THE LIE TRANSFORM METHOD FOR PERTURBATIONS OF CONTRAVARIANT ANTISYMMETRIC TENSOR FIELDS AND ITS APPLICATIONS TO HAMILTONIAN DYNAMICS

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Abstract. By means of the Schouten calculus for contravariant antisymmetric tensor fields, we apply the Lie transform method to investigate smooth deformations of tensor fields and, in particular, to perturbations of Hamiltonian systems generated by deformations of the Poisson bracket. Using results by Karasev and Vorobiev on the computation of Poisson cohomology we describe infinitesimal generators for the Lie transformations. We give applications to perturbed Euler equations on six dimensional Lie coalgebras and to Hamiltonian systems on Poisson manifolds equipped with Dirac brackets.

1. Introduction

The method of Lie transforms is a powerful and general procedure to deal with perturbed tensor fields of arbitrary type depending on a small parameter. This method initially developed for vector fields by Deprit [3], Kamel [7], Hernard [6], and others, provides recursive routines to transform the perturbation of the original system into another one which has convenient y more tractable properties.

The basic idea is to use a parametrized family of near-identity transformations defined by the evolution operator of a vector field depending on the parameter. To obtain a transformed tensor field with some desirable properties, the terms in the formal Taylor expansion on the parameter of the vector field must satisfy a set of recursive homological equations whose solvability depends on the properties we want for the transformed tensor field. Of course, in this case, the convergence of the entire corresponding expansion is not guaranteed.

Here, we give an intrinsic presentation of the Lie transform method for perturbations of contravariant antisymmetric tensor fields by using the Schouten bracket calculus. This method is applied to more general perturbations than the standard Hamiltonian ones for a fixed symplectic structure. The possibility of the applicability of the Lie transform method is

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related with the solvability of homological equations given in terms of the Schouten bracket. In particular, in the case of Poisson tensors of constant rank, an efficient method for solving homological equations was suggested in [9]. Here, we apply the Lie transform method to perturbations of Hamiltonian systems generated by deformations of the Poisson bracket, looking for families of near identity diffeomorphisms to transform the perturbation of the Poisson bracket into a perturbation of the Hamiltonian function for the unperturbed Hamiltonian system. When such a transformation exists, on a second step we can apply some of the available methods in the Hamiltonian perturbation theory (the averaging method, the KAM theory, etcetera).

Finally we give two applications related with perturbed Euler equations on six dimensional Lie coalgebras with perturbations of Hamiltonian systems on Poisson manifolds equipped with Dirac brackets.

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2. THE SCHOUTEN BRACKET AND THE LIE TRANSFORM

Let $M$ a smooth manifold of dimension $m$. Denote by $V^p(M)$ the space of contravariant antisymmetric $p$-tensor fields on $M$. In particular, $V^0(M)$ and $V^1(M)$ correspond to the space of smooth functions and smooth vector fields on $M$, respectively. On the Grassman algebra $V(M) = \bigoplus_{i=0}^{m} V^i(M)$ with the usual wedge product $\wedge$, the Schouten bracket $[,]$ is defined as the unique local type operator extending the Lie derivative and having the following properties for all $f \in V^0(M); X, Y \in V^1(M); A \in V^p(M), B \in V^q(M), C \in V^r(M)$ [13]:

$$[[ , ]]: V^p(M) \times V^q(M) \to V^{p+q-1}(M)$$

$$[X, f] = L_X f, \quad (2.2)$$

$$[X, Y] = [X, Y], \quad (2.3)$$

$$[A, B] = (-1)^{pq}[B, A], \quad (2.4)$$

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{pq+q}B \wedge [A, C] \quad (2.5)$$

Moreover, the Schouten bracket satisfies the graded Jacobi identity:

$$(-1)^{pr}[[A, B], C] + (-1)^{pq}[[B, C], A] + (-1)^{rq}[[C, A], B] = 0. \quad (2.6)$$

