Universality and stationarity of the I-Love relation for self-bound stars

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The emergence of the I-Love-Q relations, revealing that the moment of inertia, the tidal Love number (deformability) and the spin-induced quadrupole moment of compact stars are, to high accuracy, interconnected in a universal way disregarding the wide variety of equations of state (EOSs) of dense matter, has attracted much interest recently. However, the physical origin of these relations is still a debatable issue. In the present paper, we focus on the I-Love relation for self-bound stars (SBSs) such as incompressible stars and quark stars. We formulate perturbative expansions for the moment of inertia, the tidal Love number (deformability) and the I-Love relation of SBSs. By comparing the respective I-Love relations of incompressible stars and a specific kind of SBSs, we show analytically that the I-Love relation is, to relevant leading orders in stellar compactness, stationary with respect to changes in the EOS about the incompressible limit. Hence, the universality of the I-Love relation is indeed attributable to the proximity of compact stars to incompressible stars, and the stationarity of the relation as unveiled here. We also discover that the moment of inertia and the tidal deformability of a SBS with finite compressibility are, to leading order in compactness, equal to their counterparts of an incompressible star with an adjusted compactness, thus leading to a novel explanation for the I-Love universal relation.

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I. INTRODUCTION

The study of the static and dynamical properties of compact stars, including both neutron stars (NSs) and quark stars (QSs), is often plagued by the uncertainties in the equation of state (EOS) of nuclear (or quark) matter (see, e.g., [1–3]), whose stable state is currently unachievable in terrestrial laboratories. On the other hand, such uncertainties also cast doubt on the feasibility of testing the validity of Einstein’s theory of general relativity with the structural characteristics of compact stars because there is no a priori way to distinguish between the effects due to the modifications in EOS and the theory of gravity. Yet, the recently discovered “I-Love-Q universal relations” [4, 5] reveal that the moment of inertia $I$, the quadrupole tidal Love number $k_2$ (or tidal deformability $\lambda$ [6, 7]) and the spin-induced quadrupole moment $Q$ of compact stars (including both NSs and QSs), after suitably scaled by the stellar mass $M$, are inter-related in an almost EOS-independent way. The discovery of the I-Love-Q relations is surprising and encouraging as well. In the light of these universal relations, even still in the absence of the exact knowledge of nuclear (or quark) matter, the possibility of identifying alternate theories of gravity (e.g., the dynamical Chern-Simons gravity [4, 5], the Eddington-inspired Born-Infeld gravity [8], the scalar-tensor theories [9–12] and the $f(R)$ gravity theories [13]) has been examined. In addition, these relations were shown to be robust and prevailing in other stellar systems, such as close binary compact stars [14] and rapidly rotating compact stars (including both unmagnetized and magnetized NSs) [15–17], and their generalizations to higher-order multipole moments induced by either tidal forces or rotation have been proposed [18, 19]. Through such relations, the measurement of either member of the I-Love-Q trio will suffice to determine the other two once the mass of a compact star is known [4, 5]. As a result, more accurate astronomical data can be extracted in the analysis of gravitational wave signals emitted at the late stage of NS-NS binary mergers [4, 5, 20, 21].

To certain extent, the I-Love-Q relations have emerged as a surprise to researchers working on compact stars, who are accustomed to the fact that the physical characteristics of NSs (or QSs), including all members of the I-Love-Q trio, usually bear obvious dependence on the EOS of nuclear (or quark) matter (see, e.g., [1, 2, 3, 22]). Yet the I-Love-Q trio are found to be related by universal formulas which hold for both NSs and QSs, and are accurate up to a few percent [4, 5]. How can they conspire to follow such EOS-insensitive formulas? In the seminal papers discovering the I-Love-Q relations, Yagi and Yunes [4, 5] have put forward two suggestions for the solution of this puzzle: (i)
the low-density region of a NS, which lies in a layer between 70% and 90% of the stellar radius $R$, contributes the most to the quantities $I$, $\lambda$ and $Q$ and the EOS there is quite unified; and (ii) the I-Love-Q relations are indeed the remnants of the no-hair theorem of black-holes. However, the subsequent finding that QSs also follow the I-Love-Q relations \cite{26, 23, 24} seems to invalidate suggestion (i) because the behavior of quark matter is completely different from that of nuclear matter in the low density regime. Furthermore, each of the I-Love-Q trio is in fact dominated by a thicker layer bounded between 0.5$R$ and 0.95$R$ \cite{22}, which comprises both high and low-density matter. Hence, the universality demonstrated in the I-Love-Q relations is unlikely to be rooted in the low-density region of compact stars. On the other hand, the I-Love-Q relations also hold nicely in the Newtonian limit \cite{4}. Thus, the no hair theorem for black-holes seems to be irrelevant to these relations.

More recently, Sham et al. \cite{23} revealed that the I-Love-Q relations of realistic compact stars, whose effective polytropic index is less than unity in the high density regime, indeed follow closely those of incompressible stars (ISs). In particular, the accuracies of these relations are found to deteriorate significantly for polytropic stars with polytropic indices greater than 1. Moreover, Chan et al. \cite{26} analytically derived the I-Love relation for relativistic ISs and showed that the formula also applies to realistic NSs and QSs with high accuracies. Therefore, they proposed that the I-Love-Q universality is the consequence of (i) the high stiffness of dense nuclear matter, and (ii) the stationarity of the I-Love-Q relations about the incompressible limit \cite{23, 24, 26}.

The main objective of the present paper is to provide a firm theoretical support to the above-mentioned observation that the I-Love-Q relations is, to a specific order of stellar compactness $C$, stationary with respect to variations in the compressibility of nuclear (or quark) matter forming a compact star about the incompressible limit (i.e., vanishing compressibility). While the said issue has been verified qualitatively by numerical data shown in \cite{23, 26}, its theoretical justification is not yet fully known. Sham et al. \cite{23} has proposed a generalized Tolman model (GTM), whose density profile $\rho(r)$ is given by

$$\rho(r) = \rho_0 (1 - \delta x^2),$$  \hspace{1cm} (1.1)

with $\rho_0$ being the central density, $x \equiv r/R$, $0 \leq \delta \leq 1$. The parameter $\delta$ vanishes for ISs and, in general, measures the compressibility of a GTM star \cite{24}. For example, near the stellar center, GTM can mimic the density distribution of a polytropic star with polytropic index $N \simeq \delta$. For such GTM stars, Sham et al. \cite{23} show that the I-Love-Q relations are stationary with respect to changes in $\delta$ about the point $\delta = 0$ (i.e., ISs) in the Newtonian limit. However, whether similar behaviors still prevail in relativistic stars constructed with realistic EOSs is not yet clearly examined theoretically.

In the present paper, we intend to develop a fully relativistic approach to the study of such universality. Our focus is the stationarity of the I-Love relation for self-bound stars (SBSs), including both ISs and QSs \cite{24, 27} (see \cite{23} for the EOS of SBSs), for their proximity to ISs. As demonstrated numerically by Chan et al. \cite{26}, the I-Love formulas for ISs and QSs are almost the same. Here we will carry out an in-depth analytic study on such similarity between these two classes of SBSs. Our target is to pinpoint the physical mechanism underlying the universality of the I-Love relation.

First of all, in order to evaluate the moment of inertia and the tidal Love number (or deformability), we need to find the hydrostatic equilibrium configuration of relativistic stars, which is governed by the Tolman-Oppenheimer-Volkov (TOV) equations \cite{30, 31}. To this end, we develop a recursive post-Minkowsian scheme to solve for the density, the pressure and the metric coefficients of SBSs. Each of these physical quantities is shown to be expressible in terms of a power series in stellar compactness, which is then used as the input to evaluate the moment of inertia and the tidal Love number (or deformability) of SBSs.

With the abovementioned expansion scheme, we succeed in finding the post-Minkowsian expansion for the moment of inertia and the Love number of QSs obeying the simple MIT bag model (see, e.g., \cite{21, 24} and \cite{23}). Both of these two quantities are expressed in terms of power series in compactness. We combine these two series to express the scaled moment of inertia as a power series in $\lambda^{-1/5}$, where $\lambda$ is the dimensionless tidal deformability \cite{4, 7, 22} and $\lambda^{-1/5}$ is, to leading order, proportional to the stellar compactness $C$. Thus, the I-Love relation of QSs is found. Comparing the I-Love relation for QSs with that of ISs, which has been obtained recently by Chan et al. \cite{26}, we find that they are identical up to first-order in $\lambda^{-1/5}$. Therefore, in spite of the difference in their EOSs, QSs and ISs actually obey the same I-Love relation up to first-order in compactness. As a result, the I-Love relations for QSs and ISs exactly coincide in the Newtonian limit.

Moreover, we consider in general a class of SBSs characterized by EOS $\rho = c_0 + c_1 p$, which is a linear function of pressure $p$ and reduces to the MIT bag model if the two positive parameters $c_0$ and $c_1$ are suitably chosen. Again we express the I-Love relation for such (linear) SBSs as Taylor series in $\lambda^{-1/5}$. We show that the I-Love relation is completely unaffected by the value of $c_0$. More importantly, to first-order in compactness such relation is also independent of $c_1$, which is indeed the inverse of the square of the sound speed and a measure of the compressibility. As the compactness of typical compact stars is usually less than 0.3, the influence of $c_1$ on the I-Love relation is expected to be small. This point is indeed verified numerically in the QS case. The analysis developed here consequently
provides a strong theoretical justification to the stationarity of the I-Love relation about the incompressible limit as discovered in [23, 26].

Finally, we generalize our study to SBSs with EOSs given by regular power series in pressure $p$ (see [24]) and show that, to respective leading orders in stellar compactness, the I-Love relation is still stationary with respect to variations in all the expansion coefficients of the EOS about the incompressible limit. Thus, the cause for the universality of the I-Love relation of realistic stars, which are characterized by sufficiently stiff EOS, is is fully exposed. In addition, we also provide a physically transparent explanation for such stationarity, which is attributable to (i) the similarity between the responses of the moment of inertia and the tidal Love number to changes in EOS, and (ii) the proper scaling in the definition of the two variables (i.e., the scaled moment of inertia $I = I/M^3$ and the dimensionless tidal deformability $\lambda$) considered in the I-Love relation. We show that to leading order of compactness $C$, the changes in $I$ and $\lambda$ induced by a non-zero compressibility of stellar EOS are reproducible by a corresponding shift in the compactness of ISs. As a result, in spite of the fact that both $I - C$ and $\lambda - C$ relations display obvious EOS-dependency, the $I - \lambda$ relation obtained from eliminating $C$ in these two relations is approximately independent of EOSs.

