THE EXISTENCE OF FULL DIMENSIONAL TORI FOR D-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper, we prove the existence of full dimensional tori for d-dimensional nonlinear Schrödinger equation with periodic boundary conditions
\[ \sqrt{-1} u_t + \Delta u + V * u \pm \epsilon |u|^2 u = 0, \quad x \in T^d, \quad d \geq 1, \]
where \( V * \) is the convolution potential. Here the radius of the invariant torus satisfies a slower decay, i.e.
\[ I_n \sim e^{-r \ln |n|}, \quad \text{as } |n| \to \infty, \]
for any \( \sigma > 2 \) and \( r \geq 1 \). This result confirms a conjecture by Bourgain [J. Funct. Anal. 229 (2005), no. 1, 62-94].

1. INTRODUCTION AND MAIN RESULTS

It is an elementary problem to understand qualitative certain aspects of the long time behavior of solutions of Hamiltonian partial differential equations (PDEs), such as Nekhoroshev stability under an \( \epsilon \)-perturbation, possible growth of higher Sobolev norm of classical solutions for \( t \to \infty \) and the existence and abundance of quasi-periodic and almost periodic motion in phase space.

An open problem was raised by Kuksin (see Problem 7.1 in [30]):

Can the full dimensional KAM tori be expected with a suitable decay for Hamiltonian partial differential equations, for example,
\[ I_n \sim e^{-C \ln |n|} \]
with some \( C > 0 \) as \( |n| \to +\infty \)?

In 2005, a pioneering work by Bourgain [12] was to construct the full dimensional tori, which are the support of almost periodic solutions, for 1-dimensional nonlinear Schrödinger equation (NLS)
\[ \sqrt{-1} u_t - u_{xx} + V * u + \epsilon |u|^4 u = 0, \quad x \in T, \]
where the radius of the invariant torus satisfies
\[ I_n \sim e^{-\sqrt{|n|}}, \quad \text{as } |n| \to \infty. \quad (1.1) \]

At the same time Bourgain pointed out that: \textbf{we do not know at this time how to prove a 2D-analogue of Theorem 1, considering for instance the cubic NLS.}

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Our basic motivation is to prove Bourgain’s conjecture that the existence of full dimensional tori for $d$-dimensional NLS
\[
\sqrt{-1}u_t + \Delta u + V \ast u \pm |u|^2 u = 0, \quad x \in \mathbb{T}^d, \quad d \geq 1. \tag{1.2}
\]
Here $V \ast$ is the convolution potential defined by
\[
\hat{V} \ast u(n) = V_n \hat{u}(n)
\]
and $\hat{u}(n)$ is the $n$-th Fourier coefficient of $u$. Furthermore, using some techniques in [19], we can obtain a better result, i.e. the radius of the full dimensional tori satisfies a much slower decay than (1.1), i.e.
\[
I_n \sim e^{-r \ln^\sigma \|n\|}, \quad \text{as} \quad \|n\| \to \infty,
\]
with any $\sigma > 2, r \geq 1$ for equation (1.2), which is closer to the conjecture by Kuksin.

To state our result, we will introduce a stronger Diophantine condition than the one given in [12] firstly. For any vector $0 \neq \ell \in \mathbb{Z}^d$ with
\[
|\ell| := \sum_{n \in \mathbb{Z}^d} |\ell_n| < \infty,
\]
define the system by
\[
n(\ell) := \left\{ n \in \mathbb{Z}^d : \text{which is repeated } |\ell_n| \text{ times} \right\}. \tag{1.3}
\]
Also we can write
\[
n(\ell) = (n^*_i(\ell))_{1 \leq i \leq |\ell|}, \quad n^*_i(\ell) \in \mathbb{Z}^d,
\]
which satisfies
\[
\|n^*_1(\ell)\| \geq \cdots \geq \|n^*_{|\ell|}(\ell)\|,
\]
where $\|\cdot\|$ denotes the usual Euclidean norm by
\[
\|n\| := \sqrt{\sum_{i=1}^d |n_i|^2}, \quad n = (n_i)_{1 \leq i \leq d} \in \mathbb{Z}^d.
\]
Let
\[
\Pi := \left\{ \omega = (\omega_n)_{n \in \mathbb{Z}^d} : \omega_n \in \left[0, (n)^{-1}\right] \right\}, \tag{1.4}
\]
where
\[
\langle n \rangle = \max \{1, \|n\|\}.
\]
Then we say a vector $\omega \in \Pi$ is strong Diophantine, if there exists a real number $0 < \gamma < 1$ such that both of the following inequalities hold true:
(1) for any $0 \neq \ell \in \mathbb{Z}^d$ with $|\ell| < \infty$, one has
\[
\left\| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n \right\| \geq \gamma \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + |\ell_n|^{\|n\|d+\gamma}}; \tag{1.5}
\]
(2) for any $0 \neq \ell \in \mathbb{Z}^d$ with $|\ell| < \infty$ and $\|n^*_1(\ell)\| < \|n^*_2(\ell)\|$, one has
\[
\left\| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n \right\| \geq \frac{\gamma^5}{100} \prod_{\|n\| \leq \|n^*_1(\ell)\|} \left( \frac{1}{1 + |\ell_n|^{\|n\|d+\gamma}} \right)^{10}, \tag{1.6}
\]
where
\[
\langle n \rangle = \max \{1, \|n\|\}.
\]
where

\[ |||x||| = \inf_{j \in \mathbb{Z}} |x - j|. \]

Now our main result is as follows:

**Theorem 1.1.** Given any \( \sigma > 2, r \geq 1 \) and any frequency vector \( \omega \in \Pi \) satisfying the strong Diophantine conditions (1.5) and (1.6), then there exists a small \( \epsilon_* > 0 \) depending on \( \sigma, r \) only, and for any \( 0 < \epsilon < \epsilon_* \), there exist some \( V \in [0,1]^{2d} \) and a constant \( \xi \in \mathbb{R} \) such that equation (1.2) has a full dimensional invariant torus \( E \) satisfying:

1. the amplitude \( I = (I_n)_{n \in \mathbb{Z}^d} \) of \( E \) restricted as

\[ c_1^2 e^{-2r\ln^r(n)} \leq |I_n| \leq C_1^2 e^{-2r\ln^r(n)}, \quad \forall n \in \mathbb{Z}^d, \]

where \( c_1 < C_1 \) are two positive constants depending on \( \sigma \) and \( r \) only;

2. the frequency on \( E \) prescribed to be \( \left( \|n\|^2 + \xi + \omega \right)_{n \in \mathbb{Z}^d} \);

3. the invariant torus \( E \) is linearly stable.

Since 1990’s, the KAM theory for infinite dimensional Hamiltonian system has been well developed to study the existence and linear stability of invariant tori for Hamiltonian PDEs. See [1, 2, 3, 17, 20, 27, 28, 29, 30, 31, 34, 35, 39] for the related works for 1-dimensional PDEs. The solutions starting from the invariant torus stay on the torus all the time, which can be considered as permanent stability. For high dimensional PDEs, the situation becomes more complicated due to the multiple eigenvalues of Laplacian operator. Bourgain [13, 14] developed a new method initiated by Craig-Wayne [20] to prove the existence of lower dimensional KAM tori for \( d \)-dimensional NLS and \( d \)-dimensional NLW with \( d \geq 1 \), based on the Newton iteration, Fröhlich-Spencer techniques, Harmonic analysis and semi-algebraic set theory. This is so-called C-W-B method. Later, Eliasson-Kuksin [23] obtained both the existence and the linear stability of KAM tori for \( d \)-dimensional NLS in a classical KAM way. Also see [5, 6, 10, 15, 24, 32, 37, 38] for example. Recently, an important progress is given by Baldi-Berti-Haus-Montalto [3], where the existence and the linear stability of Cantor families of small amplitude time quasi-periodic solutions for water wave equation are constructed. Here the main difficulties are the fully nonlinear nature of the gravity water waves equations and the fact that the linear frequencies grow just in a sublinear way at infinity. See [4, 6, 8, 9] for the related problem.

In the above works, the obtained KAM tori are of low (finite) dimension which are the support of the quasi-periodic solutions. Biasco-Massetti-Procesi [11] pointed out that the constructed quasi-periodic solutions are not typical in the sense that the low dimensional tori have measure zero for any reasonable measure on the infinite dimensional phase space. It is natural at this point to find the full dimensional tori which are the support of the almost periodic solutions. The first result on the existence of almost periodic solutions for Hamiltonian PDEs was proved by Bourgain in [16] using C-W-B method. Later, Pöschel [36] (also see [25] by Geng-Xu) constructed the almost periodic solutions for 1-dimensional NLS by the classical KAM method. The basic idea to obtain these almost periodic solutions is by perturbing the quasi-periodic ones. That is why the action \( I = (I_n)_{n \in \mathbb{Z}} \) must satisfy some very strong compactness properties.
The first try to obtain the existence of full dimensional tori with a slower decay was given by Bourgain [12], who proved that 1-dimensional NLS has a full dimensional KAM torus of prescribed frequencies with the actions of the tori obeying the estimate (1.1). Different from [16] and [36], Bourgain [12] treated all Fourier modes at once, which caused a much worse small denominator problem. To this end, Bourgain took advantage of two key facts. Let \((n_i)_{i \geq 1}\) be a finite set of modes, \(|n_1| \geq |n_2| \geq \cdots\) and
\[
n_1 - n_2 + n_3 - \cdots = 0. \tag{1.7}
\]
Then the following inequality holds
\[
\sum_{i \geq 1} \sqrt{|n_i|} - 2 \sqrt{|n_1|} \geq \frac{1}{4} \sum_{i \geq 3} \sqrt{|n_i|}. \tag{1.8}
\]
Furthermore, there is also a relation
\[
n_1^2 - n_2^2 + n_3^2 - \cdots = o(1) \tag{1.9}
\]
in the case of a ‘near’ resonance. The conditions (1.7) and (1.9) implies that the first two biggest indices \(|n_1|\) and \(|n_2|\) can be controlled by other indices, i.e.
\[
|n_1| + |n_2| \leq C (|n_3| + |n_4| + \cdots), \tag{1.10}
\]
unless \(n_1 = n_2\). The inequalities (1.8) and (1.10) are essential to control the small divisor.

Recently, Cong-Liu-Shi-Yuan [18] generalized Bourgain’s result from \(\theta = 1/2\) to any \(0 < \theta < 1\) in a classical KAM way, where the actions of the tori satisfying
\[
I_n \sim e^{-|n|\theta}, \quad \theta \in (0, 1).
\]
The authors also proved the obtained tori are stable in a sub-exponential long time. Another important progress is given by Biasco-Massetti-Procesi [11], who proved the existence and linear stability of almost periodic solution for 1-dimensional NLS by constructing a rather abstract counter-term theorem for infinite dimensional Hamiltonian system.

A problem raised by Bourgain in [12] is that: do there exist the full dimensional tori for 2-dimensional NLS which satisfy the estimate (1.1)? As Bourgain pointed out that a main difficulty may be that (1.7) + (1.9) \(\Rightarrow\) (1.10) holds true only in 1-dimensional case. In other words, let \((n_i)_{i \geq 1}\) with \(n_i \in \mathbb{Z}^d, d \geq 2\) satisfying \(|n_1| \geq |n_2| \geq \cdots\). Assume that
\[
n_1 - n_2 + n_3 - \cdots = 0
\]
and
\[
|n_1|^2 - |n_2|^2 + |n_3|^2 - \cdots = o(1),
\]
one can not obtain the following inequality
\[
|n_1| + |n_2| \leq C (|n_3| + |n_4| + \cdots) \tag{1.11}
\]
even if \(n_1 \neq n_2\). As mentioned earlier, the inequality (1.11) (also see (1.10) for 1-dimensional case) is important to obtain a suitable bound of the solution of homological equation. To overcome such a difficulty, we firstly observe that in the case \(|n_1^*(\ell)| = |n_2^*(\ell)|\), combining with momentum conservation (2.3), the inequality (1.11) still holds true. If \(|n_1^*(\ell)| < |n_2^*(\ell)|\), we will introduce the nonresonant conditions (1.6), which seem like the second Melnikov conditions when dealing with the existence of lower dimensional tori. Note that the righthand of (1.6) only contains
the terms $(\|n_i^*(t)\|)_{i\geq 3}$. We will prove most of $\omega$ satisfy the nonresonant conditions (1.5) and (1.6) in Lemma 4.8.

Recently, Cong [19] improved the previous results in the sense that the action of the obtained full dimensional tori for 1-dimensional NLS satisfies

$$I_n \sim e^{-r \ln^n |n|}, \quad n \to \infty$$

(1.12)

with any $\sigma > 2$ and $r \geq 1$. Using some techniques in [19], we prove the existence of full dimensional tori for high dimensional NLS with a better estimate than Bourgain conjectured in [12].

Finally, we will give some more remarks.

**Remark 1.1.** When infinite systems of coupled harmonic oscillators with finite-range couplings are considered, Pöschel in [33] proved the existence of full dimensional tori for infinite dimensional Hamiltonian system with spatial structure of short range couplings consisting of connected sets only, where the action satisfies (1.12). In particular, in the simplest case, i.e. only nearest neighbour coupling, the result can be optimized for any $\sigma > 1$. Of course, Hamiltonian PDEs are not of short range, and do not contain such spatial structure. So our result may be optimal.

**Remark 1.2.** The Momentum Conservation (2.3) is very important in our paper. One reason is that the condition (2.3) guarantees there are no small divisor when $|k| + |k'| = 2$. The other one is that we have to control the frequency shift by so-called $\rho$-Töplitz-Lipschitz property for the Hamiltonian (see Definition 2.7), which generalizes the idea given by Geng-Xu-You in [24]. Hence our method can not be applied to $d$-dimensional nonlinear wave equation directly even to 1-dimensional nonlinear wave equation under periodic boundary conditions.

**Remark 1.3.** Recently, there are some important progresses on long time stability and possible growth of Sobolev norm for high dimensional Hamiltonian PDEs on irrational tori. See ([7, 21, 22, 26]) for example. It is natural to ask whether our method can be applied to this case.

**Remark 1.4.** An interesting result was proven by Biasco-Massetti-Procesi in [11]. The authors constructed almost periodic solutions of 1-dimensional NLS, which have Sobolev regularity both in time and space. This is the first result of this kind in KAM theory for PDEs.

2. The Norm of the Hamiltonian

In this section, we will introduce some notations and definitions firstly.

Fixed $\sigma > 2$ and given any $\rho \geq 0$, define the Banach space $\mathcal{F}_{\sigma, \rho}$ of all complex-valued sequences $q = (q_n)_{n \in \mathbb{Z}^d}$ with

$$\|q\|_{\sigma, \rho} = \sup_{n \in \mathbb{Z}^d} |q_n| e^{\rho \ln^n |n|} < \infty,$$

where

$$|n| = \max \left\{ 2^{10}, \|n\| \right\}.$$

**Remark 2.1.** Define another norm $\|\cdot\|_{\sigma, \rho}'$ by

$$\|q\|_{\sigma, \rho}' = \sup_{n \in \mathbb{Z}^d} |q_n| e^{\rho \ln^n \langle n \rangle}.$$
Then it is easy to see that the norms \(\|\cdot\|_{\sigma,\rho}\) and \(\|\cdot\|'_{\sigma,\rho}\) are equivalent. The reason why we use the norm \(\|\cdot\|\) is that some estimates are easy to obtain. See the proof of Lemma 4.1 for example.

Consider the Hamiltonian \(R(q, \bar{q})\) with the form of

\[
R(q, \bar{q}) = \sum_{a, k, k' \in \mathbb{N}Z^d} R_{akk'} M_{akk'},
\]

where

\[
M_{akk'} = \prod_{n \in \mathbb{Z}^d} I_n(0)^{a_n q_n^k q_{n}^{'k}}
\]

is the so-called monomial, \(R_{akk'}\) is the corresponding coefficient and \(I_n(0)\) is considered as the initial data. Given a monomial \(M_{akk'}\), define by

\[
\text{supp} M_{akk'} = \text{supp} (a,k,k') := \{ n \in \mathbb{Z}^d : a_n + k_n + k'_n \neq 0 \}.
\]

Furthermore, we always assume that each monomial \(M_{akk'}\) in \(R(q, \bar{q})\) satisfies:

1. **Mass Conservation**
   \[
   \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) = 0; \tag{2.2}
   \]

2. **Momentum Conservation**
   \[
   \sum_{n \in \mathbb{Z}^d} (k_n - k'_n)n = 0. \tag{2.3}
   \]

Similar as (1.3), we define the system by

\[
n(a, k, k') := \{ n \in \mathbb{Z}^d : \text{which is repeated } 2|a_n| + |k_n| + |k'_n| \text{ times} \},
\]

and write

\[
n(a, k, k') = (n^*_i(a, k, k'))_{i \geq 1}, \quad n^*_i(a, k, k') \in \mathbb{Z}^d, \tag{2.4}
\]

which satisfies

\[
||n^*_i(a, k, k')|| \geq \cdots \geq ||n^*_m(a, k, k')||
\]

and \(m = 2|a| + |k| + |k'|\).

**Remark 2.2.** Here we always assume that

\[
2|a| + |k| + |k'| \geq 4,
\]

since the cubic NLS is considered.

Before defining the norm of the Hamiltonian \(R(q, \bar{q})\), we introduce the following lemma firstly:

**Lemma 2.1.** Fixing \(\sigma > 2\) and given any \(a, k, k' \in \mathbb{N}Z^d\), assume Momentum Conservation (2.3) is satisfied. Then one has

\[
\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2\ln^\sigma |n^*_i(a, k, k')| \geq \frac{1}{3} \sum_{i \geq 3} \ln^\sigma |n^*_i(a, k, k')|. \tag{2.5}
\]

**Proof.** Based on Momentum Conservation (2.3) and the triangle inequality, one has

\[
|n^*_i(a, k, k')| \leq \sum_{i \geq 2} |n^*_i(a, k, k')|. \tag{2.6}
\]
Note that
\[ \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| = \sum_{i \geq 1} \ln^\sigma |n_i^*(a, k, k')|. \] (2.7)

Using (2.6) and (2.7), the inequality (2.5) follows from
\[ \sum_{i \geq 2} \ln^\sigma |n_i^*(a, k, k')| - \ln^\sigma \left( \sum_{i \geq 2} |n_i^*(a, k, k')| \right) \geq \frac{1}{2} \sum_{i \geq 3} \ln^\sigma |n_i^*(a, k, k')|. \] (2.8)

In view of (4.5) in Remark 4.2 and by induction, we finish the proof of (2.8). □

**Definition 2.2.** Consider the Hamiltonian \( R(q, \bar{q}) \) with the form of (2.1). Fixed \( \sigma > 2, r \geq 1 \) and given any \( \rho \geq 0 \), define
\[ \| R \|_{\sigma, \rho} = \sup_{a, k, k' \in \mathbb{N} \mathbb{Z}^d} |R_{akk'}| e^{\rho \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2 \ln^\sigma |n_i^*(a, k, k')| \right)} \] (2.9)
and
\[ \| R \|_{r, \rho}^* := \sum_{a, k, k' \in \mathbb{N}^d} |R_{akk'}| e^{-2r \sum_{n \in \mathbb{Z}^d} a_n \ln^\sigma |n|} e^{-\rho \sum_{n \in \mathbb{Z}^d} (k_n + k'_n) \ln^\sigma |n|}. \] (2.10)

**Remark 2.3.** The indices \( \sigma > 2 \) and \( r \geq 1 \) will be fixed during the KAM iteration while the index \( \rho \) will change. For simplicity, denote by
\[ \| \cdot \|_{\sigma, \rho} := \| \cdot \|_{\rho} \]
and
\[ \| \cdot \|_{r, \rho}^* := \| \cdot \|_{\rho}^*. \]
In fact, in the \( s \)-th KAM iterative step (also in some technical lemmas), one can consider \( \sigma = 2.001, r = 1 \) for example and \( \rho_s \) will be given at the beginning of Subsection 3.3.

