Literature survey on low rank approximation of matrices

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Abstract

Low rank approximation of matrices has been well studied in literature. Singular value decomposition, QR decomposition with column pivoting, rank revealing QR factorization (RRQR), Interpolative decomposition etc are classical deterministic algorithms for low rank approximation. But these techniques are very expensive ($O(n^3)$ operations are required for $n \times n$ matrices). There are several randomized algorithms available in the literature which are not so expensive as the classical techniques (but the complexity is not linear in $n$). So, it is very expensive to construct the low rank approximation of a matrix if the dimension of the matrix is very large. There are alternative techniques like Cross/Skeleton approximation which gives the low-rank approximation with linear complexity in $n$. In this article we review low rank approximation techniques briefly and give extensive references of many techniques.

**Keywords:** Singular value decomposition, Rank, QR decomposition, Spectral norm, Frobenius norm, Complexity, Interpolative decomposition, Subset selection, Randomized algorithm, Sub-sampling, Random Projection, Cross/Skeleton decomposition, Pseudoskeleton approximation, Pseudoinverse, Maximal volume, Adaptive cross approximation, Pivot.

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1 Introduction

The low rank matrix approximation is approximating a matrix by one whose rank is less than that of the original matrix. The goal of this is to obtain more compact representations of the data with limited loss of information. Let $A$ be $m \times n$ matrix, then the low rank approximation (rank $k$) of $A$ is given by

$$A_{m \times n} \approx B_{m \times k}C_{k \times n}.$$ 

The low rank approximation of the matrix can be stored and manipulated more economically than the matrix itself. One can see from the above approximation that only $k(m + n)$ entries have to be stored instead of $mn$ entries of the original matrix $A$.

The low rank approximation of a matrix appears in many applications. The list of applications includes image processing [57, 166], data mining [51, 146], noise reduction, seismic inversion, latent semantic indexing [167], principal component analysis (PCA) [58, 127], machine-learning [102, 120, 161], regularization for ill-posed problems, statistical data analysis applications, DNA microarray data, web search model and so on. The low rank approximation of matrices also plays a very important role in tensor decompositions [52, 70, 90, 91, 92, 93, 101, 125].

Because of the interplay of rank and error there are basically two types of problems related to the low rank approximation of a matrix; fixed-precision approximation problem and fixed-rank approximation problem (we do not use this nomenclature in this article). In the fixed-precision approximation problem, for a given matrix $A$ and a given tolerance $\epsilon$, one wants to find a matrix $B$ with rank $k = k(\epsilon)$ such that $\|A - B\| \leq \epsilon$ in an appropriate matrix norm. On the contrary, in the

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fixed-rank approximation problem, one looks for a matrix $B$ with fixed rank $k$ and an error $\|A - B\|$ as small as possible.

The low rank approximation problem is well studied in the numerical linear algebra community. There are very classical matrix decompositions which gives low rank approximation. Singular value decomposition (SVD) is the best known. It has wide applications in many areas. It provides the true rank and gives the best low rank approximation of a matrix [39, 67, 147, 150]. QR decomposition with column pivoting [144], rank revealing QR factorization [23, 67, 68, 69, 77] and interpolative decomposition [5, 35, 105, 110] are other useful techniques. These techniques require $O(mnk)$ arithmetic operations to get a rank $k$ approximation by at least $k$ passes (the number of times that the entire data is read) through the input matrix. It is not easy to access the data in many applications with very large data. So these methods become unsuitable for large scale data matrices.

Alternatives for these classical algorithms are randomized algorithms for low rank approximations [42, 51, 89, 115, 138, 139]. The complexity of these algorithms is at most sublinear in the size $m \times n$ and they only require one or two passes of the input matrix. The main idea of these randomized algorithms is to compute an approximate basis for the range space of the matrix $A$ using a random selection of columns/rows of $A$ and project $A$ onto the subspace spanned by this basis. We sketch it here:

Let $k$ be the target rank (the aim is to obtain a rank $k$ approximation). Choose a number of samples larger than $k$, i.e $s = k + p$. The randomized low rank approximation constructs the approximation in the following way.

**Step 1:** Form lower dimensional matrix $X$ by the $s$ selected row and/or columns.

**Step 2:** Compute an approximate orthonormal basis $Q = [q_1, q_2, ..., q_k]$ for the range of $X$.

**Step 3:** Construct the low rank approximation $\tilde{A}$ by projecting $A$ onto the space spanned by the basis $Q$: $\tilde{A} = QQ^T A$.

In step 1, the columns/rows can be chosen in different ways: by subsampling of the input matrix or by using random projections. The matrix $X$ formed by these columns is expected to be very close to $A$ in a sense that the basis of the range of $X$ covers the range of $A$ well. The orthonormal basis consisting of $k$ linearly independent vectors can be obtained using exact methods since the size of $X$ is very small. These techniques are relatively insensitive to the quality of randomness and produce high accurate results. The probability of failure is negligible. Using the orthonormal basis $Q$ one can approximate the standard factorizations like SVD, QR etc [51].

There are other approximation techniques available in the literature like cross/skeleton decompositions [2, 62, 74, 151]. Their complexity is of order $O(k^2(m + n))$ and they use only $k(m + n)$ entries from the original matrix to construct a rank $k$ approximation of the matrix. These methods are also very useful in data sparse representation of the higher order tensors. The algorithms that construct different data tensor formats use low rank approximations of matrices at different levels of their construction. These are obtained by the cross/skeleton approximations inexpensively (linear in $m$ and $n$) which also gives the data sparse representation with linear complexity [52, 90, 125].

The main motivation of this paper is to give a brief description of the techniques which are available in the literature. The paper gives an overview of the existing classical deterministic algorithms, randomized algorithms and finally cross/skeleton approximation techniques which have great advantage of being able to handle really large data appearing in applications.

In section 2 the classical algorithms like singular value decomposition, pivoted QR factorization, rank revealing QR factorization (RRQR) are described briefly with relevant references. More emphasize is given to the subset selection problem and interpolative decomposition (these play a big role in skeleton/cross approximation or $CUR$ decomposition which will be discussed in section 3). Various randomized algorithms are also described. In section 3 various versions of so called cross/skeleton approximation techniques are described. The algorithms are given in detail and the computational complexity of them are derived (linear in $n$).

For simplicity of the presentation we consider only the matrix of real numbers. Frobenius norm of a $m \times n$ matrix $A = (a_{ij})$ is defined as the square root of the sum of the absolute squares of its elements i.e
∥A∥_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.

The spectral norm of the matrix A is defined as largest singular value of A, i.e

∥A∥_2 = \sigma_{max}.

Here \sigma_{max} is the largest singular value of the matrix A.

2 Classical techniques and randomized algorithms

2.1 Singular value decomposition

This is a powerful technique in linear algebra. It gives the best low rank approximation of a matrix. As mentioned in [144], the singular value decomposition is the creme de la creme of rank-reducing decompositions—the decomposition that all others try to be at.

Singular Value Decomposition factorizes \( A \in \mathbb{R}^{m \times n} \) (where \( m > n \)), into the matrices \( U \), \( S \) and \( V^T \), where \( V^T \) is the transpose of a matrix \( V \). The SVD factorization does not require square matrices, therefore \( m \), the number of rows, does not have to equal \( n \), the number of columns.

\[
A_{m \times n} = U_{m \times m} S_{m \times n} V_{n \times n}^T. \tag{1}
\]

In this \( U \) and \( V \) are orthogonal matrices, therefore, all of the columns of \( U \) and \( V \) are orthogonal to one another. The matrix \( S \) is a \( m \times n \) rectangular diagonal matrix whose entries are in descending order, \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \), along the main diagonal.

\[
S = \begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_n \\
0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Note: If \( A \) is a complex matrix, then the singular value decomposition of a matrix \( A \) is

\[
A_{m \times n} = U_{m \times m} S_{m \times n} V_{n \times n}^*.
\]

where \( V^* \) is the conjugate transpose of \( V \) and \( U, V \) are unitary matrices.

Thin SVD

Since \( m > n \), one can represent the SVD of \( A \) as

\[
A_{m \times n} = U_{m \times n} S_{n \times n} V_{n \times n}^T.
\]

Here \( S_{n \times n} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \). This representation is called thin SVD of \( A \).

Low-Rank Approximation

Singular value decomposition gives the rank of a matrix. The number of nonzero singular values of \( A \) is the rank of the matrix \( A \). Let the rank of \( A \) be \( r = \text{min}(m, n) \), then its SVD reads

\[
A = U_{m \times r} S_{r \times r} V_{r \times n}^T,
\]

\[
A = [u_1 u_2 \ldots u_r]
\begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_r
\end{bmatrix}
\begin{bmatrix}
v_1^T \\
v_2^T \\
\vdots \\
v_r^T
\end{bmatrix},
\]

\[
A = \sum_{i=1}^{r} \sigma_i u_i v_i^T.
\]
where \( u_1, u_2, \ldots, u_r \) are columns of \( U_{m \times r} \) and \( v_1, v_2, \ldots, v_r \) are columns of \( V_{n \times r} \). One can see that the matrix \( A \) is represented by the sum of outer products of vectors. The matrix approximation by a low rank matrix is possible using SVD.

The rank \( k \) approximation (also called as truncated or partial SVD) of \( A \), \( A_k \) where \( k < r \), is given by zeroing out the \( r - k \) trailing singular values of \( A \), that is

\[
A_k = U_{m \times k}(S_k)_{k \times k}V_{k \times n}^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T.
\]

Here \( S_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \), \( U_{m \times k} = [u_1, u_2, \ldots, u_k] \) and \( V_{k \times n}^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix} \). Then one can see that

\[
A_k = U_{m \times k}U_{k \times m}^T A \quad \text{and} \quad A_k = A V_{n \times k}V_{k \times n}^T = A \left( \sum_{i=1}^{k} v_i v_i^T \right),
\]

i.e \( A_k \) is the projection of the \( A \) onto the space spanned by the top \( k \) singular vectors of \( A \). The following theorem states that the above approximation is the best rank \( k \) approximation in both Frobenius and spectral norm.

**Theorem: Eckart-Young theorem**

Let \( A_k \) be the rank-\( k \) approximation of \( A \) achieved by SVD-truncation as above. Then \( A_k \) is the closest rank-\( k \) matrix to \( A \), i.e.

\[
\|A - A_k\|_F \leq \|A - B\|_F
\]

where \( B \)'s are rank-\( k \) matrices.

The minimal error is given by the Euclidean norm of the singular values that have been zeroed out in the process

\[
\|A - A_k\|_F = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_r^2}
\]

where \( \|\cdot\|_F \) is Frobenius norm.

**Remark:** SVD also gives the best low rank approximation in spectral norm:

\[
\|A - A_k\|_2 = \min_{\text{rank}(B)=k} \|A - B\|_2 = \sigma_{k+1}.
\]

**Algorithms and computational complexity**

The SVD of a matrix \( A \) is typically computed numerically by a two-step procedure. In the first step, the matrix is reduced to a bidiagonal matrix. This takes \( O(mn^2) \) floating-point operations (flops), and the second step is to compute the SVD of the bidiagonal matrix. The second step takes \( O(n) \) iterations, each costing \( O(n) \) flops. Therefore the overall cost is \( O(mn^2) \). If \( A \) is a square matrix, then SVD algorithm requires \( O(n^3) \) flops \[67, 150\].

Alternatively we can obtain the rank \( k \) approximation directly by obtaining partial SVD. The partial SVD can be obtained by computing partial QR factorization and post process the factors \[81\]. This technique requires only \( O(kmn) \) flops. Krylov subspace methods like, Lanczos methods for certain large sparse symmetric matrices and Arnoldi (unsymmetric Lanczos methods) for unsymmetric matrices can be used to compute SVD \[67\]. The straight forward algorithm of Lanczos bidiagonalization has the problem of loss of orthogonality between the computed Lanczos vectors. Lanczos with complete reorthogonalization (or performing only local orthogonalization at every Lanczos steps), block Lanczos algorithms are practical Lanczos procedures \[67\]. Details of efficient algorithms for
large sparse matrices can be found in [133], chapter 4 of [13] and [83, 84, 143]. The algorithms are available in the packages SVD PACK [11, 12], PROPACK [100].