The paper is organized as follows. In Section II a recursive perturbation scheme is established to solve the TOV equations [30, 31] for SBSs. Various physical quantities describing the equilibrium configuration of SBSs (such as density, pressure and metric coefficients) are expanded in terms of power series in compactness. With these expansions, in Sections III and IV we formulate the perturbative expansions for the moment of inertia and the Love number for SBSs, respectively. In Section V we find an analytic formula relating the moment of inertia and the Love number for SBSs described by EOS linear in pressure. In particular, we compare the I-Love relations of ISs and QSs (or other SBSs characterized by linear EOS) and show that the relations in these two cases are identical to first-order in stellar compactness. In Section VI we consider the I-Love relation of SBSs with EOSs depending on the pressure nonlinearly and show that the relation is, to relevant orders of stellar compactness, still stationary to changes in all the parameters specifying such nonlinear EOSs about the incompressible limit. As a result, the robustness of the stationarity property of the I-Love relation is established in general. In Section VII we further discuss and elaborate the physical origin of such stationarity by comparing the respective changes in the scaled moment of inertia $I$ and the dimensionless tidal deformability $\lambda$ due to a specific change in EOS. We conclude our paper in Section VIII with some discussions. Besides, for readers who are interested in the accuracies of the post-Minkowsian expansions developed here, we summarize the relevant numerical results for the QS case in Appendix A of our paper. Unless otherwise stated explicitly, geometric units in which $G = c = 1$ are adopted.

II. HYDROSTATIC EQUILIBRIUM OF SBS

A. TOV equations

The hydrostatic equilibrium of a relativistic, non-rotating compact star made of a perfect fluid is governed by the TOV equations [30, 31]:

$$\frac{dp}{dr} = -\frac{(m + 4\pi r^3 \rho)(\rho + p)}{r^2(1 - 2m/r)} ,$$

$$\frac{dv}{dr} = -\frac{2}{\rho + p} \frac{dp}{dr} ,$$

where $\rho(r)$, $p(r)$ and $\nu(r)$ are the energy density, pressure and the metric coefficient at a circumferential radius $r$, respectively, and $m(r) = 4\pi \int_0^r \rho(r')r'^2 dr'$ is the gravitational mass enclosed within radius $r$ [30, 31]. The pressure vanishes at the stellar surface where $r = R$. Outside the star, the spacetime metric is given by the Schwarzschild metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) ,$$

where $t$, $r$, $\theta$, $\phi$ are the standard Schwarzschild coordinates, and $e^{\nu} = e^{-\lambda} = 1 - 2M/r$ (see, e.g., [32, 33]). Inside the star, while $e^{\lambda(r)}$ is directly given by $e^{\lambda} = 1/[1 - 2m(r)/r]$, $e^{\nu(r)}$ has to be obtained by solving the TOV equations listed above subject to the boundary condition $e^{\nu(R)} = 1 - 2M/R$. 
where \(c_n\) (\(n = 0, 1, 2, \cdots\)) are real parameters characterizing the EOS. We also assume that the above EOS can be truncated at a certain order with good accuracy and \(p \geq 0\) for any given \(p \geq 0\). Physically speaking, \(c_0 > 0\) and \(c_1 = (dp/dp)p=0 \geq 0\) are, respectively, the energy density and a measure of the compressibility of the stellar matter at zero pressure. In particular, for ISs \(c_n = 0\), where \(n = 1, 2, 3, \cdots\). On the other hand, for QSs obeying the simple linear MIT bag model (see, e.g., \([27, 29]\)), \(c_n = 0\) for \(n > 1\). However, if other factors (such as the mass of strange quarks and finite temperature effect) are also considered, the EOS of QSs can still be approximated by a finite polynomial as given by \((2.4)\) (see, e.g., \([34, 38]\)).

In order to work out the I-Love relation for a SBS whose EOS is given by \((2.4)\), in the following discussion we formulate a perturbative solution for the TOV equations. First of all, we expand \(p, \rho, I \equiv m(r)/R^3\) and \(\tilde{I} \equiv I(r=R) = M/R^3\) in terms of Taylor series in stellar compactness \(C \equiv M/R\) as follows:

\[
\begin{align*}
\rho(x;C) &= p_0(x) + p_1(x)C + p_2(x)C^2 + \cdots, \\
p(x;C) &= \rho_0(x) + \rho_1(x)C + \rho_2(x)C^2 + \cdots, \\
I(x;C) &= I_0(x) + I_1(x)C + I_2(x)C^2 + \cdots, \\
\tilde{I}(C) &= \tilde{I}_0 + \tilde{I}_1C + \tilde{I}_2C^2 + \cdots,
\end{align*}
\]

where \(x = r/R\) is the normalized radial coordinate, \(p_n(x), \rho_n(x)\) and \(I_n(x)\) are functions of \(x\), and \(\tilde{I}_n = I_n(x = 1)\) are constant coefficients for \(n = 0, 1, 2, \cdots\).

In terms of \(I\) and \(\tilde{I}\), the TOV equation \((2.1)\) can be rewritten as:

\[
\frac{dp}{dx} = -\frac{C(\tilde{I} + 4\pi x^2 p)(\rho + p)}{x(\tilde{I} - 2C \tilde{I})}.
\]

By expanding both sides of \((2.9)\) into power series of \(C\) and taking EOS \((2.4)\) into account, the functions \(p_n(x)\) and \(\rho_n(x)\) can be solved recursively. The explicit results for \(p(x;C)/c_0\) and \(\rho(x;C)/c_0\) up to \(C^2\)-term are listed below for illustration:

\[
\begin{align*}
\frac{p(x;C)}{c_0} &= \frac{C}{2} \left(1 - x^2\right) + \frac{C^2}{5} \left(1 - x^2\right) \left[(5 + c_1) - c_1 x^2\right] + \cdots, \\
\frac{\rho(x;C)}{c_0} &= 1 + \frac{C}{2} \left(1 - x^2\right) c_1 + \frac{C^2}{20} \left(1 - x^2\right) \left[(20c_1 + 4c_1^2 + 5c_0c_2) - (4c_1^2 + 5c_0c_2) x^2\right] + \cdots,
\end{align*}
\]

and the expansions for \(I(x;C)\) and \(\tilde{I}(C)\) follow directly from their definitions. We note here that (i) in general both \(p_n(x)\) and \(\rho_n(x)\) are \(n\)-th degree polynomials in \(x^2\); and (ii) \(\rho(x;C)\) tends to \(c_0\) in the zero-compactness limit; and (iii) to first-order in \(C\), \(\rho(x;C)\) is in fact given by the GTM proposed in Refs. \([23, 24]\).

Likewise, the expansion for the metric coefficient \(e' \equiv \sum_{n=0}^{\infty} (e')_n(x)C^n\) can also be found using the TOV equation \((2.1)\):

\[
e'(x;C) = 1 - (3 - x^2)C + \frac{3}{20}(5 - c_1)(1 - x^2)^2 C^2 + \cdots.
\]

It agrees with the Minkowskian metric in the zero-\(C\) limit and, remarkably, is independent of \(c_0\). Besides, to first-order in \(C\), \(e'\) also does not depend on the EOS of the self-bound matter. Hence, the metric profile \(e'\) for SBSs is universal in the low-compactness regime.

Lastly, the mass-radius \((M-R)\) relation of SBSs can also found by eliminating \(C\) from the following pair of relations:

\[
\begin{align*}
M &= CR, \\
R &= \left[\frac{C}{\tilde{I}(C)}\right]^{1/2}.
\end{align*}
\]
$$M = \frac{4}{3} \pi c_0 R^3 \left[ 1 + \frac{1}{5} c_1 \tilde{I}_0 R^2 + \frac{1}{35} (14 c_1 + 3 c_1^2 + 2 c_0 c_2) \tilde{I}_0^2 R^4 + \cdots \right],$$

with $\tilde{I}_0 = 4\pi c_0/3$. Note that the first term in the above equation corresponds to the Newtonian result for ISs, while other higher order terms account for the general relativistic corrections due to the compressibility of the self-bound matter. We have verified the validity and gauged the accuracies of the perturbative expansions obtained in this section by applying them to QSs described by the simple MIT bag model. While the detailed result of such investigation can be found in Ref. [39], we also include a short summary of the relevant information in Appendix A.

### III. MOMENT OF INERTIA

The moment of inertia $I$ of SBSs in the slowly rotating limit (see, e.g., [36, 40, 41]) can be obtained from the perturbative solution for the stellar equilibrium configuration derived in Section III as follows. First of all, the angular velocity of the local inertial frame, $\Lambda(x)$, due to the frame-dragging effect of a compact star (e.g., a SBS) rotating uniformly at a unit angular velocity is governed by the differential equation [40, 41]:

$$\frac{d}{dx} \left( x^4 \frac{d\Lambda}{dx} \right) + 4x^3 \frac{dj}{dx} (\Lambda - 1) = 0,$$

where $0 \leq x \leq 1$ and $j(x) = e^{-(\lambda + \nu)/2}$. Outside the rotating star, where $x > 1$, $\Lambda(x)$ satisfies another differential equation

$$\frac{d}{dx} \left( x^4 \frac{d\Lambda}{dx} \right) = 0,$$

which is much simpler and can be readily integrated to yield the result $\Lambda = 2I/r^3$ (see [40, 41] for the details). Hence, the normalized moment of inertia $a \equiv I/MR^2$ is given by $a = \hat{\Lambda}/(2C)$ with $\hat{\Lambda} \equiv \Lambda(x = 1)$ being the surface value of $\Lambda$.