More details about the Schouten bracket calculus, can be found in [10, 13]. Given a diffeomorphism $\gamma : M \to M$ and $A \in V^p(M), p \geq 1$, the pull-back $\gamma^*(A) \in V^p(M)$ of $A$ under $\gamma$, is defined by the usual formula,

$$(\gamma^* A)(dg_1, \ldots, dg_p)(x) = A(d(g_1 \circ \gamma^{-1}), \ldots, d(g_p \circ \gamma^{-1}))(\gamma(x)), \quad (2.7)$$
for all \( g_i \in V^0(M) \), \( i = 1, 2, \ldots, p \). The pull-back \( \gamma^* \) has the following properties

\[
\gamma^*(A \wedge B) = \gamma^*A \wedge \gamma^*B, \tag{2.8}
\]
\[
\gamma^*[A, B] = [[\gamma^*A, \gamma^*B]]. \tag{2.9}
\]

for all \( A \in V^p(M) \) and \( B \in V^q(M) \). Pick a \( A_0 \in V^p(M) \) and consider a deformation given by a smooth \( \varepsilon \)-family \( A_\varepsilon \in V^p(M) \) with \( A_0 = A \mid_{\varepsilon=0} \). Consider an \( \varepsilon \)-family of diffeomorphisms \( \gamma_\varepsilon \), \( x \mapsto y = \gamma_\varepsilon(x) \) defined by the solution of the Cauchy problem:

\[
\frac{d}{d\varepsilon} \gamma_\varepsilon = X_\varepsilon(\gamma_\varepsilon), \tag{2.10}
\]
\[
\gamma_0(x) = x, \quad x \in M.
\]

Here \( X_\varepsilon \) is a smooth \( \varepsilon \)-depending vector field. It follows from the “variation of parameters formula”

\[
\frac{d}{d\varepsilon} \gamma_\varepsilon^* A_\varepsilon = \gamma_\varepsilon^* [X_\varepsilon + \frac{\partial}{\partial \varepsilon} A_\varepsilon], \tag{2.11}
\]

that if the vector field \( X_\varepsilon \) satisfies the homological equation

\[
[X_\varepsilon, A_\varepsilon] = -\frac{\partial}{\partial \varepsilon} A_\varepsilon, \tag{2.12}
\]

then \( \gamma_\varepsilon^* A_\varepsilon = A_0 \). Such deformation \( A_\varepsilon \) of \( A_0 \) is called trivial.

If the homological equation \eqref{2.12} is solvable, then any \( \varepsilon \)-family of tensor fields of the form

\[
B_\varepsilon = [A_\varepsilon, C], \tag{2.13}
\]

is transformed by the near identity diffeomorphisms \( \gamma_\varepsilon \) into the \( \varepsilon \)-family

\[
\gamma_\varepsilon^* B_\varepsilon = [A, \gamma_\varepsilon^* C], \tag{2.14}
\]

where the deformation comes now from smooth changes of the \( q \)-tensor \( C \).

The application of the above observation to deformations of tensor fields of the form given before meet some difficulties. For example, the solvability of the homological equation \eqref{2.12} implies that all elements of the \( \varepsilon \)-parametrized smooth curve \( A_\varepsilon \in V^p(M) \) share with \( A_0 \) the same topological or geometric properties. In practice, this conditions are not satisfied globally and hence, the solution of the homological equation \eqref{2.12} does not exists or it exists only on some restricted domains. To deal with those problems, in the frame of the averaging theory we use the construction of approximate solutions for the homological equation with an error up to some given power of the small parameter \( \varepsilon \).
In fact, if the Taylor expansion of $A_\varepsilon$ and $X_\varepsilon$ is given by
\begin{equation}
A_\varepsilon = \sum_{i \geq 0} \frac{\varepsilon^i}{i!} A_i, \tag{2.15}
\end{equation}
and
\begin{equation}
X_\varepsilon = \sum_{i \geq 0} \frac{\varepsilon^i}{i!} X_i, \tag{2.16}
\end{equation}
and we substitute these expressions in the homological equation (2.12), we obtain the recursive set of homological equations for vector fields $X_0, X_1, \ldots, X_k, \ldots$:
\begin{equation}
[X_0, A_0] = -A_1 \tag{2.17}
\end{equation}
\begin{equation}
[X_1, A_0] = -[X_0, A_1] - A_2 \tag{2.18}
\end{equation}
\begin{equation}
[X_2, A_0] = -2[X_1, A_1] - [X_0, A_2] - A_3 \tag{2.19}
\end{equation}
\begin{equation}
\vdots
\end{equation}
\begin{equation}
[X_k, A_0] = - \sum_{i=1}^{k} \binom{k}{i} [X_{k-i}, A_i] - A_{k+1} \tag{2.19}
\end{equation}
\begin{equation}
\vdots
\end{equation}
Note that these equations can be written in the recursive form $[X_k, A_0] = F(A_1, \ldots, A_{k+1})$. From the discussion above we derive the following result.

