Remarks on the general Funk-Radon transform and thermoacoustic tomography

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1 Introduction

The generalized Funk transform is a integral transform acting from densities on a manifold $X$ to functions defined on a family $\Sigma$ of hypersurfaces in $X$. The dual operator $M^\circ$ is again a Funk transform which is defined for densities on the manifold $\Sigma$. We state two-side estimates for the operator $M$ and a description of the range of the Funk transform operator $M$ and approximation theorem for the kernel of the dual operator. This operator is similar to the integral transform related to a double fibration in the sense of Guillemin [4] and some results can be extracted from his theory. Other results can be generalized for the general double fibration.

2 Geometry

Let $X$ and $\Sigma$ be smooth $n$-manifolds, $n > 1$ and $F$ be a smooth closed hypersurface in $X \times \Sigma$ such that

(i) the projections $p : F \to X$ and $\pi : F \to \Sigma$ have rank $n$. This condition implies that the sets $F(\sigma) = \pi^{-1}(\sigma), \sigma \in \Sigma$ and $F(x) = p^{-1}(x), x \in X$ are hypersurfaces in $X$, respectively, in $\Sigma$. We call $F$ incidence manifold. It can be defined locally by the equation $I(x, \sigma) = 0$, where $I$ is a smooth function in $X \times \Sigma$ such that $dI \neq 0$. If the hypersurface $F$ is cooriented in $X \times \Sigma$, a function $I$ can be chosen globally; we call it incidence function. If $F$ is not cooriented, one can choose local incidence functions $I_\alpha, \alpha \in A$ such that $I_\beta = \pm I_\alpha$ in the domain, where both functions $I_\alpha, I_\beta$ are defined. The additional condition is

(ii) the mapping $q : F \to G^{n-1}(X)$ is a local diffeomorphism, where $G^{n-1}(X) = \cup_x G^{n-1}_x$ and $G^{n-1}_x$ means the manifold of $n - 1$-subspaces in the tangent space $T_x$ of $X$ at $x; q(x, \sigma) \equiv (x, H)$, where $H$ denotes the tangent hyperplane to $F(\sigma)$ at $x$. It follows that for any point $x \in X$ and any tangent hyperplane $H \subset T_x$ there is locally only one hypersurface $F(\sigma)$ that contains $x$ and is tangent to $H$. 

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Proposition 2.1 The conditions (i-ii) are equivalent to the inequality \( \det \Phi \neq 0 \) in \( F \), where

\[
\Phi = \begin{pmatrix}
\frac{\partial^2 I}{\partial x_1 \partial \sigma_1} & \ldots & \frac{\partial^2 I}{\partial x_i \partial \sigma_n} & \frac{\partial I}{\partial x_1} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 I}{\partial x_n \partial \sigma_1} & \ldots & \frac{\partial^2 I}{\partial x_n \partial \sigma_n} & \frac{\partial I}{\partial x_n} \\
\frac{\partial I}{\partial \sigma_1} & \ldots & \frac{\partial I}{\partial \sigma_n} & I
\end{pmatrix},
\]

where \( x_1, \ldots, x_n \) and \( \sigma_1, \ldots, \sigma_n \) are local coordinates in \( X \) and in \( \Sigma \), respectively.

Proof. Suppose that \( \det \Phi \neq 0 \). Then also \( d_x I \neq 0 \) and \( d_\sigma I \neq 0 \) which implies (i). Choose a point \((x_0, \sigma_0)\) and take a tangent vector \( \theta \) to \( F(\sigma_0) \) at \( x_0 \). The vector \((\theta, 0)\) is tangent to \( F \) at \((x_0, \sigma_0)\) and the map \( q \) is well defined. Change the coordinates \( x_i \) and \( \sigma_i \), \( i = 1, \ldots, n \) in such a way that \( \partial I(x_0, \sigma_0)/\partial \sigma_i = \partial I(x_0, \sigma_0)/\partial x_i = 0, i = 2, \ldots, n \) at this point. We have then

\[
\det \Phi = \frac{\partial I}{\partial x_1} \frac{\partial I}{\partial \sigma_1} \det \Psi, \quad \Psi = \left\{ \frac{\partial^2 I}{\partial x_i \partial \sigma_j} \right\}_{i,j=2}^n.
\]

The inequality \( \det \Psi \neq 0 \) implies that the forms \( \partial d_x I(x_0, \sigma)/\partial \sigma_2, \ldots, \partial d_x I(x_0, \sigma)/\partial \sigma_n \) are independent. This means that the fields \( \partial/\partial \sigma_2, \ldots, \partial/\partial \sigma_n \) do not move the point \( x_0 \) but rotate the tangent hyperplane to \( F(\sigma_0) \) at \( x_0 \) whereas the field \( \partial/\partial \sigma_1 \) move the point \( x_0 \). This yields (ii). The inverse statement can be proved on the same lines. \( \square \)

It follows that the properties (i-ii) are symmetric with respect to \( X \) and \( \Sigma \). The next condition is not symmetric:

(iii) the projection \( p : F \to X \) is proper. If \( F \) satisfies (ii) and (iii) and the hypersurface \( F(x) \) is not empty for a point \( x \in X \), then the mapping \( q : F(x) \to G^{n-1}_x \) is surjective, since the manifold \( G^{n-1}_x \) is connected.