In the following we generalize the perturbative scheme proposed recently by Chan et al. [26], which was originally targeted at the moment of inertia of ISs, to find the moment of inertia of SBSs. Firstly, it follows directly from (3.1) and the regularity boundary condition of $\Lambda$ at $x = 0$ that

$$\frac{d\Lambda}{dx} = -\frac{1}{x^3 j} \int_0^x 4x^3 \frac{dj}{dx} (\Lambda - 1) dx' .$$

In order to solve (3.3) for $0 \leq x \leq 1$, we expand $\Lambda(x)$ and $j(x)$ in power series of $C$,

$$\Lambda(x; C) = \Lambda_0(x) + \Lambda_1(x) C + \Lambda_2(x) C^2 + \cdots ,$$

$$j(x; C) = j_0(x) + j_1(x) C + j_2(x) C^2 + \cdots ,$$

where $j_n(x)$, $n = 0, 1, 2, \cdots$, can be found directly from the expansions of $e^\lambda$ and $e^\nu$ obtained in Section III. Substituting these expansions into (3.3) and noting that (i) in the Newtonian limit

$$\lim_{C \to 0} j(x) = j_0(x) = 1 ;$$

$$\lim_{C \to 0} \Lambda(x) = \Lambda_0(x) = 0 ;$$

and (ii) $\Lambda_{n+1} = -(d\Lambda_{n+1}/dx)/3$ holds at $x = 1$, which is the consequence of the continuity $\Lambda$ and $d\Lambda/dx$ across the stellar surface, we can recursively find $\Lambda_{n+1}$ from $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$, and in turn arrive at the expansion of $\Lambda$ (see [26] for the details):

$$\Lambda(x; C) = \frac{2C}{5} (5 - 3x^2) + \frac{C^2}{350} \left[ 35(3 + c_1) + 126(5 - c_1)x^2 - 15(33 - 5c_1)x^4 \right] + \cdots ,$$

which holds generally for a uniformly rotating SBS in the slow rotation limit. It can be noted from [38] that $\Lambda_n$ is in general an $n$-th degree polynomial in $x^2$.

With this analytic expression for $\Lambda(x; C)$, we can find the normalized moment of inertia $a$ for an arbitrary SBS:

$$a(C) = \frac{2}{5} \left[ 1 + \frac{2C}{35} (15 - c_1) + \frac{2C^2}{1575} (795 - 80c_1 - 7c_1^2 - 20c_0c_2) + \cdots \right] .$$
In the zero-$C$ limit, $a = 2/5$, which is just the Newtonian result of a uniform sphere. With the expressions for $R$, $M$ and $a$ given respectively in (2.13), (2.14) and (3.9), the moment of inertia $I = aMR^2$ of a SBS can be straightforwardly found. On the other hand, in the I-Love-Q relations, the scaled moment of inertia ${I}/M^3 = a/C^2$ is considered \[4, 5\]. Hence, Eq. (3.9) actually provides a simple way to find $I$ perturbatively for SBSs.

IV. TIDAL DEFORMATION

In this section we outline the method of evaluating the tidal Love number (or deformability) of SBSs from the stellar equilibrium configuration obtained Section IV. Consider a non-rotating compact star acted on by an external tidal field $E_{ij}$ \[6, 21\]. As a result, the star acquires a quadrupole moment $Q_{ij}$ given by

$$Q_{ij} = -k_2 \left( \frac{2R^5}{3} \right) E_{ij} \equiv -\lambda E_{ij},$$

where $k_2$ (dimensionless) and $\lambda$ are called the tidal Love number and deformability, respectively. In the I-Love-Q relations, the dimensionless tidal deformability $\lambda \equiv \lambda/M^3 = 2k_2/(3C^5)$ is considered \[4, 5\].

We follow the formulation developed in \[6, 21\] to evaluate the tidal Love number $k_2$. First of all, the logarithmic derivative of the metric perturbation $H = H_0 = H_2$ (see \[42, 43\] for the conventions of the metric functions), $y \equiv rH'(r)/H(r)$, is considered. The following nonlinear ordinary differential equation governs the variation of $y(r)$ inside the star \[6, 21, 22\]:

$$r \frac{dy(r)}{dr} + y(r)^2 + y(r)e^{\lambda(r)} \left\{ 1 + 4\pi r^2 [p(r) - \rho(r)] \right\} + r^2 Q(r) = 0,$$

where

$$Q(r) = 4\pi e^{\lambda(r)} \left[ 5\rho(r) + 9p(r) + \frac{\rho(r) + p(r)}{c_s^2(r)} \right] - 6\frac{e^{\lambda(r)}}{r^2} - \left[ \frac{d\nu(r)}{dr} \right]^2,$$

$c_s \equiv \sqrt{dp/\rho}$ is the speed of sound, and $y(r = 0) = 0$. After solving (4.2) inside the star to evaluate the surface value of the logarithmic derivative of the metric perturbation, $y_R \equiv y(R^+) = y(R^-) - 4\pi R^3 \rho[R^-]/M$, the tidal Love number $k_2$ can then be found from the following formula \[6, 21, 22\]:

$$k_2(C, y_R)^2 \frac{8}{5} C^5 (1 - 2C)^2 [2C(y_R - 1) + 2] \left\{ 2C[4(y_R + 1)C^4 + (6y_R - 4)C^3 + (26 - 22y_R)C^2 + 3(5y_R - 8) - 3y_R + 6] + 3(1 - 2C^2)^2 [2C(y_R - 1) - y_R + 2] \log(1 - 2C)^{-1} \right\}^{-1}.$$

In order to solve the nonlinear differential equation (4.2) for SBSs, as suggested recently by Chan et al. \[26\], we substitute the following power series expansion of $y$,

$$y(x) = y_0(x) + y_1(x)C + y_2(x)C^2 + y_3(x)C^3 + \cdots,$$

and the series expansions of other relevant physical quantities obtained previously into the equation. By matching the coefficients of equal powers of $C$ in the resultant equation, we find a set of coupled linear first-order ordinary differential equations \[26\]:

$$xy_0'(x) + y_0(x)^2 + y_0(x) - 6 = 0,$$

$$xy_1'(x) + [1 + 2y_0(x)]y_1(x) - x^2 y_0(x) + 3(c_1 + 1)x^2 = 0,$$

which can be solved recursively by imposing the boundary conditions $y_0(0) = 2$ and $y_n(0) = 0$ for $n = 1, 2, \cdots$ as follows:

$$y_0(x) = 2;$$

$$y_1(x) = -\frac{3c_1 + 1}{7} x^2;$$

etc. From these solutions we can then find $y_R$, and hence $k_2$ from (4.4):

$$k_2 = (1 - 2C)^2 \frac{3}{4} + \frac{3C}{28} (12 + c_1) + \frac{C^2}{2940} (1210 - 230c_1 - 37c_1^2 - 140c_0c_2) + \cdots.$$

(4.10)
V. I-LOVE RELATION FOR LINEAR EOS

In the present paper, we aim to understand why the I-Love relation is so insensitive to variation in the EOS as long as the star in consideration is sufficiently stiff. To this end, we compare two representative members of SBSs with EOSs given by (24), namely, ISSs and QSs. It is obvious that ISSs are the simplest case of SBSs with $c_0 = 0$ for $n \geq 1$ and $c_0$ being the constant density of a star. Of course, ISSs are also the stiffest stars and, as shown in [20], their I-Love relation can accurately approximate those of realistic stars. On the other hand, in the MIT bag model for the EOS of quark matter [24, 25], if the effects of non-zero quark masses and tidal deformations are omitted, the EOS then takes the linear form:

$$\rho = 4B + 3p,$$  (5.1)

where $B > 0$ is called the bag constant. It is a special case of SBSs with $c_0 = 4B$, $c_1 = 3$ and $c_n = 0$ for $n \geq 2$. As verified numerically in [23, 26], the I-Love relations of ISSs and QSs are almost identical, especially in the Newtonian limit. Here we look into the interrelationship between the I-Love relations of these two kinds of SBSs with the perturbative scheme developed in the previous sections.

Using the abovementioned series expansion method for ISSs and QSs, we express the scaled moment of inertia $\tilde{I}$ and the dimensionless tidal deformability $\tilde{\lambda}$ in terms of the stellar compactness $C$. Moreover, we can also eliminate $\tilde{C}$ from the expressions for $\tilde{I}$ and $\tilde{\lambda}$ and relate $\tilde{I}$ to $\tilde{\lambda}$ directly. The results for ISSs are summarized as follows:

$$\tilde{I} = \frac{2}{5\tilde{C}^2} + \frac{12}{35\tilde{C}} + \frac{212}{525} + \frac{632}{1155} + \tilde{C} + \frac{703744}{2857125} C^2 + \frac{251264}{20125} C^3 + \frac{121542272}{60913125} C^4 + \cdots,$$  (5.2)

$$\tilde{\lambda}_C = \frac{4}{7} \frac{9C}{C} - \frac{121C^2}{479C^3} + \frac{196375C^4}{106708125} - \frac{59041871509888}{2433435125} C^4 + \cdots,$$  (5.3)

$$\tilde{I} = \frac{2}{5\tilde{C}^2} + \frac{44}{35\tilde{C}} + \frac{17452}{11025} + \frac{31936}{33957} \xi + \frac{1053343875C^2}{2433435125} - \frac{59041871509888}{339641200144} C^4 + \cdots,$$  (5.4)

where $\xi \equiv (2\tilde{\lambda})^{-1/5}$ and, for convenience, the dimensionless compactness-scaled tidal deformability

$$\tilde{\lambda}_C \equiv \frac{3C^5\tilde{\lambda}}{2(1-2C)^2} = \frac{k_2}{(1-2C)^2}$$  (5.5)

is introduced. We note that (i) both $\tilde{I}$ and $\tilde{\lambda}$ are independent of the density of the star in these equations, and (ii) the above results are identical to what have been obtained in [20] using the Schwarzschild constant-density solution for the TOV equations [44] as the input to evaluate the moment of inertia and tidal Love number.

On the other hand, we can similarly find the associated formulas for QSs:

$$\tilde{I} = \frac{2}{5\tilde{C}^2} + \frac{48}{175\tilde{C}} + \frac{656}{2625} + \frac{40408}{400125} C - \frac{883424}{20125} C^2 - \frac{121542272}{60913125} C^4 + \cdots,$$  (5.6)

$$\tilde{\lambda}_C = \frac{3}{4} \frac{45C}{C} - \frac{187C^2}{1141589C^3} + \frac{27391383C^4}{37429427307C^5} - \frac{469553810716163C^6}{54076522500} - \frac{783539839762500}{53930839762500} C^4 + \cdots,$$  (5.7)

$$\tilde{I} = \frac{2}{5\tilde{C}^2} + \frac{44}{35\tilde{C}} + \frac{87134}{55125} + \frac{1783052\xi}{1929375} + \frac{181657348C^2}{15209021953125} - \frac{1797421853576C^4}{27761427688380608668\xi} + \cdots,$$  (5.8)

It is remarkable that in all these expansions for ISSs (or QSs) the dependency on $c_0$ (or the bag constant $B$) disappears.

