Manifold-based time series forecasting

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Abstract

Prediction for high dimensional time series is a challenging task due to the curse of dimensionality problem. Classical parametric models like ARIMA or VAR require strong modeling assumptions and time stationarity and are often overparametrized. This paper offers a new flexible approach using recent ideas of manifold learning. The considered model includes linear models such as the central subspace model and ARIMA as particular cases. The proposed procedure combines manifold denoising techniques with a simple nonparametric prediction by local averaging. The resulting procedure demonstrates a very reasonable performance for real-life econometric time series. We also provide a theoretical justification of the manifold estimation procedure.

Keywords: time series prediction, manifold learning, manifold denoising, ergodic Markov chain, non-mixing Markov chain

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1 Introduction

We consider the problem of time series forecasting, which finds plenty of applications in different areas such as economics, geology, physics, system fault, and planning tasks. The inability to make a consistent prediction may have a strong influence on developing companies and even countries, while a correct forecast can partially neutralize the crucial consequences. For example, it was possible to know in advance about a stock market crash [19], to forecast a natural disaster [31], or compute electricity costs for a certain period of time [1].

Classical forecasting methods such as ARMA [56], ARIMA (see e.g. [9]), ARFIMA [23], GARCH [8] and LSTM [25] among many others are based on parametric modeling. The main drawbacks and limitations of such modeling is that it requires very restrictive parametric assumptions and time homogeneity over the whole observation period. Modern time series prediction techniques use more flexible nonparametric or semiparametric models and recent advances in machine learning. We mention Bayesian methods [3], Lasso [2], kernel-based prediction [20], deep learning [29], and reinforcement learning [39] among many others. However, for multivariate time series, all the mention approaches suffer from curse of dimensionality problem: the used models become quickly overparametrized as the dimension grows. As a remedy, one or another dimensionality reduction technique assuming that the observed high-dimensional time series have nevertheless a low-dimensional structure. Once the low-dimensional structure is recovered, one can make a more accurate forecast. For this purpose, different manifold learning methods and dimension reduction techniques can be used. For instance, usual PCA is useful for linear latent factor models [28]. For nonlinear latent factor models one can use local PCA [49] and kernel PCA [37]. Local linear models were also considered in [30, 50]. In [38], the authors studied a central subspace model, which is similar to the problem of an effective dimension reduction subspace estimation (see e.g. [26]) in i.i.d. setup. In [42], the authors used Diffusion maps [11] to reduce dimensionality. One can also use another classical dimension reduction methods such as Isomap [51], LLE [43], LTSA [58], Laplacian Eigenmaps [5], Hessian Eigenmaps [12], and T-SNE [52].

Unfortunately, the mentioned dimension reduction methods assume that the observa-
tions lie precisely on a smooth manifold and often show poor performance when deal with noisy inputs. One way to overcome this issue is to use functional PCA (FPCA) \[53, 44\] which directly works with curves instead of vectors. Another approach is to project the data onto a manifold using manifold denoising methods \[24, 55, 36, 40\]. In our work, we assume that the data from a sliding window lie in a vicinity of a low-dimensional manifold. We use recently proposed manifold denoising methods \[36\] and \[40\] to project the data onto the manifold to capture the data structure. After that, we use a weighted k-nearest neighbours predictor which is a very simple but yet efficient method for time series forecasting \[31, 57\]. To make the k-NN method exploit the low-dimensional structure, we use the weights based on pairwise distances between the projected data points.

For theoretical analysis, we introduce a latent variable model with manifold structure. There is a vast of literature concerning identification of linear dynamical systems (see, for instance, \[32, 10, 45, 46, 59\]) but the non-linear case we consider is not studied so well. Our main contribution is that we obtain non-asymptotic upper bounds on accuracy of manifold reconstruction in two scenarios. The first scenario is the case of mixing time series. There are a lot of papers (e.g. \[48, 33, 27, 32\]) which consider mixing time series. In particular, in \[10, 45\], the authors study stable linear dynamical systems. The second scenario we consider is the case of non-mixing time series. In this situation, the analysis is more technically involving. In \[46\], the authors introduce martingale small ball condition and prove upper bounds on the least squares estimator which are also valid for unit root autoregressive model. In \[59\], the authors extend the results of \[46\] to the case of an autoregressive model with linear constraints. In our work, we adapt the martingale small ball condition for non-linear setup.

The rest of this paper is organised as follows. In Section 2 we introduce a latent variable model with manifold structure. In Section 3 we describe our methodology for time series forecasting. Then we present the performance of our method in time series forecasting in Section 4. Finally, in Section 5 we provide theoretical upper bounds on the accuracy of manifold estimation in our model. Proofs of the main theoretical results can be found in 6, auxiliary results are moved to Appendix.
Notations

Throughout the paper, boldfaced letters are reserved for matrices. Vectors and scalars are written in regular font. For any matrix $A$, $\|A\|$ stands for its operator norm, and $\|A\|_F$ is the Frobenius norm of $A$. The notation $f(n) \lesssim g(n)$ means that there exists an absolute constant $c > 0$, such that $f(n) \leq cg(n)$ for all $n$. The relation $f(n) \asymp g(n)$ is equivalent to $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$. Next, for any set $A$ and any $x \in \mathbb{R}^D$, $d(x, A) = \inf_{y \in A} \|x - y\|$ denotes the Euclidean distance from $x$ to $A$.

2 Statistical model

We assume that we observe a multivariate time series $Y_1, \ldots, Y_T \in \mathbb{R}^D$, which follows the model

$$Y_t = X_t + \varepsilon_t, \quad 1 \leq t \leq T,$$

(1)

where $X_t$ is a Markov chain on a hidden $d$-dimensional manifold $M^*$, $d < D$, and $\varepsilon_1, \ldots, \varepsilon_t$ are independent zero-mean innovations. We give some examples where a model with a hidden low-dimensional structure appears.

Example 1 (central subspace model, [38]). Let $g : \mathbb{R}^d \to \mathbb{R}^p$ be a smooth function and let $\Phi$ be a $(p \times d)$ matrix. The central subspace model is given by the formula

$$Z_t = g(\Phi^T Z_{t-1}) + \xi_t, \quad 1 \leq t \leq T.$$

Take $X_t = (Z_{t-1}, g(\Phi^T Z_{t-1})) \in \mathbb{R}^{2p}$, $Y_t = (Z_{t-1}, Z_t)$, $\varepsilon_t = (0, \xi_t)$. Then $Y_t = X_t + \varepsilon_t$ and $X_t$ lies on the graph of $g \circ \Phi^T$, which is a $d$-dimensional submanifold in $\mathbb{R}^D$ with $D = 2p$.

Example 2. A natural extension of the central subspace model is

$$Z_t = g(\pi_M(Z_{t-1})) + \xi_t, \quad 1 \leq t \leq T.$$

As before, $g : \mathbb{R}^d \to \mathbb{R}^D$ is a a smooth function and $M$ is an unknown smooth $d$-dimensional submanifold in $\mathbb{R}^p$. Assume $M$ is such that there is a global diffeomorphism $\varphi : \mathbb{R}^p \to \mathbb{R}^d$, which isometrically maps $M$ into $\mathbb{R}^d$. Then for any $z \in \mathbb{R}^p$ there exists $u \in \mathbb{R}^d$ such that $\pi_M(z) = \varphi^{-1}(u)$. Consider $X_t = (Z_{t-1}, g(\pi_M(Z_{t-1}))) \in \mathbb{R}^{2p}$, $Y_t = (Z_{t-1}, Z_t)$, $\varepsilon_t = (0, \xi_t)$. 


Then $X_t$ lies on the graph of $g \circ \varphi^{-1}$, which is a $d$-dimensional submanifold in $\mathbb{R}^D$ with $D = 2p$.

**Example 3 (univariate autoregressive model).** A standard univariate autoregressive model of order $\tau$ is given by

$$Z_t = \sum_{i=1}^{\tau} a_i Z_{t-i} + \xi_t, \quad 1 \leq t \leq T.$$  

Fix $D > \tau$ and apply a sliding window technique: $Y_t = (Z_t, \ldots, Z_{t-D+1}) \in \mathbb{R}^D$. Then the autoregressive model can be rewritten as $Y_t = AY_{t-1} + \varepsilon_t$, where $\varepsilon_t = (\xi_t, 0, \ldots, 0) \in \mathbb{R}^D$ and

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_k & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{D \times D}.$$

In this case, $X_t = AY_{t-1}$ lies on $\text{Im}(A)$. Since $\text{rank}(A) < D$, $\text{Im}(A)$ is a linear subspace in $\mathbb{R}^D$ of dimension $\text{rank}(A)$.

We have to impose some regularity conditions on the underlying manifold $\mathcal{M}^*$. One of the bottlenecks in nonlinear manifold estimation is high curvature of the manifold (see, for example, [6]). To overcome this issue, we have to require that $\mathcal{M}^*$ is smooth enough. We assume that

$$\mathcal{M}^* \in \mathcal{M}_d^d \doteq \left\{ \mathcal{M} \subset \mathbb{R}^D : \mathcal{M} \text{ is a compact, connected manifold without a boundary, } \mathcal{M} \in \mathcal{C}^2, \mathcal{M} \subseteq B(0, R), \right. \quad (A1)$$

$$\text{Vol}(\mathcal{M}) \leq V, \text{reach}(\mathcal{M}) \geq \kappa, \dim(\mathcal{M}) = d < D \left. \right\}.$$  

The reach of a manifold $\mathcal{M}$ is defined as a supremum of such $r$ that any point $y \in \mathbb{R}^D$, such that $d(y, \mathcal{M}) \leq r$, has a unique Euclidean projection onto $\mathcal{M}$. The assumption $(A1)$ is ubiquitous in manifold learning literature (see e.g. [35, 18, 17, 15, 1]).

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We also have to require some properties of the underlying Markov chain \(\{X_t : 1 \leq t \leq T\}\). It is a common assumption in manifold learning literature (e.g. [18, 17, 1, 40]) that, for any \(t\), the marginal density of \(X_t\) is bounded away from zero. In our setup, this would imply exponential ergodicity of the Markov chain \(\{X_t\}\). Let \(\mathbb{P}_t\) be the marginal distribution of \(X_t\), and assume that the Markov chain \(\{X_t\}\) has a stationary distribution \(\pi\). We require the following: there exist \(A > 0\) and \(\rho \in (0, 1]\) such that, for any \(t \in \{1, \ldots, T\}\), the measure \(\mathbb{P}_t\) satisfies
\[
\|\mathbb{P}_t - \pi\|_{TV} \leq A^2(1 - \rho)^t,
\] (A2)
where \(\| \cdot \|_{TV}\) is the total variation distance.

Unfortunately, it is not always the case that a latent Markov chain is exponentially mixing. In our work, besides the case of ergodic Markov chain \(\{X_t : 1 \leq t \leq T\}\), we also consider the situation of non-mixing Markov chain. The next assumption is a relaxation of (A2) and admits the absence of mixability. Let \(\mathcal{F}_t\) be a sigma-algebra generated by \(X_1, \ldots, X_t\) for \(t \in \{1, \ldots, T\}\) and put \(\mathcal{F}_0\) the trivial sigma-algebra. We require the following: there exist \(k \in \mathbb{N}, h_0 > 0,\) and \(p_1 \geq p_0 > 0\) such that, for any \(t \in \{0, \ldots, T-k\}, h \in (0, h_0),\) and \(x \in \mathcal{M}^*\), it holds
\[
\Pr(X_{t+j} \in B(x, h) | \mathcal{F}_t) \leq p_1 h^d.
\] (A3)
Assumption (A3) is similar to the martingale small ball condition introduced in [46] and it is far less restrictive than (A2). In particular, it admits some periodic Markov chains.