3 The Funk transform

Consider a hypersurface \( F \) as above defined by a incidence function \( I \) that fulfils (i). Define the Funk (or Minkowski-Funk-Radon) transform for densities \( f \) in \( X \) with compact support by means of the integral

\[
Mf(\sigma) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|I(\cdot, \sigma)| \leq \varepsilon} f = \int_{F(\sigma)} \frac{f}{d_x I}, \quad \sigma \in \Sigma,
\]

where \( \omega = f/d_x I \) is a \( n-1 \)-form such that \( d_x I \wedge \omega = f \). This form is defined up to a term \( d_x I \wedge \chi \), where \( \chi \) is a \( n-2 \)-form. Therefore the restriction of \( \omega \) to the curve \( F(\sigma) \) is a well-defined density. This density does not change, if we replace \( I \) by \(-I\). Therefore the
Funk transform is well defined. The function $Mf$ is also continuous. Suppose that the condition (iii) is fulfilled. If a density $f$ is supported in a compact set $K \subset X$, then $Mf$ is supported in the compact set $\Lambda = \pi(p^{-1}(K)) \subset \Sigma$.

**Example 1.** Let $X$ and $\Sigma$ be unit spheres in Euclidean 3-spaces and the hypersurface $F$ be defined by the global incidence function $I(x, \sigma) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$. The operator $M$ coincides with the classical Minkowski-Funk transform, [3]. The Funk transform can be also defined on projective planes $X\cong P_2 = S_2/\mathbb{Z}_2$, $\Sigma = P^2$, if we take $\pm I$ as local incidence functions. The dual operator $M^\circ$ coincides with $M$ through the natural isomorphism $X\cong \Sigma$.

**Example 2.** Let $(X, g)$ be a Riemannian 2-manifold with boundary and $\Sigma$ be the family of closed geodesic curves $\gamma$. Take a density $f = f dS$, where $dS$ is the Riemannian area form and $f$ is a continuous function with compact support. Then we can write the geodesic integral transform as the Funk transform

$$Mf(\gamma) = \int_\gamma f ds,$$

if we take an incidence function $I$ for the family $F$ such that $|\nabla_x I| = 1$. Any smooth weight function $w = w(x, \sigma)$ can be included, by replacing the function $I$ to $w^{-1}I$.

## 4 Above estimates

The scale of Sobolev $L_2$-norms $\|\cdot\|^{\alpha}, \alpha \in \mathbb{R}$ is defined for functions supported in an arbitrary compact set $K \subset X$. Fix a volume form $dX$ in $X$ and define $\|f\|^\alpha = \|f_0\|^{\alpha_0}$ for a density $f = f_0 dX$ with support in $K$. Denote by $H^K_\alpha(K, \Omega)$, $H^\alpha_\omega(X)$ the space of densities (distributions), respectively, of (generalized) functions supported in $K$ with finite norm $\|\cdot\|_\alpha$. For an arbitrary compact set $\Lambda \subset \Sigma$ we define the spaces $H^\alpha_\Lambda(\Sigma, \Omega)$, $H^\alpha(\Sigma)$ in the same way.

**Proposition 4.1** For any family $F$ that fulfils (i-ii), an arbitrary compact set $K \subset X$, any smooth function $\varepsilon$ in $\Sigma$ with compact support and any real $\alpha$ the inequality holds

$$\|\varepsilon Mf\|^{\alpha + \frac{n-1}{2}} \leq C_\alpha \|f\|^\alpha$$

for $f \in H^K_\alpha(X, \Omega)$.

**Proof.** The Funk transform can be expressed as an oscillatory integral

$$Mf(\sigma) = \int_K \int_\mathbb{R} \exp\left(2\pi i \tau I(x, \sigma)\right) f(x) d\tau.$$

The critical set of the phase function $\tau I(x, \sigma)$ is the hypersurface $F(\sigma)$ and the condition $d_\sigma I \neq 0$ implies that the phase function is non-degenerate. The corresponding conic Lagrange manifold is

$$L = \{(x, \sigma, \xi, \rho) \in T^* (X \times \Sigma), I(x, \sigma) = 0, \rho = \tau d_\sigma I, \xi = \tau d_\sigma I, \tau \neq 0\}.$$
Lemma 4.2  

Rank of the matrix \( \frac{\partial(x,\xi)}{\partial(\sigma,\rho)} \) is equal to \( 2n \) in any point of \( \Lambda \).

**Proof of Lemma.** Suppose that the rank is less than \( 2n \). Then there exists a vector \( t = (\delta x, \delta \sigma, \delta \xi, \delta \rho) \) in \( T^* (X \times \Sigma) \) tangent to \( L \) such that \( \delta \sigma = 0, \delta \rho = 0 \). This yields

\[
\begin{align*}
dI (\delta x) &= 0, \delta \rho = \delta \tau d_\sigma I + \tau d_\sigma^2 I (\delta x), \\
\delta \xi &= \delta \tau d_x I + \tau d_x^2 I (\delta x)
\end{align*}
\]

for a tangent vector \( \delta \tau \) to \( \mathbb{R} \). The first line implies that the vector \( (\tau \delta x, \delta \tau) \) fulfills the equation \( (\tau \delta x, \delta \tau) \Phi = 0 \). By Proposition 2.1 this vector vanishes, that is \( \delta x = 0, \delta \tau = 0 \). The second line gives \( \delta \xi = 0 \).