First of all, we examine the accuracy of the above post-Minkowskian expansions of the I-Love relation for ISSs and QSs. As shown in Fig. 1 where the I-Love relation obtained numerically for ISSs and QSs are compared with the 7-term post-Minkowskian expansions (5.4) and (5.8), both of these two expansions can accurately reproduce the I-Love relation for these two kinds of SBSs with the relative error $E = |\tilde{I}_{\text{series}} - \tilde{I}_{\text{data}}|/\tilde{I}_{\text{data}}$ being less than 0.01 in all cases. In fact, unless for stars close to the maximum compactness $C_m$ ($C_m = 4/9$ for ISSs and QSs, respectively), $E$ is less than 0.001. Furthermore, it can be readily observed from Fig. 1 that the difference between the I-Love relations for ISSs and QSs (respectively denoted by the continuous and dashed lines) is almost indiscernible. This observation is in good agreement with the discovery reported in [23, 26], which reveals the stationarity of the I-Love relation about the IS limit.
TABLE I: A comparison of the post-Minkowskian expansion of the \( \bar{I} - C \), \( \bar{\lambda} - C \) and \( \bar{I} - \bar{\lambda} \) relations for ISs (see (5.2), (5.3) and (5.4)) and QSs (see (5.10), (5.11) and (5.8)). For each relation, the relative difference between the coefficients of the \( n \)th term in the post-Minkowskian expansions for these two cases, i.e., (the coefficient for QS case)/(the corresponding coefficient for IS case) − 1, is shown for \( n = 0, 1, 2, \ldots, 6 \).

| relation | \( n = 0 \) | 1 | 2 | 3 | 4 | 5 | 6 |
|----------|------------|---|---|---|---|---|---|
| \( \bar{I} - C \) | 0 | -0.20 | -0.381 | -0.635 | -1.05 | -1.769 | -3.03 |
| \( \bar{\lambda} - C \) | 0 | 0.25 | -0.845 | 22.8 | 12.9 | 13.0 | 14.9 |
| \( \bar{I} - \bar{\lambda} \) | 0 | 0 | -0.00144 | -0.0174 | -0.223 | 1.90 | 2.76 |

The similarity between the I-Love relations for ISs and QSs is intriguing. In Table I we compare the corresponding expansion coefficients in the \( \bar{I} - C \), \( \bar{\lambda} - C \) and \( \bar{I} - \bar{\lambda} \) relations for these two kinds of SBSs. Comparing (5.2) with (5.10) and (5.3) with (5.11), we can see that for these two kinds of SBSs, the respective leading first terms in the post-Minkowskian expansion for the \( \bar{I} - C \) and \( \bar{\lambda} - C \) relations are the same, whereas as shown in (5.4) and (5.8) the leading two orders of the post-Minkowskian expansion for the \( \bar{I} - \bar{\lambda} \) relation are identical. Besides, barring one exceptional case, the magnitude of the relative difference of the corresponding coefficients in these two cases (ISs and QSs) grows with the order, and such differences for the \( \bar{I} - \bar{\lambda} \) relation are usually much smaller than their counterparts in the \( \bar{I} - C \) and \( \bar{\lambda} - C \) relations.

The physical consequences of the abovementioned observations are further demonstrated numerically in Figs. 2 and 3. In Fig. 2 the logarithm of the relative difference between the scaled moment of inertia of QSs and ISs, \( I_{QS} \) and \( I_{IS} \), is plotted against the compactness \( C \). It is clearly shown that the analytic result (continuous line) obtained from the leading 7-term post-Minkowskian expansion given by (5.2) and (5.6) agree nicely with the numerical result (denoted by circles). As the leading terms (referred to as the zeroth-order term hereafter) in (5.2) and (5.6) cancel the contribution of each other, the relative difference in \( \bar{I} \) vanishes in the Newtonian limit. However, the relative difference grows gradually with \( C \). In fact, the relative difference is larger than 0.01 for \( C > 0.1 \). Such situation also prevails in Fig. 3 where the relative difference between \( \lambda_{QS} \) and \( \lambda_{IS} \), which are obtained from the \( \lambda - C \) relations (see (5.3) and (5.5), is plotted against the compactness \( C \). In this case the relative difference is much larger than 0.01 for \( C > 0.1 \) and grows beyond 0.1 for \( C > 0.15 \). In both Figs. 2 and 3 we have also decomposed the relative difference into the contributions due to the 1st, 2nd, \ldots, 6th post-Minkowskian correction terms in the \( \bar{I} - C \) and \( \bar{\lambda} - C \) relations for ISs and QSs. It is clearly shown that the first-order post-Minkowskian correction term (the dot-dashed line) in fact contributes the most to the relative difference of \( \bar{I} \) and \( \bar{\lambda} \). As a result, the first-order post-Minkowskian correction term in the \( \bar{I} - C \) and \( \bar{\lambda} - C \) relations can readily account for difference in \( \bar{I} \) and \( \bar{\lambda} \) between QSs and ISs observed previously (see, e.g., (1)[22]).

However, when the \( \bar{I} - C \) and \( \bar{\lambda} - C \) relations are combined together to eliminate the variable \( C \) so as to express \( \bar{I} \) directly in terms of \( \bar{\lambda} \), the EOS-dependency on the first-order post-Minkowskian correction term of the I-Love relation surprisingly disappears (see (5.3), (5.8) and Table I). In Fig. 4 we show \( \log_{10}(|I_{QS} - I_{IS}|/I_{IS}) \) versus \( \log_{10} \bar{\lambda} \). Again the analytical result (the continuous line) obtained from (5.4) and (5.8) can well approximate the numerical data (circles). In comparison with Figs. 2 and 3 the relative difference is now much smaller in most situations. It is always less than 0.01 and decreases rapidly with increasing \( \bar{\lambda} \) (i.e., smaller compactness). Unless for QSs close to the maximum compactness, the relative difference between ISs and QSs is less than 0.001, which is ten times less than the typical value of its counterparts shown in Figs. 2 and 3. To further understand the smallness of such difference, in Fig. 4 we decompose \( |I_{QS} - I_{IS}|/I_{IS} \) into respective contributions arising from the leading six order post-Minkowskian expansions. Unlike the situations shown in Figs. 2 and 3 where the relative difference is dominated by the first-order expansion, in this case the first-order post-Minkowskian correction term vanishes identically, which is exactly the reason why the relative difference becomes so small. Instead, for stars with \( \log_{10} \bar{\lambda} > 5 \), the second-order correction term, which is less than \( 10^{-4} \), contributes the most to the relative difference. On the other hand, for more compact stars, the contributions due to higher order post-Minkowskian corrections overtake the second-order one. However, even for QSs close to the maximum compactness limit, the relative difference is still bounded by \( 10^{-2} \).

In order to obtain a more thorough understanding of these results, we consider a general linear EOS \( \rho = c_0 + c_1 p \) for SBSs and carry out the perturbative scheme to find the formulas for the \( \bar{I} - C \), \( \bar{\lambda} - C \) and \( \bar{I} - \bar{\lambda} \) relations. In Table II we show the expansion coefficients of these relations. We note that the expansion coefficients are, as mentioned previously, independent of \( c_0 \). The general terms in \( \bar{I} - C \) and \( \bar{\lambda} - C \) expansions are given by \( c_i^n C^n \) (\( m = 0, 1, \ldots \))
TABLE II: The coefficients of $c_i^n \bar{\lambda}^n$-term ($m = 0, 1, \cdots , 6$) in the series expansions of $\bar{I}$ (with $n = -2, -1, \cdots , 4$) and $\bar{\lambda}_C$ (with $n = 1, 2, \cdots , 6$) are shown for a SBS with a linear EOS given by $\rho = c_0 + c_1 p$. Similarly, the coefficients of $c_i^n \bar{\lambda}^{n/2}$-term ($m = 0, 1, \cdots , 6$ and $n = 4, -3, \cdots , 2$) in the series expansion of $\bar{I}$ are also tabulated.

| $\bar{I}$ | $c_0^0$ | $c_1^0$ | $c_2^0$ | $c_3^0$ | $c_4^0$ | $c_5^0$ |
|---|---|---|---|---|---|---|
| $C^{-2}$ | 4.000 × 10^{-1} | 0. | 0. | 0. | 0. | 0. |
| $C^{-1}$ | 3.429 × 10^{-1} | -2.286 × 10^{-2} | 0. | 0. | 0. | 0. |
| $C^0$ | 4.038 × 10^{-1} | -4.063 × 10^{-2} | -3.556 × 10^{-3} | 0. | 0. | 0. |
| $C^1$ | 5.472 × 10^{-1} | -6.612 × 10^{-2} | -1.431 × 10^{-2} | -7.454 × 10^{-4} | 0. | 0. |
| $C^2$ | 8.035 × 10^{-1} | -1.069 × 10^{-1} | -4.219 × 10^{-2} | -4.757 × 10^{-3} | -1.852 × 10^{-4} | 0. |
| $C^3$ | 1.243 | -1.735 × 10^{-1} | -1.105 × 10^{-1} | -1.998 × 10^{-2} | -1.625 × 10^{-3} | -5.106 × 10^{-5} |
| $C^4$ | 1.995 | -2.824 × 10^{-1} | -2.727 × 10^{-1} | -6.977 × 10^{-2} | -8.833 × 10^{-3} | -5.704 × 10^{-4} | -1.507 × 10^{-5} |

| $\bar{\lambda}_C$ | $c_0^0$ | $c_1^0$ | $c_2^0$ | $c_3^0$ | $c_4^0$ | $c_5^0$ |
|---|---|---|---|---|---|---|
| $C^0$ | 7.500 × 10^{-1} | 0. | 0. | 0. | 0. | 0. |
| $C^1$ | -1.286 | 1.071 × 10^{-1} | 0. | 0. | 0. | 0. |
| $C^2$ | 4.116 × 10^{-1} | -7.823 × 10^{-2} | -1.259 × 10^{-2} | 0. | 0. | 0. |
| $C^3$ | -4.232 × 10^{-2} | -2.102 × 10^{-1} | -3.112 × 10^{-2} | -2.060 × 10^{-3} | 0. | 0. |
| $C^4$ | -1.907 × 10^{-1} | -4.630 × 10^{-1} | -8.699 × 10^{-2} | -9.656 × 10^{-3} | -4.432 × 10^{-4} | 0. |
| $C^5$ | -4.933 × 10^{-1} | -1.014 | -2.348 × 10^{-1} | -3.664 × 10^{-2} | -3.166 × 10^{-3} | -1.139 × 10^{-4} |
| $C^6$ | -1.152 | -2.213 | -6.172 × 10^{-1} | -1.248 × 10^{-1} | -1.571 × 10^{-2} | -1.097 × 10^{-3} | -3.254 × 10^{-5} |

with $n = -2, -1, \cdots$ for the former and $n = 0, 1, \cdots$ for the latter. In addition, it can be seen that

$$
\frac{1}{\bar{I}} \left( \frac{\partial \bar{I}}{\partial c_1} \right)_C = -0.05715 C + O(C^2), \tag{5.9}
$$

$$
\frac{1}{\bar{\lambda}} \left( \frac{\partial \bar{\lambda}}{\partial c_1} \right)_C = -0.1428 C + O(C^2). \tag{5.10}
$$

