Lie Pseudogroups à la Cartan: Structure Equations

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Abstract

We present a modern formulation of Élie Cartan’s structure theory for Lie pseudogroups. We introduce the notions of a Cartan algebroid and its realizations as the global, coordinate-free structures that encode Cartan’s notion of the structure equations of a Lie pseudogroup. We then present a modern proof of the Second Fundamental Theorem for Lie pseudogroups, the starting point of Cartan’s theory which, in our modern language, states that any Lie pseudogroup is equivalent to the pseudogroup of local symmetries of a realization of a Cartan algebroid. Our aim with this paper is to shed further light on this classical work of Cartan and help in creating a bridge between his work on Lie pseudogroups and modern Lie theory.

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1 Introduction

In two classical papers [3,4], dating back to 1904-05, Élie Cartan introduced a structure theory for Lie pseudogroups, building on the then recent work of Sophus Lie. As part of this work, Cartan introduced several tools that have, with the years, become fundamental in differential geometry, including the theory of exterior differential systems, G-structures, the equivalence problem, and, to some extent, the very notion of differential forms. These tools, as it turned out, became much more influential than the original problem itself – the study of Lie pseudogroups, the main reason being the gap between the notions and ideas that Cartan introduced and the mathematical language that he had at his disposal. Chern and Chevalley, in an obituary to Cartan (8, 1952), wrote: “We touch here a branch of mathematics which is very rich in results but which very badly needs clarification of its foundations.” Singer and Sternberg, in their work titled The Infinite Groups of Lie and Cartan Part I (The Transitive Groups) (12, 1965), wrote: “We must confess that we find most of these papers extremely rough going and we certainly cannot follow all arguments in detail. The best procedure is to guess at the theorems, then prove them, then go back to Cartan.”

In this paper, using modern differential geometric language, and in particular notions coming from Lie theory and from the theory of geometric PDEs, we revisit Cartan’s work and present a global, coordinate-free formulation of his theory of Lie pseudogroups, with the aim of providing further insight on a work that has proven to be so rich in content. Here, we would like to emphasize that our goal is not to promote Cartan’s theory nor make claims concerning its strengths and weaknesses as compared to other theories, but only to translate his ideas as faithfully as possible into modern language. The previous quotation of Singer and Sternberg is quite an accurate description of the work involved. However, while Singer and Sternberg and many other authors (see survey of the existing literature ahead) depart from Cartan to then construct theories that deviate from his writings, we claim that we stay much closer to the original work.

This paper is based on parts of the PhD thesis of the second author ([47]), which was supervised by the first author. Modifications have been made as compared to [47].

Lie Pseudogroups

Let us begin by recalling the notion of a Lie pseudogroup. Let $M$ be a manifold and let

$$\text{Diff}_{\text{loc}}(M) := \{ \phi : U \to V \mid U, V \subset M \text{ open subsets}, \phi \text{ a diffeomorphism } \}$$

be the set of all locally defined diffeomorphisms of $M$. Recall that:

Definition 1.1. A pseudogroup on a manifold $M$ is a subset $\Gamma \subset \text{Diff}_{\text{loc}}(M)$ that satisfies:

1. Group-like axioms:
   1) if $\phi, \phi' \in \Gamma$ and $\text{Im}(\phi') \subset \text{Dom}(\phi)$, then $\phi \circ \phi' \in \Gamma$,
   2) if $\phi \in \Gamma$, then $\phi^{-1} \in \Gamma$,
   3) $\text{id}_M \in \Gamma$.
2. Sheaf-like axioms:
   1) if $\phi \in \Gamma$ and $U \subset \text{Dom}(\phi)$ is open, then $\phi|_U \in \Gamma$,
   2) if $\phi \in \text{Diff}_{\text{loc}}(M)$ and $\{U_i\}_{i \in I}$ is an open cover of $\text{Dom}(\phi)$ s.t. $\phi|_{U_i} \in \Gamma \ \forall \ i \in I$, then $\phi \in \Gamma$.  

2
An orbit of $\Gamma$ is an equivalence class of points of $M$ given by the equivalence relation
\[ x \sim y \text{ if and only if } \exists \phi \in \Gamma \text{ such that } \phi(x) = y. \]

A pseudogroup is transitive if it has a single orbit, otherwise it is intransitive.

Remark 1.2. Many of Cartan’s examples of pseudogroups arise in the following way: given a subset $\Gamma_0 \subset \text{Diff}_{\text{loc}}(M)$ satisfying the group-like axioms, there exists a smallest pseudogroup $\langle \Gamma_0 \rangle$ on $M$ containing $\Gamma_0$ which we call the pseudogroup generated by $\Gamma_0$. It is obtained by “imposing” the sheaf-like axioms, similar to sheafification in the theory of sheaves.

A Lie pseudogroup, loosely speaking, is a pseudogroup that is defined as the set of solutions of a system of partial differential equations (PDEs). The precise definition uses the language of jet groupoids and will be given at a later stage (Definition 3.2 in Section 3). For the moment, it suffices to keep in mind that Lie pseudogroups arise in differential geometry as the local symmetries of geometric structures. For example, the set of local symplectomorphisms of a symplectic manifold $(M, \omega)$, i.e. all locally defined diffeomorphisms $\phi \in \text{Diff}_{\text{loc}}(M)$ that satisfy the partial differential equation $\phi^* \omega = \omega$, is a Lie pseudogroup.

The work of Lie and Cartan, and a large part of the literature that followed, was restricted to the local study of Lie pseudogroups, i.e. Lie pseudogroups on open subsets of Euclidean spaces. In this case, given a Lie pseudogroup $\Gamma$ on $\mathbb{R}^n$ or an open subset thereof, one typically introduces coordinates $(x, y, \ldots)$ on the copy of $\mathbb{R}^n$ on which the elements of $\Gamma$ are applied, coordinates $(X, Y, \ldots)$ on the copy of $\mathbb{R}^n$ in which the elements of $\Gamma$ take value, and, with respect to these coordinates, every element of $\Gamma$ is represented by its component functions
\[ X = X(x, y, \ldots), \quad Y = Y(x, y, \ldots), \ldots \]

Here are two simple but already very interesting (see Section 3.3) examples of Lie pseudogroups cited from [5]:

Example 1.3. The diffeomorphisms of $\mathbb{R}^2 \setminus \{y = 0\}$ of the form
\[ X = x + ay, \quad Y = y, \]
parametrized by a real number $a \in \mathbb{R}$, generate (see Remark 1.2) a non-transitive Lie pseudogroup. It is characterized as the set of local solutions of the system of partial differential equations
\[ \frac{\partial X}{\partial x} = 1, \quad \frac{\partial X}{\partial y} = \frac{X - x}{y}, \quad Y = y. \]

Example 1.4. The locally defined diffeomorphisms of $\mathbb{R}^2 \setminus \{y = 0\}$ of the form
\[ X = f(x), \quad Y = \frac{y}{f'(x)}, \]
parametrized by a function $f \in \text{Diff}_{\text{loc}}(\mathbb{R})$ (locally defined diffeomorphisms of $\mathbb{R}$), generate a transitive Lie pseudogroup. This pseudogroup is characterized as the set of local solutions of the system of partial differential equations
\[ \frac{\partial X}{\partial x} = \frac{y}{Y}, \quad \frac{\partial X}{\partial y} = 0, \quad \frac{\partial Y}{\partial y} = \frac{Y}{y}. \]
Cartan’s Approach to Lie Pseudogroups

The study of Lie pseudogroups was initiated by Sophus Lie in a three volume monograph written in collaboration with Friedrich Engel ([27], 1888-1893). In this work, Lie concentrated on the special class of Lie pseudogroups of finite type. These are, loosely speaking, Lie pseudogroups whose elements are parametrized by a finite number of real variables (e.g. Example 1.3). They are substantially simpler to handle because they are (locally) encoded by their finite dimensional space of parameters which inherits the structure of a (local) Lie group. In fact, Lie’s work on this “special case” marked the birth of Lie group theory.

Lie’s key idea was to study these objects by means of their induced set of infinitesimal transformations, or, in modern terms, to study a Lie group by means of its associated Lie algebra of invariant vector fields. Cartan sought to extend Lie’s ideas to the general case and to develop a theory that also encompasses Lie pseudogroups of infinite type. These are, loosely speaking, Lie pseudogroups whose elements may be also parameterized by arbitrary functions (e.g. Example 1.4). While a direct generalization of Lie’s construction of a Lie algebra of vector fields proved difficult, Cartan showed that one can associate an infinitesimal structure with a Lie pseudogroup by passing to the dual picture of differential forms. Cartan’s approach is depicted in the following diagram:

\[
\begin{array}{ccc}
\text{Lie pseudogroup} & \xrightarrow{\text{2nd Fundamental Theorem}} & \text{Lie pseudogroup in normal form} \\
& & \xrightarrow{\text{3rd Fundamental Theorem}} & \text{Infinitesimal structure}
\end{array}
\]

In what he called the Second Fundamental Theorem, Cartan showed that any Lie pseudogroup can be replaced by an equivalent pseudogroup that is in normal form. The notion of equivalence of pseudogroups is what Cartan regarded as the correct notion for an “isomorphism” of pseudogroups. Loosely speaking, it means that there is a bijection between the pseudogroups that preserves the group-like structure. Of course, to make this precise, one has to take care of the sheaf-like structure, and the precise definition will be given Section 3.1.

The main component in Cartan’s Second Fundamental Theorem is the notion of normal form, whose novelty is the use of differential forms (in contrast to Lie’s use of vector fields). According to Cartan, a pseudogroup is in normal form if it is characterized as the set of local symmetries of the following special type of geometric structure (see also Section 2.1 for more details): a collection of functions \(I_1, \ldots, I_n\) and linearly independent 1-forms \(\omega_1, \ldots, \omega_r\), with \(\omega_1 = dI_1, \ldots, \omega_n = dI_n\), that satisfy a set of structure equations of the form

\[
d\omega_i + \frac{1}{2}c^{jk}_i \omega_j \wedge \omega_k = a^{\lambda j}_i \pi_\lambda \wedge \omega_j,
\]

where the coefficients \(c^{jk}_i\) and \(a^{\lambda j}_i\) are functions of the invariants \(I_1, \ldots, I_n\) (and, hence, also invariants) and \(\pi_1, \ldots, \pi_p\) are auxiliary 1-forms that complete the \(\omega_i\)’s to a coframe. These structure equations are, of course, a generalization of the familiar Maurer-Cartan structure equations of a Lie group. Indeed, the Maurer-Cartan forms of a Lie group satisfy the special case of the structure equations in which the right-hand side is zero and the coefficients \(c^{jk}_i\) are constant, namely the structure constants of the Lie algebra of the Lie group (see Example 2.3 for more details).

As in the case of Lie groups, Cartan interpreted the structure functions \(c^{jk}_i\) and \(a^{\lambda j}_i\) as the infinitesimal structure associated with the Lie pseudogroup and posed the following integrability

\footnote{Here, and throughout the paper, we use the Einstein summation convention.}
problem, known as the realization problem: starting with a set of functions $c_{ij}^{kl}$ (antisymmetric in the top indices) and $a_{ij}^{kl}$, do they arise as the infinitesimal structure of a Lie pseudogroup? In the special case in which the $a_{ij}^{kl}$'s are zero and the $c_{ij}^{kl}$'s are constant, the answer to the problem is given by Lie’s well-known Third Fundamental Theorem: if the constants $c_{ij}^{kl}$ satisfy the Jacobi identity

$$c_{ij}^{kl} c_{km}^{ij} + c_{ij}^{ml} c_{jm}^{lk} + c_{ij}^{mk} c_{jm}^{lj} = 0,$$

i.e., if they are the structure constants of a Lie algebra, then they are the structure constants of the Lie algebra of some Lie group. In the general case, Cartan identified a more intricate set of equations that play the role of the Jacobi identity (see Equations (C1)-(C3) in Theorem 2.2), and gave a partial solution to the realization problem in what he called the Third Fundamental Theorem for Lie pseudogroups: if the initial data is involutive, then local solutions exist in the real-analytic category (see Theorem 2.31). The main ingredient of his proof is an analytic tool that he developed for this very purpose, a tool that has evolved into the modern day theory of Exterior Differential Systems. It is interesting to note that there have been no improvements on Cartan’s results to date, namely there is no Third Fundamental Theorem in the smooth category nor one of a global nature as in the case of Lie groups.

Outline of the Paper

In this paper, we present a modern formulation of Cartan’s theory and a modern proof of Cartan’s Second Fundamental Theorem.

In Section 2, we introduce a global, coordinate-free formulation of Cartan’s structure theory. Cartan’s infinitesimal structure, which is encoded in the structure functions $c_{ij}^{kl}$ and $a_{ij}^{kl}$, will take the form a Lie algebroid-like structure we call a Cartan algebroid, and Cartan’s notion of a pseudogroup in normal form, which correspond to the “integrating” structure, will be encoded in the notion of a realization of a Cartan algebroid. We will study some of the important properties of these structures and discuss a proof of Cartan’s Third Fundamental Theorem. We conclude the section by introducing an alternative but, in a sense, equivalent definition for the notion of a Cartan algebroid, one which deviates slightly from Cartan, but which is simpler to handle and closer in nature to the well-known notion of a Lie algebroid. We believe that this alternative description of Cartan’s infinitesimal structure will more easily lend itself to modern Lie-theoretic methods.

In Section 3, after giving the precise definition of a Lie pseudogroup and of the notion of Cartan equivalence of pseudogroups, we present a modern proof of Cartan’s Second Fundamental Theorem. In the modern language of Section 2, the theorem states that any Lie pseudogroup is equivalent to the pseudogroup of local symmetries of a realization of a Cartan algebroid. In the two appendices to the paper, we recall some important notions that are needed in our proof. In Appendix A.1, we recall the language of jet groupoids and algebroids, which allows one to talk about the “defining equations” of a Lie pseudogroup in a coordinate-free fashion. We place a special emphasis on Cartan’s key idea – a Lie pseudogroup is encoded by its canonical Cartan form. This point of view motivates the notion of a Lie-Pfaffian groupoid, an axiomatization of the notion of a Lie pseudogroup that was developed in [10] (also to appear in [13]) and which we recall in Appendix A.2. This abstract framework will allow us to obtain a clean and conceptual proof of Cartan’s Second Fundamental Theorem.

For readers that are unfamiliar with the material covered in the appendix, the recommended order for reading this paper is as follows:

Section 2 ⇒ Appendix ⇒ Section 3
A Brief Survey of the Existing Literature

As mentioned above, this paper is the result of our effort to read Cartan’s classical work on Lie pseudogroups, and understand his original ideas as faithfully as possible. Thus, our main sources are Cartan’s papers [3, 4] from 1904-1905, and two later papers [5, 6] of Cartan from 1937 that present the same material in a more concise (and, in our opinion, more accessible) form. Of course, our work also builds upon the work of many mathematicians that continued in Lie and Cartan’s footsteps, many of which were motivated, like us, by the challenge of “understanding Cartan”. The first big step in modernizing Cartan’s work is attributed to Charles Ehresmann. Some of Ehresmann’s important contributions to this field are: the modern definition of a pseudogroup, the theory of jet spaces, and the introduction of Lie groupoids into the theory (see [26] for a historical account). Ehresmann’s work marked the beginning of the “modern era” of Lie pseudogroups, a renewed interest in the subject that had its peak in the 1950-1970’s, but which continues until this very day. A full account of the literature that appeared on this subject over the years deserves a paper of its own. We mention here a small selection of this literature, with an emphasis on papers that were particularly influential to our work (some of which will also be mentioned throughout this paper).

The structure equations of a Lie pseudogroup have been studied from various perspectives in [25, 33, 24, 39, 21, 42, 16, 31, 32, 19, 28, 45, 37]. In particular, proofs of Cartan’s Second Fundamental Theorem can be found in [25, 24, 16, 19, 45] and a proof of Cartan’s Third Fundamental Theorem can be found in [21]. Throughout the years, people have also taken several new approaches in the study of Lie pseudogroups. These include: the formal theory of Lie (\(F\))-groups and Lie (\(F\))-algebras [23, 24]; the infinitesimal point of view of sheaves of Lie algebras of vector fields [38, 42]; the study of Lie pseudogroups via the defining equations of their infinitesimal transformations, known as Lie equations [31, 32]; an approach using Milnor’s infinite dimensional Lie groups [20]; and the point of view of infinite jet bundles [36] (and [46] compares this approach with Cartan’s theory).

Prerequisites

In this paper, we assume familiarity with the theory of Lie groupoids and algebroids. Possible reference on the subject are [30, 34, 12]. We also assume a basic familiarity with the notion of jets (mainly the definition and their local coordinate description). There are innumerable reference on this subject, e.g. [41, 35, 1]. See also [47] for a concise review of these subjects.

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2 Cartan Algebroids, Realizations and Pseudogroups in Normal Form

In this section, we introduce the following two notions that encode the main structures in Cartan’s theory of Lie pseudogroups: Cartan algebroids, on the infinitesimal side, and their realizations, on the global side. In this modern language, Cartan’s realization problem, which
was already mentioned in the introduction, becomes the problem of whether a Cartan algebroid admits a realization. As a prelude, we recall the realization problem in more detail, as it can be found in Cartan’s work.

2.1 Cartan’s Realization Problem

Let \( N \geq n \geq 0 \) be two integers, and denote the coordinates on \( \mathbb{R}^n \) by \((x_1, ..., x_n)\) and those on \( \mathbb{R}^N \) by \((x_1, ..., x_n, x_{n+1}, ..., x_N)\). We denote the projection by

\[
I = (I_1, ..., I_n) : \mathbb{R}^N \to \mathbb{R}^n, \quad I_a(x) = x_a.
\]

Roughly speaking, Cartan’s realization problem asks: given an “infinitesimal structure” on \( \mathbb{R}^n \), does there exist a “realization” on \( \mathbb{R}^N \) consisting of \( r \) differential 1-forms? Therefore, we also fix \( r \) such that \( N \geq r \geq n \), and we set \( p := N - r \). The precise formulation of the problem, as found in Cartan’s work, is as follows:

**Problem 2.1 (Cartan’s Realization Problem).** Let

\[
c_{jk}^i, a_{ij}^\lambda \in C^\infty(U), \quad (1 \leq i, j, k \leq r, 1 \leq \lambda \leq p),
\]

be functions on an open subset \( U \subset \mathbb{R}^n \) such that

1. \( c_{jk}^i = -c_{kj}^i \),

2. the matrices \( A^\lambda = (a_{ij}^\lambda) \) are linearly independent.

Find a set of linearly independent 1-forms \( \omega_1, ..., \omega_r \in \Omega^1(V) \) on an open subset \( V \subset \mathbb{R}^N \) satisfying \( I(V) = U \), with

\[
\omega_1 = dI_1|V, \ldots, \omega_n = dI_n|V,
\]

that satisfy the following property: there exists another set of 1-forms \( \pi_1, ..., \pi_p \in \Omega^1(V) \) such that \( \{\omega_1, ..., \omega_r, \pi_1, ..., \pi_p\} \) is a coframe of \( V \) and

\[
d\omega_i + \frac{1}{2} c_{jk}^i \omega_j \wedge \omega_k = a_{ij}^\lambda \pi_\lambda \wedge \omega_j,
\]

where \( c_{jk}^i, a_{ij}^\lambda \) are viewed as functions on \( V \) that are constant along the fibers of \( I \).

An immediate consequence of (2.2), together with (2.1), is that

\[
e_{jk}^i = 0 \quad \text{and} \quad a_{ij}^\lambda = 0, \quad \forall \ 1 \leq i \leq n.
\]

We call the initial data of the realization problem, i.e. functions \( (c_{jk}^i, a_{ij}^\lambda) \) on an open subset \( U \subset \mathbb{R}^n \) that satisfy properties 1 and 2, as well as condition (C0), an **almost Cartan data.** We call a solution of the realization problem \( (I, \omega_i) \) a **realization** of the almost Cartan data.

Equations (2.2) are called the **structure equations.**

In what Cartan calls the **third fundamental theorem**, he gives a partial solution to the realization problem (see Theorem 2.34 for the statement in modern language). Cartan’s first step in solving the realization problem was to identify the following set of necessary conditions for admitting a realization:
Theorem 2.2. If an almost Cartan data \((c^j_i, a^k_j)\) on \(\mathbb{R}^n\) admits a realization, then there exist functions
\[

\nu^{jk}_\lambda, \xi^{ij}_\lambda, \epsilon^{mn}_\lambda \in C^\infty(\mathbb{R}^n) \quad (1 \leq j, k \leq r, 1 \leq \lambda, \eta, \mu \leq p),
\]
with \(\nu^{jk}_\lambda = -\nu^{kj}_\lambda, \epsilon^{mn}_\lambda = -\epsilon^{nm}_\lambda\), such that
\[
a^m_i a^{nj}_m - a^m_i a^{nj}_m = a^j_i \lambda^m_\lambda, \quad (C1)
\]
\[
c^{mj}_i c^{kl}_m + c^{nj}_i c^{lj}_m + c^{ml}_i c^{jk}_m + \left( \frac{\partial c^{kl}_m}{\partial x_j} + \frac{\partial c^{lj}_m}{\partial x_k} + \frac{\partial c^{jk}_m}{\partial x_i} \right) = a^j_i \lambda^m_\lambda + a^j_i \lambda^k_\lambda + a^j_i \nu^{kl}_\lambda \quad (C2)
\]
\[
a^\lambda_i e^{mjk} - a^\lambda_i m e^{nj}_i + a^\lambda_i l e^{jk}_i + \left( \frac{\partial a^\lambda_i k}{\partial x_j} - \frac{\partial a^\lambda_i l}{\partial x_j} \right) = a^\lambda_i \nu^{jk}_\lambda - a^\mu_i \nu^{jk}_\lambda, \quad (C3)
\]
where terms that involve \(\partial/\partial x_j\) with \(j > n\) are understood to be zero.

Proof. Let \((I_a, \omega_i)\) be a realization, and choose \(\pi_1, ..., \pi_p\) such that \(\{\omega_1, ..., \omega_r, \pi_1, ..., \pi_p\}\) is a coframe of \(\mathbb{R}^N\) and such that (2.2) is satisfied. Decompose \(d\pi_\lambda\) in terms of the coframe:
\[
d\pi_\lambda = \frac{1}{2} \nu^{jk}_\lambda \omega_j \wedge \omega_k + \xi^{ij}_\lambda \pi_\mu \wedge \omega_j + \frac{1}{2} \epsilon^{mn}_\lambda \pi_\eta \wedge \pi_\mu,
\]
where \(\nu^{jk}_\lambda, \xi^{ij}_\lambda, \epsilon^{mn}_\lambda \in C^\infty(\mathbb{R}^N)\) are the coefficients. Then, differentiate (2.2) and replace all appearances of \(d\pi_\lambda\) by this decomposition. This gives three sets of equations. Finally, (C1) – (C3), which are equations on \(\mathbb{R}^n\), are obtained by restricting these equations to the slice \(\{x_{n+1} = 0, ..., x_n = 0\}\) (or to any other section of the projection \(I\)). \(\square\)

We call an almost-Cartan data satisfying (C1)–(C3), for some set of functions \(\nu^{jk}_\lambda, \xi^{ij}_\lambda, \epsilon^{mn}_\lambda\), a **Cartan data**. The latter theorem can be rephrased as: an almost Cartan data that admits a realization is a Cartan data.

Any realization has an associated pseudogroup of local symmetries,
\[
\Gamma(I_a, \omega_i) := \{ \phi \in \text{Diff}_{loc}(V) \mid \phi^* I_a = I_a, \phi^* \omega_i = \omega_i \}.
\]
In general, this pseudogroup can be “too small”, namely there is no guarantee that the induced pseudogroup consists of more than just the identity diffeomorphism of \(V\) (and its restrictions to open subsets). A pseudogroup \(\Gamma\) on an open subset \(V\) of a Euclidean space is said to be in **normal form** if it is the pseudogroup of local symmetries of a realization \((I_a, \omega_i)\) on \(V\) and if its orbits are the fibers of \(I\). Theorem 2.19, which we will see later on and which is due to Cartan, gives sufficient conditions for the pseudogroup induced by a realization to be in normal form, i.e. for its orbits to coincide with the fibers of \(I\).

In the remainder of Section 2 we present a global, coordinate-free formulation of the structures that underly Cartan’s structure theory (Cartan data, realization, structure equations, ...). The following simple and familiar example of the realization problem may serve as a good guide in this process:

**Example 2.3** (Lie groups and Lie algebras). Consider the realization problem in the case \(n = 0\) and \(p = 0\). An almost Cartan data is simply a set of constants \(c^j_k\) \((1 \leq i, j, k \leq r)\) that are anti-symmetric in the upper indices. Fixing an \(r\)-dimensional vector space \(g\) and a basis \(X^1, ..., X^r\), this data can be encoded in an anti-symmetric bilinear operation on \(g\):
\[
\lbrack, \rbrack : g \times g \rightarrow g, \quad [X^j, X^k] = c^j_k X^l.
\]
Conditions (C1) and (C3) are vacuous and condition (C2) reduces to the Jacobi identity
\[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \quad \forall X, Y, Z \in \mathfrak{g}.

Hence, a Cartan data in this case is the same thing as a Lie algebra.

A Lie group integrating \( \mathfrak{g} \) induces a solution to the realization problem as follows: any Lie group \( G \) with Lie algebra \( \mathfrak{g} \) comes with a canonical \( \mathfrak{g} \)-valued 1-form, the Maurer-Cartan form:
\[ \Omega = \Omega_{MC} \in \Omega^1(G; \mathfrak{g}), \quad \Omega_g = (dL_g^{-1})_g : T_gG \to T_eG = \mathfrak{g}. \]

It satisfies two main properties: 1) the Maurer-Cartan equation
\[ d\Omega + \frac{1}{2} [\Omega, \Omega] = 0, \]
and 2) it is pointwise an isomorphism. Decomposing \( \Omega \) as \( \Omega = \omega_i X^i \), with \( \omega_i \in \Omega^1(G) \), the Maurer-Cartan equation becomes
\[ d\omega_i + \frac{1}{2} c_{jk}^i \omega_j \wedge \omega_k = 0, \]
and the second property is equivalent to requiring that \( \{\omega_1, ..., \omega_r\} \) be a coframe of \( G \). Thus, we obtain a realization (where the projection \( I \) is simply the map from \( G \) to a point). We remark that any realization on an open subset \( V \subset \mathbb{R}^r \) induces a local Lie group structure on \( V \) (see e.g. [15], pp. 368-369). Hence, in this simple case, the realization problem is closely related to the problem of integrating a Lie algebra to a Lie group.

Finally, one can prove that the pseudogroup induced by the realization,
\[ \Gamma(G, \Omega) = \{ \phi \in \text{Diff}_{\text{loc}}(G) \mid \phi^*\Omega = \Omega \}, \]
is the pseudogroup generated by left translations. It is in normal form, since its single orbit is \( G \) itself.

### 2.2 Realizations (Structure Equations) and Pseudogroups in Normal Form

We begin with Cartan's very basic idea: pseudogroups realized as the set of local symmetries of a system of functions and 1-forms. Globally, we start with a surjective submersion \( I : P \to N \), a vector bundle \( C \to N \) and a \( C \)-valued 1-form \( \Omega \in \Omega^1(P; I^*C) \). Such data induces a pseudogroup on \( P \),
\[ \Gamma(P, \Omega) := \{ \phi \in \text{Diff}_{\text{loc}}(P) \mid \phi^*I = I, \phi^*\Omega = \Omega \}. \quad (2.3) \]

Note that the first condition ensures that the second makes sense. One would like to understand the first order consequences of the defining equations of this pseudogroup (e.g. “\( \phi^*(d\Omega) = d\phi^*\Omega \)”). This becomes easier when \( C \) is endowed with extra structure. This structure is captured by the notion of an almost Lie algebroid and is hidden in the considerations of Cartan.

#### 2.2.1 Almost Lie Algebroids and the Maurer-Cartan Expression

**Definition 2.4.** An almost Lie algebroid over a manifold \( N \) is a vector bundle \( C \to N \) equipped with a vector bundle map \( \rho : C \to TN \) (“the anchor”) and a bilinear antisymmetric map \([\cdot, \cdot] : \Gamma(C) \times \Gamma(C) \to \Gamma(C) \) (“the bracket”) satisfying the Leibniz identity
\[ [\alpha, f\beta] = f[\alpha, \beta] + L_{\rho(\alpha)}(f)\beta, \quad \forall \alpha, \beta \in \Gamma(C), \ f \in C^\infty(N), \]
and
\[ \rho([\alpha, \beta]) = [\rho(\alpha), \rho(\beta)], \quad \forall \alpha, \beta \in \Gamma(C). \]

An almost Lie algebroid \( C \) is transitive if \( \rho : C \to TN \) is surjective.
Example 2.5. The best known example of an almost Lie algebroid is a Lie algebroid: an almost Lie algebroid whose bracket satisfies the Jacobi identity.

Definition 2.6. Given an almost Lie algebroid $C$ over $N$ and a surjective submersion $I : P \to N$, a 1-form $\Omega \in \Omega^1(P; I^*C)$ is called **anchored** if

$$\rho \circ \Omega = dI. \tag{2.4}$$

The anchored condition on $\Omega$ ensures that, although the expression $d\Omega$ does not make sense globally, the Maurer-Cartan type expression “$d\Omega + \frac{1}{2}[\Omega, \Omega]$” does. The construction is the same as for Lie algebroids: let $I : P \to N$ be a surjective submersion, $C \to N$ an almost Lie algebroid and $\nabla : \mathfrak{X}(N) \times \Gamma(C) \to \Gamma(C)$ a connection on $C$. The connection induces a de Rham-type operator

$$d_{\nabla} : \Omega^*(P; I^*C) \to \Omega^{*+1}(P; I^*C)$$

on the space of $C$-valued forms defined by the usual formula

$$(d_{\nabla} \Omega)(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i (I^* \nabla) X_i (\Omega(X_0, \ldots, \hat{X}_i, \ldots, X_p))$$

$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \Omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),$$

where $\Omega \in \Omega^p(P; I^*C)$ and $X_0, \ldots, X_p \in \mathfrak{X}(P)$. Note that $(d_{\nabla})^2 = 0$ if and only if $\nabla$ is flat. With the connection, one also associates the $C$-torsion of $\nabla$, the tensor $[\cdot, \cdot]_{\nabla} \in \Gamma(\text{Hom}(\Lambda^2 C, C))$ defined at the level of sections by $[\alpha, \beta]_{\nabla} = [\alpha, \beta] - \nabla_{\rho(\alpha)} \beta + \nabla_{\rho(\beta)} \alpha$ for all $\alpha, \beta \in \Gamma(C)$. The torsion, in turn, induces a graded bracket,

$$[\cdot, \cdot]_{\nabla} : \Omega^p(P; I^*C) \times \Omega^q(P; I^*C) \to \Omega^{p+q}(P; I^*C), \tag{2.4}$$

which is defined by the following wedge-like formula:

$$[\Omega, \Omega']_{\nabla}(X_1, \ldots, X_{p+q}) =$$

$$\sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) [\Omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)}), \Omega'(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})]_{\nabla},$$

where $S_{p,q}$ is the group of $(p, q)$-shuffles.

Proposition 2.7. If $\Omega \in \Omega^1(P; I^*C)$ is anchored, then the Maurer-Cartan 2-form

$$MC_\Omega := d_{\nabla} \Omega + \frac{1}{2} [\Omega, \Omega]_{\nabla} \in \Omega^2(P; I^*C)$$

is independent of the choice of connection.

