On Time-Dependant Symmetries of Schrödinger Equation

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We show that the number of symmetry operators of order not higher than q of the nonstationary n-dimensional \((n \leq 4)\) Schrödinger equation (SE) with nonvanishing potentials is finite and does not exceed that of free SE with zero potentials for arbitrary \(q = 0, 1, 2, \ldots\). This result is applied for the determination of the general form of time dependance of the symmetry operators of SE with time-independant potentials.

Introduction

In this paper we consider (nonstationary n-dimensional) Schrödinger equation (SE) of the form

\[
L \Psi \equiv (i \partial_t - \frac{1}{2}((p - eA(r, t))^2 + V(r, t))) \Psi = 0
\]  

(1)

where \(p_a = -i \partial / \partial x_a \equiv -i \partial^a, \; a = 1, \ldots, n\); \(p^2 \equiv \sum_{a=1}^{n} p_a p_a, \; r = (x_1, \ldots, x_n)\); \(t, x_a \in \mathbb{R}\); bold letters denote n-dimensional vectors.

We will study the properties of the symmetry operators of \([L, Q]\), i.e., of the linear differential operators of the form

\[
Q = \sum_{j=0}^{q} [\ldots [F^{a_1, \ldots, a_j}(r, t), p_{a_1}] + \ldots p_{a_j}] + \sum_{j=0}^{q} b^{a_1, \ldots, a_j} \partial^{a_1} \ldots \partial^{a_j},
\]  

(2)

such that

\[
[L, Q] = 0.
\]  

(3)

Here \([A, B]_+ = AB + BA, \; F^{a_1, \ldots, a_j} \) and \(b^{a_1, \ldots, a_j}\) are symmetric with respect to the indices \(a_1, \ldots, a_j\); here and below the summation over the repeated indices of the type \(a_1, \ldots, a_j\) from 1 to \(n\) is understood; the term with \(j = 0\) is simply \(F(r, t)\).

The paper is organized as follows. In Section 1 we evaluate the number of linearly independant time-dependant and time-independant symmetry operators of order not higher than \(q\) of SE, and in Section 2 we apply these results to establish the criterion of existence of time-dependant symmetries for SE with time-independant potentials. Here and below we assume \(n \leq 4\).

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1 The number of symmetries

Let us temporarily restrict ourselves to the case $A = 0$. Then the substitution of (2) into (3) yields (cf. [2]) the following equations:

\[ \partial (a_1^{q+1} F_{a_1^{a_1}, \ldots, a_1^{a_q}} + 1 F_{a_1^{a_1}, \ldots, a_1^{a_q}}) = 0, \] (4)

\[ \partial (a_q F_{a_1^{a_1}, \ldots, a_q - 1} + 2 F_{a_1^{a_1}, \ldots, a_q}) = 0, \] (5)

\[ \partial (a_{q-m} F_{a_1^{a_1}, \ldots, a_{q-m-1}} + 2 F_{a_1^{a_1}, \ldots, a_{q-m}} + \sum_{j=0}^{[(m-1)/2]} \frac{2(-1)^{j+1}(q-m+2j+1)!}{(q-m+1)!(2j+1)!} F_{a_1^{a_1}, \ldots, a_{q-m}, b_{2j+1}} \partial^{b_{2j+1}} V = 0, \] (6)

\[ \partial (a_{q-m} F_{a_1^{a_1}, \ldots, a_{q-m-1}} + 2 F_{a_1^{a_1}, \ldots, a_{q-m}} + \sum_{j=0}^{[(m-1)/2]} \frac{2(-1)^{j+1}(q-m+2j+1)!}{(q-m+1)!(2j+1)!} F_{a_1^{a_1}, \ldots, a_{q-m}, b_{2j+1}} \partial^{b_{2j+1}} V = 0. \] (7)

\[ s \] denotes here the integer part of the number $s$ and $(a_1, \ldots, a_k)$ denotes the symmetrisation with respect to the indices $a_1, \ldots, a_k$.

At first, let us show that the general solution of the system (4)–(7) may not contain arbitrary functions of $r$.