**Remark 2.4.** In view of (2.10), it is easy to show that
\[ \| R \|_{r, \rho}^* := \sup_{\| q \|_{r, \rho, \infty} \leq 1} \sum_{a, k, k' \in \mathbb{N}^d} |R_{akk'}| M_{akk'}, \] (2.11)
with
\[ I_n(0) := e^{-2r \ln^\sigma |n|}. \] (2.12)

Therefore we have

**Lemma 2.3.** Given two Hamiltonian \( F, G \), then the following estimate holds true
\[ \| F \cdot G \|_{\rho}^* \leq \| F \|_{\rho}^* \cdot \| G \|_{\rho}^*. \] (2.13)

**Proof.** The inequality (2.13) deduces from (2.11) directly. □

**Lemma 2.4.** Given any \( \rho, \delta \geq 0 \), then one has
\[ \| R \|_{\rho + \delta}^* \leq \| R \|_{\rho}^*. \] (2.14)

**Proof.** The inequality (2.14) follows from (2.10) in Definition 2.2. □
Lemma 2.5. \textbf{(Hamiltonian Vector Field)} Given $\sigma > 2, r \geq 1, \rho \geq 0, \delta > 0$ satisfying $\rho + \delta < r$, assume the Hamiltonian $R(q, \bar{q})$ with the form of (2.1) satisfying $\|R\|_\rho < \infty$. Then one has

\[
\sup_{\|q\|_{\sigma, \rho + \delta} \leq 1} \|X_R\|_{\sigma, \rho + \delta} \leq \exp \left\{ 10d \left( \frac{2000d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{10d}{\delta} \right)^{\frac{1}{d+1}} \right\} \right\} \|R\|_\rho. \tag{2.15}
\]

In particular, if $\delta \geq \frac{r}{2}$, one gets

\[
\sup_{\|q\|_{\sigma, r} \leq 1} \|X_R\|_{\sigma, r} \leq C_1(d) \|R\|_\rho, \tag{2.16}
\]

where $C_1(d)$ is a universal constant depending on $d$ only.

\textbf{Proof.} Fixing any $\mathbf{j} \in \mathbb{Z}^d$, it suffices to estimate the upper bound for

\[ \left| \frac{\partial R}{\partial q_j} e^{(\rho + \delta) \ln^\sigma |\mathbf{j}|} \right|. \]

In view of (2.1), one has

\[
\frac{\partial R}{\partial q_j} = \sum_{a, k, k' \in \mathbb{N}^d} R_{a k k'} \left( \prod_{n \neq \mathbf{j}} I_n(0)^{a_n k_n k'_n} \right) \left( k_j I_j(0) a_j k_j^{-1} q_j^{-1} j \right). \]

Based on (2.9), one has

\[
|R_{a k k'}| \leq \|R\|_\rho e^{p \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| \right) + 2 \ln^\sigma |n^e (a, k, k')|}. \tag{2.17}
\]

The conditions $\|q\|_{\sigma, \rho + \delta} \leq 1$ implies

\[
|q_n| \leq e^{- (\rho + \delta) \ln^\sigma |n|}, \quad n \in \mathbb{Z}^d. \tag{2.18}
\]

Using (2.12), (2.17), (2.18) and $\rho + \delta < r$, one has

\[
\left| \frac{\partial R}{\partial q_j} e^{(\rho + \delta) \ln^\sigma |\mathbf{j}|} \right| \leq C \|R\|_\rho,
\]

where

\[
C := \sum_{a, k, k' \in \mathbb{N}^d} k_j e^{-\delta \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2 \rho \ln^\sigma |n^e (a, k, k')| + 2(\rho + \delta) \ln^\sigma |\mathbf{j}|}. \]

Then the inequality (2.15) will follow from

\[
C \leq \exp \left\{ 10d \left( \frac{2000d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{10d}{\delta} \right)^{\frac{1}{d+1}} \right\} \right\}. \tag{2.19}
\]

Now we will estimate (2.19) in the following two cases:

\textbf{Case 1.} $|\mathbf{j}| \leq |n^e_3 (a, k, k')|$. Then one has

\[
C \leq \sum_{a, k, k' \in \mathbb{N}^d} k_j e^{-\frac{r}{2} \sum_{i \geq 1} \ln^\sigma |n^e_i (a, k, k')|}. \]

Note that

\[
k_j \leq \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \leq \sum_{i \geq 1} \ln^\sigma |n^e_i (a, k, k')|, \tag{2.20}
\]
and we have

\[
\sum_{a,k,k' \in \mathbb{N}^d} k_j e^{-\frac{4}{3} \sum_{i \geq 1} \ln^8 [n_i^*(a,k,k')]}
\]

\[
\leq \sum_{a,k,k' \in \mathbb{N}^d} \left( \sum_{i \geq 1} \ln^8 [n_i^*(a,k,k')] \right) e^{-\frac{4}{3} \sum_{i \geq 1} \ln^8 [n_i^*(a,k,k')]}
\]

\[
\leq \frac{12}{e^\delta} \sum_{a,k,k' \in \mathbb{N}^d} e^{-\frac{4}{3} \sum_{i \geq 1} \ln^8 [n_i^*(a,k,k')]} \quad \text{(in view of (4.7))}
\]

\[
\leq \frac{12}{e^\delta} \prod_{n \in \mathbb{Z}^d} \left( 1 - e^{-\frac{4}{3} \ln^8 [n]} \right)^{-1} \prod_{n \in \mathbb{Z}^d} \left( 1 - e^{-\frac{4}{3} \ln^8 [n]} \right)^{-2} \quad \text{(by (4.9))}
\]

\[
\leq \frac{12}{e^\delta} \exp \left\{ 3 \left( \frac{1600d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{8d}{\delta} \right)^{\frac{1}{d}} \right\} \right\}, \quad (2.21)
\]

where the last inequality is based on (4.10) in Lemma 4.6.

**Case 2.**

\[
|j| > |n_3^*(a,k,k'|.
\]

If \( k_j \geq 3 \), then one has \(|j| \leq |n_3^*(a,k,k')|\) which is in **Case 1**. Hence we always assume \( k_j \leq 2 \). Then using (2.5), we have

\[
C \leq 2 \left| \sum_{a,k,k' \in \mathbb{N}^d} e^{-\frac{4}{3} \sum_{i \geq 3} \ln^8 [n_i^*(a,k,k')]} \right|. \quad (2.22)
\]

For simplicity, we denote that \( n_i = n_i^*(a,k,k') \) below. Note that if \( (n_i)_{i \geq 1} \) is given, then \( n(a,k,k') \) is specified, and hence \( n(a,k,k') \) is specified up to a factor of

\[
\prod_{n \in \mathbb{Z}^d} \left( 1 + l_n^2 \right), \quad (2.23)
\]

where

\[
l_n = \# \{ j : n_j = n \}.
\]

In view of \(|j| > |n_3|^\) again, one has \( j \in \{ n_1, n_2 \} \) and

\[
l_{n_1} + l_{n_2} \leq 2,
\]

which implies

\[
\prod_{n \in \mathbb{Z}^d} \left( 1 + l_n^2 \right) \leq 5. \quad (2.24)
\]
Furthermore if \((n_i)_{i \geq 3}\) and \(j\) are given, \(n_1\) and \(n_2\) are uniquely determined. Then using (2.23) and (2.24) one has

\[
(2.22) \quad \leq 10 \left| \sum_{(n_i)_{i \geq 3}} \prod_{n \in \mathbb{Z}^d}^{n \leq |n_3|} (1 + l_n^2) e^{-\frac{4}{7} \sum_i \ln |n_i|} \right| 
\]

\[
\leq 10 \left( \sum_{(n_i)_{i \geq 3}} e^{-\frac{4}{7} \sum_i \ln |n_i|} \right) \cdot \sup_{(n_i)_{i \geq 3}} \left( \prod_{|n| \leq |n_3|} (1 + l_n^2) e^{-\frac{4}{7} \sum_i \ln |n_i|} \right) 
\]

\[
\leq 10 \exp \left\{ 3 \left( \frac{1600d}{\delta^2} \right) \cdot \exp \left\{ d \left( \frac{8d}{\delta} \right)^{\frac{1}{\sigma}} \right\} \right\} \times \exp \left\{ 6d \left( \frac{16}{\delta} \right)^{\frac{1}{\sigma}} \cdot \exp \left\{ \left( \frac{8}{\delta} \right)^{\frac{1}{\sigma}} \right\} \right\},
\]

where the last inequality is based on (4.9), (4.10) and (4.16).

In view of (2.21) and (2.25), we finish the proof of (2.19).

When \(\delta \geq \frac{\sigma}{2}\), taking

\[
C_1(d) = \exp \left\{ 10d (8000d^d \cdot e^{20d^\delta}) \right\},
\]

we finish the proof of (2.16) by using \(\sigma > 2\) and \(r \geq 1\).

Furthermore, we have

**Lemma 2.6.** Given \(\sigma > 2, r \geq 1, \rho \geq 0, \delta > 0\) satisfying \(\rho + \delta < r\), assume the Hamiltonian \(R(q, \bar{q})\) with the form of (2.1) satisfying \(\|R\|_\rho < \infty\). Then for any \(n, l \in \mathbb{Z}^d\), we have

\[
\left\| \frac{\partial^2 R}{\partial q_n \partial q_l} \right\|_{\rho + \delta}^* \leq C \cdot \|R\|_\rho,
\]

where the constant \(C\) is given by

\[
C = \left( \frac{12}{\epsilon \delta} \right)^2 \cdot \exp \left\{ 3600d \left( \frac{d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{12d}{\delta} \right)^{\frac{1}{\sigma}} \right\} \right\}.
\]

**Proof.** The proof of this lemma will be given in Appendix.

**Definition 2.7.** Fix \(\sigma > 2\) and \(r \geq 1\). For any \(\rho \geq 0\), the Hamiltonian \(R(q, \bar{q})\) is called \(\rho\)-Töplitz-Lipschitz if the following limits exist

\[
\lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{n+t} \partial q_{m-t}}, \quad \lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{n+tl} \partial q_{m+tl}}, \quad \lim_{t \to \infty} \frac{\partial^2 R}{\partial \bar{q}_{n+tl} \partial \bar{q}_{m-t}}, \quad \lim_{t \to \infty} \frac{\partial^2 R}{\partial \bar{q}_{n+tl} \partial \bar{q}_{m+tl}},
\]
for any fixed $n, m, l \in \mathbb{Z}^d$. Moreover, there exists $K > 0$, such that when $|t| > K$, there exists a universal positive constant $C$ such that the function $R(q, \bar{q})$ satisfies

$$
\left\| \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} - \lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} \right\|_\rho^* \leq \frac{C}{|t|}, \quad (2.27)
$$

$$
\left\| \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} - \lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} \right\|_\rho^* \leq \frac{C}{|t|}, \quad (2.28)
$$

$$
\left\| \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} - \lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{m+n} \partial q_{m+l}} \right\|_\rho^* \leq \frac{C}{|t|}, \quad (2.29)
$$

**Lemma 2.8. (Poisson Bracket)** Let $\sigma > 2, \rho > 0$ and

$$
0 < \delta_1, \delta_2 < \min\left\{ \frac{1}{4} \rho, 3 - 2\sqrt{2} \right\}.
$$

Then one has

$$
\|\{R_1, R_2\}\|_\rho \leq C \|R_1\|_{\rho - \delta_1} \|R_2\|_{\rho - \delta_2}, \quad (2.30)
$$

where the constant $C$ is given by

$$
C = \frac{1}{\delta_2} \cdot \exp\left\{ 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp\left\{ d \left( \frac{24d}{\delta_1} \right)^{\frac{1}{d-1}} \right\} \right\}.
$$

**Proof.** The proof of this lemma will be given in Appendix. \qed

Next, we will estimate the symplectic transformation $\Phi_F$ induced by the Hamiltonian function $F$. Actually, we have

**Lemma 2.9. (Hamiltonian Flow)** Let $\rho > 0$ and

$$
0 < \delta < \min\left\{ \frac{1}{4} \rho, 3 - 2\sqrt{2} \right\}.
$$

Assume further

$$
\frac{2e}{\delta} \cdot \exp\left\{ 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp\left\{ d \left( \frac{24d}{\delta} \right)^{\frac{1}{d-1}} \right\} \right\} \|F\|_{\rho - \delta} < \frac{1}{2}, \quad (2.31)
$$

Then for any Hamiltonian function $H$, we get

$$
\|H \circ \Phi_F\|_\rho \leq \left( 1 + \frac{4e}{\delta} \cdot \exp\left\{ 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp\left\{ d \left( \frac{24d}{\delta} \right)^{\frac{1}{d-1}} \right\} \right\} \|F\|_{\rho - \delta} \right) \|H\|_{\rho - \delta}, \quad (2.32)
$$

**Proof.** Firstly, we expand $H \circ \Phi_F$ into the Taylor series

$$
H \circ \Phi_F = \sum_{n \geq 0} \frac{1}{n!} H^{(n)}, \quad (2.33)
$$

where $H^{(n)} = \{H^{(n-1)}, F\}$ and $H^{(0)} = H$.

We will estimate $\|H^{(n)}\|_\rho$ by using Lemma 2.8 again and again:

$$
\|H^{(n)}\|_\rho \leq \left( \exp\left\{ 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp\left\{ d \left( \frac{24d}{\delta} \right)^{\frac{1}{d-1}} \right\} \right\} \|F\|_{\rho - \delta} \right)^n \left( \frac{2n}{\delta} \right)^n \|H\|_{\rho - \delta}, \quad (2.34)
$$
In view of (2.33), (2.34) and the following inequality
\[ j^j < j!e^j, \]
one has
\[ \|H \circ \Phi F\|_\rho \leq \left(1 + \frac{4e}{\delta} \cdot \exp \left\{ 3 \left( \frac{14400d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{24d}{\delta} \right) \cdot \exp \left\{ \frac{1}{\delta} \right\} \right\} \right\} \|F\|_\rho, \]
which finishes the proof of (2.32).

\[ \square \]

3. KAM iteration

3.1. Derivation of homological equations. According to the basic idea of KAM theory (see [12] for example), the proof of Main Theorem employs the rapidly converging iteration scheme of Newton type to deal with small divisor problems introduced by Kolmogorov, involving the infinite sequence of coordinate transformations. At the s-th step of the scheme, a Hamiltonian \( H_s = N_s + R_s \) is considered, as a small perturbation of some normal form \( N_s \). A transformation \( \Phi_s \) is set up so that
\[ H_s \circ \Phi_s = N_{s+1} + R_{s+1} \]
with another normal form \( N_{s+1} \) and a much smaller perturbation \( R_{s+1} \). We drop the index \( s \) of \( H_s, N_s, R_s, \Phi_s \) and shorten the index \( s + 1 \) as +.

Now consider the Hamiltonian \( H \) of the form
\[ H = N + R, \quad (3.1) \]
where
\[ N = \sum_{n \in \mathbb{Z}^d} (\|n\|^2 + \bar{V}_n) |q_n|^2, \]
and
\[ R = R_0 + R_1 + R_2 \]
with \( |\bar{V}_n| \leq 2 \) for all \( n \in \mathbb{Z}^d \),
\[ R_0 = \sum_{a,k,k' \in \mathbb{Z}^d} \sum_{\text{supp } k \cap \text{supp } k' = \emptyset} R_{akk'} M_{akk'}, \]
\[ R_1 = \sum_{n \in \mathbb{Z}^d} J_n \left( \sum_{a,k,k' \in \mathbb{Z}^d} \sum_{\text{supp } k \cap \text{supp } k' = \emptyset} R_{akk'}^{(n)} M_{akk'} \right), \]
\[ R_2 = \sum_{n,m \in \mathbb{Z}^d} J_n J_m \left( \sum_{a,k,k' \in \mathbb{Z}^d} \sum_{\text{no assumption}} R_{akk'}^{(n,m)} M_{akk'} \right). \]
Here \( J_n = |q_n|^2 - I_n(0) \) for any \( n \in \mathbb{Z}^d \). We desire to eliminate the terms \( R_0, R_1 \) in (3.1) by the coordinate transformation \( \Phi \), which is obtained as the time-1 map
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Let $X_F|_{t=1}$ of a Hamiltonian vector field $X_F$ with $F = F_0 + F_1$. Let $F_0$ (resp. $F_1$) has the form of $R_0$ (resp. $R_1$), that is

$$F_0 = \sum_{a,k,k' \in \mathbb{N} \cap \text{supp } k \cap \text{supp } k' = \emptyset} F_{akk'} \mathcal{M}_{akk'},$$

$$F_1 = \sum_{n \in \mathbb{Z}^d} J_n \left( \sum_{a,k,k' \in \mathbb{N} \cap \text{supp } k \cap \text{supp } k' = \emptyset} F_{akk'}^{(n)} \mathcal{M}_{akk'} \right),$$

and the homological equations become

$$\{N,F\} + R_0 + R_1 = [R_0] + [R_1],$$

where

$$[R_0] = \sum_{a \in \mathbb{N}^{d}} R_{a00,00},$$

and

$$[R_1] = \sum_{n \in \mathbb{Z}^d} J_n \sum_{a \in \mathbb{N}^{d}} R_{a00}^{(n)} \mathcal{M}_{a00}.$$  

Define $\tilde{\omega} = (\tilde{\omega}_n)_{n \in \mathbb{Z}^d}$ with

$$\tilde{\omega}_n = \sum_{a \in \mathbb{N}^{d}} R_{a00}^{(n)} \mathcal{M}_{a00},$$

is the so-called frequency shift. The solutions of the homological equations (3.4) are given by

$$F_{akk'} = \frac{R_{akk'}}{\sum_{n \in \mathbb{Z}^d} (k_n - k'_n) (||n||^2 + V_n)},$$

where

$$F_{akk'}^{(m)} = \frac{R_{akk'}^{(m)}}{\sum_{n \in \mathbb{Z}^d} (k_n - k'_n) (||n||^2 + V_n)},$$

and the new Hamiltonian $H_+$ has the form

$$H_+ = H \circ \Phi$$

$$= N + \{N,F\} + R_0 + R_1$$

$$+ \int_0^1 \{(1-t)\{N,F\} + R_0 + R_1, F\} \circ X_F^t \ dt + R_2 \circ X_F^1,$$

$$= N_+ + R_+,$$

where

$$N_+ = N + [R_0] + [R_1],$$

and

$$R_+ = \int_0^1 \{(1-t)\{N,F\} + R_0 + R_1, F\} \circ X_F^t \ dt + R_2 \circ X_F^1.$$
3.2. The new norm. To estimate the solutions of the homological equation (3.4), it is convenient to define a new norm for the Hamiltonian $R$ with the form of (3.1) by

$$\| R \|_\rho^+ = \max \left\{ \| R_0 \|_\rho^+, \| R_1 \|_\rho^+, \| R_2 \|_\rho^+ \right\},$$

where

$$\| R_0 \|_\rho^+ = \sup_{a, k, k' \in \mathbb{Z}^d} e^{\rho \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + n_n) \ln |n| + 2 \ln^\sigma |n_1^* (a, k, k')| \right)},$$

(3.11)

$$\| R_1 \|_\rho^+ = \sup_{a, k, k' \in \mathbb{Z}^d, m \in \mathbb{Z}^d} e^{\rho \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + n_n) \ln |n| + 2 \ln^\sigma |m| - 2 \ln^\sigma |n_1^* (a, k, k'; m)| \right)},$$