Finally, we describe assumptions about \(\varepsilon_1, \ldots, \varepsilon_T\). A random vector \(\xi \in \mathbb{R}^D\) is called sub-Gaussian with parameter \(\sigma^2\) if
\[
\sup_{\|u\|=1} \mathbb{E} e^{\lambda u^T(\xi - \mathbb{E}\xi)} \leq e^{\lambda^2 \sigma^2/2}, \quad \forall \lambda \in \mathbb{R}.
\] We assume that \(\varepsilon_1, \ldots, \varepsilon_T\) are independent zero-mean sub-Gaussian errors:
\[
\mathbb{E}\varepsilon_t = 0, \quad \varepsilon_t \in \text{SG}(\sigma_t^2), \quad \forall t \in \{1, \ldots, T\}.
\] (A4)

In our paper, we study an empirical risk minimizer (ERM)
\[
\hat{\mathcal{M}} \in \arg\min_{\mathcal{M} \in \mathcal{M}} \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}).
\] (2)
In other words, the manifold $\widehat{M}$ is the best fitting manifold based on the observed data. We focus on the case when the manifold dimension $d$ is known. Otherwise, one can add a regularisation term enforcing small dimension of the estimated manifold:

$$\widehat{M} \in \arg\min_{M \in \bigcup_{d=1}^{D-1} \mathcal{M}} \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, M) + \lambda \dim(M).$$

(3)

The ERM $\widehat{M}$ from (2) is a nice object for theoretical study but in practice one cannot perform a minimisation over a set of manifolds. Instead, one can try to approximate the target functional and mimic the ERM manifold. In Section 3 we discuss two approximation techniques which lead to computationally efficient manifold denoising algorithms.

3 Methodology

Assume, we observe a multivariate time series $Z_1, \ldots, Z_T \in \mathbb{R}^p$ and our goal is to make one step ahead forecast $\widehat{Z}_{T+1}$. On the first step, we use a sliding window technique for data preprocessing. We fix an integer $b$, construct a collection of patches $\{Y_t = (Z_{t-1}, Z_{t-2}, \ldots, Z_{t-b}) \in \mathbb{R}^{bp} : b+1 \leq t \leq T\}$ and consider a set of pairs $S_T = \{(Y_t, Z_t) : b+1 \leq t \leq T\}$. We assume that high-dimensional vectors $Y_{b+1}, \ldots, Y_{T+1} \in \mathbb{R}^D$, $D = bp$, lie around a low-dimensional manifold. We exploit recent advances in manifold learning, described further in this section, to project the patches $Y_t$’s onto a manifold in the patch space. These methods, in fact, mimic the estimate (2) or (3) and then return the projections $\widehat{X}_{b+1}, \ldots, \widehat{X}_{T+1}$ of $Y_{b+1}, \ldots, Y_{T+1}$ onto the manifold $\widehat{M}$, respectively. After that, our forecast is determined by the weighted $k$-nearest neighbors rule:

$$\widehat{Z}_{T+1} = Z_T + \frac{\sum_{t=b+1}^{T} w_t (Z_t - Z_{t-1})}{\sum_{t=b+1}^{T} w_t},$$

where the weights are defined by the formula

$$w_t = e^{-(T+1-t)/\tau} K\left(\frac{||\widehat{X}_{T+1} - \widehat{X}_t||}{h_k}\right),$$

(4)

where $h_k$ is the $k$-th smallest value amongst $||\widehat{X}_{T+1} - \widehat{X}_{b+1}||, \ldots, ||\widehat{X}_{T+1} - \widehat{X}_T||$, $\tau > 0$ is a discounting parameter and $K(\cdot)$ is a localising kernel. In our work, we use Epanechnikov
kernel $\mathcal{K}(u) = \frac{3}{4}(1 - u^2)_+$ but one can choose other kernels.

If one eagers to make several step ahead prediction, i.e. to construct an estimate of $Z_{T+m}$, $m > 1$, then he can use a widespread technique. Sequentially make one step ahead forecasts $\hat{Z}_{T+1}, \ldots, \hat{Z}_{T+m}$ adding the new forecast to the data after each step.

### 3.1 Manifold learning techniques

In this section, we briefly describe how to approximate the target functional (2) or (3) in order to mimic the (penalised) ERM. In practice, a manifold $\mathcal{M}$ is often associated with a point cloud $\{U_1, \ldots, U_T\}$ on it. Given observations $Y_1, \ldots, Y_T$, one can use a nonparametric smoothing technique to approximate the squared distances $d^2(Y_t, \mathcal{M})$ using the point cloud:

$$d^2(Y_t, \mathcal{M}) \approx \sum_{j=1}^{T} w_{tj} \|Y_t - U_j\|^2,$$

where $w_{tj}, 1 \leq j \leq T$, are some localising weights. Then

$$\sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \approx \sum_{t,j=1}^{T} w_{tj} \|Y_t - U_j\|^2. \tag{5}$$

The choice of $w_{tj}$’s plays an important role in performance of an algorithm. In [40], the authors introduce the algorithm called SAME (from structure-adaptive manifold estimation). For each $t$, they associate $U_t$ with a projection of $Y_t$ onto $\mathcal{M}$ and introduce a projector $\Pi_t$ onto the tangent space at the point $U_t$. Then they take the localising weights of the form

$$w_{tj} = \frac{1}{T h^d} \mathcal{K}_0 \left( \frac{\|\Pi_t(Y_t - Y_j)\|^2}{h^2} \right) \mathbb{1} \left( \|Y_t - Y_j\| < \tau_0 \right), \tag{6}$$

where $\mathcal{K}_0 : \mathbb{R} \to \mathbb{R}_+$ is a smooth kernel, $\int \mathcal{K}_0(t)dt = 1$. Given $\Pi_1, \ldots, \Pi_T$, the values of $U_1, \ldots, U_T$, minimising the approximated target functional (5), are equal to

$$\hat{U}_t = \frac{\sum_{j=1}^{T} w_{tj} Y_j}{\sum_{j=1}^{T} w_{tj}},$$

where the weights $w_{tj}$ are computed according to (6). After that, the obtained values $\hat{U}_1, \ldots, \hat{U}_T$ are used to update the projectors $\Pi_1, \ldots, \Pi_T$ and then the procedure repeats. After several iterations, the projection estimates $\hat{X}_1, \ldots, \hat{X}_T$, used in (4) are put to $U_1, \ldots, U_T$. The pseudocode of SAME is given in Algorithm 1.
Algorithm 1 SAME, [40]

1: The initial guesses $\hat{\Pi}_1^{(0)}, \ldots, \hat{\Pi}_T^{(0)}$ of projectors onto tangent spaces, dimension of the manifold $d$, the number of iterations $K + 1$, an initial bandwidth $h_0$, the threshold $\tau_0$ and constants $a > 1$ and $\gamma > 0$ are given.

2: for $k$ from 0 to $K$ do

3: Compute the weights $w_{tj}^{(k)}$ according to the formula

$$w_{tj}^{(k)} = \frac{1}{Th^d} K_0 \left( \frac{\|\hat{\Pi}_{t}^{(k)} (Y_t - Y_j)\|^2 / h_k^2}{\|Y_t - Y_j\|} \right) 1(\|Y_t - Y_j\| \leq \tau_0), \quad 1 \leq t, j \leq T.$$ 

4: Compute the estimates

$$\hat{U}_{t}^{(k)} = \left( \frac{\sum_{j=1}^{n} w_{tj}^{(k)} Y_j}{\sum_{j=1}^{n} w_{tj}^{(k)}} \right), \quad 1 \leq t \leq T.$$ 

5: If $k < K$, for each $T$ from 1 to $T$, define a set $J_{t}^{(k)} = \{ j : \|\hat{U}_{t}^{(k)} - \hat{U}_{j}^{(k)}\| \leq \gamma h_k \}$ and compute the matrices

$$\hat{\Sigma}_{t}^{(k)} = \sum_{j \in J_{t}^{(k)}} (\hat{U}_{j}^{(k)} - \hat{U}_{t}^{(k)}) (\hat{U}_{j}^{(k)} - \hat{U}_{t}^{(k)})^T, \quad 1 \leq t \leq T.$$ 

6: If $k < K$, for each $i$ from 1 to $n$, define $\hat{\Pi}_{i}^{(k+1)}$ as a projector onto a linear span of eigenvectors of $\hat{\Sigma}_{i}^{(k)}$, corresponding to the largest $d$ eigenvalues.

7: If $k < K$, set $h_{k+1} = a^{-1} h_k$.

return the estimates $\hat{X}_1 = \hat{U}_1^{(K)}, \ldots, \hat{X}_T = \hat{U}_T^{(K)}.$

The algorithm SAME uses the manifold’s dimension $d$ as an input parameter. If $d$ is not known in advance, one can use (3) instead of (2). In this case, the squared distances are also approximated according to (5) but the weights $w_{tj}$ are computed as follows:

$$w_{tj} = K_0 \left( \frac{\|U_t - U_j\|}{h} \right).$$ 

The main challenge is to deal with the discrete second term. Fortunately, Lemma 3.1 in [36] helps to overcome this issue. Let $V$ be an open subset in $T_{U_t} M$ such that $0 \in V$ and let $\mathcal{E}_t : V \to \mathbb{R}^D$ be an exponential map of $M$ at $U_t$. Then

$$\dim(M) = \|\nabla \mathcal{E}_t(0)\|^2, \quad 1 \leq t \leq T.$$
Here and further, $\| \cdot \|_F$ stands for the Frobenius norm. Similarly the squared distances, the squared norm of the gradient of the exponential map can be approximated according to the formula
\[
\| \nabla E_t(0) \|_F^2 \approx \sum_{j=1}^{T} w_{tj} \frac{\| U_t - U_j \|^2}{h^2}.
\]
Thus, the dimension of $\mathcal{M}$ can be approximated by
\[
\dim(\mathcal{M}) \approx \frac{1}{T} \sum_{t,j=1}^{T} w_{tj} \frac{\| U_t - U_j \|^2}{h^2},
\]
and, for the target functional (3), we have
\[
\frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \approx \frac{1}{T} \sum_{t,j=1}^{T} w_{tj} \| Y_t - U_j \|^2 + \frac{\lambda}{T h^2} \sum_{t,j=1}^{T} w_{tj} \| U_t - U_j \|^2. \tag{7}
\]
The algorithm LDMM in [36] uses the split Bregman iteration [22] to find a local minimum of (7). The pseudocode of LDMM is given in Algorithm 2 below.
Algorithm 2 LDMM, [36]

1: A matrix $\mathbf{Y} = (Y_1, \ldots, Y_T)^T \in \mathbb{R}^{T \times D}$ of noisy observations and positive numbers $h, \lambda, \mu$ are given.

2: Initial guess: $\mathbf{U}^{(0)} = \mathbf{Y} \in \mathbb{R}^{T \times D}$, $r^{(0)} = 0 \in \mathbb{R}^{T \times D}$.

3: while not converge do

4: Compute the weight matrix $\mathbf{W}^{(k)} = \left( w^{(k)}_{ij} : 1 \leq t, j \leq T \right)$, where

   $w^{(k)}_{ij} = e^{-\|U^{(k)}_t - U^{(k)}_j\|^2/h^2}$.