By this Lemma the projections of \( L \) to \( T^* (X) \) and to \( T^* (\Sigma) \) are submersions. In other terms, \( L \) is locally the graph of a canonical transformation. The symbol \( a (x, \sigma, \xi, \rho) = 1 \) is a homogeneous function of \( \xi, \rho \) of order 0. The order \( m \) of the Fourier integral operator \( M \) satisfies the equation

\[
m + \frac{\dim (X \times \Sigma)}{4} - \frac{\dim (X \times \Sigma)}{2} = 0,
\]

where \( \dim (X \times \Sigma) = 2n \) and \( \Lambda = \frac{1}{2} \) is the number of variables \( \tau \). This yields \( m = (1 - n) / 2 \), which means that the functional

\[
I (\psi) = \int_X \int_{\Sigma} \int_{\mathbb{R}} \exp (2\pi i \tau I (x, \sigma)) \psi (x, \sigma) d\sigma d\tau
\]

defined for a smooth densities \( \psi \) in \( X \times \Sigma \) with compact support, is a distribution of the class \( \mathcal{I}^{1-n/2} (X \times \Sigma, L) \) in the sense of Definition 25.4.9 of [6]. By Corollary 25.3.2 the operator \( \varepsilon M \) defines a continuous map \( H^\alpha (X, \Omega) \to H^{\alpha + (n-1)/2} (\Sigma) \) for any real \( \alpha \), where \( \Lambda = \text{supp} e \).

**Corollary 4.3** If \( F \) fulfills (i-iii), the Funk transform \( M \) can be extended to a bounded operator \( H^\alpha (X, \Omega) \to H^{\alpha + (n-1)/2} (\Sigma) \) for any \( \alpha, K \) and \( \Lambda = \pi (p^{-1} (K)) \).

Due to (iii), we have \( \text{supp} Mf \subset \Lambda \) and the cutoff factor \( \varepsilon \) can be dropped out.

## 5  Dual Funk transform

Let \( \varphi \) be a density in \( \Sigma \) with compact support. Define the dual Funk transform as follows

\[
M^\circ \varphi (x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{I(x,\cdot)|\leq \varepsilon} \varphi = \int_{F(x)} \frac{\varphi}{d\sigma I}.
\]

If \( F \) satisfies (iii), then the manifold \( F (x) \) is compact for any \( x \in X \) and the integral (2) is well defined for any continuous density \( \varphi \) in \( \Sigma \). The natural pairing

\[
(f, \phi) \mapsto \langle f, \phi \rangle = \int_X f \phi
\]

is well defined for a densities \( f \) and functions \( \phi \) on \( X \) provided one of them has compact support.

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Proposition 5.1  The operator \(-M^\circ\) is dual to \(M\).

Proof. We have

\[
\langle Mf, \varphi \rangle = \int_\Sigma M(f) \overline{\varphi} = \int_\Sigma \varphi \int_{F(\sigma)} \frac{f}{d_x I} = \int_F \frac{f \wedge \varphi}{d_\sigma I} = -\int_F f \wedge \frac{\varphi}{d_\sigma I},
\]

since \(dI = d_x I + d_\sigma I = 0\) on \(F\). The right-hand side equals

\[
-\int_X f \int_{F(x)} \frac{\varphi}{d_\sigma I} = -\int_X f M^\circ(\varphi) = -\langle f, M^\circ \varphi \rangle.
\]

\(\blacksquare\)

6  Backprojection and two side estimates

Definition. Fix some area forms \(dX\) in \(X\) and \(d\Sigma\) in \(\Sigma\). The back projection operator

\[M^* : g \mapsto M^\circ(gd\Sigma) dX\]

transforms functions defined in \(\Sigma\) to densities in \(X\).

Definition. We say that points \(x, y \in X\) are conjugate with respect to \(F\), if \(x \neq y\) and the form \(d_\sigma I(x, \sigma) \wedge d_\sigma I(y, \sigma)\) defined in \(\Sigma\) vanishes.

Theorem 6.1  If a family \(F\) has no conjugate points, fulfils (i-ii) and the condition:

(iv) the projection \(q : \pi^{-1}(\Lambda) \to G^{n-1}(K)\) is surjective for some sets \(K \subseteq X, \Lambda \subseteq \Sigma\), then for arbitrary cutoff function \(\varepsilon\) such that \(\varepsilon = 1\) in \(\Lambda\) and any \(\alpha > \beta\) the estimate

\[\|f\|_{\alpha} \leq C_\alpha \|\varepsilon Mf\|^\alpha \|f\|_{\beta}^{(n-1)/2} + C_\beta \|f\|_{\beta}\]

holds for the Funk transform of densities \(f\) supported in \(K\), where \(C_\alpha\) and \(C_\beta\) do not depend on \(f\).

Lemma 6.2  The composition \(M^* \varepsilon M\) is an elliptic PDO in \(K\) of order \(1 - n\).

