An abstract proof of the $L^2$-singular dichotomy for orbital measures on Lie algebras and groups

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Abstract. Let $G$ be a compact, connected simple Lie group and $g$ its Lie algebra. It is known that if $\mu$ is any $G$-invariant measure supported on an adjoint orbit in $g$, then for each integer $k$, the $k$-fold convolution product of $\mu$ with itself is either singular or in $L^2$. This was originally proven by computations that depended on the Lie type of $g$, as well as properties of the measure. In this note, we observe that the validity of this dichotomy is a direct consequence of the Duistermaat-Heckman theorem from symplectic geometry and that, in fact, any convolution product of (even distinct) orbital measures is either singular or in $L^{2+\epsilon}$ for some $\epsilon > 0$. An abstract transference result is given to show that the $L^2$-singular dichotomy holds for certain of the $G$-invariant measures supported on conjugacy classes in $G$.

1. Introduction

Let $G$ be a compact, connected simple Lie group and $g$ its Lie algebra. A classical result due to Ragozin \[11\] states that the convolution product of dimension of $g$, non-trivial orbital measures (meaning, the $G$-invariant probability measures supported on the adjoint orbits in $g$), is absolutely continuous with respect to the Lebesgue measure on $g$. Equivalently, the convolution product measure has a Radon-Nikodym derivative in $L^1$. In a series of papers, one of the authors, with various co-authors, significantly improved upon this result, ultimately determining for each orbital measure the minimum integer $k$ such that its $k$-fold convolution product is absolutely continuous. Furthermore, it was shown that a dichotomy holds: either the $k$-fold product is purely singular to Lebesgue measure or its Radon-Nikodym derivative is in $L^1 \cap L^2(g)$. The proof involved detailed case-by-case analysis for each Lie type and orbital measure, see [8] and [10]. This dichotomy was also shown to hold for a $k$-fold product of any invariant probability measure supported on a conjugacy class in the Lie group ([7]), and was subsequently extended to products of distinct orbital measures in the Lie algebra $su(n)$ in [15].

Previously, Ricci and Stein ([12], [13]) had given an abstract argument to show that if the $k$-fold convolution product has Radon-Nikodym derivative in $L^1$, then it also belongs to $L^{1+\epsilon}$ for some $\epsilon > 0$. But there was no suggestion in their proof that


\[ \varepsilon \text{ could be as large as 1. In this note, we see that the fact that a convolution product of (possibly distinct) orbital measures on } g \text{ is absolutely continuous if and only if it is in } L^2(g) \text{ can be deduced abstractly from the Duistermaat-Heckman theorem from symplectic geometry. Moreover, if such a product is absolutely continuous, then it is actually in } L^2(\varepsilon)(g) \text{ for every } \varepsilon < 2\text{rank}_g/(\dim g - \text{rank } g). \]

We also show how the Dooley-Wildeberger wrapping map \[ \mathcal{W} \] can be used to transfer the dichotomy result to the class of invariant measures supported on conjugacy classes in \( G \) that are the image under the exponential map of adjoint orbits of the same dimension. Other than for the Lie groups \( G = SU(n) \) (where all orbital measures on the group have this additional property), it remains open if all products of any (distinct) orbital measures on the Lie groups satisfy the \( L^2 \)-singular dichotomy.

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2. Duistermaat-Heckman Theorem and the Dichotomy for Orbital Measures on the Lie Algebra

Let \( G \) be a compact, connected simple Lie group with maximal torus \( T \) and let \( g \) and \( t \) be the corresponding Lie algebras. Assume \( X \in g \). By the adjoint orbit \( O_X \subseteq g \) we mean the orbit generated by \( X \) under the adjoint action of the associated compact Lie group \( G \),

\[ O_X = \{ \text{Ad}(g)X : g \in G \}. \]

As observed in \[ \text{[3]} \], any adjoint orbit \( O \) is a symplectic manifold, under the identification with the co-adjoint orbit, with the 2-form given by \( \Omega_Z([X,Z],[Y,Z]) = (Z, [X,Y]) \) for \( Z \in O \) and \( X,Y \in g \). Here \([\cdot,\cdot]\) denotes the Lie bracket. This 2-form is invariant under the \( G \)-action and with the moment map \( \phi : O \to g \) given by inclusion, \((O,\Omega,\phi)\) is a Hamiltonian \( G \)-space.