As a low rank approximation method the singular value decomposition has few drawbacks. It is expensive to compute if the dimension of the matrix is very large. In many applications it is sufficient to have orthonormal bases for the fundamental subspaces, something which the singular value decomposition provides. In other applications, however, it is desirable to have natural bases that consist of the rows or columns of the matrix. Here we describe such matrix decompositions.

2.2 Pivoted QR decomposition

Let \( A_{m \times n} \) be a rank deficient matrix \((m > n)\) with rank \( \gamma \). A pivoted QR decomposition with column pivoting has the form

\[
AP = QR
\]

where \( P \) is a permutation matrix, \( Q \) is orthonormal and \( R \) is upper triangular matrix. In exact arithmetic,

\[
AP = Q \begin{bmatrix} R_{11}^{(\gamma)} & R_{12}^{(\gamma)} \\ 0 & 0 \end{bmatrix}
\]

where \( R_{11}^{(\gamma)} \) is \( \gamma \times \gamma \) upper triangular matrix with rank \( \gamma \), \( Q \in \mathbb{R}^{m \times n} \) and \( P \in \mathbb{R}^{n \times n} \). In floating point arithmetic one may obtain

\[
AP = Q \begin{bmatrix} R_{11}^{(\gamma)} & R_{12}^{(\gamma)} \\ 0 & 0 \\ R_{22}^{(\gamma)} \end{bmatrix}
\]

(3)

such that \( \| R_{22}^{(\gamma)} \| \) is small.

A rank \( k \) approximation to any matrix \( A \) can be obtained by partitioning the decomposition \( AP = QR \). Let \( B = AP \) and write

\[
B = [B_1^{(k)} B_2^{(k)}] = [Q_1^{(k)} Q_2^{(k)}] \begin{bmatrix} R_{11}^{(k)} & R_{12}^{(k)} \\ 0 & 0 \end{bmatrix},
\]

(4)

where \( B_1^{(k)} \) has \( k \) columns. Then our rank \( k \) approximation is

\[
\hat{B}^{(k)} = Q_1^{(k)} \begin{bmatrix} R_{11}^{(k)} \\ 0 \end{bmatrix}.
\]

Therefore

\[
B - \hat{B}^{(k)} = Q_2^{(k)} \begin{bmatrix} 0 \\ R_{22}^{(k)} \end{bmatrix}.
\]

The approximation \( \hat{B}^{(k)} \) reproduces the first \( k \) columns of \( B \) exactly. Since \( Q_2^{(k)} \) is orthogonal, the error in \( \hat{B}^{(k)} \) as an approximation to \( B \) is

\[
\| B - \hat{B}^{(k)} \| = \| R_{22}^{(k)} \|.
\]

(5)

This is called truncated pivoted QR decomposition to \( A \). The permutation matrix is determined by column pivoting such that \( R_{11}^{(k)} \) is well conditioned and \( R_{22}^{(k)} \) is negligible (the larger entries of \( R \) are moved to the upper left corner and the smallest entries are isolated in the bottom submatrix).

This decomposition is computed by a variation of orthogonal triangularization by Householder transformations [67, 68, 144]. The algorithm described in [67, 144] requires \( O(kmn) \) flops. This algorithm is effective in producing a triangular factor \( R \) with small \( \| R_{22}^{(k)} \| \), very little is known in theory about its behavior and it can fail on some matrices (look at example 1 of [23] and also in [53]).

Similar decompositions like pivoted Cholesky decompositions, pivoted QLP decomposition and UTV decompositions can be found in [144].
Rank revealing QR factorization

As we have seen above, the column pivoting QR decomposition is a cheaper alternative to SVD. This factorization works in many cases but may also fail sometimes [23, 53]. The most promising alternative to SVD is the so-called rank revealing QR factorization.

Definition (RRQR): Given a matrix $A_{m \times n}(m \geq n)$ and an integer $k(k \leq n)$, assume partial QR factorizations of the form

$$AP = QR = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$  

(6)

where $Q \in \mathbb{R}^{m \times n}$ is an orthonormal matrix, $R \in \mathbb{R}^{n \times n}$ is block upper triangular, $R_{11} \in \mathbb{R}^{k \times k}, R_{12} \in \mathbb{R}^{k \times n-k}, R_{22} \in \mathbb{R}^{n-k \times n-k}$ and $P \in \mathbb{R}^{n \times n}$ is a permutation matrix. The above factorization is called RRQR factorization if it satisfies

$$\frac{\sigma_k(A)}{p(k, n)} \leq \sigma_{\min}(R_{11}) \leq \sigma_k(A)$$

(7)

$$\sigma_{k+1}(A) \leq \sigma_{\max}(R_{22}) \leq p(k, n)\sigma_{k+1}(A)$$

(8)

where $p(k, n)$ is a lower degree polynomial in $k$ and $n$.

In the above definition $\sigma_{\min}$ is the minimum singular value and $\sigma_{\max}$ is the maximum singular value. RRQR was defined by Chan in [23] (similar ideas were proposed independently in [61]). A constructive proof of the existence of a RRQR factorization of an arbitrary matrix $A_{m \times n}$ with numerical rank $r$ is given in [85]. Much research on RRQR factorizations has yielded improved results for $p(k, n)$. There are several algorithms to compute the RRQR factorization [23, 34, 67, 77]. The computational complexity of these algorithms are slightly larger than the standard QR decomposition algorithm. The values of $p(k, n)$ and the complexities of different algorithms were tabulated in [17].

Different applications of RRQR like subset selection problems, total least-squares problems including low rank approximation have been discussed in [24]. The low rank approximation of the matrix $A$ can be obtained by neglecting the submatrix $R_{22}$ in RRQQ factorization of $A$. It has been shown that matrix approximations derived from RRQR factorizations are almost as good as those derived from truncated SVD approximations.

The singular value and pivoted QR decompositions are not good for large and sparse matrices. The problem is that the conventional algorithms for computing these decomposition proceed by transformations that quickly destroy the sparsity of matrix $A$. Different algorithms for the efficient computation of truncated pivoted QR approximations to a sparse matrix without losing the sparsity of the matrix $A$ are proposed in [14, 114]. Some more references on structure preserving RRQR factorization algorithms are given in [24].

2.3 Interpolative decomposition

Interpolative decompositions (ID’s) (also called as CX decomposition) are closely related to pivoted QR factorizations and are useful for representing low rank matrices in terms of linear combinations of their columns [35, 105, 115]. Interpolative decomposition of a matrix completely rely on the column subset selection. Before defining the interpolative decomposition, a brief description is given below on the subset selection problem.

Subset selection problem

Subset selection is a method for selecting a subset of columns from a real matrix, so that the subset represents the entire matrix well and is far from being rank deficient. Given a $m \times n$ matrix $A$ and an integer $k$, subset selection attempts to find the $k$ most linearly independent columns that best represents the information in the matrix.

The mathematical formulation of the subset selection problems is: Determine a permutation matrix $P$ such that

$$AP = (A_1 A_2),$$

where

$$AP = (A_1 A_2),$$

(9)
1. $A_1$ is $m \times k$ matrix containing $k$ linearly independent columns such that smallest singular value is as large as possible. That is for some $\gamma$
\[
\frac{\sigma_k(A)}{\gamma} \leq \sigma_k(A_1) \leq \sigma_k(A). \tag{10}
\]

2. the $n-k$ columns of $A_2$ (redundant columns) are well represented by $k$ columns of $A_1$. That is
\[
\min_{Z \in \mathbb{R}^{k \times n-k}} \|A_1 Z - A_2\|_2 \text{ is small.}
\]
\[\text{i.e for some } \gamma, \quad \sigma_{k+1}(A) \leq \min_{Z \in \mathbb{R}^{k \times n-k}} \|A_1 Z - A_2\|_2 \leq \gamma \sigma_{k+1}(A). \tag{11}\]

**Remark:** $Z$ is a matrix responsible for representing the columns of $A_2$ in terms of the columns of $A_1$.

More detailed information and an equivalent definition to the subset selection problem is given in section 2.4. The subset selection using singular value decomposition has been addressed in [67, 69]. Many subset selection algorithms use a QR decomposition (which was discussed in last subsection) to find the most representative columns [20]. There are several randomized algorithms for this problem [17, 18, 36]. The strong RRQR algorithm by Gu and Eisenstat [77] gives the best deterministic approximation to the two conditions (10) and (11) of the subset selection problem. The details are given below.

As described in the last subsection the RRQR factorization of $A_{m \times n}$ is represented by
\[
AP = QR = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}.
\]

This gives the permutation matrix $P$ such that $AP = (A_1, A_2)$ where $A_1$ (the matrix with most important $k$ columns of $A$) and $A_2$ (with the redundant columns) are given by
\[
A_1 = Q \begin{bmatrix} R_{11} \\ 0 \end{bmatrix} \quad \text{and} \quad A_2 = Q \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}
\]
with
\[
\sigma_i(A_1) \geq \frac{\sigma_i(A)}{\sqrt{1 + f^2 k(n-k)}}, 1 \leq i \leq k
\]
\[
\min_{Z} \|A_1 Z - A_2\|_2 \leq \sigma_{k+1}(A) \sqrt{1 + f^2 k(n-k)}.
\]

In the above inequalities $f \geq 1$ is a tolerance supplied by some user. The Gu and Eisenstat algorithm also guarantees that $|\langle R_{11}^{-1} R_{12} \rangle_{ij}| \leq 1$ for $1 \leq i \leq k, 1 \leq j \leq n-k$. One can extend this algorithm for wide and fat matrices where $m < n$ and $k = m$. The computational complexity of this algorithm is $O(mn^2)$.

**Remark:** So from the strong RRQR algorithm we can see the following.

As described in subsection 2.2 the truncated RRQR of $A$ is $AP \simeq QR_{11} R_{12}$. Now we can write it as $AP \simeq QR_{11} [I_{k \times k}, R_{11}^{-1} R_{12}]$, where $QR_{11}$ is matrix which contains $k$ linearly independent columns. From theorem 3.2 of [77], one can see that $\| [I_{11} R_{12}] \|_2 \leq \sqrt{1 + f^2 k(n-k)}$. Therefore $QR_{11} [I_{11} R_{12}] P^T$ is an approximation to the matrix $A$. As we have seen in section 2.2 the error in the approximation is
\[
\|A - QR_{11} [I_{11} R_{12}] P^T\|_2 = \|R_{22}\|_2 \leq \sigma_{k+1}(A) \sqrt{1 + f^2 k(n-k)}.
\]
This subset selection problem is also studied widely in randomized setting. We postpone the discussion of these techniques to subsection 2.4.

**Remark:** In most of the deterministic algorithms for subset selection problem, the error estimates are given in spectral norm. The error estimates in both spectral and Frobenius norm are presented for several randomized algorithms in the literature which is the subject of next subsection.

**Definition (ID):** Let $A_{m \times n}$ be a matrix of rank $k$. There exists an $m \times k$ matrix $B$ whose columns constitute a subset of the columns of $A$, and $k \times n$ matrix $P$, such that

1. some subset of the columns of $P$ makes up $k \times k$ identity matrix,
2. $P$ is not too large (no entry of $P$ has an absolute value greater than 1), and
3. $A_{m \times n} = B_{m \times k}P_{k \times n}$.

Moreover, the decomposition provides an approximation

$$A_{m \times n} \approx B_{m \times k}P_{k \times n}$$

when the exact rank of $A$ is greater than $k$, but the $(k+1)^{st}$ greatest singular value of $A$ is small. The approximation quality of the Interpolative decomposition is described in the following Lemma \[105\] \[115\]. One can also look at \[71\] \[72\] for similar results.

**Lemma:** Suppose that $m$ and $n$ are positive integers, and $A$ is $m \times n$ matrix. Then for any positive integer $k$ with $k \leq m$ and $k \leq n$, there exist a $k \times n$ matrix $P$, and a $m \times k$ matrix $B$ whose columns constitute a subset of the columns of $A$, such that

1. some subset of the columns of $P$ makes up $k \times k$ identity matrix,
2. no entry of $P$ has an absolute value greater than 1,
3. $\|P_{k \times n}\|_2 \leq \sqrt{k(n-k)} + 1$,
4. the least (that is, $k$ the greatest) singular value of $P$ is at least 1, 5. $A_{m \times n} = B_{m \times k}P_{k \times n}$ when $k = m$ and $k = n$, and
6. when $k < m$ and $k < n$,

$$\|A_{m \times n} - B_{m \times k}P_{k \times n}\|_2 \leq \sqrt{k(n-k)} + 1 \sigma_{k+1},$$

where $\sigma_{k+1}$ is the $(k+1)^{st}$ greatest singular value of $A$.