Proof. Let $\nabla$ and $\nabla'$ be two connections on $C$ and set $\eta := \nabla - \nabla' \in \Omega^1(N; \text{Hom}(C, C))$. Let $p \in P$ and $X, Y \in T_pP$. The sum of the following two equations vanishes if $\Omega$ is anchored:

$$\eta(d_{\nabla} \Omega - d_{\nabla'} \Omega)(X, Y) = \eta(d(I(X)) \Omega(Y)) - \eta(d(I(Y)) \Omega(X))$$

$$([\Omega, \Omega]_{\nabla} - [\Omega, \Omega]_{\nabla'})(X, Y) = -\eta(\rho \circ \Omega(X)) \Omega(Y) + \eta(\rho \circ \Omega(Y)) \Omega(X)$$

Remark 2.8. From now on we suppress $\nabla$ from the notation and write $d\Omega + \frac{1}{2} [\Omega, \Omega]$ when $\Omega$ is anchored.
Intuitively, $\text{MC}_\Omega$ measures the failure of $\Omega : TP \to C$ to be a morphism of almost Lie algebroids. For example, when $P = N$ and $I$ is the identity,

$$\text{MC}_\Omega(X,Y) = -\Omega([X,Y]) + [\Omega(X),\Omega(Y)], \quad \forall X,Y \in \mathfrak{X}(N).$$

When $\Omega$ is pointwise surjective, we have the following useful formula:

**Lemma 2.9.** Let $\Omega \in \Omega^1(P; I^* \mathcal{C})$ be anchored and pointwise surjective. Given any $\alpha \in \Gamma(\mathcal{C})$, there exists $X_\alpha \in \mathfrak{X}(P)$ such that

$$\Omega(X_\alpha) = I^* \alpha.$$

Given a pair $\alpha, \beta \in \Gamma(\mathcal{C})$, and $X_\alpha, X_\beta \in \mathfrak{X}(P)$ as above,

$$\text{MC}_\Omega(X_\alpha, X_\beta) = -\Omega([X_\alpha, X_\beta]) + I^*[\alpha, \beta].$$

**Proof.** An $X_\alpha$ as in the statement can be obtained by choosing a splitting of the short exact sequence of vector bundles $0 \to \ker(\Omega) \to TP \Omega \to I^* \mathcal{C} \to 0$. By the anchored condition:

$$\left( d\Omega + \frac{1}{2} [\Omega, \Omega] \right)(X_\alpha, X_\beta) = (I^* \nabla)_{\Omega(X_\beta)}(\Omega(X_\alpha)) - (I^* \nabla)_{\Omega(X_\alpha)}(\Omega(X_\beta)) - \Omega([X_\alpha, X_\beta]) + I^*[\alpha, \beta] - I^*(\nabla_{\rho(\alpha)} \beta) + I^*(\nabla_{\rho(\beta)} \alpha). \quad \square$$

### 2.2.2 Almost Cartan Algebroids

In order to make sense of structure equations globally, one needs a little more than an almost Lie algebroid.

**Definition 2.10.** An almost Cartan algebroid over a manifold $N$ is a pair $(\mathcal{C}, \sigma)$ consisting of a transitive almost Lie algebroid $\mathcal{C} \to N$ and a vector subbundle $\sigma \subset \text{Hom}(\mathcal{C}, \mathcal{C})$ such that $T(\mathcal{C}) \subset \text{Ker} \rho$ for all $T \in \sigma$ (i.e. $\sigma \subset \text{Hom}(\mathcal{C}, \text{Ker} \rho)$), where $\rho$ is the anchor of $\mathcal{C}$.

**Remark 2.11.** The vector bundle $\sigma$ is a tableau bundle in the sense of Definition A.31, and hence we can talk about its prolongations and associated Spencer cohomology. These play an important role in the theory, and in particular in questions of formal and real-analytic integrability (see e.g. Theorem 2.19).

**Example 2.12.** Locally, we are back to Cartan: almost Cartan algebroids locally correspond to the notion of an almost Cartan data (Section 2.1) and they are encoded by functions $c_{jk}^i$ and $a_{ij}^k$. Adapting to the notation of Section 2.1:

- $N = U \subset \mathbb{R}^n$, an open subset.
- $\mathcal{C} \to N$ is the trivial vector bundle of rank $r$ (where $r \geq n$) with trivializing frame $\{e^1, \ldots, e^r\}$ and endowed with the almost Lie algebroid structure determined by

$$\rho(e^i) = \frac{\partial}{\partial x_i} \quad \text{for} \quad 1 \leq i \leq n, \quad \rho(e^i) = 0 \quad \text{for} \quad i > n,$$

and

$$[e^j, e^k] = c_{jk}^i e^i.$$

The fact that $\rho$ is a Lie algebra homomorphism is equivalent to the condition $c_{jk}^i = 0$ for $i \leq n$ (first part of condition (C0)).
• $\sigma \to N$ is the trivial vector bundle of rank $p$ with trivializing frame denoted by \{ $t^1, ..., t^p$ \}. Each element of the frame acts on $C$ by

$$t^i(e^j) = a^i_{\lambda} e^j,$$

and, extending by linearity, we obtain a map $\sigma \to \text{Hom}(C, C)$. The injectivity of this map is equivalent to Cartan's condition that, at each point of $\mathbb{R}^n$, the matrices $A^\lambda = (a^i_{\lambda})$ are linearly independent. The condition $\sigma \subset \text{Hom}(C, \ker \rho)$ is equivalent to the condition $a^i_{j\lambda} = 0$ for $i \leq n$ (second part of condition $(C0)$).

### 2.2.3 Isomorphism and Gauge Equivalence

There is an obvious notion of isomorphism of almost Cartan algebroids. First note that, given two vector bundles $C$ and $C'$ over $N$ and a vector subbundle $\sigma \subset \text{Hom}(C, C)$, a vector bundle isomorphism $\psi : C \to C'$ maps $\sigma$ into $\text{Hom}(C', C')$ by conjugation, i.e.

$$\psi(\sigma) := \{ \psi \circ T \circ \psi^{-1} \mid S \in \sigma \} \subset \text{Hom}(C', C').$$

**Definition 2.13.** Two almost Cartan algebroids $(C, \sigma)$ and $(C', \sigma')$ over $N$ are isomorphic if there exists a vector bundle isomorphism $\psi : C \to C'$ such that

1. $\psi(\sigma) = \sigma'$,
2. $\psi([\alpha, \beta]) = [\psi(\alpha), \psi(\beta)], \forall \alpha, \beta \in \Gamma(C)$,
3. $\rho \circ \psi = \rho'$.

However, this notion of an isomorphism turns out to be too strong, and the slightly weaker notion of gauge equivalence turns out to be the relevant one in the theory. The main evidence for this will come in Section 3 where we will see that the construction of a Cartan algebroid and a realization out of a Lie pseudogroup, both in the general algorithm as well as in examples, depends on a choice, and different choices lead to gauge equivalent structures. We will now define gauge equivalence of almost Cartan algebroids, and later show that realizations and Cartan algebroids behave well under such transformations.

Given an almost Cartan algebroid $(C, \sigma)$, a choice of a vector bundle map $\eta : C \to \sigma$ induces a new bracket $[\cdot, \cdot]^{\eta}$ on $C$,

$$[\alpha, \beta]^{\eta} := [\alpha, \beta] + \eta(\alpha)(\beta) - \eta(\beta)(\alpha), \quad \forall \alpha, \beta \in \Gamma(C).$$

We denote by $C^{\eta}$ the vector bundle $C$ equipped with the new bracket $[\cdot, \cdot]^{\eta}$ but with the same anchor $\rho$.

**Lemma 2.14.** Let $(C, \sigma)$ be an almost Cartan algebroid over $N$ and let $\eta \in \Gamma(\text{Hom}(C, \sigma))$. Then $(C^{\eta}, \sigma)$ is an almost Cartan algebroid over $N$.

**Proof.** We only need to verify that $C^{\eta}$ is an almost Lie algebroid. The Leibniz identity is clear, and

$$\rho(\alpha, \beta)^{\eta} = [\rho(\alpha), \rho(\beta)] + \rho(\eta(\alpha)(\beta)) - \rho(\eta(\beta)(\alpha)), \quad \forall \alpha, \beta \in \Gamma(C),$$

because $\sigma \subset \text{Hom}(C, \ker \rho)$.

**Definition 2.15.** Two almost Cartan algebroids $(C, \sigma)$ and $(C', \sigma')$ over $N$ are gauge equivalent if there exists a vector bundle map $\eta : C \to \sigma$ s.t. $(C', \sigma')$ is isomorphic to $(C^{\eta}, \sigma)$.

Clearly, gauge equivalence defines an equivalence relation on the set of almost Cartan algebroids.

**Remark 2.16.** Locally, the notion of gauge equivalence already appears in [21].
2.2.4 Realizations

Given an almost Cartan algebroid \((C, \sigma)\) and a surjective submersion \(I : P \to N\), the vector bundle \(\sigma\) and its inclusion in \(\text{Hom}(C, C)\) allow us to define a second wedge-like operation

\[
\wedge : \Omega^p(P; I^*\sigma) \times \Omega^q(P; I^*C) \to \Omega^{p+q}(P; I^*C)
\]

defined by:

\[
(\eta \wedge \phi)(X_1, \ldots, X_{p+q}) = \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \eta(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\phi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\]

Definition 2.17. A realization of an almost Cartan algebroid \((C, \sigma)\) is a pair \((P, \Omega)\) consisting of a surjective submersion \(I : P \to N\) and a pointwise-surjective anchored 1-form

\[
\Omega \in \Omega^1(P; I^*C)
\]
such that, for some 1-form \(\Pi \in \Omega^1(P; I^*\sigma)\),

\[
d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega \tag{2.5}
\]

and

\[
(\Omega, \Pi) : TP \xrightarrow{\sim} I^*(C \oplus \sigma) \tag{2.6}
\]
is vector bundle isomorphism. Equation (2.5) is called the structure equation.

The data of a realization \((P, \Omega)\), as we saw, induces a pseudogroup, the pseudogroup \(\Gamma(P, \Omega)\) of its local symmetries as defined in (2.3).

Definition 2.18. A pseudogroup \(\Gamma\) on \(P\) is said to be in normal form if there exists a realization \((P, \Omega)\) of an almost Cartan algebroid \((C, \sigma)\) over \(N\) such that \(\Gamma = \Gamma(P, \Omega)\) and such that the orbits of \(\Gamma\) coincide with the fibers of \(I : P \to N\).

The following theorem, due to Cartan, gives a criteria for when a pseudogroup induced by a realization is in normal form. Its proof uses the theory of exterior differential systems and the Cartan-Kähler Theorem, which is only valid in the real-analytic category and hence the assumption of real-analyticity. There is no known version of the theorem in the smooth category. In this theorem we see the first appearance of the Spencer cohomology of \(A\) and, in particular, the notion of involutivity (Definition A.37).

Theorem 2.19 (Cartan [3, 5], see also Kumpera [21]). Let \((P, \Omega)\) be a real-analytic realization of a real-analytic almost Cartan algebroid \((C, \sigma)\) (i.e., all manifolds and maps are real-analytic). If the tableau bundle \(\sigma\) is involutive, then the orbits of \(\Gamma(P, \Omega)\) coincide with the fibers of \(I : P \to N\), and hence \(\Gamma(P, \Omega)\) is in normal form.

Example 2.20. Locally, a realization of an almost Cartan algebroid corresponds to Cartan’s notion of a realization, as we saw in Section 2.1. Continuing from Example 2.12

- \(P = V \subset \mathbb{R}^N\) is an open subset with coordinates \((x_1, \ldots, x_N)\), and \(I : P \to N\) is the projection onto the first \(n\) coordinates,

\[
I = (I_1, \ldots, I_n) : \mathbb{R}^N \to \mathbb{R}^n, \quad I_a(x) = x_a.
\]
• Ω and Π can be decomposed as
  \[ \Omega = \omega_i I^i e^i, \quad \Pi = \pi_\lambda I^\lambda, \]
  with \( \omega_i, \pi_\lambda \in \Omega^1(P) \). The anchored condition on Ω is equivalent to
  \[ \omega_1 = dx_1, \ldots, \omega_n = dx_n. \]

  Equation (2.5) becomes
  \[ d\omega_i + \frac{1}{2} c^{jk}_i \omega_j \wedge \omega_k = a^\lambda_i \pi_\lambda \wedge \omega_j, \quad (2.7) \]
  where \( c^{jk}_i \) and \( a^\lambda_i \) are functions on \( \mathbb{R}^n \) viewed as functions on \( \mathbb{R}^N \) that are constant along the fibers of \( I \). Condition (2.6) is equivalent to \( \{ \omega_1, ..., \omega_r, \pi_1, ..., \pi_p \} \) being a coframe.

• The induced pseudogroup on \( \mathbb{R}^N \) is
  \[ \Gamma(P, \Omega) = \{ \phi \in \text{Diff}_{\text{loc}}(\mathbb{R}^N) \mid \phi^* I_a = I_a, \phi^* \omega_i = \omega_i \}. \quad (2.8) \]

As we mentioned above, realizations of almost Cartan algebroids behave well under gauge equivalence:

**Proposition 2.21.** If \((P, \Omega)\) is a realization of an almost Cartan algebroid \((C, \sigma)\), then it is a realization of any gauge equivalent almost Cartan algebroid \((C^\eta, \sigma)\), where \( \eta \in \Gamma(\text{Hom}(C, \sigma)) \). Moreover, if \( \Pi \in \Omega^1(P; I^* \sigma) \) is a choice for the realization of \((C, \sigma)\) as in Definition 2.17 then \( \Pi^\eta \in \Omega^1(P; I^* \sigma) \) defined by
  \[ \Pi^\eta(X) = \Pi(X) + (I^* \eta)(\Omega(X)), \quad \forall X \in \mathfrak{X}(P), \]
  is a choice for the realization of \((C^\eta, \sigma)\).

**Proof.** Given \( \alpha \in \Gamma(C) \) and \( S \in \Gamma(\sigma) \), we write \( X_\alpha, X_S \in \mathfrak{X}(P) \) for the unique vector fields that satisfy \( (\Omega, \Pi)(X_\alpha) = I^* \alpha \) and \( (\Omega, \Pi)(X_S) = I^* S \). One now easily checks that
  \[ d\Omega + \frac{1}{2} [\Omega, \Omega]^\eta = \Pi^\eta \wedge \Omega \]
is satisfied by applying both sides of the equation on all pairs of the type \( (X_\alpha, X_{\alpha'}), (X_\alpha, X_S), (X_S, X_{\alpha'}) \). The formula for \( \Pi^\eta \) implies that the vector fields \( X_\alpha^\eta := X_\alpha - X_{\eta(\alpha)} \) and \( X_S^\eta := X_S \in \mathfrak{X}(P) \) satisfy \( (\Omega, \Pi^\eta)(X_\alpha^\eta) = I^* \alpha \) and \( (\Omega, \Pi^\eta)(X_S^\eta) = I^* S \), from which we deduce that \( (\Omega, \Pi^\eta) : TP \rightarrow I^*(C \oplus \sigma) \) is an isomorphism.

### 2.2.5 A Dual Point of View on Realizations

It is useful to keep in mind the following “dual” point of view of the notion of a realization, in which information is retained in the inverse of the pair \((\Omega, \Pi)\) rather than in \((\Omega, \Pi)\) itself.

Let \((P, \Omega)\) be a realization of an almost Cartan algebroid \((C, \sigma)\) over \( N \). Given a choice of \( \Pi \) as in Definition 2.17 the inverse of the isomorphism (2.8) is the map
  \[ (\Omega, \Pi)^{-1} : I^*(C \oplus \sigma) \xrightarrow{\sim} TP. \quad (2.9) \]
Intuitively, one should view this map as an “infinitesimal action” of the object \( C \oplus \sigma \) on the surjective submersion \( I : P \to N \). This “action map” can be decomposed as the sum of the two vector bundle maps

\[
\Psi_{C,I} : I^*C \to TP \quad \text{and} \quad \Psi_{\sigma,\Pi} : I^*\sigma \to TP.
\]

By abuse of notation, we also denote the maps at the level of sections by this same notation

\[
\Psi_{C,I} : \Gamma(C) \to \mathfrak{x}(P), \quad \alpha \mapsto X_\alpha = (\Omega, \Pi)^{-1}(I^*\alpha),
\]

\[
\Psi_{\sigma,\Pi} : \Gamma(\sigma) \to \mathfrak{x}(P), \quad S \mapsto X_S = (\Omega, \Pi)^{-1}(I^*S).
\]

Thus, \( X_\alpha, X_S \in \mathfrak{x}(P) \), which should be thought of as the fundamental vector fields of the “infinitesimal action”, are the unique vector fields satisfying

\[
\Omega(X_\alpha) = I^*\alpha, \quad \Pi(X_\alpha) = 0,
\]

\[
\Omega(X_S) = 0, \quad \Pi(X_S) = I^*S.
\]

**Lemma 2.22.** Let \( (P, \Omega) \) be a realization of an almost Cartan algebroid \( (C, \sigma) \) and fix a choice of \( \Pi \). Then,

\[
\Omega([X_\alpha, X_{\alpha'}]) = I^*[\alpha, \alpha'],
\]

\[
\Omega([X_\alpha, X_S]) = I^*S(\alpha),
\]

\[
\Omega([X_S, X_{S'})] = 0,
\]

for all \( \alpha, \alpha' \in \Gamma(C) \) and \( S, S' \in \Gamma(\sigma) \). In particular, \( \text{Ker} \, \Omega \subset TP \) is an involutive distribution.

**Proof.** Follows directly from the structure equation \([2.5]\) together with lemma \([2.9]\). \( \square \)

The fact that \( \text{Ker} \, \Omega \) is an involutive distribution is one first consequence of the structure equations. Another important consequence is:

**Lemma 2.23.** Let \( (P, \Omega) \) be a realization of an almost Cartan algebroid \( (C, \sigma) \). The map

\[
\Psi_{\sigma} = \Psi_{\sigma,\Pi} : I^*\sigma \to TP
\]

is independent of the choice of \( \Pi \). Thus, there is a canonical isomorphism

\[
\Psi_{\sigma} : I^*\sigma \xrightarrow{\cong} \text{Ker} \, \Omega.
\]

**Proof.** Fix a choice of \( \Pi \). We must show that if \( \Pi' \) is another such choice, then \( \Pi'(X_S) = I^*S \), or equivalently, that \( \Pi'(X_S)(I^*\alpha) = \Pi(X_S)(I^*\alpha) \) for any \( \alpha \in \Gamma(C) \). Subtracting the structure equations for \( \Pi \) and \( \Pi' \) from each other, we see that \( (\Pi' - \Pi) \wedge \Omega = 0 \). Thus, for any \( \alpha \in \Gamma(C) \),

\[
0 = ((\Pi' - \Pi) \wedge \Omega)(X_S, X_\alpha)
= \Pi'(X_S)(\Omega(X_\alpha)) - \Pi'(X_\alpha)(\Omega(X_S)) - \Pi(X_S)(\Omega(X_\alpha)) + \Pi(X_\alpha)(\Omega(X_S))
= \Pi'(X_S)(I^*\alpha) - \Pi(X_S)(I^*\alpha).
\]

Thus, in the notation \( X_\alpha \) and \( X_S \), one should keep in mind that \( X_\alpha \) depends on the choice of \( \Pi \), while \( X_S \) does not.

**Remark 2.24.** While condition \([2.5]\) in the definition of a realization is rather natural, condition \([2.6]\) is less so. In the examples of realizations coming from Lie pseudogroups, this condition is always satisfied. Going beyond Lie pseudogroups, it would be interesting to relax this condition in two possible directions: 1) weaken the definition of a realization by dropping condition \([2.6]\) or requiring a weaker condition; 2) in the dual point of view, requiring of \([2.4]\) to be an isomorphism is like requiring of the “infinitesimal action” to be free (injectivity) and transitive (surjectivity), and one may relax these conditions.
2.2.6 Freedom in Choosing $\Pi$

In the definition of a realization (Definition 2.17), we require the existence of a 1-form $\Pi$, which is, in general, not unique. The ambiguity in the choice of $\Pi$ can be be described in terms of the 1st prolongation of $\sigma$ (Definition A.33). For simplicity, let us assume that the first prolongation $\sigma^{(1)}$ is of constant rank. Fixing a $\Pi \in \Omega^1(P;I^*\sigma)$ that satisfies (2.5) and (2.6) as a “reference point” (which also fixes a choice of the maps (2.10)), we have a vector bundle isomorphism

$$I^*\text{Hom}(C,\sigma) \cong \left\{ \hat{\xi} \in \text{Hom}(TP; I^*\sigma) \mid \hat{\xi}(X_s) = 0 \; \forall S \in \Gamma(\sigma) \right\}, \quad \xi \mapsto \hat{\xi}, \quad (2.12)$$

where $\hat{\xi}$ is uniquely determined by the conditions

$$\hat{\xi}(X_\alpha) = \xi(I^*\alpha) \quad \forall \alpha \in \Gamma(C), \quad (2.13)$$

$$\hat{\xi}(X_S) = 0 \quad \forall S \in \Gamma(\sigma). \quad (2.14)$$

The isomorphism (2.12) restricts to the isomorphism

$$I^*\sigma^{(1)} \cong \left\{ \hat{\xi} \in \text{Hom}(TP; I^*\sigma) \mid \hat{\xi}(X_s) = 0 \; \forall S \in \Gamma(\sigma) \text{ and } \hat{\xi} \wedge \Omega = 0 \right\}. \quad (2.15)$$

At the level of sections, we obtain a linear isomorphism between sections $\xi \in \Gamma(I^*\sigma^{(1)})$ and 1-forms $\hat{\xi} \in \Omega^1(P;I^*\sigma)$ that satisfy both (2.14) and $\hat{\xi} \wedge \Omega = 0$. From now on, we write $\xi = \hat{\xi}$.

**Proposition 2.25.** Let $(P,\Omega)$ be a realization of an almost Cartan algebroid $(C,\sigma)$ and assume that $\sigma^{(1)}$ is of constant rank. The subspace of $\Omega^1(P;I^*\sigma)$ consisting of elements $\Pi$ satisfying (2.5) and (2.6) is an affine space modeled on $\Gamma(I^*\sigma^{(1)})$.

**Proof.** Fix a choice of $\Pi \in \Omega^1(P;I^*\sigma)$ satisfying (2.5) and (2.6). Given any other choice $\Pi'$, Lemma 2.23 implies that the difference $\Pi' - \Pi \in \Omega^1(P;I^*\sigma)$ satisfies (2.13) and $\hat{\xi} \wedge \Omega = 0$. Hence, $\Pi' - \Pi \in \Gamma(I^*\sigma^{(1)})$.

Conversely, let $\xi \in \Gamma(I^*\sigma^{(1)})$. We claim that $\Pi + \xi$ satisfies conditions (2.5) and (2.6). Because $\xi$ satisfies $\hat{\xi} \wedge \Omega = 0$, it follows that $\Pi + \xi \in \Omega^1(P;I^*\sigma)$ satisfies (2.5). Moreover, the composition

$$(\Omega, \Pi + \xi) \circ (\Omega, \Pi)^{-1}(I^*\alpha, I^*S) = (I^*\alpha, I^*S + \xi(I^*\alpha))$$

is a vector bundle automorphism of $I^*(C \oplus \sigma)$, which implies that $(\Omega, \Pi + \xi)$ satisfies (2.6). \hfill $\Box$

2.3 Cartan Algebroids

While the notion of an almost Cartan algebroid allows us to talk about structure equations, the correct underlying infinitesimal structure is more subtle. This is already clear in Cartan’s local picture, where the coefficients $c^k_i$ and $a^k_i$s suffice to write down the structure equations, but the fact that they come from structure equations implies that they should form a Cartan data (Theorem 2.2). This bring us to the notion of a Cartan algebroid.

The following notion is standard for Lie algebroids, but it also makes sense for almost Lie algebroids. Let $C$ be an almost Lie algebroid over $N$. A $C$-connection on a vector bundle $\sigma$ over $N$ is a bilinear operation

$$\nabla : \Gamma(C) \times \Gamma(\sigma) \to \Gamma(\sigma), \quad (\alpha, T) \mapsto \nabla_\alpha(T)$$

satisfying

$$\nabla_{f\alpha}(T) = f\nabla_\alpha(T), \quad \nabla_\alpha(fT) = f\nabla_\alpha(T) + L_{\rho(\alpha)}(f)T,$$

for all $\alpha \in \Gamma(C)$, $T \in \Gamma(\sigma)$ and $f \in C^\infty(N)$. In the following definition, we will also use the fact that $\text{Hom}(C,C)$ is a bundle of Lie algebras with the fiberwise commutator bracket $[T,S] = T \circ S - S \circ T$. 

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Definition 2.26. A Cartan algebroid is an almost Cartan algebroid $(\mathcal{C}, \xi)$ over $N$ such that:

1. $\sigma \subset \text{Hom}(\mathcal{C}, \mathcal{C})$ is closed under the commutator bracket.

2. There exists $t \in \Gamma(\text{Hom}(\Lambda^2 \mathcal{C}, \sigma))$, $(\alpha, \beta) \mapsto t_{\alpha,\beta}$, such that for all $\alpha, \beta, \gamma \in \Gamma(\mathcal{C})$,
   \begin{equation}
   [[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = t_{\alpha,\beta}(\gamma) + t_{\beta,\gamma}(\alpha) + t_{\gamma,\alpha}(\beta). \tag{2.16}
   \end{equation}

3. There exists a $\mathcal{C}$-connection $\nabla$ on $\sigma$ such that for all $\alpha, \beta \in \Gamma(\mathcal{C})$, $T \in \Gamma(\sigma)$,
   \begin{equation}
   T([\alpha, \beta]) - [T(\alpha), \beta] - [\alpha, T(\beta)] = \nabla_{\beta}(T)(\alpha) - \nabla_{\alpha}(T)(\beta). \tag{2.17}
   \end{equation}

Thus, $t$ controls the failure of the Jacobi identity and $\nabla$ controls the failure of $\sigma$ to act on $\mathcal{C}$ by derivations. Condition 1 can be restated as the condition that $\sigma$ must be a bundle of Lie algebras. It is interesting to note that if we were to relax the definition of an almost Cartan algebroid and only require of $\sigma$ to be a vector subbundle of $\text{Hom}(\mathcal{C}, \mathcal{C})$, and not necessarily of $\text{Hom}(\mathcal{C}, \text{Ker } \rho)$, then the fact that $\sigma$ actually lies in $\text{Hom}(\mathcal{C}, \text{Ker } \rho)$ would follow from condition 3 in the above definition. Indeed, replacing $\beta$ by $f\beta$ in this condition, where $f \in C^\infty(N)$, one sees that $L_{\rho(T(\alpha))}(f)\beta = 0$, which implies that $\rho \circ T$ must vanish for all $T \in \Gamma(\sigma)$.

Example 2.27. Locally, a Cartan algebroid is the same thing as a Cartan data. Continuing from Examples 2.12 and 2.20:

- Condition (1) is equivalent to the existence of functions $\epsilon_{\eta \mu}^\lambda$ on $N$ such that
  \begin{equation*}
  [t_{\eta}, t_{\mu}] = \epsilon_{\eta \mu}^\lambda t_{\lambda}.
  \end{equation*}
  This is precisely Equation (C1) in Theorem 2.2.

- The bundle map $t : \Lambda^2 \mathcal{C} \to \sigma$ can be written as
  \begin{equation*}
  t(e_i, e_j) = \nu_{ij}^\lambda t_{\lambda},
  \end{equation*}
  and a straightforward computation shows that condition (2) is equivalent to (C2).

- A $\mathcal{C}$-connection on $\sigma$ is determined by
  \begin{equation*}
  \nabla_{e_i}(t_{\mu}) = \xi_{\mu i}^\lambda t_{\lambda},
  \end{equation*}
  and we readily verify that condition (3) is equivalent to (C3).

Cartan algebroids behave well under gauge equivalences:

Lemma 2.28. Let $(\mathcal{C}, \sigma)$ be a Cartan algebroid over $N$ and let $\xi \in \Gamma(\text{Hom}(\mathcal{C}, \sigma))$ be a gauge equivalence. Then $(\mathcal{C}^\xi, \sigma)$ is again a Cartan algebroid over $N$. Furthermore, if an almost Cartan algebroid is gauge equivalent to a Cartan algebroid, then it is a Cartan algebroid.

Proof. We know that $(\mathcal{C}^\xi, \sigma)$ is an almost Cartan algebroid and we must verify that it is a Cartan algebroid by checking the three conditions of Definition 2.26. The first condition is immediately satisfied. For the other two, choose $t$ and $\nabla$ for the Cartan algebroid $(\mathcal{C}, \sigma)$. A straightforward computation shows that the remaining conditions are satisfied with $t^\xi : \Lambda^2 \mathcal{C}^\xi \to \sigma$ and $\nabla^\xi : \Gamma(\mathcal{C}^\xi) \times \Gamma(\sigma) \to \Gamma(\sigma)$ defined by

\begin{align*}
  t_{\alpha,\beta}^\xi := t_{\alpha,\beta} - \nabla_{\alpha}(\xi(\beta)) + \nabla_{\beta}(\xi(\alpha)) - [\xi(\alpha), \xi(\beta)] + \xi([\alpha, \beta]) \\
  - \xi(\xi(\beta)(\alpha)) + \xi(\xi(\alpha)(\beta)),
\end{align*}

\begin{align*}
  \nabla^\xi(S) := \nabla_S(S) + [\xi(\alpha), S] + \xi(S(\alpha)).
\end{align*}

The second assertion follows from the fact that gauge equivalence is an equivalence relation. \(\square\)
2.3.1 Freedom in Choosing $t$ and $\nabla$

Similar to realizations, the freedom in the choice of $t$ and $\nabla$ in the definition of a Cartan algebroid $(\mathcal{C}, \sigma)$ can be expressed in terms of the Spencer complex of $\sigma$.

**Proposition 2.29.** Let $(\mathcal{C}, \sigma)$ be a Cartan algebroid over $N$.

1. The subspace of $\Gamma(\text{Hom}(\Lambda^2\mathcal{C}, \sigma))$ consisting of elements $t$ satisfying (2.16) is an affine space modeled on $\Gamma(Z^{0,2}(\sigma))$.

2. For each $S \in \Gamma(\sigma)$, the subspace of $\Gamma(\text{Hom}(\mathcal{C}, \sigma))$ consisting of elements $\nabla(S)$ satisfying (2.17) is an affine space modeled on $\Gamma(\sigma^{(1)})$.

**Proof.** By (2.16), the difference of two choices $t$ and $t'$ satisfies $\delta(t' - t) = 0$, where $\delta$ is the coboundary operator (A.59). Conversely, given a choice of a $t$ and $\xi \in \Gamma(Z^{0,2}(\sigma))$, clearly (2.16) is satisfied when replacing $t$ by $t + \xi$. This proves item 1. Similarly, in item 2, given two choices $\nabla$ and $\nabla'$ and $S \in \Gamma(\sigma)$, $\delta(\nabla(S)) = 0$ by (2.17). Conversely, given a choice of $\nabla$ and $\xi \in \Gamma(\sigma^{(1)})$, (2.17) is satisfied when replacing $\nabla(S)$ by $\nabla(S) + \xi$. \hfill $\square$

2.3.2 The Need for Cartan Algebroids for the Existence of Realizations

In the local picture, we saw that if an almost Cartan data admits a realization, then it is a Cartan data (Theorem 2.2). In the modern picture, this translates to:

**Theorem 2.30.** (necessary conditions for the existence of a realization) If an almost Cartan algebroid admits a realization, then it is a Cartan algebroid.