The product of orbits, \( O_X \times O_Y \), is also a Hamiltonian \( G \)-space with 2-form equal to the sum of the 2-forms on \( O_X \) and \( O_Y \) and moment map \( \phi(Z_1,Z_2) = Z_1 + Z_2 \) for \( Z_1 \in O_X \) and \( Z_2 \in O_Y \).

**Definition 1.** By the orbital measure \( \mu_X \) on \( g \) we mean the unique \( G \)-invariant probability measure, supported on \( O_X \), given by

\[ \int_g f dg = \int_G f(\text{Ad}(g)X) dg \]

for all continuous, compactly supported functions \( f \) on \( g \). Here \( dg \) is the Haar measure on \( G \).

When viewed on \( O_X \), \( \mu_X \) is the Liouville measure on \( O_X \) (up to normalization). Viewed on \( g \), the orbital measure is the pushforward measure under the moment map and hence the Duistermaat-Heckman measure. The product measure \( \mu_X \times \mu_Y \) is the Liouville measure on \( O_X \times O_Y \) and the pushforward under the moment map is the convolution of the orbital measures, \( \mu_X \ast \mu_Y \).

The well known Duistermaat-Heckman theorem states the following.

**Theorem 1.** (c.f. \[ \text{[5, 6]} \]) Suppose \( N \) is a compact Hamiltonian \( T \)-space with \( T \) a torus and proper moment map \( \phi \). The pushforward of the Liouville measure
on $N$ is a polynomial on each connected component of the regular values of $\phi$, of degree at most $\dim N/2 - \dim T$.

We will fix a positive Weyl chamber and let $t^0$ denote the interior of this set. Let $\Phi^+$ be the set of positive roots and denote by $\pi$ the $G$-invariant function given by

$$\pi(H) = \prod_{\alpha \in \Phi^+} \alpha(H) \text{ for } H \in t.$$  

The following change of variables formula is well known (in more generality).

**Proposition 1.** (c.f. [4]) Let $M = O_{X_1} \times \cdots \times O_{X_L}$, where $X_i \in \mathfrak{g}$ and $\phi : M \to g$ is the addition map. Denote by $\mu_M$ the Liouville measure on $M$, $\mu_M = \mu_{X_1} \times \cdots \times \mu_{X_L}$. Assume that the Duistermaat-Heckman measure on $\phi(M)$, $\mu_X \star \cdots \star \mu_{X_L}$, is an absolutely continuous measure. Then the manifold $N = \phi^{-1}(t^0)$ is a Hamiltonian $T$-space and if $\mu_N$ is its Liouville measure, there is a constant $c > 0$ such that

$$\int_M f \circ \phi d\mu_M = c \int_N f \circ \phi \ |\pi \circ \phi| \ d\mu_N$$

for all $G$-invariant Borel functions $f$.

**Remark 1.** The assumption that $\mu_{X_1} \star \cdots \star \mu_{X_L}$ is absolutely continuous is equivalent to saying that $O_{X_1} + \cdots + O_{X_L} = \phi(M)$ has positive Lebesgue measure in $\mathfrak{g}$, see [9] or [11].

As $N$ is not compact we cannot apply the Duistermaat-Heckman theorem directly. Rather, we will instead consider the compact symplectic orbifolds $N_\varepsilon$ formed by taking the symplectic cut in each root vector direction, thereby obtaining nested, compact sets whose union over all $\varepsilon > 0$ is $N$. An easy limiting argument shows that

$$\int_N f \circ \phi \ |\pi \circ \phi| \ d\mu_N = \lim_{\varepsilon \to 0} \int_{N_\varepsilon} f \circ \phi \ |\pi \circ \phi| \ d\mu_{N_\varepsilon}.$$

Denote by $\nu_M$ the pushforward of $\mu_M$ under $\phi$ and by $\nu_{N_\varepsilon}$ the pushforward of the Liouville measure $\mu_{N_\varepsilon}$ on $N_\varepsilon$. As the Duistermaat-Heckman theorem holds for orbifolds ([11]), we can apply it to $\nu_{N_\varepsilon}$ to deduce that there are functions, $P_\varepsilon$, which are locally polynomial and of bounded degree, such that

$$\int_{\mathfrak{g}} f(X) d\nu_M(X) = \int_M f \circ \phi d\mu_M = c \lim_{\varepsilon \to 0} \int_{N_\varepsilon} f \circ \phi \ |\pi \circ \phi| \ d\mu_{N_\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \int_{t^0} f(H) |\pi(H)| d\nu_{N_\varepsilon}(H)$$

$$= \lim_{\varepsilon \to 0} \int_{t^0} f(H) |\pi(H)| P_\varepsilon(H) dH.$$

On the other hand, if $R$ denotes the Radon-Nikodym derivative of $\nu_M$, then the Weyl integration formula gives that for $G$-invariant functions $f$,

$$\int_{\mathfrak{g}} f(X) d\nu_M(X) = \int_{\mathfrak{g}} f(X) R(X) dX = \int_{t^0} f(H) R(H) |\pi(H)|^2 dH.$$