**Proposition 1.** Let $A_\varepsilon \in V^p(M)$ a smooth deformation of the contravariant antisymmetric $p$-tensor field $A_0$ and $X_0, X_1, \ldots, X_k, \ldots$ vector fields satisfying the set of recursive homological equations (2.17), (2.18) and (2.19) in a domain $D$. Then, for each tensor field $C \in V^q(M)$, the $\varepsilon$-family of vector fields $[A_\varepsilon, C]$ is transformed up to $O(\varepsilon^k)$ into the $\varepsilon$-family of tensor fields $[A_0, C_\varepsilon]$ where
\begin{equation}
C_\varepsilon = C + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} [X_{i-1}, C] + O(\varepsilon^k). \tag{2.20}
\end{equation}

3. **Solvability of the homological equation for perturbed Poisson tensors**

In order to apply the above techniques to Hamiltonian dynamics on Poisson manifolds, we consider, in the sequel, perturbations of the form $A_\varepsilon = [\Psi_\varepsilon, H]$, where $\Psi_\varepsilon \in V^2(M)$ and $H \in V^0(M)$. In this case, the set of recursive equations (2.17), (2.18) and (2.19) take the special form,
\begin{equation}
[X, \Psi] = \Phi, \tag{3.1}
\end{equation}
for tensors fields $\Psi, \Phi \in V^2(M)$. Such a class of homological equations has been studied in [9] for the case when $\Psi$ is a Poisson tensor of constant rank and its symplectic foliation is a fibration. We recall here some basic facts about Poisson manifolds and the results given in [9].
A Poisson manifold is a pair $(M, \Psi)$, where $\Psi \in V^2(M)$ and $[[\Psi, \Psi]] = 0$. The distinguished quantities in a Poisson manifold are: the Casimir functions $k \in V^0(M)$ with $[[\Psi, k]] = 0$; the Hamiltonian vector fields defined as $X_f = [[\Psi, f]]$ for each $f \in V^0(M)$. The Poisson vector fields $X \in V^1(M)$ or infinitesimal automorphisms are defined as $[[\Psi, X]] = 0$. On a Poisson manifold, the operation between each pair of functions $f, g \in V^0(M)$ given by

$$\{f, g\} = [[[[\Psi, f]], g]],$$  \hspace{1cm} (3.2)

is called the Poisson bracket on $M$.

At each point $x \in M$, the dimension of the subspace in $T_x M$ generated by the vectors $[[\Psi, f]](x)$ is even and equals to the rank of the 2-tensor field $\Psi$. If the rank of $\Psi$ is constant on $M$, then the Hamiltonian vector fields define an integrable distribution $\mathcal{D}$ in the sense of Frobenius. Each leaf $\mathcal{L}$ of the regular foliation is a symplectic manifold whose symplectic form is a 2-form $\omega_\mathcal{L}$ given by the formula

$$\omega_\mathcal{L}(X_f, X_g) = \{f, g\}. \hspace{1cm} (3.3)$$

This means that field $\Psi$ defines on each symplectic leaf $\mathcal{L}$, an isomorphism between $\Lambda^1(\mathcal{L})$ and $V^1(\mathcal{L})$. For a more comprehensive study of Poisson manifolds see [8, 10, 13].

We assume that rank $\Psi = 2r$ and the symplectic foliation is a fibration and the space of leaves is an open domain in $\mathbb{R}^{m-2r}$. Then, the symplectic foliation is given by the level sets of $m - 2r$ independent Casimir functions $k_1, \ldots, k_{m-2r}$.

Recall that tensors fields on $M$, tangent to symplectic leaves are called vertical. A smooth field of $k$-forms along the leaves is called a vertical $k$-form on $M$.