Really, let us consider the following system, which consists from equation (4) and the following differential consequences of equations (5), (6):

\[ \partial (a_{q+1} \partial^{a_1^{a_1}, \ldots, a_{q+1}} F_{a_1^{a_1}, \ldots, a_q}) = 0, \] (8)

\[ \partial (a_q \partial^{a_1^{a_1}, \ldots, a_q+1} F_{a_1^{a_1}, \ldots, a_q}) = -2 \partial (a_q \partial^{a_1^{a_1}, \ldots, a_q+2} F_{a_1^{a_1}, \ldots, a_q+1}) \] (9)

\[ \times \sum_{k=0}^{[(q-j-2)/2]} \frac{2(-1)^{k+1}(j+2k+2)!}{(j+2k+1)!(2k+1)!} F_{a_1^{a_1}, \ldots, a_j, b_{2k+1}} \partial^{b_{2k+1}} V, \]

\[ j = 0, \ldots, q - 2. \]

The general solution of the system (4) (and hence, the general solution of (4)–(7)) may be represented in the form

\[ F_{a_1^{a_1}, \ldots, a_q} = F_0^{a_1^{a_1}, \ldots, a_q} + G^{a_1^{a_1}, \ldots, a_q}, \quad j = 0, \ldots, q - 2. \] (10)

where $F_0^{a_1^{a_1}, \ldots, a_q}$ is general solution of the corresponding homogeneous equation

\[ \partial (a_{q-j} \partial^{a_1^{a_1}, \ldots, a_{q-j+1}} F_{a_1^{a_1}, \ldots, a_{q-j+1}}) = 0, \] (11)

i.e. the generalized Killing tensor of rank $j$ and order $q - j + 1$ which is (at least for $n \leq 4$) some polynomial of order $q + 1$ with respect to $r$; the coefficients of this polynomial are arbitrary functions of $t$ (in fact, they are not arbitrary, vide the analysis below), and $G^{a_1^{a_1}, \ldots, a_q}$ is some particular solution of the corresponding inhomogeneous equation from (4).
The representation (10) is valid for $j = q - 1, q$ too, but in this case (as it follows from (4) and (8)) we have

$$G^{a_1, \ldots, a_j} = 0, \quad j = q - 1, q.$$  

(12)

The substitution of (10) (taking into account (11)) into (9) yields the following equations:

$$\partial^{(a_{q+1}} \partial^{a_{j+1}} G^{a_1, \ldots, a_j)} = -2\partial^{(a_{q+1}} \partial^{a_{j+1}} G^{a_1, \ldots, a_{j+1})} -$$

$$-\partial^{(a_{q+1}} \partial^{a_{j+1}} \partial^{a_{j+2}} \sum_{k=0}^{[(q-j-2)/2]} \frac{2((-1)^{k+1}(j+2k+2)!)}{(j+2)!2(2k+1)!} \times$$

$$\times (F_{0}^{a_1, \ldots, a_j}, b_1, \ldots, b_{2k+1} + G^{a_1, \ldots, a_j}, b_1, \ldots, b_{2k+1}) \partial^{b_1} \ldots \partial^{b_{2k+1}} V, \quad j = 0, \ldots, q - 2.$$  

(13)

Let us require (this is obviously possible) that the quantities $G^{a_1, \ldots, a_j}$ must satisfy not only (13) but stronger relations

$$\partial^{(a_{q+1}} \partial^{a_{j+1}} G^{a_1, \ldots, a_j)} = -2\partial^{(a_{q+1}} \partial^{a_{j+1}} G^{a_1, \ldots, a_{j+1})} -$$

$$-\partial^{(a_{q+1}} \partial^{a_{j+1}} \partial^{a_{j+2}} \sum_{k=0}^{[(q-j-2)/2]} \frac{2((-1)^{k+1}(j+2k+2)!)}{(j+2)!2(2k+1)!} \times$$

$$\times (F_{0}^{a_1, \ldots, a_j}, b_1, \ldots, b_{2k+1} + G^{a_1, \ldots, a_j}, b_1, \ldots, b_{2k+1}) \partial^{b_1} \ldots \partial^{b_{2k+1}} V, \quad j = 0, \ldots, q - 2.$$  

(14)

and vanish if the right hand sides of (14) are equal to zero. The comparison of (14) with (4)–(6) yields the following equations:

$$\partial^{(a_{q-m}} F_{0}^{a_1, \ldots, a_{q-m-1})} + 2\partial^{(a_1, \ldots, a_{q-m})} = 0, \quad m = 0, \ldots, q - 1$$  

