DIMENSION EXPANDERS VIA QUIVER REPRESENTATIONS

MARKUS REINEKE

Abstract. We relate the notion of dimension expanders to quiver representations and their general subrepresentations, and use this relation to establish sharp existence results.

1. Introduction

Let $F$ be a field, and let $\varepsilon > 0$ be a real number. An $\varepsilon$-expander is a tuple $(V, T_1, \ldots, T_k)$, consisting of a finite-dimensional $F$-vector space $V$, together with linear operators $T_1, \ldots, T_k$ on $V$, such that, for all subspaces $U \subset V$ of dimension $\dim U \leq \frac{1}{2} \dim V$, we have

$$\dim(U + \sum_{i=1}^k T_i(U)) \geq (1 + \varepsilon) \dim U.$$ 

This is a linear algebra analogue of the notion of expander graph [7]. It is proven in [8] for fields of characteristic zero, and in [1, 2] for finite fields, that there exist $k$ and a fixed $\varepsilon > 0$ such that $\varepsilon$-expanders $(V, T_1, \ldots, T_k)$ exist for all dimensions of the $F$-vector space $V$.

In the present article, we sharpen this existence result and determine the optimal expansion coefficient $\varepsilon$ for $F$ an algebraically closed field.

Theorem 1.1. Let $F$ be an algebraically closed field, let $k \geq 2$, and define

$$\varepsilon_k = (k + 1 - \sqrt{k^2 - 2k + 5})/2.$$ 

Then there exist $\varepsilon$-expanders $(V, T_1, \ldots, T_k)$ in all dimensions of $V$ if and only if $\varepsilon \leq \varepsilon_k$.

We will derive this result from a description of dimension vectors of subrepresentations of general representations of generalized Kronecker quivers; in particular, our proof will be non-constructive. However, this technique allows us to also cover the case of unbalanced dimension expanders of [5], and to speculate on potential generalizations of the notion of dimension expanders to arbitrary quivers.

We will review the necessary quiver techniques in Section 2. In Section 3 we describe the dimension vectors of general representations of generalized Kronecker quivers. The applications to dimension expanders are derived in Section 4. Finally, in Section 5 we discuss potential generalizations to arbitrary quivers.

Acknowledgments: The author is grateful to M. Bertozzi and K. Martinez for many helpful comments on this manuscript. This work is supported by the DFG SFB-TRR 191 “Symplectic structures in geometry, algebra and dynamics”.

1
2. Recollections on quiver representations

From now on, let $F$ be an algebraically closed field. Let $Q$ be a finite quiver with set of vertices $Q_0$ and arrows written $\alpha : i \to j$ for $i, j \in Q_0$, which we assume to be acyclic. We define the Euler form of $Q$ on $\mathbb{Z}Q_0$ by
\[
\langle d, e \rangle = \sum_{i \in Q_0} d_ie_i - \sum_{\alpha : i \to j} d_ie_j
\]
for $d = (d_i)_i$ and $e = (e_i)_i$ in $\mathbb{Z}Q_0$. We consider the category $\text{rep}_F Q$ of finite dimensional $F$-representations of $Q$, which is an abelian $F$-linear hereditary finite length category. Its Grothendieck group identifies with $\mathbb{Z}Q_0$ by associating to a representation $V$ its dimension vector $\text{dim} V$, and its homological Euler form is given by the Euler form:
\[
\dim \text{Hom}(V, W) - \dim \text{Ext}^1(V, W) = \langle \text{dim} V, \text{dim} W \rangle
\]
for all representations $V$ and $W$.