We have a correspondance $\Psi^\sharp$ between vertical forms and vertical contravariant antisymmetric tensor fields on $M$ given by the expression

$$(\Psi^\sharp \alpha)(df_1, \ldots, df_r) = \alpha(X_{f_1}, \ldots, X_{f_r}), \quad f_1, \ldots, f_r \in V^0(M). \hspace{1cm} (3.4)$$

where $\alpha$ is a smooth section of $r$-forms of the bundle $\pi$.

A necessary condition for the solvability of (3.1) is

$$[[\Psi, \Phi]] = 0, \hspace{1cm} (3.5)$$

This means that the tensor field $\Phi$ is a 2-cocycle of the Poisson-Lichnerowicz coboundary operator $\delta_\Psi$ on the Grassmann algebra of contravariant antisymmetric tensor fields defined by $\delta_\Psi(A) = [[\Psi, A]]$ for $A \in V^p(M)$, $p = 0, 1, \ldots, \dim M$. Moreover, the solvability of (3.1) is equivalent to the condition that $\Phi$ is a 2-coboundary for the operator $\delta_\Psi$.

**Remark 2.** In terms of the coboundary operator $\delta_\Psi$, equation (3.1) for an infinitesimal generator $X$ of the Lie transformation (2.17)–(2.19) is written as $\delta_\Psi X = \Phi$. On the other hand, in terms of the Lie derivative $L_X$ along $X$, this equation takes the form $L_X \Psi = \Phi$. In the averaging method and
normal forms \[1, 3\], such a type of equations are usually called homological equations.

If a 2-tensor field \( \Phi \) is a 2-coboundary of \( D \), we have

\[
[\Psi, X](dk_i) = \Phi(k_i), \quad i = 1, \ldots, m - 2r, \tag{3.6}
\]
or

\[
\Psi(L_X k_i) = \Phi(k_i), \quad i = 1, \ldots, m - 2r \tag{3.7}
\]
i.e., \( \Phi(k_i), i = 1, \ldots, r \) have to be Hamiltonian vector fields with respect to the Poisson tensor \( \Psi \). This last condition is also sufficient. To see that, suppose \( \Phi(k_i) = \Psi(h_i), i = 1, \ldots, m - 2r \) and consider vector fields \( V_i \) dual to the Casimir functions, i.e., \( L_{V_i} k_j = \delta_{ij}^l \), \( i, j = 1, \ldots, m - 2r \). Then

\[
(\Phi - \sum_{i=1}^r [\Psi, h_i V_i])(k_j) = 0,
\]
and such tensor becomes a vertical 2-tensor in the bundle \( \pi \). Taking into account that \( \Psi^\sharp \) is an isomorphism, we can assure the existence of a smooth section \( \alpha \) of 2-forms on \( \pi \) such that

\[
\Psi^\sharp \alpha = \Phi - \sum_{i=1}^r [\Psi, h_i V_i]. \tag{3.8}
\]

Moreover, by condition (3.5) we have

\[
0 = [\Psi, \Psi^\sharp \alpha] = \Psi^\sharp d\alpha, \tag{3.9}
\]
and \( \alpha \) is a closed vertical 2-forms. If we suppose that each symplectic leaf is simply connected, then \( \alpha = d\beta \) on symplectic leaves, \( \Psi^\sharp \alpha = [\Psi, \Psi^\sharp \beta] \) and the vector field on \( M \)

\[
X = \Psi^\sharp \beta + \sum_{i=1}^r h_i V_i \tag{Poisson vector field}, \tag{3.10}
\]
satisfies the homological equation (3.5).

We now recall the following theorem given in [9].

**Proposition 3.** If the Poisson tensor of a Poisson manifold \((M, \Psi)\) has constant rank \(2r\) and its symplectic foliation is a fibration by simply connected leaves, then the homological equation (3.5) is solvable if and only if the vector fields \( \Phi(k_j) \) are Hamiltonian vector fields with respect to \( \Psi \) for \( j = 1, \ldots, m - 2r \). In this case, all solutions of (3.4) are of the form (3.10).