The algorithms to compute the ID are computationally expensive. The algorithms described in \[71\] can be used to compute the Interpolative decomposition. In \[115\] a randomized algorithm has been proposed. The authors have constructed the interpolative decomposition under weaker conditions than those in above Lemma. The computational complexity of this algorithm is $O(kmn \log(n))$. This decomposition have also been studied in \[115\]. The details of a software package of ID algorithms can be found in \[117\] and the applications of ID in different applications can be found in \[32\] \[28\].

**CUR decomposition**

Interpolative decomposition can also be used to obtain the independent rows of a matrix. So two ID’s can be combined to construct the matrix using the subcollection of its columns and rows which is called CUR decomposition.

A Given matrix $A_{m \times n}$ is decomposed into a product of three matrices $C, U$ and $R$, where $C$ consists of small number of actual columns of $A$, $R$ consists of a small number of actual rows of $A$ and $U$ is a small carefully constructed matrix that guarantees that the product $CUR$ is close to $A$.

This decomposition is also known as skeleton decomposition \[32\]. Let $A_{m \times n}$ be a matrix with rank $r$. One can reconstruct the matrix $A$ by choosing $C_{m \times r}$ with $r$ columns and $R_{r \times n}$ with $r$ rows of the matrix such that the intersection matrix $W_{r \times r}$ is nonsingular, the corresponding CUR decomposition is

$$A = CUR, \quad \text{with} \quad U = W^{-1}.$$  

This decomposition provides a low rank approximation $A_k \simeq CUR$ when the number of selected rows and columns is less than $r$. The proper choice of these columns and rows and the matrix $U$ was the subject of study in the literature.
In [14] a Gram-Schmidt based pivoted QR algorithm is proposed. The matrices $C$ and $R$ were obtained by applying the algorithm to the matrices $A$, $A^T$ respectively. The matrix $U$ has been obtained such that (see section 4 of [14])

$$\|A - CUR\|_F^2 = \min.$$ 

In [24] a $CUR$ decomposition has been developed which is called pseudoskeleton approximation. The $k$ columns in $C$ and $k$ rows in $R$ were chosen such that their intersection $W_{k \times k}$ has maximum volume (the maximum determinant among all $k \times k$ submatrices of $A$). The complete description of this class of algorithms is given in Section 3. Randomized algorithms for $CUR$ decomposition are discussed in the following subsection.

2.4 Randomized algorithms

There are several randomized algorithms to obtain the low rank approximation of a matrix in the literature. As described in section 1 (step 1), a lower dimensional matrix $X$ can be constructed by selecting $s$ columns/rows. These columns/rows can be chosen in different ways by subsampling of the given matrix or by random projection. First we look at subsampling based randomized algorithms for the subset selection problem and for the $CUR$ decomposition and we discuss the random projection method. We also discuss randomized SVD.

(I). Sampling based methods

Column subset selection problem (CSSP)

We have seen the mathematical formulation of the subset selection problem in section 2.3. There is an equivalent definition to the subset selection problem in the literature. Here we define the column subset selection problem.

Definition (CSSP): Given a matrix $A_{m \times n}$ and a positive integer $k$ as the number of columns of $A$ forming a matrix $C \in \mathbb{R}^{m \times k}$ such that the residual $\|A - PC\|_\xi$ is minimized over all possible $\binom{n}{k}$ choices for the matrix $C$. Here $PC = CC^\dagger$ ($C^\dagger$ is Moore-Penrose pseudoinverse of $C$) denotes the projection onto the $k$ dimensional space spanned by the columns of $C$ and $\xi = 2$ or $F$ denotes the spectral norm or Frobenius norm.

This seems to be a very hard optimization problem, finding $k$ columns out of $n$ columns such that $\|A - PC\|_\xi$ is minimum. It requires $O(n^k)$ time and thus to find the optimal solution we require $O(n^{k^2} n m k)$. So obtaining the approximation is prohibitively slow if the data size is large. The NP-hardness of the CSSP (assuming $k$ is a function of $n$) is an open problem [13]. So research is focused on computing approximation solutions to CSSP.

Let $A_k$ be the best low rank $k$ approximation. Therefore $\|A - A_k\|_\xi$ provides a lower bound for $\|A - PC\|_\xi$ for $\xi = F, 2$ and for any choice of $C$. So most of the algorithms have been proposed in the literature to select $k$ columns of $A$ such that the matrix $C$ satisfies

$$\|A - A_k\|_\xi \leq \|A - PC\|_\xi \leq p(k, n) \|A - A_k\|_\xi$$

for some function $p(k, n)$.

As we have seen in the previous section, the strong RRQR algorithm (deterministic algorithm) gives spectral norm bounds. From the definition of RRQR there exists a permutation matrix $\Pi \in \mathbb{R}^{n \times n}$ (look at equation (6), please note that symbol for permutation matrix is changed here). Let $\Pi_k$ denote the first $k$ columns of this permutation matrix $\Pi$. If $C = A \Pi_k$ is $m \times k$ matrix consisting of $k$ columns of $A$ ($C$ corresponds to $Q \begin{bmatrix} R_{11} & \cdot \\ 0 & \cdot \end{bmatrix}$ in definition of RRQR), then from the equations (5) and (8) one can see (proof is very simple and similar to the proof of Lemma 7.1 in [20])

$$\|A - PC\|_2 = \|R_{22}\|_2 \leq p(k, n) \sigma_{k+1}(A).$$
That is, any algorithm that constructs an RRQR factorization of the matrix $A$ with provable guarantees also provides provable guarantees for the CSSP \cite{17}.

Several randomized algorithms have been proposed to this problem. In these methods, few columns (more than the target rank $k$) of $A$ are selected randomly according to a probability distribution which was obtained during the preprocessing of the matrix and then the low rank approximation is obtained using classical techniques from linear algebra. One such type of method, a fast Monte-Carlo algorithm for finding a low rank approximation, has been proposed in \cite{59, 60}. This algorithm gives an approximation very close to SVD by sampling the columns and rows of the matrix $A$ with only two passes through the data. It is based on selecting a small subset of important columns of $A$, forming a matrix $C$ such that the projection of $A$ on the subspace spanned by the columns of $C$ is as close to $A$ as possible. A brief description of the algorithm is given below.

A set of $s$ columns ($s > k$, where $k$ is the target rank) were chosen randomly, each according to a probability distribution proportional to their magnitudes (squared $l_2$ norms of the columns). Let $S$ be the matrix obtained by writing these $s$ columns as columns. An orthogonal set of $k$ vectors in the span of these $s$ columns have been obtained. These orthogonal vectors are the top $k$ left singular vectors of the matrix $S$ (which were obtained from the SVD of a $s \times s$ matrix formed by sampling the rows according to a probability distribution). The rank $k$ approximation to $A$ is obtained by projecting $A$ on the span of these orthogonal vectors.

The rank $k$ approximation $D^*$ of the matrix $A$ (within a small additive error) may be computed such that

$$
\|A - D^*\|_F^2 \leq \|A - A_k\|_F^2 + \epsilon \|A\|_F^2,
$$

holds with probability at least $1 - \delta$. Here $\delta$ is the failure probability, $\epsilon$ is an error parameter and the randomly chosen columns $s = \text{poly}(k, 1/\epsilon)$ (a polynomial in $k$ and $1/\epsilon$). Here $A_k$ denotes the best rank $k$ approximation of $A$. It requires $O(ms^2 + s^3)$ complexity, where $s = O\left(\max(k^4\epsilon^{-2}, k^2\epsilon^{-4})\right)$. The matrix $D^*$ can be explicitly constructed in $O(kmn)$ time. The additive error $\epsilon \|A\|_F^2$ could be arbitrarily large (for the matrices with sufficiently large $\|A\|_F^2$) compared to error $\|A - A_k\|_F^2$. This kind of sampling method may not perform well in some cases \cite{43}.

In \cite{45} a modified version of the algorithm proposed in \cite{59, 60} has been discussed. In \cite{43} Deshpande et. al. generalized the work in \cite{59, 60}. They have proved that the additive error in (12) drops exponentially by adaptive sampling and presented a multipass algorithm for low rank approximation. They have shown that it is possible to get $(1 + \epsilon)$ relative or multiplicative approximation (look at (14)). They have generalized the sampling approach using volume sampling (i.e picking $k$-subsets of the columns of any given matrix with probabilities proportional to the squared volumes of the simplicies defined by them) to get a multiplicative approximation (look at (14)) instead of additive approximation (look at (12)). They have proved the following existence result. There exists (using volume sampling) exactly $k$ columns in any $m \times n$ matrix $A$ such that

$$
\|A - D^*\|_F \leq \sqrt{(k + 1)} \|A - A_k\|_F,
$$

where $D^*$ (this may not coincide with the $D^*$ in (12)) is the projection onto the span of these $k$ columns.

They also have proved (existence result) that there exists $k + k(k+1)/\epsilon$ rows whose span contains the rows of a rank-$k$ matrix $D^*$ such that

$$
\|A - D^*\|_F^2 \leq (1 + \epsilon) \|A - A_k\|_F^2.
$$

In \cite{12}, Deshpande et. al. improved the existence result in eq (14) and developed an efficient algorithm. They have used an adaptive sampling method to approximate the volume sampling method and developed an algorithm which finds $k$ columns of $A$ such that

$$
\|A - D^*\|_F \leq \sqrt{(k + 1)!} \|A - A_k\|_F.
$$

The computational complexity of this algorithm is $O(mnk + kn)$. This algorithm requires multipasses through the data and also maintains the sparsity of $A$. 


In [17], Boutsidis et al. proposed a two stage algorithm to select exactly $k$ columns from a matrix. In the first stage (randomized stage), the algorithm randomly selects $O(k \ln k)$ columns of $V_k^T$, i.e. of the transpose of the $n \times k$ matrix consisting of the top $k$ right singular vectors of $A$, according to a probability distribution that depends on information in the top-$k$ right singular subspace of $A$. Then in the second stage (the deterministic stage), $k$ columns have been selected from the set of columns of $V_k^T$ using deterministic column selection procedure. The computational complexity of this algorithm is $O(min(mn^2, m^2n))$. It has been proved that the algorithm returns a $m \times k$ matrix $C$ consisting of exactly $k$ columns of $A$ (rank of $A$ is $\rho$) such that with probability at least 0.7:

$$
\| A - P_C A \|_2 \leq O \left( k^{3/4} \log^{1/2} (k/\rho) \right) \| A - A_k \|_2 ,
$$
$$
\| A - P_C A \|_F \leq O \left( k^{1/2} \right) \| A - A_k \|_F .
$$

They have compared the approximation results with best existing results for CSSP. They have shown that (from the above equations) the estimate in spectral norm is better than the existing result ($\| A - P_C A \|_2 \leq O(\sqrt{T + k(n - k)}) \| A - A_k \|_2$ [77]) by a factor of $n^{1/4}$ and worse than the best existence result (13) by a factor $O(k \log k)$ in Frobenius norm.

In [11], Deshpande and Rademacher have proposed an efficient algorithm for volume sampling. They have selected $k$ columns such that

$$
\| A - P_C A \|_2 \leq \sqrt{(k + 1)(n - k)} \| A - A_k \|_2
$$
$$
\| A - P_C A \|_F \leq \sqrt{(k + 1)} \| A - A_k \|_F
$$

with $O(kmn^2 \log n)$ arithmetic operations ($\omega$ is the exponent of arithmetic complexity of matrix multiplication). This improves the $O(k \sqrt{\log k})$ approximation of Boutsidis et al. in [17] for the Frobenius norm case.

In the very recent articles by Boutsidis et al. [19] and Guruswami et al. [79], these estimates have been further improved. This problem has been also studied in [6, 36, 48, 58, 82, 112, 114, 131, 137, 157, 159] and also in a PhD thesis by Civril [37]. In [31, 63, 64, 108, 158], a similar kind of work has been studied (streaming algorithms).