Proof of Lemma. Write \(f = f_0 dX\), where \(f_0\) is a function supported in \(K\) and calculate

\[
\begin{align*}
\frac{M^* \varepsilon Mf}{dX}(y) &= \int_{F(y)} \frac{\varepsilon(\sigma)}{d_\sigma I} \int_{F(\sigma)} \frac{f(x)}{d_y I(x, \sigma)}
\end{align*}
\]

since \(dI = d_x I + d_\sigma I = 0\) in \(F\). We can write the right-hand side as \(\int A(y, x) f(x)\), where

\[
A(y, x) = -\int_{F(x) \cap F(y)} \frac{\varepsilon(\sigma)}{d_\sigma I(y, \sigma) \wedge d_\sigma I(x, \sigma)}.
\]
The dominator does not vanishes for \( y \neq x \), since there is no conjugate points. The quotient \( Q \) in (3) is well defined as \( n - 2 \)-form up to an additive term \( d_\sigma I (y, \sigma) \wedge S + d_\sigma I (x, \sigma) \wedge R \), where \( S \) and \( R \) are some \( n - 3 \)-forms. The integral of this term along the smooth manifold \( F (x) \cap F (y) \) vanishes and the function \( A (y, x) \) is a well defined and smooth, except for the diagonal. Near the diagonal we can write \( I (y, \sigma) = I (x, \sigma) + \sum (y - x_i) \partial I (x, \sigma) / \partial x_i + O (|y - x|^2) \) and

\[
d_\sigma I (x, \sigma) \wedge d_\sigma I (y, \sigma) = \sum_i (y_i - x_i) d_\sigma I (x, \sigma) \wedge d_\sigma \frac{\partial I (x, \sigma)}{\partial x_i} + O (|y - x|^2).
\]

The forms \( d_\sigma I (x, \sigma) \wedge d_\sigma \partial I (x, \sigma) / \partial x_i, i = 1, ..., n \) do not vanish and are linearly independent, since of Proposition 2.1. Therefore the product \( d_\sigma I (x, \sigma) \wedge d_\sigma I (y, \sigma) \) is bounded by \( c |x - y| \) from below as \( y \to x \). Therefore we have \( A (y, x) = a (y) |x - y|^{-1} + O (1) \) near the diagonal, where \( a \) is a smooth positive function. This implies that \( M^* \varepsilon M \) is a classical integral operator on \( K \) with weak singularity, moreover it is a pseudodifferential operator of order \( 1 - n \). It is an elliptic operator, since the symbol \( a \) is positive.

**Proof of Theorem.** The support of the function \( M^* \varepsilon M f \) is contained in the compact set \( p (\text{supp} \varepsilon) \subset X \). By Proposition 4.1 \( M^* \) is \( (n - 1) / 2 \)-smoothing operator, which yields

\[
\| M^* \varepsilon M f \|^\alpha + n - 1 \leq C \| \varepsilon M f \|^\alpha (n - 1) / 2 .
\]

By Lemma 6.2 the operator \( M^* \varepsilon M \) is elliptic of order \( n - 1 \), therefore the standard inequality holds

\[
\| f \|^\alpha \leq C_\alpha \| M^* \varepsilon M f \|^\alpha + n - 1 + C_\beta \| f \|^\beta
\]

for an arbitrary \( \beta \) and some constants \( C_\alpha, C_\beta \). Taking in account (5) yields (3).

**Corollary 6.3** The eigenvalues \( \lambda_k \) of the operator \( M^* \varepsilon M \) numbered in decreasing order satisfy the estimate

\[
ck^{(1-n)/2} \leq \lambda_k \leq Ck^{(1-n)/2} , \ k \geq k_0 .
\]

For the Radon transform the eigenvalues are calculated in [10].

**Corollary 6.4** Suppose that for some \( \beta < \alpha \) the equation \( M f = 0, f \in H^\beta_K (X, \Omega) \) implies \( f = 0 \). Then the two-side estimate holds:

\[
c_\alpha \| f \|^\alpha \leq \| \varepsilon M f \|^\alpha + n - 1 / 2 \leq C_\alpha \| f \|^\alpha .
\]

**Proof.** The right-hand side inequality follows from Proposition 4.1. Suppose that the left-hand side estimate does hold for no \( c_\alpha \). Then there exists a sequence \( \{ f_k \} \subset H^\alpha_K (X, \Omega) \) such that

\[
\| f_k \|_\alpha \geq k \| M f_N \|_{\alpha + 1 / 2} \| f_k \|_\beta = 1, k = 1, 2, \ldots
\]

The inequality (3) implies that \( \| f_k \|_\alpha \leq 2C_\beta \) for \( k > 2C_\alpha \) and \( \| M f_k \|_{\alpha + 1 / 2} \to 0 \). Because the imbedding \( H^\alpha_K (X, \Omega) \to H^\beta_K (X, \Omega) \) is compact, we can choose a subsequence (denote
it again \{f_k\}) such that f_k \rightarrow g in \(H^\beta_K(X, \Omega)\). By Proposition \ref{prop:approximation} \(\|Mf_k - Mg\|_{\beta+(n-1)/2} \rightarrow 0\), which implies \(Mg = 0\). By the condition \(g = 0\); it follows that \(\|f_k\|_\beta \rightarrow 0\) in contradiction with (7).)