(3.12)

and

$$\| R_2 \|_\rho^+ = \sup_{a, k, k' \in \mathbb{Z}^d, m_1, m_2 \in \mathbb{Z}^d} e^{\rho \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + n_n) \ln |n| + 2 \ln^\sigma |m_1| + 2 \ln^\sigma |m_2| - 2 \ln^\sigma |n_1^* (a, k, k'; m_1, m_2)| \right)},$$

(3.13)

noting that

$$| n_1^* (a, k, k'; m) | = \max \{| n_1^* (a, k, k'), | m | \};$$

and

$$| n_1^* (a, k, m_1, m_2) | = \max \{| n_1^* (a, k, k'), | m_1 |, | m_2 | \}.$$

Then one has the following estimates:

**Lemma 3.1.** Given any $\rho, \delta > 0$ and a Hamiltonian $R$, one has

$$\| R \|_{\rho + \delta}^+ \leq \exp \left\{ 10d \left( \frac{10}{\delta} \right)^{\frac{1}{\rho}} \cdot \exp \left\{ \left( \frac{10}{\delta} \right)^{\frac{1}{\rho}} \right\} \right\} \| R \|_{\rho},$$

(3.14)

and

$$\| R \|_{\rho + \delta}^+ \leq \frac{64}{e^2 \delta^2} \| R \|_{\rho}^+.$$

(3.15)

**Proof.** The details of the proof will be given in the Appendix. \hfill ∎

**Remark 3.1.** For $R_1$ (also $F_1$), one needs to define the system $n(a, k, k'; m)$ by

$$n(a, k, k'; m) := n(a + e_m, k, k'),$$

where $e_m$ is the unit vector with all components equal to zero but the $m$-th one equal to 1. In another word, one can think that $J_m$ is replaced by $I_m(0)$ in $R_1$ and the responding coefficient has the same upper bound. Similarly for $R_2$, one needs to define the system $n(a, k, k'; m_1, m_2)$ by

$$n(a, k, k'; m_1, m_2) := n(a + e_{m_1} + e_{m_2}, k, k').$$

Abusing the notations, we will denote

$$n(a, k, k'; m) = n(a, k, k')$$

and

$$n(a, k, k'; m_1, m_2) = n(a, k, k')$$

for simplicity in the next subsection.
3.3. **KAM Iteration.** Now we give the precise set-up of iteration parameters. For any \( \sigma > 2 \), let \( s \geq 0 \) be the \( s \)-th KAM step.

\[
\begin{align*}
\delta_s &= \frac{\rho_0}{(s + 4) \ln^2 (s + 4)}, \quad \text{with } \rho_0 = \frac{3 - 2\sqrt{2}}{100}, \\
\rho_{s+1} &= \rho_s + 3 \delta_s, \\
\epsilon_s &= \epsilon_0 \left( \frac{2}{s} \right), \quad \text{which dominates the size of the perturbation}, \\
\lambda_s &= \epsilon_0^{0.01}, \\
\gamma_{s+1} &= \frac{1}{2s} \lambda_s \gamma_s, \quad \text{with } \gamma_0 = \lambda_0, \\
\kappa_{s+1} &= \kappa_s + \frac{1}{s^2 (s + 1)^2}, \quad \text{with } \kappa_0 = 0, \\
D_s &= \{(q_n)_{n \in \mathbb{Z}^d} : \frac{1}{2} + \delta_s \leq |q_n| e^{-\ln^2 |n|} \leq 1 - \delta_s \}.
\end{align*}
\]

**Remark 3.2.** Then one has

\[
\sum_{s \geq 0} \delta_s \leq \frac{5}{3} \rho_0,
\]

\[
\rho_0 \leq \rho_s \leq 6 \rho_0 < 3 - 2\sqrt{2}, \quad \forall \ s \geq 0,
\]

\[
\delta_s < \min \left\{ \frac{1}{4} \rho_s, 3 - 2\sqrt{2} \right\}, \quad \forall \ s \geq 0.
\]

Denote the complex cube of size \( \lambda > 0 \):

\[
C_\lambda (\bar{V}) = \left\{ (V_n)_{n \in \mathbb{Z}^d} \in \mathbb{C}^{2^d} : |V_n - \bar{V}_n| \leq \lambda \right\}.
\]

**Lemma 3.2.** Suppose \( H_s = N_s + R_s \) is real analytic on \( D_s \times C_{\gamma_s} (V_s^*) \), where

\[
N_s = \sum_{n \in \mathbb{Z}^d} \left( ||n||^2 + \bar{V}_{n,s} \right) |q_n|^2
\]

is a normal form with

\[
\bar{V}_{n,s} = \tilde{V}_s + \hat{V}_{n,s},
\]

where \( \tilde{V}_s \) does not depend on \( n \) and satisfies

\[
|\tilde{V}_s| \leq \sum_{i=0}^{s} \epsilon_i^{0.5},
\]

and \( \tilde{V}_s = (\tilde{V}_{n,s})_{n \in \mathbb{Z}^d} \) satisfies

\[
\tilde{V}_s (V_s^*) = \omega, \\
|\tilde{V}_{n,s}| \leq \left( 2 \sum_{i=0}^{s} \epsilon_i^{0.5} \right) \frac{1}{||n||},
\]

\[
\left\| \frac{\partial \tilde{V}_s}{\partial V} - Id \right\|_{l^\infty \rightarrow l^\infty} < d_s \epsilon_0^{\frac{1}{n}}.
\]

Assume that \( R_s = R_{0,s} + R_{1,s} + R_{2,s} \) satisfies

\[
||R_{0,s}||_{\rho_s}^+ \leq \epsilon_s := \epsilon_{0,s},
\]

\[
||R_{1,s}||_{\rho_s}^+ \leq \epsilon_0^{0.6} := \epsilon_{1,s},
\]

\[
||R_{2,s}||_{\rho_s}^+ \leq (1 + d_s) \epsilon_0 := \epsilon_{2,s}.
\]
Moreover, assume that $R_s$ satisfies $\rho_s$-Töplitz-Lipschitz property and for any fixed $n, m, l \in \mathbb{Z}^d$, there exists $K_s > 0$, such that when $|t| > K_s$

\[
\left\| \frac{\partial^2 R_{i,s}}{\partial q_{i+n+t} \partial q_{m-t}} \right\|_{\rho_s} \leq \frac{\epsilon_{i,s}}{|t|^3},
\]

(3.24)

\[
\left\| \frac{\partial^2 R_{i,s}}{\partial q_{i+n+t} \partial \hat{q}_{m-t}} \right\|_{\rho_s} \leq \frac{\epsilon_{i,s}}{|t|^3},
\]

(3.25)

\[
\left\| \frac{\partial^2 R_{i,s}}{\partial \hat{q}_{i+n+t} \partial \hat{q}_{m-t}} \right\|_{\rho_s} \leq \frac{\epsilon_{i,s}}{|t|^3},
\]

(3.26)

for $i = 0, 1, 2$.

Then for all $V \in \mathcal{C}_{n'}(V^*)$ satisfying $\tilde{V}_s(V) \in \mathcal{C}_{\lambda_s}(\omega)$, there exists a real analytic symplectic coordinate transformations $\Phi_{s+1} : D_{s+1} \to D_s$ satisfying

\[
\|\Phi_{s+1} - id\|_{\sigma,r} \leq \epsilon_s^{0.5},
\]

(3.27)

\[
\|D\Phi_{s+1} - Id\|_{(\sigma,r) \to (\sigma,r)} \leq \epsilon_s^{0.5},
\]

(3.28)

such that for $H_{s+1} = H_s \circ \Phi_{s+1} = N_{s+1} + R_{s+1}$, the same assumptions as above are satisfied with 's + 1' in place of 's', where $\mathcal{C}_{n+1}(V^*_{s+1}) \subset \tilde{V}_s^{-1}(\mathcal{C}_{\lambda_s}(\omega))$ and

\[
\|\tilde{V}_{s+1} - \tilde{V}_s\|_\infty \leq \epsilon_s^{0.5},
\]

(3.29)

\[
\|V^*_{s+1} - V^*_{s}\|_\infty \leq 2\epsilon_s^{0.5}.
\]

(3.30)

**Proof.** **Step. 1. Truncation step.**

We would like to estimate $F_0$ and $F_1$, which are the solutions of the homological equation (see (3.2) and (3.3)). Without loss of generality, we only consider $F_0$ below. In the step $s \to s + 1$, there is saving of a factor

\[
A_1 := e^{-\delta_s (\sum_{n \in \mathbb{Z}^d} (2n_a + k_n + k'_n) \ln^r |n| - 2 \ln^r |n'_i(a,k,k')|)}.
\]

Using (2.5) in Lemma 2.1, one has

\[
A_1 \leq e^{-\frac{1}{2} \delta_s \sum_{i \geq 3} \ln^r |n'_i(a,k,k')|}.
\]

Recalling after this step, we need

\[
\|R_{0,s+1}\|_{\rho_{s+1}} \leq \epsilon_{s+1}
\]

and

\[
\|R_{1,s+1}\|_{\rho_{s+1}} \leq \epsilon_{s+1}^{0.6}.
\]

Consequently, in $R_{0,s}$ and $R_{1,s}$, it suffices to eliminate the nonresonant monomials $M_{a,k,k'}$ for which

\[
e^{-\frac{1}{2} \delta_s \sum_{i \geq 3} \ln^r |n'_i(a,k,k')|} \geq \epsilon_{s+1}.
\]

That is to say

\[
\sum_{i \geq 3} \ln^r |n'_i(a,k,k')| \leq 2(s+4) \ln^2(s+4) \rho_0 \ln \frac{1}{\epsilon_{s+1}} := B_s.
\]

(3.31)

Hence we finished the truncation step.

**Step. 2. Estimate the lower bound of the small divisor.**
For any $n(a, k, k')$ the divisor

$$
\frac{1}{\sum_{n \in \mathbb{Z}^d} (k_n - k'_n)(\|n\|^2 + V_n)}
$$

will appear when estimating $F_0$ by (3.7). If

$$
\|n^*_2(a, k, k')\| = \|n^*_3(a, k, k')\|, \quad (3.32)
$$

we will use the nonresonant conditions (1.5); otherwise we will use the nonresonant conditions (1.6). Now we will prove there is a lower bound on the right hand side of (1.5) and (1.6) respectively where the condition (3.31) satisfies.

Firstly we assume that the condition (3.32) holds. In view of momentum conservation (2.3) and (3.32), one has

$$
\|n^*_2(a, k, k')\| \leq \sum_{i \geq 2} \|n^*_i(a, k, k')\| \leq 2 \sum_{i \geq 3} \|n^*_i(a, k, k')\|,
$$

which implies

$$
|n^*_1(a, k, k')| \leq 2 \sum_{i \geq 3} |n^*_i(a, k, k')|, \quad (3.33)
$$

Then we have

$$
\sum_{n \in \mathbb{Z}^d} |k_n - k'_n| \ln^\sigma |n| \leq \sum_{i \geq 1} \ln^\sigma |n^*_i(a, k, k')| \leq \ln^\sigma \left( 2 \sum_{i \geq 3} |n^*_i(a, k, k')| \right) + 2 \sum_{i \geq 3} \ln^\sigma |n^*_i(a, k, k')| \quad \text{(by (3.32) and (3.33))}
$$

$$
\leq 5 \sum_{i \geq 3} \ln^\sigma |n^*_i(a, k, k')|,
$$

where the last inequality follows from (4.5) in Remark 4.2.

Let

$$
N_s = B_s^{\frac{\sigma - 1}{2\sigma}} \cdot (\ln B_s)^{-\frac{2}{\sigma}}, \quad (3.34)
$$
and then one has
\[
\prod_{\mathbf{n} \in \mathbb{Z}^d \text{ satisfies (3.31)}} \frac{1}{1 + |k_\mathbf{n} - k'_\mathbf{n}|^3}^{\beta}
\]
\[
= \exp \left\{ \sum_{|\mathbf{n}| \leq N_s \text{ satisfies (3.31)}} \ln \left( \frac{1}{1 + |k_\mathbf{n} - k'_\mathbf{n}|^3} \right) \right\}
\times \exp \left\{ \sum_{|\mathbf{n}| > N_s \text{ satisfies (3.31)}} \ln \left( \frac{1}{1 + |k_\mathbf{n} - k'_\mathbf{n}|^3} \right) \right\}
\geq \exp \left\{ -10d \sum_{|\mathbf{n}| \leq N_s \text{ satisfies (3.31)}} |k_\mathbf{n} - k'_\mathbf{n}|^\sigma \ln |\mathbf{n}| \right\}
\times \exp \left\{ -10d \sum_{|\mathbf{n}| > N_s \text{ satisfies (3.31)}} |k_\mathbf{n} - k'_\mathbf{n}| \ln |\mathbf{n}| \right\}
\geq \exp \left\{ -50d \cdot (3N_s)^d B_s^{\sigma} - 50d B_s (\ln N_s)^{1-\sigma} \right\} \quad \text{(in view of (3.31))}
\geq \exp \left\{ -100 \cdot 6^d B_s (\ln B_s)^{1-\sigma} \right\}, \quad (3.35)
\]
where the last inequality is based on (3.34).
Furthermore, by (3.31) one has
\[
100 \cdot 6^d B_s (\ln B_s)^{1-\sigma}
= 100 \cdot 6^d \left( \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \ln \frac{1}{\epsilon_{s+1}} \right) \left( \ln \left( \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \ln \frac{1}{\epsilon_{s+1}} \right) \right)^{1-\sigma}
= \ln \frac{1}{\epsilon_s} \cdot \left( 100 \cdot 6^d \cdot \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \frac{3}{2} \right)
\times \left( \ln \left( \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \frac{3}{2} \right)^{s+1} \ln \frac{1}{\epsilon_0} \right)^{1-\sigma}. \quad (3.36)
\]
Note that
\[
\ln \left( \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \frac{3}{2} \right)^{s+1} \ln \frac{1}{\epsilon_0} \geq (s+1) \ln \frac{3}{2} + \ln \ln \frac{1}{\epsilon_0}, \quad (3.37)
\]
and then one has
\[
\left( 100 \cdot 6^d \cdot \frac{2(s+4) \ln^2(s+4)}{\rho_0} \cdot \frac{3}{2} \right) \left( (s+1) \ln \frac{3}{2} + \ln \ln \frac{1}{\epsilon_0} \right)^{1-\sigma} \leq 0.01 \quad (3.38)
\]
where using \( \sigma > 2 \) and \( \epsilon_0 \) is sufficiently small depending on \( d \) only. Furthermore by (3.35)-(3.38), we have
\[
\prod_{\mathbf{n} \in \mathbb{Z}^d \text{ satisfies (3.31)}} \frac{1}{1 + |k_\mathbf{n} - k'_\mathbf{n}|^3} \geq \exp \left\{ -0.01 \cdot \ln \frac{1}{\epsilon_s} \right\} = \epsilon_s^{0.01}. \quad (3.39)
\]
Following the proof of (3.39), one has
\[
\prod_{n \in \mathbb{Z}^d} \left( 1 + |k_n - k'_n|^3 \right)^{d+7} \geq \epsilon_s^{0.01}. \tag{3.40}
\]

**Step. 3. The solutions of homological equation.**

In view of (3.16) and noting that \( \hat{V}_s \) does not depend on \( n \), one has
\[
\sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \left( \|n\|^2 + \hat{V}_n \right) = \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \left( \|n\|^2 + \hat{V}_n \right). \tag{3.41}
\]
By (3.18), one gets
\[
\left| \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \left( \|n\|^2 + \hat{V}_n \right) \right| = \left| \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \left( \|n\|^2 + \hat{V}_n \right) \right| \geq \left| \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \omega_n \right|. \tag{3.42}
\]
Using (3.39)-(3.42), we have
\[
\left| \sum_{n \in \mathbb{Z}^d} (k_n - k'_n) \left( \|n\|^2 + \hat{V}_n \right) \right| \leq \epsilon_s^{0.01}. \tag{3.43}
\]
Recalling \( \lambda_s = \epsilon_s^{0.01} \), if \( V \in C_\lambda(V_s) \), the estimate (3.43) remains true when substituting \( V \) for \( \hat{V}_s \). Moreover, there is analyticity on \( C_\lambda(V_s) \). The transformations \( \Phi_{s+1} \) is obtained as the time-1 map \( X_{F_s} \) of the Hamiltonian vector field \( X_{F_s} \) with \( F_s = F_{0,s} + F_{1,s} \). Using (3.39) and (3.40) we get
\[
\| F_{i,s} \|_{\rho_s} \leq \gamma^{-1} \epsilon_s^{0.01} \| R_{i,s} \|_{\rho_s}^+, \quad i = 0, 1. \tag{3.44}
\]
In view of (3.15) in Lemma 3.1, (3.21), (3.22) and (3.44), we get
\[
\| F_{0,s} \|_{\rho_s} \leq \frac{64}{\epsilon_s^{0.52}} \| F_{0,s} \|_{\rho_s}^+, \quad \epsilon_s^{0.95}, \tag{3.45}
\]
and
\[
\| F_{1,s} \|_{\rho_s} \leq \frac{64}{\epsilon_s^{0.52}} \| F_{1,s} \|_{\rho_s}^+, \quad \epsilon_s^{0.55}. \tag{3.46}
\]
Based on (3.45) and (3.46) one has
\[
\| F_s \|_{\rho_s} \leq \epsilon_s^{0.55} \tag{3.47}
\]
and
\[
\sup_{q \in D_s} \| X_{F_s} \|_{\sigma,r} \leq \epsilon_s^{0.52}. \tag{3.48}
\]
which follows from (2.16) in Lemma 2.5.
we have $\Phi_{s+1} : D_{s+1} \to D_s$ with
\[
\|\Phi_{s+1} - id\|_{\sigma,r} \leq \sup_{q \in D_s} \|X_F\|_{\sigma,r} < \epsilon_s^{0.52},
\] (3.49)
which is the estimate (3.27). Moreover, from (3.49) we get
\[
\sup_{q \in D_s} \|DX_F - Id\|_{(\sigma,r) \to (\sigma,r)} \leq \frac{1}{d_s} \epsilon_s^{0.52} < \epsilon_s^{0.5},
\]
and thus the estimate (3.28) follows.

\textbf{Step. 4. The estimate of the remainder terms.}

Now we will estimate $R_{s+1}$. Recalling (3.10), $R_{s+1} = R_{2,s} + \mathcal{R}_s$, where
\[
\mathcal{R}_s = \int_0^1 \{ (1 - t) \{ N_s, F_s \} + R_{0,s} + R_{1,s} \} \circ X_F^t dt + \int_0^1 \{ R_{2,s} \} \circ X_F^t dt.
\] (3.50)
Rewrite
\[
R_{s+1} = R_{0,s+1} + R_{1,s+1} + R_{2,s+1},
\]
and
\[
\mathcal{R}_s = \int_0^1 \{ R_{0,s} \} \circ X_F^t dt
\] (3.51)
\[
+ \int_0^1 \{ R_{1,s} \} \circ X_F^t dt
\] (3.52)
\[
+ \int_0^1 \{ R_{2,s} \} \circ X_F^t dt
\] (3.53)
\[
+ \int_0^1 (1 - t) \{ \{ N_s, F_s \} \circ X_F^t dt.
\]

Firstly we consider the term (3.50). Note that the term (3.50) contributes to $R_{0,s+1}, R_{1,s+1}, R_{2,s+1}$ and we get
\[
\left\| \int_0^1 \{ R_{0,s} \} \circ X_F^t dt \right\|_{\rho_{\gamma} + 2\delta_s}
\]
\[
= \left\| \sum_{n \geq 1} \frac{1}{n!} \{ \{ \ldots \{ R_{0,s} \} \ldots , F_s \} \} \right\|_{\rho_{\gamma} + 2\delta_s}
\]
\[
\leq \frac{4e}{\delta_s} \exp \left\{ 3 \cdot \left( \frac{14400d}{\delta_s^2} \right)^d \cdot \exp \left\{ d \left( \frac{24d}{\delta_s^2} \right)^{\frac{1}{24d}} \right\} \right\} \| F_s \|_{\rho_{\gamma} + \delta_s} \| R_{0,s} \|_{\rho_{\gamma} + \delta_s}
\]
where the last inequality follows from the proof of Lemma 2.9.