Uniqueness of the Radon-Nikodym derivative implies that up to a normalization constant, $R = \lim_{\varepsilon \to 0} P_\varepsilon/|\pi|$ on $t^0$. We further note that $P = \lim_{\varepsilon \to 0} P_\varepsilon$ is compactly supported on $t$ as $R$ is supported on the compact set $\phi(M)$. Properties of the
moment map ensure that there are only finitely many connected components of the set of regular elements of $\phi$, hence $P$ is bounded.

Using the Weyl integration formula again, we can compute the $L^2$ norm of $R$ obtaining

$$\int_\mathfrak{g} |R|^2 \, dX = \int_\mathfrak{t} |P(H)|^2 \, dH < \infty.$$ 

Thus the assumption that $\nu_M$ is absolutely continuous guarantees that its Radon-Nikodym derivative is in $L^1 \cap L^2(g)$. Since a product of orbital measures is known to be either purely singular or absolutely continuous (c.f. [9] or [11]) this gives an abstract proof of the $L^2$-singular dichotomy:

**Corollary 1.** Let $X_i \in \mathfrak{g}$ for $i = 1, \ldots, L$. Then $\mu_{X_1} \ast \cdots \ast \mu_{X_L}$ is either singular or its Radon-Nikodym derivative is in $L^1 \cap L^2(g)$.

### 3. The Dichotomy for Orbital Measures on the Group

It is natural to ask if there is a similar dichotomy result for orbital measures on the compact Lie group, where by an orbital measure in this setting we mean the invariant probability measure $\mu_x$, supported on the conjugacy class $C_x$ generated by $x$. This measure integrates according to the rule

$$\int_G f \, d\mu_x = \int_G f(Ad(g)x) \, dg \text{ for } f \in C(G).$$

In [7] it was shown that the $L^2$-singular dichotomy does hold for convolutions of a given orbital measure. This was proven by using representation theory and the Peter-Weyl theorem to calculate the $L^2(G)$ norm for such measures. (In fact, the strategy of the earlier work on the $L^2$-singular dichotomy was to do these calculations first for orbital measures on the group and then use the answers to study the analogous problem on the Lie algebra.)

Applying this technique to convolutions of different orbital measures is cumbersome, however, and has been done only for $G = SU(n)$ in [15]. Instead, in this note we will take the opposite approach and will see how to use the Lie algebra dichotomy to deduce the dichotomy for certain convolution products of (possibly distinct) orbital measures on the group.

This transference argument will rely upon the wrapping map $\Psi$ introduced in [2]. Given a measure $\mu$ compactly supported on $\mathfrak{g}$, we define a measure $\Psi(\mu)$ on $G$ by

$$\int_G f \, d\Psi(\mu) = \int_\mathfrak{g} j f \circ \exp \, d\mu \text{ for } f \in C(G)$$

where $j$ is the analytic square root of the determinant of the exponential map satisfying $j(0) = 1$;

$$j(H) = \prod_{\alpha \in \Phi^+} \frac{\sin(\alpha(H)/2)}{\alpha(H)/2} \text{ for } H \in \mathfrak{t}.$$ 

It was shown in [2] that $\Psi(\mu * \nu) = \Psi(\mu) * \Psi(\nu)$ and it is easy to see from the definition that if $\mu \in L^1(g)$ with Radon-Nikodym derivative $F$, then $\Psi(\mu) \in L^1(G)$ and has Radon Nikodym derivative $\Psi(F)$. In the lemma below we list some further properties of $\Psi$.

**Notation:** Let $\Gamma = \{\exp^{-1}(e)\} \cap \mathfrak{t}$ ($e$ being the identity in $G$).
Lemma 1. (1) If \( f \in C_\infty(g) \) is \( G \)-invariant, then \( \Psi(jf) \) is \( G \)-invariant, and
\[
\Psi(jf)(\exp H) = \sum_{\alpha \in \Gamma} f(H + \gamma) \text{ for all } H \in t.
\]

(2) For any \( H \in t \), \( \Psi(\mu_H) = j(H)\mu_{\exp H} \). More generally, if \( X_i \in t \) and \( x_i = \exp X_i \), then \( \Psi(\mu_{X_1} \cdots \mu_{X_L}) = \prod_{i=1}^L j(X_i)\mu_{x_1} \cdots \mu_{x_L} \).