(15)

and

$$\dot{F}_{0} = -\dot{G} + \sum_{k=0}^{[(q-2)/2]} (-1)^{k}(F_{0}^{b_1, \ldots, b_{2k+1} + G^{b_1, \ldots, b_{2k+1})} \partial^{b_1} \ldots \partial^{b_{2k+1}} V.$$  

(16)

Since the quantities $G^{a_1, \ldots, a_j}$ by construction may not contain arbitrary elements at all (except those which enter in the quantities $F_{0}^{a_1, \ldots, a_{j}}$, of course) and the quantities $F_{0}^{a_1, \ldots, a_{j}}$ may contain only arbitrary functions of $t$, we really have proved that the general solution of the system (4)–(7) may not contain arbitrary functions of $t$.

Let us mention that the representation (10) for the solution of the system (4)–(7), in which the quantities $F_{0}^{a_1, \ldots, a_{j}}$ and $G^{a_1, \ldots, a_{j}}$ satisfy (11), (14), (15), (16), could be written a priori without turning to (10).

The substitution of the expressions for the quantities $F_{0}^{a_1, \ldots, a_{j}}$ via generalized Killing tensors with time-dependant coefficients into (13) and (16) after equating the coefficients at linearly independant generalized Killing tensors will evidently yield the system of first order linear ordinary differential equations (ODEs) with respect to $t$ for these coefficients. Thus, these coefficients may not be arbitrary
functions of the time \( t \), since the general solution of this system of ODEs may contain only arbitrary constants at most in the same number that unknown functions.

Thus, we have proved that the general solution of (4)–(7) contains neither arbitrary functions of \( t \) nor arbitrary functions of \( r \), but only arbitrary constants, and the number of these constants does not exceed the number \( \hat{N}^n_q \) of coefficients of the corresponding generalized Killing tensors. Since the system of equations (13) and (14) is linear, these arbitrary constants will enter in the quantities \( F_{a_1,\ldots,a_j} \) (and hence in \( F^{a_1,\ldots,a_j} \)) linearly. Therefore, their number coincides with the number of linearly independent symmetry operators of (4).

Summing up all the above statements, we see that the number of linearly independent symmetry operators of (4) for arbitrary given potential \( V \) does not exceed

\[
\hat{N}^n_q = \sum_{j=0}^{q} S^n_{j,q} = \frac{(q+n)!(q+n+1)!}{q!(q+1)!n!(n+1)!},
\]

where \( S^n_{j,q} = \frac{(j+n-1)!(q+n)!}{n!(n-1)!j!(q+j)!} \) is the number of arbitrary constants in the general time-independent solution of (11) [2].

The number \( \hat{N}^n_q \) for \( n = 1, 2, 3 \) coincides with the number of linearly independent symmetry operators of SE with \( V = 0, A = 0 \) (and with \( V = \omega r^2, A = 0 \) too) of order not higher than \( q \), found in [3]. Moreover, it is straightforward to check that this coincidence takes place for all \( n = 1, \ldots, 4 \).

Now let us return to the general case of \( A \neq 0 \). Reasoning similarly to the above, we observe that the number of symmetries of (4) of order not higher than \( q \) again does not exceed \( \hat{N}^n_q \), since the general structure of equations for the quantities \( F^{a_1,\ldots,a_j} \), which follow from (8) is similar to (4)–(7): (4) again holds true and instead of (5)–(7) we have

\[
\begin{align*}
\partial^{(a_q} F^{a_1,\ldots,a_{q-1})} + 2 F^{a_1,\ldots,a_q} + \ldots &= 0, \\
\partial^{(a_q-m} F^{a_1,\ldots,a_{q-m-1})} + 2 F^{a_1,\ldots,a_{q-m}} + \ldots &= 0, \\
F + \ldots &= 0,
\end{align*}
\]

where dots denote some terms we need not to know explicitly; their structure is analogous to that of (5)–(7). In particular, for \( m \)-th equation, \( m = 1, \ldots, q \), these terms include only \( F^{a_1,\ldots,a_{q-m+1}} \), \ldots \( F^{a_1,\ldots,a_q} \) but not \( F^{a_1,\ldots,a_j} \) with \( j = 0, \ldots, q-m-1 \). Therefore, we may again represent the general solution of (8) in the form (10), where now the quantities \( G^{a_1,\ldots,a_j} \) satisfy modified version of the inhomogeneous equations (14) and vanish if the right hand side of the corresponding equation is zero. Repeating once more the above considerations, we obtain the following