In view of
\[
\delta_s = \frac{\rho_0}{(s + 4) \ln^2 (s + 4)}
\]
and by (4.7), there exists a constant $C(d)$ depending on $d$ only such that

$$
\frac{4e}{\delta_s} \exp \left\{ 3 \left( \frac{14400d}{\delta_s^2} \right)^d \cdot \exp \left\{ d \left( \frac{24d}{\delta_s} \right)^{\frac{s}{s-1}} \right\} \right\}
\leq \exp \left\{ \exp \left\{ C(d) \left( (s+4) \ln^2 (s+4) \right)^{\frac{1}{s-1}} \right\} \right\}
\leq \exp \left\{ 0.01 \cdot |\ln \epsilon_0| \cdot \exp \left\{ s \ln \frac{3}{2} \right\} \right\} = \epsilon_s^{0.01},
$$

(3.54)

where the last inequality is based on $\sigma > 2$ and $\epsilon_0$ is very small depend on $d$. Using (3.21), (3.47) and (3.54), one has

$$
\left| \int_0^1 \{ R_0, s, F_s \} \circ X_{\rho_s}^t dt \right|_{\rho_s + 2\delta_s} \leq \epsilon_s^{1.54}.
$$

(3.55)

Following the proof of (3.54), we have

$$
\exp \left\{ 10d \left( \frac{10}{\delta_s} \right)^{\frac{s}{s-1}} \cdot \exp \left\{ \left( \frac{10}{\delta_s} \right)^{\frac{s}{s-1}} \right\} \right\} \leq \epsilon_s^{0.01}.
$$

(3.56)

Consequently from (3.14), (3.55) and (3.56), one has

$$
\left| \int_0^1 \{ R_0, s, F_s \} \circ X_{\rho_s}^t dt \right|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{1.53}.
$$

(3.57)

Secondly, we consider the term (3.51) and write

$$
(3.51) = \sum_{n \geq 1} \frac{1}{n!} \left\{ \cdots \{ R_{1,s}, F_s \}, F_s, \cdots, F_s \right\}
= \sum_{n \geq 1} \frac{1}{n!} \left\{ \cdots \{ R_{1,s}, F_{0,s} \}, F_s, \cdots, F_s \right\}
\quad \text{\text{n-fold}}
\quad \text{\text{(n-1)-fold}}
\quad \text{\text{(n-1)-fold}}
+ \{ R_{1,s}, F_{1,s} \}
+ \sum_{n \geq 2} \frac{1}{n!} \left\{ \cdots \{ R_{1,s}, F_{1,s} \}, F_s, \cdots, F_s \right\}.
$$

(3.58)

(3.59)

(3.60)

Note that (3.58) and (3.60) contribute to $R_{0+}, R_{1+}, R_{2+}$, and (3.59) contributes to $R_{1+}, R_{2+}$. Moreover, following the proof of (3.57), one has

$$
|| (3.58) ||_{\rho_s + 3\delta_s}^+, || (3.60) ||_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{1.53},
$$

(3.61)

$$
|| (3.59) ||_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{1.1}.
$$

(3.62)
Thirdly, we consider the term (3.52) and write

\[
(3.52) = \sum_{n \geq 1} \frac{1}{n!} \left\{ \cdots \left\{ R_{2,s}, F_0, \cdots, F_s, \cdots, F_s \right\} \right\}^{n-\text{fold}}
\]

\[
= \{ R_{2,s}, F_0, s \} + \{ R_{2,s}, F_1, s \} + \frac{1}{n!} \left\{ \cdots \left\{ R_{2,s}, F_0, s \right\}, F_s, \cdots, F_s \right\}^{(n-1)-\text{fold}}
\]

\[
+ \{ \{ R_{1,s}, F_1, s \}, F_s \} + \frac{1}{n!} \left\{ \cdots \left\{ R_{2,s}, F_1, s \right\}, F_s, \cdots, F_s \right\}^{(n-1)-\text{fold}}
\]

Note that (3.63) and (3.66) contribute to \( R_{1+s} \), (3.65) and (3.67) contribute to \( R_{0+s} \), (3.52) contributes to \( R_{1+s} \), and (3.64) contributes to \( R_{2+s} \).

Similarly, one has

\[
\left\| (3.63) \right\|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{0.93},
\]

\[
\left\| (3.65) \right\|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{1.53},
\]

and

\[
\left\| (3.64) \right\|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{0.53}.
\]

Finally, we consider the term (3.53) and write

\[
(3.53) = \sum_{n \geq 2} \frac{1}{n!} \left\{ \cdots \left\{ N_s, F_s, \cdots, F_s \right\} \right\}^{n-\text{fold}}
\]

\[
= \sum_{n \geq 2} \frac{1}{n!} \left\{ \cdots \left\{ -R_{0,s} - R_{1,s} + [R_{0,s}] + [R_{1,s}], F_s, \cdots, F_s \right\} \right\}^{(n-1)-\text{fold}}
\]

\[
= \sum_{n \geq 2} \frac{1}{n!} \left\{ \cdots \left\{ -R_{0,s} + [R_{0,s}], F_s, \cdots, F_s \right\} \right\}^{(n-1)-\text{fold}}
\]

\[
+ \sum_{n \geq 2} \frac{1}{n!} \left\{ \cdots \left\{ -R_{1,s} + [R_{1,s}], F_0, s, \cdots, F_s \right\} \right\}^{(n-2)-\text{fold}}
\]

\[
+ \{ -R_{1,s} + [R_{1,s}], F_1, s \} + \frac{1}{n!} \left\{ \cdots \left\{ -R_{1,s} + [R_{1,s}], F_1, s \right\}, F_s, \cdots, F_s \right\}^{(n-1)-\text{fold}}
\]

Note that (3.71), (3.72) and (3.74) contribute to \( R_{0+s} \), (3.71), (3.72) contributes to \( R_{1+s} \), (3.73) contributes to \( R_{2+s} \). Moreover, one has

\[
\left\| (3.71) \right\|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{1.53}
\]

and

\[
\left\| (3.73) \right\|_{\rho_s + 3\delta_s}^+ \leq \epsilon_s^{0.93}.
\]
Consequently we get
\[
\|R_{0,s+1}\|_{\rho_{s+1}}^{+} \leq 10\epsilon_s^{1.53} \leq \epsilon_{s+1},
\]
\[
\|R_{1,s+1}\|_{\rho_{s+1}}^{+} \leq 10\epsilon_s^{0.93} \leq \epsilon_{s+1}^{0.6}
\]
and
\[
\|R_{2,s+1}\|_{\rho_{s+1}}^{+} \leq (1 + d_s)\epsilon_0 + \epsilon_s^{0.5} \leq (1 + d_{s+1})\epsilon_0,
\]
which are just the assumptions (3.21)-(3.23) at stage \(s + 1\).

**Step 5.** \((\rho_s + 2\delta_s)-\text{Töplitz-Lipschitz property of the solution of homological equation}\)

Recall the solutions of homological equation are given in (3.2) and (3.3). Without loss of generality, it suffices to prove that given any \(n, m, l \in \mathbb{Z}^d\) and \(t \in \mathbb{Z}\) the following limit exists
\[
\lim_{t \to \infty} \frac{\partial^2 F_{0,s}}{\partial q_{n+tl} \partial \bar{q}_{m+tl}} = 0
\]
and there exists \(K_{s+1}\) such that when \(|t| > K_{s+1}\)
\[
\left\| \frac{\partial^2 F_{0,s}}{\partial q_{n+tl} \partial \bar{q}_{m+tl}} - \lim_{t \to \infty} \frac{\partial^2 F_{0,s}}{\partial q_{n+tl} \partial \bar{q}_{m+tl}} \right\|_{\rho_{s+2\delta_s}} \leq \epsilon_s^{0.95} \frac{|t|}{|t|}. 
\] (3.77)

For any \(a, k, k' \in \mathbb{N}^d\), we suppose that
\[
\prod_{i \geq 3} [n_i^*(a, k, k')] \leq \epsilon_s^{-0.01},
\]
which follows the proof of (3.39).

Note that if
\[
|t| \geq 2 \left( \epsilon_s^{-0.02} + \|n\|^2 + \|m\|^2 \right), 
\] (3.78)
then one has
\[
|n + tl|, |m + tl| > \epsilon_s^{-0.01} \geq |n_i^*(a, k, k')|.
\]

In view of momentum conservation (2.3), we have
\[
\frac{\partial^2 M_{akk'}}{\partial q_{n+tl} \partial \bar{q}_{m+tl}} = 0
\]
unless
\[
k_{n+tl} = k'_{m+tl} = 1.
\]

Here we assume that
\[
n + tl = n_i^*(a, k, k'), \quad m + tl = n_2^*(a, k, k').
\]

Furthermore, we have
\[
\|n + tl\|^2 - \|m + tl\|^2 = 2t (n - m) \cdot 1 + \|n\|^2 - \|m\|^2. 
\] (3.79)

**Case 1.**
\[
(n - m) \cdot 1 \neq 0.
\]

In view of (3.78) and (3.79), one has
\[
\sum_{j \in \mathbb{Z}^d} (k_j - k'_j) \left( |j|^2 + \bar{V}_{j,s} \right) \geq |t|. 
\] (3.80)
Following the proof of (3.54), one gets

\[
\left( \frac{12}{\epsilon \delta_s} \right)^2 \cdot \exp \left\{ \left( \frac{3600d}{\delta_s^2} \right)^d \cdot \exp \left\{ d \left( \frac{12d}{\delta_s} \right) \right\} \right\} \leq \epsilon_s^{-0.01}. \tag{3.81}
\]

Then we have

\[
\left| \left| \frac{\partial^2 F_{0,k,s}}{\partial q_{n+t \bar{q}_{m-t} t} \partial q_{m-t} t} \right| \right|_{\rho_s + 2\delta_s}^* = \left| \left| \sum_{a,k,k' \in \mathbb{N}^d} R_{akk'} \frac{\partial^2 M_{akk'}}{\partial q_{n+t \bar{q}_{m-t} t} \partial q_{m-t} t} \right| \right|_{\rho_s + 2\delta_s}^* \leq \frac{\| R_0 \|_{\rho_s + 2\delta_s}^*}{|t|} \quad \text{by (3.80)}
\]

\[
\leq \frac{\epsilon_s^{-0.01} \| R_0 \|_{\rho_s + \delta_s}}{|t|} \quad \text{by (2.26) in Lemma 2.6 and (3.81)}
\]

\[
\leq \frac{\epsilon_s^{0.98}}{|t|},
\]

where the last inequality uses (3.15) and (3.21).

Hence one has

\[
\lim_{t \to \infty} \frac{\partial^2 F_{0,s}}{\partial q_{n+t \bar{q}_{m-t} t}} = 0
\]

and

\[
\left| \left| \frac{\partial^2 F_{0,s}}{\partial q_{n+t \bar{q}_{m-t} t} \partial q_{m-t} t} \right| \right|_{\rho_s + 2\delta_s}^* = \frac{\epsilon_s^{0.98}}{|t|}.
\]

**Case 2.**

\[
(n - m) \cdot 1 = 0.
\]

In view of (3.79), one has

\[
\lim_{t \to \infty} \sum_{j \in \mathbb{Z}^d} (k_j - k_j') \left( \| j \|^2 + \bar{V}_{j,s} \right) = \sum_{j \in \mathbb{Z}^d} (k_j - k_j') \left( \| j \|^2 + \bar{V}_{j,s} \right) + \| n \|^2 - \| m \|^2 = A^{-1}
\]

and then

\[
\lim_{t \to \infty} F_{akk'} = A \cdot \lim_{t \to \infty} R_{akk'}.
\]
Furthermore, we have

\[
\left| F_{akk'} - \lim_{t \to \infty} F_{akk'} \right| \\
\leq \left| \left( \frac{1}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right) R_{akk'} \right| \\
+ \left| \left( \frac{1}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right) \left( R_{akk'} - \lim_{t \to \infty} R_{akk'} \right) \right| \\
= \left| \left( \frac{\tilde{V}_{n+tl,s} - \tilde{V}_{m+tl,s}}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right) R_{akk'} \right| \\
+ \left| \left( \frac{1}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right) \left( R_{akk'} - \lim_{t \to \infty} R_{akk'} \right) \right|. 
\]

In view of (3.19), one has

\[
\left| \tilde{V}_{n+tl,s} - \tilde{V}_{m+tl,s} \right| \leq \frac{4}{|t|}. \tag{3.82}
\]

Using (3.43) and (3.82) one gets

\[
\left| \frac{\tilde{V}_{n+tl,s} - \tilde{V}_{m+tl,s}}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right| \leq \frac{4\epsilon^{-0.02} |R_{akk'}|}{|t|}. \tag{3.83}
\]

By (3.43) again, we have

\[
\left| \left( \frac{1}{\sum_{j \in \mathbb{Z}^d}(k_j - k'_j)(\|j\|^2 + \tilde{V}_j,s)} \right) \left( R_{akk'} - \lim_{t \to \infty} R_{akk'} \right) \right| \leq \epsilon^{-0.01} \left| R_{akk'} - \lim_{t \to \infty} R_{akk'} \right|. \tag{3.84}
\]

Using (3.24), (3.83) and (3.84), we obtain

\[
\left\| \frac{\partial^2 F_{0,s}}{\partial \theta_{n+tl} \partial \bar{\theta}_{m-tl}} - \lim_{t \to \infty} \frac{\partial^2 F_{0,s}}{\partial \theta_{n+tl} \partial \bar{\theta}_{m-tl}} \right\|_{\rho - 2\delta} \leq \frac{\epsilon^{0.95}}{|t|}. 
\]

Finally, we point out that in view of momentum conservation (2.3), one has

\[
\frac{\partial^2 M_{akk'}}{\partial \theta_{n+tl} \partial \bar{\theta}_{m+tl}} = 0, \tag{3.85}
\]

if

\[
\|n - m\| \geq \epsilon^{-0.01} + 1. \tag{3.86}
\]

**Step 6.** \(\rho_{s+1}^\text{Töplitz-Lipschitz property of the remainder term}\)

It suffices to prove that \(\{F_{0,s}, R_{0,s}\}\) satisfies \(\rho_{s+1}^\text{Töplitz-Lipschitz property.}\) For simplicity, we write \(F_{0,s} := F\) and \(R_{0,s} := R.\)
Given any \( n, m \in \mathbb{Z}^d \) and \( t \in \mathbb{Z} \), one has

\[
\frac{\partial^2 \{F, R\}}{\partial q_{n+t} \partial q_{m+t}} = \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = 0.
\]

Without loss of generality we only need to show the terms (3.87) and (3.91) satisfies

\( \rho_{s+1} \)-Töplitz-Lipschitz property.

Now we will consider (3.87) firstly. In view of (3.85), one has

\[
\lim_{t \to \infty} \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} = F_{jm}^{11}, \quad \lim_{t \to \infty} \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} = R_{nj}^{11}.
\]

Then one has

\[
\lim_{t \to \infty} \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right).
\]

Next we will estimate

\[
\sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right) = \lim_{t \to \infty} \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial^2 F}{\partial q_{m+t} \partial q_{j+t}} \cdot \frac{\partial^2 R}{\partial q_{j+t} \partial q_{n+t}} \right).
\]
In view of (3.96) and (3.97), one has

\[
\|(3.98)\|^*_{\rho_{s+1}} = \left\| \sum_{j \in Z^d} \left( \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j} \cdot \frac{\partial^2 R}{\partial \eta_j + t \partial \eta_n + t} - F_{jm}^{11} R_{nj}^{11} \right) \right\|^*_{\rho_{s+1}}
\]

\[
\leq \left\| \sum_{j \in Z^d} \left( \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j + t} - F_{jm}^{11} \right) R_{nj}^{11} \right\|^*_{\rho_{s+1}}
\]

\[
+ \left\| \sum_{j \in Z^d} \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j} \cdot \left( \frac{\partial^2 R}{\partial \eta_j + t \partial \eta_n + t} - R_{nj}^{11} \right) \right\|^*_{\rho_{s+1}}
\]

\[
\leq \sum_{j \in Z^d} \left\| \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j + t} - F_{jm}^{11} \right\|^*_{\rho_{s+1}} \left\| R_{nj}^{11} \right\|^*_{\rho_{s+1}}
\]

\[
+ \sum_{j \in Z^d} \left\| \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j} \right\|^*_{\rho_{s+1}} \left\| \frac{\partial^2 R}{\partial \eta_j + t \partial \eta_n + t} - R_{nj}^{11} \right\|^*_{\rho_{s+1}},
\]

where the last inequality is based on (2.13) in Lemma 2.3.

On one hand, using (3.77) one has

\[
\left\| \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j + t} - F_{jm}^{11} \right\|^*_{\rho_{s+1}} \leq \frac{\epsilon_s}{|t|}.
\]

and

\[
\left\| \frac{\partial^2 R}{\partial \eta_j + t \partial \eta_n + t} - R_{nj}^{11} \right\|^*_{\rho_{s+1}} \leq \frac{\epsilon_s}{|t|}.
\]

which follows (3.24) and (2.14) in Lemma 2.4.

On the other hand, using (2.26) in Lemma 2.6 one gets

\[
\left\| \frac{\partial^2 F}{\partial \eta_m + t \partial \eta_j} \right\|^*_{\rho_{s+1}} \leq \epsilon_s^{0.95}
\]

and

\[
\left\| R_{nj}^{11} \right\|^*_{\rho_{s+1}} \leq \epsilon_s^{0.95},
\]

which follows from

\[
\left\| \frac{\partial^2 R}{\partial \eta_n + t \partial \eta_j + t} \right\|^*_{\rho_{s+1}} \leq \epsilon_s^{-0.01} \|R\|_{\rho_s} \leq \epsilon_s^{0.95}.
\]

Hence, we have

\[
\left\| (3.98) \right\|^*_{\rho_{s+1}} \leq \frac{\epsilon_s^{1.90}}{|t|}.
\]

Next we will estimate

\[
\sum_{j \in Z^d} \left( \frac{\partial^2 F}{\partial \eta_n + t \partial \eta_m + t \partial \eta_j} \cdot \frac{\partial R}{\partial \eta_j} \right) = \lim_{l \to \infty} \sum_{j \in Z^d} \left( \frac{\partial^2 F}{\partial \eta_n + t \partial \eta_m + t \partial \eta_j} \cdot \frac{\partial R}{\partial \eta_j} \right).
\]
Firstly one has
\[
\| (3.101) \|^\ast_{\rho_1 + 1} = \left\| \sum_{j \in \mathbb{Z}^d} \left( \frac{\partial}{\partial q_j} \left( \frac{\partial^2 F}{\partial \eta_{n+1} \partial \eta_{m+1}} - F_{11}^{nm} \right) \right) \cdot \frac{\partial R}{\partial \eta_j} \right\|^\ast_{\rho_1 + 1} 
\]
\[
\leq \sup_{j \in \mathbb{Z}^d} \left( \left\| \frac{\partial R}{\partial \eta_j} \right\|_{\rho_1 + 1} e^{\rho_1 + 1 \ln^\ast |j|} \right) \left( \sum_{j \in \mathbb{Z}^d} e^{-\rho_1 + 1 \ln^\ast |j|} \right) \left\| \frac{\partial}{\partial \eta_j} \left( \frac{\partial^2 F}{\partial \eta_{n+1} \partial \eta_{m+1}} - F_{11}^{nm} \right) \right\|^\ast_{\rho_1 + 1},
\]
(3.102)

where the last inequality is based on (2.13).