**Remark 4.** Under the assumptions of Proposition 3 for any element \( \Psi_\varepsilon \) of a smooth \( \varepsilon \)-family of Poisson tensors, \( [\Psi_\varepsilon, \Psi_\varepsilon] = 0 \), and \( k_\varepsilon \) a smooth \( \varepsilon \)-dependent Casimir function, \( [\Psi_\varepsilon, k_\varepsilon] = 0 \), then

\[
[\frac{d}{d\varepsilon} \Psi_\varepsilon, k_\varepsilon] = -[\Psi_\varepsilon, \frac{d}{d\varepsilon} k_\varepsilon],
\]
and the homological equation (2.12) is solvable over an open domain in \( \mathbb{R}^{2r} \).
4. Perturbations of Hamiltonian Systems Generated by Deformations of Poisson Brackets

On the Poisson manifold \((M, \Psi)\), let us consider a Hamiltonian system
\[
\dot{y} = [\Psi, H](y), \quad y \in M,
\]  
(4.1)
and a perturbed system of the form
\[
\dot{y} = [\Psi_{\varepsilon}, H](y), \quad y \in M,
\]  
(4.2)
where \(\Psi_{\varepsilon}\) is a smooth \(\varepsilon\)-family of 2-tensors fields \(\Psi_{\varepsilon}\) with \(\Psi_{0} = \Psi\).

In general, the perturbation vector field
\[
X_{\text{pert}} = [\Psi_{\varepsilon} - \Psi, H](y), \quad y \in M,
\]
associated to (4.2) is not a Hamiltonian vector field with respect to the initial Poisson structure \(\Psi\). In fact, \(X_{\text{pert}}\) can be transversal to the symplectic leaves of \(\Psi\) as we will see in the applications given below.

Finding conditions under which one can transform the perturbed part of the Hamiltonian system (4.2) generated by deformations of the Poisson bracket into another perturbed system where the perturbation comes from smooth changes in the Hamiltonian function \(H\), is a relevant question for the applications. When such transformations exist, we say that the transformed system is in normal form. Therefore, if system (4.2) can be taken into normal form, one can apply standard methods in the perturbation theory for Hamiltonian systems, see for example [12].

As an illustration of the above ideas we present two examples.

4.1. Perturbations of Euler systems on Lie-coalgebras. On the 6-dimensional Euclidean space \(\mathbb{R}^6 = \mathbb{R}^3_y \times \mathbb{R}^3_z\) with coordinates \(y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)\) consider the diagonal matrix
\[
\eta = \text{diag}(\eta_1, \eta_2, \eta_3) = \begin{bmatrix}
\eta_1 & 0 & 0 \\
0 & \eta_2 & 0 \\
0 & 0 & \eta_3
\end{bmatrix},
\]  
(4.3)
and the \(\varepsilon\)-family of Poisson tensors that have relevance in the rigid body motion [2]
\[
\Psi_{\eta,\varepsilon}(df, dg) = (\eta(y) \times \nabla_y f + \eta(z) \times \nabla_z f) \cdot \nabla_y g + (\eta(z) \times \nabla_y f + \varepsilon \eta(z) \times \nabla_z f) \cdot \nabla_z g.
\]  
(4.4)
for some functions \(f, g \in V^0(\mathbb{R}^6)\). The 2-tensor field (4.4) is a linear Poisson tensor and arbitrary \(\eta\) and can be written in the form
\[
\Psi_{\eta,\varepsilon} = \Psi_{\eta,0} + \varepsilon \Phi_{\eta}
\]  
(4.5)
with
\[
\Phi_{\eta}(df, dg) = (\eta(y) \times \nabla_z f) \cdot \nabla_z g.
\]
For the given values of \(\eta\) and \(\varepsilon\), below, there corresponds the Lie-Poisson bracket on the coalgebra \(g^*\) of the following six dimensional Lie algebras:
η ∈ \mathfrak{gl}(n)

| \eta | ε > 0, | \varepsilon(3)^* |
|------|---------|----------------|
| (1, 1, 1) | \varepsilon = 0, | \varepsilon(3)^* |
| (1, 1, -1) | \varepsilon > 0, | \varepsilon(3)^* |
| (1, 1, -1) | \varepsilon = 0, | \varepsilon(3)^* |
| (1, 1, 1) | \varepsilon < 0, | \varepsilon(3)^* |