**Randomised CUR**

As described in section 2.3, CUR decomposition gives low rank approximation explicitly expressed in terms of a small number of columns and rows of the matrix $A$. CUR decomposition problem has been widely studied in the literature. This problem has a close connection with the column subset selection problem. One can obtain the CUR decomposition by using column subset selection on $A$ and on $A^T$ to obtain the matrices $C$ and $R$ respectively. But this will double the error in the approximation. Most of the existing CUR algorithms uses column subset selection procedure to choose the matrix $C$.

In [16], Drineas et al. have proposed a linear time algorithm to approximate the CUR decomposition. $c$ columns of $A$ and $r$ rows of $A$ are randomly chosen according to a probability distribution to obtain the matrices $C_{m \times c}$ consisting of chosen $c$ columns, $R_{r \times n}$ consisting of chosen $r$ rows. A $c \times r$ matrix $U$ has been obtained using $C$ and $R$. They have shown that for given $k$, by choosing $O(\log(1/\delta)e^{-4})$ columns of $A$ to construct $C$ and $O(k\delta^{-2}e^{-2})$ rows of $A$ to construct $R$, the resulting CUR decomposition satisfies the additive error bound with probability at least $1 - \delta$

$$
\| A - CUR \|_2 \leq \| A - A_k \|_2 + \epsilon \| A \|_2 .
$$

By choosing $O(k\log(1/\delta)e^{-4})$ columns of $A$ to construct $C$ and $O(k\delta^{-2}e^{-2})$ rows of $A$ to construct $R$, the resulting CUR decomposition satisfies the additive error bound with probability at least $1 - \delta$

$$
\| A - CUR \|_F \leq \| A - A_k \|_F + \epsilon \| A \|_F .
$$
Here $\epsilon$ is the error parameter and $\delta$ is the failure probability. The complexity of the algorithm is $O(mc^2 + nr + c^2r + c^3)$, which is linear in $m$ and $n$. This algorithm needs very large number of rows and columns to get good accuracy.

In [17], Drineas et. al. developed an improved algorithm. $c$ columns and $r$ rows were chosen randomly by subsampling to construct the matrices $C$ and $R$ respectively and $U$ is the weighted Moore-Penrose inverse of the intersection between the matrices $C$ and $R$. For given $k$, they have shown that there exists randomized algorithms such that exactly $c = O(k^2\log(1/\delta)\epsilon^{-2})$ columns of $A$ are chosen to construct $C$, then exactly $r = O(c^2\log(1/\delta)\epsilon^{-2})$ rows of $A$ are chosen to construct $R$, such that with probability at least $1 - \delta$,

$$\|A - \text{CUR}\|_F \leq (1 + \epsilon) \|A - A_k\|_F.$$

This algorithms requires $O(kmn)$ complexity (since the construction of sampling probabilities depends on right singular vectors of $A$). In [113], the columns and rows were chosen randomly according to a probability distribution formed by normalized statistical leverage scores (based on right singular values of $A$). This algorithm takes $A$, $k$ and $\epsilon$ as input and uses column subset selection procedure with $c = O(k \log k \epsilon^{-2})$ columns of $A$ to construct $C$ and with $r = O(k \log k \epsilon^{-2})$ rows of $A$ to construct $R$. The matrix $U$ is given by $U = C^T R \hat{R}$. This algorithm requires $O(kmn)$ complexity. In [154] [155], an improved algorithm has been proposed to obtain CUR decomposition with in shorter time compared to the existing relative error CUR algorithms [18] [13].

The applicability of CUR decomposition in various fields can be found in [4] [119] [149]. The generalization of CUR decomposition to Tensors has been described in [21].

(II). Random projection based methods

The random projection method for low rank approximation of a matrix is based on the idea of random projection. In random projection, the original $d$ dimensional data is projected to a $k$ dimensional ($k << d$) subspace by post-multiplying a $k \times d$ random matrix $\Omega$ (a matrix whose entries are independent random variables of some specified distribution). The idea of random mapping is based on the Johnson-Lindenstrauss lemma which says any set of $n$ points in the $d$ dimensional Euclidean space can be embedded into $k$ dimensional Euclidean space such that the distance between the points is approximately preserved [4, 87].

The choice of the random matrix $\Omega$ plays an important role in the random projection. There are several possible choices for $\Omega$. The Bernoulli random matrix (with matrix entries 1 or -1 with an equal probability of each), Gaussian random matrix (with matrix entries have zero mean and unit variance normal distribution) are among the choices for $\Omega$. The details of several other choices for random matrices were discussed in [1] [112].

The idea of the random projection based algorithms for low rank approximation $\tilde{A}$ of a matrix $A_{m \times n}$ is given below [81] [112] [139]. Let $k$ be the target rank, $s$ be the number of samples.

Step 1. Consider a random matrix $\Omega_{n \times s}$.
Step 2. Obtain the product $Y_{m \times s} = A\Omega$.
Step 3. Compute an approximate orthonormal basis $Q_{m \times k}$ for the range of $Y$ via SVD.
Step 4. Finally obtain $\tilde{A} = QQ^T A$.

In [138], a structured random matrix has been considered with $s = O(k/\epsilon)$ columns and low rank approximation has been obtained such that

$$\|A - \tilde{A}\|_F \leq (1 + \epsilon) \|A - A_k\|_F.$$

holds with high probability. The complexity of this algorithm is $O(Mk/\epsilon + (m + n)k^2/\epsilon^2)$, where $M$ is the number of non zero elements in $A$ and it requires 2 passes over the data. In [81], a standard Gaussian matrix has been considered as $\Omega$ with $s = k + p$ columns, where $p \geq 2$ an oversampling parameter. The algorithm gives low rank approximation such that

$$\|A - \tilde{A}\|_F^2 \leq (1 + \frac{k}{p-1}) \|A - A_k\|_F^2.$$
holds with high probability. The complexity of this algorithm is $O(mns + ms^2)$.

This algorithm was further improved by coupling a form of the power iteration method with random projection method \cite{S1,136}. $Y$ in this modified algorithm is $Y = (AA^T)^q A \Omega$, where $q$ is an iteration parameter. This provides the improved error estimates of the form

$$
\|A - \bar{A}\|_F^2 \leq (1 + \frac{k}{p-1})^{1/2q+1} \|A - A_k\|_F^2
$$

with an extra computational effort.

One can look at \cite{S1} for the error estimates in spectral norm. The matrix multiplication $A \Omega$ requires $O(mns)$ operations in the above algorithms. Some special structured matrices like $\Omega = DHS$ (details can be found in \cite{112}) and subsampled random Fourier transform (SRFT) matrices requires $O(mn \log s)$ complexity \cite{S1}. Complete analysis of the random projection methods can be found in \cite{S1}. Random projection method also has been studied in \cite{78,123,130,136,160,164}.

(III). Randomized SVD

The classical algorithms to compute SVD become very expensive as the data size increases and also they require $O(k)$ passes over the data. As explained in the introduction one can approximate SVD using the randomized algorithms with less computational cost and fewer passes over the data. It is a two stage procedure. In the first stage random sampling is used to obtain a reduced matrix whose range approximates the range of $A$. The reduced matrix is factorized in the second stage. This can be done in simple three steps \cite{S1}.

1. Form $B = Q^T A$, which gives the low rank approximation $A_k = QB$, where $Q = [q_1, q_2, \ldots, q_k]$ is orthonormal basis obtained in step 3 in the Random projection methods.
2. Compute an SVD of the small matrix: $B = U \Sigma V^T$.
3. Set $U = Q \tilde{U}$.

This approximates the SVD with the same rank as the basis matrix $Q$. The efficient implementation and approximation error of this procedure can be found in section 5 of \cite{S1}. This scheme is well suited for sparse and structured matrices. If the singular values of $A$ decay slowly then power iterations (with $q = 1$ or 2) were used (to form $Y$ in the random projection methods) to improve the accuracy \cite{S1,136}. This gives the truncated SVD (rank $k$) such that

$$
\|A - U \Sigma_k V^T\|_2 \leq \sigma_{k+1} + \left[1 + 4 \sqrt{\min\{m,n\}} \right]^{1/2q+1} \sigma_{k+1}
$$

holds with high probability. Here $k$ satisfies $2 \leq k \leq 0.5 \min\{m,n\}$. The total cost of this algorithm to obtain rank $k$ SVD including the operation count to obtain $Q$ is $O(mn \log(k) + k^2(m + n))$.

Randomized SVD have also been studied and used in many applications \cite{41,50,80,107,116,160,164}.

Some other randomized algorithms for low rank approximation of a matrix have been proposed in \cite{2} (sparsification), \cite{16,152}. Performance of different randomized algorithms have been compared in \cite{16,118}.

2.5 Some other techniques

Non negative matrix factorization (NMF)

Non negative matrix factorization of a given non negative matrix $A_{m \times n}$ (i.e all the matrix entries $a_{ij} \geq 0$) is finding two non negative matrices $W_{m \times k}$ and $H_{k \times n}$ such that $WH$ approximates $A$. The chosen $k$ is much smaller than $m$ and $n$. In general it is not possible to obtain $W$ and $H$ such that $A = WH$. So NMF is only an approximation. This problem can be stated formally as follows.

**Definition (NMF problem):** Given a non negative matrix $A_{m \times n}$ and a positive integer $k < \min\{m,n\}$, find non negative matrices $W_{m \times k}$ and $H_{k \times n}$ to minimize the functional

$$
f(W,H) = \frac{1}{2} \|A - WH\|_F^2.
$$

13
This is a nonlinear optimization problem. This factorization has several applications in image processing, text mining, financial data, chemometric and blind source separating etc. Generally, the factors $W$ and $H$ are naturally sparse, so they require very less storage. This factorization has some disadvantages too. The optimization problem defined above is convex in either $W$ or $H$, but not in both $W$ and $H$, which means that the algorithms can only, if at all, guarantee the convergence to a local minimum [99]. The factorization is also not unique (different algorithms gives different factorizations).

Such a factorization was first introduced in [129] and the article [103] about NMF became popular. There are several algorithms available in the literature. Multiplicative update algorithm [103, 104], projected gradient method [109], alternating least squares method [29] and several other algorithms described in [13, 27, 65, 94, 95] and [99] are among the algorithms for NMF. The non-negative tensor factorizations are described in [6] and several algorithms for both non-negative matrix and tensor factorizations with applications can be found in the book [30].

Semidiscrete matrix decomposition (SDD)

A semidiscrete decomposition (SDD) expresses a matrix as weighted sum of outer products formed by vectors with entries constrained to be in the set $S = \{-1, 0, 1\}$. The SDD approximation ($k$ term SDD approximation) of an $m \times n$ matrix $A$ is a decomposition of the form [90]

$$A_k = \sum_{i=1}^{k} d_i y_i^T y_i.$$

Here each $x_i$ is an $m -$vector with entries from $S = \{-1, 0, 1\}$, each $y_i$ is a $n -$vector with entries from the set $S$ and each $d_i$ is a positive scalar.

The columns of $X_k, Y_k$ do not need to be linearly independent. The columns can repeated multiple times. This $k$ term SDD approximation requires very less storage compared to truncated SVD but it may require large $k$ for accurate approximation. This approximation has applications in image compression and data mining. This approximation was first introduced in [124] in the contest of image compression and different algorithms have been proposed in [90, 97]. A detailed description of SDD approximation with applications in data mining can be found in the book [146] and some other applications can be found in [111, 132].

Nyström Method

The Nyström approximation is closely related to CUR approximation. Different from CUR, Nyström methods are used for approximating the symmetric positive semidefinite matrices (large kernel matrices arise in integral equations). The Nyström method has been widely used in machine learning community. The Nyström method approximates the matrix only using a subset of its columns. These columns are selected by different sampling techniques. The approximation quality depends on the selection of the good columns. A brief description of the Nyström approximation is given below.