**Theorem 1** The number of linearly independent symmetry operators of \( n \)-dimensional (\( n \leq 4 \)) SE (4) with any fixed potentials \( V, A \) of order not higher than \( q \) does not exceed \( \hat{N}^n_q \), i.e., the number of linearly independent symmetry operators of SE with \( V = 0, A = 0 \) of order not higher than \( q \).
It is also interesting to evaluate the number of time-independent symme-
tries of (1) for the case when $V = V(r)$, $A = A(r)$, i.e., the number of
time-independent operators $Q$ of the form (2), which commute with the Hamiltonian

$$H \equiv \frac{1}{2}((p - eA(r))^2 + V(r)).$$

(19)

In this case (we again temporarily set $A = 0$) the equations (5)–(7) read

$$\partial^{(a_q} F^{a_1, \ldots, a_{q-1})} = 0,$$

$$\partial^{(a_q-m} F^{a_1, \ldots, a_{q-m-1})} =$$

$$= - \sum_{j=0}^{[(m-1)/2]} \frac{2(-1)^{j+1}(q-m+2j+1)!}{(q-m+1)!(2j+1)!} F^{a_1, \ldots, a_{q-m}, b_1, \ldots, b_{2j+1}} \partial^{b_1} \ldots \partial^{b_{2j+1}} V,$$

$$m = 1, \ldots, q - 1;$$

$$\sum_{j=0}^{[(q-1)/2]} (-1)^{j+1} F^{a_1, \ldots, a_{2j+1}} \partial^{a_1} \ldots \partial^{a_{2j+1}} V = 0.$$  

(20)

Obviously, we may represent the solution of (20) and (4)) in the following form:

$$F^{a_1, \ldots, a_{q-l}} = \tilde{F}^{a_1, \ldots, a_{q-l}, l} + \tilde{G}^{a_1, \ldots, a_{q-l}, l}, \quad l = 0, \ldots, q.$$  

(21)

where now $\tilde{F}^{a_1, \ldots, a_{q-l}}$ is general solution of the corresponding homogeneous equation

$$\partial^{(a_{q-l+1}} F^{a_1, \ldots, a_{q-l})} = 0,$$

i.e. the generalized Killing tensor of rank $q-l$ and order 1 [3] which is (for $n \leq 4$)
some polynomial of order $q-l$ with respect to $r$, containing $K_n = \frac{(j+n-1)!((j+n)!}{j!(j+1)!} m_n$ arbitrary constants [3], and $\tilde{G}^{a_1, \ldots, a_{q-l}}$ is particular solution of the corresponding inhomogeneous equation from (20) ($\tilde{G}^{a_1, \ldots, a_j} = 0$, $j = q - 1, q$ in virtue of (4) and of the first line of (20)).

In complete analogy with the above it is clear that the general solution of the system (20) and (4) may depend at most from

$$\tilde{N}^n = \sum_{j=0}^{q} K^j_n,$$

(23)

arbitrary constants, which again enter in it linearly; it really depends on $\tilde{N}^n$

constants for, e.g., $V = 0, A = 0$.

The explicit calculations show that

$$\tilde{N}^1 = q + 1, \tilde{N}^n = \frac{(q+n+1)!}{q!(2n-1)!} P_n(q), n = 2, 3, 4,$$

where $P_2(q) = 1, P_3(q) = 2q + 5, P_4(q) = 5q^2 + 30q + 42.$  

(24)

In complete analogy with above considerations, we may return to the general case, when $A \neq 0$. Thus, the following statement holds true:

**Theorem 2** For any fixed $V(r)$ and $A(r)$ the number of time-independent operators of the form (4) of order not higher than $q$, which commute with $H$ (19), does not exceed $\tilde{N}^n$, i.e., the same number for $V = 0, A = 0$, for $n = 1, \ldots, 4$.  