**Proof.** This proof is essentially a global version of the proof of Theorem 2.2. Choose $\Pi \in \Omega^1(P; \Gamma\mathcal{C})$ as in definition 2.17. The 2-form $d\Omega + \frac{1}{2}[[\Omega, \Omega], \Pi] \in \Omega^2(P; \Gamma\mathcal{C})$ vanishes, as well as its “differential consequence” $d_{\nabla}(d\Omega + \frac{1}{2}[[\Omega, \Omega], \Pi]) \in \Omega^3(P; \Gamma\mathcal{C})$, for any choice of $\nabla$. Applying $d_{\nabla}(d\Omega + \frac{1}{2}[[\Omega, \Omega], \Pi]) = 0$ on a triple of vector fields of type $X_\alpha, X_\alpha', X_\alpha'' \in \mathfrak{X}(P)$ and using Lemmas 2.9 and 2.22 implies the identity

$$I^*([[[\alpha, \alpha'], \alpha''], \alpha]''] + [[[\alpha', \alpha''], \alpha'], \alpha])] =$$

$$\Pi((X_{\alpha}, X_{\alpha'})(I^*\alpha'') + \Pi((X_{\alpha'}, X_{\alpha''}))(I^*\alpha) + \Pi((X_{\alpha''}, X_{\alpha}))(I^*\alpha'),$$

applying it on $X_\alpha, X_\alpha', X_S$ implies

$$I^*(S([\alpha, \alpha']) - [S(\alpha), \alpha'] - [\alpha, S(\alpha')]) = \Pi((X_{\alpha'}, X_S))(I^*\alpha) - \Pi((X_\alpha, X_S))(I^*\alpha'),$$

and applying it on $X_\alpha, X_S, X_{S'}$ implies

$$I^*(S' \circ S(\alpha) - S \circ S'(\alpha)) = \Pi((X_S, X_{S'}))(I^*\alpha).\quad (2.18)$$

The three latter equations are equalities in $\Gamma(I^*\mathcal{C})$. Choosing a local section $\eta$ of $I : P \rightarrow N$ with domain $U \subset N$ and precomposing each of the equations with $\eta$ produces the three conditions in definition 2.26 but restricted to $U$, where the maps $t$ and $\nabla$ at a point $x \in U$ are given by

$$(t_{\alpha, \alpha'})_x = \Pi((X_\alpha, X_{\alpha'}))_{\eta(x)},$$

$$(\nabla_\alpha(S))_x = \Pi((X_\alpha, X_S))_{\eta(x)}.\quad (2.19)$$

The fact that $\Pi(X_\alpha) = \Pi(X_{\alpha'}) = 0$ and $\Pi(X_S) = I^*S$ implies that $t$ defines a tensor and $\nabla$ a connection. A standard partition of unity argument produces a global $t$ and $\nabla$. \hfill $\square$
**Remark 2.31.** We introduced the notion of an almost Cartan algebroid as the minimal structure that is needed in order to define the notion of a realization. However, this theorem shows that the relevant structure is actually that of a Cartan algebroid. Thus, from now on we will talk about realizations of Cartan algebroids rather than of almost Cartan algebroids.

**Corollary 2.32.** If \((P, \Omega)\) is a realization of \((C, \sigma)\), then for any choice of \(\Pi, t\) and \(\nabla\),

\[
\begin{align*}
((\alpha, \alpha') \mapsto \Pi([X_\alpha, X_{\alpha'}]) - I^*t_{\alpha, \alpha'}) &\in \Gamma(Z^{0,2}(\sigma)), \\
(\alpha \mapsto \Pi([X_\alpha, X_S]) - I^*\nabla_\alpha(S)) &\in \Gamma(\sigma^{(1)}), \\
\Pi([X_S, X_{S'}]) + I^*[S, S'] &= 0
\end{align*}
\]

for all \(S, S' \in \sigma\).

**Proof.** This proof of this corollary is contained in the proof of Theorem 2.30. The third equation in the statement of the corollary is precisely (2.18). The first two equations in the statement follow from Proposition (2.29) together with (2.19), first locally by choosing a local section of \(I : P \to N\), and then globally by a standard partition of unity argument.

**Lemma 2.23** together with the third identity in **Lemma 2.22** directly imply that the bundle of Lie algebras \(\sigma\) of a Cartan algebroid \((C, \sigma)\) acts canonically on all realizations:

**Proposition 2.33.** Let \((P, \Omega)\) be a realization of a Cartan algebroid \((C, \sigma)\) over \(N\). The canonical vector bundle map

\[
\Psi_\sigma : I^*\sigma \to TP
\]

defines a Lie algebroid action of \(\sigma\) on \(I : P \to N\). Moreover, the action is infinitesimally free (i.e. \(\Psi_\sigma\) is injective) and its image is \(\text{Ker } \Omega \subset TP\).

### 2.3.3 The Third Fundamental Theorem

In this modern formulation, Cartan’s realization problem (Problem 2.1) becomes the question of whether a Cartan algebroid admits a realization. In the following theorem, Cartan provides a partial solution to this problem. The main ingredient in his proof is the Cartan-Kähler Theorem (as in Theorem 2.19).

**Theorem 2.34.** (the third fundamental theorem, Cartan [3, 5], see also Kumpera [21]) Let \((C, \sigma)\) be a real-analytic Cartan algebroid over \(N\) (i.e. all manifolds and maps are analytic). If the tableau bundle \(\sigma\) is involutive, then every \(x \in N\) has a neighborhood \(U \subset N\) such that the restricted Cartan algebroid \((C_U, \sigma_U)\) over \(U\) (see Example 2.41) admits a realization.

**Remark 2.35.** Note that the existence of local solutions to the realization problem trivially implies the existence of a global solution, since realizations of \((C_U, \sigma_U)\) and \((C_V, \sigma_V)\), with \(U, V \subset N\) open subsets, induce a realization of \((C_{U \cup V}, \sigma_{U \cup V})\) by simply taking the disjoint union of the two realizations. More interesting is the question of whether there exists a global realization \((P, \Omega)\) with \(P\) connected. This global problem is still open in the analytic case, while in the smooth case both the local and global problems are open. These problems have proven to be very difficult ones, and, at least in the smooth category, they may require new ideas and possibly new analytic tools, such as an analogue of the Cartan-Kähler theorem in the smooth setting. We hope that this modern formulation will provide new insights into this fascinating problem. In Chapter 7 of [47] (and see also [48]), we propose one possible new approach for tackling the realization problem, one which is based on a reformulation of the problem that will be discussed in the Section 2.5.
In [21], Kumpera presents a proof of Theorem 2.34 whose main ingredients are the theory of exterior differential systems and the Cartan-Kähler Theorem (a possible reference for these tools is [2]). It is along the lines of Cartan’s proofs, but presented in a rigorous and clear fashion. Let us explain the general idea of the proof. Let \((\mathcal{C}, \sigma)\) be a Cartan algebroid over \(N\).

Since we are looking for local solutions, we may assume that \(N\) is an open subset of \(\mathbb{R}^n\). Let \(r\) and \(p\) be the ranks of \(\mathcal{C}\) and \(\sigma\), respectively, and let \(pr : \mathbb{R}^{r+p} \rightarrow \mathbb{R}^n\) be the projection onto the first \(n\) coordinates. Set \(P := pr^{-1}(N) \subset \mathbb{R}^{r+p}\) and denote the restriction of \(pr\) to \(P\) by \(I : P \rightarrow N\).

Given \(q \in P\), we would like to find a 1-form \(\Omega \in \Omega^1(P; I^*\mathcal{C})\) defined locally around \(q\) (thus we may shrink \(P\) to an arbitrarily small open neighborhood of \(q\), and consequently shrink \(N\) to \(I(P)\)) such that \((P, \Omega)\) is a realization of \((\mathcal{C}, \sigma)\).

The main idea is to consider the bundle of “anchored frames” of \(P\). More precisely, recall that \(\rho : \mathcal{C} \rightarrow TN\) is the anchor of the almost Lie algebroid \(\mathcal{C}\) and let us also denote by \(\rho : \mathcal{C} \oplus \sigma \rightarrow TN\) the map \((\alpha, T) \mapsto \rho(\alpha)\). Consider the following space of “anchored frames”, i.e. linear isomorphisms that are anchored:

\[
\text{Fr}(P) := \{ \xi : T_pP \cong (\mathcal{C} \oplus \sigma)_{I(p)} \mid p \in P \text{ and } \rho \circ \xi = dI|_{T_pP} \}.
\]

We denote the natural projection from \(\text{Fr}(P)\) to \(P\) by \(\pi\) and the composition \(I \circ \pi\) also by \(I\).

The bundle of “anchored frames” \(\text{Fr}(P)\) comes equipped with two tautological 1-forms, one with values in \(\mathcal{C}\) and one with values in \(\sigma\):

\[
\begin{align*}
\Pi &\in \Omega^1(\text{Fr}(P); I^*\mathcal{C}), & \Pi_{\xi} &= \xi^\mathcal{C} \circ d\pi, \\
\Pi &\in \Omega^1(\text{Fr}(P); I^*\sigma), & \Pi_{\xi} &= \xi^\sigma \circ d\pi.
\end{align*}
\]

Here \(\xi^\mathcal{C}\) and \(\xi^\sigma\) denote the \(\mathcal{C}\) and \(\sigma\) components of \(\xi \in \text{Fr}(P)\), respectively. On \(\text{Fr}(P)\), we have the following 2-form:

\[
d\Pi + \frac{1}{2} [\Pi, \Pi] - \Pi \wedge \Pi \in \Omega^2(\text{Fr}(P); I^*\mathcal{C}). \tag{2.20}
\]

The key observation is that a local solution to the realization problem is the same thing as a local section \(\eta\) of \(\pi : \text{Fr}(P) \rightarrow P\) that pulls-back (2.20) to zero. Indeed, if this is the case, then \(\eta^*\Pi \in \Omega^1(P; I^*\mathcal{C})\) satisfies the structure equation as well as the coframe condition, and is, hence, a solution.

The main challenge is to construct such a local section \(\eta\). The strategy taken in [21] (and by Cartan) is to consider the exterior differential system on \(\text{Fr}(P)\) spanned by the components of the vector bundle-valued 2-form (2.20). Integral manifolds of dimension \(r + p\) of this exterior differential system that project diffeomorphically to \(P\) correspond to the desired local sections. One proves that if \(\sigma\) is involutive, then the assumptions of the Cartan-Kähler theorem are satisfied, and, hence, such integral manifolds exist. The three integrability conditions in the definition of a Cartan algebroid play a crucial role in the proof.
2.4 Examples

The most important source of examples of Cartan algebroids and realizations is Cartan’s Second Fundamental Theorem, which is the subject of Section 3. We will see that any Lie pseudogroup gives rise to a Cartan algebroid and a realization, and, more generally, any Lie-Pfaffian groupoid that is standard and admits an integral Cartan-Ehresmann connection gives rise to a Cartan algebroid and a realization. In Section 3.3 we also compute two explicit examples that are due to Cartan. In this section, we discuss some other examples and general constructions.

Example 2.36 (Truncated Lie algebras). In [42], Singer and Sternberg study the notion of a transitive Lie algebra sheaf (which is, at least morally, the infinitesimal counterpart of a transitive Lie pseudogroups) and show that it gives rise to an algebraic structure which they call a truncated Lie algebra (Definition 4.1 in [42]). Truncated Lie algebras are the same thing as Cartan algebroids over a point modulo gauge equivalence.

Example 2.37 (Abstract Atiyah sequences). Transitive Lie algebroids $\mathcal{A}$ over a manifold $N$ are the same thing as Cartan algebroids $(\mathcal{A},0)$ over $N$. Transitive Lie algebroids are also known as “abstract Atiyah sequences” for the following reason. Given a principal $G$-bundle $\pi : P \to N$ (where $G$ is a Lie group acting from the left), one has an associated exact sequence of vector bundles over $N$ known as the “Atiyah sequence of $P$”,

$$0 \to P[\mathfrak{g}] \to TP/G \xrightarrow{d\pi} TN \to 0,$$

where $\mathfrak{g}$ is the Lie algebra of $G$ and $P[\mathfrak{g}] = (P \times \mathfrak{g})/G$. The middle term

$$A(P) := TP/G$$

has the structure of a transitive Lie algebroid: the anchor is induced by $d\pi$, while the bracket comes from the Lie bracket of vector fields on $P$ and the identification $\Gamma(A(P)) = \mathfrak{X}(P)^G$ (see [30], Section 3.2, for more details). The relevance of this sequence comes from the fact that connections on $P$ are the same thing as splittings of the sequence, while the curvature of a connection appears as the failure of the splitting to preserve the Lie brackets. The quotient map $TP \to TP/G$ induces a tautological form

$$\Omega \in \Omega^1(P; A(P)),$$

with which $(P, \Omega)$ becomes a realization of the Cartan algebroid $(A(P), 0)$.

In general, given a general transitive Lie algebroid $\mathcal{A}$ over $N$, there is an exact sequence

$$0 \to \text{Ker}(\rho) \to A \xrightarrow{\rho} TN \to 0$$

called an “abstract Atiyah sequence”. The question of the existence of a principal bundle $P$ so that $\mathcal{A}$ is isomorphic to $A(P)$ is equivalent to the integrability of $\mathcal{A}$ as a Lie algebroid. Hence, the integrability of the Lie algebroid $\mathcal{A}$ is also closely related to the existence of a realization of the Cartan algebroid $(A, 0)$; the only difference is that a general realization $P$ might have an induced action of $\mathfrak{g}$, but this action may fail to integrate to an action of $G$.

Example 2.38 (Lie groups as pseudogroups). Changing a bit the point of view of the previous example, any Lie group $G$ can be realized as a pseudogroup in normal form by making it act freely and properly on a space $P$. To be more precise, assume that $\pi : P \to N$ is a principal $G$-bundle, then the left multiplication by elements in $G$ induces a pseudogroup $\Gamma_{G,P}$ on $P$. To see that this is a pseudogroup in normal form, we just use the Lie algebroid $A(P)$ from the
previous example and the associated tautological form $\Omega$ in (2.21). It is not difficult to see that $\Gamma_{G,P}$ is characterized by the invariance of $\pi$ and $\Omega$. Note that up to Cartan equivalences, the choice of $P$ is not so important. If $Q$ is another principal $G$-bundle, then $\Gamma_{G,P}$ and $\Gamma_{G,Q}$ admit a common isomorphic prolongation, namely $\Gamma_{G,P \times Q}$ (along the canonical projections from $Q \times P$ to $P$ and $Q$, respectively). The simplest choice for $P$ would be $P = G$ with the left action of $G$. Here, $N$ is a point, $A(P)$ is the Lie algebra $g$ of $G$ and

$$\Omega \in \Omega^1(G; g)$$

is the Maurer-Cartan form. Note that this simple case is precisely Example 2.3.

**Example 2.39.** Here is a general construction of Cartan algebroids that underlies Cartan’s proof of the Second Fundamental Theorem (see Section 3.2.3). Start with a Lie algebroid $A$ over $N$ and a connection $\nabla : \mathfrak{X}(N) \times \Gamma(A) \to \Gamma(A)$. Define

$$C = C(A) := TM \oplus A, \quad \sigma := \text{Hom}(TM, A).$$

The bracket of $C$ is defined by

$$[(X, \alpha), (Y, \beta)]_\nabla := ([X, Y], [\alpha, \beta]_\nabla + \nabla_X(\beta) - \nabla_Y(\alpha)),$$

which uses the $A$-torsion of $\nabla$,

$$[\alpha, \beta]_\nabla = [\alpha, \beta] - \nabla_{\rho(\alpha)}\beta + \nabla_{\rho(\beta)}\alpha,$$

the anchor of $C$ is just the projection onto the first component, $\sigma$ becomes a bundle of Lie algebras when endowed with the Lie bracket

$$[T, S] := T \circ \rho \circ S - S \circ \rho \circ T$$

($\rho$ is the anchor of $A$), and it can be realized as a subbundle

$$\sigma \hookrightarrow \text{Hom}(C, C), \quad T \mapsto \hat{T},$$

by setting

$$\hat{T}(\alpha, X) := (0, T(\rho(\alpha) - X)), \quad \forall \alpha \in \Gamma(C), \ X \in \mathfrak{X}(M).$$

**Proposition 2.40.** The pair $(C(A), \sigma)$ in the above example is a Cartan algebroid. Up to gauge equivalence, it is independent of the choice of a connection $\nabla$.

**Proof.** First, given two connections $\nabla$ and $\nabla'$ as in the example, we would like to show that their induced almost Cartan algebroids are gauge equivalent. Indeed, the difference of the two connections induces a vector bundle map $(\nabla' - \nabla) : A \to \text{Hom}(TM, A)$, which, in turn, induces a vector bundle map from $C = TM \oplus A$ to $\sigma = \text{Hom}(TM, A)$ by acting trivially on $TM$. This is the desired gauge equivalence (simple computation).

There are two ways to prove that $(C, \sigma)$ is a Cartan algebroid – a direct proof and an indirect one. The direct proof is to show that $(C, \sigma)$ is a Cartan algebroid by verifying the axioms of Definition 2.26. A straightforward computation shows that axiom 1 is satisfied. Indeed, the difference of the two connections induces a vector bundle map $(\nabla' - \nabla) : A \to \text{Hom}(TM, A)$, which, in turn, induces a vector bundle map from $C = TM \oplus A$ to $\sigma = \text{Hom}(TM, A)$ by acting trivially on $TM$. This is the desired gauge equivalence (simple computation).

There are two ways to prove that $(C, \sigma)$ is a Cartan algebroid – a direct proof and an indirect one. The direct proof is to show that $(C, \sigma)$ is a Cartan algebroid by verifying the axioms of Definition 2.26. A straightforward computation shows that axiom 1 is satisfied. Indeed, the difference of the two connections induces a vector bundle map $(\nabla' - \nabla) : A \to \text{Hom}(TM, A)$, which, in turn, induces a vector bundle map from $C = TM \oplus A$ to $\sigma = \text{Hom}(TM, A)$ by acting trivially on $TM$. This is the desired gauge equivalence (simple computation).

The idea is straightforward (but the computations are a bit tedious): compute the left hand side of the equations in both axioms and try to decompose the resulting expressions so as to obtain a suitable $\hat{t}$ and $\nabla$. We will write down explicit solutions, i.e. expressions for $\hat{t}$ and $\nabla$ that are obtained in this way, and leave it as an
exercising to check that these satisfy axioms 2 and 3. To write down \( \bar{\iota} \), we first define the tensors

\[
R_{X,Y}(\alpha) := \nabla_{[X,Y]}\alpha - \nabla_X \nabla_Y \alpha + \nabla_Y \nabla_X \alpha,
\]

\[
R_{\alpha,\beta}(X) := \nabla_X [\alpha, \beta] - [\nabla_X \alpha, \beta] - [\alpha, \nabla_X \beta] + \nabla_{\rho(\nabla_X \alpha)\beta} - \nabla_{\rho(\nabla_X \beta)\alpha} - \nabla_{[X, \rho(\alpha)]\beta} + \nabla_{[X, \rho(\beta)\alpha]}.\]

Then, for all \( X, Y, Z \in \mathfrak{X}(M) \) and \( \alpha, \beta, \gamma \in \Gamma(A) \),

\[
\bar{\iota}_{(X, \alpha), (Y, \beta)}(Z, \gamma) := (0, R_{\alpha, \beta}(\rho(\gamma) - Z) + \frac{1}{2} R_{\rho(\gamma) - Z, \rho(\alpha) - X}(\beta) + \frac{1}{2} R_{\rho(\beta) - Y, \rho(\gamma) - Z}(\alpha)).
\]

To write down \( \nabla \), we choose a torsion-free connection on \( TM \), which (by abuse of notation) we denote by \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \). Thus, \( [X, Y] = \nabla_X Y - \nabla_Y X \) for all \( X, Y \in \mathfrak{X}(M) \).

\[
(\nabla_{(X, \alpha)} \bar{T})(Y, \beta) := [(X, \alpha), \bar{T}(Y, \beta)]_{\nabla} + (0, T(\rho(\nabla_{\rho(\alpha) - X}\beta) - \nabla_{\rho(\beta)}\rho(\beta) + \nabla_X Y),
\]

for all \( X, Y \in \mathfrak{X}(M) \), \( \alpha, \beta \in \Gamma(A) \) and \( T \in \Gamma(\sigma) \).

Alternatively, as an indirect proof, we can construct a realization of \( (\mathcal{C}(A), \sigma) \) by noting that this example is a simple case of the construction described in Section 3.2.3 (we recommend returning to this example after reading that section). Assuming that \( A \) integrates to a Lie groupoid \( \mathcal{G} \) (in fact, it suffices to have an integration to a local groupoid which always exists), \( (\mathcal{C}(A), \sigma) \) is precisely the almost Cartan algebroid constructed out of the of Lie-Pfaffian groupoid \( (J^1 \mathcal{G}, \omega) \), where \( \omega \in \Omega^1(J^1 \mathcal{G}; t^* A) \) is the Cartan form. Note that a linear connection on \( A \) is the same thing as a choice of a splitting of \( \mathcal{G} \). Finally, \( (J^1 \mathcal{G}, \omega) \) admits an integral Cartan-Ehresmann connection, which is the same thing as a section of the affine bundle \( pr : J^2 \mathcal{G} \to J^1 \mathcal{G} \), and the proposition follows from Theorem 3.2.3.

**Example 2.41.** (Restrictions of Cartan Algebroids) A Cartan algebroid \( (\mathcal{C}, \sigma) \) over \( N \) can be restricted to any submanifold \( S \subset N \), giving rise to a Cartan algebroid \( (\mathcal{C}_S, \sigma_S) \) over \( S \). Here, \( \sigma_S := \sigma|_S \) (the restriction of the vector bundle to \( S \)), \( \mathcal{C}_S := \{ \alpha \in \mathcal{C} \mid \rho(\alpha) \in TS \} \) and the bracket is uniquely determined by

\[
[\alpha|_S, \beta|_S] = [\alpha, \beta]|_S,
\]

for all \( \alpha, \beta \in \Gamma(\mathcal{C}) \). A realization \( (P, \Omega) \) of \( (\mathcal{C}, \sigma) \) induces a realization \( (P_S, \Omega_S) \) of \( (\mathcal{C}_S, \sigma_S) \) by taking the restrictions \( P_S := I^{-1}(S), I_S := I|_{P_S} \) and \( \Omega_S := \Omega|_S \). This restriction operation underlies Cartan’s trick of restricting a realization to a complete transversal that will be discussed in Section 3.2.4.

### 2.5 An Alternative Approach to Cartan Algebroids: Cartan Pairs

We conclude this section by presenting an alternative but equivalent point of view on Cartan algebroids and the realization problem that is more intuitive and which has the advantage that the formulas (e.g., (2.10) and (2.17)) become substantially simpler. We show that Cartan algebroids up to gauge equivalence are the same thing as Cartan pairs up to isomorphism, and that realizations of one induce realizations of the other.
2.5.1 Cartan Pairs

Let $A$ be an almost Lie algebroid over $N$. The Jacobiator $\text{Jac}_A \in \Gamma(\text{Hom}(\Lambda^3 A, A))$ of $A$ is defined at the level of sections by

$$\text{Jac}_A(\alpha, \beta, \gamma) := [[\alpha, \beta], \gamma] + [[[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta], \ \forall \alpha, \beta, \gamma \in \Gamma(A).$$

We say that a vector subbundle $\sigma \subset A$ is involutive if $[\Gamma(\sigma), \Gamma(\sigma)] \subset \Gamma(\sigma)$.

**Definition 2.42.** A Cartan pair over a manifold $N$ is a pair $(A, \sigma)$ consisting of a transitive almost Lie algebroid $(A, [\cdot, \cdot], \rho)$ over $N$ and an involutive vector subbundle $\sigma \subset \text{Ker } \rho \subset A$ such that

$$\text{Jac}_A \equiv 0 \pmod{\sigma} \quad (2.22)$$

Condition (2.22) can be rephrased as saying that $\text{Jac}_A$, applied on any three sections of $A$, must take values in $\sigma$. When $\sigma = 0$, a Cartan pair is simply a transitive Lie algebroid.

A Cartan pair $(A, \sigma)$ has an associated vector bundle map

$$\iota : \sigma \to \text{Hom}(A/\sigma, A/\sigma), \quad (2.23)$$

which is defined at the level of sections by

$$\iota(T)(\text{pr}(\alpha)) := \text{pr}([T, \alpha]), \quad \forall T \in \Gamma(\sigma), \ \alpha \in \Gamma(A),$$

where $\text{pr} : A \to A/\sigma$ is the quotient map. Note that the map is well defined (i.e. the formula does not depend on the representative $\alpha$) because $\sigma$ is involutive, and the right hand side is indeed $C^\infty(N)$-linear in both the $\alpha$ and $T$ slots because $\sigma$ is killed by both $\rho$ and $\text{pr}$. Equipping $\text{Hom}(A/\sigma, A/\sigma)$ with the commutator bracket, we have that:

**Lemma 2.43.** Let $(A, \sigma)$ be a Cartan pair. The map (2.23) preserves the brackets, i.e.

$$\iota([S, T]) = [\iota(S), \iota(T)], \quad \forall S, T \in \Gamma(\sigma).$$

**Proof.** Applying (2.22) on $\alpha \in \Gamma(A), S, T \in \Gamma(\sigma),$

$$0 = \text{pr}([[S, T], \alpha] + [[T, \alpha], S] + [[\alpha, S], T])$$

$$= \iota([S, T])(\alpha) - \iota([S], \iota(T))(\alpha) \quad \square$$

**Definition 2.44.** A Cartan pair $(A, \sigma)$ is said to be standard if the map (2.23) is injective.

**Lemma 2.45.** If a Cartan pair $(A, \sigma)$ is standard, then $\sigma$ is a bundle of Lie algebras.

**Proof.** Since $\iota$ preserves the bracket, then $\iota(\sigma) \subset \text{Hom}(A/\sigma, A/\sigma)$ is closed under the bracket. Now, since the bracket of $\iota(\sigma)$ satisfies the Jacobi identity, it follows that $\iota \circ \text{Jac}_\sigma = 0$, and so $\text{Jac}_\sigma = 0$ by injectivity of $\iota$. \square

2.5.2 Isomorphism of Cartan Pairs

The notion of an isomorphism of Cartan pairs is slightly more subtle than that of Cartan algebroids.

**Definition 2.46.** Two Cartan pairs $(A, \sigma)$ and $(A', \sigma')$ over $N$ are isomorphic if there exists a vector bundle isomorphism $\psi : A \to A'$ such that $\psi(\sigma) = \sigma'$,

$$\psi([\alpha, \beta]) \equiv [\psi(\alpha), \psi(\beta)] \pmod{\sigma}, \quad \forall \alpha, \beta \in \Gamma(A), \quad (2.24)$$

and $\rho \circ \psi = \rho'$.

It is straightforward to check that, as a consequence of (2.24), an isomorphism $\psi$ between two Cartan pairs commutes with the two maps $\iota$ and $\iota'$ (defined in (2.23)) induced by each of the Cartan pairs.
2.5.3 Cartan Algebroids as Cartan Pairs

Up to isomorphism, on the one side, and gauge equivalence, on the other, Cartan pairs and Cartan algebroids are the same thing. We start by constructing a Cartan pair out of a Cartan algebroid \((\mathcal{C}, \sigma)\) over \(N\). The construction is analogous to the construction of a non-abelian extension of a Lie algebroid \([29]\), Chapter 4, Section 3), and depends on a choice of \(t : \Lambda^2 \mathcal{C} \to \sigma\) and \(\nabla : \Gamma(\mathcal{C}) \times \Gamma(\sigma) \to \Gamma(\sigma)\) as in Definition 2.26 of a Cartan algebroid. Let \((t, \nabla)\) be such a pair. We set
\[
A := \mathcal{C} \oplus \sigma
\]
and equip it with an anchor induced by the anchor of \(\mathcal{C}\),
\[
\rho : A \to TN, \quad \rho(\alpha, S) := \rho(\alpha),
\]
and a bracket
\[
[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)
\]
defined by
\[
[(\alpha, S), (\beta, T)] := ([\alpha, \beta] + S(\beta) - T(\alpha), -t_{\alpha,\beta} + \nabla_\alpha T - \nabla_\beta S + [S, T]),
\]
for all \(\alpha, \beta \in \Gamma(\mathcal{C})\), \(S, T \in \Gamma(\sigma)\).

**Proposition 2.47.** Let \((\mathcal{C}, \sigma)\) be a Cartan algebroid over \(N\). The induced pair \((\mathcal{C} \oplus \sigma, \sigma)\), for a fixed choice of \((t, \nabla)\), is a standard Cartan pair. Moreover, up to isomorphism, the resulting Cartan pair is independent of the choice of \((t, \nabla)\).

**Proof.** Fix a choice of \(t\) and \(\nabla\). Clearly \(\sigma \subset \text{Ker} \, \rho\), and it is straightforward to verify that \(A\) is an almost Lie algebroid. So, we are only left with checking that, for any \(\alpha, \beta, \gamma \in \Gamma(\mathcal{C})\) and \(S, T, U \in \Gamma(\sigma)\),
\[
\begin{align*}
\text{pr} \circ \text{Jac}_A\left(\left(\alpha, 0\right), (\beta, 0), (\gamma, 0)\right) & = [[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] - t_{\alpha, \beta}(\gamma) - t_{\beta, \gamma}(\alpha) - t_{\gamma, \alpha}(\beta) = 0, \\
\text{pr} \circ \text{Jac}_A\left(\left(\alpha, 0\right), (\beta, 0), (0, T)\right) & = -T([\alpha, \beta]) + [T(\alpha), \beta] + [\alpha, T(\beta)] + \nabla_\beta(T)(\alpha) - \nabla_\alpha(T)(\beta) = 0, \\
\text{pr} \circ \text{Jac}_A\left(\left(\alpha, 0\right), (0, S), (0, T)\right) & = [S, T](\alpha) - S(T(\alpha)) + T(S(\alpha)) = 0, \\
\text{pr} \circ \text{Jac}_A\left(\left(0, S\right), (0, T), (0, U)\right) = 0.
\end{align*}
\]

Given another choice \((t', \nabla')\), the identity map \(\text{id} : \mathcal{C} \oplus \sigma \to \mathcal{C} \oplus \sigma\) gives an isomorphism between the Cartan pair obtained using \((t, \nabla)\) and that obtained using \((t', \nabla')\).

**Remark 2.48.** If we relax the notion of a Cartan algebroid by requiring for there to be a map \(\sigma \to \text{Hom}(\mathcal{C}, \mathcal{C})\) rather than an inclusion \(\sigma \subset \text{Hom}(\mathcal{C}, \mathcal{C})\), then we will obtain Cartan pairs that are not necessarily standard. In the case of Cartan pairs, it is much more natural to impose the *standard* property separately rather than add it to the initial definition. In the case of a Cartan algebroid, we chose to impose the stronger property in order to be consistent with Cartan’s local picture.