On one hand, for any \( j \in \mathbb{Z}^d \) one gets
\[
\left\| \frac{\partial R}{\partial \eta_j} \right\|_{\rho_1 + 1} e^{\rho_1 + 1 \ln^\ast |j|}
= \sup_{\|q\|_{\sigma, \rho_1 + 1} \leq 1} \left\| \frac{\partial R}{\partial \eta_j} \right\|_{\rho_1 + 1} e^{\rho_1 + 1 \ln^\ast |j|}
\leq \sup_{\|q\|_{\sigma, \rho_1 + 1} \leq 1} \| X_R \|_{\sigma, \rho_1 + 1}
\leq \exp \left\{ 10d \left( \frac{10000d}{\delta_s^2} \right) \cdot \exp \left\{ d \left( \frac{20d}{\delta_s} \right) \frac{1}{\sigma - 1} \right\} \right\} \| R \|_{\rho_1 + 28s}
\leq \epsilon_s^{0.95},
\]
(3.104)

where the last inequality is based on (3.15), (3.21) and (3.81). Therefore one has
\[
(3.102) \leq \epsilon_s^{0.95}.
\]

On the other hand, denote
\[
\frac{\partial^2 F}{\partial \eta_{n+1} \partial \eta_{m+1}} - F_{11}^{nm} = \sum_{a,k,k' \in \mathbb{N}^d} B_{akk'} \prod_{w \in \mathbb{Z}^d} I_w(0)^{a_w} q_w^{k_w} q_w^{k'w}.
\]

It follows easily that
\[
\frac{\partial}{\partial \eta_j} \left( \frac{\partial^2 F}{\partial \eta_{n+1} \partial \eta_{m+1}} - F_{11}^{nm} \right) = \sum_{a,k,k' \in \mathbb{N}^d} k_j B_{akk'} \prod_{w \in \mathbb{Z}^d} I_w(0)^{a_w} q_w^{k_w} - q_j^{k_w} q_j^{k'w}.
\]
Here we assume that \(|k_j| \geq 1\). Otherwise,
\[
\frac{\partial}{\partial \eta_j} \left( \frac{\partial^2 F}{\partial \eta_{n+1} \partial \eta_{m+1}} - F_{11}^{nm} \right) = 0.
\]
Then one has
\[ e^{-\left(\rho_s + \frac{\delta_s}{2}\right) \ln^* |j|} \left\| \frac{\partial^2 F}{\partial q_{n+i} \partial q_{m+t}} - F_{nm}^{11} \right\|_{\rho_s+1}^* \]
\[ = \sum_{a,k,k' \in \mathbb{Z}^d} |k_j B_{akk'}| \exp \left\{ \sum_{\omega \in \mathbb{Z}^d} \left( 2r a_w + \rho_{s+1} (k_w + k'_w) \right) \ln^* |n| \right\} \cdot e^{\frac{\delta_s}{2} \ln^* |j|} \]
\[ \leq \sum_{a,k,k' \in \mathbb{Z}^d} |k_j B_{akk'}| \exp \left\{ \sum_{\omega \in \mathbb{Z}^d} \left( 2r a_w + \left( \rho_s + \frac{5}{2} \delta_s \right) (k_w + k'_w) \right) \ln^* |n| \right\} \]
\[ \leq \sup_{a,k,k' \in \mathbb{Z}^d} \left( \sum_{\omega \in \mathbb{Z}^d} (k_w + k'_w) \ln^* |w| \right) \exp \left\{ -\frac{1}{2} \delta_s \sum_{\omega \in \mathbb{Z}^d} (k_w + k'_w) \ln^* |w| \right\} \]
\[ \times \left( \sum_{a,k,k' \in \mathbb{Z}^d} |B_{akk'}| \exp \left\{ -\sum_{\omega \in \mathbb{Z}^d} \left( 2r a_w + (\rho_s + 2\delta_s) (k_w + k'_w) \right) \ln^* |w| \right\} \right) \]
\[ \leq \left( \frac{2}{\epsilon \delta_s} \right) \cdot \left( \left\| \frac{\partial^2 F}{\partial q_{n+i} \partial q_{m+t}} - F_{nm}^{11} \right\|_{\rho_s+2\delta_s}^* \right) \]
where the last inequality uses (4.7) and (2.10) in Definition 2.2.

Thus we have
\[ (3.103) \quad \leq \left( \frac{2}{\epsilon \delta_s} \right) \cdot \left( \left\| \frac{\partial^2 F}{\partial q_{n+i} \partial q_{m+t}} - F_{nm}^{11} \right\|_{\rho_s+2\delta_s}^* \right) \sum_{j \in \mathbb{Z}^d} e^{-\frac{\delta_s}{2} \ln^* |j|} \]
\[ \leq \left( \frac{2}{\epsilon \delta_s} \right) \cdot \frac{\epsilon_s^{0.95}}{|t|} \cdot \left( \frac{12d}{\delta_s} \right)^d \cdot \exp \left\{ d \left( \frac{4d}{\delta_s} \right)^{\frac{1}{d}} \right\} \]
\[ \leq \frac{\epsilon_s^{0.93}}{|t|}, \]
where the last inequality is based on (3.54).

Therefore, we have
\[ \| (3.101) \|_{\rho_{s+1}}^* \leq \frac{\epsilon_s^{1.88}}{|t|} \leq \frac{\epsilon_{s+1}}{|t|}. \]

**Step 7. The frequency shift.**

In view of (3.9), the new normal form \( N_{s+1} \) is given by
\[ N_{s+1} = N_s + [R_{0,s}] + [R_{1,s}]. \]  (3.105)

Note that \([R_{0,s}](by (3.5)) is a constant which does not affect the Hamiltonian vector field. Moreover, in view of (3.6), we denote by
\[ \omega_{j,s} = |j|^2 + \tilde{V}_{j,s} + \sum_{a \in \mathbb{Z}^d} B_{a00}^{(d)} M_{a00}, \]  (3.106)

where the term
\[ \sum_{a \in \mathbb{Z}^d} B_{a00}^{(d)} M_{a00} \]
is the so-called frequency shift which will be estimated below. For any \( j \in \mathbb{Z}^d \), one has
\[
|\beta_{a00}^{(j)}| \leq \|R_{1,s+1}^{+}\|_{\rho_{s+1}} e^{2\rho_{s+1}\left(\sum_{n \in \mathbb{Z}^d} a_n \ln^n|n| + \ln^n|j| - \ln^n|j|_{n}(a,0,0,j)\) \right)}
\leq e^{0.6} \cdot e^{2\rho_{s+1}\left(\sum_{n \in \mathbb{Z}^d} a_n \ln^n|n| + \ln^n|j| - \ln^n|j|_{n}(a,0,0,j)\) \right)}.
\]

In view of (2.12) and (3.107), we obtain
\[
\sum_{a \in \mathbb{N}^d} \beta_{a00}^{(j)}M_{a00} \leq e^{0.6} \sum_{a \in \mathbb{N}^d} e^{2\rho_{s+1}\left(\sum_{n \in \mathbb{Z}^d} a_n \ln^n|n| + \ln^n|j| - \ln^n|j|_{n}(a,0,0,j)\) \right)} \cdot e^{-r\sum_{n \in \mathbb{Z}^d} 2a_n \ln^n|n|}
\leq e^{0.6} \prod_{n \in \mathbb{Z}^d} \left(1 - e^{-r\ln^n|n|}\right)^{-1} \quad \text{(by Lemma 4.5)}
\leq e^{0.55} \exp \left\{ \left(\frac{100d}{r^2}\right) \cdot \exp \left\{ d \cdot \left(\frac{2d}{r}\right)^{\frac{1}{r_0}}\right\} \right\} \quad \text{(by Lemma 4.6)}
\leq e^{0.55} s+1 \cdot . \quad (3.108)
\]

Recall that
\[
[R_1] = \sum_{j \in \mathbb{Z}^d} J^{(j)} \sum_{a \in \mathbb{N}^d} \beta_{a00}^{(j)}M_{a00},
\]
and then
\[
\frac{\partial^2 [R_1]}{\partial q_n \partial \bar{q}_n} = \sum_{a \in \mathbb{N}^d} B_{a00}^{(n)}M_{a00}.
\]

Note that
\[
\frac{\partial^2 [R_1]}{\partial q_n \partial \bar{q}_n} = \frac{\partial^2 R_{s+1}}{\partial q_n \partial \bar{q}_n} \bigg|_{q'=\hat{q}=0}
\]
and then one has
\[
\lim_{\|n\| \to \infty} \sum_{a \in \mathbb{N}^d} B_{a00}^{(n)}M_{a00} \text{ exists}
\]
and
\[
\left\| \sum_{a \in \mathbb{N}^d} B_{a00}^{(n)}M_{a00} - \lim_{\|n\| \to \infty} \sum_{a \in \mathbb{N}^d} B_{a00}^{(n)}M_{a00} \right\|_{\rho_{s+1}} \leq e^{0.5} s+1 \cdot . \quad (3.109)
\]

Hence let
\[
\hat{V}_{s+1} := \hat{V}_s + \lim_{\|n\| \to \infty} \sum_{a \in \mathbb{N}^d} B_{a00}^{(n)}M_{a00},
\]
which does not depend on \( n \) and satisfies (3.17) for \( s+1 \).

Let
\[
\hat{V}_{j,s+1} = \hat{V}_{j,s} + \left( \sum_{a \in \mathbb{N}^d} B_{a00}^{(j)}M_{a00} - \lim_{\|j\| \to \infty} \sum_{a \in \mathbb{N}^d} B_{a00}^{(j)}M_{a00} \right),
\]
and (3.19) with \( s + 1 \) satisfies by (3.109).

Next, if \( V \in C_{\frac{\lambda s}{10}} (V_s^*) \), by using Cauchy’s estimate implies

\[
\sum_{n \in \mathbb{Z}^d} \left| \frac{\partial \hat{V}_{j,s+1}}{\partial V_n} (V) \right| \leq \frac{2}{\eta_s} \left\| \hat{V}_s \right\|_{\infty} < 10\eta_s^{-1} \quad \text{(by (3.19))},
\]

(3.110)

and let \( X \in C_{\frac{\lambda s}{10} \eta_s} (V_s^*) \), then

\[
\left\| \hat{V}_s (X) - \omega \right\|_{\infty} = \left\| \hat{V}_s (X) - \hat{V}_s (V_s^*) \right\|_{\infty} \leq \sup_{C_{\frac{\lambda s}{10} \eta_s} (V_s^*)} \left\| \frac{\partial \hat{V}_s}{\partial V} \right\|_{1=\to 1} \cdot \| X - V_s^* \|_{\infty} \leq 10\eta_s^{-1} \cdot \frac{1}{10} \lambda s \eta_s \quad \text{(in view of (3.110))}.
\]

that is

\[
\hat{V}_s \left( C_{\frac{\lambda s}{10} \eta_s} (V_s^*) \right) \subseteq C_{\lambda s} (\omega).
\]

By (3.108), we have

\[
\left| \hat{V}_{j,s+1} - \hat{V}_{j,s} \right| < \epsilon_{s+1}^{0.6} \cdot \exp \left\{ 18 \cdot \exp \left\{ 4 \frac{\pi}{\eta} \right\} \right\} \epsilon_{s+1}^{0.59},
\]

which verifies (3.29). Further applying Cauchy’s estimate on \( C_{\frac{\lambda s}{10} \eta_s} (V_s^*) \), one gets

\[
\sum_{n \in \mathbb{Z}^d} \left| \frac{\partial \hat{V}_{j,s+1}}{\partial V_n} - \frac{\partial \hat{V}_{j,s}}{\partial V_n} \right| \leq \left\| \hat{V}_{s+1} - \hat{V}_s \right\|_{\infty} \leq \frac{\epsilon_{s+1}^{0.59}}{\lambda s \eta_s}.
\]

(3.111)

Since

\[
\eta_{s+1} = \frac{1}{20} \lambda_s \eta_s,
\]

hence one has

\[
\lambda_s \eta_s = 20 \prod_{i=0}^{s} \left( \frac{1}{20} \lambda_i \right) = 20 \prod_{i=0}^{s} \left( \frac{1}{20} \epsilon_{i}^{0.01} \right) \geq 20 \prod_{i=0}^{s} \epsilon_{i}^{0.02} \geq 20 \epsilon_{s+1}^{0.04}.
\]

(3.112)

On \( C_{\frac{\lambda s}{10} \eta_s} (V_s^*) \) and for any \( j \in \mathbb{Z}^d \), we deduce from (3.111), (3.112) and the assumption (3.20) that

\[
\sum_{n \in \mathbb{Z}^d} \left| \frac{\partial \hat{V}_{j,s+1}}{\partial V_n} - \hat{V}_{j,s} + \frac{\partial \hat{V}_{j,s}}{\partial V_n} - \delta_{jn} \right| \leq \sum_{n \in \mathbb{Z}^d} \left| \frac{\partial \hat{V}_{j,s+1}}{\partial V_n} - \frac{\partial \hat{V}_{j,s}}{\partial V_n} \right| + \sum_{n \in \mathbb{Z}^d} \left| \frac{\partial \hat{V}_{j,s}}{\partial V_n} - \delta_{jn} \right| \leq \epsilon_{s+1}^{0.5} + d_s \epsilon_0^{\frac{d}{s}} < d_s \epsilon_0^{\frac{d}{s+1}}.
\]
and consequently
\[
\left| \frac{\partial \hat{V}_{s+1}}{\partial V} - \text{Id} \right|_{l \to l} < d_{s+1} \epsilon_0^m, \tag{3.113}
\]
which verifies (3.20) for \( s + 1 \).

Finally, we will freeze \( \omega \) by invoking an inverse function theorem. Consider the following functional equation
\[
\hat{V}_{s+1}(X) = \omega, \quad X \in C_{\mu}^\lambda \eta_s(V^*_s), \tag{3.114}
\]
from (3.113) and the standard inverse function theorem implies (3.114) having a solution \( V^*_{s+1} \), which verifies (3.18) for \( s + 1 \). Rewriting (3.114) as
\[
V^*_{s+1} - V^*_s = (I - \hat{V}_{s+1})(V^*_{s+1}) - (I - \hat{V}_s)(V^*_s) + (\hat{V}_s - \hat{V}_{s+1})(V^*_s), \tag{3.115}
\]
and by using (3.111) and (3.113) one has
\[
\|V^*_{s+1} - V^*_s\|_{\infty} \leq (1 + d_{s+1}) \epsilon_0^m \|V^*_{s+1} - V^*_s\|_{\infty} + \epsilon_{s+1}^{0.59} < \lambda \eta_s,
\]
where the last inequality is based on (3.112), which verifies (3.30) and completes the proof of the iterative lemma. \( \square \)

3.4. Convergence. We are now in a position to prove the convergence. To apply iterative lemma with \( s = 0 \), set
\[
V^*_0 = \omega, \quad \hat{V}_0 = \text{Id}, \quad \epsilon_0 = C \epsilon, \tag{3.116}
\]
and consequently (3.18)-(3.23) with \( s = 0 \) are satisfied. Hence applying the iterative lemma, we obtain a decreasing sequence of domains \( D_s \times C_{\eta_s}(V^*_s) \) and a sequence of transformations
\[
\Phi^* = \Phi_1 \circ \cdots \circ \Phi_s : D_s \times C_{\eta_s}(V^*_s) \to D_0 \times C_{\eta_0}(V_0),
\]
such that \( H \circ \Phi^* = N_s + R_s \) for \( s \geq 1 \). Moreover, the estimates (3.27)-(3.30) hold. Thus we can show \( V^*_s \) converge to a limit \( V^* \) with the estimate
\[
\|V^* - \omega\|_{\infty} \leq \sum_{s=0}^{\infty} 2 \epsilon_{s+1}^{0.5} < \epsilon_0^{0.4},
\]
and \( \Phi^* \) converge uniformly on \( D_s \times \{V^*_s\} \), where
\[
D_s = \left\{ (q_n)_{n \in \mathbb{Z}^d} : \frac{2}{3} \leq |q_n| e^{r \ln^+ |n|} \leq \frac{5}{6} \right\},
\]
to \( \Phi : D_s \times \{V^*_s\} \to D_0 \) with the estimates
\[
\|\Phi - \text{Id}\|_{\sigma,r} \leq \epsilon_0^{0.4},
\]
\[
\|D\Phi - \text{Id}\|_{(\sigma,r) \to (\sigma,r)} \leq \epsilon_0^{0.4}.
\]
Hence
\[
H_s = H \circ \Phi = N_s + R_{2,s}, \tag{3.117}
\]
where
\[
N_s = \sum_{n \in \mathbb{Z}^d} \left( \|n\|^2 + \xi + \omega_n \right) |g_n|^2, \tag{3.118}
\]
\[
\xi = \sum_{s \geq 0} \hat{V}_s,
\]
By (2.16), the Hamiltonian vector field $X_{R^2,^\ast}$ is a bounded map from $H^{\sigma, r}$ into $H^{\sigma, r}$, and we get an invariant torus $T$ with frequency $(\|n\|^2 + \xi + \omega_n)_{n \in \mathbb{Z}^d}$ for $X_{H^\ast}$.

### 3.5. Proof of Theorem 1.1.

**Proof.** Expand $u$ into Fourier series by

$$u = \sum_{n \in \mathbb{Z}^d} q_n \phi_n(x),$$

where

$$\phi_n(x) = \frac{1}{(2\pi)^{d/2}} e^{\sqrt{-1} \sum_{i=1}^d n_i x_i}.$$

Then the Hamiltonian of equation (1.2) is given by

$$H(q, \bar{q}) = N(q, \bar{q}) + R(q, \bar{q}),$$

where

$$N(q, \bar{q}) = \sum_{n \in \mathbb{Z}^d} \left(\|n\|^2 + V_n\right) |q_n|^2$$

and

$$R(q, \bar{q}) = \epsilon \sum_{\|a\|=0, \|k\|+\|k'\|=4} B_{akk'} M_{akk'}.$$

Here

$$B_{akk'} = \frac{1}{(2\pi)^d},$$

if mass conservation (2.2) and momentum conservation (2.3) satisfy; otherwise

$$B_{akk'} = 0.$$

Then one has

$$\|R\|_{\rho_0} \leq \frac{\epsilon}{(2\pi)^d} := \epsilon_0.$$

Then the assumptions (3.116) in iterative lemma for $s = 0$ hold. Applying iterative lemma, $\Phi(T)$ is the desired invariant torus for the Hamiltonian (3.120). Moreover, we deduce the torus $\Phi(T)$ is linearly stable from the fact that (3.117) is a normal form of order 2 around the invariant torus. □

### 4. Appendix

#### 4.1. Technical Lemma.

**Lemma 4.1.** Given any $\sigma > 2$, there exists a constant $c(\sigma) > e^3$ depending on $\sigma$ only such that

$$\ln^\sigma (x + y) - \ln^\sigma x - \frac{1}{2} \ln^\sigma y \leq 0, \; \text{for} \; c(\sigma) \leq y \leq x.$$

(4.1)
Proof. Write $x = ty$ with $t \geq 1$, and then

$$
\ln^\sigma(x + y) - \ln^\sigma x - \frac{1}{2} \ln^\sigma y
= \ln^\sigma(ty + y) - \ln^\sigma(ty) - \frac{1}{2} \ln^\sigma y
= \ln^\sigma y \left( \left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - \left( 1 + \frac{\ln t}{\ln y} \right)^\sigma - \frac{1}{2} \right).
$$

To prove (4.1), it suffices to show that

$$
\sup_{t \geq 1, y \geq c(\sigma)} \left( \left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - \left( 1 + \frac{\ln t}{\ln y} \right)^\sigma \right) \leq \frac{1}{2}. \quad (4.2)
$$

Using differential mean value theorem, one has

$$
\left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - \left( 1 + \frac{\ln t}{\ln y} \right)^\sigma \leq \frac{\sigma}{\ln y} \left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - 1 \ln \left( \frac{1 + t}{t} \right).
$$

**Case 1.**

$$
\ln \left( \frac{1 + t}{t} \right) \leq 1.
$$

Then one has

$$
\frac{\sigma}{\ln y} \left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - 1 \ln \left( \frac{1 + t}{t} \right) \leq \frac{\sigma \cdot 2^{\sigma - 1}}{\ln y} \cdot \ln 2,
$$

where using

$$
\ln \left( \frac{1 + t}{t} \right) \leq 2.
$$

Taking

$$
c_1(\sigma) = \exp \left\{ \sigma \cdot 2^\sigma \cdot \ln 2 \right\}, \quad (4.3)
$$

we finish the proof of (4.2) in view of $y \geq c_1(\sigma)$.