For $d \in \mathbb{N}Q_0$, we fix $F$-vector spaces $V_i$ of dimension $d_i$ for all $i \in Q_0$. We define the representation space
\[
R_d(Q) = \bigoplus_{\alpha : i \to j} \text{Hom}_F(V_i, V_j),
\]
whose points $(f_\alpha)_\alpha$ we identify with the corresponding representation of $Q$ on the vector spaces $V_i$. On the $F$-vector space $R_d(Q)$, the reductive linear algebraic group
\[
G_d = \prod_{i \in Q_0} \text{GL}(V_i)
\]
aracts linearly via
\[
(g_i)_i (f_\alpha)_\alpha = (g_j f_\alpha g_i^{-1})_{\alpha : i \to j}
\]
such that the $G_d$-orbits $O_V$ in $R_d(Q)$ naturally correspond to the isomorphism classes $[V]$ of $F$-representations $V$ of $Q$ of dimension vector $d$.

For $e \leq d$ componentwise, the subset of $R_d(Q)$ of all representations $V$ admitting a subrepresentation of dimension vector $e$ is Zariski-closed. Therefore, almost all representations in $R_d(Q)$ (that is, those in a Zariski-dense subset) admit a subrepresentation of dimension vector $e$ if and only if all representations in $R_d(Q)$ do so. In this case, we write $e \hookrightarrow d$. There is a recursive numerical criterion for this notion due to Schofield in characteristic zero, generalized to positive characteristic by Crawley-Boevey:

**Theorem 2.1** ([9, 3]). We have $e \hookrightarrow d$ if and only if $\langle e', d - e \rangle \geq 0$ for all $e' \hookrightarrow e$.

3. General subrepresentations of representation of generalized Kronecker quivers

Our main result in this section, which will directly apply to dimension expanders, is a non-recursive description of the relation $e \hookrightarrow d$ for generalized Kronecker quivers $K(m)$, given by two vertices $1, 2$, and $m \geq 2$ arrows from $1$ to $2$. We prepare this description by some preliminary results.

**Lemma 3.1.** For dimension vectors of $K(m)$, we have $(e_1, e_2) \hookrightarrow (d_1, d_2)$ if and only if $(d_2 - e_2, d_1 - e_1) \hookrightarrow (d_2, d_1)$.
Proof. For a representation \( V \) of \( K(m) \) given by an \( m \)-tuple \( f_1, \ldots, f_m : V_1 \to V_2 \) of linear maps, we denote by \( V^* \) the representation \( f_1^*, \ldots, f_m^* : V_2^* \to V_1^* \). This obviously defines a duality on \( \text{rep}_F K(m) \). Assume that a general representation \( V \) of dimension vector \( d \) admits a subrepresentation \( U \) of dimension vector \( e \). Then, dually, a general representation \( V^* \) of dimension vector \( (d_2, d_1) \) admits a factor representation of dimension vector \( (e_2, e_1) \), whose kernel is a subrepresentation of dimension vector \( (d_2 - e_2, d_1 - e_1) \). This finishes the proof.

We now extend the Euler form \( \langle \_ , \_ \rangle \) of \( Q = K(m) \) to \( \mathbb{R}Q_0 \). We fix \( 0 \neq d = (d_1, d_2) \in \mathbb{N}Q_0 \) such that \( 0 \geq \langle d, d \rangle = d_1^2 + d_2^2 - md_1d_2 \). In particular, \( d_1, d_2 \geq 1 \), and
\[
(m - \sqrt{m^2 - 4})/2 \leq d_2/d_1 \leq (m + \sqrt{m^2 - 4})/2 =: \beta.
\]
For fixed \( x \in [0, d_1] \), we consider the function
\[
q_x(y) = \langle (x, y), (d_1 - x, d_2 - y) \rangle
\]
on \([0, d_2]\), and denote by \( c_d \) the smaller of its two zeroes. The explicit form
\[
c_d(x) = \left( mx + d_2 - \sqrt{(mx - d_2)^2 + 4x(d_1 - x)} \right)/2
\]
shows existence. In particular, \( c_d(x) \leq (mx + d_2)/2 \), and \( mx + d_2 - c_d(x) \) is the larger zero of \( q_x \). We have \( q_x(y) \geq 0 \) for \( c_d(x) \leq y \leq mx + d_2 - c_d(x) \), and \( q_x(y) \leq 0 \) otherwise. We have the following estimate:

**Lemma 3.2.** If \( \langle d, d \rangle \leq 0 \), we have \( d_2/d_1 \cdot x \leq c_d(x) \leq \min(mx, d_2) \).

**Proof:** We have
\[
q_x(d_2/d_1 \cdot x) = \langle x/d_1 \cdot d, d - x/d_1 \cdot d \rangle = x(d_1 - x)/d_1^2 \cdot \langle d, d \rangle \leq 0
\]
by assumption. Thus the first inequality follows, since \( c_d(x) \leq (mx + d_2)/2 \), once we know that \( d_2/d_1 \cdot x \leq (mx + d_2)/2 \). If \( d_2/d_1 \leq m/2 \), this holds trivially. Otherwise, we use \( d_2 \leq md_1 \) to estimate
\[
(d_2/d_1 - m/2)2x \leq (d_2/d_1 - m/2)d_1 = d_2/2 + (d_2 - md_1)/2 \leq d_2/2,
\]
and again the desired estimate follows.

For the second inequality, we calculate
\[
q_x(mx) = \langle x \cdot (1, m), d - x \cdot (1, m) \rangle = x(d_1 - x) \geq 0,
\]
thus
\[
c_d(x) \leq mx \leq mx + d_2 - c_d(x),
\]
which finishes the proof.

**Lemma 3.3.** If \( d_2 > \beta d_1 \) and \( e \hookrightarrow d \), then \( e_2 > \beta e_1 \).

**Proof:** \( e \hookrightarrow d \) implies \( \langle e, d - e \rangle \geq 0 \) by Schofield’s criterion, thus \( e_2 \geq c_d(e_1) \) by definition of \( c_d \). It thus suffices to prove that \( c_d(x) > \beta x \) provided \( d_2 > \beta d_1 \). Since \( \beta^2 - m\beta + 1 = 0 \), we have
\[
q_x(\beta x) = x(d_1 - (m - \beta)d_2) < xd_1(1 - m\beta + \beta^2) = 0,
\]
from which we can conclude \( c_d(x) > \beta x \) provided \( \beta x \leq mx + d_2 - c_d(x) \). But
\[
\beta x < mx \leq mx + d_2 - c_d(x)
\]
since \( c_d(x) \leq d_2 \).

We can now derive the main result of this section:
Proposition 3.4. If $Q = K(m)$ is the $m$-arrow Kronecker quiver $\bullet \Rightarrow \bullet$ and $(d, d) \leq 0$, then for $e \leq d$ the following are equivalent:

1. $e \rightarrow d$,
2. $\langle e, d - e \rangle \geq 0$,
3. $e_2 \geq c_d(e_1)$.

Proof. Without loss of generality, we assume $d_1 \leq d_2$ using the duality Lemma 3.1. Obviously (1) implies (2) implies (3): if $e \rightarrow d$ then, by Schofield’s criterion applied to $e' = e$, we have $\langle e, d - e \rangle \geq 0$, and by definition of the function $c_d$, this implies $e_2 \geq c_d(e_1)$. Conversely, assume that this inequality holds, and let $e' \rightarrow e$. We want to prove that $(e', d - e) \geq 0$; then $e \rightarrow d$ follows from Schofield’s criterion. We first assume $(e, e) \geq 1$, thus $e_2 > \beta e_1$ or $e_2 < (m - \beta)e_1$. Since $e_2 \geq c_d(e_1) \geq d_2/d_1 e_1 \geq e_1$ by assumption and Lemma 3.2, we have $e_2 > \beta e_1$ since $\beta \geq 1$. By Lemma 3.3, we find $e'_2 > \beta e'_1$, and thus

$\langle e', d - e \rangle = e'_1 (d_1 - e_1 - m (d_2 - e_2)) + e'_2 (d_2 - e_2) > e'_1 (d_1 - e_1 - (m - \beta)(d_2 - e_2))$.