For the different cases above, the Casimir functions are

\begin{align*}
  k_1^{\eta, \varepsilon}(y, z) &= \eta(y) \cdot z, \\
  k_2^{\eta, \varepsilon}(y, z) &= \frac{1}{2} \varepsilon \eta(y) \cdot y + \frac{1}{2} \eta(z) \cdot z,
\end{align*}

and its regular symplectic leaves for \( \varepsilon > 0 \) are the regular coadjoint orbits on \( \mathfrak{g}^* \) and diffeomorphic to \( \Sigma \times \Sigma \) where

\[ \Sigma = \{ \xi \in \mathbb{R}^3 : \eta(\xi) \cdot \xi = \text{const} \} . \]

In the limit \( \varepsilon \to 0 \), the regular symplectic leaves of \( \Psi_{\eta, 0} \) are diffeomorphic to \( T^* \Sigma \). If \( \Sigma \) is a 2-dimensional sphere or a cylinder, then the symplectic leaves of \( \Psi_{\varepsilon} \) and \( \Psi_0 \) have different topological structure and the homological equation (2.12) has no global solutions.

For each diagonal matrix \( \eta \) and smooth function \( H(y, z) \in V^0(\mathbb{R}^6) \), let us consider the Hamiltonian system on \( (\mathbb{R}^6, \Psi_{\eta, 0}) \)

\begin{align*}
  \dot{y} &= \eta(y) \times \nabla_y H + \eta(z) \times \nabla_z H, \\
  \dot{z} &= \eta(z) \times \nabla_y H, \quad (4.8)
\end{align*}

and the \( \varepsilon \)-perturbed system of the form

\begin{align*}
  \dot{y} &= \eta(y) \times \nabla_y H + \eta(z) \times \nabla_z H + \varepsilon \eta(y) \times \nabla_z H, \\
  \dot{z} &= \eta(z) \times \nabla_y H + \varepsilon \eta(y) \times \nabla_z H, \quad (4.10)
\end{align*}

where \( \varepsilon \in (-a, a) \). The Hamiltonian systems (4.8), (4.9) and (4.10), (4.11) are usually called Euler systems.

The vector fields associated to (4.8), (4.9) and (4.10), (4.11) take the form \([ \Psi_{\eta, 0}, H] \) and \([ \Psi_{\eta, \varepsilon}, H] \), respectively. The homological equation (2.12) has a solution \( X_{\varepsilon} \) on the domain

\[ \mathcal{N} = \{ (y, z) \in \mathbb{R}^6 : (\eta(y) \times \eta(z) \cdot (y \times z) \neq 0 \} , \]

given by

\[ X_{\varepsilon} = \frac{\eta(y) \cdot y}{2 \eta(y) \times \eta(z) \cdot (y \times z)} \eta(y) \times (z \times y) \frac{\partial}{\partial z}, \quad (4.12) \]
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with flow
\[ \gamma_\varepsilon(y, z) = \left( y, \alpha z + (1 - \alpha) \frac{\eta(y) \cdot z}{\eta(y) \cdot y} y \right). \tag{4.13} \]

Here,
\[ \alpha = \left[ 1 + \frac{\varepsilon(\eta(y) \cdot y)^2}{(\eta(y) \times \eta(z) \cdot (y \times z))} \right]^{\frac{1}{2}}, \tag{4.14} \]
and the perturbation in system (4.10), (4.11) is transformed under \( \gamma_\varepsilon \) into a Hamiltonian perturbation relative to the structure \( \Psi_{\eta,0} \), which takes the form
\[ \dot{y} = \eta(y) \times \nabla_y H_\varepsilon + \eta(z) \times \nabla_z H_\varepsilon, \tag{4.15} \]
\[ \dot{z} = \eta(z) \times \nabla_y H_\varepsilon, \tag{4.16} \]

where
\[ H_\varepsilon(y, z) = H \left( y, \alpha z + (1 - \alpha) \frac{\eta(y) \cdot z}{\eta(y) \cdot y} y \right). \tag{4.17} \]

If instead of solving the equation (2.12) and thinking of \( \Psi_{\eta,\varepsilon} \) as an exact homotopy of \( \Psi_{\eta,0} \), we look for an homotopy up to \( O(\varepsilon^k) \), then we can show that the conditions for their solvability only requires that unperturbed tensor \( \Psi_{\eta,0} \) to satisfy the conditions of Proposition 3. In our case, \( \Psi_{\eta,0} \) has maximum rank in the set \( D = \{ (y, z) \in \mathbb{R}^6 : \eta(z) \neq 0 \} \) and the symplectic foliation is a trivial fibration over an open subset of \( \mathbb{R}^2 \), hence each equation in (2.19) has a solution on \( D \). In particular,
\[ X_0 = \frac{1}{2\eta(z) \cdot z} \eta(y) \times (z \times y) \frac{\partial}{\partial z}. \tag{4.18} \]

This example was considered in [5].