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive semidefinite matrix (SPSD). Let $C_{n \times m}$ be a matrix consists of $m \ (<< n)$ randomly selected columns of $A$ as columns. Now the matrix $A$ can be rearranged such that $C$ and $A$ are written as

$$C = \begin{bmatrix} W \\ S \end{bmatrix} \text{ and } A = \begin{bmatrix} W \\ S \end{bmatrix} S^T B,$$

where $W \in \mathbb{R}^{m \times m}$, $S \in \mathbb{R}^{(n-m) \times m}$ and $B \in \mathbb{R}^{(n-m) \times (n-m)}$. Since $A$ is SPSD, $W$ is also a SPSD.
For $k (k \leq m)$, the rank $k$ Nyström approximation is defined by

$$\tilde{A}_k = CW_k^\dagger C^T$$

where $W_k$ is the best rank $k$ approximation of $W$ and $W_k^\dagger$ is the pseudoinverse of $W_k$. $W_k^\dagger = \sum_{i=1}^{k} \sigma_i^{-1}U(i)(U(i))^T$, where $\sigma_i$ is the $i^{th}$ singular value of $W$ and $U(i)$ is the $i^{th}$ column of the matrix $U$ in the SVD of $W$. The computational complexity is $O(nmk + m^3)$ which is much smaller than the complexity $O(n^3)$ of direct SVD. Using $W^\dagger$ instead of $W_k^\dagger$ gives more accurate approximation with higher ranks than $k$.

The Nyström method was first introduced in [156]. They have selected the random columns using uniform sampling without replacement. A new algorithm has been proposed and theoretically analyzed in [49]. The columns have been selected randomly using non-uniform probability distribution and the error estimates of the Nyström approximation were presented. By choosing $O(k/\epsilon^4)$ columns of $A$, the authors have shown that

$$\|A - CW_k^\dagger C^T\|_\xi \leq \|A - A_k\|_\xi + \epsilon \sum_{i=1}^{n} A_{ii}^2\xi,$$

where $\xi = 2$, $F$ and $A_k$ is the best rank $k$ approximation of $A$.

These estimates have been further improved in [66, 148, 155, 162]. A detailed comparison of the existing algorithms and error estimates have been discussed in [66]. The ensemble Nyström method has been proposed in [33, 140]. Adaptive sampling techniques are used to select the random columns. In [106], a new algorithm which combines the randomized low rank approximation techniques [81] and the Nyström method was proposed. In [122], the details how the Nyström method can be applied to find the SVD of general matrices were shown.

### 3 Cross/Skeleton approximation techniques

In this section we discuss in detail the cross algorithms which gives the low rank approximation of a matrix $A_{m \times n}$. In these algorithms the approximation of a matrix is obtained using the crosses formed by the selected columns and rows of the given matrix. The computational complexity of these algorithms is linear in $m, n$ and they use a small portion of the original matrix.

As described in the last section cross/skeleton approximation of a matrix $A$ is given by $A \simeq CGR$, where $C_{m \times k}, R_{k \times n}$ consists of selected $k$ columns and $k$ rows of $A$ and $G = M^{-1}$, where $M_{k \times k}$ is the submatrix on the intersection of the crosses formed by selected rows and columns from $A$. In [72], it has been shown that one can obtain a rank $k$ approximation within an accuracy $\epsilon$ such that

$$\|A - CM^{-1}R\|_2 = O\left(\epsilon \|A\|_2^2 \|M^{-1}\|_2\right)$$

provided that $M$ is nonsingular.

If $M$ is ill conditioned or if $M$ is singular then $CM^{-1}R$ will not approximate $A$. So, the accuracy of the approximation depends on the choice of $M$. A good choice for $M$ is the maximum volume submatrix i.e, the submatrix $M$ has determinant with maximum modulus among all $k \times k$ submatrices of $A$ [71]. Since the search for this submatrix is NP-complete problem [38], it is not feasible even for moderate values of $m, n$ and $k$. In practice, such a submatrix $M$ can be replaced by matrices that can be computed by the techniques, like adaptive cross approximation [7, 8, 9], skeleton decomposition with suboptimal maximum volume submatrix [70, 71] and pseudoskeleton decomposition [72, 74]. The adaptive cross approximation (ACA) technique constructs the approximation adaptively and the columns and rows are iteratively added until an error criterion is reached. In pseudoskeleton decomposition the matrix $G$ is not necessarily equal to $M^{-1}$ and even not necessarily nonsingular. The applications of these techniques can be found in [8, 9, 125, 151]. Here we describe all these techniques in detail.
### 3.1 Skeleton decomposition

Consider a matrix $A$ of order $m \times n$. As described above the matrix $A$ is approximated by $A \approx CGR$, where $C$ and $R$ contains $k$ selected columns and rows respectively and $G = M^{-1}$, where $M = A(I,J)$ is of order $k \times k$, a submatrix on the intersection of the selected columns and rows ($I$, $J$ are indices of rows and columns respectively). As explained above, obtaining the maximum volume submatrix $M$ is very difficult, so we replace it by a quasi-optimal maximal volume submatrix. It has been discussed in [70, 71], how to find such a matrix. An algorithm has been developed (named as "maxvol" in [20]) and complexity of this algorithm has been shown to be $O(mk)$. The algorithm takes a $m \times k$ matrix as input and gives $k$ row indices such that the intersection matrix $M$ has almost maximal volume as output. To construct a rank $k$ skeleton approximation of a given matrix $A$ using maxvol algorithm we follow the steps given below.

**Step 1:** Compute $k$ columns of $A$ given by the indices $J = (j^{(1)}, j^{(2)}, \ldots, j^{(k)})$ and store them in a matrix. Let it be $C$. i.e., $C = A(:,J)$ is of order $m \times k$.

**Step 2:** Now we find a good matrix by using the maxvol procedure on $C$ [70] that gives $k$ row indices $I = (i^{(1)}, i^{(2)}, \ldots, i^{(k)})$ such that the corresponding intersection matrix, say $M$ has almost maximal volume.

**Step 3:** Store the $k$ rows in a matrix $R = A(I,:) \text{ (} R \text{ is of order } k \times n \text{)}$.

**Step 4:** Therefore the skeleton decomposition is $A \approx CGR$ where $G = M^{-1}$, $M = A(I,J)$ is of order $k \times k$.

**Remark:** If the column indices $J = (j^{(1)}, j^{(2)}, \ldots, j^{(k)})$ at the beginning were badly chosen (maybe because of some random strategy), this approximation might not be very good and the inverse of $M$ might be unstable (nearly singular matrix). Even if the ranks are over estimated the inverse of $M$ will also be unstable [122].

To overcome this, after getting good row indices $I$, one can use the maxvol procedure for the row matrix $R$ to optimize the choice of the columns and even alternate further until the determinant of $M$ stays almost constant and the approximation is fine [125, 151]. In the case of over estimation of ranks, some remedy is still possible to get good accuracy [125].

**Computational complexity:** The complexity of the algorithm is $O((m+n)k^2)$, and only $k(m+n)$ of the original entries of $A$ have to be computed. The storage required for the approximation is $k(m+n)$. We can see from the algorithm that only few original matrix entries have been used in the final approximation.

A quasi-optimal error estimate for skeleton approximation of matrix has been derived in [75]. It has been shown that if the matrix $M$ has maximal in modulus determinant among all $k$ by $k$ submatrices of $A$ then

$$
\| A - CM^{-1}R \|_\infty \leq (k + 1)^2 \min_{\text{rank } B \leq k} \| A - B \|_\infty.
$$

Where $\| A \|_\infty$ defined as the largest entry in the absolute value of the matrix $A$ (sup-norm).

### 3.2 Pseudoskeleton approximation

As explained earlier, the intersection matrix $M$ need not be invertible in the pseudoskeleton approximation. The theory of pseudoskeleton approximation has been studied in [72, 73, 74]. For example, $G$ can be chosen as the pseudoinverse of $M$. Here we describe a similar algorithm presented in [52], which can be used for the treatment of higher dimensional arrays. To construct pseudoskeleton approximation of $A_{m \times n}$ with rank $r \leq \min(m,n)$, we proceed as follows.

Instead of choosing $r$ columns from the matrix $A$, we choose $k$ ($k > r$) random columns given by the indices $J = (j^{(1)}, j^{(2)}, \ldots, j^{(k)})$ from the given matrix $A$ and store them in a matrix, let it be $C$, therefore $C = A(:,J) \in \mathbb{R}^{m \times k}$. Now we use maximum volume submatrix procedure on $C$ [70] to find $k$ row indices $I = (i^{(1)}, i^{(2)}, \ldots, i^{(k)})$ corresponding to $C$. Store the $k$ row indices in a matrix
Different applications of pseudoskeleton approximation can be found in \[22, 165\].

Some terms in the complexity involves \(m\) and \(n\). Since \(k \ll m, n\) and \(r \ll k\) they do not dominate. So the overall complexity of the algorithm is linear in \(m, n\).

Pseudoskeleton approximation is used to construct different tensor decomposition formats \([52, 123]\). Different applications of pseudoskeleton approximation can be found in \([22, 165]\).
Error Analysis: In [72, 74], the error in the pseudoskeleton approximation has been studied. It has been shown that if the matrix $A$ is approximated by rank $r$ within an accuracy $\epsilon$ then there exists a choice of $r$ columns and $r$ rows i.e $C$ and $R$ and the intersection matrix $M$ such that $A \simeq CGR$ satisfying

$$\|A - CGR\|_2 \leq \epsilon \left(1 + 2\sqrt{mr} + 2\sqrt{nr}\right).$$

Here the columns of $C$ and rows of $R$ are chosen such that their intersection $M$ has maximal volume [71].

In [28] sublinear randomized algorithm for skeleton decomposition has been proposed. Uniformly sampling $l \simeq r \log \left(\max(m, n)\right)$ rows and columns are considered to construct a rank $r$ skeleton approximation. The computational complexity of the algorithm is shown to be $O(l^3)$ and the following error estimate has been proved.

Suppose $A \simeq X_1A_{11}Y_1^T$ where $X_1, Y_1$ have $r$ orthonormal columns (not necessarily singular vectors of $A$) and $A_{11}$ is not necessarily diagonal. Assume $X_1$ and $Y_1$ are incoherent, then

$$\|A - CGR\|_2 = O\left(\frac{\sqrt{nm} \cdot l}{l} \|A - X_1A_{11}Y_1^T\|_2\right)$$

holds with high probability.

3.3 Adaptive Cross Approximation (ACA)

Adaptive cross approximation has been introduced in [7, 9]. In contrast to the pseudoskeleton method here the rows and columns are chosen adaptively such that in each step a rank one approximation is added to the approximant. We try to keep the notation used when it was first investigated thoroughly, so we prefer the language of functions here instead of matrices identifying both by a regular, sufficiently dense grid in $[0, 1]^2$, i.e.

$$A = (a_{ij}), \quad a_{ij} = f \left(\frac{i - 1}{n - 1}, \frac{j - 1}{n - 1}\right)$$

for indices $1 \leq i, j \leq n$ and a given function $f : [0, 1]^2 \rightarrow \mathbb{R}$.

We are basically concerned with the questions: How to approximate $f$ by something like

$$f \sim \sum_{i=1}^{k} g_i \otimes h_i,$$

i.e., by a finite sum of tensor products of one-dimensional functions (here we write $(g \otimes h)(x, y) = g(x)h(y)$)? And how good is this approximation?

The first famous result in this direction is due to Schmidt [141], who gave a complete answer in the case $f \in L_2$. A standard reference for questions in this area is [26], a nice survey can be found in [25].

Now a very special choice of functions $g, h$ in (16) is considered, namely the restriction of $f$ itself to certain lines. In the discrete setting, that means we only allow columns and rows of the matrix $A$ to be building blocks.

Let $f : [0, 1]^2 \rightarrow \mathbb{R}$, then the recursion $R_0(x, y) = f(x, y)$ and

$$R_k(x, y) = R_{k-1}(x, y) - \frac{R_{k-1}(x, y_k)R_{k-1}(x_k, y)}{R_{k-1}(x_k, y_k)} \quad \text{for } k \in \mathbb{N},$$

with points $1 \leq x_k, y_k \leq n$ chosen such that $R_{k-1}(x_k, y_k) \neq 0$, is the heart of the two-dimensional cross approximation, compare [9, 142]. So in each step a pivot $(x_k, y_k)$ with $f(x_k, y_k) \neq 0$ is chosen (the process of choosing these pivots is called pivoting) and the corresponding row and column is
used to add another rank 1 approximation for the remainder $R_{k-1}$. After $k$ iteration steps the approximant $S_k(x,y) = f(x,y) - R_k(x,y)$ is calculated. $S_k(x,y)$ takes the form

$$S_k(x,y) = \sum_{i,j=1}^{k} (M_k)_{ij}^{-1} f(x_i, y_j) f(x_j, y)$$

where $(M_k)_{ij} = f(x_i, y_j)$, $i,j = 1, 2, \ldots, k$.