**Remarks.** Mukhometov’s result \cite{Mukhometov} implies the estimate \(\|f\|_0 \leq C \|Mf\|_1\) for the case \(n = 2\). An estimate of this kind for more general situation was obtained by Sharafutdinov \cite{Sharafutdinov}, Ch. IV. Inequalities for Sobolev norms are well known for the Radon transform. Estimates for shift derivatives of order \(\alpha + 1/2\) \((n = 2)\) were obtained by several authors. Natterer \cite{Natterer} has shown that (6) holds also for angular derivatives. For the attenuated Radon transform see Rullgard \cite{Rullgard}.

Our approach is similar to that of Lavrent’ev and Bukhgeim \cite{Lavrentev-Bukhgeim}, where the composition \(M^*M\) was described as an integral operator in the local case. Guillemin \cite{Guillemin1, Guillemin2} has defined the ‘generalized Radon transform’ \(R\) for an arbitrary double fibration. This transform is treated as an elliptic Fourier integral operator and \(R^*R\) is shown to be a pseudodifferential elliptic operator under the ‘Bolker condition’. This condition is equivalent to absence of conjugate points in our situation. More details are given in the paper of T. Quinto \cite{Quinto}.

### 7 Range conditions and approximation

Let \(K\) be a compact set in \(X\) and \(\alpha \in \mathbb{R}\). We define \(H^\alpha(K, \Omega)\) to be the dual space of \(H^{-\alpha}_K(X)\) and use the notation \(\|\cdot\|_\alpha\) for the norm in \(H^\alpha(K, \Omega)\). The trace operator \(H^\alpha_L(X, \Omega) \rightarrow H^\alpha(K, \Omega)\) is well defined and is open for an arbitrary compact set \(L \subset X\) such that \(K \in L\), since \(H^{-\alpha}_K(X)\) is a subspace of \(H^{-\alpha}_L(X)\). Therefore \(H^\alpha(K, \Omega)\) can be realized as the quotient space of \(H^\alpha_L(X, \Omega)\) modulo the kernel of the trace operator. The last one consists of densities \(f\) supported in \(L \setminus K\). For any \(\beta > \alpha\) we have the operator \(\eta' : H^\beta(K, \Omega) \rightarrow H^\alpha(K, \Omega)\), which is dual to the natural imbedding \(\eta : H^{-\alpha}_K(X) \rightarrow H^{-\beta}_K(X)\). If the boundary of \(K\) is smooth, the imbedding \(\eta\) has dense image and \(\eta'\) is injective. Then we can define the intersection \(H^\infty(K, \Omega) = \cap \alpha H^\alpha(K, \Omega)\); any density \(f \in H^\infty(K, \Omega)\) is smooth in the interior of \(K\). Similarly, we define \(H^\alpha(K) = (H^\alpha_K(X, \Omega))'\).

Suppose that the incidence manifold \(F\) fulfills (i-iii). If a density \(f\) is supported in a compact set \(K\), then the support of \(Mf\) is contained in the compact set \(\Lambda = \pi(p^{-1}(K))\). The hypersurface \(F(x)\) is compact for any point \(x \in X\) and the dual transform \(M^*\) is well defined for all continuous densities in \(\Sigma\). Moreover, it can be extended to a continuous operator \(M^* : H^\alpha(\Lambda, \Omega) \rightarrow H^{\alpha+(n-1)/2}(K)\) for any \(\alpha\) by means of the duality

\[
\langle M^*g, f \rangle = -\langle g, Mf \rangle, \quad f \in H^{-\alpha-(n-1)/2}(X, \Omega), \quad g \in H^{\alpha}(\Lambda, \Omega).
\]

By Proposition \ref{prop:approximation} we have \(Mf \in H^{-\alpha}(\Sigma)\), hence the right-hand side is well defined. If \(\alpha \geq 0\), the function \(M^*g\) defined by this formula is equal to the integral (2), which has sense, at least, for almost all \(x \in K\).

**Theorem 7.1** Suppose that \(F\) satisfies (i-iii) and has no conjugate points. Then for any \(K \in X\) and arbitrary \(\alpha \in \mathbb{R} \cup \{\infty\}\) the image of the Funk operator

\[
M : H^\alpha_K(X, \Omega) \rightarrow H^{\alpha+(n-1)/2}_\Lambda(\Sigma), \quad \Lambda = \pi(p^{-1}(K))
\]

is
is closed and coincides with the subspace of functions \( \varphi \in H^{\alpha+(n-1)/2}_\Lambda (\Sigma) \) such that \( \int g \varphi = 0 \) for any solution \( g \in H^{-\alpha-(n-1)/2}_\Lambda (\Lambda, \Omega) \) of the equation

\[
M^o g(x) = 0, x \in K. \tag{8}
\]

**Proof.** The image of \( M \) is closed by Theorem 6.1 thereby it coincides with the polar of the kernel of the dual operator \( M^o \). ▶

**Theorem 7.2** If \( F \) fulfills (i-iii) and has no conjugate points. Then for any set \( K \subseteq X \) with smooth boundary and arbitrary real \( \alpha \in \mathbb{R} \) any density \( g \in H^\alpha (\Lambda, \Omega), \Lambda = \pi (p^{-1}(K)) \) that fulfills (3) can be approximated by solutions \( h \in H^\infty (\Lambda, \Omega) \cap \Lambda H^\alpha (\Lambda, \Omega) \).