**Case 2.**

$$
\frac{\ln(1 + t)}{\ln y} > 1.
$$

Then one has

$$
\frac{\sigma}{\ln y} \left( 1 + \frac{\ln(1 + t)}{\ln y} \right)^\sigma - 1 \ln \left( \frac{1 + t}{t} \right)
\leq \frac{\sigma \cdot 2^{\sigma - 1}}{\ln y} \cdot \ln \left( 1 + t - t \right) \cdot \ln \left( \frac{1 + t}{t} \right)
\leq \frac{\sigma \cdot 2^{\sigma - 1}}{\ln y} \cdot \left( \frac{\sigma - 1}{e} \right)^\sigma,
$$

where noting that

$$
\max_{t \geq 1} \left( \ln^{\sigma - 1}(1 + t) \cdot \ln \left( \frac{1 + t}{t} \right) \right) \leq 2 \left( \frac{\sigma - 1}{e} \right)^\sigma := c^*(\sigma).
$$

Taking

$$
c_2(\sigma) = \exp \left\{ (\sigma \cdot 2^\sigma \cdot c^*(\sigma))^\# \right\}, \quad (4.4)
$$

we finish the proof of (4.2) in view of $y \geq c_2(\sigma)$.

Finally letting

$$
c(\sigma) = \max\{c_1(\sigma), c_2(\sigma)\},
$$
we finish the proof of (4.1).

\textbf{Remark 4.1.} In view of (4.3) and (4.4), one has
\[ c_1(\sigma) \to 2^8 \]
and
\[ c_2(\sigma) \to \exp\left\{ \frac{4}{\sqrt{e}} \right\} \]
as \( \sigma \to 2 \). Hence we can take
\[ c(\sigma) := 2^{10}. \]

\textbf{Remark 4.2.} In view of (4.1), one has
\[ \ln^7 x + \ln^7 y - \ln^7 (x+y) \geq \frac{1}{2} \ln^7 y \geq 0. \quad (4.5) \]

\textbf{Lemma 4.2.} For \( \sigma > 2 \) and \( \delta \in (0,1) \), let \( f_{\sigma,\delta}(x) = e^{-\delta x^{\sigma}+x} \), then we have
\[ \max_{x \geq 0} f_{\sigma,\delta}(x) \leq \exp\left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{\sigma}} \right\}. \quad (4.6) \]

\textbf{Proof.} Since \[ f'_{\sigma,\delta}(x) = e^{-\delta x^{\sigma}+x} (-\delta \sigma x^{\sigma-1} + 1), \]
we have
\[ f'_{\sigma,\delta}(x) = 0 \iff x = \left( \frac{1}{\delta \sigma} \right)^{\frac{1}{\sigma}}. \]
Then one has
\[ \max_{x \geq 0} f_{\sigma,\delta}(x) = f_{\sigma,\delta}\left( \left( \frac{1}{\delta \sigma} \right)^{\frac{1}{\sigma}} \right) \leq \exp\left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{\sigma}} \right\} , \]
where the last inequality uses \( \sigma > 2 \). \( \square \)

\textbf{Lemma 4.3.} For \( p \geq 1 \) and \( \delta \in (0,1) \), let
\[ g_{p,\delta}(x) = x^p e^{-\delta x}, \]
then the following inequality holds
\[ \max_{x \geq 0} g_{p,\delta}(x) \leq \left( \frac{p}{e \delta} \right)^p. \quad (4.7) \]

\textbf{Proof.} Since \[ g'_{p,\delta}(x) = px^{p-1} e^{-\delta x} - \delta x^p e^{-\delta x}, \]
then we have
\[ g'_{p,\delta}(x) = 0 \iff x = \frac{p}{\delta}. \]
Then one has
\[ \max_{x \geq 0} g_{p,\delta}(x) = g_{p,\delta}\left( \frac{p}{\delta} \right) = \left( \frac{p}{e \delta} \right)^p. \]
\( \square \)

\textbf{Lemma 4.4.} For \( \sigma > 2 \) and \( \delta \in (0,1) \), we have
\[ \sum_{j \geq 1} e^{-\delta \ln^\sigma j} \leq \frac{6}{\delta} \cdot \exp\left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{\sigma}} \right\} . \quad (4.8) \]
Proof. Obviously, one has
\[ \sum_{j \geq 1} e^{-\delta \ln^\sigma j} \]
\[ \leq 2 + \int_1^{+\infty} e^{-\delta \ln^\sigma x} \, dx \]
\[ = 2 + \int_0^{+\infty} e^{-\delta y^\sigma} \, dy \]
\[ \leq 2 \exp \left\{ \left( \frac{2}{\delta} \right)^{\frac{1}{\sigma - 1}} \right\} \cdot \int_0^{+\infty} e^{-\frac{1}{2} \delta y} \, dy \quad \text{(by (4.6) and } \sigma > 2) \]
\[ \leq 2 \exp \left\{ \left( \frac{2}{\delta} \right)^{\frac{1}{\sigma - 1}} \right\} \cdot \left( 1 + \int_1^{+\infty} e^{-\frac{1}{2} \delta y} \, dy \right) \]
\[ \leq \frac{6}{\delta} \exp \left\{ \left( \frac{2}{\delta} \right)^{\frac{1}{\sigma - 1}} \right\}. \]

\[ \square \]

Lemma 4.5. For \( \sigma > 2 \) and \( \delta \in (0, 1) \), we have the following inequality
\[ \sum_{a \in \mathbb{N}^d} e^{-\delta \sum_{n \in \mathbb{Z}^d} a_n \ln^\sigma |n|} \leq \prod_{n \in \mathbb{Z}^d} \frac{1}{1 - e^{-\delta \ln^\sigma |n|}}. \tag{4.9} \]

Proof. By a direct calculation, one has
\[ \sum_{a \in \mathbb{N}^d} e^{-\delta \sum_{n \in \mathbb{Z}^d} a_n \ln^\sigma |n|} \leq \prod_{n \in \mathbb{Z}^d} \left( \sum_{a_n \in \mathbb{N}} e^{-\delta a_n \ln^\sigma |n|} \right) = \prod_{n \in \mathbb{Z}^d} \frac{1}{1 - e^{-\delta \ln^\sigma |n|}}. \]

\[ \square \]

Lemma 4.6. For \( \sigma > 2 \) and \( 0 < \delta \ll 1 \), then we have
\[ \prod_{n \in \mathbb{Z}^d} \frac{1}{1 - e^{-\delta \ln^\sigma |n|}} \leq \exp \left\{ \left( \frac{100d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{2d}{\delta} \right)^{\frac{1}{\sigma - 1}} \right\} \right\}. \tag{4.10} \]

Proof. If \( x \in (0, 3 - 2\sqrt{2}) \), one has
\[ \frac{1}{1 - e^{-x}} \leq \frac{1}{x^2} \tag{4.11} \]
and
\[ \ln \left( \frac{1}{1 - x} \right) \leq \sqrt{x}. \tag{4.12} \]

Let
\[ N = \exp \left\{ \left( \ln \left( 3 + 2\sqrt{2} \right) \delta^{-1} \right)^{\frac{1}{2}} \right\}. \]

If \( |n| > N \), then one has
\[ -\delta \ln^\sigma |n| \leq \ln \left( 3 - 2\sqrt{2} \right), \]
which implies
\[ e^{-\delta \ln^\sigma |n|} \leq 3 - 2\sqrt{2}. \tag{4.13} \]
Hence we have
\[
\prod_{n \in \mathbb{Z}^d} \frac{1}{1 - e^{-\delta \ln^p |n|}}
= \left( \prod_{|n| \leq N} \frac{1}{1 - e^{-\delta \ln^p |n|}} \right) \left( \prod_{|n| > N} \frac{1}{1 - e^{-\delta \ln^p |n|}} \right)
\leq \left( \prod_{|n| \leq N} \frac{1}{1 - e^{-\delta}} \right) \left( \prod_{|n| > N} \frac{1}{1 - e^{-\delta \ln^p |n|}} \right).
\]

On one hand, using (4.11) we have
\[
\prod_{|n| \leq N} \frac{1}{1 - e^{-\delta}} \leq \left( \frac{1}{\delta^2} \right)^{d(2N+1)^d} \leq \left( \frac{1}{\delta} \right)^{6^d N^d}.
\]
(4.14)

On the other hand, by (4.12) and (4.13) one has
\[
\prod_{|n| > N} \frac{1}{1 - e^{-\delta \ln^p |n|}}
= \exp \left\{ \sum_{|n| > N} \ln \left( \frac{1}{1 - e^{-\delta \ln^p |n|}} \right) \right\}
\leq \exp \left\{ \sum_{|n| > N} e^{-\frac{\delta}{2} \ln^p |n|} \right\}
\leq \exp \left\{ \left( 1 + 2 \sum_{j \geq 1} e^{-\frac{\delta}{2j} \ln^p j} \right)^d \right\}
\leq \exp \left\{ \left( \frac{20d}{\delta} \right)^d \cdot \exp \left\{ d \left( \frac{2d}{\delta} \right)^{\frac{1}{d}} \right\} \right\},
\]
(4.15)
where the last inequality is based on (4.8) in Lemma 4.4.

Combining (4.14) and (4.15), we finish the proof of (4.10) using \(0 < \delta \ll 1\).

\[\square\]

**Lemma 4.7.** For \(\sigma > 2\), \(\delta \in (0, 1)\), \(p = 1, 2\) and \(a = (a_n)_{n \in \mathbb{Z}^d} \in \mathbb{N}^{2^d}\), then we have
\[
\prod_{n \in \mathbb{Z}^d} (1 + a_n^p) e^{-2\delta a_n \ln^p |n|} \leq \exp \left\{ 3dp \left( \frac{p}{\delta} \right)^{\frac{1}{p-1}} \cdot \exp \left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{\sigma-1}} \right\} \right\}.
\]
(4.16)

**Proof.** Let
\[ f_{p,\sigma,\delta}(x) = x^p e^{-\delta \ln^p x}, \]
and one has
\[
\max_{x \geq 1} f_{p,\sigma,\delta}(x) \leq \exp \left\{ p \left( \frac{p}{\delta} \right)^{\frac{1}{p-1}} \right\}.
\]
(4.17)
In fact
\[ f'_{p,\sigma,\delta}(x) = px^{p-1} e^{-\delta \ln^p x} + x^{p-1} e^{-\delta \ln^p x} \left( -\delta \sigma \ln^{\sigma-1} x \right). \]
and then 
\[ f'_{p,\sigma,\delta}(x) = 0 \iff x = \exp \left\{ \left( \frac{p}{\sigma \delta} \right)^{\frac{1}{\sigma^2}} \right\}. \]

Hence one has 
\[ \max_{x \geq 1} f_{p,\sigma,\delta}(x) \leq f \left( \exp \left\{ \left( \frac{p}{\sigma \delta} \right)^{\frac{1}{\sigma^2}} \right\} \right) \leq \exp \left\{ \frac{p}{\delta} \right\}. \]

Note that 
\[ \prod_{n \in \mathbb{Z}^d} (1 + a_n^2) e^{-2\delta a_n \ln^\sigma |n|} = \prod_{\substack{n \in \mathbb{Z}^d \atop a_n \geq 1}} (1 + a_n^2) e^{-2\delta a_n \ln^\sigma |n|}. \]

Then we can assume \( a_n \geq 1 \) for \( \forall n \in \mathbb{Z}^d \) in what follows. Thus one has 
\[ (1 + a_n^2) e^{-2\delta a_n \ln^\sigma |n|} \leq (2a_n) e^{-2\delta a_n \ln^\sigma |n|} \]
\[ = \frac{1}{\ln^\sigma |n|} \cdot (2a_n \ln^\sigma |n|)^p e^{-2\delta a_n \ln^\sigma |n|} \]
\[ \leq \exp \left\{ \frac{p}{\delta} \right\}, \]
where the last inequality is based on (4.17) and using the fact that 
\[ \frac{1}{\ln^\sigma |n|} \leq 1. \]

On the other hand, for \( p = 1, 2 \) one has 
\[ (1 + a_n^p) e^{-2\delta a_n \ln^\sigma |n|} \leq 1 \]
(4.19)

when \( \ln^\sigma |n| \geq \delta^{-1} \).

Therefore, we have 
\[ \prod_{n \in \mathbb{Z}^d} (1 + a_n^p) e^{-2\delta a_n \ln^\sigma |n|} \]
\[ = \left( \prod_{\ln^\sigma |n| < \delta^{-1}} (1 + a_n^p) e^{-2\delta a_n \ln^\sigma |n|} \right) \left( \prod_{\ln^\sigma |n| \geq \delta^{-1}} (1 + a_n^p) e^{-2\delta a_n \ln^\sigma |n|} \right) \]
\[ \leq \prod_{\ln^\sigma |n| < \delta^{-1}} (1 + a_n^p) e^{-2\delta a_n \ln^\sigma |n|} \quad \text{(in view of (4.19))} \]
\[ \leq \prod_{\ln^\sigma |n| < \delta^{-1}} \exp \left\{ \frac{p}{\delta} \right\} \quad \text{(in view of (4.18))} \]
\[ \leq \left( \exp \left\{ \frac{p}{\delta} \right\} \right)^{3d \exp \left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{2}} \right\}} \]
\[ = \exp \left\{ 3dp \left( \frac{p}{\delta} \right)^{\frac{1}{\sigma^2}} \cdot \exp \left\{ \left( \frac{1}{\delta} \right)^{\frac{1}{2}} \right\} \right\}, \]
which finishes the proof of (4.16). \( \square \)
4.2. Measure Estimate.

**Lemma 4.8.** Let the set \( \Pi \), which is defined by (1.4), with probability measure. Then there exists a subset \( \Pi_\gamma \subset \Pi \) satisfying

\[
\text{meas } \Pi_\gamma \leq C\gamma, \tag{4.20}
\]

where \( C \) is a positive constant, such that for any \( \omega \in \Pi \setminus \Pi_\gamma \), the inequalities (1.5) and (1.6) holds.

**Proof.** Define the resonant set \( R_1 \) by

\[
R_1 = \bigcup_{\ell \in \mathbb{Z}^d \setminus 0} R_{\ell,j}, \tag{4.21}
\]

where

\[
R_{\ell,j} = \left\{ \omega : \left| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n + j \right| < \gamma \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + |\ell_n|^3 \langle n \rangle^{d+4}} \right\},
\]

for any \( 0 \neq \ell \in \mathbb{Z}^d \) with \( |\ell| < \infty \). Let

\[
m(\ell) = \min \{ \langle n \rangle : \ell_n \neq 0 \},
\]

and one has

\[
\text{meas } R_{\ell,j} \leq \gamma \cdot m(\ell) \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + |\ell_n|^3 \langle n \rangle^{d+4}}. \tag{4.22}
\]

In view of (1.4), one has

\[
\left| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n \right| \leq |\ell| \leq \prod_{n \in \mathbb{Z}^d, \ell_n \neq 0} |\ell_n| \langle n \rangle. \tag{4.23}
\]

Hence if

\[
|j| > 2 \prod_{n \in \mathbb{Z}^d, \ell_n \neq 0} |\ell_n| \langle n \rangle, \tag{4.24}
\]

then one has

\[
\left| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n + j \right| \geq |j| - \left| \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n \right| \geq 1, \tag{4.25}
\]

which implies the set \( R_{\ell,j} \) is empty. Therefore we always assume that

\[
|j| \leq 2 \prod_{n \in \mathbb{Z}^d, \ell_n \neq 0} |\ell_n| \langle n \rangle. \tag{4.26}
\]

Furthermore, we have

\[
\text{meas } R_1 \leq \gamma \sum_{\ell \in \mathbb{Z}^d} \sum_{j \text{ satisfies } (4.26)} m(\ell) \left( \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + |\ell_n|^3 \langle n \rangle^{d+4}} \right), \tag{4.27}
\]
and then
\[ \sum_{\ell \in \mathbb{Z}^d} \sum_{j \text{ satisfies (4.26)}} m(\ell) \left( \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + \| \ell_n^3(n) \|^{d+4}} \right) \]
\[ = \sum_{m \geq 1} \sum_{\ell \in \mathbb{Z}^d, m(\ell) \geq m, \ell_j \neq 0} m \left( \prod_{n \in \mathbb{Z}^d} \frac{1}{1 + \| \ell_n^3(n) \|^{d+4}} \right) \]
\[ \leq 4 \sum_{m \geq 1} m \prod_{n \in \mathbb{Z}^d, (n) \geq m} \left( \sum_{0 \neq \ell_n \in \mathbb{Z}} \frac{1}{\| \ell_n^2(n) \|^{d+3}} \right) \]
\[ \leq 4 \left( \sum_{0 \neq i \in \mathbb{Z}} i^{-2} \right) \left( \sum_{m \geq 1} m \prod_{n \in \mathbb{Z}^d, (n) \geq m} \frac{1}{\| n \|^{d+3}} \right) \]
\[ \leq 4 \left( \sum_{0 \neq i \in \mathbb{Z}} i^{-2} \right) \left( \sum_{n \in \mathbb{Z}^d} \frac{1}{\| n \|^{d+2}} \right) \]
\[ \leq C_1, \quad (4.28) \]
where \( C_1 \) is an absolutely positive constant. Combining (4.27) and (4.28), one has
\[ \operatorname{meas} \mathcal{R}_1 \leq C_1 \gamma. \quad (4.29) \]

Define the resonant set
\[ \mathcal{R}_2 = \bigcup_{\ell \notin \mathbb{Z}^d} \tilde{\mathcal{R}}_{\ell,j}, \quad (4.30) \]
where the resonant set \( \tilde{\mathcal{R}}_{\ell,j} \) is given by
\[ \tilde{\mathcal{R}}_{\ell,j} = \left\{ \omega : \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n + j < \frac{\gamma^3}{16} \prod_{n \notin \mathbb{Z}^d, \| n \|} \left( \frac{1}{1 + \| \ell_n^3(n) \|^{d+7}} \right)^4 \right\}, \quad (4.31) \]
for any \( 0 \neq \ell \in \mathbb{Z}^d \) satisfying \( 0 < \| \ell \| < \infty \) and
\[ \| n^*_{\ell}(\ell) \| > \| n^*_{\ell}(\ell) \|. \]
As (4.22), one has
\[ \operatorname{meas} \tilde{\mathcal{R}}_{\ell,j} \leq \frac{\gamma^3}{16} \cdot m(\ell) \prod_{n \notin \mathbb{Z}^d, \| n \|} \left( \frac{1}{1 + \| \ell_n^3(n) \|^{d+7}} \right)^4. \quad (4.32) \]
Without loss of generality, we assume that
\( n_1 = n^*_{\ell}(\ell), \quad n_2 = n^*_{\ell}(\ell) \). Then one has
\[ \sum_{n \in \mathbb{Z}^d} \ell_n \omega_n = \left( \sum_{n \notin \mathbb{Z}^d, n \neq n_1, n_2} \ell_n \omega_n \right) + \sigma_{n_1} \omega_{n_1} + \sigma_{n_2} \omega_{n_2}, \quad (4.33) \]
where \(\sigma_{n_1}, \sigma_{n_2} \in \{-1, 1\}\). Define \(\tilde{\ell} = (\tilde{\ell}_n)_{n \in \mathbb{Z}^d}\) by
\[
\tilde{\ell}_n = \ell_n, \quad \text{if } n \neq n_1, n_2;
\]
and
\[
\tilde{\ell}_n = 0, \quad \text{if } n = n_1, n_2.
\]

If \(\tilde{\ell} = 0\), using **Momentum Conversation** (2.3) one has
\[
\sigma_{n_1} \cdot \sigma_{n_2} = -1 \tag{4.34}
\]
and
\[
n_1 = n_2, \tag{4.35}
\]
which implies \(\ell = 0\). Therefore we always assume that \(\tilde{\ell} \neq 0\).