Hence, both $\bar{I}$ and $\bar{\lambda}$ are independent of the variation in $c_1$ in the Newtonian limit where $C \to 0$. However, we expect that their dependency on $c_1$ is more obvious for relativistic stars. In fact, neither of these two derivatives vanishes to first-order in the compactness.

On the other hand, $C$ can be eliminated from the $\bar{I} - C$ and $\bar{\lambda}_C - C$ relations to yield the $\bar{I} - \bar{\lambda}$ relation. As shown in Table III, the general term of such an expansion is $c_m^n \bar{\lambda}^{n/2}$-term, where $m = 0, 1, \cdots$ and $n = -4, -3, \cdots , 2$. We also find that

$$
\frac{1}{\bar{I}} \left( \frac{\partial \bar{I}}{\partial c_1} \right)_\bar{\lambda} = -0.002061 \bar{\lambda}^{-2/5} + O(\bar{\lambda}^{-3/5}) ,
$$

$$
= -0.002720 C^2 + O(C^3), \tag{5.11}
$$

where the second line follows from the fact that $\bar{\lambda} = 0.5 C^{-5} [1 + O(C)]$ (see Table III). Eq. (5.11) clearly demonstrates the reason why the I-Love relation is so insensitive to the value of $c_1$, which is a measure of the compressibility of the stellar matter. The logarithmic derivative of $\bar{I}$ with respect to $c_1$ at a constant $\lambda$, $(\partial \ln \bar{I} / \partial c_1)_\bar{\lambda}$, is approximately equal to $-0.002720 C^2$. In comparison with the results shown in (5.9) and (5.10), the dependency of the I-Love relation on the parameter $c_1$ (or compressibility) is much weaker than that of the $\bar{I} - C$ and $\bar{\lambda} - C$ relations. In particular, to first-order in $C$, $(\partial \ln \bar{I} / \partial c_1)_\bar{\lambda}$ vanishes. This clearly explains why the I-Love relation is so insensitive to the value of $c_1$ ($c_1 = 0, 3$ for ISs and QSs respectively) as observed here and other previous publications [7, 22, 24].
VI. GENERAL CASE

The insensitivity of the I-Love relation to the parameters characterizing the EOS is generic. In general, we consider a SBS with EOS given by (2.3). In the following we shall show that the I-Love relation is, to the $j$th-order post-Minkowskian correction, independent of the parameters $c_j$ about the incompressible limit. Without loss of generality, we assume for the moment that for $j = 1, 2, 3, \ldots$, $c_j = 0$ if $j \neq n$, where $n$ is a given positive integer. Therefore, all physical quantities can be considered as functions of $c_n$ and are expandable in terms of Taylor series in $c_n$. For example, $\rho(x; C; c_n)$ and $p(x; C; c_n)$ are expanded as:

$$\rho(x; C; c_n) = \rho^{(0)}(x; C) + \rho^{(1)}(x; C)c_n + \rho^{(2)}(x; C)c_n^2 + \cdots ,$$

$$p(x; C; c_n) = p^{(0)}(x; C) + p^{(1)}(x; C)c_n + p^{(2)}(x; C)c_n^2 + \cdots .$$

The physical meaning of $\rho^{(0)}(x; C)$ and $p^{(0)}(x; C)$ are obvious. They are the density and pressure profiles, respectively, of an IS with constant density $c_0$ and compactness $C$. In the following we shall show that to first-order in $c_n$ and $C^n$, the I-Love relation of such SBSs is independent of $c_n$. In other words, to $C^n$, the I-Love relation is stationary with respect to variation in $c_n$ about the incompressible limit where $c_n = 0$ (see (6.31) for the corresponding mathematical formula). To this end, we shall only keep terms up to first-order in $c_n$ and ignore its higher-order terms in the following calculations.

First of all, we note that to first order in $C$, $p^{(0)}(x; C) = c_0 C(1 - x^2)/2$, and hence it follows from the relevant EOS that

$$[\rho^{(1)}(x; C)]_C = \alpha_n [f_n(x) - 3\dot{I}_n],$$

where the notation $[g(C)]_C$ is introduced hereafter to signify the leading term in the Taylor expansion of $g$ as a function of $C$,

$$\alpha_n = (c_0 C/2)^n,$$  

$$f_n(x) = (1 - x^2)^n,$$  

$$\dot{I}_n = \int_0^1 f_n(x)x^2 \, dx = \sqrt{\pi} \Gamma(n + 1)/(4\Gamma(n + 5/2),$$

with $\Gamma(z)$ being the standard $\Gamma$-function (see, e.g., [42]). It is worthwhile to remark that in (6.3) there are two contributions to $[\rho^{(1)}(x; C)]_C$, namely, $\alpha_n f_n(x)$ and $-3\alpha_n \dot{I}_n$. While the former is the density change due to the finite compressibility of the EOS, the latter is introduced to keep the mass, the radius and hence the compactness of the star unchanged in the process of switching on $c_n$.

Next, we consider the moment of inertia given by (3.1) and expand $\Lambda(x; C; c_n)$ and $j(x; C; c_n)$ as follows:

$$\Lambda(x; C; c_n) = \Lambda^{(0)}(x; C) + \Lambda^{(1)}(x; C)c_n + \Lambda^{(2)}(x; C)c_n^2 + \cdots ,$$

$$j(x; C; c_n) = j^{(0)}(x; C) + j^{(1)}(x; C)c_n + j^{(2)}(x; C)c_n^2 + \cdots .$$

As mentioned above, $\Lambda^{(0)}(x; C)$ and $j^{(0)}(x; C)$ are the corresponding physical quantities for ISs, which are given by:

$$\Lambda^{(0)}(x; C) = \frac{2}{5} (5 - 3x^2) C + \frac{1}{70} (21 + 126x^2 - 99x^4) C^2 + \cdots ,$$

$$j^{(0)}(x; C) = 1 + \frac{3C(1 - x^2)}{2} + 3C(1 - x^2) + \cdots .$$

On the other hand, since

$$\frac{dj(x; C)}{dx} = -\frac{1}{2} \left[ 1 - \frac{\nu + \lambda}{2} + \frac{1}{2} \left( \frac{\nu + \lambda}{2} \right)^2 + \cdots \right] \frac{d(\nu + \lambda)}{dx}$$

and $d(\nu + \lambda)/dx = 8\pi(p + \rho)/(1 - 2m/r)$, it is straightforward to show from the TOV equations and (6.3) that

$$\left[ \frac{dj^{(1)}(x; C)}{dx} \right]_C = \frac{3C\alpha_n}{c_0} \left[ 3\dot{I}_n - f_n(x) \right] x ,$$

and

$$\left[ \frac{d\Lambda^{(1)}(x; C)}{dx} \right]_C = \frac{12C\alpha_n}{x^4 c_0} \int_0^x x'^4 \left[ 3\dot{I}_n - f_n(x') \right] dx' .$$
At \( x = 1, \Lambda = -\Lambda'/3 = 2C^3I \), we have

\[
\left[ \Lambda^{(1)} \right]_C (x = 1) = \frac{4C}{c_0} \int_0^1 x^4[\rho^{(1)}(x; C)]_C \, dx,
\]

\[
= -\frac{6C\alpha}{5c_0} \left( \frac{n}{2n+5} \right) \frac{\sqrt{\pi} \Gamma(n+1)}{\Gamma(n+5/2)} ,
\]

and hence

\[
[\bar{I}^{(1)}]_C = \frac{2}{c_0C^2} \int_0^1 x^4[\rho^{(1)}(x; C)]_C \, dx ,
\]

\[
= -\frac{3C\alpha}{5c_0C^2} \left( \frac{n}{2n+5} \right) \frac{\sqrt{\pi} \Gamma(n+1)}{\Gamma(n+5/2)} .
\]

We note that

\[
\bar{I}^{(0)} = \frac{1}{C^2} \left[ \frac{2}{5} + \frac{12}{35} C + \frac{212}{525} C^2 + \frac{362}{1155} C^3 + \frac{703744}{875875} C^4 + \frac{251264}{202125} C^5 + \frac{121542272}{60913125} C^6 + \cdots \right] ,
\]

which is the dimensionless moment of inertia of ISs.