**Proof.** Let \( \text{Sol}^\beta \) denote the space of solutions of (3) in the class \( H^\beta (\Lambda, \Omega) \). We show first that \( g \) can be approximated by elements of \( \text{Sol}^\beta \) for any \( \beta > \alpha \). It is sufficient to check that any functional \( \phi \) on \( H^\alpha (\Lambda, \Omega) \) that is equal to zero on \( \text{Sol}^\beta \) also vanishes on \( g \). The dual space is isomorphic to \( H^{-\alpha}_\Lambda (\Sigma) \), which implies \( \phi \in H^{-\alpha}_\Lambda (\Sigma) \). By Corollary 4.3 the Funk transform defines the continuous operator

\[
M_\beta : H^{-\beta-(n-1)/2}_K (X, \Omega) \to H^{-\beta}_\Lambda (\Sigma) .
\]

By Theorem 7.1 the range of this operator is closed and coincides with the polar set of \( \text{Sol}^\beta \). It follows that, \( \phi = M \psi \) for a density \( \psi \in H^{-\beta-(n-1)/2}_K (X, \Omega) \). By Theorem 6.1 we have \( \psi \in H^{-\alpha-(n-1)/2}_K (X, \Omega) \) and can write

\[
\langle \phi, g \rangle = \langle M \psi, g \rangle = - \langle \psi, M^o g \rangle = 0,
\]

since the function \( M^o g \) is well defined as element of \( H^{\alpha+(n-1)/2}_K (X) \). This yields that \( g \) is contained in the closure of the space \( \text{Sol}^\beta \).

Now we approximate \( g \) by elements of the space \( H^\infty (\Lambda, \Omega) \). Let \( \| \cdot \|^\alpha_\Lambda \) be the norm in the space \( H^\alpha (\Lambda, \Omega) \), which is dual to the norm \( \| \cdot \|^{-\alpha}_\Lambda \). We may assume that the norm \( \| \cdot \|^\alpha_\Lambda \) is monotone increasing function of \( \alpha \). Take an arbitrary \( \varepsilon \) and choose a function \( h_1 \in H^{\alpha+1}_\Lambda (\Lambda, \Omega) \) such \( \| h_1 - g \|^\alpha_\Lambda < \varepsilon/2 \), then we choose a function \( h_2 \in H^{\alpha+2}_\Lambda (\Lambda, \Omega) \) such \( \| h_2 - h_1 \|^\beta_\Lambda < \varepsilon/4 \) and so on. We obtain a sequence \( \{ h_k \} \) such that \( \| h_{k+1} - h_k \|^\beta_\Lambda < 2^{-k-1} \varepsilon \) for \( k = 1, 2, \ldots \). This sequence converges to an element \( h \) in any space \( H^\beta (\Lambda, \Omega) ; \beta > \alpha \). It follows that \( h \in H^\infty (\Lambda, \Omega) \) and \( \| h - g \|^\alpha_\Lambda \leq \varepsilon \). ▶

**Corollary 7.3** Under conditions of Theorem 7.1 it is sufficient to check the equation \( \int g \varphi = 0 \) for densities \( g \in H^\infty (\Lambda, \Omega) \) that satisfies (3).

### 8 Thermoacoustic tomography

We apply the above results for the thermo/opto/photoacoustic geometry. First, consider the case of complete acquisition geometry. Let \( X \) be the open unit ball in an Euclidean
space $E$, $S_R$ be the sphere of radius $R > 1$ and $\Sigma = S_R \times \mathbb{R}$. The manifold $F \subset X \times \Sigma$ is given by the equation $I(x; y, r) = |y - x| - r = 0, y \in S_R, 0 < r$. The manifold $F$ obviously fulfils (i), (iii) and has no conjugate points. Check that the condition (ii) is also satisfied. It is sufficient to check that, the sphere $F(y, r)$ can not be tangent to $F(z, s)$ at a point $x \in X$, if the points $(y, r)$ and $(z, s)$ are sufficiently close in $\Sigma$. The condition $|y + z| > 2$ is sufficient for this. The Funk operator

$$Mf(y, r) = \int_{|x - y| = r} f_0(x) \, dS, \ f = f_0 \, dx$$

is the spherical integral transform, where $dS$ is the Euclidean surface area form on spheres. The kernel of dual transform $M^o$ consists of densities $\varphi = \phi dS dr$ in $\Sigma$ such that

$$0 = \int_{F(x)} \varphi \, d\_\sigma I = \int \frac{\phi dS dr}{d(|x - y| - r)} = -\int_{F(x)} \phi dS,$$

for any $x \in X$. Theorem 7.2 yields

**Corollary 8.1** For any compact set $K \subset X$ with smooth boundary and arbitrary $\alpha \in \mathbb{R} \cup \{\infty\}$ the range of the Funk operator $M : H^\alpha_K (X, \Omega) \to H^\alpha_{n+1/2} (\Sigma)$ coincides with the set of functions $g$ in $\Sigma$ such that

$$\int_{\Sigma} g \varphi = 0$$

for any density $\varphi = \phi dS dr$ such that $\phi \in C^\infty (\Sigma)$ and

$$\int_{F(x)} \phi dS = 0, \ x \in K.$$  

**Remark.** For the operator $M$ acting on $C^\infty$-densities the range conditions were given in the papers [11], [1], [2]. The conditions of [1] and [2] give full description of the range of $M$, but have implicit form.