5: Compute matrices $\mathbf{D}^{(k)} = \text{diag}(d^{(k)}_1, \ldots, d^{(k)}_T)$ and $\mathbf{L}^{(k)} = \mathbf{D}^{(k)} - \mathbf{W}^{(k)}$, where

   $d^{(k)}_t = \sum_{j=1}^n w^{(k)}_{ij}$.

6: Solve the following linear matrix equation with respect to $V \in \mathbb{R}^{T \times D}$:

   $$(\mathbf{L}^{(k)} + \mu \mathbf{W}^{(k)}) V = \mu \mathbf{W}^{(k)} (U^t - r^t),$$

7: Update $\mathbf{U}^{(k)}$ by solving the least-squares problem

   $$\mathbf{U}^{(k+1)} \in \arg\min_{V'} \| \mathbf{Y} - V' \|_F^2 + \frac{\lambda}{\mu h^2} \| V - V' + r^{(k)} \|_F^2,$$

   which is given by the formula

   $$\mathbf{U}^{(k+1)} = \left( \mathbf{Y} + \frac{\lambda}{\mu h^2} (V + r^{(k)}) \right) / \left( 1 + \frac{\lambda}{\mu h^2} \right).$$

8: Update $r$:

   $$r^{(k+1)} = r^{(k)} + V - \mathbf{U}^{(k+1)}.$$

9: Put $k \leftarrow k + 1$.

10: return $\hat{\mathbf{X}}_1 = \mathbf{U}^{(k)}_1, \ldots, \hat{\mathbf{X}}_T = \mathbf{U}^{(k)}$.
4 Numerical Experiments

In this section, we illustrate performance of the algorithms described in Section 3. The algorithms LDMM and SAME are applied sequentially with the weighted nearest neighbours method for econometric multivariate time series forecasting. The algorithms are compared to the weighted nearest neighbours method without the manifold reconstruction step and ARIMA. We release the code with experiments on [GitHub]. We use the data provided by the Russian Presidential Academy of National Economy and Public Administration. It represents various quantities characterizing the living standard of the Russian people from January 1999 to October 2018. The data contains four components, among which there is a strong correlation between the first two and the last two ones, and moreover, it has seasonability. As the series becomes multivariate, the sliding window becomes multivariate too, and therefore, the data points fed into the LDMM and SAME input consist of four windows corresponding to univariate time series. Thus, the reconstruction of the manifold proceeds at once for all components of the series, which certainly provides additional information that allows improving the quality of prediction.

The hyperparameters for all algorithms are tuned only for the one step ahead prediction simultaneously for all components and then used in all other simulations. For LDMM, we take the heat kernel \( K_0(t) = e^{-t^2/4} \) with bandwidth \( h^2 = 0.001 \), the hyperparameters \( \lambda \) and \( \mu \) are set to \( h^2/7 \) and 1500, respectively, and the number of iterations is 7. The width of the sliding window is 11 and the number of the nearest neighbours in ascending order of the component number is 30, 10, 7, 7. The algorithm SAME has a different set of hyperparameters. We use the width of the sliding window equal to 11, \( \tau_0 = 1.0 \), the number of iterations is set to 21 and the numbers of nearest neighbours are 9, 21, 21, 15 for the first, the second, the third, and the fourth components, respectively. The time discount factor \( \tau \) (see (4)), involved in all weighted k-NN based algorithms, is set to 20. For the ARIMA algorithm, the hyperparameters are \((p, d, q) = (6, 1, 0)\).

The results of prediction are collected in Table 1. Plots of the predictions are shown in Figures 1–4 in Appendix D. First, from Table 1 one can observe that LDMM and SAME improve the predictions of the weighted nearest neighbors method in all the cases. Second, one can notice that the performance of LDMM and SAME is comparable to ARIMA and
often is even better. However, ARIMA does not particularly react to sudden leaps in the series components and minimizes the error by tending to its mean. This behavior should be taken into account by practitioners when choosing an algorithm, because forecasting of such jumps may be extremely crucial in certain tasks.

| Lookfront (months) | Algorithm | Component 1 | Component 2 | Component 3 | Component 4 |
|--------------------|-----------|-------------|-------------|-------------|-------------|
| 1                  | SAME      | 1.2         | 1.9         | 1.5         | 1.7         |
|                    | LDMM      | 1.6         | 2.4         | 1.5         | 1.4         |
|                    | k-NN      | 1.7         | 3.9         | 2.4         | 1.8         |
|                    | ARIMA     | 0.8         | 2.1         | 1.4         | 1.6         |
| 2                  | SAME      | 1.5         | 2.5         | 2.1         | 3.9         |
|                    | LDMM      | 1.4         | 2.9         | 2.0         | 4.3         |
|                    | k-NN      | 1.8         | 4.4         | 3.0         | 4.3         |
|                    | ARIMA     | 1.8         | 2.5         | 1.9         | 3.8         |
| 3                  | SAME      | 1.9         | 3.3         | 2.8         | 5.1         |
|                    | LDMM      | 2.8         | 4.0         | 2.9         | 4.9         |
|                    | k-NN      | 3.0         | 7.1         | 4.2         | 5.8         |
|                    | ARIMA     | 1.8         | 3.2         | 2.0         | 4.9         |
| 4                  | SAME      | 2.0         | 3.6         | 3.6         | 5.6         |
|                    | LDMM      | 3.0         | 3.9         | 3.5         | 5.0         |
|                    | k-NN      | 3.1         | 7.2         | 4.5         | 6.0         |
|                    | ARIMA     | 2.0         | 3.3         | 2.1         | 5.1         |

Table 1: The RMSEs of predictions for multidimensional time series.

5 Theoretical results

In this section, we provide theoretical guarantees on the empirical risk minimiser [2]. We are concerned with the question how well the manifold $\hat{\mathcal{M}}$, learned from a single trajectory $Y_1, \ldots, Y_T$, fits other trajectories. For this purpose, we introduce a trajectory $Y'_1, \ldots, Y'_T$ which has the same joint distribution as $Y_1, \ldots, Y_T$ and is independent of $Y_1, \ldots, Y_T$. For any $M \in \mathcal{M}_{\mathbb{R}^d}$, we characterise its performance by the expected average squared distance...
over the test trajectory $Y'_1, \ldots, Y'_T$:
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y'_t, \mathcal{M}).
\]
We are interested in the generalisation ability of ERM $\hat{\mathcal{M}}$ and in upper bounds on the excess risk
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y'_t, \hat{\mathcal{M}}) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y'_t, \mathcal{M}^*).
\]
Our first result concerns the case of ergodic hidden Markov chains.

**Theorem 1.** Assume (A1), (A2), and (A4). Then, for the ERM (2), it holds that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y'_t, \hat{\mathcal{M}}) - \inf_{\mathcal{M} \in \mathcal{M}_d} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y'_t, \mathcal{M}) \lesssim \begin{cases} 
\sqrt{\frac{D \log T}{T \log(1/(1-\rho))}}, & d < 4, \\
\frac{\sqrt{D \log^{3/2} T}}{T \log(1/(1-\rho))}, & d = 4, \\
\left(\frac{T \log T}{T \log(1/(1-\rho))}\right)^{2/d}, & d > 4.
\end{cases}
\]

The result of Theorem 1 improves the results of [35] and [15], where the authors obtained the rates $\tilde{O}(T^{-1/(d+4)})$ and $\tilde{O}(T^{-2/(d+4)})$, respectively, in i.i.d. setup.

For the case of non-mixing Markov chains, we provide the following system identification result.

**Theorem 2.** Assume (A1), (A3), and (A4). Assume that the normal and the tangent components of the noise are independent. Let $\sigma_1, \ldots, \sigma_T$ be such that there exists a constant $c \in (0, 1)$ such that
\[
8 \frac{T}{T} \sum_{t=1}^{T} \sigma_t^2 + 64 D \sigma_{\text{max}}^2 \leq \frac{p_0}{8k} \left(\frac{c}{4}\right)^{d+2}.
\]
Then, with probability at least $1 - 8/T$, we have
\[
\frac{1}{T} \sum_{t=1}^{T} d^2(X_t, \hat{\mathcal{M}}) \lesssim \psi_T + \frac{T}{T} \sum_{t=1}^{T} \sigma_t^2 + \frac{T}{T} \sum_{t=1}^{T} \sigma_{\text{max}}^4 \left(\frac{T}{T} \sum_{t=1}^{T} \sigma_t^2\right)^{4/d},
\]
where $\sigma_{\text{max}} = \max_{1 \leq t \leq T} \sigma_t$ and
\[
\psi_T = \begin{cases} 
\frac{1}{T} \sqrt{\sum_{t=1}^{T} \sigma_t^2}, & d < 4, \\
\frac{T}{T} \sqrt{\sum_{t=1}^{T} \sigma_t^2}, & d = 4, \\
T^{-2/d} \sqrt{\frac{\sum_{t=1}^{T} \sigma_t^2}{T}}, & d > 4.
\end{cases}
\]
6 Proofs

This section contains proofs of main results.

6.1 Proof of Theorem 1

From the definition of $\hat{M}$, we have

$$\frac{1}{T} \sum_{t=1}^{T} E d^2(Y'_t, \hat{M}) - \inf_{M \in \mathcal{M}} \frac{1}{T} \sum_{t=1}^{T} E d^2(Y'_t, M)$$

$$\leq \frac{1}{T} \sum_{t=1}^{T} \left( E d^2(Y'_t, \hat{M}) - d^2(Y_t, \hat{M}) \right) - \inf_{M \in \mathcal{M}} \frac{1}{T} \sum_{t=1}^{T} \left( E d^2(Y'_t, M) - d^2(Y_t, M) \right)$$

$$\leq 2 \mathbb{E} \sup_{M \in \mathcal{M}} \frac{1}{T} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - E d^2(Y_t, M) \right) \right| .$$

The proof of Theorem 1 is given in three steps. Let $\mathbb{E}$ be the expectation with respect to $X_1, \varepsilon_1, \ldots, X_T, \varepsilon_T$, where $X_1$ is generated with respect to the stationary measure $\pi$. On the first step, we control the discrepancy between $\mathbb{E} \sup_{M \in \mathcal{M}} \frac{1}{T} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - E d^2(Y_t, M) \right) \right|$ and

$$\mathbb{E} \sup_{M \in \mathcal{M}} \frac{1}{T} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - E d^2(Y_t, M) \right) \right| .$$

Lemma 1.

$$\mathbb{E} \sup_{M \in \mathcal{M}} \frac{1}{T} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - E d^2(Y_t, M) \right) \right| \leq \mathbb{E} \sup_{M \in \mathcal{M}} \frac{1}{T} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - E d^2(Y_t, M) \right) \right|$$

$$+ \frac{6A}{T^2} \left( 128 \sigma_{\max}^2 D + 16R^2 \right),$$

where $\sigma_{\max} = \max_{1 \leq t \leq T} \sigma_t$.

The proof of Lemma 1 is moved to Appendix B.1. If the initial state $X_1$ of the Markov chain is drawn from the distribution $\pi$ then $X_1, \ldots, X_T$ are identically distributed random elements though still dependent. Let $K$ be an integer to be specified later and split the set $\{1, \ldots, T\}$ into blocks $B_1, \ldots, B_K$, where

$$B_k = \{k, k + K, k + 2K, \ldots\}, \quad \forall k \in \{1, \ldots, K\},$$
and the size of each block is either \( T/K \) or \( T/K \). Then we have

\[
\mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right) \right| \\
\leq \sum_{k=1}^{K} \mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t \in B_k} \left( d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right) \right|.
\]

The following result shows that one can replace \( X_t \)'s inside one block \( B_k \) by i.i.d. copies \( \tilde{X}_1, \ldots, \tilde{X}_T \) drawn from the stationary distribution \( \pi \).