We now turn to the problem of tidal deformability and consider the function \( y(r) \) governed by (6.12). Similar to the series expansion of \( I \), we assume \( y \) can be written as a power series in \( c_n \)

\[
y(x; C; c_n) = y^{(0)}(x; C) + y^{(1)}(x; C)c_n + y^{(2)}(x; C)c_n^2 + y^{(3)}(x; C)c_n^3 + \cdots
\]

(6.17)

We substitute (6.17) and the series expansions of other quantities into (6.12). The zeroth-order term leads to

\[
r \frac{dy^{(0)}(r)}{dr} + y^{(0)}(r)^2 + y^{(0)}(r)e^{\lambda^{(0)}(r)}[1 + 4\pi r^2(p^{(0)}(r) - c_0)] + r^2 Q^{(0)}(r) = 0,
\]

(6.18)

where

\[
Q^{(0)}(r; C) = 4\pi e^{\lambda^{(0)}(r)} \left[ 5c_0 + 9p^{(0)}(r) \right] - \frac{6e^{\lambda^{(0)}(r)}}{r^2} - \left[ \frac{d\nu^{(0)}(r)}{dr} \right]^2 ,
\]

(6.19)

with \( \lambda^{(0)}(r) \) and \( \nu^{(0)}(r) \) being the metric coefficients of ISs. \( y^{(0)} \) is nothing but the solution of \( y \) for ISs and in Section \( \text{IV} \) we have already shown that \( y^{(0)}(x, C) = \sum_{i=0}^{\infty} y^{(i)}(0)(x)C^i \), where

\[
x \frac{dy^{(0)}(x)}{dx} + y^{(0)}(x)^2 + y^{(0)}(x) - 6 = 0,
\]

(6.20)

\[
x \frac{dy^{(1)}(x)}{dx} + [1 + 2y^{(0)}(x)]y^{(1)}(x) - x^2 y^{(0)}(x) + 3x^2 = 0,
\]

(6.21)

and higher-order equations can be similarly obtained. The leading coefficients are \( y^{(0)}(x) = 2, y^{(1)}(x) = -x^2/7, \) and higher-order solution can be obtained recursively.

Now we pay special attention to the leading order term in \( y^{(1)} \) (i.e., \([y^{(1)}]_C\)), which satisfies

\[
r \left[ \frac{dy^{(1)}}{dr} \right]_C + 5 \left[ y^{(1)} \right]_C + 2 \left[ \left( e^{\lambda} + 4\pi r^2 e^{\lambda(p - \rho)} \right)^{(1)} \right]_C + r^2 \left[ Q^{(1)} \right]_C = 0.
\]

(6.22)

Taking into account the fact that \( c_2^2 = 1/(nc_n p^{n-1}) \), it can be shown that the third and the fourth terms in the LHS of the above equation are first- and zero-order in \( C \), respectively. To leading order in \( C \), the former is then negligible, while the latter is given by:

\[
r^2 \left[ Q^{(1)} \right]_C = \frac{6\alpha}{c_0} \frac{\pi}{2}(1 - x^2)^{n-1} .
\]

(6.23)

From this result and (6.22) we show that

\[
\left[ \hat{y}^{(1)} \right]_C \equiv \left[ y^{(1)}(x = 1) \right]_C = -\frac{15\alpha}{c_0} \int_0^1 x^4 f_n(x) \, dx .
\]

(6.24)
After solving (6.2) by the method mentioned above, $k_2$ can be obtained from $y_R$ by expanding (4.4) into power series of $C$ [22]:

$$k_2 = \frac{2 - y_R}{2(y_R + 3)} + \frac{5C \left(y_R^2 + 2y_R - 6\right)}{2(y_R + 3)^2} - \frac{5C^2 \left(11y_R^3 + 66y_R^2 + 52y_R - 204\right)}{14(y_R + 3)^3} + \cdots \quad (6.25)$$

By virtue of (6.24) and (6.25), the leading correction term in $k_2(1)$ is given by

$$\left[k_2(1)\right]_c = -\frac{3}{4} \left(\frac{5}{6} \frac{\hat{\rho}(1)}{\hat{y}} + \frac{15\alpha n \hat{\rho}_n}{2c_0}\right)\quad (6.26)$$

where the second term inside the bracket on the RHS of the equation accounts for the correction in the discontinuity of $y$ across the stellar surface due to the presence of $\rho(1)$. We can also rewrite the above equation as follows:

$$\left[k_2(1)\right]_c = -\frac{75}{8c_0} \int_0^1 x^4\rho(1)(x; C)_c\, dx,$$

$$= -\frac{45\alpha n}{16c_0} \left(\frac{n}{2n + 5}\right) \sqrt{\pi \Gamma(n + 1)} \Gamma(n + 5/2) \quad (6.27)$$

where the similarity between $[\tilde{\lambda}(1)]_c(x = 1)$ (see (6.14)) and $[k_2(1)]_c$ is explicitly shown. Since $\tilde{\lambda} \equiv 2k_2/(3C^5)$, we have

$$\tilde{\lambda}(0) = \frac{1}{2C^5} - \frac{20}{7C^4} + \frac{2515}{441C^3} - \frac{51550}{11319C^2} + \frac{3347350}{3090087C} + \frac{4326424}{6491827} + \frac{368458100}{943855727} C + \cdots \quad (6.28)$$

$$\tilde{\lambda}(1) = \frac{25}{4c_0C^5} \int_0^1 x^4\rho(1)(x; C)_c\, dx,$$

$$= -\frac{15}{8C^5} \left(\frac{\alpha n}{c_0}\right) \left(\frac{n}{2n + 5}\right) \sqrt{\pi \Gamma(n + 1)} \Gamma(n + 5/2) \quad (6.29)$$

By elementary theory of calculus, it is easy to show that

$$\left(\frac{\partial I}{\partial c_n}\right)_{\tilde{\lambda}} = \left(\frac{\partial \tilde{I}}{\partial c_n}\right)_{\tilde{\lambda}} - \left(\frac{\partial \lambda}{\partial c_n}\right)_{\tilde{\lambda}} \left(\frac{\partial \hat{\lambda}}{\partial \tilde{\lambda}}\right)_{\tilde{\lambda}} \left(\frac{\alpha n}{\tilde{\lambda}}\right)_{c_n = 0}.$$

Under the approximations $\tilde{I} = \tilde{I}(0) + [\tilde{I}(1)]_c$ and $\tilde{\lambda} = \tilde{\lambda}(0) + [\tilde{\lambda}(1)]_c$, it is straightforward to show from (6.30), the explicit expressions of $\tilde{I}(0)$, $[\tilde{I}(1)]_c$, $\tilde{\lambda}(0)$ and $[\tilde{\lambda}(1)]_c$ obtained above that to order of $c_n^{n-1} C^n$,

$$\frac{1}{\tilde{I}} \left[\left(\frac{\partial \tilde{I}}{\partial c_n}\right)_{\tilde{\lambda}}\right]_{c_n = 0} = 0. \quad (6.31)$$

In order words, to order $C^n$, the I-Love relation is stationary with respect to changes in $c_n$ about the incompressible limit where $c_n = 0$. In fact, Eq. (6.11) is merely a special case of (6.31) with $n = 1$.

VII. PHYSICAL INTERPRETATION

In this section we discuss the physical mechanism underlying (6.31) and hence the stationarity of the I-Love relation. First of all, we note that both $[\tilde{I}(1)]_c$ and $[\tilde{\lambda}(1)]_c$ are proportional to the integral $\int_0^1 x^4\rho(1)(x; C)_c\, dx$, which is nothing but the change of the Newtonian moment of inertia of the star upon the introduction of finite compressibility to the EOS. This once again highlights the Newtonian nature of the observed I-Love universality. Secondly, we can also see that to leading order in compactness,

$$\frac{[\tilde{I}(1)]_c}{\tilde{I}(0)(C)} = \frac{5}{c_0} \int_0^1 x^4\rho(1)(x; C)_c\, dx \quad (7.1)$$

$$\frac{[\tilde{\lambda}(1)]_c}{\tilde{\lambda}(0)(C)} = \frac{25}{2c_0} \int_0^1 x^4\rho(1)(x; C)_c\, dx \quad (7.2)$$
where $\bar{I}^{(0)}(C)$ and $\bar{\lambda}^{(0)}(C)$ are the scaled moment of inertia and the tidal deformability of an IS with the same compactness $C$. Therefore, as noted in some previous studies (see, e.g., [22]), for SBSs (such as QSs) both $\bar{I}$ and $\bar{\lambda}$ demonstrate an observable EOS-dependency, which is characterized by $c_n$ in the present situation. However, it is interesting to find that

$$\frac{\{\bar{I}^{(1)}\|C\}}{\bar{I}^{(0)}} = \frac{2}{5},$$

(7.3)

which is actually independent of the functional form of $[\rho^{(1)}(x;C)]_c$.

On the other hand, if the compactness of an IS varies from $C$ to $C + \delta C$, where $|\delta C| \ll 1$, the associated change in $\bar{I}$ and $\bar{\lambda}$, namely $\delta \bar{I}$ and $\delta \bar{\lambda}$, are respectively given by (to leading orders in $\delta C$ and $C$)

$$\frac{\delta \bar{I}}{\bar{I}^{(0)}} = -2 \left(\frac{\delta C}{C}\right),$$

(7.4)

$$\frac{\delta \bar{\lambda}}{\bar{\lambda}^{(0)}} = -5 \left(\frac{\delta C}{C}\right),$$

(7.5)

by virtue of the fact that $\bar{I}^{(0)} \propto C^{-2}$ and $\bar{\lambda}^{(0)} \propto C^{-5}$ in the Newtonian limit. As a result, we have

$$\frac{\{\delta \bar{I}/\bar{I}^{(0)}\}}{\{\delta \bar{\lambda}/\bar{\lambda}^{(0)}\}} = \frac{2}{5}.$$  

(7.6)

Then it follows directly from (7.1) - (7.6) that the values of $\bar{I}$ and $\bar{\lambda}$ of a SBS with compactness $C$ and finite compressibility can be obtained from the their counterparts of an IS with a modified compactness $C + \delta C$ if

$$\left(\frac{\delta C}{C}\right) = -\frac{5 c_n}{2x_0} \int_0^1 x^4 [\rho^{(1)}(x;C)]_c dx.$$  

(7.7)

In other words, for an IS the respective effects of (i) introducing finite compressibility, and (ii) properly adjusting the stellar compactness on $\bar{I}$ (or $\bar{\lambda}$) are indistinguishable in the low compactness limit provided that (7.7) holds. As a result, when $\bar{I}$ and $\bar{\lambda}$ of SBSs with finite compressibility are directly linked to each other by eliminating $C$ in the $\bar{I} - C$ and $\bar{\lambda} - C$ relations, their mutual dependency is almost identical to that of ISs to leading order in compactness. This provides a physically transparent interpretation of the I-Love universality.

