write
\[
\mathcal{L}_X \sigma' = f_3 \eta + f_4 u.
\]
Then there are functions \( g_3 \) and \( g_4 \) such that
\[
f_3 = X(g_3) + c_3, \quad f_4 + 2g_3 = X(g_4).
\]
In the last expression, we do not need a constant, since we can alter \( g_3 \) by a constant summand.) Now computation shows that for \( \sigma = \sigma' - g_3\eta - g_4 u \), we have \( \mathcal{L}_X \sigma = c_3\eta \). Summing up, we have obtained
\[
\Omega = \eta \wedge u \wedge \sigma, \quad \sigma(X) = 1, \quad \mathcal{L}_X \sigma = c_3\eta.
\]

We prepare a useful lemma.

**Lemma 3.1.** If \( \mathcal{L}_Xw = a\Omega \) for some 3-form \( w \) and a constant \( a \), then \( a = 0 \) and the form \( w \) is invariant by \( X \).

**Proof.** The proof is immediate by taking the integral over \( M \).

We are going to show that in fact the manifold \( M \) is a quotient of a 3-dimensional Lie group. For this we need to compute \( d\eta \) and \( d\sigma \). First of all let
\[
d\eta = f_5\eta \wedge u + f_6\eta \wedge \sigma + f_7u \wedge \sigma.
\]
Then since \( \iota_X d\eta = -f_6\eta - f_7u = -2u \), we have \( f_6 = 0 \) and \( f_7 = 2 \), that is, \( d\eta = f_5\eta \wedge u + 2u \wedge \sigma \). Now
\[
\mathcal{L}_X (\sigma \wedge d\eta) = c_3\eta \wedge d\eta + \sigma \wedge \mathcal{L}_X d\eta
= c_3\eta \wedge d\eta + \sigma \wedge (-2du)
= 2c_3\eta \wedge u \wedge \sigma - 2\sigma \wedge \eta \wedge u
= 2(c_3 - 1)\eta \wedge u \wedge \sigma.
\]

By lemma 3.1 we have
\[
c_3 = 1 \quad \text{and} \quad \mathcal{L}_X(\sigma \wedge d\eta) = 0.
\]

On the other hand, we have
\[
\mathcal{L}_X (\sigma \wedge d\eta) = \mathcal{L}_X (f_5\sigma \wedge \eta \wedge u) = \mathcal{L}_X (f_5\Omega) = X(f_5)\Omega.
\]
Thus by Lemma 3.1 we have \( X(f_5) = 0 \), that is, \( f_5 \) is a constant, say \( c_5 \).

In summary we obtained:
\[
\mathcal{L}_X \sigma = \eta, \quad d\eta = c_5\eta \wedge u + 2u \wedge \sigma.
\]

An unknown constant \( c_5 \) will be shown to be zero in the way of computing \( d\sigma \).

Let
\[
d\sigma = f_8\eta \wedge u + f_9\eta \wedge \sigma + f_{10}u \wedge \sigma.
\]
Since \( \iota_X d\sigma = -f_9\eta - f_{10}u \), we have \( f_9 = -1 \) and \( f_{10} = 0 \). That is, \( d\sigma = f_8\eta \wedge u - \eta \wedge \sigma \). Then we have
\[
\mathcal{L}_X (\sigma \wedge d\sigma) = \eta \wedge d\sigma + \sigma \wedge \mathcal{L}_X d\sigma = 0 + \sigma \wedge d\eta
= \sigma \wedge (c_5\eta \wedge u + 2u \wedge \sigma) = c_5\sigma \wedge \eta \wedge u = c_5\Omega.
\]
Again by Lemma 3.1 we conclude that \( c_5 = 0 \).

On the other hand,
\[
\mathcal{L}_X (\sigma \wedge d\sigma) = \mathcal{L}_X (f_8\sigma \wedge \eta \wedge u) = \mathcal{L}_X (f_8\Omega) = X(f_8)\Omega.
\]
This implies \( X(f_8) = 0 \). That is, \( f_8 \) is a constant \( c_8 \).
Summing up, one gets
\[ du = \eta \wedge u, \quad d\eta = 2u \wedge \sigma, \quad d\sigma = c_8 \eta \wedge u - \eta \wedge \sigma. \]

Letting \( \hat{\sigma} = \sigma - c_8 u \), we obtain a final conclusion.

**Lemma 3.2.** On the manifold \( M \), there are three 1-forms \( \eta, u, \) and \( \hat{\sigma} \) such that
\[
\begin{align*}
\Omega &= \eta \wedge u \wedge \hat{\sigma}, \\
d\eta &= 2u \wedge \hat{\sigma}, \\
du &= \eta \wedge u, \\
d\hat{\sigma} &= -\eta \wedge \hat{\sigma}, \\
\eta(X) &= 0, \\
u(X) &= 0, \\
\hat{\sigma}(X) &= 1.
\end{align*}
\]

This lemma says that the manifold \( M \) is the quotient space of the universal cover of the Lie group \( \text{SL}(2, \mathbb{R}) \) by a cocompact lattice, and the vector field \( X \) generates the horocycle flow. But the horocycle flow is shown not to be parameter rigid by \([FF]\). So this case also leads to a contradiction, and we have done with the proof of Theorem 1.2.

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