Hence, if \(\omega \in \Pi \setminus \mathcal{R}_1\) (where \(\mathcal{R}_1\) is defined in (4.21)) and
\[
\|n^*_2(\ell)\| \geq \frac{4}{\gamma} \prod_{n \in \mathbb{Z}^d \setminus |n| \leq |n^*_2(\ell)|} \left(1 + |\ell_n|^3(n)^{d+4}\right), \tag{4.36}
\]
then one has
\[
\left|\sum_{n \in \mathbb{Z}^d} \ell_n \omega_n + j\right| \geq \left|\sum_{n \in \mathbb{Z}^d} \ell_n \omega_n\right| \geq \left|\sum_{n \in \mathbb{Z}^d, n \neq n_1, n_2} \ell_n \omega_n\right| - |||\sigma_{n_1} \omega_{n_1} + \sigma_{n_2} \omega_{n_2}||| \quad \text{(by (4.33))}
\]
\[
\geq \gamma \prod_{n \in \mathbb{Z}^d, n \neq n_1, n_2} \frac{1}{1 + |\ell_n|^3(n)^{d+4}} - \frac{2}{2}
\]
\[
\geq \frac{\gamma}{2} \prod_{n \in \mathbb{Z}^d, n \neq n_1, n_2} \frac{1}{1 + |\ell_n|^3(n)^{d+4}},
\]
where the last inequality is based on (4.36). Therefore we always assume
\[
\|n^*_2(\ell)\| < \frac{4}{\gamma} \prod_{n \in \mathbb{Z}^d, n \neq n_1, n_2} \left(1 + |\ell_n|^3(n)^{d+4}\right) := A(\ell). \tag{4.37}
\]

In view of the following two inequalities
\[
\|n^*_1(\ell)\| \leq \sum_{i \geq 2} \|n^*_i(\ell)\|,
\]
and
\[
\sum_{i \geq 2} \|n^*_i(\ell)\| = \sum_{n \in \mathbb{Z}^d, n \neq n_1, n_2} |\ell_n| \cdot \|n\| \leq \prod_{n \in \mathbb{Z}^d, n \neq n_1} \left(1 + |\ell_n|^3(n)^{d+4}\right),
\]
one has
\[ \|n_1^{*}(\ell)\| < A(\ell) \left( \prod_{n \in \mathbb{Z}^d} (1 + |\ell_n|^3(n)^{d+1}) \right) := B(\ell). \tag{4.38} \]
Hence if
\[ |j| > 2 \left( \prod_{n \in \mathbb{Z}^d, n \neq 0} |\ell_n|n \right) A(\ell) B(\ell), \]
then one has
\[ \left| \sum_{n \in \mathbb{Z}^d} \ell_n^{n_1} + j \right| \geq |j| - \left| \sum_{n \in \mathbb{Z}^d} \ell_n^{n_1} \right| \geq 1, \tag{4.39} \]
which implies the set $\tilde{R}_{\ell,j}$ is empty. Therefore we always assume that
\[ |j| \leq 2 \left( \prod_{n \in \mathbb{Z}^d, n \neq 0} |\ell_n|n \right) A(\ell) B(\ell). \tag{4.40} \]
In view of (4.32), (4.37), (4.38) and following the proof of (4.29), one has
\[ \text{meas } R_2 \leq C_2 \gamma, \tag{4.41} \]
where $C_2$ is a positive constant.
Letting
\[ \Pi_\gamma = R_1 \bigcup R_2, \]
then one has
\[ \text{meas } \Pi_\gamma \leq C \gamma, \tag{4.42} \]
and for any $\omega \in \Pi \setminus \Pi_\gamma$, the inequalities (1.5) and (1.6) holds.
\[ \square \]

4.3. Proof of Lemma 2.6.

Proof. Without loss of generality, it suffices to prove
\[ \left\| \frac{\partial^2 R}{\partial q_m \partial q_n} \right\|_{\rho + \delta} \leq \left( \frac{12}{e \delta} \right)^2 \exp \left\{ \left( \frac{3600d}{\delta^2} \right)^d \exp \left\{ d \left( \frac{12d}{\delta} \right)^{\frac{1}{\sigma}} \right\} \right\} \|R\|_{\rho}. \tag{4.43} \]
For any $m, l \in \mathbb{Z}^d$, one has
\[ \frac{\partial^2 R}{\partial q_m \partial q_n} = \sum_{a, k, k' \in \mathbb{N}^d} k_mk_1R_{akk'} \prod_{n \in \mathbb{Z}^d} I(0)^{a,l}q^{-\epsilon_m - \epsilon_{k,k'}}. \]
In view of (2.10), one obtains
\[
\left\| \frac{\partial^2 R}{\partial \eta_m \partial \eta_l} \right\|_{\rho + \delta}^* = \sum_{a,k,k' \in \mathbb{N}^d} |k_m k_l R_{a,k,k'}| \exp \left\{ - \sum_{n \in \mathbb{Z}^d} 2 \alpha_n \ln^\sigma [n] \right\} \times \exp \left\{ - (\rho + \delta) \sum_{n \in \mathbb{Z}^d} \left( k_n + k'_n \right) \ln^\sigma [n] + (\rho + \delta) \left( \ln^\sigma [m] + \ln^\sigma [1] \right) \right\} \leq \sum_{a,k,k' \in \mathbb{N}^d} |k_m k_l R_{a,k,k'}| \exp \left\{ -(\rho + \delta) \left( \sum_{n \in \mathbb{Z}^d} \left( 2 \alpha_n + k_n + k'_n \right) \ln^\sigma [n] - \ln^\sigma [m] - \ln^\sigma [1] \right) \right\},
\]
where the last inequality uses $r > \rho + \delta$.

In view of (2.9), one has
\[
|R_{a,k,k'}| \leq \|R\|_{\rho} \exp \left\{ \rho \left( \sum_{n \in \mathbb{Z}^d} \left( 2 \alpha_n + k_n + k'_n \right) \ln^\sigma [n] - 2 \ln^\sigma [n_1^a(a,k,k')] \right) \right\}
\]
and then
\[
\left\| \frac{\partial^2 R}{\partial \eta_m \partial \eta_l} \right\|_{\rho + \delta}^* \leq \mathcal{C} \|R\|_{\rho},
\]
where
\[
\mathcal{C} = \sum_{a,k,k' \in \mathbb{N}^d} k_m k_l \exp \left\{ -\delta \left( \sum_{i \geq 1} \ln^\sigma [n_i(a,k,k')] \right) + \delta \left( \ln^\sigma [m] + \ln^\sigma [1] \right) \right\}.
\]
To prove the inequality (4.43) holds, it suffices to show that
\[
\mathcal{C} \leq \left( \frac{12}{\epsilon \delta} \right)^2 \cdot \exp \left\{ \left( \frac{3600d}{\delta^2} \right)^d \cdot \exp \left\{ d \left( \frac{12d}{\delta} \right)^{1/3} \right\} \right\}, \quad (4.44)
\]
which will be discussed in the following three cases:

**Case 1.**
\[
\max \{|m|, |1\} \leq |n_2^a(a,k,k')|.
\]

Then one has
\[
- \delta \left( \sum_{i \geq 1} \ln^\sigma [n_i(a,k,k')] \right) + \delta \left( \ln^\sigma [m] + \ln^\sigma [1] \right) \leq - \frac{\delta}{3} \left( \sum_{i \geq 1} \ln^\sigma [n_i(a,k,k')] \right), \quad (4.45)
\]
Using (2.20) and (4.45), we have

\[
C \leq \sum_{a, k, k' \in \mathbb{N}^d} \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right)^2 \exp \left\{ -\frac{\delta}{3} \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right) \right\} \\
\leq \sup \left\{ \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right)^2 \exp \left\{ -\frac{\delta}{6} \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right) \right\} \right\} \\
\times \sum_{a, k, k' \in \mathbb{N}^d} \exp \left\{ -\frac{\delta}{6} \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right) \right\} \\
\leq \left( \frac{12}{e^\delta} \right)^2 \exp \left\{ \left( \frac{3600d}{\delta^2} \right)^d \exp \left\{ d \left( \frac{12d}{\delta^2} \right)^{\frac{1}{d-1}} \right\} \right\} ,
\]

(4.46)

where the last inequality follows from (4.7) in Lemma 4.3, (4.9) in Lemma 4.5 and (4.10) in Lemma 4.6.

**Case 2.**

\[
\min\{ |m|, |l| \} > |n_3^* (a, k, k')|.
\]

In this case, one has

\[
|k_m| + |k_l| \leq 2.
\]

(4.47)

Without loss of generality, we assume

\[
n_1^* (a, k, k') = 1, \quad n_2^* (a, k, k') = m.
\]

In view of (4.47), we thus obtain

\[
C \leq 5 \sum_{a, k, k' \in \mathbb{N}^d, n_1^* (a, k, k') = 1, n_2^* (a, k, k') = m} \exp \left\{ -\delta \left( \sum_{i \geq 1} \ln^\sigma [n_i^* (a, k, k')] \right) \right\} \\
\leq 5 \exp \left\{ \left( \frac{100d}{\delta^2} \right)^d \exp \left\{ d \left( \frac{2d}{\delta^2} \right)^{\frac{1}{d-1}} \right\} \right\} ,
\]

(4.48)

where the last inequality follows from (4.9) in Lemma 4.5 and (4.10) in Lemma 4.6.

**Case 3.**

\[
|m| > |n_3^* (a, k, k')|, \quad |l| \leq |n_2^* (a, k, k')|
\]
or

\[
|l| > |n_1^* (a, k, k')|, \quad |m| \leq |n_2^* (a, k, k')|
\]

Without loss of generality, we assume

\[
|m| > n_3^* (a, k, k'), \quad |l| \leq |n_2^* (a, k, k')|
\]

and \( m = n_1^* (a, k, k') \). In this case one has

\[
|k_m| \leq 2,
\]

and then

\[
C \leq 2 \sum_{a, k, k' \in \mathbb{N}^d, n_1^* (a, k, k') = m} k_l \exp \left\{ -\delta \left( \sum_{i \geq 2} \ln^\sigma [n_i^* (a, k, k')] \right) \right\} .
\]
Following the proof of Case 1., we have
\[ C \leq \left( \frac{8}{\epsilon \delta} \right) \cdot \exp \left\{ \left( \frac{1600d}{\delta^2} \right)^d \cdot \exp \left\{ 2^{d-1} \right\} \right\}. \] (4.49)

In view of (4.46), (4.48) and (4.49), we finish the proof of (4.44).

\[ \square \]

4.4. Proof of Lemma 2.8.

Proof. Let
\[ R_1(q, \bar{q}) = \sum_{a,k,k' \in \mathbb{N}^d} b_{akk'} \mathcal{M}_{akk'} \]
and
\[ R_2(q, \bar{q}) = \sum_{A,K,K' \in \mathbb{N}^d} B_{AKK'} \mathcal{M}_{AKK'}. \]
Then one has
\[ \{ R_1, R_2 \} = \sum_{a,k,k',A,K,K' \in \mathbb{N}^d} b_{akk'} B_{AKK'} \{ \mathcal{M}_{akk'}, \mathcal{M}_{AKK'} \}, \]
where
\[ \{ \mathcal{M}_{akk'}, \mathcal{M}_{AKK'} \} = i \sum_{j \in \mathbb{Z}^d} \left( \prod_{n \neq j} I_n(0)^{a_n + A_n q_n^{k_n} + K_n q_n^{k'_n} + K_n} \right) \times \left( (k_j K_j' - k'_j K_j) I_j(0)^{a_j + A_j q_j^{k_j} + K_j - 1} (k_j K_j' - 1) \right). \]
Given any \( \alpha, \kappa, \kappa' \in \mathbb{N}^d \), the coefficient of the monomial \( \mathcal{M}_{akk'} \) in \( \{ R_1, R_2 \} \) is given by
\[ B_{akk'} = i \sum_{j \in \mathbb{Z}^d} \sum_{\ast} \sum_{\ast \ast} (k_j K_j' - k'_j K_j) b_{akk'} B_{AKK'} \] (4.50)
where
\[ \sum_{\ast} = \sum_{a,A \in \mathbb{N}^d}, \]
and
\[ \sum_{\ast \ast} = \sum_{k,k',K,K' \in \mathbb{N}^d} \text{ when } n \neq j, k_n + K_n = \kappa_n, k'_n + K'_n = \kappa'_n \text{ and } n = j, k_n + K_n - 1 = \kappa_n, k'_n + K'_n - 1 = \kappa'_n. \]

To estimate (2.30), some simple facts are given firstly:
1. If \( j \notin \text{ supp } (k + k') \cap \text{ supp } (K + K') \), then
\[ k_j K_j' - k'_j K_j = 0. \]
Hence we always assume \( j \in \text{ supp } (k + k') \cap \text{ supp } (K + K') \), which implies
\[ |j| \leq \min \left\{ |n_j^1(a, k, k')|, |n_j^1(A, K, K')| \right\}. \] (4.51)
Since
\[ \text{ supp } (\alpha, \kappa, \kappa') \subset \text{ supp } (a, k, k') \cup \text{ supp } (A, K, K'), \]
the following inequality always holds
\[ |n_j^1(\alpha, \kappa, \kappa')| \leq \max \left\{ |n_j^1(a, k, k')|, |n_j^1(A, K, K')| \right\}. \] (4.52)
Combining (4.51) and (4.52), one has
\[
\ln^\sigma |j| + \ln^\sigma [n^*_i(a, \kappa, \kappa')] - \ln^\sigma [n^*_i(a, k, k')] - \ln^\sigma [n^*_i(A, K, K')] \leq 0. \tag{4.53}
\]
2. Note that
\[
\sum_{i \geq 1} \ln^\sigma [n^*_i(a, k, k')] \geq \sum_{i \geq 1} \ln^\sigma [n^*_i(a, k, k')] \geq \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n), \tag{4.54}
\]
where using \(\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \geq 4\).

Based on (4.54), we obtain
\[
\sum_{n \in \mathbb{Z}^d} (k_n + k'_n)(K_n + K'_n) \leq \left( \sum_{i \geq 3} \ln^\sigma [n^*_i(a, k, k')] \right) \left( \sum_{i \geq 1} \ln^\sigma [n^*_i(A, K, K')] \right), \tag{4.55}
\]
3. It is easy to see that
\[
\sum_{n \in \mathbb{Z}^d} (2a_n + \kappa_n + \kappa'_n) = \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) + \sum_{n \in \mathbb{Z}^d} (2A_n + K_n + K'_n) - 2 \tag{4.56}
\]
and
\[
\sum_{i \geq 1} \ln^\sigma [n^*_i(a, \kappa, \kappa')] = \left( \sum_{i \geq 1} \ln^\sigma [n^*_i(a, k, k')] \right) + \left( \sum_{i \geq 1} \ln^\sigma [n^*_i(A, K, K')] \right) - 2\ln^\sigma |j|. \tag{4.57}
\]
In view of (2.9) and (2.5) in Lemma 2.1, one has
\[
|b_{\alphaKK'}| \leq ||R_1||_{p-\delta_1} e^{\delta (\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2\ln^\sigma [n^*_i(a, k, k')])} e^{-\frac{1}{2} \delta_1 (\sum_{i \geq 3} \ln^\sigma [n^*_i(a, k, k')])}, \tag{4.58}
\]
and
\[
|B_{AKK'}| \leq ||R_2||_{p-\delta_2} e^{\delta (\sum_{n \in \mathbb{Z}^d} (2A_n + K_n + K'_n) \ln^\sigma |n| - 2\ln^\sigma [n^*_i(A, K, K')])} e^{-\frac{1}{2} \delta_2 (\sum_{i \geq 3} \ln^\sigma [n^*_i(A, K, K')])}. \tag{4.59}
\]
Using (4.57), substitution of (4.58) and (4.59) in (4.50) gives
\[
|B_{\alphaKK'}| \leq ||R_1||_{p-\delta_1} ||R_2||_{p-\delta_2} \sum_{j \in \mathbb{Z}^d} \sum_{*} \sum_{**} |k_j K'_j - k_j K_j| \]
\[
\times e^{\delta (\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2\ln^\sigma [n^*_i(a, k, k')])} \]
\[
\times e^{\delta (\sum_{n \in \mathbb{Z}^d} (2A_n + K_n + K'_n) \ln^\sigma |n| - 2\ln^\sigma [n^*_i(A, K, K')])} \]
\[
\times e^{-\frac{1}{2} \delta_1 (\sum_{i \geq 3} \ln^\sigma [n^*_i(a, k, k')])} \cdot e^{-\frac{1}{2} \delta_2 (\sum_{i \geq 3} \ln^\sigma [n^*_i(A, K, K')])}.
\]
where
\[
A = \sum_{j \in \mathbb{Z}^d} \sum_{*} \sum_{**} |k_j K'_j - k_j K_j| \]
\[
\times e^{\delta (\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln^\sigma |n| - 2\ln^\sigma [n^*_i(a, k, k')])} \]
\[
\times e^{-\frac{1}{2} \delta_1 (\sum_{i \geq 3} \ln^\sigma [n^*_i(a, k, k')])} \cdot e^{-\frac{1}{2} \delta_2 (\sum_{i \geq 3} \ln^\sigma [n^*_i(A, K, K')])}.
\]
To show (2.30) holds, it suffices to prove
\[
A \leq \frac{1}{\delta_2} \cdot \exp \left( 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp \left( d \left( \frac{24d}{\delta_1} \right)^{\frac{d}{2}} \right) \right). \tag{4.60}
\]
Now we will prove the inequality (4.60) holds in the following two cases:

Case. 1. $[n_1^*(a, \kappa, \kappa')] \leq [n_1^*(A, K, K')]$.

Case. 1.1. $[j] \leq [n_3^*(a, k, k')]$.  \hfill (4.61)

Using (4.61), one has

$$e^{2p(\ln^n |j| - \ln^n |n_1^*(a, k, k')|)} \leq e^{2p(\ln^n |n_3^*(a, k, k')| - \ln^n |n_1^*(a, k, k')|)} \leq e^{\delta_1 (\ln^n |n_3^*(a, k, k')| - \ln^n |n_1^*(a, k, k')|)}$$ \hfill (4.62)

where the last inequality is based on $0 < \delta_1 \leq \frac{1}{2} \rho$, and then

$$e^{2p(\ln^n |j| + \ln^n |n_1^*(a, k, k')| - \ln^n |n_1^*(A, K, K')|)} \leq e^{\delta_1 (\sum_{i \geq 3} \ln^n |n_1^*(a, k, k')|)}$$ \hfill (4.63)

which follows from (4.52) and (4.62).