**Lemma 2.** It holds that

\[
\mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t=1}^{T} \left( d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right) \right| \\
\leq TA(1 - \rho)^K + \sum_{k=1}^{K} \mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t \in B_k} \left( d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right) \right|,
\]

where the expectation \( \mathbb{E} \) is taken with respect to \( \{ \tilde{X}_t, \varepsilon_t : t \in B_k \} \) and \( \tilde{X}_t, t \in B_k, \) are i.i.d. copies of \( X_t, t \in B_k. \)

The proof of Lemma 2 is moved to Appendix B.2. Finally, we control

\[
\mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t \in B_k} \left( d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right) \right|
\]

for each block \( B_k. \)

**Lemma 3.** Assume that \( B_k = \{ t_1, \ldots, t_{N_k} \} \), where \( N_k \) is the cardinality of \( B_k \). Then, for any \( k \in \{ 1, \ldots, K \} \), it holds that

\[
\mathbb{E} \sup_{M \in \mathcal{M}_d^d} \left| \sum_{t \in B_k} d^2(Y_t, M) - \mathbb{E}d^2(Y_t, M) \right| \leq \begin{cases} 
D \left( \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{D N_k} \right), & d < 4, \\
D \left( \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{D N_k} \right) \log \psi_k^{-1}, & d = 4, \\
D \left( \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{D N_k} \right) \psi_k^{1-4/d}, & d > 4,
\end{cases}
\]

where

\[
\psi_k = \frac{R \sqrt{D N_k} + D \sqrt{\sum_{t \in B_k} \sigma_t^2}}{RN_k + \sqrt{D} \sum_{t \in B_k} \sigma_t}.
\]
Proof of Lemma 3 can be found in Appendix B.3. Take \( K = \lceil \log(1/TA)/\log(1 - \rho) \rceil \).

Then

\[
\tilde{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \sum_{t \in B_k} d^2(Y_t, \mathcal{M}) - \tilde{E}d^2(Y_t, \mathcal{M}) \right| \lesssim \begin{cases} \sqrt{\frac{DT \log(1/(1 - \rho))}{\log T}}, & d < 4, \\ \sqrt{DT \log(1/(1 - \rho)) \log T}, & d = 4, \\ \left( \frac{T \log(1/(1 - \rho))}{\log T} \right)^{1 - 2/d}, & d > 4, \end{cases}
\]

and the claim of Theorem 1 follows from Lemmata 1, 2, and 3.

### 6.2 Proof of Theorem 2

For any \( t \in \{1, \ldots, T\} \), let \( \Pi_t \) be the projector onto \( \mathcal{T}_{X_t} \mathcal{M}^* \). Denote \( \varepsilon_t = \Pi_t \varepsilon_t \) and \( \varepsilon_t^+ = (I - \Pi_t) \varepsilon_t \). Then, for all \( \mathcal{M} \in \mathcal{M}_d \), it holds that

\[
d^2(Y_t, \mathcal{M}) = d^2(X_t + \varepsilon_t^\parallel, \mathcal{M}) + 2a_{\mathcal{M}, t} \varepsilon_t^+ + \| \varepsilon_t^+ \|^2, \quad \forall t \in \{1, \ldots, T\},
\]

where \( a_{\mathcal{M}, t} = X_t + \varepsilon_t^\parallel - \pi_{\mathcal{M}} (X_t + \varepsilon_t^\parallel) \). On the other hand, due to the Cauchy-Schwartz inequality, we have

\[
d^2(Y_t, \mathcal{M}^*) \leq (1 + h^{-1}) d^2(X_t + \varepsilon_t^\parallel, \mathcal{M}^*) + (1 + h) \| \varepsilon_t^+ \|^2, \quad \forall t \in \{1, \ldots, T\},
\]

where \( h \) is a parameter to be specified later. The inequalities (8) and (9) yield

\[
d^2(X_t + \varepsilon_t^\parallel, \mathcal{M}) \leq 2a_{\mathcal{M}, t}^T \varepsilon_t^+ + d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}^*) + (1 + h^{-1})d^2(X_t + \varepsilon_t^\parallel, \mathcal{M}^*) + h \| \varepsilon_t^+ \|^2.
\]

The fact that the reach of \( \mathcal{M}^* \) is not less than \( \mathcal{M} \) implies that a sphere of radius \( \mathcal{M} \) rolls freely over the surface of \( \mathcal{M}^* \). Thus, if \( \| \varepsilon_t^\parallel \| \leq \mathcal{M} \), we have \( d(X_t + \varepsilon_t^\parallel, \mathcal{M}^*) \leq 2\| \varepsilon_t^\parallel \|^2 / \mathcal{M} \).

Then, since \( \mathcal{M}^* \subset B(0, R) \), it holds that

\[
d(X_t + \varepsilon_t^\parallel, \mathcal{M}^*) \leq \frac{2\| \varepsilon_t^\parallel \|^2}{\mathcal{M}} + 2R \mathbb{1} (\| \varepsilon_t^\parallel \| \geq \mathcal{M})
\]

and we obtain

\[
\sum_{t=1}^{T} d^2(X_t + \varepsilon_t^\parallel, \mathcal{M}) \leq 2 \sum_{t=1}^{T} a_{\mathcal{M}, t}^T \varepsilon_t^+ + \sum_{t=1}^{T} \left( d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}^*) \right)

+ \sum_{t=1}^{T} \left( (1 + h^{-1}) \left( \frac{2\| \varepsilon_t^\parallel \|^2}{\mathcal{M}} + 2R \mathbb{1} (\| \varepsilon_t^\parallel \| \geq \mathcal{M}) \right)^2 + h \| \varepsilon_t^+ \|^2 \right).
\]
Lemma 4 guarantees that the probability of this event is close to 1. Let 

\[ \left| d(X_t + \varepsilon_t^i, \mathcal{M}) - d(X_t, \mathcal{M}) \right| \leq \| \Pi_t^* - \Pi_t^M \| \| \varepsilon_t^i \| + \frac{2 \| \varepsilon_t^i \|^2}{\kappa} + 2R \mathbb{I}(\| \varepsilon_t^i \| \geq \kappa). \]

Then, for any \( t \in \{1, \ldots, T\} \), we have

\[ d^2(X_t, \mathcal{M}) \leq 2d^2(X_t + \varepsilon_t^i, \mathcal{M}) + 2 \left( \| \Pi_t^* - \Pi_t^M \| \| \varepsilon_t^i \| + \frac{2 \| \varepsilon_t^i \|^2}{\kappa} + 2R \mathbb{I}(\| \varepsilon_t^i \| \geq \kappa) \right)^2. \]

(11)

**Lemma 4.** Assume that there exists a constant \( c \in (0, 1) \) such that

\[ \frac{8}{T} \sum_{i=1}^{T} \sigma_i^2 + 64D\sigma_{\text{max}}^2 \leq \frac{p_0}{8k} \left( \frac{\kappa}{4} \right)^{d+2}. \]

Then, for the ERM \( \hat{\mathcal{M}} \), defined in (2), it holds that

\[ \mathbb{P} \left( \max_{x \in \mathcal{M}^*} d(x, \hat{\mathcal{M}}) \geq c\kappa \right) \leq \frac{4^{d+1}V}{(c\kappa)^d} e^{-\frac{p_0(c\kappa)^d}{12}[T/k]} + \frac{1}{T}. \]

The proofs of auxiliary results related to the proof of Theorem 2 are moved to Appendix C. We assume that \( T \) is large enough, so it holds \( \frac{4^{d+1}V}{(c\kappa)^d} e^{-\frac{p_0(c\kappa)^d}{12}[T/k]} \leq 1/T \).

From now on, we can restrict our attention on the event when \( \hat{\mathcal{M}} \subset \mathcal{M}^* + \mathcal{B}(0, c\kappa) \). Lemma 4 guarantees that the probability of this event is close to 1. Let \( \{ Z_j^* \in \mathcal{M} : 1 \leq j \leq N \} \) be the maximal (2h)-packing on \( \mathcal{M}^* \), where \( h \) is a parameter to be specified later. Split the manifold \( \mathcal{M}^* \) into \( N \) disjoint subsets \( \{ A_j : 1 \leq j \leq N \} \), such that \( \mathcal{B}(Z_j^*, h) \subseteq A_j \subseteq \mathcal{B}(Z_j^*, 2h) \). The existence of such partition follows from the fact that, on one hand, for any \( i \neq j \), the balls \( \mathcal{B}(Z_i^*, h) \) and \( \mathcal{B}(Z_j^*, h) \) do not intersect and, on the other hand, for any \( x \in \mathcal{M}^* \) there exists \( j \in \{1, \ldots, N\} \) such that \( x \in \mathcal{B}(Z_j^*, 2h) \). For any \( x \in \mathcal{M}^* \) and \( r > 0 \), denote \( F_r(x) = \mathcal{B}(x, r) \cap (\{ x \} + T_x^\perp \mathcal{M}^*) \) and \( F_j = \bigcup_{x \in A_j} F_{r(x)}(x) \). Thus, \( \mathcal{M}^* + \mathcal{B}(0, c\kappa) \) is the union of the disjoint sets \( F_j, 1 \leq j \leq N \). Fix any \( j \in \{1, \ldots, N\} \). For each \( \mathcal{M} \in \mathcal{M}_d \), we can construct its locally linear approximation. Namely, let \( Z_j^M \in \mathcal{M} \cap F_j \) be such that the projection of \( Z_j^M \) is equal to \( Z_j^* \) and denote the projector onto the tangent space \( T_{Z_j^M} \mathcal{M} \) by \( \Pi_j^M \). We write \( \Pi_j^* \) instead of \( \Pi_j^{M*} \) for brevity.

Consider any \( \mathcal{M} \in \mathcal{M}_d \). Due to Theorem 4.18 in [11], for any \( j \in \{1, \ldots, N\} \) and for any \( X_t \in A_j \), we have \( d_H \left( \mathcal{M} \cap F_j, (\{ Z_j^M \} + T_{Z_j^M} \mathcal{M}) \cap F_j \right) \leq 2h^2/\kappa \). Then it holds that

\[ \left| d(X_t, \mathcal{M}) - d(X_t, \{ Z_j^M \} + T_{Z_j^M} \mathcal{M}) \right| \leq d_H \left( \mathcal{M} \cap F_j, (\{ Z_j^M \} + T_{Z_j^M} \mathcal{M}) \cap F_j \right) \leq \frac{Ch^2}{\kappa}. \]
for some absolute constant $C$. Using the Cauchy-Schwartz inequality and Theorem 4.18 in [14], we obtain

$$d^2(X_t, \mathcal{M}) \geq \frac{1}{2} d^2(X_t, \{Z_j^M\} + \mathcal{T}_Z^M) - \frac{C^2 h^4}{\kappa^2}$$

$$\geq \frac{1}{4} d^2(\pi_{\{Z_j^T\}} + Z_j^T, \{Z_j^M\} + \mathcal{T}_Z^M) - \frac{(C^2 + 2) h^4}{\kappa^2}$$

$$= \frac{1}{4} d^2(\pi_{Z_j} + \Pi_j^*, (X_t - Z_j^*), \{Z_j^M\} + \mathcal{T}_Z^M) - \frac{(C^2 + 2) h^4}{\kappa^2}$$

$$\geq \| (I - \Pi_j^M) \Pi_j^* (X_t - Z_j^*) \|^2 - \frac{h^4}{\kappa^2}.$$ 

Using Lemma 3.5 in [7], we obtain

$$\sum_{X_t \in A_j} \| \tilde{\Pi}_t^* - \Pi_j^M \|^2 \lesssim \sum_{X_t \in A_j} \| \Pi_j^* - \Pi_j^M \|^2 + \frac{h^2}{\kappa^2}$$

$$= \sum_{X_t \in A_j} \| (I - \Pi_j^M) \Pi_j^* \|^2 + \frac{h^2}{\kappa^2}.$$ 

Let $u_1, \ldots, u_d$ be an orthonormal basis in $\mathcal{T}_Z^M$. Then

$$\| (I - \Pi_j^M) \Pi_j^* \|^2 \leq d \max_{u \in \{u_1, \ldots, u_d\}} \| (I - \Pi_j^M) \Pi_j^* u \|^2.$$ 

We prove an upper bound for the right hand side using the following result.