In the matrix form, the columns of $C$ and rows of $R$ are iteratively added and the approximation of $A_{n \times n}$ takes the form (compare with pseudoskeleton approximation)

$$A \simeq CGR, \text{ where } G = M_k^{-1}.$$ 

The cross approximation has nice properties like interpolation property and rank property. For any function $f : [0,1]^2 \to \mathbb{R}$, we have the interpolation property

$$R_k(x,y) = 0, \text{ as long as } x = x_i \lor y = y_i, \text{ for at least one } i \in \{1, \ldots, n\}.$$ 

That means, on the chosen crosses this procedure is exact. The next result takes an a priori knowledge about structural properties of the underlying function into account. We say that a function $f$ has separation rank $k$, if one can represent it as

$$f(x,y) = \sum_{i=1}^{k} g_i(x) h_i(y)$$

and there is no such representation with reduced summing order. This is just the continuous analog to the rank of a matrix. We call the following the rank property: If $f$ has separation rank $k$ cross approximation reproduces $f$ after $k$ steps exactly, that means

$$R_k = f - S_k = 0 \text{ on } [0,1]^2.$$ 

A matrix version of this result was first proved in [9] (Lemma 7). There also has been some effort to the error analysis: In [112] was stated

$$|R_k(x,y)| \leq (k + 1)^2 E(f, G)_{C([0,1]^2)},$$

where $E(f, G)_{C([0,1]^2)} = \inf_{g \in G} \|f - g\|_{\infty}$ is the error of best approximation of $f$ in

$$G = \left\{ g = \sum_{i=1}^{k} \varphi_i(x) \psi_i(y), \varphi_i, \psi_i \in C([0,1]) \right\}$$

measured in the sup-norm (compare with the matrix version of this result stated at the end of section 3.1). Similar results can be found in [7, 8]. In [18] a very special choice of pivots is crucial, namely the maximal volume concept, i.e. $(x_1, y_1), \ldots, (x_k, y_k)$ are chosen, such that

$$\left| \det (f(x_i, y_j))_{i,j=1}^{k} \right|$$

is maximal under all possible choices of the points. This is of course not practical and since one wants to keep the pivots of the previous steps untouched, a good alternative is the partial pivoting. Here the positions in one direction are chosen by some strategy (for example completely random) and in the second direction the maximum in modulus of the remainder is taken on a line. This is still inexpensive and leads to good numerical results, see [7, 8, 9, 112].

Practically it is expensive to update the whole remainder at each iteration step. As described in [7, 9] the approximation of the form $S_k = \sum_{i=1}^{k} \alpha_i u_i(x)v_i(y)$ can be obtained without updating the whole remainder.
If we set \( u_k(x) = R_{k-1}(x, y_k) \) and \( v_k(y) = R_{k-1}(x_k, y) \), then using (17) we get

\[
S_k(x, y) = f(x, y) - R_k(x, y) = f(x, y) - \left( R_{k-1}(x, y) - \frac{u_k(x)}{u_k(x_k)} v_k(y) \right) \\
= f(x, y) - \left( R_0(x, y) - \frac{u_1(x)}{u_1(x_1)} v_1(y) - \cdots - \frac{u_k(x)}{u_k(x_k)} v_k(y) \right) \\
= \sum_{i=1}^{k} \frac{u_i(x)}{u_i(x_i)} v_i(y), \tag{19}
\]

where we realize that it has the desired tensor product structure of separated variables (compare with (16)). By a similar calculation one can even derive the explicit formulas

\[
u_k(x) = f(x, y_k) - \sum_{i=1}^{k-1} \frac{u_i(x)}{u_i(x_i)} v_i(y_k), \tag{20}
\]

and

\[
v_k(y) = f(x_k, y) - \sum_{i=1}^{k-1} \frac{u_i(x_k)}{u_i(x_i)} v_i(y), \tag{21}
\]

see also [7].

For matrices that means instead of storing \( n^2 \) values one needs to call the function less than \( 2kn \) times and store this reduced amount of data.

Here we describe the construction of rank-\( k \) approximation of the matrix \( A = (a_{ij}) \) using ACA (described in [7, 9]).

**Notation:** In the equations (20) and (21), the \( u_k(x) \) and \( v_k(y) \) are functions of \( x \) and \( y \) respectively. In the matrix case, \( u_k = (u_k(1), ..., u_k(n))^T \) represents a column vector and \( v_k = (v_k(1), ..., v_k(n)) \) denotes a row vector.

**Details and operation count of the algorithm**

We choose \( y_1 \) randomly at the beginning. In the first step we choose the column vector \( u_1 \) with entries \( u_1(i) = A(i, y_1), i = 1, 2, ..., n \) and find the maximum element index from \(|u_1|\). \(^2\)

Let the maximum element index from \(|u_1|\) be \( x_1 \) and let \( \delta_1 = u_1(x_1) \). Now find the corresponding row vector \( v_1 \) with entries \( v_1(i) = A(x_1, i), i = 1, 2, ..., n \).

Now we find the maximum element index from \(|v_1|\) (the index should not be \( y_1 \)) and let it be \( y_2 \). In the second step (here we know the pivot element \( y_2 \)), we find the vector \( u_2 \) with entries (see equation 20)

\[
u_2(i) = A(i, y_2) - ((u_1(i) v_1(y_2))/\delta_1), i = 1, 2, ..., n.
\]

Let the maximum element index in \(|u_2|\) (the index should not be \( x_1 \)) be \( x_2 \) and let \( \delta_2 = u_2(x_2) \).

Now it is easy to find the vector \( v_2 \) with (see equation 21)

\[
v_2(i) = A(x_2, i) - (u_1(x_2) v_1(i))/\delta_1, i = 1, 2, ..., n.
\]

So, here we are doing \( n \) multiplications, \( n \) subtractions and 1 division for \( u_2 \) and \( v_2 \) respectively. The total number of operations at second step are \( 4n + 2 \).

Let \( y_3 \) \(^3\) be the maximum element index in \(|v_2|\). In the third step, we find the vector \( u_3 \) corresponding to known \( y_3 \). Therefore the vector

\[
u_3(i) = A(i, y_3) - ((u_1(i) v_1(y_3))/\delta_1 - (u_2(i) v_2(y_3))/\delta_2), i = 1, 2, ..., n.
\]

\(^2\)[u] = ([u(1)], ..., [u(n)])

\(^3\)In general the index at kth step should not be any index obtained at previous steps
Let the maximum element index from $|u_3|$ be $x_3$ and denote $\delta_3 = u_3(x_3)$. Then the entries of $v_3$ are given by

$$v_3(i) = A(x_3, i) - \left(\left(\frac{(u_1(x_3)v_1(i))}{\delta_1} - \frac{(u_2(x_3)v_2(i))}{\delta_2}\right)\right), \quad i = 1, 2, \ldots, n.$$  

So, here we are doing $2n$ multiplications, $n$ additions, $n$ subtractions and 2 divisions for $u_3$ and $v_3$ respectively. The total number of operations at this step are $8n + 4$.

Similarly we can find for other chosen rows/columns. Therefore at the $k$’th step, we do $(k-1)(4n+2)$ operations.

The total number of operations in the algorithm is

$$0 + 4n + 2 + 8n + 4 + \cdots + (k-1)(4n+2) = (4n + 2)(1 + 2 + \cdots + k - 1)$$

$$= (4n + 2)\frac{(k - 1)k}{2}$$

$$= (2n + 1)(k^2 - k)$$

$$= 2nk^2 + k^2 - 2kn + k.$$  

Therefore, the complexity of this algorithm is $O(k^2n)$, which is linear in $n (k \ll n)$. In particular, if a matrix is of dimension $m \times n$, the number of operations required to construct $S_k = \sum_{i=1}^k \frac{1}{\delta_k}u_k v_k$ is $O(k^2(m + n))$, while the storage required for the approximation $S_k$ is of order $k(m + n)$.

An algorithm called Cross-2D has been described in [126]. The extension of adaptive cross approximation to higher order tensors has been discussed in [7, 121] and different applications of ACA can be found in [3, 8, 9, 55, 80, 131, 135, 163].

We conclude this section with a remark on the performance of cross/skeleton approximation techniques on the matrices with non smooth data. These matrices generally arises from the discretization of the singular functions in some applications [7, 9, 126]. The cross/skeleton approximation algorithms require larger ranks to approximate the matrices with non smooth data [7, 126] compared to the ranks required to approximate the matrices with smooth data.

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References

[1] D. Achlioptas, Database-friendly random projections, Proc. ACM Symp. on the Principles of Database Systems, 274-281, 2001.  

[2] D. Achlioptas and F. McSherry, Fast computing of low rank matrix approximations, J. ACM, 54, 2, 9, 2007.  

[3] A. Aminfar, S. Ambikasaran and E. Darve, A fast block low-rank dense solver with applications to finite element matrices, arXiv:1403.5337v1, 2014  

[4] I. Ari, U. Simsekli, A. T. Cemgil and L. Akarun, Large scale polyphonic music transcription using randomized matrix decompositions, Proceedings of Signal Processing Conference (EUSIPCO), 2020-2024, 2012.  

[5] I. Ari, A. T. Cemgil and L. Akarun, Probabilistic interpolative decomposition, IEEE, Int., workshop on Machine learning and signal processing, 2012.  

[6] H. Arai, C. Maung and H. Schweitzer, Optimal column subset selection by A-star search, Proceedings of 29th AAAI conference on Artificial Intelligence, 2015.  

[7] K. Ari, U. Simsekli, A. T. Cemgil and L. Akarun, Large scale polyphonic music transcription using randomized matrix decompositions, Proceedings of Signal Processing Conference (EUSIPCO), 2020-2024, 2012.  

[8] I. Ari, A. T. Cemgil and L. Akarun, Probabilistic interpolative decomposition, IEEE, Int., workshop on Machine learning and signal processing, 2012.  

[9] H. Arai, C. Maung and H. Schweitzer, Optimal column subset selection by A-star search, Proceedings of 29th AAAI conference on Artificial Intelligence, 2015.  

[10] K. Ari, U. Simsekli, A. T. Cemgil and L. Akarun, Large scale polyphonic music transcription using randomized matrix decompositions, Proceedings of Signal Processing Conference (EUSIPCO), 2020-2024, 2012.  

[11] I. Ari, A. T. Cemgil and L. Akarun, Probabilistic interpolative decomposition, IEEE, Int., workshop on Machine learning and signal processing, 2012.  

[12] H. Arai, C. Maung and H. Schweitzer, Optimal column subset selection by A-star search, Proceedings of 29th AAAI conference on Artificial Intelligence, 2015.  

[13] K. Ari, U. Simsekli, A. T. Cemgil and L. Akarun, Large scale polyphonic music transcription using randomized matrix decompositions, Proceedings of Signal Processing Conference (EUSIPCO), 2020-2024, 2012.  

[14] I. Ari, A. T. Cemgil and L. Akarun, Probabilistic interpolative decomposition, IEEE, Int., workshop on Machine learning and signal processing, 2012.  