A word of caution about the validity of the I-Love universality is in order. While Eqs. (7.1) and (7.2) follow closely from the physical content of moment of inertia and tidal deformation, Eqs. (7.4) and (7.5) are actually the consequence of the judicious definitions (or the proper normalization) of $\bar{I}$ and $\bar{\lambda}$, which lead to the dependency $\bar{I}^{(0)} \propto C^{-2}$ and $\bar{\lambda}^{(0)} \propto C^{-5}$ in the Newtonian limit. For example, if the tidal Love number $k_2 = 3C^5\lambda/2$ is considered instead of the dimensionless tidal deformability $\bar{\lambda}$ in the I-Love relation, the universality will no longer prevail.

VIII. CONCLUSION AND DISCUSSION

In the present paper we study the physical mechanism underlying the universality demonstrated in the I-Love relation. Motivated by the recent numerical observation that the I-Love relation of realistic compact stars (including NSs and QSs) can be well approximated by that of ISs [23, 26], we carry out an in-depth investigation on this relation for QSs and other SBSs as well. To set the stage for such analysis, we establish a systematic perturbative scheme to evaluate the moment of inertia and the Love number (tidal deformability) of SBSs. Both $\bar{I}$ and $\bar{\lambda}$ are written in terms of power series of $C_n$. It can be seen from such expansions how these two physical quantities depend on the EOS and compactness of a SBS star. In general, each of the parameters $c_n$ in EOS (2.4) can lead to a fractional deviation in $\bar{I}$ (or $\bar{\lambda}$) from its IS counterpart, which is to leading order given by $c_n C^n$ and becomes noticeable (of order $10\%$) for relativistic stars (see Figs. 2 and 3). However, we further show analytically that, due to a cancellation between the corresponding terms in the $\bar{I} - C$ and $\bar{\lambda} - C$ relations, the I-Love relation is, to order $C^n (n = 1, 2, 3, \cdots)$, stationary around the incompressible limit upon variation of the parameter $c_n$. The universality of the I-Love relation is hence attributable to such stationarity. Therefore, the perturbative scheme established here indeed provides an independent corroboration of the numerical results obtained in Refs. [23, 26].

As the parameter $c_n$ is essentially a measure of the compressibility of the stellar matter, it is clearly shown here that high stiffness of EOS is crucial to the universality of the relation. It is worth of remark that Yagi et al. [25] have suggested the validity of the elliptical isodensity approximation as a key factor affecting the universal I-Love relation.
In particular, they found numerically that the variation of the eccentricity of an isodensity surface inside a slowly rotating Newtonian star is small for stiff polytropic stars and QSs and becomes exact in the incompressible limit [25]. Hence, their empirical finding is actually in agreement with the analytical study reported here.

In the present paper we also pinpoint the physical origin of the universality and the stationary for the I-Love relation of SBSs. We show that, as far as $I$ and $\lambda$ are concerned, the effects due to finite compressibility, which is rooted in the EOS, and a proper renormalization of the stellar compactness (see (7.7)) are equivalent to leading order in compactness. Such equivalence leads to the observed universality and stationarity, and is the joint consequence of (i) the similarity in the responses of $I$ and $\lambda$ to variations in the EOS (see (7.1) and (7.2)), and (ii) the judicious normalization of $I$ and $\lambda$. While the former is attributable to the nature of the two relevant physical quantities, the latter is indeed a clever choice engineered for the I-Love universality to hold. We expect that our analysis established here could be generalized to study other systems that demonstrate similar universality (see, e.g., [7, 14–19, 24]).

On the other hand, we have to stress that the stationarity of the I-Love relation about the incompressible limit holds only up to order $C^n$ ($n = 1, 2, 3 \cdots$) for variations in the parameter $c_n$. Consequently, we expect that the accuracy of such a “universal” relation worsens for stars with large compactness. This point is verified in Fig. 4 of the present paper for QSs and similar conclusion has also been obtained previously for realistic NSs [23, 26]. However, as the compactness of stable NSs and QSs is usually less than 0.3, the I-Love relation of realistic compact stars does not deviate much from its incompressible counterpart even in the extreme relativistic limit.

As a byproduct of the present paper, we have found analytically the universal structure of QSs and other SBSs as well, including the density profile, the mass-radius relation, the moment of inertia and the tidal Love number, by employing the compactness as an expansion parameter. Each of these analytic expressions for QSs is expected to be of interest to astrophysicists (see, e.g., [1, 2, 22, 46]).

Throughout the present paper we have focused our attention on the I-Love relation of SBSs, whose energy density is non-zero at the stellar surface, because of their proximity to ISs. We show that the I-Love relation of SBSs is still close to that of ISs when relativistic effects are included. We also provide analytic methods to evaluate quantitatively such effects with post-Minkowsian expansion. Whether and how our analysis established here could be generalized to realistic NSs with vanishing surface density is a challenging issue that is beyond the scope of the present paper. We hope that our work reported here could trigger more other investigations along a similar direction. On the other hand, we also note that Sham et al. [23] have already used the GTM, whose density profile is given by (1.1), to study the I-Love relation in the Newtonian case. They showed that the I-Love relation remains almost unchanged (with relative error less than 0.001) as the parameter $\delta$ in (1.1) goes from 0 to 1 (see Fig. 5 in their paper). In particular, the I-Love relation are exactly stationary with respect to changes in $\delta$ about the two points $\delta = 0$ and $\delta = 1$. While the former (i.e., $\delta = 0$) corresponds to ISs, the latter (i.e., $\delta = 1$) actually reduces to the Tolman VII model, which has been proposed by Lattimer and Prakash [1] as an approximately universal density profile for realistic NSs. Moreover, for $0 < \delta < 1$, the GTM could also nicely reproduce the leading behavior of the density profile of SBSs as given by (1.1). Inasmuch as the I-Love relation is considered in the Newtonian limit, ISs, SBSs and realistic NSs are deemed equivalent to each other. We therefore hold a positive view on the possibility of generalizing our method to realistic NSs.

Lastly, in the “I-Love-Q universal relations” discovered in Refs. [4, 5] the spin-induced quadrupole moment $Q$ (with suitable normalization) of compact stars is also a member of the trio which display universal behavior. However, in order to performed a detailed analysis on the physical nature of the I-Love relation without further ado, we have not addressed the issue of the spin-induced quadrupole in the present paper. Nevertheless, we expect that our method can be generalized to handle the case of the spin-induced quadrupole without much difficulty. As a matter of fact, in the Newtonian limit, the spin-induced quadrupole moment $Q$ (with suitable normalization) is directly proportional to the tidal quadrupole moment, which is measured by the tidal Love number, whose universal behavior is well studied here. Therefore, we expect that the universal behavior of the spin-induced quadrupole can be explained in a way similar to the case of the other two members of the trio. The work in such an direction is underway and relevant details will be reported elsewhere in due course.

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Appendix A: Perturbative expansion for Quark Stars

To illustrate and gauge the accuracy of the perturbative expansion developed in the present paper, we apply the method to study the stellar structure of QSs obeying the simple MIT bag model (5.1). For such a linear EOS, it is
readily shown that under the transformations \( \bar{\rho} = \rho/B, \bar{\tilde{\rho}} = \rho/B, \bar{r} = B^{1/2}r, \bar{R} = B^{1/2}R \) and \( \bar{M} = B^{1/2}M \) the TOV equations are independent of \( B \) and hence acquire a scale-invariant form.

The following shows the leading expansion of \( \bar{\rho}, \bar{\tilde{\rho}} \) and \( \epsilon^\nu \):

\[
\bar{\rho}(x) = 4 + 6 \left( 1 - x^2 \right) C + \frac{12}{5} \left( 1 - x^2 \right) \left( 8 - 3x^2 \right) C^2 + \cdots, \tag{A1}
\]

\[
\bar{\tilde{\rho}}(x) = 2 \left( 1 - x^2 \right) C + \frac{4}{5} \left( 1 - x^2 \right) \left( 8 - 3x^2 \right) C^2 + \cdots, \tag{A2}
\]

\[
\epsilon^\nu(x) = 1 - \left( 3 - x^2 \right) C + \frac{3}{10} \left( 1 - x^2 \right)^2 C^2 + \cdots. \tag{A3}
\]

For reference, analytic expressions of \( \bar{\rho}_n(x) \equiv \rho_n(x)/B \) and \( (\epsilon^\nu)_n(x) \) for \( 0 \leq n \leq 6 \) are given in Tables III and IV respectively. We have numerically verified that for stable QSs (i.e., \( \mathcal{C} < 0.275 \)), all of these series tend to their exact values as the order of expansion goes to infinity. In general, the percentage error of an expansion with a fixed order grows towards the stellar center with strong gravity. Besides, we also find that the \( m \)-th order diagonal Padé approximant about the point \( \mathcal{C} = 0 \) (\( m = 1, 2, \cdots \)) for \( \bar{\rho} \) and \( \epsilon^\nu \) constructed from their respective \( 2m \)-th order partial sums (see, e.g., [17] for the theory and the construction of Padé approximants) can significantly improve their rate of convergence.