**Lemma 5.** Assume that $\{X_t : 1 \leq t \leq T\} \subset \mathcal{M}^*$ satisfies (A3) and let $h \leq 2\kappa$. Then it holds that

$$p_0 h^d \left[ \sum_{t=1}^T \mathbb{1} (X_t \in B(x, 2h)) \right] \leq 2d^2 \mathcal{V} h^d e^{- \frac{p_0 h^d}{12T} \frac{[T/k]}{k}}.$$ 

We will choose $h \gtrsim (k \log T/T)^{1/d}$ with a sufficiently large hidden constant, so we assume that $\frac{2d^V h^d}{k} e^{- \frac{p_0 h^d}{12T} \frac{[T/k]}{k}} < 1/T$. Due to Lemma 5, with probability at least $1 - 1/T$, for each $u \in \{u_1, \ldots, u_d\}$ there are $\gtrsim p_0 h^d [T/k]$ points amongst $\{X_1, \ldots, X_T\} \cap A_j$ such that

$$\| (I - \Pi_j^M) \Pi_j^* u \|^2 h^2 \lesssim \| (I - \Pi_j^M) \Pi_j^* (X_t - Z_j^*) \|^2, \quad \forall u \in \{u_1, \ldots, u_d\}.$$ 

Thus,

$$p_0 h^d \left[ \frac{T}{k} \right] \| (I - \Pi_j^M) \Pi_j^* \|^2 h^2 \lesssim \sum_{X_t \in A_j} d^2(X_t, \mathcal{M}).$$
Since, according to Lemma [5], $A_j$ contains at most $3p_1(2h)^d T/2$ points, we have

$$
\sum_{X_t \in A_j} \|\tilde{\Pi}_i^* - \tilde{\Pi}_t^j\|^2 \lesssim \sum_{X_t \in A_j} \|\Pi_i^* - \Pi_t^j\|^2 + \frac{h^2}{\kappa^2}
\lesssim \sum_{X_t \in A_j} \|\Pi_i^* - \Pi_t^j\|^2 + \frac{h^2}{\kappa^2} \lesssim \frac{p_0}{k \kappa^1} \sum_{X_t \in A_j} d^2(X_t, \mathcal{M}).
$$

Plugging this bound into (10) and (11), we obtain that

$$
d^2(X_t, \mathcal{M}) \lesssim \sum_{t=1}^T \frac{p_0}{k \kappa^1} \frac{\|\varepsilon_t\|^2 d^2(X_t, \mathcal{M})}{h^2} + \sum_{t=1}^T \left( d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}^*) \right) + \sum_{t=1}^T a_{\mathcal{M}, t} \varepsilon_t^1 + \sum_{t=1}^T \left( \frac{\|\varepsilon_t\|^4}{h \kappa^2} + R^2 \mathbb{1} \left( \|\varepsilon_t\| \geq \kappa \right) + h \|\varepsilon_t\|^2 + \frac{h^4}{\kappa^2} \right)
$$
on an event with probability at least $1 - 3/2$. Plug the ERM $\hat{\mathcal{M}}$ into the last expression:

$$
\sum_{t=1}^T d^2(X_t, \hat{\mathcal{M}}) \lesssim \sum_{t=1}^T \frac{p_0}{k \kappa^1} \frac{\|\varepsilon_t\|^2 d^2(X_t, \hat{\mathcal{M}})}{h^2} + \sup_{\mathcal{M} \in \mathcal{M}_d} \sum_{t=1}^T a_{\mathcal{M}, t} \varepsilon_t^1 + \sum_{t=1}^T \left( \frac{\|\varepsilon_t\|^4}{h \kappa^2} + R^2 \mathbb{1} \left( \|\varepsilon_t\| \geq \kappa \right) + h \|\varepsilon_t\|^2 + \frac{h^4}{\kappa^2} \right).
$$

The following large deviation inequalities will be useful.

**Lemma 6.** Let $\delta \in (0, 1)$. The following inequalities hold with probability at least $1 - \delta$:

1) $\max_{1 \leq t \leq T} \|\varepsilon_t\| \leq 4 \sigma_{\text{max}} \sqrt{d} + 2 \sigma_{\text{max}} \sqrt{2 \log(T/\delta)}$, where $\sigma_{\text{max}} = \max_{1 \leq t \leq T} \sigma_t$.

2) $\sum_{t=1}^T \|\varepsilon_t^1\| \leq 4 \sqrt{(D - d)T \sum_{t=1}^T \sigma_t^2} + 2 \sqrt{2 \log(1/\delta) \sum_{t=1}^T \sigma_t^2}$.

3) $\frac{1}{T} \sum_{t=1}^T \|\varepsilon_t^1\|^2 \leq \frac{4}{T} \sum_{t=1}^T \sigma_t^2 + 64 d \sigma_{\text{max}}^2 + \frac{32}{T} \sum_{t=1}^T \sigma_t^4 \left( \log \frac{1}{\delta} \vee \sqrt{\log \frac{1}{\delta}} \right)$.

4) $\frac{1}{T} \sum_{t=1}^T \|\varepsilon_t^1\|^2 \leq \frac{4}{T} \sum_{t=1}^T \sigma_t^2 + 64(D - d) \sigma_{\text{max}}^2 + \frac{32}{T} \sum_{t=1}^T \sigma_t^4 \left( \log \frac{1}{\delta} \vee \sqrt{\log \frac{1}{\delta}} \right)$.

5) $\frac{1}{T} \sum_{t=1}^T \mathbb{1} \left( \|\varepsilon_t\| \geq c \kappa \right) < \frac{2 e^d}{T} \sum_{t=1}^T e^{-\frac{c^2 \sigma_t^2}{4 T}} + \frac{2 \log(1/\delta)}{T}$.
Choose \( h \asymp \sigma_{\max} \sqrt{\frac{\log T}{p_0}} \sqrt{\log T} \vee (\log T/T)^{1/d} \). Then it holds that
\[
\frac{1}{T} \sum_{t=1}^T d^2(X_t, \widehat{M}) \lesssim \frac{1}{T} \sup_{M \in \mathcal{M}_d^2} \sum_{t=1}^T a_{M,t}^T e_t + \frac{D (\sigma_{\max} \sqrt{\log T} \vee (\log T/T)^{1/d})}{T} \sum_{t=1}^T \sigma_t^2 \\
+ \frac{(\sigma_{\max}^4 \log^2 T \vee (\log T/T)^{4/d})}{\kappa^2}
\]
with probability at least \( 1 - 7/T \).

It remains to bound the first term in the right hand side. Note that, due to the conditions of the theorem, for any fixed \( \mathcal{M} \in \mathcal{M}_d^2 \), the vectors \( \varepsilon_{1}, \ldots, \varepsilon_{T} \) are independent of \( a_{M,1}, \ldots, a_{M,T} \). Then \( \left( \sum_{t=1}^T a_{M,t}^T e_t^+ | X_1, \varepsilon_{1}, \ldots, X_T, \varepsilon_T \right) \) is a sub-Gaussian process. Moreover, for any \( \mathcal{M}, \mathcal{M}' \in \mathcal{M}_d^2 \), \( \left( \sum_{t=1}^T a_{M,t}^T e_t^+ | X_1, \varepsilon_{1}, \ldots, X_T, \varepsilon_T \right) \) is a sub-Gaussian random variable with variance proxy
\[
\sum_{t=1}^T \|a_{M,t} - a_{M',t}\|^2 \sigma_t^2 \leq \sum_{t=1}^T d_H^2(\mathcal{M}, \mathcal{M}') \sigma_t^2 \leq \sum_{t=1}^T 4R^2 \sigma_t^2.
\]
Here we used the fact that
\[
\|a_{M,t} - a_{M',t}\| = \|\pi_M(X_t + \varepsilon_t) - \pi_{M'}(X_t + \varepsilon_t)\| \leq d_H(\mathcal{M}, \mathcal{M}').
\]
This also yields
\[
\mathbb{E} \sup_{\mathcal{M}, \mathcal{M}' \in \mathcal{M}_d^2, d_H(\mathcal{M}, \mathcal{M}') \leq \gamma} \sum_{t=1}^T (a_{M,t} - a_{M',t})^T e_t^+ \leq \mathbb{E} \sup_{\mathcal{M}, \mathcal{M}' \in \mathcal{M}_d^2, d_H(\mathcal{M}, \mathcal{M}') \leq \gamma} \sum_{t=1}^T \|a_{M,t} - a_{M',t}\| \|e_t^+\| \\
\leq \mathbb{E} \sup_{\mathcal{M}, \mathcal{M}' \in \mathcal{M}_d^2, d_H(\mathcal{M}, \mathcal{M}') \leq \gamma} \sum_{t=1}^T d_H(\mathcal{M}, \mathcal{M}') \|e_t^+\| \leq \mathbb{E} \gamma \sum_{t=1}^T \|e_t^+\| \lesssim \gamma \sum_{t=1}^T \sigma_t \sqrt{D - d}.
\]
Using the chaining technique, we obtain
\[
\mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d^2} \sum_{t=1}^T a_{M,t}^T e_t^+ \lesssim \gamma \sum_{t=1}^T \sigma_t \sqrt{D - d} + R \sqrt{\sum_{t=1}^T \sigma_t^2} \int_0^\infty \sqrt{\log N(\mathcal{M}_d^2, d_H, \varepsilon)} d\varepsilon.
\]

Theorem 9 in [18] claims that
\[
\mathcal{N}(\mathcal{M}_d^2, d_H, u) \leq c_1 \left( \frac{D}{d} \right)^{(c_2/\kappa)^D} \exp \left\{ \frac{2^{D/2} (D - d)(c_2/\kappa)^D u^{-d/2}}{4} \right\},
\]
where the constant $c_2$ depends only on $\kappa$ and $d$. This yields

$$
\mathbb{E} \sup_{M \in \#^d} \sum_{t=1}^{T} a_{M,t}^T \varepsilon_{t}^T \lesssim \gamma \sum_{t=1}^{T} \sigma_t \sqrt{D - d} + \frac{2^{d/4} R \sqrt{D - d}}{\kappa^{D/2}} \sqrt{\sum_{t=1}^{T} \sigma_t^2} \int_{\gamma}^{2R} u^{-d/4} du
$$

$$
\lesssim \gamma \sqrt{(D - d)T} \sqrt{\sum_{t=1}^{T} \sigma_t^2} + \frac{2^{d/4} R \sqrt{D - d}}{\kappa^{D/2}} \sqrt{\sum_{t=1}^{T} \sigma_t^2} \int_{\gamma}^{2R} u^{-d/4} du.
$$