Proof. (1) is shown in [2]. To prove (2) we simply note that as \( j \) is \( G \)-invariant, the definition of the orbital measure implies that for any \( G \)-invariant, continuous function \( f \) on \( G \), we have
\[
\int_G f d\Psi(\mu_H) = \int_g j(X) f \circ \exp(X) d\mu_H = \int_G j(H) f \circ \exp(Ad(g)H) dg
\]
\[
= j(H) \int_G f(Ad(g)\exp H) dg = j(H) \int_G f d\mu_{\exp H}.
\]

We will deduce the dichotomy property from the following result that may be of independent interest.

Proposition 2. Let \( K \) be a compact subset of \( g \) and \( 1 \leq p < \infty \). There is a constant \( C = C(K, p) \) such that
\[
\|\Psi(jf)\|_{L^p(G)} \leq C \left\| j^{2/p} f \right\|_{L^p(g)}
\]
for all \( G \)-invariant, Borel functions supported on \( K \).

Proof. As in [2], we will normalize the Haar measures \( dg \) on \( G \) and \( dt \) on \( T \) to have norm one and normalize the Lebesgue measures \( dX \) on \( g \) and \( dH \) on \( t \) so that if \( U \) is a neighbourhood of \( 0 \) in \( g \) (respectively \( t \) ) on which the exponential map is injective, then for continuous \( f \) on \( G \) (or \( T \) ) we have
\[
\int_U f \circ \exp X |j(X)|^2 dX = \int_{\exp U} f(g) dg
\]
(respectively, \( \int_U f \circ \exp H dH = \int_{\exp U} f(t) dt \).

Let \( |\Delta(\exp H)|^2 = |j\pi(H)|^2 \). The Weyl integration formula gives
\[
\|\Psi(jf)\|_{L^p(G)}^p = \frac{1}{|W|} \int_T |\Delta(t)|^2 |\Psi(jf(t))|^p dt.
\]

Let \( t_\Gamma \) be a fundamental domain for \( \Gamma \) in \( t \). Our choice of normalization and Lemma [1] shows that
\[
\int_T |\Delta|^2 |\Psi(jf)|^p dt = \int_{t_\Gamma} |\Delta(\exp H)|^2 |\Psi(jf) \circ \exp H|^p dH
\]
\[
= \int_{t_\Gamma} |\Delta(\exp H)|^2 \sum_{\gamma \in \Gamma} f(H + \gamma) |\Psi(jf) \circ \exp H|^{p-1} dH
\]
\[
\leq \sum_{\gamma} \int_{t_\Gamma} |\Delta(\exp H)|^2 |f(H + \gamma)| |\Psi(jf) \circ \exp H|^{p-1} dH.
\]
Since \( \exp(H + \gamma) = \exp H \) for \( \gamma \in \Gamma \) and \( f \) is supported on the compact set \( K \), this sum is equal to
\[
\int_{\cup\mathcal{K}} |\Delta(\exp H)|^2 |f(H)| |\Psi(j f) \circ \exp H|^{p - 1} dH.
\]
Applying Holder’s inequality with conjugate indices \( p, p' \), it follows that this integral is bounded by
\[
\left( \int_{\cup\mathcal{K}} |\Delta(\exp H)|^2 |f(H)|^p dH \right)^{1/p} \left( \int_{\cup\mathcal{K}} |\Delta(\exp H)|^2 |\Psi(j f) \circ \exp H|^p dH \right)^{1/p'} := I_1 \cdot I_2.
\]
Another application of the Weyl integration formula gives that
\[
I_1 = \left( \int_{\mathfrak{g}} |j(X)|^2 |f(X)|^p dX \right)^{1/p} = \|j^{2/p} f\|_{L^p(\mathfrak{g})}
\]
while Lemma 1(1) gives
\[
I_2 = \left( \int_{\cup\mathcal{K}} |\Delta(\exp H)|^2 |\sum_{\gamma \in \Gamma} f(H + \gamma)|^p dH \right)^{1/p'}.
\]
Since \( \Gamma \) is a discrete set, there are only finitely many \( \gamma \in \Gamma \) such that \( H + \gamma \in K \) for some \( H \in K \). Hence for a constant \( C \) (which may change from one occurrence to another)
\[
I_2 \leq \left( C \int |j(X)|^2 |f(X)|^p dX \right)^{1/p'} \leq \left( C \int |j(X)|^2 |f(X)|^p dH \right)^{1/p'} = C \|j^{2/p} f\|_{L^p(\mathfrak{g})}^{p/p'}.
\]
Combining the bounds on \( I_1, I_2 \) gives the desired result. \( \square \)

**Corollary 2.** Assume that \( x_i = \exp X_i \) and \( \dim C_{x_i} = \dim O_{X_i} \) for \( i = 1, \ldots, L \). Then \( \mu_{x_1} \cdots \mu_{x_L} \) is either singular with respect to Haar measure on \( G \) or its Radon-Nikodym derivative is in \( L^2(G) \).