In the other direction, a standard Cartan pair \((A, \sigma)\) induces a Cartan algebroid \((A/\sigma, \sigma)\). The construction depends on a choice of a splitting of the short exact sequence
\[
0 \to \sigma \to A \xrightarrow{\text{pr}} A/\sigma \to 0.
\]
We equip the vector bundle \( A/\sigma \) with the bracket

\[
[\cdot, \cdot] : \Gamma(A/\sigma) \times \Gamma(A/\sigma) \to \Gamma(A/\sigma), \quad [\alpha, \beta] := \text{pr}([\xi(\alpha), \xi(\beta)]),
\]

and the anchor \( \rho : A/\sigma \to TN, \quad \rho(\alpha) := \rho(\xi(\alpha)). \)

Note that the bracket depends on the choice of \( \xi \), but the anchor does not. Since the Cartan pair is standard, we have an inclusion \( \sigma \hookrightarrow \text{Hom}(A/\sigma, A/\sigma). \)

**Proposition 2.49.** Let \( (A, \sigma) \) be a standard Cartan pair and let \( \xi : A/\sigma \to A \) be a choice of a splitting \( \text{(2.27)} \). The pair \( (A/\sigma, \sigma) \) equipped with the structure defined above is a Cartan algebroid. Moreover, up to gauge equivalence, the resulting Cartan algebroid is independent of the choice of \( \xi \).

**Proof.** We must show the existence of a vector bundle map \( t : \Lambda^2(A/\sigma) \to \sigma, \) and an \( A/\sigma \)-connection \( \nabla : \Gamma(A/\sigma) \times \Gamma(\sigma) \to \Gamma(\sigma) \) as in Definition 2.26. Let us denote by \( \eta : A \to \sigma \) the left splitting induced by the right splitting \( \xi \). We define \( t \) and \( \nabla \) by

\[
t_{\alpha,\beta}(\gamma) := -\eta([\xi(\alpha), \xi(\beta)]), \quad \nabla_\alpha(T) := \eta(\xi(\alpha), T).
\]

By the definition of a Cartan pair, \( \text{pr} \circ \text{Jac}_A = 0 \). Applying this equality to the triples \((\xi(\alpha), \xi(\beta), \xi(\gamma))\) and \((\xi(\alpha), \xi(\beta), T)\), where \( \alpha, \beta, \gamma \in \Gamma(A/\sigma) \) and \( T \in \Gamma(\sigma) \), a straightforward computation shows that \((2.16)\) and \((2.17)\) are satisfied, and Lemma 2.43 implies the first axiom of Definition 2.26. For the final assertion, given two splittings \( \xi \) and \( \xi' \), the difference \((\xi' - \xi) : A/\sigma \to \sigma \) defines the desired gauge equivalence. \( \square \)

The above constructions define the following correspondence:

**Theorem 2.50.** There is a \( 1-1 \) correspondence (given by the constructions above) between Cartan algebroids over \( N \) up to gauge equivalence (Definition 2.15) and Cartan pairs over \( N \) up to isomorphism (Definition 2.46).

**Proof.** Given a Cartan algebroid \((C, \sigma)\) and a gauge equivalence \( \eta : C \to \sigma \), the map \( \psi : (C \oplus \sigma) \to (C^{\eta} \oplus \sigma), \quad (\alpha, T) \mapsto (\alpha, T - \eta(\alpha)) \), defines an isomorphism between the Cartan pair induced by \( (C, \sigma) \) and the Cartan pair induced by the gauge equivalent one \((C^{\eta}, \sigma)\). We omit the remaining details, which are straightforward to verify. \( \square \)

### 2.5.4 Realizations of Cartan Pairs

The notion of a realization takes a more elegant form in the Cartan pair picture. The existence of realizations of a Cartan algebroid is equivalent to the existence of a realization of its induced Cartan pair, and vice versa.

**Definition 2.51.** A realization of a Cartan pair \((A, \sigma)\) over \( N \) is a pair \((P, \Omega)\) consisting of a surjective submersion \( I : P \to N \) and an anchored 1-form \( \Omega \in \Omega^1(P, I^* A) \), such that

\[
d\Omega + \frac{1}{2}[\Omega, \Omega] \equiv 0 \pmod{\sigma} \quad (2.28)
\]

and such that \( \Omega \) is pointwise an isomorphism.

The proof of the following proposition is straightforward:
Proposition 2.52. Given a realization \((P, \Omega)\) of a Cartan algebroid \((C, \sigma)\) with a fixed choice of \(\Pi\) as in Definition 2.17, the pair \((P, (\Omega, \Pi))\) is a realization of the induced standard Cartan pair \((C \oplus \sigma, \sigma)\). Conversely, given a realization \((P, \Omega)\) of a standard Cartan pair \((A, \sigma)\), the pair \((P, \text{pr} \circ \Omega)\) is a realization of the induced Cartan algebroid \((A/\sigma, \sigma)\).

Remark 2.53. In the case of a Cartan pair \((A, 0)\), thus \(A\) is a transitive Lie algebroid, one can obtain a solution to the realization problem by integrating the Lie algebroid to a Lie groupoid (when the Lie algebroid is integrable), in which case the Maurer-Cartan form on any source fiber of the Lie groupoid defines a solution. In [11], the authors present a method for integrating Lie algebroids to Lie groupoids by constructing a Lie groupoid out of the space of so called \(A\)-paths of the Lie algebroid. A large part of this construction not rely on the fact that the Lie algebroid one starts with satisfies the Jacobi identity. The point of view of Cartan pairs – “transitive Lie algebroids that satisfy the Jacobi identity modulo \(\sigma\)” – suggests a new method for tackling the realization problem: imitating the construction in [11] and pinpointing the precise role of the Jacobi identity along the way. In [47] (Chapter 7, and see also [48]), yet another method for solving the realization problem in the case of a transitive Lie algebroid is introduced, one which identifies the precise role of the Jacobi identity. Using the point of view of Cartan pairs, one may also attempt to use this method in tackling Cartan’s realization problem.

3 Cartan’s Second Fundamental Theorem

Cartan’s Second Fundamental Theorem states that:

Theorem 3.1. (the second fundamental theorem) Any Lie pseudogroup is Cartan equivalent to a pseudogroup in normal form (Definition 2.18).

In other words, up to Cartan equivalence, the theorem reduces the study of Lie pseudogroups to the study of pseudogroups in normal form, and hence to the study of Cartan algebroids and their realizations.

In this section, after recalling the definitions of a Lie pseudogroup (Definition 3.2) and of Cartan equivalence of pseudogroups (Definition 3.7), we present a modern proof of the Second Fundamental Theorem. This proof is the result of our endeavor to understand Cartan’s constructions (in [3, 5]) more conceptually and in a global, coordinate-free fashion. In our proof, we use the language of jet groupoids and algebroids and, more abstractly, the language of Lie-Pfaffian groupoids and algebroids. These are recalled in Appendices A.1 and A.2. The Lie-Pfaffian groupoid framework isolates the essential properties of a Lie pseudogroup and, consequently, proofs become substantially simpler and more transparent. For the reader that is not familiar with these objects, we recommend reading the appendix before reading this section.

In the course of our work, we found Cartan’s examples of the Second Fundamental Theorem to be a useful guide in understanding the general ideas. In Section 3.3, we cite two such examples and run them through the machinery of the modern proof to “rediscover” Cartan’s formulas.

3.1 Lie Pseudogroups and Cartan Equivalence

Intuitively, a Lie pseudogroup is: “a pseudogroup that is defined by a system of partial differential equations”. The language of jet groupoids and algebroids allows us to make this definition precise.
3.1.1 Jet Groupoids and Algebroids

Let us recall the main ingredients in the framework of jet groupoids and algebroids. We refer to the reader to the appendix for more details. With any manifold \( M \), we associate a tower of jet groupoids

\[
\ldots \xrightarrow{\varepsilon} J^3 M \xrightarrow{\varepsilon} J^2 M \xrightarrow{\varepsilon} J^1 M \xrightarrow{\varepsilon} J^0 M,
\]

where \( J^k M := M \), the \( k \)-th jet groupoid of \( M \), is the Lie groupoid whose space of arrows consists of all \( k \)-jets \( j^k_x \phi \) of locally defined diffeomorphisms \( \phi \in \text{Diff}_{\text{loc}}(M) \) of \( M \), and the projections \( \pi : J^k M \to J^{k-1} M \), \( j^k_x \phi \to j^{k-1}_x \phi \), are surjective Lie groupoid morphisms and submersions. The Lie algebroid of \( J^k M \), the \( k \)-th jet algebroid of \( M \), is denoted by \( A^k M \), and the induced projections by \( l : A^k M \to A^{k-1} M \). The \( k - 1 \)-th jet algebroid \( A^{k-1} M \) is canonically a representation of the \( k \)-th jet groupoid \( J^k M \). The action is given by conjugation (see (A.45)), and the resulting representation is called the adjoint representation.

The kernel \( \sigma^k M = \ker l \subset A^k M \) of each projection, called the \( k \)-th symbol space of \( M \), plays an important role in the theory. Its elements are canonically identified with vector-valued homogeneous polynomials of degree \( k \) on \( M \), i.e. there is a canonical isomorphism \( A^k M \cong S^k T^* M \otimes TM \).

For our purposes, the most important piece of structure of a jet groupoid is its Cartan form – a tautological 1-form \( \omega \in \Omega^1(J^k M; t^* A^{k-1} M) \) that takes values in the adjoint representation, which has the key property that it is multiplicative (see (A.47)). The pair \((J^k M, \omega)\), i.e. the \( k \)-th jet groupoid equipped with the Cartan form, has the structure of a Lie-Pfaffian groupoid (Definition A.39). For practically all purposes, the Lie-Pfaffian groupoid structure is all one needs when working with jet groupoids.

At the infinitesimal level, the Cartan form on \( J^k M \) linearizes (via (A.51)) to a connection-like operator \( D : \mathcal{X}(M) \times \Gamma(A^k M) \to \Gamma(A^{k-1} M) \) on \( A^k M \) called the Spencer operator. The pair \((A^k M, D)\) has the structure of a Lie-Pfaffian algebroid (Definition A.44).

3.1.2 Lie Pseudogroups

With any pseudogroup \( \Gamma \) on \( M \), we associate the tower

\[
\ldots \xrightarrow{\varepsilon} J^3 \Gamma \xrightarrow{\varepsilon} J^2 \Gamma \xrightarrow{\varepsilon} J^1 \Gamma \xrightarrow{\varepsilon} J^0 \Gamma,
\]

a subsequence of (3.1), where

\[
J^k \Gamma := \{ j^k_x \phi | \phi \in \Gamma, x \in \text{Dom}(\phi) \} \subset J^k M,
\]

the \( k \)-th jet groupoid of \( \Gamma \), is a subgroupoid of the \( k \)-th jet groupoid of \( M \), and the maps \( \pi : J^{k+1} \Gamma \to J^k \Gamma \), the restrictions of the above projections, are surjective groupoid morphisms.

In general, the sequence may fail to be smooth in the sense that the \( J^k \Gamma \)'s may fail to be submanifolds of the \( J^k M \)'s (if they are, then they are automatically Lie subgroupoids), and the projections may fail to be submersions. If \( J^k \Gamma \) is a Lie subgroupoid for some \( k \), then it has an associated Lie subalgebroid

\[
A^k \Gamma := A(J^k \Gamma) \subset A^k M
\]

of the \( k \)-th jet algebroid of \( M \). In this case, we may define the \( k \)-th symbol space of \( \Gamma \) to be

\[
\sigma^k \Gamma := \sigma^k M \cap A^k \Gamma.
\]

In general, \( \sigma^k \Gamma \) is not a vector bundle (it may fail to be of constant rank) but a discrete vector bundle (see Section A.1.4), and it is a vector bundle if and only if it is of constant rank.
Definition 3.2. A Lie pseudogroup of (at least) order $k > 0$ on a manifold $M$ is a pseudogroup $\Gamma$ on $M$ satisfying:

1. For any $\phi \in \text{Diff}_{\text{loc}}(M)$, if $j^k_x \phi \in J^k \Gamma$ for all $x \in \text{Dom}(\phi)$, then $\phi \in \Gamma$.

2. (a) $J^k \Gamma \subset J^k M$ is a Lie subgroupoid,
   (b) $J^{k-1} \Gamma \subset J^{k-1} M$ is a Lie subgroupoid,
   (c) $\pi : J^k \Gamma \to J^{k-1} \Gamma$ is a submersion (hence $\sigma^k \Gamma$ is of constant rank),
   (d) $(\sigma^k \Gamma)^{(1)}$ is of constant rank.

A Lie pseudogroup is of finite type if $(\sigma^k \Gamma)^{(l)} = 0$ for some $l > 0$, and otherwise it is of infinite type.

In this definition, the $k$-th jet groupoid $J^k \Gamma \subset J^k M$ should be interpreted as a system of PDEs and axiom 1 should be interpreted as the condition that $\Gamma$ consists of its full set of local solutions. Thus, one may study Lie pseudogroups by studying this special class of systems of PDEs. This was Lie’s original approach. Axiom 2 is a set of regularity conditions that allow us to study these systems of PDEs geometrically. In particular, axioms 2a and 2b ensure that $J^k \Gamma$ and $J^{k-1} \Gamma$ have associated Lie algebroids, which we denote by $A^k \Gamma$ and $A^{k-1} \Gamma$, respectively, and axiom 2c implies that the projection

$$l : A^k \Gamma \to A^{k-1} \Gamma$$

is surjective (and, hence, the symbol space $\sigma^k \Gamma$ is of constant rank).

Remark 3.3. While axiom 1 is standard, the regularity conditions one imposes on the system, i.e. axiom 2, vary in the literature (e.g. compare axioms 1 and 2 in Section 3 of [16], Definition IV.1 in [24] and Definition 3.1 in [37]). In our definition, we impose sufficient conditions so as to allow us detach $J^k \Gamma$, “the defining system of PDEs”, from its ambient jet groupoid $J^k M$, and to handle it abstractly as a Lie-Pfaffian groupoid, as we will explain. These conditions, however, are sufficient but not necessary. Another possible definition would be to simply replace axiom 2 by the condition that $J^k \Gamma$ have the structure of a Lie-Pfaffian groupoid. We chose the current form of the definition in order to keep the definition as explicit as possible.

In Section 3.3, we will revisit Cartan’s examples of Lie pseudogroups that we saw in the introduction, Examples 1.3 and 1.4 (one of finite type and one of infinite type), and explicitly compute their jet groupoids and algebroids, and their associated Cartan forms, Spencer operators, etc. As we also mentioned in the introduction, Lie pseudogroups arise in differential geometry as the local symmetries of geometric structures. As a general class of examples to keep in mind:

Example 3.4. The pseudogroup $\Gamma$ of local automorphisms of any integrable $G$-structure is a Lie pseudogroup of order 1. Pseudogroups of local symmetries of foliations, symplectic structures, complex structures and integral affine structures are just a few of the examples that arise in this way (see [13, 10] for more on $G$-structures). The assumption of integrability ensures that $\Gamma$ is “large enough”. In fact, such pseudogroups are always transitive since these structures are locally homogeneous (they have a local normal form). The 1st jet groupoid $J^1 \Gamma$, in this case, is canonically isomorphic to the gauge groupoid of the $G$-structure (recall that any principal $G$-bundle $P \to M$ gives rise to a gauge groupoid $\text{Gauge}(P) \rightrightarrows M$, with $\text{Gauge}(P) = P \times P/G$). We also note that the restriction of the Cartan form on $J^1 M$ to $J^1 \Gamma$ is precisely the lift of the tautological form of the $G$-structure.
3.1.3 Lie Pseudogroups as Lie-Pfaffian Groupoids

In Appendix A.2, we reviewed the notion of a Lie-Pfaffian groupoid, an abstract notion that encodes the essential structure of the jet groupoids that play the role of the defining equations of Lie pseudogroups. Let us explain this in more detail.

The regularity conditions that we have imposed in Definition 3.2 of a Lie pseudogroup $\Gamma \subset \text{Diff}_{\text{loc}}(M)$ ensure that the Cartan form of the ambient jet groupoid $J^k M$ restricts nicely to the “defining system of PDEs” $J^k \Gamma$. Indeed:

**Proposition 3.5.** Let $\Gamma$ be a Lie pseudogroup on $M$ of order $k$. The adjoint representation restricts to a representation $A^{k-1} \Gamma$ of $J^k \Gamma$ (which we call the adjoint representation of $J^k \Gamma$), and the Cartan form restricts to a 1-form on $J^k \Gamma$ with values in this representation, 

$$\omega = \omega|_{J^k \Gamma} \in \Omega^1(J^k \Gamma; t^* A^{k-1} \Gamma),$$

which is pointwise surjective and multiplicative (we call $\omega$ the Cartan form of $J^k \Gamma$).

**Proof.** The first assertion follows directly from the defining formula (A.45) of the adjoint representation. Next, denoting the kernel of $\omega$ on $J^k \Gamma$ by $C_\omega = \text{Ker} \omega \subset T J^k \Gamma$, we construct an Ehresmann connection on $s : J^k \Gamma \rightarrow M$ by choosing a splitting of $d\pi : T J^k \Gamma \rightarrow \pi^* T J^{k-1} \Gamma$ and composing it at each point $j_k^x \phi \in J^k \Gamma$ with $(d(j_k^x \phi))_x$. This induces a decomposition $T J^k \Gamma = H \oplus \text{Ker} ds$, where $\omega$ kills the horizontal component $H$ (by the definition of $\omega$) and maps the second component surjectively onto $t^* A^{k-1} \Gamma$ by axiom 2c of a Lie pseudogroup. Finally, since $\omega$ is the restriction of a multiplicative form, it is multiplicative. \[\□\]

Thus, to check that the pair $(J^k \Gamma, \omega)$ is a Lie-Pfaffian groupoid, we are only left with checking axioms 1 and 2 of Definition A.39. Axiom 1 follows from the existence of a splitting of $ds|_{C_\omega} : C_\omega \rightarrow s^* TM$, as explained in the proof of the above Proposition, while axiom 2 is verified as in Example A.41 where we showed that a jet groupoid of a manifold, equipped with its Cartan form, is a Lie-Pfaffian groupoid.

Note also that Proposition A.29 together with axiom 1 of the definition of a Lie pseudogroup, implies that there is a bijection

$$\Gamma \rightarrow \text{Bis}_{\text{loc}}(J^k \Gamma, \omega), \quad \phi \mapsto j_k^\phi,$$

identifying the generalized pseudogroup $\text{Bis}_{\text{loc}}(J^k \Gamma, \omega)$ of local holonomic bisections with $\Gamma$ (see also Section A.2.2). Thus, modulo this map, the two objects are one and the same.

3.1.4 Cartan Equivalence of Pseudogroups

Haefliger, in his work on the transverse structure of foliations ([18]), recast the notion of a pseudogroup in the framework of Lie groupoids. He observed that pseudogroups are the same thing as effective étale Lie groupoids. Recall that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called étale if its source map $s : \mathcal{G} \rightarrow M$ (and hence its target map $t : \mathcal{G} \rightarrow M$) is a local diffeomorphism, i.e. each arrow $g \in \mathcal{G}$ has an open neighborhood $U$ such that $s|_U$ is a diffeomorphism onto its image. An étale groupoid is called effective if for any pair of local bisections $b$ and $b'$ with a common domain, $t \circ b = t \circ b'$ implies $b = b'$. Haefliger’s correspondence is as follows: given a pseudogroup $\Gamma$ on $M$, one constructs the groupoid

$$\text{Germ}(\Gamma) \rightrightarrows M$$
whose space of arrows consists of all germs of elements of $\Gamma$. We denote an arrow germ $\text{germ}_x \phi$, i.e.
the germ at $x \in \text{Dom}(\phi)$ of $\phi \in \Gamma$, and the structure maps are:

$$
s(\text{germ}_x \phi) = x, \quad t(\text{germ}_x \phi) = \phi(x), \quad 1_x = \text{germ}_x \text{id},
$$

$$
\text{germ}_{\phi(x)} \phi' \cdot \text{germ}_x \phi = \text{germ}_x (\phi' \circ \phi), \quad (\text{germ}_x \phi)^{-1} = \text{germ}_{\phi(x)} \phi^{-1}.
$$

Every element $\phi \in \Gamma$ gives rise to a local bisection $b_\phi$ of $\text{Germ}(\Gamma)$ defined by $b_\phi(x) = \text{germ}_x \phi$ for all $x \in \text{Dom}(\phi)$, and the smooth structure of $\text{Germ}(\Gamma)$ (with a possibly non-Hausdorff nor second countable topology) is uniquely determined by the requirement that each such local bisection is a diffeomorphism onto its image. With this structure, $\text{Germ}(\Gamma) \rightrightarrows M$ becomes an effective étale Lie groupoid. In the reverse direction, any effective étale Lie groupoid $\mathcal{G} \rightrightarrows M$ induces the pseudogroup

$$
\Gamma(\mathcal{G}) := \{ \phi_b = t \circ b | b \text{ local bisection of } \mathcal{G} \} \subset \text{Diff}_{\text{loc}}(M).
$$

To summarize:

**Proposition 3.6.** Let $M$ be a manifold. There is a 1-1 correspondence between

$$
\{ \text{pseudogroups } \Gamma \text{ on } M \} \longleftrightarrow \{ \text{effective étale Lie groupoids } \mathcal{G} \rightrightarrows M \}
$$

given by $\Gamma \mapsto \text{Germ}(\Gamma)$ in the right direction and $\mathcal{G} \mapsto \Gamma(\mathcal{G})$ in the left.

The proof is straightforward. For more on this correspondence, see also [18] (Chapter I, Section 6) and [5] (Example 5.23).

The correct notion of an “isomorphism” between pseudogroups is not at all obvious. Consider the pseudogroup on $\mathbb{R}$ generated (see Remark 1.2) by

$$
\{ \phi : \mathbb{R} \to \mathbb{R}, x \mapsto x + a | a \in \mathbb{R} \},
$$

and the pseudogroup on $\mathbb{R}^2$ generated by

$$
\{ \phi : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x+a,y) | a \in \mathbb{R} \}.
$$

Intuitively, these two pseudogroups should be “isomorphic”, because there is a bijection between their generators that preserves the group-like structures. Thus, one would expect that the notion of “isomorphism” should be flexible enough to allow us to identify these two pseudogroups, and, in general, to identify pseudogroups that act on manifolds of varying dimension.

In [5] (p. 1336), Cartan writes: "The notion of an 'abstract group' does not lend itself to the theory of infinite Lie pseudogroups with the same level of purity as it does in the finite case, and it is for this reason that it has been proven difficult to find a simple analytic characterization for the notion of isomorphism. It is remarkable that M. Vessiot and I were simultaneously led to the same definition of an isomorphism of two Lie pseudogroups."

Cartan’s notion of “isomorphism”, which we call Cartan equivalence, is best formulated in terms of Haefliger’s point of view on pseudogroups. In the following definition, $\mathcal{G} \ltimes P \rightrightarrows P$ denotes the action groupoid associated with an action of $\mathcal{G}$ on a surjective submersion $\pi : P \to M$.

**Definition 3.7.** A pseudogroup $\tilde{\Gamma}$ on $P$ is an isomorphic prolongation of a pseudogroup $\Gamma$ on $M$ along a surjective submersion $\pi : P \to M$ if there exist an action of $\text{Germ}(\Gamma) \rightrightarrows M$ on $\pi : P \to M$ and an isomorphism of Lie groupoids

$$
\text{Germ}(\tilde{\Gamma}) \cong \text{Germ}(\Gamma) \ltimes P.
$$

A pseudogroup $\Gamma$ on $M$ is Cartan equivalent to a pseudogroup $\Gamma'$ on $M'$ if they admit a common isomorphic prolongation. We write $\Gamma \sim \Gamma'$. 

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The notion of Cartan equivalence is illustrated in the following diagram:

\[ \tilde{\Gamma} \bowtie P \]

\[ \Gamma \bowtie M \to M' \bowtie \Gamma' \]

Remark 3.8. Cartan makes a distinction between two types of prolongations: *holoédrique* and *mériédrique* prolongation. The former corresponds to the above notion of an *isomorphic prolongation*, while the latter is like a “covering map” of pseudogroups (see [5], p. 1336).

Proposition 3.9. Cartan equivalence defines an equivalence relation on the set of all pseudogroups.

Proof. Symmetry and reflexivity are clear. For transitivity, let \( \Gamma, \Gamma', \Gamma'' \) be pseudogroups on \( M, M', M'' \), respectively, such that \( \Gamma \sim \Gamma' \) and \( \Gamma' \sim \Gamma'' \), i.e. \( \Gamma \) and \( \Gamma' \) have a common isomorphic prolongation on \( P \), and \( \Gamma' \) and \( \Gamma'' \) on \( P' \). Using this data, we may construct an isomorphic prolongation of \( \Gamma \) and \( \Gamma' \) on the fibered product \( Q := P \times_{M'} P' \).

\[ P \times_{M'} P' \]

\[ P \to M \]

\[ P' \to M' \]

\[ M'' \]

Define an action of \( \text{Germ}(\Gamma) \) on \( Q \) as follows: let \( \phi \in \Gamma, x \in \text{Dom}(\phi) \) and \( (p, p') \in Q \) such that \( p \) projects to \( x \). The isomorphism \( \text{Germ}(\Gamma) \rtimes P \cong \text{Germ}(\Gamma') \rtimes P \) maps the pair \((\text{germ}_x(\phi), p)\) to a pair \((\text{germ}_x(\phi'), p)\). Set \( \text{germ}_x(\phi) \cdot (p, p') := (\text{germ}_x(\phi') \cdot p, \text{germ}_x(\phi') \cdot p') \). Similarly, we can define an action of \( \text{Germ}(\Gamma'') \) on \( Q \), and it is straightforward to verify that \( \text{Germ}(\Gamma) \rtimes Q \cong \text{Germ}(\Gamma') \rtimes Q \).

3.1.5 Cartan Equivalence of Generalized Pseudogroups

The notions that we introduced for pseudogroups generalize naturally to the setting of generalized pseudogroups (see Section A.2.2). Given a generalized pseudogroup \( \Gamma \) on \( G \Rightarrow M \), we construct its germ groupoid \( \text{Germ}(\Gamma) \Rightarrow M \), which is an étale groupoid, but not necessarily effective. This is enough, however, to define the notion of an *isomorphic prolongation* and *Cartan equivalence* of generalized pseudogroups precisely as in Definition 3.7. This notion of equivalence allows us to compare generalized pseudogroups that act on different Lie groupoids. Since pseudogroups are special cases, then it also allows us to compare pseudogroups with generalized pseudogroups.

3.2 Proof of the Second Fundamental Theorem

We now turn to the proof of Theorem 3.1. As we have seen, a Lie pseudogroup \( \Gamma \), say of order \( k \), gives rise to a Lie-Pfaffian groupoid \((J^k \Gamma, \omega)\), which, in turn, encodes \( \Gamma \) as its generalized pseudogroup of local solutions. A Lie-Pfaffian groupoid that arises in this way satisfies the following two properties:

(a) it is standard, i.e. its symbol map is injective (see Section A.2.4),
(b) it admits an integral Cartan-Ehresmann connection (see Section A.2.8).

The main ingredients of the proof of Theorem 3.1 are the following two general results about Lie-Pfaffian groupoids, which we explain and prove in Sections 3.2.1 and 3.2.3:

1. The Canonical Prolongation: Any Lie-Pfaffian groupoid \((G, \omega)\) has an associated pseudogroup on the total space \(G\), which we call the canonical prolongation. If \((G, \omega)\) satisfies property (a), then the canonical prolongation is characterized as those elements in \(\text{Diff}_{\text{loc}}(G)\) that preserve the target map \(t : G \to M\) and \(\omega \in \Omega^1(G; t^*E)\) (Theorem 3.12).

2. Constructing a Realization: Any Lie-Pfaffian groupoid \((G, \omega)\) satisfying properties (a) and (b) gives rise to a Cartan algebroid and a realization whose induced pseudogroup of local symmetries is precisely the canonical prolongation (Theorem 3.23).

The proof of Theorem 3.1 based on these ingredients, will then be given in Section 3.2.5.

### 3.2.1 Ingredient 1: The Canonical Prolongation

Let \(G\) be a Lie groupoid over \(M\). As usual, \(s\) and \(t\) denote the source and target maps. Any local bisection \(b \in \text{Bis}_{\text{loc}}(G)\), say with

\[
\phi_b := t \circ b : U \to V, \quad U, V \subset M,
\]

induces a locally defined diffeomorphism \(\psi_b \in \text{Diff}_{\text{loc}}(G)\) given by:

\[
\psi_b : s^{-1}(U) \to s^{-1}(V), \quad g \mapsto g \cdot b(g)^{-1}.
\]

We call \(\psi_b\) the prolongation of \(b\). From its definition, it follows that \(s \circ \psi_b = \phi_b \circ s\), i.e. \(\psi_b \in \text{Diff}_{\text{loc}}(G)\) covers \(\phi_b \in \text{Diff}_{\text{loc}}(M)\) along the source map.

Now, let \((G, \omega)\) be a Lie-Pfaffian groupoid over \(M\), and recall that \(\text{Bis}_{\text{loc}}(G, \omega)\) denotes the local solutions of \((G, \omega)\), i.e. all local bisections \(b \in G\) that satisfy \(b^* \omega = 0\). We call the pseudogroup on \(G\) generated (see Remark 1.2) by all prolongations of elements of \(\text{Bis}_{\text{loc}}(G, \omega)\),

\[
\text{prol}(\text{Bis}_{\text{loc}}(G, \omega)) := \langle \{ \psi_b \mid b \in \text{Bis}_{\text{loc}}(G, \omega) \} \rangle \subset \text{Diff}_{\text{loc}}(G),
\]

the canonical prolongation of \(\text{Bis}_{\text{loc}}(G, \omega)\).

**Proposition 3.10.** Let \((G, \omega)\) be a Lie-Pfaffian groupoid. The pseudogroup \(\text{prol}(\text{Bis}_{\text{loc}}(G, \omega))\) is an isomorphic prolongation of the generalized pseudogroup \(\text{Bis}_{\text{loc}}(G, \omega)\) along the source map \(s : G \to M\), and hence the two are Cartan equivalent.

**Proof.** Given the action of the germ groupoid \(\text{Germ}(\text{Bis}_{\text{loc}}(G, \omega)) \cong M\) on \(s : G \to M\) by \(\text{germ}_b \cdot g = g \cdot b(g)^{-1}\), it is straightforward to check that

\[
\text{Germ}(\text{Bis}_{\text{loc}}(G, \omega)) \times G \cong \text{Germ}(\text{prol}(\text{Bis}_{\text{loc}}(G, \omega))))), \quad (\text{germ}_b, g) \mapsto \text{germ}_g \psi_b,
\]

is an isomorphism of Lie groupoids. \(\Box\)

**Example 3.11.** (Lie pseudogroups) Let \(\Gamma\) be a Lie pseudogroup on \(M\) of order \(k\), and let \((J^k \Gamma, \omega)\) be its induced Lie-Pfaffian groupoid. The canonical prolongation of \((J^k \Gamma, \omega)\) is the classical k-th prolongation \(I^k\) of the pseudogroup \(\Gamma\). Indeed, writing out \(I^k\), any \(\phi \in \Gamma\), say \(\phi : U \to V\) with \(U, V \subset M\), induces a locally defined diffeomorphism \(\phi^k \in \text{Diff}_{\text{loc}}(J^k \Gamma)\) given by

\[
\phi^k : s^{-1}(U) \to s^{-1}(V), \quad j^k \phi' \mapsto j^k \phi' \cdot (j^k \phi)^{-1}, \quad (3.3)
\]

generating the pseudogroup

\[
\Gamma^k := \langle \{ \phi^k \mid \phi \in \Gamma \} \rangle \subset \text{Diff}_{\text{loc}}(J^k \Gamma).
\]

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In what Cartan calls *The Fundamental Theorem* (see pp. 1337-1339 in [5]), he shows that the \( k \)-th prolongation of a Lie pseudogroup of order \( k \) from the above example is characterized as the pseudogroup preserving a collection of functions and 1-forms. In the setting of Lie-Pfaffian groupoids, the theorem takes the following form:

**Theorem 3.12.** Let \((G, \omega)\) be a Lie-Pfaffian groupoid and assume that it is standard. Then

\[
\text{prol}(\text{Bis}_{\text{loc}}(G, \omega)) = \{ \phi \in \text{Diff}_{\text{loc}}(G) \mid \phi^* t = t, \phi^* \omega = \omega \}.
\]

*Proof.* We first prove the right inclusion, i.e. that \( \psi_b^* t = t \) and \( \psi_b^* \omega = \omega \) for any \( b \in \text{Bis}_{\text{loc}}(G, \omega) \).