[15] H. Arai, C. Maung and H. Schweitzer, Optimal column subset selection by A-star search, Proceedings of 29th AAAI conference on Artificial Intelligence, 2015.
[7] M. Bebendorf, Adaptive Cross Approximation of Multivariate Function, Constr. Approx., 34, 149-179, 2011.
[8] M. Bebendorf, Hierarchical Matrices, Lecture Notes in Computer Science and Engineering, Springer, Vol. 63.
[9] M. Bebendorf, Approximation of boundary element matrices, Numer. Math., 86, 565-589, 2000.
[10] M. Bebendorf and S. Rjasanow, Adaptive low rank approximation of collocation matrices, Computing, 70, 1-24, 2003.
[11] M. W. Berry, Large scale sparse singular value decomposition, Int. J. of Supercomputer Apl., 6(1),13-49, 1992.
[12] M. W. Berry, SVPACK: A FORTRAN 77 software library for the sparse singular value decomposition, Technical Report: UT-CS-92-159, Department of Computer Science, University of Tennessee.
[13] M. W. Berry, D. Mezher, B. Philippe and A. Sameh, Parallel algorithms for singular value decomposition, Handbook on parallel computing and statistics, 2005.
[14] M. W. Berry, S. A. Pulatova and G. W. Stewart, Computing sparse reduced-rank approximations to sparse matrices, ACM Trans. on Math. Software, 32(2), 252-269, 2005.
[15] M. W. Berry, M. Browne, A. N. Langville, V. P. Pauca and R. J. Plemmons, Algorithms and applications for approximate non negative factorization, Comp. Stat. and Data Anal., 2006.
[16] S. Bhojanapalli, P. Jain and S. Sanghavi, Tighter low rank approximation via sampling the leverage element, arXiv:1410.3886, 2014.
[17] C. Boutsidis, M. W. Mahoney and P. Drineas, On selecting exactly k columns from a matrix, Manuscript 2008.
[18] C. Boutsidis, M. W. Mahoney and P. Drineas, An improved approximation algorithm for the column subset selection problem, arXiv:0812.4293v2 [cs.DS] 12 May 2010.
[19] C. Boutsidis, P. Drineas and M. Magdon-Ismail, Near-optimal column-based matrix reconstruction, arXiv:1103.0995v3 [cs.DS] 21 Jun 2013.
[20] M. E. Broadbent, M. Brown and K. Penner, Subset algorithms: Randomized vs Deterministic, SIAM Undergraduate Research Online, 3, May 2010. (Faculty advisors: I.C.F. Ipsen and R. Rehman).
[21] C. Caiafa and A. Cichocki, Generalizing the column-row matrix decomposition to multi-way arrays, Lin. Alg. and its Appl., 433(3), 557–573, 2010.
[22] L. Carin, Fast electromagnetic solvers for large scale naval scattering problems, Technical Report, 2008.
[23] T. F. Chan, Rank revealing QR-Factorizations, Lin. Alg. and Appl., 88/89, 67-82, 1987.
[24] T. F. Chan and P. C. Hansen, Some applications of the rank revealing QR factorization, SIAM J. Sci. Stat. Comp., 13(3), 727-741, 1992.
[25] E. W. Cheney, The best approximation of multivariate functions by combinations of univariate ones, Approximation theory IV (College station, Tex.), Academic Press, New York, 1983.
[26] E. W. Cheney, W. A. Light, Approximation theory in tensor product spaces, Lecture notes in Mathematics, 1169, Springer-Verlag, 1985. 3.3

[27] E. C. Chi and T. G. Kolda, On tensors, sparsity and non negative factorizations, SIAM J. Matrix Anal. and Appl., 33(4), 1272-1299, 2012. 2.5

[28] J. Chiu and L. Demanet, Sublinear randomized algorithms for skeleton decompositions, SIAM J. of Matrix Anal. and Appli., 34 (3), 1361-1383, 2013. 3.2

[29] A. Cichocki and R. Zdunek, Regularized alternating least squares algorithms for non negative matrix/Tensor factorization, Advances in Neural Networks, LNCS, 4493, 793-802, 2007. 2.5

[30] A. Cichocki, R. Zdunek, A. H. Phan and S. Amari, Non negative matrix and tensor factorizations, John Wiley, 2009. 2.5

[31] K. L. Clarkson and D. P. Woodruff, Numerical linear algebra in streaming model, In STOC, 205-214, 2009. 2.4

[32] R. Compton and S. Osher, Hybrid regularization for MRI reconstruction with static field inhomogeneity correction, Inverse problems and Imaging, 7(4), 1215-1233, 2013. 2.3

[33] C. Cortes, Sanjiv Kumar, M. Mohri and A. Talwalker, Very large-scale low rank approximation. 2.4

[34] S. Chandrasekaran and I. C. F. Ipsen, On rank-revealing factorizations, SIAM J. Matrix Anal. Appl., 15, 592-622, 1994. 2.2

[35] H. Cheng, Z. Gimbutas, P.-G. Martinsson and V. Rokhlin, On the compression of low rank matrices, SIAM J. Sci. Comput., 26 (4), 1389–1404, 2005. 1, 2.3

[36] A. Civril and M. Magdon-Ismail, Column Subset Selection via Sparse Approximation of SVD, Theoretical Computer Science, 421, 1-14, 2012. 2.4

[37] A. Civril, Column Subset Selection for approximating data matrices, PhD thesis, http://www.cs.rpi.edu/~magdon/LFDlabpublic.html/Theses/civril_ali/AliRPIthesis.pdf 2.4

[38] A. Civril and M. M. Ismail, On selecting a maximum volume submatrix of a matrix and related problems, Theo. Comp. Sci., 410, 4801-4811, 2009. 3

[39] B. N. Datta, Numerical linear algebra and applications, SIAM, 2010. 11

[40] S. Dasgupta and A. Gupta, An elementary proof of the Johnson-Lindenstrauss lemma, Random structures and Algorithms, 22 (1), 60-65, 2003. 2.4

[41] V. Dehdari and C. V. Deutsch, Applications of randomized methods for decomposing and simulating from large covariance matrices, Geostatistics, Oslo 2012, 15-26, 2012. 2.4

[42] A. Deshpande and S. Vempala, Adaptive sampling and fast low-rank matrix approximation, Approximation, Randomization and Combinatorial Optimization, Algorithms and Techniques, Lecture notes in Comp. Sci., 4110, 292-303, 2006. 1, 2.4

[43] A. Deshpande, L. Rademacher, S. Vempala and G. Wang, Matrix approximation and projective clustering via volume sampling, Theory of Computing, 2, 225-247, 2006. 2.4

[44] A. Deshpande and L. Rademacher, Efficient volume sampling for row/column subset selection, arXiv:1004.4057v1 [cs.DS], 2010. 2.4

[45] P. Drineas, R. Kannan and M. W. Mahoney, Fast Monte Carlo algorithms for matrices II: Computing a low rank approximation to a matrix, SIAM J. Comp., 36 (1), 158-183, 2006. 2.4
[64] M. Ghashami, E. Liberty, J. M. Philips and D. P. Woodruff, Frequent directions: Simple and deterministic matrix sketching, arXiv:1501.01711v2, Apr 2015.

[65] N. Gillis, The why and how of non negative matrix factorizations, arXiv:1401.5226v1, Jan 2014.

[66] A. Gittens and M. W. Mahoney, Revisiting the Nyström method for improved large scale machine learning, JMLR W&CP, 28(3), 567-575, arXiv:1303.1849v2, 2013.

[67] G. H. Golub, C. F. Van Loan, Matrix Computations, John Hopkins Uni. Press, 4th edition, 2013.

[68] G. H. Golub and P. Businger, Linear least squares solutions by householder transformations, Numer. Math., 1, 269-276, 1965.

[69] G. H. Golub, G. W. Stewart and V. Klema, Rank degeneracy and least squares problems, Technical report STAN-CS-76-559, Comp. Sci. Dept., Stanford Uni., 1976.

[70] S. A. Goreinov, I. V. Oseledets, D. V. Savostyanov, E. E. Tyrtyshnikov and Z. L. Zamarashkin, How to find a good submatrix, Research Report 08-10, Kowloon Tong, Hong Kong: ICM HKBU, 2008 [World Scientific Publishers, Singapore, 247-256, 2010].

[71] S. A. Goreinov and E. E. Tyrtyshnikov, The maximal-volume concept in approximation by low-rank matrices, Contemporary Mathematics, 208, 47-51, 2001.

[72] S. A. Goreinov, E. E. Tyrtyshnikov and N. L. Zamarashkin, A theory of pseudo-skeleton approximation, Lin. Alg. Appl., 261, 1-21, 1997.

[73] S. A. Goreinov, N. L. Zamarashkin and E. E. Tyrtyshnikov, Pseudo-skeleton approximations by matrices of maximal volume, Mathematical Notes, 62 (4), 515-519, 1997.

[74] S. A. Goreinov, E. E. Tyrtyshnikov and N. L. Zamarashkin, Pseudoskeleton approximations of matrices, Rep. Russian Acad. Sci., 342(2), 151-152, 1995.

[75] S. A. Goreinov and E. E. Tyrtyshnikov, Quasioptimality of skeleton approximation of a matrix in the Chebyshev norm, Doklady Mathematics, 83(3), 374-375, 2011.

[76] L. Grasedyck, D. Kressner and C. Tobler, A literature survey of low rank tensor approximation techniques, arXiv:1302.7121, 2013.

[77] M. Gu and S. C. Eisenstat, Efficient algorithms for computing a strong rank-revealing QR factorization, SIAM J. of Sci. Comp., 17(4), 848-869, 1996.

[78] M. Gu, Subspace iteration randomization and singular value problems, arXiv:1408.2208, 2014.

[79] V. Guruswami and K. Sinop, Optimal Column-Based Low-Rank Matrix Reconstruction, arXiv:1104.1732v4 [cs.DS] 4 Jan 2012.

[80] W. Hackbusch, Hierarchische matrizen, Algorithmen und Analysis, Springer Berlin, 2009.

[81] N. Halko, P. G. Martinsson and J. A. Tropp, Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions, SIAM Review, 53(2), 217-288, 2011.

[82] S. Har-Peled, Low rank approximation in linear time, arXiv:1410.8802[CS.CG], 31 Oct 2014.

[83] V. Hernandez, J. E. Roman and A. Tomas, A robust and efficient parallel SVD solver based on restarted Lanczos Bidiagonalization, Electr. Trans. on Num. Anal., 31, 68-85, 2008.
[84] V. Hernandez, J. E. Roman and A. Tomas, Restarted Lanczos bidiagonalization for the SVD in SLEPc, Tech. Re. STR-8, http://slepc.upv.es. 2.1

[85] H. P. Hong and C. T. Pan, Rank-revealing QR factorizations and singular value decomposition, Math. Comp., 58(197), 213–232, 1992. 2.2

[86] H. Ji and Y. Li, GPU accelerated randomized singular value decomposition and its application in image compression. 2.4

[87] W. B. Johnson and J. Lindenstrauss, Extension of Lipschitz mapping into Hilbert spaces, Proc. of modern analysis and probability, Contemporary Mathematics, 26, 189-206, 1984. 2.4

[88] I. T. Jolliffe, Principal component analysis, Springer series in Statistics, 2002. I

[89] R. Kannan and S. Vempala, Spectral algorithms, Foundations and trends in theoretical computer science, 4(3-4), 157-288, 2008. I

[90] B. N. Khoromskij, $O(d \log N)$—quantics approximation of $N – d$ tensors in high dimensional numerical modelling, Constr. Approx., 34(2), 257-280, 2011. I

[91] B. N. Khoromskij, Tensor numerical methods for higher dimensional partial differential equations: Basic theory and Initial applications, ESAIM: Proceedings and Surveys, 48, 1-28, 2014. I

[92] B. N. Khoromskij, Tensor structured numerical methods in scientific computing: Survey on recent advances, Chemometric Intell. Lab. Syst., 110,1-19, 2012. I

[93] B. N. Khoromskij and V. Khoromskaia, Multigrid accelerated tensor approximation of function related multidimensional arrays, SIAM J. of Sci. Comp., 31(4), 3002-3026, 2009. I

[94] J. Kim and H. Park, Towards faster non negative matrix factorization: A new algorithms and comparisons, IEEE International conference on Data mining (ICDM), 2008. 2.5

[95] J. Kim and H. Park, Fast non negative matrix factorization, An active-set-like method and comparisons, SIAM J. of Sci. Comp., 33(6), 3261-3281, 2011. 2.5

[96] T. G. Kolda and D. P. O’Leary, A semidiscrete matrix decomposition for latent semantic indexing in infection retrieval, ACM Trans. Inf. Syt., 16(4), 322-346, 1998. 2.7

[97] T. G. Kolda and D. P. O’Leary, Computation and uses of the semi discrete matrix decomposition, Tech. Rep. CS-TR-4012 and UMIACS-TR-99-22, Department of Computer Science, University of Maryland, College Park M. D, 1999. 2.5

[98] S. Kurz, O. Rain and S. Rjasanow, The adaptive cross approximation technique for the $3 – D$ boundary element method, IEEE transactions on magnetics, 38(2), 421-424, 2002. 3.3