TABLE III: The coefficient of the \( x^j \)-term in the polynomials \( \bar{\rho}_n \equiv \rho_n/B \) for \( 0 \leq n \leq 6 \). There is no odd power term and the coefficient vanishes if \( j > 2n \).

| \( x^0 \) | \( x^2 \) | \( x^4 \) | \( x^6 \) | \( x^8 \) | \( x^{10} \) | \( x^{12} \) |
|------|------|------|------|------|------|------|
| \( \bar{\rho}_0 \) | 4 | | | | | |
| \( \bar{\rho}_1 \) | 6 | | | | | |
| \( \bar{\rho}_2 \) | \( \frac{96}{5} \) | \( \frac{-132}{5} \) | \( 36 \) | | | |
| \( \bar{\rho}_3 \) | \( \frac{10863}{175} \) | \( \frac{-18036}{175} \) | \( 1269 \) | \( 26 \) | | |
| \( \bar{\rho}_4 \) | \( \frac{7047}{35} \) | \( \frac{-71331}{175} \) | \( 1845 \) | \( \frac{-10797}{175} \) | \( 768 \) | | |
| \( \bar{\rho}_5 \) | \( \frac{882097127}{14347500} \) | \( \frac{-509998846}{346875} \) | \( \frac{1594989}{1250} \) | \( \frac{-24492844}{500} \) | \( 273333 \) | | |
| \( \bar{\rho}_6 \) | \( \frac{186826212171}{87587500} \) | \( \frac{-485894355907}{87587500} \) | \( \frac{2463177573}{481250} \) | \( \frac{-132119397}{6125} \) | \( \frac{45389929}{122500} \) | | |

On the other hand, we note that for a given order of post-Minkowskian expansion, the fractional error of (A1) is usually several times larger than that of (A3). In fact, the fractional error of the former is almost ten times of that of the latter (see Ref. [39] for details). Hence, it is desirable to find a way to express \( \bar{\rho} \) in terms of \( \epsilon^\nu \), which has a series expansion with higher precision for the same order of expansion. To this end, we find from the TOV equations an exact expression for \( \bar{\rho} \),

\[
\bar{\rho} = 3(1 - 2C)^2 e^{-2\nu} + 1, \tag{A4}
\]

where the boundary conditions \( \epsilon^\nu(x = 1) = 1 - 2C \) and \( \bar{\rho}(x = 1) = 4 \) have been used to fix some constants. Therefore, \( \bar{\rho} \) can be found directly from \( \epsilon^\nu \), i.e., [A3]. Even at the center of the star, the percentage error in \( \bar{\rho} \) obtained from the sixth-order post-Minkowskian expansion of \( \epsilon^\nu \) and (A3) is less than 5%. If, instead, a 3-3 diagonal Padé approximant for \( \epsilon^\nu \) is constructed from its sixth-order post-Minkowskian expansion and (A3) is used in tandem to find \( \bar{\rho} \), the percentage error of \( \bar{\rho} \) is much smaller than 1% throughout the whole star.

Based on stellar profile obtained above, the mass \( M \) and the radius \( R \) of QSs can be readily found as follows:

\[
R = \Theta \left( \frac{3C}{16\pi B} \right)^{1/2}, \tag{A5}
\]

\[
M = \Theta \left( \frac{3C^3}{16\pi B} \right)^{1/2}. \tag{A6}
\]
TABLE IV: The coefficient of the \(x^j\)-term in the polynomials \((e^n)\) for \(0 \leq n \leq 6\). There is no odd power term and the coefficient vanishes if \(j > 2n\).

| \((e^n)\) | \(x^0\) | \(x^2\) | \(x^4\) | \(x^6\) | \(x^8\) | \(x^{10}\) | \(x^{12}\) |
|---------|---------|---------|---------|---------|---------|---------|---------|
| \((e^0)\) | 1       |         |         |         |         |         |         |
| \((e^1)\) | −3      | 1       |         |         |         |         |         |
| \((e^2)\) | \(\frac{-3}{10}\) | \(\frac{-3}{5}\) | \(\frac{-3}{10}\) |         |         |         |         |
| \((e^3)\) | \(\frac{27}{175}\) | \(\frac{-123}{350}\) | \(\frac{6}{25}\) | \(\frac{-3}{70}\) |         |         |         |
| \((e^4)\) | \(\frac{-441}{1400}\) | \(\frac{1}{70}\) | \(\frac{579}{700}\) | \(\frac{-183}{350}\) | \(\frac{-1}{40}\) |         |         |
| \((e^5)\) | \(\frac{-2426307}{1347500}\) | \(\frac{3361973}{2695000}\) | \(\frac{23007}{8750}\) | \(\frac{-43917}{24500}\) | \(\frac{-20}{69}\) | \(\frac{1}{11}\) |         |
| \((e^6)\) | \(\frac{-2343752949}{350350000}\) | \(\frac{86948269}{15925000}\) | \(\frac{32187921}{8625625}\) | \(\frac{-774423}{122500}\) | \(\frac{-43917}{24500}\) | \(\frac{-20}{69}\) | \(\frac{-5604}{2002000}\) |

where

\[
\Theta = \left[ \frac{16\pi}{(3I)} \right]^{1/2} = 1 - \frac{3}{10}C - \frac{939}{1400}C^2 - \frac{21977}{14000}C^3 - \frac{32997091}{8624000}C^4 - \frac{5428534499}{5605600000}C^5 - \frac{181195465167}{71344000000}C^6 + \cdots.
\]

Figure 5 shows \(\bar{R}\) as a function \(\bar{M}\), which is obtained by combining (A5), (A6) and (A7). It is clearly shown there that the sixth-order post-Minkowsian expansion of \(\Theta\) (the continuous solid line) can accurately reproduce the exact numerical result (the dots) as long as \(\bar{M}\) is less than its maximum value beyond which QSs become unstable.

Similarly, we can also find the normalized moment of inertia \(a\):

\[
a = \frac{2}{5} + \frac{48}{175}C + \frac{656}{2625}C^2 + \frac{40408}{202125}C^3 - \frac{883424}{21896875}C^4 - \frac{24137984}{25265625}C^5 - \frac{339641200144}{83755546875}C^6 + \cdots.
\]

Figure 6 compares the values of the normalized moment of inertia \(a\) obtained from the sixth-order post-Minkowsian expansion (the continuous solid line) and numerical integration (the dots), respectively. We see that the agreement between the perturbative and numerical results is almost perfect unless the compactness is close to the stability limit \(C_{\text{max}} = 0.275\).

We note that an empirical formula

\[
a = 2(1 + 0.677C)/5
\]

has been proposed previously by fitting the numerical data of the moment of inertia of QSs constructed with several different quark matter EOS, which are basically the MIT bag model \((5.1)\) plus some minor corrections \((28, 40)\). Comparing the analytic result in (A8), namely \(a = 2(1 + 0.686C + 0.625C^2 + \cdots)/5\), with the empirical one, we see that the empirical formula (shown by the dashed straight line in Fig. 6) is a good approximation at low compactness because to order \(C\) it is almost identical to our analytical result. However, as shown in Fig. 6, it fails to follow the nonlinear behavior of the normalized moment of inertia for QSs with large compactness, say, \(C > 0.2\), reflecting the importance of the higher order terms in (A8) in such situation.

Lastly, we express the tidal Love number \(k_2\) in terms of the sixth-order post-Minkowsian expansion

\[
k_2 = (1 - 2C)^2 \left\{ \frac{3}{4} \frac{45C}{28} + \frac{187C^2}{2940} - \frac{1141589C^3}{1131900} + \frac{273911383C^4}{103002900} - \frac{374294273707C^5}{54076522500} - \frac{6469553810716163C^6}{353930839762500} + \cdots \right\},
\]

and accordingly a plot of \(k_2\) versus \(C\) is given in Fig. 7. Again we see that the agreement between the numerical and
perturbative results is nice.

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FIG. 1: (Upper panel) $\log_{10} \tilde{I}$ is plotted against $\log_{10} \tilde{\lambda}$ for ISs and QSs. The numerical data (circles for ISs and squares for QSs) are compared with the analytical results obtained from the 7-term post-Minkowskian expansions (5.4) (for ISs, the continuous line) and (5.8) (for QSs, the dashed line). (Lower panel) The logarithm of the relative error between the numerical data and series expansion results, $E \equiv \left| \frac{\tilde{I}_{\text{series}}}{\tilde{I}_{\text{data}}} - 1 \right|$, is shown against $\log_{10} \tilde{\lambda}$ (with circles for ISs and squares for QSs).
FIG. 2: The logarithm of the relative difference between the scaled moment of inertia of Qs and Is, $\bar{I}_{QS}$ and $\bar{I}_{IS}$, is plotted against the compactness $C$. The analytic results obtained from the leading 7-term post-Minkowskian expansion given by (5.2) and (5.6) (denoted by the continuous line) agree nicely with the numerical result (denoted by circles). The relative difference is also analyzed in terms of individual contributions due to the 1st, 2nd, ..., 6th post-Minkowskian correction terms in (5.2) and (5.6). It is clearly shown that the first-order post-Minkowskian correction term (the dot-dashed line) is close to the total contribution (the continuous line) of the leading six correction terms.
FIG. 3: The logarithm of the relative difference between the dimensionless tidal deformability of QSs and ISs, $\bar{\lambda}_{QS}$ and $\bar{\lambda}_{IS}$, which are obtained from the $\lambda_c - C$ relations (see (5.3) and (5.7)) and (5.5), is plotted against the compactness $C$. The analytic results obtained from the leading 7-term post-Minkowskian expansion given by (5.3) and (5.7) (denoted by the continuous line) agree nicely with the numerical result (denoted by circles). The relative difference is also analyzed in terms of individual contributions due to the 1st, 2nd, ..., 6th post-Minkowskian correction terms in (5.3) and (5.7). It is clearly shown that the first-order post-Minkowskian correction term (the dot-dashed line) is close to the total contribution (the continuous line) of the leading six correction terms.
FIG. 4: The logarithm of the relative difference between the scaled moment of inertia of QSs and ISs, $\bar{I}_{QS}$ and $\bar{I}_{IS}$, is plotted against $\log_{10}\bar{\lambda}$. The analytic results obtained from the leading 7-term post-Minkowskian expansion given by (5.4) and (5.8) (denoted by the continuous line) agree nicely with the numerical result (denoted by circles). The relative difference is also analyzed in terms of individual contributions due to the 2nd, ..., 6th post-Minkowskian correction terms in (5.4) and (5.8). In this case the contribution arising from the first-order post-Minkowskian correction term vanishes and is not shown in the figure.
FIG. 5: $\bar{R}$ is plotted against $\bar{M}$ by combining (A5), (A6) and the sixth-order post-Minkowsian expansion of $\Theta$ (see (A7)). The perturbative expansion (the continuous solid line) can accurately reproduce the exact numerical result (the dots) for $\bar{M}$ less than the stability limit beyond which QSs become unstable.

FIG. 6: The values of the normalized moment of inertia $a$ obtained from the sixth-order post-Minkowsian expansion (the continuous solid line) and numerical integration (the dots), respectively, are plotted against compactness $C$ and compared with each other. For reference, we also include in the figure the value of $a$ obtained from the empirical formula (A9) (the dashed straight line).
FIG. 7: The values of the Love number $k_2$ obtained from the sixth-order post-Minkowsian expansion shown in (A10) (the continuous solid line) and numerical integration (the dots), respectively, are plotted against compactness $\mathcal{C}$ and compared with each other.