The first equality is clear from the defining formula (3.2) of \( \psi_b^* \). The second equality relies on the multiplicativity of \( \omega \). Let \( g \in G \) and \( X \in T_gG \), then

\[
\begin{align*}
(\psi_b^* \omega)_g(X) &= (m^* \omega)_{(g, b(\eta(g)))} (X, di \circ db \circ ds(X)) \\
&= \omega_g(X) + g \cdot (i^* \omega)_{b(\eta(g))} (db \circ ds(X)) \\
&= \omega_g(X) - \psi_b(g) \cdot (b^* \omega)_{\eta(g)} (ds(X)),
\end{align*}
\]

where \( m, i \) and \( s \) are the multiplication, inverse and source maps, respectively. Here, the definition of \( \psi_b^* \) is used in the first equality, the multiplicativity property (A.48) in the second and the following general identity for multiplicative forms in the third:

\[
- (i^* \omega)_g = g^{-1} \cdot \omega_g, \quad \forall g \in G, \tag{3.4}
\]

The latter is obtained by applying (A.48) on a pair \((X, di(X))\) with \( X \in T_gG \).

For the left inclusion, we have to prove that if \( \phi \in \text{Diff}_{\text{loc}}(G) \) satisfies \( \phi^* t = t \) and \( \phi^* \omega = \omega \), then, locally, \( \phi = \psi_b \) for some \( b \in \text{Bis}_{\text{loc}}(G, \omega) \). Locally here mean that, for every \( g \in \text{Dom}(\phi) \), there exists \( b \in \text{Bis}_{\text{loc}}(G, \omega) \) with \( s(g) \in \text{Dom}(b) \) and an open neighborhood \( U \subset \text{Dom}(\phi) \) of \( g \) with \( s(U) \subset \text{Dom}(b) \) such that \( \phi|_U = \psi_b|_U \). Spelling out \( \psi_b \), this equality becomes

\[
b(s(h)) = \phi(h)^{-1} \cdot h, \quad \forall h \in U. \tag{3.5}
\]

To prove this, we consider the map

\[
H = H_{\phi} : \text{Dom}(\phi) \to G, \quad h \mapsto \phi(h)^{-1} \cdot h,
\]

which is well defined because \( \phi^* t = t \), and choose a local section \( \eta \) of the source map \( s : G \to M \) with \( g \in \text{Im}(\eta) \) and \( \text{Im}(\eta) \subset \text{Dom}(\phi) \). We set \( b := H \circ \eta \). The left inclusion now follows from the following three claims that we will prove: 1) \( H^* \omega = 0 \) (and hence \( b^* \omega = 0 \)); 2) if we sufficiently shrink the domain of \( b \) around \( s(g) \), then \( b \) is a bisection; and 3) \( H \) is locally constant along the \( s \)-fibers (and hence (3.5) holds for a small enough neighborhood \( U \) of \( g \)).

1) Let \( X \in T_G|_{\text{Dom}(\phi)} \), then

\[
(H^* \omega)_g(X) = (m^* \omega)_{(\phi(g)^{-1} \cdot g)} (di \circ d\phi(X), X) \\
= (i^* \omega)_{\phi(g)} (d\phi(X)) + \phi(g)^{-1} \cdot \omega_g(X) \\
= -\phi(g)^{-1} \cdot (\phi^* \omega)_g(X) + \phi(g)^{-1} \cdot \omega_g(X) \\
= 0,
\]

where the definition of \( H \) was used in the first equality, the multiplicativity property in the second, (3.4) in the third and the assumption that \( \phi^* \omega = \omega \) in the fourth.

2) From the definition of \( H \), it follows that \( b = H \circ \eta \) is a local section of the source map \( s \). We are left to show that \( t \circ b \) is a diffeomorphism, where we are allowed to shrink the domain of
b to an arbitrarily small neighborhood of s(g). By the inverse function theorem, it is sufficient to check that (d(t ∘ b)) s(g) is a linear isomorphism, or, by dimension count, that it is injective. Since b is a section of s, it is injective, so it is enough to check that Ker dt ∩ Im db = {0}. Now, since H∗ω = 0, then Im db ⊂ Ker ω, and since Ker ω ∩ Ker dt = Ker ω ∩ Ker ds (by the definition of a Lie-Pfaffian groupoid), then Ker dt ∩ Im db = Ker ds ∩ Im db. But Ker ds ∩ Im db = {0}, because b is a section of s, and so we are done.

3) The map H is locally constant along the s-fibers if and only if dH(X) = 0 for all X ∈ Ker ds|Dom(φ). Let X ∈ Ker ds|Dom(φ). Note that since H∗ω = 0 and s ∘ H = s, then dH(X) ∈ Ker ω ∩ Ker ds. Now, since we are assuming that (G, ω) is standard (i.e. its symbol map is injective), the vanishing of dH(X) follows from:

\[ \partial(dR_{g^{-1}}(dH(X)))(Y) = \delta \omega(dH(X), db \circ (d\phi_b)^{-1})(Y) \]
\[ = \delta \omega(dH(X), dH((d(\eta \circ \phi_b^{-1}))(Y)) \]
\[ = (H^* \delta \omega)(X, (d(\eta \circ \phi_b^{-1}))(Y)) \]
\[ = 0, \]

for all Y ∈ Tt(M), where \( \phi_b = t ∘ b \). Here, in the first equality Lemma [3.51] was used, while the last equality follows from the fact that

\[ H^*ω = 0 \quad \Rightarrow \quad H^*δω = 0. \] (3.6)

The latter is a consequence of the fact that \( \delta \omega = d\nabla \omega|_{\text{Ker } \omega} \), together with the basic fact that the pullback operation commutes with the differential, which, in the case of vector bundle-valued forms, means that \( H^*d\nabla \omega = (H^* \nabla)H^*\omega \), where \( \nabla \) is a choice of a connection on the coefficients of \( \omega \) and \( H^*\nabla \) is the pull-back connection. \( \square \)

**Remark 3.13.** This proof is a nice illustration of the advantage of working with the abstract framework of Lie-Pfaffian groupoids. In the main application, when the Lie-Pfaffian groupoid is \( (J^k \Gamma, \omega) \) with \( \Gamma \) a Lie pseudogroup, this theorem can also be proven by induction on the order of the jets (this is done e.g. in Theorem 4.1 of [10] in the case of transitive pseudogroups). However, the above proof is substantially simpler, avoiding the need to work directly with jets and using solely the Cartan form and its few essential properties.

### 3.2.2 Aside: Restricting to a Transversal

We now show that Theorem 3.12 can be “improved” by restricting the canonical prolongation to a complete transversal, thus obtaining a “smaller” prolongation than the canonical one. This trick is employed by Cartan in examples (and even mentioned in his proof of the Second Fundamental Theorem) in order to reduce the dimension of the realization he constructs out of a given Lie pseudogroup. We will see two examples of this in Section 3.3.

Let \( (G, \omega) \) be a Lie-Pfaffian groupoid over \( M \) and let \( N \subset M \) be any submanifold. Because the orbits of the canonical prolongation \( \text{prol}(\text{Bis}_{\text{loc}}(G, \omega)) \) are contained in the t-fibers of \( G \), then we may restrict each of its elements to the submanifold \( G_N := t^{-1}(N) \), thus obtaining a pseudogroup on \( G_N \) which we denote by

\[ \text{prol}(\text{Bis}_{\text{loc}}(G, \omega))|_N := \{ \phi|_{G_N} | \phi \in \text{prol}(\text{Bis}_{\text{loc}}(G, \omega)) \} \subset \text{Diff}_{\text{loc}}(G_N). \]

For the purpose of proving the second fundamental theorem with this pseudogroup replacing the canonical one, we would like Proposition 3.10 and Theorem 3.12 to hold also here. This is the case when the submanifold \( N \) is “nice” in the following sense:
**Definition 3.14.** Let $\mathcal{G}$ be a Lie groupoid over $M$ with Lie algebroid $A$. A **transversal** to $\mathcal{G}$ is a submanifold $N \subset M$ that intersects the orbits of $\mathcal{G}$ transversely, i.e.

$$TN + \rho(A)|_{N} = TM|_{N}.$$ 

A transversal is **complete** if it intersects each orbit at least once.

An important consequence of this condition is that if $N$ is a complete transversal of a Lie groupoid $\mathcal{G}$, then the restriction of the source map $s_{N} := s|_{\mathcal{G}N} : \mathcal{G}N \to M$ is a surjective submersion (the restriction of the target map $t_{N} := t|_{\mathcal{G}N} : \mathcal{G}N \to M$ is a surjective submersion for any submanifold $N$). The proof of the following proposition is a straightforward adaptation of the proof of Proposition 3.10.

**Proposition 3.15.** Let $(\mathcal{G}, \omega)$ be a Lie-Pfaffian groupoid over $M$ and let $N \subset M$ be a complete transversal of $\mathcal{G}$. The pseudogroup $\text{prol}(\text{Bis}_{\text{loc}}(\mathcal{G}, \omega))|_{N} \subset \text{Diff}_{\text{loc}}(\mathcal{G}N)$ is an isomorphic prolongation of the generalized pseudogroup $\text{Bis}_{\text{loc}}(\mathcal{G}, \omega)$ along $s_{N} : \mathcal{G}N \to M$, and hence the two are Cartan equivalent.

Moreover, the condition of complete transversality also ensures that Theorem 3.12 continues to hold for the restriction. Writing $\omega_{N} := \omega|_{\mathcal{G}N}$, we have that:

**Theorem 3.16.** Let $(\mathcal{G}, \omega)$ be a Lie-Pfaffian groupoid over $M$ and assume that it is standard, and let $N \subset M$ be a complete transversal of $\mathcal{G}$. Then

$$\text{prol}(\text{Bis}_{\text{loc}}(\mathcal{G}, \omega))|_{N} = \{ \phi \in \text{Diff}_{\text{loc}}(\mathcal{G}N) \mid \phi^{*}t_{N} = t_{N}, \ \phi^{*}\omega_{N} = \omega_{N} \}.$$ 

**Proof.** Observe that in Theorem 3.12 (and its proof), $\mathcal{G}$ plays a double role: 1) it is the Lie groupoid underlying the Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$ whose generalized pseudogroup of local solutions $\text{Bis}_{\text{loc}}(\mathcal{G}, \omega)$ we consider, and 2) it is the space on which the classical prolongation $\text{prol}(\text{Bis}_{\text{loc}}(\mathcal{G}, \omega))$ acts. In the theorem we are currently proving, $\mathcal{G}$ remains unchanged in the first role and is replaced by $\mathcal{G}N$ in the second role. Modulo this replacement (and replacing $s$, $t$ and $\omega$ by their restrictions $s_{N}$, $t_{N}$ and $\omega_{N}$), the proof of Theorem 3.12 can be copied verbatim. As explained above, the role of the condition of complete transversality is to ensure that $s_{N}$ is a surjective submersion.

**Remark 3.17.** The double role played by $\mathcal{G}$, as explained in the above proof, suggests a more conceptual framework for this theorem, namely that of a Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$ acting on a “Pfaffian bundle” $(P, \theta)$, i.e. a surjective submersion $\mu : P \to N$ equipped with a vector-bundle valued 1-form $\theta$ (satisfying certain conditions). In our case, the “Pfaffian bundle” is $t_{N} : \mathcal{G}N \to N$ equipped with $\omega_{N}$. This data is all that is needed in order to make sense of the “prolongation” of the generalized pseudogroup $\text{Bis}_{\text{loc}}(\mathcal{G}, \omega)$ to a pseudogroup on $P$, and to prove that the latter is characterized as the local symmetries of the “Pfaffian bundle”, generalizing Theorem 3.16. This direction is currently being investigated in [7].

**Remark 3.18.** Ideally, to obtain the “smallest” prolongation, one would like to choose a transversal $N$ that crosses each orbit precisely once. In this case, $N$ can be regarded as the orbit space of $\Gamma$, but one which is obtained by choosing a slice rather than by taking a quotient. In fact, this is what Cartan does in local coordinates (as we will see in the examples of Section 3.3). In the global setting, however, this is only possible if the orbits are “nice enough”. For example, if the Lie pseudogroup is transitive, one takes $N$ to be a point in $M$. In [16], the authors study Cartan’s structure theory in the case of transitive Lie pseudogroups, and, in particular, they prove Theorem 3.12 in the case of $N$ being a point.
3.2.3 Ingredient 2: Constructing a Realization

The second main ingredient of the proof of the Second Fundamental Theorem is to construct a realization of a Cartan algebroid out of the data of a Lie-Pfaffian groupoid. We present this construction from two slightly different – but equivalent – points of view.

In this section, we will simply provide a recipe for constructing a realization. We begin by explaining how to construct an almost Cartan algebroid. We then construct a pair \((P, \Omega)\), as in the definition of a realization, and in Theorem 3.23, the main result of this section, we show that the existence of an auxiliary form \(\Pi\) satisfying the required conditions is equivalent to the existence of an integral Cartan-Ehresmann connection on the Lie-Pfaffian groupoid. This theorem, therefore, gives a geometric interpretation of the notion of a realization.

The recipe we describe here – which, in local coordinates, coincides with Cartan’s construction of a realization – will depend on choices and may seem somewhat arbitrary. In the next section, as an aside, we will explain how Cartan’s structure equations are essentially the pullback of the canonical Maurer-Cartan equation of a prolongation of a Lie-Pfaffian groupoid, which was studied in [40].

Constructing an Almost Cartan Algebroid

The almost Cartan algebroid induced by a Lie-Pfaffian groupoid is constructed purely out of its infinitesimal counterpart, the associated Lie-Pfaffian algebroid. Thus, let \((A, D)\) be a Lie-Pfaffian algebroid over \(M\) (Definition A.44), and assume that it is standard. Thus, \(A\) and \(E\) are Lie algebroids over \(M\), \(l : A \to E\) is a surjective Lie algebroid map, and

\[
D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E)
\]

is an \(l\)-connection. The construction will depend on a splitting of the short exact sequence of vector bundles

\[
0 \to \sigma \xrightarrow{\xi} A \xrightarrow{l} E \to 0.
\]  
(3.7)

Here, \(\sigma = \sigma(A)\) is the symbol space of \((A, D)\).

From this data, we construct an almost Cartan algebroid as follows. We set

\[
\mathcal{C} := TM \oplus E.
\]  
(3.8)

The bracket of \(\mathcal{C}\) depends on the choice of the splitting (3.7). Such a splitting induces a linear connection on \(E\) defined by

\[
\nabla^\xi : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad \nabla^\xi_X(\alpha) := D_X(\xi \circ \alpha),
\]  
(3.9)

and we can consider its torsion

\[
c^\xi \in \Gamma(\text{Hom}(\Lambda^2 E, E)), \quad c^\xi(\alpha, \beta) := [\alpha, \beta] - \nabla^\xi_{\rho(\alpha)} \beta - \nabla^\xi_{\rho(\beta)} \alpha,
\]  
(3.10)

where \(\rho\) is the anchor of \(E\). The bracket of \(\mathcal{C}\),

\[
[\cdot, \cdot]^\xi : \Gamma(\mathcal{C}) \times \Gamma(\mathcal{C}) \to \Gamma(\mathcal{C}),
\]  
(3.11)

is defined by

\[
[(X, \alpha), (Y, \beta)]^\xi := ([X, Y], c^\xi(\alpha, \beta) + \nabla^\xi_X(\beta) - \nabla^\xi_Y(\alpha)).
\]  
(3.12)
The anchor, which is independent of the splitting, is simply the projection
\[ \rho : C \to TM, \quad (X, \alpha) \mapsto X. \quad (3.13) \]

It is straightforward to verify that \( C \) is a transitive almost Lie algebroid. For the vector subbundle \( \sigma \subset \text{Hom}(C, C) \), we take the symbol space, where the inclusion is given by composing the symbol map \( (A, 67) \) (which is injective by the assumption that \( (A, D) \) is standard) with the inclusion
\[ \text{Hom}(TM, E) \hookrightarrow \text{Hom}(C, C), \quad T \mapsto \left( \hat{T} : (X, \alpha) \mapsto (0, T(\rho(\alpha) - X)) \right). \quad (3.14) \]

Indeed, \( \hat{T} \) takes values in \( \ker \rho \), and hence \( \sigma \subset \text{Hom}(C, \ker \rho) \). Note that if we equip \( \text{Hom}(TM, E) \) with the bracket \( \{ T, S \} := T \circ \rho \circ S - S \circ \rho \circ T \) and \( \text{Hom}(C, C) \) with the commutator bracket, then \( (3.14) \) becomes a Lie algebra map.

**Remark 3.19.** To generalize this construction to non-standard Lie-Pfaffian algebroids, we would need to relax the definition of a Cartan algebroid by requiring there to be a map \( \sigma \to \text{Hom}(C, C) \) rather than an inclusion.

**Proposition 3.20.** The pair \( (C, \sigma) \) defined above is an almost Cartan algebroid. Up to gauge equivalence, it is independent of the choice of splitting \( \xi \).

**Proof.** We have already seen that \( (C, \sigma) \) is an almost Cartan algebroid. We are left with showing that, for any two choices of splittings \( \xi : E \to A \) and \( \xi' : E \to A \), the resulting almost Cartan algebroids are gauge equivalent. Taking the difference, we get a map \( (\xi - \xi') : E \to \sigma \), which we can interpret as a gauge equivalence by letting it act trivially on the first component, i.e.
\[ (\xi - \xi') : C = TM \oplus E \to \sigma, \quad (X, \alpha) \mapsto (\xi - \xi')(\alpha). \]

It is now straightforward to verify that gauge transforming the almost Cartan algebroid \( (C, \sigma) \) with bracket \( \{ \cdot, \cdot \}^{\xi} \) by \( \xi - \xi' \) yields the almost Cartan algebroid \( (C_{\xi'}, \sigma) \) with bracket \( \{ \cdot, \cdot \}^{\xi'} \). \( \square \)

**Remark 3.21.** Note that Propositions \( 2.21 \) together with Proposition \( 3.20 \) implies that if we manage to construct a realization of \( (C, \sigma) \), then it will be independent of the choice of \( \xi \).

**Remark 3.22.** In fact, the more canonical construction is that of the Cartan pair rather than of the Cartan algebroid (see Section \( 2.5 \) and, in particular, Theorem \( 2.50 \)). The Cartan consists of the pair \( (TM \oplus A, \sigma) \), and does require the choice of a splitting \( \xi \). We have chosen the Cartan algebroid point of view in order to remain closer to Cartan.

**Constructing a Realization**

Next, we construct a realization of the almost Cartan algebroid \( (C, \sigma) \) obtained from the Lie-Pfaffian algebroid \( (A, D) \) of a standard Lie-Pfaffian groupoid \( (G, \omega) \). By Theorem \( 2.30 \) if a realization exists, then \( (C, \sigma) \) is a Cartan algebroid. The realization will be the pair \( (G, \Omega) \) consisting of the target map \( t : G \to M \) and the sum
\[ \Omega = (dt, \omega) \in \Omega^1(G; t^* C). \]
For \((\mathcal{G}, \omega)\) to be a realization of \((C, \sigma)\), we must prove the existence of a \(\Pi \in \Omega^1(\mathcal{G}; t^*\sigma)\) that satisfies both the structure equation (2.5) and condition (2.6). Recall that for any Lie-Pfaffian groupoid, a right splitting \(H : s^*TM \to C_\omega\) of the short exact sequence

\[
0 \to t^*\sigma \to C_\omega \xrightarrow{ds} s^*TM \to 0,
\]

is called a Cartan-Ehresmann connection. Cartan-Ehresmann connections are in particular Ehresmann connections, i.e. right splittings of the short exact sequence

\[
0 \to t^*A \cong T^*\mathcal{G} \to T\mathcal{G} \xrightarrow{dt} s^*TM \to 0,
\]

or, equivalently, left splittings known as connection 1-forms. The latter are elements of \(\Omega^1(\mathcal{G}; t^*E) \oplus \Omega^1(\mathcal{G}; t^*\sigma)\).

Cartan-Ehresmann connections, therefore, correspond precisely to those connection 1-forms whose first component in this decomposition is the Cartan form \(\omega\). We thus have a map

\[
H \mapsto \Pi = \Pi_H
\]

associating a 1-form \(\Pi = \Pi_H \in \Omega^1(\mathcal{G}; t^*\sigma)\) with a Cartan-Ehresmann connection \(H\) on \(\mathcal{G}\). Recall that a Cartan-Ehresmann \(H\) is said to be integral if \(H^*\delta\omega = 0\).

**Theorem 3.23.** Let \((\mathcal{G}, \omega)\) be a standard Lie-Pfaffian groupoid over \(M\). The map (3.18) defines a 1-1 correspondence between

\[
\text{Cartan-Ehresmann connections } H \text{ on } \mathcal{G} \longleftrightarrow \Pi \in \Omega^1(\mathcal{G}; t^*\sigma) \text{ satisfying (2.6)},
\]

that restricts to a 1-1 correspondence between

\[
\text{integral Cartan-Ehresmann connections } H \text{ on } \mathcal{G} \longleftrightarrow \Pi \in \Omega^1(\mathcal{G}; t^*\sigma) \text{ satisfying (2.6) and (2.5)},
\]

Thus, the pair \((\mathcal{G}, \Omega)\), consisting of the target map \(t : \mathcal{G} \to M\) and \(\Omega = (dt, \omega)\), is a realization of the almost Cartan algebroid \((C, \sigma)\) (which is then a Cartan algebroid) if and only if \((\mathcal{G}, \omega)\) admits an integral Cartan-Ehresmann connection.

**Proof.** We begin with (3.19). Let \(H\) be a Cartan-Ehresmann connection and let \(\Pi = \Pi_H \in \Omega^1(\mathcal{G}; t^*\sigma)\) be 1-form induced by the map (3.18). First note that

\[
(\Omega, \Pi) = (dt, \omega, \Pi) : T\mathcal{G} \xrightarrow{\cong} t^*(TM \oplus E \oplus \sigma)
\]

is pointwise surjective, since \(dt\) is surjective onto \(t^*TM\) and \((\omega, \Pi)\) restricts to the Maurer-Cartan form on \(T^*\mathcal{G}\), which is then surjective onto \(t^*(E \oplus \sigma) \cong t^*A\). So, by dimension count, \((\Omega, \Pi)\) is pointwise an isomorphism and (2.6) is satisfied. In fact, we can explicitly describe the inverse of (3.21), which will serve us in the second part of the proof. Let us denote the map at the level of sections that is induced by the inverse of (3.21) by:

\[
\mathfrak{X}(M) \to \mathfrak{X}(\mathcal{G}), \ X \mapsto Y_X; \quad \Gamma(E) \to \mathfrak{X}(\mathcal{G}), \ a \mapsto Y_a; \quad \Gamma(\sigma) \to \mathfrak{X}(\mathcal{G}), \ S \mapsto Y_S.
\]
Thus, \( Y_X, Y_\alpha, Y_S \in \mathfrak{X}(\mathcal{G}) \) are the unique vector fields that satisfy:

\[
\begin{align*}
  dt(Y_X) &= t^*X, & \omega(Y_X) &= 0, & \Pi(Y_X) &= 0, \\
  dt(Y_\alpha) &= 0, & \omega(Y_\alpha) &= t^*\alpha, & \Pi(Y_\alpha) &= 0, \\
  dt(Y_S) &= 0, & \omega(Y_S) &= 0, & \Pi(Y_S) &= t^*S. \\
\end{align*}
\]  

(3.22)

Given a section \( \alpha \in \Gamma(A) \), we denote the induced right invariant vector field by \( \tilde{\alpha} \in \mathfrak{X}(\mathcal{G}) \). Also recall that there is canonical isomorphism

\[
\psi : s^*TM \xrightarrow{\cong} t^*TM
\]

of vector bundles over \( \mathcal{G} \) which is equal to \( dt \circ H \) (Lemma A.42). We define the map:

\[
\mathfrak{X}(M) \to \mathfrak{X}(J^k \Gamma), \quad X \mapsto X^H = H \circ (dt \circ H)^{-1}(t^*X).
\]

One now readily verifies that

\[
\begin{align*}
  Y_X &= X^H, & Y_\alpha &= \tilde{\alpha} - \rho(\alpha)H, & Y_S &= \tilde{S}, \\
\end{align*}
\]

(3.24)

In the other direction, choose \( \Pi \in \Omega^1(\mathcal{G}; t^*\sigma) \) that satisfies (2.6), so we have an isomorphism (3.21). This induces a Cartan-Ehresmann connection \( H : s^*TM \to T\mathcal{G} \) as follows: denote the restriction of the inverse of (3.21) to \( t^*TM \) by \( H' : t^*TM \to T\mathcal{G} \) and set \( H = H' \circ \psi \), where \( \psi \) is the isomorphism (3.22). Indeed,

\[
ds \circ H = ds \circ H' \circ \psi = \psi^{-1} \circ dt \circ H' \circ \psi = \psi^{-1} \circ \psi = \text{id}.
\]

It is easy to see that this construction is inverse to (3.18).

We move on to (3.20). Let \( H \) be an integral Cartan-Ehresmann connection. We must show that the induced \( \Pi \) satisfies the structure equation (2.5). For this, it is enough to verify that the expression

\[
d\Omega + \frac{1}{2}[\Omega, \Omega] - \Pi \wedge \Omega = d(dt, \omega) + \frac{1}{2}[(dt, \omega), (dt, \omega)] - \Pi \wedge (dt, \omega)
\]

(3.25)

vanishes when applied to all possible pairs of the type (3.24). In the following computations, we use Lemma 2.9 to evaluate the Maurer-Cartan expression \( MC_\Omega = d\Omega + \frac{1}{2}[\Omega, \Omega] \):

1. \( (d\Omega + \frac{1}{2}[\Omega, \Omega] - \Pi \wedge \Omega)(Y_X, Y_{X'}) \)

\[
\begin{align*}
  &= (d(dt, \omega) + \frac{1}{2}[(dt, \omega), (dt, \omega)] - \Pi \wedge (dt, \omega))(Y_X, Y_{X'}) \\
  &= -(dt([X^H, X'^H]), \omega([X^H, X'^H]) + t^*([X, 0], (X', 0))] \\
  &= -(t^*[X, X'], 0) + t^*([X, X'], 0) = 0
\end{align*}
\]

\( Y_X, Y_{X'} \) are \( t \)-related to \( X, X' \)

2. \( (d\Omega + \frac{1}{2}[\Omega, \Omega] - \Pi \wedge \Omega)(Y_\alpha, Y_{\alpha'}) \)

\[
\begin{align*}
  &= -(d(dt(Y_\alpha), Y_{\alpha}), \omega([Y_X, Y_\alpha])) + t^*([X, 0], (0, \alpha]) \\
  &= -(0, \omega([X^H, \tilde{\alpha}]) - \omega([X^H, \rho(\alpha)H])) + t^* (0, \nabla_X^G(\alpha)) = 0
\end{align*}
\]

\( Y_\alpha \) is \( t \)-related to \( 0 \)

\[
= t^*\nabla_X^G(\alpha) \text{ by Lemma A.45,} \quad H \text{ is integral}
\]

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3. \((d\Omega + \frac{1}{2} [\Omega, \Omega] - \Pi \wedge \Omega)(Y_X, Y_S)\)
\[= -(dt([Y_X, Y_S]), \omega([Y_X, Y_S])) + \Pi(Y_S)((dt, \omega)(Y_X))\]
\[= -(0, \omega([X^H, S])) + t^*(0, S(-X)) = 0\]
\[= t^* D_X(S) = -t^* S(X)\]

4. \((d\Omega + \frac{1}{2} [\Omega, \Omega] - \Pi \wedge \Omega)(Y_{\alpha'}, Y_{\alpha'})\)
\[= -(dt([Y_{\alpha'}, Y_{\alpha'}]), \omega([Y_{\alpha'}, Y_{\alpha'}])) + \Pi(Y_{\alpha'})((dt, \omega)(Y_{\alpha}))\]
\[= -(0, \omega([\xi(\alpha'), \xi(\alpha')]) - \omega([\rho(\alpha')^H, \xi(\alpha')]) + \omega([\rho(\alpha')^H, \xi(\alpha')]))\]
\[= t^*\xi(\alpha') = t^*\psi_{\rho(\alpha)}(\alpha) = t^*\psi_{\rho(\alpha)}(\alpha)\]
\[+ (0, \omega([\rho(\alpha)^H, \rho(\alpha)^H])) - t^*(0, c^X(\alpha, \alpha')) = 0\]

5. \((d\Omega + \frac{1}{2} [\Omega, \Omega] - \Pi \wedge \Omega)(Y_{\alpha}, Y_s)\)
\[= -(dt([Y_{\alpha}, Y_s]), \omega([Y_{\alpha}, Y_s])) + \Pi(Y_s)((dt, \omega)(Y_{\alpha}))\]
\[= -(0, \omega([\xi(\alpha), S]) - \omega([\rho(\alpha)^H, S])) + t^*(0, S(\rho(\alpha))) = 0\]
\[= d\sigma((\xi(\alpha), S)) = 0 = -t^* S(\rho(\alpha))\]

6. \((d\Omega + \frac{1}{2} [\Omega, \Omega] - \Pi \wedge \Omega)(Y_s, Y_s')\)
\[= -(dt([Y_s, Y_s']), \omega([Y_s, Y_s']))\]
\[= -(0, \omega([S, S'])) = 0\]
\[= d\sigma((S, S')) = 0\]

Conversely, let \(\Pi \in \Omega^1(\mathcal{G}; t^*\sigma)\) satisfy (2.3) and (2.4). Thus, \(\Pi\) induces a Cartan-Ehresmann connection \(H\) on \(\mathcal{G}\). Let \(X, X' \in \mathfrak{X}(M)\) and set \(X^H := H \circ \psi^{-1}(t^*X)\) and \(X'^H := H \circ \psi^{-1}(t^*X')\).
Using (2.4),
\[0 = (d\Omega + \frac{1}{2} [\Omega, \Omega] - \Pi \wedge \Omega)(X^H, X'^H)\]
\[= -(dt([X^H, X'^H]), \omega([X^H, X'^H])) + t^*[X(0), (X', 0)]\]
\[= -(0, \delta\omega(H \circ \psi^{-1}(X), H \circ \psi^{-1}(X')))\]

We conclude that \(\delta\omega(H(\cdot), H(\cdot)) = 0\) and, hence, that \(H\) is integral. \(\square\)

3.2.4 Aside: the Maurer-Cartan Equation of a Lie-Pfaffian Groupoid

The above construction of a realization may seem somewhat arbitrary. However, as we now explain, modulo the choices we made, Cartan's structure equations are essentially the same thing as the canonical Maurer-Cartan equation associated with the notion of prolongation of a Lie-Pfaffian groupoid, which was studied in [40]. This observation provides for yet another interpretation of the notion of a realization.