**Proof.** It was seen in [9] that under the assumption that \( x_i = \exp X_i \) and \( \dim C_{x_i} = \dim O_{X_i} \), \( \mu_{x_1} \cdots \mu_{x_L} \) is absolutely continuous (on \( G \)) if and only if \( \mu_{x_1} \cdots \mu_{x_L} \) is absolutely continuous (on \( \mathfrak{g} \)). In the previous section, under the latter assumption we saw that the Radon-Nikodym derivative of \( \mu_{x_1} \cdots \mu_{x_L} \) is a \( G \)-invariant function which on \( \mathfrak{t}^0 \) has the form \( P/|\pi| \), where \( P \) is a compactly supported, bounded function.

The hypothesis that \( \dim O_{X_i} = \dim C_{x_i} \) guarantees that \( \sin \alpha(X_i) = 0 \) only if \( \alpha(X_i) = 0 \) for \( \alpha \in \Phi \). Hence \( j(X_i) \neq 0 \) for all \( i \). Lemma 1(2) implies that \( \mu_{x_1} \cdots \mu_{x_L} \) has Radon-Nikodym derivative \( C \Psi(P/|\pi|) \) for \( C = \prod_i j(X_i)^{-1} \). Hence it suffices to show that \( \Psi(P/|\pi|) = \Psi(j P/\Delta \circ \exp) \in L^2(G) \). By the previous theorem
\[
\left\| \Psi \left( \frac{j P}{\Delta \circ \exp} \right) \right\|_{L^2(G)}^2 \leq C \left\| \frac{j P}{\Delta \circ \exp} \right\|_{L^2(\mathfrak{g})}^2 \leq C \int_{\mathfrak{g}} \left\| \frac{P}{\pi} \right\|_{L^2(\mathfrak{g})}^2 \leq C \int_{\mathfrak{t}} |P|^2,
\]
and the latter integral is finite as \( P \) is compactly supported and bounded. \( \square \)
4. The $L^{2+\varepsilon}$ argument

In fact, we can prove that if a convolution of orbital measures on $g$ is absolutely continuous, then its Radon-Nikodym derivative is actually in $L^p(g)$ for some $p > 2$.

**Proposition 3.** If $\mu = \mu_X \ast \cdots \ast \mu_X$ is absolutely continuous with respect to Lebesgue measure on $g$, then its Radon Nikodym derivative belongs to $L^{2+\varepsilon}(g)$ for every $\varepsilon < 2\text{rank}_g/(\dim g - \text{rank}_g)$.

**Proof.** This will follow from a result of Stanton and Tomas [14] that states $\int_T |\Delta|^{-\varepsilon}$ is finite for such $\varepsilon$.

From Section 2 we know that the Radon-Nikodym derivative of $\mu$ is equal to $P/|\pi|$ on $t^0$, for $P$ bounded and compactly supported. Let $t_T$ be a fundamental domain for $\Gamma$ in $t$ that is precompact and $K$ be the compact support of $P$. Since $|\pi| \geq |\Delta \circ \exp|$ we have

$$\int_g \frac{|P/\pi|^{2+\varepsilon}}{\pi} = \int_{t^0} \frac{|P|^{2+\varepsilon}}{|\pi|^{-\varepsilon}} \leq \int_t \frac{|P(H)|^{2+\varepsilon}}{|\Delta(\exp H)|} dH = \sum_{\gamma \in \Gamma} \int_{t_T} \frac{|P(H + \gamma)|^{2+\varepsilon}}{|\Delta(\exp H)|} dH$$

As $\Gamma$ is discrete there can only be finitely many $\gamma \in \Gamma$ such that $K \cap (t_T - \gamma)$ is not empty. Thus similar reasoning to the proof of Prop. 2 coupled with the Stanton-Tomas result, shows that the sum above is finite and hence $\mu \in L^{2+\varepsilon}(g)$. \qed

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