We extract some explicit range conditions from Corollary 8.1. For an arbitrary $x \in K$, the manifold $F(x)$ is the intersection of the cone surface $|y - x| = r$ with the cylinder $\Sigma$. This intersection is contained in the hyperplane

$$P(x) = \{y, s; 2 \langle x, y \rangle + s = |x|^2 + R^2\} \subset E \times \mathbb{R},$$

where we set $s = r^2$. Thus, the condition (11) means vanishing of integrals of $\phi dS$ over intersections of $\Sigma$ with the hyperplanes $P(x), x \in K$. Suppose that $\phi$ is a polynomial in $s : \phi(y, s) = \sum \phi_k(y) (s - R^2)^k$ and have

$$\int_{F(x)} \phi dS = \sum_k \int_{S} \phi_k(y) (|x|^2 - 2 \langle x, y \rangle)^k dS(y).$$

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Set $x = tz$ for $|z| = 1$ and $0 \leq t < 1$ and develop the right-hand side in powers of $t$:

\[
\int_{F(x)} \phi dS = \sum \int_{S} \phi_k(y) (t^2 - 2t \langle z, y \rangle)^k dS
\]

\[
= \int \phi_0(y) dS - t \int 2\phi_1(y) \langle z, y \rangle dS + t^2 \int [\phi_1(y) + 4\phi_2(y) \langle z, y \rangle^2] dS
\]

\[
- t^3 \int [4\phi_2(y) \langle z, y \rangle + 8\phi_3(y) \langle z, y \rangle^3] dS + t^4 \int [\phi_2(y) + 12\phi_3(y) \langle z, y \rangle^2 + 16\phi_4(y) \langle z, y \rangle^4] dS
\]

\[+ \ldots = 0\]

The right-hand side vanishes for all $t$, which yields the system of equations

\[
\int \phi_0(y) dS = 0,
\]

\[
\int \phi_1(y) \langle z, y \rangle dS = 0,
\]

\[
\int [4\phi_2(y) \langle z, y \rangle^2 + \phi_1(y)] dS = 0,
\]

\[
\int [2\phi_3(y) \langle z, y \rangle^3 + \phi_2(y) \langle z, y \rangle] dS = 0,
\]

\[
\int [16\phi_4(y) \langle z, y \rangle^4 + 12\phi_3(y) \langle z, y \rangle^2 + \phi_2(y)] dS = 0,
\]

\[
\ldots
\]

**Corollary 8.2** Any solution $(\phi_0, \phi_1, \phi_2, \ldots)$ of this system such that $\phi_j = 0$ for all $j > k$ for some $k$ yields a function $\phi(y, s) = \sum_0^k \phi_j(y) (s - R^2)^j$ that is a polynomial in $s$ of order $k$, fulfils [11] and is orthogonal to the range of $M$.

There are many solutions of this form, since the system has triangle form with diagonal terms

\[
\int \phi_j(y) \langle z, y \rangle^j dS, |z| = 1, j = 0, 1, \ldots, k.
\]

To solve these equation we only need to fix the moments of $\phi_j$ of degree $j$. There are only $\binom{n+j-1}{j} \binom{1}{n-1}$ linearly independent $j$-moments, hence one can find infinitely many independent solutions which are finite sums of harmonics. In particular, we can take for $\phi_0$ any function on the sphere with zero average, an arbitrary function $\phi_1$ with zero linear moments of $\phi_1$ and set $\phi_k = 0$ for $k > 1$ etc. The range conditions of S. Patch [11] are apparently contained in [10] for polynomial $\phi$. 

10
9 Partial scan and Kaczmarz method

In the case of the partial scan geometry the analysis is more complicated. Let again \( X \) be the open ball of radius 1 in \( E \) and \( \Sigma_\delta = S_\delta \times \mathbb{R}_+ \), where \( S_\delta = \{ y; |y| = R, y_1 > -\delta \} \) for some \( \delta > 0 \). The manifold \( F_\delta \) is defined in \( X \times \Sigma_\delta \) by the same incidence function \( I \), that is, \( F_\delta(y, r) \) is the sphere of radius \( r \) with the center \( y \in S_\delta \). The manifold \( F_\delta \) fulfills (i),(ii) and has no conjugate points, but does not fulfil the condition (iii). On the other hand, \( F_\delta \) satisfies (iv) for the unit half-ball \( K = \{ x, |x| \leq 1, x_1 \geq 0 \} \) and \( \Lambda = \{ y, r : R - 1 \leq r \leq R + 1 \} \). Consider the Funk transform

\[
M : H^\alpha_K(\Omega) \to H^{\alpha+(n-1)/\alpha}_{\Lambda}(\Sigma)
\]

(12)
defined as in \([9]\). Take a smooth function \( \varepsilon_0 \geq 0 \) on \( \mathbb{R} \) supported in the interval \((-\delta, \infty)\) such that \( \varepsilon_0(t) = 1 \) for \( 0 \leq t \leq 1 \). Set \( \varepsilon(y) = \varepsilon_0(y_1) \) and consider the operator \( M^*\varepsilon M \).