Given any Lie-Pfaffian groupoid \((\mathcal{G}, \omega)\), one can construct its so-called classical prolongation
\[\tilde{\mathcal{G}} \rightrightarrows M\]
It is defined as the subgroupoid of the first jet groupoid $J^1\mathcal{G} \rightarrow M$ which consists of all elements $j^2_! b$ that pull-back both $\omega$ and its differential $d\omega$ to zero. We say that the classical prolongation is smoothly defined (or 1-integrable) if the projection $\pi : \tilde{\mathcal{G}} \rightarrow \mathcal{G}$, $j^1_! b \mapsto b(x)$, admits a global section and the symbol space of $(\mathcal{G}, \omega)$ is of constant rank. This condition implies that $\tilde{\mathcal{G}} \rightarrow M$ is a Lie subgroupoid and $\pi : \tilde{\mathcal{G}} \rightarrow \mathcal{G}$ is an affine bundle. In this case, we may equip $\tilde{\mathcal{G}}$ with the restriction of the Cartan form of $J^1(\mathcal{G})$, which we denote by $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}; t^* A)$, and the pair $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Lie-Pfaffian groupoid. (We refer the reader to Appendix A.2 for more details on the classical prolongation, which is denoted there by $P_\omega(\mathcal{G})$ rather than by $\tilde{\mathcal{G}}$).

Now, using the Spencer operator $D : \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(E)$ induced by $\omega$, we can construct a Maurer-Cartan expression for the form $\tilde{\omega}$ in complete analogy to the construction in Section 2.2. The differential of $\tilde{\omega}$,

$$d_\omega \tilde{\omega} \in \Omega^2(\tilde{\mathcal{G}}; t^* E),$$

is defined by the Koszul-type formula

$$d_\omega \tilde{\omega}(X, Y) := (t^* D)_X (\tilde{\omega}(Y)) - (t^* D)_Y (\tilde{\omega}(X)) - l \circ \tilde{\omega}([X, Y]), \quad \forall X, Y \in \mathfrak{X}(\tilde{\mathcal{G}}).$$

The torsion of $D$,

$$\{., .\}_\omega \in \Gamma(\text{Hom}(\Lambda^2 A, E))$$

is defined by

$$\{\alpha, \beta\}_\omega := [\alpha, \beta] - D_\rho(\alpha) \beta + D_\rho(\beta) \alpha, \quad \forall \alpha, \beta \in \Gamma(A),$$

and we may use this pairing to define a graded bracket

$$\{., .\}_\omega : \Omega^p(\tilde{\mathcal{G}}; t^* A) \times \Omega^q(\tilde{\mathcal{G}}; t^* A) \rightarrow \Omega^{p+q}(\tilde{\mathcal{G}}; t^* E)$$

by the usual wedge-like formula. In particular,

$$\frac{1}{2} \{\tilde{\omega}, \tilde{\omega}\}_\omega(X, Y) = \{\tilde{\omega}(X), \tilde{\omega}(Y)\}_\omega.$$

Putting these two objects together, we obtain the Maurer-Cartan expression

$$d_\omega \tilde{\omega} + \frac{1}{2} \{\tilde{\omega}, \tilde{\omega}\}_\omega \in \Omega^2(\tilde{\mathcal{G}}; t^* E).$$

Note that, in contrast to the construction in Section 2.2, this construction does not require the choice of a connection. This, however, comes at the cost of “going one level down”, in the sense that, while $\tilde{\omega}$ is a form with values in $A$, its Maurer-Cartan expression takes values in $E$. In [40] (Theorem 6.2.17 together with Proposition 6.2.41 in [40]), it was proven that:

**Theorem 3.24.** Let $(\mathcal{G}, \omega)$ be a Lie-Pfaffian groupoid and assume that its classical prolongation is smoothly defined. Then

$$d_\omega \tilde{\omega} + \frac{1}{2} \{\tilde{\omega}, \tilde{\omega}\}_\omega = 0. \quad (3.26)$$

**Remark 3.25.** In [40], the abstract notion of a Lie prolongation of a Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$ is introduced, and it is proven that, in some sense, the classical prolongation is universal amongst all Lie prolongation. A Lie prolongation is, roughly speaking, a Lie-Pfaffian groupoid $(\tilde{\mathcal{G}}, \tilde{\omega})$, together with a map $p : (\tilde{\mathcal{G}}, \tilde{\omega}) \rightarrow (\mathcal{G}, \omega)$, that satisfies certain compatibility conditions between $\omega$ and $\tilde{\omega}$. It is then proven that these compatibility conditions are equivalent to the above Maurer-Cartan equation, which can be interpreted as a compatibility condition between $\tilde{\omega}$ and the Spencer operator $D$ of $\omega$.  

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Now, the construction of a realization that was described in the previous section can be reinterpreted as follows: Let \((G, \omega)\) be a Lie-Pfaffian groupoid and assume that its classical prolongation is smooth. Note that a section of the projection \(\pi : \tilde{G} \to G\) is the same thing as an integral Cartan-Ehresmann connection (see Appendix A.2). Choosing such a section, say \(\eta : G \to \tilde{G}\), we can pull-back the Cartan form to obtain a 1-form
\[\eta^*\tilde{\omega} \in \Omega^1(G; t^*A).\]
Furthermore, choosing a splitting \((3.7)\) induces a decomposition \(A = E \oplus \sigma\), and, accordingly, \(\eta^*\tilde{\omega}\) decomposes into two components. It is not hard to see that the first component is precisely \(\omega \in \Omega^1(G; t^*E)\). The second component, which we denote by \(\Pi \in \Omega^1(G; t^*\sigma)\), is precisely the 1-form which is obtained by the map \((3.18)\) from the Cartan-Ehresmann connection corresponding to our choice of splitting \(\eta\).

So, we have made two choices of splittings to obtain Cartan’s construction. Finally, as we have seen in the previous section, Cartan chooses to complete the forms \((\omega, \Pi)\) to a “coframe” of \(G\) by including the 1-form \(dt \in \Omega^1(G; t^*TM)\), and to view \(\omega\) and \(dt\) as a single 1-form \(\Omega = (dt, \omega) \in \Omega^1(G; t^*(TM \oplus E))\). It is now not difficult to check, by the same methods that were used in the proof of Theorem 3.23, that the Maurer-Cartan equation \((3.26)\) for \(\tilde{\omega}\) is equivalent to the structure equation \((2.5)\) for the pair \((\Omega, \Pi)\).

3.2.5 Proof of the Second Fundamental Theorem

From the two “ingredients” in Sections 3.2.1 and 3.2.3, we can extract the following corollary that will be needed for the proof of the Second Fundamental Theorem (see Section 3.2.4 for the definitions of the induced realization and Cartan algebroid):

**Corollary 3.26.** Let \((G, \omega)\) be a standard Lie-Pfaffian groupoid on \(M\) that admits an integral Cartan-Ehresmann connection. The induced pair \((G, \Omega)\) is a realization of the induced Cartan algebroid \((C, \sigma)\), and its associated pseudogroup \(\Gamma(G, \Omega)\) of local symmetries is precisely the canonical prolongation of \(\text{Bis}_{\text{loc}}(G, \omega)\),
\[\Gamma(G, \omega) = \text{prol}(\text{Bis}_{\text{loc}}(G, \omega)),\] and hence it is in normal form and Cartan equivalent to the generalized pseudogroup \(\text{Bis}_{\text{loc}}(G, \omega)\) of local solutions of \((G, \omega)\).

**Proof.** This is an immediate consequence of Proposition 3.10, Theorem 3.12, and Theorem 3.23. Note that, in \((3.27)\), we are using the simple fact that, for \(\phi \in \text{Diff}_{\text{loc}}(G)\), \(\phi^*t = t\) if and only if \(\phi^*t = t\) and \(\phi^*dt = dt\). \(\square\)

This corollary can also be “improved” by restricting to a complete transversal of \(G\), as explained in Section 3.2.2. Using the notation from Example 2.41 for the restrictions of a Cartan algebroid and a realization, we have that:

**Corollary 3.27.** Let \((G, \omega)\) be a standard Lie-Pfaffian groupoid on \(M\) that admits an integral Cartan-Ehresmann connection, and let \(N \subset M\) be a complete transversal of \(G\). The induced pair \((G_N, \Omega_N)\) is a realization of the induced Cartan algebroid \((C_N, \sigma_N)\), and its associated pseudogroup \(\Gamma(G_N, \Omega_N)\) of local symmetries is precisely the canonical prolongation of \(\text{Bis}_{\text{loc}}(G, \omega)\),
\[\Gamma(G_N, \omega_N) = \text{prol}(\text{Bis}_{\text{loc}}(G, \omega)),\] and hence it is in normal form and Cartan equivalent to the generalized pseudogroup \(\text{Bis}_{\text{loc}}(G, \omega)\) of local solutions of \((G, \omega)\).
Proof. In the previous proof, replace Proposition 3.10 and Theorem 3.12 with their “complete transversal counterparts”, Proposition 3.15 and Theorem 3.16 and use the fact that Cartan algebroids and realizations can be restricted to submanifolds (Example 2.41).

Finally, we can prove Theorem 3.1 by applying either of these two corollaries:

Proof (Theorem 3.1). Let \( \Gamma \) be a Lie pseudogroup on \( M \) of order \( k \). The pair \((J^k \Gamma, \omega)\), consisting of the \( k \)-jet groupoid of \( \Gamma \) and its Cartan form, is a standard Lie-Pfaffian groupoid, and \( \Gamma \) is Cartan equivalent to its generalized pseudogroup of local solutions \( \text{Bis}_{\text{loc}}(J^k \Gamma, \omega) \) (see Section 3.1.3). By Lemma A.59 \((J^k \Gamma, \omega)\) admits an integral Cartan-Ehresmann connection. Thus, we may apply Corollary 3.26 to obtain the realization \((J^k \Gamma, \Omega)\) of the Cartan algebroid \((TM \oplus A^{k-1} \Gamma, \sigma^k \Gamma)\), and the associated pseudogroup \( \Gamma(J^k \Gamma, \Omega) \) is in normal form and Cartan equivalent to \( \text{Bis}_{\text{loc}}(J^k \Gamma, \omega) \), which, in turn, is Cartan equivalent to \( \Gamma \). Using Corollary 3.27 instead, we may also replace this realization by its restriction to any complete transversal to the orbits of \( \Gamma \).

3.3 Cartan’s Examples

We conclude this section by looking at two examples from Cartan’s work ([5], pp. 1344-1347) of the construction underlying the Second Fundamental Theorem. In each example, we start by citing Cartan, showing how he constructs a Lie pseudogroup \( \tilde{\Gamma} \) in normal form out of a given Lie pseudogroup \( \Gamma \). We then apply the algorithm of our proof, computing the Cartan algebroid and realization induced by Cartan’s initial pseudogroup and arriving at the same formulas. While the exposition here is complete, some computations are performed in a concise manner, and we refer the reader to [47] for more details.

3.3.1 Example 1 - Cartan

Cartan: “Let \( \Gamma \) be the pseudogroup of homographic transformations in one variable

\[
X = \frac{ax+b}{cx+d} \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0.
\] (3.29)

We know that the defining equation of the pseudogroup is

\[
X'X''' - \frac{3}{2} (X'')^2 = 0.
\]

We set

\[
X' = u, \quad X'' = v,
\]

and we have the system

\[
dX = \omega_1 = udx, \quad du = vdx, \quad dv = \frac{3}{2} \frac{v^2}{u} dx.
\]

We have

\[
d\omega_1 = du \wedge dx = \frac{du - vdx}{u} \wedge udx = \frac{du - vdx}{u} \wedge \omega_1.
\]

The form \( \frac{du - vdx}{u} \) is thus invariant, we denote it by \( \omega_2 \),

\[
\omega_2 = \frac{du}{u} - \frac{v}{u} dx.
\]
We compute
\[ d\omega_2 = -\frac{1}{u} dv \wedge dx + \frac{v}{u^3} du \wedge dx = \left( -\frac{1}{u^2} dv + \frac{v}{u^3} du \right) \wedge \omega_1 \]
\[ = \left( -\frac{1}{u^2} (dv - \frac{3v^2}{2} du) + \frac{v}{u^3} (du - vdx) \right) \wedge \omega_1, \]
from which we obtain the new invariant form
\[ \omega_3 = -\frac{1}{u^2} dv + \frac{v}{u^3} du + \frac{1}{2} \frac{v^2}{u^4} dx. \]

We compute
\[ d\omega_3 = \frac{1}{u^3} du \wedge dv + \frac{v}{u^3} du \wedge dx - 3 \frac{v^2}{2u^4} du \wedge dx = \omega_3 \wedge \omega_2. \]

The structure equations are
\[ d\omega_1 = \omega_2 \wedge \omega_1, \quad d\omega_2 = \omega_3 \wedge \omega_1, \quad d\omega_3 = \omega_3 \wedge \omega_2. \quad (3.30) \]

Thus, starting with a pseudogroup $\Gamma$ on $\mathbb{R}$ (with coordinate $x$) Cartan constructs a realization on $\mathbb{R}^3 \setminus \{ u = 0 \}$ (with coordinates $x, u, v$) consisting of the 1-forms $\omega_1, \omega_2, \omega_3$. We may now compute the induced pseudogroup $\tilde{\Gamma}$ on $\mathbb{R}^3 \setminus \{ u = 0 \}$. It is generated by the transformations:
\[ \tilde{x} = \frac{ax + b}{cx + d}, \quad \tilde{u} = \frac{(cx + d)^2}{ad - bc}, \quad \tilde{v} = \frac{v(cx + d)^4 + 2uc(cx + d)^3}{(ad - bc)^2}, \]
where $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. It is clearly an isomorphic prolongation of (and hence Cartan equivalent to) $\Gamma_0$.

3.3.2 Example 1 - Revisited

We consider again the pseudogroup $\Gamma$ on $M = \mathbb{R}$ defined in [3.29]. Let $x$ be the coordinate on $M$. The pseudogroup is generated by the following locally defined diffeomorphisms (that are defined where $cx + d \neq 0$):
\[ \phi : x \mapsto \frac{ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{R} \text{ with } ad - bc \neq 0. \]

The first three derivatives of $\phi$ are
\[ \frac{\partial \phi}{\partial x} = \frac{ad - bc}{(cx + d)^2}, \quad \frac{\partial^2 \phi}{\partial x^2} = -2c \frac{ad - bc}{(cx + d)^3}, \quad \frac{\partial^3 \phi}{\partial x^3} = 3 \frac{\partial^2 \phi}{\partial x^2} \left( \frac{\partial \phi}{\partial x} \right)^{-1}. \quad (3.31) \]

One can check that the third equation is the defining equation of $\Gamma$, i.e. $\Gamma$ is of order 3. We must compute $\mathcal{J}^2 \Gamma$ and $\mathcal{J}^3 \Gamma$. For the former, it is not difficult to show that $\mathcal{J}^2 \Gamma = \mathcal{J}^2 M$, on which we have the coordinates
\[ \mathcal{J}^2 \Gamma = \mathcal{J}^2 M = \{ (X, x, u, v) \mid X, x, u, v \in \mathbb{R}, u \neq 0 \}, \]
where a jet $\mathcal{J}^2_x \phi$ is mapped to the coordinates $(\phi(x), x, \frac{\partial \phi}{\partial x}(x), \frac{\partial^2 \phi}{\partial x^2}(x))$. The source map is $s(X, x, u, v) = x$, and hence the Lie algebroid $A^2 \Gamma = A^2 M$ has a global frame
\[ \partial_X(x) := \frac{\partial}{\partial X}(x, x, 1, 0), \quad \partial_u(x) := \frac{\partial}{\partial u}(x, x, 1, 0), \quad \partial_v(x) := \frac{\partial}{\partial v}(x, x, 1, 0), \]
\[ 45 \]
and the bracket is readily computed to be
\[ [\partial_X, \partial_u] = 0, \quad [\partial_X, \partial_v] = 0, \quad [\partial_u, \partial_v] = \partial_v, \]
and the anchor is
\[ \rho : A^2M \to TM, \quad \partial_X \mapsto \frac{\partial}{\partial x}, \quad \partial_u \mapsto 0, \quad \partial_v \mapsto 0. \]

Turning to \( J^3\Gamma \), the third equation in (3.31) shows that each jet in \( J^2\Gamma \) uniquely extends to a jet in \( J^3\Gamma \). This implies that there is an isomorphism of Lie groupoids given by the projection \( \pi : J^3\Gamma \xrightarrow{\cong} J^2\Gamma \) (\( \Gamma \) is of finite type). Next, one readily computes the Cartan form \( \omega \in \Omega^1(J^3\Gamma; t^* A^2\Gamma) \), which takes the following form:
\[
\omega = (dX - u dx) t^* \partial_X + \frac{1}{u} (du - v dx) t^* \partial_u + \frac{1}{u^2} (dv - \frac{v}{u} du - \frac{1}{2} \frac{v^2}{u} dx) t^* \partial_v
\]
(it is remarkable that the formulas for the components of the Cartan form precisely coincide with formulas that Cartan obtains using various tricks and manipulations, e.g. see (3.30)). The Spencer operator \( D : \Gamma(A^3\Gamma) \to \Omega^1(M; A^2\Gamma) \) is
\[
D : \partial_X \mapsto 0, \quad \partial_u \mapsto -dx \otimes \partial_X, \quad \partial_v \mapsto -dx \otimes \partial_u.
\]

With this data, we can compute the induced Cartan algebroid and its realization. First,
\[
C = TM \oplus A^2\Gamma,
\]
for which we take the frame (as before, we make these choices to conform with Cartan’s choices)
\[
e^1 = -\partial_X, \quad e^2 = \partial_u, \quad e^3 = -\partial_v, \quad e^4 = \frac{\partial}{\partial x} + \partial_X.
\]
In this example, \( \sigma = 0 \). The bracket on \( C \) is canonical, since there is no choice in splitting the projection from \( A^3\Gamma \) to \( A^2\Gamma \). Thus, the connection (3.39) coincides with the Spencer operator \( D \) and the associated torsion (3.10) is determined by
\[
c(\partial_X, \partial_u) = \partial_X, \quad c(\partial_X, \partial_v) = \partial_u, \quad c(\partial_u, \partial_v) = \partial_v.
\]
From this, we compute the bracket of \( C \),
\[
[e^1, e^2] = e^1, \quad [e^1, e^3] = e^2, \quad [e^1, e^4] = 0,
[e^2, e^3] = e^3, \quad [e^2, e^4] = 0, \quad [e^3, e^4] = 0,
\]
and the anchor
\[
\rho : C \to TM, \quad e^1 \mapsto 0, \quad e^2 \mapsto 0, \quad e^3 \mapsto 0, \quad e^4 \mapsto \frac{\partial}{\partial x}.
\]
The induced realization \((J^3\Gamma, \Omega)\) of \((C, 0)\) consists of the target map \( t : J^3\Gamma \to M \) and the extended Cartan form \( \Omega = (dt, \omega) \), which, in terms of our choice of a frame, decomposes as
\[
\Omega = \omega_1 t^* e^1 + \omega_2 t^* e^2 + \omega_3 t^* e^3 + \omega_4 t^* e^4,
\]
with
\[
\omega_1 = u dx, \quad \omega_2 = \frac{1}{u} (du - v dx), \quad \omega_3 = -\frac{1}{u^2} (dv - \frac{v}{u} du - \frac{1}{2} \frac{v^2}{u} dx), \quad \omega_4 = dX.
\]
In this case, \( \Pi = 0 \), and
\[
\Omega : J^3\Gamma \rightarrow t^*C \quad \text{and} \quad d\Omega + \frac{1}{2}[\Omega, \Omega] = 0,
\]
or, in terms of components, \( \omega_1, \omega_2, \omega_3, \omega_4 \) is a coframe of \( J^3\Gamma \) and
\[
\begin{align*}
d\omega_1 + \omega_1 \wedge \omega_2 &= 0, \\
d\omega_2 + \omega_1 \wedge \omega_3 &= 0, \\
d\omega_3 + \omega_2 \wedge \omega_3 &= 0, \\
d\omega_4 &= 0.
\end{align*}
\]
Restricting to the complete transversal \( X = 0 \), we have that \( \omega_4 = 0 \), and we recover Cartan’s forms and structure equations.

### 3.3.3 Example 2 - Cartan

Cartan: “Let \( \Gamma \) be the pseudogroup on \( \mathbb{R}^2 \) whose elements are given by
\[
X = f(x), \quad Y = \frac{y}{f'(x)},
\]
where \( f \) is an arbitrary function of \( x \) and \( f' \) its derivative (nowhere vanishing). The defining equations are
\[
dX = \frac{y}{Y}dx, \quad dY = udx + \frac{y}{y}dy =: \omega_2,
\]
they are of 1st order. We set \( Y = 1 \) on the right hand side of both equation, and obtain
\[
\omega_1 = ydx, \quad \omega_2 = udx + \frac{1}{y}dy,
\]
with the structure equations
\[
d\omega_1 = \omega_2 \wedge \omega_1, \quad d\omega_2 = \pi \wedge \omega_1,
\]
where \( \pi = \frac{1}{y}du \) (mod \( dx \)). We remark here that the pseudogroup \( \Gamma \) is the isomorphic prolongation of the pseudogroup \( X = f(x) \), where the defining equation is \( dX = udx \), with
\[
\omega_1 = udx, \quad d\omega_1 = \pi \wedge \omega_1.
\]  

Here, Cartan starts with a pseudogroup \( \Gamma \) on \( \mathbb{R}^2 \) (with coordinates \( x, y \)), or, to be more precise, on \( \mathbb{R}^2 \setminus \{ y = 0 \} \) (otherwise the equations are ill-defined). He then constructs a realization on \( \mathbb{R}^3 \setminus \{ y = 0 \} \) (with coordinates \( x, y, u \)) consisting of the 1-forms \( \omega_1, \omega_2 \). To write the structure equations, he introduces the auxiliary form \( \pi \). A computation now shows that the induced pseudogroup in normal form \( \tilde{\Gamma} \) on \( \mathbb{R}^3 \setminus \{ y = 0 \} \) is generated by
\[
\tilde{x} = f(x), \quad \tilde{y} = \frac{y}{f'(x)}, \quad \tilde{u} = \frac{uf'(x) + f''(x)}{(f'(x))^2}, \quad f \in \text{Diff}_{\text{loc}}(\mathbb{R}).
\]
As before, it is clearly an isomorphic prolongation of \( \Gamma \).

In this example (as in the previous one), Cartan simplifies the expressions by setting the target variable \( Y \) to the fixed value 1. This is an instance of the simplification obtained by
restricting to a complete transversal, as explained in Section 3.2.2. Cartan uses this simplification to reduce the dimension of the space on which the isomorphic prolongation acts, thus obtaining a smaller isomorphic prolongation. However, one may also skip this simplification to obtain the canonical prolongation. Indeed, prior to the simplification of setting $Y = 1$, we had the 1-forms
\[ \omega_1 = \frac{y}{Y} dx, \quad \omega_2 = u dx + \frac{Y}{y} dy. \] (3.35)
Adding to this data the projection functions
\[ I_1 = X, \quad I_2 = Y, \]
and their differentials
\[ \omega_3 = dX, \quad \omega_4 = dY, \] (3.36)
the structure equations are
\[
\begin{align*}
    d\omega_1 &= \frac{1}{Y} (\omega_2 - \omega_4) \wedge \omega_1, \\
    d\omega_2 &= \frac{1}{Y} \omega_4 \wedge \omega_2 + \pi \wedge \omega_1, \\
    d\omega_3 &= 0, \\
    d\omega_4 &= 0,
\end{align*}
\]
with
\[
\pi = \frac{Y}{y} du - \frac{u}{y} dY \mod dx.
\]
The isomorphic prolongation on $\mathbb{R}^5 \setminus \{y = 0 \text{ or } Y = 0\}$ (with coordinates $x, y, X, Y, u$) is
\[
\begin{align*}
    \bar{x} &= f(x), \\
    \bar{y} &= \frac{y}{f'(x)}, \\
    \bar{X} &= X, \\
    \bar{Y} &= Y, \\
    \bar{u} &= \frac{uf'(x) + Yf''(x)}{(f'(x))^2}, \\
    f &\in \text{Diff}_{\text{loc}}(\mathbb{R}).
\end{align*}
\]
The restriction to the orbit $\{X = 0, Y = 1\}$ is precisely Cartan's isomorphic prolongation.

3.3.4 Example 2 - Revisited
Consider again the pseudogroup $\Gamma$ on $M = \mathbb{R}^2 \setminus \{y = 0\}$ defined in (3.32). Let $(x, y)$ be coordinates on $M$. The pseudogroup is generated by
\[
\phi : (x, y) \mapsto (\phi_x(x, y), \phi_y(x, y)) = (f(x), \frac{y}{f'(x)}), \\
\phi &\in \text{Diff}_{\text{loc}}(\mathbb{R}).
\] (3.37)
The pseudogroup is transitive and, hence, $J^0 \Gamma = J^0 M$ and $A^0 \Gamma = A^0 M$. On $J^0 \Gamma$ we have coordinates
\[
J^0 \Gamma = \{(X, Y, x, y) \mid y \neq 0 \text{ and } Y \neq 0\},
\]
where $(x, y)$ are the source coordinates and $(X, Y)$ the target. We denote the induced frame on the Lie algebroid $A^0 \Gamma$ by
\[
\partial_X, \partial_Y \in \Gamma(A^0 \Gamma).
\]
The bracket and anchor of $A^0 \Gamma$ are
\[
[\partial_X, \partial_Y] = 0 \quad \text{and} \quad \rho : A^0 \Gamma \rightarrow TM, \quad \partial_X \mapsto \frac{\partial}{\partial x}, \quad \partial_Y \mapsto \frac{\partial}{\partial y}.
\]
The first derivatives of the elements of $\Gamma$ are
\[
\begin{align*}
    \frac{\partial \phi_x}{\partial x} &= \frac{y}{\phi_y}, \\
    \frac{\partial \phi_y}{\partial y} &= 0, \\
    \frac{\partial \phi_y}{\partial x} &= \frac{f''(x)}{(f'(x))^2}, \\
    \frac{\partial \phi_y}{\partial y} &= \frac{\phi_x}{y},
\end{align*}
\] (3.38)
from which we deduce that
\[ J^1 \Gamma = \{(X, Y, x, y, u) \mid y \neq 0 \text{ and } Y \neq 0\}, \]
where a jet \( j^1_{(x,y)} \phi \) is assigned the coordinates \((\phi_x(x,y), \phi_y(x,y), x, y, \frac{\partial \phi_x}{\partial x}(x,y))\). The source map is given by \( s(X, Y, x, y, u) = (x, y) \), and so the Lie algebroid \( A^1 \Gamma \) has a global frame \( e_X, e_Y, e_u \), where
\[
e_X(x,y) = \frac{\partial}{\partial x}(x,y,x,y,0), \quad e_Y(x,y) = \frac{\partial}{\partial y}(x,y,x,y,0), \quad e_u(x,y) = \frac{\partial}{\partial u}(x,y,x,y,0).\]
The projection is given by
\[ d\pi : A^1 \Gamma \to A^0 \Gamma, \quad e_X \mapsto \partial X, \quad e_Y \mapsto \partial Y, \quad e_u \mapsto 0. \tag{3.39} \]
Next, computing the Cartan form \( \omega \in \Omega^1(J^1 \Gamma; t^* A^0 \Gamma) \), we readily find that
\[
\omega = (dX - \frac{y}{Y} dx) t^* \partial X + (dY - u dx - \frac{Y}{y} dy) t^* \partial Y,
\]
and the Spencer operator \( D : \Gamma(A^1 \Gamma) \to \Omega^1(M; A^0 \Gamma) \) takes the form
\[
D : e_X \mapsto 0, \quad e_Y \mapsto \frac{1}{y} (dx \otimes \partial X - dy \otimes \partial Y), \quad e_u \mapsto -dx \otimes \partial Y.
\]
With this data, we can set out to compute the induced Cartan algebroid and its realization. First, for the Cartan algebroid,
\[ C = TM \oplus A^0 \Gamma. \]
One natural frame for this vector bundle would be \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \partial X, \partial Y \). Cartan’s formulas, however, correspond to the frame
\[ e^1 = -\partial X, \quad e^2 = -\partial Y, \quad e^3 = \partial X + \frac{\partial}{\partial x}, \quad e^4 = \partial Y + \frac{\partial}{\partial y}. \]
For the tableau bundle
\[ \sigma = \text{Ker} (d\pi : A^1 \Gamma \to A^0 \Gamma), \]
we choose the frame \( t = e_u \). The bracket of \( C \) depends on a choice of a splitting of \( (3.39) \), for which we choose
\[ \xi : A^0 \Gamma \to A^1 \Gamma, \quad \partial X \mapsto e_X, \quad \partial Y \mapsto e_Y. \]
The induced connection \( \nabla^\xi \) on \( A^0 \Gamma \) defined in \( (3.9) \) is given by
\[ \nabla^\xi_{\partial_j/\partial x}(\partial X) = 0, \quad \nabla^\xi_{\partial_j/\partial y}(\partial X) = 0, \quad \nabla^\xi_{\partial_j/\partial x}(\partial Y) = \frac{1}{y} \partial X, \quad \nabla^\xi_{\partial_j/\partial y}(\partial Y) = -\frac{1}{y} \partial Y, \]
and the torsion \( c^\xi \) of \( \nabla^\xi \) defined in \( (3.10) \) is given by
\[ c^\xi(\partial X, \partial Y) = -\frac{1}{y} \partial X. \]
The bracket (3.12) on \( C \) is then
\[
[e^1, e^2] = \frac{1}{y} e^1, \quad [e^1, e^3] = 0, \quad [e^1, e^4] = -\frac{1}{y} e^1,
\]
\[
[e^2, e^3] = 0, \quad [e^2, e^4] = \frac{1}{y} e^2, \quad [e^3, e^4] = 0,
\]
and the anchor (3.13) is
\[
\rho : C \to TM, \quad e^1 \mapsto 0, \quad e^2 \mapsto 0, \quad e^3 \mapsto \frac{\partial}{\partial x}, \quad e^4 \mapsto \frac{\partial}{\partial y}.
\]

The action of \( \sigma \) on \( C \) is
\[
t(e^1) = e^2, \quad t(e^2) = 0, \quad t(e^3) = 0, \quad t(e^4) = 0.
\]

Thus, writing \([e^j, e^k] = \sum_{i=1}^4 c_{jk}^i e^i \) and \( t(e^j) = \sum_{i=1}^4 a_j^i e^i \), the non-zero structure functions are
\[
c_{12}^1 = \frac{1}{y}, \quad c_{14}^1 = -\frac{1}{y}, \quad c_{23}^2 = \frac{1}{y}, \quad a_2^4 = 1.
\]

Finally, we describe the realization \((J^1 \Gamma, \Omega)\) of \((C, \sigma)\). The realization consists of the target map \( t : J^1 \Gamma \to M \) and the extended Cartan form \( \Omega = (dt, \omega) \). In terms of the frame of \( C \), \( \Omega \) decomposes as
\[
\Omega = \omega_1 t^* e_1 + \omega_2 t^* e_2 + \omega_3 t^* e_3 + \omega_4 t^* e_4,
\]
with
\[
\omega_1 = \frac{y}{Y} dx, \quad \omega_2 = u dx + \frac{Y}{y} dy, \quad \omega_3 = dX, \quad \omega_4 = dY.
\]

These are precisely the forms (3.35) and (3.36), and, when restricting to the orbit \( X = 0, Y = 1 \), these are precisely Cartan’s forms (3.33).