Note that if $j, a, k, k'$ are specified, and then $A, K, K'$ are uniquely determined. In view of (4.55) and (4.63), we have

$$A \leq \sum_{a, k, k' \in \mathbb{N}^{d}} \left( \sum_{i \geq 1} \ln^n |n_1^*(a, k, k')| \right) \left( \sum_{i \geq 3} \ln^n |n_1^*(A, K, K')| \right)$$

$$\times e^{-\frac{1}{2} \delta_1 (\sum_{i \geq 1} \ln^n |n_1^*(a, k, k')|)} e^{-\frac{1}{2} \delta_1 (\sum_{i \geq 3} \ln^n |n_1^*(A, K, K')|)}$$

$$\leq \frac{48}{e^{2 \delta_1 \delta_2}} \sum_{a, k, k' \in \mathbb{N}^{d}} e^{-\frac{1}{2} \delta_1 (\sum_{n \in \mathbb{Z}^d} (2a_n + k_n) \ln^n |n|)}$$

$$= \frac{48}{e^{2 \delta_1 \delta_2}} \sum_{a, k, k' \in \mathbb{N}^{d}} e^{-\frac{1}{2} \delta_1 (\sum_{n \in \mathbb{Z}^d} 2a_n \ln^n |n|)} \left( \sum_{k \in \mathbb{N}^{d}} e^{-\frac{1}{2} \delta_1 (\sum_{n \in \mathbb{Z}^d} k_n \ln^n |n|)} \right)^2$$

$$\leq \frac{48}{e^{2 \delta_1 \delta_2}} \prod_{n \in \mathbb{Z}^d} \left( 1 - e^{-\frac{1}{2} \delta_1 \ln^n |n|} \right)^{-1} \left( 1 - e^{-\frac{1}{2} \delta_1 \ln^n |n|} \right)^{-2}$$

(which is based on Lemma 4.5)

$$\leq \frac{1}{\delta_2} \cdot \exp \left\{ 3 \left( \frac{14400 d}{\delta_1} \right)^d \exp \left\{ d \left( \frac{24d}{\delta_1} \right)^{-\frac{1}{2}} \right\} \right\},$$

where the last inequality uses (4.10) in Lemma 4.6.

Case. 1.2. $[j] > [n_3^*(a, k, k')]$.  \hfill (4.64)

In this case, we have

$$j \in \{ n_1^*(a, k, k'), n_3^*(a, k, k') \}. \hfill (4.65)$$

Furthermore, if $2a_j + k_j + k'_j > 2$, then $|j| \leq [n_3^*(a, k, k')]$, we are in Case. 1.1.

Hence in what follows, we always assume $2a_j + k_j + k'_j \leq 2,$
which implies
\[ k_j + k_j' \leq 2. \] (4.66)

For simplicity, for \( i \geq 1 \) denote by
\[ n_i = n^*_i(a, k, k') \]
and
\[ N_i = n^*_i(A, K, K'). \]

From (4.53), (4.65) and (4.66), it follows that
\[ A \leq 2 \sum_{a,k,k' \in \mathbb{Z}^d} (K_{n_1} + K'_{n_1} + K_{n_2} + K'_{n_2}) \cdot e^{-\frac{1}{2}(\delta_1(\sum_{i \geq 3} \ln^r |n_i|) + \delta_2(\sum_{i \geq 3} \ln^r |N_i|))}. \]

In view of (4.54) and (4.56), we have
\[ \sum_{n \in \mathbb{Z}^d} (2\alpha_n + \kappa_n + \kappa'_n) \leq \left( \sum_{i \geq 3} \ln^r |n_i| \right) + \left( \sum_{i \geq 3} \ln^r |N_i| \right). \] (4.67)

Moreover, note that for any \( j \in \mathbb{Z}^d \),
\[ K_j + K'_j \leq \kappa_j + \kappa'_j - k_j - k'_j + 2 \leq \kappa_j + \kappa'_j + 2. \] (4.68)

Using (4.67) and (4.68), one has
\[ A \leq 2 \sum_{a,k,k' \in \mathbb{Z}^d} (\kappa_{n_1} + \kappa'_{n_1} + \kappa_{n_2} + \kappa'_{n_2} + 4) \]
\[ \times e^{-\frac{1}{2}\delta_1(\sum_{i \geq 3} \ln^r |n_i|)} e^{-\frac{1}{2}\delta_2(\sum_{n \in \mathbb{Z}^d} (2\alpha_n + \kappa_n + \kappa'_n))}, \] (4.69)
where \( \delta = \min\{\delta_1, \delta_2\} \).

**Remark 4.3.** Firstly, note that \( \{n_1, n_2\} \cap \text{supp } M_{\alpha \kappa \kappa'} \neq \emptyset \). Thus \( n_1 \) (or \( n_2 \)) ranges in a set of cardinality no more than
\[ \# \text{supp } M_{\alpha \kappa \kappa'} \leq \sum_{n \in \mathbb{Z}^d} (2\alpha_n + \kappa_n + \kappa'_n). \] (4.70)

Secondly, if \( (n_i)_{i \geq 3} \) and \( n_1 \) (resp. \( n_2 \)) is specified, then \( n_2 \) (resp. \( n_1 \)) is determined uniquely. Thirdly, if \( (n_i)_{i \geq 1} \) is given, then \( n(a, k, k') \) is specified, and hence \( n(a, k, k') \) is specified up to a factor of
\[ \prod_{n \in \mathbb{Z}^d} (1 + l_n^2), \] (4.71)
where
\[ l_n = \# \{j : n_j = n\}. \]

Since \( |n_1| \geq |n_2| > |n_3| \), one has
\[ \prod_{n \in \mathbb{Z}^d} (1 + l_n^2) \leq 5. \] (4.72)
Following (4.69)-(4.72), we thus obtain
\[
\mathcal{A} \leq 60 \sum_{(\mathbf{n})_{i \geq 3}} \prod_{\mathbf{m} \in \mathbb{Z}^d} (1 + l_{\mathbf{m}}^2) e^{-\frac{1}{8} \delta_1 \left( \sum_{i \geq 3} \ln^* |n_i| \right)}
\times \left( \sum_{\mathbf{n} \in \mathbb{Z}^d} (2\alpha_\mathbf{n} + \kappa_\mathbf{n} + \kappa_\mathbf{n}') e^{-\frac{1}{8} \delta (\sum_{\mathbf{n} \in \mathbb{Z}^d} (2\alpha_\mathbf{n} + \kappa_\mathbf{n} + \kappa_\mathbf{n}'))} \right)
\leq \frac{480}{e^d} \sum_{(\mathbf{n})_{i \geq 3}} \prod_{\mathbf{m} \in \mathbb{Z}^d} (1 + l_{\mathbf{m}}^2) e^{-\frac{1}{8} \delta_1 \sum_{i \geq 3} \ln^* |n_i|} \text{ (by (4.7))}
\leq \frac{480}{e^d} \sup_{(\mathbf{n})_{i \geq 3}} \left( \prod_{\mathbf{m} \in \mathbb{Z}^d} (1 + l_{\mathbf{m}}^2) e^{-\frac{1}{8} \delta_1 \sum_{i \geq 3} \ln^* |n_i|} \right)
\times \sum_{(\mathbf{n})_{i \geq 3}} e^{-\frac{1}{8} \delta_1 \sum_{i \geq 3} \ln^* |n_i|}.
\]

By (4.16), one has
\[
\sup_{(\mathbf{n})_{i \geq 3}} \left( \prod_{\mathbf{m} \in \mathbb{Z}^d} (1 + l_{\mathbf{m}}^2) e^{-\frac{1}{8} \delta_1 \sum_{i \geq 3} |n_i|} \right) \leq \exp \left\{ 6d \left( \frac{32}{\delta_1} \right)^{\frac{d}{1}} \cdot \exp \left\{ \left( \frac{1}{\delta_1} \right)^{\frac{1}{8}} \right\} \right\}.
\]

In view of (4.9) and (4.10), we have
\[
\sum_{(\mathbf{n})_{i \geq 3}} e^{-\frac{1}{8} \delta_1 \sum_{i \geq 3} \ln^* |n_i|} \leq \exp \left\{ \left( \frac{6400d}{\delta_1^2} \right)^d \cdot \exp \left\{ d \left( \frac{16d}{\delta_1} \right)^{\frac{1}{8}} \right\} \right\}.
\]

By (4.73), (4.74) and (4.75), we finish the proof of (4.60).

\textbf{Case 2.} $|\mathbf{n}_1^* (\alpha, \kappa, \kappa')| > |\mathbf{n}_1^* (A, K, K')|$. 

In view of (4.52), one has $\mathbf{n}_1^* (a, k, k') = \mathbf{n}_1^* (\alpha, \kappa, \kappa')$. Hence, $\mathbf{n}_2^* (a, k, k')$ is determined by $\mathbf{n}_1^* (a, k, k')$ and $\mathbf{n}_1^* (a, k, k')_{i \geq 3}$. Similar as Case 1.2, we have
\[
\mathcal{A} \leq \frac{1}{\delta_2} \cdot \exp \left\{ 3 \left( \frac{14400d}{\delta_1^2} \right)^d \cdot \exp \left\{ d \left( \frac{24d}{\delta_1} \right)^{\frac{1}{8}} \right\} \right\}.
\]

\[ \square \]

4.5. Proof of Lemma 3.1.

\textit{Proof.} Firstly, we will prove the inequality (3.14). Fixed $a, k, k' \in \mathbb{N}^{d}$, consider the monomial
\[
\mathcal{M}_{akk'} = \prod_{\mathbf{n} \in \mathbb{Z}^d} I_{\mathbf{n}}(0)^{a_{\mathbf{n}} k_{\mathbf{n}} k'_{\mathbf{n}}}
\]
satisfying $k_{\mathbf{n}} k'_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$. It is easy to see that $\mathcal{M}_{akk'}$ comes from some parts of the terms $\mathcal{M}_{\alpha \kappa \kappa'}$ with no assumption for $\kappa$ and $\kappa'$. Write $\mathcal{M}_{\alpha \kappa \kappa'}$ in the form of
\[
\mathcal{M}_{\alpha \kappa \kappa'} = \mathcal{M}_{ablv} \equiv \prod_{\mathbf{n} \in \mathbb{Z}} I_{\mathbf{n}}(0)^{a_{\mathbf{n}} k_{\mathbf{n}} k'_{\mathbf{n}}}
\]
where
\[ b_n = \min \{ \kappa_n, \kappa_n' \}, \quad l_n = \kappa_n - b_n, \quad l'_n = \kappa_n' - b_n \]
and then \( l_n' = 0 \) for all \( n \in \mathbb{Z}^d \).

Express the term
\[ \prod_{n \in \mathbb{Z}^d} f_n^{b_n} = \prod_{n \in \mathbb{Z}^d} (I_n(0) + J_n)^{b_n} \]
by the monomials of the following form
\[ \prod_{n \in \mathbb{Z}^d} I_n(0)^{b_n}, \]
\[ b_m J_m(0)^{b_m - 1} J_m \left( \prod_{n \in \mathbb{Z}^d, n \neq m} I_n(0)^{b_n} \right), \quad m \in \mathbb{Z}^d, \]
and
\[ \left( \prod_{n \in \mathbb{Z}^d, |n| < |m|} I_n(0)^{b_n} \right) \left( b_m I_m(0)^{b_m - 1} J_m \right) \left( \prod_{n \in \mathbb{Z}^d, |n| \geq |m|} I_n(0)^{b_n} \right) \times \left( b_{m_2} I_{m_2}(0)^{b_{m_2} - 1} J_{m_2} \right) \left( \prod_{|n| > |m_2|} I_n^{b_n} \right) \quad |m_1| \leq |m_2|, \quad 0 \leq r \leq b_{m_2} - 1. \]

Now we will estimate the bounds for the coefficients respectively. For any given \( n \in \mathbb{Z}^d \),
\[ I_n(0)^{a_n} q_n^{k_n} q_n'^{k_n'} = \sum_{b_n=\min(\kappa_n, \kappa_n')} I_n(0)^{a_n + b_n} q_n^{\kappa_n - b_n} q_n'^{\kappa_n' - b_n}. \tag{4.76} \]
Hence, one has
\[ a_n = \alpha_n + b_n, \tag{4.77} \]
and
\[ k_n = \kappa_n - b_n, \quad k'_n = \kappa_n' - b_n. \tag{4.78} \]
Therefore, if \( 0 \leq \alpha_n \leq a_n \) is chosen, so \( b_n, \kappa_n, \kappa_n' \) are determined.

On the other hand, by (2.9) we have
\[
|B_{\alpha \kappa \kappa'}| \leq \| R \|_p e^{\rho \left( \sum_{n \in \mathbb{Z}^d} \left( 2a_n + \kappa_n + \kappa_n' \right) \ln |n| - 2 \ln \left( n_1^* (\alpha, \kappa, \kappa') \right) \right)}
\[
= \| R \|_p e^{\rho \left( \sum_{n \in \mathbb{Z}^d} \left( 2a_n + (k_n + a_n - \kappa_n) + (k_n + a_n - \kappa_n) \right) \ln |n| - 2 \ln \left( n_1^* (\alpha, \kappa, \kappa') \right) \right)}
\[
\text{ (in view of (4.77) and (4.78))}
\[
= \| R \|_p e^{\rho \left( \sum_{n \in \mathbb{Z}^d} \left( 2a_n + k_n + k_n' \right) \ln |n| - 2 \ln \left( n_1^* (\alpha, k, k') \right) \right)}. \tag{4.79}
\]
Using (4.76), one has
\[
|B_{\alpha \kappa \kappa'}| \leq \| R \|_p \prod_{n \in \mathbb{Z}^d} (1 + a_n) e^{\rho \left( \sum_{n \in \mathbb{Z}^d} \left( 2a_n + k_n + k_n' \right) \ln |n| - 2 \ln \left( n_1^* (\alpha, k, k') \right) \right)}. \tag{4.79}
\]
In view of (3.11) and (4.79), we have

$$\| R_0 \|_{\rho+\delta} \leq C \| R \|_\rho,$$

where

$$C = \prod_{n \in \mathbb{Z}^d} (1 + a_n) e^{-\delta \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k_n') \right) \ln |n| - 2 \ln |n^* (a, k, k')|}.$$

(4.81)

Now it suffices to prove that

$$C \leq \exp \left\{ 10d \left( \frac{10}{\delta} \right)^{\frac{1}{m^*}} \cdot \exp \left\{ \left( \frac{10}{\delta} \right)^{\frac{1}{2}} \right\} \right\}$$

(4.82)

in the following three cases.

**Case 1.** \( |n^* (a, k, k')| = |n^*_3 (a, k, k')| \).

Using (2.5), one has

$$C \leq \prod_{n \in \mathbb{Z}^d} (1 + a_n) e^{-\frac{1}{2} \delta \sum_{i \geq 1} \ln^* |n^* (a, k, k')|}$$

$$\leq \prod_{n \in \mathbb{Z}^d} (1 + a_n) e^{-\frac{1}{2} \delta \sum_{i \geq 1} \ln^* |n^* (a, k, k')|}$$

$$= \prod_{n \in \mathbb{Z}^d} (1 + a_n) e^{-\frac{1}{2} \delta \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k_n') \ln |n|}$$

$$\leq \prod_{n \in \mathbb{Z}^d} \left( 1 + a_n \right) e^{-\frac{1}{2} \delta \sum_{n \in \mathbb{Z}^d} \ln^* |n|}$$

$$\leq \exp \left\{ 3d \left( \frac{6}{\delta} \right)^{\frac{1}{m^*}} \cdot \exp \left\{ \left( \frac{6}{\delta} \right)^{\frac{1}{2}} \right\} \right\},$$

(4.83)

where the last inequality is based on (4.16).

**Case 2.** \( |n^*_1 (a, k, k')| > |n^*_2 (a, k, k')| = |n^*_3 (a, k, k')| \).

In this case, we have

$$C = \prod_{|n| \leq |n^*_2 (a, k, k')|} (1 + a_n) e^{-\delta \left( \sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k_n') \ln |n| - 2 \ln |n^* (a, k, k')| \right)}$$

$$\leq \prod_{|n| \leq |n^*_2 (a, k, k')|} (1 + a_n) e^{-\frac{1}{2} \delta \sum_{i \geq 1} \ln^* |n^* (a, k, k')|} \quad \text{by (2.5)}$$

$$\leq \prod_{|n| \leq |n^*_2 (a, k, k')|} (1 + a_n) e^{-\frac{1}{2} \delta \sum_{i \geq 1} \ln^* |n^* (a, k, k')|}$$

$$\leq \exp \left\{ 3d \left( \frac{8}{\delta} \right)^{\frac{1}{m^*}} \cdot \exp \left\{ \left( \frac{8}{\delta} \right)^{\frac{1}{2}} \right\} \right\},$$

(4.84)

where the last inequality follows from the proof of (4.83).

**Case 3.** \( |n^*_2 (a, k, k')| > |n^*_3 (a, k, k')| \).
In this case, we have
\[
C \leq 2 \prod_{|n| \leq |n|_1} (1 + a_n) e^{-\delta (\sum_{n \in \mathbb{Z}} (2a_n + k_n + k'_n) \ln |n| - 2 \ln |n|_1 (a, k, k'))}
\]
\[
\leq 2 \prod_{|n| \leq |n|_1} (1 + a_n) e^{-\delta \sum_{i \geq 3} \ln |n|_1 (a, k, k')}
\]
\[
\leq 2 \exp \left\{ 3d \left( \frac{4}{\delta} \right)^{\frac{1}{\sqrt{\tau}}} \cdot \exp \left\{ \left( \frac{4}{\delta} \right)^{\frac{1}{2}} \right\} \right\},
\]
where the last inequality follows from the proof of (4.83).

In view of (4.83), (4.84) and (4.85), we finish the proof of (4.82). Similarly, we get
\[
\| R_1 \|_{p+\delta} + \| R_2 \|_{p+\delta} \leq \exp \left\{ 10d \left( \frac{10}{\delta} \right)^{\frac{1}{\sqrt{\tau}}} \cdot \exp \left\{ \left( \frac{10}{\delta} \right)^{\frac{1}{2}} \right\} \right\} \| R \|_p,
\]
Then we have
\[
\| R \|_{p+\delta} \leq \exp \left\{ 10d \left( \frac{10}{\delta} \right)^{\frac{1}{\sqrt{\tau}}} \cdot \exp \left\{ \left( \frac{10}{\delta} \right)^{\frac{1}{2}} \right\} \right\} \| R \|_p.
\]
On the other hand, the coefficient of $M_{abl't}$ increases by at most a factor
\[
\left( \sum_{n \in \mathbb{Z}^d} (\alpha_n + b_n) \right)^2.
\]
Then one has
\[
\| R \|_{p+\delta} \leq \| R \|_p \left( \sum_{n \in \mathbb{Z}^d} (\alpha_n + b_n) \right)^2 e^{-\delta (\sum_{n \in \mathbb{Z}^d} (2a_n + k_n + k'_n) \ln |n| - 2 \ln |n|_1 (a, k, k'))}
\]
\[
\leq \| R \|_p \left( 2 \sum_{i \geq 3} \ln |n|_1 (a, k, k') \right)^2 e^{-\delta \sum_{i \geq 3} \ln |n|_1 (a, k, k')} (\text{in view of (4.54)})
\]
\[
\leq \frac{64}{e^{\delta^2}} \| R \|_p^2,
\]
where the last inequality is based on Lemma 4.3 with $p = 2$.

\[
\square
\]

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