Since $d_2 \leq \beta d_1$, we have $d_2 - e_2 < \beta (d_1 - e_1)$, thus $d_1 - e_1 > (m - \beta)(d_2 - e_2)$, proving the claim. Now we assume $(e, e) \leq 0$. By Lemma 3.2, we have

$e'_2 \geq c_e(e'_1) \geq e_2/e_1 \cdot e'_1$,

and thus

$\langle e', d - e \rangle = e'_1 (d_1 - e_1 - m (d_2 - e_2)) + e'_2 (d_2 - e_2) \geq e'_1/e_1 \cdot (e_1 (d_1 - e_1) - m e_1 (d_2 - e_2)) + e_2 (d_2 - e_2) = e'_1/e_1 \cdot (e, d - e) \geq 0$,

again proving the claim.

4. Application to dimension expanders

We generalize the definition of dimension expanders of Section 1 to a notion of expander representation. Proposition 3.4 then almost immediately yields sharp existence results.

Definition 4.1. Let $0 < \delta < 1$ and $\varepsilon > 0$, and let $V$ and $W$ be non-zero finite-dimensional $F$-vector spaces. We call a representation $f_1, \ldots, f_m : V \rightarrow W$ of $K(m)$ a $(\delta, \varepsilon)$-expander representation if for all subspaces $0 \neq U \subset V$ such that $\dim_{\dim V} U \leq \delta$, we have

$$\dim \sum_{k=1}^{m} f_k(U) \geq (1 + \varepsilon) \cdot \frac{\dim W}{\dim V} \cdot \dim U.$$ 

The following lemma translates existence of expander representations to properties of dimension vectors of subrepresentations of general representations:

Lemma 4.2. For fixed integers $m, d_1, d_2 \geq 1$ and real numbers $0 < \delta < 1$, $\varepsilon > 0$, there exists a $(\delta, \varepsilon)$-expander representation of $K(m)$ of dimension vector $(d_1, d_2)$ if and only if for all $(e_1, e_2) \rightarrow (d_1, d_2)$ such that $e_1 \leq \delta \cdot d_1$, we have

$$e_2 \geq (1 + \varepsilon) \cdot \frac{d_2}{d_1} \cdot e_1.$$

Proof. Assume there exists such an expander representation $M$ given by $f_1, \ldots, f_m : V \rightarrow W$, and assume that $(e_1, e_2) \rightarrow (d_1, d_2)$. Then in particular $M$ admits a subrepresentation of dimension vector $(e_1, e_2)$, that is, there exists a subspace $U \subset V$ of dimension $e_1$ such that $\sum_k f_k(U)$ is of dimension at most $e_2$. On the other hand,
$\sum_k f_k(U)$ is at least of dimension $(1 + \varepsilon) \cdot \frac{d}{d_1} \cdot \dim U$. The claimed inequality for $e_2$ follows. Conversely, assume that the numerical condition is satisfied. Then the set $S_{(e_1, e_2)} \subset R_{(d_1, d_2)}(K(m))$ of representations admitting a subrepresentation of dimension vector $(e_1, e_2)$ is a proper Zariski-closed subset whenever $e_2 < (1 + \varepsilon) \cdot \frac{d}{d_1} \cdot e_1$. Thus the union of all these finitely many proper closed subsets is again a proper subset, and any representation in its complement is a $(\delta, \varepsilon)$-expander representation by definition.