Appendix

A.1 Jet Groupoids and Algebroids

In this appendix, we review the framework of jet groupoids and algebroids. This framework allows for a geometric formulation of the notion of a system of PDEs in the setting of pseudogroups and, accordingly, of a Lie pseudogroup. We place a special emphasis on the role and properties of the Cartan form in encoding the essential structure. This will lead us in Appendix A.2 to the notion of a Lie-Pfaffian groupoid, an axiomatization of the notion of a jet groupoid, which, in the spirit of Cartan, isolates the key role of the Cartan form.

A.1.1 Jet Groupoids

Let \( M \) be a manifold. For each integer \( k \geq 0 \), the \( k \)-th jet groupoid of \( \text{Diff}_{\text{loc}}(M) \) (or, for brevity, of \( M \)), denoted by \( J^k M \rightrightarrows M \), is a Lie groupoid whose space of arrows consists of all \( k \)-jets of all locally defined diffeomorphisms of \( M \),
\[
J^k M := \{ J_x^k \phi \mid \phi \in \text{Diff}_{\text{loc}}(M), \ x \in \text{Dom}(\phi) \}, \quad (A.40)
\]
and whose structure maps are

\[ s(j^k_x \phi) = x, \quad t(j^k_x \phi) = \phi(x), \quad 1_x = j^k_x(id_M), \]
\[ j^k_{\phi(x)} \phi' \cdot j^k_x \phi = j^k_x (\phi' \circ \phi), \quad (j^k_x \phi)^{-1} = j^k_x (\phi(x))^{-1}. \]

Thus, the \( k \)-th jet groupoid encodes the \( k \)-th order Taylor polynomials of locally defined diffeomorphisms of \( M \). For example, the 0-th jet groupoid \( J^0M \rightrightarrows M \) encodes the source and target points. It is canonically isomorphic to the pair groupoid \( M \times M \rightrightarrows M \) by the map \( j^0_0 \phi \mapsto (\phi(x), x) \). The smooth structure of \( J^kM \) is defined as usual for jet spaces (in fact, \( J^kM \) is an open subset of the space of \( k \)-jets of all smooth maps from \( M \) to \( M \)). Any \( \phi \in \text{Diff}_{\text{loc}}(M) \) induces a local bisection \( j^k\phi \) of \( J^kM \rightrightarrows M \), called a \textit{local holonomic bisection}, whose domain is the domain of \( \phi \) and which maps \( x \mapsto j^k_x \phi \). The subset of local holonomic bisections (inside the set of all local bisections) is, one may say, the most important piece of structure of a jet groupoid. When one studies Lie subgroupoids of a jet groupoid, which play the role of PDEs (e.g. the defining PDEs of Lie pseudogroups), local holonomic bisections play the role of local solutions.

The jet groupoids of \( M \) form a sequence of Lie groupoids,
\[ ... \xrightarrow{\pi} J^3M \xrightarrow{\pi} J^2M \xrightarrow{\pi} J^1M \xrightarrow{\pi} J^0M, \] (A.41)
where the projections, the Lie groupoid morphisms
\[ \pi : J^{k+1}M \to J^kM, \quad j^{k+1}_x \phi \mapsto j^k_x \phi, \] (A.42)
are surjective submersions.

\textbf{Remark A.28.} Continuing with the philosophy of Section [A.2.2] passing from pseudogroups on a manifold \( M \) to generalized pseudogroups on a Lie groupoid \( G \rightrightarrows M \), we may construct the \( k \)-th jet groupoid \( J^kG \rightrightarrows M \) of \( G \), consisting of \( k \)-jets \( j^k_b \) of local bisections \( b \in \text{Bis}_{\text{loc}}(G) \). The jet groupoid \( J^kM \) is the special case corresponding to the pair groupoid \( M \times M \rightrightarrows M \). All the notions explained here and in Section [A.1] generalize without extra effort to this setting.

\section*{A.1.2 Jet Algebroids and the Adjoint Representation}

At the infinitesimal level, we have the Lie algebroids of the jet groupoids \( J^kM \rightrightarrows M \), which we denote by \( A^kM = A(J^kM) \). These fit in a sequence of Lie algebroids,
\[ ... \xrightarrow{l} A^3M \xrightarrow{l} A^2M \xrightarrow{l} A^1M \xrightarrow{l} A^0M, \] (A.43)
where the projections, induced by the projections (A.42), are surjective Lie algebroid morphisms. A well-known fact is that \( A^kM \) is canonically isomorphic to the \textit{k-th jet algebroid of} \( X(M) \) (or of \( M \)), the Lie algebroid \( J^kTM \) consisting of \( k \)-jets of vector fields on \( M \). The isomorphism
\[ J^kTM \cong A^kM, \] (A.44)
is given by mapping a \( k \)-jet \( j_x^k X \) of a vector field \( X \in X(M) \) at \( x \in M \) to the vector \( \frac{d}{dt}_{t=0} j_x^k \varphi^X_x \), where \( \varphi^X_x \) is the flow of \( X \). In particular, \( A^0M \) is canonically isomorphic to \( TM \). We refer the reader to [16], [22] or [47] for further details. We, thus, refer to \( A^kM \) also as the \( k \)-th jet algebroid of \( M \).

For every \( k > 0 \), there is a canonical action of the \( k \)-th jet groupoid \( J^kM \) on the \((k-1)\)-th jet algebroid \( A^{k-1}M \), given by conjugation. Namely, an element \( j^k_x \phi \in J^kM \) acts on the fiber \((A^{k-1}M)_x \) by:
\[ j^k_x \phi \cdot \frac{d}{dt}_{t=0} j^{k-1}_x \psi = \frac{d}{dt}_{t=0} j^{k-1}_x (\phi_0 \psi \circ \phi^{-1}), \] (A.45)
where \( j^{k-1}_x \psi \) is a curve representing an element of \((A^{k-1} M)_x\). With this action, \( A^{k-1} M \) is a representation of \( J^k M \). It is called the adjoint representation. At the infinitesimal level, this induces the adjoint representation \( A^{k-1} M \) of \( A^k M \) for every \( k > 0 \) (see e.g. \([9]\)).

A.1.3 The Cartan Form
Every jet groupoid \( J^k M \rightrightarrows M \), with \( k > 0 \), comes equipped with a tautological form

\[
\omega = \omega^k \in \Omega^1(J^k M; t^* A^{k-1} M),
\]

(A.46)
called the Cartan form of \( J^k M \). It is a 1-form on the total space of the jet groupoid with values in the adjoint representation. It is defined at a point \( j^k_\phi \in J^k M \) by the formula

\[
\omega_{j^k_\phi} := dR_{(j^{k-1}_\phi)_{-1}} \circ (d \pi - (d(j^{k-1}_\phi))_x \circ ds)|_{j^k_\phi},
\]

(A.47)
which is depicted in the following diagram:

\[
\begin{array}{c}
J^k M \\
\downarrow \pi \\
J^{k-1} M \\
\downarrow s \\
\downarrow j^{k-1}_\phi \\
M.
\end{array}
\]

Indeed, the image of the map \( d \pi - (d(j^{k-1}_\phi))_x \circ ds \) at a point \( j^k_\phi \) is tangent to the \( s \)-fiber of \( J^{k-1} M \) at the point \( j^{k-1}_\phi \), and then right translation maps it to the fiber of the Lie algebroid \( A^{k-1} M \) at the point \( t(j^k_\phi) = \phi(x) \). By restricting \( \omega \) to \( \text{Ker} \, ds \), we see that \( \omega \) is pointwise surjective. The kernel of the Cartan form, a non-involutive distribution, is called the Cartan distribution and is denoted by \( C_\omega := \text{Ker} \, \omega \subset TG \). The main property of the Cartan form is that it detects holonomic sections (for the proof see e.g. Proposition 1.3.3 in \([47]\)).

**Proposition A.29.** Let \( M \) be a manifold. A local bisection \( b \) of \( J^k M \rightrightarrows M \) is holonomic, i.e. of the form \( b = j^k \phi \) for some \( \phi \in \text{Diff}_{\text{loc}}(M) \), if and only if \( b^* \omega = 0 \).

In fact, all jet spaces carry a tautological form, defined similarly to the Cartan form of a jet groupoid, and these satisfy a property similar to the one in the above proposition. However, the Cartan form of a jet groupoid satisfies an additional property, known as multiplicativity, that reflects its compatibility with the Lie groupoid structure of the jet groupoid. Recall that a differential form \( \omega \in \Omega^*(\mathcal{G}, t^* E) \) on a Lie groupoid \( \mathcal{G} \rightrightarrows M \) with values in a representation \( E \rightrightarrows M \) is said to be multiplicative if

\[
(m^* \omega)_{(g,h)} = (pr_1^* \omega)_{(g,h)} + g \cdot (pr_2^* \omega)_{(g,h)}, \quad \forall (g, h) \in \mathcal{G}_2,
\]

(A.48)
where \( \mathcal{G}_2 \subset \mathcal{G} \times \mathcal{G} \) is the space of composable arrows and \( m, pr_i : \mathcal{G}_2 \to \mathcal{G} \) are the multiplication and projection maps. For the proof of this property, see \([14]\) (Proposition 3.4) or \([17]\) (Proposition 2.4.3). To summarize:

**Proposition A.30.** Let \( M \) be a manifold. For any \( k > 0 \), the Cartan form of \( J^k M \rightrightarrows M \),

\[
\omega \in \Omega^1(J^k M; t^* A^{k-1} M),
\]

is a pointwise-surjective multiplicative 1-form on \( J^k M \) with values in the adjoint representation.
A.1.4 The Spencer Operator

At the infinitesimal level, the Cartan form of $J^k M$ induces the **Spencer operator** of $A^k M$, an operator

$$D = D^k : \mathfrak{X}(M) \times \Gamma(A^k M) \to \Gamma(A^{k-1} M), \quad (X, \alpha) \mapsto D_X(\alpha), \quad \text{(A.49)}$$

satisfying the connection-like properties

$$D_{fX}(\alpha) = fD_X(\alpha), \quad D_X(f\alpha) = fD_X(\alpha) + X(f)l(\alpha), \quad \text{(A.50)}$$

for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(A^k M)$, where recall that $l : A^k M \to A^{k-1} M$ is the projection. It is obtained from the Cartan form $\omega$ by differentiation:

$$D_X(\alpha)_x := \frac{d}{de} \big|_{e = 0} \varphi^\alpha_x(x)^{-1} \cdot \omega(d\varphi^\alpha_x(X_x)), \quad \text{(A.51)}$$

for all $x \in M$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(A^k M)$, where $\varphi^\alpha_x$ is the flow of bisections of $J^k M$ associated with $\alpha$. Alternatively, under the identification (A.44), the Spencer operator on $A^k M$ corresponds to the classical Spencer operator on the Lie algebroid $J^k TM$ of $k$-jets of vector fields on $M$, which is defined purely out of infinitesimal data (e.g., see Section 1.1.4 in [40]). The multiplicativity property (A.48) translates into a property known as **infinitesimal multiplicativity** ($IM$ in short). We will explain this property in the more general context of Lie-Pfaffian algebroids in Section A.2 (in particular, see (A.63)). With this, the Spencer operator becomes what is known as an **IM-form**, the infinitesimal counterpart of a multiplicative form.

A.1.5 The Symbol Space

The kernel of the projection $l : A^k M \to A^{k-1} M$ (where $k > 0$),

$$\sigma^k M := \text{Ker} (l : A^k M \to A^{k-1} M), \quad \text{(A.52)}$$

is called the **symbol space** of $A^k M$. It plays a key role in the theory. Being the kernel of a Lie algebroid morphism, the symbol space has the structure of a bundle of Lie algebras, which, one can show, is abelian for $k > 1$. For notational purposes, we set $\sigma^0 M := A^0 M = TM$.

The restriction of the Spencer operator to the symbol space gives a canonical inclusion

$$\sigma^k M \hookrightarrow \text{Hom}(TM, \sigma^{k-1} M), \quad T \mapsto (\hat{T} : X \mapsto D_X(T)). \quad \text{(A.53)}$$

For $k > 1$, this identifies $\sigma^k M$ with the symmetric part of $\text{Hom}(TM, \sigma^{k-1} M)$, in the sense that

$$\sigma^k M \cong \{ \hat{T} \in \text{Hom}(TM, \sigma^{k-1} M) \mid \hat{T}(X)(Y) = \hat{T}(Y)(X) \quad \forall X, Y \in TM \}, \quad \text{(A.54)}$$

where the equation on the right-hand side uses the inclusion $\sigma^{k-1} M \hookrightarrow \text{Hom}(TM, \sigma^{k-2} M)$. For $k = 1$,

$$\sigma^1 M \cong \text{Hom}(TM, TM). \quad \text{(A.55)}$$

We note that, if we equip $\text{Hom}(TM, \sigma^{k-1})$ with the zero bracket for $k > 1$ and the commutator bracket for $k = 1$, then these become isomorphisms of Lie algebroids. Applying (A.54) and (A.55) inductively, we obtain the well-known isomorphism

$$\sigma^k M \cong S^k T^* M \otimes TM, \quad \text{(A.56)}$$

which identifies the symbol space of $A^k M$ with the space of vector-valued monomials of degree $k$ on $M$ (this identification is usually described in local coordinates, see e.g. p. 21 in [47]).
A.1.6 Aside: Tableau Bundles and the Spencer Cohomology

The inclusion (A.53) of the symbol space as a vector subbundle of a Hom-bundle is an instance of the abstract notion of a tableau bundle. The important notions of prolongation, the Spencer cohomology and involutivity, that one typically associates with the symbol space, can be defined purely in terms of its tableau bundle structure. To construct and define these notions, it is sufficient to work with the following discrete version of the notion of a vector bundle, which allows us to include non-smooth vector bundles into the picture, such as kernels, images and cokernels of vector bundle maps.

Given a manifold $M$, a discrete vector bundle over $M$ is a disjoint union of vector spaces indexed by $M$, i.e. a space $E = \sqcup_{x \in M} E_x$ where $\{E_x\}_{x \in M}$ is a collection of vector spaces. A discrete vector subbundle of a discrete vector bundle $F$ over $M$ is a subset $E \subset F$, such that $E_x \subset F_x$, the fiber of $E$ over $x$, is a vector subspace for each $x \in M$. Given two discrete vector bundles $E$ and $F$ over $M$, a map $\partial : E \to F$ covering the identity is a discrete vector bundle map if it restricts to a linear map on each fiber. A discrete vector subbundle can be viewed as a discrete vector bundle together with an injective discrete vector bundle map.

**Definition A.31.** Let $E, F$ be discrete vector bundles over a manifold $M$. A tableau bundle relative to $(E, F)$ is a discrete vector bundle $\sigma$ over $M$ together with a discrete vector bundle map

$$\partial : \sigma \to \text{Hom}(E, F).$$

**(A.57)**

**Example A.32.** The symbol space $\sigma^k M$, together with (A.53), is a tableau bundle relative to $(TM, \sigma^{k-1} M)$. Note that, in this case, the map (A.57) is injective.

We denote a tableau bundle by $(\sigma, \partial)$, or simply by $\sigma$ when it is clear what the map $\partial$ is. While the map $\partial$ is injective in most applications, such as in the previous example, we will see in Section A.2 and later in Section 3 that a great deal of the theory does not rely on this property and that, in the setting of Lie-Pfaffian groupoids, it is natural to consider non-injective maps.

**Definition A.33.** Let $(\sigma, \partial)$ be a tableau bundle relative to $(E, F)$. The 1st prolongation of $\sigma$ is the tableau bundle given by the discrete vector subbundle

$$\sigma^{(1)} \subset \text{Hom}(E, \sigma),$$

whose fiber at $x \in M$ is

$$\sigma_x^{(1)} := \{ \xi \in \text{Hom}_x(E, \sigma) \mid \partial(\xi(u))(v) = \partial(\xi(v))(u) \ \forall \ u, v \in E_x \}.$$

**Remark A.34.** Even when the initial data of a tableau bundle is smooth, i.e. $E$, $F$ and $\sigma$ are vector bundles and $\partial$ a vector bundle map, its 1st prolongation may fail to be smooth. This is the main reason we resort to the language of discrete vector bundles. However, we note that $\sigma^{(1)}$ is the kernel of the map (A.59) defined below (with $m = 1$ and $l = 0$), and, hence, if the data of the tableau bundle is smooth, then $\sigma^{(1)}$ is smooth (i.e. a vector subbundle of $\text{Hom}(E, \sigma)$) if and only if it is of constant rank.

Since the 1st prolongation is again a tableau bundle, we can continue and define the higher prolongations in the following inductive manner:

**Definition A.35.** Let $(\sigma, \partial)$ be a tableau bundle relative to $(E, F)$. The $l$-th prolongation of $\sigma$, with $l > 0$ an integer, is the discrete vector subbundle $\sigma^{(l)} \subset \text{Hom}(E, \sigma^{(l-1)})$ defined by

$$\sigma^{(l)} := (\sigma^{(l-1)})^{(1)},$$

where we set $\sigma^{(-1)} := F$ and $\sigma^{(0)} := \sigma$. 54
Example A.36. By (A.54) and (A.55), we see that
\[ \sigma^k M = (\sigma^{k-1} M)^{(1)} = (\sigma^{k-2} M)^{(2)} = \ldots = \text{Hom}(TM, TM)^{(k-1)}. \]

A tableau bundle \((\sigma, \partial)\) and its prolongations fit into the following sequence of cochain complexes called the **Spencer complex** of \((\sigma, \partial)\):

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\sigma & \hookrightarrow \text{Hom}(E, F) & \sigma^{(1)} & \hookrightarrow \text{Hom}(E, \sigma) & \delta \rightarrow \text{Hom}(\Lambda^2 E, F) \\
\sigma^{(2)} & \hookrightarrow \text{Hom}(E, \sigma^{(1)}) & \delta \rightarrow \text{Hom}(\Lambda^2 E, \sigma) & \delta \rightarrow \text{Hom}(\Lambda^3 E, F) \\
\sigma^{(3)} & \hookrightarrow \text{Hom}(E, \sigma^{(2)}) & \delta \rightarrow \text{Hom}(\Lambda^2 E, \sigma^{(1)}) & \delta \rightarrow \text{Hom}(\Lambda^3 E, \sigma) & \delta \rightarrow \text{Hom}(\Lambda^4 E, F) \\
\end{array}
\]

The coboundary operator \(\delta : \text{Hom}(\Lambda^m E, \sigma^{(l)}) \rightarrow \text{Hom}(\Lambda^{m+1} E, \sigma^{(l-1)})\) is defined by
\[
\delta(\xi)(u_0, \ldots, u_m) := \sum_{i=0}^{m} (-1)^i \partial(\xi(u_0, \ldots, \hat{u}_i, \ldots, u_m))(u_i),
\]

where \(\hat{u}_i\) denotes the removal of the \(i\)-th term. It is straightforward to verify that \(\delta \circ \delta = 0\).

We denote the cocycles at \(\text{Hom}(\Lambda^m E, \sigma^{(l)})\) by \(Z^{l,m}(\sigma, \partial)\), the coboundaries by \(B^{l,m}(\sigma, \partial)\), and the resulting cohomology group by
\[
H^{l,m}(\sigma, \partial) = \frac{Z^{l,m}(\sigma)}{B^{l,m}(\sigma)}.
\]

The resulting cohomology theory is called the **Spencer cohomology** of \((\sigma, \partial)\). Note that, by definition, \(H^{l,1}(\sigma, \partial) = 0\) for all \(l \geq 0\).

**Definition A.37.** Let \(r \geq 1\) be an integer. A tableau bundle \(\sigma\) is said to be \(r\)-acyclic if
\[
H^{l,m}(\sigma, \partial) = 0 \quad \forall \ 1 \leq m \leq r, \ 0 \leq l,
\]
and involutive if it is \(r\)-acyclic for all \(r \geq 1\).

**Remark A.38.** The notion of a tableau was introduced by Cartan in the context of PDEs and his theory of Exterior Differential Systems. Cartan gave a different definition of the notion of involutivity (see e.g. [2]). In [43], Spencer introduced the Spencer complex and cohomology in the context of deformations of geometric structures, and Guillemin, Singer and Sternberg conjectured and Serre proved ([17] [42]) the equivalence between Cartan’s notion of involutivity and involutivity in terms of the Spencer cohomology.
A.2 Lie-Pfaffian Groupoids and Algebroids

The essential structure of a jet groupoid is encoded in its Cartan form. For example, Proposition A.29 shows us that the notion of a holonomic section can be formulated purely in terms of the Cartan form. In [40], Cartan’s approach of studying Lie pseudogroups by means of their Cartan form was formalized in the abstract notions of a Lie-Pfaffian groupoid and, its infinitesimal counterpart, a Lie-Pfaffian algebroid. These structures isolate the minimal set of ingredients that one needs. As a result, constructions and proofs become substantially simpler and more transparent as compared to working directly with jets. For example, it is shown in [40] that the construction of prolongations of systems of PDEs and the proof of a formal integrability theorem can be formulated in terms of this structure alone, allowing one to avoid the messy computations and book-keeping that one typically encounters. In this section, we review these notions, which we use in Section 3 in the proof of Cartan’s Second Fundamental Theorem.

A.2.1 Lie-Pfaffian Groupoids

Let $G \Rightarrow M$ be a Lie groupoid with source and target maps $s$ and $t$, respectively. Recall that a differential form $\omega \in \Omega^*(G; t^*E)$ on $G$ with values in a representation $E \to M$ of $G$ is said to be multiplicative if it satisfies (A.48).

**Definition A.39.** A Lie-Pfaffian groupoid over a manifold $M$ is a Lie groupoid $G \Rightarrow M$ equipped with a pointwise-surjective multiplicative 1-form (the Cartan form) $\omega \in \Omega^1(G; t^*E)$ with values in a representation $E \to M$ of $G$, such that:

1. $C_\omega + \text{Ker } ds = T^sG$ (“s-transversality”),
2. $C_\omega \cap \text{Ker } dt = C_\omega \cap \text{Ker } ds$,

where $C_\omega := \text{Ker } \omega$ (the Cartan distribution). A (local) holonomic bisection (or a (local) solution) of $(G, \omega)$ is a (local) bisection $\eta$ of $G$ satisfying $\eta^*\omega = 0$. The set of local holonomic bisections is denoted by $\text{Bis}_{\text{loc}}(G, \omega)$.

**Remark A.40.** Axiom 1 is equivalent to requiring that $ds|_{C_\omega} : C_\omega \to s^*TM$ (A.60) be pointwise-surjective. A consequence of axiom 2 is that the Lie algebroid $A$ of $G$ induces a Lie algebroid structure on $E$, namely the unique Lie algebroid structure with which the restriction $\omega|_A : A \to E$ is a Lie algebroid map (see Proposition 6.1.8 in [40]).

**Example A.41 (Jet groupoids).** Given a manifold $M$ and an integer $k > 0$, the pair $(J^kM, \omega)$, consisting of the jet groupoid $J^kM \Rightarrow M$ and the Cartan form $\omega \in \Omega^1(J^kM; t^*A^{k-1}M)$ with values in the adjoint representation $A^{k-1}M$, is a Lie-Pfaffian groupoid. Due to Proposition A.30, we are only left to verify axioms 1 and 2. Axiom 1 follows from the existence of pointwise splittings of (A.60) at each point $j^k_x \phi$ given by $(d(j^k \phi))_x$, where $\phi$ is some representative. For
axiom 2, we note that there is a (unique) isomorphism $s^*TM \cong t^*TM$ of vector bundles over $J^kM$ with which the diagram

\[
\begin{array}{ccc}
    s^*TM & \cong & t^*TM \\
    \downarrow{ds} & & \downarrow{dt} \\
    C_\omega & & C_\omega \\
\end{array}
\]

commutes given at a point $j^k_x \phi$ by $(d\phi)_x$. This implies that $C_\omega \cap \text{Ker} \ ds = C_\omega \cap \text{Ker} \ dt$. Note that the isomorphism (A.61) defines an action with which $TM$ becomes a representation of $J^kM$. By Proposition [A.29] we also see that

\[
\text{Diff}_{\text{loc}}(M) \to \text{Bis}_{\text{loc}}(J^kM, \omega), \quad \phi \mapsto j^k\phi,
\]

is a bijection. From this, it is easy to see that the pseudogroup $\text{Diff}_{\text{loc}}(M)$ is Cartan equivalent to the generalized pseudogroup $\text{Bis}_{\text{loc}}(J^kM, \omega)$.

Our main examples of interest of a Lie-Pfaffian groupoid are “nice” Lie subgroupoids of jet groupoids that play the role of the defining equations of Lie pseudogroups. These will be discussed in Section 3.1.3.

In the above examples of Lie-Pfaffian groupoids, we saw that $TM$ is canonically a representation of any jet groupoid. This is, in fact, a general feature of Lie-Pfaffian groupoids:

**Lemma A.42.** Let $(\mathcal{G}, \omega)$ be a Lie-Pfaffian groupoid. There exists a unique isomorphism $s^*TM \cong t^*TM$ with which the following diagram commutes:

\[
\begin{array}{ccc}
    s^*TM & \cong & t^*TM \\
    \downarrow{ds} & & \downarrow{dt} \\
    C_\omega & & C_\omega \\
\end{array}
\]

This defines an action with which $TM$ becomes a representation of $\mathcal{G}$.

**Proof.** The isomorphism is given by choosing a splitting $H : s^*TM \to C_\omega$ of (A.60) and composing it with $dt$. It is independent of the choice, since the difference of any two connections takes values in $C_\omega \cap \text{Ker} \ ds$, which is killed by $dt$ by axiom 2.

\[\square\]

### A.2.2 Generalized Pseudogroups

The set of local holonomic bisections of a Lie-Pfaffian groupoid has a structure resembling that of a pseudogroup, only instead of locally defined diffeomorphisms of a manifold, it consists of local bisections of a Lie groupoid. Such objects are called **generalized pseudogroups**.

To be more precise, let $\mathcal{G} \equiv M$ be a Lie groupoid and let $\text{Bis}_{\text{loc}}(\mathcal{G})$ denote its set of local bisections. Given $b \in \text{Bis}_{\text{loc}}(\mathcal{G})$, we write $\phi_b := t \circ b \in \text{Diff}_{\text{loc}}(M)$ for the induced locally defined diffeomorphism. Recall that 1) if two local bisections $b, b' \in \text{Bis}_{\text{loc}}(\mathcal{G})$ satisfy $\text{Im}(\phi_{b'}) \subset \text{Dom}(b)$, then we define their product $b \cdot b' \in \text{Bis}_{\text{loc}}(\mathcal{G})$ by $(b \cdot b')(x) := b(\phi_{b'}(x)) \cdot b'(x)$; 2) the inverse $b^{-1} \in \text{Bis}_{\text{loc}}(\mathcal{G})$ of a local bisection $b \in \text{Bis}_{\text{loc}}(\mathcal{G})$ is defined by $b^{-1}(x) := b(\phi_b^{-1}(x))^{-1}$; and 3) there is an identity bisection $1 \in \text{Bis}_{\text{loc}}(\mathcal{G})$ assigning to each point its corresponding unit, i.e. $1(x) = 1_x$. With these operations, we can define a **generalized pseudogroup** on $\mathcal{G}$ to be a subset $\Gamma \subset \text{Bis}_{\text{loc}}(\mathcal{G})$ satisfying the group-like and sheaf-like axioms precisely as in Definition 1.1 (replacing the composition of diffeomorphisms by the product of local bisections, etc.). A pseudogroup on $M$ is then the same thing as a generalized pseudogroup on the pair groupoid $M \times M \equiv M$. 57
Thus, our main example of a generalized pseudogroup is the generalized pseudogroup of local holonomic bisections of a Lie-Pfaffian groupoid. In particular:

**Example A.43.** By Proposition A.29, for any $k > 0$, there is a bijection $\text{Diff}_{\text{loc}}(M)$ and the generalized pseudogroup $\text{Bis}_{\text{loc}}(J^k M, \omega)$ given by $\phi \mapsto j^k \phi$.

We refer the reader to [47] (Section 3.6) for more details on generalized pseudogroups and for some simple examples.

**A.2.3 Lie-Pfaffian Algebroids**

The infinitesimal counterpart of a Lie-Pfaffian groupoid is a Lie-Pfaffian algebroid. Let $A \rightarrow M$ and $E \rightarrow M$ be vector bundles and let $l : A \rightarrow E$ be a surjective vector bundle map. An $l$-connection $D$ on $A$ is a bilinear map

$$D : \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(E)$$

satisfying the connection-like properties

$$D_{fX}(\alpha) = f D_X(\alpha), \quad D_X(f \alpha) = f D_X(\alpha) + X(f) l(\alpha),$$

for all $X \in \mathfrak{X}(M), \alpha \in \Gamma(A)$ and $f \in C^\infty(M)$. Equivalently, we can view $D$ as a linear map $D : \Gamma(A) \rightarrow \Omega^1(M; E)$ satisfying the Leibniz condition $D(f \alpha) = f D(\alpha) + df \otimes l(\alpha)$. If $A$ and $E$ are Lie algebroids and $l : A \rightarrow E$ is a surjective Lie algebroid map, then an $l$-connection induces an $A$-connection on $E$, $\nabla^D : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$, defined by

$$\nabla^D_\alpha(\beta) = [l(\alpha), \beta] + D_{\rho(\beta)}(\alpha), \quad \forall \alpha \in \Gamma(A), \beta \in \Gamma(E).$$

Fixing a section $\alpha \in \Gamma(A)$, the connection induces a Lie derivative operation on $E$-valued forms,

$$\mathcal{L}_\alpha^D : \Omega^*(M; E) \rightarrow \Omega^*(M; E).$$

We will only need the formula for 1-forms, which is

$$\mathcal{L}_\alpha^D \omega(X) = \nabla^D_\alpha(\omega(X)) - \omega([\rho(\alpha), X]), \quad \forall \omega \in \Omega^1(M; E).$$

For the formula for arbitrary degrees, see [14].

**Definition A.44.** A Lie-Pfaffian algebroid over a manifold $M$ consists of a pair of Lie algebroids $A$ and $E$ over $M$, a surjective Lie algebroid map $l : A \rightarrow E$ and an $l$-connection (called the Spencer operator)

$$D : \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(E),$$

such that

$$D([\alpha, \beta]) = \mathcal{L}_\alpha^D(D(\beta)) - \mathcal{L}_\beta^D(D(\alpha)), \quad \forall \alpha, \beta \in \Gamma(A). \quad (A.63)$$

A (local) holonomic section of $(A, D)$ is a (local) section $\alpha \in \Gamma(A)$ such that $D(\alpha) = 0$.

**Remark A.45.** Due to (A.63), the $A$-connection $\nabla^D$ is flat and $E$ is a representation of $A$. 58
A Lie-Pfaffian groupoid \((G, \omega)\) induces a Lie-Pfaffian algebroid as its infinitesimal counterpart as follows. We denote the Lie algebroid of \(G\) by \(A = A(G)\), and set \(l : A \rightarrow E\) to be the restriction of \(\omega \in \Omega^1(G; t^*E)\) to \(A\). The form \(\omega\) induces an \(l\)-connection

\[
D = D_\omega : \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(E)
\]

by the differentiation formula \((A.51)\). The following differentiation/integrability theorem is due to [10] (Theorem 6.1.23 and Proposition 6.1.25):

**Theorem A.46.** If \((G, \omega)\) is a Lie-Pfaffian groupoid, then \((A(G), D_\omega)\) is a Lie-Pfaffian algebroid. Conversely, if \((A, D)\) is a Lie-Pfaffian algebroid such that \(A\) is integrable and \(G\) is the \(s\)-simply connected integration of \(A\), then there is a unique Lie-Pfaffian groupoid \((G, \omega)\) integrating \((A, D)\).