[99] A. N. Langville, C. D. Meyer, R. Albright, J. Cox and D. Duling, Algorithms, Initializations and convergence for the non negative matrix factorization, Manuscript. 2.6

[100] R. M. Larsen, Lanczos bidiagonalization with partial reorthogonalization, http://soi.stanford.edu/~rmunk/PROPACK. 2.1

[101] L. D. Lathauwer, B. De. Moor, and J. Vandewalle, A multilinear singular value decomposition, SIAM J. of Matrix Anal. Aprr., 21, 1253-1278, 2000. I

[102] J. Lee, S. Kim, G. Lebanon and Y. Singer, Matrix approximation under local low rank assumption, arXiv:1301.3192, 2013. I

[103] D. D. Lee and H. S. Seung, Learning the parts of objects by non negative factorization, Nature, 401, 788-791, 1999. 2.5
[104] D. D. Lee and H. S. Seung, Algorithms for non negative matrix factorizations, Advances in Neural Information Processing Systems (NIPS), 13, 556-562, 2001. 2.3

[105] E. Liberty, F. Woolfe, P. G. Martinsson, V. Rokhlin and M. Tygert, Randomized algorithms for the low rank approximation of matrices, PNAS, 104(51), 20167-20172, 2007. 1, 2.3, 2.3

[106] M. Li, J. T. Kwok and B. L. Lu, Making large-scale Nyström approximation possible, Proceeding of 27th Int. Conf. on Machine Learning, Haifa, Israel, 2010. 2.5

[107] M. Li, W. Bi, J. T. Kwok and B. L. Lu, Larger Nystrom kernel matrix approximation using randomized SVD, IEEE tran. of Neural Networks and learning systems, Vol 26(1), January 2015. 2.4

[108] E. Liberty, Simple and deterministic matrix sketching, In proceedings of 19th ACM conference on knowledge, discovery and data mining, arXiv:1206.0594, June 2012. 2.4

[109] C. J. Lin, Projected gradient method for non negative matrix factorization, Neural Computation, 19(10), 2756-2779, 2007. 2.3

[110] A. Lucas, M. Stalzer and J. Feo, Parallel implementation of a fast randomized algorithm for the decomposition of low rank matrices, arXiv:1205.3830. 1, 2.3

[111] Y. Luo, An improved semidiscrete matrix decompositions and its application in Chinese information retrieval, Applied Mech. and Mate., Vol: 241-244, 3121-3124, 2013. 2.5

[112] M. W. Mahoney, Randomized algorithms for matrices and data, arXiv:1104.5557v3 [cs.DS], 2011. 2.4, 2.4

[113] M. W. Mahoney and P. Drineas, CUR matrix decompositions for improved data analysis, PNAS, 106(3), 697-702, 2009. 2.4

[114] C. Maung and H. Schweitzer, Pass-efficient unsupervised feature selection, Adv. in Neural Inf. Proc. Sys., 26, 1628-1636, 2013. 2.4

[115] P. G. Martinsson, V. Rokhlin and M. Tygert, A randomized algorithm for the decomposition of matrices, Applied and Comp. Har. Anal., 30, 47-68, 2011. 1, 2.3, 2.3

[116] P. G. Martinsson, A. Szlam and M. Tygert, Normalized power iterations for computation of SVD, NIPS workshop on low-rank methods for large-scale machine learning, 2010. 2.3

[117] P. G. Martinsson, V. Rokhlin, Y. Shkolnisky and M. Tygert, ID: A software package for low rank approximation of matrices via interpolative decompositions, Version 0.3, 2013. cims.nyu.edu/~tygert/id_doc3.pdf 2.3

[118] A. K. Menon and C. Elkan, Fast algorithms for approximating the SVD, ACM transaction knowledge discovery from data, Vol 5, 2, 13, 2011. 2.4

[119] N. Mitrovic, M. T. Asif, U. Rasheed, J. Dauwels and P. Jaillet, CUR decomposition for compression and compressed sensing of large-scale traffic data, Manuscript. 2.4

[120] K. P. Murphy, Machine learning: A probabilistic perspective, 2012. 1

[121] K. K. Naraparaju and J. Schneider, Generalized cross approximation for 3d-tensors, Comp. Vis. Sci., 14(3), 105-115, 2011. 2.3

[122] A. Nemtsov, A. Averbuch and A. Schlar, Matrix compression using the Nyström method, Manuscript [http://www.cs.tau.ac.il/~amir1/PS/Subsampling.pdf 2.5

[123] N. H. Nguyen, T. T. Do and T. T. Tran, A fast and efficient algorithm for low rank approximation of a matrix, STOC’09, 215-224, 2009. 2.4
[124] D. P. O’Leary, S. Peleg, Digital Image compression by outer product expansion, IEEE Trans. Commun., 31, 441-444, 1983. [2.5]

[125] I. V. Oseledets and E. E. Tyrtyshnikov, TT-cross approximation for multidimensional arrays, Lin. Alg. Appl. 432(1), 70-88, 2010. [1] [3] [5.1] [3.2]

[126] I. V. Oseledets, D. V. Savostyanov and E. E. Tyrtyshnikov, Tucker dimensionality reduction of the three dimensional arrays in linear time, SIAM J. of Matrix Anal. Appl., 30(3), 939-956, 2008. [3.3]

[127] H. Park and L. Elden, Matrix rank reduction for data analysis and feature extraction, Technical report, Tr 03-015, University of Minnesota. [1]

[128] X. M. Pan, J. G. Wei, Z. Peng and X. Q. Sheng, A fast algorithm for multiscale electromagnetic problems using interpolative decompositions and multilevel fast multipole algorithm, Radio Science, 47(1), 2012. [2.3]

[129] P. Paatero and U. Tapper, Positive matrix factorizations, A nonnegative factor model with optimal utilization of error estimates of data values, Environmetrics, 5, 111-126, 1994. [2.5]

[130] C. H. Papadimitriou, P. Raghavan, H. Tamaki and S. Vempala, Latent Semantic Indexing: A probabilistic analysis, Journal of Computer and System Sciences, 61(2), 217-235, 2000. [2.4]

[131] Y. Pi, H. Peng, S. Zhou and Zhihua Zhang, A scalable approach to column based low rank approximation, Proc. of the 23rd International Joint Conference on Artificial Intelligence, 1600-1606. [2.4]

[132] W. Qiang, W. XiaoLong and G. Yi, A study of semi discrete decomposition for LSI in automated text categorization, LNCS, 3248, 606-615, 2005. [2.5]

[133] S. Rajmanickam, Efficient algorithms for sparse singular value decomposition, Thesis, University of Florida, [http://www.cise.ufl.edu/~srajam/Srajamanickam_S.pdf] (2.1)

[134] S. Rjasanow and O. Steinbach, The fast solution of boundary integral equations, Springer, 2007. [3.3]

[135] B. Rogus, The adaptive cross approximation algorithm applied to electromagnetic scattering by bodies of revolution, Duquesne University, 2008. [3.3]

[136] V. Rokhlin, A. Szlam and M. Tygert, A randomized algorithm for principal component analysis, SIAM J. Matrix Anal. App., 31(3), 1100-1124. [2.4] [2.4]

[137] M. Rudelson and R. Vershynin, Sampling from large matrices: An approach through geometric functional analysis, J. Asso. Comp. Mach., 54, 2007. [2.4]

[138] T. Sarlos, Improved approximation algorithms for large matrices via random projections, Proc. of the 47th annual IEEE foundations of computer science (FOCS), 143-152,2006. [1] [2.4]

[139] B. J. Saap, Randomized algorithms for low rank matrix decomposition, Technical Report, Computer and Information Science University of Pennsylvania, May 2011. [1] [2.4]

[140] Sanjiv Kumar, M. Mohri and A. Talwalkar, Sampling methods for Nyström method, Journal of Machine Learning Research, 13, 981-1006, 2012. [2.5]

[141] E. Schmidt, Zur theorie der linearen und nicht linearen integralgleichungen, I., Math., Ann., 63, 433-476, 1907. [3.3]

[142] J. Schneider, Error estimates for two-dimensional Cross Approximation, J. Approx. Theory, 162(9), 1685-1700, 2010. [3.3]
[143] H. D. Simon and H. Zha, Low rank matrix approximation using the Lanczos bidiagonalization process with applications, SIAM J. of Sci. Comp., 21(6), 2257-2274, 2000. 2.1

[144] G. W. Stewart, Matrix Algorithms, Vol 1: Basic decompositions, SIAM. 1 2.1 2.2

[145] G. W. Stewart, Four algorithms for the efficient computation of truncated pivoted QR approximations to a sparse matrix, Numer. Math., 83, 313-323, 1999. 2.2 2.3

[146] D. Skillicorn, Understanding complex datasets, Data mining with matrix decompositions, Chapman & Hall/CRC, 2007. 1 2.5

[147] V. Sundarapandian, Numerical Linear Algebra, Prentice Hall of India Pvt.Ltd. 1

[148] A. Talwalkar and A. Rostamizadeh, Matrix coherence and Nyström method, In Proceedings of the 26th Conference in Uncertainty in Artificial Intelligence, 2010. 2.6

[149] C. Thurau, K. Kersting, C. Bauckhage, Deterministic CUR for improved large-scale data analysis: An empirical study, Proceeding of 12th SIAM International Conference on Data Mining, 684-695, 2012. 2.4

[150] L. N. Trefethen and D. Bau III: Numerical Linear Algebra, Prentice Hall. 1 2.1

[151] E. E. Tyrtyshnikov, Incomplete Cross approximation in the Mosaic-skeleton method, Computing, 64(4), 367-380, 2000. 1 3 3.1

[152] Sh. Ubary, A. Mazumdar and Y. Saad, Low rank approximation using error correcting coding matrices, Proceeding of 32nd international conference on Machine Learning, JMLR: W & CP, 37, 2015. 2.3

[153] S. Vempala, The random projection method, DIMACS, AMS, 2004. 2.4

[154] S. Wang and Z. Zhang and J. Li, A scalable CUR matrix decomposition algorithm: Lower time complexity and tighter bound, arXiv:1210.1461, 2012. 2.1

[155] S. Wang and Z. Zhang, Improving CUR matrix decomposition and the Nyström approximation via adaptive sampling, Journal of Machine Learning Research, 14, 2549-2589, 2013. 2.4 2.5

[156] C. K. I. Williams and M. Seeger, Using the Nyström method to speed up kernel machines, Advances in neural information processing system 2000, MIT press, 2001. 2.6

[157] R. Witten and E. Candes, Randomized algorithms for low-rank matrix factorizations: Sharp performance bounds, arXiv:1308.5697v1 [cs.NA], 2013. 2.4

[158] D. P. Woodruff, Low rank approximation lower bounds in row-update streams, In Proceedings of the 27th Annual Conference on Advances in Neural Information Processing Systems, 2014. 2.4

[159] F. Woolfe, E. Liberty, V. Rokhlin and Mark Tygert, A fast randomized algorithm for the approximation of matrices, Appl. Comput. Harmon. Anal. 25, 335–366, 2008. 2.4

[160] H. Xiang and J. Zou, Regularization with randomized SVD for large scale discrete inverse problems, November 2013. 2.4

[161] J. Ye, Generalized low rank approximation of matrices, Machine Learning, 61(1-3), 167-191, 2005. 1

[162] K. Zhang, I. W. Tsang and J. T. Kwok, Improved Nyström low rank approximation and error analysis, ICML, 2008. 2.5
[163] K. Zhao, M. N. Vouvakis and Jin-Fa Lee, The adaptive cross approximation algorithm for accelerated method of moments computations of EMC problems, IEEE transactions on electromagnetic compatibility, 47(4), 763-773, 2005 3.3

[164] J. Zhang, J. Erway, X. Hu, Q. Zhang and R. Plemmons, Randomized SVD methods in hyperspectral imaging, Jour. of Elec. and Comp. Engg., Article ID: 409357, 2012. 2.4

[165] X. Zhu and W. Lin, Randomized pseudo-skeleton approximation and its applications in electromagnetics, Electronics Letters, 47(10), 590-592, 2011. 3.2 3.2

[166] http://perception.csl.illinois.edu/matrix-rank/home.html 1

[167] http://web.eecs.utk.edu/research/lsi/