This allows us to establish the following sharp existence result:

**Theorem 4.3.** Fix an integer $m \geq 1$, real numbers $0 < \delta < 1$ and $\varepsilon > 0$, and a rational $\alpha$ such that $\alpha^2 - m\alpha + 1 < 0$ and $m\delta + \alpha - 2\alpha\delta > 0$. Define

$$
\varepsilon_m(\alpha, \delta) = \frac{m\delta + \alpha - 2\alpha\delta - \sqrt{(m\delta - \alpha)^2 + 4\delta(1 - \delta)}}{2\alpha\delta} > 0.
$$

Then there exist $(\delta, \varepsilon)$-expander representations of $K(m)$ for all dimension vectors $(d_1, d_2)$ such that $d_2/d_1 = \alpha$ if and only if $\varepsilon \leq \varepsilon_m(\alpha, \delta)$.

**Proof.** The assumptions on $\alpha$ ensure that $\varepsilon_m(\alpha, \delta) > 0$ by a straightforward calculation. We consider dimension vectors $d$ such that $d_2/d_1 = \alpha$; in particular $(d, d) < 0$. By the previous lemma and Proposition 3.4, we have:

There exists a $(\delta, \varepsilon)$-expander representation of $K(m)$ of dimension vector $d$ if and only if $e_2 \geq (1 + \varepsilon) \cdot e_1$ for all $e_1 \leq \delta d_1$ and all $e_2 \geq c_d(e_1)$, or, equivalently, if $[c_d(x)] \geq (1 + \varepsilon) \cdot x$ for all integral $x \leq \delta \cdot d_1$.

This implies:

There exist $(\delta, \varepsilon)$-expander representations of $K(m)$ for all dimension vectors $(d_1, d_2)$ such that $d_2/d_1 = \alpha$ if and only if $[c_d(x)] \geq (1 + \varepsilon) \cdot x$ for all dimension vectors $d = (d_1, d_2)$ such that $d_2/d_1 = \alpha$ and all integral $x \leq \delta d_1$.

The function $c_d(x)$ is concave on the interval $[0, d_2]$ since, by a straightforward calculation, its second derivative equals

$$
c''_d(x) = \frac{2(d, d)}{((mx - d_2)^2 + 4x(d_1 - x))^{3/2}},
$$

which is negative by assumption. Thus, in the interval $[0, \delta d_1]$, the fraction $c_d(x)/x$ attains its minimum at $\delta d_1$. For $\rho \in [0, 1]$, we have $c_d(\rho d_1)/(\alpha \rho d_1) = 1 + \varepsilon_m(\alpha, \rho)$, thus in the interval $[0, \delta]$, the function $\varepsilon_m(\alpha, \rho)$ of $\rho$ attains its minimum at $\rho = \delta$. We thus find that the above existence condition is equivalent to

$$
[(1 + \varepsilon_m(\alpha, \rho)) \cdot \alpha \rho d_1] \geq (1 + \varepsilon) \cdot \rho d_1
$$

for all $d_1$ such that $\alpha d_1$ is integral and all $\rho \in [0, \delta]$ such that $\rho d_1$ is integral. This is clearly equivalent to $\varepsilon_m(\alpha, \rho) \geq \varepsilon$ for all $\rho \in [0, \delta]$, and this in turn to $\varepsilon_m(\alpha, \delta) \geq \varepsilon$. This finishes the proof.

This result immediately implies Theorem 1.1 as the special case $m = k + 1$, $\alpha = 1$, $\delta = 1/2$. Namely, in a general representation of $K(k + 1)$ of dimension vector $(d, d)$, the map representing the first arrow is invertible, thus w.l.o.g. the identity, and id, $T_1, \ldots, T_k : V \to V$ defines an expander representation if and only if $(V, T_1, \ldots, T_k)$ is a dimension expander; moreover, $\varepsilon_{k+1}(1, 1/2) = \varepsilon_k$. 
5. Potential generalizations

We finish with a few remarks suggesting further directions.