4.2. Deformations of Dirac brackets. Let \( M \) be a smooth manifold and \( w_\varepsilon \) a smooth \( \varepsilon \)-family of nondegenerate closed 2-forms (symplectic structures) on \( M \) for \( \varepsilon \in (-a, a) \). Denote by \( \Psi_\varepsilon \) the nondegenerate Poisson structure on \( M \) associated to \( w_\varepsilon \). Suppose we have a set of functionally independent functions \( A^1_\varepsilon, \ldots, A^r_\varepsilon \in V^0(M) \), smoothly depending on \( \varepsilon \). Denote
\[ \Delta(\varepsilon) = ((\Delta_{ij})) \equiv \left( \left[ \left[ \Psi_\varepsilon, A^i_\varepsilon \right], A^j_\varepsilon \right] \right), \]
and assume that condition \( \det(\Delta) \neq 0 \) holds everywhere on \( M \), for all \( \varepsilon \). For every \( \varepsilon \in (-a, a) \), define the Dirac tensor by the standard formula (see, for example [8]),
\[ \Psi^\text{DIR}_\varepsilon = \Psi_\varepsilon + \frac{1}{2} \sum_{(i,j)} \Delta_{ij}(\varepsilon) [\Psi_\varepsilon, A^i_\varepsilon] \wedge [\Psi_\varepsilon, A^j_\varepsilon], \tag{4.19} \]

where \( \Delta^{is} \Delta_{sj} = \delta^i_j \). Let us consider the family of Poisson brackets (4.19) as a deformation of the "unperturbed" Dirac structure \( \Psi^\text{DIR}_0 \). The symplectic
leaves \( S_\varepsilon \) of \([1.19]\) coincide with the level sets of the constraint functions \( A_i^\varepsilon \),
\[
S_\varepsilon = \{ A_1^\varepsilon = \text{const}, \ldots, A_r^\varepsilon = \text{const} \}.
\] (4.20)
In this case, the rank is constant and the forms \( \dot{w}_\varepsilon = \frac{dw_\varepsilon}{d\varepsilon} \) are closed for every value of the parameter \( \varepsilon \).

Now, suppose that there exists a smooth family of 1-forms \( \theta_\varepsilon \) on \( M \) such that
\[
\dot{w}_\varepsilon = d\theta_\varepsilon \mod(\mathcal{F}_2^S(M)), \quad \varepsilon \in (-a,a).
\] (4.21)
Here \( \mathcal{F}_2^S(M) \) is the subspace of the 2-forms on \( M \) vanishing along the leaves \( S_\varepsilon \). So, in particular the 2-forms \( \dot{w}_\varepsilon \) are exact on each symplectic leaf \( S_\varepsilon \).

Notice that if in addition to this condition the symplectic foliation \( (4.20) \) is a trivial fibration by simply connected leaves, then \( (4.21) \) holds. As is known \([4]\), Dirac structures possesses a maximum set of Poisson vector fields which are transversal to the symplectic leaves. In fact, the vector fields
\[
Z_i^\varepsilon = \sum_{j=1}^r \Delta_{ji} [\Psi_\varepsilon, A_j^\varepsilon], \quad i = 1, \ldots, r,
\] (4.22)
constitute a set of independent transversal Poisson vector fields. The Poisson manifolds having such property are called transversally constant \([13]\) or transversally maximal \([4]\). Then, the homological equation \( (2.12) \) is solvable on \( M \) for all \( \varepsilon \in (-a,a) \) and the corresponding time-dependent solution can be represented by
\[
X_\varepsilon = \sum_{(i,j)} \Delta_{ij} \dot{A}_j [\Psi_\varepsilon, A_i^\varepsilon] - \Psi_{\varepsilon}^{\text{DIR}} \theta_\varepsilon,
\] (4.23)
where \( \dot{A}_j^\varepsilon = \frac{dA_j^\varepsilon}{d\varepsilon} \).

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