Choosing

$$
\gamma = \begin{cases} 
0, & d < 4, \\
\log \frac{T}{\sqrt{T}}, & d = 4, \\
T^{-2/d}, & d > 4,
\end{cases}
$$

we obtain

$$
\frac{1}{T} \mathbb{E} \sup_{M \in \#^d} \sum_{t=1}^{T} a_{M,t}^T \varepsilon_{t}^T \lesssim \begin{cases} 
\frac{1}{T} \sqrt{(D - d) \sum_{t=1}^{T} \sigma_t^2}, & d < 4, \\
\log \frac{T}{\sqrt{T}} \sqrt{(D - d) \sum_{t=1}^{T} \sigma_t^2}, & d = 4, \\
T^{-2/d} \sqrt{(D - d) \sum_{t=1}^{T} \sigma_t^2}, & d > 4.
\end{cases}
$$

Finally, Azuma-Hoeffding inequality yields that

$$
\sup_{M \in \#^d} \sum_{t=1}^{T} a_{M,t}^T \varepsilon_{t}^T - \mathbb{E} \sup_{M \in \#^d} \sum_{t=1}^{T} a_{M,t}^T \varepsilon_{t}^T \lesssim R \sqrt{(D - d) \sum_{t=1}^{T} \sigma_t^2 \log(T)}
$$

with probability at least $1 - 1/T$. Thus, with probability at least $1 - 8/T$, it holds that

$$
\frac{1}{T} \sum_{t=1}^{T} d^2(X_t, \hat{M}) \lesssim \psi_T + \frac{D \left( \sigma_{\max} \sqrt{\log T} \vee (\log T/T)^{1/d} \right)}{T} \sum_{t=1}^{T} \sigma_t^2 + \frac{\left( \sigma_{\max}^4 \log^2 T \vee (\log T/T)^{4/d} \right)}{\kappa^2},
$$

where

$$
\psi_T = \begin{cases} 
\frac{1}{T} \sqrt{\sum_{t=1}^{T} \sigma_t^2}, & d < 4, \\
\log \frac{T}{\sqrt{T}} \sqrt{\sum_{t=1}^{T} \sigma_t^2}, & d = 4, \\
T^{-2/d} \sqrt{\sum_{t=1}^{T} \sigma_t^2}, & d > 4.
\end{cases}
$$
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A Technical tools

Lemma 7. Let $P$ and $Q$ be probability measures on a set $X$ and let $f : X \to \mathbb{R}$ be a function such that the second moments $E_P f^2$ and $E_Q f^2$ with respect to $P$ and $Q$ are finite. Then

$$E_P f - E_Q f \leq \sqrt{(E_P f^2 + E_Q f^2) \|P - Q\|_{TV}}.$$  

Proof. The claim of Lemma 7 follows from the Cauchy-Schwartz inequality

$$\left|E_P f - E_Q f\right| \leq \int_X |f(x)| \cdot |dP(x) - dQ(x)|$$

$$\leq \sqrt{\int_X f^2(x) \cdot |dP(x) - dQ(x)| \int_X |dP(x) - dQ(x)|} \leq \sqrt{(E_P f^2 + E_Q f^2) \|P - Q\|_{TV}}.$$

Lemma 8. Let $M \in \mathcal{M}_d^d$ and let $\varepsilon_t$ be a sub-Gaussian random vector in $\mathbb{R}^D$ with parameter $\sigma_t^2$. Assume that the Markov Chain $\{X_t : 1 \leq t \leq T\}$ on $M^* \in \mathcal{M}_d^d$ has a stationary measure $\pi$ and denote a distribution of $X_t$ by $P_t$. Let $Y_t = X_t + \varepsilon_t$ and denote the convolutions of $P_t$ and $\pi$ with the distribution of $\varepsilon_t$ by $\tilde{P}_t$ and $\tilde{\pi}$ respectively. Then, for any $t \in \{1, \ldots, T\}$, we have

$$\left|E_{\tilde{P}_t} d^2(Y_t, M) - E_{\tilde{\pi}} d^2(Y_t, M)\right| \leq (128\sigma_t^2 D + 16R^2) \|P_t - \pi\|_{TV}^{1/2}.$$  

Proof. Denote the convolutions of $P_t$ and $\pi$ with the distribution of $\varepsilon_t$ by $\tilde{P}_t$ and $\tilde{\pi}$ respectively. Then, due to Lemma 7, we have

$$\left|E_{\tilde{P}_t} d^2(Y_t, M) - E_{\tilde{\pi}} d^2(Y_t, M)\right| \leq \sqrt{E_{\tilde{P}_t} d^4(Y_t, M) + E_{\tilde{\pi}} d^4(Y_t, M)} \sqrt{\|\tilde{P}_t - \tilde{\pi}\|_{TV}}.$$  

Since the total variation distance between convolutions $\tilde{P}_t$ and $\tilde{\pi}$ is not greater than $\|P_t - \pi\|_{TV}$, it remains to prove

$$\sqrt{E_{\tilde{P}_t} d^4(Y_t, M) + E_{\tilde{\pi}} d^4(Y_t, M)} \leq (128\sigma_t^2 D + 16R^2).$$

[59] Yao Zheng and Guang Cheng. Finite time analysis of vector autoregressive models under linear restrictions, 2018.
Using the inequality \((a + b)^4 \leq 8a^4 + 8b^4\), we obtain
\[
\mathbb{E}_{\hat{P}} d_t^4(Y_t, M) = \mathbb{E}_{\hat{P}} \min_{x \in M} \|X_t + \varepsilon_t - x\|^4 \\
\leq \mathbb{E}_{\hat{P}} \left( \min_{x \in M} 8\|X_t - x\|^4 + 8\|\varepsilon_t\|^4 \right) = 8\mathbb{E}_{\hat{P}} d_t^4(X_t, M) + 8\mathbb{E}\|\varepsilon_t\|^4.
\] (12)

Note that
\[
\mathbb{E}_{\hat{P}} d_t^4(X_t, M) \leq d_H^4(M^*, M) \leq (2R)^4,
\] (13)
where the last inequality follows from the fact that, by definition of \(M^d\), M and \(M^*\) are contained in \(B(0, R)\).

Finally, consider \(\mathbb{E}\|\varepsilon_t\|^2\).
\[
\mathbb{E}\|\varepsilon_t\|^2 = \int_0^{+\infty} \mathbb{P}(\|\varepsilon_t\| > v^{1/4}) \, dv = \int_0^{+\infty} \mathbb{P}(\max_{\|u\|=1} u^T \varepsilon_t > v^{1/4}) \, dv.
\]
From the proof of Theorem 1.19 in [11], we have
\[
\mathbb{E}\|\varepsilon_t\|^2 = \int_0^{+\infty} \mathbb{P}(\max_{\|u\|=1} u^T \varepsilon_t > v^{1/4}) \, dv \leq \int_0^{+\infty} \min \left\{ 1, 6D e^{-\frac{v}{8\sigma_t^2}} \right\} \, dv
\]
\[
= (8\sigma_t^2 D \log 6)^2 + 2 \int_{8\sigma_t^2 D \log 6}^{+\infty} 6D e^{-\frac{v}{8\sigma_t^2}} t \, dv
\]
\[
= (8\sigma_t^2 D \log 6)^2 + 2 \int_0^{+\infty} e^{-\frac{v}{8\sigma_t^2}} (v + 8\sigma_t^2 D \log 6) \, dv
\]
\[
= 64\sigma_t^2 D^2 \log^2 6 + 128\sigma_t^4 D \log 6 + 256\sigma_t^4 < 2^{10}\sigma_t^4 D^2.
\] (14)

The inequalities (12), (13) and (14) imply \(\mathbb{E}_{\hat{P}} d_t^4(Y_t, M) \leq 2^{13}\sigma_t^4 D^2 + 2^7 R^4\). Similarly, one can show that \(\mathbb{E}_{\hat{\pi}} d_t^4(Y_t, M) \leq 2^{13}\sigma_t^4 D^2 + 2^7 R^4\). The inequality
\[
\sqrt{\mathbb{E}_{\hat{P}} d_t^4(Y_t, M) + \mathbb{E}_{\hat{\pi}} d_t^4(Y_t, M)} \leq \sqrt{2^{14}\sigma_t^4 D^2 + 2^8 R^4} \leq 128\sigma_t^2 D + 16R^2.
\]

\[\square\]

**Lemma 9.** Assume that \(\{X_t : 1 \leq t \leq T\} \subset M^*\) satisfies [A3]. Fix any \(x \in M^*, h < h_0\).

Then
\[
\mathbb{P} \left( \sum_{t=1}^T \mathbb{1}(X_t \in B(x, h)) \leq \frac{p_0 d |T/k|}{2} \right) \leq e^{-\frac{p_0 d |T/k|}{4}}.
\]
where we introduced $\xi_j = \mathbb{1}(\exists t \in \{(j-1)k+1, \ldots, jk\} : X_t \in \mathcal{B}(x, h)), 1 \leq j \leq \lfloor T/k \rfloor$. Due to (A3), we have
\[ \mathbb{E}(\xi_j \mid \mathcal{F}_{(j-1)k}) \geq \max_{(j-1)k \leq t \leq jk} \mathbb{P}(X_t \in \mathcal{B}(x, h) \mid \mathcal{F}_{(j-1)k}) \geq \frac{1}{k} \sum_{(j-1)k \leq t \leq jk} \mathbb{P}(X_t \in \mathcal{B}(x, h) \mid \mathcal{F}_{(j-1)k}) \geq \rho_0 h^d. \]

Applying the martingale Bernstein inequality (see [16], (1.6)), we obtain
\[ \mathbb{P} \left( \sum_{j=1}^{\lfloor T/k \rfloor} \xi_j < \frac{p_0 h^d}{2} \left( \frac{\lfloor T/k \rfloor}{k} \right) \right) \leq \exp \left\{ -\frac{\lfloor T/k \rfloor (p_0 h^d)^2/8}{p_0 h^d(1 - p_0 h^d) + p_0 h^d/2} \right\} \leq e^{-\frac{p_0 h^d}{12} \lfloor T/k \rfloor}. \]