The characterization of dimension vectors \( \mathbf{e} \leftrightarrow \mathbf{d} \) of subrepresentations of general representation by the single quadratic equation \( \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0 \) of Proposition 3.4 is special to the quivers \( K(m) \). Namely, we have the following

**Example 5.1.** For the complete bipartite three-vertex quiver \( \bullet \Rightarrow \bullet \Leftarrow \bullet \), the dimension vector \( \mathbf{d} = (3, 6, 5) \) is a Schur root (even belonging to the fundamental domain), and \( \mathbf{e} = (3, 5, 1) \) fulfills \( \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0 \) (even > 0), but \( \mathbf{e} \nleftrightarrow \mathbf{d} \).

It is natural to ask whether the explicit dimension expanders constructed in [8] using representations of \( \text{SL}_2(\mathbb{Z}) \), and in [1] using monotone expanders, are already \( \varepsilon_k \)-expanders for the optimal expansion coefficients \( \varepsilon_k \).

In another direction, dimension expanders were used in [4] to construct non-hyperfinite families of representations of generalized Kronecker quivers, and it would be interesting to know whether the present methods yield new insights about such families.

Representations \( f_1, \ldots, f_m : V_1 \to V_2 \) such that \( \dim \sum \delta_k \mathbf{f}(U) > \frac{\dim V_2}{\dim V_1} \dim U \) for all proper non-zero subspaces \( U \subset V_1 \) are stable in the sense of Geometric Invariant Theory [6], thus the \((\delta, \varepsilon)\)-expander property might be viewed as a quantitative form of stability. This point of view suggests a generalization to arbitrary quivers:

**Definition 5.2.** Let \( Q \) be a finite quiver, and let \( \Theta \in (\mathbb{R}Q_0)^* \) be a stability function for \( Q \). Let \( \mathbf{d} \in \mathbb{N}Q_0 \) be a dimension vector for \( Q \) such that \( \Theta(\mathbf{d}) = 0 \), and let \( 0 < \delta < 1 \) and \( \varepsilon > 0 \) be reals. We call a representation \( V \) of \( Q \) of dimension vector \( \dim V = \mathbf{d} \) a \((\delta, \varepsilon)\)-expander relative to \( \Theta \) if for all subrepresentations \( U \subset V \) such that \( \dim U \leq \delta \cdot \dim V \), we have \( \Theta(\dim U) \leq -\varepsilon \cdot \dim U \).

Then it is natural to ask for uniform expansion:

**Question 5.3.** For which \( \Theta, \delta \) and \( \varepsilon \) is it true that there exist \((\delta, \varepsilon)\)-expander representations relative to \( \Theta \) for all (resp. all sufficiently large) dimension vectors \( \mathbf{d} \) such that \( \Theta(\mathbf{d}) = 0 \)?

**References**

[1] Bourgain, Jean, *Expanders and dimensional expansion*. C. R. Math. Acad. Sci. Paris 347 (2009), no. 7–8, 357–362.

[2] Bourgain, Jean; Yehudayoff, Amir, *Expansion in \text{SL}_2(\mathbb{R}) and monotone expanders*. Geom. Funct. Anal. 23 (2013), no. 1, 1–41.

[3] Crawley-Boevey, William, *Subrepresentations of general representations of quivers*. Bull. London Math. Soc. 28 (1996), no. 4, 363–366.

[4] Eckert, Sebastian, *(Extended) Kronecker quivers and amenability*. arXiv:2011.02040

[5] Guruswami, Venkatesan; Resch, Nicolas; Xing, Chaoping, *Lossless dimension expanders via linearized polynomials and subspace designs*. Combinatorica 41 (2021), no. 4, 545–579.

[6] King, Alastair, *Moduli of representations of finite dimensional algebras*. Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.

[7] Lubotzky, Alexander, *Expander graphs in pure and applied mathematics*. Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 1, 113–162.

[8] Lubotzky, Alexander; Zelmanov, Efim, *Dimension Expanders*. J. Algebra 319 (2008), no. 2, 730–738.

[9] Schofield, Aidan, *General representations of quivers*. Proc. London Math. Soc. (3) 65 (1992), no. 1, 46–64.