5
If we suppose that the quantities $F^{a_1, \ldots, a_j}$, $A, V$ are "generalized" distributions (e.g., in Colombeau’s sense), both our theorems hold true, since all the necessary derivatives exist (of course, as "generalized" distributions) and hence we may apply our results to even such singular potentials as Dirac delta-functions and other distributions.

Let us also mention that our results exploit ad hoc adjusted ideas of Cartan’s theory of compatibility of overdetermined systems of partial differential equations for the system (4) and (18).

2 Time-dependant Symmetries

In this section we consider the case when both $V$ and $A$ are time-independant.

The symmetry operators of the form (2) of (1) may be considered as the elements of Lie algebra $\text{Sym}$ of all the generalized symmetries of SE (cf. [1]), since the Lie algebra (with respect to the usual commutator $[A, B] = AB - BA$) $\text{Lin}$ of all the operators of the form (2) may be homomorphically mapped (with zero kernel) to the Lie algebra $S$ (with respect to Lie bracket) of all the differential operators of the form

$$K = \eta(r, t, \Psi, \ldots, \Psi^{(q)}) \partial/\partial \Psi$$

by mapping $Q(2)$ into

$$Q \rightarrow Q' = \left( \sum_{j=0}^{q} b^{a_1, \ldots, a_j} \partial^{a_1} \ldots \partial^{a_j} \Psi \right) \partial/\partial \Psi$$

(26)

Here $\Psi^{(q)}$ denotes the set of derivatives of $\Psi$ with respect to $r$ of order $q$.

Let $W$ be the image of $\text{Lin}$ under (26). For such a $W$ the dimensions $v^{(q)}$ of the spaces of symmetry operators of order not higher than $q$ $V^{(q)} = W \cap \text{Sym}^{(q)}$ (as in [1], we set $V = W \cap \text{Sym}$) are finite and do not exceed $N^n_q$, as we have established in the Theorem 1 above. Since in the case considered (1) admits Lie symmetry $\partial/\partial t$ and if $R$ is symmetry operator of (1), then so does $\partial R/\partial t$, all the conditions of Theorem 1 from [5] are fulfilled. Therefore, we may seek for all the linearly independant symmetry operators of (1) of order $q$ in the form:

$$R = \exp(\lambda t) \sum_{k=0}^{m} C_k \frac{t^k}{k!}$$

(27)

where $\lambda \in \mathbb{C}$ and $C_k$ are some differential operators from $\text{Lin}$ with time-independant coefficients, $m \leq v^{(q)} - 1$.

Moreover, applying Theorem 2 from [6], we immediately obtain the following result, which holds true for $n \leq 4$:

$^{4}$the products and derivatives of these "generalized" distributions again are some "generalized" distributions.
Theorem 3 \( SE(1) \) with time-independent potentials \( V \) and \( A \) possesses time-dependent symmetry operators if and only if it possesses (at least one) symmetry operator of the form

\[
Q = \exp(\lambda t)K_0, \lambda \in \mathbb{C}, \lambda \neq 0
\]

or of the form

\[
Q = K_0 + tK_1, \ K_1 \neq 0,
\]

where \( K_0, K_1 \) are differential operators from \( \text{Lin} \) with time-independent coefficients.

Now let us write down the commutation relations for \( C_k \), obtained as a result of substitution of (27) into (3):

\[
[H, C_m] = i\lambda C_m; \quad [H, C_l] = i\lambda C_l + iC_{l+1}, l = 0, \ldots, m - 1.
\]

Having excluded \( C_l, l = 1, \ldots, m \), we may rewrite (30) simply as

\[
(-i\text{ad}_H - \lambda)^m C_0 = 0,
\]

where \( \text{ad}_H R \equiv [H, R] \). Similarly, for \( K_0 \) and \( K_1 \) we have

\[
[H, K_0] = i\lambda K_0 \text{ for the case (28)}
\]

\[
[H, K_0] = iK_1, \quad [H, K_1] = 0 \text{ for the case (29)}.
\]

For the case (29) from (33) it follows that

\[
[H, [H, K_0]] = 0,
\]

and hence by analogy with the theory of mastersymmetries of integrable equations [7] it is natural to call \( K_0 \) the mastersymmetry of Schrödinger equation (1).

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