Similarly,
\[ \mathbb{P} \left( \sum_{t=1}^{T} \mathbb{1}(X_t \in \mathcal{B}(x, h)) > \frac{3p_1 h^d T}{2} \right) \leq \mathbb{P} \left( \sum_{t=1}^{\lfloor T/k \rfloor} \mathbb{1}(X_t \in \mathcal{B}(x, h)) < \frac{3p_1 h^d T}{2} \right) \]
\[ \leq \mathbb{P} \left( \sum_{j=1}^{\lfloor T/k \rfloor} \sum_{t=(j-1)k+1}^{jk} \mathbb{1}(X_t \in \mathcal{B}(x, h)) > \frac{3p_1 h^d T}{2k} \right) \leq \mathbb{P} \left( \sum_{j=1}^{\lfloor T/k \rfloor} \eta_j > \frac{3p_1 h^d \lfloor T/k \rfloor}{2} \right), \]
where $\eta_j = \mathbb{1}(\forall t \in \{(j-1)k+1, \ldots, jk\} : X_t \in \mathcal{B}(x, h)), 1 \leq j \leq \lfloor T/k \rfloor$. Condition (A3) yields
\[ \mathbb{E}(\eta_j \mid \mathcal{F}_{(j-1)k}) \leq \min_{(j-1)k \leq t \leq jk} \mathbb{P}(X_t \in \mathcal{B}(x, h) \mid \mathcal{F}_{(j-1)k}) \leq \frac{1}{k} \sum_{(j-1)k \leq t \leq jk} \mathbb{P}(X_t \in \mathcal{B}(x, h) \mid \mathcal{F}_{(j-1)k}) \leq \rho_1 h^d, \]
and, using the martingale Bernstein inequality we obtain
\[
\mathbb{P}\left(\sum_{j=1}^{\lceil T/k \rceil} \xi_j > \frac{3p_1 h_d}{2} \left\lceil \frac{T}{k} \right\rceil \right) \leq \exp\left\{ - \frac{|T/k| (p_1 h_d)^2 / 8}{p_1 h_d (1 - p_1 h_d) + p_1 h_d / 2} \right\} \leq e^{-\frac{p_1 h_d}{12} |T/k|}.
\]

\[\square\]

B Proofs related to Theorem 1

B.1 Proof of Lemma 1

It holds that
\[
\mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \right| 
\leq \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) \right| + \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) \right|.
\]

Due to Lemma 8 and the spectral gap condition (A2), we have
\[
\sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) \right| \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \mathbb{E} d^2(Y_t, \mathcal{M}) - \mathbb{E} d^2(Y_t, \mathcal{M}) \right| \leq \frac{2A(128\sigma^2 \max D + 16R^2)}{T(1 - \sqrt{1 - \rho})},
\]

where \( \sigma_{\max} = \max_{1 \leq t \leq T} \sigma_t \). Using the inequality \( 1 - \sqrt{1 - \rho} \geq \rho / 2 \), we obtain
\[
\sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) \right| \leq \frac{2A(128\sigma^2 \max D + 16R^2)}{T \rho}.
\]

Similarly,
\[
\mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \right| - \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \right| 
\leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \mathbb{E} d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}) \right| - \mathbb{E} \sup_{\mathcal{M} \in \mathcal{M}_d} \left| \mathbb{E} d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}) \right|.
\]
Applying Lemma 7 and using the inequalities

$$\mathbb{E} \sup_{M \in \mathcal{M}_d} \left[ d^2(Y_t, \mathcal{M}) - \mathcal{E}d^2(Y_t, \mathcal{M}) \right] \leq \mathbb{E} \sup_{M \in \mathcal{M}_d} \left[ 2\|\varepsilon_t\|^2 + 2\mathbb{E}\|\varepsilon_t\|^2 + 4d^2_H(\mathcal{M}, \mathcal{M}') \right]^2$$

$$\leq \mathbb{E} \left[ 2\|\varepsilon_t\|^2 + 2\mathbb{E}\|\varepsilon_t\|^2 + 4(2R)^2 \right]^2 \leq 16\mathbb{E}\|\varepsilon_t\|^4 + 16\mathbb{E}\|\varepsilon_t\|^4 + 2^9R^4 \leq 2^{14}\sigma^4tD^2 + 2^9R^4,$$

where the last inequality follows from (14), we conclude

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \sup_{M \in \mathcal{M}_d} \left| \mathcal{E}d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}) \right| - \mathbb{E} \sup_{M \in \mathcal{M}_d} \left| \mathcal{E}d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}) \right|$$

$$\leq \sum_{t=1}^{T} \frac{2A(128\sigma^2_tD + 16R^2)}{T} (1 - \rho)^{t/2} \leq \frac{2A(128\sigma^2_{\max}D + 16R^2)}{T(1 - \sqrt{1 - \rho})} \leq \frac{4A(128\sigma^2_{\max}D + 16R^2)}{T\rho}.$$ 

Thus,

$$\mathbb{E} \sup_{M \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \right|$$

$$- \mathbb{E} \sup_{M \in \mathcal{M}_d} \left| \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}d^2(Y_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) \right| \leq \frac{6A(128\sigma^2_{\max}D + 16R^2)}{T\rho}.$$

**B.2 Proof of Lemma 2**

Fix any $k \in \{1, \ldots, K\}$ and consider the block $B_k = \{t_1, t_2, \ldots, t_{N_k}\}$, where $t_1 < t_2 < \cdots < t_{N_k}$. Let $\mathbb{P}$ stand for the measure, corresponding to the case when $X_t$'s are generated from the stationary measure $\pi$, and let $\tilde{\mathbb{P}}$ stand for the measure, corresponding to the case, when $X_t$'s are replaced by their independent copies $\tilde{X}_1, \ldots, \tilde{X}_T$. Corollary F.3.4 in [13] and the spectral gap condition yield

$$\|\mathbb{P} - \tilde{\mathbb{P}}\|_{TV} = \sup_{A_1, \ldots, A_{N_k}} \left| \mathbb{P} \left( X_{t_1} \in A_1, \ldots, X_{t_{N_k}} \in A_{N_k} \right) - \prod_{j=1}^{N_k} \tilde{\mathbb{P}} \left( X_{t_j} \in A_j \right) \right|$$

$$\leq A(1 - \rho)^K + \sup_{A_1, \ldots, A_{N_k}} \left| \mathbb{P} \left( X_{t_1} \in A_1, \ldots, X_{t_{N_k-1}} \in A_{N_k-1} \right) \cdot \mathbb{P} \left( X_{t_{N_k}} \in A_{N_k} \right) - \prod_{j=1}^{N_k} \tilde{\mathbb{P}} \left( X_{t_j} \in A_j \right) \right|$$

$$\leq A(1 - \rho)^K + \sup_{A_1, \ldots, A_{N_k-1}} \left| \mathbb{P} \left( X_{t_1} \in A_1, \ldots, X_{t_{N_k-1}} \in A_{N_k-1} \right) - \prod_{j=1}^{N_k-1} \tilde{\mathbb{P}} \left( X_{t_j} \in A_j \right) \right| .$$

Repeating the same trick $T - 1$ times, we obtain

$$\|\mathbb{P} - \tilde{\mathbb{P}}\|_{TV} \leq N_k A(1 - \rho)^K.$$
Thus,
\[
\sum_{k=1}^{K} \mathbb{E} \sup_{\mathcal{M} \in \mathcal{A}^d} \left| \sum_{t \in B_k} (d^2(Y_t, \mathcal{M}) - \mathbb{E}d^2(Y_t, \mathcal{M})) \right| \\
\leq \sum_{t=1}^{T} N_k A(1 - \rho)^K + \sum_{k=1}^{K} \mathbb{E} \sup_{\mathcal{M} \in \mathcal{A}^d} \left| \sum_{t \in B_k} \left( d^2(Y_t, \mathcal{M}) - \mathbb{E}d^2(Y_t, \mathcal{M}) \right) \right| \\
= TA(1 - \rho)^K + \sum_{k=1}^{K} \tilde{\mathbb{E}} \sup_{\mathcal{M} \in \mathcal{A}^d} \left| \sum_{t \in B_k} \left( d^2(Y_t, \mathcal{M}) - \mathbb{E}d^2(Y_t, \mathcal{M}) \right) \right|.
\]

### B.3 Proof of Lemma 3

Let \( \{\xi_t : t \in B_k\} \) be i.i.d. Rademacher random variables. Introduce the Rademacher complexity of the block \( B_k \):
\[
\mathcal{R}_k(\mathcal{A}^d) = \mathbb{E}\mathbb{E}_{\xi} \sup_{\mathcal{M} \in \mathcal{A}^d} \left| \sum_{t \in B_k} \xi_t d^2(Y_t, \mathcal{M}) \right|.
\]

The standard symmetrization argument (see, for instance, [21]) yields
\[
\tilde{\mathbb{E}} \sup_{\mathcal{M} \in \mathcal{A}^d} \left| \sum_{t \in B_k} d^2(Y_t, \mathcal{M}) - \tilde{\mathbb{E}}d^2(Y_t, \mathcal{M}) \right| \leq 2\mathcal{R}_k(\mathcal{A}^d).
\]

Note that, for any \( \mathcal{M}, \mathcal{M}' \in \mathcal{A}^d \), it holds
\[
|d(Y_t, \mathcal{M}) - d(Y_t, \mathcal{M}')| \leq d_H(\mathcal{M}, \mathcal{M}'). \tag{15}
\]

Indeed,
\[
d(Y_t, \mathcal{M}) = \min_{x \in \mathcal{M}} \|Y_t - x\| \leq \|Y_t - \pi_{\mathcal{M}'}(Y_t)\| + \min_{x \in \mathcal{M}} \|\pi_{\mathcal{M}'}(Y_t) - x\| \leq d(Y_t, \mathcal{M}') + d_H(\mathcal{M}, \mathcal{M}').
\]

Similarly, one can prove the inequality \( d(Y_t, \mathcal{M}') \leq d(Y_t, \mathcal{M}') + d_H(\mathcal{M}, \mathcal{M}') \), and therefore, (15) holds. Then
\[
|d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}')| = |d(Y_t, \mathcal{M}) - d(Y_t, \mathcal{M}')| (d(Y_t, \mathcal{M}) + d(Y_t, \mathcal{M}')) \leq 2d_H(\mathcal{M}, \mathcal{M}') (\|\varepsilon_t\| + 2R),
\]

where the last inequality holds due to the fact that
\[
d(Y_t, \mathcal{M}) = \min_{x \in \mathcal{M}} \|X_t + \varepsilon_t - x\| \leq \min_{x \in \mathcal{M}} \|X_t - x\| + \|\varepsilon_t\| \leq d_H(\mathcal{M}, \mathcal{M}^*) + \|\varepsilon_t\| \leq 2R + \|\varepsilon_t\|.
\]

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Also, note that
\[
\sqrt{\sum_{t \in B_k} (d^2(Y_t, \mathcal{M}) - d^2(Y_t, \mathcal{M}'))^2} \leq \sqrt{2d_H^2(\mathcal{M}, \mathcal{M}')} \sum_{t \in B_k} (||\varepsilon_t|| + 2R)^2
\leq d_H(\mathcal{M}, \mathcal{M}') \sqrt{2 \sum_{t \in B_k} (||\varepsilon_t|| + 2R)^2}.
\]

Applying the chaining technique (see [47], Lemma A.3), we obtain the following form of the Dudley’s integral:

\[
\mathcal{R}_k(\mathcal{M}_d^d) \lesssim \mathbb{E} \left( \gamma \sum_{t \in B_k} (2R + ||\varepsilon_t||) + \sqrt{2 \sum_{t \in B_k} (||\varepsilon_t|| + 2R)^2} \int_{\gamma}^{2R} \sqrt{\log \mathcal{N}(\mathcal{M}_d^d, d_H, u) du} \right), \quad \forall \gamma \in [0, 2R],
\]

(16)

where \(\mathcal{N}(\mathcal{M}_d^d, d_H, u)\) is the \(u\)-covering number of \(\mathcal{M}_d^d\) with respect to the Hausdorff distance \(d_H\). Theorem 9 in [18] claims that

\[
\log \mathcal{N}(\mathcal{M}_d^d, d_H, u) \leq c_1 \left( \frac{D}{d} \right)^{\frac{c_2}{d}} \exp \left\{ 2^{d/2}(D - d)(c_2/d^D)u^{-d/2} \right\},
\]

where the constant \(c_2\) depends only on \(\mathcal{M}^d\) and \(d\). Using the inequality \(\left( \frac{D}{d} \right)^{\frac{c_2}{d}} \leq (eD/d)^d\), we obtain

\[
\log \mathcal{N}(\mathcal{M}_d^d, d_H, u) \leq \log c_1 + d(c_2/d^D) \log \frac{eD}{d} + 2^{d/2}(D - d)(c_2/d^D)u^{-d/2}.
\]