**Example A.47** (Jet algebroids). Corresponding to Example \(A.41\) the jet algebroid \(A^k M\) of a manifold \(M\), equipped with the Spencer operator \(D : \mathfrak{X}(M) \times \Gamma(A^k M) \rightarrow \Gamma(A^{k-1} M)\), is a Lie-Pfaffian algebroid.

The following lemma provides a useful formula for computing the Spencer operator induced by a Lie-Pfaffian groupoid:

**Lemma A.48.** Let \((G, \omega)\) be a Lie-Pfaffian groupoid, and let \((A, D)\) be its Lie-Pfaffian algebroid. Then

\[
\omega([\hat{X}, \tilde{\alpha}]) = t^*(DX(\alpha)),
\]

(A.64)

for all \(\alpha \in \Gamma(A)\) and \(X \in \mathfrak{X}(M)\), where \(\tilde{\alpha} \in \mathfrak{X}(G)\) is the right invariant vector field induced by \(\alpha\) and \(\hat{X} \in \mathfrak{X}(G)\) is a lift of \(X\) that satisfies \(dt(\hat{X}) = X\) and \(\omega(\hat{X}) = 0\).

**Proof.** First note that a lift \(\hat{X}\) always exist by Lemma \(A.42\) together with the fact that \((A.60)\) admits splittings. Let us write \(x = t(g)\). One easily shows that \(\varphi_\alpha^g(x) = \varphi_\alpha^g(x) \cdot g\), and hence, by replacing \(g\) with a curve representing \(X_g\), we see that \(d\varphi_\alpha^g(X_g) = dm(d\varphi_\alpha^g(X_g), X_g)\). Applying \(\omega\) on both sides and using the multiplicativity of \(\omega\),

\[
\omega(d\varphi_\alpha^g(X_g)) = \omega(d\varphi_\alpha^g(X_g)) + \varphi_\alpha^g(x) \cdot \omega(X_g)).
\]

With these identities, the right hand side of \((A.51)\) can be re-expressed as

\[
\varphi_\alpha^g(x)^{-1} \cdot \omega(d\varphi_\alpha^g(X_g)) = g \cdot \varphi_\alpha^g(g)^{-1} \cdot \omega(d\varphi_\alpha^g(X_g)).
\]

Using this,

\[
DX(\alpha)_x = \frac{d}{d\epsilon}_{\epsilon=0} \varphi_\alpha^g(x)^{-1} \cdot \omega(d\varphi_\alpha^g(X_g)) = \frac{d}{d\epsilon}_{\epsilon=0} g \cdot \varphi_\alpha^g(g)^{-1} \cdot \omega(d\varphi_\alpha^g(X_g))
\]

\[=
\lim_{\epsilon \rightarrow 0} g \cdot \varphi_\alpha^g(g)^{-1} \cdot \omega(d\varphi_\alpha^g(X_g)) - \omega(\hat{X}_g)
\]

\[=
\lim_{\epsilon \rightarrow 0} g \cdot \varphi_\alpha^g(g)^{-1} \cdot \omega(d\varphi_\alpha^g(\hat{X}_g)) - \omega(\varphi_\alpha^g \cdot \hat{X}_g(g))
\]

\[=
\lim_{\epsilon \rightarrow 0} g \cdot \varphi_\alpha^g(g)^{-1} \cdot \omega(\varphi_\alpha^g(\hat{X}_g(g))) - \omega(\varphi_\alpha^g \cdot \hat{X}_g(g))
\]

\[=
- \omega( \frac{d}{d\epsilon}_{\epsilon=0} \varphi_\alpha^g \cdot \hat{X}_g(g) ) = \omega([\hat{X}, \tilde{\alpha}])
\]

In the fourth equality, \(\hat{X}_g\) can be replaced by \(d\varphi_\alpha^g \cdot \hat{X}_g(g)\) since they coincide in the limit. \(\square\)
A.2.4  The Symbol Space and the Symbol Map

Let \((A,D)\) be a Lie-Pfaffian algebroid over \(M\). The kernel of \(l,\)
\[
\sigma = \sigma(A,D) := \text{Ker } (l : A \to E) \subset A \tag{A.65}
\]
is called the symbol space of \((A,D)\). Being the kernel of a Lie algebroid morphism, it has
the structure of a bundle of Lie algebras. Given a Lie-Pfaffian groupoid \((G,\omega)\) with associated
Lie-Pfaffian algebroid \((A,D)\), we define its symbol space to be the symbol space of \((A,D)\), thus
\[
\sigma = \sigma(G,\omega) := \sigma(A,D). \tag{A.66}
\]

The restriction of the Spencer operator to the symbol space induces a map
\[
\partial = \partial_D: \sigma(A) \to \text{Hom}(TM,E), \quad T \mapsto (\hat{T} : X \mapsto D_X(T)), \tag{A.67}
\]
called the symbol map. The pair \((\sigma,\partial)\) is a tableau bundle, in the sense of Definition A.31,
and hence we can construct its prolongations and Spencer cohomology (see Section A.1.6).

Definition A.49. A Lie-Pfaffian algebroid \((A,D)\) is standard if its symbol map is injective.
A Lie-Pfaffian groupoid \((G,\omega)\) is standard if its associated Lie-Pfaffian algebroid is standard.

Example A.50. The Lie-Pfaffian groupoids coming from jet groupoids (Example A.41) and
from Lie pseudogroups (Section 3.1.3) are standard, as well as their associated Lie-Pfaffian
algebroids.

A.2.5  The Differential of the Cartan Form

The main problem in the study of PDEs is that of integrability, i.e. the existence of solutions.
Obstructions to integrability are obtained by looking at what are known as prolongations of
PDEs, which are first and higher order differential consequences of the equations. We also think
of the construction of prolongations as the construction of formal solutions of the equations (see
Remark A.57). In the framework of Lie-Pfaffian groupoids, prolongations are encoded in the
differential of the Cartan form, as we now explain.

While the differential of a vector bundle-valued form is not canonically defined, its restriction
to the kernel of the form is, and it is precisely this part that contains the relevant information.
Let \((G,\omega)\) be a Lie-Pfaffian groupoid, with \(\omega \in \Omega^1(G; t^*E)\) and \(E\) a representation of \(G\).
A choice of a connection \(\nabla\) on \(E\) induces a de-Rham type operator \(d_\nabla\) on the space of \(E\)-valued forms (by the usual Koszul-type formula), and we denote the restriction of \(d_\nabla\omega \in \Omega^2(G; t^*E)\)
to the Cartan distribution \(C_\omega\), the kernel of \(\omega\), by
\[
\delta \omega := d_\nabla \omega|_{C_\omega} : \Lambda^2 C_\omega \to t^*E. \tag{A.68}
\]
At the level of sections,
\[
\delta \omega(X,Y) = -\omega([X,Y]), \quad \forall X, Y \in \Gamma(C_\omega).
\]
As a corollary of Lemma A.48 we obtain the following formula that relates the symbol map
with the differential of the Cartan form:

Lemma A.51. Let \((G,\omega)\) be a Lie-Pfaffian groupoid over \(M\). Then
\[
\delta \omega(T^r, \hat{X}) = t^*(\delta(T)(X)), \tag{A.69}
\]
for all $T \in \Gamma(\sigma)$ and $X \in \mathfrak{X}(M)$, where $\tilde{T} \in \mathfrak{X}(G)$ is the right invariant vector field induced by $T$ and $\tilde{X} \in \mathfrak{X}(G)$ is a lift of $X$ that satisfies $dt(\tilde{X}) = X$ and $\omega(\tilde{X}) = 0$. Because $\delta \omega$ is a tensor, \hfill \text{(A.69)} \text{ holds pointwise.}

The prolongation of a Lie-Pfaffian groupoid, which we call the classical prolongation, is the space of all first order solutions. Recall that a local solution (or a local holonomic bisection) $b \in \text{Bis}_{\text{loc}}(G, \omega)$ of a Lie-Pfaffian groupoid $(G, \omega)$ over $M$ is a bisection $b$ that satisfies

$$b^* \omega = 0. \quad \text{(A.70)}$$

As a consequence of \text{(A.70)}, the first order approximation of a local solution $b$ at a point $x \in \text{Dom}(b) \subset M$, i.e., its differential $\xi := (db)_x : T_x M \to T_{b(x)} G$, satisfies the equations

$$\xi^* \omega = 0 \quad \text{and} \quad \xi^* \delta \omega = 0. \quad \text{(A.71)}$$

We call a linear map $\xi = (db)_x : T_{s(x)} M \to T_{g(x)} G$ satisfying \text{(A.71)}, with $b \in \text{Bis}_{\text{loc}}(G)$ and $x \in \text{Dom}(b) \subset M$, a 1st order solution of $(G, \omega)$. Of course, there may be first order solutions that don’t arise as the differentials of local solutions, but the existence of a first order solutions is a necessary condition for the existence of local solutions.

**Definition A.52.** Let $(G, \omega)$ be a Lie-Pfaffian groupoid. The space of all 1st order solutions,

$$P_\omega(G) := \{ \xi = (db)_x | b \in \text{Bis}_{\text{loc}}(G), \ x \in \text{Dom}(b), \xi^* \omega = 0, \xi^* \delta \omega = 0 \},$$

is called the classical prolongation of $(G, \omega)$.

Relaxing the two conditions in \text{(A.71)} one by one, we get two inclusions,

$$P_\omega(G) \subset J^1_{\omega} G \subset J^1 G,$$

where

$$J^1_{\omega}(G) := \{ \xi = (db)_x | b \in \text{Bis}_{\text{loc}}(G), \ x \in \text{Dom}(b), \xi^* \omega = 0 \} \quad \text{(A.72)}$$

is called the partial prolongation of $(G, \omega)$, and

$$J^1 G = \{ (db)_x | b \in \text{Bis}_{\text{loc}}(G), \ x \in \text{Dom}(b) \} \quad \text{(A.73)}$$

is precisely the 1st jet groupoid of local bisections of $G$ (since first jets $j^1_{\omega} b$ of local bisections are canonically identified with the differentials $(db)_x$ of local bisections). The classical prolongation inherits its structure from these ambient spaces, as we explain in the next two sections.

**A.2.6 The Partial Prolongation and its Affine Structure**

Let us examine more closely the partial prolongation $J^1_{\omega} G$ of a Lie-Pfaffian groupoid $(G, \omega)$. Recall first that the 1st jet groupoid $J^1 G \Rightarrow M$ of a Lie groupoid $G \Rightarrow M$, as defined in \text{(A.73)} (and see also Remark \text{A.28}), is a Lie groupoid over $M$, generalizing the 1st jet groupoid of a manifold. The source and target maps send $(db)_x$ to $x$ and $\phi_b(x)$, where $\phi_b = t \circ b$, multiplication is induced by the composition of local bisections, i.e., $(db')_y.(db)_x := (d(b' \cdot b))_x$, and the inverse and unit maps are induced by the inverse operation on bisections and the identity bisection. The smooth structure is the usual one for jet spaces, and the projection,

$$\pi : J^1 G \to G, \quad (db)_x \mapsto b(x), \quad \text{(A.74)}$$
is a surjective Lie groupoid morphism and a submersion. The Cartan form,
\[ \omega \in \Omega^1(J^1G; t^*A), \]  
(85)
which takes with values in the Lie algebroid \( A \) of \( G \), is defined by the formula (c.f. (47))
\[ \omega_{(db)x} := dR_{b(x)}^{-1} \cdot (d\pi - (db)_x \circ ds)_{(db)x}. \]
Also here, \( A \) is a representation of \( J^1G \), the adjoint representation (the action is given by (45)), and \( \omega \) is multiplicative. With this structure, \( (J^1G, \omega) \) is a Lie-Pfaffian groupoid.

The partial prolongation \( J^1_s G \subset J^1G \) inherits this Lie-Pfaffian groupoid structure. The main step in showing this is to show that this inclusion is smooth. This follows from the following important observation: the restriction of the projection (76), which we also denote by
\[ \pi : J^1_G \to \mathcal{G}, \]  
(76)
has the structure of an affine bundle modeled on \( t^*\text{Hom}(TM, \sigma) \), where \( \sigma \) is the symbol space of \( (\mathcal{G}, \omega) \). To describe this structure, recall that, by Lemma 42 and (60), we have canonical isomorphisms \( s^*TM \cong t^*TM \) and \( C_\omega \cap \text{Ker} \, ds \cong t^*\sigma \) of vector bundles over \( \mathcal{G} \). Together, these give the identification
\[ t^*\text{Hom}(TM, \sigma) \cong \text{Hom}(s^*TM, C_\omega \cap \text{Ker} \, ds). \]  
(77)
Let us first describe the affine space structure of a single fiber of (76) over a point \( g \in \mathcal{G} \). The difference \( (db)'_{s(g)} - (db)_{s(g)} \) of two points in the fiber is a linear map \( T_{s(g)}M \to T_gG \) which takes values in \( C_\omega \cap \text{Ker} \, ds \), and hence, modulo (77), belongs to \( \text{Hom}(s_{(g)})(TM, \sigma) \). Conversely, if \( (db)_{s(g)} \) is in this fiber and \( \zeta \in \text{Hom}(s_{(g)})(TM, \sigma) \), then the sum \( (db)_{s(g)} + \zeta \) is again in this fiber, where axiom 2 of Definition 39 ensures that the composition of \( (db)_{s(g)} + \zeta \) with \( dt \) is a linear isomorphism, and hence \( (db)_{s(g)} + \zeta \) comes from a local bisection. We thus have a collection of affine spaces parametrized by \( \mathcal{G} \). These, in turn, glue smoothly to an affine bundle, since (76) has a smooth global section (smooth as a section of (74)). Indeed, a section of (76) is the same thing as a splitting of (60), which always exists by axiom 1 of Definition 39 (see also Remark 40).

Finally, equipped with the restriction of (75), which we also denote by
\[ \omega \in \Omega^1(J^1_G; t^*A), \]  
(78)
it is not hard to show that the partial prolongation is a Lie-Pfaffian groupoid over \( M \). For example, the fact that the restrictions of the multiplication and inverse maps are well-defined and that the unit map is surjective is a consequence of the multiplicativity of \( \omega \). We refer to Section 8.2.3 in [40] for more details. To summarize:

**Proposition A.53.** Let \( (\mathcal{G}, \omega) \) be a Lie-Pfaffian groupoid. The partial prolongation \( (J^1_s G, \omega) \) is a Lie-Pfaffian groupoid and the projection (70) is an affine bundle modeled on \( t^*\text{Hom}(TM, \sigma) \) and a morphism of Lie groupoids.

### A.2.7 The Classical Prolongation and its Affine Structure

While the partial prolongation \( J^1_s G \) of a Lie-Pfaffian groupoid \( (\mathcal{G}, \omega) \) is always smooth, the classical prolongation \( P_s(\mathcal{G}) \) may fail to be, or, more precisely, it may fail to be an affine subbundle of \( J^1_s(\mathcal{G}) \). Understanding when it is smooth is a first step in the problem of integrability. When it is smooth, it is not hard to show that it is a Lie subgrouppoid of \( J^1_s G \) and a Lie-Pfaffian groupoid, when equipped with the restriction of (78).
Consider the projection 
\[ \pi : P_\omega(G) \to G, \] 
the restriction of \( (A.79) \), and recall that \( \sigma^{(1)} \) denotes the 1st prolongation of the symbol space \( \sigma \) (see Section \( A.14 \)).

**Lemma A.54.** Let \((G, \omega)\) be a Lie-Pfaffian groupoid. If the fiber of \( (A.79) \) at \( g \in G \) is non-empty, then it is an affine subspace of the respective fiber of \( (A.76) \), modeled on the vector subspace \( \sigma^{(1)}_{t(g)} \subset \text{Hom}_{t(g)}(TM, \sigma) \).

**Proof.** Fix \( g \in G \). To simplify notation, we treat the identifications \( s^*TM \cong t^*TM \), \( C_\omega \cap \ker ds \cong t^*\sigma \) and \( (A.77) \) as equalities. First, let \( \xi, \xi' \in P_\omega(G) \) in the fiber over \( g \). We prove that, for every \( X, Y \in T_{1(g)}M \) the difference \( \xi' - \xi \), which is a priori an element of \( \text{Hom}_{t(g)}(TM, \sigma) \), is an element of \( \sigma^{(1)} \):

\[ \partial((\xi' - \xi)(X))(Y) = \delta \omega((\xi' - \xi)(X), \xi(Y)) = \delta \omega(\xi(X), \xi(Y)) \]
\[ = \delta \omega(\xi(X), (\xi - \xi')(Y)) + \delta \omega((\xi' - \xi)(Y), \xi'(X)) = \partial((\xi' - \xi)(Y))(X). \]

Lemma \( A.51 \) was used in the first and last equality, the fact that \( \xi^*\delta \omega = \xi'^*\delta \omega = 0 \) (since they are elements of \( P_\omega(G) \)) in the second and third, and anti-symmetry of \( \delta \omega \) in the fourth.

Next, let \( \xi \in P_\omega(G) \) in the fiber over \( g \) and let \( \zeta \in \sigma^{(1)}_{t(g)} \). We know already that \( \xi + \zeta \in J_1^2G \), in the fiber over \( g \), and we prove that it is in \( P_\omega(G) \). Clearly, \( (\xi + \zeta)^*\omega = 0 \). Furthermore, \( (\xi + \zeta)^*\delta \omega = 0 \), since for every \( X, Y \in T_{1(g)}M \),

\[ \delta \omega((\xi + \zeta)(X), (\xi + \zeta)(Y)) \]
\[ = \delta \omega(\xi(X), \xi(Y)) + \delta \omega(\zeta(X), \xi(Y)) - \delta \omega(\zeta(Y), \xi(X)) + \delta \omega(\zeta(X), \zeta(Y)) \]
\[ = 0, \]

where the first term vanishes because \( \xi^*\delta \omega = 0 \), the sum of the second and third are equal to \( \partial((\zeta(X))(Y) - \partial((\zeta(Y))(X)) \), which vanish because \( \xi \in \sigma^{(1)} \), and the third vanishes because \( C_\omega \cap \ker ds \) is involutive. \( \square \)

So we can conclude that:

**Proposition A.55.** Let \((G, \omega)\) be a Lie-Pfaffian groupoid over \( M \). The pair \((P_\omega(G), \omega)\) is a Lie-Pfaffian groupoid and the projection \( (A.79) \) is an affine bundle modeled on \( t^*\sigma^{(1)} \) and a morphism of Lie groupoids if and only if \( \sigma^{(1)} \) is of constant rank and \( (A.79) \) has a smooth global section (smooth as a section of \( (A.76) \)).

**Remark A.56.** In \([40]\), it is shown that \( (A.79) \) is, in fact, a morphism of Lie-Pfaffian groupoids, in a sense that they make precise. Furthermore, they define an abstract notion of a Lie prolongation, which, roughly speaking, is a morphism \( p : (\tilde{G}, \tilde{\omega}) \to (G, \omega) \) of Lie-Pfaffian groupoids such that \( \tilde{\omega} \) “extends” \( \omega \). It is then proven that \( (A.79) \) is a Lie prolongation and that it is “universal” (see Proposition 6.2.42 in \([40]\) for the precise statement).

**Remark A.57.** In the study of formal integrability, the classical prolongation of a Lie-Pfaffian groupoid \((G, \omega)\) is also called the \( 1st \) prolongation. If it is smooth, i.e., it is a Lie-Pfaffian groupoid and the projection is an affine bundle, then we can proceed and construct its classical prolongation, which is called the \( 2nd \) prolongation of \((G, \omega)\). Its elements correspond to \( 2nd \) order solutions of \((G, \omega)\). Proceeding inductively (where at each step there may be obstructions to smoothness), we obtain, at the \( k \)-th step, the \( k-th \) prolongation of \((G, \omega)\) consisting of \( k-th \)
order solutions. If there are no obstructions and we can continue indefinitely, then the inverse limit of the resulting infinite tower of prolongations is called the \( \infty \) prolongation of \( (\mathcal{G}, \omega) \) and its elements correspond to formal solutions. In this case, \( (\mathcal{G}, \omega) \) is said to be formally integrable. Theorem 6.3.13 in [40] gives criteria for formal integrability.

A.2.8 Cartan-Ehresmann Connections

In the previous section, we saw that the existence of sections of the projection \( \pi : \mathcal{P}_k(\mathcal{G}) \to \mathcal{G} \) is an obstruction to prolongation and to formal integrability. Geometrically, such sections can be interpreted as special type of connections. We first note that sections of \( \pi : \mathcal{J}^k_2\mathcal{G} \to \mathcal{G} \) are the same thing as splittings of the vector bundle map \( \mathcal{A.50} \), i.e. splittings \( H \) of \( ds : T\mathcal{G} \to s^*TM \) that satisfy the condition \( H^*\omega = 0 \). In other words, they are are Ehresmann connections that take value in the Cartan distribution. These, in turn, correspond to sections of the classical prolongation if and only if they satisfy the extra condition \( H^*\delta\omega = 0 \). This motivates the following terminology:

**Definition A.58.** Let \( (\mathcal{G}, \omega) \) be a Lie-Pfaffian groupoid over \( M \). A **Cartan-Ehresmann connection** is a splitting

\[
H : s^*TM \to C_\omega
\]

of \( \mathcal{A.50} \). It is said to be **integral** if \( H^*\delta\omega = 0 \).

Cartan-Ehresmann connections will play an important role in our proof of Cartan’s Second Fundamental Theorem (Section 3). In particular, the following fact will be of importance:

**Lemma A.59.** Let \( \Gamma \) be a Lie pseudogroup on \( M \) of order \( k \). The Lie-Pfaffian groupoid \( (\mathcal{J}^k\Gamma, \omega) \) admits an integral Cartan-Ehresmann connection.

**Remark A.60.** Since the first prolongation \( (\sigma^k\Gamma)^{(1)} \) of the symbol space of \( (\mathcal{J}^k\Gamma, \omega) \) is of constant rank by assumption (see Definition 3.3), then \( (\mathcal{J}^k\Gamma, \omega) \) satisfies the conditions of Proposition \( \mathcal{A.32} \) and, hence, its classical prolongation \( \mathcal{P}_x(\mathcal{J}^k\Gamma, \omega) \) is a Lie-Pfaffian groupoid.

**Proof.** The key idea is that the classical prolongation \( \mathcal{P}_x(\mathcal{J}^k\Gamma) \) is the intersection of two affine subbundles of an affine bundle, and, in general, the intersection of two affine subbundles is again an affine bundle if and only if the intersection is non-empty in each fiber and if the intersection of the modeling vector bundles is of constant rank (see Proposition 1.1.6 in [37]). Indeed, \( \mathcal{P}_x(\mathcal{J}^k\Gamma) \) is the intersection of \( \mathcal{J}^1(\mathcal{J}^k\Gamma) \to \mathcal{J}^k\Gamma \) (the first jets of sections of the source map of \( \mathcal{J}^k\Gamma \)) and the restriction of \( \pi : \mathcal{J}^{k+1}M \to \mathcal{J}^kM \) to \( \mathcal{J}^k\Gamma \subset \mathcal{J}^kM \). Both of these are affine subbundles of the restriction of \( \pi : \mathcal{J}^1(\mathcal{J}^kM) \to \mathcal{J}^kM \) to \( \mathcal{J}^k\Gamma \). Now, by the general fact above, this intersection is an affine bundle because the intersection of the modeling vector bundles is precisely \( (\sigma^k\Gamma)^{(1)} \), which is assumed to be of constant rank, and each fiber over \( j^k_\phi \in \mathcal{J}^k\Gamma \) contains at least one point \( j^{k+1}_\phi \), where \( \phi \in \Gamma \) is some representative of \( j^k_\phi \in \mathcal{J}^k\Gamma \) (thus, the main ingredient in this proof is that the PDE \( \mathcal{J}^k\Gamma \) contains a solution through each point). Finally, since \( \mathcal{P}_x(\mathcal{J}^k\Gamma) \to \mathcal{J}^k\Gamma \) has sections (any affine bundle does), then integral Cartan-Ehresmann connections exist.

**References**

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor’kova, I. S. Krasil’shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. Symmetries and conservation laws for differential equations of mathematical physics, volume 182 of Translations of Mathematical Monographs. American Mathematical Society, Providence,
RI, 1999. Edited and with a preface by Krasil’shchik and Vinogradov, Translated from the 1997 Russian original by Verbovetsky [A. M. Verbovetski˘ı] and Krasil’shchik.

[2] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991.

[3] Élie Cartan. Sur la structure des groupes infinis de transformation. *Ann. Sci. École Norm. Sup. (3)*, 21:153–206, 1904.

[4] Élie Cartan. Sur la structure des groupes infinis de transformation (suite). *Ann. Sci. École Norm. Sup. (3)*, 22:219–308, 1905.

[5] Élie Cartan. La structure des groupes infinis. *Séminaire de Math.*, 4e année, 1936-1937.

[6] Élie Cartan. La structure des groupes infinis (suite). *Séminaire de Math.*, 4e année, 1936-1937.

[7] Francesco Cattafi. Thesis in progress. Thesis.

[8] Shiing-Shen Chern and Claude Chevalley. Obituary: Elie Cartan and his mathematical work. *Bull. Amer. Math. Soc.*, 58:217–250, 1952.

[9] M. Crainic and R. L. Fernandes. Secondary characteristic classes of Lie algebroids. In *Quantum field theory and noncommutative geometry*, volume 662 of *Lecture Notes in Phys.*, pages 157–176. Springer, Berlin, 2005.

[10] Marius Crainic. Mastermath course differential geometry (lecture notes), (available for download at http://www.staff.science.uu.nl/~crain101/dg-2015/main10.pdf), 2015.

[11] Marius Crainic and Rui Loja Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.

[12] Marius Crainic and Rui Loja Fernandes. Lectures on integrability of Lie brackets. In *Lectures on Poisson geometry*, volume 17 of *Geom. Topol. Monogr.*, pages 1–107. Geom. Topol. Publ., Coventry, 2011.

[13] Marius Crainic and Maria Amelia Salazar. Pfaffian groupoids, In preparation.

[14] Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Math. Z.*, 279(3-4):939–979, 2015.

[15] Werner Greub, Stephen Halperin, and Ray Vanstone. *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Pure and Applied Mathematics, Vol. 47-II.

[16] Victor Guillemin and Shlomo Sternberg. Deformation theory of pseudogroup structures. *Mem. Amer. Math. Soc. No.*, 64:80, 1966.

[17] Victor W. Guillemin and Shlomo Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.*, 70:16–47, 1964.

[18] André Haefliger. Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. *Comment. Math. Helv.*, 32:248–329, 1958.
[19] Niky Kamran. Contributions to the study of the equivalence problem of Élie Cartan and its applications to partial and ordinary differential equations. *Acad. Roy. Belg. Cl. Sci. Mém. Collect.* 8° (2), 45(7):122, 1989.

[20] Niky Kamran and Thierry Robart. An infinite-dimensional manifold structure for analytic Lie pseudogroups of infinite type. *Int. Math. Res. Not.*, (34):1761–1783, 2004.

[21] A. Kumpera. A theorem on Cartan pseudogroups. In *Topologie et Géométrie Différentielle (Séminaire Ch. Ehresmann, Vol. VI, 1964)*, page 12. Inst. Henri Poincaré, Paris, 1964.

[22] A. Kumpera. Invariants différentiels d’un pseudogroupe de Lie. i. *J. Differential Geometry*, 10(2):289–345, 1975.

[23] Masatake Kuranishi. On the local theory of continuous infinite pseudo groups. I. *Nagoya Math. J.*, 15:225–260, 1959.

[24] Masatake Kuranishi. On the local theory of continuous infinite pseudo groups. II. *Nagoya Math. J.*, 19:55–91, 1961.

[25] Paulette Libermann. Sur le problème d’équivalence de certaines structures infinitésimales. *Ann. Mat. Pura Appl. (4)*, 36:27–120, 1954.

[26] Paulette Libermann. Charles Ehresmann’s concepts in differential geometry. In *Geometry and topology of manifolds*, volume 76 of *Banach Center Publ.*, pages 35–50. Polish Acad. Sci., Warsaw, 2007.

[27] Sophus Lie and Friedrich Engel. *Theorie der Transformationsgruppen*, volume 1-3. B.G. Teubner, Leipzig, 1888-93.

[28] Ian G. Lisle and Gregory J. Reid. Cartan structure of infinite Lie pseudogroups. In *Geometric approaches to differential equations (Canberra, 1995)*, volume 15 of *Austral. Math. Soc. Lect. Ser.*, pages 116–145. Cambridge Univ. Press, Cambridge, 2000.

[29] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[30] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[31] Bernard Malgrange. Equations de Lie. I. *J. Differential Geometry*, 6:503–522, 1972. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

[32] Bernard Malgrange. Equations de Lie. II. *J. Differential Geometry*, 7:117–141, 1972.

[33] Y. Matsushima. Pseudo-groupes de Lie transitifs. In *Séminaire Bourbaki, Vol. 3*, pages Exp. No. 118, 183–196. Soc. Math. France, Paris, 1995.

[34] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[35] Peter J. Olver. *Equivalence, invariants, and symmetry*. Cambridge University Press, Cambridge, 1995.
[36] Peter J. Olver and Juha Pohjanpelto. Maurer-Cartan forms and the structure of Lie pseudo-groups. *Selecta Math. (N.S.)*, 11(1):99–126, 2005.

[37] Peter J. Olver, Juha Pohjanpelto, and Francis Valiquette. On the structure of Lie pseudo-groups. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 5:Paper 077, 14, 2009.

[38] A. M. Rodrigues. The first and second fundamental theorems of Lie for Lie pseudo groups. *Amer. J. Math.*, 84:265–282, 1962.

[39] A. M. Rodrigues. On Cartan pseudo groups. *Nagoya Math. J.*, 23:1–4, 1963.

[40] Maria Amelia Salazar. Pfaffian groupoids, 2013. Thesis.

[41] D. J. Saunders. *The geometry of jet bundles*, volume 142 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1989.

[42] I. M. Singer and Shlomo Sternberg. The infinite groups of Lie and Cartan. I. The transitive groups. *J. Analyse Math.*, 15:1–114, 1965.

[43] D. C. Spencer. Deformation of structures on manifolds defined by transitive, continuous pseudogroups. I. Infinitesimal deformations of structure. *Ann. of Math. (2)*, 76:306–398, 1962.

[44] Shlomo Sternberg. *Lectures on differential geometry*. Chelsea Publishing Co., New York, second edition, 1983. With an appendix by Sternberg and Victor W. Guillemin.

[45] Olle Stormark. *Lie’s structural approach to PDE systems*, volume 80 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000.

[46] Francis Valiquette. Structure equations of Lie pseudo-groups. *J. Lie Theory*, 18(4):869–895, 2008.

[47] Ori Yudilevich. Lie pseudogroups à la cartan from a modern perspective (thesis).

[48] Ori Yudilevich. The role of the Jacobi identity in solving the Maurer–Cartan structure equation. *Pacific J. Math.*, 282(2):487–510, 2016.