**Proposition 9.1** The operator (12) is injective for any \( \delta > 0 \) and arbitrary \( \alpha > 1/2 \). The inequality (6) holds for the operator \( M^*\varepsilon M \) and any \( \alpha > 1/2 \).

\( \blacksquare \) We prove that the equation \( Mf = 0 \) in \( \Sigma_\varepsilon \) for a density \( f \in H^\alpha_K(\Omega) \) implies \( f \neq 0 \). This condition means that the spherical means of \( f \) vanish for spheres centered at points \( y \in S_\delta \). By the Lin-Pinkus theorem \([3]\) this implies that either \( f = 0 \) or a non-trivial harmonic polynomial \( h \) vanishes on \( S_\delta \) (the continuity condition for \( f \) in \([3]\) can be weakened). The last option is impossible, since \( S_\delta \) is strictly convex and \( \Delta h \) cannot vanish identically near \( S_\delta \). The estimate (6) now follows from Theorem 6.1 and Corollary 6.4.

\( \blacksquare \)

We show that the Kaczmarz method can be adapted for inversion of the operator (12) as well as for a general Funk operator \( M \). Let \( K, \Lambda \) be compact manifolds, occasionally with boundaries, \( F \) be a closed hypersurface in \( K \times \Lambda \) that fulfills (i) and (ii). We want to find a solution \( f \in H^0(\Omega) \) of the equation

\[
Mf = \varphi
\]

(13)
for a function \( \varphi \in H^0(\Lambda) \). By Proposition 4.1 the left-hand side is contained in the space \( H^{(n-1)/2}(\Lambda) \subset H^0(\Lambda) \).

**Example 3.** Take for \( K \) the closed unit ball in \( E \), for \( \Lambda \) the product \( \bar{S}_\delta \times [R - 1, R + 1] \) and for \( F \) the manifold of spheres as above. The conditions (i0) and (ii) are fulfilled.

Fix a volume form \( dX \) in \( K \), a volume form \( d\Sigma \) in \( \Lambda \) and consider the operator \( MM^* : H^0(\Lambda) \to H^0(\Lambda) \). It is non-positive; set \( R = -MM^* + \theta I \), where \( I \) is the identity operator and \( \theta > 0 \). The operator \( R \) is self-adjoint, positive and invertible. Following [10], we choose a real parameter \( \omega \) and set \( Q = I - \omega MM^*R^{-1}M \). We use the notation \( \|\cdot\| = \|\cdot\|^0 \).

**Lemma 9.2** We have \( \|Qg\| < \|g\| \) for \( 0 < \omega < 2 \) and any \( g \in H^0(K, \Omega) \) such that \( Mg \neq 0 \).
Proof. We have by Proposition 5.1

\[ \|Qg\|^2 = \|g\|^2 - 2\omega \langle g, M^* R^{-1} Mg \rangle + \omega^2 \langle M^* R^{-1} Mg, M^* R^{-1} Mg \rangle \]

\[ = \|g\|^2 - 2\omega \langle Mg, R^{-1} Mg \rangle - \omega^2 \langle R^{-1} Mg, MMR^{-1} Mg \rangle \]

\[ = \|g\|^2 - 2\omega \langle Mg, R^{-1} Mg \rangle + \omega^2 \langle R^{-1} Mg, Mg \rangle - \omega^2 \langle R^{-1} Mg, R^{-1} Mg \rangle \]

\[ \leq \|g\|^2 - \omega (2 - \omega) \langle Mg, R^{-1} Mg \rangle, \]

since \( \langle R^{-1} Mg, R^{-1} Mg \rangle \geq 0 \). The term \( \langle Mg, R^{-1} Mg \rangle \) is positive, if \( Mg \neq 0 \). ▶

Take an arbitrary density \( f^0 \) and construct the sequence \( f^k, k = 1, 2, ... \) by means of the recurrent formula

\[ f^{k+1} = f^k + \omega M^* R^{-1} (\varphi - Mf^k). \]

**Theorem 9.3** If \( M \) is injective and \( \varphi \) fulfils the range conditions, we have \( f^k \to f \), where \( f \) is a solution of (13).

Proof. We have

\[ Q (f^k - f) = f^k - \omega M^* R^{-1} Mf^k - f + \omega M^* R^{-1} Mf \]

\[ = f^k + \omega M^* R^{-1} (\varphi - Mf^k) - f = f^{k+1} - f. \]

It follows that

\[ \|f^{k+1} - f\| < \|f^k - f\| < ... < \|f^0 - f\| \]

and \( f^k \to g \) strongly in \( L_2 (K, \Omega) \). We have \( \|Q (g - f)\| = \|g - f\| \), which yields \( g = f \) by Lemma 9.2 ▶

Another inversion method is developed by Popov and Sushko [12].

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