In (16), choose

\[
\gamma = \begin{cases} 
0, & d < 4, \\
\psi_k, & d = 4, \\
\psi_k^{1-4/d}, & d > 4.
\end{cases}
\]

where

\[
\psi_k = \frac{R \sqrt{DN_k} + D \sqrt{\sum_{t \in B_k} \sigma_t^2}}{RN_k + \sqrt{D} \sum_{t \in B_k} \sigma_t}.
\]

Then

\[
\mathcal{R}_k(\mathcal{M}_d^d) \lesssim \begin{cases} 
D \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{DN_k}, & d < 4, \\
D \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{DN_k} \log \psi_k, & d = 4, \\
D \sqrt{\sum_{t \in B_k} \sigma_t^2} + R \sqrt{DN_k} \psi_k^{1-4/d}, & d > 4.
\end{cases}
\]

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C Proofs related to Theorem 2

C.1 Proof of Lemma 4

Assume that there exists $x_0 \in \mathcal{M}^*$ such that $d(x_0, \mathcal{M}^*) > c\kappa$. Then, for any $x \in \mathcal{B}(x_0, c\kappa/2)$, it holds that $d(x, \mathcal{M}) > c\kappa/2$. Due to Lemma 5 with probability at least $1 - \frac{4dV}{(c\kappa)^d} e^{-\frac{p_0(c\kappa/4)^d}{12}|T/k|}$, the ball $\mathcal{B}(x_0, c\kappa/2)$ contains at least $0.5p_0(c\kappa/4)^d|T/k|$ points.

On this event, for any $\mathcal{M} \in \mathcal{M}^d$, we have

$$\frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) = \frac{1}{T} \sum_{t=1}^{T} \min_{x \in \mathcal{M}} \|Y_t - x\|^2 > \frac{1}{T} \sum_{t=1}^{T} \left( \min_{x \in \mathcal{M}} \frac{1}{2} \|X_t - x\|^2 - \|\varepsilon_t\|^2 \right)$$

$$= \frac{1}{2T} \sum_{t=1}^{T} d^2(X_t, \mathcal{M}) - \frac{1}{T} \sum_{t=1}^{T} \|\varepsilon_t\|^2 > 2p_0 \left( \frac{c\kappa}{4} \right)^{d+2} - \frac{1}{T} \sum_{t=1}^{T} \|\varepsilon_t\|^2.$$

On the other hand,

$$\frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}^*) \leq \frac{1}{T} \sum_{t=1}^{T} \|\varepsilon_t\|^2.$$

Lemma 6 implies

$$\frac{1}{T} \sum_{t=1}^{T} \|\varepsilon_t\|^2 \leq \frac{8}{T} \sum_{t=1}^{T} \sigma_t^2 + 64D\sigma_{\text{max}}^2 + \frac{64}{T} \left[ \sum_{t=1}^{T} \sigma_t^4 \left( \log \frac{1}{\delta} \vee \sqrt{\log \frac{1}{\delta}} \right) \right].$$

Choose $\delta = 1/T$. If $T$ is large enough then, with probability at least $1 - \frac{4dV}{(c\kappa)^d} e^{-\frac{p_0(c\kappa/4)^d}{12}|T/k|} - 1/T$, we have

$$\frac{1}{T} \sum_{t=1}^{T} \|\varepsilon_t\|^2 < \frac{p_0}{4k} \left( \frac{c\kappa}{4} \right)^{d+2},$$

which yields

$$\frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}) > \frac{1}{T} \sum_{t=1}^{T} d^2(Y_t, \mathcal{M}^*)$$

simultaneously for all $\mathcal{M} \in \mathcal{M}^d$ such that $\max_{x \in \mathcal{M}^*} d(x, \mathcal{M}) \geq c\kappa$. 

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C.2 Proof of Lemma 5

Let $\mathcal{N}(M^*, h)$ be the minimal $h$-net of $M^*$ with respect to the Euclidean distance. Then Lemma 9 yields

$$P \left( \exists x \in M^* : \sum_{t=1}^{T} 1 \left( X_t \in B(x, 2h) \right) < \frac{p_0 h^d}{2} \left\lfloor \frac{T}{k} \right\rfloor \right) \leq |\mathcal{N}(M^*, h)| e^{-\frac{p_0 h^d}{12} \left\lfloor \frac{T}{k} \right\rfloor}.$$

Similarly,

$$P \left( \exists x \in M^* : \sum_{t=1}^{T} 1 \left( X_t \in B(x,h/2) \right) > \frac{3p_1 h^d T}{2} \right) \leq |\mathcal{N}(M^*, h)| e^{-\frac{p_1 h^d}{12} \left\lceil \frac{T}{k} \right\rceil}.$$

To finish the proof, note that Lemma 2.5 in [7] implies that, for any $h \leq 2\kappa$ a Euclidean ball $B(x,h)$, $x \in M^*$, contains a ball of radius $h/2$ with respect to the geodesic distance on $M^*$. Since the volume of $M^*$ is at most $V$, it can be covered with $(2^d V)/h^d$ Euclidean balls of radius $h$.

C.3 Proof of Lemma 6

Proof of (1).

Fix any $t \in \{1, \ldots, T\}$. Theorem 1.19 in [41] implies that, with probability at least $1 - \delta/T$, we have

$$\|\epsilon_t\| \leq 4\sigma_t \sqrt{d} + 2\sigma_t \sqrt{2 \log(T/\delta)}.$$

The union bound yields the assertion of the lemma.

Proof of (2).

Using the $\epsilon$-net argument (see [41], Theorem 1.19), we obtain

$$\sum_{t=1}^{T} \|\epsilon_t^+\| = \max_{u_1, \ldots, u_T \in B(0,1)} \sum_{t=1}^{T} u_t^T \epsilon_t^+ \leq \max_{u_t \in \mathcal{N}_1^{T/2}, \ldots, u_T \in \mathcal{N}_1^T} \sum_{t=1}^{T} u_t^T \epsilon_t^+.$$

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where $N_{t/2}^t$, $1 \leq t \leq T$, is the minimal $1/2$-net of $B(0,1) \cap T_{X_t}^\perp \mathcal{M}^*$. It is known that $|N_{t/2}^t| \leq 6^{D-d}$. For any $t$ and any $u_t \in N_{t/2}^t$, $u_t^T \varepsilon_t^\perp$ is a sub-Gaussian random variable with parameter $\sigma_t^2$. The Hoeffding inequality and the union bound yield that, for any $u > 0$,

$$
\Pr \left( \sum_{t=1}^T \| \varepsilon_t^\perp \| \geq u \right) \leq 6^{(D-d)T} \exp \left\{ - \frac{u^2}{8 \sum_{t=1}^T \sigma_t^2} \right\},
$$

and the claim of the lemma follows.

**Proof of (3).**

The $\varepsilon$-net argument (see [41], Theorem 1.19) yields

$$
\| \varepsilon_t^\perp \|^2 = \max_{u \in B(0,1)} (u^T \varepsilon_t^\perp)^2 \leq 4 \max_{u \in N_{t/2}^t} (u^T \varepsilon_t^\perp)^2,
$$

where $N_{t/2}^t$, $1 \leq t \leq T$, is the minimal $1/2$-net of $B(0,1) \cap T_{X_t}^\perp \mathcal{M}^*$. Then

$$
\frac{1}{T} \sum_{t=1}^T \| \varepsilon_t^\perp \|^2 \leq \frac{4}{T} \max_{u_1 \in N_{1/2}^1, \ldots, u_T \in N_{1/2}^T} \sum_{t=1}^T (u_t^T \varepsilon_t^\perp)^2.
$$

Lemma 1.12 in [41] implies that $(u_t^T \varepsilon_t^\perp)^2$ is sub-Exponential random variable $\text{SE}(16\sigma_t^2, 16\sigma_t^2)$ (see [54], Definition 2.7 for definition of sub-exponential random variable). Then, for any $u_1 \in N_{1/2}^1, \ldots, u_T \in N_{1/2}^T$, $\sum_{t=1}^T (u_t^T \varepsilon_t^\perp)^2$ is sub-Exponential random variable $\text{SE} \left( 16 \sqrt{\sum_{t=1}^T \sigma_t^4}, 16\sigma_{\max}^2 \right)$. Using a concentration inequality for sub-exponential random variables (see [54], Proposition 2.9), we obtain that

$$
\frac{4}{T} \sum_{t=1}^T (u_t^T \varepsilon_t^\perp)^2 \leq \frac{4}{T} \sum_{t=1}^T \sigma_t^2 + 32 \left( \sigma_{\max}^2 \log \frac{1}{\delta} \vee \sqrt{\sum_{t=1}^T \sigma_t^4 \log \frac{1}{\delta}} \right)
$$

with probability at least $1 - \delta$. The union bound implies that, with probability at least
1 − δ, it holds that
\[
\frac{4}{T} \max_{u_1 \in \mathcal{N}_1/2, \ldots, u_T \in \mathcal{N}_T/2} \sum_{t=1}^{T} (u_t^T \varepsilon_t)^2 
\leq \frac{4}{T} \sum_{t=1}^{T} \sigma_t^2 + 32 \frac{T}{T} \left( \sigma_{\max}^2 \left( dT \log 6 + \log \frac{1}{\delta} \right) \vee \left( dT \log 6 + \log \frac{1}{\delta} \right) \sum_{t=1}^{T} \sigma_t^4 \right) 
\leq \frac{4}{T} \sum_{t=1}^{T} \sigma_t^2 + 64 \sigma_{\max}^2 + \frac{32}{T} \left( \sum_{t=1}^{T} \sigma_t^4 \left( \log \frac{1}{\delta} \vee \sqrt{\log \frac{1}{\delta}} \right) \right).
\]

**Proof of (4).**

The proof of (4) is similar to the proof of (3).

**Proof of (5).**

Using the large deviation inequality for the norm of sub-Gaussian random vector (see [41], the proof of Theorem 1.19), we obtain \( P \left( \| \varepsilon_t \| \geq c \kappa \right) \leq 6^d e^{-\frac{\kappa^2}{4dT}} \). The Bernstein’s inequality for Bernoulli random variables yields that, for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), it holds that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{I} \left( \| \varepsilon_t \| \geq c \kappa \right) \leq \frac{6^d}{T} \sum_{t=1}^{T} e^{-\frac{c^2 \kappa^2}{4dT}} + \sqrt{\frac{2 \cdot 6^d \log(1/\delta)}{T} \sum_{t=1}^{T} e^{-\frac{c^2 \kappa^2}{4dT}}} + \frac{2 \log(1/\delta)}{3T} 
\leq \frac{2 \cdot 6^d}{T} \sum_{t=1}^{T} e^{-\frac{c^2 \kappa^2}{4dT}} + \frac{2 \log(1/\delta)}{T}.
\]

**D  Plots of the predictions**
Figure 1: Prediction of the first component of multidimensional time series

Figure 2: Prediction of the second component of multidimensional time series
Figure 3: Prediction of the third component of multidimensional time series

Figure 4: Prediction of the fourth component of multidimensional time series